Large deviations for level sets of branching Brownian motion and Gaussian free fields

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Dedicated to the memory of Professor V.N. Sudakov

Summary. We study deviation probabilities for the number of high positioned particles in branching Brownian motion, and confirm a conjecture of Derrida and Shi [10]. We also solve the corresponding problem for the two-dimensional discrete Gaussian free field. Our method relies on an elementary inequality for inhomogeneous Galton–Watson processes.

Keywords. Branching Brownian motion, Gaussian free field, large deviation.

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1 Introduction

Consider the model of one-dimensional Branching Brownian Motion (BBM): Initially a particle starts at the origin and performs standard (one-dimensional)

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Brownian motion. After a random exponential time of parameter 1, the particle splits into two particles; they perform independent Brownian motions. Each of the particles splits into two after an exponential time. We assume that the exponential random variables and the Brownian motions are independent. The system goes on indefinitely.

Let $X_{\max}(t)$ denote the rightmost position in the BBM at time t. McKean [14] proves that the distribution function of $X_{\max}(t)$ satisfies the F-KPP equation (Fisher [11], Kolmogorov, Petrovskii and Piskunov [12]), from which it follows that

$$\lim_{t\to\infty} \frac{X_{\max}(t)}{t} = 2^{1/2},$$

in probability. Further order developments can be found in Bramson [5] and [6]. For an account of general properties of BBM, see Bovier [4].

The following large deviation estimate for $X_{\max}(t)$ is known (see [15], [8]): for $x > 2^{1/2}$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(X_{\max}(t) \geq xt\Big) = -\psi(x)\,, \tag{1.1}$$

where

$$\psi(x) := \frac{x^2}{2} - 1\,.$$

For $x > 0$ and $t > 0$, let $N(t, x)$ denote the number of particles, in the BBM, alive at time t and positioned in $[tx,\, \infty)$. It is well-known (Biggins [1]) that for $0 < x < 2^{1/2}$,

$$\lim_{t\to\infty} \frac{\log N(t,x)}{t} = 1 - \frac{x^2}{2}, \qquad \text{a.s.} \tag{1.2}$$

Theorem 1.1. *Let $x > 0$ and $(1 - \frac{x^2}{2})^+ < a < 1$. We have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(N(t,x) \geq \mathrm{e}^{at}) = -I(a,\, x)\,,$$

where

$$I(a,\, x) := \frac{x^2}{2(1-a)} - 1\,.$$

Theorem 1.1 gives an affirmative answer to a conjecture by Derrida and Shi [10]. The conjecture was motivated by a problem for the N-BBM, which is a BBM with the additional criterion that the number of particles in the system should never exceed N (whenever the number is more than N, the particle at the leftmost position is removed from the system). Let $X^{(N)}_{\max}(t)$ denote the rightmost position in the N-BBM at time t. It is known ([10]) that

$$\psi_N(x) := -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(X^{(N)}_{\max}(t) \geq xt),$$

exists. In [10], it is proved that Theorem 1.1 implies the following estimate for $\psi_N(x)$:

Theorem 1.2. *For $x > 2^{1/2}$, we have*

$$\limsup_{N\to\infty} \frac{\log[\psi_N(x) - \psi(x)]}{\log N} \leq -\Big(\frac{x^2}{2} - 1\Big).$$

The inequality in Theorem 1.2 is conjectured in [10] to be an equality.

The rest of the paper is as follows. In Section 2, we present an inequality for inhomogeneous Galton–Watson processes. This inequality will be used in Section 3 for the proof of Theorem 1.1, and in Section 4 to establish the corresponding result for two-dimensional Gaussian free fields.

2 An inequality for inhomogeneous Galton–Watson processes

Let $(Z_n,\, n \geq 0)$ be an inhomogeneous Galton–Watson process, the reproduction law at generation n being denoted by ν_n.[3] More precisely,

$$Z_{n+1} = \sum_{k=1}^{Z_n} \nu_n^{(k)}, \qquad n \geq 0\,,$$

[3] We write, indifferently, a probability measure ν_n on $\{0, 1, 2, \ldots\}$ and a random variable whose distribution is ν_n.

where $\nu_n^{(i)}$, $i \ge 1$, are independent copies of ν_n, and are independent of everything up to generation n. Let

$$m_n := \mathbb{E}(\nu_n).$$

We assume $0 < m_n < \infty$, for $n \ge 0$.

Proposition 2.1. *Let $\alpha > 1$ and $n \ge 1$. For all $0 \le i < n$, we assume the existence of $\lambda_i > 0$ such that*

$$\mathbb{E}(\mathrm{e}^{\lambda_i \nu_i}) \le \mathrm{e}^{\alpha \lambda_i m_i}. \tag{2.1}$$

Then for all $\delta > 0$ and all integer $\ell \ge 1$,

$$\mathbb{P}\Big(Z_n \ge \max\Big\{ \ell,\, (\alpha+\delta)^n\, \ell \max_{0\le i<n} \prod_{j=i}^{n-1} m_j \Big\} \,\Big|\, Z_0 = \ell \Big) \le n \exp\Big(- \frac{\delta \ell}{\alpha+\delta} \min_{0\le i<n} \lambda_i + \max_{0\le i<n} \lambda_i \Big).$$

We say some words about forthcoming applications of the proposition to BBM (in Section 3) and to Gaussian free fields (in Section 4). In both applications, $\alpha + \delta$ is taken to be as close to 1 as possible, whereas ℓ is taken to be $\mathrm{e}^{\varepsilon n}$ with $\varepsilon > 0$ that can be as small as possible (so that ℓ is sufficiently large to compensate $\min_{0\le i<n} \lambda_i$ on the right-hand side, but sufficiently small in front of $\max_{0\le i<n} \prod_{j=i}^{n-1} m_j$ on the left-hand side). Roughly speaking, Proposition 2.1 says that if (2.1) is satisfied with appropriate λ_i, then starting at $Z_0 = \ell$, the inhomogeneous Galton–Watson process exceeds $\max\{\ell,\, \mathrm{e}^{(1+o(1))n} \max_{0\le i<n} \prod_{j=i}^{n-1} m_j\}$ at generation n with very small probability.

Proof of Proposition 2.1. Let $\ell \ge 1$ be an integer. For notational simplification, we write $\mathbb{P}^\ell(\,\cdot\,) := \mathbb{P}(\,\cdot \,|\, Z_0 = \ell)$.

Let $b_i \ge 1$, $1 \le i \le n$, be integers. We have, for $1 \le i < n$,

$$\mathbb{P}^\ell(Z_{i+1} \ge b_{i+1}) \le \mathbb{P}^\ell(Z_i \ge b_i) + \mathbb{P}^\ell\Big(\sum_{k=1}^{b_i} \nu_i^{(k)} \ge b_{i+1} \Big),$$

whereas for $i = 0$, the inequality simply says $\mathbb{P}^\ell(Z_1 \geq b_1) \leq \mathbb{P}^\ell(\sum_{k=1}^\ell \nu_0^{(k)} \geq b_1)$. By Chebyshev's inequality,

$$\mathbb{P}^\ell\Big(\sum_{k=1}^{b_i} \nu_i^{(k)} \geq b_{i+1}\Big) \leq \mathrm{e}^{-\lambda_i\, b_{i+1}} [\mathbb{E}(\mathrm{e}^{\lambda_i \nu_i})]^{b_i},$$

which, by assumption (2.1), is bounded by $\exp(-\lambda_i\, b_{i+1} + \alpha b_i \lambda_i m_i)$. Hence

$$\mathbb{P}^\ell(Z_{i+1} \geq b_{i+1}) \leq \mathbb{P}^\ell(Z_i \geq b_i) + \exp(-\lambda_i\, b_{i+1} + \alpha\lambda_i m_i b_i).$$

Let $\delta > 0$. We choose $b_0 := \ell$ and, by induction,

$$b_{i+1} := \max\{\lfloor(\alpha+\delta)m_i b_i\rfloor,\, \ell\}, \qquad 0 \leq i \leq n-1.$$

Then $\alpha\lambda_i m_i b_i = \frac{\alpha\lambda_i}{\alpha+\delta}(\alpha+\delta)m_i b_i \leq \frac{\alpha\lambda_i}{\alpha+\delta}(1+b_{i+1}) \leq \lambda_i + \frac{\alpha\lambda_i}{\alpha+\delta} b_{i+1}$, so

$$\begin{aligned} -\lambda_i\, b_{i+1} + \alpha\lambda_i m_i b_i &\leq -\lambda_i\, b_{i+1} + \lambda_i + \frac{\alpha\lambda_i}{\alpha+\delta} b_{i+1} \\ &= -\frac{\delta\lambda_i}{\alpha+\delta} b_{i+1} + \lambda_i \\ &\leq -\frac{\delta\lambda_i}{\alpha+\delta}\ell + \lambda_i\,. \end{aligned}$$

Consequently, we have, for $1 \leq i \leq n-1$,

$$\mathbb{P}^\ell(Z_{i+1} \geq b_{i+1}) \leq \mathbb{P}^\ell(Z_i \geq b_i) + \exp\Big(-\frac{\delta\lambda_i}{\alpha+\delta}\ell + \lambda_i\Big),$$

whereas $\mathbb{P}^\ell(Z_1 \geq b_1) \leq \mathrm{e}^{-\lambda_0\, b_1}[\mathbb{E}(\mathrm{e}^{\lambda_0\nu_0})]^\ell \leq \exp(-\frac{\delta\lambda_0}{\alpha+\delta}\ell + \lambda_0)$. Summing over i, we obtain:

$$\begin{aligned} \mathbb{P}^\ell(Z_n \geq b_n) &\leq \sum_{i=0}^{n-1} \exp\Big(-\frac{\delta\lambda_i}{\alpha+\delta}\ell + \lambda_i\Big) \\ &\leq n\exp\Big(-\frac{\delta\ell}{\alpha+\delta}\min_{0\leq i\leq n-1}\lambda_i + \max_{0\leq i\leq n-1}\lambda_i\Big). \end{aligned}$$

By induction in n, $b_n \leq \max\{\ell,\, \max_{0\leq i\leq n-1}[(\alpha+\delta)^{n-i}\prod_{j=i}^{n-1} m_j]\ell\}$, which is bounded by $\max\{\ell,\, (\alpha+\delta)^n \ell \max_{0\leq i\leq n-1}\prod_{j=i}^{n-1} m_j\}$. The proposition follows immediately. $\square$

3 Proof of Theorem 1.1

The proof of the theorem relies on the following elementary result, which explains the presence of the constant $I(a,\, x) := \frac{x^2}{2(1-a)} - 1$ in the theorem.

