OHSAWA-TAKEGOSHI EXTENSION THEOREM FOR COMPACT KÄHLER MANIFOLDS AND APPLICATIONS

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ABSTRACT. Our main goal in this article is to prove an extension theorem for sections of the canonical bundle of a weakly pseudoconvex Kähler manifold with values in a line bundle endowed with a possibly singular metric. We also give some applications of our result.

1. Introduction

The L^2 extension theorem by Ohsawa-Takegoshi is a tool of fundamental importance in algebraic and analytic geometry. After the crucial contribution of [OT87, Ohs88], this result has been generalized by many authors in various contexts, including [Man93], [Dem00], [Siu04], [Ber05], [Che11], [Yi12], [ZGZ12], [Blo13], [GZ15a], [Dem15], [BL16].

In this article we treat yet another version of the extension theorem in the context of Kähler manifolds. We first state a consequence of our main result; we remark that a version of it was conjectured by Y.-T. Siu in the framework of his work on the invariance of plurigenera.

Theorem 1.1. Let (X, ω) be a Kähler manifold and $\operatorname{pr}: X \to \Delta$ be a proper holomorphic map to the ball $\Delta \subset \mathbb{C}^1$ centered at 0 of radius R. Let (L,h) be a holomorphic line bundle over X equipped with a hermitian metric (maybe singular) $h = h_0 e^{-\varphi_L}$ such that $i\Theta_h(L) \geq 0$ in the sense of currents, where h_0 is a smooth hermitian metric and φ_L is a quasi-psh function over X. We suppose that $X_0 := \operatorname{pr}^{-1}(0)$ is smooth of codimension 1, and that the restriction of h to X_0 is not identically ∞ .

Let $f \in H^0(X_0, K_{X_0} \otimes L)$ be a holomorphic section in the multiplier ideal defined by the restriction of h to X_0 . Then there exists a section $F \in H^0(X, K_X \otimes L)$ whose restriction to X_0 is equal to f, and such that the following optimal estimate holds

(1)
$$\frac{1}{\pi R^2} \int_X |F|^2_{\omega,h} dV_{X,\omega} \le \int_{X_0} |f|^2_{\omega,h} dV_{X_0,\omega}.$$

We note that the volume form $|F|_{\omega,h}^2 dV_{X,\omega}$ is independent of choice of the metric ω , and $dV_{X_0,\omega}$ is the volume form on X_0 induced by the metric $\omega|_{X_0}$.

If the manifold X is isomorphic to the product $X_0 \times \Delta$ and if the line bundle L is trivial, then it is clear how to construct F. If not, the existence of an extension verifying the estimate above is quite subtle, and it has many important applications. The result above is proved by combining the arguments in [Blo13], [GZ15a] and [Yi12]. Comparing to [Blo13, GZ15a], the new input here is that we allow the

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metric h of L to be singular, while the ambient manifold is only assumed to be Kähler. This general context leads to rather severe difficulties, mainly due to the loss of positivity in the process of regularizing the metric h which adds to the intricate relationship between the several parameters involved in the proof. We use here the arguments in [Yi12] to overcome the difficulties.

Before stating the main result of this paper in its most general form and explaining the main ideas in the proof, we note the following consequence of Theorem 1.1 by an idea of H. Tsuji. It is a generalization of [BP10, Thm 0.1] to arbitrary compact Kähler families, which follows from our main theorem and the arguments in [GZ15a, Cor 3.7].

Theorem 1.2. Let $p: X \to Y$ be a fibration between two compact Kähler manifolds. Let $L \to X$ be a line bundle endowed with a metric (maybe singular) $h = h_0 e^{-\varphi_L}$ such that $i\Theta_h(L) \geq 0$ in the sense of currents, where h_0 is a smooth hermitian metric and φ_L is a quasi-psh function over X. Suppose that there exists a generic point $z \in Y$ and a section $u \in H^0(X_z, mK_{X/Y} + L)$ such that

$$\int_{X_z} |u|_{\omega,h}^{\frac{2}{m}} dV_{X_z,\omega} < +\infty.$$

Then the line bundle $mK_{X/Y} + L$ admits a metric with positive curvature current. Moreover, this metric equals to the fiberwise m-Bergman kernel metric on the generic fibers of p.

We note that the original proof of the theorem above in the projective case does not go through in the Kähler case. This is due to the fact that in [BP10, Thm 0.1] the authors are using in an essential manner the existence of Zariski dense open subsets of X.

We will state next our general version of Theorem 1.1; prior to this, we introduce some auxiliary weights, following [Blo13], [GZ15a].

Notations 1. Given $\delta > 0$ and $A \in \mathbb{R}$, let $c_A(t)$ be a positive smooth function on $(-A, +\infty)$ such that $\int_{-A}^{+\infty} c_A(t)e^{-t}dt < +\infty$. Set

$$u(t) = -\ln(\frac{c_A(-A)e^A}{\delta} + \int_{-A}^t c_A(t_1)e^{-t_1}dt_1),$$

and

$$s(t) = \frac{\int_{-A}^{t} e^{-u(t_1)} dt_1 + \frac{c_A(-A)e^A}{\delta^2}}{e^{-u(t)}}.$$

Then u(t) and s(t) satisfy the ODE equations:

(2)
$$(s(t) + \frac{(s'(t))^2}{u''(t)s(t) - s''(t)})e^{u(t) - t} = \frac{1}{c_A(t)}$$

and

(3)
$$s'(t) - s(t)u'(t) = 1.$$

We suppose moreover that

(4)
$$e^{-u(t)} \ge c_A(t)s(t) \cdot e^{-t}$$
 for every $t \in (-A, +\infty)$.

Remark 2. If $c_A(t) \cdot e^{-t}$ is decreasing, then (4) is automatically satisfied. Moreover, by the construction of u(t), s(t), we know that

(5)
$$\lim_{t \to +\infty} u(t) = -\ln(\frac{c_A(-A)e^A}{\delta} + \int_{-A}^{+\infty} c_A(t_1)e^{-t_1}dt_1) < +\infty$$

and

$$(6) |s(t)| \le C_1|t| + C_2$$

for two constants C_1, C_2 independent of t.

In this set-up, by combining the arguments in [GZ15a] and [Yi12], we can prove the main result of the present paper:

Theorem 1.3. Let (X, ω) be a weakly pseudoconvex n-dimensional Kähler manifold and E be a vector bundle of rank r endowed with a smooth metric h_E . Let $Z \subset X$ be the zero locus of $v \in H^0(X, E)$. We assume that Z is smooth of codimension r and $|v|_{h_E}^{2r} \leq e^A$ for some $A \in \mathbb{R}$. Set $\Psi(z) := \ln |v|_{h_E}^{2r}$.

