ON $(4,2)$-CHOOSABLE GRAPHS

JIXIAN MENG, GREGORY J. PULEO, AND XUDING ZHU

Abstract. A graph G is called (a,b)-choosable if for any list assignment L which assigns to each vertex v a set $L(v)$ of a permissible colours, there is a b-tuple L-colouring of G. An $(a,1)$-choosable graph is also called a-choosable. In the pioneering paper on list colouring of graphs by Erdős, Rubin and Taylor [2], 2-choosable graphs are characterized. Confirming a special case of a conjecture in [2], Tuza and Voigt [3] proved that 2-choosable graphs are $(2m,m)$-choosable for any positive integer m. On the other hand, Voigt [6] proved that if m is an odd integer, then these are the only $(2m,m)$-choosable graphs; however, when m is even, there are $(2m,m)$-choosable graphs that are not 2-choosable. A graph is called 3-choosable-critical if it is not 2-choosable, but all its proper subgraphs are 2-choosable. Voigt conjectured that for every positive integer m, all bipartite 3-choosable-critical graphs are $(4m,2m)$-choosable. In this paper, we determine which 3-choosable-critical graphs are $(4,2)$-choosable, refuting Voigt's conjecture in the process. Nevertheless, a weaker version of the conjecture is true: we prove that there is an even integer k such that for any positive integer m, every bipartite 3-choosable-critical graph is $(2km,km)$-choosable. Moving beyond 3-choosable-critical graphs, we present an infinite family of non-3-choosable-critical graphs which have been shown by computer analysis to be $(4,2)$-choosable. This shows that the family of all $(4,2)$-choosable graphs has rich structure.

1. Introduction

Multiple list colouring of graphs was introduced in the 1970s by Erdős, Rubin and Taylor [2]. A *list assignment* is a function L which assigns to each vertex v a set of permissible colours $L(v)$. A *b-tuple colouring* of a graph G is a function f that assigns to each vertex v a set $f(v)$ of b colours so that $f(u) \cap f(v) = \emptyset$ for any edge uv of G. Given a list assignment L of G, a *b-tuple L-colouring* of G, also called an (L,b)-colouring of G, is a b-tuple colouring f of G with $f(v) \subseteq L(v)$ for all $v \in V(G)$. We say G is (L,b)*-colourable* if there is a b-tuple L-colouring of G, and say G is (a,b)*-choosable* if G is (L,b)-colourable for any list assignment L with $|L(v)| = a$ for all v. A $(a,1)$-choosable graph is also called *a-choosable.* The *choice number* $\mathrm{ch}(G)$ of a graph G is the smallest integer a such that G is a-choosable. List colouring of graphs has been studied extensively in the literature; see [4] for a survey.

The family of 2-choosable graphs was characterized by Erdős, Rubin and Taylor [2]. These graphs have very simple structure. We define the *core* of a graph G to be the graph obtained by iteratively deleting vertices of degree 1. It is easy to see that a graph is 2-choosable if and only if its core is 2-choosable. It was proved in [2] that a graph G is 2-choosable if and only if its core is K_1 or an even

Date: October 5, 2017.

This research supported partially by grant CNSF 11571319.

cycle or $\Theta_{2,2,2p}$ for some positive integer p, where $\Theta_{r,s,t}$ is the graph consisting of two end vertices u and v joined by three internally vertex-disjoint paths containing r, s, and t edges respectively. Erdős, Rubin, and Taylor [2] conjectured that if a graph is (a,b)-choosable, then it is (am,bm)-choosable for every positive integer m; Tuza and Voigt [3] confirmed a special case of this conjecture by proving that all 2-choosable graphs are $(2m,m)$-choosable for all m, but the conjecture is otherwise open. Moreover, Voigt [6] proved that if m is an odd integer, then these are the only $(2m,m)$-choosable graphs.

When m is even, the family of $(2m,m)$-choosable graphs has much richer structure. A *b-tuple a-colouring* of a graph G is a b-tuple colouring f of G with $f(v) \subseteq \{1,2,\ldots,a\}$ for each v. We say G is *(a,b)-colourable* if such a colouring exists. Alon, Tuza, and Voigt [1] showed that if a graph G is (a,b)-colourable, then there is a positive integer k_G such that G is (ak_Gm, bk_Gm)-choosable for all m. In particular, for any bipartite graph G, there is a positive integer m_G such that G is $(2m_G, m_G)$-choosable.

This paper is devoted to the study of $(4m,2m)$-choosability. In particular, we are interested in the question of which graphs are $(4,2)$-choosable.

A graph G is called *3-choosable-critical* if G is not 2-choosable, but any proper subgraph is 2-choosable. The family of 3-choosable-critical graphs is characterized by Voigt [6]:

Theorem 1.1 (Voigt [6])**.** *A graph is 3-choosable-critical if and only if it is one of the following:*

(a) two vertex-disjoint even cycles joined by a path,
(b) two even cycles with exactly one vertex in common,
(c) a $\Theta_{2r,2s,2t}$-graph or $\Theta_{2r-1,2s-1,2t-1}$-graph with $r \geq 1$ and $s,t > 1$,
(d) a $\Theta_{2,2,2,2t}$-graph with $t \geq 1$,
(e) an odd cycle.

Voigt conjectured that for any positive integer m, all bipartite 3-choosable-critical graphs are $(4m,2m)$-choosable. In this paper, we prove the following characterization of the $(4,2)$-choosable 3-choosable-critical graphs, which refutes Voigt's conjecture:

Theorem 1.2. *A 3-choosable-critical graph is $(4,2)$-choosable if and only if it is one of the following:*

(a) two vertex-disjoint even cycles joined by a path,
(b) two even cycles with exactly one vertex in common,
(c) a $\Theta_{2,2s,2t}$-graph or $\Theta_{1,2s-1,2t-1}$-graph with $s,t > 1$,
(d) $\Theta_{2,2,2,2}$.

(Note that $\Theta_{2,2,2,2} \cong K_{2,4}$.) In particular, among the bipartite 3-choosable-critical graphs, when r,s,t have the same parity and $\min\{r,s,t\} \geq 3$, the graph $\Theta_{r,s,t}$ fails to be $(4,2)$-choosable, and when $t > 1$, the graph $\Theta_{2,2,2,2t}$ fails to be $(4,2)$-choosable.

Nevertheless, a weaker version of Voigt's conjecture is true:

Theorem 1.3. *There is a fixed integer k such that for every positive integer m, every bipartite 3-choosable-critical graph is $(4km,2km)$-choosable.*

	v_1	v_2	v_3
$L(v_i)$	abcd	abef	adeg
X_i	abcd	ef	adg
$\hat{X}_i$	cd		dg

FIGURE 1. Example computations of X_i and $\hat{X}_i$. Here $A = \{\mathrm{a}\}$.

The paper is structured as follows. In Section 2 we introduce the main lemmas and definitions needed for the proof of Theorem 1.2. In Section 3 we collect some more useful lemmas of a more technical nature.

In Section 4 we prove that theta graphs of the form $\Theta_{2,2s,2t}$ and $\Theta_{1,2s-1,2t-1}$ are $(4,2)$-choosable. In Section 5 we apply these results to show that if G consists of two vertex-disjoint even cycles joined by a path or two even cycles sharing a vertex, then G is $(4,2)$-choosable. Tuza and Voigt have already shown [5] that $K_{2,4}$ is $(4m,2m)$-choosable for all m, so this completes the positive direction of Theorem 1.2.

In Section 6 we present list assignments showing that $\Theta_{3,3,3}$, $\Theta_{4,4,4}$, and $\Theta_{2,2,2,4}$ are not $(4,2)$-choosable; a quick argument given in that section shows that the larger theta graphs also fail to be $(4,2)$-choosable. This completes the characterization of the $(4,2)$-choosable 3-choosable-critical graphs.

