Corners always scatter

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1 Abstract

We study time harmonic scattering for the Helmholtz equation in $\mathbb{R}^n$. We show that certain penetrable scatterers with rectangular corners scatter every incident wave nontrivially. Even though these scatterers have interior transmission eigenvalues, the relative scattering (a.k.a. far field) operator has a trivial kernel and cokernel at every real wavenumber.

2 Introduction

The diffraction of light around corners and edges, and through slits, provided the first evidence for the wave nature of light. The diffraction patterns caused by plane waves incident on corners or edges and were among the first scattered waves to be calculated [14]. Geometric optics expansions for scattered waves [9] reveal the presence of scattered waves in regions where the simple theory of optics does not. Much of our understanding of classical electromagnetism is based on these patterns. This is why a stealth airplane is built to minimize the scattering from corners and edges.

Although the single frequency inverse scattering problem has a unique solution, the wave scattered from a single incident wave does not contain enough information to determine an obstacle or a penetrable scatterer. In many cases, the same scattered wave might have been scattered by a scatterer supported on a smaller set. In this paper we will show that, a penetrable scatterer whose support contains a right angle corner as an extreme point of its convex hull

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will scatter any incident wave nontrivially. This result has the following consequence. Suppose one scatterer contains such a right angle corner and a second scatterer does not contain the corner point in its convex hull. Then the two scatterers have no scattered wave in common. The ranges of their scattering operators are disjoint.

The same is not true for a compactly supported obstacle. A square with sidelength π has Dirichlet eigenfunctions

$$\begin{aligned} v(x,y) &= 4\sin(nx)\sin(my) \\ &= e^{i(nx+my)} + e^{-i(nx+my)} - e^{i(nx-my)} - e^{-i(nx-my)} \end{aligned}$$

which means that a sum of four plane waves incident on this sound soft obstacle produces no scattered wave. Even though the obstacle has corners, it is invisible to this incident pattern.

For a penetrable scatterer, the *interior transmission eigenvalues* play the same role that the Dirichlet eigenvalues play for the sound soft obstacle. Any compactly supported L^∞ scatterer with positive contrast has infinitely many *interior transmission eigenvalues*. This implies the existence of wavenumbers k for which there exist L^2 incident waves defined on the support of the scatterer, which produce no scattered wave.

In the spherically symmetric case, the existence of such wavenumbers has been known for a long time [6, 7]. In this case, the corresponding incident waves extend to $\mathbb{R}^n$ as Herglotz wavefunctions, so the classical relative scattering operator has a nontrivial kernel. This is significant because many reconstruction algorithms in inverse scattering theory, such as the linear sampling method of Colton and Kirsch [4], and the factorization method of Kirsch [10], will work correctly only if the kernel and cokernel of the relative scattering operator is trivial.

The existence of finitely many interior transmission eigenvalues for general (non-spherically symmetric) scatterers with positive contrast was first shown in [11] in 2008, extended to infinitely many in [3] in 2010, and generalized to higher order operators in [8]. If the support contains a right angle corner, we prove that these incident waves cannot extend to any open neighborhood of the corner. The interpretation is that these incident waves could only be produced by sources located on the boundary of the scatterer, but not by any combination of sources located outside an open neighborhood of the scatterer. One particular corollary is that the linear sampling and factorizations methods, which utilize only Herglotz wavefunctions as incident waves, will work successfully for such scatterers.

Our analysis relies on two new theorems that are of independent interest. We give a new construction of the so-called complex geometric optics solutions for the Helmholtz equation, combining the techniques of Agmon-Hormander [1] and Ruiz [13] to work in L^p based Besov spaces. This allows us to improve the local regularity of these solutions without sacrificing the decay as a function of complex frequency.

The second theorem states the the Laplace transform of a harmonic polynomial cannot vanish identically on its complex characteristic variety $\{\zeta \,\big|\, \zeta\cdot\zeta = 0\}$. This is a generalization of the well-known fact that the Fourier Transform of the solution to a homogeneous constant coefficient partial differential equation is supported on the real characteristic variety of the differential operator, so that it cannot vanish on that set unless it is identically zero. Although the support statement cannot be true for the Laplace transform because it is an analytic function, we show, in the special case of the Laplacian, that only the zero harmonic polynomial can vanish identically on this variety. A proof of this theorem for a general second order elliptic operator with constant coefficients would remove the restriction of our results to right angle corners.

The classical scattering of time harmonic waves by a penetrable medium can be modeled by the Helmholtz equation

$$(\Delta + k^2 n^2)u = 0 \quad \text{in } \mathbb{R}^n,$$

where $n(x)$ denotes the index of refraction. In this model, we seek the total wave as

$$u = v^0 + u^+$$

where v^0 is the *incident wave* and u^+ the outgoing *scattered wave*. This means that

$$(\Delta + k^2)v^0 = 0 \quad \text{in } \mathbb{R}^n \tag{1}$$

and therefore that

$$(\Delta + k^2)u^+ = k^2 m(v^0 + u^+) \tag{2}$$

We assume that the *contrast* m, defined by

$$n^2 = 1 - m,$$

is compactly supported. The relative scattering operator maps the asymptotics of Herglotz incident waves to the asymptotics of scattered waves. A Herglotz incident wave is defined to be a solution to (1) of the form

$$v^0(x) = \int_{S^{n-1}} g_0(\theta) e^{ik\theta\cdot x} d\sigma(\theta),$$

for some $g_0 \in L^2(S^{n-1})$. The Herglotz incident waves can be characterized as the solutions to (1) whose Fourier transforms belong to the Besov space $B^{-1/2}_{2,\infty}(\mathbb{R}^n)$ [1][1]. These incident waves have well-defined asymptotics at infinity

$$v^0(r\theta) \sim \frac{e^{ikr}}{(ikr)^{\frac{n-1}{2}}} g_0(\theta) + \frac{e^{-ikr}}{(-ikr)^{\frac{n-1}{2}}} g_0(-\theta).$$

[1] We will give these defnitions and make use of these norms in section 6. See also[15, 17]

The scattered wave u^+ also has asymptotics at infinity

$$u^+(r\theta) \sim \frac{e^{ikr}}{(ikr)^{\frac{n-1}{2}}}\alpha^+(\theta)$$

and the relative scattering operator $S(k)$ maps

$$S(k) : L^2(S^{n-1}) \ni g_0 \mapsto \alpha^+ \in L^2(S^{n-1}).$$

For each k, the operator $S(k)$ is compact and normal; it never has a bounded inverse, but a number of methods in inverse scattering succeed only if the kernel, and hence cokernel, of $S(k)$ are trivial. If the contrast $m(x)$ in (2) is compactly supported, then a nontrivial kernel implies that k^2 is an *interior transmission eigenvalue* (ITE) for any domain Ω that contains the support of m in its interior. This means that there are nontrivial u^+ and v^0 satisfying

$$(\Delta + k^2)v^0 \quad = \quad 0 \quad \text{in } \Omega \tag{3}$$

$$(\Delta + k^2(1-m))u^+ \quad = \quad k^2 m v^0 \quad \text{in } \Omega \tag{4}$$

$$u^+|_{\partial\Omega} = 0, \qquad \tfrac{\partial u^+}{\partial \nu}|_{\partial\Omega} = 0 \tag{5}$$

In general, the eigenfunctions u^+ belong to $H_0^2(\Omega)$, and therefore extend to all of $\mathbb{R}^n$ as a function which is zero outside Ω. The waves v^0, in general, are only known to satisfy $v^0 \in L^2(\Omega)$ and $\Delta v^0 \in L^2(\Omega)$. We will refer to v_0 as an *interior incident wave*, to emphasize that it is only defined in Ω. In particular, the ITE's depend on both m and Ω. If $m(x)|_\Omega > 0$, there exist infinitely many real ITE's. However, for the same scatterer m, and a slightly larger domain Ω_1, there may exist no real ITE's, because the interior incident waves may not extend to Ω_1.

In the spherically symmetric case ($m = m(|x|)$), every interior incident wave v^0 extends to $\mathbb{R}^n$ as a spherical harmonic times a Bessel function, which is a Herglotz wavefunction, so that the relative scattering operator genuinely has a nontrivial kernel and cokernel. We will say that k is a *non-scattering wavenumber* whenever the relative scattering operator $S(k)$ has a nontrivial kernel. Although all scatterers with positive contrasts have infinitely many real ITE's (with Ω equal to the support of the contrast), no non-spherically symmetric scatterers are known to have *non-scattering wavenumbers*.

In this paper, we show that, if the contrast $m(x)$ is the characteristic function of an n-dimensional rectangle times a smooth function which is nonzero at at least one corner of the rectangle, any non-scattering interior incident wave v^0 does not extend, as a solution to (1), to any open neighborhood of the corner. In particular, no such scatterer can have non-scattering wavenumbers.

3 All Corners Scatter

The theorem below applies to scatterers whose support contains a corner (a standard right angle corner) as an extreme point of its convex hull (i.e. there exists a hyperplane which touches the support of the scatterer at precisely that corner). We describe this condition in item i) below by stating that m is the product of a smooth function and the characteristic function of a rectangle.

Theorem 3.1. *Suppose that $k \neq 0$, that K is an n-dimensional rectangle, and*

i) $m = \chi_K \varphi(x)$ with $\varphi \in C^\infty(\mathbb{R}^n)$ and $\varphi(x_0) \neq 0$ where x_0 is a corner of K

ii) the pair (u^+, v^0) are interior transmission eigenfunctions of m in $\Omega = \operatorname{supp} m$, i.e solutions to (3-4-5)

then v^0 cannot be extended as an incident wave(i.e a solution to (1)) to any open neighborhood of the corner.

Corollary 3.2. *A scatterer m which satisfies item i) has no non-scattering wavenumbers.*

Proof. If the kernel of $S(k)$ is nontrivial, then there is a Herglotz wavefunction v^0 satisfying (1) in $\mathbb{R}^n$, and an outgoing u^+ satisfying (2) in $\mathbb{R}^n$ with vanishing far field α^+. Rellich's lemma and unique continuation [5] guarantee that u^+ vanishes outside the support of m. It follows from the fact that $m \in L^\infty$ and $v^0 \in L^2$ that $u^+ \in H^2_{loc}(\mathbb{R}^n)$, and therefore the restriction of u^+ and its first derivative to $\partial\Omega$ must vanish. Hence the pair (u^+, v^0) are interior transmission eigenfunctions in Ω, but v^0 extends past the corner, contradicting Theorem 3.1. ☐

We summarize our proof of Theorem 3.1 in the following paragraph. We will make use of some *complex geometric optics* solutions to the homogeneous version of (4). Specifically, if we multiply equation (4) by any solution w to

$$\big(\Delta + k^2(1-m)\big)w = 0 \tag{6}$$

and integrate by parts, using the fact that u^+ and its first derivatives vanish on $\partial\Omega$, we see that

$$\int_K wk^2mv^0 = 0 \tag{7}$$

Theorem 3.3 below shows that we may choose w to be exponentially decaying as we move into Ω from the corner, so that the main contribution to the integral occurs at the corner. If v^0 could be extended to a neighborhood of the corner, its Taylor series would necessarily begin with a harmonic polynomial (Lemma 3.4), and the dominant term in the integral would come from the decaying exponential times that harmonic polynomial. This would then imply that the Laplace transform of this harmonic polynomial vanished on the complex characteristic variety associate to the Laplacian, and we devote Section 7 to the proof of Theorem 3.5, which says that this cannot be so.

