Metric in the moduli of SU(2) monopoles from spectral curves and Gauss-Manin connection in disguise.

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We show here that from the metric of the manifold M_2^0, i.e., the reduced moduli of $SU(2)$ 2-monopoles in Yang-Mills-Higgs theory, one can recover the respective moduli of spectral curves using the method Gauss-Manin connection in disguise. We also claim that this process can be done conversely to find the metric of M_k^0 , $k > 2$. This is a thirty years old problem that we hope to shed some light in it.

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I. INTRODUCTION

The study of instantons and monopoles in three and four dimension are among the most compreensive research areas at the interface of physics and mathematics. From the physics point of view, they relate to solitons, dualities and non-perturbative Yang-Mills theories. From the mathematical point of view, it involves knowledge of Analysis, Differential Geometry, Algebraic Geometry and Twistor theory. In this work, we add elements of Hodge theory to relate the metric of the moduli space of charge k monopoles M_k^0 and its spectral curves in SU(2) Yang-Mills-Higgs (YMH) theory in three spacial dimensions (static monopoles).

YMH monopoles in three dimensions are equivalent to instanton solutions of Yang-Mills theory in Euclidian four dimensions constrained by the fact that gauge fields do not depend on the fourth direction. In this equivalence, the Higgs field is the fourth component of the gauge field in four dimensions. In this way, the twistor methods applied in instantons were also adapted to YMH monopoles[1].

In a upcoming work[2], revising the metric of M_2^0 reduced moduli of 2-monopoles, the present author noticed that the the moduli of enhanced elliptic curves obtained from the self-dual metric should correspond to the moduli of spectral curves of 2-monopoles in its enhanced version. Notice that the spectral curve of a k-monopole is an algebraic curve of genus $(k-1)^2$[1] and for k=2, the spectral curve is an elliptic curve[3].

The moduli of enhanced elliptic curves appears from the metric of M_2^0 using the method Gauss-Manin connection in disguise[4–6] which shows that the set of Darboux-Halphen differential equations obtained from self-duality of the metric of M_2^0 are a vector field in the moduli of an enhanced elliptic curve.

In here, the guess made in[2] is proved and new claims are made to extend this result. Using Gauss-Manin connection in disguise[4–6], one can lift the families of spectral curves of k-monopoles to its enhanced version. The vector fields in the moduli of the enhanced spectral curve should define the curvature equations of M_k^0, since there is a homomorphism between the moduli of spectral curves of k-monopoles and the moduli of k-monopoles itself. In this case, the metric of M_k^0 will be written as quasi-homogeneous polynomial expressions in terms of modular-type functions attached to the spectral curve and weighted by the algebraic group of the moduli of the enhanced spectral curve[7].

In sections II and III we review basic elements of SU(2) monopoles and spectral curves following closely some original articles[1,3,8] and reference books[9,10]. In section IV we review the program Gauss-Manin connection in disguise and in section V we show how the moduli of spectral curves of 2-monopoles emerge from the metric of M_2^0. In VI we summarize this article and comment about the cases $k > 2$ and the issue of different parametrization of universal families of curves and their respective weights of attached modular-type functions to such curves.

II. *K*-MONOPOLES

A k-monopole or BPS-monopole of charge k in Yang-Mills-Higgs theory is a static soliton in $\mathbb{R}^3$ that is a solution of the Bogomolny equation[11]:

$$F = \star D\phi, \qquad \text{with} \tag{1}$$
$$F := \mathrm{d}A + A \wedge A \quad \text{and} \quad D := \mathrm{d} + A,$$

where A is the gauge field or connection form on a principal SU(2)-bundle over $\mathbb{R}^3$, F is its curvature 2-form or field strength, D is the covariant exterior derivative or connection and the Higgs field ϕ is a section of the associated $\mathfrak{su}(2)$-bundle. $\star$ is the Hodge dual operation and the Bogomolny equation (1) is part of the self-duality equations for the related instantons in four dimensions.

