BERNSTEIN-SATO POLYNOMIALS FOR PROJECTIVE HYPERSURFACES WITH WEIGHTED HOMOGENEOUS ISOLATED SINGULARITIES

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Abstract. We present a quite efficient method to compute the roots of Bernstein-Sato polynomial of a homogeneous polynomial if the associated projective hypersurface has only weighted homogeneous isolated singularities (so that its local Bernstein-Sato polynomials are uniquely determined by weights) and if a certain condition is satisfied. In the three variable case, the last condition holds except for polynomials of quite special type (that is, extremely degenerated ones) as far as calculated. The computation of roots is reduced to that of the Hilbert series of graded Milnor algebras, which can be done instantly by computers (unless the degree is huge), although it takes much longer to get the Bernstein-Sato polynomial itself (that is, with multiplicities) using a computer program in general. For the proof of the formula, we prove the E_2-degeneration of the pole order spectral sequence. We have a simpler formula for the roots of Bernstein-Sato polynomials if the projective hypersurfaces have only ordinary double points or the Tjurina number is relatively small.

Introduction

Let f be a homogeneous polynomial of n variables with $n \geqslant 3$ and $d := \deg f \geqslant 3$. Put $Z := \{f = 0\} \subset Y := \mathbb{P}^{n-1}$. Let $b_f(s)$ be the Bernstein-Sato polynomial of f. Set

$$\mathcal{R}_f := \big\{\alpha \in \mathbb{Q} \mid b_f(-\alpha) = 0\big\},$$

and similarly for $\mathcal{R}_{h_z}$ replacing f by a local defining function h_z of $(Z, z) \subset (Y, z)$ $(z \in Z)$. Define the set of roots of Bernstein-Sato polynomial *supported at the origin* $\mathcal{R}_f^0$ by

$$\mathcal{R}_f^0 := \mathcal{R}_f \setminus \mathcal{R}_Z \quad \text{with} \quad \mathcal{R}_Z := \textstyle\bigcup_{z\in \mathrm{Sing}\, Z} \mathcal{R}_{h_z} \subset \mathcal{R}_f. \tag{1}$$

Here the last inclusion follows from the equality $b_{h_z}(s) = b_{f,y}(s)$ with $b_{f,y}(s)$ the local Bernstein-Sato polynomial of f at $y \in \mathbb{C}^n \setminus \{0\}$ with $[y] = z$ in $Y = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$. (This follows from the assertion that $b_{f,y}(s)$ depends only on $f^{-1}(0)$, see Remark (4.2)(i) below.)

Set $F_f := f^{-1}(1) \subset \mathbb{C}^n$ (the Milnor fiber of f). We have the pole order filtration P on each monodromy eigenspace $H^j(F_f, \mathbb{C})_\lambda$ $(\lambda \in \mathbb{C}^*)$. Recall the following.

Theorem 1 ([Sa4, Theorem 2]). *Assume $\alpha \notin \mathcal{R}_Z$. If α satisfies the condition*

$$\alpha \notin \mathcal{R}_Z + \mathbb{Z}_{<0}, \tag{2}$$

then

$$\alpha \in \mathcal{R}_f^0 \iff \mathrm{Gr}_P^p H^{n-1}(F_f, \mathbb{C})_\lambda \neq 0 \ \big(p = [n-\alpha],\ \lambda = e^{-2\pi i\alpha}\big). \tag{3}$$

If condition (2) *does not hold, then only the implication* $\Longleftarrow$ *holds in* (3).

Remark 1. For the moment there is no example such that the implication $\Longrightarrow$ in (3) fails without assuming condition (2). This problem is related to the *asymptotic expansion* of local sections of $\mathcal{D}_X[s]f^s$ via the V-filtration, and is rather complicated, see Remark (4.2)(iii) below.

Theorem 1 gives a partial generalization of a well-known theorem in the isolated singularity case asserting that the Steenbrink spectral numbers coincide with the roots of the microlocal Bernstein-Sato polynomial $\widetilde{b}_f(s) := b_f(s)/(s+1)$ up to a sign by forgetting the multiplicities in the case f is a weighted homogeneous polynomial, see (1.9.2) below.

We have the *pole order spectral sequence* associated with the *pole order filtration* on the (algebraic) *microlocal Gauss-Manin complex* of f. Set

$$R := \mathbb{C}[x_1, \dots, x_n],$$

with $x_1, \dots, x_n$ the coordinates of $\mathbb{C}^n$. This is a graded ring with $\deg x_i = 1$. Let $\Omega^\bullet$ be the complex of the exterior products of the Kähler differentials of R over $\mathbb{C}$ so that Ω^p is a free R-module of rank $\binom{n}{p}$. This complex has anti-commuting two differentials d and $\mathrm{d}f\wedge$ preserving the grading (up to the shift by $d = \deg f$ in the case of $\mathrm{d}f\wedge$), where each component of the complex is graded with $\deg x_i = \deg \mathrm{d}x_i = 1$.

In this paper we assume the *isolated singularity condition*:

$$\text{(IS)} \qquad \sigma_Z := \dim \Sigma = 0 \quad \text{with} \quad \Sigma := \operatorname{Sing} Z.$$

We have the vanishing

$$H_{\mathrm{d}}^j\big(H_{\mathrm{d}f\wedge}^\bullet \Omega^\bullet\big) = 0 \quad \text{unless} \quad j = n-1 \quad \text{or} \quad n, \tag{4}$$

where $H_{\mathrm{d}f\wedge}^\bullet$ means that the cohomology with respect to the differential $\mathrm{d}f\wedge$ is taken, and similarly for $H_{\mathrm{d}}^\bullet$. These are members of the E_2-term of the spectral sequence associated with the double complex with anti-commuting two differentials d and $\mathrm{d}f\wedge$ on $\Omega^\bullet$, see also (1.1.1) below. The latter has been studied in [Di1], although the relation to the Gauss-Manin system and the Brieskorn modules was not mentioned there. Note that the *usual* (that is, non-microlocal) pole order spectral sequence was considered there, but its E_2-degeneration is equivalent to that of the *microlocal* one (see [DiSa3, Corollary 4.7]), and the latter is related to the pole order filtration as in (1.2.8) below. It has been observed in many examples by A. Dimca and G. Sticlaru [DiSt2] that the pole order spectral sequence degenerates at E_2 if the singularities of Z are isolated and weighted homogeneous. We have the following.

Proposition 1. *Under the assumption* (IS)*, the E_2-degeneration of the pole order spectral sequence is equivalent to each of the following two conditions*:

$$\dim H_{\mathrm{d}}^n\big(H_{\mathrm{d}f\wedge}^\bullet \Omega^\bullet\big) = \dim H^{n-1}(F_f, \mathbb{C}). \tag{5}$$

$$\dim H_{\mathrm{d}}^{n-1}\big(H_{\mathrm{d}f\wedge}^\bullet \Omega^\bullet\big) = \dim H^{n-2}(F_f, \mathbb{C}). \tag{6}$$

If these equivalent conditions are satisfied, then the pole order spectrum $\mathrm{Sp}_P(f)$ (*see* [DiSa3]) *is given by the difference of the Hilbert series of the graded $\mathbb{C}$-vector spaces*

$$M^{(2)} := H_{\mathrm{d}}^n\big(H_{\mathrm{d}f\wedge}^\bullet \Omega^\bullet\big), \quad N^{(2)} := H_{\mathrm{d}}^{n-1}\big(H_{\mathrm{d}f\wedge}^\bullet \Omega^\bullet\big)(-d),$$

with variable t of the Hilbert series replaced by $t^{1/d}$ (where (m) for $m \in \mathbb{Z}$ denotes the shift of grading so that $E(m)_k := E_{m+k}$ for any graded module E), and moreover, in the notation of (3)*, we have the canonical isomorphisms*

$$M_k^{(2)} = \mathrm{Gr}_P^p H^{n-1}(F_f, \mathbb{C})_\lambda \quad \big(\alpha = \tfrac{k}{d},\ p = [n-\alpha],\ \lambda = e^{-2\pi i\alpha}\big). \tag{7}$$

In this paper we prove the following.

Theorem 2. *Under the assumption* (IS)*, assume further*

(W) *Every singularity of Z is analytically defined by a weighted homogeneous polynomial locally on Y, see* (1.8) *below.*

Then the pole order spectral sequence degenerates at E_2 so that (7) *holds.*

This is quite nontrivial even in the case $n = 3$. Without condition (W), Theorem 2 never holds, see [DiSa3, Theorem 5.2]. The assumption (W) is equivalent to that every singularity of Z is *quasihomogeneous*, that is, $h_z \in (\partial h_z)$ $(\forall\, z \in \Sigma)$, where h_z is as in (1), and (∂h_z) is the Jacobian ideal generated by the partial derivatives of h_z, see [SaK] (and (1.8) below). Note that $\mathcal{R}_{h_z}$ can be determined only by the *weights* in the weighted homogeneous isolated singularity case, see (1.9.2–3) and (A.1) below. This is *quite different* from the non-quasihomogeneous isolated hypersurface singularity case where we need a computer program to determine the local Bernstein-Sato polynomials of Z in general.

Combining Theorem 2 with [DiSa3], [Sa4], we can show the following.

Corollary 1. *Under the assumptions of Theorem* 2*, any* $\alpha \in \mathcal{R}_f^0$ *satisfying condition* (2) *in Theorem* 1 *can be detected by using the Hilbert series of the graded* $\mathbb{C}$*-vector space* $M^{(2)}$ *in Proposition* 1*.*

Set

$$M := H^n_{\mathrm{d}f\wedge}\Omega^\bullet, \quad N := H^{n-1}_{\mathrm{d}f\wedge}\Omega^\bullet(-d).$$

These are graded $\mathbb{C}$-vector spaces. Let y be a sufficiently general linear combination of the coordinates x_i of $\mathbb{C}^n$, and $M' \subset M$ be the y-torsion part. Set $M'' := M/M'$. Note that M' is a finite dimensional graded $\mathbb{C}$-vector subspace of M, and is independent of the choice of a sufficiently general y, see for instance [DiSa3].

Theorem 3. *Under the assumptions of Theorem* 2*, we have the injectivity of the composition of canonical morphisms*

$$M' \hookrightarrow M \twoheadrightarrow M^{(2)}.$$

From Theorems 1, 2 and 3, we can deduce

Corollary 2. *Under the hypotheses of Theorem* 2*, assume further* $M'_k \ne 0$ *for some* k*. Then* $M_k^{(2)} \ne 0$*, and hence* $\frac{k}{d} \in \mathcal{R}_f$*.*

These do not hold if the assumption (W) of Theorem 2 is unsatisfied, see [Sa6].

Let $\tau_Z := \sum_{z\in \mathrm{Sing}\, Z} \tau_{h_z}$ with τ_{h_z} the *Tjurina number* of a local defining function h_z, that is, $\tau_{h_z} = \dim \mathcal{O}_{Y,z}/\big((\partial h_z), h_z\big)$. Under the assumption of Theorem 2, this coincides with the Milnor number $\mu_{h_z} := \dim \mathcal{O}_{Y,z}/(\partial h_z)$. We have the inequalities (see for instance [DiSa3]):

$$\tag{8} \begin{array}{l} \mu''_k := \dim M''_k \leqslant \tau_Z, \quad \nu_k := \dim N_k \leqslant \tau_Z \quad (\forall\, k \in \mathbb{Z}), \\ \text{where the equalities hold if } \; k \geqslant nd. \end{array}$$

The differential d of $\Omega^\bullet$ induces $\mathrm{d}^{(1)} : N_{k+d} \to M_k$ $(k \in \mathbb{Z})$ so that $M_k^{(2)}$, $N_{k+d}^{(2)}$ are respectively its cokernel and kernel. It is easy to calculate the dimensions of M_k, N_{k+d} by using computers, and moreover, under the assumption (W), we have the following:

$$\tag{9} \text{The composition } \; N_{k+d} \xrightarrow{\mathrm{d}^{(1)}} M_k \twoheadrightarrow M''_k \; \text{ is } \textit{injective} \text{ if } \; \tfrac{k}{d} \notin \mathcal{R}_Z.$$

This follows from [DiSa3, Theorem 5.3 and Remark 5.6(i)] together with the assertion that the Hodge and pole order filtrations coincide in the case of *weighted homogeneous isolated singularities,* see (1.9.2) below.

Remark 2. In order to determine $\mathcal{R}_f^0$, we do *not have to calculate* $\mathrm{d}^{(1)} : N_{k+d} \to M_k$ in (9) for $\frac{k}{d} \in \mathcal{R}_Z$ because of the last inclusion $\mathcal{R}_Z \subset \mathcal{R}_f$ in (1). In particular, the information of $H^{n-2}(F_f, \mathbb{C})$ (which is given by the kernel of $\mathrm{d}^{(1)}$) is unnecessary to determine $\mathcal{R}_f^0$.

Let $(\partial f) \subset R$ be the *Jacobian ideal* generated by the partial derivatives of f. Then M is identified with the *Milnor algebra* $R/(\partial f)$ up to the shift of grading by $-n$. It is well known that the $\nu_k := \dim N_k$ are expressed by the $\mu_k := \dim M_k$; more precisely

$$\tag{10} \nu_k = \mu_k - \gamma_k \quad (\forall\, k \in \mathbb{Z}) \qquad \text{with} \qquad \textstyle\sum_{k\in\mathbb{Z}} \gamma_k\, t^{\,k} = \big(t + \cdots + t^{\,d-1}\big)^n,$$

see [DiSa3, Formula (3)]. (This follows from the assertion that the Euler characteristic of a bounded complex of finite dimensional vector spaces is independent of the differential.)

To deal with the difficulty associated with condition (2) in Theorem 1, define the *critical set* of f by

$$\mathrm{CS}(f) := \big\{k \in \mathbb{Z} \;\big|\; \tfrac{k}{d} \in [\alpha_Z,\, n-2-\alpha_Z] \cap (\mathcal{R}_Z + \mathbb{Z}_{<0}) \setminus \mathcal{R}_Z\big\} \subset \mathbb{Z},$$

where $\alpha_Z := \min \mathcal{R}_Z$. (Note that $\mathrm{CS}(f)$ is determined only by $\mathcal{R}_Z$.) We have the following.

Theorem 4. *Assume condition* (W) *in Theorem* 2 *hold and moreover*

$$\mu_k > \nu_{k+d} \quad \textit{for any} \quad k \in \mathrm{CS}(f). \tag{11}$$

Then $\mathcal{R}_f^0$ can be determined by the μ_k $(k \in \mathbb{Z})$. More precisely, we have the inclusion

$$\mathcal{R}_f^0 \subset \big[\tfrac{n}{d}, n\big) \cap \tfrac{1}{d}\,\mathbb{Z}, \tag{12}$$

and for $\frac{k}{d} \in \big(\big[\frac{n}{d}, n\big) \cap \frac{1}{d}\,\mathbb{Z}\big) \setminus \mathcal{R}_Z$, we have $\frac{k}{d} \in \mathcal{R}_f^0$ if and only if $\mu_k > \nu_{k+d}$.

Here we must have the inclusion $\mathrm{CS}(f) \subset \big[n, \infty\big)$ in order that condition (11) hold (since $\mu_k = 0$ for $k < n$). This inclusion is satisfied if $\alpha_Z \geqslant \frac{n-1}{d}$ (since $n-1 \notin \mathrm{CS}(f)$ in the case $\frac{n-1}{d} = \alpha_Z \in \mathcal{R}_Z$). Note that the last inequality holds if $\frac{n-1}{d} \leqslant 1$ (for instance, if $n = 3$ and $d \geqslant 2$) according to [Ch], [dFEM, Corollary 3.6], [MuPo, Section 26]. (Here the semicontinuity theorem as in [St3], [Va2] does not seem to be enough according to a discussion with J. Stevens.) Note also that $\big[\frac{n}{d}, \alpha_Z\big) \cap \frac{1}{d}\,\mathbb{Z} \subset \mathcal{R}_f$ by (4.2.6) below.

Condition (11) in Theorem 4 is a kind of device invented to *avoid* the difficulty related to condition (2) as is explained in Remark 1. It is expected that condition (11) hold, for instance, in the projective line arrangement case. However, it does *not* hold, for instance, in the case $f = x^4y^2z + z^7$ (or $f = x^6y^3 - z^9$), where Z is *reducible* and $\chi(U) = 1$ with $U := Y \setminus Z$, see Example (5.4) below. Such examples are restricted so far to the case of *extremely degenerated* curves (see (1.11) below for definition). We have $\chi(U) = 0$ or 1 for extremely degenerated curves, but we do not know any example such that $\chi(U) = 0$ and condition (11) is unsatisfied. An extremely degenerated curve satisfying the condition $\mathcal{R}_Z \cap \frac{1}{d}\,\mathbb{Z} \neq \{1\}$ is always *reducible*. We have the following.

Question 1. Are there any examples such that condition (11) does *not* hold and Z is *irreducible* in the case $n = 3$?

For the moment we do not know any examples as in Question 1, and it seems quite interesting if there are really such examples. The condition $\mu_k > \nu_{k+d}$ in (11) would be satisfied easily if $|\chi(U)|$ is not sufficiently small, see (2.9.2) below. So $|\chi(U)|$ must be very small, and for this, the Milnor number of Z must be quite large, see (2.9.4) and Remark (5.5) below. It is rather difficult to construct examples satisfying the conditions that $\alpha_Z < \frac{n}{d}$ and $|\chi(U)|$ is very small with $n = 3$, except for extremely degenerated curves, because of the condition that the singularities of Z must be *weighted homogeneous*, where *"semi-weighted-homogeneous"* is not enough (for instance, $f = (yz - x^2)^3 - y^6$). It is known that the latter is equivalent to a μ-constant deformation of a weighted homogeneous polynomial.

Set

$$\delta_k := \mu_k - \nu_{k+d}, \qquad \mathrm{Supp}\,\{\delta_k\} := \{k \in \mathbb{Z} \mid \delta_k \neq 0\}. \tag{13}$$

The latter is called the *support* of $\{\delta_k\}$. Assuming (W), we have by Theorem 2 and (9)

$$\delta_k \leqslant \mu_k^{(2)} := \dim M_k^{(2)} \;\; (\forall\, k \in \mathbb{Z}), \;\text{ where the equality holds if } \tfrac{k}{d} \notin \mathcal{R}_Z. \tag{14}$$

In any calculated examples, the following is observed:

$$\begin{gathered} \mathrm{Supp}\,\{\delta_k\} \setminus d\,\mathcal{R}_Z = \{k_{\min}, \dots, k_{\max}\} \setminus d\,\mathcal{R}_f \quad \text{in} \quad \mathbb{Z}, \\ \text{with} \;\; k_{\min} := \min\big(\mathrm{Supp}\,\{\delta_k\} \setminus d\,\mathcal{R}_Z\big) \;\text{ and similarly for }\; k_{\max}. \end{gathered} \tag{15}$$

Here it is also possible that $\mathrm{Supp}\,\{\delta_k\}\setminus d\,\mathcal{R}_Z=\emptyset$. This may sometime occur especially when $|\chi(U)|\leqslant 1$. We say that $\mathrm{Supp}\,\{\delta_k\}$ is *discretely connected outside* $d\,\mathcal{R}_Z\cap\mathbb{Z}$ if (15) holds or $\mathrm{Supp}\,\{\delta_k\}\setminus d\,\mathcal{R}_Z=\emptyset$. The discrete connectedness of $\mathrm{Supp}\,\{\delta_k\}$ *in* $\mathbb{Z}$ may fail only at $k\in d\,\mathbb{Z}$ and only in the case $|\chi(U)|\leqslant 1$ for the moment, see Examples (5.2) and (5.7) below.

Question 2. Is $\mathrm{Supp}\,\{\delta_k\}$ discretely connected outside $d\,\mathcal{R}_Z\cap\mathbb{Z}$ at least when condition (W) is satisfied?

This seems to hold always as far as calculated. Setting $\mu'_k:=\dim M'_k$, $\mu''_k:=\dim M''_k$, we have the decomposition

$$\delta_k=\delta'_k+\delta''_k\quad\text{with}\quad \delta'_k:=\mu'_k,\quad \delta''_k:=\mu''_k-\nu_{k+d}\quad(k\in\mathbb{Z}),\tag{16}$$

so that $\{\delta'_k\}$, $\{\delta''_k\}$ have *symmetries* with centers $nd/2$ and $(n-1)d/2$ respectively, that is,

$$\delta'_k=\delta'_{nd-k}\,,\qquad \delta''_k=\delta''_{(n-1)d-k}\,.\tag{17}$$

(These follow from $\mu'_k=\mu'_{nd-k}$, $\mu''_k=\tau_Z-\nu_{nd-k}$, see [DiSa3, Corollaries 2 and 3].)

It is known that the support of $\{\delta'_k\}=\{\mu'_k\}$ is discretely connected for $n=3$, according to [DiPo] (generalizing some arguments in [BrKa]). However, this does not hold if $n=4$ and condition (W) is *unsatisfied*, according to Aldo Conca, see [Sti] (and Remark (5.8) below). We have $\delta''_k\geqslant 0$ for $\frac{k}{d}\notin\mathcal{R}_Z$ by (9) assuming condition (W), although we may have $\delta''_k=0$ ($\forall\,k\in\mathbb{N}$), see Example (5.4) below. (See also Remark (4.11)(v) below for good examples.)

