JULIA SETS AS BURIED JULIA COMPONENTS

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ABSTRACT. Let f be a rational map with degree $d \geq 2$ whose Julia set is connected but not equal to the whole Riemann sphere. It is proved that there exists a rational map g such that g contains a buried Julia component on which the dynamics is quasiconformally conjugate to that of f on the Julia set if and only if f does not have parabolic basins and Siegel disks. If such g exists, then the degree can be chosen such that $\deg(g) \leq 7d - 2$. In particular, if f is a polynomial, then g can be chosen such that $\deg(g) \leq 4d + 4$. Moreover, some quartic and cubic rational maps whose Julia sets contain buried Jordan curves are also constructed.

CONTENTS

1. INTRODUCTION

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map with degree at least two. The *Julia set* $J(f)$ of f is the set of the points which fail to be normal in the sense of Montel. Or equivalently, $J(f)$ is the closure of the repelling periodic points of f. The complement of $J(f)$ is the *Fatou set* of f which we denote by $F(f)$. Each connected component of $J(f)$ (resp. $F(f)$) is called a *Julia* (resp. *Fatou*) *component*. It was

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known that the Julia components can exhibit several kinds of shapes: the singletons, Jordan curves and some other complex topologies.

A Julia component (or a point on the Julia set) is called *buried* provided it is disjoint with the boundary of any Fatou component. In particular, buried Julia components cannot occur in the polynomials since the Julia set coincides with the boundary of the unbounded Fatou component. The first example of buried Julia component was constructed by McMullen in [McM88]. Consider the family of rational maps which is given by

$$f_{c,\lambda}(z) = z^\ell + c + \frac{\lambda}{z^m}, \text{ where } \ell \geq 2, m \geq 1 \text{ and } c, \lambda \in \mathbb{C}.$$

McMullen proved that if $c = 0$, $1/\ell + 1/m < 1$ and $\lambda \in \mathbb{C} \setminus \{0\}$ is small enough, then $J(f_{c,\lambda})$ is a Cantor set of circles and contains some buried Julia components which are Jordan curves (see also [DLU05]). In particular, the same phenomenon occurs if $c \neq 0$ is small (see [XQY14]). In [PT00], Pilgrim and Tan studied an example which is slightly different from $f_{c,\lambda}$:

$$\widetilde{f}_{-1,\lambda}(z) = \frac{1}{z} \circ (z^2 - 1) \circ \frac{1}{z} + \frac{\lambda}{z^3} = \frac{z^2}{1 - z^2} + \frac{\lambda}{z^3}, \text{ where } \lambda \in \mathbb{C}.$$

They proved that if $\lambda \neq 0$ is small enough, then $J(\widetilde{f}_{-1,\lambda})$ contains a Julia component which is homeomorphic to the Julia of $z \mapsto z^2 - 1$. Moreover, the Julia components of $J(\widetilde{f}_{-1,\lambda})$ which are not eventually periodic are buried Jordan curves.

Let f be a rational map with degree at least two. Beardon proved that the Julia set $J(f)$ has buried components if it is disconnected and every component of the Fatou set $F(f)$ has finite connectivity [Bea91b]. Then Qiao proved that $J(f)$ has buried components is equivalent to it is disconnected and $F(f)$ has no completely invariant component [Qia95]. Although the existence of buried Julia components which are singletons was known very early, the first specific example was given in [Qia95] by applying Beardon's criterion. He showed that the Julia set of Herman's example contains some buried components which are singletons.

In 2008 a specific example in [BDGR08] shows that if $c \neq 0$ is the center of a hyperbolic component of the Multibrot set M_q, then $J(f_{c,\lambda})$ contains both Jordan curves and singletons which are buried Julia components, where $q = \ell = m \geq 3$ and $\lambda \neq 0$ is small enough (see also [GMR13]).

At the time when McMullen gave the first example of buried Julia components [McM88], he raised also the following question: Does there exist a buried Julia component of a rational map with degree less than 5 which is not a singleton? Until 2015, Godillon constructed a family of *cubic* rational maps[1]

$$F_\lambda(z) = \frac{(1-\lambda)[(1 - 4\lambda + 6\lambda^2 - \lambda^3)z - 2\lambda^3]}{(z-1)^2[(1 - \lambda - \lambda^2)z - 2\lambda^2(1-\lambda)]}, \text{ where } \lambda \in \mathbb{C}, \tag{1}$$

and proved that $J(F_\lambda)$ contains a buried Julia component which is neither a Jordan curve nor a singleton if $\lambda \neq 0$ is small enough [God15]. In particular, his buried Julia component is homeomorphic to the Julia set of $z \mapsto 1/(z-1)^2$, which has a super-attracting periodic orbit $0 \mapsto 1 \mapsto \infty \mapsto 0$ with period 3.

For the buried *points* in the Julia sets of rational maps, the problem of existence has not been solved completely. Makienko conjectured that the Julia set of a rational map f has buried points if and only if there is no completely invariant component of the Fatou set of $f^{\circ 2}$ (see [Mak87]). One can refer to [Mor97], [Qia97], [Mor00], [SY03], [CM^{2}R09], [CMT13] and the references therein for the progress.

[1] The quadratic rational map cannot contain any buried Julia component since its Julia set is either connected or totally disconnected.

1.1. **Statement of the main results.** Still in [McM88], McMullen asked the following question: Let J_0 be a periodic Julia component of a rational map f. Does there exist another rational map g such that the restriction of g on $J(g)$ is quasiconformally conjugate to f on J_0? He answered this question affirmatively and actually, in complex dynamics, how to decompose a dynamical systems is an important problem.

In this article, we are interested in the inverse question of McMullen: Could any connected Julia set appear as a buried Julia component of a higher degree rational map? As a complete answer to this question, we prove the following result.

Theorem 1.1. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map with degree at least two whose Julia set $J(f) \neq \widehat{\mathbb{C}}$ is connected. Then there is a rational map g such that g has a buried Julia component on which g is quasiconformally conjugate to f on $J(f)$ if and only if f does not have parabolic basins and Siegel disks.*

If such g exists, then the degree of g can be chosen such that $\deg(g) \leq 7d - 2$, where $d = \deg(f)$. In particular, $\deg(g) \leq 4d + 4$ if f is a polynomial.

Theorem 1.1 implies that a rational map can contain buried Julia components which are 'almost' arbitrary. Indeed, the assumptions in Theorem 1.1 allow the presence of critical points on the buried Julia components. The upper bound of the degree of g is not sharp in general. In fact, if $J(f)$ is not a dendrite (see §§3, 4), then $\deg(g)$ can be chosen at most $7d - 6$ (and at most $4d$ if f is a polynomial).

The proof of Theorem 1.1 is based on successive perturbations and quasiconformal surgery. See Figure 1 for an example, which illustrates the process of perturbations[2] of the map $z \mapsto z^2 - 1$.

A parameter $c \in \mathbb{C}$ (resp. $b \in \mathbb{C}$) is called the *center* of a hyperbolic component of the Mandelbrot set (resp. Multibrot set M_q) if the critical point 0 is a periodic point with period $p \geq 1$ under the iterate of $P_c(z) = z^2 + c$ (resp. $z \mapsto z^q + b$ with $q \geq 3$). In this article, we consider the singular perturbation of this kind of unicritical polynomials and prove the following result.

Theorem 1.2. *Let $c \neq 0$ (resp. $b \neq 0$) be the center of a hyperbolic component of the Mandelbrot set (resp. Multibrot set M_q with $q \geq 3$). If $\lambda \neq 0$ (resp. $\mu \neq 0$) is small enough, then the Julia sets of*

$$f_\lambda(z) = z^2 + c + \frac{\lambda}{(z-c)^2} \quad \text{and} \quad g_\mu(z) = z^q + b + \frac{\mu}{z-b} \tag{2}$$

contain some buried Julia components which are Jordan curves.

This answers a question of McMullen proposed in [McM88] by simple examples. See Figure 2 for two specific examples. A similar singular perturbation of $z \mapsto z^q + b$ with the form $z \mapsto z^q + b + \mu/z^q$ was considered in [BDGR08], where $b \neq 0$ is the center of a hyperbolic component of the Multibrot set M_q with $q \geq 3$. However, the perturbation there is made at the *critical point* 0. In order to obtain the buried Julia components, that perturbation makes the degree of the resulting rational map at least 6. Here our perturbation is made at the *critical value*. Hence the degree of the perturbed rational map can be reduced to 4 (if $q = 3$).

On the other hand, a singular perturbation of $z \mapsto z^2 + c$ with the form $z \mapsto z^2 + c + \lambda/z^2$ was considered in [Mar08], where $c \neq 0$ is the center of a hyperbolic component of the Mandelbrot set. Similarly, the perturbation is made also at the

[2]Note that in Figure 1, the degree of the last rational map is 12. However, by the proof of a special case of Theorem 1.1 in §3, there should exist a rational map with degree 8 whose Julia set contains a buried copy of basilica. This kind of rational maps exist indeed. But in order to obtain the buried Julia component, the second parameter μ there should be chosen extremely small. In this case we cannot obtain a good picture.

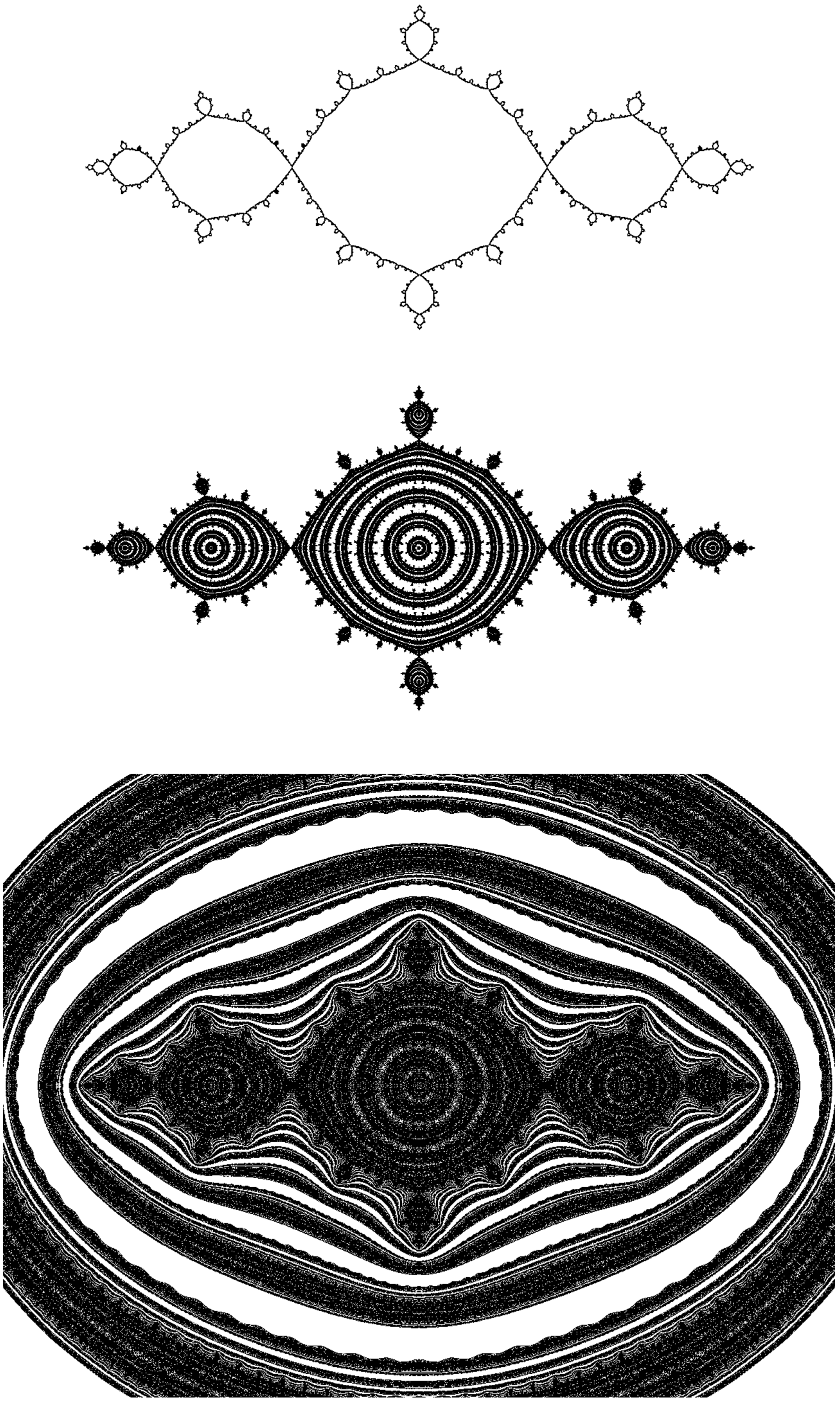

Figure 1: The Julia sets of $z \mapsto z^2 - 1$, $z \mapsto z^2 - 1 + \frac{\lambda}{z^3}$ and $z \mapsto \frac{1}{z} \circ (\frac{1}{z} \circ (z^2 - 1 + \frac{\lambda}{z^3}) \circ \frac{1}{z} + \frac{\mu}{z^7}) \circ \frac{1}{z}$ (from the top down), where $\lambda = 10^{-10}$ and $\mu = 10^{-14}$. The middle and bottom Julia sets, respectively, contain a semi-buried and buried component which are homeomorphic to the top Julia set: the basilica.

critical point 0. The degree of the resulting rational map is 4 but this map has connected Julia set and hence cannot contain any buried Julia components. See also [GMR13] for a singular perturbation of $z \mapsto z^2 + c$ on the corresponding bounded super-attracting cycle by adding one pole to each point in the cycle.

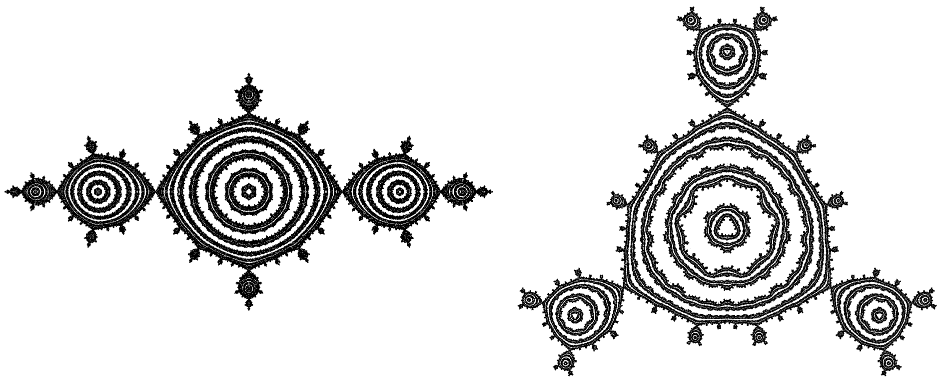

Figure 2: The Julia sets of two quartic rational maps $f_\lambda(z) = z^2 - 1 + \lambda/(z+1)^2$ and $g_\mu(z) = z^3 + \mathrm{i} + \mu/(z - \mathrm{i})$, where $\lambda = 10^{-7}$ and $\mu = 10^{-4}$. Both of these two Julia sets contain some buried Julia components which are Jordan curves.

Godillon's example in (1) is a cubic rational map but the construction depends heavily on the complicated combinatorics which are encoded by a weighted dynamical tree. Here we adopt the idea in the proof of Theorem 1.1 and the construction of Theorem 1.2, to give a simple example of cubic rational map such that the corresponding Julia set contains a buried Julia component which is not a Jordan curve nor a singleton.

Theorem 1.3. *Let $a \notin \{0, -(3+\sqrt{5})/2\}$ be the center of a hyperbolic component of $Q_a(z) = 1 + a/z^2$. If $\nu \neq 0$ is small enough, then the Julia set of the cubic rational map*

$$f_\nu(z) = 1 + \frac{a}{z^2} + \frac{(1 + a + \sqrt{-a})\,\nu}{z - 1 - \nu} \tag{3}$$

contains a buried Julia component which is homeomorphic to $J(Q_a)$ and some buried Julia components which are Jordan curves.

