On transversal and 2-packing numbers in straight line systems on $\mathbb{R}^2$

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Abstract

A *linear system* is a pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a finite family of subsets on a ground set X, and it satisfies that $|A \cap B| \leq 1$ for every pair of

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distinct subsets $A, B \in \mathcal{F}$. As an example of a linear system are the straight line systems, which family of subsets are straight line segments on $\mathbb{R}^2$. By τ and ν_2 we denote the size of the minimal transversal and the 2–packing numbers of a linear system respectively. A natural problem is asking about the relationship of these two parameters; it is not difficult to prove that there exists a quadratic function f holding $\tau \leq f(\nu_2)$. However, for straight line system we believe that $\tau \leq \nu_2 - 1$. In this paper we prove that for any linear system with 2-packing numbers ν_2 equal to $2, 3$ and 4, we have that $\tau \leq \nu_2$. Furthermore, we prove that the linear systems that attains the equality have transversal and 2-packing numbers equal to 4, and they are a special family of linear subsystems of the projective plane of order 3. Using this result we confirm that all straight line systems with $\nu_2 \in \{2, 3, 4\}$ satisfies $\tau \leq \nu_2 - 1$.

Key words. Linear systems, straight line systems, transversal, 2–packing, projective plane.

1 Introduction

A *set system* is a pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a finite family of subsets on a ground set X. A set system can be also thought of as a hypergraph, where the elements of X and $\mathcal{F}$ are called *vertices* and *hyperedges* respectively.

Definition 1.1. *A subset $T \subset X$ is called a* transversal *of $(X, \mathcal{F})$, if it intersects all the sets of $\mathcal{F}$. The* transversal number *of $(X, \mathcal{F})$, denoted by $\tau(X, \mathcal{F})$, is the smallest possible cardinality of a transversal of $(X, \mathcal{F})$.*

Transversal numbers have been studied in the literature in many different contexts and names. For example with the name of *piercing number* and *covering number* (see [1, 2, 3, 10, 12, 14, 15]).

A system $(X, \mathcal{F})$ is a λ-*Helly system*, if $\mathcal{F}$ satisfies the λ-*Helly property*, that is, if every subfamily $\mathcal{F}' \subset \mathcal{F}$ has the property that any $(\lambda+1)$-tuple of $\mathcal{F}'$ is intersecting, then $\mathcal{F}'$ is intersecting. Examples of λ-Helly systems are families of convex sets in $\mathbb{R}^\lambda$ and the systems arriving from a λ-hypergraph as following: Let G be a λ-hypergraph, and consider the set $V(G)$ and the family $\mathbb{I}$ of maximal independent subset of vertices of G (where $I \subset V(G)$ is independent, if there is no edge $e \in E(G)$ such that $e \subset I$). We associate to the λ-hypergraph G the following set system $(\mathbb{I}, V^*)$, where $V^* = \{v^* \mid v \in V(G)\}$ and $v^* = \{S \in \mathbb{I} \mid v \in S\}$. Then it is not difficult to see that $\tau(\mathbb{I}, V^*)$ is the *chromatic number* $\chi(G)$ of G. Furthermore, the system $(\mathbb{I}, V^*)$ is a λ-Helly system.

Definition 1.2. *A set system $(X, \mathcal{F})$ is called a* linear system*, if it satisfies $|A \cap B| \leq 1$ for every pair of distinct subsets $A, B \in \mathcal{F}$.*

Note that any linear system $(X, \mathcal{F})$ is a 2-Helly system and therefore its transversal number $\tau(X, \mathcal{F})$ can be regarded as the chromatic number of the 3-hypergraph G, such that $V(G) = \mathcal{F}$ and $\{A, B, C\} \in E(G)$, if and only if, $A \cap B \cap C = \emptyset$.

Definition 1.3. *A subset $R \subseteq \mathcal{F}$ is called a* 2-packing *of a set system $(X, \mathcal{F})$, if the elements of R are triplewise disjoint. The 2-packing number of $(X, \mathcal{F})$, denoted by $\nu_2(X, \mathcal{F})$, is the greatest possible number of a 2-packing of $(X, \mathcal{F})$.*

Note that for a linear system its 2-packing number $\nu_2(X, \mathcal{F})$ can be regarded as the *clique number* $\omega(G)$ of the 3-hypergraph G described above. So, for linear systems $(X, \mathcal{F})$ we have:

$$\lceil \nu_2(X, \mathcal{F})/2 \rceil \leq \tau(X, \mathcal{F}) \leq \frac{\nu_2(\nu_2 - 1)}{2},$$

since any maximum 2–packing of $(X, \mathcal{F})$ induces at most $\frac{\nu_2(\nu_2-1)}{2}$ double points (points incident to two lines). In general the transversal number $\tau(X, \mathcal{F})$ of a λ-Helly system can be arbitrarily large even if $\nu_\lambda(X, \mathcal{F})$ is small.

There are many interesting works studying the relationship between $\tau(X, \mathcal{F})$ and $\nu_\lambda(X, \mathcal{F})$, and of course recording the problem of giving a bound of $\tau(X, \mathcal{F})$ in terms of a function of $\nu_2(X, \mathcal{F})$ (see [1]). For linear systems in a more general context there are bounds to transversal number [6, 9].

In this paper we denote linear systems by $(P, \mathcal{L})$, where the elements of P and $\mathcal{L}$ are called *points* and *lines* respectively.

We study some specific linear systems called *straight line systems*, which are defined below. Some results of this kind of linear systems related with this work appears in [14].

Definition 1.4. *A* straight line representation *on $\mathbb{R}^2$ of a linear system $(P, \mathcal{L})$ maps each point $x \in P$ to a point $p(x)$ of $\mathbb{R}^2$, and each line $F \in \mathcal{L}$ to a straight line segment $l(F)$ of $\mathbb{R}^2$ in such way that for each point $x \in P$ and line $F \in \mathcal{L}$ we have $p(x) \in l(F)$, if and only if, $x \in F$, and for each pair of distinct lines $F, H \in \mathcal{F}$ we have $l(F) \cap l(H) = \{p(x) : x \in F \cap H\}$. A* straight line system *$(P, \mathcal{L})$ is a linear system, such that it has a straight line representation on $\mathbb{R}^2$.*

The main result of this work is set in the following theorem:

Theorem 1.1. *Let $(P, \mathcal{L})$ be a straight line system with $|\mathcal{L}| > \nu_2(P, \mathcal{L})$. If $\nu_2(P, \mathcal{L}) \in \{2, 3, 4\}$, then $\tau(P, \mathcal{L}) \leq \nu_2(P, \mathcal{L}) - 1$.*

We believe that Theorem 1.1 is true in general, that is $\tau(P, \mathcal{L}) \leq \nu_2(P, \mathcal{L}) - 1$, for $\nu_2(P, \mathcal{L}) \geq 2$, which seems to be extremely difficult to prove. For the cases where the 2-packing number is equal to 2 or 3 its proof is easy (see propositions 2.1 and 2.2), and the interesting case is when $\nu_2 = 4$.

