p-ADIC ANALOGUES OF HYPERGEOMETRIC IDENTITIES

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ABSTRACT. We prove many congruences modulo p^2 for the truncated hypergeometric series in a unified way. For example, for any odd prime p and two p-integers α, β, we have the p-adic Gauss identity

$$\sum_{k=0}^{p-1} \frac{(\alpha)_k(\beta)_k}{(1)_k^2} \equiv -\frac{\Gamma_p(1-\alpha-\beta)}{\Gamma_p(1-\alpha)\Gamma_p(1-\beta)} \pmod{p^2},$$

provided that p is less than the sum of the least non-negative residues of α and β modulo p. Furthermore, the p-adic analogues of some common hypergeometric identities, including the balanced ${}_4F_3$ transformation and Whipple's ${}_7F_6$ transformation, are established. We also confirm a conjecture of Deines et al.

1. Introduction

Define the *hypergeometric series*

$$ {}_{m+1}F_m\left[\begin{matrix}\alpha_0 & \alpha_1 & \ldots & \alpha_m \\ & \beta_1 & \ldots & \beta_m\end{matrix}\middle|\, z\right] := \sum_{k=0}^{\infty} \frac{(\alpha_0)_k(\alpha_1)_k\cdots(\alpha_m)_k}{(\beta_1)_k\cdots(\beta_m)_k}\cdot\frac{z^k}{k!}, \tag{1.1}$$

where $\alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_m, z \in \mathbb{C}$ and

$$(\alpha)_k = \begin{cases}\alpha(\alpha+1)\cdots(\alpha+k-1), & \text{if } k \geq 1,\\ 1, & \text{if } k = 0.\end{cases}$$

It is easy to see that (1.1) absolutely converges whenever $|z| < 1$, or $|z| = 1$ and $\Re(\beta_1 + \cdots + \beta_m) > \Re(\alpha_0 + \cdots + \alpha_m)$. If (1.1) is convergent, it is also called a *hypergeometric function*. The hypergeometric functions play a very important role in mathematics. A reason is that many common mathematical functions, such as e^x, $\log x$, $\arcsin x$, $\arctan x$, can be expressed in the form of a hypergeometric series. The explicit expressions of a few hypergeometric functions are known. For example, a well-known result due to Gauss says that

$${}_2F_1\left[\begin{matrix}\alpha & \beta \\ & \gamma\end{matrix}\middle|\, 1\right] = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{1.2}$$

2010 *Mathematics Subject Classification.* Primary 33C05; Secondary 05A10, 11A07, 11B65, 11S80, 33B15, 33C20.

Key words and phrases. truncated hypergeometric series; p-adic gamma function; hypergeometric transformation.

provided that $\Re(\gamma) > \Re(\alpha+\beta)$, where $\Gamma(\cdot)$ is the gamma function. Furthermore, there are some transformations between the hypergeometric functions. For example, we have the Euler transformation

$$ {}_2F_1\left[\begin{matrix}\alpha & \beta \\ & \gamma\end{matrix}\middle| z\right] = (1-z)^{-\alpha}\cdot {}_2F_1\left[\begin{matrix}\alpha & \gamma-\beta \\ & \gamma\end{matrix}\middle| \frac{z}{z-1}\right]. \tag{1.3}$$

For $n = 0, 1, 2, \ldots$, define *the truncated hypergeometric function*

$$ {}_{m+1}F_m\left[\begin{matrix}\alpha_0 & \alpha_1 & \ldots & \alpha_m \\ & \beta_1 & \ldots & \beta_m\end{matrix}\middle| z\right]_n := \sum_{k=0}^{n}\frac{(\alpha_0)_k(\alpha_1)_k\cdots(\alpha_m)_k}{(\beta_1)_k\cdots(\beta_m)_k}\cdot\frac{z^k}{k!}.$$

That is, a truncated hypergeometric function is the summation of the first finitely many terms in a hypergeometric series. Unlike the original hypergeometric functions, few explicit formulas are known for the truncated hypergeometric functions. On the other hand, in the recent years, the arithmetic properties of the truncated hypergeometric series were widely studied. For example, Ahlgren and Ono [2] proved that for any odd prime p,

$$ {}_4F_3\left[\begin{matrix}\frac12 & \frac12 & \frac12 & \frac12 \\ & 1 & 1 & 1\end{matrix}\middle| 1\right]_{p-1} \equiv a(p) \pmod{p^2}, \tag{1.4}$$

where $a(p)$ is the p-th Fourier coefficient of $\eta(2z)^4\eta(4z)^2 \in S_4(\Gamma_0(8))$ and $\eta(\cdot)$ is the Dedekind eta-function. Ahlgren and Ono's result solves a conjectured congruence concerning the Apéry numbers. Subsequently, Kilbourn [12] showed that (1.4) is still valid if p^2 is replaced by p^3. Furthermore, it has been shown that the truncated hypergeometric functions are closely related to the Gaussian hypergeometric functions [1, 7, 26], which is a finite field analogue of the original hypergeometric functions.

In [21, 22], Mortenson proved that for any prime $p \geq 5$,

$$\begin{aligned} {}_2F_1\left[\begin{matrix}\frac12 & \frac12 \\ & 1\end{matrix}\middle| 1\right]_{p-1} &\equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \quad {}_2F_1\left[\begin{matrix}\frac13 & \frac23 \\ & 1\end{matrix}\middle| 1\right]_{p-1} \equiv \left(\frac{-3}{p}\right) \pmod{p^2}, \\ {}_2F_1\left[\begin{matrix}\frac14 & \frac34 \\ & 1\end{matrix}\middle| 1\right]_{p-1} &\equiv \left(\frac{-2}{p}\right) \pmod{p^2}, \quad {}_2F_1\left[\begin{matrix}\frac16 & \frac56 \\ & 1\end{matrix}\middle| 1\right]_{p-1} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \end{aligned} \tag{1.5}$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol. The above congruences confirms several conjectures of Rodriguez-Villegas [31], which are motivated by the Calabi-Yau manifolds. A key ingredient of Mortenson's proofs is the Gross-Koblitz formula, which can transfer a p-adic gamma function to a Gauss sum. Moreover, in [23], Mortenson established a general frame to study the supercongruences concerning the truncated hypergeometric functions by using the Gross-Koblitz formula and the Gaussian hypergeometric functions. Also, an elementary proof of those congruences in (1.5) was given in [37].

On the other hand, in [34], Sun found that Mortenson's congruences can be extended to a unified form. For any $\alpha \in \mathbb{Q}$ which is p-integral (i.e. the denominator of α is prime to p), let $\langle\alpha\rangle_p$ be the least non-negative residue of α modulo p, i.e., the integer lying in $\{0, 1, \ldots, p-1\}$

such that $\langle\alpha\rangle_p \equiv \alpha \pmod p$. Sun proved that for any p-integral $\alpha \in \mathbb{Q}$,

$$ {}_2F_1\begin{bmatrix}\alpha & 1-\alpha \\ & 1\end{bmatrix}\bigg|\, 1\bigg]_{p-1} \equiv (-1)^{\langle-\alpha\rangle_p} \pmod{p^2}. \tag{1.6}$$

It is not difficult to verify that Mortenson's congruences in (1.5) are the special cases of (1.6) when $\alpha = 1/2, 1/3, 1/4, 1/6$. In fact, Sun obtained the following general result:

$$ {}_2F_1\left[\begin{matrix}\alpha & 1-\alpha \\ & 1\end{matrix}\,\middle|\, z\right]_{p-1} \equiv (-1)^{\langle-\alpha\rangle_p}\,{}_2F_1\left[\begin{matrix}\alpha & 1-\alpha \\ & 1\end{matrix}\,\middle|\, 1-z\right]_{p-1} \pmod{p^2}. \tag{1.7}$$

As we shall see later, (1.7) can be viewed as a p-adic analogue of a special case of the linear transformation due to Pfaff [3, Eq. (2.3.14)]

$$ {}_2F_1\left[\begin{matrix}-n & \beta \\ & \gamma\end{matrix}\,\middle|\, z\right] = \frac{(\gamma-\beta)_n}{(\gamma)_n}\cdot {}_2F_1\left[\begin{matrix}-n & \beta \\ & \beta-\gamma-n+1\end{matrix}\,\middle|\, 1-z\right], \tag{1.8}$$

where $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. For more supercongruences involving the truncated hypergeometric functions, the readers may read [2, 4, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 27, 28, 29, 32, 33, 35, 36, 37, 38, 39, 40, 41].

In this paper, we shall focus on the relation between the congruences concerning the truncated hypergeometric functions and the identities concerning the original hypergeometric functions. In particular, we can show that many hypergeometric identities have its p-adic analogue, i.e., a congruence modulo p^2 concerning the corresponding truncated hypergeometric function.

First, substituting $\gamma = 1$ in the Gauss identity (1.2), we get

$$ {}_2F_1\left[\begin{matrix}\alpha & \beta \\ & 1\end{matrix}\,\middle|\, 1\right] = \frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)}. \tag{1.9}$$

In order to give a p-adic analogue of (1.9), we need the p-adic gamma function. For a prime p, let $\mathbb{Z}_p$ denote the ring of all p-adic integers and let

$$\mathbb{Z}_p^\times := \{a \in \mathbb{Z}_p : a \text{ is prime to } p\}.$$

For each $\alpha \in \mathbb{Z}_p$, define the p-adic order $\nu_p(\alpha) := \max\{n \in \mathbb{N} : p^n \mid \alpha\}$ and the p-adic norm $|\alpha|_p := p^{-\nu_p(\alpha)}$. Define the p-adic gamma function $\Gamma_p(\cdot)$ by

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1\le j<n \\ (j,p)=1}} j, \qquad n = 1, 2, 3, \ldots,$$

and

$$\Gamma_p(\alpha) = \lim_{\substack{|\alpha-n|_p\to 0 \\ n\in\mathbb{N}}} \Gamma_p(n), \qquad \alpha \in \mathbb{Z}_p.$$

In particular, we set $\Gamma_p(0) = 1$. Throughout the whole paper, we only need to use the most basic properties of Γ_p, and all of them can be found in [25]. For example, we know that

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } |x|_p = 1, \\ -1, & \text{if } |x|_p > 1. \end{cases}$$

Now we can give the following p-adic analogue of (1.9).

Theorem 1.1. *Let p be an odd prime and $\alpha, \beta \in \mathbb{Z}_p$. If $\langle -\alpha\rangle_p + \langle -\beta\rangle_p < p$, then*

$$ {}_2F_1\begin{bmatrix}\alpha & \beta \\ & 1\end{bmatrix}\bigg|\,1\bigg]_{p-1} \equiv -\frac{\Gamma_p(1-\alpha-\beta)}{\Gamma_p(1-\alpha)\Gamma_p(1-\beta)} \pmod{p^2}. \tag{1.10}$$

And if $\langle -\alpha\rangle_p + \langle -\beta\rangle_p \ge p$, then

$$ {}_2F_1\begin{bmatrix}\alpha & \beta \\ & 1\end{bmatrix}\bigg|\,1\bigg]_{p-1} \equiv \big(\alpha+\beta+\langle -\alpha\rangle_p + \langle -\beta\rangle_p - p\big)\cdot\frac{\Gamma_p(1-\alpha-\beta)}{\Gamma_p(1-\alpha)\Gamma_p(1-\beta)} \pmod{p^2}. \tag{1.11}$$

For example, substituting $\alpha = 1/3$ and $\beta = 1/4$ in (1.10), we have

$$ {}_2F_1\begin{bmatrix}\frac13 & \frac14 \\ & 1\end{bmatrix}\bigg|\,1\bigg]_{p-1} \equiv -\frac{\Gamma_p(\frac{5}{12})}{\Gamma_p(\frac23)\Gamma_p(\frac34)} \pmod{p^2}$$

for any prime $p \equiv 1 \pmod 4$. Also, (1.6) can be deduced from (1.10) since $\Gamma_p(\alpha)\Gamma_p(1-\alpha) = (-1)^{\langle -\alpha\rangle_p - 1}$. Furthermore, we mention that the negative sign in the right side of (1.10) is due to $\Gamma_p(1) = -1$. In fact, as we shall see soon, if the "$p-1$" in the right subscript of the left side of (1.10) is replaced by "$\langle -\gamma\rangle_p$" , then we may get a complete p-adic analogue of (1.2).

Before we introduce more results, let us briefly describe the way to prove (1.10). Our first observation is that if $\alpha = -\langle -\alpha\rangle_p$, then the truncated hypergeometric function is actually an original hypergeometric function. At this moment, the desired congruence easily follows from the identity (1.9). Thus for general $\alpha \in \mathbb{Z}_p$, we only need to prove

$$\begin{aligned}&\frac1p\left({}_2F_1\begin{bmatrix}\alpha & \beta \\ & 1\end{bmatrix}\bigg|\,1\bigg]_{p-1} - {}_2F_1\begin{bmatrix}-a & \beta \\ & 1\end{bmatrix}\bigg|\,1\bigg]\right)\\ \equiv&\frac1p\left(\frac{\Gamma_p(1+a-\beta)}{\Gamma_p(1+a)\Gamma_p(1-\beta)} - \frac{\Gamma_p(1-\alpha-\beta)}{\Gamma_p(1-\alpha)\Gamma_p(1-\beta)}\right) \pmod{p},\end{aligned} \tag{1.12}$$

where $a = \langle -\alpha\rangle_p$. Obviously (1.12) is just a congruence modulo p, which can be easily deduced from the combinatorial identity

$$\sum_{k=0}^{b}\binom{b}{k}\sum_{j=1}^{k}\frac{(-1)^j}{j}\cdot\binom{a}{k-j} = -\binom{a+b}{b}\sum_{j=a+1}^{a+b}\frac1j \tag{1.13}$$

where $b = \langle -\beta\rangle_p$. Furthermore, clearly (1.13) is an equivalent form of

$$\frac{d}{dx}\left({}_2F_1\begin{bmatrix}-a+x & -b \\ & 1\end{bmatrix}\bigg|\,1\bigg]\right)\bigg|_{x=0} = \frac{d}{dx}\left(\frac{\Gamma(1+a+b-x)}{\Gamma(1+a-x)\Gamma(1+b)}\right)\bigg|_{x=0}.$$

Notice that there are two steps in the above proof of (1.10). One is that the congruence can be directly deduced from the corresponding hypergeometric identity whenever $\alpha = -\langle -\alpha\rangle_p$. Thus the original congruence modulo p^2 is reduced to a new congruence modulo p. The next step is that by taking the derivatives of the hypergeometric identity in the variable α, we may get a combinatorial identity which easily implies the desired congruence modulo p. Then the proof is complete. The above two steps both heavily depend on the corresponding hypergeometric identity. In fact, with help of the same idea, in most cases, if at least one of $\alpha_0,\ldots,\alpha_m,\beta_1,\ldots,\beta_m$ is variable in a hypergeometric identity, then we may get a congruence modulo p^2 involving the corresponding truncated hypergeometric function.

In the next section, we shall systematically introduce our main results of this paper, including the p-adic analogues of some quadratic transformations of ${}_2F_1$, the transformation of balanced ${}_4F_3$ series, and Whipple's transformation on ${}_7F_6$ series. Throughout the whole paper, we always assume that p is an odd prime.

2. Main results

First, let us consider the p-adic analogue of the linear and quadratic transformations of the ${}_2F_1$ series. Substituting $\beta=1-\alpha$ in the quadratic transformation ([3, Eq. (3.1.3)])

$$ {}_2F_1\left[\begin{matrix}\alpha & \beta\\ & \frac12+\frac12(\alpha+\beta)\end{matrix}\middle|\, z\right]={}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12\beta\\ & \frac12+\frac12(\alpha+\beta)\end{matrix}\middle|\, 4z(1-z)\right], \tag{2.1}$$

we may get

$$ {}_2F_1\left[\begin{matrix}\alpha & 1-\alpha\\ & 1\end{matrix}\middle|\, z\right]={}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12-\frac12\alpha\\ & 1\end{matrix}\middle|\, 4z(1-z)\right]. \tag{2.2}$$

The identity (2.2) has the following p-adic analogue.

Theorem 2.1. *Suppose that $\alpha\in\mathbb{Z}_p$ and $\langle-\alpha\rangle_p$ is even. Then for any $z\in\mathbb{Z}_p$,*

$$ {}_2F_1\left[\begin{matrix}\alpha & 1-\alpha\\ & 1\end{matrix}\middle|\, z\right]_{p-1}\equiv{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12-\frac12\alpha\\ & 1\end{matrix}\middle|\, 4z(1-z)\right]_{p-1}\pmod{p^2}. \tag{2.3}$$

For example, for each prime $p\equiv 1,3 \pmod 5$, we have

$$ {}_2F_1\left[\begin{matrix}\frac25 & \frac35\\ & 1\end{matrix}\middle|\, z\right]_{p-1}\equiv{}_2F_1\left[\begin{matrix}\frac15 & \frac3{10}\\ & 1\end{matrix}\middle|\, 4z(1-z)\right]_{p-1}\pmod{p^2}.$$

A similar example is the Clausen identity [3, p. 116, Ex. 13]

$$ \left({}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12\beta\\ & \frac12+\frac12(\alpha+\beta)\end{matrix}\middle|\, z\right]\right)^2={}_3F_2\left[\begin{matrix}\alpha & \beta & \frac12(\alpha+\beta)\\ & \alpha+\beta & \frac12+\frac12(\alpha+\beta)\end{matrix}\middle|\, z\right]. \tag{2.4}$$

Combining (2.4) with (2.1), we get

$$ \left({}_2F_1\left[\begin{matrix}\alpha & \beta\\ & \frac12+\frac12(\alpha+\beta)\end{matrix}\middle|\, z\right]\right)^2={}_3F_2\left[\begin{matrix}\alpha & \beta & \frac12(\alpha+\beta)\\ & \alpha+\beta & \frac12+\frac12(\alpha+\beta)\end{matrix}\middle|\, 4z(1-z)\right]. \tag{2.5}$$

In particular,

$$ \left({}_2F_1\left[\begin{matrix}\alpha & 1-\alpha\\ & 1\end{matrix}\middle|\, z\right]\right)^2={}_3F_2\left[\begin{matrix}\alpha & 1-\alpha & \frac12\\ & 1 & 1\end{matrix}\middle|\, 4z(1-z)\right] \tag{2.6}$$

We also have the p-adic analogue of (2.6) as follows.

Theorem 2.2. *Let $\alpha, z \in \mathbb{Z}_p$.*

$$\left({}_2F_1\left[\begin{matrix}\alpha & 1-\alpha \\ & 1\end{matrix}\middle| z\right]_{p-1}\right)^2 \equiv {}_3F_2\left[\begin{matrix}\alpha & 1-\alpha & \frac{1}{2} \\ & 1 & 1\end{matrix}\middle| 4z(1-z)\right]_{p-1} \pmod{p^2}. \tag{2.7}$$

It follows from Theorems 2.1 and 2.2 that

Corollary 2.1. *Suppose that $\alpha, z \in \mathbb{Z}_p$ and $\langle -\alpha\rangle_p$ is even.*

$$\left({}_2F_1\left[\begin{matrix}\frac{1}{2}\alpha & \frac{1}{2}-\frac{1}{2}\alpha \\ & 1\end{matrix}\middle| z\right]_{p-1}\right)^2 \equiv {}_3F_2\left[\begin{matrix}\alpha & 1-\alpha & \frac{1}{2} \\ & 1 & 1\end{matrix}\middle| z\right]_{p-1} \pmod{p^2}. \tag{2.8}$$

However, most transformations of the ${}_2F_1$ series, unlike (2.1) and (2.4), will involve a factor of the form $(1-z)^{-\alpha}$, $(1-z/2)^{-\alpha}$ or $(1-z)^{-\frac{1}{2}\alpha}$. The main difficulty, which we must meet, is how to give the p-adic analogues of such factors. Setting $\gamma = 1$ in the Euler transformation (1.3), we have

$${}_2F_1\left[\begin{matrix}\alpha & \beta \\ & 1\end{matrix}\middle| z\right] = (1-z)^{-\alpha}\cdot {}_2F_1\left[\begin{matrix}\alpha & 1-\beta \\ & 1\end{matrix}\middle| \frac{z}{z-1}\right]. \tag{2.9}$$

Unfortunately, we don't know what the p-adic analogue of (2.9) is. On the other hand, we can give a partial p-adic analogue of

$${}_2F_1\left[\begin{matrix}\alpha & \alpha \\ & 1\end{matrix}\middle| z\right] = (1-z)^{-\alpha}{}_2F_1\left[\begin{matrix}\alpha & 1-\alpha \\ & 1\end{matrix}\middle| \frac{z}{z-1}\right], \tag{2.10}$$

which is evidently a special case of (2.9). For $\alpha \in \mathbb{Z}_p$, define

$$\lambda_p(\alpha) := \frac{\langle -\alpha\rangle_p - \langle -\alpha\rangle_{p^2}}{p}\cdot (p-1) - \langle -\alpha\rangle_p.$$

Theorem 2.3. *Let $\alpha \in \mathbb{Z}_p$ and $z \in \mathbb{Z}_p^\times$. Suppose that $z-1$ is prime to p. Then*

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}\alpha & \alpha \\ & 1\end{matrix}\middle| z\right]_{p-1} - z^{1-\lambda_p(\alpha)}{}_2F_1\left[\begin{matrix}\alpha & \alpha \\ & 1\end{matrix}\middle| \frac{1}{z}\right]_{p-1}\\ \equiv&(1-z)^{1-\lambda_p(\alpha)}{}_2F_1\left[\begin{matrix}\alpha & 1-\alpha \\ & 1\end{matrix}\middle| \frac{z}{z-1}\right]_{p-1} \pmod{p^2}.\end{aligned} \tag{2.11}$$

In fact, here $(1-z)^{-\lambda_p(\alpha)}$ in (2.11) can be viewed as a p-adic analogue of $(1-z)^{-\alpha}$ in (2.10). For example, for any $p \equiv 1 \pmod 3$, we have

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}\frac{1}{3} & \frac{1}{3} \\ & 1\end{matrix}\middle| z\right]_{p-1} - z^{1+\frac{1}{3}p(p-1)}{}_2F_1\left[\begin{matrix}\frac{1}{3} & \frac{1}{3} \\ & 1\end{matrix}\middle| \frac{1}{z}\right]_{p-1}\\ \equiv&(1-z)^{1+\frac{1}{3}p(p-1)}{}_2F_1\left[\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}\middle| \frac{z}{z-1}\right]_{p-1} \pmod{p^2}.\end{aligned}$$

On the other hand, if z is replaced by z^{-1} in (2.10), we may get

$$ {}_2F_1\left[\begin{matrix}\alpha & 1-\alpha\\ & 1\end{matrix}\middle| \frac{z}{z-1}\right]_{p-1} \equiv (-1)^{\langle-\alpha\rangle_p}{}_2F_1\left[\begin{matrix}\alpha & 1-\alpha\\ & 1\end{matrix}\middle| \frac{1}{1-z}\right]_{p-1} \pmod{p^2}, $$

which is apparently an equivalent form of (1.7).

Let us turn another quadratic transformations. Substituting $\beta=\alpha$ in the transformation [3, Eq. (3.1.9)]

$$ {}_2F_1\left[\begin{matrix}\alpha & \beta\\ & \alpha-\beta+1\end{matrix}\middle| z\right] = (1+z)^{-\alpha}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12+\frac12\alpha\\ & \alpha-\beta+1\end{matrix}\middle| \frac{4z}{(1+z)^2}\right], \tag{2.12} $$

we have

$$ {}_2F_1\left[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\middle| z\right] = (1+z)^{-\alpha}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12+\frac12\alpha\\ & 1\end{matrix}\middle| \frac{4z}{(1+z)^2}\right]. \tag{2.13} $$

Theorem 2.4. *Suppose that $\alpha\in\mathbb{Z}_p$ and $z\in\mathbb{Z}_p^\times$ with $z+1$ is prime to p.*

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\middle| z\right]_{p-1} + z^{1-\lambda_p(\alpha)}{}_2F_1\left[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\middle| \frac1z\right]_{p-1}\\ \equiv&(1+z)^{1-\lambda_p(\alpha)}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12+\frac12\alpha\\ & 1\end{matrix}\middle| \frac{4z}{(1+z)^2}\right]_{p-1} \pmod{p^2}. \end{aligned} \tag{2.14} $$

We give an explanation on (2.14). Replacing z by z^{-1} in (2.13), we get

$$ z^{-\alpha}{}_2F_1\left[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\middle| \frac1z\right] = (1+z)^{-\alpha}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12+\frac12\alpha\\ & 1\end{matrix}\middle| \frac{4z}{(1+z)^2}\right]. \tag{2.15} $$

Hence (2.14) factly can be viewed as a p-adic of (2.13) $+\, z\cdot$ (2.15). In fact, in the p-adic analogue of a quadratic transformation

$$ {}_2F_1\left[\begin{matrix}* & *\\ & 1\end{matrix}\middle| z\right] = \big(1-\upsilon(z)\big)^{-\alpha}{}_2F_1\left[\begin{matrix}* & *\\ & 1\end{matrix}\middle| \Omega(z)\right] $$

where Ω is a quadratic rational function, there will are two terms appearing in the left side: one is concerning z, and the other is concerning $\varrho(z)$, where ϱ is a rational function such that $\Omega\big(\varrho(z)\big)=\Omega(z)$.

Letting $y=4z/(1+z)^2$, clearly we have $y/(y-1)=-4z/(1-z)^2$. So combining (2.10) and (2.13), we obtain that

$$ {}_2F_1\left[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\middle| z\right] = (1-z)^{-\alpha}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12-\frac12\alpha\\ & 1\end{matrix}\middle| -\frac{4z}{(1-z)^2}\right]. \tag{2.16} $$

Theorem 2.5. *Suppose that $\alpha \in \mathbb{Z}_p$ with $\langle -\alpha\rangle_p$ is even, and that $z \in \mathbb{Z}_p^\times$ with $z-1$ is prime to p. Then*

$$\begin{aligned}&{}_2F_1\begin{bmatrix}\alpha & \alpha \\ & 1\end{bmatrix}\!\!\left|\, z\right]_{p-1} - z^{1-\lambda_p(\alpha)}{}_2F_1\begin{bmatrix}\alpha & \alpha \\ & 1\end{bmatrix}\!\!\left|\, \frac{1}{z}\right]_{p-1}\\ \equiv&(1-z)^{1-\lambda_p(\alpha)}{}_2F_1\begin{bmatrix}\frac12\alpha & \frac12-\frac12\alpha \\ & 1\end{bmatrix}\!\!\left|\, -\frac{4z}{(1-z)^2}\right]_{p-1} \pmod{p^2}.\end{aligned}\tag{2.17}$$

Consider the quadratic transformation

$${}_2F_1\left[\begin{matrix}\alpha & \beta \\ & 2\beta\end{matrix}\,\middle|\, z\right] = \left(1-\frac{z}{2}\right)^{-\alpha}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12+\frac12\alpha \\ & \frac12+\beta\end{matrix}\,\middle|\, \frac{z^2}{(2-z)^2}\right].\tag{2.18}$$

Replacing β by $1/2$ in (2.18), we have

$${}_2F_1\left[\begin{matrix}\alpha & \frac12 \\ & 1\end{matrix}\,\middle|\, z\right] = \left(1-\frac{z}{2}\right)^{-\alpha}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12+\frac12\alpha \\ & 1\end{matrix}\,\middle|\, \frac{z^2}{(z-2)^2}\right].\tag{2.19}$$

Theorem 2.6. *Let $\alpha, z \in \mathbb{Z}_p$. Suppose that both $z-1$ and $z-2$ are prime to p.*

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}\alpha & \frac12 \\ & 1\end{matrix}\,\middle|\, z\right]_{p-1} + (1-z)^{1-\lambda_p(\alpha)}{}_2F_1\left[\begin{matrix}\alpha & \frac12 \\ & 1\end{matrix}\,\middle|\, \frac{z}{z-1}\right]_{p-1}\\ \equiv&2\left(1-\frac{z}{2}\right)^{1-\lambda_p(\alpha)}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12+\frac12\alpha \\ & 1\end{matrix}\,\middle|\, \frac{z^2}{(z-2)^2}\right]_{p-1} \pmod{p^2}.\end{aligned}\tag{2.20}$$

Note that $y/(y-1) = z^2/(4z-4)$ if $y = z^2/(z-2)^2$. It follows from (2.10) and (2.19) that

$${}_2F_1\left[\begin{matrix}\alpha & \frac12 \\ & 1\end{matrix}\,\middle|\, z\right] = (1-z)^{-\frac12\alpha}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12-\frac12\alpha \\ & 1\end{matrix}\,\middle|\, -\frac{z^2}{4z-4}\right].\tag{2.21}$$

However, the p-adic analogue of (2.21) looks a little different.

