LINEAR STABILITY OF ROTATING BLACK HOLES: OUTLINE OF THE PROOF

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Abstract. After a brief introduction to the black hole stability problem, we outline our recent proof of the linear stability of the non-extreme Kerr geometry.

Contents

1. Introduction

Black holes are peculiar, mysterious and probably the most exciting objects in our universe. They arise mathematically from particular solutions of Einstein's equations of General Relativity (GR). The first solutions describing black holes were found by Karl Schwarzschild in 1915 in his quest to describe stars by spherically symmetric and static solutions of the Einstein equations. Shortly afterward, it was discovered that these solutions exhibit amazing physical properties, the most interesting of which is that nothing, not even light, can escape from their interior. In essence, this property defines a black hole. Recently, astronomers have found strong evidence that black holes are ubiquitous in the sense that they lie at the center of most galaxies. These developments have brought black holes into the spotlight of astronomical research. This has also inspired much interest in the study of theoretical properties of black holes. In particular, the problem of stability of black holes is currently a question of high interest. As Frolov and Novikov put it [17, page 143], this is "one of the few truly outstanding problems that remain in the field of black hole perturbations." We here report on recent work, partly carried out at the Center of Mathematical Sciences and

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Applications at Harvard University, which shows that rotating black holes are indeed stable under small perturbations (linear stability).

2. The Kerr Black Hole

In General Relativity, gravity is described geometrically in terms of the curvature of space-time. Thus space-time is modelled mathematically by a four-dimensional Lorentzian manifold $(\mathscr{M}, g)$ of signature $(+ - - -)$ (for a more detailed introduction see [6] or the textbooks on GR [1, 18, 23, 21]). In GR, Newton's gravitational law is replaced by the Einstein equations

$$R_{jk} - \frac{1}{2}\, R\, g_{jk} = 8\pi\kappa\, T_{jk}\;,$$

where R_{jk} is the Ricci tensor, R is scalar curvature, and κ denotes the gravitational constant. Here T_{jk} is the energy-momentum tensor which describes the distribution of matter in space-time.

A rotating black hole is described by the celebrated Kerr geometry. It is a solution of the vacuum Einstein equations discovered in 1963 by Roy Kerr. In the so-called Boyer-Lindquist coordinates, the Kerr metric is given by (see [3, 19])

$$ds^2 = \frac{\Delta}{U}\,(dt - a \sin^2\vartheta\, d\varphi)^2 - U\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) - \frac{\sin^2\vartheta}{U}\Big(a\, dt - (r^2+a^2)d\varphi\Big)^2,$$

where

$$U = r^2 + a^2\cos^2\vartheta, \qquad \Delta = r^2 - 2Mr + a^2\;,$$

and the coordinates $(t, r, \vartheta, \varphi)$ are in the range

$$-\infty < t < \infty, \quad M + \sqrt{M^2 - a^2} < r < \infty, \quad 0 < \vartheta < \pi, \quad 0 < \varphi < 2\pi\;.$$

Here the parameters M and aM describe the mass and the angular momentum of the black hole. It is easily verified that in the case $a = 0$, one recovers the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\;.$$

For the Kerr metric to describe a physical black hole we need to assume that $M^2 > a^2$, giving a bound for the angular momentum relative to the mass. In this so-called *non-extreme case*, the hypersurface

$$r = r_1 := M + \sqrt{M^2 - a^2}$$

defines the *event horizon* of the black hole. For $r < r_1$, the radial coordinate r becomes time, whereas t becomes a spatial coordinate. Since time always propagates to the future, the event horizon can be regarded as the "boundary of no escape." Here we shall only consider the region $r > r_1$ outside the event horizon. The coefficients of the Kerr metric are independent of t and φ, showing that the space-time geometry is *stationary* and *axi-symmetric*. One of the key features of the Kerr geometry is the existence of an *ergosphere*, a region which lies outside the event horizon and in which the vector field $\frac{\partial}{\partial t}$ is *space-like*.

It is a major open problem whether the Kerr geometry is stable under small perturbations. This question is of physical relevance because if black holes were unstable, they would decay and should therefore most probably not be observable in our universe. A possible scenario is that a rotating black hole loses more and more angular momentum by emitting gravitational radiation. If this were the case, the Kerr black hole with $a > 0$ would be unstable, because asymptotically for large times it would settle down to the static and spherically symmetric Schwarzschild solution with $a = 0$.

