ON THE SHADOWING AND LIMIT SHADOWING PROPERTIES

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Abstract. In this paper, we deal with the relation between the shadowing property and the limit shadowing property. We prove that for any continuous self-map f of a compact metric space, if f has the limit shadowing property, then the restriction of f to the non-wandering set satisfies the shadowing property. As an application, we prove the equivalence of the two shadowing properties for equicontinuous maps.

1. Introduction

Shadowing has been the subject of much interest in the qualitative study of dynamical systems [3, 20], and various kinds of shadowing property have been defined in the course of such studies so far. The limit shadowing property introduced in [8] is one of the variants of shadowing property, which focuses on the possibility of asymptotic shadowing of pseudo orbits whose one-step errors are converging to zero, and it is a subject of ongoing research [5, 6, 7, 11, 12, 14, 17, 19, 21, 22].

In [21, 22], the limit shadowing properties are examined in relation to the notion of hyperbolicity and stability. The set of Ω-stable diffeomorphisms of a smooth closed manifold is characterized as the C^1-interior of the set of diffeomorphisms satisfying the limit shadowing property [21]. The s-limit shadowing property is a stronger property than the limit shadowing property defined in combination with the shadowing property. It has been proved to be a C^0-dense property on the space of continuous self-maps (resp. continuous surjections) of a compact topological manifold [17]. The two-sided limit shadowing property of homeomorphisms is a bilateral version of the limit shadowing property, and its consequences are given, for example, in [6, 7, 14, 19]. It is, in fact, a much stronger property than the limit shadowing property and a sufficient condition for chaos [14, 19]. In [5], the limit shadowing property is exploited to characterize the ω-limit sets in topologically hyperbolic systems in terms of the notions such as internal chain transitivity. Furthermore, in [11, 12], the limit shadowing property is studied together with other types of shadowing properties like average shadowing property (ASP) and asymptotic average shadowing property (AASP). AASP is a stochastic version of the limit shadowing property, yet there is a large class of homeomorphisms satisfying both of the shadowing and limit shadowing properties but not AASP [12].

First, we give the formal definitions of the standard and limit shadowing properties. Throughout this paper, we deal with a continuous map $f : X \to X$ on a compact metric space (X, d). For $\delta > 0$, a sequence $(x_i)_{i\geq 0}$ of points in X is a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) \leq \delta$ for all $i \geq 0$. Then, for given $\epsilon > 0$, a δ-pseudo orbit $(x_i)_{i\geq 0}$ is said

2010 *Mathematics Subject Classification.* 37C50; 54H20; 37B20.

Key words and phrases. limit shadowing property; shadowing property; non-wandering set; equicontinuity.

to be ϵ-*shadowed* by $x \in X$ if $d(x_i, f^i(x)) \le \epsilon$ for all $i \ge 0$. We say that f has the *shadowing property* if for every $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit of f is ϵ-shadowed by some point of X. A sequence $(x_i)_{i\ge0}$ of points in X is a *limit pseudo orbit* of f if $\lim_{i\to\infty} d(f(x_i), x_{i+1}) = 0$. We say that f has the *limit shadowing property* if for any limit pseudo orbit $(x_i)_{i\ge0}$ of f, there is $y \in X$ such that $\lim_{i\to\infty} d(x_i, f^i(y)) = 0$. Such y is called a *limit shadowing point* of $(x_i)_{i\ge0}$.

An interesting problem is to reveal the relation between the (standard) shadowing property and the limit shadowing property. In [20], an example of circle homeomorphism (or diffeomorphism) satisfying the limit shadowing property but not the shadowing property is given. By taking the direct product of such diffeomorphisms, we obtain diffeomorphisms of arbitrary dimensional torus with the limit shadowing property but not the shadowing property. We also give such a simple example below.

Example 1.1. Let $X = \{0, 1, 2\} \cup \{s_n : n \in \mathbb{Z}\} \cup \{t_n : n \in \mathbb{Z}\}$, where $(s_n)_{n\in\mathbb{Z}}$ and $(t_n)_{n\in\mathbb{Z}}$ are sequences of real numbers satisfying the following properties.

(1) $s_n < s_{n+1}$ and $t_n < t_{n+1}$ for all $n \in \mathbb{Z}$.
(2) $\lim_{n\to-\infty} s_n = 0$, $\lim_{n\to+\infty} s_n = \lim_{n\to-\infty} t_n = 1$, and $\lim_{n\to+\infty} t_n = 2$.

