COMPARISON ANGLES AND VOLUME

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Abstract. We introduce a new geometric invariant called the obtuse constant of spaces with curvature bounded below, defined in terms of comparison angles. We first find relations between this invariant and volume. We discuss the case of maximal obtuse constant equal to $\pi/2$, where we prove some rigidity for spaces. Although we consider Alexandrov spaces with curvature bounded below, the results are new even in the Riemannian case.

1. Introduction

In the present paper, we introduce a new geometric invariant called the obtuse constant of a space, defined in terms of comparison angles. We investigate its properties and relate it to volume.

For this invariant, there are some historical backgrounds. For positive integer n and $D, v > 0$, let $\mathcal{M}(n, D, v)$ denote the family of n-dimensional closed Riemannian manifolds with sectional curvature ≥ -1, diameter $\leq D$ and volume $\geq v$. In [2], Cheeger proved that for every $M \in \mathcal{M}(n, D, v)$, the length of every periodic closed geodesic has length $\geq \ell_{n,D}(v) > 0$ for some uniform constant $\ell_{n,D}(v)$. In [3], Grove and Petersen extended Cheeger's argument as follows: There are positive constants $\delta = \delta_{n,D}(v)$ and $\epsilon = \epsilon_{n,D}(v)$ such that for every $M \in \mathcal{M}(n, D, v)$ and for every distinct $p, q \in M$ with distance $|p, q| < \delta$, either q is ϵ-regular to p, or p is ϵ-regular to q. Those results were keys to control local geometry of the space, and brought a significant results, topological finiteness of Riemannian manifolds (see [2], [3],[4]).

In this paper, we do not need to restrict ourselves to Riemannian manifolds. Let M be a complete Alexandrov space with curvature $\geq \kappa$. For three points p, q, x of M, $\tilde{\angle}_\kappa pqx$ denotes the comparison angle in the κ-plane $\mathbb{M}_\kappa$ at the point corresponding to q. Let $\mathcal{A}(n, D, v)$ denote the family of n-dimensional compact Alexandrov spaces with curvature ≥ -1, diameter $\leq D$ and volume $\geq v$. Grove and Petersen's result mentioned above still holds for Alexandrov spaces (see [8]). This

Date: October 3, 2017.

2010 *Mathematics Subject Classification.* 53C20, 53C21, 53C23.

Key words and phrases. comparison angle; volume; Alexandrov space; ideal boundary.

This work was supported by JSPS KAKENHI Grant Numbers 26287010, 15H05739, 15K17529.

implies that for every $M \in \mathcal{A}(n, D, v)$ and for every distinct $p, q \in M$ with $|p, q| < \delta$ there exists a point $x \in M$ such that either $\tilde{\angle}_\kappa xpq > \pi/2 + \epsilon$ or $\tilde{\angle}_\kappa xqp > \pi/2 + \epsilon$. However such a point x is assumed to be close to those points p or q in general. As we see later, if one can take such a point x relatively far away from p or q, it will be useful in some situations.

The above is a motivation to our invariants, which we are going to define in detail. First we suppose that M is compact. Let $R = \mathrm{rad}(M)$ be the radius of M:

$$R = \inf_{p \in M} \sup_{q \in M} |p, q|.$$

For $p \neq q \in M$, set

$$\mathrm{ob}_\kappa(p; q) := \sup_{x \in B(p, R/2)^c} \tilde{\angle}_\kappa xpq - \pi/2,$$

which we call the *obtuse constant of* $\{p, q\}$ *at* p, and define the *obtuse constant at* p *and* q by

$$\mathrm{ob}_\kappa(p, q) := \max\{\mathrm{ob}_\kappa(p; q), \mathrm{ob}_\kappa(q; p)\}$$

Finally we define the *obtuse constant* $\mathrm{ob}(M)$ of M as

$$\mathrm{ob}(M) := \liminf_{|p,q| \to 0} \mathrm{ob}_\kappa(p, q)$$

Note that $0 \leq \mathrm{ob}(M) \leq \pi/2$ and that $\mathrm{ob}(M)$ does not depend on the choice of the lower curvature bound κ.

Theorem 1.1. *There exists a uniform positive constant* $\epsilon_{n,D}(v)$ *such that*

$$\mathrm{ob}(M) > \epsilon_{n,D}(v)$$

for every $M \in \mathcal{A}(n, D, v)$.

More precisely, there exists also a positive constant $\delta_{n,D}(v)$ *such that if* $M \in \mathcal{A}(n, D, v)$ *and* p, q *are distinct points of* M *with* $|p, q| < \delta_{n,D}(v)$, *then* $\mathrm{ob}_{-1}(p, q) > \epsilon_{n,D}(v)$.

This generalizes the result of Grove and Petersen as stated before.

The converse to Theorem 1.1 is also true. Let $\mathcal{A}(n, D)$ denote the family of n-dimensional compact Alexandrov spaces with curvature ≥ -1 and diameter $\leq D$. Notice that the obtuse constant is rescaling invariant. Therefore for $M \in \mathcal{A}(n, D)$ it is natural to compare $\mathrm{ob}(M)$ with the *normalized volume* by the diameter defined as

$$\tilde{v}(M) := \frac{\mathrm{vol}(M)}{(\mathrm{diam}\,(M))^n}.$$

Theorem 1.2. *There exists a positive continuous function* $C_{n,D}(\epsilon)$ *with* $\lim_{\epsilon \to 0} C_{n,D}(\epsilon) = 0$ *such that for every* $M \in \mathcal{A}(n, D)$, *we have*

$$\mathrm{ob}(M) < C_{n,D}(\tilde{v}(M)).$$

In the case of nonnegative curvature, as an immediate consequence of Theorems 1.1 and 1.2, we have

Corollary 1.3. *There exist positive continuous functions* $\epsilon_n(t)$ *and* $C_n(t)$ *with* $\lim_{t\to 0}\epsilon_n(t) = \lim_{t\to 0} C_n(t) = 0$ *such that for every compact Alexandrov* n*-space* M *of nonnegative curvature, we have*

$$\epsilon_n(\tilde{v}(M)) \le \mathrm{ob}(M) \le C_n(\tilde{v}(M)).$$

From Theorems 1.1, 1.2 and Corollary 1.3, we conclude that there is a strong relation between the obtuse constant and the normalized volume.

Next we discuss the noncompact case. Suppose that M is noncompact complete Alexandrov space with curvature $\ge \kappa$ $(\kappa \le 0)$. Set

$$\mathrm{ob}_{\kappa,\infty}(p,q) := \limsup_{x\to\infty} \max\{\tilde{\angle}_\kappa xpq, \tilde{\angle}_\kappa xqp\} - \pi/2,$$

which we call the *obtuse constant at* p *and* q *from infinity.* We define the *obtuse constant* $\mathrm{ob}_\infty(M)$ *of* M *from infinity* as

$$\mathrm{ob}_\infty(M) := \liminf_{|p,q|\to 0} \mathrm{ob}_{\kappa,\infty}(p,q).$$

Clearly the obtuse constant from infinity does not depend on the choice of the lower curvature bound, and we also have $0 \le \mathrm{ob}_\infty(M) \le \pi/2$.

In the geometry of complete noncompact spaces with nonnegative curvature, the notion of asymptotic cone or volume growth rate plays an important role. For instance, any complete noncompact Riemannian manifold with nonnegative curvature having maximal volume growth is known to be diffeomorphic to an Euclidean space.

Let M be an n-dimensional complete noncompact Alexandrov space with curvature ≥ 0, and for any fixed $p \in M$, let

$$v_\infty(M) := \lim_{R\to\infty} \frac{\mathrm{vol}B(p,R)}{R^n}$$

be the volume growth rate of M.

As a noncompact version of Theorems 1.1 and 1.2, We have the following:

Theorem 1.4. *There exist continuous increasing functions* ϵ_n *and* C_n *with* $\epsilon_n(0) = C_n(0) = 0$ *such that for every complete noncompact Alexandrov* n*-space with nonnegative curvature, we have*

$$\epsilon_n(v_\infty(M)) \le \mathrm{ob}_\infty(M) \le C_n(v_\infty(M)).$$

In particular, $v_\infty(M) = 0$ *if and only if* $\mathrm{ob}_\infty(M) = 0$.

Finally we consider the maximal case of the obtuse constants equal to $\pi/2$. We need to define a variant of the notion on the injectivity radius. Let M be an Alexandrov space with curvature bounded below having no singularities, in the sense that $\mathcal{S}(D(M)) = \emptyset$, where $D(M)$ denotes the double of M. Let us denote by 1-inj(M) the supremum of

$r \geq 0$ such that for every $p \in M$ and every direction $\xi \in \Sigma_p$ at p there exists a minimal geodesic γ starting from p in the direction of at least one of ξ or the opposite $-\xi$ (if any) of length $\geq r$. We call 1-inj(M) the *one-side injectivity radius of* M. It should be noted that if $p \in \partial M$ and $\xi \in \Sigma_p \setminus \partial\Sigma_p$, then the opposite $-\xi$ does not exist, and therefore there always exists a minimal geodesic in the direction ξ of length $\geq r$. We have the following rigidity:

Theorem 1.5. *If a compact Alexandrov space M with curvature $\geq \kappa$ and radius R has* $\mathrm{ob}(M) = \pi/2$*, then* $\mathcal{S}(D(M)) = \emptyset$ *and* $1\text{-inj}(M) \geq R/2$*.*

In particular if $\mathrm{ob}(M) = \pi/2$, then M is a C^0-Riemannian manifold possibly with totally geodesic boundary (see [7]).

