Long term dynamics for the restricted N-body problem with mean motion resonances and crossing singularities

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Abstract

We consider the long term dynamics of the restricted N-body problem, modeling the dynamics of an asteroid that moves in the gravitational field of the Solar system. We deal with the case of a mean motion resonance with a planet. The asteroid evolution is computed by standard techniques of perturbation theory. We focus on the critical case where the trajectory of the asteroid crosses the one of some planet during the evolution. This produces a singularity in the corresponding equations of motion. We prove that the related vector field can be extended to two locally Lipschitz-continuous vector fields and we define generalized solutions, continuous but not differentiable, going beyond the crossing singularities. Moreover, we prove that the long term evolution of the 'signed' orbit distance (Gronchi and Tommei 2007) between the asteroid and the planet whose trajectory is crossed is differentiable in a neighborhood of the crossing times. We conclude by a numerical comparison of the long term and the full evolutions in the case of planet crossing asteroids belonging to the 'Alinda' and 'Toro' classes (Milani et al. 1989). This work extends the results in (Gronchi and Tardioli 2013) to the resonant case.

1 Introduction

The role of the minimal distance between the trajectories of an asteroid and a planet is crucial in the study of possible Earth impactors. Actually, a small value of this quantity, that we denote by d_{min}, is a necessary condition for an impact. We study the evolution of d_{min} in the framework of the restricted N-body problem, modeling the motion of an asteroid in the gravitational field generated by the Sun and some planets.

It is well known since Poincaré that for $N \geq 3$ the N-body problem is not integrable. In particular, the evolutions of near-Earth asteroids (NEAs) have short Lyapunov times, beyond which the orbit computed by numerical techniques and the true orbit are completely uncorrelated [13]. However, we can obtain statistical information on the evolution by considering a normal form of the Hamiltonian of the problem, where we try to filter out the short periodic oscillations. More precisely, we would like to eliminate the dependence on the mean anomalies of the asteroid and the planets from the Hamiltonian. Outside mean motion resonances this procedure corresponds to averaging Hamilton's equations over the fast angles [1]. In case of resonances, we must keep in the normal form also the resonant combinations of the angles.

In both cases, the intersections between the trajectories of the asteroid and the planets correspond to singularities; namely, a singularity occurs whenever $d_{min} = 0$. Since the trajectory of a near-Earth asteroid (NEA) is likely to cross the trajectory of the Earth, we cannot avoid to deal with these singularities.

After the preliminary study by Lidov and Ziglin [8], in the case of orbits uniformly close to a circular orbit, the problem of averaging over crossing orbits was studied in [5]. Here the authors assumed the orbits of the planets being circular and coplanar, and excluded mean motion resonances and close approaches with them. In [4] the results were extended to the case of non-zero eccentricities and inclinations. In these works, the distance was computed through a Taylor expansion at the mutual nodes. These results were improved in [7] where the main singular term is expanded at the minimum distance points (see Section 4) and where it is proved that the averaged vector field admits two different Lipschitz-continuous extensions in a neighborhood of almost every crossing configuration. The latter property allows us to define a generalized solution, representing the secular evolution of the asteroid, that is continuous but not differentiable at crossings. Moreover, one can suitably choose the sign of d_{min} and obtain a map $\tilde{d}_{min}$ that is differentiable in a neighborhood of almost all crossing configurations [6]. The secular evolution of $\tilde{d}_{min}$ along the generalized solutions turns out to be differentiable in a neighborhood of the singularity.

The basic model considered in these works comes from the averaging principle. Therefore, it is supposed that the dynamics is not affected by mean motion resonances. However, the population of resonant NEAs is not negligible. Moreover, mean motion resonances are considered responsible for a relatively fast change in the orbital elements leading some asteroids to cross the planet trajectories [14]. Hence it is important to extend the analysis to such asteroids.

For the resonant case, the averaging process suffers the presence of small divisors. Hence, the dependence on the fast angles cannot be completely eliminated, and the terms corresponding to the resonant combination of the anomalies still appear in the resonant normal form $\mathcal{H}'$, see (7). We observe that the averaged Hamiltonian $\overline{\mathcal{H}}$ considered in [7] is still present in the expression of $\mathcal{H}'$. However, a new term, denoted by $\mathcal{H}_{res}$, appears in $\mathcal{H}'$. This term is singular at orbit crossings and needs to be studied.

Another difference with the non-resonant case is that the semimajor axis is not constant, and the number of state variables to consider in the equations is 6. In particular, we cannot study the singularities just applying the same techniques.

We will prove that, despite these differences, we can obtain results analogous to [7]. The vector field of the resonant normal form admits two different locally Lipschitz-continuous extensions and we can define a generalized solution, continuous but not differentiable, going beyond the crossing singularities. The long term evolution of the map $\tilde{d}_{min}$ along these generalized solutions is differentiable in a neighborhood of crossings. The analysis of the singularity is performed in two different ways, depending if the crossed planet is the one in mean motion resonance with the asteroid or not. More precisely, inspired by the classification in [9], we consider two classes of orbital behavior for planet crossing resonant asteroids: the *Toro* class, corresponding to asteroids in resonance with the planet whose trajectory is crossed, and the *Alinda* class, whose asteroids are in resonance with a planet different from the crossed one.

The article is organized as follows. In Section 2 we use standard perturbation theory to derive the equations of the long term dynamics for a given mean motion resonance. In Section 3 we recall the definition of the signed orbit distance $\tilde{d}_{min}$. The main result is stated and proved in Section 4. In Section 5 we define the generalized solutions and prove the regularity of the evolution of $\tilde{d}_{min}$. We conclude with some numerical examples in Section 6, showing the agreement between the long term evolution and the full evolution in a statistical sense.

2 The equations for the long term evolution

We consider the differential equations

$$\ddot{\boldsymbol{r}} = -\mathtt{k}^2 \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} + \mathtt{k}^2 \sum_{j=1}^{N-2} \mu_j \left(\frac{\boldsymbol{r}_j - \boldsymbol{r}}{|\boldsymbol{r}_j - \boldsymbol{r}|^3} - \frac{\boldsymbol{r}_j}{|\boldsymbol{r}_j|^3} \right), \tag{1}$$

where $\boldsymbol{r}$ describes, in heliocentric coordinates, the motion of a massless asteroid under the gravitational attraction of the Sun and $N-2$ planets. The heliocentric motions of the planets $\boldsymbol{r}_j = \boldsymbol{r}_j(t)$ are known functions of the time t. Moreover, $\mathtt{k} = \sqrt{\mathcal{G} m_0}$ is Gauss's constant, $\mu_j = m_j/m_0$ with m_0 the mass of the Sun and m_j the mass of the j-th planet. Equations (1) can be written in Hamiltonian form as

$$\dot{\boldsymbol{p}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{r}}, \qquad \dot{\boldsymbol{r}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} = \boldsymbol{p},$$

with Hamiltonian

$$\mathcal{H}(\boldsymbol{p}, \boldsymbol{r}, t) = \frac{|\boldsymbol{p}|^2}{2} - \frac{\mathtt{k}^2}{|\boldsymbol{r}|} - \mathtt{k}^2 \sum_{j=1}^{N-2} \mu_j \left(\frac{1}{d_j(\boldsymbol{r}, t)} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}_j(t)}{|\boldsymbol{r}_j(t)|^3} \right). \tag{2}$$

In (2) $d_j = |\boldsymbol{r}_j - \boldsymbol{r}|$ stands for the distance between the asteroid and the j-th planet. We use Delaunay's elements (L, G, Z, l, g, z) defined by

$$\begin{aligned} L &= \mathtt{k}\sqrt{a}, & \ell &= \mathtt{n}(t - t_0), \\ G &= \mathtt{k}\sqrt{a(1-e^2)}, & g &= \omega, \\ Z &= \mathtt{k}\sqrt{a(1-e^2)}\cos I, & z &= \Omega, \end{aligned}$$

where $a, e, I, \Omega, \omega, t_0$ represent semimajor axis, eccentricity, inclination, longitude of the ascending node, argument of perihelion, and epoch of passage at perihelion. For the definition of ℓ we use the mean motion

$$\mathtt{n} = \frac{\mathtt{k}^4}{L^3}.$$

In these coordinates, the Hamiltonian (2) can be written as

$$\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1,$$

with $\epsilon = \mu_5$,

$$\mathcal{H}_0 = -\frac{\mathtt{k}^4}{2L^2},$$

and

$$\mathcal{H}_1 = \sum_{j=1}^{N-2} \mathcal{H}_1^{(j)}, \qquad \mathcal{H}_1^{(j)} = -\mathtt{k}^2 \frac{\mu_j}{\mu_5} \left(\frac{1}{d_j} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}_j}{|\boldsymbol{r}_j|^3} \right), \tag{3}$$

and $\boldsymbol{r}_j = \boldsymbol{r}_j(t)$. Note that in (3)

$$\mathcal{H}_1 = \mathcal{H}_1(L, G, Z, \ell, g, z, t).$$

To eliminate the dependence on time in $\mathcal{H}_1$ we overextend the phase space. We assume that the planets move on quasi-periodic orbits with three independent frequencies $\mathtt{n}_j, \mathtt{g}_j, \mathtt{s}_j$. This is the case considered by Laplace (see for example [10]), where the mean semi-major axis a_j is constant and the mean value of the mean anomaly ℓ_j grows linearly with time, i.e. up to a phase, $\ell_j = \mathtt{n}_j t$. Here $\mathtt{n}_j$ is the mean motion of planet j. Moreover, every planet is characterized by two more frequencies $\mathtt{g}_j, \mathtt{s}_j$, describing the slow motions of the other mean orbital elements. We introduce the angles

$$\ell_j = \mathtt{n}_j t + \ell_j(0), \quad g_j = \mathtt{g}_j t + g_j(0), \quad z_j = \mathtt{s}_j t + z_j(0)$$

and their conjugate variables L_j, G_j, Z_j.

Note that these variables do not correspond to the Delaunay's elements of planet j, since they are functions of the orbital elements of the asteroid and planet j. We use the following notation:

$$\begin{aligned} \boldsymbol{\ell} &= (\ell, \ell_1 \dots, \ell_N), \quad & \boldsymbol{g} &= (g, g_1 \dots, g_N), \quad & \boldsymbol{z} &= (z, z_1 \dots, z_N), \\ \boldsymbol{\ell}_j &= (\ell, \ell_j), & \boldsymbol{g}_j &= (g, g_j), & \boldsymbol{z}_j &= (z, z_j) \end{aligned}$$

and analogously we define $\boldsymbol{L}, \boldsymbol{G}, \boldsymbol{Z}, \boldsymbol{L}_j, \boldsymbol{G}_j, \boldsymbol{Z}_j$.

The dynamics in this overextended phase space is determined by the autonomous Hamiltonian

$$\tilde{\mathcal{H}} = -\frac{\mathtt{k}^4}{2L^2} + \sum_{j=1}^{N-2} (\mathtt{n}_j L_j + \mathtt{g}_j G_j + \mathtt{s}_j Z_j) + \epsilon \tilde{\mathcal{H}}_1(L, G, Z, \boldsymbol{\ell}, \boldsymbol{g}, \boldsymbol{z}),$$

where

$$\tilde{\mathcal{H}}_1 = \sum_{j=1}^{N-2} \tilde{\mathcal{H}}_1^{(j)}, \qquad \tilde{\mathcal{H}}_1^{(j)} = -\mathtt{k}^2 \frac{\mu_j}{\mu_5} \left(\frac{1}{\tilde{d}_j} - \frac{\boldsymbol{r} \cdot \tilde{\boldsymbol{r}}_j}{|\tilde{\boldsymbol{r}}_j|^3} \right),$$

with

$$\tilde{\boldsymbol{r}}_j = \tilde{\boldsymbol{r}}_j(\ell_j, g_j, z_j), \qquad \tilde{d}_j = |\tilde{\boldsymbol{r}}_j - \boldsymbol{r}|.$$

Here we are assuming that $\boldsymbol{r}_j$ evolves according to Laplace's solution for the planetary motions, and we write it as a function of its frequencies, denoted by $\tilde{\boldsymbol{r}}_j$. Hereafter we shall omit

the symbol 'tilde', to simplify the notation.
The frequencies $\mathtt{g}_j$ and $\mathtt{s}_j$ are of order ϵ [10]. In order to study the secular dynamics, we would like to eliminate all the frequencies corresponding to the fast angles $\boldsymbol{\ell}$. In case of a mean motion resonance with a planet this is not possible.