Theorem 2.7. *Let $\alpha, z \in \mathbb{Z}_p$. If $\langle -\alpha\rangle_p$ is even and $z-1$ is prime to p, then*

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}\alpha & \frac12 \\ & 1\end{matrix}\,\middle|\, z\right]_{p-1} + (1-z)^{\langle-\alpha\rangle_p}{}_2F_1\left[\begin{matrix}\alpha & \frac12 \\ & 1\end{matrix}\,\middle|\, \frac{z}{z-1}\right]_{p-1}\\ \equiv&2(1-z)^{\frac12\langle-\alpha\rangle_p}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12-\frac12\alpha \\ & 1\end{matrix}\,\middle|\, \frac{z^2}{4z-4}\right]_{p-1} \pmod{p^2}.\end{aligned}\tag{2.22}$$

Evidently in (2.22), the p-adic analogue of $(1-z)^{-\alpha}$ is $(1-z)^{\langle-\alpha\rangle_p}$, rather than $(1-z)^{-\lambda_p(\alpha)}$. For example, for prime $p \equiv 1 \pmod 5$,

$${}_2F_1\left[\begin{matrix}\frac15 & \frac12 \\ & 1\end{matrix}\,\middle|\, z\right]_{p-1} + (1-z)^{\frac{p-1}{5}}{}_2F_1\left[\begin{matrix}\frac15 & \frac12 \\ & 1\end{matrix}\,\middle|\, \frac{z}{z-1}\right]_{p-1} \equiv 2(1-z)^{\frac{p-1}{10}}{}_2F_1\left[\begin{matrix}\frac1{10} & \frac25 \\ & 1\end{matrix}\,\middle|\, \frac{z^2}{4z-4}\right]_{p-1} \pmod{p^2}.$$

Since $y^2/(y-2)^2 = 4z^2/(1+z^2)^2$ if $y = 4z/(1+z)^2$, combining (2.13) and (2.19), we get

$${}_2F_1\left[\begin{matrix}\alpha & \alpha \\ & 1\end{matrix}\,\middle|\, z^2\right] = (1+z)^{-2\alpha}{}_2F_1\left[\begin{matrix}\alpha & \frac12 \\ & 1\end{matrix}\,\middle|\, \frac{4z}{(1+z)^2}\right].\tag{2.23}$$

Theorem 2.8. *Suppose that $\alpha, z \in \mathbb{Z}_p$ with $z(1+z)$ is prime to p.*

$$
\begin{aligned}
&{}_2F_1\bigg[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\bigg|\, z^2\bigg]_{p-1} + z^{1-2\lambda_p(\alpha)}{}_2F_1\bigg[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\bigg|\, \frac{1}{z^2}\bigg]_{p-1}\\
\equiv&(1+z)^{1-2\lambda_p(\alpha)}{}_2F_1\bigg[\begin{matrix}\alpha & \frac{1}{2}\\ & 1\end{matrix}\bigg|\, \frac{4z}{(1+z)^2}\bigg]_{p-1} \pmod{p^2}.
\end{aligned}
\tag{2.24}
$$

Replacing z by $4z/(1+z)$ in (2.20) and applying (2.14), we obtain that

$$
\begin{aligned}
&{}_2F_1\bigg[\begin{matrix}\alpha & \frac{1}{2}\\ & 1\end{matrix}\bigg|\, \frac{4z}{(1+z)^2}\bigg]_{p-1} + \left(\frac{1-z}{1+z}\right)^{2-2\lambda_p(\alpha)}{}_2F_1\bigg[\begin{matrix}\alpha & \frac{1}{2}\\ & 1\end{matrix}\bigg|\, -\frac{4z}{(1-z)^2}\bigg]_{p-1}\\
\equiv&\frac{2\cdot(1+z^2)^{1-\lambda_p(\alpha)}}{(1+z)^{2-2\lambda_p(\alpha)}}{}_2F_1\bigg[\begin{matrix}\frac{1}{2}\alpha & \frac{1}{2}+\frac{1}{2}\alpha\\ & 1\end{matrix}\bigg|\, \frac{4z^2}{(1+z^2)^2}\bigg]_{p-1}\\
\equiv&\frac{2}{(1+z)^{2-2\lambda_p(\alpha)}}\left({}_2F_1\bigg[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\bigg|\, z^2\bigg]_{p-1} + z^{2-2\lambda_p(\alpha)}{}_2F_1\bigg[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\bigg|\, \frac{1}{z^2}\bigg]_{p-1}\right) \pmod{p^2}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&{}_2F_1\bigg[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\bigg|\, z^2\bigg]_{p-1} - z^{1-2\lambda_p(\alpha)}{}_2F_1\bigg[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\bigg|\, \frac{1}{z^2}\bigg]_{p-1}\\
\equiv&(1-z)^{1-2\lambda_p(\alpha)}{}_2F_1\bigg[\begin{matrix}\alpha & \frac{1}{2}\\ & 1\end{matrix}\bigg|\, -\frac{4z}{(1-z)^2}\bigg]_{p-1} \pmod{p^2}.
\end{aligned}
\tag{2.25}
$$

Of course, although the above process requires both $1+z$ and $1+z^2$ is prime to p, (2.25) is still valid for those exceptional z's, since (2.25) can be proved directly in the similar way as (2.24).

Now we have listed all p-adic quadratic transformations in this paper. In particular, substituting the special values of α and z in the above p-adic transformations, we may obtain some simple consequences. For example, substituting $z = 1/2$ in (2.3) and applying (1.10), we may get

$$
{}_2F_1\bigg[\begin{matrix}\alpha & 1-\alpha\\ & 1\end{matrix}\bigg|\, \frac{1}{2}\bigg]_{p-1} \equiv {}_2F_1\bigg[\begin{matrix}\frac{1}{2}\alpha & \frac{1}{2}-\frac{1}{2}\alpha\\ & 1\end{matrix}\bigg|\, 1\bigg]_{p-1} \equiv -\frac{\Gamma_p(\frac{1}{2})}{\Gamma_p(1-\frac{1}{2}\alpha)\Gamma_p(\frac{1}{2}+\frac{1}{2}\alpha)} \pmod{p^2} \tag{2.26}
$$

which was firstly proved by Liu [14]. More similar examples will be listed in Section 10. Further, we can prove several congruences modulo p^2 concerning ${}_2F_1\bigg[\begin{matrix}\frac{1}{4} & \frac{1}{4}\\ & 1\end{matrix}\bigg|\, z\bigg]_{p-1}$ and ${}_2F_1\bigg[\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ & 1\end{matrix}\bigg|\, z\bigg]_{p-1}$ for some special values of z, e.g.,

$$
{}_2F_1\bigg[\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ & 1\end{matrix}\bigg|\, -\frac{1}{3}\bigg]_{p-1} \equiv -\left(\frac{2}{p}\right)\cdot\frac{3\Gamma_p(\frac{4}{3})}{2\Gamma_p(\frac{3}{2})\Gamma_p(\frac{5}{6})} \pmod{p^2}
$$

for any prime $p \equiv 1 \pmod 3$.

Let us consider the identities involving ${}_3F_2$ series. The first one is the Watson identity [3, Theorem 3.5.5 (i)]

$$ {}_3F_2\left[\begin{matrix}\alpha & \beta & \gamma \\ & 2\beta & \frac{1}{2}(\alpha+\gamma+1)\end{matrix}\middle|\, 1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\beta)\Gamma(\frac{1}{2}+\frac{1}{2}(\alpha+\gamma))\Gamma(\frac{1}{2}+\beta-\frac{1}{2}(\alpha+\gamma))}{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)\Gamma(\frac{1}{2}+\frac{1}{2}\gamma)\Gamma(\frac{1}{2}+\beta-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}+\beta-\frac{1}{2}\gamma)}. \tag{2.27} $$

Substituting $\gamma = 1-\alpha$ in (2.27), we get

$$ {}_3F_2\left[\begin{matrix}\alpha & 1-\alpha & \beta \\ & 1 & 2\beta\end{matrix}\middle|\, 1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\beta)\Gamma(\beta)}{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha)\Gamma(1-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}+\beta-\frac{1}{2}\alpha)\Gamma(\beta+\frac{1}{2}\alpha)}. \tag{2.28} $$

Theorem 2.9. *Let $\alpha, \beta \in \mathbb{Z}_p$. Suppose that $\langle -\beta\rangle_p < p/2$ and $(2\beta)_{p-1} \not\equiv 0 \pmod{p^2}$. If $\langle -\alpha\rangle_p$ is even, then*

$$ {}_3F_2\left[\begin{matrix}\alpha & 1-\alpha & \beta \\ & 1 & 2\beta\end{matrix}\middle|\, 1\right]_{p-1} \equiv -\frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{2}+\beta)\Gamma_p(\beta)}{\Gamma_p(\frac{1}{2}+\frac{1}{2}\alpha)\Gamma_p(1-\frac{1}{2}\alpha)\Gamma_p(\frac{1}{2}+\beta-\frac{1}{2}\alpha)\Gamma_p(\beta+\frac{1}{2}\alpha)} \pmod{p^2}. \tag{2.29} $$

On the other hand, if $\langle -\alpha\rangle_p$ is odd, then

$$ {}_3F_2\left[\begin{matrix}\alpha & 1-\alpha & \beta \\ & 1 & 2\beta\end{matrix}\middle|\, 1\right]_{p-1} \equiv 0 \pmod{p^2}. \tag{2.30} $$

Notice that two requirements $\langle -\beta\rangle_p < p/2$ and $(2\beta)_{p-1} \not\equiv 0 \pmod{p^2}$ of Theorem 2.9 are both necessary. In fact, if $\langle -\beta\rangle_p > p/2$, then for any $\langle -2\beta\rangle_p < k \le \langle -\beta\rangle_p$, we shall have $(\beta)_k/(2\beta)_k$ is not p-integral. Also, the condition $(2\beta)_{p-1} \not\equiv 0 \pmod{p^2}$ can prevent a high power of p appearing in the denominators of the left sides of (2.29) and (2.30). As an example of Theorem 2.9, we have

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ & 1 & \frac{2}{3}\end{matrix}\middle|\, 1\right]_{p-1} \equiv -\frac{\Gamma_p(\frac{1}{3})\Gamma_p(\frac{1}{2})\Gamma_p(\frac{5}{6})}{\Gamma_p(\frac{3}{4})^2\Gamma_p(\frac{7}{12})^2} \pmod{p^2} $$

for any prime $p \equiv 1 \pmod{12}$ and

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ & 1 & \frac{2}{3}\end{matrix}\middle|\, 1\right]_{p-1} \equiv 0 \pmod{p^2} $$

for any prime $p \equiv 7 \pmod{12}$.

The next is Dixon's well-poised sum [3, Theorem 3.4.1]

$$ {}_3F_2\left[\begin{matrix}\alpha & \beta & \gamma \\ & \alpha-\beta+1 & \alpha-\gamma+1\end{matrix}\middle|\, 1\right] = \frac{\Gamma(\frac{1}{2}\alpha+1)\Gamma(\alpha-\beta+1)\Gamma(\alpha-\gamma+1)\Gamma(\frac{1}{2}\alpha-\beta-\gamma+1)}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}\alpha-\beta+1)\Gamma(\frac{1}{2}\alpha-\gamma+1)\Gamma(\alpha-\beta-\gamma+1)}. \tag{2.31} $$

Letting $\gamma = \alpha$ in (2.31), we have

$$ {}_3F_2\left[\begin{matrix}\alpha & \alpha & \beta \\ & 1 & \alpha-\beta+1\end{matrix}\middle|\, 1\right] = \frac{\Gamma(1+\frac{1}{2}\alpha)\Gamma(1+\alpha-\beta)\Gamma(1-\frac{1}{2}\alpha-\beta)}{\Gamma(\alpha+1)\Gamma(1-\frac{1}{2}\alpha)\Gamma(1-\beta)\Gamma(1+\frac{1}{2}\alpha-\beta)}. \tag{2.32} $$

Theorem 2.10. *Let $\alpha, \beta \in \mathbb{Z}_p$. Suppose that p^2 doesn't divide $(\alpha-\beta+1)_{p-1}$.*

(1) Suppose that $\langle -\alpha\rangle_p$ is even and $\langle -\alpha\rangle_p \le \langle -\beta\rangle_p < (p-\langle -\alpha\rangle_p)/2$. Then

$$ {}_3F_2\left[\begin{matrix}\alpha & \alpha & \beta \\ & 1 & \alpha-\beta+1\end{matrix}\middle| 1\right]_{p-1} \equiv -\frac{2\Gamma_p(1+\frac{1}{2}\alpha)\Gamma_p(1+\alpha-\beta)\Gamma_p(1-\frac{1}{2}\alpha-\beta)}{\Gamma_p(1+\alpha)\Gamma_p(1-\frac{1}{2}\alpha)\Gamma_p(1-\beta)\Gamma_p(1+\frac{1}{2}\alpha-\beta)} \pmod{p^2}. \tag{2.33}$$

(2) Suppose that $\langle -\alpha\rangle_p$ is odd and $\max\{\langle -\alpha\rangle_p, (p-\langle -\alpha\rangle_p)/2\} \le \langle -\beta\rangle_p < (p+\langle -\alpha\rangle_p)/2$. Then

$$ {}_3F_2\left[\begin{matrix}\alpha & \alpha & \beta \\ & 1 & \alpha-\beta+1\end{matrix}\middle| 1\right]_{p-1} \equiv 0 \pmod{p^2}. \tag{2.34}$$

(3) Suppose that $\langle -\alpha\rangle_p$ is odd and $\langle -\alpha\rangle_p \le \langle -\beta\rangle_p < (p-\langle -\alpha\rangle_p)/2$. Then

$$ {}_3F_2\left[\begin{matrix}\alpha & \alpha & \beta \\ & 1 & \alpha-\beta+1\end{matrix}\middle| 1\right]_{p-1} \equiv -\frac{(\alpha+\langle -\alpha\rangle_p)\cdot\Gamma_p(1+\frac{1}{2}\alpha)\Gamma_p(1+\alpha-\beta)\Gamma_p(1-\frac{1}{2}\alpha-\beta)}{\Gamma_p(1+\alpha)\Gamma_p(1-\frac{1}{2}\alpha)\Gamma_p(1-\beta)\Gamma_p(1+\frac{1}{2}\alpha-\beta)} \pmod{p^2}. \tag{2.35}$$

For example,

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ & 1 & \frac{7}{6}\end{matrix}\middle| 1\right]_{p-1} \equiv -\frac{2\Gamma_p(\frac{5}{12})\Gamma_p(\frac{7}{6})\Gamma_p(\frac{5}{4})}{\Gamma_p(\frac{2}{3})\Gamma_p(\frac{3}{4})\Gamma_p(\frac{11}{12})\Gamma_p(\frac{3}{2})} \pmod{p^2}$$

for any prime $p \equiv 5 \pmod{12}$ and

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ & 1 & \frac{7}{6}\end{matrix}\middle| 1\right]_{p-1} \equiv 0 \pmod{p^2}$$

for any prime $p \equiv 11 \pmod{12}$.

Third, the Pfaff-Saalschütz theorem [3, Theorem 2.2.6] says that

$$ {}_3F_2\left[\begin{matrix}-n & \alpha & \beta \\ & \gamma & \delta\end{matrix}\middle| 1\right] = \frac{(\gamma-\alpha)_n(\gamma-\beta)_n}{(\gamma)_n(\gamma-\alpha-\beta)_n}, \tag{2.36}$$

where $n \in \mathbb{N}$ and $\gamma+\delta = \alpha+\beta+1-n$. In particular, setting $\gamma = 1$ in (2.36), we get

$$ {}_3F_2\left[\begin{matrix}-n & \alpha & \beta \\ & 1 & \alpha+\beta-n\end{matrix}\middle| 1\right] = \frac{(1-\alpha)_n(1-\beta)_n}{n!\cdot(1-\alpha-\beta)_n}. \tag{2.37}$$

Theorem 2.11. *Let $\alpha, \beta, \gamma \in \mathbb{Z}_p$. Suppose that $\max\{\langle -\alpha\rangle_p, \langle -\beta\rangle_p\} \le \langle -\gamma\rangle_p$ and $(\gamma)_{p-1}$ is not divisible by p^2. If $\langle -\alpha\rangle_p + \langle -\beta\rangle_p < \langle -\gamma\rangle_p$, then*

$$ {}_3F_2\left[\begin{matrix}\alpha & \beta & \gamma-\alpha-\beta \\ & 1 & \gamma\end{matrix}\middle| 1\right]_{p-1} \equiv -\frac{\Gamma_p(1+\alpha-\gamma)\Gamma_p(1+\beta-\gamma)\Gamma_p(1-\alpha-\beta)}{\Gamma_p(1-\alpha)\Gamma_p(1-\beta)\Gamma_p(1-\gamma)\Gamma_p(1+\alpha+\beta-\gamma)} \pmod{p^2}. \tag{2.38}$$

If $\langle -\alpha\rangle_p + \langle -\beta\rangle_p > p$, *then*

$$ {}_3F_2\begin{bmatrix}\alpha & \beta & \gamma-\alpha-\beta \\ & 1 & \gamma\end{bmatrix}\!\!\left.\vphantom{\begin{bmatrix}1\\1\end{bmatrix}}\right|\, 1\Big]_{p-1} \equiv 0 \pmod{p^2}. \tag{2.39}$$

For example,

$$ {}_3F_2\left[\begin{matrix}\frac14 & \frac13 & \frac1{12} \\ & 1 & \frac23\end{matrix}\,\middle|\, 1\right]_{p-1} \equiv -\frac{\Gamma_p(\frac{5}{12})\Gamma_p(\frac{7}{12})}{\Gamma_p(\frac13)\Gamma_p(\frac34)\Gamma_p(\frac{11}{12})} \pmod{p^2}$$

for any prime $p \equiv 1 \pmod{12}$ and

$$ {}_3F_2\left[\begin{matrix}\frac14 & \frac13 & \frac1{12} \\ & 1 & \frac23\end{matrix}\,\middle|\, 1\right]_{p-1} \equiv 0 \pmod{p^2}$$

for any prime $p \equiv 7 \pmod{12}$. We also mention that the special case $\gamma = 1$ of (2.39) was proved by Pan and Zhang in [30].

We also have the following transformation due to Kummer [3, Corollary 3.3.5]:

$$ {}_3F_2\left[\begin{matrix}\alpha & \beta & \gamma \\ & \delta & \epsilon\end{matrix}\,\middle|\, 1\right] \equiv \frac{\Gamma(\delta)\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\delta-\alpha)\Gamma(\delta+\epsilon-\beta-\gamma)} \cdot {}_3F_2\left[\begin{matrix}\alpha & \epsilon-\beta & \epsilon-\gamma \\ & \epsilon & \epsilon+\epsilon-\beta-\gamma\end{matrix}\,\middle|\, 1\right]. \tag{2.40}$$

Setting $\epsilon = 1$, we get

$$ {}_3F_2\left[\begin{matrix}\alpha & \beta & \gamma \\ & 1 & \delta\end{matrix}\,\middle|\, 1\right] = \frac{\Gamma(\delta)\Gamma(1+\delta-\alpha-\beta-\gamma)}{\Gamma(\delta-\alpha)\Gamma(1+\delta-\beta-\gamma)} \cdot {}_3F_2\left[\begin{matrix}\alpha & 1-\beta & 1-\gamma \\ & 1 & 1+\delta-\beta-\gamma\end{matrix}\,\middle|\, 1\right]. \tag{2.41}$$

Theorem 2.12. *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_p$. *Suppose that*

(i) $\max\{\langle -\alpha\rangle_p, \langle -\beta\rangle_p, \langle -\gamma\rangle_p\} \le \langle -\delta\rangle_p$; *(ii)* $\langle -\alpha\rangle_p + \langle -\beta\rangle_p + \langle -\gamma\rangle_p < p + \langle -\delta\rangle_p$;

(iii) $\langle -\beta\rangle_p + \langle -\gamma\rangle_p \ge \langle -\delta\rangle_p$; *(iv)* $(\delta)_{p-1}, (1+\delta-\beta-\gamma)_{p-1} \not\equiv 0 \pmod{p^2}$.

Then

$$\begin{aligned} & {}_3F_2\left[\begin{matrix}\alpha & \beta & \gamma \\ & 1 & \delta\end{matrix}\,\middle|\, 1\right]_{p-1} \\ \equiv & \frac{\Gamma_p(\delta)\Gamma_p(1+\delta-\alpha-\beta-\gamma)}{\Gamma_p(\delta-\alpha)\Gamma_p(1+\delta-\beta-\gamma)} \cdot {}_3F_2\left[\begin{matrix}\alpha & 1-\beta & 1-\gamma \\ & 1 & 1+\delta-\beta-\gamma\end{matrix}\,\middle|\, 1\right]_{p-1} \pmod{p^2}. \end{aligned} \tag{2.42}$$

For example, for any prime $p \equiv 1 \pmod 6$,

$$ {}_3F_2\left[\begin{matrix}\frac13 & \frac12 & \frac23 \\ & 1 & \frac56\end{matrix}\,\middle|\, 1\right]_{p-1} \equiv \frac{\Gamma_p(\frac56)\Gamma_p(\frac13)}{\Gamma_p(\frac12)\Gamma_p(\frac23)} \cdot {}_3F_2\left[\begin{matrix}\frac13 & \frac12 & \frac13 \\ & 1 & \frac23\end{matrix}\,\middle|\, 1\right]_{p-1} \pmod{p^2}.$$

Whipple also found a quadratic transformation concerning the ${}_3F_2$ series [3, Eq. (3.1.15)]:

$$\begin{aligned} & {}_3F_2\left[\begin{matrix}\alpha & \beta & \gamma \\ & \alpha-\beta+1 & \alpha-\gamma+1\end{matrix}\,\middle|\, z\right] \\ =& (1-z)^{-\alpha}{}_3F_2\left[\begin{matrix}\alpha-\beta-\gamma+1 & \frac12\alpha & \frac12\alpha+\frac12 \\ & \alpha-\beta+1 & \alpha-\gamma+1\end{matrix}\,\middle|\, -\frac{4z}{(1-z)^2}\right]. \end{aligned} \tag{2.43}$$

Letting $\gamma = \alpha$ in (2.43), we have

$$ {}_3F_2\left[\begin{matrix}\alpha & \alpha & \beta \\ & 1 & \alpha-\beta+1\end{matrix}\,\middle|\, z\right] = (1-z)^{-\alpha}{}_3F_2\left[\begin{matrix}1-\beta & \frac{1}{2}\alpha & \frac{1}{2}\alpha+\frac{1}{2} \\ & 1 & \alpha-\beta+1\end{matrix}\,\middle|\, -\frac{4z}{(1-z)^2}\right]. \tag{2.44}$$

Theorem 2.13. *Let $\alpha, \beta \in \mathbb{Z}_p$ and $z \in \mathbb{Z}_p^\times$ with $z-1$ is prime to p. Suppose that*

(i) $\langle -\alpha\rangle_p \le \langle -\beta\rangle_p$; (ii) $p + \langle -\alpha\rangle_p > 2\langle -\beta\rangle_p$; (iii) $(\alpha-\beta+1)_{p-1} \not\equiv 0 \pmod{p^2}$.

Then

$$\begin{aligned} &{}_3F_2\left[\begin{matrix}\alpha & \alpha & \beta \\ & 1 & \alpha-\beta+1\end{matrix}\,\middle|\, z\right]_{p-1} + (-z)^{1-\lambda_p(\alpha)}{}_3F_2\left[\begin{matrix}\alpha & \alpha & \beta \\ & 1 & \alpha-\beta+1\end{matrix}\,\middle|\, \frac{1}{z}\right]_{p-1} \\ \equiv&(1-z)^{1-\lambda_p(\alpha)}{}_3F_2\left[\begin{matrix}1-\beta & \frac{1}{2}\alpha & \frac{1}{2}\alpha+\frac{1}{2} \\ & 1 & \alpha-\beta+1\end{matrix}\,\middle|\, -\frac{4z}{(1-z)^2}\right]_{p-1} \pmod{p^2}. \end{aligned}\tag{2.45}$$

Let us turn to the ${}_4F_3$ series. The well-known Whipple formula is a basic transformation between two balanced ${}_4F_3$ series, which says

$$ {}_4F_3\left[\begin{matrix}-n & \alpha & \beta & \gamma \\ & \delta & \epsilon & \rho\end{matrix}\,\middle|\, 1\right] = \frac{(\delta-\alpha)_n(\epsilon-\alpha)_n}{(\delta)_n(\epsilon)_n}\cdot {}_4F_3\left[\begin{matrix}-n & \alpha & \rho-\beta & \rho-\gamma \\ & \rho & 1+\alpha-n-\delta & 1+\alpha-n-\epsilon\end{matrix}\,\middle|\, 1\right], \tag{2.46}$$

where $n \in \mathbb{N}$ and $\alpha+\beta+\gamma-n+1 = \delta+\epsilon+\rho$. Setting $\rho = 1$ in (2.46), we get

$$ {}_4F_3\left[\begin{matrix}-n & \alpha & \beta & \gamma \\ & 1 & \delta & \epsilon\end{matrix}\,\middle|\, 1\right] = \frac{(\delta-\alpha)_n(\epsilon-\alpha)_n}{(\delta)_n(\epsilon)_n}\cdot {}_4F_3\left[\begin{matrix}-n & \alpha & 1-\beta & 1-\gamma \\ & 1 & 1+\alpha-n-\delta & 1+\alpha-n-\epsilon\end{matrix}\,\middle|\, 1\right], \tag{2.47}$$

where $\alpha+\beta+\gamma-n = \delta+\epsilon$. The p-adic analogue of (2.47) is a little complicated. Under several assumptions, we have a congruence between two truncated ${}_4F_3$ functions.

Theorem 2.14. *Let $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{Z}_p$. Suppose that*

(i) $\langle -\alpha\rangle_p \le \min\{\langle -\delta\rangle_p, \langle -\epsilon\rangle_p\}$, $\langle -\beta\rangle_p \le \langle -\delta\rangle_p$ and $\langle -\gamma\rangle_p \le \langle -\epsilon\rangle_p$;

(ii) $\langle -\alpha\rangle_p + \langle -\beta\rangle_p + \langle -\gamma\rangle_p \le \langle -\delta\rangle_p + \langle -\epsilon\rangle_p$;

(iii) $\langle -\beta\rangle_p + \langle -\gamma\rangle_p \ge \max\{\langle -\delta\rangle_p, \langle -\epsilon\rangle_p\}$;

(iv) $(\delta)_{p-1}$, $(\epsilon)_{p-1}$, $(1+\delta-\beta-\gamma)_{p-1}$ and $(1+\epsilon-\beta-\gamma)_{p-1}$ are not divisible by p^2.

Let $\rho = \delta+\epsilon-\alpha-\beta-\gamma$. Then

$$\begin{aligned} {}_4F_3\left[\begin{matrix}\rho & \alpha & \beta & \gamma \\ & 1 & \delta & \epsilon\end{matrix}\,\middle|\, 1\right]_{p-1} \equiv& \frac{\Gamma_p(\beta+\gamma-\delta)\Gamma_p(\beta+\gamma-\epsilon)\Gamma_p(\delta)\Gamma_p(\epsilon)}{\Gamma_p(\delta-\rho)\Gamma_p(\epsilon-\rho)\Gamma_p(\delta-\alpha)\Gamma_p(\epsilon-\alpha)} \\ &\cdot {}_4F_3\left[\begin{matrix}\rho & \alpha & 1-\beta & 1-\gamma \\ & 1 & 1+\delta-\beta-\gamma & 1+\epsilon-\beta-\gamma\end{matrix}\,\middle|\, 1\right]_{p-1} \pmod{p^2}. \end{aligned}\tag{2.48}$$

Although many conditions are involved in Theorem 2.14, we emphasize that all those conditions are necessary, to make both sides of (2.48) p-integral and not divisible by p. Also, by modifying the conditions in the above theorem, we may get a result on the divisibility of the truncated ${}_4F_3$ function.

Theorem 2.15. *Let $\alpha \in \mathbb{Z}_p^\times$ and $\beta, \gamma, \delta, \epsilon \in \mathbb{Z}_p$. Suppose that*

(i) $\langle -\alpha\rangle_p \le \min\{\langle -\delta\rangle_p, \langle -\epsilon\rangle_p\}$, $\langle -\beta\rangle_p \le \langle -\delta\rangle_p$ *and* $\langle -\gamma\rangle_p \le \langle -\epsilon\rangle_p$;

(ii) $\langle -\alpha\rangle_p + \langle -\beta\rangle_p + \langle -\gamma\rangle_p \ge p + \max\{\langle -\delta\rangle_p, \langle -\epsilon\rangle_p\}$;

(iii) $(\delta)_{p-1}, (\epsilon)_{p-1} \not\equiv 0 \pmod{p^2}$.

Then

$$ {}_4F_3\left[\begin{matrix}\rho & \alpha & \beta & \gamma \\ & 1 & \delta & \epsilon\end{matrix}\,\middle|\, 1\right]_{p-1} \equiv 0 \pmod{p^2}, \tag{2.49}$$

where $\rho = \delta + \epsilon - \alpha - \beta - \gamma$.

We need to give an explanation on the relation between Theorems 2.11, 2.12, 2.14 and 2.15. Evidently (2.38) and (2.39) can be easily deduced from (2.48) and (2.49) respectively, by substituting $\gamma = \epsilon = 1$ in Theorems 2.14 and 2.15. On the other hand, notice that (2.41) is also an consequence of (2.47), if α, β, γ, δ and ϵ are all fixed and n tends to $+\infty$. However, clearly such a operation is invalid for the p-adic analogues. Seemingly (2.12) can't be deduced from (2.48) directly. Of course, Theorem 2.12 can be proved via the same discussion in the proof of Theorem 2.14.