In the noncompact case, we have

Theorem 1.6. *If a complete noncompact n-dimensional Alexandrov space M with curvature $\geq \kappa$ has* $\mathrm{ob}_\infty(M) = \pi/2$*, then* $\mathcal{S}(D(M)) = \emptyset$ *and* $1\text{-inj}(M) = \infty$*.*

Suppose in addition that M has nonempty boundary. Then M is homeomorphic to the Euclidean half space $\mathbb{R}^n_+$, and any distinct two points of ∂M are on a line of M which is contained in ∂M.

In the case of nonnegative curvature, we have the following result. To state it, we define the notion of the *weak one-side injectivity radius*, abbreviated by $1\text{–inj}^*(M)$, of an Alexandrov space M as the supremum of those $r \geq 0$ such that for every $p \in M$ and $\xi \in \Sigma_p(M)$, there is a minimal geodesic of length $\geq r$ either in the direction ξ or in the opposite direction $-\xi$ if it exists. Note that the existence of the opposite $-\xi$ is not assumed here even in the case of $p \in M \setminus \partial M$, in contrast with one-side injectivity radius.

Theorem 1.7. *Let M be a complete noncompact n-dimensional Alexandrov space with nonnegative curvature. Suppose that* $\mathrm{ob}_\infty(M) = \pi/2$*. Then we have the following.*

1. *If M has no boundary, then* $1\text{-inj}^*(M(\infty)) \geq \pi/2$*;*
2. *If M has nonempty boundary, then M is isometric to $\mathbb{R}^n_+$.*

Note that the estimate $1\text{-inj}^*(M(\infty)) \geq \pi/2$ in Theorem 1.7 (1) is sharp, because there is a surface of revolution of nonnegative curvature satisfying $\mathrm{ob}_\infty(M) = \pi/2$ and the length of $M(\infty)$ is equal to π (see Example 6.4). We also cannot expect $1\text{-inj}(M(\infty)) \geq \pi/2$ in Theorem 1.7 (1). Namely if we replace $1\text{-inj}^*(M(\infty))$ by $1\text{-inj}(M(\infty))$, we have a counter example (see Remark 6.6).

As a consequence, we conclude that our results provide new insights for comparison angles and volume even in the Riemannian case. It should also be noted that one can define the obtuse constants for general metric space as metric invariants.

The organization of the present paper is as follows: After preliminaries about Alexandrov spaces in Section 2, we prove Theorem 1.1 in Section 3. Here the key is to construct a gradient-like curve for a DC-function, which is not a semi-concave function. For the proof of Theorem 1.2, we apply the Lipschitz submersion theorem in [15], which is carried out in Section 4. To prove Theorem 1.4, we consider the convergence to the asymptotic cone, and apply ideas of the proof of Theorems 1.1 and 1.2. This is done in Section 5. In Section 6, we discuss the case when the obtuse constants attain the maximum value $\pi/2$, where we obtain the rigidity results, Theorems 1.5, 1.6 and 1.7, together with the example showing that Theorem 1.7 is sharp (see Theorem 6.5). In Section 7, we consider another notion of κ-obtuse constant from infinity, which does depend on the choice of the lower curvature bound κ of a noncompact space. This gives more restriction on the space and we have a strong rigidity in the case of nonnegative curvature, which might be of independent interest.

2. Preliminaries

In this paper, $|x,y|$ denotes the distance between two points x,y in a metric space. An isometric embedding from an interval to a metric space is called a *minimal geodesic*. Furthermore, a fixed minimal geodesic between two points x and y is sometimes denoted by xy. For $\kappa \in \mathbb{R}$, we denote by $\mathbb{M}_\kappa$ the simply-connected complete surface of constant curvature κ, which is called the κ-plane. For distinct three points x,y,z in a metric space, we denote by $\tilde{\triangle}_\kappa xyz$ a geodesic triangle in $\mathbb{M}_\kappa$ with the length of three sides $|x,y|$, $|y,z|$ and $|z,x|$, where $|x,y|+|y,z|+|z,x| < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$. Vertices of $\tilde{\triangle}_\kappa xyz$ will be denoted by $\tilde{x},\tilde{y},\tilde{z}$. Furthermore, the angle of $\tilde{\triangle}_\kappa xyz$ at $\tilde{x}$ is denoted by $\tilde{\angle}_\kappa yxz$ and is called the *κ-comparison angle* of x,y,z at x.

2.1. **Basics of Alexandrov spaces.** Let us recall the definition of Alexandrov spaces, following [1]. An *Alexandrov space M of curvature $\geq \kappa$* is a locally complete metric space satisfying the following:

(1) for any two points in M, there exists a minimal geodesic joining them;
(2) every point has a neighborhood U such that for any two minimal geodesics xy, xz contained in U with the same starting point x, and for any $s \in xy$ and $t \in xz$, we have
$$|s,t| \geq |\tilde{s},\tilde{t}|.$$
Here, $\tilde{s} \in \tilde{x}\tilde{y}$ and $\tilde{t} \in \tilde{x}\tilde{z}$ are taken in the comparison triangle $\tilde{\triangle}_\kappa xyz = \triangle\tilde{x}\tilde{y}\tilde{z}$ with $|x,s| = |\tilde{x},\tilde{s}|$ and $|x,t| = |\tilde{x},\tilde{t}|$.

When an Alexandrov space is complete as a metric space, due to [1], the property (2) holds globally.

From the definition, the monotonicity of comparison angle holds for an Alexandrov space, that is, for two geodesics xy and xz in an Alexandrov space M of curvature $\geq \kappa$ as above, and $s \in xy - \{x\}$, $t \in xz - \{x\}$, we have

$$\tilde{\angle}_\kappa yxz \leq \tilde{\angle}_\kappa sxt. \tag{2.1}$$

In particular, the limit

$$\angle(xy, xz) := \lim_{xy \ni s \to x,\, xz \ni t \to x} \tilde{\angle}_\kappa sxt$$

always exists. It is called the *angle* between xy and xz. When the geodesics xy and xz are fixed, we write $\angle yxz = \angle(xy, xz)$. By the definition of the angle, we obtain

$$\angle yxz \geq \tilde{\angle}_\kappa yxz. \tag{2.2}$$

When an Alexandrov space is complete, (2.1) and (2.2) are also true for any geodesics.

From now on, M denotes an Alexandrov space of curvature $\geq \kappa$. Furthermore, we assume that M has at least two points. For a point $x \in M$, let us set Γ_x the set of all non-trivial geodesics starting from x. It is known that the angle $\angle$ is a pseudo-distance function on Γ_x. The completion of the metric space induced from $(\Gamma_x, \angle)$ is called the *space of directions at p (in M)* which is denoted by $\Sigma_x = \Sigma_x M$. The distance function on Σ_x is written as $\angle$, the same symbol as the angle. An element of Σ_x is called a direction at x. Furthermore, for geodesics $xy, xz \in \Gamma_x$, $\angle yxz = 0$ if and only if $xy \subset xz$ or $xz \subset xy$ as the images of geodesics. In particular, any Alexandrov space does not admit a branching geodesic. The equivalent class of xy is denoted by $\uparrow_x^y$. Let $\Uparrow_x^y$ denote the set of all directions of geodesics from x to y.

It is known that the Lebesgue covering dimension of M is the same as the Hausdorff dimension of it, which is called the *dimension of M* and is written as $\dim M$ ([1], [12]). From now on, we assume that $\dim M < \infty$. This assumption implies that, the space of directions Σ_x at $x \in M$ is compact and becomes an Alexandrov space of curvature ≥ 1 and of dimension equal to $\dim M - 1$. Here, we used a convention that the metric space of two points with distance π is regarded as an Alexandrov space of curvature ≥ 1 and dimension zero.

A point $p \in M$ is *regular* if Σ_p is isometric to the standard unit sphere of constant curvature one. For $\delta > 0$, $p \in M$ is *δ-strained* if there exists a collection $\{(a_i, b_i)\}_{1 \leq i \leq n}$ of pairs of points, where $n = \dim M$, such that

$$\tilde{\angle}_\kappa a_i p b_i > \pi - \delta, \qquad \tilde{\angle}_\kappa a_i p a_j > \pi/2 - \delta,$$
$$\tilde{\angle}_\kappa b_i p b_j > \pi/2 - \delta, \qquad \tilde{\angle}_\kappa a_i p b_j > \pi/2 - \delta,$$

hold for all $i < j$. Such a collection $\{(a_i, b_i)\}$ is called a *δ-strainer at p*. Let us denote by δ-str.rad(p) the supremum of $\min\{|p, a_i|, |p, b_i|\}_i$, where the supremum runs over all δ-strainers at p, which is called the *δ-strained radius at p.*

The set $\mathcal{R}_\delta(M)$ of all δ-strained points in M is known to have full measure in the n-dimensional Hausdorff measure $\mathcal{H}^n$ ([1], [7]). In particular, $\mathcal{R}(M) = \bigcap_{\delta>0} \mathcal{R}_\delta(M)$ also has full measure and is dene in M. A point in $\mathcal{R}(M)$ is said to be regular. It is known that p is regular if and only if Σ_p is isometric to the sphere of constant curvature one. A point in $M \setminus \mathcal{R}_\delta(M)$ (resp. in $M \setminus \mathcal{R}(M)$) is said to be δ-singular (resp. singular). The set of all singular points is denoted by $\mathcal{S}(M)$.

Let $\mathrm{Cut}(x)$ be the cut locus of $x \in M$ which is defined by

$$\mathrm{Cut}(x) = \{y \in M \mid \text{For all } z \neq y, \text{ we have } |xz| < |xy| + |yz|\}.$$

It is known that $\mathrm{Cut}(x)$ has zero measure in $\mathcal{H}^n$ ([7]).