The monopole solution has also to satisfy the finite action condition $\int |F|^2 < \infty$ and the boundary condition

$$|\phi| = 1 - \frac{k}{2r} + O(r^{-2}) \quad \text{as} \ \ r \to \infty, \tag{2}$$

where the charge k is an integer number.

A clever treatment to monopoles was given by Hitchin[1] where he applied the twistor methods in the space $\tilde{\mathbf{T}}$ of oriented straight lines (geodesics) in $\mathbb{R}^3$. $\tilde{\mathbf{T}}$ has a holomorphic structure given by cross product and $\tilde{\mathbf{T}} \equiv \mathbf{T}\mathbb{P}_1(\mathbb{C})$ the holomorphic tangent bundle of the projective line.

Then the solutions of Bogomolny equations were restated in terms of complex geometry of $\tilde{\mathbf{T}}$ where the spectral curves were introduced[1,12].

First consider E the rank 2 complex vector bundle on $\mathbb{R}^3$ associated to the principal SU(2) bundle and by associating to each oriented line l the space of sections s over l such that

$$(u^j D_j - i\phi)s = 0, \tag{3}$$

along l, one defines a bundle $\tilde{E}$ on $\tilde{\mathbf{T}}$. u is the unit tangent vector pointing (in the positive direction) along the oriented line l. It follows from Bogomolny equations that $\tilde{E}$ has a natural holomorphic structure[1]. Conversely, from a holomorphic vector bundle $\tilde{E}$ on $\tilde{\mathbf{T}}$ one reconstructs the solution (A, ϕ) to Bogomolny equations. But not all section s of $\tilde{E}_l$ satisfy the boundary conditions, which is the vanishing of s at both ends of l.

III. SPECTRAL CURVE OF A K-MONOPOLE

For each oriented line l, the space of solutions (3) which decay at $+\infty$ is one-dimensional. This space is a holomorphic line bundle and a subbundle of $\tilde{E}_l$ and it belongs to a class of ansätz $\mathcal{A}_k$ according to the charge k of the monopole[8]. Furthermore, the set of lines for which equation (3) has a solution decaying to zero at both ends forms a compact algebraic curve S in $\mathbf{T}\mathbb{P}_1(\mathbb{C})$. S is called the spectral curve and it has genus $(k-1)^2$.

Inhomogeneous coordinates (η, ζ)on $\mathbb{P}_1(\mathbb{C})$ gives local coordinates on $\tilde{\mathbf{T}} : (\eta, \zeta) \to \eta\partial/\partial\zeta$. S in terms of such local coordinates is given by

$$p(\eta, \zeta) = \eta^k + a_1(\zeta)\eta^{k-1} + \ldots a_k(\zeta) = 0, \tag{4}$$

where $a_i(\zeta)$ is a polinomial of degree 2i.

The polynomial $p(\eta, \zeta)$ is preserved by an antiholomorphic involution $\tau(\eta, \zeta) = (-\overline{\eta}/\overline{\zeta}^2, -\overline{\zeta}^{-1})$, a real structure on $\mathbf{T}\mathbb{P}_1(\mathbb{C})$. Therefore $p(\eta, \zeta)$ depends on $(k+1)^2 - 1$ real parameters. Since S is constrained by its genus, the parameter space has dimension

$$(k+1)^2 - 1 - (k-1)^2 = 4k - 1. \tag{5}$$

This is the dimension of the moduli space of k-monopoles M_k. Out of these parameters, the center of mass position of a k-monopole in $\mathbb{R}^3$ can be translated to the origin and the remaining parameter space corresponds to the reduced moduli space M_k^0 with dimension $4k - 4$.

A. Spectral curve for k= 2

This case was extensively studied by Hurtubise[3]. The spectral curve S is an elliptic curve of genus 1. The real structure of S imposes via Weierstrass $\mathfrak{p}$-function the complex structure τ of the corresponding torus $\mathbb{C}/\Lambda$ to be purely imaginary and its corresponding lattice Λ to be rectangular.

