

ON THE WEAK SOLUTIONS TO THE 3D INVISCID QUASI-GEOSTROPHIC SYSTEM

MATTHEW D. NOVACK

ABSTRACT. The purpose of this note is to study the weak solutions to the inviscid quasi-geostrophic system for initial data belonging to Lebesgue spaces. We give a global existence result as well as detail the connections between several different notions of weak solutions. In addition, we give a condition under which the energy of the system is conserved.

1. INTRODUCTION AND MAIN RESULTS

We study the 3-D inviscid quasi-geostrophic system (QG)

$$\begin{cases} \partial_t(\Delta\Psi) + \overline{\nabla}^\perp\Psi \cdot \overline{\nabla}(\Delta\Psi) = f_L & t > 0, \quad z > 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 & (QG)_L \\ \partial_t(\partial_\nu\Psi) + \overline{\nabla}^\perp\Psi \cdot \overline{\nabla}(\partial_\nu\Psi) = f_\nu & t > 0, \quad z = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 & (QG)_\nu \end{cases}$$

supplied with an initial data Ψ_0 . Here $\Psi(t, z, x) : [0, \infty) \times \mathbb{R}_+^3$ is the stream function for the geostrophic flow, and f_L and f_ν are forcing terms. We use the notations

$$\begin{aligned} \Delta\Psi &= \partial_{x_1x_1}\Psi + \partial_{x_2x_2}\Psi + \partial_{zz}\Psi, & \overline{\nabla}\Psi &= (0, \partial_{x_1}\Psi, \partial_{x_2}\Psi) \\ \overline{\nabla}^\perp\Psi &= (0, -\partial_{x_2}\Psi, \partial_{x_1}\Psi), & \partial_\nu\Psi &= -\partial_z\Psi(t, 0, x). \end{aligned}$$

The system is used to study stratified flows in which the Coriolis force is balanced with the pressure and serves as a model in simulations of large-scale atmospheric and oceanic circulation.

The purpose of this work is to study the existence and properties of various types of weak solutions to this system. We provide global existence results for initial data belonging to Lebesgue spaces, and determine conditions under which a weak solution conserves the energy of the system. Much mathematical research has been focused on this system and its variants. Beale and Bourgeois [3] and Desjardins and Grenier [7] derived the system from physical principles. Puel and Vasseur [14] first proved the global existence of weak solutions in the case of L^2 initial data, using a projection operator to reformulate the problem. In the case when $\Delta\Psi_0 \equiv 0$, (QG) reduces to the well-studied inviscid surface quasi-geostrophic equation, which has received considerable attention due to its similarities with the important systems of fluid mechanics (see Constantin, Majda, and Tabak [6], Garner, Held, Pierrehumbert, and Swanson [9], among others). Weak solutions were constructed in L^2 by Resnick [15]. Marchand [12] first gave a proof of the existence of global weak solutions when the initial data is not in L^2 but rather L^p for any $p > \frac{4}{3}$.

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1.1. The Reformulated System. A crucial tool in our analysis will be a reformulation of (QG) . We draw inspiration from Puel and Vasseur [14], who used a reformulation to obtain their global existence result. The physical system as written is analogous to the vorticity form of the Euler equations with an additional boundary condition. However, one may consider the following reformulation, in which $\text{curl}(Q)$ acts as a Lagrange multiplier similar to the gradient of the pressure in the Euler equations:

$$\begin{cases} \partial_t(\nabla\Psi) + \overline{\nabla}^\perp\Psi : \overline{\nabla}(\nabla\Psi) = \text{curl } Q + \nabla F & z > 0 \\ \text{curl } Q \cdot \nu = 0, \quad \partial_\nu F = f_\nu & z = 0 \\ \Delta F = f_L & z > 0. \end{cases} \quad (\text{rQG})$$

Formally, taking the divergence of (rQG) gives $(QG)_L$, and taking the trace gives $(QG)_\nu$. To obtain (rQG) from (QG) , one must invert the divergence operator coupled with a Neumann boundary condition. While providing a link between the two formulations will be an important part of our analysis (see Theorem 1.3), let us proceed from the perspective of (rQG) for the time being. Following Puel and Vasseur [14], we define the notion of weak solutions to (rQG) .

Definition 1.1 (Weak Solutions to (rQG)). *Let T, R be fixed, $\phi \in C^\infty(\mathbb{R}^4)$ compactly supported in $(-T, T) \times (-R, R)^3$, and F be such that $\Delta F = f_L$, $\partial_\nu F = f_\nu$. A weak solution Ψ to (rQG) with forcing f_ν, f_L on $(0, T) \times \mathbb{R}_+^3$ must satisfy*

$$\begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \nabla \phi + \overline{\nabla}^\perp \Psi : \overline{\nabla} \nabla \phi \right) \cdot \nabla \Psi + \nabla \phi \cdot \nabla F \right) dx dz dt \\ & = \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0, z, x) \cdot \nabla \Psi(0, z, x) dx dz \end{aligned}$$

for all R, ϕ . For the weak formulation to make sense, we require $\nabla \Psi, \overline{\nabla}^\perp \Psi \otimes \nabla \Psi \in L^1_{loc}([0, T] \times \mathbb{R}_+^3)$.

We remark that the definition of weak solutions contains no information about $\text{curl}(Q)$. Indeed, the choice of test functions formally encodes the fact that inverting the divergence operator is unique only up to the curl of a vector field.

1.2. Statement of Main Results. We begin with the global existence of weak solutions to (rQG) .

Theorem 1.1. *Suppose that $p \in (\frac{4}{3}, \infty]$ and $q \in (\frac{6}{5}, 3]$. Let $\theta \in L^p(\mathbb{R}^2)$, $\omega \in L^q(\mathbb{R}_+^3)$, $f_L \in L^1([0, T]; L^q(\mathbb{R}_+^3))$, and $f_\nu \in L^1([0, T]; L^p(\mathbb{R}^2))$ for all $T > 0$. Then there exists a global weak solution $\nabla \Psi$ on $(0, \infty) \times \mathbb{R}_+^3$ to (rQG) with forcing f_ν, f_L such that $\Delta \Psi|_{t=0} = \omega$ and $\partial_\nu \Psi|_{t=0} = \theta$. In addition, there exists a constant C such that for all $T > 0$, Ψ satisfies the following bound:*

$$\begin{aligned} & \|\nabla \Psi\|_{L^\infty([0, T]; L^{\frac{3p}{2}}(\mathbb{R}_+^3) + L^{\frac{3q}{3-q}}(\mathbb{R}_+^3))} + \|\Delta \Psi\|_{L^\infty([0, T]; L^q(\mathbb{R}_+^3))} + \|\partial_\nu \Psi\|_{L^\infty([0, T]; L^p(\mathbb{R}^2))} \\ & \leq C \left(\|\omega\|_{L^q} + \|\theta\|_{L^p} + \|f_L\|_{L^1([0, T]; L^q(\mathbb{R}_+^3))} + \|f_\nu\|_{L^1([0, T]; L^p(\mathbb{R}^2))} \right). \end{aligned}$$

Let us give a simple explanation for the restrictions on p and q . In order for the nonlinear term $\bar{\nabla} \cdot (\bar{\nabla}^\perp \Psi \otimes \nabla \Psi)$ to be well-defined as a distribution from integration by parts, we need $\nabla \Psi \in L^2(\mathbb{R}_+^3)$ (at least locally). If $\Delta \Psi_0 \in L^{\frac{6}{5}}(\mathbb{R}_+^3)$ and $\partial_\nu \Psi_0 \in L^{\frac{4}{3}}(\mathbb{R}^2)$, then solving the elliptic boundary value problem gives $\nabla \Psi_0 \in L^2(\mathbb{R}_+^3)$, hence the restrictions on q and p . If $q = 3$ or $p = \infty$, the corresponding Lebesgue norm on $\nabla \Psi$ is actually the standard *BMO* norm in the space of functions of bounded mean oscillation; for simplicity's sake we employ this abbreviation.

The following theorem addresses the conservation of the energy $\|\nabla \Psi(t)\|_{L^2(\mathbb{R}_+^3)}$ in the case of no forcing. Here $\dot{B}_{3,\infty}^\alpha(\mathbb{R}^2)$ is the usual homogenous Besov space.

Theorem 1.2. *Let $\nabla \Psi$ be a weak solution to (rQG) with no forcing such that*

$$\nabla \Psi \in C([0, T]; L^2(\mathbb{R}_+^3)) \cap L^3([0, T] \times [0, \infty); \dot{B}_{3,\infty}^\alpha(\mathbb{R}^2))$$

for some $\alpha > \frac{1}{3}$. Then $\|\nabla \Psi(t)\|_{L^2(\mathbb{R}_+^3)} = \|\nabla \Psi_0\|_{L^2(\mathbb{R}_+^3)}$ for $t \in [0, T]$.

In the case $\Delta \Psi_0 \equiv 0$, the system reduces to SQG and one has the equality

$$\|\nabla \Psi(t)\|_{L^2(\mathbb{R}_+^3)} = \|\partial_\nu \Psi(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^2)}.$$

The quantity $\|\partial_\nu \Psi(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^2)}$ is actually the Hamiltonian of the system in this case; see Resnick [15] or Buckmaster, Shkoller, Vicol [4]. Buckmaster, Shkoller, and Vicol provide a proof of the non-uniqueness of weak solutions below a certain regularity threshold. Conversely, Isett and Vicol [11] prove that when $\partial_\nu \Psi \in L_{t,x}^3$, the Hamiltonian is conserved.

1.3. Relations between (QG) and (rQG). It is interesting to consider whether weak solutions to (rQG) might be weak solutions to (QG), and vice versa. In this section we address this question, therein justifying our use of the reformulated system. We define two classes of weak solutions to (QG); the first is the more standard notion of weak solution, while the second incorporates the Calderón commutator used in the existence proofs of Marchand [12] and Resnick [15].

