EXTENSIONS OF AMENABLE GROUPS BY RECURRENT GROUPOIDS

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ABSTRACT. We show that the amenability of a group acting by homeomorphisms can be deduced from a certain local property of the action and recurrency of the orbital Schreier graphs. This applies to a wide class of groups, the amenability of which was an open problem, as well as unifies many known examples to one general proof. In particular, this includes Grigorchuk's group, Basilica group, the full topological group of Cantor minimal system, groups acting on rooted trees by bounded automorphisms, groups generated by finite automata of linear activity growth, groups that naturally appear in holomorphic dynamics.

1. INTRODUCTION

M. Day introduced the class EG of *elementary amenable groups* in [Day57] as the class of all groups that can be constructed from finite and abelian using operations of passing to a subgroup, quotient, group extensions, and direct limits (the fact that the class of amenable groups is closed under these operations was already proved by J. von Neumann). He notes that at that time no examples of amenable groups that do not belong to the class EG were known.

The first example of an amenable group not belonging to EG was the Grigorchuk group of intermediate growth [Gri83]. An example of a group which can not be constructed from groups of sub-exponential growth (which, in some sense, can be also considered as an "easy case" of amenability) is the basilica group introduced in [GŻ02]. Its amenability was proved in [BV05] using asymptotic properties of random walks on groups. These methods were then generalized in [BKN10] and [AAV13] for a big class of groups acting on rooted trees. The first examples of finitely generated infinite simple amenable groups (which also can not belong to EG) were constructed in [JM12].

A common feature of all known examples of non-elementary amenable groups is that they are defined as groups of homeomorphisms of the Cantor set (or constructed from such groups).

The aim of this paper is to show a general method of proving amenability for a wide class of groups acting on topological spaces. This class contains many new examples of groups, whose amenability was an open question. It also contains all of the mentioned above non-elementary amenable groups as simple examples.

We show that amenability of a group of homeomorphisms can be deduced from a combination of local topological information about the homeomorphisms and global information about the orbital Schreier graphs. Namely, we prove the following

amenability condition (see Theorem 3.1). If G is a group acting by homeomorphisms on a topological space $\mathcal{X}$, then by $[[G]]$ we denote the full topological group of the action, i.e., the group of all homeomorphisms h of $\mathcal{X}$ such that for every $x \in \mathcal{X}$ there exists a neighborhood of x such that restriction of h to that neighborhood is equal to restriction of an element of G. For $x \in \mathcal{X}$ the *group of germs* of G at x is the quotient of the stabilizer of x by the subgroup of elements acting trivially on a neighborhood of x.

Theorem 1.1. *Let G and H be groups of homeomorphisms of a compact topological space $\mathcal{X}$, and G is finitely generated. Suppose that the following conditions hold.*

(1) *The full group $[[H]]$ is amenable.*
(2) *For every element $g \in G$, the set of points $x \in X$ such that g does not coincide with an element of H on any neighborhood of x is finite.*
(3) *For every point $x \in \mathcal{X}$ the Schreier graph of the action of G on the orbit of x is recurrent.*
(4) *For every $x \in \mathcal{X}$ the group of germs of G at x is amenable.*

Then the group G is amenable. Moreover, the group $[[G]]$ is amenable.

A key tool of the proof of the theorem is the following fact (see Theorem 2.8).

Theorem 1.2. *Let G be a finitely generated group acting on a set X. If the graph of the action of G on X is recurrent, then there exists an $(\mathbb{Z}/2\mathbb{Z}) \wr_X G$-invariant mean on $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$.*

The last two sections of the paper are devoted to showing different examples of applications of Theorem 1.1. At first we consider the case when the group $[[H]]$ is locally finite. Examples of such groups $[[H]]$ are given by *block-diagonal direct limits* of symmetric groups defined by a *Bratteli diagram*. The corresponding groups G satisfying the conditions of Theorem 1.1 are generated by *homeomorphisms of bounded type*.

The class of groups generated by homeomorphisms of bounded type includes some known examples of non-elementary amenable groups (groups of bounded automata [BKN10] and topological full group of minimal homeomorphisms of the Cantor set [JM12]), as well as groups whose amenability was an open question (for instance groups of arbitrary bounded automorphisms of rooted trees).

The last section describes some examples of application of Theorem 1.1 in the case when H is not locally finite. For example, one can use Theorem 1.1 twice: prove amenability of $[[H]]$ using it, and then construct new amenable groups G using H. This way one gets a simple proof of the main result of [AAV13]: that groups generated by finite automata of linear activity growth are amenable. Two other examples from Section 5 are new, and are groups naturally appearing in holomorphic dynamics. One is a holonomy group of the stable foliation of the Julia set of a Hénnon map, the other is the iterated monodromy group of a mating of two quadratic polynomials.

2. Amenable and recurrent G-sets

2.1. **Amenable actions.**

Definition 2.1. Let G be a discrete group. An action of G on a set X is said to be *amenable* if there exists an *invariant mean* on X. Here an invariant mean is a map μ from the set of all subsets of X to $[0, 1]$ such that μ is finitely additive, $\mu(X) = 1$, and $\mu(g(A)) = \mu(A)$ for all $A \subset X$ and $g \in G$.

A group G is amenable if and only if its action on itself by left multiplication is amenable. Note that the above definition of amenability is different from another definition of amenability of an action, due to Zimmer, see [Zim84].

The following criteria of amenability are proved in the same way as the corresponding classical criteria of amenability of groups.

Theorem 2.2. *Let G be a discrete group acting on a set X. Then the following conditions are equivalent.*

(1) *The action of G on X is amenable.*
(2) **Reiter's condition.** *For every finite subset $S \subset G$ and for every $\epsilon > 0$ there exists a non-negative function $\phi \in \ell^1(X)$ such that $\|\phi\|_1 = 1$ and $\|g \cdot \phi - \phi\|_1 < \epsilon$ for all $g \in S$.*
(3) **Følner's condition.** *For every finite subset $S \subset G$ and for every $\epsilon > 0$ there exists a finite subset $F \subset X$ such that $\sum_{g \in S} |gF \Delta F| < \epsilon |F|$.*

Note that if G is finitely generated, it is enough to check conditions (2) and (3) for one generating set S. If the conditions of (3) hold, then we say that F is (S, ϵ)-Følner set, and that F is ϵ-invariant with respect to the elements of S.

The following proposition is well known and follows from the fact that a group is amenable if and only if it admits an action which is amenable and Zimmer amenable, see [rose].

Proposition 2.3. *Let G be a group acting on a set X. If the action is amenable and for every $x \in X$ the stabilizer G_x of x in G is an amenable group, then the group G is amenable.*

2.2. **Recurrent actions.** Let G be a finitely generated group acting transitively on a set X. Choose a measure μ on G such that support of μ is a finite generating set of G and $\mu(g) = \mu(g^{-1})$ for all $g \in G$. Consider then the Markov chain on X with transition probability from x to y equal to $p(x, y) = \sum_{g \in G, g(x) = y} \mu(g)$.

The Markov chain is called *recurrent* if the probability of ever returning to x_0 after starting at x_0 is equal to 1 for some (and hence for every) $x_0 \in X$.

It is well known (see [Woe00, Theorems 3.1, 3.2]) that recurrence of the described Markov chain does not depend on the choice of the measure μ, if the measure is symmetric, and has finite support generating the group. We say that the *action* of G on X is *recurrent* if the corresponding Markov chain is recurrent. Every recurrent action is amenable, see [Woe00, Theorem 10.6].

For a finite symmetric generating set S the *Schreier graph* $\Gamma(X, G, S)$ is the graph with the set of vertices identified with X, the set of edges is $S \times X$, where an edge (s, x) connects x to $s(x)$.

A transitive action of G on X is recurrent if and only if the simple random walk on the Schreier graph $\Gamma(X, G, S)$ is recurrent.

The following fact is also a corollary of [Woe00, Theorem 3.2].

Lemma 2.4. *If the action of a finitely generated group G on a set X is recurrent, and $H < G$ is a subgroup, then the action of H on every its orbit on X is also recurrent.*

The following theorem is a straightforward corollary of the Nash-Williams criterion (see [NW59] or [Woe00, Corollary 2.20]).

Theorem 2.5. *Let Γ be a connected graph of uniformly bounded degree with set of vertices V. Suppose that there exists an increasing sequence of finite subsets $F_n \subset V$ such that $\bigcup_{n \geq 1} F_n = V$, ∂F_n are disjoint subsets and*

$$\sum_{n \geq 1} \frac{1}{|\partial F_n|} = \infty,$$

where ∂F_n is the set of vertices of F_n adjacent to the vertices of $V \setminus F_n$. Then the simple random walk on Γ is recurrent.

We will also use a characterization of transience of a random walk on a locally finite connected graph (V, E) in terms of electrical network. The *capacity* of a point $x_0 \in V$ is the quantity defined by

$$cap(x_0) = \inf \left\{ \left(\sum_{(x,x') \in E} |a(x) - a(x')|^2 \right)^{1/2} \right\}$$

where the infimum is taken over all finitely supported functions $a : V \to \mathbb{C}$ with $a(x_0) = 1$. We will use the following

Theorem 2.6 ([Woe00], Theorem 2.12). *The random walk on a locally finite connected graph (V, E) is transient if and only if $cap(x_0) > 0$ for some (and hence for all) $x_0 \in V$.*

2.3. **A mean on $\bigoplus_X \mathbb{Z}_2$ invariant with respect to $\mathbb{Z}_2 \wr_X G$.** Let G be a discrete group acting transitively on a set X. Let $\{0, 1\}^X$ be the set of all subsets of X considered as abelian group with multiplication given by the symmetric difference of sets. It is naturally isomorphic to the Cartesian product $\mathbb{Z}_2^X$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. The group G acts naturally on the group $\mathbb{Z}_2^X$ by automorphisms.

Denote by $\mathcal{P}_f(X)$ the subgroup of $\mathbb{Z}_2^X = \{0, 1\}^X$ which consists of all finite subsets of X, i.e., the subgroup $\bigoplus_X \mathbb{Z}_2$ of $\mathbb{Z}_2^X$. It is obviously invariant under the action of G.

Consider the restricted wreath product $\mathbb{Z}_2 \wr_X G \cong G \ltimes \mathcal{P}_f(X)$. Its action on $\mathcal{P}_f(X)$ is given by the formula

$$(g, E)(F) = g(E \Delta F)$$

for $E, F \in \mathcal{P}_f(X)$ and $g \in G$.

The Pontryagin dual of $\mathcal{P}_f(X)$ is the compact group $\mathbb{Z}_2^X$, with the duality given by the pairing $\phi(E, \omega) = exp(i\pi \sum_{j \in E} \omega_j)$, $E \in \mathcal{P}_f(X)$, $\omega \in \mathbb{Z}_2^X$. Fix a point $p \in X$ and denote by $L_2(\mathbb{Z}_2^X, \mu)$ the Hilbert space of functions on $\mathbb{Z}_2^X$ with the Haar probability measure μ. Denote by $A_p = \{(\omega_x)_{x \in X} \in \mathbb{Z}_2^X : \omega_p = 0\}$ be the cylinder set which fixes ω_p as zero.