In Section 7, we prove Theorem 1.3. In Section 8, we present some non-3-choosable-critical graphs and briefly discuss the computer analysis that demonstrates that these graphs are $(4,2)$-choosable. We close with a conjectured characterization of the $(4,2)$-choosable graphs.

2. Paths and Damage

Definition 2.1. When P is an n-vertex path with vertices $v_1,\dots,v_n$ in order, and L is a list assignment on P, we define sets $X_1,\dots,X_n$ by

$$\begin{aligned} X_1 &= L(v_1), \\ X_i &= L(v_i) - X_{i-1} \qquad (i \in \{2,\dots,n\}). \end{aligned}$$

We also define the quantity $S_L(P)$ by

$$S_L(P) = \sum_{i=1}^{n} |X_i|.$$

Lemma 2.2. *Let P be an n-vertex path and let L be a list assignment on P such that $|L(v_1)|, |L(v_n)| \geq 2m$ and $|L(v_i)| = 4m$ for $i \in \{2,\dots,n-1\}$. The path P is $(L,2m)$-colourable if and only if $S_L(P) \geq 2mn$.*

Proof. We use induction on n. The claim is trivial for $n = 1$. Assume that $n \geq 2$ and the claim holds for $n' < n$. Let $P' = P - v_n$, and observe that if $X'_1, \dots X'_{n-1}$ are computed as above for P', then $X'_i = X_i$ for all $i \in \{1,\dots,n-1\}$. Since $|X_1| \geq 2m$ and $|X_i| + |X_{i-1}| \geq |L(v_i)| \geq 4m$ for $i \in \{2,\dots,n-1\}$, we have $\sum_{i=1}^{n-1} |X'_i| \geq 2(n-1)m$.

First assume $S_L(P) \geq 2nm$. We shall prove that P is $(L,2m)$-colourable. We determine a $2m$-set of colours $\phi(v_n)$ to be assigned to v_n as follows: when $|X_n| \geq$

$2m$, let $\phi(v_n)$ be any $2m$-subset contained in X_n; when $|X_n| < 2m$, let $\phi(v_n)$ be any $2m$-subset of $L(v_n)$ containing X_n.

Let L^* be the restriction of L to P', except that $L^*(v_{n-1}) = L(v_{n-1}) - \phi(v_n)$. When $|X_n| \geq 2m$, we have $S_{L^*}(P') = \sum_{i=1}^{n-1}|X_i| \geq 2(n-1)m$, since $\phi(v_n) \cap X_{n-1} = \emptyset$; when $|X_n| < 2m$, we have $S_{L^*}(P') \geq \sum_{i=1}^{n}|X_i| - |\phi(v_n)| \geq 2(n-1)m$, since $\phi(v_n) \supseteq X_n$. Either way, by the induction hypothesis, P' has an $(L^*, 2m)$-colouring, which extends to an $(L, 2m)$-colouring of P by assigning $\phi(v_n)$ to v_n.

For the other direction, let ϕ be an $(L, 2m)$-colouring of P. Let L^* be the restriction of L to P', except that $L^*(v_{n-1}) = L(v_{n-1}) - \phi(v_n)$, and let $X_1^*, \dots X_{n-1}^*$ be computed for L^*. Since ϕ is an $(L^*, 2m)$-colouring of P', the induction hypothesis implies that

$$\sum_{i=1}^{n-1}|X_i^*| = S_{L^*}(P') \geq 2(n-1)m.$$

It is easy to verify that $X_i = X_i^*$ for $i = 1, 2, \dots, n-2$, and $|X_{n-1}| + |X_n| \geq |X_{n-1}^*| + |\phi(v_n)| \geq |X_{n-1}^*| + 2m$. Hence $S_L(P) \geq 2nm$. $\square$

Our typical strategy for showing that a graph G is $(4m, 2m)$-choosable is as follows: identify a set of vertices X such that $G - X$ is a linear forest (disjoint union of paths), and find a precolouring of X such that each path P in $G - X$ satisfies $S_{L^*}(P) \geq 2m|V(P)|$, where L^* is obtained from L by removing from each vertex of $G - X$ the colours used on its neighbors in X. Provided that the degree-2 vertices of $G - X$ have no neighbors in X, Lemma 2.2 then guarantees that we can extend the precolouring of X to the rest of the graph, as desired.

In order to carry out this strategy, we need to know how $S_L(P)$ changes when colours are removed from the endpoints of P. We will be particularly interested in the case where P has an odd number of vertices. Before stating the results, we set up some more notation.

Definition 2.3. If L is a list assignment on an n-vertex path P and S, T are sets of colours, we define $L \ominus (S, T)$ to be the list assignment obtained from L by deleting all colours in S from $L(v_1)$, all colours in T from $L(v_n)$, and leaving all other lists unchanged.

Definition 2.4. Let L be a list assignment on an n-vertex path P, where n is odd. Define

$$A = \bigcap_{x \in V(P)} L(x).$$

Let

$$\hat{X}_1 = \{c \in L(v_1) - A \colon \text{ the smallest index } i \text{ for which } c \notin L(v_i) \text{ is even}\}.$$
$$\hat{X}_n = \{c \in L(v_n) - A \colon \text{ the largest index } i \text{ for which } c \notin L(v_i) \text{ is even}\}.$$

See Figure 1 for an example of $\hat{X}_1$ and $\hat{X}_n$.

Observation 2.5. If P is an n-vertex path, where n is odd, then for any list assignment on P, we have $\hat{X}_n = X_n - A$.

Lemma 2.6. *Let L be a list assignment on an n-vertex path P, where n is odd. For any sets of colours S, T, we have*

$$S_{L \ominus (S,T)}(P) = S_L(P) - \left(\left|(A \cup \hat{X}_1) \cap S\right| + \left|(A \cup \hat{X}_n) \cap T\right| - |A \cap S \cap T|\right).$$

	v_1	v_2	v_3
L	abcd	abef	adeg
X_i	abcd	ef	adg
$\hat{X}_i$	cd		dg

	v_1	v_2	v_3
$L \ominus (S,T)$	cd	abef	de
X_i	cd	abef	d

FIGURE 2. Example computations for $L \ominus (S,T)$ when $(S,T) = (\text{ab}, \text{ag})$.

Proof. It suffices to consider the effect of deleting just one colour c. First we consider deleting the colours in T from $L(v_n)$. Clearly, if $c \notin X_n$ then deleting the colour c from $L(v_n)$ has no effect on $S_L(P)$, since it does not change any X_i. On the other hand, if $c \in X_n = A \cup \hat{X}_n$, then deleting the colour c from $L(v_n)$ decreases $S_L(P)$ by exactly 1.

Next we consider deleting a colour c from $L(v_1)$. Here, unlike with $L(v_n)$, the changes in X_1 can "ripple" through later X_i, as shown in Figure 2. If $c \notin X_1 = L(v_1)$, then deleting c from $L(v_1)$ clearly does not change any X_i, hence does not change $S_L(P)$.

Now suppose $c \in X_1 - A$. Deleting c from $L(v_1)$ causes c to be removed from X_1. However, if $c \in L(v_2)$, we gain c in X_2. Now this may cause us to lose c in X_3, gain c in X_4, and so forth. The process continues until we reach an index i for which $c \notin L(v_i)$. If i is odd, then we lose c from the sets $X_1, X_3, \ldots, X_{i-2}$ and gain c in the sets $X_2, X_4, \ldots, X_{i-1}$. So there is no net change in $S_L(P)$. If i is even, then we lose c from the sets $X_1, X_3, \ldots, X_{i-1}$ and gain c in the sets $X_2, X_4, \ldots, X_{i-2}$. So $S_L(P)$ has decreased by 1.

