SHARP EIGENVALUE ESTIMATES ON DEGENERATING SURFACES

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Abstract. We consider the first non-zero eigenvalue λ_1 of the Laplacian on hyperbolic surfaces for which one disconnecting collar degenerates and prove that $8\pi\nabla\log(\lambda_1)$ essentially agrees with the dual of the differential of the degenerating Fenchel-Nielson length coordinate. As a corollary of our analysis we can improve previous results of Schoen, Wolpert, Yau [27] and Burger [5] to obtain estimates with optimal error rates.

1. Introduction and results

Let M be a closed oriented surface of genus $\gamma \geq 2$ (always assumed to be connected) and let g be a hyperbolic (i.e. Gauss curvature $K_g \equiv -1$) metric on M. Let σ^1 be a simple closed geodesic in (M,g) which decomposes M into two connected components M^+ and M^-. We consider surfaces for which the length $\ell_1 = L_g(\sigma^1)$ is small compared to the length of any other simple closed geodesic in (M,g). In this case the first eigenvalue of the Laplacian on (M,g) turns out to be small and to essentially only depend on ℓ_1 and the genera of $M^\pm$.

The asymptotic behaviour of small eigenvalues on degenerating surfaces was first considered by Schoen, Wolpert and Yau in [27]. They studied surfaces with bounded negative curvature $-c \leq K_g \leq -\tilde{c} < 0$ and proved in particular that if the collapsing geodesics decompose M into $n+1$ connected components then precisely n eigenvalues $0 < \lambda_1 \leq \ldots \leq \lambda_n$ tend to zero, with the rate of convergence being linear with respect to the (sum of the) lengths of the corresponding geodesics. Their results apply in particular to the setting of one collapsing disconnecting geodesic σ^1 we described above and in this case yield that

$$c\ell_1 \leq \lambda_1 \leq C\ell_1 \tag{1.1}$$

while

$$\lambda_2 \geq \tilde{c} > 0, \tag{1.2}$$

for constants $c, \tilde{c} > 0$ and $C < \infty$ that depend, apart from the genus, only on a lower bound on the lengths of the simple closed geodesics different from σ^1, or equivalently on a lower bound $\hat{\delta} > 0$ for the injectivity radius on $M \setminus \mathcal{C}(\sigma^1)$. Here and in the following $\mathcal{C}(\sigma^1)$ denotes the collar neighbourhood around σ^1 described by the collar lemma that we recall in Lemma A.1.

Remark 1.1. We note that (1.1) and (1.2) imply in particular that λ_1 is simple provided $\ell_1 = L_g(\sigma^1) \leq \ell_0$ for a suitably small constant $\ell_0 = \ell_0(\hat{\delta}, \gamma) \leq \ell_0(\gamma)$.

A refined picture of the behaviour of small eigenvalues on degenerating hyperbolic surfaces was then given by Burger in [4] and [5], who compared the small eigenvalues of $-\Delta_g$ on M with the eigenvalues $\widehat{\lambda}_j$ of the Laplacian of a weighted graph that is associated to the set of collapsing geodesics. In [4] he established that $\frac{\lambda_j}{\widehat{\lambda}_j} \to \frac{1}{2\pi^2}$, $1 \leq j \leq n$, as the surface collapses and subsequently refined this convergence result in [5] by giving both a lower bound (of order

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$O(\sqrt{\ell}))$ and an upper bound (of order $O(\ell \log \ell))$ on the resulting errors. We note that in the setting we consider here his result from [5] yields that

$$C_{top} - C\sqrt{\ell_1} \le \frac{\lambda_1}{\ell_1} \le C_{top} + C\ell_1|\log(\ell_1)| \tag{1.3}$$

where C_{top} is given in terms of the genera $\gamma^\pm$ of the connected components $M^\pm$ of $M \setminus \sigma^1$

$$C_{top} = \frac{-\chi(M)}{2\pi^2\chi(M^-)\chi(M^+)} = \frac{2(\gamma-1)}{2\pi^2(1-2\gamma^+)\cdot(1-2\gamma^-)}. \tag{1.4}$$

We remark that the upper bound in (1.3) can be obtained directly from comparing with a function that is linear on the collar $\mathcal{C}(\sigma^1)$ (or alternatively a function that solves the corresponding ODE on the collar) and constant on the rest of the surface while the proof of the lower bound is far more involved and does not yield the same order of the error.

We note that (1.3) implies in particular that if g and $\tilde{g}$ are two metrics which satisfy the assumptions above for geodesics σ^1 and $\tilde{\sigma}^1$ of the same length ℓ_1 and connected components $M^\pm$ and $\tilde{M}^\pm$ of the same genera $\gamma^\pm$ then

$$\ell_1^{-1}|\lambda_1(M,g) - \lambda_1(M,\tilde{g})| \le C\sqrt{\ell_1}. \tag{1.5}$$

It is natural to ask whether the lower bound in (1.3) and hence also the above estimate can be improved to $O(\ell_1|\log(\ell_1)|)$ and, more importantly, whether such an estimate would be optimal, respectively whether one can derive an estimate of the form (1.5) with optimal error rates.

In the present work we will give positive answers to both of these questions and indeed derive both C^0- and C^1-estimates with sharp error bounds. Most of our analysis is quite different from the methods in [5] as we use a dynamic approach and consider the variation of the eigenvalues induced by a change of the geometry of (M,g), or to be more precise by a change of the Fenchel-Nielson coordinates. We then obtain C^0-bounds, such as refinements of (1.5) and (1.3), only as corollary of our C^1-bounds.

We remark that bounds on some derivatives of small eigenvalues have been obtained previously by Batchelor [2] who considered the change of the small eigenvalues induced by a change of the length of the collapsing geodesics, so in our case $\frac{\partial \lambda_1}{\partial \ell_1}$, though his error estimates are only of order $O(\frac{1}{|\log(\ell)|})$ and would thus in particular not allow for any improvement of (1.3).

To state our first main result, we recall that we may extend any given simple closed geodesic σ^1 in a closed oriented hyperbolic surface (M,g) to a collection $\mathcal{E} = \{\sigma^1, \ldots, \sigma^{3(\gamma-1)}\}$ of simple closed geodesics in (M,g) that decompose the surface into pairs of pants. We also recall that we can and will choose this collection so that the length of all geodesics σ^j is bounded from above by a constant $\bar{L}$ that depends only on the genus and an upper bound on $L_g(\sigma^1)$, compare also Lemma A.4, so in the situation of Remark 1.1, by some $\bar{L} = \bar{L}(\gamma)$.

The geometry of (M,g) is determined by the corresponding Fenchel-Nielson coordinates, i.e. the set of length parameters $\ell_i = L_g(\sigma^i)$ together with a set of twist parameters ψ_i that describe the way in which the pairs of pants are glued together. We refer the reader to Appendix A.2 for a summary of the results on hyperbolic surfaces that we use in the present paper and in particular to Remark A.3 for the precise definition of the twist coordinates we use.

The idea that λ_1 essentially only depends on the length of σ^1 if ℓ_1 is small compared to the lengths of the other simple closed geodesics of (M,g) can be quantified as follows:

Theorem 1.2. *Let (M,g) be a closed oriented hyperbolic surface of genus $\gamma \ge 2$ and let σ^1 be a simple closed geodesic which disconnects M into two connected components. We let $\hat{\delta} > 0$ be a lower bound on the injectivity radius $\inf_{M \setminus \mathcal{C}(\sigma^1)} \mathrm{inj}_g(p)$ away from the collar around σ^1 and suppose that $\ell_1 \le \ell_0$, for $\ell_0 = \ell_0(\hat{\delta}, \gamma) > 0$ as in Remark 1.1.*

Let $\sigma^2, \dots, \sigma^{3(\gamma-1)}$ be simple closed geodesics so that $\mathcal{E} = \{\sigma^1, \dots, \sigma^{3(\gamma-1)}\}$ decomposes (M,g) into pairs of pants, which we can furthermore assume to be chosen so that $L_g(\sigma^j) \leq \bar{L} = \bar{L}(\gamma)$ for every j. Then the first non-zero eigenvalue $\lambda = \lambda_1$ of $-\Delta_g$ has the following dependence on the corresponding Fenchel-Nielson length and twist coordinates ℓ_i and ψ_i:

There exists a constant C depending only on $\hat{\delta}$ and the genus of M so that

$$\left|\frac{\partial \lambda}{\partial \ell_1} - \frac{\lambda}{\ell_1}\right| \leq C\ell_1 |\log(\ell_1)| \text{ and } \left|\frac{\partial \lambda}{\partial \ell_j}\right| \leq C\ell_1^2 \text{ for } j \neq 1$$

while a change of the twist coordinates can only change the first eigenvalue by

$$\left|\frac{\partial \lambda}{\partial \psi_1}\right| \leq C\ell_1^4, \qquad \text{respectively} \qquad \left|\frac{\partial \lambda}{\partial \psi_j}\right| \leq C\ell_1^2 \quad \text{for } j = 2, \dots, 3(\gamma-1). \tag{1.6}$$

Here and in the following $\mathcal{C}(\sigma^1)$ denotes the collar neighbourhood of σ^1 described by the Collar lemma A.1 which we recall in the appendix and $\mathrm{inj}_g(p)$ stands for the injectivity radius in $p \in (M,g)$. We also note that the facts that λ is simple and invariant under pull-back by diffeomorphisms guarantee that the above derivatives are well defined, compare also Lemma 2.1.

As a consequence of the C^1-bounds on λ stated in the above result we immediately obtain the following refinement of the result of Burger [5] which not only gives the desired improvement of the lower bound in (1.3) but furthermore establishes a quantitative version of the idea that λ essentially only depends on ℓ_1 which is *sharp* as we shall see in Theorem 1.4 below.

Corollary 1.3. *Let M be a closed oriented surface of genus $\gamma \geq 2$, let $\bar{\sigma}$ be a simple closed curve that disconnects M into two connected components of genera $\gamma^{\pm}$ and let C_{top} be given by (1.4). Then there exists a function $f\colon (0, 2\mathrm{arsinh}(1)) \to \mathbb{R}^+$ that depends only on $\gamma^{\pm}$ and satisfies*

$$\left|\frac{f(\ell)}{\ell} - C_{top}\right| \leq C\ell |\log(\ell)|$$

and for any $\hat{\delta} > 0$ there exists a constant $C = C(\hat{\delta}, \gamma)$ such that the following holds true.

Let g be any hyperbolic metric on M for which $\mathrm{inj}_g(x) \geq \hat{\delta}$ on $M \setminus \mathcal{C}(\sigma^1)$, σ^1 the unique geodesic in (M,g) that is homotopic to $\bar{\sigma}$, and for which $\ell := L_g(\sigma^1) < 2\mathrm{arsinh}(1)$. Then the first eigenvalue $\lambda(M,g) = \lambda_1(M,g)$ of $-\Delta_g$ satisfies

$$|\lambda(M,g) - f(\ell)| \leq C\ell^2. \tag{1.7}$$

In particular, for metrics $g, \tilde{g}$ for which the lengths of the corresponding geodesics σ^1 and $\tilde{\sigma}^1$ agree, we have that

$$|\lambda(M,g) - \lambda(M,\tilde{g})| \leq C\ell^2. \tag{1.8}$$

For surfaces of genus at least 3 this result is sharp as we shall prove

Theorem 1.4. *For every genus $\gamma \geq 3$ and every number $\hat{\delta} > 0$ there exist constants $\bar{c} > 0$ and $\ell_0 > 0$ so that the following holds true. Let M be a closed oriented surface of genus γ and let $\bar{\sigma}$ be a simple closed curve that disconnects M into two connected components of genera $\gamma^{\pm}$.*

Then there exist families of hyperbolic metrics $(g_\ell)_{\ell \in (0,\ell_0)}$ and $(\tilde{g}_\ell)_{\ell \in (0,\ell_0)}$ satisfying the assumptions of Corollary 1.3 for the fixed constant $\hat{\delta} > 0$ and with $L_{\tilde{g}_\ell}(\sigma^1_{\tilde{g}_\ell}) = \ell = L_{g_\ell}(\sigma^1_{g_\ell})$ for which

$$|\lambda(M,g_\ell) - \lambda(M,\tilde{g}_\ell)| \geq \bar{c} \cdot \ell^2.$$

We will obtain the above results based on an essentially explicit characterisation of $\nabla\lambda$ that we state in Theorem 1.5 below and on a careful analysis of the tensors that induce a change of the Fenchel-Nielson length respectively twist coordinates.

In order to state these results in detail we need to introduce some more notation and recall some well-known properties of hyperbolic surfaces as well as results on holomorphic quadratic differentials obtained in the joint works [22] of Topping and the second author respectively [23] of Topping, Zhu and the second author.

Given a closed orientable surface M of genus $\gamma \geq 2$ we denote by $\mathcal{M}_{-1}$ the set of smooth hyperbolic metrics on M and recall that the tangent space to $T_g\mathcal{M}_{-1}$ splits orthogonally

$$T_g\mathcal{M}_{-1} = \{L_X g, X \in \Gamma(TM)\} \oplus H(g)$$

into the directions generated by the pull-back by diffeomorphisms and the *horizontal space* $H(g)$ which can be characterised as space of symmetric $(0,2)$ tensors that are both divergence- and trace-free. An equivalent characterisation of the horizontal space which will be more suitable for our analysis is to view $H(g) = \mathrm{Re}\,(\mathcal{H}(M,g))$ as the real part of the complex vector space

$$\mathcal{H}(M,g) := \{\Psi : \text{ holomorphic quadratic differentials on } (M,g)\}$$

whose (complex) dimension is $3(\gamma-1)$.

Given a collection $\mathcal{E} = \{\sigma^i\}_{i=1}^{3(\gamma-1)}$ of simple closed geodesics which decomposes (M,g) into pairs of pants we consider $\mathbb{C}$-linear maps

$$\partial\ell_j : \mathcal{H}(M,g) \to \mathbb{C}$$

which can be seen as $\mathbb{C}$-linear derivatives of the length coordinates $\ell_j = L_g(\sigma^j)$ and which are defined as follows, compare also [22, Remark 4.1] and [28]: We view $\mathcal{H}(M,g)$ as real vector space with complex structure J and identify $\mathcal{H}(M,g)$ with a subspace of $T_g\mathcal{M}_{-1}$ via the isomorphism $\Phi \mapsto \mathrm{Re}\,(\Phi)$. We then define

$$\partial\ell_j(\Phi) := \tfrac{1}{2}\big(d\ell_j(\Phi) - \mathrm{i}d\ell_j(J\Phi)\big) \tag{1.9}$$

where $d\ell_j : T_g\mathcal{M}_{-1} \to \mathbb{R}$ is given by $d\ell_j(k) = \frac{d}{d\varepsilon}|_{\varepsilon=0} L_{g_\varepsilon}(\sigma^j(\varepsilon))$ where g_ε is a smooth curve of metrics in $\mathcal{M}_{-1}$ so that $\partial_\varepsilon|_{\varepsilon=0} g_\varepsilon = k$ and $\sigma^j(\varepsilon)$ is the unique simple closed geodesic in (M, g_ε) homotopic to σ^j.

The work of Wolpert, in particular [28, Thm 3.7], assures that the codimension of $\ker(\partial\ell_1, \ldots \partial\ell_k)$ is k for all disjoint simple closed geodesics $\sigma^1, \ldots, \sigma^k$, see also the joint work [23] of Topping, Zhu and the second author for an alternative proof of this fact in case that $\sigma^1, \ldots, \sigma^k$ are the simple closed geodesics of length no more than $2\eta_1 = 2\eta_1(\gamma)$. We denote by $\{\tilde\Theta^1, \ldots, \tilde\Theta^k\}$ the basis of $(\ker(\partial\ell_1, \ldots, \partial\ell_k))^\perp$ which is dual to $\{\partial\ell_j\}_{j=1}^k$, compare (2.22), and set $\tilde\Omega^j = -\frac{\tilde\Theta^j}{\|\tilde\Theta^j\|_{L^2}}$. The fine properties of this renormalised dual basis $\tilde\Omega^1, \ldots, \tilde\Omega^k$ of $(\ker(\partial\ell_1, \ldots, \partial\ell_k))^\perp$ were analysed in the joint work [22] of Topping and the second author. The results of [22], in particular [22, Lemma 4.5] which we recall in Section 2.2, guarantee that up to error terms of order $O(\ell_j^{3/2})$ each of these elements is concentrated on the corresponding collar $\mathcal{C}(\sigma^j)$ and there given essentially as a constant multiple of dz^2. As explained in [22] the error rate of $\ell_j^{3/2}$ is optimal and agrees with the error rates obtained by Wolpert in [31, Lemma 3.12] for the closely related basis of $\mathcal{H}(M,g)$ that corresponds to the differentials (there denoted by θ_{σ_j}) which describe the L^2-gradients of ℓ_j in the sense that $\partial\ell_j(\Psi) = \langle\Psi, \theta_{\sigma_j}\rangle$ instead of the dual basis for which $\partial\ell_j(\Theta^i) = \delta_j^i$ considered here later on.

Our last main result about the properties of the first eigenvalue, which at the same time will be the basis of the proofs of both Theorems 1.2 and 1.4, establishes that the gradient of $\lambda = \lambda_1$ is essentially given in terms of the element $\tilde\Theta^1 \in \ker(\partial\ell_1)^\perp$ which is dual to the differential $\partial\ell_1$ of the degenerating length coordinate.

Theorem 1.5. *Let (M,g) be a closed oriented hyperbolic surface, let σ^1 be a disconnecting simple closed geodesic and let $\partial\ell_1$ be the complex differential of the corresponding length coordinate introduced in (1.9).*

Then if $L_g(\sigma^1)$ is small compared to the injectivity radius on $M \setminus \mathcal{C}(\sigma^1)$ then the L^2-gradient of the first eigenvalue $\lambda_1 \colon \mathcal{M}_{-1} \to \mathbb{R}$ of $-\Delta_g$ is essentially determined by the element $\tilde{\Theta}^1$ of $\ker(\partial\ell_1)^\perp$ for which $\partial\ell_1(\tilde{\Theta}^1) = 1$, namely

$$\nabla \log(\lambda) \sim \tfrac{1}{8\pi}\mathrm{Re}\,(\tilde{\Theta}^1)$$

holds true in the following sense: Suppose that $\ell_1 = L_g(\sigma^1) \leq \ell_0$ for $\ell_0 = \ell_0(\hat{\delta},\gamma)$ chosen as in Remark 1.1 and $\hat{\delta}$ as usual a lower bound on inj_g on $M \setminus \mathcal{C}(\sigma^1)$. Then there exists a number $\alpha > 0$ with

$$|\alpha - \tfrac{1}{8\pi}| \leq C\ell_1 |\log(\ell_1)| \tag{1.10}$$

such that, for a constant C that depends only on the genus of M and on $\hat{\delta}$

$$\|\nabla \log(\lambda) - \alpha \mathrm{Re}\,(\tilde{\Theta}^1)\|_{L^\infty(M,g)} \leq C\ell_1. \tag{1.11}$$

We will obtain the proof of this theorem by combining two different types of results: On the one hand we derive energy estimates for the first eigenfunction, and this part is similar to existing approaches, including [5]. These results will be stated in Section 2.1 and proved later on in Section 3. On the other hand we crucially use the fine properties of holomorphic quadratic differentials on degenerating surfaces as developed in [22] and [23], as well as the uniform Poincaré estimate for holomorphic quadratic differentials that was proven in [21] by Topping and the second author. We recall the relevant results from [21, 22, 23] in Section 2.2. Based on these results we will then be able to give the proof of the above Theorem 1.5 in Section 2.3.

While Theorem 1.5 forms the basis of the proofs of our other main results, we will need further results on the fine properties of holomorphic quadratic differentials in order to prove Theorems 1.2 and 1.4, and as a consequence also Corollary 1.3. In particular, we need to study dual bases of $\mathcal{H}(M,g)$ that are associated to a full set of simple closed geodesics $\{\sigma^j\}_{j=1,\ldots,3(\gamma-1)}$ which decomposes M into pairs of pants, respectively the corresponding Fenchel-Nielson coordinates.

To be more precise, we shall need that for any such decomposing set of geodesics the dual basis $\{\Theta^j\}_{j=1}^{3(\gamma-1)}$ to $\{\partial\ell_j\}_{j=1}^{3(\gamma-1)}$ of the whole space $\mathcal{H}(M,g)$, whose existence is ensured by [28, Theorem 3.7], still satisfy estimates with optimal error rates as obtained in [22] for the elements $\tilde{\Theta}^j$ described above, respectively in [31, Lemma 3.12] for the related elements of $\mathcal{H}(M,g)$ that describe the gradient of ℓ_j. We will furthermore need to consider elements Ψ^j, Λ^j of $\mathcal{H}(M,g)$ that are dual to the (real differentials of the) Fenchel-Nielson coordinates ψ_j, ℓ_j in the sense that for every i,j

$$d\ell_j(\mathrm{Re}\,(\Lambda^i)) = \delta_j^i = d\psi_j(\mathrm{Re}\,(\Psi^i)) \text{ and } d\psi_j(\mathrm{Re}\,(\Lambda^i)) = 0 = d\ell_j(\mathrm{Re}\,(\Psi^i)). \tag{1.12}$$

As we shall see, also these bases can be well controlled and, for small ℓ_j, can be characterised essentially explicitly in terms of the Θ^j, respectively the $\tilde{\Omega}^j$ from [22].

These results will be stated in detail in Section 2.4 and proven in Section 4. They can be summarised by the following proposition which is partly based on results from [22] and [28].

Proposition 1.6. *Let (M,g) be any closed oriented hyperbolic surface of genus γ and let $\mathcal{E} = \{\sigma^1,\ldots,\sigma^{3(\gamma-1)}\}$ be any set of simple closed geodesics which decomposes M into pairs of pants. Let $\{\ell_i,\psi_i\}$ be the corresponding Fenchel-Nielson coordinates and let $\partial\ell_i$ be the $\mathbb{C}$-linear differentials introduced in (1.9). Then*

(I) The dual basis $\{\Theta^j\}_{j=1}^{3(\gamma-1)}$ *to* $(\partial\ell_1,\ldots,\partial\ell_{3(\gamma-1)})$ *and the corresponding renormalised elements* $\Omega^j=-\frac{\Theta^j}{\|\Theta^j\|_{L^2(M,g)}}$ *satisfy the same sharp error estimates as obtained in* [22] *(there for* $\tilde{\Omega}^j$*), see Lemma 2.13 and Corollary 2.14 for details.*

(II) The element Ψ^j *which generates a Dehn-twist around* σ^j*, compare* (1.12)*, satisfies*

$$\Psi^j\in(\ker(\partial\ell_j))^\perp \text{ and } \partial\ell_j(\Psi^j)\in i\mathbb{R}\setminus\{0\} \tag{1.13}$$

and, for indices for which ℓ_j *is small, is such that*

$$\frac{\Psi^j}{\|\Psi^j\|_{L^2}}\sim -i\Omega^j$$

as is made precise in Lemma 2.19.

(III) The duals Λ^j *of the (real) differentials of the Fenchel-Nielson lengths coordinates introduced in* (1.12) *are so that*

$$\Lambda^j\sim\tfrac{1}{2}\Theta^j$$

up to error terms whose L^∞*-norm is of order* $O(\ell_j)$*, compare* (2.18)*, which agrees with the sharp error rates on the* Θ^j *from part (I).*

As remarked above, the existence of a dual basis was already obtained by Wolpert in [28], and that paper furthermore establishes that the element generating a Dehn-twist is given as a complex multiple of the gradient of the corresponding length coordinate from which the characterisation (1.13) of Dehn-twists can be derived. We will nonetheless include short proofs of these two facts in Section 4, not only to make the paper selfcontained but more importantly as we shall need to work with the dual basis $\{\Theta^j\}$ also in the more general setting of punctured surfaces and as the tools on which the proofs of these facts are based are also used directly in the proofs of our main results on eigenvalues of the Laplacian.

We also remark that bases of the space of holomorphic quadratic differentials which are related to Fenchel-Nielson and other choices of coordinates on Teichmüller space have been considered by many authors and we refer in particular to the works of Masur [17], Yamada [33, 34], Wolpert [29, 30, 31, 32] and the recent work of Mazzeo-Swoboda [18] and the references therein for an overview of existing results. While the bases previously considered were often characterised in terms of the gradient of the coordinates, here we follow the approach of the paper [22] of Topping and the second author and consider bases that are obtained as dual bases as this is needed to consider the dependence of a function on the coordinates as done in Theorem 1.2.

Remark 1.7. The results of Schoen, Wolpert and Yau [27], Burger [5] and Batchelor [2] apply to more general settings of several degenerating collars and so does our our analysis of holomorphic quadratic differentials and Fenchel-Nielson coordinates carried out in Section 4. The refined analysis of small eigenvalues in this more general setting will be addressed in future work. It would furthermore be of interest to know whether (1.8) is sharp also for surfaces of genus 2 and whether the error rate of $\ell_1|\log\ell_1|$ for the dependence (1.7) of the first eigenvalue on the degenerating length coordinate ℓ_1 is optimal.

We remark that the study of eigenvalues of the Laplacian on manifolds has a long and fruitful history. We recall in particular that the work of Cheeger [10] establishes that the first eigenvalue of the Laplacian on any manifold is bounded from below by $\frac{1}{4}h^2(M,g)$, while Buser [8] obtained an upper bound on λ_1 of $2\sqrt{K}(\dim(M)-1)h(M,g)+10h^2(M,g)$, $-(\dim(M)-1)K$ a lower bound on the Ricci-curvature and $h(M,g)$ the Cheeger isoperimetric constant, compare also [16]. Properties of eigenvalues on Riemannian manifolds in general, and hyperbolic surfaces in particular, and their relations to other topics such as Selberg's eigenvalue conjecture (see e.g. [25]) and minimal surfaces (see [12]), have been considered by many authors. We refer in particular to the books of Buser [6] and Bergeron [3] for an overview of results on eigenvalues on hyperbolic surfaces and note that the asymptotic behaviour of small eigenvalues has been considered also by Grotowski, Huntley and Jorgenson in [13], that Colbois and Colin de Verdière used the study of eigenvalues on weighted graphs to obtain multiplicity results for eigenvalues on hyperbolic surfaces [11] and that the question of how many eigenvalues of $-\Delta_g$ on a hyperbolic surface of genus γ can be smaller than $\frac{1}{4}$ has been addressed in particular by [7], [26] and [19].

Outline of the paper: In Section 2 we give the proofs of our main results which are based on energy estimates that are derived in Section 3 and on results on dual-bases of the Fenchel-Nielson coordinates which are proven in the last Section 4 of this paper.

2. Proofs of the main results

In the first two parts of this section we collect properties of the first eigenfunction, proved later on in Section 3, and well known properties of quadratic differentials as well as the results from [21, 22, 23] on the properties of the dual basis $\tilde{\Theta}^j$ and the Poincaré-inequality, all of which are needed to give the proof of our first main result Theorem 1.5 in the subsequent Section 2.3. In Section 2.4 we then state in detail the properties of the dual bases Θ^j, Λ^j and Ψ^j of the space of holomorphic quadratic differentials that we outlined in Proposition 1.6 and that we prove in Section 4. These results then allow us to give the proof of our other main results in the subsequent sections: we prove Theorem 1.2 in Section 2.5, Corollary 1.3 in Section 2.6 and finally Theorem 1.4 in Section 2.7.

2.1. Properties of the first eigenfunction.

We first recall that the gradient of an eigenvalue, considered as a function on the set $\mathcal{M}_{-1}$ of all smooth hyperbolic metrics, is determined by the holomorphic part of the Hopf-differential of the corresponding eigenfunction, namely

Lemma 2.1. *Let (M,g) be a hyperbolic surface for which the k-th eigenvalue λ_k is simple, k any element of $\mathbb{N}$. Let u_k be the corresponding eigenfunction, normalised to have $\|u_k\|_{L^2(M,g)} = 1$. Then the L^2-gradient of $\lambda_k \colon \mathcal{M}_{-1} \to \mathbb{R}$ is given by*

$$\nabla\lambda_k(g) = -\tfrac{1}{2}\mathrm{Re}\,(P_g^{\mathcal{H}}(\Phi(u_k,g))$$

where $\Phi(u,g)$ denotes the Hopf-differential of u, given in local isothermal coordinates (x,y) of (M,g) as

$$\Phi(u,g) = (|u_x|^2 - |u_y|^2 - 2\mathrm{i}u_x u_y)dz^2, \qquad z = x + \mathrm{i}y, \tag{2.1}$$

and $P_g^{\mathcal{H}}$ is the $L^2(M,g)$-orthogonal projection from the space of L^2-quadratic differentials onto the space of holomorphic quadratic differentials $\mathcal{H}(M,g)$.

A proof of this lemma is provided later on in Section 3.1 while more details about the space of holomorphic quadratic differentials will be provided in Section 2.2.

We recall that the (real part of the) Hopf-differential describes the L^2-gradient of the Dirichlet-energy with respect to variations of the metric and remark that for *real-valued* functions as considered here

$$\|\Phi(v,g)\|_{L^1(K,g)} = 2\|dv\|^2_{L^2(K,g)} \text{ for every } K \subset M, \tag{2.2}$$

see Remark 3.2 for more details. As we shall from now on always assume that eigenfunctions are normalised to have $\|u\|_{L^2(M,g)} = 1$, we thus have that

$$\|\Phi(u,g)\|_{L^1(M,g)} = 2\lambda. \tag{2.3}$$

We recall that Remark 1.1 ensures that the first eigenvalue is simple in the situations considered in our main results, allowing us to apply the above lemma for $\lambda = \lambda_1$ and the corresponding (normalised) eigenfunction $u = u_1$.

We also recall that u is characterised as minimiser of the Rayleigh-quotient

$$\lambda_1 = \min\left\{\frac{\|dv\|^2_{L^2(M,g)}}{\|v\|^2_{L^2(M,g)}} : v \in H^1(M,g) \text{ so that } \int_M v dv_g = 0\right\}$$

and will use that u satisfies the following energy estimates which are proven in Section 3.2.

Lemma 2.2. *There exist constants C_0 and $\delta_2 \in (0, \operatorname{arsinh}(1))$ depending only on the genus $\gamma \geq 2$ so that the following holds true for any closed oriented hyperbolic surface (M,g) of genus γ and any number $\bar{\delta} \in (0, \delta_2]$.*

Suppose that $\operatorname{inj}(M,g) \leq \bar{\delta}$ and that all simple closed geodesics $\sigma^1, \ldots, \sigma^k$ of length no more then $2\bar{\delta}$ are so that $M \setminus \sigma^j$ is disconnected. Then the first eigenfunction u of $-\Delta_g$ (as always normalised by $\|u\|_{L^2(M,g)} = 1$) satisfies the estimate

$$\|du\|^2_{L^2(\delta\text{-thick}(M,g))} \leq \frac{C_0}{\delta}\lambda_1^2 \text{ for every } 0 < \delta \leq \bar{\delta}.$$

Here and in the following we denote by $\delta\text{-thick}(M,g) := \{p \in M : \operatorname{inj}_g(p) \geq \delta\}$ while $\delta\text{-thin}(M,g) := M \setminus \delta\text{-thick}(M,g)$.

We shall furthermore see that on collars around short simple closed geodesics the *angular* energy decays rapidly towards the centre of the collar, allowing us to obtain in particular the following bounds on weighted angular energies.

Lemma 2.3. *There exist universal constants $C_{1,2}$ and $\delta_3 > 0$ so that the following holds true for any closed oriented hyperbolic surface (M,g) and any eigenfunction u of $-\Delta_g$ to an eigenvalue $\lambda \in \mathbb{R}$. Let σ be a simple closed geodesic of length $\ell < 2\operatorname{arsinh}(1)$ and let $(s,\theta) \in (-X(\ell), X(\ell)) \times S^1$ be the corresponding collar coordinates in which the metric takes the form $g = \rho^2(ds^2 + d\theta^2)$, compare Lemma A.1. Then*

$$\int_{-X(\ell)}^{X(\ell)} \int_{S^1} |u_\theta|^2 \rho^{-4} ds d\theta \leq C_1 \|du\|^2_{L^2(\delta_3\text{-thick}(\mathcal{C}(\sigma))} + C_1 \lambda^2 X(\ell)$$

and

$$\int_{-X(\ell)}^{X(\ell)} \int_{S^1} |u_\theta|^2 \rho^{-2} ds d\theta \leq C_2 \|du\|^2_{L^2(\delta_3\text{-thick}(\mathcal{C}(\sigma))} + C_2 \lambda^2 \|u\|^2_{L^\infty(M,g)}. \tag{2.4}$$

In particular, if (M,g) satisfies the assumptions of Lemma 2.2 for some $\bar{\delta} \in (0, \operatorname{arsinh}(1))$ and geodesics $\{\sigma^j\}_{j=1}^k$ and if u is the first eigenfunction of $-\Delta_g$ then the angular energies on the corresponding collars $\mathcal{C}(\sigma^j)$ are bounded by

$$\int_{-X(\ell_j)}^{X(\ell_j)} \int_{S^1} |u_\theta|^2 \rho^{-4} ds d\theta \leq C(\bar{\delta}, \gamma) \cdot \lambda_1^2 + C_1 \lambda_1^2 \cdot \ell_j^{-1} \tag{2.5}$$

and

$$\int_{-X(\ell_j)}^{X(\ell_j)} \int_{S^1} |u_\theta|^2 \rho^{-2} ds d\theta \leq C(\bar{\delta}, \gamma) \cdot \lambda_1^2. \tag{2.6}$$

We furthermore recall, compare Appendix A.4

Remark 2.4. There exists a constant C_3 depending at most on the genus of M so that the following holds true: Let (M,g) be a closed hyperbolic surface whose shortest simple closed geodesic σ is such that $M \setminus \sigma$ is disconnected. Then the (normalised) first eigenfunction u of $-\Delta_g$ is bounded by $\|u\|_{L^\infty(M,g)} \leq C_3$.

2.2. Properties of holomorphic quadratic differentials.

Here we recall some standard properties of holomorphic quadratic differentials as well as results on $\mathcal{H}(M,g)$ from the joint works [21, 22] and [23] of Topping (respectively Topping, Zhu) and the second author that will be used in the proof of Theorem 1.5. We note that alternatively we could also use other bases of $\mathcal{H}(M,g)$ such as the gradient basis of the length coordinates considered by Wolpert in [29, 31] as basis of our work.

We recall that a quadratic differential is a complex tensor Ψ which is given in local isothermal coordinates (x,y) as $\Psi = \psi \cdot dz^2$, $z = x + iy$. Here ψ is a complex function which, for elements of $\mathcal{H}(M,g)$, is furthermore asked to be holomorphic. Using the normalisation that $|dz^2|_g = 2\rho^{-2}$ for $g = \rho^2(dx^2+dy^2)$ we may write the (hermitian) L^2-inner product on the space of quadratic differentials locally as

$$\langle \Psi, \Phi \rangle_{L^2} = \int \psi \cdot \bar{\phi} |dz^2|_g^2 dv_g = 4 \int \psi \cdot \bar{\phi} \rho^{-2} dx dy.$$

In particular

$$\langle \mathrm{Re}\,(\Psi), \mathrm{Re}\,(\Phi) \rangle_{L^2(M,g)} = \tfrac{1}{2} \mathrm{Re}\, \langle \Psi, \Phi \rangle_{L^2(M,g)}, \tag{2.7}$$

and we recall that this relation implies that the projection P_g^H from the space of symmetric real $(0,2)$-tensors onto $H(g) = \mathrm{Re}\,(\mathcal{H}(M,g))$ and the projection $P_g^{\mathcal{H}}$ from the space of L^2-quadratic differentials onto $\mathcal{H}(M,g)$ are related by

$$P_g^H(\mathrm{Re}\,(\Phi)) = \mathrm{Re}\,(P_g^{\mathcal{H}}(\Phi)).$$

We furthermore recall from [22, Proposition 4.10] that for any quadratic differential Υ

$$\|P_g^{\mathcal{H}}(\Upsilon)\|_{L^1(M,g)} \le C \|\Upsilon\|_{L^1(M,g)} \tag{2.8}$$

for a constant C that depends only on the genus.