2.2. **The first variation formula.** Let us consider $|p, \cdot\,|$ the distance function from a point p in an Alexandrov space M. The first variation formula for $|p, \cdot\,|$ holds as well as Riemannian cases, that is,

$$\lim_{rq \ni x \to q} \frac{|p,x| - |p,q|}{|x,q|} = -\cos \angle(\Uparrow_q^p, \uparrow_q^r).$$

Remark 2.1. Distance functions and semiconcave functions are fundamental tools to study Alexandrov spaces. Perelman and Petrunin gave a theory of gradient flows of general semiconcave functions ([9], [11]). On the other hands, the difference of distance functions $|p, \cdot\,| - |q, \cdot\,|$ is not semiconcave, and is contained in a class of functions, so-called DC-functions. For (general) DC-functions, there is no reasonable theory of gradient flows. A key of this paper is studying a gradient-like flow of the differences of distance functions.

3. Proof of Theorem 1.1

As indicated in Section 2, $\mathcal{R}(M)$ and $\mathcal{S}(M)$ denote the regular set and the singular set of M respectively.

First we need the following.

Sublemma 3.1. *There exists a positive number $\sigma_0 = \sigma_0(n, D, v)$ such that for any point p of every space M in $\mathcal{A}(n, D, v)$ there is an open metric ball B of radius $\geq \sigma_0$ in M such that*

(1) $|p, B| > \mathrm{rad}\,(M)/2$;
(2) *B is homeomorphic to an open disk D^n.*

Proof. Suppose Sublemma 3.1 does not hold. Then there are sequences $\sigma_i \to 0$, $M_i \in \mathcal{A}(n, D, v)$ and $p_i \in M_i$ not satisfying the conclusion of the sublemma. Namely there are no metric ball B_i of radius $\geq \sigma_i$ satisfying the conclusions (1), (2) for (M_i, x_i). By the compactness of $\mathcal{A}(n, D, v)$, we may assume that (M_i, p_i) converges to (M, p) in

$\mathcal{A}(n, D, v)$. Take $q \in \mathcal{R}(M)$ with $|p, q| > \frac{2}{3}\mathrm{rad}\,(M)$. If $\sigma > 0$ is small enough,

(1) $B(q, \sigma)$ is almost isometric to a σ-ball in $\mathbb{R}^n$;
(2) if $q_i \in M_i$ is chosen as $q_i \to q$, $B(q_i, \sigma)$ is also almost isometric to a σ-ball in $\mathbb{R}^n$.

Now it turns out that $B(q_i, \sigma)$ is homeomorphic to D^n and

$$|p_i, B(q_i, \sigma)| > \mathrm{rad}\,(M_i)/2,$$

which is a contradiction. $\square$

The proof of Theorem 1.1 is as follows. For every $M \in \mathcal{A}(n, D, v)$ and $p \neq q \in M$, we set for simplicity

$$d := \mathrm{diam}\,(M), \qquad \delta := |p, q|.$$

Notice that $d/2 \leq R \leq d$. We assume $\delta \leq d/100$. Let $B = B(x_0, \sigma_0)$ be the metric ball determined in the previous lemma for p. To prove Theorem 1.1, it suffices to show

Assertion 3.2. *There is a point z in B such that either $\tilde{\angle} pqz \geq \pi/2 + \epsilon_0$ or $\tilde{\angle} qpz \geq \pi/2 + \epsilon_0$ for some uniform constant $\epsilon_0 = \epsilon_{n,D}(v) > 0$.*

Proof. Let us consider the function $f : B \to \mathbb{R}$ defined by

$$f(x) = |p, x| - |q, x|.$$

Constructing a gradient-like curve of f, we shall find a required point $z \in B$. Suppose that

$$\tilde{\angle} pqx \leq \pi/2 + \epsilon_0 \text{ and } \tilde{\angle} qpx \leq \pi/2 + \epsilon_0 \tag{3.3}$$

for all $x \in B$ and some $\epsilon_0 > 0$. Later we shall find such an explicit constant ϵ_0 which yields contradiction.

Under the above situation, we have

Sublemma 3.3. *There exists a uniform positive number $\epsilon_0 = \epsilon_{n,D}(v)$ and $\delta_0 = \delta_{n,D}(v)$ such that if $\delta \leq \delta_0$, for every $x \in B$ we have*

$$\tilde{\angle} pxq > \frac{\sinh \delta}{2 \sinh d}. \tag{3.4}$$

Proof. We may assume that $\tilde{\angle} xpq \leq \tilde{\angle} xqp$. Let $\tilde{w}$ be the nearest point on the geodesic segment $\tilde{p}\tilde{x}$ from $\tilde{q}$. Set $\tilde{\theta} := \tilde{\angle} pxq$. From the law of sines, we have

$$\sinh |\tilde{q}, \tilde{w}| = \sinh |q, x| \sin \tilde{\theta} \leq \tilde{\theta} \sinh d \tag{3.5}$$

$$\sinh |\tilde{q}, \tilde{w}| = \sinh \delta \sin \tilde{\angle} qpx, \tag{3.6}$$

which implies

$$\tilde{\theta} \geq \frac{\sinh \delta}{\sinh d} \sin \tilde{\angle} qpx.$$

On the other hand, by the area comparison theorem, the area $A(\tilde{\triangle}_{-1} pqz)$ of the comparison triangle of $\triangle pqx$ in the hyperbolic plane is less than the area $A(\tilde{\triangle}_0 pqx)$ of the comparison triangle in the Euclidean plane. It follows that

$$A(\tilde{\triangle}_{-1} pqx) \leq |p, x|\delta/2 \leq \delta D/2.$$

Let $\epsilon_0 = \epsilon_{n,D}(v)$ be a uniform positive constant which will be determined later. From (3.3), we have $\tilde{\angle} xqp \leq \pi/2 + \epsilon_0$. It follows from the Gauss-Bonnet formula that

$$A(\tilde{\triangle}_{-1} pqx) = \pi - \tilde{\angle} xpq - \tilde{\angle} xqp - \tilde{\theta} \geq \pi/2 - \tilde{\angle} xpq - \epsilon_0 - \tilde{\theta}.$$

Therefore,

$$\frac{\pi}{2} + \epsilon_0 \geq \tilde{\angle} xpq \geq \pi/2 - (\epsilon_0 + \tilde{\theta} + \delta D/2).$$

Here we take ϵ_0 and δ_0 such that

$$\epsilon_0 < 1/10, \ \ \delta \leq \delta_0 \leq \epsilon_0/D.$$

Since d has a positive lower bound $d_0 = d(n, D, v)$, if δ_0 is chosen so that

$$\frac{\sinh \delta_0}{\sinh d_0} \leq \epsilon_0,$$

we may assume that $\tilde{\theta} < \epsilon_0/2$. It follows that

$$\frac{\pi}{2} + \epsilon_0 \geq \tilde{\angle} xpq \geq \pi/2 - 2\epsilon_0,$$

and hence

$$\tilde{\theta} \geq \frac{\sinh \delta}{\sinh d} \cos 2\epsilon_0 > \frac{\sinh \delta}{2 \sinh d}.$$

□

Sublemma 3.4. *f is regular on B. More precisely, for every $x \in B$, there is $\xi \in \Sigma_x$ such that*

(1) $df(\xi) > \frac{1}{3}\left(\frac{\sinh \delta}{2\sinh d}\right)^2$;
(2) $df(\xi) > \frac{\sinh \delta}{3 \sinh d}$ *if* $x \in B \cap \mathcal{R} \setminus (Cut(p) \cup Cut(q))$.

Proof. By the first variation formula, for every $v \in \Sigma_x$, we have

$$f'(v) = -\cos|v, \Uparrow_x^p| + \cos|v, \Uparrow_x^q|.$$

By Sublemma 3.3

$$\angle(\Uparrow_x^p, \Uparrow_x^q) \geq \tilde{\angle} pxq > \frac{\sinh \delta}{2 \sinh d}.$$

First take a direction $\xi \in \Uparrow_x^q$. Then it follows from $\delta \ll d$ that

$$df(\xi) > -\cos\left(\frac{\sinh \delta}{2 \sinh d}\right) + 1 > \frac{1}{3}\left(\frac{\sinh \delta}{2 \sinh d}\right)^2.$$

Next for any $x \in B \cap \mathcal{R} \setminus (Cut(p) \cup Cut(q))$, let $\xi_p \in \Sigma_x$, $\xi_q \in \Sigma_x$ be the unique directions from x to p and x to q respectively. Since Σ_x is isometric to the unit sphere, it is possible to find a $\xi \in \Sigma_x$ such that $\angle(\xi, \xi_q) = \pi/2$ and $\angle(\xi, \xi_p) = \pi/2 + \angle(\xi_p, \xi_q)$. Then we have

$$df(\xi) = -\cos\angle(\xi, \xi_p) = \sin\angle(\xi_p, \xi_q).$$

If $\angle(\xi_p, \xi_q) < \pi - \frac{\sinh\delta}{2\sinh d}$, then $df(\xi) > \frac{\sinh\delta}{3\sinh d}$. Suppose $\angle(\xi_p, \xi_q) \geq \pi - \frac{\sinh\delta}{2\sinh d}$. Then letting $v :=\uparrow_x^q$, we obtain

$$df(v) > 1 > \frac{\sinh\delta}{3\sinh d}.$$

$\square$

Next we construct a gradient-like curve $c(t)$ of f starting from $c(0) = x_0$. First we may assume that $x_0 \in B \cap \mathcal{R} \setminus (Cut(p) \cup Cut(q))$. For any $y \in B$, take a direction $V_y \in \Sigma_y$ in such a way that

$$\begin{aligned} df(V_y) &> \frac{1}{3}\left(\frac{\sinh\delta}{2\sinh d}\right)^2 && \text{if } y \notin B \cap \mathcal{R} \setminus (Cut(p) \cup Cut(q)), \\ df(V_y) &> \frac{\sinh\delta}{3\sinh d} && \text{if } y \in B \cap \mathcal{R} \setminus (Cut(p) \cup Cut(q)). \end{aligned}$$

Let $c_1 : [0, \ell_1] \to B$ be a unit speed curve starting from x_0 such that $c_1'(0) = V_{x_0}$ and

$$f(c_1(\ell_1)) \geq f(c_1(0)) + \ell_1 \frac{\sinh\delta}{4\sinh d}.$$

In general one cannot expect that $c_1(\ell_1) \in B \cap \mathcal{R} \setminus (Cut(p) \cup Cut(q))$. From this reason, we take a unit speed curve $d_1 : [0, r_1] \to B$ starting from $c_1(\ell_1)$ such that

(1) $df(d_1'(0)) > \frac{1}{3}\left(\frac{\sinh\delta}{2\sinh d}\right)^2$;
(2) $r_1 < \ell_1$;
(3) $f(d_1(r_1)) \geq f(d_1(0)) + \frac{r_1}{3}\left(\frac{\sinh\delta}{2\sinh d}\right)^2$;
(4) $d_1(r_1) \in B \cap \mathcal{R} \setminus (Cut(p) \cup Cut(q))$.