Finally, suppose $c \in X_1 \cap A$. Deleting c from $L(v_1)$ causes the same ripple process described above, terminating when we try to delete c from X_n (since n is odd). If $c \notin T$, then as before, this causes $S_L(P)$ to decrease by 1. However, if $c \in T$, then we have *already* deleted c from X_n, so in this step we really gain and lose c an equal number of times. Thus, when $c \in A \cap S \cap T$, deleting c from both endpoints of L decreases $S_L(P)$ by exactly 1, but such colours are double-counted in the sum $|(A \cup \hat{X}_1) \cap S| + |(A \cup \hat{X}_n) \cap T|$. The final term $|A \cap S \cap T|$ corrects for this overcount. $\square$

Together, Lemma 2.2 and Lemma 2.6 allow us to ignore the details of the list assignment and focus on the sets $\hat{X}_1, \hat{X}_n, A$, as described below.

Definition 2.7. For a pair of colour sets S, T, the *damage* of (S,T) with respect to L and P is written $\text{dam}_{L,P}(S,T)$ and defined by

$$\text{dam}_{L,P}(S,T) = S_L(P) - S_{L \ominus (S,T)}(P).$$

Lemma 2.6 shows that if P has an odd number of vertices, then given a pair S, T of colour sets, the damage $\text{dam}_{L,P}(S,T)$ just depends on $\hat{X}_1, \hat{X}_n$ and A, and in particular

$$\begin{aligned} \text{dam}_{L,P}(S,T) &= |(A \cup \hat{X}_1) \cap S| + |(A \cup \hat{X}_n) \cap T| - |A \cap S \cap T| \\ &= |\hat{X}_1 \cap S| + |\hat{X}_n \cap T| + |A \cap (S \cup T)|. \end{aligned} \tag{1}$$

In the example of Figure 2, we have $\text{dam}_{L,P}(S,T) = 2$.

Lemma 2.8. *Let G be a graph, and let $X \subseteq V(G)$ be a set of vertices such that every component of $G - X$ is a path with an odd number of vertices. Assume that* *for each component P of $G-X$, only the two end vertices of P have neighbors in X. Let L be a list assignment on G with $|L(v)| = 4m$ for all $v \in V(G)$. The graph G is $(L, 2m)$-colourable if and only if $G[X]$ has an $(L, 2m)$-colouring ϕ such that for every path P in $G-X$ with vertices $v_1, \ldots, v_n$ in order, the following conditions hold:*

(i) $|L(v_1) \cap \phi(N_X(v_1))| \leq 2m$,

(ii) $|L(v_n) \cap \phi(N_X(v_n))| \leq 2m$, *and*

(iii) $\operatorname{dam}_{L,P}(\phi(N_X(v_1)), \phi(N_X(v_n))) \leq S_L(P) - 2mn$.

Proof. Clearly, G is $(L, 2m)$-colourable if and only if $G[X]$ has an $(L, 2m)$-colouring ϕ that extends to G, i.e., extends to each of the paths P in $G-X$. For each path P of $G-X$, we show that ϕ extends to P if and only if ϕ satisfies conditions (i)–(iii). Conditions (i) and (ii) are clearly necessary, so it suffices to show that when Conditions (i) and (ii) hold, ϕ extends to P if and only if Condition (iii) holds. This follows from Lemma 2.2 and Lemma 2.6. □

3. Technical Lemmas

To apply Lemma 2.8, we need to find lower bounds for $S_L(P)$ and upper bounds for $\operatorname{dam}_{L,P}(S,T)$. In this section, we collect some technical lemmas regarding such bounds.

Lemma 3.1. *If L is a list assignment on an n-vertex path P, where n is odd and $|L(v_i)| = 4m$ for all i, then*

$$S_L(P) = 2nm - 2m + \sum_{\substack{k \text{ even} \\ k<n}} |X_{k-1} - L(v_k)| + |X_n|.$$

Proof. We use induction on n. When $n = 1$, the sum is empty and $2nm - 2m = 0$, so the claim is just $S_L(P) = |X_1|$, which is clearly true. Assume that $n > 1$ and the claim holds for smaller odd n. Let $P' = P - \{v_{n-1}, v_n\}$ and let L' be the restriction of L to P', so that $S_L(P) = S_{L'}(P') + |X_{n-1}| + |X_n|$. Applying the induction hypothesis to P' yields

$$S_L(P) = \left(2nm - 6m + \sum_{\substack{k \text{ even} \\ k<n-2}} |X_{k-1} - L(v_k)| + |X_{n-2}|\right) + |X_{n-1}| + |X_n|$$

Observe that

$$\begin{aligned} |X_{n-1}| &= |L(v_{n-1}) - X_{n-2}| \\ &= |L(v_{n-1})| - |X_{n-2}| + |X_{n-2} - L(v_{n-1})| \\ &= 4m - |X_{n-2}| + |X_{n-2} - L(v_{n-1})| \end{aligned}$$

Combining these terms with the terms from $S_{L'}(P')$ gives the desired expression for $S_L(P)$. □

Lemma 3.2. *If L is a list assignment on an n-vertex path P, where n is odd and $|L(v_i)| = 4m$ for all i, then*

$$S_L(P) \geq 2nm - 2m + |\hat{X}_1| + |\hat{X}_n| + |A|.$$

Proof. By the definition of $\hat{X}_1$, every element of $\hat{X}_1$ appears in a set of the form $X_{k-1} - L(v_k)$ where k is even. Thus, the claim follows from Lemma 3.1, since $|X_n| = |\hat{X}_n| + |A|$. □

Lemma 3.3. *If L is a list assignment on an n-vertex path P, where n is odd and $|L(v_i)| = 4m$ for all i, then $S_L(P) \geq 2nm + 2m$.*

Proof. This follows immediately from the definition $S_L(P) = \sum_{i=1}^{n} |X_i|$ and the observations that $|X_1| = |L(v_1)| = 4m$ and that $|X_i| + |X_{i+1}| \geq 4m$ for $i > 1$. □

4. $(4,2)$-choosable Theta Graphs

In this section, we show that $\Theta_{r,s,t}$ is $(4,2)$-choosable if r, s, t have the same parity and $\min\{r, s, t\} \leq 2$. In Section 6, we will show that $\min\{r, s, t\} \geq 3$ implies that $\Theta_{r,s,t}$ is not $(4,2)$-choosable. As we are only concerned with $(4,2)$-choosability, we will tacitly assume that all list assignments considered in this section have $|L(v)| = 4$ for all $v \in V(G)$.

We first use an observation of Voigt to restrict to the case where r, s, t are even.

Lemma 4.1 (Lemma 4.3 of Voigt [6])**.** *Let G be a graph, let $v \in V(G)$, and let G' be obtained from G by deleting v and merging its neighbors. If G is $(4m, 2m)$-choosable, then G' is $(4m, 2m)$-choosable.*

The transformation used in Lemma 4.1 was first used in [2], which observed that if G is 2-choosable, then G' is also 2-choosable. Voigt [6] made the stronger observation that if G is $(2m, m)$-choosable, then G' is also $(2m, m)$-choosable. While Voigt imposed the additional assumption that $d(v) = 2$, this assumption is not necessary.

Proof of Lemma 4.1. We may assume that $d(v) \geq 2$, as otherwise G' is just a subgraph of G. Let v' be the merged vertex in G', and let L' be a list assignment on G' such that $|L'(w)| = 4m$ for all $w \in V(G')$. Define a list assignment L on G as follows:

$$L(w) = \begin{cases} L'(v'), & \text{if } w = v \text{ or } w \in N(v), \\ L'(w), & \text{otherwise.} \end{cases}$$

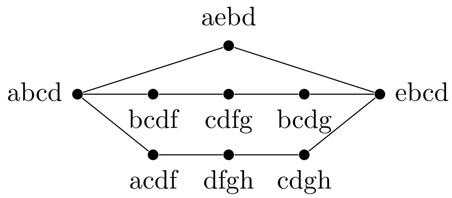
Since G is $(4m, 2m)$-choosable, it has some proper L-colouring ϕ. For all $w \in N(v)$, we have $\phi(w) \cap \phi(v) = \emptyset$. Since $L(w) = L(v)$ and since $\phi(w), \phi(v) \subseteq L(v)$ with $|\phi(w)| + |\phi(v)| = |L(v)|$, this implies that $\phi(w) = L(v) - \phi(v)$ for all $w \in N(v)$. We define an L'-colouring ϕ' of G' by putting $\phi'(v') = L(v) - \phi(v)$ and putting $\phi'(w) = \phi(w)$ for all $w \in V(G') - v'$. Since ϕ was a proper L-colouring, we see that ϕ' is a proper L'-colouring. As L' was arbitrary, we conclude that G' is $(4m, 2m)$-choosable. □

Corollary 4.2. *If $\Theta_{2,2r,2s}$ is $(4,2)$-choosable, then $\Theta_{1,2r-1,2s-1}$ is $(4,2)$-choosable.*

Proof. Applying the operation of Lemma 4.1 to a vertex v of degree 3 transforms $\Theta_{2,2r,2s}$ into $\Theta_{1,2r-1,2s-1}$. □

It therefore suffices to show that $\Theta_{2,2r,2s}$ is $(4,2)$-choosable for all $r, s \geq 1$. Similar techniques will allow us to deal with cycles sharing a vertex or joined by a path.