Another basic formula on the hypergeometric series is Whipple's ${}_7F_6$ transformation [3, Theorem 3.4.5]:

$$\begin{aligned} &{}_7F_6\left[\begin{matrix}\alpha & 1+\frac{1}{2}\alpha & \beta & \gamma & \delta & \epsilon & \rho \\ & \frac{1}{2}\alpha & \alpha-\beta+1 & \alpha-\gamma+1 & \alpha-\delta+1 & \alpha-\epsilon+1 & \alpha-\rho+1\end{matrix}\,\middle|\, 1\right] \\ =&\frac{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\gamma+1)\Gamma(\alpha-\delta+1)\Gamma(\alpha-\beta-\gamma-\delta+1)}{\Gamma(\alpha+1)\Gamma(\alpha-\beta-\gamma+1)\Gamma(\alpha-\beta-\delta+1)\Gamma(\alpha-\gamma-\delta+1)} \\ &\cdot {}_4F_3\left[\begin{matrix}\alpha-\epsilon-\rho+1 & \beta & \gamma & \delta \\ & \beta+\gamma+\delta-\alpha & \alpha-\epsilon+1 & \alpha-\rho+1\end{matrix}\,\middle|\, 1\right]. \end{aligned} \tag{2.50}$$

Setting $\rho = \alpha$ in (2.50), we have

$$\begin{aligned} &{}_7F_6\left[\begin{matrix}\alpha & \alpha & \frac{1}{2}\alpha+1 & \beta & \gamma & \delta & \epsilon \\ & 1 & \frac{1}{2}\alpha & \alpha-\beta+1 & \alpha-\gamma+1 & \alpha-\delta+1 & \alpha-\epsilon+1\end{matrix}\,\middle|\, 1\right] \\ =&\frac{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\gamma+1)\Gamma(\alpha-\delta+1)\Gamma(\alpha-\beta-\gamma-\delta+1)}{\Gamma(\alpha+1)\Gamma(\alpha-\beta-\gamma+1)\Gamma(\alpha-\beta-\delta+1)\Gamma(\alpha-\gamma-\delta+1)} \\ &\cdot {}_4F_3\left[\begin{matrix}1-\epsilon & \beta & \gamma & \delta \\ & 1 & \alpha-\epsilon+1 & \beta+\gamma+\delta-\alpha\end{matrix}\,\middle|\, 1\right]. \end{aligned} \tag{2.51}$$

Of course, the p-adic analogues of (2.51) become more complicated.

Theorem 2.16. *Let $\alpha,\beta,\gamma,\delta,\epsilon\in\mathbb{Z}_p$. Suppose that*

(i) $\langle-\beta\rangle_p<\langle-\alpha\rangle_p\le\min\{\langle-\gamma\rangle_p,\langle-\delta\rangle_p,\langle-\epsilon\rangle_p\}$*;*

(ii) $p+\langle-\alpha\rangle_p>\max\{\langle-\gamma\rangle_p+\langle-\epsilon\rangle_p,\langle-\delta\rangle_p+\langle-\epsilon\rangle_p,\langle-\beta\rangle_p+\langle-\gamma\rangle_p+\langle-\delta\rangle_p\}$*;*

(iii) $\langle-\alpha\rangle_p/\alpha=\langle-\beta\rangle_p/\beta$*.*

(iv) $(\alpha-\beta+1)_{p-1}$, $(\alpha-\gamma+1)_{p-1}$, $(\alpha-\delta+1)_{p-1}$, $(\alpha-\epsilon+1)_{p-1}$, $(\beta+\gamma+\delta-\alpha)_{p-1}$ *are not divisible by* p^2*.*

Then

$$\begin{aligned}&{}_7F_6\left[\begin{matrix}\alpha&\alpha&1+\frac12\alpha&\beta&\gamma&\delta&\epsilon\\&1&\frac12\alpha&\alpha-\beta+1&\alpha-\gamma+1&\alpha-\delta+1&\alpha-\epsilon+1\end{matrix}\,\middle|\,1\right]_{p-1}\\ \equiv&\frac{\langle-\alpha\rangle_p}{\langle-\alpha\rangle_p-\langle-\beta\rangle_p}\cdot\frac{\Gamma_p(\alpha-\beta+1)\Gamma_p(\alpha-\gamma+1)\Gamma_p(\alpha-\delta+1)\Gamma_p(\alpha-\beta-\gamma-\delta+1)}{\Gamma_p(\alpha+1)\Gamma_p(\alpha-\beta-\gamma+1)\Gamma_p(\alpha-\beta-\delta+1)\Gamma_p(\alpha-\gamma-\delta+1)}\\ &\cdot{}_4F_3\left[\begin{matrix}1-\epsilon&\beta&\gamma&\delta\\&1&\alpha-\epsilon+1&\beta+\gamma+\delta-\alpha\end{matrix}\,\middle|\,1\right]_{p-1}\pmod{p^2}.\end{aligned}\tag{2.52}$$

(2.52) is actually a very curious congruence. In fact, under the assumptions of Theorem 2.16, if $\langle-\alpha\rangle_p\le2\langle-\beta\rangle_p$, then we may have

$$\frac{(\alpha)_k^2(1+\frac12\alpha)_k(\beta)_k(\gamma)_k(\delta)_k(\epsilon)_k}{(\frac12\alpha)_k(\alpha-\beta+1)_k(\alpha-\gamma+1)_k(\alpha-\delta+1)_k(\alpha-\epsilon+1)_k}\notin\mathbb{Z}_p$$

for any $\langle-\alpha\rangle_p-\langle-\beta\rangle_p\le k\le\langle-\beta\rangle_p$. Fortunately, even if $\langle-\alpha\rangle_p\le2\langle-\beta\rangle_p$, the truncated ${}_7F_6$ function in (2.52) is still p-integral and (2.52) is also valid.

We also give an explanation on the condition $\langle-\alpha\rangle_p/\alpha=\langle-\beta\rangle_p/\beta$. Assume that $\beta=r/d$ and $\alpha=sr/d$ where $(r,d)=1$ and $1\le r<d/s$. If $1\le t<d/s$ and p is a prime with $tp\equiv r\pmod d$, then clearly $\langle-\beta\rangle_p=(tp-r)/d$ and $\langle-\alpha\rangle_p=(stp-sr)/d$. Then in such a case, we have $\langle-\alpha\rangle_p/\langle-\beta\rangle_p=\alpha/\beta=s$. For example, for any $p\equiv1\pmod{12}$,

$$\begin{aligned}&{}_7F_6\left[\begin{matrix}\frac23&\frac23&\frac43&\frac1{12}&\frac34&\frac34&\frac34\\&1&\frac13&\frac{19}{12}&\frac{11}{12}&\frac{11}{12}&\frac{11}{12}\end{matrix}\,\middle|\,1\right]_{p-1}\\ \equiv&\frac87\cdot\frac{\Gamma_p(\frac32)\Gamma_p(\frac{11}{12})^2\Gamma_p(\frac1{12})}{\Gamma_p(\frac53)\Gamma_p(\frac56)^2\Gamma_p(\frac16)}\cdot{}_4F_3\left[\begin{matrix}\frac14&\frac1{12}&\frac34&\frac34\\&1&\frac{11}{12}&\frac{11}{12}\end{matrix}\,\middle|\,1\right]_{p-1}\pmod{p^2}.\end{aligned}$$

On the other hand, the truncated ${}_7F_6$ function can be divisible by p^2 under another assumptions.

Theorem 2.17. *Let $\alpha,\beta,\gamma,\delta,\epsilon\in\mathbb{Z}_p$. Suppose that*

(i) $\langle-\alpha\rangle_p\le\min\{\langle-\beta\rangle_p,\langle-\gamma\rangle_p,\langle-\delta\rangle_p,\langle-\epsilon\rangle_p\}$*;*

(ii) $p+\langle-\alpha\rangle_p>\max\{\langle-\beta\rangle_p+\langle-\gamma\rangle_p,\langle-\beta\rangle_p+\langle-\delta\rangle_p,\langle-\beta\rangle_p+\langle-\epsilon\rangle_p,\langle-\gamma\rangle_p+\langle-\delta\rangle_p\}$*;*

(iii) $2p-1+\langle-\alpha\rangle_p\le\langle-\beta\rangle_p+\langle-\gamma\rangle_p+\langle-\delta\rangle_p+\langle-\epsilon\rangle_p$*;*

(iv) $(\alpha-\beta+1)_{p-1},(\alpha-\gamma+1)_{p-1},(\alpha-\delta+1)_{p-1},(\alpha-\epsilon+1)_{p-1}\not\equiv0\pmod{p^2}$*.*

Then

$$
{}_7F_6\left[\begin{matrix}\alpha & \alpha & 1+\frac12\alpha & \beta & \gamma & \delta & \epsilon \\ & 1 & \frac12\alpha & \alpha-\beta+1 & \alpha-\gamma+1 & \alpha-\delta+1 & \alpha-\epsilon+1\end{matrix}\,\middle|\,1\right]_{p-1}\equiv 0 \pmod{p^2}. \tag{2.53}
$$

Furthermore, we have a different p-analogue of (2.51) provided that the truncated ${}_7F_6$ function is divisible by p, but not by p^2.

Theorem 2.18. *Let $\alpha,\beta,\gamma,\delta,\epsilon\in\mathbb{Z}_p$. Suppose that*

(i) $\langle-\alpha\rangle_p\le\min\{\langle-\beta\rangle_p,\langle-\gamma\rangle_p,\langle-\delta\rangle_p,\langle-\epsilon\rangle_p\}$;

(ii) $p+\langle-\alpha\rangle_p>\max\{\langle-\beta\rangle_p+\langle-\gamma\rangle_p+\langle-\delta\rangle_p,\ \min\{\langle-\beta\rangle_p,\langle-\gamma\rangle_p,\langle-\delta\rangle_p\}+\langle-\epsilon\rangle_p\}$;

(iii) $(\alpha-\beta+1)_{p-1}$, $(\alpha-\gamma+1)_{p-1}$, $(\alpha-\delta+1)_{p-1}$, $(\alpha-\epsilon+1)_{p-1}$ *and* $(\beta+\gamma+\delta-\alpha)_{p-1}$ *are not divisible by* p^2.

Then

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix}\alpha & \alpha & \frac12\alpha+1 & \beta & \gamma & \delta & \epsilon \\ & 1 & \frac12\alpha & \alpha-\beta+1 & \alpha-\gamma+1 & \alpha-\delta+1 & \alpha-\epsilon+1\end{matrix}\,\middle|\,1\right]_{p-1}\\
\equiv&\big(\alpha+\langle-\alpha\rangle_p\big)\cdot\frac{\Gamma_p(\alpha-\beta+1)\Gamma_p(\alpha-\gamma+1)\Gamma_p(\alpha-\delta+1)\Gamma_p(\alpha-\beta-\gamma-\delta+1)}{\Gamma_p(\alpha+1)\Gamma_p(\alpha-\beta-\gamma+1)\Gamma_p(\alpha-\beta-\delta+1)\Gamma_p(\alpha-\gamma-\delta+1)}\\
&\cdot{}_4F_3\left[\begin{matrix}1-\epsilon & \beta & \gamma & \delta \\ & 1 & \alpha-\epsilon+1 & \beta+\gamma+\delta-\alpha\end{matrix}\,\middle|\,1\right]_{p-1}\pmod{p^2}.
\end{aligned}\tag{2.54}
$$

The importance of Whipple's ${}_7F_6$ transformation is that (2.50) has many interesting consequences. For example, combining (2.50) with (2.36), we get the Dougall's formula

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix}\alpha & 1+\frac12\alpha & \beta & \gamma & \delta & \epsilon & -n \\ & \frac12\alpha & \alpha-\beta+1 & \alpha-\gamma+1 & \alpha-\delta+1 & \alpha-\epsilon+1 & \alpha+n+1\end{matrix}\,\middle|\,1\right]\\
=&\frac{(\alpha+1)_n(\alpha-\beta-\gamma+1)_n(\alpha-\beta-\delta+1)_n(\alpha-\gamma-\delta+1)_n}{(\alpha-\beta+1)_n(\alpha-\gamma+1)_n\Gamma(\alpha-\delta+1)_n(\alpha-\beta-\gamma-\delta+1)_n},
\end{aligned}\tag{2.55}
$$

where $n\in\mathbb{N}$ and $n=\beta+\gamma+\delta+\epsilon-2\alpha-1$. In particular,

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix}\alpha & \alpha & 1+\frac12\alpha & \beta & \gamma & \delta & -n \\ & 1 & \frac12\alpha & \alpha-\beta+1 & \alpha-\gamma+1 & \alpha-\delta+1 & \alpha+n+1\end{matrix}\,\middle|\,1\right]\\
=&\frac{(\alpha+1)_n(\alpha-\beta-\gamma+1)_n(\alpha-\beta-\delta+1)_n(\alpha-\gamma-\delta+1)_n}{(\alpha-\beta+1)_n(\alpha-\gamma+1)_n\Gamma(\alpha-\delta+1)_n(\alpha-\beta-\gamma-\delta+1)_n},
\end{aligned}\tag{2.56}
$$

where $n=\beta+\gamma+\delta-\alpha-1$. We have two p-adic analogues of (2.56).

Theorem 2.19. *Let $\alpha,\beta,\gamma,\delta\in\mathbb{Z}_p$. Suppose that*

(i) $\langle-\beta\rangle_p<\langle-\alpha\rangle_p\le\min\{\langle-\gamma\rangle_p,\langle-\delta\rangle_p\}$;

(ii) $\langle-\beta\rangle_p+\langle-\gamma\rangle_p+\langle-\delta\rangle_p<p$;

(iii) $\langle-\alpha\rangle_p/\alpha=\langle-\beta\rangle_p/\beta$;

(iv) $(\alpha-\beta+1)_{p-1}$, $(\alpha-\gamma+1)_{p-1}$, $(\alpha-\delta+1)_{p-1}$ *and* $(\beta+\gamma+\delta)_{p-1}$ *are not divisible by* p^2.

Let $\epsilon=\alpha+1-\beta-\gamma-\delta$. *Then*

$$\begin{aligned}&{}_7F_6\begin{bmatrix}\alpha & \alpha & 1+\frac{1}{2}\alpha & \beta & \gamma & \delta & \epsilon \\ & 1 & \frac{1}{2}\alpha & \alpha-\beta+1 & \alpha-\gamma+1 & \alpha-\delta+1 & \alpha-\epsilon+1\end{bmatrix}\,1\Bigg]_{p-1}\\ \equiv&-\frac{\alpha}{\alpha-\beta}\cdot\frac{\Gamma_p(1-\beta-\gamma)\Gamma_p(1-\beta-\delta)\Gamma_p(1-\gamma-\delta)}{\Gamma_p(1-\beta)\Gamma_p(1-\gamma)\Gamma_p(1-\delta)\Gamma_p(1-\beta-\gamma-\delta)}\\ &\cdot\frac{\Gamma_p(\alpha-\beta+1)\Gamma_p(\alpha-\gamma+1)\Gamma_p(\alpha-\delta+1)\Gamma_p(\alpha-\beta-\gamma-\delta+1)}{\Gamma_p(\alpha+1)\Gamma_p(\alpha-\beta-\gamma+1)\Gamma_p(\alpha-\beta-\delta+1)\Gamma_p(\alpha-\gamma-\delta+1)}\pmod{p^2}. \end{aligned}\tag{2.57}$$

Theorem 2.20. *Let* $\alpha,\beta,\gamma,\delta\in\mathbb{Z}_p$. *Suppose that*

(i) $\langle-\alpha\rangle_p\leq\min\{\langle-\beta\rangle_p,\langle-\gamma\rangle_p,\langle-\delta\rangle_p,\langle-\epsilon\rangle_p\}$;

(ii) $\langle-\beta\rangle_p+\langle-\gamma\rangle_p+\langle-\delta\rangle_p<p$;

(iii) $(\alpha-\beta+1)_{p-1}$, $(\alpha-\gamma+1)_{p-1}$, $(\alpha-\delta+1)_{p-1}$ *and* $(\beta+\gamma+\delta)_{p-1}$ *are not divisible by* p^2.

Let $\epsilon=\alpha+1-\beta-\gamma-\delta$. *Then*

$$\begin{aligned}&{}_7F_6\begin{bmatrix}\alpha & \alpha & \frac{1}{2}\alpha+1 & \beta & \gamma & \delta & \epsilon \\ & 1 & \frac{1}{2}\alpha & \alpha-\beta+1 & \alpha-\gamma+1 & \alpha-\delta+1 & \alpha-\epsilon+1\end{bmatrix}\,1\Bigg]_{p-1}\\ \equiv&-\big(\alpha+\langle-\alpha\rangle_p\big)\cdot\frac{\Gamma_p(1-\beta-\gamma)\Gamma_p(1-\beta-\delta)\Gamma_p(1-\gamma-\delta)}{\Gamma_p(1-\beta)\Gamma_p(1-\gamma)\Gamma_p(1-\delta)\Gamma_p(1-\beta-\gamma-\delta)}\\ &\cdot\frac{\Gamma_p(\alpha-\beta+1)\Gamma_p(\alpha-\gamma+1)\Gamma_p(\alpha-\delta+1)\Gamma_p(\alpha-\beta-\gamma-\delta+1)}{\Gamma_p(\alpha+1)\Gamma_p(\alpha-\beta-\gamma+1)\Gamma_p(\alpha-\beta-\delta+1)\Gamma_p(\alpha-\gamma-\delta+1)}\pmod{p^2}. \end{aligned}\tag{2.58}$$

Furthermore, letting $n\to+\infty$ in (2.56), we may get

$${}_5F_4\left[\begin{matrix}\alpha & \alpha & 1+\frac{1}{2}\alpha & \beta & \gamma \\ & 1 & \frac{1}{2}\alpha & \alpha-\beta+1 & \alpha-\gamma+1\end{matrix}\,\middle|\,1\right]=\frac{\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1-\beta-\gamma)}{\Gamma(1+\alpha)\Gamma(1-\beta)\Gamma(1-\gamma)\Gamma(1+\alpha-\beta-\gamma)}.\tag{2.59}$$

(2.59) has the following p-adic analogues.

Theorem 2.21. *Let* $\alpha,\beta,\gamma\in\mathbb{Z}_p$. *Suppose that*

(i) $\langle-\beta\rangle_p<\langle-\alpha\rangle_p\leq\langle-\gamma\rangle_p<(p+\langle-\alpha\rangle_p)/2$;

(ii) $\langle-\beta\rangle_p+\langle-\gamma\rangle_p<p$;

(iii) $\langle-\alpha\rangle_p/\alpha=\langle-\beta\rangle_p/\beta$;

(iv) $(\alpha-\beta+1)_{p-1}$, $(\alpha-\gamma+1)_{p-1}$ *are not divisible by* p^2.

Then

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix}\alpha & \alpha & 1+\frac{1}{2}\alpha & \beta & \gamma\\ & 1 & \frac{1}{2}\alpha & \alpha-\beta+1 & \alpha-\gamma+1\end{matrix}\middle|\, 1\right]_{p-1}\\
\equiv&-\frac{\alpha}{\alpha-\beta}\cdot\frac{\Gamma_p(1+\alpha-\beta)\Gamma_p(1+\alpha-\gamma)\Gamma_p(1-\beta-\gamma)}{\Gamma_p(1+\alpha)\Gamma_p(1-\beta)\Gamma_p(1-\gamma)\Gamma_p(1+\alpha-\beta-\gamma)} \pmod{p^2}.
\end{aligned}\tag{2.60}
$$

Theorem 2.22. *Let $\alpha,\beta,\gamma\in\mathbb{Z}_p$. Suppose that*

(i) $\langle-\alpha\rangle_p\leq\min\{\langle-\beta\rangle_p,\langle-\gamma\rangle_p\}$; *(ii) $(\alpha-\beta+1)_{p-1}$, $(\alpha-\gamma+1)_{p-1}$ are not divisible by p^2.*

If $p\leq\langle-\beta\rangle_p+\langle-\gamma\rangle_p<p+\langle-\alpha\rangle_p$, then

$$
{}_5F_4\left[\begin{matrix}\alpha & \alpha & 1+\frac{1}{2}\alpha & \beta & \gamma\\ & 1 & \frac{1}{2}\alpha & \alpha-\beta+1 & \alpha-\gamma+1\end{matrix}\middle|\, 1\right]_{p-1}\equiv 0 \pmod{p^2}.\tag{2.61}
$$

And if $\langle-\beta\rangle_p+\langle-\gamma\rangle_p<p$, then

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix}\alpha & \alpha & 1+\frac{1}{2}\alpha & \beta & \gamma\\ & 1 & \frac{1}{2}\alpha & \alpha-\beta+1 & \alpha-\gamma+1\end{matrix}\middle|\, 1\right]_{p-1}\\
\equiv&-(\alpha+\langle-\alpha\rangle_p)\cdot\frac{\Gamma_p(1+\alpha-\beta)\Gamma_p(1+\alpha-\gamma)\Gamma_p(1-\beta-\gamma)}{\Gamma_p(1+\alpha)\Gamma_p(1-\beta)\Gamma_p(1-\gamma)\Gamma_p(1+\alpha-\beta-\gamma)} \pmod{p^2}.
\end{aligned}\tag{2.62}
$$

Notice that Theorem 2.10 is factly the consequence of Theorems 2.21 and 2.22, by substituting $\beta=\alpha/2$. Of course, Theorems 2.21 and 2.22 also can be deduced from Theorems 2.16, 2.17 and 2.18 by setting $\epsilon=1+\alpha-\delta$. However, this will produce some unnecessary additional requirements concerning β and γ.

Let us see another application of (2.51). Letting ϵ tend to $-\infty$ in (2.51), we may get

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix}\alpha & \frac{1}{2}\alpha+1 & \alpha & \beta & \gamma & \delta\\ & \frac{1}{2}\alpha & 1 & \alpha-\beta+1 & \alpha-\gamma+1 & \alpha-\delta+1\end{matrix}\middle|\, -1\right]\\
=&\frac{\Gamma(\alpha-\gamma+1)\Gamma(\alpha-\delta+1)}{\Gamma(\alpha+1)\Gamma(\alpha-\gamma-\delta+1)}\cdot{}_3F_2\left[\begin{matrix}1-\beta & \gamma & \delta\\ & 1 & \alpha-\beta+1\end{matrix}\middle|\, 1\right]_p.
\end{aligned}\tag{2.63}
$$

Theorem 2.23. *Let $\alpha,\beta,\gamma,\delta\in\mathbb{Z}_p$. Suppose that*

(i) $\langle-\beta\rangle_p<\langle-\alpha\rangle_p\leq\min\{\langle-\gamma\rangle_p,\langle-\delta\rangle_p\}$;

(ii) $p+\langle-\alpha\rangle_p>\langle-\gamma\rangle_p+\langle-\delta\rangle_p$;

(iii) $\langle-\alpha\rangle_p/\alpha=\langle-\beta\rangle_p/\beta$;

(iv) $(\alpha-\beta+1)_{p-1}$, $(\alpha-\gamma+1)_{p-1}$, $(\alpha-\delta+1)_{p-1}$ are not divisible by p^2.

Then

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix}\alpha & \alpha & \frac{1}{2}\alpha+1 & \beta & \gamma & \delta\\ & 1 & \frac{1}{2}\alpha & \alpha-\beta+1 & \alpha-\gamma+1 & \alpha-\delta+1\end{matrix}\middle|\, -1\right]_{p-1}\\
\equiv&\frac{\alpha}{\alpha-\beta}\cdot\frac{\Gamma_p(\alpha-\beta+1)\Gamma_p(\alpha-\gamma+1)}{\Gamma_p(\alpha+1)\Gamma_p(\alpha-\beta-\gamma+1)}\cdot{}_3F_2\left[\begin{matrix}1-\delta & \beta & \gamma\\ & 1 & \alpha-\delta+1\end{matrix}\middle|\, 1\right]_{p-1} \pmod{p^2}.
\end{aligned}\tag{2.64}
$$

Theorem 2.24. *Let $\alpha,\beta,\gamma,\delta\in\mathbb{Z}_p$. Suppose that*

(i) $\langle-\alpha\rangle_p\le\min\{\langle-\beta\rangle_p,\langle-\gamma\rangle_p,\langle-\delta\rangle_p\}$;

(ii) $p+\langle-\alpha\rangle_p>\max\{\langle-\beta\rangle_p+\langle-\gamma\rangle_p,\langle-\beta\rangle_p+\langle-\delta\rangle_p,\langle-\gamma\rangle_p+\langle-\delta\rangle_p\}$;

(iii) $(\alpha-\beta+1)_{p-1}$, $(\alpha-\gamma+1)_{p-1}$ $(\alpha-\delta+1)_{p-1}$ *are not divisible by* p^2.

Then

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix}\alpha & \alpha & \frac12\alpha+1 & \beta & \gamma & \delta\\ & 1 & \frac12\alpha & \alpha-\beta+1 & \alpha-\gamma+1 & \alpha-\delta+1\end{matrix}\middle|\,-1\right]_{p-1}\\
&\equiv\big(\alpha+\langle-\alpha\rangle_p\big)\cdot\frac{\Gamma_p(\alpha-\beta+1)\Gamma_p(\alpha-\gamma+1)}{\Gamma_p(\alpha+1)\Gamma_p(\alpha-\beta-\gamma+1)}\cdot{}_3F_2\left[\begin{matrix}1-\delta & \beta & \gamma\\ & 1 & \alpha-\delta+1\end{matrix}\middle|\,1\right]_{p-1}\pmod{p^2}.
\end{aligned}\tag{2.65}
$$

Fortunately, we can set $\delta=1+\alpha-\gamma$ in (2.64) and (2.65) now. Choose $\delta\in\mathbb{Z}_p$ such that neither $(\gamma)_{p-1}$ nor $(\alpha-\gamma+1)_{p-1}$ is divisible by p^2, and

$$
\langle\gamma\rangle_p=\begin{cases}\frac12(p+\langle-\alpha\rangle_p-1), & \text{if } \langle-\alpha\rangle_p \text{ is even},\\ \frac12(p+\langle-\alpha\rangle_p)-1, & \text{if } \langle-\alpha\rangle_p \text{ is odd}.\end{cases}
$$

Substituting $\delta=1+\alpha-\gamma$ in (2.64) and (2.65) and applying Theorem 1.1, we obtain that

Corollary 2.2. *Let $\alpha,\beta\in\mathbb{Z}_p$. (1) Suppose that $\langle-\beta\rangle_p<\langle-\alpha\rangle_p$, $\langle-\alpha\rangle_p/\alpha=\langle-\beta\rangle_p/\beta$ and p^2 doesn't divide $(\alpha-\beta+1)_{p-1}$. Then*

$$
{}_4F_3\left[\begin{matrix}\alpha & \alpha & \frac12\alpha+1 & \beta\\ & 1 & \frac12\alpha & \alpha-\beta+1\end{matrix}\middle|\,-1\right]_{p-1}\equiv-\frac{\alpha}{\alpha-\beta}\cdot\frac{\Gamma_p(\alpha-\beta+1)}{\Gamma_p(\alpha+1)\Gamma_p(1-\beta)}\pmod{p^2}.\tag{2.66}
$$

(2) Suppose that $\langle-\alpha\rangle_p\le\langle-\beta\rangle_p\le(p+\langle-\alpha\rangle_p-1)/2$ and $(\alpha-\beta+1)_{p-1}\not\equiv0\pmod{p^2}$. Then

$$
{}_4F_3\left[\begin{matrix}\alpha & \alpha & \frac12\alpha+1 & \beta\\ & 1 & \frac12\alpha & \alpha-\beta+1\end{matrix}\middle|\,-1\right]_{p-1}\equiv-\big(\alpha+\langle-\alpha\rangle_p\big)\cdot\frac{\Gamma_p(\alpha-\beta+1)}{\Gamma_p(\alpha+1)\Gamma_p(1-\beta)}\pmod{p^2}.\tag{2.67}
$$

All above p-adic transformations will be proved in a unified way. In fact, with help of the same idea, we can prove the following result, which confirms a conjecture of Deines, Fuselier, Long, Swisher and Tu [6].

Theorem 2.25. *Let $p\equiv1\pmod 4$ be a prime. Then*

$$
{}_2F_1\left[\begin{matrix}\frac12 & \frac12\\ & 1\end{matrix}\middle|\,-1\right]_{p-1}\equiv p^2\cdot{}_3F_2\left[\begin{matrix}1 & 1 & 1\\ & \frac32 & \frac32\end{matrix}\middle|\,-1\right]_{p-1}\pmod{p^2}.\tag{2.68}
$$

The remainder sections will be organized as follows. In Section 3, we shall firstly give two key lemmas, which are frequently used in the whole paper. Then in Section 4, we shall prove a generalization of Theorem 1.1, which is also a complete p-adic analogue of (1.2).

From Section 5 to Section 9, the proofs of those linear and quadratic p-adic transformations of ${}_2F_1$ series will be given successively. In particular, we shall sketch the proof of Theorem 2.13 in Section 5, since its proof is very similar as Theorem 2.4. Furthermore, in Section 10, we shall consider some special cases of those quadratic p-adic transformations.

In Section 11, the proof of Theorem 2.9 will be given. Of course, as we have mentioned, Theorem2.11 is the consequence of Theorems 2.14 and 2.15, and Theorem 2.10 easily follows from Theorems 2.21 and 2.22. In Section 12, we shall prove Theorems 2.14 and 2.15, as well as Theorem 2.12.

In Sections 13 and 14, we shall complete the proofs of those p-adic Whipple's ${}_7F_6$ transformation, and explain how to deduce Theorems 2.19 and 2.20 from Theorems 2.16 and 2.18. However, the proofs of Theorems 2.21-2.24 are factly very similar as Theorems 2.16-2.18. So we won't give the detailed proofs of Theorems 2.21-2.24 here. Finally, in the last section, Theorem 2.25 will be proved.

3. Two auxiliary lemmas

In this section, we introduce two auxiliary lemmas, which are the key ingredients of almost all proofs in this paper. The first one is nearly trivial.

Lemma 3.1. *Suppose that $P(x), Q(x) \in \mathbb{Z}_p[x]$ are polynomials over $\mathbb{Z}_p$ and let $f(x) = P(x)/Q(x)$. Suppose that $\alpha \in \mathbb{Z}_p$ and $p \nmid Q(\alpha)$. Then for any $s \in \mathbb{Z}_p$, we have the p-adic expansion*

$$f(\alpha+sp) = f(\alpha) + f'(\alpha)\cdot sp + \frac{f''(\alpha)}{2!}\cdot s^2p^2 + \cdots + \frac{f^{(k)}(\alpha)}{k!}\cdot s^kp^k + \cdots,$$

where $f^{(k)}(x)$ denotes the k-th ordinary derivative of $f(x)$ as a rational function, not the p-adic derivative.