When analyzing the stability of the Kerr solution, one must specify which type of perturbations are considered. Perturbing in the class of vacuum solutions corresponds to perturbations by *gravitational radiation*. But one can also consider solutions with matter, for example by considering the Einstein-Maxwell equations or the Einstein equations coupled to other matter fields. For the stability problem, one considers the Cauchy problem for initial data which is close to a Kerr solution (in a suitable weighted Sobolev norm). Then the question is whether the resulting solution settles down to a Kerr solution asymptotically for large time. Due to the nonlinearity of the Einstein equations, this problem is very difficult. As a first step, one can linearize the equations around a Kerr solution and analyze the long-time behavior of the resulting linear wave equations. This is referred to as the *linear stability problem* for the Kerr black hole.

The problem of linear stability of the Schwarzschild black hole has a long history. It goes back to the study by Regge-Wheeler [20] who showed that an integral norm of the perturbation of each angular mode is bounded uniformly in time. Decay of these perturbations was first proved in [16]. For the Kerr black hole, linear stability has been an open problem for many years. We now state our recent result on the linear stability of the Kerr black hole and give an outline of the proof.

3. The Teukolsky Equation and its Separation

The problem of linear stability of black holes can be stated mathematically as the question of whether solutions of massless linear wave equations in the Kerr geometry decay in time. The different types of equations are characterized systematically in the Newman-Penrose formalism by their spin s, taking the possible values $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. From the physical point of view, the most interesting cases are $s = 1$ (Maxwell field) and $s = 2$ (gravitational waves). A general framework for analyzing the equations of arbitrary spin in the Kerr geometry is due to Teukolsky [22], who showed that the massless equations of any spin can be rewritten as a single wave equation for a complex scalar field ϕ. The Teukolsky equation reads

$$\begin{aligned}\Bigg(\frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} - \frac{1}{\Delta} \Big\{ (r^2 + a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \varphi} - (r - M)\, s \Big\}^2 - 4s\, (r + ia \cos\vartheta) \frac{\partial}{\partial t} \\ + \frac{\partial}{\partial \cos\vartheta} \sin^2\vartheta \frac{\partial}{\partial \cos\vartheta} + \frac{1}{\sin^2\vartheta} \Big\{ a \sin^2\vartheta \frac{\partial}{\partial t} + \frac{\partial}{\partial \varphi} + is \cos\vartheta \Big\}^2 \Bigg) \phi = 0 \,.\end{aligned}$$

We consider the Cauchy problem for the Teukolsky equation. Thus we seek a solution ϕ of the Teukolsky equation for given initial data

$$\phi|_{t=0} = \phi_0 \qquad \text{and} \qquad \partial_t \phi|_{t=0} = \phi_1 \,.$$

Being a linear hyperbolic PDE, the Cauchy problem for the Teukolsky equation has unique global solutions. Also, taking smooth initial data, the solution is smooth for all time. Our main interest is to show that solutions decay for large time. In order to avoid specifying decay assumptions at the event horizon and at spatial infinity, we restrict attention to compactly supported initial data outside the event horizon,

$$\phi_0, \phi_1 \in C^\infty_0\big((r_1, \infty) \times S^2 \big) \,. \tag{3.1}$$

Since the Kerr geometry is axisymmetric, the Teukolsky equation decouples into separate equations for each azimuthal mode. Therefore, the solution of the Cauchy problem is obtained by solving the Cauchy problem for each azimuthal mode and taking the sum of the resulting solutions. With this in mind, we restrict attention to the Cauchy problem for a single azimuthal mode, i.e.

$$\phi_0(r,\vartheta,\varphi) = e^{-ik\varphi}\, \phi_0^{(k)}(r,\vartheta)\,, \qquad \phi_1(r,\vartheta,\varphi) = e^{-ik\varphi}\, \phi_1^{(k)}(r,\vartheta) \tag{3.2}$$

for given $k \in \mathbb{Z}/2$ (if s is half integer, then so is k). The main result of [15] is stated as follows:

Theorem 3.1. *Consider a non-extreme Kerr black hole of mass M and angular momentum aM with $M^2 > a^2 > 0$. Then for any $s \geq \frac{1}{2}$ and any $k \in \mathbb{Z}/2$, the solution of the Teukolsky equation with initial data of the form* (3.1) *and* (3.2) *decays to zero in $L^\infty_{\text{loc}}((r_1,\infty)\times S^2)$.*

This theorem establishes in the dynamical setting that the non-extreme Kerr black hole is linearly stable.

