ON LAMINAR GROUPS, TITS ALTERNATIVES, AND CONVERGENCE GROUP ACTIONS ON S^2

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ABSTRACT. Following previous work of the second author, we establish more properties of groups of circle homeomorphisms which admit invariant laminations. A certain type of such groups—so-called pseudo-fibered groups—have previously been conjectured to be closely related to fundamental groups of closed hyperbolic 3-manifolds. We clarify this conjecture, and explain the necessity of various conditions. We then prove that torsion-free pseudo-fibered groups satisfy a Tits alternative. We conclude by proving that a purely hyperbolic pseudo-fibered group acts on the 2-sphere as a convergence group. This leads to a conjectural characterization of the fundamental group of a fibered hyperbolic 3-manifold as a pseudo-fibered group with some additional properties.

Keywords. Tits alternative, laminations, circle homeomorphisms, Fuchsian groups, fibered 3-manifolds, pseudo-Anosov surface homeomorphism.

MSC classes: 20F65, 20H10, 37C85, 37E10, 57M60.

1. INTRODUCTION

Thurston [24] showed that if M is an atoroidal 3-manifold admitting a taut foliation, then $\pi_1(M)$ acts faithfully on S^1 with a pair of dense invariant laminations. The result was generalized by Calegari-Dunfield [6] and one can find a complete treatment in [5]. Motivated by these results, the second author studied groups acting faithfully on S^1 with prescribed types and numbers of invariant laminations, in the process giving a new characterization of Fuchsian groups [1]. In the same paper, he asked if one can characterize fibered 3-manifold groups in a similar way. More precisely, call a subgroup G of $\mathrm{Homeo}^+(S^1)$ *pseudo-fibered* if its action on S^1 admits two invariant, very full, loose laminations with distinct endpoints. See Section 2 for the precise definitions. Then, in the language of this paper, and having made some necessary modifications, [1] proposed the following

Conjecture 1.1 (Promotion of Pseudo-Fibering)**.** *Let G be a finitely-generated torsion-free pseudo-fibered group which does not split as a nontrivial free product. Then there are three possibilities.*

1. *G is elementary,* i.e.*virtually abelian.*
2. *G is a Fuchsian group.*
3. *G is a closed hyperbolic 3-manifold group.*

In Section 3, we explain the differences between this conjecture and the unmodified conjecture of [1]. In particular, we provide examples which indicate the simultaneous necessity of the modifiers "torsion-free" and "no nontrivial splitting." We also explain what we expect the meaning of "is" is. In the rest of the paper, we collect evidence for the conjecture.

In Section 4 we establish our first main result, a Tits alternative for torsion-free pseudo-fibered groups:

Theorem A. Let G be a torsion-free pseudo-fibered group. Each subgroup of G either contains a non-abelian free subgroup or is virtually abelian.

This is proved in Section 4.2 by studying how two elements of a (torsion-free) pseudo-fibered group interact with each other dynamically. In particular, we show a kind of dynamical alternative for elements of a pseudo-fibered group:

Theorem B. Let G be a torsion-free pseudo-fibered group. Let $g, h \in G$. Then Per_g and Per_h are either equal or disjoint.

Theorem A will follow from Theorem B by applying the ping-pong lemma, and Hölder's theorem that a group acting faithfully and freely on $\mathbb{R}$ is necessarily abelian.

In Sections 5.1 and 5.2, we study pseudo-fibered groups with more structure, inspired by Fuchsian groups and fibered hyperbolic 3-manifold groups, respectively. Both sections should be considered as part of Fenley's program which generalizes the work of Cannon-Thurston [8] considerably from the viewpoint of pseudo-Anosov flows (see [13]).

Our main result of Section 5.1 is to connects the pseudo-fibered group action on S^1 with the convegence group action on S^2. Note that similar idea has been carried out in Fenley's program (see [13], [14], and also compare [15], [23]).

Theorem C. Let G be a pseudo-fibered group which is purely hyperbolic. Then G acts on S^2 as a convergence group.

For the definition of "purely hyperbolic", see Section 2. We expect that Theorem C can be strengthened. Indeed, when G is a purely hyperbolic pseudo-fibered group, the second author has previously conjectured that G is a Fuchsian group, hence acts on S^1 as a convergence group [1]. Recall that the work of many authors (e.g. [25], [9] and [16]) shows that a group acts on S^1 as a convergence group if and only if it is topologically conjugate to a Fuchsian group.

In Section 5.2, we discuss the case that G is an extension of a purely hyperbolic group by a pseudo-Anosov-like element. While we are unable to prove an analog of Theorem C in this case, we nevertheless expect that such a group G also acts on S^2 as a convergence group, and propose a possible approach to proving this. Furthermore, in this extended case, these actions are expected to be *uniform* convergence group actions. If this were true, then Cannon's conjecture [7] and a result of Bowditch [3] would imply that G is a hyperbolic 3-manifold group. These observations are reflected in Conjecture 1.1 and discussed more thoroughly in Section 5.2.

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2. Preliminaries

We briefly review and motivate several definitions regarding laminations on the circle. Two pairs (a,b) and (c,d) of distinct points of the circle S^1 are said to be *linked* if each connected component of $S^1 \setminus \{a,b\}$ contains precisely one of c,d. They are called *unlinked* if they are not linked. Let $\mathscr{M}$ denote the set of all unordered pairs of two distinct points of S^1, i.e., $\mathscr{M} = (S^1 \times S^1 - \Delta)/(x,y) \sim (y,x)$ where Δ is the diagonal $\{(x,x) : x \in S^1\}$. A *lamination* of S^1 is a closed subset of $\mathscr{M}$ whose elements are pairwise unlinked. Given a lamination Λ, an element (a,b) of Λ is called a *leaf*, and the points a,b are called the *endpoints* of the leaf (a,b) (or just endpoints of Λ if there is no possible confusion). Two laminations have *distinct endpoints* if their sets of endpoints are disjoint. A lamination Λ is called *dense* if the set of endpoints of Λ is a dense subset of S^1.

Any subgroup G of $\mathrm{Homeo}^+(S^1)$ has an induced action on $\mathscr{M}$. We say that a lamination Λ is *G-invariant* if the G-action on $\mathscr{M}$ preserves Λ set-wise. A discrete subgroup G of $\mathrm{Homeo}^+(S^1)$ is called *laminar* if it admits a dense G-invariant lamination.

Let $\mathbb{D}$ denote the closed unit disk in $\mathbb{C}$ where the interior is equipped with the Poincaré metric, i.e., $\mathbb{D} = \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$. A lamination Λ' of $\mathbb{D}$ is a set of chords with disjoint interiors such that there exists a lamination Λ in $S^1 = \partial \mathbb{D}$ where the chords in Λ' can be obtained by connecting the endpoints of the leaves of Λ.

As noted in Construction 2.4 of [5], the set of laminations on S^1 and the set of geodesic laminations of $\mathbb{H}^2$ are in one-to-one correspondence up to isotopy relative to $S^1 = \partial_\infty \mathbb{H}^2$. Hence, we freely switch our viewpoint between these two without further mentioning. A *gap* of a lamination Λ is the closure of a connected component of $\mathbb{H}^2 \setminus \Lambda$ in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}$.

We recall some key properties of laminations from [1] (also compare [4]) .

Definition 2.1. *A lamination Λ is said to be*

- ***totally disconnected*** *if no open subset of the disk is foliated by Λ,*
- ***very full*** *if each gap is a finite-sided ideal polygon in the disk, and*
- ***loose*** *if no two leaves share an endpoint unless they are edges of the same (necessarily unique) gap.*

For every element f of $\mathrm{Homeo}^+(S^1)$, let $\mathrm{Fix}_f \subset S^1$ denote the set of all fixed points of f. Let Per_f denote the set of all periodic points of f, where a point p of S^1 is *periodic* for f if the orbit of p under f is finite. Thus $\mathrm{Fix}_f \subset \mathrm{Per}_f$.

We first give names to particular types of homeomorphisms of S^1 in the following definition as in [1]. For the first four types of homeomorphisms, compare [19] where Möbius-like elliptic, Möbius-like parabolic, Möbius-like hyperbolic, and Möbius-like homeomorphisms are defined.