Let L be a line bundle on X equipped with a singular metric $h := h_0 \cdot e^{-\varphi}$ such that $i\Theta_h(L) \geq \gamma$ for some continuous (1,1)-form γ , where h_0 is a smooth metric on L. We assume that there exists a sequence of analytic approximations $\{\varphi_k\}_{k=1}^{\infty}$ of φ such that ¹

(7)
$$i\Theta_{h_0 \cdot e^{-\varphi_k}}(L) \ge \gamma - \frac{\omega}{k}.$$

We suppose that there exists a continuous function a(t) on $(-A, +\infty]$, such that $0 < a(t) \le s(t)$ and

(8)
$$a(-\Psi)(\gamma + id'd''\Psi) + id'd''\Psi \ge 0.$$

Then for every $f \in H^0(Z, K_X \otimes L \otimes \mathcal{I}(\varphi|_Z))^2$, there exists a $F \in H^0(X, K_X \otimes L)$ such that $F|_{Z} = f$ and

(9)
$$\int_X c_A(-\Psi)|F|_{\omega,h}^2 dV_{X,\omega} \le e^{-\lim_{t \to +\infty} u(t)} \int_Z \frac{|f|_{\omega,h}^2}{|\Lambda^r(dv)|^2} dV_{Z,\omega},$$

where the weight $|\Lambda^r(ds)|^2$ is defined as the unique function such that

$$\int_{Z} \frac{G}{|\Lambda^r(dv)|^2} dV_{Z,\omega} = \lim_{m \to +\infty} \int_{-m-1 \le \ln|v|_{h_E}^{2r} \le -m} \frac{G}{|v|_{h_E}^{2r}} dV_{X,\omega} \qquad \textit{for every } G \in C^{\infty}(X).$$

Remark 3. As already pointed out in [GZ15a], by taking $E = \operatorname{pr}^* \mathcal{O}_{\Delta}$, $v = \operatorname{pr}^* z$, $A = \ln R^2$, $c_A(t) \equiv 1$ and letting $\delta \to +\infty$, Theorem 1.3 implies Theorem 1.1.

We comment next a few results at the foundation of Theorem 1.3. The original Ohsawa-Takegoshi extension theorem [OT87] deals with the local case, i.e. X is a pseudoconvex domain in \mathbb{C}^n . The potential applications of this type of results in global complex geometry become apparent shortly after the article [OT87] appeared, and to this end it was necessary to rephrase it in the context of manifolds. As far as we are aware, the first global version is due to L. Manivel [Man93]. We quote here a simplified version of his result.

¹If X is compact, such approximation always exists, cf. [Dem12, Chapter 13]

 $^{{}^2\}mathcal{I}(\varphi|_Z)$ is the multiplier ideal sheaf on Z associated to the weight $\varphi|_Z$

Theorem 1.4. [Man93, Thm 2] Let X be a n-dimensional Stein manifold, and E be a holomorphic vector bundle over X of rank r with a smooth metric h_E . Let $Y \subset X$ be the zero locus of $s \in H^0(X, E)$. We assume that Y is smooth and of codimension r. Let Ω be a (1,1)-closed semi-positive form on X such that

$$\Omega \otimes \mathrm{Id}_E \geq i\Theta_{h_E}(E)$$

in the sense of Griffiths, i.e., $\Omega \otimes \operatorname{Id}_E - i\Theta_{h_E}(E)$ is semipositive on the vectors $\xi \otimes s \in T_X \otimes \operatorname{End}(E)$ for every $\xi \in T_X$ and $s \in \operatorname{End}(E)$.

Let (L, h_0) be a line bundle on X equipped with a smooth metric h_0 , such that there exists a constant $\alpha > 0$ satisfying

$$i\Theta_{h_0}(L) \ge \alpha\Omega - rid'd'' \ln|s|_{h_E}^2$$
.

Then for every $f \in H^0(Y, K_Y \otimes L \otimes (\det E)^{-1})$, there exists a section $F \in H^0(X, K_X \otimes L)$ such that $F|_{Y} = f \wedge (\wedge^r ds)$ and

(10)
$$\int_{X} \frac{|F|_{\omega,h_0}^2}{|s|_{h_E}^{2r-2} (1+|s|_{h_E}^2)^{\beta}} dV_{X,\omega} \le C \int_{Y} |f|_{\omega,h_0}^2 dV_{Y,\omega},$$

where C is a numerical constant depending only on r, α and β .

Remark 4. Theorem 1.4 can be easily generalized to the case when X is a weakly pseudoconvex Kähler manifold and the weight function $|s|_{h_E}^{2r-2}(1+|s|_{h_E}^2)^{\beta}$ can be ameliorated by $|s|_{h_E}^{2r}(\ln|s|_{h_E})^2$, cf. [Dem12, Thm 12.6].

One of the important limitations of Theorem 1.4 is that the metric h_0 is assumed to be smooth. Indeed this is unfortunate, given that in the usual set-up of algebraic geometry one has to deal with extension problems for canonical forms with values in pseudo-effective line bundles. A famous example is the invariance of plurigenera for projective manifolds ([Siu04]): one needs an extension theorem under the hypothesis that the metric h_0 has arbitrary singularities. We remark that the proof of the extension theorem used in the article mentioned above is confined to the case of projective manifolds. Thus, in order to generalize [Siu04] to compact Kähler manifolds, the first step would be to allow the metric h_0 in Theorem 1.4 to have arbitrary singularities.

Among the very few results in this direction we mention the important work of L. Yi. In order to keep the discussion simple, we restrict ourselves to the setup in Theorem 1.1. Let $\mathcal{I}_{+}(h) := \lim_{\delta \to 0^{+}} \mathcal{I}(h^{1+\delta})$. L. Yi [Yi12] established Theorem 1.1 ³ for sections f which belong to the augmented multiplier ideal sheaf $\mathcal{I}_{+}(h)$. Guan and Zhou [GZ15b] (cf. also Hiep [Hie14]) showed that $\mathcal{I}_{+}(h) = \mathcal{I}(h)$. Thus, the conjunction of these two results as well as the optimal extension [GZ15a] establish Theorem 1.1. The proof of our main theorem is mainly based on the arguments in [GZ15a] and [Yi12].

Remark 5. In the situation of Theorem 1.3, if we take the weight function $c_A(t) \equiv 1$, then we have Theorem 1.1. There is another weight function which might be useful. If we take $c_A(t) = \frac{e^t}{(t+A+c)^2}$ for some constant c > 0, thanks to Remark 2, (4) is satisfied. Using this weight function, [GZ15a, Thm 3.16] proved an optimal estimate version of Theorem 1.4 and its remark 4. Thanks to Theorem 1.3, we know that [GZ15a, Thm 3.16] is also true for weakly pseudoconvex Kähler manifolds under the approximation assumption (7).

 $^{^{3}}$ [Yi12] proved it in a more general setting.

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2. Proof of Theorem 1.3

Proof of Theorem 1.3. The constants C_1, C_2, \cdots below are all independent of k. The proof follows closely [GZ15a] and [Yi12]. To begin with, we introduce several notations. In the setting of Theorem 1.3, for every $m \in \mathbb{R}$ fixed, we can define a C^1 -function b_m on \mathbb{R} such that

$$b_m(t) = t \text{ for } t \ge -m$$
 and $b''_m(t) = \mathbf{1}_{\{-m-1 \le t \le -m\}}.$

Then

(11)
$$b_m(t) \ge t$$
 and $b_m(t) \ge -m-1$ for every $t \in \mathbb{R}^1$.