Lemma 3.1. *Let $x > 0$ and $(1 - \frac{x^2}{2})^+ < a < 1$. We have, for any $t > 0$,*

$$\frac{1}{t} \sup_{s\in(0,\,t),\; y\le xt:\; (t-s)-\frac{(xt-y)^2}{2(t-s)}=at} \Big(s - \frac{y^2}{2s}\Big) \;=\; -I(a,\, x)\,. \tag{3.1}$$

$$\frac{1}{t} \sup_{s\in(0,\,t),\; y\in\mathbb{R},\; z\ge xt:\; (t-s)-\frac{(z-y)^2}{2(t-s)}\ge at} \Big(s - \frac{y^2}{2s}\Big) \;=\; -I(a,\, x)\,. \tag{3.2}$$

Proof. Clearly, (3.2) is a consequence of (3.1): It suffices to observe that for given $(s,\, z)$, the supremum in $y \in \mathbb{R}$ is the supremum in $y \in (-\infty,\, z]$.

The proof of (3.1) is elementary: The maximizer is $s_* = \frac{(1-a)[x^2-2(1-a)]}{x^2-2(1-a)^2}t$, $y_* = \frac{x}{1-a}s_*$, which is the unique root of the gradient of the Lagrangian, and the supremum is not reached at the boundary. $\square$

We often use the elementary Gaussian tail estimate:

$$\mathbb{P}(|\mathscr{N}| \ge x) \le \exp\Big(-\frac{x^2}{2\,\mathrm{Var}(\mathscr{N})}\Big), \qquad x \ge 0,$$

for all mean-zero non-degenerate Gaussian random variable $\mathscr{N}$. As a consequence, for $x \in \mathbb{R}$ and $y \ge 0$,

$$\mathbb{P}(|\mathscr{N} - x| \le y) \le \exp\Big(-\frac{x^2}{2\,\mathrm{Var}(\mathscr{N})} + \frac{|x|\,y}{\mathrm{Var}(\mathscr{N})}\Big). \tag{3.3}$$

3.1 Lower bound

The strategy of the lower bound in Theorem 1.1 is as follows: Let $\varepsilon > 0$. Let $s_* = \frac{(1-a)[x^2-2(1-a)]}{x^2-2(1-a)^2}t$ and $y_* = \frac{x}{1-a}s_*$ be the maximizer in (3.1) of Lemma 3.1. Let the BBM reach $[y_*,\, \infty)$ at time s_* (which, by (1.1), happens with probability at least $\exp[-(1+\varepsilon)(\frac{y_*^2}{2s_*} - s_*)] = \mathrm{e}^{-(1+\varepsilon)I(a,\,x)t}$ for all sufficiently large t), then after time s_* the system behaves "normally" in the sense that by (1.2), with probability at least $1-\varepsilon$ for all sufficiently large t, the number of descendants positioned in $[xt, \infty)$ at time t of the particle positioned in $[y_*, \infty)$ at time s_* is at least $\exp\{(1-\varepsilon)[(t-s_*)-\frac{(xt-y_*)^2}{2(t-s_*)}]\}$ (which is $\mathrm{e}^{(1-\varepsilon)at}$); note that the condition $0 < \frac{xt-y_*}{t-s_*} < 2^{1/2}$ in (1.2) is automatically satisfied. Consequently, for all sufficiently large t,

$$\mathbb{P}\Big(N(t,x) \geq \mathrm{e}^{(1-\varepsilon)at}\Big) \geq (1-\varepsilon)\, \mathrm{e}^{-(1+\varepsilon)I(a,\,x)t} .$$

Since $\varepsilon > 0$ can be as small as possible, this yields the lower bound in Theorem 1.1. $\square$

3.2 Upper bound

Let $\frac{1}{2} < \delta < 1$. We discretize time by splitting time interval $[0, t]$ into intervals of length t^δ: Let $s_i := it^\delta$ for $0 \leq i \leq M := t^{1-\delta}$. For notational simplification, we treat M as an integer (upper integer part should be used for a rigorous treatment; a similar remark applies later when we discretize space).

We first throw away some uninteresting situations. Let $C > 0$ be a constant, and let $E_1(t)$ denote the event that all the particles in the BBM lie in $[-Ct, Ct]$ at time s_i, for all $1 \leq i \leq M$. The expected number of particles that fall out of the interval is bounded by $\sum_{i=1}^M \mathrm{e}^{s_i}\, \mathbb{P}(\sup_{u\in[0,\, s_i]} |B(u)| \geq Ct)$, where $(B(u),\, u \geq 0)$ denotes a standard one-dimensional Brownian motion. We choose and fix the constant $C > 0$ (whose value depends on a and x) such that this expected number is $o(\mathrm{e}^{-I(a,\,x)t})$, $t \to \infty$. By the Markov inequality,

$$\mathbb{P}(E_1(t)^c) = o(\mathrm{e}^{-I(a,\,x)t}), \qquad t \to \infty .$$

Let $E_2(t)$ be the event that for all $0 \leq i < M$, any particle in the BBM alive at time s_i has a total number of descendants fewer than $t^2 \mathrm{e}^{t^\delta}$ at time s_{i+1}. This number has the geometric distribution of parameter $\mathrm{e}^{-(s_{i+1}-s_i)} = \mathrm{e}^{-t^\delta}$, i.e., it equals k with probability $(1-\mathrm{e}^{-t^\delta})^{k-1}\mathrm{e}^{-t^\delta}$ for all integers $k \ge 1$. By the Markov inequality again, we have

$$\mathbb{P}(E_2(t)^c) \le \sum_{i=0}^{M-1} \mathrm{e}^{s_i} \sum_{k \ge t^2 \mathrm{e}^{t^\delta}} (1-\mathrm{e}^{-t^\delta})^{k-1}\mathrm{e}^{-t^\delta} = o(\mathrm{e}^{-I(a,\,x)t}), \qquad t \to \infty\,.$$

Consequently, for $t \to \infty$,

$$\mathbb{P}(N(t,x) \ge \mathrm{e}^{at}) \le \mathbb{P}(N(t,x) \ge \mathrm{e}^{at},\ E_1(t),\ E_2(t)) + o(\mathrm{e}^{-I(a,\,x)t})\,. \tag{3.4}$$

We now discretize space. Let $\delta' \in (0,\,\delta)$. [Later, we are going to assume $\delta' < 2\delta - 1$.] Let $\varepsilon > 0$ be a small constant (which will ultimately go to 0). Space interval $[-Ct,\, Ct]$ is split into intervals of length $t^{\delta'}$: Let $x_k := kt^{\delta'}$ for $-t^{1-\delta'} \le k \le t^{1-\delta'}$. We call $f : \{s_i,\, 0 \le i \le M\} \to \{x_j := jt^{\delta'},\, -Ct^{1-\delta'} \le j \le Ct^{1-\delta'}\}$ a path if

$$f(0) = 0, \qquad f(s_M) \ge (x-\varepsilon)t\,.$$

The total number of paths is bounded by $(2Ct^{1-\delta'}+1)^{t^{1-\delta}} = \mathrm{e}^{o(t)}$, $t \to \infty$.

Consider the BBM. For $1 \le i \le M$, a particle at time s_i is said to follow the path f until time s_i if for all $0 \le j \le i$, the ancestor of the particle at time s_j lies in $[f(s_j) - t^{\delta'},\, f(s_j) + t^{\delta'}]$. Let

$$Z_i(f) := \text{number of particles following the path } f \text{ until time } s_i\,.$$

On the event $E_1(t)$, we have (using the fact that $xt - t^{\delta'} \ge (x-\varepsilon)t$ for all large t)

$$N(t,x) \le \sum_f Z_M(f) \le \#(\text{paths}) \max_f Z_M(f),$$

where $\sum_f$ and $\max_f$ denote sum and maximum, respectively, over all possible paths f, and $\#(\text{paths})$ stands for the total number of paths.