The complex geometric optics solutions we use go back to [16]. There have been many improvements since then, but none provide enough local regularity to show that their contributions to the integral in (7) are dominated by the Laplace transform of the harmonic polynomial. Therefore, we give a new construction in Section 5, combining the L^p techniques in [13] with the geometric L^2 based constructions in [1] to prove

Theorem 3.3. *Suppose that $m(x)$ satisfies i) in Theorem 3.1. For any bounded domain D, and any $2 \leq p < \infty$, there exist constants C and r such that if $\rho \in \mathbb{C}^n$ and satisfies $\rho \cdot \rho = 0$ and $|\rho| > r$, there exists w satisfying (6) in D of the form*

$$w = e^{-x\cdot\rho}(1+\psi) \tag{8}$$

with

$$\|\psi\|_{L^p(D)} \leq \frac{C}{|\rho|} \tag{9}$$

It is the statement $2 \leq p < \infty$ that differentiates Theorem 3.3 from previous constructions. We will need to choose $p > n$, while maintaining the first power of $|\rho|$ in the denominator for our proof to succeed.

The simple lemma below notes that the first term in the Taylor series of an incident wave at an interior point is a harmonic polynomial.

Lemma 3.4. *Suppose that $v^0 \not\equiv 0$ and x_0 is in an open set where $(\Delta + k^2)v^0 = 0$. Then the lowest order homogeneous polynomial in the Taylor series for v^0 at x_0 is harmonic.*

Proof. The function v^0 is real analytic at x_0, so its Taylor expansion doesn't vanish. We call the lowest order polynomial P_N and v^{N+1} is the remainder.

$$\begin{aligned} v^0(x) &= P^N(x-x_0) + v^{N+1}(x) \\ \Delta v^0(x) &= \Delta P^N(x-x_0) + \Delta v^{N+1}(x) \\ &= Q^{N-2}(x-x_0) + q^{N-1}(x) \end{aligned}$$

where P^N and Q^{N-2} are homogeneous polynomials of degree N and $N-2$ respectively, and

$$\begin{aligned} \left|v^{N+1}(x)\right| &\leq c\,|x-x_0|^{N+1} \\ \left|q^{N-1}(x)\right| &\leq c\,|x-x_0|^{N-1}. \end{aligned}$$

We may assume that $N \geq 2$ as all polynomials of degree less than two are harmonic. In this case, it follows from

$$\Delta v^0 = -k^2 v_0$$

that

$$\left|Q^{N-2}(x-x_0)\right| = \left|-q^{N-1}(x) - k^2(P^N + v^{N+1})\right| \leq c\,|x-x_0|^{N-1},$$

but Q^{N-2} is homogeneous of order $N-2$, so must be zero. □

The final main ingredient, which we will prove in Section 7, concerns the Laplace transform of a homogeneous harmonic polynomial, i.e.

$$\widehat{P}(\rho) := \int_{x>0} e^{-x\cdot\rho} P^N(x)dx$$

where the notation $x > 0$ means that every component of x is greater than 0. We also use the notation $\frac{1}{\rho}$ to denote the vector in $\mathbb{C}^n$ whose components are the reciprocals of the components of ρ.

Theorem 3.5. *The Laplace transform of a nonzero degree N homogeneous harmonic polynomial on $\mathbb{R}^n$ is a degree $N+n$ homogeneous polynomial $Q^{N+n}(\frac{1}{\rho})$ of the reciprocals of the transform variables. If $n \geq 3$, it cannot vanish identically on any open subset of the variety $\rho \cdot \rho = 0$. If $n = 2$, it cannot vanish identically on both an open subset of $\rho_1 = i\rho_2$ and an open subset of $\rho_1 = -i\rho_2$.*

Theorem 3.1 is now a fairly direct consequence.

Proof of Theorem 3.1. Without loss of generality, we will assume that the rectangle is located in the positive orthant $\{x_j > 0\}$, that $x = 0$ is the corner at which m doesn't vanish, and that $m(0) = 1$. We choose $\rho \in \mathbb{C}^n$ satisfying $\rho \cdot \rho = 0$ and such that the real part of each component $\rho_j > \frac{1}{2\sqrt{n}}$. This guarantees that for each x in the positive orthant

$$-\Re x \cdot \rho < -\tau|x||\rho|$$

with $\tau = \frac{1}{2\sqrt{n}}$. Note that the set of ρ that satisfy this condition is an open subset of the variety $\rho \cdot \rho = 0$. Hence, for $n \geq 3$, the Laplace transform of any harmonic polynomial does not vanish at at least one such ρ. For $n = 2$, we note that, an open subset of $\rho \cdot \rho = 0$ contains an open subset of either $\rho_1 = i\rho_2$ or an open subset of $\rho_1 = -i\rho_2$. Our harmonic polynomial cannot vanish on both. If it vanishes on one of these, we change ρ to its complex conjugate $\overline{\rho}$, which is in the other, and has the same real part.

We insert the w from Theorem 3.3, with this ρ into (7), obtaining

$$0 = \int_{\Omega} e^{-x\cdot\rho}(1+\psi)mv_0. \tag{10}$$

Outside a disk of radius ϵ of the corner, the contribution is exponentially small

$$\left| \int_{\Omega\setminus N_\epsilon} e^{-x\cdot\rho}(1+\psi)mv^0 \right| \leq e^{-\tau\epsilon|\rho|}||1+\psi||_2||mv^0||_2 \leq Ce^{-\tau\epsilon|\rho|}$$

if we choose τ large enough. Inside the ϵ neighborhood, we expand v^0 as in Lemma 3.4, obtaining

$$\int_{N_\epsilon} e^{-x\cdot\rho}(1+\psi)m\big(P^N(x) + v^{N+1}(x)\big)$$

We now rewrite (10) as

$$\int_{N_\epsilon} e^{-x\cdot\rho} m P^N = \int_{N_\epsilon} e^{-x\cdot\rho} m Q^{N+1}\widetilde{v}^{N+1} + \int_{N_\epsilon} e^{-x\cdot\rho}\psi m(P^N + Q^{N+1}\widetilde{v}^{N+1}) + \int_{\Omega\setminus N_\epsilon} e^{-x\cdot\rho}(1+\psi) m v^0 \quad (11)$$

where we have rewritten $v^{N+1} = -Q^{N+1}\widetilde{v}^{N+1}$ as a homogeneous polynomial times an analytic function $\widetilde{v}^{N+1}$. Note that $\widetilde{v}^{N+1}$ remains bounded in N_ϵ because v_0 is analytic in a full neighborhood of the corner point. The following lemma tells us how the first two terms on right hand side of (11) decay as $|\rho| \to \infty$.

Lemma 3.6. *Let $R^N(x)$ be a homogeneous polynomial of degree N and $\Re\rho_j > 0$ for all $j = 1, \ldots, n$. Then, for any $f \in L^p$*

$$\left|\int_{x>0} e^{-x\cdot\rho} R^N(x) f(x) dx\right| \le C\,|\rho|^{-(N+n)+n/p}\,\|f\|_{L^p}$$

Proof. Let $\rho = s\cdot\theta$, where $\theta \in \mathbb{C}^n$, $|\theta| = 1$ and $s > 0$. Then

$$\begin{aligned}
\int_{x>0} e^{-sx\cdot\theta} R^N(x) f(x) dx &= \frac{1}{s^{N+n}}\int_{y>0} e^{-y\cdot\theta} R^N(y) f\left(\tfrac{y}{s}\right) dy \\
&\le \frac{1}{s^{N+n}}\left\|e^{-y\cdot\theta} R^N(y)\right\|_{L^q}\left\|f\left(\tfrac{y}{s}\right)\right\|_{L^p} \\
&= \frac{C_{\theta,n,q,R}}{s^{N+n}} s^{n/p}\,\|f\|_{L^p}
\end{aligned}$$

□

The lemma gives us a bound on the first two terms on the right hand side of (11)

$$\left|\int_{N_\epsilon} e^{-x\cdot\rho} m Q^{N+1}\widetilde{v}^{N+1}\right| + \left|\int_{N_\epsilon} e^{-x\cdot\rho}\psi m(P^N + Q^{N+1}\widetilde{v}^{N+1})\right| \le \frac{C}{|\rho|^{N+n+1}}\left\|m\widetilde{v}^{N+1}\right\|_{L^\infty} + \frac{C}{|\rho|^{N+n-n/p}}\left\|\psi m\widetilde{v}^{N+1}\right\|_{L^p}, \quad (12)$$

which combines with (9) to yield

$$\le \left\|m\widetilde{v}^{N+1}\right\|_{L^\infty}\left(\frac{C}{|\rho|^{N+n+1}} + \frac{C}{|\rho|^{N+n-n/p}}\cdot\frac{C}{|\rho|}\right) \le \frac{C}{|\rho|^{N+n+(1-n/p)}}$$

Theorem 3.3 allows us to choose any $2 \le p < \infty$, say $p = 2n$, so the right hand side of (12) is bounded by[2]

$$\le \frac{C}{|\rho|^{N+n+1/2}}.$$

[2]This is where we make essential use of the L^p estimates with $p > 2$ for ψ in Theorem 3.3. We need $1 - \frac{n}{p}$ to be positive in order to show that these terms are dominated by the Laplace transform of the harmonic polynomial, which is bounded from below by $|\rho|^{-(N+n)}$.

and consequently

$$\left| \int_{N_\epsilon} e^{-x\cdot\rho} m P^N \right| \leq \frac{C}{|\rho|^{N+n+1/2}} + Ce^{-\tau|\rho|\epsilon} \tag{13}$$

Because $m(x) - 1$ vanishes at $x = 0$, we have

$$\begin{aligned} \left| \int_{x>0} e^{-x\cdot\rho} P^N(x)\big(m(x) - 1\big)dx \right| &= \left| \int_{x>0} e^{-x\cdot\rho} \widetilde{Q}^{N+1}(x)\widetilde{m}(x)dx \right| \\ &\leq \frac{C}{|\rho|^{N+n+1}} \|\widetilde{m}\|_{L^\infty} \end{aligned}$$

combining with equation (13) implies

$$\left| \int_{N_\epsilon} e^{-x\cdot\rho} P^N \right| \leq \frac{C}{|\rho|^{n+N+1/2}} + Ce^{-\tau|\rho|\epsilon}$$

On the other hand, Theorem 3.5 tells us that

$$\left| \int_{x>0} e^{-x\cdot\rho} P^N(x)dx \right| \geq \frac{C}{|\rho|^{N+n}}$$

with C nonzero after a suitable choice of $\theta = \rho/|\rho|$, and consequently that

$$\left| \int_{N_\epsilon} e^{-x\cdot\rho} P^N(x)dx \right| \geq \frac{C}{|\rho|^{N+n}} - Ce^{-\tau|\rho|\epsilon}$$

Hence we arrive at the contradiction that

$$\frac{C}{|\rho|^{N+n}} \leq \left| \int_{N_\epsilon} e^{-x\cdot\rho} P^N(x)dx \right| \leq \frac{C}{|\rho|^{N+n+1/2}}$$

for all large $|\rho|$ and the theorem is proved. □

It remains to prove Theorem 3.3 and Theorem 3.5, which are the subjects of Section 5 and Section 7.

