On diregular digraphs with degree two and excess two

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Abstract

An important topic in the design of efficient networks is the construction of $(d, k, +\epsilon)$-digraphs, i.e. k-geodetic digraphs with minimum out-degree $\geq d$ and order $M(d, k) + \epsilon$, where $M(d, k)$ represents the Moore bound for degree d and diameter k and $\epsilon > 0$ is the (small) excess of the digraph. Previous work has shown that there are no $(2, k, +1)$-digraphs for $k \geq 2$. In a separate paper, the present author has shown that any $(2, k, +2)$-digraph must be diregular for $k \geq 2$. In the present work, this analysis is completed by proving the nonexistence of diregular $(2, k, +2)$-digraphs for $k \geq 3$ and classifying diregular $(2, 2, +2)$-digraphs up to isomorphism.

Keywords: Degree/diameter problem, Digraphs, Excess, Extremal digraphs
2000 MSC: 05C35, 05C20, 90C35

1. Introduction

An important topic in the design of interconnection networks is the directed degree/diameter problem: what is the largest possible order $N(d, k)$ of a digraph G with maximum out-degree d and diameter $\leq k$? A simple inductive argument shows that for $0 \leq l \leq k$ the number of vertices at distance l from a fixed vertex v is bounded above by d^l. Therefore, a natural upper bound for the order of such a digraph is the so-called *Moore bound* $M(d, k) = 1 + d + d^2 + ... + d^k$. A digraph that attains this upper bound is called a *Moore digraph*. It is easily seen that a digraph G is Moore if and only if it is out-regular with degree d, has diameter k and is k-geodetic, i.e. for any two vertices u, v there is at most one $\leq k$-path from u to v.

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases $d = 1$ or $k = 1$ (the Moore digraphs are directed $(k+1)$-cycles and complete digraphs K_{d+1} respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree d, diameter $\leq k$ and order $M(d, k) - \delta$ for small $\delta > 0$; this is equivalent to relaxing the k-geodecity requirement in the conditions for a digraph to be Moore. δ is known as the *defect* of the digraph. The reader is referred to the survey [4] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the k-geodecity requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed k: what is the smallest possible order of a k-geodetic digraph G with minimum out-degree $\geq d$? A k-geodetic digraph with minimum out-degree $\geq d$ and order $M(d, k)+\epsilon$ is said to be a $(d, k, +\epsilon)$-digraph or to have *excess* ϵ. It was shown in [6] that there are no diregular $(2, k, +1)$-digraphs for $k \geq 2$. In 2016 it was shown in [5] that digraphs with excess one must be diregular and that there are no $(d, k, +1)$-digraphs for $k = 2, 3, 4$ and sufficiently large d. In a

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separate paper [7], the present author has shown that $(2, k, +2)$-digraphs must be diregular with degree $d = 2$ for $k \geq 2$. In the present paper, we classify the $(2, 2, +2)$-digraphs up to isomorphism and show that there are no diregular $(2, k, +2)$-digraphs for $k \geq 3$, thereby completing the proof of the nonexistence of digraphs with degree $d = 2$ and excess $\epsilon = 2$ for $k \geq 3$. Our reasoning and notation will follow closely that employed in [3] for the corresponding result for defect $\delta = 2$.

2. Preliminary results

We will let G stand for a $(2, k, +2)$-digraph for arbitrary $k \geq 2$, i.e. G has minimum out-degree $d = 2$, is k-geodetic and has order $M(2, k) + 2$. We will denote the vertex set of G by $V(G)$. By the result of [7], G must be diregular with degree $d = 2$ for $k \geq 2$. The distance $d(u, v)$ between vertices u and v is the length of the shortest path from u to v. Notice that $d(u, v)$ is not necessarily equal to $d(v, u)$. $u \to v$ will indicate that there is an arc from u to v. We define the in- and out-neighbourhoods of a vertex u by $N^-(u) = \{v \in V(G) : v \to u\}$ and $N^+(u) = \{v \in V(G) : u \to v\}$ respectively; more generally, for $0 \leq l \leq k$, the set $\{v \in V(G) : d(u, v) = l\}$ of vertices at distance exactly l from u will be denoted by $N^l(u)$. For $0 \leq l \leq k$ we will also write $T_l(u) = \cup_{i=0}^{l} N^i(u)$ for the set of vertices at distance $\leq l$ from u. The notation $T_{k-1}(u)$ will be abbreviated by $T(u)$.

