CLASSIFICATION OF L^p AF ALGEBRAS

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ABSTRACT. We define spatial L^p AF algebras for $p \in [1, \infty) \setminus \{2\}$, and prove the following analog of the Elliott AF algebra classification theorem. If A and B are spatial L^p AF algebras, then the following are equivalent:

- A and B have isomorphic scaled preordered K₀-groups.
- $A \cong B$ as rings.
- $A \cong B$ (not necessarily isometrically) as Banach algebras.
- A is isometrically isomorphic to B as Banach algebras.
- A is completely isometrically isomorphic to B as matrix normed Banach algebras.

As background, we develop the theory of matrix normed L^p operator algebras, and show that there is a unique way to make a spatial L^p AF algebra into a matrix normed L^p operator algebra. We also show that any countable scaled Riesz group can be realized as the scaled preordered K_0 -group of a spatial L^p AF algebra.

1. Introduction

In a well known paper [7] of 1976, Elliott gave a complete classification of approximately finite dimensional (AF) C*-algebras. He showed that two AF C*-algebras A_1 and A_2 are isomorphic if and only if their scaled preordered K_0 -groups $(K_0(A_1), K_0(A_1)_+, \Sigma(A_1))$ and $(K_0(A_2), K_0(A_2)_+, \Sigma(A_2))$ are isomorphic. Moreover, the work of Effros, Handelman, and Shen showed (see [5] and [6]) that any countable scaled Riesz group (G, G_+, Σ) can be realized as the scaled preordered K_0 -group of an AF C*-algebra.

In a series of papers (see [16], [17], [18], and [19]), the first author introduced and studied L^p analogs of the uniformly hyperfinite (UHF) algebras and L^p analogs of the Cuntz algebras. One result of [17] is that two spatial L^p UHF algebras are isomorphic if and only if they have the same supernatural number. This result is analogous to the result of Glimm [10], that two UHF C*-algebras are isomorphic if and only if they have the same supernatural number. (This is a special case, done earlier, of Elliott's AF classification theorem.)

It is therefore natural to ask if there are L^p analogs of AF algebras which can be classified by their scaled preordered K_0 groups. In this paper, we show that the algebras that we call the spatial L^p AF algebras provide a positive answer to this question. In Theorem 10.20, we show that two spatial L^p AF algebras are completely isometrically isomorphic (as matricial L^p operator algebras) if and only if their scaled ordered K_0 groups are isomorphic. We further show that,

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as in the C*-algebra case, given any scaled countable Riesz group (G, G_+, Σ) , there exists a spatial L^p AF algebra A such that the scaled preordered K_0 group $(K_0(A), K_0(A)_+, \Sigma(A))$ is isomorphic to (G, G_+, Σ) . We also show that spatial L^p AF algebras have unique L^p matrix norms.

We don't list examples. Theorem 10.17 and Theorem 10.22 show that for each $p \in [1, \infty)$ there is a one to one correspondence between isomorphism classes of AF C*-algebras and spatial L^p AF algebras, so the examples are "the same". Although we don't address this issue here, constructions like the C*-algebra of a locally finite discrete abelian group, which give AF C*-algebras, give L^p operator algebras which are AF in some sense but are not spatial L^p AF algebras.

In a forthcoming paper we will prove that the ideal structure of a spatial L^p AF algebra is determined by K-theory in the same way as for an AF C*-algebra. We will prove that, like a C*-algebra, a spatial L^p AF algebra is incompressible in the sense that any contractive homomorphism to some other Banach algebra can be factored as a quotient map followed by an isometric homomorphism. (In particular, contractive injective homomorphisms from spatial L^p AF algebras are isometric.) We will also study the isometries and automorphisms of a spatial L^p AF algebra. The results will be quite different from what happens with AF C*-algebras.

A spatial L^p AF algebra is the direct limit of a direct system of semisimple finite dimensional L^p operator algebras in which the connecting maps are contractive homomorphisms having the property that the image of the identity is a spatial partial isometry in the sense of Definition 6.4 of [16]. In the context of L^p operator algebras, where in general we do not require the homomorphisms between L^p operator algebras to be unital, that is the best possible form for our maps.

To make sense of uniqueness of L^p matrix norms and completely isometric isomorphism, we develop the basics of the theory of matrix normed Banach algebras and matricial L^p operator algebras.

The arguments used for the classification of spatial L^p AF algebras are similar to the ones used for the classification of AF C*-algebras. However, to be able to carry out these arguments, background material needs to be developed. Much of it is fairly elementary, and for this part the novelty is putting it together in the right way. There are several somewhat more substantial ingredients, including a structure theorem for contractive representations of C(X) on an L^p space (Theorem 4.5), the recognition that, in connection with nonunital maps between unital algebras, idempotents must be required to be hermitian (contractivity is not good enough; see Section 6), and what to require of approximate identities of idempotents in order to get a unique suitable norm on the unitization (see Proposition 9.9). We also have to prove that the direct limit of L^p operator algebras is again an L^p operator algebra.

The paper is organized as follows. In Section 2 we recall L^p operator algebras and give some preliminary results on their representations on L^p -spaces. In Section 3 we introduce matricial (matrix normed) L^p operator algebras and discuss their representations on L^p spaces. This material is needed to define L^p operator algebras that have unique L^p matrix norms, which we examine in Section 4. Most of the L^p operator algebras in this article have unique L^p matrix norms, including the matrix algebra M_p^p and the algebra C(X) for a compact metric space X.

Sections 5 and 7 deal with direct sums and direct limits of (matricial) L^p operator algebras, while Section 6 contains material on hermitian idempotents, including a

characterization of hermitian idempotents in an L^p operator algebra in terms of multiplication operators.

In Section 8 we introduce our building blocks (the spatial semisimple finite dimensional L^p operator algebras), and the appropriate homomorphisms between them, the spatial homomorphisms. We characterize spatial homomorphisms in terms of block diagonal homomorphisms. In Section 9 we define spatial L^p AF algebras, show that every spatial L^p AF algebra is an L^p operator algebra as in [16], and that it has unique L^p matrix norms.

Section 10 contains our main result. We give a complete classification of spatial L^p AF algebras using the scaled preordered K_0 group, and show that, as in the C*-algebra case, any countable scaled Riesz group can be realized as the scaled preordered K_0 group of a spatial L^p AF algebra.

Shortly after posting this paper on the airXiv, E. Gardella informed us of his work with Lupini on the uniqueness of the matricial norm structure for L^p analogs of groupoid C*-algebras. (See [9].) Spatial L^p AF algebras are examples of such algebras; see Subsection 7.2 in [9]. We were unaware of the work of Gardella and Lupini while preparing this manuscript and we refer the reader to their paper for a different proof of the uniqueness of the L^p matrix norms.

We use the following standard notation throughout the paper.

Notation 1.1. If E is a Banach space, then L(E) denotes the Banach algebra of all bounded linear operators on E, with the operator norm.

Notation 1.2. If (X, \mathcal{B}, μ) is a measure space, and $E \subset X$ is measurable, then $\mu|_E$ denotes the measure on E gotten by restricting μ to the σ -algebra of measurable subsets of E.

We also recall that an idempotent in a ring is an element e satisfying $e^2 = e$.

2. L^p operator algebras

In this section we define L^p operator algebras, and state some of the standard results about L^p operator algebras and their representations. These results are basic for the rest of the paper.

The following definitions are based on Definition 1.1 and Definition 1.17 of [19].

Definition 2.1. Let $p \in [1, \infty)$. An L^p operator algebra is a Banach algebra such that there exists a measure space (X, \mathcal{B}, μ) and an isometric isomorphism from A to a norm closed subalgebra of $L(L^p(X, \mu))$.

Definition 2.2. Let $p \in [1, \infty)$.

- (1) A representation of an L^p operator algebra A (on $L^p(Y, \nu)$) is a continuous homomorphism $\pi: A \to L(L^p(Y, \nu))$ for some measure space (Y, \mathcal{C}, ν) .
- (2) The representation π is *contractive* if $\|\pi(a)\| \leq \|a\|$ for all $a \in A$, and isometric if $\|\pi(a)\| = \|a\|$ for all $a \in A$.
- (3) We say that the representation $\pi \colon A \to L(L^p(Y, \nu))$ is separable if $L^p(Y, \nu)$ is separable, and that A is separably representable if it has a separable isometric representation.
- (4) We say that π is σ -finite if ν is σ -finite, and that A is σ -finitely representable if it has a σ -finite isometric representation.
- (5) We say that π is nondegenerate if

$$\pi(A)L^p(Y,\nu) = \operatorname{span}(\{\pi(a)\xi \colon a \in A \text{ and } \xi \in L^p(Y,\nu)\})$$

is dense in $L^p(Y,\nu)$. We say that A is nondegenerately (separably) representable if it has a nondegenerate (separable) isometric representation, and nondegenerately σ -finitely representable if it has a nondegenerate σ -finite isometric representation.

The following fact about the restriction of an operator looks obvious (and the proof is easy), but it is the sort of statement that should not be taken for granted outside of the context of C*-algebras. The condition ||f|| = 1 is necessary; see Example 2.5.

Lemma 2.3. Let E be a Banach space, let $a \in L(E)$, and let $f \in L(E)$ be an idempotent with ||f|| = 1 such that af = a. Then $||a||_{fE}|| = ||a||$.

Proof. It is obvious that $||a||_{fE}|| \le ||a||$. For the reverse inequality, let $\varepsilon > 0$, choose $\xi \in E$ such that $||\xi|| \le 1$ and $||a\xi|| > ||a|| - \varepsilon$, and set $\eta = f\xi$. Then $\eta \in fE$, $||\eta|| \le 1$, and $||a\eta|| = ||a\xi|| > ||a|| - \varepsilon$.

The main application of Lemma 2.3 is the next result, which we will use repeatedly in the following sections.

Corollary 2.4. Let A be a unital Banach algebra in which ||1|| = 1. Let E be a Banach space, and let $\pi: A \to L(E)$ be a nonzero representation. Set $F = \pi(1)E$. Then there is a unital representation $\pi_0: A \to L(F)$ such that $\pi_0(a)\xi = \pi(a)\xi$ for all $a \in A$ and $\xi \in F$. If $||\pi(1)|| = 1$, then π_0 is contractive if and only if π is contractive and π_0 is isometric if and only if π is isometric.

Proof. The existence of π_0 follows from the equation $\pi(1)\pi(a)\pi(1) = \pi(a)$ for all $a \in A$. If π is contractive then $\|\pi(1)\| \le 1$. If $\|\pi(1)\| = 1$, taking $f = \pi(1)$ in Lemma 2.3 gives $\|\pi_0(a)\| = \|\pi(a)\|$ for all $a \in A$.

Example 2.5. Lemma 2.3 fails without ||f|| = 1 and Corollary 2.4 fails without ||1|| = 1. Take $f \in L(E)$ to be any idempotent with ||f|| > 1. For example, take $p \in (1, \infty)$, take E to be \mathbb{C}^2 with $||\cdot||_p$, and take $f = (\frac{1}{0} \frac{1}{0})$. Then $||f|| = 2^{1-1/p}$. For Lemma 2.3 take a = f. Then $a|_{fE}$ is the identity operator, so $||a|_{fE}|| = 1 < ||a||$. For Corollary 2.4 take $A = \mathbb{C} f \subset L(E)$ with the operator norm and take π to be the identity representation. Then π is isometric but π_0 is not.

Proposition 2.6 (Proposition 1.25 of [19]). Let $p \in [1, \infty)$, and let A be a separable L^p operator algebra. Then A is separably representable. If A is nondegenerately representable, then A is separably nondegenerately representable.

Since we will only use separable L^p operator algebras in this paper, we need only deal with separable L^p spaces. Lemma 2.7 implies that we can always assume that the measures are σ -finite.

Lemma 2.7. Let $p \in [1, \infty)$. Let (X, \mathcal{B}, μ) be a measure space such that $L^p(X, \mu)$ is separable. Then there exists a σ -finite measure space (Y, \mathcal{C}, ν) such that $L^p(X, \mu)$ is isometrically isomorphic to $L^p(Y, \nu)$.

Proof. See the Corollary to Theorem 3 in Section 15 of [13]. \Box

The following result will be used often enough that we restate it here.

Proposition 2.8. Let $p \in [1, \infty) \setminus \{2\}$. Let (X, \mathcal{B}, μ) be a measure space and let $e \in L(L^p(X, \mu))$ be an idempotent. Then $||e|| \leq 1$ if and only if there exists a

measure space (Y, \mathcal{C}, ν) and an isometric bijection from $L^p(Y, \nu)$ to the range of e. Moreover, if $L^p(X, \mu)$ is separable, then ν can be chosen to be σ -finite.

Proof. This is part of Theorem 3 in Section 17 of [13].

Proposition 2.9. Let A be a unital L^p operator algebra in which ||1|| = 1. Then A has an isometric unital representation on an L^p space. If A is separable then the L^p space can be chosen to be separable and to be the L^p space of a σ -finite measure space.

Proof. Let (X, \mathcal{B}, μ) be a measure space such that there is an isometric representation $\rho \colon A \to L(L^p(X,\mu))$. Then $e = \rho(1)$ is an idempotent in $L(L^p(X,\mu))$ with $\|e\| = 1$. Set $E = \operatorname{ran}(e)$. Then ρ induces an isometric unital homomorphism $\rho_0 \colon A \to L(E)$ by Corollary 2.4. By Proposition 2.8, there is a measure space (Y, \mathcal{C}, ν) such that E is isometrically isomorphic to $L^p(Y, \nu)$. The first part of the conclusion follows. For the second part, if A is separable then we may require that $L^p(X, \mu)$ be separable by Proposition 2.6. Then E must be separable. Proposition 2.8 implies that E is isometrically isomorphic to the L^p space of a σ -finite measure space.

3. Matrix normed algebras and matricial L^p operator algebras

We will mostly work with ordinary L^p operator algebras, but for some results we will need the matrix normed version introduced here (Definition 3.18). We also need the analogs of Proposition 2.6 and Proposition 2.9 for matricial L^p operator algebras; see Proposition 3.19 and Proposition 3.20.

Matrix normed spaces (operator spaces of various kinds) are well known, but we have not seen a general definition of a matrix normed algebra. We therefore give one here (Definition 3.2). The conditions on the matrix norms seem to be the minimal "reasonable" conditions. Condition (1) essentially says that submatrices have smaller norm. We first describe our (fairly standard) notation for matrices.

Notation 3.1. Let $n \in \mathbb{Z}_{>0}$. Then M_n denotes the algebra of $n \times n$ complex matrices (without any specific norm being assumed). For $j, k \in \{1, 2, ..., n\}$, we let $e_{j,k}$ denote the corresponding standard matrix unit of M_n . For any complex algebra A,

we identify the algebra
$$M_n(A)$$
 with $M_n \otimes A$ via $(a_{j,k})_{1 \leq j,k \leq n} \mapsto \sum_{j,k=1}^n e_{j,k} \otimes a_{j,k}$.

For $x \in M_n$ and $a \in M_n(A)$, the products xa and ax are defined in the obvious way, so that $x(y \otimes b) = xy \otimes b$ and $(y \otimes b)x = yx \otimes b$ for $y \in M_n$ and $b \in A$.

Definition 3.2. A matrix normed algebra is a complex algebra A equipped with algebra norms $\|\cdot\|_n$ on $M_n(A)$ for all $n \in \mathbb{Z}_{>0}$, satisfying the following:

(1) For any $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$, any injective functions

$$\sigma,\tau\colon\{1,2,\ldots,m\}\to\{1,2,\ldots,n\},$$

and any $a = (a_{i,k})_{1 \leq i,k \leq n} \in M_n(A)$, we have

$$\|(a_{\sigma(j),\tau(k)})_{1 \le j,k \le m}\|_m \le \|a\|_n.$$

(2) For any $n \in \mathbb{Z}_{>0}$, any $a \in M_n(A)$, and any $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, if we set $s = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in M_n$, then

$$||as||_n$$
, $||sa||_n \le \max(|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|) ||a||_n$.

(3) For any $m, n \in \mathbb{Z}_{>0}$, $a \in M_m(A)$, and $b \in M_n(A)$, we have $\|\operatorname{diag}(a,b)\|_{m+n} = \max(\|a\|_m, \|b\|_n)$.

We abbreviate $\|\cdot\|_1$ to $\|\cdot\|$. If A is complete in $\|\cdot\|$, we call A a matrix normed Banach algebra.

Remark 3.3. In Definition 3.2, if A is unital and ||1|| = 1, or even if A has an approximate identity which is bounded by 1, condition (2) follows from condition (3) and submultiplicativity of $||\cdot||_n$.

Remark 3.4. In Definition 3.2, the inequality $\|\operatorname{diag}(a,b)\|_{m+n} \ge \max(\|a\|_m, \|b\|_n)$ in condition (3) follows from condition (1).

Lemma 3.5. Let A be a matrix normed algebra, let $n \in \mathbb{Z}_{>0}$, and let $a = (a_{j,k})_{1 \leq j,k \leq n} \in M_n(A)$. Then

$$\max_{1 \le j,k \le n} \|a_{j,k}\| \le \|a\|_n \le \sum_{j,k=1}^n \|a_{j,k}\|.$$

Proof. The first inequality follows from Definition 3.2(1). We prove the second inequality. First, two applications of condition (1), taking σ and τ there to be permutations, show that permuting the rows and also permuting the columns of a matrix does not change its norm. Using this fact at the second step and condition (3) at the first step, we get

$$||a_{j,k}|| = ||\operatorname{diag}(a_{j,k}, 0, \dots, 0)||_n = ||e_{j,j}ae_{k,k}||_n.$$

Apply this to the relation $a = \sum_{j,k=1}^{m} e_{j,j} a e_{k,k}$ to complete the proof.

Corollary 3.6. Let A be a matrix normed Banach algebra. Then $M_n(A)$ is complete for all $n \in \mathbb{Z}_{>0}$.

Proof. This is immediate from Lemma 3.5.

For clarity, we state the standard definitions related to completely bounded maps.

Definition 3.7. Let A and B be matrix normed algebras, and let $\varphi \colon A \to B$ be a linear map. For $n \in \mathbb{Z}_{>0}$, write $\varphi^{(n)}$ or $\mathrm{id}_{M_n} \otimes \varphi$ for the map $M_n(A) \to M_n(B)$ determined by $(a_{j,k})_{1 \leq j,k \leq n} \mapsto (\varphi(a_{j,k}))_{1 \leq j,k \leq n}$. Then:

- (1) We set $\|\varphi\|_{cb} = \sup_{n \in \mathbb{Z}_{>0}} \|\varphi^{(n)}\|$. If $\|\varphi\|_{cb} < \infty$, we say that φ is completely bounded.
- (2) We say that φ is completely contractive if $\|\varphi\|_{cb} \leq 1$.
- (3) We say that φ is *completely isometric* if $\varphi^{(n)}$ is isometric (not necessarily surjective) for all $n \in \mathbb{Z}_{>0}$.
- (4) We say that φ is a completely isometric isomorphism if φ is completely isometric and bijective.

Definition 3.8. Let A be a matrix normed algebra.

- (1) Let B be a subalgebra of A. For $n \in \mathbb{Z}_{>0}$, we define the norm $\|\cdot\|_n$ on $M_n(B)$ to be the restriction to $M_n(B)$ of the given norm on $M_n(A)$.
- (2) Let $J \subset A$ be a closed ideal. For $n \in \mathbb{Z}_{>0}$, we define the norm $\|\cdot\|_n$ on $M_n(A/J)$ to be the quotient norm coming from the obvious identification of $M_n(A/J)$ with $M_n(A)/M_n(J)$.

Lemma 3.9. Let A be a matrix normed algebra.

- (1) Let $B \subset A$ be a subalgebra. Then the norms in Definition 3.8(1) make B a matrix normed algebra, and the inclusion map is completely isometric.
- (2) Let $J \subset A$ be a closed ideal. Then the norms in Definition 3.8(2) make A/J a matrix normed algebra, and the quotient map is completely contractive.

Proof. Part (1) is immediate.