Let s, u be the two functions defined in the introduction. Set $\chi_m(z) := -b_m \circ \Psi$, $\eta_m(z) := s \circ \chi_m$ and $\phi_m(z) := u \circ \chi_m$. Thanks to (5) and (6), we have

$$(12) |\phi_m(z)| \le C_1$$

and

$$(13) |\eta_m(z)| \le C_2 |\chi_m(z)| + C_3 \le C_2 \cdot \min\{2r|\ln|v|_{h_E}|, m+1\} + C_3.$$

Set $\lambda_m(z) := \frac{(s')^2}{u''s-s''} \circ \chi_m$, $h_k := h_0 \cdot e^{-\varphi_k}$ and $\widetilde{h}_{m,k} := h_k \cdot e^{-\Psi - \phi_m}$. By (2), we have

$$(14) c_A(\chi_m) \cdot e^{-\chi_m + \phi_m} = (\eta_m + \lambda_m)^{-1}.$$

The proof of the theorem is divided by three steps.

Step 1: Construction of smooth extension

We construct in this step a smooth section $\tilde{f} \in C^{\infty}(X, K_X \otimes L)$ extending f such that $(D''\tilde{f})(z) = 0$ for every $z \in Z$ and

(15)
$$\int_{X} \frac{|D''\widetilde{f}|_{\omega,h_{0}}^{2}}{|v|_{h_{E}}^{2r}(\ln|v|_{h_{E}})^{2}} \cdot e^{-(1+\sigma)\varphi} dV_{X,\omega} \le C_{1} \cdot \int_{Z} \frac{|f|_{\omega,h}^{2}}{|\Lambda^{r}(dv)|^{2}} dV_{Z,\omega}$$

for some constant $\sigma > 0$.

In fact, let (U_i) be a small Stein cover of X and let (χ_i) be a partition of unity subordinate to (U_i) . Thanks to [GZ15b], there exists a $\sigma > 0$, such that

$$\int_{U_i \cap Z} |f|_{\omega, h_0}^2 e^{-(1+\sigma)\varphi} dV_{Z,\omega} \le 2 \int_{U_i \cap Z} |f|_{\omega, h}^2 dV_{Z,\omega}.$$

Applying the local Ohsawa-Takegoshi extension theorem (cf. for example [Dem12, Thm 12.6]) to the weight $e^{-(1+\sigma)\varphi}$ on U_i , we obtain a holomorphic section f_i on U_i such that

(16)
$$\int_{U_i} \frac{|f_i|_{\omega,h_0}^2}{|v|_{h_E}^{2r} (\ln|v|_{h_E})^2} \cdot e^{-(1+\sigma)\varphi} dV_{X,\omega} \le C_2 \cdot \int_{U_i \cap Z} \frac{|f|_{\omega,h}^2}{|\Lambda^r(dv)|^2} dV_{Z,\omega}.$$

Set
$$\widetilde{f} := \sum_{i} \chi_i \cdot f_i$$
. Then

$$(D''\widetilde{f})|_{U_j} = D''(\sum_i \chi_i \cdot (f_i - f_j)) = \sum_i (\overline{\partial}\chi_i) \cdot (f_i - f_j) \text{ on } U_j.$$

Combining this with (16), (15) is proved. We have also $(D^{"}\widetilde{f})(z) = 0$ for every $z \in \mathbb{Z}$.

Step 2: L^2 estimate

Set $g_m := D''((1 - b'_m \circ \Psi) \cdot \widetilde{f})$. We claim that

Claim: There exists a sequence $\{a_m\}_{m=1}^{+\infty}\subset\mathbb{N}$ tending to $+\infty$, γ_m and β_m such that

(17)
$$D''\gamma_m + (\frac{m}{a_m})^{\frac{1}{2}}\beta_m = g_m, \qquad \lim_{m \to +\infty} \frac{m}{a_m} = 0,$$

and

(18)
$$\frac{\overline{\lim}}{m \to +\infty} \left(\int_{X} \frac{|\gamma_{m}|_{\omega,\widetilde{h}_{m,a_{m}}}^{2}}{\eta_{m} + \lambda_{m}} dV_{X,\omega} + C \int_{X} |\beta_{m}|_{\omega,\widetilde{h}_{m,a_{m}}}^{2} dV_{X,\omega} \right) \\
\leq e^{-\lim_{t \to +\infty} u(t)} \cdot \int_{Z} \frac{|f|_{\omega,h}^{2}}{|\Lambda^{r}(dv)|^{2}} dV_{Z,\omega}$$

for some uniform constant C > 0. The proof of the claim combines the estimates in [GZ15a] and [Yi12]. We postpone the proof of the claim in Lemma 2.1 and first finish the proof of the theorem.

We use (18) to estimate $\int_X c_A(-b_m \circ \Psi) \cdot |\gamma_m|_{\omega,h_{a_m}}^2 dV_{X,\omega}$. By (11) and (14), we have

$$c_A(-b_m \circ \Psi) \cdot e^{\Psi + \phi_m} = c_A(\chi_m) \cdot e^{\Psi + \phi_m} \le (\eta_m + \lambda_m)^{-1}.$$

Therefore

(19)
$$\int_{X} c_{A}(-b_{m} \circ \Psi) \cdot |\gamma_{m}|_{\omega, h_{a_{m}}}^{2} dV_{X, \omega} \leq \int_{X} \frac{|\gamma_{m}|_{\omega, \widetilde{h}_{m, a_{m}}}^{2}}{(\eta_{m} + \lambda_{m})} dV_{X, \omega}.$$

Combining this with (18), we get

$$(20) \quad \overline{\lim}_{m \to +\infty} \int_X c_A(-b_m \circ \Psi) |\gamma_m|_{\omega, h_{a_m}}^2 dV_{X,\omega} \le e^{-\lim_{t \to +\infty} u(t)} \cdot \int_Z \frac{|f|_{\omega, h_0}^2 e^{-\varphi_k}}{|\Lambda^r(dv)|^2} dV_{Z,\omega}.$$

Thanks to (20), by passing to a subsequence, we can assume that the sequence

$$\{\gamma_m - (1 - b'_m \circ \Psi)\widetilde{f}\}_{k=1}^{+\infty}$$

converges weakly (in the weak L^2 -sense) to a section $F \in L^2(X, K_X \otimes L)$.

Step 3: Final conclusion

We first prove that F is holomorphic and satisfies (9). In fact, thanks to (12) and (18), we have

(21)
$$\int_{Y} |\beta_m|_{\omega, h_{a_m}}^2 e^{-\Psi} dV_{X,\omega} \le C_3$$

for some uniform constant C_3 . Since $\frac{m}{a_m}$ tends to 0, (17) and (21) imply that $D^{''}(\gamma_m - (1 - b_m' \circ \Psi)\tilde{f})$ tends to 0. Therefore $F \in H^0(X, K_X \otimes L)$.

