COMPACT SETS IN THE FREE TOPOLOGY

MERIC AUGAT, SRIRAM BALASUBRAMANIAN[1], AND SCOTT MCCULLOUGH[2]

ABSTRACT. Subsets of the set of g-tuples of matrices that are closed with respect to direct sums and compact in the free topology are characterized. They are, in a dilation theoretic sense, contained in the hull of a single point.

1. INTRODUCTION

Given positive integers n, g, let $M_n(\mathbb{C})^g$ denote the set of g-tuples of $n \times n$ matrices. Let $M(\mathbb{C})^g$ denote the sequence $(M_n(\mathbb{C})^g)_n$. A subset E of $M(\mathbb{C})^g$ is a sequence $(E(n))$ where $E(n) \subset M_n(\mathbb{C})^g$. The free topology [AM14] has as a basis free sets of the form $G_\delta = (G_\delta(n))$, where

$$G_\delta(n) = \{X \in M_n(\mathbb{C})^g : \|\delta(X)\| < 1\},$$

and δ is a (matrix-valued) free polynomial. Agler and McCarthy [AM14] prove the remarkable result that a bounded free function on a basis set G_δ is uniformly approximable by polynomials on each smaller set of the form

$$K_{s\delta} = \{X \in M(\mathbb{C})^g : \|\delta(X)\| \leq s\}, \quad 0 \leq s < 1.$$

For the definitive treatment of free function theory, see [KVV14].

Sets $E \subset M(\mathbb{C})^g$ naturally arising in free analysis ([AM15, BMV, BKP16, HKN14, KV, KŠ, Pas14, Voi10] is a sampling of the references) are typically closed with respect to direct sums in the sense that if $X \in E(n)$ and $Y \in E(m)$, then

$$X \oplus Y = \left(\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \ldots, \begin{pmatrix} X_g & 0 \\ 0 & Y_g \end{pmatrix} \right) \in E(n+m).$$

Theorem 1.1 below, characterizing free topology compact sets that are closed with respect to directs sums, is the main result of this article. A tuple $Y \in M_n(\mathbb{C})^g$ **polynomially dilates** to a tuple $X \in M_N(\mathbb{C})^g$ if there is an isometry $V : \mathbb{C}^n \to \mathbb{C}^N$ such that for all free polynomials p,

$$p(Y) = V^* p(X) V.$$

2010 *Mathematics Subject Classification.* 47L07, 52A05 (Primary); 46N10, 46L07, 32F17 (Secondary).

[1]Supported by the New Faculty Initiative Grant (MAT/15-16/836/NFIG/SRIM) of IIT Madras.

[2]Research supported by the NSF grant DMS-1361501.

An **ampliation** of X is a tuple of the form $I_k \otimes X$, for some positive integer k. The **dilation hull** of $X \in M(\mathbb{C})^g$ is the set of all $Y \in M(\mathbb{C})^g$ that dilate to an ampliation of X.

Theorem 1.1. *A subset E of $M(\mathbb{C})^g$ that is closed with respect to direct sums is compact if and only if it is contained in the polynomial dilation hull of an $X \in E$.*

Corollary 1.2. *If $E \subset M(\mathbb{C})^g$ is closed with respect to direct sums and is compact in the free topology, then there exists a non-zero free polynomial p such that E is a subset of the zero set of p; i.e., $p(Y) = 0$ for all $Y \in E$. In particular, there is an N such that for $n \geq N$ the set $E(n)$ has empty interior.*

Proof. By Theorem 1.1, there is an n and $X \in E(n)$ such that each $Y \in E$ polynomially dilates to an ampliation of X. Choose a nonzero scalar free polynomial p such that $p(X) = 0$ (using the fact that the span of $\{w(X) : w \text{ is a word}\}$ is a subset of the finite dimensional vector space $M_n(\mathbb{C})$). It follows that $p(Y) = 0$ for all Y. Hence E is a subset of the zero set of p. It is well known (see for instance the Amistur-Levitzki Theorem [Row80]) that the zero set p in $M_n(\mathbb{C})^g$ must have empty interior for sufficiently large n. □

The authors thank Igor Klep for a fruitful correspondence which led to this article. The proof of Theorem 1.1 occupies the remainder of this article.

2. The proof of Theorem 1.1

Proposition 2.1. *Suppose $E \subset M(\mathbb{C})^g$ is nonempty and closed with respect to direct sums. If for each $X \in E$ there is a matrix-valued free polynomial δ and a $Y \in E$ such that*

$$\|\delta(X)\| < \|\delta(Y)\|,$$

then E is not compact in the free topology.

