
WORMS AND SPIDERS: REFLECTION CALCULI AND ORDINAL NOTATION SYSTEMS

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To the memory of Professor Grigori Mints.

Abstract

We give a general overview of ordinal notation systems arising from reflection calculi, and extend the to represent impredicative ordinals up to those representable using Buchholz-style collapsing functions.

1 Introduction

I had the honor of receiving the Gödel Centenary Research Prize in 2008 based on work directed by my doctoral advisor, Grigori ‘Grisha’ Mints. The topic of my dissertation was *dynamic topological logic*, and while this remains a research interest of mine, in recent years I have focused on studying polymodal provability logics. These logics have proof-theoretic applications and give rise to ordinal notation systems, although previously only for ordinals below the Feferman-Shütte ordinal, Γ_0 . I last saw Professor Mints in the *First International Wormshop* in 2012, where he asked if we could represent the Bachmann-Howard ordinal, $\psi(\varepsilon_{\Omega+1})$, using provability logics. It seems fitting for this volume to once again write about a problem posed to me by Professor Mints.

Notation systems for $\psi(\varepsilon_{\Omega+1})$ and other ‘impredicative’ ordinals are a natural step in advancing Beklemishev’s Π_1^0 ordinal analysis¹ to relatively strong theories

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¹The Π_1^0 ordinal of a theory T is a way to measure its ‘consistency strength’. A different measure, more widely studied, is its Π_1^1 ordinal; we will not define either in this work, but the interested reader may find details in [4] and [33], respectively.

of second-order arithmetic, as well as systems based on Kripke-Platek set theory. Indeed, Professor Mints was not the only participant of the Wormshop interested in representing impredicative ordinals within provability algebras. Fedor Pakhomov brought up the same question, and we had many discussions on the topic. At the time, we each came up with a different strategy for addressing it. These discussions inspired me to continue reflecting about the problem the next couple of years, eventually leading to the ideas presented in the latter part of this manuscript.

1.1 Background

The Gödel-Löb logic GL is a modal logic in which $\Box\varphi$ is interpreted as ‘ φ is derivable in T ’, where T is some fixed formal theory such as Peano arithmetic. This may be extended to a polymodal logic GLP_ω with one modality $[n]$ for each natural number n , as proposed by Japaridze [27]. The modalities $[n]$ may be given a natural proof-theoretic interpretation by extending T with new axioms or infinitary rules. However, GLP_ω is not an easy modal logic to work with, and to this end Dashkov [14] and Beklemishev [7, 6] have identified a particularly well-behaved fragment called the *reflection calculus* (RC), which contains the dual modalities $\langle n \rangle$, but does not allow one to define $[n]$.

Because of this, when working within RC , we may simply write n instead of $\langle n \rangle$. With this notational convention in mind, of particular interest are *worms*, which are expressions of the form

$$m_1 \dots m_n \top,$$

which can be read as

It is m_1 -consistent with T that it is m_2 consistent with T that ... that T is m_n -consistent.

In [26], Ignatiev proved that the set of worms of GLP_ω is well-ordered by consistency strength and computed their order-type. Beklemishev has since shown that transfinite induction along this well-order may be used to give an otherwise finitary proof of the consistency of Peano arithmetic [4].

Indeed, the order-type of the set of worms in RC_ω is ε_0 , an ordinal which already appeared in Gentzen’s earlier proof of the consistency of PA [21]. Moreover, as Beklemishev has observed [5], worms remain well-ordered if we instead work in RC_Λ (or GLP_Λ), where Λ is an arbitrary ordinal. The worms of RC_Λ give a notation system up to the Feferman-Schütte ordinal Γ_0 , considered the upper bound of predicative mathematics.

This suggests that techniques based on reflection calculi may be used to give a proof-theoretic analysis of theories of strength Γ_0 , the focus of an ongoing research

project. However, if worms only provide notations for ordinals below Γ_0 , then these techniques cannot be applied to ‘impredicative’ theories, such as Kripke-Platek set theory with infinity, whose proof-theoretic ordinal is much larger and is obtained by ‘collapsing’ an uncountable ordinal.

1.2 Goals of the article

The goal of this article is to give a step-by-step and mostly self-contained account of the ordinal notation systems that arise from reflection calculi. Sections 2-5 are devoted to giving an overview of known, ‘predicative’ notation systems, first for ε_0 and then for Γ_0 . However, our presentation is quite a bit different from those available in the current literature. In particular, it is meant to be ‘minimalist’, in the sense that we only prove results that are central to our goal of comparing the reflection-based ordinal notations to standard proof-theoretic ordinals. Among other things, we sometimes do not show that the notation systems considered are computable.

The second half presents new material, providing impredicative notation systems based on provability logics. We first introduce *impredicative worms*, which give a representation system for $\psi(e^{\Omega+1}1)$, an ordinal a bit larger than the Bachmann-Howard ordinal. Then we introduce *spiders*, which are used to represent ordinals up to $\psi_0\Omega^\omega 1$ in Buchholz-style notation [11]. Here, $\Omega^\omega 1$ is the first fixed point of the aleph function; unlike the predicative systems discussed above, these notation systems also include notations for several uncountable ordinals. The latter are then ‘collapsed’ in order to represent countable ordinals much larger than Γ_0 .

Although our focus is on notations arising from the reflection calculi and not on proof-theoretic interpretations of the provability operators, we precede each notation system with an informal discussion on such interpretations. These discussions are only given as motivation; further details may be found in the references provided. We also go into detail discussing the ‘traditional’ notation systems for each of the proof-theoretical ordinals involved before discussing the reflection-based version, and thus this text may also serve as an introduction of sorts to ordinal notation systems.

1.3 Layout of the article

- §2: Review of the basic definitions and properties of the reflection calculus RC and the transfinite provability logic GLP.
- §3: Introduction to *worms* and their order-theoretic properties.

- §4: Computation of the order-type of worms with finite entries, and a brief overview of their interpretation in the language of Peano arithmetic.
- §5: Computation of the order-type of worms with ordinal entries, and an overview of their interpretation in the language of second-order arithmetic.
- §6: Introduction and analysis of impredicative worms, obtained by introducing an uncountable modality and its collapsing function.
- §7: Introduction to *spiders*, variants of worms interpreted using the aleph function and its collapses.
- §8: Concluding remarks.

2 The reflection calculus

Provability logics are modal logics for reasoning about Gödel’s provability operator and its variants [10]. One uses $\Box\varphi$ to express ‘ φ is provable in T ’; here, T may be Peano arithmetic, or more generally, any sound extension of elementary arithmetic (see Section 4.1 below). The dual of \Box is $\Diamond = \neg\Box\neg$, and we may read $\Diamond\varphi$ as ‘ φ is consistent with T ’. This unimodal logic is called *Gödel-Löb logic*, which Japaridze extended to a polymodal variant with one modality $[n]$ for each natural number in [28], further extended by Beklemishev to allow one modality for each ordinal in [5].

The resulting polymodal logics have some nice properties; for example, they are decidable, provided the modalities range over some computable linear order. However, there are also some technical difficulties when working with these logics; most notoriously, they are incomplete for their relational semantics, and their topological semantics are quite complex [9, 18, 15, 25].

Fortunately, Dashkov [14] and Beklemishev [6, 7] have shown that for proof-theoretic applications, it is sufficient to restrict to a more manageable fragment of Japaridze’s logic called the *Reflection Calculus* (RC). Due to its simplicity relative to Japaridze’s logic, we will perform all of our modal reasoning directly within RC.

2.1 Ordinal numbers and well-orders

(Ordinal) reflection calculi are polymodal systems whose modalities range over a set or class of ordinal numbers, which are canonical representatives of well-orders. Recall that if A is a set (or class), a *preorder* on A is a transitive, reflexive relation $\preceq \subseteq A \times A$. The preorder \preceq is *total* if, given $a, b \in A$, we always have that $a \preceq b$ or $b \preceq a$, and *antisymmetric* if whenever $a \preceq b$ and $b \preceq a$, it follows that $a = b$. A total,

antisymmetric preorder is a *linear order*. We say that $\langle A, \preccurlyeq \rangle$ is a *pre-well-order* if \preccurlyeq is a total preorder and every non-empty $B \subseteq A$ has a minimal element (i.e., there is $m \in B$ such that $m \preccurlyeq b$ for all $b \in B$). A *well-order* is a pre-well-order that is also linear. Note that pre-well-orders are not the same as well-quasiorders (the latter need not be total). Pre-well-orders will be convenient to us because, as we will see, worms are pre-well-ordered but not linearly ordered.

Define $a \prec b$ by $a \preccurlyeq b$ but $b \not\preccurlyeq a$, and $a \approx b$ by $a \preccurlyeq b$ and $b \preccurlyeq a$. The next proposition may readily be checked by the reader:

Proposition 2.1. *Let $\langle A, \preccurlyeq \rangle$ be a total preorder. Then, the following are equivalent:*

1. \preccurlyeq is a pre-well-order;
2. if $a_0, a_1, \dots \subseteq A$ is any infinite sequence, then there are $i < j$ such that $a_i \preccurlyeq a_j$;
3. there is no infinite descending sequence

$$a_0 \succ a_1 \succ a_2 \succ \dots \subseteq A;$$

4. if $B \subseteq A$ is such that for every $a \in A$,

$$(\forall b \prec a (b \in B)) \rightarrow a \in B,$$

then $B = A$.

We use the standard interval notation for preorders: $(a, b) = \{x : a \prec x \prec b\}$, $(a, \infty) = \{x : a \prec x\}$, etc. With this, we are ready to introduce ordinal numbers as a special case of a well-ordered set. Their formal definition is as follows:

Definition 2.2. *Say that a set A is transitive if whenever $B \in A$, it follows that $B \subseteq A$. Then, a set ξ is an ordinal if ξ is transitive and $\langle \xi, \in \rangle$ is a strict well-order.*

When ξ, ζ are ordinals, we write $\xi < \zeta$ instead of $\xi \in \zeta$ and $\xi \leq \zeta$ if $\xi < \zeta$ or $\xi = \zeta$. The class of ordinal numbers will be denoted **Ord**. We will rarely appeal to Definition 2.2 directly; instead, we will use some basic structural properties of the class of ordinal numbers as a whole. First, observe that **Ord** is itself a (class-sized) well-order:

Lemma 2.3. *The class **Ord** is well-ordered by \leq , and if $\Theta \subseteq \mathbf{Ord}$ is a set, then Θ is an ordinal if and only if Θ is transitive.*

Thus if ξ is any ordinal, then $\xi = \{\zeta \in \text{Ord} : \zeta < \xi\}$, and $0 = \emptyset$ is the least ordinal. For $\xi \in \text{Ord}$, define $\xi + 1 = \xi \cup \{\xi\}$; this is the least ordinal greater than ξ . It follows from these observations that any natural number is an ordinal, but there are infinite ordinals as well; the set of natural numbers is itself an ordinal and denoted ω . More generally, new ordinals can be formed by taking successors and unions:

Lemma 2.4.

1. *If ξ is any ordinal, then $\xi + 1$ is also an ordinal. Moreover, if $\zeta < \xi + 1$, it follows that $\zeta \leq \xi$.*
2. *If Θ is a set of ordinals, then $\lambda = \bigcup \Theta$ is an ordinal. Moreover, if $\xi < \lambda$, it follows that $\xi < \theta$ for some $\theta \in \Theta$.*

These basic properties will suffice to introduce the reflection calculus, but later in the text we will study ordinals in greater depth. A more detailed introduction to the ordinal numbers may be found in a text such as [29].

2.2 The reflection calculus

The modalities of reflection calculi are indexed by elements of some set of ordinals Λ . Alternately, one can take Λ to be the class of all ordinals, obtaining a class-sized logic. Formulas of RC_Λ are built from the grammar

$$\top \mid \phi \wedge \psi \mid \langle \lambda \rangle \phi,$$

where $\lambda < \Lambda$ and ϕ, ψ are formulas of RC_Λ ; we may write $\lambda\phi$ instead of $\langle \lambda \rangle \phi$, particularly since RC_Λ does not contain expressions of the form $[\lambda]\phi$. The set of formulas of RC_Λ will be denoted \mathcal{L}_Λ , and we will simply write \mathcal{L}_{RC} and RC instead of \mathcal{L}_{Ord} , RC_{Ord} . Propositional variables may also be included, but we will omit them since they are not needed for our purposes. Note that this strays from convention, since the variable-free fragment is typically denoted RC^0 . Reflection calculi derive

sequents of the form $\phi \Rightarrow \psi$, using the following rules and axioms:

$$\begin{array}{lll}
 \phi \Rightarrow \phi & \phi \Rightarrow \top & \frac{\phi \Rightarrow \psi \quad \psi \Rightarrow \theta}{\phi \Rightarrow \theta} \\
 \phi \wedge \psi \Rightarrow \phi & \phi \wedge \psi \Rightarrow \psi & \frac{\phi \Rightarrow \psi \quad \phi \Rightarrow \theta}{\phi \Rightarrow \psi \wedge \theta} \\
 \lambda\lambda\phi \Rightarrow \lambda\phi & \frac{\phi \Rightarrow \psi}{\lambda\phi \Rightarrow \lambda\psi} & \\
 \lambda\phi \Rightarrow \mu\phi & \text{for } \mu \leq \lambda; & \\
 \lambda\phi \wedge \mu\psi \Rightarrow \lambda(\phi \wedge \mu\psi) & \text{for } \mu < \lambda. &
 \end{array}$$

Let us write $\phi \equiv \psi$ if $\text{RC}_\Lambda \vdash \phi \Rightarrow \psi$ and $\text{RC}_\Lambda \vdash \psi \Rightarrow \phi$. Then, the following equivalence will be useful to us:

Lemma 2.5. *Given formulas ϕ and ψ and ordinals $\mu < \lambda$,*

$$(\lambda\phi \wedge \mu\psi) \equiv \lambda(\phi \wedge \mu\psi).$$

Proof. The left-to-right direction is an axiom of RC . For the other direction we observe that $\lambda\mu\psi \Rightarrow \mu\psi$ is derivable using the axioms $\lambda\mu\psi \Rightarrow \mu\mu\psi$ and $\mu\mu\psi \Rightarrow \mu\psi$, from which the desired derivation can easily be obtained. \square

Reflection calculi enjoy relatively simple relational semantics, where formulas have truth values on some set of points X , and each expression $\lambda\varphi$ is evaluated using an accessibility relation \succ_λ on X .

Definition 2.6. *An RC_Λ -frame is a structure $\mathfrak{F} = \langle X, \langle \succ_\lambda \rangle_{\lambda < \Lambda} \rangle$ such that for all $x, y, z \in X$ and all $\mu < \lambda < \Lambda$,*

- (i) *if $x \succ_\mu y \succ_\mu z$ then $x \succ_\mu z$,*
- (ii) *if $z \succ_\mu x$ and $z \succ_\lambda y$ then $y \succ_\mu x$, and*
- (iii) *if $x \succ_\lambda y$ then $x \succ_\mu y$.*

The valuation on \mathfrak{F} is the unique function $\llbracket \cdot \rrbracket_{\mathfrak{F}} : \mathcal{L}_{\Lambda} \rightarrow 2^X$ such that

$$\begin{aligned} \llbracket \perp \rrbracket_{\mathfrak{F}} &= \emptyset \\ \llbracket \neg \phi \rrbracket_{\mathfrak{F}} &= X \setminus \llbracket \phi \rrbracket_{\mathfrak{F}} \\ \llbracket \phi \wedge \psi \rrbracket_{\mathfrak{F}} &= \llbracket \phi \rrbracket_{\mathfrak{F}} \cap \llbracket \psi \rrbracket_{\mathfrak{F}} \\ \llbracket \lambda \phi \rrbracket_{\mathfrak{F}} &= \{x \in X : \exists y \prec_{\lambda} x (y \in \llbracket \phi \rrbracket_{\mathfrak{F}})\}. \end{aligned}$$

We may write $(\mathfrak{F}, x) \models \psi$ instead of $x \in \llbracket \psi \rrbracket_{\mathfrak{F}}$. As usual, ϕ is satisfied on \mathfrak{F} if $\llbracket \phi \rrbracket_{\mathfrak{F}} \neq \emptyset$, and true on \mathfrak{F} if $\llbracket \phi \rrbracket_{\mathfrak{F}} = X$.

Theorem 2.7. *For any class or set of ordinals Λ , RC_{Λ} is sound for the class of RC_{Λ} -frames.*

Proof. The proof proceeds by a standard induction on the length of a derivation and we omit it. \square

In fact, Dashkov proved that RC_{ω} is also complete for the class of RC_{ω} -frames [14];² it is very likely that his result can be generalized to full RC over the ordinals, either by adapting his proof or by applying reduction techniques as in [8]. However, we remark that only soundness will be needed for our purposes.

2.3 Transfinite provability logic

The reflection calculus was introduced as a restriction of Japaridze's logic GLP_{ω} [27], which itself was extended by Beklemishev to full GLP [5], containing one modality for each ordinal number. Although we will work mostly within the reflection calculus, for historical reasons it is convenient to review the logic GLP .

The (variable-free) language of GLP is defined by the following grammar:

$$\top \mid \perp \mid \phi \wedge \psi \mid \phi \rightarrow \psi \mid \langle \lambda \rangle \phi.$$

Note that in this language we can define negation (as well as other Boolean connectives), along with $[\lambda]\phi = \neg \langle \lambda \rangle \neg \phi$.

The logic GLP_{Λ} is then given by the following rules and axioms:

- (i) all propositional tautologies,
- (ii) $[\lambda](\phi \rightarrow \psi) \rightarrow ([\lambda]\phi \rightarrow [\lambda]\psi)$ for all $\lambda < \Lambda$,

²Beware that RC_{ω} in our notation is not the same as RC_{ω} in [7].

- (iii) $[\lambda]([\lambda]\phi \rightarrow \phi) \rightarrow [\lambda]\phi$ for all $\lambda < \Lambda$,
- (iv) $[\mu]\phi \rightarrow [\lambda]\phi$ for $\mu < \lambda < \Lambda$,
- (v) $\langle \mu \rangle \phi \rightarrow [\lambda]\langle \mu \rangle \phi$ for $\mu < \lambda < \Lambda$,
- (vi) modus ponens and
- (vii) necessitation for each $[\xi]$.

The reader may recognize axiom (iii) as Löb's axiom [32], ostensibly absent from RC; it is simply not expressible there. However, it was proven by Dashkov that GLP is conservative over RC, in the following sense:

Theorem 2.8. *If $\phi, \psi \in \mathcal{L}_{\text{RC}}$, then $\text{RC} \vdash \phi \Rightarrow \psi$, if and only if $\text{GLP} \vdash \phi \rightarrow \psi$.*

Proof. That $\text{RC} \vdash \phi \Rightarrow \psi$ implies $\text{GLP} \vdash \phi \rightarrow \psi$ is readily proven by induction on the length of a derivation; one need only verify that, for $\mu < \lambda$,

$$\text{GLP} \vdash \langle \lambda \rangle \phi \wedge \langle \mu \rangle \psi \rightarrow \langle \lambda \rangle (\phi \wedge \langle \mu \rangle \psi),$$

using the GLP axiom (v).

The other direction was proven for RC_ω by Dashkov in [14]. To extend to modalities over the ordinals, assume that $\text{GLP} \vdash \phi \rightarrow \psi$. Then, there are finitely many modalities appearing in the derivation of $\vdash \phi \rightarrow \psi$, hence $\text{GLP}_\Theta \vdash \phi \rightarrow \psi$ for some finite set Θ . But GLP_Θ readily embeds into GLP_ω (see [8]), and thus we can use the conservativity of GLP_ω over RC_ω to conclude that $\text{RC} \vdash \phi \Rightarrow \psi$. \square

As we have mentioned, full GLP (with propositional variables), or even GLP_2 , is incomplete for its relational semantics. Without propositional variables, Ignatiev has built a relational model in which every consistent formula of variable-free GLP_ω is satisfied [26], and Joosten and I extended this to variable-free GLP over the ordinals. However, these models are infinite, and even $1\top$ cannot be satisfied on any finite relational model validating variable-free GLP. On the other hand, every worm has a relatively small RC-model, as we will see below.

3 Worms and consistency orderings

Worms are expressions of RC (or GLP) representing iterated consistency assertions. Ignatiev first observed that the worms in GLP_ω are well-founded [26]. The order-types of worms in GLP_2 were then studied by Boolos [10], and in full GLP by Beklemishev [5] and further by Joosten and I in [20], this time working in RC. Moreover,

this particular well-order has surprising proof-theoretical applications: Beklemishev has used transfinite induction along the RC_ω worms to prove the consistency of Peano arithmetic and compute its Π_1^0 ordinal [4].

In this section we will review the ordering between worms and show that it is well-founded. Let us begin with some preliminaries.

3.1 Basic definitions

Definition 3.1. A worm is any RC formula of the form

$$\mathfrak{w} = \lambda_1 \dots \lambda_n \top,$$

with each λ_i an ordinal and $n < \omega$ (including the ‘empty worm’, \top). The class of worms is denoted \mathbb{W} .

If Λ is a set or class of ordinals and each $\lambda_i \in \Lambda$, we write $\mathfrak{w} \sqsubset \Lambda$. The set of worms \mathfrak{v} such that $\mathfrak{v} \sqsubset \Lambda$ is denoted \mathbb{W}_Λ .

‘Measuring’ worms is the central theme of this work. Let us begin by giving notation for some simple measurements, such as the length and the maximum element of a worm.

Definition 3.2. If $\mathfrak{w} = \lambda_1 \dots \lambda_n \top$, then we set $\#\mathfrak{w} = n$ (i.e., $\#\mathfrak{w}$ is the length of \mathfrak{w}). Define $\min \mathfrak{w} = \min_{i \in [1, n]} \lambda_i$, and similarly $\max \mathfrak{w} = \max_{i \in [1, n]} \lambda_i$. The class of worms \mathfrak{w} such that $\mathfrak{w} = \top$ or $\mu \leq \min \mathfrak{w}$ will be denoted $\mathbb{W}_{\geq \mu}$. We define $\mathbb{W}_{> \mu}$ analogously.

These give us some idea of ‘how big’ a worm is, but what we are truly interested in is in ordering worms by their *consistency strength*:

Definition 3.3. Given an ordinal λ , we define a relation \triangleleft_λ on \mathbb{W} by $\mathfrak{v} \triangleleft_\lambda \mathfrak{w}$ if and only if $\text{RC} \vdash \mathfrak{w} \Rightarrow \lambda \mathfrak{v}$. We also define $\mathfrak{v} \trianglelefteq_\mu \mathfrak{w}$ if $\mathfrak{v} \triangleleft_\mu \mathfrak{w}$ or $\mathfrak{v} \equiv \mathfrak{w}$.

Instead of $\triangleleft_0, \trianglelefteq_0$ we may simply write $\triangleleft, \trianglelefteq$. As we will see, these orderings have some rather interesting properties. Let us begin by proving some basic facts about them:

Lemma 3.4. Let $\mu \leq \lambda$ be ordinals and $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$ be worms. Then:

1. if $\mathfrak{w} \neq \top$ and $\mu < \min \mathfrak{w}$, then $\top \triangleleft_\mu \mathfrak{w}$,
2. if $\mathfrak{v} \triangleleft_\lambda \mathfrak{w}$, then $\mathfrak{v} \triangleleft_\mu \mathfrak{w}$, and
3. if $\mathfrak{u} \triangleleft_\mu \mathfrak{v}$ and $\mathfrak{v} \triangleleft_\mu \mathfrak{w}$, then $\mathfrak{u} \triangleleft_\mu \mathfrak{w}$.

Proof. For the first item, write $\mathfrak{w} = \lambda \mathfrak{v}$, so that $\lambda \geq \mu$. Then, $\mathfrak{v} \Rightarrow \top$ is an axiom of RC, from which we can derive $\lambda \mathfrak{v} \Rightarrow \lambda \top$ and from there use the axiom $\lambda \top \Rightarrow \mu \top$.

For the second item, if $\mathfrak{v} \triangleleft_\lambda \mathfrak{w}$, then by definition, $\mathfrak{w} \Rightarrow \lambda \mathfrak{v}$ is derivable. Using the axiom $\lambda \mathfrak{v} \Rightarrow \mu \mathfrak{v}$, we see that $\mathfrak{w} \Rightarrow \mu \mathfrak{v}$ is derivable as well, that is, $\mathfrak{v} \triangleleft_\mu \mathfrak{w}$.

Transitivity simply follows from the fact that $\text{RC} \vdash \mu \mu \mathfrak{u} \Rightarrow \mu \mathfrak{u}$, so that if $\mathfrak{u} \triangleleft_\mu \mathfrak{v}$ and $\mathfrak{v} \triangleleft_\mu \mathfrak{w}$, we have that $\text{RC} \vdash \mathfrak{w} \Rightarrow \mu \mathfrak{v} \Rightarrow \mu \mu \mathfrak{u} \Rightarrow \mu \mathfrak{u}$, so $\mathfrak{u} \triangleleft_\mu \mathfrak{w}$. \square

3.2 Computing the consistency orders

The definition of $\mathfrak{v} \triangleleft_\lambda \mathfrak{w}$ does not suggest an obvious algorithm for deciding whether it holds or not. Fortunately, it can be reduced to computing the ordering between smaller worms; in this section, we will show how this is done. Let us begin by proving that \triangleleft_μ is always irreflexive. To do this, we will use the following frames.

Definition 3.5. Let $\mathfrak{w} = \lambda_n \dots \lambda_0 \top$ be any worm (note that we are using a different enumeration from that in Definition 3.1). Define a frame $\mathfrak{F}(\mathfrak{w}) = \langle X, \langle \succ_\lambda \rangle_{\lambda < \Lambda} \rangle$ as follows.

First, set $X = [0, n+1] \subseteq \mathbb{N}$. To simplify notation below, let $\lambda_{n+1} = 0$. Then, define $x \succ_\eta y$ if and only if:

1. $x > y$ and for all $i \in [y, x)$, $\lambda_i \geq \eta$, or
2. $x \leq y$ and for all $i \in [x, y]$, $\lambda_i > \eta$.

Although this might not be obvious from the definition, these frames are indeed RC-frames.

Lemma 3.6. Given any worm \mathfrak{w} , $\mathfrak{F}(\mathfrak{w})$ is an RC-frame.

Proof. We must check that $\mathfrak{F}(\mathfrak{w})$ satisfies each item of Definition 2.6.

(i) Suppose that $x \succ_\eta y \succ_\eta z$. If $x > y$, consider three sub-cases.

- a. If $y > z$, from $x \succ_\eta y \succ_\eta z$ we see that for all $i \in [z, y) \cup [y, x) = [z, x)$, $\lambda_i \geq \eta$, so that $x \succ_\eta z$.
- b. If $z \in [y, x)$, from $[z, x) \subseteq [y, x)$ and $x \succ_\eta y$ we obtain $\lambda_i \geq \eta$ for all $i \in [z, x)$, so $x \succ_\eta z$.
- c. If $z \geq x$, from $[x, z] \subseteq [y, z]$ and $y \succ_\eta z$ we obtain $\lambda_i > \eta$ for all $i \in [x, z]$, hence $x \succ_\eta z$.

The cases where $x \leq y$ are analogous.

(ii). As in the previous item, we must consider several cases. Suppose that $\mu < \eta$, $z \succ_\mu x$ and $z \succ_\eta y$. If $z > x$, we consider three subcases.

- a. If $y \leq x$, then from $z \succ_\eta y$ and $[y, x] \subseteq [y, z]$ we obtain $\lambda_i \geq \eta > \mu$ for all $i \in [y, x]$, hence $y \succ_\mu x$.
- b. If $y \in (x, z]$, then from $[x, y] \subseteq [x, z]$ and $z \succ_\mu x$ we obtain $\lambda_i \geq \mu$ for all $i \in [x, y]$, hence $y \succ_\mu x$.
- c. If $y > z$, then from $z \succ_\mu x$ we have that $\lambda_i \geq \mu$ for all $i \in [x, z]$, while from $z \succ_\eta y$ it follows that for all $i \in [z, y)$, $\lambda_i > \eta > \mu$, giving us $y \succ_\mu x$.

Cases where $z \leq x$ are similar.

(iii). That \succ_μ is monotone on μ is obvious from its definition. \square

Thus to prove that \triangleleft_μ is irreflexive, it suffices to show that there is $x \in [0, n+1]$ such that $(\mathfrak{F}(\mathfrak{w}), x) \models \lambda_{n-1} \dots \lambda_0 \top$ but $(\mathfrak{F}(\mathfrak{w}), x) \not\models \lambda_n \dots \lambda_0 \top$, as then by setting $\mu = \lambda_n$ and $\mathfrak{v} = \lambda_{n-1} \dots \lambda_0 \top$ we see that $\mathfrak{v} \not\triangleleft_\mu \mathfrak{v}$. The following lemma will help us find such an x .

Lemma 3.7. *Let $\mathfrak{w} = \lambda_n \dots \lambda_0 \top$ be a worm, and for any $i \in [0, n+1]$, define $\mathfrak{w}[i]$ recursively by $\mathfrak{w}[0] = \top$ and $\mathfrak{w}[i+1] = \lambda_i \mathfrak{w}[i]$. Then:*

1. $(\mathfrak{F}(\mathfrak{w}), i) \models \mathfrak{w}[i]$, and
2. if $x \in [0, i)$, then $(\mathfrak{F}(\mathfrak{w}), x) \not\models \mathfrak{w}[i]$.

Proof. The first claim is easy to check from the definition of $\mathfrak{F}(\mathfrak{w})$, so we focus on proving the second by induction on i . The base case is vacuously true as $[0, 0) = \emptyset$. Otherwise, assume the claim for i , and consider $x \in [0, i+1)$; we must show that $(\mathfrak{F}(\mathfrak{w}), x) \not\models \mathfrak{w}[i+1] = \lambda_i \mathfrak{w}[i]$, which means that for all $y \triangleleft_{\lambda_i} x$, $(\mathfrak{F}(\mathfrak{w}), y) \not\models \mathfrak{w}[i]$. Note that we cannot have that $y \in [i, n+1]$, as in this case $y \geq i \geq x$; but obviously $\lambda_i \not\prec \lambda_i$, so that $y \not\triangleleft_{\lambda_i} x$. It follows that $y \in [0, i)$, and we can apply the induction hypothesis to $\mathfrak{w}[i]$. \square

Lemma 3.8. *Given any ordinal μ and any worm \mathfrak{v} , we have that $\mathfrak{v} \not\triangleleft_\mu \mathfrak{v}$.*

Proof. Let μ be any ordinal, \mathfrak{v} be any worm, and consider the RC-frame $\mathfrak{F}(\mu\mathfrak{v})$. If $n = \#\mathfrak{v}$, observe that $(\mu\mathfrak{v})[n] = \mathfrak{v}$, hence by Lemma 3.7, $(\mathfrak{F}(\mu\mathfrak{v}), n) \models \mathfrak{v}$ but $(\mathfrak{F}(\mu\mathfrak{v}), n) \not\models \mu\mathfrak{v}$; it follows from Theorem 2.7 that $\mathfrak{v} \not\triangleleft_\mu \mathfrak{v}$. \square

Thus the worm orderings are irreflexive. Next we turn our attention to a useful operation between worms. Specifically, worms can be regarded as strings of symbols, and as such we can think of concatenating them.

