Spectral Properties of Fractional Differentiation Operators

M. V. Kukushkin [1]

[1] *Institute of Applied Mathematics and Automation, Nalchik, Russia, kukushkinmv@rambler.ru*

Abstract

In this paper we consider the fractional differentiation operators in a variety of senses. We show that the strong accretive property is the common property of fractional differentiation operators. Also we prove that the sectorial property holds for operators of the second order with fractional derivative in the junior members. We explore the location of spectrum and resolvent sets of operators and show that the sum of operator with its conjugated operator has a discrete spectrum. We prove that there is two-sided estimate for eigenvalues of operators of the second order with fractional derivative in the junior members.

Keywords: Fractional derivative; embedding theorems; energetic space; energetic inequality; fractional integral; strong accretive operator; positive defined operator.

MSC 35D30; 35D35; 47F05; 47F99.

1 Introduction

The term accretive applicable to a linear operator T acting in the Hilbert space H were introduced by Friedrichs in the work [1], and means that the operator has the following property — a numeric domain of values $\Theta(T)$ is a subset of the right half-plane i.e.

$$\mathrm{Re}\langle Tu, u\rangle_H \geq 0,\ u \in \mathfrak{D}(T).$$

Accepting a notation [2] we assume that Ω — convex domain of n — dimensional Euclidean space, P is a fixed point of the boundary $\partial\Omega$, $Q(r, \vec{\mathbf{e}})$ is an arbitrary point of Ω; we denote by $\vec{\mathbf{e}}$ is a unit vector having the direction from P to Q, using r is the Euclidean distance between points P and Q. We will consider classes of Lebesgue $L_p(\Omega)$, $1 \leq p < \infty$ complex valued functions. In polar coordinates summability f on Ω of degree p, means that

$$\int\limits_{\Omega} |f(Q)|^p dQ = \int\limits_{\omega} d\chi \int\limits_{0}^{d(\vec{\mathbf{e}})} |f(Q)|^p r^{n-1} dr < \infty, \tag{1}$$

where $d\chi$ — is the element of the solid angle the surface of a unit sphere in n-dimensional space and ω — surface of this sphere, $d := d(\vec{\mathbf{e}})$ — is the length of segment of ray going from point P in the direction $\vec{\mathbf{e}}$ within the domain Ω. Without lose of generality, we consider only those directions of $\vec{\mathbf{e}}$ for which the inner integral on the right side of equality (1) exists and is finite, is well known that this is almost all directions. Denote $\vec{\mathrm{e}}_k$, $1 \leq k \leq n$ — ort on n — dimensional Euclidean space, and define the difference attitude $\triangle_k^{-h} v = [v(Q + \vec{\mathrm{e}}_k h) - v(Q)]/h$. Notation $\mathrm{Lip}\,\lambda$, $0 < \lambda \leq 1$ means the set of functions satisfying the Holder-Lipschitz condition

$$\mathrm{Lip}\,\lambda := \left\{\rho(Q):\ |\rho(Q) - \rho(P)| \leq M r^{\lambda},\ P, Q \in \bar{\Omega}\right\}.$$

The operator of fractional differentiation in the sense of Kipriyanov defined in [3] by formal expression

$$\mathfrak{D}^{\alpha}(Q) = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_{0}^{r} \frac{[f(Q) - f(P + \vec{\mathbf{e}}t)]}{(r-t)^{\alpha+1}} \left(\frac{t}{r}\right)^{n-1} dt + C_n^{(\alpha)} f(Q) r^{-\alpha},\ P \in \partial\Omega, \tag{2}$$

where $C_n^{(\alpha)} = (n-1)!/\Gamma(n-\alpha)$, according to theorem 2 [3] acting as follows

$$\mathfrak{D}^{\alpha} :\overset{0}{W_p^l}(\Omega) \to L_q(\Omega),\ lp \le n,\ 0 < \alpha < l - \frac{n}{p} + \frac{n}{q},\ p \le q < \frac{np}{n-lp}. \tag{3}$$

In the case when in the condition (3) we have the strict inequality $q > p$, for sufficiently small $\delta > 0$ the next inequality holds

$$\|\mathfrak{D}^{\alpha} f\|_{L_q(\Omega)} \le \frac{K}{\delta^{\nu}} \|f\|_{L_p(\Omega)} + \delta^{1-\nu} \|f\|_{L_p^l(\Omega)}, \tag{4}$$

where

$$\nu = \frac{n}{l}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{\alpha+\beta}{l}.$$

The constant K independents on $\delta,\ f$ and point $P \in \partial\Omega$ by which construct the operator $\mathfrak{D}^{\alpha};\ \beta$ — an arbitrarily small fixed positive number. We consider $(0 < \alpha < 1)$. Using the terminology of [4], the left-side, right-side fractional derivatives in the sense of Marsho on the segment we will denote respectively via $\mathbf{D}_{a+}^{\alpha},\ \mathbf{D}_{b-}^{\alpha}$; classes of functions representable by the fractional integral on the segment we will denote respectively by $I_{a+}^{\alpha}(L_p(a,b)),\ I_{b-}^{\alpha}(L_p(a,b)),\ 1 \le p \le \infty$. Denote $\operatorname{diam}\Omega = \mathfrak{d};\ C, C_i = \text{const},\ i \in \mathbb{N}_0$. We use for inner product of point $P = (P_1, P_2, ..., P_n)$ and $Q = (Q_1, Q_2, ..., Q_n)$ of n — dimensional Euclidean space a contracted notations $P \cdot Q = P^i Q_i = \sum_{i=1}^{n} P_i Q_i$, denote $|P - Q|$ —- Euclidean distance between P and Q. As usually denote $D_i u$ — the generalized derivative of function u with respect to coordinate variable with index $1 \le i \le n$. Everywhere further, if not stated otherwise we use the notations of [2], [3], [4].

We introduce the classes of functions representable by the fractional integral in the direction of $\vec{\mathbf{e}}$

$$\mathfrak{I}_{0+}^{\alpha}(L_p) := \left\{ u:\ u(Q) = \frac{1}{\Gamma(\alpha)} \int\limits_0^r \frac{g(P + t\vec{\mathbf{e}})}{(r-t)^{1-\alpha}} \left(\frac{t}{r}\right)^{n-1} dt,\ g \in L_p(\Omega),\ 1 \le p \le \infty \right\}, \tag{5}$$

$$\mathrm{I}_{d-}^{\alpha}(L_p) = \left\{ u:\ u(Q) = \frac{1}{\Gamma(\alpha)} \int\limits_r^{d(\vec{\mathbf{e}})} \frac{g(P + t\vec{\mathbf{e}})}{(t-r)^{1-\alpha}} dt,\ g \in L_p(\Omega),\ 1 \le p \le \infty \right\}. \tag{6}$$

Integral operators in (5),(6) we will call the left-sided, right-sided operator of fractional integration in the direction. We define the family of operators $\psi_{\varepsilon}^{+},\ \varepsilon > 0$ as follows

$$\mathfrak{D}(\psi_{\varepsilon}^{-}) \subset L_p(\Omega),\ (\psi_{\varepsilon}^{-} f)(Q) = \begin{cases} \displaystyle\int\limits_{r+\varepsilon}^{d(\vec{\mathbf{e}})} \frac{f(P + \vec{\mathbf{e}}r) - f(P + \vec{\mathbf{e}}t)}{(t-r)^{\alpha+1}} dt,\ 0 \le r \le d - \varepsilon, \\ \displaystyle\frac{f(Q)}{\alpha}\left(\frac{1}{\varepsilon^{\alpha}} - \frac{1}{(d-r)^{\alpha}}\right),\quad d - \varepsilon < r \le d. \end{cases} \tag{7}$$

Following [4, c.181] we define a truncated fractional derivative in the sense of Marchaud

$$(\mathbf{D}_{d-,\varepsilon}^{\alpha} f)(Q) = \frac{1}{\Gamma(1-\alpha)} f(Q)(d-r)^{-\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} (\psi_{\varepsilon}^{-} f)(Q). \tag{8}$$

Right-hand Marchaud derivative will be understood as a limit by norm of space $L_p(\Omega),\ 1 \le p < \infty$ of the truncated fractional derivative

$$\mathbf{D}_{d-}^{\alpha} f = \lim_{\substack{\varepsilon \to 0 \\ (L_p)}} \mathbf{D}_{d-,\varepsilon}^{\alpha} f.$$

Similarly to the one-dimensional case (13.1) [4, p.181] we can define the operator ψ_{ε}^{+} and left-hand Marchaud derivative $\mathbf{D}_{0+}^{\alpha}$. Define the next operator as follows

$$(\mathfrak{D}_{0+}^{\alpha} f)(Q) = r^{1-n}\left(\mathbf{D}_{0+}^{\alpha} \varrho f\right)(Q),\ \varrho(P + \vec{\mathbf{e}}t) = t^{n-1},\ f \in \mathfrak{I}_{0+}^{\alpha}(L_p).$$

We need several auxiliary propositions, which we will present in the next section.