Before outlining the proof of this theorem, we remark that in the case $a = 0$ of the Schwarzschild geometry, the above result was already obtained in the paper [10]; this was our starting point for attacking the problem with $a > 0$.

Restricting attention to a fixed azimuthal mode, the Teukolsky equation becomes

$$\begin{aligned}\bigg(\frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} &- \frac{1}{\Delta} \left\{ (r^2+a^2)\, \frac{\partial}{\partial t} - iak - (r-M)\, s \right\}^2 - 4s\,(r + ia\cos\vartheta)\, \frac{\partial}{\partial t} \\ &+ \frac{\partial}{\partial \cos\vartheta}\, \sin^2\vartheta\, \frac{\partial}{\partial\cos\vartheta} + \frac{1}{\sin^2\vartheta} \left\{ a\sin^2\vartheta\, \frac{\partial}{\partial t} - ik + is\cos\vartheta \right\}^2 \bigg) \phi = 0\,.\end{aligned}$$

This equation can be further separated. Namely, making the separation ansatz

$$\phi(t,r,\vartheta) = \frac{1}{\sqrt{r^2+a^2}}\, e^{-i\omega t}\, X(r)\, Y(\vartheta)\,,$$

the Teukolsky equation gives rise to the coupled system of ODEs

$$\mathcal{R}_\omega X(r) = -\lambda X(r)\,, \qquad \mathcal{A}_\omega Y(\vartheta) = \lambda Y(\vartheta)\,, \tag{3.3}$$

where λ is a separation constant, and the radial operator $\mathcal{R}_\omega$ as well as the angular operator $\mathcal{A}_\omega$ are given by

$$\begin{aligned}\mathcal{R}_\omega = &-\frac{(r^2+a^2)^2}{\Delta} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial_u^2 \sqrt{r^2+a^2}}{\sqrt{r^2+a^2}} \right) \\ &+ \frac{1}{\Delta} \Big(-i\omega\,(r^2+a^2) - iak - (r-M)\, s \Big)^2 - 4isr\omega + 4k\, a\omega \end{aligned}\tag{3.4}$$

$$\mathcal{A}_\omega = -\frac{\partial}{\partial\cos\vartheta}\, \sin^2\vartheta\, \frac{\partial}{\partial\cos\vartheta} + \frac{1}{\sin^2\vartheta} \Big(-a\omega\sin^2\vartheta + k - s\cos\vartheta \Big)^2\,. \tag{3.5}$$

Here the time separation reflects the fact that the Kerr metric is stationary. However, as the Kerr metric is only axisymmetric (and not spherically symmetric), the separation of the ϑ-dependence does not correspond to a symmetry of space-time. This separability goes back to the discovery by Carter for the scalar wave equation [2].

We point out that, in contrast to the φ-dependence (3.2), the separation of the ϑ-dependence cannot be performed for the initial data. The reason is that the angular operator $\mathcal{A}_\omega$ depends on ω. Therefore, the decomposition into ϑ-modes makes it necessary to restrict attention to fixed ω, meaning that the time dependence is that of a plane wave. However, decomposing the wave into a superposition of plane waves $e^{-i\omega t}$ would make it necessary to know the entire time evolution.

4. Hamiltonian Formulation and Integral Representations

In order to analyze the dynamics of the Teukolsky wave, it is useful to work with contour integrals over the resolvent, as we now outline. In preparation, we write the Teukolsky equation in Hamiltonian form. To this end, we introduce the two-component wave function

$$\Psi = \sqrt{r^2+a^2} \begin{pmatrix} \phi \\ i\partial_t \phi \end{pmatrix}$$

and write the Teukolsky equation as

$$i\partial_t \Psi = H\Psi \,, \tag{4.1}$$

where H is a second-order spatial differential operator. We consider H as an operator on a Hilbert space $\mathcal{H}$ with the domain

$$\mathcal{D}(H) = C^\infty_0\big((r_1,\infty)\times S^2, \mathbb{C}^4\big) \,.$$

We point out that the operator H is not symmetric on the Hilbert space $\mathcal{H}$. However, we choose the scalar product on $\mathcal{H}$ as a suitable weighted Sobolev scalar product in such a way that the operator $H - H^*$ is bounded.

