THE SINGULAR FIBRE OF THE HITCHIN MAP

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Abstract. We give a description of the singular fibre of the Hitchin map on the moduli space of L-twisted Higgs pairs of rank 2 with fixed determinant bundle, when the corresponding spectral curve has any singularity of type A_{m-1}. In particular, we prove directly that this fibre is connected.

1. Introduction

Let X be a compact Riemann surface of genus g and let $L \to X$ be a holomorphic line bundle. An *L-twisted Higgs pair* or *Hitchin pair* is a pair (E, φ), where $E \to X$ is a holomorphic vector bundle and $\varphi \in H^0(X, \mathrm{End}(E) \otimes L)$ is an L-twisted endomorphism. In particular, if K denotes the canonical line bundle of X, then a K-twisted Higgs pair is a Higgs bundle.

Denote by $\mathcal{M}_L^\Lambda$ the moduli space of semistable L-twisted Higgs pairs of rank 2 with fixed determinant bundle $\Lambda \to X$ and $\mathrm{tr}(\varphi) = 0$. The Hitchin map

$$\begin{aligned} \mathcal{H}\colon \mathcal{M}_L^\Lambda &\longrightarrow H^0(X, L^2) \\ (E,\varphi) &\longmapsto \det(\varphi) \end{aligned}$$

is a proper map. It plays a central role in many important aspects of Higgs bundle theory. Two examples are integrable systems (see, e.g., Hitchin [13], Bottacin [6], Markman [16] and Donagi–Markman [9]) and the study of special representations of surface groups (now known as Hitchin representations) initiated by Hitchin [14]. More recently, the Hitchin map played a crucial role, for example, in the work of Kapustin and Witten [15], Frenkel and Witten [10] and Ngô [18, 19].

For a section $s \in H^0(X, L^2)$, let X_s be the spectral curve given by the zero locus of s inside the total space of L. Then $X_s \to X$ is a double cover. Note that if $s = \det(\varphi)$ then the preimage of $x \in X$ corresponds to the eigenvalues of $\varphi_x\colon E_x \to E_x \otimes L_x$.

The spectral curve X_s is smooth if and only if the divisor of zeros of s is reduced. It is an essential feature of the integrable systems picture for the Hitchin map that its fibre over such a generic s is an abelian variety. In the present case this abelian variety is the Prym variety of the double cover $X_s \to X$, i.e. the part of the Jacobian of X_s which is anti-invariant under the natural involution induced by exchanging the sheets of the double cover.

Date: November 13, 2011.

2000 *Mathematics Subject Classification.* Primary 14H60, 14H40; Secondary 14D20.

Members of VBAC (Vector Bundles on Algebraic Curves). Partially supported by CRUP through Acção Integrada Luso-Espanhola no. E-38/09 and by the FCT (Portugal) with EU (COMPETE) and national funds through the projects PTDC/MAT/099275/2008 and PTDC/MAT/098770/2008, and through Centro de Matemática da Universidade do Porto (PEst-C/MAT/UI0144/2011, first author) and Centro de Matemática da Universidade de Trás-os-Montes e Alto Douro (PEst-OE/MAT/UI4080/2011, second author).

At the other extreme, the most special fibre of the Hitchin map is the fibre over zero, which is known as the nilpotent cone. The nilpotent cone is singular and has a complicated structure: it contains the fixed point locus of the natural $\mathbb{C}^*$-action on $\mathcal{M}_L^\Lambda$ (given by multiplication on the Higgs field) and has an irreducible component for each component of this fixed point locus. In particular, the nilpotent cone contains the moduli space of bundles as the locus $\varphi = 0$. Since the nilpotent cone encodes the topology of the moduli space, it has been studied extensively (see e.g. [5] and references therein).

Our main goal in this paper is to give a description of the remaining special fibres of the Hitchin map, corresponding to the case of $s = \det(\varphi)$ being non-zero and having at least one multiple zero. Here there are two cases to distinguish, corresponding to whether the spectral curve is irreducible or not.

When the spectral case is irreducible, we use the correspondence between Higgs pairs on X and rank one torsion free sheaves on X_s to show that the fibre of the Hitchin map is essentially the compactification by rank 1 torsion free sheaves of the Prym of the double cover $X_s \to X$ (see Theorem 6.1 below for the detailed statement). In order to prove this, we make use of the the compactification of the Jacobian of X_s by the parabolic modules of Cook [7, 8] in order to describe the fibre correctly. One advantage of this compactification is that it fibers over the Jacobian of the normalization of X_s, as opposed to the compactification by rank one torsion free sheaves.

In the case of reducible spectral curve, we resort to a direct description of the fibre as a stratified space (Theorem 7.7 below). This approach seems to us simpler than attempting to use the spectral curve approach.

All together, our results allow us to prove the following main result (Theorem 8.1). Again, the description of the fibre of the Hitchin map via parabolic modules is an essential ingredient in the proof.

Theorem. *Assume that* $\deg(L) > 0$. *For any* $s \in H^0(X, L^2)$, *the fibre of the Hitchin map* $\mathcal{H} : \mathcal{M}_L^\Lambda \to H^0(X, L^2)$ *is connected. Moreover, if* $s \neq 0$, *the dimension of the fibre is* $\dim(\mathcal{H}^{-1}(s)) = d_L + g - 1$.

We should point out that if $(2, d) = 1$, the degree of L is greater than or equal to $2g-2$ and $L^2 \neq K^2$, then the moduli space is known to be irreducible and then the connectedness of the generic fibre can be used to prove connectedness of all fibres (see Proposition 3.7 below). Thus, it is important to notice that our results apply independently of the degree of the twisting line bundle L (as opposed to most other studies of the moduli space of twisted Higgs pairs carried out so far).

We also point out that Frenkel and Witten [10, Sec. 5.2.2] proved the connectedness of the fibre in the particular case of irreducible spectral curve having only simple nodes as singularities. Their argument makes implicit use of the parabolic line bundles introduced by Bhosle in [4]. In the case of simple nodes parabolic line bundles are the same as parabolic modules but, in general, they are different objects.

It seems quite likely that our relatively explicit description of the fibre could be useful as a tool for further study of geometric and topological properties of the singular fibre and we hope to come back to this question on another occasion.

To finish this introduction, we give an outline of the organization of the paper. In Sections 2 and 3 we give some background on Higgs pairs and the spectral curve and in Section 4 we review the theory of line bundles and the Prym variety in the case of singular irreducible spectral curve. Then, in Section 5, we introduce the parabolic modules and give some results, which are used in Section 6 to describe the fibre of the Hitchin map in the case of irreducible spectral curve. In Section 7 we deal with the case of reducible spectral curve. Finally in Section 8.1 we put everything together to prove our main theorem.

Acknowledgments. We thank U. Bhosle, N. Hitchin and T. Pantev for useful discussions. We also thank an anonymous referee for helpful comments and in particular for pointing out that the connectedness result could be extended to also cover the fibre of the Hitchin map over zero.

2. Higgs pairs

Let X be a compact Riemann surface of genus $g \geqslant 2$ and let $K = K_X$ be the canonical bundle of X. Denote by $\operatorname{Jac}(X) = \operatorname{Pic}^0(X)$ the Jacobian of X, which parametrizes degree 0 holomorphic line bundles on X; we also write $\operatorname{Jac}^d(X) = \operatorname{Pic}^d(X)$.

Definition 2.1. Let L be a fixed holomorphic line bundle over X. An *L-twisted Higgs pair* of type (n, d) over X is a pair (V, φ), where V is a holomorphic vector bundle over X, with $\operatorname{rk}(V) = n$ and $\deg(V) = d$, and φ is a global holomorphic section of $\operatorname{End}(V) \otimes L$, called the *Higgs field*. A *Higgs bundle* is a K-twisted Higgs pair.

Throughout the paper we shall make the following assumption (see Remark 8.2 for comments on the case $\deg(L) = 0$).

Assumption 2.2. The line bundle $L \to X$ satisfies $\deg(L) > 0$.

Definition 2.3. Two L-twisted Higgs pairs (V, φ) and (V', φ') are *isomorphic* if there is a holomorphic isomorphism $f : V \to V'$ such that $\varphi' f = (f \otimes 1_L)\varphi$.

Using GIT, Nitsure [20] constructed the moduli space $\mathcal{M}_L(n, d)$ of S-equivalence classes of rank n and degree d semistable L-twisted Higgs pairs over X. In this paper we shall only be concerned with the case $n = 2$. In this case, the definition of stability of L-twisted Higgs pairs takes the following form.

Definition 2.4. An L-twisted Higgs pair (V, φ) of type $(2, d)$ is:
- *stable* if $\deg(N) < d/2$ for any line bundle $N \subset V$ such that $\varphi(N) \subset NL$.
- *semistable* if $\deg(N) \leqslant d/2$ for any line bundle $N \subset V$ such that $\varphi(N) \subset NL$.
- *polystable* if is semistable and for any line bundle $N_1 \subset V$ such that $\varphi(N_1) \subset N_1 L$ and $\deg(N_1) = d/2$, there is another line bundle $N_2 \subset V$ such that $\varphi(N_2) \subset N_2 L$ and $V = N_1 \oplus N_2$.

The notion of S-equivalence for Higgs pairs is defined analogously to the case of vector bundles and, as in that case, each S-equivalence class contains a unique polystable representative.

The following is known from [5, Theorem 1.2] (see also Nitsure [20, Proposition 7.4] and Simpson [22, Theorem 11.1]): if $(n, d) = 1$, $\deg(L) \geqslant 2g - 2$, and either $L = K$ or $L^n \neq K^n$, then the moduli space $\mathcal{M}_L(n, d)$ is smooth (and irreducible) with

$$\dim \mathcal{M}_L(n, d) = n^2 \deg(L) + 1 + \dim H^1(X, L).$$

There is a map

$$\begin{aligned} p : \mathcal{M}_L(n, d) &\longrightarrow \operatorname{Jac}^d(X) \times H^0(X, L) \\ (V, \varphi) &\longmapsto (\Lambda^n V, \operatorname{tr}(\varphi)). \end{aligned}$$

For each fixed $\Lambda \in \mathrm{Jac}^d(X)$, we define the *moduli space of L-twisted Higgs pairs of type (n,d) with fixed determinant* Λ to be the subvariety of $\mathcal{M}_L(n,d)$ given by the fibre of p over $(\Lambda, 0)$:

$$\mathcal{M}_L^\Lambda(n,d) = p^{-1}(\Lambda, 0).$$

Accordingly, we say that (V,φ) is an *L-twisted Higgs pair with fixed determinant* Λ, if $\Lambda^2 V \cong \Lambda$ and $\mathrm{tr}(\varphi) = 0$.

Nitsure [20, Theorem 7.5] proved that in rank two, the moduli space is connected for any L. His proof goes through unchanged in the fixed determinant case, so we have the following result.

Proposition 2.5. *For any L, d and Λ, the moduli spaces $\mathcal{M}_L(2,d)$ and $\mathcal{M}_L^\Lambda(2,d)$ are connected.* □

Henceforth, we shall assume that $n = 2$ and also drop n and d from the notation, writing simply $\mathcal{M}_L$ and $\mathcal{M}_L^\Lambda$ for the rank two moduli spaces.

The Hitchin map is defined by taking the characteristic polynomial of φ. Thus, on the rank two, fixed determinant moduli space, it is given as follows:

Definition 2.6. The *Hitchin map* on $\mathcal{M}_L^\Lambda$ is the map

$$\begin{array}{rccc} \mathcal{H}: & \mathcal{M}_L^\Lambda & \longrightarrow & H^0(X, L^2) \\ & (V,\varphi) & \longmapsto & \det(\varphi). \end{array}$$

3. The spectral curve

In this section we recall the construction and basic properties of the spectral curve; see [3, 12, 13] for details. The following notation will be used for the remainder of the paper: we denote the degree of the line bundle L by

$$d_L = \deg(L) > 0$$

and write

$$D_s = \mathrm{div}(s) \in \mathrm{Sym}^{2d_L}(X)$$

for the divisor of a section $s \in H^0(X, L^2)$.

3.1. The spectral curve and Higgs pairs.

We begin by reviewing the construction of the spectral curve X_s associated to a section $s \in H^0(X, L^2)$. Consider the complex surface T given by the total space of the line bundle L, and let $\pi : T \to X$ be the projection. The pullback $\pi^* L$ of L to its total space has a tautological section

$$\lambda \in H^0(T, \pi^* L)$$

defined by $\lambda(x) = x$.

Definition 3.1. Let $s \in H^0(X, L^2)$. The *spectral curve* X_s associated to s is the zero scheme in the surface T of the section

$$\lambda^2 + \pi^* s \in H^0(T, \pi^* L^2).$$

The following simple observations (cf. Section 3 of [3]) will be of relevance later.

Remark 3.2. The spectral curve X_s is always reduced, but it may be singular and reducible. In fact, it is smooth if and only if s only has simple zeros and it is irreducible if and only if s is not the square of a section of L. Note that if $D_s = 2\widetilde{D}$ for some divisor $\widetilde{D}$, then $L \cong \mathcal{O}(\widetilde{D}) \otimes N$ where N is a 2-torsion point of the Jacobian and, in this situation, X_s is reducible if and only if $N = \mathcal{O}$.