Set

$$\widetilde{\alpha}_Z:=\min\widetilde{\mathcal{R}}_Z\quad\text{with}\quad \widetilde{\mathcal{R}}_Z:=\textstyle\bigcup_{z\in\mathrm{Sing}\,Z}\widetilde{\mathcal{R}}_{h_z}\subset\mathcal{R}_Z.$$

Here $\widetilde{\mathcal{R}}_{h_z}$ is the roots of the reduced Bernstein-Sato polynomial $\widetilde{b}_{h_z}(s):=b_{h_z}(s)/(s+1)$ up to a sign with h_z as in (1). We have a positive answer to Question 2 in the ordinary double point case or in the case the Tjurina number τ_Z is relatively small where the condition is given in terms of the γ_k and $\widetilde{\alpha}_Z$ (see Theorem (4.10) below for a more precise statement):

Theorem 5. *Assume $d\geqslant n$ and either all the singularities of Z are ordinary double points or condition* (W) *holds together with the strict inequality*

$$\tau_Z<\min\bigl(2\gamma_{m_0},\,\gamma_{m_1}\bigr)\ \ \bigl(m_0:=\lceil d\widetilde{\alpha}_Z\rceil,\ m_1:=\min\bigl(\lceil\tfrac{d(n-1)}{2}\rceil,\lceil d(\widetilde{\alpha}_Z+1)\rceil\bigr)\bigr),\tag{18}$$

where γ_k is as in (10) *and $\lceil\beta\rceil:=\min\{k\in\mathbb{Z}\mid k\geqslant\beta\}$ for $\beta\in\mathbb{R}$ in general. Then we have a positive answer to Question* 2*, and moreover* (15) *holds with*

$$\begin{aligned}&k_{\min}=n,\quad k_{\max}=n(d-1)-\min(d,\beta_f),\\&\beta_f:=\min\{k\in\mathbb{N}\mid\mu'_k\neq 0\}-n\quad(\textit{see also }(23)\textit{ below}).\end{aligned}\tag{19}$$

The hypothesis (18) implies that $\widetilde{\alpha}_Z>\frac{n}{d}$. Indeed, we have $\gamma_{m_0}\geqslant 2$ unless $\tau_Z=1$ (where $\widetilde{\alpha}_Z=(n-1)d/2>\frac{n}{d}$). Theorem 5 is not necessarily very efficient for *explicit* calculations of $\mathcal{R}_f^0$, since we have to examine whether condition (18) holds after the computation of the γ_k and β_f (using (A.3) below). It seems better to use Corollary 3 below calculating the δ_k (using (A.4) below) for explicit calculations.

For the relation with $|\chi(U)|$, we have the following for any $k\in\{1,\dots,d-1\}$:

$$(-1)^{n-1}\chi(U)=\textstyle\sum_{j=0}^{n-1}\gamma_{jd+k}-\mu_Z=\sum_{j=1}^{n-1}\gamma_{jd}-\mu_Z-(-1)^n,\tag{20}$$

see (2.9) below. Here $\mu_Z=\tau_Z$ under the assumption (W). Condition (18) then implies that $|\chi(U)|$ cannot be small (unless d is very small). It seems interesting whether the conclusion of Theorem 5 holds by assuming only $\tau_Z<\gamma_d$, $\widetilde{\alpha}_Z>\frac{n}{d}$ together with condition (W).

If Z has only singularities of type A,D,E, we have

$$\tfrac{d(n-2)}{2}<d\widetilde{\alpha}_Z\leqslant\tfrac{d(n-1)}{2}\quad\text{so that}\quad m_1=\lceil\tfrac{d(n-1)}{2}\rceil,\tag{21}$$

and the inequality (18) is quite often satisfied for $n \geqslant 4$ (where $\widetilde{\alpha}_Z > 1 \geqslant \frac{n}{d}$), but not very often for $n = 3$. If furthermore Z has only ordinary double points, we have

$$\widetilde{\mathcal{R}}_Z = \{\widetilde{\alpha}_Z\} = \{\tfrac{n-1}{2}\}, \quad m_0 = m_1 = \lceil \tfrac{d(n-1)}{2} \rceil, \tag{22}$$

and the inequality (18) does not necessarily hold (even though the conclusion of Theorem 5 is true). Indeed, γ_{m_1} coincides with Arnold's number $A_{n-1}(d)$ so that the inequality $\tau_Z \leqslant \gamma_{m_1}$ holds in this case by [Va2], although the *equality* may occur, see Remark (4.11)(ii) below.

In general we have

$$\beta_f = \min\bigl\{k \in \mathbb{N} \mid \exists\, g \in R_k \setminus (\partial f)_k \ \text{ such that } \ \tilde{g} \in \mathcal{O}_Y(k) \otimes_{\mathcal{O}_Y} (\partial f)^{\sim}\bigr\}, \tag{23}$$

where $\tilde{g}$ is the section of $\mathcal{O}_Y(k)$ associated with $g \in R_k$, and $(\partial f)^{\sim} \subset \mathcal{O}_Y$ is the ideal sheaf associated with the graded ideal $(\partial f) \subset R$, see [Ha2]. In the ordinary double point case, the condition for $\tilde{g}$ in (23) is equivalent to that $\tilde{g}|_{\mathrm{Sing}\, Z} = 0$. In the latter case, it does not seem easy to construct an example with $\min(d, \beta_f) \ne \beta_f$, that is, $d < \beta_f$, when $n < 5$ (see Example (5.7) below for the case $n = 5$, $d = 3$, where $M' = 0$, that is, $\beta_f = +\infty$). Theorem 5 is proved by using a bound of k with $\nu_k = 0$, which is shown in [Di2], [DiSt1] (in the ordinary double point case) and [DiSa4] (in general), see (4.10) below.

Practically a fundamental problem seems to be the following.

Question 3. Does n belong to $\mathrm{Supp}\,\{\delta_k\} \setminus d\,\mathcal{R}_f$?

This depends on each polynomial f. In the case $\alpha_Z \,(= \min(\widetilde{\alpha}_Z, 1)) > \frac{n}{d}$, we have a positive answer by Remark (4.2)(iv) together with Theorem 2 and (9) (see also Remark (4.11)(v) below). If Questions 2 and 3 have positive answers so that $k_{\min}$ in (15) coincides with n, then we may determine $\mathcal{R}_f^0$. More precisely, Theorem 2 implies the following.

Corollary 3. *Assume condition* (W) *holds, and moreover* (15) *is satisfied with*

$$k_{\min} = n, \quad k_{\max} \geqslant \max\bigl(d\,\mathcal{R}_Z \cap \mathbb{Z}\bigr) - d. \tag{24}$$

Then

$$d\,\mathcal{R}_f^0 = \{n, \dots, k_{\max}\} \setminus d\,\mathcal{R}_Z.$$

The last condition of (24) is needed to show that $\frac{k}{d} \notin \mathcal{R}_f^0$ for $k > k_{\max}$ (using Theorem 1). It may be replaced by the condition that $k_{\max} \geqslant d-1$ when $n = 3$. As for the first condition $k_{\min} = n$ in (24), we have always $k_{\min} \geqslant n$, where the strict inequality can occur (so that Question 3 has a *negative* answer) *only if* $\alpha_Z \leqslant \frac{n}{d}$ by using Remark (4.2)(iv) below (for instance, if $f = x^4y^2z + z^7$), see the remark after Question 3. *If $k_{\min} > n$, we cannot determine whether $\frac{k}{d} \in \mathcal{R}_f^0$ for $k \in [n, k_{\min}-1]$ with $\frac{k}{d} \in \bigl(\mathcal{R}_Z + \mathbb{Z}_{<0}\bigr) \setminus \mathcal{R}_Z$* (because of condition (2) in Theorem 1). Note that it does not matter whether $\alpha_Z > \frac{n}{d}$ or not, if $k_{\min} = n$ (see for instance Remark (4.11)(v) below). It is conjectured that the hypothesis of Corollary 3 is satisfied whenever $\alpha_Z > \frac{n}{d}$ assuming condition (W) (and $n = 3$ if necessary). If these conditions are satisfied(or if $|\chi(U)|$ is not small), it is recommended to see whether the hypothesis of Corollary 3 holds by using (A.4) below which calculates the δ_k explicitly.

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In Section 1 we review some basics of pole order spectral sequences associated with the algebraic microlocal Gauss-Manin systems, and prove Proposition 1 and also Theorem (1.5) about the compatibility of certain self-duality isomorphisms. In Section 2 we study the fundamental exact sequences of vanishing cycles, and prove Theorem (2.2) which is a key to the proof of Theorem 2. In Section 3 we calculate the filtered twisted de Rham complexes associated to weighted homogeneous polynomials with isolated singularities. In Section 4 we prove Theorems 2, 3 and 4. In Section 5 we calculate some examples explicitly. In Appendix we explain how to use the computer programs Macaulay2 and Singular for explicit calculations of roots of Bernstein-Sato polynomials.

1. Microlocal pole order spectral sequences

In this section we review some basics of pole order spectral sequences associated with the algebraic microlocal Gauss-Manin systems, and prove Proposition 1 and also Theorem (1.5) about the compatibility of certain self-duality isomorphisms.

1.1. Algebraic microlocal Gauss-Manin systems. For a homogeneous polynomial f, we have the *algebraic microlocal Gauss-Manin complex* $\widetilde{C}_f^{\bullet}$ defined by

$$\widetilde{C}_f^j := \Omega^j[\partial_t, \partial_t^{-1}] \quad \text{with differential} \quad \mathrm{d} - \partial_t \, \mathrm{d}f\wedge, \tag{1.1.1}$$

where $\Omega^{\bullet}$ is as in the introduction. It is a complex of graded $\mathbb{C}[t]\langle\partial_t, \partial_t^{-1}\rangle$-modules with

$$\deg t = -\deg \partial_t = d, \tag{1.1.2}$$

and the actions of t, ∂_t^i are defined by

$$t(\omega\partial_t^k) = (f\omega)\partial_t^k - k\,\omega\partial_t^{k-1}, \quad \partial_t^i(\omega\partial_t^k) = \omega\partial_t^{k+i} \quad (\omega \in \Omega^j,\, i \in \mathbb{Z}). \tag{1.1.3}$$

The *algebraic microlocal Gauss-Manin systems* are defined by

$$\widetilde{\mathcal{G}}_f^j := H^{n-j}\widetilde{C}_f^{\bullet} \qquad (j \in [0, \sigma_Z + 1]),$$

where $\sigma_Z := \dim \operatorname{Sing} Z$ as in the introduction. These are free graded $\mathbb{C}[\partial_t, \partial_t^{-1}]$-modules of finite type, and there are canonical isomorphisms

$$\widetilde{\mathcal{G}}_{f,k}^j = \widetilde{H}^{n-1-j}(F_f, \mathbb{C})_{\mathbf{e}(-k/d)} \qquad \big(\forall\, k \in \mathbb{Z}\big), \tag{1.1.4}$$

where $\mathbf{e}(\alpha) := \exp(2\pi i\alpha)$ for $\alpha \in \mathbb{Q}$, $F_f := f^{-1}(1)$ (which is viewed as the Milnor fiber of f), and $E_\lambda := \operatorname{Ker}(T_s - \lambda)$ in $E := \widetilde{H}^{\bullet}(F_f, \mathbb{C})$ with T_s the semisimple part of the monodromy T, see [DiSa2], [DiSa3], etc.

1.2. Pole order spectral sequences. In the notation of (1.1), we have the filtration P' on $\widetilde{C}_f^{\bullet}$ defined by

$$P'_i\,\widetilde{C}_f^j := \bigoplus\nolimits_{k \leqslant i+j} \Omega^j \partial_t^k. \tag{1.2.1}$$

This is an exhaustive increasing filtration, and induces the filtration P' on the Gauss-Manin systems $\mathcal{G}_f^j$ compatible with the grading. Since $\deg \partial_t = -d$, we have

$$\Omega^j \partial_t^k = \Omega^j(kd), \tag{1.2.2}$$

where (m) for $m \in \mathbb{Z}$ denotes the shift of grading as in Proposition 1. Set

$$P'^{\,i} = P'_{-i}.$$

We have

$$\operatorname{Gr}^i_{P'}\widetilde{C}_f^{\bullet} = K_f^{\bullet}((n-i)d). \tag{1.2.3}$$

Here $K_f^{\bullet}$ is the *Koszul complex* defined by

$$K_f^j := \Omega^j((j-n)d) \quad \text{with differential} \quad \mathrm{d}f\wedge. \tag{1.2.4}$$

We have the *microlocal pole order spectral sequence*

$${}_{P'}\widetilde{E}_1^{i,j-i} = H^j \operatorname{Gr}^i_{P'}\widetilde{C}_f^{\bullet} \Longrightarrow \widetilde{\mathcal{G}}_f^{j+n} \,\big(= H^j\widetilde{C}_f^{\bullet}\big). \tag{1.2.5}$$

In the notation of Proposition 1, we have

$$
{}_{P'}\widetilde{E}_r^{i,j-i} = \begin{cases} \big(H^j_{\mathrm{d}f\wedge}\Omega^\bullet\big)((j-i)d) = \big(H^jK_f^\bullet\big)((n-i)d) & \text{if } r=1, \\ H^j_{\mathrm{d}}\big(H^\bullet_{\mathrm{d}f\wedge}\Omega^\bullet\big)((j-i)d) & \text{if } r=2. \end{cases} \tag{1.2.6}
$$

Set

$$
P := P'[1], \quad \text{that is,} \quad P^i := P'^{\,i+1} \quad (i\in\mathbb{Z}). \tag{1.2.7}
$$

The isomorphisms in (1.1.4) induce the filtered isomorphisms

$$
\big(\widetilde{\mathcal{G}}^j_{f,k}, P\big) = \big(\widetilde{H}^{n-1-j}(F_f,\mathbb{C})_{\mathbf{e}(-k/d)}, P\big) \qquad \big(\forall\, k\in[1,d]\big), \tag{1.2.8}
$$

where P on the right-hand side is the *pole order filtration*, see [Di1], [DiSa2, Section 1.8].

By the definition (1.2.1) we have the filtered graded isomorphisms

$$
\partial_t^k : \big(\widetilde{C}_f^\bullet, P'\big) \xrightarrow{\sim} \big(\widetilde{C}_f^\bullet(-kd), P'[-k]\big) \quad (k\in\mathbb{Z}). \tag{1.2.9}
$$

Recall that $(P'[m])^i = P'^{i+m}$, $(P'[m])_i = P'_{i-m}$, and $G(m)_k = G_{m+k}$ for any graded module $G_\bullet$ and $m\in\mathbb{Z}$ in general.

By (1.2.9) we get the graded isomorphisms of spectral sequences for $k\in\mathbb{Z}$:

$$
\partial_t^k : {}_{P'}\widetilde{E}_r^{i,j-i} \xrightarrow{\sim} {}_{P'}\widetilde{E}_r^{i-k,j-i+k}(-kd) \quad (r\geqslant 1), \tag{1.2.10}
$$

which are compatible with the differentials d_r of the spectral sequences.

1.3. Isolated singularity case. In the notation of the introduction, assume $Z\subset Y$ has only isolated singularities, that is,

$$
\sigma_Z := \dim\Sigma = 0 \quad \text{with} \quad \Sigma := \operatorname{Sing} Z. \tag{1.3.1}
$$

Set

$$
M := H^nK_f^\bullet, \quad N := H^{n-1}K_f^\bullet, \tag{1.3.2}
$$

where $k_f^\bullet$ is as in (1.2.4). These are graded $\mathbb{C}$-vector spaces.

Let y be a sufficiently general linear combination of coordinates x_i of $\mathbb{C}^n$. Then M, N are finitely generated graded $\mathbb{C}[y]$-modules. It is known that N is y-torsion-free (see, for instance, [DiSa3]). Set

$$
M' := M_{\mathrm{tor}}, \quad M'' := M_{\mathrm{free}} = M/M_{\mathrm{tor}}, \tag{1.3.3}
$$

where M_{tor} denotes the y-torsion part of M. (The latter is independent of y as long as y is sufficiently general). These are also finitely generated graded $\mathbb{C}[y]$-modules. Set

$$
\begin{aligned} \widetilde{M} &:= M[y^{-1}] = M\otimes_{\mathbb{C}[y]}\mathbb{C}[y,y^{-1}] \;\big(= \widetilde{M}'' := M''[y^{-1}]\big), \\ \widetilde{N} &:= N[y^{-1}] = N\otimes_{\mathbb{C}[y]}\mathbb{C}[y,y^{-1}]. \end{aligned} \tag{1.3.4}
$$

These are finitely generated free graded $\mathbb{C}[y,y^{-1}]$-modules. By the grading, we have the direct sum decompositions

$$
M = \textstyle\bigoplus_{k\in\mathbb{Z}} M_k, \quad N = \bigoplus_{k\in\mathbb{Z}} N_k, \quad \widetilde{M} = \bigoplus_{k\in\mathbb{Z}} \widetilde{M}_k, \quad \widetilde{N} = \bigoplus_{k\in\mathbb{Z}} \widetilde{N}_k, \text{ etc.}
$$

By definition there are isomorphisms

$$
y : \widetilde{N}_k \xrightarrow{\sim} \widetilde{N}_{k+1}, \quad y : \widetilde{M}_k \xrightarrow{\sim} \widetilde{M}_{k+1} \quad (k\in\mathbb{Z}), \tag{1.3.5}
$$

together with natural inclusions

$$
N_k \hookrightarrow \widetilde{N}_k, \quad M''_k \hookrightarrow \widetilde{M}''_k = \widetilde{M}_k \quad (k\in\mathbb{Z}), \tag{1.3.6}
$$

inducing isomorphisms for $k \gg 0$, since N and M'' are y-torsion-free. In the notation of [DiSa3, 5.1] we have

$$H^{-1}({}^{s\prime}K^\bullet_{f'}) = \widetilde{N}, \quad H^0({}^{s\prime}K^\bullet_{f'}) = \widetilde{M}. \tag{1.3.7}$$

In the notation of (1.2), set

$$M^{(r)} := {}_{P'}\widetilde{E}_r^{n,0}, \quad N^{(r)} := {}_{P'}\widetilde{E}_r^{n,-1} \quad (r \geqslant 1). \tag{1.3.8}$$

Using the differentials d_r of the microlocal pole order spectral sequence (1.2.5) together with the isomorphisms in (1.2.10), we get the morphisms of graded $\mathbb{C}$-vector spaces of degree $-rd$:

$$\mathrm{d}^{(r)} : N^{(r)} \to M^{(r)} \quad \text{for any } \ r \geqslant 1, \tag{1.3.9}$$

such that $N^{(r)}$, $M^{(r)}$ are respectively the kernel and cokernel of $\mathrm{d}^{(r-1)}$ for any $r \geqslant 2$, and are independent of $r \gg 0$ (that is, $\mathrm{d}^{(r)} = 0$ for $r \gg 0$). More precisely, (1.3.9) is given by the composition (using the isomorphisms (1.2.10)):

$$ {}_{P'}\widetilde{E}_r^{n,0} \xrightarrow{\mathrm{d}_r} {}_{P'}\widetilde{E}_r^{n+r,-r-1} \xrightarrow{\partial_t^r} {}_{P'}\widetilde{E}_r^{n,-1}(-rd). \tag{1.3.10}$$

By (1.2.3) or (1.2.6), we have for $r = 1$

$$M^{(1)} = M, \quad N^{(1)} = N, \tag{1.3.11}$$

and $\mathrm{d}^{(1)} : N \to M$ is induced by the differential d of the de Rham complex $(\Omega^\bullet, \mathrm{d})$. By (1.2.6) we get moreover

$$M^{(2)} = H^n_{\mathrm{d}}\big(H^\bullet_{\mathrm{d}f\wedge}\Omega^\bullet\big), \quad N^{(2)} = H^{n-1}_{\mathrm{d}}\big(H^\bullet_{\mathrm{d}f\wedge}\Omega^\bullet\big)(-d). \tag{1.3.12}$$

These $M^{(r)}$, $N^{(r)}$, and $\mathrm{d}^{(r)}$ are equivalent to the microlocal pole order spectral sequence because of the isomorphisms (1.2.9–10). Set

$$M^{(\infty)} := M^{(r)}, \quad N^{(\infty)} := N^{(r)} \quad (r \gg 0). \tag{1.3.13}$$

Here $\mu_k^{(r)} := \dim M_k^{(r)}$, $\nu_k^{(r)} := \dim N_k^{(r)}$ are finite and non-increasing for $r \geqslant 1$ with k fixed. Hence they are stationary for $r \gg 0$ with k fixed.

Note that the shift of the grading by $-d$ for $N^{(2)}$ in (1.3.12) comes from the definition of the Koszul complex $K_f^\bullet$ where the differential $\mathrm{d}f\wedge$ preserves the grading.

1.4. Weighted homogeneous isolated singularity case. Assume the isolated singularity condition (1.3.1) together with condition (W) in Theorem 2, see also (1.8) below. In the notation of (1.3), set as in [DiSa3, 5.1]:

$$h := f/y^d \quad \text{on} \quad Y' := Y \setminus \{y = 0\} \cong \mathbb{C}^{n-1}. \tag{1.4.1}$$

Note that h is a defining function of $Z \setminus \{y = 0\}$ in Y', and we have $\Sigma = \mathrm{Sing}\, Z \subset Y'$ (since y is sufficiently general).