The following lemma was proved in [JM12, Lemma 3.1].

Lemma 2.7. *Let G acts transitively on a set X and choose a point $p \in X$. The following are equivalent:*

(i) *There exist a net of unit vectors $\{f_n\} \in L_2(\{0,1\}^X, \mu)$ such that for every $g \in G$*

$$\|g \cdot f_n - f_n\|_2 \to 0 \ and \ \|f_n \cdot \chi_{A_p}\|_2 \to 1.$$

(ii) *The action of $\mathbb{Z}_2 \wr_X G$ on $\mathcal{P}_f(X)$ is amenable.*
(iii) *The action of G on $\mathcal{P}_f(X)$ admits an invariant mean giving full weight to the collection of sets containing p.*

We say that a function $f \in L_2(\mathbb{Z}_2^X, \mu)$ is *p.i.r.* if it is a product of random independent variables, i.e., there are functions $f_x \in \mathbb{R}^{\mathbb{Z}_2}$ such that $f(\omega) = \prod_{x \in X} f_x(\omega_x)$.
In other words, if we consider $L_2(\mathbb{Z}_2^X, \mu)$ as the infinite tensor power of the Hilbert space $L_2(\mathbb{Z}_2, m)$ with unit vector 1, where $m(\{0\}) = m(\{1\}) = 1/2$, the condition p.i.r. means that f is an elementary tensor in $L_2(\mathbb{Z}_2^X, \mu)$.

Theorem 2.8. *Let G be a finitely generated group acting transitively on a set X. There exists a sequence of p.i.r. functions $\{f_n\}$ in $L_2(\mathbb{Z}_2^X, \mu)$ that satisfy condition (i) in Lemma 2.7 if and only if the action of G on X is recurrent.*

Proof. Denote by (X, E) the Schreier graph of the action of G on X with respect to S. Suppose that the random walk on (X, E) is recurrent. By Theorem 2.6 there exists $a_n = (a_{x,n})_x$ a sequence of finitely supported functions such that $a_{x_0,n} = 1$ and $\sum_{x \sim x'} |a_{x,n} - a_{x',n}|^2 \to 0$. Replacing all values $a_{x,n}$ that are greater than 1 (smaller than 0) by 1 (respectively 0) does not increase the differences $|a_{x,n} - a_{x',n}|$, hence we may assume that $0 \leq a_{x,n} \leq 1$. For $0 \leq t \leq 1$ consider the unit vector $\xi_t \in L_2(\{0,1\}, m)$

$$(\xi_t(0), \xi_t(1)) = (\sqrt{2}\cos(t\pi/4), \sqrt{2}\sin(t\pi/4)).$$

Define $f_{x,n} = \xi_{1-a_{x,n}}$ and $f_n = \bigotimes_{x \in X} f_{x,n}$. We have to show that $\langle gf_n, f_n \rangle \to 1$ for all $g \in \Gamma$. It is sufficient to show this for $g \in S$. Then

$$\langle gf_n, f_n \rangle = \prod_x \langle f_{x,n}, f_{gx,n} \rangle = \prod_x \cos\frac{\pi}{4}(a_{x,n} - a_{gx,n}) \geq$$

$$\prod_x e^{-\frac{\pi^2}{16}(a_{x,n} - a_{gx,n})^2} \geq e^{-\frac{\pi^2}{16} \sum_{x \sim x'} |a_{x,n} - a_{x',n}|^2}$$

and the last value converges to 1. We used that $\cos(x) \geq e^{-x^2}$ if $|x| \leq \pi/4$.

Let us prove the other direction of the theorem. Define the following pseudo-metric on the unit sphere of $L_2(\{0,1\}, m)$ by

$$d(\xi, \eta) = \inf_{\omega \in \mathbb{C}, |\omega|=1} \|\omega\xi - \eta\| = \sqrt{2 - 2|\langle\xi, \eta\rangle|}.$$

Assume that there is a sequence of p.i.r. functions $\{f_n\}$ in $L_2(\mathbb{Z}_2^X, \mu)$ that satisfy condition (i) of Lemma 2.7. Write $f_n(\omega) = \prod_{x \in X} f_{n,x}(\omega_x)$. We can assume that the product is finite. Replacing $f_{n,x}$ by $f_{n,x}/\|f_{n,x}\|$ we can assume that $\|f_{n,x}\|_{L_2(\mathbb{Z}_2, m)} = 1$. Define $a_{x,n} = d(f_{x,n}, 1)$. It is straightforward that $(a_{x,n})_{x \in X}$ has finite support and

$$\lim_n a_{x_0,n} = d(\sqrt{2}\delta_0, 1) > 0.$$

Moreover for every $g \in G$

$$|\langle gf_n, f_n \rangle| = \prod_x |\langle f_{n,x}, f_{n,gx} \rangle| = \prod_x (1 - d(f_{n,x}, f_{n,gx})^2/2) \leq e^{-\sum_x d(f_{n,x}, f_{n,gx})^2/2},$$

which by assumption goes to 1 for every $g \in S$. By definition of the Schreier graph and the triangle inequality for d,

$$\sum_{(x,x') \in E} |a_{x,n} - a_{x',n}|^2 = \sum_{g \in S} \sum_x |a_{x,n} - a_{gx,n}|^2 \leq \sum_{g \in S} \sum_x d(f_x, f_{gx})^2 \to 0.$$

This proves that $cap(x_0) = 0$ in (X, E), and hence by Theorem 2.6 that the random walk on (X, E) is recurrent.   □

3. Amenability of groups of homeomorphisms

3.1. Groupoids.
Let G be a group acting faithfully by homeomorphisms on a topological space $\mathcal{X}$. A *germ* of the action is an equivalence class of pairs $(g, x) \in G \times \mathcal{X}$, where two germs (g_1, x_1) and (g_2, x_2) are equal if and only if $x_1 = x_2$, and there exists a neighborhood U of x_1 such that $g_1|_U = g_2|_U$. The set of all germs of the action of G on $\mathcal{X}$ is a *groupoid*. Denote by $\mathsf{o}(g, x) = x$ and $\mathsf{t}(g, x) = g(x)$ the *origin* and *target* of the germ. A composition $(g_1, x_1)(g_2, x_2)$ is defined if $g_2(x_2) = x_1$, and then it is equal to $(g_1 g_2, x_2)$. The inverse of a germ (g, x) is the germ $(g, x)^{-1} = (g^{-1}, g(x))$.

The groupoid of germs has a natural topology defined by the basis of open sets of the form $\{(g, x) : x \in U\}$, where $g \in G$ and $U \subset \mathcal{X}$ is open. For a given groupoid $\mathcal{G}$ of germs of an action on $\mathcal{X}$, and for $x \in \mathcal{X}$, the *isotropy group*, or *group of germs* $\mathcal{G}_x$ is the group of all germs $\gamma \in \mathcal{G}$ such that $\mathsf{o}(\gamma) = \mathsf{t}(\gamma) = x$. If $\mathcal{G}$ is the groupoid of germs of the action of G on $\mathcal{X}$, then the isotropy group $\mathcal{G}_x$ is the quotient of the stabilizer G_x of x by the subgroup of elements of G that act trivially on a neighborhood of x.

The *topological full group* of a groupoid of germs $\mathcal{G}$, denoted $[[\mathcal{G}]]$ is the set of all homeomorphisms $F : \mathcal{X} \longrightarrow \mathcal{X}$ such that all germs of F belong to $\mathcal{G}$. The *(orbital) Schreier graph* $\Gamma(x, G)$ is the Schreier graph of the action of G on the G-orbit of x.

3.2. Amenability of groups.

Theorem 3.1. *Let G be a finitely generated group of homeomorphisms of a topological space $\mathcal{X}$, and $\mathcal{G}$ be its groupoid of germs. Let $\mathcal{H}$ be a groupoid of germs of homeomorphisms of $\mathcal{X}$. Suppose that the following conditions hold.*

(1) *The group $[[\mathcal{H}]]$ is amenable.*
(2) *For every generator g of G the set of points $x \in \mathcal{X}$ such that $(g, x) \notin \mathcal{H}$ is finite. We say that $x \in \mathcal{X}$ is* singular *if there exists $g \in G$ such that $(g, x) \notin \mathcal{H}$.*
(3) *For every singular point $x \in \mathcal{X}$ the orbital Schreier graph $\Gamma(x, G)$ is recurrent.*
(4) *The isotropy groups $\mathcal{G}_x$ are amenable.*

Then the group G is amenable.

Proof. After replacing $\mathcal{H}$ by $\mathcal{H} \cap \mathcal{G}$, we may assume that $\mathcal{H} \subset \mathcal{G}$. Let S be a finite symmetric generating set of G. Let Σ be the set of points $x \in \mathcal{X}$ such that there exists $g \in S$ such that $(g, x) \notin \mathcal{H}$. Let V be the union of the G-orbits of the elements of Σ. By assumption 2 every G-orbit is a union of a finite number of $\mathcal{H}$-orbits. Since the set Σ is finite, the set V is a union of a finite number of $\mathcal{H}$-orbits.

Lemma 3.2. *The set V contains all singular points of $\mathcal{X}$.*

Proof. Let $g = g_1 g_2 \cdots g_n$ be a representation of g as a product of generators $g_i \in S$. The germ (g, x) is then equal to the product of germs

$$(g_1, g_2 \cdots g_n(x)) \cdot (g_2, g_3 \cdots g_n(x)) \cdots (g_n, x)$$

If all these germs belong to $\mathcal{H}$, then (g, x) belongs to $\mathcal{H}$. Therefore, $(g, x) \notin \mathcal{H}$ only if $x \in \Sigma \cup g_n^{-1} \Sigma \cup (g_{n-1} g_n)^{-1} \Sigma \cup \cdots (g_2 \cdots g_n)^{-1} \Sigma$. $\qquad\square$

Let $A \subset V$ be an $\mathcal{H}$-orbit transversal. For every $v \in V$ there exists a unique element A that belongs to the same $\mathcal{H}$-orbit as v. Let us denote it by $\alpha(v)$. Choose a germ $\delta_v \in \mathcal{H}$ such that $\mathsf{o}(\delta_v) = \alpha(v)$ and $\mathsf{t}(\delta_v) = v$. For $g \in G$ and $v \in V$ denote by g_v the element of $\mathcal{G}$ defined by

$$(1) \qquad\qquad g_v = \delta_{g(v)}^{-1} (g, v) \delta_v,$$

and note that it satisfies the cocycle relation $(g g')_v = g_{g'(v)} g'_v$.