The last condition is possible because $Cut(p) \cup Cut(q) \cup \mathcal{S}$ has measure zero (See [7], [1]). Next let $c_2 : [0, \ell_2] \to B$ be a unit speed curve starting from $d_1(r_1)$ such that $c_2'(0) = V_{d_1(r_1)}$ and

$$f(c_2(\ell_2)) \geq f(c_2(0)) + \ell_2 \frac{\sinh\delta}{4\sinh d}.$$

Repeating this procedure, we obtain a rectifiable curve $c_1 \cup d_1 \cup c_2 \cup d_2 \cup c_3 \cup d_3 \cup \cdots$. Since $f \leq \delta$ on B, it is easy to see that there is a positive integer N such that

$$c := c_1 \cup d_1 \cup \cdots \cup c_N \cup d_N : [0, \ell_1 + r_1 + \cdots \ell_N + r_N] \to B$$

finally reach the boundary ∂B. Set $\ell := \sum_{i=1}^N \ell_i$, $r := \sum_{i=1}^N r_i$. Obviously, we have $\ell + r \geq \sigma_0$. It follows from $\ell \geq r$ that $\ell \geq \sigma_0/2$. Note that

$$f(c(\ell + r)) > f(x_0) + \ell \frac{\sinh\delta}{2\sinh d}.$$

Set $z := c(\ell + r)$ for simplicity. Then

$$f(z) = |p,z| - |q,z| > |p,x_0| - |q,x_0| + \ell\frac{\sinh\delta}{2\sinh d}. \tag{3.7}$$

Recall the assumption (3.3). First we assume $f(x_0) \le 0$ for simplicity. From $\tilde\angle qpx_0 \le \pi/2 + \epsilon_0$, the law of cosines implies

$$\cosh|p,x_0|\cosh\delta - \cosh|q,x_0| \ge -\sin\epsilon_0\sinh\delta\sinh|p,x_0|,$$

which implies

$$\cosh|p,x_0| - \cosh|q,x_0| \ge -2\delta(\epsilon_0+\delta)\cosh d.$$

Therefore together with $f(x_0) \le 0$, the mean value theorem implies that

$$|p,x_0| - |q,x_0| \ge -2\delta(\epsilon_0+\delta)\frac{\cosh d}{\sinh d/4}. \tag{3.8}$$

Similarly from $\tilde\angle pqz \le \pi/2 + \epsilon_0$, we have

$$\cosh|p,z| - \cosh|q,z| \le 2\delta(\epsilon_0+\delta)\cosh d,$$

which implies

$$|p,z| - |q,z| \le 2\delta(\epsilon_0+\delta)\frac{\cosh d}{\sinh d/4}. \tag{3.9}$$

It follows from $\ell \ge \sigma_0/2$, (3.7), (3.8) and (3.9) that

$$|p,z| - |q,z| \ge \delta\left(\sigma_0/4 - 2(\epsilon_0+\delta)\frac{\cosh d}{\sinh d/4}\right).$$

Together with (3.9), this implies

$$\sigma_0 < 16(\epsilon_0+\delta)\frac{\cosh d}{\sinh d/4}.$$

Here we denote by $SC(D)$ the maximum of the function $\varphi(t) = \frac{\sinh t/4}{\cosh t}$ on $[0, D]$. If we set

$$\epsilon_0 = SC(D)\sigma_0/20, \qquad \delta_0 \le \epsilon_0/10,$$

and assume $\delta \le \delta_0$, we have a contradiction.

The case when $f(x_0) \ge 0$ is similar. This completes the proof of Theorem 1.1. $\square$

4. Collapsing case

In this section, we prove Theorem 1.2 by contradiction.

We say that a surjective map $f : M \to X$ between Alexandrov spaces is an ϵ-almost Lipschitz submersion if

(1) it is an ϵ-approximation;

(2) for every $p, q \in M$, we have

$$\left| \frac{|f(p), f(q)|}{|p, q|} - \sin\theta \right| < \epsilon,$$

where θ denotes the infimum of $\angle qpx$ when x runs over the fiber $f^{-1}(f(p))$.

We recall the following result from [15, Theorem 0.2 and Lemma 4.19].

Theorem 4.1. *For given positive integer m and $\mu_0 > 0$ there are $\delta = \delta_m > 0$ and $\epsilon = \epsilon_m(\mu_0) > 0$ satisfying the following: Let X be an m-dimensional complete Alexandrov space with curvature ≥ -1 and with δ-str-rad$(X) > \mu_0$. Then if the Gromov-Hausdorff distance between X and a complete Alexandrov space M with curvature ≥ -1 is less than ϵ, then there exists a map $f : M \to X$ such that*

(1) *it is a $\tau(\delta, \epsilon)$-almost Lipschitz submersion;*
(2) *it is $(1 - \tau(\delta, \epsilon))$-open in the sense that for every $p \in M$ and $x \in X$ there exists a point $q \in f^{-1}(x)$ such that $|f(p), f(q)| \geq (1 - \tau(\epsilon, \delta))|p, q|$.*

Here $\tau(\delta, \epsilon)$ is a positive constant depending only on n,μ_0 and δ, ϵ satisfying $\lim_{\delta,\epsilon\to 0} \tau(\delta, \epsilon) = 0$.

Proof of Theorem 1.2. Suppose it is not true. Then there would exist a sequence M_i in $\mathcal{A}(n, D)$ with $\tilde{v}(M_i) \to 0$ and $\mathrm{ob}(M_i) > c > 0$ for some uniform constant c. When $\mathrm{diam}\,(M_i) \to 0$, we rescale the metric so that $\mathrm{diam}\,(M_i) = 1$ with respect to the new metric. Then since $\mathrm{vol}(M_i) \to 0$, passing to a subsequence, we may assume that M_i collapses to a lower dimensional Alexandrov space X with $\dim X \geq 1$. Let $m = \dim X$, and take a regular point x_0 of X and small $\epsilon_0 > 0$ such that the $B(x_0, r_0) \subset R_\delta(X)$ with $\delta < \delta_m$ and that the δ-strain radius of $B(x_0, r_0)$ is greater than a constant $\mu_0 > 0$. Applying Theorem 4.1 to $B := B(x_0, r_0)$, we have a $\tau(\delta, \epsilon)$-almost Lipschitz submersion $f : U_i \to B$. By the coarea formula (see [5] for instance), we obtain

$$\int_{U_i} C_n(f, p) d\mathcal{H}^n(p) = \int_B \mathcal{H}^{n-m}(f^{-1}(x))\, d\mathcal{H}^m(x),$$

where $C_n(f, p)$ denotes the coarea factor at p. Since f is $\tau(\epsilon, \delta)$-almost Lipschitz submersion, we see that $|C_n(f, p) - 1| < \tau(\epsilon, \delta)$. Let B_0 be the set of points $x \in B$ such that $\mathcal{H}^{n-m}(f^{-1}(x)) > 0$. It follows that B_0 is dense in B. For $x \in B_0$ and for every $\nu > 0$, one can take distinct points p and q in $f^{-1}(x)$ which are sufficiently close to each other. Then for every $x \in B(p, R/2)^c$ take a minimal geodesic $\gamma : [0, d] \to M$ from p to x. Lemma 4.11 of [15] shows that $\angle(\uparrow_p^q, H_p) < \tau(\epsilon, \delta)$, where $H_p \subset \Sigma_p$ denotes the horizontal directions p defined in [15]. Since $\gamma'(0)$ is in an almost horizontal direction, we conclude that $|\angle xpq - \pi/2| < \tau(\epsilon, \delta)$.

Similarly we have $|\angle xqp - \pi/2| < \tau(\epsilon, \delta)$. This implies

$$|\tilde{\angle} xpq - \pi/2| < \tau(\epsilon, \delta), \qquad |\tilde{\angle} xqp - \pi/2| < \tau(\epsilon, \delta).$$

Similarly we have $|\tilde{\angle} ypq - \pi/2| < \tau(\epsilon, \delta)$ and $|\tilde{\angle} yqp - \pi/2| < \tau(\epsilon, \delta)$ for every $x \in B(q, R/2)^c$, and therefore $\mathrm{ob}_\kappa(p, q) < \tau(\epsilon, \delta)$. This completes the proof of Theorem 1.2. □

Problem 4.2. *Probably, the fiber $f^{-1}(x)$ has positive $(n-m)$-dimensional Hausdorff measure for all $x \in X$ in the situation of Theorem 4.1.*

Proof of Corollary 1.3. The conclusion follows from Theorems 1.1 and 1.2. The desired functions ϵ_n and C_n in the conclusion are defined as follows, for instance. We construct only ϵ_n. Let

$$\mathcal{A} := \left\{ M \;\middle|\; \begin{array}{l} M \text{ is an } n\text{-dimensional compact} \\ \text{Alexandrov space of nonnegative curvature} \end{array} \right\}$$

and set

$$\epsilon_n'(\tilde{v}) := \inf \{ \mathrm{ob}(M) \mid M \in \mathcal{A} \text{ with } \tilde{v}(M) \geq \tilde{v} \}$$

for $\tilde{v} > 0$. Then, ϵ_n' satisfies

$$\epsilon_n'(\tilde{v}(M)) \leq \mathrm{ob}(M)$$

for every $M \in \mathcal{A}$. Furthermore, by Theorem 1.1, $\epsilon_n'(\tilde{v}) > 0$ for any $\tilde{v} > 0$. From Theorem 1.2, we have

$$\lim_{\tilde{v} \to 0} \epsilon_n'(\tilde{v}) = 0.$$

Note that the problem of maximizing $\tilde{v}(M)$ in $\mathcal{A}$ is equivalent to the problem of maximizing the usual volume in the restricted class of M's whose diameter is one, because $\mathrm{ob}(M)$ and $\tilde{v}(M)$ are scale invariants. Since a maximizing sequence in the latter class has a convergent subsequence, there is a maximal value of $\tilde{v}(M)$ in $\mathcal{A}$, say $\tilde{v}_{n,\max}$.