As $\{\varphi_k\}_{k=1}^{+\infty}$ is a decreasing sequence, for every $k_0 \in \mathbb{N}$ fixed, we have

(22)
$$\int_{X} c_{A}(-b_{m} \circ \Psi) |\gamma_{m}|_{\omega, h_{k_{0}}}^{2} dV_{\omega, X} \leq \int_{X} c_{A}(-b_{m} \circ \Psi) |\gamma_{m}|_{\omega, h_{k}}^{2} dV_{\omega, X}$$

for every $k \geq k_0$. Combining this with (20), we get

$$\varliminf_{m \to +\infty} \int_X c_A(-b_m \circ \Psi) |\gamma_m|^2_{\omega,h_{k_0}} dV_{X,\omega} \leq e^{-\lim_{t \to +\infty} u(t)} \cdot \int_Z \frac{|f|^2_{\omega,h_0} e^{-\varphi_k}}{|\Lambda^r(dv)|^2} dV_{Z,\omega}$$

Applying Fatou's lemma to the above inequality, we obtain

$$\int_{X} c_{A}(-\Psi)|F|_{\omega,h_{k_{0}}}^{2} dV_{\omega,X} \leq e^{-\lim_{t \to +\infty} u(t)} \int_{Z} \frac{|f|_{\omega,h}^{2}}{|\Lambda^{r}(dv)|^{2}} dV_{Z,\omega},$$

and (9) is proved by letting $k_0 \to +\infty$.

Let $\{U_i\}$ be a Stein cover of X. To finish the proof of the theorem, it remains to prove that $F|_{U_i\cap Z}=f$ for every i. Since β_m is $\overline{\partial}$ -closed, on the Stein open set U_i , we can find a function w_m such that $\overline{\partial} w_m=\beta_m$ and

$$\int_{U_i} |w_m|_{\omega, h_{a_m}}^2 e^{-\Psi} dV_{X,\omega} \le C_4 \int_{U_i} |\beta_m|_{\omega, h_{a_m}}^2 e^{-\Psi} dV_{X,\omega} \le C_4 \cdot C_3.$$

for some uniform constant C_4 . Then

$$F_m := (1 - b'_m \circ \Psi) \cdot \widetilde{f} - \gamma_m - (\frac{m}{a_m})^{\frac{1}{2}} \cdot w_m$$

is a holomorphic function on U_i and $F_m \rightharpoonup F$ on U_i . As $F_m|_{U_i \cap Z} = f$ by construction, we know that $F|_{U_i \cap Z} = f$. The theorem is proved.

We complete here the proof of Theorem 1.3 by establishing the claim in Step 2.

Lemma 2.1. The claim in Theorem 1.3 is true.

Proof. Step 1: Approximation

Since b_m is not smooth, we construct first a smooth approximation of b_m . Let m, k be two fixed constants. Set

$$v_{\epsilon}(t) := \int_{-\infty}^{t} \int_{-\infty}^{t_1} \frac{1}{1 - 2\epsilon} \mathbf{1}_{\{-m-1+\epsilon < s < -m-\epsilon\}} * \rho_{\frac{\epsilon}{4}} ds dt_1$$
$$- \int_{-\infty}^{0} \int_{-\infty}^{t_1} \frac{1}{1 - 2\epsilon} \mathbf{1}_{\{-m-1+\epsilon < s < -m-\epsilon\}} * \rho_{\frac{\epsilon}{4}} ds dt_1$$

where $\rho_{\frac{\epsilon}{4}}$ is the kernel of convolution satisfying $\operatorname{supp}(\rho_{\frac{\epsilon}{4}}) \subset (-\frac{\epsilon}{4}, \frac{\epsilon}{4})$. It is easy to check that $v_{\epsilon}(t)$ is a smooth approximation of $b_m(t)$. Set

$$\eta_{\epsilon} := s(-v_{\epsilon} \circ \Psi), \qquad \phi_{\epsilon} := u(-v_{\epsilon} \circ \Psi), \qquad \widetilde{h}_{\epsilon} := h_k \cdot e^{-\Psi - \phi_{\epsilon}}$$

and

$$B_{\epsilon} := [\eta_{\epsilon} i \Theta_{\widetilde{h}_{m,k}} - i \partial \overline{\partial} \eta_{\epsilon} - i (\lambda_{\epsilon})^{-1} \partial \eta_{\epsilon} \wedge \overline{\partial} \eta_{\epsilon}, \Lambda_{\omega}],$$

where Λ_{ω} is the contraction with respect to ω . Then $\eta_{\epsilon}, \phi_{\epsilon}, B_{\epsilon}$ tend to $\eta_m, \phi_m, B_{m,k}$.

Step 2: L^2 estimate

By using the estimates in [GZ15a, page 1180], we know that

(23)
$$B_{m,k} := \left[\eta_m(i\Theta_{\widetilde{h}_{m,k}}(L)) - id'd''\eta_m - \lambda_m^{-1}id'\eta_m \wedge d''\eta_m, \Lambda_\omega \right]$$

satisfies

$$B_{m,k} \ge (b_m'' \circ \Psi) \cdot [\partial \Psi \wedge \overline{\partial} \Psi, \Lambda_\omega] - \frac{\eta_m}{k} \operatorname{Id}.$$

Combining this with (13), we have

$$B_{m,k} \ge (b_m'' \circ \Psi) \cdot [\partial \Psi \wedge \overline{\partial} \Psi, \Lambda_\omega] - \frac{C \cdot m}{k} \operatorname{Id}.$$

for some uniform constant C. Therefore, for every form $\alpha \in C_c^{\infty}(X, \wedge^{n,1}T_X^* \otimes L)$, we have ⁴

(24)
$$\|(\eta_{\epsilon} + \lambda_{\epsilon})^{\frac{1}{2}} (D'')^* \alpha \|_{\widetilde{h}_k}^2 + \|(\eta_{\epsilon})^{\frac{1}{2}} D'' \alpha \|_{\widetilde{h}_k}^2 \ge \langle B_{\epsilon} \alpha, \alpha \rangle_{\widetilde{h}_k}.$$

and

(25)
$$\langle (B_{\epsilon} + \frac{C \cdot m}{k} \operatorname{Id}) \alpha, \alpha \rangle_{\widetilde{h}_{\epsilon}} \ge (v_{\epsilon}^{"} \circ \Psi) \langle [\partial \Psi \wedge \overline{\partial} \Psi, \Lambda_{\omega}] \alpha, \alpha \rangle_{\widetilde{h}_{\epsilon}}$$

By applying a standard L^2 -estimate (cf. appendix), we can find γ_{ϵ} and β_{ϵ} such that

(26)
$$D''\gamma_{\epsilon} + (\frac{m}{k})^{\frac{1}{2}}\beta_{\epsilon} = g_{m}$$

and

(27)
$$\int_{X} \frac{|\gamma_{\epsilon}|^{2}_{\omega,\widetilde{h}_{\epsilon}}}{\eta_{\epsilon} + \lambda_{\epsilon}} dV_{X,\omega} + \frac{1}{2C} \int_{X} |\beta_{\epsilon}|^{2}_{\omega,\widetilde{h}_{\epsilon}} dV_{X,\omega}$$
$$\leq \int_{X} \langle (B_{\epsilon} + \frac{2C \cdot m}{k})^{-1} g_{m}, g_{m} \rangle_{\omega,\widetilde{h}_{m,k}} dV_{X,\omega}.$$

By letting $\epsilon \to 0$, we can find $\gamma_{m,k}$ and $\beta_{m,k}$, such that