To prove Theorem 1.1 we use the following theorem, which is one of the main results of this work.

Theorem 1.2. *Let $(P, \mathcal{L})$ be a linear system with $|\mathcal{L}| > 4$. If $\nu_2(P, \mathcal{L}) = 4$, then $\tau(P, \mathcal{L}) \leq 4$. Moreover, if $\tau(P, \mathcal{L}) = \nu_2(P, \mathcal{L}) = 4$, then $(P, \mathcal{L})$ is a linear subsystem of Π_3.*

It is important to say that this problems is closely related with the Hadwiger-Debrunner (p, q)–property for linear set systems $(P, \mathcal{L})$ with $p = \nu_2(P, \mathcal{L}) + 1$ and $q = 3$. A family of sets has the (p, q) property, if among any p members of the family some q have a nonempty intersection. In this contest, our results states that, if $(P, \mathcal{L})$ is a linear system satisfying the $(\nu_2(P, \mathcal{L}) + 1, 3)$ property, for $\nu_2(P, \mathcal{L}) = 2, 3, 4$, then $\tau(P, \mathcal{L}) \leq \nu_2(P, \mathcal{L})$. For more information about the Hadwiger-Debrunner (p, q)–property see [4, 5].

Theorem 2.1 states that any linear system $(P, \mathcal{L})$ with $\nu_2(P, \mathcal{L}) = 4$ and $|\mathcal{L}| > 4$ is such that $\tau(P, \mathcal{L}) \leq 4$, giving a characterization to those linear systems which transversal number is 4. Furthermore, we prove that these linear systems have not a straight line representation on $\mathbb{R}^2$.

It is worth noting that such linear systems $(P, \mathcal{L})$ where $\nu_2(P, \mathcal{L}) = \tau(P, \mathcal{L}) = 4$ are certain linear subsystems of the projective plane of order 3 (Figure 1).

Recall that a *finite projective plane* (or merely *projective plane*) is a linear system satisfying that any pair of points have a common line, any pair of lines have a common point and there exist four points in general position (there are not three collinear points). It is well known that, if $(P, \mathcal{L})$ is a projective plane then there exists a number $q \in \mathbb{N}$, called *order of projective plane*, such that every point (line, resp.) of $(P, \mathcal{L})$ is incident to exactly $q + 1$ lines (points, resp.), and $(P, \mathcal{L})$ contains exactly $q^2 + q + 1$ points (lines, resp.). In addition it is well known that projective planes of order q exist when q is a power prime. In this work we denote by Π_q the projective plane of order q. For more information about the existence and the unicity of projective planes see, for instance, [4, 5].

Concerning the transversal number of projective planes it is well known that every line in Π_q is a transversal, then $\tau(\Pi_q) \leq q+1$. On the other hand $\tau(\Pi_q) \geq q + 1$ since a transversal with less than q points cannot exist by a counting argument (recall that every point in Π_q is incident to exactly $q+1$

lines and the total number of lines is equal to q^2+q+1). Now, related to the 2-packing number, since projective planes are dual systems, this parameter coincides with the cardinality of an *oval*, which is the maximum number of points in general position (no three of them collinear), and it is equal to $q+1$ when q is odd (see for example [5]). Consequently, for projective planes Π_q of odd order q we have that $\tau(\Pi_q) = \nu_2(\Pi_q) = q+1$.

In this work we prove, beyond of Theorem 1.1, if $(P, \mathcal{L})$ is a linear system satisfying $|\mathcal{L}| > \nu_2(P, \mathcal{L})$ with $\nu_2(P, \mathcal{L}) \in \{2, 3, 4\}$, then $\tau(P, \mathcal{L}) \leq \nu_2(P, \mathcal{L})$; and that every projective plane Π_q of odd order satisfies $\tau(\Pi_q) = \nu_2(\Pi_q) = q+1$. Furthermore, it is not difficult to prove that, if $(P, \mathcal{L})$ is a 2-uniform linear system (a simple graph) with $|\mathcal{L}| > \nu_2(P, \mathcal{L})$, then $\tau(P, \mathcal{L}) \leq \nu_2(P, \mathcal{L})-1$. It is tempting to conjecture that any linear system $(P, \mathcal{L})$ with $|\mathcal{L}| > \nu_2(P, \mathcal{L})$ satisfies $\tau(P, \mathcal{L}) \leq \nu_2(P, \mathcal{L})$. Unfortunately that is not true, in [11] proved, using probabilistic methods the existence of k-uniform linear systems $(P, \mathcal{L})$ for infinitely many k´s and $n = |P|$ large enough, which transversal number is $\tau(P, \mathcal{L}) = n - o(n)$. This k-uniform linear systems has 2-packing number upper bounded by $\frac{2n}{k}$, therefore $\nu_2(P, \mathcal{L}) < \tau(P, \mathcal{L})$. Moreover, this implies that $\tau \leq \lambda\nu_2$ does not hold for any positive λ.

2 Results

Before continuing we give some basic concepts and standard notation although many of them can be applied for general set systems. Let $(P, \mathcal{L})$ be a linear system and $p \in P$ be a point. We use $\mathcal{L}_p$ to denote the set of lines incident to p. The *degree* of p is defined as $deg(p) = |\mathcal{L}_p|$, the maximum degree overall points of the linear systems is denoted by $\Delta(P, \mathcal{L})$ and the set of points of degree at least k is denoted by X_k, this is $X_k = \{p \in P : deg(p) \geq k\}$. A point of degrees 2 and 3 is called *double point* and *triple point* respectively. Finally, a linear system $(P, \mathcal{L})$ is called r-*regular*, if every point of P has degree r, and $(P, \mathcal{L})$ is called k-*uniform*, if every line of $\mathcal{L}$ has exactly k points.

The following is a trivial observation that will be used later on in order to avoid annoying cases.

Remark 2.1. *A linear system* $(P,\mathcal{L})$ *satisfies* $\Delta(P,\mathcal{L}) \leq 2$*, if and only if,* $\nu_2(P,\mathcal{L}) = |\mathcal{L}|$.

Note that for linear systems $(P,\mathcal{L})$ with $|\mathcal{L}| > \nu_2(P,\mathcal{L})$ the meaning of $\nu_2(P,\mathcal{L}) = n$ is that, on the one hand there is at least one set of n lines inducing no triple points, and on the other hand any set of $(n+1)$ lines induces a triple point. In the next propositions 2.1 and 2.2 we prove that any linear system $(P,\mathcal{L})$ with $|\mathcal{L}| > \nu_2(P,\mathcal{L})$ is such that $\tau(P,\mathcal{L}) \leq \nu_2(P,\mathcal{L}) - 1$, for $\nu_2(P,\mathcal{L}) = 2$ and $\nu_2(P,\mathcal{L}) = 3$ respectively; consequently, Theorem 1.1 holds for $\nu_2(P,\mathcal{L}) = 2$, and $\nu_2(P,\mathcal{L}) = 3$. In [14] we proved that, if $(P,\mathcal{L})$ is a straight line system with the property that, if any 4 members of $\mathcal{L}$ have a triple point, then $\tau(P,\mathcal{L}) \leq 2$, that is, if $(P,\mathcal{L})$ is a straight line systems with $|\mathcal{L}| > 4$ and $2 \leq \nu_2(P,\mathcal{L}) \leq 3$, then $\tau(P,\mathcal{L}) \leq 2$, which is also a consequence of the propositions 2.1 and 2.2 proved below.