Factoring out six parameters from translation action and $SO(3)$ action from the polynomial (4) for the spectral curve of 2-monopoles, two real parameters remain:

$$\eta^2 = r_1\zeta^3 - r_2\zeta^2 - r_1\zeta,\ r_i \in \mathbb{R}, r_1 \geq 0 \tag{6}$$

The genus constrain is enforced by matching the above equation to a normal form of the elliptic curve.

First notice that when $r_1 = 0$, the spectral curve degenerates to two $k = 1$ spectral curves.

$$\eta = i\frac{\pi}{2}\zeta \quad \text{and} \quad \eta = -i\frac{\pi}{2}\zeta. \tag{7}$$

In this case $r_2 = \pi^2/4$, as we show below (14), and there is no free parameter. This is the case where a 2-monopole is simply a superposition of two 1-monopoles both centered at the origin of $\mathbb{R}^3$[13,14] and it agrees with the fact that the dimension of the reduced moduli M_1^0 is zero. The spectral curves (7) are two complex lines tangent to two different points in $\mathbb{P}_1 \equiv S^2$. The symmetries of the spectral curve determines a symmetry of the monopole. In this case, the isotropy group $S^1 \times \mathbb{Z}_2$ of the 2 $k = 1$ spectral curves corresponds to the axial symmetry of the two 1-monopoles solution and the exchanging the two 1-monopoles. The six parameters earlier factored out map to the positions in $\mathbb{R}^3$ of the centers of the two 1-monopoles.

For $r_1 > 0$, the spectral curve can be reparametrized to:

$$\tilde{\eta}^2 = 4\tilde{\zeta}^3 - g_2(\Lambda)\tilde{\zeta} - g_3(\Lambda),\ \text{where,} \tag{8}$$

$$\tilde{\eta} = \eta(4/r_1)^{1/2}, \quad \tilde{\zeta} = \zeta - \tfrac{r_2}{3r_1},$$

$$g_2(\Lambda) = 60G_4(\Lambda) = 12(r_2/3r_1)^2 + 4 \quad \text{and,}$$

$$g_3(\Lambda) = 140G_6(\Lambda) = 8(r_2/3r_1)^3 + 4(r_2/3r_1). \tag{9}$$

and G_4 and G_6 are Eisenstein series of weight 4 and 6, respectively, functions of the retangular lattice Λ with real generator $l_r = \sqrt{4r_1}$ and imaginary generator l_i. A homothetic scaling of the lattice transform g_2 and g_3:

$$g_i(m\Lambda) = m^{-2i} g_i(\Lambda),\ i = 2, 3, \quad m \in \mathbb{R}^* \tag{10}$$

and the polinomial (8) is preserved if we reparametrize $(\tilde{\eta}, \tilde{\zeta})$ to absorb such scaling

$$\tilde{\eta} \to m^{-3}\tilde{\eta} \quad \text{and} \quad \tilde{\zeta} \to m^{-2}\tilde{\zeta}. \tag{11}$$

Therefore we should consider modular functions such as $I = 27g_3^2/g_2^3$. This function will be invariant to scaling of the lattice Λ and it will only depend on the ratio of the generators $\tau = l_i/l_r$, a purely imaginary number. From (8), I depends on the ratio $(r_1/r_2)^2$. Notice that in the limit $r_1 \to 0$, the discriminant of the elliptic curve $\Delta = g_2^3 - 27g_3^2 = 0$ and $I(\tau) = 1$. This corresponds to the limit $\tau \to i\infty$. In order to proceed showing that $r_2 = \pi^2/4$ in this limit, we explore I near 1.