2 Analogs of some known lemmas and theorems

We have a following Lemma on boundedness of fractional integration operators in the direction.

Theorem 1. *Operators of fractional integration are bounded in $L_p(\Omega),\ 1\le p<\infty$.*

Proof. We prove that under the assumptions of this lemma we have the estimates

$$\|\mathfrak{I}^{\alpha}_{0+}u\|_{L_p(\Omega)}\le C\|u\|_{L_p(\Omega)},\ \|\mathrm{I}^{\alpha}_{d-}u\|_{L_p(\Omega)}\le C\|u\|_{L_p(\Omega)},\ C=\mathfrak{d}^{\alpha}/\Gamma(\alpha+1). \tag{9}$$

Let us prove the first estimate (9), the proof of the second estimate is absolutely analogous. Using the generalized Minkowski inequality, we have

$$\|\mathfrak{I}^{\alpha}_{0+}u\|_{L_p(\Omega)}=\frac{1}{\Gamma(\alpha)}\left(\int\limits_{\Omega}\left|\int\limits_{0}^{r}\frac{g(P+t\vec{\mathbf{e}})}{(r-t)^{1-\alpha}}\left(\frac{t}{r}\right)^{n-1}dt\right|^{p}dQ\right)^{1/p}=$$

$$=\frac{1}{\Gamma(\alpha)}\left(\int\limits_{\Omega}\left|\int\limits_{0}^{r}\frac{g(P+(r-\tau)\vec{\mathbf{e}})}{\tau^{1-\alpha}}\left(\frac{r-\tau}{r}\right)^{n-1}d\tau\right|^{p}dQ\right)^{1/p}\le\frac{1}{\Gamma(\alpha)}\left(\int\limits_{\Omega}\left(\int\limits_{0}^{\mathfrak{d}}\frac{|g(P+(r-\tau)\vec{\mathbf{e}})|}{\tau^{1-\alpha}}d\tau\right)^{p}dQ\right)^{1/p}\le$$

$$\le\frac{1}{\Gamma(\alpha)}\int\limits_{0}^{\mathfrak{d}}\tau^{\alpha-1}d\tau\left(\int\limits_{\Omega}|g(P+(r-\tau)\vec{\mathbf{e}})|^{p}dQ\right)^{1/p}\le\frac{\mathfrak{d}^{\alpha}}{\Gamma(\alpha+1)}\|u\|_{L_p(\Omega)}.$$

□

Theorem 2. *Assume $f\in L_p(\Omega)$ and exists a limit $\lim\limits_{\varepsilon\to0}\psi^{+}_{\varepsilon}f$ in the sense of norm $L_p(\Omega),\ 1\le p<\infty$. Then $f\in\mathrm{I}^{\alpha}_{d-}(L_p)$.*

Proof. The proof is based on the proof of theorem 13.2 [4, p.183] we must only repeat the sufficient part of proof theorem 13.2 applying multidimensional case. □

Theorem 3. *Let $f=\mathrm{I}^{\alpha}_{d-}\psi,\ 1\le p<\infty$. Then $\mathbf{D}^{\alpha}_{d-}f=\lim\limits_{\substack{\varepsilon\to0\\(L_p)}}\mathbf{D}^{\alpha}_{d-,\varepsilon}f=\psi$.*

Proof. The proof is based on the proof of theorem 13.1 [4, p.182] we must only repeat the proof theorem 13.1 applying multidimensional case. □

Theorem 4. *Let $\rho\in\mathrm{Lip}\,\lambda,\ \alpha<\lambda\le1,\ f\in H^1_0(\Omega)$, then $\rho f\in\mathrm{I}^{\alpha}_{d-}(L_2)$.*

Proof. At first assume $f\in C_0^{\infty}(\Omega)$. Consider

$$\|\psi^{-}_{\varepsilon_1}f-\psi^{-}_{\varepsilon_2}f\|_{L_2(\Omega)}\le\left(\int\limits_{\omega}d\chi\int\limits_{0}^{d-\varepsilon_1}\left|\int\limits_{r+\varepsilon_1}^{r+\varepsilon_2}\frac{(\rho f)(Q)-(\rho f)(P+\vec{\mathbf{e}}t)}{(t-r)^{\alpha+1}}dt\right|^{2}r^{n-1}dr\right)^{\frac12}+$$

$$+\left(\int\limits_{\omega}d\chi\int\limits_{d-\varepsilon_1}^{d-\varepsilon_2}\left|\int\limits_{r+\varepsilon_1}^{d}\frac{(\rho f)(Q)}{(t-r)^{\alpha+1}}dt-\int\limits_{r+\varepsilon_2}^{d}\frac{(\rho f)(Q)-(\rho f)(P+\vec{\mathbf{e}}t)}{(t-r)^{\alpha+1}}dt\right|^{2}r^{n-1}dr\right)^{\frac12}+$$

$$+\left(\int\limits_{\omega}d\chi\int\limits_{d-\varepsilon_2}^{d}\left|\int\limits_{r+\varepsilon_1}^{d}\frac{(\rho f)(Q)}{(t-r)^{\alpha+1}}dt-\int\limits_{r+\varepsilon_2}^{d}\frac{(\rho f)(Q)}{(t-r)^{\alpha+1}}dt\right|^{2}r^{n-1}dr\right)^{\frac12}=$$

$$=I_1+I_2+I_3.$$

Note, since $f \in C_0^\infty(\Omega)$ then for sufficient small $\varepsilon_1 : \ f(Q) = 0, \, r > d - \varepsilon_1$. This implies that $I_2 + I_3 = 0$. Making the change of variable in I_1, we have

$$I_1 = \left(\int\limits_\omega d\chi \int\limits_0^d \left| \int\limits_{\varepsilon_1}^{\varepsilon_2} \frac{(\rho f)(Q) - (\rho f)(P + \vec{\mathbf{e}}[t + r])}{t^{\alpha+1}} dt \right|^2 r^{n-1} dr \right)^{\frac{1}{2}} .$$

Using the generalized Minkowski inequality we get

$$I_1 \leq \int\limits_{\varepsilon_2}^{\varepsilon_1} t^{-\alpha-1} \left(\int\limits_\omega d\chi \int\limits_0^d \left| (\rho f)(Q) - (\rho f)(Q + \vec{\mathbf{e}}t) \right|^2 r^{n-1} dr \right)^{\frac{1}{2}} dt \leq$$

$$\leq C \int\limits_{\varepsilon_2}^{\varepsilon_1} t^{\lambda-\alpha-1} dt \leq \frac{C}{\lambda - \alpha} \left(\varepsilon_1^{\lambda-\alpha} - \varepsilon_2^{\lambda-\alpha} \right), \; C > 0.$$

In consequence of theorem 1 we have completing the proof inclusion $\rho f \in \mathrm{I}_{d-}^\alpha(L_2)$.