4 Estimates for Fundamental Solutions

The proof of Theorem 3.3 will rely on an estimate of the solution to

$$P_\rho(D)\psi := (\Delta - 2\rho\cdot\nabla)\psi = f. \tag{14}$$

Although we will work in different norms, we follow the outline in [13] and begin by estimating the convolution $||\chi_\varepsilon * g||_{L^\infty}$ where χ is a Schwartz class function,

$$g(\xi) = \frac{1}{P_\rho(\xi)} \qquad \text{and} \qquad \chi_\varepsilon(\xi) = \frac{1}{\varepsilon^n}\chi\left(\frac{\xi}{\varepsilon}\right)$$

We will prove these estimates for a fairly general P, using a geometric approach similar to that in [1]. The key properties of the symbol $P(\xi)$ are the codimension of its characteristic variety (the set $\mathscr{M} = P^{-1}(0)$) and the order to which it vanishes as $\xi \to \mathscr{M}$. The dimension of $\mathscr{M}$ tells us the behavior of the solutions to the homogeneous differential equation, while the order of vanishing tells us the behavior of the the particular solutions $G * f$. In the case of equation (14), the codimension is 2 and P vanishes simply on $\mathscr{M}$.

Theorem 4.1. *Suppose that $\chi(x) \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ and $\chi_\varepsilon(x) := \varepsilon^{-n}\chi(\frac{x}{\varepsilon})$. If $P(\xi)$ satisfies*

i) $P : \mathbb{R}^n \to \mathbb{R}^k$ *is smooth*

ii) $\mathscr{M} = P^{-1}\{0\}$ *is compact*

iii) $DP_{|\mathscr{M}}$ *has constant rank, and*

iv) $\liminf_{|\xi|\to\infty} |P| \geq B > 0$

then

a) $\mathscr{M}$ *is a smooth embedded codimension k manifold in $\mathbb{R}^n$*

b) $\|\chi_\varepsilon * \delta_{\mathscr{M}}\|_{L^\infty} \leq \frac{C}{\varepsilon^k}$

c) If P is real or complex valued ($k = 1$ or 2), then

$$\left\| \chi_\varepsilon * \frac{1}{P} \right\|_{L^\infty} \leq \frac{C}{\varepsilon}.$$

Moreover, if $k \geq 2$ and F is a complex valued function satisfying $|F(P)| \leq \frac{1}{|P|}$ then

$$\|\chi_\varepsilon * F(P)\|_{L^\infty} \leq \frac{C}{\varepsilon}.$$

Remark 4.2. We define

$$\langle \delta_{\mathscr{M}}, \phi \rangle := \int_{\mathscr{M}} \phi d\sigma_{\mathscr{M}}$$

where $d\sigma_{\mathscr{M}}$ is the natural element of surface area on $\mathscr{M}$.

Remark 4.3. If $k \geq 2$ then $\frac{1}{P} \in L^1_{loc}$ is a well defined distribution on the whole of $\mathbb{R}^n$. If $k = 1$ we will use the principal value

$$\langle \frac{1}{P}, \phi \rangle := \int_{N_\delta(\mathscr{M})} (\phi(y) - \phi(m(y))) \frac{dy}{P(y)} + \int_{\mathbb{R}^n \setminus N_\delta(\mathscr{M})} \phi(y) \frac{dy}{P(y)},$$

where $N_\delta(\mathscr{M})$ is a neighborhood of $\mathscr{M}$ and $m(y)$ associates with each $y \in N_\delta(\mathscr{M})$ the closest point in $\mathscr{M}$. Both of which are described more explicitly in the proposition below.

The following proposition recalls some immediate consequences of the implicit function theorem. We don't include a proof.

Proposition 4.4. *Suppose that i), ii) and iii) in Theorem 4.1 are satisfied. Then*

A) $DP|_{\mathscr{M}}$ *has full rank* k

B) $\mathscr{M}$ *is a smooth compact embedded submanifold of* $\mathbb{R}^n$

C) $\exists\ \delta > 0$ *and a Lipschitz constant* L_δ *such that writing*

$$N_\delta(\mathscr{M}) = \{x \in \mathbb{R}^n \mid d(x, \mathscr{M}) \leq \delta\},$$

every $x \in N_\delta(\mathscr{M})$ *has a unique closest point* $m(x)$ *in* $\mathscr{M}$*. The map*

$$\eta : N_\delta(\mathscr{M}) \to \mathscr{M} \times B^k_\delta(0)$$

defined by

$$\eta(x) = \left(m(x), |x - m(x)| \, \frac{DP_{m(x)}\,(x - m(x))}{\left|DP_{m(x)}\,(x - m(x))\right|} \right) \tag{15}$$

is a global diffeomorphism from $N_\delta(\mathscr{M})$ *onto* $\mathscr{M} \times B^k_\delta(0)$*. Both* η *and* η^{-1} *are Lipshitz with uniform constant* L_δ*.*

D) Every point $m \in \mathscr{M}$ *has a* δ*-neighborhood* $U_\delta(m) \subset \mathscr{M}$ *that is diffeomorphic to a ball in* $\mathbb{R}^{n-k}$*, i.e.*

$$\psi_m : U_\delta(m) := B^n_\delta(m) \cap \mathscr{M} \to B^{n-k}_\delta(0).$$

Both ψ_m *and* ψ_m^{-1} *are Lipschitz with uniform constant* L_δ*.*

Two corollaries (also stated without proof) are:

Corollary 4.5. *For* $x \in \mathbb{R}^n$*,*

$$\operatorname{Area}\left(B^n_r(x) \cap \mathscr{M}\right) := \int_{\mathscr{M} \cap B^n_r(x)} d\sigma_{\mathscr{M}} \leq C_\delta r^{n-k}$$

Corollary 4.6. *For* $x \in N_\delta(\mathscr{M})$*,*

$$|P(x)| \geq C_\delta d(x, \mathscr{M}).$$

We are going to use diffeomorphisms to rewrite integrals over manifolds as integrals over Euclidean balls, where we can do some explicit calculations. Since our integrals will involve convolutions with Schwartz class functions, we need to describe the properties that the pullbacks of such functions inherit.

Definition 4.7. A family of ε-mollifiers, $\chi_\varepsilon(x, y)$, defined on $\Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ satisfies

i) $\sup_{x \in \Omega_1} \int_{\Omega_2} |\chi_\varepsilon(x, y)| \, dy \leq C$

ii) $|\chi_\varepsilon(x,y)| \leq \frac{C_N}{\varepsilon^n}\left(\frac{\varepsilon}{|x-y|}\right)^N$ for all $N \in \mathbb{N}$

iii) $|\nabla_y \chi_\varepsilon(x,y)| \leq \frac{C_N}{\varepsilon^{n+1}}\left(\frac{\varepsilon}{|x-y|}\right)^N$ for all $N \in \mathbb{N}$

Lemma 4.8. *If $\chi \in \mathscr{S}$, then*

$$\chi_\varepsilon(x,y) := \frac{1}{\varepsilon^n}\chi\left(\frac{x-y}{\varepsilon}\right)$$

is a family of ε-mollifiers defined on $\Omega_1 \times \Omega_2 = \mathbb{R}^n \times \mathbb{R}^n$.

Definition 4.9. The pullback of a family of ε-mollifiers is defined[3] to be

$$\psi^*\chi_\varepsilon(x,y) := \chi_\varepsilon(\psi(x),\psi(y)). \tag{16}$$

The next lemma explains why we need to work with general ε-mollifiers.

Lemma 4.10. *If ψ and ψ^{-1} are uniformly Lipschitz diffeomorphisms, then the pullback of a family of ε-mollifiers is a family of ε-mollifiers.*

Proof. Let L_1 and L_2 be the Lipschitz constants for ψ and ψ^{-1}, respectively. For i), we estimate

$$\begin{aligned}
\sup_{x\in\psi^{-1}(\Omega_1)} \int_{\psi^{-1}(\Omega_2)} \chi_\varepsilon(\psi(x),\psi(y))dy &= \sup_{x\in\Omega_1}\int_{\Omega_2}\chi_\varepsilon(x,y)\frac{dy}{\det(D\psi(y))} \\
&\leq \sup_{x\in\Omega_1} L_2^n \int_{\Omega_2}\chi_\varepsilon(x,y)dy
\end{aligned}$$

Next

$$|\chi_\varepsilon(\psi(x),\psi(y))| \leq \frac{C_N}{\varepsilon^n}\left(\frac{\varepsilon}{|\psi(x)-\psi(y)|}\right)^N \leq \frac{C_N L_2^N}{\varepsilon^n}\left(\frac{\varepsilon}{|x-y|}\right)^N.$$

Finally, for iii),

$$\begin{aligned}
|\nabla_y\chi_\varepsilon(\psi(x),\psi(y))| &= |D\psi\cdot\nabla_v\chi_\varepsilon(u,v)|_{\substack{u=\psi(x)\\ v=\psi(y)}} \\
&\leq L_1\frac{C_N}{\varepsilon^{n+1}}\left(\frac{\varepsilon}{|\psi(x)-\psi(y)|}\right)^N \leq \frac{C_N L_1 L_2^N}{\varepsilon^{n+1}}\left(\frac{\varepsilon}{|x-y|}\right)^N
\end{aligned}$$

□

Proposition 4.11. *Let χ_ε be a family of ε-mollifiers defined on $\Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and $\mathscr{M}$ a compact embedded submanifold of $\mathbb{R}^n$ of codimension k. Then*

$$\sup_{x\in\Omega_1}\int_{\mathscr{M}\cap\Omega_2}|\chi_\varepsilon(x,m)d\sigma_{\mathscr{M}}(m)| \leq \frac{C}{\varepsilon^k}$$

for small ε.