Proof. By hypothesis, for each $X \in E$ there is a matrix-valued polynomial δ_X and $Y_X \in E$ such that $\|\delta_X(X)\| < 1 < \|\delta_X(Y_X)\|$. The collection $\mathcal{G} = \{G_{\delta_X} : X \in E\}$ is an open cover of E. Suppose $S \subset E$ is a finite. Observe that for each $X \in S$, $Y_X \in E \setminus G_{\delta_X}$. Since E is closed with respect to direct sums, $Z = \oplus_{X \in S} Y_X \in E$. On the other hand, for a fixed $W \in S$,

$$\|\delta_W(Z)\| \geq \|\delta_W(Y_W)\| > 1.$$

Thus $Z \notin G_{\delta_W}$ and therefore $Z \in E$ but $Z \notin \cup_{X \in S} G_{\delta_X}$. Thus $\mathcal{G}$ admits no finite subcover of E and therefore E is not compact. □

The following lemma is a standard result.

Lemma 2.2. *Suppose $X, Y \in M(\mathbb{C})^g$. The tuple Y polynomially dilates to an ampliation of X if and only if*

$$\|\delta(Y)\| \le \|\delta(X)\|$$

for every free matrix-valued polynomial δ.

Proof. Let $\mathcal{P}$ denote the set of scalar free polynomials in g variables. Given a tuple $Z \in M_n(\mathbb{C})^g$, let $\mathcal{S}(Z) = \{p(Z) : p \in \mathcal{P}\} \subset M_n(\mathbb{C})$. The set $\mathcal{S}(Z)$ is a unital operator algebra. Let m and n denote the sizes of Y and X respectively. The hypotheses thus imply that the unital homomorphism $\lambda : \mathcal{S}(X) \to \mathcal{S}(Y)$ given by $\lambda(p(X)) = p(Y)$ is well defined and completely contractive. Thus by Corollary 7.6 of [Pau02], it follows that there exists a completely positive map $\varphi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ extending λ. By Choi's Theorem [Pau02], there exists an M and, for $1 \le j \le M$, mappings $W_j : \mathbb{C}^m \to \mathbb{C}^n$ such that $\sum W_j^* W_j = I$ and

$$\varphi(T) = \sum W_j^* T W_j.$$

Let W denote the column matrix with entries W_i. With this notation, $\varphi(T) = W^*(I_M \otimes T)W$. In particular, W is an isometry, since $I = \varphi(I) = W^*W$. Moreover, for polynomials p,

$$p(Y) = \varphi(p(X)) = W^*(I_M \otimes p(X))W$$

and the proof of the reverse direction is complete.

To prove the converse, suppose there is a N and an isometry V such that for all free scalar polynomials p,

$$p(Y) = V^* p(I_N \otimes X)V = V^*[I_N \otimes p(X)]V.$$

Thus for all matrix free polynomials δ, say of size $d \times d$ (without loss of generality δ can be assumed square),

$$\delta(Y) = [V \otimes I_d]^*[I_N \otimes \delta(X)][V \otimes I_d].$$

It follows that $\|\delta(Y)\| \le \|\delta(X)\|$. □

Proof of Theorem 1.1. If for each $X \in E$ there is a $Y \in E$ that does not polynomially dilate to an ampliation of X, then, by Lemma 2.2, for each $X \in E$ there is a $Y \in E$ and a matrix-valued polynomial δ_X such that $\|\delta_X(X)\| < \|\delta_X(Y)\|$. An application of Proposition 2.1 shows E is not compact.

To prove the converse, suppose there exists $X \in E$ such that every $Y \in E$ polynomially dilates to an ampliation of X. Let $\mathcal{G}$ be an open cover of E. There is a $G \in \mathcal{G}$ and a matrix valued free polynomial δ such that $X \in G_\delta \subset G$. Since Y polynomially dilates to an ampliation of X, it follows that $\|\delta(Y)\| \le \|\delta(X)\| < 1$. Hence $Y \in G_\delta \subset G$ and therefore $E \subset G$. □

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Meric Augat, Department of Mathematics, University of Florida, Gainesville

E-mail address: mlaugat@ufl.edu

Sriram Balasubramanian, Department of Mathematics, IIT Madras, Chennai - 600036, India

E-mail address: bsriram@iitm.ac.in

Scott McCullough, Department of Mathematics, University of Florida, Gainesville

E-mail address: sam@math.ufl.edu