(28)
$$D''\gamma_{m,k} + (\frac{m}{k})^{\frac{1}{2}}\beta_{m,k} = g_m$$

and

(29)
$$\int_{X} \frac{\left|\gamma_{m,k}\right|_{\omega,\widetilde{h}_{m,k}}^{2}}{\eta_{m} + \lambda_{m}} dV_{X,\omega} + \frac{1}{2C} \int_{X} \left|\beta_{m,k}\right|_{\omega,\widetilde{h}_{m,k}}^{2} dV_{X,\omega}$$
$$\leq \int_{X} \left\langle \left(B_{m,k} + \frac{2C \cdot m}{k}\right)^{-1} g_{m}, g_{m}\right\rangle_{\omega,\widetilde{h}_{m,k}} dV_{X,\omega}.$$

Step 3: Final conclusion

We first estimate the right hand side of (29). By the construction of g_m and (25), we have

$$(30) \qquad \int_{X} \langle (B_{m,k} + \frac{2C \cdot m}{k})^{-1} g_{m}, g_{m} \rangle_{\omega, \widetilde{h}_{m,k}} dV_{X,\omega}$$

$$\leq \int_{X} (b_{m}^{"} \circ \Psi) \cdot |\widetilde{f}|_{\omega, \widetilde{h}_{m,k}}^{2} dV_{\omega} + \frac{C \cdot k}{m} \int_{X} (1 - b_{m}^{'} \circ \Psi) |D^{"} \widetilde{f}|_{\omega, \widetilde{h}_{m,k}}^{2} dV_{X,\omega}.$$
Since $(1 - b_{m}^{'} \circ \Psi)(z) = 0$ on $\{\Psi \geq -m\}$, we have
$$\int_{X} (1 - b_{m}^{'} \circ \Psi) |D^{"} \widetilde{f}|_{\omega, \widetilde{h}_{m,k}}^{2} dV_{X,\omega} \leq \int_{\{\Psi \leq -m\}} |D^{"} \widetilde{f}|_{\omega, \widetilde{h}_{m,k}}^{2} dV_{X,\omega}.$$

We use the following key estimate [Yi12, Lemma 3.1]: by Hölder inequality, we have

(31)
$$\int_{\{\Psi < -m\}} \frac{|D''\widetilde{f}|_{\omega,h}^2}{|v|_{h_E}^{2r}} dV_{X,\omega}$$

 $^{^{4}}$ We refer to [GZ15a, 5.1] for a detailed calculus.

$$\leq (\int_{\{\Psi \leq -m\}} \frac{|D''\widetilde{f}|_{\omega,h_0}^2 e^{-(1+\sigma)\varphi}}{|v|_{h_E}^{2r} (\ln|v|_{h_E})^2} dV_{X,\omega})^{\frac{1}{1+\sigma}} \cdot (\int_{\{\Psi \leq -m\}} \frac{|D''\widetilde{f}|_{\omega,h_0}^2 (\ln|v|_{h_E})^{\frac{2}{\sigma}}}{|v|_{h_E}^{2r}} dV_{X,\omega})^{\frac{\sigma}{1+\sigma}}.$$

As $D''\widetilde{f} = 0$ on Z by construction, we have

$$\lim_{m \to +\infty} \int_{\{\Psi \le -m\}} \frac{|D''\widetilde{f}|_{\omega,h_0}^2 (\ln |v|_{h_E})^{\frac{2}{\sigma}}}{|v|_{h_E}^{2r}} dV_{X,\omega} = 0.$$

Combining this with (31) and (15), we obtain

$$\lim_{m \to +\infty} \int_{\{\Psi < -m\}} \frac{|D''\widetilde{f}|_{\omega,h}^2}{|v|_{h_F}^{2r}} dV_{X,\omega} = 0.$$

As a consequence, we can find a sequence $a_m \to +\infty$ such that

(32)
$$\lim_{m \to +\infty} \frac{m}{a_m} = 0 \quad \text{and} \quad \lim_{m \to +\infty} \frac{a_m}{m} \int_{\{\Psi < -m\}} |D''\widetilde{f}|^2_{\omega,\widetilde{h}_{m,a_m}} dV_{X,\omega} = 0.$$

Applying (32) to (30), we obtain

$$(33) \qquad \overline{\lim}_{m \to +\infty} \int_{X} \langle (B_{m,a_{m}} + \frac{2C \cdot m}{a_{m}})^{-1} g_{m}, g_{m} \rangle_{\omega, \widetilde{h}_{m,a_{m}}} dV_{X,\omega}$$

$$\leq \overline{\lim}_{m \to +\infty} \int_{X} (b_{m}^{"} \circ \Psi) \cdot |\widetilde{f}|_{\omega, \widetilde{h}_{m,k}}^{2} dV_{X,\omega} \leq e^{-\lim_{t \to +\infty} u(t)} \cdot \int_{Z} \frac{|f|_{\omega,h}^{2}}{|\Lambda^{r}(dv)|^{2}} dV_{Z,\omega}.$$

Finally, we take $\gamma_m = \gamma_{m,a_m}$ and $\beta_m = \beta_{m,a_m}$ in (28). Then (29) and (33) imply

$$\overline{\lim}_{m \to +\infty} \left(\int_{X} |\gamma_{m}|_{\omega,\widetilde{h}_{m,a_{m}}}^{2} (\eta_{m} + \lambda_{m})^{-1} dV_{X,\omega} + \frac{1}{2C} \int_{X} |\beta_{m}|_{\omega,\widetilde{h}_{m,a_{m}}}^{2} dV_{X,\omega} \right)$$

$$\leq e^{-\lim_{t\to+\infty}u(t)} \cdot \int_{Z} \frac{|f|_{\omega,h}^{2}}{|\Lambda^{r}(dv)|^{2}} dV_{Z,\omega}.$$

The lemma is proved.

3. Applications

3.1. Some direct applications. As pointed out in Remark 3 in the introduction, Theorem 1.3 implies that

Corollary 3.1. Let (X, ω) be a Kähler manifold with a proper map $\operatorname{pr}: X \to \Delta$ to a ball $\Delta \subset \mathbb{C}^1$ centered at 0 of radius R. Let (L,h) be a holomorphic line bundle over X equipped with a hermitian metric (maybe singular) h such that $i\Theta_h(L) \geq 0$. Suppose that $X_0 := \operatorname{pr}^{-1}(0)$ is a smooth subvariety of codimension 1. Let $f \in H^0(X_0, K_{X_0} + L)$. Then there exists a section $F \in H^0(X, K_X + L)$ such that

(34)
$$\frac{1}{\pi R^2} \int_X |F|^2_{\omega,h} dV_{X,\omega} \le \int_{X_0} |f|^2_{\omega,h} dV_{X_0,\omega}$$

and $F|_{X_0} = \operatorname{pr}^*(dt) \wedge f$, where t is the standard coordinate of \mathbb{C}^1 .