It is easily seen that for any vertex u of G, there are exactly two distinct vertices that are at distance $\geq k + 1$ from u. For any $u \in V(G)$, we will write $O(u)$ for the set of these vertices and call such a set an *outlier set* and its elements *outliers* of u. Notice that $O(u) = V(G) - T_k(u)$. An elementary counting argument shows that in a diregular $(2, k, +2)$-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex u can *reach* a vertex v if $v \notin O(u)$.

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

Lemma 1. *For $k \geq 2$, let u and v be distinct vertices such that $N^+(u) = N^+(v) = \{u_1, u_2\}$. Then $u_1 \in O(u_2), u_2 \in O(u_1)$ and there exists a vertex x such that $O(u) = \{v, x\}, O(v) = \{u, x\}$.*

Proof. Suppose that u can reach v by a $\leq k$-path. Then $v \in T(u_1) \cup T(u_2)$. As $N^+(v) = N^+(u)$, it follows that there would be a $\leq k$-cycle through v, contradicting k-geodecity. If $O(u) = \{v, x\}$, then $x \neq v$ and $x \notin T(u_1) \cup T(u_2)$, so that v cannot reach x by a $\leq k$-path. Similarly, if u_1 can reach u_2 by a $\leq k$-path, then we must have $\{u, v\} \cap T(u_1) \neq \varnothing$, which is impossible. □

Lemma 2. *For $k \geq 2$, there exists a pair of vertices u, v with $|N^+(u) \cap N^+(v)| = 1$.*

Proof. Suppose for a contradiction that there is no such pair of vertices. Define a map $\phi : V(G) \to V(G)$ as follows. Let u^+ be an out-neighbour of a vertex u and let $\phi(u)$ be the in-neighbour of u^+ distinct from u. By our assumption, it is easily verified that ϕ is a well-defined bijection with no fixed points and with square equal to the identity. It follows that G must have even order, whereas $|V(G)| = M(2, k) + 2$ is odd. □

u, v will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of $T_k(u)$ according to the scheme $N^+(u) = \{u_1, u_2\}, N^+(u_1) = \{u_3, u_4\}, N^+(u_2) = \{u_5, u_6\}, N^+(u_3) = \{u_7, u_8\}, N^+(u_4) = \{u_9, u_{10}\}$ and so on, with the same convention for the vertices of $T_k(v)$, where we will assume that $u_2 = v_2$.

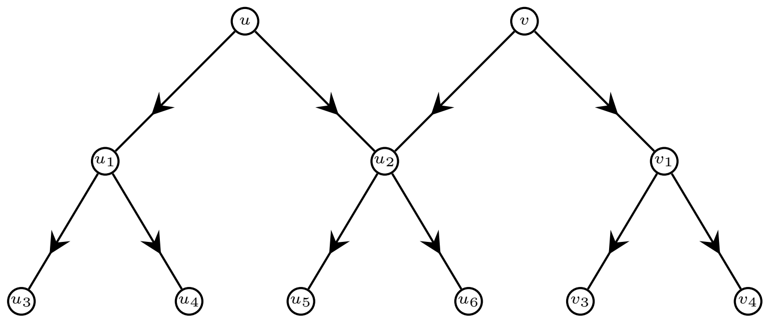

Figure 1: The vertices u and v

3. Classification of $(2, 2, +2)$-digraphs

We begin by classifying the $(2, 2, +2)$-digraphs up to isomorphism. We will prove the following theorem.

Theorem 1. *There are exactly two diregular $(2, 2, +2)$-digraphs, which are displayed in Figures 2 and 5.*

Let G be an arbitrary diregular $(2, 2, +2)$-digraph. G has order $M(2, 2) + 2 = 9$. By Lemma 2, G contains a pair of vertices (u, v) such that $|N^+(u) \cap N^+(v)| = 1$; we will assume that $u_2 = v_2$, so that we have the situation shown in Figure 1.

We can immediately deduce some information on the possible positions of v and v_1 in $T_2(u)$.

Lemma 3. *If $v \notin O(u)$, then $v \in N^+(u_1)$. If $v_1 \notin O(u)$, then $v_1 \in N^+(u_1)$.*

Proof. $v \notin T(u_2)$ by 2-geodecity. $v \neq u$ by construction. If we had $v = u_1$, then there would be two distinct ≤ 2-paths from u to u_2. Also $v_1 \notin \{u\} \cup T(u_2)$ by 2-geodecity and by assumption $u_1 \neq v_1$. □

Since v and v_1 cannot both lie in $N^+(u_1)$ by 2-geodecity, we have the following corollary.