Proposition 2.1. *If* $(P,\mathcal{L})$ *is any linear system with* $\nu_2(P,\mathcal{L}) = 2$ *and* $|\mathcal{L}| > 2$*, then* $\tau(P,\mathcal{L}) = 1$.

Proof. As any set of three lines has a common point then by 2–Helly property all lines of $\mathcal{L}$ have a common point, that is $\tau(P,\mathcal{L}) = 1$. ■

It is worth noting that the converse of Proposition 2.1 is also true, that is, any linear system $(P,\mathcal{L})$ with $\tau(P,\mathcal{L}) = 1$ satisfies $\nu_2(P,\mathcal{L}) = 2$.

Next we establish an analogous statement to Proposition 2.1 concerning linear systems, which 2–packing number is three.

Proposition 2.2. *If* $(P,\mathcal{L})$ *is any linear system with* $\nu_2(P,\mathcal{L}) = 3$ *and* $|\mathcal{L}| > 3$*, then* $\tau(P,\mathcal{L}) = 2$.

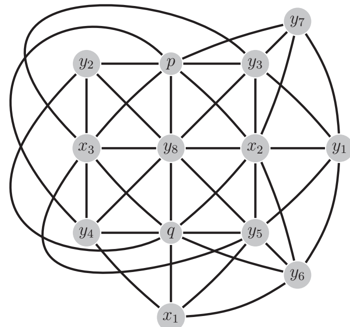


Figure 1:

Proof. Recall that $\nu_2(P, \mathcal{L}) = 3$ implies that any set of four lines induces a triple point. By Remark 2.1, $\Delta(P, \mathcal{L}) \geq 3$, thus the set of points of degree at least 3, X_3, is not empty. If $|X_3| \geq 2$ we can easily find a set of four lines inducing no triple point (take two distinct points in X_3, and two lines inciding at each). If $|X_3| = 1$, let $p \in P$ be the only point with $deg(p) \geq 3$. Assume that there is another point $q \in P$, $q \neq p$, such that $deg(q) = 2$, otherwise $|\mathcal{L} \setminus \mathcal{L}_p| \leq 1$ and the statement holds true. Now consider $\mathcal{L}'' = \mathcal{L} \setminus (\mathcal{L}_p \cup \mathcal{L}_q)$. Note that $\mathcal{L}'' = \emptyset$, otherwise we can take four lines (two in $\mathcal{L}_p$, one in $\mathcal{L}_q$ and one more in $\mathcal{L}''$) inducing no triple point; a contradiction to the hypothesis $\nu_2(P, \mathcal{L}) = 3$. Hence, the set $\{p, q\}$ is a transversal, and $\tau(P, \mathcal{L}) = 2$ as stated.
■

In view of Propositions 2.1 and 2.2 it is tempting to try to prove that any linear system $(P, \mathcal{L})$ with $\nu_2(P, \mathcal{L}) = 4$ satisfies $\tau(P, \mathcal{L}) \leq 3$. However, as we stated in the introduction the projective plane $\Pi_3 = (P, \mathcal{L})$ of order 3 (Figure 1) satisfies $\nu_2(\Pi_3) = \tau(\Pi_3) = 4$.

The main work of this paper is to prove that any straight line system $(P, \mathcal{L})$ with $\nu_2(P, \mathcal{L}) = 4$, and $|\mathcal{L}| > 4$ is such that $\tau(P, \mathcal{L}) \leq 3$. To prove this we use Theorem 2.1, that is we prove that any linear system $(P, \mathcal{L})$ with $\nu_2(P, \mathcal{L}) = 4$, and $|\mathcal{L}| > 4$ is such that $\tau(P, \mathcal{L}) \leq 4$, giving a characterization to those linear systems, which transversal number is 4, and we prove that these linear systems have not a straight line representation on $\mathbb{R}^2$.

Definition 2.1. *A* linear subsystem $(P', \mathcal{L}')$ *of a linear system* $(P, \mathcal{L})$ *satisfies that for any line* $l' \in \mathcal{L}'$ *there exists a line* $l \in \mathcal{L}$ *such that* $l' = l \cap P'$*. The* linear subsystem induced *by a set of lines* $\mathcal{L}' \subseteq \mathcal{L}$ *is the linear subsystem* $(P', \mathcal{L}')$ *where* $P' = \bigcup_{l \in \mathcal{L}'} l$*.*

One of the main results of this paper estates the following:

Theorem 2.1. *Let* $(P, \mathcal{L})$ *be a linear system with* $|\mathcal{L}| > 4$*. If* $\nu_2(P, \mathcal{L}) = 4$*, then* $\tau(P, \mathcal{L}) \leq 4$*. Moreover, if* $\tau(P, \mathcal{L}) = \nu_2(P, \mathcal{L}) = 4$*, then* $(P, \mathcal{L})$ *is a linear subsystem of* Π_3*.*

In order to prove Theorem 2.1 we analyze different cases related to the maximum degree of the linear system. Note that by Remark 2.1, a linear system $(P, \mathcal{L})$ satisfying the hypothesis of Theorem 2.1 is such that $\Delta(P, \mathcal{L}) > 2$. In Lemma 2.1 below we prove that linear systems with $\nu_2(P, \mathcal{L}) = 4$, and $\Delta(P, \mathcal{L}) \geq 5$ are such that $\tau(P, \mathcal{L}) \leq 3$. The remaining cases, $\Delta(P, \mathcal{L}) = 3$ and $\Delta(P, \mathcal{L}) = 4$ are the cases for which there are linear systems satisfying $\nu_2(P, \mathcal{L}) = \tau(P, \mathcal{L}) = 4$. We handle those cases in Section 3 and Section 4 respectively. In each case we describe all linear systems $(P, \mathcal{L})$ satisfying $\nu_2(P, \mathcal{L}) = \tau(P, \mathcal{L}) = 4$.

Before proceeding to the next section we will prove that linear systems $(P, \mathcal{L})$ with $\nu_2(P, \mathcal{L}) = 4$, and $\Delta(P, \mathcal{L}) \geq 5$ are such that $\tau(P, \mathcal{L}) = 2$, except for a particular case, which satisfies $\tau(P, \mathcal{L}) = 3$.