Let now $\mathcal{C}(\sigma)$ be a collar around a simple closed geodesic σ in (M,g) described by the Collar lemma A.1 of Keen-Randol that we recall in the appendix. We we will often use that on $\mathcal{C}(\sigma)$ we may represent any $\Upsilon \in \mathcal{H}(M,g)$ by its Fourier series in collar coordinates (s,θ)

$$\Upsilon = \sum_{n=-\infty}^{\infty} b_n(\Upsilon) e^{n(s+i\theta)} dz^2, \qquad b_n(\Upsilon) = b_n(\Upsilon, \mathcal{C}(\sigma)) \in \mathbb{C}$$

and that on $\mathcal{C}(\sigma)$ we may split Υ orthogonally into its principal part $b_0(\Upsilon)dz^2$ and its collar decay part $\Upsilon - b_0(\Upsilon)dz^2$. Hence, for any $\Upsilon, \Psi \in \mathcal{H}(M,g)$

$$\begin{aligned} \langle \Upsilon, \Psi \rangle_{L^2(\mathcal{C}(\sigma))} &= b_0(\Upsilon) \cdot \overline{b_0(\Psi)} \|dz^2\|^2_{L^2(\mathcal{C}(\sigma))} + \langle \Upsilon - b_0(\Upsilon)dz^2, \Psi - b_0(\Psi)dz^2 \rangle_{L^2(\mathcal{C}(\sigma))} \\ &= b_0(\Upsilon) \cdot \overline{b_0(\Psi)} \|dz^2\|^2_{L^2(\mathcal{C}(\sigma))} + \langle \Upsilon, \Psi - b_0(\Psi)dz^2 \rangle_{L^2(\mathcal{C}(\sigma))}, \end{aligned} \tag{2.9}$$

where here and in the following we sometimes abbreviate $b_0(\Psi) = b_0(\Psi, \mathcal{C}(\sigma))$ respectively $b_0^i(\Upsilon) = b_0(\Upsilon, \mathcal{C}(\sigma^i))$ if it is clear from the context that we work on a fixed collar respectively on collars around a fixed collection $\{\sigma^i\}$ of simple closed geodesics. We will also use the convention that norms over $\mathcal{C}(\sigma)$ are always computed with respect to the hyperbolic metric $g = \rho^2(ds^2 + d\theta^2)$.

We recall the standard fact, see e.g. [23, Lemma 2.2] or [29], that the collar decay part of any holomorphic quadratic differential $\Upsilon \in \mathcal{H}(M,g)$ on a collar $\mathcal{C}(\sigma)$ around a simple closed geodesics of length $L_g(\sigma) \le 2\mathrm{arsinh}(1)$ decays rapidly along that collar in the sense that there exist universal numbers $\delta_4 \in (0, \mathrm{arsinh}(1))$ and C so that

$$\|\Upsilon - b_0(\Upsilon)dz^2\|_{L^\infty(\delta\text{-thin}(\mathcal{C}(\sigma)))} \le C\delta^{-2} e^{-\pi/\delta} \|\Upsilon\|_{L^2(\delta_4\text{-thick}(\mathcal{C}(\sigma)))} \quad \text{for every } 0 < \delta \le \delta_4. \tag{2.10}$$

Conversely, for every $\delta > 0$ we may bound an arbitrary element $\Upsilon \in \mathcal{H}(M,g)$ by

$$\|\Upsilon\|_{L^\infty(\delta\text{-thick}(M,g))} \le C_\delta \|\Upsilon\|_{L^1(\frac{\delta}{2}\text{-thick}(M,g))}, \tag{2.11}$$

in particular

$$\|\Upsilon\|_{L^\infty(\delta\text{-thick}(M,g))} \leq C_\delta \|\Upsilon\|_{L^2(M,g)}, \tag{2.12}$$

where C_δ depends on δ and the genus. Indeed, [24, Lemma 2.6] ensures that (2.12) holds true for $C_\delta = C\delta^{-1/2}$, C depending only on the genus, and indeed also with the L^∞-norm on the left hand side replaced by the C^k-norm (then with C depending additionally on k).

We also recall that the collar regions around disjoint geodesics are disjoint, that the arsinh(1) thin part of a hyperbolic surface is always contained in the union of the collars around the simple closed geodesics of length less than 2arsinh(1), that such geodesics are always disjoint and that their number is no more than $3(\gamma-1)$. We refer to Appendix A.2 for an overview of relevant results about hyperbolic surfaces.

If $\{\sigma^1,\ldots,\sigma^k\}$ is the set of all simple closed geodesics of (M,g) of length no more than some constant $2\eta < 2\text{arsinh}(1)$ we hence have that, as observed in [23, Lemma 2.4],

$$\|w\|_{L^\infty(M,g)} \leq C_\eta \|w\|_{L^1(M,g)} \tag{2.13}$$

for all elements $w \in W_\eta := \{\Upsilon \in \mathcal{H}(M,g) : b_0(\Upsilon, \mathcal{C}(\sigma^j)) = 0,\ 1 \leq j \leq k\}$. Here and in the following all constants are allowed to depend on the genus in addition to the indicated dependences unless explicitly said otherwise.

We also recall the well-known fact that along a curve $(g(t))_t$ of hyperbolic metrics with $g(0) = g$ and $\partial_t g(0) = \text{Re}\,\Upsilon$ for $\Upsilon \in \mathcal{H}(M,g)$ the evolution of the length $\ell(t)$ of the simple closed geodesic $\sigma_t \subset (M, g(t))$ homotopic to σ_0 is given by

$$\tfrac{d}{dt}\,\ell = -\tfrac{2\pi^2}{\ell}\text{Re}\,(b_0(\Upsilon, \mathcal{C}(\sigma_0))) \qquad \text{at } t = 0, \tag{2.14}$$

see e.g. [22, Remark 4.12] or [28]. So, as observed in [22, Remark 4.1], if we select any k disjoint simple closed geodesics σ^j in (M,g) we have

$$\ker(\partial\ell_1,\ldots,\partial\ell_k) = \{\Upsilon \in \mathcal{H}(M,g) : b_0^j(\Upsilon) = b_0(\Upsilon, \mathcal{C}(\sigma^j)) = 0 \text{ for } j = 1,\ldots,k\}, \tag{2.15}$$

where $\partial\ell_j$ is defined as in (1.9) and thus given by

$$\partial\ell_j(\Upsilon) = \tfrac{1}{2}(-\tfrac{2\pi^2}{\ell_j}\text{Re}\,(b_0^j(\Upsilon)) + \mathrm{i}\tfrac{2\pi^2}{\ell_j}\text{Re}\,(b_0^j(\mathrm{i}\Upsilon))) = -\tfrac{\pi^2}{\ell_j}b_0^j(\Upsilon). \tag{2.16}$$

In particular for $\{\sigma^1,\ldots,\sigma^k\}$ chosen as above as the set of geodesics of length $\leq 2\eta$ we have that $W_\eta = \ker(\partial\ell_1,\ldots,\partial\ell_k)$ and recall that as a consequence of [28, Theorem 3.7],

$$\text{codim}(W_\eta) = \text{codim}(\ker(\partial\ell_1,\ldots,\partial\ell_k)) = k,$$

see also [23] for an alternative proof in case that η is sufficiently small.

The fine properties of the elements of $W_\eta^\perp$, η small, were then analysed in [22] and we shall use in particular the following version of [22, Lemma 4.5], compare [31, Lemma 3.12] for a closely related result on the corresponding gradient basis.

Lemma 2.5 (Contents of [22, Lemma 4.5])**.** *For any genus $\gamma \geq 2$ there exists a number $\eta_1 \in (0, \text{arsinh}(1))$ so that for every $\bar\eta \in (0,\eta_1]$ the following holds true for a constant C that depends only on $\bar\eta$ and the genus:*

Let (M,g) be a closed oriented hyperbolic surface of genus γ and let $\{\sigma^1,\ldots,\sigma^k\}$ be the set of all simple closed geodesics in (M,g) of length no more than $2\bar\eta$. Define

$$W = W_{\bar\eta} := \{\Upsilon \in \mathcal{H}(M,g) \mid \partial\ell_j(\Upsilon) = 0, \quad j = 1,\ldots,k\},$$

$\partial\ell_j$ the differentials of the length coordinates associated to σ^j, compare (1.9).

Then there exists a (unique) basis $\tilde\Omega^1,\ldots,\tilde\Omega^k$ of $W^\perp$, normalised by $\|\tilde\Omega^j\|_{L^2(M,g)} = 1$, so that

$$b_0(\tilde\Omega^j, \mathcal{C}(\sigma^i)) = 0 \quad \textit{for } i \neq j \textit{ while } b_0(\tilde\Omega^j, \mathcal{C}(\sigma^j)) \in \mathbb{R}^+$$

and each $\tilde\Omega^j$ *is concentrated essentially only on the corresponding collar in the sense that*

$$\|\tilde\Omega^j\|_{L^\infty(M\setminus\mathcal{C}(\sigma^j),g)} \le C\ell_j^{3/2} \tag{2.17}$$

while on this collar

$$\|\tilde\Omega^j - b_0(\tilde\Omega^j,\mathcal{C}(\sigma^j))dz^2\|_{L^\infty(\mathcal{C}(\sigma^j),g)} \le C\ell_j^{3/2} \tag{2.18}$$

and

$$1 - C\ell_j^3 \le b_0(\tilde\Omega^j,\mathcal{C}(\sigma^j))\|dz^2\|_{L^2(\mathcal{C}(\sigma^j),g)} \le 1. \tag{2.19}$$

Furthermore, the $\tilde\Omega^j$*'s are nearly orthogonal in the sense that for* $i \neq j$

$$|\langle\tilde\Omega^i,\tilde\Omega^j\rangle_{L^2(M,g)}| \le C\ell_i^{3/2}\ell_j^{3/2} \tag{2.20}$$

and satisfy

$$\|\tilde\Omega^j\|_{L^\infty(M,g)} \le C\ell_j^{-1/2} \text{ with } \|\tilde\Omega^j\|_{L^\infty(\delta\text{-thick}(M,g))} \le C_\delta\ell_j^{3/2} \tag{2.21}$$

for any $\delta > 0$*, where* C_δ *depends on* δ*,* $\bar\eta$ *and the genus.*

We can view the $\tilde\Omega^j$ as renormalisations $\tilde\Omega^j = -\frac{\tilde\Theta^j}{\|\tilde\Theta^j\|_{L^2(M,g)}}$ of the dual basis $\{\tilde\Theta^j\}$ of $W^\perp$ to $\{\partial\ell_j\}$, i.e. of the elements

$$\tilde\Theta^j \in W^\perp \text{ for which } \delta_i^j = \partial\ell_i(\tilde\Theta^j) = -\tfrac{\pi^2}{\ell_j}b_0(\tilde\Theta^j,\mathcal{C}(\sigma^i)). \tag{2.22}$$

We note that by construction

$$\tilde\Theta^j = -\frac{\ell_j}{\pi^2 b_0(\tilde\Omega^j,\mathcal{C}(\sigma^j))}\tilde\Omega^j. \tag{2.23}$$

Remark 2.6. We note that a short argument (which is carried out in Appendix A.3) shows that, after possibly reducing $\eta_1 = \eta_1(\gamma)$, we obtain that in the setting of Lemma 2.5 furthermore

$$b_0(\tilde\Omega^j,\mathcal{C}(\sigma^j))\|dz^2\|_{L^2(\mathcal{C}(\sigma^j),g)} \ge \tfrac12 \text{ for every } j = 1,\dots,k \tag{2.24}$$

and in the following we shall always use Lemma 2.5 for $\eta_1 = \eta_1(\gamma)$ chosen in this way.

Combined with (A.12) this implies in particular that

$$\|\tilde\Theta^j\|_{L^2(M,g)} \le C\ell_j\|dz^2\|_{L^2(\mathcal{C}(\sigma^j))} \le C\ell_j^{-1/2}.$$

We note that the principal part of $\tilde\Theta^j$ on $\mathcal{C}(\sigma^j)$ is determined by (2.22) and recall that principal and collar decay parts are L^2-orthogonal, compare (2.9). This allows us to obtain a far more refined bound on $\|\tilde\Theta^j\|_{L^2}^2$ as a direct consequence of the above result from [22], while the expression for $\|dz^2\|_{L^1}$ from (A.10) furthermore gives a bound on the L^1-norm of $\tilde\Theta^j$, namely

Corollary 2.7. *In the setting of Lemma 2.5 the elements* $\tilde\Theta^j$ *characterised by* (2.22) *satisfy*

$$\big|\|\tilde\Theta^j\|_{L^2(M,g)} - \tfrac{\ell_j}{\pi^2}\|dz^2\|_{L^2(\mathcal{C}(\sigma^j),g)}\big| \le C\ell_j^{5/2}$$

and

$$\big|\|\tilde\Theta^j\|_{L^1(M,g)} - 8\pi\big| \le C\ell_j. \tag{2.25}$$

These results from [22] and [23] will be sufficient to prove that in the setting of Theorem 1.5 the gradient of $\lambda = \lambda_1$ is essentially a real multiple of $\tilde\Theta^1$. To estimate the size of $\|\nabla\lambda\|_{L^1}$, and hence the factor α in Theorem 1.5, we will furthermore need the following uniform Poincaré estimate for quadratic differentials proven in the joint work [21] of Topping and the second author

Theorem 2.8 (Theorem 1.1 of [21])**.** *For any genus $\gamma \geq 2$ there exists a constant $C_\gamma < \infty$ so that for any closed oriented hyperbolic surface (M,g) of genus γ the distance of any L^2-quadratic differential Υ from its holomorphic part is bounded by*

$$\|\Upsilon - P_g^{\mathcal{H}}(\Upsilon)\|_{L^1(M,g)} \leq C_\gamma \|\bar\partial \Upsilon\|_{L^1(M,g)}.$$

We note that it is crucial for our application that C_γ is a topological constant, depending only on the genus and not on geometric quantities such as the diameter of (M,g).

2.3. **Proof of Theorem 1.5.**

Before we give the proof of our first main result about the first eigenvalue $\lambda = \lambda_1$ of $-\Delta_g$, we note that as a corollary of the energy estimates stated in Section 2.1 we obtain

Lemma 2.9. *Let (M,g) be as in Lemma 2.2, i.e. so that for some $0 < \bar\delta \leq \delta_2 < \operatorname{arsinh}(1)$ each of the simple closed geodesics $\sigma^1, \dots, \sigma^k$ with length $L_g(\sigma^k) = \ell_k \leq 2\bar\delta$ is so that $M \setminus \sigma^j$ is disconnected. Then there exists a constant C depending only on $\bar\delta$ and the genus so that the following estimates are satisfied for the Hopf-differential $\Phi(u,g)$ of the normalised first eigenfunction u and for $\delta := \min(\frac{1}{2}\bar\delta, \delta_4)$, $\delta_4 \in (0, \operatorname{arsinh}(1))$ the universal constant from (2.10):*

For every $\Upsilon \in \mathcal{H}(M,g)$ and every subset F of $\bar\delta$-thick(M,g), in particular for $F = M \setminus \bigcup_j \mathcal{C}(\sigma^j)$, we have

$$|\langle \Upsilon, \Phi(u,g)\rangle_{L^2(F,g)}| \leq C\lambda^2 \|\Upsilon\|_{L^2(\delta\text{-}thick(M,g))} \tag{2.26}$$

while for every $j = 1, \dots, k$

$$|\langle \Upsilon - b_0(\Upsilon, \sigma^j) dz^2, \Phi(u,g)\rangle_{L^2(\mathcal{C}(\sigma^j))}| \leq C\lambda^2 \|\Upsilon\|_{L^2(\delta\text{-}thick(M,g))} \tag{2.27}$$

$$|\mathrm{Re}\,(\langle i dz^2, \Phi(u,g)\rangle_{L^2(\mathcal{C}(\sigma^j))})| \leq C\lambda^2 \ell_j^{-1}. \tag{2.28}$$

Proof of Lemma 2.9. Let $F \subset \bar\delta$-thick(M,g). Using the bounds on the energy of u, and consequently the L^1-norm of Φ, on $\bar\delta$-thick(M,g) from Lemma 2.2 we may bound

$$|\langle \Upsilon, \Phi(u,g)\rangle_{L^2(F,g)}| \leq C\lambda^2 \|\Upsilon\|_{L^\infty(\bar\delta\text{-thick}(M,g))} \leq C\lambda^2 \|\Upsilon\|_{L^2(\delta\text{-thick}(M,g))},$$

see (2.11) for the last step, as claimed in (2.26). Here and in the following $C = C(\bar\delta, \gamma)$.

Applying this estimate for $F = \delta$-thick$(\mathcal{C}(\sigma^j))$ and using once more the bound on $\|\Phi\|_{L^1}$ from Lemma 2.2 (as well as that $|dz^2|_g = 2\rho^{-2} \leq 2\pi^2\delta^{-2}$ on δ-thick(M,g), see (A.6)) then gives

$$\begin{aligned}
|\langle \Upsilon - b_0^j(\Upsilon) dz^2, \Phi\rangle_{L^2(\mathcal{C}(\sigma^j))}| &\leq |\langle \Upsilon - b_0^j(\Upsilon) dz^2, \Phi\rangle_{L^2(\delta\text{-thin}(\mathcal{C}(\sigma^j)))}| + |\langle \Upsilon, \Phi\rangle_{L^2(\delta\text{-thick}(\mathcal{C}(\sigma^j)))}| \\
&\quad + |b_0^j(\Upsilon)| \cdot \|dz^2\|_{L^\infty(\delta\text{-thick}(\mathcal{C}(\sigma^j)))} \cdot \|\Phi\|_{L^1(\delta\text{-thick}(\mathcal{C}(\sigma^j)))} \\
&\leq C\lambda^2 (\|\Upsilon\|_{L^2(\delta\text{-thick}(M,g))} + |b_0^j(\Upsilon)|) \\
&\qquad + |\langle \Upsilon - b_0^j(\Upsilon) dz^2, \Phi\rangle_{L^2(\delta\text{-thin}(\mathcal{C}(\sigma^j)))}| \\
&\leq C\lambda^2 \|\Upsilon\|_{L^2(\delta\text{-thick}(M,g))} + |\langle \Upsilon - b_0^j(\Upsilon) dz^2, \Phi\rangle_{L^2(\delta\text{-thin}(\mathcal{C}(\sigma^j)))}|
\end{aligned} \tag{2.29}$$

where we used in the last step that the principal part is L^2-orthogonal to the collar decay part on every subcylinder of the collar so, as $\delta \leq \frac{1}{2}\delta_2 \leq \frac{1}{2}\operatorname{arsinh}(1)$,

$$|b_0^j(\Upsilon)| \leq \|dz^2\|^{-1}_{L^2(\frac{1}{2}\operatorname{arsinh}(1)\text{-thick}(\mathcal{C}(\sigma^j)))} \|\Upsilon\|_{L^2(\delta\text{-thick}(M,g))} \leq C\|\Upsilon\|_{L^2(\delta\text{-thick}(M,g))}.$$

To bound the remaining term we split δ-thin$(\mathcal{C}(\sigma^j))$ into regions of injectivity radius $\mathrm{inj}_g(p) \in [2^{-k-1}\delta, 2^{-k}\delta)$. On such regions we can bound $\|\Phi\|_{L^1} \leq C\|du\|^2_{L^2} \leq C\lambda^2 (2^{-k-1}\delta)^{-1}$ thanks to

Lemma 2.2 while $\Upsilon - b_0^j(\Upsilon)dz^2$ is controlled by (2.10). Combined this gives

$$\begin{aligned}|\langle \Upsilon - b_0^j(\Upsilon)dz^2, \Phi\rangle_{L^2(\delta\text{-thin}(\mathcal{C}(\sigma^j)))}| &\le C\sum_{k\ge 0} e^{-2^k\pi/\delta}\cdot(2^{-k}\delta)^{-3}\lambda^2\|\Upsilon\|_{L^2(\delta\text{-thick}(\mathcal{C}(\sigma^j))}\\ &\le C\lambda^2\|\Upsilon\|_{L^2(\delta\text{-thick}(\mathcal{C}(\sigma^j))},\end{aligned}$$

and inserting this into (2.29) gives the second claim (2.27) of the lemma.

Finally, we note that by (2.7) and (2.1)

$$\begin{aligned}|\mathrm{Re}\,\langle \mathrm{i}dz^2, \Phi(u,g)\rangle_{L^2(\mathcal{C}(\sigma^j))}| &= 2|\langle \mathrm{Re}\,(\Phi), ds\otimes d\theta + d\theta\otimes ds\rangle_{L^2(\mathcal{C}(\sigma^j))}|\\ &\le C\cdot\int_{\mathcal{C}(\sigma^j)} |u_s|\cdot|u_\theta|\rho^{-4}dv_g.\end{aligned}$$

This expression can be bounded using a combination of the angular energy estimate (2.5) from Lemma 2.3 and of the usual energy estimates of Lemma 2.2, this time applied on $\frac{1}{2}\ell_j$-thick$(M,g)\supset \mathcal{C}(\sigma^j)$, to obtain the final claim (2.28) of the lemma as

$$\begin{aligned}\int_{\mathcal{C}(\sigma^j)} |u_s|\cdot|u_\theta|\rho^{-4}dv_g &= \int_{-X(\ell_j)}^{X(\ell_j)}\int_{S^1} |u_s|\cdot|u_\theta|\rho^{-2}dsd\theta\\ &\le \left(\int_{-X(\ell_j)}^{X(\ell_j)}\int_{S^1}|u_s|^2dsd\theta\right)^{\frac12}\left(\int_{-X(\ell_j)}^{X(\ell_j)}\int_{S^1}|u_\theta|^2\rho^{-4}dsd\theta\right)^{\frac12}\\ &\le C\|du\|_{L^2(\frac12\ell_j\text{-thick}(M,g))}\cdot\left[\lambda^2(1+\ell_j^{-1})\right]^{\frac12}\\ &\le C\ell_j^{-1}\lambda^2.\end{aligned}$$

□

We are now in a position to give the proof of our first main result – Theorem 1.5. We shall split this proof into three steps, showing first that $\nabla\log(\lambda)$ is essentially given by a complex multiple $(\alpha+\mathrm{i}\tilde{c})\tilde{\Theta}^1$ of $\tilde{\Theta}^1$, then proving that $\tilde{c}$ is small, i.e. that the factor $(\alpha+\mathrm{i}\tilde{c})$ is essentially real and finally estimating the size of α. Each of these steps relies on a combination of general results for holomorphic quadratic differentials from [21, 22, 23] that we recalled in the previous section and on properties specific to the first eigenfunction, such as the energy estimates that we stated in Section 2.1 and the resulting estimates collected in Lemma 2.9.

So let (M,g) be as in the theorem so that $\mathrm{inj}\,|_{M\setminus\mathcal{C}(\sigma^1)}\ge\hat{\delta}$ and so that $\ell_1 = L_g(\sigma^1)\le\ell_0(\hat{\delta},\gamma)$, which we recall is chosen in a way that guarantees that the first eigenvalue is simple and hence that its gradient is well defined and described by Lemma 2.1. We set $\bar{\eta}=\min(\eta_1,\hat{\delta},\delta_2)$, where $\eta_1=\eta_1(\gamma)$ is the constant from Lemma 2.5 while $\delta_2=\delta_2(\gamma)$ is the constant from Lemmas 2.2 and 2.9.

We first consider the case that $\ell_1 > 2\bar{\eta}$. We will later see that $\partial\ell_1$ is non-trivial, compare Lemma 2.13, so also in this case $\mathrm{codim}(\ker(\partial\ell_1)) = 1$ and $\tilde{\Theta}^1$ is well defined. The claim then trivially follows as the constant C in the theorem is allowed to depend on $\hat{\delta}$ and the genus (and hence $\bar{\eta}$) and since a combination of Lemma 2.1 and (2.11) (applied for $\delta=\bar{\eta}\le\mathrm{inj}(M,g)$) allows us to estimate

$$\|\nabla\log(\lambda)\|_{L^\infty(M,g)}\le C\lambda^{-1}\|P_g^{\mathcal{H}}(\Phi)\|_{L^\infty(M,g)}\le C\lambda^{-1}\|P_g^{\mathcal{H}}(\Phi)\|_{L^1(M,g)}\le C\lambda^{-1}\|\Phi\|_{L^1(M,g)}\le C$$

for $C=C(\bar{\eta},\gamma)$, where we applied (2.8) in the penultimate step and (2.3) in the last step.

We may thus from now on assume without loss of generality that $\ell_1\le 2\bar{\eta}=2\min(\eta_1,\hat{\delta},\delta_2)$. We apply Lemma 2.5 with this $\bar{\eta}$, whose choice guarantees that σ^1 is the only simple closed geodesic of length less than $2\bar{\eta}$. As $\ker(\partial\ell_1)^\perp$ is spanned by the element $\tilde{\Omega}^1$ described in this lemma, we may thus write

$$P^{\mathcal{H}}(\Phi(u,g)) = b\cdot\tilde{\Omega}^1 + P_g^{\ker(\partial\ell_1)}(\Phi(u,g))$$

for some $b\in\mathbb{C}$ that is analysed later.

In a first step we combine the estimates for elements of $\ker(\partial\ell_1)$ from [23] that we recalled in (2.13) with the energy estimate of Lemma 2.2 and the resulting bounds on inner products of Φ from Lemma 2.9, to show

Lemma 2.10. *Suppose that (M,g) is as in Theorem 1.5 with $\ell_1 \le 2\bar{\eta} := 2\min(\eta_1,\hat{\delta},\delta_2)$. Then the orthogonal projection of the Hopf-differential $\Phi(u,g)$ of the normalized first eigenfunction u onto $\ker(\partial\ell_1)$ is bounded by*

$$\|P_g^{\ker(\partial\ell_1)}(\Phi(u,g))\|_{L^\infty(M,g)} \le C\lambda^2$$

where C depends only on the lower bound $\hat{\delta}>0$ on $\mathrm{inj}_g|_{M\setminus\mathcal{C}(\sigma^1)}$ and the genus of M.

Proof. We set $w = P_g^{\ker(\partial\ell_1)}(\Phi)$ and recall that $b_0(\cdot,\mathcal{C}(\sigma^1))=0$ for any element of $\ker(\partial\ell_1)$, so in particular for w, compare (2.16). We can thus apply Lemma 2.9 (for $\bar{\delta}=\bar{\eta}$) to obtain

$$\begin{aligned}\|w\|^2_{L^2(M,g)} &= \langle w,\Phi(u,g)\rangle_{L^2(M,g)} = \langle w-b_0^1(w)dz^2,\Phi(u,g)\rangle_{L^2(\mathcal{C}(\sigma^1))} + \langle w,\Phi(u,g)\rangle_{L^2(M\setminus\mathcal{C}(\sigma^1),g)}\\ &\le C\lambda^2\|w\|_{L^2(M,g)}.\end{aligned}$$

Combined with (2.13) this yields

$$\|w\|_{L^\infty(M,g)} \le C\cdot\|w\|_{L^2(M,g)} \le C\lambda^2$$

where C depends only on the genus and on $\hat{\delta}$. □

From Lemma 2.10 we thus obtain that $P_g^{\mathcal{H}}(\Phi)$ is, up to a well controlled error term, a complex multiple of $\tilde{\Omega}^1$. In a next step we show that this factor is almost real which, as we shall see later on, is equivalent to proving that Dehn-twists on $\mathcal{C}(\sigma^1)$ do not have a significant effect on the first eigenvalue.

Lemma 2.11. *Suppose that (M,g) is as in Theorem 1.5 with $\ell_1 \le 2\bar{\eta} := 2\min(\eta_1,\hat{\delta},\delta_2)$. Then we have*

$$|\mathrm{Im}\,\langle P_g^{\mathcal{H}}(\Phi(u,g)),\tilde{\Omega}^1\rangle_{L^2(M,g)}| \le C\ell_1^{1/2}\lambda^2,\ \text{ where } C=C(\hat{\delta},\gamma).$$

As in the proof of the previous Lemma 2.10 we use two different types of tools for the proof of this lemma, one given by a general result about the structure of the space of holomorphic quadratic differentials, this time the properties of the elements $\tilde{\Omega}^j$ from [22] that we recalled in Lemma 2.5, and properties of the first eigenfunction from Lemma 2.9.

Proof of Lemma 2.11. We note that the $\langle P_g^{\mathcal{H}}(\Phi(u,g)),\tilde{\Omega}^1\rangle_{L^2(M,g)} = \langle\Phi(u,g),\tilde{\Omega}^1\rangle_{L^2(M,g)}$ is essentially given by the inner product of $\Phi=\Phi(u,g)$ and the principal part $b_0^1(\tilde{\Omega}^1)dz^2$ of $\tilde{\Omega}^1$ on $\mathcal{C}(\sigma^1)$; to be more precise, estimate (2.21) from Lemma 2.5 and Lemma 2.9 (applied for $\bar{\delta}=\bar{\eta}$) allow us to bound

$$\begin{aligned}&|\langle\Phi,\tilde{\Omega}^1\rangle_{L^2(M,g)} - \langle\Phi,b_0^1(\tilde{\Omega}^1)dz^2\rangle_{L^2(\mathcal{C}(\sigma^1))}|\\ &\quad= |\langle\Phi,\tilde{\Omega}^1\rangle_{L^2(M\setminus\mathcal{C}(\sigma^1),g)} + \langle\Phi,\tilde{\Omega}^1-b_0^1(\tilde{\Omega}^1)dz^2\rangle_{L^2(\mathcal{C}(\sigma^1))}|\\ &\quad\le C\|\tilde{\Omega}^1\|_{L^2(\delta\text{-thick}(M,g))}\cdot\lambda^2 \le C\ell_1^{3/2}\lambda^2\end{aligned} \tag{2.30}$$

where $\delta=\min(\frac{1}{2}\bar{\eta},\delta_4(\gamma))$, $\delta_4(\gamma)$ from (2.10), depends only on $\hat{\delta}$ and the genus and hence so does the constant C.

Since the principal part of $\tilde\Omega^1$ on $\mathcal{C}(\sigma^1)$ is real, Lemma 2.9 furthermore gives

$$|\mathrm{Im}\langle\Phi, b_0^1(\tilde\Omega^1)dz^2\rangle_{L^2(\mathcal{C}(\sigma^1))}| = |b_0^1(\tilde\Omega^1)|\cdot|\mathrm{Re}\langle\Phi, idz^2\rangle_{L^2(\mathcal{C}(\sigma^1))}| \le C\ell_1^{3/2}\cdot C\lambda^2\ell_1^{-1} \le C\ell_1^{1/2}\lambda^2$$

where we used (A.14) in the penultimate step. Combined with (2.30) this yields the claim of the lemma. $\square$

At this stage we thus know that we can write

$$\nabla\log(\lambda) = -P_g^H(\mathrm{Re}\,(\frac{1}{2\lambda}\Phi(u,g)) = \mathrm{Re}\,(P^{\mathcal{H}}(-\frac{1}{2\lambda}\Phi(u,g))) = \mathrm{Re}\,(\alpha\tilde\Theta^1 + R) \tag{2.31}$$

for the real multiple $\tilde\Theta^1$ of $\tilde\Omega^1$ given by (2.23), a real number α and an error term of the form

$$R = \mathrm{i}c_0\tilde\Omega^1 - \frac{1}{2\lambda}P_g^{\ker(\partial\ell_1)}\Phi(u,g), \text{ for some } c_0\in\mathbb{R}, \tag{2.32}$$

where we note that $|c_0| = \frac{1}{2\lambda}|\mathrm{Im}\,\langle\Phi,\tilde\Omega^1\rangle| \le C\ell_1^{1/2}\lambda$ thanks to Lemma 2.11. Since $\|\tilde\Omega^1\|_{L^\infty(M,g)} \le C\ell_1^{-1/2}$ and since the second term in (2.32) is controlled by Lemma 2.10 this implies that

$$\|R\|_{L^\infty(M,g)} \le C\lambda \le C\ell_1$$

which establishes the claim (1.11) of the theorem.

To prove the remaining claim (1.10) of Theorem 1.5 we now show that the coefficient α in (2.31) satisfies $|\alpha| = \frac{1}{8\pi} + O(\ell_1\log(\ell_1))$ and, for sufficiently small ℓ_1, also $\alpha > 0$.

To prove the estimate on the absolute value of α we note that since $\|\tilde\Theta^1\|_{L^1(M,g)} = 8\pi + O(\ell_1)$, compare (2.25), while $\|\Phi\|_{L^1(M,g)} = 2\lambda$, compare (2.3), the above bound on the error term implies that

$$|\alpha| = \|\tilde\Theta^1\|^{-1}_{L^1(M,g)}\big[\tfrac{1}{2\lambda}\|P^{\mathcal{H}}(\Phi)\|_{L^1(M,g)} + O(\ell_1)\big] = \frac{1}{(8\pi + O(\ell_1))}\cdot\frac{\|P^{\mathcal{H}}(\Phi)\|_{L^1(M,g)}}{\|\Phi\|_{L^1(M,g)}} + O(\ell_1).$$

It thus suffices to prove that

$$\big|\|P_g^{\mathcal{H}}(\Phi)\|_{L^1(M,g)} - \|\Phi\|_{L^1(M,g)}\big| \le C\lambda^2|\log(\ell_1)|$$

which follows from the following lemma.

Lemma 2.12. *Let (M,g) be as in Theorem 1.5 with $\ell_1 \le 2\bar\eta := 2\min(\eta_1,\hat\delta,\delta_2)$. Then the Hopf-differential of the first eigenfunction u, chosen as always with $\|u\|_{L^2(M,g)} = 1$, satisfies*

$$\|\Phi - P_g^{\mathcal{H}}(\Phi)\|_{L^1(M,g)} \le C\lambda^2|\log(\ell_1)|,$$

C depending only on γ and the lower bound $\hat\delta$ on $\mathrm{inj}\,|_{M\setminus\mathcal{C}(\sigma^1)}$.

The crucial ingredient in the proof of this lemma is the uniform Poincaré estimate for quadratic differentials from the joint work [21] of P. Topping and the second author that we recalled in Theorem 2.8.

Proof of Lemma 2.12. To derive the lemma from Theorem 2.8 we need to prove that $\Phi = \Phi(u,g)$ is *almost holomorphic* in the sense that the estimate

$$\|\bar\partial\Phi\|_{L^1(M,g)} \le C\lambda^2|\log(\ell_1)|$$

holds true for a constant C that depends only on the genus and on $\hat\delta$.