If H acted on a finite-dimensional vector space, the Cauchy problem for the equation (4.1) with initial data Ψ_0 could be solved with the Cauchy integral formula by

$$\Psi(t) = -\frac{1}{2\pi i} \oint_\Gamma e^{-i\omega t}\, \big(H-\omega\big)^{-1}\, \Psi_0 \, d\omega \,, \tag{4.2}$$

where Γ is a contour which encloses all eigenvalues of H (note that this formula holds for any matrix H, even if it is not diagonalizable). It turns out that in our infinite-dimensional setting, a similar formula holds. Namely, using the fact that $H - H^*$ is a bounded operator, we prove that the resolvent $R_\omega := (H-\omega)^{-1}$ exists if ω lies outside a strip enclosing the real axis (see [15, Lemma 4.1]):

Lemma 4.1. *For every ω with*

$$|\operatorname{Im}\omega| > c\,,$$

the resolvent $R_\omega = (H-\omega)^{-1}$ exists and is bounded by

$$\|R_\omega\| \leq \frac{1}{|\operatorname{Im}\omega| - c}\,.$$

When forming contour integrals, one must always make sure to stay outside the strip $|\operatorname{Im}\omega| \leq c$, making it impossible to work with closed contours enclosing the spectrum. However, we can work with unbounded contours in the following way (see [15, Corollary 5.3]):

Proposition 4.2. *For any integer $p \geq 1$, the solution of the Cauchy problem for the Teukolsky equation with initial data $\Psi|_{t=0} = \Psi_0 \in \mathcal{D}(H)$ has the representation*

$$\Psi(t) = -\frac{1}{2\pi i} \int_C e^{-i\omega t}\, \frac{1}{(\omega+3ic)^p}\, \Big(R_\omega\, \big(H+3ic\big)^p\, \Psi_0\Big)\, d\omega \,, \tag{4.3}$$

where C is the contour

$$C = \big\{\omega \;\big|\; \operatorname{Im}\omega = 2c\big\} \cup \big\{\omega \;\big|\; \operatorname{Im}\omega = -2c\big\} \tag{4.4}$$

with counter-clockwise orientation.

Here the factor $(\omega + 3ic)^{-p}$ gives suitable decay for large $|\omega|$ and ensures that the integral converges in the Hilbert space $\mathcal{H}$.

5. A Spectral Decomposition of the Angular Teukolsky Operator

The next task is to use the separation of variables in the integrand of our contour integral representation (4.3). Regarding the angular equation (3.3) as an eigenvalue equation, we are led to consider the angular operator $\mathcal{A}_\omega$ in (3.5) as an operator on the Hilbert space

$$\mathcal{H}_k := L^2(S^2) \cap \{e^{-ik\varphi}\, \Theta(\vartheta) \mid \Theta : (0, \pi) \to \mathbb{C}\}$$

with dense domain $\mathcal{D}(\mathcal{A}_\omega) = C^\infty(S^2) \cap \mathcal{H}_k$. Unfortunately, the parameter ω is not real but lies on the contour (4.4). As a consequence, the operator $\mathcal{A}_\omega$ in (3.5) is not symmetric, because its adjoint is given by

$$\mathcal{A}_\omega^* = \mathcal{A}_{\overline{\omega}} \neq \mathcal{A}_\omega \,.$$

The operator $\mathcal{A}_\omega$ is not even a normal operator, making it impossible to apply the spectral theorem in Hilbert spaces. Indeed, $\mathcal{A}_\omega$ does not need to be diagonalizable, because there might be Jordan chains. On the other hand, in order to make use of the separation of variables, we must decompose the initial data into angular modes. This can be achieved by decomposing the angular operator into invariant subspaces of bounded dimension, as is made precise in the following theorem (see [14, Theorem 1.1]):