Let $\mathcal{G}|_A$ (resp. $\mathcal{H}|_A$) be the set of germs $\gamma \in \mathcal{G}$ (resp. $\gamma \in \mathcal{H}$) with the target and the origin in A. Note that $\mathcal{H}|_A$ is the disjoint union of the isotropy groups $\mathcal{H}_a$ for $a \in A$, and that $g_v \in \mathcal{G}|_A$ for all $g \in G$ and $v \in V$. Consider the quotient $Z = \mathcal{G}|_A / \mathcal{H}|_A$ of $\mathcal{G}|_A$ defined by the right action of $\mathcal{H}|_A$, i.e., two germs $\gamma_1, \gamma_2 \in \mathcal{G}|_A$ are equivalent if there exists $\gamma \in \mathcal{H}$ such that $\gamma_2 = \gamma_1 \gamma$. Note that then $\mathsf{t}(\gamma_1) = \mathsf{t}(\gamma_2)$ and $\mathsf{o}(\gamma_1) = \mathsf{o}(\gamma_2)$, hence the maps $\mathsf{t} : Z \longrightarrow A$ and $\mathsf{o} : Z \to A$ are well defined.

Let $\mathcal{P}$ be the set of functions $\phi : V \longrightarrow Z$ such that $t(\phi(v)) = \alpha(v)$ for all $v \in V$ and of *finite support*, i.e., such that the values of $\phi(v)$ are trivial (i.e., belong to $\mathcal{H}_{\alpha(v)}$) for all but a finite number of values $v \in V$.

For $\phi \in \mathcal{P}$, $g \in G$ and $v \in V$, define $g(\phi)(v) = g_{g^{-1}v} \cdot (\phi(g^{-1}v))$. By assumption 2 of the theorem, $g(\phi)$ belongs to $\mathcal{P}$ and this defines an action of G on $\mathcal{P}$ by the cocycle relation.

Proposition 3.3. *There exists a G-invariant mean on $\mathcal{P}$.*

Proof. If we decompose $V = \bigcup_i V_i$ as a finite union of G-orbits, we get a decomposition of $\mathcal{P}$ as a direct product of $\mathcal{P}_i$ where $\mathcal{P}_i$ are the restrictions of elements of $\mathcal{P}$ to V_i, and G acts diagonally. It is therefore enough to prove the proposition for the case when G acts transitively on V.

For every pair of elements $a, b \in A$, choose an element $f_{a,b} \in \mathcal{G}$ such that $\mathsf{o}(f_{a,b}) = a$ and $\mathsf{t}(f_{a,b}) = b$. We also assume that $f_{a,a}$ is the identity of $\mathcal{G}_a$. For every $\gamma \in \mathcal{G}|_A$ consider the element $\widetilde{\gamma} \in \coprod_{a \in A} \mathcal{G}_a$ (disjoint union) defined by $\widetilde{\gamma} = f_{\mathsf{t}(\gamma), \mathsf{o}(\gamma)} \gamma \in \mathcal{G}_{\mathsf{o}(\gamma)}$. We also denote by $\widetilde{\cdot}$ the induced map $\mathcal{G}|_A / \mathcal{H}|_A \to \coprod_{a \in A} \mathcal{G}_a / \mathcal{H}_a$.

For every $\phi \in \mathcal{P}$, consider the map $\psi \colon V \to \coprod_{a \in A} \mathcal{G}_a / \mathcal{H}_a$ defined by $\psi(v) = \widetilde{\phi(v)}$. The map $\phi \mapsto \psi$ allows to identify $\mathcal{P}$ with the set $\widetilde{\mathcal{P}}$ of functions ψ from V to $\coprod_{a \in A} \mathcal{G}_a / \mathcal{H}_a$ such that $\psi(v) = \mathcal{H}_{\alpha(v)}$ for all but finitely many v's.

One easily checks that the action of G on $\widetilde{\mathcal{P}}$ using this identification is given by

$$(2) \qquad\qquad (g \cdot \psi)(gv) = f_{\alpha(gv), \mathsf{o}(\psi(v))} g_v f_{\alpha(v), \mathsf{o}(\psi(v))}^{-1} \psi(v),$$

where g_v is given by (1). If the germ (g, v) belongs to $\mathcal{H}$, then $\alpha(v) = \alpha(gv)$ and $g_v \in \mathcal{H}_{\alpha(v)}$. If additionally $\psi(v)$ is trivial (i.e. is equal to $\mathcal{H}_{\alpha(v)}$), then so is $(g \cdot \psi)(gv)$ by our choice of $f_{a,a} = 1$.

By Lemma 2.7 and Theorem 2.8 there exists a G-invariant mean on $\mathcal{P}_f(V)$ giving full weight to the collection of sets containing a given point $p \in V$. Note that since the mean is G-invariant, finitely additive, and the action of G on V is transitive,

the mean gives full weight to the collection of sets containing any given finite subset of V. In particular, it gives full weight to the collection of sets containing Σ.

It follows then from Theorem 2.2 that for every $\epsilon > 0$ there exists a finite subset $\mathcal{F}$ of $\mathcal{P}_f(V)$ which is ϵ-invariant under the action of elements of S, such that every element of $\mathcal{F}$ contains Σ. Moreover, since G preserves cardinalities of elements of $\mathcal{P}_f(V)$, we can choose $\mathcal{F}$ consisting of sets of the same cardinality N.

Fix $x_0 \in A$. Let us assume at first that $\mathcal{G}_{x_0}/\mathcal{H}_{x_0}$ is infinite. Let R be the finite subset of $\mathcal{G}_{x_0}$ defined by

$$R = \{f_{\alpha(gv),x_0} g_v f_{\alpha(v),x_0}^{-1}, v \in \cup_{B \in \mathcal{F}} B, g \in S\}.$$

Since the group $\mathcal{G}_{x_0}$ is amenable, for every $\epsilon > 0$ there exists a subset F of $\mathcal{G}_{x_0}/\mathcal{H}_{x_0}$ such that $|\gamma F \Delta F| \leq \epsilon/N$ for all $\gamma \in R$. Since $\mathcal{G}_{x_0}/\mathcal{H}_{x_0}$ is infinite we may assume that F does not contain the trivial element $\mathcal{H}_{x_0}$.

Let $\widehat{\mathcal{F}}$ be the set of functions $V \longrightarrow \coprod_{a \in A} \mathcal{G}_a/\mathcal{H}_a$ such that there exists $B \in \mathcal{F}$ such that $\phi(v) = \mathcal{H}_{\alpha(v)}$ for $v \notin B$, and $\phi(v) \in F$ for $v \in B$. Then $\widehat{\mathcal{F}}$ is split into a disjoint union of sets $\widehat{\mathcal{F}}_B$ of functions with support equal to $B \in \mathcal{F}$. We use here the fact that F does not contain the trivial element $\mathcal{H}_{x_0}$ of $\mathcal{G}_{x_0}/\mathcal{H}_{x_0}$. For each $B \in \mathcal{F}$ we have

$$\left|\widehat{\mathcal{F}}_B\right| = |F|^{|B|} = |F|^N,$$

hence

$$\left|\widehat{\mathcal{F}}\right| = |\mathcal{F}| \cdot |F|^N.$$

The number of sets $B \in \mathcal{F}$ such that $g(B) \notin \mathcal{F}$ for some $g \in S$ is not larger than $\epsilon|\mathcal{F}|$. It follows that the number of functions $\phi \in \widehat{\mathcal{F}}$ with support equal to such sets is not larger than

$$\epsilon|\mathcal{F}| \cdot |F|^{N-1} = \epsilon \left|\widehat{\mathcal{F}}\right|.$$

Let $g \in S$. Suppose that $B, g(B) \in \mathcal{F}$, and let $\psi \in \widehat{\mathcal{F}}_B$. Take $v \notin B$. Then $\psi(v) = \mathcal{H}_{\alpha(v)}$, and by our assumption that $\Sigma \subset B$, $g_v \in \mathcal{H}_{\alpha(v)}$. By (2) the support of $g(\psi)$ is therefore a subset of $g(B)$. It follows that $g(\psi)$ does not belong to $\widehat{\mathcal{F}}$ if and only if there exists $v \in B$ such that $f_{\alpha(gv),x_0} g_v f_{\alpha(v),x_0}^{-1} \psi(v) \notin F$. It follows that the cardinality of the set of elements $\psi \in \widehat{\mathcal{F}}_B$ such that $g(\psi) \notin \widehat{\mathcal{F}}$ is at most

$$\sum_{v \in B} |f_{\alpha(gv),x_0} g_v f_{\alpha(v),x_0}^{-1} F \setminus F||F|^{N-1} < \epsilon|F|^N.$$

It follows that the cardinality of the set of functions $\phi \in \widehat{\mathcal{F}}$ such that $g(\phi) \notin \widehat{\mathcal{F}}$ for some $g \in S$ is not greater than

$$(\epsilon + \epsilon \cdot |S|) \cdot \left|\widehat{\mathcal{F}}\right|.$$

Since ϵ is an arbitrary positive number, it follows that the action of G on $\mathcal{P}$ is amenable.

The case when $\mathcal{G}_{x_0}/\mathcal{H}_{x_0}$ is finite can be reduced to the infinite case, for instance, by the following trick. Replace $\mathcal{G}_a$ by $\mathcal{G}_a \times \mathbb{Z}$, and define $\tilde{\mathcal{P}}$ as the set of maps $V \longrightarrow \coprod_{a \in A}(\mathcal{G}_a \times \mathbb{Z})/\mathcal{H}_a$ of finite support, where support is defined in the same way as before. Here $\mathcal{H}_a$ is considered to be the subgroup of $\mathcal{H}_a \times \{0\} < \mathcal{G}_a \times \mathbb{Z}$. Define the action of G on $\tilde{\mathcal{P}}$ by the same formulae (1) and (2) as the action of G on $\mathcal{P}$. Then, by the same arguments as above, there exists a G-invariant mean on

$\tilde{\mathcal{P}}$. The projection $\coprod_{a \in A} \mathcal{G}_a \times \mathbb{Z} \longrightarrow \coprod a \in A\mathcal{G}_a$ induces a surjective G-equivariant map $\tilde{\mathcal{P}} \longrightarrow \mathcal{P}$, which implies that $\mathcal{P}$ has a G-invariant mean. $\qquad\square$

In order to prove amenability of G, it remains to show that for every $\phi \in \mathcal{P}$ the stabilizer G_ϕ of ϕ in G is amenable, see Proposition 2.3. We will use a modification of an argument of Y. de Cornulier from [dC13, Proof of Theorem 4.1.1]. Suppose that it is not amenable. It follows from Lemma 2.4 that for every $v \in V$ the action of every subgroup of G on the orbit of v has an invariant mean. Consequently, by Proposition 2.3, the stabilizer $G_{\phi,v}$ of v in G_ϕ is non-amenable. It follows by induction, that the intersection K of the pointwise fixator of the support of ϕ with G_ϕ is non-amenable. Since K fixes all points $y \in \operatorname{supp} \phi$, we have a homomorphism from K to the direct product of the isotropy groups $\mathcal{G}_y$ for $y \in \operatorname{supp} \phi$. If g belongs to the kernel of this homomorphism, then all germs of g at points of V belong to $\mathcal{H}$, hence $g \in [[\mathcal{H}]]$, by Lemma 3.2. It follows that K is an extension of a subgroup of $[[\mathcal{H}]]$ by a subgroup of a finite direct product of isotropy groups $\mathcal{G}_y$. But this implies that K is amenable, which is a contradiction. $\qquad\square$

Remark. Note that since conditions on the elements of G in Theorem 3.1 are local, it follows from the theorem that $[[\mathcal{G}]]$ is amenable if $\mathcal{X}$ is compact.