In the present setting, the fundamental result on the relation between the spectral curve and Higgs pairs can be formulated as follows (cf. [3, Proposition 3.6] and [13]):

Theorem 3.3. *Let $s \in H^0(X, L^2)$ be such that the spectral curve X_s is irreducible. Then there is a bijective correspondence between isomorphism classes of torsion-free sheaves of rank 1 on X_s and isomorphism classes of L-twisted Higgs pairs (V, φ) of rank 2, where $\varphi : V \to V \otimes L$ is a homomorphism with $\det(\varphi) = s$ and $\operatorname{tr}(\varphi) = 0$. The correspondence is given by associating to such a sheaf $\mathcal{F}$ on X_s, the sheaf $\pi_*\mathcal{F}$ on X and the homomorphism $\pi_*\mathcal{F} \to \pi_*\mathcal{F} \otimes L \cong \pi_*(\mathcal{F} \otimes \pi^*L)$ given by multiplication by the canonical section $\lambda \in H^0(X_s, \pi^*L)$.*

This theorem will be the main tool to describe the fibre of $\mathcal{H}$ over a non-zero section $s \in H^0(X, L^2)$.

3.2. **The generic fibre of the Hitchin map.** The material of this subsection is well-known but we include it for completeness. Let $s \in H^0(X, L^2)$ be a section with simple zeros. Then X_s is smooth and we have a double cover

$$\pi : X_s \longrightarrow X, \tag{3.1}$$

of X, ramified over D_s. Since X_s is smooth, the torsion free sheaf $\mathcal{F}$ on X_s, which corresponds to (V, φ) under Theorem 3.3, is in fact a line bundle F.

By the Riemann–Hurwitz formula, the genus of X_s is

$$g(X_s) = 2g - 1 + d_L. \tag{3.2}$$

Moreover, since π has discrete fibres, we have $H^i(X_s, \mathcal{F}) \cong H^i(X, \pi_*\mathcal{F})$, $i = 0, 1$, for any coherent sheaf $\mathcal{F}$ on X_s. Hence, from the Riemann-Roch theorem and from (3.2), it follows that

$$\deg(\pi_*\mathcal{F}) = \deg(\mathcal{F}) - d_L.$$

Let $\mathcal{T}_\pi \subset \operatorname{Jac}^{d+d_L}(X_s)$ be the space of line bundles on X_s such that the determinant of their push forward under π is $\Lambda \in \operatorname{Jac}^d(X)$:

$$\mathcal{T}_\pi = \{F \in \operatorname{Jac}^{d+d_L}(X_s) \mid \det(\pi_*F) \cong \Lambda\}. \tag{3.3}$$

We will now recall the definition of Prym variety. Consider a connected double cover

$$p : X' \longrightarrow X \tag{3.4}$$

of X. Given a divisor $E = \sum n_i q_i$ in X', the norm $\operatorname{Nm}_p(E)$ of E is the divisor in X defined by $\sum n_i p(q_i)$. In terms of line bundles, this gives rise to the norm map

$$\operatorname{Nm}_p : \operatorname{Jac}(X') \longrightarrow \operatorname{Jac}(X)$$

which is a group homomorphism.

Definition 3.4. The *Prym variety of X'*

$$\operatorname{Prym}_p(X') = \{N \in \operatorname{Jac}(X') \mid \operatorname{Nm}_p(N) \cong \mathcal{O}_{X'}\}$$

is the kernel of the norm map with respect to p.

Remark 3.5. It is proved by Mumford in [17] that if $p : X' \to X$ in (3.4) is a ramified double cover, then $\operatorname{Prym}_p(X')$ is connected, whereas if p is a non-trivial unramified cover, then $\operatorname{Prym}_p(X')$ has two connected components. In the latter case, the Prym variety of X' is, sometimes, defined just as the connected component of the kernel of Nm_p which contains the identity. It is, however, important to note that we define the Prym to be the full kernel of Nm_p.

We will be interested in $\mathrm{Prym}_\pi(X_s)$, where π is the double cover (3.1). For any line bundle F in X_s, it is known (see e.g. [3]) that

$$\det(\pi_* F) \cong \mathrm{Nm}_\pi(F) \otimes L^{-1} \tag{3.5}$$

so we can rewrite $\mathcal{T}_\pi$ defined in (3.3) as

$$\mathcal{T}_\pi = \{F \in \mathrm{Jac}^{d+d_L}(X_s) \mid \mathrm{Nm}_\pi(F) \cong \Lambda L\}.$$

For each choice of a fixed element F_0 of $\mathcal{T}_\pi$, we therefore obtain a non-canonical isomorphism

$$\begin{aligned} \mathcal{T}_\pi &\cong \mathrm{Prym}_\pi(X_s) \\ F &\mapsto F F_0^{-1}. \end{aligned} \tag{3.6}$$

Proposition 3.6. *Let $s \in H^0(X, L^2)$ have simple zeros. Then there is an isomorphism between $\mathcal{H}^{-1}(s)$ and $\mathcal{T}_\pi$.*

Proof. Theorem 3.3 gives a bijection between $\mathcal{T}_\pi$ and isomorphism classes of pairs Higgs pairs (V, φ) with fixed determinant Λ. It remains to see that any (V, φ) thus obtained is stable. Suppose that there is a line subbundle $N \subset V = \pi_* F$ such that $\varphi(N) \subset NL$. Then N is an eigenbundle of φ. If the corresponding eigenvalue is $\lambda \in H^0(X, L)$, then the other eigenvalue is $-\lambda$ and hence $\det(\varphi) = -\lambda^2$. It follows that X_s is not smooth (in fact, not even irreducible), so there is no such N. The Higgs pair (V, φ) is therefore stable. $\square$

The Prym variety is a principally polarized abelian variety of dimension $g(X_s) - g = d_L + g - 1$. Hence it follows from Proposition 3.6 and (3.6) that the dimension of the generic fibre of $\mathcal{H}$ over s is

$$\dim \mathcal{H}^{-1}(s) = d_L + g - 1. \tag{3.7}$$

In this generic case s has simple zeros and the double cover (3.1) is ramified over D_s, thus it follows from (3.6) that $\mathcal{H}^{-1}(s)$ is connected.

Under certain conditions on the degree of L, the connectedness of the generic fibre of the Hitchin map actually implies that all fibres are connected. To be precise, we have the following result.

Proposition 3.7. *Assume that $(2, d) = 1$ and that moreover $\deg(L) \geqslant 2g - 2$ and $L^2 \neq K^2$. Then the fibres of the Hitchin map are connected.*

Proof. By Stein factorization ([11, Corollary III.11.15]), the Hitchin map factors as

$$\mathcal{H} = g \circ \mathcal{H}' \colon \mathcal{M}_L^\Lambda \xrightarrow{\mathcal{H}'} Y \xrightarrow{g} H^0(X, L^2),$$

where $\mathcal{H}'$ has connected fibres and g is a finite morphism. Since the generic fibre of $\mathcal{H}$ is connected, so is the generic fibre of g. The hypotheses of the proposition guarantee that the moduli space is irreducible by [5, Theorem 1.2]. Hence $Y = \mathcal{H}'(\mathcal{M}_L^\Lambda)$ is irreducible and we can apply Zariski's main Theorem ([11, Corollary III.11.14]) and deduce that all fibres of g are connected. The result follows. $\square$

Remark 3.8. The previous proposition holds more generally for the moduli spaces $\mathcal{M}_L(n, d)$ and $\mathcal{M}_L^\Lambda(n, d)$, under the conditions $(n, d) = 1$, $\deg(L) \geqslant 2g-2$ and $L^n \neq K^n$. This is because by the independent work of Bottacin (Theorems 4.7.2 and 4.8.4 of [6]) and Markman (Theorem 8.5 and Remark 8.7 of [16]) the generic fibre of the Hitchin map is connected, and the moduli spaces are irreducible by [5, Theorem 1.2].

4. Line bundles on an irreducible singular spectral curve

We now move on to give a description of the fibre of the Hitchin map $\mathcal{H}$ over $s \in H^0(X, L^2)$ when X_s is singular. Thus we consider the situation when the section s has multiple zeros (cf. Remark 3.2). There are two different cases to be considered:

(1) The singular spectral curve X_s is irreducible. This case will be studied in the present section, Section 5 and Section 6.
(2) The spectral curve X_s is singular and has two irreducible components. This case will be studied in Section 7.

Recall that we write $D_s = \operatorname{div}(s)$. Note that case (2) occurs exactly when $D_s = 2\widetilde{D}$ and $\mathcal{O}(\widetilde{D}) \cong L$. Thus, in the remainder of this section and in Sections 5 and 6 we shall consider the case when $s \in H^0(X, L^2)$ has a multiple zero, and moreover assume that if $D_s = 2\widetilde{D}$ for some $\widetilde{D}$, then the line bundle L is not isomorphic to $\mathcal{O}(\widetilde{D})$. The spectral curve X_s is hence singular and irreducible.

4.1. **The Jacobian.** Suppose then that $q \in X$ is a zero of s with multiplicity m. Consider a local coordinate z on X centered in q such that X_s is defined locally by the equation

$$x^2 - z^m = 0.$$

Let $p = \pi^{-1}(q) \in X_s$. If $m > 1$, p is a singular double point of X_s; it is a simple singularity of type A_{m-1} (see for instance Chapter 2, Sec. 8 of [2]).

The normalization

$$\widetilde{\pi} : \widetilde{X}_s \longrightarrow X_s$$

is then a smooth curve and $\widetilde{\pi}$ is an isomorphism outside of $\widetilde{\pi}^{-1}(X_s^{\mathrm{sing}})$, where X_s^{sing} denotes the singular locus of X_s. If m is even, then p is a node (ordinary, if $m = 2$) and $\widetilde{\pi}^{-1}(p) = \{p_1, p_2\}$ with $p_1 \neq p_2$. If $m \geqslant 3$ is odd, p is a cusp and $\widetilde{\pi}^{-1}(p) = p_1$.

The following result is well-known. We include the proof for the convenience of the reader.

Proposition 4.1. *Suppose that X_s has r_1 nodes of types A_{m_i-1}, $i = 1, \dots, r_1$ (m_i even) and, that it has r_2 cusps of types $A_{m'_j-1}$, $j = 1, \dots, r_2$ (m'_j odd). Then there is a short exact sequence*

$$0 \longrightarrow (\mathbb{C}^*)^{r_1} \times \mathbb{C}^{\sum_{i=1}^{r_1}(m_i-2)/2 + \sum_{j=1}^{r_2}(m'_j-1)/2} \longrightarrow \operatorname{Jac}(X_s) \longrightarrow \operatorname{Jac}(\widetilde{X}_s) \longrightarrow 0.$$

Proof. Assume first that the curve X_s has only one singularity. We consider the cases of a node and of a cusp separately.

Let p be a singular point of X_s and $p \in U$ for an open $U \subset X_s$, with local equation $x^2 - z^m = 0$, where $m \geqslant 2$ is even. Around p, X_s is reducible and we write

$$U = U_1 \cup U_2$$

for the decomposition of U into the two irreducible components. The component U_1 (resp. U_2) has then defining equation $x - z^{m/2} = 0$ (resp. $x + z^{m/2} = 0$). The coordinate ring of U is

$$\mathbb{C}[x, z]/(x^2 - z^m).$$

The desingularization of U is given by two disjoint copies of $\mathbb{C}$ which we denote by $\mathbb{C}_1$ and $\mathbb{C}_2$, and the normalization map is

$$f = f_1 \cup f_2 : \mathbb{C}_1 \cup \mathbb{C}_2 \longrightarrow U_1 \cup U_2$$

defined by

$$f(w_1)=f_1(w_1)=(w_1^{m/2},w_1),\ w_1\in\mathbb{C}_1\quad\text{and}\quad f(w_2)=f_2(w_2)=(-w_2^{m/2},w_2),\ w_2\in\mathbb{C}_2.$$

The corresponding map $\phi=\phi_1\oplus\phi_2$ between the coordinate rings is

$$\begin{aligned}\phi:\mathbb{C}[x,z]/(x^2-z^m)&\longrightarrow\mathbb{C}_1[t_1]\oplus\mathbb{C}_2[t_2],\\ g(x,z)&\longmapsto g(t_1^{m/2},t_1)\oplus g(-t_2^{m/2},t_2).\end{aligned}\tag{4.1}$$

If we write

$$g(x,z)=\sum a_{ij}x^iz^j\tag{4.2}$$

then

$$\phi_1(g(x,z))(t_1)=\sum a_{ij}t_1^{\frac{m}{2}i+j}\qquad\text{and}\qquad\phi_2(g(x,z))(t_2)=\sum(-1)^ia_{ij}t_2^{\frac{m}{2}i+j}.$$

As a result, if $\phi_i(g(x,z))^{(k)}(0)$ denotes the k-th derivative of $\phi_i(g(x,z))$ at 0, we obtain

$$\phi_1(g(x,z))^{(k)}(0)=\phi_2(g(x,z))^{(k)}(0)\in\mathbb{C}$$

for every $k=0,\dots,(m-2)/2$, and conclude that

$$U\cong\operatorname{Spec}\{(f_1,f_2)\in\mathbb{C}_1[t_1]\oplus\mathbb{C}_2[t_2]\mid f_1^{(k)}(0)=f_2^{(k)}(0),\ k=0,\dots,(m-2)/2\}.\tag{4.3}$$