Theorem 5.1. *Let $U \subset \mathbb{C}$ be the strip*

$$|\mathrm{Im}\,\omega| < 3c \,.$$

Then there is a positive integer N and a family of bounded linear operators Q_n^ω on $\mathcal{H}_k$ defined for all $n \in \mathbb{N} \cup \{0\}$ and $\omega \in U$ with the following properties:

- *(i) The image of the operator Q_0^ω is an N-dimensional invariant subspace of $\mathcal{A}_k$.*
- *(ii) For every $n \geq 1$, the image of the operator Q_n^ω is an at most two-dimensional invariant subspace of $\mathcal{A}_k$.*
- *(iii) The Q_n^ω are uniformly bounded in $\mathrm{L}(\mathcal{H}_k)$, i.e. for all $n \in \mathbb{N} \cup \{0\}$ and $\omega \in U$,*

$$\|Q_n^\omega\| \leq c_2$$

for a suitable constant $c_2 = c_2(s, k, c)$ (here $\|\cdot\|$ denotes the sup*-norm on $\mathcal{H}_k$).*

- *(iv) The Q_n^ω are idempotent and mutually orthogonal in the sense that*

$$Q_n^\omega\, Q_{n'}^\omega = \delta_{n,n'}\, Q_n^\omega \qquad \text{for all } n, n' \in \mathbb{N} \cup \{0\} \,.$$

- *(v) The Q_n^ω are complete in the sense that for every $\omega \in U$,*

$$\sum_{n=0}^\infty Q_n^\omega = \mathbb{1} \tag{5.1}$$

with strong convergence of the series.

6. Invariant Disk Estimates for the Complex Riccati Equation

In order to locate the spectrum of $\mathcal{A}_\omega$, we use detailed ODE estimates. The operators Q_n^ω are then obtained similar to (4.2) as Cauchy integrals,

$$Q_n^\omega := -\frac{1}{2\pi i} \oint_{\Gamma_n} s_\lambda \, d\lambda \,, \qquad n \in \mathbb{N}_0 \,,$$

where the contour Γ_n encloses the corresponding spectral points, and $s_\lambda = (\mathcal{A}_\omega - \lambda)^{-1}$ is the resolvent of the angular operator. What makes the analysis achievable is the fact that $\mathcal{A}_\omega$ is an ordinary differential operator. Transforming the angular equation in (3.3) into Sturm-Liouville form

$$\left(-\frac{d^2}{du^2} + V(u) \right) \phi = 0 \,, \tag{6.1}$$

(where $u = \vartheta$ and $V \in C^\infty((0,\pi), \mathbb{C})$ is a complex potential), the resolvent s_λ can be represented as an integral operator whose kernel is given explicitly in terms of suitable fundamental solutions ϕ_L^{D} and ϕ_R^{D},

$$s_\lambda(u,u') = \frac{1}{w(\phi_L^{\mathrm{D}}, \phi_R^{\mathrm{D}})} \times \begin{cases} \phi_L^{\mathrm{D}}(u)\, \phi_R^{\mathrm{D}}(u') & \text{if } u \leq u' \\ \phi_L^{\mathrm{D}}(u')\, \phi_R^{\mathrm{D}}(u) & \text{if } u' < u \,, \end{cases} \tag{6.2}$$

where $w(\phi_L^{\mathrm{D}}, \phi_R^{\mathrm{D}})$ denotes the Wronskian.

The main task is to find good approximations for the solutions of the Sturm-Liouville equation (6.1) with rigorous error bounds which must be uniform in the parameters ω and λ. These approximations are obtained by "glueing together" suitable WKB, Airy and parabolic cylinder functions. The needed properties of these special functions are derived in [12]. In order to obtain error estimates, we combine several methods:

(a) Osculating circle estimates (see [14, Section 6])
(b) The T-method (see [13, Section 3.2])
(c) The κ-method (see [13, Section 3.3])

The method (a) is needed in order to separate the spectral points of $\mathcal{A}_\omega$ (gap estimates). The methods (b) and (c) are particular versions of *invariant disk* estimates as derived for complex potentials in [11] (based on previous estimates for real potentials in [8] and [5]). These estimates are also needed for the analysis of the radial equation, see Section 7 below. We now explain the basic idea behind the invariant disk estimates.