Let us define a step function $\epsilon_n'' : (0, \tilde{v}_{n,\max}] \to [0, \pi/2]$ by

$$\epsilon_n''(\tilde{v}) := \epsilon_n'(\tilde{v}_{n,\max}/k) \text{ if } \tilde{v} \in (\tilde{v}_{n,\max}/k, \tilde{v}_{n,\max}/(k-1)]$$

which bounds ϵ_n' from below. Furthermore, we consider the piecewise linear function connecting points $(\tilde{v}_{n,\max}/(k-1), \epsilon_n''(\tilde{v}_{n,\max}/k))$'s. Then, the function ϵ_n satisfies the desired condition of the conclusion of Corollary 1.3. □

5. Volume growth and obtuse constant from infinity

This section is devoted to prove Theorem 1.4.

In this section, let M denote noncompact complete Alexandrov n-space of nonnegative curvature. As written in the introduction, we discuss about a relation between the volume growth rate

$$v_\infty(M) = \lim_{R \to \infty} \frac{\mathrm{vol} B(x, R)}{R^n}$$

and the obtuse constant from infinity.

Proof of Theorem 1.4. We first prove that when $\mathrm{ob}_\infty(M)$ is small, so is $v_\infty(M)$. Let us fix $\epsilon_0 > 0$ so that

$$\mathrm{ob}_\infty(M) < \epsilon_0.$$

From this, there exist distinct nearby two points $p, q \in M$ such that

$$\mathrm{ob}_{0,\infty}(p, q) < \epsilon_0.$$

For $z \in M \setminus \{p, q\}$, we set

$$\epsilon(z) = \max\{\tilde{\angle}_0 qpz, \tilde{\angle}_0 pqz\} - \pi/2.$$

Then, we have $\limsup_{z\to\infty} \epsilon(z) < \epsilon_0$.

If $v_\infty(M) = 0$, we have nothing to do. Suppose that $v_\infty(M) > 0$. Then, the asymptotic cone

$$(M_\infty, x_\infty) = \lim_{R\to\infty} \left(\frac{1}{R}M, x\right)$$

of M is n-dimensional, where $x \in M$ is any reference point. By an argument similar to the proof of Sublemma 3.1, there exists $\sigma_0 = \sigma_0(v_\infty(M)) > 0$ satisfying that for any $R > 0$, there exists an almost regular metric ball $B = B(p_0, \sigma_0 R)$ with $B \subset B(x, 2R)\setminus B(x, R)$. Here, a metric ball $B(p_1, r_1)$ is almost regular if each point of it is δ-strained for sufficiently small $\delta > 0$ and it is almost isometric to the Euclidean r_1-ball.

We now apply the argument in the proof of Theorem 1.1. Consider the function $f(z) = |p, z| - |q, z|$ defined on B. Let $\delta := |p, q|$. In a way similar to Sublemma 3.3, we have

$$\tilde{\angle} pzq \geq \delta/2R$$

for every $z \in B$. This yields that f is $(\delta/2R)^2$-regular on B and $\delta/3R$-regular on $B \cap \mathcal{R}(M) \setminus (\mathrm{Cut}(p) \cup \mathrm{Cut}(q))$. By the law of cosines,

$$|p, z|^2 \leq \delta^2 + |q, z|^2 + 6\delta R \sin \epsilon(z). \tag{5.10}$$

It follows that

$$|p, z| - |q, z| \leq \delta^2/2R + 3\delta\epsilon(z) \leq \delta^2/2R + 4\delta\epsilon_0, \tag{5.11}$$

for large enough R. In a way similar to the proof of Theorem 1.1, one can construct a gradient like curve $c : [0, \ell + r] \to B$ from $c(0) = p_0$ to $w := c(1) \in \partial B$ such that

$$f(w) - f(p_0) \geq (\delta/3R)(\sigma_0 R/2) = \delta\sigma_0/6. \tag{5.12}$$

It follows from (5.11) and (5.12) that $\sigma_0 \leq 3\delta/R + 24\epsilon_0 < 25\epsilon_0$ for large enough R. This argument also shows that the radius of any almost regular metric ball B in M with $B \subset B(x, 2R) \setminus B(x, R)$ is less than $25\epsilon_0 R$. The following Sublemma 5.1 then implies that $v_\infty(M) < \tau_n(\epsilon_0)$. This completes the proof of a half of statements in Theorem 1.4.

Sublemma 5.1. *There exists a positive function $\tau_n(\delta)$ such that $\lim_{\delta\to 0}\tau_n(\delta) = 0$ satisfying the following: Let (M,x) be a pointed complete noncompact n-dimensional Alexandrov space with nonnegative curvature, and suppose that for any sufficiently large $R > 0$, the radius of almost regular metric ball with $B \subset B(x,2R)\setminus B(x,R)$ is less than δR. Then we have*

$$v_\infty(M) \le \tau_n(\delta).$$

Proof. We prove the sublemma by contradiction. Suppose that the sublemma does not hold. Then we have a sequence (M_i, x_i) of pointed complete noncompact n-dimensional Alexandrov space with nonnegative curvature such that

(1) $\liminf_{i\to\infty} v_\infty(M_i) > 0$;
(2) if $B_i \subset M_i$ is any almost regular metric ball with $B_i \subset B(x_i,2R_i)\setminus B(x_i,R_i)$ and $R_i \to \infty$, then the radius of B_i is less than $\delta_i R_i$, where $\delta_i \to 0$.

Passing to a subsequence, we may assume that $(\frac{1}{R_i}M_i, x_i)$ converges to a complete noncompact pointed Alexandrov space (M_∞, x_∞) of nonnegative curvature. From the condition (1) above, we have $\dim M_\infty = n$. Take an almost regular metric ball B_∞ in M_∞ of radius, say σ_∞, such that $B_\infty \subset B(x_\infty,2)\setminus B(x_\infty,1)$. It is now possible to take a regular balls $\hat{B}_i$ of M_i converging to B_∞ under the convergence $(\frac{1}{R_i}M_i, x_i) \to (M_\infty, x_\infty)$. Note $\hat{B}_i \subset B(x_i,2R_i)\setminus B(x_i,R_i)$ and the radius of $\hat{B}_i$ is at least $R_i\sigma_\infty/2$ with respect to the original metric. This is a contradiction. □

Let us continue the proof of Theorem 1.4. The rest of the statement which has not been proven yet is: when $v_\infty(M)$ is small, so is $\mathrm{ob}_\infty(M)$. Let us prove it by contradiction. Let us assume that there exists a sequence $\{M_i\}$ of noncompact complete nonnegatively curved Alexandrov n-spaces such that $v_\infty(M_i) \to 0$ and $\inf_i \mathrm{ob}_\infty(M_i) > 0$. Let us fix reference points $p_i \in M_i$. From the definition of the volume growth rate, there exists a sequence of positive numbers R_i such that

$$\frac{\mathrm{vol}(B(p_i,R_i))}{R_i^n} - v_\infty(M_i) < i^{-1}.$$

Hence, the unit ball in the scaled space $\frac{1}{R_i}M_i$ around p_i is volume-collapsing. Here, we may assume $R_i \to \infty$ as $i \to \infty$. Therefore, extracting a subsequence, we may assume that $(\frac{1}{R_i}M_i, p_i)$ converges to a noncompact complete Alexandrov space (X, p_0) of nonnegative curvature whose dimension is less than n.

Let us take an almost regular domain V in X. Then for sufficiently large i and for some domain U_{R_i} in $\frac{1}{R_i}M_i$ we have an almost Lipschitz submersion $f_i : U_{R_i} \to V$. By an argument similar to the proof of Theorem 1.2, we can choose a point $y_0 \in V$ such that its fiber $f_i^{-1}(y_0)$

contains close distinct points p, q. Then, from [15], we obtain $\epsilon(R)$ with $\lim_{R\to\infty}\epsilon(R) = 0$ such that for any point $z \in M_i$ with $|p, z| \geq R_i$,

$$|\tilde{\angle}_0 pqz - \pi/2| < \epsilon(R_i),\ |\tilde{\angle}_0 qpz - \pi/2| < \epsilon(R_i).$$

It contradicts to $\inf_i \mathrm{ob}_\infty(M_i) > 0$. This completes the proof of Theorem 1.4. □

6. Maximal cases

In this section, let us discuss the maximal case of the obtuse constants equal to $\pi/2$. We need to define a variant of the notion on the injectivity radius. Let M be an Alexandrov space with curvature bounded below having no singularities, in the sense that $\mathcal{S}(D(M)) = \emptyset$, where $D(M)$ denotes the double of M. Let us denote by 1-inj(M) the supremum of $r \geq 0$ satisfying the following:

(1) for every $p \in M \setminus \partial M$ and $\xi \in \Sigma_p$ (resp. for every $p \in \partial M$ and $\xi \in \partial\Sigma_p$), there exists a minimal geodesic γ starting from p in the direction ξ or $-\xi$ of length $\geq r$;
(2) for every $p \in \partial M$ and $\xi \in \Sigma_p \setminus \partial\Sigma_p$, there exists a minimal geodesic in the direction ξ of length $\geq r$;

We call 1-inj(M) the *one-side injectivity radius of M*.