Let $a' \in (0,\, a)$. Since $\#(\text{paths}) = \mathrm{e}^{o(t)}$ (for $t \to \infty$), it follows that for all sufficiently large t (say $t \ge t_0$), on the event $\{N(t,x) \ge \mathrm{e}^{at}\} \cap E_1(t)$, there exists a path f such that $Z_M(f) \geq \mathrm{e}^{a't}$. Accordingly, for $t \geq t_0$,

$$\begin{aligned}\mathbb{P}(N(t,x) \geq \mathrm{e}^{at},\ E_1(t),\ E_2(t)) &\leq \sum_f \mathbb{P}(Z_M(f) \geq \mathrm{e}^{a't},\ E_2(t)) \\ &\leq \mathrm{e}^{o(t)} \max_f \mathbb{P}(Z_M(f) \geq \mathrm{e}^{a't},\ E_2(t))\,.\end{aligned}$$

In view of (3.4), and since a' can be as close to a as possible, the proof of the upper bound in Theorem 1.1 is reduced to showing the following: For $x > 0$ and $(1 - \frac{x^2}{2})^+ < a < 1$,

$$\limsup_{t\to\infty} \frac{1}{t} \max_f \log \mathbb{P}(Z_M(f) \geq \mathrm{e}^{at},\ E_2(t)) \leq -I(a,\, x)\,, \tag{3.5}$$

with $I(a,\, x) := \frac{x^2}{2(1-a)} - 1$ as before. [The meaning of a has slightly changed: It is, in fact, a'.]

To bound $\mathbb{P}(Z_M(f) \geq \mathrm{e}^{at},\ E_2(t))$, we distinguish two situations. A path f is said to be good if there exists $i \in [1,\, M) \cap \mathbb{Z}$ such that

$$(t - s_i) - \frac{(f(s_M) - f(s_i))^2}{2(t - s_i)} \geq (a - \varepsilon)t\,. \tag{3.6}$$

It is said to be bad if it is not good.

When the path f is good, it is easy to bound $\mathbb{P}(Z_M(f) \geq \mathrm{e}^{at},\ E_2(t))$; we can even drop $E_2(t)$ in this case: Let $i \in [1,\, M) \cap \mathbb{Z}$ be as in (3.6); since $\{Z_M(f) \geq \mathrm{e}^{at}\} \subset \{Z_i(f) \geq 1\}$, we have

$$\mathbb{P}(Z_M(f) \geq \mathrm{e}^{at}) \leq \mathbb{E}[Z_i(f)] \leq \mathrm{e}^{s_i}\, \mathbb{P}(|B(s_i) - f(s_i)| \leq t^{\delta'}), \tag{3.7}$$

with $(B(s),\, s \geq 0)$ denoting, as before, a standard Brownian motion. Since $\delta' < \delta < 1$, $s_i = it^\delta$ and $f(s_i) = O(t)$, it follows from (3.3) that

$$\mathbb{P}(|B(s_i) - f(s_i)| \leq t^{\delta'}) \leq \exp\Big(- \frac{f(s_i)^2}{2s_i} + o(t)\Big),$$

uniformly in i and in f. This yields that

$$
\begin{aligned}
& \mathbb{P}(Z_M(f) \geq \mathrm{e}^{at}) \\
\leq\ & \exp\Big(s_i - \frac{f(s_i)^2}{2s_i} + o(t)\Big) \\
\leq\ & \exp\Big\{ \sup_{s\in(0,t),\, y\in\mathbb{R},\, z\geq(x-\varepsilon)t:\ (t-s)-\frac{(z-y)^2}{2(t-s)}\geq(a-\varepsilon)t} \Big(s - \frac{y^2}{2s}\Big) + o(t)\Big\}.
\end{aligned}
$$

By (3.2) of Lemma 3.1, the supremum equals $-I(a-\varepsilon,\, x-\varepsilon)t$, as long as $\varepsilon > 0$ is sufficiently small such that $x > \varepsilon$ and that $(1 - \frac{(x-\varepsilon)^2}{2})^+ < a - \varepsilon$. Hence, uniformly in all good paths f,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Z_M(f) \geq \mathrm{e}^{at}) \leq -I(a-\varepsilon,\, x-\varepsilon)\,.$$

Since $I(a-\varepsilon,\, x-\varepsilon)$ can be as close to $I(a,\, x)$ as possible, this will settle the case of good paths f. To prove (3.5), it suffices to check that, uniformly in all bad paths f,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Z_M(f) \geq \mathrm{e}^{at},\ E_2(t)) = -\infty\,. \tag{3.8}$$

Let $\varepsilon' \in (0,\, \varepsilon)$. For any path f, define

$$\tau = \tau(f,\, t) := \inf\{i:\ 1 \leq i \leq M,\ Z_i(f) \geq \mathrm{e}^{\varepsilon' t}\}, \qquad \inf\varnothing := \infty\,.$$

On the event $\{Z_M(f) \geq \mathrm{e}^{at}\}$, we have $\tau < \infty$, and $Z_\tau(f) \leq t^2 \mathrm{e}^{t^\delta} \mathrm{e}^{\varepsilon' t}$ on the event $\{\tau < \infty\} \cap E_2(t)$. Hence

$$
\begin{aligned}
& \mathbb{P}(Z_M(f) \geq \mathrm{e}^{at},\ E_2(t)) \\
\leq\ & \mathbb{P}(Z_M(f) \geq \mathrm{e}^{at},\ Z_\tau(f) \leq t^2 \mathrm{e}^{t^\delta + \varepsilon' t},\ E_2(t)) \\
\leq\ & \sum_{i=1}^{M} \mathbb{P}(Z_M(f) \geq \mathrm{e}^{at},\ \mathrm{e}^{\varepsilon' t} \leq Z_i(f) \leq t^2 \mathrm{e}^{t^\delta + \varepsilon' t},\ E_2(t)) \\
=\ & \sum_{i=1}^{M} \sum_{\ell=\mathrm{e}^{\varepsilon' t}}^{t^2 \mathrm{e}^{t^\delta + \varepsilon' t}} \mathbb{P}(Z_M(f) \geq \mathrm{e}^{at},\ Z_i(f) = \ell,\ E_2(t))\,. \qquad (3.9)
\end{aligned}
$$

Let us have a close look at the probability $\mathbb{P}(Z_M(f) \geq e^{at},\ Z_i(f) = \ell,\ E_2(t))$ on the right-hand side, for $1 \leq i \leq M$ and $e^{\varepsilon' t} \leq \ell \leq t^2 e^{t^\delta + \varepsilon' t}$. The sequence $Z_{i+j}(f)$, for $0 \leq j \leq M-i$, can be written as $Z_{j+1}(f) = \sum_{k=1}^{Z_j(f)} \nu_k^{(j)}$, where for each j, $\nu_k^{(j)}$, $k \geq 1$, would be i.i.d. if the particle at time s_{i+j} were exactly positioned at $f(s_{i+j})$ rather than only lying in the interval $[f(s_{i+j}) - t^{\delta'},\ f(s_{i+j}) + t^{\delta'}]$. However, $\nu_k^{(j)}$ is stochastically smaller than or equal to $\widetilde{\nu}^{(j)}$, the number of particles in a BBM, starting at position $f(s_{i+j})$, that lie in $[f(s_{i+j+1}) - 2t^{\delta'},\ f(s_{i+j+1}) + 2t^{\delta'}]$ at time t^δ. So we can make a coupling for $(Z_{i+j}(f),\ 0 \leq j \leq M-i)$ and a new process $(\widetilde{Z}_{i+j}(f),\ 0 \leq j \leq M-i)$, which satisfies $\widetilde{Z}_{j+1}(f) = \sum_{k=1}^{\widetilde{Z}_j(f)} \widetilde{\nu}_k^{(j)}$, where for each j, $\widetilde{\nu}_k^{(j)}$, $k \geq 1$, are i.i.d. having the law of $\widetilde{\nu}^{(j)}$, such that $Z_{i+j}(f) \leq \widetilde{Z}_{i+j}(f)$ for all $0 \leq j \leq M-i$. Since $(\widetilde{Z}_{i+j}(f),\ 0 \leq j \leq M-i)$ is an inhomogeneous Galton–Watson process, we can apply Proposition 2.1.

Write $\Delta f(s_{i+j}) := f(s_{i+j+1}) - f(s_{i+j}) = O(t)$. Note that by (3.3),

$$\begin{aligned}
\mathbb{E}(\widetilde{\nu}^{(j)}) &= e^{t^\delta}\, \mathbb{P}(|B(t^\delta) - \Delta f(s_{i+j})| \leq 2t^{\delta'}) \\
&\leq \exp\Big(t^\delta - \frac{(\Delta f(s_{i+j}))^2}{2t^\delta} + O(t^{1+\delta'-\delta})\Big) \\
&=: m_j\,,
\end{aligned}$$

with $O(t^{1+\delta'-\delta})$ being uniform in i, j and f. In order to apply Proposition 2.1, we need to bound $\max_{0 \leq k < M-i} \prod_{j=k}^{M-i-1} m_j$, as well as to find a convenient λ_k satisfying condition (2.1) in Proposition 2.1.