This gives also an answer to the question of McMullen in terms of different cubic rational maps from Godillon's. See Figure 3 for a specific example. The family $Q_a(z) = 1 + a/z^2$, where $a \in \mathbb{C} \setminus \{0\}$ was first studied by Lyubich in [Lyu86, §2.4]. Note that Q_a has two critical points $0, \infty$ and exactly one critical orbit

$$0 \overset{(2)}{\longmapsto} \infty \overset{(2)}{\longmapsto} 1 \overset{(1)}{\longmapsto} 1 + a \longmapsto \cdots \quad \text{and} \quad \pm\sqrt{-a} \overset{(1)}{\longmapsto} 0.$$

Lyubich asked whether there exist $a \neq 0$ such that Q_a has a Herman ring. In 1987 Shishikura gave a negative answer to this question [Shi87] and later another proof was provided by Bamón and Bobenrieth [BB99] (see also [Yan17]).

Let us compare our example with Godillon's in (1). His formula is a bit complicated but can be written as family with only one free critical point hence one can draw the bifurcation locus in the complex plane. Our formula is simpler but there are several free critical points.

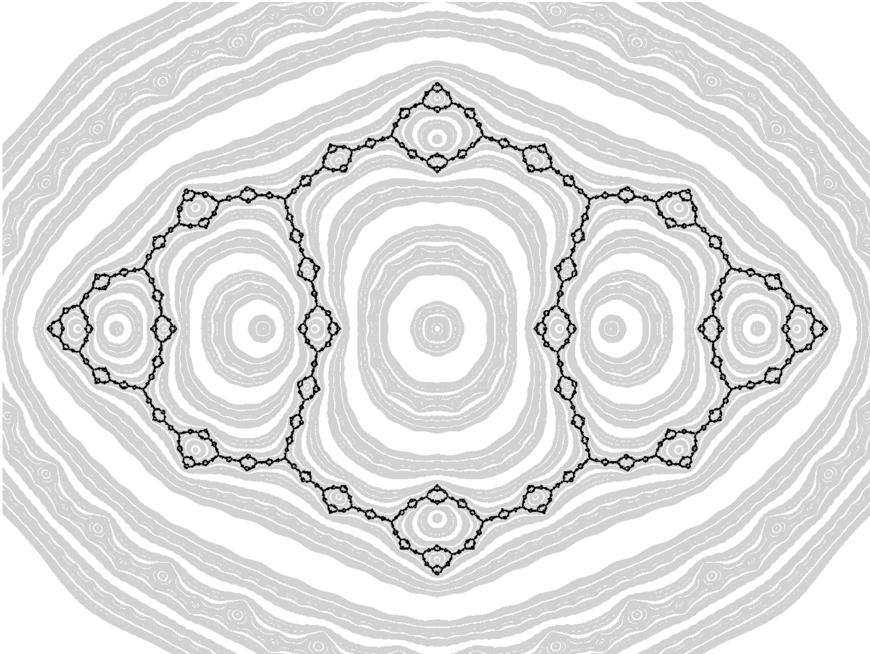

Figure 3: The Julia set of cubic rational map $f_\nu(z) = 1 - 1/z^2 + \nu/(z - 1 - \nu)$, where $\nu = 10^{-5}$. This Julia set contains some buried Julia components which are Jordan curves (some gray parts) and contains also a buried Julia component which is homeomorphic to the Julia set of $z \mapsto 1 - 1/z^2$ (the black part).

1.2. **The idea of the proofs.** Let us give a sketch of the proof of Theorem 1.1. For the sufficiency, suppose that f has a connected Julia set and that it does not have parabolic basins and Siegel disks. A standard quasiconformal surgery guarantees that one can assume that all periodic Fatou components of f are super-attracting and f is post-critically finite in the Fatou set (see [CG93, p. 106]). The proof will be divided into two main steps: The first one is to perturb all super-attracting periodic orbits (except one) and obtain a quasiregular map F (if f has at least two disjoint super-attracting periodic orbits). Then this map F is conjugate to a rational map h by quasiconformal surgery principle. The second step is to perturb the rest one and obtain a quasiregular map H which can be conjugate to a rational map g (if f has exactly one super-attracting periodic orbits we omit the first step). We will prove that the Julia set of the first rational map h has a 'semi-buried' Julia component. Then we show that the second rational map g contains a fully buried Julia component which is a copy of $J(f)$.

For the necessity in Theorem 1.1 we use a proof of contradiction. Since topological conjugacy preserves the multiplier at indifferent periodic points, one can exclude the existence of parabolic periodic points in the buried Julia components. If f has a Siegel disk U and let $\varphi : J(f) \to J_0(g)$ be a quasiconformal conjugacy defined from a neighborhood of $J(f)$ to that of a Julia component $J_0(g)$ of g. It is easy to see that U contains an annular neighborhood A such that $\varphi(A \cap U)$ is contained in a rotation domain of g. However, this is impossible if $\varphi(J(f))$ is a buried Julia component of g. For details, see §3.

As stated above, the constructions in Theorems 1.2 and 1.3 are inspired by the proof of Theorem 1.1. Perturbing at the critical values not only provides the annulus-to-disk dynamical behavior but also reduces the degrees as much as possible. For details, see §6 and §7.

Notations. We collect some notations which will be used throughout of this paper. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively, be the set of natural numbers, integers, real numbers and complex numbers. For $r>0$, we use $\mathbb{D}(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$ to denote the round disk centered at $a \in \mathbb{C}$ with radius r, and $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r\}$ be the boundary of $\mathbb{D}(0,r)$. For $0 < r < 1$, we denote the annulus $\mathbb{A}_r := \{z \in \mathbb{C} : r < |z| < 1\}$.

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2. Semi-buried Julia components

In this section, we will operate the first quasiconformal surgery on the given rational map f and obtain a new rational map h. This map h has a 'semi-buried' Julia component on which the restriction of h is quasiconformally conjugate to f on the Julia set $J(f)$.

Let f be a rational map with degree at least two whose Julia set is connected. Further, we assume that the Fatou set of f is non-empty and does not contain parabolic basins and Siegel disks. This means that all the periodic Fatou components of f are attracting or super-attracting. By a standard quasiconformal surgery, see [CG93, Theorem 5.1, p. 106] for example, we assume in the following that f is a post-critically finite rational map in the Fatou set. This means that all the periodic Fatou components of f are super-attracting and each critical point in the Fatou set will be iterated to the super-attracting cycle eventually.

Let $O_1, O_2, \ldots, O_n$ be the collection of different cycle of super-attracting periodic Fatou components of f. It is well known that $1 \le n \le 2d-2$ (see [Shi87] or [Bea91a, Theorem 9.4.1]). In the rest of this section, we assume that $n \ge 2$ and the case for $n = 1$ will be handled in §4 and §5.

For $1 \le i \le n$, let $p_i \ge 1$ be the (minimal) period of O_i and a_i a super-attracting periodic point of f contained in O_i. For $1 \le j \le p_i$, let $B_{i,j}$ be the Fatou component containing the point $f^{\circ(j-1)}(a_i)$. Hence we have $O_i = \{B_{i,j} : 1 \le j \le p_i\}$ for all $1 \le i \le n$. Moreover, we use $\mathcal{A}_i$ to denote the super-attracting basin of f containing O_i (note that each $B_{i,j}$ is connected but $\mathcal{A}_i$ may be disconnected). Without loss of generality, we assume that $a_1 = 0$ and $a_n = \infty$.

2.1. **Cutting along the equipotentials I.**

For performing the quasiconformal surgery, we need to divide $\widehat{\mathbb{C}}$ into several pieces on which a quasiregular map F will be piecewisely defined. This partition comes from some equipotential curves of f in the super-attracting periodic orbits.

According to Böttcher's theorem, each super-attracting periodic orbit O_i provides the Riemann mappings $\phi_{i,j} : B_{i,j} \to \mathbb{D}$, where $1 \le i \le n-1$ and $1 \le j \le p_i$, such that $\phi_{i,j}(f^{\circ(j-1)}(a_i)) = 0$ and the following diagram commutes:

$$\begin{tikzcd}
B_{i,1} \arrow[r, "f"] \arrow[d, "\phi_{i,1}"] & B_{i,2} \arrow[r, "f"] \arrow[d, "\phi_{i,2}"] & \cdots \arrow[r, "f"] \arrow[d] & B_{i,p_i} \arrow[r, "f"] \arrow[d, "\phi_{i,p_i}"] & B_{i,1} \arrow[d, "\phi_{i,1}"] \\
\mathbb{D} \arrow[r, "z \mapsto z^{d_{i,1}}"] & \mathbb{D} \arrow[r, "z \mapsto z^{d_{i,2}}"] & \cdots \arrow[r, "z \mapsto z^{d_{i,p_i-1}}"] & \mathbb{D} \arrow[r, "z \mapsto z^{d_{i,p_i}}"] & \mathbb{D},
\end{tikzcd}$$

where[3] $d_{i,p_i} \geq 2$ and $d_{i,j} \geq 1$ are positive integers for $1 \leq j \leq p_i - 1$. For convenience, we denote

$$d_i := \prod_{j=1}^{p_i} d_{i,j} \geq 2. \tag{4}$$

Hence $f^{\circ p_i} : B_{i,j} \to B_{i,j}$ is holomorphically conjugate to $z \mapsto z^{d_i}$ from $\mathbb{D}$ to itself, where $1 \leq i \leq n-1$ and $1 \leq j \leq p_i$.

An *equipotential curve* (or *equipotential* in short) γ in $B_{i,j}$ is the preimage by $\phi_{i,j}$ of an Euclidean circle in $\mathbb{D}$ centered at 0. The radius of this circle is called the *level* of γ and is denoted by $L_{i,j}(\gamma) \in (0,1)$, i.e. $\gamma = \{z \in B_{i,j} : |\phi_{i,j}(z)| = L_{i,j}(\gamma)\}$.

We will define the corresponding potentials and levels in the basin $B_{n,j}$ (actually is its conformal image of the conjugacy map) with $1 \leq j \leq p_n$ in the next section since they will not be used in the present section. The definitions will be modified slightly such that the levels of the equipotentials in $B_{n,j}$ are contained in $(1, +\infty)$ since $\infty \in B_{n,1}$.

Let γ be a Jordan curve contained in $\widehat{\mathbb{C}} \setminus \{\infty\}$. We use $D(\gamma)$ to denote the connected component of $\widehat{\mathbb{C}} \setminus \gamma$ which does not contain ∞. For two disjoint connected compact sets γ_1 and γ_2 in $\widehat{\mathbb{C}} \setminus \{\infty\}$ which are not singletons, we use $A(\gamma_1, \gamma_2)$ to denote the unique annular component of $\widehat{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2)$. Moreover, $A(\gamma_1, \gamma_2)$ is biholomorphically equivalent to a standard annulus $\mathbb{A}_r := \{z \in \mathbb{C} : r < |z| < 1\}$ with $r \in (0,1)$ whose *conformal modulus* is $\mathrm{mod}(\mathbb{A}_r) = (1/2\pi)\log(1/r)$.

Lemma 2.1 (holomorphic covering from disk to disk)**.** *For each $1 \leq i \leq n-1$, let $m_i \geq 1$ be an integer satisfying $m_i > d_{i,1}/(d_i - 1)$. Then there exist equipotentials*[4] *$\gamma_{i,1}$, γ_{i,p_i+1}, $\xi_{i,1} \subset B_{i,1}$, $\gamma_{i,2} \subset B_{i,2}$ and a holomorphic branched mapping $F|_{D(\xi_{i,1})} : D(\xi_{i,1}) \to \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_{i,2})$ satisfying the following conditions:*

(a) *$L_{i,1}(\gamma_{i,p_i+1}) < L_{i,1}(\xi_{i,1}) < L_{i,1}(\gamma_{i,1})$;*
(b) *$F(a_i) = \infty$ and $F : D(\xi_{i,1}) \setminus \{a_i\} \to \mathbb{C} \setminus \overline{D}(\gamma_{i,2})$ is a degree m_i covering map; and*
(c) *$F(\xi_{i,1}) = \gamma_{i,2}$ and $F(\gamma_{i,p_i+1}) = \eta_i$, where η_i is an analytic Jordan curve in $B_{n,1}$ separating ∞ from $\partial B_{n,1}$.*

Proof. Without loss of generality, we prove this lemma only for $i = 1$ since the other cases are completely similar. According to the normalization, we have $a_1 = 0$. For each small $r \in (0,1)$, let $\gamma_{1,1}$, γ_{1,p_1+1} be the equipotentials in $B_{1,1}$ such that $L_{1,1}(\gamma_{1,1}) = r$ and $L_{1,1}(\gamma_{1,p_1+1}) = r^{d_1}$. For $2 \leq j \leq p_1$, we denote $\gamma_{1,j} := f^{\circ(j-1)}(\gamma_{1,1})$. Then $\gamma_{1,j}$ is an equipotential in $B_{1,j}$ for all $1 \leq j \leq p_1$. See Figure 4 for a partial illustration.

Let $\xi_{1,1}$ be an equipotential in $B_{1,1}$ such that $L_{1,1}(\xi_{1,1}) = s \in (r^{d_1}, r)$. Hence $L_{1,1}(\gamma_{1,p_1+1}) < L_{1,1}(\xi_{1,1}) < L_{1,1}(\gamma_{1,1})$. Recall that $\phi_{1,1} : D(\xi_{1,1}) \to \mathbb{D}(0,s)$ is the restriction of the Böttcher map. For $m_1 \geq 1$, we define $Q_{m_1}(z) = z^{m_1}/s^{m_1} : \mathbb{D}(0,s) \to \mathbb{D}$. Let $\psi_1 : \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_{1,2}) \to \mathbb{D}$ be a conformal map such that $\psi_1(\infty) = 0$. Define

$$F := \psi_1^{-1} \circ Q_{m_1} \circ \phi_{1,1}.$$

Then $F : D(\xi_{1,1}) \to \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_{1,2})$ is a holomorphic branched mapping with degree m_1, $F(0) = \infty$ and 0 is the unique possible critical point. Since

$$Q_{m_1} \circ \phi_{1,1}(\gamma_{1,p_1+1}) = \{z \in \mathbb{C} : |z| = r^{d_1 m_1}/s^{m_1}\} \subset \mathbb{D} \tag{5}$$

[3]We assume that $d_{i,p_i} \geq 2$ and $d_{i,j} \geq 1$ for $1 \leq j \leq p_i - 1$ rather than $d_{i,1} \geq 2$ and $d_{i,j} \geq 1$ for $2 \leq j \leq p_i$ since this normalization can reduce the degree of the new rational map after the surgery as much as possible. See Lemma 2.1.

[4]If $p_i = 1$ for some $1 \leq i \leq n-1$, then $B_{i,2} = B_{i,1}$ and $\gamma_{i,2}$ is regarded as a curve in $B_{i,1}$.

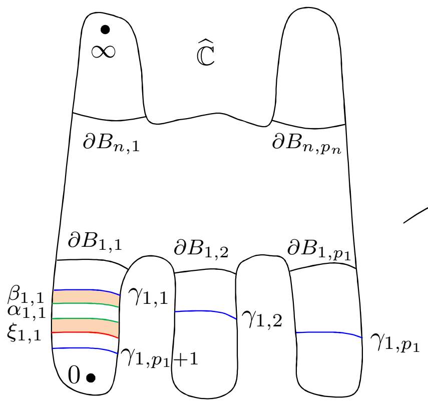

Figure 4: The sketch of the definition of the map $F|_{D(\xi_{1,1})} : D(\xi_{1,1}) \to \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_{1,2})$. For further references, some other equipotentials are drawn also in the super-attracting basins and the boundaries of immediate basins are also marked. For simplicity, we only draw the first super-attracting orbit $B_{1,1}, \cdots, B_{1,p_1}$.

is an Euclidean circle, it follows that $\eta_1 := F(\gamma_{1,p_1+1}) = \psi_1^{-1}\{z : |z| = r^{d_1 m_1}/s^{m_1}\}$ is an analytic Jordan curve separating ∞ from $\gamma_{1,2}$.