Definition 1.2 (Weak Solutions to (QG)). *Let T, R be fixed, $\phi \in C^\infty(\mathbb{R}^4)$ compactly supported in $(-T, T) \times (0, R) \times (-R, R)^2$, and $\bar{\phi} \in C^\infty(\mathbb{R}^3)$ compactly supported in $(-T, T) \times (-R, R)^2$. A weak solution Ψ to (QG) on $(0, T) \times \mathbb{R}_+^3$ with forcing f_ν, f_L must satisfy*

$$\begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left((\partial_t \phi + \bar{\nabla}^\perp \Psi \cdot \bar{\nabla} \phi) \Delta \Psi + \phi f_L \right) dx dz dt \\ (1.1) \quad & = \int_0^\infty \int_{\mathbb{R}^2} \phi(0, z, x) \Delta \Psi(0, z, x) dx dz \end{aligned}$$

and

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^2} \left((\partial_t \bar{\phi} + \bar{\nabla}^\perp \Psi(t, 0, x) \cdot \bar{\nabla} \bar{\phi}) \partial_\nu \Psi(t, x) + \bar{\phi} f_\nu \right) dx dt \\ (1.2) \quad & = \int_{\mathbb{R}^2} \bar{\phi}(0, x) \partial_\nu \Psi(0, x) dx \end{aligned}$$

for all $R, \phi, \bar{\phi}$. For the weak formulation to make sense, we require $\Delta\Psi, \bar{\nabla}^\perp\Psi\Delta\Psi \in L^1_{loc}([0, T] \times \mathbb{R}^3_+)$ and $\partial_\nu\Psi, \bar{\nabla}^\perp\Psi\partial_\nu\Psi \in L^1_{loc}([0, T] \times \mathbb{R}^2)$.

For functions of two variables, $\bar{\Lambda}\theta = \sqrt{-\Delta}(\theta)$ and $\bar{\Lambda}^{-1}$ is the corresponding inverse operator. In addition,

$$\mathcal{R}^\perp\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta)$$

is the rotated vector of Riesz transforms. The commutator $[A, B]$ of two operators is $AB - BA$. In the following definition, we use the commutator result of Marchand [12] to define a notion of weak solution for (QG) for low levels of integrability. Marchand's results concerning boundedness and convergence of the commutator are stated in the preliminaries. For the sake of brevity we suppress for now issues concerning the frequency support of $\partial_\nu\Psi$; these are also addressed in the preliminaries.

Definition 1.3 (Weak Solutions to (QG) with Commutator). *Let T, R be fixed, $\phi \in C^\infty(\mathbb{R}^4)$ compactly supported in $(-T, T) \times (0, R) \times (-R, R)^2$, and $\bar{\phi} \in C^\infty(\mathbb{R}^3)$ compactly supported in $(-T, T) \times (-R, R)^2$. Let $\Psi : [0, T] \times \mathbb{R}^3_+ \rightarrow \mathbb{R}$ be given and Ψ_1 and Ψ_2 be defined for all $t \in [0, T]$ by the boundary value problems*

$$\begin{cases} \Delta\Psi_1 = 0 \\ \partial_\nu\Psi_1 = \partial_\nu\Psi \end{cases} \quad \begin{cases} \Delta\Psi_2 = \Delta\Psi \\ \partial_\nu\Psi_2 = 0. \end{cases}$$

We define $\left(\bar{\nabla}^\perp\Psi(t, 0, x)\partial_\nu\Psi(t, x)\right)_C$ as a distribution by (and use the notation $(\cdot)_C$ to specify that we are using the commutator formulation)

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2} \left(\bar{\nabla}^\perp\Psi(t, 0, x)\partial_\nu\Psi(t, x)\right)_C \cdot \bar{\nabla}\bar{\phi} \, dx \, dt &:= \frac{1}{2} \int_0^T \int_{\mathbb{R}^2} \mathcal{R}^\perp\partial_\nu\Psi_1[\bar{\Lambda}, \bar{\nabla}\bar{\phi}](\bar{\Lambda}^{-1}\partial_\nu\Psi_1) \, dx \, dt \\ &+ \int_0^T \int_{\mathbb{R}^2} \left(\bar{\nabla}^\perp\Psi_2(t, 0, x)\partial_\nu\Psi_1(t, x)\right)_C \cdot \bar{\nabla}\bar{\phi} \, dx \, dt \end{aligned}$$

and say that Ψ is a weak solution to (QG) with commutator on $(0, T) \times \mathbb{R}^3_+$ with forcing f_ν , f_L if

$$\begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t\phi + \bar{\nabla}^\perp\Psi \cdot \bar{\nabla}\phi \right) \Delta\Psi + \phi f_L \right) \, dx \, dz \, dt \\ &= \int_0^\infty \int_{\mathbb{R}^2} \phi(0, z, x) \Delta\Psi(0, z, x) \, dx \, dz \end{aligned}$$

and

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^2} \left(\partial_t\bar{\phi}\partial_\nu\Psi + \left(\bar{\nabla}^\perp\Psi\partial_\nu\Psi\right)_C \cdot \bar{\nabla}\bar{\phi} + \bar{\phi}f_\nu \right) \, dx \, dt \\ &= \int_{\mathbb{R}^2} \bar{\phi}(0, x) \partial_\nu\Psi(0, x) \, dx \end{aligned}$$

for all $T, R, \phi, \bar{\phi}$. For the weak formulation to make sense, we require $\partial_\nu\Psi(t) \in L^p(\mathbb{R}^2)$ for all time t and some $p \in (\frac{4}{3}, 2]$ and $\bar{\nabla}^\perp\Psi_2\partial_\nu\Psi \in L^1_{loc}([0, T] \times \mathbb{R}^2)$.

See the preliminaries for Marchand's convergence result regarding the commutator and other details.

We now connect the weak solutions of Definition 1.1, Definition 1.2, and Definition 1.3.

Theorem 1.3. (1) Assume that $\Delta\Psi \in L^\infty([0, T]; L^q(\mathbb{R}_+^3))$ for $q \in [\frac{3}{2}, 3]$ and $\partial_\nu\Psi \in L^\infty([0, T]; L^p(\mathbb{R}^2))$ for $p \in [2, \infty]$. Then $\nabla\Psi$ satisfies Definition 1.1 if and only if $\nabla\Psi$ satisfies Definition 1.2.
(2) Assume that $\Delta\Psi \in L^\infty([0, T]; L^q(\mathbb{R}_+^3))$ for $q \in [\frac{3}{2}, 3]$ and $\partial_\nu\Psi \in L^\infty([0, T]; L^p(\mathbb{R}^2))$ for $p \in (\frac{4}{3}, 2]$. Assume in addition that

$$p \geq \frac{2q}{3(q-1)}.$$

Then $\nabla\Psi$ satisfies Definition 1.1 if and only if $\nabla\Psi$ satisfies Definition 1.3.

(3) Assume that $\Delta\Psi \in L^\infty([0, T]; L^q(\mathbb{R}_+^3))$ for $q \in [\frac{3}{2}, 3]$ and $\partial_\nu\Psi \in L^\infty([0, T]; L^p \cap L^r(\mathbb{R}^2))$ for $p \in (\frac{4}{3}, 2]$, $r \in [2, \infty]$. Then $\nabla\Psi$ satisfies Definition 1.2 if and only if $\nabla\Psi$ satisfies Definition 1.3.

Theorem 1.3 complements the existence result in Theorem 1.1 nicely. Indeed, imposing that the initial data $\nabla\Psi_0$, $\Delta\Psi_0$, and $\partial_\nu\Psi_0$ all belong to L^2 , then we recover the result of Puel and Vasseur [14]. Imposing $\Delta\Psi_0 \equiv 0$ and $\partial_\nu\Psi_0 \in L^p(\mathbb{R}^2)$, we recover the result of Marchand [12].

It is interesting to note that if the initial data satisfies $\Delta\Psi_0 \in L^{\frac{6}{5}}(\mathbb{R}_+^3)$ and $\partial_\nu\Psi_0 \equiv 0$ to remove the boundary condition, trace theory would give $\bar{\nabla}^\perp\Psi_0|_{z=0} \in L^{\frac{4}{3}}(\mathbb{R}^2)$ (see Lemma 2.3), corresponding precisely to the lower limit of integrability in the proof of Marchand. Conversely, imposing that $\Delta\Psi_0 \equiv 0$ and $\partial_\nu\Psi_0 \in L^{\frac{4}{3}}(\mathbb{R}^2)$ to eliminate the transport equation for $z > 0$, Lemma 2.5 ensures that $\nabla\Psi_0 \in L^2(\mathbb{R}_+^3)$. In addition, one can see from the proof of Theorem 1.3 that

$$p \geq \frac{2q}{3(q-1)}$$

is the minimum integrability needed to define the nonlinear terms in both $(QG)_L$ and $(QG)_\nu$. Thus, the conditions on p and q correspond in a natural way and appear to be the sharpest possible afforded by the structure of the system. Furthermore, our analysis combines the reformulation (rQG) of Vasseur and Puel and the commutator of Marchand. In conjunction with the correspondence between the conditions on p and q , this naturally connects the two approaches.

2. PRELIMINARIES

We collect several definitions and known results as well as state and prove the elliptic estimates necessary for the proof of our main theorems. Let us begin with the results of Marchand [12].

2.1. Definitions and Previous Results.

Lemma 2.1 (Calderón Commutator). (1) For $f \in L^p(\mathbb{R}^2)$, $p \in (\frac{4}{3}, 2]$, and $\phi \in \mathcal{D}((0, T) \times \mathbb{R}^2)$, $\bar{\nabla} \cdot (f\mathcal{R}^\perp f)$ is defined as a distribution by

$$\langle \phi, \bar{\nabla} \cdot (f\mathcal{R}^\perp f) \rangle := \frac{1}{2} \int_{\mathbb{R}^2} \mathcal{R}^\perp f [\bar{\Lambda}, \bar{\nabla} \phi] (\bar{\Lambda}^{-1} f).$$

If $f \in L^2(\mathbb{R}^2)$ is such that \hat{f} is zero in a neighborhood of the origin, then

$$\int_{\mathbb{R}^2} f \mathcal{R}^\perp f \cdot \bar{\nabla} \phi = -\frac{1}{2} \int_{\mathbb{R}^2} \mathcal{R}^\perp f [\bar{\Lambda}, \bar{\nabla} \phi] (\bar{\Lambda}^{-1} f).$$

- (2) Let $p \in (\frac{4}{3}, \infty]$ and $\{\theta_\epsilon(t, x)\}_{\epsilon>0} \subset L^\infty([0, T]; L^p(\mathbb{R}^2))$ be a sequence of functions such that θ_ϵ converges weakly- $*$ to $\theta(t, x) \in L^\infty([0, T]; L^p(\mathbb{R}^2))$, T fixed. Then the following holds in the sense of distributions:

$$\lim_{\epsilon \rightarrow 0} \bar{\nabla} \cdot (\theta_\epsilon \mathcal{R}^\perp \theta_\epsilon) = \bar{\nabla} \cdot (\theta \mathcal{R}^\perp \theta).$$

Here it is understood that for $p \geq 2$, $\bar{\nabla} \cdot (\theta_\epsilon \mathcal{R}^\perp \theta_\epsilon)$ is defined by integration by parts, whereas for $p \leq 2$, we use the commutator.