Let ϕ be a solution of the Sturm-Liouville equation (6.1) with a complex potential V. Then the function y defined by

$$y = \frac{\phi'}{\phi}$$

is a solution of the Riccati equation

$$y' = V - y^2 \,. \tag{6.3}$$

Conversely, given a solution y of the Riccati equation, a corresponding fundamental system for the Sturm-Liouville equation is obtained by integration. With this in mind, it suffices to construct a particular approximate solution $\tilde{y}$ and to derive rigorous error estimates. The invariant disk estimates are based on the observation that the Riccati flow maps disks to disks (see [11, Sections 2 and 3]). In fact, denoting the center of the disk by $m \in \mathbb{C}$ and its radius by $R > 0$, we get the flow equations

$$R' = -2R \,\mathrm{Re}\, m$$
$$m' = V - m^2 - R^2 \,.$$

Clearly, this system of equations is as difficult to solve as the original Riccati equation (6.3). But suppose that m is an approximate solution in the sense that

$$R' = -2R \,\mathrm{Re}\, m + \delta R$$
$$m' = V - m^2 - R^2 \;+\; \delta m \,,$$

with suitable error terms δm and δR, then the Riccati flow will remain inside the disk provided that its radius grows sufficiently fast, i.e. (see [11, Lemma 3.1])

$$\delta R \geq |\delta m| \,.$$

This is the starting point for the invariant disk method. In order to reduce the number of free functions, it is useful to solve the linear equations in the above system of ODEs by integration. For more details we refer the reader to [11, 13].

7. Separation of the Resolvent and Contour Deformations

The next step is to use the spectral decomposition of the angular operator in Theorem 5.1 in the integral representation of the solution of the Cauchy problem. More specifically, inserting (5.1) into (4.3) gives

$$\Psi(t) = -\frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} e^{-i\omega t} \, \frac{1}{(\omega + 3ic)^p} \left(R_\omega \, Q_n^\omega \, (H + 3ic)^p \, \Psi_0 \right) d\omega \,. \tag{7.1}$$

At this point, the operator product $R_\omega Q_n^\omega$ can be expressed in terms of solutions of the radial and angular ODEs (3.3) which arise in the separation of variables (see [15, Theorem 7.1]). Namely, the operator Q_ω^n maps onto an invariant subspace of $\mathcal{A}_\omega$ of dimension at most N, and it turns out that the operator product $R_\omega \, Q_n^\omega$ leaves this subspace invariant. Therefore, choosing a basis of this invariant subspace, the PDE $(H - \omega) R_\omega Q_\omega^n = Q_\omega^n$ can be rewritten as a radial ODE involving matrices of rank at most N. The solution of this ODE can be expressed explicitly in terms of the resolvent of the radial ODE. In order to compute this resolvent, it is useful to also transform the radial ODE into Sturm-Liouville form (6.1). To this end, we introduce the Regge-Wheeler coordinate $u \in \mathbb{R}$ by

$$\frac{du}{dr} = \frac{r^2 + a^2}{\Delta} \,,$$

mapping the event horizon to $u = -\infty$. Then the radial ODE takes again the form (6.1), but now with u defined on the whole real axis. Thus the resolvent can be written as an integral operator with kernel given in analogy to (6.2) by

$$s_\omega(u, v) = \frac{1}{w(\acute{\phi}, \grave{\phi})} \times \left\{ \begin{array}{ll} \acute{\phi}(u)\, \grave{\phi}(v) & \text{if } v \geq u \\ \grave{\phi}(u)\, \acute{\phi}(v) & \text{if } v < u \,, \end{array} \right.$$

where $\acute{\phi}$ and $\grave{\phi}$ form a specific fundamental system for the radial ODE. The solutions $\acute{\phi}$ and $\grave{\phi}$ are constructed as Jost solutions, using methods of one-dimensional scattering theory (see [4] and [15, Section 6], [5, Section 3]).