4. Homeomorphisms of bounded type

We start with description of groupoids $\mathcal{H}$ such that the full group $[[\mathcal{H}]]$ is locally finite, which is in some sense the simplest class of amenable groups. We will use then such groupoids as base for application of Theorem 3.1 and construction of non-elementary amenable groups.

4.1. **Bratteli diagrams.** A *Bratteli diagram* $\mathsf{D} = ((V_i)_{i \geq 1}, (E_i)_{i \geq 1}, \mathsf{o}, \mathsf{t})$ is defined by two sequences of finite sets $(V_i)_{i=1,2,\ldots}$ and $(E_i)_{i=1,2,\ldots}$, and sequences of maps $\mathsf{o} : E_i \longrightarrow V_i$, $\mathsf{t} : E_i \longrightarrow V_{i+1}$. We interpret the sets V_i as sets of vertices of the diagram partitioned into *levels*. Then E_i is the set of edges connecting vertices of the neighboring levels V_i and V_{i+1}. See [Bra72] for their applications in theory of C^*-algebras. A *path of length n*, where n is a natural number or ∞, in the diagram D is a sequence of edges $e_i \in E_i$, $1 \leq i \leq n$, such that $\mathsf{t}(e_i) = \mathsf{o}(e_{i+1})$ for all i. Denote by $\Omega_n(\mathsf{D}) = \Omega_n$ the set of paths of length n. We will write Ω instead of Ω_∞.

The set Ω is a closed subset of the direct product $\prod_{i \geq 1} E_i$, and thus is a compact totally disconnected metrizable space. If $w = (a_1, a_2, \ldots, a_n) \in \Omega_n$ is a finite path of D, then we denote by $w\Omega$ the set of all paths beginning with w. Let $w_1 = (a_1, a_2, \ldots, a_n)$ and $w_2 = (b_1, b_2, \ldots, b_n)$ be elements of Ω_n such that $\mathsf{t}(a_n) = \mathsf{t}(b_n)$. Then for every infinite path $(a_1, a_2, \ldots, a_n, e_{n+1}, \ldots)$, the sequence $(b_1, b_2, \ldots, b_n, e_{n+1}, e_{n+2}, \ldots)$ is also a path. The map

$$(3) \qquad T_{w_1,w_2} : (a_1, a_2, \ldots, a_n, e_{n+1}, \ldots) \mapsto (b_1, b_2, \ldots, b_n, e_{n+1}, e_{n+2}, \ldots)$$

is a homeomorphism $w_1\Omega \longrightarrow w_2\Omega$.

Denote by $\mathcal{T}_{w_1,w_2}$ the set of germs of the homeomorphisms T_{w_1,w_2}. It is naturally identified with the set of all pairs $((e_i)_{i \geq 1}, (f_i)_{i \geq 1}) \in \Omega^2$ such that $e_i = a_i$ and $f_i = b_i$ for all $1 \leq i \leq n$, and $e_i = f_i$ for all $i > n$.

Let $\mathcal{T}(\mathsf{D})$ (or just $\mathcal{T}$) be the groupoid of germs of the semigroup generated by the transformations of the form T_{w_1,w_2}. It can be identified with the set of all pairs of cofinal paths, i.e., pairs of paths $(e_i)_{i \geq 1}$, $(f_i)_{i \geq 1}$ such that $e_i = f_i$ for all

i big enough. The groupoid structure coincides with the groupoid structure of an equivalence relation: the product $(w_1, w_2) \cdot (w_3, w_4)$ is defined if and only if $w_2 = w_3$, and then it is equal to (w_1, w_4). Here $\mathsf{o}(w_1, w_2) = w_2$ and $\mathsf{t}(w_1, w_2) = w_1$. It follows from the definition of topology on a groupoid of germs that topology on $\mathcal{T}$ is given by the basis of open sets of the form $\mathcal{T}_{w_1, w_2}$. We call $\mathcal{T}$ the *tail equivalence groupoid* of the Bratteli diagram D. More on the tail equivalence groupoids, their properties, and relation to C^*-algebras, see [ER06].

Let us describe the topological full group $[[\mathcal{T}]]$. By compactness of Ω, for every $g \in [[\mathcal{T}]]$ there exists n such that for every $w_1 \in \Omega_n$ there exist a path $w_2 \in \Omega_n$ such that $g(w) = T_{w_1, w_2}(w)$ for all $w \in w_1\Omega$. Then we say that *depth* of g is at most n. For every $v \in V_{n+1}$ denote by Ω_v the set of paths $w \in \Omega_n$ ending in v. Then the group of all elements $g \in [[\mathcal{T}]]$ of depth at most n is naturally identified with the direct product $\prod_{v \in V_{n+1}} \mathrm{Symm}(\Omega_v)$ of symmetric groups on the sets Ω_v. Namely, if $g = (\pi_v)_{v \in V_{n+1}} \in \prod_{v \in V_{n+1}} \mathrm{Symm}(\Omega_v)$ for $\pi_v \in \mathrm{Symm}(\Omega_v)$, then g acts on Ω by the rule $g(w_0 w) = \pi_v(w_0)w$, where $w_0 \in \Omega_v$ and $w_0 w \in \Omega$. Let us denote the group of elements of $[[\mathcal{T}]]$ of depth at most n by $[[\mathcal{T}]]_n$. We obviously have $[[\mathcal{T}]]_n \leq [[\mathcal{T}]]_{n+1}$ and $[[\mathcal{T}]] = \bigcup_{n \geq 1} [[\mathcal{T}]]_n$. In particular, $[[\mathcal{T}]]$ is locally finite (i.e., every its finitely generated subgroup is finite).

The embedding $[[\mathcal{T}]]_n \hookrightarrow [[\mathcal{T}]]_{n+1}$ is *block diagonal* with respect to the direct product decompositions $\prod_{v \in V_{n+1}} \mathrm{Symm}(\Omega_v)$ and $\prod_{v \in V_{n+2}} \mathrm{Symm}(\Omega_v)$. For more on such embeddings of direct products of symmetric and alternating groups and their inductive limits see [LP05, LN07].

4.2. **Homeomorphisms of bounded type.** Let D be a Bratteli diagram. Recall that we denote by Ω_v, where v is a vertex of D, the set of paths of D ending in v (and beginning in a vertex of the first level of D), and by $w\Omega$ we denote the set of paths whose beginning is a given finite path w.

Definition 4.1. Let $F : \Omega \longrightarrow \Omega$ be a homeomorphism. For $v \in V_i$ denote by $\alpha_v(F)$ the number of paths $w \in \Omega_v$ such that $F|_{w\Omega}$ is not equal to a transformation of the form $T_{w,u}$ for some $u \in \Omega_v$.

The homeomorphism F is said to be *of bounded type* if $\alpha_v(F)$ is uniformly bounded and the set of points $w \in \Omega$ such that the germ (F, w) does not belong to $\mathcal{T}$ is finite.

It is easy to see that the set of all homeomorphisms of bounded type form a group.

Theorem 4.2. *Let D be a Bratteli diagram. Let G be a group acting faithfully by homeomorphisms of bounded type on $\Omega(\mathsf{D})$. If the groupoid of germs of G has amenable isotropy groups, then the group G is amenable.*

Proof. It is enough to prove the theorem for finitely generated groups G. Since $[[\mathcal{T}]]$ is locally finite, it is amenable. Therefore, by Theorem 3.1 applied with $\mathcal{H} = \mathcal{T} \cap \mathcal{G}$, where $\mathcal{G}$ is the groupoid of germs of G, it is enough to prove that the orbital Schreier graphs of the action of G on Ω are recurrent.

Let $w \in \Omega$, and let S be a finite generating set of G. Consider the Schreier graph of the orbit $G(w)$ of w. For $F \subset G(w)$ denote by $\partial_S F$ the set of elements $x \in F$ such that $g(x) \notin F$ for some $g \in S$.

Lemma 4.3. *There exists an increasing sequence of finite subsets $F_n \subset G(w)$ such that $\partial_S F_n$ are disjoint and $|\partial_S F_n|$ is uniformly bounded.*

Proof. The orbit $G(w)$ is the union of a finite number of $\mathcal{T}$-orbits. Let $P \subset \Gamma(S, w)$ be a $\mathcal{T}$-orbit transversal. For $u = (e_i)_{i \geq 1} \in P$ denote by $F_{n,u}$ the set of paths of the form $(a_1, a_2, \ldots, a_n, e_{n+1}, e_{n+2}, \ldots)$. It is a finite subset of $G(w)$. Let $F_n = \bigcup_{u \in P} F_{n,u}$. Then $\bigcup_{n \geq 1} F_n = G(w)$.

Let $s \in S$. The number of paths $v = (a_1, a_2, \ldots, a_n, e_{n+1}, e_{n+2}, \ldots) \in F_{n,u}$ such that $s(v) \notin F_{n,u}$ is not greater than $\alpha_{\mathsf{o}(e_{n+1})}(s)$, see Definition 4.1. It follows that $|\partial_S F_n|$ is not greater than $|P| \cdot |S| \cdot \max\{\alpha_v(s) \ : \ s \in S, v \in \bigcup V_i\}$, which is finite. We can assume that $\partial_S F_n$ are disjoint by taking a subsequence. $\square$

Applying Theorem 2.5, we conclude that the graph $\Gamma(S, w)$ is recurrent. Note that this proof of recurrence of the graphs of actions of homeomorphisms of bounded type is essentially the same as the proof of [Bon07, Theorem V.24]. $\square$

In the remaining subsections, we give some examples of applications of Theorem 4.2.

4.3. **Groups acting on rooted trees and bounded automata.** Let us start with the case when the Bratteli diagram D is such that every level V_i contains only one vertex. Then the diagram is determined by the sequence $\mathsf{X} = (E_1, E_2, \ldots)$ of finite sets of edges. We will denote $\Omega_n = E_1 \times E_2 \times \cdots \times E_n$ by X^n, and $\Omega = \prod_{i=1}^{\infty} E_i$ by X^ω. The disjoint union $\mathsf{X}^* = \bigsqcup_{n \geq 0} \mathsf{X}^n$, where X^0 is a singleton, has a natural structure of a rooted tree. The sets X^n are its levels, and two paths $v_1 \in \mathsf{X}^n$ and $v_2 \in \mathsf{X}^{n+1}$ are connected by an edge if and only if v_2 is a continuation of v_1, i.e., if $v_2 = v_1 e$ for some $e \in E_{n+1}$.