$$\frac{1 - I(\tau)}{27} = \frac{64}{j(\tau)} = 2^{12}3^3(q - 744q^2 + \dots) \quad \text{where,} \tag{12}$$

$$j(\tau) = \frac{1728g_2^3}{\Delta} \text{ is the Klein modular function and } \ q = \exp(2\pi i\tau).$$

From (9),

$$\frac{1 - I}{27} = \frac{r^4(\frac{1}{4} + r^2)}{(1 + 3r^2)^3}, \quad \text{with } r = \frac{r_1}{r_2}. \tag{13}$$

We see that the limit $r_1, r \to 0$ coincides with $\tau \to i\infty$ or $q \to 0$. Near this limit, we keep only the first term of the Eisenstein series $G_4(\tau)$ of the normalized lattice. From (9):

$$g_2 = \frac{1}{16r_1^2}\frac{64}{3}r_2^2(1 + 3r^2) = \frac{1}{16r_1^2}60G_4(\tau) \xrightarrow{q \to 0} \frac{1}{16r_1^2}\frac{(2\pi)^4}{12}. \tag{14}$$

$$\text{Therefore,} \quad r_2 \xrightarrow{q \to 0} \pi^2/4. \tag{15}$$

Hence, the point $(r_1, r_2) = (0, \pi^2/4)$ corresponds to the singular point $(\Delta = 0)$ of the real elliptic curve S factored out by $SO(3)$ action and $\mathbb{R}^3$ translations, corresponding to $\tau = i\infty$. The total space of parameters has one real dimension and it corresponds to purely imaginary τ, $-i\tau \in \mathbb{R}_{\geq 0} \cup \infty$.

IV. GAUSS MANIN CONNECTION IN DISGUISE

In[15,16] Movasati realized that the Ramanujan relations between Eisenstein series can be computed using the Gauss-Manin connection of families of elliptic curves. Later in a private communication, Pierre Deligne called this the Gauss-Manin connection in disguise. Since then the method Gauss-Manin connection in disguise has been applied in many families of algebraic curves and relating them to differential equations and automorphic forms or modular-type functions[4,7,17,18].

Our interest are in finding differential equations in the universal families of spectral curves of k-monopoles, which are algebraic curves of genus $(k-1)^2$. The method developed for the elliptic curve[6,16] still need to be thought through for spectral curves because of the reality condition, but the general argument is that the reality condition is lifted for the sake of finding the Gauss-Manin connection and the respective vector field and later the reality condition is imposed.

We present here the two known cases for families of elliptic curves as geometric expressions of Ramanujan and Darboux-Halphen differential equations. In both cases, the idea is to define the moduli of enhanced elliptic curve that extends the one-parameter family of elliptic curves by including information about its Hodge structure. In sequence, one calculates its Gauss-Manin connection and finds the appropriate vector field.

A. Ramanujan differential equations

We extend the one-parameter family of elliptic curves (8). Recall that the first de Rham cohomology $H^1_{\mathrm{dR}}(E)$ of an elliptic curve E is a two-dimensional vector space. The moduli $\mathtt{T}_\mathtt{R}$ of pairs $(E, [\alpha, \omega])$, where $\alpha, \omega \in H^1_{\mathrm{dR}}(E)$ are a basis of the cohomology classes of holomorphic 1-forms in E with α a regular differential 1-form on E and ω such that $\langle \alpha, \omega \rangle = 1$. Therefore $\mathtt{T}_\mathtt{R}$ is a three-dimensional space and it has a corresponding universal family of elliptic curves

$$E_t : y^2 = 4(x - t_1)^3 - t_2(x - t_1) - t_3, \tag{16}$$

with $\alpha = [\frac{dx}{y}]$, $\omega = [\frac{xdx}{y}]$ and the moduli $\mathtt{T}_\mathtt{R}$ can be expressed as

$$\mathtt{T}_\mathtt{R} := \{(t_1, t_2, t_3) \in \mathbb{C}^3 | \Delta = t_2^3 - 27t_3^2 \neq 0\}. \tag{17}$$