We need to find a sufficient condition to guarantee that η_1 is contained in $B_{n,1}$. Since the degree of the restriction of f on $B_{1,1}$ is $d_{1,1}$ and $a_1 = 0$, the map f can be written near the origin as

$$f(z) = f(0) + b_1 z^{d_{1,1}} + O(z^{d_{1,1}+1}),$$

where $b_1 \neq 0$ is a constant depending only on f. If $r > 0$ is sufficiently small, then there exists a constant $C_1 > 0$ independent on r such that $D(\gamma_{1,2}) = f(D(\gamma_{1,1}))$ is a Jordan disk centered at $f(0)$ with radius about $C_1 r^{d_{1,1}}$. More specifically, r can be chosen small enough such that $\mathbb{D}(f(0), C_1 r^{d_{1,1}}/2) \subset D(\gamma_{1,2}) \subset \mathbb{D}(f(0), 2C_1 r^{d_{1,1}})$. Then there exists a constant $C_2 > 0$ depending on C_1 but independent on $r > 0$ such that the conformal modulus

$$\mathrm{mod}(\mathbb{D} \setminus \overline{D}(\psi_1(\partial B_{n,1}))) = \mathrm{mod}(A(\gamma_{1,2}, \partial B_{n,1})) \leq \frac{1}{2\pi} \log \frac{1}{r^{d_{1,1}}} + C_2.$$

On the other hand, by Koebe's distortion theorem, there exists a constant $C_3 > 1$ independent on $r > 0$ such that

$$C_3^{-1}|w_2| \leq |w_1| \leq C_3|w_2| \text{ for all } w_1, w_2 \in \psi_1(\partial B_{n,1}).$$

This means that there exists a constant $C_4 > 0$ independent on $r > 0$ such that for any $w \in \psi_1(\partial B_{n,1})$, we have

$$\log|w| \geq \log r^{d_{1,1}} - C_4.$$

In order to guarantee that $\eta_1 \subset B_{n,1}$, by (5), it is sufficient to obtain the inequality

$$\log \frac{r^{d_1 m_1}}{s^{m_1}} \leq \log r^{d_{1,1}} - C_4.$$

Since d_1, $d_{1,1}$, m_1 are positive integers, $r > 0$ can be chosen arbitrarily small and $s \in (r^{d_1}, r)$ can be chosen arbitrarily close to r, it means that we only need to guarantee that

$$\frac{r^{d_1 m_1}}{r^{m_1}} < r^{d_{1,1}}.$$

This is equivalent to $(d_1 - 1)m_1 > d_{1,1}$, i.e. $m_1 > d_{1,1}/(d_1 - 1)$, as desired. $\square$

Remark. Since $1 \le d_{i,1} \le d_i$ and $d_i \ge 2$ for all $1 \le i \le n-1$, the inequality $m_i > d_{i,1}/(d_i - 1)$ is always satisfied if we set $m_i = 3$.

If $d_{i,1} = 1$ and $d_i = 2$ for some $1 \le i \le n-1$, then m_i can be chosen to be 2. This observation will allow us to construct a family of quartic rational functions whose Julia sets contain buried components which are Jordan curves. See §6.

Denote $W := \widehat{\mathbb{C}} \setminus \cup_{i=1}^{n-1} D(\gamma_{i,1})$. We define $F|_W := f|_W$. Then $F|_{\gamma_{i,1}} : \gamma_{i,1} \to \gamma_{i,2}$ is a degree $d_{i,1}$ covering map for all $1 \le i \le n-1$.

2.2. **Holomorphic covering from an annulus to a disk I.** Note that the map F has been defined on the Riemann sphere except on the annulus $A(\xi_{i,1}, \gamma_{i,1})$, where $1 \le i \le n-1$. In order to use the quasiconformal surgery principle, we need to consider a holomorphic covering from an annulus to a disk.

The first statement of the following lemma was stated in [God15, Lemma A.3] and a sketch of the proof was given. For completeness, we include a detailed proof here and add a complementary result.

Lemma 2.2 (holomorphic covering from annulus to disk). *Let ℓ and m be two positive integers. Then there exists a holomorphic branched covering map $\psi : \mathbb{A}_r \to \mathbb{D}$ with degree $\ell + m$ such that*

(a) *ψ has $\ell + m$ critical points in $\mathbb{A}_r$;*
(b) *ψ can be extended continuously to $\partial \mathbb{A}_r$ by a degree ℓ covering $\psi|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \partial\mathbb{D}$ and a degree m covering $\psi|_{\mathbb{T}_r} : \mathbb{T}_r \to \partial\mathbb{D}$; and*
(c) $\operatorname{mod}(\mathbb{A}_r) < \frac{2}{\pi} \log 2$.

Further, suppose that $\ell = m$ and $K > 1$. Then $\psi : \mathbb{A}_r \to \mathbb{D}$ can be chosen such that

(d) *ψ has exactly 2 critical values at $\pm 1/K$;*
(e) $\operatorname{mod}(\mathbb{A}_r) < \frac{1}{m\pi} \log(2K+1)$*; and*
(f) *$\psi^{-1}(\Gamma) \subset \mathbb{A}_r$ is a connected compact set separating 0 from ∞, where $\Gamma \subset \mathbb{D}$ is a connected compact set connecting $1/K$ with $-1/K$.*

Proof. The proof is based on the study of the dynamical properties of the McMullen maps

$$g_\lambda(z) = z^\ell + \lambda/z^m, \text{ where } \lambda > 0.$$

Except 0 and ∞, it is easy to see that g_λ has $\ell + m$ critical points and the critical values of g_λ are

$$v_j = \Big(1 + \frac{\ell}{m}\Big)\Big(\frac{m}{\ell}\lambda\Big)^{\frac{\ell}{\ell+m}} e^{\frac{\ell j}{\ell+m} 2\pi \mathrm{i}}, \text{ where } 1 \le j \le \ell + m. \tag{6}$$

Note that v_j may be equal to v_k if $1 \le j \ne k \le \ell + m$. Let $v := v_{\ell+m} > 0$ be the modulus of each v_j, where $1 \le j \le \ell + m$.

Let $K > 1$ be a constant. By Riemann-Hurwitz's formula, it follows that $g_\lambda^{-1}(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}(0, Kv))$ consists of two Jordan disks V_0 and V_∞ such that

(a) $0 \in V_0$ and $\infty \in V_\infty$;
(b) $g_\lambda|_{\partial V_\infty} : \partial V_\infty \to \mathbb{T}_{Kv}$ is a covering map with degree ℓ; and
(c) $g_\lambda|_{\partial V_0} : \partial V_0 \to \mathbb{T}_{Kv}$ is a covering map with degree m.

Therefore, $g_\lambda : A(\partial V_0, \partial V_\infty) \to \mathbb{D}(0, Kv)$ is a holomorphic branched covering map with degree $\ell + m$. For all $\lambda > 0$, then it is easy to check that

$$|z| < \Big(\frac{m}{\ell}\lambda\Big)^{\frac{1}{\ell+m}} \Big(K + \frac{(K+1)\ell}{m}\Big)^{\frac{1}{\ell}} \text{ for every } z \in \partial V_\infty,$$

and

$$|z| > \Big(\frac{m}{\ell}\lambda\Big)^{\frac{1}{\ell+m}} \Big(\frac{\ell}{m}\Big)^{\frac{1}{m}} \Big(K + 1 + \frac{K\ell}{m}\Big)^{-\frac{1}{m}} \text{ for every } z \in \partial V_0.$$

Since $x \mapsto \frac{1}{x}\log(K+(K+1)x)$ is decreasing on $[1,+\infty)$ for all $K>1$, this means that

$$\begin{aligned}&2\pi \bmod(A(\partial V_0, \partial V_\infty))\\ &< \frac{1}{\ell}\log\Big(K+\frac{(K+1)\ell}{m}\Big)+\frac{1}{m}\log\Big(K+\frac{(K+1)m}{\ell}\Big)\\ &\le \frac{1}{\ell}\log\big(K+(K+1)\ell\big)+\frac{1}{m}\log\big(K+(K+1)m\big)\le 2\log(2K+1).\end{aligned}$$

In particular, if $K=3/2$, then we have $\bmod(A(\partial V_0,\partial V_\infty))<\frac{2}{\pi}\log 2$. Moreover, if $\ell=m\ge 1$, then

$$2\pi \bmod(A(\partial V_0,\partial V_\infty))<\frac{2}{m}\log(2K+1).$$

Let $0<r<1$ be the number such that $\bmod(\mathbb{A}_r)=\bmod(A(\partial V_0,\partial V_\infty))$. Then there exist two conformal mappings $\psi_1:\mathbb{A}_r\to A(\partial V_0,\partial V_\infty)$ and $\psi_2:\mathbb{D}(0,Kv)\to\mathbb{D}:z\mapsto z/(Kv)$. Then $\psi:=\psi_2\circ g_\lambda\circ\psi_1:\mathbb{A}_r\to\mathbb{D}$ is the required holomorphic function. Indeed, the statement (f) holds since $g_\lambda^{-1}([-v,v])\subset A(\partial V_0,\partial V_\infty)$ is a connected curve separating 0 from ∞. □

Remark. Lemma 2.2 is similar to the key lemma in [PT99, Lemma 2.1] (see also [BF14, Lemma 7.47]) about the annulus-to-disk branched coverings. However, here the covering from annulus to disk is required to be holomorphic. We will use the properties (d), (e) and (f) of Lemma 2.2 in §5.

Let us continue the construction of F. We will use the properties (a), (b) and (c) of Lemma 2.2 to prove the following result.

Lemma 2.3. *For each $1\le i\le n-1$, there exist two equipotentials $\alpha_{i,1}$, $\beta_{i,1}$ in $B_{i,1}$, an equipotential $\eta_{i,2}$ in $B_{i,2}$ and a holomorphic branched covering map $F|_{A(\alpha_{i,1},\beta_{i,1})}:A(\alpha_{i,1},\beta_{i,1})\to D(\eta_{i,2})$ with degree $d_{i,1}+m_i$ such that*

(a) *$L_{i,1}(\xi_{i,1})<L_{i,1}(\alpha_{i,1})<L_{i,1}(\beta_{i,1})<L_{i,1}(\gamma_{i,1})$ and $L_{i,2}(\eta_{i,2})<L_{i,2}(\gamma_{i,2})$;*
(b) *$F|_{A(\alpha_{i,1},\beta_{i,1})}$ has $d_{i,1}+m_i$ critical points in $A(\alpha_{i,1},\beta_{i,1})$; and*
(c) *$F|_{A(\alpha_{i,1},\beta_{i,1})}$ can be extended continuously to $\alpha_{i,1}\cup\beta_{i,1}$ by a degree m_i covering $F|_{\alpha_{i,1}}:\alpha_{i,1}\to\eta_{i,2}$ and a degree $d_{i,1}$ covering $F|_{\beta_{i,1}}:\beta_{i,1}\to\eta_{i,2}$.*

Proof. By the first half result of Lemma 2.2, it is sufficient to prove the existence of $\alpha_{i,1}$ and $\beta_{i,1}$ in $B_{i,1}$ such that $L_{i,1}(\xi_{i,1})<L_{i,1}(\alpha_{i,1})<L_{i,1}(\beta_{i,1})<L_{i,1}(\gamma_{i,1})$ and $A(\alpha_{i,1},\beta_{i,1})\ge\frac{2}{\pi}\log 2$. See Figure 4. This is obvious if we choose the level $r>0$ of $\gamma_{i,1}$ in the proof of Lemma 2.1 small enough. □

Now F is defined on the Riemann sphere except on the annuli $A(\xi_{i,1},\alpha_{i,1})$ and $A(\beta_{i,1},\gamma_{i,1})$, where $1\le i\le n-1$. Since all of the connected components of the boundaries of these annuli, together with their images $\gamma_{i,2}$ and $\eta_{i,2}$, are quasicircles (actually are analytic curves), one can make an interpolation such that the resulting map F satisfies

(a) $F|_{A(\xi_{i,1},\alpha_{i,1})}:A(\xi_{i,1},\alpha_{i,1})\to A(\gamma_{i,2},\eta_{i,2})$ is a degree m_i covering map;
(b) $F|_{A(\beta_{i,1},\gamma_{i,1})}:A(\beta_{i,1},\gamma_{i,1})\to A(\eta_{i,2},\gamma_{i,2})$ is a degree $d_{i,1}$ covering map; and
(c) $F|_{A(\xi_{i,1},\alpha_{i,1})}$ and $F|_{A(\beta_{i,1},\gamma_{i,1})}$ are local quasiconformal.

2.3. **Uniformization I.** Now we have a quasiregular map F defined from the Riemann sphere to itself whose dynamics is sketched in Figure 4. We need to find a quasiconformal homeomorphism to conjugate F to a rational map. For this, we will apply Shishikura's fundamental lemma for quasiconformal surgery.

Lemma 2.4 (Fundamental lemma for qc surgery, [Shi87])**.** *Let $G:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a quasiregular map. Suppose that there exist an open set $E\subset\widehat{\mathbb{C}}$ and an integer $N\ge 0$ satisfying the following two conditions:*

(a) *$G(E) \subset E$; and*
(b) *$\partial G/\partial \overline{z} = 0$ holds on E and a.e. on $\widehat{\mathbb{C}} \setminus G^{-N}(E)$.*

Then there exists a quasiconformal map $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\varphi \circ G \circ \varphi^{-1}$ is rational.

The lemma above established firstly by Shishikura although its original statement is more general. The reader can refer to [Shi87, §3], [Bea91a, Lemma 9.6.2] and [BF14, Proposition 5.2] for a proof and more details.

Corollary 2.5. *There is a quasiconformal map $\varphi_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that*

$$h := \varphi_1 \circ F \circ \varphi_1^{-1}$$

is rational map satisfying $\varphi_1(0) = 0$, $\varphi_1(\infty) = \infty$ and $\deg(h) = d + \sum_{i=1}^{n-1} m_i$.

Proof. Recall the definition of the equipotential $\gamma_{i,j}$ in §2.1 and Lemma 2.1, where $1 \le i \le n-1$ and $1 \le j \le p_i + 1$. For the quasiregular map F, we define an open set

$$E_1 := \bigcup_{j=1}^{p_n} B_{n,j} \cup \bigcup_{i=1}^{n-1} \bigcup_{j=2}^{p_i+1} D(\gamma_{i,j}). \tag{7}$$

According to Lemma 2.1, we have $F(D(\gamma_{i,p_i+1})) \subset B_{n,1}$ for all $1 \le i \le n-1$. This means that $F(E_1) \subset E_1$ since $F(D(\gamma_{i,j})) = D(\gamma_{i,j+1})$ for $1 \le i \le n-1$ and $2 \le j \le p_i$. On the other hand, $A(\xi_{i,1}, \alpha_{i,1}) \cup A(\beta_{i,1}, \gamma_{i,1})$ is contained in $F^{-1}(D(\gamma_{i,2}))$ and F is analytic except on the annuli $A(\xi_{i,1}, \alpha_{i,1})$ and $A(\beta_{i,1}, \gamma_{i,1})$, where $1 \le i \le n-1$. Therefore, we have $\partial F/\partial \overline{z} = 0$ on E_1 and a.e. on $\widehat{\mathbb{C}} \setminus F^{-1}(E_1)$. The result then follows immediately by Lemma 2.4. □

2.4. **The semi-buried property.** By the construction of surgery, h has a cycle of super-attracting periodic Fatou components $\varphi_1(O_n) = \{\varphi_1(B_{n,j}) : 1 \le j \le p_n\}$. Recall that $\mathcal{A}_n$ is the super-attracting basin of f containing O_n. In this subsection we show that the boundary of $\varphi_1(\mathcal{A}_n)$ is 'semi-buried'.

Definition (Semi-buried Julia components)**.** Let $J_0 \ne \widehat{\mathbb{C}}$ be a Julia component of a rational map R and U a connected component of $\widehat{\mathbb{C}} \setminus J_0$. If J_0 is disjoint with the boundary of any Fatou component of R in U, then J_0 is called *semi-buried* from U. In particular, J_0 is a buried Julia component if and only if it is semi-buried from any component of $\widehat{\mathbb{C}} \setminus J_0$.

Let $\mathcal{V} := F(f) \setminus \mathcal{A}_n$, where $F(f)$ is the Fatou set of f. Since we have assumed that $n \ge 2$ in this section, it means that $\mathcal{V} \ne \emptyset$. Recall that $\varphi_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the quasiconformal mapping introduced in Corollary 2.5.