For $z \in \Sigma$, set

$$\Xi_{h_z} := \Omega^{n'}_{Y'_{\mathrm{an}},z}/\mathrm{d}h_z\wedge\Omega^{n'-1}_{Y'_{\mathrm{an}},z} \quad (n' := \dim Y' = n-1). \tag{1.4.2}$$

These are finite dimensional vector spaces of dimension μ_z, where μ_z is the Milnor number of (h_z, z) which coincides with the Tjurina number τ_z $(z \in \Sigma)$ under the assumption (W) in Theorem 2. We have the canonical isomorphism

$$\Xi^{\mathrm{alg}}_{h_z} \xrightarrow{\sim} \Xi_{h_z}, \tag{1.4.3}$$

where $\Xi_{h_z}^{\rm alg}$ is defined in the same way as Ξ_{h_z} with $\Omega^\bullet_{Y'_{\rm an},z}$ replaced by $\Omega^\bullet_{Y',z}$. (Indeed, $\mathcal{O}_{Y'_{\rm an},z}$ is flat over $\mathcal{O}_{Y',z}$ so that the tensor product of $\mathcal{O}_{Y'_{\rm an},z}$ over $\mathcal{O}_{Y',z}$ is an exact functor. Applying this to the exact sequence defining $\Xi_{h_z}^{\rm alg}$, that is,

$$\Omega^{n'-1}_{Y',z}\xrightarrow{{\rm d}h\wedge}\Omega^{n'}_{Y',z}\to\Xi_{h_z}^{\rm alg}\to 0,$$

we get the isomorphism

$$\Xi_{h_z}=\Xi_{h_z}^{\rm alg}\otimes_{\mathcal{O}_{Y',z}}\mathcal{O}_{Y'_{\rm an},z}.$$

We have a canonical morphism $\Xi_{h_z}^{\rm alg}\to\Xi_{h_z}$ induced by the inclusion $\mathcal{O}_{Y',z}\hookrightarrow\mathcal{O}_{Y'_{\rm an},z}$. So the assertion follows by suing a finite filtration on $\Xi_{h_z}^{\rm alg}$ such that its graded quotients are $\mathbb{C}$, since $\mathbb{C}\otimes_{\mathcal{O}_{Y',z}}\mathcal{O}_{Y'_{\rm an},z}=\mathbb{C}$.)

Fixing y, we have the isomorphisms compatible with (1.3.5)

(1.4.4) $$\widetilde{N}_k=\bigoplus\nolimits_{z\in\Sigma}\Xi_{h_z},\quad \widetilde{M}_k=\bigoplus\nolimits_{z\in\Sigma}\Xi_{h_z}\quad(\forall\,k\in\mathbb{Z}).$$

This follows from the argument in [DiSa3, 5.1] by using the isomorphisms in (1.3.7), see also the proof of Theorem (1.5) below.

By the theory of Gauss-Manin connections on Brieskorn lattices (see [Br]) in the weighted homogeneous polynomial case, we have the finite direct sum decompositions

(1.4.5) $$\Xi_{h_z}=\bigoplus\nolimits_{\alpha\in\mathbb{Q}}\Xi_{h_z}^\alpha\quad(z\in\Sigma).$$

defined by

(1.4.6) $$\Xi_{h_z}^\alpha:={\rm Ker}(\partial_t t-\alpha)\subset\Xi_{h_z}\quad(\alpha\in\mathbb{Q},\,z\in\Sigma),$$

which is independent of the choice of analytic local coordinates. (Note that we have a well-defined action of $\partial_t t$ on Ξ_{h_z}, since h_z is contained in the ideal generated by its partial derivatives, see (3.2.2) below.)

Theorem 1.5. *Under the isomorphisms in* (1.4.4)*, the duality between $\widetilde{N}_k$ and $\widetilde{M}_{nd-k}$ induced from the self-duality isomorphism for the Koszul complex in* [DiSa3, Theorem 1] (*using the graded local duality as in* [DiSa3, 1.1.4 and 1.7.3]) *is identified up to a constant multiple with the direct sum of the canonical self-duality of Ξ_{h_z} for $z\in\Sigma$.*

Proof. The self-duality isomorphism used in the proof of [DiSa3, Theorem 1] is a canonical one, and is induced by the canonical graded isomorphism

$$\Omega^j={\rm Hom}_R(\Omega^{n-j},\Omega^n),$$

where $\Omega^n[n]$ is the graded dualizing complex, see also [Ei], [Ha1], etc. (There is no shift of grading here, and it appears in the duality isomorphisms in [DiSa3, Theorem 1].)

The duality isomorphisms are compatible with the localization by y in (1.3.4) (see also [DiSa3, 5.1]). Here we consider the graded modules only over $\mathbb{C}[y]$ or $\mathbb{C}[y,y^{-1}]$ as in [DiSa3, Remark 1.7], and then consider the graded duals also over $\mathbb{C}[y]$ or $\mathbb{C}[y,y^{-1}]$. The graded dual over $\mathbb{C}[y]$ is defined by using the graded dualizing complex $(\mathbb{C}[y]{\rm d}y)[1]$, where the degree of complex is shifted by 1, and the degree of grading is also shifted by 1 because of ${\rm d}y$. We have the same with $\mathbb{C}[y]$ replaced by $\mathbb{C}[y,y^{-1}]$.

As for the relation with the graded local duality as in [DiSa3, 1.1.4 and 1.7.3], we can take graded free generators $\{u_i\}$, $\{v_i\}$ of M'', N over $\mathbb{C}[y]$ such that $u_i\in M''_{k_i}$, $v_i\in N_{nd+1-k_i}$, with $k_i\in\mathbb{Z}_{>0}$, and $\mathbb{C}[y]u_i$ is orthogonal to $\mathbb{C}[y]v_j$ for $i\ne j$. (Here the shift of grading by 1 comes from the degree of ${\rm d}y$ in the above remark.) We can get the u_i by using the filtration on M_k defined by $M_k\cap y^pM''$ $(p\in\mathbb{Z})$, and similarly for the v_i. Then $\{y^{p-k_i}u_i\}$ and $\{y^{k_i-p-nd-1}u_i\}$ for $p\gg 0$ are $\mathbb{C}$-bases of $\widetilde{M}_p=M''_p$ and $\widetilde{N}_{-p}=(\widetilde{N}/N)_{-p}$, which are orthogonal to each other under the pairing given by [DiSa3, 1.1.4 and 1.7.3].

By the above argument, it is enough to consider the graded $\mathbb{C}[y,y^{-1}]$-dual of $\widetilde{M}$ instead of the $\mathbb{C}$-dual of the M''_p for $p \gg 0$. In particular, we can neglect the shift of grading in the self-duality isomorphism (by using the isomorphisms in (1.3.5)) for the proof of Theorem (1.5).

Consider the blow-up along the origin

$$\pi : \widetilde{X} \to X := \mathbb{C}^n.$$

(This blow-up is used only for the *coordinate change* as is explained below. Here we may assume $y = x_n$ replacing x_n if necessary.)

Let $\widetilde{X}'' \subset \widetilde{X}$ be the complement of the *total* transform of $\{y=0\} \subset X$. We have the isomorphism

$$\widetilde{X}'' = \mathbb{C}^{n'} \times \mathbb{C}^* \quad \text{with} \quad n' = n-1. \tag{1.5.1}$$

(This must be distinguished from the isomorphism $\widetilde{X}'' = X \setminus \{y=0\} \cong \mathbb{C}^{n'} \times \mathbb{C}^*$, where the effect of the blow-up is neglected, since the coordinates x_i are used here instead of the x'_i defined below.)

In the notation of (1.3), set $x'_i := x_i/y$ $(i \in [1,n'])$, and

$$R' := \mathbb{C}[x'_1, \dots, x'_{n'}].$$

Then

$$\pi^* f|_{\widetilde{X}''} = y^d h \quad \text{with} \quad h := f/y^d \in R', \tag{1.5.2}$$

see also (1.4.1). This decomposition is compatible with the decomposition (1.5.1), where the x'_i and y are identified with the coordinates of $\mathbb{C}^{n'}$ and $\mathbb{C}^*$ respectively. Since

$$\partial_y(y^d h) = dy^{d-1}h, \quad \partial_{x'_i}(y^d h) = y^d h_i \quad \text{with} \quad h_i := \partial_{x'_i} h \in R',$$

the localized Koszul complex $\widetilde{K}^\bullet_{y^d h}$ for the partial derivatives of $y^d h$ on $\widetilde{X}'' = Y' \times \mathbb{C}^*$ can be identified, up to a non-zero constant multiple, with the shifted mapping cone

$$C\big(h : K^\bullet_h[y,y^{-1}] \to K^\bullet_h[y,y^{-1}]\big)[-1], \tag{1.5.3}$$

where $K^\bullet_h[y,y^{-1}]$ is the scalar extension by $\mathbb{C} \hookrightarrow \mathbb{C}[y,y^{-1}]$ of the Koszul complex $K^\bullet_h$ defined by

$$h_i \in \mathrm{End}(R') \quad (i \in [1,n']).$$

Note that $y^d \mathrm{d}y$ and y^d can be omitted in the above descriptions, since they express only the shift of grading, and can be neglected by using the isomorphisms in (1.3.5) as is explained above. (Indeed, only y has non-zero degree, since $\deg y = 1$ and $\deg h = \deg h_i = \deg x'_i = 0$.)

By (1.4.3) there is a quasi-isomorphism (or an isomorphism in the derived category)

$$K^\bullet_h[y,y^{-1}] \xrightarrow{\sim} \textstyle\bigoplus_{z\in\Sigma} \Xi_{h_z}[y,y^{-1}][-n']. \tag{1.5.4}$$

Consider the filtration G on $\widetilde{K}^\bullet_{y^d h}$ such that

$$\begin{aligned} \mathrm{Gr}^0_G \widetilde{K}^\bullet_{y^d h} &\cong K^\bullet_h[y,y^{-1}] \big(\xrightarrow{\sim} \textstyle\bigoplus_{z\in\Sigma} \Xi_{h_z}[y,y^{-1}][-n']\big), \\ \mathrm{Gr}^1_G \widetilde{K}^\bullet_{y^d h} &\cong K^\bullet_h[y,y^{-1}][-1] \big(\xrightarrow{\sim} \textstyle\bigoplus_{z\in\Sigma} \Xi_{h_z}[y,y^{-1}][-n'-1]\big), \end{aligned} \tag{1.5.5}$$

where $\widetilde{K}^\bullet_{y^d h}$ is identified with the mapping cone (1.5.3) (and (1.5.4) is used for the second isomorphisms in the derived category).

By condition (W) in Theorem 2, we have the vanishing of the morphisms

$$h : \Xi_{h_z}[y,y^{-1}] \to \Xi_{h_z}[y,y^{-1}] \quad (\forall\, z \in \Sigma).$$

So the filtration G on $\widetilde{K}^\bullet_{y^dh}$ *splits* in the bounded derived category of graded $R'[y,y^{-1}]$-modules $D^b(R'[y,y^{-1}])_{\rm gr}$, and there is a (non-canonical) isomorphism

$$\widetilde{K}^\bullet_{y^dh} \cong \mathrm{Gr}^0_G \widetilde{K}^\bullet_{y^dh} \oplus \mathrm{Gr}^1_G \widetilde{K}^\bullet_{y^dh} \quad \text{in } D^b(R'[y,y^{-1}])_{\rm gr}, \tag{1.5.6}$$

compatible with the filtration G so that the Gr^k_G of (1.5.6) are the identity morphisms $(k=0,1)$. Here we use the isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(M,M') = \mathrm{Hom}_{D^b(\mathcal{A})}(M,M'),$$

for objects M, M' of an abelian category $\mathcal{A}$ in general.

The self-duality isomorphism for $\widetilde{K}^\bullet_{y^dh}$ is compatible with the filtration G, and induces a duality between

$$\mathrm{Gr}^0_G \widetilde{K}^\bullet_{y^dh} \quad \text{and} \quad \mathrm{Gr}^1_G \widetilde{K}^\bullet_{y^dh},$$

which can be identified, up to a non-zero constant multiple and also a shift of grading, with the canonical self-duality of $K^\bullet_h[y,y^{-1}]$ by the above argument. Moreover the last self-duality is the *scalar extension* by $\mathbb{C} \hookrightarrow \mathbb{C}[y,y^{-1}]$ of the self-duality of the Koszul complex $K^\bullet_h$. So the assertion follows (since the above induced duality between $\mathrm{Gr}^0_G \widetilde{K}^\bullet_{y^dh}$ and $\mathrm{Gr}^1_G \widetilde{K}^\bullet_{y^dh}$ is sufficient for the proof of Theorem (1.5)). This finishes the proof of Theorem (1.5).

Remarks 1.6 (i) The self-duality of Ξ_{h_z} is given by using the so-called residue pairing (see for instance [Ha1]) which is compatible with the direct sum decompositions (1.4.5), and implies the duality between $\Xi^\alpha_{h_z}$ and $\Xi^{n'-\alpha}_{h_z}$.

(ii) In the notation of the introduction, N_k and M''_{dn-k} are orthogonal subspaces to each other by the duality in Theorem (1.5). This is compatible with Corollary 2.

1.7. Proof of Proposition 1. The assertion follows from the filtered isomorphisms (1.2.8) together with the identification of the pole order spectral sequence with the $M^{(r)}$, $N^{(r)}$, $\mathrm{d}^{(r)}$ in (1.3). This finishes the proof of Proposition 1.

For the convenience of the reader, we review here some basics of weighted homogeneous polynomials and related things.

1.8. Weighted homogeneous polynomials. We say that h is a *weighted homogeneous polynomial* with positive weights (w_i) for a coordinate system (y_i) of $Y := \mathbb{C}^n$, if h is a linear combination of $\prod_i y_i^{a_i}$ with $\sum_i w_i a_i = 1$, where $w_i \in \mathbb{Q}_{>0}$, $a_i \in \mathbb{N}$. We have

$$\lambda^m h(y_1,\dots,y_n) = h(\lambda^{m_1}y_1,\dots,\lambda^{m_n}y_n) \quad (\lambda \in \mathbb{C}^*), \tag{1.8.1}$$

where $m_i := mw_i \in \mathbb{N}$ with m the smallest positive integer satisfying $mw_i \in \mathbb{N}$, and (1.8.1) holds with λ replaced by $g \in \mathcal{O}_{Y,0}$. This implies that condition (W) in Theorem 2 is independent of the choice of a defining function h_z.

We also have

$$h = v(h) \quad \text{with} \quad v := \textstyle\sum_i w_i\, y_i \partial_{y_i} \in \Theta_Y. \tag{1.8.2}$$

This implies that h is *quasihomogeneous*, that is, $h \in (\partial h)$ with the notation in a remark after Theorem 2. The converse holds in the isolated singularity case, see [SaK].

1.9. Steenbrink spectrum and Bernstein-Sato polynomials. Let h be a weighted homogeneous polynomial of n variables with weights $w_1,\dots,w_n$ as above. The *Steenbrink spectrum* $\mathrm{Sp}(h) = \sum_{i=1}^{\mu_h} t^{\alpha_{h,i}}$ and the *spectral numbers* $\alpha_{h,i} \in \mathbb{Q}$ are defined by

$$\begin{gathered} 0 < \alpha_{h,1} \leqslant \cdots \leqslant \alpha_{h,\mu_h} < n, \\ \#\bigl\{i \mid \alpha_{h,i} = \alpha\bigr\} = \dim \mathrm{Gr}^p_F H^{n-1}(F_h,\mathbb{C})_\lambda \bigl(= \dim \Xi^\alpha_h\bigr), \end{gathered} \tag{1.9.1}$$

where $p := [n-\alpha]$, $\lambda := \exp(-2\pi i\alpha)$, and μ_h is the Milnor number of h, see [St2], [ScSt]. (Here Ξ_h^α is as in (1.4).) We have moreover

$$b_h(s) = (s+1)\,\big[\textstyle\prod_{i=1}^{\mu_h}(s+\alpha_{h,i})\big]_{\mathrm{red}}, \tag{1.9.2}$$

where $\big[\prod_j(s+\beta_j)^{m_j}\big]_{\mathrm{red}} := \prod_j(s+\beta_j)$ for $\beta_i \ne \beta_j$ $(i \ne j)$ and $m_j \in \mathbb{Z}_{>0}$ in general. This equality can be proved by combining [Ma1] and [ScSt], [Va1] (which show that the Bernstein-Sato polynomial and the Hodge filtration on the Milnor cohomology can be obtained by using the Brieskorn lattice [Br]), see also [Sat], [St1]. It is also well known that

$$\mathrm{Sp}(h) = \textstyle\prod_{j=1}^n (t^{w_j}-t)/(1-t^{w_j}). \tag{1.9.3}$$

This implies the *symmetry of spectral numbers*

$$\alpha_{h,i} = \alpha_{h,j} \qquad (i+j = \mu_h+1). \tag{1.9.4}$$

Taking the limit of (1.9.3) for $t \to 1$, we also get

$$\mu_h = \textstyle\prod_{j=1}^n \big(\frac{1}{w_j}-1\big). \tag{1.9.5}$$

This is well known in the Brieskorn-Pham type case, that is, if $w_i = a_i^{-1}$ with $a_i \in \mathbb{N}$.

Remark 1.10. We can prove (1.9.3) by using the Koszul complex $K_h^\bullet$ as in (1.2.4) with f replaced by h. Indeed, the denominator of the right hand side of (1.9.3) gives the Hilbert series of the graded ring R with $\deg x_i = w_i$, and the numerator corresponds to the shift of the grading of each component of the Koszul complex (since $\deg \partial_{x_i} h = 1 - w_i$), where the complex is viewed as the associated single complex of an n-ple complex associated with the multiplications by partial derivatives $\partial_{x_i} h$ $(i \in [1,n])$, see for instance [Se, Section IV.2].

We can use (1.9.3) for an explicit computation using Macaulay2, see (A.1) below.

1.11. Extremely degenerated reduced curves. We say that a reduced curve Z on $\mathbb{P}^2$ is *extremely degenerated*, if, choosing some coordinates x,y,z of $\mathbb{C}^3$, its defining polynomial f is a linear combination of monomials $x^iy^jz^k$ satisfying $ai+bj+ck=0$ for some fixed $a,b,c \in \mathbb{Q}$ with at least two of them nonzero. Since Z is assumed to be reduced, f must contain two monomials $x^iy^jz^k$, $x^{i'}y^{j'}z^{k'}$ with $i,j,k' \in \{0,1\}$ up to a permutation of variables. Indeed, f must contain certain monomials corresponding to the conditions that f is not divisible by x^2, y^2, z^2, and two of the monomials must coincide by the condition that Z is extremely degenerated (except for the case $a=b$, $c=0$ up to a permutation of x, y, z). This notion is closely related to (1.12.2) below, where $\mathrm{vol}(\Gamma_f) = \mathrm{len}(\partial\Gamma_d \cap \Gamma_f) = 0$ in the case of extremely degenerated curves (with assumptions in (1.12) satisfied).

1.12. A formula for $\chi(U)$. Assume $n=3$ and

$$\mathrm{Sing}\,Z \subset \Sigma' := \big\{[1{:}0{:}0], [0{:}1{:}0], [0{:}0{:}1]\big\} \subset \mathbb{P}^2. \tag{1.12.1}$$

Let Γ_d be the convex hull of $\Gamma_d^0 := \big\{(d,0,0),(0,d,0),(0,0,d)\big\} \subset \mathbb{R}^3$. Let Γ_f be the Newton polygon of f, that is, the convex hull of the points of $\mathbb{N}^3$ corresponding to the monomials appearing in f. We have the equalities (using (2.9.5) below for the first equality):

$$\begin{aligned}\chi(U) &= d^2 - 3d + 3 - \mu_Z \\ &= 2\,\mathrm{vol}(\Gamma_f) - \mathrm{len}(\partial\Gamma_d \cap \Gamma_f) + |\Gamma_d^0 \cap \Gamma_f|,\end{aligned} \tag{1.12.2}$$

if f is *non-degenerate for singular points of Z*. Moreover the last equality of (1.12.2) is replaced by the inequality $\leqslant$ in the general case. The last hypothesis means that, for any 1-dimensional face σ of Γ_f *not contained in* $\partial\Gamma_d$, f_σ/g is *reduced* for some monomial g, where $f_\sigma := \sum_{\nu\in\sigma} a_\nu x^\nu$ if $f = \sum_\nu a_\nu x^\nu$. The volume $\mathrm{vol}(\Gamma_f)$ is defined by the standard measure on the minimal affine space E containing Γ_f such that the volume of $E/(\mathbb{Z}^3 \cap E)$ is 1 (similarly for $\mathrm{len}(\Gamma_f \cap \partial\Gamma_d)$). We have the strict inequality $>$ in certain degenerate cases (for instance, $\chi(U) = 0$ if $f = (x^2 - yz)^2 + y^4$). The above assertion around (1.12.2) is equivalent to

$$\mu_Z = 2\mathrm{vol}(\Gamma_d \setminus \Gamma_f) - \mathrm{len}(\partial\Gamma_d \setminus \Gamma_f) + |\Gamma_d^0 \setminus \Gamma_f|, \tag{1.12.3}$$

in the non-degenerate case together with the inequality $\geqslant$ in the general case.