One of main results of this section is to prove the following, which is a detailed version of Theorem 1.5.

Theorem 6.1. *If a compact Alexandrov space M with curvature $\geq \kappa$ and radius R has* $\mathrm{ob}(M) = \pi/2$*, then* $\mathcal{S}(D(M)) = \emptyset$ *and* 1-inj$(M) \geq R/2$.

Moreover we have the following in this case:

(1) *For every $p \in \partial M$ and every $\xi \in \Sigma_p$ there exists a minimal geodesic in the direction ξ of length $\geq R/2$.*
(2) *For every $p \in M \setminus \partial M$ and every $\xi \in \Sigma_p$ there exists a minimal geodesic in the direction ξ of certain length > 0.*

In particular, M is a C^0-Riemannian manifold possibly with totally geodesic boundary.

We begin with

Lemma 6.2. *If* $\mathrm{ob}(M) = \pi/2$*, then*

(1) *M has no singular points except boundary points, in the sense that $\mathcal{S}(D(M)) = \emptyset$, and M is a C^0-Riemannian manifolds;*
(2) 1-inj$(M) \geq R/2$.

Moreover, the conclusion (1) *of Theorem 6.1 holds. In particular, ∂M is totally geodesic in M.*

Proof. Let M be as in the assumption, and $p \in M$. We first show that $\mathrm{diam}\,(\Sigma_p) = \pi$. Take $\xi, \eta \in \Sigma_p$ with $\angle(\xi, \eta) = \mathrm{diam}\,(\Sigma_p)$, and suppose that $\angle(\xi, \eta) < \pi$. Let us take sequences $x_i, y_i \in M$ such that

$|p, x_i| = |p, y_i| \to 0$, $\uparrow_p^{x_i} \to \xi$ and $\uparrow_p^{y_i} \to \eta$. Since $\mathrm{ob}(M) = \pi/2$, there exists a point $z_i \in M$ such that one of the following holds:

(1) $\tilde{\angle} x_i y_i z_i \ge \pi - \epsilon_i$ and $|y_i, z_i| \ge R/2$;
(2) $\tilde{\angle} y_i x_i z_i > \pi - \epsilon_i$ and $|x_i, z_i| \ge R/2$,

where $\epsilon_i \to 0$. By extracting a subsequence and by replacing x_i and y_i if necessarily, we may assume (1) holds for all i. Under the convergence of $(|p, x_i|^{-1} M, p)$ to the tangent cone (T_pM, o_p), the sequence of broken geodesics $x_i y_i z_i$ converges to a ray starting from ξ through η. Now we can take a direction $\zeta \in \Sigma_p$ along the ray satisfying $\angle(\xi, \zeta) > \angle(\xi, \eta)$. Since this is a contradiction, we have $\mathrm{diam}\,(\Sigma_p) = \pi$.

By the splitting theorem, T_pM is isometric to the product of the line ℓ through ξ, η and the space T' of vectors perpendicular to ℓ. Let $\Lambda \subset \Sigma_p$ denote the set of directions tangent to T'. Then, we have $\mathrm{diam}\,(\Lambda) = \pi$. Indeed, if $\bar{\xi}, \bar{\eta} \in \Lambda$ attain the diameter of Λ, then taking sequences $\bar{x}_i, \bar{y}_i \to p$ with $|\bar{x}_i, p| = |\bar{y}_i, p|$ so that $\uparrow_p^{\bar{x}_i} \to \bar{\xi}$ and $\uparrow_p^{\bar{y}_i} \to \bar{\eta}$, we have a point $\bar{z}_i$ in a way similar to the above argument. Then, the limit ray of $\bar{x}_i \bar{y}_i \bar{z}_i$ (or $\bar{y}_i \bar{x}_i \bar{z}_i$) under the convergence $(|p, \bar{x}_i|^{-1} M, p) \to (T_pM, o)$, is contained in T'. The existence of such a ray enforces that $\mathrm{diam}\,(\Lambda) = \pi$, and T' is isometric to a product $\mathbb{R} \times T''$. Repeating this argument, finally we obtain that T_pM is isometric to a product $\mathbb{R}^{n-1} \times L$, where L is $\mathbb{R}_+$ or $\mathbb{R}$ depending on whether p is a boundary point or not.

Since the space of directions is maximal at every point in M, due to [7], such an M admits a C^1-smooth atlas in the usual sense together with C^0-Riemannian metric which is compatible to the original distance function.

For every $p \in M \setminus \partial M$ and $\xi \in \Sigma_p$, take sequences $p_i, q_i \in M$ such that $|p, p_i|, |p, q_i| \to 0$, $\uparrow_p^{p_i} \to \xi$ and $\uparrow_p^{q_i} \to -\xi$. Since $\mathrm{ob}(M) = \pi/2$, there exist points $x_i, y_i \in M$ such that one of the following (1), (2) and one of the following (3), (4) hold:

(1) $\tilde{\angle} p p_i x_i \ge \pi - \epsilon_i$ and $|p_i x_i| \ge R/2$;
(2) $\tilde{\angle} p_i p x_i > \pi - \epsilon_i$ and $|p, x_i| \ge R/2$;
(3) $\tilde{\angle} p q_i y_i \ge \pi - \epsilon_i$ and $|q_i, y_i| \ge R/2$;
(4) $\tilde{\angle} q_i p y_i > \pi - \epsilon_i$ and $|p, y_i| \ge R/2$.

where $\epsilon_i \to 0$. If (1) or (4) holds for infinitely many i, then letting $i \to \infty$, we have a minimal geodesic from p in the direction ξ of length $\ge R/2$, and if (2) or (3) holds for infinitely many i, then we have a minimal geodesic from p in the direction $-\xi$ of length $\ge R/2$.

Next suppose $p \in \partial M$ and $\xi \in \Sigma_p$. If ξ is an interior direction, then there is no opposite direction to ξ. Take p_i with $|p, p_i| \to 0$ and $\uparrow_p^{p_i} \to \xi$. It follows that there exists x_i such that $\tilde{\angle} p p_i x_i > \pi - \epsilon_i$ and $|p_i, x_i| \ge R/2$ with $\lim \epsilon_i = 0$. Therefore the broken geodesic $p p_i x_i$ converges to a minimal geodesic in the direction ξ of length $\ge R/2$. Next sssume $\xi \in \partial \Sigma_p$ and take a sequence of interior directions $\xi_i \in \Sigma_p \setminus \partial \Sigma_p$ converging to ξ. Then the sequence of minimal geodesics of length $\geq R/2$ tangent to ξ_i converges to a minimal geodesic tangent to ξ of length $\geq R/2$. Thus we conclude that 1-inj$(M) \geq R/2$. This completes the proof. $\square$

Proof of Theorem 6.1. To complete the proof, in view of Lemma 6.2, it suffices to prove the conclusion (2). For every $p \in M \setminus \partial M$ and $\xi \in \Sigma_p$, let $c(t)$ be the quasi-geodesic such that $c(0) = p$ and $c'(0) = \xi$ (see [9] and [10]). Take $\epsilon_i \to 0$, and set $q_i = c(\epsilon_i)$. Suppose that

$$(*)_i \begin{cases} \text{there is a minimal geodesic } \gamma_i \text{ emanating from } q_i \text{ with} \\ \gamma_i'(0) = c'(\epsilon_i), L(\gamma_i) \geq R/2. \end{cases}$$

Then the limit γ_∞ of γ_i is a minimal geodesic from p in the direction ξ. Therefore we may assume that $(*)_i$ does not hold for any large i. By the assumption, we have a minimal geodesic σ_i starting from q_i in the direction $-c'(\epsilon_i)$ of length $\geq R/2$. Note that a geodesic and a quasi-geodesic having the same direction at a point must coincide. Thus we have $\sigma_i(s) = c(\epsilon_i - s)$ for $0 \leq s \leq \epsilon_i$. In particular c is minimal on $[0, \epsilon_i]$.

$\square$

In the noncompact case, by an argument similar to the proof of Theorem 6.1, we get the following.

Theorem 6.3. *If a noncompact Alexandrov n-space M of curvature $\geq \kappa$ has $\mathrm{ob}_\infty(M) = \pi/2$, then $\mathcal{S}(D(M)) = \emptyset$ and* 1-inj$(M) = \infty$.

Moreover we have the following in this case:

1. *For every $p \in \partial M$ and every $\xi \in \Sigma_p$ there exists a geodesic ray in the direction ξ;*
2. *For every $p \in M \setminus \partial M$ and every $\xi \in \Sigma_p$ there exists a minimal geodesic in the direction ξ of certain length > 0.*

Suppose additionally that M has nonempty boundary. Then M is homeomorphic to $\mathbb{R}^n_+$, and any distinct two points of ∂M are on a line of M which is contained in ∂M.