We prove part (2). Let $\pi: A \to A/J$ be the quotient map. Complete contractivity of π is immediate. For Definition 3.2(1), let $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$, let

$$\sigma, \tau \colon \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$$

be injective functions, and let $x = (x_{j,k})_{1 \le j,k \le n} \in M_n(A/J)$. Let $\varepsilon > 0$. Choose $a = (a_{j,k})_{1 \le j,k \le n} \in M_n(A)$ such that $\pi^{(n)}(a) = x$ and $||a||_n < ||x||_n + \varepsilon$. Since $\pi^{(m)}$ is contractive, we have

$$\|(x_{\sigma(j),\tau(k)})_{1 \le j,k \le m}\|_{m} \le \|(a_{\sigma(j),\tau(k)})_{1 \le j,k \le m}\|_{m} \le \|a\|_{n} < \|x\|_{n} + \varepsilon.$$

The proofs of Definition 3.2(2) and the inequality

$$\|\operatorname{diag}(a,b)\|_{m+n} \le \max(\|a\|_m, \|b\|_n)$$

in Definition 3.2(3) are similar. Equality in Definition 3.2(3) now follows from Remark 3.4. $\hfill\Box$

Definition 3.10. Let $n \in \mathbb{Z}_{>0}$. A matrix $s \in M_n$ is a permutation matrix if there exists a bijection $\sigma \colon \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ such that $s = \sum_{j=1}^n e_{\sigma(j), j}$. The matrix s is a complex permutation matrix if there exist a bijection $\sigma \colon \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in S^1 = \{z \in \mathbb{C} \colon |z| = 1\}$ such that $s = \sum_{j=1}^n \lambda_j e_{\sigma(j), j}$.

The complex permutation matrices form a group.

Lemma 3.11. Let A be a matrix normed algebra and fix $n \in \mathbb{Z}_{>0}$. Let $a \in M_n(A)$, and let $s \in M_n$ be a complex permutation matrix. Interpret as and sa as in Notation 3.1. Then $||as||_n = ||sa||_n = ||a||_n$.

Proof. Since s^{-1} is also a complex permutation matrix, it suffices to prove that $||as||_n \leq ||a||_n$ and $||sa||_n \leq ||a||_n$. Since a complex permutation matrix is a product of a permutation matrix and a diagonal matrix with diagonal entries in S^1 , it suffices to prove these inequalities for these two kinds of matrices separately. For the first kind, apply Definition 3.2(1). For the second kind, apply Definition 3.2(2).

Definition 3.12. Let $m, n \in \mathbb{Z}_{>0}$ and let

$$\sigma: \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \to \{1, 2, \dots, mn\}$$

be a bijection. We let $\theta_{\sigma} \colon M_m \otimes M_n \to M_{mn}$ be the unique algebra isomorphism such that for $i, j \in \{1, 2, ..., m\}$ and $k, l \in \{1, 2, ..., n\}$, we have $\theta_{\sigma}(e_{i,j} \otimes e_{k,l}) = e_{\sigma(i,k), \sigma(j,l)}$.

The standard choice of bijection is the one given by $\sigma(j,l) = j + m(l-1)$ for j = 1, 2, ..., m and l = 1, 2, ..., n.

Definition 3.13. Let A be a matrix normed algebra and let $m \in \mathbb{Z}_{>0}$. We define matrix norms on $M_m(A)$ as follows. For $n \in \mathbb{Z}_{>0}$, choose some bijection

$$\sigma_n: \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \to \{1, 2, \dots, nm\},\$$

and use it (and Notation 3.1) to get the isomorphism $\theta_{\sigma_n} \otimes id_A : M_n(M_m(A)) \to M_{nm}(A)$. For $a \in M_n(M_m(A))$, we then use the matrix norms on A to define

$$||a||_n = ||(\theta_{\sigma_n} \otimes \mathrm{id}_A)(a)||_{nm}.$$

Lemma 3.14. In Definition 3.13, the matrix norms are independent of the choice of $(\sigma_n)_{n\in\mathbb{Z}_{>0}}$, and make $M_m(A)$ a matrix normed algebra.

Proof. Independence of $(\sigma_n)_{n \in \mathbb{Z}_{>0}}$ follows from Lemma 3.11, and the fact that one gets a matrix normed algebra follows easily from Definition 3.2.

Definition 3.15. Let (X, \mathcal{B}, μ) be a measure space, and let $A \subset L(L^p(X, \mu))$ be a closed subalgebra. We equip A with the matrix norms coming from the identification of $M_n(A)$ with a closed subalgebra of $L(L^p(\{1, 2, ..., n\} \times X, \nu \times \mu))$, in which ν is counting measure on $\{1, 2, ..., n\}$.

Notation 3.16. For any set S and any $p \in [1, \infty]$, we give $l^p(S)$ the usual meaning (using counting measure on S), and we set (as usual) $l^p = l^p(\mathbb{Z}_{>0})$. For $d \in \mathbb{Z}_{>0}$ we let $l^p_d = l^p(\{1, 2, \ldots, d\})$. We further let $M^p_d = L(l^p_d)$ with the usual operator norm, and we algebraically identify M^p_d with M_d in the standard way. For $a \in M^p_d$, we write the norm as $\|a\|_p$. We equip M^p_d with the matrix norms as in Definition 3.15.

Lemma 3.17. Let (X, \mathcal{B}, μ) be a measure space, and let $A \subset L(L^p(X, \mu))$ be a closed subalgebra. Then A is a matrix normed algebra with the matrix norms of Definition 3.15. Moreover for $n \in \mathbb{Z}_{>0}$, $a \in M_n(A)$, and $x \in M_n$, with products as in Notation 3.1, we have $||ax||_n$, $||xa||_n \le ||x||_p ||a||_n$. Furthermore, for $m \in \mathbb{Z}_{>0}$ the matrix norms on $M_m(A)$ from Definition 3.13 agree with those gotten from the obvious inclusion

$$M_m(A) \to L(L^p(\{1, 2, \dots, m\} \times X, \nu \times \mu)),$$

in which ν is counting measure on $\{1, 2, ..., m\}$.

Proof. All parts are easy.

Definition 3.18. Let $p \in [1, \infty)$. A matricial L^p operator algebra is a matrix normed Banach algebra A such that there exists a measure space (X, \mathcal{B}, μ) and a completely isometric isomorphism from A to a norm closed subalgebra of $L(L^p(X, \mu))$.

Using the terminology from Definition 2.2, we say that a matricial L^p operator algebra A is separably representable if it has a separable completely isometric representation. We say that A is σ -finitely representable if it has a σ -finite completely isometric representation. We say that A is nondegenerately (separably) representable if it has a nondegenerate (separable) completely isometric representation, and nondegenerately σ -finitely representable if it has a nondegenerate σ -finite completely isometric representation.

Proposition 3.19. Let $p \in [1, \infty)$, and let A be a separable matricial L^p operator algebra. Then A is separably representable. If A is nondegenerately representable, then A is separably nondegenerately representable.

Proof. For $n \in \mathbb{Z}_{>0}$ let ν_n be counting measure on $\{1, 2, ..., n\}$. Let S be a countable dense subset of A, and for $n \in \mathbb{Z}_{>0}$ define

$$S_n = \{b \in M_n(A) : b_{j,k} \in S \text{ for } j, k = 1, 2, \dots, n\},\$$

which is a countable dense subset of $M_n(A)$.

By hypothesis, there exist a measure space (X, \mathcal{B}, μ) and a completely isometric representation $\rho: A \to L(L^p(X, \mu))$, which we can take to be nondegenerate when A is nondegenerately representable. For any $m, n \in \mathbb{Z}_{>0}$ and $b \in S_n$, choose

$$\xi_{n,b,m} = (\xi_{n,b,m}^{(j)})_{1 \le j \le n} \in L^p(\{1, 2, \dots, n\} \times X, \nu_n \times \mu)$$

such that

$$\|\xi_{n,b,m}\|_p = 1$$
 and $\|(\mathrm{id}_{M_n} \otimes \rho)(b)\xi_{n,b,m}\| > \|b\| - \frac{1}{m}$.

By the argument used in the proof of Proposition 1.25 of [19] there exists a separable closed sublattice $F_{n,b,m}$ of $L^p(X,\mu)$ containing $\xi_{n,b,m}^{(1)}$, $\xi_{n,b,m}^{(2)}$, ..., $\xi_{n,b,m}^{(2)}$ and such that $\rho(A)F_{n,b,m} \subset F_{n,b,m}$. Moreover, $F_{n,b,m}$ is isomorphic to $L^p(Y_{n,b,m},\nu_{n,b,m})$ for some measure space $(Y_{n,b,m},\nu_{n,b,m})$. Furthermore, if ρ is nondegenerate then $F_{n,b,m}$ can be chosen to satisfy $\overline{\text{span}}(\rho(A)F_{n,b,m}) = F_{n,b,m}$. The map defined by $\pi_{n,b,m}(a) = \rho(a)|_{F_{n,b,m}}$ is a completely contractive representation of A on a separable L^p -space, which is nondegenerate if ρ is nondegenerate. Since $F_{n,b,m}$ contains $\xi_{n,b,m}^{(1)}$, $\xi_{n,b,m}^{(2)}$, ..., $\xi_{n,b,m}^{(2)}$, we get $\|(\mathrm{id}_{M_n}\otimes\pi_{n,b,m})(b)\| > \|b\| - \frac{1}{m}$ for every $m \in \mathbb{Z}_{>0}$. Now let π be the L^p direct sum of the representations $\pi_{n,b,m}$ for $m,n \in \mathbb{Z}_{>0}$ and $b \in S_n$, as in Definition 1.23 of [19]. Then π is a completely contractive representation on a separable L^p space. We have $\|(\mathrm{id}_{M_n}\otimes\pi)(b)\| = \|b\|$ for all $n \in \mathbb{Z}_{>0}$ and $b \in S_n$, so density of S_n in $M_n(A)$ implies that π is completely isometric. Moreover, by Lemma 1.24 in [19], π is nondegenerate if ρ is nondegenerate.

Proposition 3.20. Let $p \in [1, \infty)$, and let A be a unital matricial L^p operator algebra in which ||1|| = 1. Then A has a completely isometric unital representation on an L^p space. If A is separable then the L^p space can be chosen to be separable and to come from a σ -finite measure space.

Proof. Let (X, \mathcal{B}, μ) be a measure space such that there is completely isometric representation $\rho_0 \colon A \to L(L^p(X, \mu))$. Then $e = \rho_0(1)$ is an idempotent in $L(L^p(X, \mu))$, and $\|e\| = 1$. By Proposition 2.8 there exists a measure space (Y, \mathcal{C}, ν) such that ran(e) is isometrically isomorphic to $L^p(Y, \nu)$. Thus, ρ_0 gives a completely isometric unital homomorphism $\rho \colon A \to L(L^p(Y, \nu))$. Moreover, if A is separable, then $L^p(X, \mu)$ can be chosen to be separable, which implies that ran(e) is also separable. To get σ -finiteness, use Lemma 2.7.

4. Unique matrix norms

We consider uniqueness of matrix norms on L^p operator algebras. Most of the L^p operator algebras we deal with will have unique L^p operator matrix norms, in the sense of Definition 4.1 below. The basic examples are M_d^p and C(X). We will show in Corollary 9.12 below that all spatial L^p AF algebras have unique L^p operator matrix norms. The proof that C(X) has unique L^p operator matrix norms uses a structure theorem (Theorem 4.5) for contractive unital representations of C(X) on L^p spaces, which also plays a key role later. To avoid technical issues, we restrict our discussion to the separable case.

Definition 4.1. Let $p \in [1, \infty) \setminus \{2\}$. Let A be a separable L^p operator algebra. We say that A has unique L^p operator matrix norms if whenever (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) are σ -finite measure spaces such that $L^p(X, \mu)$ and $L^p(Y, \nu)$ are separable, $\pi \colon A \to L(L^p(X, \mu))$ and $\sigma \colon A \to L(L^p(Y, \nu))$ are isometric representations, and $\pi(A)$ and $\sigma(A)$ are given the matrix normed structures of Definition 3.15, then $\sigma \circ \pi^{-1} \colon \pi(A) \to \sigma(A)$ is completely isometric.

When A is unital and ||1|| = 1, in Definition 4.1 we need only consider unital isometric representations.

Lemma 4.2. Let $p \in [1, \infty) \setminus \{2\}$. Let A be a unital separable L^p operator algebra in which ||1|| = 1. Assume that whenever (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) are σ -finite measure spaces such that $L^p(X, \mu)$ and $L^p(Y, \nu)$ are separable, $\pi \colon A \to L(L^p(X, \mu))$ and $\sigma \colon A \to L(L^p(Y, \nu))$ are unital isometric representations, and $\pi(A)$ and $\sigma(A)$ are given the matrix normed structures of Definition 3.15, then $\sigma \circ \pi^{-1} \colon \pi(A) \to \sigma(A)$ is completely isometric. It follows that A has unique L^p operator matrix norms.

Proof. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces such that $L^p(X, \mu)$ and $L^p(Y, \nu)$ are separable, and let $\pi \colon A \to L(L^p(X, \mu))$ and $\sigma \colon A \to L(L^p(Y, \nu))$ be isometric representations.

The operator $e = \pi(1)$ is an idempotent in $L(L^p(X,\mu))$ with ||e|| = 1. Set $E = \operatorname{ran}(e)$. Then π induces a unital homomorphism $\pi_0 \colon A \to L(E)$, which is isometric by Corollary 2.4. By Proposition 2.8, there is a measure space $(X_0, \mathcal{B}_0, \mu_0)$ such that E is isometrically isomorphic to $L^p(X_0, \mu_0)$. Since E is separable, we may require that μ_0 be σ -finite. Similarly, $\operatorname{ran}(\sigma(1))$ is isometrically isomorphic to a separable L^p space $L^p(Y_0, \nu_0)$ in which ν_0 is σ -finite, and σ induces an isometric unital homomorphism $\sigma_0 \colon A \to L(L^p(Y_0, \nu_0))$. In particular, $\sigma_0 \circ \pi_0^{-1} \colon \pi_0(A) \to \sigma_0(A)$ is isometric.

Let $n \in \mathbb{Z}_{>0}$. We take the norms on $M_n(\pi(A))$, $M_n(\pi_0(A))$, $M_n(\sigma(A))$, and $M_n(\sigma_0(A))$ to be as in Definition 3.15. Define

$$\rho_0 = \sigma_0 \circ \pi_0^{-1} \colon \pi_0(A) \to \sigma_0(A) \subset L(L^p(Y_0, \nu_0))$$

and

$$\rho = \sigma \circ \pi_0^{-1} \colon \pi_0(A) \to \sigma(A) \subset L(L^p(Y, \nu)).$$

Then

$$(\mathrm{id}_{M_n}\otimes\rho)(1)=1_{M_n}\otimes\sigma(1)\in L\big(l_n^p\otimes_p L^p(Y,\nu)\big),$$

so $\|(\mathrm{id}_{M_n}\otimes\rho)(1)\|=1$. The hypothesis implies that $\mathrm{id}_{M_n}\otimes\rho_0$ is isometric. So Corollary 2.4 implies that $\mathrm{id}_{M_n}\otimes\rho$ is isometric.

Similarly, the map $\mathrm{id}_{M_n}\otimes(\pi\circ\pi_0^{-1})\colon M_n(\pi_0(A))\to M_n(\pi(A))$ is isometric. Therefore $\mathrm{id}_{M_n}\otimes(\sigma\circ\pi^{-1})\colon M_n(\pi(A))\to M_n(\sigma(A))$ is isometric. This completes the proof.

Proposition 4.3. Let $p \in [1, \infty) \setminus \{2\}$. Let M_d^p be as in Notation 3.16. Then every nonzero contractive unital representation of M_d^p on a separable L^p space is completely isometric.

Proof. Let (X, \mathcal{B}, μ) be a measure space such that $L^p(X, \mu)$ is separable, and let $\rho \colon M^p_d \to L(L^p(X, \mu))$ be a contractive unital representation. By Lemma 2.7, we can assume that (X, \mathcal{B}, μ) is σ -finite. Theorem 7.2 of [16] provides a σ -finite measure space $(Z, \mathcal{C}, \lambda)$ and a bijective isometry

$$u: l_d^p \otimes L^p(Z, \lambda) \to L^p(X, \mu)$$

such that for all $a \in M_d^p$ we have $\rho(a) = u(a \otimes 1)u^{-1}$. For $n \in \mathbb{Z}_{>0}$, it is easy to see that

$$1_{M_n} \otimes u \colon l_n^p \otimes l_d^p \otimes L^p(Z,\lambda) \to l_n^p \otimes L^p(X,\mu)$$

is a bijective isometry such that

$$(1_{M_n} \otimes \rho)(b) = (1_{M_n} \otimes u)(b \otimes 1)(1_{M_n} \otimes u)^{-1}$$

for all $b \in M_n(M_d^p)$. It is now immediate that ρ_n is isometric.

Corollary 4.4. Let $p \in [1, \infty) \setminus \{2\}$. The algebra M_d^p of Notation 3.16 has unique L^p operator matrix norms.

Proof. Combine Lemma 4.2 and Proposition 4.3.

Next, we give a structure theorem for any contractive unital representation of C(X) on an L^p space.

Theorem 4.5. Let $p \in [1, \infty) \setminus \{2\}$. Let X be a compact metrizable space, let (Y, \mathcal{C}, ν) be a σ -finite measure space, and let $\pi \colon C(X) \to L(L^p(Y, \nu))$ be a contractive unital homomorphism. Let $\mu \colon L^\infty(Y, \nu) \to L(L^p(Y, \nu))$ be the representation of $L^\infty(Y, \nu)$ on $L^p(Y, \nu)$ by multiplication operators. Then there exists a unital C*-algebra homomorphism $\varphi \colon C(X) \to L^\infty(Y, \nu)$ such that $\pi = \mu \circ \varphi$.

Proof. We claim that the range of π is contained in the range of μ . It suffices to prove that if $f \in C(X)$ is real valued and satisfies $\|f\| < \pi$, then $\pi(f)$ is in the range of μ . Let f be such a function. For $\lambda \in \mathbb{R}$, the function $w_{\lambda} = \exp(i\lambda f)$ is invertible in C(X) and satisfies $\|w_{\lambda}\| = \|w_{\lambda}^{-1}\| = 1$. Therefore $\pi(w_{\lambda})$ is a bijective isometry in $L(L^{p}(Y,\nu))$. By Lemma 6.16 of [16], the operator $\pi(w_{\lambda})$ is a spatial isometry in the sense of Definition 6.4 of [16]. In particular, it has a spatial system $(E_{\lambda}, F_{\lambda}, S_{\lambda}, g_{\lambda})$ as there. By Lemma 6.22 of [16], we have $S_{\lambda} = S_{0}$ for all $\lambda \in \mathbb{R}$. Now $\pi(w_{0}) = 1$, so, in the notation of Definition 6.3 of [16] and Definition 5.4 of [16], the operator $\pi(w_{0})$ has the spatial system $(Y, Y, \mathrm{id}_{C/\mathcal{N}(\nu)}, 1)$. The uniqueness statement in Lemma 6.6 of [16] now implies that $S_{\lambda} = \mathrm{id}_{C/\mathcal{N}(\nu)}$ for all $\lambda \in \mathbb{R}$. Therefore $\pi(w_{\lambda})$ is a multiplication operator; in fact, $\pi(w_{\lambda}) = \mu(g_{\lambda})$ for all $\lambda \in \mathbb{R}$. Now let log be the holomorphic branch which is real on $(0, \infty)$ and defined on $\mathbb{C} \setminus (-\infty, 0]$. We have

$$\operatorname{sp}(g_1) = \operatorname{sp}(\pi(w_1)) \subset \operatorname{sp}(w_1) \subset \mathbb{C} \setminus (-\infty, 0]$$

and

$$\pi(f) = \pi(-i\log(w_1)) = -i\log(\pi(w_1)) = -i\log(\mu(q_1)) = \mu(-i\log(q_1)).$$

Thus $\pi(f)$ is in the range of μ , as claimed.

It follows that there is a contractive homomorphism $\varphi \colon C(X) \to L^{\infty}(Y, \nu)$ such that $\pi = \mu \circ \varphi$. Obviously φ is unital. It follows from Proposition A.5.8 of [2] that φ is a C*-algebra homomorphism.

We don't need the following proposition, but it is an interesting result which follows from the machinery we have developed.

Proposition 4.6. Let X be a compact metrizable space, and let $p \in [1, \infty) \setminus \{2\}$. Then C(X) has unique L^p operator matrix norms. They are given as follows. Let $a \in M_n(C(X))$. Interpret a as a continuous function $a: X \to M_n$. Equip $M_n = M_n^p$ with the norm $\|\cdot\|_p$ from Notation 3.16. Then $\|a\|_n = \sup_{x \in X} \|a(x)\|_p$.

Proof. Choose a σ -finite Borel measure μ on X such that $\mu(U) > 0$ for every nonempty open set $U \subset X$. Represent C(X) on $L^p(X,\mu)$ as multiplication operators. It is then easy to check that the matrix norms from Definition 3.15 are equal to the matrix norms in the statement of the proposition.