The Gauss-Manin connection of the above family E_t, written in the basis (α, ω) is given by

$$\nabla \begin{pmatrix} \alpha \\ \omega \end{pmatrix} = A \begin{pmatrix} \alpha \\ \omega \end{pmatrix}, \tag{18}$$

where

$$A = \frac{1}{\Delta}\begin{pmatrix} -\frac{3}{2}t_1\beta - \frac{1}{12}d\Delta & \frac{3}{2}\beta \\ \Delta dt_1 - \frac{1}{6}t_1 d - (\frac{3}{2}t_1^2 + \frac{1}{8}t_2)\beta & \frac{3}{2}t_1\beta\Delta + \frac{1}{12}d\Delta \end{pmatrix}, \tag{19}$$

$$\beta = 3t_3dt_2 - 2t_2dt_3.$$

In $\mathtt{T_R}$ there is a unique vector field R such that[6]

$$\nabla_R(\alpha) = -\omega, \quad \nabla_R(\omega) = 0. \tag{20}$$

The vector field R is given by the Ramanujan differential equations

$$\begin{cases} \dot{t}_1 = t_1^2 - \frac{1}{12}t_2, \\ \dot{t}_2 = 4t_1t_2 - 6t_3, \\ \dot{t}_3 = 6t_1t_3 - \frac{1}{3}t_2^2. \end{cases} \tag{21}$$

R has been called Ramanujan vector field, modular vector field and lately, Gauss-Manin connection in disguise.

B. Darboux-Halphen differential equations

In this case, the enhanced elliptic curve is given by a triple $(E, (P, Q), \omega)$, where E is an elliptic curve, $\omega \in H^1_{\mathrm{dR}}(E)\backslash F^1$, and P and Q are a pair of points of E that generates the 2-torsion subgroup with the Weil pairing $e(P, Q) = -1$. The points P and Q are given by $(T_1, 0)$ and $(T_2, 0)$. In here, the torsion data is necessary because the modular group , or group of lattice equivalence, of this enhanced curve is the congruence subgroup $\Gamma(2) \subset SL_2(\mathbb{Z})$, which has index $[SL_2(\mathbb{Z}) : \Gamma(2)] = 6$. The torsion data choose one out of six enhanced elliptic curves with same (E, ω) pairs.

For each choice of ω, there is a unique regular differential 1-form in the Hodge filtration $\omega_1 \in F^1$, such that $\langle\omega, \omega_1\rangle = 1$ and ω, ω_1 together form a basis of $H^1_{\mathrm{dR}}(E)$. The corresponding universal family of elliptic curves is given by

$$E_T : \quad y^2 - 4(x - T_1)(x - T_2)(x - T_3) = 0, \tag{22}$$

and moduli $\mathtt{T_H} = \{(T_1, T_2, T_3) \in \mathbb{C}^3 |\, T_1 \neq T_2 \neq T_3\}$.

In fact, this universal family patches together all six enhanced elliptic curves, separated by singularity borders $T_i = T_j$, due its symmetry under permutation of T_1, T_2 and T_3. Hence, it is a six-fold cover of the enhanced elliptic curve (E, ω) that is isomorphic to the enhanced elliptic curve $(E, [\alpha, \omega])$ for the full modular group $SL_2(\mathbb{Z})$.

The Gauss-Manin connection of the family of elliptic curves E_T written in the basis $\frac{dx}{y}$, $\frac{xdx}{y}$ is given as bellow:

$$\nabla \begin{pmatrix} \frac{dx}{y} \\ \frac{xdx}{y} \end{pmatrix} = A \begin{pmatrix} \frac{dx}{y} \\ \frac{xdx}{y} \end{pmatrix} \tag{23}$$

where

$$\begin{aligned} A = \quad & \frac{dT_1}{2(T_1 - T_2)(T_1 - T_3)} \begin{pmatrix} -T_1 & 1 \\ T_2T_3 - T_1(T_2 + T_3) & T_1 \end{pmatrix} + \\ & \frac{dT_2}{2(T_2 - T_1)(T_2 - T_3)} \begin{pmatrix} -T_2 & 1 \\ T_1T_3 - T_2(T_1 + T_3) & T_2 \end{pmatrix} + \\ & \frac{dT_3}{2(T_3 - T_1)(T_3 - T_2)} \begin{pmatrix} -T_3 & 1 \\ T_1T_2 - T_3(T_1 + T_2) & T_3 \end{pmatrix}. \end{aligned}$$