Then, X is a compact subset of $[0, 2]$. Define $f : X \to X$ by

(3) $f(y) = y$ for $y \in \{0, 1, 2\}$.
(4) $f(s_n) = s_{n+1}$ and $f(t_n) = t_{n+1}$ for all $n \in \mathbb{Z}$.

Then, f is an expansive homeomorphism. For any limit pseudo orbit $(x_i)_{i\ge0}$ of f, it is easy to see that $\lim_{i\to\infty} x_i = y$ for some $y \in \{0, 1, 2\}$, and this implies that $\lim_{i\to\infty} d(x_i, f^i(y)) = 0$ since y is a fixed point. Hence, f has the limit shadowing property, and it is also easy to see that f does not have the shadowing property. Let $g = f^{\mathbb{N}} : X^{\mathbb{N}} \to X^{\mathbb{N}}$ be the direct product of countably many copies of f. Then, f is a homeomorphism of a Cantor space satisfying the limit shadowing property but not the shadowing property. We have $\Omega(g) = \{0, 1, 2\}^{\mathbb{N}} = Fix(g)$, a Cantor space.

Also in [20], it is proved that for circle homeomorphisms, the shadowing property always implies the limit shadowing property, and the same implication holds true for c-expansive maps including expansive homeomorphisms (see [4, 5, 13]). It is rather difficult to construct a continuous map satisfying the shadowing property but not the limit shadowing property, but in [9], such an example is given, while the equivalence of the two shadowing properties is proved for a certain class of interval maps.

The above facts may indicate that the limit shadowing property is weaker than the shadowing property at least intuitively. However, in [11], as a (partial) converse, it is proved that if a continuous map with the limit shadowing property is chain transitive, then it satisfies the shadowing property. As the main result of this paper, through a generalization of the result for the chain recurrent case (Lemma 2.1), we prove a basic relation between the two shadowing properties. To state the result, we give some definitions and notations.

Given a continuous map $f : X \to X$, a finite sequence of points $(x_i)_{i=0}^k$ in X (where k is a positive integer) is called a δ-*chain* of f if $d(f(x_i), x_{i+1}) \le \delta$ for every $0 \le i \le k-1$. We say that f is *chain transitive* if for any $x, y \in X$ and $\delta > 0$, there is a δ-chain $(x_i)_{i=0}^k$ of f such that $x_0 = x$ and $x_k = y$. A δ-chain $(x_i)_{i=0}^k$ of f is said to be a δ-*cycle* of f if $x_0 = x_k$, and a point $x \in X$ is a *chain recurrent point* for f if for any $\delta > 0$, there is a δ-cycle $(x_i)_{i=0}^k$ of f with $x_0 = x_k = x$. We denote by $CR(f)$ the set of chain recurrent points for f. A point $x \in X$ is said to be *minimal* if the restriction of f to the orbit closure $\overline{O_f(x)} = \overline{\{f^n(x) : n \geq 0\}}$ is minimal, and *non-wandering* if for every neighborhood U of x, we have $f^n(U) \cap U \neq \emptyset$ for some $n > 0$. We denote by $M(f)$ (resp. $\Omega(f)$) the set of minimal (resp. non-wandering) points for f. Note that $M(f) \subset \Omega(f) \subset CR(f)$. It holds that $M(f) = M(f^m)$ for every $m \in \mathbb{N}$ (see, for example, [18]).

Now we state the theorem.

Theorem 1.1. *Let $f : X \to X$ be a continuous map with the limit shadowing property. Then, $CR(f) = \Omega(f) = \overline{M(f)}$, and $f|_{\Omega(f)}$ satisfies the shadowing property.*

This theorem enables us to apply the developed theory of the shadowing property to continuous maps enjoying the limit shadowing property. As an example, we obtain the following corollary (cf. [10, 15]).

Corollary 1.1. *Let $f : X \to X$ be a continuous map with the limit shadowing property. Then, for every $x \in \Omega(f)$ and every $\epsilon > 0$, there exists $y \in \Omega(f)$ with $d(x, y) < \epsilon$ such that y is a periodic point for f, or $f|_{\overline{O_f(y)}}$ is conjugate to an odometer.*