In view of Lemma 4.2, it remains to show that if (Y,\mathcal{C},ν) is a σ -finite measure space such that $L(L^p(Y,\nu))$ is separable, and $\pi\colon C(X)\to L(L^p(Y,\nu))$ is an isometric unital homomorphism, then π is completely isometric. Let $\rho\colon L^\infty(Y,\nu)\to L(L^p(Y,\nu))$ be the representation given by multiplication operators. Then ρ is isometric, so we can identify $L^\infty(Y,\nu)$ with its image under ρ , and thus make $L^\infty(Y,\nu)$ a matricial L^p operator algebra using the matrix norms of Definition 3.15. For $n\in\mathbb{Z}_{>0}$, identify $M_n(L^\infty(Y,\nu))$ with the algebra of L^∞ functions from Y to M_n . It is easy to check that the norms on $M_n(L^\infty(Y,\nu))$ are given by $\|a\|_n = \text{ess sup } \|a(y)\|_p$.

Now let Z be the maximal ideal space of $L^{\infty}(Y,\nu)$, and let $\gamma \colon L^{\infty}(Y,\nu) \to C(Z)$ be the Gelfand transform, which is an isomorphism. Define matrix norms on C(Z) in the same way as on C(X) in the statement of the proposition. For every $f \in L^{\infty}(Y,\nu)$, the essential range of f is the range of f. It follows that for every $f \in M_n(L^{\infty}(Y,\nu))$, the essential range of f is equal to the range of f is completely isometric.

Apply Theorem 4.5 to π . We get a unital C*-algebra homomorphism $\varphi \colon C(X) \to L^{\infty}(Y,\nu)$ such that $\pi = \rho \circ \varphi$. Moreover, φ is injective. There is a continuous function $h \colon Z \to X$ such that $(\gamma \circ \varphi)(f) = f \circ h$ for all $f \in C(X)$. Injectivity of $\gamma \circ \varphi$ implies surjectivity of h. It is now immediate that $\gamma \circ \varphi$ is completely isometric. Since γ and ρ are completely isometric, we conclude that π is completely isometric.

5. Direct sums

In this section we show that direct sum of a family of (matricial) L^p operator algebras is also a (matricial) L^p operator algebra.

Definition 5.1. If $((X_i, \mathcal{B}_i, \mu_i))_{i \in I}$ is a family of measure spaces, then the measure space $(X, \mathcal{B}, \mu) = \coprod_{i \in I} (X_i, \mathcal{B}_i, \mu_i)$ is determined by taking $X = \coprod_{i \in I} X_i$,

$$\mathcal{B} = \{ E \subset X : E \cap X_i \in \mathcal{B}_i \text{ for all } i \in I \},$$

and
$$\mu(E) = \sum_{i \in I} \mu_i(E \cap X_i)$$
 for $E \in \mathcal{B}$.

Definition 5.2. Whenever $N \in \mathbb{Z}_{>0}$ and A_1, A_2, \ldots, A_N are Banach algebras, we make $\bigoplus_{k=1}^{N} A_k$ a Banach algebra by giving it the obvious algebra structure and the norm

$$\|(a_1, a_2, \dots, a_N)\| = \max(\|a_1\|, \|a_2\|, \dots, \|a_N\|)$$

for $a_1 \in A_1, a_2 \in A_2, \ldots, a_N \in A_N$. If A_1, A_2, \ldots, A_N are matrix normed Banach algebras, we define matrix norms on $\bigoplus_{k=1}^N A_k$ by

$$||(a_1, a_2, \dots, a_N)||_n = \max(||a_1||_n, ||a_2||_n, \dots, ||a_N||_n)$$

for $n \in \mathbb{Z}_{>0}$ and $a_1 \in M_n(A_1), a_2 \in M_n(A_2), \ldots, a_N \in M_n(A_N)$.

Lemma 5.3. Let $N \in \mathbb{Z}_{>0}$. Let A_1, A_2, \ldots, A_N be matrix normed Banach algebras. Then $\bigoplus_{k=1}^{N} A_k$, as in Definition 5.2, is a matrix normed Banach algebra.

Proof. The proof is easy, and is omitted.

Lemma 5.4. Let the notation be as in Definition 5.2. Let B be a Banach algebra, and for $k=1,2,\ldots,N$ let $\varphi_k\colon B\to A_k$ be a homomorphism. Define $\varphi\colon B\to \bigoplus_{k=1}^N A_k$ by $\varphi(b)=\left(\varphi_1(b),\varphi_2(b),\ldots,\varphi_N(b)\right)$ for $b\in B$. Then φ is contractive if and only if φ_k is contractive for $k=1,2,\ldots,N$. If A_1,A_2,\ldots,A_N are matrix normed Banach algebras, then φ is completely contractive if and only if φ_k is completely contractive for $k=1,2,\ldots,N$.

Proof. The proof is immediate.

Lemma 5.5. Let the notation be as in Definition 5.2. Let $S \subset \{1, 2, ..., N\}$. Then $\bigoplus_{k \in S} A_k$ is an ideal in $\bigoplus_{k=1}^N A_k$, and the obvious map

$$\bigoplus_{k=1}^{N} A_k / \bigoplus_{k \in S} A_k \to \bigoplus_{k \notin S} A_k$$

is completely isometric when the quotient is given the matrix norms of Definition 3.8(2).

Proof. The proof is easy, and is omitted.

Lemma 5.6. Let A be a matrix normed Banach algebra. Let $n \in \mathbb{Z}_{>0}$, and let $\varphi \colon \bigoplus_{k=1}^n A \to M_n(A)$ be the map $\varphi(a_1, a_2, \dots, a_n) = \operatorname{diag}(a_1, a_2, \dots, a_n)$ for $a_1, a_2, \dots, a_n \in A$. Then φ is completely isometric.

Proof. Let $r \in \mathbb{Z}_{>0}$. Let σ be the standard bijection of Definition 3.12, with r in place of n, and let θ_{σ} be as there. For $a \in \bigoplus_{k=1}^{n} A$ the matrix $[(\theta_{\sigma} \otimes id_{A}) \circ (id_{M_{r}} \otimes id_{A})]$ is block diagonal. So iteration of condition (3) in Definition 3.2 shows that

 φ)](a) is block diagonal. So iteration of condition (3) in Definition 3.2 shows that $(\theta_{\sigma} \otimes \mathrm{id}_{A}) \circ (\mathrm{id}_{M_{r}} \otimes \varphi)$ is isometric. Lemma 3.14 implies that $\theta_{\sigma} \otimes \mathrm{id}_{A}$ is isometric. So $\mathrm{id}_{M_{r}} \otimes \varphi$ is isometric.

Lemma 5.7. Let $p \in [1, \infty)$. In Definition 5.2, if A_1, A_2, \ldots, A_N are L^p operator algebras, then so is $A = \bigoplus_{k=1}^N A_k$. If A_1, A_2, \ldots, A_N are matricial L^p operator algebras, then A is a matricial L^p operator algebra.

Proof. We give the proof for L^p operator algebras; the matricial case is essentially the same. Suppose that $\rho_k \colon A_k \to L(L^p(X_k, \mu_k))$ is an isometric representation for $k = 1, 2, \ldots, N$. Let $X = \coprod_{k=1}^N X_k$ and μ be as in Definition 5.1. Then $L^p(X, \mu)$ is

the
$$L^p$$
 direct sum $\bigoplus_{k=1}^N L^p(X_k, \mu_k)$. Define $\rho \colon \bigoplus_{k=1}^N A_k \to L(L^p(X, \mu))$ by

$$\rho(a_1, a_2, \dots, a_N) = \rho_1(a_1) \oplus \rho_2(a_2) \oplus \dots \oplus \rho_N(a_N)$$

for $a_1 \in A_1, \ a_2 \in A_2, \ \dots, \ a_N \in A_N$. Clearly ρ is an isometric representation of $\bigoplus_{k=1}^N A_k$.

6. Hermitian idempotents

The right kind of idempotent to consider in an L^p operator algebra for $p \neq 2$ is what might be called a "spatial idempotent", that is, one which is a spatial partial isometry in the sense of Definition 6.4 of [16]. We develop some of the basic theory in this section. Such idempotents can be characterized as those which are hermitian in the sense of Definition 6.4 below, a much older notion (see [20]). Although we will not need the general theory of hermitian elements of a Banach algebra, it seems appropriate to make the connection with the older concept.

We formalize the following terminology for idempotents.

Definition 6.1 (Definition 4.1.1 of [1]). Let A be a ring (not necessarily unital), and let $e, f \in A$ be idempotents. We say that f dominates e, written $f \geq e$ or $e \leq f$, if fe = ef = e. We say that e and f are orthogonal if ef = fe = 0.

Even if A is a C*-algebra, the notation $e \leq f$ need not agree with the usual C*-algebraic order. Among other things, e and f need not be selfadjoint. If e and f happen to be projections in a C*-algebra, then our notation does agree with the usual C*-algebraic order. Orthogonality need not be the same as the version of orthogonality for C*-algebras implicit in the remark after Definition 4.1.1 of [1].

Definition 6.2 (Definition 2.6.1 of [15]). Let A be a unital Banach algebra in which ||1|| = 1. Let $a \in A$. Then the numerical range W(a) is the set of all numbers $\omega(a) \in \mathbb{C}$ for linear functionals ω on A such that $||\omega|| = \omega(1) = 1$.

Theorem 6.3 (Theorem 2.6.7 of [15]). Let A be a unital Banach algebra in which ||1|| = 1, and let $a \in A$. Then the following are equivalent:

- (1) $W(a) \subset \mathbb{R}$.
- (2) $\|\exp(i\lambda a)\| = 1$ for all $\lambda \in \mathbb{R}$.
- (3) $\|\exp(i\lambda a)\| \le 1$ for all $\lambda \in \mathbb{R}$.
- (4) With the limit being taken over $\lambda \in \mathbb{R}$, $\lim_{\lambda \to 0} |\lambda|^{-1} (\|1 i\lambda a\| 1) = 0$.

Proof. The equivalence of conditions (1), (2), and (4) is in Theorem 2.6.7 of [15]. That (2) implies (3) is trivial. That (3) implies (2) follows from ||1|| = 1 and $\exp(i\lambda a)^{-1} = \exp(-i\lambda a)$.

Definition 6.4 (see Definition 2.6.5 of [15] and the preceding discussion). Let A be a unital Banach algebra in which ||1|| = 1, and let $a \in A$. We say that a is hermitian if a satisfies the equivalent conditions of Theorem 6.3. If a is also an idempotent, we call it a hermitian idempotent.

Remark 6.5. Let A be a unital Banach algebra in which ||1|| = 1. Then clearly 0 and 1 are hermitian idempotents. Also, in any unital Banach algebra, if e is a hermitian idempotent, then so is 1 - e. Indeed, if $\lambda \in \mathbb{R}$ then

 $\| \exp(i\lambda(1-e))\| = \|e + \exp(i\lambda)(1-e)\| = \| \exp(i\lambda) \exp(-i\lambda e)\| = \| \exp(-i\lambda e)\| = 1,$ as desired.

A hermitian idempotent in a C*-algebra is simply a projection. (See Proposition 3.3.3 in [4], observing that a nonzero hermitian idempotent has norm 1 by Lemma 6.6 below.)

The following result gives the characterization we use most often.

Lemma 6.6. Let A be a unital Banach algebra in which ||1|| = 1. Let $e \in A$ be an idempotent. Define a homomorphism $\beta_e : \mathbb{C} \oplus \mathbb{C} \to A$ by $\beta_e(\lambda_1, \lambda_2) = \lambda_1 e + \lambda_2 (1 - e)$ for $\lambda_1, \lambda_2 \in \mathbb{C}$. Then e is hermitian in the sense of Definition 6.4 if and only if, when $\mathbb{C} \oplus \mathbb{C}$ is normed as in Definition 5.2, the homomorphism β_e is contractive.

Proof. We use the characterization (3) of Theorem 6.3.

First suppose that β_e is contractive. Then for $\lambda \in \mathbb{R}$ we have

$$\|\exp(i\lambda e)\| = \|\beta((\exp(i\lambda), 1))\| \le \|(\exp(i\lambda), 1)\| = 1.$$

For the converse, suppose e is hermitian, and let $\lambda_1, \lambda_2 \in \mathbb{C}$. We need to prove

(6.1)
$$\|\beta_e((\lambda_1, \lambda_2))\| \le \max(|\lambda_1|, |\lambda_2|).$$

This relation is trivial if $\lambda_1 = \lambda_2 = 0$.

Next, suppose $|\lambda_1| \leq |\lambda_2|$ and $\lambda_2 \neq 0$. Multiplying by λ_2^{-1} , we reduce to the case $\lambda_2 = 1$. Write $\lambda_1 = \rho \exp(i\theta)$ with $\theta \in \mathbb{R}$ and $0 \leq \rho \leq 1$. Define

$$\alpha_1 = \theta + \arccos(\rho)$$
 and $\alpha_2 = \theta - \arccos(\rho)$.

Then one checks that $(\lambda_1, 1) = \frac{1}{2} [(\exp(i\alpha_1), 1) + (\exp(i\alpha_2), 1)].$ So

$$\|\beta_e((\lambda_1, 1))\| = \left\|\frac{1}{2}\left[\exp(i\alpha_1 e) + \exp(i\alpha_2 e)\right]\right\| \le \frac{1}{2}\left(\|\exp(i\alpha_1 e)\| + \|\exp(i\alpha_2 e)\|\right) \le 1,$$

which is (6.1).

Finally, suppose $|\lambda_2| \leq |\lambda_1|$ and $\lambda_1 \neq 0$. Using Remark 6.5, we can apply the case of (6.1) already done to 1-e, with (λ_2, λ_1) in place of (λ_1, λ_2) . This gives (6.1) for e and (λ_1, λ_2) .

Lemma 6.7. Let A and B be unital Banach algebras such that $||1_A|| = 1$ and $||1_B|| = 1$. Let $\varphi \colon A \to B$ be a contractive unital homomorphism, and let $e \in A$ be a hermitian idempotent. Then $\varphi(e) \in B$ is a hermitian idempotent.

Proof. The proof is immediate from Lemma 6.6.

Lemma 6.8. Let $N \in \mathbb{Z}_{>0}$, and let A_1, A_2, \ldots, A_N be unital Banach algebras whose identities have norm one. Set $A = \bigoplus_{k=1}^{N} A_k$, equipped with the norm in Definition 5.2, and for $k = 1, 2, \ldots, N$ let e_k be a hermitian idempotent in A_k . Then (e_1, e_2, \ldots, e_N) is a hermitian idempotent in A.

Proof. The proof is immediate from Lemma 6.6.

We are interested in hermitian idempotents in L^p operator algebras. Given $p \in [1, \infty) \setminus \{2\}$, the following lemma gives a characterization of hermitian idempotents in $L(L^p(X, \mu))$, for a σ -finite measure space (X, \mathcal{B}, μ) .

Lemma 6.9. Let $p \in [1, \infty) \setminus \{2\}$. Let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $e \in L(L^p(X, \mu))$ be an idempotent. Then the following are equivalent:

- (1) e is a hermitian idempotent.
- (2) e is a spatial partial isometry in the sense of Definition 6.4 of [16].
- (3) There is a measurable subset $E \subset X$ such that e is multiplication by χ_E .

Proof. Lemma 6.18 in [16] shows that (2) and (3) are equivalent.

It is obvious that (3) implies (1). For the converse, assume that e is a hermitian idempotent. Thus the homomorphism $\beta_e \colon \mathbb{C} \oplus \mathbb{C} \to L(L^p(X,\mu))$ of Lemma 6.6 is unital and contractive. Set $Y = \{0,1\}$, and define $f \in C(Y)$ by f(0) = 1 and f(1) = 0. Let ρ be the representation of $L^{\infty}(X,\mu)$ on $L^p(X,\mu)$ by multiplication operators. By Theorem 4.5, there exists a unital *-homomorphism $\varphi \colon C(Y) \to L^{\infty}(X,\mu)$ such that $\beta_e = \rho \circ \varphi$. Since $\varphi(f)$ is an idempotent in $L^{\infty}(X,\mu)$, there is a measurable set $E \subset X$ such that $\varphi(f) = \chi_E$.

Corollary 6.10. Let $p \in [1, \infty) \setminus \{2\}$, let $d \in \mathbb{Z}_{>0}$, and let $e \in M_d^p$ be an idempotent. Then e is hermitian if and only if e is a diagonal matrix with entries in $\{0,1\}$.

Proof. Identify $M_d^p = L(l_d^p)$. Then the statement is immediate from the equivalence of (1) and (3) in Lemma 6.9.

We now give several counterexamples. It isn't enough to require that $||e|| \le 1$ and $||1-e|| \le 1$ to get a hermitian idempotent, even if A is a σ -finitely representable unital L^p operator algebra. There is an example in M_2^p .

Lemma 6.11. Let $p \in [1, \infty)$. Define $e \in M_2^p$ by $e = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\|e\|_p = \|1 - e\|_p = 1$, but if $p \neq 2$ then e is not a hermitian idempotent.

Proof. Let $\alpha, \beta \in \mathbb{C}$. Let q > 1 be such that $\frac{1}{p} + \frac{1}{q} = 1$. Applying Hölder's inequality to $(1,1) \in l_2^q$ and $(\alpha,\beta) \in l_2^p$, we get

$$|\alpha + \beta| \le 2^{1-1/p} (|\alpha|^p + |\beta|^p)^{1/p}.$$

Use this inequality at the third step, to get

$$||e(\alpha,\beta)||_p = ||(\frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta))||_p = 2^{1/p} \left| \frac{\alpha+\beta}{2} \right| \le (|\alpha|^p + |\beta|^p)^{1/p} = ||(\alpha,\beta)||_p.$$

Since $\alpha, \beta \in \mathbb{C}$ are arbitrary, this shows that $||e||_p \leq 1$. Obviously, we have $||e||_p \geq 1$ because $e^2 = e$ and $e \neq 0$.

The same argument applies to 1 - e. (Or else take $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and use the fact that s is an invertible isometry with $ses^{-1} = 1 - e$.)

It follows from Corollary 6.10 that e is not hermitian.

One can also explicitly show that e is not hermitian. For example, suppose p < 2. Letting β_e be as in Lemma 6.6, one can explicitly show that $\|\beta_e((1,i))(1,0)\|_p = 2^{1/p-1/2} > 1 = \|(1,0)\|_p$.

We don't define hermitian idempotents in a nonunital Banach algebra, since whether an idempotent is hermitian depends on the norm used on the unitization, even for L^p operator algebras, as is shown by the following example.

Example 6.12. Let $p \in [1, \infty) \setminus \{2\}$. Let X be a second countable locally compact Hausdorff space with a nontrivial compact open set K. Let μ a measure on X with full support. Denote by ψ the representation of $C_0(X)$ on $L^p(X,\mu)$ given by multiplication operators. Let e be the idempotent in Lemma 6.11. Define $\rho \colon C_0(X) \to L(l_2^p \otimes_p L^p(X,\mu))$ by $\rho(f) = e \otimes \psi(f)$. Then ρ is isometric. Take as unitization the subagebra $\rho(C_0(X)) \oplus \mathbb{C}1$ of $L(l_2^p \otimes_p L^p(X,\mu))$. Let $\chi_K \in C_0(X)$ be the characteristic function of K. Then $\psi(\chi_K)$ is a hermitian idempotent in $L(L^p(X,\mu))$, but $\rho(\chi_K)$ is not a hermitian idempotent because e is not a hermitian idempotent.

Lemma 6.13. Let $p \in [1, \infty) \setminus \{2\}$. Let A and B be unital σ -finitely representable L^p operator algebras with $||1_A|| = 1$ and $||1_B|| = 1$, and let $\psi \colon A \to B$ be a contractive homomorphism such that $\psi(1)$ is a hermitian idempotent in B. Let $e \in A$ be a hermitian idempotent. Then $\psi(e)$ is a hermitian idempotent in B.

We don't know to what extent the hypotheses can be weakened. But some hypothesis is necessary. Let $p \in [1, \infty) \setminus \{2\}$. By Lemma 6.11 there is a nonhermitian idempotent $e \in M_2^p$ such that $\|e\|_p = 1$. The homomorphism $\mathbb{C} \to M_2^p$ defined by $\lambda \mapsto \lambda e$ is contractive but sends the hermitian idempotent 1 to the nonhermitian idempotent e.