Corollary 1. *$O(u) \cap \{v, v_1\} \neq \varnothing$.*

We will call a pair of vertices (u, v) with a single common out-neighbour *bad* if at least one of

$$O(u) \cap \{v_1, v_3\} = \varnothing, O(u) \cap \{v_1, v_4\} = \varnothing, O(v) \cap \{u_1, u_3\} = \varnothing, O(v) \cap \{u_1, u_4\} = \varnothing.$$

holds. Otherwise such a pair will be called *good*.

Lemma 4. *There is a unique $(2, 2, +2)$-digraph containing a bad pair.*

Proof. Assume that there exists a bad pair (u, v). Without loss of generality, $O(u) \cap \{v_1, v_3\} = \varnothing$. By Lemma 3 we can set $v_1 = u_3$. By 2-geodecity $v_3 = u$. We cannot have $v_4 = v_3 = u$, so v_4 must be an outlier of u. By Corollary 1 it follows that $O(u) = \{v, v_4\}$.

Consider the vertex u_1. By Lemma 3, if $u_1 \notin O(v)$, then $u_1 \in N^+(v_1)$. However, as $v_1 = u_3$, there would be a 2-cycle through u_1. Hence $u_1 \in O(v)$. As $O(u) = \{v, v_4\}$, we have $V(G) = \{u, u_1, u_2, u_3 = v_1, u_4, u_5, u_6, v, v_4\}$ and $O(v) = \{u_1, u_4\}$. As neither u nor v lies in $T(u_1)$, we must also have $u_2 \in O(u_1)$. As u_1 can reach u_1, v_1, u_4, u and v_4, it follows that without loss of generality we either have $O(u_1) = \{u_2, v\}$ and $N^+(u_4) = \{u_5, u_6\} = N^+(u_2)$ or $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. In either case, (v, u_1) is a good pair.

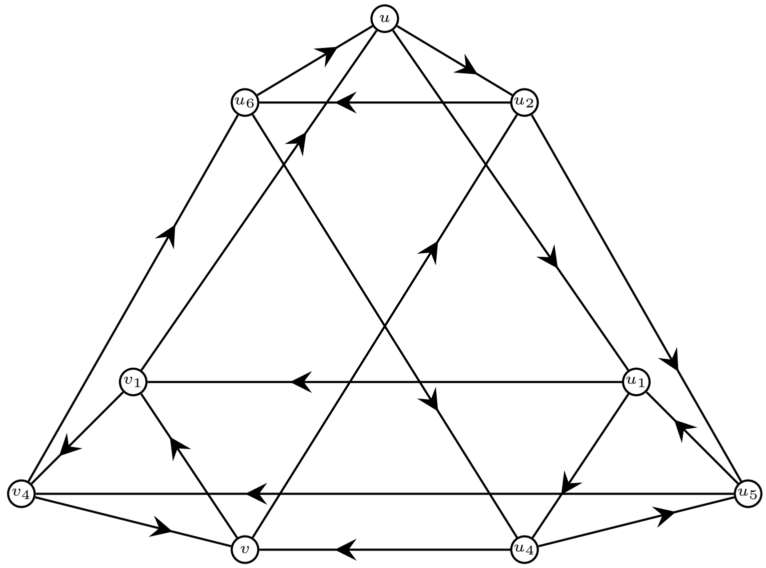

Figure 2: The unique $(2, 2, +2)$-digraph containing a bad pair

Suppose firstly that $N^+(u_2) = N^+(u_4)$. Then v is an outlier of u and u_1. As each vertex is the outlier of exactly two vertices, v_1 must be able to reach v by a ≤ 2-path. Hence $v_4 \rightarrow v$. Likewise u_2 can reach v, so without loss of generality $u_5 \rightarrow v$. Suppose that $O(u_2) \cap \{u, u_1\} = \varnothing$. As u and v have a common out-neighbour, we must have $u_6 \rightarrow u$. Since $u \rightarrow u_1$, by 2-geodecity we must have $u_5 \rightarrow u_1$. However, this is a contradiction, as v and u_1 also have a common out-neighbour. Therefore, at least one of u, u_1 is an outlier of u_2. By Lemma 1 u_4 is an outlier of u_2. Therefore either $O(u_2) = \{u, u_4\}$ or $O(u_2) = \{u_1, u_4\}$. If $O(u_2) = \{u, u_4\}$, then u_2 must be able to reach u_1, v_1 and v_4. $u_5 \rightarrow v$ and $v \rightarrow v_1$, so $v_1 \in N^+(u_6)$. As $u_1 \rightarrow v_1$, we must have $N^+(u_5) = \{v, u_1\}$. As v and u_1 have a common out-neighbour, this violates 2-geodecity. Hence $O(u_2) = \{u_1, u_4\}$ and u_2 can reach u, v_1 and v_4. As $v \rightarrow v_1$, $v_1 \in N^+(u_6)$. As $v_1 \rightarrow v_4$, it follows that $N^+(u_5) = \{v, v_4\}$. However, $v_4 \rightarrow v$, so this again violates 2-geodecity.