Decomposing an arbitrary function using Littlewood-Paley projections allows one to use the commutator only for the high-frequency piece. To avoid cumbersome Besov space notations, we suppress these details and will write

$$\mathcal{R}^\perp \theta[\bar{\Lambda}, \bar{\nabla} \phi] (\bar{\Lambda}^{-1} \theta)$$

for any L^p function with $p \in (\frac{4}{3}, 2]$. We refer the reader to Marchand [12] for further details and proofs.

For the proof of Theorem 1.2, we shall need several identities, definitions, and notations concerning Littlewood-Paley decompositions and Besov spaces. The homogenous Besov spaces $\dot{B}_{3,\infty}^\alpha(\mathbb{R}^2)$ are defined via the usual bi-infinite sequence of homogenous Littlewood-Paley decompositions. Here $\{\gamma_\epsilon\}_{\epsilon>0}$ is a sequence of compactly supported, radially symmetric approximate identities. For a function $u : \mathbb{R}^2 \rightarrow \mathbb{R}^n$, we define $u^\epsilon := u * \gamma_\epsilon$.

Proposition 2.2. (1) For L_{loc}^1 functions f and g ,

$$\int_{\mathbb{R}^2} g(f^\epsilon)^\epsilon = \int_{\mathbb{R}^2} g^\epsilon f^\epsilon$$

(2) The following commutator identity holds:

$$(f \cdot g)^\epsilon(x) - f^\epsilon g^\epsilon(x) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (f(x - \bar{x}) - f(x)) \phi_\epsilon(\bar{x}) (g(x - \bar{x}) - g(x - x')) \phi_\epsilon(x') dx' d\bar{x}$$

(3) For $\alpha \in (0, 1)$ and $u \in \dot{B}_{3,\infty}^\alpha(\mathbb{R}^2)$, there exists C independent of u such that for all $|y| > 0$,

$$\|u(\cdot - y) - u(\cdot)\|_{L^3(\mathbb{R}^2)} \leq C y^\alpha \|u\|_{\dot{B}_{3,\infty}^\alpha(\mathbb{R}^2)}$$

and

$$\|\bar{\nabla} u^\epsilon\|_{L^3(\mathbb{R}^2)} \leq C \epsilon^{\alpha-1} \|u\|_{\dot{B}_{3,\infty}^\alpha(\mathbb{R}^2)}.$$

Proof. (1) follows immediately from a change of variables and the radial symmetry of the mollifier. For (2), we can write

$$\begin{aligned} (f \cdot g)^\epsilon(x) - f^\epsilon g^\epsilon(x) &= \int_{\mathbb{R}^2} f(x - \bar{x}) g(x - \bar{x}) \phi_\epsilon(\bar{x}) d\bar{x} \\ &\quad - \int_{\mathbb{R}^2} f(x - \bar{x}) \phi_\epsilon(\bar{x}) d\bar{x} \cdot \int_{\mathbb{R}^2} g(x - x') \phi_\epsilon(x') dx' \\ &= \int_{\mathbb{R}^2} f(x - \bar{x}) \phi_\epsilon(\bar{x}) \left(g(x - \bar{x}) - \int_{\mathbb{R}^2} g(x - x') \phi_\epsilon(x') dx' \right) d\bar{x} \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (f(x - \bar{x}) - f(x)) \phi_\epsilon(\bar{x}) (g(x - \bar{x}) - g(x - x')) \phi_\epsilon(x') dx' d\bar{x} \end{aligned}$$

Statements and proofs of (3) can be found in the text of Bahouri, Chemin, and Danchin [2]. \square

The homogenous Sobolev spaces are defined by

$$\dot{W}^{1,r}(\mathbb{R}_+^3) := \{u \in \mathcal{D}'(\mathbb{R}_+^3) | \nabla u \in L^r(\mathbb{R}_+^3)\}$$

with norm

$$\|u\|_{\dot{W}^{1,r}(\mathbb{R}_+^3)} = \|\nabla u\|_{L^r(\mathbb{R}_+^3)}.$$

Strictly speaking, for the norm to be well-defined and for the following inequality to hold, we consider equivalence classes of distributions which differ by an additive constant. Let us recall the classical Escobar inequality for the half-space \mathbb{R}_+^3 [8].

Lemma 2.3. Suppose that $q \in [1, 3)$, and $u \in \dot{W}^{1,q}(\mathbb{R}_+^3)$. Then

$$\|u|_{z=0}\|_{L^{\frac{2q}{3-q}}(\mathbb{R}^2)} \leq C(q)\|u\|_{\dot{W}^{1,q}(\mathbb{R}_+^3)}$$

2.2. Elliptic Estimates. We now specify the appropriate Lebesgue spaces and obtain the corresponding bounds for the solution to the Poisson problem with Neumann boundary data in the upper half space. While the results are standard, we include proofs for the sake of completeness. We also include a technical lemma which will be useful in the proof of Theorem 1.1.

Lemma 2.4. Given $f \in L^q(\mathbb{R}_+^3)$ for $q \in (\frac{6}{5}, 3]$, there exists a unique $u \in \dot{W}^{1,\frac{3q}{3-q}}(\mathbb{R}_+^3)$ ($\nabla u \in BMO$ if $q = 3$) such that

$$\begin{cases} -\Delta u = f & z > 0 \\ \partial_\nu u = 0 & z = 0 \end{cases}$$

with

$$\|\nabla u\|_{L^{\frac{3q}{3-q}}(\mathbb{R}_+^3)} \leq C(q)\|f\|_{L^q(\mathbb{R}_+^3)}, \quad q < 3$$

or

$$\|\nabla u\|_{BMO(\mathbb{R}_+^3)} \leq C(q)\|f\|_{L^q(\mathbb{R}_+^3)}, \quad q = 3.$$

Proof. Let us begin with the case $q = 3$. Applying the operator whose symbol is $\frac{i\xi}{|\xi|^2}$ (we ignore constants coming from the Fourier transform) to

$$f_E(z, x) = \begin{cases} f(z, x) & z > 0 \\ 0 & z \leq 0 \end{cases}$$

gives a curl free vector field in $BMO(\mathbb{R}^3)$ which is in fact the gradient of a function u_E (see, for example, Temam [16]). Then applying the same operator to

$$f_{E,r}(z, x) = \begin{cases} 0 & z > 0 \\ f(-z, x) & z \leq 0 \end{cases}$$

yields a vector field in $BMO(\mathbb{R}^3)$ which is again the gradient of a function $u_{E,r}$. Putting $u = u_E + u_{E,r}$, it is clear that $-\Delta u = f$ in \mathbb{R}_+^3 and

$$\partial_\nu u = \partial_\nu u_E + \partial_\nu u_{E,r} = \partial_\nu u_E - \partial_\nu u_E = 0.$$

The bound follows from the boundedness of the multiplier operator from $L^3(\mathbb{R}^3)$ to $BMO(\mathbb{R}^3)$.

We use the generalized Lax-Milgram theorem for Banach spaces to show the existence as well as the bound for $q < 3$. Define $X := \dot{W}^{1, \frac{a}{a-1}}(\mathbb{R}_+^3)$ and $Y := \dot{W}^{1, a}(\mathbb{R}_+^3)$. Define $B : X \times Y \rightarrow \mathbb{R}$ by

$$B(u, v) = \int_{\mathbb{R}_+^3} \nabla u \cdot \nabla v$$

and $F(v) : Y \rightarrow \mathbb{R}$ by

$$F(v) = \int_{\mathbb{R}_+^3} v f.$$

Choosing $a = \frac{3q}{4q-3}$ gives that $v \in L^{\frac{q}{q-1}}(\mathbb{R}_+^3)$ by Sobolev embedding, and thus F is well-defined and continuous. Continuity of B follows from Hölder's inequality. We must show B to be non-degenerate, i.e.

$$\sup_{u \in X} B(u, v) > 0 \quad \forall v \in Y$$

and coercive, i.e.

$$\inf_{\substack{u \in X \\ \|u\|=1}} \sup_{\substack{v \in Y \\ \|v\|=1}} B(u, v) \geq \gamma > 0.$$

To show coercivity, we begin by fixing $u \in X$ with $\|u\|_{\dot{W}^{1, \frac{a}{a-1}}} = 1$. The ideal choice for ∇v would be $\nabla u |\nabla u|^{\frac{a}{a-1}-2}$. Of course, this may not be the gradient of a function. Therefore, let us define the operator \mathbb{P}_∇ for Schwartz vector fields $s : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$\begin{aligned} \widehat{\mathbb{P}_\nabla(s)}(\xi) &= \left(\frac{\langle \hat{s}(\xi), \xi \rangle}{|\xi|^2} \xi \right) \\ &= \left(\sum_{i=1}^3 \hat{s}_i(\xi) \frac{\xi_i \xi_j}{|\xi|^2} \right), \quad j = 1, 2, 3. \end{aligned}$$

Recalling that the symbol for the j^{th} Riesz transform \mathcal{R}_j is $-\frac{i\xi_j}{|\xi|}$, \mathbb{P}_∇ is a linear combination of compositions of Riesz transforms. We then extend \mathbb{P}_∇ by density as a bounded operator from $(L^r(\mathbb{R}^3))^3$ to itself for all $r \in (1, \infty)$. In addition, for s_1 scalar valued, s_2 vector valued Schwarz functions, examining the symbol of \mathbb{P}_∇ shows that

$$\langle \nabla s_1, \mathbb{P}_\nabla s_2 \rangle = \langle \nabla s_1, s_2 \rangle.$$