Let $f \in H_0^1(\Omega)$, then there exists a sequence $\{f_n\} \subset C_0^\infty(\Omega), \; f_n \xrightarrow{H_0^1} f, \; \rho f_n \xrightarrow{L_2} \rho f$. According proven above $\rho f_n = \mathrm{I}_{d-}^\alpha \varphi_n, \; \{\varphi_n\} \in L_2(\Omega)$, therefore

$$\mathrm{I}_{d-}^\alpha \varphi_n \xrightarrow{L_2} \rho f. \tag{10}$$

We will show that exist $\varphi \in L_2(\Omega), \; \varphi_n \xrightarrow{L_2} \varphi$. Note that in consequence of theorem 2 we have $\mathbf{D}_{b-}^\alpha \rho f_n = \varphi_n$. Thus introduce the notation $f_{n+m} - f_n = c_{n,m}$, we get

$$\|\varphi_{n+m} - \varphi_n\|_{L_2(\Omega)} \leq \frac{\alpha}{\Gamma(1-\alpha)} \left(\int\limits_\omega d\chi \int\limits_0^d \left| \int\limits_r^d \frac{(\rho c_{n,m})(Q) - (\rho c_{n,m})(P + \vec{\mathbf{e}}t)}{(t - r)^{\alpha+1}} dt \right|^2 r^{n-1} dr \right)^{\frac{1}{2}} +$$

$$+ \frac{1}{\Gamma(1-\alpha)} \left(\int\limits_\omega d\chi \int\limits_0^d \left| \frac{(\rho c_{n,m})(Q)}{(d - r)^\alpha} \right|^2 r^{n-1} dr \right)^{\frac{1}{2}} = I_1 + I_2.$$

Let's evaluate I_1

$$\frac{\Gamma(1-\alpha)}{\alpha} I_1 = \left(\int\limits_\omega d\chi \int\limits_0^d \left| \int\limits_0^{d-r} \frac{(\rho c_{n,m})(Q) - (\rho c_{n,m})(Q + \vec{\mathbf{e}}t)}{t^{\alpha+1}} dt \right|^2 r^{n-1} dr \right)^{\frac{1}{2}} \leq$$

$$\leq C \left\{ \int\limits_\omega d\chi \int\limits_0^d \left(\int\limits_0^{d-r} \frac{|c_{n,m}(Q) - c_{n,m}(Q + \vec{\mathbf{e}}t)|}{t^{\alpha+1}} dt \right)^2 r^{n-1} dr \right\}^{\frac{1}{2}} +$$

$$+ M \left\{ \int\limits_\omega d\chi \int\limits_0^d \left(\int\limits_0^{d-r} \frac{|c_{n,m}(Q + \vec{\mathbf{e}}t)|}{t^{1+\alpha-\lambda}} dt \right)^2 r^{n-1} dr \right\}^{\frac{1}{2}} = I_{11} + I_{12}, \; C = \sup_{Q \in \Omega} |\rho(Q)|.$$

Consider I_{11}. Using the generalized Minkowski inequality, then represent the function under the integral in terms of the derivative in the direction of $\vec{\mathbf{e}}$, we get

$$I_{11} C^{-1} \leq \int\limits_0^{\mathfrak{d}} t^{-\alpha-1} \left(\int\limits_\omega d\chi \int\limits_0^d \left| c_{n,m}(Q) - c_{n,m}(Q + \vec{\mathbf{e}}t) \right|^2 r^{n-1} dr \right)^{\frac{1}{2}} dt =$$

$$= \int\limits_0^{\mathfrak{d}} t^{-\alpha-1} \left(\int\limits_{\omega} d\chi \int\limits_0^d \left| \int\limits_0^t c'_{n,m}(Q+\vec{\mathbf{e}}\tau)d\tau \right|^2 r^{n-1}dr \right)^{\frac{1}{2}} dt.$$

Using the Cauchy-Schwarz inequality, the Fubini's theorem, we have

$$I_{11}C^{-1} \leq \int\limits_0^{\mathfrak{d}} t^{-\alpha-1} \left(\int\limits_{\omega} d\chi \int\limits_0^d r^{n-1}dr \int\limits_0^t \left|c'_{n,m}(Q+\vec{\mathbf{e}}\tau)\right|^2 d\tau \int\limits_0^t d\tau \right)^{\frac{1}{2}} dt =$$

$$= \int\limits_0^{\mathfrak{d}} t^{-\alpha-1/2} \left(\int\limits_0^t d\tau \int\limits_{\Omega} \left|c'_{n,m}(Q+\vec{\mathbf{e}}\tau)\right|^2 dQ \right)^{\frac{1}{2}} dt \leq \frac{\mathfrak{d}^{1-\alpha}}{1-\alpha} \|c'_{n,m}\|_{L_2(\Omega)}.$$

Consider I_{12}. Analogously to the previous reasoning we have a following estimate

$$I_{12}M^{-1} \leq \int\limits_0^{\mathfrak{d}} t^{\lambda-\alpha-1} \left(\int\limits_{\omega} d\chi \int\limits_0^d |c_{n,m}(Q+\vec{\mathbf{e}}t)|^2 r^{n-1}dr \right)^{\frac{1}{2}} dt \leq \frac{\mathfrak{d}^{\lambda-\alpha}}{\lambda-\alpha} \|c'_{n,m}\|_{L_2(\Omega)}.$$

Consider I_2, represent the function under the integral in terms of the derivative in the direction of $\vec{\mathbf{e}}$

$$\Gamma(1-\alpha)C^{-1}I_2 \leq \left(\int\limits_{\omega} d\chi \int\limits_0^d |c_{n,m}(Q)|^2 (d-r)^{-2\alpha} r^{n-1}dr \right)^{\frac{1}{2}} =$$

$$= \left(\int\limits_{\omega} d\chi \int\limits_0^d (d-r)^{-2\alpha} \left| \int\limits_r^d c'_{n,m}(P+\vec{\mathbf{e}}t)dt \right|^2 r^{n-1}dr \right)^{\frac{1}{2}}.$$

Using the generalized Minkowski inequality, then applying the obvious estimate, we have

$$\Gamma(1-\alpha)C^{-1}I_2 \leq \left\{ \int\limits_{\omega} \left[\int\limits_0^d c'_{n,m}(P+\vec{\mathbf{e}}t) \left(\int\limits_0^t (d-r)^{-2\alpha} r^{n-1}dr \right)^{\frac{1}{2}} dt \right]^2 d\chi \right\}^{\frac{1}{2}} \leq$$

$$\leq \left\{ \int\limits_{\omega} \left[\int\limits_0^d c'_{n,m}(P+\vec{\mathbf{e}}t) t^{(n-1)/2} \left(\int\limits_0^t (d-r)^{-2\alpha}dr \right)^{\frac{1}{2}} dt \right]^2 d\chi \right\}^{\frac{1}{2}}.$$

Using the Cauchy-Schwarz inequality, we get

$$\Gamma(1-\alpha)C^{-1}I_2 \leq \left\{ \int\limits_{\omega} \left[\int\limits_0^d |c'_{n,m}(P+\vec{\mathbf{e}}t)|^2 t^{n-1}dt \int\limits_0^d d\tau \int\limits_0^{\tau} (d-r)^{-2\alpha}dr \right] d\chi \right\}^{\frac{1}{2}} =$$

$$= \left\{ \int\limits_{\omega} \left[\int\limits_0^d |c'_{n,m}(P+\vec{\mathbf{e}}t)|^2 t^{n-1}dt \int\limits_0^d (d-r)^{1-2\alpha}dr \right] d\chi \right\}^{\frac{1}{2}} \leq \frac{\mathfrak{d}^{2(1-\alpha)}}{\sqrt{2(1-\alpha)}} \|c'_{n,m}\|_{L_2(\Omega)}.$$

From fundamental property of the sequence $\{c'_{n,m}\}$ in the sense of norm $L_2(\Omega)$, follows that $I_1, I_2 \to 0$. Hence sequence $\{\varphi_n\}$ is fundamental and in consequence of completeness property of the space $L_2(\Omega)$ exists a limit of sequence $\{\varphi_n\}$, a some function $\varphi \in L_2(\Omega)$. Since by lemma **??**, the operator of fractional differentiation is boundary acting in the space $L_2(\Omega)$, then

$$\mathrm{I}^{\alpha}_{d-}\varphi_n \xrightarrow{L_2} \mathrm{I}^{\alpha}_{d-}\varphi.$$

This implies with respect to (10), that $\rho f = \mathrm{I}^{\alpha}_{d-}\varphi$. $\square$

3 Strong accretiveness property

The following theorem establishes the strong accretive property (see [5, p. 352]) for the operator of fractional differentiation in the sense of Kipriyanov acting in the complex weight space of Lebesgue summable with squared functions.