Definition 2.2. *An element f of $\mathrm{Homeo}^+(S^1)$ is said to be*

- ***elliptic*** *if f has no fixed points,*
- ***parabolic*** *if f has a unique fixed point,*
- ***hyperbolic*** *if f has two fixed points, one attracting and one repelling,*
- ***Möbius-like*** *if f is conjugate in $\mathrm{Homeo}^+(S^1)$ to an element of $\mathrm{PSL}_2(\mathbb{R})$,*
- ***pseudo-Anosov-like*** *or* ***p-A-like*** *if f is not hyperbolic and some positive power f^n has a positive, even number of fixed points alternating between attracting and repelling, and*
- ***properly pseudo-Anosov-like*** *or* ***properly p-A-like*** *if f is pseudo-Anosov-like and non-elliptic.*

Thus f is p-A-like if and only if a positive power of f is properly p-A-like. Here a fixed point p of the homeomorphism f is *attracting* if there exists an interval $I \ni p$ containing no other fixed points such that $f(I) \subsetneq I$. Similarly, a fixed point q is *repelling* if there exists an interval $J \ni q$ containing no other fixed points such that $f(J) \supsetneq J$. For a p-A-like homeomorphism $f \in \mathrm{Homeo}^+(S^1)$, the set of boundary leaves of the convex hull of the attracting fixed points of a properly p-A-like power f^n is called the **attracting polygon** of f. Similarly, f has a **repelling polygon**.

Definition 2.3. *Let Λ be a lamination. A leaf $l \in \Lambda$ is said to be* ***visible*** *from a point $p \in S^1$ if one can connect l to p by a geodesic of $\mathbb{H}^2$ (i.e., there exists a geodesic ray from a point on l to p) which does not intersect any leaf of Λ in $\mathbb{H}^2$.*

Observe that if p is an endpoint of a gap in a very full, loose lamination, the set of leaves visible from p is precisely the set of edges of the gap.

Now we outline some examples of laminar groups mentioned in the introduction. Let S be a closed hyperbolic surface, and ϕ be a pseudo-Anosov homeomorphism of S. Let M be the mapping torus $S \times [0,1]/(x,1) \sim (\phi(x),0)$. Then one can construct a faithful action of $\pi_1(M)$ on S^1 in the following way. Note $\pi_1(M)$ is isomorphic to $\pi_1(S) \rtimes \mathbb{Z}$. The deck transformation action of $\pi_1(S)$ on $\mathbb{H}^2$ extends continuously to a faithful action on $\partial\mathbb{H}^2$. Let $\widetilde{\phi} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of ϕ to the universal cover of S. Since S has finite area, $\widetilde{\phi}$ is a quasi-isometry, hence extends to a homeomorphism on $\partial\mathbb{H}^2$. Considering this homeomorphism on $\partial\mathbb{H}^2$ as a generator of $\mathbb{Z}$, this defines an action

$$\rho : \pi_1(M) = \pi_1(S) \rtimes \mathbb{Z} \to \mathrm{Homeo}^+(\partial\mathbb{H}^2) = \mathrm{Homeo}^+(S^1).$$

Then $\rho(\pi_1(M))$ is laminar, since it fixes both the stable and unstable laminations of ϕ. In fact, ρ is faithful.

Definition 2.4. *A finitely generated laminar group G is said to be* ***fibered*** *if G is topologically conjugate to $\rho(\pi_1(M))$ where ρ and M are as in the previous paragraph, and the conjugacy takes the G-invariant lamination to one of the invariant laminations of the monodromy of M.*

Definition 2.5. *A finitely generated laminar group G is said to be* ***pseudo-fibered*** *if it preserves a pair of very full loose invariant laminations Λ_1, Λ_2 with distinct endpoints, and each nontrivial element of G has at most countably many fixed points in S^1. We also say $(G, \Lambda_1, \Lambda_2)$ is a pseudo-fibered triple.*

Pseudo-fibered groups were first studied in [1], although they had not yet been given a name.

Theorem 2.6 (see Section 8 of [1])**.** *Let $(G, \Lambda_1, \Lambda_2)$ be a pseudo-fibered triple. Let $g \in G$. Then*

(1) *g is either Möbius-like or pseudo-Anosov-like, and*
(2) *if g is p-A-like, then for some $i, j = 1, 2$ with $i \neq j$, Λ_i contains the attracting polygon of g, and Λ_j contains the repelling polygon of g.*

Furthermore, [1] shows that if G is torsion-free, then all Möbius-like elements are hyperbolic elements. The following proposition justifies the term "pseudo-fibered".

Proposition 2.7. *A fibered group G is pseudo-fibered.*

Proof. We only need to worry about the cardinality of the set of fixed points of each element. But this is not a problem due to Theorem 5.5 of [10] which asserts that for a given pseudo-Anosov surface homeomorphism h, any lift of a strictly positive power of h has finitely many fixed points on $\partial_\infty \mathbb{H}^2$, alternating between attracting and repelling. □

In fact the proof of Theorem 5.5 in [10] (pp. 85-87) shows Theorem 2.6 in the case of fibered groups. Any lift of a strictly positive power of h falls into one of the three cases. Case 1 and Case 2 correspond to properly pseudo-Anosov-like elements and Case 3 corresponds to hyperbolic elements in the sense of Definition 2.2. In Case 1, the attracting repelling polygons of the p-A-like element have 3 or more sides and in Case 2, those polygons are degenerate, i.e., there are exactly two attracting fixed points and two repelling fixed points.

We remark that the "pseudo" in "pseudo-fibered group" intentionally carries two different connotations. The first, as in Theorem 2.6, indicates that some elements are pseudo-Anosov-like. The second, as in Conjecture 1.1, indicates that pseudo-fibered groups are (conjecturally) not far from fibered groups.

In the second half of the paper (including Theorem C), we study a special class of pseudo-fibered groups.

Definition 2.8. *Let G be a pseudo-fibered group. G is called* ***purely hyperbolic*** *if it has no pseudo-Anosov-like elements, and called* ***mixed*** *if it contains at least one hyperbolic element and at least one pseudo-Anosov-like element.*

Definition 2.9. *A (mixed) pseudo-fibered group G is said to be* ***tame*** *if the following condition holds: for any two elements g, h of G, gh is hyperbolic if both g, h are hyperbolic.*

Theorem C says that a purely hyperbolic pseudo-fibered group acts on the sphere as a convergence group. In general, a group G acting on a compactum X is called a discrete convergence group if the following holds: for any infinite sequence of distinct elements (g_i) of G, there exists a subsequence (g_{i_j}) of (g_i) and two points $a, b \in X$ not necessarily distinct such that g_{i_j} converges to the constant map with value a uniformly on every compact subset of $X \setminus \{b\}$, and $g_{i_j}^{-1}$ converges to the constant map with value b uniformly on every compact subset of $X \setminus \{a\}$. Since we only deal with discrete convergence groups in this paper, we will omit the word discrete, and simply call it a convergence group.

As mentioned before, when X is S^1, being a convergence group is equivalent to being (conjugate to) a Fuchsian group. This result is known as the Convergence Group Theorem [25, 16, 9], and the same statement for indiscrete convergence groups was proved in [18].