Continuity of the operator ensures that this property remains true for vector fields in X and Y . We define $u_E(z, x) = u(|z|, x)$ to be the symmetric extension of u over the plane $z = 0$. With this definition,

$$(2.1) \quad \partial_z u_E(z, x) = -\partial_z u_E(-z, x)$$

and

$$(2.2) \quad \overline{\nabla} u_E(z, x) = \overline{\nabla} u_E(-z, x).$$

We apply \mathbb{P}_∇ to the extended vector field $\nabla u_E |\nabla u_E|^{\frac{a}{a-1}-2}$. Using the symmetry and anti-symmetry of the Riesz transforms and $\nabla u_E |\nabla u_E|^{\frac{a}{a-1}-2}$ with respect to reflection over the plane $z = 0$, it is simple to check that

$$(2.3) \quad \partial_z \mathbb{P}_\nabla \left(\nabla u_E |\nabla u_E|^{\frac{a}{a-1}-2} \right) (z, x) = -\partial_z \mathbb{P}_\nabla \left(\nabla u_E |\nabla u_E|^{\frac{a}{a-1}-2} \right) (-z, x)$$

and

$$(2.4) \quad \bar{\nabla} \mathbb{P}_{\nabla} \left(|\nabla u_E| |\nabla u_E|^{\frac{a}{a-1}-2} \right) (z, x) = \bar{\nabla} \mathbb{P}_{\nabla} \left(|\nabla u_E| |\nabla u_E|^{\frac{a}{a-1}-2} \right) (-z, x).$$

We set $\nabla v(z, x) = \frac{1}{\|\mathbb{P}_{\nabla}\|} \mathbb{P}_{\nabla}(|\nabla u_E| |\nabla u_E|^{\frac{a}{a-1}-2})|_{z \geq 0}$. By direct computation, $\|\nabla v\|_{L^a(\mathbb{R}_+^3)} \leq 1$, and using (2.1), (2.2), (2.3), and (2.4) gives that

$$\begin{aligned} \int_{\mathbb{R}_+^3} \nabla u \cdot \nabla v &= \frac{1}{2\|\mathbb{P}_{\nabla}\|} \int_{\mathbb{R}^3} \nabla u_E \cdot \mathbb{P}_{\nabla}(|\nabla u_E| |\nabla u_E|^{\frac{a}{a-1}-2}) \\ &= \frac{1}{2\|\mathbb{P}_{\nabla}\|} \int_{\mathbb{R}^3} |\nabla u_E|^{\frac{a}{a-1}} \\ &= \frac{1}{\|\mathbb{P}_{\nabla}\|}. \end{aligned}$$

Thus the coercivity is shown with $\gamma = \frac{1}{\|\mathbb{P}_{\nabla}\|}$. Non-degeneracy follows from switching u and v and repeating the argument. Therefore, the conditions of Lax-Milgram are met, and we have the existence of a solution u to the variational problem, as well as the gradient bound on u in terms of f . Then, taking v to be compactly supported in \mathbb{R}_+^3 shows that $-\Delta u = f$ in the sense of distributions. Now, taking $v \in \mathcal{D}(\mathbb{R}^3)$ shows that $\partial_\nu u$ is well defined as a distribution by

$$\int_{\mathbb{R}_+^3} \nabla u \cdot \nabla v + v \Delta u =: \int_{\mathbb{R}^2} v \partial_\nu u$$

and is equal to zero. □

For the following lemma we use the space

$$\mathring{W}_{\Delta}^{1,p}(\mathbb{R}_+^3) := \{u \in \mathring{W}^{1,p}(\mathbb{R}_+^3) | \Delta u = 0 \text{ in } \mathcal{D}'(\mathbb{R}_+^3)\}$$

with norm

$$\|u\|_{\mathring{W}_{\Delta}^{1,p}(\mathbb{R}_+^3)} = \|\nabla u\|_{L^p(\mathbb{R}_+^3)}$$

Lemma 2.5. Given $g \in L^p(\mathbb{R}^2)$ for $p \in (\frac{4}{3}, \infty]$, there exists $u \in \mathring{W}_{\Delta}^{1, \frac{3p}{2}}(\mathbb{R}_+^3)$ solving

$$\begin{cases} \Delta u = 0 & z > 0 \\ \partial_\nu u = g & z = 0 \end{cases}$$

with

$$\|\nabla u\|_{L^{\frac{3p}{2}}(\mathbb{R}_+^3)} \leq C(p) \|g\|_{L^p(\mathbb{R}^2)}, \quad p > \infty$$

or

$$\|\nabla u\|_{BMO(\mathbb{R}_+^3)} \leq C(p) \|g\|_{L^p(\mathbb{R}^2)}, \quad p = \infty.$$

Proof. Let us begin with the case $p = \infty$. Applying the Poisson kernel $\mathcal{P}(z, x)$ to $g(x)$ gives a harmonic function in \mathbb{R}_+^3 . Considering the vector field

$$v(z, x) = -(\mathcal{P}(z, \cdot) * g(\cdot)(x), \mathcal{R}_1 \mathcal{P}(z, \cdot) * g(\cdot)(x), \mathcal{R}_2 \mathcal{P}(z, \cdot) * g(\cdot)(x)),$$

it is clear that v is curl free and is thus the gradient of a harmonic function u with $\partial_\nu u = g$. The bound follows from noting that the Riesz transforms are bounded from $L^\infty(\mathbb{R}^2)$ to $BMO(\mathbb{R}^2)$ and $\|\mathcal{P}(z, \cdot) * g(\cdot)(x)\|_{L^\infty(\mathbb{R}^2)} \leq \|g(x)\|_{L^\infty(\mathbb{R}^2)}$ for all z .

We use again the Lax-Milgram theorem for $p < \infty$. Define $X := \dot{W}_\Delta^{1, \frac{3p}{2}}(\mathbb{R}_+^3)$ and $Y := \dot{W}^{1, \frac{3p}{3p-2}}(\mathbb{R}_+^3)$. Let $B : X \times Y \rightarrow \mathbb{R}$ be defined by

$$B(u, v) = \int_{\mathbb{R}_+^3} \nabla u \cdot \nabla v$$

and $F : Y \rightarrow \mathbb{R}$ be defined by

$$F(v) = \int_{\mathbb{R}^2} v|_{z=0} g.$$

By Lemma 2.3, we have that $v|_{z=0} \in L^{\frac{p}{p-1}}(\mathbb{R}^2)$, and therefore F is well-defined and continuous. Continuity of B follows from Hölder's inequality. As before, we are tasked with showing the coercivity and non-degeneracy of B . Making use of the \mathbb{P}_∇ operator, the details follow as in the previous lemma and are omitted. The existence of u and the gradient bound in terms of g are provided by the Lax-Milgram theorem. Taking v compactly supported in \mathbb{R}_+^3 shows that indeed $\Delta u = 0$. We then again have that $\partial_\nu u$ is well-defined as a distribution from integration by parts and satisfies $\partial_\nu u = g$. \square

Lemma 2.6. Let $\{g_\epsilon\}_{\epsilon>0}$ be a bounded sequence of functions in $L^p(\mathbb{R}^2)$ for $p > \frac{4}{3}$. Let $u_\epsilon(z, x) : \mathbb{R}_+^3 \rightarrow \mathbb{R}$ be the solution to

$$\begin{cases} \Delta u_\epsilon = 0 & z > 0 \\ \partial_\nu u_\epsilon = g_\epsilon & z = 0 \end{cases}$$

Then there exists u such that up to a subsequence, ∇u_ϵ converges strongly to ∇u in $L_{loc}^2(\mathbb{R}_+^3)$.

Proof. We first extract a subsequence which we shall continue to call $\{g_\epsilon\}$ in an abuse of notation that converges weakly- $*$ to g in $L^p(\mathbb{R}^2)$. Applying Lemma 2.5 to g_ϵ gives that u_ϵ converges weakly- $*$ to u in $\dot{W}^{1, \frac{3p}{2}}(\mathbb{R}_+^3)$, where u solves the Laplace equation with Neumann data g . Because $p > \frac{4}{3}$, we have that $\frac{3p}{2} > 2$, and therefore $\{\nabla u_\epsilon\}$ is a weakly- $*$ convergent sequence in $L_{loc}^2(\mathbb{R}_+^3)$. Note that ∇u_ϵ is harmonic for all ϵ and thus for fixed z ,

$$\nabla u_\epsilon(z, x) = -(\mathcal{P}(z, \cdot) * g_\epsilon(\cdot)(x), \mathcal{R}_1 \mathcal{P}(z, \cdot) * g_\epsilon(\cdot)(x), \mathcal{R}_2 \mathcal{P}(z, \cdot) * g_\epsilon(\cdot)(x)).$$

Furthermore, it is clear that $\mathcal{P} * g$ (and $(\mathcal{R} \mathcal{P} * g)$) belongs to $W^{k, \frac{3p}{2}}((z_0, \infty) \times \mathbb{R}^2)$ for any $k \in \mathbb{N}$ and fixed $z_0 > 0$. Then by the Rellich-Kondrachov theorem, $\nabla u_\epsilon|_{z>z_0}$ converges strongly to $\nabla u|_{z>z_0}$ in $L_{loc}^2((z_0, \infty) \times \mathbb{R}^2)$ for fixed $z_0 > 0$. Thus given $R > 0$ and fixing $0 < \delta < R$, we can write

$$\begin{aligned} & \limsup_{\epsilon \rightarrow 0} \int_0^R \int_{B_R(0)} |\nabla u_\epsilon(z, x) - \nabla u(z, x)|^2 dx dz \\ &= \limsup_{\epsilon \rightarrow 0} \left(\int_0^\delta \int_{B_R(0)} |\nabla u_\epsilon(z, x) - \nabla u(z, x)|^2 dx dz \right. \\ & \quad \left. + \int_\delta^R \int_{B_R(0)} |\nabla u_\epsilon(z, x) - \nabla u(z, x)|^2 dx dz \right) \\ &\leq \sup_{\epsilon > 0} \int_0^\delta \int_{B_R(0)} |\nabla u_\epsilon(z, x) - \nabla u(z, x)|^2 dx dz \\ &\leq \sup_{\epsilon > 0} \|\nabla u_\epsilon - \nabla u\|_{L^{\frac{3p}{2}}(\mathbb{R}_+^3)}^2 \|\mathcal{X}_{\{[0, \delta] \times B_R(0)\}}\|_{L^{\frac{3p}{3p-4}}(\mathbb{R}_+^3)}^2 \end{aligned}$$

$$\leq C (R^2 \delta)^{\frac{3p-4}{3p}}$$

after applying the uniform bound on g_ϵ in $L^p(\mathbb{R}^2)$ and Hölder's inequality. Considering that $p > \frac{4}{3}$, the final expression approaches zero as δ decreases to 0, proving the claim. \square

3. PROOF OF THEOREM 1.1

We now have the estimates necessary for the proof of the main theorem. Here we assume that $q \in (\frac{6}{5}, 3]$ and $p \in (\frac{4}{3}, \infty]$.