In the parameter space of the family of elliptic curves E_T there is a unique vector field H, such that

$$\nabla_H(\frac{dx}{y}) = -\frac{xdx}{y}, \ \nabla_H(\frac{xdx}{y}) = 0. \tag{24}$$

The vector field H is given by the Darboux-Halphen differential equation

$$\begin{cases} \dot{T}_1 = T_1(T_2 + T_3) - T_2T_3, \\ \dot{T}_2 = T_2(T_1 + T_3) - T_1T_3, \\ \dot{T}_3 = T_3(T_1 + T_2) - T_1T_2. \end{cases} \tag{25}$$

H has been called Darboux-Halphen vector field,and lately, Gauss-Manin connection in disguise.

C. $\mathtt{T_R}$ and $\mathtt{T_H}$

There is a algebraic morphism between the moduli $f : \mathtt{T_H} \longrightarrow \mathtt{T_R}$ given by a match between the elliptic curves (18) and (23):

$$(T_1, T_2, T_3) \to (T, -4 \sum_{1\leq i<j\leq 3} (T - T_i)(T - T_j), 4(T - T_1)(T - T_2)(T - T_3)), \qquad (26)$$

$$\text{where} \quad T = \frac{T_1 + T_2 + T_3}{3}$$

Since the permutations of T_1, T_2 and T_3 in $\mathtt{T_H}$ are mapped to the same point in $\mathtt{T_R}$, f is a six to one map, but if we restrict to the region $T_1 < T_2 < T_3$, f is an isomorphism.

V. FROM THE METRIC OF THE MODULI SPACE OF 2-MONOPOLES TO THE SPECTRAL CURVE

In[9], Atiyah and Hitchin showed that the reduced moduli M_2^0 of 2-monopoles is a four dimensional hyperkähler manifold and an anti-self-dual Einstein manifold. Since M_2^0 admits $SO(3)$ isometry, the metric is a Bianch IX ([19]). This is consequence of the hyperkähler structure of M_2^0 which has an S^2-parameter family of complex structures: if I, J, K are covariant constant complex structures in M_2^0 then $aI + bJ + cK$ is also a covariant constant complex structure in M_2^0 given that $a^2 + b^2 + c^2 = 1$, $(a, b, c) \in \mathbb{R}^3$. The SO(3) isometry rotates this S^2 in a standard way. Following[9] (Chapter 8,9) and[2,20], the 4-dimensional Bianchi IX metric is cast in the form

$$ds^2 = (abc)^2 d\rho^2 + a^2(\sigma_1)^2 + b^2(\sigma_2)^2 + c^2(\sigma_3)^2, \qquad (27)$$

where a, b, c are real functions of ρ which parametrize translations orthogonal to SO(3) orbits and σ_i are the $SO(3)$ invariant 1-forms dual to the standard basis X_1, X_2, X_3 of its Lie algebra. They obey the structure equation

$$d\sigma_i = -\sigma_j \wedge \sigma_k, \qquad (28)$$

for all cyclic permutations (i, j, k) of $(1, 2, 3)$.