The latter assertion follows from a well-known formula for Milnor numbers as in [Ko] (together with the finite determinacy of hypersurface isolated singularities as in [Br]). We have to decompose $\overline{\Gamma_d \setminus \Gamma_f}$ into three closed subsets Γ'_1, Γ'_2, Γ'_3 so that

$$\partial\Gamma_f \subset \textstyle\bigcup_{i=1}^3 \partial\Gamma'_i, \quad \Gamma'_i \cap \Gamma'_j = \partial\Gamma'_i \cap \partial\Gamma'_j \;\; (i \ne j),$$

where $\Gamma'_i = \emptyset$ if and only if

$$\Gamma_f \cap \{\nu_i = d\} \ne \emptyset \;\; \text{or} \;\; \Gamma_f \cap \{\nu_i = d-1\} \ne \emptyset, \Gamma_f \subset \{\nu_k \geqslant 1\} \; (\exists\, k \ne i).$$

Note that $\Gamma_f \cap \{\nu_k = 1\}$ is *one* point if $\Gamma_f \subset \{\nu_k \geqslant 1\}$ (using (1.12.1)). In the case $\Gamma'_i \ne \emptyset$, $\Gamma'_j \ne \emptyset$, and $\Gamma_f \subset \{\nu_k \geqslant 1\}$ with $\{i,j,k\} = \{1,2,3\}$, the boundary between Γ'_i, Γ'_j is given by *choosing* a line passing through $\Gamma_f \cap \{\nu_k = 1\}$ and intersecting with $\partial\Gamma_d \cap \{\nu_k = 0\} \setminus \Gamma_d^0$. (This line may be chosen so that it is contained in $\{\nu_i = \mathrm{const}\}$ or $\{\nu_j = \mathrm{const}\}$.) The separation of Γ'_i and Γ'_j is trivial if $\Gamma_f \not\subset \{\nu_k \geqslant 1\}$. If Γ'_i corresponds to a singular point $z \in Z$ (that is, if $\Gamma'_i \not\subset \{\nu_j \leqslant 1\}$ $(\forall\, j \ne i)$), then Kouchnirenko's number $2S - a - b + 1$ for a non-degenerate convenient holomorphic function h defining (Z, z) coincides with

$$\mu_i^K := 2\,\mathrm{vol}(\Gamma'_i) - \mathrm{len}(\partial\Gamma_d \cap \Gamma'_i) + |\Gamma_d^0 \cap \Gamma'_i|,$$

which is *independent* of choices of the above lines. (Here some convexity condition on the complement of Γ'_i in $\mathbb{R}^2_{\geqslant 0}$ can be forgotten as a consequence of it.) Some calculations are needed in the case $\Gamma_f \cap \{\nu_i = d\} = \emptyset$, $\Gamma_f \cap \{\nu_i = d-1\} \ne \emptyset$, as is seen in the example: $f = x^4 y + y^4 z$. Here $\Gamma'_i = \emptyset$ (where some calculation is needed for some other Γ'_j containing $\{\nu_k \leqslant 1\}$) or Γ'_i corresponds to a smooth point of Z (where $\Gamma'_i \subset \{\nu_j \leqslant 1\}$, $\mu_i^K = 0$).

2. Fundamental exact sequences of vanishing cycles

In this section we study the fundamental exact sequences of vanishing cycles, and prove Theorem (2.2) which is a key to the proof of Theorem 2.

2.1. Calculation of the Milnor cohomology. In the notation and assumption of the introduction, set

$$n' := \dim Y = n - 1, \quad U := Y \setminus Z,$$

with $j : U \hookrightarrow Y$ the canonical inclusion. Set

$$\Lambda_d := \big\{\lambda \in \mathbb{C}^* \mid \lambda^d = 1\big\}. \tag{2.1.1}$$

Let $H^\bullet(F_f, \mathbb{C})_\lambda$, $H_c^\bullet(F_f, \mathbb{C})_\lambda$ be the λ-eigenspaces of the monodromy T. It is well known that $T^d = id$, and hence $H^\bullet(F_f, \mathbb{C})_\lambda = 0$ for $\lambda \notin \Lambda_d$ (by using the geometric monodromy).

For $\lambda \in \Lambda_d$, we have the rank 1 local system L_λ on U such that

$$H^\bullet(U, L_\lambda) = H^\bullet(F_f, \mathbb{C})_\lambda, \quad H_c^\bullet(U, L_\lambda) = H_c^\bullet(F_f, \mathbb{C})_\lambda, \tag{2.1.2}$$

see for instance [DiSa3, 3.1.1]. More precisely, let $\pi : \widetilde{X} \to X$ be the blow-up at the origin, and set $\tilde{f} := \pi^* f$. The exceptional divisor E is identified with Y. Define

$$L_\lambda := \psi_{\tilde{f},\lambda} \mathbb{C}_{\widetilde{X}}|_U. \tag{2.1.3}$$

This is justified by [BuSa1, Theorem 4.2], see also [BuSa2, Section 1.6].

The second isomorphism of (2.1.2) follows from the first isomorphism by using the self-duality of the nearby cycle sheaves $\psi_{\tilde{f}}\mathbb{C}_{\widetilde{X}}$, which implies the isomorphisms

$$L_\lambda^\vee = L_{\lambda^{-1}} \ \ (\lambda \in \Lambda_d), \qquad L_1 = \mathbb{C}_U, \tag{2.1.4}$$

see also [BuSa2, Sections 1.3–4]. The shifted local system $\bigoplus_{\lambda\in\Lambda_d} L_\lambda[n']$ naturally underlie a mixed Hodge module on U by (2.1.3), and the isomorphisms in (2.1.2) are compatible with mixed Hodge structures.

Note that the rank 1 local systems L_λ are uniquely determined by the local monodromies at smooth points of Z, which are given by the multiplication by λ^{-1}. Indeed, in the notation of (1.4.1), L_λ is uniquely determined by its restriction over $U\cap Y'$ (using the direct image by $U\cap Y' \hookrightarrow U$), and we have the isomorphism

$$L_\lambda|_{U\cap Y'} = h^*E_{\lambda^{-1}}, \tag{2.1.5}$$

where $E_{\lambda^{-1}}$ is a rank 1 local system on $\mathbb{C}^*$ with monodromy λ^{-1}. (We can verify that $L_\lambda|_{U\cap Y'}\otimes h^*E_\lambda$ is extendable as a rank 1 local system over $Y'\setminus\Sigma$, and then over $Y' = \mathbb{C}^{n'}$ since $\dim Y' \geqslant 2$.)

Note that (2.1.5) also implies (2.1.4) as well as the isomorphisms

$$\psi_{h,\eta}L_\lambda|_{U\cap Y'} = \psi_{h,\lambda\eta}\mathbb{C}_{U\cap Y'} \qquad (\forall\, \eta\in\mathbb{C}^*). \tag{2.1.6}$$

Indeed, we have more generally

$$\psi_{h,\lambda}\mathcal{F} = \psi_{h,1}(\mathcal{F}\otimes h^*E_{\lambda^{-1}}) \quad \text{for any } \ \mathcal{F}\in D_c^b(\mathbb{C}_{U\cap Y'}). \tag{2.1.7}$$

It is well known (see [Mi] and also [DiSa1, 2.1.2], etc.) that if $\dim \mathrm{Sing}\, f^{-1}(0) = 1$, then

$$\widetilde{H}^j(U, L_\lambda) = \widetilde{H}^\bullet(F_f,\mathbb{C})_\lambda = 0 \quad \text{unless } \ j\in\{n'-1, n'\}, \tag{2.1.8}$$

with $\widetilde{H}^j(U,L_\lambda) := H^j(U,L_\lambda)$ for $\lambda\neq 1$. (Recall that $L_1 = \mathbb{C}_U$, see (2.1.4).)

For $z\in\Sigma = \mathrm{Sing}\, Z$ (see (1.3.1)), there are distinguished triangles

$$(\mathbb{R}j_*L_\lambda)_z \to \psi_{h,1}(L_\lambda|_{U\cap Y'})_z \xrightarrow{N} \psi_{h,1}(L_\lambda|_{U\cap Y'})_z(-1) \xrightarrow{+1}. \tag{2.1.9}$$

The associated long exact sequence is strictly compatible with the Hodge filtration F coming from the theory of mixed Hodge modules [Sa1] (by using for instance [Sa2, Proposition 1.3]).

By (2.1.4) for $\eta = 1$ together with condition (W) in Theorem 2, we have the vanishing of N in the long exact sequences associated with the distinguished triangles in (2.1.9). Using this, we can prove the canonical isomorphisms for $z\in\Sigma$

$$(R^ij_*L_k)_z \cong \begin{cases} H^{n'-1}(F_{h_z})_\lambda & \text{if } \ i = n'-1,\\ H^{n'-1}(F_{h_z})_\lambda(-1) & \text{if } \ i = n',\\ 0 & \text{if } \ i\in[2,n'-2] \ \text{ or } \ i > n',\end{cases} \tag{2.1.10}$$

where (p) for $p\in\mathbb{Z}$ denotes the Tate twist in general (see [De1]), F_{h_z} is the Milnor fiber of h_z, and the coefficient field $\mathbb{C}$ of the cohomology is omitted to simplify the notation. These isomorphisms are compatible with the Hodge filtration F and the weight filtration W of mixed Hodge structures.

Actually we can also construct the isomorphisms in (2.1.10) by using another method. Indeed, let $\mathbb{Q}_{h,X} := a_X^*\mathbb{Q}\in D^b\mathrm{MHM}(X)$ with $a_X : X\to pt$ the structure morphism. Let $i_0 : \{0\}\hookrightarrow X := \mathbb{C}^n$, $j_0 : X\setminus\{0\}\hookrightarrow X$ be natural inclusions. Using [BuSa1, Theorem 4.2] together with the distinguished triangle

$$(i_0)_*i_0^!\varphi_f\mathbb{Q}_{h,X} \to \varphi_f\mathbb{Q}_{h,X} \to (j_0)_*j_0^*\varphi_f\mathbb{Q}_{h,X} \xrightarrow{+1},$$

we can prove Theorem (2.2) below by showing the self-dual exact sequences of mixed Hodge structures (see also [DiSa1, 2.1]):

$$\begin{aligned}(2.1.11)\qquad & 0 \to H^{n'-1}(F_f)_{\neq 1} \to \textstyle\bigoplus_{z\in\Sigma} H^{n'-1}(F_{h_z})^{T_l}_{\neq 1} \xrightarrow{\rho^\vee} H^{n'}_c(F_f)_{\neq 1}\\ & \to H^{n'}(F_f)_{\neq 1} \xrightarrow{\rho} \textstyle\bigoplus_{z\in\Sigma} H^{n'-1}(F_{h_z})^{T_l}_{\neq 1}(-1) \to H^{n'+1}_c(F_f)_{\neq 1} \to 0,\end{aligned}$$

$$\begin{aligned}(2.1.12)\qquad & 0 \to H^{n'-1}(F_f)_{1} \to \textstyle\bigoplus_{z\in\Sigma} H^{n'-1}(F_{h_z})^{T_l}_{1} \xrightarrow{\rho^\vee} H^{n'}_c(F_f)_{1}(-1)\\ & \to H^{n'}(F_f)_{1} \xrightarrow{\rho} \textstyle\bigoplus_{z\in\Sigma} H^{n'-1}(F_{h_z})^{T_l}_{1}(-1) \to H^{n'+1}_c(F_f)_{1}(-1) \to 0,\end{aligned}$$

where E^{T_l} denotes the invariant part by the action of the local system monodromy T_l on $E := H^{n'-1}(F_{h_z})$. These are called the *fundamental exact sequences of vanishing cycles*. Since this proof is rather long, it is omitted in this version. In the proof of Theorem (2.2) in (2.4–5) below, we will construct long exact sequence which should correspond to the above ones (except for the case $\lambda = 1$, $n = 3$). For the moment it is *still unclear* whether these two canonical isomorphisms *really coincide* (even though this is very much expected to hold).

Theorem 2.2. *In the notation and assumption of* (2.1) *together with the assumption* (W) *in Theorem* 2, *there is a canonical isomorphism compatible with the Hodge filtration* F :

$$(2.2.1)\qquad \mathrm{Coker}\Big(H^{n'}(F_f)_\lambda \xrightarrow{\rho} \bigoplus_{z\in\Sigma} H^{n'-1}(F_{h_z})_\lambda(-1)\Big) = \begin{cases} H^{n'-1}(F_f)^\vee_{\lambda^{-1}}(-n') & \text{if } \lambda\neq 1,\\ H^{n'-1}(F_f)^\vee_{1}(-n'-1) & \text{if } \lambda = 1,\end{cases}$$

where F_f is the Milnor fiber of f as in the introduction, ρ is a canonical morphism, and the coefficient field $\mathbb{C}$ of the cohomology is omitted to simplify the notation. Moreover ρ can be identified via isomorphisms as in (2.1.2) *and* (2.1.10) *with the following canonical morphism*:

$$(2.2.2)\qquad H^{n'}(U, L_\lambda) \to \textstyle\bigoplus_{z\in\Sigma} (R^{n'} j_* L_k)_z.$$

Remark 2.3. By the first injective morphisms in (2.1.11–12), $H^{n'-1}(F_f)_{\neq 1}$ and $H^{n'-1}(F_f)_1$ are identified respectively with subspaces of

$$\textstyle\bigoplus_{z\in\Sigma} H^{n'-1}(F_{h_z})_{\neq 1}, \quad \bigoplus_{z\in\Sigma} H^{n'-1}(F_{h_z})_1,$$

which are *pure* $\mathbb{Q}$*-Hodge structures of weight* $n'-1$ *and* n' under the assumption (W) in Theorem 2, see [St1]. In particular, polarizations of Hodge structures induce self-duality isomorphisms of $\mathbb{Q}$-Hodge structures

$$(2.3.1)\qquad H^{n'-1}(F_f)_{\neq 1} \cong H^{n'-1}(F_f)^\vee_{\neq 1}(1-n'), \quad H^{n'-1}(F_f)_1 \cong H^{n'-1}(F_f)^\vee_1(-n').$$

These are closely related to the difference in the Tate twist on the right-hand side of (2.2.1) for $\lambda \neq 1$ and $\lambda = 1$.

In this paper we give a proof of Theorem (2.2) (which is closely related to [Di1], [Kl]) by dividing it into three cases.

2.4. Proof of Theorem (2.2) for $\lambda \neq 1$. Set

$$Y^\circ := Y \setminus \Sigma \ \text{ with } \ j' : Y^\circ \hookrightarrow Y \ \text{ the inclusion.}$$

Let L'_λ be the zero extension of L_λ over Y°. There is a distinguished triangle

$$(2.4.1)\qquad j'_! L'_\lambda \to \mathbb{R} j'_* L'_\lambda \to \textstyle\bigoplus_{z\in\Sigma} (\mathbb{R} j_* L_\lambda)_z \xrightarrow{+1},$$

together with isomorphisms

$$(2.4.2)\qquad j'_! L'_\lambda = j_! L_\lambda, \quad \mathbb{R} j'_* L'_\lambda = \mathbb{R} j_* L_\lambda.$$

These induce exact sequences

$$
\begin{aligned}
(2.4.3)\qquad & 0 \to H^{n'-1}(U,L_\lambda) \to \textstyle\bigoplus_{z\in\Sigma}(R^{n'-1}j_*L_\lambda)_z \to H_c^{n'}(U,L_\lambda)\\
&\to H^{n'}(U,L_\lambda) \to \textstyle\bigoplus_{z\in\Sigma}(R^{n'}j_*L_\lambda)_z \to H_c^{n'+1}(U,L_\lambda) \to 0,
\end{aligned}
$$

which are essentially self-dual by replacing λ with λ^{-1}.

By (2.1.2), (2.1.10), these give essentially self-dual exact sequences

$$
\begin{aligned}
(2.4.4)\qquad & 0 \to H^{n'-1}(F_f)_\lambda \to \textstyle\bigoplus_{z\in\Sigma} H^{n'-1}(F_{h_z})_\lambda \xrightarrow{\rho^\vee} H_c^{n'}(F_f)_\lambda\\
&\to H^{n'}(F_f)_\lambda \xrightarrow{\rho} \textstyle\bigoplus_{z\in\Sigma} H^{n'-1}(F_{h_z})_\lambda(-1) \to H_c^{n'+1}(F_f)_\lambda \to 0.
\end{aligned}
$$

It is expected that their direct sum over $\lambda\in\Lambda_d\setminus\{1\}$ would be identified with (2.1.11). Applying Poincaré duality to the last term of (2.4.4), we get Theorem (2.2) in the case $\lambda\ne 1$.

2.5. Proof of Theorem (2.2) for $\lambda=1$, $n'\geqslant 3$. We assume for the moment *only* $\lambda=1$. Set

$$Z^\circ := Z\setminus\Sigma \quad\text{with}\quad j_Z : Z^\circ \hookrightarrow Z \quad\text{the inclusion.}$$

We have the distinguished triangle

$$(2.5.1)\qquad \mathbb{Q}_{Y^\circ} \to \mathbb{R}j''_*\mathbb{Q}_U \to \mathbb{Q}_{Z^\circ}(-1)[-1] \xrightarrow{+1},$$

where $j'' : U \hookrightarrow Y^\circ$ is the inclusion, and $Y^\circ := Y\setminus\Sigma$ with $j' : Y^\circ \hookrightarrow Y$ is as in (2.2). Applying $\mathbb{R}j'_*$ to (2.5.1), we get the distinguished triangle

$$(2.5.2)\qquad \mathbb{R}j'_*\mathbb{Q}_{Y^\circ} \to \mathbb{R}j_*\mathbb{Q}_U \to \mathbb{R}(j_Z)_*\mathbb{Q}_{Z^\circ}(-1)[-1] \xrightarrow{+1}.$$

These naturally underlie distinguished triangles of complexes of mixed Hodge modules.

Setting $m:=|\Sigma|$, we have

$$(2.5.3)\qquad H^i(Y^\circ) = \begin{cases} \mathbb{Q}(-j) & \text{if } i=2j \text{ with } j\in[0,n'-1],\\ \bigoplus^{m-1}\mathbb{Q}(-n') & \text{if } i=2n'-1,\\ 0 & \text{otherwise.}\end{cases}$$

$$(2.5.4)\qquad \widetilde{H}^i(U)=0 \quad\text{unless}\quad i\in\{n'-1,\,n'\}, \text{ see (2.1.8).}$$

Assume $n'\geqslant 3$ from now on. The distinguished triangle (2.5.2) implies the isomorphisms for $z\in\Sigma$:

$$(2.5.5)\qquad (R^ij_*\mathbb{Q}_U)_z = (R^{i-1}(j_Z)_*\mathbb{Q}_{Z^\circ})_z(-1) \qquad (i\in[1,2n'-3]),$$

as well as the short exact sequences of mixed Hodge structures

$$(2.5.6)\qquad 0\to H^{i+1}(U)\to H^i(Z^\circ)(-1)\to H^{i+2}(Y^\circ)\to 0 \quad (i\in\mathbb{N}),$$

where the coefficient field $\mathbb{Q}$ of the cohomology is omitted to simplify the notation. Indeed, the restriction morphisms $H^i(Y^\circ)\to H^i(U)$ vanish for any $i>0$ (where we use (2.5.4) together with $2n'-1>n'$ in the case $i=2n'-1$ in (2.5.3)).

Define the *primitive part* by

$$H^i_{\rm prim}(Z^\circ) := \mathrm{Ker}\big(H^i(Z^\circ)\to H^{i+2}(Y^\circ)(1)\big) \quad\text{for}\quad i\in\{n'-2,n'-1\},$$

where the morphism on the right-hand side is the Gysin morphism appearing in (2.5.6). (This is compatible with the standard definition of the primitive part.) Note that the same definition does not work very well for $i=n'$ if $n'=3$, see (2.5.3). We are only interested in the case $i\in\{n'-2,n'-1\}$ for the exact sequence (2.5.10) explained below.

By (2.5.6) we get the isomorphisms

$$H^i_{\rm prim}(Z^\circ)=H^{i+1}(U)(1)\qquad \bigl(i\in\{n'-2,n'-1\}\bigr). \tag{2.5.7}$$

Define the *primitive part* for the cohomology with compact supports by

$$H^i_{c,{\rm prim}}(Z^\circ):=H^{2n'-2-i}_{\rm prim}(Z^\circ)^\vee(1-n')\qquad \bigl(i\in\{n'-1,n'\}\bigr).$$

By (2.5.7) this vanishes for $i=n'-2$, and we have

$$H^i_{c,{\rm prim}}(Z^\circ)=H^{2n'-1-i}(U)^\vee(-n')=H^{i+1}_c(U)\qquad \bigl(i\in\{n'-1,n'\}\bigr). \tag{2.5.8}$$

On the other hand, there is a distinguished triangle

$$(j_Z)_!\mathbb{Q}_{Z^\circ}\to\mathbb{R}(j_Z)_*\mathbb{Q}_{Z^\circ}\to(\mathbb{R}(j_Z)_*\mathbb{Q}_{Z^\circ})|_\Sigma\xrightarrow{+1}, \tag{2.5.9}$$

underlying naturally a distinguished triangle of complexes of mixed Hodge modules. By (2.6) below, this induces a self-dual exact sequence of mixed Hodge structures

$$\begin{aligned}0\to H^{n'-2}_{\rm prim}(Z^\circ)\to\textstyle\bigoplus_{z\in\Sigma}(R^{n'-2}(j_Z)_*\mathbb{Q}_{Z^\circ})_z\to H^{n'-1}_{c,{\rm prim}}(Z^\circ)\\ \to H^{n'-1}_{\rm prim}(Z^\circ)\to\textstyle\bigoplus_{z\in\Sigma}(R^{n'-1}(j_Z)_*\mathbb{Q}_{Z^\circ})_z\to H^{n'}_{c,{\rm prim}}(Z^\circ)\to 0,\end{aligned} \tag{2.5.10}$$

assuming $n'\geqslant 3$. Combined with (2.1.10), (2.5.5), and (2.5.7–8), this gives a self-dual exact sequence of mixed Hodge structures

$$\begin{aligned}0\to H^{n'-1}(U)\to\textstyle\bigoplus_{z\in\Sigma}H^{n'-1}(F_{h_z})_1\to H^{n'}_c(U)(-1)\\ \to H^{n'}(U)\to\textstyle\bigoplus_{z\in\Sigma}H^{n'-1}(F_{h_z})_1(-1)\to H^{n'+1}_c(U)(-1)\to 0,\end{aligned} \tag{2.5.11}$$

(which is expected to be identified with (2.1.12) in this case). Theorem (2.2) then follows in the case $\lambda=1$ and $n'\geqslant 3$ by using Poincaré duality for U together with (2.1.2).