Consider now only functions $g(x,z)$ as in (4.2) which do not vanish at p, i.e., with $a_{00}\neq 0$. Then

$$\phi_1(g(x,z))(0)=\phi_2(g(x,z))(0)\in\mathbb{C}^*$$

and also, if $m\geqslant 4$, then

$$\phi_1(g(x,z))^{(k)}(0)=\phi_2(g(x,z))^{(k)}(0)\in\mathbb{C}$$

for every $k=0,\dots,(m-2)/2$. Let $(\mathbb{C}^*\times\mathbb{C}^{(m-2)/2})_p$ be the skyscraper sheaf supported at p and consider the evaluation map $\widetilde{\pi}_*\mathcal{O}^*_{\widetilde{X}_s}\to(\mathbb{C}^*\times\mathbb{C}^{(m-2)/2})_p$ at p given, for each open U, by

$$f\mapsto 0$$

if $p\notin U$, and

$$(f_1,f_2)\mapsto(f_1(p_1)/f_2(p_2),f_1'(p_1)-f_2'(p_2),\dots,f_1^{((m-2)/2)}(p_1)-f_2^{((m-2)/2)}(p_2))$$

if $p\in U$. This has kernel the image of the sheaf map $\phi:\mathcal{O}^*_{X_s}\to\widetilde{\pi}_*\mathcal{O}^*_{\widetilde{X}_s}$ given by $\phi(U)=\mathrm{Id}$ if $p\notin U$ and by (4.1) if $p\in U$. Hence we have the exact sequence

$$0\longrightarrow\mathcal{O}^*_{X_s}\xrightarrow{\phi}\widetilde{\pi}_*\mathcal{O}^*_{\widetilde{X}_s}\longrightarrow(\mathbb{C}^*\times\mathbb{C}^{(m-2)/2})_p\longrightarrow 0.$$

We now consider cusps in the spectral curve. If p is a singular point of X_s and U the open set of X_s containing p with local equation $x^2-z^m=0$, with $m\geqslant 3$ odd, then U is now irreducible and one sees analogously to the previous case that there is a map $\phi:\mathbb{C}[x,z]/(x^2-z^m)\to\mathbb{C}[t]$ between the corresponding coordinate rings of p and of $p_1=\widetilde{\pi}^{-1}(p)$. ϕ is given by

$$\phi(g(x,z))=g(t^m,t^2)\tag{4.4}$$

and

$$U\cong\operatorname{Spec}\{f\in\mathbb{C}[t]\mid f'(0)=f'''(0)=\cdots=f^{(m-2)}(0)=0\}.$$

Consider the skyscraper sheaf $(\mathbb{C}^{(m-1)/2})_p$ supported at p and again the evaluation $\widetilde{\pi}_*\mathcal{O}^*_{\widetilde{X}_s} \to (\mathbb{C}^{(m-1)/2})_p$ at p given, for each open U, by

$$f \mapsto 0$$

if $p \notin U$, and

$$f \mapsto (f'(p), f'''(p), \ldots, f^{(m-2)}(p))$$

if $p \in U$. This has kernel the image of the sheaf map $\phi : \mathcal{O}^*_{X_s} \to \widetilde{\pi}_*\mathcal{O}^*_{\widetilde{X}_s}$, given by $\phi(U) = \mathrm{Id}$ if $p \notin U$ and by (4.4) if $p \in U$. We have thus the following short exact sequence

$$0 \longrightarrow \mathcal{O}^*_{X_s} \xrightarrow{\phi} \widetilde{\pi}_*\mathcal{O}^*_{\widetilde{X}_s} \longrightarrow (\mathbb{C}^{(m-1)/2})_p \longrightarrow 0.$$

In general, if X_s has more singularities, each of which of type A_{m-1}, we have the short exact sequence of sheaves on X_s

$$0 \longrightarrow \mathcal{O}^*_{X_s} \longrightarrow \widetilde{\pi}_*\mathcal{O}^*_{\widetilde{X}_s} \longrightarrow T \longrightarrow 0$$

where T is a skyscraper sheaf supported at X_s^{sing} such that, over each singular point, it is given as one of the above types. Taking the corresponding cohomology sequence, we obtain

$$0 \longrightarrow \bigoplus_{p \in X_s^{\mathrm{sing}}} T_p \xrightarrow{\delta} H^1(X_s, \mathcal{O}^*_{X_s}) \xrightarrow{\widetilde{\pi}^*} H^1(X_s, \widetilde{\pi}_*\mathcal{O}^*_{\widetilde{X}_s}) \longrightarrow 0. \tag{4.5}$$

Since $\widetilde{\pi} : \widetilde{X}_s \to X_s$ is a finite morphism, we have an isomorphism

$$H^1(X_s, \widetilde{\pi}_*\mathcal{O}^*_{\widetilde{X}_s}) \cong H^1(\widetilde{X}_s, \mathcal{O}^*_{\widetilde{X}_s})$$

hence, restricting sequence (4.5) to degree zero line bundles, we obtain

$$0 \longrightarrow \bigoplus_{p \in X_s^{\mathrm{sing}}} T_p \xrightarrow{\delta} \mathrm{Jac}(X_s) \xrightarrow{\widetilde{\pi}^*} \mathrm{Jac}(\widetilde{X}_s) \longrightarrow 0. \tag{4.6}$$

Since we are assuming that X_s has r_1 nodes with types A_{m_i-1}, $i = 1, \ldots, r_1$ and, that it has r_2 cusps of types $A_{m'_j-1}$, $j = 1, \ldots, r_2$, then sequence (4.6) becomes

$$0 \longrightarrow (\mathbb{C}^*)^{r_1} \times \mathbb{C}^{\sum_{i=1}^{r_1}(m_i-2)/2 + \sum_{j=1}^{r_2}(m'_j-1)/2} \xrightarrow{\delta} \mathrm{Jac}(X_s) \xrightarrow{\widetilde{\pi}^*} \mathrm{Jac}(\widetilde{X}_s) \longrightarrow 0$$

and this finishes the proof. $\square$

4.2. **The Prym.** Consider the double cover,

$$\overline{\pi} = \pi \circ \widetilde{\pi} : \widetilde{X}_s \longrightarrow X$$

of X:

$$\begin{tikzcd} \widetilde{X}_s \arrow[dd, "\overline{\pi}"'] \arrow[dr, "\widetilde{\pi}"] & \\ & X_s \arrow[dl, "\pi"] \\ X. & \end{tikzcd}$$

We have the corresponding Prym variety (see Definition 3.4), $\mathrm{Prym}_{\overline{\pi}}(\tilde{X}_s) \subset \mathrm{Jac}(\tilde{X}_s)$. The purpose of this section is to identify the short exact sequence obtained from the one of Proposition 4.1 by restricting to this Prym. We continue under the hypothesis of that proposition, assuming that X_s has r_1 nodes with types A_{m_i-1}, $i = 1, \dots, r_1$, and r_2 cusps of types $A_{m'_j-1}$, $j = 1, \dots, r_2$.

Let

$$D_s = \sum_{i=1}^{r_1} m_i q_i + \sum_{j=1}^{r_2} m'_j q_j \tag{4.7}$$

be the decomposition of the divisor of $s \in H^0(X, L^2)$ into its even and odd parts. Notice that $\overline{\pi} : \widetilde{X}_s \to X$ is ramified exactly over the odd part of the divisor D_s.

Remark 4.2. Since $\deg(D_s) = 2d_L$ is even, then r_2 must be even.

Define the divisor D'_s as

$$D'_s = \sum_{i=1}^{r_1} \frac{m_i}{2} q_i + \sum_{j=1}^{r_2} \frac{m'_j - 1}{2} q_j. \tag{4.8}$$

Then we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X_s} \longrightarrow \widetilde{\pi}_* \mathcal{O}_{\widetilde{X}_s} \longrightarrow \mathcal{O}_{\pi^{-1}(D'_s)} \longrightarrow 0 \tag{4.9}$$

where D'_s is defined in (4.8) and $\pi^{-1}(D'_s)$ is the obvious divisor in X_s.

Remark 4.3. It follows from Proposition 4.1 that

$$\dim \operatorname{Jac}(X_s) = \dim \operatorname{Jac}(\widetilde{X}_s) + \deg(D'_s)$$

hence, denoting by $g(X_s)$ the arithmetic genus of X_s and $g(\widetilde{X}_s)$ the genus of $\widetilde{X}_s$,

$$g(X_s) = g(\widetilde{X}_s) + \deg(D'_s).$$

Lemma 4.4. *Let F be a line bundle over X_s. Then* $\det(\overline{\pi}_* \widetilde{\pi}^* F) \cong \det(\pi_* F) \otimes \mathcal{O}_X(D'_s)$.

Proof. Tensoring the short exact sequence (4.9) by F, noticing that $F \otimes \widetilde{\pi}_* \mathcal{O}_{\widetilde{X}_s} \cong \widetilde{\pi}_* \widetilde{\pi}^* F$, and then applying the push forward by π (which has discrete fibres), we obtain the short exact sequence

$$0 \longrightarrow \pi_* F \longrightarrow \overline{\pi}_* \widetilde{\pi}^* F \longrightarrow \mathcal{O}_{D'_s} \longrightarrow 0$$

on X. The result follows by taking determinants. $\square$

In analogy with (3.3), we define

$$\mathcal{T}_\pi = \{F \in \operatorname{Jac}^{d+d_L}(X_s) \mid \det(\pi_* F) \cong \Lambda\} \subset \operatorname{Jac}^{d+d_L}(X_s). \tag{4.10}$$

From Lemma 4.4, the restriction to $\mathcal{T}_\pi$ of the map $\widetilde{\pi}^* : \operatorname{Jac}^{d+d_L}(X_s) \to \operatorname{Jac}^{d+d_L}(\widetilde{X}_s)$ of Proposition 4.1 (which holds for any degree) takes values in

$$\widetilde{\mathcal{T}}_{\overline{\pi}} = \{F \in \operatorname{Jac}^{d+d_L}(\widetilde{X}_s) \mid \det(\overline{\pi}_* F) \cong \Lambda(D'_s)\}.$$

If F and F' are two line bundles over X_s such that $\widetilde{\pi}^* F \cong \widetilde{\pi}^* F'$, then $\det(\overline{\pi}_* \widetilde{\pi}^* F) = \det(\overline{\pi}_* \widetilde{\pi}^* F')$, so, again from Lemma 4.4, $\det(\pi_* F) = \det(\pi_* F')$. This shows that the fibre of the restriction of $\widetilde{\pi}^*$ to $\mathcal{T}_\pi$ is the same as the fibre of $\widetilde{\pi}^* : \operatorname{Jac}^{d+d_L}(X_s) \to \operatorname{Jac}^{d+d_L}(\widetilde{X}_s)$ in Proposition 4.1. Hence, we have the sequence

$$0 \longrightarrow (\mathbb{C}^*)^{r_1} \times \mathbb{C}^{\sum_{i=1}^{r_1}(m_i-2)/2 + \sum_{j=1}^{r_2}(m'_j-1)/2} \longrightarrow \mathcal{T}_\pi \xrightarrow{\widetilde{\pi}^*} \widetilde{\mathcal{T}}_{\overline{\pi}} \longrightarrow 0.$$

Recall that the spectral curve X_s depends on the line bundle L and on the section $s \in H^0(X, \mathcal{O}(D_s))$ with $L^2 \cong \mathcal{O}(D_s)$. Conversely, if we have D_s and a double cover $\pi : X_s \to X$ ramified over D_s, we recover L since $\pi_* \mathcal{O}_{X_s} \cong \mathcal{O}_X \oplus L^{-1}$ (cf. Lemma 17.2

of [2]). In the same way, we recover the line bundle $\widetilde{L}$ over X, associated to the double cover $\overline{\pi} : \widetilde{X}_s \to X$.

Using the short exact sequence (4.9) and the definitions of L and $\widetilde{L}$, we obtain the following diagram with exact rows and column:

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathcal{O}_X \arrow[r] & \pi_*\mathcal{O}_{X_s} \arrow[r] \arrow[d] & L^{-1} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{O}_X \arrow[r] & \overline{\pi}_*\mathcal{O}_{\widetilde{X}_s} \arrow[r] \arrow[d] & \widetilde{L}^{-1} \arrow[r] & 0 \\ & & \mathcal{O}_{D'_s} \arrow[d] & & \\ & & 0. & & \end{tikzcd}$$

From this it follows that

$$\widetilde{L} \cong L(-D'_s). \tag{4.11}$$

Now, the space $\mathcal{T}_\pi$ is clearly isomorphic to $P_\pi \subset \mathrm{Jac}(X_s)$ defined by

$$P_\pi = \{F \in \mathrm{Jac}(X_s) \mid \det(\pi_*F) \cong L^{-1}\} \tag{4.12}$$

and, from (4.11) and Lemma 4.4,

$$\widetilde{\pi}^*(P_\pi) = \{F \in \mathrm{Jac}(\widetilde{X}_s) \mid \det(\overline{\pi}_*F) \cong \widetilde{L}^{-1}\}.$$

But, since $\det(\overline{\pi}_*N) \cong \mathrm{Nm}_{\overline{\pi}}(N) \otimes \widetilde{L}^{-1}$ (cf. (3.5)), we have

$$\widetilde{\pi}^*(P_\pi) \cong \{F \in \mathrm{Jac}(\widetilde{X}_s) \mid \mathrm{Nm}_\pi(F) \cong \mathcal{O}_X\} = \mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s).$$

We have thus proved the following result.