Proposition 2.6. *The set $\varphi_1(J(f))$ is a semi-buried Julia component of h. In particular, $\varphi_1(J(f))$ is semi-buried from every component of $\varphi_1(\mathcal{V})$.*

Proof. We first show that $J' := \varphi_1(J(f))$ is contained in the Julia set of h. Indeed, for any $z_0 \in J'$ and any open neighborhood U of z_0, there is a point $z_1 \in U \cap \varphi_1(\mathcal{A}_n)$ since $J(f) = \partial \mathcal{A}_n$. Since z_1 will be iterated into the periodic orbit O_n eventually while the orbit of z_0 is contained in J', it means that $\{h^{\circ k}\}_{k \in \mathbb{N}}$ is not equi-continuous in U. Therefore, z_0 is contained in the Julia set of h and hence $J' \subset J(h)$.

Next we prove that for each component V of $\varphi_1(\mathcal{V})$, there exists a sequence of Julia components $\{J_k\}_{k \in \mathbb{N}}$ of h in V which converges to ∂V in the Hausdorff metric. Without loss of generality, we only consider the case that $V \subset \varphi_1(\mathcal{A}_i)$ for $i = 1$ since the rest case is completely similar. By Lemma 2.1, $F(D(\xi_{1,1}) \setminus \{0\}) = \mathbb{C} \setminus \overline{D}(\gamma_{1,2})$ is a covering map with degree $m_1 \ge 1$ and $F(\gamma_{1,p_1+1}) = \eta_1 \subset B_{n,1}$, it follows that the annulus $\varphi_1(A(\gamma_{1,p_1+1}, \xi_{1,1}))$ contains a Julia component J_0 of h separating 0 from ∞. On the other hand, by the surgery construction in §2.2, it follows that the annulus $\varphi_1(A(\xi_{1,1},\gamma_{1,1}))$ is contained in a Fatou component U_0 of h and it separates 0 from ∞ also.

Since $F^{\circ p_1} = f^{\circ p_1} : A(\gamma_{1,1},\partial B_{1,1}) \to A(\gamma_{1,p_1+1},\partial B_{1,1})$ is a covering map with degree d_1, it follows that the annulus $\varphi_1(A(\gamma_{1,1},\partial B_{1,1}))$ contains a Julia component $J_1 := h^{-p_1}(J_0)\cap\varphi_1(A(\gamma_{1,1},\partial B_{1,1}))$ and a Fatou component $U_1 := h^{-p_1}(U_0)\cap\varphi_1(A(\gamma_{1,1},\partial B_{1,1}))$ such that both J_1 and U_1 separate 0 from ∞. Inductively, one can obtain a sequence of Julia components $\{J_k\}_{k\geq 1}$ and a sequence of Fatou components $\{U_k\}_{k\geq 1}$ in $\varphi_1(A(\gamma_{1,1},\partial B_{1,1}))$ such that $h^{\circ p_1}(J_k) = J_{k-1}$, $h^{\circ p_1}(U_k) = U_{k-1}$ and each J_k and U_k separates 0 from ∞.

Recall that the levels of $\gamma_{1,1}$ and γ_{1,p_1+1} in $B_{1,1}$ are r and r^{d_1} respectively, where $r > 0$ is small enough. This means that for every $z \in \varphi_1^{-1}(J_k\cup U_k)$, the level of z in $B_{1,1}$ satisfies

$$1 > L_{1,1}(z) = |\phi_{1,1}(z)| \geq \sqrt[d_1^{k-1}]{r}, \text{ where } k \in \mathbb{N}.$$

Since both $\varphi_1^{-1}(J_k)$ and $\varphi^{-1}(U_k)$ separate 0 from ∞, it means that the Hausdorff distance between $\partial B_{1,1}$ and $\varphi_1^{-1}(J_k)$ (resp. $\varphi^{-1}(U_k)$) tends to zero as $k\to\infty$. Equivalently, the sequence of Julia components $\{J_k\}_{k\in\mathbb{N}}$ of h in $\varphi_1(B_{1,1})$ converges to $\varphi_1(\partial B_{1,1})$ in the Hausdorff metric.

Now we show that $J' = \varphi_1(J(f))$ is a Julia component of h. Let J'' be the Julia component of h containing J'. Suppose that there exists a point $z_0 \in J''\setminus J'$. Then we have $z_0 \in \varphi_1(\mathcal{V})$. By iterating z_0 several times if necessary, we assume that $z_0 \in \varphi_1(B_{1,1})$. However this is a contradiction since $\varphi_1(B_{1,1})$ contains a sequence of Julia components $\{J_k\}_{k\in\mathbb{N}}$ which converges to $\varphi_1(\partial B_{1,1})$ in the Hausdorff metric. This means that z_0 does not exist and J' is a semi-buried Julia component of h.

As a connected subset of the Julia component J', we know that $\varphi_1(\partial B_{1,1})$ is semi-buried from $\varphi_1(B_{1,1})$. By considering the preimages of $\varphi_1(B_{1,1})$ under h, it follows that ∂V is semi-buried from V for every component V of $\varphi_1(\mathcal{A}_1)$. In particular, $\varphi_1(J(f))$ is a semi-buried Julia component of h. □

Remark. By the construction of F, it follows that $f|_{J(f)} : J(f)\to J(f)$ is conjugate to $h|_{\varphi_1(J(f))} : \varphi_1(J(f)) \to \varphi_1(J(f))$ by a restriction of a quasiconformal mapping, where $\varphi_1(J(f))$ is a semi-buried Julia component of h.

3. From semi-buried to buried

In this section, we perform the surgery of the second stage on f based on h and prove that the semi-buried Julia component can be transferred to a fully buried component. According to Proposition 2.6, $\varphi_1(J(f))$ is semi-buried from every component of $\varphi_1(\mathcal{V})$. Hence a natural idea is to perform the surgery in the immediate super-attracting basins $\varphi_1(O_n)$. Note that the properties of h are quite different from that of f since h is no longer post-critically finite in the Fatou set and the Julia set of h is not connected.

By definition we know that $\varphi_1(E_1)$ is contained in the super-attracting basin of ∞, where E_1 is defined in (7). For $1\leq i\leq n-1$ and $1\leq j\leq p_i$, we use $U_{i,j}$ to denote the Fatou component of h containing $\varphi_1(f^{\circ(j-1)}(a_i)) \in \varphi_1(B_{i,j})$. For saving the notations, we still use O_n and $B_{n,j}$ etc, respectively, to denote the super-attracting periodic orbit $\varphi_1(O_n)$ and the immediate super-attracting basin $\varphi_1(B_{n,j}) \ni h^{\circ(j-1)}(\infty)$ of h, etc (i.e. the quasiconformal prefix φ_1 is omitted).

3.1. Cutting along the equipotentials II.

According to Böttcher's theorem, the super-attracting periodic orbit O_n provides the Riemann mappings $\phi_{n,j} : B_{n,j} \to \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$, where $1\leq j\leq p_n$, such that $\phi_{n,j}(h^{\circ(j-1)}(\infty)) = \infty$ and the following diagram commutes:

$$\begin{array}{ccccccccc}
B_{n,1} & \xrightarrow{h} & B_{n,2} & \xrightarrow{h} & \cdots & \xrightarrow{h} & B_{n,p_n} & \xrightarrow{h} & B_{n,1} \\
\downarrow{\scriptstyle \phi_{n,1}} & & \downarrow{\scriptstyle \phi_{n,2}} & & \downarrow & & \downarrow{\scriptstyle \phi_{n,p_n}} & & \downarrow{\scriptstyle \phi_{n,1}} \\
\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}} & \xrightarrow{z\mapsto z^{d_{n,1}}} & \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}} & \xrightarrow{z\mapsto z^{d_{n,2}}} & \cdots & \xrightarrow{z\mapsto z^{d_{n,p_n-1}}} & \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}} & \xrightarrow{z\mapsto z^{d_{n,p_n}}} & \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}},
\end{array}$$

where $d_{n,p_n} \geq 2$ and $d_{n,j} \geq 1$ are positive integers for $1 \leq j \leq p_n - 1$. For convenience, we denote $d_n := \prod_{j=1}^{p_n} d_{n,j} \geq 2$. Hence $h^{\circ p_n} : B_{n,j} \to B_{n,j}$ is holomorphically conjugate to $z \mapsto z^{d_n}$ from $\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$ to itself for $1 \leq j \leq p_n$.

An *equipotential* γ in $B_{n,j}$ is the preimage by $\phi_{n,j}$ of an Euclidean circle in $\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$ centered at 0. The radius of this circle is called the *level* of γ and is denoted by $L_{n,j}(\gamma) \in (1, +\infty)$. The *level* of a point $z \in B_{n,j}$ (is contained in some equipotential curve) can be defined similarly as $L_{n,j}(z) := |\phi_{n,j}(z)| \in (1, +\infty)$. Recall that $m_1 \geq 1$ is an integer introduced in Lemma 2.1.

Lemma 3.1 (holomorphic covering from disk to disk II)**.** *Let $\ell \geq 1$ be an integer satisfying $\ell > (d_{n,1} + \frac{d_n}{m_1})/(d_n - 1)$. Then there exist equipotentials*[5] *$\gamma_{n,1}$, γ_{n,p_n+1}, $\xi_{n,1} \subset B_{n,1}$, $\gamma_{n,2} \subset B_{n,2}$ and a holomorphic branched mapping $H|_{\widehat{\mathbb{C}}\setminus\overline{D}(\xi_{n,1})} : \widehat{\mathbb{C}}\setminus\overline{D}(\xi_{n,1}) \to \widehat{\mathbb{C}}\setminus\overline{D}(\gamma_{n,2})$ (if $p_n \geq 2$) or*[6] *$H|_{\widehat{\mathbb{C}}\setminus\overline{D}(\xi_{n,1})} : \widehat{\mathbb{C}}\setminus\overline{D}(\xi_{n,1}) \to D(\gamma_{n,2})$ (if $p_n = 1$) satisfying the following conditions:*

(a) *$H(\infty) = 0$ and $H : \mathbb{C}\setminus\overline{D}(\xi_{n,1}) \to \widehat{\mathbb{C}}\setminus(\overline{D}(\gamma_{n,2}) \cup \{0\})$ (or $H : \mathbb{C}\setminus\overline{D}(\xi_{n,1}) \to D(\gamma_{n,2})\setminus\{0\}$) is a degree ℓ covering map;*

(b) *$H(\xi_{n,1}) = \gamma_{n,2}$ and $H(\gamma_{n,p_n+1}) = \eta_{1,1}$, where $\eta_{1,1}$ is an analytic Jordan curve in $U_{1,1}$ separating 0 from $\partial U_{1,1}$;*

(c) *h maps $\eta_{1,1}$ to a curve $\kappa_{n,1}$; and*

(d) *$L_{n,1}(\gamma_{n,1}) < L_{n,1}(\xi_{n,1}) < L_{n,1}(\gamma_{n,p_n+1}) < L_{n,1}(z)$ for all $z \in \kappa_{n,1}$.*

Proof. The proof is a bit similar to that of Lemma 2.1. Without loss of generality, we assume that $p_n \geq 2$ since the proof for $p_n = 1$ is similar. For each large $R \in (1, +\infty)$, let $\gamma_{n,1}$, γ_{n,p_n+1} be the equipotentials in $B_{n,1}$ such that $L_{n,1}(\gamma_{n,1}) = R$ and $L_{n,1}(\gamma_{n,p_n+1}) = R^{d_n}$. For $2 \leq j \leq p_n$, we denote $\gamma_{n,j} := h^{\circ(j-1)}(\gamma_{n,1})$. Then $\gamma_{n,j}$ is an equipotential in $B_{n,j}$ for all $1 \leq j \leq p_n$. See Figure 5 for a partial illustration.

Let $\xi_{n,1}$ be an equipotential in $B_{n,1}$ such that $L_{n,1}(\xi_{n,1}) = S \in (R, R^{d_n})$. Recall that $\phi_{n,1} : \widehat{\mathbb{C}}\setminus\overline{D}(\xi_{n,1}) \to \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}(0, S)$ is the restriction of the Böttcher map. For $\ell \geq 1$, we define $Q_\ell(z) = z^\ell/S^\ell : \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}(0, S) \to \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$. Let $\psi_n : \widehat{\mathbb{C}}\setminus\overline{D}(\gamma_{n,2}) \to \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$ be a conformal map such that $\psi_n(0) = \infty$. Define

$$H := \psi_n^{-1} \circ Q_\ell \circ \phi_{n,1}.$$

Then $H : \widehat{\mathbb{C}}\setminus\overline{D}(\xi_{n,1}) \to \widehat{\mathbb{C}}\setminus\overline{D}(\gamma_{n,2})$ is a holomorphic branched mapping with degree ℓ, $F(\infty) = 0$ and ∞ is the unique critical point. Since

$$Q_\ell \circ \phi_{n,1}(\gamma_{n,p_n+1}) = \{z \in \mathbb{C} : |z| = R^{d_n\ell}/S^\ell\} \subset \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}} \tag{8}$$

is an Euclidean circle, it follows that $\eta_{1,1} := H(\gamma_{n,p_n+1}) = \psi_n^{-1}(\{z : |z| = R^{d_n\ell}/S^\ell\})$ is an analytic Jordan curve separating 0 from $\gamma_{n,2}$.

We need to find a sufficient condition to guarantee that $\eta_{1,1}$ is contained in $U_{1,1}$ and $L_{n,1}(\gamma_{n,p_n+1}) < L_{n,1}(z)$, where $z \in \kappa_{n,1} := h(\eta_{1,1})$. Note that $\kappa_{n,1}$ is not

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[5] If $p_n = 1$, then $B_{n,2} = B_{n,1}$ and $\gamma_{n,2}$ is regarded as an equipotential in $B_{n,1}$.

[6] Recall that $D(\gamma_{n,2})$ is the connected component of $\widehat{\mathbb{C}}\setminus\gamma_{n,2}$ which does not contain ∞. If $p_n \geq 2$, then $D(\gamma_{n,2})$ is a Jordan disk which contains neither ∞ nor 0. See Figure 5.

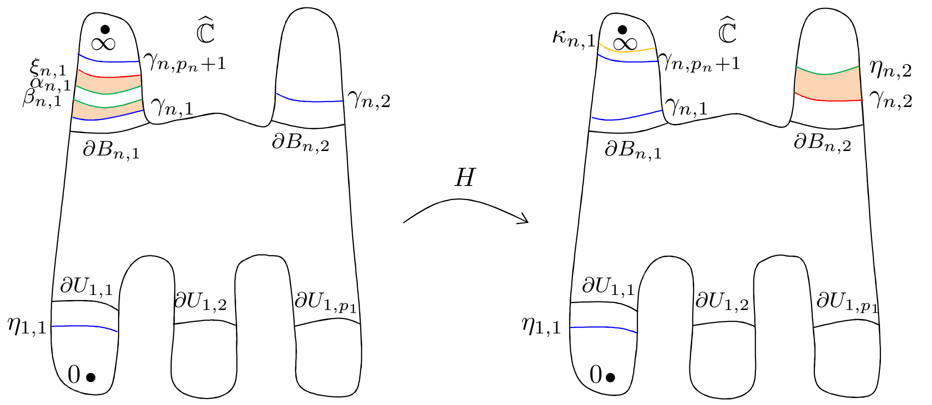

Figure 5: The sketch of the definition of $H|_{\widehat{\mathbb{C}}\setminus\overline{D}(\xi_{n,1})} : \widehat{\mathbb{C}} \setminus \overline{D}(\xi_{n,1}) \to \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_{n,2})$, where $p_n \geq 2$. For further references, some other equipotentials are drawn also in the super-attracting basins of ∞. The boundaries of immediate basins of the cycle O_n and parts of their first preimages are marked also.

necessarily an equipotential in $B_{n,1}$. Since the degree of the restriction of h on $B_{n,1}$ is $d_{n,1}$, the map h can be written near the infinity as

$$h(z) = h(\infty) + b_n/z^{d_{n,1}} + O(1/z^{d_{n,1}+1}),$$

where $b_n \neq 0$ is a constant depending only on h. If $R > 1$ is large enough, then $D(\gamma_{n,2}) = h(\widehat{\mathbb{C}} \setminus \overline{D}(\gamma_{n,1}))$ is a Jordan disk centered at $h(\infty)$ with radius about $C_1/R^{d_{n,1}}$, where $C_1 > 0$ is a constant independent on R. More specifically, R can be chosen large enough such that $\mathbb{D}(h(\infty), C_1/(2R^{d_{n,1}})) \subset D(\gamma_{n,2}) \subset \mathbb{D}(h(\infty), 2C_1/R^{d_{n,1}})$.