To this end we recall that the antiholomorphic derivative of the Hopf-differential of maps from a surface to an arbitrary Riemannian manifolds is bounded in terms of the tension field, so in our situation simply by the Laplacian. To be more precise, working in local isothermal coordinates (x,y), $z=x+\mathrm{i}y$, we may write

$$\bar\partial\Phi=\frac{1}{2}\left(\partial_x\phi+\mathrm{i}\partial_y\phi\right)d\bar z\otimes dz^2=(u_{xx}+u_{yy})\cdot(u_x-\mathrm{i}u_y)d\bar z\otimes dz^2$$

and hence estimate

$$|\bar\partial\Phi|_g\leq\rho^2|\Delta_g u|\cdot\rho|du|_g\cdot|dz|_g^3=2\sqrt{2}|\Delta_g u|\cdot|du|_g.$$

Since our eigenfunction u is uniformly bounded, c.f. Remark 2.4, with $\|u\|_{L^2}=1$ we thus have (2.33)

$$\begin{aligned}\|\bar\partial\Phi\|_{L^1(M,g)}&\leq 2\sqrt{2}\lambda\int|u|\cdot|du|_g dv_g\leq 2\sqrt{2}\lambda\Big[\|du\|_{L^2(M\setminus\mathcal{C}(\sigma^1))}+\|u\|_{L^\infty(M,g)}\int_{\mathcal{C}(\sigma^1)}|du|_g dv_g\Big]\\&\leq C\lambda^2+C\lambda\int_{\mathcal{C}(\sigma^1)}|du|_g dv_g\end{aligned}$$

where we used the assumption that $M\setminus\mathcal{C}(\sigma^1)\subset\hat\delta$-thick$(M,g)$ and the energy estimate of Lemma 2.2 in the last step. To obtain a bound of $C\ell_1|\log(\ell_1)|$ for the last integral instead of just the trivial bound of $C\lambda^{1/2}\leq C\ell_1^{1/2}$, we note that du is small near the ends of the collar while regions near the centre of the collar have small volume. We thus split the collar into subsets

$$C_k:=\{p\in\mathcal{C}(\sigma^1):2^{k-1}\ell_1\leq\mathrm{inj}_g(p)<2^k\ell_1\}\quad 0\leq k\leq\bar K$$

whose total number is bounded by $|\bar K|\leq C|\log(\ell_1)|$ as the injectivity radius is bounded from above uniformly on collars around geodesics of length $\ell\leq 2\mathrm{arsinh}(1)$, compare (A.6). Combining the bound on $\mathrm{Area}_g(C_k)\leq\mathrm{Area}_g(2^k\ell_1\text{-thin}(\mathcal{C}(\sigma^1))\leq C2^k\ell_1$, compare (A.5), with the energy estimate from Lemma 2.2 gives that for every k

$$\int_{C_k}|du|_g dv_g\leq\mathrm{Area}_g(C_k)^{1/2}\|du\|_{L^2(2^{k-1}\ell_1\text{-thick}(\mathcal{C}(\sigma^1)))}\leq C\cdot(2^k\ell_1)^{1/2}\cdot(\frac{1}{2^k\ell_1}\lambda^2)^{1/2}\leq C\lambda.$$

Thus (2.33) reduces to

$$\|\bar\partial\Phi\|_{L^1(M,g)}\leq C\lambda^2+C\bar K\lambda^2\leq C\lambda^2|\log(\ell_1)|$$

which is the bound that we needed to derive the lemma from Theorem 2.8. $\square$

Having thus established that the coefficient α in (2.31) is so that $|\alpha|=\frac{1}{8\pi}+O(\ell_1)$ we finally complete the proof of Theorem 1.5 by showing that $\alpha>0$ for sufficiently small ℓ_1. Intuitively this is clear as one expects the eigenvalue to decrease as ℓ_1 itself decreases as this causes the collar to stretch out. To obtain an analytical proof we note that since the real part of the error term R described in (2.32) is orthogonal to $\mathrm{Re}\,(\tilde\Theta^1)=-\frac{\ell_1}{\pi^2 b_0^1(\tilde\Omega^1)}\mathrm{Re}\,(\tilde\Omega^1)$, compare also (2.7), we have

$$\begin{aligned}\langle\mathrm{Re}\,(\tilde\Omega^1),\mathrm{Re}\,(\Phi)\rangle&=\langle\mathrm{Re}\,(\tilde\Omega^1),\mathrm{Re}\,(P^{\mathcal{H}}\Phi)\rangle=\langle\mathrm{Re}\,(\tilde\Omega^1),-2\lambda\alpha\mathrm{Re}\,(\tilde\Theta^1)\rangle\\&=2\lambda\frac{\ell_1}{\pi^2 b_0^1(\tilde\Omega^1)}\|\mathrm{Re}\,(\tilde\Omega^1)\|^2_{L^2(M,g)}\cdot\alpha=\lambda\frac{\ell_1}{\pi^2 b_0^1(\tilde\Omega^1)}\cdot\alpha.\end{aligned}\tag{2.34}$$

As $0<b_0^1(\tilde\Omega^1)\leq C\ell_1^{3/2}$ for some universal constant C, compare (A.14), we thus need to prove that this inner product is positive or equivalently that $\langle\mathrm{Re}\,(\tilde\Omega^1),\mathrm{Re}\,(\Phi)\rangle>-|\langle\mathrm{Re}\,(\tilde\Omega^1),\mathrm{Re}\,(\Phi)\rangle|$.

To see that this holds true (for ℓ_1 sufficiently small) we first note that (2.34), combined with the bound on $|\alpha|$ that we already derived, yields that

$$-|\langle\mathrm{Re}\,(\tilde\Omega^1),\mathrm{Re}\,(\Phi)\rangle|\leq-\lambda|\frac{1}{8\pi}+O(\ell_1)|\frac{\ell_1}{\pi^2 C\ell_1^{3/2}}\quad\leq-(\tilde c+O(\ell_1))\ell_1^{-1/2}\lambda\tag{2.35}$$

for some universal $\tilde{c} > 0$. The main term in this inner product is however bounded *below* by

$$\langle \mathrm{Re}\,(b_0^1(\tilde\Omega^1)dz^2), \mathrm{Re}\,(\Phi)\rangle_{L^2(\mathcal{C}(\sigma^1))} = 2b_0^1(\tilde\Omega^1)\int(|u_s|^2 - |u_\theta|^2)\rho^{-4}dv_g$$
$$\geq -2b_0^1(\tilde\Omega^1)\int_{-X(\ell_1)}^{X(\ell_1)}\int_{S^1}|u_\theta|^2\rho^{-2}dsd\theta \geq -C\ell_1^{3/2}\lambda^2$$

thanks to the angular energy estimate (2.6) from Lemma 2.3. Combining Lemma 2.9 with the bound (2.21) on $\tilde\Omega^1$ hence allows us to conclude that

$$\langle \mathrm{Re}\,(\tilde\Omega^1), \mathrm{Re}\,(\Phi)\rangle \geq -C\ell_1^{3/2}\lambda^2 - |\langle \mathrm{Re}\,(\tilde\Omega^1 - b_0^1(\tilde\Omega^1)dz^2), \mathrm{Re}\,(\Phi)\rangle_{L^2(\mathcal{C}(\sigma^1))}|$$
$$- |\langle \mathrm{Re}\,\tilde\Omega^1, \mathrm{Re}\,\Phi\rangle_{L^2(M\setminus\mathcal{C}(\sigma^1))}|$$
$$\geq -C\ell_1^{3/2}\lambda^2 - C\lambda^2\|\tilde\Omega^1\|_{L^2(\delta\text{-thick}(M,g))} \geq -C\ell_1^{3/2}\lambda^2$$

which is of course strictly larger than the expression in (2.35) for small ℓ_1. This completes the proof that $\alpha > 0$ for small ℓ_1 and hence the proof of Theorem 1.5.

2.4. Holomorphic quadratic differentials associated to the Fenchel-Nielson coordinates.

While the known results on holomorphic quadratic differentials (from [21, 22, 23] or alternatively from [31]) are sufficient to establish Theorem 1.5, to prove our other main results we require similar results for bases of holomorphic quadratic differentials that are dual to a full set of Fenchel-Nielson coordinates. In this section we collect these new results which will be proven later on in Section 4.

Given any collection $\mathcal{E} = \{\sigma^1, \dots, \sigma^{3(\gamma-1)}\}$ of simple closed geodesics which decompose an oriented closed hyperbolic surface (M, g) into pairs of pants we denote by (ℓ_j, ψ_j) the corresponding Fenchel-Nielson coordinates, compare Appendix A.2. Furthermore we let $\eta \in (0, \mathrm{arsinh}(1))$ and $\bar{L} < \infty$ be so that

$$\mathcal{E} \text{ contains all simple closed geodesics } \sigma \text{ of } (M,g) \text{ of length } L_g(\sigma) \leq 2\eta \tag{2.36}$$

and

$$L_g(\sigma) \leq \bar{L} \text{ for every } \sigma \in \mathcal{E}. \tag{2.37}$$

We want to consider elements $\Lambda^1, \dots, \Lambda^{3(\gamma-1)}$ of $\mathcal{H}(M, g)$ which induce only a change of the corresponding length coordinates in the sense that

$$d\ell_i(\mathrm{Re}\,\Lambda^j) = \delta_i^j \text{ and } d\psi_i(\mathrm{Re}\,\Lambda^j) = 0 \text{ for every } i = 1, \dots, 3(\gamma-1), \tag{2.38}$$

as well as elements $\Psi^1, \dots, \Psi^{3(\gamma-1)}$ which only change the corresponding twist coordinate

$$d\ell_i(\mathrm{Re}\,\Psi^j) = 0 \text{ and } d\psi_i(\mathrm{Re}\,\Psi^j) = \delta_i^j \text{ for } i = 1, \dots, 3(\gamma-1), \tag{2.39}$$

where we use the sign-convention for the twist coordinates from Remark A.3.

We recall that [28, Theorem 3.7] establishs that the map introduced in (2.42) is an isomorphism and hence that we can obtain a basis $\{\Theta^j\}_{j=1}^{3(\gamma-1)}$ of the whole space $\mathcal{H}(M, g)$ which is dual to the complex differentials $\partial\ell_j$, i.e. is so that

$$\partial\ell_i(\Theta^j) = -\frac{\pi^2}{\ell_j}b_0(\Theta^j, \mathcal{C}(\sigma^i)) = \delta_i^j. \tag{2.40}$$

In the present work we will need that the renormalised elements

$$\Omega^j := -\frac{\Theta^j}{\|\Theta^j\|_{L^2(M,g)}} \tag{2.41}$$

still satisfy estimates (2.17)–(2.20) with the sharp error rate of $O(\ell_j^{3/2})$ and constants that depend only on η, $\bar{L}$ and the genus. To be more precise, in Section 4.2 we will show

Lemma 2.13. *Let (M,g) be any closed oriented hyperbolic surface of genus $\gamma \geq 2$ and let $\mathcal{E} = \{\sigma^1,\ldots,\sigma^{3(\gamma-1)}\}$ be any set of simple closed geodesics that decompose (M,g) into pairs of pants. We denote by $\{\Theta^j\}_{j=1}^{3(\gamma-1)}$ the dual basis correspesponding to the isomormophism*

$$\mathcal{H}(M,g) \ni \Upsilon \mapsto \big(\partial\ell_1(\Upsilon),\ldots,\partial\ell_{3(\gamma-1)}(\Upsilon)\big) \in \mathbb{C}^{3(\gamma-1)} \tag{2.42}$$

and by $\{\Omega^j\}_{j=1}^{3(\gamma-1)}$ its renormalised elements, see (2.41).

After reordering $\mathcal{E}$ so that the simple closed geodesics of length no more than $2\bar{\eta} := 2\min(\eta,\eta_1)$, η_1 as in Lemma 2.5, are given by $\sigma^1,\ldots,\sigma^k$, $k \in \{0,\ldots,3(\gamma-1)\}$, we furthermore let $\{\tilde{\Omega}^j\}_{j=1}^k$ respectively $\{\tilde{\Theta}^j\}_{j=1}^k$ be the bases of $\ker(\partial\ell_1,\ldots,\partial\ell_k)^\perp$ from Lemma 2.5 respectively Corollary 2.7.

Then the following claims hold true for a constant C that depends only on the genus and the numbers η, $\bar{L}$ from (2.36) and (2.37):

For every $j = 1,\ldots,3(\gamma-1)$

$$\|\Theta^j\|_{L^2(M,g)} \leq C\ell_j^{-1/2} \tag{2.43}$$

while for $j = 1,\ldots,k$ furthermore $\Theta^j \sim \tilde{\Theta}^j$ and $\Omega^j \sim \tilde{\Omega}^j$ in the sense that

$$\Theta^j = \tilde{\Theta}^j + v^j \text{ for some } v^j \in \ker(\partial\ell_1,\ldots,\partial\ell_k) \tag{2.44}$$

with

$$\|v^j\|_{L^\infty(M,g)} \leq C\ell_j, \tag{2.45}$$

respectively

$$\Omega^j = a_j\tilde{\Omega}^j + w^j \text{ for some } a_j \in \mathbb{R}^+ \text{ and } w^j \in \ker(\partial\ell_1,\ldots,\partial\ell_k) \tag{2.46}$$

for which

$$|1-a_j| \leq C\ell_j^3 \text{ while } \|w^j\|_{L^\infty(M,g)} \leq C\ell_j^{3/2}. \tag{2.47}$$

As we shall see, the key step in the proof of the above lemma will be to establish (2.43), and note that this upper bound on the dual basis is equivalent to establishing bounds on the inverse of the isomorphism (2.42).

As an immediate consequence of Lemma 2.5 we obtain that the renormalised elements Ω^j of this basis of $\mathcal{H}$ satisfy the same estimates as obtained in [22] for $\tilde{\Omega}^j$, respectively in [31] for the gradient elements, namely

Corollary 2.14. *In the setting of Lemma 2.13 there exist constants C and $\varepsilon_1 > 0$ that depend only on the genus and the constants η and $\bar{L}$ for which the assumptions (2.36) and (2.37) are satisfied so that the estimates (2.17)–(2.21) of Lemma 2.5 remain valid also for the basis $\{\Omega^j\}$ of $\mathcal{H}(M,g)$, i.e. for every $j = 1,\ldots,3(\gamma-1)$ and $i \neq j$ we have*

$$\|\Omega^j\|_{L^\infty(M\setminus\mathcal{C}(\sigma^j),g)} \leq C\ell_j^{3/2}, \tag{2.48}$$

$$\|\Omega^j - b_0(\Omega^j,\mathcal{C}(\sigma^j))dz^2\|_{L^\infty(M,g)} \leq C\ell_j^{3/2}, \tag{2.49}$$

$$1 - C\ell_j^3 \leq b_0(\Omega^j,\mathcal{C}(\sigma^j))\|dz^2\|_{L^2(\mathcal{C}(\sigma^j),g)} \leq 1, \tag{2.50}$$

$$|\langle\Omega^i,\Omega^j\rangle_{L^2(M,g)}| \leq C\ell_i^{3/2}\ell_j^{3/2}. \tag{2.51}$$

Furthermore we have a uniform lower bound on

$$b_0(\Omega^j,\mathcal{C}(\sigma^j))\|dz^2\|_{L^2(\mathcal{C}(\sigma^j),g)} \geq \varepsilon_1 > 0. \tag{2.52}$$

Remark 2.15. The uniform lower bound on the principal parts (2.52), together with (2.9), implies in particular that the estimate

$$\| \sum c_j \Omega^j \|^2_{L^2(M,g)} \geq \varepsilon_1^2 \cdot \sum |c_j|^2 \text{ for all } c_j \in \mathbb{C}$$

holds true for the constant $\varepsilon_1 > 0$ obtained in the above corollary.

Remark 2.16. We note that Θ^j and Ω^j are related by

$$\Theta^j = -\frac{\ell_j}{\pi^2 b_0(\Omega^j, \mathcal{C}(\sigma^j))}\Omega^j \tag{2.53}$$

and that also the estimates from Corollary 2.7 remain valid for the Θ^j. To be more precise, we first note that combining (2.50) and (2.52) with (A.11) and (A.12), respectively using (2.40) gives that

$$c\ell_j^{3/2} \leq b_0^j(\Omega^j) \leq C\ell_j^{3/2} \text{ while } b_0^j(\Theta^j) = -\frac{\ell_j}{\pi^2} \tag{2.54}$$

for some $c = c(\eta, \bar{L}, \gamma) > 0$ and $C = C(\bar{L})$. In particular

$$\|\Theta^j\|_{L^2(M,g)} \leq C\ell_j^{-1/2} \tag{2.55}$$

holds true for a constant $C = C(\eta, \bar{L}, \gamma)$ while for every $\delta > 0$ there exists $C_\delta = C(\delta, \eta, \bar{L}, \gamma)$ so that

$$\|\Omega^j\|_{L^\infty(\delta\text{-thick}(M,g))} \leq C_\delta \ell_j^{3/2} \text{ respectively } \|\Theta^j\|_{L^\infty(\delta\text{-thick}(M,g))} \leq C_\delta \ell_j, \tag{2.56}$$

which, as observed in [22], is sharp. Conversely, the global L^∞-norms of Ω^j are

$$\|\Omega^j\|_{L^\infty(M,g)} \leq C\ell_j^{-1/2}, \tag{2.57}$$

compare (2.21). Finally, we note that combining (2.54), (2.48) and (2.49) with (2.9) gives $|\|\Theta^j\|^2_{L^2(M,g)} - \frac{\ell_j^2}{\pi^4}\|dz^2\|^2_{L^2(\mathcal{C}(\sigma^j))}| \leq C\ell_j^2$, which, when combined with the expression for $\|dz^2\|_{L^2}$ from (A.10), yields

$$|\|\Theta^j\|^2_{L^2(M,g)} - \frac{32\pi}{\ell_j}| \leq C\ell_j^2. \tag{2.58}$$

Remark 2.17. As we may extend an arbitrary collection $\{\sigma^1, \ldots, \sigma^k\}$ of disjoint simple closed geodesics to a collection which decomposes (M, g) into pairs of pants which satisfies (2.37) for some $\bar{L} = \bar{L}(\gamma, \max_{j=1,\ldots,k}(L_g(\sigma^j)))$, compare Lemma A.4, the above results ensure in particular that Lemma 2.5 remains valid also without the smallness assumption on the ℓ_j's.

In addition to the basis $\{\Omega^j\}$ obtained above, we will also require bases $\{\Psi^j\}$ and $\{\Lambda^j\}$ which are dual to the Fenchel-Nielson coordinates as described in (2.38) and (2.39). While (2.14) and (2.16) imply in particular that $d\ell_j(\frac{1}{2}\mathrm{Re}\,\Theta^i) = \delta^i_j$ for every $1 \leq i, j \leq 3(\gamma - 1)$, these elements will in general not leave the twist coordinates invariant. However, we shall see that the elements $\Lambda^j \in \mathcal{H}(M, g)$ which have the same effect on the length coordinates, but do not change the twist coordinates, differ from $\frac{1}{2}\Theta^j$ by an error term that is only of order $O(\ell_j)$. As $\|\Theta^j\|_{L^2}$ is of order $\ell_j^{-1/2}$ this again corresponds to an error rate of $O(\ell_j^{3/2})$ for the corresponding renormalised elements of unit L^2-norm. In Section 4.4 we will prove

Lemma 2.18. *Let (M, g) be any closed oriented hyperbolic surface of genus γ and let $\mathcal{E} = \{\sigma^1, \ldots, \sigma^{3(\gamma-1)}\}$ be any collection of simple closed geodesics which decomposes M into pairs of pants and let $\eta \in (0, \operatorname{arsinh}(1))$ and $\bar{L} < \infty$ be so that (2.36) and (2.37) are satisfied.*

Then the elements Λ^j which induce only a change of the length coordinate ℓ_j as specified in (2.38) are given by

$$\Lambda^j = \tfrac{1}{2}\Theta^j + \sum_k \mathrm{i} \cdot c^j_k \Omega^k$$

for coefficients $c^j_k \in \mathbb{R}$ *which satisfy*

$$|c^j_k| \le C\ell_j \ell_k^{3/2} \text{ for every } j,k = 1,\dots,3(\gamma-1). \tag{2.59}$$

In particular

$$\|\Lambda^j - \tfrac{1}{2}\Theta^j\|_{L^\infty(M,g)} \le C\ell_j. \tag{2.60}$$

In Section 4.3 we shall furthermore prove the following result which establishes that the elements which induce Dehn-twists are so that, again up to error terms of order $O(\ell_j^{3/2})$,

$$\frac{\Psi^j}{\|\Psi^j\|_{L^2(M,g)}} \sim -\mathrm{i}\Omega^j.$$

Lemma 2.19. *Let (M,g) be any closed oriented hyperbolic surface of genus γ and let $\mathcal{E} = \{\sigma^1,\dots,\sigma^{3(\gamma-1)}\}$ be any decomposing set of simple closed geodesics. Then there exists a constant C that depends only on the genus and the numbers $\eta \in (0,\operatorname{arsinh}(1))$ and $\bar{L} < \infty$ for which (2.36) and (2.37) are satisfied so that the following holds true for the elements Ψ^j which generate Dehn-twists as described in (2.39):*

$$\Psi^j \perp \ker(\partial\ell_j) \tag{2.61}$$

are so that

$$\frac{\Psi^j}{\|\Psi^j\|_{L^2(M,g)}} = -a_j\mathrm{i}\Omega^j + \mathrm{i}\sum_{k\neq j} c^j_k\Omega^k \text{ for some } a_j \in \mathbb{R}^+, c^j_k \in \mathbb{R} \tag{2.62}$$

for coefficients

$$|c^j_k| \le C\ell_k^{3/2}\ell_j^{3/2} \text{ and } |1-a_j| \le C\ell_j^3, \tag{2.63}$$

in particular

$$\Big\|\tfrac{\Psi^j}{\|\Psi^j\|_{L^2(M,g)}} + \mathrm{i}\Omega^j\Big\|_{L^\infty(M,g)} \le C\ell_j^{3/2}. \tag{2.64}$$

Furthermore

$$\Big|\|\Psi^j\|_{L^2(M,g)} - 8\pi\|dz^2\|^{-1}_{L^2(\mathcal{C}(\sigma^j),g)}\Big| \le C\ell_j^{9/2}. \tag{2.65}$$

We note that (2.65) implies in particular that

$$\|\Psi^j\|_{L^2(M,g)} \le C\ell_j^{3/2}. \tag{2.66}$$

We will obtain the above result, including the characterisation (2.61) which also follows from [28], by combining an explicit description of the elements Ψ^j given later in Lemma 4.2 with the properties of our renormalised dual basis $\{\Omega^j\}$.

2.5. **Proof of Theorem 1.2.**

Based on the results of the previous section as well as Theorem 1.5 we can now discuss the dependence of the first eigenvalue $\lambda = \lambda_1$ on a full set of Fenchel-Nielson coordinates.

So let (M,g) be a closed oriented hyperbolic surface of genus $\gamma \ge 2$ and let σ^1 be a simple closed geodesic for which $M \setminus \sigma^1$ is disconnected. We let $\hat{\delta} > 0$ be so that $M \setminus \mathcal{C}(\sigma^1) \subset \hat{\delta}$-thick$(M,g)$ and assume, as in the theorem, that $\ell_1 \le \ell_0(\hat{\delta},\gamma)$ in order to ensure that $\nabla\lambda$ is well defined and described by Theorem 1.5. We then extend σ^1 to a collection $\mathcal{E} = \{\sigma^1,\dots,\sigma^{3(\gamma-1)}\}$ of geodesics, chosen so that $L_g(\sigma^j) \le \bar{L} = \bar{L}(\gamma,\ell_0)$, which decompose M into pairs of pants, compare Lemma A.4.

We may furthermore assume without loss of generality that $\ell_1 < 2\hat{\delta} < 2\operatorname{arsinh}(1)$ as for larger values of ℓ_1 the claims trivially follow from the upper bounds on the L^2-norms of $\nabla\lambda$, Ψ^j and Λ^j that can be obtained from Theorem 1.5 respectively Lemmas 2.18 and 2.19. We hence obtain that the only simple closed geodesic in (M, g) of length less than $2\hat{\delta}$ is σ^1 which implies in particular that any collection $\mathcal{E}$ as obtained above satisfies assumption (2.36) for η chosen to be $\eta < \hat{\delta}$.

We first claim that we may write

$$\nabla\lambda = \alpha\lambda \mathrm{Re}\,(\Theta^1) + \tilde{c}\cdot \mathrm{Re}\,(\mathrm{i}\Omega^1) + \sum_{j\geq 2}\mathrm{Re}\,(d_j\Omega^j) \tag{2.67}$$

for α as in (1.10) and

$$\tilde{c}\in\mathbb{R} \text{ satisfying } |\tilde{c}|\leq C\ell_1^{5/2} \text{ and coefficients } d_j\in\mathbb{C} \text{ with } |d_j|\leq C\ell_1^2. \tag{2.68}$$

Here and in the following Ω^j and Θ^j are the elements of the corresponding bases of the full space $\mathcal{H}(M, g)$ described in Lemma 2.13, while $\tilde{\Omega}^1$ and $\tilde{\Theta}^1$ stand for the elements of $\ker(\partial\ell_1)^\perp$ described in Lemma 2.5 and the subsequent corollary.

To see that $\nabla\lambda$ is of the above form we recall from Section 2.3, see in particular (2.31) and (2.32), that

$$\nabla\lambda = \alpha\lambda\mathrm{Re}\,(\tilde{\Theta}^1) + \mathrm{Re}\,(\mathrm{i}\lambda c_0\tilde{\Omega}^1 + w)$$

for $w = -\frac{1}{2}P_g^{\ker(\partial\ell_1)}(\Phi)$ which satisfies $\|w\|_{L^\infty(M,g)}\leq C\lambda^2\leq C\ell_1^2$ and $c_0\in\mathbb{R}$ with $|c_0|\leq C\ell_1^{1/2}\lambda\leq C\ell_1^{3/2}$ and where α satisfies (1.10). Here and in the following we repeatedly use that $\lambda\leq C\ell_1$, compare (1.1). We then recall that the elements $\tilde{\Theta}^1$, $\tilde{\Omega}^1$ and Θ^1, Ω^1 are related by the formulas (2.44) and (2.46) from Lemma 2.13 and that $\ker(\partial\ell_1)$ is spanned by $\{\Omega^j\}_{j\geq 2}$. This allows us to write $\nabla\lambda$ in the form (2.67) for the same number $\alpha = \frac{1}{8\pi} + O(\ell_1)$, a number $\tilde{c}\in\mathbb{R}$ which is bounded by $|\tilde{c}|\leq C\cdot c_0\lambda\leq C\ell_1^{5/2}$ and for d_j which must be so that $\|\sum_{j\geq 2} d_j\Omega^j\|_{L^\infty(M,g)}\leq C\ell_1\cdot\lambda + \|w\|_{L^\infty(M,g)}\leq C\ell_1^2$. Hence, Remark 2.15 implies that indeed $|d_j|\leq C\ell_1^2$ for every j as claimed in (2.68) above.

In the following we will use (2.67) and (2.68) to estimate the change of the first eigenvalue λ induced by a change of the Fenchel-Nielson coordinates.

We begin by considering the derivatives with respect to the twist coordinates, which corresponds to the change of λ along a curve of metrics that evolves by $\partial_t g = \mathrm{Re}\,(\Psi^j)$, where Ψ^j is characterised by (2.39) and, according to Lemma 2.19, is orthogonal to $\ker(\partial\ell_j) = \mathrm{span}\{\Omega^i\}_{i\neq j} = \mathrm{span}\{\Theta^i\}_{i\neq j}$.

For $j\neq 1$ we thus obtain that

$$\frac{\partial\lambda}{\partial\psi_j} = \langle\nabla\lambda, \mathrm{Re}\,(\Psi^j)\rangle = \langle\mathrm{Re}\,(d_j\Omega^j), \mathrm{Re}\,(\Psi^j)\rangle.$$

We note that $\|\Psi^j\|_{L^2}\leq C$, compare (2.66), so the bound on d_j obtained in (2.68) yields

$$\left|\frac{\partial\lambda}{\partial\psi_j}\right|\leq C\|\Psi^j\|_{L^2}|d_j|\leq C\ell_1^2$$

as claimed in (1.6). Here and in the following C is allowed to depend on $\hat{\delta}$ and the genus (and so also on $\bar{L} = \bar{L}(\gamma, \ell_0(\hat{\delta}, \gamma))$) and all norms are computed over all of (M, g) unless specified otherwise.

For $j = 1$ we obtain by the same argument that

$$\frac{\partial\lambda}{\partial\psi_1} = \|\Psi^1\|_{L^2}\cdot\left[\alpha\lambda\langle\mathrm{Re}\,(\Theta^1), \mathrm{Re}\,(\tfrac{\Psi^1}{\|\Psi^1\|_{L^2}})\rangle + \tilde{c}\langle\mathrm{Re}\,(\mathrm{i}\Omega^1), \mathrm{Re}\,(\tfrac{\Psi^1}{\|\Psi^1\|_{L^2}})\rangle\right].$$

Inserting $\Theta^1 = -\|\Theta^1\|_{L^2} \cdot \Omega^1$ and the upper bounds (2.55) and (2.66) on the L^2-norms of Θ^1 respectively Ψ^1 as well as the bound (2.68) on $\tilde{c}$ into this estimate gives

$$\begin{aligned}\left|\frac{\partial\lambda}{\partial\psi_1}\right| &\le C\ell_1^{3/2}\left[\alpha\lambda\|\Theta^1\|_{L^2}\cdot\left|\langle \mathrm{Re}\,(\Omega^1), \mathrm{Re}\,(\tfrac{\Psi^1}{\|\Psi^1\|_{L^2}})\rangle\right| + |\tilde{c}|\right]\\ &\le C\ell_1^2|\langle \mathrm{Re}\,(\Omega^1), \mathrm{Re}\,(\tfrac{\Psi^1}{\|\Psi^1\|_{L^2}})\rangle| + C\ell_1^4.\end{aligned}\tag{2.69}$$

We now recall from Lemma 2.19 that we may write

$$\tfrac{\Psi^1}{\|\Psi^1\|_{L^2(M,g)}} = -a_1 \mathrm{i}\Omega^1 + \mathrm{i}\sum_{i\ge 2} c_i^1\Omega^i, \quad a_1, c_i^j \in \mathbb{R} \text{ with } |c_i^1| \le C\ell_1^{3/2}.$$

We note that the real part of the main term $-\mathrm{i}a_1\Omega^1$ is orthogonal to $\mathrm{Re}\,(\Omega^1)$, compare (2.7), and recall that Ω^1 is nearly orthogonal to the other Ω^j as described in (2.51). We thus obtain the bound

$$|\langle \mathrm{Re}\,(\Omega^1), \mathrm{Re}\,(\tfrac{\Psi^1}{\|\Psi^1\|_{L^2}})\rangle| \le \sum_{i\ge 2}|c_i^1||\langle \Omega^1, \Omega^i\rangle| \le C\ell_1^3$$

which, when inserted into (2.69), gives the estimate $|\frac{\partial\lambda}{\partial\psi_1}| \le C\ell_1^4$ claimed in the theorem.

To discuss the dependence of λ on the length coordinates, we write $\nabla\lambda$ as in (2.67) to see that

$$\frac{\partial\lambda}{\partial\ell_j} = \langle \nabla\lambda, \mathrm{Re}\,\Lambda^j\rangle = -\alpha\lambda\|\Theta^1\|_{L^2}\cdot\langle \mathrm{Re}\,\Omega^1, \mathrm{Re}\,\Lambda^j\rangle + R_j \tag{2.70}$$

for a remainder term R_j which, thanks to (2.68), is bounded by

$$\begin{aligned}|R_j| &\le |\tilde{c}|\cdot|\langle \mathrm{Re}\,\mathrm{i}\Omega^1, \mathrm{Re}\,\Lambda^j\rangle| + \sum_{k\ge 2}|\langle \mathrm{Re}\,(d_k\Omega^k), \mathrm{Re}\,\Lambda^j\rangle|\\ &\le C\ell_1^{5/2}|\mathrm{Re}\,\langle \mathrm{i}\Omega^1, \Lambda^j\rangle| + C\ell_1^2\cdot\max_{k\ge 2}|\langle\Omega^k, \Lambda^j\rangle|.\end{aligned}\tag{2.71}$$

We furthermore recall from Lemma 2.18 that

$$\Lambda^j = \tfrac{1}{2}\Theta^j + \sum_k \mathrm{i}\cdot c_k^j\Omega^k \text{ for real coefficients } |c_k^j| \le C\ell_j\ell_k^{3/2}. \tag{2.72}$$

Since $\|\Theta^1\|_{L^2} \le C\ell_1^{-1/2}$, see (2.55), we note that also $\|\Lambda^1\|_{L^2} \le C\ell_1^{-1/2} + C\ell_1 \le C\ell_1^{-1/2}$ while $\|\Lambda^j\|_{L^2} \le C = C(\hat{\delta}, \gamma)$ for $j \ge 2$.

In case that $j \ne 1$ we can use this last remark to bound the remainder term by $|R_j| \le C\ell_1^2$ and then combine the bounds (2.51) on the inner product of the Ω^i's with (2.72) to estimate also the absolute value of the main term in (2.70) by

$$\begin{aligned}\alpha\lambda\|\Theta^1\|_{L^2}|\langle \mathrm{Re}\,\Omega^1, \mathrm{Re}\,\Lambda^j\rangle| &\le C\ell_1^{-1/2}\lambda|\langle\Omega^1, \Lambda^j\rangle|\\ &\le C\ell_1^{1/2}\cdot\left[\|\Theta^j\|_{L^2(M,g)}\cdot|\langle\Omega^1, \Omega^j\rangle| + \sum_{k\ne 1}|c_k^j||\langle\Omega^1, \Omega^k\rangle| + |c_1^j|\right]\\ &\le C\ell_1^{1/2}\sum_{i\ne 1}|\langle\Omega^1, \Omega^i\rangle| + C\ell_1^{1/2}|c_1^j| \le C\ell_1^2.\end{aligned}$$

Combined with (2.70) and the already derived bound of $|R_j| \le C\ell_j^2$ this yields the bound $|\frac{\partial\lambda}{\partial\ell_j}| \le C\ell_1^2$, $j \ne 1$, claimed in the theorem.

Having thus established that the first eigenvalue essentially only depends on ℓ_1, as claimed in the theorem, we finally turn to the analysis of $\frac{\partial\lambda}{\partial\ell_1}$. In this case the remainder term R_1 obtained in (2.70) and (2.71) is bounded by

$$|R_1| \le C\ell_1^{5/2}|\langle \mathrm{Re}\,(\mathrm{i}\Omega^1, \tfrac{1}{2}\Theta^1\rangle| + C\ell_1^{5/2}\sum_k |c_k^1|\cdot|\langle\Omega^1, \Omega^k\rangle| + C\ell_1^2\max_{k\ge 2}|\langle\Omega^k, \tfrac{1}{2}\Theta^1\rangle| + C\ell_1^2\sum_k |c_k^1|.$$

Since Θ^1 is a real multiple of Ω^1 the first term vanishes. Using the bound (2.51) on the inner products of the Ω^j's, the bound (2.55) on $\|\Theta^1\|_{L^2}$ and the estimates on the coefficients c_j^1 from (2.72) we may thus conclude that

$$|R_1| \le 0 + C\ell_1^{5/2}\ell_1^{5/2} + C\ell_1^2\ell_1^{-1/2}\ell_1^{3/2} + C\ell_1^2\ell_1 \le C\ell_1^3.$$

To analyse the main term in (2.70) we note that, by (2.72),

$$\begin{aligned}|\langle \mathrm{Re}\,\Omega^1, \mathrm{Re}\,\Lambda^1\rangle + \tfrac{1}{4}\|\Theta^1\|_{L^2}| &= |\langle \mathrm{Re}\,\Omega^1, \mathrm{Re}\,\Lambda^1 + \tfrac{1}{2}\|\Theta^1\|_{L^2}\mathrm{Re}\,\Omega^1\rangle| = |\langle \mathrm{Re}\,\Omega^1, \mathrm{Re}\,(\Lambda^1 - \tfrac{1}{2}\Theta^1)\rangle| \\ &= |\sum_k c_k^1\langle \mathrm{Re}\,\Omega^1, \mathrm{Re}\,\mathrm{i}\Omega^k\rangle| = |\sum_{k\neq 1} c_k^1\langle \mathrm{Re}\,\Omega^1, \mathrm{Re}\,\mathrm{i}\Omega^k\rangle| \\ &\le C\ell_1^{5/2}\end{aligned}$$

where we use the orthogonality of $\mathrm{Re}\,(\Omega^1)$ and $\mathrm{Re}\,(\mathrm{i}\Omega^1)$ in the penultimate and (2.51) and (2.72) in the last step. Inserting these two bounds into (2.70) thus gives

$$|\tfrac{\partial\lambda}{\partial\ell_1} - \tfrac{1}{4}\alpha\lambda\|\Theta^1\|_{L^2}^2| \le C\ell_1^3 + C\ell_1^{5/2}\alpha\lambda\|\Theta^1\|_{L^2} \le C\ell_1^3.$$

We finally remark that

$$|\tfrac{1}{4}\alpha\lambda\|\Theta^1\|_{L^2}^2 - \frac{\lambda}{\ell_1}| \le C\ell_1|\log(\ell_1)|$$

as (1.10) implies $\alpha = \frac{1}{8\pi} + O(\ell_1|\log(\ell_1)|)$ while (2.58) assures that $\|\Theta^1\|_{L^2}^2 = \frac{32\pi}{\ell_1} + O(\ell_1^2)$. So indeed

$$|\tfrac{\partial\lambda}{\partial\ell_1} - \tfrac{\lambda}{\ell_1}| \le C\ell_1|\log(\ell_1)|$$

as claimed in the theorem.