Lemma 2.1. *Any linear system* $(P, \mathcal{L})$ *with* $\nu_2(P, \mathcal{L}) = 4$*, and* $\Delta(P, \mathcal{L}) \geq 5$ *satisfies* $\tau(P, \mathcal{L}) \leq 3$*.*

Proof. Recall that $\nu_2(P, \mathcal{L}) = 4$ implies that any set of five lines induces a triple point. Consider $p \in X_5$, and define $\mathcal{L}' = \mathcal{L} \setminus \mathcal{L}_p$. Let $(P', \mathcal{L}')$ be the linear subsystem induced by $\mathcal{L}'$. Note that $|\mathcal{L}'| \geq 2$, otherwise $\nu_2(P, \mathcal{L}) \leq 3$, a contradiction to the hypothesis $\nu_2(P,\mathcal{L})=4$. If $\mathcal{L}'=\{l_1,l_2\}$, then $\{p,l_1\cap l_2\}$ is a minimum transversal of $(P,\mathcal{L})$, if $l_1\cap l_2\neq\emptyset$, or else (when $l_1\cap l_2=\emptyset$) the linear system satisfies $\tau(P,\mathcal{L})=3$. On the other hand, if $|\mathcal{L}'|\geq 3$ we claim that $\nu_2(P',\mathcal{L}')=2$ from which it follows by Proposition 2.1 that $\tau(P',\mathcal{L}')=1$, therefore $\tau(P,\mathcal{L})=2$. To verify the claim, suppose on the contrary that there are a set of three lines $\{l_1,l_2,l_3\}$ of $\mathcal{L}'$ inducing no triple point. This set of three lines induces at most three double points. By the Pigeonhole Principle there are at least two lines $l,l'\in\mathcal{L}_p$, which do not contain any of these double points, then the set $\{l,l',l_1,l_2,l_3\}$ induces no triple point; a contradiction to the hypothesis $\nu_2(P,\mathcal{L})=4$. ■

3 The case when $\Delta(P,\mathcal{L})=3$

We begin this section by introducing some terminology, which will simplify the description of linear systems $(P,\mathcal{L})$ with $\Delta(P,\mathcal{L})=3$, and $\nu_2(P,\mathcal{L})=\tau(P,\mathcal{L})=4$.

Definition 3.1. *Given a linear system $(P,\mathcal{L})$, and a point $p\in P$, the linear system obtained from $(P,\mathcal{L})$ by* deleting point p *is the linear system $(P',\mathcal{L}')$ induced by $\mathcal{L}'=\{l\setminus\{p\}:l\in\mathcal{L}\}$. Given a linear system $(P,\mathcal{L})$ and a line $l\in\mathcal{L}$, the linear system obtained from $(P,\mathcal{L})$ by* deleting line l *is the linear system $(P',\mathcal{L}')$ induced by $\mathcal{L}'=\mathcal{L}\setminus\{l\}$.*

It is important to state that in the rest of this paper we consider linear systems $(P,\mathcal{L})$ without points of degree one because, if $(P,\mathcal{L})$ is a linear system which has all lines with at least two points of degree 2 or more, and $(P',\mathcal{L}')$ is the linear system obtained from $(P,\mathcal{L})$ by deleting all points of degree one, then they are essentially the same linear system because it is not difficult to prove that transversal and 2-packing numbers of both coincide.

Definition 3.2. *Let $(P',\mathcal{L}')$ and $(P,\mathcal{L})$ be two linear systems. $(P',\mathcal{L}')$ and*

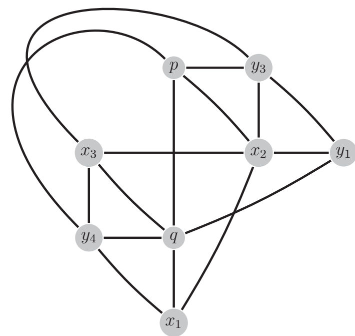


Figure 2:

$(P, \mathcal{L})$ are isomorphic, and we write $(P', \mathcal{L}') \simeq (P, \mathcal{L})$, if after deleting vertices of degree 1 or 0 from both, the systems $(P', \mathcal{L}')$ and $(P, \mathcal{L})$ are isomorphic as hypergraphs.

Definition 3.3. *Consider any point k, and any line l of Π_3, such that $k \notin l$. We define $\mathcal{C}_{3,4}$ to be the linear system obtained from Π_3 by:*

i) deleting point k, and its four incident lines,

ii) deleting line l and its four points.

The linear system $\mathcal{C}_{3,4} = (P_{\mathcal{C}_{3,4}}, \mathcal{L}_{\mathcal{C}_{3,4}})$ just defined is a 3-regular and 3-uniform linear system with eight points, and eight lines, described as:

$$\begin{aligned} P_{\mathcal{C}_{3,4}} &= \{p, q, x_1, x_2, x_3, y_1, y_3, y_4\}, \\ \mathcal{L}_{\mathcal{C}_{3,4}} &= \{\{p, y_1, y_3\}, \{x_2, x_3, y_1\}, \{q, y_1, y_4\}, \{x_1, x_3, y_4\}, \\ &\quad \{p, q, x_1\}, \{x_1, x_2, y_3\}, \{q, x_3, y_3\}, \{p, x_2, y_4\}\}. \end{aligned}$$

and depicted in Figure 2. In the next Proposition 3.1 and Lemma 3.1 we prove that if $(P, \mathcal{L})$ satisfies $\nu_2(P, \mathcal{L}) = 4$ and $\Delta(P, \mathcal{L}) = 3$, then $\tau(P, \mathcal{L}) \leq 4$; moreover the equality holds only if $(P, \mathcal{L}) \simeq \mathcal{C}_{3,4}$.

Proposition 3.1. $\nu_2(\mathcal{C}_{3,4}) = \tau(\mathcal{C}_{3,4}) = 4$.

Proof. Since the set of lines

$$\{\{p, x_2, y_4\}, \{q, x_3, y_3\}, \{x_1, x_3, y_4\}, \{x_1, x_2, y_3\}\}$$

induces no triple point, then $\nu_2(\mathcal{C}_{3,4}) \geq 4$. On the other hand, it is not difficult to prove that any set of five lines in $\mathcal{C}_{3,4}$ induces a triple point. Thus $\nu_2(\mathcal{C}_{3,4}) = 4$.

Since $\{x_1, x_2, y_1, y_4\}$ is a transversal, then $\tau(\mathcal{C}_{3,4}) \leq 4$. On the other hand, it is easy to check that there is no transversal on three points. Thus $\tau(\mathcal{C}_{3,4}) = 4$. ■

Lemma 3.1. *Let $(P, \mathcal{L})$ be a linear system with $\nu_2(P, \mathcal{L}) = 4$, and $\Delta(P, \mathcal{L}) = 3$. If $(P, \mathcal{L}) \not\simeq \mathcal{C}_{3,4}$, then $\tau(P, \mathcal{L}) \leq 3$.*