Definition 3.9. *Let $\mathfrak{v} = \xi_1 \dots \xi_n \top$ and $\mathfrak{w} = \zeta_1 \dots \zeta_m \top$ be worms. Then, define*

$$\mathfrak{vw} = \xi_1 \dots \xi_n \zeta_1 \dots \zeta_m \top$$

Often we will want to put an extra ordinal between the worms, and we write $\mathfrak{v} \lambda \mathfrak{w}$ for $\mathfrak{v}(\lambda \mathfrak{w})$.

Lemma 3.10. *If $\mathfrak{w}, \mathfrak{v}$ are worms and $\mu < \min \mathfrak{w}$, then $\mathfrak{w} \mu \mathfrak{v} \equiv \mathfrak{w} \wedge \mu \mathfrak{v}$.*

Proof. By induction on $\#\mathfrak{w}$. If $\mathfrak{w} = \top$, the claim becomes $\mu \mathfrak{v} \equiv \top \wedge \mu \mathfrak{v}$, which is obviously true. Otherwise, we write $\mathfrak{w} = \lambda \mathfrak{u}$ with $\lambda > \mu$, and observe that by Lemma 2.5,

$$\lambda \mathfrak{u} \wedge \mu \mathfrak{v} \equiv \lambda(\mathfrak{u} \wedge \mu \mathfrak{v}) \stackrel{\text{IH}}{\equiv} \lambda(\mathfrak{u} \mu \mathfrak{v}) = \mathfrak{w} \mu \mathfrak{v}. \quad \square$$

Thus we may “pull out” the initial segment of a worm, provided the following element is a lower bound for this initial segment. In general, for any ordinal λ , we can pull out the maximal initial segment of \mathfrak{w} which is bounded below by λ ; this segment is the λ -head of \mathfrak{w} , and what is left over (if anything) is its λ -body.

Definition 3.11. *Let λ be an ordinal and $\mathfrak{w} \in \mathbb{W}_{\geq \lambda}$. We define $h_\lambda(\mathfrak{w})$ to be the maximal initial segment of \mathfrak{w} such that $\lambda < \min h_\lambda(\mathfrak{w})$, and define $b_\lambda(\mathfrak{w})$ as follows: if λ appears in \mathfrak{w} , then we set $b_\lambda(\mathfrak{w})$ to be the unique worm such that $\mathfrak{w} = h_\lambda(\mathfrak{w}) \lambda b_\lambda(\mathfrak{w})$. Otherwise, set $b_\lambda(\mathfrak{w}) = \top$.*

We may write h, b instead of h_0, b_0 . We remark that our notation is a variant from that used in [20], where our h_λ would be denoted $h_{\lambda+1}$.

Lemma 3.12. *Given a worm $\mathfrak{w} \neq \top$ and an ordinal $\mu \leq \min \mathfrak{w}$,*

1. $h_\mu(\mathfrak{w}) \in \mathbb{W}_{> \mu}$,
2. $\#h_\mu(\mathfrak{w}) \leq \#\mathfrak{w}$, with equality holding only if $\mu < \min \mathfrak{w}$, in which case $h_\mu(\mathfrak{w}) = \mathfrak{w}$;
3. $\#b_\mu(\mathfrak{w}) < \#\mathfrak{w}$, and
4. $\mathfrak{w} \equiv h_\mu(\mathfrak{w}) \wedge \mu b_\mu(\mathfrak{w})$.

Proof. The first two claims are immediate from the definition of h_μ . For the third, this is again obvious in the case that μ occurs in \mathfrak{w} , otherwise we have that $b_\mu(\mathfrak{w}) = \top$ and by the assumption that $\mathfrak{w} \neq \top$ we obtain $\#b_\mu(\mathfrak{w}) < \#\mathfrak{w}$.

The fourth claim is an instance of Lemma 3.10 if μ appears in \mathfrak{w} , otherwise $\mathfrak{w} = h_\mu(\mathfrak{w})$ and we use Lemma 3.4 to see that $\mathfrak{w} \Rightarrow \mu\top = \mu b_\mu(\top)$ is derivable. \square

With this we can reduce relations between worms to those between their heads and bodies.

Lemma 3.13. *If $\mathfrak{w}, \mathfrak{v} \neq \top$ are worms and $\mu \leq \min \mathfrak{w}\mathfrak{v}$, then*

1. $\mathfrak{w} \triangleleft_\mu \mathfrak{v}$ whenever

(a) $\mathfrak{w} \trianglelefteq_\mu b_\mu(\mathfrak{v})$, or

(b) $b_\mu(\mathfrak{w}) \triangleleft_\mu \mathfrak{v}$ and $h_\mu(\mathfrak{w}) \triangleleft_{\mu+1} h_\mu(\mathfrak{v})$, and

2. $\text{RC} \vdash \mathfrak{v} \Rightarrow \mathfrak{w}$ whenever $b_\mu(\mathfrak{w}) \triangleleft_\mu \mathfrak{v}$ and $\text{RC} \vdash h_\mu(\mathfrak{v}) \Rightarrow h_\mu(\mathfrak{w})$.

Proof. For the first claim, if $\mathfrak{w} \trianglelefteq_\mu b_\mu(\mathfrak{v})$, then by Lemma 3.12.4 we have that $\mathfrak{v} \Rightarrow \mu b_\mu(\mathfrak{v})$, that is, $b_\mu(\mathfrak{v}) \triangleleft_\mu \mathfrak{v}$. By transitivity we obtain $\mathfrak{w} \triangleleft_\mu \mathfrak{v}$. If $b_\mu(\mathfrak{w}) \triangleleft_\mu \mathfrak{v}$ and $h_\mu(\mathfrak{w}) \triangleleft_{\mu+1} h_\mu(\mathfrak{v})$, reasoning in RC we have that

$$\mathfrak{v} \Rightarrow h_\mu(\mathfrak{v}) \wedge \mathfrak{v} \Rightarrow \langle \mu + 1 \rangle h_\mu(\mathfrak{w}) \wedge \mu b_\mu(\mathfrak{w}) \equiv \langle \mu + 1 \rangle h_\mu(\mathfrak{w}) \mu b_\mu(\mathfrak{w}) \Rightarrow \mu \mathfrak{w},$$

and $\mathfrak{w} \triangleleft_\mu \mathfrak{v}$, as needed.

For the second, if $b_\mu(\mathfrak{w}) \triangleleft_\mu \mathfrak{v}$ and $\text{RC} \vdash h_\mu(\mathfrak{v}) \Rightarrow h_\mu(\mathfrak{w})$, we have that

$$\mathfrak{v} \Rightarrow h_\mu(\mathfrak{v}) \wedge \mathfrak{v} \Rightarrow h_\mu(\mathfrak{w}) \wedge \mu b_\mu(\mathfrak{w}) \equiv \mathfrak{w}. \quad \square$$

As we will see, Lemma 3.13 gives us a recursive way to compute \trianglelefteq_μ . This recursion will allow us to establish many of the fundamental properties of \trianglelefteq_μ , beginning with the fact that it defines a total preorder.

Lemma 3.14. *Given worms $\mathfrak{v}, \mathfrak{w}$ and $\mu \leq \min(\mathfrak{w}\mathfrak{v})$, exactly one of $\mathfrak{w} \trianglelefteq_\mu \mathfrak{v}$ or $\mathfrak{v} \triangleleft_\mu \mathfrak{w}$ occurs.*

Proof. That they cannot simultaneously occur follows immediately from Lemma 3.8, since \triangleleft_μ is irreflexive.

To show that at least one occurs, proceed by induction on $\#\mathfrak{w} + \#\mathfrak{v}$. To be precise, assume inductively that whenever $\#\mathfrak{w}' + \#\mathfrak{v}' < \#\mathfrak{w} + \#\mathfrak{v}$ and $\mu \leq \min(\mathfrak{w}'\mathfrak{v}')$ is arbitrary, then either $\mathfrak{w}' \trianglelefteq_\mu \mathfrak{v}'$ or $\mathfrak{v}' \triangleleft_\mu \mathfrak{w}'$. If either $\mathfrak{v} = \top$ or $\mathfrak{w} = \top$, then the claim is immediate from Lemma 3.4.

Otherwise, let $\lambda = \min(\mathfrak{w}\mathfrak{v})$, so that $\lambda \geq \mu$. If $\mathfrak{w} \trianglelefteq_\lambda b_\lambda(\mathfrak{v})$, then by Lemma 3.13, $\mathfrak{w} \triangleleft_\lambda \mathfrak{v}$, and similarly if $\mathfrak{v} \trianglelefteq_\lambda b_\lambda(\mathfrak{w})$, then $\mathfrak{v} \triangleleft_\lambda \mathfrak{w}$. On the other hand, if neither occurs then by the induction hypothesis we have that $b_\lambda(\mathfrak{v}) \triangleleft_\lambda \mathfrak{w}$ and $b_\lambda(\mathfrak{w}) \triangleleft_\lambda \mathfrak{v}$.

Since λ appears in either \mathfrak{w} or \mathfrak{v} , by Lemma 3.12.2 we have that

$$\#h_\lambda(\mathfrak{w}) + \#h_\lambda(\mathfrak{v}) < \#\mathfrak{w} + \#\mathfrak{v},$$

so that by the induction hypothesis, either $h_\lambda(\mathfrak{w}) \triangleleft_\mu h_\lambda(\mathfrak{v})$, $h_\lambda(\mathfrak{w}) \equiv h_\lambda(\mathfrak{v})$, or $h_\lambda(\mathfrak{v}) \triangleleft_\lambda h_\lambda(\mathfrak{w})$. If $h_\lambda(\mathfrak{w}) \triangleleft_\lambda h_\lambda(\mathfrak{v})$, we may use Lemma 3.13.1 to see that $\mathfrak{w} \triangleleft_\lambda \mathfrak{v}$, so that by Lemma 3.4, $\mathfrak{w} \triangleleft_\mu \mathfrak{v}$. Similarly, if $h_\lambda(\mathfrak{v}) \triangleleft_\lambda h_\lambda(\mathfrak{w})$, we obtain $\mathfrak{v} \triangleleft_\mu \mathfrak{w}$. If $h_\lambda(\mathfrak{w}) \equiv h_\lambda(\mathfrak{v})$, then Lemma 3.13.2 yields both $\mathfrak{w} \Rightarrow \mathfrak{v}$ and $\mathfrak{w} \Rightarrow \mathfrak{v}$, i.e., $\mathfrak{w} \equiv \mathfrak{v}$. \square

Corollary 3.15. *If $\text{RC} \vdash \mathfrak{w} \Rightarrow \mathfrak{v}$, then $\mathfrak{v} \trianglelefteq \mathfrak{w}$.*

Proof. Towards a contradiction, suppose that $\text{RC} \vdash \mathfrak{w} \Rightarrow \mathfrak{v}$ but $\mathfrak{v} \not\trianglelefteq \mathfrak{w}$. By Lemma 3.14, $\mathfrak{w} \triangleleft \mathfrak{v}$. Hence $\mathfrak{v} \Rightarrow \mathfrak{w} \Rightarrow 0\mathfrak{v}$, and $\mathfrak{v} \triangleleft \mathfrak{v}$, contradicting the irreflexivity of \triangleleft . \square

Moreover, the orderings \triangleleft_λ , \triangleleft_μ coincide on $\mathbb{W}_{\geq \max\{\lambda, \mu\}}$:

Lemma 3.16. *Let $\mathfrak{w}, \mathfrak{v}$ be worms and $\mu, \lambda \leq \min(\mathfrak{w}\mathfrak{v})$. Then, $\mathfrak{w} \triangleleft_\mu \mathfrak{v}$ if and only if $\mathfrak{w} \triangleleft_\lambda \mathfrak{v}$.*

Proof. Assume without loss of generality that $\mu \leq \lambda$. One direction is already in Lemma 3.4. For the other, assume towards a contradiction that $\mathfrak{w} \triangleleft_\mu \mathfrak{v}$ but $\mathfrak{w} \not\triangleleft_\lambda \mathfrak{v}$. Then, by Lemma 3.14, $\mathfrak{v} \trianglelefteq_\lambda \mathfrak{w}$ and thus $\mathfrak{v} \trianglelefteq_\mu \mathfrak{w}$, so that $\mathfrak{v} \trianglelefteq_\mu \mathfrak{w} \triangleleft_\mu \mathfrak{v}$, contradicting the irreflexivity of \triangleleft_μ (Lemma 3.8). \square

With this we can give an improved version of Lemma 3.13, that will be more useful to us later.

Theorem 3.17. *The relation \trianglelefteq_λ is a total preorder on $\mathbb{W}_{\geq \lambda}$, and for all $\mu \leq \lambda$ and $\mathfrak{w}, \mathfrak{v} \in \mathbb{W}_{\geq \lambda}$ with $\mathfrak{w}, \mathfrak{v} \neq \top$,*

1. $\mathfrak{w} \triangleleft_\mu \mathfrak{v}$ if and only if
 - (a) $\mathfrak{w} \trianglelefteq_\mu b_\lambda(\mathfrak{v})$, or
 - (b) $b_\lambda(\mathfrak{w}) \triangleleft_\mu \mathfrak{v}$ and $h_\lambda(\mathfrak{w}) \triangleleft_\mu h_\lambda(\mathfrak{v})$, and
2. $\mathfrak{w} \trianglelefteq_\mu \mathfrak{v}$ if and only if
 - (a) $\mathfrak{w} \trianglelefteq_\mu b_\lambda(\mathfrak{v})$, or
 - (b) $b_\lambda(\mathfrak{w}) \triangleleft_\mu \mathfrak{v}$ and $h_\lambda(\mathfrak{w}) \trianglelefteq_\mu h_\lambda(\mathfrak{v})$.

Proof. Totality is Lemma 3.14. Let us prove item 2; the proof of item 1 is similar. If (2a) holds, then by Lemma 3.16, $\mathfrak{w} \trianglelefteq_\lambda b_\lambda(\mathfrak{v})$, so that by Lemma 3.13.1, $\mathfrak{w} \triangleleft_\lambda \mathfrak{v}$, and once again by Lemma 3.16, $\mathfrak{w} \triangleleft_\mu \mathfrak{v}$. If (2b) holds, then by Lemma 3.16 we obtain $b_\lambda(\mathfrak{w}) \triangleleft_\lambda \mathfrak{v}$ and $h_\lambda(\mathfrak{w}) \trianglelefteq_{\lambda+1} h_\lambda(\mathfrak{v})$. If $h_\lambda(\mathfrak{w}) \triangleleft_{\lambda+1} h_\lambda(\mathfrak{v})$, we may use Lemma 3.13.1 to obtain $\mathfrak{w} \triangleleft_\lambda \mathfrak{v}$. Otherwise, by Lemma 3.13.2, we see that $\text{RC} \vdash \mathfrak{v} \Rightarrow \mathfrak{w}$, which by Corollary 3.15 gives us $\mathfrak{w} \triangleleft_\lambda \mathfrak{v}$. In either case, $\mathfrak{w} \triangleleft_\mu \mathfrak{v}$.

For the other direction, assume that (2a) and (2b) both fail. Then by Lemma 3.14 together with Lemma 3.16, we have that $b_\lambda(\mathfrak{v}) \triangleleft_\lambda \mathfrak{w}$ and either $\mathfrak{v} \trianglelefteq_\lambda b_\lambda(\mathfrak{w})$ or $h_\lambda(\mathfrak{v}) \triangleleft_{\lambda+1} h_\lambda(\mathfrak{w})$. In either case $\mathfrak{v} \triangleleft_\lambda \mathfrak{w}$, and thus $\mathfrak{w} \triangleleft_\mu \mathfrak{v}$. \square

Before continuing, it will be useful to derive a few straightforward consequences of Theorem 3.17.

Corollary 3.18. *Every $\phi \in \mathcal{L}_{\text{RC}}$ is equivalent to some $\mathfrak{w} \in \mathbb{W}$. Moreover, we can take \mathfrak{w} so that every ordinal appearing in \mathfrak{w} already appears in ϕ .*

Proof. By induction on the complexity of ϕ . We have that \top is a worm and for $\phi = \lambda\psi$, by induction hypothesis we have that $\psi \equiv \mathfrak{v}$ for some worm \mathfrak{v} with all modalities appearing in ψ and hence $\phi \equiv \lambda\mathfrak{v}$.

It remains to consider an expression of the form $\psi \wedge \phi$. Using the induction hypothesis, there are worms $\mathfrak{w}, \mathfrak{v}$ equivalent to ϕ, ψ , respectively, so that $\psi \wedge \phi \equiv \mathfrak{w} \wedge \mathfrak{v}$. We proceed by a secondary induction on $\#\mathfrak{w} + \#\mathfrak{v}$. Note that the claim is trivial if either $\mathfrak{w} = \top$ or $\mathfrak{v} = \top$, so we assume otherwise.

Let μ be the least ordinal appearing either in \mathfrak{w} or in \mathfrak{v} , so that

$$\psi \wedge \phi \equiv (h_\mu(\mathfrak{w}) \wedge h_\mu(\mathfrak{v})) \wedge (\mu b_\mu(\mathfrak{w}) \wedge \mu b_\mu(\mathfrak{v})).$$

By induction hypothesis, $h_\mu(\mathfrak{w}) \wedge h_\mu(\mathfrak{v}) \equiv \mathfrak{u}_1$ for some $\mathfrak{u}_1 \in \mathbb{W}_{\mu+1}$ with all modalities occurring in $\phi \wedge \psi$. Meanwhile, either $b_\mu(\mathfrak{w}) \triangleleft_\mu b_\mu(\mathfrak{v})$, $b_\mu(\mathfrak{w}) \equiv b_\mu(\mathfrak{v})$ or $b_\mu(\mathfrak{v}) \triangleleft_\mu b_\mu(\mathfrak{w})$. In the first case,

$$\mu b_\mu(\mathfrak{v}) \Rightarrow \mu \mu b_\mu(\mathfrak{w}) \Rightarrow \mu b_\mu(\mathfrak{w}),$$

and in the second $\mu b_\mu(\mathfrak{v}) \Rightarrow \mu b_\mu(\mathfrak{w})$; in either case, $\mu b_\mu(\mathfrak{w}) \wedge \mu b_\mu(\mathfrak{v}) \equiv \mu b_\mu(\mathfrak{v})$. Similarly, if $b_\mu(\mathfrak{v}) \triangleleft_\mu b_\mu(\mathfrak{w})$, then $\mu b_\mu(\mathfrak{w}) \wedge \mu b_\mu(\mathfrak{v}) \equiv \mu b_\mu(\mathfrak{w})$. In either case,

$$\mu b_\mu(\mathfrak{w}) \wedge \mu b_\mu(\mathfrak{v}) \equiv \mu b_\mu(\mathfrak{u}_0)$$

for some worm $\mathfrak{u}_0 \in \{\mathfrak{w}, \mathfrak{v}\}$, and thus

$$\phi \wedge \psi \equiv (h_\mu(\mathfrak{w}) \wedge h_\mu(\mathfrak{v})) \wedge (\mu b_\mu(\mathfrak{w}) \wedge \mu b_\mu(\mathfrak{v})) \equiv \mathfrak{u}_1 \wedge \mu \mathfrak{u}_0 \equiv \mathfrak{u}_1 \mu \mathfrak{u}_0. \quad \square$$

Below, we remark that $\mathfrak{w} \sqsubset \mu$ is equivalent to $\max \mathfrak{w} < \mu$.

Corollary 3.19. *Let μ be an ordinal and $\top \neq \mathfrak{w} \in \mathbb{W}$. Then,*

1. *if $\mathfrak{w} \neq \top$ and $\mu < \max \mathfrak{w}$ then $\mu \top \triangleleft \mathfrak{w}$,*
2. *if $\mathfrak{w} \neq \top$ and $\mu \leq \max \mathfrak{w}$ then $\mu \top \trianglelefteq \mathfrak{w}$, and*
3. *if $\mathfrak{w} \sqsubset \mu$ then $\mathfrak{w} \triangleleft \mu \top$.*

Proof. For the first claim, proceed by induction on $\#\mathfrak{w}$. Write $\mathfrak{w} = \lambda \mathfrak{v}$ and consider two cases. If $\lambda \leq \mu$, by induction on length, $\mu \top \triangleleft \mathfrak{v}$, so $\mu \top \triangleleft \mathfrak{v} \triangleleft \mathfrak{w}$. Otherwise, $\lambda > \mu$, so from $\mathfrak{v} \Rightarrow \top$, $\lambda \top \Rightarrow \mu \top$, and Lemma 3.10 we obtain

$$\mathfrak{w} \Rightarrow \lambda \top \wedge \mu \top \Rightarrow \lambda \mu \top \Rightarrow 0 \mu \top.$$

The second claim is similar. Again, write $\mathfrak{w} = \lambda \mathfrak{v}$. If $\mu > \lambda$, we have inductively that $\mu \top \trianglelefteq \mathfrak{v} \triangleleft \mathfrak{w}$. Otherwise, $\mu \leq \lambda$, in which case

$$\mathfrak{w} \Rightarrow \lambda \top \Rightarrow \mu \top,$$

and we may use Corollary 3.15.

For the third, we proceed once again by induction on $\#\mathfrak{w}$. The case for $\mathfrak{w} = \top$ is obvious. Otherwise, let $\eta = \min \mathfrak{w}$. Then, by the induction hypothesis, $h_\eta(\mathfrak{w}) \triangleleft \mu \top = h_\eta(\mu \top)$, while also by the induction hypothesis $b_\eta(\mathfrak{w}) \triangleleft \mu \top$, hence $\mathfrak{w} \triangleleft \mu \top$ by Theorem 3.17. \square

3.3 Well-orderedness of worms

We have seen that \trianglelefteq_μ is a total preorder, but in fact we have more; it is a pre-well-order. We will prove this using a Kruskal-style argument [31]. It is very similar to Beklemishev's proof in [5], although he uses normal forms for worms. Here we will use our 'head-body' decomposition instead.

Theorem 3.20. *For any ordinal λ and any $\eta \leq \lambda$, \triangleleft_η is a pre-well-order on $\mathbb{W}_{\geq \lambda}$.*

Proof. We have already seen that \mathbb{W}_λ is total in Theorem 3.17, so it remains to show that there are no infinite \triangleleft_η -descending chains. We will prove this by contradiction, assuming that there is such a chain.

Let \mathfrak{w}_0 be any worm such that \mathfrak{w}_0 is the first element of some infinite descending chain $\mathfrak{w}_0 \triangleright_\eta \mathfrak{v}_1 \triangleright_\eta \mathfrak{v}_2 \triangleright_\eta \dots$ and $\#\mathfrak{w}_0$ is minimal among all worms that can be the

first element of such a chain. Then, for $i > 0$, choose \mathfrak{w}_i recursively by letting it be a worm such that there is an infinite descending chain

$$\mathfrak{w}_0 \triangleright_\eta \mathfrak{w}_1 \triangleright_\eta \dots \triangleright_\eta \mathfrak{w}_i \triangleright_\eta \mathfrak{v}_{i+1} \triangleright_\eta \dots,$$

and such that $\#\mathfrak{w}_i$ is minimal among all worms with this property (where \mathfrak{w}_j is already fixed for $j < i$). Let $\vec{\mathfrak{w}}$ be the resulting chain.

Now, let $\mu \geq \eta$ be the least ordinal appearing in $\vec{\mathfrak{w}}$, and define $h(\vec{\mathfrak{w}})$ to be the sequence

$$h_\mu(\mathfrak{w}_0), h_\mu(\mathfrak{w}_1), \dots, h_\mu(\mathfrak{w}_i), \dots$$

Let j be the first natural number such that μ appears in \mathfrak{w}_j . By Lemma 3.12.2, $h_\mu(\mathfrak{w}_i) = \mathfrak{w}_i$ for all $i < j$, while $\#h_\mu(\mathfrak{w}_j) < \#\mathfrak{w}_j$, so by the minimality of $\#\mathfrak{w}_j$, $h(\vec{\mathfrak{w}})$ is not an infinite decreasing chain. Hence for some k , $h_\mu(\mathfrak{w}_k) \sqsupseteq_\eta h_\mu(\mathfrak{w}_{k+1})$.

Next, define $b(\vec{\mathfrak{w}})$ to be the sequence

$$\mathfrak{w}_0, \dots, \mathfrak{w}_{k-1}, b_\mu(\mathfrak{w}_k), \mathfrak{w}_{k+2}, \mathfrak{w}_{k+3}, \dots$$

In other words, we replace \mathfrak{w}_k by $b_\mu(\mathfrak{w}_k)$ and skip \mathfrak{w}_{k+1} . By the minimality of $\#\mathfrak{w}_k$, this cannot be a decreasing sequence, and hence $b_\mu(\mathfrak{w}_k) \sqsubseteq_\eta \mathfrak{w}_{k+2} \triangleleft_\eta \mathfrak{w}_{k+1}$.

It follows from Theorem 3.17 that $\mathfrak{w}_k \sqsubseteq_\eta \mathfrak{w}_{k+1}$, a contradiction. We conclude that there can be no decreasing sequence, and \triangleleft_η is well-founded, as claimed. \square

One consequence of worms being pre-well-ordered is that we can assign them an ordinal number measuring their order-type. In the next section we will make this precise.

3.4 Order-types on a pre-well-order

As we have mentioned, any well-order may be canonically represented using an ordinal number. To do this, if $\mathfrak{A} = \langle A, \preceq \rangle$ is any pre-well-order, for $a \in A$ define

$$o(a) = \bigcup_{b \prec a} (o(b) + 1).$$

Observe that o is strictly increasing, in the following sense:

Definition 3.21. Let $\langle A, \preceq_A \rangle, \langle B, \preceq_B \rangle$ be preorders, and $f: A \rightarrow B$. We say that f is strictly increasing if

1. for all $x, y \in A$, $x \preceq_A y$ implies $f(x) \preceq_B f(y)$, and
2. for all $x, y \in A$, $x \prec_A y$ implies $f(x) \prec_B f(y)$.

We note that if \prec_A is total, then there are other equivalent ways of defining strictly increasing maps:

Lemma 3.22. *If $\langle A, \preceq_A \rangle, \langle B, \preceq_B \rangle$ are total preorders and $f: A \rightarrow B$, then the following are equivalent:*

1. *f is strictly increasing;*
2. *for all $x, y \in A$, $x \preceq_A y$ if and only if $f(x) \preceq_B f(y)$;*
3. *for all $x, y \in A$, $x \prec_A y$ if and only if $f(x) \prec_B f(y)$.*

Proof. Straightforward, using the fact that $a \prec_A b$ if and only if $b \not\preceq_A a$, and similarly for \prec_B . \square

Then, the map o can be characterized as the only strictly increasing, initial map $f: A \rightarrow \text{Ord}$, where $f: A \rightarrow B$ is *initial* if whenever $b \prec_B f(a)$, it follows that $b = f(a')$ for some $a' \prec_A a$:

Lemma 3.23. *Let $\langle A, \preceq \rangle$ be a pre-well-order. Then,*

1. *for all $x, y \in A$, $x \prec y$ if and only if $o(x) < o(y)$, and*
2. *$o: A \rightarrow \text{Ord}$ is an initial map.*

The proof proceeds by transfinite induction along \prec and we omit it, as is the case of the proof of the following:

Lemma 3.24. *Let $\langle A, \preceq \rangle$ be a pre-well-order. Suppose that $f: A \rightarrow \text{Ord}$ satisfies*

1. *$x \prec y$ implies that $f(x) < f(y)$,*
2. *$x \preceq y$ implies that $f(x) \leq f(y)$, and*
3. *if $\xi \in f[A]$ then $\xi \subseteq f[A]$.*

Then, $f = o$.

Observe that $o(a) = o(b)$ implies that $a \preceq b$ and $b \preceq a$, i.e. $a \approx b$. Let us state this explicitly for the case of worms.

Lemma 3.25. *If $\mathfrak{w}, \mathfrak{v}$ are worms such that $o(\mathfrak{w}) = o(\mathfrak{v})$, then $\mathfrak{w} \equiv \mathfrak{v}$.*

Proof. Reasoning by contrapositive, assume that $\mathfrak{w} \neq \mathfrak{v}$. Then by Lemma 3.14, either $\mathfrak{w} \triangleleft \mathfrak{v}$, which implies that $o(\mathfrak{w}) < o(\mathfrak{v})$, or $\mathfrak{v} \triangleleft \mathfrak{w}$, and hence $o(\mathfrak{v}) < o(\mathfrak{w})$. In either case, $o(\mathfrak{w}) \neq o(\mathfrak{v})$. \square

Computing $o(\mathfrak{w})$ will take some work, but it is not too difficult to establish some basic relationships between $o(\mathfrak{w})$ and the ordinals appearing in \mathfrak{w} .

Lemma 3.26. *Let $\mathfrak{w} \neq \top$ be a worm and μ an ordinal. Then,*

1. *if $\mu \leq \max \mathfrak{w}$, then $\mu \leq o(\mu\top) \leq o(\mathfrak{w})$, and*
2. *if $\max \mathfrak{w} < \mu$, then $o(\mathfrak{w}) < o(\mu\top)$.*

Proof. First we proceed by induction on μ to show that $\mu \leq o(\mu\top)$. Suppose that $\eta < \mu$. Then by Corollary 3.19, $\eta\top \triangleleft \mu\top$, while by the induction hypothesis $\eta \leq o(\eta\top)$, and hence $\eta \leq o(\eta\top) < o(\mu\top)$. Since $\eta < \mu$ was arbitrary, $\mu \leq o(\mu\top)$. That $o(\mu\top) \leq o(\mathfrak{w})$ if $\mu \leq \max \mathfrak{w}$ follows from Corollary 3.19, since $\mu\top \trianglelefteq \mathfrak{w}$.

The second claim is immediate from Corollary 3.19.3. \square

Let us conclude this section by stating a useful consequence of the fact that $o: \mathbb{W} \rightarrow \text{Ord}$ is initial.

Corollary 3.27. *For every ordinal ξ there is a worm $\mathfrak{w} \trianglelefteq \xi\top$ such that $\xi = o(\mathfrak{w})$.*

Proof. By Lemma 3.26, $\xi \leq o(\xi\top)$, so this is a special case of Lemma 3.23.2. \square

4 Finite worms

In the previous section we explored some basic properties of o , but they are not sufficient to compute $o(\mathfrak{w})$ for a worm \mathfrak{w} . In this section we will provide an explicit calculus for $o \upharpoonright \mathbb{W}_\omega$ (where \upharpoonright denotes domain restriction). \mathbb{W}_ω is a particularly interesting case-study in that it has been used by Beklemishev for a Π_1^0 ordinal analysis of Peano arithmetic. Before we continue, it will be illustrative to sketch the relationship between \mathbb{W}_ω and PA.