Remark 2.10. *There is a well-known equivalent definition of a convergence group action. G acting on X is called a convergence group if the diagonal action of G on $X \times X \times X \setminus \Delta$ is properly discontinuous, where Δ is the set of triples of points of X which are not all distinct. See, for instance,* [26]*. If the diagonal action on the set of distinct triples is also cocompact, then G is called a uniform convergence group.*

Remark 2.11. *When X is S^1, it is easy to see that the uniform convergence in the definition of a convergence group can be replaced by the pointwise convergence. However, it is important not to conflate the notions of "uniform convergence group" and "uniform convergence."*

Remark 2.12. *When X is S^2, the analogue of Convergence Group Theorem is not true, i.e., not every discrete convergence group action on S^2 comes from a Kleinian group. For instance, one can just start with a Fuchsian representation of a surface group into $\mathrm{PSL}_2(\mathbb{C})$, and quotient the lower hemisphere to a single point. On the other hand, it is a famous open problem whether or not all uniform convergence group actions on S^2 come from Kleinian groups. See Section 5.2 for a related discussion.*

Finally, the idea of a rainbow, first described in [1], will be useful throughout this paper:

Definition 2.13. *A lamination Λ is said to have a* rainbow *at a point $p \in S^1$ if there is a sequence of leaves $(l_i) = ((a_i, b_i))$ of Λ such that (a_i) and (b_i) converge to p from opposite sides. Such a sequence (l_i) is called a rainbow at p in Λ.*

A rainbow is a particularly nice way of approximating a point $p \in S^1$ by leaves of a lamination. Clearly endpoints of leaves do not admit rainbows. On the other hand, an observation we shall use later is that for a very full lamination Λ, these approximations exist for every point that is not an endpoint of a leaf:

Lemma 2.14 (Rainbow Lemma, Theorem 5.3 of [1])**.** *Let Λ be a very full lamination of S^1. Every point $p \in S^1$ is either an endpoint of a leaf of Λ, or there is a rainbow in Λ at p. These two possibilities are mutually exclusive.* □

2.1. **Semi-conjugacy destroys pseudo-fiberedness.** We remark in this subsection that semi-conjugacy appears to be irrelevant to the study of pseudo-fibered groups. This is to be expected in the context of our promotion of pseudo-fibering conjecture, since semi-conjugacy does not preserve convergence actions. In particular, our Theorem A should be seen as distinct from Margulis's Tits alternative for minimal subgroups of $\mathrm{Homeo}^+(S^1)$, a point we explain now.

A continuous surjective map $f : S^1 \to S^1$ is said to be **monotone** if the preimage of each point is connected. For a group G, two actions ρ and $\mu : G \to \mathrm{Homeo}^+(S^1)$ are said to be semi-conjugate (or ρ is semi-conjugate to μ) if there exists a monotone map f such that $f \circ \rho = \mu \circ f$. For many aspects of the theory of groups of circle homeomorphisms, it is enough to consider the actions up to semi-conjugacy.

A classical theorem of Poincaré says that every subgroup of $\mathrm{Homeo}^+(S^1)$ either has a finite orbit or is semiconjugate to a minimal action (meaning every orbit is dense). Furthermore, a theorem of Margulis says that subgroups of $\mathrm{Homeo}^+(S^1)$ that act minimally either contain F_2 as a subgroup or are abelian [20]. (For more details regarding both of these results, as well as a general introduction to group actions on S^1, see [17].) Hence, if one studies group actions up to semi-conjugacy, then the Tits alternative is not particularly new. On the other hand, pseudo-fibered triples cannot be studied up to semi-conjugacy, since the laminations do not behave well under semi-conjugacy. More precisely, we have:

Proposition 2.15. *Let $f : S^1 \to S^1$ be a monotone map which is not a homeomorphism, and (Λ_1, Λ_2) a pair of laminations. At most one of the pairs, (Λ_1, Λ_2) and $(f(\Lambda_1), f(\Lambda_2))$, can be a pair of very full loose laminations with disjoint endpoint sets.*

Proof. Since f is not injective, there is a point $p \in S^1$ such that $I := f^{-1}(p)$ has non-empty interior. Let $\hat{p}$ be an endpoint of I. Recall that the Rainbow Lemma says that for each $p \in S^1$ and a very full lamination Λ, either p is an endpoint of a leaf or there is a rainbow at p in Λ.

Suppose (Λ_1, Λ_2) is a pair of very full loose laminations with disjoint endpoint sets. In particular, there must be a rainbow at $\hat{p}$ in Λ_i for at least one of the $i = 1$ or 2. But the image of a rainbow at $\hat{p}$ under f is an infinite set of leaves of $f(\Lambda_i)$ which share a common endpoint. Hence, $f(\Lambda_i)$ cannot be loose.

For the other direction, suppose $(f(\Lambda_1), f(\Lambda_2))$ is a pair of very full loose laminations with disjoint endpoint sets. From the above argument, we know that there is no rainbow at $\hat{p}$ in Λ_i for each i. But this means $\hat{p}$ is an endpoint of some leaf in both Λ_1 and Λ_2, hence they cannot have disjoint endpoint sets. □

Furthermore, there are examples of pseudo-fibered triples whose actions are not minimal. Indeed, it is easy to construct examples of pseudo-fibered groups with finite orbits, and in the next section, we construct examples of pseudo-fibered groups whose actions are neither minimal nor have finite orbits. These examples imply that the scope of our Theorem A is distinct from the Tits alternative of Margulis, and that it would be interesting to more thoroughly unravel the relationship between (non)minimal actions, finite orbits, and Conjecture 1.1.

3. Free products, torsion, and promotion of pseudo-fibering

Theorems A and C can be seen as evidence for the following conjecture formulated by the second author in [1].

Conjecture 3.1 (Promotion of Pseudo-Fibering)**.** *Let G be a torsion-free pseudo-fibered group. Then there are three possibilities.*

1. *G is elementary.*
2. *G is a Fuchsian group.*
3. *G is a closed hyperbolic 3-manifold group.*

Here, G is said to be elementary if it is virtually abelian. This definition makes sense due to Theorem A, which asserts that every non-elementary pseudo-fibered group contains a non-abelian free subgroup. In the second case, "is" means that G is topologically conjugate to a Möbius group action (as usual, we consider $\mathrm{PSL}_2(\mathbb{R})$ as a subgroup of $\mathrm{Homeo}^+(S^1)$), as in the statement of the convergence subgroup theorem. In the last possibility, "is" simply means isomorphic as abstract groups. What we really have in mind though is that such a group is virtually a fibered group in the sense of Section 2. In the last section, we discuss a potential approach to this third possibility under additional assumptions.

One may notice that the above conjecture is slightly stronger than Conjecture 1.1, since it lacks the assumption about free indecomposability. The following construction explains why we added this assumption to Conjecture 1.1.

Theorem 3.2. *Let G,H be any finite cyclic groups. Then $G*H$ embeds into* $\mathrm{Homeo}^+(S^1)$ *as a pseudo-fibered group.*

Proof. This construction is adapted from a construction in [2] that yields a faithful action of any free product of subgroups of $\mathrm{Homeo}^+(S^1)$ on a new circle, which blows down onto each of the original circles. We content ourselves with a brief review of the ideas of [2], and a description of how to additionally construct invariant laminations.

To construct an action of $G*H$ on S^1, begin by forming two pointed copies of S^1 called S^1_G and S^1_H, with marked points both denoted 1. Let G act on S^1_G as a finite rotation subgroup, and H act on S^1_H as a finite rotation subgroup. The G-orbit of the point marked 1 in S^1_G is now a copy of G, and the H-orbit of 1 in S^1_H is a copy of H. Mark all of these points accordingly, wedge S^1_G and S^1_H together at the points marked by 1, blow up all of the marked points, and consistenly label one of the endpoints of the blow-up intervals. The resulting "seed", called Γ_0, is in Figure 1, where we have $G = \langle a \mid a^4 = 1\rangle$ and $H = \langle b \mid b^3 = 1\rangle$. Now generate an infinite graph Γ'_∞ on which $G*H$ acts faithfully. As in Figure 2, write $\Gamma'_\infty = \Lambda_0 \cup \Gamma_\infty$ where Λ_0 is the orbit of the blown-up intervals in Γ_0, and Γ_∞ is everything else. The order completion $\overline{\Gamma_\infty}$ is S^1, and Λ_0 is a discrete lamination on this circle. This proves

Lemma 3.3. *Let G,H be any finite cyclic groups. Then there exists an injective homomorphism $\rho : G*H \to$* $\mathrm{Homeo}^+(S^1)$ *such that $\rho(G*H)$ admits a discrete invariant lamination.* □

Now one can easily add more leaves to Λ_0 to construct a $G*H$-invariant very full and loose lamination Λ_1. For example, in the left circle of the seed Γ_0, first take a polygon which has one vertex in each connected component of the complement of the dotted segments and is invariant under the action of G. Then in the region between this polygon and an element of Λ_0 in Γ_0, add infinitely many triangles to make the lamination very full and loose in that region. Now fill the other such regions so that lamination becomes G-invariant.

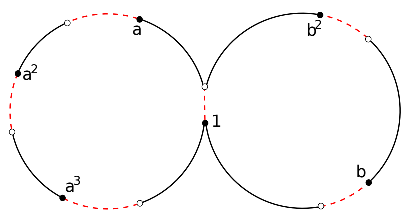

FIGURE 1. The seed graph Γ_0.