Proof. Let p and q be two points of P such that $deg(p) = 3$ and $deg(q) = \max\{deg(x) : x \in P \setminus \{p\}\}$. Assume $\deg(q) = 3$, otherwise the statement holds true, since the set of lines $\mathcal{L} \setminus \{l\}$ with $l \in \mathcal{L}_p$ induces no triple point, and as $|\mathcal{L} \setminus \mathcal{L}_p| \leq 2$, then $\tau(P, \mathcal{L}) \leq 3$ as Lemma 3.1 states. Let $(P'', \mathcal{L}'')$ be the linear subsystem induced by $\mathcal{L}'' = \mathcal{L} \setminus (\mathcal{L}_p \cup \mathcal{L}_q)$. Suppose that $|\mathcal{L}''| \geq 3$. We claim that $\nu_2(P'', \mathcal{L}'') = 2$ from which it follows by Proposition 2.1 that $\tau(P'', \mathcal{L}'') = 1$. Hence Lemma 3.1 is proven in this case. To verify the claim suppose to the contrary that there exists a set of three lines $\{l_1, l_2, l_3\}$ of $\mathcal{L}''$ inducing no triple points. This set of three lines induces at most three double points $X = \{x_1, x_2, x_3\}$. Since $\Delta(P, \mathcal{L}) = 3$, by the Pigeonhole Principle there are at least two lines $l_4, l_5 \in \mathcal{L}_p \cup \mathcal{L}_q$, which do not contain any point of X. Therefore, the set $\{l_1, l_2, l_3, l_4, l_5\}$ induces no triple point in $(P, \mathcal{L})$, a contradiction to the hypothesis $\nu_2(P, \mathcal{L}) = 4$. Suppose that $|\mathcal{L}''| \leq 2$. Assume that $\mathcal{L}'' = \{l_1, l_2\}$ with $l_1 \cap l_2 = \emptyset$, otherwise the statement holds true. We claim that every line $l \in (\mathcal{L}_p \cup \mathcal{L}_q) \setminus (\mathcal{L}_p \cap \mathcal{L}_q)$ satisfies $l \cap l_1 \neq \emptyset$, and $l \cap l_2 \neq \emptyset$. To verify the claim suppose to the contrary that there exists a line $l \in (\mathcal{L}_p \cup \mathcal{L}_q) \setminus (\mathcal{L}_p \cap \mathcal{L}_q)$, such that $l \cap l_2 = \emptyset$. Without loss of generality assume that $l \in \mathcal{L}_p$. By Pigeonhole Principle there are at least two lines

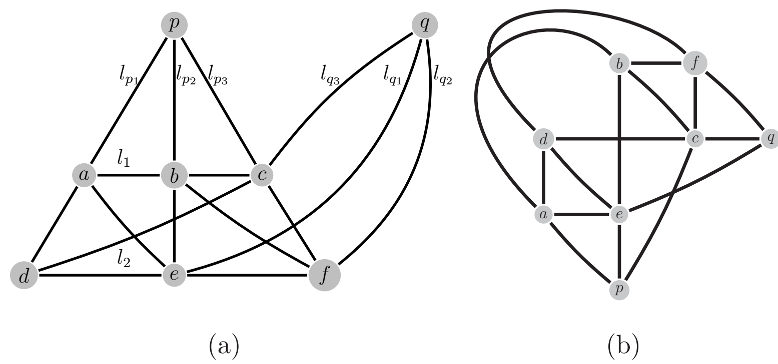


Figure 3:

$l_{q_1}, l_{q_2} \in \mathcal{L}_q$, such that $l \cap l_1 \cap l_{q_1} = \emptyset$, and $l \cap l_1 \cap l_{q_2} = \emptyset$. Therefore, the set $\{l, l_1, l_2, l_{q_1}, l_{q_2}\}$ induces no triple points, a contradiction to the hypothesis $\nu_2(P, \mathcal{L}) = 4$. Let $\mathcal{L}_p = \{l_{p_1}, l_{p_2}, l_{p_3}\}$, and $\mathcal{L}_q = \{l_{q_1}, l_{q_2}, l_{q_3}\}$.

Case 1: Suppose that $\mathcal{L}_p \cap \mathcal{L}_q \neq \emptyset$. Let $\{a\} = l_{p_1} \cap l_1$, $\{b\} = l_{p_2} \cap l_1$, $\{c\} = l_{p_1} \cap l_2$, $\{d\} = l_{p_2} \cap l_2$, where $l_{p_1}, l_{p_2} \in \mathcal{L}_p \setminus (\mathcal{L}_p \cap \mathcal{L}_q)$, then $\{a, d, q\}$ is a transversal, and the statement holds true.

Case 2: Suppose that $\mathcal{L}_p \cap \mathcal{L}_q = \emptyset$. Let $\{a\} = l_{p_1} \cap l_1$, $\{b\} = l_{p_2} \cap l_1$, $\{c\} = l_{p_3} \cap l_1$, $\{d\} = l_{p_1} \cap l_2$, $\{e\} = l_{p_2} \cap l_2$, and $\{f\} = l_{p_3} \cap l_2$. As $l_{q_i} \cap l_j \neq \emptyset$, for $i = 1, 2, 3$ and $j = 1, 2$, then given $l_{q_i} \in \mathcal{L}_q$ there exists $l_{p_{s_i}}, l_{p_{r_i}} \in \mathcal{L}_p$, $l_{p_{s_i}} \neq l_{p_{r_i}}$, such that $l_{q_i} \cap l_{p_{r_i}} \cap l_1 \neq \emptyset$, and $l_{q_i} \cap l_{p_{s_i}} \cap l_2 \neq \emptyset$ (since l_{q_i} induces a triple point on the 2-packing $\{l_1, l_2, l_{p_{r_i}}, l_{p_{s_i}}\}$, $\{l_1, l_2, l_{p_{s_i}}, l_{p_{t_i}}\}$, and $\{l_1, l_2, l_{p_{r_i}}, l_{p_{t_i}}\}$, where $\mathcal{L}_p = \{l_{p_{r_i}}, l_{p_{s_i}}, l_{p_{t_i}}\}$). Let $A_i = \{l_{p_{r_i}}, l_{p_{s_i}}\}$ be the set of such lines of l_{q_i}. By linearly we have that $|A_i \cap A_j| = 1$, for $1 \leq i < j \leq 3$, and $A_1 \cap A_2 \cap A_3 = \emptyset$, where A_1, A_2 and A_3 are the corresponding set of lines of l_{q_1}, l_{q_2} and l_{q_3} respectively. Therefore, either $l_{q_1} \ni a, e$, $l_{q_2} \ni b, f$, and $l_{q_3} \ni d, c$ or $l_{q_1} \ni a, f$, $l_{q_2} \ni b, d$, and $l_{q_3} \ni c, e$. Without loss of generality assume that $l_{q_1} \ni a, e$, $l_{q_2} \ni b, f$ and $l_{q_3} \ni d, c$ (in the other case we obtain the same linear system, namely the resultant linear systems are isomorphic).

If all three intersections $l_{q_1} \cap l_{p_3}$, $l_{q_2} \cap l_{p_1}$ and $l_{q_3} \cap l_{p_2}$ are empty, then

$(P, \mathcal{L}) \simeq \mathcal{C}_{3,4}$, otherwise one of three sets $\{b, d, l_{q_1} \cap l_{p_3}\}$, $\{a, f, l_{q_3} \cap l_{p_2}\}$, $\{c, e, l_{q_2} \cap l_{p_1}\}$ provides a three point transversal. Therefore, the set of points $\{b, d, l_{q_1} \cap l_{p_3}\}$ or $\{a, f, l_{q_3} \cap l_{p_2}\}$ or $\{c, e, l_{q_2} \cap l_{p_1}\}$ is a transversal of $(P, \mathcal{L})$. Hence, $\tau(P, \mathcal{L}) \leq 3$ as Lemma 3.1 states. ■

4 The case when $\Delta(P, \mathcal{L}) = 4$

As in the previous section we begin this section by introducing some terminology to describe linear systems $(P, \mathcal{L})$ with $\Delta(P, \mathcal{L}) = 4$, and $\nu_2(P, \mathcal{L}) = \tau(P, \mathcal{L}) = 4$.