In the following we shall assume that there is only one mean motion resonance with a planet and no close approaches occur. To expose our result we shall consider a $|h_5^*| : |h^*|$ mean motion resonance with Jupiter given by

$$h^*\mathtt{n} + h_5^*\mathtt{n}_5 = 0 \quad \text{for some } (h^*, h_5^*) \in \mathbb{Z}^2. \tag{4}$$

A mean motion resonance with another planet can be treated in a similar way. We denote by

$$\boldsymbol{\varphi} = (\boldsymbol{\ell}, \boldsymbol{g}, \boldsymbol{z}), \quad \boldsymbol{\varphi}_j = (\boldsymbol{\ell}_j, \boldsymbol{g}_j, \boldsymbol{z}_j)$$

the vectors of the angles and by

$$\mathbf{I} = (\boldsymbol{L}, \boldsymbol{G}, \boldsymbol{Z}), \quad \mathbf{I}_j = (\boldsymbol{L}_j, \boldsymbol{G}_j, \boldsymbol{Z}_j)$$

the corresponding vectors of the actions.
We use the Lie method [10] to search for a suitable canonical transformation close to the identity, that is we search for a function $\chi = \chi(\mathbf{I}', \boldsymbol{\varphi}')$ such that the inverse transformation is

$$\Phi_\chi^\epsilon(\mathbf{I}', \boldsymbol{\varphi}') = (\mathbf{I}, \boldsymbol{\varphi}),$$

where Φ_χ^t is the Hamiltonian flow associated to χ. The function χ is selected so that the transformed Hamiltonian $\mathcal{H}' = \mathcal{H} \circ \Phi_\chi^\epsilon$ depends, at least at first order, on as less fast angular variables as possible. Using a formal expansion in ϵ we have

$$\mathcal{H}' = \mathcal{H} \circ \Phi_\chi^\epsilon = \mathcal{H} + \epsilon\{\mathcal{H}, \chi\} + O(\epsilon^2) = \mathcal{H}_0 + \epsilon(\mathcal{H}_1 + \{\mathcal{H}_0, \chi\}) + O(\epsilon^2).$$

In the resonant case we search for a solution χ of the equation

$$\mathcal{H}_1 + \{\mathcal{H}_0, \chi\} = f \tag{5}$$

for some function $f = f(\mathbf{I}', h^*\ell' + h_5^*\ell_5', \boldsymbol{g}', \boldsymbol{z}')$. To solve (5) we restrict to the case where no orbit crossings with the planets occur. We shall see in the next sections how we can deal with the case of crossings.

We develop

$$\mathcal{H}_1 = \sum_{j=1}^{N-2} \mathcal{H}_1^{(j)}$$

in Fourier's series of the fast angles:

$$\mathcal{H}_1^{(j)} = \sum_{(h,h_j)\in\mathbb{Z}^2} \widehat{\mathcal{H}}_{(h,h_j)}^{(j)} e^{i(h\ell + h_j\ell_j)}.$$

Here

$$\widehat{\mathcal{H}}_{(h,h_j)}^{(j)} = \widehat{\mathcal{H}}_{(h,h_j)}^{(j)}(L, G, Z, \boldsymbol{g}_j, \boldsymbol{z}_j) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \mathcal{H}_1^j e^{-i(h\ell + h_j\ell_j)} d\ell d\ell_j \tag{6}$$

are the Fourier coefficients. We observe that $\widehat{\mathcal{H}}^{(j)}_{(h,h_j)}$ are defined also in case of orbit crossings, since the integral in (6) converges (see e.g. [7]).
Moreover, we can write χ as

$$\chi = \sum_{j=1}^{N-2} \chi^{(j)}, \qquad \chi^{(j)} = \chi^{(j)}(L', G', Z', \boldsymbol{\ell}'_j, \boldsymbol{g}'_j, \boldsymbol{z}'_j)$$

and search for the coefficients

$$\widehat{\chi}^{(j)}_{(h,h_j)} = \widehat{\chi}^{(j)}_{(h,h_j)}(L', G', Z', \boldsymbol{g}'_j, \boldsymbol{z}'_j)$$

in the Fourier series development

$$\chi^{(j)} = \sum_{(h,h_j)\in\mathbb{Z}^2} \widehat{\chi}^{(j)}_{(h,h_j)} e^{i(h\ell' + h_j\ell'_j)}.$$

Inserting these Fourier developments into (5) we obtain

$$\mathcal{H}_1 + \{\mathcal{H}_0, \chi\} = \sum_{j=1}^{N-2} \Big(\mathcal{H}_1^{(j)} - \frac{\partial \mathcal{H}_0}{\partial \mathbf{I}} \cdot \frac{\partial \chi^{(j)}}{\partial \boldsymbol{\varphi}} \Big),$$

where

$$\mathcal{H}_1^{(j)} - \frac{\partial \mathcal{H}_0}{\partial \mathbf{I}} \cdot \frac{\partial \chi^{(j)}}{\partial \boldsymbol{\varphi}} = \sum_{(h,h_j)\in\mathbb{Z}^2} \big[\widehat{\mathcal{H}}^{(j)}_{(h,h_j)} - i(h\mathtt{n} + h_j\mathtt{n}_j)\widehat{\chi}^{(j)}_{(h,h_j)} \big] e^{i(h\ell' + h_j\ell'_j)}.$$

This expression suggests to choose the function f in (5) in the following form:

$$f = \sum_{j=1}^{N-2} f_j,$$

where $f_5 = f_5(\mathbf{I}'_5, h^*\ell' + h^*_5\ell'_5, \boldsymbol{g}'_5, \boldsymbol{z}'_5)$ and $f_j = f_j(\mathbf{I}'_j, \boldsymbol{g}'_j, \boldsymbol{z}'_j)$ for $j \neq 5$. This can be accomplished by choosing

$$\widehat{\chi}^{(j)}_{(h,h_j)} = \frac{\widehat{\mathcal{H}}^{(j)}_{(h,h_j)}}{i(h\mathtt{n} + h_j\mathtt{n}_j)}$$

when the denominator does not vanish. Hence, we exclude the case $(h, h_j) = (0, 0)$ and the resonant case $(h, h_5) = n(h^*, h^*_5)$ for some $n \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, for which we assume that the corresponding Fourier coefficient of χ vanishes. With this choice we have

$$\begin{aligned} f_5 &= \widehat{\mathcal{H}}^{(5)}_{(0,0)} + \sum_{n\in\mathbb{Z}^*} \widehat{\mathcal{H}}^{(5)}_{n(h^*,h^*_5)} e^{in(h^*\ell' + h^*_5\ell'_5)}, \\ f_j &= \widehat{\mathcal{H}}^{(j)}_{(0,0)} \qquad \text{for } j \neq 5. \end{aligned}$$

The new Hamiltonian is

$$\mathcal{H}' = \mathcal{H}_0 + \epsilon(\overline{\mathcal{H}}_1 + \mathcal{H}_{res}) + O(\epsilon^2), \tag{7}$$

where, denoting by $\Re(z)$ the real part of $z \in \mathbb{C}$ and considering only a finite number n_{max} of harmonics, we have

$$\overline{\mathcal{H}}_1 = \sum_{j=0}^{N-2} \widehat{\mathcal{H}}^{(j)}_{(0,0)}, \qquad \mathcal{H}_{res} = 2\Re\left(\sum_{n=0}^{n_{max}} \widehat{\mathcal{H}}^{(5)}_{n(h^*,h_5^*)} e^{in(h^*\ell'+h_5^*\ell_5')}\right)$$

because $\overline{\widehat{\mathcal{H}}^{(5)}_{(h,h_5)}} = \widehat{\mathcal{H}}^{(5)}_{(-h,-h_5)}$. Moreover, it is easy to see that

$$\begin{aligned}\widehat{\mathcal{H}}^{(j)}_{(0,0)} &= \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \mathcal{H}_1^{(j)} d\ell d\ell_j = -\frac{\mathtt{k}^2\mu_j}{(2\pi)^2\mu_5}\int_{\mathbb{T}^2}\left(\frac{1}{d_j} - \frac{\boldsymbol{r}\cdot\boldsymbol{r}_j}{|\boldsymbol{r}_j|^3}\right) d\ell d\ell_j = \\ &= -\frac{\mathtt{k}^2\mu_j}{(2\pi)^2\mu_5}\int_{\mathbb{T}^2}\frac{1}{d_j} d\ell d\ell_j,\end{aligned}$$

being null the average of the indirect perturbation (see [3]). We observe that in the coefficient $\widehat{\mathcal{H}}^{(5)}_{n(h^*,h_5^*)}$ the term corresponding to the indirect perturbation does not vanish. We can write, dropping the apexes,

$$\begin{aligned}\overline{\mathcal{H}}_1 &= \sum_{j=0}^{N-2} \frac{C_j}{(2\pi)^2}\int_{\mathbb{T}^2}\frac{1}{d_j} d\ell d\ell_j, \\ \mathcal{H}_{res} &= \frac{2C_5}{(2\pi)^2}\sum_{n=0}^{n_{max}}\left[I_5^{c,n}\cos n(h^*\ell + h_5^*\ell_5) + I_5^{s,n}\sin n(h^*\ell + h_5^*\ell_5)\right],\end{aligned}$$

where

$$\begin{aligned}C_j &= -\frac{\mathtt{k}^2\mu_j}{\mu_5} = -\frac{\mathtt{k}^2 m_j}{m_5}, \\ I_5^{c,n} &= \int_{\mathbb{T}^2}\left(\frac{1}{d_5} - \frac{\boldsymbol{r}\cdot\boldsymbol{r}_5}{|\boldsymbol{r}_5|^3}\right)\cos n(h^*\ell + h_5^*\ell_5) d\ell d\ell_5, \\ I_5^{s,n} &= \int_{\mathbb{T}^2}\left(\frac{1}{d_5} - \frac{\boldsymbol{r}\cdot\boldsymbol{r}_5}{|\boldsymbol{r}_5|^3}\right)\sin n(h^*\ell + h_5^*\ell_5) d\ell d\ell_5.\end{aligned}$$

Moreover, since the new Hamiltonian does not depend on ℓ_j for $j \neq 5$ we have

$$\mathcal{H}_0(L, L_5, G_1, \ldots, G_N, Z_1, \ldots, Z_N) = -\frac{\mathtt{k}^4}{2L^2} + \mathtt{n}_5 L_5 + \sum_{j=1}^{N-2}(\mathtt{g}_j G_j + \mathtt{s}_j Z_j).$$