Let $\widetilde{\eta}_{1,1} \subset U_{1,1}$ be a sufficiently small round circle separating 0 from $\partial U_{1,1}$ with radius $r > 0$. There exists a constant $C_2 > 0$ depending on C_1 but independent on the large $R > 0$ and small $r > 0$ such that the conformal modulus

$$\operatorname{mod}(D(\psi_n(\widetilde{\eta}_{1,1})) \setminus \overline{\mathbb{D}}) = \operatorname{mod}(A(\widetilde{\eta}_{1,1}, \gamma_{n,2})) \leq \frac{1}{2\pi} \log \frac{R^{d_{n,1}}}{r} + C_2.$$

By Koebe's distortion theorem, there exists a constant $C_3 \geq 1$ independent on large $R > 1$ and small $r > 0$ such that

$$C_3^{-1}|w_2| \leq |w_1| \leq C_3|w_2| \text{ for all } w_1, w_2 \in \psi_n(\widetilde{\eta}_{1,1}).$$

This means that there exists a constant $C_4 > 0$ independent on $R > 0$ such that for any $w \in \psi_n(\widetilde{\eta}_{1,1})$, we have

$$\log|w| \leq \log \frac{R^{d_{n,1}}}{r} + C_4.$$

In order to guarantee that $\eta_{1,1} = H(\gamma_{n,p_n+1}) \subset U_{1,1}$, by (8), it is sufficient to obtain the inequality

$$\log \frac{R^{d_n \ell}}{S^\ell} \geq \log \frac{R^{d_{n,1}}}{r} + C_4. \tag{9}$$

Since the local degree of h at 0 is m_1 and $h(0) = \infty$, it means that h can be written near the origin as

$$1/h(z) = b_0 z^{m_1} + O(z^{m_1+1}),$$

where $b_0 \neq 0$ is a constant depending only on h. In order to guarantee that $L_{n,1}(\gamma_{n,p_n+1}) < L_{n,1}(z)$ for all $z \in \kappa_{n,1} = h(\eta_{1,1})$, it is sufficient to obtain the inequality

$$\log \frac{1}{r^{m_1}} > \log R^{d_n} + C_5, \tag{10}$$

where $C_5 > 0$ is a constant depending on h but independent on large R and small r. Note that d_n, $d_{n,1}$, m_1, ℓ are positive integers and $S \in (R, R^{d_n})$. Since $R > 1$ can be arbitrarily large (and hence $r > 0$ should be sufficiently small) and S can be arbitrarily close to R, by (9) and (10), it is sufficient to guarantee that

$$\frac{R^{d_n \ell}}{R^\ell} > \frac{R^{d_{n,1}}}{r} \quad \text{and} \quad \frac{1}{r} > R^{\frac{d_n}{m_1}}.$$

This is equivalent to $(d_n - 1)\ell - d_{n,1} > \frac{d_n}{m_1}$, i.e. $\ell > (d_{n,1} + \frac{d_n}{m_1})/(d_n - 1)$, as desired. □

Remark. Since $d_{n,1} \leq d_n$, $d_n \geq 2$ and $m_1 \geq 1$, the inequality $\ell > (d_{n,1} + \frac{d_n}{m_1})/(d_n - 1)$ is always satisfied if we set $\ell = 5$. Moreover, ℓ can be chosen as 3 if $m_1 \geq 3$.

One can obtain a similar result as Lemma 3.1 if ℓ is chosen such that $\ell > (d_{n,1} + \frac{d_n}{m_i})/(d_n - 1)$ for some $1 \leq i \leq n - 1$.

Denote $W := D(\gamma_{n,1})$. We define $H|_W := h|_W$. Then $H|_{\gamma_{n,1}} : \gamma_{n,1} \to \gamma_{n,2}$ is a degree $d_{n,1}$ covering map and $H|_W$ maps W onto $\widehat{\mathbb{C}}$.

3.2. **Annulus-to-disk and annulus-to-annulus coverings II.** Similar to the construction of F, we also need to construct some holomorphic branched covering maps from annulus to disk and quasiregular covering maps from annulus to annulus for H. Similar to Lemma 2.3, the following result is an immediate corollary of Lemma 2.2. See Figure 5.

Lemma 3.2. *There exist two equipotentials $\alpha_{n,1}$, $\beta_{n,1}$ in $B_{n,1}$, an equipotential $\eta_{n,2}$ in $B_{n,2}$ and a holomorphic branched covering map $H|_{A(\alpha_{n,1},\beta_{n,1})} : A(\alpha_{n,1}, \beta_{n,1}) \to D(\eta_{n,2})$ (if $p_n \geq 2$) or $H|_{A(\alpha_{n,1},\beta_{n,1})} : A(\alpha_{n,1}, \beta_{n,1}) \to \widehat{\mathbb{C}} \setminus \overline{D}(\eta_{n,2})$ (if $p_n = 1$) with degree $d_{n,1} + \ell$ such that*

(a) *$L_{n,1}(\xi_{n,1}) > L_{n,1}(\alpha_{n,1}) > L_{n,1}(\beta_{n,1}) > L_{n,1}(\gamma_{n,1})$, $L_{n,2}(\eta_{n,2}) > L_{n,2}(\gamma_{n,2})$;*
(b) *$H|_{A(\alpha_{n,1},\beta_{n,1})}$ has $d_{n,1} + \ell$ critical points in $A(\alpha_{n,1}, \beta_{n,1})$; and*
(c) *$H|_{A(\alpha_{n,1},\beta_{n,1})}$ can be extended continuously to $\alpha_{n,1} \cup \beta_{n,1}$ by a degree ℓ covering $H|_{\alpha_{n,1}} : \alpha_{n,1} \to \eta_{n,2}$ and a degree $d_{n,1}$ covering $H|_{\beta_{n,1}} : \beta_{n,1} \to \eta_{n,2}$.*

Now H is defined on the Riemann sphere except on the annuli $A(\xi_{n,1}, \alpha_{n,1})$ and $A(\beta_{n,1}, \gamma_{n,1})$. Since all of the connected components of the boundaries of these two annuli, together with their images $\gamma_{n,2}$ and $\eta_{n,2}$, are quasicircles, one can make an interpolation such that the resulting map H satisfies

(a) $H|_{A(\xi_{n,1},\alpha_{n,1})} : A(\xi_{n,1}, \alpha_{n,1}) \to A(\gamma_{n,2}, \eta_{n,2})$ is a degree ℓ covering map;
(b) $H|_{A(\beta_{n,1},\gamma_{n,1})} : A(\beta_{n,1}, \gamma_{n,1}) \to A(\eta_{n,2}, \gamma_{n,2})$ is a degree $d_{n,1}$ covering map; and
(c) $H|_{A(\xi_{n,1},\alpha_{n,1})}$ and $H|_{A(\beta_{n,1},\gamma_{n,1})}$ are local quasiconformal.

3.3. **Uniformization II.** Now we have a quasiregular map H defined from the Riemann sphere to itself whose dynamics is sketched in Figure 5. As before, we will apply Shishikura's fundamental lemma for quasiconformal surgery to conjugate H to a rational map.

Corollary 3.3. *There is a quasiconformal map $\varphi_2 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that*

$$g := \varphi_2 \circ H \circ \varphi_2^{-1}$$

is rational map satisfying $\varphi_2(0) = 0$, $\varphi_2(\infty) = \infty$ and $\deg(g) = d + \ell + \sum_{i=1}^{n-1} m_i$.

Proof. Recall that $\gamma_{n,j}$ is an equipotential defined in $B_{n,j}$, where $1 \leq j \leq p_n+1$ (In particular, γ_{n,p_n+1} is contained in $B_{n,1}$). For the quasiregular map H, we define an open set

$$E_2 := \begin{cases} D(\eta_{1,1}) \cup (\widehat{\mathbb{C}} \setminus \overline{D}(\gamma_{n,p_n+1})) \cup \bigcup_{j=2}^{p_n} D(\gamma_{n,j}) & \text{if } p_n \geq 2, \\ D(\eta_{1,1}) \cup (\widehat{\mathbb{C}} \setminus \overline{D}(\gamma_{n,2})) & \text{if } p_n = 1. \end{cases} \tag{11}$$

According to Lemma 3.1, we have $H(\widehat{\mathbb{C}} \setminus \overline{D}(\gamma_{n,p_n+1})) \subset D(\eta_{1,1})$ and $H(D(\eta_{1,1})) \subset \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_{n,p_n+1})$. This means that $H(E_2) \subset E_2$. On the other hand, $A(\xi_{n,1}, \alpha_{n,1}) \cup A(\beta_{n,1}, \gamma_{n,1})$ is contained in $H^{-1}(D(\gamma_{n,2}))$ if $p_n \geq 2$ and is contained in $H^{-1}(\widehat{\mathbb{C}} \setminus \overline{D}(\gamma_{n,2}))$ if $p_n = 1$. Note that H is analytic except on the annuli $A(\xi_{n,1}, \alpha_{n,1})$ and $A(\beta_{n,1}, \gamma_{n,1})$. Therefore, we have $\partial H / \partial \overline{z} = 0$ on E_2 and a.e. on $\widehat{\mathbb{C}} \setminus H^{-1}(E_2)$. The result then follows immediately from Lemma 2.4. □

Remark. The rational map g has a super-attracting cycle $0 \leftrightarrow \infty$ with period 2.

3.4. **The buried property.** Recall that $\varphi_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and $\varphi_2 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are quasiconformal mappings introduced in Corollaries 2.5 and 3.3.

Proposition 3.4. *The set $\varphi_2 \circ \varphi_1(J(f))$ is a buried Julia component of g.*

Proof. Applying a similar argument to the first paragraph in the proof of Proposition 2.6, we know that $\varphi_2(J(h))$ (which is not connected) is contained in the Julia set of g. In particular, $J' := \varphi_2 \circ \varphi_1(J(f))$ is contained in $J(g)$.

Similar to the proof of Proposition 2.6, we show that there exists a sequence of Julia components $\{J_k\}_{k \in \mathbb{N}}$ of g in $\varphi_2(B_{n,1})$ which converges to $\varphi_2(\partial B_{n,1})$ in the Hausdorff metric. In the following we assume the period of $B_{n,1}$ is $p_n \geq 2$ without loss of generality (the case for $p_n = 1$ is completely similar). By Lemma 3.1, $H : \mathbb{C} \setminus \overline{D}(\xi_{n,1}) \to \widehat{\mathbb{C}} \setminus (\overline{D}(\gamma_{n,2}) \cup \{0\})$ is a covering map with degree $\ell \geq 1$ and $H(\gamma_{n,p_n+1}) = \eta_{1,1} \subset U_{1,1}$, it follows that the annulus $\varphi_2(A(\gamma_{n,p_n+1}, \xi_{n,1}))$ contains a Julia component J_0 of g separating ∞ from $\varphi_2(\partial B_{n,1})$. On the other hand, by the surgery construction in §3.2, it follows that the annulus $\varphi_2(A(\xi_{n,1}, \gamma_{n,1}))$ is contained in a Fatou component W_0 of g and it separates ∞ from $\varphi_2(\partial B_{n,1})$ also.

Since $H^{\circ p_n} = h^{\circ p_n} : A(\gamma_{n,1}, \partial B_{n,1}) \to A(\gamma_{n,p_n+1}, \partial B_{n,1})$ is a covering map with degree d_n, it follows that the annulus $\varphi_2(A(\gamma_{n,1}, \partial B_{n,1}))$ contains a Julia component $J_1 := g^{-p_n}(J_0) \cap \varphi_2(A(\gamma_{n,1}, \partial B_{n,1}))$ and a Fatou component $W_1 := g^{-p_n}(W_0) \cap \varphi_2(A(\gamma_{n,1}, \partial B_{n,1}))$ such that both J_1 and W_1 separate ∞ from $\varphi_2(\partial B_{n,1})$. Inductively, one can obtain a sequence of Julia components $\{J_k\}_{k \geq 1}$ and a sequence of Fatou components $\{W_k\}_{k \geq 1}$ in $\varphi_2(A(\gamma_{n,1}, \partial B_{n,1}))$ converging to $\varphi_2(\partial B_{n,1})$ as $k \to \infty$ such that $g^{\circ p_n}(J_k) = J_{k-1}$, $g^{\circ p_n}(W_k) = W_{k-1}$ and each J_k and W_k separates ∞ from $\varphi_2(\partial B_{n,1})$.

Let U be a Fatou component of h. Then there exists $j = j(U) \in \mathbb{N}$ such that $h^{\circ j}(U) = B_{n,1}$. This means that there exist a sequence of Julia components $\{J_k^U\}_{k \in \mathbb{N}}$ and a sequence of Fatou components $\{W_k^U\}_{k \in \mathbb{N}}$ in $\varphi_2(U)$ such that they converge to $\varphi_2(\partial U)$ in the Hausdorff metric as $k \to \infty$. Recall that $\mathcal{V} = F(f) \setminus \mathcal{A}_n$ is the union of the Fatou components which are not iterated to the cycle O_n eventually. By Proposition 2.6, for each component V of $\varphi_1(\mathcal{V})$, there is a sequence of Fatou components $\{U_l\}_{l \geq 1}$ of h in V converging to ∂V in the Hausdorff metric. Therefore, the sequence of Julia components $\{J_0^{U_l}\}_{l \in \mathbb{N}}$ and Fatou components $\{W_0^{U_l}\}_{l \in \mathbb{N}}$ in $\varphi_2(V)$ converge to $\varphi_2(\partial V)$ as $l \to \infty$.

Up to now, we have proved that for each component V of $\varphi_2 \circ \varphi_1(F(f))$, there exist a sequence of different Julia components $\{J_i\}_{i \in \mathbb{N}}$ and a sequence of different Fatou components $\{W_i\}_{i \in \mathbb{N}}$ in V such that they converge to ∂V in the Hausdorff metric as $i \to \infty$. This means that $J' = \varphi_2 \circ \varphi_1(J(f))$ is a Julia component of g

and J' is semi-buried from every component of $\varphi_2 \circ \varphi_1(F(f))$. Hence J' is a buried Julia component of g. □

Remark. By the construction of H, it follows that $f|_{J(f)} : J(f) \to J(f)$ is quasicomformally conjugate to the restriction of g on the buried Julia component $\varphi_2 \circ \varphi_1(J(f))$. Actually, one can obtain the same result by performing a 'big' surgery by combining the surgeries in §2 and in the present section.

We now prove Theorem 1.1 under the assumption that f has at least two attracting periodic orbits.

Proof of Theorem 1.1 in the case $n \geq 2$. The sufficiency of the first assertion follows from Proposition 3.4. For the necessity, we assume that f contains a parabolic periodic point or a Siegel disk. Suppose that there is a rational map g such that g has a buried Julia component J' on which g is conjugate to f on $J(f)$ by a restriction of a quasiconformal map φ. Since topological conjugacy preserves the multiplier at indifferent periodic points (see [Naĭ82] or [Pér97]), it follows that g has also a parabolic periodic point on J' if f does. But this is impossible since J' is a buried Julia component. On the other hand, suppose that f has a Siegel disk U with period $p \geq 1$. Note that the quasiconformal map φ is defined in a neighborhood of $J(f)$. This means that U contains an annular neighborhood A such that $\varphi(A \cap U)$ is contained in a rotation domain of g. However, this is also impossible since $\varphi(J(f))$ is a buried Julia component of g.

Now we only need to verify the statements on the degrees. By Corollary 3.3, the proof is reduced to find a upper bound of $d + \ell + \sum_{i=1}^{n-1} m_i$ as small as possible. By Lemmas 2.1 and 3.1, the integers m_i and ℓ should satisfy

$$m_i > \frac{d_{i,1}}{d_i - 1} \quad \text{and} \quad \ell > \frac{d_{n,1} + \frac{d_n}{m_1}}{d_n - 1},$$

where $1 \leq i \leq n-1$. Since $d_{i,1} \leq d_i$ and $d_i \geq 2$ for all $1 \leq i \leq n$, we have

$$\frac{d_{i,1}}{d_i - 1} \leq \frac{d_i}{d_i - 1} \leq 2 \quad \text{and} \quad \frac{d_{n,1} + \frac{d_n}{m_1}}{d_n - 1} \leq \frac{d_n}{d_n - 1} \cdot \frac{m_1 + 1}{m_1} \leq 2 + \frac{2}{m_1}.$$

This means that we can choose $m_i = 3$ and $\ell = 3$, where $1 \leq i \leq n-1$. Since $n \leq 2d - 2$, the degree of g is at most $d + 3 + 3(2d - 3) = 7d - 6$. In particular, if f is a polynomial, then $n \leq d$ and we have $\deg(g) \leq d + 3 + 3(d - 1) = 4d$. □

4. The first case of exactly one attracting cycle

In this section, we assume that f is a rational map with degree $d \geq 2$ whose Julia set is connected. Furthermore, f is assumed to be post-critically finite in the Fatou set and have exactly one cycle of periodic super-attaracting basins. Therefore, all the Fatou components of f will be iterated onto the periodic Fatou components containing this cycle eventually. Let O be this cycle and $p \geq 1$ its period.