Definition 4.1. *Given a linear system $(P, \mathcal{L})$, we will call a* triangle $\mathcal{T}$ *of $(P, \mathcal{L})$ as the linear system induced by three points in general position (non collinear) and three lines induced by them.*

Definition 4.2. *Consider the projective plane Π_3 and a triangle $\mathcal{T}$ of Π_3. Define $\mathcal{C} = (P_{\mathcal{C}}, \mathcal{L}_{\mathcal{C}})$ be the linear system obtained from Π_3 by deleting $\mathcal{T}$.*

The linear system $\mathcal{C} = (P_{\mathcal{C}}, \mathcal{L}_{\mathcal{C}})$ just defined has ten points, and ten lines, described as:

$$
\begin{aligned}
P_{\mathcal{C}} &= \{p, q, x_1, x_2, x_3, y_1, y_2, y_3, y_4, y_5\}, \\
\mathcal{L}_{\mathcal{C}} &= \{\{p, y_1, y_2, y_3\}, \{q, y_1, y_4, y_5\}, \{x_1, x_2, y_3, y_5\}, \{x_1, x_3, y_2, y_4\}, \{p, x_2, y_4\}, \\
&\quad \{p, x_3, y_5\}, \{p, q, x_1\}, \{q, x_2, y_2\}, \{q, x_3, y_3\}, \{x_2, x_3, y_1\}\},
\end{aligned}
$$

and depicted in Figure 4.

Below we present as a remark some proprieties of $\mathcal{C}$.

Remark 4.1.

- $3 \leq deg(x) \leq 4$, *for every $x \in P_{\mathcal{C}}$,*

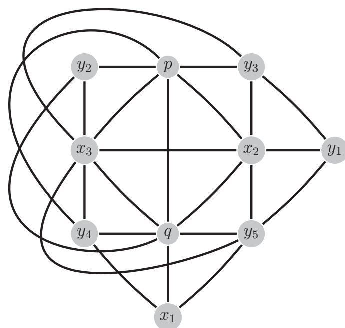


Figure 4:

- $3 \leq |l| \leq 4$, *for every* $l \in \mathcal{L}_{\mathcal{C}}$,
- $deg(x) = 4$, *if and only if,* x *is adjacent to every* $y \in P_{\mathcal{C}} \setminus \{x\}$,
- $|l| = 4$, *if and only if,* $l \cap l' \neq \emptyset$, *for every* $l' \in \mathcal{L}_{\mathcal{C}} \setminus \{l\}$,
- *there are no three collinear vertices of degree four,*
- *for every* $l \in \mathcal{L}_{\mathcal{C}}$ *there exists at most one line* $l' \in \mathcal{L}_{\mathcal{C}} \setminus \{l\}$, *such that* $l \cap l' = \emptyset$.

Definition 4.3. *We define* $\mathcal{C}_{4,4}$ *to be the family of linear systems* $(P, \mathcal{L})$ *with* $\nu_2(P, \mathcal{L}) = 4$, *such that:*

i) $\mathcal{C}$ *is a linear subsystem of* $(P, \mathcal{L})$,

ii) $(P, \mathcal{L})$ *is a linear subsystem of* Π_3,

this is $\mathcal{C}_{4,4} = \{(P, \mathcal{L}) : \mathcal{C} \subseteq (P, \mathcal{L}) \subseteq \Pi_3$ *and* $\nu_2(P, \mathcal{L}) = 4\}$.

In the next Proposition 4.1 and Lemma 4.1 we prove that if $(P, \mathcal{L})$ satisfies $\nu_2(P, \mathcal{L}) = 4$ and $\Delta(P, \mathcal{L}) = 4$, then $\tau(P, \mathcal{L}) \leq 4$; moreover the equality holds only if $(P, \mathcal{L}) \in \mathcal{C}_{4,4}$.

Before continuing we need some notation for the understand the remainder of this paper. Let $(P', \mathcal{L}')$ be a linear subsystem of a linear system $(P, \mathcal{L})$, then we denote $\mathcal{L} \setminus \mathcal{L}'$ as $\{l \in \mathcal{L} : l' \nsubseteq l, l' \in \mathcal{L}'\}$.

Proposition 4.1. *If $(P, \mathcal{L}) \in \mathcal{C}_{4,4}$, then $\tau(P, \mathcal{L}) = 4$.*

Proof.

As any line of $(P, \mathcal{L})$ of size four is a transversal of $(P, \mathcal{L})$ (since any line of size four is a transversal of Π_3), then $\tau(P, \mathcal{L}) \leq 4$. Suppose that $(P, \mathcal{L})$ does not have a transversal of cardinality 4, then there is a transversal $\{a, b, c\}$. Since there are 4 points of degree 4 in $(P_{\mathcal{C}}, \mathcal{L}_C)$, by Pigeonhole Principle at least one of them does not belong $\{a, b, c\}$, denote this point by x. Since $|\mathcal{L}_x| = 4$, then at least one $l \in \mathcal{L}_x$ is not pierced by $\{a, b, c\}$. ■

Lemma 4.1. *Let $(P, \mathcal{L})$ be a linear system with $\nu_2(P, \mathcal{L}) = \Delta(P, \mathcal{L}) = 4$, and $|\mathcal{L}| \geq 4$. If $(P, \mathcal{L}) \notin \mathcal{C}_{4,4}$, then $\tau(P, \mathcal{L}) \leq 3$.*

Proof.

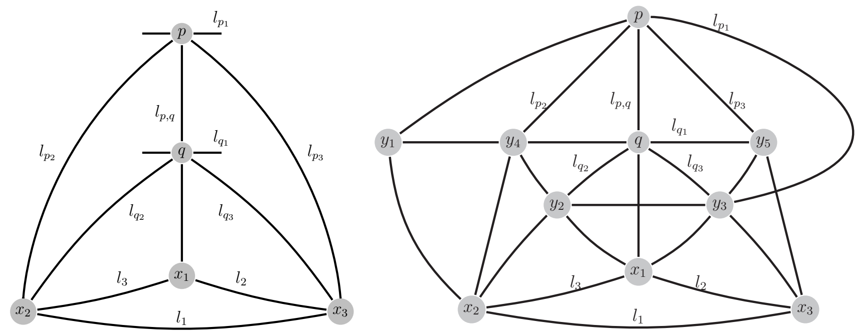
Recall that $\nu_2(P, \mathcal{L}) = 4$ implies that any set of five lines induces a triple point. Let p and q be two points of P, such that $deg(p) = 4$, and $deg(q) = \max\{deg(x) : x \in P \setminus \{p\}\}$. Assume that $deg(q) = 4$, otherwise the statement holds true, since if $deg(q) \leq 2$ the set of lines $\mathcal{L} \setminus \{l, l'\}$, with $l, l' \in \mathcal{L}_p$, induces no triple point, and as $|\mathcal{L} \setminus \mathcal{L}_p| \leq 2$, then $\tau(P, \mathcal{L}) \leq 3$. On the other hand, if $deg(q) = 3$, then the linear system $(P', \mathcal{L}')$ induced by $\mathcal{L}' = \mathcal{L} \setminus \{l_p\}$, with $l_p \in \mathcal{L}_p$, satisfies $\tau(P', \mathcal{L}') \leq 3$, by Lemma 3.1. Furthermore there exists a transversal T of $(P', \mathcal{L}')$ containing the point p (see proof of Lemma 3.1), and therfore T is a transversal of $(P, \mathcal{L})$.