We now introduce the resonant angle σ through the canonical transformation

$$\begin{pmatrix}\sigma \\ \sigma_5\end{pmatrix} = A\begin{pmatrix}\ell \\ \ell_5\end{pmatrix}, \quad \begin{pmatrix}S \\ S_5\end{pmatrix} = A^{-T}\begin{pmatrix}L \\ L_5\end{pmatrix},$$

with

$$A = \begin{pmatrix} h^* & h_5^* \\ 0 & 1/h^*\end{pmatrix}.$$

Since $\ell, \ell_5 \in \mathbb{R}/(2\pi)$ and the entries of the matrix A are rational, the transformation is well defined taking

$$\sigma \in \mathbb{R}/(2\pi), \quad \sigma_5 \in \mathbb{R}/(2\pi/h^*).$$

With this choice, the inverse transformation could be multivalued. This fact will not affect our analysis, since our aim is to filter out the angles (ℓ, ℓ_5).
In these new variables, neglecting the terms $O(\epsilon^2)$, the Hamiltonian is transformed into

$$\mathcal{H} = \mathcal{H}_0 + \epsilon(\overline{\mathcal{H}}_1 + \mathcal{H}_{res}), \tag{8}$$

in which, dropping the apexes,

$$\begin{aligned}
\mathcal{H}_0(S, S_5, G_1, \ldots, G_N, Z_1, \ldots, Z_N) &= -\frac{\mathtt{k}^4}{2(h^*S)^2} + \mathtt{n}_5(h_5^*S + S_5/h^*) + \sum_{j=1}^{N-2}(\mathtt{g}_j G_j + \mathtt{s}_j Z_j),\\
\mathcal{H}_{res}(S, G, Z, \sigma, \boldsymbol{g}_5, \boldsymbol{z}_5) &= \frac{2C_5}{(2\pi)^2}\sum_{n=0}^{n_{max}}(I_5^{c,n}\cos n\sigma + I_5^{s,n}\sin n\sigma),\\
\overline{\mathcal{H}}_1(S, G, Z, \boldsymbol{g}, \boldsymbol{z}) &= \sum_{j=1}^{N-2}\frac{C_j}{(2\pi)^2}\int_{\mathbb{T}^2}\frac{1}{d_j(\ell, \ell_j)}d\ell d\ell_j.
\end{aligned}$$

Since the Hamiltonian does not depend on σ_5, the value of S_5 will remain constant and we will treat it as a parameter. Calling $\mathcal{Y} = (S, G, Z, \sigma, g, z)$ we consider the equations for the motion of the asteroid given by

$$\dot{\mathcal{Y}} = \mathbb{J}_3 \nabla_{\mathcal{Y}} \mathcal{H}, \tag{9}$$

where

$$\mathbb{J}_3 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

is the symplectic identity of order 6. In components, system (9) is written as

$$\begin{aligned}
\dot{S} &= -\frac{\partial \mathcal{H}}{\partial \sigma} = -\epsilon\frac{\partial \mathcal{H}_{res}}{\partial \sigma},\\
\dot{\sigma} &= \frac{\partial \mathcal{H}}{\partial S} = \frac{h^*\mathtt{k}^4}{(h^*S)^3} + \mathtt{n}_5 h_5^* + \epsilon\left(\frac{\partial \mathcal{H}_{res}}{\partial S} + \frac{\partial \overline{\mathcal{H}}_1}{\partial S}\right),\\
\dot{G} &= -\frac{\partial \mathcal{H}}{\partial g} = -\epsilon\left(\frac{\partial \mathcal{H}_{res}}{\partial g} + \frac{\partial \overline{\mathcal{H}}_1}{\partial g}\right),\\
\dot{g} &= \frac{\partial \mathcal{H}}{\partial G} = \epsilon\left(\frac{\partial \mathcal{H}_{res}}{\partial G} + \frac{\partial \overline{\mathcal{H}}_1}{\partial G}\right),\\
\dot{Z} &= -\frac{\partial \mathcal{H}}{\partial z} = -\epsilon\left(\frac{\partial \mathcal{H}_{res}}{\partial z} + \frac{\partial \overline{\mathcal{H}}_1}{\partial z}\right),\\
\dot{z} &= \frac{\partial \mathcal{H}}{\partial Z} = \epsilon\left(\frac{\partial \mathcal{H}_{res}}{\partial Z} + \frac{\partial \overline{\mathcal{H}}_1}{\partial Z}\right).
\end{aligned}$$

where $\mathcal{H}_{res}$, $\overline{\mathcal{H}}_1$ are functions of $(S, G, Z, \sigma, \boldsymbol{g}_5, \boldsymbol{z}_5)$ and $(S, G, Z, \boldsymbol{g}, \boldsymbol{z})$ respectively. Since $\epsilon C_j = -\mathtt{k}^2 \mu_j$, we get

$$
\begin{aligned}
\dot{S} &= \frac{\mathtt{k}^2}{(2\pi)^2} 2\mu_5 \sum_{n=0}^{n_{max}} n \left(I_5^{s,n} \cos n\sigma - I_5^{c,n} \sin n\sigma \right), \\
\dot{\sigma} &= \frac{h^* \mathtt{k}^4}{(h^* S)^3} + \mathtt{n}_5 h_5^* \\
&\quad - \frac{\mathtt{k}^2}{(2\pi)^2} \Big\{ 2\mu_5 \sum_{n=0}^{n_{max}} \Big(\frac{\partial I_5^{c,n}}{\partial S} \cos n\sigma + \frac{\partial I_5^{s,n}}{\partial S} \sin n\sigma \Big) + \sum_{j=1}^{N-2} \mu_j \frac{\partial}{\partial S} \int_{\mathbb{T}^2} \frac{1}{d_j} d\ell d\ell_j \Big\}, \\
\dot{G} &= \frac{\mathtt{k}^2}{(2\pi)^2} \Big\{ 2\mu_5 \sum_{n=0}^{n_{max}} \Big(\frac{\partial I_5^{c,n}}{\partial g} \cos n\sigma + \frac{\partial I_5^{s,n}}{\partial g} \sin n\sigma \Big) + \sum_{j=1}^{N-2} \mu_j \frac{\partial}{\partial g} \int_{\mathbb{T}^2} \frac{1}{d_j} d\ell d\ell_j \Big\}, \\
\dot{g} &= -\frac{\mathtt{k}^2}{(2\pi)^2} \Big\{ 2\mu_5 \sum_{n=0}^{n_{max}} \Big(\frac{\partial I_5^{c,n}}{\partial G} \cos n\sigma + \frac{\partial I_5^{s,n}}{\partial G} \sin n\sigma \Big) + \sum_{j=1}^{N-2} \mu_j \frac{\partial}{\partial G} \int_{\mathbb{T}^2} \frac{1}{d_j} d\ell d\ell_j \Big\}, \\
\dot{Z} &= \frac{\mathtt{k}^2}{(2\pi)^2} \Big\{ 2\mu_5 \sum_{n=0}^{n_{max}} \Big(\frac{\partial I_5^{c,n}}{\partial z} \cos n\sigma + \frac{\partial I_5^{s,n}}{\partial z} \sin n\sigma \Big) + \sum_{j=1}^{N-2} \mu_j \frac{\partial}{\partial z} \int_{\mathbb{T}^2} \frac{1}{d_j} d\ell d\ell_j \Big\}, \\
\dot{z} &= -\frac{\mathtt{k}^2}{(2\pi)^2} \Big\{ 2\mu_5 \sum_{n=0}^{n_{max}} \Big(\frac{\partial I_5^{c,n}}{\partial Z} \cos n\sigma + \frac{\partial I_5^{s,n}}{\partial Z} \sin n\sigma \Big) + \sum_{j=1}^{N-2} \mu_j \frac{\partial}{\partial Z} \int_{\mathbb{T}^2} \frac{1}{d_j} d\ell d\ell_j \Big\}.
\end{aligned}
$$

The derivatives of $\mathcal{H}_{res}$ and $\overline{\mathcal{H}}_1$ are not defined at orbit crossings with the planets. In the following Sections we shall discuss how system (9) can be used also in case of orbit crossings.

3 The orbit distance

We recall here some facts and notations from [6], [7]. Let (E, v), (E', v') be two sets of orbital elements, where E, E' describe the trajectories of the asteroid and one planet, v, v' describe the position of these bodies along them. We denote by μ' the ratio of the mass of this planet to the mass of the Sun. We also introduce the two-orbit configuration $\mathcal{E} = (E, E')$ and the vector of parameters along the orbits $V = (v, v')$. We denote by $\mathcal{X} = \mathcal{X}(E, v)$ and $\mathcal{X}' = \mathcal{X}'(E', v')$ the Cartesian coordinates of the asteroid and the planet respectively. For every given $\mathcal{E}$, we denote by $V_h(\mathcal{E})$ a local minimum point of the function

$$V \mapsto d^2(\mathcal{E}, V) = |\mathcal{X}(E, v) - \mathcal{X}'(E', v')|^2.$$

We introduce the local minimum maps

$$\mathcal{E} \mapsto d_h(\mathcal{E}) = d(\mathcal{E}, V_h),$$

and the orbit distance

$$\mathcal{E} \mapsto d_{min}(\mathcal{E}) = \min_h d_h(\mathcal{E}).$$

We shall consider non-degenerate configurations $\mathcal{E}$, i.e such that all the critical points of the map $V \mapsto d(\mathcal{E}, V)$ are non-degenerate. In this way, we can always choose a neighborhood $\mathcal{W}$ of $\mathcal{E}$ where the maps d_h do not have bifurcations. A crossing configuration is a two-orbit configuration $\mathcal{E}_c$ such that $d(\mathcal{E}_c, V_h(\mathcal{E}_c)) = 0$ where $V_h(\mathcal{E}_c)$ is the corresponding minimum point. The maps d_h and d_{min} are singular at crossing configurations, and their derivatives in general do not exists. Anyway, it is possible to obtain analytic maps in a neighborhood of a crossing configuration $\mathcal{E}_c$ by suitably choosing a sign for these maps. We summarize here the procedure to deal with this singularity for d_h; the procedure for d_{min} is the same. Let $V_h = (v_h, v'_h)$ be a local minimum point of d^2 and let $\mathcal{X}_h = \mathcal{X}_h(E, v_h)$ and $\mathcal{X}'_h = \mathcal{X}'_h(E', v'_h)$. We introduce the vectors tangent to the trajectories defined by E, E' at these points

$$\tau_h = \frac{\partial \mathcal{X}}{\partial v}(E, v_h), \quad \tau'_h = \frac{\partial \mathcal{X}'}{\partial v'}(E', v'_h)$$

and their cross product $\tau^*_h = \tau'_h \times \tau_h$. Both vectors τ_h, τ'_h are orthogonal to $\Delta_h = \mathcal{X}'_h - \mathcal{X}_h$, so that τ^*_h is parallel to Δ_h, see Figure 1.

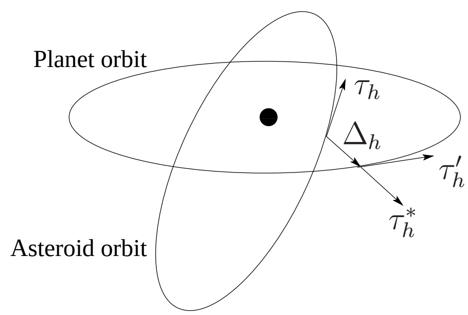

Figure 1: The vectors τ^*_h, Δ_h.