Proof of Theorem 1.1. Let $\{\gamma_\epsilon\}_{\epsilon>0}$ be a sequence of compactly supported approximate identities in \mathbb{R}^2 and $\{\Gamma_\epsilon\}_{\epsilon>0}$ a sequence of compactly supported approximate identities in \mathbb{R}^3 . We define truncated versions of the initial data and forcing by

$$\omega_{T_\epsilon} = \omega \mathcal{X}_{\{|\omega| < \frac{1}{\epsilon}, |(z,x)| < \frac{1}{\epsilon}\}}, \quad \theta_{T_\epsilon} = \theta \mathcal{X}_{\{|\theta| < \frac{1}{\epsilon}, |x| < \frac{1}{\epsilon}\}},$$

with $f_{L,T_\epsilon}(t)$ and $f_{\nu,T_\epsilon}(t)$ defined analogously for each time $t \geq 0$. Then we regularize by putting

$$\omega_\epsilon = \Gamma_\epsilon * \omega_{T_\epsilon}, \quad \theta_\epsilon = \gamma_\epsilon * \theta_{T_\epsilon}, \quad f_{L,\epsilon}(t) = \Gamma_\epsilon * f_{L,T_\epsilon}(t), \quad f_{\nu,\epsilon}(t) = \gamma_\epsilon * f_{\nu,T_\epsilon}(t),$$

ensuring that ω_ϵ , θ_ϵ , $f_{L,\epsilon}$, and $f_{\nu,\epsilon}$ are compactly supported in space, C^∞ functions. Following the method in [13], there exists a unique, spacially smooth solution Ψ_ϵ to the regularized system

$$\begin{cases} \partial_t(\Delta \Psi_\epsilon) + \overline{\nabla}^\perp \Psi_\epsilon \cdot \overline{\nabla}(\Delta \Psi_\epsilon) = f_{L,\epsilon} & t > 0, \quad z > 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \\ \partial_t(\partial_\nu \Psi_\epsilon) + \overline{\nabla}^\perp \Psi_\epsilon \cdot \overline{\nabla}(\partial_\nu \Psi_\epsilon) + \epsilon(-\overline{\Delta})^{\frac{1}{2}} \partial_\nu \Psi_\epsilon = f_{\nu,\epsilon} & t > 0, \quad z = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \end{cases}$$

with initial data

$$\Delta \Psi_\epsilon|_{t=0} = \omega_\epsilon, \quad \partial_\nu \Psi_\epsilon|_{t=0} = \theta_\epsilon.$$

We decompose $\Psi_\epsilon = \Psi_{\epsilon,1} + \Psi_{\epsilon,2}$ as follows:

$$\begin{cases} \Delta \Psi_{\epsilon,1} = 0 \\ \partial_\nu \Psi_{\epsilon,1} = \partial_\nu \Psi_\epsilon \end{cases} \quad \begin{cases} \Delta \Psi_{\epsilon,2} = \Delta \Psi_\epsilon \\ \partial_\nu \Psi_{\epsilon,2} = 0. \end{cases}$$

The existence of $\Psi_{\epsilon,1}$ and $\Psi_{\epsilon,2}$ follow from applying Lemma 2.5 and Lemma 2.4 with $g = \partial_\nu \Psi_\epsilon$ and $f = -\Delta \Psi_\epsilon$, respectively. In addition, we have the following bounds for $t \in [0, T]$, which follow from Lemma 2.5 and Lemma 2.4, the divergence-free nature of the flow for $z \geq 0$, and the well-known maximum principle for SQG (see [12]). The constant C is fixed throughout and independent of time. In fact C depends only on Lemma 2.3, Lemma 2.4, and Lemma 2.5, which in turn depend only on p and q . Recall that for $q = 3$ or $q = \infty$, the BMO norm is substituted in for the appropriate Lebesgue norm.

$$\begin{aligned} \|\nabla \Psi_{\epsilon,1}(t)\|_{L^{\frac{3p}{2}}(\mathbb{R}_+^3)} &\leq C \|\partial_\nu \Psi_{\epsilon,1}(t)\|_{L^p(\mathbb{R}^2)} \\ &\leq C (\|\theta_\epsilon\|_{L^p(\mathbb{R}^2)} + \|f_{\nu,\epsilon}\|_{L^1([0,T];L^p(\mathbb{R}^2))) \\ (3.1) \quad &\leq C (\|\theta\|_{L^p(\mathbb{R}^2)} + \|f_\nu\|_{L^1([0,T];L^p(\mathbb{R}^2))) \end{aligned}$$

and

$$\begin{aligned} \|\nabla \Psi_{\epsilon,2}(t)\|_{L^{\frac{3q}{3-q}}(\mathbb{R}_+^3)} &\leq C \|\Delta \Psi_{\epsilon,2}(t)\|_{L^q(\mathbb{R}_+^3)} \\ &\leq C (\|\omega_\epsilon\|_{L^q(\mathbb{R}_+^3)} + \|f_{L,\epsilon}\|_{L^1([0,T];L^q(\mathbb{R}_+^3))) \end{aligned}$$

$$(3.2) \quad \leq C \left(\|\omega\|_{L^q(\mathbb{R}_+^3)} + \|f_L\|_{L^1([0,T];L^q(\mathbb{R}_+^3))} \right).$$

Define $F_\epsilon : [0, \infty) \times \mathbb{R}_+^3 \rightarrow \mathbb{R}$ for all time by

$$\begin{cases} \Delta F_\epsilon = f_{L,\epsilon} & z > 0 \\ \partial_\nu F_\epsilon = f_{\nu,\epsilon} - \epsilon(-\overline{\Delta})^{\frac{1}{2}} \partial_\nu \Psi_\epsilon & z = 0 \end{cases}$$

Integrating $\nabla \Psi_\epsilon$ by parts with a smooth, compactly supported test function $\phi(t, z, x)$, we have

$$\begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left((\partial_t \nabla \phi + \overline{\nabla}^\perp \Psi_\epsilon : \overline{\nabla} \nabla \phi) \cdot \nabla \Psi_\epsilon + \nabla \phi \cdot \nabla F_\epsilon \right) dx dz dt \\ & = \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left((\partial_t \phi + \overline{\nabla}^\perp \Psi_\epsilon \cdot \overline{\nabla} \phi) \Delta \Psi_\epsilon + \phi \Delta F_\epsilon \right) dx dz dt \\ & \quad - \int_0^T \int_{\mathbb{R}^2} \left((\partial_t \phi + \overline{\nabla}^\perp \Psi_\epsilon \cdot \overline{\nabla} \phi) \partial_\nu \Psi_\epsilon + \phi \partial_\nu F_\epsilon \right) dx dt \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0, z, x) \cdot \nabla \Psi_\epsilon(0, z, x) dx dz & = - \int_0^\infty \int_{\mathbb{R}^2} \phi(0, z, x) \Delta \Psi_\epsilon(0, z, x) dx dz \\ & \quad + \int_{\mathbb{R}^2} \phi(0, 0, x) \partial_\nu \Psi_\epsilon(0, 0, x) dx \end{aligned}$$

Using that Ψ_ϵ is a solution to the regularized system, the right hand sides of the above equalities are in fact equal, and therefore the left hand sides are equal as well, i.e.

$$(3.3) \quad \begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left((\partial_t \nabla \phi + \overline{\nabla}^\perp \Psi_\epsilon : \overline{\nabla} \nabla \phi) \cdot \nabla \Psi_\epsilon + \nabla \phi \cdot \nabla F_\epsilon \right) dx dz dt \\ & = \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0, z, x) \cdot \nabla \Psi_\epsilon(0, z, x) dx dz \end{aligned}$$

To pass to the limit in (3.3), we use (3.1) and (3.2) to detail the spaces in which the sequences $\{\Psi_{1,\epsilon}\}, \{\Psi_{2,\epsilon}\}$ are pre-compact. Throughout, $T > 0$ is fixed, and weak-* convergence is abbreviated simply as weak convergence.