Proof of Lemma 6.13. We may assume that there is a σ -finite measure space (Y, \mathcal{C}, ν) such that B is a unital subalgebra of $L(L^p(Y, \nu))$. Lemma 6.9 provides a measurable subset $E \subset X$ such that $\psi(1)$ is multiplication by χ_E . By Corollary 2.4, we may view ψ as a unital contractive homomorphism from A to $L(L^p(E, \nu|_E))$.

Let $\beta_e \colon \mathbb{C} \oplus \mathbb{C} \to A$ be as in Lemma 6.6. Then β_e is contractive, so $\psi \circ \beta_e$ is contractive. By Lemma 6.7, it follows that $\psi(e)$ is a hermitian idempotent in $L(L^p(E,\nu|_E))$. Lemma 6.9 provides a measurable subset $F \subset E$ such that $\psi(e)$ is multiplication by χ_F . Another application of Lemma 6.9 implies that $\psi(e)$ is a hermitian idempotent in $L(L^p(Y,\nu))$. So $\psi(e)$ is hermitian in B by Lemma 6.6. \square

Corollary 6.14. Let $p \in [1, \infty) \setminus \{2\}$. Let (X, \mathcal{B}, μ) be a σ -finite measure space, let $N \in \mathbb{Z}_{>0}$, and let $e_1, e_2, \ldots, e_N \in L(L^p(X, \mu))$ be orthogonal hermitian idempotents. Then:

- (1) There exist disjoint measurable sets $E_1, E_2, \ldots, E_N \subset X$ such that e_k is multiplication by χ_{E_k} for $k = 1, 2, \ldots, N$.
- (2) $\sum_{k=1}^{N} e_k$ is a hermitian idempotent.
- (3) For every $\xi \in L^p(X,\mu)$, we have $\|\xi\|_p^p = \sum_{k=1}^N \|e_k\xi\|_p^p$.
- (4) The map $\beta \colon \mathbb{C}^N \to L(L^p(X,\mu))$, given by $\beta(\lambda_1,\lambda_2,\ldots,\lambda_N) = \sum_{k=1}^N \lambda_k e_k$ for $\lambda_1,\lambda_2,\ldots,\lambda_N \in \mathbb{C}$, is a contractive homomorphism.

Proof. Let $\rho: L^{\infty}(X, \mu) \to L(L^p(X, \mu))$ be the representation by multiplication operators. Lemma 6.9 provides measurable sets $F_1, F_2, \ldots, F_N \subset X$ such that $e_k = \rho(\chi_{F_k})$ for $k = 1, 2, \ldots, N$. Since $e_j e_k = 0$ for $j \neq k$, we have $\mu(F_j \cap F_k) = 0$ for $j \neq k$. So there exist disjoint measurable sets $E_k \subset F_k$ for $k = 1, 2, \ldots, N$ such that $\mu(F_k \setminus E_k) = 0$. Then $\rho(\chi_{E_k}) = \rho(\chi_{F_k})$. This proves (1). Part (2) follows

from Lemma 6.9 because setting $E = \bigcup_{k=1}^{N} E_k$ gives $\sum_{k=1}^{N} e_k = \rho(\chi_E)$. Part (3) is

immediate. For (4), define
$$\beta_0 : \mathbb{C}^N \to L^{\infty}(X,\mu)$$
 by $\beta_0(\lambda_1,\lambda_2,\ldots,\lambda_N) = \sum_{k=1}^N \lambda_k \chi_{E_k}$ for $\lambda_1,\lambda_2,\ldots,\lambda_N \in \mathbb{C}$. Then β_0 and ρ are contractive homomorphisms, and $\beta = \rho \circ \beta_0$.

Lemma 6.15. Let $p \in [1, \infty) \setminus \{2\}$. Let A be a separable unital L^p operator algebra, let $N \in \mathbb{Z}_{>0}$, and let $e_1, e_2, \ldots, e_N \in A$ be orthogonal hermitian idempotents such that $\sum_{k=1}^{N} e_k = 1$. Assume $a \in A$ satisfies $ae_k = e_k a$ for $k = 1, 2, \ldots, N$. Then

$$||a|| = \max(||e_1ae_1||, ||e_2ae_2||, \dots, ||e_Nae_N||).$$

Proof. We prove this when N=2, $e_1=e$, and $e_2=1-e$. The general case follows by induction using Corollary 6.14. We may assume that $e\neq 0$.

It is immediate from Lemma 6.6 that ||e|| = 1. Proposition 2.9 provides an isometric unital representation π of A on a separable L^p space $L^p(X,\mu)$. By Lemma 2.7 we may assume that μ is σ -finite. Lemma 6.7 implies that $\pi(e)$ is a hermitian idempotent, so Lemma 6.9 provides a measurable set $E \subset X$ such that $\pi(e)$ is multiplication by χ_E . Thus $\pi(a)$ commutes with multiplication by χ_E . With respect to the L^p direct sum decomposition $L^p(X,\mu) = L^p(E,\mu) \oplus_p L^p(X \setminus E,\mu)$, we get $\pi(a) = \pi(eae) \oplus \pi((1-e)a(1-e))$. So

$$||a|| = ||\pi(a)||$$

$$= \max (||\pi(eae)||, ||\pi((1-e)a(1-e))||) = \max (||eae||, ||(1-e)a(1-e)||).$$

This completes the proof.

It follows that if A is a separable unital L^p operator algebra and $e \in A$ is a central hermitian idempotent, then $A = eAe \oplus (1-e)A(1-e)$, normed as in Definition 5.2. We don't know whether this is true for more general Banach algebras.

Lemma 6.16. In Definition 5.2, suppose that A_1, A_2, \ldots, A_N are separable unital L^p operator algebras whose identities have norm 1, and that they have unique

 L^p operator matrix norms. Then $A = \bigoplus_{k=1}^n A_k$, normed as in Definition 5.2, has unique L^p operator matrix norms.

Proof. For $k=1,2,\ldots,N$ choose some unital isometric representation of A_k on a separable L^p space of a σ -finite measure, and equip A_k with the matrix norms on its image under this representation as in Definition 3.15. Then make A a matrix normed algebra as in Definition 5.2. In view of Lemma 4.2, it suffices to prove that if (X,\mathcal{B},μ) is a σ -finite measure space with $L^p(X,\mu)$ separable, $n\in\mathbb{Z}_{>0}$, and $\pi\colon A\to L(L^p(X,\mu))$ is a unital isometric representation, then $\mathrm{id}_{M_n}\otimes\pi\colon M_n(A)\to L(l_p^n\otimes_p L^p(X,\mu))$ is isometric.

For $k=1,2,\ldots,N$, we identify A_k with its image in A, and we let f_k be the identity of A_k . Set $Z=\{1,2,\ldots,N\}$. Let $\varphi\colon C(Z)\to A$ be the unital homomorphism determined by $\varphi(\chi_{\{k\}})=f_k$ for $k=1,2,\ldots,N$. It is obvious

from the definition of the norm on A that φ is isometric. Then $\pi \circ \varphi$ is isometrically ric, from which it easily follows that $(\pi \circ \varphi)(\chi_{\{k\}})$ is a hermitian idempotent for $k=1,2,\ldots,N$. Let $E_1,E_2,\ldots,E_N\subset X$ be the disjoint sets corresponding to the idempotents $\left((\pi\circ\varphi)(\chi_{\{k\}})\right)_{k=1}^N$ as in Corollary 6.14(1). Since $\pi\circ\varphi$ is unital,

we can assume without loss of generality that $\bigcup_{k=1}^{N} E_k = X$. For k = 1, 2, ..., N, let

 $\pi_k \colon A_k \to L(L^p(E_k, \mu))$ be the unital representation gotten as in Corollary 2.4 from $\pi|_{A_k}$. Corollary 2.4 and the definition of the norm on A imply that π_k is isometric.

It is immediate that $l_n^p \otimes_p L^p(X,\mu)$ is the L^p direct sum of the spaces $l_n^p \otimes_p L^p(E_k,\mu)$

for
$$k = 1, 2, ..., N$$
. It follows that if $a = (a_1, a_2, ..., a_N) \in \bigoplus_{k=1}^{N} M_n(A_k) = M_n(A)$,

then

$$\|(\mathrm{id}_{M_n}\otimes\pi)(a)\|$$

$$= \max (\|(\mathrm{id}_{M_n} \otimes \pi_1)(a_1)\|, \|(\mathrm{id}_{M_n} \otimes \pi_2)(a_2)\|, \ldots, \|(\mathrm{id}_{M_n} \otimes \pi_N)(a_N)\|).$$

The hypotheses imply that $id_{M_n} \otimes \pi_k$ is isometric for k = 1, 2, ..., N. Definition 5.2 therefore implies that $id_{M_n} \otimes \pi$ is isometric.

The following lemma will be used in connection with representations of nonunital spatial L^p AF algebras.

Lemma 6.17. Let $p \in [1, \infty) \setminus \{2\}$. Let (X, \mathcal{B}, μ) be a σ -finite measure space such that $L^p(X,\mu)$ is separable. Let $e_1,e_2,\ldots\in L(L^p(X,\mu))$ be idempotents, and take $e_0 = 0$. Assume that, for all $n \in \mathbb{Z}_{>0}$, $||e_n|| \le 1$ and e_{n-1} is a hermitian idempotent in $e_n L(L^p(X,\mu))e_n$. Then there are an idempotent $e \in L(L^p(X,\mu))$, a σ -finite measure space (Y, \mathcal{C}, ν) such that $L^p(Y, \nu)$ is separable, an isometric linear map $s: L^p(Y, \nu) \to L^p(X, \mu)$, and measurable subsets $Y_1, Y_2, \ldots \subset Y$, such that:

- (1) $||e|| \le 1$. (2) $eL^p(X,\mu) = \overline{\bigcup_{n=1}^{\infty} e_n L^p(X,\mu)}$.
- (3) For every $n \in \mathbb{Z}_{>0}$, e_n is a hermitian idempotent in $eL(L^p(X,\mu))e$.
- (4) $\operatorname{ran}(s) = eL^p(X, \mu).$

$$(5) Y = \coprod_{n=1}^{\infty} Y_n.$$

(6) For every $n \in \mathbb{Z}_{>0}$, $sL^p(Y_n, \nu|_{Y_n}) = (e_n - e_{n-1})L^p(X, \mu)$.

We do not assume that e_n is a hermitian idempotent in $L(L^p(X,\mu))$, and the conclusion does not claim that e is a hermitian idempotent in $L(L^p(X,\mu))$.

Proof of Lemma 6.17. For $n \in \mathbb{Z}_{>0}$ define $E_n = e_n L^p(X, \mu) \subset L^p(X, \mu)$. Set $E = \bigcup_{n=0}^{\infty} E_n$. For $n \in \mathbb{Z}_{>0}$, use Proposition 2.8 to find a σ -finite measure space $(Z_n, \mathcal{D}_n, \lambda_n)$ such that $L^p(Z_n, \lambda_n)$ is isometrically isomorphic to E_n . Also, define $\pi_n : L(E_n) \to L(L^p(X,\mu))$ by $\pi_n(a)\xi = ae_n\xi$ for $a \in L(E_n)$ and $\xi \in L^p(X,\mu)$. Since $e_n a e_n \xi = a e_n \xi$ for $a \in L(E_n)$, one checks easily that π_n is a (nonunital) homomorphism. An application of Corollary 2.4 shows that π_n is isometric. Thus, $L(L^p(Z_n,\lambda_n))$ is isometrically isomorphic to $e_nL(L^p(X,\mu))e_n$.

For $m, n \in \mathbb{Z}_{>0}$ with $m \le n$, a similar argument shows that the analogous map $\pi_{n,m} \colon L(E_m) \to L(E_n)$ is an isometric homomorphism. It follows from Remark 6.5 that $e_m - e_{m-1}$ is a hermitian idempotent in $e_m L(L^p(X,\mu))e_m$, and then it follows from Lemma 6.13 and induction on n that $e_m - e_{m-1}$ is a hermitian idempotent in $e_n L(L^p(X,\mu))e_n$.

For $n \in \mathbb{Z}_{>0}$, Corollary 6.14(3), applied to $e_1 - e_0$, $e_2 - e_1$, ..., $e_n - e_{n-1}$, shows that for every $\xi \in E_n$ we have

(6.2)
$$\|\xi\|_p^p = \sum_{k=1}^n \|(e_k - e_{k-1})\xi\|_p^p.$$

For $n \in \mathbb{Z}_{>0}$, use Proposition 2.8 to find a σ -finite measure space $(Y_n, \mathcal{C}_n, \nu_n)$ and an isometric isomorphism $s_n \colon L^p(Y_n, \lambda_n) \to (e_n - e_{n-1})L^p(X, \mu)$. Following the notation of Definition 5.1, set $(Y, \mathcal{C}, \nu) = \coprod_{n=1}^{\infty} (Y_n, \mathcal{C}_n, \nu_n)$. By (6.2), the map

from
$$L^p\left(\prod_{k=1}^n (Y_k, \mathcal{C}_k, \nu_k)\right)$$
 to $e_n L^p(X, \mu)$, given by

$$(\eta_1, \eta_2, \dots, \eta_n) \mapsto \sum_{k=1}^n s_k(\eta_k),$$

is an isometric isomorphism. Combining these for $n \in \mathbb{Z}_{>0}$ and extending by continuity, we get an isometric linear map $s \colon L^p(Y,\nu) \to L^p(X,\mu)$, whose range must be equal to E.

We now have the objects s, (Y, \mathcal{C}, ν) , and $Y_1, Y_2, \ldots \subset Y$ of the conclusion, as well as parts (5) and (6). If we use E in place of $eL^p(X, \mu)$, we also have parts (2) and (4) of the conclusion.

Now let $\xi \in L^p(X, \mu)$. For $n \in \mathbb{Z}_{>0}$ we use $||e_n|| \le 1$ and (6.2) to get

$$\|\xi\|_p^p \ge \|e_n \xi\|_p^p = \sum_{k=1}^n \|(e_k - e_{k-1})\xi\|_p^p.$$

Therefore $\sum_{k=1}^{\infty} \|(e_k - e_{k-1})\xi\|_p^p$ converges. For $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$, we get

$$\|(e_n - e_m)\xi\|_p^p = \sum_{k=m+1}^n \|(e_k - e_{k-1})\xi\|_p^p$$
, so $(e_n\xi)_{n\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in

 $L^p(X,\mu)$. Thus $\lim_{n\to\infty} e_n\xi$ exists. Call this limit $e\xi$. By taking suitable limits, one checks that e is linear, $||e|| \le 1$, $e^2 = e$, and $\operatorname{ran}(e) = E$. We now have all the required objects for the conclusion, and all the conditions except (3).

For (3), use continuity to get $||e_n\xi||_p^p + ||(e-e_n)\xi||_p^p = ||\xi||_p^p$ for all $\xi \in E$ and $n \in \mathbb{Z}_{>0}$. From this, it is easy to see that the map $(\lambda_1, \lambda_2) \mapsto \lambda_1 e_n + \lambda_2 (e-e_n)$ is a contraction from $\mathbb{C} \oplus \mathbb{C}$ to $eL(L^p(X, \mu))e$. Apply Lemma 6.6.

7. Direct limits

The main result in this section is that the direct limit of matricial L^p operator algebras is also a matricial L^p operator algebra. Moreover, if each algebra in the system has unique L^p operator matrix norms, and the connecting maps of the direct

system are isometric, then the direct limit also has unique L^p operator matrix norms.

Definition 7.1. Let I be an infinite directed set. A (completely) contractive direct system of Banach algebras indexed by I is a pair $((A_i)_{i\in I}, (\varphi_{j,i})_{i\leq j})$ consisting of a family $(A_i)_{i\in I}$ of (matrix normed) Banach algebras and a family $(\varphi_{j,i})_{i\leq j}$ of (completely) contractive homomorphisms $\varphi_{j,i}\colon A_i\to A_j$ for $i,j\in I$ with $i\leq j$, such that $\varphi_{i,i}=\mathrm{id}_{A_i}$ for all $i\in I$ and $\varphi_{k,j}\circ\varphi_{j,i}=\varphi_{k,i}$ whenever $i,j,k\in I$ satisfy $i\leq j\leq k$. We say that the system is unital if A_i is unital for all $i\in I$ and $\varphi_{j,i}$ is unital for all $i,j\in I$ with $i\leq j$.

In the contractive case, the *direct limit* $\varinjlim_{i} A_{i}$ of this direct system is the Banach algebra direct ("inductive") limit, as constructed in Section 3.3 of [1].

In the completely contractive case, for $n \in \mathbb{Z}_{>0}$ we use Lemma 3.5 to identify $M_n\left(\varinjlim_i A_i\right)$ with $\varinjlim_i M_n(A_i)$ up to isomorphism of topological algebras. Then we equip $M_n\left(\varinjlim_i A_i\right)$ with the norm obtained by applying the contractive case to $\varinjlim_i M_n(A_i)$. Lemma 7.2 below shows that we do indeed get a matrix normed Banach algebra this way.

Lemma 7.2. Let $((A_i)_{i \in I}, (\varphi_{j,i})_{i \leq j})$ be a completely contractive direct system of matrix normed Banach algebras. Then $\varinjlim_i A_i$ is a matrix normed Banach algebra, and for every $j \in I$ the standard homomorphism $\varphi_j \colon A_j \to \varinjlim_i A_i$ is completely contractive.

Proof. The statement about complete contractivity follows from the identification of the matrix norms.

For every $n \in \mathbb{Z}_{>0}$ and $i \in I$ identify $B_i^{(n)} = M_n(A_i)$ with $M_n \otimes A_i$, and let $\varphi_{j,i}^{(n)} = \mathrm{id}_{M_n} \otimes \varphi_{j,i} \colon B_i^{(n)} \to B_j^{(n)}$ and $\varphi_i^{(n)} = \mathrm{id}_{M_n} \otimes \varphi_i \colon B_i^{(n)} \to M_n(A)$ be the maps induced by $\varphi_{j,i}$ and φ_i . Set $B^{(n)} = \bigcup_{i \in I} \varphi_i^{(n)} (B_i^{(n)})$. We claim that the norms

on $B^{(n)}$, for $n \in \mathbb{Z}_{>0}$, obtained by viewing $B^{(n)}$ as a subalgebra of the Banach algebra direct limit $\varinjlim_{i} B_{i}^{(n)}$, are a system of matrix norms as in Definition 3.2.

Let $b \in B^{(n)}$, and choose $i_0 \in I$ and $a \in B_{i_0}^{(n)}$ such that $\varphi_{i_0}^{(n)}(a) = b$. Let σ and τ be injective functions as in Definition 3.2(1). Since $M_m\left(\varinjlim_i A_i\right) = \varinjlim_i B_i^{(m)}$, using

Definition 3.2(1) in $M_n(A_i) = B_i^{(n)}$ we have

$$\begin{aligned} \left\| (b_{\sigma(j),\tau(k)})_{1 \le j,k \le m} \right\|_m &= \lim_i \left\| \varphi_{i,i_0}^{(m)} \left((a_{\sigma(j),\tau(k)})_{1 \le j,k \le m} \right) \right\|_m \\ &\le \lim_i \left\| \varphi_{i,i_0}^{(n)} \left((a_{j,k})_{1 \le j,k \le n} \right) \right\|_n = \|b\|_n. \end{aligned}$$

Moreover, given $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, if we set $s = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, we have $\varphi_{i,i_0}(sa) = s\varphi_{i,i_0}(a)$, so, using Definition 3.2(2) on $M_n(A_i)$,

$$||sb||_n = \lim_i ||\varphi_{i,i_0}(sa)||_n = \lim_i ||s\varphi_{i,i_0}(a)||_n$$

$$\leq \max \left(|\lambda_1|, |\lambda_2|, \dots, |\lambda_n| \right) \lim_i \|\varphi_{i,i_0}(a)\|_n = \max \left(|\lambda_1|, |\lambda_2|, \dots, |\lambda_n| \right) \|b\|_n.$$

Similarly $||bs||_n \le \max(|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|) ||b||_n$.