Theorem 5. *Let $n \geq 2$, $\rho(Q)$ is non-negative real function in class* $\mathrm{Lip}\lambda$, $\lambda > \alpha$. *Then for the operator of fractional differentiation in the sense of Kipriyanov the inequality of a strong accretiveness holds*

$$\mathrm{Re}\langle f, \mathfrak{D}^{\alpha} f\rangle_{L_2(\Omega,\rho)} \geq \frac{1}{\lambda^2}\|f\|^2_{L_2(\Omega,\rho)},\ f \in H_0^1(\Omega). \tag{11}$$

Proof. First we assume that f is real. For $f \in C_0^{\infty}(\Omega)$ consider the following difference in which the second summand exists due to theorem 3 [2]

$$\rho(Q)f(Q)(\mathfrak{D}^{\alpha}f)(Q) - \frac{1}{2}\left(\mathfrak{D}^{\alpha}(\rho f^2)\right)(Q) =$$

$$= \frac{\alpha}{2\Gamma(1-\alpha)}\int\limits_0^r \frac{\rho(Q)[f(P+\vec{\mathbf{e}}r) - f(P+\vec{\mathbf{e}}t)]^2}{(r-t)^{\alpha+1}}\left(\frac{t}{r}\right)^{n-1}dt + \frac{C_n^{(\alpha)}}{2}\rho(Q)|f(Q)|^2 r^{-\alpha}dr \geq 0.$$

Therefore

$$\rho(Q)f(Q)(\mathfrak{D}^{\alpha}f)(Q) \geq \frac{1}{2}\left(\mathfrak{D}^{\alpha}(\rho f^2)\right)(Q). \tag{12}$$

Integrating the left and right sides of inequality (12), then using a Fubini theorem we get the next inequality

$$\int\limits_0^{d(\vec{\mathbf{e}})} f(Q)(\mathfrak{D}^{\alpha}f)(Q)\rho(Q)r^{n-1}dr \geq \frac{1}{2}\int\limits_0^{d(\vec{\mathbf{e}})}(\mathfrak{D}^{\alpha}(\rho f^2))(Q)r^{n-1}dr =$$

$$= \frac{\alpha}{2\Gamma(1-\alpha)}\int\limits_0^{d(\vec{\mathbf{e}})} t^{n-1}dt\int\limits_t^{d(\vec{\mathbf{e}})}\frac{(\rho f^2)(Q) - (\rho f^2)(P+\vec{\mathbf{e}}t)}{(r-t)^{\alpha+1}}dr + \frac{C_n^{(\alpha)}}{2}\int\limits_0^{d(\vec{\mathbf{e}})}|f(Q)|^2\rho(Q)r^{n-1-\alpha}dr =$$

$$= -\frac{1}{2}\int\limits_0^{d(\vec{\mathbf{e}})}(\mathbf{D}^{\alpha}_{d(\vec{\mathbf{e}})-}\rho f^2)(Q)r^{n-1}dr + \frac{C_n^{(\alpha)}}{2}\int\limits_0^{d(\vec{\mathbf{e}})}|f(Q)|^2\rho(Q)r^{n-1-\alpha}dr +$$

$$+\frac{1}{2\Gamma(1-\alpha)}\int\limits_0^{d(\vec{\mathbf{e}})}|f(Q)|^2\rho(Q)r^{n-1}(d(\vec{\mathbf{e}})-r)^{-\alpha}dr = I.$$

Rewrite the first summand of the last sum using the formula of fractional integration of exponential function (2.44) [4, c.47], we get

$$\int\limits_0^d(\mathbf{D}^{\alpha}_{d-}\rho f^2)(Q)r^{n-1}dr = \frac{(n-1)!}{\Gamma(n-\alpha)\Gamma(\alpha)}\int\limits_0^d(\mathbf{D}^{\alpha}_{d-}\rho f^2)(Q)dr\int\limits_0^r\frac{t^{n-1-\alpha}}{(r-t)^{1-\alpha}}dt.$$

Note that in consequence of lemma **??**, theorem 3 we have $\rho f^2 = \mathrm{I}^{\alpha}_{d-}\left(\mathbf{D}^{\alpha}_{d-}\rho f^2\right)$. Using the Fubini theorem, we get

$$\frac{(n-1)!}{\Gamma(n-\alpha)\Gamma(\alpha)}\int\limits_0^d(\mathbf{D}^{\alpha}_{d-}\rho f^2)(Q)dr\int\limits_0^r\frac{t^{n-1-\alpha}}{(r-t)^{1-\alpha}}dt = \frac{(n-1)!}{\Gamma(n-\alpha)}\int\limits_0^d\left[\mathrm{I}^{\alpha}_{d-}\left(\mathbf{D}^{\alpha}_{d-}\rho f^2\right)\right](P+\vec{\mathbf{e}}t)t^{n-1-\alpha}dt =$$

$$= C_n^{(\alpha)} \int\limits_0^d \left(\rho f^2\right)(Q) r^{n-1-\alpha} dr.$$

Therefore, we have

$$I = \frac{1}{2\Gamma(1-\alpha)} \int\limits_0^{d(\vec{\mathbf{e}})} |f(Q)|^2 \rho(Q) r^{n-1} (d(\vec{\mathbf{e}}) - r)^{-\alpha} dr \geq \frac{\mathfrak{d}^{-\alpha}}{2\Gamma(1-\alpha)} \int\limits_0^{d(\vec{\mathbf{e}})} |f(Q)|^2 \rho(Q) r^{n-1} dr.$$

Finally for any direction $\vec{\mathbf{e}}$, we get the inequality

$$\int\limits_0^{d(\vec{\mathbf{e}})} f(Q)(\mathfrak{D}^\alpha f)(Q)\rho(Q) r^{n-1} dr \geq \frac{\mathfrak{d}^{-\alpha}}{2\Gamma(1-\alpha)} \int\limits_0^{d(\vec{\mathbf{e}})} |f(Q)|^2 \rho(Q) r^{n-1} dr.$$

Integrating the left and right sides of the last inequality we get

$$\langle f, \mathfrak{D}^\alpha f \rangle_{L_2(\Omega,\rho)} \geq \frac{1}{\lambda^2} \|f\|^2_{L_2(\Omega,\rho)},\ f \in C_0^\infty(\Omega),\ \lambda^2 = 2\Gamma(1-\alpha)\mathfrak{d}^\alpha. \tag{13}$$