- (1) $\{\partial_\nu \Psi_{\epsilon,1}\}$ is bounded in $L^\infty([0, T]; L^p(\mathbb{R}^2))$ and we can pass to a weakly convergent subsequence.
- (2) $\{\Psi_{\epsilon,2}\}$ is bounded in $L^\infty([0, T]; \dot{W}^{2,q}(\mathbb{R}_+^3))$ and we can pass to a weakly convergent subsequence.
- (3) Consider the embedding of $L^p(\mathbb{R}^2)$ into $L_{loc}^2(\mathbb{R}_+^3)$ which sends $g \in L^p(\mathbb{R}^2)$ to $\nabla u \in L_{loc}^2(\mathbb{R}_+^3)$ for u solving the Laplace equation with Neumann data g . From Lemma 2.6, this embedding is compact. Similarly, consider the embedding of $L^q(\mathbb{R}_+^3)$ into $L_{loc}^2(\mathbb{R}_+^3)$ which sends $f \in L^q(\mathbb{R}_+^3)$ to $\nabla u \in L^{\frac{3q}{3-q}}(\mathbb{R}_+^3)$ solving the Poisson problem with data f and zero Neumann data. From the Rellich-Kondrachov theorem, this embedding is also compact. Therefore, we define the Banach space X of gradients with divergence bounded in $L^q(\mathbb{R}_+^3)$ and the vertical component of the trace at $z = 0$ bounded in $L^p(\mathbb{R}^2)$. Then $\{\nabla \Psi_\epsilon\}$ is a bounded sequence in $L^\infty([0, T]; X)$. From (3.3), $\partial_t \nabla \Psi_\epsilon$ is bounded $L^\infty([0, T]; W_{loc}^{-2,\infty}(\mathbb{R}_+^3))$. We have that $L_{loc}^2(\mathbb{R}_+^3)$ embeds continuously into $W_{loc}^{-2,\infty}(\mathbb{R}_+^3)$ and X embeds compactly into $L_{loc}^2(\mathbb{R}_+^3)$. As (1) and (2) show that $\nabla \Psi_\epsilon$

converges weakly in $L^\infty([0, T]; X)$, the Aubin-Lions lemma [1] can be applied to conclude that $\nabla \Psi_\epsilon$ converges strongly in $L^\infty([0, T]; L^2_{loc}(\mathbb{R}^3_+))$.

Let $\nabla \Psi$ be the limit of $\nabla \Psi_\epsilon$ with convergence in the spaces specified in (1)-(3). By (1) and integration by parts,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi \cdot \nabla F_\epsilon \, dx \, dz \, dt &= \lim_{\epsilon \rightarrow 0} \left(- \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \phi f_{L,\epsilon} \, dx \, dz \, dt \right. \\ &\quad \left. + \int_0^T \int_{\mathbb{R}^2} \phi \left(f_{\nu,\epsilon} - \epsilon(-\overline{\Delta})^{\frac{1}{2}} \partial_\nu \Psi_\epsilon \right) \, dx \, dt \right) \\ &= \lim_{\epsilon \rightarrow 0} \left(- \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \phi f_{L,\epsilon} + \int_0^T \int_{\mathbb{R}^2} \left(\phi f_{\nu,\epsilon} - \epsilon(-\overline{\Delta})^{\frac{1}{2}} \phi \partial_\nu \Psi_{\epsilon,1} \right) \right) \\ &= - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \phi f_L + \int_0^T \int_{\mathbb{R}^2} \phi f_\nu \\ &= \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi \cdot \nabla F \end{aligned}$$

for F solving the boundary value problem $\Delta F = f_L$ and $\partial_\nu F = f_\nu$. Second, by (3),

$$\lim_{\epsilon \rightarrow 0} \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\overline{\nabla}^\perp \Psi_\epsilon : \overline{\nabla} \nabla \phi \right) \cdot \nabla \Psi_\epsilon \, dx \, dz \, dt = \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\overline{\nabla}^\perp \Psi : \overline{\nabla} \nabla \phi \right) \cdot \nabla \Psi \, dx \, dz \, dt$$

In addition, it is immediate that

$$\lim_{\epsilon \rightarrow 0} \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0, z, x) \cdot \nabla \Psi_\epsilon(0, z, x) \, dx \, dz = \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0, z, x) \cdot \nabla \Psi(0, z, x) \, dx \, dz$$

and

$$\lim_{\epsilon \rightarrow 0} \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \partial_t \nabla \phi \cdot \nabla \Psi_\epsilon \, dx \, dz \, dt = \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \partial_t \nabla \phi \cdot \nabla \Psi \, dx \, dz \, dt.$$

Passing to the limit in (3.3), we have that

$$\begin{aligned} &- \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \nabla \phi + \overline{\nabla}^\perp \Psi : \overline{\nabla} \nabla \phi \right) \cdot \nabla \Psi + \nabla \phi \cdot \nabla F \right) \, dx \, dz \, dt \\ &= \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0, z, x) \cdot \nabla \Psi(0, z, x) \, dx \, dz \end{aligned}$$

and thus Ψ satisfies Definition 1.1. The bound in the statement of the theorem follows from (3.1) and (3.2), completing the proof. \square

4. PROOF OF THEOREM 1.2

Proof of Theorem 1.2. Define for all time

$$(\nabla \Psi^\epsilon)^\epsilon(z, x) := (\nabla \Psi(z, \cdot) * \gamma_\epsilon) * \gamma_\epsilon(x);$$

that is, we convolve $\nabla \Psi$ with a mollifier γ_ϵ in x only, z by z . The extra mollification is for passage onto the nonlinear term later. Strictly speaking, $(\nabla \Psi^\epsilon)^\epsilon$ is not an admissible test function; it lacks compact support in space and time, and differentiability in z and t . However, let us proceed formally for the time being, and assume that $(\nabla \Psi^\epsilon)^\epsilon$ is admissible

and that $\nabla \Psi$ is differentiable in time. Multiplying (rQG) by $(\nabla \Psi^\epsilon)^\epsilon$ and integrating in space and from time 0 to t , we obtain

$$(4.1) \quad \begin{aligned} E_\epsilon(t) - E_\epsilon(0) &:= \int_0^\infty \int_{\mathbb{R}^2} \nabla \Psi^\epsilon(t) \cdot \nabla \Psi^\epsilon(t) dx dz - \int_0^\infty \int_{\mathbb{R}^2} \nabla \Psi^\epsilon(0) \cdot \nabla \Psi^\epsilon(0) dx dz \\ &= -2 \int_0^t \int_0^\infty \int_{\mathbb{R}^2} \left(\bar{\nabla}^\perp \Psi : \bar{\nabla}(\nabla \Psi^\epsilon)^\epsilon \right) \cdot \nabla \Psi dx dz d\tau \end{aligned}$$

We can now apply Proposition 2.2(1) to the right hand side to move the mollifier over, introduce the commutator between multiplication and mollification, and rewrite the nonlinear terms using tensor notation, obtaining

$$\begin{aligned} E_\epsilon(t) - E_\epsilon(0) &= -2 \int_0^t \int_0^\infty \int_{\mathbb{R}^2} \left\langle \left(\bar{\nabla}^\perp \Psi \otimes \nabla \Psi \right)^\epsilon - \left(\bar{\nabla}^\perp \Psi^\epsilon \otimes \nabla \Psi^\epsilon \right), \bar{\nabla} \nabla \Psi^\epsilon \right\rangle dx dz d\tau \\ &\quad - 2 \int_0^t \int_0^\infty \int_{\mathbb{R}^2} \left\langle \bar{\nabla}^\perp \Psi^\epsilon \otimes \nabla \Psi^\epsilon, \bar{\nabla} \nabla \Psi^\epsilon \right\rangle dx dz d\tau \end{aligned}$$

Integrating by parts in x for fixed z and τ gives that the second term is equal to zero. Applying Proposition 2.2(2) z by z with $f = \bar{\nabla}^\perp \Psi$ and $g = \nabla \Psi$ to the first term, we have

$$\begin{aligned} E_\epsilon(t) - E_\epsilon(0) &= -2 \int_0^t \int_0^\infty \iiint_{(\mathbb{R}^2)^3} \left\langle \left(\bar{\nabla}^\perp \Psi(x - \bar{x}) - \bar{\nabla}^\perp \Psi(x) \right) \otimes \right. \\ &\quad \left. (\nabla \Psi(x - \bar{x}) - \nabla \Psi(x - x')), \bar{\nabla} \nabla \Psi^\epsilon(x) \right\rangle \cdot \phi_\epsilon(\bar{x}) \phi_\epsilon(x') d\bar{x} dx' dx dz d\tau. \end{aligned}$$

Using the fact that the approximate identities have integral 1 in x for each fixed z , noting that $\text{supp } \gamma_\epsilon \subset B_\epsilon(0)$, and applying Proposition 2.2(3) z by z yields

$$\begin{aligned} |E_\epsilon(t) - E_\epsilon(0)| &\leq C \int_0^t \int_0^\infty \|\bar{\nabla}^\perp \Psi(z, \tau, \cdot)\|_{\dot{B}_{3,\infty}^\alpha(\mathbb{R}^2)} \|\nabla \Psi(z, \tau, \cdot)\|_{\dot{B}_{3,\infty}^\alpha(\mathbb{R}^2)}^2 \epsilon^{3\alpha-1} dz d\tau \\ &\leq C \epsilon^{3\alpha-1} \|\nabla \Psi\|_{L^3([0,T] \times [0,\infty); \dot{B}_{3,\infty}^\alpha(\mathbb{R}^2))}^3 \end{aligned}$$

which approaches 0 as $\epsilon \rightarrow 0$ if $\alpha > \frac{1}{3}$.

We must now account for that fact that $(\nabla \Psi^\epsilon)^\epsilon$ is not an admissible test function. Replacing Ψ (which is a well-defined function in $C([0, T]; L^6(\mathbb{R}_+^3))$ by Sobolev embedding) with

$$\Psi_\eta := \left(\mathcal{X}_{\{|(x,z)| \leq \frac{1}{\eta}, \tau \leq T-\eta\}} \Psi \right) * \Gamma_\eta$$

for Γ_η a space-time mollifier in \mathbb{R}^4 ensures compact support and differentiability in z and t . Then after mollifying as before in x , we can use $(\nabla(\Psi_\eta)^\epsilon)^\epsilon$ as a test function. It is well known that (4.1) holds when differentiability in time is replaced with $C([0, T]; L^2(\mathbb{R}_+^3))$. Passing to the limit in η first and then in ϵ gives that

$$\|\nabla \Psi(t)\|_{L^2(\mathbb{R}_+^3)}^2 - \|\nabla \Psi(0)\|_{L^2(\mathbb{R}_+^3)}^2 = \lim_{\epsilon \rightarrow 0} E_\epsilon(t) - E_\epsilon(0) = 0,$$

completing the proof. □

5. PROOF OF THEOREM 1.3

We divide up the proof into parts (1), (2) and (3).

Proof of Theorem 1.3(1). The first step shows that integration by parts is valid for the reformulated equation, and the second step then integrates by parts to prove the claim.