Lastly, given $b_1 \in B^{(m)}$ and $b_2 \in B^{(n)}$, there exist $i_0 \in I$, $a_1 \in B_{i_0}^{(m)}$, and $a_2 \in B_{i_0}^{(n)}$ such that $b_1 = \varphi_{i_0}^{(m)}(a_1)$ and $b_2 = \varphi_{i_0}^{(n)}(a_2)$. Therefore, $\operatorname{diag}(b_1, b_2) = \varphi_{i_0}^{(m+n)}(\operatorname{diag}(a_1, a_2))$ and

$$\begin{split} \left\| \operatorname{diag}(b_1, b_2) \right\|_{m+n} &= \lim_{i} \left\| \varphi_{i, i_0}^{(m+n)}(\operatorname{diag}(a_1, a_2)) \right\|_{m+n} \\ &= \lim_{i} \left\| \operatorname{diag} \left(\varphi_{i, i_0}^{(m)}(a_1), \; \varphi_{i, i_0}^{(n)}(a_2) \right) \right\|_{m+n} \\ &= \lim_{i} \max \left(\left\| \varphi_{i, i_0}^{(m)}(a_1) \right\|_{m}, \; \left\| \varphi_{i, i_0}^{(n)}(a_2) \right\|_{n} \right) = \max(\|b_1\|_{m}, \|b_2\|_{n}). \end{split}$$

This completes the proof of the claim.

Since $B^{(n)}$ is dense in $M_n\left(\varinjlim_i A_i\right)$ for all $n \in \mathbb{Z}_{>0}$, the conditions of Definition 3.2 for $\varinjlim_i A_i$ follow by continuity.

Theorem 7.3. Let $p \in [1, \infty)$. Let $((A_i)_{i \in I}, (\varphi_{j,i})_{i \leq j})$ be a contractive direct system of L^p operator algebras. Then $\varinjlim_i A_i$ is an L^p operator algebra.

Theorem 7.4. Let $p \in [1, \infty)$. Let $((A_i)_{i \in I}, (\varphi_{j,i})_{i \leq j})$ be a completely contractive direct system of matricial L^p operator algebras. Then $\varinjlim_i A_i$ is a matricial L^p operator algebra.

The proofs are essentially the same. We prove Theorem 7.3 here. We describe the modifications for the proof of Theorem 7.4 afterwards.

Proof of Theorem 7.3. The statement is trivial if I has a largest element. Otherwise, let \mathcal{U}_0 be the collection of all subsets of I of the form $\{i \in I: i \geq i_0\}$ for $i_0 \in I$. These sets are nonempty. Since I is directed, the intersection of any finite collection of them contains another one. Since I has no largest element, for every $i \in I$ there is $S \in \mathcal{U}_0$ such that $i \notin S$. Therefore there is a free ultrafilter \mathcal{U} on I which contains \mathcal{U}_0 .

By definition, for every $i \in I$ there exists a measure space $(X_i, \mathcal{B}_i, \mu_i)$ and an isometric representation $\rho_i \colon A_i \to L(L^p(X_i, \mu_i))$. Let M be the Banach space ultraproduct $\Big(\prod_{i \in I} L^p(X_i, \mu_i)\Big) / \mathcal{U}$ (Definition 2.1 of [12]). By Theorem 3.3(ii) of [12],

there exists a measure space (X, \mathcal{B}, μ) such that M is isometrically isomorphic to $L^p(X, \mu)$. So it suffices to find an isometric representation of $\varinjlim A_i$ on M.

Let B be the algebraic direct limit of the algebras A_i , and for $i \in I$ let $\varphi_i \colon A_i \to B$ be the homomorphism associated to the direct system. Equip B with the direct limit seminorm, and let $A = \varinjlim_i A_i$ be the completion of $B/\{b \in B \colon ||b|| = 0\}$, with the obvious isometric map $\kappa \colon B \to A$.

We will construct an isometric representation γ of B on M. (It will not be injective; rather, its kernel will be $\{b \in B : ||b|| = 0\}$.) Let $x \in B$. Choose $i \in I$ and $a \in A_i$ such that $\varphi_i(a) = x$. We give an associated operator $y_l \in L(L^p(X_l, \mu_l))$ for each $l \in I$. If $l \geq i$, set $y_l = \rho_l(\varphi_{l,i}(a))$. Otherwise, set $y_l = 0$. Clearly $||y_l|| \leq ||a||$ for all $l \in I$, so the ultraproduct of operators (Definition 2.2 of [12]) gives an

operator $y = (y_l)_{\mathcal{U}} \in L(M)$ such that $||y|| = \lim_{\mathcal{U}} ||y_l||$. Since $J = \{l \in I : l \geq i\}$ is cofinal in I, we have

$$\lim_{l \in I} ||y_l|| = \lim_{l \in J} ||\rho_l(\varphi_{l,i}(a))|| = \lim_{l \in J} ||\varphi_{l,i}(a)|| = ||x||.$$

The choice of \mathcal{U} ensures that $\lim_{\mathcal{U}} \|y_l\| = \lim_{l \in I} \|y_l\|$. Therefore $\|y\| = \|x\|$.

We claim that y does not depend on the choices of i and $a \in A_i$. To prove this, suppose that $j \in I$ and $b \in A_j$ also satisfy $\varphi_j(b) = x$. Let $z_l \in L(L^p(X_l, \mu_i))$ for $l \in I$, and $z = (z_l)_{\mathcal{U}} \in L(M)$, be defined in the same way as y_l above, but using j and b in place of i and a. Choose $k \in I$ such that $k \geq i$, $k \geq j$, and $\varphi_{k,i}(a) = \varphi_{k,j}(b)$. Then $z_l = y_l$ for all $l \in I$ with $l \geq k$, and $\{l \in I : l \geq k\} \in \mathcal{U}$, so z = y. The claim is proved.

It follows that there is a well defined isometric map $\gamma \colon B \to L(M)$ such that if $x \in B$ and $i \in I$ and $a \in A_i$ satisfy $\varphi_i(a) = x$, then $\gamma(x)$ is the element y constructed above. Using directedness of I, it is easy to prove that γ is a homomorphism. Since γ is isometric, we have $\gamma(x) = 0$ whenever ||x|| = 0, and there exists a unique isometric homomorphism $\rho \colon A \to L(M)$ such that $\rho(\kappa(x)) = \gamma(x)$ for all $x \in B$. The existence of ρ shows that A is an L^p operator algebra.

Proof of Theorem 7.4. We describe the differences from the proof of Theorem 7.3. We choose the maps ρ_i in the proof of Theorem 7.3 to be completely isometric, not just isometric. Let $m \in \mathbb{Z}_{>0}$ and let ν be counting measure on $\{1, 2, \ldots, m\}$. One can check that the obvious map gives an isometric isomorphism

$$\left(\prod_{i\in I} L^p(\{1,2,\ldots,m\}\times X_i,\,\nu\times\mu_i)\right)\bigg/\mathcal{U}\to L^p(\{1,2,\ldots,m\}\times X,\,\nu\times\mu)\right).$$

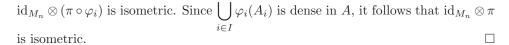
(This is a direct computation from the definitions.) Using the standard isomorphism $M_m\left(\varinjlim_i A_i\right) \cong \varinjlim_i M_m(A_i)$, the argument used in the proof of Theorem 7.3 to show that γ is isometric now shows that $\mathrm{id}_{M_m} \otimes \gamma$ is isometric. Since this is true for all $m \in \mathbb{Z}_{>0}$, we conclude that γ , hence also ρ , is completely isometric. \square

Proposition 7.5. Let $p \in [1, \infty)$, and let $((A_i)_{i \in I}, (\varphi_{j,i})_{i \leq j})$ be as in Theorem 7.3. Suppose further that I is countable, that for all $i \in I$ the algebra A_i is separable and has unique L^p operator matrix norms, and that for all $i, j \in I$ with $i \leq j$, the map $\varphi_{j,i}$ is isometric. Then $A = \varinjlim A_i$ has unique L^p operator matrix norms.

They are obtained by equipping A_i with its unique L^p operator matrix norms for $i \in I$ and, for each $n \in \mathbb{Z}_{>0}$, giving $M_n(A)$ the norm coming from the contractive case of Definition 7.1 applied to $\varinjlim_{i} M_n(A_i)$.

Proof. The hypotheses imply that A is separable. For $i \in I$, equip A_i with its unique L^p operator matrix norms. The hypotheses imply that if $j \in I$ and $j \geq i$, then $\varphi_{j,i}$ is completely isometric. Equip A with the matrix norms in the statement. Then A is an L^p operator algebra by Theorem 7.4.

Let (X, \mathcal{B}, μ) be a σ -finite measure space with $L^p(X, \mu)$ separable, and let $\pi \colon A \to L(L^p(X, \mu))$ be isometric. We show that π is completely isometric. Let $n \in \mathbb{Z}_{>0}$. For $i \in I$, let $\varphi_i \colon A_i \to A$ be the homomorphism coming from the direct system. Then φ_i is isometric, so $\pi \circ \varphi_i$ is isometric. The hypothesis on A_i implies that



8. Spatial semisimple finite dimensional L^p operator algebras

In this section we introduce our setup by giving the definitions of spatial semisimple finite dimensional L^p operator algebras and spatial homomorphisms. We also give a characterization of spatial homomorphisms between spatial semisimple finite dimensional L^p operator algebras in terms of block diagonal homomorphisms (Lemma 8.20), and show that any spatial semisimple finite dimensional L^p operator algebra has unique L^p operator matrix norms.

Definition 8.1. Let B be a unital Banach algebra, and let $b, c \in B$. We say that b and c are isometrically similar if there is an invertible isometry $s \in B$ such that $c = sbs^{-1}$.

If A is also a Banach algebra, and $\varphi, \psi \colon A \to B$ are linear maps, we say that φ and ψ are isometrically similar if there is an invertible isometry $s \in B$ such that $\psi(a) = s\varphi(a)s^{-1}$ for all $a \in A$.

We state some immediate properties.

Proposition 8.2. Let A and B be Banach algebras, with B unital, and let $\varphi, \psi \colon A \to B$ be isometrically similar linear maps.

- (1) If φ is contractive then so is ψ .
- (2) If φ is isometric then so is ψ .
- (3) If A and B are matrix normed (Definition 3.2) then $\mathrm{id}_{M_n} \otimes \varphi$ and $\mathrm{id}_{M_n} \otimes \psi$ are isometrically similar.
- (4) If A and B are matrix normed and φ is completely bounded, then so is ψ .
- (5) If A and B are matrix normed and φ is completely contractive, then so is ψ .
- (6) If A and B are matrix normed and φ is completely isometric, then so is ψ .

Proof. The only part requiring proof is (3). For this part, we use Definition 3.2(3) to see that if $s \in B$ is an invertible isometry, then so is $1 \otimes s \in M_n \otimes B$ for any $n \in \mathbb{Z}_{>0}$.

Proposition 8.3. Let B be a unital Banach algebra, and let $e, f \in B$ be isometrically similar idempotents. If e is hermitian (Definition 6.4), so is f.

Proof. The corresponding homomorphisms in Lemma 6.6 are isometrically similar. Apply Proposition 8.2(1).

Definition 8.4. Let $p \in [1,\infty) \setminus \{2\}$. Let A be a unital σ -finitely representable L^p operator algebra, let $d \in \mathbb{Z}_{>0}$, and let $\varphi \colon M^p_d \to A$ be a homomorphism (not necessarily unital). We say that φ is *spatial* if $\varphi(1)$ is a hermitian idempotent (Definition 6.4) and φ is contractive. The zero homomorphism is allowed as a choice of φ .

It isn't enough to merely require that φ be contractive.

Example 8.5. Let $p \in [1, \infty) \setminus \{2\}$. Then there is a contractive homomorphism $\varphi \colon \mathbb{C} \to M_2^p$ which is not spatial. To construct one, let e be the nonhermitian idempotent from Lemma 6.11. Define $\varphi(\lambda) = \lambda e$ for $\lambda \in \mathbb{C}$. Then φ is clearly contractive, but not spatial because e is not hermitian.

Lemma 8.6. Let $p \in [1, \infty) \setminus \{2\}$, let $d \in \mathbb{Z}_{>0}$, and let $s \in M_d^p$. Then s is an invertible isometry if and only if s is a complex permutation matrix (Definition 3.10).

Proof. It is obvious that complex permutation matrices are invertible isometries. Conversely, assume that s is an invertible isometry. It follows from Lemma 6.16 of [16] that s is spatial, and it is easily seen from the definitions (Definition 6.3 and Definition 6.4 of [16]) that s is a complex permutation matrix.

The next lemma shows that every spatial homomorphism between two matrix algebras is isometrically similar to a block diagonal homomorphism.

Lemma 8.7. Let $p \in [1, \infty) \setminus \{2\}$. Let $d, m \in \mathbb{Z}_{>0}$, and let $\psi \colon M_d^p \to M_m^p$ be a homomorphism (not necessarily unital). Then the following are equivalent:

- (1) ψ is spatial.
- (2) There exist $k \in \mathbb{Z}_{>0}$ with $0 \le kd \le m$ such that ψ is isometrically similar to the homomorphism $a \mapsto \operatorname{diag}(a, a, \dots, a, 0)$, the block diagonal matrix in which a occurs k times and 0 is the zero element of M^p_{m-kd} .

Proof. It is easy to check that (2) implies (1). So assume (1).

First assume that ψ is unital. Then m = kd for some $k \in \mathbb{Z}_{>0}$. The implication from (4) to (8) in Theorem 7.2 in [16] provides a σ -finite measure space (Y, \mathcal{C}, ν) and a bijective isometry

$$u: l_d^p \otimes_p L^p(Y, \nu) \to l_{kd}^p$$

such that for all $a \in M_d^p$ we have $\psi(a) = u(a \otimes 1)u^{-1}$. The space $L^p(Y, \nu)$ must have dimension k, so it is isometrically isomorphic to l_k^p . There is a bijection

$$\{1, 2, \dots, d\} \times \{1, 2, \dots, k\} \to \{1, 2, \dots, m\}$$

such that the corresponding isomorphism of $L(l_d^p \otimes_p l_k^p)$ with $L(l_m^p)$ sends $a \otimes 1$ to $\operatorname{diag}(a,a,\ldots,a)$. This allows us to identify $s=u^{-1}$ with an invertible isometry in $L(l_m^p) \cong M_m^p$ such that $s\psi(a)s^{-1} = \operatorname{diag}(a,a,\ldots,a)$ for all $a \in M_d^p$.

Now consider the general case. Assume that ψ is spatial. So ψ is contractive and $\psi(1)$ is spatial. By Lemma 6.9, there exists a measurable subset $E \subset \{1,2,\ldots,m\}$ such that $\psi(1)$ is multiplication by χ_E on l_m^p . By conjugating by a permutation matrix, we can assume that $E = \{1,2,\ldots,n\}$ for some $n \in \{1,2,\ldots,m\}$. Identify l_n^p with $l^p(E) \subset l_m^p$, and let $\iota \colon L(l_n^p) \to L(l_m^p)$ be $\iota(b) = b \oplus 0$ for $b \in L(l_n^p)$. (In matrix form, this is $\iota(b) = \operatorname{diag}(b,0)$.) There is a homomorphism $\varphi \colon L(l_d^p) \to L(l_n^p)$ such that $\psi(a) = \iota(\varphi(a)) = \varphi(a) \oplus 0$ for all $a \in L(l_d^p)$, and φ is a unital homomorphism from M_d^p to M_n^p which is contractive by Corollary 2.4. By the case done above, there exist $k \in \mathbb{Z}_{>0}$ such that n = kd and an invertible isometry $s_0 \in M_n^p$ such that $s_0\varphi(a)s_0^{-1} = \operatorname{diag}(a,a,\ldots,a)$ for all $a \in L(l_d^p)$. Then $s = \operatorname{diag}(s_0,1)$, with 1 being the identity of M_{m-n}^p , is an invertible isometry such that for all $a \in L(l_d^p)$ we have $s\psi(a)s^{-1} = \operatorname{diag}(a,a,\ldots,a,0)$.

To define a spatial L^p AF algebra, we need to first define its building blocks, the spatial semisimple finite dimensional L^p operator algebras.

Definition 8.8. Let $p \in [1, \infty) \setminus \{2\}$. A Banach algebra A is called a *spatial semisimple finite dimensional* L^p *operator algebra* if there are $N \in \mathbb{Z}_{>0}$ and $d_1, d_2, \ldots, d_N \in \mathbb{Z}_{>0}$ such that A is isometrically isomorphic to $\bigoplus_{k=1}^N M_{d_k}^p$, endowed with the norm as in Definition 5.2.

Remark 8.9. Let $p \in [1, \infty) \setminus \{2\}$. To simplify the notation in our proofs, if A is a spatial semisimple finite dimensional L^p operator algebra, we will omit the isometric isomorphism and simply write $A = \bigoplus_{k=1}^{N} M_{d_k}^p$ with $N, d_1, d_2, \ldots, d_N \in \mathbb{Z}_{>0}$.

Lemma 8.10. Let $p \in [1, \infty) \setminus \{2\}$. Let A be a spatial semisimple finite dimensional L^p operator algebra, and let $J \subset A$ be an ideal. Then A/J, with the quotient norm, is a spatial semisimple finite dimensional L^p operator algebra.

Proof. Use the notation in Remark 8.9. Then there is a subset $S \subset \{1, 2, ..., N\}$ such that, as an algebra, $A/J = \bigoplus_{k \in S} M_{d_k}^p$. The quotient norm agrees with the norm

on
$$\bigoplus_{k \in S} M_{d_k}^p$$
 by Lemma 5.5. \square

Lemma 8.11. Let $p \in [1, \infty) \setminus \{2\}$, and let $A = \bigoplus_{k=1}^{N} M_{d_k}^p$ be a spatial semisimple

finite dimensional L^p operator algebra. Let $s = (s_1, s_2, \ldots, s_N) \in A$. Then s is an invertible isometry if and only if s_k is a complex permutation matrix for $k = 1, 2, \ldots, N$.

Proof. This is immediate from Lemma 8.6 and the definition of the norm on A. \square

Lemma 8.12. Let $p \in [1, \infty) \setminus \{2\}$, and let A be a spatial semisimple finite dimensional L^p operator algebra. Then A is an L^p operator algebra with unique L^p operator matrix norms, obtained by combining Definition 5.2 and Definition 3.15.

Proof. That A is an L^p operator algebra follows from Lemma 5.7. That A has unique L^p operator matrix norms follows from Corollary 4.4 and Lemma 6.16. \square

The maps we will consider between spatial semisimple finite dimensional L^p operator algebras are the spatial homomorphisms.

Definition 8.13. Let $p \in [1, \infty) \setminus \{2\}$, and let $A = \bigoplus_{k=1}^{N} M_{d_k}^p$ be a spatial semisimple finite dimensional L^p operator algebra. Let B be a σ -finitely representable unital

finite dimensional L^p operator algebra. Let B be a σ -finitely representable unital L^p operator algebra, and let $\varphi \colon A \to B$ be a homomorphism. We say that φ is spatial if for $k = 1, 2, \ldots, N$, the restriction of φ to the summand $M_{d_k}^p$ is spatial in the sense of Definition 8.4.

Lemma 8.14. Let $p \in [1, \infty) \setminus \{2\}$, and let A be a spatial semisimple finite dimensional L^p operator algebra. Let B be a σ -finitely representable unital L^p operator algebra, and let $\varphi \colon A \to B$ be a homomorphism. Then φ is spatial if and only if $\varphi(1)$ is a hermitian idempotent (Definition 6.4) and φ is contractive.

Proof. We can assume without loss of generality that B is a unital subalgebra of $L(L^p(Y,\nu))$ for some σ -finite measure space (Y,\mathcal{C},ν) . Assume also that $A=\bigoplus_{l=1}^N M_{d_l}^p$.

For $k=1,2,\ldots,N$, let $\iota_k\colon M^p_{d_k}\to \bigoplus_{l=1}^N M^p_{d_l}$ be the inclusion of the k-th summand

into A. Let $\rho: L^{\infty}(Y, \nu) \to L(L^p(Y, \nu))$ be the representation by multiplication operators.

Suppose that φ is spatial. For $k=1,2,\ldots,N$, the homomorphism $\varphi|_{M_{d_k}}$ is spatial, so $e_k=\varphi(\iota_k(1_{M_{d_k}}))$ is a hermitian idempotent. The idempotents e_1,e_2,\ldots,e_N are clearly orthogonal, so Corollary 6.14(2) implies that $\varphi(1)=\sum_{k=1}^N e_k$ is a hermitian idempotent.