Denoting by $\hat{\tau}^*_h$, $\hat{\Delta}_h$ the corresponding unit vectors, we consider the local minimal distance with sign

$$\tilde{d}_h = (\hat{\tau}^*_h \cdot \hat{\Delta}_h) d_h.$$

This map is analytic in a neighborhood of most crossing configurations. Actually, this smoothing procedure fails in case the vectors τ_h, τ'_h are parallel.
Finally, given a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$ without bifurcations of d_h, we write $\mathcal{W} = \mathcal{W}^- \cup \Sigma \cup \mathcal{W}^+$, where

$$\Sigma = \mathcal{W} \cap \{\tilde{d}_h(\mathcal{E}) = 0\}, \quad \mathcal{W}^+ = \mathcal{W} \cap \{\tilde{d}_h(\mathcal{E}) > 0\}, \quad \mathcal{W}^- = \mathcal{W} \cap \{\tilde{d}_h(\mathcal{E}) < 0\}.$$

4 Extraction of the singularities

Let $\mathcal{E}_c$ be a crossing configuration and suppose that the trajectories are described by the element $E = (S, G, Z, g, z)$; denote by y_i any component of the vector E. We choose the mean anomalies as parameters along the trajectory so that $V = (\ell, \ell')$. The first step of our analysis is to consider, for each $\mathcal{E}$ in a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$, the Taylor expansion of $V \mapsto d(\mathcal{E}, V)$ in a neighborhood of $V_h = V_h(\mathcal{E})$, i.e.

$$d^2(\mathcal{E}, V) = d_h^2(\mathcal{E}) + (V - V_h) \cdot \mathcal{A}_h(V - V_h) + \mathcal{R}_3^{(h)}(\mathcal{E}, V),$$

and define the approximated distance

$$\delta_h(\mathcal{E}, V) = \sqrt{d_h^2(\mathcal{E}) + (V - V_h) \cdot \mathcal{A}_h(V - V_h)}, \tag{10}$$

where

$$\mathcal{A}_h = \begin{bmatrix} |\tau_h|^2 + \frac{\partial^2 \mathcal{X}}{\partial v^2}(E, v_h) \cdot \Delta_h & -\tau_h \cdot \tau_h' \\ -\tau_h \cdot \tau_h' & |\tau_h'|^2 + \frac{\partial^2 \mathcal{X}'}{\partial v'^2}(E', v_h') \cdot \Delta_h \end{bmatrix}$$

is positive definite except for tangent crossings.

To study the crossing singularities we distinguish between the case where the asteroid trajectory crosses the trajectory of Jupiter itself and the case where it crosses the trajectory of another planet. In the second case the crossing singularity appears only in the average terms $\frac{\partial \overline{\mathcal{H}}_1}{\partial y_i}$, so that the component $\frac{\partial H}{\partial \sigma}$ is regular. In the first case the terms $I_5^{s,n}$ and $I_5^{c,n}$ are continuous but their derivatives $\frac{\partial I_5^{s,n}}{\partial y_i}$, $\frac{\partial I_5^{c,n}}{\partial y_i}$ can be singular.

We obtain the following results.

Theorem 1. *Let $\mathcal{E}_c$ be a non-degenerate crossing configuration with a planet. Then, there exists a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$ such that*

(a) for each $i = 1 \dots 5$ the maps

$$\mathcal{W}^{\pm} \ni \mathcal{E} \mapsto \frac{\mu' \mathtt{k}^2}{(2\pi)^2} \frac{\partial}{\partial y_i} \int_{\mathbb{T}^2} \frac{1}{d(\mathcal{E}, V)} dV,$$

corresponding to the derivative of $\overline{\mathcal{H}}_1$ with respect to y_i, can be extended to two Lipschitz-continuous maps

$$\mathcal{E} \mapsto \epsilon \left(\frac{\partial \overline{\mathcal{H}}_1}{\partial y_i} \right)_h^{\pm} (\mathcal{E})$$

defined on $\mathcal{W}$;

(b) the following relation holds in $\mathcal{W}$:

$$\epsilon \left(\frac{\partial \overline{\mathcal{H}}_1}{\partial y_i} \right)_h^{-} - \epsilon \left(\frac{\partial \overline{\mathcal{H}}_1}{\partial y_i} \right)_h^{+} = -\frac{\mu' \mathtt{k}^2}{\pi} \left[\frac{\partial}{\partial y_i} \left(\frac{1}{\sqrt{\det(\mathcal{A}_h)}} \right) \tilde{d}_h + \frac{1}{\sqrt{\det(\mathcal{A}_h)}} \frac{\partial \tilde{d}_h}{\partial y_i} \right].$$

Proof. We can show this result by following the same steps as in [7, Theorem 4.2], replacing R by $-\epsilon\overline{\mathcal{H}}_1$.

$\square$

Theorem 2. *Let $\mathsf{h} = (h^*, h_5^*)$ and $\mathcal{E}_c$ be a non-degenerate crossing configuration with Jupiter. Then, there exists a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$ such that, for every $n > 0$,*

(a) *for each $i = 1 \dots 5$ the maps*

$$\mathcal{W}^{\pm} \ni \mathcal{E} \mapsto \frac{\partial}{\partial y_i} \int_{\mathbb{T}^2} \left(\frac{1}{d(\mathcal{E}, V)} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}_5}{|\boldsymbol{r}_5|^3} \right) \cos(n\mathsf{h} \cdot V) dV, \tag{11}$$

$$\mathcal{W}^{\pm} \ni \mathcal{E} \mapsto \frac{\partial}{\partial y_i} \int_{\mathbb{T}^2} \left(\frac{1}{d(\mathcal{E}, V)} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}_5}{|\boldsymbol{r}_5|^3} \right) \sin(n\mathsf{h} \cdot V) dV, \tag{12}$$

corresponding to the derivatives of $I_5^{c,n}$, $I_5^{s,n}$ with respect to y_i, can be extended to Lipschitz-continuous maps

$$\mathcal{E} \mapsto \left(\frac{\partial I_5^{c,n}}{\partial y_i} \right)_h^{\pm} (\mathcal{E}), \qquad \mathcal{E} \mapsto \left(\frac{\partial I_5^{s,n}}{\partial y_i} \right)_h^{\pm} (\mathcal{E})$$

defined on $\mathcal{W}$;

(b) *the following relations hold in $\mathcal{W}$:*

$$\left(\frac{\partial I_5^{c,n}}{\partial y_i} \right)_h^{-} - \left(\frac{\partial I_5^{c,n}}{\partial y_i} \right)_h^{+} = 4\pi \cos(n\mathsf{h} \cdot V_h) \left[\frac{\partial}{\partial y_i} \left(\frac{1}{\sqrt{\det(\mathcal{A}_h)}} \right) \tilde{d}_h + \frac{1}{\sqrt{\det(\mathcal{A}_h)}} \frac{\partial \tilde{d}_h}{\partial y_i} \right],$$

$$\left(\frac{\partial I_5^{s,n}}{\partial y_i} \right)_h^{-} - \left(\frac{\partial I_5^{s,n}}{\partial y_i} \right)_h^{+} = 4\pi \sin(n\mathsf{h} \cdot V_h) \left[\frac{\partial}{\partial y_i} \left(\frac{1}{\sqrt{\det(\mathcal{A}_h)}} \right) \tilde{d}_h + \frac{1}{\sqrt{\det(\mathcal{A}_h)}} \frac{\partial \tilde{d}_h}{\partial y_i} \right].$$

Before giving a proof of Theorem 2 we state some consequences, which will be relevant in the following.

We define the following locally Lipschitz-continuous maps, extending the vector field in (9) in a neighborhood of the crossing singularity,

$$\mathcal{W} \times S^1 \ni (\mathcal{E}, \sigma) \mapsto \left(\frac{\partial \mathcal{H}}{\partial y_i} \right)_h^{\pm} (\mathcal{E}, \sigma) := \frac{\partial \mathcal{H}_0}{\partial y_i}(\mathcal{E}) + \epsilon \left(\frac{\partial \overline{\mathcal{H}}_1}{\partial y_i} \right)_h^{\pm} (\mathcal{E}) + \epsilon \left(\frac{\partial \mathcal{H}_{res}}{\partial y_i} \right)_h^{\pm} (\mathcal{E}, \sigma),$$

with $\mathcal{H}_0$, $\mathcal{H}_{res}$ as in (8), and

$$\epsilon \left(\frac{\partial \mathcal{H}_{res}}{\partial y_i} \right)_h^{\pm} (\mathcal{E}, \sigma) = -\frac{2\mu_5 k^2}{(2\pi)^2} \sum_{n=0}^{n_{max}} \left(\left(\frac{\partial I_5^{c,n}}{\partial y_i} \right)_h^{\pm} (\mathcal{E}) \cos(n\sigma) + \left(\frac{\partial I_5^{s,n}}{\partial y_i} \right)_h^{\pm} (\mathcal{E}) \sin(n\sigma) \right).$$

Moreover, we consider the map

$$\mathcal{W} \times S^1 \ni (\mathcal{E}, \sigma) \mapsto \mathrm{Diff}_h \left(\frac{\partial \mathcal{H}}{\partial y_i} \right) (\mathcal{E}, \sigma) := \left(\frac{\partial \mathcal{H}}{\partial y_i} \right)_h^{-} (\mathcal{E}, \sigma) - \left(\frac{\partial \mathcal{H}}{\partial y_i} \right)_h^{+} (\mathcal{E}, \sigma).$$

Corollary 1. *If $\mathcal{E}_c$ corresponds to a crossing configuration with a planet different from Jupiter, then the following relation holds in $\mathcal{W}$:*

$$\begin{aligned}\mathrm{Diff}_h\left(\frac{\partial \mathcal{H}}{\partial y_i}\right) &= \epsilon\left(\frac{\partial \overline{\mathcal{H}}_1}{\partial y_i}\right)_h^- - \epsilon\left(\frac{\partial \overline{\mathcal{H}}_1}{\partial y_i}\right)_h^+ \\ &= -\frac{\mu' \mathtt{k}^2}{\pi}\left[\frac{\partial}{\partial y_i}\left(\frac{1}{\sqrt{\det(\mathcal{A}_h)}}\right)\tilde{d}_h + \frac{1}{\sqrt{\det(\mathcal{A}_h)}}\frac{\partial \tilde{d}_h}{\partial y_i}\right].\end{aligned}$$

Corollary 2. *If $\mathcal{E}_c$ corresponds to a crossing configuration with Jupiter, then the following relation holds in $\mathcal{W}$*

$$\begin{aligned}\mathrm{Diff}_h\left(\frac{\partial \mathcal{H}}{\partial y_i}\right) &= \epsilon\left(\frac{\partial \overline{\mathcal{H}}_1}{\partial y_i}\right)_h^- - \epsilon\left(\frac{\partial \overline{\mathcal{H}}_1}{\partial y_i}\right)_h^+ + \epsilon\left(\frac{\partial \mathcal{H}_{res}}{\partial y_i}\right)_h^- - \epsilon\left(\frac{\partial \mathcal{H}_{res}}{\partial y_i}\right)_h^+ = \\ -\frac{2\mu' \mathtt{k}^2}{\pi}&\left[\sum_{n=0}^{n_{max}}\cos\big(n(\sigma - \mathsf{h}\cdot V_h)\big) + \frac{1}{2}\right]\left[\frac{\partial}{\partial y_i}\left(\frac{1}{\sqrt{\det(\mathcal{A}_h)}}\right)\tilde{d}_h + \frac{1}{\sqrt{\det(\mathcal{A}_h)}}\frac{\partial \tilde{d}_h}{\partial y_i}\right].\end{aligned}$$

Remark 1. *Theorem 2-(a) yields that the component $\frac{\partial H}{\partial \sigma}$ is locally Lipschitz-continuous. In the following, with the notation $\left(\frac{\partial H}{\partial \sigma}\right)_h^{\pm}$ we shall mean $\frac{\partial H}{\partial \sigma}$, and $\mathrm{Diff}_h\left(\frac{\partial \mathcal{H}}{\partial \sigma}\right) = 0$.*

4.1 Proof of Theorem 2-(a)

We shall prove the result only for the maps (11), the proof for (12) being similar. Since we assume that Jupiter cannot collide with the Sun, the term $\mathbf{r}_5$ will never vanish, so that we study only the derivatives

$$\frac{\partial}{\partial y_i}\int_{\mathbb{T}^2}\frac{1}{d(\mathcal{E}, V)}\cos(n\mathsf{h}\cdot V)dV$$

for a fixed value of $n \in \mathbb{N}$. We shall refer to some estimates and results proved in [7]. For the reader convenience we gather them in Appendix A.