Proposition 4.5. *Suppose that the spectral curve $X_s \xrightarrow{\pi} X$ has r_1 nodes with types A_{m_i-1}, $i = 1, \dots, r_1$ and, that it has r_2 cusps of types $A_{m'_j-1}$, $j = 1, \dots, r_2$. Let $\tilde{\pi}\colon \tilde{X}_s \to X_s$ be the normalization map and let $\bar{\pi} = \pi \circ \tilde{\pi}\colon \tilde{X}_s \to X$. Then there is a short exact sequence*

$$0 \longrightarrow (\mathbb{C}^*)^{r_1} \times \mathbb{C}^{\sum_{i=1}^{r_1}(m_i-2)/2+\sum_{j=1}^{r_2}(m'_j-1)/2} \longrightarrow P_\pi \xrightarrow{\widetilde{\pi}^*} \mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s) \longrightarrow 0, \tag{4.13}$$

where P_π was defined in (4.12).

5. Compactified Jacobian and parabolic modules

Theorem 3.3 tells us that when X_s is singular, we must take into account not only line bundles on X_s, but also rank 1 torsion-free sheaves on X_s (the case of a spectral curve with a single node goes back to Hitchin's paper [13]). The space of these objects with degree $d + d_L$ provides a natural compactification $\overline{\mathrm{Jac}^{d+d_L}(X_s)}$ of $\mathrm{Jac}^{d+d_L}(X_s)$. Furthermore, as X_s lies inside the complex surface given by the total space T of L, its singularities are planar. From [1] and [21, Theorem A], this is equivalent to $\overline{\mathrm{Jac}^{d+d_L}(X_s)}$ being irreducible and, therefore, $\mathrm{Jac}^{d+d_L}(X_s)$ is dense in $\overline{\mathrm{Jac}^{d+d_L}(X_s)}$. Of course, the compactification of

$\mathrm{Jac}^{d+d_L}(X_s)$ by rank 1, degree $d+d_L$ torsion-free sheaves is determined by the one of $\mathrm{Jac}(X_s)$ since all these spaces are isomorphic.

We shall analyse here the effect of the compactification of $\mathrm{Jac}(X_s)$ on the bundle $\widetilde{\pi}^*:\mathrm{Jac}(X_s)\to\mathrm{Jac}(\widetilde{X}_s)$ of Proposition 4.1, when X_s has singularities of type A_{m-1}.

In order to do this we shall use *parabolic modules*. These objects were first defined by Rego [21] and they were intensively studied by Cook [7, 8]. Cook's work generalizes to arbitrary rank and for any A,D,E singularity the work of Bhosle [4] for ordinary nodes.

5.1. **Parabolic modules for one ordinary node.** For simplicity and motivation, let us start with the case of a unique ordinary node ($m=2$), where we have the sequence

$$0\longrightarrow\mathbb{C}^*\longrightarrow\mathrm{Jac}(X_s)\xrightarrow{\widetilde{\pi}^*}\mathrm{Jac}(\widetilde{X}_s)\longrightarrow 0 \tag{5.1}$$

which is a particular case of Proposition 4.1. Let $p\in X_s$ be the node and $\{p_1,p_2\}=\widetilde{\pi}^{-1}(p)$. The above sequence tells us that a line bundle $\mathcal{F}$ over X_s is determined by a pair (F,λ) consisting by a line bundle F over the normalization $\widetilde{X}_s$ and a non-zero scalar $\lambda\in\mathbb{C}^*$, such that $\mathcal{F}|_{X\smallsetminus\{p\}}\cong\widetilde{\pi}_*F|_{X\smallsetminus\{p\}}$ and $\mathcal{F}_p$ is given by the identification of F_{p_1} with F_{p_2} via λ. In other words, $\mathcal{F}$ fits in the sequence

$$0\longrightarrow\mathcal{F}\longrightarrow\widetilde{\pi}_*F\longrightarrow F_{p_1}\oplus F_{p_2}/U_\lambda(F)\longrightarrow 0 \tag{5.2}$$

where $U_\lambda(F)$ is the 1-dimensional subspace of $F_{p_1}\oplus F_{p_2}$ generated by $(1,\lambda)$. Recall from (4.3) that

$$\mathcal{O}_{X_s,p}\cong\{(f_1,f_2)\in\mathbb{C}_1[t_1]\oplus\mathbb{C}_2[t_2]\mid f_1(0)=f_2(0)\}.$$

Since

$$\mathcal{F}_p\cong\{(s_1,s_2)\in\mathbb{C}_1[t_1]\oplus\mathbb{C}_2[t_2]\mid s_2(0)=\lambda s_1(0)\},$$

we see that $\mathcal{F}$ is indeed an invertible $\mathcal{O}_{X_s}$-module.

This motivates the consideration of all pairs

$$(F,U_\lambda(F))$$

with $F\in\mathrm{Jac}(\widetilde{X}_s)$ and $U_\lambda(F)$ a 1-dimensional subspace of $F_{p_1}\oplus F_{p_2}$. If $U_\lambda(F)$ is generated by $(1,\lambda)$ with $\lambda\in\mathbb{C}^*$, then these are the pairs (F,λ) mentioned in the previous paragraph. But now we are allowing all 1-dimensional subspaces and these are parametrized by $\mathbb{P}^1$. The subspaces spanned by $(1,0)$ and $(0,1)$ correspond, respectively, to $\lambda=0$ and $\lambda=\infty$ in the compactification of $\mathbb{C}^*$ by $\mathbb{P}^1$, so we now allow $\lambda\in\mathbb{C}\cup\{\infty\}$. Such pairs were considered by Bhosle in [4, Definition 2.1] who called them *generalized parabolic line bundles* on $\widetilde{X}_s$. If

$$\mathrm{PMod}_2(\widetilde{X}_s)$$

denotes the moduli space of such pairs $(F,U_\lambda(F))$ (cf. Remark 5.2 below) then the projection on the first coordinate

$$\mathrm{pr}_1:\mathrm{PMod}_2(\widetilde{X}_s)\longrightarrow\mathrm{Jac}(\widetilde{X}_s)$$

gives $\mathrm{PMod}_2(\widetilde{X}_s)$ the structure of a $\mathbb{P}^1$-bundle over $\mathrm{Jac}(\widetilde{X}_s)$:

$$\mathbb{P}^1\longrightarrow\mathrm{PMod}_2(\widetilde{X}_s)\xrightarrow{\mathrm{pr}_1}\mathrm{Jac}(\widetilde{X}_s).$$

Again from $(F,U_\lambda(F))$ we construct a rank 1 torsion-free sheaf $\mathcal{F}$ on X_s, by taking $\mathcal{F}$ in the exact sequence

$$0\longrightarrow\mathcal{F}\longrightarrow\widetilde{\pi}_*F\longrightarrow F_{p_1}\oplus F_{p_2}/U_\lambda(F)\longrightarrow 0.$$

If $\lambda \in \mathbb{C}^*$, this is precisely the construction in (5.2). If $\lambda = 0$, then

$$\mathcal{F}_p \cong \{(s_1, s_2) \in \mathbb{C}_1[t_1] \oplus \mathbb{C}_2[t_2] \mid s_2(0) = 0\}$$

and, if $\lambda = \infty$, then

$$\mathcal{F}_p \cong \{(s_1, s_2) \in \mathbb{C}_1[t_1] \oplus \mathbb{C}_2[t_2] \mid s_1(0) = 0\}.$$

In any case $\mathcal{F}_p$ is a 2-dimensional $\mathcal{O}_{X_s,p}$-module and therefore $\mathcal{F}$ is not invertible. So, using this construction, consider the map

$$\tau : \mathrm{PMod}_2(\widetilde{X}_s) \longrightarrow \overline{\mathrm{Jac}(X_s)}$$

defined by

$$\tau(F, U_\lambda(F)) = \mathcal{F}.$$

From [4, Theorem 3] we know that τ is a surjective morphism and

$$\tau|_{\tau^{-1}(\mathrm{Jac}(X_s))} : \tau^{-1}(\mathrm{Jac}(X_s)) \longrightarrow \mathrm{Jac}(X_s)$$

is an isomorphism. Hence, via $\tau|_{\tau^{-1}(\mathrm{Jac}(X_s))}$, the morphism $\mathrm{pr}_1|_{\tau^{-1}(\mathrm{Jac}(X_s))}$ is identified with $\widetilde{\pi}^*$ and the $\mathbb{P}^1$-bundle pr_1 is a fibrewise compactification of the $\mathbb{C}^*$-bundle $\widetilde{\pi}^*$ in (5.1):

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] & \mathrm{PMod}_2(\widetilde{X}_s) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\tau"] & \mathrm{Jac}(\widetilde{X}_s) \\ & \overline{\mathrm{Jac}(X_s)}. & \end{tikzcd}$$

Moreover, the fibre of τ over $\overline{\mathrm{Jac}(X_s)} \setminus \mathrm{Jac}(X_s)$ consists of two points which are *not* mapped, through pr_1, to the same point of $\mathrm{Jac}(\widetilde{X}_s)$. This shows that $\overline{\mathrm{Jac}(X_s)}$ does not fibre over $\mathrm{Jac}(\widetilde{X}_s)$ through pr_1 and τ. In Example 5.4 below, we will develop this observation in a more general setting.

5.2. **Parabolic modules for A_{m-1} singularities.** Let us now consider the generalization of this construction for the other kinds of singularities we are considering.

Definition 5.1. Suppose that X_s has r_1 nodes $p_1, \ldots, p_{r_1}$, with p_i of type A_{m_i-1} with m_i even and r_2 cusps $q_1, \ldots, q_{r_2}$, with q_j of type $A_{m'_j-1}$ with m'_j odd. For each $i = 1, \ldots, r_1$ let $\widetilde{\pi}^{-1}(p_i) = \{p^i_1, p^i_2\}$ and for each $j = 1, \ldots, r_2$, let $\widetilde{\pi}^{-1}(q_j) = \{\tilde{q}_j\}$. A *parabolic module on $\widetilde{X}_s$ on X_s* is a pair $(F, \underline{U}(F))$ where:

(1) $F \in \mathrm{Jac}(\widetilde{X}_s)$;
(2) $\underline{U}(F) = (U_1(F), \ldots, U_{r_1}(F), U'_1(F), \ldots, U'_{r_2}(F))$ where:
- for each $i = 1, \ldots, r_1$, $U_i(F)$ is an $m_i/2$-dimensional subspace of the vector space $(F_{p^i_1} \oplus F_{p^i_2})^{m_i/2}$ which is also a $\mathcal{O}_{X_s,p_i}$-module via $\widetilde{\pi}_*$;
- for each $j = 1, \ldots, r_2$, $U'_j(F)$ is an $(m'_j - 1)/2$-dimensional subspace of the vector space $F_{\tilde{q}_j}^{m'_j-1}$ which is also a $\mathcal{O}_{X_s,q_j}$-module via $\widetilde{\pi}_*$.

Remark 5.2. Definition 5.1 is the special case of singularities of type A_{m-1} of the general definition of Cook [7, 8]. For ordinary nodes only, a parabolic module is a generalized parabolic line bundle in the sense of Bhosle in [4]. Notice that, in the above definition, the condition for a subspace to be a $\mathcal{O}_{X_s,p}$-module via $\widetilde{\pi}_*$, is always satisfied for ordinary nodes. If we let

$$\underline{m} = (m_1, \ldots, m_{r_1}, m'_1, \ldots, m'_{r_2}),$$

then Cook constructed the moduli space

$$\mathrm{PMod}_{\underline{m}}(\widetilde{X}_s)$$

of parabolic modules on $\widetilde{X}_s$, associated to r_1+r_2 singularities of X_s of type indexed by $\underline{m}$.

From Proposition 4.1 we have the bundle

$$(\mathbb{C}^*)^{r_1}\times\mathbb{C}^{\sum_{i=1}^{r_1}(m_i-2)/2+\sum_{j=1}^{r_2}(m'_j-1)/2}\xrightarrow{\delta}\mathrm{Jac}(X_s)\xrightarrow{\widetilde{\pi}^*}\mathrm{Jac}(\widetilde{X}_s). \tag{5.3}$$

As in the case of a ordinary node, consider the projection on the first factor

$$\mathrm{pr}_1:\mathrm{PMod}_{\underline{m}}(\widetilde{X}_s)\longrightarrow\mathrm{Jac}(\widetilde{X}_s),\qquad (F,\underline{U}(F))\mapsto F.$$

Cook showed that the fibre of this projection is a product

$$\prod_{i=1}^{r_1}P(A_{m_i-1})\times\prod_{j=1}^{r_2}P(A_{m'_j-1})$$

of r_1+r_2 closed subschemes, $P(A_{m_i-1})$ or $P(A_{m'_j-1})$, of a certain Grassmannian (depending on the type of the corresponding singularity). These subschemes are, in general, quite complicated, but it is proved in [8, Proposition 2, p.46] that $P(A_{m-1})$ is *connected* for every m.

Also, there is a finite morphism

$$\tau:\mathrm{PMod}_{\underline{m}}(\widetilde{X}_s)\longrightarrow\overline{\mathrm{Jac}(X_s)}$$

such that $\tau(F,\underline{U}(F))$ is the kernel of the restriction

$$\widetilde{\pi}_*F\longrightarrow\bigoplus_{i=1}^{r_1}(F_{p_1^i}\oplus F_{p_2^i})^{m_i/2}/U_i(F)\oplus\bigoplus_{j=1}^{r_2}F_{\tilde{q}_j}^{m'_j-1}/U'_j(F).$$

The following is proved in Theorem 4.4.1 of [7].