We now introduce some notation for various parts of theta graphs; Figure 3 shows $\Theta_{2,4,4}$, as a reference.



j	c_j	c'_j
0	a	e
1	b	b
2	c	c
3	d	d

FIGURE 3. The graph $\Theta_{2,4,4}$, with a list assignment and an associated coupling. From top to bottom, the internal paths are P^0, P^1, and P^2.

Definition 4.3. The vertices of degree 3 in a theta graph are called u and v. The *internal paths* of a theta graph are the paths in $G - \{u, v\}$; the endpoints of the internal paths are the neighbors of u and v.

Fix a list assignment L, and let $L(u) = \{c_0, c_1, c_2, c_3\}$ and $L(v) = \{c'_0, c'_1, c'_2, c'_3\}$, where the colours are indexed so that $c'_j = c_j$ whenever $c_j \in L(u) \cap L(v)$. Note that this indexing implies that $\{c_i, c'_i\} \cap \{c_j, c'_j\} = \emptyset$ whenever $i \neq j$.

Definition 4.4. For a fixed indexing of $L(u)$ and $L(v)$, a *couple* is a tuple of the form (c_j, c'_j) for $j \in \{0, 1, 2, 3\}$. When we write a couple, we suppress the parentheses and simply write $c_jc'_j$. A *pair* is a tuple (S, T) with $S \subseteq L(u)$, $T \subseteq L(v)$, and $|S| = |T| = 2$. A *simple pair* is a pair (S, T) such that for all $c_j \in S$, we also have $c'_j \in T$. A *simple solution* is a simple pair (S, T) such that $\mathrm{dam}_{L,P}(S, T) \leq S_L(P) - 2|V(P)|$ for all internal paths P.

Observe that the definition of a couple and a simple pair depends on the indexing of the colours of $L(u)$ and colours in $L(v)$. A simple solution can be interpreted as a precolouring of $\{u, v\}$ which extends (via Lemma 2.8) to all internal paths of the theta graph. With any fixed indexing of $L(u)$ and $L(v)$, we first try to find a simple solution. We show that a simple solution exists unless L has a very specific form. Then we address this form as a special case.

Equation (1) implies that if S, T is a simple pair, then

$$\mathrm{dam}_{L,P}(S, T) = \sum_{c_j \in S} \mathrm{dam}_{L,P}(\{c_j\}, \{c'_j\}).$$

In other words, when (S, T) is a simple pair, we can simply calculate the damage of each couple in (S, T) independently, and add them together to obtain $\mathrm{dam}_{L,P}(S, T)$. Moreover, for each j, we have $\mathrm{dam}_{L,P}(\{c_j\}, \{c'_j\}) \in \{0, 1, 2\}$.

Definition 4.5. When L is a list assignment on a theta graph,

- The couple $c_jc'_j$ is *heavy* for the internal path P if $\mathrm{dam}_{L,P}(\{c_j\}, \{c'_j\}) = 2$;
- The couple $c_jc'_j$ is *light* for the internal path P if $\mathrm{dam}_{L,P}(\{c_j\}, \{c'_j\}) = 1$;
- The couple $c_jc'_j$ is *safe* for the internal path P if $\mathrm{dam}_{L,P}(\{c_j\}, \{c'_j\}) = 0$.

Definition 4.6. When L is a list assignment on a theta graph, we say that an internal path P *blocks* a pair (S, T) if $\mathrm{dam}_{L,P}(S, T) > S_L(P) - 2|V(P)|$, i.e., if we cannot extend the partial colouring $\phi(u) = S$, $\phi(v) = T$ to all vertices of P.

Example 4.7. For the list assignment shown in Figure 3, the couple ae is heavy for P^0, safe for P^1, and light for P^2. The path P^2 blocks the simple pair (ac, ec).

Now we count how many simple pairs are blocked by each internal path. It will be helpful to prove this lemma for more general theta graphs than $\Theta_{r,s,t}$.

Lemma 4.8. *Let $r_1, \ldots, r_k$ be positive integers, and let L be a list assignment on $\Theta_{2r_1,\ldots,2r_k}$. Each internal path P blocks at most 2 simple pairs, and if P blocks 2 simple pairs, then $S_L(P) = 2|V(P)| + 2$, and P has one heavy couple and two light couples.*

Proof. Let P be any internal path, and let $n = |V(P)|$. By the $m = 1$ case of Lemma 3.3, $S_L(P) \geq 2n + 2$. If $S_L(P) \geq 2n + 4$, then P does not block any simple pairs, since for any pair (S, T), we have $\mathrm{dam}_{L,P}(S, T) \leq 4$. Hence it suffices to consider $S_L(P) \in \{2n + 2, 2n + 3\}$.

We first argue that in both cases, P has at most 2 heavy couples. If $c_j c'_j$ is a heavy couple, then by Equation (1) we have

$$|\hat{X}_1 \cap \{c_j\}| + |\hat{X}_n \cap \{c'_j\}| + |A \cap \{c_j, c'_j\}| = 2.$$

In particular, since $\{c_i, c'_i\} \cap \{c_j, c'_j\} = \emptyset$ whenever $i \neq j$, we see that if P has 3 heavy couples, then $|\hat{X}_1| + |\hat{X}_n| + |A| \geq 6$. By the $m = 1$ case of Lemma 3.2, this implies that $S_L(P) \geq 2n + 4$.

If $S_L(P) = 2n+3$, then P blocks the simple pair (S, T) only if $\mathrm{dam}_{L,P}(S, T) = 4$, i.e., if both couples used in (S, T) are heavy. Since P has at most 2 heavy couples, this implies that P blocks at most 1 simple pair.

If $S_L(P) = 2n+2$, then P blocks the simple pair (S, T) if and only if $\mathrm{dam}_{L,P}(S, T) \geq 3$, i.e., if one of the couples in (S, T) is heavy and the other is not safe. Lemma 3.2 implies that if P has 2 heavy couples, then P has no light couple, since that would imply that $|\hat{X}_1| + |\hat{X}_n| + |A| \geq 5$. In particular, if P has 2 heavy couples, then it blocks at most 1 simple pair. Likewise, if P has 1 heavy couple, then P has at most 2 light couples. The desired conclusion follows. □

Now we specialize to the $\Theta_{r,s,t}$ case.