Let us turn the derivative of the p-adic gamma function. For a continuous function $f : \mathbb{Z}_p \to \mathbb{Z}_p$, define the p-adic derivative

$$f'(\alpha) := \lim_{|x-\alpha|_p\to 0} \frac{f(x)-f(\alpha)}{x-\alpha}.$$

Clearly if the above limit exists, then

$$f'(\alpha) = \lim_{n\to\infty} \frac{f(\alpha+p^n)-f(\alpha)}{p^n}.$$

Let $\Gamma_p'(x)$ denote the p-adic derivative of $\Gamma_p(x)$. The following lemma establishes a connection between the p-adic derivative of $\Gamma_p(x)$ and the common derivative of $\Gamma(x)$.

Lemma 3.2. *Let $s_1,\ldots,s_m,t_1,\ldots,t_n$ be some p-integral rational numbers with $s_1+\cdots+s_m=t_1+\cdots+t_n$. Then for any $\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_n\in\mathbb{Z}_p$,*

$$\begin{aligned}&\frac{\Gamma_p(\alpha_1+s_1p)\cdots\Gamma_p(\alpha_m+s_mp)}{\Gamma_p(\beta_1+t_1p)\cdots\Gamma_p(\beta_n+t_np)}-\frac{\Gamma_p(\alpha_1)\cdots\Gamma_p(\alpha_m)}{\Gamma_p(\beta_1)\cdots\Gamma_p(\beta_n)}\\ \equiv&(-1)^{\delta}p\cdot\frac{d}{dx}\bigg(\frac{\Gamma(a_1+s_1x)\cdots\Gamma(a_m+s_mx)}{\Gamma(b_1+t_1x)\cdots\Gamma(b_n+t_nx)}\bigg)\bigg|_{x=0}\pmod{p^2},\end{aligned}\tag{3.1}$$

where $a_i=p-\langle-\alpha_i\rangle_p$, $b_j=p-\langle-\beta_j\rangle_p$ and $\delta=a_1+\cdots+a_m-b_1-\cdots-b_n$.

Proof. We know that

$$\frac{\Gamma_p'(\alpha)}{\Gamma_p(\alpha)}\equiv\Gamma_p'(0)+H_{p-\langle-\alpha\rangle_p-1}\pmod p\tag{3.2}$$

for each $\alpha\in\mathbb{Z}_p$, where

$$H_n:=\sum_{k=1}^n\frac{1}{k}.$$

For the sake of completeness, here we give a proof of (3.2). It is easy to check that

$$\Gamma_p(\alpha+p^n)\equiv\Gamma_p(\alpha)\pmod{p^n}$$

for any $n\in\mathbb{N}$. Choose a sufficiently large integer n such that

$$\Gamma_p'(\alpha)\equiv\frac{\Gamma_p(\alpha+p^n)-\Gamma_p(\alpha)}{p^n}\pmod p\qquad\text{and}\qquad\Gamma_p'(0)\equiv\frac{\Gamma_p(p^n)-1}{p^n}\pmod p.$$

Let $a=p^{n+1}-\langle-\alpha\rangle_{p^{n+1}}$. Then

$$\begin{aligned}\frac{\Gamma_p'(\alpha)}{\Gamma_p(\alpha)}\equiv&\frac{1}{\Gamma_p(a)}\cdot\frac{\Gamma_p(a+p^n)-\Gamma_p(a)}{p^n}=\frac{1}{p^n}\cdot\Bigg(\Gamma_p(p^n)\prod_{\substack{1\le j<a\\(j,p)=1}}\frac{j+p^n}{j}-1\Bigg)\\ \equiv&\frac{\Gamma_p(p^n)-1}{p^n}+\sum_{\substack{1\le j<p^{n+1}-\langle-\alpha\rangle_{p^{n+1}}\\(j,p)=1}}\frac{1}{j}\equiv\Gamma_p'(0)+\sum_{1\le j<p-\langle-\alpha\rangle_p}\frac{1}{j}\pmod p.\end{aligned}$$

On the other hand, we know (cf. [3, Theorem 1.2.5]) that

$$\frac{\Gamma'(x)}{\Gamma(x)}=-\frac{1}{x}-\gamma_0+\sum_{n=1}^{\infty}\bigg(\frac{1}{n}-\frac{1}{n+x}\bigg),\tag{3.3}$$

where γ_0 is the Euler constant. So we have

$$\frac{\Gamma_p'(\alpha)}{\Gamma_p(\alpha)}-\Gamma_p'(0)\equiv\frac{\Gamma'(a)}{\Gamma(a)}+\gamma_0\pmod p\tag{3.4}$$

where $a=p-\langle-\alpha\rangle_p$.

Clearly

$$\begin{aligned}
&\frac{\Gamma_p(\alpha_1+s_1p)\cdots\Gamma_p(\alpha_m+s_mp)}{\Gamma_p(\beta_1+t_1p)\cdots\Gamma_p(\beta_n+t_np)}\\
\equiv&\frac{(\Gamma_p(\alpha_1)+\Gamma_p'(\alpha_1)\cdot s_1p)\cdots(\Gamma_p(\alpha_m)+\Gamma_p'(\alpha_m)\cdot s_mp)}{(\Gamma_p(\beta_1)+\Gamma_p'(\beta_1)\cdot t_1p)\cdots(\Gamma_p(\beta_n)+\Gamma_p'(\beta_n)\cdot t_np)}\\
\equiv&\frac{\Gamma_p(\alpha_1)\cdots\Gamma_p(\alpha_m)}{\Gamma_p(\beta_1)\cdots\Gamma_p(\beta_n)}\bigg(1+p\sum_{i=1}^m s_i\cdot\frac{\Gamma_p'(\alpha_i)}{\Gamma_p(\alpha_i)}-p\sum_{j=1}^n t_j\cdot\frac{\Gamma_p'(\beta_j)}{\Gamma_p(\beta_j)}\bigg)\pmod{p^2}.
\end{aligned}$$

Since $s_1+\cdots+s_m=t_1+\cdots+t_n$, in view of (3.4), we have

$$\begin{aligned}
\sum_{i=1}^m s_i\cdot\frac{\Gamma_p'(\alpha_i)}{\Gamma_p(\alpha_i)}-\sum_{j=1}^n t_j\cdot\frac{\Gamma_p'(\beta_j)}{\Gamma_p(\beta_j)}=&\sum_{i=1}^m s_i\bigg(\frac{\Gamma_p'(\alpha_i)}{\Gamma_p(\alpha_i)}-\Gamma_p'(0)\bigg)-\sum_{j=1}^n t_j\bigg(\frac{\Gamma_p'(\beta_j)}{\Gamma_p(\beta_j)}-\Gamma_p'(0)\bigg)\\
\equiv&\sum_{i=1}^m s_i\bigg(\frac{\Gamma'(a_i)}{\Gamma(a_i)}+\gamma_0\bigg)-\sum_{j=1}^n t_j\bigg(\frac{\Gamma'(b_j)}{\Gamma(b_j)}+\gamma_0\bigg)\\
=&\sum_{i=1}^m s_i\cdot\frac{\Gamma'(a_i)}{\Gamma(a_i)}-\sum_{j=1}^n t_j\cdot\frac{\Gamma'(b_j)}{\Gamma(b_j)}\pmod{p}.
\end{aligned}$$

So

$$\begin{aligned}
&\frac{\Gamma_p(\alpha_1+s_1p)\cdots\Gamma_p(\alpha_m+s_mp)}{\Gamma_p(\beta_1+t_1p)\cdots\Gamma_p(\beta_n+t_np)}-\frac{\Gamma_p(\alpha_1)\cdots\Gamma_p(\alpha_m)}{\Gamma_p(\beta_1)\cdots\Gamma_p(\beta_n)}\\
\equiv&p\cdot\frac{\Gamma_p(a_1)\cdots\Gamma_p(a_m)}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)}\bigg(\sum_{i=1}^m s_i\cdot\frac{\Gamma'(a_i)}{\Gamma(a_i)}-\sum_{j=1}^n t_j\cdot\frac{\Gamma'(b_j)}{\Gamma(b_j)}\bigg)\\
=&(-1)^\delta p\cdot\frac{d}{dx}\bigg(\frac{\Gamma(a_1+s_1x)\cdots\Gamma(a_m+s_mx)}{\Gamma(b_1+t_1x)\cdots\Gamma(b_n+t_nx)}\bigg)\bigg|_{x=0}\pmod{p^2}.
\end{aligned}$$

□

4. The p-adic analogue of Gauss' identity

In this section, we shall give the detailed proof of Theorem 1.1. In fact, we have the following complete p-adic analogue of Gauss' identity (1.2).

Theorem 4.1. *Let $\alpha,\beta\in\mathbb{Z}_p$ and $\gamma\in\mathbb{Z}_p^\times$ with $\langle-\alpha\rangle_p,\langle-\beta\rangle_p\le\langle-\gamma\rangle_p$.*

(i) If $\langle-\alpha\rangle_p+\langle-\beta\rangle_p\le\langle-\gamma\rangle_p$, then

$$ {}_2F_1\left[\begin{matrix}\alpha & \beta\\ & \gamma\end{matrix}\middle|\,1\right]_{\langle-\gamma\rangle_p}\equiv\frac{\Gamma_p(\gamma)\Gamma_p(\gamma-\alpha-\beta)}{\Gamma_p(\gamma-\alpha)\Gamma_p(\gamma-\beta)}\pmod{p^2}. \tag{4.1}$$

(ii) If $\langle -\alpha\rangle_p + \langle -\beta\rangle_p > \langle -\gamma\rangle_p$*, then*

$$ {}_2F_1\left[\begin{matrix}\alpha & \beta\\ & \gamma\end{matrix}\middle|\, 1\right]_{\langle -\gamma\rangle_p} \equiv (\gamma+\langle -\gamma\rangle_p-\alpha-\langle -\alpha\rangle_p-\beta-\langle -\beta\rangle_p)\cdot\frac{\Gamma_p(\gamma)\Gamma_p(\gamma-\alpha-\beta)}{\Gamma_p(\gamma-\alpha)\Gamma_p(\gamma-\beta)} \pmod{p^2}. \quad (4.2)$$

Proof. Let $a = \langle -\alpha\rangle_p$, $b = \langle -\beta\rangle_p$ and $c = \langle -\gamma\rangle_p$. Since $a \le c$, by (1.2), we have

$$ {}_2F_1\left[\begin{matrix}-a & \beta\\ & \gamma\end{matrix}\middle|\, 1\right]_c = {}_2F_1\left[\begin{matrix}-a & \beta\\ & \gamma\end{matrix}\middle|\, 1\right] = \frac{\Gamma(\gamma)\Gamma(\gamma+a-\beta)}{\Gamma(\gamma+a)\Gamma(\gamma-\beta)}.$$

First, assume that $a + b \le c$. Since $\gamma \not\equiv -j \pmod p$ for any $0 \le j < a$, we have

$$\frac{\Gamma(\gamma)}{\Gamma(\gamma+a)} = \prod_{j=0}^{a-1}\frac{1}{\gamma+j} = (-1)^a\cdot\frac{\Gamma_p(\gamma)}{\Gamma_p(\gamma+a)}.$$

Similarly,

$$\frac{\Gamma(\gamma+a-\beta)}{\Gamma(\gamma-\beta)} = (-1)^a\cdot\frac{\Gamma_p(\gamma+a-\beta)}{\Gamma_p(\gamma-\beta)}.$$

Thus

$$ {}_2F_1\left[\begin{matrix}-a & \beta\\ & \gamma\end{matrix}\middle|\, 1\right]_c = \frac{\Gamma_p(\gamma)\Gamma_p(\gamma+a-\beta)}{\Gamma_p(\gamma+a)\Gamma_p(\gamma-\beta)}.$$

Let

$$\Psi(x) = {}_2F_1\left[\begin{matrix}-a+x & \beta\\ & \gamma\end{matrix}\middle|\, 1\right]_c, \qquad \psi(x) = {}_2F_1\left[\begin{matrix}-a+x & -b\\ & p-c\end{matrix}\middle|\, 1\right].$$

Clearly $\psi(x)$ is also a polynomial in x. And we also have $\Psi'(0) \equiv \psi'(0) \pmod p$, since $(\beta)_k \equiv (-b)_k \pmod p$ and $(\gamma)_k \equiv (p-c)_k \pmod p$ for each $0 \le k \le b$. Let $s = (\alpha + a)/p$. By Lemma 3.1,

$$ {}_2F_1\left[\begin{matrix}\alpha & \beta\\ & \gamma\end{matrix}\middle|\, 1\right]_c = \Psi(sp) \equiv \Psi(0) + \Psi'(0)\cdot sp \equiv \Psi(0) + \psi'(0)\cdot sp \pmod{p^2}.$$

On the other hand, using (1.2) and Lemma 3.2, we also get

$$\begin{aligned} sp\cdot\psi'(0) =& p\cdot\frac{d}{dx}\left({}_2F_1\left[\begin{matrix}-a+sx & -b\\ & p-c\end{matrix}\middle|\, 1\right]\right)\Bigg|_{x=0}\\ =&\frac{\Gamma(p-c)}{\Gamma(p-c+b)}\cdot p\cdot\frac{d}{dx}\left(\frac{\Gamma(p-c+a+b-sx)}{\Gamma(p-c+a-sx)}\right)\Bigg|_{x=0}\\ \equiv&\frac{\Gamma_p(\gamma)}{\Gamma_p(\gamma-\beta)}\left(\frac{\Gamma_p(\gamma-\beta+a-sp)}{\Gamma_p(\gamma+a-sp)} - \frac{\Gamma_p(\gamma-\beta+a)}{\Gamma_p(\gamma+a)}\right) \pmod{p^2},\end{aligned}$$

i.e.,

$$\begin{aligned} &{}_2F_1\left[\begin{matrix}\alpha & \beta\\ & \gamma\end{matrix}\middle|\, 1\right]_c - \frac{\Gamma_p(\gamma)}{\Gamma_p(\gamma-\beta)}\cdot\frac{\Gamma_p(\gamma-\beta-\alpha)}{\Gamma_p(\gamma-\alpha)}\\ \equiv&{}_2F_1\left[\begin{matrix}-a & \beta\\ & \gamma\end{matrix}\middle|\, 1\right]_c - \frac{\Gamma_p(\gamma)}{\Gamma_p(\gamma-\beta)}\cdot\frac{\Gamma_p(\gamma-\beta+a)}{\Gamma_p(\gamma+a)} = 0 \pmod{p^2}.\end{aligned}$$

(4.1) is concluded.

Next, assume that $a+b \geq c+1$. Since p divides $\gamma-\beta+c-b$, now we have

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix}-a & \beta \\ & \gamma\end{matrix}\middle| 1\right]_c = \frac{\Gamma(\gamma)\Gamma(\gamma+a-\beta)}{\Gamma(\gamma+a)\Gamma(\gamma-\beta)} =&(\gamma-\beta+c-b)\cdot\frac{\Gamma_p(\gamma)\Gamma_p(\gamma+a-\beta)}{\Gamma_p(\gamma+a)\Gamma_p(\gamma-\beta)}\\
\equiv&(\gamma-\beta+c-b)\cdot\frac{\Gamma_p(\gamma)\Gamma_p(\gamma-\alpha-\beta)}{\Gamma_p(\gamma-\alpha)\Gamma_p(\gamma-\beta)} \pmod{p^2}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{d}{dx}\left({}_2F_1\left[\begin{matrix}-a+x & \beta \\ & \gamma\end{matrix}\middle| 1\right]_c\right)\Bigg|_{x=0} \equiv&\frac{d}{dx}\left({}_2F_1\left[\begin{matrix}p-a+x & -b \\ & p-c\end{matrix}\middle| 1\right]_c\right)\Bigg|_{x=0}\\
=&\frac{\Gamma(p-c)}{\Gamma(p-c+b)}\cdot\frac{d}{dx}\left(\frac{\Gamma(a+b-c-x)}{\Gamma(a-c-x)}\right)\Bigg|_{x=0} \pmod{p^2}.
\end{aligned}
$$

Note that

$$
\frac{d}{dx}\left(\frac{\Gamma(a+b-c-x)}{\Gamma(a-c-x)}\right)\Bigg|_{x=0} = \frac{d\big((a-c-x)_b\big)}{dx}\Bigg|_{x=0} = -\prod_{\substack{0\leq j<b\\ j\neq c-a}}(a-c+j) = \frac{(-1)^{b-1}\Gamma_p(a+b-c)}{\Gamma_p(a-c)}.
$$

It follows from Lemma 3.1 that

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix}\alpha & \beta \\ & \gamma\end{matrix}\middle| 1\right]_c - {}_2F_1\left[\begin{matrix}-a & \beta \\ & \gamma\end{matrix}\middle| 1\right]_c \equiv& -sp\cdot\frac{\Gamma_p(p-c)\Gamma_p(a+b-c)}{\Gamma_p(p-c+b)\Gamma_p(a-c)}\\
\equiv& -sp\cdot\frac{\Gamma_p(\gamma)\Gamma_p(\gamma-\alpha-\beta)}{\Gamma_p(\gamma-\beta)\Gamma_p(\gamma-\alpha)} \pmod{p^2}.
\end{aligned}
$$

Thus (4.2) is also concluded since $sp=\alpha+a$. □

We mention that by using the similar discussions, in the other theorems, we also may replace ${}_{m+1}F_m\left[\begin{matrix}* & * & * & \ldots & * \\ & 1 & * & \ldots & *\end{matrix}\middle| z\right]_{p-1}$ by ${}_{m+1}F_m\left[\begin{matrix}* & * & * & \ldots & * \\ & \gamma & * & \ldots & *\end{matrix}\middle| z\right]_{\langle-\gamma\rangle_p}$.

Furthermore, we have the following p-adic analogue of (1.8), which also generalizes both (1.7) and (4.1).

Theorem 4.2. *Let $\alpha\in\mathbb{Z}_p$ and $\beta,\gamma\in\mathbb{Z}_p^\times$ with $\langle-\alpha\rangle_p+\langle-\beta\rangle_p\leq\langle-\gamma\rangle_p$. Then*

$$
{}_2F_1\left[\begin{matrix}\alpha & \beta \\ & \gamma\end{matrix}\middle| z\right]_{\langle-\gamma\rangle_p} \equiv \frac{\Gamma_p(\gamma)\Gamma_p(\gamma-\alpha-\beta)}{\Gamma_p(\gamma-\alpha)\Gamma_p(\gamma-\beta)}\cdot{}_2F_1\left[\begin{matrix}\alpha & \beta \\ & \alpha+\beta-\gamma+1\end{matrix}\middle| 1-z\right]_{\langle\gamma-\alpha-\beta-1\rangle_p} \pmod{p^2}. \tag{4.3}
$$

Proof. Let $a=\langle-\alpha\rangle_p$, $b=\langle-\beta\rangle_p$, $c=\langle-\gamma\rangle_p$ and $d=\langle\gamma-\alpha-\beta-1\rangle_p$. Clearly $d=p+a+b-c-1$ and $d\geq\max\{a,b\}$. Let

$$
\Psi(x)={}_2F_1\left[\begin{matrix}-a+x & \beta \\ & \gamma\end{matrix}\middle| z\right]_c,\qquad \Omega(x)=\frac{\Gamma_p(\gamma)\Gamma_p(\gamma+a-\beta-x)}{\Gamma_p(\gamma+a-x)\Gamma_p(\gamma-\beta)}
$$

and

$$\Phi(x) = {}_2F_1\left[\begin{matrix}-a+x & \beta \\ & 1-a+\beta-\gamma+x\end{matrix}\middle| 1-z\right]_d.$$

In view of (1.8), we have

$$\Psi(0) = {}_2F_1\left[\begin{matrix}-a & \beta \\ & \gamma\end{matrix}\middle| z\right] = \frac{\Gamma_p(\gamma)\Gamma_p(\gamma+a-\beta)}{\Gamma_p(\gamma+a)\Gamma_p(\gamma-\beta)}\cdot{}_2F_1\left[\begin{matrix}-a & \beta \\ & 1-a+\beta-\gamma\end{matrix}\middle| 1-z\right] = \Omega(0)\Phi(0).$$

Let

$$\omega(x) = \frac{\Gamma(p-c)}{\Gamma(p+b-c)}\cdot\frac{\Gamma(p+a+b-c-x)}{\Gamma(p+a-c-x)}$$

and

$$\phi(x) = {}_2F_1\left[\begin{matrix}-a+x & -b \\ & 1-a-b+c-p+x\end{matrix}\middle| 1-z\right].$$

Clearly

$$\Phi(0)\equiv\phi(0) \pmod p,\qquad \Phi'(0)\equiv\phi'(0) \pmod p.$$

We also have

$$\Omega(0)\equiv\omega(0) \pmod p,\qquad \Omega'(0)\equiv\omega'(0) \pmod p$$

by Lemma 3.2. Hence

$$\begin{aligned}\Psi'(0) \equiv& \frac{d}{dx}\left({}_2F_1\left[\begin{matrix}-a+x & -b \\ & p-c\end{matrix}\middle| z\right]\right)\bigg|_{x=0} \equiv \frac{d\big(\omega(x)\phi(x)\big)}{dx}\bigg|_{x=0}\\ =&\omega'(0)\phi(0)+\omega(0)\phi'(0) \equiv \frac{d\big(\Omega(x)\Phi(x)\big)}{dx}\bigg|_{x=0} \pmod p.\end{aligned}$$

Letting $s=(\alpha+a)/p$, we get

$$\Psi(sp)\equiv\Psi(0)+sp\cdot\Psi'(0)\equiv\Omega(0)\Phi(0)+sp\cdot\frac{d\big(\Omega(x)\Phi(x)\big)}{dx}\bigg|_{x=0}\equiv\Omega(sp)\Phi(sp) \pmod{p^2}.$$

□

5. The p-adic quadratic ${}_2F_1$ transformation I: $z\to 4z(1-z)$ and $z\to 4z/(1+z)^2$

In this section, we shall prove Theorems 2.1, 2.2, 2.4 and 2.13. The proof of Theorem 2.1 is simple.

Proof of Theorem 2.1. Let $a=\langle-\alpha\rangle_p$. Let

$$\Psi(x,z) = {}_2F_1\left[\begin{matrix}-a-x & 1+a+x \\ & 1\end{matrix}\middle| z\right]_{p-1}$$

and

$$\Phi(x,z) = {}_2F_1\left[\begin{matrix}-\frac12(a+x) & \frac12+\frac12(a+x) \\ & 1\end{matrix}\middle| 4z(1-z)\right]_{p-1}$$

Since a is even, by (2.2),

$$\Psi(0,z) = {}_2F_1\left[\begin{matrix}-a & 1+a \\ & 1\end{matrix}\middle| z\right] = {}_2F_1\left[\begin{matrix}-\frac{1}{2}a & \frac{1}{2}+\frac{1}{2}a \\ & 1\end{matrix}\middle| 4z(1-z)\right] = \Phi(0,z).$$

Let

$$F(z) = \frac{\partial \Psi(x,z)}{\partial x}\bigg|_{x=0} \quad \text{and} \quad G(z) = \frac{\partial \Phi(x,z)}{\partial x}\bigg|_{x=0}.$$

It suffices to show that

$$[z^n]F(z) \equiv [z^n]G(z) \pmod p \tag{5.1}$$

for each $0 \le n \le 2p-2$, where $[z^n]F(z)$ denotes the coefficient of z^n in $F(z)$. Suppose that $p \le n \le 2p-2$. Note that $\binom{k}{n-k}$ vanishes for each $0 \le k < n/2$ and $(-a/2)_k = 0$ for each $a/2 < k \le p-1$. It is easy to see that

$$[z^n]G(z) =(-1)^n \sum_{k=0}^{p-1}\binom{k}{n-k}(-4)^k \cdot \frac{(\frac{1}{2}+\frac{1}{2}a)_k}{k!} \cdot \frac{d}{dx}\left(\frac{(-\frac{1}{2}(a+x))_k}{(1)_k}\right)\bigg|_{x=0}.$$

Also, if $(p-a+1)/2 \le k \le p-1$, then

$$\left(\frac{1}{2}+\frac{a}{2}\right)_k = \prod_{j=0}^{k-1}\frac{1+a+2j}{2} \equiv 0 \pmod p.$$

Thus we always have $[z^n]G(z) \equiv 0 \pmod p$ for those $p \le n \le 2p-2$. On the other hand, for any $0 \le n \le p-1$, clearly we have

$$\begin{aligned}[z^n]F(z) =& \frac{\partial}{\partial x}\left([z^n]{}_2F_1\left[\begin{matrix}-a-x & 1+a+x \\ & 1\end{matrix}\middle| z\right]\right)\bigg|_{x=0} \\ =& \frac{\partial}{\partial x}\left([z^n]{}_2F_1\left[\begin{matrix}-\frac{1}{2}(a+x) & \frac{1}{2}+\frac{1}{2}(a+x) \\ & 1\end{matrix}\middle| 4z(1-z)\right]\right)\bigg|_{x=0} = [z^n]G(z).\end{aligned}$$

Thus (5.1) is derived. $\square$

The proof of Theorem 2.4 is a little more complicated, since $\lambda_p(\alpha)$ is involved now. Clearly we can't take the derivative of $z^{-\lambda_p(a+x)}$. So we need the following lemma.

Lemma 5.1. *Suppose that $P(x), Q(x) \in \mathbb{Z}_p[x]$ are polynomials over $\mathbb{Z}_p$ with $Q(0) \in \mathbb{Z}_p^\times$. Let $f(x) = P(x)/Q(x)$. Suppose that $z \in \mathbb{Z}_p^\times$ and $s \in \mathbb{Z}$. Then*

$$z^{1+s(p-1)}f(sp) - zf(0) \equiv s\big(z^p f(p) - zf(0)\big) \pmod{p^2}.$$

Proof. Note that

$$z^{s(p-1)} - 1 = \big(1+(z^{p-1}-1)\big)^s - 1 \equiv s(z^{p-1}-1) \pmod{p^2}.$$

So by Lemma 3.1,

$$\begin{aligned} z^{1+s(p-1)}f(sp)-zf(0) =& z^{1+s(p-1)}\cdot\big(f(sp)-f(0)\big)+\big(z^{1+s(p-1)}-z\big)\cdot f(0)\\ \equiv& z^{1+s(p-1)}\cdot spf'(0)+s(z^p-z)\cdot f(0)\\ \equiv& z^p\cdot s(f(p)-f(0))+s(z^p-z)\cdot f(0)=s\big(z^pf(p)-zf(0)\big) \pmod{p^2}. \end{aligned}$$

□

Proof of Theorem 2.4. Trivially

$${}_2F_1\left[\begin{matrix}\alpha & \beta\\ & 1\end{matrix}\middle|\, z\right]_{p-1}\equiv {}_2F_1\left[\begin{matrix}-\langle-\alpha\rangle_{p^2} & -\langle-\beta\rangle_{p^2}\\ & 1\end{matrix}\middle|\, z\right]_{p-1} \pmod{p^2}.$$

Without loss of generality, assume that $\alpha=-\langle-\alpha\rangle_{p^2}$. Let $a=\langle-\alpha\rangle_p$ and $s=-(\alpha+a)/p$. clearly $a+s(p-1)=-\lambda_p(\alpha)$. Let

$$\Psi_1(x)={}_2F_1\left[\begin{matrix}-a-x & -a-x\\ & 1\end{matrix}\middle|\, z\right]_{p-1},\quad \Psi_2(x)=z^a{}_2F_1\left[\begin{matrix}-a-x & -a-x\\ & 1\end{matrix}\middle|\, \frac{1}{z}\right]_{p-1},$$

and let

$$\Phi(x)=(1+z)^a{}_2F_1\left[\begin{matrix}-\frac12(a+x) & -\frac12(a+x)+\frac12\\ & 1\end{matrix}\middle|\, \frac{4z}{(1+z)^2}\right]_{p-1}.$$

Note that either $a/2$ or $a/2-1/2$ is a non-negative integer. By (2.13),

$$\begin{aligned} \Psi_1(0)+z\cdot\Psi_2(0)=&{}_2F_1\left[\begin{matrix}-a & -a\\ & 1\end{matrix}\middle|\, z\right]+z^{a+1}{}_2F_1\left[\begin{matrix}-a & -a\\ & 1\end{matrix}\middle|\, \frac1z\right]\\ =&(1+z)^{a+1}{}_2F_1\left[\begin{matrix}-\frac{a}{2} & -\frac{a}{2}+\frac12\\ & 1\end{matrix}\middle|\, \frac{4z}{(1+z)^2}\right]=(1+z)\Phi(0). \end{aligned}\tag{5.2}$$

On the other hand, since $0\le a\le p-1$, we also have

$$\begin{aligned} (1+z)^{p+1}\Phi(p)=&(1+z)^{a+p+1}{}_2F_1\left[\begin{matrix}-\frac12(a+p) & -\frac12(a+p)+\frac12\\ & 1\end{matrix}\middle|\, \frac{4z}{(1+z)^2}\right]\\ =&{}_2F_1\left[\begin{matrix}-a-p & -a-p\\ & 1\end{matrix}\middle|\, z\right]+z^{a+p+1}{}_2F_1\left[\begin{matrix}-a-p & -a-p\\ & 1\end{matrix}\middle|\, \frac1z\right]. \end{aligned}$$

Clearly $(-a-p)_k$ is divisible by p for each $a+1\le k\le p-1$. Note that

$$\frac{(-a)_k}{(1)_k}=(-1)^k\cdot\binom{a}{k}=(-1)^k\cdot\binom{a}{a-k}=(-1)^a\cdot\frac{(-a)_{a-k}}{(1)_{a-k}}\tag{5.3}$$

for any $0 \le k \le a$. Hence

$$\begin{aligned}
&{}_2F_1\left[\begin{matrix}-a-p & -a-p\\ & 1\end{matrix}\middle| z\right]\\
\equiv&\sum_{k=0}^{a}\frac{(-a-p)_k^2}{(1)_k}\cdot\frac{z^k}{k!}+\sum_{k=0}^{a}\frac{(-a-p)_{p+k}^2}{(1)_{p+k}}\cdot\frac{z^{p+k}}{(p+k)!}\\
=&\sum_{k=0}^{a}\frac{(-a-p)_k^2}{(1)_k}\cdot\frac{z^k}{k!}+\sum_{k=0}^{a}\frac{(-a-p)_k^2}{(1)_k}\cdot\frac{z^{p+a-k}}{k!}\\
\equiv&{}_2F_1\left[\begin{matrix}-a-p & -a-p\\ & 1\end{matrix}\middle| z\right]_{p-1}+z^{a+p}{}_2F_1\left[\begin{matrix}-a-p & -a-p\\ & 1\end{matrix}\middle| \frac{1}{z}\right]_{p-1} \pmod{p^2}.
\end{aligned}\tag{5.4}$$

It follows that

$$(1+z)^p\Phi(p)\equiv\Psi_1(p)+z^p\Psi_2(p) \pmod{p^2}.$$

By Lemma 5.1 and (5.2), we get that

$$\begin{aligned}
(1+z)^{1+s(p-1)}\Phi(sp)-\Phi(0)\equiv&s\cdot\big((1+z)^p\Phi(p)-(1+z)\Phi(0)\big)\\
\equiv&s\cdot\big(\Psi_1(p)-\Psi_1(0)\big)+s\cdot\big(z^p\Psi_2(p)-z\Psi_2(0)\big)\\
\equiv&\big(\Psi_1(sp)-\Psi_1(0)\big)+\big(z^{sp}\Psi_2(sp)-z\Psi_2(0)\big) \pmod{p^2},
\end{aligned}$$

i.e.,

$$(1+z)^{1+s(p-1)}\Phi(sp)\equiv\Psi_1(sp)+z^{s(p-1)+1}\Psi_2(sp) \pmod{p^2}.$$

□

Below we just sketch the proofs of Theorems 2.2 and 2.13, since they are very similar as the proofs of Theorems 2.1 and 2.4 respectively.