We present here a short overview of the notions and results on groups acting on level-transitive rooted trees. For more details, see [GNS00, Section 6] and [Nek10]. Denote by $\operatorname{Aut} \mathsf{X}^*$ the automorphism group of the rooted tree X^*. The tree X^* is level-transitive, i.e., $\operatorname{Aut} \mathsf{X}^*$ acts transitively on each of the levels X^n. Denote by $\mathsf{X}^*_{(n)}$ the tree of finite paths of the "truncated" diagram defined by the sequence $\mathsf{X}_{(n)} = (E_{n+1}, E_{n+2}, \ldots)$.

For every $g \in \operatorname{Aut} \mathsf{X}^*$ and $v \in \mathsf{X}^n$ there exists an automorphism $g|_v \in \operatorname{Aut} \mathsf{X}^*_{(n)}$ such that

$$g(vw) = g(v)g|_v(w)$$

for all $w \in \mathsf{X}^*_{(n)}$. The automorphism $g|_v$ is called the *section* of g at v.

We have the following obvious properties of sections:

(4) $$(g_1 g_2)|_v = g_1|_{g_2(v)} g_2|_v, \qquad g|_{v_1 v_2} = g|_{v_1}|_{v_2}$$

for all $g_1, g_2, g \in \operatorname{Aut} \mathsf{X}^*$, $v, v_1 \in \mathsf{X}^*$, $v_2 \in \mathsf{X}^*_{(n)}$, where n is the length of v_1.

The relation between groups acting on rooted trees and the corresponding full topological groups of groupoids of germs is much closer that in the general case of groups acting by homeomorphisms on the Cantor set.

Proposition 4.4. *Let G be a group acting on a locally connected rooted tree X^*. Let $\mathcal{G}$ be the groupoid of germs of the corresponding action on the boundary X^ω of the tree. Then $[[\mathcal{G}]]$ is amenable if and only if G is amenable.*

Proof. Since $G \leq [[\mathcal{G}]]$, amenability of $[[\mathcal{G}]]$ implies amenability of G. Let us prove the converse implication. Suppose that G is amenable. It is enough to prove that every finitely generated subgroup of $[[\mathcal{G}]]$ is amenable. Let $S \subset [[\mathcal{G}]]$ be a finite set. There exists a level X^n of the tree X^* such that for every $v \in \mathsf{X}^n$ and g_i restriction of the action of g_i onto $v\mathsf{X}^*_{(n)}$ is equal to restriction of an element of G. Then every

element $g \in S$ permutes the cylindrical sets $v\mathsf{X}^*_{(n)}$ for $v \in \mathsf{X}^n$ and acts inside each of these sets as an element of G. It follows that $\langle S \rangle$ contains a subgroup of finite index which can be embedded into the direct product of a finite number of quotients of subgroups of G. Consequently, amenability of G implies amenability of $\langle S \rangle$. $\square$

Definition 4.5. An automorphism $g \in \operatorname{Aut} \mathsf{X}^*$ is said to be *finitary* of *depth* at most n if all sections $g|_v$ for $v \in \mathsf{X}^n$ are trivial.

It is an easy corollary of (4) that the set of finitary automorphisms and the set of finitary automorphisms of depth at most n are groups. The latter group is finite, hence the group of all finitary automorphisms of X^* is locally finite. It is also easy to see that the groupoid of germs of the action of the group of finitary automorphisms on X^ω coincides with the tail equivalence groupoid $\mathcal{T}$ of the diagram D.

Definition 4.6. Let $g \in \operatorname{Aut} \mathsf{X}^*$. Denote by $\alpha_n(g)$ the number of paths $v \in \mathsf{X}^n$ such that $g|_v$ is non-trivial. We say that $g \in \operatorname{Aut} \mathsf{X}^*$ is *bounded* if the sequence $\alpha_n(g)$ is bounded.

If $g \in \operatorname{Aut} \mathsf{X}^*$ is bounded, then there exists a finite set $P \subset \Omega$ of infinite paths such that $g|_v$ is non-finitary only if v is a beginning of some element of P.

The following is proved in [Nek10, Theorem 3.3].

Theorem 4.7. *Let G be a group of automorphisms of a locally finite rooted tree T. If G contains a non-abelian free subgroup, then either there exists a free non-abelian subgroup $F \le G$ and a point w of the boundary ∂T of the tree such that the stabilizer of w in F is trivial, or there exists $w \in \partial T$ and a free non-abelian subgroup of the group of germs $\mathcal{G}_w$.*

In particular, if the orbits of the action of G on ∂T have sub-exponential growth, and the groups of germs $\mathcal{G}_w$ do not contain free subgroups (e.g., if they are finite), then G does not contain a free subgroup. We get therefore the following corollary of Theorems 4.2 and 4.7. (The proof of the fact that there is no freely acting subgroup of G under conditions of Theorem 4.8 is the same as the proof of [Nek10, Theorem 4.4], and also follow from Lemma 4.3.)

Theorem 4.8. *Let G be a subgroup of the group of bounded automorphisms of X^*.*
   (1) If the isotropy groups $\mathcal{G}_w$ are amenable, then the group G is amenable.
   (2) If the isotropy groups $\mathcal{G}_w$ have no free subgroups, then the group G has no free subgroups.

In many cases it is easy to prove that the groups of germs $\mathcal{G}_w$ are finite. Namely, the following proposition is straightforward (see also the proof of [Nek10, Theorem 4.4]).

Proposition 4.9. *Suppose that the sequence $|E_i|$ is bounded. Let G be a group generated by bounded automorphisms of X^*. If for every generator g of G there exists a number n such that the depth of every finitary section $g|_v$ is less than n, then the groups of germs $\mathcal{G}_w$ for $w \in \mathsf{X}^\omega$ are locally finite. Consequently, the group G is amenable.*

Amenability of groups satisfying the conditions of Proposition 4.9 answers a question posed in [Nek10]. Below we show some concrete examples of groups satisfying the conditions of Proposition 4.9.

4.3.1. *Finite automata of bounded activity growth.*

Definition 4.10. Suppose that the sequence $\mathsf{X} = (E_1, E_2, \ldots) = (X, X, \ldots)$ is constant, so that $\mathsf{X}^*_{(n)}$ does not depend on n. An automorphism $g \in \operatorname{Aut} \mathsf{X}^*$ is *finite-state* if the set $\{g|_v \ : \ v \in \mathsf{X}^*\} \subset \operatorname{Aut} \mathsf{X}^*$ is finite.

The sequence $\alpha_n(g)$ from Definition 4.6 for finite-state automorphisms was studied by S. Sidki in [Sid00]. He showed that it is either bounded, or grows either as a polynomial of some degree $d \in \mathbb{N}$, or grows exponentially (in fact, he showed that the series $\sum_{n \geq 0} \alpha_n(g) x^n$ is rational). For each d the set $P_d(\mathsf{X}^*)$ of automorphisms of X^* for which $\alpha_n(g)$ grows as a polynomial of degree at most d is a subgroup of $\operatorname{Aut} \mathsf{X}^*$. He showed later in [Sid04] that these groups of *automata of polynomial activity growth* do not contain free non-abelian subgroups.

For different examples of subgroups of the groups of finite automata of bounded activity, see [BKN10, Section 1.D] and references therein.

It is easy to see that finite-state bounded automorphisms of the tree X^* satisfy the conditions of Proposition 4.9. This proves, therefore, the following.

Theorem 4.11. *The groups of finite automata of bounded activity growth are amenable.*

This theorem is the main result of the paper [BKN10]. It was proved there by embedding all finitely generated groups of finite-state bounded automorphisms into a sequence of self-similar "mother groups" M_d, and then using self-similarity structure on M_d and studying entropy of the random walk on M_d. Similar technique was applied in [AAV13] to prove that the group of automata with at most linear activity growth is also amenable. We will prove this fact later using Theorem 3.1.

One of examples of groups generated by finite automata of bounded activity growth is the Grigorchuk group [Gri80]. It is the first example of a group of intermediate growth, and also the first example of a non-elementary amenable group. It was used later to construct the first example of a finitely presented non-elementary amenable group in [Gri98]. An uncountable family of generalizations of the Grigorchuk group were studied in [Gri85]. They all satisfy the conditions of Proposition 4.9, and all have sub-exponential growth.

Another example is the *Basilica group*, which is the iterated monodromy group of the complex polynomial $z^2 - 1$, and is generated by two automorphisms a, b of the binary rooted tree given by the rules

$$a(0v) = 1v, \qquad a(1v) = 0b(v),$$
$$b(0v) = 0v, \qquad b(1v) = 1a(v).$$

It is easy to see that $\alpha_n(a)$ and $\alpha_n(b)$ are equal to 1 for all n, therefore, by Proposition 4.9, the Basilica group is amenable.

It was defined for the first time in [GŻ02] as a group defined by a three-state automaton. The authors showed that this group has no free subgroups, and asked if it is amenable. They also showed that this group can not be constructed from groups of sub-exponential growth using elementary operations preserving amenability (it is not *sub-exponentially amenable*). Amenability of the Basilica group was proved, using self-similarity and random walks, in [BV05]. It follows from the results of [Nek05] and Theorem 4.11 that iterated monodromy groups of sub-hyperbolic polynomials are amenable.

4.3.2. *Groups of Neumann-Segal type.* The conditions of Definition 4.10 are very restrictive. In particular, the set of all finite-state automorphisms of X^* is countable, whereas the set of all bounded automorphisms is uncountable. There are many interesting examples of groups generated by bounded but not finite-state automorphisms of X^*. For example, the groups from the uncountable family of Grigorchuk groups [Gri85] are generated by bounded automorphisms of the binary rooted trees. They are of sub-exponential growth.

An uncountable family of groups of non-uniformly exponential growth was constructed in [Bri09]. All groups of the family (if the degrees of the vertices of the tree are bounded) satisfy the conditions of Proposition 4.9, hence are amenable. Amenability of these groups were proved in [Bri09] using the techniques of [BV05].

Other examples are given by a construction used by P. Neumann in [Neu86] and D. Segal in [Seg01].

Let us describe a general version of D. Segal's construction. Let (G_i, X_i), $i = 0, 1, \ldots$, be a sequence of groups acting transitively on finite sets X_i. Let $a_{i,j} \in G_i$ for $1 \leq j \leq k$ be sequences of elements such that $a_{i,1}, a_{i,2}, \ldots, a_{i,k}$ generate G_i. Choose also points $x_i, y_i \in X_i$. Define then automorphisms $\alpha_{i,j}, \beta_{i,j}$, $i = 0, 1, \ldots$, of the tree $\mathsf{X}^*_{(j)}$ for the sequence $\mathsf{X}_{(j)} = (X_i, X_{i+1}, \ldots)$ given by the following recurrent rules:

$$\alpha_{i,j}(xw) = a_{i,j}(x)w,$$

$$\beta_{i,j}(xw) = \begin{cases} x_i\beta_{i+1,j}(w) & \text{if } x = x_i, \\ y_i\alpha_{i+1,j}(w) & \text{if } x = y_i, \\ xw & \text{otherwise}, \end{cases}$$

where $w \in \mathsf{X}^*_{(j+1)}$ and $x \in X_i$.