2.6. **Proof of Corollary 1.3.**

Let M be a closed surface of genus $\gamma \ge 2$ and let $\bar\sigma$ be a simple closed curve that disconnects M into two connected components. Let $\bar{\mathcal{E}}_1 = \{\bar\sigma_1^j\}_{j=1}^{3(\gamma-1)}$ be a collection of disjoint simple closed curves which decomposes M into pairs of pants and which is chosen so that $\bar\sigma_1^1 = \bar\sigma$.

We let $\bar{\mathcal{E}}_i = \{\bar\sigma_i^j\}_{j=1}^{3(\gamma-1)}$, $i = 2, \ldots, N$, be collections of simple closed curves in M as chosen in Lemma A.5 (for $k = 1$); i.e. we ask that each $\bar\sigma_i^1 = \bar\sigma_1^1 = \bar\sigma$, that each $\bar{\mathcal{E}}_i$ decomposes M into pairs of pants and that these collections are so that any other disconnecting set of curves on M that contains a curve σ^1 that is homotopic to $\bar\sigma$ can be pulled-back to give one of the $\bar{\mathcal{E}}_i$'s.

To define a function f as in the corollary we introduce Fenchel-Nielson coordinates associated with one of these collections, say with $\bar{\mathcal{E}}_1$, and let $(g_\ell)_{\ell\in(0,2\mathrm{arsinh}(1))}$ be a family of hyperbolic metrics on M for which the first Fenchel-Nielson length coordinate is $\ell_1 = \ell \in (0, 2\mathrm{arsinh}(1))$ while all other Fenchel-Nielson coordinates are given by fixed numbers $\ell_j \equiv c_j$ and $\psi_j \equiv \tilde c_j$.

We denote the geodesics in (M, g_ℓ) that are homotopic to the curves $\bar\sigma_i^j$ by $\sigma_i^j(\ell)$ and let $\mathcal{E}_i(\ell) = \{\sigma_i^j(\ell)\}_{j=1}^{3(\gamma-1)}$. We furthermore write for short $\sigma(\ell) = \sigma_i^1(\ell)$ for the geodesic that is homotopic to $\bar\sigma$ (and which thus has length ℓ) and denote by $\mathcal{C}(\sigma(\ell))$ the corresponding collar in (M, g_ℓ).

To begin with we claim that on $M \setminus \mathcal{C}(\sigma(\ell))$ the injectivity radius inj_{g_ℓ} is bounded away from zero by a constant $\delta_0 > 0$ that depends only on the genus (having fixed the numbers c_j): To see this we recall that if the injectivity radius in a point $p \in (M \setminus \mathcal{C}(\sigma(\ell)), g_\ell)$ is equal to some $\delta \in (0, \mathrm{arsinh}(1))$ then this point must be in a collar around a geodesic $\tilde\sigma \subset (M, g_\ell)$ of length no more than 2δ. This geodesic either agrees with one of the $\sigma_1^j(\ell) \in \mathcal{E}_1(\ell)$, $j \neq 1$, in which case $\delta \ge \frac{1}{2}\min(c_j)$, or it must intersect at least one of the $\sigma_1^j(\ell) \in \mathcal{E}_1(\ell)$ (as a pair of pants does not contain any simple closed geodesics), in which case its length is bounded below by the width of the corresponding collar, see (A.2). As the width of $\mathcal{C}(\sigma(\ell))$ is at least $2\mathrm{arsinh}(1)$, we must thus have that $j \geq 2$ and hence that in this second case $\delta \geq \frac{w_j}{2} = \operatorname{arsinh}(\sinh^{-1}(\frac{c_j}{2}))$ is bounded away from zero in terms of $\max(c_j)$.

We also note that since the collections $\bar{\mathcal{E}}_i$ (and the Fenchel-Nielson coordinates $\ell_j = c_j$ and $\phi_j = \tilde{c}_j$ with respect to $\bar{\mathcal{E}}_1$) are fixed we also have an upper bound $\bar{L} = \bar{L}(\gamma)$ on the lengths of all geodesics $\sigma_i^j(\ell)$ in (M, g_ℓ) which are homotopic to one of the simple closed curves in $\bigcup_i \bar{\mathcal{E}}_i$. Hence the assumptions (2.36) and (2.37) are satisfied for constants $\eta(\gamma)$ and $\bar{L}(\gamma)$ for each of the metrics g_ℓ and each of the associated collections $\mathcal{E}_i(\ell) := \{\sigma_i^j(\ell)\}_{j=1}^{3(\gamma-1)}$, $i = 1, \ldots, N(\gamma)$, of simple closed geodesics.

These observations allow us to derive Corollary 1.3 from Theorem 1.2 as follows. Let

$$f(\ell) := \lambda(M, g_\ell)$$

and note that Theorem 1.2, applied for the Fenchel-Nielson coordinates associated with the collection $\bar{\mathcal{E}}_1$ for which all coordinates except ℓ_1 are constant along $(g_\ell)_\ell$, yields that

$$\left| \frac{d}{d\ell} \frac{f(\ell)}{\ell} \right| = \ell^{-1} \left| \frac{\partial \lambda}{\partial \ell_1} - \frac{\lambda}{\ell} \right| \leq C \; |\log(\ell)| \text{ for } 0 < \ell < \ell_0,$$

where $\ell_0 = \ell_0(\delta_0, \gamma)$ is as in Remark 1.1 and depends only on the genus.

Using that the result (1.3) of Burger implies in particular that $\frac{f(\ell)}{\ell}$ converges to the constant C_{top} defined in (1.4) as $\ell \to 0$, we thus obtain that

$$\left| \frac{f(\ell)}{\ell} - C_{top} \right| \leq C\ell \, |\log(\ell)|$$

holds true, initially for $\ell \in (0, \ell_0)$, and as this estimate trivially holds true for larger values of ℓ, thus indeed for $\ell \in (0, 2\operatorname{arsinh}(1))$ as claimed.

Let now g be any hyperbolic metric on M which satisfies the assumptions of the corollary. We extend the geodesic σ in (M, g) to a disconnecting set $\{\hat{\sigma}^j\}_{j=1}^{3(\gamma-1)}$, $\hat{\sigma}^1 = \sigma$, of simple closed geodesics which we recall can be chosen so that $L_g(\hat{\sigma}^j) \leq \bar{L} = \bar{L}(\gamma)$, compare Lemma A.4 in the appendix. Lemma A.5 then assures that there exists an $i \in \{1, \ldots, N(\gamma)\}$ and a diffeomorphism $\tilde{f} \colon M \to M$ which maps $\bar{\sigma}_i^j \in \bar{\mathcal{E}}_i$ to $\hat{\sigma}^j$ for every $j = 1, \ldots, 3(\gamma - 1)$.

We then consider the Fenchel-Nielson coordinates $(\tilde{\ell}_j, \tilde{\psi}_j)$ associated to $\bar{\mathcal{E}}_i$ of both $\tilde{f}^* g$ and the element $g_{\ell = L_g(\sigma)}$ of the family of metrics considered above. We note that the length coordinates ℓ_j, $j \geq 2$, of both $\tilde{f}^* g$ and g_ℓ are bounded away from zero by $2 \min(\delta_0, \hat{\delta})$, where $\hat{\delta}$ is the constant from the corollary while $\delta_0 = \delta_0(\gamma)$ is the lower bound on $\operatorname{inj}_{g_\ell}$ on $M \setminus \mathcal{C}(\sigma(\ell))$ obtained above, and from above by a constant $\bar{L}$ that depends only on the genus. The same thus holds true also for a curve of metrics $(g(t))_{t \in [0,1]}$ which interpolates between $\tilde{f}^* g$ and g_ℓ in the sense that its Fenchel-Nielson coordinates are $\tilde{\ell}_j(g(t)) = t\tilde{\ell}_j(\tilde{f}^* g) + (1-t)\tilde{\ell}_j(g_\ell)$ and likewise for the twist coordinates.

Arguing as in the first part of the proof we can hence obtain a uniform lower bound $\hat{\delta}_0$ on the injectivity radius $\operatorname{inj}_{g(t)}$ on $M \setminus \mathcal{C}(\sigma(t))$ in terms of $\max_{j \geq 2}(\tilde{\ell}_j(f^* g), \tilde{\ell}_j(g_\ell)) \leq \bar{L}(\gamma)$ and $\min_{j \geq 2}(\tilde{\ell}_j(\tilde{f}^* g), \tilde{\ell}(g_\ell)) \geq \min(2\hat{\delta}, 2\delta_0)$.

As the constant $\ell_0 = \ell_0(\hat{\delta}_0, \gamma)$ from Remark 1.1 depends only on the genus and on the assumed lower bound $\hat{\delta}$ on inj_g on $M \setminus \mathcal{C}(\sigma)$ we note that for $\ell \geq \ell_0$ the claim of the Corollary is trivially true. For smaller values of ℓ we may apply Theorem 1.2 (with $\hat{\delta}_0$ instead of $\hat{\delta}$) along the curve $g(t)$ to conclude that indeed

$$|\lambda(M, g) - f(\ell)| = |\lambda(M, \tilde{f}^* g) - f(\ell)| \leq C\ell^2 \text{ for some } C = C(\hat{\delta}, \gamma).$$

2.7. **Proof of Theorem 1.4.**

Let M be a surface of genus $\gamma \geq 3$. To prove that the error estimates we derived in the previous sections are sharp in the sense stated in Theorem 1.4, we consider metrics g on M for which we not only have a disconnecting geodesic σ^1 of very small length ℓ_1 but a further disconnecting geodesic σ^2 whose length is quite small, though contained in a fixed interval. We shall then see that a change of the associated length coordinate ℓ_2 results in a change of the eigenvalue of order ℓ_1^2.

Lemma 2.20. *Let M be a closed oriented surface of genus $\gamma \geq 3$ and let $\hat{\delta} \in (0, \operatorname{arsinh}(1))$ be any fixed number. Then there exist constants $\hat{\ell}, \hat{\eta} \in (0, \frac{1}{2}\operatorname{arsinh}(1))$ and $\hat{c} > 0$ depending only on γ and $\hat{\delta}$ so that the following holds true.*

Let g be a hyperbolic metric on M so that (M, g) contains two disjoint simple closed geodesics σ^1 and σ^2, both of which are so that $M \setminus \sigma^{1,2}$ is disconnected, whose lengths are $\ell_1 = L_g(\sigma^1) \in (0, \hat{\ell})$ and $\ell_2 = L_g(\sigma^2) \in [2\hat{\eta}, 4\hat{\eta}]$. Suppose furthermore that $\operatorname{inj}_g \geq \hat{\delta}$ on $M \setminus (\mathcal{C}(\sigma^1) \cup \mathcal{C}(\sigma^2))$.

Then
$$\frac{\partial \lambda}{\partial \ell_2} \geq \hat{c} \cdot \ell_1^2$$
holds true for every set of Fenchel-Nielson coordinates that is obtained by extending σ^1, σ^2 to a set of geodesics $\mathcal{E} = \{\sigma^j\}_{j=1}^{3(\gamma-1)}$ which decomposes (M, g) into pairs of pants and for which (2.37) is satisfied for some $\bar{L} = \bar{L}(\gamma)$.

We note that Theorem 1.4 is a direct consequence of this lemma as we may e.g. choose the families of metrics $g_\ell^{1,2}$ so that all of their Fenchel-Nielson coordinates different from ℓ_2 agree while $\ell_2(g_\ell^1) = 2\hat{\eta}$ and $\ell_2(g_\ell^2) = 4\hat{\eta}$.

Proof of Lemma 2.20. Let $\hat{\delta} \in (0, \operatorname{arsinh}(1))$ be any given number and assume that g is a metric which satisfies the assumptions of the lemma for numbers $\hat{\eta}$ and $\hat{\ell}$ that are still to be determined. We denote the connected component of $M \setminus \sigma^1$ which contains σ^2 by M_1 and the other connected component by M_2. We further split up $M_1 \setminus \sigma^2$ into its connected components $M_{1,\pm}$ with the convention that $\partial M_{1,-} = \sigma^1 \cup \sigma^2$. We let u be the first eigenfunction, for which we can assume without loss of generality that $\|u\|_{L^2} = 1$ and that $\int_{M_1^{\hat{\eta}}} u dv_g \geq 0$, where $M_1^{\hat{\eta}} := M_1 \cap \hat{\eta}\text{-thick}(M, g)$.

As a first step towards the proof of Lemma 2.20 we show that this integral is bounded away from zero in the following sense

Claim 1: There exist numbers $\eta_2, c_2 > 0$ depending only on $\hat{\delta}$ and the genus so that for any $0 < \hat{\eta} < \eta_2$ we may determine $\bar{\ell} = \bar{\ell}(\hat{\eta}, \hat{\delta}, \gamma)$ in a way that ensures that
$$\int_{M_1^{\hat{\eta}}} u dv_g \geq c_2 > 0$$
for every metric g which satisfies the assumptions of Lemma 2.20 for some $\ell_1 \in (0, \bar{\ell})$ and $\ell_2 \in [2\hat{\eta}, 4\hat{\eta}]$.

Proof of Claim 1. We prove this claim by exploiting that for small values of ℓ_1 (and fixed $\hat{\eta}$) the function u is essentially constant on the $\hat{\eta}$-thick part of the two connected components $M_{1,2}$ of $M \setminus \sigma^1$.

So let g be a metric for which the assumptions of Lemma 2.20 are satisfied for some $\ell_1 \in (0, \bar{\ell})$ and $\ell_2 \in [2\hat{\eta}, 4\hat{\eta}]$ for some $\hat{\eta} \leq \eta_2$, where $\eta_2 = \eta_2(\hat{\delta}, \gamma)$ and $\bar{\ell} = \bar{\ell}(\hat{\eta}, \hat{\delta}, \gamma)$ are to be determined. We will in particular choose these numbers so that $\eta_2 < \frac{\hat\delta}{4} < \frac{1}{4}\text{arsinh}(1)$ and $\bar\ell < 2\hat\eta$ which ensures that σ^1 is the only simple closed geodesic of length less than $2\hat\eta$ and hence that $\hat\eta$-thick(M,g) consists of only two connected components, namely $M_{1,2}^{\hat\eta} := M_{1,2} \cap \hat\eta\text{-thick}(M,g)$.

We now set $D_{1,2} := \int_{M_{1,2}^{\hat\eta}} u dv_g$ and note that since $\int_M u dv_g = 0$

$$\begin{aligned} ||D_1| - |D_2|| &\leq \left| \int_{\hat\eta\text{-thick}(M,g)} u dv_g \right| = \left| \int_{\hat\eta\text{-thin}(M,g)} u dv_g \right| \\ &\leq \text{Area}(\hat\eta\text{-thin}(M,g)) \|u\|_{L^\infty(M,g)} \leq C\hat\eta \end{aligned} \tag{2.73}$$

for a constant C that depends only on the genus, see Remark 2.4 and (A.5) for the last step.

Furthermore, a combination of the standard Poincaré estimate applied on $M_i^{\hat\eta}$ and the energy estimate from Lemma 2.2 allow us to estimate

$$\|u - ⨍_{M_i^{\hat\eta}} u dv_g\|_{L^2(M_i^{\hat\eta},g)} \leq C_{\hat\eta} \cdot \|du\|_{L^2(M_i^{\hat\eta},g)} \leq C_{\hat\eta}\lambda \leq C_{\hat\eta}\ell_1 \leq C_{\hat\eta}\bar\ell \tag{2.74}$$

where the constant $C_{\hat\eta}$ depends on $\hat\eta$ in addition to the usual dependence of all constants on the genus and the fixed number $\hat\delta$.

For a suitable choice of $\bar\ell$ (which we stress is allowed to depend on $\hat\eta$) the above expression hence does not give a significant contribution to $\|u\|_{L^2} = 1$, nor does $\|u\|_{L^2(\hat\eta\text{-thin}(M,g))} \leq C\hat\eta^{1/2}$ if $\hat\eta$ is sufficiently small. Hence

$$\begin{aligned} 1 = \|u\|_{L^2(M,g)} &\leq \sum_{i=1,2} \|u - ⨍_{M_i^{\hat\eta}} u dv_g\|_{L^2(M_i^{\hat\eta},g)} + \frac{|D_i|}{\sqrt{\text{Area}(M_i^{\hat\eta})}} + \|u\|_{L^2(\hat\eta\text{-thin}(M,g))} \\ &\leq C \cdot (|D_1| + |D_2|) + C \cdot \hat\eta^{1/2} + C_{\hat\eta}\bar\ell \leq C \cdot \min_{i=1,2} |D_i| + C \cdot \hat\eta^{1/2} + C_{\hat\eta}\bar\ell \end{aligned}$$

where the last estimate is due to (2.73).

Choosing first the upper bound η_2 on $\hat\eta$ small enough to ensure that the penultimate term in the above expression is no more than $\frac{1}{4}$ and then, for each given $\hat\eta \leq \eta_2$, selecting $\bar\ell$ so that also the final term in the estimate above is no more than $\frac{1}{4}$ gives the desired uniform lower bound on $\int_{M_1^{\hat\eta}} u dv_g = |D_1|$. $\square$

Having thus established this claim we can now prove Lemma 2.20 as follows:

Let g be as in the lemma where $\hat\ell$ and $\hat\eta$ are still to be determined and are chosen in particular so that $\hat\eta \leq \eta_2$, $\hat\ell \leq \bar\ell(\hat\eta, \hat\delta, \gamma)$ for the constants obtained in claim 1.

We then let $g(t)$, $t \in (-\varepsilon, \varepsilon)$, be a curve of metrics for which $g(0) = g$ and $\partial_t g = -\text{Re}\,(\Lambda^2)$, Λ^2 as in (2.38). Then, by construction,

$$-\frac{\partial\lambda}{\partial\ell_2} = \frac{d}{dt}\lambda(g(t)) = \frac{d}{dt}\min\{Q_{g(t)}(v) : \int v dv_{g(t)} = 0\},$$

where here and in the following all time derivatives are evaluated in $t = 0$, where $Q_g(v) := \frac{\|dv\|^2_{L^2(M,g)}}{\|v\|^2_{L^2(M,g)}}$ and where we recall that λ is differentiable as we may assume that $\hat\ell$ is chosen to be less than the constant $\ell_0(\hat\eta, \gamma)$ from Remark 1.1. So as u is a minimiser of $Q_{g(0)}$ amongst all functions with $\int_M v dv_g = 0$, and as $\partial_t g$ is trace-free, we have that for every $w \in H^1(M,g)$

$$\begin{aligned} \frac{d}{dt}\lambda(g(t)) &= \frac{d}{dt}Q_{g(t)}(u + t(w - \bar{w}_{g(t)})) \\ &= \frac{d}{dt}\int |d(u+tw)|^2_{g(t)} dv_{g(t)} - 2\lambda \int u \cdot (w - \bar{w}_g) dv_g \\ &= \frac{d}{dt}\int |d(u+tw)|^2_{g(t)} dv_{g(t)} - 2\lambda \int u \cdot w dv_g \end{aligned} \tag{2.75}$$

where here and in the following all integrals are computed over all of M unless specified otherwise, where we write for short $\bar{w}_{g(t)} = \fint_M w dv_{g(t)}$ and where we used in the last step that $\int u dv_g = 0$.

To establish the desired upper bound of $-c\lambda^2$ for $\frac{d}{dt}\lambda(g(t))$ we will apply this for a function w that is constructed as follows:

Let $(s,\theta) \in (-X(\ell_2), X(\ell_2)) \times S^1$ be collar coordinates on $\mathcal{C}(\sigma^2)$ whose orientation is chosen so that $s > 0$ on $M_{1,+} \cap \mathcal{C}(\sigma^2)$, let $a := (\int_{-X(\ell_2)}^{X(\ell_2)} \rho^{-1}(t)dt)^{-1}$ and define $w \in H^1(M,g)$ by

$$w(p) := \begin{cases} \lambda & \text{on } M_{1,+} \setminus \mathcal{C}(\sigma^2) \\ f(s) := a \cdot \lambda \int_{-X(\ell_2)}^{s} \rho^{-1}(t)dt & \text{for } p = (s,\theta) \in \mathcal{C}(\sigma^2) \\ 0 & \text{else} \end{cases}.$$

As the area of the set $M_{1,+} \setminus \mathcal{C}(\sigma^2)$ is bounded below by a universal constant we know that $\int w dv_g \geq c\lambda$, $c > 0$ a universal constant. We also observe that $\text{supp}(w) \subset M_1^{\hat{\eta}}$ and recall that $\|u - \fint_{M_1^{\hat{\eta}}} u dv_g\|_{L^2(M_1^{\hat{\eta}},g)} \leq C_{\hat{\eta}}\hat{\ell}$, compare (2.74), and that $\fint_{M_1^{\hat{\eta}}} u dv_g$ is bounded from below by claim 1.

The second term in (2.75) is thus bounded by

$$-2\lambda \int u \cdot w dv_g \leq -2\lambda \fint_{M_1^{\hat{\eta}}} u dv_g \cdot \int w dv_g + C_{\hat{\eta}}\hat{\ell} \cdot \lambda \|w\|_{L^2(M,g)} \leq -(c_3 - C_{\hat{\eta}}\hat{\ell}) \cdot \lambda^2, \tag{2.76}$$

for a constant $c_3 > 0$ that depends only on $\hat{\delta}$ and the genus. We note that for $\hat{\ell}$ chosen sufficiently small (depending on $\hat{\eta}$) this gives precisely the type of upper bound $-c\ell_1^2$ that we want to prove for $\frac{d}{dt}\lambda$.

It thus remains to establish an upper bound on the change of the energy

$$\frac{d}{dt}\int |d(u+tw)|^2_{g(t)} dv_{g(t)} = \frac{d}{dt}\int |d(u+tw)|^2_g dv_g + \frac{d}{dt}\int |du|^2_{g(t)} dv_{g(t)}$$

which is small compared to ℓ_1^2.

We note that on the collar $\mathcal{C}(\sigma^2)$ the two terms above have the opposite effect on the energy: while the addition of tw to u can increase the "s-energy" $\int |u_s|^2$ (computed with respect to the fixed metric g), the change of the metric, which corresponds to stretching out the collar $\mathcal{C}(\sigma^2)$, decreases this very part of the energy. As we shall see below this allows us to prove that the energy cannot increase substantially and hence that the rate of change of λ is indeed bounded from above by $-c\lambda^2$.

To be more precise, we first note that (A.1) implies that the coefficient a in the definition of w is bounded by $|a| \leq C\ell_2^2$. Hence

$$\begin{aligned}\frac{d}{dt}\int |d(u+tw)|_g^2 dv_g &= 2\int du\cdot dw\, dv_g = 2a\cdot\lambda\int_{\mathcal{C}(\sigma^2)} u_s\rho^{-1}dsd\theta \\ &\leq 2a\cdot\lambda\cdot(4\pi X(\ell_2))^{1/2}\cdot\left(\int_{\mathcal{C}(\sigma^2)}\rho^{-2}|u_s|^2dsd\theta\right)^{1/2} \\ &\leq C\lambda\ell_2^{3/2}\left(\int_{\mathcal{C}(\sigma^2)}\rho^{-2}|u_s|^2dsd\theta\right)^{1/2}.\end{aligned}$$

On the other hand, the change of the energy induced by the evolution of g is given by

$$\begin{aligned}\frac{d}{dt}\int |du|_{g(t)}^2 dv_{g(t)} &= -\tfrac{1}{2}\langle\partial_t g, \mathrm{Re}\,(\Phi(u,g))\rangle = \tfrac{1}{2}\langle \mathrm{Re}\,(\Lambda^2), \mathrm{Re}\,(\Phi(u,g))\rangle \\ &= \tfrac{1}{4}\mathrm{Re}\,\langle\Lambda^2,\Phi(u,g)\rangle = \tfrac{1}{4}\mathrm{Re}\,\langle\tfrac{1}{2}\Theta^2,\Phi(u,g)\rangle + R_1\end{aligned} \tag{2.77}$$

where Lemma 2.18 allows us to estimate the error term R_1 as

$$|R_1| = \tfrac{1}{4}|\sum_k c_k^2 \mathrm{Re}\,\langle \mathrm{i}\Omega^k,\Phi\rangle| \leq C\sum_k \ell_2\ell_k^{3/2}|\mathrm{Re}\,\langle\mathrm{i}\Omega^k,\Phi\rangle|.$$

We note that since $\mathrm{inj}_g \geq \hat{\delta}$ away from the collars around $\sigma^{1,2}$, we may apply Lemma 2.9 with $\bar{\delta} = \frac{1}{2}\hat{\delta} > 2\hat{\eta}$ to bound these inner products. For $k \geq 3$ this immediately gives $|\mathrm{Re}\,\langle\mathrm{i}\Omega^k,\Phi\rangle| \leq C\lambda^2$. For $k = 1,2$ we can use in addition to Lemma 2.9 that the real numbers $b_0^k(\Omega^k)$ are bounded by $|b_0^k(\Omega^k)| \leq C\ell_k^{3/2}$, see (2.54) and hence that

$$|\mathrm{Re}\,\langle\mathrm{i}\Omega^k,\Phi\rangle| \leq C\lambda^2 + |b_0^k(\Omega^k)|\cdot|\mathrm{Re}\,\langle \mathrm{i}dz^2,\Phi\rangle| \leq C\lambda^2 + C\ell_k^{3/2}\lambda^2\ell_k^{-1} \leq C\lambda^2.$$

Combined this gives a bound on the error term of $|R_1| \leq C\ell_2\lambda^2$.

We then recall that, up to well controlled errors, Θ^2 is concentrated on $\mathcal{C}(\sigma^2)$ and there described by its principal part $b_0^2(\Theta^2)dz^2 = -\frac{\ell_2}{\pi^2}dz^2$. Hence we may estimate the main term in (2.77) as

$$\begin{aligned}\tfrac{1}{4}\mathrm{Re}\,\langle\tfrac{1}{2}\Theta^2,\Phi\rangle &= \tfrac{1}{8}b_0^2(\Theta^2)\mathrm{Re}\,\langle dz^2,\Phi\rangle_{L^2(\mathcal{C}(\sigma^2))} + R_2 \\ &= \tfrac{1}{4}b_0^2(\Theta^2)\int_{\mathcal{C}(\sigma^2)}(|u_s|^2-|u_\theta|^2)\rho^{-2}dsd\theta + R_2 \\ &\leq -\frac{\ell_2}{4\pi^2}\int_{\mathcal{C}(\sigma^2)}|u_s|^2\rho^{-2}dsd\theta + C\ell_2\lambda^2 + R_2\end{aligned}$$

where we applied the angular energy estimates from Lemma 2.3 in the last step and where the error term R_2 is bounded by

$$|R_2| \leq |\langle\Theta^2,\Phi\rangle_{L^2(\mathcal{C}(\sigma^1))}| + |\langle\Theta^2 - b_0^2(\Theta^2)dz^2,\Phi\rangle_{L^2(\mathcal{C}(\sigma^2))}| + |\langle\Theta^2,\Phi\rangle_{L^2(M\setminus(\mathcal{C}(\sigma^1)\cup\mathcal{C}(\sigma^2))}|.$$

Since $b_0^1(\Theta^2) = 0$ we may bound this term using Lemma 2.9 (where we note that the resulting number δ depends only on $\hat{\delta}$) as well as (2.56) as

$$|R_2| \quad \leq C\lambda^2\|\Theta^2\|_{L^2(\delta\text{-thick}(M,g))} \leq C\ell_2\lambda^2.$$

Altogether we hence obtain that the change of the energy is

$$\begin{aligned}\frac{d}{dt}\int |d(u+tw)|_{g(t)}^2 dv_{g(t)} &\leq C\lambda\ell_2^{3/2}\left(\int_{\mathcal{C}(\sigma^2)}\rho^{-2}|u_s|^2dsd\theta\right)^{\frac{1}{2}} - \frac{\ell_2}{4\pi^2}\int_{\mathcal{C}(\sigma^2)}|u_s|^2\rho^{-2}dsd\theta + C\ell_2\lambda^2 \\ &\leq C\ell_2\lambda^2.\end{aligned}$$

Combined with (2.76) we conclude that

$$\frac{d}{dt}\lambda(g(t)) \leq -\lambda^2(c_3 - C_{\hat{\eta}}\hat{\ell}) + C\ell_2\lambda^2 \leq -\tfrac{1}{2}c_3\lambda^2$$

where the last inequality is obtained by first choosing $\hat{\eta}$ sufficiently small to ensure that $C\ell_2\lambda^2 \leq 4C\hat{\eta}\lambda^2 \leq \frac{1}{4}c_3\lambda^2$ and then finally determining $\hat{\ell}$ so that also $C_{\hat{\eta}}\hat{\ell} \leq \frac{1}{4}c_3$.

□

3. Proof of the properties of the first eigenfunction

In this section we prove the results on the first eigenfunction that we stated in Section 2.1 and used in the previous sections to prove the main results about the first eigenvalue.

3.1. The gradient of eigenvalues.

Let (M,g) be a hyperbolic surface and let $\lambda_0 = 0 < \lambda_1 \le \dots$ be the eigenvalues of $-\Delta_g$. We recall that given an element $k \in T_g\mathcal{M}_{-1}$, a real analytic path of metrics $g^{(\varepsilon)}$ with $g^{(0)} = g$ and $\partial_\varepsilon|_{\varepsilon=0} g^{(\varepsilon)} = k$ and any index $m \in \mathbb{N}$, there is a real analytic family of pairs of eigenvalues $\lambda^{(\varepsilon)}$ of $-\Delta_{g^{(\varepsilon)}}$ and corresponding eigenfunctions $u^{(\varepsilon)}$ with $\lambda^{(0)} = \lambda_m = \lambda_m(g)$, compare [6, Thm. 14.9.3]. We note that if the m-th eigenvalue of (M,g) is simple then for ε small $\lambda^{(\varepsilon)}$ is also the m-th eigenvalue $\lambda_m(g^{(\varepsilon)})$ of $(M, g^{(\varepsilon)})$ and hence $\varepsilon \mapsto \lambda_m(g^{(\varepsilon)})$ is differentiable. Lemma 2.1 is thus an immediate consequence of the following standard result.

Lemma 3.1. *Let $(g^{(\varepsilon)})_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ be a smooth family of hyperbolic metrics on M and let $(\lambda,u)^{(\varepsilon)}$ be a smooth family of pairs of eigenvalues of $-\Delta_{g^{(\varepsilon)}}$ and corresponding eigenfunctions, which in the following are always normalised to have $\|u^{(\varepsilon)}\|_{L^2(M,g)} = 1$. Then*

$$\frac{d\lambda^{(\varepsilon)}}{d\varepsilon} = \langle -\frac{1}{2}\mathrm{Re}\,(P^{\mathcal{H}}_{g^{(\varepsilon)}}(\Phi(u^{(\varepsilon)}, g^{(\varepsilon)})), \partial_\varepsilon g^{(\varepsilon)}\rangle_{L^2(M,g^{(\varepsilon)})}, \tag{3.1}$$

where $\Phi(u,g)$ denotes the Hopf-differential of u introduced in (2.1) and $P^{\mathcal{H}}_g$ is the $L^2(M,g)$-orthogonal projection from the space of L^2-quadratic differentials onto the space of holomorphic quadratic differentials $\mathcal{H}(M,g)$.

We note that formulas for the derivative of eigenvalues have been considered by many authors, see e.g. [29, Lemma 3.1]. Here we include a derivation of the above formula to make the presentation selfcontained.

Remark 3.2. We recall that the Hopf-differential describes the L^2-gradient of the Dirichlet-energy, namely for any metric g on M, any symmetric $(0,2)$ tensor k and any map $u\colon M \to (N, g_N)$ to some target manifold N we have

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} \int |du|^2_{g+\varepsilon k} dv_{g+\varepsilon k} = -\frac{1}{2}\langle \mathrm{Re}\,(\Phi(u,g)), k\rangle_{L^2(M,g)}. \tag{3.2}$$

We note that for real valued functions we can use local isothermal coordinates (x,y) for (M,g) and (A.10) to compute

$$|\Phi(v,g)|^2_g = ||v_x|^2 - |v_y|^2 - 2\mathrm{i} v_x \cdot v_y|^2 \cdot |dz^2|^2_g = (|v_x|^2 + |v_y|^2)^2 4\rho^{-4} = 4|dv|^4_g,$$

and hence to obtain that $\|\Phi\|_{L^1(K,g)} = 2\|dv\|_{L^2(K,g)}$ for every $K \subset M$ as observed in (2.2).

Proof of Lemma 3.1. We first remark that it suffices to prove the claim for $\varepsilon = 0$ and in case that $(g^{(\varepsilon)})$ is a *horizontal curve*, i.e. so that $\partial_\varepsilon g^{(\varepsilon)} \in H(g^{(\varepsilon)}) = \mathrm{Re}\,\mathcal{H}(M, g^{(\varepsilon)})$ for all ε. To see the later, we remark that if $g^{(\varepsilon)}$ is not horizontal, then we can choose a smooth family of diffeomorphisms $f_\varepsilon\colon M \to M$ with $f_{\varepsilon=0} = \mathrm{id}$ so that $\tilde{g}^{(\varepsilon)} = f_\varepsilon^* g^{(\varepsilon)}$ is a horizontal curve, compare e.g. [9, Lemma 2.2]. As the spectrum of $-\Delta_g$ is invariant under the pull-back by diffeomorphisms, the left-hand side of (3.1) is the same for both families of metrics. We furthermore note that since the space of Lie-derivatives $\{L_X g\}$ is orthogonal to $H(g)$ and since for $\varepsilon = 0$ the tensor $\partial_\varepsilon g^{(\varepsilon)} - \partial_\varepsilon \tilde{g}^{(\varepsilon)} = L_X g$ for some X, also the right-hand sides of (3.1) agree.