The next step is to deform the contour in the integral representation (7.1). Standard arguments show that the integrand in (7.1) is holomorphic on the resolvent set (i.e. for all ω for which the resolvent R_ω in (4.3) exists). Thus the contour may be deformed as long as it does not cross singularities of the resolvent. Therefore, it is crucial to show that the integrand in (7.1) is meromorphic and to determine its pole structure. Here we make essential use of Whiting's mode stability result [24] which states, in our context, that every summand in (7.1) is holomorphic off the real axis. In order to make use of this mode stability, we need to interchange the integral in (7.1) with the infinite sum. To this end, we derive estimates which show that the summands in (7.1) decay for large in n uniformly in ω. Here we again use ODE techniques, in the same spirit as described above for the angular equation (see [15, Section 10]). In this way, we can move the contour in the lower half plane arbitrarily close to the real axis. Moreover, the contour in the upper half plane may be moved to infinity. We thus obtain the integral representation (see [15, Corollary 10.4])

$$\Psi(t) = -\frac{1}{2\pi i} \sum_{n=0}^\infty \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}-i\varepsilon} \frac{e^{-i\omega t}}{(\omega+3ic)^p} \left(R_{\omega,n}\, Q_n^\omega \left(H+3ic\right)^p \Psi_0 \right) d\omega \,.$$

The remaining issue is that the integrands in this representation might have poles on the real axis. These so-called *radiant modes* are ruled out by a causality argument (see [15, Section 11]). We thus obtain the following result (see [15, Theorem 12.1]).

Theorem 7.1. *For any $k \in \mathbb{Z}/2$, there is a parameter $p > 0$ such that for any $t < 0$, the solution of the Cauchy problem for the Teukolsky equation with initial data*

$$\Psi|_{t=0} = e^{-ik\varphi}\, \Psi_0^{(k)}(r,\vartheta) \qquad \textit{with} \qquad \Psi_0^{(k)} \in C^\infty(\mathbb{R}\times S^2, \mathbb{C}^2)$$

has the integral representation

$$\begin{aligned}
&\Psi(t,u,\vartheta,\varphi) \\
&= -\frac{1}{2\pi i}\, e^{-ik\varphi} \sum_{n=0}^\infty \int_{-\infty}^\infty \frac{e^{-i\omega t}}{(\omega+3ic)^p} \left(R^-_{\omega,n}\, Q_n^\omega \big(H+3ic\big)^p\, \Psi_0^{(k)} \right)(u,\vartheta)\, d\omega \,,
\end{aligned} \tag{7.2}$$

where $R^-_{\omega,n}\Psi := \lim_{\varepsilon\searrow 0} \left(R_{\omega-i\varepsilon,n}\Psi\right)$. Moreover, the integrals in (7.2) all exist in the Lebesgue sense. Furthermore, for every $\varepsilon > 0$ and $u_\infty \in \mathbb{R}$, there is N such that for all $u < u_\infty$,

$$\sum_{n=N}^\infty \int_{-\infty}^\infty \left\| \frac{1}{(\omega+3ic)^p} \left(R^-_{\omega,n}\, Q_n^\omega \left(H+3ic\right)^p \Psi_0^{(k)} \right)(u) \right\|_{L^2(S^2)} d\omega < \varepsilon \,. \tag{7.3}$$

8. Proof of Decay

Theorem 3.1 is a direct consequence of the integral representation (7.2) in Theorem 7.1. Namely, combining the estimate (7.3) with Sobolev methods, one can make the contributions for large n pointwise arbitrarily small. On the other hand, for each of the angular modes $n = 0, \ldots, N-1$, the desired pointwise decay as $t \to -\infty$ follows from the Riemann-Lebesgue lemma. For details we refer to [15, Section 12].

9. Concluding Remarks

We first point out that the integral representation of Theorem 7.1 is a suitable starting point for a detailed analysis for the dynamics of the solutions of the Teukolsky equation. In particular, one can study decay rates (similar as worked out for massive Dirac waves in [5]) and derive uniform energy estimates outside the ergosphere (similar as for scalar waves in [9]). Moreover, using the methods in [7], one could analyze superradiance phenomena for wave packets in the time-dependent setting.

Clearly, the next challenge is to prove *nonlinear stability* of the Kerr geometry. This will make it necessary to refine our results on the linear problem, for example by deriving weighted Sobolev estimates and by analyzing the k-dependence of our estimates. Moreover, it might be useful to combine our methods and results with microlocal techniques.

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