Recall that $M := t^{1-\delta}$. We have, for $0 \leq k < M-i$,

$$\begin{aligned}
\prod_{j=k}^{M-i-1} m_j &= \exp\Big((M-i-k)t^\delta - \frac{1}{2t^\delta} \sum_{j=k}^{M-i-1} (\Delta f(s_{i+j}))^2 + O(t^{2+\delta'-2\delta})\Big) \\
&= \exp\Big((M-i-k)t^\delta - \frac{1}{2t^\delta} \sum_{j=k}^{M-i-1} (\Delta f(s_{i+j}))^2 + o(t)\Big),
\end{aligned}$$

as long as $2 + \delta' - 2\delta < 1$ (which is equivalent to $\delta' < 2\delta - 1$), which we take for granted from now on. By the Cauchy–Schwarz inequality,

$$\sum_{j=k}^{M-i-1} (\Delta f(s_{i+j}))^2 \geq \frac{(f(s_M) - f(s_{i+k}))^2}{M-i-k} .$$

Recall that $s_j := jt^\delta$ and that $Mt^\delta = t$. Hence

$$\begin{aligned}\prod_{j=k}^{M-i-1} m_j &\leq \exp\Big((M-i-k)t^\delta - \frac{(f(s_M) - f(s_{i+k}))^2}{2(M-i-k)t^\delta} + o(t)\Big) \\ &= \exp\Big((t - s_{i+k}) - \frac{(f(s_M) - f(s_{i+k}))^2}{2(t - s_{i+k})} + o(t)\Big) .\end{aligned}$$

If f is a bad path, then by definition of good paths in (3.6), $(t - s_{i+k}) - \frac{(f(s_M)-f(s_{i+k}))^2}{2(t-s_{i+k})} < (a - \varepsilon)t$ for all k. Thus

$$\max_{0\leq k<M-i} \prod_{j=k}^{M-i-1} m_j \leq \mathrm{e}^{(a-\varepsilon)t+o(t)} . \tag{3.10}$$

In order to apply Proposition 2.1, we still need to find a convenient λ_k satisfying condition (2.1) in the proposition. Let $\alpha > 1$. There exists $r > 0$ sufficiently small such that $\mathrm{e}^y \leq 1 + \alpha y$ for all $y \in [0, r]$. On the event $E_2(t)$, we have $\widetilde{\nu}^{(j)} \leq t^2 \mathrm{e}^{t^\delta}$ by definition. Let $\lambda_j := \mathrm{e}^{-2t^\delta}$. Then $\lambda_j \widetilde{\nu}^{(j)} \leq r$ for all sufficiently large t (and we will be working with such large t); hence $\mathrm{e}^{\lambda_j \widetilde{\nu}^{(j)}} \leq 1 + \alpha\lambda_j \widetilde{\nu}^{(j)}$, which yields that

$$\mathbb{E}(\mathrm{e}^{\lambda_j \widetilde{\nu}^{(j)}}) \leq 1 + \alpha\lambda_j \mathbb{E}(\widetilde{\nu}^{(j)}) \leq 1 + \alpha\lambda_j m_j \leq \mathrm{e}^{\alpha\lambda_j m_j} .$$

In words, condition (2.1) of Proposition 2.1 is satisfied with the choice of $\lambda_j := \mathrm{e}^{-2t^\delta}$. Applying Proposition 2.1 to $n := M - i$, we see that for all sufficiently large t and uniformly in $1 \leq i \leq M$ and $\mathrm{e}^{\varepsilon' t} \leq \ell \leq t^2 \mathrm{e}^{t^\delta + \varepsilon' t}$ (recalling that $\varepsilon' < \varepsilon$ and $\varepsilon' < a$)

$$\mathbb{P}(Z_M(f) \geq \mathrm{e}^{at},\ E_2(t) \,|\, Z_i(f) = \ell) \leq M \exp(-c\,\ell\,\mathrm{e}^{-2t^\delta}) ,$$

where $c > 0$ is an unimportant constant that does not depend on t. A fortiori, $\mathbb{P}(Z_M(f) \geq \mathrm{e}^{at},\ E_2(t),\ Z_i(f) = \ell) \leq M \exp(-c\,\ell\,\mathrm{e}^{-2t^\delta})$. By (3.9), we obtain

$$\mathbb{P}(Z_M(f) \geq \mathrm{e}^{at},\ E_2(t)) \leq M^2\, t^2 \mathrm{e}^{t^\delta + \varepsilon' t} \exp(-c\,\ell\,\mathrm{e}^{-2t^\delta}).$$

This yields (3.8), and completes the proof of the upper bound in Theorem 1.1. $\square$

4 Application to discrete Gaussian free fields

Let $V_N := \{1, \ldots, N\}^2$, and ∂V_N be the inner boundary of V_N which is the set of points in V_N having a nearest neighbour outside. Consider the two-dimensional discrete Gaussian free field (GFF) $\Phi = (\Phi(x),\, x \in V_N)$ in V_N with zero boundary conditions as follows: Φ is a collection of jointly mean-zero Gaussian random variables with $\Phi(x) = 0$ for $x \in \partial V_N$ and with covariance given by the discrete Green's function

$$G_N(x,\, y) := \mathbb{E}_x\Big(\sum_{i=0}^{\tau_{\partial V_N}} \mathbf{1}_{\{S_i = y\}} \Big), \qquad x,\, y \in V_N \backslash \partial V_N\,,$$

where $(S_i,\, i \geq 0)$ is a two-dimensional simple random walk on $\mathbb{Z}^2$, $\tau_{\partial V_N}$ the first time the walk hits ∂V_N, and $\mathbb{E}_x$ is expectation with respect to $\mathbb{P}_x$ under which $\mathbb{P}_x(S_0 = x) = 1$.

In the rest of the paper, we write

$$\gamma := \Big(\frac{2}{\pi}\Big)^{1/2}. \tag{4.1}$$

This constant originates from the fact that $G_N(0,\, 0) = \gamma^2 \log N + O(1)$, $N \to \infty$ (Lawler [13], Theorem 1.6.6). The maximum of Φ on V_N was studied by Bolthausen, Deuschel and Giacomin [3], who proved that

$$\lim_{N\to\infty} \frac{1}{\log N} \max_{x \in V_N} \Phi(x) = 2\gamma, \qquad \text{in probability.}$$

[It is possible to have a further development for $\max_{x\in V_N} \Phi(x)$ until constant order of magnitude; see Bramson, Ding and Zeitouni [7].] Daviaud [9] was interested in the intermediate level sets

$$\mathscr{H}_N(\eta) := \{x \in V_N : \Phi(x) \geq 2\gamma\eta \log N\}, \qquad 0 < \eta < 1,$$

and proved that for all $0 < \eta < 1$,

$$\#\mathscr{H}_N(\eta) = N^{2(1-\eta^2)+o(1)}, \qquad \text{in probability,}$$

where $\#\mathscr{H}_N(\eta)$ denotes the cardinality of $\mathscr{H}_N(\eta)$. Recently, Biskup and Louidor [2] established the scaling limit of $\mathscr{H}_N(\eta)$ upon an encoding via a point measure.

We study the deviation probability $\mathbb{P}(\#\mathscr{H}_N(\eta) \geq N^{2a})$, for $1-\eta^2 < a < 1$.

Theorem 4.1. *Let $\eta \in (0, 1)$ and $a \in (1-\eta^2, 1)$. We have*

$$\mathbb{P}(\#\mathscr{H}_N(\eta) \geq N^{2a}) = N^{-J(a,\,\eta)+o(1)}, \qquad N \to \infty\,,$$

where

$$J(a, \eta) := \frac{2\eta^2}{1-a} - 2\,.$$

To prove Theorem 4.1, let us introduce a useful decomposition. Let $D \subset V_N$ be a square. Define

$$h_D(x) := \mathbb{E}\big(\Phi(x)\,|\,\mathscr{F}_{\partial D}\big), \qquad x \in D,$$

where $\mathscr{F}_A := \sigma(\Phi(x),\, x \in A)$ for all $A \subset V_N$, and ∂D denotes the inner boundary of D. Let

$$\Phi^D(x) := \Phi(x) - h_D(x), \qquad x \in D. \tag{4.2}$$

Then $(\Phi^D(x),\, x \in D)$ is independent of $\mathscr{F}_{\partial D \cup D^c}$; in particular, $(\Phi^D(x),\, x \in D)$ and $(h_D(x),\, x \in D)$ are independent. Moreover, $(\Phi^D(x),\, x \in D)$ is a GFF in D in the sense that it is a mean-zero Gaussian field vanishing on ∂D

with covariance $\mathrm{Cov}(\Phi^D(x),\, \Phi^D(y)) = \mathbb{E}_x(\sum_{i=0}^{\tau_{\partial D}} \mathbf{1}_{\{S_i=y\}})$, for $x,\, y \in D \backslash \partial D$, where $\tau_{\partial D}$ is the first hitting time at the inner boundary ∂D by the simple random walk (S_i).

Write x_D for the centre of D. Let

$$\phi_D := \mathbb{E}(\Phi(x_D) \,|\, \mathscr{F}_{\partial D}) = h_D(x_D).$$

[Degenerate case: $\phi_D = \Phi(x)$ if $D = \{x\}$.] We frequently use an elementary inequality: By Bolthausen, Deuschel and Giacomin [3] p. 1687,

$$\mathrm{Var}(h_D(x) - \phi_D) \le 2 \sup_{y \in \partial D} [a(x-y) - a(x_D - y)],$$

where $a(z) := \sum_{n=0}^{\infty} [\mathbb{P}_0(S_n = 0) - \mathbb{P}_0(S_n = z)]$ with $(S_n,\, n \ge 0)$ denoting as before a simple random walk on $\mathbb{Z}^2$. Since $a(z) = \gamma^2 \log |z| + O(1)$, $|z| \to \infty$ ([13], Theorem 1.6.2), there exists a constant $c_1 > 0$ (independent of N) such that for all square $D \subset V_N$,

$$\mathrm{Var}(h_D(x) - \phi_D) \le c_1\,, \qquad x \in D\,. \tag{4.3}$$

It is possible to estimate $\mathrm{Var}(\phi_D)$. Let $\gamma := (\frac{2}{\pi})^{1/2}$ as in (4.1). By equation (7) and Lemma 1 of Bolthausen, Deuschel and Giacomin [3], there exists a constant $c_2 > 0$ such that for all square $D \subset V_N$ with side length m,

$$\mathrm{Var}(\phi_D) \le \gamma^2 \log(\frac{N}{m}) + c_2, \tag{4.4}$$

and for any $0 < \delta < \frac{1}{2}$, there exists $c_3(\delta) > 0$ such that for all square $D \subset V_N$ with $\mathrm{dist}(x_D,\, V_N^c) \ge \delta N$,

$$\mathrm{Var}(\phi_D) \ge \gamma^2 \log(\frac{N}{m}) - c_3(\delta). \tag{4.5}$$

[Degenerate case: $m := 1$ if D is a singleton.]