Let $a=\langle-\alpha\rangle_p$. First, Theorem 2.2 can be easily deduced from

$$\begin{aligned}
&[z^n]\frac{d}{dx}\left(\left({}_2F_1\left[\begin{matrix}-a-x & 1+a+x\\ & 1\end{matrix}\middle| z\right]_{p-1}\right)^2\right)\Bigg|_{x=0}\\
\equiv&[z^n]\frac{d}{dx}\left({}_3F_2\left[\begin{matrix}-a-x & 1+a+x & \frac12\\ & 1 & 1\end{matrix}\middle| 4z(1-z)\right]_{p-1}\right)\Bigg|_{x=0} \pmod{p}
\end{aligned}\tag{5.5}$$

for any $0\le n\le 2p-2$. Of course, the proof of (5.5) is very similar as the one of (5.1).

Next, it is not difficult to check that

$$\sum_{k=0}^{a}\frac{(-a-p)_{p+k}^2(\beta)_{p+k}}{(1)_{p+k}^2(1-\beta-a-p)_{p+k}}\cdot z^k=\frac{(\beta)_p(-\beta-p-a+1)_a}{(-\beta-p-a+1)_p(\beta)_a}\sum_{k=0}^{a}\frac{(-a-p)_k^2(\beta)_k}{(1)_k^2(1-\beta-a-p)_k}\cdot z^{a-k}.$$

It follows that

$$\begin{aligned}
&{}_3F_2\begin{bmatrix}-a-p & -a-p & \beta \\ & 1 & 1-\beta-a-p\end{bmatrix}\,z\Bigg] \\
\equiv&{}_3F_2\begin{bmatrix}-a-p & -a-p & \beta \\ & 1 & 1-\beta-a-p\end{bmatrix}\,z\Bigg]_{p-1} \\
&+(-z)^{a+p}{}_3F_2\begin{bmatrix}-a-p & -a-p & \beta \\ & 1 & 1-\beta-a-p\end{bmatrix}\,\frac{1}{z}\Bigg]_{p-1} \pmod{p^2}.
\end{aligned}$$

Then Theorems 2.13 can be proved by using the same discussions in the proof of Theorem 2.4.

6. The p-adic quadratic ${}_2F_1$ transformation II: $z \to z^2/(z-2)^2$

In order to prove Theorem 2.6, we need the following lemma.

Lemma 6.1. *Suppose that* $0 \le a \le p-1$ *and* $\beta \in \mathbb{Z}_p$. *Let* $t = (\beta + \langle -\beta\rangle_p)/p$. *Then*

$$(1-t)\sum_{k=0}^{a}\frac{(-a-p)_{p+k}(\beta)_{p+k}}{(1)_{p+k}^2}\cdot z^k \equiv t(1-z)^a\sum_{k=0}^{a}\frac{(-a-p)_{p+k}(1-\beta)_{p+k}}{(1)_{p+k}^2}\cdot\frac{z^k}{(z-1)^k} \pmod{p^2}. \tag{6.1}$$

Proof. Let $b = \langle -\beta\rangle_p$. Clearly for any $0 \le k \le a$, we have

$$\begin{aligned}
\frac{(-a-p)_{p+k}}{(1)_{p+k}} =&\frac{\Gamma_p(-a+k)\Gamma_p(1)}{\Gamma_p(-a-p)\Gamma_p(p+k+1)}\cdot\frac{-p}{p} \\
\equiv&-\frac{\Gamma_p(-a+k)\Gamma_p(1)}{\Gamma_p(-a)\Gamma_p(k+1)}\left(1+p\cdot\frac{\Gamma_p'(-a)}{\Gamma_p(-a)}-p\cdot\frac{\Gamma_p'(k+1)}{\Gamma_p(k+1)}\right) \\
\equiv&-\frac{(-a)_k}{(1)_k}-p\cdot\frac{(-a)_k}{(1)_k}\cdot(H_{p-a-1}-H_k) \pmod{p^2}.
\end{aligned}$$

If $k \le b$, then

$$\begin{aligned}
\frac{(\beta)_{p+k}}{(1)_{p+k}} =&\frac{\Gamma_p(\beta+p+k)\Gamma_p(1)}{\Gamma_p(\beta)\Gamma_p(p+k+1)}\cdot\frac{tp}{p} \\
\equiv&t\cdot\frac{\Gamma_p(\beta+k)\Gamma_p(1)}{\Gamma_p(\beta)\Gamma_p(k+1)}\cdot\left(1+p\cdot\frac{\Gamma_p'(\beta+k)}{\Gamma_p(\beta+k)}-p\cdot\frac{\Gamma_p'(k+1)}{\Gamma_p(k+1)}\right) \\
\equiv&t\cdot\frac{(\beta)_k}{(1)_k}+tp\cdot\frac{(\beta)_k}{(1)_k}\cdot\left(H_{p-1-b}+\sum_{j=0}^{k-1}\frac{1}{j+\beta}-H_k\right) \pmod{p^2},
\end{aligned}$$

by noting that

$$H_{p-1-b+k} = H_{p-1-b}+\sum_{j=1}^{k}\frac{1}{p-1-b+j} \equiv H_b+\sum_{j=1}^{k}\frac{1}{j+\beta-1} \pmod{p}.$$

And if $k\geq b+1$, we also have

$$\begin{aligned}\frac{(\beta)_{p+k}}{(1)_{p+k}}=&\frac{\Gamma_p(\beta+p+k)\Gamma_p(1)}{\Gamma_p(\beta)\Gamma_p(p+k+1)}\cdot\frac{tp}{p}\cdot\frac{(t+1)p}{p}\equiv(t+1)\cdot\frac{(\beta)_k}{(1)_k}\\ \equiv&t\cdot\frac{(\beta)_k}{(1)_k}+tp\cdot\frac{(\beta)_k}{(1)_k}\cdot\bigg(H_{p-1-b}+\sum_{j=0}^{k-1}\frac{1}{j+\beta}-H_k\bigg)\pmod{p^2}.\end{aligned}$$

Now by (1.3),

$$\begin{aligned}\sum_{k=0}^{a}\frac{(-a)_k(\beta)_k}{(1)_k^2}\cdot z^k=&{}_2F_1\bigg[\begin{matrix}-a&\beta\\&1\end{matrix}\bigg|\,z\bigg]=(1-z)^a{}_2F_1\bigg[\begin{matrix}-a&1-\beta\\&1\end{matrix}\bigg|\,\frac{z}{z-1}\bigg]\\ =&(1-z)^a\sum_{k=0}^{a}\frac{(-a)_k(1-\beta)_k}{(1)_k^2}\cdot\frac{z^k}{(z-1)^k}.\end{aligned}$$

Also, $\langle 1-\beta\rangle_p=p-1-b$ and $H_b\equiv H_{p-1-b}\pmod p$. So it suffices to show that

$$\begin{aligned}&\sum_{k=0}^{a}\frac{(-a)_k(\beta)_k}{(1)_k^2}\cdot z^k\cdot\bigg(\sum_{j=0}^{k-1}\frac{1}{j+\beta}-2H_k\bigg)\\ =&(1-z)^a\sum_{k=0}^{a}\frac{(-a)_k(1-\beta)_k}{(1)_k^2}\cdot\frac{z^k}{(z-1)^k}\cdot\bigg(\sum_{j=0}^{k-1}\frac{1}{j+1-\beta}-2H_k\bigg).\end{aligned}\tag{6.2}$$

Note that

$$\frac{d}{dx}\bigg(\frac{(\beta+x)_k}{(1+2x)_k}\bigg)\bigg|_{x=0}=\frac{(\beta)_k}{(1)_k}\cdot\bigg(\sum_{j=0}^{k-1}\frac{1}{\beta+j}-\sum_{j=0}^{k-1}\frac{2}{1+j}\bigg).$$

Clearly (6.2) immediately follows from

$$\frac{d}{dx}\bigg({}_2F_1\bigg[\begin{matrix}-a&\beta+x\\&1+2x\end{matrix}\bigg|\,z\bigg]\bigg)\bigg|_{x=0}=\frac{d}{dx}\bigg((1-z)^a{}_2F_1\bigg[\begin{matrix}-a&1-\beta+x\\&1+2x\end{matrix}\bigg|\,\frac{z}{z-1}\bigg]\bigg)\bigg|_{x=0}.$$

□

In particular, substituting $\beta=1/2$ in (6.1), we get

Corollary 6.1. *Suppose that* $0\leq a\leq p-1$. *Then*

$$\sum_{k=0}^{a}\frac{(-a-p)_{p+k}(\frac12)_{p+k}}{(1)_{p+k}^2}\cdot z^k\equiv(1-z)^a\sum_{k=0}^{a}\frac{(-a-p)_{p+k}(\frac12)_{p+k}}{(1)_{p+k}^2}\cdot\frac{z^k}{(z-1)^k}\pmod{p^2}.\tag{6.3}$$

We are ready to prove Theorem 2.6. Let $a=\langle-\alpha\rangle_p$ and $s=-(\alpha+a)/p$. Let

$$\Psi_1(x)={}_2F_1\bigg[\begin{matrix}-a-x&\frac12\\&1\end{matrix}\bigg|\,z\bigg]_{p-1},\qquad\Psi_2(x)=(1-z)^a{}_2F_1\bigg[\begin{matrix}-a-x&\frac12\\&1\end{matrix}\bigg|\,\frac{z}{z-1}\bigg]_{p-1},$$

and let

$$\Phi(x) = \left(1 - \frac{z}{2}\right)^a \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2}(a+x) & \frac{1}{2} - \frac{1}{2}(a+x) \\ & 1 \end{matrix}\middle| \frac{z^2}{(z-2)^2}\right]_{p-1}.$$

Since either $a/2$ and $a/2 - 1/2$ is an non-negative integer, by (2.19),

$$\begin{aligned}\Psi_1(0) + (1-z)\Psi_2(0) =& {}_2F_1\left[\begin{matrix} -a & \frac{1}{2} \\ & 1 \end{matrix}\middle| z\right] + (1-z)^{a+1}{}_2F_1\left[\begin{matrix} -a & \frac{1}{2} \\ & 1 \end{matrix}\middle| \frac{z}{z-1}\right] \\ =& 2\left(1 - \frac{z}{2}\right)^{a+1} \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2}a & \frac{1}{2} - \frac{1}{2}a \\ & 1 \end{matrix}\middle| \frac{z^2}{(z-2)^2}\right] = (2-z) \cdot \Phi(0).\end{aligned}$$

On the other hand, by Lemma 5.1, we have

$$\begin{aligned}&\big(\Psi_1(sp) - \Psi_1(0)\big) + (1-z)^{1+s(p-1)} \cdot \big(\Psi_2(sp) - \Psi_2(0)\big) \\ \equiv& s\big(\Psi_1(p) - \Psi_1(0)\big) + s\big((1-z)^p \cdot \Psi_2(p) - (1-z) \cdot \Psi_2(0)\big) \pmod{p^2}\end{aligned}$$

and

$$\left(1 - \frac{z}{2}\right)^{1+s(p-1)} \cdot \Phi(sp) - \left(1 - \frac{z}{2}\right) \cdot \Phi(0) \equiv s\left(1 - \frac{z}{2}\right)^p \cdot \Phi(p) - s\left(1 - \frac{z}{2}\right) \cdot \Phi(0) \pmod{p^2}.$$

It suffices to show that

$$\Psi_1(p) + (1-z)^p \cdot \Psi_2(p) \equiv 2\left(1 - \frac{z}{2}\right)^p \cdot \Phi(p) \pmod{p^2}.$$

Note that either $(a+p)/2$ or $(a+p-1)/2$ is an integer lying in $[0, p-1]$ now. So we have

$$\begin{aligned}2\left(1 - \frac{z}{2}\right)^p \cdot \Phi(p) =& 2\left(1 - \frac{z}{2}\right)^{a+p} \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2}(a+p) & \frac{1}{2} - \frac{1}{2}(a+p) \\ & 1 \end{matrix}\middle| \frac{z^2}{(z-2)^2}\right] \\ =& {}_2F_1\left[\begin{matrix} -a-p & \frac{1}{2} \\ & 1 \end{matrix}\middle| z\right] + (1-z)^{a+p}{}_2F_1\left[\begin{matrix} -a-p & \frac{1}{2} \\ & 1 \end{matrix}\middle| \frac{z}{z-1}\right] \\ \equiv& \Psi_1(p) + (1-z)^p\Psi_2(p) \pmod{p^2},\end{aligned}$$

where the last step follows from (6.3). □

7. The p-adic quadratic ${}_2F_1$ transformation III: $z \to 4\sqrt{z}/(1+\sqrt{z})^2$

Let us turn to the proof of Theorem 2.8. Let $a = \langle -\alpha \rangle_p$. Let

$$\Psi_1(x) = {}_2F_1\left[\begin{matrix} -a-x & -a-x \\ & 1 \end{matrix}\middle| z^2\right]_{p-1}, \quad \Psi_2(x) = z^{2a-1}{}_2F_1\left[\begin{matrix} -a-x & -a-x \\ & 1 \end{matrix}\middle| \frac{1}{z^2}\right]_{p-1}$$

and

$$\Phi(x) = (1+z)^{2a-1} \cdot {}_2F_1\left[\begin{matrix} -a-x & \frac{1}{2} \\ & 1 \end{matrix}\middle| \frac{4z}{(1+z)^2}\right]_{p-1}.$$

Now we need a lemma which is similar as Corollary 6.1.

Lemma 7.1.

$$\begin{aligned}&2\sum_{k=0}^{a}\frac{(-a-p)_{p+k}(\frac{1}{2})_{p+k}}{(1)_{p+k}^2}\cdot\left(\frac{4z}{(1+z)^2}\right)^k\\ \equiv&-\frac{\Phi(0)}{4^{p-1}(1+z)^{2a-1}}-\frac{p}{(1+z)^{2a}}\cdot\frac{\Psi_1'(0)+z\cdot\Psi_2'(0)}{2}\pmod{p^2}.\end{aligned}\tag{7.1}$$

Proof. According to the discussions in the proof of Lemma 6.1, we know that

$$\frac{(-a-p)_{p+k}(\frac{1}{2})_{p+k}}{(1)_{p+k}^2}\equiv-\frac{(-a)_k(\frac{1}{2})_k}{2\cdot(1)_k^2}\left(1+p\left(H_{p-1-a}+H_{\frac{p-1}{2}}+\sum_{j=0}^{k-1}\frac{1}{j+\frac{1}{2}}-2H_k\right)\right)\pmod{p^2}$$

for any $0\le k\le a$. Let

$$\phi(x)={}_2F_1\left[\begin{matrix}-a & \frac{1}{2}+x\\ & 1+2x\end{matrix}\middle|\,\frac{4z}{(1+z)^2}\right].$$

Clearly for $0\le k\le a$,

$$\frac{d}{dx}\left(\frac{(\frac{1}{2}+x)_k}{(1+2x)_k}\right)\bigg|_{x=0}=\frac{(\frac{1}{2}+x)_k}{(1+2x)_k}\sum_{j=0}^{k-1}\left(\frac{1}{j+\frac{1}{2}}-\frac{2}{j+1}\right).$$

Furthermore, we have $H_{p-1-a}\equiv H_a\pmod p$ and

$$H_{\frac{p-1}{2}}\equiv\frac{2(2^{1-p}-1)}{p}\equiv\frac{4^{1-p}-1}{p}\pmod p,$$

since

$$2^p=2+\sum_{j=1}^{p-1}\frac{p}{j}\cdot\binom{p-1}{j-1}\equiv2+p\cdot(H_{p-1}-H_{\frac{p-1}{2}})\equiv2-p\cdot H_{\frac{p-1}{2}}\pmod{p^2}.$$

Hence

$$\sum_{k=0}^{a}\frac{(-a-p)_{p+k}(\frac{1}{2})_{p+k}}{(1)_{p+k}^2}\cdot\left(\frac{4z}{(1+z)^2}\right)^k\equiv-\frac{\phi(0)}{2}\cdot(4^{1-p}+p\cdot H_a)-\frac{p}{2}\cdot\phi'(0)\pmod{p^2}.$$

On the other hand, we have the following general form of (2.23) [3, Eq. (3.1.11)]

$${}_2F_1\left[\begin{matrix}\alpha & \beta\\ & \alpha-\beta+1\end{matrix}\middle|\,z^2\right]=(1+z)^{-2\alpha}{}_2F_1\left[\begin{matrix}\alpha & \alpha-\beta+\frac{1}{2}\\ & 2\alpha-2\beta+1\end{matrix}\middle|\,\frac{4z}{(1+z)^2}\right].\tag{7.2}$$

Hence

$$\phi(x)=\frac{1}{(1+z)^{2a}}\cdot{}_2F_1\left[\begin{matrix}-a & -a-x\\ & 1+x\end{matrix}\middle|\,z^2\right].$$

Note that for any $0\le k\le a$, we have

$$\frac{d}{dx}\left(\frac{(-a-x)_k}{(1+x)_k}\right)\bigg|_{x=0}=\frac{(-a)_k}{(1)_k}\sum_{j=0}^{k-1}\left(\frac{1}{a-j}-\frac{1}{j+1}\right)=\frac{(-a)_k}{(1)_k}\cdot(H_a-H_{a-k}-H_k).$$

Also, clearly

$$\frac{d}{dx}\bigg(\frac{(-a-x)_k}{(1)_k}\bigg)\bigg|_{x=0}=\frac{(-a)_k}{(1)_k}\cdot(H_a-H_{a-k}).$$

Thus

$$\begin{aligned}
&-2\sum_{k=0}^{a}\frac{(-a-p)_{p+k}(\frac12)_{p+k}}{(1)_{p+k}^2}\cdot\bigg(\frac{4z}{(1+z)^2}\bigg)^k\\
\equiv&4^{1-p}\phi(0)+\frac{p}{(1+z)^{2a}}\cdot\frac{d}{dx}\bigg({}_2F_1\bigg[\begin{matrix}-a & -a-x\\ & 1\end{matrix}\bigg|\,z^2\bigg]+z^{2a}{}_2F_1\bigg[\begin{matrix}-a & -a-x\\ & 1\end{matrix}\bigg|\,\frac{1}{z^2}\bigg]\bigg)\bigg|_{x=0}\\
\equiv&\frac{\Phi(0)}{4^{p-1}(1+z)^{2a-1}}+\frac{p}{(1+z)^{2a}}\cdot\frac{\Psi_1'(0)+z\cdot\Psi_2'(0)}{2}\pmod{p^2}.
\end{aligned}$$

□

Let us return the proof of Theorem 2.8. By (2.23), we have

$$\Psi_1(0)=z\cdot\Psi_2(0)=(1+z)\cdot\Phi(0),$$

whence

$$\Psi_1(0)+z^2\cdot\Psi_2(0)=(1+z)^2\cdot\Phi(0).$$

According to Lemma 5.1, it suffices to show that

$$(1+z)^{2p}\cdot\Phi(p)\equiv\Psi_1(p)+z^{2p}\cdot\Psi_2(p)\pmod{p^2}.$$

In view of (2.23), (5.4) and (7.1), we have

$$\begin{aligned}
&{}_2F_1\bigg[\begin{matrix}-a-p & \frac12\\ & 1\end{matrix}\bigg|\,\frac{4z}{(1+z)^2}\bigg]_{p-1}\\
=&{}_2F_1\bigg[\begin{matrix}-a-p & \frac12\\ & 1\end{matrix}\bigg|\,\frac{4z}{(1+z)^2}\bigg]-\bigg(\frac{4z}{(1+z)^2}\bigg)^p\sum_{k=0}^{a}\frac{(-a-p)_{p+k}(\frac12)_{p+k}}{(1)_{p+k}^2}\cdot\bigg(\frac{4z}{(1+z)^2}\bigg)^k\\
\equiv&\frac{\Psi_1(p)+z^{2p+1}\Psi_2(p)}{(1+z)^{2a+2p}}+\frac{2z^p\cdot\Phi(0)}{(1+z)^{2p+2a-1}}+\frac{z^p\cdot p\big(\Psi_1'(0)+z\cdot\Psi_2'(0)\big)}{(1+z)^{2a+2p}}\pmod{p^2}.
\end{aligned}$$

Clearly

$$z^p\cdot p\big(\Psi_1'(0)+z\cdot\Psi_2'(0)\big)\equiv z\cdot\big(\Psi_1(p)-\Psi_1(0)\big)+z^{2p}\cdot\big(\Psi_2(p)-\Psi_2(0)\big)\pmod{p^2}.$$

Therefore

$$\begin{aligned}
(1+z)^{2p}\cdot\Phi(p)\equiv&\Psi_1(p)+z^{2p}\cdot\Psi_2(p)+2z^p\cdot\Phi(0)-\frac{z\cdot\Psi_1(0)+z^{2p}\cdot\Psi_2(0)}{1+z}\\
=&\Psi_1(p)+z^{2p}\cdot\Psi_2(p)-\frac{z(z^{p-1}-1)^2\cdot\Psi_1(0)}{1+z}\\
\equiv&\Psi_1(p)+z^{2p}\cdot\Psi_2(p)\pmod{p^2}.
\end{aligned}$$

□

8. The p-adic linear ${}_2F_1$ transformation.

Let us consider Theorem 2.3, i.e., the p-adic analogue of the linear transformation (2.9). Let $a = \langle -\alpha \rangle_p$. Let

$$\Psi_1(x) = {}_2F_1\left[\begin{matrix} -a-x & -a-x \\ & 1 \end{matrix}\middle| z\right]_{p-1}, \quad \Psi_2(x) = z^a {}_2F_1\left[\begin{matrix} -a-x & -a-x \\ & 1 \end{matrix}\middle| \frac{1}{z}\right]_{p-1},$$

and let

$$\Phi(x) = (1-z)^a {}_2F_1\left[\begin{matrix} -a-x & 1+a+x \\ & 1 \end{matrix}\middle| \frac{z}{z-1}\right]_{p-1}.$$

Then $\Psi_1(0) = \Phi(0)$ by (2.10). Moreover, by (5.3), we also have

$$\Psi_1(0) = \sum_{k=0}^{a} \frac{(-a)_k^2}{(1)_k^2} \cdot z^k = \sum_{k=0}^{a} \frac{(-a)_{a-k}^2}{(1)_{a-k}^2} \cdot z^k = z^a \sum_{k=0}^{a} \frac{(-a)_k^2}{(1)_k^2} \cdot \frac{1}{z^k} = \Psi_2(0).$$

So

$$\Psi_1(0) - z \cdot \Psi_2(0) = (1-z) \cdot \Phi(0).$$

Clearly

$$\begin{aligned}
&\frac{d}{dx}\left((1-z)^a {}_2F_1\left[\begin{matrix} -a-x & 1+a \\ & 1 \end{matrix}\middle| \frac{z}{z-1}\right]_{p-1}\right)\Bigg|_{x=0} \\
\equiv&\frac{d}{dx}\left((1-z)^a {}_2F_1\left[\begin{matrix} -a-x & 1+a-p \\ & 1 \end{matrix}\middle| \frac{z}{z-1}\right]\right)\Bigg|_{x=0} \\
=&\frac{(1-z)^a}{p}\left({}_2F_1\left[\begin{matrix} -a-p & 1+a-p \\ & 1 \end{matrix}\middle| \frac{z}{z-1}\right] - {}_2F_1\left[\begin{matrix} -a & 1+a-p \\ & 1 \end{matrix}\middle| \frac{z}{z-1}\right]\right) \\
=&\frac{1}{p}\left(\frac{1}{(1-z)^p} \cdot {}_2F_1\left[\begin{matrix} -a-p & p-a \\ & 1 \end{matrix}\middle| z\right] - {}_2F_1\left[\begin{matrix} -a & p-a \\ & 1 \end{matrix}\middle| z\right]\right) \pmod p.
\end{aligned}$$

For $0 \le k \le p-1$, in view of (3.2), we have

$$(-a-p)_k(p-a)_k = \frac{\Gamma_p(-a+k-p)}{\Gamma_p(-a-p)} \cdot \frac{\Gamma_p(-a+k+p)}{\Gamma_p(-a+p)} \equiv \frac{\Gamma_p(-a+k)^2}{\Gamma_p(-a)^2} = (-a)_k^2 \pmod{p^2}.$$

And if $0 \le k \le a$,

$$\begin{aligned}
\frac{(-a-p)_{p+k}(p-a)_{p+k}}{(1)_{p+k}^2} =&\frac{\Gamma_p(-a+k)}{\Gamma_p(-a-p)} \cdot \frac{\Gamma_p(2p-a+k)}{\Gamma_p(p-a)} \cdot \frac{\Gamma_p(1)^2}{\Gamma_p(p+k+1)^2} \cdot \frac{(-p)\cdot p}{p^2} \\
\equiv& -\frac{(-a)_k^2}{(1)_k^2} \cdot \big(1 + 2p(H_{p-1-a+k} - H_k)\big) \pmod{p^2}.
\end{aligned}$$

On the other hand, we have

$$\frac{d}{dx}\left(\frac{(-a-x)_k}{(1)_k}\right)\Bigg|_{x=0} = \frac{(-a)_k}{(1)_k} \sum_{j=0}^{k-1} \frac{1}{a-j} = \frac{(-a)_k}{(1)_k} \cdot (H_a - H_{a-k}).$$

It follows that

$$\begin{aligned}&\frac{(-a)_k^2}{(1)_k^2}\cdot(H_{p-1-a+k}-H_k)\equiv\frac{(-a)_k^2}{(1)_k^2}\cdot(H_{a-k}-H_k)\\=&\frac{d}{dx}\bigg(\frac{(-a)_{a-k}(-a-x)_{a-k}}{(1)_{a-k}^2}-\frac{(-a)_k(-a-x)_k}{(1)_k^2}\bigg)\bigg|_{x=0}\pmod p.\end{aligned}$$

Note that

$$\begin{aligned}\Psi_1'(0)=&2\cdot\frac{d}{dx}\bigg({}_2F_1\bigg[\begin{matrix}-a & -a-x\\ & 1\end{matrix}\bigg|\,z\bigg]_{p-1}\bigg)\bigg|_{x=0}\\=&2\cdot\frac{d}{dx}\bigg((1-z)^a{}_2F_1\bigg[\begin{matrix}-a & 1+a+x\\ & 1\end{matrix}\bigg|\,\frac{z}{z-1}\bigg]_{p-1}\bigg)\bigg|_{x=0}\end{aligned}$$

and

$$\begin{aligned}\Psi_2'(0)=&2z^a\cdot\frac{d}{dx}\bigg({}_2F_1\bigg[\begin{matrix}-a & -a-x\\ & 1\end{matrix}\bigg|\,\frac{1}{z}\bigg]\bigg)\bigg|_{x=0}\\=&2z^a\cdot\frac{d}{dx}\bigg(\sum_{k=0}^{a}\frac{(-a)_{a-k}(-a-x)_{a-k}}{(1)_{a-k}^2}\cdot\frac{1}{z^{a-k}}\bigg)\bigg|_{x=0}.\end{aligned}$$