We are left with the case $O(u_1) = \{u_2, u_6\}$ and $N^+(u_4) = \{v, u_5\}$. Then $v_1 \in O(u_2)$, as neither v nor u_1 lies in $T(u_2)$. Observe that u_2 and u_4 have a single common out-neighbour, so by Corollary 1 $O(u_2) \cap \{u_4, v\} \neq \varnothing$. Therefore either $O(u_2) = \{v_1, u_4\}$ or $O(u_2) = \{v_1, v\}$. Suppose firstly that $O(u_2) = \{v_1, u_4\}$. Then $N^2(u_2) = \{u, v, u_1, v_4\}$. As $N^+(u_4) = \{v, u_5\}$, $u_5 \not\rightarrow v$, so $u_6 \rightarrow v$. As $N^+(u) \cap N^+(v) \neq \varnothing$, $u_5 \rightarrow u$. $u \rightarrow u_1$, so necessarily $N^+(u_6) = \{v, u_1\}$. However, $v_1 \in N^+(u_1) \cap N^+(v)$, contradicting 2-geodecity.

Hence $O(u_2) = \{v_1, v\}$ and $N^2(u_2) = \{u, u_1, u_4, v_4\}$. As $u_4 \rightarrow u_5$, $u_5 \not\rightarrow u_4$. Thus $u_6 \rightarrow u_4$. Now $u_1 \rightarrow u_4$ and $u \rightarrow u_1$ implies that $N^+(u_5) = \{u_1, v_4\}$ and $N^+(u_6) = \{u, u_4\}$. Finally we must have $N^+(v_4) = \{v, u_6\}$. This gives us the $(2, 2, +2)$-digraph shown in Figure 2.

□

We can now assume that all pairs given by Lemma 2 are good. Let us fix a pair (u, v) with a single common out-neighbour. It follows from Corollary 1 and the definition of a good pair that $v_1 \in O(u)$; otherwise $O(u)$ would contain v, v_3 and v_4, which is impossible. Likewise $u_1 \in O(v)$.

Considering the positions of v_3 and v_4, we see that there are without loss of generality four possibilities: 1) $u = v_3, u_4 = v_4$, 2) $u = v_3, O(u) = \{v_1, v_4\}$, 3) $N^+(u_1) = N^+(v_1)$ and 4) $u_3 = v_3, O(u) = \{v_1, v_4\}$. A suitable relabelling of vertices shows that case 4 is equivalent to case 1.a) below, so we will examine cases 1 to 3 in turn.

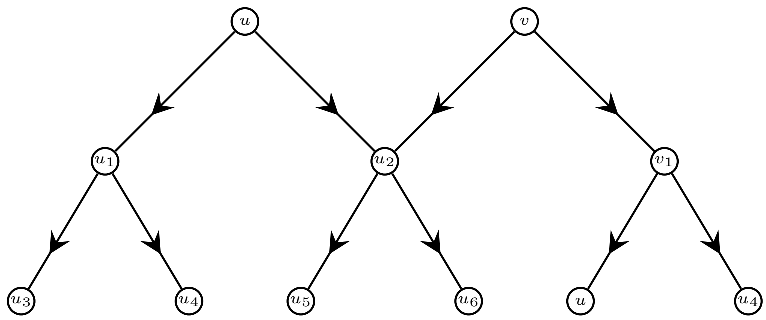

Figure 3: Case 1 configuration

Case 1: $\mathbf{u = v_3, u_4 = v_4}$

Depending upon the position of v, we must either have $O(u) = \{v_1, v\}$ and $O(v) = \{u_1, u_3\}$ or $v = u_3$.

Case 1.a): $\mathbf{O(u) = \{v_1, v\}, O(v) = \{u_1, u_3\}}$

In this case $V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\}$. u_1 and v_1 have a single common out-neighbour, namely u_4, so, as we are assuming all such pairs to be good, we have $u_3 \in O(v_1), u \in O(u_1)$. By 2-geodecity, $N^+(u_4) \subset \{u_5, u_6, v\}$, so without loss of generality either $N^+(u_4) = \{u_5, u_6\}$ or $N^+(u_4) = \{u_5, v\}$.