Corollary 6.14(1) provides disjoint measurable sets $E_1, E_2, \ldots, E_N \subset Y$ such that $e_k = \rho(\chi_{E_k})$ for $k = 1, 2, \ldots, N$. Set $E = \bigcup_{k=1}^N E_k$. We can identify $L^p(Y, \nu)$ with the L^p direct sum $L^p(Y \setminus E, \nu) \oplus_p \bigoplus_{k=1}^N L^p(E_k, \nu)$. For $l = 1, 2, \ldots, N$, let $\varphi_l \colon M_{d_l}^p \to L(L^p(Y, \nu))$ be $\varphi_l(a) = \rho(\chi_{E_l}) \varphi(\iota_l(a)) \rho(\chi_{E_l})$ for $a \in M_{d_l}^p$. If $a_k \in M_{d_k}^p$ for $k = 1, 2, \ldots, N$, then

$$\varphi(a_1, a_2, \dots, a_N) = \sum_{k=1}^N \varphi(\iota_k(a_k)) = \sum_{k=1}^N \varphi(\iota_k(1_{M_{d_k}})) \varphi(\iota_k(a_k)) \varphi(\iota_k(1_{M_{d_k}}))$$
$$= \sum_{k=1}^N \rho(\chi_{E_k}) \varphi(\iota_k(a_k)) \rho(\chi_{E_k}) = \sum_{k=1}^N \varphi_k(a_k).$$

Since the sets E_j are disjoint,

$$\|\varphi(a_1, a_2, \dots, a_N)\| = \max_{1 \le k \le N} \|\varphi_k(a_k)\| \le \max_{1 \le k \le N} \|a_k\| = \|(a_1, a_2, \dots, a_N)\|,$$

so $\varphi \colon A \to B$ is contractive.

Conversely assume that φ is contractive and $\varphi(1)$ is a hermitian idempotent. For $k=1,2,\ldots,N$, it is obvious from Definition 5.2 and Lemma 6.6 that $\iota_k(1_{M_{d_k}})$ is a

hermitian idempotent in $\bigoplus_{k=1}^{N} M_{d_k}^p$. Therefore $\varphi(\iota_k(1_{M_{d_k}}))$ is a hermitian idempotent

in B by Lemma 6.13. Also, $\varphi \circ \iota_k$ is contractive because ι_k and φ are. So $\varphi|_{M_{d_k}}$ is spatial.

Corollary 8.15. Let $p \in [1, \infty) \setminus \{2\}$. Let A be a spatial semisimple finite dimensional L^p operator algebra, let B and C be unital σ -finitely representable L^p operator algebras, let $\varphi \colon A \to B$ be a spatial homomorphism, and let $\psi \colon B \to C$ be a contractive homomorphism such that $\psi(1)$ is a hermitian idempotent in C. Then $\psi \circ \varphi$ is spatial.

Proof. Lemma 8.14 implies that $\psi \circ \varphi$ is contractive. It follows from Lemma 8.14 and Lemma 6.13 that $(\psi \circ \varphi)(1)$ is a hermitian idempotent in C. So $\psi \circ \varphi$ is spatial by Lemma 8.14.

Corollary 8.16. Let $p \in [1, \infty) \setminus \{2\}$. Let A be a spatial semisimple finite dimensional L^p operator algebra, let B be a unital σ -finitely representable L^p operator algebra, and let $\varphi, \psi \colon A \to B$ be isometrically similar homomorphisms. Then φ is spatial if and only if ψ is spatial.

Proof. Use Lemma 8.14, Proposition 8.2(1), and Lemma 8.3. \Box

The following definition is standard, but is given here for reference.

Definition 8.17. Let $A = \bigoplus_{j=1}^{M} M_{c_j}$ be a finite direct sum of full matrix algebras.

(1) Let $d \in \mathbb{Z}_{>0}$, and let $\varphi: A \to M_d$ be a homomorphism. Then for $j = 1, 2, \ldots, M$ the *j-th partial multiplicity* of φ is defined to be

$$m_j(\varphi) = \operatorname{rank}(\varphi(1_{M_{c_j}}))/c_j.$$

(2) Let $B = \bigoplus_{k=1}^{N} M_{d_k}$ be another finite direct sum of full matrix algebras, and let $\varphi \colon A \to B$ be a homomorphism. For j = 1, 2, ..., M and k = 1, 2, ..., N, we denote by $m_{k,j}(\varphi)$ the j-th partial multiplicity of the composition of φ with the projection map $B \to M_{d_k}$. We call $m(\varphi) = (m_{k,j}(\varphi))_{k,j}$ the partial multiplicity matrix of φ . We use analogous notation for direct sums indexed by finite sets not of the form $\{1, 2, ..., M\}$.

Next we define block diagonal homomorphisms between finite direct sums of full matrix algebras.

Definition 8.18. Let $p \in [1, \infty) \setminus \{2\}$. Let $A = \bigoplus_{j=1}^{M} M_{c_j}$ be a finite direct sum of full matrix algebras.

(1) A unital homomorphism $\varphi \colon A \to M_d$ is said to be *block diagonal* if there exist $n \ge 1$ and $r(1), r(2), \ldots, r(n) \in \{1, 2, \ldots, M\}$, satisfying $\sum_{k=1}^{n} c_{r(k)} = d$, such that

$$\varphi(a_1, a_2, \dots, a_M) = \begin{pmatrix} a_{r(1)} & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_{r(2)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{r(3)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{r(n)} \end{pmatrix}.$$

- (2) A nonunital homomorphism $\varphi \colon A \to M_d$ is block diagonal if its unitization $\bigoplus_{j=1}^M M_{c_j} \oplus \mathbb{C} \to M_d \text{ is block diagonal.}$
- (3) Let $B = \bigoplus_{j=1}^{N} M_{d_j}$ be a finite direct sum of full matrix algebras. Then a homomorphism $\varphi \colon A \to B$ is block diagonal if for k = 1, 2, ..., N the homomorphism $\varphi_k \colon A \to M_{d_k}$, given by the composition of φ and the projection map $B \to M_{d_k}$, is block diagonal.

We list some properties of block diagonal homomorphisms.

Lemma 8.19. Let $p \in [1, \infty) \setminus \{2\}$.

(1) Let A, B, and C be finite direct sums of full matrix algebras, and let $\varphi \colon A \to B$ and $\psi \colon B \to C$ be homomorphisms. Then $m(\psi \circ \varphi) = m(\psi)m(\varphi)$.

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- (2) Let A and B be spatial semisimple finite dimensional L^p operator algebras. Then every block diagonal homomorphism $\varphi \colon A \to B$ is spatial.
- (3) If φ is as in Definition 8.18(1), then

$$m_j(\varphi) = \operatorname{card}(\{k \in \{1, 2, \dots, n\} : r(k) = j\}).$$

- (4) Let $A = \bigoplus_{j=1}^{M} M_{c_j}$ and $B = \bigoplus_{k=1}^{N} M_{d_k}$ be finite direct sums of full matrix algebras, and let m be an $N \times M$ matrix with entries in $\mathbb{Z}_{\geq 0}$. Then the following are equivalent:
 - (a) There exists a block diagonal homomorphism $\varphi \colon A \to B$ such that $m(\varphi) = m$.
 - (b) There exists a homomorphism $\varphi \colon A \to B$ such that $m(\varphi) = m$.

(c) For
$$k = 1, 2, ..., N$$
, we have $\sum_{j=1}^{M} m_{k,j} c_j \le d_k$.

- (5) The composition of two block diagonal homomorphisms is block diagonal.
- (6) Let A_1, A_2, B_1, B_2 be finite direct sums of full matrix algebras, and let $\varphi_1 \colon A_1 \to B_1$ and $\varphi_2 \colon A_2 \to B_2$ be block diagonal homomorphisms. Then $\varphi_1 \oplus \varphi_2 \colon A_1 \oplus A_2 \to B_1 \oplus B_2$ is block diagonal.
- (7) Let $A = \bigoplus_{j=1}^{M} M_{c_j}$ and $B = \bigoplus_{k=1}^{N} M_{d_k}$ be finite direct sums of full matrix algebras, let $\varphi \colon A \to B$ be a block diagonal homomorphism, and let $r \in \mathbb{Z}_{>0}$. Make the identifications $M_r \otimes A = \bigoplus_{j=1}^{M} M_{rc_j}$ and $M_r \otimes B = \bigoplus_{k=1}^{N} M_{rd_k}$, by using on each summand the isomorphism θ_{σ} of Definition 3.12 with σ taken to be the standard choice of bijection as given there. Then $\mathrm{id}_{M_r} \otimes \varphi$ is block diagonal.
- (8) Let A and B be spatial semisimple finite dimensional L^p operator algebras, and let $\varphi \colon A \to B$ be block diagonal. Then φ is completely contractive.

Proof. We first prove (2). Write
$$B = \bigoplus_{k=1}^{N} M_{d_k}^p$$
, and for $k = 1, 2, ..., N$ let $\pi_k \colon B \to M_{d_k}^p$ be the projection map. Block diagonal maps to $M_{d_k}^p$ are clearly contractive, so

 $M_{d_k}^p$ be the projection map. Block diagonal maps to $M_{d_k}^p$ are clearly contractive, so $\pi_k \circ \varphi$ is contractive. Thus φ is contractive by Lemma 5.4. For $k=1,2,\ldots,N$, the matrix $(\pi_k \circ \varphi)(1)$ is diagonal with entries in $\{0,1\}$. So $(\pi_k \circ \varphi)(1)$ is a hermitian idempotent by Corollary 6.10. Now $\varphi(1)$ is a hermitian idempotent by Lemma 6.8. Use Lemma 8.14.

Part (8) follows from part (7) and part (2).

Everything else is either well known or immediate.

Lemma 8.20. Let
$$p \in [1, \infty) \setminus \{2\}$$
, let $A = \bigoplus_{j=1}^{L} M_{c_j}^p$ and $B = \bigoplus_{k=1}^{N} M_{d_k}^p$ be spatial

semisimple finite dimensional L^p operator algebras, and let $\varphi \colon A \to B$ be a homomorphism. Then φ is spatial if and only if φ is isometrically similar to a block diagonal homomorphism.

Proof. If φ is isometrically similar to a block diagonal homomorphism, then φ is spatial by Lemma 8.19(2) and Corollary 8.16.

Conversely, assume that φ is spatial. Since the projection map $\pi_k \colon B \to M_{d_k}^p$ is contractive and $\pi_k(1)$ is a hermitian idempotent, Corollary 8.15 implies that $\pi_k \circ \varphi$ is spatial. Therefore, it is enough to prove the claim when $B = M_d^p$ for some $d \in \mathbb{Z}_{>0}$.

For $j=1,2,\ldots,M$ let $\iota_j\colon M^p_{c_j}\to A$ be the inclusion map. Since $\varphi\circ\iota_j$ is spatial (by Lemma 8.14 and Corollary 8.15), it follows from Corollary 6.14 that there are disjoint subsets

$$E_1, E_2, \ldots, E_M \subset \{1, 2, \ldots, d\}$$

such that $(\varphi \circ \iota_j)(1_{M^p_{c_j}})$ is multiplication by χ_{E_j} for $j=1,2,\ldots,M$. Let ρ be the representation of $C(\{1,2,\ldots,d\})$ on l^p_d by multiplication operators. Set $d_0=0$ and choose a permutation σ of $\{1,2,\ldots,d\}$ and numbers $d_1,d_2,\ldots,d_M\in\{1,2,\ldots,d\}$ such that for $j=1,2,\ldots,M$ we have $\sigma(E_j)=(d_{j-1},d_j]\cap \mathbb{Z}$. Let $s_0\in M^p_d$ be the corresponding permutation matrix, satisfying $s_0\rho(\chi_{E_j})s_0^{-1}=\rho(\chi_{\sigma(E_j)})$ for $j=1,2,\ldots,M$. Corollary 8.16 implies that the map $a\mapsto s_0\varphi(a)s_0^{-1}$ is spatial.

For $1 \leq j \leq M$ make the obvious identification

$$\rho(\chi_{(d_{j-1}, d_j] \cap \mathbb{Z}}) M_d^p \rho(\chi_{(d_{j-1}, d_j] \cap \mathbb{Z}}) = M_{d_j - d_{j-1}}^p.$$

Since

$$s_0(\varphi \circ \iota_j)(1)s_0^{-1} = \rho(\chi_{(d_{j-1}, d_j] \cap \mathbb{Z}}),$$

by Corollary 2.4 the formula $s_0(\varphi \circ \iota_j)(\cdot)s_0^{-1}$ defines a contractive unital homomorphism $\psi_j \colon M_{c_j}^p \to M_{d_j-d_{j-1}}^p$. It follows from Lemma 8.7 that there is a complex permutation matrix $s_j \in M_{d_j-d_{j-1}}^p$ such that $a \mapsto s_j \psi_j(a) s_j^{-1}$ is a block diagonal homomorphism from $M_{c_j}^p$ to $M_{d_j-d_{j-1}}^p$ for $1 \le j \le M$.

Set $s = [\operatorname{diag}(s_1, s_2, \dots, s_M, 1_{d-d_M})] \cdot s_0$, which is a complex permutation matrix in M_d^p . Since

$$s_0\varphi(a_1, a_2, \dots, a_M)s_0^{-1} = \operatorname{diag}(\psi_1(a_1), \psi_2(a_2), \dots, \psi_M(a_M), 0_{d-d_M})$$

for $a=(a_1,a_2,\ldots,a_M)\in\bigoplus_{j=1}^L M_{c_j}^p$, it follows that $a\mapsto s\varphi(a)s^{-1}$ is block diagonal.

9. Spatial L^p AF algebras

We define spatial L^p AF algebras and show that any spatial L^p AF algebra is a separable nondegenerately representable L^p operator algebra.

Definition 9.1. Let $p \in [1, \infty) \setminus \{2\}$. A spatial L^p AF direct system is a contractive direct system with index set $\mathbb{Z}_{\geq 0}$ (that is, a pair $((A_m)_{m \in \mathbb{Z}_{\geq 0}}, (\varphi_{n,m})_{0 \leq m \leq n})$ as in Definition 7.1), which satisfies the following additional conditions:

- (1) For every $m \in \mathbb{Z}_{\geq 0}$, the algebra A_m is a spatial semisimple finite dimensional L^p operator algebra (Definition 8.8).
- (2) For all $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$, the map $\varphi_{n,m}$ is a spatial homomorphism (Definition 8.13).

We further say that a Banach algebra A is a spatial L^p AF algebra if it is isometrically isomorphic to the direct limit of a spatial L^p AF direct system.

Definition 9.2. Let $p \in [1, \infty) \setminus \{2\}$. Let $((A_m)_{m \in \mathbb{Z}_{\geq 0}}, (\varphi_{n,m})_{0 \leq m \leq n})$ be a spatial L^p AF direct system (Definition 9.1). We make $A = \varinjlim_{m} (A_m, \varphi_{n,m})$ into a matricial L^p operator algebra via Lemma 8.12 and Theorem 7.4.

The matrix norms on A a priori depend on how A is realized as a direct limit. We will show in Theorem 9.12 that in fact they are independent of the realization.

Lemma 9.3. Let $p \in [1, \infty) \setminus \{2\}$. Let $r \in \mathbb{Z}_{>0}$ and let $((A_m)_{m \in \mathbb{Z}_{\geq 0}}, (\varphi_{n,m})_{0 \leq m \leq n})$ be a spatial L^p AF direct system. Then

$$((M_r(A_m))_{m\in\mathbb{Z}_{>0}}, (\mathrm{id}_{M_r}\otimes\varphi_{n,m})_{0\leq m\leq n})$$

is a spatial L^p AF direct system.

Proof. Using Definition 3.15 and Definition 3.13, for any $d \in \mathbb{Z}_{>0}$ we see that $M_r(M_d^p)$ is isometrically isomorphic to M_{rd}^p , via a map as in Definition 3.12. Therefore $M_r(A_m)$ is a spatial semisimple finite dimensional L^p operator algebra for all $m \in \mathbb{Z}_{>0}$. Lemma 8.20 implies that $\varphi_{n,m}$ is isometrically similar to a block diagonal homomorphism. It follows from Lemma 8.19(7) and Proposition 8.2(3) that the maps $\mathrm{id}_{M_r} \otimes \varphi_{n,m}$ are isometrically similar to block diagonal homomorphisms. Now use Lemma 8.20.

Corollary 9.4. Let $p \in [1, \infty) \setminus \{2\}$. Let $((A_m)_{m \in \mathbb{Z}_{\geq 0}}, (\varphi_{n,m})_{0 \leq m \leq n})$ be a spatial L^p AF direct system. Let $A = \varinjlim_{m} (A_m, \varphi_{n,m})$ be the direct limit, equipped with

the matricial L^p operator algebra structure of Definition 9.2. Let $r \in \mathbb{Z}_{>0}$. Then $M_r(A)$ is a spatial L^p AF algebra.

Proof. This is immediate from Lemma 9.3.

Lemma 9.5. Let $p \in [1, \infty) \setminus \{2\}$. Let $N \in \mathbb{Z}_{>0}$ and for k = 1, 2, ..., N let $\left((A_m^{(k)})_{m \in \mathbb{Z}_{\geq 0}}, (\varphi_{n,m}^{(k)})_{0 \leq m \leq n} \right)$ be a spatial L^p AF direct system (Definition 9.1). Then

$$\left(\left(\bigoplus_{k=1}^{N} A_{m}^{(k)} \right)_{m \in \mathbb{Z}_{>0}}, \left(\bigoplus_{k=1}^{N} \varphi_{n,m}^{(k)} \right)_{0 \le m \le n} \right)$$

is a spatial L^p AF direct system.

Proof. Obviously $\bigoplus_{k=1}^{N} A_m^{(k)}$ is a spatial semisimple finite dimensional L^p operator

algebra for all $m \in \mathbb{Z}_{>0}$. By Lemma 8.20, a direct system of spatial semisimple finite dimensional L^p operator algebras is a spatial L^p AF direct system if and only if its maps are all isometrically similar to block diagonal maps. It follows from Lemma 8.19(6) that the direct sum of maps isometrically similar to block diagonal maps is again isometrically similar to a block diagonal map.

Corollary 9.6. Let $p \in [1, \infty) \setminus \{2\}$. Then the direct sum of finitely many spatial L^p AF algebras is again a spatial L^p AF algebra.

Proof. This is immediate from Lemma 9.5.

Definition 9.7. Let A be a Banach algebra, and let $e = (e_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of idempotents in A which is nondecreasing, that is, for $n \in \mathbb{Z}_{>0}$ we have $e_n \leq e_{n+1}$ in the sense of Definition 6.1. Set $e_0 = 0$ (by convention), and let $\theta_e \colon C_c(\mathbb{Z}_{>0}) \to A$

be the unique homomorphism such that $\theta_e(\chi_{\{n\}}) = e_n - e_{n-1}$ for all $n \in \mathbb{Z}_{>0}$. We equip $C_c(\mathbb{Z}_{>0})$ with the norm $\|\cdot\|_{\infty}$, and when we refer to $\|\theta_e\|$, or demand that θ_e be contractive or bounded, we use this norm.

Proposition 9.8. Let $p \in [1, \infty) \setminus \{2\}$. Let A be a separable L^p operator algebra, and let $e = (e_n)_{n \in \mathbb{Z}_{>0}}$ and θ_e be as in Definition 9.7. Assume that this sequence is an approximate identity for A, and that θ_e is contractive. Then there are a σ -finite measure space (Y, \mathcal{C}, ν) , with $L^p(Y, \nu)$ separable, and an isometric nondegenerate representation $\pi \colon A \to L(L^p(Y, \nu))$, such that $\pi(e_n)$ is a hermitian idempotent in $L(L^p(Y, \nu))$ for all $n \in \mathbb{Z}_{>0}$.

Proof. Use Proposition 2.6 to find a σ -finite measure space (X, \mathcal{B}, μ) such that $L^p(X, \mu)$ is separable and an isometric representation ρ of A on $L^p(X, \mu)$. It is clear from contractivity of θ_e and Lemma 6.6 that if $n \in \mathbb{Z}_{>0}$ then $||e_n|| = 1$ and (taking $e_0 = 0$) that e_{n-1} is a hermitian idempotent in $e_n A e_n$.

Apply Lemma 6.17 to the idempotents $\rho(e_n)$ for $n \in \mathbb{Z}_{>0}$. In the rest of the proof, we use the notation of Lemma 6.17. Set $E = eL^p(X,\mu)$. Then s is an invertible isometry from $L^p(Y,\nu)$ to E. Moreover, the map $a \mapsto \rho(a)|_E$ defines a homomorphism from A to L(E), with $\|\rho(a)|_E\| = \|\rho(a)\|$ by Lemma 2.3. Now the representation $\pi \colon A \to L(L^p(Y,\nu))$, defined by $\pi(a) = s^{-1}[\rho(a)|_E]s$, is nondegenerate and isometric. Moreover, for $n \in \mathbb{Z}_{>0}$, the operator $\pi(e_n)$ is multiplication

by the characteristic function of $\bigcup_{k=1}^{\infty} Y_n$ and is hence a hermitian idempotent in $L(L^p(Y,\nu))$.