2.6. Proof of the exact sequence (2.5.10). Define the *non-primitive part* by

$$\begin{aligned}H^i_{c,np}(Z^\circ)&:={\rm Im}\bigl(H^i_c(Y^\circ)\to H^i_c(Z^\circ)\bigr)\qquad \bigl(i\in\{n'-1,n'\}\bigr),\\ H^i_{np}(Z^\circ)&:={\rm Im}\bigl(H^i(Y^\circ)\to H^i(Z^\circ)\bigr)\qquad \bigl(i\in\{n'-1,n'-2\}\bigr),\end{aligned}$$

For $i=n'-1$, we have a canonical isomorphism

$$H^{n'-1}_{c,np}(Z^\circ)\xrightarrow{\sim}H^{n'-1}_{np}(Z^\circ), \tag{2.6.1}$$

by using the canonical isomorphism $H^{n'-1}_c(Y^\circ)\xrightarrow{\sim}H^{n'-1}(Y^\circ)$.

For $i=n'$, a similar argument implies the injectivity of the canonical isomorphism

$$H^{n'}_{c,np}(Z^\circ)\hookrightarrow H^{n'}(Z^\circ). \tag{2.6.2}$$

There are moreover canonical isomorphisms

$$\begin{aligned}H^i_{c,{\rm prim}}(Z^\circ)&=H^i_c(Z^\circ)/H^i_{c,np}(Z^\circ)\qquad \bigl(i\in\{n'-1,n'\}\bigr),\\ H^i_{\rm prim}(Z^\circ)&=H^i(Z^\circ)/H^i_{np}(Z^\circ)\qquad \bigl(i\in\{n'-1,n'-2\}\bigr).\end{aligned} \tag{2.6.3}$$

Indeed, the first isomorphism easily follows from the definition (since the dual of the kernel of a morphism is the cokernel of the dual morphism in general), and the second follows from the bijectivity of the composition

$$H^i(Y^\circ)\to H^i(Z^\circ)\to H^{i+2}(Y^\circ)(1)\qquad \bigl(i\in\{n'-1,n'-2\}\bigr).$$

By (2.6.1) and (2.6.3) for $i=n'-1$, we get the exactness of the middle part of (2.5.10) using the long exact sequence associated with (2.5.9). By the self-duality of (2.5.10) together with the first isomorphism of (2.6.3) for $i=n'$ and the injectivity of (2.6.2), it now remains to show the surjectivity of the morphism to $H^{n'}_{c,{\rm prim}}(Z^\circ)$ in (2.5.10).

Since $H^{n'}_{c,\mathrm{prim}}(Z^\circ)$ has weights $\leqslant n'$ (see [De1], [Sa1]), it is enough to show

$$H^{n'}(Z^\circ)/H^{n'}_{c,np}(Z^\circ) \text{ has weights} > n', \tag{2.6.4}$$

by using the long exact sequence associated with (2.5.9) together with the injectivity of (2.6.2). However, (2.6.4) easily follows from (2.5.3–4) and (2.5.6). This finishes the proof of the exact sequence (2.5.10).

2.7. Proof of Theorem (2.2) for $\lambda = 1$, $n' = 2$. In this case, Z is a curve on $Y = \mathbb{P}^2$. Using the distinguished triangle (2.5.2) together with the snake lemma, we can get the commutative diagram of exact sequences

$$\begin{array}{ccccc} H^2(U) & \hookrightarrow & H^1(Z^\circ)(-1) & \twoheadrightarrow & H^3(Y^\circ) \\ \downarrow\rho & & \downarrow & & \cap \\ \bigoplus_z (R^2 j_*\mathbb{Q}_U)_z & \hookrightarrow & \bigoplus_z (R^1(j_Z)_*\mathbb{Q}_{Z^\circ}(-1))_z & \twoheadrightarrow & \bigoplus_z (R^3 j'_*\mathbb{Q}_{Y^\circ})_z \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \operatorname{Coker}\rho & \hookrightarrow & H^2_c(Z^\circ)(-1) & \twoheadrightarrow & H^4_c(Y^\circ) \end{array} \tag{2.7.1}$$

where the direct sums are taken over $z \in \Sigma$, see also [Di1, Ch. 6, 3.14] and Remark (2.8) below. (As for the right vertical exact sequence, note that $\dim H^3(Y^\circ) = m-1$, $\dim H^4_c(Y^\circ) = 1$.)

We have moreover the canonical isomorphism

$$\operatorname{Coker}\rho = H^1(U)^\vee(-3), \tag{2.7.2}$$

using the dual of the isomorphism

$$H^1(U) = \operatorname{Coker}\big(H^0(Y^\circ) \to H^0(Z^\circ)\big)(-1). \tag{2.7.3}$$

The latter follows from (2.5.6) for $i = 0$ by using the bijectivity of the composition

$$H^0(Y^\circ) \to H^0(Z^\circ) \to H^2(Y^\circ)(1).$$

So (2.2.1) also follows in the case $\lambda = 1$, $n' = 2$. This finishes the proof of Theorem (2.2).

Remark 2.8. In the case $n' = 2$, let r_z and r_Z be respectively the number of local and global irreducible components of (Z, z) and Z. Then

$$\dim(R^2 j_*\mathbb{Q}_U)_z = r_z - 1, \quad \dim(R^1(j_Z)_*\mathbb{Q}_{Z^\circ})_z = r_z, \quad \dim(R^3 j'_*\mathbb{Q}_{Y^\circ})_z = 1. \tag{2.8.1}$$

$$\dim H^1(U) = r_Z - 1, \quad \dim H^0(Z^\circ) = r_Z, \quad \dim H^0(Y^\circ) = \dim H^2(Y^\circ) = 1. \tag{2.8.2}$$

2.9. Relation with the Euler characteristics. Set $U := Y \setminus Z$. By (2.1.2) we have

$$\textstyle\sum_i (-1)^i \dim H^i(F_f, \mathbb{C})_\lambda = \chi(U) \quad \big(\lambda \in \Lambda_d\big). \tag{2.9.1}$$

In the case the pole order spectral sequence *degenerates at* E_2 as in Theorem 2, we then get

$$\textstyle\sum_{j\in\mathbb{Z}} \big(\mu_{k+jd} - \nu_{k+jd+d}\big) = \begin{cases} (-1)^{n'}\chi(U) & \text{if } k \in \mathbb{Z}\setminus d\,\mathbb{Z}, \\ (-1)^{n'}\big(\chi(U) - 1\big) & \text{if } k \in d\,\mathbb{Z}. \end{cases} \tag{2.9.2}$$

Here $n' = n - 1$. On the other hand, it is well known that $\chi(U) = n' + 1 - \chi(Z)$ (since $\chi(\mathbb{P}^{n'}) = n' + 1$), and for a *smooth* hypersurface $V \subset \mathbb{P}^{n'}$ of degree d, we have

$$\chi(Z) = \chi(V) - (-1)^{n'-1}\mu_Z \quad \text{with} \quad \mu_Z := \textstyle\sum_{z\in\Sigma} \mu_z, \tag{2.9.3}$$

by using a one-parameter family $\{f + sg\}_{s\in\mathbb{C}}$ with $V = \{g = 0\}$, where μ_z is the Milnor number of h_z, see [De2]. (Here we may assume $V \cap \operatorname{Sing} Z = \emptyset$ since $\operatorname{Sing} Z$ is isolated.) So we get

$$\chi(U) = \chi(\mathbb{P}^{n'} \setminus V) - (-1)^{n'}\mu_Z. \tag{2.9.4}$$

These imply (20) in the introduction. (Indeed, (2.9.2) holds in the isolated singularity case, where the left-hand side is replaced by $\sum_{j\in\mathbb{Z}}\gamma_{k+jd}$, and U by $\mathbb{P}^{n'}\setminus V$.)

In the case $n'=2$, we get

$$\chi(U)=(d-1)(d-2)+1-\mu_Z. \tag{2.9.5}$$

since a well-known formula for the genus of smooth plane curves implies that

$$\chi(\mathbb{P}^2\setminus V)=(d-1)(d-2)+1. \tag{2.9.6}$$

3. Calculation of twisted de Rham complexes

In this section we calculate the filtered twisted de Rham complexes associated to weighted homogeneous polynomials with isolated singularities.

3.1. Twisted de Rham complexes Let h be a weighted homogeneous polynomial for a local coordinate system $(y_1,\dots,y_{n'})$ of $(Y,0):=(\mathbb{C}^{n'},0)$ (as a complex manifold) with positive weights $w_1,\dots,w_{n'}$, see (1.8). We assume moreover that $h^{-1}(0)$ has an *isolated singularity* at 0. We take and fix

$$\alpha\in\mathbb{Q}\cap(0,1].$$

Consider the *twisted de Rham complex*

$$K^\bullet(h,\alpha):=\Omega_{Y,0}^\bullet[h^{-1}]h^{-\alpha}\cong\big(\Omega_{Y,0}^\bullet[h^{-1}];\mathrm{d}-\alpha\tfrac{\mathrm{d}h}{h}\wedge\big), \tag{3.1.1}$$

where $\Omega_Y^\bullet$ is the analytic de Rham complex. Define the *pole order filtration* P by

$$P^kK^i(h,\alpha):=\Omega_{Y,0}^i\,h^{-\alpha+k-i}\qquad(i,k\in\mathbb{Z}), \tag{3.1.2}$$

so that

$$\mathrm{Gr}_P^kK^\bullet(h,\alpha)\cong(\Omega_{Y,0}^\bullet/h\Omega_{Y,0}^\bullet;\mathrm{d}h\wedge). \tag{3.1.3}$$

Set $P_k=P^{-k}$ as usual. Note that $K^\bullet(h,\alpha)$ is a *graded complex* with degrees defined by

$$\deg y_i=\deg\mathrm{d}y_i=w_i,\quad \deg h=\deg\mathrm{d}h=1,\quad \deg h^{-\alpha}=-\alpha, \tag{3.1.4}$$

and P is compatible with the grading. Note that the $H^i\big(K^\bullet(h,\alpha)\big)$ $(i\in\mathbb{Z})$ are graded $\mathbb{C}$-vector spaces, and have the induced filtration P compatible with the grading.

Set $\lambda:=\exp(-2\pi i\alpha)$. We have the isomorphism

$$H^{n'}\big(K^\bullet(h,\alpha)\big)=H^{n'-1}(F_h,\mathbb{C})_\lambda(-1). \tag{3.1.5}$$

This follows from a generalization of (2.1.10) with L_λ replaced by $h^*E_{\lambda^{-1}}$ as in (2.1.5), where h is a weighted homogeneous polynomial for some local coordinates of a complex manifold Y'. Indeed, $\mathcal{O}_Y[h^{-1}]h^{-\alpha}$ is a regular holonomic $\mathcal{D}_Y$-module corresponding to $\mathbb{R}j_*h^*E_{\lambda^{-1}}[n']$ by the de Rham functor DR_Y so that we have a canonical isomorphism in the derived category

$$K^\bullet(h,\alpha)=\mathbb{R}j_*h^*E_{\lambda^{-1}}, \tag{3.1.6}$$

where $E_{\lambda^{-1}}$ is as in the explanation after (2.1.5), and j is the inclusion of the complement of $h^{-1}(0)$.

3.2. Brieskorn modules. In the above notation and assumption, define the Brieskorn module H_h'' (see [Br]) by

$$H_h'':=\Omega_{Y,0}^{n'}/\mathrm{d}h\wedge\mathrm{d}\Omega_{Y,0}^{n'-2}, \tag{3.2.1}$$

where $n'=\dim Y$. This is a *graded module* with degrees defined by (3.1.4).

The actions of t and ∂_tt on H_h'' are defined by

$$t[\omega]:=[h\omega],\quad \partial_tt[\omega]:=\mathrm{d}\eta\quad\text{with}\quad \mathrm{d}h\wedge\eta=h\omega, \tag{3.2.2}$$

where $\omega \in \Omega_{Y,0}^{n'}$, $\eta \in \Omega_{Y,0}^{n'}$. It is well known that H_h'' is a free $\mathbb{C}\{t\}$-module of rank μ with μ the Milnor number of h. For the definition of $\partial_t t$, we use a property of the weighted homogeneous polynomial that h is contained in the Jacobian ideal $(\partial h) \subset \mathcal{O}_{Y,0}$, see (1.8). More precisely, using (1.8.2), it is easy to show that

$$\partial_t t[\omega] = \beta[\omega] \quad \text{for} \;\; [\omega] \in H_{h,\beta}'', \tag{3.2.3}$$

(see the proof of Proposition (3.3) below), where $H_{h,\beta}'' \subset H_h''$ is the degree β part of graded module. (It is then easy to show the t-torsion-freeness of H_h'' by using (3.2.3) in the weighted homogeneous polynomial case, since $H_{h,\beta}'' = 0$ for $\beta \leqslant 0$.)

Proposition 3.3. *In the above notation and assumption* (*in particular,* $\alpha \in (0,1]$), *we have*

$$\begin{gathered} H^{n'}\big(K^\bullet(h,\alpha)\big)_\beta = 0 \quad (\forall\, \beta \neq 0), \quad \textit{that is,} \\ H^{n'}\big(K^\bullet(h,\alpha)\big) = H^{n'}\big(K^\bullet(h,\alpha)\big)_0, \end{gathered} \tag{3.3.1}$$

and there are canonical isomorphisms

$$\iota_k : H_{h,\alpha+k}'' \xrightarrow{\sim} P_{k-n'} H^{n'}\big(K^\bullet(h,\alpha)\big) \qquad (k \in \mathbb{N}), \tag{3.3.2}$$

satisfying

$$\iota_{k+i} \circ t^i = \iota_k \quad \textit{in} \;\; H^{n'}\big(K^\bullet(h,\alpha)\big) \qquad (k, i \in \mathbb{N}), \tag{3.3.3}$$

where $t^i : H_{h,\alpha+k}'' \hookrightarrow H_{h,\alpha+k+i}''$ *is induced by the action of* t *on* H_h'' *defined in* (3.2.2).

Proof. Set

$$\omega_0 := \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n'}, \quad \eta_0 := \textstyle\sum_i (-1)^{i-1} w_i y_i \, \mathrm{d}y_1 \wedge \cdots \wedge \widehat{\mathrm{d}y_i} \wedge \cdots \wedge \mathrm{d}y_{n'}.$$

The latter is the inner derivation (or contraction) of ω_0 by the vector field v in (1.8.2) so that

$$\mathrm{d}h \wedge g\,\eta_0 = hg\,\omega_0 \quad \text{for} \;\; g \in \mathcal{O}_{Y,0}. \tag{3.3.4}$$

Set $\beta_0 := \sum_i w_i = \deg \omega_0$. We can easily verify that

$$\mathrm{d}(g\,\eta_0) = \beta g\,\omega_0 \quad \text{for} \;\; g \in (\mathcal{O}_{Y,0})_{\beta-\beta_0} \;\; \big(\text{that is,} \;\; g\omega_0 \in (\Omega_{Y,0}^{n'})_\beta\big), \tag{3.3.5}$$

by using the equality $v(g) = \beta g$ for $g \in (\mathcal{O}_{Y,0})_\beta$ together with $\sum_i w_i \partial_{y_i} y_i = v + \beta_0$.

The assertion (3.3.1) then follows from (3.3.4–5) by the definition of the differential of $K^\bullet(h,\alpha)$ in (3.1.1). (Note that (3.2.3) also follows from (3.3.4–5).)

Define the canonical morphisms ι_k in (3.3.2) by

$$\iota_k[\omega] = [\omega h^{-\alpha-k}] \in H^{n'}\big(K^\bullet(h,\alpha)\big)_0 \quad \text{for} \;\; \omega \in (\Omega_{Y,0}^{n'})_{\alpha+k}. \tag{3.3.6}$$

Then (3.3.3) and the surjectivity of ι_k in (3.3.2) follow from the definition. For the well-definedness of (3.3.6), we have to show that

$$\iota_k[\omega] = 0 \quad \text{if} \quad \omega = \mathrm{d}h \wedge \mathrm{d}\xi \quad \text{with} \quad \xi \in (\Omega_{Y,0}^{n'-2})_{\alpha+k-1}. \tag{3.3.7}$$

Set $\eta := \mathrm{d}\xi$. Then

$$\mathrm{d}(\eta\, h^{-\alpha-k+1}) = -(\alpha+k-1)\mathrm{d}h \wedge \eta = -(\alpha+k-1)\omega. \tag{3.3.8}$$

Here we may assume $\alpha + k - 1 \neq 0$, since we have $\mathrm{d}\xi = 0$ if $\xi \in (\Omega_{Y,0}^{n'-2})_0$. (Note that ξ may be nonzero if $n' = 2$.) So (3.3.7) follows from (3.3.8).

It now remains to show the injectivity of (3.3.2). For $k \gg 0$, this follows from the surjectivity of (3.3.2) by using (3.1.5). Indeed, the pole order filtration P is exhaustive, and we have the isomorphisms

$$H_{\alpha+k}'' = H^{n'-1}(F_h)_{\mathbf{e}(-\alpha)} \quad \text{if} \;\; k \gg 0, \tag{3.3.9}$$

compatible with the morphism $t: H''_{\alpha+k} \to H''_{\alpha+k+1}$ (where $\mathbf{e}(-\alpha) := \exp(-2\pi i\alpha)$), see [Br] (and also Remark (3.4) below). This implies the injectivity for any k by using (3.3.3) together with the injectivity of the action of t on H''_h, see the remark after (3.2.3). This finishes the proof of Proposition (3.3).

Remark 3.4. (i) The action of ∂_t^{-1} on H''_h can be defined in a compatible way with (3.2.2), and the Gauss-Manin system G_h is the localization of the Brieskorn module H''_h by the action of ∂_t^{-1}. It is a graded $\mathbb{C}\{t\}\langle\partial_t, \partial_t^{-1}\rangle$-module, and (3.2.3) holds with H''_h replaced by G_h. We have moreover the canonical isomorphisms

$$G_{h,\alpha} = H^{n'-1}(F_h)_{\mathbf{e}(-\alpha)}, \tag{3.4.1}$$

compatible with the morphism $\partial_t : G_{h,\alpha} \to G_{h,\alpha-1}$. Note that the condition $N = 0$ is equivalent to that $\partial_t t = \alpha$ on $H''_{h,\alpha}$. So the actions of t and ∂_t^{-1} on $H''_{h,\alpha}$ (and also on $G_{h,\alpha}$ $(\alpha \neq 0)$) can be identified with each other up to a non-zero constant multiple.

(ii) We have the canonical isomorphisms (see [ScSt], [Va1]):

$$F^{n'-1-k}H^{n'-1}(F_h)_{\mathbf{e}(-\alpha)} = H''_{h,\alpha+k} \qquad \big(\alpha \in (0,1],\ k \in \mathbb{N}\big), \tag{3.4.2}$$

using the canonical isomorphism (3.3.9) together with the action of t or ∂_t^{-1} on H''_h (by using $N = 0$ as is explained above).

4. Proof of the main theorems

In this section we prove Theorems 2, 3 and 4.

4.1. Bernstein-Sato polynomials. Let $f : X \to \mathbb{C}$ be a holomorphic function on a complex manifold in general. Let $i_f : X \hookrightarrow X \times \mathbb{C}$ be the graph embedding by f, and t be the coordinate of $\mathbb{C}$. We have the canonical inclusion

$$M := \mathcal{D}_X[s]f^s \hookrightarrow \mathcal{B}_f := (i_f)_*^{\mathcal{D}}\mathcal{O}_X, \tag{4.1.1}$$

where the last term is the direct image of $\mathcal{O}_X$ as a $\mathcal{D}$-module, and is freely generated by $\mathcal{O}_X[\partial_t]$ over the canonical generator $\delta(f-t)$. Indeed, f^s and s can be identified respectively with $\delta(t-f)$ and $-\partial_t t$, see [Ma1], [Ma2]. We have the filtration V of Kashiwara [Ka2] and Malgrange [Ma2] on $\mathcal{B}_f$ along $t = 0$ indexed decreasingly by $\mathbb{Q}$ so that $\partial_t t - \alpha$ is nilpotent on $\mathrm{Gr}_V^\alpha \mathcal{B}_f$. This induces the filtration V on M and on M/tM so that

$$b_f(-\alpha) \neq 0 \iff \mathrm{Gr}_V^\alpha(M/tM) \neq 0, \tag{4.1.2}$$

since $b_f(s)$ is the minimal polynomial of the action of $-\partial_t t$ on M/tM by definition. (Here we assume that $b_f(s)$ exists by shrinking X if necessary.) Note that the support of $\mathrm{Gr}_V^\alpha(M/tM)$ is a *closed* analytic subset, and the above construction is compatible with the pull-back by smooth morphisms. These together with Remark (4.2)(i) below imply the last inclusion in (1) in the introduction.