Let $(P'', \mathcal{L}'')$ be the linear subsystem induced by $\mathcal{L}'' = \mathcal{L} \setminus (\mathcal{L}_p \cup \mathcal{L}_q)$. Suppose that $|\mathcal{L}''| \leq 2$. Assume that $\mathcal{L}'' = \{l_1, l_2\}$ with $l_1 \cap l_2 = \emptyset$, otherwise the statement holds true. Proceeding as the proof of Lemma 3.1, it can be proven that every line $l \in (\mathcal{L}_p \cup \mathcal{L}_q) \setminus (\mathcal{L}_p \cap \mathcal{L}_q)$ satisfies $l \cap l_1 \neq \emptyset$, and $l \cap l_2 \neq \emptyset$. Without loss of generality assume that there exists a line $l_q \in \mathcal{L}_q \setminus (\mathcal{L}_p \cap \mathcal{L}_q)$, such that $l_q \cap l_1 \cap l_{p_1} \neq \emptyset$ and $l_q \cap l_2 \cap l_{p_2} \neq \emptyset$, where $l_{p_1}, l_{p_2} \in \mathcal{L}_p \setminus (\mathcal{L}_p \cap \mathcal{L}_q)$. Then the set $\{l_1, l_2, l_{p_3}, l_{p_4}, l_q\}$, where $l_{p_3}, l_{p_4} \in \mathcal{L}_p \setminus \{l_{p_1}, l_{p_2}\}$, induces no triple point, a contradiction to the hypothesis $\nu_2(P, \mathcal{L}) = 4$. Suppose that $|\mathcal{L}''| \geq 3$. Assume $\nu_2(P'', \mathcal{L}'') \geq 3$, otherwise, if $\nu_2(P'', \mathcal{L}'') = 2$ from which it follows by Proposition 2.1 that $\tau(P'', \mathcal{L}'') = 1$, therefore $\tau(P, \mathcal{L}) \leq 3$. Let $\{l_1, l_2, l_3\}$ be a set of three lines of $\mathcal{L}''$ inducing no triple point. This set of three lines induces at most three double points $X = \{x_1, x_2, x_3\}$. Assume that three lines of $\mathcal{L}_p$, and three lines of $\mathcal{L}_q$ each inside at a point in X, otherwise there exist two lines of $l_4, l_5 \in \mathcal{L}_p \cup \mathcal{L}_q$, which do not contain any point of X (by the definition of $deg(q)$), therefore the set of five lines $\{l_1, l_2, l_3, l_4, l_5\}$ induces no triple point, a contradiction to the hypothesis $\nu_2(P, \mathcal{L}) = 4$.

We claim that there exists one line containing p, q and x, for some $x \in X$. To verify the claim suppose the contrary. Let $\mathcal{L}_p = \{l_{p_1}, l_{p_2}, l_{p_3}, l_{p_4}\}$, and $\mathcal{L}_q = \{l_{q_1}, l_{q_2}, l_{q_3}, l_{q_4}\}$, with $(\mathcal{L}_p \setminus \{l_{p_4}\}) \cap (\mathcal{L}_q \setminus \{l_{q_4}\}) = \emptyset$. Since three lines of $\mathcal{L}_p$, and three lines of $\mathcal{L}_q$ are each incident to a point of X, then without loss of generality suppose that $l_{p_i}, l_{q_i} \ni x_i$, for i=1,2,3, and $\{x_1\} = l_2 \cap l_3$, $\{x_2\} = l_3 \cap l_1$ and $\{x_3\} = l_1 \cap l_2$. Then the set $\{l_1, l_{p_1}, l_{p_2}, l_{q_1}, l_{q_3}\}$ induces no triple point, a contradiction to the hypothesis $\nu_2(P, \mathcal{L}) = 4$. Assume that $l_{p,q} \ni x_1$ and $l_{p_i}, l_{q_i} \ni x_i$, for $i = 2, 3$ (see Figure 5(a)), where $l_{p_4} = l_{q_4} = l_{p,q}$. Consider the lines l_{p_1} and l_{q_1}, and the following 2-packing sets:

$$\begin{gathered}\mathcal{L}_1 = \{l_1, l_2, l_{q_2}, l_{pq}\},\ \mathcal{L}_2 = \{l_1, l_3, l_{q_3}, l_{pq}\},\\ \mathcal{L}_3 = \{l_1, l_2, l_{p_2}, l_{pq}\},\ \mathcal{L}_4 = \{l_1, l_3, l_{p_3}, l_{pq}\}.\end{gathered}$$

The line l_{p_1} induces a triple point on $\mathcal{L}_1$ and $\mathcal{L}_2$, consequently there must exist intersections $\{y_2\} = l_2 \cap l_{q_2}$ and $\{y_3\} = l_3 \cap l_{q_3}$, with $y_2, y_3 \in l_{p_1}$, otherwise there exists a set of five lines $\mathcal{L}_1 \cup \{l_{p_1}\}$ or $\mathcal{L}_2 \cup \{l_{p_1}\}$ inducing no triple point, a contradiction to the hypothesis $\nu_2(P, \mathcal{L}) = 4$. Analogously, the line l_{q_1} induces a triple point on $\mathcal{L}_3$, and $\mathcal{L}_4$. Therefore there must exist



(a) (b)

Figure 5:

intersections $\{y_4\} = l_2 \cap l_{p_2}$, and $\{y_5\} = l_3 \cap l_{p_3}$ with $y_4, y_5 \in l_{p_2}$. Finally, as the following set of five lines $\{l_1, l_2, l_3, l_{p_1}, l_{q_1}\}$ induces a triple point, there must exists the intersection point $\{y_1\} = l_1 \cap l_{p_1} \cap l_{q_1}$. It is not difficult to prove that the resultant linear system $(P, \mathcal{L})$ (Figure 5(b)) is isomorphic to linear system $\mathcal{C}$. Therefore there exists at least one line $l \in \mathcal{L} \setminus \mathcal{L}_{\mathcal{C}}$. We claim that each line $l \in \mathcal{L} \setminus \mathcal{L}_{\mathcal{C}}$ is a line of Π_3, hence $(P, \mathcal{L}) \in \mathcal{C}_{4,4}$, contradicting the hypothesis $(P, \mathcal{L}) \notin \mathcal{C}_{4,4}$. Before this note that $|\mathcal{L} \setminus \mathcal{L}_{\mathcal{C}}| \leq 3$ (therefore $|\mathcal{L}| \leq |\mathcal{L}_{\Pi_3}| = 13$) since every line of $\mathcal{L} \setminus \mathcal{L}_{\mathcal{C}}$ induces a triple point on the 2-packing $\{l_2, l_3, l_{p_2}, l_{p_3}\}$, consequently each line of $\mathcal{L} \setminus \mathcal{L}_{\mathcal{C}}$ is incident to one point of $\{x_1, y_4, y_5\}$.