We also assume that the Fatou set of f has at least two connected components. The case that f has exactly one connected component will be discussed in the next section (i.e. the dendrite case). Under these assumptions, f cannot be a polynomial since otherwise it has at least two super-attracting cycles or its Fatou set has exactly one connected component.

If f has exactly two Fatou components, then $f^{\circ 2}$ is conjugate to a polynomial. Hence f is conjugate to $z \mapsto z^{-d}$ with $d \geq 2$ and its Julia set is a round circle. It is known from [McM88] (see also [DLU05], [BDGR08], [GMR13], [HP12], [QYY15] and [FY15]) that there exists a rational map g whose Julia set is a set of Cantor circles and contains a buried Julia component which is a Jordan curve. Moreover, on this buried Jordan curve g is quasiconformally conjugate to $z \mapsto z^{-d}$ on the unit circle. Therefore, in the rest of this section we always assume that f has at least three Fatou components. By [Bea91a, Theorem 5.6.2], this implies that f has infinitely many Fatou components.

We denote the immediate super-attracting basin of f as $B_1 \mapsto B_2 \mapsto \cdots \mapsto B_p \mapsto B_1$. Let $U_1 \neq B_p$ be a preimage of B_1. Without loss of generality, we assume that $\infty \in B_1$ is contained in the super-attracting periodic orbit, $0 \in U_1$ and $f(0) = \infty$.

One may find that actually we are in a similar case as §3. The rational map h there and the rational map f here both have exactly one super-attracting basin. The difference is that the Julia set of f is connected while the Julia set of h is disconnected. However, from the construction of the surgery in §3, we know that the connectivity of the Julia set is not an obstruction.

Let $d_0 \geq 2$ be the degree of the restriction $f^{\circ p}|_{B_1} : B_1 \to B_1$ and denote $d_{0,1} := \deg(f|_{B_1} : B_1 \to B_2)$. We denote $m_1 := \deg(f|_{U_1} : U_1 \to B_1) \geq 1$. The proof of the following proposition is completely similar to that of Proposition 3.4 (compare Lemma 3.1) and we omit the details.

Proposition 4.1. *There exists a rational map g with degree $\deg(g) = d + \ell$, where $\ell \geq 1$ is the minimal integer satisfying*

$$\ell > \frac{d_{0,1} + \frac{d_0}{m_1}}{d_0 - 1},$$

such that its Julia set contains a buried Julia component on which the dynamics is quasiconformally conjugate to that of f on $J(f)$.

We now prove Theorem 1.1 under the assumption that f has exactly one attracting periodic orbit and the Fatou set contains infinitely many Fatou components.

Proof of Theorem 1.1 in the case $n = 1$ (the non-dendrite case). The first assertion follows from Proposition 4.1. We only needs to verify the statements on the degrees. Since $d_0 \geq d_{0,1}$, $d_0 \geq 2$ and $m_1 \geq 1$, we have

$$\frac{d_{0,1} + \frac{d_0}{m_1}}{d_0 - 1} \leq \frac{d_0}{d_0 - 1} \cdot \frac{m_1 + 1}{m_1} \leq 4.$$

This means that $\ell \leq 5$. The degree of g is at most $d + 5 \leq 4d$ if $d \geq 2$. □

5. The second case of exactly one attracting cycle: dendrite

The Julia set of a rational map f is called a *dendrite* if the Julia set is connected and the Fatou set of f is non-empty and contains exactly one component. In this section, we assume that f is a rational map whose Julia set is a dendrite. Since the unique Fatou component of f is completely invariant, without loss of generality, we suppose that f is a polynomial with degree $d \geq 2$ and 0 is contained in the filled-in Julia set of f.

Let B be the unique super-attracting basin of f centered at the infinity. We will construct a quasiregular map G based on f such that the straightened map of G contains a buried Julia component which is homeomorphic to $J(f)$. Let $\phi : B \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be the Böttcher map conjugating $f : B \to B$ holomorphically to $z \mapsto z^d$ from $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ to itself, where $\phi(\infty) = \infty$. Recall that an *equipotential* γ in B is the preimage by ϕ of an Euclidean circle in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ centered at 0. In this section, the *level* of an equipotential γ in B will be denoted by $L(\gamma) \in (1, +\infty)$.

Lemma 5.1. *Let $m \geq 5$ be an integer. There exist five different equipotentials γ_1, γ_2, ξ_1, ξ_2, α_1 in B and a holomorphic branched mapping $G|_{\widehat{\mathbb{C}} \setminus \overline{D}(\xi_1)} : \widehat{\mathbb{C}} \setminus \overline{D}(\xi_1) \to \widehat{\mathbb{C}} \setminus \overline{D}(\alpha_1)$ satisfying the following conditions:*

(a) *$L(\gamma_1) < L(\alpha_1) < L(\xi_1) < L(\gamma_2) < L(\xi_2)$;*

(b) $G(\infty) = \infty$ *and* $G : \mathbb{C} \setminus \overline{D}(\xi_1) \to \mathbb{C} \setminus \overline{D}(\alpha_1)$ *is a degree* m *covering map; and*
(c) $G(\xi_1) = \alpha_1$, $G(\gamma_2) = \xi_2$ *and* $f(\gamma_1) = \gamma_2$.

Proof. Let $R > 1$ and let S_1, S_2, S_3 be the following constants satisfying $R < S_1 < S_2 < R^d < S_3$, where

$$S_1 = R^{(d+4)/5}, \quad S_2 = R^{(4d+1)/5} \quad \text{and} \quad S_3 = S_1 R^{dm}/S_2^m > R^d. \tag{12}$$

We define γ_1, α_1, ξ_1, γ_2 and ξ_2 as the equipotentials in B with the levels R, S_1, S_2, R^d and S_3 respectively. See Figure 6.

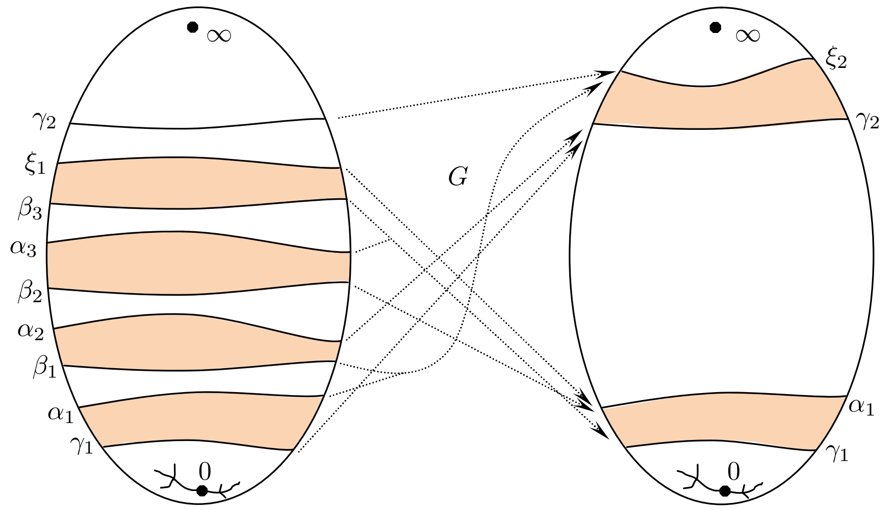

Figure 6: The sketch of the definition of the map $G|_{\widehat{\mathbb{C}}\setminus\overline{D}(\xi_1)} : \widehat{\mathbb{C}}\setminus\overline{D}(\xi_1) \to \widehat{\mathbb{C}}\setminus\overline{D}(\alpha_1)$. For further references, some other equipotentials are drawn also in the basin of infinity and the dendrite is also marked.

Define $Q_m(z) = S_1 z^m / S_2^m : \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}(0, S_2) \to \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}(0, S_1)$. Then $Q_m(\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}(0, R^d)) = \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}(0, S_3)$. Recall that $\phi : B \to \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$ is the Böttcher map conjugating $f : B \to B$ holomorphically to $z \mapsto z^d$, where $\phi(\infty) = \infty$. Then $G := \phi^{-1} \circ Q_m \circ \phi : \widehat{\mathbb{C}}\setminus\overline{D}(\xi_1) \to \widehat{\mathbb{C}}\setminus\overline{D}(\alpha_1)$ is the required holomorphic branched covering map. □

As the surgery performed before, we need to construct some annulus-to-disk branched covering map and annulus-to-annulus covering maps. Recall that $m \geq 5$ is an integer introduced in Lemma 5.1.

Lemma 5.2. *There exist two different equipotentials* α_2, β_2 *in* B *and a holomorphic covering mapping* $G|_{A(\alpha_2,\beta_2)} : A(\alpha_2, \beta_2) \to A(\gamma_2, \alpha_1)$ *with degree* m *such that:*

(a) $L(\alpha_1) < L(\alpha_2) < L(\beta_2) < L(\xi_1)$; *and*
(b) $G|_{A(\alpha_2,\beta_2)}$ *can be extended continuously to* $\alpha_2 \cup \beta_2$ *by two degree* m *covering maps* $G|_{\alpha_2} : \alpha_2 \to \gamma_2$ *and* $G|_{\beta_2} : \beta_2 \to \alpha_1$.

Proof. By (12), we have

$$\operatorname{mod}(A(\alpha_1, \xi_1)) = \frac{3(d-1)}{10\pi} \log R \quad \text{and} \quad \operatorname{mod}(A(\alpha_1, \gamma_2)) = \frac{2(d-1)}{5\pi} \log R. \tag{13}$$

This means that one can choose an essential subannulus[7] $A(\alpha_2,\beta_2)$ of $A(\alpha_1,\xi_1)$ such that $G|_{A(\alpha_2,\beta_2)} : A(\alpha_2,\beta_2) \to A(\gamma_2,\alpha_1)$ is a holomorphic covering mapping with degree $m \geq 5$. Moreover, α_2 and β_2 can be chosen such that $L(\alpha_1) < L(\alpha_2) < L(\beta_2) < L(\xi_1)$. See Figure 6. $\square$

Remark. Actually, the annulus $A(\alpha_2,\beta_2)$ can be chosen such that it is 'well-located' in $A(\alpha_1,\xi_1)$, i.e. one can require that the conformal modulus of $A(\alpha_1,\alpha_2)$, $A(\alpha_2,\beta_2)$ and $A(\beta_2,\xi_1)$ are comparable.

Lemma 5.3. *There exist* 3 *equipotentials* β_1, α_3, β_3 *in* B *and two holomorphic branched covering maps* $G|_{A(\alpha_1,\beta_1)} : A(\alpha_1,\beta_1) \to \widehat{\mathbb{C}} \setminus \overline{D}(\xi_2)$ *with degree* $d+m$ *and* $G|_{A(\alpha_3,\beta_3)} : A(\alpha_3,\beta_3) \to D(\gamma_1)$ *with degree* $2m$ *such that*

(a) $L(\alpha_1) < L(\beta_1) < L(\alpha_2) < L(\beta_2) < L(\alpha_3) < L(\beta_3) < L(\xi_1)$;
(b) $G|_{A(\alpha_1,\beta_1)}$ *has* $d+m$ *critical points in* $A(\alpha_1,\beta_1)$ *and* $G|_{A(\alpha_1,\beta_1)}$ *can be extended continuously to* $\alpha_1 \cup \beta_1$ *by a degree* d *covering* $G|_{\alpha_1} : \alpha_1 \to \xi_2$ *and a degree* m *covering* $G|_{\beta_1} : \beta_1 \to \xi_2$*;*
(c) $G|_{A(\alpha_3,\beta_3)}$ *has* $2m$ *critical points in* $A(\alpha_3,\beta_3)$ *and* $G|_{A(\alpha_3,\beta_3)}$ *can be extended continuously to* $\alpha_3 \cup \beta_3$ *by two degree* m *covering maps* $G|_{\alpha_3} : \alpha_3 \to \gamma_1$ *and* $G|_{\beta_3} : \beta_3 \to \gamma_1$*; and*
(d) $G|_{A(\alpha_3,\beta_3)}$ *has exactly* 2 *different critical values which are contained in the Julia set of* f *and* $G^{-1}(J(f)) \cap A(\alpha_3,\beta_3)$ *is a connected compact set separating* α_3 *and* β_3.

Proof. As remarked at the end of the proof of Lemma 5.2, $A(\alpha_2,\beta_2)$ can be chosen such that conformal modulus of $A(\alpha_1,\alpha_2)$, $A(\alpha_2,\beta_2)$ and $A(\beta_2,\xi_1)$ are comparable. Letting $R > 1$ large enough such that $\mathrm{mod}(A(\alpha_1,\alpha_2))$ is at least $\frac{2}{\pi}\log 2$. Then the existence of $G|_{A(\alpha_1,\beta_1)} : A(\alpha_1,\beta_1) \to \widehat{\mathbb{C}} \setminus \overline{D}(\xi_2)$ satisfying the first two conditions is guaranteed by the first half statement of Lemma 2.2. Hence it is sufficient to prove the existence of $G|_{A(\alpha_3,\beta_3)} : A(\alpha_3,\beta_3) \to D(\gamma_1)$.

Let $\psi_1 : D(\gamma_1) \to \mathbb{D}$ be a conformal isomorphism such that $0 \in \psi_1(J(f))$. Let $b_0 \in \psi_1(J(f))$ be the point such that

$$|b_0| = \sup\{|z| : z \in \psi_1(J(f))\}.$$

Since $\mathrm{mod}(\mathbb{D} \setminus \psi_1(J(f))) = \mathrm{mod}(D(\gamma_1) \setminus J(f)) = \frac{1}{2\pi}\log R$, it follows that $|b_0| \geq 1/R$. Without loss of generality, we assume that $b_0 > 0$. Let $\psi_2 : \mathbb{D} \to \mathbb{D}$ be an isomorphism such that

$$\psi_2(z) = \frac{z - b_1}{1 - b_1 z}, \text{ where } b_1 = \frac{1 - \sqrt{1 - b_0^2}}{b_0} \in (0,1).$$

Then ψ_2 maps 0, b_0 to $-b_1$, b_1 respectively. Moreover, since $b_0 \geq 1/R$, we have

$$b_1 \geq R - \sqrt{R^2 - 1} \geq 1/(2R).$$

By the second conclusion of Lemma 2.2, there exists a holomorphic branched covering map $\psi : \mathbb{A}_r \to \mathbb{D}$ with degree $2m$ such that

(1) ψ has exactly $2m$ critical points in $\mathbb{A}_r$ and exactly 2 critical values at $\pm b_1$;
(2) ψ can be extended continuously to $\partial\mathbb{A}_r$ by two degree m covering maps $\psi|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \partial\mathbb{D}$ and $\psi|_{\mathbb{T}_r} : \mathbb{T}_r \to \partial\mathbb{D}$;
(3) $\mathrm{mod}(\mathbb{A}_r) < \frac{1}{m\pi}\log(\frac{2}{b_1}+1) \leq \frac{1}{m\pi}\log(4R+1)$; and
(4) $\psi^{-1}(\psi_1(J(f))) \subset \mathbb{A}_r$ is a connected compact set separating 0 from ∞.

[7] An annulus A_1 is called an *essential subannulus* of the annulus A_2 if A_1 separates two boundaries of A_2.

Therefore, to prove the existence of $G|_{A(\alpha_3,\beta_3)}: A(\alpha_3,\beta_3)\to D(\gamma_1)$ satisfying (c) and (d), it is sufficient to guarantee that $\mathrm{mod}(A(\alpha_3,\beta_3))\geq \frac{1}{m\pi}\log(4R+1)$.