By the same arguments as in [BP10, A.1], Corollary 3.1 implies the following result:

Corollary 3.2. Let (X, ω) be a Kähler manifold and $\operatorname{pr}: X \to \Delta$ be a proper map to the ball $\Delta \subset \mathbb{C}^1$ centered at 0 of radius R. Let L be a holomorphic line bundle over X equipped with a hermitian metric (maybe singular) $h = h_0 \cdot e^{-\varphi}$ such that $i\Theta_h(L) \geq 0$ in the sense of current, where h_0 is a smooth metric and φ is a quasi-psh function on X. Suppose that $X_0 := \operatorname{pr}^{-1}(0)$ is smooth of codimension 1. Let $f \in H^0(X_0, mK_{X_0} \otimes L)$. We suppose that

$$\int_{X_0} |f|_{\omega,h}^{\frac{2}{m}} dV_{X_0,\omega} < +\infty$$

and there exists a $F \in H^0(X, mK_X \otimes L)$ such that

$$F|_{X_0} = f \otimes \operatorname{pr}^*(dt^{\otimes m})$$
 and $\int_X |F|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega} < +\infty,$

where t is the standard coordinate of \mathbb{C}^1 . Then there exists a $\widetilde{F} \in H^0(X, mK_X \otimes L)$ such that

$$(35) \ \widetilde{F}|_{X_0} = f \otimes \operatorname{pr}^*(dt^{\otimes m}) \qquad and \qquad \frac{1}{\pi R^2} \int_X |\widetilde{F}|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega} \leq \int_{X_0} |f|_{\omega,h}^{\frac{2}{m}} dV_{X_0,\omega}.$$

Proof. The proof given here follows closely [BP10, A.1]. Set

$$C_1 := \int_{X_0} |f|_{\omega,h}^{\frac{2}{m}} dV_{X_0,\omega}$$
 and $C_2 := \frac{1}{\pi R^2} \int_X |F|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega}.$

If $C_2 \leq C_1$, then F satisfies (35) and the corollary is proved. If $C_1 < C_2$, since F is holomorphic, we can apply Corollary 3.1 with weight

$$\varphi_1 := \frac{m-1}{m} \ln |F|_{\omega,h_0}^2 + \frac{1}{m} \varphi$$

on the line bundle $(m-1)K_X \otimes L$, and obtain a new extension F_1 of f satisfying

(36)
$$\frac{1}{\pi R^2} \int_X \frac{|F_1|_{\omega,h}^2}{|F|_{\omega,h}^{\frac{2(m-1)}{m}}} dV_{X,\omega} \le \int_{X_0} \frac{|f|_{\omega,h}^2}{|f|_{\omega,h}^{\frac{2(m-1)}{m}}} dV_{X_0,\omega} = \int_{X_0} |f|_{\omega,h}^{\frac{2}{m}} dV_{X_0,\omega}.$$

By Hölder's inequality, we have

$$\frac{1}{\pi R^2} \int_X |F_1|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega} \le \left(\frac{1}{\pi R^2} \int_X |F|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega}\right)^{1-\frac{1}{m}} \cdot \left(\frac{1}{\pi R^2} \int_X \frac{|F_1|_{\omega,h}^2}{|F|^{\frac{2(m-1)}{m}}} dV_{X,\omega}\right)^{\frac{1}{m}}.$$

Combining with (36), we have

(37)
$$\frac{1}{\pi R^2} \int_X |F_1|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega} \le (C_2)^{1-\frac{1}{m}} (C_1)^{\frac{1}{m}}.$$

We can repeat the same argument with F replaced by F_1 , etc. We obtain thus a sequence $\{F_i\}_{i=1}^{+\infty} \subset H^0(X, mK_X \otimes L)$, and

(38)
$$\frac{1}{\pi R^2} \int_X |F_i|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega} \le \left(\frac{1}{\pi R^2} \int_X |F_{i-1}|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega}\right) \cdot (C_1)^{\frac{1}{m}}$$

If there exists an $i \in \mathbb{N}$ such that F_i satisfies (35), then Corollary (3.2) is proved. If not, thanks to (38), we have

(39)
$$\frac{1}{\pi R^2} \int_X |F_i|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega} \searrow C_1.$$

By passing to a subsequence, F_i tends to a section $\widetilde{F} \in H^0(X, mK_X \otimes L)$, and $\widetilde{F}|_{Z} = f$. By Fatou lemma, (39) implies that

$$\frac{1}{\pi R^2} \int_X |\widetilde{F}|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega} \le C_1.$$

Corollary 3.2 is proved.

3.2. Positivity of m-relative Bergman Kernel metric. We first recall the definition of m-relative Bergman Kernel metric (cf. [BP10, A.2], [BP08], [Kaw85], [Tsu07]). Let $p: X \to Y$ be a surjective map between two smooth manifolds and let (L, h_L) be a line bundle on X equipped with a hermitian metric h_L . Let $x \in X$ be a point on a smooth fiber of p. We first define a hermitian metric h on $-(mK_{X/Y} + L)_x$ by

(40)
$$\|\xi\|_h^2 := \sup \frac{|\xi(\tau(x))|^2}{\left(\int_{X_{p(x)}} |\tau|_{\omega,h_L}^{\frac{2}{m}} dV_{X_{p(x)},\omega}\right)^m},$$

where the "sup" is taken over all sections $\tau \in H^0(X_{p(x)}, mK_{X/Y} + L)$. The *m*-relative Bergman Kernel metric $h_{X/Y}^{(m)}$ on $mK_{X/Y} + L$ is defined to be the dual of h.

Although the construction of the metric $h_{X/Y}^{(m)}$ is fiberwise and only defined on the smooth fibers, by using the positivity of direct image arguments, [BP10, Thm 0.1] proved that:

Theorem 3.3. [BP10, Thm 0.1] Let $p: X \to Y$ be a fibration between two projective manifolds, and let ω be a Kähler metric on X. Let $L \to X$ be a line bundle endowed with a metric (maybe singular) h such that $i\Theta_h(L) \ge 0$. Suppose that there exists a generic point $z \in Y$ and a section $u \in H^0(X_z, mK_{X/Y} + L)$ such that

$$\int_{X_z} |u|_{\omega,h}^{\frac{2}{m}} dV_{X_z,\omega} < +\infty.$$

Then the line bundle $mK_{X/Y} + L$ admits a metric with positive curvature current. Moreover, this metric equals to the fiberwise m-Bergman kernel metric (with respect to h) on the generic fibers of p.

An alternative proof of Theorem 3.3 is given by using the optimal extension proved in [GZ15a, Thm 2.1, Cor 3.7]. We should remark that, if φ_L has arbitrary singularity, the proof of in [BP10, Thm 0.1] uses the existence of ample line on X. Therefore the assumption that p is a projective map is essential in the proof of Theorem 3.3 in [BP10, Thm 0.1]. However, as pointed out by M. Păun, since the optimal extension proved in Corollary 3.2 is without projectivity assumption, we can use Corollary 3.2 to generalize Theorem 3.3 to arbitrary compact Kähler fibrations, by using the same arguments in [GZ15a, Cor 3.7]. For the reader's convenience, we give the proof of this generalization in this subsection.

To begin with, we first prove the following lemma, which uses the recent important result [GZ15b].

Lemma 3.4. Let φ be a psh function on a Stein open set U. Set:

$$\mathcal{I}_m(\varphi)_x := \{ f \in \mathcal{O}_x | \int_{U_-} |f|^{\frac{2}{m}} e^{-\frac{\varphi}{m}} < +\infty \}.$$

Then $\mathcal{I}_m(\varphi)$ is a coherent sheaf.