Proposition 5.3. *The restriction $\tau|_{\tau^{-1}(\mathrm{Jac}(X_s))}$ gives an isomorphism $\tau^{-1}(\mathrm{Jac}(X_s))\cong\mathrm{Jac}(X_s)$.*

We conclude that, under the identification of the previous proposition, $\mathrm{pr}_1|_{\tau^{-1}(\mathrm{Jac}(X_s))}$ is a *fibrewise compactification* of the bundle $\widetilde{\pi}^*$ in (5.3),

$$\begin{array}{ccccc}\prod_{i=1}^{r_1}P(A_{m_i-1})\times\prod_{j=1}^{r_2}P(A_{m'_j-1}) & \longrightarrow & \mathrm{PMod}_{\underline{m}}(\widetilde{X}_s) & \xrightarrow{\mathrm{pr}_1} & \mathrm{Jac}(\widetilde{X}_s)\\ & & \Big\downarrow{\scriptstyle\tau} & & \\ & & \overline{\mathrm{Jac}(X_s)}. & & \end{array} \tag{5.4}$$

In conclusion, one can say that $\mathrm{PMod}_m(\widetilde{X}_s)$ *is a compactification of* $\mathrm{Jac}(X_s)$ *which fibres over* $\mathrm{Jac}(\widetilde{X}_s)$. This should be contrasted with the fact that the fibre of τ over $\overline{\mathrm{Jac}(X_s)}\smallsetminus\mathrm{Jac}(X_s)$ consists of a finite number of points which may *not* be mapped to the same point of $\mathrm{Jac}(\widetilde{X}_s)$ by pr_1. The following is an example of this phenomenon. It will be important in the proof of Theorem 6.3 below.

Example 5.4. Let $m \geqslant 2$ be even, $p \in X_s$ be the only singularity of type A_{m-1} and $\widetilde{\pi}^{-1}(p) = \{p_1, p_2\}$. Consider the trivial bundle $\mathcal{O}_{\widetilde{X}_s}$ over $\widetilde{X}_s$ and let $\mathbb{C}_{p_i}$ be its fibre over p_i. Let

$$(\mathcal{O}_{\widetilde{X}_s}, U_0(\mathcal{O}_{\widetilde{X}_s})) \in \mathrm{PMod}_m(\widetilde{X}_s)$$

where $U_0(\mathcal{O}_{\widetilde{X}_s})$ is defined by

$$U_0(\mathcal{O}_{\widetilde{X}_s}) = \{(v_1^1, v_2^1, \ldots, v_{m-1}^{m/2}, v_m^{m/2}) \in (\mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2})^{m/2} \mid v_i^{i/2} = 0 \text{ if } i \text{ is even}\}.$$

$U_0(\mathcal{O}_{\widetilde{X}_s})$ is then a subspace of $(\mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2})^{m/2}$ which is also a $\mathcal{O}_{X_s,p}$-module and

$$(\mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2})^{m/2} / U_0(\mathcal{O}_{\widetilde{X}_s}) = \mathbb{C}_{p_2}^{m/2}.$$

By definition, $\tau(\mathcal{O}_{\widetilde{X}_s}, U_0(\mathcal{O}_{\widetilde{X}_s})) = \mathcal{F}$ fits in

$$0 \longrightarrow \mathcal{F} \longrightarrow \widetilde{\pi}_* \mathcal{O}_{\widetilde{X}_s} \longrightarrow \underbrace{\mathcal{O}_{\widetilde{X}_s,p_2} \oplus \cdots \oplus \mathcal{O}_{\widetilde{X}_s,p_2}}_{m/2 \text{ summands}} \longrightarrow 0 \tag{5.5}$$

thus, over an open set $U \subset X_s$ which contains p,

$$\mathcal{F}(U) \cong \{(s_1, s_2) \in \mathbb{C}_1[t_1] \oplus \mathbb{C}_2[t_2] \mid s_2(0) = s_2'(0) = \cdots = s_2^{((m-2)/2)}(0) = 0\}. \tag{5.6}$$

Now, consider the divisor

$$E = \frac{m}{2} p_1 - \frac{m}{2} p_2$$

over $\widetilde{X}_s$, the line bundle $\mathcal{O}_{\widetilde{X}_s}(E)$ and let

$$(\mathcal{O}_{\widetilde{X}_s}(E), U_\infty(\mathcal{O}_{\widetilde{X}_s}(E)) \in \mathrm{PMod}_m(\widetilde{X}_s),$$

where $U_\infty(\mathcal{O}_{\widetilde{X}_s}(E))$ is given by

$$U_\infty(\mathcal{O}_{\widetilde{X}_s}(E)) = \{(v_1^1, v_2^1, \ldots, v_{m-1}^{m/2}, v_m^{m/2}) \in (\mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2})^{m/2} \mid v_{i-1}^{i/2} = 0 \text{ if } i \text{ is even}\}.$$

Then $U_\infty(\mathcal{O}_{\widetilde{X}_s}(E))$ is a subspace of $(\mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2})^{m/2}$ which is also a $\mathcal{O}_{X_s,p}$-module and

$$(\mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2})^{m/2} / U_\infty(\mathcal{O}_{\widetilde{X}_s}(E)) = \mathbb{C}_{p_1}^{m/2}.$$

So, if we consider $\mathcal{F}'$ such that it fits in

$$0 \longrightarrow \mathcal{F}' \longrightarrow \widetilde{\pi}_* \mathcal{O}_{\widetilde{X}_s} \longrightarrow \underbrace{\mathcal{O}_{\widetilde{X}_s,p_1} \oplus \cdots \oplus \mathcal{O}_{\widetilde{X}_s,p_1}}_{m/2 \text{ summands}} \longrightarrow 0, \tag{5.7}$$

then, over U,

$$\mathcal{F}'(U) \cong \{(s_1, s_2) \in \mathbb{C}_1[t_1] \oplus \mathbb{C}_2[t_2] \mid s_1(0) = s_1'(0) = \cdots = s_1^{((m-2)/2)}(0) = 0\}. \tag{5.8}$$

Tensoring the sequence (5.7) by $\widetilde{\pi}_* \mathcal{O}_{\widetilde{X}_s}(E)$ we conclude that

$$\tau(\mathcal{O}_{\widetilde{X}_s}(E), U_\infty(\mathcal{O}_{\widetilde{X}_s}(E))) = \mathcal{F}' \otimes \widetilde{\pi}_* \mathcal{O}_{\widetilde{X}_s}(E).$$

Now, from (5.6) and (5.8) one has

$$\mathcal{F}' \otimes \widetilde{\pi}_* \mathcal{O}_{\widetilde{X}_s}(E) \cong \mathcal{F}$$

and thus

$$\tau(\mathcal{O}_{\widetilde{X}_s}, U_0(\mathcal{O}_{\widetilde{X}_s})) = \tau(\mathcal{O}_{\widetilde{X}_s}(E), U_\infty(\mathcal{O}_{\widetilde{X}_s}(E))) = \mathcal{F}. \tag{5.9}$$

But

$$\mathrm{pr}_1(\mathcal{O}_{\widetilde{X}_s}, U_0(\mathcal{O}_{\widetilde{X}_s})) = \mathcal{O}_{\widetilde{X}_s}$$

while
$$\mathrm{pr}_1(\mathcal{O}_{\widetilde{X}_s}(E), U_\infty(\mathcal{O}_{\widetilde{X}_s}(E))) = \mathcal{O}_{\widetilde{X}_s}(E).$$
Hence $\overline{\mathrm{Jac}(X_s)}$ does not fibre over $\mathrm{Jac}(\widetilde{X}_s)$ via pr_1 and τ.

6. The fibre of the Hitchin map for irreducible singular spectral curve

In the two preceding sections we have been analysing the structure of the Jacobian of X_s and its compactification by rank 1 torsion-free sheaves. Now we come back to our goal: the description of the fibre $\mathcal{H}^{-1}(s)$.

For the remainder of this section, let $s \in H^0(X, L^2)$ be such that the spectral curve X_s is singular and irreducible. Thus $D_s = \mathrm{div}(s)$ has at least one multiple point, but s is not a square of a section of L (cf. Remark 3.2).

6.1. **Description of the fibre.** As already mentioned, in order to have a description of the fibre of $\mathcal{H}$ over s, and taking into account Theorem 3.3 and Proposition 3.6, we now have to consider also rank 1 torsion-free sheaves over X_s. Let $\overline{P_\pi} \subset \overline{\mathrm{Jac}(X_s)}$ denote the compactification of P_π obtained by taking its closure inside $\overline{\mathrm{Jac}(X_s)}$. Then P_π is dense in $\overline{P_\pi}$.

Consider also the closure $\overline{\mathcal{T}_\pi} \subset \overline{\mathrm{Jac}^{d+d_L}(X_s)}$ of $\mathcal{T}_\pi$, defined in (4.10), induced by the compactification of $\mathrm{Jac}^{d+d_L}(X_s)$. Again, $\overline{\mathcal{T}_\pi}$ is a torsor for $\overline{P_\pi}$, the isomorphism $\overline{\mathcal{T}_\pi} \to \overline{P_\pi}$ being given by $\mathcal{F} \mapsto \mathcal{F} \otimes L_0^{-1}$ where L_0 is a fixed element of $\mathcal{T}_\pi$.

Theorem 6.1. *Let $s \in H^0(X, L^2)$ such that X_s is singular and irreducible. Then $\mathcal{H}^{-1}(s)$ is isomorphic to $\overline{\mathcal{T}_\pi}$.*

Proof. From the definition of $\mathcal{T}_\pi$, it is clear that $\mathcal{T}_\pi \subset \mathcal{H}^{-1}(s)$. So elements in $\mathcal{T}_\pi$ are in one to one correspondence with L-twisted Higgs pairs (V, φ) such that $\Lambda^2 V \cong \Lambda$, $\det(\varphi) = s$ and $\mathrm{tr}(\varphi) = 0$. All these are closed conditions, therefore if $\mathcal{F} \in \overline{\mathcal{T}_\pi}$, the L-twisted Higgs pair (V, φ) obtained from $\mathcal{F}$ will also satisfy the same conditions. Finally, notice that, as in the smooth case, there cannot exist a φ-invariant line subbundle $N \subset V = \pi_*\mathcal{F}$, because that would contradict the irreducibility of X_s. This ensures that (V, φ) is stable, hence $\overline{\mathcal{T}_\pi} \subset \mathcal{H}^{-1}(s)$.

Conversely, let $(V, \varphi) \in \mathcal{H}^{-1}(s)$. It is then identified with a rank 1 torsion-free sheaf $\mathcal{F} \in \overline{\mathrm{Jac}^{d+d_L}(X_s)}$ such that $\det(\pi_*\mathcal{F}) \cong \Lambda$. Now, let $\mathcal{F}' \in \overline{\mathrm{Jac}(X_s)}$ represent the point corresponding to $\mathcal{F}$ under the isomorphism $\overline{\mathrm{Jac}(X_s)} \simeq \overline{\mathrm{Jac}^{d+d_L}(X_s)}$. Then $\mathrm{pr}_1(\tau^{-1}(\mathcal{F}')) \subset \mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s)$ where pr_1 and τ are the morphisms in (5.4). But pr_1 is a bundle which is a fibrewise compactification of the bundle $\widetilde{\pi}^*$, so, when suitably restricted, it also compactifies the bundle (4.13). Hence $\mathcal{F}' \in \overline{P_\pi}$ i.e. $\mathcal{F} \in \overline{\mathcal{T}_\pi}$. □

The dimension of the fibre of $\mathcal{H}$ in the case treated here can be easily computed.

Proposition 6.2. *Let $s \in H^0(X, L^2)$ be such that X_s is irreducible. Then* $\dim(\mathcal{H}^{-1}(s)) = d_L + g - 1$.

Proof. If X_s is smooth, this is just (3.7). Thus it suffices to consider the case when X_s is singular and irreducible. The ramification divisor of $\overline{\pi} : \widetilde{X}_s \to X$ is $\widetilde{D}_s = \sum_{j=1}^{r_2} q_j$. Therefore, by the Riemann-Hurwitz' formula, the genus of $\widetilde{X}_s$ is
$$g(\widetilde{X}_s) = 2g - 1 + r_2/2, \tag{6.1}$$
where we recall that r_2 is even (see Remark 4.2).

From (4.13),

$$\dim \overline{P_\pi} = \dim P_\pi = \dim \mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s) + \sum_{i=1}^{r_1} m_i/2 + \sum_{j=1}^{r_2}(m'_j - 1)/2$$

and $\dim \mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s) = g(\widetilde{X}_s) - g$. Hence from (6.1),

$$\dim \mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s) = g - 1 + r_2/2.$$

Therefore, from Theorem 6.1, the dimension of the fibre of $\mathcal{H}$ over s is

$$\dim \overline{\mathcal{T}_\pi} = \dim \overline{P_\pi} = g - 1 + \frac{1}{2}\left(\sum_{i=1}^{r_1} m_i + \sum_{i=1}^{r_2} m'_j\right) = d_L + g - 1.$$

□

6.2. **Connectedness of the fibre.** Now we can prove that the fibre of $\mathcal{H}$ is connected.

Theorem 6.3. *Assume* $\deg(L) > 0$ *and let* $s \in H^0(X, L^2)$ *be such that* X_s *is irreducible. Then the fibre of* $\mathcal{H} : \mathcal{M}_L^\Lambda \to H^0(X, L^2)$ *over* s *is connected.*

Proof. If $D_s = \mathrm{div}(s)$ has no multiple points, $\mathcal{H}^{-1}(s) = \mathcal{T}_\pi \cong \mathrm{Prym}_\pi(X_s)$ hence connected. (Here we use that $X_s \to X$ is ramified because $\deg(L) > 0$, cf. Remark 3.5.)