Suppose that $f \in H_0^1(\Omega)$. There is a sequence $\{f_k\} \in C_0^\infty(\Omega)$ such that $f_k \xrightarrow{W_2^1} f$. The conditions imposed on the weight function ρ implies the equivalence of norms $L_2(\Omega)$ and $L_2(\Omega,\rho)$, hence $f_k \xrightarrow{L_2(\Omega,\rho)} f$. Using the smoothness of weight function ρ, the embedding of spaces $L_p(\Omega)$, $p \geq 1$, and the inequality (5) of [3], we get the following estimate $\|\mathfrak{D}^\alpha f\|_{L_2(\Omega,\rho)} \leq C_1 \|\mathfrak{D}^\alpha f\|_{L_q(\Omega)} \leq C_2 \|f\|^2_{W_2^1(\Omega)}$, $2 < q < 2n/(2\alpha - 2 + n)$, $C_i > 0$, $(i = 1, 2)$. Therefore $\mathfrak{D}^\alpha f_k \xrightarrow{L_2(\Omega,\rho)} f$. Hence from the continuity properties of the inner product in the Hilbert space, we get

$$\langle f_k, \mathfrak{D}^\alpha f_k \rangle_{L_2(\Omega,\rho)} \to \langle f, \mathfrak{D}^\alpha f \rangle_{L_2(\Omega,\rho)}.$$

Passing to the limit in the left and right side of inequality (13), we obtain the inequality (11) in the real case. Now consider the case when f is complex-valued. Note that following obvious equality is true

$$\mathrm{Re}\langle f, \mathfrak{D}^\alpha f \rangle_{L_2(\Omega,\rho)} = \langle u, \mathfrak{D}^\alpha u \rangle_{L_2(\Omega,\rho)} + \langle v, \mathfrak{D}^\alpha v \rangle_{L_2(\Omega,\rho)}, \tag{14}$$

$$u = \mathrm{Re} f,\ v = \mathrm{Im} f.$$

The inequality (11) follow now from equality (14). □

4 Sectorial property

Consider a uniformly elliptic operator with fractional derivative in the Kipriyanov sense in lower terms and real-valued coefficients defined formal expression

$$Lu := -D_j(a^{ij} D_i u) + p\, \mathfrak{D}^\alpha u = f \in L_2(\Omega), \tag{15}$$

$$\mathfrak{D}(L) = H^2(\Omega) \cap H_0^1(\Omega),$$

$$a_{ij}(Q) \in C^1(\bar{\Omega}),\ a^{ij}\xi_i\xi_j \geq k_0|\xi|^2,\ k_0 > 0. \tag{16}$$

$$p(Q) \in \mathrm{Lip}\,\lambda,\ (0 < \alpha < \lambda),\ 0 < p_0 < p(Q) < p_1. \tag{17}$$

The following Lemma establishes a property of the closure for operator L.

Lemma 1. *The operator L has a closure $\tilde{L}$, $\mathfrak{D}(\tilde{L}) \subset H_0^1(\Omega)$.*

Proof. Choose an arbitrary $\varepsilon > 0$. In consequence of (14),(16), theorem 5 it is easy to show that for $\{u_n\} \subset \mathfrak{D}(L)$ the next two-sided estimate holds

$$2k_0 \|u_n\|^2_{L_2^1(\Omega)} + \frac{2p_0}{\lambda^2} \|u_n\|^2_{L_2(\Omega)} \leq 2\mathrm{Re}\langle u_n, Lu_n \rangle_{L_2(\Omega)} \leq \frac{1}{\varepsilon} \|Lu_n\|^2_{L_2(\Omega)} + \varepsilon \|u_n\|^2_{L_2(\Omega)}. \tag{18}$$

In consequence of theorem 3.4 [5, p. 337], from lower estimate (18) follows that operator L has a closure. Let $u \in \mathfrak{D}(\tilde{L})$, then by definition exists a sequence $\{u_n\} \subset \mathfrak{D}(L)$ such that $u_n \xrightarrow{L_2} u$, $\{Lu_n\}$ - is fundamental sequence in the sense of the norm $L_2(\Omega)$. Hence the inequality (18) implies that a sequence $\{u_n\}$ is fundamental in the sense of the norm $W_2^1(\Omega)$. Note the assumption relative of domain Ω implies that $H_0^2(\Omega) \subset H_0^1(\Omega)$. In consequence of the completeness property of the space $H_0^1(\Omega)$, we have the inclusion $u \subset H_0^1(\Omega)$. □

We have the following theorem describing the spectral properties of the closed operator $\tilde{L}$.

Theorem 6. *Operator $\tilde{L}$ is strongly accretive, the numerical range of values $\tilde{L}$ belongs to sector*

$$\mathfrak{S} := \{\zeta \in \mathbb{C} : \ |\arg(\zeta - \gamma)| \le \theta\}, \tag{19}$$

where θ and γ defined by the coefficients of operator L. For all $\zeta \in \mathbb{C} \setminus \mathfrak{S}$ operator $\tilde{L} - \zeta$ has a closed range of values,

$$\mathrm{nul}(\tilde{L} - \zeta) = 0, \ \mathrm{def}(\tilde{L} - \zeta) = \mu, \ \mu = \mathrm{const}. \tag{20}$$

In the case $\mu = 0$, we have a following estimate for norm of resolvent

$$\|(\tilde{L} - \zeta)^{-1}\| \le 1/\mathrm{dist}(\zeta, \mathfrak{S}), \ \zeta \in \mathbb{C} \setminus \mathfrak{S}. \tag{21}$$

Proof. We can get a next inequality from the condition (16) by using the Green's formula

$$\mathrm{Re}\langle f_n, Lf_n\rangle_{L_2(\Omega)} \ge k_0 \|f_n\|^2_{L_2^1(\Omega)} + \mathrm{Re}\langle f_n, \mathfrak{D}^\alpha f_n\rangle_{L_2(\Omega,p)}, \ \{f_n\} \subset \mathfrak{D}(L). \tag{22}$$

Assume $f \in \mathfrak{D}(\tilde{L})$, passing to the limit in the left and right side of inequality (22), using lemma 1 and the continuity property of the inner product we get

$$\mathrm{Re}\langle f, \tilde{L}f\rangle_{L_2(\Omega)} \ge k_0 \|f\|^2_{L_2^1(\Omega)} + \mathrm{Re}\langle f, \mathfrak{D}^\alpha f\rangle_{L_2(\Omega,p)}, \ f \in \mathfrak{D}(\tilde{L}). \tag{23}$$

Applying theorem 5 we can rewrite the previous inequality in the following form

$$\mathrm{Re}\langle f, \tilde{L}f\rangle_{L_2(\Omega)} \ge k_0 \|f\|^2_{L_2^1(\Omega)} + \frac{1}{\lambda^2}\|f\|^2_{L_2(\Omega,p)}, \ f \in \mathfrak{D}(\tilde{L}). \tag{24}$$

Therefore the inequality of a strong accretiveness for operator $\tilde{L}$ follows from (24)

$$\mathrm{Re}\langle f, \tilde{L}f\rangle_{L_2(\Omega)} \ge \frac{1}{\mu^2}\|f\|^2_{L_2(\Omega)}, \ f \in \mathfrak{D}(\tilde{L}), \ \mu = k_0 + \lambda^{-2}p_0. \tag{25}$$

Consider imaginary component of the form generated by the operator L. For $f \in \mathfrak{D}(L)$ we get

$$\left|\mathrm{Im}\langle f, Lf\rangle_{L_2(\Omega)}\right| \le 2\left|\int\limits_\Omega a^{ij} D_i u D_j v dQ\right| + \left|\langle u, \mathfrak{D}^\alpha v\rangle_{L_2(\Omega,p)} - \langle v, \mathfrak{D}^\alpha u\rangle_{L_2(\Omega,p)}\right| = I_1 + I_2.$$

Using the Cauchy-Schwarz inequality for a sum, than the Jung's inequality, we have

$$a^{ij} D_i u D_j v \le P(Q)|Du||Dv| \le P(Q)\left(|Du|^2 + |Dv|^2\right), \ P(Q) = \left(\sum_{i,j=1}^{n} |a_{ij}(Q)|^2\right)^{1/2}. \tag{26}$$