Hence

$$\begin{aligned}\Phi'(0)=&(1-z)^a\cdot\frac{d}{dx}\bigg({}_2F_1\bigg[\begin{matrix}-a & 1+a+x\\ & 1\end{matrix}\bigg|\,\frac{z}{z-1}\bigg]_{p-1}+{}_2F_1\bigg[\begin{matrix}-a-x & 1+a\\ & 1\end{matrix}\bigg|\,\frac{z}{z-1}\bigg]_{p-1}\bigg)\bigg|_{x=0}\\\equiv&\frac{\Psi_1'(0)}{2}+\frac{1}{p}\bigg(\frac{1-z^p}{(1-z)^p}\cdot{}_2F_1\bigg[\begin{matrix}-a & -a\\ & 1\end{matrix}\bigg|\,z\bigg]-{}_2F_1\bigg[\begin{matrix}-a & p-a\\ & 1\end{matrix}\bigg|\,z\bigg]\bigg)\\&-\frac{2z^p}{(1-z)^p}\cdot\frac{d}{dx}\bigg(\sum_{k=0}^{a}\frac{(-a)_{a-k}(-a-x)_{a-k}}{(1)_{a-k}^2}\cdot z^k-\sum_{k=0}^{a}\frac{(-a)_k(-a-x)_k}{(1)_k^2}\cdot z^k\bigg)\bigg|_{x=0}\\\equiv&\frac{(1-z)^p+z^p}{(1-z)^p}\cdot\Psi_1'(0)-\frac{z^p\cdot\Psi_2'(0)}{(1-z)^p}+\frac{1-z^p-(1-z)^p}{p\cdot(1-z)^p}\cdot{}_2F_1\bigg[\begin{matrix}-a & -a\\ & 1\end{matrix}\bigg|\,z\bigg]\pmod p.\end{aligned}$$

Thus

$$\begin{aligned}&(1-z)^p\cdot\Phi(p)\equiv(1-z)^p\cdot p\Phi'(0)+(1-z)^p\cdot\Phi(0)\\\equiv&p\big(\Psi_1'(0)-z^p\cdot\Psi_2'(0)\big)+\big(1-z^p-(1-z)^p\big)\cdot\Psi_1(0)+(1-z)^p\cdot\Phi(0)\\\equiv&\big(\Psi_1(p)-\Psi_1(0)\big)-z^p\cdot\big(\Psi_2(p)-\Psi_2(0)\big)+(1-z^p)\cdot\Psi_1(0)\\=&\Psi_1(p)-z^p\cdot\Psi_2(p)\pmod{p^2}.\end{aligned}\tag{8.1}$$

By Lemma 5.1, Theorem 2.3 is concluded. $\square$

9. The p-adic quadratic ${}_2F_1$ transformation IV: $z \to -4z/(1-z)^2$ and $z \to z^2/(4z-4)$

In this section, we shall complete the proof of the last two p-adic quadratic transformations: Theorem 2.5 and Theorem 2.7

Proof of Theorem 2.5. Assume that $\alpha = -\langle -\alpha\rangle_{p^2}$ and let $a = \langle -\alpha\rangle_p$. Let

$$\Psi_1(x) = {}_2F_1\left[\begin{matrix}-a-x & -a-x\\ & 1\end{matrix}\middle|\, z\right]_{p-1}, \quad \Psi_2(x) = z^a \cdot {}_2F_1\left[\begin{matrix}-a-x & -a-x\\ & 1\end{matrix}\middle|\, \frac{1}{z}\right]_{p-1}$$

and

$$\Phi(x) = (1-z)^a \cdot {}_2F_1\left[\begin{matrix}-\frac{1}{2}(a+x) & \frac{1}{2}+\frac{1}{2}(a+x)\\ & 1\end{matrix}\middle|\, -\frac{4z}{(1-z)^2}\right]_{p-1}.$$

In view of (5.3), we have $\Psi_1(0) = \Psi_2(0)$. It follows from (2.16) that $\Psi_1(0) = \Phi(0)$, i.e.,

$$\Psi_1(0) - z \cdot \Psi_2(0) = (1-z) \cdot \Phi(0).$$

Clearly

$$\Phi(p) \equiv \Phi(0) + p \cdot \Phi'(0) \equiv \Phi(0) + \big(\Phi(0) - \Phi(-p)\big) \pmod{p^2}.$$

Note that $(p-1-a)/2$ is an integer lying $\{0, 1, \ldots, p-1\}$ now. By (2.10) and (2.16),

$$\begin{aligned}\Phi(-p) =&(1-z)^a \cdot {}_2F_1\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}(p-1-a) & -\frac{1}{2}(p-1-a)\\ & 1\end{matrix}\middle|\, -\frac{4z}{(1-z)^2}\right]\\ =&(1-z)^{2a+1-p} \cdot {}_2F_1\left[\begin{matrix}a+1-p & a+1-p\\ & 1\end{matrix}\middle|\, z\right]\\ =&(1-z)^a \cdot {}_2F_1\left[\begin{matrix}p-a & a+1-p\\ & 1\end{matrix}\middle|\, \frac{z}{z-1}\right]\\ =&\Omega(-p) \equiv 2\Omega(0) - \Omega(p) \pmod{p^2},\end{aligned}$$

where

$$\Omega(x) = (1-z)^a {}_2F_1\left[\begin{matrix}-a-x & a+1+x\\ & 1\end{matrix}\middle|\, \frac{z}{z-1}\right]_{p-1}.$$

On the other hand, in view of (8.1), we have

$$(1-z)^p \cdot \Omega(p) \equiv \Psi_1(p) - z^p \cdot \Psi_2(p) \pmod{p}.$$

Noting that $\Omega(0) = \Psi_1(0)$ by (2.10), we get

$$(1-z)^p \cdot \Phi(-p) \equiv 2(1-z)^p \cdot \Psi_1(0) - \Psi_1(p) + z^p \cdot \Psi_2(p) \pmod{p^2},$$

i.e.,

$$(1-z)^p \cdot \Phi(p) \equiv \Psi_1(p) - z^p \cdot \Psi_2(p) \pmod{p^2}.$$

It follows from Lemma 5.1 that

$$\Psi_1(sp) - z^{1+s(p-1)} \cdot \Psi_2(sp) \equiv (1-z)^{1+s(p-1)} \cdot \Phi(sp) \pmod{p^2}$$

where $s = -(\alpha + a)/p$. $\square$

Proof of Theorem 2.7. Assume that $\alpha = -\langle -\alpha\rangle_{p^2}$ and let $a = \langle -\alpha\rangle_p$. Let

$$\Psi_1(x,y) = {}_2F_1\left[\begin{matrix} -a+x & \frac{1}{2}(1-y) \\ & 1 \end{matrix}\middle|\, z\right],$$

$$\Psi_2(x,y) = (1-z)^a {}_2F_1\left[\begin{matrix} -a+x & \frac{1}{2}(1-y) \\ & 1 \end{matrix}\middle|\, \frac{z}{z-1}\right],$$

and

$$\Phi(x) = (1-z)^{\frac{a}{2}} \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2}(a-x) & \frac{1}{2}+\frac{1}{2}(a-x) \\ & 1 \end{matrix}\middle|\, \frac{z^2}{4z-4}\right].$$

In view of (2.21),

$$\Psi_1(0,0)+\Psi_2(0,0) = 2(1-z)^{\frac{a}{2}} \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2}a & \frac{1}{2}+\frac{1}{2}a \\ & 1 \end{matrix}\middle|\, \frac{z^2}{4z-4}\right] = 2\Phi(0).$$

Furthermore, applying (2.21) reversely, we get

$$\begin{aligned} 2\Phi(p) =& 2(1-z)^{\frac{a}{2}} \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}+\frac{1}{2}(p-1-a) & -\frac{1}{2}(p-1-a) \\ & 1 \end{matrix}\middle|\, \frac{z^2}{4z-4}\right]_{p-1} \\ =& (1-z)^{a-\frac{p-1}{2}} \cdot \big(\Omega_1(0)+\Omega_2(0)\big), \end{aligned}$$

where

$$\Omega_1(y) = {}_2F_1\left[\begin{matrix} a+1-p & \frac{1}{2}(1-y) \\ & 1 \end{matrix}\middle|\, z\right]_{p-1}$$

and

$$\Omega_2(y) = (1-z)^{p-1-a} {}_2F_1\left[\begin{matrix} a+1-p & \frac{1}{2}(1-y) \\ & 1 \end{matrix}\middle|\, \frac{z}{z-1}\right]_{p-1}.$$

By (2.10), clearly

$$\Omega_1(y) = \Omega_2(-y).$$

Thus

$$\Omega_1(0)-\Omega_1(p) \equiv -p\cdot\Omega_1'(0) = p\cdot\Omega_2'(0) \equiv \Omega_2(p)-\Omega_2(0) \pmod{p^2}.$$

It also follows from (2.10) that

$$\begin{aligned} \Omega_1(p) =& {}_2F_1\left[\begin{matrix} \frac{1}{2}(1-p) & a+1-p \\ & 1 \end{matrix}\middle|\, z\right] \\ =& (1-z)^{\frac{p-1}{2}} \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}(1-p) & p-a \\ & 1 \end{matrix}\middle|\, \frac{z}{z-1}\right] = (1-z)^{\frac{p-1}{2}-a}\cdot\Psi_2(p,p). \end{aligned}$$

Similarly, we have $\Omega_2(p) = (1-z)^{\frac{p-1}{2}-a}\cdot\Psi_1(p,p)$, too. On the other hand, clearly

$$\Psi_i(p,p)-\Psi_i(0,0) \equiv p\cdot\frac{\partial\Psi_i(x,0)}{\partial x}\bigg|_{x=0} + p\cdot\frac{\partial\Psi_i(0,y)}{\partial y}\bigg|_{y=0} \pmod{p^2}, \qquad i=1,2.$$

Note that

$$\Psi_1(0,y) = {}_2F_1\left[\begin{matrix}-a & \frac{1}{2}(1-y)\\ & 1\end{matrix}\middle| z\right] = (1-z)^a{}_2F_1\left[\begin{matrix}-a & \frac{1}{2}(1+y)\\ & 1\end{matrix}\middle| \frac{z}{z-1}\right] = \Psi_2(0,-y)$$

by (2.10). Therefore

$$\begin{aligned}2\Phi(p) \equiv&(1-z)^{a-\frac{p-1}{2}}\cdot\big(\Omega_1(p)+\Omega_2(p)\big) \equiv \Psi_1(p,p)+\Psi_2(p,p)\\ \equiv&\sum_{i=1}^{2}\left(\Psi_i(0,0)+p\cdot\frac{\partial\Psi_i(x,0)}{\partial x}\bigg|_{x=0}+p\cdot\frac{\partial\Psi_i(0,y)}{\partial y}\bigg|_{y=0}\right)\\ =&\sum_{i=1}^{2}\left(\Psi_i(0,0)+p\cdot\frac{\partial\Psi_i(x,0)}{\partial x}\bigg|_{x=0}\right)\equiv\Psi_1(p,0)+\Psi_2(p,0) \pmod{p^2}.\end{aligned}$$

Thus letting $s=(\alpha+a)/p$, we get

$$2\Phi(sp)\equiv\Psi_1(sp,0)+\Psi_2(sp,0) \pmod{p^2}.$$

□

10. Special values of α and z

Apparently $\lambda_p(\alpha)$ and $\langle-\alpha\rangle_p$ have the same parity. First, substituting $z=2$ in Theorem 2.7, we have

$${}_2F_1\left[\begin{matrix}\alpha & \frac{1}{2}\\ & 1\end{matrix}\middle| 2\right]_{p-1} \equiv (-1)^{\frac{1}{2}\langle-\alpha\rangle_p}{}_2F_1\left[\begin{matrix}\frac{1}{2}\alpha & \frac{1}{2}-\frac{1}{2}\alpha\\ & 1\end{matrix}\middle| 1\right]_{p-1} \pmod{p^2}$$

provided that $\langle-\alpha\rangle_p$ is even. Thus by (1.10), we obtain that

Theorem 10.1. *Suppose that $\alpha\in\mathbb{Z}_p$ with $\langle-\alpha\rangle_p$ is even. Then*

$${}_2F_1\left[\begin{matrix}\alpha & \frac{1}{2}\\ & 1\end{matrix}\middle| 2\right]_{p-1} \equiv (-1)^{1+\frac{1}{2}\langle-\alpha\rangle_p}\cdot\frac{\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{1}{2}+\frac{1}{2}\alpha)\Gamma_p(1-\frac{1}{2}\alpha)} \pmod{p^2}. \tag{10.1}$$

For example, for any prime $p\equiv 1 \pmod 3$,

$${}_2F_1\left[\begin{matrix}\frac{1}{3} & \frac{1}{2}\\ & 1\end{matrix}\middle| 2\right]_{p-1} \equiv (-1)^{\frac{p+5}{6}}\cdot\frac{\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{2}{3})\Gamma_p(\frac{5}{6})} \pmod{p^2}.$$

Similarly, we have

Theorem 10.2. *Suppose that $\alpha\in\mathbb{Z}_p$ and $\langle-\alpha\rangle_p$ is even.*

$${}_2F_1\left[\begin{matrix}\alpha & \alpha\\ & 1\end{matrix}\middle| -1\right]_{p-1} \equiv -\frac{2\Gamma_p(1+\frac{1}{2}\alpha)}{\Gamma_p(1+\alpha)\Gamma_p(1-\frac{1}{2}\alpha)} \pmod{p^2}. \tag{10.2}$$

In fact, the special case $\alpha=1/2$ was firstly proved by Coster and van Hamme [5], and an alternative proof was given in [6].

Proof. Substituting $z=-1$ in (2.11), we get

$$\begin{aligned}\big(1-(-1)^{1-\lambda_p(\alpha)}\big)\cdot {}_2F_1\begin{bmatrix}\alpha & \alpha \\ & 1\end{bmatrix}\,-1\bigg]_{p-1} =& 2^{1-\lambda_p(\alpha)}{}_2F_1\begin{bmatrix}\alpha & 1-\alpha \\ & 1\end{bmatrix}\,\frac{1}{2}\bigg]_{p-1}\\ \equiv& -\frac{2^{1-\lambda_p(\alpha)}\cdot\Gamma_p(\frac{1}{2})}{\Gamma_p(1-\frac{1}{2}\alpha)\Gamma_p(\frac{1}{2}+\frac{1}{2}\alpha)} \pmod{p^2}.\end{aligned}$$

Clearly $(-1)^{1-\lambda_p(\alpha)}=-1$ now. It suffices to show that

$$\frac{\Gamma_p(1+\frac{1}{2}\alpha)}{\Gamma_p(1+\alpha)\Gamma_p(1-\frac{1}{2}\alpha)}\equiv\frac{2^{-\lambda_p(\alpha)}\cdot\Gamma_p(\frac{1}{2})}{\Gamma_p(1-\frac{1}{2}\alpha)\Gamma_p(\frac{1}{2}+\frac{1}{2}\alpha)} \pmod{p^2}. \tag{10.3}$$

Let $A=\langle-\alpha\rangle_{p^2}$ or $A=\langle-\alpha\rangle_{p^2}+p^2$ according to whether $\langle-\alpha\rangle_{p^2}$ is even or odd. Then

$$\begin{aligned}\frac{\Gamma_p(1+\frac{1}{2}\alpha)}{\Gamma_p(1+\alpha)\Gamma_p(1-\frac{1}{2}\alpha)}\equiv& -\frac{\Gamma_p(1)\Gamma_p(1-\frac{1}{2}A)}{\Gamma_p(1-A)\Gamma_p(1+\frac{1}{2}A)}\\ =& -\prod_{\substack{1\le j\le A\\ j\not\equiv A \pmod p}}(j-A)\cdot\prod_{\substack{1\le j\le A\\ j\not\equiv \frac{1}{2}A \pmod p}}\frac{1}{j-\frac{1}{2}A} \pmod{p^2},\end{aligned}$$

and

$$\begin{aligned}\frac{\Gamma_p(\frac{1}{2})}{\Gamma_p(1-\frac{1}{2}\alpha)\Gamma_p(\frac{1}{2}+\frac{1}{2}\alpha)}\equiv& -\frac{\Gamma_p(1)\Gamma_p(\frac{1}{2})}{\Gamma_p(1+\frac{1}{2}A)\Gamma_p(\frac{1}{2}-\frac{1}{2}A)}\\ =& -\prod_{\substack{1\le j\le \frac{1}{2}A\\ j\not\equiv 0 \pmod p}}\frac{1}{j}\cdot\prod_{\substack{1\le j\le \frac{1}{2}A\\ j\not\equiv \frac{1}{2}(A+1) \pmod p}}\left(j-\frac{A+1}{2}\right) \pmod{p^2}.\end{aligned}$$

It is not difficult to check that

$$\prod_{\substack{1\le j\le A\\ j\not\equiv A \pmod p}}(j-A)\prod_{\substack{1\le j\le A\\ j\not\equiv \frac{1}{2}A \pmod p}}\frac{2}{2j-A}=2^{\Delta}\prod_{\substack{1\le j\le \frac{1}{2}A\\ j\not\equiv 0 \pmod p}}\frac{1}{j}\prod_{\substack{1\le j\le \frac{1}{2}A\\ j\not\equiv \frac{1}{2}(A+1) \pmod p}}\frac{2j-1-A}{2},$$

where

$$\Delta=\sum_{\substack{1\le j\le A\\ j\not\equiv \frac{1}{2}A \pmod p}}1-\sum_{\substack{1\le j\le \frac{1}{2}A\\ j\not\equiv 0 \pmod p}}1+\sum_{\substack{1\le j\le \frac{1}{2}A\\ j\not\equiv \frac{1}{2}(A+1) \pmod p}}1.$$

Let $a=\langle-\alpha\rangle_p$. We have

$$\begin{aligned}\Delta =&\left(A-\frac{A-a}{p}\right)-\left(\frac{A}{2}-\frac{A-a}{2p}\right)+\left(\frac{A}{2}-\frac{A-a}{2p}\right)\\ =&A-\frac{A-a}{p}\equiv-\lambda_p(\alpha) \pmod{p(p-1)}.\end{aligned}$$

Thus (10.3) is concluded. □

Let us consider the congruences concerning ${}_2F_1\begin{bmatrix}\frac14 & \frac34 \\ & 1\end{bmatrix} z\Big]_{p-1}$ modulo p^2. We have the following results.

Theorem 10.3. *(1) Suppose that* $p\equiv 1 \pmod 3$. *Then*

$$ {}_2F_1\left[\begin{matrix}\frac14 & \frac34 \\ & 1\end{matrix}\,\middle|\, -\frac13\right]_{p-1} \equiv -\left(\frac{2}{p}\right)\cdot\frac{3\Gamma_p(\frac43)}{2\Gamma_p(\frac32)\Gamma_p(\frac56)} \pmod{p^2}. \tag{10.4}$$

Also,

$$ {}_2F_1\left[\begin{matrix}\frac14 & \frac14 \\ & 1\end{matrix}\,\middle|\, 4\right]_{p-1} \equiv -\left(\frac{2}{p}\right)\cdot\frac{3^{\lambda_p(\frac14)+1}\Gamma_p(\frac43)}{2\Gamma_p(\frac32)\Gamma_p(\frac56)} \pmod{p^2}. \tag{10.5}$$

(2) Suppose that $p\equiv 1 \pmod 4$. *Then*

$$ {}_2F_1\left[\begin{matrix}\frac14 & \frac34 \\ & 1\end{matrix}\,\middle|\, \frac89\right]_{p-1} \equiv -\left(\frac{6}{p}\right)\cdot\frac{\Gamma_p(\frac12)}{\Gamma_p(\frac34)^2} \pmod{p^2}, \tag{10.6}$$

$$ {}_2F_1\left[\begin{matrix}\frac14 & \frac34 \\ & 1\end{matrix}\,\middle|\, -8\right]_{p-1} \equiv -\left(\frac{2}{p}\right)\cdot\frac{\Gamma_p(\frac12)}{\Gamma_p(\frac34)^2} \pmod{p^2} \tag{10.7}$$

and

$$ {}_2F_1\left[\begin{matrix}\frac14 & \frac14 \\ & 1\end{matrix}\,\middle|\, -8\right]_{p-1} \equiv (-1)^{\frac{p+3}{4}}\cdot\frac{\Gamma_p(\frac12)}{\Gamma_p(\frac34)^2} \pmod{p^2} \tag{10.8}$$

Proof. Here we only prove (10.3), since the proofs of other congruences are very similar. However, the direct proof of (10.3) is not easy, since the hypergeometric function just involves constants. So we have to prove a stronger result:

$$ {}_2F_1\left[\begin{matrix}\frac14-\frac{sp}{4} & \frac34-\frac{sp}{4} \\ & 1+\frac{sp}{2}\end{matrix}\,\middle|\, -\frac13\right]_{p-1} \equiv \left(\frac{2}{p}\right)^{s-1}\cdot\frac{8^{\frac{s(p-1)}{2}}}{9^{\frac{s(p-1)}{2}}}\cdot\frac{3\Gamma_p(\frac43)\Gamma_p(1+\frac{sp}{2})}{2\Gamma_p(\frac32)\Gamma_p(\frac56+\frac{sp}{2})} \pmod{p^2} \tag{10.9}$$

for any integer n. We also need the hypergeometric identity

$$ {}_2F_1\left[\begin{matrix}\alpha & \frac12+\alpha \\ & \frac32-2\alpha\end{matrix}\,\middle|\, -\frac13\right] = \frac{8^{-2\alpha}}{9^{-2\alpha}}\cdot\frac{\Gamma(\frac43)\Gamma(\frac32-2\alpha)}{\Gamma(\frac32)\Gamma(\frac43-2\alpha)}. \tag{10.10}$$

Let

$$\Psi(x) = {}_2F_1\left[\begin{matrix}\frac14-\frac14x & \frac34-\frac14x \\ & 1+\frac12x\end{matrix}\,\middle|\, -\frac13\right]_{p-1}.$$

It follows from (10.10) that

$$\Psi(p) = {}_2F_1\left[\begin{matrix}\frac{1-p}{4} & \frac{3-p}{4} \\ & 1+\frac{p}{2}\end{matrix}\,\middle|\, -\frac13\right] = \frac{8^{\frac{p-1}{2}}}{9^{\frac{p-1}{2}}}\cdot\frac{\Gamma(\frac43)\Gamma(\frac32+\frac{p-1}{2})}{\Gamma(\frac32)\Gamma(\frac43+\frac{p-1}{2})}$$

and

$$\Psi(3p) = {}_2F_1\left[\begin{matrix}\frac{1-3p}{4} & \frac{3-3p}{4} \\ & 1+\frac{3p}{2}\end{matrix}\,\middle|\, -\frac13\right] = \frac{8^{\frac{3p-1}{2}}}{9^{\frac{3p-1}{2}}}\cdot\frac{\Gamma(\frac43)\Gamma(\frac32+\frac{3p-1}{2})}{\Gamma(\frac32)\Gamma(\frac43+\frac{3p-1}{2})}.$$

By Lemma 3.1,

$$\Psi(sp)-\Psi(p)\equiv(s-1)p\cdot\Psi'(0)\equiv\frac{s-1}{2}\cdot\big(\Psi(3p)-\Psi(p)\big)\pmod{p^2}.$$

Since $p\equiv 1\pmod 3$, we have

$$\frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{4}{3}+\frac{p-1}{2})}\cdot\frac{\Gamma(\frac{3}{2}+\frac{p-1}{2})}{\Gamma(\frac{3}{2})}=\frac{3}{p}\cdot\frac{\Gamma_p(\frac{4}{3})}{\Gamma_p(\frac{4}{3}+\frac{p-1}{2})}\cdot\frac{p}{2}\cdot\frac{\Gamma_p(\frac{3}{2}+\frac{p-1}{2})}{\Gamma_p(\frac{3}{2})},$$

$$\frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{4}{3}+\frac{3p-1}{2})}\cdot\frac{\Gamma(\frac{3}{2}+\frac{3p-1}{2})}{\Gamma(\frac{3}{2})}=\frac{9}{4p}\cdot\frac{\Gamma_p(\frac{4}{3})}{\Gamma_p(\frac{4}{3}+\frac{3p-1}{2})}\cdot\frac{3p}{4}\cdot\frac{\Gamma_p(\frac{3}{2}+\frac{3p-1}{2})}{\Gamma_p(\frac{3}{2})}.$$

Let

$$\Omega(x)=\frac{3\Gamma_p(\frac{4}{3})\Gamma_p(1+\frac{x}{2})}{2\Gamma_p(\frac{3}{2})\Gamma_p(\frac{5}{6}+\frac{x}{2})}$$

Viewing $\Omega(x)$ as a function over $\mathbb{Z}_p$, we also have

$$\Omega(sp)-\Omega(p)=(s-1)p\cdot\Omega'(p)\equiv\frac{s-1}{2}\cdot\big(\Omega(3p)-\Omega(p)\big)\pmod{p^2}.$$

Let

$$\lambda=\left(\frac{2}{p}\right)\cdot\frac{8^{\frac{p-1}{2}}}{9^{\frac{p-1}{2}}}.$$

Clearly $\lambda\equiv 1\pmod p$. So

$$\begin{aligned}\lambda^s\Omega(sp)-\lambda\Omega(p)=&(\lambda^{s-1}-1)\cdot\lambda\Omega(sp)+\lambda\cdot\big(\Omega(sp)-\Omega(p)\big)\\ \equiv&\frac{s-1}{2}\cdot\big((\lambda^2-1)\cdot\lambda\Omega(3p)+\lambda\cdot(\Omega(3p)-\Omega(p))\big)\\ =&\frac{s-1}{2}\cdot\big(\lambda^3\Omega(3p)-\lambda\Omega(p)\big)\pmod{p^2}.\end{aligned}$$

It follows that

$$\begin{aligned}\Psi(sp)\equiv&\Psi(p)+\frac{s-1}{2}\cdot\big(\Psi(3p)-\Psi(p)\big)\\ =&\left(\frac{2}{p}\right)\left(\lambda\Omega(p)+\frac{s-1}{2}\cdot\big(\lambda^3\Omega(3p)-\lambda\Omega(p)\big)\right)\equiv\left(\frac{2}{p}\right)\cdot\lambda^s\Omega(sp)\pmod{p^2}.\end{aligned}$$

□

11. The p-adic analogue of Watson's identity

In this section, we shall give the proof of Theorem 2.9. Let $a=\langle-\alpha\rangle_p$ and $b=\langle-\beta\rangle_p$. Further, without loss of generality, we may assume that $\beta=p^2-\langle-\beta\rangle_{p^2}$. Let $s=(\alpha+a)/p$.