4.1 First-order arithmetic

Expressions of RC_ω have a natural proof-theoretical interpretation in first-order arithmetic. We will use the language Π_ω of first-order arithmetic containing the signature

$$\{0, 1, +, \cdot, 2^{\cdot}, =\}$$

so that we have symbols for addition, multiplication, and exponentiation, as well as Boolean connectives and quantifiers ranging over the natural numbers. Elements of Π_ω are *formulas*. The set of all formulas where all quantifiers are *bounded*, that is, of the form $\forall x < t \phi$ or $\exists x < t \phi$ (where t is any term), is denoted Δ_0 . A formula

of the form $\exists x_n \forall x_{n-1} \dots \delta(x_1, \dots, x_n)$, with $\delta \in \Delta_0$, is Σ_n , and a formula of the form $\forall x_n \exists x_{n-1} \dots \delta(x_1, \dots, x_n)$ is Π_n . These classes are extended modulo provable equivalence, so that every formula falls into one of them. Note that the negation of a Σ_n formula is Π_n and vice-versa.

To simplify notation we may assume that some additional function symbols are available, although these are always definable from the basic arithmetical operations. In particular, we assume that we have for each n a function $\langle x_1, \dots, x_n \rangle$ coding a sequence as a single natural number.

In order to formalize provability within arithmetic, we fix some Gödel numbering mapping a formula $\psi \in \Pi_\omega$ to its corresponding Gödel number $\ulcorner \psi \urcorner$, and similarly for terms and sequences of formulas, which can be used to represent derivations. We also define the *numeral* of $n \in \mathbb{N}$ to be the term

$$\bar{n} = 0 + \underbrace{1 + \dots + 1}_{n \text{ times}}.$$

In order to simplify notation, we will often identify ψ with $\ulcorner \psi \urcorner$.

We will assume that every theory T contains classical predicate logic, is closed under modus ponens, and that there is a Δ_0 formula $\text{Proof}_T(x, y)$ which holds if and only if x codes a derivation in T of a formula coded by y . Using Craig's trick, any theory with a computably enumerable set of axioms is deductively equivalent to one in this form, so we do not lose generality by these assumptions.

If ϕ is a natural number (supposedly coding a formula), we use $\Box_T \phi$ as shorthand for $\exists y \text{Proof}_T(y, \bar{\phi})$. We also write $\Box_T \phi(\dot{x}_0, \dots, \dot{x}_n)$ as short for $\exists \psi (\psi = \phi(\bar{x}_0, \dots, \bar{x}_n) \wedge \Box_T \psi)$. To get started on proving theorems about arithmetic, we need a minimal ‘background theory’. This will use Robinson’s arithmetic \mathbf{Q} enriched with axioms for the exponential; call the resulting theory \mathbf{Q}^+ . To be precise, \mathbf{Q}^+ is axiomatized by classical first-order logic with equality, together with the following:

- $\forall x (x + 0 = x)$
- $\forall x (x \neq 0 \leftrightarrow \exists y x = y + 1)$
- $\forall x \forall y (x + 1 = y + 1 \rightarrow x = y)$
- $\forall x \forall y (x + (y + 1) = (x + y) + 1)$
- $\forall x (x \times 0 = 0)$
- $\forall x \forall y (x \times (y + 1) = (x \times y) + x)$
- $2^0 = 1$
- $\forall x (2^{x+1} = 2^x + 2^x)$

Aside from these basic axioms, the following schemes will be useful in axiomatizing many theories of interest to us. Let Γ to denote a set of formulas. Then, the induction schema for Γ is defined by

$$\text{IF: } \phi(0) \wedge \forall x (\phi(x) \rightarrow \phi(x + 1)) \rightarrow \forall x \phi(x), \quad \text{where } \phi \in \Gamma.$$

Elementary arithmetic is the first-order theory

$$\text{EA} = \mathbf{Q}^+ + \mathbf{I}\Delta_0,$$

and *Peano arithmetic* is the first-order theory

$$\text{PA} = \mathbf{Q}^+ + \mathbf{III}_\omega.$$

As usual, $\Diamond_T\phi$ is defined as $\neg\Box_T\neg\phi$, and this will be used to interpret the RC -modality 0. Other modalities can be interpreted as stronger notions of consistency. For this purpose it is very useful to consider the provability predicates $[n]_T$, where $[n]_T$ is a natural first-order formalization of “provable from the axioms of T together with some true Π_n sentence”. More precisely, let \mathbf{True}_{Π_n} be the standard partial truth-predicate for Π_n formulas, which is itself of complexity Π_n (see [24] for information about partial truth definitions within EA). Then, we define

$$[n]_T\varphi \leftrightarrow \exists\pi (\mathbf{True}_{\Pi_n}(\pi) \wedge \Box_T(\pi \rightarrow \varphi)).$$

Definition 4.1. *Given a theory T , we then define $\cdot_T: \mathcal{L}_{\text{RC}} \rightarrow \Pi_\omega$ given recursively by*

- (i) $\top_T = \top$,
- (ii) $(\phi \wedge \psi)_T = \phi_T \wedge \psi_T$, and
- (iii) $(n\phi)_T = \langle n \rangle_T \phi_T$.

The next theorem follows from the arithmetical completeness of GLP_ω proven by Ignatiev [26] together with the conservativity of GLP_ω over RC_ω (Theorem 2.8).

Theorem 4.2. *Let T be any sound, representable extension of PA. Given a formula ϕ of RC_ω , $\text{RC}_\omega \vdash \phi$ if and only if $T \vdash \phi_T$.*

We remark that Japaridze first proved a variant of this result, where $[n]_T$ is defined using iterated ω -rules [27]. A similar interpretation will be discussed in Section 5.2 in the context of second-order arithmetic. However, the interpretation we have sketched using proof predicates has been used by Beklemishev to provide a consistency proof of Peano arithmetic as well as a Π_1^0 ordinal analysis. Here we will briefly sketch the consistency proof; for details, see [4].

The first step is to represent Peano arithmetic in terms of n -consistency:

Theorem 4.3. *It is provable in EA that*

$$\text{PA} \equiv \text{EA} + \{\langle n \rangle_{\text{EA}} \top : n < \omega\}.$$

This is a reformulation of a result of Kreisel and Lévy [30], although they used *primitive recursive arithmetic* in place of EA. The variant with EA is due to Beklemishev.

The consistency proof will be realized mostly within a ‘finitary base theory’, EA^+ , which is only a bit stronger than EA. To describe it, first define the *superexponential*, denoted 2_m^n , to be the function given recursively by (i) $2_0^n = 2^n$ and (ii) $2_{m+1}^n = 2^{2_m^n}$. Thus, 2_m^1 denotes an exponential tower of m 2’s. Then, we let EA^+ be the extension of EA with an axiom stating that the superexponential function is total. With this, we may enunciate Beklemishev’s *reduction rule*:

Theorem 4.4. *If $\mathfrak{w} \sqsubset \omega$ is any worm, then EA^+ proves that*

$$(\forall \mathfrak{v} \triangleleft \mathfrak{w} \, (\Diamond_{\text{EA}} \mathfrak{v}_{\text{EA}})) \rightarrow \Diamond_{\text{EA}} \mathfrak{w}_{\text{EA}}.$$

This extends a previous result by Schmerl [35]. Meanwhile, the reader may recognize this as the premise of the *transfinite induction scheme* for worms. To be precise, if $\phi(x), x \prec y$ are arithmetical formulas, then the transfinite induction scheme for ϕ along \prec is given by:

$$\text{TI}_{\prec}(\phi) = \left(\forall x \, ((\forall y \prec x \, \phi(y)) \rightarrow \phi(x)) \right) \rightarrow \forall x \, \phi(x).$$

If Γ is a set of formulas, then $\text{TI}_{\prec}(\Gamma)$ is the scheme $\{\text{TI}_{\prec}(\phi) : \phi \in \Gamma\}$.

Observe that $\Diamond_{\text{EA}} \phi \in \Pi_1$ independently of ϕ ; with this in mind, we obtain the following as an immediate consequence of Theorem 4.4:

Theorem 4.5. $\text{EA}^+ + \text{TI}_{\triangleleft|\mathbb{W}_\omega}(\Pi_1) \vdash \Diamond_{\text{PA}} \top$.

In words, we can prove the consistency of Peano arithmetic using EA^+ and transfinite induction along $\langle \mathbb{W}_\omega, \triangleleft \rangle$. In fact, we use only one instance of transfinite induction for a predicate $\phi(x)$ expressing “ $x \sqsubset \omega$ and $\Diamond_{\text{EA}} x_{\text{EA}}$ ”.

Compare this to Gentzen’s work [21], where he proves the consistency of Peano arithmetic with transfinite induction up to the ordinal ε_0 . In the remainder of this section, we will see how finite worms and ε_0 are closely related.

4.2 The ordinal ε_0

The ordinal ε_0 is naturally defined by extending the arithmetical operations of addition, multiplication and exponentiation to the transfinite. In view of Lemma 2.4, we may have to consider not only successor ordinals, but also unions of ordinals. Fortunately, these operations are exhaustive.

Lemma 4.6. *Let ξ be an ordinal. Then, exactly one of the following occurs:*

- (i) $\xi = 0$;
- (ii) there exists ζ such that $\xi = \zeta + 1$, in which case we say that ξ is a successor;
or
- (iii) $\xi = \bigcup_{\zeta < \xi} \zeta$, in which case we say that ξ is a limit.

Thus we may recursively define operations on the ordinals if we consider these three cases. For example, ordinal addition is defined as follows:

Definition 4.7. *Given ordinals ξ, ζ , we define $\xi + \zeta$ by recursion on ζ as follows:*

1. $\xi + 0 = \xi$
2. $\xi + (\zeta + 1) = (\xi + \zeta) + 1$
3. $\xi + \zeta = \bigcup_{\vartheta < \zeta} (\xi + \vartheta)$, for ζ a limit ordinal.

Ordinal addition retains some, but not all, of the properties of addition on the natural numbers; it is associative, but not commutative. For example, $1 + \omega = \omega < \omega + 1$, and more generally $1 + \xi = \xi < \xi + 1$ whenever ξ is infinite. We also have a form of subtraction, but only on the left:

Lemma 4.8. *If $\zeta < \xi$ are ordinals, there exists a unique η such that $\zeta + \eta = \xi$.*

The proof follows by a standard transfinite induction on ξ . We will denote this unique η by $-\zeta + \xi$. It will be convenient to spell out some of the basic properties of left-subtraction:

Lemma 4.9. *Let α, β, γ be ordinals. Then:*

- (i) $-0 + \alpha = \alpha$ and $-\alpha + \alpha = 0$;
- (ii) if $\alpha \leq \beta$ and $-\alpha + \beta \leq \gamma$ then $-\alpha + (\beta + \gamma) = (-\alpha + \beta) + \gamma$;
- (iii) if $\alpha + \beta \leq \gamma$ then $-\beta + (-\alpha + \gamma) = -(\alpha + \beta) + \gamma$;
- (iv) if $\alpha \leq \beta \leq \alpha + \gamma$ then $-\beta + (\alpha + \gamma) = -(-\alpha + \beta) + \gamma$.

Proof. These properties are proven using the associativity of addition and the fact that $-\mu + \lambda$ is unique. We prove only (iii) as an example. Observe that

$$\begin{aligned} (\alpha + \beta) + (-\beta + (-\alpha + \gamma)) &= \alpha + (\beta + (-\beta + (-\alpha + \gamma))) \\ &= \alpha + (-\alpha + \gamma) = \gamma; \end{aligned}$$

but $-(\alpha + \beta) + \gamma$ is the unique η such that $(\alpha + \beta) + \eta = \gamma$, so we conclude that (iii) holds. The other properties are proven similarly. \square

The definition of addition we have given can be used as a template to generalize other arithmetical operations. Henceforth, if $\langle \mu_\xi \rangle_{\xi < \lambda}$ is an increasing sequence of ordinals, we will write $\lim_{\xi < \lambda} \mu_\xi$ instead of $\bigcup_{\xi < \lambda} \mu_\xi$.

Definition 4.10. *Given ordinals ξ, ζ , we define $\xi \cdot \zeta$ by recursion on ζ as follows:*

1. $\xi \cdot 0 = 0$,
2. $\xi \cdot (\zeta + 1) = \xi \cdot \zeta + \xi$, and
3. $\xi \cdot \zeta = \lim_{\vartheta < \zeta} \xi \cdot \vartheta$, for ζ a limit ordinal.

Similarly, we define ξ^ζ by:

1. $\xi^0 = 1$,
2. $\xi^{\zeta+1} = \xi^\zeta \cdot \xi$, and
3. $\xi^\zeta = \lim_{\vartheta < \zeta} \xi^\vartheta$, for ζ a limit ordinal.

Addition, multiplication and exponentiation give us our first examples of *normal functions*. These are functions that are increasing and continuous, in the following sense:

Definition 4.11. *A function $f: \text{Ord} \rightarrow \text{Ord}$ is normal if:*

1. *whenever $\xi < \zeta$, it follows that $f(\xi) < f(\zeta)$, and*
2. *whenever λ is a limit ordinal, $f(\lambda) = \lim_{\xi < \lambda} f(\xi)$.*

Normal functions are particularly nice to work with. Among other things, they have the following property, proven by an easy transfinite induction:

Lemma 4.12. *If $f: \text{Ord} \rightarrow \text{Ord}$ is normal, then for every ordinal ξ , $\xi \leq f(\xi)$.*

Of course this does not rule out the possibility that $\xi = f(\xi)$, and in fact the identity function is an example of a normal function. As we have mentioned, the elementary arithmetical functions give us further examples:

Lemma 4.13. *Let α be any ordinal. Then, the functions $f, g, h: \text{Ord} \rightarrow \text{Ord}$ given by*

1. $f(\xi) = \alpha + \xi$,
2. $g(\xi) = (1 + \alpha) \cdot \xi$,

$$3. h(\xi) = (2 + \alpha)^\xi$$

are all normal.

Note, however, that the function $\xi \mapsto \xi + \alpha$ is not normal in general, and neither are $\xi \mapsto 0 \cdot \xi$, $\xi \mapsto 1^\xi$. But $\xi \mapsto \omega^\xi$ is normal, and this function is of particular interest, since it is the basis of the Cantor normal form representation of ordinals (similar to a base- n representation of natural numbers), where we write

$$\xi = \omega^{\alpha_n} + \dots + \omega^{\alpha_0}$$

with the α_i 's non-decreasing. Moreover, the ordinals of the form ω^β are exactly the *additively indecomposable* ordinals; that is, non-zero ordinals that cannot be written as the sum of two smaller ordinals. Let us summarize some important properties of this function:

Lemma 4.14. *Let $\xi \neq 0$ be any ordinal. Then:*

1. *There are ordinals α, β such that $\xi = \alpha + \omega^\beta$. The value of β is unique.*
2. *We can take $\alpha = 0$ if and only if, for all $\gamma, \delta < \xi$, we have that $\gamma + \delta < \xi$.*

We call this the *Cantor decomposition* of ξ . Cantor decompositions can often be used to determine whether $\xi < \zeta$:

Lemma 4.15. *Given ordinals $\xi = \alpha + \omega^\beta$ and $\zeta = \gamma + \omega^\delta$,*

- | | |
|--|--|
| <ol style="list-style-type: none"> 1. $\xi < \zeta$ if and only if | <ol style="list-style-type: none"> 2. $\xi \leq \zeta$ if and only if |
| <ol style="list-style-type: none"> (a) $\xi \leq \gamma$, or (b) $\alpha < \zeta$ and $\beta < \delta$, and | <ol style="list-style-type: none"> (a) $\xi \leq \gamma$, or (b) $\alpha < \zeta$ and $\beta \leq \delta$. |

Note, however, that this decomposition is only useful when $\beta < \xi$ or $\gamma < \zeta$, which as we will see is not always the case. In particular, the ordinal ε_0 is the first ordinal such that $\varepsilon_0 = \omega^{\varepsilon_0}$. Roughly, it is defined by beginning with 0 and closing under the operation $\langle \alpha, \beta \rangle \mapsto \alpha + \omega^\beta$. Since many proof-theoretical ordinals are defined by taking the closure under a family of functions, it will be convenient to formalize such a closure with some generality.

The general scheme is to consider a family of ordinal functions f_1, \dots, f_n , then considering the least ordinal ξ such that $f_i(\alpha_1, \dots, \alpha_m) < \xi$ whenever each $\alpha_i < \xi$. To simplify our presentation, let us make a few preliminary observations:

1. The functions f_i may be partial or total. Since a total function is a special case of a partial function, we may in general consider $f_i: \text{Ord}^m \dashrightarrow \text{Ord}$ (where $f: A \dashrightarrow B$ indicates that f is a partial function).
2. We may have functions with fixed or variable arity. Given a class A , let $A^{<\omega}$ denote the class of finite sequences $\langle a_1, \dots, a_m \rangle$ with $m < \omega$ and each $a_i \in A$. An ordinal function with fixed arity m may be regarded as a partial function on $\text{Ord}^{<\omega}$, whose domain is $\text{Ord}^m \subseteq \text{Ord}^{<\omega}$. Thus without loss of generality, we may assume that all partial functions have variable arity.
3. We may represent the family f_1, \dots, f_n as a single function by setting

$$f(i, \alpha_1, \dots, \alpha_m) = f_i(\alpha_1, \dots, \alpha_m).$$

Note that this idea can also be used to represent infinite families of functions as a single function.

Thus we may restrict our discussion to ordinals closed under a single partial function of variable arity, and will do so in the next definition.

Definition 4.16. *Let $f: \text{Ord}^{<\omega} \dashrightarrow \text{Ord}$ be a partial function. Given a set of ordinals Θ , define $f[\Theta]$ to be the set of all ordinals λ such that there exist $\mu_1, \dots, \mu_n \in \Theta$ (possibly with $n = 0$) such that $\lambda = f(\mu_1, \dots, \mu_n)$.*

For $n < \omega$, define inductively $\Theta_0^f = \Theta$ and $\Theta_{n+1}^f = \Theta_n^f \cup f[(\Theta_n^f)]$. Then, define

$$\overline{\Theta}^f = \bigcup_{n < \omega} \Theta_n^f.$$

The set $\overline{\Theta}^f$ is the *closure of Θ under f* , and indeed behaves like a standard closure operation:

Lemma 4.17. *Let $f: \text{Ord}^{<\omega} \dashrightarrow \text{Ord}$ and let Θ be any set of ordinals. Then,*

1. $\Theta \cup f[(\overline{\Theta}^f)] \subseteq \overline{\Theta}^f$,
2. if $\Theta \cup \Xi^f \subseteq \Xi$ then $\overline{\Theta}^f \subseteq \Xi$, and
3. for any ordinal λ , $\lambda \in (\overline{\Theta}^f) \setminus \Theta$ if and only if there are $\mu_1, \dots, \mu_n \in \overline{\Theta}^f \setminus \{\lambda\}$ with $\lambda = f(\mu_1, \dots, \mu_n)$.

Proof. For the first item, note that if $\lambda_1, \dots, \lambda_n \in \overline{\Theta}^f$ then $\lambda_1, \dots, \lambda_n \in \Theta_m^f$ for m large enough and hence $f(\lambda_1, \dots, \lambda_n) \in \Theta_{m+1}^f \subseteq \overline{\Theta}^f$. The second follows by showing

inductively that $\Theta_n^f \subseteq \Xi$ for all n , hence $\overline{\Theta}^f \subseteq \Xi$. For the third, assume otherwise, and consider $\Xi = \overline{\Theta}^f \setminus \{\lambda\}$. One can readily verify that $\Theta \cup f[\Xi] \subseteq \Xi$, contradicting the previous item. \square

With this, we are ready to define the ordinal ε_0 . Below, recall that we are following the standard set-theoretic convention that $1 = \{0\}$.

Definition 4.18. Define $\text{Cantor}: \text{Ord}^2 \rightarrow \text{Ord}$ by $\text{Cantor}(\alpha, \beta) = \alpha + \omega^\beta$. Then, we define

$$\varepsilon_0 = \overline{1}^{\text{Cantor}}.$$

As promised, ε_0 is the first fixed-point of the function $\xi \mapsto \omega^\xi$:

Theorem 4.19. The set ε_0 is an ordinal and satisfies the identity $\varepsilon_0 = \omega^{\varepsilon_0}$. Moreover, if $0 < \xi < \varepsilon_0$, there are $\alpha, \beta < \xi$ such that $\xi = \alpha + \omega^\beta$.

Proof. First we will show that if $0 < \xi \in \varepsilon_0$, then there are $\alpha, \beta < \xi$ such that $\xi = \alpha + \omega^\beta$. By Lemma 4.17.3, there are $\alpha, \beta \in \varepsilon_0$ with $\alpha, \beta \neq \xi$ and such that $\xi = \alpha + \omega^\beta$. Since $\omega^\beta > 0$ it follows that $\alpha < \xi$, and since $\beta \leq \omega^\beta \leq \xi$ it follows that $\beta \leq \xi$; but $\beta \neq \xi$, so $\beta < \xi$.

Now, since every element of ε_0 is an ordinal, in view of Lemma 2.3, in order to show that ε_0 is also an ordinal it suffices to show that if $\xi < \zeta \in \varepsilon_0$, then $\xi \in \varepsilon_0$. We proceed by induction on ζ with a secondary induction on ξ . Write $\zeta = \alpha + \omega^\beta$ and $\xi = \gamma + \omega^\delta$ with $\alpha, \beta \in \varepsilon_0 \cap \zeta$. Since $\xi < \zeta$, by Lemma 4.15, we have that either $\xi \leq \alpha$ or $\gamma < \zeta$ and $\delta < \beta$. In the first case, our induction hypothesis applied to $\alpha < \zeta$ gives us $\xi \in \varepsilon_0$, in the second the secondary induction hypothesis on $\gamma < \xi$ gives us $\gamma \in \varepsilon_0$ and the induction hypothesis on $\beta < \zeta$ gives us $\delta \in \varepsilon_0$, hence $\xi = \alpha + \omega^\beta \in \varepsilon_0$. \square

4.3 Order-types of finite worms

Our work on elementary ordinal operations and the ordinal ε_0 will suffice to compute the order-types of ‘finite’ worms, i.e., worms where every entry is finite. In order to give a calculus for these order-types, we will need to consider, in addition to concatenation, ‘promotion’ (\uparrow) and ‘demotion’ (\downarrow) operations on worms. Below, let us write $\mathcal{L}_{\geq \lambda}$ for the sublanguage of \mathcal{L}_{RC} which only contains modalities $\xi \geq \lambda$.

Definition 4.20. Let $\phi \in \mathcal{L}_{\text{RC}}$ and λ be an ordinal. We define $\lambda \uparrow \phi$ to be the result of replacing every ordinal ξ appearing in ϕ by $\lambda + \xi$. Formally, $\lambda \uparrow \top = \top$, $\lambda \uparrow (\phi \wedge \psi) = (\lambda \uparrow \phi) \wedge (\lambda \uparrow \psi)$, and $\lambda \uparrow \mu\phi = (\lambda + \mu)(\lambda \uparrow \phi)$.

If $\phi \in \mathcal{L}_{\geq \lambda}$, we similarly define $\lambda \downarrow \phi$ by replacing every occurrence of ξ by $-\lambda + \xi$.

The relationship between \uparrow and \downarrow is analogous to that between ordinal addition and subtraction. The following are all straightforward consequences of Lemma 4.9 and we omit the proofs.

Lemma 4.21. *Let α, β be ordinals and $\phi \in \mathcal{L}_{\text{RC}}$. Then,*

- (i) $0 \uparrow \phi = \phi$;
- (ii) $\alpha \uparrow (\beta \uparrow \phi) = (\alpha + \beta) \uparrow \phi$;
- (iii) if $\phi \in \mathcal{L}_{\geq \beta + \alpha}$ then $\alpha \downarrow (\beta \downarrow \phi) = (\beta + \alpha) \downarrow \phi$;
- (iv) if $\alpha \leq \beta$ then $\alpha \downarrow (\beta \uparrow \phi) = (-\alpha + \beta) \uparrow \phi$, and
- (v) if $\alpha \leq \beta$ and $\phi \in \mathcal{L}_{\geq -\alpha + \beta}$ then $\alpha \uparrow \phi \in \mathcal{L}_{\geq \beta}$ and

$$\beta \downarrow (\alpha \uparrow \phi) = (-\alpha + \beta) \downarrow \phi.$$

The operation $\phi \mapsto \lambda \uparrow \phi$ is particularly interesting in that it provides a sort of self-embedding of RC:

Lemma 4.22. *Let α, β be ordinals and $\phi, \psi \in \mathcal{L}_{\text{RC}}$. If $\phi \Rightarrow \psi$ is derivable in RC, then so is $(\lambda \uparrow \phi) \Rightarrow (\lambda \uparrow \psi)$.*

Proof. By induction on the length of a derivation of $\phi \Rightarrow \psi$; intuitively, one replaces every formula θ appearing in the derivation by $\lambda \uparrow \theta$. The details are straightforward and left to the reader. \square

The promotion operator gives us an order-preserving transformation on the class of worms:

Lemma 4.23. *Given a worm $\mathfrak{w} \in \mathbb{W}_{\geq \mu}$ and an ordinal λ , the following are equivalent:*

- (i) $\mathfrak{w} \triangleleft_{\mu} \mathfrak{v}$;
- (ii) $\lambda \uparrow \mathfrak{w} \triangleleft_{\mu} \lambda \uparrow \mathfrak{v}$, and
- (iii) $\lambda \uparrow \mathfrak{w} \triangleleft_{\lambda} \lambda \uparrow \mathfrak{v}$.

Proof. The equivalence between (ii) and (iii) is immediate from Lemma 3.16, so we focus on the equivalence between (i) and (iii).

If $\mathfrak{w} \triangleleft_{\mu} \mathfrak{v}$, then $\mathfrak{w} \triangleleft \mathfrak{v}$, so RC derives $\mathfrak{v} \Rightarrow 0\mathfrak{w}$. By Lemma 4.22, RC also derives $(\lambda \uparrow \mathfrak{v}) \Rightarrow \lambda(\lambda \uparrow \mathfrak{w})$, that is, $(\lambda \uparrow \mathfrak{w}) \triangleleft_{\lambda} (\lambda \uparrow \mathfrak{v})$.

Conversely, if $\lambda \uparrow \mathfrak{w} \triangleleft_\lambda \lambda \uparrow \mathfrak{v}$, assume towards a contradiction that $\mathfrak{w} \not\triangleleft_\mu \mathfrak{v}$, so that by Lemma 3.14, $\mathfrak{v} \trianglelefteq_\mu \mathfrak{w}$. Again by Lemma 4.22, $(\lambda \uparrow \mathfrak{v}) \trianglelefteq_\lambda (\lambda \uparrow \mathfrak{w})$, so $(\lambda \uparrow \mathfrak{v}) \trianglelefteq_\lambda (\lambda \uparrow \mathfrak{w}) \triangleleft_\lambda (\lambda \uparrow \mathfrak{v})$, contradicting irreflexivity. \square

Lemma 4.23 is useful for comparing worms; if we wish to settle whether $\lambda \uparrow \mathfrak{w} \triangleleft \lambda \uparrow \mathfrak{v}$, then it suffices to check whether $\mathfrak{w} \triangleleft \mathfrak{v}$. More generally, we obtain the following variant of Theorem 3.17. Below, recall that we write h, b instead of h_0, b_0 .

Lemma 4.24. *Given worms $\mathfrak{w}, \mathfrak{v} \neq \top$,*

1. *$\mathfrak{w} \triangleleft \mathfrak{v}$ if and only if*

- (a) *$\mathfrak{w} \trianglelefteq b(\mathfrak{v})$, or*
- (b) *$b(\mathfrak{w}) \triangleleft \mathfrak{v}$ and $1 \downarrow h(\mathfrak{w}) \triangleleft 1 \downarrow h(\mathfrak{v})$;*

2. *$\mathfrak{w} \trianglelefteq \mathfrak{v}$ if and only if*

- (a) *$\mathfrak{w} \trianglelefteq b(\mathfrak{v})$, or*
- (b) *$b(\mathfrak{w}) \triangleleft \mathfrak{v}$ and $1 \downarrow h(\mathfrak{w}) \trianglelefteq 1 \downarrow h(\mathfrak{v})$.*

If all entries of $\mathfrak{v} \neq \top$ are natural numbers, $1 \downarrow h(\mathfrak{w})$ will be ‘smaller’ than \mathfrak{w} . To be precise, it will have a smaller 1-norm, defined as follows:

Definition 4.25. *We define $\|\cdot\|_1 : \mathbb{W}_\omega \rightarrow \omega$ recursively by*

- 1. *$\|\top\|_1 = 0$;*
- 2. *if $\mathfrak{w} \neq \top$ and $\min \mathfrak{w} = 0$,*

$$\|\mathfrak{w}\|_1 = \|h(\mathfrak{w})\|_1 + \|b(\mathfrak{w})\|_1 + 1;$$

- 3. *if $\mathfrak{w} \neq \top$ and $\min \mathfrak{w} > 0$,*

$$\|\mathfrak{w}\|_1 = \|1 \downarrow \mathfrak{w}\|_1 + 1.$$

Recall that we use h and b as shorthands for h_0, b_0 .

Lemma 4.26. *For every worm $\mathfrak{w} \sqsubset \omega$ with $\mathfrak{w} \neq \top$,*

- 1. *$\|b(\mathfrak{w})\|_1 < \|\mathfrak{w}\|_1$, and*
- 2. *$\|1 \downarrow h(\mathfrak{w})\|_1 < \|\mathfrak{w}\|_1$.*

Proof. For the first claim, note that if 0 appears in \mathfrak{w} then $\|b(\mathfrak{w})\|_1 + 1 \leq \|\mathfrak{w}\|_1$. If 0 does not appear, $\|b(\mathfrak{w})\|_1 = 0 < \|\mathfrak{w}\|_1$.

For the second, if $h(\mathfrak{w}) = \top$ then once again $\|1 \downarrow h(\mathfrak{w})\|_1 = 0 < \|\mathfrak{w}\|_1$, and if $h(\mathfrak{w}) \neq \top$ then

$$\|1 \downarrow h(\mathfrak{w})\|_1 + 1 = \|h(\mathfrak{w})\|_1 \leq \|\mathfrak{w}\|_1,$$

so $\|1 \downarrow h(\mathfrak{w})\|_1 < \|h(\mathfrak{w})\|_1 \leq \|\mathfrak{w}\|_1$. \square

We remark that there are other possible ways to define $\|\cdot\|_1$ that would also satisfy Lemma 4.26; for example, we can define $\|\mathfrak{w}\|'_1 = \#\mathfrak{w} + \max \mathfrak{w}$, or

$$\|m_1 \dots m_n \top\|''_1 = \sum_{i=1}^n (m_i + 1).$$

However, these definitions do not generalize well to worms with transfinite entries, which will be the focus of Section 5. On the other hand, our norm $\|\cdot\|_1$ can be applied to transfinite worms with only a minor modification.