Hence

$$I_1 \le P\|f\|^2_{L_2^1(\Omega)}, \ P = \sup_{Q\in\Omega} |P(Q)|. \tag{27}$$

Applying the inequality (4) we get

$$I_2/p_1 \le \|u\|_{L_2(\Omega)}\|\mathfrak{D}^\alpha v\|_{L_2(\Omega)} + \|v\|_{L_2(\Omega)}\|\mathfrak{D}^\alpha u\|_{L_2(\Omega)} \le C\left\{\|u\|_{L_2(\Omega)}\|\mathfrak{D}^\alpha v\|_{L_q(\Omega)} + \|v\|_{L_2(\Omega)}\|\mathfrak{D}^\alpha u\|_{L_q(\Omega)}\right\} \le$$

$$\leq C\|u\|_{L_2(\Omega)}\left\{\frac{K}{\delta^{\nu}}\|v\|_{L_2(\Omega)}+\delta^{1-\nu}\|v\|_{L_2^1(\Omega)}\right\}+$$

$$+C\|v\|_{L_2(\Omega)}\left\{\frac{K}{\delta^{\nu}}\|u\|_{L_2(\Omega)}+\delta^{1-\nu}\|u\|_{L_2^1(\Omega)}\right\},\ 2<q<\frac{2n}{2\alpha-2+n}.$$

$$I_2\leq\frac{1}{\varepsilon}\left(\|u\|^2_{L_2(\Omega)}+\|v\|^2_{L_2(\Omega)}\right)+\varepsilon\left(\frac{p_1KC}{\sqrt{2}\delta^{\nu}}\right)^2\left(\|u\|^2_{L_2(\Omega)}+\|v\|^2_{L_2(\Omega)}\right)+\frac{\varepsilon}{2}\left(p_1C\delta^{1-\nu}\right)^2\left(\|u\|^2_{L_2^1(\Omega)}+\|v\|^2_{L_2^1(\Omega)}\right)=$$

$$=\left(\varepsilon\left(\frac{p_1KC}{\sqrt{2}\delta^{\nu}}\right)^2+\frac{1}{\varepsilon}\right)\|f\|^2_{L_2(\Omega)}+\frac{\varepsilon}{2}\left(p_1C\delta^{1-\nu}\right)^2\|f\|^2_{L_2^1(\Omega)},\ C=(\operatorname{mess}\Omega)^{\frac{q-2}{q}}. \tag{28}$$

Note (27), (28) and applying the same of the proof (23) reasoning, finally we have the following estimate

$$\left|\operatorname{Im}\langle f,\tilde{L}f\rangle_{L_2(\Omega)}\right|\leq C_1\|f\|^2_{L_2(\Omega)}+C_2\|f\|^2_{L_2^1(\Omega)},\ f\in\mathfrak{D}(\tilde{L})$$

$$C_1=\varepsilon\left(\frac{p_1KC}{\delta^{\nu}\sqrt{2}}\right)^2+\frac{1}{\varepsilon},\ C_2=\frac{\varepsilon}{2}\left(p_1C\delta^{1-\nu}\right)^2+P.$$

In consequence of (25) for arbitrary $k>0$ the next inequality holds

$$\operatorname{Re}\langle f,\tilde{L}f\rangle_{L_2(\Omega)}-k\left|\operatorname{Im}\langle f,\tilde{L}f\rangle_{L_2(\Omega)}\right|\geq(k_0-kC_2)\|f\|^2_{L_2^1(\Omega)}+\left(\frac{1}{\mu^2}-kC_1\right)\|f\|^2_{L_2(\Omega)}.$$

Choose $k=k_0/C_2$, we get

$$\left|\operatorname{Im}\langle f,(\tilde{L}-\gamma)f\rangle_{L_2(\Omega)}\right|\leq\frac{1}{k}\operatorname{Re}\langle f,(\tilde{L}-\gamma)f\rangle_{L_2(\Omega)},\ \gamma=\frac{1}{\mu^2}-kC_1,$$

The last inequality implies that the numerical range of values $\Theta(\tilde{L})$ belongs to the sector with top in γ and half-angle $\theta=\arctan(1/k)$. In consequence of theorem 3.2 [5, c.336] we come to conclusions that $\mathfrak{R}(\tilde{L}-\zeta)$ is closed space for any $\zeta\in\mathbb{C}\setminus\mathfrak{S}$, relations (20),(21) holds. □

5 Conjugate operators

Information about the image of Kipriyanov operator gives the theorems 1,2 [3], however, the disadvantage of research in this area is the lack of any information about the defect. The purpose of this section is to construct such extension of Kipriyanov operator, which has zero defect. In this section the indices in the notation of Sobolev spaces associated ratios (3).

Lemma 2. *The operator $\mathfrak{D}^{\alpha}$ is a contraction of operator $\mathfrak{D}^{\alpha}_{0+}$, exactly $\mathfrak{D}^{\alpha}\subset\mathfrak{D}^{\alpha}_{0+}$.*

Proof. We will show that the next equality holds

$$(\mathfrak{D}^{\alpha}f)(Q)=\left(\mathfrak{D}^{\alpha}_{0+}f\right)(Q),\ f\in\overset{0}{W}{}^l_p(\Omega). \tag{29}$$

This implies from the following obvious conversions

$$r^{n-1}\mathfrak{D}^{\alpha}v=\frac{\alpha}{\Gamma(1-\alpha)}\int\limits_0^r\frac{[v(Q)-v(P+\vec{\mathbf{e}}t)]}{(r-t)^{\alpha+1}}t^{n-1}dt+\frac{C_n^{(\alpha)}}{\Gamma(1-\alpha)}v(Q)r^{n-1-\alpha}=$$

$$=\frac{\alpha}{\Gamma(1-\alpha)}\int\limits_0^r\frac{[r^{n-1}v(Q)-t^{n-1}v(P+\vec{\mathbf{e}}t)]}{(r-t)^{\alpha+1}}dt-v(Q)\frac{\alpha}{\Gamma(1-\alpha)}\int\limits_0^r\frac{[r^{n-1}-t^{n-1}]}{(r-t)^{\alpha+1}}dt+\frac{(n-1)!}{\Gamma(n-\alpha)}v(Q)r^{n-1-\alpha}=$$

$$=(\mathbf{D}^{\alpha}_{0+}t^{n-1}v)(Q)-\frac{\alpha v(Q)}{\Gamma(1-\alpha)}\sum_{i=0}^{n-2}r^{n-2-i}\int\limits_0^r\frac{t^i}{(r-t)^{\alpha}}dt+\frac{(n-1)!}{\Gamma(n-\alpha)}v(Q)r^{n-1-\alpha}-\frac{1}{\Gamma(1-\alpha)}v(Q)r^{n-1-\alpha}=$$

$$= (\mathbf{D}^{\alpha}_{0+}t^{n-1}v)(Q) - I_1 + I_2 - I_3. \tag{30}$$

Conduct the following conversions, using the formula of fractional integration of the exponential function, we get

$$I_1 = \frac{\alpha v(Q)}{\Gamma(1-\alpha)} r^{n-2} \int\limits_0^r \frac{1}{(r-t)^{\alpha}} dt + \frac{\alpha v(Q)}{\Gamma(1-\alpha)} \sum_{i=1}^{n-2} r^{n-2-i} \int\limits_0^r \frac{t^i}{(r-t)^{\alpha}} dt =$$

$$= v(Q) \frac{\alpha}{\Gamma(2-\alpha)} r^{n-1-\alpha} + v(Q)\alpha \sum_{i=1}^{n-2} r^{n-2-i} (I^{1-\alpha}_{0+} t^i)(r) =$$

$$= v(Q) \frac{\alpha}{\Gamma(2-\alpha)} r^{n-1-\alpha} + v(Q)\alpha \sum_{i=1}^{n-2} r^{n-1-\alpha} \frac{i!}{\Gamma(2-\alpha+i)}.$$