The other result of this paper concerns the notion of equicontinuity. A map $f : X \to X$ is said to be *equicontinuous* if for any $\epsilon > 0$, there is $\delta > 0$ such that $d(x, y) \leq \delta$ implies $\sup_{n \geq 0} d(f^n(x), f^n(y)) \leq \epsilon$ for all $x, y \in X$. It is known that if an equicontinuous map f is surjective, then f is a homeomorphism, and f^{-1} is also equicontinuous (cf. [2, 16]). When $f : X \to X$ is an equicontinuous map (or an equicontinuous homeomorphism), there is a metric D equivalent to d on X such that $D(f(x), f(y)) \leq D(x, y)$ (or $= D(x, y)$) for all $x, y \in X$. In fact, $D : X \times X \to [0, \infty)$ defined by $D(x, y) = \sup_{n \geq 0} d(f^n(x), f^n(y))$ (or $\sup_{n \in \mathbb{Z}} d(f^n(x), f^n(y))$) is such a metric. Every equicontinuous homeomorphism $f : X \to X$ is known to satisfy $X = M(f)$ (see [16]). An equicontinuous map $f : X \to X$ satisfies the shadowing property iff every $x \in X$ is a *chain continuity point* for f (see [1]).

As an application of Theorem 1.1, we prove the equivalence of the two shadowing properties for equicontinuous maps (including equicontinuous homeomorphisms).

Theorem 1.2. *Let $f : X \to X$ be an equicontinuous map. Then, the following three properties are equivalent.*

(1) *f has the limit shadowing property.*
(2) *f has the shadowing property.*
(3) *$\dim \Omega(f) = 0$, or equivalently, $\Omega(f)$ is totally disconnected.*

This theorem generalizes a result of [4] proving that every equicontinuous homeomorphism of a totally disconnected space (e.g. odometer) satisfies the shadowing property and the limit shadowing property.

This paper consists of three sections. We prove Theorem 1.1 in Section 2. Theorem 1.2 is proved in Section 3.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we need a lemma which is a generalization of [11, Theorem 7.3]. For a continuous map $f : X \to X$ and $S \subset X$, we say that f has the *shadowing* *property around* S if for any $\epsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit of f contained in S is ϵ-shadowed by some point of X, and we say that f has the *limit shadowing property around* S if every limit pseudo orbit of f contained in S has a limit shadowing point in X.

Lemma 2.1. *Let $f : X \to X$ be a continuous map and let $S \subset X$ be a compact f-invariant subset such that $CR(f|_S) = S$. If f has the limit shadowing property around S, then f has the shadowing property around S.*

Proof. For $\delta > 0$, we decompose S into a disjoint union of so-called δ-chain components. Precisely, for any $\delta > 0$, we define a relation $\sim_\delta$ on S as follows. For $x, y \in S$, $x \sim_\delta y$ iff there are a δ-chain $(x_i)_{i=0}^k \subset S$ of f with $x_0 = x$ and $x_k = y$, and a δ-chain $(y_i)_{i=0}^l \subset S$ of f with $y_0 = y$ and $y_l = x$. Since every point of S is chain recurrent for $f|_S$, $\sim_\delta$ is an equivalence relation and, moreover, we can show that $x \sim_\delta f(x)$ for every $x \in S$, and $x \sim_\delta y$ for all $x, y \in S$ with $d(x, y) < \delta$. Hence, every equivalence class C with respect to $\sim_\delta$ is clopen in S and f-invariant, i.e., $f(C) \subset C$. Each equivalence class is called a δ-chain component, therefore S is decomposed into finitely many δ-chain components. Note that if $\delta_1 < \delta_2$, then each δ_1-chain component is contained in a δ_2-chain component, and hence every δ_2-chain component is a disjoint union of δ_1-chain components.

Now, assume that f does not have the shadowing property around S. Then, there is $\epsilon > 0$ such that for every integer $n \geq 1$, we can take an n^{-1}-chain $\gamma_n = (x_i^{(n)})_{i=0}^{l_n} \subset S$ of f which is not ϵ-shadowed by any point of X, meaning that for every $y \in X$, there is $0 \leq i \leq l_n$ such that $d(x_i^{(n)}, f^i(y)) > \epsilon$. Let us consider the k^{-1}-chain decomposition of S for every integer $k \geq 1$. Here, note that for any given $\delta > 0$, since every δ-chain component is clopen in S and f-invariant, for sufficiently large n, γ_n is contained in a δ-chain component. Hence, we can take a 1-chain component $C(1)$ so that $\gamma_n \subset C(1)$ for infinitely many n. Then, since $C(1)$ is a disjoint union of 2^{-1}-chain components, we can choose a 2^{-1}-chain component $C(2) \subset C(1)$ so that $\gamma_n \subset C(2)$ for infinitely many n. Proceeding inductively, we obtain a sequence $(C(k))_{k=1}^\infty$ and an increasing sequence of integers $(n_k)_{k=1}^\infty$ such that $C(k)$ is a k^{-1}-chain component, $C(k+1) \subset C(k)$, and $\gamma_{n_k} \subset C(k)$ for every $k \geq 1$. Then, for every $k \geq 1$, since $\gamma_{n_k} \cup \gamma_{n_{k+1}} \subset C(k)$, we can take a k^{-1}-chain $\beta_k = (y_i)_{i=0}^{m_k} \subset S$ of f such that $y_0 = x_{l_{n_k}}^{(n_k)}$ and $y_{m_k} = x_0^{(n_{k+1})}$. By concatenating obtained chains, we get the following limit pseudo orbit:

$$\alpha = \gamma_{n_1}\beta_1\gamma_{n_2}\beta_2\gamma_{n_3}\beta_3\gamma_{n_4}\beta_4 \cdots = (x_i)_{i \geq 0} \subset S.$$

However, by the choice of γ_n, we have $\limsup_{i \to \infty} d(x_i, f^i(y)) \geq \epsilon$ for every $y \in X$, which contradicts that f has the limit shadowing property around S. Thus, f has the shadowing property around S. $\square$

Now, let us prove Theorem 1.1.

Proof of Theorem 1.1. We first prove that $CR(f) \subset \overline{M(f)}$. Let $x \in CR(f)$. Then, for every integer $n \geq 1$, there is an n^{-1}-cycle $\gamma_n = (x_i^{(n)})_{i=0}^{m_n}$ of f with $x_0^{(n)} = x_{m_n}^{(n)} = x$. By concatenating them, we obtain a limit pseudo orbit $\alpha = \gamma_1\gamma_2\gamma_3 \cdots$. Take a limit shadowing point $y \in X$ of α. Then, it is easy to see that $x \in \omega(y)$. Note that $f|_{\omega(y)}$ is chain transitive (so chain recurrent) and f has the limit shadowing property around

$\omega(y)$. Hence, by Lemma 2.1, f has the shadowing property around $\omega(y)$, and thus for any $\epsilon > 0$, we can take $\delta > 0$ such that every δ-pseudo orbit of f contained in $\omega(y)$ is ϵ-shadowed by some point of X. For such δ, since $f|_{\omega(y)}$ is chain transitive, there is a δ-cycle $\gamma = (x_i)_{i=0}^m \subset \omega(y)$ of f with $x_0 = x_m = x$. By concatenating γ's, we obtain a periodic δ-pseudo orbit $\beta = \gamma\gamma\gamma\cdots \subset \omega(y)$. The rest of the proof is identical to that of [18, Theorem 1]. Take an ϵ-shadowing point $z \in X$ of β. Then, $\overline{O_{f^m}(z)} \subset B_\epsilon(x) = \{w \in X : d(x, w) \le \epsilon\}$ is a compact f^m-invariant subset, and therefore $\overline{O_{f^m}(z)} \cap M(f^m) \neq \emptyset$. For any $p \in \overline{O_{f^m}(z)} \cap M(f^m)$, we have $p \in B_\epsilon(x)$ and $p \in M(f^m) = M(f)$. Since $\epsilon > 0$ is arbitrary, it holds that $x \in \overline{M(f)}$, proving $CR(f) \subset \overline{M(f)}$. Now, we have $CR(f) = \Omega(f) = \overline{M(f)}$, implying that $CR(f|_{\Omega(f)}) = \Omega(f)$. Since f has the limit shadowing property around $\Omega(f)$, from Lemma 2.1, it follows that f has the shadowing property around $\Omega(f)$. In this case, we can apply the argument as seen in the proof of [3, Theorem 3.4.2] or [18, Lemma 1] to conclude that $f|_{\Omega(f)}$ has the shadowing property. □

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. First, we prove the following lemma.

Lemma 3.1. *Let $f : X \to X$ be an equicontinuous homeomorphism. Then, the following three properties are equivalent.*

(1) *f has the limit shadowing property.*
(2) *f has the shadowing property.*
(3) *$\dim X = 0$, or equivalently, X is totally disconnected.*

A remark is needed before the proof of Lemma 3.1. (1) $\Rightarrow$ (2) in Lemma 3.1 is an immediate corollary of Theorem 1.1. (2) $\Leftrightarrow$ (3) is an already known fact on the shadowing property (see [18]). (3) $\Rightarrow$ (1) can be verified by a simple combination of the results given in [4], which are (A) Every equicontinuous homeomorphism $f : X \to X$ with $\dim X = 0$ satisfies the h-shadowing property, and (B) h-shadowing property implies the s-limit shadowing property, and then the limit shadowing property. However, we provide a self-contained proof of (3) $\Rightarrow$ (1) below. For the purpose, we need the following lemma.