Corollary 4.9. *Let r, s, t be positive integers, and let L be a list assignment on $\Theta_{2r,2s,2t}$. If L has no simple solution, then each simple pair (S, T) is blocked by exactly one internal path. In particular, each couple $c_j c'_j$ is heavy for at most one internal path.*

Proof. There are 6 simple pairs, and each of the three internal paths blocks at most 2 of them; this proves the first part. If the couple $c_j c'_j$ is heavy for two different internal paths P and Q, then since P and Q each have two light couples, there is some couple $c_k c'_k$ that is light for both P and Q. Now the pair $(\{c_j, c_k\}, \{c'_j, c'_k\})$ is blocked by both P and Q, contradicting the first part of the corollary. □

We now must handle the case where L has no simple solution. First we refine our notation. By Corollary 4.9, we may reindex $L(u)$ and $L(v)$ so that for all $j \in \{0, 1, 2\}$, the couple $c_j c'_j$ is heavy for P^j. (By simultaneously permuting the labels in $L(u)$ and $L(v)$, this maintains the original property that $c'_j = c_j$ whenever $c_j \in L(u) \cap L(v)$.) With this new notation, we have the following further consequence of Corollary 4.9:

Corollary 4.10. *Let r, s, t be positive integers, and let L be a list assignment on $\Theta_{2r,2s,2t}$. If L has no simple solution, then $c_3 c'_3$ is light for all internal paths P^i, and one of the two following situations must hold:*

(a) c_1c_1' *is light for* P^0, c_2c_2' *is light for* P^1, *and* c_0c_0' *is light for* P^2, *or*
(b) c_2c_2' *is light for* P^0, c_0c_0' *is light for* P^1, *and* c_1c_1' *is light for* P^2.

Proof. As in Corollary 4.9, since there is no simple solution, each internal path blocks 2 simple pairs. Thus, by Lemma 4.8, each internal path has one heavy couple and two light couples, and therefore has exactly one safe couple. For each $j \in \{0,1,2\}$, let $\pi(j)$ be the unique index in $\{0,1,2,3\}$ such that $c_{\pi(j)}c'_{\pi(j)}$ is safe for P^j. We will show that π is a permutation of $\{0,1,2\}$ having no fixed points. It is clear that $\pi(j) \neq j$ for all $j \in \{0,1,2\}$, since c_jc_j' is heavy for P^j.

First we argue that π is an injection. Suppose that $\pi(i) = \pi(j)$ for some $i \neq j$. Since c_jc_j' is heavy for P^j and since P^j has one heavy couple and two light couples, it follows that c_ic_i' is light for P^j. Likewise, c_jc_j' is light for P^i. By Lemma 4.8, we have $S_L(P^i) - 2|V(P^i)| = S_L(P^j) - 2|V(P^j)| = 2$, so the simple pair $(\{c_i, c_j\}, \{c_i', c_j'\})$ is blocked by both P^i and P^j, contradicting Corollary 4.9.

Next we argue that $\pi(j) \neq 3$ for all j. If $\pi(j) = 3$, then for both $i \in \{0,1,2\} - j$, the couple c_ic_i' is light for P^j. Since π is an injection, there is some $i \in \{0,1,2\} - j$ with $\pi(i) \neq j$, so that c_jc_j' is light for P^i. Now the simple pair $(\{c_i, c_j\}, \{c_i', c_j'\})$ is blocked by both P^i and P^j, again contradicting Corollary 4.9.

Thus π is a permutation of $\{0,1,2\}$ with no fixed points. This implies that π is a 3-cycle. If $\pi = (0\ 2\ 1)$ then situation (a) holds, and if $\pi = (0\ 1\ 2)$ then situation (b) holds. □

Corollary 4.11. *If* r, s *are positive integers, then* $\Theta_{2,2r,2s}$ *is* $(4,2)$*-choosable.*

Proof. Let $G = \Theta_{2,2r,2s}$, and let L be any list assignment on G. We must show that G is $(L,2)$-colourable. If L has a simple solution, then there is nothing more to show, so we may assume that L does not have a simple solution. Let P^0, P^1, and P^2 be the internal paths of G, with $|V(P^0)| = 1$. We may choose the indexing of P^1 and P^2 so that situation (a) of Corollary 4.10 holds (this is the case, for example, in Figure 3). For each $i \in \{0,1,2\}$, we write $\hat{X}_1^i$, $\hat{X}_n^i$, and A^i to refer to the sets $\hat{X}_1$, $\hat{X}_n$, and A calculated for P^i.

Since $|V(P^0)| = 1$, we know that $\hat{X}_1^0 = \hat{X}_n^0 = \emptyset$. Hence, since c_0c_0' is heavy for P_0, we must have $c_0 \neq c_0'$. Hence $c_0 \notin L(u) \cap L(v)$ and $c_0' \notin L(u) \cap L(v)$.

Now consider P^2. Since $c_0 \neq c_0'$ and c_0c_0' is light for P^2, we must have either $c_0 \notin A^2 \cup \hat{X}_1^2$ or $c_0' \notin A^2 \cup \hat{X}_n^2$. By symmetry, we may assume that $c_0 \notin A^2 \cup \hat{X}_1^2$. Let $S = \{c_0, c_3\}$ and let $T = \{c_2', c_3'\}$.

We check that $\mathrm{dam}_{L,P}(S,T) \leq 2$ for each internal path P. By Equation (1), for each i we have

$$\begin{aligned}
\mathrm{dam}_{L,P^i}(S,T) &= |(A^i \cup \hat{X}_1^i) \cap S| + |(A^i \cup \hat{X}_n^i) \cap T| - |A^i \cap S \cap T| \\
&\leq |(A^i \cup \hat{X}_1^i) \cap \{c_0\}| + |(A^i \cup \hat{X}_n^i) \cap \{c_2'\}| + \\
&\qquad |(A^i \cup \hat{X}_1^i) \cap \{c_3\}| + |(A^i \cup \hat{X}_n^i) \cap \{c_3'\}| - |A^i \cap \{c_3\} \cap \{c_3'\}| \\
&= |(A^i \cup \hat{X}_1^i) \cap \{c_0\}| + |(A^i \cup \hat{X}_n^i) \cap \{c_2'\}| + \mathrm{dam}_{L,P^i}(\{c_3\}, \{c_3'\}).
\end{aligned}$$

Since the couple c_3c_3' is light for all internal paths, we have $\mathrm{dam}_{L,P^i}(\{c_3\}, \{c_3'\}) = 1$ for all P^i, so that

$$\mathrm{dam}_{L,P^i}(S,T) \leq |(A^i \cup \hat{X}_1^i) \cap \{c_0\}| + |(A^i \cup \hat{X}_n^i) \cap \{c_2'\}| + 1.$$

Each term of this sum is clearly at most 1, so to show that $\mathrm{dam}_{L,P^i}(S,T) \le 2$, it suffices to show that one of the terms is 0 for each i.

Since c_2c_2' is safe for P^0, we have $c_2' \notin A^0 \cup \hat{X}^0_n$, so $\mathrm{dam}_{L,P^0}(S,T) \le 2$. Likewise, since c_0c_0' is safe for P^1, we have $c_0 \notin A^1 \cup \hat{X}^1_1$, so $\mathrm{dam}_{L,P^1}(S,T) \le 2$. By assumption, $c_0 \notin A^2 \cup \hat{X}^2_1$, so we also have $\mathrm{dam}_{L,P^2}(S,T) \le 2$. □

5. Even Cycles Sharing a Vertex or Joined by a Path

In this section, we show that if G consists of two cycles sharing a single vertex or two vertex-disjoint cycles joined by a path, then G is $(4,2)$-choosable. In fact, one can show that these graphs are $(4m,2m)$-choosable for all m; in the interest of brevity, we prove only the $(4,2)$-choosability case, which allows us to reuse some tools from the previous section. As before, whenever L is a list assignment, we tacitly assume $|L(v)| = 4$ for all $v \in V(G)$.

Definition 5.1. Let P be a path with an odd number of vertices, let L be a list assignment on P, and let W be a set of 4 colours. An *L-bad W-set* for P is a set $S \subseteq W$ of 2 colours such that $\mathrm{dam}_{L,P}(S,S) > S_L(P) - 2|V(P)|$. When L is understood, we abbreviate "L-bad W-set" to "bad W-set".