The proof of Theorem 4.1 uses the same ideas as the proof of Theorem 1.1 in Section 3, with some appropriate modifications. Again, for the sake of clarity, we prove the upper and the lower bounds in distinct paragraphs.

The proof is based on the following elementary fact: For $0 < \eta < 1$ and $1-\eta^2 < a < 1$,

$$\sup_{(s,b,z):\, 0<s<1,\, z\ge\eta,\, s-\frac{(z-b)^2}{s}\ge a} [(1-s) - \frac{b^2}{1-s}] = -\Big(\frac{\eta^2}{1-a} - 1\Big) . \tag{4.6}$$

[This is (3.2) of Lemma 3.1 after a linear transform. The maximizer is $s^* := \frac{a\eta^2}{\eta^2-(1-a)^2}$, $b^* := \frac{[\eta^2-(1-a)]\eta}{\eta^2-(1-a)^2}$, $z^* = \eta$.]

As in the proof for BBM, for notational simplification, we treat several counting quantities (such as $(\log N)^{1-\delta}$ and N^{1-s_i} below) as integers.

4.1 Upper bound

Let $\frac{5}{6} < \delta < 1$. Let $L = L(N) := (\log N)^{1-\delta}$. Let $s_0 := 1 > s_1 > ... > s_L := 0$ with $s_i - s_{i+1} = (\log N)^{-(1-\delta)}$.

For $0 \le i < L$, let $\mathscr{D}_{s_i}(N)$ denote the partition of N^{2-2s_i} squares of side length N^{s_i} of V_N. [In particular, $\mathscr{D}_{s_0}(N) = \{V_N\}$, the singleton V_N.] Let $\mathscr{D}_{s_L}(N) := \{\{x\}, x \in V_N\}$, the family of singletons of V_N. [So for $D = \{x\} \in \mathscr{D}_{s_L}(N)$, $\phi_D = \Phi(x)$.] Let $C > 0$ be a constant, and let

$$\mathscr{E}_1(N) := \Big\{ |\phi_D| \le C \log N, \ \forall 1 \le i \le L, \forall D \in \mathscr{D}_{s_i}(N) \Big\} .$$

This is the analogue for GFF of the event $E_1(t)$ in Section 3.2. Since $\mathrm{Var}(\phi_D) \le \gamma^2 \log N + c_2$ (see (4.4)), we can choose $C > 0$ sufficiently large such that

$$\mathbb{P}(\mathscr{E}_1(N)^c) = o(N^{-J(a,\eta)}), \qquad N \to \infty .$$

Let $\varrho \in (\frac{1}{2}, \frac{3}{2} - \delta)$. Let

$$\mathscr{E}_2(N) := \Big\{ \max_{B \in \mathtt{ch}(D)} |h_D(x_B) - \phi_D| \le (\log N)^\varrho, \ \forall 1 \le i < L, \forall D \in \mathscr{D}_{s_i}(N) \Big\} ,$$

where x_B denotes as before the centre of the square B, and for all $D \in \mathscr{D}_{s_i}(N)$ with $1 \le i < L$,

$$\mathtt{ch}(D) := \{B \subset D \text{ with } B \in \mathscr{D}_{s_{i+1}}(N)\} . \tag{4.7}$$

[In words, the elements in $\mathtt{ch}(D)$ play the role of children in the genealogical tree of BBM.] By (4.3), $\mathrm{Var}(h_D(x) - \phi_D) \le c_1$ for all $x \in D$, which allows to see that

$$\mathbb{P}(\mathscr{E}_2(N)^c) = o(N^{-J(a,\eta)}), \qquad N \to \infty .$$

Consequently, the following analogue for GFF of (3.4) holds: for $N \to \infty$,

$$\begin{aligned} & \mathbb{P}(\#\mathscr{H}_N(\eta) \ge N^{2a}) \\ \le \ & \mathbb{P}(\#\mathscr{H}_N(\eta) \ge N^{2a},\ \mathscr{E}_1(N),\ \mathscr{E}_2(N)) + o(N^{-J(a,\eta)}) . \end{aligned} \tag{4.8}$$

Let us discretize space. Let $\varepsilon > 0$ be a small constant such that $a - \varepsilon > 1 - (\eta - \varepsilon)^2$. Let $\delta' \in (0, \varrho)$. Space interval $[-C \log N, C \log N]$ is split into intervals of length $(\log N)^{\delta'}$. We call $g : \{s_i, 0 \le i \le L\} \to \{\frac{j}{(\log N)^{1-\delta'}}, -C(\log N)^{1-\delta'} \le j \le C(\log N)^{1-\delta'}\}$ a path if

$$g(s_0) = 0, \qquad g(s_L) \ge \eta - \varepsilon .$$

The total number of paths is $N^{o(1)}$ when $N \to \infty$.

Define sets of squares $\mathbf{Z}_0(g) := \{V_N\}$ (the singleton) and for $1 \le i \le L$,

$$\mathbf{Z}_i(g) := \Big\{ D \in \mathscr{D}_{s_i}(N) : |\phi_{D_k} - g(s_k) 2\gamma \log N| \le 2\gamma (\log N)^{\delta'},\ \forall 1 \le k \le i \Big\} ,$$

where D_k denotes the unique square in $\mathscr{D}_{s_k}(N)$ containing D (so $D_i = D$ for $D \in \mathscr{D}_{s_i}(N)$). We write

$$Z_i(g) := \#\mathbf{Z}_i(g), \qquad 0 \le i \le L ,$$

the cardinality of $\mathbf{Z}_i(g)$. On the event $\mathscr{E}_1(N)$, we have $\#\mathscr{H}_N(\eta) \le \sum_g Z_L(g)$, where $\sum_g$ sums over all possible paths g.

Let $a' \in (0, a)$. For all sufficiently large N,

$$\begin{aligned} & \mathbb{P}(\#\mathscr{H}_N(\eta) \ge N^{2a},\ \mathscr{E}_1(N),\ \mathscr{E}_2(N)) \\ \le \ & \#(\text{paths}) \max_g \mathbb{P}(Z_L(g) \ge N^{2a'},\ \mathscr{E}_2(N)) , \end{aligned}$$

where $\max_g$ denotes maximum over all possible paths g, and #(paths) stands for the total number of paths, which is $N^{o(1)}$ when $N \to \infty$. In view of (4.8), the proof of the upper bound in Theorem 4.1 is reduced to showing the following: For $0 < \eta < 1$ and $1 - \eta^2 < a < 1$,

$$\limsup_{N\to\infty} \frac{1}{\log N} \max_g \log \mathbb{P}(Z_L(g) \geq N^{2a},\ \mathscr{E}_2(N)) \leq -J(a,\, \eta)\,, \tag{4.9}$$

with $J(a,\, \eta) := \frac{2\eta^2}{1-a} - 2$ as before.

A path g is said to be good if there exists $i \in [1,\, L) \cap \mathbb{Z}$ such that

$$s_i - \frac{[g(s_i) - g(s_L)]^2}{s_i} \geq a - \varepsilon\,. \tag{4.10}$$

[Since $a - \varepsilon > 1 - (\eta - \varepsilon)^2$, it is clear that $g(s_i) \neq 0$ in this case.] The path is said to be bad if it is not good.

Let g be a good path. Let $i \in [1,\, L) \cap \mathbb{Z}$ be as in (4.10). We have the following analogue for GFF of (3.7):

$$\mathbb{P}(Z_L(g) \geq N^{2a}) \leq \sum_{D \in \mathscr{D}_{s_i}(N)} \mathbb{P}\Big\{|\phi_D| \geq |g(s_i)| 2\gamma \log N - 2\gamma (\log N)^{\delta'}\Big\}\,.$$

Since $g(s_i) \neq 0$, we have $|g(s_i)|2\gamma \log N - 2\gamma(\log N)^{\delta'} \geq 0$ by definition of g. By (4.4), $\mathrm{Var}(\phi_D) \leq (1 - s_i)\gamma^2 \log N + c_2$, so for $D \in \mathscr{D}_{s_i}(N)$,

$$\begin{aligned} & \mathbb{P}\Big\{|\phi_D| \geq |g(s_i)| 2\gamma \log N - 2\gamma (\log N)^{\delta'}\Big\} \\ \leq\ & \exp\Big(- \frac{(|g(s_i)|2\gamma \log N - 2\gamma(\log N)^{\delta'})^2}{2[(1 - s_i)\gamma^2 \log N + c_2]}\Big) \\ =\ & \exp\Big(- \frac{2g^2(s_i)}{1 - s_i} \log N + o(\log N)\Big)\,, \end{aligned}$$

uniformly in $i \in [1,\, L) \cap \mathbb{Z}$ (recalling that $\frac{1}{2} < \delta < 1$ and that $0 < \delta' < \delta$). Since $\#\mathscr{D}_{s_i}(N) = N^{2(1-s_i)}$, this yields

$$\begin{aligned} \mathbb{P}(Z_L(g) \geq N^{2a}) \ &\leq\ \exp\Big([2(1 - s_i) - \frac{2g^2(s_i)}{1 - s_i}] \log N + o(\log N)\Big) \\ &\leq\ \exp\Big(2 \sup_{(s,\, b,\, z)} [(1 - s) - \frac{b^2}{1 - s}] \log N + o(\log N)\Big), \end{aligned}$$

the supremum being over (s, b, z) satisfying $0 < s < 1$, $z \geq \eta - \varepsilon$ and $s - \frac{(z-b)^2}{s} \geq a - \varepsilon$. By (4.6), we get that uniformly in good paths g,

$$\limsup_{N\to\infty} \frac{1}{\log N} \log \mathbb{P}(Z_L(g) \geq N^{2a}) \leq -J(a-\varepsilon, \eta-\varepsilon) .$$

As such, the proof of (4.9) is reduced to checking that

$$\lim_{N\to\infty} \frac{1}{\log N} \max_{g \text{ bad path}} \log \mathbb{P}(Z_L(g) \geq N^{2a}, \mathscr{E}_2(N)) = -\infty . \tag{4.11}$$

Let, for $0 \leq i < L$ and $D \in \mathscr{D}_{s_i}(N)$,

$$\nu_i^{(D)} := \sum_{B \in \mathtt{ch}(D)} \mathbf{1}_{\{|\phi_B - g(s_{i+1}) 2\gamma \log N| \leq 2\gamma (\log N)^{\delta'}\}}, \tag{4.12}$$

where $\mathtt{ch}(D)$ is as in (4.7). Then

$$Z_{i+1}(g) = \sum_{D \in \mathbf{Z}_i(g)} \nu_i^{(D)}, \qquad 0 \leq i < L . \tag{4.13}$$

This gives a branching-type process, except that there is lack of independence. So we are going to replace $\nu_i^{(D)}$ by something slightly different.