Proposition 4.12. *The group* $G = \langle \alpha_{0,1}, \ldots, \alpha_{0,k}, \beta_{0,1}, \ldots, \beta_{0,k} \rangle$ *is amenable if and only if the group generated by the sequences* $(a_{1,j}, a_{2,j}, \ldots) \in \prod_{i=1}^{\infty} G_i$, *for* $j = 1, \ldots, k$, *is amenable.*

Proof. The group generated by the sequences $(a_{1,j}, a_{2,j}, \ldots)$ is isomorphic to the group generated by $\beta_{0,1}, \beta_{0,2}, \ldots, \beta_{0,k}$. Consequently, if this group is non-amenable, then G is non-amenable too.

The automorphisms $\alpha_{i,j}$ are finitary. All sections of $\beta_{i,j}$ are finitary except for the sections in finite beginnings of the sequence $x_i x_{i+1} \ldots$. It follows that $\beta_{i,j}$ are bounded, and the groups of germs the action of G on X^ω are isomorphic to the group of germs of G at the point $x_0 x_1 \ldots$. It also follows from the description of the sections that the group of germs of G at $x_0 x_1 \ldots$ is a quotient of $\langle \beta_{0,1}, \beta_{0,2}, \ldots, \beta_{0,k} \rangle$. Then Theorem 4.8 finishes the proof. $\square$

The examples considered by P. Neumann in [Neu86] are similar, but they are finite-state, so their amenability follows from Theorem 4.11. The main examples of D. Segal [Seg01] are non-amenable (they are constructed for G_i equal to $PSL(2, p_i)$ for an increasing sequence of primes p_i).

A. Woryna in [Wor11] and E. Fink in [Fin12] consider the case when G_i are cyclic groups (of variable order). A. Woryna uses the corresponding group G to compute the minimal size of a topological generating set of the profinite infinite wreath product of groups G_i. E. Fink shows that if the orders of G_i grow sufficiently fast, then the group G is of exponential growth, but does not contain free subgroups. Proposition 4.12 immediately implies that such groups are amenable (for any sequence of cyclic groups).

I. Bondarenko used in [Bon10] the above construction in the general case to study number of topological generators of the profinite infinite wreath product of permutation groups.

4.3.3. *Iterated monodromy groups of polynomial iterations.* Two uncountable families of groups generated by bounded automorphisms were studied in [Nek07] in relation with holomorphic dynamics. Let us describe one of them. For a sequence $w = x_1 x_2 \ldots$, $x_i \in \{0, 1\}$ define $s(w) = x_2 x_3 \ldots$, and let $\alpha_w, \beta_w, \gamma_w$ be automorphisms of $\{0, 1\}^*$ defined by

$$\alpha_w(0v) = 1v, \qquad \gamma_w(0v) = 0v, \qquad \beta_w(0v) = 0\alpha_{s(w)}(v),$$
$$\alpha_w(1v) = 0\gamma_{s(w)}(v), \qquad \gamma_w(1v) = 1\beta_{s(w)}(v), \qquad \beta_w(1v) = 1v,$$

if $x_1 = 0$, and

$$\beta_w(0v) = 0v, \qquad \beta_w(1v) = 1\alpha_{s(w)}(v),$$

if $x_1 = 1$, where $s(w) = x_2 x_3 \ldots$.

The automorphisms $\alpha_w, \beta_w, \gamma_w$ are obviously bounded. Note that $\alpha_w, \beta_w, \gamma_w$ are finite-state if and only if w is eventually periodic. It is easy to see that the groups of germs of the action of the group $R_w = \langle \alpha_w, \beta_w, \gamma_w \rangle$ on the boundary of the binary tree are trivial. Therefore, it follows from Theorem 4.8 that the groups G_w are amenable.

It is shown in [Nek07] that the sets of isomorphism classes of groups R_w, for $w \in \{0, 1\}^{\mathbb{Z}}$ are countable, and that the map $w \mapsto R_w$ is a homeomorphic embedding of the Cantor space of infinite binary sequences into the space of three-generated groups.

The groups R_w are iterated monodromy groups of sequences of polynomials of the form $f_n(z) = 1 - \frac{z^2}{p_n^2}$, where $p_n \in \mathbb{C}$ satisfy $p_n = 1 - \frac{1}{p_{n+1}^2}$, $n = 0, 1, \ldots$. In general, let $f_1, f_2, \ldots$ be a sequence of complex polynomials seen as maps

$$\mathbb{C} \xleftarrow{f_1} \mathbb{C} \xleftarrow{f_2} \mathbb{C} \xleftarrow{f_3} \cdots .$$

Denote by P_n the union of the sets of critical values (i.e., values at critical points) of the polynomials $f_n \circ f_{n+1} \circ \cdots \circ f_{n+k}$ for all $k \geq 0$. Note that $P_{n+1} \subseteq f_n^{-1}(P_n)$. We say that the sequence is *post-critically finite* if the set P_1 is finite. In this case, we get a sequence of covering maps

$$M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \xleftarrow{f_3} \cdots$$

where $M_1 = \mathbb{C} \setminus P_1$ and $M_n = (f_1 \circ \cdots \circ f_{n-1})^{-1}(M_1)$. Choose a basepoint $t \in M_1$, and consider the fundamental group $\pi_1(M_1, t)$.

The disjoint union

$$T = \{t\} \cup \bigsqcup_{n \geq 1} (f_1 \circ \cdots \circ f_n)^{-1}(t)$$

has a natural structure of a rooted tree, where t is the root, and a vertex $z \in (f_1 \circ \cdots \circ f_n)^{-1}(t)$ is connected to $f_n(z) \in (f_1 \circ \cdots \circ f_{n-1})^{-1}(t)$.

The fundamental group $\pi_1(M_1, t)$ acts on each level of the tree by the monodromy action. Taken together, these actions become an action of $\pi_1(M_1, t)$ on the tree T by automorphisms. Denote by $IMG(f_1, f_2, \ldots)$ the quotient of $\pi_1(M_1, t)$ by the kernel of the action. This group is called the *iterated monodromy group* of the sequence. For example, Basilica group is the iterated monodromy group of the constant sequence $f_n(z) = z^2 - 1$. Here $P_1 = \{0, -1\}$. Iterated monodromy groups

of such sequences of polynomials were studied in the article [Nek09]. The following statement is a direct corollary of its results.

Proposition 4.13. *Suppose that $|P_n|$ is a bounded sequence. Then there exists an isomorphism of the tree T with a tree X^* conjugating $IMG(f_1, f_2, \ldots)$ with a group acting on X^* by bounded automorphisms.*

It is also not hard to understand the structure of the groups of germs of the action of $IMG(f_1, f_2, \ldots)$ on the boundary of the tree. In particular, one can show for many sequences (using Proposition 4.9) that they are finite.

Example 4.14. Consider an arbitrary sequence $f_n(z)$, $n \geq 0$, such that $f_n(z) = z^2$ or $f_n(z) = 1 - z^2$. Then the sequence is post-critically finite with the post-critical sets P_n subsets of $\{0, 1\}$. The iterated monodromy group of such a sequence is generated by automorphisms a_w, b_w, for some sequence $w = x_1 x_2 \ldots \in \{0, 1\}^\infty$, where

$$a_w(0v) = 1v, \quad a_w(1v) = 0b_{s(w)}(v), \quad b_w(0v) = 0v, \quad b_w(1v)1a_{s(w)}(v),$$

if $x_1 = 0$, and

$$a_w(0v) = 1v, \quad a_w(1v) = 0a_{s(w)}(v), \quad b_w(0v) = 0v, \quad b_w(1v)1b_{s(w)}(v),$$

otherwise. All these groups are amenable by Proposition 4.9.

4.4. **Bratteli-Vershik transformations and minimal homeomorphisms.** A *Bratteli-Vershik* diagram is a Bratteli diagram $\mathsf{D} = ((V_n)_{n \geq 1}, (E_n)_{n \geq 1}, \mathsf{o}, \mathsf{t})$ together with a linear order on each of the sets $\mathsf{t}^{-1}(v)$, $v \in \bigcup_{n \geq 2} V_n$. A path $v \in \Omega_n$, for $n = 1, 2, \ldots, \infty$, is said to be *minimal* (resp. *maximal*) if it consists only of minimal (resp. maximal) edges. Note that for every vertex $v \in \bigcup_{n \geq 1} V_n$ of a Bratteli-Vershik diagram there exist unique maximal and minimal paths $v_{\max}, v_{\min} \in \Omega_v$. Let $(e_1, e_2, \ldots) \in \Omega$ be a non-maximal path in the diagram. Let n be the smallest index such that e_n is non-maximal. Let e'_n be the next edge after e_n in $\mathsf{t}^{-1}(\mathsf{t}(e_n))$, and let $(e'_1, e'_2, \ldots, e'_{n-1})$ be the unique minimal path in $\Omega_{\mathsf{o}(e'_n)}$. Define then

$$a(e_1, e_2, \ldots) = (e'_1, e'_2, \ldots, e'_{n-1}, e'_n, e_{n+1}, e_{n+2}, \ldots).$$

The map a is called the *adic transformation* of the Bratteli-Vershik diagram.

The adic transformation is a continuous map from the set of non-maximal elements of Ω to Ω. In fact, it is easy to see that it is a homeomorphism from the set of non-maximal paths to the set of non-minimal paths. The inverse map is the adic transformation defined by the opposite ordering of the diagram. If the diagram has a unique maximal and a unique minimal infinite paths, then we can extend a to a homeomorphism $\Omega \longrightarrow \Omega$ by mapping the maximal path to the minimal path.

The following theorem is proved in [HPS92].

Theorem 4.15. *Every minimal homeomorphism of the Cantor set (i.e., a homeomorphism for which every orbit is dense) is topologically conjugate to the adic transformation defined by a Bratteli-Vershik diagram.*

See also [Med06] where Vershik-Bratteli diagrams are used to describe *aperiodic* homeomorphisms, i.e., homeomorphisms without finite orbits.

It follows directly from the definition of the adic transformation a that $\alpha_v(a) = 1$ for every vertex v of the diagram, and that germs of a belong to $\mathcal{T}$ for all points $w \in \Omega$ except for the unique maximal path. It is also obvious that the a has no

fixed points, hence the groups of germs of the group generated by a are trivial. Using Theorem 4.2, we get a proof of following result of [JM12] by K. Juschenko and N. Monod.