Consequently

$$r^{-n+1+\alpha}(I_1 + I_3)/v(Q) = \frac{1}{\Gamma(2-\alpha)} + \alpha \sum_{i=1}^{n-2} \frac{i!}{\Gamma(2-\alpha+i)} = \frac{2}{\Gamma(3-\alpha)} + \alpha \sum_{i=2}^{n-2} \frac{i!}{\Gamma(2-\alpha+i)} =$$

$$= \frac{3!}{\Gamma(4-\alpha)} + \alpha \sum_{i=3}^{n-2} \frac{i!}{\Gamma(2-\alpha+i)} = \frac{(n-2)!}{\Gamma(n-1-\alpha)} + \alpha \frac{(n-2)!}{\Gamma(n-\alpha)} = \frac{(n-1)!}{\Gamma(n-\alpha)}. \tag{31}$$

The equality (29) follows from (30),(31). The proof of the fact of difference the operators $\mathfrak{D}^{\alpha}$ and $\mathfrak{D}^{\alpha}_{0+}$ implies from the following reasoning. Let $f \in \mathfrak{I}^{\alpha}_{0+}\varphi,\ \varphi \in L_p(\Omega)$, then in consequence of theorem 2 we have $\mathfrak{D}^{\alpha}_{0+}\mathfrak{I}^{\alpha}_{0+}\varphi = \varphi$. Hence $\mathfrak{I}^{\alpha}_{0+}(L_p) \subset \mathfrak{D}\left(\mathfrak{D}^{\alpha}_{0+}\right)$. Given the above, it remains to note that

$$\exists f \in \mathfrak{I}^{\alpha}_{0+}(L_p),\ f(Q) \neq 0,\ Q \in \partial\Omega,$$

in the same time

$$\forall f \in \mathfrak{D}(\mathfrak{D}^{\alpha}),\ f(Q) = 0,\ Q \in \partial\Omega.$$

□

Theorem 7. *The operators*

$$\mathfrak{D}^{\alpha}_{0+},\ \mathfrak{D}(\mathfrak{D}^{\alpha}_{0+}) = \mathfrak{I}^{\alpha}_{0+}(L_p);\ \mathbf{D}^{\alpha}_{d-},\ \mathfrak{D}(\mathbf{D}^{\alpha}_{d-}) = \mathrm{I}^{\alpha}_{d-}(L_p), \tag{32}$$

are closed.

Proof. According to the definition of the closed operator, is required to prove that from conditions

$$g_n \xrightarrow{L_p} g,\ \{g_n\} \subset \mathfrak{D}(\mathbf{D}^{\alpha}_{d-}),\ \mathbf{D}^{\alpha}_{d-}g_n \xrightarrow{L_p} g^* \in L_p, \tag{33}$$

follows

$$g \in \mathfrak{D}(\mathbf{D}^{\alpha}_{d-}),\ \mathbf{D}^{\alpha}_{d-}g = g^*. \tag{34}$$

Assume the conditions (33) are satisfied, then from the theorem 3 follows that $\mathbf{D}^{\alpha}_{d-}g_n = \mathbf{D}^{\alpha}_{d-}\mathrm{I}^{\alpha}_{d-}\psi_n = \psi_n$, hence $\psi_n \xrightarrow{L_p} g^*$, and in consequence of the theorem 1 we have a limit $\mathrm{I}^{\alpha}_{d-}\psi_n \xrightarrow{L_p} \mathrm{I}^{\alpha}_{d-}g^*$. Since $g_n \xrightarrow{L_p} g$ than $g = \mathrm{I}^{\alpha}_{d-}g^*$. Note the theorem 3, we get (34) from the last equality. □

Theorem 8. *The following equality holds*

$$\mathfrak{D}^{\alpha^*}_{0+} = \mathbf{D}^{\alpha}_{d-}, \tag{35}$$

where

$$\mathfrak{D}(\mathfrak{D}^{\alpha}_{0+}) = \mathfrak{I}^{\alpha}_{0+}(L_p),\ \mathfrak{D}(\mathbf{D}^{\alpha}_{d-}) = \mathrm{I}^{\alpha}_{d-}(L_{p'}),\ 1/p + 1/p' \leq 1.$$

Proof. We will show that the next equality holds

$$(\mathfrak{D}_{0+}^{\alpha}f,g)_{L_2(\Omega)}=(f,\mathbf{D}_{d-}^{\alpha}g)_{L_2(\Omega)},\tag{36}$$
$$f\in\mathfrak{I}_{0+}^{\alpha}\left(L_p\right),\, g\in \mathrm{I}_{d-}^{\alpha}\left(L_{p'}\right).$$

The proof we will conduct for the real linear space of functions. The given strengthening of conditions does not restrict the generality, since the operators are linear with respect to the operation of complex conjugation. Note that in consequence of theorem 3 the next equalities holds: $\mathfrak{D}_{0+}^{\alpha}\left(\mathfrak{I}_{0+}^{\alpha}\varphi\right)=\varphi\in L_p(\Omega)$, $\mathbf{D}_{d-}^{\alpha}\left(\mathrm{I}_{d-}^{\alpha}\psi\right)=\psi\in L_{p'}(\Omega)$. Given the above and in consequence of theorem 1, we get that left and right side of (36) exists and finite. Using the Fubini's theorem we can perform the following conversions

$$(\mathfrak{D}_{0+}^{\alpha}f,g)_{L_2(\Omega)}=\int\limits_{\omega}d\chi\int\limits_{0}^{d}\varphi(P+\vec{\mathbf{e}}r)\left(\mathrm{I}_{d-}^{\alpha}\psi\right)(Q)r^{n-1}dr=\frac{1}{\Gamma(\alpha)}\int\limits_{\omega}d\chi\int\limits_{0}^{d}\varphi(P+\vec{\mathbf{e}}r)r^{n-1}dr\int\limits_{r}^{d}\frac{\psi(P+\vec{\mathbf{e}}t)}{(t-r)^{1-\alpha}}dt=$$
$$=\frac{1}{\Gamma(\alpha)}\int\limits_{\omega}d\chi\int\limits_{0}^{d}\psi(P+\vec{\mathbf{e}}t)t^{n-1}dt\int\limits_{0}^{t}\frac{\varphi(P+\vec{\mathbf{e}}r)}{(t-r)^{1-\alpha}}\left(\frac{r}{t}\right)^{n-1}dr=\int\limits_{\omega}d\chi\int\limits_{0}^{d}\psi(Q)\left(\mathfrak{I}_{0+}^{\alpha}\varphi\right)(Q)r^{n-1}dr=$$
$$=(f,\mathbf{D}_{d-}^{\alpha}g)_{L_2(\Omega)}.\tag{37}$$

The equality (36) is proved. From the equality (36) follows $\mathfrak{D}(\mathbf{D}_{d-}^{\alpha})\subset\mathfrak{D}(\mathfrak{D}_{0+}^{\alpha^*})$, $\mathfrak{R}\left(\mathfrak{D}_{0+}^{\alpha^*}\right)=L_{p'}$. We will show that $\mathfrak{D}(\mathfrak{D}_{0+}^{\alpha^*})\subset\mathfrak{D}(\mathbf{D}_{d-}^{\alpha})$ thus completing the proof. In accordance with the definition of conjugate operator, for all elements $f\in\mathfrak{D}(\mathfrak{D}_{0+}^{\alpha})$ and pars of elements $g\in\mathfrak{D}\left(\mathfrak{D}_{0+}^{\alpha^*}\right)$, $g^*\in\mathfrak{R}\left(\mathfrak{D}_{0+}^{\alpha^*}\right)$, the integral equality holds

$$\left\langle\mathfrak{D}_{0+}^{\alpha}f,g\right\rangle_{L_2(\Omega)}=\left\langle f,g^*\right\rangle_{L_2(\Omega)}.\tag{38}$$

Suppose $f=\mathfrak{I}_{0+}^{\alpha}\varphi,\ \varphi\in L_p(\Omega)$. Using the Fubini's theorem and performing the conversion similar to (37), we have

$$\left\langle\mathfrak{D}_{0+}^{\alpha}f,g-\mathrm{I}_{d-}^{\alpha}g^*\right\rangle_{L_2(\Omega)}=0.$$

In consequence of theorem 3 the image of operator $\mathfrak{D}_{0+}^{\alpha}$ coincides with the space $L_p(\Omega)$. Hence the functionals corresponding to the elements $(g-\mathrm{I}_{d-}^{\alpha}g^*)\in L_{p'}$ equals zero. Consequently, according to isometrically isomorphic correspondence $L_p^*\leftrightarrow L_{p'}$, the elements $g-\mathrm{I}_{d-}^{\alpha}g^*$ equals zero. It implies that $\mathfrak{D}(\mathfrak{D}_{0+}^{\alpha^*})\subset\mathfrak{D}(\mathbf{D}_{d-}^{\alpha})$.