Let $\mathcal{E}_c$ be a non-degenerate crossing configuration. Let us choose two neighborhoods $\mathcal{W}$ of $\mathcal{E}_c$ and $\mathcal{U}$ of $(\mathcal{E}_c, V_h(\mathcal{E}_c))$, as in Lemma 1 in the Appendix. We get the result by proving that the function

$$F(\mathcal{E}) = \int_{\mathbb{T}^2}\cos(n\mathsf{h}\cdot V)\frac{\partial}{\partial y_i \partial y_j}\frac{1}{d(\mathcal{E}, V)}dV$$

is uniformly bounded in $\mathcal{W}\setminus\Sigma$. To investigate the crossing singularity we can restrict the integral in the definition of $F(\mathcal{E})$ to the set

$$\mathcal{D} = \{V \in \mathbb{T}^2 :\ (V - V_h)\cdot\mathcal{A}_h(V - V_h) \le r^2\}$$

for some $r > 0$. Let us consider the Taylor expansion

$$\cos(n\mathsf{h} \cdot V) = \cos(n\mathsf{h} \cdot V_h) - n \sin(n\mathsf{h} \cdot V_h)\mathsf{h} \cdot (V - V_h) + \mathcal{R}_2^{(h)},$$

where

$$\mathcal{R}_2^{(h)} = \mathcal{R}_2^{(h)}(\mathcal{E}, V)$$

is the remainder in integral form, so that in $\mathcal{U}$ we have

$$|\mathcal{R}_2^{(h)}| \leq C|V - V_h|^2 \tag{13}$$

for some $C > 0$. Using the approximated distance δ_h defined in (10) we can write $F(\mathcal{E})$ as sum of four terms:

$$F = F_1 + F_2 + F_3 + F_4,$$

where

$$\begin{aligned}
F_1 &= \cos(n\mathsf{h} \cdot V_h) \int_{\mathcal{D}} \frac{\partial^2}{\partial y_i \partial y_j} \frac{1}{d(\mathcal{E}, V)} dV, \\
F_2 &= -n \sin(n\mathsf{h} \cdot V_h) \int_{\mathcal{D}} \mathsf{h} \cdot (V - V_h) \frac{\partial^2}{\partial y_i \partial y_j} \left(\frac{1}{d(\mathcal{E}, V)} - \frac{1}{\delta_h(\mathcal{E})} \right) dV, \\
F_3 &= -n \sin(n\mathsf{h} \cdot V_h) \int_{\mathcal{D}} \mathsf{h} \cdot (V - V_h) \frac{\partial^2}{\partial y_i \partial y_j} \frac{1}{\delta_h(\mathcal{E})} dV, \\
F_4 &= \int_{\mathcal{D}} \mathcal{R}_2^{(h)} \frac{\partial}{\partial y_i \partial y_j} \frac{1}{d(\mathcal{E}, V)} dV.
\end{aligned}$$

We prove that each term F_i is bounded by a constant independent on $\mathcal{E}$. In the following we will denote by C_k, $k = 1, \dots, 10$, some positive constants, not necessarily corresponding to the ones in Appendix A.
The boundedness of F_1 comes trivially from (19).

From the relation

$$\frac{\partial}{\partial y_i \partial y_j} \frac{1}{d} = \frac{3}{4} \frac{1}{d^5} \frac{\partial d^2}{\partial y_i} \frac{\partial d^2}{\partial y_j} - \frac{1}{2} \frac{1}{d^3} \frac{\partial^2 d^2}{\partial y_i \partial y_j}$$

and the estimates (17),(20),(22) we obtain

$$\left| \frac{\partial}{\partial y_i \partial y_j} \frac{1}{d} \right| \leq C_1 \left[\frac{1}{d^5} (d_h + |V - V_h|)^2 + \frac{1}{d^3} \right] \leq \frac{C_2}{(d_h^2 + |V - V_h|^2)^{3/2}}.$$

Using (13) and (21) yields the boundedness of F_4:

$$\left| \int_{\mathcal{D}} \mathcal{R}_2^{(h)} \frac{\partial}{\partial y_i \partial y_j} \frac{1}{d(\mathcal{E}, V)} dV \right| \leq C_3 \int_{\mathcal{D}} \frac{dV}{d_h + |V - V_h|} \leq C_4.$$

To show the boundedness of F_2 we just need to prove that

$$\left|\frac{\partial^2}{\partial y_i \partial y_j}\left(\frac{1}{d}-\frac{1}{\delta_h}\right)\right| \leq \frac{C_5}{d_h^2+|V-V_h|^2}, \tag{14}$$

so that

$$\left|\int_{\mathcal{D}} \mathsf{h}\cdot(V-V_h)\frac{\partial^2}{\partial y_i \partial y_j}\left(\frac{1}{d}-\frac{1}{\delta_h}\right)dV\right| \leq C_6\int_{\mathcal{D}}\frac{dV}{d_h+|V-V_h|} \leq C_7.$$

Using $d^2=\delta_h^2+\mathcal{R}_3^{(h)}$ we get

$$\begin{aligned}\frac{\partial^2}{\partial y_i \partial y_j}\left(\frac{1}{d}-\frac{1}{\delta_h}\right) &= \frac{3}{4}\left(\frac{1}{d^5}\frac{\partial d^2}{\partial y_i}-\frac{1}{\delta_h^5}\frac{\partial \delta_h^2}{\partial y_i}\right)\frac{\partial \delta_h^2}{\partial y_j}+\frac{1}{2}\left(\frac{1}{d^3}-\frac{1}{\delta_h^3}\right)\frac{\partial^2\delta_h^2}{\partial y_i \partial y_j}\\ &\quad +\frac{3}{4}\frac{1}{d^5}\frac{\partial d^2}{\partial y_i}\frac{\partial \mathcal{R}_3}{\partial y_j}-\frac{1}{2}\frac{1}{d^3}\frac{\partial^2\mathcal{R}_3}{\partial y_i \partial y_j}.\end{aligned}$$

We prove that each of the four terms in the previous sum satisfies an estimate like (14). For the second term we use estimates (22),(23), for the third (20),(24), and for the last (25). To estimate the first term we note that

$$\left(\frac{1}{d^5}\frac{\partial d^2}{\partial y_i}-\frac{1}{\delta_h^5}\frac{\partial \delta_h^2}{\partial y_i}\right)\frac{\partial \delta_h^2}{\partial y_j} = \left(\frac{1}{d^5}-\frac{1}{\delta_h^5}\right)\frac{\partial \delta_h^2}{\partial y_i}\frac{\partial \delta_h^2}{\partial y_j}+\frac{1}{d^5}\frac{\partial \mathcal{R}_3}{\partial y_i}\frac{\partial \delta_h^2}{\partial y_j}$$

and use

$$\left|\frac{1}{d^5}-\frac{1}{\delta_h^5}\right| \leq \left|\frac{1}{d}-\frac{1}{\delta_h}\right|\left|\frac{1}{d^4}+\frac{1}{d^3\delta_h}+\frac{1}{d^2\delta_h^2}+\frac{1}{d\delta_h^3}+\frac{1}{\delta_h^4}\right|.$$

We can conclude using (17),(20),(24),(26).

Now we show the boundedness of F_3. We write

$$\begin{aligned}\int_{\mathcal{D}} \mathsf{h}\cdot(V-V_h)\frac{\partial^2}{\partial y_i \partial y_j}\frac{1}{\delta_h} &= \frac{3}{4}\int_{\mathcal{D}} \mathsf{h}\cdot(V-V_h)\frac{1}{\delta_h^5}\frac{\partial \delta_h^2}{\partial y_i}\frac{\partial \delta_h^2}{\partial y_j}dV\\ &\quad -\frac{1}{2}\int_{\mathcal{D}} \mathsf{h}\cdot(V-V_h)\frac{1}{\delta_h^3}\frac{\partial^2\delta_h^2}{\partial y_i \partial y_j}dV\end{aligned} \tag{15}$$

and study the two integrals separately. To estimate the first we use (10) to get

$$\frac{\partial \delta_h^2}{\partial y_j} = \frac{\partial d_h^2}{\partial y_j}-2\frac{\partial V_h}{\partial y_j}\cdot\mathcal{A}_h(V-V_h)+(V-V_h)\cdot\frac{\partial \mathcal{A}_h}{\partial y_j}(V-V_h),$$

so that

$$\begin{aligned}
\frac{\partial \delta_h^2}{\partial y_i}\frac{\partial \delta_h^2}{\partial y_j} &= \frac{\partial d_h^2}{\partial y_i}\frac{\partial d_h^2}{\partial y_j} - 2\left(\frac{\partial d_h^2}{\partial y_i}\frac{\partial V_h}{\partial y_j} + \frac{\partial d_h^2}{\partial y_j}\frac{\partial V_h}{\partial y_i}\right)\cdot \mathcal{A}_h(V-V_h) \\
&+ \frac{\partial d_h^2}{\partial y_i}(V-V_h)\cdot\frac{\partial \mathcal{A}_h}{\partial y_j}(V-V_h) + \frac{\partial d_h^2}{\partial y_j}(V-V_h)\cdot\frac{\partial \mathcal{A}_h}{\partial y_i}(V-V_h) \\
&+ 4\left[\frac{\partial V_h}{\partial y_i}\cdot\mathcal{A}_h(V-V_h)\right]\left[\frac{\partial V_h}{\partial y_j}\cdot\mathcal{A}_h(V-V_h)\right] \\
&- 2\left[\frac{\partial V_h}{\partial y_i}\cdot\mathcal{A}_h(V-V_h)\right]\left[(V-V_h)\cdot\frac{\partial \mathcal{A}_h}{\partial y_j}(V-V_h)\right] \\
&- 2\left[\frac{\partial V_h}{\partial y_j}\cdot\mathcal{A}_h(V-V_h)\right]\left[(V-V_h)\cdot\frac{\partial \mathcal{A}_h}{\partial y_i}(V-V_h)\right] \\
&+ \left[(V-V_h)\cdot\frac{\partial \mathcal{A}_h}{\partial y_i}(V-V_h)\right]\left[(V-V_h)\cdot\frac{\partial \mathcal{A}_h}{\partial y_j}(V-V_h)\right].
\end{aligned}$$

Then we use the change of variables $\xi = \mathcal{A}_h^{1/2}(V-V_h)$ and polar coordinates $\xi = \rho(\cos\theta, \sin\theta)$. We distinguish between terms with even and odd degree in $(V-V_h)$. First we consider the ones with even degree: the term of degree 2 is estimated as follows

$$\begin{aligned}
&\left|\int_{\mathcal{D}} \mathsf{h}\cdot(V-V_h)\frac{1}{\delta_h^5}\left(\frac{\partial d_h^2}{\partial y_i}\frac{\partial V_h}{\partial y_j} + \frac{\partial d_h^2}{\partial y_j}\frac{\partial V_h}{\partial y_i}\right)\cdot\mathcal{A}_h(V-V_h)dV\right| = \\
&\left|\int_{\mathcal{D}} 2\tilde{d}_h\left(\frac{\partial \tilde{d}_h}{\partial y_i}\frac{\partial V_h}{\partial y_j} + \frac{\partial \tilde{d}_h}{\partial y_j}\frac{\partial V_h}{\partial y_i}\right)\cdot\mathcal{A}_h(V-V_h)\mathsf{h}\cdot(V-V_h)\frac{1}{\delta_h^5}dV\right| = \\
&2\frac{d_h}{\sqrt{\det\mathcal{A}_h}}\int_0^r \frac{\rho^3}{(d_h^2+\rho^2)^{5/2}}d\rho\left|\sum_{|\gamma|=2} b_\gamma\int_0^{2\pi}(\cos\theta)^{\gamma_1}(\sin\theta)^{\gamma_2}d\theta\right| \le \\
&2\frac{d_h}{\sqrt{\det\mathcal{A}_h}}\frac{C_8}{d_h} \le C_9,
\end{aligned}$$