Step One : First, we extend the Sobolev function Ψ to \mathbb{R}^3 , denoting the extended function by Ψ_E . Let $\{\Gamma_\epsilon\}_{\epsilon>0}$ be a sequence of approximate identities in \mathbb{R}^3 . Define

$$\nabla \Psi_{E,\epsilon} := \nabla \Psi_E * \Gamma_\epsilon$$

for $\epsilon > 0$. By assumption, we have

$$\Delta \Psi \in L^\infty([0, T]; L^q(\mathbb{R}_+^3)), \quad \partial_\nu \Psi \in L^\infty([0, T]; L^p(\mathbb{R}^2))$$

for $q \in [\frac{3}{2}, 3]$ and $p \in [2, \infty]$. Combined with the elliptic estimates in Lemma 2.4 and Lemma 2.5, this ensures that integration by parts for $\nabla \Psi_{E,\epsilon}$ is valid, and thus for ϕ compactly supported in \mathbb{R}_+^3 and time,

$$\begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left((\partial_t \nabla \phi + \bar{\nabla}^\perp \Psi_{E,\epsilon} : \bar{\nabla} \nabla \phi) \cdot \nabla \Psi_{E,\epsilon} + \nabla \phi \cdot \nabla F \right) dx dz dt \\ & = \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left((\partial_t \phi + \bar{\nabla}^\perp \Psi_{E,\epsilon} \cdot \bar{\nabla} \phi) \Delta \Psi_{E,\epsilon} + \phi \Delta F \right) dx dz dt \\ & \quad - \int_0^T \int_{\mathbb{R}^2} \left((\partial_t \phi + \bar{\nabla}^\perp \Psi_{E,\epsilon} \cdot \bar{\nabla} \phi) \partial_\nu \Psi_{E,\epsilon} + \phi \partial_\nu F \right) dx dt \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0, z, x) \cdot \nabla \Psi_\epsilon(0, z, x) dx dz &= - \int_0^\infty \int_{\mathbb{R}^2} \phi(0, z, x) \Delta \Psi_\epsilon(0, z, x) dx dz \\ &\quad + \int_{\mathbb{R}^2} \phi(0, 0, x) \partial_\nu \Psi_\epsilon(0, 0, x) dx. \end{aligned}$$

We now argue that passing to the limit is justified in each identity. We have that Lemma 2.4, Lemma 2.5 give that $\nabla \Psi \in L^{\frac{3q}{3-q}}(\mathbb{R}_+^3) + L^{\frac{3p}{2}}(\mathbb{R}_+^3)$ for all time. Noticing that since $q \geq \frac{3}{2}$ and $p \geq 2$, we have that

$$\frac{3q}{3-q} \geq 3, \quad \frac{3p}{2} \geq 3,$$

and the following convergences follow:

$$\Delta \Psi_{E,\epsilon} \rightarrow \Delta \Psi \quad \text{in} \quad L^2\left([0, T]; L_{loc}^{\frac{3}{2}}(\mathbb{R}_+^3)\right)$$

$$\nabla \Psi_{E,\epsilon} \rightarrow \nabla \Psi \quad \text{in} \quad L^2([0, T]; L_{loc}^3(\mathbb{R}_+^3)).$$

Furthermore, using Hölder's inequality shows that for each fixed time, $\bar{\nabla}^\perp \Psi \Delta \Psi \in L_{loc}^1(\mathbb{R}_+^3)$ and $\bar{\nabla}^\perp \Psi \otimes \nabla \Psi \in L_{loc}^1(\mathbb{R}_+^3)$. Therefore,

$$\bar{\nabla}^\perp \Psi_{E,\epsilon} \Delta \Psi_{E,\epsilon} \rightarrow \bar{\nabla}^\perp \Psi \Delta \Psi \quad \text{in} \quad L^1([0, T]; L_{loc}^1(\mathbb{R}_+^3))$$

$$\bar{\nabla}^\perp \Psi_{E,\epsilon} \otimes \nabla \Psi_{E,\epsilon} \rightarrow \bar{\nabla}^\perp \Psi \otimes \nabla \Psi \quad \text{in} \quad L^1([0, T]; L_{loc}^1(\mathbb{R}_+^3))$$

Finally, we have that

$$\frac{2q}{3-q} \geq 2,$$

and Lemma 2.3 gives $\nabla \Psi_2|_{z=0} \in L^{\frac{2q}{3-q}}(\mathbb{R}^2)$. Recalling that $\partial_\nu \Psi \in L^2(\mathbb{R}^2)$ and $\bar{\nabla}^\perp \Psi_1 = -\mathcal{R}^\perp \partial_\nu \Psi$, applying Hölder again gives $\bar{\nabla}^\perp \Psi \partial_\nu \Psi \in L^1_{loc}(\mathbb{R}^2)$. It therefore follows that

$$\partial_\nu \Psi_{E,\epsilon} \rightarrow \partial_\nu \Psi \quad \text{in} \quad L^2([0, T]; L^2_{loc}(\mathbb{R}^2))$$

$$\bar{\nabla}^\perp \Psi_{E,\epsilon}|_{z=0} \rightarrow \bar{\nabla}^\perp \Psi|_{z=0} \quad \text{in} \quad L^2([0, T]; L^2_{loc}(\mathbb{R}^2))$$

and

$$\bar{\nabla}^\perp \Psi_{E,\epsilon} \partial_\nu \Psi_{E,\epsilon} \rightarrow \bar{\nabla}^\perp \Psi \partial_\nu \Psi \quad \text{in} \quad L^1([0, T]; L^1_{loc}(\mathbb{R}^2))$$

Letting ϵ tend to 0 shows that

$$\begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left((\partial_t \nabla \phi + \bar{\nabla}^\perp \Psi : \bar{\nabla} \nabla \phi) \cdot \nabla \Psi + \nabla \phi \cdot \nabla F \right) dx dz dt \\ & = \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left((\partial_t \phi + \bar{\nabla}^\perp \Psi \cdot \bar{\nabla} \phi) \Delta \Psi + \phi \Delta F \right) dx dz dt \\ & - \int_0^T \int_{\mathbb{R}^2} \left((\partial_t \phi + \bar{\nabla}^\perp \Psi \cdot \bar{\nabla} \phi) \partial_\nu \Psi + \phi \partial_\nu F \right) dx dt. \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0, z, x) \cdot \nabla \Psi(0, z, x) dx dz = - \int_0^\infty \int_{\mathbb{R}^2} \phi(0, z, x) \Delta \Psi(0, z, x) dx dz \\ & + \int_{\mathbb{R}^2} \phi(0, 0, x) \partial_\nu \Psi(0, 0, x) dx. \end{aligned} \tag{5.2}$$

Step Two : Let us start by assuming that $\nabla \Psi$ satisfies Definition 1.1. Then we have that

$$\begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left((\partial_t \nabla \phi + \bar{\nabla}^\perp \Psi : \bar{\nabla} \nabla \phi) \cdot \nabla \Psi + \nabla \phi \cdot \nabla F \right) dx dz dt = \\ & \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0, z, x) \cdot \nabla \Psi(0, z, x) dx dz, \end{aligned}$$

i.e. the left hand side of (5.1) is equal to the left hand side of (5.2). Choosing ϕ to be compactly supported in $[-T, T] \times \mathbb{R}^3_+$ gives that

$$\begin{aligned} & \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left((\partial_t \phi + \bar{\nabla}^\perp \Psi \cdot \bar{\nabla} \phi) \cdot \Delta \Psi + \phi \Delta F \right) dx dz dt \\ & = - \int_0^\infty \int_{\mathbb{R}^2} \phi(0, z, x) \Delta \Psi(0, z, x) dx dz, \end{aligned}$$

and therefore $\nabla \Psi$ satisfies (1.1).

To show that $\nabla \Psi$ satisfies (1.2), choose $\bar{\phi}$ to be a test function compactly supported in $[-T, T] \times \mathbb{R}^2$. Let $\gamma(z)$ be a smooth function of one variable supported in a ball of radius 1 with $\gamma(0) = 1$. Let $\gamma_n(z) = \gamma(nz)$. Define $\phi_n(t, z, x) = \gamma_n(z) \bar{\phi}(t, x)$. Then $\bar{\nabla} \phi_n$, $\partial_t \phi_n$, and ϕ_n converge to 0 in \mathbb{R}^3_+ (both pointwise and in any Lebesgue space). We have that the right

hand side of (5.1) is equal to the right hand side of (5.2). Then plugging in ϕ_n as a test function, letting n tend to infinity, and passing to the limit shows that

$$(5.3) \quad \begin{aligned} & - \int_0^T \int_{\mathbb{R}^2} \left(\left(\partial_t \bar{\phi} + \bar{\nabla}^\perp \Psi \cdot \bar{\nabla} \bar{\phi} \right) \partial_\nu \Psi + \bar{\phi} \partial_\nu F \right) dx dt \\ & = \int_{\mathbb{R}^2} \bar{\phi}(0, 0, x) \partial_\nu \Psi(0, x) dx. \end{aligned}$$

Now assume for the other direction that Ψ verifies Definition 1.2. Then for ϕ compactly supported in \mathbb{R}_+^3 (and time) and $\bar{\phi}$ compactly supported in \mathbb{R}^2 (and time),

$$(5.4) \quad \begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \phi + \bar{\nabla}^\perp \Psi \cdot \bar{\nabla} \phi \right) \Delta \Psi + \phi f_L \right) dx dz dt \\ & = \int_0^\infty \int_{\mathbb{R}^2} \phi(0, z, x) \Delta \Psi(0, z, x) dx dz \end{aligned}$$

and

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^2} \left(\left(\partial_t \bar{\phi} + \bar{\nabla}^\perp \Psi(t, 0, x) \cdot \bar{\nabla} \bar{\phi} \right) \partial_\nu \Psi(t, x) + \bar{\phi} f_\nu \right) dx dt \\ & = \int_{\mathbb{R}^2} \bar{\phi}(0, x) \partial_\nu \Psi(0, x) dx. \end{aligned}$$

Before proceeding we show that (5.4) holds for ϕ compactly supported in \mathbb{R}^3 rather than \mathbb{R}_+^3 . Let ϕ be compactly supported in \mathbb{R}^3 and time. Using $\gamma_n(z)$ as defined previously, define

$$\phi_n(t, z, x) = (1 - \gamma_n(z)) \phi(t, z, x).$$

Then ϕ_n is compactly supported in \mathbb{R}_+^3 and $\bar{\nabla} \phi_n$, $\partial_t \phi_n$, and ϕ_n converge to $\bar{\nabla} \phi$, $\partial_t \phi$, and ϕ respectively, both pointwise in \mathbb{R}_+^3 and in any Lebesgue space. Therefore

$$\begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \phi + \bar{\nabla}^\perp \Psi \cdot \bar{\nabla} \phi \right) \Delta \Psi + \phi f_L \right) dx dz dt \\ & = \lim_{n \rightarrow \infty} - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \phi_n + \bar{\nabla}^\perp \Psi \cdot \bar{\nabla} \phi_n \right) \Delta \Psi + \phi_n f_L \right) dx dz dt \\ & = \lim_{n \rightarrow \infty} \int_0^\infty \int_{\mathbb{R}^2} \phi_n(0, z, x) \Delta \Psi(0, z, x) dx dz \\ & = \int_0^\infty \int_{\mathbb{R}^2} \phi(0, z, x) \Delta \Psi(0, z, x) dx dz, . \end{aligned}$$

We have then that the right hand side of (5.1) is equal to the right hand side of (5.2), showing then that the left hand side of (5.1) is equal to the left hand side of (5.2). Therefore, $\nabla \Psi$ satisfies Definition 1.1 and is a weak solution to (rQG) . □

Proof of Theorem 1.3(2). As in part (1), the proof is split up into two steps.