Consider two squares $B \subset D$ in V_N. Let $\Phi^D(x) := \Phi(x) - h_D(x)$, $x \in D$, as in (4.2). Define

$$\phi_B^D := \mathbb{E}(\Phi^D(x_B) \,|\, \mathscr{F}_{\partial B}^D) ,$$

where $\mathscr{F}_{\partial B}^D := \sigma(\Phi^D(y), y \in \partial B)$, and x_B is as before the centre of B. Then ϕ_B^D is independent of $h_D(x_B)$, and

$$\phi_B^D = \phi_B - h_D(x_B). \tag{4.14}$$

We now replace $\nu_i^{(D)}$ (defined in (4.12)) by

$$\widetilde{\nu}_i^{(D)} := \sum_{B \in \mathtt{ch}(D)} \mathbf{1}_{\{|\phi_B^D - (g(s_{i+1}) - g(s_i)) 2\gamma \log N| \leq 4\gamma (\log N)^{\delta'} + (\log N)^{\varrho}\}}.$$

Conditionally on $\mathbf{Z}_i(g)$, the random variables $\widetilde{\nu}_i^{(D)}$, for $D \in \mathbf{Z}_i(g)$, are independent.

On the event $\mathscr{E}_2(N)$, we have $\nu_i^{(D)} \le \widetilde{\nu}_i^{(D)}$, which implies that

$$Z_i(g) \le \widetilde{Z}_i(g), \qquad \forall\, 0 \le i \le L,$$

where $\widetilde{Z}_0(g) := 1$ and for $0 \le i < L$,

$$\widetilde{Z}_{i+1}(g) = \sum_{\ell=1}^{\widetilde{Z}_i(g)} \widetilde{\nu}_i^{(\ell)},$$

with $\widetilde{\nu}_i^{(\ell)}$, $\ell \ge 1$, denoting independent copies of $\widetilde{\nu}_i^{(D)}$, which are independent of $\widetilde{Z}_i(g)$. As such, $(\widetilde{Z}_i(g),\, 0 \le i \le L)$ is an inhomogeneous Galton–Watson process.

Let us estimate $\mathrm{Var}(\phi_B^D)$ on the right-hand side. We may assume that $d(D,\, \partial V_N) \ge \frac{N}{4}$; otherwise we may consider a GFF Φ defined on $[-N,\, 2N]^2$ instead of V_N in the following computations of variances (because $\mathrm{Var}(\phi_B^D)$ only depends on the GFF in D).

Recall from (4.14) that $\phi_B = \phi_B^D + h_D(x_B)$, the random variables ϕ_B^D and $h_D(x_B)$ being independent. So $\mathrm{Var}(\phi_B^D) = \mathrm{Var}(\phi_B) - \mathrm{Var}(h_D(x_B))$. To estimate $\mathrm{Var}(h_D(x_B))$, we use $\mathrm{Var}(X) - \mathrm{Var}(Y) = \mathrm{Var}(X-Y) + 2\mathrm{Cov}(X-Y,\, Y)$ and $|\mathrm{Cov}(X-Y, Y)| \le [\mathrm{Var}(X-Y)\,\mathrm{Var}(Y)]^{1/2}$, as well as the fact $\mathrm{Var}(h_D(x_B) - \phi_D) \le c_1$ (see (4.3)), to see that

$$|\mathrm{Var}(h_D(x_B)) - \mathrm{Var}(\phi_D)| \le c_1 + 2\sqrt{c_1\,\mathrm{Var}(\phi_D)}. \tag{4.15}$$

Since $\mathrm{Var}(\phi_D) = \gamma^2(1-s_i)\log N + O(1)$ and $\mathrm{Var}(\phi_B) = \gamma^2(1-s_{i+1})\log N + O(1)$ uniformly in i, B and D (see (4.4) and (4.5); this is why we need to assume that $d(D,\, \partial V_N) \ge \frac{N}{4}$), we get

$$\begin{aligned} \mathrm{Var}(\phi_B^D) &= \mathrm{Var}(\phi_B) - \mathrm{Var}(h_D(x_B)) \\ &= \gamma^2(s_i - s_{i+1})\log N + O((\log N)^{1/2}) \qquad (4.16) \\ &= (1 + O(\frac{1}{(\log N)^{\delta - \frac12}}))\gamma^2(s_i - s_{i+1})\log N\,, \qquad (4.17) \end{aligned}$$

uniformly in $0 \le i < L$.

We now estimate $\mathbb{E}(\widetilde{\nu}_i)$, where $\widetilde{\nu}_i$ denotes a random variable having the distribution of $\widetilde{\nu}_i^{(D)}$ (for any $D \in \mathscr{D}_{s_i}(N)$). Applying (3.3) to $x = (g(s_{i+1}) - g(s_i))2\gamma \log N$ and $y = 4\gamma(\log N)^{\delta'} + (\log N)^{\varrho}$, and using (4.17) (noting that $\#\mathtt{ch}(D) = N^{2(s_i - s_{i+1})}$), we arrive at: uniformly in $0 \le i < L$,

$$\begin{aligned}\mathbb{E}(\widetilde{\nu}_i) \;\; &\le \;\; N^{2(s_i - s_{i+1})} \exp\Big(- \frac{2[g(s_i) - g(s_{s+i})]^2}{s_i - s_{i+1}} \log N \\ &\qquad + O((\log N)^{\frac{5}{2} - 2\delta}) + O((\log N)^{1+\varrho-\delta})\Big).\end{aligned}$$

Note that $(\log N)^{1+\varrho-\delta} = O((\log N)^{\frac{5}{2}-2\delta})$ (because $\varrho < \frac{3}{2} - \delta$). Recalling $s_L = 0$ and $L = (\log N)^{1-\delta}$, we obtain, uniformly in $1 \le j < L$,

$$\prod_{i=j}^{L-1} \mathbb{E}(\widetilde{\nu}_i) \le N^{2s_j - 2\sum_{i=j}^{L-1} [g(s_i) - g(s_{s+i})]^2/(s_i - s_{i+1}) + o(1)}.$$

[This is where the condition $\delta > \frac{5}{6}$ is needed.] By the Cauchy–Schwarz inequality, $\sum_{i=j}^{L-1} \frac{[g(s_i) - g(s_{s+i})]^2}{s_i - s_{i+1}} \ge \frac{[g(s_j) - g(s_L)]^2}{s_j}$. Since g is a bad path, this yields the following analogue for GFF of (3.10):

$$\max_{1 \le j < L} \prod_{i=j}^{L-1} \mathbb{E}(\widetilde{\nu}_i) \le N^{2(a-\varepsilon) + o(1)}.$$

On the other hand, for any $\varepsilon' \in (0, \varepsilon)$ and all sufficiently large N,

$$\mathbb{P}(\widetilde{Z}_L(g) \ge N^{2a}) \le \sum_{i=1}^{L} \sum_{\ell = N^{\varepsilon'}}^{N^{2(s_{i-1} - s_i) + \varepsilon'}} \mathbb{P}(\widetilde{Z}_L(g) \ge N^{2a}, \ \widetilde{Z}_i(g) = \ell).$$

[This is the analogue for GFF of (3.9).] To apply Proposition 2.1 to $\mathbb{P}(\widetilde{Z}_L(g) \ge N^{2a} \,|\, \widetilde{Z}_i(g) = \ell)$, we need to find the corresponding λ_j (notation of the proposition): Since $\widetilde{\nu}_j \le N^{2(s_j - s_{j+1})} = \mathrm{e}^{2(\log N)^{\delta}}$, we can take $\lambda_j := \mathrm{e}^{-3(\log N)^{\delta}}$ (in place of 3, any constant greater than 2 will do the job). Applying Proposition 2.1 to $n := L - i$, we see that for all sufficiently large N and uniformly in $1 \le i \le L$ and $N^{\varepsilon'} \le \ell \le N^{2(s_{i-1} - s_i) + \varepsilon'}$,

$$\mathbb{P}(\widetilde{Z}_L(g) \ge N^{2a} \,|\, \widetilde{Z}_i(g) = \ell) \le L \exp(-c\,\ell\,\mathrm{e}^{-3(\log N)^{\delta}}),$$

where $c > 0$ is an unimportant constant. This yields that $\mathbb{P}(\widetilde{Z}_L(g) \geq N^{2a}) \leq L^2 N^{2(s_{i-1}-s_i)+\varepsilon'} \exp(-c\, N^{\varepsilon'}\, \mathrm{e}^{-3(\log N)^\delta})$. Since $Z_L(g) \leq \widetilde{Z}_L(g)$ on $\mathscr{E}_2(N)$, this yields (4.11), and completes the proof of the upper bound in Theorem 4.1. $\square$

4.2 Lower bound

Let $0 < \eta < 1$, $0 < b < \eta$, $\varepsilon > 0$. Let $0 < \zeta < 1$.