Set

$$G_i\mathcal{B}_f := t^{-i}M \subset \mathcal{B}_f[t^{-1}] \qquad (i \in \mathbb{Z}).$$

We have the canonical inclusion $\mathcal{B}_f \hookrightarrow \mathcal{B}_f[t^{-1}]$, inducing the isomorphisms

$$\mathrm{Gr}_V^\alpha \mathcal{B}_f \xrightarrow{\sim} \mathrm{Gr}_V^\alpha\big(\mathcal{B}_f[t^{-1}]\big) \qquad (\alpha > 0). \tag{4.1.3}$$

This gives an increasing filtration G on $\mathrm{Gr}_V^\alpha \mathcal{B}_f$ for $\alpha > 0$ so that

$$b_f(-\alpha - i) \neq 0 \iff \mathrm{Gr}_i^G \mathrm{Gr}_V^\alpha \mathcal{B}_f \neq 0 \qquad \text{for each } \alpha \in (0,1],\ i \in \mathbb{N}. \tag{4.1.4}$$

Indeed, by the isomorphisms

$$t^i : \mathrm{Gr}_i^G \mathrm{Gr}_V^\alpha\big(\mathcal{B}_f[t^{-1}]\big) \xrightarrow{\sim} \mathrm{Gr}_0^G \mathrm{Gr}_V^{\alpha+i}\big(\mathcal{B}_f[t^{-1}]\big) \qquad (\alpha \in (0,1],\ i \in \mathbb{Z}), \tag{4.1.5}$$

we get the canonical isomorphisms

$$\mathrm{Gr}_i^G \mathrm{Gr}_V^\alpha \mathcal{B}_f = \mathrm{Gr}_V^{\alpha+i}(M/tM) \qquad (\alpha \in (0,1],\ i \in \mathbb{Z}). \tag{4.1.6}$$

For $\alpha \in (0,1]$, the filtration G on $\mathrm{Gr}_V^\alpha \mathcal{B}_f$ is a finite filtration such that

$$\mathrm{Gr}_i^G \mathrm{Gr}_V^\alpha \mathcal{B}_f = 0 \quad \text{unless} \quad i \in \mathbb{N}. \tag{4.1.7}$$

This is closely related to the negativity of the roots of $b_f(s)$, see [Ka1].

For any $\beta \in \mathbb{Q}$, we have the canonical decomposition

$$\mathcal{B}_f/V^\beta \mathcal{B}_f = \textstyle\bigoplus_{\alpha<\beta} \mathrm{Gr}_V^\alpha \mathcal{B}_f. \tag{4.1.8}$$

by using the action of $s = -\partial_t t$. This is obtained by taking the inductive limit of the decomposition of $V^\gamma \mathcal{B}_f/V^\beta \mathcal{B}_f$ for $\gamma \to -\infty$, and is compatible with the canonical surjections

$$\mathcal{B}_f/V^\beta \mathcal{B}_f \twoheadrightarrow \mathcal{B}_f/V^{\beta'} \mathcal{B}_f \qquad (\beta > \beta'). \tag{4.1.9}$$

(This means that (4.1.9) corresponds via (4.1.8) to the canonical surjection associated to shrinking the index set of direct sums.)

We then get the *asymptotic expansion* for any $\xi \in \mathcal{B}_f$:

$$\xi \sim \textstyle\sum_{\alpha \geqslant \alpha_\xi} \xi^{(\alpha)} \quad \text{with} \quad \xi^{(\alpha)} \in \mathrm{Gr}_V^\alpha \mathcal{B}_f,\ \alpha_\xi \in \mathbb{Q}. \tag{4.1.10}$$

This means that the following equality holds for any $\beta > \alpha_\xi$ via the isomorphism (4.1.8):

$$\xi \mod V^\beta \mathcal{B}_f = \textstyle\bigoplus_{\alpha_\xi \leqslant \alpha < \beta} \xi^{(\alpha)}. \tag{4.1.11}$$

We have to *control* this expansion at least in the case $\xi = f^s$ in order to determine $b_f(s)$. Indeed, we have the decomposition of f^s modulo $V^\beta \mathcal{B}_f$ in a compatible way with the action of $\mathcal{D}_X[s]$ by using the action of $\mathbb{C}[s]$ for any $\beta > \alpha_f$ as in (4.1.11), and *the direct summands of f^s generate $\mathcal{D}_X[s]f^s$ modulo $V^\beta \mathcal{B}_f$ over $\mathcal{D}_X[s]$*. So the determination of the *leading term* of the expansion is *not* enough to calculate $b_f(s)$. A similar phenomenon is known for the Gauss-Manin systems in the isolated singularity case, and this is the reason for which the determination of $b_f(s)$ is so complicated. In the weighted homogeneous isolated singularity case, this problem does *not* occur for the Gauss-Manin systems. However, it *does* occur for $\mathcal{B}_f$ in the non-isolated homogeneous singularity case, since the situation is quite different in the case of $\mathcal{B}_f$.

Remarks 4.2. (i) In the above notation, it is well known that $b_f(s)$ depends only on the divisor $D := f^{-1}(0)$. Indeed, we have a line bundle L over X corresponding to D, and the Euler field corresponding to the natural $\mathbb{C}^*$-action on L and the ideal sheaf of the zero-section of L are sufficient to determine the Bernstein-Sato polynomial by using the filtration V of Kashiwara and Malgrange along the zero-section on the direct image of $\mathcal{O}_X$ as a $\mathcal{D}$-module by the canonical section $i_D : X \hookrightarrow L$ defined by D (where i_D is locally identified with the graph embedding if a local defining function of D is chosen).

(ii) Under the hypotheses of Theorem 2, the weight filtration W on $\mathrm{Gr}_V^\alpha \mathcal{B}_f$ for $\alpha \in (0,1)$ has a symmetry with center $n' = n-1$, and we can show

$$\mathrm{Gr}_i^W \mathrm{Gr}_V^\alpha \mathcal{B}_f = 0 \ \ (i \ne n') \quad \text{if} \ \ \alpha \in (0,1) \setminus \tfrac{1}{d}\mathbb{Z}. \tag{4.2.1}$$

For the proof we use a well-known assertion that the local system monodromy of $\psi_f \mathbb{C}_X|_{C_j \setminus \{0\}}$ coincides with T^{-d}, where C_j is an irreducible component of $\mathrm{Sing}\, f^{-1}(0)$, and T is the monodromy associated with the nearby cycle functor ψ_f. (The last assertion can be shown by using the blow-up of X along the origin as in the proof of Theorem (1.5).) For the proof of (4.2.1), we also need a Verdier type extension theorem (which involves only with the *unipotent* monodromy part as in [Sa1]). We apply this to $\psi_{y,1}$, $\varphi_{y,1}$, where y is as in (1.3). Using the above relation between the two monodromies, it implies $\psi_{y,1}\mathrm{Gr}_V^\alpha \mathcal{B}_f = 0$ $(\alpha \notin \frac{1}{d}\mathbb{Z})$.

This means that there is no nontrivial extension. Combining this with the semisimplicity of $\mathrm{Gr}_i^W\mathrm{Gr}_V^\alpha\mathcal{B}_f$ together with a remark after (2.1.1), we get (4.2.1).

The above argument implies moreover that $\mathrm{Gr}_V^\alpha\mathcal{B}_f$ for $\alpha\in(0,1)\setminus\frac{1}{d}\mathbb{Z}$ is a direct sum of copies of

$$\mathcal{D}_{C_j}/\mathcal{D}_{C_j}(y\partial_y-\beta)\quad\text{for}\quad\beta=-d\alpha,$$

where C_j is as above, and y is identified with a coordinate of C_j. Since this holds also for each $\mathrm{Gr}_i^G\mathrm{Gr}_V^\alpha\mathcal{B}_f$ with G as in (4.1.4), we conclude that

$$\mathcal{R}_f^0\in\tfrac{1}{d}\mathbb{Z}. \tag{4.2.2}$$

(Note that this is a quite nontrivial assertion.)

(iii) With the notation and assumption of Remark (ii) above, assume $\alpha\in(0,1)\cap\frac{1}{d}\mathbb{Z}$. Then

$$\begin{gathered}\mathrm{Gr}_i^W\mathrm{Gr}_V^\alpha\mathcal{B}_f=0\quad\text{for}\quad|i-n'|>1,\\ \operatorname{supp}\mathrm{Gr}_i^W\mathrm{Gr}_V^\alpha\mathcal{B}_f\subset\{0\}\quad\text{for}\quad|i-n'|=1.\end{gathered}\tag{4.2.3}$$

If the asymptotic expansion of some $\xi\in F_0\mathcal{B}_f=\mathcal{O}_X\delta(t-f)$ in (4.1.10) has a nonzero direct factor satisfying

$$\xi^{(\alpha)}\in W_{n'-1}\mathrm{Gr}_V^\alpha\mathcal{B}_f\setminus\mathcal{D}_X[s](F_0\mathrm{Gr}_V^\alpha\mathcal{B}_f)\quad\big(\alpha\in(0,1)\cap\tfrac{1}{d}\mathbb{Z}\big), \tag{4.2.4}$$

then this would give an example where the implication $\Longrightarrow$ in (3) fails without assuming condition (2) in Theorem 1. Note that the direct image of $\mathrm{Gr}_V^\alpha\mathcal{B}_f$ by the projection to $S=\mathbb{C}$ defined by the function y in (1.3) contains direct factors isomorphic to

$$E:=\mathcal{D}_S/\mathcal{D}_S(y\partial_y y),$$

where y is identified with the natural coordinate of $S=\mathbb{C}$. We have the isomorphism

$$\mathrm{DR}(E)_0=C\big(\mathrm{Gr}_{V_y}\partial_y:\psi_{y,1}E:=\mathrm{Gr}_{V_y}^1E\to\varphi_{y,1}E:=\mathrm{Gr}_{V_y}^0E\big)_0,$$

with V_y the filtration of Kashiwara and Malgrange along $y=0$, and the image of $\mathrm{Gr}_{V_y}\partial_y$ coincides with the subspace obtained by taking $\varphi_{y,1}$ of the intersection of E with the direct image of $W_{n'-1}\mathrm{Gr}_V^\alpha\mathcal{B}_f$.

(iv) By a calculation of multiplier ideals in [Sa4, Proposition 1] and using [BuSa1], we get

$$\dim\mathrm{Gr}_V^\alpha\mathcal{O}_X=\tbinom{k-1}{n-1}\quad\text{for}\quad\alpha=\tfrac{k}{d}\in\big[\tfrac{n}{d},\alpha_Z\big), \tag{4.2.5}$$

where $\mathcal{O}_X=\mathrm{Gr}_0^F\mathcal{B}_f$. By (1.2.5), (1.2.8), (1.3.8), this implies

$$M_k^{(\infty)}\ne0\quad\text{for}\quad\tfrac{k}{d}\in\big[\tfrac{n}{d},\alpha_Z\big). \tag{4.2.6}$$

Indeed, $\mathrm{Gr}_V^\alpha\mathrm{Gr}_0^F\mathcal{B}_f=\mathrm{Gr}_0^F\mathrm{Gr}_V^\alpha\mathcal{B}_f$ is supported at the origin for $\alpha\in\big[\frac{n}{d},\alpha_Z\big)$, and we have the inclusion $F^p\subset P^p$ $(\forall\,p\in\mathbb{Z})$ together with the vanishing $P^nH^{n-1}(F_f,\mathbb{C})=0$. (Note that $\alpha_Z=\min\big(\widetilde{\alpha}_Z,1\big)\leqslant1$.)

Theorem 4.3. *In the notation and assumption of* (1.4) (*including the assumption* (W) *in Theorem* 2), *we have the equality*

$$\textstyle\sum_{k\in\mathbb{N}}\dim_{\mathbb{C}}\mathrm{Coker}\big(r_k:M_k\to\bigoplus_{z\in\Sigma}\Xi_{h_z}^{k/d}\big)=\dim_{\mathbb{C}}H^{n'-1}(F_f,\mathbb{C}), \tag{4.3.1}$$

where r_k is the composition of

$$M_k\to\widetilde{M}_k\quad\textit{and}\quad\widetilde{M}_k\to\textstyle\bigoplus_{z\in\Sigma}\Xi_{h_z}^{k/d},$$

induced by (1.3.6) *and* (1.4.4–5) *respectively.*

Proof. The assertion follows from Theorem (2.2) and Proposition (3.3) together with [Di1], [DiSa2]. In the notation of [DiSa2, Section 1.8], set

$$\mathcal{L}_\lambda = \mathcal{S}^i \qquad \big(i = 0, \dots, d-1,\ \lambda = \mathbf{e}(i/d)\big),$$

with $\mathbf{e}(\alpha) := \exp(2\pi i\alpha)$. This is a locally free $\mathcal{O}_Y$-module of rank 1 having a canonical meromorphic connection so that the localization $\mathcal{L}_\lambda(*Z)$ of $\mathcal{L}_\lambda$ along Z is a regular holonomic $\mathcal{D}_Y$-module with

$$\mathrm{DR}(\mathcal{L}_\lambda) = \mathbb{R}j_*L_\lambda.$$

The restriction of $\mathcal{L}_\lambda$ to the smooth points of Z is the Deligne extension with residue i/d, see also [BuSa2, 1.4.1]. On $Y' := Y \setminus \{u = 0\}$, there is an isomorphism of regular holonomic $\mathcal{D}_{Y'}$-modules

$$\mathcal{L}_\lambda(*Z)|_{Y'} \cong \mathcal{O}_{Y'}(*Z')h^{i/d} \quad \text{inducing} \quad \mathcal{L}_\lambda|_{Y'} \cong \mathcal{O}_{Y'}h^{i/d},$$

with $h := f/u^d$, $Z' := Z \cap Y'$. We have the pole order filtration P on $\mathcal{L}(*Z)$ defined by

$$P_j\big(\mathcal{L}_\lambda(*)\big) = \begin{cases} \mathcal{L}_\lambda((j+1)Z) & \text{if } j \geqslant 0, \\ 0 & \text{if } j < 0, \end{cases}$$

which induces the pole order filtration P on the Milnor cohomology $H^{n-1}(F_f)_\lambda$ by taking the induced filtration on the de Rham complex (where the shift of filtration by $n-1$ occurs), see [DiSa2, Section 1.8], [Di1]. By using the Bott vanishing theorem, the filtered de Rham complex can be described by using the differential d_f on $\Omega^\bullet$ defined by

$$\mathrm{d}_f\,\omega := f\mathrm{d}\omega - \tfrac{k}{d}\,\mathrm{d}f\wedge\omega \qquad (\omega \in \Omega_k^\bullet),$$

(see [Do, 4.1.1] and also [Di1, Ch. 6, 1.18]). We have moreover the isomorphism

$$\mathrm{Gr}_P^{n-1-q} H^{n-1}(F_f, \mathbb{C})_\lambda \cong M^{(2)}_{(q+1)d-i}\,, \tag{4.3.2}$$

using the E_2-degeneration of the pole order spectral sequence in Theorem 2 together with [Di1, Ch. 6, Theorem 2.9]. Here $[\omega] \in M^{(2)}_{(q+1)d-i}$ with $\omega \in \Omega^n_{(q+1)d-i}$ corresponds to

$$\big[i_E(\omega/f^{q+1})f^{i/d}\big] \in \mathrm{Gr}_P^{n-1-q} H^{n-1}(U, L_\lambda) \qquad \big(\lambda = \mathbf{e}(i/d)\big),$$

by using the isomorphism (2.1.2), where i_E denotes the contraction with the Euler field $E := \sum_j x_i\partial_{x_i}$, and $f^{i/d}$ is a symbol denoting a generator of $\mathcal{L}_\lambda(i) \cong \mathcal{O}_Y$ (which is unique up to a non-zero constant multiple, and is closely related to the above differential d_f).

The target of (2.2.2) is calculated by Proposition (3.3). (Note that there is a shift of filtration by 1 coming from (3.1.5).) The graded pieces of the pole order filtration P of restriction morphism (2.2.2) is then induced up to a non-zero constant multiple by the morphism r_k in (4.3.1). Here the source of Gr_P^{n-1-q} of (2.2.2) can be identified with $M^{(2)}_{(q+1)d-i}$ as above, and r_k can be defined by using the contraction with the Euler field together with the substitution $y = 1$ after restricting to

$$Y' := \{y \neq 0\} \subset Y = \mathbb{P}^{n-1}.$$

Note that this is compatible with the blow-up construction in the proof of Theorem (1.5), where the Euler field E becomes the vector field $y\partial_y$ using the product structure of the affine open piece of the blow-up.

We have the strict compatibility of (2.2.2) with the pole order filtration P by Remark (4.4) below. (Here F is the Hodge filtration. We need Proposition (3.3) together with (3.1.5) and (3.4.2) to show $F = P$ on the target.) So $\mathrm{Gr}_P^\bullet$ of the cokernel of (2.2.2) coincides with the cokernel of $\mathrm{Gr}_P^\bullet$ of (2.2.2), and the latter is given by the cokernel of r_k by the above argument. We thus get (4.3.1). This finishes the proof of Theorem (4.3).

Remark 4.4. Let $\phi : (V; F, P) \to (V'; F, P)$ be a morphism of bifiltered vector spaces with $F \subset P$ and $F = P$ on V'. If ϕ is strictly compatible with F, then it is so with P.

Indeed, the strict compatibility for F means $\phi(F^pV) = F^pV' \cap \phi(V)$, and

$$\phi(F^pV) \subset \phi(P^pV) \subset P^pV' \cap \phi(V) = F^pV' \cap \phi(V).$$

We thus get $\phi(P^pV) = P^pV' \cap \phi(V)$, that is, ϕ is also strictly compatible with P.

From Theorems (1.5) and (4.3), we can deduce the following.

Theorem 4.5. *We have*

$$\sum_{q\in\mathbb{N}} \dim N_{q+d} \cap \big(\textstyle\bigoplus_{z\in\Sigma} \Xi_{h_z}^{q/d}\big) = \dim H^{n-2}(F_f, \mathbb{C}), \tag{4.5.1}$$

where the intersection on the left-hand side is taken in $\widetilde{N}_{k+d}$ via the isomorphism (1.4.4) *using the function y.*

Proof. By (1.3.6) and (1.4.4–5), we have the inclusions

$$V_1 := N_{q+d} \quad\subset\quad V := \widetilde{N}_{q+d} \quad\supset\quad V_2 := \textstyle\bigoplus_{z\in\Sigma} \Xi_{h_z}^{q/d}, \tag{4.5.2}$$

where $p + q + d = nd$. By the graded local duality as is explained in [DiSa3, 1.1] together with the compatibility of duality isomorphisms in Theorem (1.5), we have

$$V^* = \widetilde{M}_p, \quad V_1^{\perp} = M''_p, \quad V_2^* = \textstyle\bigoplus_{z\in\Sigma} \Xi_{h_z}^{p/d}, \tag{4.5.3}$$

where V^* denotes the dual vector space of V. Using the diagram of the nine lemma, we get a surjection of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & V_1^{\perp} & \to & V^* & \to & V_1^* & \to & 0 \\ & & \twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow & & \\ 0 & \to & V_1^{\perp}/V_1^{\perp}\cap V_2^{\perp} & \to & V_2^* & \to & (V_1\cap V_2)^* & \to & 0 \end{array} \tag{4.5.4}$$

which implies the exact sequence

$$V_1^{\perp} \to V_2^* \to (V_1 \cap V_2)^* \to 0. \tag{4.5.5}$$

By Theorem (1.5) this gives the exact sequence

$$M''_p \xrightarrow{r_p} \textstyle\bigoplus_{z\in\Sigma} \Xi_{h_z}^{p/d} \to \big(N_{q+d} \cap \big(\bigoplus_{z\in\Sigma} \Xi_{h_z}^{q/d}\big)\big)^* \to 0. \tag{4.5.6}$$

The assertion (4.5.1) then follows from Theorem (4.3).

4.6. Proof of Theorem 2. By the pole order spectral sequence, we have the inequality

$$\sum_{k\in\mathbb{N}} \dim \mathrm{Ker}\big(\mathrm{d}^{(1)} : N_{k+d} \to M_k\big) \geqslant \dim H^{n-2}(F_f, \mathbb{C}), \tag{4.6.1}$$

where the equality holds if and only if the spectral sequence degenerates at E_2.

On the other hand, the proof of [DiSa3, Theorem 5.2] actually proves the inequality

$$\sum_{k\in\mathbb{N}} \dim \mathrm{Ker}\big(\mathrm{d}^{(1)} : N_{k+d} \to M_k\big) \leqslant \sum_{k\in\mathbb{N}} \dim N_{k+d} \cap \big(\textstyle\bigoplus_{z\in\Sigma} \Xi_{h_z}^{k/d}\big). \tag{4.6.2}$$

So the E_2-degeneration follows from (4.6.1–2) and Theorem (4.5). This finishes the proof of Theorem 2.