Lemma 3.2. *Let $f : X \to X$ be an equicontinuous map. If $\dim X = 0$, then for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that any δ-pseudo orbit $(x_i)_{i\ge 0}$ of f is ϵ-shadowed by x_0, that is, $d(x_i, f^i(x_0)) \le \epsilon$ for every $i \ge 0$.*

Proof. We can assume that $d(f(x), f(y)) \le d(x, y)$ for every $x, y \in X$. Since $\dim X = 0$, for any given $\epsilon > 0$, X is decomposed into a disjoint union of clopen subsets $A_1, \dots, A_m \subset X$ such that $\operatorname{diam} A_k \le \epsilon$ for every $1 \le k \le m$. Take $0 < \delta < \min_{1\le k<l\le m} d(A_k, A_l)$ and suppose that $(x_i)_{i\ge 0}$ is a δ-pseudo orbit of f. We prove by induction on i that the following property holds for every $i \ge 1$.

$$(P_i) \qquad d(f^{i-j}(x_j), f^{i-j-1}(x_{j+1})) \le \delta \quad (0 \le \forall j \le i-1).$$

When $i = 1$, (P_1) is just $d(f(x_0), x_1) \le \delta$, which is true. Suppose that (P_i) holds for some $i \ge 1$. Then, we have

$$d(f^{i+1-j}(x_j), f^{i-j}(x_{j+1})) \le d(f^{i-j}(x_j), f^{i-j-1}(x_{j+1})) \le \delta \quad (0 \le \forall j \le i-1)$$

and $d(f(x_i), x_{i+1}) \le \delta$, implying that (P_{i+1}) holds, and so completing the induction. Now, for every $i \ge 1$, from (P_i) and the choice of δ, it follows that $\{f^{i-j}(x_j) : 0 \le j \le i\} \subset A_{k(i)}$ for some $1 \le k(i) \le m$, especially $f^i(x_0), x_i \in A_{k(i)}$. Since $\operatorname{diam} A_{k(i)} \le \epsilon$, we have $d(x_i, f^i(x_0)) \le \epsilon$ for each $i \ge 1$, and this proves the lemma. □

Then, let us prove Lemma 3.1.

Proof of Lemma 3.1. (1) $\Rightarrow$ (2): Since $f : X \to X$ is an equicontinuous homeomorphism, we have $X = M(f) = \Omega(f)$. Hence, (1) $\Rightarrow$ (2) is implied by Theorem 1.1.
(2) $\Leftrightarrow$ (3): This equivalence is already known (see [18, Theorem 4]).
(3) $\Rightarrow$ (1): We can assume that $d(f(x), f(y)) = d(x, y)$ for every $x, y \in X$. Let us suppose that $(x_i)_{i\ge0}$ is a limit pseudo orbit of f and prove that $(x_i)_{i\ge0}$ has a limit shadowing point. Since $\lim_{i\to\infty} d(f(x_i), x_{i+1}) = 0$, by Lemma 3.2, we can take a sequence of integers $0 \le i_1 < i_2 < \cdots$ such that for each $n \ge 1$, we have

$$d(x_{i_n+j}, f^j(x_{i_n})) \le 2^{-n} \quad (\forall j \ge 0). \tag{a}$$

Put $y_n = f^{-i_n}(x_{i_n})$ for each $n \ge 1$. Then, we have

$$\begin{aligned} d(y_n, y_{n+1}) &= d(f^{-i_n}(x_{i_n}), f^{-i_{n+1}}(x_{i_{n+1}})) \\ &= d(f^{i_{n+1}-i_n}(x_{i_n}), x_{i_{n+1}}) \\ &\le 2^{-n} \quad \text{(by taking } j = i_{n+1} - i_n \text{ in (a))} \end{aligned}$$

for every $n \ge 1$. This implies that $(y_n)_{n\ge1}$ is a Cauchy sequence of points in X, and thus there is $y \in X$ such that $\lim_{n\to\infty} y_n = y$. We have