while for the term of degree 4 we note that

$$\begin{aligned}
&\left|\int_{\mathcal{D}} \frac{1}{\delta_h^5}\mathsf{h}\cdot(V-V_h)\left[\frac{\partial V_h}{\partial y_j}\cdot\mathcal{A}_h(V-V_h)\right]\left[(V-V_h)\cdot\frac{\partial \mathcal{A}_h}{\partial y_i}(V-V_h)\right]dV\right| = \\
&\frac{1}{\sqrt{\det\mathcal{A}_h}}\int_0^r \frac{\rho^5}{(d_h^2+\rho^2)^{5/2}}d\rho\left|\sum_{|\gamma|=4} c_\gamma\int_0^{2\pi}(\cos\theta)^{\gamma_1}(\sin\theta)^{\gamma_2}d\theta\right| \le C_{10}
\end{aligned}$$

for some functions b_γ, c_γ, uniformly bounded in $\mathcal{W}\backslash\Sigma$, and for $\gamma = (\gamma_1,\gamma_2) \in (\mathbb{N}\cup\{0\})^2$. The terms with odd degree in $(V-V_h)$ vanish, as can be shown by similar computations, using

$$\int_0^{2\pi}(\cos\theta)^{\gamma_1}(\sin\theta)^{\gamma_2}d\theta = 0$$

for $\gamma_1+\gamma_2$ odd.
To estimate the second integral in (15) we proceed in a similar way, using

$$\begin{aligned}\frac{\partial^2\delta_h^2}{\partial y_i\partial y_j} &= \frac{\partial^2 d_h^2}{\partial y_i\partial y_j} - 2\frac{\partial^2 V_h}{\partial y_i\partial y_j}\cdot\mathcal{A}_h(V-V_h) - 2\frac{\partial V_h}{\partial y_j}\cdot\frac{\partial\mathcal{A}_h}{\partial y_i}(V-V_h)\\ &\quad - 2\frac{\partial V_h}{\partial y_i}\cdot\frac{\partial\mathcal{A}_h}{\partial y_j}(V-V_h) + \left[(V-V_h)\cdot\frac{\partial^2\mathcal{A}_h}{\partial y_i\partial y_j}(V-V_h)\right].\end{aligned}$$

4.2 Proof of Theorem 2-(b)

As before, we shall denote by C_k, $k=11,12$ some positive constants independent on $\mathcal{E}$. We shall also refer to the estimates and results in Appendix A.
Using the approximated distance δ_h defined in (10) we write, for $\mathcal{E}\in\mathcal{W}\setminus\Sigma$,

$$\begin{aligned}\int_{\mathbb{T}^2}\frac{1}{d(\mathcal{E},V)}\cos(n\mathsf{h}\cdot V)dV &= \int_{\mathbb{T}^2}\left(\frac{1}{d(\mathcal{E},V)}-\frac{1}{\delta_h(\mathcal{E},V)}\right)\cos(n\mathsf{h}\cdot V)dV\\ &\quad+\int_{\mathbb{T}^2}\frac{1}{\delta_h(\mathcal{E},V)}\cos(n\mathsf{h}\cdot V)dV.\end{aligned}$$

From estimate (27) the map

$$\mathcal{W}\setminus\Sigma\ni\mathcal{E}\mapsto\frac{\partial}{\partial y_i}\int_{\mathbb{T}^2}\left(\frac{1}{d(\mathcal{E},V)}-\frac{1}{\delta_h(\mathcal{E},V)}\right)\cos(n\mathsf{h}\cdot V)dV$$

can be extended continuously to $\mathcal{W}$. To estimate the derivatives with respect to y_i of the second term we write

$$\begin{aligned}\int_{\mathbb{T}^2}\frac{1}{\delta_h(\mathcal{E},V)}\cos(n\mathsf{h}\cdot V)dV &= \int_{\mathbb{T}^2}\frac{\cos(n\mathsf{h}\cdot V)-\cos(n\mathsf{h}\cdot V_h)}{\delta_h(\mathcal{E},V)}dV\\ &+ \cos(n\mathsf{h}\cdot V_h)\int_{\mathbb{T}^2}\frac{1}{\delta_h(\mathcal{E},V)}dV.\end{aligned}$$

We prove that also the map

$$\mathcal{W}\setminus\Sigma\ni\mathcal{E}\mapsto\frac{\partial}{\partial y_i}\int_{\mathbb{T}^2}\frac{\cos(n\mathsf{h}\cdot V)-\cos(n\mathsf{h}\cdot V_h)}{\delta_h(\mathcal{E},V)}dV \tag{16}$$

admits a continuous extension to $\mathcal{W}$. We note that

$$\begin{aligned}\frac{\partial}{\partial y_i}\frac{\cos(n\mathsf{h}\cdot V)-\cos(n\mathsf{h}\cdot V_h)}{\delta_h(\mathcal{E},V)} &= \frac{\sin(n\mathsf{h}\cdot V_h)n\mathsf{h}\cdot\frac{\partial V_h}{\partial y_i}}{\delta_h(\mathcal{E},V)}\\ &\quad-[\cos(n\mathsf{h}\cdot V)-\cos(n\mathsf{h}\cdot V_h)]\frac{\partial}{\partial y_i}\frac{1}{\delta_h(\mathcal{E},V)}.\end{aligned}$$

By (18), (28) the first addendum is summable. For the second, by (20) we get

$$\left|\frac{\partial}{\partial y_i}\frac{1}{\delta_h}\right| = \left|\frac{1}{2\delta_h^3}\frac{\partial \delta_h^2}{\partial y_i}\right| \leq \frac{C_{11}}{d_h^2 + |V - V_h|^2}.$$

From

$$|\cos(n\mathsf{h}\cdot V) - \cos(n\mathsf{h}\cdot V_h)| \leq C_{12}|V - V_h|$$

we can conclude using estimate (21). Hence also the map (16) can be extended continuously to $\mathcal{W}$.
Therefore, we just need to extend the maps

$$\begin{aligned}&\mathcal{W}^{\pm} \ni \mathcal{E} \mapsto \frac{\partial}{\partial y_i}\left[\cos(n\mathsf{h}\cdot V_h)\int_{\mathbb{T}^2}\frac{1}{\delta_h(\mathcal{E},V)}dV\right] = \\ &-\sin(n\mathsf{h}\cdot V_h)n\mathsf{h}\cdot\frac{\partial V_h}{\partial y_i}\int_{\mathbb{T}^2}\frac{1}{\delta_h(\mathcal{E},V)}dV + \cos(n\mathsf{h}\cdot V_h)\frac{\partial}{\partial y_i}\int_{\mathbb{T}^2}\frac{1}{\delta_h(\mathcal{E},V)}dV.\end{aligned}$$

From (18) the first term is regular so that we can use Lemma 3 to conclude the proof.

Remark 2. *If $\mathcal{E}_c$ is an orbit configuration with two crossings, assuming that $d_h(\mathcal{E}_c) = 0$ for $h = 1, 2$, we can extract the singularity by considering the approximated distances δ_1, δ_2 and considering* $1/d$ *as sum of the three terms* $(1/d - 1/\delta_1 - 1/\delta_2)$, $1/\delta_1$, $1/\delta_2$.

5 Generalized solutions and evolution of the orbit distance

Following [7, Sections 5-6] we can construct generalized solutions patching classical solutions defined in the domain $\mathcal{W}^+$ with classical solutions defined on $\mathcal{W}^-$ and vice-versa. Let $(E(t), \sigma(t))$, with $E(t) = (S(t), G(t), Z(t), g(t), z(t))$, represent the evolution of the asteroid according to (9). In a similar way we introduce $E'(t)$ a known function of time representing the evolution of the trajectory of the planet. Setting $\mathcal{E}(t) = (E(t), E'(t))$ we let $T(\mathcal{Y})$ be the set of times t_c such that $d_{min}(\mathcal{E}(t_c)) = 0$ and suppose that it has no accumulation points.

We say that $\mathcal{Y}(t)$ is a *generalized solution* of (9) if it is a classical solution for $t \notin T(\mathcal{Y})$ and for each $t_c \in T(\mathcal{Y})$ there exist finite values of

$$\lim_{t\to t_c^+}\dot{\mathcal{Y}}(t), \qquad \lim_{t\to t_c^-}\dot{\mathcal{Y}}(t).$$

In order to construct a generalized solution we consider a solution $\mathcal{Y}(t)$ of the Cauchy problem given by (9) with a non crossing initial condition $\mathcal{Y}(t_0)$. Suppose that it is defined on a maximal interval J such that $\sup J = t_c \in T(\mathcal{Y})$ and that $\mathcal{Y}(t) \in \mathcal{W}^+$ as $t \to t_c$. Suppose that the crossing is occurring with a planet different from Jupiter (resp. Jupiter itself). Applying Theorem 1-(a) (resp. Theorems 1-(a) and 2-(a)) we have that there exists

$$\lim_{t\to t_c^-} \dot{\mathcal{Y}}(t) = \dot{\mathcal{Y}}_c$$

and the solution can be extended beyond t_c considering the Cauchy problem

$$\dot{\mathcal{Y}} = \mathbb{J}_3(\nabla_{\mathcal{Y}}\mathcal{H})^+, \quad \mathcal{Y}(\tau) = \mathcal{Y}_\tau$$

for some $\tau \to t_c$, so that we call $\mathcal{Y}(t_c) = \mathcal{Y}_c$. Using again Theorem 1-(a) (resp. Theorems 1-(a) and 2-(a)), we can extend the solution beyond the singularity considering the new Cauchy problem

$$\dot{\mathcal{Y}} = \mathbb{J}_3(\nabla_{\mathcal{Y}}\mathcal{H})^-, \quad \mathcal{Y}(t_c) = \mathcal{Y}_c.$$

whose solution fulfills, from Corollary 1 (resp. Corollary 2)

$$\lim_{t\to t_c^-} \dot{\mathcal{Y}}(t) = \dot{\mathcal{Y}}_c - \mathrm{Diff}_h\left(\nabla_{\mathcal{Y}}\mathcal{H}\right)(\mathcal{E}(t_c), V).$$

Note that the evolution of the orbital elements according to a generalized solution is continuous but not differentiable in a neighborhood of a crossing singularity. More precisely, the evolution of the elements (G, Z, σ, g, z) is only Lipschitz-continuous while the evolution of S is C^1, since $\frac{\partial\mathcal{H}}{\partial\sigma}$ is continuous at orbit crossings.
Once a generalized solution $\mathcal{Y}(t) = (E(t), \sigma(t))$ is defined, we can consider the evolution of the distance $\tilde{d}_h(\mathcal{E}(t))$. Let us define

$$\bar{d}_h(t) = \tilde{d}_h(\mathcal{E}(t))$$

and suppose that it is defined in an interval containing a crossing time t_c corresponding to a non-degenerate crossing configuration. We have the following

Proposition 1. *Let $\mathcal{Y}(t)$ be a generalized solution of (9) and $\mathcal{E}(t)$ be defined as above. Suppose that t_c is a crossing time such that $\mathcal{E}_c = \mathcal{E}(t_c)$ is a non-degenerate crossing configuration. Then there exists an open interval $I \ni t_c$ such that $\bar{d}_h \in C^1(I, \mathbb{R})$.*

Proof. We choose the interval I such that $\mathcal{E}(I) \in \mathcal{W}$ with $\mathcal{W}$ defined in Theorem 1 (resp. 2) and suppose that $\mathcal{E}(t) \in \mathcal{W}^+$ for $t < t_c$ and $\mathcal{E}(t) \in \mathcal{W}^-$ for $t > t_c$. We can compute, for $t \neq t_c$,

$$\begin{aligned}\dot{\bar{d}}_h(t) &= \nabla_{\mathcal{E}}\tilde{d}_h(\mathcal{E}(t)) \cdot \dot{\mathcal{E}}(t) = \nabla_E\tilde{d}_h(\mathcal{E}(t)) \cdot \dot{E}(t) + \nabla_{E'}\tilde{d}_h(\mathcal{E}(t)) \cdot \dot{E}'(t) \\ &= \nabla_E\tilde{d}_h(\mathcal{E}(t)) \cdot \left(-\frac{\partial\mathcal{H}}{\partial\sigma}, -\frac{\partial\mathcal{H}}{\partial g}, -\frac{\partial\mathcal{H}}{\partial z}, \frac{\partial\mathcal{H}}{\partial G}, \frac{\partial\mathcal{H}}{\partial Z}\right)^T + \nabla_{E'}\tilde{d}_h(\mathcal{E}(t)) \cdot \dot{E}'(t).\end{aligned}$$