Note that γ_1, ξ_1, α_i and β_i, where $1\leq i\leq 3$ are equipotentials. By (13), it is sufficient to guarantee that

$$\frac{2}{\pi}\log 2+\frac{2(d-1)}{5m\pi}\log R+\frac{1}{m\pi}\log(4R+1)<\frac{3(d-1)}{10\pi}\log R.$$

Since $R>1$ can be chosen large enough and m, d are positive integers, we only need to obtain the inequality

$$\frac{2(d-1)}{5m}+\frac{1}{m}<\frac{3(d-1)}{10}.$$

This inequality is equivalent to $(3m-4)(d-1)>10$. This is true since $d\geq 2$ and $m\geq 5$. $\square$

Define $G|_{D(\gamma_1)}:=f|_{D(\gamma_1)}$. Then G is defined on the Riemann sphere except on the annuli $A(\gamma_1,\alpha_1)$, $A(\beta_1,\alpha_2)$, $A(\beta_2,\alpha_3)$, $A(\beta_3,\xi_1)$. Since all of the connected components of the boundaries of these four annuli, together with their images γ_1, α_1, γ_2 and ξ_2, are quasicircles, one can make an interpolation such that the resulting map G satisfies

(a) $G|_{A(\gamma_1,\alpha_1)}: A(\gamma_1,\alpha_1)\to A(\gamma_2,\xi_2)$ is a degree d covering map;
(b) $G|_{A(\beta_1,\alpha_2)}: A(\beta_1,\alpha_2)\to A(\xi_2,\gamma_2)$ is a degree m covering map;
(c) $G|_{A(\beta_2,\alpha_3)}: A(\beta_2,\alpha_3)\to A(\alpha_1,\gamma_1)$ is a degree m covering map;
(d) $G|_{A(\beta_3,\xi_1)}: A(\beta_3,\xi_1)\to A(\gamma_1,\alpha_1)$ is a degree m covering map; and
(e) The four annular mapping above are local quasiconformal.

Now we have a quasiregular map G defined from the Riemann sphere to itself whose dynamics is sketched in Figure 6.

Corollary 5.4. *There is a quasiconformal map $\varphi:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that $g:=\varphi\circ G\circ\varphi^{-1}$ is rational map satisfying $\varphi(0)=0$, $\varphi(\infty)=\infty$ and $\deg(g)=d+2m$.*

Proof. For the quasiregular map G, we define an open set $E:=\widehat{\mathbb{C}}\setminus\overline{D}(\gamma_2)$. According to Lemma 5.1, we have $G(E)\subset E$. By Lemmas 5.1, 5.2 and 5.3, and the annulus-to-annulus construction of G, it follows that G is holomorphic outside of $G^{-2}(\overline{E})$. This means that $\partial G/\partial\overline{z}=0$ on E and a.e. on $\widehat{\mathbb{C}}\setminus G^{-2}(E)$. Then the result follows immediately from Lemma 2.4. $\square$

Proposition 5.5. *The set $\varphi(J(f))$ is a buried Julia component of g and the restriction of g on $\varphi(J(f))$ is quasiconformally conjugate to f on $J(f)$.*

Proof. The proof is similar to that of Propositions 2.6 and 3.4. Hence we only give a sketch of the proof here. Firstly $\varphi(J(f))$ is contained in the Julia set of g by a similar argument as in the first paragraph of the proof of Proposition 2.6. Then we show that there exists a sequence of Julia components $\{J_k\}_{k\in\mathbb{N}}$ of g in $\varphi(D(\gamma_2))$ which converges to $\varphi(J(f))$ in the Hausdorff metric. By the annulus-to-annulus construction of G above, it follows that the image of the annuli $A(\gamma_1,\alpha_1)$, $A(\beta_1,\alpha_2)$, $A(\beta_2,\alpha_3)$ and $A(\beta_3,\xi_1)$ under φ are contained in the Fatou set of g. According to Lemma 5.3(d), the annulus $\varphi(A(\alpha_3,\beta_3))\subset\varphi(A(\gamma_1,\gamma_2))$ contains a Julia component J_0 of g separating ∞ from $\varphi(J(g))$. Then $\varphi(A(\gamma_1,\alpha_1))$ is contained in a Fatou component U_0 of g which separates ∞ and $\varphi(J(g))$.

Since $f: A(\gamma_1,J(f))\to A(\gamma_2,J(f))$ is a covering map with degree d, it follows that the annulus $\varphi(A(\gamma_1,J(f)))$ contains a Julia component $J_1:=G^{-1}(J_0)\cap\varphi(A(\gamma_1,J(f)))$ and a Fatou component $U_1:=G^{-1}(U_0)\cap\varphi(A(\gamma_1,J(f)))$ such that both J_1 and U_1 separate ∞ from $\varphi(J(f))$. Inductively, one can obtain a sequence of Julia components $\{J_k\}_{k\geq 1}$ and a sequence of Fatou components $\{U_k\}_{k\geq 1}$ in

$\varphi(A(\gamma_1, J(f)))$ converging to $\varphi(J(f))$ such that $G(J_k) = J_{k-1}$, $G(U_k) = U_{k-1}$ and each J_k and U_k separates ∞ from $\varphi(J(f))$. This means that $\varphi(J(f))$ is a buried Julia component of g. □

Proof of Theorem 1.1 in the case $n = 1$ (the dendrite case). The first assertion follows from Proposition 5.5. For the statements on the degrees, by Lemmas 5.1, 5.2 and 5.3, we can choose $m = 5$ and then $\deg(g) = d + 10 \leq 4d + 4$ for all $d \geq 2$. □

Proof of Theorem 1.1. The sufficiency of the first statement has been proved respectively in Propositions 3.4, 4.1 and 5.5 by dividing the arguments into three cases. Next to the proof of these propositions the results on the degrees have been proved. The necessity of the first statement has been proved at the end of §3. □

6. The quartic Julia sets containing buried Jordan curves

In this section, we aim to construct some quartic rational maps whose Julia sets contain buried components which are Jordan curves. Such kind of the examples was never constructed before. Except Godillon's example of cubic rational map [God15], the known rational maps whose buried Julia components are Jordan curves have at least degree five. Note that Godillon's example is degree three but his construction requires constructing weighted dynamical trees and solving Hurwitz's problem, which is very complicated. We present here a much simpler construction, which is based on the observation of Lemma 2.1.

Recall that in §2.1, $B_{i,1} \mapsto \cdots \mapsto B_{i,p_i}$ is a cycle of super-attracting basins of the given rational map f, where $i \geq 1$ is an index. In order to construct a semi-buried Julia component, the degree of the holomorphic branched covering F from a disk to another (see Lemma 2.1) is required to satisfy $m_i > d_{i,1}/(d_i - 1)$, where $d_{i,1} = \deg(f : B_{i,1} \to B_{i,2}) \geq 1$ and $d_i = \deg(f^{\circ p_i} : B_{i,1} \to B_{i,1}) \geq 2$. Here the construction of F can be seen as a small perturbation.

As remarked before, if $d_{i,1} = 1$ and $d_i = 2$, then m_i can be chosen as 2. If $d_{i,1} = 1$ and $d_i \geq 3$, then m_i can be chosen as 1. Therefore, it is natural to consider the following two families:

$$f_\lambda(z) = z^2 + c + \frac{\lambda}{(z-c)^2} \quad \text{and} \quad g_\mu(z) = z^q + b + \frac{\mu}{z-b},$$

where $\lambda \neq 0$ (resp. $\mu \neq 0$) and $c \neq 0$ (resp. $b \neq 0$) is the center of a hyperbolic component of the Mandelbrot set (resp. the Multibrot set M_q with $q \geq 3$). This means that the perturbations are made at the *critical values*, but not at the *critical point* 0.

The main content in this section is to prove that if $\lambda \neq 0$ (resp. $\mu \neq 0$) is small enough, then the Julia set of f_λ (resp. g_μ) contains some buried Jordan curves and a semi-buried Julia component which is homeomorphic to the Julia set of $z \mapsto z^2 + c$ (resp. $z \mapsto z^q + b$). We only give the proof of Theorem 1.2 for f_λ since the argument for g_μ is completely similar.

6.1. **The position of the critical orbits.** In the rest of this section, we always assume that $\lambda \neq 0$ is sufficiently small. We use B_λ and T_λ, respectively, to denote the immediate attracting basin of f_λ containing ∞ and the Fatou component of f_λ containing the critical value c. Let $p \geq 2$ be the minimal period of the critical point of $P_c(z) = z^2 + c$, i.e. $P_c^{\circ p}(0) = 0$ and $P_c^{\circ p}(c) = c$.

Let $X(\lambda) > 0$ and $Y(\lambda) > 0$ be two numbers depending on λ. We denote $X(\lambda) \preceq Y(\lambda)$ if there exists a constant $C_0 > 0$ independent on λ such that $C_0 X(\lambda) \leq Y(\lambda)$. Moreover, we use $X(\lambda) \asymp Y(\lambda)$ to denote the two numbers which are comparable, i.e. there exist two constants C_1, $C_2 > 0$ independent on λ such that $C_1 Y(\lambda) \leq X(\lambda) \leq C_2 Y(\lambda)$.

Lemma 6.1. *There exists $\delta > 0$ such that if $0 < |\lambda| < \delta$, then there exist two Fatou components A_λ and U_λ such that*

(a) *A_λ contains exactly three critical points $\{c_j^\lambda : 1 \le j \le 3\}$ of f_λ such that $c_j^\lambda \in \{z : |\lambda|^{11/30} \le |z-c| \le |\lambda|^{3/10}\} \subset A_\lambda$; In particular, $\lim_{\lambda\to 0} c_j^\lambda = c$;*

(b) *U_λ contains exactly one critical point c_0^λ of f_λ such that $c_0^\lambda \in \overline{\mathbb{D}}(0, |\lambda|^{2/3}) \subset U_\lambda$; In particular, $\lim_{\lambda\to 0} c_0^\lambda = 0$; and*

(c) *$f_\lambda^{\circ p}(A_\lambda) = T_\lambda$ and $f_\lambda(U_\lambda) = T_\lambda$.*

In particular, f_λ is hyperbolic.

Proof. A direct calculation shows that

$$f_\lambda'(z) = 2\,\frac{z(z-c)^3 - \lambda}{(z-c)^3}.$$

It is easy to see that f_λ has 6 critical points: ∞, c, c_0^λ, c_j^λ with $1 \le j \le 3$, where

$$c_0^\lambda = -\frac{\lambda}{c^3} + o(|\lambda|) \quad \text{and} \quad c_j^\lambda = c + \Big(\frac{\lambda}{c}\Big)^{1/3} e^{2\pi \mathrm{i} j/3} + o(|\lambda|^{1/3}).$$

Note that $\lim_{\lambda\to 0} c_0^\lambda = 0$ and $\lim_{\lambda\to 0} c_j^\lambda = c$ for $1 \le j \le 3$. Since ∞ is a super-attracting fixed point of f_λ, there exists a constant $R > 1$ such that $\widehat{\mathbb{C}} \setminus \mathbb{D}(0,R)$ is contained in B_λ. This means that there exists a constant $a_1 > 0$ independent on small λ such that $\mathbb{D}(c, a_1|\lambda|^{1/2})$ is contained in T_λ.

For any $z \in \overline{\mathbb{D}}(0, |\lambda|^{2/3})$, we have $|f_\lambda(z) - c| \asymp |\lambda|$. Since $|\lambda| \ll |\lambda|^{1/2}$ and $c_0^\lambda \in \mathbb{D}(0, |\lambda|^{2/3})$ if λ is small enough, it means that $f_\lambda(c_0^\lambda) \in T_\lambda$ and c_0^λ escapes to ∞ under the iterate of f_λ for small λ. For $|\lambda|^{11/30} \le |z-c| \le |\lambda|^{3/10}$ and $1 \le k \le p-1$, we have

$$|f_\lambda^{\circ k}(z) - P_c^{\circ k}(c)| \preceq |\lambda|^{4/15} \quad \text{and} \quad |f_\lambda^{\circ p}(z) - c| \preceq |\lambda|^{8/15}.$$

Since $|\lambda|^{8/15} \ll |\lambda|^{1/2}$ and $|\lambda|^{11/30} < |c_j^\lambda - c| < |\lambda|^{3/10}$ for all $1 \le j \le 3$ if λ is small enough, it means that $f_\lambda^{\circ p}(c_j^\lambda) \in T_\lambda$ provided λ is sufficiently small. Therefore, all the critical points of f_λ escape to ∞ and hence f_λ is hyperbolic. □

Remark. A small perturbation of a hyperbolic rational map is not necessarily still hyperbolic. For example, let $F(z) = z^2$ and $F_\lambda(z) = z^2 + \lambda/z^2$, where $\lambda \in \mathbb{C}$. There exists a sequence $\lambda_n \to 0$ as $n \to \infty$ such that F_λ is not hyperbolic. See [DG08].

6.2. **The Julia set moves holomorphically.** According to [Mil06, Corollary 4.12], $J(P_c)$ is the boundary of the basin of infinity. If $\lambda \ne 0$ is sufficiently small, it is reasonable to expect that the dynamics near $J(P_c)$ would not be changed too much. Indeed, we will use holomorphic motion to show that $J(P_c)$ moves holomorphically if λ varies in a small neighborhood of the origin.

Definition (Holomorphic motion, [MSS83], [Lyu83])**.** Let E be a subset of $\widehat{\mathbb{C}}$, a map $h : \mathbb{D} \times E \to \widehat{\mathbb{C}}$ is called a *holomorphic motion* of E parameterized by the unit disk $\mathbb{D}$ and with base point 0 if

(a) For every $z \in E$, $\lambda \mapsto h(\lambda, z)$ is holomorphic for λ in $\mathbb{D}$;

(b) For every $\lambda \in \mathbb{D}$, $z \mapsto h(\lambda, z)$ is injective on E; and

(c) $h(0, z) = z$ for all $z \in E$.

The unit disk $\mathbb{D}$ in the above definition can be replaced by any connected complex manifold. See [McM94, §4.1].

Lemma 6.2 (The λ-Lemma, [MSS83])**.** *A holomorphic motion $h : \mathbb{D} \times E \to \widehat{\mathbb{C}}$ of E has a unique extension to a holomorphic motion of $\overline{E}$. The extension is a continuous map. Moreover, for each $\lambda \in \mathbb{D}$, the map $h(\lambda, \cdot) : E \to \widehat{\mathbb{C}}$ extends to a quasiconformal map of $\widehat{\mathbb{C}}$ to itself.*

Let B_0 be the basin of infinity of $f_0 := P_c$ and recall that B_λ is the immediate attracting basin of ∞ of f_λ. Let $\delta > 0$ be the constant introduced in Lemma 6.1 and define $\Lambda := \{\lambda \in \mathbb{C} : |\lambda| < \delta\}$.

Lemma 6.3. *There is a holomorphic motion $h : \Lambda \times J(f_0) \to \mathbb{C}$ parameterized by Λ and with base point 0 such that $h(\lambda, \partial B_0) = \partial B_\lambda$ for all $\lambda \in \Lambda$.*

Proof. We first prove that every repelling periodic point of f_0 moves holomorphically in Λ. Let $z_0 \in J(f_0)$ be such a point with period p_0. For small λ, the map f_λ is a small perturbation of f_0. By implicit function theorem, there is a neighborhood U_0 of 0 such that z_0 becomes a repelling point z_λ of f_λ with the same period p_0, for all $\lambda \in U_0$. On the other hand, by Lemma 6.1, f_λ has exactly one attracting cycle for all $\lambda \in \Lambda \setminus \{0\}$. This means that each repelling cycle of f_λ moves holomorphically throughout $\Lambda \setminus \{0\}$ (see [McM94, Theorem 4.2]). In particular, there is a holomorphic motion of $J(f_{\lambda_0})$ parameterized by $\Lambda \setminus \{0\}$ and with any given base point λ_0.