We may thus assume without loss of generality that $(g^{(\varepsilon)})_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ is horizontal. Writing for short $g = g^{(\varepsilon=0)}$ we now let $m \in \mathbb{N}$ be so that $\lambda^{(0)} = \lambda^{(\varepsilon=0)}$ is the m-th (non-zero) eigenvalue of

$-\Delta_g$ (counted with multiplicity such that the $m-1$-th eigenvalue is strictly smaller than $\lambda^{(0)}$). Let $v_0 \equiv \text{Area}(M,g)^{-\frac{1}{2}}, v_1, \dots, v_{m-1}$ be an orthonormal set of eigenfunctions to the eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_{m-1}$ of $-\Delta_g$ and recall that $\lambda^{(0)} = \lambda_m$ is given by

$$\lambda^{(0)} = \min \left\{ \|dv\|^2_{L^2(M,g)} : \|v\|_{L^2(M,g)} = 1 \text{ and } \langle v, v_i \rangle_{L^2(M,g)} = 0, \quad i = 0, \dots, m-1 \right\}. \tag{3.3}$$

Since $\partial_\varepsilon g^{(\varepsilon)}$ is trace-free, and thus $\partial_\varepsilon dv_{g^{(\varepsilon)}} = 0$, we have that for every ε

$$\|u^{(\varepsilon)}\|_{L^2(M,g)} = \|u^{(\varepsilon)}\|_{L^2(M,g^{(\varepsilon)})} = 1 \text{ and } \int_M u^{(\varepsilon)} dv_g = \int_M u^{(\varepsilon)} dv_{g^{(\varepsilon)}} = 0.$$

We set

$$w^{(\varepsilon)} = u^{(\varepsilon)} - \sum_{i=0}^{m-1} \langle u^{(\varepsilon)}, v_i \rangle v_i \text{ and } v^{(\varepsilon)} = \frac{w^{(\varepsilon)}}{\|w^{(\varepsilon)}\|_{L^2(M,g)}}$$

and note that since $\langle u^{(\varepsilon)}, v_i \rangle_{L^2(M,g)} = O(\varepsilon)$, we have $\|w^{(\varepsilon)}\|^2_{L^2(M,g^{(\varepsilon)})} = \|u^{(\varepsilon)}\|^2_{L^2(M,g^{(\varepsilon)})} + O(\varepsilon^2) = 1 + O(\varepsilon^2)$. Hence

$$v^{(\varepsilon)} = a^{(\varepsilon)} u^{(\varepsilon)} + \sum_{i=0}^{m-1} b_i^{(\varepsilon)} v_i \text{ for coefficients satisfying } a^{(\varepsilon)} = 1 + O(\varepsilon^2), \quad b_i^{(\varepsilon)} = O(\varepsilon),$$

in particular $\|dv^{(\varepsilon)}\|^2_{L^2(M,g)} = \|du^{(\varepsilon)}\|^2_{L^2(M,g)} + O(\varepsilon^2)$ as the v_i are eigenfunctions of $-\Delta_g$. As the maps $v^{(\varepsilon)}$ are admissible in (3.3) we thus have that

$$0 = \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_M |dv^{(\varepsilon)}|^2_g dv_g = \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_M |du^{(\varepsilon)}|^2_g dv_g.$$

Combined with (3.2) we obtain that

$$\begin{aligned} \frac{d}{d\varepsilon}|_{\varepsilon=0} \lambda^{(\varepsilon)} &= \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_M |du^{(\varepsilon)}|^2_g dv_g + \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_M |du^{(\varepsilon=0)}|^2_{g^{(\varepsilon)}} dv_{g^{(\varepsilon)}} \\ &= 0 - \frac{1}{2} \langle \text{Re}\, \Phi(u,g), \partial_\varepsilon|_{\varepsilon=0} g^{(\varepsilon)} \rangle_{L^2(M,g)} \end{aligned}$$

which agrees with (3.1) since the curve g is horizontal. □

3.2. Proof of the energy estimates.

As we have seen in Section 2 we need two different types of energy estimates for the first eigenfunction, one assuring that the energy on the thick part is of order λ^2, compare Lemma 2.2, and the other giving enough control on the angular component of the energy to be able to bound weighted angular energies as specified in Lemma 2.3. In this section we give the proofs of these two results.

To begin with we note that the angular energy of general eigenfunctions is controlled by

Lemma 3.3. *Let g be a hyperbolic metric on M, let u be an eigenfunction of $-\Delta_g$ to some eigenvalue λ, normalised to $\|u\|_{L^2(M,g)} = 1$, and let $\mathcal{C}(\sigma)$ be a collar around a simple closed geodesic σ of length $\ell \leq 2\text{arsinh}(1)$. We let $(s,\theta) \in (-X(\ell), X(\ell)) \times S^1$ be the collar coordinates on $\mathcal{C}(\sigma)$ and set*

$$\vartheta(s) := \int_{\{s\} \times S^1} |u_\theta|^2 d\theta.$$

Then for any $\Lambda \geq 0$ and any s with $|s| \leq X(\ell) - (\Lambda + 1)$ we have

$$\vartheta(s) \leq 2 \cdot e^{-\Lambda} \|du\|^2_{L^2(\mathscr{C}_{|s|+\Lambda+1} \setminus \mathscr{C}_{|s|+\Lambda})} + \frac{1}{2}\lambda^2 \int_{\mathscr{C}_{|s|+\Lambda+1}} \rho^4 e^{-|s-q|} u^2 d\theta dq \tag{3.4}$$

where we write for short $\mathscr{C}_Y$ for the subset of the collar $\mathcal{C}(\sigma)$ that is given by the cylinder $\mathscr{C}_Y = (-Y,Y) \times S^1$ in collar coordinates.

In particular, there exist universal constants $\delta_0 > 0$ and $C < \infty$ so that

$$\vartheta(s) \leq Ce^{-(X(\ell)-|s|)}\|du\|^2_{L^2(\delta_0\text{-}thick(\mathcal{C}(\sigma)))} + C\lambda^2 \int_{-X(\ell)}^{X(\ell)} \int_{S^1} \rho^4 e^{-|s-q|} u^2 d\theta dq \tag{3.5}$$

holds true for every $s \in [-X(\ell)+1, X(\ell)-1]$.

The proof of this lemma is obtained by well known arguments that have been used in particular in many works in the analysis of bubbling for harmonic maps. Here we closely follow [15].

Proof of Lemma 3.3. We remark that

$$\begin{aligned}\vartheta''(s) &= 2\int_{\{s\}\times S^1} |u_{\theta s}|^2 - u_{ss}u_{\theta\theta} d\theta \\ &= 2\int_{\{s\}\times S^1} |u_{\theta\theta}|^2 + |u_{s\theta}|^2 d\theta - 2\int_{\{s\}\times S^1} u_{\theta\theta}\rho^2 \Delta_g u d\theta \\ &\geq \int_{\{s\}\times S^1} |u_{\theta\theta}|^2 d\theta - \int_{\{s\}\times S^1} \rho^4 |\Delta_g u|^2 d\theta.\end{aligned}$$

Applying Wirtinger's inequality, we thus obtain that ϑ obeys the differential inequality

$$\vartheta''(s) - \vartheta(s) \geq -\int_{\{s\}\times S^1} \rho^4 |\Delta_g u|^2 d\theta = -\lambda^2 \int_{\{s\}\times S^1} \rho^4 u^2 d\theta =: -h(s).$$

As the solutions of the corresponding differential equation $f'' - f = -h$ are given by

$$f_{A_+,A_-}(s) = A_+ e^{s-Y_+} + A_- e^{-s-Y_-} + \frac{1}{2}\int_{-Y_-}^{Y_+} e^{-|s-q|} h(q) dq, \qquad A_\pm \in \mathbb{R}$$

we obtain by the maximum principle that if $Y_\pm$ and $A_\pm$ are so that $\vartheta(\pm Y_\pm) \leq f_{A_+,A_-}(\pm Y_\pm)$, and thus in particular if $\vartheta(\pm Y_\pm) \leq A_\pm$ for non-negative $A_\pm$, then

$$\vartheta(s) \leq f_{A_+,A_-}(s) \text{ on } [-Y_-, Y_+]. \tag{3.6}$$

To prove (3.4) for given $\Lambda \geq 0$ and $|s| \leq X(\ell) - (\Lambda+1)$ we choose $Y_\pm \in [|s|+\Lambda, |s|+\Lambda+1)$ so that

$$\vartheta(\pm Y_\pm) \leq \int_{|s|+\Lambda<|q|<|s|+\Lambda+1} \vartheta(q) dq \leq \|du\|^2_{L^2(\mathscr{C}_{|s|+\Lambda+1}\setminus \mathscr{C}_{|s|+\Lambda})},$$

where we continue to use the shorthand $\mathscr{C}_Y = (-Y, Y) \times S^1$. We may hence apply (3.6) with $A_- = A_+ = \|du\|^2_{L^2(\mathscr{C}_{|s|+\Lambda+1}\setminus\mathscr{C}_{|s|+\Lambda})}$ to obtain that

$$\vartheta(s) \leq (e^{-(Y_+-s)} + e^{-(s+Y_-)})\|du\|^2_{L^2(\mathscr{C}_{|s|+\Lambda+1}\setminus\mathscr{C}_{|s|+\Lambda})} + \frac{1}{2}\lambda^2 \int_{-Y_-}^{Y_+}\int_{S^1} \rho^4 e^{-|s-q|} u^2 d\theta dq$$

which yields the first claim (3.4) of the lemma.

If we set $\Lambda = X(\ell) - |s| - 1$ then the estimate (3.4) we have just proven yields

$$\vartheta(s) \leq 2e^{-(X(\ell)-|s|)+1}\|du\|^2_{L^2(\mathscr{C}_{X(\ell)}\setminus\mathscr{C}_{X(\ell)-1})} + \frac{1}{2}\lambda^2 \int_{-X(\ell)}^{X(\ell)}\int_{S^1} \rho^4 e^{-|s-q|} u^2 d\theta dq.$$

Since (A.9) implies that there exists a universal constant $\delta_0 > 0$ so that $\mathscr{C}_{X(\ell)} \setminus \mathscr{C}_{X(\ell)-1} \subset \delta_0\text{-thick}(M,g)$ we thus obtain also the second claim (3.5) of the lemma. $\square$

This lemma implies the estimates on the weighted angular energies of Lemma 2.3 as follows.

Proof of Lemma 2.3. Let $\alpha > 0$, later chosen to be $\alpha = 2$ or $\alpha = 4$, and let $\mathcal{C}(\sigma)$ be a collar around a simple closed geodesic of length $\ell \leq 2\text{arsinh}(1)$. We first remark that the universal lower bound (A.9) on $\rho(X(\ell)-1)$ allows us to bound

(3.7)
$$\int_{\mathscr{C}_{X(\ell)}\setminus\mathscr{C}_{X(\ell)-1}} \rho^{-\alpha}|u_\theta|^2 ds d\theta \leq \rho(X(\ell)-1)^{-\alpha}\|du\|^2_{L^2(\mathscr{C}_{X(\ell)}\setminus\mathscr{C}_{X(\ell)-1})} \leq C_\alpha \|du\|^2_{L^2(\delta_0\text{-thick}(\mathcal{C}(\sigma)))}$$

for a constant C_α depending only on α. Here $\delta_0 > 0$ is chosen as in the proof of Lemma 3.3 above so that $\mathscr{C}_{X(\ell)} \setminus \mathscr{C}_{X(\ell)-1} \subset \delta_0\text{-thick}(\mathcal{C}(\sigma))$.

Multiplying the estimate (3.5) of the above Lemma 3.3 with $\rho^{-\alpha}$ and integrating over $(-X(\ell)+1, X(\ell)-1)$ furthermore gives

(3.8)
$$\begin{aligned}\int_{\mathscr{C}_{X(\ell)-1}} |u_\theta|^2\rho^{-\alpha} ds d\theta \leq & C \cdot \int_{-X(\ell)+1}^{X(\ell)-1} \rho^{-\alpha}(s) e^{-(X(\ell)-|s|)} ds \cdot \|du\|^2_{L^2(\delta_0\text{-thick}(\mathcal{C}(\sigma)))} \\ & + C\lambda^2 \int_{-X(\ell)}^{X(\ell)}\int_{S^1} u(q,\theta)^2\rho^4(q)\Big(\int_{-X(\ell)+1}^{X(\ell)-1} e^{-|s-q|}\rho^{-\alpha}(s)ds\Big) d\theta dq.\end{aligned}$$

Since $|\partial_s \rho^{-1}| \leq 1$, compare (A.7), and since ρ is bounded from above on the collars we consider, we have that for every $s, q \in (-X(\ell), X(\ell))$

$$\rho^{-1}(s) \leq |\rho^{-1}(s) - \rho^{-1}(q)| + \rho^{-1}(q) \leq |s-q| + \rho^{-1}(q) \leq C\rho^{-1}(q)(1+|s-q|).$$

We may thus estimate the integrals appearing in (3.8) by

$$\int_{-X(\ell)+1}^{X(\ell)-1} \rho^{-\alpha}(s) e^{-(X(\ell)-|s|)} ds \leq C_\alpha \rho^{-\alpha}(X(\ell)) \int_{-X(\ell)+1}^{X(\ell)-1} (1+|X(\ell)-s|)^\alpha e^{-(X(\ell)-|s|)} ds \leq C_\alpha$$

respectively

$$\int_{-X(\ell)+1}^{X(\ell)-1} e^{-|s-q|}\rho^{-\alpha}(s) ds \leq C\rho^{-\alpha}(q)\int_{-X(\ell)+1}^{X(\ell)-1} (1+|q-s|)^\alpha e^{-|s-q|} ds \leq C_\alpha \rho^{-\alpha}(q).$$

Combined with (3.7) and (3.8) this implies that

$$\int_{\mathscr{C}_{X(\ell)}} |u_\theta|^2\rho^{-\alpha} ds d\theta \leq C_\alpha \|du\|^2_{L^2(\delta_0\text{-thick}(\mathcal{C}(\sigma)))} + C\lambda^2 \int_{\mathscr{C}_{X(\ell)}} |u|^2 \rho^{4-\alpha} ds d\theta. \tag{3.9}$$

The second integral in the above formula can be written equivalently as $\int_{\mathcal{C}(\sigma)} \rho^{\alpha-2} u^2 dv_g$. In case that $\alpha = 2$ it is hence is bounded by $\|u\|^2_{L^2(M,g)} = 1$, so claim (2.4) immediately follows from (3.9). If $\alpha = 4$ we obtain from (3.9) that

$$\begin{aligned}\int_{\mathscr{C}_{X(\ell)}} \rho^{-4}|u_\theta|^2 ds d\theta \leq & C\|du\|^2_{L^2(\delta_0\text{-thick}(\mathcal{C}(\sigma)))} + C\lambda^2 \int_{\mathscr{C}_{X(\ell)}} u^2 ds d\theta \\ \leq & C\|du\|^2_{L^2(\delta_0\text{-thick}(\mathcal{C}(\sigma)))} + C\lambda^2 X(\ell)\cdot \|u\|^2_{L^\infty(M,g)}\end{aligned}$$

as claimed. Finally, we remark that the second part of the lemma concerning the angular energies of the first eigenfunction in the setting of Lemma 2.2 is an immediate consequence of the first part of the lemma, Lemma 2.2 and Remark 2.4. □

We now turn to the proof of the estimates on the energy on the thick part of the surface that we stated in Lemma 2.2. For this we shall use the following version of the standard Poincaré inequality for functions, a proof of which is included in Appendix A.4.

Lemma 3.4. *Let (M,g) be a closed oriented hyperbolic surface and suppose that $\delta \in (0, \frac{1}{2}\text{arsinh}(1))$ is so that $\text{inj}(M,g) \leq \delta$. Let M_1^δ be the closure of a connected component of $\{p : \text{inj}_g(p) > \delta\}$, whose boundary components are denoted by $\partial M_1^\delta = \gamma^1 \sqcup \ldots \sqcup \gamma^{k_1}$ and suppose that $v \in H^1(M,g)$ is so that*

$$v \equiv 0 \text{ on at least one } \gamma^j.$$

Then we may estimate

$$\|v\|^2_{L^2(M_1^\delta,g)} + \delta^{-1}\|v\|^2_{L^2(\partial M_1^\delta,g)} \leq \frac{C}{\delta}\|dv\|^2_{L^2(M_1^\delta,g)}$$

for a constant C that depends only on the genus of M.

We note that since $\delta < \frac{1}{2}\mathrm{arsinh}(1) < \mathrm{arsinh}(1)$ the boundary curves γ^i of M_1^δ are contained in collars around simple closed geodesics $\sigma^1, \ldots, \sigma^{k_1}$ of length $L_g(\sigma^j) = \ell_j < 2\delta$ and given by $\{X_\delta(\ell_j)\}\times S^1$, X_δ as in (A.3), in collar coordinates (whose orientation we can choose accordingly, selecting two sets of collar coordinates in case that two of the curves $\gamma^j \neq \gamma^i$ are in the collar around the same geodesic $\sigma^j = \sigma^i$). As $\rho(X_\delta) \leq \delta \leq \pi\rho(X_\delta)$, compare (A.6), we thus note that the claim on the trace $u|_{\partial M_i^\delta}$ made in the lemma is equivalent to

$$\int_{\{X_\delta(\ell_i)\}\times S^1} |v|^2 d\theta \leq \frac{C}{\delta}\|dv\|^2_{L^2(M_1^\delta,g)} \text{ for every } i = 1, \ldots, k_1. \tag{3.10}$$

Proof of Lemma 2.2. Let (M, g) be so that the assumptions of the lemma are satisfied for a number $\bar{\delta} \in (0, \delta_2]$ where (A.4) allows us to choose $\delta_2 = \delta_2(\gamma) \in (0, \mathrm{arsinh}(1))$ so that

$$X(\ell) - X_\delta(\ell) \geq \Lambda_0 + 2$$

holds true for every $\ell \leq 2\delta_2$ and $\delta \in (\frac{1}{2}\ell, \delta_2]$ where $\Lambda_0 = \Lambda_0(\gamma) \geq 0$ is a constant that is chosen later on.

We first note that it suffices to prove the claim for $\delta \in [\mathrm{inj}(M, g), \bar{\delta}]$ as the estimate for smaller values of δ follows from the case that $\delta = \mathrm{inj}(M, g)$.

Given such a $\delta \in [\mathrm{inj}(M, g), \bar{\delta}]$ we note that the assumptions of the lemma guarantee that the set of simple closed geodesics $\{\sigma^i\}_{i=1}^k$ of length no more than 2δ is non-empty and contains only geodesics σ for which $M \setminus \sigma$ is disconnected. Since the length of these geodesics is less than $2\mathrm{arsinh}(1)$, the σ^j are furthermore pairwise disjoint so $M \setminus \bigcup_{i=1}^k \sigma^i$ has $k+1$ connected components which we denote by M_i. We furthermore note that by construction $M_i^\delta = \delta\text{-thick}(M, g) \cap \overline{M_i}$ are the closures of the connected components of $\{p : \mathrm{inj}_g(p) > \delta\}$ and remark that $\delta\text{-thin}(M, g) \subset \bigcup_{i=1}^k \mathcal{C}(\sigma^i)$.

The basic idea of the proof is that if too much energy was concentrated on M_i^δ then a function which is constant on (most of) M_i^δ, but agrees with u up to a constant on each of the connected components of $M \setminus M_i^\delta$, would have a smaller Rayleigh-quotient than u, contradicting the fact that u is a first eigenfunction. To make this idea precise, we associate to each M_i^δ the numbers $\mu_i = \mu_i(\delta)$ which are determined by

$$\|du\|^2_{L^2(M_i^\delta,g)} = \mu_i \frac{\lambda^2}{\delta}, \qquad i = 1, \ldots, k+1. \tag{3.11}$$

After reordering we may assume without loss of generality that $\mu_i \leq \mu_1$, $i = 2, \ldots, k+1$, so to establish the claim of the lemma we need to show that $\mu_1 \leq C_0$ for a constant C_0 that depends only on the genus.

We let $\gamma^1, \ldots, \gamma^{k_1}$ be the boundary curves of M_1^δ. As the injectivity radius is equal to δ on ∂M_1^δ, each γ^i must lie in a collar around a geodesic σ^{j_i} of the collection of $\{\sigma^j\}$ obtained above. The assumption that $M \setminus \sigma^j$ is disconnected for each j ensures that $j_i \neq j_k$ for $i \neq k$ as well as that each of the connected components $\widetilde{M}^i$ of $M \setminus M_1^\delta$ is adjacent to precisely one γ^i. We may thus assume without loss of generality that γ^i is contained in the closure of $\mathcal{C}(\sigma^i) \cap \widetilde{M}_i$. In collar coordinates (chosen with suitable orientation) γ^i then corresponds to the curve $\{X_\delta(\ell_i)\} \times S^1$, $\ell_i = L_g(\sigma^i)$, while $M_1^\delta \cap \mathcal{C}(\sigma^i)$ corresponds to the cylinder $[X_\delta(\ell_i), X(\ell_i)) \times S^1$. We note that the choice of δ_2 made above guarantees that $X(\ell_i) - X_\delta(\ell_i) \geq \Lambda_0 + 2 \geq 2$.

We will later consider the Rayleigh-quotient of $v = \tilde{u} - \fint_M \tilde{u}$ where $\tilde{u} \in H^1(M,g) \cap C^0(M,g)$ is obtained as modification of u as follows: We let $c_i := \fint_{\gamma_i} u dS_g = (2\pi)^{-1} \int_{\{X_\delta(\ell_i)\}\times S^1} u d\theta$ and define $\tilde{u}$ so that $\tilde{u} - u$ is constant on each connected component $\widetilde{M}^j$ of $M \setminus M_1^\delta$ while $\tilde{u} \equiv c_1$ on all of M_1^δ except for the cylinders $K_j = [X_\delta(\ell_j), X_\delta(\ell_j)+1] \times S^1 \subset \mathcal{C}(\sigma^j)$ on which we interpolate, namely

$$\tilde{u}(p) = \begin{cases} c_1 & \text{for } p \in M_1^\delta \setminus \bigcup_{j=1}^{k_1} K_j \\ c_1 + (X_\delta(\ell_j) + 1 - s) \cdot \big[u(X_\delta(\ell_j), \theta) - c_j\big] & \text{for } p = (s,\theta) \in K_j \subset \mathcal{C}(\sigma^j) \\ u + (c_1 - c_j) & \text{for } p \in \widetilde{M}_j. \end{cases}$$

We first note that

$$\|du\|^2_{L^2(M,g)} - \|d\tilde{u}\|^2_{L^2(M,g)} = \|du\|^2_{L^2(M_1^\delta,g)} - \sum_j \|d\tilde{u}\|^2_{L^2(K_j,g)} = \mu_1 \frac{\lambda^2}{\delta} - \sum_j \|d\tilde{u}\|^2_{L^2(K_j,g)}. \tag{3.12}$$

The choice of $\delta_2 \geq \delta$ guarantees that $|s| \leq X(\ell_j) - (\Lambda_0 + 1)$ on the cylinders K_j on which we interpolate, so we may apply the angular energy estimate (3.4) from Lemma 3.3 with $\Lambda = \Lambda_0 \geq 0$ to obtain

$$\begin{aligned} \|d\tilde{u}\|^2_{L^2(K_j,g)} &\leq \int_{\{X_\delta(\ell_j)\}\times S^1} |u - c_j|^2 + |u_\theta|^2 d\theta \leq 2\vartheta(X_\delta(\ell_j)) \\ &\leq 4e^{-\Lambda_0} \|du\|^2_{L^2(\mathscr{C}_{X_\delta(\ell_j)+\Lambda_0+1} \setminus \mathscr{C}_{X_\delta(\ell_j)+\Lambda_0})} + \lambda^2 \int_{\mathscr{C}_{X_\delta(\ell_j)+\Lambda_0+1}} \rho^4 e^{-|X_\delta(\ell_j)-q|} u^2 d\theta dq \\ &\leq 4e^{-\Lambda_0} \|du\|^2_{L^2(\mathcal{C}(\sigma^j)\cap\delta\text{-thick}(M,g))} + C\lambda^2 \rho^4 (X_\delta(\ell_j) + \Lambda_0 + 1) \|u\|^2_{L^\infty(M,g)}. \end{aligned}$$

We finally choose Λ_0 so that $k + 1 \leq 3(\gamma - 1) + 1 \leq \frac{1}{8} e^{\Lambda_0}$ where we recall that $k+1$ is the number of connected components of $\{p : \mathrm{inj}_g(p) > \delta\}$. As the estimates (A.6) and (A.8) from the appendix furthermore allow us to bound $\rho(X_\delta(\ell_j) + \Lambda_0 + 1) \leq e^{\Lambda_0+1}\rho(X_\delta(\ell_j)) \leq C\delta$ and as u is uniformly bounded, c.f. Remark 2.4, we hence obtain

$$\begin{aligned} \sum_{i=1}^{k_1} \|d\tilde{u}\|^2_{L^2(K_i,g)} &\leq 4e^{-\Lambda_0} \|du\|^2_{L^2(\delta\text{-thick}(M,g))} + C\lambda^2\delta^4 = 4e^{-\Lambda_0} \sum_{i=1}^{k+1} \mu_i \cdot \frac{\lambda^2}{\delta} + C\lambda^2\delta^4 \\ &\leq 4(k+1)e^{-\Lambda_0} \mu_1 \frac{\lambda^2}{\delta} + C\lambda^2\delta^4 \leq \frac{1}{2}\mu_1 \frac{\lambda^2}{\delta} + C\lambda^2\delta^4, \end{aligned} \tag{3.13}$$

where the numbers μ_i are as in (3.11) and where we used the assumption that $\mu_i \leq \mu_1$ in the last step.

On the one hand, we can combine this estimate with (3.12) to obtain that

$$\|d\tilde{u}\|^2_{L^2(M,g)} \leq \|du\|^2_{L^2(M,g)} - \frac{1}{2}\mu_1 \frac{\lambda^2}{\delta} + C\lambda^2\delta^4 = \lambda \cdot \big[1 - (\frac{\mu_1 \lambda}{2\delta} - C\lambda\delta^4)\big] \tag{3.14}$$

for a constant C that depends only on the genus.

On the other hand, (3.13) also allows us to estimate the L^2-norm of $u - \tilde{u}$: Since $u - \tilde{u} \equiv 0$ on γ^1 and

$$\|d(u - \tilde{u})\|^2_{L^2(M_1^\delta,g)} \leq 2\|du\|^2_{L^2(M_1^\delta,g)} + 2\|d\tilde{u}\|^2_{L^2(M_1^\delta,g)} \leq C\mu_1 \frac{\lambda^2}{\delta} + C\lambda^2\delta^4$$

we may apply the Poincaré estimate stated in Lemma 3.4 to bound

$$\|u - \tilde{u}\|^2_{L^2(M_1^\delta,g)} \leq C\mu_1\lambda^2\delta^{-2} + C\delta^3\lambda^2 \tag{3.15}$$

as well as the trace estimate (3.10) obtained as a consequence of this lemma to estimate

$$\begin{aligned}\|u-\tilde{u}\|^2_{L^2(M\setminus M_1^\delta,g)} &= \sum_i |c_i - c_1|^2 \mathrm{Area}(\widetilde{M}^i, g)\\ &\leq C\sum_i \int_{\{X_\delta(\ell_i)\}\times S^1} |u-\tilde{u}|^2 d\theta \leq \frac{C}{\delta}\|d(u-\tilde{u})\|^2_{L^2(M_1^\delta,g)}\\ &\leq C\mu_1\lambda^2\delta^{-2} + C\delta^3\lambda^2.\end{aligned} \tag{3.16}$$

We now set $v = \tilde{u} - (\tilde{u})_M$ where we write for short $(\tilde{u})_M := ⨍ \tilde{u} dv_g$ and note that since $\int u dv_g = 0$ we have

$$|(\tilde{u})_M| \leq C\|u-\tilde{u}\|_{L^2(M,g)}.$$

Since u and hence also v is bounded uniformly and since $\|u\|_{L^2(M,g)} = 1$ we thus get

$$\begin{aligned}1 - \|v\|^2_{L^2(M,g)} &\leq \|u+v\|_{L^2(M,g)} \cdot \|u-v\|_{L^2(M,g)} \leq C(\|u-\tilde{u}\|_{L^2(M,g)} + \|(\tilde{u})_M\|_{L^2(M,g)})\\ &\leq C\|u-\tilde{u}\|_{L^2(M,g)}.\end{aligned}$$

Inserting (3.15) and (3.16) into this estimate hence gives a bound of

$$\|v\|^2_{L^2(M,g)} \geq 1 - C[\mu_1^{1/2}\lambda\delta^{-1} + \delta^{3/2}\lambda]$$

which we may combine with the estimate (3.14) on $\|d\tilde{u}\|^2_{L^2} = \|dv\|^2_{L^2}$ to reach

$$\frac{\|dv\|^2_{L^2(M,g)}}{\|v\|^2_{L^2(M,g)}} \leq \lambda \cdot \frac{1-(\frac{\mu_1\lambda}{2\delta} - C\lambda\delta^4)}{1 - C(\delta^{-1}\mu_1^{1/2}\lambda + \delta^{3/2}\lambda)}.$$

Since this quotient can be no smaller than the first eigenvalue λ we must thus have that

$$\mu_1 \leq C \cdot \frac{2\delta}{\lambda} \cdot [\lambda\delta^4 + \delta^{-1}\mu_1^{1/2}\lambda + \delta^{3/2}\lambda] \leq \frac{1}{2}\mu_1 + C(\delta_2^5 + \delta_2^{5/2} + 1),$$

which gives the desired uniform upper bound on μ_1 and hence completes the proof of the lemma.

□

4. Holomorphic quadratic differentials and Fenchel Nielson coordinates

In this section we provide the proofs of the results on holomorphic quadratic differentials which we stated in Section 2.4 and used in Section 2.5 to prove some of our main results. In Section 4.2 we analyse the dual basis $\{\Theta^j\}$ and prove the assertions of Lemma 2.13 as well as Corollary 2.14 on the properties of the Θ^j respectively of the renormalised elements $\Omega^j = -\frac{\Theta^j}{\|\Theta^j\|_{L^2(M,g)}}$. Section 4.3 is then concerned with the analysis of the elements Ψ^j which generate Dehn-twists and hence the proof of Lemma 2.19, while the properties of the elements Λ^j which induce a change of only the length coordinates are analysed in Section 4.4 where we prove Lemma 2.18.

4.1. Existence of a dual basis to $\partial\ell_i$.

Let $\mathcal{E} = \{\sigma^i\}_{i=1}^{3(\gamma-1)}$ be a decomposing collection of simple closed geodesics in a hyperbolic surface (M,g), i.e. a collection of disjoint geodesics which decomposes (M,g) into pairs of pants. We recall that Theorem 3.7 of [28] ensures that the map $\Upsilon \mapsto \big(\partial\ell_1(\Upsilon), \ldots, \partial\ell_{3(\gamma-1)}(\Upsilon)\big)$ introduced in (2.42) is an isomorphism.

As we want to use this result later on in Section 4.2 not just for closed hyperbolic surfaces, but also for surfaces which are obtained as limits of a degenerating sequence of such surfaces, we consider in this section more generally hyperbolic surfaces with a finite (possibly empty) set of punctures and provide a short proof of this result in this setting which at the same time introduces the properties of the elements associated to the twist coordinates that we use later on.

So let (M,g) be a complete oriented hyperbolic surface (as always assumed to be connected) with $k \in \mathbb{N}_0$ punctures. We recall from [23, Lemma A.11] that the space

$$\mathcal{H}(M,g) := \{\Upsilon \text{ holomorphic quadratic differential on } M \text{ with } \|\Upsilon\|_{L^2(M,g)} < \infty\}$$

agrees with the space of meromorphic quadratic differentials on $(\overline{M}, \bar{\mathfrak{c}})$ ($\mathfrak{c}$ the corresponding complex structure and $(\overline{M}, \bar{\mathfrak{c}})$ its extension obtained by filling in the punctures) which have at most simple poles in the punctures and are holomorphic on $M \subset \overline{M}$. The dimension of $\mathcal{H}(M,g)$ is $\dim_{\mathbb{C}} \mathcal{H}(M,g) = \hat{k} := 3(\gamma-1)+k$, where γ is the genus of $\overline{M}$. We note that this number $\hat{k}$ agrees with the number of simple closed geodesics needed to decompose (M,g) into pairs of pants [6, Thm. 4.4.6] (where some of the pants will be degenerate if $k>0$, i.e. so that one or more of their three boundary components has length zero).

We use the following lemma, see also [28, Theorem 3.7]

Lemma 4.1. *Let (M,g) be a complete hyperbolic surface of genus γ with $k \in \mathbb{N}_0$ punctures. Then for any decomposing set $\{\sigma^1, \ldots, \sigma^{\hat{k}}\}$, $\hat{k} = 3(\gamma-1)+k$, of disjoint simple closed geodesics the map*

$$\mathcal{H}(M,g) \ni \Upsilon \mapsto \big(\partial\ell_1(\Upsilon), \ldots, \partial\ell_{\hat{k}}(\Upsilon)\big) \in \mathbb{C}^{\hat{k}} \tag{4.1}$$

is an isomorphism.

The main tool for the proof of this lemma is the following characterisation of elements generating Dehn-twists which is proved later on in Section 4.3.

Lemma 4.2. *Let (M,g) be a complete hyperbolic surface of genus γ with $k \in \mathbb{N}_0$ punctures, and let $\{\sigma^j\}_{j=1}^{\hat{k}}$, $\hat{k} = 3(\gamma-1)+k$, be a set of disjoint simple closed geodesics that decomposes (M,g) into pairs of pants.*

Then the element $\Psi^j \in \mathcal{H}(M,g)$ that generates only a Dehn-twist around σ^j, i.e. is characterised by (2.39), can be obtained as projection

$$\Psi^j = P_g^{\mathcal{H}}(K^j)$$

of any quadratic differential K^j on M that is zero outside of $\mathcal{C}(\sigma^j)$ and on $\mathcal{C}(\sigma^j)$ given in collar coordinates by $\mathrm{i}\rho^2(s)\xi'(s)(ds+\mathrm{i}d\theta)^2$ for a function $\xi \in C^\infty([-X(\ell_j), X(\ell_j)])$ which is equal to $\pm\frac{1}{2}$ in a neighbourhood $\pm X(\ell_j)$. Here the orientation of the collar coordinates is chosen as explained in Remark A.3.

Furthermore for every $\Upsilon \in \mathcal{H}(M,g)$

$$\langle \Upsilon, \Psi^j \rangle = 8\pi \mathrm{i} b_0(\Upsilon, \mathcal{C}(\sigma^j)) \tag{4.2}$$

so in particular

$$b_0(\Psi^j, \mathcal{C}(\sigma^j)) = -\frac{\mathrm{i}}{8\pi}\|\Psi^j\|^2_{L^2(M,g)} \quad \text{and} \quad \Psi^j \in (\ker(\partial\ell_j))^\perp \tag{4.3}$$

while additionally

$$\mathrm{Re}\,(b_0(\Psi^j, \mathcal{C}(\sigma^i))) = 0 \text{ for every } i = 1, \ldots, \hat{k}. \tag{4.4}$$

Proof of Lemma 4.1. Since $\partial\ell_j(\Upsilon) = -\frac{\pi^2}{\ell_j} b_0(\Upsilon, \mathcal{C}(\sigma^j))$, compare (2.16), the claim that (4.1) is an isomorphism is equivalent to

$$\mathcal{H}(M,g) \ni \Upsilon \mapsto \big(b_0(\Upsilon, \mathcal{C}(\sigma^i))\big)_{i=1}^{\hat{k}} \in \mathbb{C}^{\hat{k}}$$

being an isomorphism. To prove this, we remark that any nontrivial *real* linear combination $\Upsilon = \sum_j \mu_j \Psi^j$ of the elements Ψ^j that are described in the previous Lemma 4.2 leads to a nontrivial change of the twist coordinates along curves of metrics moving by $\partial_t g = \mathrm{Re}\,(\sum \mu_j \Psi^j)$.