Theorem 4.16. *Let a be a minimal homeomorphism of the Cantor set. Then the topological full group of the groupoid of germs of $\langle a \rangle$ is amenable.*

It follows from the results of [Med06] that the same result is true for all aperiodic homeomorphisms of the Cantor set. Theorem 4.16 provides the first examples of simple finitely generated amenable groups. In fact, there are uncountably many pairwise non-isomorphic full groups of minimal homeomorphisms of the Cantor set, see [JM12].

4.4.1. *One-dimensional tilings.* Adic transformations and subgroups of their full groups naturally appear in the study of one-dimensional aperiodic tilings, see [BJS10]. We present here one example, called *Fibonacci tiling*.

Consider the endomorphism ψ of the free monoid $\langle a, b \rangle$ defined by $a \mapsto ab$, $\qquad b \mapsto a$. Then $\psi^n(a)$ is beginning of $\psi^{n+1}(a)$, so that we can pass to the limit, and get a right-infinite sequence

$$\psi^\infty(a) = abaababaabaababaababaabaababaabaab\ldots$$

Let $\mathcal{F}$ be the set of bi-infinite sequences $w \in \{a, b\}^{\mathbb{Z}}$ over the alphabet $\{a, b\}$ such that every finite subword of w is a sub-word of $\psi^\infty(a)$. The set $\mathcal{F}$ is obviously invariant under the shift, which we will denote by τ. Here the shift acts by the rule

$$\ldots x_{-2}x_{-1}.x_0x_1\ldots \mapsto \ldots x_{-1}x_0.x_1x_2,$$

where dot shows the place between the coordinate number -1 and coordinate number 0.

Every letter b in a sequence $w \in \mathcal{F}$ is uniquely grouped with the previous letter a. Replace each group ab by a, and each of the remaining letters a by b. Denote the new sequence by $\sigma(w) = \ldots y_{-2}y_{-1}.y_0y_1\ldots$, so that y_0 corresponds to the group that contained x_0. It follows from the definition of $\mathcal{F}$ that $\sigma(w) \in \mathcal{F}$.

Symbol $\alpha(w)$ of a sequence $\ldots x_{-2}x_{-1}.x_0x_1\ldots \in \mathcal{F}$ is an element of $\{a_0, a_1, b\}$ defined by the following conditions:

(1) $\alpha(w) = a_0$ if $x_{-1}x_0 = aa$;
(2) $\alpha(w) = a_1$ if $x_{-1}x_0 = ba$;
(3) $\alpha(w) = b$ if $x_0 = b$.

Itinerary of $w \in \mathcal{F}$ is the sequence $\alpha(w)\alpha(\sigma(w))\alpha(\sigma^2(w))\ldots$. We have the following description of $\mathcal{F}$, which easily follows from the definitions.

Proposition 4.17. *A sequence $w \in \mathcal{F}$ is uniquely determined by its itinerary. A sequence $x_1x_2\ldots \in \{a_0, a_1, b\}^\infty$ is an itinerary of an element of $\mathcal{F}$ if and only if $x_ix_{i+1} \in \{a_0a_1, a_1a_0, a_1b, ba_0, ba_1\}$. The shift τ acts on $\mathcal{F}$ in terms of itineraries by the rules:*

$$\tau(a_0 w) = bw, \quad \tau(bw) = a_1\tau(w)$$

$$\tau(a_1 w) = \begin{cases} b\tau(w) & \text{if } w \text{ starts with } a_0, \\ a_0\tau(w) & \text{if } w \text{ starts with } b, \end{cases}$$

Restrictions τ_{a_0}, τ_{a_1}, τ_b of τ to the cylindrical sets of sequences starting with a_0, a_1, b, respectively, generate a *self-similar inverse semigroup* in the sense of [Nek06]. The range of each of these transformations does not intersect with the domain.

FIGURE 1. Tiles of Penrose tilings and grouping of tiles

Therefore, we may define homeomorphisms $\alpha_0, \alpha_1, \beta$ equal to transformations $\tau_{a_0} \cup \tau_{a_0}^{-1}$, $\tau_{a_1} \cup \tau_{a_1}^{-1}$, $\tau_b \cup \tau_b^{-1}$ extended identically to transformations of the space $\mathcal{F}$.

It is easy to see that the orbital Schreier graphs of the action of the group $\langle \alpha_0, \alpha_1, \beta \rangle$ on $\mathcal{F}$ coincide with the corresponding two-sided sequences. Namely, the Schreier graph of a point $\ldots x_{-1}.x_0 x_1 \ldots$ is isomorphic to a chain of vertices $(\ldots, w_{-1}, w_0, w_1, \ldots)$, where $w_0 = w$, and the edge (w_i, w_{i+1}) corresponds to the generator α_0, α_1, or β if and only if $x_{i-1} x_i$ is equal to aa, ba, or ab, respectively. It follows from Theorem 4.16 that the group $\langle \alpha_0, \alpha_1, \beta \rangle$ is amenable.

4.5. **An example related to the Penrose tilings.** Consider isosceles triangles formed by two diagonals and a side, and two sides and a diagonal of a regular pentagon. Mark their vertices by white and black dots, and orient one of the sides as it is shown on Figure 1.

A *Penrose tiling* is a tiling of the plane by such triangles obeying *matching rules*: if two triangles have a common vertex, then this vertex must be marked by dots of the same color; if two triangle have a common side, then either they both are not oriented, or they are both oriented and orientations match.

One can show, just considering all possible sufficiently big finite Penrose tilings, that tiles of any Penrose tilings can be grouped into blocks as on Figure 1. The blocks are triangles similar to the original tiles (with similarity coefficient $(1 + \sqrt{5})/2$), and the new tiles also form a Penrose tiling, which we call *inflation* of the original tiling.

Given a Penrose tiling with a marked tile, consider the sequence of iterated inflations of the tiling, where in each tiling the tile containing the original marked tile is marked. Let $x_1 x_2 \ldots$ be the corresponding *itinerary*, where $x_i \in \{a, b, c\}$ is the letter describing position of the marked tile of the $(i-1)$st inflation in the tile of the ith inflation according to the rule shown on Figure 1. A sequence is an itinerary of a marked Penrose tiling if and only if it does not contain a subword ba. More on the inflation and itineraries of the Penrose tilings, see [GS87].

Given a marked Penrose tiling with itinerary $w = x_1 x_2 \ldots$, denote by $L(w)$, $S(w)$, $M(w)$ the itineraries of the same tiling in which a neighboring tile is marked, where the choice of the neighbor is shown on Figure 1. One can show by considering sufficiently big finite patches of the Penrose tilings (see also [BGN03, Nek02]) that the transformations L, S, and M on the space of itineraries is given by the rules

$$S(aw) = cw, \quad S(bw) = bM(w), \quad S(cw) = aw, \quad L(cw) = cL(w),$$

$$L(aw) = \begin{cases} bS(w) & \text{if } w \text{ starts with } a, \\ aM(w) & \text{otherwise,} \end{cases} \quad L(bw) = \begin{cases} bS(w) & \text{if } w \text{ starts with } b, \\ aS(w) & \text{if } w \text{ starts with } c, \end{cases}$$

$$M(aw) = aL(w), \quad M(bw) = cw, \quad M(cw) = \begin{cases} cM(w) & \text{if } w \text{ starts with } a, \\ bw & \text{otherwise.} \end{cases}$$
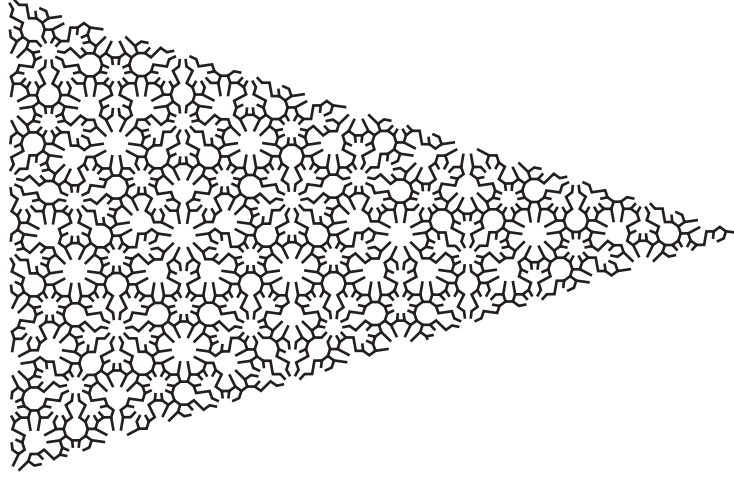
FIGURE 2. A part of the Schreier graph of $\langle L', S', M' \rangle$

The orbital Schreier graphs of the action of the group $\langle L, S, M \rangle$ coincide then with the graphs dual to the Penrose tilings (except in the exceptional cases when the tilings have a non-trivial symmetry group; then the dual graphs are finite coverings of the Schreier graphs).

Let us redefine the transformations L, S, M, trivializing the action on some cylindrical sets, so that in some cases they correspond to moving the marking to a neighboring tile, but sometimes they correspond to doing nothing. Namely, define new transformations L', S', M' by the rules.

$$S'(aw) = cw, \quad S'(bw) = bM'(w), \quad S'(cw) = aw,$$

$$L'(aw) = \begin{cases} bS'(w) & \text{if } w \text{ starts with } a, \\ aw & \text{otherwise,} \end{cases}$$

$$L'(bw) = \begin{cases} bS'(w) & \text{if } w \text{ starts with } b, \\ aS'(w) & \text{if } w \text{ starts with } c, \end{cases} \quad L'(cw) = cL'(w),$$

$$M'(aw) = aw, \quad M'(bw) = cw, \quad M'(cw) = \begin{cases} cM'(w) & \text{if } w \text{ starts with } a, \\ bw & \text{otherwise.} \end{cases}$$

Then the Schreier graphs of $\langle L', S', M' \rangle$ are subgraphs of the graphs dual to the Penrose tilings. A piece of such Schreier graph is shown on Figure 2.

It is easy to see that M', and hence S' are finitary. It follows that L' is bounded. Further analysis shows that the groups of germs are finite, hence the group $\langle L', S', M' \rangle$ is amenable.

5. Unbounded examples

In the previous section we applied Theorem 3.1 in the case when the full group $[[\mathcal{H}]]$ is locally finite. Here we present some other examples of applications of Theorem 3.1.

5.1. **Linearly and quadratically growing automata.** Recall that $P_d(X)$, for $d = 1, 2, \ldots$, denotes the group of finite-state automorphisms g of X^* such that $\alpha_n(g)$ is bounded by a polynomial of degree d. In particular, $P_1(\mathsf{X})$ is the group of finite-state automorphisms of *linear activity growth*. Here $\alpha_n(g)$, as before, is the number of words v of length n such that $g|_v \neq 1$.

For example, consider the group generated by the following two transformations of $\mathbb{Z}$.

$$a : n \mapsto n + 1, \qquad b : 2^m(2k+1) \mapsto 2^m(2k+3), \quad b(0) = 0,$$

where $n, k, m \in \mathbb{Z}$ and $m \geq 0$.