[3] It seems natural to include a factor of $\det(D\psi)$ in (16), to treat $\chi_\varepsilon dx_1 \wedge \cdots \wedge dx_n$ as an n-form. We don't add the factor because it makes the proof of Lemma 4.10 slightly longer.

Proof. We may assume that $\mathscr{M} \subset \Omega_2$. Let δ be the uniform constant in Proposition 4.4. Fix $x \in \Omega_1$ and assume that $\varepsilon < \delta$. According to ii) in Definition 4.7 we have

$$\left| \int_{\mathscr{M} \cap \{m | |x-m| \geq \delta\}} \chi_\varepsilon(x,m) d\sigma_{\mathscr{M}}(m) \right| \leq \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{\delta} \right)^N \text{area}(\mathscr{M}).$$

On the other hand

$$\begin{aligned} \left| \int_{\mathscr{M} \cap \{m | |x-m| \leq \varepsilon\}} \chi_\varepsilon(x,m) d\sigma_{\mathscr{M}}(m) \right| &\leq \frac{C_0}{\varepsilon^n} \text{area}\left(\mathscr{M} \cap B_\varepsilon^n(x)\right) \\ &\leq \frac{C_0}{\varepsilon^n} L_\delta^{n-k} \varepsilon^{n-k} = \frac{C_0 L_\delta^{n-k}}{\varepsilon^k}, \end{aligned}$$

where L_δ is the Lipschitz constant. To estimate the remaining part of the integral, we use local coordinates ψ, based at $m(x)$, the point on $\mathscr{M}$ closest to x, as described in Proposition 4.4 D). Let $\Psi = \psi^{-1}$. Then

$$\left| \int_{\mathscr{M} \cap \{m | \varepsilon < |x-m| < \delta\}} \chi_\varepsilon(x,m) d\sigma_{\mathscr{M}}(m) \right| = \left| \int_{B_\delta^{n-k}(0) \setminus B_\varepsilon^{n-k}(0)} \Psi^* \chi_\varepsilon \Psi^* d\sigma_{\mathscr{M}} \right|.$$

Because

$$\begin{aligned} |\chi_\varepsilon(x,m)| &\leq \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{(|x-m(x)|^2 + |m(x)-m|^2)^{1/2}} \right)^N \\ &\leq \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{|m(x)-m|} \right)^N = \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{\rho} \right)^N, \end{aligned}$$

where $\rho = |m(x) - m|$, we may use polar coordinates centered at $m(x)$ to see that

$$\begin{aligned} \int_{B_\delta^{n-k}(0) \setminus B_\varepsilon^{n-k}(0)} |\Psi^* \chi_\varepsilon \Psi^* d\sigma_{\mathscr{M}}| &\leq L_\delta^{n-k} \int_{S^{n-k-1}} d\sigma_{S^{n-k-1}} \int_\varepsilon^\delta \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{\rho} \right)^N \rho^{n-k-1} d\rho \\ &\leq L_\delta^{n-k} \omega_{n-k-1} C_N \varepsilon^{N-n} \frac{|\delta^{n-k-N} - \varepsilon^{n-k-N}|}{|n-k-N|} \\ &\leq L_\delta^{n-k} \omega_{n-k-1} C_N \frac{\varepsilon^{-k}}{|n-k-N|} \end{aligned}$$

where S_{n-k-1} is the unit sphere in $\mathbb{R}^{n-k}$ and ω_{n-k-1} its surface measure. The claim follows by taking $N > n - k$. $\square$

Remark 4.12. In the proof of Proposition 4.11, when considering $x \in N_\delta(\mathscr{M})$, we only required that the mollifier satisfy

$$|\chi_\varepsilon(x,y)| \leq \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{|m(x) - m(y)|} \right)^N.$$

We will use this observation in the proof of Proposition 4.13 below, which will finish the proof of Theorem 4.1.

Proposition 4.13. *Let χ_ε be a family of ε-mollifiers, $\mathscr{M}$, P and $k \geq 2$ satisfy the conditions in Theorem 4.1, and $F : \mathbb{R}^k \to \mathbb{C}$ satisfy $|F(P)| \leq \frac{C}{|P|}$. Then, for sufficiently small ε,*

$$\left| \int_{\mathbb{R}^n} \chi_\varepsilon(x,y) F(P(y)) dy \right| \leq \frac{C}{\varepsilon}.$$

If $k = 1$ then

$$\left| \int_{\mathbb{R}^n} \frac{\chi_\varepsilon(x,y)}{P(y)} dy \right| \leq \frac{C}{\varepsilon},$$

where $\frac{1}{P}$ is defined by principal value as in Remark 4.3.

Proof. We assume that $\varepsilon < \frac{\delta}{2}$, with δ the constant in Proposition 4.4 C). Because $|F(P)| \leq \frac{C}{|P|} \leq \frac{C_\delta}{\varepsilon}$ on $N_\delta(\mathscr{M}) \setminus N_\varepsilon(\mathscr{M})$ and $\leq C_\delta$ outside $N_\delta(\mathscr{M})$,

$$\int_{\mathbb{R}^n \setminus N_\varepsilon(\mathscr{M})} |\chi_\varepsilon F(P)|\, dy \leq \sup_{y \in \mathbb{R}^n \setminus N_\varepsilon(\mathscr{M})} |F(P(y))|\, \|\chi_\varepsilon\|_{L^1} \leq \frac{C}{\varepsilon}.$$

For the moment, we restrict to the case that $k = \operatorname{codim}(\mathscr{M}) \geq 2$, so that $F(P) \in L^1(\mathbb{R}^n)$. If $x \notin N_\delta(\mathscr{M})$, then

$$\sup_{y \in N_\epsilon(\mathscr{M})} |\chi_\varepsilon(x,y)| \leq \left(\frac{\varepsilon}{\delta/2} \right)^N \frac{C_N}{\varepsilon^n}$$

so that

$$\sup_{x \notin N_\delta(\mathscr{M})} \int_{N_\varepsilon(\mathscr{M})} |\chi_\varepsilon F(P)|\, dy \leq \int_{N_\varepsilon(\mathscr{M})} |F(P)|\, dy \left(\frac{\varepsilon}{\delta/2} \right)^N \frac{C_N}{\varepsilon^n}.$$

and choosing $N \geq n-1$ shows that this bounded by a constant over ε.

If $x \in N_\delta(\mathscr{M})$, we can use the diffeomorphism η and its inverse H, described in C) of Proposition 4.4 to obtain

$$\begin{aligned} &\sup_{x \in N_\delta(\mathscr{M})} \int_{N_\varepsilon(\mathscr{M})} |\chi_\varepsilon(x,y) F(P(y))|\, dy \\ &\qquad = \sup_{u \in \mathscr{M} \times B_\delta^k(0)} \int_{\mathscr{M} \times B_\varepsilon^k(0)} |H^* \chi_\varepsilon(u,v) F(P(H(v)))| \frac{d\sigma_{\mathscr{M}}(m) ds}{|\det(D\eta)|} \end{aligned}$$

where $v = (m,s) \in \mathscr{M} \times B_\varepsilon^k(0)$. Because $|F(P(H(s)))| \leq \frac{C}{|P(y)|} \leq \frac{C}{|s|}$ here and $|\det(D\eta)|$ is bounded from below by the the n-th power of the Lipschitz constant L_2, this is bounded by

$$\leq C L_2^{-n} \int_{B_\varepsilon^k(0)} \left(\sup_{u \in \mathscr{M} \times B_\delta^k(0)} \int_{\mathscr{M}} |H^* \chi_\varepsilon|\, d\sigma_{\mathscr{M}} \right) \frac{1}{|s|} ds. \tag{17}$$

For each fixed s,

$$|H^* \chi_\varepsilon| \leq \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{|u - (m,s)|} \right)^N,$$

so according to Remark 4.12 we can apply Proposition 4.11 to the manifold $\mathscr{M} \times \{s\}$ to show that the quantity in brackets in (17) satisfies

$$\sup_{u \in \mathscr{M} \times B^k_\delta(0)} \int_{\mathscr{M}} |H^* \chi_\varepsilon| \, d\sigma_{\mathscr{M}} \leq \frac{C}{\varepsilon^k}.$$

This implies the estimate

$$\sup_{x \in N_\delta(\mathscr{M})} \int_{N_\varepsilon(\mathscr{M})} |\chi_\varepsilon(x,y) F(P(y))| \, dy \leq \int_{B^k_\varepsilon(0)} \frac{C}{\varepsilon^k} \frac{ds}{|s|} = \frac{C}{\varepsilon^k} \cdot \varepsilon^{k-1},$$

which completes the proof in the codimension 2 case.