First, assume that a is even. Let

$$\Psi(x)={}_3F_2\left[\begin{matrix}-a+x & 1+a-x & \beta\\ & 1 & 2\beta\end{matrix}\,\middle|\,1\right]_{p-1}$$

and

$$\Phi(x)=\frac{\Gamma_p(\frac12)\Gamma_p(\frac12+\beta)\Gamma_p(\beta)}{\Gamma_p(\frac12-\frac{a-x}{2})\Gamma_p(1+\frac{a-x}{2})\Gamma_p(\beta-\frac{a-x}{2})\Gamma_p(\frac12+\beta+\frac{a-x}{2})}.$$

By (2.28),

$$\Psi(0)={}_3F_2\left[\begin{matrix}-a & 1+a & \beta\\ & 1 & 2\beta\end{matrix}\middle|\,1\right]=\frac{\Gamma(\frac12)\Gamma(\frac12+\beta)\Gamma(\beta)}{\Gamma(\frac12-\frac12a)\Gamma(1+\frac12a)\Gamma(\beta-\frac12a)\Gamma(\frac12+\beta+\frac12a)}.$$

Now we have

$$\frac{\Gamma(\frac12+\beta)}{\Gamma(\frac12+\beta+\frac12a)}=\prod_{j=0}^{\frac12a-1}\frac{1}{\frac12+\beta+j}=(-1)^{\frac12a}\cdot\frac{\Gamma_p(\frac12+\beta)}{\Gamma_p(\frac12+\beta+\frac12a)}.$$

since $b<p/2$. Similarly,

$$\frac{\Gamma(\frac12)}{\Gamma(\frac12+\frac12a)}=(-1)^{\frac12a}\cdot\frac{\Gamma_p(\frac12)}{\Gamma_p(\frac12+\frac12a)},\qquad\frac{\Gamma(\beta)}{\Gamma(\beta-\frac12a)}=(-1)^{\frac12a}\cdot\frac{\Gamma_p(\beta)}{\Gamma_p(\beta-\frac12a)}.$$

It follows that

$$\Psi(0)=-\frac{\Gamma_p(\frac12)\Gamma_p(\frac12+\beta)\Gamma_p(\beta)}{\Gamma_p(\frac12-\frac12a)\Gamma_p(1+\frac12a)\Gamma_p(\beta-\frac12a)\Gamma_p(\frac12+\beta+\frac12a)}=-\Phi(0).$$

Let

$$\psi(x)={}_3F_2\left[\begin{matrix}-a+x & 1+a-p-x & \beta\\ & 1-\frac12p & 2\beta\end{matrix}\middle|\,1\right].$$

Though $\psi(x)$ is factly not a polynomial in x, clearly $\psi(0)\equiv\Psi(0)\pmod p$. Also, we have

$$\begin{aligned}\Psi'(0)\equiv&\frac{d}{dx}\left({}_3F_2\left[\begin{matrix}-a+x & 1+a-p-x & \beta\\ & 1-\frac12p & 2\beta\end{matrix}\middle|\,1\right]_{p-1}\right)\Bigg|_{x=0}\\ =&\frac{d}{dx}\left({}_3F_2\left[\begin{matrix}-a+x & 1+a-p & \beta\\ & 1-\frac12p & 2\beta\end{matrix}\middle|\,1\right]+{}_3F_2\left[\begin{matrix}-a & 1+a-p-x & \beta\\ & 1-\frac12p & 2\beta\end{matrix}\middle|\,1\right]\right)\Bigg|_{x=0}\\ =&\frac{d}{dx}\left({}_3F_2\left[\begin{matrix}-a+x & 1+a-p-x & \beta\\ & 1-\frac12p & 2\beta\end{matrix}\middle|\,1\right]\right)\Bigg|_{x=0}=\psi'(0)\pmod p.\end{aligned}\tag{11.1}$$

According to (2.27),

$$\psi(x)=\frac{\Gamma(\frac12)\Gamma(\frac12+\beta)\Gamma(1-\frac p2)\Gamma(\beta+\frac p2)}{\Gamma(\frac{1-a+x}{2})\Gamma(1+\frac{a-p-x}{2})\Gamma(\beta+\frac{1+a-x}{2})\Gamma(\beta+\frac{p-a+x}{2})}.$$

It follows from (3.3) that

$$\begin{aligned}\frac{\psi'(0)}{\psi(0)} =&\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{1-a}{2}}+\frac{1}{n+\beta+\frac{p-a}{2}}-\frac{1}{n+1+\frac{a-p}{2}}-\frac{1}{n+\beta+\frac{1+a}{2}}\right)\\ =&-\frac{1}{2}\sum_{j=1}^{\frac{p-1}{2}-a}\left(\frac{1}{j+\frac{a-p}{2}}+\frac{1}{j+\beta+\frac{a-1}{2}}\right).\end{aligned}$$

Hence

$$\Psi'(0)\equiv-\frac{\Psi(0)}{2}\sum_{j=1}^{\frac{p-1}{2}-a}\left(\frac{1}{j+\frac{a-p}{2}}+\frac{1}{j+\beta+\frac{a-1}{2}}\right) \pmod p.$$

On the other hand, let

$$\phi(x)=\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{p+1}{2}-b)\Gamma(p-b)}{\Gamma(\frac{p+1-a+x}{2})\Gamma(1+\frac{a-x}{2})\Gamma(p-b-\frac{a-x}{2})\Gamma(\frac{p+1+a-x}{2}-b)}.$$

In view of Lemma 3.2 and (3.3),

$$\Phi'(0)\equiv\phi'(0)=\frac{\phi(0)}{2}\sum_{j=1}^{\frac{p-1}{2}-a}\left(\frac{1}{j+\frac{a}{2}}+\frac{1}{j+\frac{p-1+a}{2}-b}\right) \pmod p.$$

Clearly

$$\phi(0)=\frac{(\frac{p+1-a}{2})_{\frac{a}{2}}(p-b-\frac{a}{2})_{\frac{a}{2}}}{\Gamma(1+\frac{a}{2})(\frac{p+1}{2}-b)_{\frac{a}{2}}}\equiv\frac{(\frac{1-a}{2})_{\frac{a}{2}}(\beta-\frac{a}{2})_{\frac{a}{2}}}{\Gamma(1+\frac{a}{2})(\frac{1}{2}+\beta)_{\frac{a}{2}}}=\Psi(0) \pmod{p^2}.$$

So

$$\Psi'(0)\equiv-\frac{\Psi(0)}{2}\sum_{j=1}^{\frac{p-1}{2}-a}\left(\frac{1}{j+\frac{a}{2}}+\frac{1}{j+\frac{p+a-1}{2}-b}\right)\equiv-\Phi'(0) \pmod p,$$

i.e.,

$$\Psi(sp)\equiv\Psi(0)+sp\cdot\Psi'(0)\equiv-\Phi(0)-sp\cdot\Phi'(0)\equiv-\Phi(sp) \pmod{p^2}.$$

Let us turn to (2.30) and assume that a is odd. Now

$$\Psi(0)=\lim_{x\to0}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\beta)\Gamma(\beta)}{\Gamma(\frac{1}{2}-\frac{a-x}{2})\Gamma(1+\frac{a-x}{2})\Gamma(\beta-\frac{a-x}{2})\Gamma(\frac{1}{2}+\beta+\frac{a-x}{2})}=0.$$

In view of (3.3) and (11.1), we have

$$\Psi'(0)\equiv\psi'(0)=-\frac{1}{2}\sum_{j=1}^{\frac{p-1}{2}-a}\left(\frac{1}{j+\frac{a-p}{2}}+\frac{1}{j+\beta+\frac{a-1}{2}}\right)\cdot\lim_{x\to0}\psi(x)=0 \pmod p.$$

Thus

$$\Psi(sp)\equiv\Psi(0)+sp\cdot\Psi'(0)\equiv0 \pmod{p^2}.$$

□

12. p-adic ${}_4F_3$ transformation

In this section, we shall prove Theorems 2.14 and 2.15. Also, we shall we give a relatively sketched proof of Theorem 2.12 at the end of this section.

Let $a = \langle -\alpha\rangle_p$, $b = \langle -\beta\rangle_p$, $c = \langle -\gamma\rangle_p$, $d = \langle -\delta\rangle_p$, $e = \langle -\epsilon\rangle_p$ and $f = \langle -\rho\rangle_p$. Let $s = (\alpha + a)/p$. First, let us consider (2.48).

Proof of Theorem 2.14. By the conditions (i) and (ii), we must have

$$f = \langle \alpha+\beta+\gamma-\delta-\epsilon\rangle_p = d+e-a-b-c.$$

Let $N = \max\{a, f\}$. Since $\max\{a, b, f\} \le d$ and $\max\{a, c, f\} \le e$ by (i) and (iii), it is easy to check that

$$\frac{(\rho)_k(\alpha)_k(\beta)_k(\gamma)_k}{k!\cdot(\delta)_k(\epsilon)_k} \equiv 0 \pmod{p^2}$$

for each $N + 1 \le k \le p - 1$. And if $1 \le k \le N$, then p doesn't divide $(\delta)_k(\epsilon)_k$. It follows that

$${}_4F_3\left[\begin{matrix}\rho & \alpha & \beta & \gamma\\ & 1 & \delta & \epsilon\end{matrix}\middle|\, 1\right]_{p-1} \equiv {}_4F_3\left[\begin{matrix}\rho & \alpha & \beta & \gamma\\ & 1 & \delta & \epsilon\end{matrix}\middle|\, 1\right]_{N} \pmod{p^2}.$$

Similarly, note that

$$\langle \beta+\gamma-\delta-1\rangle_p = p+d-b-c-1 \ge \max\{a, f, \langle \gamma-1\rangle_p\}$$

and

$$\langle \beta+\gamma-\epsilon-1\rangle_p = p+e-b-c-1 \ge \max\{a, f, \langle \beta-1\rangle_p\},$$

by (i) and (iii). We also have

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}\rho & \alpha & 1-\beta & 1-\gamma\\ & 1 & 1+\delta-\beta-\gamma & 1+\epsilon-\beta-\gamma\end{matrix}\middle|\, 1\right]_{p-1}\\ \equiv&{}_4F_3\left[\begin{matrix}\rho & \alpha & 1-\beta & 1-\gamma\\ & 1 & 1+\delta-\beta-\gamma & 1+\epsilon-\beta-\gamma\end{matrix}\middle|\, 1\right]_{N} \pmod{p^2}.\end{aligned}$$

Let $\varrho_a = \delta + \epsilon + a - \beta - \gamma$. Let

$$\Psi(x) = {}_4F_3\left[\begin{matrix}\varrho_a - x & -a+x & \beta & \gamma\\ & 1 & \delta & \epsilon\end{matrix}\middle|\, 1\right]_{N}$$

and

$$\Phi(x) = {}_4F_3\left[\begin{matrix}\varrho_a - x & -a+x & 1-\beta & 1-\gamma\\ & 1 & 1+\delta-\beta-\gamma & 1+\epsilon-\beta-\gamma\end{matrix}\middle|\, 1\right]_{N}.$$

By (2.47),

$$\Psi(0) = \frac{(\delta-\varrho_a)_a(\epsilon-\varrho_a)_a}{(\delta)_a(\epsilon)_a}\cdot\Phi(0).$$

By (ii) and (iii),

$$\langle \epsilon - \varrho_a \rangle_p = \langle \beta + \gamma - \delta - a \rangle_p = p + d - b - c - a \leq p - a.$$

Hence

$$(\epsilon - \varrho_a)_a = (-1)^a \cdot \frac{\Gamma_p(\beta + \gamma - \delta)}{\Gamma_p(\epsilon - \varrho_a)}.$$

Similarly, we have

$$\max\left\{ \langle \delta - \varrho_a \rangle_p, \langle \delta \rangle_p, \langle \epsilon \rangle_p \right\} \leq p - a.$$

It follows that

$$\Psi(0) = \frac{\Gamma_p(\beta + \gamma - \delta)}{\Gamma_p(\delta - \varrho_a)} \cdot \frac{\Gamma_p(\beta + \gamma - \epsilon)}{\Gamma_p(\epsilon - \varrho_a)} \cdot \frac{\Gamma_p(\delta)}{\Gamma_p(\delta + a)} \cdot \frac{\Gamma_p(\epsilon)}{\Gamma_p(\epsilon + a)} \cdot \Phi(0) = \frac{\Upsilon_0}{\Omega(0)} \cdot \Phi(0),$$

where

$$\Upsilon_0 = \Gamma_p(\beta + \gamma - \delta)\Gamma_p(\beta + \gamma - \epsilon)\Gamma_p(\delta)\Gamma_p(\epsilon)$$

and

$$\Omega(x) = \Gamma_p(\delta - \varrho_a + x)\Gamma_p(\epsilon - \varrho_a + x)\Gamma_p(\delta + a - x)\Gamma_p(\epsilon + a - x).$$

Recall that $f = \langle -\varrho_a \rangle_p$. By Lemma 3.2, we have

$$\begin{aligned}\frac{\Upsilon_0 \cdot \Phi(sp)}{\Omega(sp)} - \frac{\Upsilon_0 \cdot \Phi(0)}{\Omega(0)} =& \left(\frac{\Upsilon_0}{\Omega(sp)} - \frac{\Upsilon_0}{\Omega(0)} \right) \cdot \Phi(sp) + \frac{\Upsilon_0}{\Omega(0)} \cdot \big(\Phi(sp) - \Phi(0)\big) \\ \equiv& sp \cdot \frac{d}{dx} \left(\frac{\upsilon_0}{\omega(x)} \right) \bigg|_{x=0} \cdot \Phi(0) + sp \cdot \frac{\Upsilon_0}{\Omega(0)} \cdot \Phi'(0) \pmod{p^2},\end{aligned}$$

where

$$\upsilon_0 = \Gamma(p + d - b - c)\Gamma(p + e - b - c)\Gamma(p - d)\Gamma(p - e)$$

and

$$\omega(x) = \Gamma(p + f - d + x)\Gamma(p + f - e - x)\Gamma(p + a - d - x)\Gamma(p + a - e - x).$$

Let

$$\psi(x, y) = {}_4F_3\left[\begin{matrix} -f - x & -a + y & -b & -c \\ & 1 & -d & -e \end{matrix} \middle| 1 \right]_N.$$

Recalling that $\max\{a, b, f\} \leq d$ and $\max\{a, c, f\} \leq e$, we obtain that

$$\begin{aligned}\Psi'(0) =& \frac{d}{dx} \left({}_4F_3\left[\begin{matrix} \varrho_a - x & -a & \beta & \gamma \\ & 1 & \delta & \epsilon \end{matrix} \middle| 1 \right]_N + {}_4F_3\left[\begin{matrix} \varrho_a & -a + x & \beta & \gamma \\ & 1 & \delta & \epsilon \end{matrix} \middle| 1 \right]_N \right) \bigg|_{x=0} \\ \equiv& \psi'_x(0, 0) + \psi'_y(0, 0) \pmod{p}.\end{aligned}$$

Let

$$\phi(x, y) = {}_4F_3\left[\begin{matrix} -f - x & -a + y & 1 + b & 1 + c \\ & 1 & 1 + b + c - d & 1 + b + c - e \end{matrix} \middle| 1 \right]_N.$$

Clearly

$$\phi(0, 0) \equiv \Phi(0) \pmod{p}, \qquad \phi'_x(0, 0) + \phi'_y(0, 0) \equiv \Phi'(0) \pmod{p}.$$

According to (2.47), we have

$$\psi'_x(0,0)=\frac{d}{dx}\left(\frac{(f-d+x)_a(f-e+x)_a}{(-d)_a(-e)_a}\right)\bigg|_{x=0}\cdot\phi(0,0)+\frac{(f-d)_a(f-e)_a}{(-d)_a(-e)_a}\cdot\phi'_x(0,0)$$

and

$$\psi'_y(0,0)=\frac{d}{dy}\left(\frac{(a-d-y)_f(a-e-y)_f}{(-d)_f(-e)_f}\right)\bigg|_{y=0}\cdot\phi(0,0)+\frac{(a-d)_f(a-e)_f}{(-d)_f(-e)_f}\cdot\phi'_y(0,0).$$

It is easy to check that

$$\frac{(f-d)_a(f-e)_a}{(-d)_a(-e)_a}=\frac{(a-d)_f(a-e)_f}{(-d)_f(-e)_f}\equiv\frac{\Upsilon_0}{\Omega(0)}\pmod p.$$

And

$$\begin{aligned}&\frac{d}{dx}\left(\frac{(f-d+x)_a(f-e+x)_a}{(-d)_a(-e)_a}\right)\bigg|_{x=0}+\frac{d}{dx}\left(\frac{(a-d-x)_f(a-e-x)_f}{(-d)_f(-e)_f}\right)\bigg|_{x=0}\\ \equiv&\frac{d}{dx}\left(\frac{\Gamma(p+f+a-d+x)\Gamma(p+f+a-e+x)\Gamma(p-d)\Gamma(p-e)}{\Gamma(p+f-d+x)\Gamma(p+f-e+x)\Gamma(p+a-d)\Gamma(p+a-e)}\right)\bigg|_{x=0}\\ &+\frac{d}{dx}\left(\frac{\Gamma(p+a+f-d-x)\Gamma(p+a+f-e-x)\Gamma(p-d)\Gamma(p-e)}{\Gamma(p+a-d-x)\Gamma(p+a-e-x)\Gamma(p+f-d)\Gamma(p+f-e)}\right)\bigg|_{x=0}\\ =&\frac{d}{dx}\left(\frac{\upsilon_0}{\omega(x)}\right)\bigg|_{x=0}\pmod p.\end{aligned}$$

It follows that

$$\begin{aligned}\Psi'(0)\equiv&\frac{d}{dx}\left(\frac{\upsilon_0}{\omega(x)}\right)\bigg|_{x=0}\cdot\phi(0,0)+\frac{\Upsilon_0}{\Omega(0)}\cdot\big(\phi'_x(0,0)+\phi'_y(0,0)\big)\\ \equiv&\frac{d}{dx}\left(\frac{\upsilon_0}{\omega(x)}\right)\bigg|_{x=0}\cdot\Phi(0)+\frac{\Upsilon_0}{\Omega(0)}\cdot\Phi'(0)\pmod p.\end{aligned}$$

Thus we get

$$\Psi(sp)-\Psi(0)\equiv sp\cdot\Psi'(0)\equiv\frac{\Upsilon_0}{\Omega(sp)}\cdot\Phi(sp)-\frac{\Upsilon_0}{\Omega(0)}\cdot\Phi(0)\pmod{p^2}.$$

□

Next, let us turn to (2.49). We need another hypergeometric identity

$$ {}_4F_3\left[\begin{matrix}-n & \alpha & \beta & \gamma\\ & 1 & \delta & \epsilon\end{matrix}\bigg|\,1\right]=\frac{(1-\alpha)_n(\epsilon-\alpha)_n}{n!\cdot(\epsilon)_n}\cdot{}_4F_3\left[\begin{matrix}-n & \alpha & \delta-\beta & \delta-\gamma\\ & \delta & \alpha-n & 1+\alpha-n-\epsilon\end{matrix}\bigg|\,1\right] \tag{12.1}$$

where $n=\delta+\epsilon-\alpha-\beta-\gamma\in\mathbb{N}$. Clearly (12.1) follows from (2.46) by setting $\delta=1$.

Proof of Theorem 2.15. Without loss of generality, we may assume that $d \geq e$. Then $\max\{b,c\} \leq d$ by the condition (i). Let

$$\Psi(x) = {}_4F_3\left[\begin{matrix} \varrho_a - x & -a+x & \beta & \gamma \\ & 1 & \delta & \epsilon \end{matrix}\middle| 1\right]_{p-1}$$

where $\varrho_a = \delta+\epsilon+a-\beta-\gamma$. Note that $\varrho_a \in \mathbb{Z}_p^\times$ by (ii). With help of (12.1),

$$\Psi(0) = \frac{(1-\varrho_a)_a(\epsilon-\varrho_a)_a}{a!\cdot(\epsilon)_a}\cdot {}_4F_3\left[\begin{matrix} \varrho_a & -a & \delta-\beta & \delta-\gamma \\ & \delta & \varrho_a - a & 1+\varrho_a-a-\epsilon \end{matrix}\middle| 1\right]. \tag{12.2}$$

If $a+b+c > p+d$, then by (i) and (ii), we have

$$\langle \beta+\gamma-\delta-a\rangle_p = 2p+d-a-b-c > p+d-b-c = \langle\beta+\gamma-\delta\rangle_p.$$

It follows that

$$(\epsilon-\varrho_a)_a = (\beta+\gamma-\delta-a)_a \equiv 0 \pmod{p}. \tag{12.3}$$

Of course, since $a \geq 1$, (12.3) also holds when $a+b+c = p+d$. Similarly, we get

$$(1-\varrho_a)_a = (1+\beta+\gamma-\delta-\epsilon-a)_a \equiv 0 \pmod{p}. \tag{12.4}$$

Let $r_1 = \nu_p((1-\varrho_a)_a)$ and $r_2 = \nu_p((\epsilon-\varrho_a)_a)$. It is easy to see that $\nu_p((\varrho_a-a)_{p-1}) = r_1$ and $\nu_p((1+\varrho_a-a-\epsilon)_{p-1}) = r_2$. Moreover, by (i) and (ii), we have

$$\langle -(\varrho_a - a)\rangle_p = d+e-b-c \geq d-b = \langle -(\delta-\beta)\rangle_p \tag{12.5}$$

and

$$\langle -(1+\varrho_a-a-\epsilon)\rangle_p = p+d-b-c-1 \geq d-c = \langle -(\delta-\gamma)\rangle_p. \tag{12.6}$$

So

$$p^{r_1+r_2-2}\cdot\frac{(\varrho_a)_k(-a)_k(\delta-\beta)_k(\delta-\gamma)_k}{(\delta)_k(\varrho_a-a)_k(1+\varrho_a-a-\epsilon)_k} \in \mathbb{Z}_p$$

for any $0 \leq k \leq p-1$. It follows from (12.2) that

$$\Psi(0) \equiv 0 \pmod{p^2}.$$

Let

$$\phi(x,y) = {}_4F_3\left[\begin{matrix} -f-x & -a+y & \delta-\beta & \delta-\gamma \\ & \delta & -f-a-x+y & 1-f-a-\epsilon-x+y \end{matrix}\middle| 1\right]_{p-1}.$$

In view of (12.1),

$$\begin{aligned}\Psi'(0)\equiv&\frac{d}{dx}\left({}_4F_3\left[\begin{matrix}-f-x & -a & \beta & \gamma\\ & 1 & \delta & \epsilon\end{matrix}\middle|1\right]+{}_4F_3\left[\begin{matrix}-f & -a+x & \beta & \gamma\\ & 1 & \delta & \epsilon\end{matrix}\middle|1\right]\right)\Bigg|_{x=0}\\
\equiv&\frac{d}{dx}\left(\frac{(1+f+x)_a(\epsilon+f+x)_a}{a!\cdot(\epsilon)_a}\cdot\phi(x,0)+\frac{(1+a-x)_f(\epsilon+a-x)_f}{f!\cdot(\epsilon)_f}\cdot\phi(0,x)\right)\Bigg|_{x=0}\\
\equiv&\frac{d}{dx}\left(\frac{(1+f+x)_a(\epsilon+f+x)_a}{a!\cdot(\epsilon)_a}+\frac{(1+a-x)_f(\epsilon+a-x)_f}{f!\cdot(\epsilon)_f}\right)\Bigg|_{x=0}\cdot\phi(0,0)\\
&+\frac{(1+f)_a(\epsilon+f)_a}{a!\cdot(\epsilon)_a}\cdot\phi_x'(0,0)+\frac{(1+a)_f(\epsilon+a)_f}{f!\cdot(\epsilon)_f}\cdot\phi_y'(0,0)\pmod p.\end{aligned}$$

Clearly (12.3) and (12.4) implies that $(1+f)_a$ and $(\epsilon+f)_a$ are the multiples of p, And using the similar discussions, we may get that p also divides both $(\epsilon+a)_f$ and $(1+a)_f$. So

$$\frac{d}{dx}\left(\frac{(1+f+x)_a(\epsilon+f+x)_a}{a!\cdot(\epsilon)_a}+\frac{(1+a-x)_f(\epsilon+a-x)_f}{f!\cdot(\epsilon)_f}\right)\Bigg|_{x=0}\equiv 0\pmod p.$$

Furthermore, since $e\geq a$ and $(\epsilon)_{p-1}\not\equiv 0\pmod{p^2}$, we must have $(\epsilon+a)_{p-1}\not\equiv 0\pmod{p^2}$. Let $r_3=\nu_p((\epsilon+f)_a)$. Then we also have $\nu_p((1-f-a-\epsilon)_a)=r_3$. In view of (12.5) and (12.6),

$$\begin{aligned}p^{r_3}\cdot\phi_x'(0,0)=&\sum_{k=0}^{a}\frac{p^{r_3}\cdot(-a)_k(\delta-\beta)_k(\delta-\gamma)_k}{(\delta)_k(-f-a)_k(1-f-a-\epsilon)_k}\cdot\frac{d\big((-f-x)_k\big)}{dx}\Bigg|_{x=0}\\
&+\sum_{k=0}^{a}\frac{(-f)_k(-a)_k(\delta-\beta)_k(\delta-\gamma)_k}{(\delta)_k(-f-a)_k(1-f-a-\epsilon)_k}\sum_{j=0}^{k-1}\left(\frac{p^{r_3}}{j-f-a}+\frac{p^{r_3}}{j+1-f-a-\epsilon}\right)\end{aligned}$$

is p-adic integral. Similarly, we also have $p^{r_3}\cdot\phi_y'(0,0)\in\mathbb{Z}_p$. Hence

$$\Psi(sp)\equiv\Psi(0)+sp\cdot\Psi'(0)\equiv 0\pmod{p^2}.$$

□

Finally, we shall prove Theorem 2.12 in a similar way as Theorem 2.14.

Proof of Theorem 2.12. Let $a=\langle-\alpha\rangle_p$, $b=\langle-\beta\rangle_p$, $c=\langle-\gamma\rangle_p$ and $d=\langle-\delta\rangle_p$. Let $M=\max\{a,b,c\}$ and $N=\max\{a,p-1-b,p-1-c\}$. Since $\max\{a,b,c\}\leq d$, $b+c\geq d$ and $a+b+c\leq p+d-1$, we have

$$\frac{(\alpha)_k(\beta)_k(\gamma)_k}{(\delta)_k}\equiv 0\pmod{p^2},\qquad\frac{(\alpha)_j(1-\beta)_j(1-\gamma)_j}{(1+\delta-\beta-\gamma)_j}\equiv 0\pmod{p^2},$$

for any $M+1\leq k\leq p-1$ and $N+1\leq j\leq p-1$. Let

$$\Psi(x)={}_3F_2\left[\begin{matrix}-a+x & \beta & \gamma\\ & 1 & \delta\end{matrix}\middle|1\right]_M,\qquad\Phi(x)={}_3F_2\left[\begin{matrix}-a+x & 1-\beta & 1-\gamma\\ & 1 & 1+\delta-\beta-\gamma\end{matrix}\middle|1\right]_N$$

Let

$$\Omega(x)=\frac{\Gamma(\delta)\Gamma(1+\delta+a-x-\beta-\gamma)}{\Gamma(\delta+a-x)\Gamma(1+\delta-\beta-\gamma)},\quad \Upsilon(x)=\frac{\Gamma_p(\delta)\Gamma_p(1+\delta+a-x-\beta-\gamma)}{\Gamma_p(\delta+a-x)\Gamma_p(1+\delta-\beta-\gamma)}.$$

It is easy to check that $\Upsilon(0)=\Omega(0)$. So by (2.41), we have $\Psi(0)=\Upsilon(0)\Phi(0)$. Now

$$\begin{aligned}\Psi'(0)\equiv&\frac{d}{dx}\left({}_3F_2\left[\begin{matrix}-a+x & -b & p-c\\ & 1 & p-d\end{matrix}\middle|\,1\right]\right)\Bigg|_{x=0}\\ =&\frac{d}{dx}\left(\frac{\Gamma(p-d)\Gamma(1+a+b+c-d-x)}{\Gamma(p-d+a-x)\Gamma(1+b+c-d)}\cdot{}_3F_2\left[\begin{matrix}-a+x & 1+b & 1+c-p\\ & 1 & 1+b+c-d\end{matrix}\middle|\,1\right]\right)\Bigg|_{x=0}\\ \equiv&\frac{d}{dx}\left(\frac{\Gamma(p-d)\Gamma(1+a+b+c-d-x)}{\Gamma(p-d+a-x)\Gamma(1+b+c-d)}\right)\Bigg|_{x=0}\cdot\Phi(0)+\frac{(1+\delta-\beta-\gamma)_a}{(\delta)_a}\cdot\Phi'(0)\pmod p.\end{aligned}$$

Hence by Lemma 3.2, we have

$$\Psi(sp)-\Psi(0)\equiv\big(\Upsilon(sp)-\Upsilon(0)\big)\Phi(sp)+\Upsilon(0)\big(\Phi(sp)-\Phi(0)\big)=\Upsilon(sp)\Phi(sp)-\Upsilon(0)\Phi(0)\pmod{p^2},$$

where $s=-(\alpha+a)/p$. We are done. □

13. p-adic Whipple's ${}_7F_6$ transformation I: Theorem 2.16

In this section, we shall prove Theorem 2.16. Of course, as we have mentioned, Theorems 2.21 and 2.23 can be proved in the same way. And at the end of this section, we shall explain how to deduce Theorem 2.19 from Theorem 2.16.