4.7. Proof of Theorem 3. The above proof of (4.6.2) shows the inequality

$$\sum_{k\in\mathbb{N}} \dim \mathrm{Ker}\big(\mathrm{d}^{(1)} : N_{k+d} \to M''_k\big) \leqslant \sum_{k\in\mathbb{N}} \dim N_{k+d} \cap \big(\textstyle\bigoplus_{z\in\Sigma} \Xi_{h_z}^{k/d}\big), \tag{4.7.1}$$

since the intersection of N_{k+d} with the kernel of $\mathrm{d}^{(1)} : \widetilde{N}_{k+d} \to \widetilde{M}_k$ is considered in [DiSa3, 5.1.2] (where the latter morphism is actually defined by the morphism in *loc. cit.*) This implies the injectivity of the composition in Theorem 3, since the failure of this injectivity implies a strict inequality in (4.6.2) by using (4.7.1). This finishes the proof of Theorem 3.

Theorem 4.8. *In the notation of the introduction and under the assumption* (IS)*, we have*

$$\alpha_f := \min \mathcal{R}_f = \min\bigl\{\alpha_Z, \tfrac{n}{d}\bigr\}. \tag{4.8.1}$$

Proof. In the notation of (4.1), we index the Hodge filtration F on $\mathcal{B}_f$ so that

$$F_0\mathcal{B}_f = \mathcal{O}_X\,\delta(t-f). \tag{4.8.2}$$

By [BuSa1], we can essentially identify the induced filtration V on it with the multiplier ideals of $D := f^{-1}(0) \subset X$ (up to the difference between left continuous and right continuous properties at jumping coefficients). This implies that α_f coincides with the log canonical threshold, that is, the minimal jumping number, of $D \subset X$. Indeed, we have

$$\mathcal{D}_X[s]f^s \subset V^\alpha\mathcal{B}_f \quad \text{if} \quad \mathcal{O}_X\,\delta(t-f) \subset V^\alpha\mathcal{B}_f.$$

So we can replace α_f, α_Z in (4.8.1) with the log canonical threshold of $D \subset X$ and that of $D \setminus \{0\} \subset X \setminus \{0\}$. Then the assertion follows from [Sa4, Theorem 2.2]. This finishes the proof of Theorem (4.8).

4.9. Proof of Theorem 4. We first show the inclusion (12). By [Sa3] and (4.2.2), we have

$$\mathcal{R}_f^0 \subset (0,n) \cap \tfrac{1}{d}\,\mathbb{Z}.$$

We show $k \geqslant n$ if $\frac{k}{d} \in \mathcal{R}_f^0$ with $k \in \mathbb{N}$.

If $\frac{k}{d} < \alpha_Z$ and $k < n$, then we have $\frac{k}{d} \notin \mathcal{R}_f$ since $\frac{k}{d} < \alpha_f = \min\bigl\{\alpha_Z, \frac{n}{d}\bigr\}$ by using Theorem (4.8). This shows that we have $k \geqslant n$ if $\frac{k}{d} < \alpha_Z$ and $\frac{k}{d} \in \mathcal{R}_f^0$.

If $\frac{k}{d} \geqslant \alpha_Z$ and $\frac{k}{d} \in \mathcal{R}_Z + \mathbb{Z}_{<0}$, then we get $k \geqslant n$ by using condition (11) which implies $\mu_k > 0$. (Note that $\mu_k = 0$ if $k < n$.)

If $\frac{k}{d} \geqslant \alpha_Z$ and $\frac{k}{d} \notin \mathcal{R}_Z + \mathbb{Z}_{<0}$, then we have as well $k \geqslant n$ by the first assertion of Theorem 1. (The latter implies $\mu_k > 0$ since we assume $\frac{k}{d} \in \mathcal{R}_f^0$).

So the inclusion (12) is proved. We now show the last assertion of Theorem 4. Take any $\frac{k}{d} \in \bigl[\frac{n}{d}, n\bigr) \setminus \mathcal{R}_Z$.

Assume $\mu_k > \nu_{k+d}$. Then we have $\frac{k}{d} \in \mathcal{R}_f^0$ by the last assertion of Theorem 1 together with Theorem 2.

Assume conversely $\mu_k = \nu_{k+d}$. (Note that we have always $\mu_k \geqslant \nu_{k+d}$ for $\frac{k}{d} \notin \mathcal{R}_Z$ by (9) in the introduction.) Then we have $\frac{k}{d} \geqslant \alpha_Z$. (Otherwise we would get $\mu_k > \nu_{k+d}$ by the calculation of multiplier ideals in [Sa4, Proposition 1], see (4.2.6).) We have furthermore $\frac{k}{d} \notin \mathcal{R}_Z + \mathbb{Z}_{<0}$. (Otherwise we would get $\mu_k > \nu_{k+d}$ by condition (11).) Then we can apply the first assertion of Theorem 1, and get $\frac{k}{d} \notin \mathcal{R}_f$. This finishes the proof of Theorem 4.

Theorem 4.10. *In the notation of the introduction, assume $d \geqslant n$ and either Z has only ordinary double points as singularities or condition* (W) *holds together with the inequalities:*

$$\begin{aligned} &\tau_Z < 2\gamma_{m_0} \quad \textit{if} \quad m_0 := \min\bigl(\bigl[d\widetilde{\alpha}_Z, \tfrac{d(n-1)}{2}\bigr] \cap \mathbb{Z} \setminus d\mathcal{R}_Z\bigr) < +\infty,\\ &\tau_Z < \gamma_{m_1} \quad \textit{if} \quad m_1 := \min\bigl(\bigl(\tfrac{d(n-1)}{2}, \tfrac{dn}{2}\bigr) \cap \mathbb{Z} \setminus d\mathcal{R}_Z\bigr) < +\infty,\\ &\tau_Z < \gamma_{m_2} \quad \textit{if} \quad m_2 := \max\bigl(\bigl[\tfrac{dn}{2}, d(n-1-\widetilde{\alpha}_Z)\bigr] \cap \mathbb{Z} \setminus d\mathcal{R}_Z\bigr) > -\infty. \end{aligned} \tag{4.10.1}$$

Then the conclusion of Theorem 5 *holds, that is,* (15) *holds with $k_{\min}$, $k_{\max}$, β_f as in* (19)*.*

Proof. By [DiSt1], [Di2] (in the ordinary double point case) and [DiSa4, Theorem 9] (in general), we have

$$\nu_k = \dim N_k = 0 \quad \text{if} \quad \tfrac{k}{d} < \widetilde{\alpha}_Z + 1. \tag{4.10.2}$$

(Note that ${}^sN_k$ in [DiSa4] mean N_{k+d} in this paper.)

By definition we have the decomposition $\delta_k=\delta'_k+\delta''_k$ $(k\in\mathbb{Z})$ satisfying

$$\begin{gathered}\delta'_k=\delta'_{dn-k},\quad \delta''_k=\delta''_{d(n-1)-k}\quad(\forall\,k),\\ \delta'_k=\mu'_k\geqslant 0\ \ (\forall\,k),\quad \delta''_k=\mu''_k-\nu_{k+d}\geqslant 0\ \ (\forall\,k\notin d\mathcal{R}_Z),\end{gathered}\tag{4.10.3}$$

where the last inequality follows from (9) in the introduction. (Note that the assumption $k\notin d\mathcal{R}_Z$ is needed only for this inequality.) We have the inequalities

$$0<\mu''_k\leqslant\tau_Z\quad(\forall\,k\geqslant n),\tag{4.10.4}$$

since μ''_k is weakly increasing (see for instance [DiSa3]) and $\mu_n=\mu''_n=1$. (For the last inequality of (4.10.4), see (8).) We then get

$$\delta_k\geqslant\delta''_k>0\quad\text{if}\quad k\in\big[n,d\widetilde{\alpha}_Z\big)\cup\big(d(n-1-\widetilde{\alpha}_Z),d(n-1)-n\big].\tag{4.10.5}$$

Indeed, the inequality $\delta''_k>0$ for $k\in\big[n,d\widetilde{\alpha}_Z\big)$ follows from (4.10.2–4), and we can use the symmetry of δ''_k in (4.10.3). (Here we may assume $\big[n,d\widetilde{\alpha}_Z\big)\neq\emptyset$, that is, $\frac{k}{d}<\widetilde{\alpha}_Z$. Indeed, the assertion (4.10.5) trivially holds if it is not satisfied.)

By the symmetries in (4.10.3) together with (9), we have

$$\begin{gathered}\mathrm{Supp}\{\delta'\}\subset\big[\beta_f+n,dn-n-\beta_f\big],\\ \mathrm{Supp}\{\delta''\}\subset\big[n,d(n-1)-n\big]\cup d\mathcal{R}_Z.\end{gathered}\tag{4.10.6}$$

Related to condition (2) in Theorem 1, note that

$$\tfrac{k}{d}\notin\mathcal{R}_Z+\mathbb{Z}_{<0}\quad\text{if}\quad k>d(n-1)-n.\tag{4.10.7}$$

Indeed, we have $\max\mathcal{R}_Z<n-1$ and $d\geqslant n$ by the assumption of Theorem (4.10).

Set as in the introduction

$$I:=\textstyle\bigoplus_k I_k\subset R\quad\text{with}\quad I_k:=\big\{g\in R_k\mid\tilde{g}\in\mathcal{O}_Y(k)\otimes_{\mathcal{O}_Y}(\partial f)^\sim\big\}.$$

Then M' is identified with $I/(\partial f)$ up to the shift of grading by n, that is,

$$I_k/(\partial f)_k=M'_{k+n},\tag{4.10.8}$$

In particular,

$$\beta_f=\min\big\{k\in\mathbb{N}\mid I_k/(\partial f)_k\neq 0\big\}.$$

We now show the inequality

$$\delta'_k=\mu'_k>0\quad\text{if}\quad k\in\big(d(n-1)-n,dn-n-\beta_f\big].\tag{4.10.9}$$

We may assume $k\in(\beta_f+n,d+n)$ by the symmetry in (4.10.3). (Here we may assume $\beta_f<d$, since the assertion (4.10.9) trivially holds otherwise.) By (4.10.8) the assertion (4.10.9) is equivalent to

$$I_k/(\partial f)_k\neq 0\quad\text{if}\quad k\in[\beta_f,d).\tag{4.10.10}$$

We have

$$\dim(\partial f)_k=\begin{cases}n & \text{if } k=d-1,\\ 0 & \text{if } k<d-1.\end{cases}\tag{4.10.11}$$

This implies that (4.10.10) is nontrivial only in the case $\beta_f=d-2$ and $\dim I_{d-2}=1$. (Indeed, $I\subset R$ is a graded ideal so that $R_jI_k\subset I_{j+k}$, and $\dim I_{d-1}>\dim(\partial f)_{d-1}=n$ if $\dim I_{d-2}>1$.) Thus (4.10.9–10) are reduced to that

$$(\partial f)_{d-1}\neq R_1\,g\quad\text{for any}\ \ g\in R_{d-2}.\tag{4.10.12}$$

Assume $(\partial f)_{d-1}=R_1\,g$ for some $g\in R_{d-2}$. Then $(\partial f)_{d-1+k}=R_{k+1}\,g$ for any $k\geqslant 1$, since (∂f) is generated by $(\partial f)_{d-1}$. This is, however, a contradiction by considering the associated ideal sheaf $(\partial f)^\sim\subset\mathcal{O}_Y$. So (4.10.12) and (4.10.9–10) are proved.

In the ordinary double point case, Theorem (4.10) now follows from (4.10.5–8). Indeed, we have the symmetry of δ''_k as in (4.10.3), and

$$\delta''_k = \mu''_k > 0 \quad \text{for} \quad k \in \big[n, d(n-1)/2\big), \tag{4.10.13}$$

since $\nu_{k+d} = 0$ for $k < d(n-1)/2$ by (4.10.2). (Note that $\widetilde{\mathcal{R}}_Z = \{d(n-1)/2\}$ in the ordinary double point case.)

In the case where condition (4.10.1) holds together with (W), it now remains in view of (4.10.5–8) to show the following strict inequalities:

$$\begin{aligned} \delta_k > 0 \quad &\text{if} \quad k \in \big[d\widetilde{\alpha}_Z, \tfrac{d(n-1)}{2}\big] \cap \mathbb{Z} \setminus d\mathcal{R}_Z, \\ \mu'_k > 0 \quad &\text{if} \quad k \in \big(\tfrac{d(n-1)}{2}, \tfrac{dn}{2}\big) \cap \mathbb{Z} \setminus d\mathcal{R}_Z, \\ \mu'_k > 0 \quad &\text{if} \quad k \in \big[\tfrac{dn}{2}, d(n-1-\widetilde{\alpha}_Z)\big] \cap \mathbb{Z} \setminus d\mathcal{R}_Z. \end{aligned} \tag{4.10.14}$$

The last two inequalities of (4.10.14) follow from the hypothesis (4.10.1) together with Remark (4.11)(i) below and the symmetry $\gamma_k = \gamma_{nd-k}$ $(\forall\, k \in \mathbb{Z})$ by using the inequality

$$\mu'_k = \gamma_k + \nu_k - \mu''_k \geqslant \gamma_k - \tau_Z \quad (\forall\, k), \tag{4.10.15}$$

which follows from (8) in the introduction together with $\nu_k \geqslant 0$ $(\forall\, k)$.

For the first inequality of (4.10.14), we may assume $\delta'_k = \mu'_k = 0$. (Indeed, $\delta_k > 0$ by (4.10.3) otherwise.) Since $\{\mu''_j\}$ is weakly increasing, $k \leqslant \frac{d(n-1)}{2}$, and $\nu_{k+d} \geqslant 0$, we then get

$$\delta_k = \mu''_k - \nu_{k+d} = \mu''_k - \big(\tau_Z - \mu''_{d(n-1)-k}\big) \geqslant 2\mu''_k - \tau_Z \geqslant 2\delta_k - \tau_Z. \tag{4.10.16}$$

(For the second equality, see a remark after (17) in the introduction.) So the first equality of (4.10.14) follows from the hypothesis (4.10.1) together with Remark (4.11)(i) below. This finishes the proof of Theorem (4.10).

Remarks 4.11. (i) For γ_k as in (10) in the introduction, the following is well known:

$$\gamma_{k-1} < \gamma_k \quad \text{if} \quad n < k \leqslant \tfrac{nd}{2}. \tag{4.11.1}$$

This can be shown, for instance, by using the hard Lefschetz property in Hodge theory, see also [Sti], etc. It can be proved also by using an elementary argument as follows.

Define $\gamma_{n,k} \in \mathbb{N}$ for $k \in \mathbb{Z}$, $n \geqslant 2$ by

$$\textstyle\sum_k \gamma_{n,k}\, t^k = \Phi_{n,d}(t) := (t - t^d)^n/(1-t)^n = \big(t + \cdots + t^{d-1}\big)^n. \tag{4.11.2}$$

Note that $\gamma_{n,n} = 1$ and $\gamma_{n,k} = 0$ $(k < n)$. (In this paper $\gamma_{n,k}$ is denoted by γ_k.) We have

$$(1-t)\,\Phi_{n,d}(t) = (t - t^d)\,\Phi_{n-1,d}(t).$$

Comparing the coefficients, we get

$$\gamma_{n,\,k} - \gamma_{n,\,k-1} = \gamma_{n-1,\,k-1} - \gamma_{n-1,\,k-d} \quad (k \in \mathbb{Z}).$$

From the condition $n < k \leqslant nd/2$, we can deduce that

$$n - 1 < k - 1, \quad \big((k-1) + (k-d)\big)/2 < (n-1)d/2.$$

These imply the strict inequality

$$\gamma_{n-1,\,k-1} < \gamma_{n-1,\,k-d},$$

using the inductive hypothesis and the symmetry with center $(n-1)d/2$:

$$\gamma_{n-1,\,k} = \gamma_{n-1,\,(n-1)d-k} \quad (k \in \mathbb{Z}).$$

So the assertion (4.11.1) follows by increasing induction on $n \geqslant 2$, since it is trivial for $n = 2$.

(ii) In the case all the singularities of Z are ordinary double points, we sometimes observe that $\mu'_k = \max(\gamma_k - \tau_Z, 0)$ $(\forall\, k)$. However, this is restricted to the case where $\tau_Z = |\mathrm{Sing}\, Z|$

is not very large compared to Arnold's number $\mathrm{Ar}_{n-1}(d)$, see Example (5.6) below. Here $\mathrm{Ar}_{n-1}(d)$ gives a bound of numbers of singular points of projective hypersurfaces of degree d in $\mathbb{P}^{n-1}$ (see[Va2]), and is defined in the notation of (4.11.2) by

$$\mathrm{Ar}_{n-1}(d) := \gamma_{n,\,[(n-1)d/2]+1}\,. \tag{4.11.3}$$

If $n=3$, we have $\gamma_{n,\,[(n-1)d/2]+1} = \gamma_{3,d+1} = d(d-1)/2$, and this upper bound is attained in the case of generic hyperplane arrangements (where $k_{\max} = 2d-2$, see [Wa1], [Sa5]).

If $n=4$, $\mathrm{Ar}_3(d)$ for $d=3,4,5,6$ are given by $4, 16, 31, 68$ (in the first two cases, the upper bound is attained by Cayley and Kummer surfaces as is well-known).

(iii) For $k \in \mathbb{N}$, we have a short exact sequence

$$0 \to \mathcal{I}(k) \to \mathcal{O}_Y(k) \to \bigoplus_{z \in \mathrm{Sing}\, Z} \mathcal{O}_{Y,z}/(\partial f)^{\sim}_z \to 0, \tag{4.11.4}$$

where $\mathcal{I}(k) := \mathcal{O}_Y(k) \otimes_{\mathcal{O}_Y} (\partial f)^{\sim} \subset \mathcal{O}_Y(k)$, and the last morphism is defined by substituting $y=1$ where y is a coordinate of $\mathbb{C}^n$ such that $\mathrm{Sing}\, Z \cap \{y=0\} = \emptyset$. It induces a long exact sequence

$$0 \to H^0(Y, \mathcal{I}(k)) \to R_k \xrightarrow{\sigma_k} \bigoplus_{z \in \mathrm{Sing}\, Z} \mathcal{O}_{Y,z}/(\partial f)^{\sim}_z \to H^1(Y, \mathcal{I}(k)) \to 0. \tag{4.11.5}$$

Since $\dim \mathcal{O}_{Y,z}/(\partial f)^{\sim}_z = \tau_z$, and $H^0(Y, \mathcal{I}(k)) = I_k$ with $I_k/(\partial f)_k = M'_{k+n}$ as in (4.10.8), we get

$$\dim(\mathrm{Im}\, \sigma_k) = \mu''_{k+n}, \quad \dim H^1(Y, \mathcal{I}(k)) = \dim(\mathrm{Coker}\, \sigma_k) = \nu_{nd-k-n}. \tag{4.11.6}$$

(iv) If there is $z \in \mathrm{Sing}\, Z$ with $\tau_z = \mu_z > 1$, it is rather easy to see that we may have

$$\mu''_k < \tau_Z, \ \text{ that is, } \ \nu_{nd-k} > 0, \ \text{ even if } \ \gamma_k \geqslant \mu_Z. \tag{4.11.7}$$

For simplicity, assume $\mathrm{Sing}\, Z = \{z\}$ and there are global coordinates $x_1, \dots, x_n$ of $\mathbb{C}^n$ such that $z \notin \{x_n = 0\}$ and moreover the $y_i := x_i/x_n$ $(i \in [1, n'])$ give local coordinates such that $h := f/x_n^d$ is a semi-weighted-homogeneous polynomial of $y_1, \dots, y_{n'}$ with isolated singularity, where $n' := n-1$. (Here (Z, z) is assumed quasihomogeneous.) Then σ_k in (4.11.5) is not necessarily surjective even if we have the inequality

$$\gamma_{k+n} \bigl(= \dim R_k\bigr) \geqslant \mu_z \bigl(= \dim \mathcal{O}^{\mathrm{an}}_{Y,z}/(\partial h)_z\bigr).$$

Indeed, $(\partial h)_z \subset \mathcal{O}^{\mathrm{an}}_{Y,z}$ does not coincide with a power of the maximal ideal of $\mathcal{O}^{\mathrm{an}}_{Y,z}$ if $\mu_z > 1$. Note that the μ''_k, ν_k are *not* determined by the weights w_i, see Examples (5.1–2) below. (This depends on the relation between the coordinates x_i of $\mathbb{C}^n$ and local coordinates y_j of (Y, z) associated with a weighted homogeneous polynomial defining (Z, z).)