In particular, such a Υ must be non-zero so, by (4.2),

$$0 \neq \|\Upsilon\|^2_{L^2(M,g)} = \sum_j \mu_j \langle \Upsilon, \Psi^j \rangle_{L^2(M,g)} = 8\pi \mathrm{i} \sum_j \mu_j b_0(\Upsilon, \mathcal{C}(\sigma^j))$$

meaning that at least one of the $b_0(\Upsilon, \mathcal{C}(\sigma^j))$ must be non-zero. As these principal parts are all elements of $\mathrm{i}\mathbb{R}$, compare (4.4), we thus obtain that

$$\mathbb{R}^{\hat{k}} \ni \mu \mapsto \left(\mathrm{i} b_0 \big(\sum_l \mu_l \Psi^l, \mathcal{C}(\sigma^j) \big) \right)_{j=1}^{\hat{k}} \in \mathbb{R}^{\hat{k}}$$

is a real isomorphism which then implies the claim. □

4.2. Estimates on the dual basis $\{\Theta^j\}_{j=1}^{3(\gamma-1)}$ to $\partial\ell_j$.

Let (M,g) be a closed oriented surface of genus γ, let $\mathcal{E}$ be any decomposing collection of simple closed geodesics and let $\{\Theta^j\}_{j=1}^{3(\gamma-1)}$ be the dual basis of the corresponding $\{\partial\ell_j\}_{j=1}^{3(\gamma-1)}$. In this section we want to derive the estimates on Θ^j and the corresponding renormalised elements Ω^j that we claimed in Lemma 2.13 and its Corollary 2.14. The key step in this proof is to show the following Lemma 4.3 which allows us to bound the principal parts of Ω^j on the corresponding collar. In the last part of the present section we will then combine this lemma with the results from [22] that we recalled in Section 2.2 to give the proof of Lemma 2.13 and its Corollary 2.14.

Lemma 4.3. *Let (M,g) be a closed oriented hyperbolic surface of genus γ and let $\mathcal{E} = \{\sigma^1, \ldots, \sigma^{3(\gamma-1)}\}$ be a decomposing collection of disjoint simple closed geodesics for which* (2.36) *and* (2.37) *are satisfied for some $\eta \in (0, \operatorname{arsinh}(1))$ and some $\bar{L} < \infty$. Then for any number $\ell_0 > 0$ there exists a constant $\varepsilon_0 > 0$ depending on ℓ_0, η, $\bar{L}$ and the genus γ such that the elements Ω^j characterised by* (2.41) *satisfy*

$$b_0(\Omega^j, \mathcal{C}(\sigma^j)) \geq \varepsilon_0 \text{ for every } j \text{ for which } L_g(\sigma^j) \geq \ell_0.$$

We will apply the above lemma mainly to estimate the principal parts of elements Ω^j corresponding to geodesics σ^j which are not too short so we do not need to be concerned with the precise dependence of ε_0 on ℓ_0 (though we shall later see that $\varepsilon_0 = C(\eta, \bar{L})\ell_0^{3/2}$ is the sharp rate).

The proof of this lemma is based in particular on the following result from [23]:

Lemma 4.4. *[Contents of* [23, Lemma 2.4]*] Let (M, g_i) be a sequence of closed oriented hyperbolic surfaces that degenerate to a punctured (possibly disconnected) hyperbolic surface (Σ, g_∞) by collapsing k geodesics $\sigma_i^j \subset (M, g_i)$, $j \in \{1, \ldots, k\}$, as described in Proposition A.2. Then, after pulling back by the diffeomorphisms $f_i \colon \Sigma \to M \setminus \cup_{j=1}^k \sigma_i^j$ introduced in Proposition A.2 the spaces*

$$W_i := \{\Upsilon \in \mathcal{H}(M, g_i) \mid b_0(\Upsilon, \mathcal{C}(\sigma_i^j)) = 0,\ j = 1, \ldots, k\} \tag{4.5}$$

converge to $\mathcal{H}(\Sigma, g_\infty)$ in the following sense:

There exists a sequence $\{w_i^j\}_{j=1}^{3(\gamma-1)-k}$ of orthonormal bases of W_i and an orthonormal basis $\{w_\infty^j\}_{j=1}^{3(\gamma-1)-k}$ of $\mathcal{H}(\Sigma, g_\infty)$ so that for every $j \in \{1, \ldots, 3(\gamma-1)-k\}$

$$f_i^* w_i^j \to w_\infty^j \text{ in } C^\infty_{loc}(\Sigma, g_\infty) \qquad \text{as } i \to \infty.$$

The above lemma implies in particular:

Corollary 4.5. *In the setting of Lemma 4.4 the following holds true. For any element $\Omega_\infty \in \mathcal{H}(\Sigma, g_\infty)$ there exists a sequence $\Omega_i \in W_i \subset \mathcal{H}(M, g_i)$ with*

$$\|\Omega_i\|_{L^2(M,g_i)} = \|\Omega_\infty\|_{L^2(\Sigma,g_\infty)} \text{ so that } f_i^* \Omega_i \to \Omega_\infty \text{ in } C^\infty_{loc}(\Sigma, g_\infty).$$

Conversely, any sequence of elements $\Omega_i \in W_i \subset \mathcal{H}(M, g_i)$ with $\|\Omega_i\|_{L^2(M,g_i)} = 1$ has a subsequence so that $f_i^*\Omega_i$ converges in $C^\infty_{loc}(\Sigma, g_\infty)$ to some $\Omega_\infty \in \mathcal{H}(\Sigma, g_\infty)$ with $\|\Omega_\infty\|_{L^2(\Sigma,g_\infty)} = 1$.

The proof of this corollary immediately follows by writing a given element $\Omega_\infty \in \mathcal{H}(\Sigma, g_\infty)$ in the form $\Omega_\infty = \sum_{j=1}^{3(\gamma-1)-k} a_j w_\infty^j$ and considering the corresponding sequence $\Omega_i = \sum_{j=1}^{3(\gamma-1)-k} a_j w_i^j$ respectively, for the second part of the Corollary, by writing a sequence Ω_i in the form $\Omega_i = \sum_{j=1}^{3(\gamma-1)-k} a_j^i w_i^j$ and passing to a subsequence for which the coefficients $(a_j^i)_i$ converge.

To prove Lemma 4.3 we will furthermore need

Lemma 4.6. *In the setting of Lemma 4.4 the following holds true. Let $\sigma_\infty \subset (\Sigma, g_\infty)$ be a simple closed geodesic and let σ_i be the (unique) simple closed geodesic in (M, g_i) which is homotopic to $(f_i)_*\sigma_\infty \subset M$. Then for any sequence $\Omega_i \in W_i$ for which*

$$f_i^*\Omega_i \to \Omega_\infty \text{ in } C^\infty_{loc}(\Sigma) \qquad \text{as } i \to \infty \tag{4.6}$$

we have

$$b_0(\Omega_i, \mathcal{C}(\sigma_i)) \to b_0(\Omega_\infty, \mathcal{C}(\sigma_\infty)) \qquad \text{as } i \to \infty.$$

Proof of Lemma 4.6. We prove this lemma in two steps, first using the local smooth convergence of the metrics to obtain C^1-convergence of the geodesics $\tilde{\sigma}_i = f_i^*\sigma_i \subset (\Sigma, f_i^*g_i)$ to σ_∞ and then in a second step using the relation (2.14) between the change of the length coordinate $d\ell(\mathrm{Re}\,(\Omega))$ and the principal part $\mathrm{Re}\,(b_0(\Omega))$ on the corresponding collar. We furthermore remark that it suffices to prove the convergence of the real parts $\mathrm{Re}\,(b_0(\Omega_i))$ as we may replace Ω_i by $\mathrm{i}\Omega_i$.

To begin with we note that there is a compact subset $K \subset \Sigma$ whose interior contains both the simple closed geodesic $\sigma_\infty \subset (\Sigma, g_\infty)$ as well as the simple closed geodesics $\tilde{\sigma}_i = f_i^*\sigma_i \subset (\Sigma, f_i^*g_i)$, for i sufficiently large. Such a set can for example be obtained by setting $K = \delta\text{-thick}(\Sigma, g)$ for $\delta < \frac{1}{2}\mathrm{arsinh}(1)$ chosen so that the length of the shortest simple closed geodesic in (Σ, g_∞) is at least 4δ. This choice of δ ensures that $2\delta\text{-thin}(\Sigma, g_\infty)$ is contained in the union of the collar neighbourhoods around the punctures, so the uniform convergence of the metrics on K assures that for i sufficiently large

$$\delta\text{-thin}(\Sigma, g) \subset \bigcup_j f_i^*(\mathcal{C}(\sigma_i^j) \setminus \sigma_i^j),$$

where $\mathcal{C}(\sigma_i^j)$ are the collars around the collapsing geodesics σ_i^j in (M, g_i) and where we recall that f_i is a diffeomorphism from Σ to $M \setminus (\bigcup_j \sigma_i^j)$. Since the collar neighbourhoods of disjoint simple closed geodesics are disjoint, this ensures that $\tilde{\sigma}_i \subset \mathcal{C}(\tilde{\sigma}_i) \subset \mathrm{int}(K)$.

As the metrics converge smoothly in K this allows us to obtain parametrisations $\gamma_i \colon S^1 \to \tilde{\sigma}_i \subset (\Sigma, f_i^*g_i)$ (proportional to arc length) that converge in $C^1(\Sigma, g_\infty)$ to a parametrisation γ_∞ of σ_∞ by a standard argument: convergence of the injectivity radii ensures that $\tilde{\sigma}_i \to \sigma_\infty$ in Hausdorff-distance (that may e.g. be computed w.r.t. g_∞). Then using the geodesic equation we note that if the tangent vector $\gamma_i'(\theta_1)$ were not close to the tangent vector at a nearby point on γ_∞ then for some $c > 0$ the point $\gamma_i(\theta_1 + c)$ could not be close to σ_∞. Appealing once more to the geodesic equation this then yields the desired C^1-convergence.

We now recall that along a curve of metrics $g(t)$ that evolves by $\partial_t g(0) = \mathrm{Re}\,\Omega$ for some $\Omega \in \mathcal{H}(M, g(0))$, the evolution of the length $L_{g(t)}(\sigma(t))$ of the simple closed geodesic $\sigma(t)$ in $(M, g(t))$ that is homotopic to $\sigma = \sigma(0)$ is determined by (2.14). As the geodesic σ minimises length in its homotopy class, we have that

$$\frac{d}{dt}|_{t=0} L_{g(t)}(\sigma(t)) = \frac{d}{dt} L_{g(t)}(\sigma(0)),$$

compare [22, Rem. 4.11], so the principal part of a holomorphic quadratic differential $\Omega \in \mathcal{H}(M, g(0))$ can be determined by

$$\mathrm{Re}\,(b_0(\Omega, \mathcal{C}(\sigma))) = -\frac{L_g(\sigma)}{2\pi^2}\frac{d}{dt}|_{t=0} L_{g+t\mathrm{Re}\,(\Omega)}(\sigma).$$

In the context of the lemma we thus find that

$$\begin{aligned}
&|\mathrm{Re}\,\big(b_0(\Omega_i, \mathcal{C}(\sigma_i)) - b_0(\Omega_\infty, \mathcal{C}(\sigma_\infty))\big)| \\
&\quad= \left|\frac{1}{2\pi^2}\big(\ell_i \cdot \tfrac{d}{dt} L_{f_i^*(g_i + t\mathrm{Re}\,_{g_i}(\Omega_i))}(f_i^*\sigma_i) - \ell_\infty \cdot \tfrac{d}{dt} L_{g_\infty + t\mathrm{Re}\,_{g_\infty}(\Omega_\infty)}(\sigma_\infty)\big)\right| \\
&\quad\le C \cdot |\ell_i - \ell_\infty| \cdot |\tfrac{d}{dt} L_{g_i + t\mathrm{Re}\,_{g_i}(\Omega_i)}(\sigma_i)| \\
&\qquad + C\left|\tfrac{d}{dt}\big(L_{f_i^*(g_i + t\mathrm{Re}\,_{g_i}(\Omega_i))}(f_i^*\sigma_i) - L_{f_i^*(g_i + t\mathrm{Re}\,_{g_i}(\Omega_i))}(\sigma_\infty)\big)\right| \\
&\qquad + C\left|\tfrac{d}{dt}\big(L_{f_i^*(g_i + t\mathrm{Re}\,_{g_i}(\Omega_i))}(\sigma_\infty) - L_{g_\infty + t\mathrm{Re}\,_{g_\infty}(\Omega_\infty)}(\sigma_\infty)\big)\right| \\
&\quad= I + II + III
\end{aligned} \tag{4.7}$$

where we remark that the real parts of holomorphic quadratic differentials need to be computed with respect to the corresponding conformal structure, where we write for short $\ell_i := L_{g_i}(\sigma_i) \to \ell_\infty := L_{g_\infty}(\sigma_\infty) > 0$, and where derivatives with respect to t are to be evaluated in $t = 0$.

The first term is bounded by

$$I \le C \cdot |\ell_i - \ell_\infty| \cdot \frac{|b_0(\Omega_i, \mathcal{C}(\sigma_i))|}{\ell_i} \le C \cdot |\ell_i - \ell_\infty| \cdot \ell_i^{1/2} \cdot \|f_i^*\Omega_i\|_{L^2(K, f_i^* g_i)} \to 0,$$

compare (A.14).

To bound the third term in (4.7) we remark that the convergence of the metrics implies that $|\gamma_\infty'|_{f_i^* g_i} \to |\gamma_\infty'|_{g_\infty} = \frac{\ell_\infty}{2\pi} > 0$. For i sufficiently large we may thus use (4.6) and the convergence of the conformal structures to conclude that

$$\begin{aligned}
III &= C\left|\int_{S^1} \frac{\mathrm{Re}\,_{f_i^* g_i}(f_i^*\Omega_i)(\gamma_\infty', \gamma_\infty')}{|\gamma_\infty'|_{f_i^* g_i}} - \frac{\mathrm{Re}\,_{g_\infty}(\Omega_\infty)(\gamma_\infty', \gamma_\infty')}{|\gamma_\infty'|_{g_\infty}}\right| \\
&\le C\|\mathrm{Re}\,_{f_i^* g_i}(f_i^*\Omega_i) - \mathrm{Re}\,_{g_\infty}(\Omega_\infty)\|_{L^\infty(K)} + C\|f_i^* g_i - g_\infty\|_{L^\infty(K)} \cdot \|f_i^*\Omega_i\|_{L^\infty(K)} \to 0
\end{aligned}$$

as $i \to \infty$. Similarly, we may bound the second term in (4.7) by

$$II \le C \cdot \|\gamma_i - \gamma_\infty\|_{C^1(K)} \cdot \|\mathrm{Re}\,_{f_i^* g_i}(f_i^*\Omega_i)\|_{C^1(K)} \to 0$$

where we may compute the norms with respect to any of the equivalent metrics, say w.r.t. g_∞. Combined we thus obtain the claim of the lemma. □

Based on the above results we can now give the

Proof of Lemma 4.3. We argue by contradiction. So let us assume that there exist positive numbers $\bar{L}$, η and ℓ_0 and a sequence of closed hyperbolic surfaces (M, g_i) together with decomposing collections $\mathcal{E}_i = \{\sigma_i^1, \ldots, \sigma_i^{3(\gamma-1)}\}$ of simple closed geodesics which satisfy (2.36) and (2.37) for $\bar{L}$ and η so that, after reordering the geodesics if necessary,

$$L_{g_i}(\sigma_i^{3(\gamma-1)}) \ge \ell_0 \text{ but } b_0(\Omega_i^{3(\gamma-1)}, \mathcal{C}(\sigma_i^{3(\gamma-1)})) \to 0$$

for the renormalised elements $\Omega_i^j = -\Theta_i^j \|\Theta_i^j\|_{L^2(M, g_i)}^{-1}$ of the dual bases $\{\Theta_i^j\}_{j=1}^{3(\gamma-1)}$ of $\mathcal{H}(M, g_i)$ corresponding to $\mathcal{E}_i$, compare (2.40).

After passing to a subsequence, and if necessary relabelling the geodesics $\{\sigma_i^j\}_{j=1}^{3(\gamma-1)-1}$, we may assume that (M, g_i) converges to a (possibly punctured and disconnected) hyperbolic surface by collapsing the $k \in \{0, \ldots, 3(\gamma-1)-1\}$ geodesics σ_i^j, $j = 1, \ldots, k$, as described in Proposition A.2.

We note that in the above description only geodesics that are contained in $\mathcal{E}_i$ can collapse since each $\mathcal{E}_i$ satisfies assumption (2.36) for some fixed $\eta > 0$.

By construction $b_0(\Omega_i^{3(\gamma-1)}, \mathcal{C}(\sigma_i^j)) = 0$ for all $j < 3(\gamma-1)$, in particular for $j = 1, \dots, k$, so $\Omega_i^{3(\gamma-1)}$ is an element of the space W_i defined in (4.5). By Corollary 4.5 we may thus pass to a subsequence to obtain that

$$f_i^* \Omega_i^{3(\gamma-1)} \to \Omega_\infty \text{ in } C^\infty_{loc}(\Sigma, g_\infty) \text{ for some } \Omega_\infty \in \mathcal{H}(\Sigma, g_\infty) \text{ with } \|\Omega_\infty\|_{L^2(\Sigma, g_\infty)} = 1.$$

We recall that since the simple closed geodesics $\sigma_i^{k+1}, \dots, \sigma_i^{3(\gamma-1)}$ are disjoint from the collars $\mathcal{C}(\sigma_i^j)$, $j = 1, \dots, k$, we may choose a compact subset $K \subset \Sigma$ as in the proof of Lemma 4.6 so that for i sufficiently large

$$\tilde{\sigma}_i^j := f_i^* \sigma_i^j \subset K \text{ for every } j = k+1, \dots, 3(\gamma-1).$$

Since the metrics $f_i^* g_i$ converge smoothly to g_∞ on K we obtain from (2.37) that for i sufficiently large also $L_{g_\infty}(\tilde{\sigma}_i^j) \leq \bar{L} + 1$ for each $k+1 \leq j \leq 3(\gamma-1)$. As there are only finitely many homotopy classes of closed curves in (Σ, g_∞) which have a representative of length no more than $\bar{L} + 1$, we may thus pass to a further subsequence in a way that ensures that for each $j = k+1, \dots, 3(\gamma-1)$ the curves $\tilde{\sigma}_i^j$, $i \in \mathbb{N}$, are homotopic to each other. We denote the simple closed geodesic in (Σ, g_∞) that belongs to this homotopy class $[\tilde{\sigma}_i^j]$ by σ_∞^j and remark that $\{\sigma_\infty^j\}_{j=k+1}^{3(\gamma-1)}$ decomposes (Σ, g_∞) into pairs of pants.

We recall that $b_0(\Omega_i^{3(\gamma-1)}, \mathcal{C}(\sigma_i^{3(\gamma-1)})) \to 0$ while by definition $b_0(\Omega_i^{3(\gamma-1)}, \mathcal{C}(\sigma_i^j)) = 0$ for $j \leq 3(\gamma-1) - 1$. Hence Lemma 4.6 implies that

$$b_0(\Omega_\infty, \mathcal{C}(\sigma_\infty^j)) = \lim_{i \to \infty} b_0(\Omega_i^{3(\gamma-1)}, \mathcal{C}(\sigma_i^j)) = 0$$

for the whole decomposing collection $\{\sigma_\infty^j\}_{j=k+1}^{3(\gamma-1)}$ of (Σ, g_∞). By Lemma 4.1 (applied on each connected components of (Σ, g_∞)) this forces $\Omega_\infty \equiv 0$ in contradiction to $\|\Omega_\infty\|_{L^2(\Sigma, g_\infty)} = 1$. ☐

We are now finally in a position to complete the

Proof of Lemma 2.13. Let (M, g) be a closed oriented hyperbolic surface and consider any decomposing collection $\mathcal{E} = \{\sigma^1, \dots, \sigma^{3(\gamma-1)}\}$ of simple closed geodesics and denote as usual by $\{\Theta^j\}$ the dual basis of the corresponding $\{\ell_j\}$ and by $\{\Omega^j\}$ the corresponding renormalised elements.

Let $\eta > 0$ and $\bar{L} < \infty$ be such that (2.36) and (2.37) hold true for $\mathcal{E}$. As Lemma 4.1 ensures that the map (2.42) is an isomorphism we can consider the corresponding dual bases $\{\Theta^j\}$ respectively $\{\Omega^j\}$ of $\mathcal{H}(M, g)$.

We set $\bar{\eta} := \min(\eta_1, \eta)$, where $\eta_1 = \eta_1(\gamma) \in (0, \operatorname{arsinh}(1))$ is as in Lemma 2.5 (and the subsequent Remark 2.6), and reorder the geodesics if necessary to ensure that the simple closed geodesics of length no more than $2\bar{\eta}$ are given by $\sigma^1, \dots \sigma^k$ for some $0 \leq k \leq 3(\gamma-1)$.

We first remark that for the elements Θ^j, $j = k+1, \dots, 3(\gamma-1)$, we may apply Lemma 4.3 (with $\ell_0 = 2\bar{\eta}$) to obtain a uniform lower bound $\varepsilon_0 = \varepsilon_0(\bar{\eta}, \gamma, \bar{L}) > 0$ on the principal parts $b_0(\Omega^j, \mathcal{C}(\sigma^j))$ and hence an upper bound on

$$\|\Theta^j\|_{L^2(M,g)} = \frac{\ell_j}{\pi^2 b_0(\Omega^j, \mathcal{C}(\sigma^j))} \|\Omega^j\|_{L^2(M,g)} \leq \frac{\bar{L}}{\pi^2 \varepsilon_0},$$

compare (2.53). So (2.43) holds for these j and as this is the only claim made about the elements Θ^j, $j \geq k+1$, we may from now on assume that $j \in \{1, \dots, k\}$ and hence $\ell_j \leq 2\bar{\eta}$.

This allows us to apply the results of [22] which we recalled in Lemma 2.5 and Corollary 2.7. So let $\{\tilde\Theta^j\}_{j=1}^k$ and $\{\tilde\Omega^j\}_{j=1}^k$ be the bases of

$$W = W_{\bar\eta} := \{\Upsilon \mid b_0(\Upsilon, \mathcal{C}(\tilde\sigma^j)) = 0 \text{ for } j = 1, \dots, k\} = \ker(\partial\ell_j, j = 1, \dots, k)$$

from this lemma respectively corollary which are dual to $(\partial\ell_1, \dots, \partial\ell_k)$.

As $\{\Omega^j\}_{j=1}^{3(\gamma-1)}$ is a basis of $\mathcal{H}(M,g)$, we can write each such $\tilde\Omega^j = \sum_{i=1}^{3(\gamma-1)} d_i^j \Omega^i$ for complex coefficients d_i^j which we claim must be so that $d_j^j \in \mathbb{R}^+$ while $d_j^i = 0$ for $i \in \{1, \dots k\}$ with $i \neq j$: Indeed the first property follows since the principal parts of Ω^j and $\tilde\Omega^j$ on $\mathcal{C}(\sigma^j)$ are both positive while $b_0(\Omega^i, \mathcal{C}(\sigma^j)) = 0$ for every $i \neq j$, while the second property follows since for $i \in \{1, \dots, k\}$ with $i \neq j$ the only element in the above expression whose principal part on $\mathcal{C}(\sigma^i)$ is non-zero is Ω^i.

We may thus write each Ω^j, $j = 1, \dots, k$, in the form

$$\Omega^j = a_j \cdot \big[\tilde\Omega^j + \sum_{m=k+1}^{3(\gamma-1)} c_m^j \Omega^m\big] \text{ for some } a_j \in \mathbb{R}^+ \text{ and } c_m^j \in \mathbb{C}, \tag{4.8}$$

and we claim that

$$|1 - a_j| \le C\ell_j^3 \text{ while } |c_m^j| \le C\ell_j^{3/2}, \quad m \in \{k+1, \dots, 3(\gamma-1)\}. \tag{4.9}$$

As $\Omega^m \in \ker(\partial\ell_1, \dots, \partial\ell_k)$, $m \geq k+1$, and hence, by (2.13), $\|\Omega^m\|_{L^\infty(M,g)} \leq C = C(\eta, \gamma)$ this will imply the two claims (2.46) and (2.47) about Ω^j made in the lemma.

To prove the claimed estimate on c_m^j we compare the principal parts in (4.8) on the corresponding collar: since $b_0(\Omega^i, \mathcal{C}(\sigma^m)) = 0$ for $i \neq m$, while, as observed above, $b_0(\Omega^m, \mathcal{C}(\sigma^m)) \geq \varepsilon_0(\bar\eta, \gamma, \bar L) > 0$, we may use (A.15) to bound

$$\begin{aligned} |c_m^j| &= \left| \frac{b_0(\tilde\Omega^j, \mathcal{C}(\sigma^m))}{b_0(\Omega^m, \mathcal{C}(\sigma^m))} \right| \leq C|b_0(\tilde\Omega^j, \mathcal{C}(\sigma^m))| \\ &\leq C\|\tilde\Omega^j\|_{L^2(\mathcal{C}(\sigma^m))} \leq C\|\tilde\Omega^j\|_{L^2(M\setminus\mathcal{C}(\sigma^j))} \leq C\ell_j^{3/2}, \end{aligned}$$

where the fact that collars around disjoint geodesics are disjoint is used in the penultimate step, while estimate (2.17) of Lemma 2.5 is used in the last step.

Having thus established the bound on c_m^j claimed in (4.9) we now turn to the analysis of a_j which is characterised by $a_j = \frac{b_0(\Omega^j, \mathcal{C}(\sigma^j))}{b_0(\tilde\Omega^j, \mathcal{C}(\sigma^j))} \in \mathbb{R}^+$. Using (2.24) we obtain an initial bound of

$$a_j = \frac{b_0(\Omega^j, \mathcal{C}(\sigma^j))}{b_0(\tilde\Omega^j, \mathcal{C}(\sigma^j))} \leq 2b_0(\Omega^j, \mathcal{C}(\sigma^j))\|dz^2\|_{L^2(\mathcal{C}(\sigma^j))} \leq 2\|\Omega^j\|_{L^2(M,g)} = 2.$$

To obtain the more precise bound on a_j claimed in (4.9) we first remark that the elements Ω^m, $m \geq k+1$, are almost orthogonal to $\tilde\Omega^j$. To be more precise, since $b_0(\Omega^m, \mathcal{C}(\sigma^j)) = 0$ while $\tilde\Omega^j$ is essentially given by $b_0(\tilde\Omega^j, \mathcal{C}(\sigma^j))dz^2$, compare (2.17) and (2.18), we may use (2.9) to write

$$\begin{aligned} |\langle\tilde\Omega^j, \Omega^m\rangle_{L^2(M,g)}| &= |\langle\tilde\Omega^j - b_0(\tilde\Omega^j, \mathcal{C}(\sigma^j))dz^2, \Omega^m\rangle_{L^2(\mathcal{C}(\sigma^j))} + \langle\tilde\Omega^j, \Omega^m\rangle_{L^2(M\setminus\mathcal{C}(\sigma^j))}| \\ &\leq C\|\tilde\Omega^j - b_0(\tilde\Omega^j, \mathcal{C}(\sigma^j))dz^2\|_{L^\infty(\mathcal{C}(\sigma^j))} + C\|\tilde\Omega^j\|_{L^\infty(M\setminus\mathcal{C}(\sigma^j))} \\ &\leq C\ell_j^{3/2}. \end{aligned}$$

Comparing the norms of the two sides of (4.8), we may hence conclude that indeed

$$\begin{aligned} |1 - a_j^2| &= \Big|\|\Omega^j\|_{L^2(M,g)}^2 - \|a_j\tilde\Omega^j\|_{L^2(M,g)}^2\Big| \\ &\leq 2|a_j|^2 \sum_m |c_m^j||\langle\tilde\Omega^j, \Omega^m\rangle_{L^2(M,g)}| + |a_j|^2\|\sum_m c_m^j\Omega^m\|_{L^2(M,g)}^2 \leq C\ell_j^3. \end{aligned}$$

By the same argument, we may write Θ^j in the form (2.44), now with leading coefficient identically 1 instead of $1+O(\ell_j^3)$, since (2.16) ensures that $b_0(\tilde{\Theta}^j, \mathcal{C}(\sigma^j)) = -\frac{\ell_j}{\pi^2} = b_0(\Theta^j, \mathcal{C}(\sigma^j))$. Writing as before $\|\Theta^j\|_{L^2(M,g)} = \frac{\ell_j}{\pi^2 b_0(\Omega^j, \mathcal{C}(\sigma^j))}\|\Omega^j\|_{L^2(M,g)}$ we may thus use (2.24) and (A.14) to prove that (2.43) holds true also for $j = 1, \dots, k$ as

$$\|\Theta^j\|_{L^2(M,g)} \leq \frac{\ell_j \cdot \|dz^2\|_{L^2(\mathcal{C}(\sigma^j))}}{a_j \pi^2 b_0(\tilde{\Omega}^j, \mathcal{C}(\sigma^j)) \cdot \|dz^2\|_{L^2(\mathcal{C}(\sigma^j))}} \leq C\ell_j \|dz^2\|_{L^2(\mathcal{C}(\sigma^j))} \leq C\ell_j^{-1/2}.$$

We finally remark that combining this bound on $\|\Theta^j\|_{L^2}$ with the bound (2.47) on w^j that we have already proven also ensures that the elements $v^j = -\|\Theta^j\|_{L^2} w^j$ in the expression (2.44) for Θ^j satisfy (2.45), which was the only claim of the lemma that we had not yet proven. □

For the sake of completeness we finally provide a brief proof of Corollary 2.14 which is an immediate consequence of the results from [22] on Ω^j collected in Lemma 2.5 and the Lemma 2.13 that we just proved.

Proof of Corollary 2.14. To begin with we recall that the upper bound in (2.50) is trivially satisfied as $\|\Omega^j\|_{L^2(M,g)} = 1$, compare (A.13). In case that $\ell_j \geq 2\bar{\eta} = 2\min(\eta, \eta_1)$, the lower bound on the principal part in (2.50) is trivially satisfied while (2.52) follows from Lemma 4.3 and (A.11). In this case also (2.48) and (2.49) are trivially satisfied thanks to (2.13).

On the other hand, for indices with $\ell_j \leq 2\bar{\eta}$, estimates (2.48) and (2.49) are a direct consequence of the corresponding estimates (2.17) and (2.18) for $\tilde{\Omega}^j$ and the relations (2.46) and (2.47) between Ω^j and $\tilde{\Omega}^j$. The lower bound of (2.50) follows from (2.19) as $b_0(\Omega^j, \mathcal{C}(\sigma^j)) = a_j b_0(\tilde{\Omega}^j, \mathcal{C}(\sigma^j))$. We may now choose $\bar{\ell} = \bar{\ell}(\bar{\eta}, \gamma) > 0$ so that (2.50) implies in particular that (2.52) holds true for $\varepsilon_1 = \frac{1}{2}$ and $\ell_j \leq \bar{\ell}$. For the remaining indices j for which we have $\ell_j \in (\bar{\ell}, 2\bar{\eta}]$ we then again combine Lemma 4.3 (now applied for $\ell_0 = \bar{\ell}$) with (A.11) to obtain (2.52).

Finally, to estimate the inner products, we combine (2.9) with the estimates (2.48) and (2.49) to obtain that for $i \neq j$

$$\begin{aligned}|\langle \Omega^i, \Omega^j \rangle_{L^2(M,g)}| \leq\ & \|\Omega^i - b_0(\Omega^i, \mathcal{C}(\sigma^i)) dz^2\|_{L^\infty(\mathcal{C}(\sigma^i))} \|\Omega^j\|_{L^1(\mathcal{C}(\sigma^i))} \\ & + \|\Omega^j - b_0(\Omega^j, \mathcal{C}(\sigma^j)) dz^2\|_{L^\infty(\mathcal{C}(\sigma^j))} \|\Omega^i\|_{L^1(\mathcal{C}(\sigma^j))} \\ & + \|\Omega^i\|_{L^2(M \setminus \mathcal{C}(\sigma^i))} \|\Omega^j\|_{L^2(M \setminus \mathcal{C}(\sigma^j))} \\ \leq & C \ell_i^{3/2} \ell_j^{3/2}.\end{aligned}$$

□

4.3. Elements of $\mathcal{H}$ representing Dehn-twist.

The goal of this section is to derive the properties of the elements Ψ^j of $\mathcal{H}(M,g)$ which we claimed in Lemma 2.19 and Lemma 4.2.

Proof of Lemma 4.2. To prove this result we first note that we can explicitly construct a smooth (non-horizontal) curve of hyperbolic metrics $g(t)$ that moves by Dehn-twists around σ^j as follows: Let $\mathcal{C}(\sigma^j)$ be the collar region around σ^j which we recall is disjoint from the collars around all other geodesics in $\mathcal{E}$. We can then cut the collar off from the surface, twist it by angle t and glue it back into the surface using the same gluing map.

The twisting on the collar can be obtained by pulling back the metric on this collar (not on the whole surface) with a diffeomorphism that is given in collar coordinates by

$$f_t \colon \mathcal{C}(\sigma^j) \to \mathcal{C}(\sigma^j), \ \ f_t(s, \theta) = (s, \theta + t\xi(s))$$

where we choose ξ such that $\xi \equiv \pm\frac{1}{2}$ for s near $\pm X(\ell_j)$. Thus near the ends of the collars we just carry out a rotation by a fixed angle $\pm\frac{1}{2}t$ so we can glue the collar back to the rest of the surface and thus obtain a metric whose twist coordinate ψ^j has increased by t, compare Remark A.3. This results in a smooth curve of complete hyperbolic metrics $(g(t))_t$ with $g(0) = g$ which evolves by

$$\partial_t g(0) = k^j = \begin{cases} 0 & \text{on } M \setminus \mathcal{C}(\sigma^j) \\ \xi'(s)\rho^2(s)(ds \otimes d\theta + d\theta \otimes ds) & \text{on } \mathcal{C}(\sigma^j). \end{cases}$$

We note that we can equivalently view k^j as the real part of the quadratic (but not holomorphic) differential K^j which is supported on $\mathcal{C}(\sigma^j)$ and there given in collar coordinates by

$$K^j = -\mathrm{i}\xi'(s)\rho^2(s)\,(ds + \mathrm{i}d\theta)^2.$$

We then recall that neither the length nor the twist coordinates of a metric can be changed by pulling back the metric with a diffeomorphism that is homotopic to the identity (and of course defined on all of M as opposed to the f_t above). Hence the horizontal part $\Psi^j := P_g^{\mathcal{H}}(K^j)$ of $\partial_t g(0) = \mathrm{Re}\,(K^j) = L_X g + \mathrm{Re}\,(\Psi^j)$ induces the same Dehn-twist as K^j so Ψ^j is the desired element of $\mathcal{H}(M, g)$.

On the one hand, this implies that evolving the metric in the direction of $\mathrm{Re}\,(\Psi^j)$ leaves all length coordinates invariant which, by (2.14), means that (4.4) must hold true.