Without of loss generality, we may assume that β, δ and ϵ are all negative integers. Let $a=\langle-\alpha\rangle_p$, $b=\langle-\beta\rangle_p$, $c=\langle-\gamma\rangle_p$, $d=\langle-\delta\rangle_p$ and $e=\langle-\epsilon\rangle_p$. Note that $b<a\le\min\{c,d,e\}$ and $\max\{c+d,c+e,d+e\}\le p+a-1$ by (i) and (ii). It is not difficult to verify that

$$\frac{(\alpha)_k^2(1+\frac12\alpha)_k(\beta)_k(\gamma)_k(\delta)_k(\epsilon)_k}{(\frac12\alpha)_k(\alpha-\beta+1)_k(\alpha-\gamma+1)_k(\alpha-\delta+1)_k(\alpha-\epsilon+1)_k}\equiv0\pmod{p^2}$$

for any $a+1\le k\le p-1$. Since $\max\{p-1-e,b,c,d\}\le p+a-1-e$ and $\max\{c,d\}\le b+c+d-a$, we obtain that

$$\frac{(1-\epsilon)_k(\beta)_k(\gamma)_k(\delta)_k}{(\alpha-\epsilon+1)_k(\beta+\gamma+\delta-\alpha)_k}\equiv0\pmod{p^2}$$

provided

$$1+\max\big\{b,\min\{c,d,p-1-e\}\big\}\le k\le p-1.$$

Let

$$\mathcal{A}_k(x)=(-a-ax)_k^2(-b-bx)_k(\gamma)_k(\delta)_k(\epsilon)_k$$

and

$$\mathcal{B}_k(x)=(1)_k^2(1-a-ax-\gamma)_k(1-a-ax-\delta)_k(1-a-ax-\epsilon)_k\prod_{\substack{1\le j\le k\\ j\ne a-b}}(j-a-ax+b+bx).$$

Let

$$\Psi(x)=\sum_{k=0}^{a-b-1}\frac{(1-\frac12a-\frac12ax)_k}{(-\frac12a-\frac12ax)_k}\cdot\frac{\mathcal{A}_k(x)}{\mathcal{B}_k(x)}+\sum_{k=a-b}^{M}\frac{(1-\frac12a-\frac12ax)_k}{(-\frac12a-\frac12ax)_k}\cdot\frac{1}{(b-a)x}\cdot\frac{\mathcal{A}_k(x)}{\mathcal{B}_k(x)},$$

where $M=\min\{c,d,e\}$. Write $\alpha=-a(1+sp)$. Clearly

$$\Psi(sp)={}_7F_6\left[\begin{matrix}\alpha & 1+\frac12\alpha & \alpha & \beta & \gamma & \delta & \epsilon\\ & \frac12\alpha & 1 & 1+\alpha-\beta & 1+\alpha-\gamma & 1+\alpha-\delta & 1+\alpha-\epsilon\end{matrix}\middle|\,1\right]_M.$$

Lemma 13.1. *$\Psi(x)=P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are two polynomials with $p\nmid Q(0)$. Furthermore, the coefficients of $P(x)$ and $Q(x)$ are all the polynomials in $a,c,\beta,\delta,\epsilon$ with integral coefficients.*

Proof. Clearly $p\nmid\mathcal{B}_k(0)$ for any $0\le k\le M$. Also, the polynomial $\mathcal{A}_k(x)$ is divisible by x^3 for those $a+1\le k\le M$. Note that

$$\frac{(1-\frac12a-\frac12ax)_k}{(-\frac12a-\frac12ax)_k}=1-\frac{2k}{a+ax}.$$

According to definition of $\Psi(x)$, we only need to prove that for each $a-b\le k\le a/2$, the constant term of the numerator of

$$\left(1-\frac{2k}{a+ax}\right)\cdot\frac{\mathcal{A}_k(x)}{\mathcal{B}_k(x)}+\left(1-\frac{2a-2k}{a+ax}\right)\cdot\frac{\mathcal{A}_{a-k}(x)}{\mathcal{B}_{a-k}(x)}$$

is zero, i.e.,

$$(a-2k)\mathcal{A}_k(0)\mathcal{B}_{a-k}(0)+(2k-a)\mathcal{A}_{a-k}(0)\mathcal{B}_k(0)=0. \tag{13.1}$$

There is nothing to do when $2b<a$, since $\mathcal{A}_k(0)=0$ for any $k>b$. So we may assume that $2b\ge a$. Since $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$, for any non-integral $z\in\mathbb{R}$, we have

$$\frac{(z)_{a-k}}{(1-a-z)_{a-k}}=\frac{\Gamma(z+a-k)\Gamma(1-a-z)}{\Gamma(z)\Gamma(1-z-k)}=(-1)^a\cdot\frac{\Gamma(1-a-z)\Gamma(z+k)}{\Gamma(1-a-z+k)\Gamma(z)}=\frac{(-1)^a(z)_k}{(1-a-z)_k},$$

i.e.,

$$(z)_k(1-a-z)_{a-k}=(-1)^a(z)_{a-k}(1-a-z)_k. \tag{13.2}$$

Clearly both sides of (13.2) are the polynomials in z. So (13.2) is factly valid for every $z\in\mathbb{R}$. Further,

$$\begin{aligned}(-b)_k\prod_{\substack{1\le j\le a-k\\ j\ne a-b}}(j-a+b)=&\lim_{z\to0}\frac{(-b-z)_k(1-a+b+z)_{a-k}}{z}\\ =&(-1)^a\lim_{z\to0}\frac{(-b-z)_{a-k}(1-a+b+z)_k}{z}=(-1)^a(-b)_{a-k}\prod_{\substack{1\le j\le k\\ j\ne a-b}}(j-a+b).\end{aligned}$$

Thus (13.1) is immediately derived. □

Let

$$\Omega_1(x) = \frac{\Gamma(1-a-ax+b+bx)}{\Gamma(1-a-ax)},$$

$$\Omega_2(x) = \frac{\Gamma(1-a-ax-\gamma)\Gamma(1-a-ax-\delta)\Gamma(1-a-ax+b+bx-\gamma-\delta)}{\Gamma(1-a-ax+b+bx-\gamma)\Gamma(1-a-ax+b+bx-\delta)\Gamma(1-a-ax-\gamma-\delta)},$$

and

$$\Phi(x) = {}_4F_3\left[\begin{matrix}1-\epsilon & -b-bx & \gamma & \delta \\ & 1 & 1-a-ax-\epsilon & a+ax-b-bx+\gamma+\delta\end{matrix}\middle|\, 1\right]_N$$

where $N = \max\{b, \min\{c, d, p-1-e\}\}$. Note that

$$\lim_{x\to 0}\Omega_1(x) = \lim_{x\to 0}\frac{\Gamma(1-ax+bx)}{\Gamma(1-ax)}\cdot\frac{(1-a-ax)(2-a-ax)\cdots(-ax)}{(1-a-ax+b+bx)\cdots(-ax+bx)} = \frac{a\cdot(1-a)_b}{a-b}.$$

So we may set

$$\Omega_1(0) = \frac{a}{a-b}\cdot(1-a)_b.$$

Recall that β, δ, ϵ are all negative integers, and $\mathcal{A}_k(x)$ is divisible by x^3 for each $k > a$. In view of (2.51), we have

$$\Psi(0) = \lim_{x\to 0}\Psi(x) = \lim_{x\to 0}\Omega_1(x)\Omega_2(x)\Phi(x) = \Omega_1(0)\Omega_2(0)\Phi(0).$$

Let

$$\begin{aligned}\Upsilon(x) =& \frac{\Gamma_p(1-a-ax+b+bx)\Gamma_p(1-a-ax-\gamma)}{\Gamma_p(1-a-ax)\Gamma_p(1-a-ax+b+bx-\gamma)}\\ &\cdot\frac{\Gamma_p(1-a-ax-\delta)\Gamma_p(1-a-ax+b+bx-\gamma-\delta)}{\Gamma_p(1-a-ax+b+bx-\delta)\Gamma_p(1-a-ax-\gamma-\delta)}.\end{aligned}$$

Since $b < a \le \min\{c, d\}$, we may get

$$(1-a)_b = (-1)^b\cdot\frac{\Gamma_p(1-a+b)}{\Gamma_p(1-a)}, \qquad \frac{\Gamma(1-a-\gamma)}{\Gamma(1-a+b-\gamma)} = (-1)^b\cdot\frac{\Gamma_p(1-a-\gamma)}{\Gamma_p(1-a+b-\gamma)}$$

and

$$\frac{\Gamma(1-a-\delta)}{\Gamma(1-a+b-\delta)} = (-1)^b\cdot\frac{\Gamma_p(1-a-\delta)}{\Gamma_p(1-a+b-\delta)}.$$

Also, since $b+c+d \le p+a-1$ by (ii), we have

$$\frac{\Gamma(1-a+b-\gamma-\delta)}{\Gamma(1-a-\gamma-\delta)} = (-1)^b\cdot\frac{\Gamma_p(1-a+b-\gamma-\delta)}{\Gamma_p(1-a-\gamma-\delta)}.$$

Thus

$$\Upsilon(0) = \frac{a-b}{a}\cdot\Omega_1(0)\Omega_2(0),$$

i.e.,

$$\Psi(0) = \frac{a}{a-b}\cdot\Upsilon(0)\Phi(0). \tag{13.3}$$

Let

$$\psi(x) = \sum_{k=0}^{a-b-1} \frac{(1-\frac{1}{2}a-\frac{1}{2}ax)_k}{(-\frac{1}{2}a-\frac{1}{2}ax)_k} \cdot \frac{\mathcal{A}_k^*(x)}{\mathcal{B}_k^*(x)} + \sum_{k=a-b}^{M} \frac{(1-\frac{1}{2}a-\frac{1}{2}ax)_k}{(-\frac{1}{2}a-\frac{1}{2}ax)_k} \cdot \frac{1}{(b-a)x} \cdot \frac{\mathcal{A}_k^*(x)}{\mathcal{B}_k^*(x)},$$

where

$$\mathcal{A}_k^*(x) = (-a-ax)_k^2(-b-bx)_k(-c)_k(-d)_k(p-e)_k$$

and

$$\mathcal{B}_k^*(x) = (1)_k^2(1-a-ax+c)_k(1-a-ax+d)_k(1-a-ax+e-p)_k \prod_{\substack{1\le j\le k\\ j\ne a-b}} (j-a-ax+b+bx).$$

By Lemma 13.1, clearly

$$\Psi'(0) \equiv \psi'(0) \pmod p. \tag{13.4}$$

Let

$$\omega_2(x) = \frac{\Gamma(1-a-ax+c)\Gamma(1-a-ax+d)\Gamma(1-a-ax+b+bx+c+d)}{\Gamma(1-a-ax+b+bx+c)\Gamma(1-a-ax+b+bx+d)\Gamma(1-a-ax+c+d)}$$

and

$$\phi(x) = {}_4F_3\left[\begin{matrix} 1+e-p & -b-bx & -c & -d \\ & 1 & 1-a-ax+e-p & a+ax-b-bx-c-d \end{matrix}\middle| 1\right].$$

By (2.51),

$$\psi(x) = \Omega_1(x)\omega_2(x)\phi(x). \tag{13.5}$$

It is easy to see that

$$\Phi(0) \equiv \phi(0) \pmod p, \qquad \Phi'(0) \equiv \phi'(0) \pmod p, \tag{13.6}$$

and

$$\omega_2(0) = \frac{(1-a+c+d)_b}{(1-a+c)_b(1-a+d)_b} \equiv \frac{(1-a-\gamma-\delta)_b}{(1-a-\gamma)_b(1-a-\delta)_b} = \Omega_2(0) \pmod p. \tag{13.7}$$

According to Lemma 3.2,

$$\Upsilon(sp) - \Upsilon(0) \equiv sp \cdot \frac{d}{dx}\left(\frac{\Gamma(p+1-a-ax+b+bx)}{\Gamma(p+1-a-ax)} \cdot \omega_2(x)\right)\bigg|_{x=0} \pmod{p^2}.$$

We have

$$\frac{\Gamma(p+1-a-ax+b+bx)}{\Gamma(p+1-a-ax)} = \Omega_1(x) \cdot \frac{(1-a-ax+b+bx)_p}{(1-a-ax)_p}.$$

Clearly

$$\lim_{x\to 0} \frac{(1-a-ax+b+bx)_p}{(1-a-ax)_p} = \frac{a-b}{a} \cdot \prod_{\substack{1\le j\le p\\ j\ne a-b}} (j-a+b) \cdot \prod_{\substack{1\le j\le p\\ j\ne a}} \frac{1}{j-a} \equiv \frac{a-b}{a} \pmod p$$

and

$$\begin{aligned}&\frac{d}{dx}\bigg(\frac{(1-a-ax+b+bx)_p}{(1-a-ax)_p}\bigg)\bigg|_{x=0}\\ =&\frac{a-b}{a}\cdot\prod_{\substack{1\le j\le p\\ j\ne a-b}}(j-a+b)\cdot\prod_{\substack{1\le j\le p\\ j\ne a}}\frac{1}{j-a}\cdot\Bigg(\sum_{\substack{1\le j\le p\\ j\ne a}}\frac{a}{j-a}-\sum_{\substack{1\le j\le p\\ j\ne a-b}}\frac{a-b}{j-a+b}\Bigg)\equiv 0 \pmod p.\end{aligned}$$

It follows that

$$\frac{d}{dx}\bigg(\frac{\Gamma(p+1-a-ax+b+bx)}{\Gamma(p+1-a-ax)}\bigg)\bigg|_{x=0}\equiv\frac{a-b}{a}\cdot\Omega_1'(0) \pmod p.$$

Hence

$$\Upsilon(sp)-\Upsilon(0)\equiv sp\cdot\frac{a-b}{a}\cdot\big(\Omega_1'(0)\cdot\omega_2(0)+\Omega_1(0)\cdot\omega_2'(0)\big) \pmod{p^2}. \tag{13.8}$$

Since

$$\Upsilon(sp)\equiv\Upsilon(0)\equiv\frac{a-b}{a}\cdot\Omega_1(0)\omega_2(0) \pmod p,$$

combining (13.4), (13.5), (13.6), (13.7) and (13.8), we obtain that

$$\begin{aligned}\Psi(sp)-\Psi(0)\equiv&sp\cdot\psi'(0)\equiv sp\cdot\big(\Omega_1'(0)\omega_2(0)\phi(0)+\Omega_1(0)\omega_2'(0)\phi(0)+\Omega_1(0)\omega_2(0)\phi'(0)\big)\\ \equiv&\frac{a}{a-b}\cdot\big(\Upsilon(sp)-\Upsilon(0)\big)\cdot\Phi(0)+\frac{a}{a-b}\cdot\Upsilon(0)\cdot(\Phi(sp)-\Phi(0))\\ \equiv&\frac{a}{a-b}\cdot\big(\Upsilon(sp)\Phi(sp)-\Upsilon(0)\phi(0)\big) \pmod{p^2}.\end{aligned}$$

It follows from (13.3) that

$$\Psi(sp)\equiv\frac{a}{a-b}\cdot\Upsilon(sp)\Phi(sp)=\frac{\alpha}{\alpha-\beta}\cdot\Upsilon(sp)\Phi(sp) \pmod{p^2}.$$

□

Finally, let us give the proof of Theorem 2.19. Let $a=\langle-\alpha\rangle_p$, $b=\langle-\beta\rangle_p$, $c=\langle-\gamma\rangle_p$, $d=\langle-\delta\rangle_p$ and $e=\langle-\epsilon\rangle_p$. By the condition (ii), clearly

$$e=p+a-1-b-c-d\ge a.$$

Note that now

$$\big\langle-(\beta+\gamma+\delta)\big\rangle_p=b+c+d>b+c=\langle-\beta\rangle_p+\langle-\gamma\rangle_p.$$

According to Theorem 2.11, we know that

$$\begin{aligned}&{}_4F_3\begin{bmatrix}1-\epsilon & \beta & \gamma & \delta\\ & 1 & \alpha-\epsilon+1 & \beta+\gamma+\delta-\alpha\end{bmatrix}\,1\Bigg]_{p-1}\\ =&{}_3F_2\begin{bmatrix}\beta & \gamma & \delta\\ & 1 & \beta+\gamma+\delta\end{bmatrix}\,1\Bigg]_{p-1}\equiv-\frac{\Gamma_p(1-\beta-\gamma)\Gamma_p(1-\beta-\delta)\Gamma_p(1-\gamma-\delta)}{\Gamma_p(1-\beta)\Gamma_p(1-\gamma)\Gamma_p(1-\delta)\Gamma_p(1-\beta-\gamma-\delta)} \pmod{p^2}.\end{aligned}$$

Thus by Theorem 2.16, we immediately get the desired result.

14. p-adic Whipple's ${}_7F_6$ transformation II: Theorems 2.17 and 2.18

Proof of Theorem 2.17. Let $a=\langle-\alpha\rangle_p$, $b=\langle-\beta\rangle_p$, $c=\langle-\gamma\rangle_p$, $d=\langle-\delta\rangle_p$ and $e=\langle-\epsilon\rangle_p$. Since $a\le\min\{b,c,d,e\}$ and $\max\{b+e,c+d\}\le p+a-1$ by (i) and (ii), we have

$$\frac{(\alpha)_k^2(\frac12\alpha+1)_k(\beta)_k(\gamma)_k(\delta)_k(\epsilon)_k}{(\frac12\alpha)_k(\alpha-\beta+1)_k(\alpha-\gamma+1)_k(\alpha-\delta+1)_k(\alpha-\epsilon+1)_k}\equiv 0 \pmod{p^2}$$

for each $a+1\le k\le p-1$. Let

$$\Psi(x)={}_7F_6\left[\begin{matrix}\alpha & \alpha & \frac12\alpha+1 & -b+x & \gamma & \delta & \epsilon\\ & 1 & \frac12\alpha & \alpha+b-x+1 & \alpha-\gamma+1 & \alpha-\delta+1 & \alpha-\epsilon+1\end{matrix}\,\middle|\,1\right]_a$$

In view of (2.51),

$$\Psi(0)=\frac{(1+\alpha)_b(1+\alpha-\gamma-\delta)_b}{(1+\alpha-\gamma)_b(1+\alpha-\delta)_b}\cdot{}_4F_3\left[\begin{matrix}1-\epsilon & -b & \gamma & \delta\\ & 1 & \alpha-\epsilon+1 & \gamma+\delta-b-\alpha\end{matrix}\,\middle|\,1\right].$$

Note that (iii) implies $p+a\le b+c+d$. As $a\le b$ and $c+d-a<p$, we get

$$(1+\alpha)_b\equiv(1+\alpha-\gamma-\delta)_b\equiv 0 \pmod p.$$

Also, since $a\le\min\{c,d\}$ and $p+a-b>\max\{c,d\}$, neither $(1+\alpha-\gamma)_b$ nor $(1+\alpha-\delta)_b$ is divisible by p. Moreover, we have $a\le e$, $p-1-e+a\ge b$ and $p-1-e\le b+c+d-a-p$ by (i), (ii) and (iii) respectively. Letting $r=\nu_p\big((1+\alpha-\gamma-\delta)_b\big)$, we obtain that

$$p^{r-1}\cdot\frac{(1-\epsilon)_k(-b)_k(\gamma)_k(\delta)_k}{(\alpha-\epsilon+1)_k(\gamma+\delta-b-\alpha)_k}$$

is p-integral for each $0\le k\le b$, i.e.,

$$p^{r-1}\cdot{}_4F_3\left[\begin{matrix}1-\epsilon & -b & \gamma & \delta\\ & 1 & \alpha-\epsilon+1 & \gamma+\delta-b-\alpha\end{matrix}\,\middle|\,1\right]\in\mathbb{Z}_p.$$

It follows that

$$\Psi(0)\equiv 0 \pmod{p^2}.$$

Let

$$\psi(x)={}_7F_6\left[\begin{matrix}-a & 1-\frac12 a & -a & -b+x & -c & -d & -e\\ & -\frac12 a & 1 & 1-a+b-x & 1-a+c & 1-a+d & 1-a+e\end{matrix}\,\middle|\,1\right].$$

By (2.51),

$$\psi(x)=\frac{(1-a)_d(1-a+c+b-x)_d}{(1-a+c)_d(1-a+b-x)_d}\cdot{}_4F_3\left[\begin{matrix}1+e & -b+x & -c & -d\\ & 1 & 1-a+e & a-c-d-b+x\end{matrix}\,\middle|\,1\right]$$

always vanishes, since $(1-a)_d=0$. Hence letting $s=(\beta+b)/p$, we have

$$\Psi(sp)\equiv\Psi(0)+sp\cdot\Psi'(0)\equiv\Psi(0)+sp\cdot\psi'(0)\equiv 0 \pmod{p^2}.$$

□

Proof of Theorem 2.18. Let $a = \langle -\alpha \rangle_p$, $b = \langle -\beta \rangle_p$, $c = \langle -\gamma \rangle_p$, $d = \langle -\delta \rangle_p$ and $e = \langle -\epsilon \rangle_p$. Let $s = (\beta + b)/p$. Let $M = \min\{b, c, d, e\}$ and $N = \max\{b, c, d\}$. In view of (ii), we have

$$(\alpha - \beta + 1)_k(\alpha - \gamma + 1)_k(\alpha - \delta + 1)_k(\alpha - \epsilon + 1)_k \not\equiv 0 \pmod{p}$$

for any $0 \le k \le M$, and

$$\frac{(\alpha)_k^2(\frac{1}{2}\alpha + 1)_k(\beta)_k(\gamma)_k(\delta)_k(\epsilon)_k}{(\frac{1}{2}\alpha)_k(\alpha - \beta + 1)_k(\alpha - \gamma + 1)_k(\alpha - \delta + 1)_k(\alpha - \epsilon + 1)_k} \equiv 0 \pmod{p^2}$$

for any $M + 1 \le k \le p - 1$. Further, since $N < b + c + d - a < p$ and $p - 1 - e \le p + a - 1 - e$,

$$\frac{(1 - \epsilon)_k(\beta)_k(\gamma)_k(\delta)_k}{(\alpha - \epsilon + 1)_k(\beta + \gamma + \delta - \alpha)_k} \equiv 0 \pmod{p^2}$$

for any $N < k \le p - 1$. Let

$$\Psi(x) = {}_7F_6\left[\begin{matrix}\alpha & \alpha & \frac{1}{2}\alpha + 1 & -b + x & \gamma & \delta & \epsilon \\ & 1 & \frac{1}{2}\alpha & \alpha + b - x + 1 & \alpha - \gamma + 1 & \alpha - \delta + 1 & \alpha + \epsilon + 1\end{matrix}\middle|\, 1\right]_M,$$

$$\Phi(x) = {}_4F_3\left[\begin{matrix}1 - \epsilon & -b + x & \gamma & \delta \\ & 1 & 1 + \alpha - \epsilon & \gamma + \delta - \alpha - b + x\end{matrix}\middle|\, 1\right]_N$$

and

$$\Omega(x) = \frac{\Gamma(1 + \alpha + b - x)\Gamma(1 + \alpha - \gamma)\Gamma(1 + \alpha - \delta)\Gamma(1 + \alpha - \delta - \gamma + b - x)}{\Gamma(1 + \alpha)\Gamma(1 + \alpha - \gamma + b - x)\Gamma(1 + \alpha - \delta + b - x)\Gamma(1 + \alpha - \delta - \gamma)}.$$

By (2.51),

$$\Psi(0) = \Omega(0)\Phi(0). \tag{14.1}$$

Recall that $a \le \min\{b, c, d\}$ and $0 < b + c + d - a < p$. It is easy to check that

$$\begin{aligned}
&\frac{\Gamma(\alpha + b + 1)}{\Gamma(\alpha + 1)} \cdot \frac{\Gamma(\alpha - \gamma + 1)}{\Gamma(\alpha - \gamma + b + 1)} \cdot \frac{\Gamma(\alpha - \delta + 1)}{\Gamma(\alpha - \delta + b + 1)} \cdot \frac{\Gamma(\alpha - \delta - \gamma + b + 1)}{\Gamma(\alpha - \delta - \gamma + 1)} \\
=&(\alpha + a) \cdot \frac{\Gamma_p(\alpha + b + 1)}{\Gamma_p(\alpha + 1)} \cdot \frac{\Gamma_p(\alpha - \gamma + 1)}{\Gamma_p(\alpha - \gamma + b + 1)} \cdot \frac{\Gamma_p(\alpha - \delta + 1)}{\Gamma_p(\alpha - \delta + b + 1)} \cdot \frac{\Gamma_p(\alpha - \delta - \gamma + b + 1)}{\Gamma_p(\alpha - \delta - \gamma + 1)}.
\end{aligned}$$

Thus we have

$$\Omega(0) = (\alpha + a) \cdot \Upsilon(0), \tag{14.2}$$

where

$$\Upsilon(x) = \frac{\Gamma_p(1 + \alpha + b - x)\Gamma_p(1 + \alpha - \gamma)\Gamma_p(1 + \alpha - \delta)\Gamma_p(1 + \alpha - \delta - \gamma + b - x)}{\Gamma({}_p1 + \alpha)\Gamma_p(1 + \alpha - \gamma + b - x)\Gamma_p(1 + \alpha - \delta + b - x)\Gamma_p(1 + \alpha - \delta - \gamma)}.$$

Applying Lemma 3.1, we have

$$\Psi(sp) - \Psi(0) \equiv sp \cdot \Psi'(0) \equiv sp \cdot \psi'(0) \pmod{p^2},$$

where

$$\begin{aligned}\psi(x)=&{}_7F_6\begin{bmatrix}-a & -a & 1-\frac{1}{2}a & -b+x & -c & -d & -e \\ & 1 & -\frac{1}{2}a & 1-a+b-x & 1-a+c & 1-a+d & 1-a+e\end{bmatrix}\!\!\left|\,1\right.\Bigg]\\ =&\frac{(1-a)_d(1-a+c+b-x)_d}{(1-a+c)_d(1-a+b-x)_d}\cdot{}_4F_3\begin{bmatrix}1+e & -b+x & -c & -d \\ & 1 & 1-a+e & a-c-d-b+x\end{bmatrix}\!\!\left|\,1\right.\Bigg]\end{aligned}$$

by (2.51). Since $1\le a\le d$, we have $\psi(x)=0$. Further, clearly

$$\Upsilon(sp)\Phi(sp)\equiv\Upsilon(0)\Phi(0)\pmod p.$$

So

$$\Psi(sp)-\Psi(0)\equiv 0\equiv(\alpha+a)\cdot\big(\Upsilon(sp)\Phi(sp)-\Upsilon(0)\Phi(0)\big)\pmod{p^2}.$$

It follows from (14.1) and (14.2) that

$$\Psi(sp)\equiv(\alpha+a)\cdot\Upsilon(sp)\Phi(sp)\pmod{p^2}.$$

□

15. A conjecture of Deines-Fuselier-Long-Swisher-Tu

Deines, Fuselier, Long, Swisher and Tu [6, Conjecture 18] conjectured

$$\sum_{k=0}^{p-1}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2\cdot(-1)^k\equiv p^2\sum_{k=\frac{p-1}{2}}^{p-1}\left(\frac{k!}{(\frac{3}{2})_k}\right)^2\cdot(-1)^k\pmod{p^2}\tag{15.1}$$

for any prime $p\equiv 1\pmod 4$. Note that $(3/2)_k$ is not divisible by p for any $0\le k<(p-1)/2$. Clearly (15.1) is an equivalent form of (2.68).

It is easy to see that

$$\frac{(1)_k}{(\frac{3}{2})_k}=\frac{(1)_{p-1}}{(1-p)_{p-1-k}}\cdot\frac{(\frac{1}{2}-p)_{p-1-k}}{(\frac{3}{2})_{p-1}}.$$

We have

$$p^2\sum_{k=\frac{p-1}{2}}^{p-1}\frac{(1)_k^2}{(\frac{3}{2})_k^2}\cdot(-1)^k=\frac{p^2\cdot(1)_{p-1}^2}{(\frac{3}{2})_{p-1}^2}\sum_{k=0}^{\frac{p-1}{2}}\frac{(\frac{1}{2}-p)_k^2}{(1-p)_k^2}\cdot(-1)^k.$$

We need the following identity due to Kummer [3, Corollary 3.1.2]:

$${}_2F_1\begin{bmatrix}\alpha & \beta \\ & \alpha-\beta+1\end{bmatrix}\!\!\left|\,-1\right.\Bigg]=\frac{\Gamma(\alpha-\beta+1)\Gamma(\frac{1}{2}\alpha+1)}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}\alpha-\beta+1)}.\tag{15.2}$$

Let

$$\Psi(x)=\sum_{k=0}^{\frac{p-1}{2}}\frac{(\frac{1}{2}-x)_k^2}{(1-x)_k^2}\cdot(-1)^k.$$

Clearly

$$\begin{aligned}\Psi'(0) =&\frac{d}{dx}\left(2\sum_{k=0}^{\frac{p-1}{2}}\frac{(\frac{1}{2}-x)_k\cdot(\frac{1}{2})_k}{(1-x)_k\cdot(1)_k}\cdot(-1)^k\right)\Bigg|_{x=0}\\ \equiv&\frac{d}{dx}\left(2\sum_{k=0}^{\frac{p-1}{2}}\frac{(\frac{1}{2}+\frac{1}{2}p-x)_k\cdot(\frac{1}{2}-\frac{1}{2}p)_k}{(1+p-x)_k\cdot(1)_k}\cdot(-1)^k\right)\Bigg|_{x=0}\\ =&2\cdot\frac{d}{dx}\left(\frac{\Gamma(1+\frac{p}{2}-x)\Gamma(\frac{5}{4}-\frac{x}{2})}{\Gamma(\frac{3}{2}-x)\Gamma(\frac{3}{4}+\frac{p}{2}-\frac{x}{2})}\right)\Bigg|_{x=0}\pmod p.\end{aligned}$$

We have

$$\begin{aligned}&\frac{d}{dx}\left(\frac{\Gamma(1+\frac{p}{2}-x)\Gamma(\frac{5}{4}-\frac{x}{2})}{\Gamma(\frac{3}{2}-x)\Gamma(\frac{3}{4}+\frac{p}{2}-\frac{x}{2})}\right)\Bigg|_{x=0}\\ =&\frac{d}{dx}\left(\frac{(\frac{3}{2}-x)_{\frac{p-1}{2}}}{(\frac{5}{4}-\frac{x}{2})_{\frac{p-1}{2}}}\right)\Bigg|_{x=0}=\frac{(\frac{3}{2})_{\frac{p-1}{2}}}{(\frac{5}{4})_{\frac{p-1}{2}}}\sum_{j=1}^{\frac{p-1}{2}}\left(\frac{1}{2j+\frac{1}{2}}-\frac{1}{j+\frac{1}{2}}\right)\\ =&{}_2F_1\left[\begin{matrix}\frac{1}{2}(1+p) & \frac{1}{2}(1-p)\\ & 1+p\end{matrix}\;\middle|\;-1\right]\cdot\sum_{j=1}^{\frac{p-1}{2}}\left(\frac{1}{2j+\frac{1}{2}}-\frac{1}{j+\frac{1}{2}}\right).\end{aligned}$$

Note that since $p\equiv 1 \pmod 4$,

$$\begin{aligned}\sum_{j=1}^{\frac{p-1}{2}}\left(\frac{1}{2j+\frac{1}{2}}-\frac{1}{j+\frac{1}{2}}\right)=&\sum_{\substack{1\le j\le\frac{p-1}{2}\\ j\ne\frac{1}{4}(p-1)}}\frac{2}{4j+1}-\sum_{j=1}^{\frac{p-3}{2}}\frac{2}{2j+1}\\ \equiv\left(\sum_{j=\frac{p+3}{4}}^{\frac{p-1}{2}}\frac{2}{4j+1-p}-\sum_{j=1}^{\frac{p-5}{4}}\frac{2}{p-(4j+1)}\right)&+\sum_{j=1}^{\frac{p-3}{2}}\frac{2}{p-(2j+1)}=\sum_{j=1}^{\frac{p-1}{2}}\frac{1}{j}\pmod p.\end{aligned}$$

Thus

$$\Psi'(0)=2\Psi(0)\cdot H_{\frac{p-1}{2}}\pmod p.$$

On the other hand, by Lemma 3.2,

$$\begin{aligned}p^2\cdot\frac{(1)_{p-1}^2}{(\frac{3}{2})_{p-1}^2}=&\frac{\Gamma_p(1+p)^2\Gamma_p(\frac{1}{2})^2}{\Gamma_p(1)^2\Gamma_p(\frac{1}{2}+p)^2}\\ \equiv&1+p\cdot\frac{d}{dx}\left(\frac{\Gamma(1+x)^2\Gamma_p(\frac{1}{2})^2}{\Gamma_p(1)^2\Gamma(\frac{p+1}{2}+x)^2}\right)\Bigg|_{x=0}=1-2pH_{p-1}\pmod{p^2},\end{aligned}$$

since $\Gamma_p(1/2)^2 = (-1)^{\langle -\frac{1}{2}\rangle_p} = 1$. Hence

$$p^2 \sum_{k=\frac{p-1}{2}}^{p-1} \frac{(1)_k^2}{(\frac{3}{2})_k^2} \cdot (-1)^k \equiv \frac{p^2(1)_{p-1}^2}{(\frac{3}{2})_{p-1}^2} \cdot \big(\Psi(0) + p \cdot \Psi'(0)\big)$$

$$\equiv (1 - 2pH_{\frac{p-1}{2}}) \cdot \Psi(0)(1 + 2pH_{\frac{p-1}{2}}) \equiv \Psi(0) \pmod{p^2}.$$

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