Lemma 5.2. *If P is a path with an odd number of vertices, L is a list assignment on P, and W is any set of 4 colours, then P has at most 2 L-bad W-sets.*

Proof. Consider the graph H obtained by adding new vertices u and v on the ends of P, and extend L to $V(H)$ by putting $L(u) = L(v) = W$. Considering H as a theta graph with P as its only internal path (as in Section 4), we see that S is a bad set for P if and only if P blocks the simple pair (S,S). By Lemma 4.8, it follows that P has at most 2 bad sets. □

Lemma 5.3. *Let Q be a path with endpoints u and v. For every list assignment L on Q, there is an injective function $h : \binom{L(u)}{2} \to \binom{L(v)}{2}$ such that for all $S \in \binom{L(u)}{2}$, the precolouring $\phi(u) = S$, $\phi(v) = h(S)$ extends to all of Q.*

Proof. We use induction on $|V(Q)|$. When $|V(Q)| = 1$ or $|V(Q)| = 2$, the claim clearly holds: when $|V(Q)| = 1$ we may take h to be the identity function, and when $|V(Q)| = 2$ it suffices that $S \cap h(S) = \emptyset$ for all S; such an h is easy to construct.

Hence we may assume that $|V(Q)| > 2$ and the claim holds for smaller paths. Let v' be the unique neighbor of v. We split Q into the u,v'-subpath Q_1 and the v',v-subpath Q_2, overlapping only at v'. Let h_1 and h_2 be the functions for Q_1 and Q_2 respectively, as guaranteed by the induction hypothesis. Composing h_2 and h_1, we see that $h_2 \circ h_1$ has the desired properties. □

We handle "two cycles sharing a vertex" as a special case of "two cycles joined by a path", considering the shared vertex as a path on 1 vertex.

Corollary 5.4. *If G is a graph consisting of two even cycles joined by a (possibly-trivial) path, then G is $(4,2)$-choosable.*

Proof. Let C and D be the cycles in G, and let $u \in V(C)$ and $v \in V(D)$ be the endpoints of the path joining C and D. Let $P = C - u$, let $R = D - v$, and let Q be the path joining u and v, so that P, Q, R are disjoint paths with $V(P) \cup V(Q) \cup V(R) = V(G)$. The situation is illustrated in Figure 4. By Lemma 5.2, the path P has at most two bad $L(u)$-sets, and the path R has at most two bad $L(v)$-sets. Let

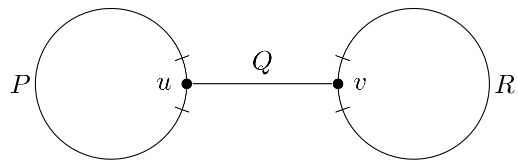
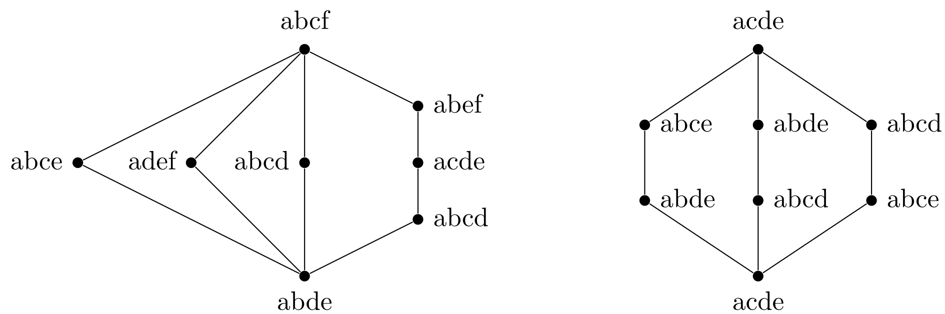


FIGURE 4. Decomposing G into P, Q, R.



(a) $\Theta_{2,2,2,4}$ (b) $\Theta_{3,3,3}$

FIGURE 5. Noncolourable list assignments for $\Theta_{2,2,2,4}$ and $\Theta_{3,3,3}$.

$h : \binom{L(u)}{2} \to \binom{L(v)}{2}$ be the injection guaranteed by Lemma 5.3. Since there are 6 ways to choose a set $S \in \binom{L(u)}{2}$, we see that there is some S such that S is not bad for P and $h(S)$ is not bad for Q. It follows that we may extend the precolouring $\phi(u) = S$, $\phi(v) = h(S)$ to all of P, Q, and R. $\square$

Tuza and Voigt have already shown [5] that $K_{2,4}$ is $(4m, 2m)$-choosable for all m, so this completes the positive direction of Theorem 1.2.

6. Non-(4, 2)-Choosable Theta Graphs

In this section, we argue that if $\min\{r, s, t\} \geq 3$, then $\Theta_{r,s,t}$ is not $(4, 2)$-choosable, and that if $t \geq 2$, then $\Theta_{2,2,2,2t}$ is not $(4, 2)$-choosable. Figure 5 shows noncolourable list assignments for $\Theta_{2,2,2,4}$ and $\Theta_{3,3,3}$.

To show that larger theta graphs are not $(4, 2)$-choosable, we again apply Lemma 4.1. In particular, the contrapositive of Lemma 4.1 states that if G' is not $(4, 2)$-choosable, then G is not $(4, 2)$ choosable either. Hence $\Theta_{4,4,4}$ is not $(4, 2)$-choosable, since $\Theta_{3,3,3}$ is obtained from $\Theta_{4,4,4}$ by applying this reduction to a vertex of degree 3.

Likewise, $\Theta_{2,2,2,2t}$ is obtained from $\Theta_{2,2,2,2t+2}$ by applying this reduction to an interior vertex of the path of length $2t + 1$; hence, since $\Theta_{2,2,2,4}$ is not $(4, 2)$-choosable, it follows by induction on t that when $t \geq 2$, the graph $\Theta_{2,2,2,2t}$ is not $(4, 2)$-choosable. Similarly, since $\Theta_{3,3,3}$ is not $(4, 2)$-choosable, no graph of the form $\Theta_{2r+1,2s+1,2t+1}$ for $r, s, t \geq 1$ is $(4, 2)$-choosable, and since $\Theta_{4,4,4}$ is not $(4, 2)$-choosable, no graph of the form $\Theta_{2r,2s,2t}$ for $r, s, t \geq 2$ is $(4, 2)$-choosable.

7. A Conjecture of Voigt

Voigt [6] conjectured that every bipartite 3-choosable-critical graph is $(4m, 2m)$-choosable for all m. We have seen that this conjecture fails for $m = 1$: there are bipartite 3-choosable-critical graphs which are not $(4, 2)$-choosable. In this section, we prove the following weaker version of Voigt's conjecture:

Theorem 7.1. *There is a fixed integer k such that for every positive integer m, every bipartite 3-choosable-critical graph is $(4km, 2km)$-choosable.*

Our proof is based on the following theorem of Alon, Tuza, and Voigt [1].

Theorem 7.2 (Alon–Tuza–Voigt [1])**.** *For every integer n there exists a number $f(n) \leq (n+1)^{2n+2}$ such that the following holds. For every graph G with n vertices and with fractional chromatic number χ^*, and for every integer M which is divisible by all integers from 1 to $f(n)$, G is $(M, M/\chi^*)$-choosable.*

Lemma 3.2 and Lemma 3.3 suggest that when n is odd, the "worst case" tuples $(A, \hat{X}_1, \hat{X}_n)$ are those satisfying $|A|+|\hat{X}_1|+|\hat{X}_n| = 4m$. The following lemma shows that any such sets can be "realized" on a path of length 3:

Lemma 7.3. *Let P be a path on 3 vertices, and let B, Y, Z be sets such that $B \cap Y = \emptyset$, $B \cap Z = \emptyset$, and $|B| + |Y| + |Z| = 4m$. There exists a list assignment L on P such that:*

- *$|L(v)| = 4m$ for all $v \in V(P_3)$, and*
- *$(A, \hat{X}_1, \hat{X}_3) = (B, Y, Z)$, and*
- *$S_L(P) = 8m$.*

Proof. Let J_1 and J_2 be sets disjoint from each other and disjoint from $B \cup Y \cup Z$ such that

$$|J_1| = 4m - |B| - |Y|,$$
$$|J_2| = 4m - |B| - |Z|.$$

Observe that

$$|B| + |J_1| + |J_2| = 8m - |B| - |Y| - |Z| = 4m.$$

Let v_1, v_2, v_3 be the vertices of P written in order, and consider the following list assignment:

$$L(v_1) = B \cup Y \cup J_1,$$
$$L(v_2) = B \cup J_1 \cup J_2,$$
$$L(v_3) = B \cup Z \cup J_2.$$

It is easy to verify that L has the desired properties. □

Lemma 7.3 allows us to obtain a partial converse of Lemma 4.1, subject to certain restrictions on the choice of the vertex v.