The self-duality equations lead to the following equation

$$\frac{2}{a}\frac{da}{d\rho} = b^2 + c^2 - a^2 - 2bc, \qquad (29)$$

and two other equations obtained by cyclic permutations of (a,b,c). Upon reparametrization

$$a^2 = \frac{\Omega_2\Omega_3}{\Omega_1}, \quad b^2 = \frac{\Omega_3\Omega_1}{\Omega_2}, \quad c^2 = \frac{\Omega_1\Omega_2}{\Omega_3}. \tag{30}$$

we obtain from (29) the three Darboux-Halphen differential equations:

$$\dot{\Omega}_i = \Omega_i(\Omega_j + \Omega_k) - \Omega_j\Omega_k, \tag{31}$$

where (i,j,k) run over cyclic permutations of (1,2,3) and the derivative (denoted by dot) is with respect to ρ.

From the discussion in IV B, the space

$$\mathtt{T}_\Omega := \{(\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3 | \Omega_1 \neq \Omega_2 \neq \Omega_3\}, \tag{32}$$

is the real section of the moduli space $\mathtt{T}_\mathtt{H}$ of the enhanced elliptic curve E_T. These curves E_T and E_t live in $\mathbb{C}^2$ and their real section are easily defined, while in $\tilde{\mathbf{T}}$, the reality condition is given by a real structure ς that acts by reverting the orientation of the lines in $\tilde{\mathbf{T}}^1$. In terms of $\tilde{\mathbf{T}}$ coordinates, it corresponds to $\varsigma(\eta, \zeta) = (-\overline{\eta}/\overline{\zeta}^2, -1/\overline{\zeta})$. Hence, a real curve in $\tilde{\mathbf{T}}$ is invariant under ς as it happens in (6). The reparametrization of the spectral curve (8) change the real structure to the conventional one of real sections in $\mathbb{C}^2$, and therefore real coefficients for the defining polynomial.

Through the algebraic morphism f in IV C, $\mathtt{T}_\Omega$ maps to the real section of the moduli $\mathtt{T}_\mathtt{R}$ of the enhanced elliptic curve E_t (16) and such elliptic curve restricted to real parameters maps to the reparametrized spectral curve (8) in $\mathbb{C}^2$,

$$t_2 \to g_2 \quad t_3 \to g_3 \tag{33}$$

up to scaling equivalence of the lattice (10,11). Therefore we recognize that $\mathtt{T}_\Omega$ is a six-fold cover of the moduli of the enhanced spectral curve, which we define below:

Definition 1. *An enhanced spectral curve $\tilde{S}_k$ of a k-monopole is the real section of the universal family of curves corresponding to the enhanced algebraic curve $(S_k^{\mathbb{C}}, \alpha)$ where $S_k^{\mathbb{C}}$ is the family of algebraic curves in $\tilde{\mathbf{T}}$ given by (4) without the real structure constraints and α is a basis of cohomology classes $H^1_{\mathrm{dR}}(S_k^{\mathbb{C}})$ of holomorphic differential 1-forms in $S_k^{\mathbb{C}}$.*

Below we summarize our findings in a form of a theorem:

Theorem 1. *The moduli of enhanced spectral curves corresponding to the moduli of $SU(2)$ monopoles of charge 2 quotient by $SO(3)$ action and $\mathbb{R}^3$ translations is given by $\mathtt{T}_\Omega$ quotient by permutations of Ω_1, Ω_2 and Ω_3. Furthermore, the self-duality curvature equation (29) corresponds to the Ramanujan vector field, upon reparametrization (30) and f morphism (26).*

Proof. This theorem is proved by f isomorphism under restriction $T_1 < T_2 < T_3$ in $\mathtt{T}_\mathtt{H}$. The quotient of the moduli of enhanced spectral curves by SO(3) action corresponds to the SO(3) isometry in M_2^0. ☐