On the other hand, we note that for every $\Upsilon \in \mathcal{H}(M, g)$

$$\begin{aligned}\langle \Upsilon, \Psi^j\rangle = \langle \Upsilon, K^j\rangle &= \mathrm{i}b_0(\Upsilon, \mathcal{C}(\sigma^j)) \int_{\mathcal{C}(\sigma^j)} \rho^2\xi' |dz^2|_g^2 dv_g = 4\mathrm{i}b_0(\Upsilon, \mathcal{C}(\sigma^j)) \int_{\mathcal{C}(\sigma^j)} \rho^{-4}\rho^2\xi' dv_g \\ &= 8\pi\mathrm{i}b_0(\Upsilon, \mathcal{C}(\sigma^j))\end{aligned}$$

as claimed in (4.2) since the other Fourier modes are orthogonal to dz^2 on every circle $\{s\} \times S^1$. Applied with $\Upsilon = \Psi^j$ this gives the first claim of (4.3), while the orthogonality of Ψ^j to $\ker(\partial\ell_j)$ follows from the above equation and the characterisation (2.15) of $\ker(\partial\ell_j)$. $\square$

Proof of Lemma 2.19. We note that Lemma 4.2 (which we have just proven) already establishes the first claim (2.61) of Lemma 2.19, i.e. that $\Psi^j \perp \ker(\partial\ell_j) = \mathrm{span}\{\Omega^i\}_{i\neq j}$. We may thus write

$$\frac{\Psi^j}{\|\Psi^j\|_{L^2(M,g)}} = -a_j \cdot \mathrm{i}(\Omega^j - P_g^{\ker\partial\ell_j}(\Omega^j)), \tag{4.10}$$

where we note that $a_j \in \mathbb{R}^+$ since $b_0^j(\Psi^j) = b_0(\Psi^j, \mathcal{C}(\sigma^j)) \in \mathrm{i}\mathbb{R}_-$, compare (4.3), while $b_0^j(\Upsilon) = 0$ for $\Upsilon \in \ker(\partial\ell_j)$. Clearly also $|a_j| \geq 1$ as the element on the left hand side has norm $1 \geq \|\Omega^j - P_g^{\ker\partial\ell_j}(\Omega^j)\|_{L^2(M,g)}$.

The trivial upper bound (A.13) on $b_0^j(\frac{\Psi^j}{\|\Psi^j\|L^2})$ yields that

$$|a_j \cdot b_0^j(\Omega^j)| \cdot \|dz^2\|_{L^2(\mathcal{C}(\sigma^j))} = |b_0^j(\tfrac{\Psi^j}{\|\Psi^j\|_{L^2(M,g)}})| \cdot \|dz^2\|_{L^2(\mathcal{C}(\sigma^j))} \leq 1$$

and hence, thanks to the estimates (2.50) and (2.52) for the principal part of Ω^j,

$$1 \leq a_j \leq \frac{1}{\max(\varepsilon_1, 1 - C\ell_j^3)} \leq 1 + C\ell_j^3,$$

resulting in the second claim of (2.63).

We then note that since the principal part of Ψ^j on each of the collars $\mathcal{C}(\sigma^k)$ is purely imaginary and since $\{\Omega^k\}_{k\neq j}$ is a basis of $\ker(\partial\ell_j)$ we may write the second term in (4.10) as

$$a_j \mathrm{i} P_g^{\ker\partial\ell_j}(\Omega^j) = \sum_{k\neq j} c_k^j \mathrm{i}\Omega^k, \tag{4.11}$$

for coefficients $c^j_k \in \mathbb{R}$, which gives the claimed expression (2.62) for $\frac{\Psi^j}{\|\Psi^j\|_{L^2}}$.

In order to prove the estimates on the coefficients c^j_k claimed in (2.63) we first prove that $\|P_g^{\ker \partial\ell_j}(\Omega^j)\|_{L^2} \leq C\ell_j^{3/2}$, which, by Remark 2.15, will give an initial bound of $|c^j_k| \leq C\ell_j^{3/2}$. To this end we note that since $b_0^j(P_g^{\ker\partial\ell_j}(\Omega^j)) = 0$ we may apply (2.9) as well as the estimates (2.48) and (2.49) on Ω^j from Corollary 2.14 to bound

$$\begin{aligned}\|P_g^{\ker\partial\ell_j}(\Omega^j)\|^2_{L^2(M,g)} &= \langle P_g^{\ker\partial\ell_j}(\Omega^j), \Omega^j\rangle_{L^2(M,g)}\\ &= \langle P_g^{\ker\partial\ell_j}(\Omega^j), \Omega^j - b_0^j(\Omega^j)dz^2\rangle_{L^2(\mathcal{C}(\sigma^j))} + \langle P_g^{\ker\partial\ell_j}(\Omega^j), \Omega^j\rangle_{L^2(M\setminus\mathcal{C}(\sigma^j))}\\ &\leq C\ell_j^{3/2}\|P_g^{\ker\partial\ell_j}(\Omega^j)\|_{L^1(M,g)}\end{aligned}$$

and hence to conclude that indeed

$$\|P_g^{\ker\partial\ell_j}(\Omega^j)\|_{L^2(M,g)} \leq C\ell_j^{3/2}. \tag{4.12}$$

To improve the obtained bound of $|c^j_k| \leq C\ell_j^{3/2}$ we now consider the inner product of (4.11) with $\Omega^k \in \ker(\partial\ell_j)$ which, thanks to the estimates on Ω^j from Corollary 2.14, yields

$$\begin{aligned}|c^j_k| \leq& |\langle\Omega^k, a_j \mathrm{i} P_g^{\ker\partial\ell_j}(\Omega^j)\rangle| + \sum_{i\neq j,k}|c^j_i|\cdot|\langle\Omega^k,\Omega^i\rangle| \leq C|\langle\Omega^k,\Omega^j\rangle| + C\sum_{i\neq j,k}\ell_j^{3/2}\ell_k^{3/2}\ell_i^{3/2}\\ \leq& C\ell_j^{3/2}\ell_k^{3/2}\end{aligned}$$

as claimed in (2.63).

We remark that the claims (2.62) and (2.63) of the lemma, which we have just now established, imply in particular that (2.64) holds true since $\|\Omega^k\|_{L^\infty(M,g)} \leq C\ell_k^{-1/2}$, compare (2.57).

It remains to show the bound (2.65) on $\|\Psi^j\|_{L^2}$ which, by (4.3), is given as

$$\|\Psi^j\|_{L^2(M,g)} = 8\pi \mathrm{i} b_0^j\big(\tfrac{\Psi^j}{\|\Psi^j\|_{L^2(M,g)}}\big) = a_j 8\pi b_0^j(\Omega^j).$$

Combining the bound (2.50) on $b_0^j(\Omega^j)$ with the estimate (2.63) on a_j yields

$$\begin{aligned}\Big|\|\Psi^j\|_{L^2(M,g)} - 8\pi\|dz^2\|^{-1}_{L^2(\mathcal{C}(\sigma^j))}\Big| &\leq 8\pi a_j|b_0^j(\Omega^j) - \|dz^2\|^{-1}_{L^2(\mathcal{C}(\sigma^j))}| + 8\pi|1-a_j|\cdot\|dz^2\|^{-1}_{L^2(\mathcal{C}(\sigma^j))}\\ &\leq C\ell_j^3\|dz^2\|^{-1}_{L^2(\mathcal{C}(\sigma^j))} \leq C\ell_j^{9/2}\end{aligned}$$

as claimed, where we used (A.10) respectively (A.11) in the last step. □

4.4. Elements dual to $d\ell$.

In this section we prove the properties of the elements Λ^j claimed in Lemma 2.18. This proof is carried out in two steps: In a first step we will construct a tensor $h^j \in T_g\mathcal{M}_{-1}$ which induces the desired change of the Fenchel-Nielson coordinates and hence projects down onto $\mathrm{Re}\,\Lambda^j$ under the projection $P_g^H\colon T_g\mathcal{M}_{-1} \to H(g) = \mathrm{Re}\,(\mathcal{H}(M,g))$ but is itself not horizontal. In a second step we will then analyse its projection using the estimates on the dual basis Θ^j to the complex differentials $\partial\ell_j$ proven in Lemma 2.13.

Given a simple closed geodesic $\sigma^j \in \mathcal{E}$ we construct such an element $h^j \in T_g\mathcal{M}_{-1}$ as follows. We decompose M into pairs of pants by cutting along the curves in $\mathcal{E}$ and consider the (closures of the) pair(s) of pants P_i for which σ^j is a boundary curve, where either $i = 1, 2$ with σ^j corresponding to one boundary curve of each of the P_i's, or $i = 1$ with σ^j corresponding to two of the boundary curves of P_1.

We decompose these one or two pairs of pants further by cutting along the seams, i.e. the shortest geodesics between the boundary curves, resulting in one respectively two pairs of identical geodesic rectangular hexagons.

On these hexagons we can consider an evolution of the metric as described in the following Lemma 4.7 where one should think of the sides $\Gamma_{a,b,c}$ as the seams of such a pair of pants P_i, while $\gamma_{a,b,c}$ correspond to (half)curves from $\mathcal{E}$ which have constant length unless they correspond to half of σ^j, compare Figure 1.

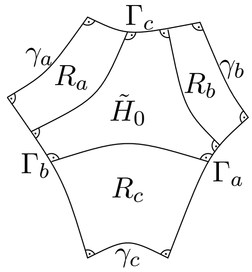

Figure 1. A rectangular hexagon corresponding to half of a pair of pants P_i. Here $\Gamma_\cdot$ are the seams of P_i and $\gamma_\cdot$ are (half)curves from $\mathcal{E}$. The collar parts $R_\cdot$ and the interior hexagon $\tilde{H}_0$ are defined in Lemma 4.7.

Lemma 4.7. *Let $\bar{L} > 0$ be any given number and let $(H_t)_t$ be a family of rectangular, geodesic hexagons in the hyperbolic plane $(\mathbb{H}, g_{\mathbb{H}})$ whose sides $\gamma_{a,t}, \Gamma_{c,t}, \gamma_{b,t}, \Gamma_{a,t}, \gamma_{c,t}, \Gamma_{b,t}$ satisfy either*

$$L_{g_{\mathbb{H}}}(\gamma_{a,t}) = a_t \equiv a, \qquad L_{g_{\mathbb{H}}}(\gamma_{b,t}) = b_t \equiv b, \qquad L_{g_{\mathbb{H}}}(\gamma_{c,t}) = c_t = \frac{1}{2}(\ell + t) \tag{4.13}$$

or

$$L_{g_{\mathbb{H}}}(\gamma_{a,t}) = a_t \equiv a, \qquad L_{g_{\mathbb{H}}}(\gamma_{b,t}) = b_t = L_{g_{\mathbb{H}}}(\gamma_{c,t}) = c_t = \frac{1}{2}(\ell + t) \tag{4.14}$$

for numbers $0 < a_t, b_t, c_t \leq \frac{1}{2}\bar{L}$. We set $c_5 := \frac{w_{\bar{L}}}{4}$ for $w_{\bar{L}}$ the width of the collar around a simple closed geodesic of length $\bar{L}$ characterised by (A.2), and consider the subset $R_a(t)$ of the collar around $\gamma_{a,t}$ given by

$$R_a(t) = \{p \in \mathcal{C}(\gamma_{a,t}) : dist(p, \partial\mathcal{C}(\gamma_{a,t})) \geq c_5 \in (0, \tfrac{w_{2a_t}}{2})\}$$

as well as the analogue subsets $R_b(t)$ and $R_c(t)$ of $\mathcal{C}(\gamma_{b,t})$ and $\mathcal{C}(\gamma_{c,t})$, see Figure 1.

Then there exists a family of diffeomorphisms $F_t \colon (H_0, g_{\mathbb{H}}) \to (H_t, g_{\mathbb{H}})$, which is generated by a smooth vector field X on H_0, so that the following holds true:

(i) F_t is an isometry from $R_a := R_a(0)$ to $R_a(t)$, and in the setting of (4.13) also from $R_b := R_b(0)$ to $R_b(t)$.
(ii) F_t maps $R_c := R_c(0)$ onto $R_c(t)$ and takes the form

$$F_t(s, \theta) = (f_{c,t}(s), \theta) \qquad \text{on } R_c$$

with respect to the corresponding collar coordinates, where $f_{c,t}(\cdot) \colon (-X(2c), X(2c)) \to (-X(2c_t), X(2c_t))$ is an odd function. In the setting of (4.14) the same property holds also for R_b.
(iii) The change $\partial_t g(0) = L_X g$ of the induced metrics $g(t) = F_t^* g_{\mathbb{H}}$ on

$$\tilde{H}_0 := H_0 \setminus (R_a \cup R_b \cup R_c)$$

is bounded by

$$\|\partial_t g(0)\|_{L^\infty(\tilde{H}_0)} \leq C\ell$$

for a constant C that depends only on $\bar{L}$.
(iv) On $\Gamma_{a,b,c}$ we have that the normal derivatives of odd order $\left(\frac{\partial}{\partial n_{\Gamma_\cdot}}\right)^{2j+1} X$, $j = 0, 1, \ldots,$ vanish identically.

Returning to the construction of a tensor $h^j \in T_g\mathcal{M}_{-1}$ that induces the desired change of the Fenchel-Nielson coordinates we note that (4.13) corresponds to having two pairs of pants adjacent to σ^j, while the case that σ^j only has one adjacent pairs of pants for which it corresponds to two boundary curves is treated by considering the case (4.14).

We furthermore remark that (iv) imposes compatibility conditions on the sides $\Gamma_{a,b,c}$ of the hexagon corresponding to seams of the pairs of pants that guarantee that the resulting tensor $\partial_t g(0) = L_X g$ can be extended to a smooth tensor on $P_{1,2}$ respectively to P_1 by symmetry. Since the function $f.$ in part (ii) of Lemma 4.7 is odd we may glue the resulting tensors on $P_{1,2}$ respectively on P_1 along σ^j to obtain a smooth tensor on the closed set $P \subset M$ which corresponds to the (union of the) pairs of pants that are adjacent to σ^j. By (i) $\partial_t g(0)$ is zero near the boundary of P so we may extend the obtained tensor by zero to the rest of M to finally obtain an element h^j of $T_g\mathcal{M}_{-1}$ which, thanks to (4.13) resp. (4.14), induces the desired change of the Fenchel-Nielson coordinates. As a result we therefore obtain

Corollary 4.8. *Let (M,g) be a hyperbolic surface, let $(\ell_i,\psi_i)_{i=1}^{3(\gamma-1)}$ be the Fenchel-Nielson coordinates corresponding to a collection $\mathcal{E}$ of disjoint simple closed geodesics which decomposes M into pairs of pants and for which assumptions (2.36) and (2.37) hold true for some $\eta \in (0, \operatorname{arsinh}(1))$ and some $\bar{L}$. Then for every $j \in \{1,\dots,3(\gamma-1)\}$ there exists a tensor $h^j \in T_g\mathcal{M}_{-1}$ such that*

$$d\ell_i(h^j) = \delta_i^j \text{ and } d\psi_i(h^j) = 0 \text{ for every } i = 1,\dots,3(\gamma-1) \tag{4.15}$$

so that on the subset

$$\mathcal{C}_{c_5}(\sigma^j) := \{p \in \mathcal{C}(\sigma^j) : dist(p, \partial\mathcal{C}(\sigma^j)) \geq c_5\},$$

$c_5 = c_5(\bar{L}) > 0$ as in Lemma 4.7, of the collar $\mathcal{C}(\sigma^j)$ the tensor h^j takes the form

$$h^j = \xi_1(s)(ds^2 - d\theta^2) + \xi_2(s)(ds^2 + d\theta^2) \tag{4.16}$$

with respect to collar coordinates (s,θ) while

$$supp(h^j) \setminus \mathcal{C}_{c_5}(\sigma^j) \subset \tilde{\eta}\text{-}thick(M,g)$$

for a number $\tilde{\eta} = \tilde{\eta}(\bar{L},\eta) > 0$ and so that for a constant $C = C(\eta,\bar{L})$

$$\|h^j\|_{L^\infty(M\setminus\mathcal{C}_{c_5}(\sigma^j))} \leq C\ell_j. \tag{4.17}$$

Before giving the proof of Lemma 4.7 we complete the

Proof of Corollary 4.8. It remains to show that the tensor $h^j \in T_g\mathcal{M}_{-1}$, which we obtained above by gluing together the tensors $\partial_t g = L_X g$ from Lemma 4.7, has the desired properties.

We have already observed that h^j induces the desired change (4.15) of the Fenchel-Nielson coordinates and note that (4.17) is an immediate consequence of part (iii) of Lemma 4.7. Furthermore $\partial_t g = L_X g$ has the desired form (4.16) on $\mathcal{C}_{c_5}(\sigma^j)$ as on this set the vector field X generating F_t has the form $X(s,\theta) = \xi(s)\frac{\partial}{\partial s}$ for some function ξ, while the metric is of course given by $g = \rho^2(s)(ds^2 + d\theta^2)$. We furthermore note that part (i) of the lemma ensures that points in $\operatorname{supp}(h^j) \cap \mathcal{C}(\sigma^i)$, $i \neq j$, have distance no more than c_5 from $\partial\mathcal{C}(\sigma^i)$ which, by (A.9), means that their injectivity radius is bounded from below by a constant depending only on c_5 and hence $\bar{L}$. Since $M \setminus \bigcup_i \mathcal{C}(\sigma^i) \subset \eta$-thick$(M,g)$, compare (2.36), this finally yields the claim on the support of h^j. $\square$

Proof of Lemma 4.7. Given a family H_t of hexagons as described in the lemma we denote by $p_i(t)$ the vertices of the hexagon H_t as shown in Figure 2 and note that since the length of the geodesics $\gamma_{a,t}$ is constant, we may assume without loss of generality that the corresponding sides of H_0 and H_t coincide, i.e. that $p_1(t) = p_1$ and $p_2(t) = p_2$ for every t where we abbreviate

$p_i = p_i(0)$. Together with the prescribed lengths a_t, b_t, c_t of the alternate sides $\gamma_{a,t}, \gamma_{b,t}$ and $\gamma_{c,t}$ this determines the subset H_t in the hyperbolic plane.

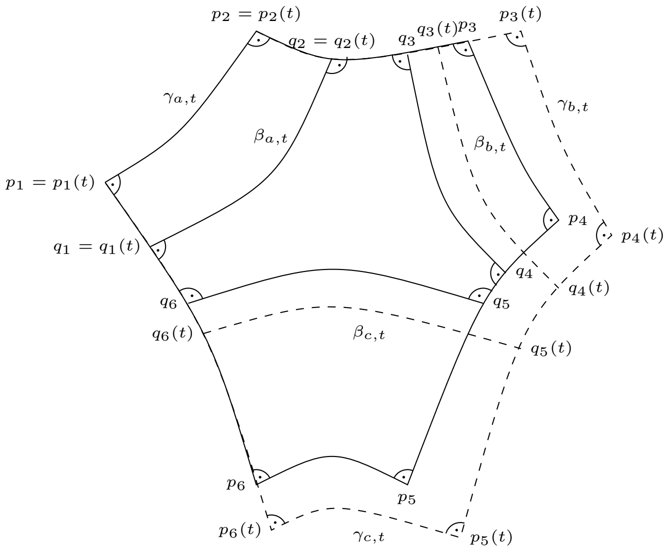

FIGURE 2. Hexagons H_0 and H_t: All lines are geodesics except $\beta_{a,b,c}$ which are curves of constant geodesic curvature.

We set $c_5 = \frac{w_{\bar{L}}}{4}$ and note that $c_5 \leq \frac{w_{2a,2b,2c}}{4} < \frac{w_{2a,2b,2c}}{2} = \operatorname{dist}(\gamma_{a,b,c}, \partial\mathcal{C}(\gamma_{a,b,c}))$ as we have assumed that $a_t, b_t, c_t \leq \frac{\bar{L}}{2}$. We may thus consider the curve $\beta_{a,t}$ of all points in the collar $\mathcal{C}(\gamma_{a,t})$ whose distance to $\partial\mathcal{C}(\gamma_{a,t})$ is c_5, i.e. whose distance to $\gamma_{a,t}$ is equal to $\frac{w_{2a_t}}{2} - c_5$. These curves $\beta_{a,t}$ meet the geodesic boundary curves $\Gamma_{b,t}$ and $\Gamma_{c,t}$ orthogonally in two points which we denote by $q_1(t)$ and $q_2(t)$. The set R_a introduced in the lemma then corresponds to the rectangular quadrangle with vertices p_1, p_2, q_2, q_1 whose boundary curves are geodesics except for β_a along which the geodesic curvature is constant, but non-zero. We introduce the analogue notation also on the collars around $\gamma_{b,t}$ and $\gamma_{c,t}$, compare Figure 2, and denote by $\tilde{H}_t$ the inner rectangular hexagon.

We furthermore remark that the collar region $\mathcal{C}(\gamma_{a,t})$ around $\gamma_{a,t}$ is isometric to $\big([0, X(2a_t)) \times [0,\pi], \rho_{2a_t}(s)^2(ds^2 + d\theta^2)\big)$, X, ρ as always given by (A.1), as it corresponds to a quarter of a collar around a simple closed geodesic of length $2a_t$ in a closed hyperbolic surface. In collar coordinates, the curve $\beta_{a,t}$ corresponds to the semicircle $\{Z_{c_5}(2a_t)\} \times [0,\pi]$, where $Z_c(\ell)$ is characterised by $\int_{Z_c(\ell)}^{X(\ell)} \rho(s)ds = c$ and hence given as $Z_c(\ell) = \frac{\ell}{2\pi}(\frac{\pi}{2} - Y_c(\ell))$ for

$$Y_c(\ell) = 2\arctan\big(e^c \tan(\tfrac{1}{2}Y_0(\ell)\big) \quad \text{where } Y_0(\ell) = \arctan\sinh(\tfrac{\ell}{2}). \tag{4.18}$$

We now construct the desired diffeomorphism $F_t \colon H_0 \to H_t$ as follows:

On the rectangular subsets $R_\cdot$ which contain a boundary curve $\gamma_\cdot$ whose length is fixed, i.e. on R_a and R_b in the setting of (4.13) respectively only on R_a in the setting of (4.14), the diffeomorphism $F_t \colon R_a \to R_a(t)$ is uniquely determined by the condition that it is an isometry.

The other rectangular sets $R_\cdot$ correspond to subsets of collars around geodesics of length $\frac{1}{2}(\ell+t)$. Here we can choose $F_t \colon R_{b,c} \to R_{b,c}(t)$ explicitly e.g. as a linear map $F_t(s,\theta) = (\frac{Z_{c_5}(\ell+t)}{Z_{c_5}(\ell)} \cdot s, \theta)$ in the corresponding collar coordinates, in which $R_\cdot(t)$ is given by $[0, Z_{c_5}(\ell+t)] \times [0,\pi]$.

It hence remains to show that we can define F_t on the inner hexagon $\tilde{H}_0$ in such a way that F_t is smooth, so that the induced change of the metric $\frac{d}{dt}|_{t=0} F_t^* g_{\mathbb{H}} = L_X g_{\mathbb{H}}$ is of order $O(\ell)$

as required in (iii) and so that X satisfies the symmetry conditions from (iv). The main step to prove this is to show that the change of all geometric quantities characterising $\tilde{H}_t$, i.e. the distances $\mathrm{dist}(q_i(t), q_{i+1}(t))$, $i = 1, \ldots, 6$ (where we set $q_7 = q_1$) as well as the geodesic curvatures of those boundary curves $\beta_{a,b,c}$ of $\tilde{H}_t$ which are not geodesics, is of order $O(\ell)$:

This trivially holds true for the curves $\beta_{a,t}$, and in case of (4.13) also for $\beta_{b,t}$, since a_t is constant and hence these curves are isometric to one-another.

For $\beta_{c,t}$ (and so in case of (4.14) by symmetry also for $\beta_{b,t}$) this can be seen as follows: We write $\beta_{c,t}$ in collar coordinates as $\{Z_{c_5}(\ell)\} \times [0, \pi]$, $Z_{c_5}(\ell) = \frac{2\pi}{\ell}(\frac{\pi}{2} - Y_{c_5}(\ell))$ where Y_{c_5} is given by (4.18) and hence satisfies in particular

$$Y_{c_5}(\ell) = e^{c_5} \tfrac{\ell}{2} + O(\ell^3) \text{ and } Y'_{c_5}(\ell) = e^{c_5} \tfrac{1}{2} + O(\ell^2). \tag{4.19}$$

As the geodesic curvature of a curve $\{s\} \times S^1$ in a collar $\mathcal{C}(\sigma)$ around a geodesic of length ℓ is given by

$$\kappa_g = \langle \nabla_{\rho^{-1}\frac{\partial}{\partial\theta}} (\rho^{-1}\tfrac{\partial}{\partial\theta}), \rho^{-1}\tfrac{\partial}{\partial s} \rangle = \rho^{-1}\Gamma^s_{\theta\theta} = -\rho^{-2}(s)\rho'(s) = -\sin(\tfrac{\ell}{2\pi}s),$$

we thus obtain that the curvature $\kappa_{c,t} = -\sin(\frac{\ell+t}{2\pi} Z_{c_5}(\ell+t)) = -\cos(Y_{c_5}(\ell+t))$ of $\beta_{c,t}$ satisfies

$$|\tfrac{d}{dt}\kappa_{c,t}| = \sin(Y_{c_5}(\ell)) \cdot |\tfrac{dY_{c_5}(\ell)}{d\ell}| \leq C\ell,$$

where here and in the following all time derivatives are evaluated in $t = 0$. As the length of $\beta_{c,t}$ is $L(\beta_{c,t}) = \pi\rho(Z_{c_5}(\ell+t)) = \frac{\ell+t}{2\sin(Y_{c_5}(\ell+t))}$ also its change is only of order

$$\left|\tfrac{d}{dt}L(\beta_{c,t})\right| = \left|\frac{\sin(Y_{c_5}(\ell)) - \ell Y'_{c_5}(\ell)\cos(Y_{c_5}(\ell+t))}{2\sin^2(Y_{c_5}(\ell))}\right| \leq C\ell,$$

compare (4.19).

The change of the lengths of the sides of $\tilde{H}_t$ which are subsets of the boundary curves $\Gamma_{a,b,c}$ of the original hexagon, and hence geodesics, can be computed using standard formulas from hyperbolic trigonometry as found e.g. in [6, p.454] as we explain in the following:

We first consider the case (4.13) in which both a_t and b_t are constant. Here we may use

$$\cosh(L(\Gamma_{c,t})) = \frac{\cosh c_t + \cosh a \cosh b}{\sinh a \sinh b}$$

to express the length of the side opposite $\gamma_{c,t}$ as

$$L(\Gamma_{c,t}) = \log\left(\cosh c_t + \cosh a\cosh b + \sqrt{(\cosh c_t + \cosh a \cosh b)^2 - \sinh^2 a \sinh^2 b}\right) \\ - \log(\sinh a \sinh b).$$

We we note that the last term is constant, while the term in the square root is bounded away from zero by $\cosh^2 a \cosh^2 b - \sinh^2 a \sinh^2 b \geq 1$ so

$$|\tfrac{d}{dt}L(\Gamma_{c,t})| \leq C\sinh(c_t) \cdot |\tfrac{d}{dt}c_t| \leq C\ell. \tag{4.20}$$

Here and in the following C denotes a generic constant that depends at most on the upper bound $\frac{1}{2}\bar{L}$ on a, b, c.

As $\mathrm{dist}(q_i(t), p_i(t)) = \frac{w}{2} - c_5$, w the width of the corresponding collar, we have

$$\mathrm{dist}(q_2(t), q_3(t)) = L(\Gamma_{c,t}) + 2c_5 - \tfrac{w_a}{2} - \tfrac{w_b}{2}$$

and hence, still considering only the setting of (4.13) where a, b and thus w_a, w_b are constant,

$$|\tfrac{d}{dt}\mathrm{dist}(q_2(t), q_3(t))| = |\tfrac{d}{dt}L(\Gamma_{c,t})| \leq C\ell.$$

We now observe that the terms in $\mathrm{dist}(q_4(t),q_5(t)) = L(\Gamma_{a,t}) + 2c_5 - \frac{w_a}{2} - \frac{w_{c_t}}{2}$ are given by

$$\begin{aligned} L(\Gamma_{a,t}) &= \operatorname{arsinh}\Big(\frac{\sinh L(\Gamma_{c,t})\sinh a}{\sinh c_t}\Big) \\ &= \log\big[\sinh L(\Gamma_{c,t})\sinh a + \sqrt{\sinh^2 L(\Gamma_{c,t})\sinh^2 a + \sinh^2 c_t}\big] - \log\sinh c_t \end{aligned}$$

respectively, using (A.2), by

$$\tfrac{w_{c_t}}{2} = \operatorname{arsinh}(\sinh^{-1} c_t) = \log\big[1+\sqrt{1+\sinh^2 c_t}\big] - \log\sinh c_t.$$

The terms $\log\sinh c_t$, whose derivative would be of order 1 rather than ℓ, thus cancel and

$$\begin{aligned} \mathrm{dist}(q_4(t),q_5(t)) = &\ -\log\big[1+\sqrt{1+\sinh^2 c_t}\big] + 2c_5 - \frac{w_a}{2} + \log(\sinh L(\Gamma_{c,t})) + \log(\sinh a) \\ &+ \log\Big[1+\sqrt{1+\sinh^2 c_t\sinh^{-2}(a)\sinh^{-2}(L(\Gamma_{c,t}))}\Big], \end{aligned}$$

where we note that $L(\Gamma_{c,t}) \geq \frac{w_a}{2}$ and hence $\sinh(a)\sinh(L(\Gamma_{c,t})) \geq \sinh(a)\cdot\sinh(\frac{w_a}{2}) = 1$.

Since $L(\Gamma_{c,t}) \geq w_{\bar L}$ while $\frac{d}{dt}L(\Gamma_{c,t})$ is controlled by (4.20), we may thus estimate

$$|\tfrac{d}{dt}\mathrm{dist}(q_4(t),q_5(t))| \leq C\cdot\sinh(c_t) + C\cdot|\tfrac{d}{dt}L(\Gamma_{c,t})| \leq C\ell.$$

The same argument also applies for $|\frac{d}{dt}\mathrm{dist}(q_6(t),q_1(t))|$.

This completes the proof that the change of all geometric quantities of $\tilde H_t$ are of order $O(\ell)$ in the case (4.13). In the setting of (4.14) we may argue similarly, now additionally using the symmetry of the hexagon H_t, to obtain the analogue estimates.

Returning to the construction of the diffeomorphism F_t on $\tilde H_0$ we note that the derived bounds imply in particular that the points $q_3(t)$ and $q_6(t)$ in the hyperbolic plane move only by $|\frac{d}{dt}q_3(t)|_{g_{\mathbb H}} + |\frac{d}{dt}q_6(t)|_{g_{\mathbb H}} \leq C\ell$ along the geodesics p_2p_3 respectively p_1p_6 of the original hexagon. We will later choose the vector field X along these sides of $\tilde H_0$ as a suitable interpolation between $X(q_1) = 0$ and $X(q_6) = O(\ell)$, respectively $X(q_2)$ and $X(q_3)$, but before that consider the curves $\beta_{c,b}$, which we parametrise by the corresponding collar coordinate θ, i.e. proportionally to arc length $|\beta_{c,t}'(\theta)| = \pi^{-1}L(\beta_{c,t})$ over the interval $[0,\pi]$. On these curves F_t is already determined by the choice of F_t on the rectangles $R_{b,c}$ which requires that $F_t(\beta_c(\theta)) = \beta_{c,t}(\theta)$ etc. Hence $X\circ\beta_c = \frac{d}{dt}|_{t=0}\beta_{c,t}$, respectively $X\circ\beta_b = \frac{d}{dt}|_{t=0}\beta_{b,t}$.

While we can control the change of the metric $L_X g$ induced by the diffeomorphisms by working directly in collar coordinates, to obtain the necessary extension of X to the interior hexagon $\tilde H_t$ we want to interpolate between the vector fields $X\circ\beta_a \equiv 0$, $X\circ\beta_b$ and $X\circ\beta_c$. So we need C^1-bounds on $X\circ\beta_b$ and $X\circ\beta_c$ which we obtain as follows. As $\beta_{c,t}$ has constant geodesic curvature $\kappa_{c,t}$, it is characterised by its initial data $\beta_{c,t}(0) = q_6(t)$ and $\beta_{c,t}'(0)$, which points in direction of the interior normal of ∂H_t at $q_6(t)$ and has length $\pi^{-1}L(\beta_{c,t})$, and the ODE

$$\nabla_{\beta_{c,t}'}\beta_{c,t}' = \kappa_{c,t}\cdot|\beta_{c,t}'|\cdot(\beta_{c,t}')^\perp.$$

Here we denote by $v^\perp$ the vector obtained by rotating v by $\pi/2$, by $\cdot'$ derivatives with respect to θ and by ∇ covariant derivatives in $(\mathbb H, g_{\mathbb H})$. We can write this equation equivalently as

$$\beta_{c,t}'' = \pi^{-1}L(\beta_{c,t})\kappa_{c,t}\cdot(\beta_{c,t}')^\perp - \Gamma(\beta_{c,t})(\beta_{c,t}',\beta_{c,t}').$$

where $\Gamma(p)(v,w) := \Gamma_{ij}^k(p)v^iw^j\frac{\partial}{\partial x^k}$ is given in terms of the Christoffel-symbols of the hyperbolic plane. Differentiation in time yields that the vector field $Y(\theta) := X(\beta_{c,t}(\theta)) = \frac{d}{dt}|_{t=0}\beta_{c,t}(\theta)$ satisfies a second order ODE of the form

$$Y'' = f_{c,1}(\theta) + f_{c,2}(\theta, Y, Y')$$

where the first term is bounded by $|f_{c,1}| \leq C|\frac{d}{dt}|_{t=0}\kappa_{c,t}| + C|\frac{d}{dt}|_{t=0}L(\beta_{c,t})| \leq C\ell$, while the second term is linear in Y and Y' and has Lipschitz-constant bounded by $C(1+L(\beta_c)^2) \leq C(\bar L)$.

Rewriting this as a system of first order linear ODEs and applying Gronwall's inequality hence allows us to bound

$$|Y(\theta)| + |Y'(\theta)| \leq C\ell + C \cdot (|Y(0)| + |Y'(0)|) \leq C\ell$$

for a constant C that depends only on the upper bound $\bar{L}$ on the sidelengths a, b, c. The same argument applies of course also for $X \circ \beta_b$.

Having thus determined the size of X on the boundary curves $\beta_{a,b,c}$ of $\tilde{H}_0$ we finally remark that the lengths of the other sides of $\tilde{H}_0$ are bounded away from zero by the constant $2c_5 > 0$ that depends only on $\bar{L}$. This allows us to extend X to a smooth vector field on all of H_t, chosen with the symmetries from (iv), for which the C^1-norm on $\tilde{H}_0$ is bounded by a fixed multiple of the C^1-norm on $\beta_a \cup \beta_b \cup \beta_c$, i.e. by $C\ell$ as claimed in (iii). □

We can now finally prove the properties of the holomorphic quadratic differentials Λ^j that are dual to the (real) differentials of the Fenchel-Nielson length coordinates.

Proof of Lemma 2.18. Let $h^j \in T_g\mathcal{M}_{-1}$ be a tensor that induces the desired change (2.38) of the Fenchel-Nielson coordinates as obtained in Corollary 4.8. As its projection $P_g^H(h^j)$ onto the horizontal space $H(g)$ differs from h^j only by a Lie-derivative, it induces the same change (2.38) of the Fenchel-Nielson coordinates. The desired element Λ^j of $\mathcal{H}(M, g)$ is hence characterised uniquely by

$$\operatorname{Re}\Lambda^j = P_g^H(h^j). \tag{4.21}$$

We then write

$$\Lambda^j - \tfrac{1}{2}\Theta^j = \sum_{k=1}^{3(\gamma-1)} \mathrm{i} c_k^j \Omega^k \tag{4.22}$$

in terms of the basis $\{\Omega^j\}$ described in Lemma 2.13. By definition Λ^j and $\frac{1}{2}\Theta^j$ induce the same change of the length coordinates, namely $d\ell_i(\Lambda^j) = \delta_i^j = d\ell_i(\frac{1}{2}\Theta^j)$, so (2.14) implies that $\operatorname{Re}(b_0^i(\Lambda^j - \frac{1}{2}\Theta^j)) = 0$ for every $i = 1, \ldots, 3(\gamma-1)$. Since $b_0^i(\Omega^k) = 0$ if $i \neq k$ while $b_0^k(\Omega^k) > 0$, we thus find that the coefficients c_k^j in the above expression must all be real, as claimed in the lemma.