Our goal now is to give an explicit calculus for computing $o(\mathfrak{w})$ if $\mathfrak{w} \sqsubset \omega$. In view of Lemma 3.24, it is sufficient to propose a candidate function for o and show that it has the required properties. Now, if we compare Lemma 4.24 with Lemma 4.15, we observe that the clauses for checking whether $\mathfrak{w} \triangleleft \mathfrak{v}$ in terms of

$$b(\mathfrak{w}), 1 \downarrow h(\mathfrak{w}), b(\mathfrak{v}), 1 \downarrow h(\mathfrak{v})$$

are analogous to the clauses for checking whether $\alpha + \omega^\beta < \gamma + \omega^\delta$ in terms of $\alpha, \beta, \gamma, \delta$, respectively. This suggests that

$$o(\mathfrak{w}) = ob(\mathfrak{w}) + \omega^{o(1 \downarrow h(\mathfrak{w}))}, \tag{1}$$

and we will use this idea to define our ‘candidate function’.

Definition 4.27. *Let $\mathfrak{v}, \mathfrak{w}$ be worms and α an ordinal.*

Then, define a map $\acute{o}: \mathbb{W}_\omega \rightarrow \mathbf{Ord}$ by

1. $\acute{o}(\top) = 0$, and
2. if $\mathfrak{w} \neq \top$ then $\acute{o}(\mathfrak{w}) = \acute{o}(b(\mathfrak{w})) + \omega^{\acute{o}(1 \downarrow h(\mathfrak{w}))}$.

First, let us check that \acute{o} is indeed a function:

Lemma 4.28. *The map \acute{o} is well-defined.*

Proof. This follows from an easy induction on $\|\mathfrak{w}\|_1$ using Lemma 4.26. \square

It remains to check that \acute{o} is strictly increasing and initial. Let us begin with the former:

Lemma 4.29. *The map $\acute{o}: \mathbb{W}_\omega \rightarrow \text{Ord}$ is strictly increasing.*

Proof. We will prove by induction on $\|\mathfrak{w}\|_1 + \|\mathfrak{v}\|_1$ that $\mathfrak{w} \triangleleft \mathfrak{v}$ if and only if $\acute{o}(\mathfrak{w}) < \acute{o}(\mathfrak{v})$. Note that $\mathfrak{w} \triangleleft \top$ is never true, nor is $\xi < \acute{o}(\top) = 0$, so we may assume that $\mathfrak{v} \neq \top$. Then, if $\mathfrak{w} = \top$ it follows that $\acute{o}(\top) = 0$, so both sides are true. Hence we may also assume that $\mathfrak{w} \neq \top$.

By Lemma 4.24, $\mathfrak{w} \triangleleft \mathfrak{v}$ if and only if either $\mathfrak{w} \leq b(\mathfrak{v})$ or $b(\mathfrak{w}) \triangleleft \mathfrak{v}$ and $1 \downarrow h(\mathfrak{w}) \triangleleft 1 \downarrow h(\mathfrak{v})$. Observe that, by the induction hypothesis,

1. $\mathfrak{w} \leq b(\mathfrak{v})$ if and only if $\acute{o}(\mathfrak{w}) \leq \acute{o}b(\mathfrak{v})$, since

$$\|\mathfrak{w}\|_1 + \|b(\mathfrak{v})\|_1 < \|\mathfrak{w}\|_1 + \|\mathfrak{v}\|_1;$$

2. $b(\mathfrak{w}) \triangleleft \mathfrak{v}$ if and only if $\acute{o}b(\mathfrak{w}) < \acute{o}(\mathfrak{v})$, since

$$\|b(\mathfrak{w})\|_1 + \|\mathfrak{v}\|_1 < \|\mathfrak{w}\|_1 + \|\mathfrak{v}\|_1,$$

and

3. $1 \downarrow h(\mathfrak{w}) \triangleleft 1 \downarrow h(\mathfrak{v})$ if and only if $\acute{o}(1 \downarrow h(\mathfrak{w})) < \acute{o}(1 \downarrow h(\mathfrak{v}))$, since

$$\|1 \downarrow h(\mathfrak{w})\|_1 + \|1 \downarrow h(\mathfrak{v})\|_1 < \|\mathfrak{w}\|_1 + \|\mathfrak{v}\|_1.$$

This implies that $\mathfrak{w} \triangleleft \mathfrak{v}$ if and only if either $\acute{o}(\mathfrak{w}) \leq \acute{o}b(\mathfrak{v})$, or $\acute{o}b(\mathfrak{w}) < \acute{o}(\mathfrak{v})$ and $\acute{o}(1 \downarrow h(\mathfrak{w})) < \acute{o}(1 \downarrow h(\mathfrak{v}))$. But by Lemma 4.15.1, the latter is equivalent to

$$\acute{o}b(\mathfrak{w}) + \omega^{\acute{o}(1 \downarrow h(\mathfrak{w}))} < \acute{o}b(\mathfrak{v}) + \omega^{\acute{o}(1 \downarrow h(\mathfrak{v}))},$$

i.e., $\acute{o}(\mathfrak{w}) < \acute{o}(\mathfrak{v})$. □

It remains to check that the range of \acute{o} is ε_0 . We will use the following lemma:

Lemma 4.30. *For all $m < \omega$, $\acute{o}(m\top) < \varepsilon_0$.*

Proof. By induction on n ; if $n = 0$ then $\acute{o}(0\top) = 0 + \omega^0 = 1 < \varepsilon_0$. Otherwise, by induction hypothesis $\acute{o}(n\top) < \varepsilon_0$, so

$$\acute{o}(\langle n+1 \rangle \top) = \omega^{\acute{o}(n\top)} < \varepsilon_0,$$

as claimed. □

Lemma 4.31. *An ordinal ξ lies in the range of \acute{o} if and only if $\xi < \varepsilon_0$.*

Proof. First, assume that $\xi < \varepsilon_0$; we must find $\mathfrak{w} \sqsubset \omega$ such that $\xi = \acute{o}(\mathfrak{w})$. Proceed by induction on ξ . If $\xi = 0$, then $\xi = \acute{o}(\top)$. Otherwise, by Theorem 4.19, $\xi = \alpha + \omega^\beta$ for some $\alpha, \beta < \xi$. By the induction hypothesis, there are worms $\mathfrak{u}, \mathfrak{v}$ such that $\alpha = \acute{o}(\mathfrak{u})$ and $\beta = \acute{o}(\mathfrak{v})$, thus

$$\acute{o}((1 \uparrow \mathfrak{v}) 0 \mathfrak{u}) = \acute{o}(\mathfrak{u}) + \omega^{\acute{o}(\mathfrak{v})} = \alpha + \omega^\beta = \xi.$$

Next we check that if $\mathfrak{w} \sqsubset \omega$, then $\acute{o}(\mathfrak{w}) < \varepsilon_0$. Fix $M > \max \mathfrak{w}$; then, by Corollary 3.19.3, $\mathfrak{w} \triangleleft M\top$, so that $\acute{o}(\mathfrak{w}) \triangleleft \acute{o}(M\top)$. But by Lemma 4.30, $\acute{o}(M\top) < \varepsilon_0$, as claimed. \square

We now have all the necessary ingredients to show that $\acute{o} = o$.

Lemma 4.32. *For all $\mathfrak{w} \sqsubset \omega$, $o(\mathfrak{w}) = \acute{o}(\mathfrak{w})$.*

Proof. By Lemma 4.28, \acute{o} is well-defined on \mathbb{W}_ω , and by Lemmas 4.29 and 4.31, it is strictly increasing and initial. By Lemma 3.24, $o = \acute{o}$ on \mathbb{W}_ω . \square

Let us conclude this section by summarizing our main results:

Theorem 4.33. *The map $o: \mathbb{W}_\omega \rightarrow \varepsilon_0$ is surjective and satisfies*

1. $o(\top) = 0$, and
2. $o((1 \uparrow \mathfrak{v}) 0 \mathfrak{w}) = o(\mathfrak{w}) + \omega^{o(\mathfrak{v})}$.

Proof. Immediate from Lemma 4.32 and the definition of \acute{o} . \square

5 Transfinite worms

We have now seen that finite worms give a notation for ε_0 , the proof-theoretic ordinal of Peano arithmetic. However, stronger theories, including many important theories of reverse mathematics, have much larger proof-theoretic strength, suggesting that RC_ω is not suitable for their Π_1^0 ordinal analysis. Fortunately, Theorem 3.20 is valid even when worms have arbitrary ordinal entries. In this section, we will extend Theorem 4.33 to all of \mathbb{W} .

5.1 Subsystems of second-order arithmetic

Let us begin by discussing proof-theoretic interpretations of RC_Λ with $\Lambda > \omega$. It will be convenient to pass to the language Π_ω^1 of second-order arithmetic. This language extends that of first-order arithmetic with new variables X, Y, Z, \dots denoting sets of natural numbers, along with new atomic formulas $t \in X$ and second-order quantifiers $\forall X, \exists X$. As is standard, we may define $X \subseteq Y$ by $\forall x(x \in X \rightarrow x \in Y)$, and $X = Y$ by $X \subseteq Y \wedge Y \subseteq X$.

When working in a second-order context, we write Π_n^0 instead of Π_n (note that these formulas could contain second-order parameters, but no quantifiers over sets). The classes Σ_n^1, Π_n^1 are defined analogously to their first-order counterparts, but using alternating second-order quantifiers and setting $\Sigma_0^1 = \Pi_0^1 = \Delta_0^1 = \Pi_\omega^0$. It is well-known that every second-order formula is equivalent to another in one of the above forms.

When axiomatizing second-order arithmetic, the focus passes from induction to *comprehension*; that is, axioms stating the existence of sets whose elements satisfy a prescribed property. Some important axioms and schemes are:

Γ -CA: $\exists X \forall x (x \in X \leftrightarrow \phi(x))$, where $\phi \in \Gamma$ and X is not free in ϕ ;

Δ_1^0 -CA: $\forall x(\pi(w) \leftrightarrow \sigma(x)) \rightarrow \exists X \forall x (x \in X \leftrightarrow \sigma(x))$, where $\sigma \in \Sigma_1^0$, $\pi \in \Pi_1^0$, and X is not free in σ or π ;

Ind: $0 \in X \wedge \forall x (x \in X \rightarrow x + 1 \in X) \rightarrow \forall x (x \in X)$.

We mention one further axiom that requires a more elaborate setup. We may represent well-orders in second-order arithmetic as pairs of sets $\Lambda = \langle |\Lambda|, \leq_\Lambda \rangle$, and define

$$\text{WO}(\Lambda) = \text{linear}(\Lambda) \wedge \forall X \subseteq |\Lambda| (\exists x \in X \rightarrow \exists y \in X \forall z \in X y \leq_\Lambda z),$$

where $\text{linear}(\Lambda)$ is a formula expressing that Λ is a linear order.

Given a set X whose elements we will regard as ordered pairs $\langle \lambda, n \rangle$, let $X_{<_\Lambda \lambda}$ be the set of all $\langle \mu, n \rangle$ with $\mu <_\Lambda \lambda$. With this, we define the *transfinite recursion* scheme by

$$\text{TR}_\phi(X, \Lambda) = \forall \lambda \in |\Lambda| \forall n (n \in X \leftrightarrow \phi(n, X_{<_\Lambda \lambda})).$$

Intuitively, $\text{TR}_\phi(X, \Lambda)$ states that X is made up of “layers” indexed by elements of Λ , and the elements of the λ^{th} layer are those natural numbers n satisfying $\phi(n, X_{<_\Lambda \lambda})$, where $X_{<_\Lambda \lambda}$ is the union of all previous layers. If Γ is a set of formulas, we denote the Γ -*transfinite recursion* scheme by

$$\Gamma\text{-TR} = \left\{ \forall \Lambda (\text{WO}(\Lambda) \rightarrow \exists X \text{TR}_\phi(X, \Lambda)) : \phi \in \Gamma \right\}.$$

Now we are ready to define some important theories:

$$\begin{aligned}
 \text{ECA}_0 &: \quad \mathbf{Q}^+ + \text{Ind} + \Delta_0^0\text{-CA}; \\
 \text{RCA}_0^* &: \quad \mathbf{Q}^+ + \text{Ind} + \Delta_1^0\text{-CA}; \\
 \text{RCA}_0 &: \quad \mathbf{Q}^+ + \text{ISigma}_1^0 + \Delta_1^0\text{-CA}; \\
 \text{ACA}_0 &: \quad \mathbf{Q}^+ + \text{Ind} + \Sigma_1^0\text{-CA}; \\
 \text{ATR}_0 &: \quad \mathbf{Q}^+ + \text{Ind} + \Pi_\omega^0\text{-TR}; \\
 \Pi_1^1\text{-CA}_0 &: \quad \mathbf{Q}^+ + \text{Ind} + \Pi_1^1\text{-CA}.
 \end{aligned}$$

These are listed from weakest to strongest. The theories RCA_0 , ACA_0 , ATR_0 and $\Pi_1^1\text{-CA}_0$, together with the theory of *weak König's lemma*, WKL_0 , are the ‘Big Five’ theories of reverse mathematics, where RCA_0 functions as a ‘constructive base theory’, and the stronger four theories are all equivalent to many well-known theorems in mathematical analysis. For a detailed treatment of these and other subsystems of second-order arithmetic, see [36].

ECA_0 (the theory of *elementary comprehension*) is the second-order analogue of elementary arithmetic, and is a bit weaker than the more standard RCA_0^* . Meanwhile, *arithmetical comprehension* (ACA_0) is essentially the second-order version of PA, and has the same proof-theoretic ordinal, ε_0 . Thus the next milestone in the Π_1^0 ordinal analysis program is naturally ATR_0 , the theory of *arithmetical transfinite recursion*. Appropriately, the constructions we will use to interpret the modalities $\langle \lambda \rangle$ for countable $\lambda > \omega$ may be carried out within ATR_0 .

5.2 Iterated ω -rules

If we wish to interpret $[\lambda]_T \phi$ for transfinite λ , we need to consider a notion of provability that naturally extends beyond ω . One such notion, which is well-studied in proof theory (see, e.g., [33]), considers infinitary derivations with the ω -rule. Intuitively, this rule has the form

$$\frac{\phi(\bar{0}) \quad \phi(\bar{1}) \quad \phi(\bar{2}) \quad \phi(\bar{3}) \quad \phi(\bar{4}) \quad \dots}{\forall x \phi(x)}$$

The parameter λ in $[\lambda]_T \phi$ denotes the nesting depth of ω -rules that may be used for proving ϕ . The notion of λ -provability is defined as follows:

Definition 5.1. *Let T be a theory of second-order arithmetic and $\phi \in \Pi_\omega^1$. For an ordinal λ , we define $[\lambda]_T \phi$ recursively if either*

(i) $\Box_T \phi$, or

(ii) there are an ordinal $\mu < \lambda$ and a formula $\psi(x)$ such that

- (a) for all $n < \omega$, $[\mu]_T \psi(\bar{n})$, and
- (b) $\Box_T(\forall x \psi(x) \rightarrow \phi)$.

This notion can be formalized by representing ω -proofs as infinite trees, as presented by Arai [2] and Girard [22]. Here we will instead use the formalization of Joosten and I [19]. We use a set P as an *iterated provability class*, whose elements are codes of pairs $\langle \lambda, \varphi \rangle$, with λ a code for an ordinal and φ a code for a formula. The idea is that we want P to be a set of pairs $\langle \lambda, \varphi \rangle$ satisfying Definition 5.1 if we set $[\lambda]_T \varphi \leftrightarrow \langle \lambda, \varphi \rangle \in P$. Thus we may write $[\lambda]_P \varphi$ instead of $\langle \lambda, \varphi \rangle \in P$.

Definition 5.2. Fix a well-order Λ on \mathbb{N} . Say that a set P of natural numbers is an iterated provability class for Λ if it satisfies the expression

$$[\lambda]_P \varphi \leftrightarrow \left(\Box_T \varphi \vee \exists \psi \exists \xi <_\Lambda \lambda \left(\forall n \ [\xi]_P \psi(\bar{n}) \wedge \Box_T(\forall x \psi(x) \rightarrow \varphi) \right) \right).$$

Let $\text{IPC}_T^\Lambda(P)$ be a Π_ω^0 formula stating that P is an iterated provability class for Λ . Then, define

$$[\lambda]_T^\Lambda \phi := \forall P \left(\text{IPC}_T^\Lambda(P) \rightarrow [\lambda]_P \phi \right).$$

Note that $[\lambda]_T^\Lambda$ is a Π_1^1 formula. Alternately, one could define $[\lambda]_T^\Lambda$ as a Σ_1^1 formula, but the two definitions are equivalent due to the following.

Lemma 5.3.

1. It is provable in ACA_0 that if Λ is a countable well-order and P, Q are both iterated provability classes for Λ , then $P = Q$.
2. It is provable in ATR_0 that if Λ is a countable well-order, then there exists an iterated provability class for Λ .

The first claim is proven by considering two IPC's P, Q and showing by transfinite induction on λ that $[\lambda]_P \phi \leftrightarrow [\lambda]_Q \phi$; this induction is readily available in ACA_0 since the expression $[\lambda]_P \phi$ is arithmetical. For the second, we simply observe that the construction of an IPC is a special case of arithmetical transfinite recursion. See [19] for more details.

If we fix a computable well-order Λ and a theory T in the language of second-order arithmetic, we can readily define $\cdot_T^\Lambda: \mathcal{L}_\Lambda \rightarrow \Pi_\omega^1$ as in Definition 4.1, but setting $(\lambda \phi)_T^\Lambda = \langle \bar{\lambda} \rangle_T^\Lambda \phi_T$. We then obtain the following:

Theorem 5.4. Let Λ be a computable well-order and T be a theory extending ACA_0 such that it is provable in T that Λ is well-ordered, and that there is a set P satisfying $\text{IPC}_T^\Lambda(P)$.

Then, for any sequent $\phi \Rightarrow \psi$ of \mathcal{L}_Λ , $\text{RC} \vdash \phi \Rightarrow \psi$ if and only if $T \vdash \phi_T^\Lambda \rightarrow \psi_T^\Lambda$.

Proof. This is proven in [19] with GLP_Λ in place of RC_Λ , and this version is obtained by observing that GLP_Λ is conservative over RC_Λ by Theorem 2.8. \square

The computability condition in Λ is included due to the fact that in the proof of Theorem 5.4, we need to be able to prove properties about Λ within T ; for example, we need for

$$\forall x \forall y (x \leq_\Lambda y \rightarrow \Box_T(\dot{x} \leq_\Lambda \dot{y}))$$

to hold. However, we can drop this condition if we allow an *oracle* for Λ ; or, more generally, for any set of natural numbers. To do this, we add a set-constant O to the language of second-order arithmetic in order to ‘feed’ information about any set of numbers into T .

To be precise, given a theory T and $A \subseteq \mathbb{N}$, define $T|A$ to be the theory whose rules and axioms are those of T together with all instances of $\bar{n} \in O$ for $n \in X$, and all instances of $\bar{n} \notin O$ for $n \notin X$. Then, for any formula ϕ , we define

$$[\lambda|X]_T^\Lambda \phi = [\lambda]_{T|X}^\Lambda \phi.$$

Its dual, $\langle \lambda|X \rangle_T^\Lambda \phi$, is defined in the usual way. With this, we obtain an analogue of Theorem 4.3 for ATR_0 , proven by Cordón-Franco, Joosten, Lara-Martín and myself in [13]:

Theorem 5.5. $\text{ATR}_0 \equiv \text{ECA}_0 + \forall \Lambda \forall X \langle \lambda|X \rangle_T^\Lambda \top$.

This result may well be the first step in a consistency proof of ATR_0 in the style of Theorem 4.5. Moreover, the proof-theoretic strength of ATR_0 is measured by the Feferman-Schütte ordinal, Γ_0 . In the rest of this section, we will see how the worm ordering relates to this ordinal.

5.3 Ordering transfinite worms

Let us extend our calculus for computing o to worms that may contain transfinite entries. In Section 4, we used the operations b, h and $1 \downarrow$ to simplify worms and compute their order-types. However, this will not suffice for transfinite worms. For example, if $\mathfrak{w} = \omega 0 \omega \top$, we have that $h(\mathfrak{w}) = \omega \top$ while $b(\mathfrak{w}) = \omega \top$, both of which are shorter than ω . However,

$$1 \downarrow (\omega \top) = \langle -1 + \omega \rangle \top = \omega \top;$$

thus, demoting by 1 will not get us anywhere. Instead, we could demote by ω , and obtain $\omega \downarrow (\omega \top) = 0 \top$, which is indeed ‘simpler’. As we will see, this is the appropriate way to decompose infinite worms:

Lemma 5.6. *Given a worm $\mathfrak{w} \neq \top$, there exist unique $\mu < \Lambda$ and worms $\mathfrak{w}_1, \mathfrak{w}_0$ such that either $\mathfrak{w}_1 = \top$ or $0 < \min \mathfrak{w}_1$ and*

$$\mathfrak{w} = \mu \uparrow (\mathfrak{w}_1 \circ \mathfrak{w}_0).$$

Proof. Take $\mu = \min \mathfrak{w}$, $\mathfrak{w}_1 = h(\mu \downarrow \mathfrak{w})$ and $\mathfrak{w}_0 = b(\mu \downarrow \mathfrak{w})$; evidently these are the only possible values that satisfy the desired equation. \square

With this we may define the norm of a worm \mathfrak{w} , which roughly corresponds to the number of operations of 0-concatenation and μ -promotion needed to construct \mathfrak{w} .

Definition 5.7. *For $\mathfrak{w} \sqsubset \text{Ord}$ we define $\|\mathfrak{w}\|$ inductively by*

1. $\|\top\| = 0$;
2. if $\mathfrak{w} \neq \top$ and $\min \mathfrak{w} = 0$, set

$$\|\mathfrak{w}\| = \|h(\mathfrak{w})\| + \|b(\mathfrak{w})\| + 1;$$

3. otherwise, let $\mu = \min \mathfrak{w} > 0$, and set

$$\|\mathfrak{w}\| = \|\mu \downarrow \mathfrak{w}\| + 1.$$

The following is obvious from Definition 5.7 and Lemma 5.6:

Lemma 5.8. *For every worm \mathfrak{w} , $\|\mathfrak{w}\| \in \mathbb{N}$ is well-defined. Moreover, if $\mathfrak{w} = \alpha \uparrow (\mathfrak{w}_1 \circ \mathfrak{w}_0)$ with $0 < \min \mathfrak{w}_1$, then $\|\mathfrak{w}_1\|, \|\mathfrak{w}_0\| < \|\mathfrak{w}\|$.*

Thus we may try to compute $o(\mathfrak{w})$ by recursion on $\|\mathfrak{w}\|$. Assuming that the identity $o(\mathfrak{w}) = ob(\mathfrak{w}) + \omega^{o(1 \downarrow h(\mathfrak{w}))}$ remains valid for transfinite worms, we only have to find a way to compute $o(\mu \uparrow \mathfrak{w})$ in terms of $o(\mathfrak{w})$. Fortunately, the map $o(\mathfrak{w}) \mapsto o(\mu \uparrow \mathfrak{w})$ is well-defined; let us denote it by σ^μ .

Lemma 5.9. *There exists a unique family of functions $\vec{\sigma} = \langle \sigma^\xi \rangle_{\xi \in \text{Ord}}$ such that $\sigma^\xi: \text{Ord} \rightarrow \text{Ord}$ and, for every ordinal ξ and every worm \mathfrak{w} , $\sigma^\xi o(\mathfrak{w}) = o(\xi \uparrow \mathfrak{w})$.*

Proof. Given ordinals ξ, ζ , we need to see that there exists a unique ordinal ϑ such that $\vartheta = o(\xi \uparrow \mathfrak{w})$ whenever $\zeta = o(\mathfrak{w})$.

First observe that, by Corollary 3.27, there is some worm \mathfrak{w}_* such that $\zeta = o(\mathfrak{w}_*)$. Since by Theorem 3.20, the class of worms is well-ordered, $o(\xi \uparrow \mathfrak{w}_*)$ is well-defined. It remains to check that if \mathfrak{w} is an arbitrary worm such that $o(\mathfrak{w}) = \xi$, then also $o(\xi \uparrow \mathfrak{w}) = o(\xi \uparrow \mathfrak{w}_*)$. But if $o(\mathfrak{w}) = o(\mathfrak{w}_*)$, by Lemma 3.25 we have that $\mathfrak{w} \equiv \mathfrak{w}_*$, and thus by Lemma 4.22, $\xi \uparrow \mathfrak{w} \equiv \xi \uparrow \mathfrak{w}_*$. The latter implies that $o(\xi \uparrow \mathfrak{w}) = o(\xi \uparrow \mathfrak{w}_*)$, as needed. \square

Lemma 5.10. *The family of functions $\vec{\sigma}$ has the following properties:*

1. σ^α is strictly increasing for all α ;
2. $\sigma^0 \xi = \xi$, and
3. $\sigma^{\alpha+\beta} = \sigma^\alpha \sigma^\beta$.

Proof. For item 1, suppose that $\xi < \zeta$. If $\xi = o(\mathfrak{w})$ and $\zeta = o(\mathfrak{v})$, then by Lemma 3.23, $\mathfrak{w} \triangleleft \mathfrak{v}$, so that by Lemma 4.23, $\alpha \uparrow \mathfrak{w} \triangleleft \alpha \uparrow \mathfrak{v}$ and thus $o(\alpha \uparrow \mathfrak{w}) < o(\alpha \uparrow \mathfrak{v})$; a similar argument shows that if $o(\mathfrak{w}) \leq o(\mathfrak{v})$, then $o(\alpha \uparrow \mathfrak{w}) \leq o(\alpha \uparrow \mathfrak{v})$. Item 2 follows from the fact that $0 \uparrow \mathfrak{w} = \mathfrak{w}$ for all \mathfrak{w} , so if $\zeta = o(\mathfrak{w})$ we have that $\sigma^0 \zeta = o(0 \uparrow \mathfrak{w}) = \zeta$.

Item 3 is immediate from Lemma 4.21.(ii), since if $o(\mathfrak{w}) = \zeta$ then $o(\beta \uparrow \mathfrak{w}) = \sigma^\beta(\zeta)$, which means that

$$o(\alpha \uparrow (\beta \uparrow \mathfrak{w})) = \sigma^\alpha \sigma^\beta(\zeta).$$

But, on the other hand, $\alpha \uparrow (\beta \uparrow \mathfrak{w}) = (\alpha + \beta) \uparrow \mathfrak{w}$, and

$$o((\alpha + \beta) \uparrow \mathfrak{w}) = \sigma^{\alpha+\beta} \zeta,$$

and we conclude that $\sigma^{\alpha+\beta} \zeta = \sigma^\alpha \sigma^\beta \zeta$. □

Observe also that if $\zeta < \varepsilon_0$, then by Theorem 4.33, there is $\mathfrak{w} \sqsubset \omega$ such that $\zeta = o(\mathfrak{w})$, and hence by Theorem 4.33, $\sigma^1 \zeta = o(1 \uparrow \mathfrak{w}) = -1 + \omega^{o(\mathfrak{w})}$ (where we subtract 1 to account for the case $\mathfrak{w} = \top$). Thus for $\zeta < \varepsilon_0$, $\sigma^1 \zeta = -1 + \omega^\zeta$. It is thus natural to conjecture that $\sigma^1 \zeta = -1 + \omega^\zeta$ for all ζ . In the next section we will discuss how a family of ordinal functions satisfying these properties can be constructed, and show that they are closely related to the Feferman-Schütte ordinal Γ_0 .

5.4 Hyperations and the Feferman-Schütte ordinal

Beklemishev has shown how provability algebras give rise to a notation system for Γ_0 . Such ordinals are usually presented using Veblen progressions [37], but alternatively they may be defined through *hyperations*, which are more convenient in our present context.

Definition 5.11. *Let f be a normal function. Then, we define the hyperation of f to be the unique family of normal functions $\langle f^\zeta \rangle_{\zeta \in \text{On}}$ such that*

$$(i) \quad f^1 = f$$

- (ii) $f^{\alpha+\beta} = f^\alpha f^\beta$ for all ordinals α, β
- (iii) $\langle f^\zeta \rangle_{\zeta \in \text{On}}$ is pointwise minimal amongst all families of normal functions satisfying the above clauses³.

It is not obvious that such a family of functions exists, but a detailed construction is given by Joosten and myself in [17]. It is also shown there that they may be computed by the following recursion:

Lemma 5.12. *Let f be a normal function such that $f(0) = 0$. Then, given ordinals λ, μ ,*

- (i) $f^0 \mu = \mu$;
- (ii) $f^{\lambda+1} \mu = f^\lambda f \mu$;
- (iii) if μ is a limit, $f^\lambda \mu = \lim_{\xi < \mu} f^\lambda \xi$;
- (iv) if λ is a limit, $f^\lambda(\mu + 1) = \lim_{\xi < \lambda} f^\xi(f^\lambda(\mu) + 1)$.

Although each function f^ξ is normal, the function $\xi \mapsto f^\xi \mu$ typically is not, even when $\mu = 0$, since if $f(0) = 0$ then it follows that $f^\xi 0 = 0$ for all ξ . However, when $f(0) > 0$ then $\xi \mapsto f^\xi 0$ is normal, and more generally, we have the following:

Lemma 5.13. *Assume that $f: \text{Ord} \rightarrow \text{Ord}$ is normal and suppose that μ is the least ordinal such that $f(\mu) > \mu$ (if it exists).*

Then, the function $\xi \mapsto f^\xi \mu$ is normal, and for all ξ , $f^\xi \upharpoonright \mu$ is the identity (where \upharpoonright denotes domain restriction).

We omit the proof which proceeds by transfinite induction using Lemma 5.12. We are particularly interested in hyperating $e(\xi) = -1 + \omega^\xi$; the family of functions $\langle e^\xi \rangle_{\xi \in \text{Ord}}$ are the *hyperexponentials*. Observe that, in view of Lemma 5.13, $e^\xi 0 = 0$ for all ξ and the function $\xi \mapsto e^\xi 1$ is normal. Aside from the clauses mentioned above, we remark that to entirely determine the value of $e^\lambda \mu$ we need the additional clause

$$e^1(\mu + 1) = \lim_{n < \omega} ((1 + e^1 \mu) \cdot n);$$

this follows directly from the definitions of ordinal exponentiation and the function e .

Aguilera and I proved the following in [1]:

³That is, if $\langle g^\zeta \rangle_{\zeta \in \text{On}}$ is a family of functions satisfying conditions (i) and (ii), then for all ordinals ξ, ζ , $f^\zeta \xi \leq g^\zeta \xi$.

Proposition 5.14. *For every ordinal $\xi > 0$, there exist unique ordinals α, β such that β is 1 or additively decomposable and $\xi = e^\alpha \beta$.*

We call α above the *degree of indecomposability* of ξ ; in particular, if ξ is already additively decomposable, then $\alpha = 0$. More generally, $e^\alpha \beta$ is always additively indecomposable if $\alpha, \beta > 0$, since

$$e^\alpha \beta = e e^{-1+\alpha} \beta = -1 + \omega^{e^{-1+\alpha} \beta} = \omega^{e^{-1+\alpha} \beta}.$$

Note that by writing β as a sum of indecomposables we may iterate this lemma and thus write any ordinal in terms of $e, +, 0$ and 1 . This form is unique if we do not allow sums of the form $\xi + \eta$ where $\xi + \eta = \eta$.