Under the hypotheses of Proposition 9.8, an L^p operator algebra has a canonical norm on its unitization.

Proposition 9.9. Let $p \in [1, \infty) \setminus \{2\}$. Let A be a separable nonunital L^p operator algebra, and let $e = (e_n)_{n \in \mathbb{Z}_{>0}}$ be as in Definition 9.7. Assume that this sequence is an approximate identity for A, and that the homomorphism θ_e of Definition 9.7 is contractive. Then there is a unique norm $\|\cdot\|$ on the unitization A^+ of A satisfying the following conditions:

- (1) $\|\cdot\|$ agrees with the given norm on $A \subset A^+$.
- (2) $\|\cdot\|$ is equivalent to the usual norm on the unitization.
- (3) A^+ is an L^p operator algebra.
- (4) Identify $C(\mathbb{Z}_{>0} \cup \{\infty\})$ with $C_0(\mathbb{Z}_{>0})^+$, and give it the usual supremum norm on $C(\mathbb{Z}_{>0} \cup \{\infty\})$. Let $\theta_e^+ : C(\mathbb{Z}_{>0} \cup \{\infty\}) \to A^+$ be the unitization of θ_e . Then θ_e^+ is contractive.

Proof. We first prove existence. Let $\pi\colon A\to L(L^p(Y,\nu))$ be as in Proposition 9.8. Extend this homomorphism to a homomorphism $\pi^+\colon A^+\to L(L^p(Y,\nu))$. Then π^+ is injective because A is not unital. Define $\|a\|=\|\pi^+(a)\|$ for $a\in A^+$. Conditions (1), (2), and (3) are immediate. It remains to prove condition (4).

By density, it suffices to prove that for $n \in \mathbb{Z}_{>0}$, if we set

$$K = \{n + 1, n + 2, \dots, \infty\},\$$

and take any function $f \in C(\mathbb{Z}_{>0} \cup \{\infty\})$ vanishing on K and any $\lambda \in \mathbb{C}$, then

(9.1)
$$\|(\pi^{+} \circ \theta_{e}^{+})(f + \lambda \chi_{K})\| \leq \max(\|f\|, |\lambda|).$$

Since Proposition 9.8 implies that $\pi(e_n)$ is a hermitian idempotent, by Lemma 6.9 there is a measurable subset $E \subset Y$ such that $\pi(e_n)$ is multiplication by χ_E on $L^p(Y,\nu)$. Then $(\pi^+ \circ \theta_e^+)(f)$ acts on $L^p(E,\nu)$ and is zero on $L^p(Y \setminus E,\nu)$, while $(\pi^+ \circ \theta_e^+)(\lambda \chi_K)$ is multiplication by λ on $L^p(Y \setminus E,\nu)$ and zero on $L^p(E,\nu)$. So (9.1) holds.

Now we prove uniqueness. Let $\|\cdot\|$ be a norm as in the statement. For $n \in \mathbb{Z}_{>0}$, $a \in e_n A e_n$, and $\lambda \in \mathbb{C}$ we prove that

(9.2)
$$||a + \lambda \cdot 1|| = \max(||a + \lambda e_n||, |\lambda|).$$

Since the right hand side of (9.2) depends only on the norm on A, and since $\bigcup_{n=1}^{\infty} e_n A e_n$ is dense in A, uniqueness will follow.

It follows from (4) that e_n is a hermitian idempotent in A^+ . Also, e_n commutes with $a + \lambda \cdot 1$ and $||1 - e_n|| = ||\theta_e^+(\chi_K)|| \le 1$. So Lemma 6.15 implies that

$$||a+\lambda \cdot 1|| = \max \left(||e_n(a+\lambda \cdot 1)e_n||, ||(1-e_n)(a+\lambda \cdot 1)(1-e_n)|| \right) = \max(||a+\lambda e_n||, |\lambda|),$$
 which is (9.2).

Proposition 9.10. Let $p \in [1, \infty) \setminus \{2\}$, and let $A = \varinjlim_{m} (A_m, \varphi_{n,m})$ be a spatial

 L^p AF algebra, expressed as a direct limit as in Definition 9.1, and with canonical maps $\varphi_n \colon A_n \to A$ for $n \in \mathbb{Z}_{>0}$. Then A is a separable nondegenerately representable L^p operator algebra. Moreover, $e = (\varphi_n(1_{A_n}))_{n \in \mathbb{Z}_{>0}}$ is a nondecreasing approximate identity of idempotents such that the corresponding homomorphism θ_e of Definition 9.7 is contractive.

Proof. Theorem 7.3 and Lemma 8.12 imply that A is an L^p operator algebra. Separability is obvious. We prove the statement about the approximate identity. By Proposition 9.8, this will imply that $A = \varinjlim_{m} (A_m, \varphi_{n,m})$ is nondegenerately representable.

For $n \in \mathbb{Z}_{>0}$, write f_n for the identity of A_n , and set $e_n = \varphi_n(f_n)$. It is clear that $||e_n|| \le 1$ (with equality unless $A_n = 0$), and that $e_n \le e_{n+1}$.

Set $e = (e_n)_{n \in \mathbb{Z}_{>0}}$, as in the statement of the theorem. Then we have

$$\lim_{n \to \infty} e_n \varphi_m(a) = \lim_{n \to \infty} \varphi_m(a) e_n = \varphi_m(a)$$

for every $m \in \mathbb{Z}_{>0}$ and $a \in A_m$. Since $\bigcup_{m \in \mathbb{Z}_{>0}} \varphi_m(A_m)$ is dense in A and $||e_n|| \le 1$

for all $n \in \mathbb{Z}_{>0}$, a standard $\frac{\varepsilon}{3}$ -argument shows that e is an approximate identity for A.

It remains to prove that θ_e is contractive. We prove by induction on n that, with $\varphi_{n,0}(f_0)$ taken to be zero, the idempotents $\varphi_{n,j}(f_j) - \varphi_{n,j-1}(f_{j-1})$ are hermitian for $j = 1, 2, \ldots, n$. For n = 1, this is just the assertion that the identity is a hermitian idempotent in A_1 . If the statement is known for n, then for $j = 1, 2, \ldots, n$ we have

$$\varphi_{n+1,j}(f_j) - \varphi_{n+1,j-1}(f_{j-1}) = \varphi_{n+1,n} (\varphi_{n,j}(f_j) - \varphi_{n,j-1}(f_{j-1})),$$

which is a hermitian idempotent by Lemma 6.13. Also

$$\varphi_{n+1, n+1}(f_{n+1}) - \varphi_{n+1, n}(f_n) = 1_{A_{n+1}} - \varphi_{n+1, n}(f_n)$$

is hermitian because $\varphi_{n+1,n}(f_n)$ is hermitian. This completes the induction.

Corollary 6.14(4) now implies that $\theta_e|_{C(\{1,2,...,n\})}$ is contractive for all $n \in \mathbb{Z}_{>0}$. It follows that θ_e is contractive.

Proposition 9.11. Let $p \in [1, \infty) \setminus \{2\}$, and let A be a spatial L^p AF algebra. Then A is isometrically isomorphic to the direct limit of a spatial L^p AF direct system in which all the connecting maps are injective.

Proof. Let $((A_m)_{m\in\mathbb{Z}_{\geq 0}}, (\varphi_{n,m})_{m\leq n})$ be a spatial L^p AF direct system such that A

is isometrically isomorphic to $\varinjlim_{m} A_{m}$. Then for $m \in \mathbb{Z}_{>0}$ we can write $A_{m} = \bigoplus_{j=1}^{N(m)} M_{d(m,j)}^{p}$

with

$$N(m) \in \mathbb{Z}_{\geq 0}$$
 and $d(m, 1), d(m, 2), \dots, d(m, N(m)) \in \mathbb{Z}_{> 0}$.

Set $J_m = \bigcup_{n=m+1}^{\infty} \operatorname{Ker}(\varphi_{n,m})$. Then J_m is a closed ideal in A_m , and A_m/J_m is a

spatial semisimple finite dimensional L^p operator algebra by Lemma 8.10.

Let $m \in \mathbb{Z}_{\geq 0}$. For $n \in \mathbb{Z}_{\geq 0}$ with $n \geq m$, Corollary 8.15 shows that the induced homomorphism $A_m \to A_n/J_n$ is spatial. Lemma 8.14 can then be used to show that the induced homomorphism $\overline{\varphi}_{n,m} \colon A_m/J_m \to A_n/J_n$ is spatial. Clearly $\overline{\varphi}_{n,m}$ is injective. Now $((A_m/J_m)_{m \in \mathbb{Z}_{\geq 0}}, (\overline{\varphi}_{n,m})_{0 \leq m \leq n})$ is a spatial L^p AF direct system whose direct limit is isometrically isomorphic to A.

Corollary 9.12. Let $p \in [1, \infty) \setminus \{2\}$, and let A be a spatial L^p AF algebra. Then A has unique L^p operator matrix norms.

Proof. By Proposition 9.11, there is a spatial L^p AF direct system $((A_m)_{m \in \mathbb{Z}_{\geq 0}}, (\varphi_{n,m})_{0 \leq m \leq n})$ with injective maps such that A is isometrically isomorphic to $\varinjlim A_m$. For $m, n \in$

 $\mathbb{Z}_{\geq 0}$ with $m \leq n$, Lemma 8.20 implies that $\varphi_{n,m}$ is isometrically similar to a block diagonal homomorphism. An injective block diagonal homomorphism is isometric, so $\varphi_{n,m}$ is isometric. The result now follows from Lemma 8.12 and Proposition 7.5. \square

10. Classification of spatial L^p AF algebras

In this section we prove our main result, the classification of spatial L^p AF algebras based on their scaled preordered K_0 groups. Moreover, as for AF algebras, we show that every countable scaled Riesz group can be realized as the scaled preordered K_0 group of a spatial L^p AF algebra. Given the theory already developed, the proofs are now essentially the same as in the C^* algebra case.

The original C* theory of AF algebras is mainly due to Bratteli [3], Elliott [7], and Effros, Handelman, and Shen [6]. As references for the entire theory, we rely on Chapter 3 of [1] and on [5]. For a more detailed discussion of Riesz groups than is needed here, see [11].

We only state the classification theorem in terms of scaled preordered K-theory. We don't discuss the connection with Bratteli diagrams, since the relation between Riesz groups and Bratteli diagrams is well known and the generalization of the AF algebra classification to spatial L^p AF algebras introduces nothing new here.

We begin by describing the relevant K-theoretic background.

Definition 10.1. A preordered abelian group is a pair (G, G_+) in which G is an abelian group and G_+ is a subset of G such that $0 \in G_+$ and $G_+ + G_+ \subset G_+$. For $\eta, \mu \in G$ we write $\eta \leq \mu$ to mean that $\mu - \eta \in G_+$.

A scaled preordered abelian group is a triple (G, G_+, Σ) such that (G, G_+) is a preordered abelian group, and Σ (the scale) is a subset of G_+ such that $0 \in \Sigma$.

If (H, H_+) is another preordered abelian group, and $f: G \to H$ is a homomorphism, then f is positive if $f(G_+) \subset H_+$. If $\Sigma \subset G_+$ and $\Gamma \subset H_+$ are scales, we say that f is contractive if $f(\Sigma) \subset \Gamma$.

The definitions are weak because they are supposed to accommodate $K_0(A)$ for any Banach algebra A. For example (using the notation of Definition 10.3 below), for the algebras $C_0(\mathbb{R}^2)$, \mathcal{O}_{∞} , and \mathcal{O}_n we get the following results for $(K_0(A), K_0(A)_+, \Sigma(A))$:

$$(\mathbb{Z}, \{0\}, \{0\}), (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}), \text{ and } (\mathbb{Z}/(n-1)\mathbb{Z}, \mathbb{Z}/(n-1)\mathbb{Z}, \mathbb{Z}/(n-1)\mathbb{Z}).$$

The scale need not be hereditary if A does not have cancellation.

We will take the K_0 -group of a Banach algebra to be as in Section 5 of [1]. (We will make very little use of the K_1 -group, and we don't recall its definition.) To start, we recall one of the standard equivalence relations on idempotents. It is called algebraic equivalence in Definition 4.2.1 of [1].

Definition 10.2. Let A be a Banach algebra. Let e and f be idempotents in A. We say that e is algebraically Murray-von Neumann equivalent to f, denoted by $e \sim f$, if there exist $x, y \in A$ such that xy = e and yx = f.

Definition 10.3. Let A be a ring. We define $M_{\infty}(A)$ to be the (algebraic) direct limit of the matrix rings $M_n(A)$ under the embeddings $a \mapsto \operatorname{diag}(a,0)$. (See Definition 5.1.1 of [1].) We define V(A) to be the abelian semigroup of algebraic Murray-von Neumann equivalence classes of idempotents in $M_{\infty}(A)$. (See Definition 5.1.2 of [1] and the discussion afterwards.)

When A is a Banach algebra, we define $(K_0(A), K_0(A)_+, \Sigma(A))$ as follows. We take $K_0(A)$ to be the usual K_0 -group of A, as in, for example, Definition 5.5.1 of [1]. (There is trouble if one uses the definition there for more general rings.) For $n \in \mathbb{Z}_{>0}$ and an idempotent $e \in M_n(A)$, we write [e] for its class in $K_0(A)$. We take $K_0(A)_+$ to be the image of V(A) in $K_0(A)$ under the map coming from 5.5.2 and Definition 5.3.1 of [1]. We take $\Sigma(A)$ to be the image under this map of the subset of V(A) consisting of the classes of idempotents in $A \subset M_{\infty}(A)$.

We warn that [e] is sometimes used for the class of e in V(A). Since $V(A) \to K_0(A)$ need not be injective, this is not the same as the class of e in $K_0(A)$.

Remark 10.4. We can rewrite the definitions of $K_0(A)_+$ and $\Sigma(A)$ as

$$K_0(A)_+ = \{[e] : e \text{ is an idempotent in } M_\infty(A)\}.$$

and

$$\Sigma(A) = \big\{[e] \colon e \text{ is an idempotent in } A\big\}.$$

Proposition 10.5. Let A be a Banach algebra. Then $(K_0(A), K_0(A)_+, \Sigma(A))$ is a scaled preordered abelian group in the sense of Definition 10.1.

Direct limits of direct systems of scaled preordered abelian groups are constructed in the obvious way.

Lemma 10.6. Let I be a directed set. For every $i \in I$ let $(G_i, (G_i)_+, \Sigma_i)_{i \in I}$ be a scaled preordered abelian group, and for $i, j \in I$ with $i \leq j$ let $g_{j,i} \colon G_i \to G_j$ be a positive contractive homomorphism. Let G be the direct limit $\varinjlim G_i$ as abelian

groups and for $i \in I$ let $g_i : G_i \to G$ be the canonical map. Set

$$G_+ = \bigcup_{i \in I} g_i((G_i)_+)$$
 and $\Sigma = \bigcup_{i \in I} g_i(\Sigma_i)$.

Then (G, G_+, Σ) is a scaled preordered abelian group and (G, G_+, Σ) is the direct limit of $(G_i, (G_i)_+, \Sigma_i)_{i \in I}$ in the category of scaled preordered abelian groups and positive contractive homomorphisms.

Proof. Without the scales, see Proposition 1.15 in [11]. The additional work for scaled preordered abelian groups is easy, and is omitted. \Box

Theorem 10.7. The assignment $A \mapsto (K_0(A), K_0(A)_+, \Sigma(A))$ is a functor from Banach algebras and homomorphisms to scaled preordered abelian groups and contractive positive homomorphisms which commutes with direct limits in which the maps are contractive.

Proof. Functoriality of $K_0(A)$ is stated after Definition 5.5.1 of [1]. Functoriality of the other two parts is clear. (Also see 5.2.1 of [1].)

The fact that $K_0(A)$ commutes with direct limits is Theorem 6.4 of [19]. The statement for $K_0(A)_+$ follows from that for V(A), which is 5.2.4 of [1]. The statement for $\Sigma(A)$ follows by the same proof, which is Propositions 4.5.1 and 4.5.2 of [1].

For the part about the scale in the following definition, we refer the reader to the beginning of Chapter 7 of [5]. Riesz groups are sometimes called dimension groups, for example in the definition at the beginning of Chapter 3 of [11].

Definition 10.8. Let (G, G_+) be a preordered abelian group. We say that G is an unperforated ordered group if:

- (1) $G_+ G_+ = G$.
- (2) $G_+ \cap (-G_+) = \{0\}.$
- (3) Whenever $\eta \in G$ and $n \in \mathbb{Z}_{>0}$ satisfy $n\eta \in G_+$, then $\eta \in G_+$.

We say that (G, G_+) is a Riesz group if, in addition:

(4) Whenever $\eta_1, \eta_2, \mu_1, \mu_2 \in G$ satisfy $\eta_j \leq \mu_k$ for $j, k \in \{1, 2\}$, then there exists $\lambda \in G$ such that $\eta_j \leq \lambda \leq \mu_k$ for $j, k \in \{1, 2\}$.

Let (G, G_+, Σ) be a scaled preordered abelian group. We say that G is a scaled Riesz group if (G, G_+) is a Riesz group, and in addition:

- (5) For every $\eta \in G_+$ there are $n \in \mathbb{Z}_{>0}$ and $\mu_1, \mu_2, \dots, \mu_n \in \Sigma$ such that $\eta = \mu_1 + \mu_2 + \dots + \mu_n$.
- (6) Whenever $\eta, \mu \in G$ satisfy $0 \le \eta \le \mu$ and $\mu \in \Sigma$, then $\eta \in \Sigma$.
- (7) For all $\eta, \mu \in \Sigma$ there is $\lambda \in \Sigma$ such that $\eta \leq \lambda$ and $\mu \leq \lambda$.

We recall for reference some standard definitions and facts. A few are restated for the L^p case.

Definition 10.9. For $N \in \mathbb{Z}_{>0}$ we make \mathbb{Z}^N a Riesz group by taking

$$(\mathbb{Z}^N)_+ = \{(\eta_1, \eta_2, \dots, \eta_N) \in \mathbb{Z}^N : \eta_k \ge 0 \text{ for } k = 1, 2, \dots, N\}.$$

Remark 10.10. The possible scales on \mathbb{Z}^N are exactly the following sets. Take $d = (d_1, d_2, \dots, d_N) \in \mathbb{Z}^N$ with $d_k > 0$ for $k = 1, 2, \dots, N$, and define

$$[0,d] = \{(\mu_1, \mu_2, \dots, \mu_N) \in (\mathbb{Z}^N)_+ : \mu_k \le d_k \text{ for } k = 1, 2, \dots, N\}.$$

See page 43 of [5].

Remark 10.11. Let $p \in [1, \infty)$. Let $A = \bigoplus_{j=1}^{M} M_{c_j}^p$ be a spatial semisimple finite

dimensional L^p operator algebra. Set $c = (c_1, c_2, \ldots, c_M)$. As in the C* algebra case (see pages 55–56 of [5]), using the notation from Definition 10.9 and Remark 10.10, we have

$$(K_0(A), K_0(A)_+, \Sigma(A)) \cong (\mathbb{Z}^M, (\mathbb{Z}^M)_+, [0, c]).$$

The map $K_0(A) \to \mathbb{Z}^M$ sends the class of an idempotent $(e_1, e_2, \dots, e_M) \in \bigoplus_{j=1}^M M_n(M_{e_j}^p)$

to

$$(\operatorname{rank}(e_1), \operatorname{rank}(e_2), \ldots, \operatorname{rank}(e_N)).$$

If $B = \bigoplus_{j=1}^{N} M_{d_j}^p$ is another spatial semisimple finite dimensional L^p operator algebra,

and $\varphi \colon A \to B$ is a homomorphism, then $\varphi_* \colon \mathbb{Z}^M \to \mathbb{Z}^N$ is given by the partial multiplicity matrix $m(\varphi)$ of Definition 8.17(2).

Lemma 10.12. Let $A = \bigoplus_{j=1}^{M} M_{c_j}^p$ and $B = \bigoplus_{k=1}^{N} M_{d_k}^p$ be spatial semisimple finite

dimensional L^p operator algebras. Let f a positive contractive homomorphism from $(K_0(A), K_0(A)_+, \Sigma(A))$ to $(K_0(B), K_0(B)_+, \Sigma(B))$. Then there exists a spatial homomorphism $\varphi \colon A \to B$ such that $\varphi_* = f$. Moreover, φ is unique up to isometric similarity, and it can be chosen to be block diagonal.