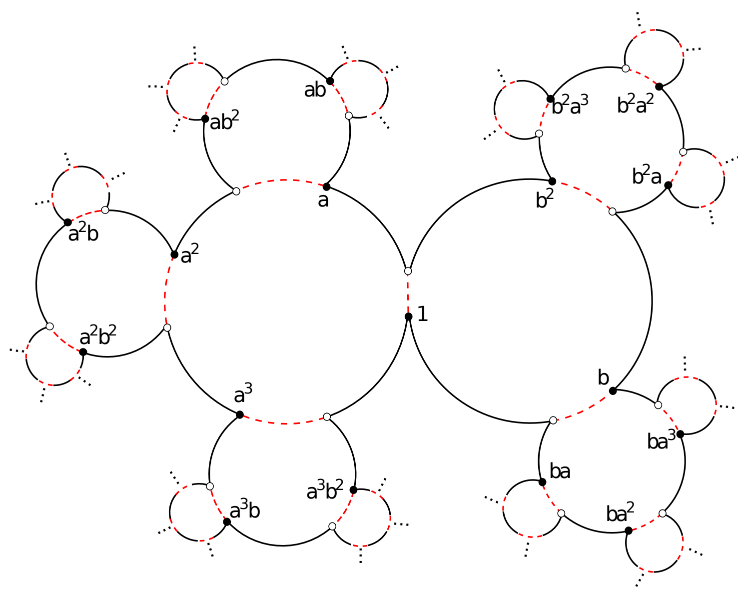

FIGURE 2. The limiting circle $S^1 = \overline{\Gamma_\infty}$ laminated by Λ_0, which is represented by the dashed red lines.

One can do the same thing for the right circle to get an H-invariant very full loose lamination, and then extend it as a $G*H$-invariant lamination Λ_1 which contains Λ_0 as a sublamination. The final result is in Figure 3. Λ_1 is obviously very full, and loose away from Λ_0. It is loose at Λ_0 because the leaves of Λ_0 are not contained in gaps—they are instead limits of gaps.

We need to construct another $G*H$-invariant very full loose lamination Λ_2, so that Λ_1 and Λ_2 have distinct endpoints. To build Λ_2, we first replace each leaf of Λ_0 with endpoints by four leaves forming, say, a rectangle such that each endpoint of the original dotted segment lies between two adjacent vertices of the rectangle. In the two regions between the rectangle and the endpoints of the original leaf in Λ_0, put infinitely many triangles to make the lamination very full and loose. These choices can obviously be made so that the endpoints of the new leaves are disjoint from Λ_1, and by working in one region at a time, we can do the construction $G*H$-invariantly, resulting in Figure 4. In the regions where all the rectangles are visible, we do the exactly same thing as when construction Γ_1 from Γ_0: take a big invariant polygon, and fill out all the complementary regions. The result is shown in Figure 5.

Finally, to show $(\rho(G*H), \Lambda_1, \Lambda_2)$ is a pseudo-fibered triple, we need to show that every element of $G*H = \rho(G*H)$ has countably many fixed points in its action on S^1. There are two ways to prove this, either using Bass-Serre theory, or the existence of even more $G*H$-invariant laminations under this action.

For the latter approach, we quote the following result:

Theorem 3.4 ([1])**.** *Every subgroup of* $\mathrm{Homeo}^+(S^1)$ *admitting three very full invariant laminations with distinct endpoint sets is Möbius-like, meaning every element is (individually) Möbius-like.* □

Since both G and H are finite, there is a large freedom to construct laminations inductively as before. Indeed, we can slightly perturb the construction of Λ_2 to get a third very full and loose invariant lamination Λ_3 with endpoints distinct from Λ_1 and Λ_2. Theorem 3.4 then implies every element of $G*H$ is Möbius-like, hence has finitely many fixed points.

Alternatively, to show every element of $G*H$ acts on S^1 with at most two fixed points, we can use Bass-Serre theory. Clearly torsion elements of $G*H$ act freely on S^1. Nontorsion elements must have their fixed points in the subset $S^1 \setminus \Gamma_\infty$, which (partly because G and H are finite) can be identified with the ends of the Bass-Serre tree for $G*H$. Standard results now imply such elements have two fixed points in S^1. □

Theorem 3.2 has an immediate corollary in the context of 3-manifold groups.

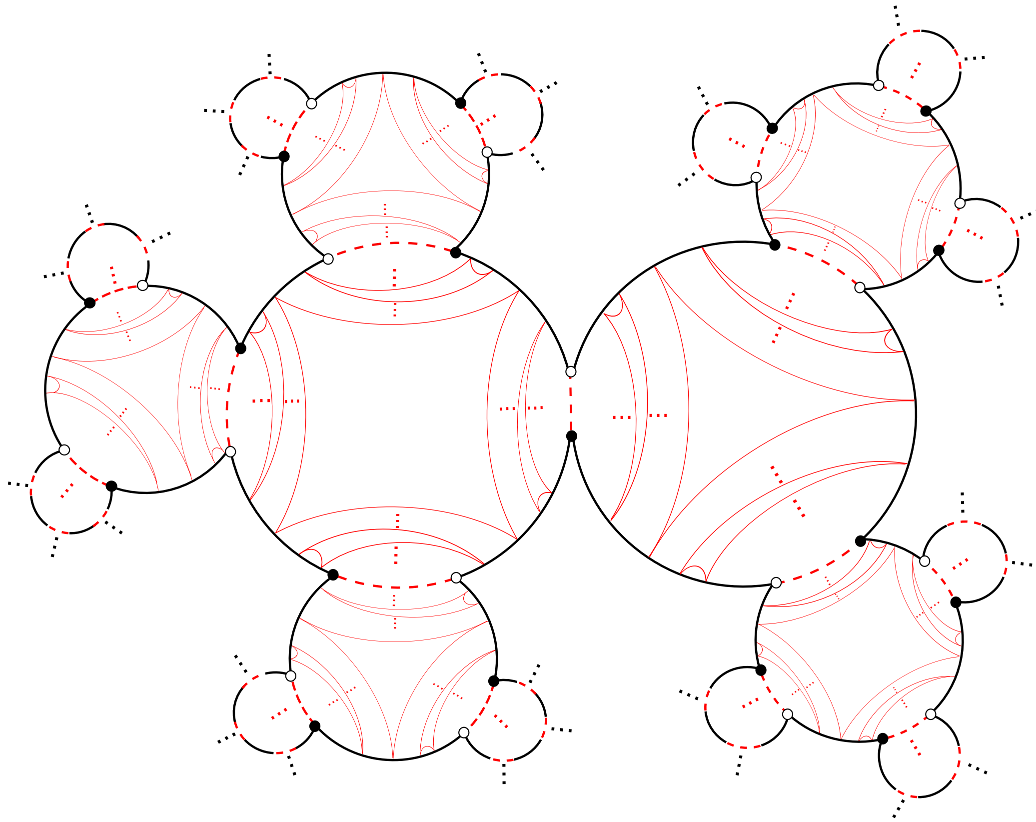

FIGURE 3. The very full and loose lamination $\Lambda_1 \supset \Lambda_0$. We have removed the markings to avoid clutter.

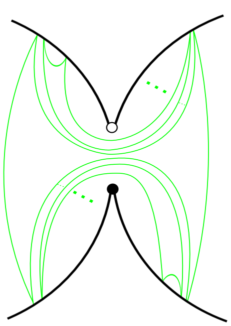

FIGURE 4. Replacing Λ_0 with rectangles and triangles.

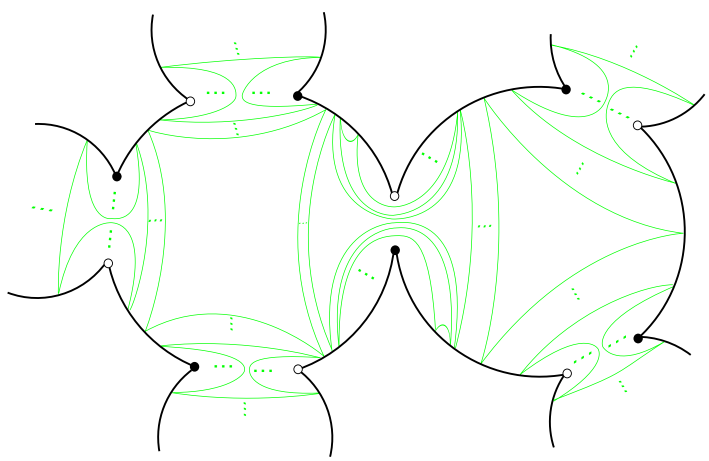

FIGURE 5. The very full and loose lamination Λ_2.