Suppose that $N^+(u_4) = \{u_5, u_6\}$. By elimination, $O(v_1) = \{v, u_3\}$. As G is diregular, every vertex is an outlier of exactly two vertices; v is an outlier of u and v_1, so both u_1 and u_2 can reach v by a ≤ 2-path. Hence $v \in N^+(u_3)$. As $v \to v_1$, we see that v_1 is an outlier of u_1; as u is also an outlier of u_1, we have $O(u_1) = \{u, v_1\}$ and $N^+(u_3) = \{v, u_2\}$. As $v \to u_2$, this is impossible.

Now consider $N^+(u_4) = \{u_5, v\}$. We now have $O(v_1) = \{u_3, u_6\}$. Thus $u_3 \in O(v) \cap O(v_1)$, so $u_3 \in T_2(u_4)$. v is not adjacent to u_3, so $u_3 \in N^+(u_5)$. u_2 and u_4 have u_5 as a unique common out-neighbour, so $u_6 \in O(u_4), v \in O(u_2)$. As $u_6 \in O(v_1) \cap O(u_4)$, u_1 can reach u_6. Hence $u_6 \in N^+(u_3)$. Neither u nor v lie in $T(u_1)$, so $u_2 \in O(u_1)$. Therefore either $O(u_1) = \{u, u_2\}$ or $O(u_1) = \{u_2, v_1\}$. If $O(u_1) = \{u, u_2\}$, then $N^+(u_3) = \{u_6, v_1\}$. u_2 can't reach v_1, since $v, u_3 \notin T(u_2)$, so $O(u_2) = \{v, v_1\}$ and $N^2(u_2) = \{u, u_1, u_3, u_4\}$. As $u_4 \to u_5$, $u_4 \in N^+(u_6)$. $u_1 \to u_4$, so $N^+(u_5) = \{u_1, u_3\}$. As $u_1 \to u_3$, this is a contradiction. Thus $O(u_1) = \{u_2, v_1\}$, so that $N^+(u_3) = \{u, u_6\}$. u_1 must have an in-neighbour apart from u, which must be either u_5 or u_6. As $u_1 \to u_3$, we have $u_1 \in N^+(u_6)$. By elimination, v and v_1 must also have in-neighbours in $\{u_5, u_6\}$. As u_1 and v_1 have a common out-neighbour, we have $N^+(u_5) = \{u_3, v_1\}, N^+(u_6) = \{u_1, v\}$. However, both u_3 and v_1 are adjacent to u, violating 2-geodecity.

Case 1.b): $\mathbf{v = u_3}$

There exists a vertex x such that $V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, x\}$, $O(u) = \{v_1, x\}$ and $O(v) = \{u_1, x\}$. As $x \in O(u) \cap O(v)$, u_1 and u_2 can reach x, so without loss of generality $x \in N^+(u_4) \cap N^+(u_5)$. As u_5 and u_4 have a common out-neighbour, $u_5 \in O(u_1)$. Also, u_1 and v_1 have u_4 as a unique common out-neighbour, so $u \in O(u_1)$ and $O(u_1) = \{u, u_5\}$. Thus $N^+(u_4) = \{x, u_6\}$. Observe that u_2 and u_4 have the out-neighbour u_6 in common. Thus $x \in O(u_2)$, whereas we already have $x \in O(u) \cap O(v)$, a contradiction.

Case 2: $\mathbf{u = v_3, O(u) = \{v_1, v_4\}}$

As v is not equal to v_1 or v_4, v must lie in $T_2(u)$. Without loss of generality, $v = u_3$. Hence