Proof. The proofs of the conclusions (1) and (2) are similar to those of Theorem 6.1, and hence omitted. We prove only the last statement. Suppose that M has nonempty boundary, and let $p, q \in \partial M$ be distinct points. By the conclusion (1), there exist geodesic rays $\gamma : [0, \infty) \to M$, $\sigma : [0, \infty) \to M$ such that $\gamma(0) = p = \sigma(|p, q|)$ and $\sigma(0) = q = \gamma(|p, q|)$. Note that γ and σ are contained in ∂M. We show that the curve $\alpha : \mathbb{R} \to \partial M$ defined by $\alpha(t) = \sigma(-t + |p, q|)$ if $t \leq 0$ and $\alpha(t) = \gamma(t)$ if $t \geq 0$, becomes a line. Let $r = \gamma(t_1)$ and $s = \sigma(t_2)$ with $t_1, t_2 > |p, q|$. From the definition, we have

$$|s, p| + |p, r| = |s, q| + |q, r|. \tag{6.13}$$

On the other hands, let β be the ray with $\beta(0) = s$ and $\beta(|s, p|) = p$. Since a geodesic between s and q is unique, the intersection of β and

σ is the geodesic sq. Let $t_3 \in (0, |s, q|)$. From the same discussion as the one to obtain (6.13), we get $|s, \beta(t_3)| + |\beta(t_3), r| = |s, p| + |p, r|$. Letting $t_3 \to 0$, we have $|s, r| = |s, p| + |p, r|$. Hence, we see that α is a line.

From the conclusion (1), for any $p \in \partial M$, the exponential map $\exp_p : T_pM \to M$ is defined and provides a homeomorphism between M and $\mathbb{R}^n_+$. This completes the proof. □

Let M be a surface of revolution with vertex p_0 homeomorphic to $\mathbb{R}^2$ having Riemannian metric

$$g = dr^2 + m(r)^2\, d\theta^2,$$

with respect to a polar coordinates (r, θ) around p_0. Note that

$$m(0) = 0, \quad m'(0) = 1, \quad m'' + Km = 0.$$

We assume that

(1) the Gaussian curvature K of M is nonnegative;
(2) the total curvature is at most π:

$$\int_M K\, dM \leq \pi.$$

Note that the ideal boundary $M(\infty)$ of M is a circle of length $2\pi - \displaystyle\int_M K\, dM \geq \pi$ (see [14]).

Example 6.4. As an example, consider the hyperboloid M_a defined by

$$z = a\sqrt{x^2 + y^2 + 1}.$$

Then its asymptotic cone $(M_a)_\infty$ is written as

$$(M_a)_\infty = \{z = a\sqrt{x^2 + y^2}\}.$$

Therefore M_a satisfies all the above assumptions when $0 \leq a \leq \sqrt{3}$.

The following Theorem 6.5 shows that Theorem 1.7 is sharp.

Theorem 6.5. *Le M be a complete open surface of revolution having nonnegative Gaussian curvature such that*

$$\int_M K\, dM \leq \pi.$$

Then $\mathrm{ob}_\infty(M) = \pi/2$.

Proof. First we recall the description of geodesics in M. Let $(r(s), \theta(s))$ be the coordinates of a unit speed geodesic $\gamma(s)$ on M, and $\zeta = \zeta(s)$ be the angle, $0 \leq \zeta \leq \pi$, between γ and the positive direction of the parallel circle $r =$ constant. The Clairaut's relation states that

$$m(r(s)) \cos\zeta(s) = \text{constant} = \nu, \tag{6.14}$$

where ν is called the Clairaut's constant of γ. Moreover we have

$$\frac{d\theta}{dr} = \frac{\theta'}{r'} = \epsilon \frac{\nu}{m(r)\sqrt{m^2(r) - \nu^2}}, \tag{6.15}$$

where $\epsilon = \pm 1$ is determined by the sign of r' (see [13, Proposition 7.1.3]).

Let $L(t)$ denote the length of geodesic sphere $S(p_0, t) := \partial B(p, t)$. Since

$$\lim_{t\to\infty} \frac{L(t)}{t} = L(M(\infty)) > 0,$$

we have $\int_1^\infty \frac{dt}{L^2(t)} < \infty$. It follows from [13, Theorem 7.2.1] that the set of poles of M coincides with a closed ball around p_0 of positive radius $r(M) > 0$. Therefore for every $p, q \in M$ if one of p, q is contained in $B(p_0, r(M))$, then obviously we have $\mathrm{ob}_{0,\infty}(p, q) = \pi/2$.

Therefore in the below, we assume that $p, q \in M \setminus B(p_0, r(M))$. Let A_p denote the set of velocity vectors $v \in \Sigma_p$ of the geodesic rays emanating from p. Let us first show that A_p contains a closed arc of length $2\pi - \int_M K\, dM \geq \pi$. Let $m(A_p)$ denote the measure of A_p. By the result due to Maeda [6], we know that

$$\inf_{p\in M} m(A_p) = 2\pi - \int_M K\, dM \geq \pi.$$

From this point of view, the claim is likely to be true. In the argument below, we confirm this.

We may assume that $(r(p), \theta(p)) = (r_0, 0)$ and $r_0 > r(M)$. Let $\xi_0 \in \Sigma_p$ (resp. $\eta_0 \in \Sigma_p$) denote the positive direction of the meridian through p (resp. the positive direction of the parallel circle through p). For each $t \in [-\pi, \pi]$, we let

$$\xi_t = \cos t \cdot \xi_0 + \sin t \cdot \eta_0$$

Denote by γ_t the geodesic from p such that $\gamma_t'(0) = \xi_t$. For each $s \in [-\pi, \pi]$, we let σ_s be the geodesic ray from p_0 that is equal to the meridian with $u(\sigma_s) = s$, and take a sequence $t_i \to \infty$ and a minimal geodesic $\mu_{s,i}$ joining p to $\sigma_s(t_i)$. When $s = \pi$, we choose $\mu_{\pi,i}$ in such a way that $0 \leq u(\mu_{\pi,i}(t)) \leq \pi$ for all $t \geq 0$. Then a subsequence of $\mu_{s,i}$ converges to a geodesic ray μ_s from p satisfying

(1) $0 \leq u(\mu_s(t_1)) < u(\mu_s(t_2)) < s$ for all $0 \leq t_1 < t_2 < \infty$;
(2) $\lim_{t\to\infty} u(\mu_s(t)) = s$.

Take $t_* \in (0, \pi]$ such that $\xi_{t_*} = \mu_\pi'(0)$. We claim that

$$t_* \geq \pi - \frac{1}{2}\int_M K\, dM \geq \pi/2. \tag{6.16}$$

Let D denote the domain bounded by the two geodesic rays γ_{t_*} and γ_{-t_*} such that $p_0 \in D$. Let $\lambda_s : [0, d_s] \to M$ be a minimal geodesic from $\gamma_{t_*}(s)$ to $\gamma_{-t_*}(s)$. Note that both γ_{t_*} and γ_{-t_*} are asymptotic to σ_π by symmetry, and hence

$$\lim_{s\to\infty} |\gamma_{\pm t_*}(s), \sigma_\pi(s)|/s = 0. \tag{6.17}$$

It follows that λ_s is contained in D for large enough $s > 0$. Let

$$\alpha_+(s) := \angle\gamma_{-t_*}(s)\gamma_{t_*}(s)p, \quad \alpha_-(s) := \angle\gamma_{t_*}(s)\gamma_{-t_*}(s)p,$$
$$\tilde\alpha_+(s) := \tilde\angle\gamma_{-t_*}(s)\gamma_{t_*}(s)p, \quad \tilde\alpha_-(s) := \tilde\angle\gamma_{t_*}(s)\gamma_{-t_*}(s)p.$$

In view of (6.17), considering 1-strainers $(p, \gamma_{t_*}(2s))$ at $\gamma_{t_*}(s)$ and $(p, \gamma_{-t_*}(2s))$ at $\gamma_{-t_*}(s)$, we have

$$\lim_{s\to\infty} |\alpha_\pm(s) - \tilde\alpha_\pm(s)| = 0.$$

Since $\lim_{s\to\infty} \tilde\alpha_\pm(s) = \pi/2$, we obtain $\lim_{s\to\infty} \alpha_\pm(s) = \pi/2$. The Gauss-Bonnet theorem then implies that

$$\begin{aligned}\int_D K\, dM &= \lim_{s\to\infty}(\alpha_+(s) + \alpha_-(s) + \angle_p(D) - \pi)\\ &= \angle_p(D) = 2(\pi - t_*) \le \int_M K\, dM.\end{aligned}$$

It follows that $t_* \ge \pi - \frac{1}{2}\int_M K\, dM \ge \pi/2$ as required.

Now we show that γ_t is a geodesic ray for each $t \in [-\pi/2, \pi/2]$. Let $\hat t$ denote the maximum of those $t \in [0, \pi/2]$ that γ_s is a geodesic ray for all $s \in [0, t]$. It suffices to show that $\hat t = \pi/2$. Suppose that $\hat t < \pi/2$. Since $t_* \ge \pi/2 > \hat t$ and both $\gamma_{\hat t}$ and $\gamma_{t_*} = \mu_\pi$ are geodesic rays, we have $0 \le \theta(\gamma_{\hat t}(s)) < \pi$ for all $s \ge 0$. By (6.14), $\theta(\gamma_t(s))$ is monotone increasing in s, and therefore there is a unique limit

$$\theta_t(\infty) := \lim_{s\to\infty} \theta(\gamma_t(s)) \in [0, \pi]$$

for every $t \in [0, \hat t]$. If $\theta_{\hat t}(\infty) = \pi$, in a way similar to (6.16) we would have $\hat t \ge \pi/2$, which is a contradiction. Thus we see $\theta_{\hat t}(\infty) < \pi$. From continuity, there is some $\tilde t \in (\hat t, \pi/2)$ such that

$$0 \le \theta(\gamma_t(s)) < \pi, \quad 0 \le \theta_t(\infty) < \pi,$$

for any $0 \le t \le \tilde t$ and all $s \ge 0$. Obviously $\theta_t(\infty)$ is continuous in $t \in [0, \tilde t]$. For $0 \le t_1 < t_2 \le \tilde t$, let $\theta_i(s) := \theta(\gamma_{t_i}(s))$, and ν_i the Clairaut's constants of γ_{t_i} for $i = 1, 2$. Since $\nu_1 < \nu_2$, the formula (6.15) implies that $d\theta_1/dr < d\theta_2/dr$, and hence $\theta_{t_1}(\infty) < \theta_{t_2}(\infty)$. Thus $\theta_t(\infty)$ is injective in $t \in [0, \tilde t]$. This yields that γ_t coincides with the geodesic ray $\mu_{\theta_t(\infty)}$ for all $t \in [0, \tilde t]$, which is a contradiction to the definition of $\hat t$. Thus we conclude that $\hat t = \pi/2$ and γ_t is a geodesic ray for every $t \in [-\pi/2, \pi/2]$ by symmetry.