If $\mathscr{M}$ is of codimension one we have the definition

$$\langle \frac{1}{P}, \phi \rangle = \int_{N_\delta(\mathscr{M})} (\phi(y) - \phi(m(y))) \frac{dy}{P(y)} + \int_{\mathbb{R}^n \setminus N_\delta(\mathscr{M})} \phi(y) \frac{dy}{P(y)}$$

and note that this agrees with $\int_{\mathbb{R}^n} \phi \frac{dy}{P}$ for all $\phi \in C_0^\infty(\mathbb{R}^n \setminus \mathscr{M})$. With this definition,

$$\frac{1}{P} * \chi_\varepsilon = \int_{\mathbb{R}^n \setminus N_\varepsilon} \chi_\varepsilon \frac{dy}{P} + \int_{N_\varepsilon(\mathscr{M})} \frac{(\chi_\varepsilon(x,y) - \chi_\varepsilon(x, m(y)))}{P(y)} dy.$$

We estimate the first integral as we did in the codimension ≥ 2 case, and rewrite the second as

$$\int_{\mathscr{M}} \left[\int_{-\varepsilon}^{\varepsilon} \frac{\chi_\varepsilon(m(x), \nu(x), m(y), \nu(y)) - \chi_\varepsilon(m(x), \nu(x), m(y), 0)}{P(m(y), \nu, y)} d\nu(y) \right] d\sigma_{\mathscr{M}}.$$

where $m(x)$ again denotes the closest point on $\mathscr{M}$, and $\nu(x)$ are the normal coordinates, given explicitly by the second component on the right hand side of equation (15). If we call the integral in brackets $\widetilde{\chi_\varepsilon}$, we see that

$$|\widetilde{\chi_\varepsilon}| \leq \frac{C_N}{\varepsilon^n} \left| \frac{\varepsilon}{|m(x) - m(y)|} \right|^N$$

so that Remark 4.12 applies here, and we may conclude that $\left| \int_{\mathscr{M}} \widetilde{\chi_\varepsilon} d\sigma_{\mathscr{M}} \right| \leq \frac{C}{\varepsilon^k}$ with $k = 1$ in this case. $\square$

We need only one application of Theorem 4.1 for our proof of Theorem 3.3. We return to (14) and set

$$g(\xi) = \frac{1}{-\xi \cdot \xi - 2i\rho \cdot \xi}$$

Proposition 4.14. *There is a constant C, depending only on the dimension n and $\chi \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$, so that*

$$\|\chi_\epsilon * g\|_\infty \leq \frac{C}{\epsilon \rho} \tag{18}$$

Proof. Let $\rho = s\Theta$ where $\Theta \in \mathbb{C}^n$ has unit norm and $s = |\rho|$. We will apply the estimate in item c from Theorem 4.1, but first we need do some scaling

$$\chi_\epsilon * g(s\eta) = -\int \chi\left(\frac{s\eta - \xi}{\epsilon}\right) \frac{1}{\xi \cdot \xi + 2is\Theta \cdot \xi} \frac{d^n\xi}{\epsilon^n}$$

letting $\sigma = s\xi$ gives

$$= -\frac{1}{s^2} \int \chi\left(\frac{\eta - \sigma}{\frac{\epsilon}{s}}\right) \frac{1}{\sigma \cdot \sigma + 2i\Theta \cdot \sigma} \frac{d^n\sigma}{(\frac{\epsilon}{s})^n}$$

$$= \frac{1}{s^2} \chi_{\frac{\epsilon}{s}} * \frac{1}{\widetilde{P}}$$

where $\widetilde{P} = -\xi \cdot \xi - 2i\Theta \cdot \xi$. According to Theorem 4.1,

$$\|\chi_\epsilon * g(s\eta)\|_\infty \leq \frac{1}{s^2} \frac{C}{\frac{\epsilon}{s}} \leq \frac{C}{s\epsilon}$$

Recalling that $s = |\rho|$, and that $\|\chi_\epsilon * g(s\eta)\|_\infty = \|\chi_\epsilon * g\|_\infty$ gives

$$\|\chi_\epsilon * g\|_\infty \leq \frac{C}{\epsilon|\rho|}$$

□

5 Proof of Theorem 3.3

In order to prove Theorem 3.3, we insert the ansatz (8) into (6) to see that ψ must satisfy

$$(\Delta - 2\rho \cdot \nabla)\psi = -k^2(1 - m)(1 + \psi) \quad \text{in } D. \tag{19}$$

We replace the right hand side of (19) using

$$Q = -k^2(1 - m)\Phi_D$$

where Φ_D is smooth, compactly supported, and identically equal to one on the bounded domain D. We seek ψ satisfying

$$(\Delta - 2\rho \cdot \nabla)\psi = Q(1 + \psi) \qquad \text{in } \mathbb{R}^n \tag{20}$$

noting that a solution to (20) in $\mathbb{R}^n$ will satisfy (19) in an open neighborhood of D. We will construct ψ by summing the series

$$\psi = \sum_{N=0}^{\infty} \psi^N \tag{21}$$

where $\psi^0 = 0$ and the remaining ψ^N satisfy

$$(\Delta - 2\rho \cdot \nabla)\psi^N = Q\psi^{N-1} \tag{22}$$

The existence of solutions to (22) and the convergence of the sum will follow from an estimate of solutions to the constant coefficient differential equation

$$P_\rho(D)\psi := (\Delta - 2\rho\cdot\nabla)\psi = f. \tag{23}$$

The simplest estimate would follow from taking the Fourier transform of both sides and dividing by the symbol $P_\rho(\xi)$. If we use the letter $g(\xi)$ to denote the reciprocal of P_ρ, we want to estimate

$$\widehat{\psi} \quad = \quad g\widehat{f}$$

or equivalently

$$\psi \quad = \quad G * f \tag{24}$$

where $*$ denotes convolution and G is the inverse Fourier transform of $(2\pi)^{-n/2}g$. A simple L^∞ estimate for g does not hold because of the zeros of P, but these affect the behavior of ψ for large x, and our goal is to prove a strong local estimate. We are willing to prove an estimate that allows ψ to grow as $x \to \infty$ in exchange for a good local estimate, i.e. L^q for large q on compact sets. We will separate the local and global behavior by writing G, the inverse Fourier transform of g, as a sum of functions G_j with compact support, and estimating each separately.

We introduce a dyadic partition of unity. Let

$$1 = \phi_0(s) + \sum_{j=1}^{\infty}\phi_j(s) \tag{25}$$

where ϕ_0 and ϕ are C^∞ even functions of $s \in \mathbb{R}$, and

$$\operatorname{supp}\phi_0 \subset [-2,2]$$
$$\operatorname{supp}\phi \subset [\tfrac{1}{2},2]$$

and

$$\Phi_j(x) := \phi(\tfrac{|x|}{2^j}) \quad \text{for } j \geq 1, \quad \Phi_0(x) = \phi_0(|x|),$$

so that

$$\operatorname{supp}\Phi_j \subset B_{2^{j+1}}(0) \setminus B_{2^{j-1}}(0), \quad \operatorname{supp}\Phi_0 \subset B_2(0).$$

We will make use of the fact that

$$\widehat{\Phi_j}(\xi) = 2^{nj}\widehat{\Phi}(2^j\xi) = \frac{\widehat{\Phi}(\frac{\xi}{\epsilon})}{\epsilon^n}$$

which makes the $\{\widehat{\Phi_j}\}$ a family of ϵ-mollifiers with $\chi = \widehat{\Phi}$ and with $\epsilon = 2^{-j}$.

We expand ψ, G, and f with respect to this partition, i.e

$$\psi = \sum\psi_j = \sum\Phi_j\psi$$
$$f = \sum f_j = \sum\Phi_j f$$
$$G = \sum G_j = \sum\Phi_j G$$

so that (24) becomes

$$\psi_m = \Phi_m \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} G_j * f_k \tag{26}$$

If we recall that the support of the convolution is a subset of the sum of the supports, we see that if $r_1 = 2^{k-1} - 2^{j+1} > 0$ or $r_2 = 2^{j-1} - 2^{k+1} > 0$, the support of $G_j * f_k$ is contained outside the ball of radius r_1 or r_2, respectively. In particular, this means that

$$\Phi_m G_j * f_k = 0$$

if

$$2^{m+1} < 2^{j-1} - 2^{k+1}$$

which will always be the case if

$$j > 3 + \max(k, m)$$

so that the second sum in (26) is finite

$$\psi_m = \Phi_m \sum_{k=0}^{\infty} \sum_{j=0}^{\max(k,m)+3} G_j * f_k.$$

Taking the Fourier transform gives

$$\widehat{\psi_m} = (2\pi)^{-n} \widehat{\Phi_m} * \sum_{k=0}^{\infty} \sum_{j=0}^{\max(k,m)+3} g_j \widehat{f_k}$$

where $g_j = \widehat{\Phi_j} * g = (2\pi)^n \widehat{G_j}$, so that

$$\left\|\widehat{\psi_m}\right\|_p \leq (2\pi)^{-n} \left\|\widehat{\Phi_m}\right\|_1 \sum_{k=0}^{\infty} \sum_{j=0}^{\max(k,m)+3} \|g_j\|_\infty \left\|\widehat{f_k}\right\|_p$$

We may now estimate the convolution $||\widehat{\Phi_j} * g||_{L^\infty}$ using (18) of Proposition 4.14 with $\chi = \Phi$ and $\epsilon = 2^{-j}$ to establish that

$$||g_j||_\infty \leq \frac{C}{|\rho|} 2^j$$

for $|\rho|$ sufficiently large, so that

$$\left\|\widehat{\psi_m}\right\|_p \leq \left\|\widehat{\Phi_m}\right\|_1 \sum_{k=0}^{\infty} \sum_{j=0}^{\max(k,m)+3} \frac{C}{|\rho|} 2^j \left\|\widehat{f_k}\right\|_p$$

Because $\widehat{\Phi_m}(\xi) = 2^{nm} \widehat{\Phi}(2^m \xi)$, its L^1 norm is the same as the L^1 norm of $\widehat{\Phi}$, which doesn't depend on m, so

$$\begin{aligned}
\left\|\widehat{\psi_m}\right\|_p &\leq \frac{C}{|\rho|}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{\max(k,m)+3} 2^j\right)\left\|\widehat{f}_k\right\|_p \\
&\leq \frac{C}{|\rho|}\sum_{k=0}^{\infty} 2^{\max(k,m)+4}\left\|\widehat{f}_k\right\|_p
\end{aligned}$$

which we rewrite as

$$\sup_m 2^{-m}\left\|\widehat{\psi_m}\right\|_p \leq \frac{C}{|\rho|}\sum_{k=0}^{\infty} 2^k\left\|\widehat{f}_k\right\|_p \tag{27}$$

with a new constant C that is 2^4 times the old one.

Our goal is to estimate the $L^q(D)$ norm of ψ on a compact set D for $q > 2$, and this is bounded by the left hand side of (27) if we choose $p < 2$ to be the dual exponent. In our application, f will be the right hand side of (22) which will have its support in D, so the sum on the right hand side of (27) will also be a finite sum, bounded by a constant times $\left\|\widehat{f}\right\|_p$. We we will have the desired bound for ψ as long as we can guarantee that the Fourier transform of f is in L^p for all $p \leq 2$.

In the special case that $p = 2$, the Plancherel inequality tells us that (27) is equivalent to

$$\sup_m\ 2^{-m}\left\|\psi_m\right\|_2 \leq \frac{C}{|\rho|}\sum_{k=0}^{\infty} 2^k\left\|f_k\right\|_2$$

This kind of estimate was used in [1] to study constant coefficient PDE's with simple characteristics, including, as the principal example, the free Helmholtz equation. The norms defined there were:

$$\begin{aligned}
\|f\|_{B^*_{k/2}} &:= \sup_{0\leq j<\infty}\frac{1}{2^{jk/2}}\left\|f_j\right\|_2 = \|\widehat{f}\|_{B^{-k/2}_{2,\infty}} \\
\|f\|_{B_{k/2}} &:= \sum_{j=0}^{\infty} 2^{jk/2}\left\|f_j\right\|_2 = \|\widehat{f}\|_{B^{k/2}_{2,1}}
\end{aligned}$$

The authors showed, in particular, that the incident waves for the Helmholtz equation in $B^*_{1/2}$ were exactly the Herglotz wave functions.