Lemma 7.4. *Let G be a graph containing a path P on 5 vertices which all have degree 2 in G, and let G' be the graph obtained by deleting the middle vertex of P and merging its neighbors. The original graph G is $(4m, 2m)$-choosable if and only if the merged graph G' is $(4m, 2m)$-choosable.*

Proof. By Lemma 4.1, it suffices to show that if G' is $(4m, 2m)$-choosable, then G is $(4m, 2m)$-choosable. Let $v_1, \ldots, v_5$ be the vertices of P, written in order. Let P' be the 3-vertex path in G' corresponding to P, and let v'_1, v'_2, v'_3 be the vertices of G', so that $v'_1 = v_1$ and $v'_3 = v_5$.

Let L be any list assignment for G such that $|L(v)| = 4m$ for all $v \in V(G)$, and let $A, \hat{X}_1, \hat{X}_5$ be computed relative to P. We will define sets B, Y, Z based on $A, \hat{X}_1, \hat{X}_5$ and apply Lemma 7.3 to obtain a list assignment L' on the shorter path P'. The definition is slightly different depending on whether $|A|+|\hat{X}_1|+|\hat{X}_5| \le 4m$: we either arbitrarily add elements or arbitrarily remove elements in order to reach the desired sum.

- When $|A|+|\hat{X}_1|+|\hat{X}_5| \le 4m$, let B, Y, Z be arbitrary supersets of $A, \hat{X}_1, \hat{X}_n$ respectively such that $B \cap Y = \emptyset$, $B \cap Z = \emptyset$, and $|B| + |Y| + |Z| = 4m$.
- When $|A| + |\hat{X}_1| + |\hat{X}_5| > 4m$, let B, Y, Z be arbitrary subsets of $A, \hat{X}_1, \hat{X}_5$ respectively, such that $|B| + |Y| + |Z| = 4m$.

In either case, we may apply Lemma 7.3 to obtain a list assignment L' on the shorter path P' such that:

- $|L'(v)| = 4m$ for all $v \in V(P')$, and
- $(A', \hat{X}'_1, \hat{X}'_3) = (B, Y, Z)$, and
- $S_{L'}(P') = 8m$.

We extend L' to all of G' by defining $L'(v) = L(v)$ for $v \notin V(P')$.

Let $G_0 = G' - V(P') = G - V(P)$, and let w, z be the neighbors of v'_1, v'_3 in G_0, respectively. Since G' is $(4m, 2m)$-choosable, Lemma 2.8 says there is a proper $(L' : 2m)$-colouring ϕ of G_0 such that $\mathrm{dam}_{L',P'}(\phi(w), \phi(z)) \le 2m$.

If $|A| + |\hat{X}_1| + |\hat{X}_n| \le 4m$, then Equation (1) yields

$$\mathrm{dam}_{L,P}(\phi(w), \phi(z)) \le \mathrm{dam}_{L',P'}(\phi(w), \phi(z)) \le 2m,$$

while if $|A| + |\hat{X}_1| + |\hat{X}_n| = 4m + c$ for some $c > 0$, then Equation (1) yields

$$\mathrm{dam}_{L,P}(\phi(w), \phi(z)) \le \mathrm{dam}_{L',P'}(\phi(w), \phi(z)) + c \le 2m + c.$$

Applying Lemma 3.3 in the first case and Lemma 3.2 in the second, we obtain

$$\mathrm{dam}_{L,P}(\phi(w), \phi(z)) \le S_L(P) - 10m.$$

Applying Lemma 2.8 in the other direction, we see that G is $(L, 2m)$-colourable. Since L was arbitrary, G is $(4m, 2m)$-choosable. □

Proof of Theorem 7.1. There are only finitely many bipartite 3-choosable-critical graphs which are minimal with respect to the reduction of Lemma 7.4. In particular, all such graphs have at most 14 vertices, the largest such graph being $\Theta_{5,5,5}$. Let f be the function given by Theorem 7.2, and let $f_{\max} = \max\{f(n)\colon\ n \le 14\}$.

By Theorem 7.2, if $k/4$ is divisible by all numbers up to $f_{\max}$, then all minimal bipartite 3-choosable-critical graphs are $(4k, 2k)$ choosable. In particular, fixing the smallest such k and applying Lemma 7.4, we see that all bipartite 3-choosable-critical graphs are $(4km, 2km)$-choosable for all m. □

8. Characterizing the $(4, 2)$-Choosable Graphs: A Conjecture

Having determined which 3-choosable-critical graphs are $(4, 2)$-choosable, the next natural step in investigating $(4, 2)$-choosability is to characterize *all* $(4, 2)$-choosable graphs, mirroring Rubin's characterization of the 2-choosable graphs [2].

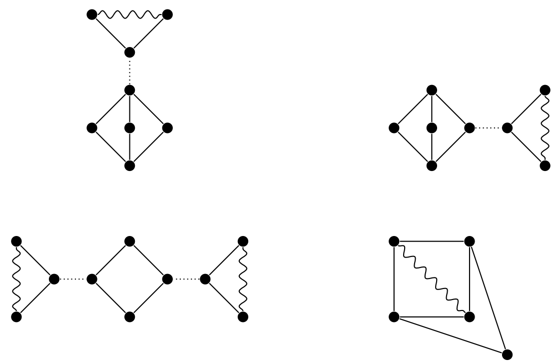


FIGURE 6. Exceptional graphs in Conjecture 8.1. Wavy lines represent paths with an arbitrary odd number of vertices. Dotted lines represent paths with an arbitrary number (any parity) of vertices, possibly 1.

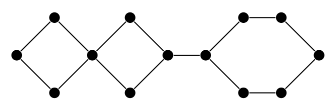


FIGURE 7. One possible realization of the lower-left graph in Figure 6.

As Theorem 1.2 shows, the $(4,2)$-choosable graphs have considerably more variety than the 2-choosable graphs, so the proof of any such characterization is likely to be much more involved than Rubin's proof.

Rubin observed that G is 2-choosable if and only if its core is 2-choosable, and the same observation holds for $(4,2)$-choosability. It clearly also suffices to consider only connected graphs, so we restrict to the case where G is connected with minimum degree at least 2.

Conjecture 8.1. *If G is a connected graph with $\delta(G) \geq 2$, then G is $(4,2)$-choosable if and only if one of the following holds:*

- *G is 2-choosable, or*
- *G is one of the 3-choosable-critical graphs listed in Theorem 1.2, or*
- *G is one of the exceptional graphs shown in Figure 6.*

Figure 6 contains some complex visual notation used to represent parameterized families of graphs; Figure 7 shows an example of how to interpret this notation.

Conjecture 8.1 is supported by substantial evidence. Through computer search, we determined that among all graphs with at most 9 vertices, only the graphs given by Conjecture 8.1 are $(4,2)$-choosable. It appears that all graphs with a larger number of vertices are either one of the $(4,2)$-choosable graphs listed in Conjecture 8.1, or contain some subgraph already known to be non-$(4,2)$-choosable.

A list of "small" minimal non-$(4,2)$-choosable graphs, each with a nonchoosable list assignment, is given in Figure 8. Each of the list assignments was found by computer search. The variety of these graphs represents a significant obstruction to any proof of Conjecture 8.1, which would seem to require a correspondingly complex structure theorem. While we believe that such a proof could be found, it would likely be quite long and beyond the scope of this paper.