We will not review Veblen progressions here; however, as these are more standard than hyperexponentials, we remark that notations using hyperexponentials or Veblen functions can be easily translated from one to the other using the following proposition. Below, φ_α denotes the Veblen functions as defined in [33].

Proposition 5.15. *Given ordinals α, β ,*

1. $e^\alpha(0) = 0$,
2. $e^1(1 + \beta) = \varphi_0(1 + \beta)$,
3. $e^{\omega^{1+\alpha}}(1 + \beta) = \varphi_{1+\alpha}(\beta)$.

The proof can be found in [17]. We have seen that every ordinal $\xi < \varepsilon_0$ can be written as a sum of the form $\alpha + \omega^\beta$ with $\alpha, \beta < \xi$. In general, it is desirable in any ordinal notation system that, if we have a notation for an additively indecomposable ξ , then we also have notations for ordinals $\alpha, \beta < \xi$ such that $\alpha + \beta = \xi$. If instead ξ is additively indecomposable, it is also convenient to have notations for α, β such that $\xi = e^\alpha \beta$ (although we cannot always guarantee that $\alpha < \xi$). The following definition captures these properties.

Definition 5.16. *Let Θ be a set of ordinals.*

1. *We say that Θ is additively reductive if whenever ξ is additively decomposable, we have that $\xi \in \Theta$ if and only if there are $\alpha, \beta \in \xi \cap \Theta$ such that $\xi = \alpha + \beta$.*
2. *We say that Θ is hyperexponentially reductive if whenever $\xi > 1$ is additively indecomposable, we have that $\xi \in \Theta$ if and only if there are $\alpha, \beta \in \Theta$ such that $\beta < \xi$ and $\xi = e^\alpha \beta$.*
3. *We say that Θ is reductive if it is additively and hyperexponentially reductive.*

Additively reductive sets of ordinals always contain Cantor decompositions of their elements and are closed under left subtraction by *arbitrary* ordinals:

Lemma 5.17. *Let Θ be an additively reductive set of ordinals such that $0 \in \Theta$. Then:*

1. *If $0 \neq \xi \in \Theta$ is arbitrary, there are ordinals α, β such that $\alpha, \omega^\beta \in \Theta$ and $\xi = \alpha + \omega^\beta$.*
2. *If $\beta \in \Theta$ and $\alpha < \beta$ (not necessarily a member of Θ), then $-\alpha + \beta \in \Theta$.*

Proof. For the first claim, if ξ is additively indecomposable there is nothing to do, since we already have that $\xi = \omega^\beta$ for some β . Otherwise, using the assumption that Θ is additively reductive, write $\xi = \gamma + \delta$ with $\gamma, \delta \in \xi \cap \Theta$.

By the induction hypothesis applied to δ , there are η, β such that $\eta, \omega^\beta \in \Theta$ and $\delta = \eta + \omega^\beta$. Again using the assumption that Θ is additively reductive, we may set $\alpha = \gamma + \eta \in \Theta$, and see that $\xi = \alpha + \omega^\beta$.

Now we prove the second item by induction on ξ . We may assume that β is additively indecomposable, since otherwise $-\alpha + \beta \in \{0, \beta\} \subseteq \Theta$. Thus we may write $\beta = \gamma + \delta$ with $\gamma, \delta \in \beta \cap \Theta$. If $\alpha \leq \gamma$, by the induction hypothesis $-\alpha + \gamma \in \Theta$, and thus $-\alpha + \beta = (-\alpha + \gamma) + \delta \in \Theta$. Otherwise, also by the induction hypothesis applied to $\delta < \xi$,

$$-\alpha + \beta = -(-\gamma + \alpha) + \delta \in \Theta. \quad \square$$

Meanwhile, hyperexponentially reductive sets of ordinals always contain hyperexponential normal forms for their elements:

Lemma 5.18. *If Θ contains 0 and is hyperexponentially reductive, then for every $\xi \in \Theta$, there are $\alpha, \beta \in \Theta$ such that $\beta = 1$ or is additively decomposable, and $\xi = e^\alpha \beta$.*

Proof. By induction on ξ ; if ξ is additively decomposable or 1 then $\xi = e^0 \xi$, otherwise there are $\alpha', \beta' \in \Theta$ with $\beta' < \xi$ such that $\xi = e^{\alpha'} \beta'$. By induction hypothesis there are $\gamma, \beta \in \Theta$ such that $\beta = 1$ or is additively decomposable and $\beta' = e^\gamma \beta$. Setting $\alpha = \alpha' + \gamma$, we see that $\xi = e^\alpha \beta$, as desired. \square

The ordinal Γ_0 can be constructed by closing $\{0, 1\}$ under addition and hyperexponentiation, or more succinctly by the function $\alpha, \beta, \gamma \mapsto e^\alpha(\beta + \gamma)$. In fact, Γ_0 is the least *hyperexponentially perfect* set, in the sense of the following definition:

Definition 5.19. Define a function $\text{HE}: \text{Ord}^3 \rightarrow \text{Ord}$ by

$$\text{HE}(\alpha, \beta, \gamma) = e^\alpha(\beta + \gamma).$$

Given a set of ordinals Θ , say that Θ is hyperexponentially closed if $2 \cup \text{HE}[\Theta] \subseteq \Theta$. We say that Θ is hyperexponentially perfect if it is reductive and hyperexponentially closed.

It is easy to see that Θ is hyperexponentially perfect if and only if it is reductive and $0, 1 \in \Theta$. Note also that hyperexponentially closed sets are closed under both addition and hyperexponentiation:

Lemma 5.20. If $0 \in \Theta$ and $\alpha, \beta \in \Theta$, then $\alpha + \beta, e^\alpha \beta \in \text{HE}[\Theta]$.

Proof. If Θ is hyperexponentially closed then by definition we have that $0 \in \Theta$, hence if $\alpha, \beta \in \Theta$, $\alpha + \beta = e^0(\alpha + \beta) \in \text{HE}[\Theta]$ and $e^\alpha \beta = e^\alpha(\beta + 0) \in \text{HE}[\Theta]$. \square

With this, we are ready to define the ordinal Γ_0 :

Theorem 5.21. Let $\Gamma_0 = \overline{2}^{\text{HE}}$. Then, Γ_0 is an ordinal and for every $\xi < \Gamma_0$ with $\xi > 1$, there are ordinals $\alpha, \beta, \gamma < \xi$ such that $\xi = e^\alpha(\beta + \gamma)$.

Proof. The proof closely mimics that of Theorem 4.19. First we will show that if $1 < \xi \in \Gamma_0$, then there are $\alpha, \beta, \gamma < \xi$ such that $\xi = e^\alpha(\beta + \gamma)$. By Lemma 4.17.3, there are $\alpha, \beta, \gamma \in \Gamma_0$ with $\alpha, \beta, \gamma \neq \xi$ and such that $\xi = e^\alpha(\beta + \gamma)$. Since $\xi \neq 0$ it follows that $\beta + \gamma \geq 1$, and since the function e^α is normal, $\beta + \gamma \leq \xi$, from which we obtain $\beta, \gamma < \xi$. Similarly, $\alpha \leq \xi$ since $e^\alpha(\beta + \gamma) \geq e^\alpha 1$ and the function $\alpha \mapsto e^\alpha 1$ is normal. Thus we also have $\alpha < \xi$.

Next we show that Γ_0 is transitive. We proceed by induction on ζ with a secondary induction on ξ to show that $\xi < \zeta \in \Gamma_0$ implies that $\xi \in \Gamma_0$. We may without loss of generality assume that $\xi, \zeta > 1$. Write $\zeta = e^\alpha(\beta + \gamma)$ with $\alpha, \beta, \gamma \in \Gamma_0 \cap \zeta$. Then, using Proposition 5.14, write $\xi = e^\lambda \mu$ with $\mu = 1$ or additively decomposable.

Now consider two cases. If $\lambda = 0$, we have that $\mu = \xi > 1$, hence ξ is additively decomposable and we can write $\xi = \nu + \eta$, with $\nu, \eta < \xi$. By the secondary induction hypothesis, $\nu, \eta \in \Gamma_0$, hence $\xi = e^0(\nu + \eta) \in \Gamma_0$.

Otherwise, $\lambda > 0$, and we consider two subcases. If $\alpha \geq \lambda$, by the induction hypothesis applied to $\alpha < \zeta$, $\lambda \in \Gamma_0$. But $\xi > 1$ and is additively indecomposable, while $\mu \leq \xi$ is 1 or additively decomposable, so $\mu < \xi$. By the secondary induction hypothesis, $\mu \in \Gamma_0$, hence $\xi = e^\lambda \mu \in \Gamma_0$. If instead $\alpha < \lambda$, we observe that $e^\lambda \mu = e^\alpha e^{-\alpha+\lambda} \mu$, and by normality of e^α , $e^{-\alpha+\lambda} \mu < \beta + \gamma$. Since $e^{-\alpha+\lambda} \mu$ is additively indecomposable, it follows that $e^{-\alpha+\lambda} \mu \leq \max\{\beta, \gamma\}$, so that by the induction hypothesis applied to $\max\{\beta, \gamma\} < \zeta$, we have that $e^{-\alpha+\lambda} \mu \in \Gamma_0$. Since $\alpha \in \Gamma_0$, $\xi = e^\alpha e^{-\alpha+\lambda} \mu \in \Gamma_0$. \square

Thus Γ_0 can be characterized as the least hyperexponentially closed ordinal, or alternatively the least hyperexponentially perfect ordinal. Later we will see that it can also be obtained using worms, by closing under o .

5.5 Order-types of transfinite worms

As in Section 4.3, our strategy for giving a calculus for computing o will be to guess a candidate function and prove that it has the required properties. Let us assume that Theorem 4.33 remains true for transfinite worms. Moreover, note that the functions e^ξ satisfy all desired properties of our functions σ^ξ . Thus we will conjecture that $e^\xi = \sigma^\xi$ for every ordinal ξ , and propose the following candidate:

Definition 5.22. *Let Λ be an ordinal, $\mathfrak{v}, \mathfrak{w} \in \mathbb{W}^\Lambda$ be worms and $\alpha < \Lambda$ an ordinal. Then, define*

1. $\hat{o}(\top) = 0$,
2. $\hat{o}(\mathfrak{w}) = \hat{o}b(\mathfrak{w}) + \omega^{\hat{o}(1 \downarrow h(\mathfrak{w}))}$ if $\mathfrak{w} \neq \top$ and $\min \mathfrak{w} = 0$,
3. $\hat{o}(\mathfrak{w}) = e^\mu \hat{o}(\mu \downarrow \mathfrak{w})$ if $\mathfrak{w} \neq \top$ and $\mu = \min \mathfrak{w} > 0$.

The next few lemmas establish that \hat{o} behaves as it should.

Lemma 5.23. *If $\mathfrak{w} \neq \top$ is any worm, then $\hat{o}(\mathfrak{w}) \neq 0$.*

Proof. If $\min \mathfrak{w} = 0$, this is obvious since $\omega^\xi > 0$ independently of ξ . Otherwise, $\hat{o}(\mathfrak{w}) = e^\mu \hat{o}(\mu \downarrow \mathfrak{w})$ with $\mu = \min \mathfrak{w} > 0$. But $\min \mu \downarrow \mathfrak{w} = 0$, so by the previous case $\hat{o}(\mu \downarrow \mathfrak{w}) \neq 0$ and hence $\hat{o}(\mathfrak{w}) = e^\mu \hat{o}(\mu \downarrow \mathfrak{w}) \neq 0$. \square

Lemma 5.24. *For any worm $\mathfrak{w} \neq \top$, $\hat{o}(\mathfrak{w}) = \hat{o}b(\mathfrak{w}) + \omega^{\hat{o}(1 \downarrow h(\mathfrak{w}))}$.*

Proof. If $\min \mathfrak{w} = 0$, there is nothing to prove. Otherwise, $\min \mathfrak{w} > 0$, so we can write $\min \mathfrak{w} = 1 + \eta$ for some η . Moreover, $h(\mathfrak{w}) = \mathfrak{w}$ and $b(\mathfrak{w}) = \top$, so $\hat{o}h(\mathfrak{w}) = \hat{o}(\mathfrak{w}) \neq 0$ and $\hat{o}b(\mathfrak{w}) = 0$. Meanwhile, $(1 + \eta) \downarrow \mathfrak{w} \neq \top$, so $\hat{o}((1 + \eta) \downarrow \mathfrak{w}) \neq 0$ and thus $e^\eta \hat{o}((1 + \eta) \downarrow \mathfrak{w}) > 0$, from which it follows that

$$-1 + \omega^{e^\eta(\hat{o}((1+\eta)\downarrow\mathfrak{w}))} = \omega^{e^\eta(\hat{o}((1+\eta)\downarrow h(\mathfrak{w}))}. \quad (2)$$

Finally, observe that

$$\hat{o}(1 \downarrow h(\mathfrak{w})) = e^\eta \hat{o}(\eta \downarrow (1 \downarrow h(\mathfrak{w}))) = e^\eta \hat{o}((1 + \eta) \downarrow h(\mathfrak{w})). \quad (3)$$

Putting all of this together,

$$\begin{aligned}
 \hat{o}(\mathfrak{w}) &= e^{1+\eta} \hat{o}((1+\eta) \downarrow \mathfrak{w}) && \text{by definition} \\
 &= 0 + ee^\eta (\hat{o}((1+\eta) \downarrow \mathfrak{w})) && \text{since } e^{1+\eta} = ee^\eta \\
 &= \hat{o}(b(\mathfrak{w})) + ee^\eta (\hat{o}((1+\eta) \downarrow \mathfrak{w})) && \text{since } \hat{o}(b(\mathfrak{w})) = 0 \\
 &= \hat{o}(b(\mathfrak{w})) + (-1 + \omega^{e^\eta (\hat{o}((1+\eta) \downarrow \mathfrak{w}))}) && \text{by definition of } e \\
 &= \hat{o}(b(\mathfrak{w})) + \omega^{e^\eta (\hat{o}((1+\eta) \downarrow \mathfrak{w}))} && \text{by (2)} \\
 &= \hat{o}(b(\mathfrak{w})) + \omega^{\hat{o}(1 \downarrow h(\mathfrak{w}))} && \text{by (3),}
 \end{aligned}$$

as claimed. \square

Lemma 5.25. *For any worm \mathfrak{w} and ordinal λ , $\hat{o}(\lambda \uparrow \mathfrak{w}) = e^\lambda \hat{o}(\mathfrak{w})$.*

Proof. If $\mathfrak{w} = \top$, then

$$\hat{o}(\lambda \uparrow \top) = \hat{o}(\top) = 0 = e^\lambda 0 = e^\lambda \hat{o}(\top).$$

Otherwise, $\mathfrak{w} \neq \top$. If $\lambda = 0$ the lemma follows from the fact that $0 \uparrow \mathfrak{w} = \mathfrak{w}$ and e^0 is the identity, and if $\min \mathfrak{w} = 0$ then $\min(\lambda \uparrow \mathfrak{w}) = \lambda$ and

$$\hat{o}(\lambda \uparrow \mu) = e^\lambda \hat{o}(\lambda \downarrow (\lambda \uparrow \mathfrak{w})) = e^\lambda \hat{o}(\mathfrak{w}).$$

If not, let $\mu = \min \mathfrak{w} > 0$, so that $\hat{o}(\mathfrak{w}) = e^\mu (\mu \downarrow \mathfrak{w})$. Observe that $\min(\lambda \uparrow \mathfrak{w}) = \lambda + \mu$. Hence,

$$\begin{aligned}
 \hat{o}(\lambda \uparrow \mathfrak{w}) &= e^{\lambda+\mu} ((\lambda + \mu) \downarrow (\lambda \uparrow \mathfrak{w})) \\
 &= e^{\lambda+\mu} (\mu \downarrow (\lambda \downarrow (\lambda \uparrow \mathfrak{w}))) \\
 &= e^\lambda e^\mu (\mu \downarrow \mathfrak{w}) \\
 &= e^\lambda \hat{o}(\mathfrak{w}),
 \end{aligned}$$

as claimed. \square

With this we can prove that \hat{o} is strictly increasing and initial.

Lemma 5.26. *The map $\hat{o}: \mathbb{W} \rightarrow \text{Ord}$ is strictly increasing.*

Proof. We proceed by induction on $\|\mathfrak{w}\| + \|\mathfrak{v}\|$ to show that $\mathfrak{w} \triangleleft \mathfrak{v}$ if and only if $\hat{o}(\mathfrak{w}) < \hat{o}(\mathfrak{v})$. If $\mathfrak{w} = \top$ the claim is immediate from Lemma 5.23, so we assume otherwise. Note that in this case $\mathfrak{w} \triangleright \top$ and $\hat{o}(\mathfrak{w}) > \hat{o}(\top)$, so we may also assume that $\mathfrak{v} \neq \top$.

Thus we consider $\mathfrak{w}, \mathfrak{v} \neq \top$, and define $\mu = \min(\mathfrak{w}\mathfrak{v})$. If $\mu = 0$, we observe that either $\|h(\mathfrak{w})\| < \|\mathfrak{w}\|$ or $\|h(\mathfrak{v})\| < \|\mathfrak{v}\|$, and we can proceed exactly as in the proof of Lemma 4.29. Thus we consider only the case for $\mu > 0$.

Note that in this case we have that

$$\|\mu \downarrow \mathfrak{w}\| + \|\mu \downarrow \mathfrak{v}\| < \|\mathfrak{w}\| + \|\mathfrak{v}\|,$$

so we may apply the induction hypothesis to $\mu \downarrow \mathfrak{w}$ and $\mu \downarrow \mathfrak{v}$. Hence we obtain:

$$\begin{aligned} \mathfrak{w} \triangleleft \mathfrak{v} &\Leftrightarrow (\mu \downarrow \mathfrak{w}) \triangleleft (\mu \downarrow \mathfrak{v}) && \text{by Lemma 4.23} \\ &\Leftrightarrow \hat{o}(\mu \downarrow \mathfrak{w}) < \hat{o}(\mu \downarrow \mathfrak{v}) && \text{by induction hypothesis} \\ &\Leftrightarrow e^\mu \hat{o}(\mu \downarrow \mathfrak{w}) < e^\mu \hat{o}(\mu \downarrow \mathfrak{v}) && \text{by normality of } e^\mu \\ &\Leftrightarrow \hat{o}\mathfrak{w} < \hat{o}\mathfrak{v} && \text{by Lemma 5.25,} \end{aligned}$$

as needed. \square

Lemma 5.27. *The map $\hat{o}: \mathbb{W} \rightarrow \text{Ord}$ is surjective.*

Proof. Proceed by induction on $\xi \in \text{Ord}$ to show that there is \mathfrak{w} with $\hat{o}(\mathfrak{w}) = \xi$. For the base case, $\xi = 0 = \hat{o}(\top)$. Otherwise, by Proposition 5.14, ξ can be written in the form $e^\alpha \beta$ with β additively decomposable or 1. Write $\beta = \gamma + \omega^\delta$, so that $\gamma, \delta < \beta \leq \xi$. By the induction hypothesis, there are worms $\mathfrak{u}, \mathfrak{v}$ such that $\hat{o}(\mathfrak{u}) = \gamma$ and $\hat{o}(\mathfrak{v}) = \delta$. Then, $\xi = \hat{o}(\alpha \uparrow ((1 \uparrow \mathfrak{v}) 0 \mathfrak{u}))$, as needed. \square

Lemma 5.28. *For every worm \mathfrak{w} , $\hat{o}(\mathfrak{w}) = o(\mathfrak{w})$.*

Proof. Immediate from Lemmas 5.26 and 5.27 using Lemma 3.24. \square

Before giving the definitive version of our calculus, let us show that the clause for $\mathfrak{w} 0 \mathfrak{v}$ can be simplified somewhat.

Lemma 5.29. *Given arbitrary worms $\mathfrak{w}, \mathfrak{v}$, $o(\mathfrak{w} 0 \mathfrak{v}) = o(\mathfrak{v}) + 1 + o(\mathfrak{w})$.*

Proof. Observe that by Lemma 5.25 together with Lemma 5.28, we have that for any worm \mathfrak{u} , $o(1 \uparrow \mathfrak{u}) = eo(\mathfrak{u}) = -1 + \omega^{o(\mathfrak{u})}$, so that

$$\omega^{o(\mathfrak{u})} = 1 + o(1 \uparrow \mathfrak{u}). \quad (4)$$

With this in mind, proceed by induction on $\#\mathfrak{v} + \#\mathfrak{w}$ to prove the lemma. First consider the case where $0 < \min \mathfrak{v}$. In this case, $h(\mathfrak{v} 0 \mathfrak{w}) = \mathfrak{v}$, so that

$$o(\mathfrak{v} 0 \mathfrak{w}) = o(\mathfrak{w}) + \omega^{o(1 \downarrow \mathfrak{v})} = o(\mathfrak{w}) + 1 + o(\mathfrak{v}),$$

where the first equality is by Definition 5.22 and the second follows from (4).

If \mathfrak{v} does contain a zero, we have that $\mathfrak{v} = h(\mathfrak{v}) \dot{0} b(\mathfrak{v})$, so that

$$\mathfrak{v} \dot{0} \mathfrak{w} = h(\mathfrak{v}) \dot{0} b(\mathfrak{v}) \dot{0} \mathfrak{w}.$$

This means that $h(\mathfrak{v} \dot{0} \mathfrak{w}) = h(\mathfrak{v})$ and $b(\mathfrak{v} \dot{0} \mathfrak{w}) = b(\mathfrak{v}) \dot{0} \mathfrak{w}$. Applying the induction hypothesis to $b(\mathfrak{v}) \dot{0} \mathfrak{w}$, we obtain

$$ob(\mathfrak{v} \dot{0} \mathfrak{w}) = o(\mathfrak{w}) + 1 + ob(\mathfrak{v}),$$

and thus

$$\begin{aligned} o(\mathfrak{v} \dot{0} \mathfrak{w}) &= ob(\mathfrak{w} \dot{0} \mathfrak{v}) + \omega^{o(1 \downarrow h(\mathfrak{v}))} \\ &\stackrel{\text{IH}}{=} o(\mathfrak{w}) + 1 + ob(\mathfrak{v}) + \omega^{o(1 \downarrow h(\mathfrak{v}))} = o(\mathfrak{w}) + 1 + o(\mathfrak{v}), \end{aligned}$$

as needed. \square

Let us put our results together to give our definitive calculus for o .

Theorem 5.30. *Let $\mathfrak{v}, \mathfrak{w}$ be worms and α be an ordinal. Then,*

1. $o(\top) = 0$,
2. $o(\mathfrak{v} \dot{0} \mathfrak{w}) = o(\mathfrak{w}) + 1 + o(\mathfrak{v})$, and
3. $o(\alpha \uparrow \mathfrak{w}) = e^\alpha o(\mathfrak{w})$.

Proof. The first item is immediate from Definition 5.22, the second from Lemma 5.29, and the third from Lemma 5.25, respectively, using the fact that $o = \hat{o}$ by Lemma 5.28. \square

Note that Theorem 5.30 can be applied to any worm \mathfrak{w} , and hence it gives a complete calculus for computing o . Next, let us see how this gives rise to a notation system for Γ_0 .

5.6 Beklemishev's predicative worms

Now we review results from [5] showing that Γ_0 is the least set definable by iteratively taking order-types of worms. Let us begin by discussing the properties of sets of worms obtained from additively reductive sets of ordinals. Recall that $\mathfrak{w} \sqsubset \Theta$ means that every ordinal appearing in \mathfrak{w} belongs to Θ .

Lemma 5.31. *Let Θ be an additively reductive set of ordinals such that $0 \in \Theta$, and let $\mathfrak{w} \sqsubset \Theta$. Then,*

1. If $\mu \in \Theta$, $\mu \uparrow \mathfrak{w} \sqsubset \Theta$, and
2. if $\mu \leq \mathfrak{w}$ is arbitrary, then $\mu \downarrow \mathfrak{w} \sqsubset \Theta$.

Proof. Suppose that $\mathfrak{w} = \lambda_1 \dots \lambda_n \top \sqsubset \Theta$. If $\mu \in \Theta$, using the fact that Θ is closed under addition, for each $i \in [1, n]$ we have that $\mu + \lambda_i \in \Theta$. Thus $\mu \uparrow \mathfrak{w} \sqsubset \Theta$.

Similarly, by Lemma 5.17.2, if μ is arbitrary then $-\mu + \lambda_i \in \Theta$ for each $i \in [1, n]$, so $\mu \downarrow \mathfrak{w} \sqsubset \Theta$. \square

Now, let us make the notion of “closing under o ” precise.

Definition 5.32. Observe that o may be regarded as a function $o: \text{Ord}^{<\omega} \rightarrow \text{Ord}$ by setting

$$o(\mu_1, \dots, \mu_n) = o(\mu_1 \dots \mu_n \top).$$

Then, given a set of ordinals Θ , if $o[\Theta] \subseteq \Theta$ we say that Θ is worm-closed, and if $\Theta = o[\Theta]$ we say that Θ is worm-perfect.

Even when Θ is not worm-perfect, sets of the form $o[\Theta]$ are rather well-behaved:

Lemma 5.33. If Θ is any set of ordinals, then $0 \in o[\Theta]$. If moreover $0 \in \Theta$, then also $1 \in o[\Theta]$, and $o[\Theta]$ is additively reductive.

Proof. Observe that $0 = o(\top)$, and $\top \sqsubset \Theta$ since \top contains no ordinals, so $0 \in o[\Theta]$. Similarly, $1 = o(0\top)$, and $0\top \sqsubset \Theta$ if $0 \in \Theta$.

Let us see that $o[\Theta]$ is additively reductive when $0 \in \Theta$. First assume that $\alpha, \beta \in o[\Theta]$. Then, there are worms $\mathfrak{u}, \mathfrak{v} \sqsubset \Theta$ such that $\alpha = o(\mathfrak{u})$ and $\beta = o(\mathfrak{v})$. If $\beta \geq \omega$, then

$$o(\mathfrak{v} \ 0 \ \mathfrak{u}) = o(\mathfrak{u}) + 1 + o(\mathfrak{v}) = \alpha + 1 + \beta = \alpha + \beta,$$

otherwise

$$o(\langle 0 \rangle^\beta \mathfrak{u}) = o(\mathfrak{u}) + \beta = \alpha + \beta,$$

where we define $\langle \lambda \rangle^n = \underbrace{\langle \lambda \rangle \dots \langle \lambda \rangle}_{n \text{ times}}$. Both $\mathfrak{v} \ 0 \ \mathfrak{u}, \langle 0 \rangle^\beta \mathfrak{u} \sqsubset \Theta$, so $\alpha + \beta \in o[\Theta]$.

Conversely, if $\xi \in o[\Theta]$ is additively decomposable, write $\xi = o(\mathfrak{w})$. Then, $\xi = ob(\mathfrak{w}) + 1 + oh(\mathfrak{w})$, and since $1 + oh(\mathfrak{w})$ is additively indecomposable, we have that $\xi \neq 1 + oh(\mathfrak{w})$ and hence $ob(\mathfrak{w}), 1 + oh(\mathfrak{w}) < \xi$. Clearly $ob(\mathfrak{w}) \in o[\Theta]$, while $1 + oh(\mathfrak{w})$ is either 1 or $oh(\mathfrak{w})$, both of which belong to $o[\Theta]$. \square

Lemma 5.34. Let Θ be any set of ordinals. Then, Θ is worm-perfect if and only if it is hyperexponentially perfect.

Proof. Assume first that Θ is worm-perfect. By Lemma 5.33, $0 \in \Theta$, thus also $1 \in \Theta$ and Θ is additively reductive. It remains to prove that $\text{HE}[\Theta] \subseteq \Theta$ and that Θ is hyperexponentially reductive.

To show that $\text{HE}[\Theta] \subseteq \Theta$, it suffices to check that $e^\alpha \beta \in \Theta$ whenever $\alpha, \beta \in \Theta$, given that we already know that Θ is closed under addition. If $\alpha, \beta \in \Theta$, since Θ is worm-perfect, there is $\mathfrak{w} \sqsubset \Theta$ such that $o(\mathfrak{w}) = \beta$. By Lemma 5.31, $\alpha \uparrow \mathfrak{w} \sqsubset \Theta$, and by Theorem 5.30, $e^\alpha \beta = o(\alpha \uparrow \mathfrak{w}) \in \Theta$.

Next we show that if $1 < \xi \in \Theta$, there are $\alpha, \beta \in \Theta$ such that $\xi = e^\alpha \beta$ and $\beta < \xi$. Since Θ is worm-perfect, $\xi = o(\mathfrak{w})$ for some $\mathfrak{w} \sqsubset \Theta$. We proceed by induction on $\|\mathfrak{w}\|$ to find suitable $\alpha, \beta \in \Theta$. We may assume that $\mathfrak{w} \neq \top$ since $\xi > 0$, and we set $\mu = \min \mathfrak{w}$. If $\mu = 0$, then $h(\mathfrak{w}), b(\mathfrak{w}) \sqsubset \Theta$, and since Θ is worm-perfect, $ob(\mathfrak{w}), oh(\mathfrak{w}) \in \Theta$. Now, if $oh(\mathfrak{w}) = \xi$, by induction on $\|h(\mathfrak{w})\|$ we see that there exist suitable $\alpha, \beta \in \Theta$. If instead $oh(\mathfrak{w}) < \xi$, this means that $\xi = ob(\mathfrak{w}) + 1 + oh(\mathfrak{w})$ is additively decomposable, contrary to our assumption.

Now consider $\mu > 0$. By Lemma 5.31, $\mu \downarrow \mathfrak{w} \sqsubset \Theta$. Hence by induction on $\|\mu \downarrow \mathfrak{w}\| < \|\mathfrak{w}\|$, we have that $o(\mu \downarrow \mathfrak{w}) = e^\eta \beta$ for some $\eta, \beta \in \Theta$ with $\beta < o(\mu \downarrow \mathfrak{w})$. It follows that

$$o(\mathfrak{w}) = e^\mu o(\mu \downarrow \mathfrak{w}) = e^\mu e^\eta \beta = e^{\mu+\eta} \beta,$$

and since Θ is closed under addition, we may set $\alpha = \mu + \eta \in \Theta$.

For the other direction, assume that Θ is hyperexponentially perfect. To show that $o[\Theta] \subseteq \Theta$, we will prove by induction on $\|\mathfrak{w}\|$ that if $\mathfrak{w} \sqsubset \Theta$, then $o(\mathfrak{w}) \in \Theta$. For the base case, if $\mathfrak{w} = \top$, then $o(\mathfrak{w}) = 0 \in \Theta$. Otherwise, let $\mu = \min \mathfrak{w}$.

If $\mu = 0$, then by induction hypothesis $oh(\mathfrak{w}), ob(\mathfrak{w}) \in \Theta$. Since also $1 \in \Theta$, then $o(\mathfrak{w}) = ob(\mathfrak{w}) + 1 + oh(\mathfrak{w}) \in \Theta$. Otherwise, $\|\mu \downarrow \mathfrak{w}\| < \|\mathfrak{w}\|$, and as before, $\mu \downarrow \mathfrak{w} \sqsubset \Theta$. It follows by the induction hypothesis that $o(\mu \downarrow \mathfrak{w}) \in \Theta$. Moreover, since μ appears in \mathfrak{w} we must have that $\mu \in \Theta$, thus $o(\mathfrak{w}) = e^\mu o(\mu \downarrow \mathfrak{w}) \in \Theta$, using the fact that Θ is hyperexponentially closed.