Finally we show that $\mathrm{ob}_{0,\infty}(p, q) \ge \pi/2 - \tau(\delta)$ with $\delta = |p, q|$ and $\lim_{\delta\to 0} \tau(\delta) = 0$. Take a minimal geodesic $\gamma : [0, \delta] \to M$ from p to q.

First assume that $r(q) = r(p)$. Since $\zeta(0) = \zeta(\delta)$ and $\zeta(\delta/2) = 0$, we have from (6.14)

$$\cos\zeta(0) = \frac{m(r(\delta/2))}{m(r(0))}, \tag{6.18}$$

where $|m(r(0)) - m(r(\delta/2))| \le \frac{\delta}{2}m' \le \frac{\delta}{2}$ because of nonnegative curvature. It follows that

$$\left|1 - \frac{m(r(\delta/2))}{m(r(0))}\right| \le \frac{\delta}{2m(r(M))}.$$

Together with (6.18), this yields

$$\zeta(0) \le \sqrt{\frac{\delta}{m(r(M))}} =: \delta_1.$$

Let $\gamma_{\pi/2}$ be the ray from p defined above. We may assume that $\angle(\gamma'_{\pi/2}(0), \gamma'(0)) = \zeta(0)$. For large enough $R > 0$, we have

$$\begin{aligned} \tilde\angle pq\gamma_{\pi/2}(R) &\ge \pi - \tilde\angle qp\gamma_{\pi/2}(R) - \tilde\angle p\gamma_{\pi/2}(R)q \\ &\ge \pi - \zeta(0) - o_R \\ &\ge \pi - \delta_1 - o_R, \end{aligned}$$

where $\lim_{R\to\infty} o_R = 0$, and hence $\mathrm{ob}_{0,\infty}(p,q) \ge \pi/2 - \delta_1$.

Next assume $r(p) < r(q)$.If $\angle(\uparrow_p^q, \xi_0) \le \pi/2$, then p and q are on a geodesic ray. In the other case, taking $p_1 \in pq$ with $r(p) = r(p_1)$, one can show that $\zeta(0) \le \delta_1$ and $\tilde\angle pq\gamma_{\pi/2}(R) \ge \pi - \delta_1 - o_R$ by a similar manner. Thus we conclude that $\mathrm{ob}_\infty(M) = \pi/2$. $\square$

Remark 6.6. Theorem 6.5 shows that the estimate $1\text{-inj}^*(M(\infty)) \ge \pi/2$ in Theorem 1.7 (1) is sharp. It should also be noted that one cannot expect $1\text{-inj}(M(\infty)) \ge \pi/2$ in Theorem 1.7 (1), because if one take $N = M \times \mathbb{R}$, where M is a non-flat open surface as in Theorem 6.5, then $\mathrm{ob}_\infty(N) = \pi/2$ and $N(\infty)$ is the spherical suspension over $M(\infty)$. Note that $N(\infty)$ has the two singular points at the vertices of the suspension since the length of the circle $M(\infty)$ is less than 2π.

Proof of Theorem 1.7. Theorem 1.7 (2) immediately follows from Theorem 6.3 and the splitting theorem. Suppose M has no boundary. For $\epsilon_i \to 0$ and $p \in M$, consider the asymptotic limit

$$\lim_{i\to\infty}(\epsilon_i M, p) = (M_\infty, o), \tag{6.19}$$

where the asymptotic cone M_∞ is the Euclidean cone over $M(\infty)$. Identify $M(\infty) = M(\infty) \times \{1\} \subset M_\infty$. For every $\xi \in M(\infty)$ and every geodesic $\gamma : [0,\delta] \to M_\infty$ from ξ, fix any $0 < a < \delta$, and set $\eta := \gamma(\delta)$ and $\xi_a = \gamma(a)$. Take sequences p_i, q_i and $x_i \in p_i q_i$ in $\epsilon_i M$ such that

$$p_i \to \xi, \ \ x_i \to \xi_a, \ \ q_i \to \eta,$$

as $i \to \infty$ under the convergence (6.19). On the interior of a minimal geodesic $p_i x_i$, take points $y_{i,\alpha}$ such that $y_{i,\alpha} \to x_i$ as $\alpha \to \infty$. From the assumption, for any sequences $R_i \to \infty$ and $o_i \to 0$, if α is large enough compared to i, one can find points z_i with $|x_i, z_i| \geq R_i/\epsilon_i$ and either

$$\tilde{\angle} z_i x_i y_{i,\alpha} > \pi - o_i \ \text{ or } \ \tilde{\angle} z_i y_{i,\alpha} x_i > \pi - o_i.$$

Letting $i \to \infty$, we obtain a geodesic ray emanating from ξ_a either in the direction $\gamma'(a)$ or in the opposite direction $-\gamma'(a)$. Then letting $a \to 0$, we conclude that there is a geodesic ray σ starting from ξ such that either $\sigma'(0) = \gamma'(0)$ or else there is the opposite direction $-\gamma'(0)$ and $\sigma'(0) = -\gamma'(0)$. Thus we have 1-inj$^*(M_\infty) = \infty$.

Now for any direction $v \in \Sigma_\xi(M(\infty)) \subset \Sigma_\xi(M_\infty)$, there is a geodesic ray σ of M_∞ starting from ξ either in the direction v or else in the direction $-v$ if $-v$ exists. For each $t \geq 0$, let $\xi_t := \uparrow_o^{\sigma(t)} \in M(\infty)$. It is easy to see that there is a unique limit $\xi' = \lim_{t\to\infty} \xi_t$ and $\angle(\xi, \xi') = \pi/2$. Thus ξ_t provides a shortest segment in $M(\infty)$ from ξ to ξ' in the direction v or $-v$ if any. This shows 1-inj$^*(M(\infty)) \geq \pi/2$ as required. $\square$

7. κ-obtuse constants from infinity

We conclude the paper with some comments on another definition of "obtuse constant from infinity" for noncompact spaces which does depend on the lower curvature bound.

Let M be a complete noncompact Alexandrov space with curvature $\geq \kappa$, and $p \neq q \in M$. Using our previous definition of $\mathrm{ob}_{\kappa,\infty}(p,q)$, set

$$\mathrm{ob}_{\kappa,\infty}(M) := \inf_{p\neq q} \mathrm{ob}_{\kappa,\infty}(p,q).$$

which we call the *κ-obtuse constant of M from infinity.* Note that $\mathrm{ob}_{\kappa,\infty}(M) \leq \mathrm{ob}_\infty(M)$. Clearly the κ-obtuse constant from infinity does depend on the choice of the lower curvature bound κ, and $\mathrm{ob}_{0,\infty}(M) \geq 0$ for $\kappa = 0$. However if $\kappa < 0$, the κ-obtuse constant from infinity could be negative. For instance, if M is the domain bounded by an ideal triangle all of whose vertexes are on the ideal boundary of the hyperbolic plane $\mathbb{H}^2(-1)$. Then $\mathrm{ob}_{-1,\infty}(M) = -\pi/2$.

This invariant seems interesting in itself. For instance, we have the following strong rigidity.

Theorem 7.1. *Let M be a complete noncompact Alexandrov n-space with nonnegative curvature satisfying $\mathrm{ob}_{0,\infty}(M) = \pi/2$. If M has no boundary, then M is isometric to the Euclidean space $\mathbb{R}^n$.*

Proof. Take $r_i \to 0$ and consider the pointed Gromov-Hausdorff convergence $(r_i M, p) \to (M_\infty, o)$, where M_∞ is the asymptotic cone, which is isometric to the the Euclidean cone $K(M(\infty))$ over the ideal boundary $M(\infty)$. By Theorem 1.4, $v_\infty(M) > 0$ and hence $\dim M_\infty = n$. It suffices to show that M_∞ is isometric to $\mathbb{R}^n$. First we show that $\operatorname{diam}(M(\infty)) = \pi$. Suppose $\operatorname{diam}(M(\infty)) < \pi$ and take $\xi, \eta \in M(\infty)$ with $|\xi, \eta| = \operatorname{diam}(M(\infty))$. We identify $M(\infty)$ as $M(\infty) \times 1 \subset M_\infty$, and take $x_i, y_i \in r_i M$ such that $x_i \to \xi$, $y_i \to \eta$ under the convergence $(r_i M, p) \to (M_\infty, o)$. From the assumption, we may assume that there is a geodesic ray γ_i emanating from x_i through y_i. Passing to a subsequence, we may also assume that γ_i converges to a geodesic ray γ_∞ in M_∞ emanating from ξ through η. Obviously we can find a point z on γ_∞ such that the direction $\zeta = \uparrow_o^z$ satisfies $|\xi, \zeta| > |\xi, \eta|$. Since this is a contradiction, we have $\operatorname{diam}(M(\infty)) = \pi$. By the splitting theorem, M_∞ is isometric to a product $M'_\infty \times \mathbb{R}$. Repeating the argument to M'_∞, we see that M'_∞ is isometric to a product $M''_\infty \times \mathbb{R}$. In this way, we conclude that M_∞ is isometric to $\mathbb{R}^n$. $\square$

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