Proof. We first prove the lemma under the assumption that φ has analytic singularities. In this case, Let $\pi: \widetilde{U} \to U$ be a resolution of singularities of φ , i.e., $\varphi \circ \pi$ can be written locally as

$$\varphi \circ \pi = \sum_{i} a_i \ln(|s_i|) + O(1),$$

where s_i are holomorphic functions on \widetilde{U} and $\bigcup_i \operatorname{Div}(s_i)$ is normal crossing. We suppose that $K_{\widetilde{U}} = K_X + \sum_i b_i \cdot E_i$ and $\sum_i a_i \cdot \operatorname{Div}(s_i) = \sum_i c_i \cdot E_i$. Let k_i be the minimal number in \mathbb{Z}^+ such that $k_i \cdot \frac{2}{m} > \frac{c_i}{m} - 2b_i - 2$. It is easy to check that $\mathcal{I}_m(\varphi) = \pi_*(\mathcal{O}(-\sum_i k_i \cdot E_i))$. Therefore $\mathcal{I}_m(\varphi)$ is a coherent sheaf.

We now prove the lemma for arbitrary psh functions. Thanks to [Dem12, 15.B], we can find a sequence of quasi-psh φ_k with analytic singularities and a sequence $\delta_k \to 0^+$, such that

(i): φ_k decrease to φ .

(ii): $\int_{\{\frac{\varphi}{m} < \frac{(1+\delta_k)\varphi_k}{m} + a_k\}} e^{-\frac{\varphi}{m}} < +\infty$ (cf. [Dem12, proof of Thm 15.3, Step 2]) for certain constant a_k .

As a consequence, we have $\mathcal{I}_m((1+\delta_k)\varphi_k)\subset\mathcal{I}_m(\varphi)$. Since we proved that

$$\mathcal{I}_m((1+\delta_k)\varphi_k)$$

are coherent, by the Noetherien property of coherent sheaf, $\bigcup_{k=1}^{+\infty} \mathcal{I}_m((1+\delta_k)\varphi_k)$ is also coherent and

$$\bigcup_{k=1}^{+\infty} \mathcal{I}_m((1+\delta_k)\varphi_k) \subset \mathcal{I}_m(\varphi).$$

To prove the lemma, it is sufficient to prove that for every $f \in \mathcal{I}_m(\varphi)$, we can find a $k \in \mathbb{N}$, such that $f \in \mathcal{I}_m((1 + \delta_k)\varphi_k)$.

Let f be a holomorphic germ of $(\mathcal{I}_m(\varphi))_x$. Then

$$\int_{U_{-}} |f|^{2} e^{-\frac{\varphi}{m} - \frac{2(m-1)\ln|f|}{m}} < +\infty,$$

for some neighborhood U_x of x. By [GZ15b], there exists some $\delta > 0$, such that

$$\int_{U_x} |f|^2 e^{-\frac{(1+\delta)\varphi}{m} - \frac{2(1+\delta)(m-1)\ln|f|}{m}} < +\infty.$$

Replacing U_x by a smaller neighborhood U'_x of x, we have

$$\int_{U'} |f|^{\frac{2}{m}} e^{-\frac{(1+\delta)\varphi}{m}} < +\infty.$$

We take a $k \in \mathbb{N}$, such that $\delta_k < \delta$. Thanks to (i) and (41), we have

$$\int_{U_x'} |f|^{\frac{2}{m}} e^{-\frac{(1+\delta_k)\varphi_k}{m}} < +\infty.$$

Therefore $f \in \mathcal{I}_m((1 + \delta_k)\varphi_k)$ and the lemma is proved.

We now generalize [BP10, Thm 0.1] to arbitrary proper Kähler fibrations. The proof is almost the same as [GZ15a, Cor 3.7].

Theorem 3.5. Let $p: X \to Y$ be a proper fibration between two Kähler manifolds and let ω be a Kähler metric on X. Let $L \to X$ be a line bundle endowed with a metric (maybe singular) $h = h_0 \cdot e^{-\varphi}$ such that $i\Theta_h(L) \geq 0$ in the sense of current, where h_0 is a smooth metric and φ is a quasi-psh function on X.

Suppose that there exists a generic point $z \in Y$ and a $u \in H^0(X_z, (K_{X/Y})^m \otimes L)$ such that

$$\int_{X_z} |u|_{\omega,h}^{\frac{2}{m}} dV_{X_z,\omega} < +\infty \qquad and \qquad u \not\equiv 0.$$

Then the line bundle $(K_{X/Y})^m \otimes L$ admits a metric with positive curvature current. Moreover, this metric equals to the fiberwise m-Bergman kernel metric (with respect to h) on the generic fibers of p.

Proof. By Lemma 3.4, $p_*((K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$ is coherent. Using [Fle81] (cf. also [BDIP02, Thm 10.7, page 47]), there exists a subvariety Z of Y of codimension at least 1 such that p is smooth on $Y \setminus Z$ and for every point $t \in Y \setminus Z$, we have

$$\dim H^0(X_t, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi)|_{X_t}) = \operatorname{rank} p_*((K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi)),$$

where $\mathcal{I}_m(\varphi)|_{X_t}$ is the restriction of the coherent sheaf $\mathcal{I}_m(\varphi)$ on X_t . By local extension theorem, we know that $\mathcal{I}_m(\varphi|_{X_t}) \subset \mathcal{I}_m(\varphi)|_{X_t}$. As a consequence, for every Stein neighborhood U of $t \in Y \setminus Z$, the fibration $p: p^{-1}(U) \to U$ and the point t satisfy the conditions in Corollary 3.2.

Let $h^{(m)}$ be the fiberwise m-Bergman kernel metric on $p^{-1}(Y \setminus Z) \to Y \setminus Z$ (cf. construction in the beginning of this subsection). For every $x \in p^{-1}(Y \setminus Z)$, we now estimate the curvature of $h^{(m)}$ near x. Let e be a local coordinate of $(K_{X/Y})^m \otimes L$ near x. Let

(42)
$$B(z) := \sup \frac{|u^0(z)|^2}{(\int_{X_{p(z)}} |u|^{\frac{2m}{m}} dV_{X_{p(z)},\omega})^m},$$

where $u = u^0 \cdot e$ and the "sup" is taken over all sections $u \in H^0(X_{p(z)}, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$. Thanks to (40), to prove that the curvature of $h^{(m)}$ is positive near x, it is sufficient to prove that $\ln B(z)$ is psh near x.