The Schreier graph of this action, and the orbital Schreier graphs of the associated group acting on a rooted tree were studied in [BH05, BCSDN12].

The following theorem was proved in [AAV13]. We present here a short proof based on Theorem 3.1.

Theorem 5.1. *For every finite alphabet X the group $P_1(X)$ is amenable.*

Proof. Let us investigate the groups of germs of finitely generated subgroups of $P_d(X)$. Let S be a finite subset of $P_d(X)$, and $G = \langle S \rangle$. We may assume that S is *state-closed*, i.e., that $g|_x \in S$ for all $x \in X$ and $g \in S$. Replacing X by X^N for some integer N, we may assume that the elements of S satisfy the following condition. For every element $g \neq 1$ of the form $g = h|_x$, for $h \in S$ and $x \in X$, there exists a unique letter $x_g \in X$ such that $g|_{x_g} = g$, for all other letters $x \in X$ the section $g|_x$ has degree of activity growth lower than that of g (in particular, it is finitary if g is bounded and trivial if g is finitary). This fact follows from the structure of automata of polynomial activity growth [Sid00], see also [BKN10, proof of Theorem 3.3].

In particular, it follows that for every $g \in P_d(X)$ the set of points $w \in \mathsf{X}^\omega$ such that the germ (g, w) is not a germ of an element of $P_{d-1}(X)$ is finite.

Let $w = x_1 x_2 \ldots \in \mathsf{X}^\omega$, and consider the isotropy group $\mathcal{G}_w$ of the groupoid of germs of G. Denote by Φ be the set of all eventually constant sequences. If w does not belong to Φ, our assumptions on S imply that the germ (g, w) of every $g \in S$ (and hence every $g \in G$) belongs to the groupoid of germs of the finitary group. This implies that $\mathcal{G}_w$ is trivial. Assume now that w belongs to Φ, say $x_n = x$ for all n large enough. The sequence $g|_{x_1 x_2 \ldots x_n}$ is eventually constant for every $g \in S$ (and hence in G by (4)). In particular if $g \in G$ fixes w, then for all $y \neq x$ the sequence $g|_{x_1 x_2 \ldots x_n y}$ is eventually constant and belongs to $P_{d-1}(X)$. We therefore get a homomorphism from the stabilizer of w in G into the wreath product $P_{d-1}(X) \wr \mathrm{Symm}(X \setminus \{x\})$ with kernel the elements of G that act trivially on a neighborhood of w. This shows that $\mathcal{G}_w$ embeds in $P_{d-1}(X) \wr \mathrm{Symm}(X \setminus \{x\})$.

Now Theorem 3.1 shows that the amenability of G will follow from amenability of $P_{d-1}(X)$ and recurrence of the orbital Schreier graphs of G. We have proved in Theorem 4.11 that $P_0(X)$ is amenable. Let us prove amenability of finitely generated subgroups of $P_1(X)$, i.e., amenability of $P_1(X)$.

If $w_0 \notin \Phi$, then w_0 is not singular in the sense of Theorem 3.1, i.e., germs of elements of $P_1(X)$ at w_0 are equal to germs of some elements of $P_0(X)$.

We therefore only have to show that the action of G on the orbit of an element w_0 of Φ is recurrent. Let $F_n \subset \Phi$ be the set of all sequences $w = x_1 x_2 \ldots \in \mathsf{X}^\omega$ in the G-orbit of w_0 such that $x_i = x_j$ for all $i, j > n$. Suppose that $w \in \partial_S F_n$ for $n > 1$. Then $w = vx^\omega$ for some $|v| = n$ and $x \in X$, and there exists $g \in S$ such

that $g(w) \notin F_n$, i.e., $g|_v(x^\omega)$ is not a constant sequence. This implies that $g|_v \neq 1$, and $g|_{vx} \neq g|_v$, hence $g|_{vx}$ is bounded. Then either $g|_v(x^\omega)$ is of the form $y_1 y^\omega$, for $y_1, y \in X$; or $g|_{vxx}$ is finitary, so that $g|_v(x^\omega)$ is of the form $y_1 y_2 y_3 x^\omega$, for $y_i \in X$. It follows that $|\partial_S F_n| \leq |X| \sum_{g \in S} \alpha_n(g)$ and that every element of $g(\partial_S F_n)$ for $g \in S$, belongs to F_{n+3}. Then $\partial_S F_{3n}$ are disjoint, $\cup_n F_n = G(\omega_0)$ and $|\partial_S F_{3n}|$ is bounded from above by a linear function, which implies by Theorem 2.5 that the Schreier graph of G on the orbit of ω_0 is recurrent. □

Note that arguments of the proof of Theorem 5.1 imply the following.

Theorem 5.2. *If $G \leq P_2(X)$ is a finitely generated group with recurrent Schreier graphs of the action on X^ω, then G is amenable.*

It was announces in [AV11] that the Schreier graphs of the "mother group" of automata of quadratic activity growth are recurrent. This implies, by Theorem 5.2, that these groups are amenable, which in turn implies amenability of $P_2(X)$ for every finite alphabet X.

On the other hand, it is shown in [AV11] that the Schreier graphs of some groups generated by automata of growth of activity of degree greater than 2 are transient.

5.2. **Holonomy groups of Hénon maps.** A *Hénon map* is a polynomial map $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ given by

$$f(x, y) = (x^2 + c - ay, x),$$

where $a, c \in \mathbb{C}$, $a \neq 0$. Note that f is invertible, and its inverse is $f^{-1}(x, y) = (y, (y^2 + c - x)/a)$.

For a given Hénon map f, let K_+ (resp. K_-) be the set of points $p \in \mathbb{C}^2$ such that $f^n(p)$ is bounded as $n \to +\infty$ (resp. as $n \to -\infty$). Let J_+ and J_- be the boundaries of K_+ and K_-, respectively. The *Julia set* of f is the set $J_+ \cap J_-$.

In some cases the Hénon map is *hyperbolic* on a neighborhood of its Julia set, i.e., the Julia set can be decomposed locally into a direct product such that f is expanding one and contracting the other factors of the decomposition. The contracting direction will be totally disconnected. See [Ish08, Ish09] for examples and more detail. In this case we have a naturally defined holonomy pseudogroup of the contracting direction of the local product decomposition of the Julia set. Fixing a subset C of the contracting direction homeomorphic to the Cantor set, we can consider the group H_C of all homeomorphisms of C belonging to the holonomy pseudogroup, or a subgroup of H_C such that its groupoid of germs coincides with the holonomy groupoid. General theory of holonomy pseudogroups of hyperbolic Hénon maps (also their generalizations of higher degree) is developed in [Ish13].

One example of computation of the holonomy groupoid is given in [Oli98]. It corresponds to the Hénon maps with parameter values $a = 0.125$, $c = -1.24$.

The corresponding group acts by homeomorphisms on the space $\{0, 1\}^\omega$ of binary infinite sequences, and is generated by transformations $\alpha, \beta, \gamma, \tau$, defined by the

following recurrent rules:

$$\alpha(0w) = 1\alpha^{-1}(w), \qquad \alpha(1w) = 0\beta(w),$$
$$\beta(0w) = 1\gamma(w), \qquad \beta(1w) = 0\tau(w),$$
$$\gamma(00w) = 11\tau^{-1}(w), \qquad \gamma(11w) = 00\tau(w),$$
$$\gamma(10w) = 10w, \qquad \gamma(01w) = 01w,$$
$$\tau(0w) = 1w, \qquad \tau(1w) = 0\tau(w).$$

Note that $G = \langle \alpha, \beta, \gamma, \tau \rangle$ does not act by homeomorphisms on the tree $\{0,1\}^*$. The recurrent rules follow from the automaton shown on [Oli98, Figure 3.8].

It is easy to check that the generators α, β have linearly growing activity (i.e., the sequence from Definition 4.1 for these homeomorphisms are bounded by a linear function). The generators γ, τ are of bounded type. The groups of germs of G are trivial. Therefore, Theorem 3.1 applied for $\mathcal{H}$ equal to the groupoid of germs of $\langle \gamma, \tau \rangle$ shows that G is amenable.

5.3. **Mating of two quadratic polynomials.** Let $f(z) = z^2 + c = z^2 - 0.2282\ldots + 1.1151\ldots i$ be the quadratic polynomial such that $f^3(0)$ is a fixed point of f. It is easy to see that the last condition means that c is a root of the polynomial $x^3 + 2x^2 + 2x + 2$. Let us compactify the complex plane $\mathbb{C}$ by a circle at infinity (adding points of the form $+\infty \cdot e^{i\theta}$) and extend the action of f to the obtained disc $\mathbb{D}$ so that f acts on circle at infinity by angle doubling. Take two copies of $\mathbb{D}$ with f acting on each of them, and then glue them to each other along the circles at infinity using the map $+\infty \cdot e^{i\theta} \mapsto +\infty \cdot e^{-i\theta}$. We get then a branched self-covering F of a sphere. This self-covering is a *Thurston map*, i.e., orbit of every critical point of F (there are two of them in this case) are finite. For more about this map in particular, its relation to the paper-folding curve, and for the general *mating* procedure, see [Mil04].

The iterated monodromy group of F was computed in [Nek08]. It is generated by the following five automorphisms of the binary tree $\{0,1\}^*$:

$$a(0w) = 1w, \qquad a(1w) = 0w,$$
$$b = (1, a), \qquad b' = (cc'abb', 1),$$
$$c = (c, b), \qquad c' = (c', b'),$$

where $g = (g_0, g_1)$ means that $g(0w) = 0g_0(w)$ and $g(1w) = 1g_1(w)$.

It is shown in [Nek08] that the subgroup generated by a, $B = bb'$, and $C = cc'$ has a subgroup of index two isomorphic to $\mathbb{Z}^2$. In particular, it is amenable, and Schreier graphs of its action on $\{0,1\}^\omega$ are recurrent. It is also shown (see [Nek08, Proposition 6.3]) that the Schreier graphs of the whole group $\mathrm{IMG}(F) = \langle a, b, c, b', c' \rangle$ coincide with the Schreier graphs of the virtually abelian subgroup $H = \langle a, B, C \rangle$. Namely, if $b(v) \neq v$, $b'(v) \neq v$, $c(v) \neq v$, or $c'(v) \neq v$, then $b(v) = B(v)$, $b'(v) = B(v)$, $c(v) = C(v)$, or $c'(v) = C(v)$, respectively. A picture showing how the Schreier graph of $\langle a, b, c \rangle$ is embedded into the Schreier graph of H can be found in [Nek08, Figure 15].

It follows from the definition of the generators of $\mathrm{IMG}(F)$ that all germs of b and b' belong to the groupoid of germs $\mathcal{H}$ of H. Consequently, all germs of c and c' except for the germ at the point $000\ldots$ belong to $\mathcal{H}$. It follows from Theorem 3.1 and Proposition 4.4 that the group $\mathrm{IMG}(F)$ is amenable.

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