Corollary 3.5. *Let M be a connected sum of two lens spaces. Then $\pi_1(M)$ admits a pseudo-fibered group action on S^1.*

It is not clear if the method of the proof of Theorem 3.2 could be generalized to the torsion-free case. Hence, it is not clear if all the assumptions in Conjecture 1.1 (namely, torsion-freeness, discreteness, and free-indecomposability) are strictly necessary. For example, it's possible that free-indecomposability follows from the other conditions. In any case, it would be preferable to have some dynamical characterization of free-indecomposability in terms of laminations, and Theorem 3.2 serves as an indication that there is no a priori reason to believe Conjecture 1.1 without at least some of these assumptions.

4. The Tits Alternative for Pseudo-Fibered Groups

Throughout this section, G will denote a torsion-free pseudo-fibered group, except where explicitly indicated elsewhere. This will allow us to apply Theorem 2.6, which we may do sometimes without mentioning.

4.1. **Proof of Theorem B.** To prove that pseudo-fibered groups satisfy the Tits alternative, we first prove Theorem B, which we recall says that two elements of a pseudo-fibered group have either equal or disjoint sets of periodic points. This can be done, for instance, by analyzing how each element of the group acts on the quotient of the circle obtained by collapsing leaves of an invariant lamination. One can show that such a quotient is a dendrite as in [27], and this point of view has its own advantages. But for our purpose, it is simpler to analyze the group action on the circle directly.

We will need a number of lemmas. The first follows immediately from the definitions, so we leave its proof to the reader:

Lemma 4.1. *Suppose a subgroup of* $\mathrm{Homeo}^+(S^1)$ *admits two dense laminations with distinct endpoints. Then each of the laminations is totally disconnected.* □

Recall that by definition, a lamination Λ is a type of closed subset of $(S^1 \times S^1 - \Delta)/(x,y) \sim (y,x)$. Thus it makes sense to talk about the neighborhood in Λ of a leaf, isolated leaves, *etc*.

Lemma 4.2. *Each leaf of a totally disconnected very full lamination is either a boundary leaf of a gap, or is the limit of an infinite sequence of gaps.*

Proof. This is a direct consequence of the definition of a very full lamination. Indeed, if a neighbourhood of a leaf meets no gaps, it must be foliated, contradicting that the lamination is totally disconnected. □

Lemma 4.3. *Let Λ be a totally disconnected very full lamination. Then Λ is loose if and only if the following conditions are satisfied:*

(1) *For each $p \in S^1$, at most finitely many leaves of Λ have p as an endpoint.*
(2) *There are no isolated leaves.*

Proof. Suppose Λ is loose. Then each $p \in S^1$ is an endpoint of at most two leaves (maybe none) of Λ. Hence condition (1) follows immediately.

For condition (2), suppose there exists a leaf L which is isolated. Let J_1 and J_2 be the connected components of the complements of the endpoints of L in S^1. The fact that L is isolated means there exists an open arc I_1 containing one endpoint of L, and another open arc I_2 containing the other endpoint of L, such that for each $i = 1,2$, there exists no leaf connecting $I_1 \cap J_i$ to $I_2 \cap J_i$. For $i = 1,2$, define Λ_i to be the set of leaves of Λ with endpoints in J_i that are visible from both endpoints of L. Both Λ_i are nonempty since Λ is dense. For each $i = 1,2$, $\Lambda_i \cup \{L\}$ is the set of boundary leaves of a gap P_i of Λ. This contradicts looseness of Λ, since both P_1 and P_2 are gaps sharing some of their vertices.

Now for the converse, assume Λ satisfies the conditions (1) and (2). Suppose p is a common endpoint of two gaps. Let L_1, L_2 be the innermost leaves ending at p. (It's possible $L_1 = L_2$, but this does not change what follows.) Then L_1 is either isolated (absurd by condition (2)), or is approximated by infinitely many leaves. Since these leaves cannot cross L_2, they must end at p, contradicting condition (1). □

For a lamination Λ, let $\mathrm{ends}(\Lambda) \subset S^1$ be the set of endpoints of all leaves of Λ.

Lemma 4.4. *Let G be a pseudo-fibered group, and let Λ be a very full loose G-invariant lamination. If g is a hyperbolic element, then the fixed points of g in S^1 are not in* $\mathrm{ends}(\Lambda)$

Proof. Let p,q be the fixed points of g. Suppose p is an endpoint of a leaf l.

If the other endpoints of l is not q, then the set $\{g^{\circ n}(l) : n \in \mathbb{Z}\}$ gives an infinite set of leaves each of which has p as an endpoint, contradicting Lemma 4.3. Hence, we may assume the other endpoint of l is q.

Let I be a connected component of $S^1 \setminus \{p,q\}$. Define Λ_I to be the set of leaves whose endpoints are in I, and visible from both p and q. Assume Λ_I is nonempty. Since $\{L\} \cup \Lambda_I$ bound a gap, say P, Λ_I must be finite. But this means there must be a leaf in Λ_I which connects p to a point in I which we already saw impossible. Now assume Λ_I is empty. This means there exists a family of infinitely many leaves contained in I which accumulate to l. But since g acts as a translation on I, this is impossible (if l' is a leaf close enough to l, then l' and gl' must be linked). □

Theorem 4.5 (Solodov [22])**.** *If G is a subgroup of* $\mathrm{Homeo}^+(\mathbb{R})$ *such that each non-trivial element has at most one fixed point, and there is no global fixed point, then $[G,G] - \{Id\}$ consists of fixed-point-free elements.*

Proof. See Step 4 in the proof of Theorem 2.2.36 in [22]. □

Lemma 4.6. *Let (G,Λ_1,Λ_2) be a pseudo-fibered triple. Suppose $g \in G$ is properly pseudo-Anosov-like. If $h \in G$ shares a fixed point p with g, then* $\mathrm{Fix}_h = \mathrm{Fix}_g$*. In particular, h is also a properly p-A-like element.*

Proof. First, we show that h cannot be hyperbolic. Since p is a fixed point of g, then by Theorem 2.6 one of Λ_1 or Λ_2 contains a gap which has p as a vertex. But then Lemma 4.4 says that p cannot be a fixed point of an hyperbolic element. Therefore, by Theorem 2.6, h must be pseudo-Anosov-like.

Without loss of generality, we may assume that p is an attracting fixed point of both g,h, and Λ_1 contains the attracting polygon P_g of g. Let q be a vertex of P_g which is connected to p by a boundary leaf l of P_g. If q is not a fixed point of h, then $\{h^{\circ n}(l) : n \in \mathbb{Z}\}$ is an infinite set of leaves which share p as a common endpoint. This is impossible by Lemma 4.3 since Λ_1 is loose. This inductively shows that all vertices (i.e., all attracting fixed points of g) are fixed by h. Theorem 2.6 says there is no leaf connecting an attracting fixed point to a repelling fixed point for a given p-A-like elements. Hence, all attracting fixed points of elements of g are attracting fixed points of h. Applying the same argument to the attracting polygon of h, one concludes that P_g is in fact the attracting polygon of h as well.

We showed the attracting polygon of g and the attracting polygon of h must coincide if p is attracting. How about the repelling polygons? Note Theorem 2.6(2) implies both the repelling polygon of g and the repelling polygon of h are contained in Λ_2.

Case 1. Suppose g (hence, also h) has at least three attracting fixed points. Since both g and h has a unique repelling fixed point between two adjacent vertices of P_g, the only way for the repelling polygons to be unlinked is that they coincide. Therefore, $\mathrm{Fix}_g = \mathrm{Fix}_h$.