$$d(y_n, y) \le 2^{-n} + 2^{-n-1} + 2^{-n-2} + \cdots = 2^{-n+1} \tag{b}$$

for any $n \ge 1$. It follows that

$$\begin{aligned} d(x_{i_n+j}, f^{i_n+j}(y)) &\le d(x_{i_n+j}, f^{i_n+j}(y_n)) + d(f^{i_n+j}(y_n), f^{i_n+j}(y)) \\ &= d(x_{i_n+j}, f^j(x_{i_n})) + d(y_n, y) \\ &\le 2^{-n} + 2^{-n+1} = 3 \cdot 2^{-n} \quad \text{(by (a) and (b))} \end{aligned}$$

for all $n \ge 1$ and $j \ge 0$. Thus, we have $\lim_{i\to\infty} d(x_i, f^i(y)) = 0$. □

Two more simple lemmas are needed for the proof of Theorem 1.2.

Lemma 3.3. *Let $f : X \to X$ be a continuous map. If f has the shadowing property, and $f|_{\Omega(f)}$ has the limit shadowing property, then f has the limit shadowing property.*

Proof. Let $(x_i)_{i\ge0}$ be a limit pseudo orbit of f. We first prove that $\lim_{i\to\infty} d(x_i, \Omega(f)) = 0$. For any given $\epsilon > 0$, since f has the shadowing property, there is $\delta > 0$ such that every δ-pseudo orbit of f is ϵ-shadowed by some point of X, and so there is an integer $i \ge 0$ and $y \in X$ such that $d(x_{i+j}, f^j(y)) \le \epsilon$ for all $j \ge 0$. Since $\lim_{j\to\infty} d(f^j(y), \Omega(f)) = 0$, we have $\limsup_{j\to\infty} d(x_{i+j}, \Omega(f)) \le \epsilon$, implying that $\limsup_{i\to\infty} d(x_i, \Omega(f)) \le \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $\lim_{i\to\infty} d(x_i, \Omega(f)) = 0$. For each $i \ge 0$, take $x_i' \in \Omega(f)$ such that $d(x_i, x_i') = d(x_i, \Omega(f))$. Then, we have $\lim_{i\to\infty} d(x_i, x_i') = 0$, and it is easy to see that $\lim_{i\to\infty} d(f(x_i'), x_{i+1}') = 0$. By the limit shadowing property of $f|_{\Omega(f)}$, there is $z \in \Omega(f)$ such that $\lim_{i\to\infty} d(x_i', f^i(z)) = 0$. It follows that $\lim_{i\to\infty} d(x_i, f^i(z)) = 0$, and this proves the lemma. □

The next lemma is proved in [16].

Lemma 3.4. [16] *Let $f : X \to X$ be an equicontinuous map. Then, the following properties hold.*

(1) $\Omega(f) = \bigcap_{n\geq 1} f^n(X)$.
(2) $f|_{\Omega(f)}$ *is an equicontinuous homeomorphism.*

As the final proof of this section, we prove Theorem 1.2.

Proof of Theorem 1.2. (2) ⇒ (1): It is well-known that if f has the shadowing property, then so $f|_{\Omega(f)}$ does (see [18, Lemma 1]). Since $f|_{\Omega(f)}$ is an equicontinuous homeomorphism by Lemma 3.4, $f|_{\Omega(f)}$ has the limit shadowing property by Lemma 3.1. Thus, by Lemma 3.3, f has the limit shadowing property.
(1) ⇒ (3): By Theorem 1.1, if f has the limit shadowing property, then $f|_{\Omega(f)}$ has the shadowing property. Since $f|_{\Omega(f)}$ is an equicontinuous homeomorphism by Lemma 3.4, $\dim \Omega(f) = 0$ by Lemma 3.1.
(3) ⇒ (2): It suffices to prove that every $x \in X$ is a chain continuity point for f. Here, for any given $x \in X$, we define $C(x) \subset X$ by

$$C(x) = \{y \in X : \forall\delta > 0\ \forall n \geq 1\ \exists\ \delta\text{-chain } (x_i)_{i=0}^k \text{ of } f \text{ s.t. } k \geq n,\ x_0 = x \text{ and } x_k = y\}.$$

Since f is equicontinuous, by [1], if $\dim C(x) = 0$, then x is a chain continuity point for f. Note that we have $C(x) \subset \bigcap_{n\geq 1} f^n(X)$ and $\bigcap_{n\geq 1} f^n(X) = \Omega(f)$ by Lemma 3.4. Thus, from $\dim \Omega(f) = 0$, it follows that $\dim C(x) = 0$, proving that every $x \in X$ is a chain continuity point for f. □

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