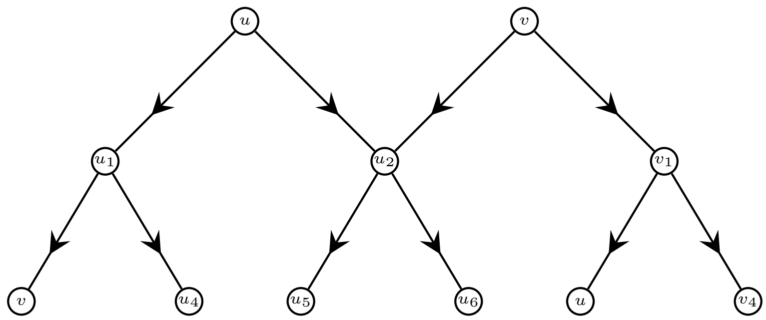

Figure 4: Case 2 configuration

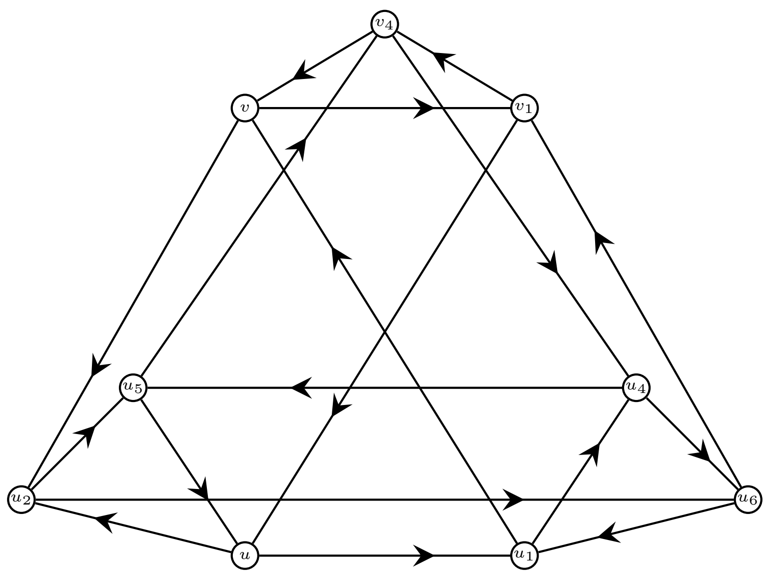

Figure 5: A second $(2, 2, +2)$-digraph

$V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, v_4\}$ and $O(v) = \{u_1, u_4\}$. We have the configuration shown in Figure 4. Hence u_1 can reach u_1, v, u_4, u_2 and v_1, so we have without loss of generality one of the following: a) $O(u_1) = \{u, v_4\}, N^+(u_4) = \{u_5, u_6\}$, b) $O(u_1) = \{u, u_5\}, N^+(u_4) = \{u_6, v_4\}$, c) $O(u_1) = \{u_5, u_6\}, N^+(u_4) = \{u, v_4\}$ or d) $O(u_1) = \{u_5, v_4\}, N^+(u_4) = \{u, u_6\}$.

Case 2.a): $\mathbf{O(u_1) = \{u, v_4\}, N^+(u_4) = \{u_5, u_6\}}$

As $v_4 \in O(u) \cap O(u_1)$, u_2 can reach v_4 and without loss of generality $v_4 \in N^+(u_5)$. $N^+(u_2) = N^+(u_4)$, so by Lemma 1 $u_2 \in O(u_4), u_4 \in O(u_2), u_5 \in O(u_6)$ and $u_6 \in O(u_5)$. Hence $u_4 \in O(v) \cap O(u_2)$, so v_1 can reach u_4, so $u_4 \in N^+(v_4)$. Neither u_5 nor u_6 lies in $N^+(v_4)$, so $O(v_1) = \{u_5, u_6\}$ and $N^+(v_4) = \{u_4, v\}$. Hence $O(v_4) = \{u, u_1\}$. Observe that $N^+(u_1) = N^+(v_4)$, so that $v \in O(u_4)$. Therefore $v \notin N^+(u_5) \cup N^+(u_6)$, yielding $O(u_2) = \{u_4, v\}, N^2(u_2) = \{v_4, v_1, u, u_1\}$. As $v_1 \to v_4$ and $N^+(u_1) = N^+(v_4)$, we must have $N^+(u_5) = \{v_4, u\}, N^+(u_6) = \{v_1, u_1\}$. This yields the digraph in Figure 5. Unlike the digraph in Figure 2, this digraph contains pairs of vertices with identical out-neighbourhoods, so the two are not isomorphic.