Our estimate in (27) may be written as

$$\|\psi\|_{\widehat{B^{-1}_{p,\infty}}} \leq \frac{C}{|\rho|}\|f\|_{\widehat{B^{1}_{p,1}}} \tag{28}$$

with

$$\|\psi\|_{\widehat{B^{-1}_{p,\infty}}} := \sup_{0\le j<\infty} \frac{1}{2^j} \left\|\widehat{\psi_j}\right\|_p$$

$$\|f\|_{\widehat{B^1_{p,1}}} := \sum_{j=0}^{\infty} 2^j \left\|\widehat{f}_j\right\|_p$$

In this notation we have proved

Proposition 5.1. *For every $f \in \widehat{B^1_{p,1}}$, there exists a $\psi \in \widehat{B^{-1}_{p,\infty}}$ satisfying (23) and the estimate (28).*

We now return to (22). We will show, in Lemma 5.2 below, that

$$Q \in \widehat{B^1_{p,1}} \qquad \text{and} \qquad ||Qg||_{\widehat{B^1_{p,1}}} \le C_Q ||g||_{\widehat{B^{-1}_{p,\infty}}} \tag{29}$$

where C_Q denotes a constant depending on Q. Combining (29) with (28) shows that

$$\left\|\psi^N\right\|_{\widehat{B^{-1}_{p,\infty}}} \le \frac{C_Q}{|\rho|} \left\|\psi^{N-1}\right\|_{\widehat{B^{-1}_{p,\infty}}} \le \left(\frac{C_Q}{|\rho|}\right)^N \|Q\|_{\widehat{B^1_{p,1}}}$$

and hence that the series (21) converges when $|\rho| > C_Q$ and thererefore that, for $2 < q = \frac{p}{p-1}$ and D contained in a ball of radius R, the sum ψ satisfies

$$\|\psi\|_{L^q(D)} \le R \|\psi\|_{\widehat{B^{-1}_{p,\infty}}} \le \frac{C}{|\rho|} \|Q\|_{\widehat{B^1_{p,1}}}$$

and establishes (9) for all $q > 2$ (and, because D is bounded, for $q < 2$ as well) and therefore proves Theorem 3.3.

It remains to prove (29). The function Q satisfies

$$Q = \prod_{i=1}^{n} (H^+(x_i) - H^+(x_i - 1)) q(x)$$

where q is smooth and supported in a ball of radius R, and $H^+(t)$ is the Heavyside function, the indicator function of the positive half line.

$$\|q\|_{\widehat{B^1_{p,1}}} = \sum_{j=0}^{\log_2 R} 2^j ||\widehat{\Phi_j} * \widehat{q}||_p \le 2R \sup_j \left\|\widehat{\Phi_j}\right\|_1 ||\widehat{q}||_p = 2R \left\|\widehat{\Phi}\right\|_1 ||\widehat{q}||_p$$

so $q \in \mathscr{F}B^1_{p,1}$. The lemma below tells us that multiplication by the Heavyside function preserves $\mathscr{F}B^1_{p,1}$ and that multiplication by smooth compactly supported q maps $\mathscr{F}B^{-1}_{p,\infty}$ to $\mathscr{F}B^1_{p,1}$. This is enough to establish (29) and finish this section.

Lemma 5.2. *Suppose that q is smooth and supported in the ball of radius R, and Θ a unit vector in $\mathbb{R}^n$. Then*

$$\|qg\|_{\widehat{B^1_{p,1}}} \leq 2R^2 \|\widehat{q}\|_1 \|g\|_{\widehat{B^{-1}_{p,\infty}}} \tag{30}$$

$$\left\| \big(H_+(x\cdot\Theta) - H^+(x\cdot\Theta - 1) \big) g(x) \right\|_{\widehat{B^1_{p,1}}} \leq C_p \|g\|_{\widehat{B^1_{p,1}}} \tag{31}$$

for $1 < p < \infty$.

Proof.

$$\|qg\|_{\widehat{B^1_{p,1}}} = \sum_{j=0}^{\infty} 2^j \left\| \widehat{(qg\Phi_j)} \right\|_p$$

Because q has compact support, the sum is finite, i.e.

$$= \sum_{j=0}^{\log_2 R} 2^j \|\widehat{q} * \widehat{g_j}\|_p$$

where g_j denotes $g * \Phi_j$

$$\begin{aligned}
&\leq \sum_{j=0}^{\log_2 R} 2^{2j} \|\widehat{q}\|_1 \left(2^{-j} \|\widehat{g_j}\|_p \right) \\
&\leq \left(\sum_{j=0}^{\log_2 R} 2^{2j} \right) \|\widehat{q}\|_1 \|g\|_{\widehat{B^{-1}_{p,\infty}}} \\
&\leq 2R^2 \|\widehat{q}\|_1 \|g\|_{\widehat{B^{-1}_{p,\infty}}}
\end{aligned}$$

which establishes (30). To prove (31)

$$\begin{aligned}
\left\| H^+(x\cdot\Theta) g \right\|_{\widehat{B^1_{p,1}}} &= \sum_{j=0}^{\infty} 2^j \left\| \widehat{H^+ g_j} \right\|_p \\
&= \sum_{j=0}^{\infty} 2^j \left\| \widehat{H^+} * \widehat{g_j} \right\|_p
\end{aligned}$$

but convoution with $\widehat{H^+}(\xi\cdot\Theta)$ is just a one dimensional Hilbert transform in the direction Θ, which is bounded from L^p to L^p for all $1 < p < \infty$, so that

$$\begin{aligned}
&\leq \sum_{j=0}^{\infty} 2^j C_p \|\widehat{g_j}\|_p \\
&\leq C_p \|g\|_{\widehat{B^1_{p,1}}} .
\end{aligned}$$

The same estimate holds for $H^+(x\cdot\Theta - 1)$ because rigid motions induce bounded maps from $\mathscr{F}B^1_{p,1}$ to itself. ∎

6 Function spaces

We have proved estimates using $\mathscr{F}B^1_{p,1}$ and $\mathscr{F}B^{-1}_{p,\infty}$ norms, but have, so far said very little about the function spaces, other than pointing out that smooth compactly supported functions times Heavyside functions belong to $\mathscr{F}B^1_{p,1}$, and that, for $p < 2$ and q the dual exponent, any $u \in \mathscr{F}B^{-1}_{p,\infty}$ is in L^q of every compact set.

These spaces are Fourier transforms of Besov-spaces, which are defined in [2, 15, 17] using the partition of unity in (25). For $s \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q \le \infty$,

$$B^s_{p,q} = B^s_{p,q}(\mathbb{R}^n) = \{f \in \mathscr{S}'(\mathbb{R}^n) \mid \|f\|_{B^s_{p,q}} < \infty\}$$

where

$$\|f\|_{B^s_{p,q}} := \left(\sum_{j=0}^{\infty} \left(R_j^s \left\| \mathscr{F}^{-1}(\Phi_j \widehat{f}) \right\|_{L^p} \right)^q \right)^{1/q}$$

with $R_j = 2^j$, and with the usual modification for $q = \infty$.

Definition 6.1. We say that $f \in \widehat{B^s_{p,q}}$ if $\widehat{f} \in B^s_{p,q}$ and write $\|f\|_{\widehat{B^s_{p,q}}} = \|\widehat{f}\|_{B^s_{p,q}}$. Note that

$$\|f\|_{\widehat{B^s_{p,q}}} = \left(\sum_{j=0}^{\infty} \left(R_j^s \left\| \widehat{f_j} \right\|_{L^p} \right)^q \right)^{1/q} = \left(\sum_{j=0}^{\infty} \left(R_j^s \left\| \widehat{\Phi_j} * \widehat{f} \right\|_{L^p} \right)^q \right)^{1/q}.$$

The fact that $\widehat{B^s_{p,q}}$ is a Banach space follows from the that corresponding fact for $B^s_{p,q}$ [17, 2.3.3]. We simply note that the Fourier transform, acting on tempered distributions, is one to one, and that convergence in the $B^s_{p,q}$ norm implies convergence as tempered distributions.

7 Proof of Theorem 3.5

In this section, we will use what are sometimes called array, or componentwise operations, as well as standard multi-index notation. If η is a vector in $\mathbb{C}^n$, and α is a multi-index (i.e. also a vector), we will use η^α to mean the product

$$\eta^\alpha = \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n}$$

When a is a scalar, η^a will denote the vector

$$\eta^a = (\eta_1^a, \dots, \eta_n^a)$$

Similarly, we will use a scalar divided by a vector, or a vector divided by a vector, to denote componentwise division, e.g.

$$\frac{1}{\eta} = \left(\frac{1}{\eta_1}, \dots, \frac{1}{\eta_n} \right),$$

We let $\sigma_k(\eta)$ denotes the k'th elementary symmetric function of $(\eta_1, \dots, \eta_n)$. The two symmetric functions we will make use of are

$$\sigma_n(\eta) = \prod_{i=1}^{n} \eta_i,$$
$$\sigma_{n-1}(\eta) = \sum_{i=1}^{n} \prod_{j \neq i} \eta_j.$$

In this section, we will use the superscript $\widehat{\ }$ to indicate that an index does not occur, so that

$$\eta_{\widehat{i}} = (\eta_1, \dots, \eta_{i-1}, \eta_{i+1}, \dots, \eta_n)$$

means the $n-1$-dimensional vector that omits the i'th component of η. We will use the notation $P_{\widehat{i}}$ and $P(\eta_{\widehat{i}})$ interchangeably to denote a polynomial which does not depend on the i'th variable.