As a first step towards the proof of the estimate (2.59) for the c_k^j claimed in the lemma, we show that

$$\bar{c} := \max_k |c_k^j| \leq C\ell_j$$

by comparing the L^2-norms of the two sides of (4.22). Remark 2.15 implies that

$$\|\Lambda^j - \tfrac{1}{2}\Theta^j\|_{L^2(M,g)}^2 = \Big\|\sum_k \mathrm{i}c_k^j\Omega^k\Big\|_{L^2(M,g)}^2 \geq \varepsilon_1^2 \cdot \sum_k |c_k^j|^2 \geq \varepsilon_1^2 \cdot \bar{c}^2 \tag{4.23}$$

where $\varepsilon_1 > 0$ is the lower bound on $|b_0^k(\Omega^k)| \cdot \|dz^2\|_{L^2(\mathcal{C}(\sigma^k))}$ obtained in (2.52). Using (4.21) as well as (4.22) we may furthermore write

$$\begin{aligned}
\|\Lambda^j - \frac{1}{2}\Theta^j\|_{L^2(M,g)}^2 &= 2\|\operatorname{Re}(\Lambda^j) - \tfrac{1}{2}\operatorname{Re}(\Theta^j)\|_{L^2(M,g)}^2 \\
&= -2\langle \operatorname{Re}(\Lambda^j - \tfrac{1}{2}\Theta^j), \tfrac{1}{2}\operatorname{Re}(\Theta^j)\rangle + 2\langle \operatorname{Re}(\Lambda^j - \tfrac{1}{2}\Theta^j), P_g^H(h^j)\rangle \\
&= -\sum_k c_k^j \langle \operatorname{Re}(\mathrm{i}\Omega^k), \operatorname{Re}(\Theta^j)\rangle + 2\sum_k c_k^j \langle \operatorname{Re}(\mathrm{i}\Omega^k), h^j\rangle \\
&\leq C \cdot \Big(\sum_k |\langle \operatorname{Re}(\mathrm{i}\Omega^k), \operatorname{Re}(\Theta^j)\rangle| + \sum_k |\langle \operatorname{Re}(\mathrm{i}\Omega^k), h^j\rangle|\Big) \cdot \bar{c}.
\end{aligned} \tag{4.24}$$

Once we bound the above sums, we thus obtain a bound that is linear in $\bar{c}$ which, combined with (4.23), will allow us to bound $\bar{c}$.

To deal with the first sum, we note that the term with $k=j$ vanishes since Θ^j is a *real* multiple of Ω^j and since $\langle \mathrm{Re}\,(\Omega^j), \mathrm{Re}\,(\mathrm{i}\Omega^j)\rangle = 0$, compare (2.7). To estimate the other terms in the first sum we use the estimate (2.51) on $\Omega^j = -\Theta^j \|\Theta^j\|^{-1}_{L^2(M,g)}$ from Corollary 2.14 and the bound (2.55) on $\|\Theta^j\|_{L^2(M,g)}$ from Remark 2.16 to obtain that for every $k \neq j$

$$|\langle \mathrm{Re}\,(\mathrm{i}\Omega^k), \mathrm{Re}\,(\Theta^j)\rangle| = \|\Theta^j\|_{L^2(M,g)} \cdot |\langle \mathrm{Re}\,(\mathrm{i}\Omega^k), \mathrm{Re}\,(\Omega^j)\rangle| \leq C\ell_j^{-1/2}\ell_j^{3/2}\ell_k^{3/2} \leq C\ell_j.$$

To estimate the terms in the second sum in (4.24) we first consider the subset $\mathcal{C}_{c_5}(\sigma^j)$ of the collar $\mathcal{C}(\sigma^j)$ introduced in Corollary 4.8 on which h^j is of the form (4.16). We recall that horizontal tensors are trace-free and observe that for every circle $\{s\} \times S^1$ contained in this set and every k we obtain

$$\begin{aligned}\int_{\{s\}\times S^1} \langle \mathrm{Re}\,(\mathrm{i}\Omega^k), h^j\rangle d\theta &= \sum_n \int_{S^1} \mathrm{Re}\,\left(b_n^j(\mathrm{i}\Omega^k)e^{n(s+\mathrm{i}\theta)}\right) \cdot \xi_1(s) 2\rho^{-4}(s) d\theta \\ &= -2\xi_1(s)\rho^{-4}(s)\mathrm{Im}(b_0^j(\Omega^k))2\pi = 0\end{aligned}$$

as the principal parts of elements Ω^k are real. In particular $\langle \mathrm{Re}\,(\mathrm{i}\Omega^k), h^j\rangle_{L^2(\mathcal{C}_{c_5}(\sigma^j))} = 0$.

As Corollary 4.8 ensures that $\mathrm{supp}(h^j) \setminus \mathcal{C}_{c_5}(\sigma^j) \subset \tilde{\eta}\text{-thick}(M,g)$, we may thus combine the bounds (4.17) for h^j and (2.56) for Ω^k that are valid on this set to estimate

$$\begin{aligned}|\langle \mathrm{Re}\,(\mathrm{i}\Omega^k), h^j\rangle_{L^2(M,g)}| &= |\langle \mathrm{Re}\,(\mathrm{i}\Omega^k), h^j\rangle_{L^2(\mathrm{supp}(h^j)\setminus \mathcal{C}_{c_5}(\sigma^j),g)}| \\ &\leq \|h^j\|_{L^1(M\setminus\mathcal{C}_{c_5}(\sigma^j))} \cdot \|\mathrm{Re}\,\Omega^k\|_{L^\infty(\tilde{\eta}\text{-thick}(M,g))} \\ &\leq C\ell_j\ell_k^{3/2}.\end{aligned} \tag{4.25}$$

Hence also the second sum in (4.24) is bounded by $C\ell_j$ so we obtain that $\|\Lambda^j - \frac{1}{2}\Theta^j\|^2_{L^2} \leq C\ell_j\bar{c}$. Combined with (4.23) this gives an initial bound of $\max_k |c_k^j| = \bar{c} \leq C\ell_j$.

For indices k for which ℓ_k is bounded away from zero by some fixed constant this already yields the bound on c_k^j claimed in the lemma. For any other index, we multiply (4.22) with $\mathrm{i}\Omega^k$ to get

$$\begin{aligned}|c_k^j| \leq& \frac{1}{2}|\mathrm{Re}\,\langle\Theta^j, \mathrm{i}\Omega^k\rangle| + 2|\langle \mathrm{Re}\,(\Lambda^j), \mathrm{Re}\,(\mathrm{i}\Omega^k)\rangle| + \bar{c}\sum_{i\neq k}|\langle\Omega^i, \Omega^k\rangle| \\ \leq& C\ell_j\ell_k^{3/2} + 2|\langle h^j, \mathrm{Re}\,(\mathrm{i}\Omega^k)\rangle| + C\ell_j\sum_{i\neq k}\ell_k^{3/2}\ell_i^{3/2} \leq C\ell_j\ell_k^{3/2},\end{aligned}$$

see (4.25) for the last step. Having thus established (2.59), we finally remark that (2.60) is an immediate consequence of this estimate and the bound (2.57) on $\|\Omega^k\|_{L^\infty(M,g)}$. □

Appendix A. Appendix

A.1. Hyperbolic collars and holomorphic quadratic differentials.

We will need the following 'Collar lemma' throughout the paper.

Lemma A.1 (Keen-Randol [20]). *Let (M,g) be a closed oriented hyperbolic surface and let σ be a simple closed geodesic of length ℓ. Then there is a neighbourhood $\mathcal{C}(\sigma)$ around σ, a so-called collar, which is isometric to the cylinder*

$$(-X(\ell), X(\ell)) \times S^1$$

equipped with the metric $\rho^2(s)(ds^2 + d\theta^2)$ where

$$\rho(s) = \rho_\ell(s) = \frac{\ell}{2\pi\cos(\frac{\ell s}{2\pi})} \qquad \text{and} \qquad X(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan\left(\sinh\left(\frac{\ell}{2}\right)\right)\right). \tag{A.1}$$

The geodesic σ then corresponds to the circle $\{(0,\theta) \mid \theta \in S^1\}$.

In the following we recall without proof several useful properties of metrics on collars that are used throughout the paper and refer to [6] as well as the appendices of [22, 23, 24] and the references therein for more information.

We first recall that the width of a collar, i.e. the distance $w_\ell := \int_{-X(\ell)}^{X(\ell)} \rho(s)ds$ between the two boundary curves, is related to the length ℓ of the central geodesic by

$$\sinh \tfrac{w_\ell}{2} \sinh \tfrac{\ell}{2} = 1. \tag{A.2}$$

Furthermore, for $\delta \in (0, \operatorname{arsinh}(1))$, the δ-thin part of a collar is given by the subcylinder

$$(-X_\delta(\ell), X_\delta(\ell)) \times S^1 \subset \mathcal{C}(\sigma), \text{ where } X_\delta(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arcsin\left(\frac{\sinh(\frac{\ell}{2})}{\sinh \delta}\right)\right) \tag{A.3}$$

for $\delta \geq \ell/2$, respectively $X_\delta(\ell) = 0$ for smaller values of δ. As explained in [22, Proposition A.2] this implies in particular the existence of a universal constant $C \in (0, \infty)$ such that

$$\frac{\pi}{\delta} - C \leq X(\ell) - X_\delta(\ell) \leq \frac{\pi^2}{2\delta} \tag{A.4}$$

holds true for every $\delta \in (0, \operatorname{arsinh}(1))$ and $0 < \ell \leq 2\delta$.

We furthermore recall from [23, Lemma A.5] that the estimate

$$\operatorname{Area}(\delta\text{-thin}(\mathcal{C}(\sigma)) \leq C\delta \tag{A.5}$$

holds true with a universal constant C for every $\delta \in (0, \operatorname{arsinh}(1))$. Since the number of simple closed geodesics with length $\ell \leq 2\operatorname{arsinh}(1)$ is bounded by $3(\gamma - 1)$ we thus find that also $\operatorname{Area}(\delta\text{-thin}(M, g)) \leq C\delta$ albeit now with a constant that depends on the genus.

It is also useful to remark that on collars around geodesics of length $\ell \in (0, 2\operatorname{arsinh}(1)]$ the injectivity radius and the conformal factor are of comparable size, to be more precise

$$\rho(s) \leq \operatorname{inj}_g(s, \theta) \leq \pi\rho(s) \text{ for all } (s, \theta) \in \mathcal{C}(\sigma) \tag{A.6}$$

c.f. [22, (A.7) and (A.9)]. As

$$\frac{d}{ds} \log \rho(s) = \frac{\ell}{2\pi} \tan \frac{\ell s}{2\pi}, \qquad \text{so} \qquad \left|\frac{d}{ds} \log \rho(s)\right| \leq \rho(s) \tag{A.7}$$

is bounded uniformly on collars $\mathcal{C}(\sigma)$ with $\ell \leq 2\operatorname{arsinh}(1)$, the conformal factors, and thus also the injectivity radii, are of comparable size on cylinders of bounded length, namely

$$\rho(s_1) \leq e^{|s_1 - s_2|}\rho(s_2) \text{ while } \operatorname{inj}_g(s_1, \theta) \leq e^{|s_1 - s_2|}\pi \operatorname{inj}_g(s_2, \theta) \text{ for all } (s_{1,2}, \theta) \in \mathcal{C}(\sigma). \tag{A.8}$$

In particular, for points that are a fixed distance from the ends of a collar we have a uniform bound on the injectivity radius

$$\pi\rho(s) \geq \operatorname{inj}_g(s, \theta) \geq c_\Lambda > 0 \text{ for all } |s| \in [X(\ell) - \Lambda, X(\ell)) \tag{A.9}$$

with $c_\Lambda > 0$ depending only on Λ.

In our analysis of holomorphic quadratic differentials we use repeatedly that on a collar $\mathcal{C}(\sigma)$ around a geodesic of length $\ell \in (0, 2\operatorname{arsinh}(1))$ we have

$$\begin{aligned} |dz^2|_g &= 2\rho^{-2}; \qquad & \|dz^2\|_{L^1(\mathcal{C}(\sigma))} &= 8\pi X(\ell); \\ \|dz^2\|_{L^\infty(\mathcal{C}(\sigma))} &= \frac{8\pi^2}{\ell^2}; \qquad & \|dz^2\|^2_{L^2(\mathcal{C}(\sigma))} &= \frac{32\pi^5}{\ell^3} - \frac{16\pi^4}{3} + O(\ell^2), \end{aligned} \tag{A.10}$$

where norms on $\mathcal{C}(\sigma)$ are always computed with respect to $g = \rho^2(ds^2 + d\theta^2)$.

We also remark that for every $\bar{L}$ there exists a constant $c_1 = c_1(\bar{L}) > 0$ so that if $\ell < \bar{L}$ then

$$\|dz^2\|_{L^2(\mathcal{C}(\sigma))} \geq c_1 \tag{A.11}$$

while an upper bound of the form

$$\|dz^2\|_{L^2(\mathcal{C}(\sigma))} \leq C\ell^{-3/2} \tag{A.12}$$

holds true for a universal constant C.

As the principal part is orthogonal to the collar decay part we may combine the above estimates with

$$|b_0(\Upsilon, \mathcal{C}(\sigma))| \cdot \|dz^2\|_{L^2(\mathcal{C}(\sigma))} \leq \|\Upsilon\|_{L^2(\mathcal{C}(\sigma))} \leq \|\Upsilon\|_{L^2(M,g)} \tag{A.13}$$

to obtain a trivial upper bound for the coefficient of the principal part on collars of

$$|b_0(\Upsilon, \mathcal{C}(\sigma))| \leq C\ell^{3/2}\|\Upsilon\|_{L^2(M,g)} \tag{A.14}$$

so in particular

$$|b_0(\Upsilon, \mathcal{C}(\sigma))| \leq C(\bar{L})\|\Upsilon\|_{L^2(\mathcal{C}(\sigma))} \leq C(\bar{L})\|\Upsilon\|_{L^2(M,g)} \tag{A.15}$$

for collars around geodesics of bounded length $L_g(\sigma) \leq \bar{L}$.

A.2. **Properties of hyperbolic surfaces.**

In Section 4.1 we make use of the following differential geometric version of the Deligne-Mumford compactness theorem:

Proposition A.2 (Deligne-Mumford compactness [14, Prop. 5.1])**.** *Let (M, g_i) be a sequence of closed hyperbolic Riemann surfaces of genus $\gamma \geq 2$ which degenerate in the sense that* $\mathrm{inj}(M, g_i) \to 0$*. Then after selection of a subsequence, (M, g_i) converges (in the sense described below) to a hyperbolic punctured Riemann surface (Σ, g_∞) where Σ is obtained as follows: There exists a collection $\mathcal{E} = \{\sigma^j, j = 1, \ldots, k\}$ of k pairwise disjoint, homotopically nontrivial, simple closed curves on M such that $\Sigma = \tilde{M} \setminus \cup_{j=1}^k q^j$ where $\tilde{M}$ is the surface obtained by pinching all curves σ^j to points q^j.*

The convergence above is as follows: For each i there exists a collection $\mathcal{E}_i = \{\sigma_i^j, j = 1, \ldots, k\}$ of k pairwise disjoint simple closed geodesics in (M, g_i) and a continuous map $\tau_i \colon M \to \tilde{M}$ with $\tau_i(\sigma_i^j) = q^j$, $j \in \{1, \ldots, k\}$, such that

- *(i)* $L_{g_i}(\sigma_i^j) \to 0$ *as $i \to \infty$ for each $j \in \{1, \ldots, k\}$*
- *(ii)* $\tau_i \colon M \setminus \cup_{j=1}^k \sigma_i^j \to \Sigma$ *is a diffeomorphism for all i and its inverse is denoted by $f_i \colon \Sigma \to M \setminus \cup_{j=1}^k \sigma_i^j$.*
- *(iii)* $f_i^* g_i \to g_\infty$ *in $C^\infty_{loc}(\Sigma)$ and $f_i^* \mathfrak{c}_i \to \mathfrak{c}_\infty$ in $C^\infty_{loc}(\Sigma)$, where $\mathfrak{c}_i$ and $\mathfrak{c}_\infty$ are the complex structures corresponding to g_i respectively g_∞.*

We furthermore use that a hyperbolic surface of genus γ can be decomposed into pairs of pants by cutting along $3(\gamma - 1)$ pairwise disjoint simple closed geodesics.

The metric on a pair of pants is uniquely determined (up to pull-back by diffeomorphisms) by the lengths of its three boundary curves. Keeping track of how the decomposing geodesics were chosen, the hyperbolic metric g on M is hence determined (up to pull-back by diffeomorphisms) by the $3(\gamma - 1)$ length coordinates $\ell_j = L_g(\sigma^j)$ of the boundary curves of the pairs of pants, or equivalently of the geodesics along which we cut, and the twist coordinates ψ_j which describe how the pairs of pants are glued together along σ^j and which are defined as follows:

Remark A.3. Let (M, g) be a closed oriented hyperbolic surface and let $\mathcal{E} = \{\sigma^j\}_{j=1}^{3(\gamma-1)}$ be a collection of disjoint simple closed geodesics that decomposes (M, g) into pairs of pants.

Suppose first that σ^j is contained in only one of the pairs of pants P_1 and hence corresponds there to two of the boundary curves γ^1 and γ^2 of P_1. In this case we consider the two seams $\Gamma_{1,2}$

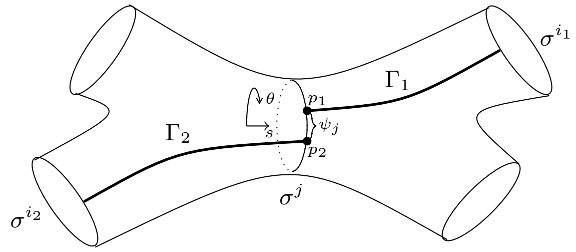

FIGURE 3. Introduction of the twist coordinate ψ_j

which are defined to be the shortest curves from γ^1 respectively from γ^2 to the other boundary curve γ^3 of P_1. In M the seams $\Gamma^{1,2}$ are thus given by geodesics that connect σ^j and γ^3 and meet both curves orthogonally and we denote by $p_{1,2}$ the points in which $\Gamma_{1,2}$ intersects σ^j. The corresponding twist coordinate is then defined to be

$$\psi_j = \theta(p_1) - \theta(p_2) \qquad \text{(modulo } 2\pi)$$

where (s, θ) are collar coordinates around σ^j which we are chosen so that $\Gamma_1 \cap \mathcal{C}(\sigma^j)$ is contained in $\{(s, \theta) : s \geq 0\}$.

In the other case that σ^j is a boundary curve of two pairs of pants P_1 and P_2, compare Figure 3, we select $\sigma^{i_{1,2}}$ to be one of the other boundary curves of $P_{1,2}$, consider the corresponding seams $\Gamma_{1,2}$ and the points $p_{1,2}$ in which they intersect σ^j and define ψ_j accordingly, again with the convention that the collar coordinates are chosen with the orientation so that $\Gamma_1 \cap \mathcal{C}(\sigma^j)$ is contained in $\{s \geq 0\}$.

Combined, the set of length and twist coordinates $\{\ell_j, \psi_j\}_{j=1}^{3(\gamma-1)}$ determine the metric g on M up to pull-back by diffeomorphisms and form the so called Fenchel-Nielson coordinates on moduli space.

We furthermore recall from [6, Theorem 4.1.1] that any set $\{\sigma^1, \ldots, \sigma^k\}$ of simple closed disjoint geodesics in (M, g) can be extended to a decomposing collection $\mathcal{E} = \{\sigma^1, \ldots, \sigma^{3(\gamma-1)}\}$ of simple closed geodesics which can and will always be chosen so that the following holds:

Lemma A.4. *Consequence of* [14, Theorem 3.7] *For any genus $\gamma \geq 2$ and any number $\bar{L}_1$ there exists a number $\bar{L}$ so that the following holds true: Let $\{\sigma^1, \ldots, \sigma^k\}$ be any set of disjoint simple closed geodesics in a hyperbolic surface (M, g) of genus γ whose lengths are $L_g(\sigma^j) \leq \bar{L}_1$, $j = 1, \ldots, k$. Then this set can be extended to a collection $\mathcal{E} = \{\sigma^1, \ldots, \sigma^{3(\gamma-1)}\}$ of disjoint simple closed geodesics that decomposes (M, g) into pairs of pants and that is chosen so that $L_g(\sigma^i) \leq \bar{L}$ for each i.*

We shall furthermore use

Lemma A.5. *Let M be a closed oriented surface of genus $\gamma \geq 2$, let $\mathcal{E}_1 = \{\sigma_1^1, \ldots, \sigma_1^{3(\gamma-1)}\}$ be any collection of disjoint simple closed curves in M that decompose M into pairs of pants and let $k \in \{0, \ldots, 3(\gamma-1)\}$ be any number. Then there exists a finite set of decomposing collections $\mathcal{E}_i = \{\sigma_i^j\}_{j=1}^{3(\gamma-1)}$ of disjoint simple closed curves, $i = 2, \ldots, N$, all of which are so that $\sigma_i^j = \sigma_1^j$ for $j = 1, \ldots k$, such that the following holds true. For any decomposing collection $\tilde{\mathcal{E}} = \{\tilde{\sigma}^1, \ldots, \tilde{\sigma}^{3(\gamma-1)}\}$ of disjoint simple closed curves in M for which $\tilde{\sigma}^j$ and σ_1^j are homotopic for $j = 1, \ldots, k$, there exists an index $i \in \{1, \ldots, N\}$ and a diffeomorphism $f \colon M \to M$ which maps $\tilde{\sigma}^j$ to σ_i^j for $j = 1, \ldots, 3(\gamma-1)$.*

We note that after cutting the surface along the geodesics σ_1^j respectively $\tilde{\sigma}^j$, $j = 1, \ldots, k$, we obtain a bounded number of surfaces Σ_i with in total $2k$ boundary curves. The number of decomposing collections $\mathcal{E}_i$ in the above lemma then corresponds to the number of different ways that the sets Σ_i can be built from pairs of pants (some containing boundary curves of Σ_i) while keeping track of which boundary curves are glued together.

A.3. **Proof of Remark 2.6.** We finally include a brief explanation why we may always assume that the constant $\eta_1 = \eta_1(\gamma) > 0$ in Lemma 2.5 is chosen small enough so that also the estimate (2.24) of Remark 2.6 holds.

Let $\eta_0 = \eta_0(\gamma)$ be a number for which Lemma 2.5 holds true. Given a hyperbolic surface (M, g) we let $\{\sigma^1, \ldots, \sigma^k\}$ be the set of geodesics of length no more than $2\eta_0$, without loss of generality assumed to be ordered by increasing length, and denote by $\bar{\Omega}^1, \ldots, \bar{\Omega}^k$ the basis of $W_{\eta_0}^{\perp}$ obtained in that lemma.

Let now $\bar{\eta} \in (0, \eta_1]$ for a number $\eta_1 \le \eta_0$ that is to be determined and let $k_1 \le k$ be so that the set of geodesics of length $2\bar{\eta}$ or less is $\{\sigma^1, \ldots, \sigma^{k_1}\}$ and let $\tilde{\Omega}^1, \ldots, \tilde{\Omega}^{k_1}$ be the basis of $W_{\bar{\eta}}$ obtained in Lemma 2.5. We note that while the $\tilde{\Omega}^j$ satisfy all of the estimates stated in Lemma 2.5, and thus in particular (2.19), the constants C in these estimates depend on $\bar{\eta}$ so that we cannot directly conclude that (2.19) implies (2.24) for sufficiently small η_1. Instead we apply (2.19) to the corresponding elements $\bar{\Omega}^1, \ldots, \bar{\Omega}^{k_1}$ of the basis of W_{η_0}, as this allows us to conclude that

$$b_0(\bar{\Omega}^j)\|dz^2\|_{L^2(\mathcal{C}(\sigma^j))} \ge 1 - C_{\eta_0} \cdot \eta_1^{3/2} \ge \tfrac{3}{4} \tag{A.16}$$

provided $\eta_1 = \eta_1(\gamma)$ is chosen sufficiently small (as η_0, and hence C_{η_0} is fixed). The elements $\tilde{\Omega}^j$ of $W_{\bar{\eta}}^{\perp}$ are now obtained from Ω^j as

$$\tilde{\Omega}^j = \frac{\bar{\Omega}^j - P_g^{W_\eta}(\bar{\Omega}^j)}{\|\bar{\Omega}^j - P_g^{W_\eta}(\bar{\Omega}^j)\|_{L^2(M,g)}}. \tag{A.17}$$

We note that we can bound $\|P_g^{W_\eta}(\bar{\Omega}^j)\|_{L^2(M,g)} \le C_{\eta_0} \ell_j^{3/2} \le C_{\eta_0} \eta_1^{3/2} \le \frac{1}{2}$ for η_1 small enough, where the first estimate follows by exactly the same argument that we used to obtain (4.12) in the proof of Lemma 2.19, except that here we use the estimates for the basis $\bar{\Omega}^k$ of W_{η_0} from Lemma 2.5 rather then Corollary 2.7. As $b_0^j(P_g^{W_\eta}(\bar{\Omega}^j)) = 0$ we thus obtain (2.24) by combining (A.16) and (A.17).

A.4. **Proof of the Poincaré estimate.**

For the sake of completeness here we provide a short proof of the particular form of the Poincaré inequality that we stated in Lemma 3.4.

Proof of Lemma 3.4. Let (M, g) and M_1^δ be as in the lemma and let $\sigma^1, \ldots, \sigma^{k_1}$ be the simple closed geodesics whose collars contain the boundary curves γ^i of M_1^δ as described in the lemma and the subsequent remark. We set $\delta_0 := \frac{1}{2}\text{arsinh}(1)$ and define $\widehat{M} := M_1^\delta \cap \delta_0\text{-thick}(M, g)$.

To prove the lemma, we will first derive estimates on each of the connected components K_j of the closure of $M_1^\delta \setminus \widehat{M}$, then derive estimates on the connected components $\widehat{M}^j$ of $\widehat{M}$ and finally combine these two cases to obtain the lemma.

To begin with we note that any point in $M_1^\delta \setminus \widehat{M}$ has injectivity radius $\text{inj}_g(p) \in [\delta, \delta_0)$. This set must thus be contained in the (disjoint) union of the collars around geodesics of length $\ell < 2\delta_0$. We note that any simple closed geodesic σ of length $\ell < 2\delta_0$ whose collar region is not disjoint from M_1^δ either agrees with one of the σ^j's $j = 1, \ldots, k_1$ (in case $\ell \le 2\delta$) or else has length $\ell \in (2\delta, 2\delta_0)$ and is fully contained in M_1^δ. We denote these later curves by $\sigma^{k_1+1}, \ldots, \sigma^{k_2}$, $k_2 \ge k_1$ and add them to the collection of the geodesics $\sigma^1, \ldots, \sigma^{k_1}$ already obtained above, where we note that k_2 is bounded in terms of the genus.

We may hence write $\overline{M_1^\delta \setminus \widehat{M}} = \bigcup_{j=1}^{k_2} K_j$, where K_j is given by $\{p \in \mathcal{C}(\sigma^j) \cap M_1^\delta : \mathrm{inj}_g(p) \leq \delta_0\}$ respectively by one of the connected components of this set in case that $\mathcal{C}(\sigma^j)$ contains two boundary curves of M_1^δ and hence $\sigma^i = \sigma^j$ for some $i \neq j$.

In collar coordinates on $\mathcal{C}(\sigma^j)$ (for $j \leq k_1$ oriented so that $\gamma^i = \{X_\delta(\ell_i)\} \times S^1$) these sets correspond to the cylinders $K_j = [X_j^-, X_j^+] \times S^1$ where $X_j^+ = X_{\delta_0}(\ell_j)$ for every j while $X_j^- = X_\delta(\ell_j)$ in case $1 \leq j \leq k_1$ respectively $X_j^- = -X_{\delta_0}(\ell_j)$ if $j > k_1$.

We note that in both cases $0 \leq X_j^+ - X_j^- \leq \frac{C}{\delta}$, C a universal constant, since for $j \leq k_1$ we may use (A.4) to bound $|X_{\delta_0}(\ell_j) - X_\delta(\ell_j)|$ while for $j > k_1$ we have $\ell_j > 2\delta$ and hence $|X_{\delta_0}(\ell_j)| \leq |X(\ell_j)| \leq \frac{C}{\delta}$ by (A.3).

We note that for any $s_{1,2} \in (-X(\ell_j), X(\ell_j))$

$$\begin{aligned} \int_{\{s_1\}\times S^1} v^2 d\theta &= \int_{S^1} \left(v(s_2,\theta) + \int_{s_2}^{s_1} v_s(s,\theta) ds \right)^2 d\theta \\ &\leq 2 \int_{\{s_2\}\times S^1} v^2 d\theta + 2(s_2 - s_1) \int_{S^1} \int_{s_1}^{s_2} |v_s|^2 d\theta. \end{aligned} \tag{A.18}$$

So for every $s_{1,2} \in [X_j^-, X_j^+]$, in particular for $s_1 = X_j^\pm$ and $s_2 = X_j^\mp$,

$$\|v\|^2_{L^2(\{s_1\}\times S^1)} \leq 2\|v\|^2_{L^2(\{s_2\}\times S^1)} + \frac{C}{\delta}\|dv\|^2_{L^2(K_j,g)}, \tag{A.19}$$

where L^2-norms on $\{s\} \times S^1$ are calculated with respect to the standard metric on S^1.

Furthermore, multiplying (A.18) with $\rho^2(s_1)$ (which we recall is bounded from above uniformly on the collar), choosing $s_2 = X_j^\pm$ and integrating over $[X_j^-, X_j^+]$ yields

$$\begin{aligned} \|v\|^2_{L^2(K_j,g)} &\leq \int_{X_j^-}^{X_j^+} \rho^2(s_1) ds_1 \left(2\|v\|^2_{L^2(\{X_j^\pm\}\times S^1)} + \frac{C}{\delta}\|dv\|^2_{L^2(K_j,g)} \right) \\ &\leq C \left(\|v\|^2_{L^2(\{X_j^\pm\}\times S^1)} + \frac{1}{\delta}\|dv\|^2_{L^2(K_j,g)} \right) \end{aligned} \tag{A.20}$$

for a universal constant C since $2\pi \int_{X_j^-}^{X_j^+} \rho^2(s_1) ds_1 \leq \mathrm{Area}(K_j) \leq \mathrm{Area}(\mathcal{C}(\sigma^j))$ is bounded uniformly.

These two estimates (A.19) and (A.20) imply that if the trace of v on one of the boundary curves of K_j satisfies an estimate of the form (3.10), then also the trace of v on the other boundary component of K_j satisfies such an estimate and furthermore $\|v\|^2_{L^2(K_j,g)}$ satisfies the desired bound of $\frac{C}{\delta}\|dv\|^2_{L^2(M_1^\delta,g)}$.

To complete the proof of the lemma we now show that the analogue holds true also for the connected components $\widehat{M}^j$ of $\widehat{M} = \delta_0\text{-thick}(M_i^\delta, g)$. Since the injectivity radius is equal to δ_0 on the boundary curves $\widehat{\gamma}_i^j$ of such a $\widehat{M}^j$, these curves correspond to circles of the form $\{\pm X_{\delta_0}(\ell_i^j)\} \times S^1$ in collar coordinates on which $\delta_0 \leq \rho \leq \pi\delta_0$. We note that in the present situation the trace-theorem takes the form

$$\|v\|^2_{L^2(\widehat{\gamma}_i^j,g)} \leq C\|v\|^2_{L^2(\{\pm X_{\delta_0}(\ell_i^j)\}\times S^1)} \leq C\|v\|^2_{H^1(\widehat{M}^j)},$$

where C is a universal constant. This may be easily checked by integrating (A.18) (with respect to s_2) over the part of the corresponding collar $\{s : \pm s \in [X_{\delta_0}(\ell_j), X(\ell_j)]\}$ that is contained in $\widehat{M}$ and using that the choice of δ_0 guarantees that $X(\ell_j) - X_{\delta_0}(\ell_j) > c > 0$ for a universal constant.

Finally we remark that the diameter of the connected components $\widehat{M}^i$ of $\widehat{M} = \delta_0\text{-thick}(M_i^\delta)$ is bounded in terms of the genus. This allows us to use the version of the Poincaré inequality as found e.g. in [1, Theorem 4.5 and Remark 4.4], to obtain that

$$\|v\|^2_{L^2(\widehat{M}^j,g)} \le C\left(\min_i \|v\|^2_{L^2(\widehat{\gamma}_i^j,g)} + \|dv\|^2_{L^2(\widehat{M}^j,g)}\right) \tag{A.21}$$

holds true for a constant C that depends only on the genus.

The proof of the lemma is now obtained by combining these estimates as follows: Starting with the connected component K_i which contains the boundary component γ^i on which v vanishes we obtain from (A.20) that $\|v\|^2_{L^2(K_i,g)} \le \frac{C}{\delta}\|dv\|_{L^2(M_i^\delta,g)}$ whereas (A.19) (applied for $s_1 = X_i^+$ and $s_2 = X_i^-$) yields the same type of bound for $\|u\|^2_{L^2(\{X_i^+\}\times S^1)}$. We can then apply the Poincaré estimate (A.21) on the connected component $\widehat{M}^{j_i}$ of δ_0-thick(M_i^δ) adjacent to K_i to also obtain an estimate of $\|v\|^2_{L^2(\widehat{M}^{j_i})} \le \frac{C}{\delta}\|dv\|^2_{L^2(M_i^\delta)}$ which in turn allows us to apply the trace estimate (3.10) to get the same type of bounds on the traces of v on all of the boundary curves of $\widehat{M}^{j_i}$. Iterating this argument results in

$$\sum_i \|v\|^2_{L^2(K_i,g)} + \|v\|^2_{L^2(\{X_i^\pm\}\times S^1)} + \|v\|^2_{L^2(\widehat{M}^i,g)} \le \frac{C}{\delta}\|dv\|^2_{L^2(M_i^\delta,g)}$$

which gives the claim of the lemma. □

For the sake of completeness we finally include a proof of the standard fact that the first eigenfunction u is bounded in terms of only the genus.

Proof of Remark 2.4. We let $\delta > 0$ be a universal constant that is chosen small enough so that δ-thin(M,g) is contained in the subsets $\mathscr{C}_{X(\ell_j)-1}$ of the collars $\mathcal{C}(\sigma^j)$ around the simple closed geodesics σ^j of length $\ell_j < 2\operatorname{arsinh}(1)$. As standard arguments imply that the H^2-norm of u over δ-thick(M,g) is bounded in terms of only δ and an upper bound on λ (which is of course bounded uniformly) we obtain a universal upper bound on the oscillation of u over each connected component of δ-thick(M,g). As the angular energies $\vartheta(s)$ on cylinders $\mathscr{C}_{X(\ell_j)-1} \subset \mathcal{C}(\sigma^j)$ considered in Lemma 3.3 are bounded by a universal constant, the oscillation over each such $\mathscr{C}_{X(\ell_j)-1}$ is bounded by

$$\begin{aligned}\operatorname*{osc}_{\mathscr{C}_{X(\ell_j)-1}} u \le& \sup_{|s_{1,2}|\le X(\ell_j)-1} \left|\fint_{\{s_1\}\times S^1} u\,d\theta - \fint_{\{s_2\}\times S^1} u\,d\theta\right| + C \sup_{|s|\le X(\ell_j)-1} \vartheta(s)^{1/2} \\ \le& C\cdot\lambda^{1/2}X(\ell_j)^{1/2} + C \le C + C\lambda^{1/2}\ell_j^{-1/2}.\end{aligned}$$

In case that the shortest simple closed geodesic in (M,g) is disconnecting, we thus obtain the desired uniform bound on u as an immediate consequence of (1.1). □

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