The second addendum is continuous while for the first we need to distinguish between crossings with Jupiter or with another planet. For the second case, we apply Corollary

1 and obtain

$$\begin{aligned}\lim_{t\to t_c^+}\dot{\tilde{d}}_h(t)-\lim_{t\to t_c^-}\dot{\tilde{d}}_h(t)&=\left[\nabla_E\tilde{d}_h\cdot\mathrm{Diff}_h\Big(-\frac{\partial\mathcal{H}}{\partial\sigma},-\frac{\partial\mathcal{H}}{\partial g},-\frac{\partial\mathcal{H}}{\partial z},\frac{\partial\mathcal{H}}{\partial G},\frac{\partial\mathcal{H}}{\partial Z}\Big)^T\right]_{t=t_c}\\&=\left[\nabla_E\tilde{d}_h\cdot\mathrm{Diff}_h\Big(0,-\frac{\partial\mathcal{H}}{\partial g},-\frac{\partial\mathcal{H}}{\partial z},\frac{\partial\mathcal{H}}{\partial G},\frac{\partial\mathcal{H}}{\partial Z}\Big)^T\right]_{t=t_c}\\&=\left[-\frac{2\mu_5\mathtt{k}^2}{\pi\sqrt{\det(\mathcal{A}_h)}}\{\tilde{d}_h,\tilde{d}_h\}\right]_{t=t_c}=0,\end{aligned}$$

where $\{,\}$ are the Poisson brackets.

For the first case, we apply Corollary 2 and get

$$\begin{aligned}\lim_{t\to t_c^+}\dot{\tilde{d}}_h(t)-\lim_{t\to t_c^-}\dot{\tilde{d}}_h(t)&=\left[\nabla_E\tilde{d}_h\cdot\mathrm{Diff}_h\left(-\frac{\partial\mathcal{H}}{\partial\sigma},-\frac{\partial\mathcal{H}}{\partial g},-\frac{\partial\mathcal{H}}{\partial z},\frac{\partial\mathcal{H}}{\partial G},\frac{\partial\mathcal{H}}{\partial Z}\right)^T\right]_{t=t_c}\\&=\left[\nabla_E\tilde{d}_h\cdot\mathrm{Diff}_h\left(0,-\frac{\partial\mathcal{H}}{\partial g},-\frac{\partial\mathcal{H}}{\partial z},\frac{\partial\mathcal{H}}{\partial G},\frac{\partial\mathcal{H}}{\partial Z}\right)^T\right]_{t=t_c}\\&=\left[-\frac{2\mu_5\mathtt{k}^2\left[\sum_{n=0}^{n_{max}}\cos\big(n(\sigma-\mathsf{h}\cdot V_h)\big)+\frac{1}{2}\right]}{\pi\sqrt{\det(\mathcal{A}_h)}}\{\tilde{d}_h,\tilde{d}_h\}\right]_{t=t_c}=0.\end{aligned}$$

$\square$

6 Numerical experiments

We compare the long term evolution coming from system (9) with the full evolution of equation (1), corresponding to the classical restricted N-body problem.

To get the evolution of the planets, we compute a planetary ephemerides database for a time span of 2000 yrs, starting at 57600 MJD with a time step of 0.5 years. The computation is performed using the FORTRAN program `orbit9` included in the `OrbFit` free software[1]. The planetary evolution at the desired time is obtained from this database by linear interpolation.

We consider two paradigmatic cases, representing the two crossing behaviors highlighted in the previous sections. For the 'Alinda' class we choose the asteroid 887(Alinda) under the influence of 5 planets, from Venus to Saturn. This asteroid is in 3 : 1 mean motion resonance with Jupiter and we will consider its crossing with the orbit of Mars. For the 'Toro' class, we consider a fictitious asteroid that we call 1685a under the influence of 3 planets: the Earth, Mars and Jupiter. This asteroid crosses the orbit of the Earth, which is in the 5 : 8 mean motion resonance with it.

We use the same algorithm as in [7] to compute the solution of system (9). This is a Runge-Kutta-Gauss method evaluating the vector field at intermediate points of

[1] `http://adams.dm.unipi.it/orbmaint/orbfit`

the time step. The time step is decreased when the trajectory of the asteroid is close to a planet crossing, in order to get exactly the crossing condition. By Theorems 1-2 we can find two locally Lipschitz-continuous extension of the vector field from both sides of the singular set Σ. The difference between the two extended fields is given by Corollary 1 for asteroid 887 (Alinda) and by Corollary 2 for asteroid 1685a. In both cases, we compute the intermediate values of the extended vector field just after the crossing, and then we correct them using Corollary 1 or Corollary 2. We use these corrected values as an approximation of the vector field at the intermediate point of the solution, see Figure 2. This algorithm avoids the computation of the vector field at the singular points, which could be affected by numerical instability.

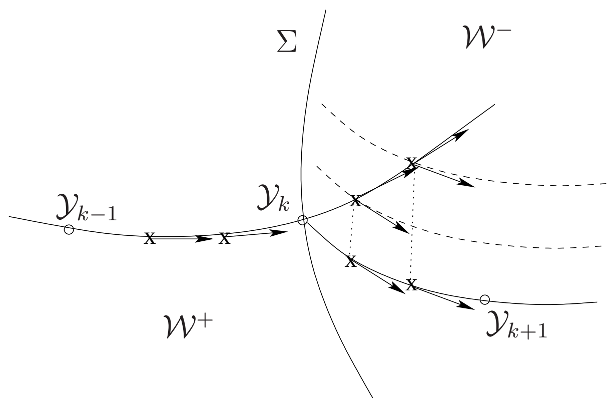

Figure 2: Runge-Kutta-Gauss method and continuation of the solution of (9) beyond the singularity.

To produce the comparison, we consider 64 possible initial conditions for system (1) corresponding to the same initial condition of system (9). For asteroid 887 (Alinda) these are produced shifting the mean anomalies in the following way. Let $\bar{\ell}_j$ and $\bar{\ell}$ be the mean anomalies of planet j and the asteroid, at the initial epoch 57600 MJD. For each planet, we consider the 64 values $\ell_j^{(k)} = \bar{\ell}_j + k\pi/64$ with $k = 0, \ldots, 63$. For every k, we compute the initial value of the mean anomaly $\ell^{(k)} = \bar{\ell} + l^{(k)}$ of the asteroid such that

$$h_5^*(\bar{\ell}_5 + k\pi/64) + h^*(\bar{\ell} + l^{(k)}) = h_5^*\bar{\ell}_5 + h^*\bar{\ell}.$$

The integration of this 64 different initial conditions is performed with the program `orbit9`. Then we consider the arithmetic mean of the 5 Keplerian elements a, e, I, Ω, ω and the critical angle $\sigma = h_5^*\ell_5 + h^*\ell$ over these evolutions and compare them with the corresponding elements coming from system (9), in which we choose $n_{max} = 3$. Figure 3 summarizes the results: the solid line corresponds to the solution of (9) while the dashed line corresponds to the arithmetic mean of the full numerical integrations. The shaded region represents the standard deviation from the arithmetic mean. The correspondence between the solutions is good. The Mars crossing singularity occurs around $t = 3786\, yr$.

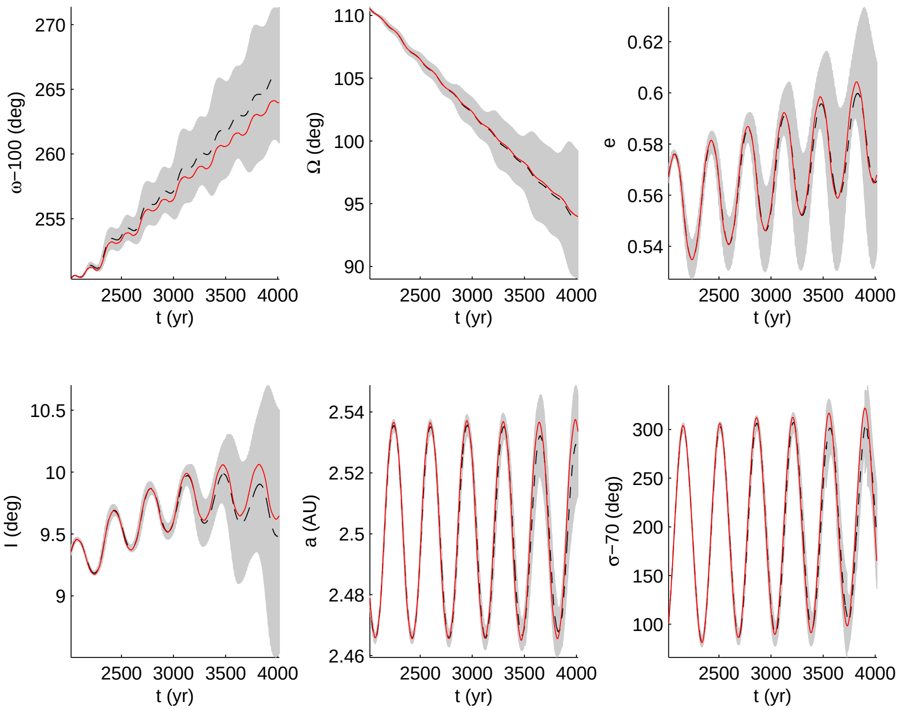

Figure 3: Asteroid 887 (Alinda): comparison between the long term evolution using $n_{max} = 3$ (solid line) and the arithmetic mean of 64 full numerical integrations (dashed line).

For asteroid 1685a we proceed in the same way, with the Earth playing the role of Jupiter. For the long term evolution we used $n_{max} = 3, 15$. In Figure 4 we show the results. Using $n_{max} = 15$ we see that the result improves very much. The Earth crossing singularity occurs around $t = 2281\, yr$. We cannot really appreciate the singularity in the evolution since we obtain very small values of the components of Diff_h.