For every fixed point z near x, we can find a section $u_1 \in H^0(X_{p(z)}, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$ such that

$$B(z) = \frac{|u_1^0(z)|^2}{(\int_{X_{p(z)}} |u_1|_{\omega,h}^{\frac{2}{m}} dV_{X_{p(z)},\omega})^m}.$$

Let Δ_r be a one dimensional radius r disc in Y centered at p(z), and Δ'_r be a one dimensional disc in X passing through z and $p(\Delta'_r) = \Delta_r$. Thanks to Proposition 3.2, there exists an extension of u_1 : $U_1 \in H^0(p^{-1}(\Delta_r), (K_X)^m \otimes L \otimes \mathcal{I}_m(\varphi))$, such that

(43)
$$\frac{1}{\pi r^2} \int_{p^{-1}(\Delta_r)} |U_1|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega} \le \int_{X_{p(z)}} |u_1|_{\omega,h}^{\frac{2}{m}} dV_{X_{p(z)},\omega}.$$

Set $\widetilde{u}_1 := U_1/(dt)^m \in H^0(p^{-1}(\Delta_r), (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$ and $\widetilde{u}_1^0 := \frac{\widetilde{u}_1}{e}$, where t is coordinate of Δ_r . By the definition of B(z), we have

$$\frac{1}{\pi r^2} \int_{\Delta_r'} \ln B(x) p^*(d't \wedge d''t) \geq \frac{1}{\pi r^2} \int_{\Delta_r'} \ln \frac{|\widetilde{u}_1^0(x)|^2}{(\int_{X_{p(x)}} |\widetilde{u}_1(x)|_{\omega,h}^{\frac{2}{m}} dV_{X_{p(z)},\omega})^m} p^*(d't \wedge d''t)$$

$$\geq \frac{1}{\pi r^2} \int_{\Delta_r'} \ln |\widetilde{u}_1^0|^2 p^*(d't \wedge d''t) - \frac{m}{\pi r^2} \ln \int_{p^{-1}(\Delta_r)} |U_1|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega}.$$

Combining this with (43) and the holomorphicity of \tilde{u}_1^0 , we obtain

$$\frac{1}{\pi r^2} \int_{\Delta_r'} \ln B(x) p^*(d't \wedge d''t) \ge \ln B(z).$$

Therefore, $\ln B(x)$ is psh in the horizontal direction. By the convexity of $\ln |u^0(x)|$ and the construction of $\ln B(x)$, $\ln B(x)$ is also psh in the fiberwise direction. Therefore $\ln B(x)$ is psh on $p^{-1}(Y \setminus Z)$ and the curvature of $h^{(m)}$ is semi-positive on $p^{-1}(Y \setminus Z)$ (in the sense of currents).

Using the arguments in [BP10, A.2], we now prove that $h^{(m)}$ can be extended to the whole X. We first express $h^{(m)}$ locally as the potential form $e^{-\varphi_{X/Y}}$, where $\varphi_{X/Y}$ is a quasi-psh function outside the subvariety $p^{-1}(Z)$. By the standard results in pluripotential theory, to prove that $h^{(m)}$ can be extended to X, it is sufficient to prove the existence of a uniform constant C such that

(44)
$$\varphi_{X/Y} \le C \qquad \text{on } X \setminus p^{-1}(Z).$$

Let U be a small open set in X. Let B be the function on $U \setminus p^{-1}(Z)$ defined by (42). Thanks to (40), to prove (44), it is equivalent to prove that B is uniformly bounded on $U \setminus p^{-1}(Z)$. For every $z \in U \setminus p^{-1}(Z)$, we can find a $u_2 \in H^0(X_{p(z)}, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$ such that

$$B(z) = |u_2^0(z)|^2$$
 and $\int_{X_{p(z)}} |u_2|_{\omega,h}^{\frac{2}{m}} dV_{X_{p(z)},\omega} = 1,$

where $u_2^0 := \frac{u_2}{e}$. Using Proposition 3.2, we can find an extension \widetilde{u}_2 of u_2 , such that

$$\int_{p^{-1}(p(U))} |\widetilde{u}_2|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega} \le C_U,$$

where the constant C_U depends only on U. By mean value inequality, we know that $|u_2^0(z)|$ is controlled by a constant depending only on C_U . The theorem is thus proved.

4. Appendix

For the reader's convenience, we give the proof of (26) and (27), which is a rather standard estimate (cf. [Dem12, Prop. 12.4, Remark 12.5], [DP03] or [Yi12]).

Set $g := g_m$, $\eta := \eta_{\epsilon}$, $B := B_{\epsilon,k}$ and $\delta := \delta_k$ for simplicity. Let Y_k be a subvariety of X such that φ_k is smooth outside Y_k . Then there exists a complete Kähler metric ω_1 on $X \setminus Y_k$. Set $\omega_s := \omega + s\omega_1$. Then ω_s is also a complete Kähler metric on $X \setminus Y_k$ for every s > 0.

We apply the twist L^2 -estimate (cf. [Dem12, 12.A, 12.B]) for the line bundle (L, \tilde{h}_k) on $(X \setminus Y_k, \omega_s)$. Thanks to (24) and [GZ15a, Lemma 4.1], for every smooth (n, 1)-form v with compact support, we have

$$\leq (\int_{X\backslash Y_{k}} \langle (B + \frac{2C\cdot m}{k})^{-1}g, g \rangle dV_{\omega_{s}}) \cdot (\|(\eta + \lambda)^{\frac{1}{2}}D''^{*}v\|_{\omega_{s}}^{2} + \frac{2C\cdot m}{k} \int_{X\backslash Y_{k}} \langle v, v \rangle dV_{\omega_{s}})$$

Set $H_1 := \|\cdot\|_{L^2}$, where the L^2 -norm $\|\cdot\|_{L^2}$ is defined with respect to the metrics ω_s and (L, \widetilde{h}_k) . Let H_2 be a Hilbert space where the norm is defined by

$$||f||_{H_2}^2 := \frac{2C \cdot m}{k} \int_{X \setminus Y_k} |f|_{\widetilde{h}_k}^2 dV_{\omega_s}.$$

By (45) and the Hahn-Banach theorem, we can construct a continuous linear map (cf. for example [Dem12, 5.A])

$$H_1 \oplus H_2 \to \mathbb{C}$$
,

which is an extension of the application

$$((\eta + \lambda)^{\frac{1}{2}}D''^*v, v) \to \langle g, v \rangle_{\omega_s}.$$

Therefore, there exist f and h such that

$$\langle g, v \rangle_{\omega_s} = \langle f, (\eta + \lambda)^{\frac{1}{2}} D''^* v \rangle_{\omega_s} + \frac{2C \cdot m}{k} \langle h, v \rangle_{\omega_s}$$

and

$$||f||_{\omega_s}^2 + \frac{2C \cdot m}{k} ||h||_{\omega_s}^2 \le \int_X (\langle B + \frac{2C \cdot m}{k})^{-1} g, g \rangle dV_{\omega_s}$$

Let $\beta := 2C(\frac{m}{k})^{\frac{1}{2}} \cdot h$ and $\gamma := (\eta + \lambda)^{\frac{1}{2}} f$. Then

$$g = D''\gamma + (\frac{m}{k})^{\frac{1}{2}}\beta$$

and

$$\left\|\frac{\gamma}{(\lambda+\eta)^{\frac{1}{2}}}\right\|_{(X\backslash Y_k,\omega_s)}^2 + \frac{1}{2C}\|\beta\|_{(X\backslash Y_k,\omega_s)}^2 \le \int_{X\backslash Y_k} \langle (B+\frac{2C\cdot m}{k})^{-1}g,g\rangle dV_{\omega_s}$$

Then (26) and (27) are proved by letting $s \to 0^+$.

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