Case 2. Now suppose g has only two attracting fixed points. We know that each connected components of $S^1 \setminus \mathrm{Fix}_g$ has exactly one repelling fixed point of g and exactly one repelling fixed point of h. Let I be a connected component of $S^1 \setminus \mathrm{Fix}_g$, r_g the repelling fixed point of g on I, r_h the repelling fixed point of h on I. Let H be the subgroup of G generated by g,h. First note that every element of H fixes the endpoints of I. From the previous arguments, we know that each element of H is p-A-like and has exactly two attracting fixed points and two repelling fixed points. This implies that each element of H has exactly one fixed point in I. Now we apply Theorem 4.5 by identifying I with $\mathbb{R}$. If $r_g \neq r_h$, H has no global fixed point in I, and the theorem says that H is abelian. But if g and h commute, then r_g and r_h coincide, a contradiction. Therefore, r_g and r_h must coincide from the beginning, and $\mathrm{Fix}_g = \mathrm{Fix}_h$ again. □

The following generalizes Lemma 4.6 for p-A-like elements that could be elliptic. We use Per_f to denote the set of all periodic points of a homeomorphism f.

Lemma 4.7. *Let g be a p-A-like element in a pseudo fibered group G. If $h \in G$ shares a periodic point p with g, then* $\mathrm{Per}_h = \mathrm{Per}_g$ *and h is also p-A-like.*

Proof. Take powers g^n and h^m such that g^n is properly p-A-like and p is fixed for h^m. Note that every periodic point of a properly p-A-like element is fixed. So $\mathrm{Fix}_{g^n} = \mathrm{Per}_{g^n} = \mathrm{Per}_g$. Applying Lemma 4.6, we get that $\mathrm{Fix}_{g^n} = \mathrm{Fix}_{h^m}$ and that h^m is properly p-A-like. So h is p-A-like and we also have $\mathrm{Per}_h = \mathrm{Fix}_{h^m}$, that agrees with $\mathrm{Fix}_{g^m} = \mathrm{Per}_g$. □

The previous lemma establishes "half" of the dynamical alternative. The other half follows from the next lemma.

Lemma 4.8. *Let G be a pseudo-fibered group. Suppose g,h are hyperbolic elements of G which share a fixed point p. Then every element of the subgroup generated by g,h has the same fixed points as g.*

Proof. Since p is fixed by g,h,g^{-1},h^{-1}, any element of the subgroup H of G generated by g,h fixes p. But Lemma 4.6 says no p-A-like element shares a fixed point of a hyperbolic element. Hence, all elements of such a subgroup must be hyperbolic.

Now we apply Theorem 4.5 to H by identifying $S^1 \setminus \{p\}$ with $\mathbb{R}$. Just as in Case 2 of the proof of Lemma 4.6, we conclude that H has a (unique) global fixed point in $S^1 \setminus \{p\}$. □

Combining Lemmas 4.7 and 4.8 with Theorem 2.6, we immediately conclude Theorem B. □

4.2. **Proof of Theorem A.** We now combine Theorem B with two known results to prove Theorem A. The first result is a very well-known tool in geometric group theory (for instance, see Ch. II.B of [11]):

Theorem 4.9 (Ping-pong lemma)**.** *Let G be a group acting on a set X. Let g_1, g_2 be elements of G. Suppose there exist disjoint nonempty subsets $X_1^+, X_1^-, X_2^+, X_2^-$ of X such that $g_i(X - X_i^-) \subset X_i^+, g_i^{-1}(X - X_i^+) \subset X_i^-$ for each $i = 1,2$. Then the subgroup generated by g_1, g_2 is free.* □

A proof of the second result we need can be found in many places, e.g. [17] or [22]:

Theorem 4.10 (Hölder)**.** *Let K be a subgroup of* $\mathrm{Homeo}^+(\mathbb{R})$ *which acts freely on $\mathbb{R}$. Then K is abelian.* □

Now the ping-pong lemma implies the first case of the Tits alternative:

Lemma 4.11. *Let G be a pseudo-fibered group and $g_1, g_2 \in G$. If* Per_{g_1} *and* Per_{g_2} *are disjoint, then there are powers of g_1 and g_2 that generate a non-abelian free subgroup of G.*

Proof. We can replace g_1 and g_2 by some powers that satisfy $\mathrm{Per}_{g_i} = \mathrm{Fix}_{g_i}$. (If g_i is hyperbolic, or properly p-A-like, no power needs to be taken. If g_i is elliptic p-A-like, take a power that is properly p-A-like). Take X_i^+ to be a neighborhood of the attracting fixed points of g_i and X_i^- to be a neighborhood of the repelling fixed points of g_i (for $i = 1,2$). Since Fix_{g_1} and Fix_{g_2} are disjoint by hypothesis, we can take X_i^+, X_i^- (for $i = 1,2$) to be all disjoint. Now let h_i be a high enough power of g_i so that $h_i(S^1 - X_i^-) \subset X_i^+$ and $h_i^{-1}(X - X_i^+) \subset X_i^-$, for $i = 1,2$. This is possible because of the dynamics of hyperbolic and properly p-A-like elements. By the Ping-Pong lemma, h_1 and h_2 generate a free subgroup. □

Hölder's theorem implies the alternative case:

Lemma 4.12. *Let G be a pseudo-fibered group, $g \in G$ and $P = \mathrm{Per}_g$. Consider the subgroup $H = \{h \in G : hP = P\}$ of G that leaves P invariant. Then:*

(1) $H = \{h \in G : \mathrm{Per}_h = P\}$
(2) *H is virtually abelian.*

Proof. Let $h \in H$. Since $hP = P$ and $P = \mathrm{Per}_g$ is finite, then $P \subset \mathrm{Per}_h$. Then by Theorem B we get that $\mathrm{Per}_h = \mathrm{Per}_g = P$. This proves the first assertion, the other inclusion being trivial.

For the second assertion, let $K \trianglelefteq H$ consist of elements that stabilize Per_g pointwise. That is,

$$K = \{h \in H \mid \mathrm{Fix}_h = \mathrm{Per}_g\}.$$

Note

$$[H : K] \leq \frac{|\mathrm{Per}_g|}{2},$$

so in particular, to show H is virtually abelian, it suffices to show K is abelian. Assertion (1) implies K acts freely on each component of $S^1 \setminus \mathrm{Per}_g$, each of which is an interval. So by Hölder's theorem, K is abelian. □

Theorem A now follows from Theorem B, since if H is a subgroup of G, then either there exist two elements of H with distinct periodic point sets, or H has a global set of periodic points. In the first case, apply Lemma 4.11; in the second case, apply Lemma 4.12. □

4.3. **Remarks on the proofs of Theorems A and B.** Conjecturally, non-elementary pseudo-fibered groups are word-hyperbolic. For word-hyperbolic groups, a stronger version of the Tits alternative holds: an infinite subgroup of a word-hyperbolic group either contains a free group of rank 2 or is virtually *cyclic*. To obtain this stronger Tits alternative, one needs to strengthen Lemma 4.12. Let H be a subgroup as in Lemma 4.12, and assume it is actually abelian. Then Ghys [17] provides an H-invariant measure on each connected component of $S^1 - \mathrm{Per}_H$. The problem is that this measure may not have full support. Even when the action of G on S^1 is minimal, it is still not clear if it can be shown that we have an invariant measure of full support on the complement of Per_H. The stronger Tits alternative would easily follow from there.

More directly, it is easy to verify that this stronger Tits alternative is equivalent to showing that the subgroup K in the proof of Lemma 4.12 is isomorphic to $\mathbb{Z}$. A stronger version of Hölder's theorem says that for any abelian subgroup of $\mathrm{Homeo}^+(\mathbb{R})$, there is a blowdown of $\mathbb{R}$ such that the induced action of the subgroup is faithful and by translations. Thus, showing K is $\mathbb{Z}$ is equivalent to showing that this translation action is not minimal.

Finally, we remark that if we knew that pseudo-Anosov-like elements really were pseudo-Anosov, the stronger Tits alternative would follow from the fact that two pseudo-Anosovs commute if and only if they are powers of some other pseudo-Anosov. This can be seen by considering the action of the mapping class group on Thurston's compactification of Teichmüller space: if two pseudo-Anosovs commute, they must fix the same axis. The mapping class group acts discretely on Teichmüller space, hence, the subgroup generated by the two pseudo-Anosovs must act discretely on the axis in Teichmüller space. Each pseudo-Anosov acts by translations on this axis, so we conclude that the subgroup the two generate is isomorphic to $\mathbb{Z}$.