Let $\mathscr{D}_\zeta(N)$ denote the partition of $N^{2-2\zeta}$ squares of side length N^ζ of V_N. For any $D \in \mathscr{D}_\zeta(N)$, let $\widetilde{D} := \{x \in D : d(x, \partial D) \geq \frac{1}{4} N^\zeta\}$ and

$$\begin{aligned} A_D &:= \Big\{\forall x \in \widetilde{D} :\, |h_D(x) - \phi_D| \leq \varepsilon \log N\Big\}, \\ B_D &:= \Big\{\sum_{x \in \widetilde{D}} \mathbf{1}_{\{\Phi^D(x) \geq 2\gamma(\eta-b)\log N\}} \geq N^{2a-\varepsilon}\Big\}. \end{aligned}$$

It is clear that if there exists $D \in \mathscr{D}_\zeta(N)$ such that $\phi_D \geq (2\gamma b + \varepsilon)\log N$ and that both A_D and B_D are realized, then we have $\#\mathscr{H}_N(\eta) \geq N^{2a-\varepsilon}$. Hence

$$\begin{aligned} &\mathbb{P}\Big(\#\mathscr{H}_N(\eta) \geq N^{2a-\varepsilon}\Big) \\ \geq\ &\mathbb{P}\Big(\exists D \in \mathscr{D}_\zeta(N) :\, \phi_D \geq (2\gamma b + \varepsilon)\log N,\ A_D \cap B_D\Big). \end{aligned} \tag{4.18}$$

By Daviaud [9], if $\frac{\eta-b}{\zeta} < 1$, then for any $D \in \mathscr{D}_\zeta(N)$ and $N \to \infty$,

$$\mathbb{P}\Big[\#\Big\{x \in \widetilde{D} :\, \Phi^D(x) \geq 2\gamma(\eta-b)\log N\Big\} \geq N^{2\zeta[1-\frac{(\eta-b)^2}{\zeta^2}]-\varepsilon}\Big] \to 1\,.$$

Hence, we have, for all sufficiently large N (say $N \geq N_0$), $\mathbb{P}(B_D) \geq \frac{1}{2}$ if

$$a \leq \zeta\Big(1 - \frac{(\eta-b)^2}{\zeta^2}\Big). \tag{4.19}$$

The events B_D, $D \in \mathscr{D}_\zeta(N)$, are i.i.d. and each B_D is independent of $(\phi_C,\, A_C)$, $C \in \mathscr{D}_\zeta(N)$. We now go back to (4.18), and use the fact that

$$\mathbb{P}\Big(\bigcup_{i=1}^n (A_i \cap B_i)\Big) \geq \min_{1\leq i\leq n} \mathbb{P}(B_i)\, \mathbb{P}\Big(\bigcup_{j=1}^n A_j\Big),$$

if each B_i is independent of $(A_j,\ 1 \le j \le n)$. As such, for $N \ge N_0$ (and for a satisfying (4.19)),

$$\mathbb{P}\Big(\#\mathscr{H}_N(\eta) \ge N^{2a-\varepsilon}\Big) \ge \frac{1}{2}\,\mathbb{P}\Big(\exists D \in \mathscr{D}_\zeta(N) : \phi_D \ge (2\gamma b + \varepsilon)\log N, A_D\Big)\,.$$

By (4.3) and the Gaussian tail, $\mathbb{P}(A_D^c) \le N^{2\zeta}\mathrm{e}^{-\varepsilon^2(\log N)^2/(2c_1)}$, uniformly in $D \in \mathscr{D}_\zeta(N)$. Hence $\mathbb{P}(\cup_{D\in\mathscr{D}_\zeta(N)} A_D^c) \le N^2\mathrm{e}^{-\varepsilon^2(\log N)^2/(2c_1)}$. Consequently, for a satisfying (4.19), any constant $c > 0$ and all sufficiently large N,

$$\begin{aligned} &\mathbb{P}\Big(\#\mathscr{H}_N(\eta) \ge N^{2a-\varepsilon}\Big) \\ \ge\ &\frac{1}{2}\,\mathbb{P}\Big(\exists D \in \mathscr{D}_\zeta(N) :\ \phi_D \ge (2\gamma b + \varepsilon)\log N\Big) - N^{-c}\,. \end{aligned} \tag{4.20}$$

The probability on the right-hand side is studied in the following lemma.

Lemma 4.2. *Let $0 \le \zeta < 1$ and $b > 1-\zeta$. Then for $N \to \infty$,*

$$\mathbb{P}\Big(\exists D \in \mathscr{D}_\zeta(N) : \phi_D \ge 2\gamma b \log N\Big) = N^{2[(1-\zeta)-\frac{b^2}{1-\zeta}]+o(1)}\,.$$

Admitting Lemma 4.2 for the moment, we are able to finish the proof of the lower bound in Theorem 4.1. Indeed, applying Lemma 4.2 to $b+\frac{\varepsilon}{2\gamma}$ in place of b, it follows from (4.20) that if $b > 1-\zeta$ and a satisfies (4.19),

$$\mathbb{P}\Big(\#\mathscr{H}_N(\eta) \ge N^{2a-\varepsilon}\Big) \ge N^{2-2\zeta-2\frac{(b+\frac{\varepsilon}{2\gamma})^2}{1-\zeta}+o(1)}, \qquad N \to \infty\,.$$

The lower bound in Theorem 4.1 follows immediately, with the optimal choice $\eta = \frac{a\eta^2}{\eta^2-(1-a)^2}$ and $b = \frac{[\eta^2-(1-a)]\eta}{\eta^2-(1-a)^2}$.

It remains to prove Lemma 4.2.

Proof of Lemma 4.2. The argument is quite standard.

The upper bound, which is not needed in the paper, follows immediately from the Markov inequality, with $\mathrm{Var}(\phi_D)$ being controlled by (4.4).

For the lower bound, we only consider those D away from ∂V_N: $D \in \mathscr{D}_\zeta(N)$ such that $D \subset V_N^*$ with $V_N^* \subset V_N$ denoting a fixed square of length

$\frac{1}{2}N$ such that $d(V_N^*, \partial V_N) \geq \frac{1}{4}N$. Denoting by $\mathscr{D}_\zeta^*(N)$ the set of such squares D. We are going to prove that

$$\mathbb{P}\Big(\exists D \in \mathscr{D}_\zeta^*(N) : \phi_D \geq 2\gamma b \log N\Big) \geq N^{2[(1-\zeta)-\frac{b^2}{1-\zeta}]+o(1)}\,. \tag{4.21}$$

Let $K \geq 1$ be a large integer. Define $\zeta_i := \zeta + (1-\zeta)\frac{i}{K}$ for $0 \leq i \leq K$. For a square $D \in \mathscr{D}_\zeta^*(N)$, let D_i be the square in $\mathscr{D}_{\zeta_i}^*(N)$ containing D (for $0 \leq i < K$; so $D_0 = D$) and $D_K := V_N^*$. For any $1 \leq i \leq K$, we write

$$\phi_{D_i} = c_D(i)\phi_D + Y_D(i),$$

where $Y_D(i)$, $1 \leq i \leq K$, is a Gaussian vector independent of ϕ_D, and

$$c_D(i) := \frac{\mathrm{Cov}(\phi_{D_i},\, \phi_D)}{\mathrm{Var}(\phi_D)}\,.$$

Since $D \subset D_i$, we can use the decomposition (4.14) and in its notation:

$$\phi_D = \phi_D^{D_i} + h_{D_i}(x_D)\,.$$

The independence of $\phi_D^{D_i}$ and ϕ_{D_i} gives

$$\begin{aligned} \mathrm{Cov}(\phi_{D_i},\, \phi_D) &= \mathrm{Cov}(\phi_{D_i},\, h_{D_i}(x_D)) \\ &= \mathrm{Var}(\phi_{D_i}) + \mathrm{Cov}(\phi_{D_i},\, h_{D_i}(x_D) - \phi_{D_i})\,. \end{aligned} \tag{4.22}$$

Let us look at the covariance expression on the right-hand side. By (4.4) and (4.5) (since $D \subset V_N^*$), for $0 \leq i \leq K$,

$$\mathrm{Var}(\phi_{D_i}) = (1-\zeta_i)\gamma^2 \log N + O(1), \qquad N \to \infty\,, \tag{4.23}$$

whereas by (4.3), $\mathrm{Var}(h_{D_i}(x_D) - \phi_{D_i}) \leq c_1$. Hence $\mathrm{Cov}(\phi_{D_i},\, h_{D_i}(x_D) - \phi_{D_i}) = O((\log N)^{1/2})$ (by Cauchy–Schwarz). Putting this and (4.23) into (4.22), we get

$$\mathrm{Cov}(\phi_{D_i},\, \phi_D) = (1-\zeta_i)\gamma^2 \log N + O((\log N)^{1/2})\,.$$

Together with (4.23) (case $i = 0$, so $D_i = D$), this yields

$$c_D(i) = \frac{\mathrm{Cov}(\phi_{D_i},\, \phi_D)}{\mathrm{Var}(\phi_D)} = \frac{1-\zeta_i}{1-\zeta} + O((\log N)^{-1/2})\,. \tag{4.24}$$