We will also use the superscript $\widehat{\ }$ to denote the Laplace transform, $\widehat{P}$, of a degree N homogeneous polynomial $P(x)$, given by

$$\widehat{P}(\rho) = \int_{x>0} e^{-\rho \cdot x} P(x) dx \tag{32}$$

where $x > 0$ means that each component $x_i > 0$. Making the substitutions $y_i = \rho_i x_i$, with ρ real and $\rho > 0$ (for the moment) we have

$$\widehat{P}(\rho) = \int_{y>0} e^{-1 \cdot y} P\left(\frac{y}{\rho}\right) \sigma_n\left(\frac{1}{\rho}\right) dy.$$

If $P = \sum_{|\alpha|=N} p_\alpha x^\alpha$, then

$$\widehat{P}(\rho) = \sum_{|\alpha|=N} p_\alpha \frac{1}{\rho^{\alpha+1}} \int_{y>0} e^{-1 \cdot y} y^\alpha dy = \sum_{|\alpha|=N} p_\alpha \left(\frac{1}{\rho}\right)^{\alpha+1} \alpha!$$

where $\alpha + 1$ is the multi-index with components $\alpha_i + 1$. Thus

$$\widehat{P}(\rho) = Q^{N+n}\left(\frac{1}{\rho}\right)$$

where $Q = Q^{N+n}$ is the homogeneous polynomial of degree $N + n$ with coefficients $q_{\alpha+1} = \alpha! p_\alpha$. The main assertion of Theorem 3.5 is that $\widehat{P}(\rho)$ does not vanish on any open subset of the variety $\rho \cdot \rho = 0$. This is equivalent to the assertion that the polynomial $Q(\eta)$ does not vanish identically on any open subset of

$$\left\{\frac{1}{\eta} \cdot \frac{1}{\eta} = 0\right\}$$

where

$$\rho \cdot \rho = \frac{1}{\eta} \cdot \frac{1}{\eta} = \frac{\sigma_{n-1}(\eta^2)}{\sigma_n(\eta^2)} = \frac{\sigma_{n-1}(\eta^2)}{\sigma_n^2(\eta)}. \tag{33}$$

If P is harmonic, $Q(\eta) = \widehat{P}(\frac{1}{\eta})$ has an additional representation.

Lemma 7.1. *If P is harmonic and homogeneous, then*

$$Q(\eta) = \widehat{P}(\frac{1}{\eta}) = \frac{\sigma_n(\eta)}{\sigma_{n-1}(\eta^2)} \sum_{i=1}^{n} \left(P_{\widehat{i}} + \eta_i Q_{\widehat{i}} \right) \tag{34}$$

where $P_{\widehat{i}}$ and $Q_{\widehat{i}}$ are homogeneous polynomials of degree $N+2n-2$, $N+2n-3$, respectively, which do not depend on η_i.

Proof. We will prove (34) on the open set where $\Re\rho > 0$ and that $\rho \cdot \rho \neq 0$. Because Q is a polynomial in η , the right hand side of (34) is also a polynomial, and the identity must hold everywhere.

We start with (32), integrate by parts, and recall that P is harmonic,

$$\begin{aligned}
\widehat{P}(\rho) &= \int_{x>0} \frac{\Delta e^{-\rho\cdot x}}{\rho\cdot\rho} P(x)dx \\
&= \frac{1}{\rho\cdot\rho} \sum_{i=1}^{n} \left(\int\limits_{\substack{x_i=0\\ x_{\widehat{i}}>0}} e^{-\rho\cdot x} \left(\rho_i P + \frac{\partial}{\partial x_i} P \right) + \int\limits_{x>0} e^{-\rho\cdot x} \Delta P \right) \\
&= \frac{1}{\rho\cdot\rho} \sum_{i=1}^{n} \left(\rho_i \int_{x_{\widehat{i}}>0} e^{-\rho_{\widehat{i}}\cdot x_{\widehat{i}}} P\Big|_{x_i=0} + \int\limits_{x_{\widehat{i}}>0} e^{-\rho_{\widehat{i}}\cdot x_{\widehat{i}}} \frac{\partial}{\partial x_i} P\Big|_{x_i=0} \right) \\
&= \frac{1}{\rho\cdot\rho} \sum_{i=1}^{n} \left(\rho_i \widetilde{P_i}\left(\frac{1}{\rho_{\widehat{i}}}\right) + \widetilde{Q_i}\left(\frac{1}{\rho_{\widehat{i}}}\right) \right)
\end{aligned}$$

where $\widetilde{P_i}$ and $\widetilde{Q_i}$ simply denote polynomials in $n-1$ variables.

Recalling (33), the polynomial Q then satisfies

$$\begin{aligned}
Q(\eta) &= \frac{(\sigma_n(\eta))^2}{\sigma_{n-1}(\eta^2)} \sum_{i=1}^{n} \left(\frac{1}{\eta_i} \widetilde{P}_i(\eta_{\widehat{i}}) + \widetilde{Q_i}(\eta_{\widehat{i}}) \right) \\
&= \frac{\sigma_n(\eta)}{\sigma_{n-1}(\eta^2)} \sum_{i=1}^{n} \left(\frac{\sigma_n(\eta)}{\eta_i} \widetilde{P}_i(\eta_{\widehat{i}}) + \sigma_n(\eta) \widetilde{Q_i}(\eta_{\widehat{i}}) \right) \\
&= \frac{\sigma_n(\eta)}{\sigma_{n-1}(\eta^2)} \sum_{i=1}^{n} \left(P_i(\eta_{\widehat{i}}) + \eta_i Q_i(\eta_{\widehat{i}}) \right),
\end{aligned}$$

so that $P_i(\eta_{\widehat{i}}) = \sigma_{n-1}(\eta_{\widehat{i}}) \widetilde{P}_i(\eta_{\widehat{i}})$ and $Q_i(\eta_{\widehat{i}}) = \sigma_{n-1}(\eta_{\widehat{i}}) \widetilde{Q_i}(\eta_{\widehat{i}})$ are the polynomials $P_{\widehat{i}}$ and $Q_{\widehat{i}}$ of (34), and the proof is finished. ∎

The irreducibility of the denominator, $\sigma_{n-1}(\eta^2)$ in (34) will play a role in several parts of our proof, so we prove this fact here:

Lemma 7.2. *If $n \geq 3$, $\sigma_{n-1}(\eta^2)$ is an irreducible polynomial. If $n = 2$, then $\sigma_{n-1}(\eta^2) = \eta_1^2 + \eta_2^2 = (\eta_1 - i\eta_2)(\eta_1 + i\eta_2)$.*

Proof. The statement for $n = 2$ is obvious. We will prove this lemma for $n \geq 3$ by induction, making use of the identity

$$\sigma_{n-1}(\eta^2) = \eta_1^2 \sigma_{n-2}(\eta_{\hat{i}}^2) + \sigma_{n-1}(\eta_{\hat{i}}^2)$$

for $\eta \in \mathbb{C}^n$. If $\sigma_{n-1}(\eta^2)$ factors, and one factor does not depend on η_1, then we must have

$$\eta_1^2 \sigma_{n-2}(\eta_{\hat{1}}^2) + \sigma_{n-1}(\eta_{\hat{1}}^2) = \sigma_{n-1}(\eta^2) = p_{\hat{1}}(\eta_1^2 q_{\hat{1}} + r_{\hat{1}})$$

where $p_{\hat{1}}, q_{\hat{1}}$, and $r_{\hat{1}}$ are non-constant polynomials independent of η_1. Equating coefficients of η_1^2 gives

$$\sigma_{n-2}(\eta_{\hat{1}}^2) = p_{\hat{1}} q_{\hat{1}},$$

which contradicts the induction hypothesis because $\sigma_{n-2}(\eta_{\hat{1}}^2) = \sigma_{m-1}(\xi^2)$ with $m = n - 1$ and $\xi = \eta_{\hat{1}} \in \mathbb{C}^m$.

On the other hand, if both factors depend on η_1, i.e.

$$\eta_1^2 \sigma_{n-2}(\eta_{\hat{1}}^2) + \sigma_{n-1}(\eta_{\hat{1}}^2) = (\eta_1 p_{\hat{1}} + r_{\hat{1}})(\eta_1 q_{\hat{1}} + s_{\hat{1}}).$$

Equating coefficients of η_1^2 again gives

$$\sigma_{n-2}(\eta_{\hat{1}}^2) = p_{\hat{1}} q_{\hat{1}}$$

and contradicts the induction hypothesis.

We finish the induction by verifying irreduciblity in the case $n = 3$. In this case,

$$\sigma_2(\eta^2) = \eta_1^2(\eta_2 + i\eta_3)(\eta_2 - i\eta_3) + \eta_2^2\eta_3^2.$$

If

$$p_{\hat{1}}(\eta_1^2 q_{\hat{1}} + r_{\hat{1}}) = \sigma_2(\eta^2) = \eta_1^2(\eta_2 + i\eta_3)(\eta_2 - i\eta_3) + \eta_2^2\eta_3^2$$

equating the coefficients of η_1^2, we see again that $p_{\hat{1}}$ must divide $(\eta_2 + i\eta_3)(\eta_2 - i\eta_3)$. Equating the coefficients of the terms that do not involve η_1^2, tells us that $p_{\hat{1}}$ also divides $\eta_2^2\eta_3^2$, but this is impossible because the two have prime factorizations without common factors.

If, on the other hand, $(\eta_1 p_{\hat{1}} + r_{\hat{1}})(\eta_1 q_{\hat{1}} + s_{\hat{1}}) = \sigma_2(\eta^2)$, expanding both sides of the equation shows that

$$\eta_1^2 p_{\hat{1}} q_{\hat{1}} + \eta_1(q_{\hat{1}} r_{\hat{1}} + p_{\hat{1}} s_{\hat{1}}) + = \eta_1^2(\eta_2 + i\eta_3)(\eta_2 - i\eta_3) + \eta_2^2\eta_3^2 + r_{\hat{1}} s_{\hat{1}}$$

then

$$p_{\hat{1}} q_{\hat{1}} = (\eta_2 + i\eta_3)(\eta_2 - i\eta_3)$$

which implies that $p_{\hat{1}}$ must be a constant multiple of either $(\eta_2+i\eta_3)$ or $(\eta_2-i\eta_3)$ and $q_{\hat{1}}$ must be a constant multiple of the other. Also

$$q_{\hat{1}} r_{\hat{1}} = -p_{\hat{1}} s_{\hat{1}}$$

so that $p_{\hat{1}}$ must divide $r_{\hat{1}}$ because it doesn't divide $q_{\hat{1}}$. However

$$r_{\hat{1}} s_{\hat{1}} = \eta_2^2 \eta_3^2$$

does not have $(\eta_2 \pm i\eta_3)$ as a factor, so this is also impossible and the proof is complete. $\square$

We will need two more propositions for the proof of Theorem 3.5. The first follows easily from the previous lemma.