Suppose now that D_s has multiple points. From (5.4) we have the morphisms τ and pr_1

$$\begin{array}{ccc} \tau^{-1}(\overline{P_\pi}) & \xrightarrow{\mathrm{pr}_1} & \mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s)\ . \\ \downarrow{\scriptstyle\tau} & & \\ \overline{P_\pi} & & \end{array} \tag{6.2}$$

Assume that $\mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s)$ is connected. This occurs if and only if the cover $\overline{\pi} : \widetilde{X}_s \to X$ is ramified, which is equivalent to saying that D_s has some point with odd multiplicity. From sequence (4.13), P_π is connected hence $\overline{P_\pi}$ is connected, because P_π is dense in $\overline{P_\pi}$ (recall that $\mathrm{Jac}(X_s)$ is dense in $\overline{\mathrm{Jac}(X_s)}$).

Suppose now that $\mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s)$ is not connected, i.e., that D_s is twice another divisor. In [17] Mumford shows that $\mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s)$ has two components, which we denote by $\mathrm{Prym}^0_{\overline{\pi}}(\widetilde{X}_s)$ and $\mathrm{Prym}^1_{\overline{\pi}}(\widetilde{X}_s)$, and that the difference between them lies in the parity of the dimension of the space of sections:

$$\mathrm{Prym}^0_{\overline{\pi}}(\widetilde{X}_s) = \{F \in \mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s) \mid \dim H^0(\widetilde{X}_s, F \otimes \overline{\pi}^* F_0) \text{ even}\}$$

and

$$\mathrm{Prym}^1_{\overline{\pi}}(\widetilde{X}_s) = \{F \in \mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s) \mid \dim H^0(\widetilde{X}_s, F \otimes \overline{\pi}^* F_0) \text{ odd}\}$$

where F_0 is a fixed square root of the canonical line bundle K of X. Moreover, given $q \in \widetilde{X}_s$ and

$$F \in \mathrm{Prym}^i_{\overline{\pi}}(\widetilde{X}_s) \tag{6.3}$$

then

$$F(q - \tilde{\sigma}(q)) \in \mathrm{Prym}^{1-i}_{\overline{\pi}}(\widetilde{X}_s), \tag{6.4}$$

where $\tilde{\sigma} : \widetilde{X}_s \to \widetilde{X}_s$ is the involution exchanging the sheets of the double cover $\overline{\pi} : \widetilde{X}_s \to X$.

Returning to P_π, we see from (4.13) that P_π has two components, because it is a bundle over a space with two connected components with connected fibre. Moreover, since pr_1 in (6.2) is a bundle which compactifies fibrewise $\widetilde{\pi}^*$ in (4.13), the space $\tau^{-1}(\overline{P_\pi})$ also has two connected components. We shall use the construction of Example 5.4 to show that the images under τ of these two components intersect in $\overline{P_\pi}$. We begin by considering two special cases.

Case 1. Suppose that $D_s = mp$ with $m = 2 \mod 4$. Let

$$F_0 = \mathrm{pr}_1^{-1}(\mathcal{O}_{\widetilde{X}_s}) \qquad \text{and} \qquad F_1 = \mathrm{pr}_1^{-1}(\mathcal{O}_{\widetilde{X}_s}((m/2)p_1 - (m/2)p_2)).$$

It follows from Example 5.4 that $\tau(F_0)$ intersects $\tau(F_1)$ because, by (5.9), we have that $\mathcal{F}$ defined in (5.5) belongs to $\tau(F_0) \cap \tau(F_1)$. Now, since $\widetilde{\pi}^{-1}(p) = \{p_1, p_2\}$, we have $\tilde{\sigma}(p_1) = p_2$. Furthermore $\mathcal{O}_{\widetilde{X}_s} \in \mathrm{Prym}^0_{\overline{\pi}}(\widetilde{X}_s)$. Thus it follows from (6.3), (6.4) and (6.2) that

$$\mathcal{O}_{\widetilde{X}_s}((m/2)p_1 - (m/2)p_2) \in \mathrm{Prym}^1_{\overline{\pi}}(\widetilde{X}_s).$$

The conclusion is that the images under τ of the two components of $\tau^{-1}(\overline{P_\pi})$ intersect. Therefore $\overline{P_\pi}$ is connected and hence the same holds for $\overline{\mathcal{T}_\pi}$.

Case 2. Suppose that $D_s = mp$ with $m = 0 \mod 4$. Take again $F_0 = \mathrm{pr}_1^{-1}(\mathcal{O}_{\widetilde{X}_s})$, but now consider $(\mathcal{O}_{\widetilde{X}_s}(p_1 - p_2), U(\mathcal{O}_{\widetilde{X}_s}(p_1 - p_2))) \in F_1 = \mathrm{pr}_1^{-1}(\mathcal{O}_{\widetilde{X}_s}(p_1 - p_2))$, where

$$\begin{aligned} U(\mathcal{O}_{\widetilde{X}_s}(p_1 - p_2)) = \{(v_1^1, v_2^1, \ldots, v_{m-1}^{m/2}, v_m^{m/2}) \in (\mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2})^{m/2} \mid \\ v_1^1 = 0 \text{ and } v_i^{i/2} = 0, \text{ for } 2 \leqslant i \leqslant m-2 \text{ even}\}. \end{aligned}$$

Then $U(\mathcal{O}_{\widetilde{X}_s}(p_1 - p_2))$ is an $m/2$-dimensional subspace of $(\mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2})^{m/2}$ and it is an $\mathcal{O}_{X_s,p}$-module. Using very similar arguments to those of Example 5.4, one sees again that $\tau(F_0)$ intersects $\tau(F_1)$. Moreover, by (6.4), F_0 and F_1 are fibres over different components of $\mathrm{Prym}_{\overline{\pi}}(\widetilde{X}_s)$. Hence the images under τ of the two components of $\tau^{-1}(\overline{P_\pi})$ intersect and we conclude that $\overline{P_\pi}$ and $\overline{\mathcal{T}_\pi}$ are connected.

The general case is proved by using the appropriate local construction at each of the singularities to construct F_0 and F_1 contained in different components of $\tau^{-1}(\overline{P}_\pi)$ whose images under τ intersect in $\overline{P}_\pi$. □

7. The fibre of the Hitchin map for reducible spectral curve

In this section we shall analyse the fibre of $\mathcal{H} : \mathcal{M}_L^\Lambda \to H^0(X, L^2)$ over the sections s such that the spectral curve X_s is reducible. We shall resort to a direct method for describing the fibre $\mathcal{H}^{-1}(s)$. We start by constructing certain fibre bundles over the Jacobian, which will be used in the analysis of the fibre of the Hitchin map. This is carried out in Section 7.1, where we also state a structure theorem about the fibre. The proof of this theorem takes up the next two sections. In Section 7.2 we construct families of Higgs pairs over these fibre bundles, thus obtaining morphisms from their total spaces to the fibre. Having established that, in Section 7.3 we show that these families give rise to a stratification of the fibre. Finally, in Section 7.4, we put the results of the preceding sections together to obtain the connectedness and dimension results for the fibre in the case of reducible spectral curve.

Recall from Remark 3.2 that X_s is reducible if and only if the section $s \in H^0(X, L^2)$ admits a square root $s' \in H^0(X, L)$. Throughout this section we fix such a square root. Thus we have

$$D_s = 2D', \tag{7.1}$$

and $L \cong \mathcal{O}(D')$, where where D_s and D' are the (effective) divisors of s and s' respectively.

7.1. **Stratification of the fibre of the Hitchin map.** First we introduce some notation. For any effective divisor D on X and any line bundle $M \in \mathrm{Jac}^m(X)$ of degree m, define subspaces of $H^0(D', M^2L\Lambda^{-1})$ as follows:

(7.2)

$$E(D, M) = \{q \in H^0(D', M^2L\Lambda^{-1}) \mid q_{|D'-D} = 0\},$$

(7.3)

$$F(D, M) = \{q \in H^0(D', M^2L\Lambda^{-1}) \mid \begin{cases} \mathrm{ord}_p(q) = D'(p) - D(p) & \text{if } 0 < D(p) \leqslant D'(p) \\ q(p) = 0 & \text{otherwise} \end{cases}\}.$$

Remark 7.1. Recall that if $\Delta = \sum n_i p_i$ is any effective divisor in X, then, choosing a local coordinate z_i centred at p_i, a global section of $\mathcal{O}_\Delta$ can be written as $\sum f_i(z)$ where $f_i(z) = \sum_{k=0}^{n_i-1} a_k z_i^k$. Thus, if we choose a local coordinate z around each $p \in \mathrm{Supp}(D')$, the space $E(D, M)$ consists of sections of the form

$$\sum_{p\in\mathrm{Supp}(D')} \sum_{k=D'(p)-D(p)}^{D'(p)-1} a_k z^k$$

and the space $F(D, M)$ consists of such sections with $a_{D'(p)-D(p)} \neq 0$ (we interpret an empty sum as being equal to zero).

Note that the space $E(D, M)$ is a linear subspace of $H^0(D', M^2L\Lambda^{-1})$, while $F(D, M)$ does not contain zero unless $D = 0$, in which case $F(D, M) = 0$. We gather some obvious observations about these spaces in the following Proposition.

Proposition 7.2. *(1) The spaces $E(D, M)$ give a filtration of $H^0(D', M^2L\Lambda^{-1})$ indexed by divisors D satisfying $0 \leqslant D \leqslant D'$. Thus*

$$D_1 \leqslant D_2 \implies E(D_1, M) \subseteq E(D_2, M).$$

Moreover, $E(D, M) = 0$ if and only if $D = 0$, and $E(D, M) = H^0(D', M^2L\Lambda^{-1})$ if and only if $D \geqslant D'$.

(2) The space $E(D, M)$ is the disjoint union

$$E(D, M) = \bigcup_{0\leqslant\overline{D}\leqslant D} F(\overline{D}, M).$$

(3) For any two effective divisors D_1 and D_2 on X, we have

$$E(D_1, M) \cap E(D_2, M) = E(\min\{D_1, D_2\}, M),$$
$$E(D_1, M) \cup E(D_2, M) = E(\max\{D_1, D_2\}, M).$$

Remark 7.3. If $D \neq 0$ then $F(D, M)$ is a linear subspace of $H^0(D', M^2L\Lambda^{-1})$ with a hyperplane removed. Thus $\mathbb{C}^*$ acts by multiplication on $F(D, M)$ and

$$\dim F(D, M)/\mathbb{C}^* = \deg(D) - 1. \tag{7.4}$$

Definition 7.4. Let D be a divisor satisfying $0 \leqslant D \leqslant D'$ and let m be an integer. We denote by

$$\mathcal{E}(D,m) \longrightarrow \mathrm{Jac}^m(X) \tag{7.5}$$

the vector bundle (constructed using the degree m Poincaré line bundle) whose fibre over $M \in \mathrm{Jac}^m(X)$ is $E(D,M)$ and by

$$\mathcal{F}(D,m) \longrightarrow \mathrm{Jac}^m(X) \tag{7.6}$$

the subbundle[1], whose fibre over M is $F(D,M)$. In particular, $\mathcal{F}(0,m) = \mathcal{E}(0,m) = \mathrm{Jac}^m(X)$.

Remark 7.5. Clearly the properties stated in Proposition 7.2 and Remark 7.3 give rise to analogous properties for $\mathcal{E}(D,m)$ and $\mathcal{F}(D,m)$.

Remark 7.6. Notice that, if d is even, the only compact $\mathcal{E}(D,m)$ is $\mathcal{E}(0,d/2)$, whereas if d is odd, the compact $\mathcal{E}(D,m)$ are those of the form $\mathcal{E}(D,(d-1)/2)$, i.e. those such that $\deg(D) = 1$.

Now we state the relation between the spaces $\mathcal{E}(D,m)$ and $\mathcal{F}(D,m)$ just introduced and the fibre of the Hitchin map.

Theorem 7.7. *Let $s \in H^0(X,L^2) \setminus \{0\}$ be such that $D_s = 2D'$ and $L = \mathcal{O}(D')$. For any integer m and any effective divisor $D \leqslant D'$ such that*

$$d/2 - \deg(D) \leqslant m \leqslant d/2, \tag{7.7}$$

there is a morphism $p\colon \mathcal{E}(D,m) \to \mathcal{H}^{-1}(s)$ with the following properties:

1. *The union of the images $p(\mathcal{E}(D,m))$ over all (D,m) satisfying the above conditions covers the fibre $\mathcal{H}^{-1}(s)$.*
2. *If m_1 satisfies (7.7) for a fixed divisor D, then so does $m_2 = d - \deg(D) - m_1$ and*
$$p(\mathcal{F}(D,m_1)) = p(\mathcal{F}(D,m_2)).$$
3. *If $D \neq 0$, then there is a fibrewise $\mathbb{C}^*$-action on $\mathcal{F}(D,m)$ and the restriction of p factors through the quotient to induce a morphism*
$$p\colon \mathcal{F}(D,m)/\mathbb{C}^* \longrightarrow \mathcal{H}^{-1}(s).$$
This morphism is an isomorphism onto its image, unless $m = (d-\deg(D))/2$. In the latter case it is generically two to one, ramified at the preimage in $\mathcal{F}(D,m)$ of the locus of line bundles $M \in \mathrm{Jac}^m(X)$ satisfying $M^2 \cong \Lambda(-D)$.
4. *If $D = 0$, then $m = d/2$ and the restriction of p to $\mathcal{F}(0,d/2)$*
$$p\colon \mathcal{F}(0,d/2) \longrightarrow \mathcal{H}^{-1}(s)$$
is generically two to one, ramified at the locus of line bundles $M \in \mathrm{Jac}^{d/2}(X) = \mathcal{F}(0,d/2)$ satisfying $M^2 \cong \Lambda$.