Case 2.b): $\mathbf{O(u_1) = \{u, u_5\}, N^+(u_4) = \{u_6, v_4\}}$

As $u_4 \to v_4$, $u_4 \notin N^+(v_4)$, so $u_4 \in O(v_1)$. Hence $u_4 \in O(v) \cap O(v_1)$, so u_2 can reach u_4. As $u_4 \to u_6$, we must have $u_5 \to u_4$. u_2 and u_4 have u_6 as a common out-neighbour, so

$v_4 \in O(u_2), u_5 \in O(u_4)$. Therefore $v_4 \in O(u) \cap O(u_2)$, so that u_6 can reach v_4, but $v_4 \notin T(u_6)$, so $N^+(u_6)$ contains an in-neighbour of v_4. $u_4 \notin N^+(u_6)$, so we must have $u_6 \rightarrow v_1$. We have $u_5 \in O(u_4) \cap O(u_1)$, so v_1 can reach u_5 and hence $v_4 \rightarrow u_5$. v_1 cannot reach u_6, as $u_2, u_4 \notin T(v_1)$, so $O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{u_5, v\}$. Now u_2 and v_4 have u_5 as a unique common out-neighbour, so $u_6 \in O(v_4), v \in O(u_2)$. Thus $O(u_2) = \{v, v_4\}$ and $N^2(u_2) = \{u_4, v_1, u, u_1\}$. Taking into account adjacencies between members of $N^2(u_2)$, it follows that $N^+(u_5) = \{u_4, u\}, N^+(u_6) = \{u_1, v_1\}$. However, (u_2, u_4) now constitutes a bad pair, contradicting our assumption.

Case 2.c): $\mathbf{O(u_1) = \{u_5, u_6\}, N^+(u_4) = \{u, v_4\}}$

As $u_4 \rightarrow v_4$, $u_4 \notin N^+(v_4)$. Hence $u_4 \in O(v) \cap O(v_1)$, implying that u_2 can reach u_4. Without loss of generality, $u_5 \rightarrow u_4$. There are three possibilities: i) $O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_5\}$, ii) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\}$ and iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$.

i) $O(v_1) = \{u_4, u_6\}, N^+(v_4) = \{v, u_5\}$

u_1 and v_4 have v as a unique common out-neighbour, so $u_4 \in O(v_4)$. However, this contradicts $v_4 \rightarrow u_5 \rightarrow u_4$.

ii) $O(v_1) = \{u_4, u_5\}, N^+(v_4) = \{v, u_6\}$

Neither u_4 nor v_1 lie in $T(u_2)$, so $v_4 \in O(u_2)$. Now observe that u_2 and v_4 have u_6 as unique common out-neighbour, so $v \in O(u_2)$, yielding $O(u_2) = \{v, v_4\}$ and $N^2(u_2) = \{u_4, u_1, u, v_1\}$. As $u_4 \rightarrow u$ and $u \rightarrow u_1$, we must have $N^+(u_5) = \{u_4, u_1\}, N^+(u_6) = \{u, v_1\}$, a contradiction, since $u_1 \rightarrow u_4$.

iii) $O(v_1) = \{u_4, v\}, N^+(v_4) = \{u_5, u_6\}$

We now have $N^+(u_2) = N^+(v_4)$, so $u_2 \in O(v_4), v_4 \in O(u_2), u_5 \in O(u_6), u_6 \in O(u_5)$. Also $N^+(u_4) = N^+(v_1)$, so $u_4 \in O(v_1), v_1 \in O(u_4)$ and $u \in O(v_4)$. $u \in O(v_4)$ implies that $u \notin N^+(u_5) \cup N^+(u_6)$, so we see that $u \in O(u_2)$ and hence $O(u_2) = \{u, v_4\}$ and $N^2(u_2) = \{u_4, u_1, v, v_1\}$. As $u_1 \rightarrow u_4$ and $u_1 \rightarrow v$, we have $N^+(u_5) = \{u_4, v\}, N^+(u_6) = \{u_1, v_1\}$. It is not difficult to show that this yields a $(2, 2, +2)$-digraph isomorphic to that in Figure 5.

Case 2.d): $\mathbf{O(u_1) = \{u_5, v_4\}, N^+(u_4) = \{u, u_6\}}$

In this case $v_4 \in O(u) \cap O(u_1)$, so u_2 can reach v_4. u_4 and v_1 have unique common out-neighbour u, so $v_4 \in O(u_4), u_6 \in O(v_1)$. If $u_6 \rightarrow v_4$, then we would have $u_4 \rightarrow u_6 \rightarrow v_4$, contradicting $v_4 \in O(u_4)$, so $u_5 \rightarrow v_4$. This also implies that $u_5 \notin N^+(v_4)$, so $u_5 \in O(v_1)$, yielding $O(v_1) = \{u_5, u_6\}$ and $N^+(v_4) = \{v, u_4\} = N^+(u_1)$. Now $v_4, u_1 \notin T(u_2)$, so $O(u_2) = \{v, u_4\}$ and $N^2(u_2) = \{v_1, v_4, u, u_1\}$. As $v_1 \rightarrow v_4$ and $v_1 \rightarrow u$, it follows that $N^+(u_5) = \{v_4, u\}, N^+(u_6) = \{u_1, v_1\}$. However, we now have paths $u_4 \rightarrow u \rightarrow u_1$ and $u_4 \rightarrow u_6 \rightarrow u_1$, which is impossible.