Finally, we show that $\Theta \subseteq o[\Theta]$. We prove by induction on ξ that if $\xi \in \Theta$, then $\xi = o(\mathfrak{w})$ for some $\mathfrak{w} \sqsubset \Theta$. If $\xi = 0$ we may take $\mathfrak{w} = \top$. If not, using the fact that Θ is hyperexponentially perfect, write $\xi = e^\alpha \beta$ with $\alpha, \beta \in \Theta$ and $\beta = 1$ or additively decomposable. If $\beta = 1$, then $\xi = e^\alpha 1 = o(\alpha \top)$. Otherwise, since Θ is additively reductive, we may write $\beta = \gamma + \delta'$ with $\gamma, \delta' \in \beta \cap \Theta$. Using Lemma 5.17 we see that $\delta = -1 + \delta' \in \Theta$. By the induction hypothesis, there are worms $\mathfrak{u}, \mathfrak{v} \sqsubset \Theta$ such that $\gamma = o(\mathfrak{u})$, $\delta = o(\mathfrak{v})$, and thus

$$\beta = \gamma + \delta' = \gamma + 1 + \delta = o(\mathfrak{u}) + 1 + o(\mathfrak{v}) = o(\mathfrak{v} \ 0 \ \mathfrak{u}).$$

But $\mathfrak{v} \ 0 \ \mathfrak{u} \sqsubset \Theta$, and thus by Lemma 5.31, $\alpha \uparrow (\mathfrak{v} \ 0 \ \mathfrak{u}) \sqsubset \Theta$, and $o(\alpha \uparrow (\mathfrak{v} \ 0 \ \mathfrak{u})) = e^\alpha \beta$, as needed. \square

With this, we obtain our worm-based characterization of Γ_0 :

Theorem 5.35. Γ_0 is the least worm-perfect set of ordinals.

Proof. Γ_0 is the least hyperexponentially perfect set, and since it is transitive and closed under addition, it is additively reductive. Hence Γ_0 is also worm-perfect, and since any worm-perfect set is hyperexponentially perfect, there can be no smaller worm-perfect set. \square

5.7 Autonomous worms and predicative ordinal notations

The map $o: \mathbb{W} \rightarrow \text{Ord}$ suggests that worms could themselves be used as modalities. This gives rise to Beklemishev's *autonomous worms* [5]:

Definition 5.36. We define the set of autonomous worms \mathbb{W} to be the least set such that $\top \in \mathbb{W}$ and, if $\mathbf{w}, \mathbf{v} \in \mathbb{W}$, then $(\mathbf{w})\mathbf{v} \in \mathbb{W}$.

The idea is to interpret autonomous worms as regular worms using o :

Definition 5.37. We define a map $\cdot^o: \mathbb{W} \rightarrow \mathbb{W}$ given recursively by

1. $\top^o = \top$
2. $((\mathbf{w})\mathbf{v})^o = (o(\mathbf{w}^o))\mathbf{v}^o$.

We then define $o: \mathbb{W} \rightarrow \text{Ord}$ by setting $o(\mathbf{w}) = o(\mathbf{w}^o)$.

As Beklemishev has noted, autonomous worms give notations for any ordinal below Γ_0 .

Theorem 5.38. If γ is any ordinal, then $\gamma < \Gamma_0$ if and only if there is $\mathbf{w} \in \mathbb{W}$ such that $\gamma = o(\mathbf{w})$.

Proof. To see that $\Gamma_0 \subseteq o[\mathbb{W}]$, it suffices in view of Theorem 5.35 to observe that $o[\mathbb{W}]$ is worm-perfect by construction.

To see that $o[\mathbb{W}] \subseteq \Gamma_0$, one proves by induction on the number of parentheses in \mathbf{w} that if Θ contains 0 and is worm-closed, then $o(\mathbf{w}) \in \Theta$. In particular, $o(\mathbf{w}) \in \Gamma_0$. \square

6 Impredicative worms

Now we turn to a possible solution to Mints' and Pakhomov's problem of representing the Bachmann-Howard ordinal using worms. This ordinal is related to *inductive definitions*, that is, least fixed points of monotone operators $F: 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$. Let us begin by reviewing these operators and their fixed points.

1	$()$	ω	$(())$	ε_0	$(((())))$
ε_1	$(((()))) (((())))$	$\varepsilon_\omega + \varepsilon_0$	$(((()))) () (((())))$	$e^{e^{e^{e^{1_1 1_1 1_1}}}}$	$((((((()))))))$

Figure 1: Some ordinals represented as autonomous worms. We use the identity $\varepsilon_\xi = e^\omega(1 + \xi)$, which is a special case of Proposition 5.15.

6.1 Inductive definitions

Let $F: 2^\mathbb{N} \rightarrow 2^\mathbb{N}$. We say that F is *monotone* if $F(X) \subseteq F(Y)$ whenever $X \subseteq Y$. For example, if $f: \mathbb{N}^{<\omega} \rightarrow \mathbb{N}$, we obtain a monotone operator by setting $F(X) = f[X]$; as we have seen in Lemma 4.17, we can reach a fixed point for such an F by iterating it ω -many times and taking the union of these iterations. More generally, any monotone operator has a least fixed point:

Definition 6.1. *Let $F: 2^\mathbb{N} \rightarrow 2^\mathbb{N}$ be monotone. We define μF to be the unique set such that:*

1. $\mu F = F(\mu F)$, and
2. If $X \subseteq \mathbb{N}$ is such that $F(X) \subseteq X$, then $\mu F \subseteq X$.

The Knaster-Tarski theorem states that the set μF is always well-defined [23]; it can always be reached “from below” by iterating F , beginning from the empty set. However, in general, we may need to iterate F far beyond ω .

Definition 6.2. *Let $F: 2^\mathbb{N} \rightarrow 2^\mathbb{N}$. For an ordinal ξ , we define an operator $F^\xi: 2^\mathbb{N} \rightarrow 2^\mathbb{N}$ inductively by*

1. $F^0(X) = X$,
2. $F^{\xi+1}(X) = F(F^\xi(X))$,
3. $F^\lambda(X) = \bigcup_{\xi < \lambda} F^\xi(X)$ for λ a limit ordinal.

These iterations eventually become constant, but the ordinal at which they stabilize can be rather large; in principle, our only guarantee is that it is countable, since at each stage before reaching a fixed point we must add at least one natural number. Below, recall that ω_1 denotes the first uncountable cardinal.

Lemma 6.3. *If $F: 2^\mathbb{N} \rightarrow 2^\mathbb{N}$ is monotone, then there is $\lambda < \omega_1$ such that $F^\lambda(\emptyset) = \mu F$.*

We omit the proof, which follows from cardinality considerations. Alternately, it is possible to construct least fixed points ‘from above’, by taking the intersection of all F -closed sets.

Lemma 6.4. *If $F: 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ is monotone, then*

$$\mu F = \bigcap \{Y \subseteq \mathbb{N} : F(Y) \subseteq Y\}.$$

Monotone operators and their fixed points can be formalized in second-order arithmetic, provided they are definable. Any formula $\phi(n, X) \in \Pi^1_\omega$ (with no other free variables) can be regarded as an operator on $2^{\mathbb{N}}$ given by $X \mapsto \{n \in \mathbb{N} : \phi(n, X)\}$. Say that a formula ϕ is in *negation normal form* if it contains no instances of \rightarrow , and \neg occurs only on atomic formulas. It is well-known that every formula is equivalent to one in negation normal form, obtained by applying De Morgan’s rules iteratively.

Definition 6.5. *Let ϕ be a formula in negation normal form and X a set-variable. We say ϕ is positive on X if ϕ contains no occurrences of $t \notin X$.*

Positive formulas give rise to monotone operators, due to the following:

Lemma 6.6. *Given a formula $\phi(n, X)$ that is positive on X , it is provable in ECA_0 that*

$$\forall X \forall Y \left(X \subseteq Y \rightarrow \forall n \left(\phi(n, X) \rightarrow \phi(n, Y) \right) \right).$$

Thus if we define $F_\phi: 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ by $F_\phi(X) = \{n \in \mathbb{N} : \phi(n, X)\}$, F_ϕ will be monotone on X whenever ϕ is positive on X . Moreover, if ϕ is arithmetical, Lemma 6.4 may readily be formalized in $\Pi^1_1\text{-CA}_0$, by defining

$$M = \left\{ n \in \mathbb{N} : \forall X \left(\forall m \left(\phi(m, X) \rightarrow m \in X \right) \rightarrow n \in X \right) \right\}.$$

Thus we arrive at the following:

Lemma 6.7. *Let $\phi(n, X)$ be arithmetical and positive on X . Then, it is provable in $\Pi^1_1\text{-CA}_0$ that there is a least set M such that, for all n ,*

$$n \in M \leftrightarrow \phi(n, M).$$

We will denote this set M by $\mu X.\phi$.

With these tools in mind, we are now ready to formalize ω -logic in second-order arithmetic.

6.2 Formalizing full ω -logic

We have discussed before how the ω -rule can be iterated along a well-order. However, we may also consider full ω -logic based on a theory T ; that is, the set of formulas that can be derived using the ω -rule and reasoning in T , regardless of the nesting depth of these ω -rules. Let us write $[\infty]_T\phi$ if ϕ is derivable in this fashion. To be precise, we want $[\infty]_T\phi$ to hold whenever:

- (i) $\Box_T\phi$,
- (ii) $\phi = \forall x\psi(x)$ and for all n , $[\infty]_T\psi(\bar{n})$, or
- (iii) there is ψ such that $[\infty]_T\psi$ and $[\infty]_T(\psi \rightarrow \phi)$.

In words, $[\infty]_T$ is closed under T and the ω -rule. This notion may be formalized using ω -trees to represent infinite derivations, as in [2, 22]. We follow a different approach, using a fixed-point construction as in [16].

Definition 6.8. *Fix a theory T , possibly with oracles. Let $\text{SPC}_T(Q)$ be a Π_1^1 formula naturally expressing that Q is the least set such that $\phi \in Q$ whenever (i) $\Box_T\phi$ holds, (ii) $\phi = \forall v\psi(v)$ and for all n , $\psi(\bar{n}) \in Q$, or (iii) there exists $\psi \in Q$ such that $\psi \rightarrow \phi \in Q$.*

Then, define

$$[\infty]_T\phi \equiv \forall Q(\text{SPC}_T(Q) \rightarrow \phi \in Q).$$

In view of Lemma 6.3, this fixed point is reached after some countable ordinal, which gives us the following:

Proposition 6.9. *Given a theory T and $\phi \in \Pi_\omega^1$, $[\infty]_T\phi$ holds if and only if $[\xi]_T\phi$ holds for some $\xi < \omega_1$.*

As before, we may also consider saturated provability operators with oracles, and we write $[\infty|A]_T\phi$ instead of $[\infty]_{T|A}\phi$. Since these provability operators are defined via a least fixed point, in view of Lemma 6.7, their existence can be readily proven in $\Pi_1^1\text{-CA}_0$.

Lemma 6.10. *Let T be any theory, possibly with oracles. Then, it is provable in $\Pi_1^1\text{-CA}_0$ that there exists a set Q such that $\text{SPC}_T(Q)$ holds.*

This notion of provability allows us to represent $\Pi_1^1\text{-CA}_0$ in terms of a strong consistency assertion, in the spirit of Theorems 4.3 and 5.5. The following is proven in [16]:

Theorem 6.11. $\Pi_1^1\text{-CA}_0 \equiv \text{ECA}_0 + \forall X \langle \infty|X \rangle_T \top$.

This suggests that studying worms which contain the modality $\langle \infty \rangle$ may be instrumental in studying theories capable of reasoning about least fixed points. In view of Proposition 6.9, we may identify $\langle \infty \rangle$ with $\langle \Omega \rangle$ for some ordinal Ω large enough so that $[\infty]_T \phi$ is equivalent to $[\Omega]_T \phi$; we can take $\Omega = \omega_1$, for example, but a large enough countable ordinal will do. In the next section, we will see how adding uncountable ordinals to our notation system allows us to provide notations for much larger countable ordinals as well.

6.3 Beyond the Bachmann-Howard ordinal

It is not hard to see that ε_0 and Γ_0 are countable; for example, it is an easy consequence of Theorem 5.38. With a bit of extra work, one can see that they are computable as well, for example representing elements of Γ_0 as in Theorem 5.38. However, this does not mean that uncountable ordinals cannot appear as a “detour” in defining proof-theoretic ordinals. Indeed, the Bachmann-Howard ordinal precisely arises by adding a symbol for an uncountable ordinal. Before continuing, let us recall a few basic properties of cardinals and cardinalities.

Definition 6.12. *Given a set A , we define $|A|$ to be the least ordinal κ such that there is a bijection $f: A \rightarrow \kappa$. If $\kappa = |\kappa|$, we say that κ is a cardinal.*

The following properties are well-known and discussed in detail, for example, in [29].

Lemma 6.13. *Let A, B be sets. Then,*

1. $|A \cup B| \leq \max\{\omega, |A|, |B|\}$;
2. *if at least one of A, B is infinite, then $|A \cup B| = \max\{|A|, |B|\}$;*
3. $|A \times B| \leq \max\{\omega, |A|, |B|\}$,
4. *if one of A, B is infinite and both are non-empty, $|A \times B| = \max\{|A|, |B|\}$, and*
5. *if $\{A_i : i \in I\}$ is a family of sets, then*

$$\left| \bigcup_{i \in I} A_i \right| \leq \max \left\{ \omega, \sup_{i \in I} |A_i|, |I| \right\}.$$

These results readily allow us to compute the cardinalities of ordinals obtained using addition and multiplication.

Lemma 6.14. *Let α, β be ordinals. Then,*

1. $|\alpha + \beta| \leq \max\{\omega, |\alpha|, |\beta|\}$;
2. $|\alpha + \beta| = \max\{|\alpha|, |\beta|\}$ if one of the two is infinite;
3. $|\alpha\beta| \leq \max\{\omega, |\alpha|, |\beta|\}$, and
4. $|\alpha\beta| = \max\{|\alpha|, |\beta|\}$ if one of the two is infinite and both are non-zero.

Proof. These claims are immediate from Lemma 6.13 if we observe that $\alpha + \beta$ is the disjoint union of α with $[\alpha, \alpha + \beta)$, and $|[\alpha, \alpha + \beta)| = |\beta|$, while $\alpha\beta$ is in bijection with $\alpha \times \beta$ (via the map $\alpha\xi + \zeta \mapsto (\zeta, \xi) \in \alpha \times \beta$). \square

Similar claims hold for the hyperexponential function:

Lemma 6.15. *Let α, β be arbitrary ordinals. Then, $|e^\alpha\beta| \leq \max\{\omega, |\alpha|, |\beta|\}$. If moreover $\beta > 0$ and $\max\{\alpha, \beta\} \geq \omega$, then $|e^\alpha\beta| = \max\{|\alpha|, |\beta|\}$.*

Proof. To bound $|e^\alpha\beta|$, we proceed by induction on α with a secondary induction on β to show that $|e^\alpha\beta| \leq \max\{\omega, |\alpha|, |\beta|\}$. We consider several cases, using Lemma 5.12. If $\alpha = 0$, then $e^0\beta = \beta$, so the claim is obviously true. If $\beta = 0$, we see that $e^\alpha 0 = 0$, so the claim holds as well. For $\alpha = 1$ and $\beta = \gamma + 1$,

$$e(\gamma + 1) = \lim_{n < \omega} (1 + e^\gamma) \cdot n \stackrel{\text{IH}}{\leq} \max\{\omega, |\alpha|, |\beta|\}.$$

If α is a limit and $\beta = 1$,

$$e^\alpha 1 = \lim_{\gamma < \alpha} e^\gamma 1 \stackrel{\text{IH}}{\leq} \max\{\omega, |\alpha|, |\beta|\}.$$

For $\alpha = \gamma + 1$ with $\gamma > 0$ we obtain

$$\begin{aligned} e^{\gamma+1}\beta &= e^\gamma e\beta \stackrel{\text{IH}}{\leq} \max\{|\alpha|, |e\beta|\} \\ &\stackrel{\text{IH}}{\leq} \max\{\omega, |\alpha|, \max\{\omega, |\alpha|, |\beta|\}\} = \max\{\omega, |\alpha|, |\beta|\}. \end{aligned}$$

If β is a limit, then we obtain

$$e^\alpha\beta = \lim_{\gamma < \beta} e^\alpha\gamma \stackrel{\text{IH}}{\leq} \max\{\omega, |\alpha|, |\beta|\}.$$

Finally, for limit α and $\beta = \delta + 1$ we obtain

$$e^\alpha(\delta + 1) = \lim_{\gamma < \alpha} e^\gamma(e^\alpha(\delta) + 1) \stackrel{\text{IH}}{\leq} \max\{\omega, |\alpha|, |\beta|\}.$$

Since this covers all cases, the result follows.

For the second claim, if $\beta > 0$, then $e^\alpha \beta \geq \max\{\alpha, \beta\}$, so $|e^\alpha \beta| \geq \max\{|\alpha|, |\beta|\}$ and we obtain the desired equality if one of the two is infinite. \square

Corollary 6.16. *If κ is an uncountable cardinal, then κ is additively indecomposable and $e^\kappa 1 = \kappa$.*

Proof. We know that $e^\kappa 1 \geq \kappa$. However, from Lemma 6.15, $|e^\xi 1| < \kappa$ whenever $\xi < \kappa$, so that $e^\xi 1 < \kappa$. But $e^\kappa 1 = \lim_{\xi < \kappa} e^\xi 1$, so $e^\kappa 1 = \kappa$, from which it also follows that $\kappa = \omega^\kappa$ and thus is additively indecomposable. \square

We have a similar situation with worms; it is very easy to infer the cardinality of $o(\mathfrak{w})$ by looking at the entries in \mathfrak{w} .

Lemma 6.17. *If $\mathfrak{w} \in \mathbb{W}$ then $|o(\mathfrak{w})| \leq |\max \omega \mathfrak{w}|$. If moreover $\mathfrak{w} \neq \top$ and $\max \mathfrak{w} \geq \omega$, then $|o(\mathfrak{w})| = |\max \mathfrak{w}|$.*

Proof. We prove by induction on $\|\mathfrak{w}\|$ that $|o(\mathfrak{w})| \leq |\max \omega \mathfrak{w}|$. For $\mathfrak{w} = \top$ this is obvious. Otherwise, let $\mu = \min \mathfrak{w}$. If $\mu = 0$, then $o(\mathfrak{w}) = ob(\mathfrak{w}) + 1 + oh(\mathfrak{w})$, so that by Lemma 6.15,

$$|o(\mathfrak{w})| = |ob(\mathfrak{w}) + 1 + oh(\mathfrak{w})| \leq \max\{\omega, |ob(\mathfrak{w})|, 1, |oh(\mathfrak{w})|\}.$$

By the induction hypothesis $|oh(\mathfrak{w})| \leq |\max \omega h(\mathfrak{w})| \leq |\max \omega \mathfrak{w}|$ and similarly for $|ob(\mathfrak{w})|$, so we obtain $|o(\mathfrak{w})| \leq |\max \omega \mathfrak{w}|$.

If $\mu > 0$, then $o(\mathfrak{w}) = e^\mu(\mu \downarrow \mathfrak{w})$. Since $\mu, \max(\mu \downarrow \mathfrak{w}) \leq \max \mathfrak{w}$ and $\|\mu \downarrow \mathfrak{w}\| < \|\mathfrak{w}\|$, we use the induction hypothesis and Lemma 6.15 once again to see that

$$|o(\mathfrak{w})| = |e^\mu o(\mu \downarrow \mathfrak{w})| \leq \max\{\omega, |\mu|, |\max(\mu \downarrow \mathfrak{w})|\} \leq |\max \omega \mathfrak{w}|.$$

The claim follows.

For the second claim, if $\mathfrak{w} \neq \top$ and $\max \mathfrak{w} \geq \omega$, then by Lemma 3.26.1, $o(\mathfrak{w}) \geq \max \mathfrak{w}$, so

$$|o(\mathfrak{w})| \geq |\max \mathfrak{w}| = |\max \omega \mathfrak{w}|,$$

and thus we obtain equality. \square

Similarly, closure under a function f does not produce many more ordinals than we had to begin with:

Lemma 6.18. *If $f: \text{Ord}^{<\omega} \dashrightarrow \text{Ord}$ and Θ is a set of ordinals, then*

$$|\Theta| \leq |\overline{\Theta}^f| \leq \max\{\omega, |\Theta|\}.$$

Proof. We inductively check that

$$|\Theta| \leq |\Theta_n^f| \leq \max\{\omega, |\Theta|\}, \quad (5)$$

from which the lemma follows using the fact that $\overline{\Theta}^f = \bigcup_{n < \omega} \Theta_n^f$.

We have that $\Theta_0^f = \Theta$, so (5) holds. Now, assume inductively that (5) holds for n . Then, $\Theta_{n+1}^f = \Theta_n^f \cup f[\Theta_n^f]$; by the induction hypothesis,

$$|\Theta| \leq |\Theta_n^f| \leq |\Theta_{n+1}^f|.$$

Now, elements of $f[\Theta]$ are of the form $f(\xi_1, \dots, \xi_m)$ with $\xi_1, \dots, \xi_m \in \Theta_n^f$; but there are at most $\max\{\omega, |\Theta_n^f|\}$ of these, so

$$|f[\Theta_n^f]| \leq \max\{\omega, |\Theta_n^f|\} \stackrel{\text{IH}}{\leq} \max\{\omega, |\Theta|\},$$

from which it follows that

$$|\Theta_{n+1}^f| = |\Theta_n^f \cup f[\Theta_n^f]| \leq \max\{\omega, |\Theta_n^f|, |f[\Theta_n^f]|\} \stackrel{\text{IH}}{\leq} \max\{\omega, |\Theta|\}. \quad \square$$

This tells us that none of the ordinal operations we have discussed so far will give rise to any uncountable ordinals. So, we may add one directly; we can then use it to produce more countable ordinals using *collapsing functions*. We shall present them using hyperexponentials rather than Veblen functions, although this change is merely cosmetic as the two define the same ordinals. It is standard to use Ω to denote a ‘big’ ordinal, which for convenience may be assumed to be ω_1 . However, we mention that, with some additional technical work, one can take $\Omega = \omega_1^{CK}$, the first non-computable ordinal [34].

Definition 6.19. Let Ω, ξ be ordinals. We simultaneously define the sets $C(\xi)$ and the ordinals $\psi(\xi)$ by induction on ξ as follows:

1. $C(\xi)$ is the least set such that
 - (a) $\Omega \in C(\xi)$,
 - (b) $C(\xi)$ is hyperexponentially closed, and
 - (c) if $\alpha \in C(\xi)$ and $\alpha < \xi$ then $\psi(\alpha) \in C(\xi)$.
2. $\psi(\xi)$ is the least λ such that $\lambda \notin C(\xi)$.

In the notation of Definition 4.16, let BH_ξ be the pair of functions $\{\text{HE}, \psi \upharpoonright \xi\}$. Then,

$$C(\xi) = \overline{\{0, 1, \Omega\}}^{\text{BH}_\xi}.$$

Thus our previous work on closures under ordinal functions readily applies to the sets $C(\xi)$. The function ψ appears in the ordinal analysis of systems such as ID_1 and Kripke-Platek set-theory with infinity [33].

Lemma 6.20. *If ξ is any ordinal, then $\psi(\xi)$ is additively indecomposable and $\psi(\xi) = e^{\psi(\xi)}1$.*

Proof. To see that $\psi(\xi)$ is additively indecomposable, we will assume otherwise and reach a contradiction. Hence, suppose that $\psi(\xi) = \alpha + \beta$ with $\alpha, \beta < \psi(\xi)$. By definition of $\psi(\xi)$ we have that $\alpha, \beta \in C(\xi)$, hence $\psi(\xi) = \alpha + \beta \in C(\xi)$, contradicting its definition.

Next we show that $\psi(\xi) = e^{\psi(\xi)}1$. By Proposition 5.14, there are α, β with β either 1 or additively decomposable such that $\psi(\xi) = e^\alpha\beta$. Since $\psi(\xi)$ is additively indecomposable we have that $\beta \neq \psi(\xi)$, and since e^α is normal, we have that $\beta < \psi(\xi)$. Now, towards a contradiction, assume that $\alpha < \psi(\xi)$; then $\alpha, \beta \in C(\xi)$ so $\psi(\xi) \in C(\xi)$, contrary to its definition. We conclude that $\alpha = \psi(\xi)$, and again since $e^{\psi(\xi)}$ is normal and $e^{\psi(\xi)}1 \geq \psi(\xi)$, that $\beta = 1$. \square

We remark that the above lemma already tells us that the *countable* ordinals we can construct using ψ are much bigger than Γ_0 ; indeed, we already have that $\Gamma_0 = \psi(0)$, and this is only scratching the surface of our notation system: ordinals such as $\psi(\Omega)$ or $\psi(e^\omega(\Omega + 1))$ are much larger. The latter is the Howard-Bachmann ordinal $\psi(\varepsilon_{\Omega+1})$, as one can readily check that $e^\omega\xi = \varepsilon_\xi$ for all ξ using Proposition 5.15.

Lemma 6.21. *Assume that Ω is such that $\Omega = e^\Omega 1$. If ξ is any ordinal, then $C(\xi)$ is hyperexponentially perfect.*

Proof. We already know that $C(\xi)$ is hyperexponentially closed, so it remains to show that it is reductive. Let $\zeta \in C(\xi)$. By Lemma 4.17.3, either $\zeta \in \{0, 1, \Omega\}$, there are $\alpha, \beta, \gamma \neq \zeta$ with $\zeta = e^\alpha(\beta + \gamma)$, or $\zeta = \psi(\alpha)$ for some $\alpha \in C(\xi) \cap \xi$. If $\zeta < 2$, there is nothing to prove, so we assume otherwise.

First assume that $\zeta = e^\alpha(\beta + \gamma)$. If ζ is additively decomposable, by Lemma 6.20, we cannot have that $\alpha > 0$, so we conclude that $\zeta = e^0(\beta + \gamma) = \beta + \gamma$, as needed. If it is additively indecomposable, since $\beta + \gamma \in C(\xi)$, then we already have that $\zeta = e^\alpha(\beta + \gamma)$ with $\alpha, \beta + \gamma \in C(\xi)$. In all other cases, ζ must be additively

indecomposable. If $\zeta = \Omega$, then $\zeta = e^\Omega 1$ and $\Omega, 1 \in C(\xi)$, and if $\zeta = \psi(\alpha)$, by Lemma 6.20, $\zeta = e^\zeta 1$, with $\zeta, 1 \in C(\xi)$. \square

The intention of the function ψ is to produce new countable ordinals from possibly uncountable ones. Let us see that this is the case:

Lemma 6.22. *Let ξ be any ordinal and $\Omega = \omega_1$. Then, $C(\xi)$ is countable and $\psi(\xi) < \Omega$.*

Proof. The first claim is an instance of Lemma 6.18, while the second is immediate from the first. \square

Observe that $\sup C(\xi) = \Gamma_{\Omega+1}$, the first hyperexponentially closed ordinal which is greater than Ω , and thus the smallest ordinal not contained in any $C(\xi)$ is $\psi(\Gamma_{\Omega+1})$. However, our worm notation will give slightly smaller ordinals. Thus it will be convenient to consider a “cut-off” version of the sets $C(\xi)$. Let us see that these cut-off versions maintain a restricted version of the minimality property of $C(\xi)$.

Lemma 6.23. *If $\mu \leq \lambda$ are ordinals such that $\Omega < \lambda$, then $C(\mu) \cap \lambda$ is the least set D such that:*

- (i) $0, 1, \Omega \in D$;
- (ii) if $\alpha, \beta, \gamma \in D$ and $e^\alpha(\beta + \gamma) < \lambda$ then $e^\alpha(\beta + \gamma) \in D$, and
- (iii) if $\alpha \in D \cap \mu$ then $\psi(\alpha) \in D$.

Proof. First we observe that $C(\mu) \cap \lambda$ indeed satisfies (i)-(iii), where for the first item we use the assumption that $\Omega < \lambda$ and for the third we use Lemma 6.22 to see that $\psi(\alpha) < \Omega < \lambda$. Now, let D be the least set satisfying (i)-(iii), and consider

$$D' = D \cup (C(\mu) \setminus \lambda).$$

One readily verifies that $0, 1, \Omega \in D'$, and that if $\alpha, \beta, \gamma \in D'$ then $e^\alpha(\beta + \gamma) \in D'$ (using the fact that $D \subseteq C(\mu) \cap \lambda \subseteq C(\mu)$ by minimality of D). Finally, if $\alpha < \mu$ and $\alpha \in D'$, then since $\mu \leq \lambda$ we have that $\alpha \in D$, and since D satisfies (iii) we have that $\psi(\alpha) \in D \subseteq D'$. But by definition $C(\mu)$ is the least set with these properties, so we obtain $C(\mu) \subseteq D'$, and hence

$$C(\mu) \cap \lambda \subseteq D' \cap \lambda = D,$$

as was to be shown. \square

We remark that the ordinal $\psi(\Gamma_{\Omega+1})$ is computable, meaning that it is isomorphic to an ordering $\langle A, \preceq \rangle$, where $A \subseteq \mathbb{N}$ and both A and \preceq are Δ_1^0 -definable; however, we will not go into details here, and instead refer the reader to a text such as [33].

6.4 Collapsing uncountable worms

Now let us turn our attention to uncountable worms. The general idea is as follows. We have seen in Theorem 5.38 that worms give us a notation system for Γ_0 if we interpret $\langle \mathfrak{w} \rangle$ as $\langle o(\mathfrak{w}) \rangle$. Meanwhile, now we have a new modality $\langle \infty \rangle$, which we can regard as $\langle \omega_1 \rangle$. Note that, by Corollary 6.16,

$$o(\langle \omega_1 \rangle \top) = e^{\omega_1} o(\langle 0 \rangle \top) = \omega_1.$$

Thus if we add the new symbol Ω representing $\langle \omega_1 \rangle$ to Beklemishev's autonomous worms, we see inductively that

$$\langle \omega_1 \rangle \top = \Omega^o = (\Omega)^o = ((\Omega))^o \dots$$

Moreover, if such operations are to be interpreted proof-theoretically using iterated ω -rules, then in view of Proposition 6.9 we have that $\langle \omega_1 \rangle \top \equiv \langle \omega_1 + \xi \rangle \top$ for any ordinal ξ . Thus we also would have, for example,

$$\langle \omega_1 \rangle \top = (\Omega)^o = ((\Omega))^o = (\Omega\Omega)^o \dots$$

This would lead to quite a wasteful notation system! Thus we will adopt the following rule: when writing an autonomous worm $(\mathfrak{w})\top$, if $o(\mathfrak{w})$ is countable, then we will take it at face-value and interpret $(\mathfrak{w})\top$ as $\langle o(\mathfrak{w}) \rangle \top$. However, if $o(\mathfrak{w})$ is uncountable, we will first “project” it to a countable ordinal, in order to represent large countable worms.