Let $\frac{1}{2} < \theta < 1$. Let

$$\begin{aligned} I_N &:= [2\gamma b \log N,\, 2\gamma b \log N + (\log N)^\theta]\,, \\ \mathscr{A}_D &:= \Big\{\phi_D \in I_N,\ \max_{1\le i\le K} |Y_D(i)| \le (\log N)^\theta\Big\} \\ &= \Big\{\phi_D \in I_N,\ \max_{1\le i\le K} |\phi_{D_i} - c_D(i)\phi_D| \le (\log N)^\theta\Big\}\,. \end{aligned}$$

Let

$$Z := \sum_{D\in\mathscr{D}_\zeta^*(N)} \mathbf{1}_{\mathscr{A}_D}.$$

For each $D \in \mathscr{D}_\zeta^*(N)$, ϕ_D is independent of $Y_D(i)$, $1 \le i \le K$. So

$$\mathbb{E}(Z) = N^{2(1-\zeta)}\,\mathbb{P}(\phi_D \in I_N)\,\mathbb{P}\Big(\max_{1\le i\le K} |Y_D(i)| \le (\log N)^\theta\Big).$$

By (4.23) (case $i = 0$), $\mathbb{P}(\phi_D \in I_N) = N^{-\frac{2b^2}{1-\zeta}+o(1)}$. On the other hand, $\mathrm{Var}(Y_D(i)) \le \mathrm{Var}(\phi_{D_i}) = O(\log N)$ (by (4.23)), so $\mathbb{P}(\max_{1\le i\le K} |Y_D(i)| \le (\log N)^\theta) \to 1$. It follows that

$$\mathbb{E}(Z) = N^{2(1-\zeta)-\frac{2b^2}{1-\zeta}+o(1)}\,. \tag{4.25}$$

We now estimate the second moment $\mathbb{E}(Z^2)$. Write

$$Z^2 = Z + \sum_{\ell=1}^{K} \sum_{F\in\mathscr{D}_{\zeta_\ell}^*(N)} \sum_{E,\,E'} \sum_{D,\,D'} \mathbf{1}_{\mathscr{A}_D\cap\mathscr{A}_{D'}}\,,$$

where $\sum_{E,\,E'}$ sums over E, $E' \in \mathscr{D}_{\zeta_{\ell-1}}^*(N)$ with E, $E' \subset F$ and $E \cap E' = \varnothing$, and $\sum_{D,\,D'}$ over D, $D' \in \mathscr{D}_\zeta^*(N)$ satisfying $D \subset E$ and $D' \subset E'$. We define

$$\widetilde{\mathscr{E}}(N) := \Big\{\forall D \in \mathscr{D}_\zeta^*(N), \max_{0\le i<K,\ E\in\mathscr{D}_{\zeta_i}^*(N) \text{ with } E\supset D} |h_E(x_D) - \phi_E| \le (\log N)^\theta\Big\}.$$

The set $\widetilde{\mathscr{E}}(N)$ plays the same role as $\mathscr{E}_2(N)$ in the proof of the upper bound. Exactly as for $\mathscr{E}_2(N)$, we have, for any constant $c > 0$, $\mathbb{P}(\widetilde{\mathscr{E}}(N)^c) = o(N^{-c})$; since $Z^2 \le N^{4(1-\zeta)}$, it follows from (4.25) that

$$\mathbb{E}(Z^2\,\mathbf{1}_{\widetilde{\mathscr{E}}(N)^c}) = o(\mathbb{E}(Z)), \qquad N \to \infty\,. \tag{4.26}$$

We have

$$\mathbb{E}(Z^2\,\mathbf{1}_{\widetilde{\mathscr{E}}(N)}) \le \mathbb{E}(Z) + \sum_{\ell=1}^{K} \sum_{F\in\mathscr{D}^*_{\zeta_\ell}(N)} \sum_{E,E'} \sum_{D,D'} \mathbb{P}(\mathscr{A}_D \cap \mathscr{A}_{D'} \cap \widetilde{\mathscr{E}}(N))\,.$$

Recall from (4.14) that $\phi_D^E = \phi_D - h_E(x_D)$. On the event $\widetilde{\mathscr{E}}(N)$, $h_E(x_D) \le \phi_E + (\log N)^\theta$, so $\phi_D^E \ge \phi_D - \phi_E - (\log N)^\theta$. On the event $\mathscr{A}_D$, $\phi_E \le c_D(\ell-1)\phi_D + (\log N)^\theta$. Consequently, on the event $\mathscr{A}_D \cap \mathscr{A}_{D'} \cap \widetilde{\mathscr{E}}(N)$, we have

$$\begin{aligned} \phi_D^E &\ge [1-c_D(\ell-1)]\phi_D - 2(\log N)^\theta \\ &\ge [1-c_D(\ell-1)]2\gamma b\log N - 2(\log N)^\theta\,, \end{aligned}$$

and $\phi_{D'}^{E'} \ge [1-c_{D'}(\ell-1)]2\gamma b\log N - 2(\log N)^\theta$ for the same reason. Furthermore, on $\mathscr{A}_D$,

$$\phi_F \ge c_D(\ell)\phi_D - (\log N)^\theta \ge c_D(\ell)2\gamma b\log N - (\log N)^\theta\,.$$

By independence of ϕ_D^E, $\phi_{D'}^{E'}$ and ϕ_F, this yields

$$\mathbb{P}(\mathscr{A}_D \cap \mathscr{A}_{D'} \cap \widetilde{\mathscr{E}}(N)) \le p_{1,N}\,p_{2,N}\,p_{3,N}\,,$$

where

$$\begin{aligned} p_{1,N} &:= \mathbb{P}\{\phi_D^E \ge [1-c_D(\ell-1)]2\gamma b\log N - 2(\log N)^\theta\}, \\ p_{2,N} &:= \mathbb{P}\{\phi_{D'}^{E'} \ge [1-c_{D'}(\ell-1)]2\gamma b\log N - 2(\log N)^\theta\}, \\ p_{3,N} &:= \mathbb{P}\{\phi_F \ge c_D(\ell)2\gamma b\log N - (\log N)^\theta\}. \end{aligned}$$

[Note that $p_{1,N} = p_{2,N}$.] By (4.16) (with s_i and s_{i+1} replaced by $\zeta_{\ell-1}$ and ζ, respectively),

$$\mathrm{Var}(\phi_D^E) = \gamma^2(\zeta_{\ell-1}-\zeta)\log N + O((\log N)^{1/2}),$$

whereas $\mathrm{Var}(\phi_F) = (1-\zeta_\ell)\gamma^2\log N + O(1)$ (case $i=\ell$ in (4.23)), and in view of the value of $c_D(i)$ in (4.24), we obtain:

$$\begin{aligned} p_{1,N} = p_{2,N} &\le \exp\Big(-\frac{2b^2}{1-\zeta}\,\frac{\ell-1}{K}\log N + O((\log N)^\theta)\Big), \\ p_{3,N} &\le \exp\Big(-\frac{2b^2}{1-\zeta}\,\frac{K-\ell}{K}\log N + O((\log N)^\theta)\Big)\,. \end{aligned}$$

Consequently,

$$\mathbb{E}(Z^2\,\mathbf{1}_{\widetilde{\mathscr{E}}(N)}) \le \mathbb{E}(Z) + \sum_{\ell=1}^{K} N^{2(1-\zeta_\ell)}\, N^{4(\zeta_\ell-\zeta)}\, N^{-\frac{4b^2}{1-\zeta}\,\frac{\ell-1}{K}-\frac{2b^2}{1-\zeta}\,\frac{K-\ell}{K}+o(1)}\,.$$

Note that $2(1-\zeta_\ell)+4(\zeta_\ell-\zeta)-\frac{4b^2}{1-\zeta}\frac{\ell-1}{K}-\frac{2b^2}{1-\zeta}\frac{K-\ell}{K} = 2[(1-\zeta)-\frac{b^2}{1-\zeta}](1+\frac{\ell}{K})+\frac{4b^2}{(1-\zeta)K}$, which is bounded by $2[(1-\zeta)-\frac{b^2}{1-\zeta}](1+\frac{1}{K})+\frac{4b^2}{(1-\zeta)K}$ (for $1\le \ell\le K$; recalling our assumption $b>1-\zeta$ which implies $(1-\zeta)-\frac{b^2}{1-\zeta}<0$). Consequently, for any $\varepsilon>0$, we can choose K sufficiently large such that

$$\mathbb{E}(Z^2\,\mathbf{1}_{\widetilde{\mathscr{E}}(N)}) \le \mathbb{E}(Z) + N^{2[(1-\zeta)-\frac{b^2}{1-\zeta}]+\varepsilon}, \qquad N\to\infty\,.$$

Together with (4.26) and (4.25), we obtain, for all sufficiently large N, $\mathbb{E}(Z^2)\le N^{2\varepsilon}\,\mathbb{E}(Z)$. By the Cauchy–Schwarz inequality,

$$\mathbb{P}(Z\ge 1) \ge \frac{(\mathbb{E}(Z))^2}{\mathbb{E}(Z^2)} \ge N^{-2\varepsilon}\,\mathbb{E}(Z)\,.$$

In view of (4.25), this yields the lower bound in Lemma 4.2. □

Remark 4.3. When $\zeta=0$, Lemma 4.2 gives the following analogue for GFF of (1.1): For $b>1$,

$$\mathbb{P}\Big(\max_{x\in V_N}\Phi(x)\ge 2\gamma b\log N\Big) = N^{2(1-b^2)+o(1)}, \qquad N\to\infty\,.$$

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