The computer analysis for the positive direction of Conjecture 8.1 is based on Lemma 2.8. Each of the graphs in Figure 6 has a small set of vertices X such that $G - X$ is a linear forest, with only the endpoints of its paths having neighbors in X. Rather than generating all list assignments for the entire graph G, it suffices to generate all list assignments for X, and for each list assignment, to generate the possible tuples $(A, \hat{X}_1, \hat{X}_n)$ for each of the paths in $G - X$. For each such tuple, we then search for a partial colouring ϕ of $G[X]$ that satisfies the hypothesis of Lemma 2.8.

However, we have not been able to find a human-readable proof that the exceptional graphs in Conjecture 8.1 are indeed $(4, 2)$-choosable, nor have we been able to prove the structure theorem alluded to above.

References

1. N. Alon, Zs. Tuza, and M. Voigt, *Choosability and fractional chromatic numbers*, Discrete Math. **165/166** (1997), 31–38, Graphs and combinatorics (Marseille, 1995). MR 1439258 (97k:05066)
2. Paul Erdős, Arthur L. Rubin, and Herbert Taylor, *Choosability in graphs*, Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979) (Winnipeg, Man.), Congress. Numer., XXVI, Utilitas Math., 1980, pp. 125–157. MR 593902 (82f:05038)
3. Zs. Tuza and M. Voigt, *Every* 2*-choosable graph is* $(2m, m)$*-choosable*, J. Graph Theory **22** (1996), no. 3, 245–252. MR 1394326 (97f:05074)
4. Zsolt Tuza, *Graph colorings with local constraints—a survey*, Discuss. Math. Graph Theory **17** (1997), no. 2, 161–228. MR 1627932 (99e:05055)
5. Zsolt Tuza and Margit Voigt, *On a conjecture of Erdős, Rubin and Taylor*, Tatra Mt. Math. Publ. **9** (1996), 69–82, Cycles and colourings '94 (Stará Lesná, 1994). MR 1402950 (98c:05070)
6. M. Voigt, *On list colourings and choosability of graphs*, 1998.

(Jixian Meng, Xuding Zhu) Department of Mathematics, Zhejiang Normal University

(Gregory J. Puleo) Department of Mathematics, University of Illinois at Urbana–Champaign

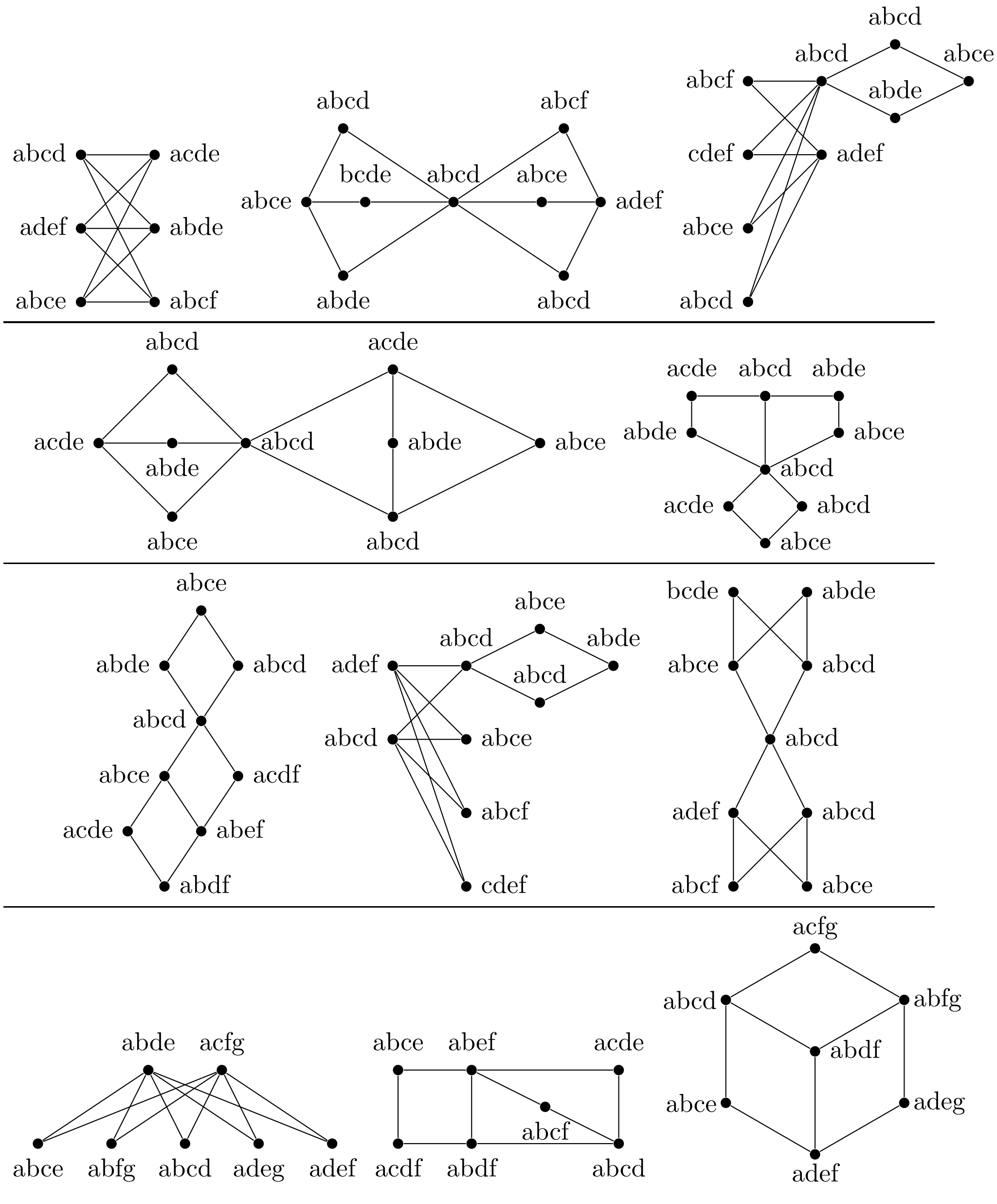


Figure 8. Some non-$(4, 2)$-choosable graphs.

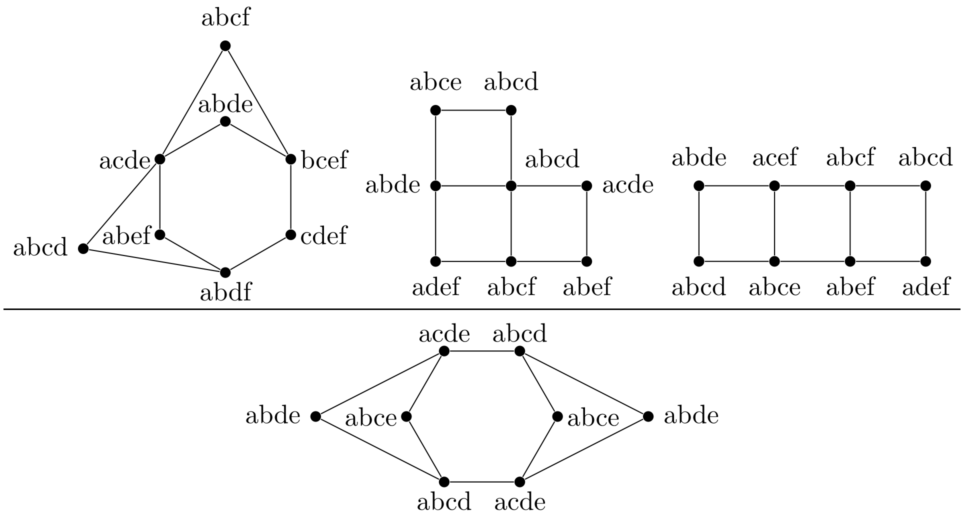


Figure 8. Some non-$(4,2)$-choosable graphs.