Of course, projections will be very similar to collapsing functions; however, given that countable ordinals are taken at face value, these projections will have the property that $\pi = \pi \circ \pi$ (thus their name). Other than that, their construction is very similar to that of ψ :

Definition 6.24. *Given a worm $\mathfrak{w} \in \mathbb{W}$ and an ordinal Ω , we define $U(\mathfrak{w}) \subseteq \text{Ord}$ and a map $\pi: \mathbb{W} \rightarrow \text{Ord}$ by induction on \mathfrak{w} along \triangleleft as follows.*

1. *Let $U(\mathfrak{w})$ be the least set of ordinals such that*

- (a) $\Omega \in U(\mathfrak{w})$,
- (b) *if $\mathfrak{u} \sqsubset U(\mathfrak{w})$ and $\mathfrak{u} \triangleleft \mathfrak{w}$ then $\pi(\mathfrak{u}) \in U(\mathfrak{w})$.*

2. *Then, set*

- (a) $\pi(\mathfrak{w}) = o(\mathfrak{w})$ *if $\mathfrak{w} \sqsubset \Omega$,*
- (b) *otherwise, set $\pi(\mathfrak{w})$ to be the least ordinal μ such that $\mu \notin U(\mathfrak{w})$.*

We will write $\pi(\mathfrak{w})$ or $\pi\mathfrak{w}$ indistinctly. Once again, we can write Definition 6.24 in the terminology of Definition 4.16 by setting

$$U(\mathfrak{w}) = \overline{\{\Omega\}}^{\pi \upharpoonright \{\mathfrak{v} : \mathfrak{v} \triangleleft \mathfrak{w}\}}.$$

Thus Lemma 6.18 gives us the following:

Lemma 6.25. *For every worm \mathfrak{w} , $U(\mathfrak{w})$ and $\pi\mathfrak{w}$ are countable.*

Throughout this section we will assume that $\Omega = \omega_1$, so that from Lemma 6.25 we obtain $\pi\mathfrak{w} < \Omega$ for all worms \mathfrak{w} . As was the case for defining ψ , with some extra technical work we can take $\Omega = \omega_1^{CK}$ instead.

Note that $U(\mathfrak{w})$ itself is not worm-closed, as it does not contain, for example, the ordinal $\Omega + 1 = o(0\Omega\top)$. However, its countable part is indeed worm-perfect. The next lemmas will establish this fact. First, we show that it is worm-closed.

Lemma 6.26. *For any worm \mathfrak{v} with $o(\mathfrak{v}) \geq \Omega$, $U(\mathfrak{v}) \cap \Omega$ is worm-closed.*

Proof. By Corollary 3.19.3, if $\mathfrak{w} \sqsubset U(\mathfrak{v}) \cap \Omega$, then $\mathfrak{w} \triangleleft \Omega\top \triangleleft \mathfrak{v}$, so that $o(\mathfrak{w}) = \pi(\mathfrak{w}) \in U(\mathfrak{v})$. But by Lemma 6.17, $o(\mathfrak{w}) < \Omega$, so $o(\mathfrak{w}) \in U(\mathfrak{v}) \cap \Omega$ as needed. \square

Recall that Lemma 6.20 states that $\psi(\xi) = e^{\psi(\xi)}1$. Next, we show that π enjoys a similar property.

Lemma 6.27. *If $o(\mathfrak{w}) \geq \Omega$, then $o(\langle \pi\mathfrak{w} \rangle \top) = \pi\mathfrak{w}$.*

Proof. Suppose not. Then, by Lemma 3.26.1, $\pi\mathfrak{w} < o(\langle \pi\mathfrak{w} \rangle \top)$, so that by Corollary 3.27, there is a worm \mathfrak{v} such that $o(\mathfrak{v}) = \pi\mathfrak{w}$. Since $o(\mathfrak{v}) < o(\langle \pi\mathfrak{w} \rangle \top)$, by Lemma 3.26.1 once again, we must have that $\mathfrak{v} \sqsubset \pi\mathfrak{w} \subseteq U(\mathfrak{w})$. But by Lemma 6.26, $\pi\mathfrak{w} = o(\mathfrak{v}) \in U(\mathfrak{w})$, contradicting the definition of $\pi\mathfrak{w}$. \square

Lemma 6.28. *For any worm \mathfrak{w} , $U(\mathfrak{w}) \cap \Omega$ is worm-perfect and*

$$U(\mathfrak{w}) \cap \Omega = U(\mathfrak{w}) \setminus \{\Omega\}.$$

Proof. For the first claim, in view of Lemma 6.26, it remains to show that if $\xi \in U(\mathfrak{w}) \cap \Omega$, then $\xi = o(\mathfrak{v})$ for some $\mathfrak{v} \sqsubset U(\mathfrak{w}) \cap \Omega$. By definition of $U(\mathfrak{w})$, if $\xi \in U(\mathfrak{w}) \cap \Omega$, then $\xi = \pi\mathfrak{u}$ for some $\mathfrak{u} \sqsubset U(\mathfrak{w})$. If $\mathfrak{u} \sqsubset \Omega$, then $\xi = \pi\mathfrak{u} = o\mathfrak{u}$. Otherwise, by Lemma 6.27, $\xi = \pi\mathfrak{u} = o(\langle \pi\mathfrak{u} \rangle \top)$.

The second claim is immediate from Lemma 6.25 and the assumption that $\Omega = \omega_1$, since $\pi\mathfrak{w} < \Omega$ for every worm \mathfrak{w} . \square

However, as we have mentioned, $U(\mathfrak{w})$ itself is not worm-closed, and neither is $o[U(\mathfrak{w})]$. Nevertheless, the latter does satisfy a bounded form of hyperexponential closure:

Lemma 6.29. *Given any worm \mathfrak{w} and ordinals α, β , if $\alpha, \beta \in o[U(\mathfrak{w})]$ and $e^\alpha \beta < e^{\Omega+1} 1$ then $e^\alpha \beta \in o[U(\mathfrak{w})]$.*

Proof. If $\alpha, \beta \in o[U(\mathfrak{w})]$ and $e^\alpha \beta < e^{\Omega+1} 1$, we may assume without loss of generality that $\beta > 0$ (since otherwise $e^\alpha \beta = 0$), so by the assumption that $o(\mathfrak{w}) < e^{\Omega+1} 1 = e^{\Omega} \omega$, we see by monotonicity that either $\alpha < \Omega$, or $\alpha = \Omega$ and $\beta < \omega$.

First assume that $\alpha < \Omega$, and let

$$\mathfrak{v} = \lambda_1 \dots \lambda_n \top \sqsubset U(\mathfrak{w})$$

be such that $\beta = o(\mathfrak{v})$. In view of Lemma 6.28, for each $\lambda \in [1, n]$, either $\lambda_i = \Omega$, in which case $\alpha + \lambda_i = \lambda_i$, or $\lambda_i \in U(\mathfrak{w}) \cap \Omega$, which since $U(\mathfrak{w}) \cap \Omega$ is worm-perfect (Lemma 6.28) gives us $\alpha + \lambda_i \in U(\mathfrak{w}) \cap \Omega \subseteq U(\mathfrak{w})$ (Lemma 5.33). Thus $\alpha + \lambda_i \in U(\mathfrak{w})$ for each i , hence $\alpha \uparrow \mathfrak{v} \sqsubset U(\mathfrak{w})$, and

$$o(\alpha \uparrow \mathfrak{v}) = e^\alpha o(\mathfrak{v}) = e^\alpha \beta.$$

Otherwise, $\alpha = \Omega$, so $\beta < \omega$ and we see that $o(\langle \Omega \rangle^\beta \top) = e^\alpha \beta$. In either case, it follows that $e^\alpha \beta \in o[U(\mathfrak{w})]$. \square

Lemma 6.30. *Suppose that $\Omega = \omega_1$. Then, given any worm \mathfrak{w} ,*

$$e^{\Omega+1} 1 = \sup \left\{ o(\mathfrak{v}) : \exists \mathfrak{w} \ (\mathfrak{v} \sqsubset U(\mathfrak{w})) \right\}.$$

Proof. Let

$$\Lambda = \sup \left\{ o(\mathfrak{v}) : \exists \mathfrak{w} \ (\mathfrak{v} \sqsubset U(\mathfrak{w})) \right\}.$$

We have that

$$e^{\Omega+1} 1 = e^{\Omega} \omega = \lim_{n < \omega} e^{\Omega} n.$$

But, $e^{\Omega} n = o(\langle \Omega \rangle^n \top)$, so $e^{\Omega+1} 1 \leq \Lambda$.

To see that $\Lambda \leq e^{\Omega+1} 1$, proceed by induction on $\|\mathfrak{v}\|$ to show that if $\mathfrak{v} \sqsubset U(\mathfrak{w})$ for some \mathfrak{w} , then $o(\mathfrak{v}) < e^{\Omega+1} 1$.

If $\mathfrak{v} = \top$ there is nothing to prove, and if $\min \mathfrak{v} = 0$ then by the induction hypothesis, $oh(\mathfrak{v}), ob(\mathfrak{v}) < e^{\Omega+1} 1$. Since the latter is additively indecomposable,

$$o(\mathfrak{v}) = ob(\mathfrak{v}) + 1 + oh(\mathfrak{v}) < e^{\Omega+1} 1.$$

Finally, if $\mu = \min \mathfrak{v} > 0$, then $o(\mathfrak{v}) = e^\mu o(\mu \downarrow \mathfrak{v})$. Consider two cases. If $\mu < \Omega$, then since by the induction hypothesis $o(\mu \downarrow \mathfrak{v}) < e^{\Omega+1}1$, we obtain

$$e^\mu o(\mu \downarrow \mathfrak{v}) < e^\mu e^{\Omega+1}1 = e^{\mu+\Omega+1}1 = e^{\Omega+1}1.$$

Otherwise, $\mu = \Omega$, but this means that $\mu \downarrow \mathfrak{v} = 0^n \top$ for some n , hence $o(\mathfrak{v}) = e^\Omega n < e^{\Omega+1}1$. \square

The above results tell us that π behaves a lot like a version of ψ that is restricted to $e^{\Omega+1}1$. Let us see that this is, in fact, the case.

Lemma 6.31. *For every worm \mathfrak{w} with $o(\mathfrak{w}) \in [\Omega, e^{\Omega+1}1]$,*

1. $C(-\Omega + o(\mathfrak{w})) \cap e^{\Omega+1}1 = o[U(\mathfrak{w})]$, and
2. $\pi(\mathfrak{w}) = \psi(-\Omega + o(\mathfrak{w}))$.

Proof. We prove both claims by induction on $o(\mathfrak{w})$. Set $C = C(-\Omega + o(\mathfrak{w}))$. First let us show that

$$C \cap e^{\Omega+1}1 \subseteq o[U(\mathfrak{w})].$$

Note that by Lemma 6.23, $C \cap e^{\Omega+1}1$ is the least set containing $0, 1, \Omega$, closed under $\alpha, \beta, \gamma \mapsto e^\alpha(\beta + \gamma)$ below $e^{\Omega+1}1$, and closed under $\psi \upharpoonright (-\Omega + o(\mathfrak{w}))$. But by Lemma 5.33, $o[U(\mathfrak{w})]$ is closed under addition and by Lemma 6.29, by hyperexponentiation below $e^{\Omega+1}1$, so we only need to check that it is closed under $\psi \upharpoonright (-\Omega + o(\mathfrak{w}))$.

If $\alpha \in o[U(\mathfrak{w})]$ and $\alpha < o(-\Omega + o(\mathfrak{w}))$, then by Lemma 5.33 we have that $\Omega + \alpha = o(\mathfrak{u})$ for some $\mathfrak{u} \sqsubset U(\mathfrak{w})$. Then, by the induction hypothesis,

$$\psi(\alpha) = \psi(-\Omega + o(\mathfrak{u})) = \pi(\mathfrak{u}) \in U(\mathfrak{w}),$$

so that $\pi(\mathfrak{u}) \top \sqsubset U(\mathfrak{w})$ and by Lemma 6.27, $\pi(\mathfrak{u}) = o(\pi(\mathfrak{u}) \top)$, as needed. Thus by the minimality of $C \cap e^{\Omega+1}1$, we conclude that $C \cap e^{\Omega+1}1 \subseteq o[U(\mathfrak{w})]$.

Next we check that

$$o[U(\mathfrak{w})] \subseteq C \cap e^{\Omega+1}1.$$

By Lemma 6.30, $o[U(\mathfrak{w})] \subseteq e^{\Omega+1}1$, so we only need to prove that $o[U(\mathfrak{w})] \subseteq C$. But, in view of Lemmas 6.21 and Lemma 5.34, C is worm-perfect. Thus to show that $o[U(\mathfrak{w})] \subseteq C$, it suffices to prove that $U(\mathfrak{w}) \subseteq C$. As before, we show that C satisfies the inductive definition of $U(\mathfrak{w})$.

Let $\mathfrak{v} \sqsubset C$ be such that $\mathfrak{v} \triangleleft \mathfrak{w}$. Once again by Lemma 6.21, we have that $o(\mathfrak{v}) \in C$. Now, if $o(\mathfrak{v}) < \Omega$, then this gives us $\pi \mathfrak{v} = o(\mathfrak{v}) \in C$. Otherwise, $-\Omega + o(\mathfrak{v}) < -\Omega + o(\mathfrak{w})$, and thus $\psi(-\Omega + o(\mathfrak{v})) \in C$. But, by the induction

hypothesis, $\psi(-\Omega + o(\mathfrak{w})) = \pi\mathfrak{w}$, so that $\pi\mathfrak{w} \in C$, as needed. By minimality of $U(\mathfrak{w})$, we conclude that $U(\mathfrak{w}) \subseteq C$ and thus $o[U(\mathfrak{w})] \subseteq C$.

Since we have shown both inclusions, we conclude that

$$o[U(\mathfrak{w})] = C \cap e^{\Omega+1}1.$$

Moreover, $\psi(-\Omega + o(\mathfrak{w}))$ is defined as the least ordinal not in $C = C(-\Omega + o(\mathfrak{w}))$, and since C is countable it is also the least ordinal not in $C \cap \Omega$. Similarly, $\pi\mathfrak{w}$ is the least ordinal not in $U(\mathfrak{w}) \cap \Omega = o[U(\mathfrak{w})] \cap \Omega$. Since these two sets are equal, it follows also that $\psi(-\Omega + o(\mathfrak{w})) = \pi\mathfrak{w}$. \square

Corollary 6.32. $\pi(\langle\Omega + 1\rangle\top) = \psi(e^{\Omega+1}1)$.

Proof. Immediate from Lemma 6.31 using the fact that

$$e^{\Omega+1}1 = e^{\Omega+1}o(\langle 0 \rangle\top) = o(\langle e^{\Omega+1} \rangle\top). \quad \square$$

6.5 Impredicative worm notations

Now let us extend Beklemishev's autonomous worms with the new modality Ω and projections of uncountable worms. Aside from the addition of Ω , the presentation is very similar to that of Section 5.7.

Definition 6.33. *Define the set of impredicative autonomous worms to be the least set W_Ω such that*

(i) $\top \in W_\Omega$, and

(ii) if $\mathfrak{w}, \mathfrak{v} \in W_\Omega$, then

(a) $(\mathfrak{w})\mathfrak{v} \in W_\Omega$, and

(b) $\Omega\mathfrak{v} \in W_\Omega$.

As before, the intention is for impredicative autonomous worms to be interpreted as standard worms. We do this via the following translation:

Definition 6.34. *We define a map $\cdot^\pi: W_\Omega \rightarrow \mathbb{W}$ given by*

1. $\top^\pi = \top$,

2. $((\mathfrak{w})\mathfrak{v})^\pi = \langle \pi(\mathfrak{w}^\pi) \rangle \mathfrak{v}^\pi$, and

3. $(\Omega\mathfrak{v})^\pi = \langle \Omega \rangle \mathfrak{v}^\pi$.

Every ordinal in $U(\langle\Omega + 1\rangle\top) \cap \Omega$ can be represented as an autonomous worm. Below, define $\mathbf{pw} = \pi(\mathbf{w}^\pi)$.

Lemma 6.35. *If $\Omega = \omega_1$, then for every ordinal $\xi \in U(\langle\Omega + 1\rangle\top) \cap \Omega$ there is $\mathbf{w} \in W_\Omega$ such that $\xi = \mathbf{pw}$.*

Proof. Using the notation of Definition 4.16, we prove by induction on n that if $\xi \in \{\Omega\}_n^\pi \cap \Omega$, then there is $\mathbf{w} \in W_\Omega$ such that $\xi = \mathbf{pw}$. If $n = 0$ there is nothing to prove, so we may assume that $n = k + 1$. Write $\xi = \pi(\mathbf{v})$ with $\mathbf{v} \sqsubset \{\Omega\}_k^\pi$. If $\mathbf{v} = \top$, then $\xi = 0 = \mathbf{p}\top$. Otherwise, we can write $\mathbf{v} = \lambda\mathbf{u}$ for some worm \mathbf{u} . By a secondary induction on the length of \mathbf{v} , we have that $\mathbf{u} = \mathbf{u}^\pi$ for some $\mathbf{u} \in W_\Omega$; meanwhile, either $\lambda = \Omega$, and $\mathbf{v} = \Omega\mathbf{u} \in W_\Omega$ satisfies

$$\mathbf{pv} = \pi(\mathbf{v}^\pi) = \pi(\langle\Omega\rangle\mathbf{u}^\pi) = \pi(\langle\Omega\rangle\mathbf{u}) = \pi(\mathbf{v}) = \xi,$$

or $\lambda < \Omega$, which means that $\lambda \in \{\Omega\}_k^\pi$, so by the induction hypothesis, $\lambda = \mathbf{pw}$ for some $\mathbf{w} \in W_\Omega$. It follows that $\xi = \mathbf{p}(\mathbf{w})\mathbf{v}$, as desired. \square

Just as autonomous worms gave us a notation system for Γ_0 , impredicative autonomous worms give us a notation system for $\psi(e^{\Omega+1}1)$.

Theorem 6.36. *If $\Omega = \omega_1$, then for every $\xi < \psi(e^{\Omega+1}1)$ there is $\mathbf{w} \in W_\Omega$ such that $\xi = \mathbf{pw}$.*

Proof. By Corollary 6.32,

$$\psi(e^{\Omega+1}1) = \pi(\langle\Omega + 1\rangle\top),$$

and the latter is, by definition, the least ordinal not belonging to $U(\langle\Omega + 1\rangle\top)$. Moreover, $\psi(e^{\Omega+1}1)$ is countable by Lemma 6.22, so we have that $\xi < \Omega$. It follows that

$$\psi(e^{\Omega+1}1) \subseteq U(\langle\Omega + 1\rangle\top) \cap \Omega;$$

thus we obtain the claim by Lemma 6.35. \square

Impredicative autonomous worms may be suitable for a consistency proof in the spirit of Theorem 4.5 for theories with proof-theoretic strength the Bachmann-Howard ordinal (or even slightly more powerful theories). Examples of such theories are the theory ID_1 of non-iterated inductive definitions, Kripke-Platek with infinity, and *parameter-free* $\Pi_1^1\text{-CA}_0$, where the Π_1^1 comprehension axiom is restricted to formulas without free set variables. However, the proof-theoretical ordinal of unrestricted $\Pi_1^1\text{-CA}_0$ is quite a bit larger, and obtained by collapsing all of the ordinals $\{\aleph_n : n < \omega\}$.

We remark that our notation system does not take the oracle in $[\infty|X]_T$ into account, and it is possible that autonomous worms with oracles would indeed give us a notation system for the proof-theoretical ordinal of $\Pi_1^1\text{-CA}_0$. However, we will not follow this route; instead, we will pass from worms to *spiders*, which will allow us to obtain notations for this, and much larger, ordinals.

7 Spiders

The problem with using iterated ω -rules to interpret $[\lambda]_T\phi$ is that **GLP** no longer applies when $\lambda \geq \omega_1$; since we have that $[\omega_1 + 1]_T\phi$ is equivalent to $[\omega_1]_T\phi$, we cannot expect the **GLP** axiom $\langle\omega_1\rangle\phi \rightarrow [\omega_1 + 1]\langle\omega_1\rangle\phi$ to hold. So the question naturally arises: what kind of (sound) provability operator could derive all true instances of $\langle\omega_1\rangle\phi$?

Well, we know that $\langle\omega_1\rangle\phi$ is equivalent to $\forall\xi < \omega_1 \langle\xi\rangle\phi$, which gives us a strategy for proving that $\langle\infty\rangle_T\phi$ holds: prove that

$$\langle 0 \rangle_T\phi, \langle 1 \rangle_T\phi, \langle 2 \rangle_T\phi, \dots, \langle \omega \rangle_T\phi, \dots, \langle \Gamma_0 \rangle_T\phi, \dots \langle \Psi(\varepsilon_{\Omega+1}) \rangle_T\phi, \dots$$

all hold, and more generally, that $\langle\xi\rangle_T\phi$ holds for all $\xi < \omega_1$. Let us sketch some ideas for formalizing this in the language of set-theory. We remark that this material is exploratory, and will be studied in detail in upcoming work.

7.1 \aleph_ξ -rules

We use \mathcal{L}_\in to denote the language of first-order set theory whose only relation symbols are \in and $=$. As we did in second-order arithmetic, we use $x \subseteq y$ as a shorthand for $\forall z(z \in x \rightarrow z \in y)$. We also use $\exists!x\phi(x)$ as the standard shorthand for “there is a unique”. Then, recall that Zermelo-Fraenkel set theory with choice, denoted ZFC, is the extension of first-order logic axiomatized by the universal closures of the following:

Extensionality: $(x \subseteq y \wedge y \subseteq x) \rightarrow y = x$;

Foundation: $\exists x\phi(x) \rightarrow \exists x(\phi(x) \wedge \forall y \in x \neg\phi(y))$, where $\phi(x)$ is an arbitrary formula in which y does not occur free;

Pair: $\exists z(x \in z \wedge y \in z)$;

Union: $\exists y \forall z \in x (z \subseteq y)$;

Powerset: $\exists y \forall z (z \subseteq x \rightarrow z \in y)$;

Separation: $\exists y \forall z (z \in y \leftrightarrow z \in x \wedge \phi(z))$, where y does not occur free in $\phi(z)$,

Collection: $\forall x \in w \exists y \phi(x, y) \rightarrow \exists z \forall x \in w \exists y \in z \phi(x, y)$, where z does not occur free in $\phi(x)$,

Infinity: $\exists w (\exists x (x \in w \wedge \forall y (y \notin x)) \wedge \forall x \in w \exists y \in w \forall z (z \in y \leftrightarrow z \in x \vee z = x))$,
and

Choice: $\forall x \in w (\exists y (y \in x) \wedge \forall y \in w (\exists z (z \in x \wedge z \in y) \rightarrow x = y))$
 $\rightarrow \exists z \forall x \in w \exists ! y (y \in x \wedge y \in z)$.

As we have stated the union and powerset axioms we may obtain sets that are too big, but we can then obtain the desired sets using separation. Observe also that the Foundation scheme states that \in is well-founded; this allows us to simply define an ordinal as a transitive set all of whose elements are transitive as well, obtaining well-foundedness for free.

This set-theoretic context will allow us to define an analogue of the ω -rule which quantifies over all elements of ω_1 ; more generally, for any cardinal κ we can define the κ -rule by

$$\frac{\langle \phi(\xi) \rangle_{\xi < \kappa}}{\forall x < \kappa \phi(x)}.$$

Of course, in order to do this we need to have names for all elements of κ , as well as κ itself. To this effect, let \mathcal{L}_\in^κ be a (possibly uncountable) extension of \mathcal{L}_\in which contains one constant c_ξ for each $\xi < \kappa$; to simplify notation, we may assume that $c_\xi = \xi$ and simply write the latter. Then, the κ -rule is readily applicable in any language extending $\mathcal{L}_\in^{\kappa+1}$. Similarly, for a theory T over \mathcal{L}_\in , let T^κ be the extension of T over \mathcal{L}_\in^κ with the axioms $\xi \in \zeta$ whenever $\xi < \zeta \leq \kappa$, and $\xi \notin \zeta$ whenever $\zeta \leq \xi \leq \kappa$.

If T is an extension of ZFC^κ , we may enrich T by operators of the form $[\kappa]_T^\lambda \phi$, meaning that ϕ is provable using κ -rules of depth at most α . Recall that if ξ is an ordinal, then \aleph_ξ denotes the ξ^{th} infinite ordinal. Then, any infinite cardinal κ may be represented in the form \aleph_β for some β , and we write $[\beta]_T^\xi \phi$ to state that ϕ may be proven by iterating \aleph_β -rules along ξ .

If we want the \aleph function to be well-defined, we must work within a cardinal that is closed under $\xi \mapsto \aleph_\xi$. Fortunately, $\xi \mapsto \aleph_\xi$ is a normal function, so we may hyperate it, and readily observe that $\aleph^\omega(0)$ is the first ordinal ξ such that $\aleph_\xi = \xi$. Thus we may assume that T is an extension of $\text{ZFC}^{\aleph^\omega(0)}$.

Definition 7.1. Let T be a theory over $\mathcal{L}_\infty^{\aleph_\omega(0)}$, α, β be ordinals, and $\phi \in \mathcal{L}_\infty^{\aleph_\omega(0)}$. Then, by recursion on β with a secondary recursion on α , we define $[\beta]_T \phi$ to hold if either

1. $\Box_T \phi$, or
2. there are a formula $\psi(x)$ and ordinals γ, η such that $\eta \leq \beta$ and either $\eta < \beta$ or $\gamma < \alpha$, and such that
 - (a) for each $\delta < \aleph_\eta$, $[\gamma]_T \psi(\delta)$, and
 - (b) $\Box_T((\forall x < \aleph_\eta \psi(x)) \rightarrow \phi)$.

As was the case with ω -rules, we have that for any β , the \aleph_β -rule saturates by $\aleph_{\beta+1}$:

Theorem 7.2. If $[\lambda]_T \phi$ for arbitrary λ , then there is $\lambda' < \aleph_{\eta+1}$ such that $[\lambda']_T \phi$.

Proof. By induction on η with a secondary induction on λ . If $\Box_T \phi$ holds then clearly $[0]_T \phi$. Otherwise, there are a formula $\psi(x)$ and ordinals γ and $\delta \leq \eta$ such that either $\delta < \eta$ or $\gamma < \lambda$, and for each $\xi < \aleph_\delta$, $[\gamma]_T \psi(\xi)$ and $\Box_T((\forall x < \aleph_\delta \psi(x)) \rightarrow \phi)$.

By the induction hypothesis, for each $\xi < \aleph_\delta$ there is

$$\lambda_\xi < \aleph_{\delta+1} \leq \aleph_{\eta+1}$$

such that $[\lambda_\xi]_T \psi(\xi)$. By Lemma 6.13, we have that

$$\lambda = \sup_{\xi < \aleph_\delta} \lambda_\xi < \aleph_{\eta+1},$$

and therefore also $\lambda + 1 < \aleph_{\eta+1}$. But then observe that $[\lambda+1]_T \phi$, as desired. \square

Thus we have a similar situation as we had when considering $\langle \omega_1 + \xi \rangle_T \phi$; any expressions of the form $\langle \aleph_{\beta+1} + \alpha \rangle_T \phi$ is equivalent to $\langle \aleph_{\beta+1} \rangle_T \phi$. Moreover, observe that $\langle \aleph_{\beta+1} \rangle_T \phi$ is in turn equivalent to $\langle \aleph_{\beta+1}^0 \rangle_T \phi$; thus we should only be interested in expressions of the form $\langle \aleph_\beta^\alpha \rangle_T \phi$ in cases when $\alpha < \aleph_{\beta+1}$. Otherwise, as we did for impredicative worms, we may collapse α to an ordinal $\psi_\beta(\alpha) < \aleph_{\beta+1}$.

In Section 7.3 we will review a version of Buchholz's ordinal notation system which achieves exactly that, and in Section 7.4 we will see how these ideas may be applied to *spiders*, which are similar to worms but based on modalities $\langle \aleph_\beta^\alpha \rangle$. However, before we continue, we remark that working with uncountable languages has some obvious drawbacks. Fortunately, this can be avoided by working with admissible ordinals rather than cardinals.

7.2 Iterated admissibles

If we work with an uncountable language then the usual proof of the validity of

$$\langle 0 \rangle_T \phi \rightarrow [1]_T \langle 0 \rangle_T \phi$$

will not go through, given that we cannot code all possible derivations as natural numbers. There is more than one way to get around this problem; one can allow only ordinals appearing in ϕ to be used in a derivation of ϕ , for example. Alternately, we can work with admissible ordinals, (many of) which are countable, instead of cardinals.

In the set-theoretical context, a Δ_0 formula is any formula ϕ of \mathcal{L}_\in such that all quantifiers appearing in ϕ are either of the form $\forall x \in y$ or $\exists x \in y$. Then, *Kripke-Platek set theory* is the subtheory KP of ZFC in which the axioms of choice, powerset and infinity are removed, and separation and collection are restricted to $\phi \in \Delta_0$.

With this in mind, we say that an ordinal α is *admissible* if \mathbb{L}_α (in Gödel's constructible hierarchy) is a model of KP. Admissible sets are studied in great detail in [3]. Moreover, an analogue of Theorem 7.2 also holds if we define:

- (i) $\omega_0^{CK} = \omega$,
- (ii) $\omega_{\xi+1}^{CK}$ to be the least admissible α such that $\omega_\xi^{CK} < \alpha$, and
- (iii) $\omega_\lambda^{CK} = \lim_{\xi < \lambda} \omega_\xi^{CK}$ for λ a limit ordinal.

This allows us to interpret $[\beta]_T^\alpha$ using a countable language by replacing the \aleph_β -rule by the ω_β^{CK} -rule,

$$\frac{\langle \phi(\xi) \rangle_{\xi < \omega_\beta^{CK}}}{\forall x < \omega_\beta^{CK} \phi(x)}.$$

Working with admissibles rather than cardinals makes the properties of collapsing functions more difficult to prove, but this has been done by Rathjen in [34]. For simplicity, in this text we will continue to work with the \aleph -function.

7.3 Collapsing the Aleph function

In this section we will review a variant of Buchholz's notation system of ordinal notations based on collapsing the aleph function [11]. The ordinals obtained appear, for example, in the proof-theoretical analysis of the theories ID_ν of iterated inductive definitions [12]. Below, define $\Omega(\xi) = -\omega + \aleph_\xi$; we will continue with this convention throughout the rest of the text.

Definition 7.3. *Given ordinals η, ξ , we simultaneously define the sets $C_\eta(\xi)$ and the ordinals $\psi_\eta(\xi)$ by induction on ξ as follows:*

1. $C_\eta(\xi)$ is the least set such that
 - (a) $2 + \Omega(\eta) \subseteq C_\eta(\xi)$;
 - (b) if $\alpha, \beta, \gamma \in C_\eta(\xi)$ then $e^\alpha(\beta + \gamma) \in C_\eta(\xi)$, and
 - (c) if $\alpha, \beta \in C_\eta(\xi)$ and $\beta < \xi$, then $\psi_\alpha(\beta) \in C_\eta(\xi)$;
2. $\psi_\eta(\xi) = \min\{\xi : \xi \notin C_\eta(\xi)\}$.