□

6 Two-sided estimation of eigenvalues for Sturm-Liouville problem

Let us recall some known facts of the theory of unbounded operators by choosing as the object the operator of fractional differentiation in the Kipriyanov sense. Consider the operator L (15). Let $L_{\mathfrak{R}}=(L+L^*)/2$ — real component of operator L. From the theory of unbounded operators it is well known, that

$$L^*=[T+p\,\mathfrak{D}^{\alpha}]^*\subset T^*+[p\,\mathfrak{D}^{\alpha}]^*,\tag{39}$$

where T — the elliptic component of operator L. Applying theorem 4, theorem 8 and lemma 2 we have

$$(p\,\mathfrak{D}^{\alpha}f,g)_{L_2(\Omega)}=(\mathfrak{D}^{\alpha}f,p\,g)_{L_2(\Omega)}=(f,\,\mathbf{D}_{d-}^{\alpha}p\,g)_{L_2(\Omega)},\ f,g\in H_0^1(\Omega),\tag{40}$$

it implies that

$$[\mathbf{D}_{d-}^{\alpha}p]\subset[p\,\mathfrak{D}^{\alpha}]^*.\tag{41}$$

Note that with assumptions relative of the domain — Ω, we have embedding $H_0^2(\Omega)\subset H_0^1(\Omega)$. Using the Green's formula and equality (40), it is easy to show that

$$\forall f,g\in H_0^2(\Omega),\,\exists\, g^*\in L_2(\Omega),\ (Lf,g)_{L_2(\Omega)}=(f,g^*)_{L_2(\Omega)}.\tag{42}$$

It implies that $H_0^2(\Omega)\subset\mathfrak{D}(L^*)$. In consequence of (39), selfadjointness of the operator T, (41), we have

$$L^*f=L^+f=(T^*+[p\,\mathfrak{D}^{\alpha}]^*)f=(T+[\mathbf{D}_{d-}^{\alpha}p])f,\ f\in H_0^2(\Omega).\tag{43}$$

Thus, from (41),(43) we have a representation for the real component of operator L

$$L_{\mathfrak{R}}=T+\frac{1}{2}(p\,\mathfrak{D}^{\alpha}+[\mathbf{D}_{d-}^{\alpha}p])=T+[p\,\mathfrak{D}^{\alpha}]_{\mathfrak{R}},\ \mathfrak{D}(L)=H_0^2(\Omega).$$

Theorem 9. *We have the following estimate for the eigenvalues of operator* $L_{\mathfrak{R}}$

$$\lambda_n(L_0) \le \lambda_n(L_{\mathfrak{R}}) \le \lambda_n(L_1),\ n \in \mathbb{N}, \tag{44}$$

where $\lambda_n(L_0), \lambda_n(L_1)$ *— accordingly eigenvalues of Sturm-Liouville operators* L_0, L_1 *with constant coefficients defined by coefficients of operator* L.

Proof. i) Note that operator $L_{\mathfrak{R}}$, $\mathfrak{D}(L_{\mathfrak{R}}) = H_0^2(\Omega)$ has domain of definition is dense in $L_2(\Omega)$; the symmetry of the operator $L_{\mathfrak{R}}$ follows from the definition; in consequently of theorem 5, $L_{\mathfrak{R}}$ is strongly coercive. Hence one is positive defined.
ii) The space $H_0^1(\Omega)$ coincides as a set of elements with energetic space $H_{L_{\mathfrak{R}}}$. Prove this fact using the Green's formula and conversion a norm of energetic space $H_{L_{\mathfrak{R}}}$

$$\|f\|^2_{H_{L_{\mathfrak{R}}}} = \int\limits_{\Omega} a^{ij} D_i f\, \overline{D_j f} dQ + \int\limits_{\Omega} \overline{f}\, [p\mathfrak{D}^{\alpha}]_{\mathfrak{R}} f dQ,\ f \in H_0^2(\Omega). \tag{45}$$

Applying inequalities (18), (26), (4) we get

$$C_0\|f\|_{H_0^1} \le \|f\|_{H_{L_{\mathfrak{R}}}} \le C_1\|f\|_{H_0^1},\ f \in H_0^2(\Omega). \tag{46}$$

From the relation (45),(46), in consequence of completeness property of spaces $H_{L_{\mathfrak{R}}}(\Omega),\ H_0^1(\Omega)$ following coincides as the elements spaces given above.
iii) Let the Sturm-Liouville operators L_0, L_1 with constant coefficients, have the form

$$L_k f = -a_k^{ij} D_{ij} f + p_k f,\ f \in H_0^2(\Omega),\ k = 0, 1.$$

Applying reasoning that was used to obtain the inequality (46) we can easy to see that exist real constant coefficients of operators L_0, L_1, that the next inequalities holds

$$\|f\|_{H_{L_0}} \le C_0\|f\|_{H_0^1},\ C_1\|f\|_{H_0^1} \le \|f\|_{H_{L_1}},\ f \in H_0^1(\Omega). \tag{47}$$

From (46),(47) we have following estimate of norms

$$\|f\|_{H_{L_0}} \le \|f\|_{H_{L_{\mathfrak{R}}}} \le \|f\|_{H_{L_1}},\ f \in H_0^1(\Omega). \tag{48}$$

In (i) we proved that the operators $L_0, L_{\mathfrak{R}}, L_1$ — positive definite. By equivalence of norm (46), estimates (47), in consequence of Rellich-Kondrashov theorem, the spaces $H_{L_0},\ H_{L_{\mathfrak{R}}},\ H_{L_1}$ is compactly embedded in $L_2(\Omega)$. Note (48) we have

$$L_0 \le L_{\mathfrak{R}} \le L_1,$$

where order relation is understood in terms of [7, p.111]. Using the theorem 5.10.1 [7, p.111] we get (44). □

7 Conclusions

In this paper was proved the theorem establishing the strong accretive property for the operator of fractional differentiation in the Kipriyanov sense acting in weighted Lebesgue space of integrable with squared functions.

References

[1] Friedrichs, K. Symmetric positive linear differential equations . *Comm. Pure Appl. Math.*, **11** (1958), 238-241.

[2] Kipriyanov, I.A. On spaces of fractionally differentiable functions. *Proceedings of the Academy Of Sciences. USSR*, **24** (1960), 665-882.

[3] Kipriyanov, I.A. The operator of fractional differentiation and the degree of elliptic operators. *Proceedings of the Academy of Sciences. USSR*, **131** (1960), 238-241.

[4] Samko S.G., Kilbas A.A., Marichev O.I. Integrals and derivatives of fractional order and some of their applications. *Minsk Science and technology*, 1987.

[5] Kato, T. Perturbation theory for linear operators. *Springer-Verlag Berlin, Heidelberg, New York*, 1966.

[6] Kukushkin M.V. Evaluation of the eigenvalues of the Sturm-Liouville problem for a differential operator with fractional derivative in the junior members. *Belgorod State University Scientific Bulletin, Math. Physics.*, **46**, N6 (2016) 29-35.

[7] Mihlin S.G. Linear partial differential equations. *Moscow Higher school*, 1977.

[8] Gilbarg D., Trudinger N.S. Eliptic partial differential equations of second order. Second edition. *Springer-Verlag Berlin, Heidelberg, New York, Tokyo*, 1983.