Proof. The proof takes up the next two sections. In Section 7.2 we construct the morphism p and show that it descends to the quotient by $\mathbb{C}^*$ when $D \neq 0$ (cf. Remarks 7.3 and 7.5). In Section 7.3 we prove the remaining properties stated. □

Remark 7.8. As will be seen from the construction, the morphism p depends on the choice of the square root s' of s.

[1]Note that this bundle is not a vector bundle.

There is a certain redundancy in the description of the fibre of Theorem 7.7. Indeed, note that in (2) of the theorem, we have $m_1 = m_2$ if and only if $m_1 = (d - \deg(D))/2$. Thus we may, by using the larger of the m_i, write

$$\mathcal{H}^{-1}(s) = \bigcup p(\mathcal{E}(D,m)) = \bigcup p(\mathcal{F}(D,m)),$$

where the union is over (D,m), with D an effective divisor $D \leqslant D'$ such that

$$(d - \deg(D))/2 \leqslant m \leqslant d/2. \tag{7.8}$$

7.2. **Construction of the morphism** p**.** Any element of $\mathcal{E}(D,m)$ is given by a pair (q,M), where $M \in \mathrm{Jac}^m(X)$ and

$$q \in E(D,M). \tag{7.9}$$

In order to construct a pair $(V,\varphi) \in \mathcal{H}^{-1}(s)$ from this data, we shall make use of the short exact sequence of complexes of sheaves $0 \to C_1^\bullet \to C_2^\bullet \to C_3^\bullet \to 0$ given by the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
C_1^\bullet : & M^2\Lambda^{-1} \arrow[r, "\mathrm{Id}"] \arrow[d, "="'] & M^2\Lambda^{-1} \arrow[d, "c"] \\
C_2^\bullet : & M^2\Lambda^{-1} \arrow[r, "c"] \arrow[d] & M^2L\Lambda^{-1} \arrow[d, "r(D')"] \\
C_3^\bullet : & 0 \arrow[r, "0"] \arrow[d] & M^2L\Lambda^{-1}|_{D'} \arrow[d] \\
 & 0 & 0,
\end{tikzcd} \tag{7.10}$$

where the map c is defined by $\psi \mapsto \sqrt{-1}\, s'\psi$ and $r(D')$ denotes the restriction to the divisor D'. The associated long exact sequence in hypercohomology shows that $r(D')$ yields an isomorphism

$$r(D')\colon \mathbb{H}^1(X, C_2^\bullet) \xrightarrow{\cong} \mathbb{H}^1(X, C_3^\bullet) = H^0(D', M^2L\Lambda^{-1}).$$

With respect to some open covering $U = \{U_a\}$ of X, choose a representative (x_{ab}, y_a) of the class $r(D')^{-1}(q) \in \mathbb{H}^1(X, C_2^\bullet)$ corresponding to q of (7.9). Then

$$\sqrt{-1}\, s' x_{ab} = y_b - y_a \tag{7.11}$$

and $y_a|_{D'} = q|_{D' \cap U_a}$. We shall construct a pair (V,φ) from (x_{ab}, y_a) as follows. We let V be the vector bundle defined by taking on each open U_a the direct sum

$$M|_{U_a} \oplus M^{-1}\Lambda|_{U_a} \tag{7.12}$$

and gluing over U_{ab} via the map

$$f_{ab} = \begin{pmatrix} 1_M & x_{ab}/2 \\ 0 & 1_{M^{-1}\Lambda} \end{pmatrix}. \tag{7.13}$$

Also over each open U_a, consider the section of $H^0(U_a, \mathrm{End}_0(M \oplus M^{-1}\Lambda) \otimes L)$ given, with respect to the decomposition (7.12), by

$$\varphi_a = \begin{pmatrix} \sqrt{-1}\, s' & y_a \\ 0 & -\sqrt{-1}\, s' \end{pmatrix}. \tag{7.14}$$

From (7.11), one has $f_{ab}\varphi_b = \varphi_a f_{ab}$, so $\{\varphi_a\}$ gives a global traceless endomorphism $\varphi : V \to V \otimes L$.

Remark 7.9. The long exact cohomology sequence associated to the vertical short exact sequence on the right of (7.10) gives an isomorphism $\mathbb{H}^1(X, C_2^\bullet) \cong H^1(X, M^2\Lambda^{-1})$. The vector bundle V is just the extension

$$0 \longrightarrow M \longrightarrow V \longrightarrow M^{-1}\Lambda \longrightarrow 0$$

given by the image of $r(D')^{-1}(q)$ under this isomorphism. The point of the preceding construction using hypercohomology is that it provides a convenient way of encoding the construction of φ.

Note also that when $D = 0$, we have $q = 0$ which gives rise to $V = M \oplus M^{-1}\Lambda$ and $\varphi = \begin{pmatrix} \sqrt{-1}\, s' & 0 \\ 0 & -\sqrt{-1}\, s' \end{pmatrix}$.

We have thus constructed a Higgs pair (V, φ) with $\det(\varphi) = s$. Moreover, by construction

$$M = \ker(\varphi - s') \subset V.$$

It remains to prove that (V, φ) is semistable. For that we need the following obvious observation.

Proposition 7.10. *Let (V, φ) be an L-twisted, rank 2 Higgs pair with $\Lambda^2 V = \Lambda$. Assume that $\det(\varphi) \in H^0(X, L^2)$ has a square root $s' \in H^0(X, L)$. Let $M_\pm = \ker(\varphi \pm s')$ and let $D = \mathrm{div}(\epsilon)$, where $\epsilon\colon M_+M_- \to \Lambda$ is induced by the injective sheaf map $M_+ \oplus M_- \hookrightarrow V$. Then*

$$M_+M_- = \Lambda(-D).$$

Moreover, M_+ and M_- are the only φ-invariant subbundles of V.

In view of this proposition, we only need to check that the semistability condition holds for the φ-invariant subbundles M and $M^{-1}\Lambda(-D)$. But this is equivalent to the assumption (7.7).

We note that our construction can clearly be carried out in families, and thus the set theoretic map $p : \mathcal{E}(s) \to \mathcal{H}^{-1}(s)$ given by

$$p([q], M) = \text{ isomorphism class of } (V, \varphi) \text{ defined above}$$

is in fact a morphism.

Finally, in order to see that when $D \neq 0$, the morphism $p\colon \mathcal{F}(D, m) \to \mathcal{H}^{-1}(s)$ descends to the quotient $\mathcal{F}(D, m)/\mathbb{C}^*$ proceed as follows: if we carry out the above construction with βq for some $\beta \in \mathbb{C}^*$ we obtain a pair $(\tilde{V}, \tilde{\varphi})$ and we get an isomorphism $g\colon (V, \varphi) \to (\tilde{V}, \tilde{\varphi})$ by defining locally, with respect to the decomposition $M|_{U_a} \oplus M^{-1}\Lambda|_{U_a}$,

$$g_a = \begin{pmatrix} \sqrt{\beta^{-1}} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix}$$

for a choice a square root $\sqrt{\beta}$.

7.3. **Conclusion of the proof of Theorem 7.7.** We start by some preliminary observations and results. Let $(V,\varphi)\in\mathcal{H}^{-1}(s)$, with $s\in H^0(X,L^2)\setminus\{0\}$ such that X_s is reducible. Then (V,φ) is a rank 2 semistable L-twisted Higgs pair, such that $\Lambda^2V\cong\Lambda$, $\operatorname{tr}(\varphi)=0$ and $\det(\varphi)=s$. Thus φ has eigenvalues

$$\pm s'\in H^0(X,L)$$

which are generically non-zero. Consider the divisor

$$D'=\operatorname{div}(\pm s'). \tag{7.15}$$

Then we have $D_s=2D'$ and

$$\deg(D')=d_L. \tag{7.16}$$

Let $M_1,M_2\subset V$ be defined by

$$M_1=\ker(\varphi-s')\qquad\text{and}\qquad M_2=\ker(\varphi+s').$$

Since, for $i=1,2$, M_i is locally-free and V/M_i is torsion-free, it follows that M_i is in fact a line subbundle of V. Moreover, on the complement of the divisor D', we have a decomposition $V=M_1\oplus M_2$ with respect to which $\varphi=\left(\begin{smallmatrix}s'&0\\0&-s'\end{smallmatrix}\right)$. Even though this does not extend over D', we shall still refer to the M_i as *eigenbundles* of φ.

Let

$$D=\operatorname{div}(\epsilon), \tag{7.17}$$

where

$$\begin{aligned}\epsilon\colon M_1M_2&\longrightarrow\Lambda^2V\\ x\otimes y&\longmapsto x\wedge y\end{aligned} \tag{7.18}$$

is induced by the inclusion $M_1\oplus M_2\hookrightarrow V$. Then by Proposition 7.10 we have $M_1M_2\cong\Lambda(-D)$ and hence for $i=1,2$

$$d/2-\deg(D)\leqslant\deg(M_i)\leqslant d/2, \tag{7.19}$$

where the second inequality follows from semistability of (V,φ).

Next we carry out a more careful analysis of φ over D'. Recalling that $\Lambda=\Lambda^2V$, we have the canonical extension

$$0\longrightarrow M_1\longrightarrow V\xrightarrow{v\mapsto v\wedge-}M_1^{-1}\Lambda\longrightarrow 0. \tag{7.20}$$

Over a small open set U in X, we can choose a splitting $V_{|U}\cong M_{1|U}\oplus(M_1^{-1}\Lambda)_{|U}$. Using the definition of M_1 and the fact $\operatorname{tr}(\varphi)=0$ we see that, with respect to this decomposition, $\varphi_{|U}$ is of the form

$$\varphi_{|U}=\begin{pmatrix}s'&q\\0&-s'\end{pmatrix} \tag{7.21}$$

for some q. We shall show that the restriction of q to D' (in the scheme theoretic sense) is independent of the choice of splitting.

Proposition 7.11. *There is a well defined section*

$$q_\varphi\in H^0(D',M_1^2L\Lambda^{-1}) \tag{7.22}$$

given by restriction of $\varphi+s'\operatorname{Id}$ to D'.

Proof. Restrict the sequence (7.20) to D'. Since $\varphi(t) = s't$ for any local section t of M_1 and s' vanishes over D', the restriction $\varphi_{|D'}$ factors through $(M_1^{-1}\Lambda)_{|D'}$. The restriction of $s'\operatorname{Id}\colon V \to VL$ similarly factors and hence so does $(\varphi + s'\operatorname{Id})_{|D'}\colon V_{|D'} \to (VL)_{|D'}$, giving a map

$$\varphi_1\colon (M_1^{-1}\Lambda)_{|D'} \to (VL)_{|D'}.$$

Now the local form (7.21) of φ shows that the composition φ_1 with the projection $(VL)_{|D'} \to (M_1^{-1}\Lambda)_{|D'}$ vanishes. Hence we have a lift of φ_1 to a map

$$q_\varphi\colon (M_1^{-1}\Lambda)_{|D'} \to (M_1L)_{|D'}$$

as claimed. □

We have that q_φ is a holomorphic section of $M_1^2L\Lambda^{-1}$, over the subscheme D' so, if $q_\varphi \neq 0$ and $\operatorname{div}(q_\varphi)$ is the corresponding divisor, then

$$\operatorname{div}(q_\varphi) \geqslant 0. \tag{7.23}$$

Lemma 7.12. *Let q_φ be the section given in Proposition 7.11, let D' be the effective divisor defined in (7.15) and let D be the effective divisor defined in (7.17). Then the following statements hold.*

(1) *If $q_\varphi \neq 0$, then $\operatorname{div}(q_\varphi) = D' - D$.*
(2) *$0 \leqslant D \leqslant D'$.*
(3) *$q_\varphi = 0$ if and only if $D = 0$.*

Proof. First note that (2) follows immediately from (1) and (7.23). In order to prove (1) and (3), we consider the local form (7.21) of φ in a neighbourhood U of a point $p \in X$. Let $k_1(p) = \operatorname{ord}_p(s') \geqslant 0$ and $k_2(p) = \operatorname{ord}_p(q) \geqslant 0$ so that, if z is a local coordinate on U centred at p, then

$$s'(z) = z^{k_1(p)} f'_s(z) \qquad \text{and} \qquad q(z) = z^{k_2(p)} f_q(z)$$

with $\operatorname{ord}_0(f'_s) = \operatorname{ord}_0(f_q) = 0$.

Let (z_1, z_2) denote the coordinates on $\mathbb{C} \oplus \mathbb{C}$. We have chosen a trivialization of V over U such that $M|_U$ is defined by the equation $z_2 = 0$, i.e., $M|_U$ is the subspace of $\mathbb{C} \oplus \mathbb{C}$ generated by the vector $(1, 0)$. Also, the other eigenbundle $M^{-1}\Lambda(-D)|_U$ is defined by the equation

$$2s'z_1 + qz_2 = 0.$$

We have the following observations:

- if $k_1(p) < k_2(p)$, then $M^{-1}\Lambda(-D)|_U$ is the subspace of $\mathbb{C} \oplus \mathbb{C}$ generated by the vector $(0, 1)$, thus $\epsilon : M_1M_2 \to \Lambda^2 V$ does not vanish in p;
- if $k_1(p) = k_2(p)$, then $M^{-1}\Lambda(-D)|_U$ is the subspace of $\mathbb{C} \oplus \mathbb{C}$ generated by the vector $(1, -2f'_s(0)f_q(0)^{-1})$, so again ϵ is not zero in p;
- if $k_1(p) > k_2(p)$, then $M^{-1}\Lambda(-D)|_U$ is the subspace of $\mathbb{C} \oplus \mathbb{C}$ generated by the vector $(1, 0)$, and now ϵ vanishes in p with $\operatorname{ord}_p(\epsilon) = k_1(p) - k_2(p) > 0$.