The moduli of $\tilde{S}_2$ does not include the point $r_1 = 0$ and $r_2 = \pi^2/4$ of zero discriminant where the curve S_2 degenerates to two S_1. This point can be mapped to a point of zero discriminant in E_T (22) and it is given by the $\tau = i\infty$ limit in the solution of the system (31), with appropriate lattice scaling to match g_2 in (9,14):

$$\Omega_i(\rho) = \frac{\pi}{r_1}\frac{\partial}{\partial\rho}\left(\log\theta_{i+1}(i\rho)\right), \quad \text{where} \quad \rho = -i\tau \quad \text{and} \tag{34}$$

$$\begin{cases} \theta_2(\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} \\ \theta_3(\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} \\ \theta_4(\tau) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} \end{cases}, \ q = e^{2\pi i\tau}, \ \tau \in \mathbb{H}. \tag{35}$$

Notice that r_1/r_2 is function of $\tau = i\rho$ and in the limit $\rho \to \infty$ we can write a explicit relation

$$r_2 \to \pi^2/4, \quad \text{and} \quad r_1 \to \pi^2 q^{1/4} \tag{36}$$

Therefore, near the limit $\rho \to \infty$ ($q << 1$) we have

$$\Omega_1 \approx -\frac{q^{1/4}}{4}, \quad \Omega_2 \approx -2q^{1/4}, \quad \Omega_3 \approx 2q^{1/4} \tag{37}$$

and the metric of M_2^0 becomes

$$ds^2 \approx 4q^{1/4}d\rho^2 + q^{3/4}(\sigma_1)^2 + \frac{q^{-1/4}}{4}\left((\sigma_3)^2 + (\sigma_2)^2\right) \tag{38}$$

The metric is singular at $q = 0$, but assintotically, two of the coefficients of the metric (27) become equal. In this case, the isometry grows to $SO(3) \times SO(2)$[21], where $SO(2)$ action corresponds to the axial symmetry of 2 1-monopole solution and it corresponds to the S^1 isotropy subgroup of the spectral curve[3]. In other words, the assintotic behavior of the metric confirms the behavior of the spectral curve at $r_1 = 0$.

Furthermore, at infinite ρ distance, the $SO(3) \times SO(2)$ orbit is a 2-torus Hopf fibration of the 3-sphere S^3, which confirms the fact that the manifold M_2^0 is an asymptotically locally Euclidean (ALE) space. This fact together with self-duality equations, characterizes it as a gravitational instanton configuration[20]. These are elements to take in consideration for finding metrics of $M_k^0, k > 2$.

VI. CONCLUSION

We hope this article pave the way for future contributions in understanding the moduli M_k^0 of SU(2) monopoles in YMH theory. Among the obstacles, there are the growing computational challenge of Gauss Manin Connection in Disguise for larger k and the need to understand the homomorphism between vector fields in the enhanced spectral curves and curvature equations in the moduli M_k^0.

One interesting aspect to be touched is the fact that the universal families of curves can be written using different choices of parametrization, which yield different set of differential equations with different algebraic group of transformations of the moduli (where lattice scaling is one of the operations)[6] (Chapter 6). In the well known case of elliptic curves, the different choices of parametrization of the universal families for the enhanced elliptic curves takes place according to the choices of congruence subgroups Γ of the modular group $SL_2(\mathbb{Z})$[6].The moduli parametrization of the enhanced curves are lifted to modular-type functions under algebraic group action in the moduli with distinct weights. The current construction of spectral curves of k-monopoles leads to Ramanujan type of parametrization with parameters with distinct scaling weights, while it is expected that the curvature equations from M_k^0 leads to Darboux-Halphen type of parametrization with parameters with same (scaling) weight. Therefore, another step to this projects is finding new modular-type functions attached to S_k curves that will play a role defining the metric of M_k^0. Such roles will depend of the symmetries of the moduli that may define the behavior of the metric under algebraic group reparametrizations of the moduli $\mathtt{T}$ of the enhanced curve. The results of M_2^0 suggest that the terms of the metric will be a (quasi-)homogeneous polynomial or rational functions of modular-type functions that are simply the coordinates of the moduli $\mathtt{T}$ satisfying a unique vector field equation in $\mathtt{T}$[7].

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