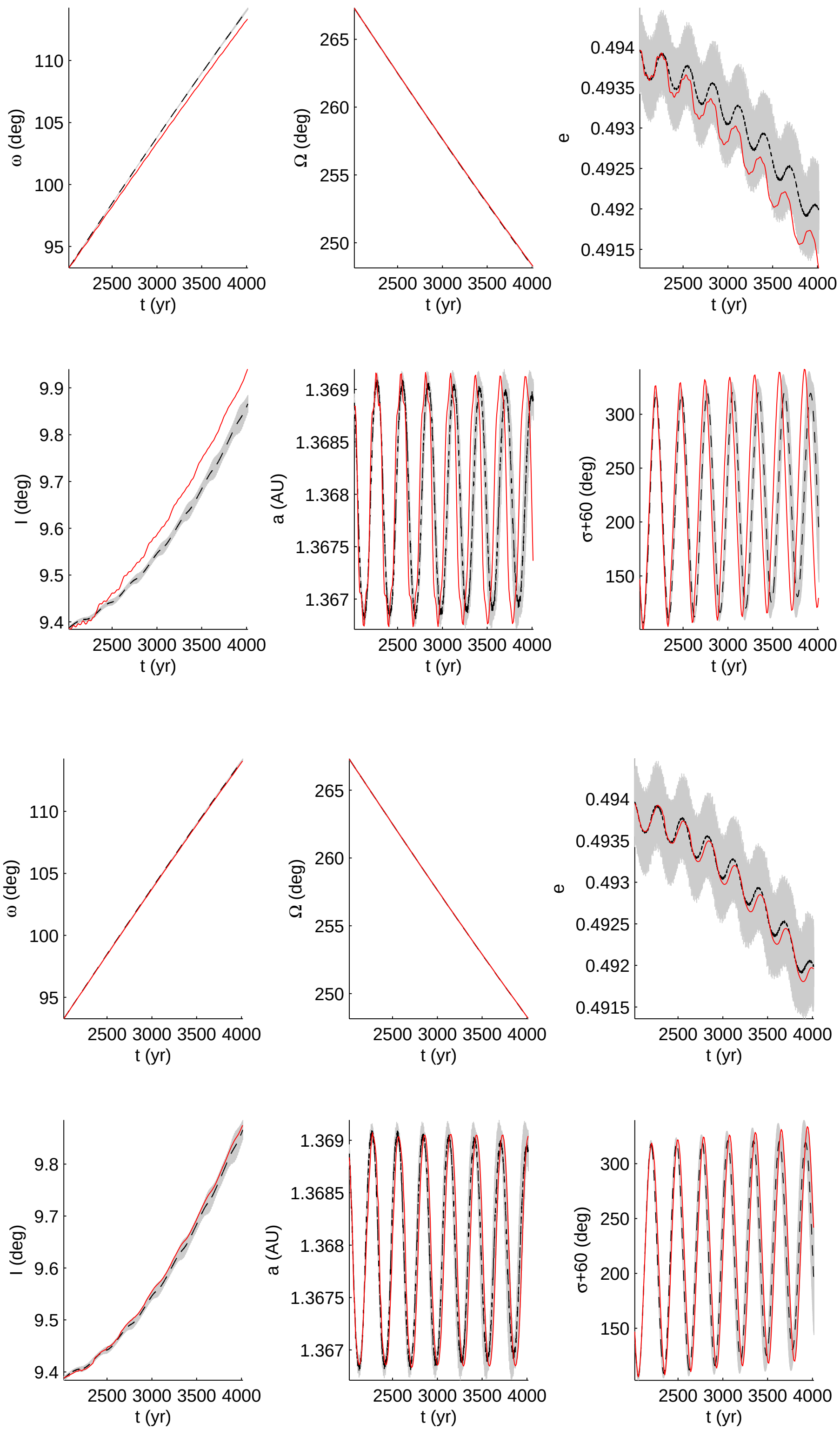

Figure 4: Asteroid 1685a: comparison between the long term evolution (solid line) and the arithmetic mean of 64 full numerical integrations (dashed line). Above $n_{max} = 3$. Below $n_{max} = 15$.

7 Conclusions

We studied the long term dynamics of an asteroid under the influence of the Sun and some planets, assuming that a mean motion resonance between the asteroid and one planet occurs. The dynamical model is obtained through standard techniques of perturbation theory. We considered the case of crossing singularities and distinguished between the case of crossings with the resonant planet or with another one. In both cases, we defined a unique generalized solution of the long term model, going beyond the singularity. This solution is continuous but in general not differentiable. We also proved that generically, in a neighborhood of a crossing time, the evolution of the (signed) orbit distance along generalized solution is differentiable, hence more regular that the long term evolution of the orbital elements.

By means of numerical experiments in some relevant cases, we showed that the model seems to approximate well the full evolution. This work extends the results in [7] to the resonant case and increases the number of NEAs whose dynamics can be studied through averaging techniques. We plan to produce numerical experiments on large scale, to describe different dynamical behaviors of the population of NEAs. However, we stress that we considered the occurrence of just one mean motion resonance, and our technique would fail in case of multiple resonances. Another critical case is represented by deep close encounters: in this case the semimajor axis is expected to suffer a drastic change [12], pushing the asteroid outside the considered resonance.

A Appendix

From the definition of the approximate distance δ_h, we have that

$$d^2(\mathcal{E}, V) = d_h^2(\mathcal{E}) + (V - V_h) \cdot \mathcal{A}_h(V - V_h) + \mathcal{R}_3^{(h)}(\mathcal{E}, V) = \delta_h^2(\mathcal{E}) + \mathcal{R}_3^{(h)}(\mathcal{E}, V).$$

We summarize below some relevant estimates and results from [7]. In the following, we shall denote by C_i, $i = 1, \ldots, 14$, some positive constant independent on $\mathcal{E}$. We first recall the following Lemmas.

Lemma 1. *There exist positive constants C_1, C_2 and a neighborhood $\mathcal{U}$ of $(\mathcal{E}_c, V_h(\mathcal{E}_c))$ such that*

$$C_1\delta_h^2 \leq d^2 \leq C_2\delta_h^2$$

holds for $(\mathcal{E}, \mathcal{V})$ in $\mathcal{U}$. Moreover, there exist positive constants C_3, C_4 and a neighborhood $\mathcal{W}$ of $\mathcal{E}_c$ such that

$$d_h^2 + C_3|V - V_h|^2 \leq \delta_h^2 \leq d_h^2 + C_4|V - V_h|^2 \tag{17}$$

holds for $\mathcal{E}$ in $\mathcal{W}$ and for every $V \in \mathbb{T}^2$.

Lemma 2. *Using the coordinate change* $\xi = \mathcal{A}_h^{1/2}(V - V_h)$ *and then polar coordinates* (ρ, θ)*, defined by* $(\rho\cos\theta, \rho\sin\theta) = \xi$*, we have*

$$\int_{\mathcal{D}} \frac{1}{\delta_h} d\ell d\ell' = \frac{1}{\sqrt{\det \mathcal{A}_h}} \int_{\mathcal{B}} \frac{1}{\sqrt{d_h^2 + |\xi|^2}} d\xi = \frac{2\pi}{\sqrt{\det \mathcal{A}_h}} (\sqrt{d_h^2 + r^2} - d_h)\,, \tag{18}$$

with $\mathcal{B} = \{\xi \in \mathbb{R}^2 : |\xi| \leq r\}$*. The term* $-2\pi d_h/\sqrt{\det \mathcal{A}_h}$ *is not differentiable at* $\mathcal{E} = \mathcal{E}_c \in \Sigma$*.*

Lemma 3. *The maps*

$$\mathcal{W}^+ \ni \mathcal{E} \mapsto \frac{\partial}{\partial y_i} \int_{\mathbb{T}^2} \frac{1}{\delta_h(\mathcal{E}, V)} dV, \quad \mathcal{W}^- \ni \mathcal{E} \mapsto \frac{\partial}{\partial y_i} \int_{\mathbb{T}^2} \frac{1}{\delta_h(\mathcal{E}, V)} dV$$

can be extended to two different analytic maps $\mathcal{G}_h^+$*,* $\mathcal{G}_h^-$ *such that, in* $\mathcal{W}$*,*

$$\mathcal{G}_h^- - \mathcal{G}_h^+ = 4\pi \left[\frac{\partial}{\partial y_i} \left(\frac{1}{\sqrt{\det(\mathcal{A}_h)}} \right) \tilde{d}_h + \frac{1}{\sqrt{\det(\mathcal{A}_h)}} \frac{\partial \tilde{d}_h}{\partial y_i} \right].$$

Moreover the following estimates hold, with $\mathcal{U}_\Sigma = \{(\mathcal{E}, V_h(\mathcal{E})) : \mathcal{E} \in \Sigma\}$:

$$\int_{\mathcal{D}} \frac{\partial^2}{\partial y_i \partial y_j} \frac{1}{d(\mathcal{E}, V)} dV \leq C_5 \quad \text{for } \mathcal{E} \text{ in } \mathcal{W}, \tag{19}$$

$$\left| \frac{\partial d^2}{\partial y_i} \right|, \left| \frac{\partial \delta_h^2}{\partial y_j} \right| \leq C_6 (d_h + |V - V_h|) \quad \text{in } \mathcal{U} \setminus \mathcal{U}_\Sigma, \tag{20}$$

$$\int_{\mathcal{D}} \frac{dV}{d_h + |V - V_h|} \leq C_7 \quad \text{for } \mathcal{E} \text{ in } \mathcal{W}, \tag{21}$$

$$\left| \frac{\partial^2 \delta_h^2}{\partial y_i \partial y_j} \right|, \left| \frac{\partial^2 d^2}{\partial y_i \partial y_j} \right| \leq C_8 \quad \text{for } \mathcal{E} \text{ in } \mathcal{W}, \tag{22}$$

$$\left| \frac{1}{d^3} - \frac{1}{\delta_h^3} \right| \leq \frac{C_9}{d_h^2 + |V - V_h|^2} \quad \text{in } \mathcal{U} \setminus \mathcal{U}_\Sigma, \tag{23}$$

$$\frac{\partial \mathcal{R}_3^{(h)}}{\partial y_i} \leq C_{10} |V - V_h|^2 \quad \text{in } \mathcal{U} \setminus \mathcal{U}_\Sigma, \tag{24}$$

$$\frac{\partial^2 \mathcal{R}_3^{(h)}}{\partial y_i \partial y_j} \leq C_{11} |V - V_h| \quad \text{in } \mathcal{U} \setminus \mathcal{U}_\Sigma, \tag{25}$$

$$\left| \frac{1}{d} - \frac{1}{\delta_h} \right| \leq C_{12} \quad \text{in } \mathcal{U} \setminus \mathcal{U}_\Sigma, \tag{26}$$

$$\left| \frac{\partial}{\partial y_i} \left(\frac{1}{d} - \frac{1}{\delta_h} \right) \right| \leq \frac{C_{13}}{d_h + |V - V_h|} \quad \text{in } \mathcal{U} \setminus \mathcal{U}_\Sigma, \tag{27}$$

$$\frac{\partial V_h}{\partial y_i} \leq C_{14} \quad \text{for } \mathcal{E} \text{ in } \mathcal{W}. \tag{28}$$

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References

[1] Arnold, V.I, Kozlov, V.V., Neishtadt, A.I.: *Mathematical Aspects of Classical and Celestial Mechanics* , Springer, 1997

[2] Dieudonné, J.: *Foundations of Modern Analysis*, Academic press, 1969

[3] Gronchi, G.F.: Theoretical and computational aspects of collision singularities in the N-body problem, PhD thesis, Univ. of Pisa (2002)

[4] Gronchi, G.F.: *Generalized averaging principle and the secular evolution of planet crossing orbits*, Cel. Mech. Dyn. Ast. 83 (2002), 97-120

[5] Gronchi, G.F., Milani, A.: *Averaging on crossing orbits*, Cel. Mech. Dyn. Ast. 71 (1998), 109-136

[6] Gronchi, G.F., Tommei, G.: *On the uncertainty of the minimal distance between two confocal Keplerian orbits*, Discrete Contin. Dyn. Syst. Ser. B 7 (2007), 755-778

[7] Gronchi, G.F., Tardioli, C: *Secular evolution of the orbit distance in the double averaged restricted three-body problem with crossing singularities*, Discrete Contin. Dyn. Syst. Ser. B 18 (2013), 1323-1344

[8] Lidov, M.L., Ziglin, S.L.: *The Analysis of Restricted Circular Twice-averaged Three Body Problem in the Case of Close Orbits*, Cel. Mech. 9/2 (1974), 151-173

[9] Milani, A., Carpino, M., Hahn, G., Nobili, A.M.: *Dynamics of Planet Crossing Asteroids: Classes of Orbital Behavior*, Icarus 78 (1989), 212-269

[10] Morbidelli, A.: *Modern Celestial Mechanics* Taylor & Francis, 2002

[11] Nesvorný, D. and Morbidelli, A.: *An analytic model of three-body mean motion resonances* Cel. Mech. Dyn. Astron. 71 (1999), 243-263

[12] Valsecchi, G.B., Milani, A., Gronchi, G.F., Chesley, S.R.: *Resonant returns to close approaches: Analytical theory* Astron. Astrophys. 408 (2003), 1179-1196

[13] Whipple, A.: *Lyapunov times of the inner asteroids*, Icarus 115 (1995), 347-353

[14] Wisdom, J.: *Chaotic behavior and the origin of the 3/1 Kirkwood gap*, Icarus 56 (1983), 51-74