Observe that (1a) could be simplified somewhat if we had defined $\Omega(0) = 2$, but our presentation will in turn simplify some expressions later. As before, it is possible to define $C_\eta(\xi)$ using the notation of Definition 4.16 and thus we can apply our previous work to these sets. Aside from the first item, which is easy to check, the following lemma summarizes the analogues of Lemmas 6.20, 6.21, and 6.22. The proofs are essentially the same and we omit them.

Lemma 7.4. *Given ordinals η, μ ,*

1. $\psi_{1+\eta}(0) = \Omega(1 + \eta)$;
2. $\psi_\eta(\mu)$ is additively indecomposable and satisfies $e^{\psi_\eta(\mu)}1 = \psi_\eta(\mu)$;
3. $C_\eta(\mu)$ is hyperexponentially perfect,
4. $|C_\eta(\mu)| = \Omega(\eta)$, and
5. $\psi_\eta(\mu) \in [\Omega(\eta), \Omega(\eta + 1))$.

The first ordinal that we cannot write using indexed collapsing functions is $\psi_0(\Omega^\omega 1)$:

Lemma 7.5. *Given ordinals $\eta < \Omega^\omega 1$ and an arbitrary ordinal μ ,*

$$\sup C_\eta(\mu) = \Omega^\omega 1.$$

Proof. To see that $\sup C_\eta(\mu) \leq \Omega^\omega 1$, we observe that $\Omega^\omega 1$ is closed under all of the operations defining $C_\eta(\mu)$:

Since $\eta < \Omega^\omega 1$, we have that $\Omega(\eta) \subseteq \Omega^\omega 1$. By Lemmas 6.14 and 6.15, we see that if $\alpha, \beta, \gamma < \Omega^\omega 1$, then

$$\kappa := |e^\alpha(\beta + \gamma)| \leq \max\{\omega, |\alpha|, |\beta|, |\gamma|\}.$$

We then have that $\kappa < \Omega^\omega 1$, so writing $\kappa = \Omega(\xi)$ for some $\xi < \Omega^\omega 1$, we observe that

$$e^\alpha(\beta + \gamma) < \Omega(\xi + 1) < \Omega^\omega 1.$$

Finally we note that if $\nu, \xi < \Omega^\omega 1$, then by Lemma 7.4.5, $\psi_\nu(\xi) < \Omega(\nu + 1) < \Omega^\omega 1$.

Now, to see that

$$\sup C_\eta(\mu) \geq \Omega^\omega 1,$$

simply consider the sequence $(\pi_n)_{n < \omega}$ given by $\pi_0 = 0$ and $\pi_{n+1} = \psi_{\pi_n}(0) \in C_\eta(\mu)$. By Lemma 7.4.1 we have that $\pi_{n+1} = \Omega(\pi_n)$ which by Lemma 5.12 converges to $\Omega^\omega 1$. \square

The ordinal $\psi_0(\Omega^\omega 1)$ is also computable, but we will not prove this here; see e.g. [11] for details. In the next section, we will present a variant of the functions ψ_ν using worm-like notations obtained from iterated \aleph_ξ -rules.

7.4 Iterated Alephs and spiders

We have seen in Theorem 5.38 that Beklemishev's autonomous worms give a notation system for all ordinals below the Feferman-Schütte ordinal Γ_0 , and in Theorem 6.36 that impredicative worms extend this to all ordinals below $\psi(e^{\Omega^{+1}} 1)$ (which becomes $\psi_0(e^{\psi_0(0)+1} 1)$ in our version of Buchholz's notation). Now let us introduce spiders, which may be used to give notations for much larger ordinals than we could with worms.

Definition 7.6. *Let Λ be either an ordinal or the class of all ordinals, and $f: \Lambda \rightarrow \Lambda$ be a normal function. We define $\langle \Lambda_f \rangle$ to be the class of all pairs of ordinals $\langle \lambda_\mu \rangle$ such that $f(\mu) + \lambda < f(\mu + 1)$, and write \mathbb{S}_f^Λ for the set of all expressions of the form*

$$\lambda_1 \dots \lambda_n \top,$$

with each $\lambda_i \in \langle \Lambda_f \rangle$. We simply write \mathbb{S} instead of $\mathbb{S}_\Omega^{\text{Ord}}$. Elements of \mathbb{S} are called spiders.

We will restrict our attention to the case where $f(\xi) = \Omega(\xi) = -\omega + \aleph_\xi$, although we state Definition 7.6 with some generality to stress that there are other possible choices for f . In a way, spiders are simply a different way to represent worms; to pass from one representation to the other, we introduce two auxiliary functions.

Definition 7.7. *Let α be any ordinal. Then, define*

- (i) $\lfloor \alpha \rfloor$ *to be the greatest ordinal such that $\Omega(\lfloor \alpha \rfloor) \leq \alpha$, and*

$$(ii) \quad \dot{\alpha} = -\Omega(\lfloor \alpha \rfloor) + \alpha.$$

This definition is sound because for any normal function f with $f(0) = 0$ and any ordinal μ , there is always a greatest ordinal ξ such that $f(\xi) \leq \mu$. The ‘translation’ between worms and spiders is the following:

Definition 7.8. *Define:*

1. $\flat: \langle \Omega^{\text{Ord}} \rangle \rightarrow \text{Ord}$ by $\flat \langle \lambda \rangle_\mu = \Omega(\mu) + \lambda$, and set $\langle \lambda \rangle_\mu \leq \langle \eta \rangle_\nu$ if and only if $\flat \langle \lambda \rangle_\mu \leq \flat \langle \eta \rangle_\nu$.
If $\mathfrak{X} = \lambda_1 \dots \lambda_n \top \in \mathbb{S}$, set $\flat \mathfrak{X} = \flat \lambda_1 \dots \flat \lambda_n \top$.
2. $\sharp: \text{Ord} \rightarrow \langle \Omega^\Lambda \rangle$ by $\sharp \lambda = \langle \lambda \rangle_{\lfloor \lambda \rfloor}$. If $\mathfrak{w} = \mu_1 \dots \mu_n \top \in \mathbb{W}$, set $\sharp \mathfrak{w} = \sharp \mu_1 \dots \sharp \mu_n \top$.

The following is then immediately verified:

Lemma 7.9. *The class functions \flat and \sharp are bijective and inverses of each other.*

With this, we can extend our worm notation to spiders.

Definition 7.10. *If $\mathfrak{X} \in \mathbb{S}$, define*

1. $O(\mathfrak{X}) = o(\flat \mathfrak{X})$,
2. $H(\mathfrak{X}) = h(\flat \mathfrak{X})$ and $B(\mathfrak{X}) = b(\flat \mathfrak{X})$,
3. $\mathfrak{X} \triangleleft \mathfrak{Y}$ if and only if $\flat \mathfrak{X} \triangleleft \flat \mathfrak{Y}$, and
4. if μ is any ordinal, $\mu \uparrow \mathfrak{X} = \sharp(\mu \uparrow \flat \mathfrak{X})$.

Alternately, we can define the head and body of a spider without first turning them into worms:

Lemma 7.11. *Given a spider \mathfrak{X} , $H(\mathfrak{X})$ is the maximum initial segment*

$$H(\mathfrak{X}) = \left\langle \lambda_1 \right\rangle_{\eta_1} \dots \left\langle \lambda_m \right\rangle_{\eta_m} \top \in \mathbb{S}$$

of \mathfrak{X} such that for all $i \in [1, m]$, either $\lambda_i \neq 0$ or $\eta_i \neq 0$.

If $H(\mathfrak{X}) = \mathfrak{X}$ then $B(\mathfrak{X}) = \top$, otherwise $B(\mathfrak{X})$ is the unique spider such that

$$\mathfrak{X} = H(\mathfrak{X}) \langle 0 \rangle_0 B(\mathfrak{X}).$$

As was the case with worms, the cardinality of $O(\mathfrak{X})$ is easy to extract from \mathfrak{X} :

Lemma 7.12. *If*

$$\mathfrak{X} = \left\langle \begin{smallmatrix} \lambda_1 \\ \eta_1 \end{smallmatrix} \right\rangle \dots \left\langle \begin{smallmatrix} \lambda_n \\ \eta_n \end{smallmatrix} \right\rangle \top \in \mathbb{S},$$

then

1. *for every $i \in [1, n]$, $\lambda_i, \eta_i \leq O(\mathfrak{X})$, and*
2. *if $|O(\mathfrak{X})| > \omega$, then $|O(\mathfrak{X})| = \Omega(\max_{i \in [1, n]} \eta_i)$.*

Proof. Immediate by applying Lemma 6.17 to $\flat\mathfrak{X}$ and observing that if $\mu > 0$, $|\Omega(\mu) + \lambda| = \Omega(\mu)$ given that $\lambda < \Omega(\mu + 1)$. \square

We can also give an analogue of \sqsubset for spiders:

Definition 7.13. *If*

$$\mathfrak{W} = \left\langle \begin{smallmatrix} \lambda_1 \\ \eta_1 \end{smallmatrix} \right\rangle \dots \left\langle \begin{smallmatrix} \lambda_n \\ \eta_n \end{smallmatrix} \right\rangle \top \in \mathbb{S}$$

and Θ is a set of ordinals, we define $\mathfrak{W} \sqsubset_{\Omega} \Theta$ if each $\lambda_i, \eta_i \in \Theta$.

With this, we are ready to ‘project’ spiders.

Definition 7.14. *Given $\mathfrak{X}, \mathfrak{Y} \in \mathbb{S}$, we define $U_{\mathfrak{Y}}(\mathfrak{X}) \subseteq \text{Ord}$ and an ordinal $\pi_{\mathfrak{Y}}\mathfrak{X}$ by induction on \mathfrak{X} along \triangleleft as follows.*

1. *Let $U_{\mathfrak{Y}}(\mathfrak{X})$ be the least set of ordinals such that if*

$$\mathfrak{U}, \mathfrak{V} \sqsubset_{\Omega} \Omega(O(\mathfrak{Y})) \cup U_{\mathfrak{Y}}(\mathfrak{X})$$

and $\mathfrak{V} \triangleleft \mathfrak{X}$, then $\pi_{\mathfrak{U}}\mathfrak{V} \in U_{\mathfrak{Y}}(\mathfrak{X})$.

2. *For any $\mathfrak{Y} \in \mathbb{S}$,*

(a) *If $\mathfrak{X} \triangleleft \langle \begin{smallmatrix} 0 \\ O(\mathfrak{Y})+1 \end{smallmatrix} \rangle \top$, set $\pi_{\mathfrak{Y}}(\mathfrak{X}) = O(\mathfrak{X})$;*

(b) *otherwise,*

$$\pi_{\mathfrak{Y}}(\mathfrak{X}) = \min\{\xi : \xi \notin U_{\mathfrak{Y}}(\mathfrak{X})\}.$$

In the remainder of this section, we will see that the functions $\pi_{\mathfrak{X}}$ behave very similarly to the functions ψ_{ν} . We begin with a simple lemma.

Lemma 7.15. *If $\mathfrak{X}, \mathfrak{Y}$ are spiders with $O(\mathfrak{X}) > 1$, then $0, 1 \in U_{\mathfrak{Y}}(\mathfrak{X})$.*

Proof. Immediate from observing that $0 = O(\top) = \pi_{\top}\top$ and $1 = O(\langle \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \rangle \top) = \pi_{\top}(\langle \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \rangle \top)$. \square

With the next few lemmas, we show that the elements of $U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ can be characterized as the order-types of suitable spiders. In the process, we obtain some useful properties of $\pi_{\mathfrak{Y}}\mathfrak{X}$.

Lemma 7.16. *If $\mathfrak{X} \sqsubset_{\Omega} U_{\mathfrak{Y}}(\mathfrak{W})$ and $\mathfrak{X} \triangleleft \mathfrak{W}$, then $O(\mathfrak{X}) \in U_{\mathfrak{Y}}(\mathfrak{W})$.*

Proof. Let $\mathfrak{X} \sqsubset_{\Omega} U_{\mathfrak{Y}}(\mathfrak{W})$ be such that $\mathfrak{X} \triangleleft \mathfrak{W}$. Since Ω is normal, for every ξ we have that $\xi \leq \Omega(\xi)$. In particular,

$$O(\mathfrak{X}) < O(\mathfrak{X}) + 1 \leq \Omega(O(\mathfrak{X}) + 1).$$

It follows that $O(\mathfrak{X}) = \pi_{\mathfrak{X}}\mathfrak{X} \in U_{\mathfrak{Y}}(\mathfrak{W})$. \square

With this, we can show that $\pi_{\mathfrak{Y}}\mathfrak{X}$ has cardinality $\Omega(O(\mathfrak{Y}))$, provided $O(\mathfrak{X})$ is large enough.

Lemma 7.17. *If $\mathfrak{X}, \mathfrak{Y}$ are spiders with $O(\mathfrak{X}) \geq \Omega(O(\mathfrak{Y}) + 1)$, then*

$$\pi_{\mathfrak{Y}}\mathfrak{X} \in [\Omega(O(\mathfrak{Y})), \Omega(O(\mathfrak{Y}) + 1)).$$

Proof. If $\xi < \Omega(O(\mathfrak{Y}))$, then by Corollary 3.27 we obtain $\mathfrak{w} \sqsubset \Omega(O(\mathfrak{Y}))$ such that $o(\mathfrak{w}) = \xi$ and observe that $\sharp\mathfrak{w} \triangleleft \mathfrak{X}$, so that by Lemma 7.16, $\xi = O(\sharp\mathfrak{w}) \in U_{\mathfrak{Y}}(\mathfrak{X})$. It follows that $\pi_{\mathfrak{Y}}\mathfrak{X} \geq \Omega(O(\mathfrak{Y}))$. Meanwhile, by Lemma 6.18, $|U_{\mathfrak{Y}}(\mathfrak{X})| \leq \omega + \Omega(O(\mathfrak{Y}))$, so $\pi_{\mathfrak{Y}}\mathfrak{X} < \Omega(O(\mathfrak{Y}) + 1)$. \square

Moreover, $\pi_{\mathfrak{Y}}\mathfrak{X}$ satisfies an analogue of Lemma 6.27:

Lemma 7.18. *If $O(\mathfrak{X}) \geq \Omega(O(\mathfrak{Y}) + 1)$, then $O(\langle \pi_{\mathfrak{Y}}\mathfrak{X} \rangle_{O(\mathfrak{Y})}) = \pi_{\mathfrak{Y}}\mathfrak{X}$.*

Proof. Analogous to the proof of Lemma 6.27, except that to reach a contradiction we use Lemma 7.12.1 to obtain a spider \mathfrak{V} such that $O(\mathfrak{V}) = \pi_{\mathfrak{Y}}\mathfrak{X}$ and all of whose entries are strictly bounded by $\pi_{\mathfrak{Y}}\mathfrak{X}$. \square

With this we can show that the elements of $U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ are the order-types of suitable spiders, as claimed.

Lemma 7.19. *Let $\mathfrak{X}, \mathfrak{Y}$ be spiders and ξ an ordinal. Then, $\xi \in U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ if and only if there is $\mathfrak{W} \sqsubset_{\Omega} U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ such that $\xi = O(\mathfrak{W})$.*

Proof. One direction is Lemma 7.16. For the other, if $\xi \in U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$, then there are $\mathfrak{U}, \mathfrak{V} \sqsubset_{\Omega} U_{\mathfrak{Y}}(\mathfrak{X})$ such that $\mathfrak{U} \triangleleft \mathfrak{X}$ and $\xi = \pi_{\mathfrak{V}}\mathfrak{U}$. If $\mathfrak{U} \triangleleft \langle \pi_{\mathfrak{V}}\mathfrak{U} \rangle_{O(\mathfrak{V})+1}^0$, then we already have $\xi = O(\mathfrak{U})$. If not, by Lemma 7.17,

$$O(\mathfrak{V}) \leq \Omega(O(\mathfrak{V})) \leq \xi < O(\mathfrak{X}),$$

so that by Lemma 7.16, $O(\mathfrak{W}) \in U_{\mathfrak{Y}}(\mathfrak{X})$, and hence $\langle O(\mathfrak{W})^\xi \rangle^\top \sqsubset_\Omega U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$.

Meanwhile, by Lemma 7.18,

$$O\left(\left\langle \pi_{\mathfrak{W}}\mathfrak{U} \right\rangle^\top\right) = \pi_{\mathfrak{W}}\mathfrak{U} = \xi,$$

as needed. \square

Lemma 7.19 is useful in showing that $U_{\mathfrak{Y}}(\mathfrak{X})$ is well-behaved. For example, it satisfies a bounded version of additive reducibility.

Lemma 7.20. *Given spiders $\mathfrak{X}, \mathfrak{Y}$ and an additively decomposable ordinal $\xi < O(\mathfrak{X})$, we have that $\xi \in U_{\mathfrak{Y}}(\mathfrak{X})$ if and only if there are $\alpha, \beta \in U_{\mathfrak{Y}}(\mathfrak{X}) \cap \xi$ such that $\xi = \alpha + \beta$.*

Proof. Analogous to the proof of Lemma 5.33. To illustrate, let us check that if $\xi \in U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ is additively decomposable, then there are $\alpha, \beta \in U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ such that $\xi = \alpha + \beta$. Using Lemma 7.19, write $\xi = O(\mathfrak{W})$ with $\mathfrak{W} \sqsubset_\Omega U_{\mathfrak{Y}}(\mathfrak{X})$. Then, by Theorem 5.30,

$$\xi = O(\mathfrak{W}) = o(\mathfrak{b}\mathfrak{W}) = ob(\mathfrak{b}\mathfrak{W}) + 1 + oh(\mathfrak{b}\mathfrak{W}).$$

Set $\alpha = ob(\mathfrak{b}\mathfrak{W})$ and $\beta = 1 + oh(\mathfrak{b}\mathfrak{W})$. Observe that $\alpha < \xi$, while β is additively indecomposable so $\beta \neq \xi$. Hence, $\alpha, \beta < \xi$.

Finally, observe that $H(\mathfrak{W}), B(\mathfrak{W}) \sqsubset_\Omega U_{\mathfrak{Y}}(\mathfrak{X})$,

$$\beta = 1 + oh(\mathfrak{b}\mathfrak{W}) = o((\mathfrak{b}H(\mathfrak{W}))0) = O\left(H(\mathfrak{W})\langle 0 \rangle\right),$$

and $H(\mathfrak{W}) \sqsubset_\Omega U_{\mathfrak{Y}}(\mathfrak{X})$; similarly, $\alpha = OB(\mathfrak{W})$, so $\alpha, \beta \in U_{\mathfrak{Y}}(\mathfrak{X})$. \square

Note that $U_{\mathfrak{Y}}(\mathfrak{X})$ is not necessarily additively reductive; however, this truncated form of additive reducibility is sufficient to obtain the conclusion of Lemma 5.17:

Lemma 7.21. *Let Θ be a set of ordinals such that $0 \in \Theta$, and λ be an ordinal such that, whenever $\xi < \lambda$ is additively reducible, then $\xi \in \Theta$ if and only if there are $\alpha, \beta < \xi$ such that $\alpha + \beta = \xi$. Then, for any ordinal ξ :*

1. *if $0 \neq \xi \in \Theta \cap \lambda$, there are ordinals α, β such that $\alpha, \omega^\beta \in \Theta$ and $\xi = \alpha + \omega^\beta$;*
2. *if $\beta \in \Theta \cap \lambda$ and $\alpha < \beta$ (not necessarily a member of Θ), then $-\alpha + \beta \in \Theta$.*

The proof is identical to that of Lemma 5.17 and we omit it. Next we see that the sets $U_{\mathfrak{Y}}(\mathfrak{X})$ are also closed under some operations related to cardinality.

Lemma 7.22. *If $\mathfrak{X}, \mathfrak{Y}$ are worms and $\xi \in U_{\mathfrak{Y}}(\mathfrak{X}) \cap \Omega(\mathfrak{X})$, then:*

1. $\lfloor \xi \rfloor \in U_{\mathfrak{Y}}(\mathfrak{X})$;
2. *if moreover $\Omega(\xi) < O(\mathfrak{X})$, then $\Omega(\xi) \in U_{\mathfrak{Y}}(\mathfrak{X})$.*

Proof. For the first claim, if ξ is at most countable, $\lfloor \xi \rfloor = 0 \in U_{\mathfrak{Y}}(\mathfrak{X})$. If not, by Lemma 7.19, $\xi = O(\mathfrak{W})$ for some $\mathfrak{W} \sqsubset_{\Omega} U_{\mathfrak{Y}}(\mathfrak{X})$, and by 7.12.2, $\eta = \lfloor \xi \rfloor$ occurs in \mathfrak{W} , hence $\eta \in U_{\mathfrak{Y}}(\mathfrak{X})$.

For the second, we observe that $\langle \overset{0}{O(\mathfrak{W})} \rangle^{\top} \triangleleft \langle \overset{0}{O(\mathfrak{W})+1} \rangle^{\top}$, so that

$$\Omega(\xi) = O \left(\left\langle \overset{0}{O(\mathfrak{W})} \right\rangle^{\top} \right) = \pi_{\mathfrak{W}} \left(\left\langle \overset{0}{O(\mathfrak{W})} \right\rangle^{\top} \right).$$

If we moreover have $\Omega(\xi) < O(\mathfrak{X})$, this gives us $\Omega(\xi) \in U_{\mathfrak{Y}}(\mathfrak{X})$. □

The following lemmas show that our work on worms can be used to study the sets $U_{\mathfrak{Y}}(\mathfrak{X})$.

Lemma 7.23. *Given spiders $\mathfrak{W}, \mathfrak{X}, \mathfrak{Y}$,*

$$\mathfrak{W} \sqsubset_{\Omega} U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$$

if and only if

$$\mathfrak{b}\mathfrak{W} \sqsubset U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X}).$$

Proof. Let

$$\mathfrak{W} = \left\langle \begin{smallmatrix} \lambda_1 \\ \eta_1 \end{smallmatrix} \right\rangle \dots \left\langle \begin{smallmatrix} \lambda_n \\ \eta_n \end{smallmatrix} \right\rangle^{\top} \in \mathbb{S}.$$

If $\mathfrak{W} \sqsubset_{\Omega} U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$, then each $\lambda_i, \eta_i \in U_{\mathfrak{Y}}(\mathfrak{X})$. By Lemma 7.22, $\Omega(\eta_i) \in U_{\mathfrak{Y}}(\mathfrak{X})$, and by Lemma 7.20, $\Omega(\eta_i) + \lambda_i \in U_{\mathfrak{Y}}(\mathfrak{X})$. Since

$$\Omega(\eta_i), \Omega(\eta_i) + \lambda_i \leq O(\mathfrak{W}) < O(\mathfrak{X}),$$

it follows that $\mathfrak{b}\mathfrak{W} \sqsubset U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$.

Conversely, if $\mathfrak{b}\mathfrak{W} \sqsubset U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$, write $\mathfrak{b}\mathfrak{W} = \mu_1 \dots \mu_n^{\top}$. By Lemma 7.22, $\lfloor \mu_i \rfloor \in U_{\mathfrak{Y}}(\mathfrak{X})$, and by Lemma 7.20 together with Lemma 7.21.2, $\dot{\mu}_i \in U_{\mathfrak{Y}}(\mathfrak{X})$. It follows from Lemma 7.9 that

$$\mathfrak{W} = \sharp \mathfrak{b}\mathfrak{W} \sqsubset_{\Omega} U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$$

by observing that $\lfloor \mu_i \rfloor, \dot{\mu}_i \leq \mu_i < O(\mathfrak{X})$. □

With this we see that the sets $U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ are almost worm-perfect.

Theorem 7.24. *Given spiders $\mathfrak{X}, \mathfrak{Y}$ and an ordinal $\xi < O(\mathfrak{X})$, $\xi \in U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ if and only if there is $\mathfrak{z} \sqsubset U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ with $\xi = o(\mathfrak{z})$.*

Proof. Given an ordinal ξ , by Lemma 7.19, $\xi \in U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ if and only if there is $\mathfrak{z} \sqsubset_{\Omega} U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ with $\xi = O(\mathfrak{z})$. But by Lemma 7.23, by setting $\mathfrak{z} = \flat \mathfrak{z}$ we see that this is equivalent to there existing $\mathfrak{z} \sqsubset U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$ with $\xi = o(\mathfrak{z})$. \square

As a consequence, we obtain that $U_{\mathfrak{Y}}(\mathfrak{z})$ is closed under bounded hyperexponentiation.

Lemma 7.25. *If $\mathfrak{X}, \mathfrak{Y}$ are worms and $\alpha, \beta \in U_{\mathfrak{Y}}(\mathfrak{X})$ are such that $e^{\alpha}\beta < O(\mathfrak{X})$, then $e^{\alpha}\beta \in U_{\mathfrak{Y}}(\mathfrak{X})$.*

Proof. We may assume that $0 < \alpha, \beta < e^{\alpha}\beta$, so that if $e^{\alpha}\beta < O(\mathfrak{X})$, then $\alpha, \beta < O(\mathfrak{X})$. By Theorem 7.24, $\beta = o(\mathfrak{v})$ for some $\mathfrak{v} = \lambda_1 \dots \lambda_n \top \sqsubset U_{\mathfrak{Y}}(\mathfrak{X}) \cap O(\mathfrak{X})$. Since $e^{\alpha}\beta$ is additively indecomposable, for each $i \in [1, n]$, $\alpha + \lambda_i \leq \alpha + \beta < e^{\alpha}\beta$, hence by Lemma 7.20, $\alpha + \lambda_i \in U_{\mathfrak{Y}}(\mathfrak{X})$. Thus $\mathfrak{w} = \alpha \uparrow \mathfrak{v} \sqsubset U_{\mathfrak{Y}}(\mathfrak{X})$, and $o(\mathfrak{w}) = e^{\alpha}\beta$, which by Theorem 7.24 implies that $e^{\alpha}\beta \in U_{\mathfrak{Y}}(\mathfrak{X})$. \square

This tells us that, below $O(\mathfrak{X})$, the sets $U_{\mathfrak{Y}}(\mathfrak{X})$ behave very similar to the sets $C_{\eta}(\lambda)$. Conversely, we can prove that the sets $C_{\eta}(\lambda)$ are ‘spider-perfect’.

Lemma 7.26. *If η, λ are ordinals and $\mathfrak{W} \sqsubset_{\Omega} C_{\eta}(\lambda)$, then $O(\mathfrak{W}) \in C_{\eta}(\lambda)$.*

Proof. Suppose that $\mathfrak{W} \sqsubset_{\Omega} C_{\eta}(\lambda)$ and $\mathfrak{W} \triangleleft \mathfrak{X}$. The set $C_{\eta}(\lambda)$ is closed under $\Omega(\cdot)$ and addition, so from $\mathfrak{W} \sqsubset_{\Omega} C$ we obtain $\flat \mathfrak{W} \sqsubset C$. But $C_{\eta}(\lambda)$ is hyperexponentially perfect, thus by Lemma 5.34 it is worm-perfect. We conclude that $O(\mathfrak{W}) = o(\flat \mathfrak{W}) \in C$. \square

Thus the functions π_{η} should closely mimic the functions ψ_{η} . However, a full translation between the two systems would go beyond the scope of the current work. Instead, we conclude with a conjecture.

Conjecture 7.27. $\psi_0 \Omega^{\omega} 1 = \pi_{\top} \left(\langle \Omega^0_{\omega 1} \rangle^{\top} \right)$.



Figure 2: An autonomous spider.

7.5 Autonomous spiders and ordinal notations

We can use autonomous spiders to produce an ordinal notation system, similar to Beklemishev's autonomous worms. We define them as follows:

Definition 7.28. *We define the set of autonomous spiders, \mathcal{S} , to be the least set such that:*

1. $\top \in \mathcal{S}$;
2. if $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathcal{S}$, then $\left(\begin{smallmatrix} \mathbf{X} \\ \mathbf{Y} \end{smallmatrix}\right) \mathbf{Z} \in \mathcal{S}$.

As with autonomous worms, each autonomous spider can be interpreted as a 'real' spider.

Definition 7.29. *We define a function $\cdot^\pi_O : \mathcal{S} \rightarrow \mathbb{S}$ by*

1. $\top^\pi_O = \top$,
2. $\left(\left(\begin{smallmatrix} \mathbf{X} \\ \mathbf{Y} \end{smallmatrix}\right) \mathbf{Z}\right)^\pi_O = \langle \pi_{\mathbf{Y}^\pi_O} \mathbf{X}^\pi_O \rangle_{O(\mathbf{Y}^\pi_O)} \mathbf{Z}^\pi_O$.

For $\mathbf{X}, \mathbf{Y} \in \mathcal{S}$ we set $O(\mathbf{X}) = O(\mathbf{X}^\pi_O)$ and $\mathbf{p}_\mathbf{Y} \mathbf{X} = \pi_{\mathbf{Y}^\pi_O} \mathbf{X}^\pi_O$.

We will often omit writing \top , so that for example $\left(\begin{smallmatrix} \top \\ \top \end{smallmatrix}\right)$ denotes $\left(\begin{smallmatrix} \top \\ \top \end{smallmatrix}\right) \top$. The proofs of the following two results are analogous to those of Lemma 6.35 and Theorem 6.36, respectively, and we omit them.

Theorem 7.30. *For any $\xi \in U_\top \left(\langle \Omega^0_{\omega(0)} \rangle \top\right)$, there exists $\mathbf{X} \in \mathcal{S}$ such that $\xi = O(\mathbf{X})$.*

Thus assuming Conjecture 7.27, the autonomous spiders indeed provide a notation system for all ordinals below $\psi_0 \Omega^\omega 1$, along with some uncountable ordinals.

8 Concluding remarks

We have developed notation systems for impredicative ordinals based on reflection calculi, thus providing a positive answer to Mints' and Pakhomov's question. These notation systems are obtained by considering strong provability operators extending a theory T . In the process, we have also given a general overview of existing notation systems based on worms.

This work is still exploratory and further developments are required to fully flesh out our proposal. First, no decision procedure is given to determine whether $O(w) < O(v)$ when w, v are impredicative autonomous worms or spiders. While such a decision procedure might be extractable from Theorem 3.17 together with procedures for more standard systems based on ψ , it would be preferable to provide deductive calculi in the style of RC. Second, the set-theoretic interpretations sketched in Section 7 are only tentative and require a rigorous treatment. I'll leave both of these points for future work.

The ultimate goal of the efforts presented here are for the computation of Π_1^0 ordinals of strong theories of second-order arithmetic. There are many more hurdles to overcome before attaining such a goal, but hopefully the ideas presented here will help to lead the way forward.

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$$\omega \quad \left(\left(\right) \right) \quad \varepsilon_0 \quad \left(\left(\left(\right) \right) \right) \quad \Gamma_0 \quad \left(\left(\left(\right) \right) \right)$$

Figure 3: Some familiar ordinals represented as autonomous spiders.

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