5. Pseudo-fibered groups and convergence group actions on S^2

5.1. **Purely hyperbolic pseudo-fibered groups.** We now restrict our attention to the special class of purely hyperbolic pseudo-fibered groups. In [1], it was conjectured that such groups are always Fuchsian, or, equivalently, convergence subgroups of $\mathrm{Homeo}^+(S^1)$. While this conjecture remains open, Theorem C shows that, as expected, purely hyperbolic pseudo-fibered groups act on the 2-sphere as convergence groups. In the next subsection, we discuss a possible generalization of Theorem C to tame mixed pseudo-fibered groups. We remark that a similar idea of relating group action on S^1 with a geometric origin to convergence group action on S^2 has been carried out in the context of pseudo-Anosov flows in Fenley's program (see Section 4 of [13]).

To prove Theorem C, we begin by reviewing the construction of [1], which is responsible for the existence of an S^2 on which a pseudo-fibered group can act, and inspired by results of Cannon and Thurston [8]. Moore's theorem [21] implies that for any pseudo-fibered triple (G,Λ_1,Λ_2), there exists a quotient map $\pi : S^1 \to S^2$, constructed by first identifying two disks laminated by Λ_1 and Λ_2 (respectively) along their common boundary S^1, and then collapsing all the gaps of the Λ_i to points. Since each lamination is G-invariant, this induces a G-action on S^2 such that π is G-equivariant. We call this map π the *Cannon-Thurston map* for the pseudofibered triple (G,Λ_1,Λ_2). For details, one can also consult Section 14 of [8]. A basic observation about this construction is

Lemma 5.1. *Let (G,Λ_1,Λ_2) be a pseudo-fibered triple. Suppose there exists a sequence $(\bar{x}_i)$ of points in $S^2(=\pi(S^1))$ which converges to $\bar{x}$, and a sequence (g_i) of elements of G such that $g_i(\bar{x}_i)$ converges to $\bar{x}'$ in S^2. Then, passing to subsequences if necessary, there exists a sequence (x_i) of points in S^1 converging to x such that $g_i(x_i)$ converges to x' in S^1, where $\bar{x}_i = \pi(x_i)$ and $\bar{x}' = \pi(x')$.*

Proof. This is straightforward because S^1 is compact, and π is continuous, surjective and G-equivariant. □

We now state a few dynamical lemmas, after which we will prove Theorem C.

Lemma 5.2. *Let G be a group acting on S^1 such that there exists a G-invariant lamination with a rainbow at $p \in S^1$. Suppose there exists a sequence (g_i) of elements of G such that for any neighborhood U of p, $g_i(U)$ intersects U nontrivially for all large i. Then p is an accumulation point of some fixed points of the elements in the sequence (g_i).*

Proof. See the proof of Proposition 7.5 in [1]. □

Lemma 5.3. *Let $(G, \Lambda_1, \Lambda_2)$ be a pseudo-fibered triple. Suppose (x_i) is a sequence of points in S^1 which converges to x, and there exists a sequence (g_i) of elements of G such that $g_i(x_i)$ converges to x'. Then either x is an accumulation points of fixed points of the sequence $(g_{i+1}^{-1} \circ g_i)$ or x' is an accumulation point of fixed points of the sequence $(g_{i+1} \circ g_i^{-1})$. Moreover, if $x_i = x$ for all i, then x' must be an accumulation point of fixed points of (any subsequence of) the sequence $(g_{i+1} \circ g_i^{-1})$.*

Proof. This is a straightforward consequence of Lemma 5.2. See, for instance, the proof of Proposition 7.6 in [1]. □

Proof of Theorem C. Suppose the negation of the conclusion. Then there exists an infinite sequence of distinct elements (g_i) of G which does not have the convergence property, *i.e.*, the set $\{g_i\}$ does not act properly discontinuously on the set of triples of distinct points of S^2. More precisely, this means that, after passing to a subsequence of (g_i), there exist three convergent sequences in S^2

$$\overline{x}_i \to \overline{x},\ \ \overline{y}_i \to \overline{y},\ \ \overline{z}_i \to \overline{z},\ \ \overline{x}_i \neq \overline{y}_i \neq \overline{z}_i \neq \overline{x}_i,\ \ \overline{x} \neq \overline{y} \neq \overline{z} \neq \overline{x},$$

and three elements $\overline{x}', \overline{y}', \overline{z}' \in S^2$ such that

$$g_i(\overline{x}_i) \to \overline{x}',\ \ g_i(\overline{y}_i) \to \overline{y}',\ \ g_i(\overline{z}_i) \to \overline{z}',\ \ \overline{x}' \neq \overline{y}' \neq \overline{z}' \neq \overline{x}'.$$

We conclude that there exist sequences and points in S^1 such that

$$x_i \to x,\ \ y_i \to y,\ \ z_i \to z,\ \ x_i \neq y_i \neq z_i \neq x_i,\ \ x \neq y \neq z \neq x$$

and three elements $x', y', z' \in S^1$ such that

$$g_i(x_i) \to x',\ \ g_i(y_i) \to y',\ \ g_i(z_i) \to z',\ \ x' \neq y' \neq z' \neq x',$$

where in our notation, p is some fixed point in the preimage of $\overline{p}$ under π for any $\overline{p} \in S^2$, in accordance with Lemma 5.1. In words, we can lift sequences exhibiting the failure of G to act as a convergence group on S^2 to sequences exhibiting the failure of G to act as a convergence group on S^1.

Since Λ_1 and Λ_2 do not share any endpoints, for each $p \in \{x, y, z\}$, there exists a rainbow at p in at least one of the Λ_i. In particular, for each $p \in \{x, y, z\}$ there exists a leaf L_p which separates p from the other two points in $\{x, y, z\} \setminus \{p\}$ (which lamination L_p belongs to is not important, and L_x, L_y, L_z are not necessarily leaves of the same lamination). Passing to a subsequence, we may assume that each of the sequences of pairs of points described by the endpoints of the leaves $(g_i(L_x))$, $(g_i(L_y))$, $(g_i(L_z))$, $(g_i^{-1}(L_x))$, $(g_i^{-1}(L_y))$, $(g_i^{-1}(L_z))$ converges to a pair of points, which are possibly not distinct.

By Lemma 5.3, either at least two of x, y, and z are accumulation points of fixed points of the sequence $(g_{i+1}^{-1} \circ g_i)$ or at least two of x', y', and z' are accumulation points of fixed points of the sequence $(g_{i+1} \circ g_i^{-1})$. Without loss of generality, we assume that x' and y' are accumulation points of fixed points of the sequence $(g_{i+1} \circ g_i^{-1})$, possibly after exchanging the roles of g_i and g_i^{-1}. Furthermore, since each element of G has exactly two fixed points, there exists a subsequence $(g_{i_j+1} \circ g_{i_j}^{-1})$ of the sequence $(g_{i+1} \circ g_i^{-1})$ such that x' and y' are the *only* accumulation points of the fixed points of the $g_{i_j+1} \circ g_{i_j}^{-1}$. By the second statement of Lemma 5.3, if $p \in S^1$ is such that $g_i(p)$ converges to p', then p' must be either x' or y'. In particular, considering the sequence $(g_i(L_z))$, which converges to some pair of possibly nondistinct points $\{e_1, e_2\}$, what we have shown implies each e_i is either x' or y'.

If $e_1 \neq e_2$, since laminations are required to be closed, the limit of the sequence of leaves $(g_i(L_z)))$ must be a leaf connecting x' to y' in the lamination Λ_i containing L_z. But since $\overline{x}' = \pi(x')$ and $\overline{y}' = \pi(y')$ are assumed to be distinct, by the definition of the Cannon-Thurston map π, their preimages cannot be connected by a leaf. This contradiction implies $e_1 = e_2$.

Let's assume that $e_1 = e_2 = x'$. We will show that x' is not distinct from both y' and z'. In the case $e_1 = e_2 = y'$, the same argument would lead us to contradict nthat y' is not distinct from both x' and z'.