To verify the claim consider the linear system $\mathcal{C}$ depicted in Figure 5(b), and l be a fixed line of $\mathcal{L} \setminus \mathcal{L}_{\mathcal{C}}$. We will prove that there exists one line $l' \in \mathcal{L}_{\Pi_3}$, such that $l' = l$. First we will prove that $l' \subseteq l$. Without losing generality assume $l \ni y_4$ (the same argument is used, if $l \ni x_1$, or $l \ni y_5$). Line l induces a triple point on the following 2-packing sets:

$$\mathcal{L}'_1 = \{l_1, l_{p_3}, l_{q_1}, l_{q_2}\}, \mathcal{L}'_2 = \{l_1, l_3, l_{q_1}, l_{p,q}\}, \mathcal{L}'_3 = \{l_3, l_{p_1}, l_{q_1}, l_{p,q}\},$$

The intersection $\{y_7\} = l_{p_3} \cap l_{q_2}$ and $\{y_8\} = l_1 \cap l_{p,q}$ must exists, as well as $y_3, y_7, y_8 \in l$ (since $\{y_3\} \in l \cap l_3 \cap l_{p_1}$), otherwise, there must exist a set of five lines $\mathcal{L}'_1 \cup \{l\}$, or $\mathcal{L}'_2 \cup \{l\}$, or $\mathcal{L}'_3 \cup \{l\}$ inducing no triple point, a contradiction to the hypothesis $\nu_2(P, \mathcal{L}) = 4$. Hence $l' \subseteq l$, where $l' = \{y_3, y_4, y_7, y_8\} \in \mathcal{L}_{\Pi_3}$ (see Figure 1). To prove that $l \subseteq l'$ is sufficient to verify that any line $\widetilde{l}$ of $\mathcal{L} \setminus \mathcal{L}_{\mathcal{C}}$ different of l satisfies $\widetilde{l} \cap l \subseteq l'$, since there are no points of degree one in l. Let $\widetilde{l}$ be a line as before. Without loss of generality assume $y_5 \in \widetilde{l}$ (the same argument is used if $l \ni x_1$). Since the line $\widetilde{l}$ induces a triple point on the 2-packing $\{l_1, l_3, l_{p_1}, l_{p,q}\}$ the intersection $\widetilde{l} \cap l_1 \cap l_{p,q}$ must exist. As $y_8 = l \cap l_1 \cap l_{p,q}$, then $y_8 = \widetilde{l} \cap l' \cap l$, therefore $\widetilde{l} \cap l \in l'$. $\blacksquare$

Proof of Theorem 2.1. Let $(P, \mathcal{L})$ be a linear system satisfying the hypothesis of Theorem 2.1. If $\Delta(P, \mathcal{L}) = 3$, then by Lemma 3.1 we have $\tau(P, \mathcal{L}) \leq 3$, unless that $(P, \mathcal{L}) \simeq \mathcal{C}_{3,4}$ where by Proposition 3.1 we have $\tau(P, \mathcal{L}) = 4$. On the other hand, if $\Delta(P, \mathcal{L}) = 4$, then by Lemma 4.1 we have $\tau(P, \mathcal{L}) \leq 3$, unless the linear system $(P, \mathcal{L}) \in \mathcal{C}_{4,4}$ whereby Proposition 4.1 we have $\tau(P, \mathcal{L}) = 4$. Finally, if $\Delta(P, \mathcal{L}) \geq 5$, by Lemma 2.1 we have $\tau(P, \mathcal{L}) \leq 3$. This concludes the proof of Theorem 2.1. $\square$

5 Proof of the Main Theorem

Before continuing with the last part of this paper we need some definitions and results.

Definition 5.1. *The* incidence graph *of a set system $(X, \mathcal{F})$, denoted by $B(X, \mathcal{F})$, is a bipartite graph with vertex set $V = X \cup \mathcal{F}$, where two vertices $x \in X$, and $F \in \mathcal{F}$ are adjacent, if and only if, $x \in F$.*

According to [13] any straight line system is *Zykov-planar* (see [17]). Zykov proposed to represent the lines of a set system by a subset of the faces of a planar map (map on $\mathbb{R}^2$). That is, a set system $(X, \mathcal{F})$ is Zykov-planar, if there exists a planar graph G (not necessarily a simple graph), such that $V(G) = X$, and G can be drawn in the plane with faces of G two-colored (say red and blue), so that there exists a bijection between the red faces of G, and the subsets of $\mathcal{F}$, such that a point x is incident with a red face, if and only if, it is incident with the corresponding subset. Walsh in [16] has shown that definition of Zykov is equivalent to the following: A set system $(X, \mathcal{F})$ is Zykov-planar, if and only if, the incidence graph $B(X, \mathcal{F})$ is planar.

Proof of Theorem 1.1. By Propositions 2.1 and 2.2 we only need to prove the case when $\nu_2 = 4$. We consider any linear system $(P, \mathcal{L})$ with $\nu_2(P, \mathcal{L}) = 4$, and $|\mathcal{L}| > 4$. Suppose that $(P, \mathcal{L}) \simeq \mathcal{C}_{3,4}$. We shall prove that $(P, \mathcal{L})$ is not Zykov-planar. Moreover, as $\mathcal{C}_{3,4}$ is a linear subsystem of $\mathcal{C} \in \mathcal{C}_{4,4}$, then any element of $\mathcal{C}_{4,4}$ is not Zykov-planar. If $(P, \mathcal{L})$ is a straight line system then $(P, \mathcal{L})$ is Zykov-planar, therefore the incidence graph $B(P, \mathcal{L})$ of $(P, \mathcal{L})$ is a planar graph, but it is not difficult to prove that $B(P, \mathcal{L})$ is not a planar graph, which is a contradiction. Therefore, there does not exist a straight line representation on $\mathbb{R}^2$ of $(P, \mathcal{L})$. On the other hand, if $(P, \mathcal{L}) \not\simeq \mathcal{C}_{3,4}$ or $(P, \mathcal{L}) \notin \mathcal{C}_{4,4}$ with $\nu_2(P, \mathcal{L}) = 4$, and $|\mathcal{L}| > 4$, then by Lemas 2.1, 3.1, and 4.1 we have $\tau(P, \mathcal{L}) \leq 3$, as Theorem 1.1 states. $\square$

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