(v) Assume $|\mathrm{Sing}\, Z| = 1$ and $w_i = a_i^{-1}$ for $a_i \in \mathbb{N}$ with $|a| := \sum_{i=1}^{n-1} a_i < (n-1)d - 1$ (for instance, $f = x^a z^{d-a} + y^b z^{d-b} + x^d + y^d$ with $a + b < 2d - 1$, where $\alpha_Z = a^{-1} + b^{-1}$). Then

$$\delta''_k > 0 \quad \text{for } \ k \in [n, (n-1)d - n]. \tag{4.11.8}$$

Indeed, using (4.11.6) together with the $\mathfrak{m}$-adic filtration on $\mathcal{O}_{Y,z}$, we can show in this case

$$\mu''_k \geqslant p_k := q_{k-n} \ \text{ with } \ q_k := \#\{\nu \in \mathbb{Z}^{n-1} \mid 0 \leqslant \nu_i \leqslant a_i - 2,\, |\nu| \leqslant k\}. \tag{4.11.9}$$

Here we have $q_k + q_{|a|-2n+1-k} = \prod_i (a_i - 1) = \mu_Z = \tau_Z$, that is, $p_k + p_{|a|+1-k} = \mu_Z$. Since $\delta''_k = \mu''_k + \mu''_{(n-1)d-k} - \mu_Z$, the assertion is reduced to

$$p_k + p_{(n-1)d-k} - \mu_Z = p_k - p_{|a|+1-(n-1)d+k} > 0.$$

5. Explicit calculations

In this section we calculate some examples explicitly.

Example 5.1. Set $\mu'_k = \dim M'_k$, $\mu_k^{(2)} = \dim M_k^{(2)}$, etc., and

$$f_1 = x^5 + y^4 z \quad \text{with} \quad h_1 = x^5 + y^4,$$

where $n = 3$, $d = 5$, $\tau_Z = 12$, and $\chi(U) = 1$, see (2.9.2). In this case Σ consists of one point $p := [0:0:1] \in \mathbb{P}^2$, and (Z, p) is defined by h_1. We have

$$\begin{array}{r|cccccccccccc}
k: & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \cdots \\
\gamma_k: & 1 & 3 & 6 & 10 & 12 & 12 & 10 & 6 & 3 & 1 & & \\
\mu_k: & 1 & 3 & 6 & 10 & 12 & 13 & 13 & 12 & 12 & 12 & 12 & \cdots \\
\nu_k: & & & & & & 1 & 3 & 6 & 9 & 11 & 12 & \cdots \\
\mu_k^{(2)}: & & & & 1 & 1 & 1 & 1 & & & & & \\
\mu''_k: & 1 & 3 & 6 & 9 & 11 & 12 & 12 & 12 & 12 & 12 & 12 & \cdots \\
\mu'_k: & & & & 1 & 1 & 1 & 1 & & & & &
\end{array} \tag{5.1.1}$$

and

$$\begin{aligned} & b_{f_1}(s) = b_{h_1}(s) \textstyle\prod_{i=6}^{9} \big(s + \frac{i}{5}\big), \\ \text{with} \quad & b_{h_1}(s) = (s+1) \textstyle\prod_{i=1}^{4} \prod_{j=1}^{3} \big(s + \frac{i}{5} + \frac{j}{4}\big). \end{aligned} \tag{5.1.2}$$

These are done by using respectively Macaulay2 and RISA/ASIR as in [Sa6]. Note that $\mu_k^{(2)} = \mu_k - \nu_{k+d}$ by (9) in the introduction, and $b_h(s)$ can be calculated also by applying (1.9.2). These two calculations both imply

$$\mathcal{R}^0_{f_1} = \big\{\tfrac{6}{5}, \tfrac{7}{5}, \tfrac{8}{5}, \tfrac{9}{5}\big\}. \tag{5.1.3}$$

This is a good example for Theorem 3, Corollary 2 and (17).

Example 5.2. Let

$$f_2 = x^5 + x^2y^3 + y^4 z \quad \text{with} \quad h_2 = x^5 + x^2y^3 + y^4,$$

where $n = 3$, $d = 5$, $\tau_Z = 12$, and $\chi(U) = 1$, see (2.9.2). Again Σ consists of one point $p := [0:0:1] \in \mathbb{P}^2$, and (Z, p) is defined by h_2. In this case, it is rather surprising that h_2 is *quasihomogeneous* with $\tau_{h_2} = \mu_{h_2} = 12$, see a remark after Theorem 2. (This cannot be generalized to polynomials of higher degrees as far as tried.) We have

$$\begin{array}{r|cccccccccccc}
k: & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \cdots \\
\gamma_k: & 1 & 3 & 6 & 10 & 12 & 12 & 10 & 6 & 3 & 1 & & \\
\mu_k: & 1 & 3 & 6 & 10 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & \cdots \\
\nu_k: & & & & & & & 2 & 6 & 9 & 11 & 12 & \cdots \\
\mu_k^{(2)}: & 1 & 1 & & 1 & 1 & & & & & & & \\
\mu'_k: & & & & & & & & & & & &
\end{array} \tag{5.2.1}$$

$$\begin{aligned} & b_{f_2}(s) = b_{h_2}(s) \textstyle\prod_{i=3}^{4} \big(s + \frac{i}{5}\big) \cdot \prod_{i=6}^{7} \big(s + \frac{i}{5}\big), \\ \text{with} \quad & b_{h_2}(s) = (s+1) \textstyle\prod_{i=1}^{4} \prod_{j=1}^{3} \big(s + \frac{i}{5} + \frac{j}{4}\big). \end{aligned} \tag{5.2.2}$$

$$\mathcal{R}^0_{f_2} = \big\{\tfrac{3}{5}, \tfrac{4}{5}, \tfrac{6}{5}, \tfrac{7}{5}\big\}. \tag{5.2.3}$$

These are compatible with Theorem 3, Corollary 2 and (17), since $M' = 0$ in this case. This together with Example (5.1) above gives a good example for Remark (4.11)(iv).

Example 5.3. Let

$$f_3 = x^5 + x^3y^2 + y^4 z \quad \text{with} \quad h_3 = x^5 + x^3y^2 + y^4,$$

where $n=3$, $d=5$, $\tau_Z=11$, and $\chi(U)=1$, see (2.9.2). As before Σ consists of one point $p:=[0:0:1]\in\mathbb{P}^2$, and (Z,p) is defined by h_3. However, h_3 is *not* quasihomogeneous with $\mu_{h_3}=12$, $\tau_{h_3}=11$ in this case. We have by RISA/ASIR

$$\begin{aligned} b_{f_3}(s)&=b_{h_3}(s)\,\bigl(s+\tfrac{4}{5}\bigr)\textstyle\prod_{i=6}^{8}\bigl(s+\tfrac{i}{5}\bigr),\\ b_{h_3}(s)&=(s+1)\textstyle\prod_{i=1}^{4}\prod_{j=1}^{3}\bigl(s+\tfrac{i}{5}+\tfrac{j}{4}-\delta_{i,4}\delta_{j,3}\bigr), \end{aligned}\tag{5.3.1}$$

where $\delta_{i,k}=1$ if $i=k$, and 0 otherwise. So we get

$$\mathcal{R}^0_{f_3}=\bigl\{\tfrac{4}{5},\tfrac{6}{5},\tfrac{7}{5},\tfrac{8}{5}\bigr\}.\tag{5.3.3}$$

In this case the pole order spectral sequence degenerates at E_3, and $\mu_k^{(3)}\neq 0$ if and only if $k\in\{4,6,7,8\}$ by calculations using computer programs based on [DiSt2], [Sa7]. We then get $\mathcal{R}^0_{f_3}$ by applying Theorem 1 since $\mathcal{R}_Z\cap\frac{1}{d}\,\mathbb{Z}=\{1\}$ in this case.

Example 5.4. Let

$$f_4=x^4y^2z+z^7\quad\text{with}\quad h_4=x^4z+z^7,\ h_4'=y^2z+z^7,$$

where $n=3$, $d=7$, $\tau_Z=\mu_Z=22+8=30$, and $\chi(U)=1$, see (2.9.2–6) and (A.1) below. We have

$k:$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\gamma_k:$	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1	
$\mu_k:$	1	3	6	10	15	21	25	28	30	31	31	30	30	30	30	30	30
$\nu_k:$								1	3	6	10	15	20	24	27	29	30
$\mu_k^{(2)}:$		1	1	1	2	2	2	2	1	1	1						
$\nu_k^{(2)}:$									1	1	1	2	1	1	1		
$\delta_k:$						1	1	1	1	1	1						

where we have $\delta_k:=\mu_k-\nu_{k+d}=\mu'_k$ in this case, see (14) in the introduction. The calculation of $\mu_k^{(2)}$, $\nu_k^{(2)}$ is made by computer programs based on [DiSt2], [Sa7].

On the other hand we get by (A.1) below and RISA/ASIR

$$\begin{aligned} 14\,\mathcal{R}_{h_4'}\subset 14\,\mathcal{R}_{h_4}=14\,\mathcal{R}_Z&=\{5\}\sqcup\{7,\dots,21\}\sqcup\{23\},\\ 14\,\mathcal{R}_{f_4}&=\{5\}\sqcup\{7,\dots,24\}\sqcup\{26\},\\ \mathcal{R}^0_{f_4}&=\bigl\{\tfrac{11}{7},\tfrac{12}{7},\tfrac{13}{7}\bigr\}. \end{aligned}$$

Here $3/7\in\bigl(\mathcal{R}_Z+\mathbb{Z}_{<0}\bigr)\setminus\mathcal{R}_Z$ with $3/7>\alpha_Z=5/14$, and $\mu_3=\nu_{10}=1$ so that condition (11) in Theorem 4 is not satisfied. (There is a similar phenomenon in the case $f=x^6y^3-z^9$.)

Remark 5.5. Let

$$f_5=x^5yz+x^4y^2z+z^7\quad\text{with}\quad h_5=x^5z+x^4z+z^7,\ h_5'=yz+y^2z+z^7.$$

Here h_5 is quasi-homogeneous at the origin, but it has also a singularity of type A_1 at $(x,z)=(-1,0)$. On the other hand, h_5' has two singularities of type A_1, one of which is the same as the A_1-singularity of h_5. So we get $\mu_Z=22+1+1=24$. A calculation by Macaulay2 shows that conditions (11) and the inclusion (12) in Theorem 4 *do* hold in this case. Indeed, the inequality $\mu_k>\nu_{k+d}$ is valid for $k=3$, and $\mathrm{CS}(f)=\{3\}$. Here $\chi(U)=7$ is quite large, see also Remark (2.9).

Example 5.6. Let

$$f_6=(x^2+y^2+z^2+w^2)^3-(x^6+y^6+z^6+w^6),$$

with $n = 4$, $d = 6$, $\tau_Z = \mu_Z = |\mathrm{Sing}\, Z| = 52$. We have

k :	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	$\cdots$
γ_k :	1	4	10	20	35	52	68	80	85	80	68	52	35	20	10	4	1		
μ_k :	1	4	10	20	35	52	68	80	85	80	68	56	53	52	52	52	52	52	$\cdots$
ν_k :												4	18	32	42	48	51	52	$\cdots$
μ''_k :	1	4	10	20	34	48	52	52	52	52	52	52	52	52	52	52	52	52	$\cdots$
μ'_k :					1	4	16	28	33	28	16	4	1						
δ_k :	1	4	10	20	35	48	50	48	43	32	17	4	1						
δ''_k :	1	4	10	20	34	44	34	20	10	4	1								

Here $I_4 = M'_8 \ne 0$ in $R_4 = \Omega_8^4$, although the latter has dimension 35 which is quite smaller than $|\mathrm{Sing}\, Z| = 52$.

Example 5.7. Let

$$f_7 = u^3 + v^3 + x^3 + y^3 + z^3 - (u + v + x + y + z)^3,$$

with $n = 5$, $d = 3$, $\tau_Z = \mu_Z = |\mathrm{Sing}\, Z| = 10$. This is a kind of a generalization of Cayley surface to the case $n = 5$. Here M' vanishes. Indeed, we have

k :	5	6	7	8	9	10	11	$\cdots$
γ_k :	1	5	10	10	5	1		
μ_k :	1	5	10	10	10	10	10	$\cdots$
ν_k :					5	9	10	$\cdots$
μ''_k :	1	5	10	10	10	10	10	$\cdots$
μ'_k :								
δ_k :	1		1					

This is compatible with a computation by RISA/ASIR showing that

$$b_{f_7}(s) = (s+1)(s+2)^2\bigl(s + \tfrac{5}{3}\bigr)\bigl(s + \tfrac{7}{3}\bigr) \quad \text{with} \quad \mathcal{R}_Z = \{1, 2\}.$$

Remark 5.8. It is known that for $f = x^5 + y^5 + yz(x^3 + z^2 w)$, $\mathrm{Supp}\{\mu'_k\}$ is not discretely connected, according to Aldo Conca, see [Sti]. Indeed, $\sum_k \mu'_k T^k = T^9 + T^{11}$ in this case (using (A.3) below). We also have $\sum_k \mu'_k T^k = T^{12} + T^{16}$ if $f = x^7 + y^7 + yz^2(x^4 + z^3 w)$. No such example is known if condition (W) is satisfied.

Appendix: Explicit computations using computers

In this Appendix we explain how to use the computer programs Macaulay2 and Singular for explicit calculations of roots of Bernstein-Sato polynomials.

A.1. Calculation of $\mathcal{R}_Z$. Using (1.9.2–3) together with the computer program Macaulay2, one can easily calculate the Steenbrink spectrum and the Bernstein-Sato polynomial of a weighted homogeneous polynomial h with weights w_i and having an isolated singularity.

To determine the weights of weighted homogeneous polynomials with isolated singularities for $n = 2$, it is enough to consider the following cases up to a permutation of variables:

$$x^i + y^j, \quad x(x^i + y^j), \quad xy(x^i + y^j), \tag{A.1.1}$$

where the weights (w_1, w_2) are given respectively by

$$\bigl(\tfrac{j}{ij}, \tfrac{i}{ij}\bigr), \quad \bigl(\tfrac{j}{(i+1)j}, \tfrac{i}{(i+1)j}\bigr), \quad \bigl(\tfrac{j}{ij+i+j}, \tfrac{i}{ij+i+j}\bigr). \tag{A.1.2}$$

Combined with (1.9.5), this implies that the Milnor numbers are respectively

$$(i-1)(j-1), \quad (i+1)(j-1)+1, \quad (i+1)(j+1). \tag{A.1.3}$$

Set $e = \mathrm{GCD}(i,j)$, $a = j/e$, $b = i/e$, and let m be either

$$ij/e \quad \text{or} \quad (i+1)j/e \quad \text{or} \quad (ij+i+j)/e,$$

depending on the above three cases. One can calculate the right-hand side of (1.9.3) with $T = t^{1/m}$ in the case $h = xy(x^3+y^2)$, for instance, by typing in Macaulay2

```
A = QQ[T]; i=3; j=2; e=gcd(i,j); a=j/e; b=i/e; m=(i*j+i+j)/e
(T^a - T^m)/(1 - T^a)*(T^b - T^m)/(1 - T^b)
```

(If one copies the input from a pdf file of this paper, then ^ should be replaced by a character from a keyboard.) In this case, the output should be

$$T^{17} + T^{15} + T^{14} + T^{13} + T^{12} + 2T^{11} + T^{10} + T^9 + T^8 + T^7 + T^5$$

(In the other two cases, the definition of m should be replaced by i*j/e or (i+1)*j/e.)

These calculations can be used to determine $\mathcal{R}_Z$. (For this it would be also possible to use RISA/ASIR if the singularities are not very complicated.)

A.2. Determination of the singularities of Z. It is sometimes difficult to see whether a semi-weighted-homogeneous polynomial h_z is quasi-homogeneous or not, see (1.8). In our case this can be verified by seeing whether the total Tjurina number τ_Z coincides with the total Milnor number μ_Z. The latter can be obtained by using the method in (A.1), and the former is given by high coefficients of the polynomial mu in (A.3) or (A.4) below. This can be seen explicitly if the separator " ; " at the end of the definition of mu is removed (where Return must be pressed).

However, it may be possible that τ_Z and μ_Z obtained by the above procedure coincide even though some singularity is *not* quasi-homogeneous. This may occur if there is a *hidden* singular point of Z. In the case of Remark (5.5), for instance, this may be verified by typing in Macaulay2 as follows:

```
R = QQ[x,y,z]; f = x^5*y*z+x^4*y^2*z+z^7;
S = R/(f); I = (radical ideal singularLocus S); decompose I
```

The output in this case should be

{ideal (z, y), ideal (z, x + y), ideal (z, x)}.

This shows that there is a hidden singular point at $[1:-1:0]$. (This may be useful to see whether the assumption $\mathrm{Sing}\, Z \subset \Sigma'$ in (1.12) is satisfied for explicit examples.)

There is another method to see whether all the singular points of Z are quasi-homogeneous by using a computer program like Singular as in [DiSt2]. Setting $h := f|_{z=1}$ after a *general coordinate change* of $\mathbb{C}^3$ such that $\mathrm{Sing}\, Z \cap \{z=0\} = \emptyset$, we can compare

$$\tau_h := \dim \mathbb{C}[x,y]/(h, h_x, h_y) \quad \text{and} \quad \mu_h := \dim \mathbb{C}[x,y]/(h^2, h_x, h_y),$$

using Singular (and also [BrSk]), see [DiSt2]. For instance, setting

$$h = x^5(x+1)z + x^4(x+1)^2z + z^7,$$

(which is the restriction of f to $y = x+1$, and defines $Z|_{x\ne y} \subset \mathbb{C}^2$), we can calculate the total Tjurina number of h for the singular points of $Z|_{x\ne y}$ by typing in the computer program Singular as follows:

```
ring R = 0, (x,z), dp; poly y=x+1;
poly f=x^5*y*z+x^4*y^2*z+z^7;
ideal J=(jacob(f),f);
vdim(groebner(J));
```

We also get the total Milnor number μ_h of h by replacing (jacob(f),f) with (jacob(f),f^2) (where f^2 should be f^(n-1) in general, see [BrSk]). The problem is then whether we have

$$\operatorname{Sing} Z \cap \{x = y\} = \emptyset.$$

This can be verified by comparing τ_h and τ_Z, where τ_Z can be obtained by μ_k for $k \gg 0$, see also the calculation of μ'_k as in (A.3) below (with f replaced by f in this subsection). Note that τ_Z is also obtained as tau in the Macaulay2 calculation in (A.3) below, which can be seen by removing ";" after the definition of tau.

A.3. Calculation of the μ'_k. We can get $\sum_k \mu'_k T^k$ in the case of Example (5.1), for instance, by typing in Macaulay2 as follows:

```
R=QQ[x,y,z]; A=frac(QQ[T]); n=3; f=x^5+y^4*z;
d=first degree f; d1=n*d-n+1; d2=n*d-n;
mus1=sub(hilbertSeries(R/(diff(x,f),diff(y,f),diff(z,f)),Order=>d1),A);
mus2=sub(hilbertSeries(R/(diff(x,f),diff(y,f),diff(z,f)),Order=>d2),A);
nu=mus1*T^n-((T^d-T)/(T-1))^n; tau=(mus1-mus2)/T^(n*d-n);
mub=tau*(T^(n*d+1)-1)/(T-1)-sub(nu,{T=>1/T})*T^(n*d);
mup=mus1*T^n-mub
```

Here mus1, mus2, mub, mup respectively mean $\mu_\bullet$ shifted(1), $\mu_\bullet$ shifted(2), $\mu''_\bullet$, $\mu'_\bullet$. In this case, the output should be $T^9+T^8+T^7+T^6$. One can see the intermediate results by removing ";" (and pressing Return). This method can be applied also to the case of Example (5.4).

A.4. Calculation of the $\delta_k = \mu_k - \nu_{k+d}$. It is also possible to get $\sum_k \delta_k T^k$, for instance, in the case of Example (5.2) by typing

```
R=QQ[x,y,z]; A=frac(QQ[T]); n=3; f=x^5+x^2*y^3+y^4*z;
d=first degree f; d2=n*d-n; d3=n*d-n+d;
mus2=sub(hilbertSeries(R/(diff(x,f),diff(y,f),diff(z,f)),Order=>d2),A);
mus3=sub(hilbertSeries(R/(diff(x,f),diff(y,f),diff(z,f)),Order=>d3),A);
nu2=mus3*T^n-((T^d-T)/(T-1))^n; delta=mus2*T^n-nu2/T^d
```

In the four variable case, Q[x,y,z] and n = 3 should be replaced respectively by Q[x,y,z,w] and n = 4, and diff(w,f) must be added in the definitions of mus2, mus3. (This is similarly for (A.3).) In this case the output should be $T^7 + T^6 + T^4 + T^3$. This is compatible with (5.2.3) by Theorem 4, where $\mathrm{CS}(f) = \emptyset$ and $\mathcal{R}_Z$ is calculated by (A.1).

We can apply the above calculation to Walther's example [Wa2] (see also [Sa5]) where f is given by

```
x*y*z*(x+3*z)*(x+y+z)*(x+2*y+3*z)*(2*x+y+z)*(2*x+3*y+z)*(2*x+3*y+4*z);
x*y*z*(x+5*z)*(x+y+z)*(x+3*y+5*z)*(2*x+y+z)*(2*x+3*y+z)*(2*x+3*y+4*z);
```

In this case, $\sum_k \delta_k T^k$ is a polynomial of degree 16 and 15 respectively. Since $\frac{16}{9} \notin \frac{1}{3}\mathbb{Z}$ and $\mathcal{R}_Z = \{\frac{2}{3}, 1, \frac{4}{3}\}$, this implies that $b_f(s)$ is not a combinatorial invariant of a hyperplane arrangement, see also [Sa5]. Note that the above first polynomial gives the same $\{\delta_k\}$ as Ziegler's (see [Zi]):

```
x*y*z*(x+y-z)*(x-y+z)*(2*x-2*y+z)*(2*x-y-2*z)*(2*x+y+z)*(2*x-y-z);
```

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