Let I_y be the closure of the connected component of $S^1 - L_z$ which contains y, and define I_z similarly for z. Take a nested sequence of closed neighborhoods (U_i) of x' such that $U_i \to x'$ as $i \to \infty$. Passing to a subsequence, one may assume that the endpoints of $g_i(L_z)$ are contained in U_i.

For each i, there are two possibilities: either $g_i(I_z) \subset U_i$, or $g_i(I_y) \subset U_i$. Suppose the former happens for infinitely many i. Since (z_i) converges to z, then for all large enough i, z_i is in I_z. Hence, $g_i(z_i) \in U_i$ for infinitely many i, so some subsequence of $g_i(z_i)$ converges to x'. But this is impossible since z' is assumed to be distinct from x', and $g_i(z_i)$ converges to z'. If instead $g_i(I_y) \subset U_i$ for infinitely many i, then we similarly contradict the assumption that $y' \neq x'$. □

We remark that this proof almost goes through to show that a purely hyperbolic pseudo-fibered group G acts as a convergence group on the circle $S^1 = \pi^{-1}(S^2)$. The only gap to consider is the case where $e_1 \neq e_2$.

5.2. **Future directions: tame mixed pseudo-fibered groups.** A theorem of Bowditch [3] says that if a group G acts on a compactum X as a uniform convergence group, then G is word-hyperbolic, and X is G-equivariantly homeomorphic to the boundary of G. Hence, the convergence group action obtained in the previous subsection can not be uniform in general, since, for example, Fuchsian groups act on the circle as purely hyperbolic pseudo-fibered groups and their boundaries are always homeomorphic to S^1, not S^2. On the other hand, one may hope to have a uniform convergence group action after adding some pseudo-Anosov-like elements. In this subsection, we formulate and provide evidence for Conjecture 5.7 on tame mixed pseudo-fibered groups. We conclude by relating it to Conjecture 1.1.

Recall that a mixed pseudo-fibered group is *tame* if the product of any two hyperbolic elements is again hyperbolic. The proof of the next lemma is immediate from our definitions.

Lemma 5.4. *Let G be a tame mixed pseudo-fibered group. Then the set H of all hyperbolic elements of G with the identity element is a normal subgroup of G.* □

We call the subgroup H of G in the previous lemma *the maximal purely hyperbolic subgroup* of G. The next lemma shows that pseudo-Anosov-like elements of G behave as expected in the context of Conjecture 1.1, *i.e.*that pseudo-Anosov-like elements of G behave like pseudo-Anosov elements of mapping class groups of surfaces.

Lemma 5.5. *Let G be a tame mixed pseudo-fibered group, and H be the the maximal purely hyperbolic subgroup of G. Then the conjugation action on H by any pseudo-Anosov-like element ϕ of G induces a nontrivial outer automorphism of H. Moreover, there is no periodic conjugacy class in H under this action.*

Proof. It suffices to prove the second statement. Let $h \in H$. Suppose $\phi^n h \phi^{-n}$ is conjugate to h for some n, *i.e.*, $\phi^n h \phi^{-n} = fhf^{-1}$ for some $f \in H$. This implies that $\mathrm{Fix}_h = f^{-1}\phi^n(\mathrm{Fix}_h)$. By the tameness condition, $f^{-1}\phi^n$ is not hyperbolic, hence, by the dynamical alternative Theorem B, a fixed point of h cannot be a fixed point of $f^{-1}\phi^n$. So $f^{-1}\phi^n$ must exchange the attracting fixed point of h with the repelling fixed point of h. This is impossible, since by definition, no p-A-like element can swap two points. □

Let G be a tame mixed pseudo-fibered group G and ϕ be a p-A-like element of G. Let G_ϕ denote the subgroup $H \rtimes \langle \phi \rangle$ of G where H is the maximal purely hyperbolic subgroup of G and ϕ acts on H by conjugation. We call G_ϕ a mapping torus subgroup of G with monodromy ϕ. We propose the following conjecture as an extension of Theorem C.

Conjecture 5.6. *Let G be a tame mixed pseudo-fibered group, and H be the the maximal purely hyperbolic subgroup of G. Then for any p-A-like element ϕ of G, the mapping torus subgroup G_ϕ with monodromy ϕ acts on S^2 as a convergence group.*

A false proof. Since G is pseudo-fibered, it has a pair of very full, loose invariant laminations Λ_1, Λ_2, and a corresponding Cannon-Thurston map $\pi : S^1 \to S^2$. Let (g_i) be an infinite sequence of distinct elements of G_ϕ and suppose (g_i) does not have the convergence group property when acting on $S^2 = \pi(S^1)$. Then there exist convergent sequences in S^2

$$x_i \to x,\ \ y_i \to y,\ \ z_i \to z,\ \ x_i \neq y_i \neq z_i \neq x_i,\ \ x \neq y \neq z \neq x$$

and three elements $x', y', z' \in S^2$ such that

$$g_i(x_i) \to x',\ \ g_i(y_i) \to y',\ \ g_i(z_i) \to z',\ \ x' \neq y' \neq z' \neq x'.$$

By definition of G_ϕ, we can write $g_i = \phi^{n_i} h_i$ for some integer n_i and an element h_i of H. Passing to subsequences if necessary, we can assume one of the following three possibilities holds:

(1) Suppose $n_i = n$ for all i. Then $h_i = \phi^{-n} \circ g_i$, so there are convergent sequences

$$h_i(x_i) \to \phi^{-n}(x'),\ \ h_i(y_i) \to \phi^{-n}(y'),\ \ h_i(z_i) \to \phi^{-n}(z').$$

Since ϕ^{-n} is a homeomorphism, $\phi^{-n}(x'), \phi^{-n}(y')$ and $\phi^{-n}(z')$ are all distinct. But this contradicts Theorem C, which says H acts on S^2 as a convergence group.

(2) Suppose $h_i = h$ for all i. Letting $h(x_i), h(y_i), h(z_i), h(x), h(y), h(z)$ play the roles of x_i, y_i, z_i, x, y, z shows the sequence (ϕ^{n_i}) does not have the convergence group property. This is also impossible, since the cyclic group $\langle \phi \rangle$ acts on S^2 as a convergence group.

(3) Finally, suppose all h_i are distinct and all n_i are distinct. Passing to subsequences, we may assume there exist points x'', y'' and z'' in S^2 such that

$$h_i(x_i) \to x'',\ \ h_i(y_i) \to y'',\ \ h_i(z_i) \to z''.$$

Since (h_i) is an infinite sequence of distinct elements of the convergence group H, at least two of x'', y'', z'' must coincide. Without loss of generality, suppose $x'' = y''$ and $n_i \to +\infty$. Then for a small compact neighborhood K of x'' in S^2 and a small neighborhood U of the attracting fixed point of ϕ, we see $\phi^{n_i}(K) \subset U$ for all large i. But since $x' \neq y'$, U can be taken small enough so that not both of x' and y' are contained in U. This is now a contradiction, since $\phi^{n_i} h_i(x_i) \to x'$ and $\phi^{n_i} h_i(y_i) \to y'$ while $h_i(x_i), h_i(y_i) \in K$ for all large i. ⊠

The only reason why this is a false proof is that in Case 3, x'' may be a fixed point of ϕ, in which case U does not exist. At the moment, we do not know how to resolve this issue. Nevertheless, we propose the following even stronger conjecture.

Conjecture 5.7. *Let G_ϕ be as in the above conjecture. The action constructed in the false proof is a uniform convergence group.*

Together with Bowditch's theorem, Conjecture 5.7 would imply that G_ϕ is hyperbolic with boundary homeomorphic to S^2. Assuming Cannon's conjecture, one would conclude that G_ϕ is a fibered 3-manifold group, thereby proving Conjecture 1.1 for such groups. However, we suspect that Conjecture 1.1 can be proved independently of Cannon's conjecture. For example, if one could strengthen the conclusion of Theorem C to show that purely hyperbolic pseudo-fibered groups act as convergence groups on the circle S^1, then it would follow from the convergence subgroup theorem that such groups are fundamental groups of hyperbolic surfaces; in turn, Lemmas 5.4, 5.5 and the Dehn-Nielsen-Baer theorem (see Theorem 8.1 of [12]) would imply that G_ϕ is the fundamental group of a fibered hyperbolic 3-manifold.

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