By definition, $D = \operatorname{div}(\epsilon)$, so we have just seen that

$$D(p) = \begin{cases} k_1(p) - k_2(p) & \text{if } k_1(p) > k_2(p) \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $D' = \mathrm{div}(s')$, i.e., $D'(p) = \mathrm{ord}_p(s') = k_1(p)$. Hence,

$$D(p) = \begin{cases} D'(p) - k_2(p) & \text{if } D'(p) > k_2(p) \\ 0 & \text{otherwise.} \end{cases} \tag{7.24}$$

Now notice that at each point $p \in \mathrm{Supp}(D')$ the section q_φ is given exactly by the value of q at p (cf. (7.21)), i.e.,

$$\mathrm{div}(q_\varphi)(p) = \mathrm{ord}_p(q) = k_2(p).$$

Hence, (7.24) proves (1).

Finally (3) also follows from (7.24). □

Remark 7.13. In the situation of irreducible spectral curve X_s, the eigenbundles are only globally well defined for the pull back of (V, φ) to X_s. However, one may still define a divisor D on X by using the locally defined eigenbundles, and it turns out that $D = D'$, in contrast to the present situation. The basic reason for this is that on the spectral curve the eigenbundles are interchanged by the involution on X_s, thus tying them more tightly to each other.

Finally we can finish the proof of Theorem 7.7. We start by proving (1). Let $(V, \varphi) \in \mathcal{H}^{-1}(s)$. In the preceding we have constructed a divisor D satisfying $0 \leqslant D \leqslant D'$ (by Lemma 7.12) and a φ-invariant line subbundle $M_1 \subset V$ such that $m = \deg(M_1)$ satisfies the bound (7.7) (by (7.19)). We also constructed $q_\varphi \in H^0(D', M_1^2 L \Lambda^{-1})$, which by Lemma 7.12 satisfies $q_\varphi \in E(D, M)$. Thus we have associated to $(V, \varphi) \in \mathcal{H}^{-1}(s)$ an element

$$\zeta(V, \varphi) = (q_\varphi, M_1) \in \mathcal{E}(D, m).$$

Analysing the construction of the morphism p given in Section 7.2, it is easy to see that

$$p(\zeta(V, \varphi)) = (V, \varphi).$$

This proves the surjectivity of p.

In order to prove the remaining claims of Theorem 7.7, let M_i be the eigenbundles of a pair (V, φ) for $i = 1, 2$. We carried out the preceding construction of $\zeta(V, \varphi)$ using the eigenbundle M_1, but we could equally well have carried it out using the eigenbundle M_2, which has degree $\deg(M_2) = d - \deg(D) - \deg(M_1)$. This proves (2) of Theorem 7.7. Finally the remaining claims of the theorem follow because it is easy to see that the choice of the eigenbundle M_1 is the only ambiguity in the construction. Thus $p\colon \mathcal{E}(D, m) \to \mathcal{H}^{-1}(s)$ can only fail to be injective if $m = d - \deg(D) - m \iff m = (d - \deg(D))/2$ and the eigenbundles are non isomorphic, in which case p is two to one.

7.4. **Connectedness and dimension of the fibre.** For a given m, let d_m be the smallest integer greater than or equal to $d/2 - m$, i.e.,

$$d_m = \lceil d/2 - m \rceil \tag{7.25}$$

and define

$$\mathcal{E}(m) = \bigcup_{\deg(D) \geqslant d_m} \mathcal{E}(D, m).$$

Then Theorem 7.7 shows that

$$\mathcal{H}^{-1}(s) = \bigcup_{d/2 - d_L \leqslant m \leqslant d/2} p(\mathcal{E}(m)) \tag{7.26}$$

Lemma 7.14. *The subspace $p(\mathcal{E}(m)) \subset \mathcal{H}^{-1}(s)$ is connected for any m satisfying $d/2 - d_L \leqslant m \leqslant d/2$.*

Proof. It suffices to see that $\mathcal{E}(m)$ is connected. But from (1) of Proposition 7.2 we have

$$\bigcup_{\deg(D)\geqslant d_m} E(D,M) = H^0(D', M^2L\Lambda^{-1}).$$

Hence $\mathcal{E}(m)$ is simply the natural vector bundle over $\mathrm{Jac}^m(X)$ with fibres $H^0(D', M^2L\Lambda^{-1})$. ☐

We now prove that the fibre of $\mathcal{H}$ over a point such that the spectral curve is reducible, is connected.

Theorem 7.15. *Let $s \in H^0(X, L^2) \smallsetminus \{0\}$ such that X_s is reducible. Then the fibre of $\mathcal{H} : \mathcal{M}_L^\Lambda \to H^0(X, L^2)$ over s is connected.*

Proof. For any m, take a divisor D of degree d_m (defined in (7.25)). Then

$$d - \deg(D) - m = [d/2]$$

and (2) of Theorem 7.7 shows that

$$p(\mathcal{E}(m)) \cap p(\mathcal{E}([d/2])) \neq \emptyset.$$

The conclusion is now immediate from (7.26) and Lemma 7.14. ☐

Now we compute the dimension of every stratum of the fibre and, since it is connected, of the fibre itself.

Proposition 7.16. *If $s \in H^0(X, L^2)$ is such that $D_s = 2D'$ and $L = \mathcal{O}(D')$, then:*

1. $\dim \mathcal{F}(0, d/2) = g$;
2. *for any effective divisor $D \leqslant D'$ and m satisfying (7.7),* $\dim \mathcal{F}(D,m) = \deg(D) + g - 1$;
3. $\dim \mathcal{H}^{-1}(s) = d_L + g - 1$.

Proof. Immediate from the definition of $\mathcal{F}(D,m)$ and Theorem 7.7. ☐

Hence, every stratum of $\mathcal{H}^{-1}(s)$ has dimension less or equal than $\deg(D') + g - 1 = d_L + g - 1$ and this upper bound is reached only by $\mathcal{F}(D', m)$ for any integer m such that $d/2 - d_L \leqslant m \leqslant d/2$.

Remark 7.17. Recall that $L \cong \mathcal{O}(D')$. Let now $\tilde{D}$ be a divisor such that $\tilde{D} > D'$, and consider $\tilde{L} \cong \mathcal{O}(\tilde{D})$. Denote by $\mathcal{M}_{\tilde{L}}^\Lambda$ the moduli space of $\tilde{L}$-twisted Higgs pairs and let $\tilde{\mathcal{H}} : \mathcal{M}_{\tilde{L}}^\Lambda \to H^0(X, \tilde{L}^2)$ be the corresponding Hitchin map. Let $\tilde{E}(D,m)$ be the space defined in the same way as $E(D,m)$ in (7.2), but now using $\tilde{D}$ instead of D'. By Remark 7.1 we see that, for an effective divisor D such that $D \leqslant D'$, there is a natural inclusion

$$E(D,M) \hookrightarrow \tilde{E}(\tilde{D} - D' + D, M),$$

which induces an inclusion

$$\mathcal{E}(D,m) \hookrightarrow \tilde{\mathcal{E}}(\tilde{D} - D' + D, m),$$

where $\tilde{\mathcal{E}}(\tilde{D} - D' + D, M)$ is the bundle over $\mathrm{Jac}^m(X)$ whose fibre over M is $\tilde{E}(\tilde{D} - D' + D, M)$, as in Definition 7.4. Now, we have a map $p : \mathcal{E}(D,m) \to \mathcal{H}^{-1}(s)$, depending on a choice of a square root s' of s (see Remark 7.8), satisfying the conditions of Theorem 7.7. Given $\tilde{s}' \in H^0(X, \tilde{L})$, we can apply Theorem 7.7 to $\tilde{\mathcal{H}}^{-1}(\tilde{s}'^2)$, to get a map $\tilde{p}$:

$\tilde{\mathcal{E}}(\tilde{D} - D' + D, m) \to \tilde{\mathcal{H}}^{-1}(\tilde{s}'^2)$, depending on the choice of $\tilde{s}'$, and satisfying all the conditions stated in that theorem. This, together with the above inclusions, induces an inclusion $\mathcal{H}^{-1}(s) \hookrightarrow \tilde{\mathcal{H}}^{-1}(\tilde{s}'^2)$.

In other words, the fibre of $\mathcal{H} : \mathcal{M}_L^\Lambda \to H^0(X, L^2)$ over $s \in H^0(X, L^2)$ such that X_s is reducible, can be identified as a subvariety of the fibre of $\tilde{\mathcal{H}} : \mathcal{M}_{\tilde{L}}^\Lambda \to H^0(X, \tilde{L}^2)$, over an $\tilde{s} \in H^0(X, \tilde{L}^2)$ such that $X_{\tilde{s}}$ is reducible and $\tilde{L}$ is a line bundle of the form $\tilde{L} \cong \mathcal{O}(\tilde{D})$ for some $\tilde{D} > D'$.

8. Main Theorem

We finish by putting everything together to obtain the following main result.

Theorem 8.1. *Let $L \to X$ be a line bundle with $\deg(L) > 0$ and let $\mathcal{H} : \mathcal{M}_L^\Lambda \to H^0(X, L^2)$ be the Hitchin map. For any section $s \in H^0(X, L^2)$, the fibre $\mathcal{H}^{-1}(s)$ is connected. Moreover, if $s \neq 0$, the dimension of the fibre is $\dim(\mathcal{H}^{-1}(s)) = d_L + g - 1$.*

Proof. Assume first that $s \neq 0$. The connectedness of $\mathcal{H}^{-1}(s)$ is immediate from Theorems 6.3, and 7.15. The dimension formula follows from (3.7), and Propositions 6.2 and 7.16.

It remains to prove that $\mathcal{H}^{-1}(0)$ is connected[2]. Assume that this is not the case. Then there exist non-empty open sets U and V in $\mathcal{M}_L^\Lambda$ such that $\mathcal{H}^{-1}(0) \subseteq U \cup V$. Since proper maps are closed, there is an open ball around zero, $W \subset H^0(X, L^2)$, such that $\mathcal{H}^{-1}(W) \subseteq U \cup V$. Hence we may write $\mathcal{H}^{-1}(W) = \tilde{U} \cup \tilde{V}$ for non-empty disjoint open sets $\tilde{U}$ and $\tilde{V}$. The connectedness of the fibres $\mathcal{H}^{-1}(s)$ for $s \neq 0$ implies, again using that $\mathcal{H}$ is closed, that $\mathcal{H}^{-1}(W \setminus \{0\})$ is connected. Hence either $\tilde{U}$ or $\tilde{V}$ is contained in the fibre over zero and is therefore a connected component of the moduli space $\mathcal{M}_L^\Lambda$. Since the moduli space is connected (Proposition 2.5) we have reached a contradiction. □

Remark 8.2. Throughout this paper we have always considered a line bundle L of positive degree. Let us say a few words about the situation for L with $\deg(L) = 0$.

Suppose that $\deg(L) = 0$ and that $s \in H^0(X, L^2)$ is non-zero. Hence, $L^2 \cong \mathcal{O}$ and $s \in \mathbb{C}^*$. Suppose that s does not admit a square root. Then the spectral curve $\pi : X_s \to X$ is a non-trivial unramified double cover. In this case, we are in the smooth generic situation, and $H^{-1}(s) \cong \mathrm{Prym}_\pi(X_s)$. However, as shown in [17], $\mathrm{Prym}_\pi(X_s)$ has two connected components, thus the fibre of $\mathcal{H}$ over s is disconnected.

On the other hand, if $s = s'^2$ for some $s' \in \mathbb{C}^*$, then $L \cong \mathcal{O}$, and X_s is a disconnected double cover of X. If $(V, \varphi) \in \mathcal{H}^{-1}(s)$, then $V \cong M \oplus \Lambda M^{-1}$ and

$$\varphi = \pm \begin{pmatrix} \sqrt{-1}s' & 0 \\ 0 & -\sqrt{-1}s' \end{pmatrix}, \quad \text{for some } M \in \mathrm{Jac}^{d/2}(X_s).$$

It follows that there is a double cover $\mathrm{Jac}^{d/2}(X_s) \coprod \mathrm{Jac}^{d/2}(X_s) \to \mathcal{H}^{-1}(s)$, ramified over pairs (V, φ) such that $V \cong M \oplus \Lambda M^{-1}$, with $M^2 \cong \Lambda$. In other words, $\mathcal{H}^{-1}(s)$ is the disjoint union of two copies of $\mathrm{Jac}^{d/2}(X_s)$ glued together along the subvariety defined by $M^2 \cong \Lambda$. Hence $\mathcal{H}^{-1}(s)$ is connected and has dimension g. Finally, notice that the image of $p : \mathcal{F}(0, d/2) \to \mathcal{H}^{-1}(s)$ in (4) of Theorem 7.7 (in the case of $\deg(L) = d_L > 0$) can be identified with the image of the above ramified double cover $\mathrm{Jac}^{d/2}(X_s) \to \mathcal{H}^{-1}(s)$ for $L \cong \mathcal{O}$. This is an example of the phenomenon described in Remark 7.17.

[2]This argument was outlined to us by the referee.

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