Proof. The homomorphism f from

$$(K_0(A), K_0(A)_+, \Sigma(A)) \cong (\mathbb{Z}^M, \mathbb{Z}_+^M, [0, [1_A]])$$

to

$$(K_0(B), K_0(B)_+, \Sigma(B)) \cong (\mathbb{Z}^N, \mathbb{Z}_+^N, [0, [1_B]])$$

is given by an $N \times M$ matrix $m = (m_{k,j})_{1 \le k \le N, 1 \le j \le M}$ with entries in \mathbb{Z} . (See Remark 10.11). One checks that positivity implies that the entries are in $\mathbb{Z}_{\ge 0}$

and that contractivity implies that $\sum_{j=1}^{M} m_{k,j} c_j \leq d_k$ for k = 1, 2, ..., N. Lemma

8.19(4) implies that there is a block diagonal homomorphism $\varphi \colon A \to B$ such that $m(\varphi) = m$, and it is clear that $\varphi_* = f$ (Remark 10.11). It follows from Lemma 8.19(2) that φ is spatial.

Now suppose $\psi \colon A \to B$ is another spatial homomorphism such that $\psi_* = f$. Lemma 8.20 implies that ψ is isometrically similar to a block diagonal homomorphism. Therefore we may assume that both φ and ψ are block diagonal homomorphisms. It is easy to check that if $n \in \mathbb{Z}_{>0}$ then two block diagonal homomorphisms from A to M_n with the same partial multiplicities are similar via a permutation matrix, and are thus isometrically similar. It is now immediate that $m(\varphi) = m(\psi)$ implies that φ and ψ are isometrically similar.

Theorem 10.13. Let (G, G_+, Σ) be a countable scaled preordered abelian group. Then (G, G_+, Σ) is a scaled Riesz group if and only if for $n \in \mathbb{Z}_{>0}$ there are scaled Riesz groups $(G_n, (G_n)_+, \Sigma_n)$, each isomorphic to a group as in Remark 10.10, and positive contractive homomorphisms $f_{n+1,n}: G_n \to G_{n+1}$, such that, if for $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$ we set

$$f_{n,m} = f_{n,n-1} \circ f_{n-1,n-2} \circ \cdots \circ f_{m+1,m},$$

then

$$(G, G_+, \Sigma) \cong \underline{\lim} \left((G_n, (G_n)_+, \Sigma_n)_{n \in \mathbb{Z}_{>0}}, (f_{n,m})_{0 \leq m \leq n} \right).$$

Proof. The statement for a countable preordered abelian group (G, G_+) is Theorem 2.2 in [6]. Using Lemma 7.1 of [5] one shows that if (G, G_+, Σ) is a scaled Riesz group then the homomorphisms in the commutative diagram in Lemma 2.1 of [6] can be chosen to be positive and contractive. With this choice of homomorphisms in Theorem 2.2 of [6], one obtains the result for a scaled Riesz group.

Corollary 10.14. Let $p \in [1, \infty) \setminus \{2\}$, and let A be a spatial L^p AF algebra. Then $(K_0(A), K_0(A)_+, \Sigma(A))$ is a scaled Riesz group.

Proof. Use Theorem 10.7, Remark 10.11, and Theorem 10.13.
$$\Box$$

For completeness, we also state the result for K_1 .

Proposition 10.15. Let $p \in [1, \infty) \setminus \{2\}$, and let A be a spatial L^p AF algebra. Then $K_1(A) = 0$.

Proof. By Definition 9.1, there is a spatial L^p AF direct system $((A_m)_{m \in \mathbb{Z}_{>0}}, (\varphi_{n,m})_{0 \leq m \leq n})$

such that
$$A \cong \varinjlim_{m} A_{m}$$
. For $m \in \mathbb{Z}_{\geq 0}$, write $A_{m} = \bigoplus_{j=1}^{N(m)} M_{c_{m,j}}^{p}$. Since $K_{1}(A_{m}) = \bigoplus_{j=1}^{N(m)} K_{1}(M_{c_{j}}^{p})$,

and since $K_1(M_n^p) = 0$ for every $n \in \mathbb{Z}_{>0}$ by Example 8.1.2(a) in [1], we obtain $K_1(A_m) = 0$. Since K_1 commutes with Banach algebra direct limits (Remark 8.1.5) in [1]), the conclusion follows.

Lemma 10.16 (Proposition 5.5.5 of [1]). Let A be a Banach algebra which has an approximate identity consisting of idempotents. Then $K_0(A)$ is naturally isomorphic to the Grothendieck group of V(A).

We can now give the main classification results.

Theorem 10.17. Let $p \in [1, \infty)$. Let (G, G_+, Σ) be a countable scaled Riesz group. Then there exists a spatial L^p AF algebra A such that

$$(K_0(A), K_0(A)_+, \Sigma(A)) \cong (G, G_+, \Sigma).$$

Proof. Choose a direct system as in Theorem 10.13. For $n \in \mathbb{Z}_{>0}$, Remark 10.11 shows that there is a spatial semisimple finite dimensional L^p operator algebra A_n such that

$$(K_0(A_n), K_0(A_n)_+, \Sigma(A_n)) \cong (G_n, (G_n)_+, \Sigma_n).$$

Lemma 10.12 provides a block diagonal homomorphism $\varphi_{n+1,\,n}\colon A_n\to A_{n+1}$ such that $(\varphi_{n+1,n})_* = f_{n+1,n}$. For $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$ set

$$\varphi_{n,m} = \varphi_{n,n-1} \circ \varphi_{n-1,n-2} \circ \cdots \circ \varphi_{m+1,m} \colon A_m \to A_n.$$

Then $\varphi_{n,m}$ is spatial by Corollary 8.15 and Lemma 8.19(2).

Define $A = \varinjlim ((A_n)_{n \in \mathbb{Z}_{\geq 0}}, (\varphi_{n,m})_{0 \leq m \leq n})$. Then A is a spatial L^p AF algebra, and

$$(K_0(A), K_0(A)_+, \Sigma(A)) \cong (G, G_+, \Sigma)$$

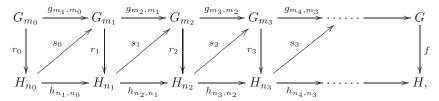
by Theorem 10.7.

The proof of Elliott's Theorem has two steps, the first of which is entirely about the category of scaled Riesz groups and positive contractive homomorphisms (and which is exactly the same in every category of algebras), and the second of which transfers the result to algebras in the appropriate category. The first step is the following lemma. Even though it is a key step in the proof, and the proof appears in a number of books, we haven't found an explicit statement of this result in the literature.

Lemma 10.18. For every $m \in \mathbb{Z}_{\geq 0}$ let $(G_m, (G_m)_+, \Sigma_m)$ and $(H_m, (H_m)_+, T_m)$ be scaled Riesz groups, each isomorphic to a group as in Remark 10.10, and for $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$ let $g_{n,m} \colon G_m \to G_n$ and $h_{n,m} \colon H_m \to H_n$ be positive contractive homomorphisms satisfying $g_{n,m} \circ g_{m,k} = g_{n,k}$ and $h_{n,m} \circ h_{m,k} = h_{n,k}$ whenever $0 \leq k \leq m \leq n$. Set

$$(G,G_+,\Sigma) = \varinjlim_{n} (G_n,(G_n)_+,\Sigma_n) \quad \text{and} \quad (H,H_+,T) = \varinjlim_{n} (H_n,(H_n)_+,T_n),$$

and let $f: G \to H$ be an isomorphism of scaled ordered groups. Then there exist $m_0, m_1, \ldots, n_0, n_1, \ldots \in \mathbb{Z}_{\geq 0}$ such that $m_0 < m_1 < \cdots$ and $n_0 < n_1 < \cdots$, and positive contractive homomorphisms $r_k \colon G_{m_k} \to H_{n_k}$ and $s_k \colon H_{n_k} \to G_{m_{k+1}}$ for $k \in \mathbb{Z}_{\geq 0}$, such that the following diagram commutes:



and such that f is the direct limit of the maps r_k for $k \in \mathbb{Z}_{\geq 0}$ and f^{-1} is the direct limit of the maps s_k for $k \in \mathbb{Z}_{\geq 0}$.

Proof. This result is contained, in slightly different language, in the proof of Theorem 7.3.2 of [1] (starting with the second diagram there). \Box

We also want the one sided version of Lemma 10.18.

Lemma 10.19. Let (G, G_+, Σ) and (H, H_+, T) be as in Lemma 10.18, and let $f: G \to H$ be a positive contractive homomorphism. Then there exist $m_0, m_1, \ldots, n_0, n_1, \ldots \in \mathbb{Z}_{\geq 0}$ such that $m_0 < m_1 < \cdots$ and $n_0 < n_1 < \cdots$, and positive contractive homomorphisms $r_k: G_{m_k} \to H_{n_k}$ for $k \in \mathbb{Z}_{\geq 0}$, such that the following diagram commutes:

$$G_{m_0} \xrightarrow{g_{m_1,m_0}} G_{m_1} \xrightarrow{g_{m_2,m_1}} G_{m_2} \xrightarrow{g_{m_3,m_2}} G_{m_3} \xrightarrow{g_{m_4,m_3}} \cdots \longrightarrow G$$

$$\downarrow r_0 \qquad \qquad \downarrow r_1 \qquad \qquad \downarrow r_2 \qquad \qquad \downarrow r_3 \qquad \qquad \downarrow r_3 \qquad \qquad \downarrow f$$

$$H_{n_0} \xrightarrow{h_{n_1,n_0}} H_{n_1} \xrightarrow{h_{n_2,n_1}} H_{n_2} \xrightarrow{h_{n_3,n_2}} H_{n_3} \xrightarrow{h_{n_4,n_3}} \cdots \longrightarrow H,$$

and such that f is the direct limit of the maps r_k for $k \in \mathbb{Z}_{\geq 0}$.

Proof. The proof is very similar to, but slightly simpler than, that of Lemma 10.18.

Theorem 10.20. Let $p \in [1, \infty)$. Let A and B be spatial L^p AF algebras, and let $f: K_0(A) \to K_0(B)$ define an isomorphism from $(K_0(A), K_0(A)_+, \Sigma(A))$ to $(K_0(B), K_0(B)_+, \Sigma(B))$. Then there is a completely isometric isomorphism $\varphi: A \to B$ such that $\varphi_* = f$.

Proof. By definition, we can write A and B as direct limits of spatial L^p AF direct systems,

$$A = \varinjlim_{m} \big((A_m)_{m \in \mathbb{Z}_{\geq 0}}, (\alpha_{n,m})_{0 \leq m \leq n} \big) \quad \text{and} \quad B = \varinjlim_{m} \big((B_m)_{m \in \mathbb{Z}_{\geq 0}}, (\beta_{n,m})_{0 \leq m \leq n} \big).$$

For $m \in \mathbb{Z}_{\geq 0}$ let $\alpha_m \colon A_m \to A$ and $\beta_m \colon B_m \to B$ be the canonical maps. Apply Lemma 10.18 with

$$(G_m, (G_m)_+, \Sigma_m) = (K_0(A_m), K_0(A_m)_+, \Sigma(A_m))$$

and

$$(H_m, (H_m)_+, T_m) = (K_0(B_m), K_0(B_m)_+, \Sigma(B_m))$$

for $m \in \mathbb{Z}_{\geq 0}$ (see Remark 10.11), with $g_{m,n} = (\alpha_{n,m})_*$ and $h_{m,n} = (\beta_{n,m})_*$ whenever $m, n \in \mathbb{Z}_{\geq 0}$ with $n \geq m$, and with

$$(G, G_+, \Sigma) = (K_0(A), K_0(A)_+, \Sigma(A))$$

and

$$(H, H_+, T) = (K_0(B), K_0(B)_+, \Sigma(B))$$

(justified by Theorem 10.7). Let

$$m_0 < m_1 < \cdots, \quad n_0 < n_1 < \cdots, \quad r_k \colon G_{m_k} \to H_{n_k}, \quad \text{and} \quad s_k \colon H_{n_k} \to G_{m_{k+1}}$$

be as in Lemma 10.18, making the diagram there commute.

We construct by induction on k spatial homomorphisms

$$\varphi_k \colon A_{m_k} \to B_{n_k}$$
 and $\psi_k \colon B_{n_k} \to A_{m_{k+1}}$

such that $(\varphi_k)_* = r_k$ and $(\psi_k)_* = s_k$ for $k \in \mathbb{Z}_{>0}$, and such that the diagram

$$(10.1) \quad A_{m_0} \xrightarrow{\alpha_{m_1,m_0}} A_{m_1} \xrightarrow{\alpha_{m_2,m_1}} A_{m_2} \xrightarrow{\alpha_{m_3,m_2}} A_{m_3} \xrightarrow{\alpha_{m_4,m_3}} \cdots \longrightarrow A$$

$$\downarrow \varphi_0 \qquad \qquad \downarrow \psi_0 \qquad \qquad \downarrow \psi_1 \qquad \qquad \downarrow \psi_2 \qquad \qquad \downarrow \psi_2 \qquad \qquad \downarrow \psi_3 \qquad \qquad \downarrow \psi_4 \qquad \qquad \downarrow \psi_4$$

commutes.

For the initial step, use the existence statement in Lemma 10.12 to choose a spatial homomorphism $\varphi_0 \colon A_{m_0} \to B_{n_0}$ such that $(\varphi_0)_* = r_0$. Use the existence statement in Lemma 10.12 to choose a spatial homomorphism $\psi_0^{(0)} \colon B_{n_0} \to A_{m_1}$ such that $(\psi_0^{(0)})_* = s_0$, and use Corollary 8.15 and the uniqueness statement in Lemma 10.12 to choose an invertible isometry $t \in A_{m_1}$ such that

$$t(\psi_0^{(0)} \circ \varphi_0)(a)t^{-1} = \alpha_{m_1, m_0}(a)$$

for all $a \in A_{m_0}$. Define ψ_0 by $\psi_0(b) = t\psi_0^{(0)}(b)t^{-1}$ for $b \in B_{n_0}$. For the induction step, suppose we have φ_k and ψ_k . Use the existence statement in Lemma 10.12 to

choose a spatial homomorphism $\varphi_{k+1}^{(0)}: A_{m_{k+1}} \to B_{n_{k+1}}$ such that $(\varphi_{k+1}^{(0)})_* = r_{k+1}$. Use the uniqueness statement in Lemma 10.12 to choose an invertible isometry $v \in B_{n_{k+1}}$ such that

$$v(\varphi_{k+1}^{(0)} \circ \psi_k)(b)v^{-1} = \beta_{n_{k+1},n_k}(b)$$

for all $b \in B_{n_k}$, and define φ_{k+1} by $\varphi_{k+1}(a) = v\varphi_{k+1}^{(0)}(a)v^{-1}$ for $a \in A_{m_{k+1}}$. The construction of ψ_{k+1} is now the same as the construction of ψ_0 in the initial step.

For $k \in \mathbb{Z}_{\geq 0}$, the map φ_k is completely contractive by Lemma 8.20, Lemma 8.19(8), and Proposition 8.2(5). Commutativity of the diagram (10.1) therefore implies the existence of a contractive homomorphism $\varphi \colon A \to B$ such that $\varphi \circ \alpha_{m_k} = \beta_{n_k} \circ \varphi_k$ for all $k \in \mathbb{Z}_{\geq 0}$, and φ must in fact be completely contractive. Similarly, we get a completely contractive homomorphism $\psi \colon B \to A$ such that $\psi \circ \beta_{n_k} = \alpha_{m_{k+1}} \circ \psi_k$ for all $k \in \mathbb{Z}_{\geq 0}$. Using the universal property of direct limits, we find that $\varphi \circ \psi = \mathrm{id}_B$ and $\psi \circ \varphi = \mathrm{id}_A$. Therefore φ and ψ are completely isometric. It is clear that $\varphi_* = f$.

Theorem 10.21. Let $p \in [1, \infty)$. Let A and B be spatial L^p AF algebras, and let $f: K_0(A) \to K_0(B)$ define a positive contractive homomorphism from $(K_0(A), K_0(A)_+, \Sigma(A))$ to $(K_0(B), K_0(B)_+, \Sigma(B))$. Then there is a completely contractive homomorphism $\varphi: A \to B$ such that $\varphi_* = f$.

Proof. The proof is a one sided version of the proof of Theorem 10.20, using Lemma 10.19 in place of Lemma 10.18. \Box

Theorem 10.22. Let $p \in [1, \infty)$. Let A and B be spatial L^p AF algebras. Then the following are equivalent:

- (1) $(K_0(A), K_0(A)_+, \Sigma(A)) \cong (K_0(B), K_0(B)_+, \Sigma(B)).$
- (2) $A \cong B$ as rings.
- (3) A is isomorphic to B (not necessarily isometrically) as Banach algebras.
- (4) A is isometrically isomorphic to B as Banach algebras.
- (5) A is completely isometrically isomorphic to B as matrix normed Banach algebras.

Proof. It is trivial that (5) implies (4), that (4) implies (3), and that (3) implies (2). Since V(A) depends only on the ring structure of A, Lemma 10.16 and Proposition 9.10 show that $K_0(A)$ depends only on the ring structure of A. It now follows directly from the definitions that $K_0(A)_+$ and $\Sigma(A)$ depend only on the ring structure of A. Thus (2) implies (1). The implication from (1) to (5) is Theorem 10.20.

References

- B. Blackadar, K-Theory for Operator Algebras, 2nd ed., MSRI Publication Series 5, Cambridge University Press, Cambridge, New York, Melbourne, 1998.
- [2] D. P. Blecher and C. Le Merdy, Operator Algebras and their Modules—an Operator Space Approach, London Mathematical Society Monographs, New Series, no. 30, Oxford Science Publications, The Clarendon Press, Oxford University Press, Oxford, 2004.
- [3] O. Bratteli, Inductive limits of finite dimensional C*-algebras, Trans. Amer. Soc. 171 (1971), 195–234.
- [4] J. B. Conway A Course in Functional Analysis, 2nd ed., Springer-Verlag Graduate Texts in Math. no. 96, Springer-Verlag, New York, Berlin, etc., 1990.
- [5] E. G. Effros, Dimensions and C*-Algebras, CBMS Regional Conf. Ser. in Math. No. 46, Amer. Math. Soc., Providence RI, 1981.

- [6] E. G. Effros, D. E. Handelman, and C. L. Shen, Dimension groups and their affine representations, Amer. J. Math. 102 (1980), 385–407.
- [7] G. A. Elliott, On the classification of inductive limits of sequences of semi-simple finite dimensional algebras, J. Algebra 38 (1976), 29-44.
- [8] R. J. Fleming and J. E. Jamison, Isometries on Banach Spaces, Vol. 1: Function Spaces, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, no. 129, Chapman & Hall/CRC, Boca Raton FL, 2003.
- [9] E. Gardella and M. Lupini, Representations of étale groupoids on L^p spaces, Adv. Math. 318 (2017), 233–278..
- [10] J. G. Glimm, On a certain class of operator algebras, Trans. Amer. Soc. 95 (1960), 318–340.
- [11] K. R. Goodearl, Partially Ordered Abelian Groups with Interpolation, Math. Surveys and Monographs no. 20, Amer. Math. Soc., Providence RI, 1986.
- [12] S. Heinrich, Ultraproducts in Banach space theory, J. reine angew. Math. 313 (1980), 72–104.
- [13] H. E. Lacey, The Isometric Theory of Classical Banach Spaces, Die Grundlehren der mathematischen Wissenchaften no. 208, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
- [14] J. Lamperti, On the isometries of certain function-spaces, Pacific J. Math. 8 (1958), 459-466.
- [15] T. W. Palmer, Banach Algebras and the General Theory of *-Algebras. Vol. I. Algebras and Banach algebras, Encyclopedia of Mathematics and its Applications no. 49, Cambridge University Press, Cambridge, 1994.
- [16] N. C. Phillips, Analogs of Cuntz algebras on L^p spaces, preprint (arXiv:1201.4196v1 [math.FA]).
- [17] N. C. Phillips, Simplicity of UHF and Cuntz algebras on L^p spaces, preprint (arXiv:1309.0115.4196v1 [math.FA]).
- [18] N. C. Phillips, Isomorphism, nonisomorphism, and amenability of L^p UHF algebras, preprint (arXiv:1309.36941 [math.FA]).
- [19] N. C. Phillips, Crossed products of L^p operator algebras and the K-theory of Cuntz algebras on L^p spaces, preprint (arXiv:1309.6406 [math.FA]).
- [20] I. Vidav, Eine metrische Kennzeichnung der selbstadjungierten Operatoren, Math. Zeitschr. Bd. 66 (1956), 121–128.

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