Step One : We assume that $p \in (\frac{4}{3}, 2]$, $q \in [\frac{3}{2}, 3]$, and

$$p \geq \frac{2q}{3(q-1)}.$$

Let us first point out the implications of the assumptions on p and q . Throughout, we use the definitions of Ψ_1 and Ψ_2 described in Definition 1.3. First, since $q \geq \frac{3}{2}$, Lemma 2.4

ensures that for all time, $\nabla \Psi_2 \in L_{loc}^3(\mathbb{R}_+^3)$, and therefore $\nabla \Psi_2 \Delta \Psi \in L_{loc}^1(\mathbb{R}_+^3)$ is well-defined by Hölder's inequality. Secondly, from Lemma 2.5, we have $\nabla \Psi_1 \in L^{\frac{3p}{2}}(\mathbb{R}_+^3)$. Thus, the assumption that $p \geq \frac{2q}{3(q-1)}$ ensures that

$$\frac{2}{3p} \leq \frac{q-1}{q},$$

and therefore $\nabla \Psi_1 \Delta \Psi \in L_{loc}^1(\mathbb{R}_+^3)$ is also well-defined by Hölder's inequality. Next, applying Lemma 2.3 to Ψ_2 gives that $\nabla \Psi_2|_{z=0} \in L^{\frac{2q}{3-q}}(\mathbb{R}^2)$. Using that $p \geq \frac{2q}{3(q-1)}$, it follows that

$$\frac{3-q}{2q} \leq \frac{p-1}{p}.$$

Therefore, $\overline{\nabla}^\perp \Psi_2|_{z=0} \partial_\nu \Psi \in L_{loc}^1(\mathbb{R}^2)$ is also well-defined from Hölder's inequality. Combined with the fact that $p > \frac{4}{3}$, we can apply Lemma 2.1, yielding that

$$\left(\overline{\nabla}^\perp \Psi \partial_\nu \Psi \right)_C$$

is well-defined as a distribution.

The proof now proceeds as before. We regularize and extend $\nabla \Psi$ to $\nabla \Psi_{E,\epsilon}$. Then

$$\begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \nabla \phi + \overline{\nabla}^\perp \Psi_{E,\epsilon} : \overline{\nabla} \nabla \phi \right) \cdot \nabla \Psi_{E,\epsilon} + \nabla \phi \cdot \nabla F \right) dx dz dt \\ &= \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \phi + \overline{\nabla}^\perp \Psi_{E,\epsilon} \cdot \overline{\nabla} \phi \right) \Delta \Psi_{E,\epsilon} + \phi \Delta F \right) dx dz dt \\ &\quad - \int_0^T \int_{\mathbb{R}^2} \left(\left(\partial_t \phi + \overline{\nabla}^\perp \Psi_{E,\epsilon} \cdot \overline{\nabla} \phi \right) \partial_\nu \Psi_{E,\epsilon} + \phi \partial_\nu F \right) dx dt \\ &= \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \phi + \overline{\nabla}^\perp \Psi_{E,\epsilon} \cdot \overline{\nabla} \phi \right) \Delta \Psi_{E,\epsilon} + \phi \Delta F \right) dx dz dt \\ &\quad - \int_0^T \int_{\mathbb{R}^2} \left(\partial_t \phi \partial_\nu \Psi_{E,\epsilon} + \left(\overline{\nabla}^\perp \Psi_{E,\epsilon} \partial_\nu \Psi_{E,\epsilon} \right)_C \cdot \overline{\nabla} \phi + \phi \partial_\nu F \right) dx dt. \end{aligned}$$

The second equality holds since the smoothness of $\nabla \Psi_{E,\epsilon}$ ensures that $\left(\overline{\nabla}^\perp \Psi_{E,\epsilon} \partial_\nu \Psi_{E,\epsilon} \right)_C$ as a distribution is equal to the regular distribution $\left(\overline{\nabla}^\perp \Psi_{E,\epsilon} \partial_\nu \Psi_{E,\epsilon} \right)$ (see Lemma 2.1). We have

$$\begin{aligned} \Delta \Psi_{E,\epsilon} &\rightarrow \Delta \Psi & \text{in} & L^2([0, T]; L^q(\mathbb{R}_+^3)) \\ \nabla \Psi_{E,\epsilon} &\rightarrow \nabla \Psi & \text{in} & L^2\left([0, T]; L_{loc}^{\frac{q}{q-1}}(\mathbb{R}_+^3)\right), \end{aligned}$$

and therefore

$$\nabla \Psi_{E,\epsilon} \Delta \Psi_{E,\epsilon} \rightarrow \nabla \Psi \Delta \Psi \quad \text{in} \quad L^1([0, T]; L_{loc}^1(\mathbb{R}_+^3))$$

and

$$\overline{\nabla}^\perp \Psi_{E,\epsilon} \otimes \nabla \Psi_{E,\epsilon} \rightarrow \overline{\nabla}^\perp \Psi \nabla \Psi \quad \text{in} \quad L^1([0, T]; L_{loc}^1(\mathbb{R}_+^3))$$

We have

$$\partial_\nu \Psi_{E,\epsilon} \rightarrow \partial_\nu \Psi \quad \text{in} \quad L^2([0, T]; L^p(\mathbb{R}^2))$$

and the weak-* convergence

$$\partial_\nu \Psi_{E,\epsilon} \rightarrow \partial_\nu \Psi \quad \text{in} \quad L^\infty([0, T]; L^p(\mathbb{R}^2)).$$

In addition,

$$\overline{\nabla}^\perp \Psi_{E,\epsilon,2}|_{z=0} \rightarrow \overline{\nabla}^\perp \Psi_2|_{z=0} \quad \text{in} \quad L^2\left([0, T]; L_{loc}^{\frac{p}{p-1}}(\mathbb{R}^2)\right),$$

and applying Lemma 2.1 since $p > \frac{4}{3}$ yields that

$$\begin{aligned} \overline{\nabla} \cdot \left(\overline{\nabla}^\perp \Psi_{E,\epsilon} \partial_\nu \Psi_{E,\epsilon} \right)_C &= \overline{\nabla} \cdot \left(\overline{\nabla}^\perp \Psi_{E,\epsilon,2} \partial_\nu \Psi_{E,\epsilon} - \partial_\nu \Psi_{E,\epsilon} \mathcal{R}^\perp \partial_\nu \Psi_{E,\epsilon} \right) \\ &\rightarrow \overline{\nabla} \cdot \left(\overline{\nabla}^\perp \Psi \partial_\nu \Psi \right)_C \quad \text{in} \quad \mathcal{D}'\left((0, T) \times \mathbb{R}^2\right). \end{aligned}$$

Passing to the limit shows that

$$\begin{aligned} & - \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \nabla \phi + \overline{\nabla}^\perp \Psi : \overline{\nabla} \nabla \phi \right) \cdot \nabla \Psi + \nabla \phi \cdot \nabla F \right) dx dz dt \\ &= \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \phi + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla} \phi \right) \cdot \Delta \Psi + \phi \Delta F \right) dx dz dt \\ & - \int_0^T \int_{\mathbb{R}^2} \left(\partial_t \phi \partial_\nu \Psi + \left(\overline{\nabla}^\perp \Psi \partial_\nu \Psi \right)_C \cdot \overline{\nabla} \phi + \phi \partial_\nu F \right) dx dt \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0, z, x) \cdot \nabla \Psi(0, z, x) dx dz &= - \int_0^\infty \int_{\mathbb{R}^2} \phi(0, z, x) \Delta \Psi(0, z, x) dx dz \\ &+ \int_{\mathbb{R}^2} \phi(0, 0, x) \partial_\nu \Psi(0, 0, x) dx, . \end{aligned} \quad (5.6)$$

Step Two : Assuming (5.5) and (5.6) hold, we can argue precisely as in the proof of Theorem 1.3(1) to prove the theorem. We refer the reader to the proof of Theorem 1.3(1) for further details. \square

Proof of Theorem 1.3(3). The claim follows immediately from the observation that $\overline{\nabla}^\perp \Psi_1 = -\mathcal{R}^\perp \partial_\nu \Psi_1$ and the claim in Lemma 2.1 that

$$\int_{\mathbb{R}^2} f \mathcal{R}^\perp f \cdot \overline{\nabla} \phi = -\frac{1}{2} \int_{\mathbb{R}^2} \mathcal{R}^\perp f [\bar{\Lambda}, \overline{\nabla} \phi] (\bar{\Lambda}^{-1} f)$$

for $f \in L^2(\mathbb{R}^2)$. \square

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(Matthew D. Novack)
 DEPARTMENT OF MATHEMATICS,
 THE UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TX 78712, USA
E-mail address: mnovack@math.utexas.edu