Proposition 7.3. *If P is harmonic and homogeneous, and $\widehat{P}(\rho)$ vanishes identically on $\{\rho \cdot \rho = 0\}$, then $\sigma_{n-1}(\eta^2)$ divides the polynomial $Q(\eta) = \widehat{P}(\frac{1}{\eta})$.*

Proof. Because $\widehat{P}$ vanishes on $\{\rho \cdot \rho = 0\}$, it folllows from (33) that Q vanishes on the set $\{\sigma_{n-1}(\eta^2) = 0\} \setminus \{\sigma_n(\eta^2) = 0\}$. Therefore, the product $\sigma_n(\eta^2)Q(\eta^2)$ vanishes on the entire variety $\{\sigma_{n-1}(\eta^2) = 0\}$, and hence must be divisible by $\sigma_{n-1}(\eta^2)$ by Hilbert's Nullstellensatz. For $n \geq 3$, $\sigma_{n-1}(\eta^2)$ is irreducible and doesn't divide $\sigma_n(\eta^2)$, so it must divide Q. In the case $n = 2$, $\sigma_{n-1}(\eta^2)$ has two factors; neither factor divides $\sigma_n(\eta^2)$, so both divide Q. $\square$

The proof of the next proposition will not be so easy,

Proposition 7.4. *$\sigma_{n-1}(\eta^2)$ cannot divide any polynomial Q of the form* (34).

but the proof of Theorem 3.5 is an immediate consequence.

Proof of Theorem 3.5. If $n \geq 3$, the hypothesis of Theorem 3.5 is that $\widehat{P}$ vanishes on an open subset of $\{\rho \cdot \rho = 0\}$, which means that Q, vanishes on an open subset of the irreducible variety $\sigma_{n-1}(\eta^2) = 0$. But this means that Q vanishes on the whole variety by [12, p.91] or [18] and that $\sigma_{n-1}(\eta^2)$ divides Q, contradicting Proposition 7.4.

If $n = 2$, we have the same hypothesis for each of the irreducible factors, $\rho_1 - i\rho_2$ and $\rho_1 + i\rho_2$, so we may conclude that each divides Q, and therefore that Q is divisible by their product $\sigma_{n-1}(\eta^2)$. $\square$

Proof of Proposition 7.4. We will make essential use of the the fact that $\sigma_{n-1}(\eta^2)$ is even in each component η_j of η.

Lemma 7.5. *Every polynomial $R(\eta)$ has a unique decomposition into a sum*

$$R(\eta) = \sum_{\tau \in \{0,1\}^n} \eta^\tau R_\tau(\eta^2)$$

where τ is a multi-index with each component equal to 0 *or* 1*. If R has the special form $R = \sum_i (\eta_i P_{\hat{i}} + Q_{\hat{i}})$, then each of the coefficients $R_\tau(\eta^2)$ has the special form*

$$R_\tau = \sum_i S_{\hat{i}}(\eta_{\hat{i}}^2) \tag{35}$$

Proof. We express R as a sum of monomials,

$$R(\eta) \quad = \quad \sum_\alpha p_\alpha \eta^\alpha$$

group the terms that are even or odd for each η_i together

$$= \quad \sum_{\tau \in \{0,1\}^n} \left(\sum_{\alpha \equiv_2 \tau} (p_\alpha \eta^\alpha) \right)$$

and remove a single power of η_i from each monomial that is odd in η_i

$$= \quad \sum_{\tau \in \{0,1\}^n} \left(\sum_{\alpha \equiv_2 \tau} (p_\alpha \eta^{\alpha - \tau}) \right) \eta^\tau \tag{36}$$

so that the summands in the parentheses contains only even powers

$$= \quad \sum_{\tau \in \{0,1\}^n} R_\tau(\eta^2) \eta^\tau$$

The explicit formula for each R_τ in (36) implies that the decomposition is unique. Suppose now that R has the special form $\sum_i (\eta_i P_{\hat{i}} + Q_{\hat{i}})$, we can first decompose each of the $Q_{\hat{i}}$ and the $P_{\hat{i}}$.

$$\begin{aligned}
\eta_i P_{\hat{i}} + Q_{\hat{i}} \quad &= \quad \eta_i \left(\sum_{\tau_{\hat{i}} \in \{0,1\}^{n-1}} P_{\tau_{\hat{i}}}(\eta_{\hat{i}}^2) \eta^{\tau_{\hat{i}}} \right) + \sum_{\tau_{\hat{i}} \in \{0,1\}^{n-1}} Q_{\tau_{\hat{i}}}(\eta_{\hat{i}}^2) \eta^{\tau_{\hat{i}}} \\
&= \quad \sum_{\tau_{\hat{i}} \in \{0,1\}^{n-1}} P_{\tau_{\hat{i}}}(\eta_{\hat{i}}^2) \left(\eta^{\tau_{\hat{i}}} \eta_i^1 \right) + \sum_{\tau_{\hat{i}} \in \{0,1\}^{n-1}} Q_{\tau_{\hat{i}}}(\eta_{\hat{i}}^2) \left(\eta^{\tau_{\hat{i}}} \eta_i^0 \right)
\end{aligned}$$

which shows that each summand $\eta_i P_{\hat{i}} + Q_{\hat{i}}$ has a decompostion where the coefficients of η^τ are indepedent of η_i. Thus the sum has coefficients which are sums of such functions. □

Lemma 7.6. *If a polynomial $S(\eta^2)$ divides $R(\eta) = \displaystyle\sum_{\tau \in \{0,1\}^n} \eta^\tau R_\tau(\eta^2)$, then S divides each R_τ.*

Proof. Suppose that

$$R(\eta) = S(\eta^2)C(\eta)$$

expand both R and C as in Lemma 7.5

$$\begin{aligned}\sum_{\tau\in\{0,1\}^n} \eta^\tau R_\tau(\eta^2) &= S(\eta^2)\sum_{\tau\in\{0,1\}^n} \eta^\tau C_\tau(\eta^2) \\ &= \sum_{\tau\in\{0,1\}^n} \eta^\tau S(\eta^2)C_\tau(\eta^2)\end{aligned}$$

and now use the uniqueness of the expansion to equate the coefficients of each monomial η^τ. □

The last ingredient necessary for the proof of Proposition 7.4 is

Proposition 7.7. *$\sigma^2_{n-1}(s)$ does not divide any polynomial of the form $T(s) = \sum T_{\hat{i}}$ unless T is identically zero.*

Before giving its proof, we use it to finish the proof of Proposition 7.4. If, as in the hypothesis of Proposition 7.4, $\sigma^2_{n-1}(\eta^2)$ divides $R = \sum(\eta_i P_{\hat{i}} + Q_{\hat{i}})$, then, according to Lemma 7.6, $\sigma^2_{n-1}(\eta^2)$ divides each of the R_τ in the expansion of Lemma 7.5, and each R_τ has the special form (35). Proposition 7.7 says that this is impossible (the variable s replaces η^2) and thus finishes the proof of Proposition 7.4. □

Proof of Proposition 7.7. We will prove the proposition by induction on the number of independent variables. We will expand all polynomials as polynomials in the single variable s_1 with coefficients that depend on the other variables. We begin with

$$\begin{aligned}\sigma_{n-1}(s) &= s_1\sigma_{n-2}(s_{\hat{1}}) + \sigma_{n-1}(s_{\hat{1}}) \\ \sigma^2_{n-1}(s) &= s_1^2\sigma^2_{n-2}(s_{\hat{1}}) + s_1 2\sigma_{n-2}(s_{\hat{1}})\sigma_{n-1}(s_{\hat{1}}) + \sigma^2_{n-1}(s_{\hat{1}})\end{aligned}$$

If a general polynomial $T(s)$ has $\sigma^2_{n-1}(s)$ as a factor, then expanding the equality $T = \sigma^2_{n-1}C$ in powers of s_1 gives

$$\sum_{k=0}^{N} s_1^k T^k(s_{\hat{1}}) = \left(s_1^2\sigma^2_{n-2}(s_{\hat{1}}) + s_1 2\sigma_{n-2}(s_{\hat{1}})\sigma_{n-1}(s_{\hat{1}}) + \sigma^2_{n-1}(s_{\hat{1}})\right)\left(\sum_{k=0}^{N-2} s_1^k C^k(s_{\hat{1}})\right)$$

Equating coefficients of powers of s_1 gives

$$T^N(s_{\hat{1}}) = \sigma^2_{n-2}(s_{\hat{1}})C^{N-2}(s_{\hat{1}}) \tag{37}$$

and, for $j = 1\ldots(N-2)$,

$$T^{N-j}(s_{\hat{1}}) = \sigma^2_{n-2}(s_{\hat{1}})C^{N-2-j}(s_{\hat{1}}) + \ldots \tag{38}$$

where the ... indicate terms involving C^k for $k > N-2-j$. We won't need to use the equations for T^1 and T^0.

Now, if T has the special form $T = \sum T_{\widehat{i}}$, with the $T_{\widehat{i}}$ independent of s_i, then each of the T^k, except T^0, will have the special form

$$T^k = \sum_{i=2}^{n} T^k_{\widehat{1},\widehat{i}}$$

where the subscripts indicate that $T^k_{\widehat{1},\widehat{i}}$ is independent of both η_1 and η_i. Thus equation (37) becomes

$$\sum_{i=2}^{n} T^N_{\widehat{1},\widehat{i}} = \sigma^2_{n-2}(s_{\widehat{1}})C^{N-2}(s_{\widehat{1}}) \tag{39}$$

but this is exactly the hypothesis of the proposition for one fewer dimension. If we let $\beta = s_{\widehat{1}}$ and $m = n-1$, then (39) becomes

$$\sum_{i=1}^{m} T^N_{\widehat{i}}(\beta) = \sigma^2_{m-1}(\beta)C^{N-2}(\beta)$$

and the induction hypothesis guarantees that C^{N-2} and T^N are both identically zero. Once we know that C^{N-2} is zero, we may conclude that the term represented by the ... in equation (38) for T^{N-1} is zero, and repeat the argument to conclude that C^{N-3} and T^{N-2} are zero. We continue in this manner to conclude that all the C^k, and therefore all the T^k are zero.

Finally, we verify the proposition in the case $n = 2$. In this case, we must check that the equality below

$$p_N x^N + q_N y^N = (x+y)^2 \sum_{k=0}^{N-2} c_k x^k y^{N-2-k}$$

is only possible if p_N, q_N, and all the c_k are zero. Equating powers of x and y give

$$\begin{aligned} p_N &= c_{N-2} \\ 0 &= 2c_{N-2} + c_{N-1} \end{aligned}$$

for $j = 2 \ldots (N-2)$

$$0 = c_{N-(j+2)} + 2c_{N-(j+1)} + c_{N-j}$$

and

$$\begin{aligned} 0 &= c_1 + 2c_0 \\ q_N &= c_0 \end{aligned}$$

Discarding the first and last equations gives the invertible tridiagonal system

$$\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ 0 & 1 & 2 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} c_{N-2} \\ c_{N-1} \\ \vdots \\ c_1 \\ c_0 \end{pmatrix}$$

whence we conclude that all the c_k are zero. This finishes the proof of the proposition. □

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