Since Λ is simply connected, there is a holomorphic map $Z : \Lambda \to \mathbb{C}$ such that $Z(\lambda) = z_\lambda$ for $\lambda \in U_0$. Let $\mathrm{Per}(f_0)$ be all repelling periodic points of f_0. Then the map $h : \Lambda \times \mathrm{Per}(f_0) \to \mathbb{C}$ defined by $h(\lambda, z_0) = Z(\lambda)$ is a holomorphic motion. Indeed, it is sufficient to verify that $h(\lambda, \cdot)$ is injective on $\mathrm{Per}(f_0)$ for any fixed $\lambda \in \Lambda \setminus \{0\}$. Suppose that there exists $\lambda_0 \in \Lambda \setminus \{0\}$ such that $h(\lambda_0, z') = h(\lambda_0, z'')$, where z' and z'' are different points contained in $\mathrm{Per}(f_0)$. Since $J(f_\lambda)$ moves holomorphically in $\Lambda \setminus \{0\}$, it means that $h(\lambda, z') = h(\lambda, z'')$ for all $\lambda \in \Lambda \setminus \{0\}$. But this is a contradiction since $h(\lambda, z') \neq h(\lambda, z'')$ if $\lambda \neq 0$ is small enough. Notice that $J(f_0) = \overline{\mathrm{Per}(f_0)}$. By Lemma 6.2, we obtain a holomorphic motion which is an extension of h, say $h : \Lambda \times J(f_0) \to \mathbb{C}$. It is obvious that $h(\lambda, J(f_0))$ is a connected subset[8] of $J(f_\lambda)$.

To finish, we show that $h(\lambda, \partial B_0) = \partial B_\lambda$ for all $\lambda \in \Lambda$. Actually, since $J(f_\lambda)$ moves holomorphically in Λ, we have $h_\lambda \circ f_0(z) = f_\lambda \circ h_\lambda(z)$ for all $z \in J(f_0)$, where $h_\lambda := h(\lambda, \cdot)$ (see [McM94, §4.1]). Let $\widetilde{B}_\lambda$ be the connected component of $\widehat{\mathbb{C}} \setminus h_\lambda(\partial B_0)$. Since $h_\lambda(\partial B_0)$ is fixed by f_λ, it means that $f_\lambda(\widetilde{B}_\lambda) = \widetilde{B}_\lambda$ or $f_\lambda(\widetilde{B}_\lambda) = \widehat{\mathbb{C}}$. If λ is small enough, then $0 \notin f_\lambda(\widetilde{B}_\lambda)$ by Lemma 6.1. This means that $f_\lambda(\widetilde{B}_\lambda) = \widetilde{B}_\lambda$ for sufficiently small λ and hence $\widetilde{B}_\lambda = B_\lambda$ is the immediate attracting basin of ∞ of f_λ. By the uniqueness of the holomorphic motion, we have $f_\lambda(B_\lambda) = B_\lambda$ for all $\lambda \in \Lambda$. This implies $h(\lambda, \partial B_0) = \partial B_\lambda$ for all $\lambda \in \Lambda$. □

Remark. The holomorphic motion has also been used in [BDGR08] and [GMR13] to prove the structure of the Julia sets still exists after perturbation for other singularly perturbed families. There the Böttcher map defined on the basin of infinity varies holomorphically is proved.

However, our proof here can be adopted to the following two more general cases: the first one is when ∞ lies in a super-attracting basin whose period is greater than one, and the second is when the corresponding basins of the perturbed maps are no longer super-attracting where the Böttcher map cannot be defined. This observation will be used in the next section.

Recall that A_λ is the Fatou component of f_λ containing 3 critical points $\{c_j^\lambda : 1 \leq j \leq 3\}$ and U_λ is the Fatou component containing the critical point c_0^λ.

Corollary 6.4. *The Fatou component A_λ is an annulus and the forward images of $f_\lambda(A_\lambda)$ are simply connected. In particular, U_λ, T_λ and B_λ are all simply connected.*

[8]We will see later that $h(\lambda, J(f_0))$ is actually a Julia component of f_λ.

Proof. By Lemmas 6.1 and 6.3, it follows that $B_\lambda \setminus \{\infty\}$ does not contain any critical points and critical values. Hence B_λ is simply connected. Note that we have the following orbit under f_λ:

$$A_\lambda \to f_\lambda(A_\lambda) \to \cdots \to U_\lambda \to T_\lambda \to B_\lambda.$$

Since $f_\lambda : T_\lambda \setminus \{c\} \to B_\lambda \setminus \{\infty\}$ is a degree two covering map, it follows that T_λ is simply connected. Similarly, $U_\lambda = f_\lambda^{\circ(p-1)}(A_\lambda)$ is simply connected. Moreover, $f_\lambda(A_\lambda)$, $\cdots$, $f_\lambda^{\circ(p-2)}(A_\lambda)$ is simply connected since they do not contain critical points (if $p \geq 3$). Since $f_\lambda : A_\lambda \to f_\lambda(A_\lambda)$ is a degree 3 branched covering and A_λ contains exactly 3 critical points, it means that A_λ is an annulus by Riemann-Hurwitz's formula. □

6.3. **The semi-buried component and buried Jordan curves.** Recall that $\Lambda = \mathbb{D}(0,\delta)$ and $h : \Lambda \times J(f_0) \to \mathbb{C}$ is a holomorphic motion introduced in Lemma 6.3. Let us fix $\lambda \in \Lambda \setminus \{0\}$ and define $h_\lambda = h(\lambda, \cdot) : J(f_0) \to \mathbb{C}$. Then

$$J_0(f_\lambda) := h_\lambda(J(f_0))$$

is contained in $J(f_\lambda)$ and $h_\lambda : J(f_0) \to J_0(f_\lambda)$ is a restriction of a quasiconformal homeomorphism by Lemma 6.2.

Let T_0 be the Fatou component of $f_0 = P_c$ containing the critical value c. Note that $h_\lambda(\partial T_0)$ and ∂T_λ are very different. In particular, ∂T_λ is compactly contained in the interior of $h_\lambda(T_0)$.

Proposition 6.5. *Let $0 < |\lambda| < \delta$. Then $J_0(f_\lambda)$ is a semi-buried Julia component of f_λ and $J(f)$ contains some buried Julia components which are Jordan curves.*

Proof. We use $\partial^+ A_\lambda$ and $\partial^- A_\lambda$ to denote the external and internal boundaries of A_λ respectively. For simplicity, we denote three annuli

$$V_1 = A(\partial^+ A_\lambda, h_\lambda(\partial T_0)),\ V_2 = A(\partial^- A_\lambda, \partial T_\lambda) \text{ and } V = A(\partial T_\lambda, h_\lambda(\partial T_0)).$$

Note that $f_\lambda^{\circ p} : V_1 \to V$ and $f_\lambda^{\circ p} : V_2 \to V$ are covering maps with degree 2 and 4 respectively. By a similar argument as in the proof of Proposition 2.6, it follows that V_1 contains a sequence of Julia components and a sequence of annular Fatou components of f_λ, such that they separate c from ∞ and converge to $h_\lambda(\partial T_0)$ in the Hausdorff metric. This means that $h_\lambda(\partial T_0)$ is semi-buried from $h_\lambda(T_0)$. Taking preimages of $h_\lambda(T_0)$ under f_λ, it is easy to see that $J_0(f_\lambda)$ is a semi-buried Julia component of f_λ.

For $j = 1, 2$, let $g_i : V \to V_j$ be the 'inverse' of $f_\lambda : V_j \to V$. Consider the set $J_{j_1 j_2 \cdots j_k \cdots} = \bigcap_{k=1}^{\infty} g_{j_k} \circ \cdots \circ g_{j_2} \circ g_{j_1}(V)$, where $(j_1, j_2, \cdots, j_k, \cdots)$ is an infinite sequence satisfying $j_k = 1, 2$. By the arguments above, each component $J_{j_1 j_2 \cdots j_k \cdots}$ is a Julia component separating c from $h_\lambda(\partial T_0)$. Since f_λ is hyperbolic, it follows that the Julia component $J_{j_1 j_2 \cdots j_k \cdots}$ is locally connected (see [TY96] and [PT00]).

Note that $V_j \subset V$ and the identity $\mathrm{id} : V_j \hookrightarrow V$ is not homotopic to a constant map. Let $J_{j_1 j_2 \cdots j_k \cdots}$ be the Julia component which is contained in the interior of V, then $J_{j_1 j_2 \cdots j_k \cdots}$ is a Jordan curve by [PT00, Lemma 2.4 (Case 2)] and [PT00, Proposition (Case 2)]. In particular, it is a buried Julia component. This finishes the proof of Proposition 6.5 and Theorem 1.2. □

Remark. In fact one can obtain a precise classification of the topology of the Julia components of f_λ. Specifically, some Julia components are semi-buried copies of $J(P_c)$, some are buried Jordan curves and some are buried singletons. One can see [BDGR08] and [GMR13] for a similar classification.

7. The cubic Julia sets containing buried Jordan curves

In this section, we aim to construct some cubic rational maps whose Julia sets contain buried Jordan curves and more importantly, we show that these cubic rational maps contain a buried Julia component which is homeomorphic to the Julia set of a quadratic rational map. The construction is inspired by combining Lemma 2.1 and Proposition 4.1.

For the quadatic rational map $Q_a(z) = 1 + a/z^2$, where $a \in \mathbb{C} \setminus \{0\}$, there are two critical points 0 and ∞ in the critical orbit of Q_a beginning with 0. Therefore, similar to the consideration in last section, a natural idea is to make a perturbation of Q_a at the critical value 1. Let $a \neq 0$ be the center of a hyperbolic component of $Q_a(z) = 1 + a/z^2$. We consider the following family:

$$f_\nu(z) = 1 + \frac{a}{z^2} + \frac{(1 + a + \sqrt{-a})\,\nu}{z - 1 - \nu},$$

where $\nu \in \mathbb{C} \setminus \{0\}$. This perturbation is made at the critical value 1 such that $f_\nu(1) = -\sqrt{-a}$. Note that $-\sqrt{-a}$ is a preimage of 0 under Q_a.

The main content in this section is to prove that if $\nu \neq 0$ is small enough, then the Julia set of f_ν contains some buried Jordan curves and a fully buried Julia component which is homeomorphic to the Julia set of $Q_a(z) = 1 + a/z^2$. In the rest of this section, we always assume that $\nu \neq 0$ is sufficiently small. Let $p \geq 3$ be the minimal period of the critical point of $Q_a(z) = 1 + a/z^2$, i.e. $Q_a^{\circ p}(0) = 0$ and $Q_a^{\circ p}(\infty) = \infty$.

Lemma 7.1. *There exists $\sigma > 0$ such that if $0 < |\nu| < \sigma$, then there exist five Fatou components A_ν, U_ν^1, U_ν^{-1}, U_ν^0 and U_ν^∞ such that*

(a) *A_ν contains exactly two critical points $\{c_j^\nu : j = 1, 2\}$ of f_ν such that $c_j^\nu \in \{z : |\nu|^{5/8} \leq |z - 1| \leq |\nu|^{3/8}\} \subset A_\nu$; In particular, $\lim_{\nu\to 0} c_j^\nu = 1$;*
(b) *U_ν^∞ contains exactly one critical point c_∞^ν of f_ν such that $\lim_{\nu\to 0} c_\infty^\nu = \infty$;*
(c) *$\overline{\mathbb{D}}(1, |\nu|^{11/8}) \subset U_\nu^1$; and*
(d) *$f_\nu^{\circ(p-2)}(A_\nu) = U_\nu^0$, $f_\nu(U_\nu^0) = U_\nu^\infty$, $f_\nu(U_\nu^\infty) = U_\nu^1$ and $f_\nu(U_\nu^1) = U_\nu^{-1}$ and $f_\nu(U_\nu^{-1}) = U_\nu^0$.*

In particular, f_ν is hyperbolic.

Proof. A direct calculation shows that

$$f_\nu'(z) = -\frac{2a(z - 1 - \nu)^2 + (1 + a + \sqrt{-a})\nu z^3}{z^3(z - 1 - \nu)^2}.$$

It is easy to see that f_ν has 4 critical points: 0, c_∞^ν, c_1^ν and c_2^ν, where

$$|c_\infty^\nu| \asymp |\nu|^{-1} \quad \text{and} \quad |c_j^\nu - 1| \asymp |\nu|^{1/2}, \text{ where } j = 1, 2.$$

Note that $\lim_{\nu\to 0} c_\infty^\nu = \infty$ and $\lim_{\nu\to 0} c_j^\nu = 1$ for $j = 1, 2$.

For any $z \in \overline{\mathbb{D}}(1, |\nu|^{11/8})$, we have $|f_\nu(z) + \sqrt{-a}| \preceq |\nu|^{3/8}$. Then

$$|f_\nu^{\circ 2}(z)| \preceq |\nu|^{3/8},\ |\nu|^{-3/4} \preceq |f_\nu^{\circ 3}(z)| \text{ and } |f_\nu^{\circ 4}(z) - 1| \preceq |\nu|^{3/2}.$$

Since $|\nu|^{3/2} \ll |\nu|^{11/8}$ if ν is small enough, it means that 1, $-\sqrt{-a}$, 0 and ∞ are contained in a cycle of periodic Fatou components[9] U_ν^1, U_ν^{-1}, U_ν^0 and U_ν^∞ respectively. In particular, $\overline{\mathbb{D}}(1, |\nu|^{11/8}) \subset U_\nu^1$. Since $|\nu|^{-3/4} \ll |\nu|^{-1}$ if ν is small enough, it means that $c_\infty^\nu \in U_\nu^\infty$.

Similarly, for $|\nu|^{5/8} \leq |z - 1| \leq |\nu|^{3/8}$ and $1 \leq k \leq p - 2$, we have

$$|f_\nu^{\circ k}(z) - Q_a^{\circ k}(1)| \preceq |\nu|^{3/8},\ |\nu|^{-3/4} \preceq |f_\nu^{\circ(p-1)}(z)| \text{ and } |f_\nu^{\circ p}(z) - 1| \preceq |\nu|^{3/2}.$$

[9]In this moment we don't know whether these four Fatou components are different.

Since $|\nu|^{3/2} \ll |\nu|^{11/8}$ and $|\nu|^{5/8} < |c_j^\nu - 1| < |\nu|^{3/8}$ for all $j = 1, 2$ if ν is small enough, it means that $f_\nu^{\circ p}(c_j^\nu) \in U_\nu^1$ provided ν is sufficiently small. Therefore, all the critical points of f_ν are attracted by an attracting cycle and hence f_ν is hyperbolic. □

Let U_0^1, U_0^{-1}, U_0^0 and U_0^∞, respectively, be the Fatou components of $f_0 := Q_a$ containing 1, $-\sqrt{-a}$, 0 and ∞. Let $\sigma > 0$ be the constant introduced in Lemma 7.1 and define $\Xi := \{\nu \in \mathbb{C} : |\nu| < \sigma\}$.

Lemma 7.2. *The five Fatou components A_ν, U_ν^1, U_ν^{-1}, U_ν^0 and U_ν^∞ are different. In particular, the latter four form a cycle of attracting periodic orbit with period 4.*

Proof. By applying a completely similar argument as Lemma 6.3, there is a holomorphic motion $h : \Xi \times J(f_0) \to \mathbb{C}$ parameterized by Ξ and with base point 0. The second statement follows from Lemma 7.1. □

Let us fix $\nu \in \Xi \setminus \{0\}$ and define $h_\nu = h(\nu, \cdot) : J(f_0) \to \mathbb{C}$. Then $J_0(f_\nu) := h_\nu(J(f_0))$ is contained in $J(f_\nu)$ and $h_\nu : J(f_0) \to J_0(f_\nu)$ is a restriction of a quasiconformal homeomorphism.

Proof of Theorem 1.3. We only give a sketch here since the proof is almost a copy of that of Proposition 6.5. We use $\partial^+ A_\nu$ and $\partial^- A_\nu$ to denote the external and internal boundaries of A_ν respectively. For simplicity, we denote three annuli

$$V_1 = A(\partial^+ A_\nu, h_\nu(\partial U_0^1)),\ V_2 = A(\partial^- A_\nu, \partial U_\nu^1) \text{ and } V = A(\partial U_\nu^1, h_\nu(\partial U_0^1)).$$

Note that $f_\nu^{\circ p} : V_1 \to V$ and $f_\nu^{\circ p} : V_2 \to V$ are both covering maps with degree 4. Similar to the proof of Proposition 6.5, one can show that $h_\nu(\partial U_0^1)$ is semi-buried from $h_\nu(U_0^1)$ first. Taking preimages of $h_\nu(U_0^1)$ under f_ν, one can show that $J_0(f_\nu)$ is a fully buried Julia component of f_ν. Then the existence of buried Julia components which are Jordan curves is completely similar to the proof of Proposition 6.5. We leave the details to the reader. □

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