Case 3: $\mathbf{N^+(u_1) = N^+(v_1)}$

It is easy to see by 2-geodecity that $V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\}$, $O(u) = \{v, v_1\}$ and $O(v) = \{u, u_1\}$. As $u_1, v_1 \notin T(u_2)$, we have $O(u_2) = \{u_3, u_4\}$ and $N^2(u_2) = \{u, u_1, v, v_1\}$. Without loss of generality, $N^+(u_5) = \{u, v_1\}, N^+(u_6) = \{v, u_1\}$. u and v have in-neighbours apart from u_5 and u_6 respectively, so without loss of generality $u_3 \rightarrow u, u_4 \rightarrow v$. Likewise, u_5 and u_6 have in-neighbours other than u_2, so, as $u_5 \rightarrow u$ and $u_6 \rightarrow v$, we must have $N^+(u_3) = \{u, u_6\}, N^+(u_4) = \{v, u_5\}$. But now we have paths $u_3 \rightarrow u \rightarrow u_1$ and $u_3 \rightarrow u_6 \rightarrow u_1$, violating 2-geodecity.

Corollary 2. *There is a unique $(2, 2, +2)$-digraph containing no bad pairs.*

This completes our analysis of diregular $(2, 2, +2)$-digraphs. As it was shown in [7] that there are no non-diregular $(2, 2, +2)$-digraphs, $(2, 2, +2)$-digraphs are now classified up to isomorphism. These conclusions have been verified computationally by Erskine [2]. It is interesting to note that neither of the $(2, 2, +2)$-digraphs are vertex-transitive, for in each case there are exactly three vertices

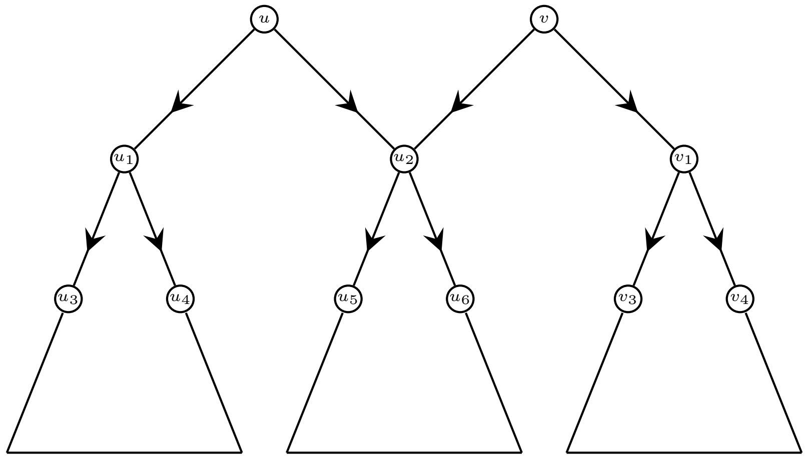

Figure 6: Configuration for $k \geq 3$

contained in two 3-cycles. However, there does exist a Cayley $(2, 2, +5)$-digraph (on the alternating group A_4), so it would be interesting to determine the smallest vertex-transitive $(2, 2, +\epsilon)$-digraphs.

4. Main result

We can now complete our analysis by showing that there are no diregular $(2, k, +2)$-digraphs for $k \geq 3$. Let G be such a digraph. By Lemma 2, G contains vertices u and v with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in Figure 6. A triangle based at a vertex x represents the set $T(x)$.

We now proceed to determine the possible outlier sets of u and v.

Lemma 5. *$v \in N^{k-1}(u_1) \cup O(u)$ and $u \in N^{k-1}(v_1) \cup O(v)$. If $v \in O(u)$, then $u_2 \in O(u_1)$ and if $u \in O(v)$, then $u_2 \in O(v_1)$.*

Proof. v cannot lie in $T(u)$, or the vertex u_2 would be repeated in $T_k(u)$. Also, $v \notin T(u_2)$, or there would be a $\leq k$-cycle through v. Therefore, if $v \notin O(u)$, then $v \in N^{k-1}(u_1)$. Likewise for the other result. If $v \in O(u)$, then neither in-neighbour of u_2 lies in $T(u_1)$, so that $u_2 \in O(u_1)$. □

Lemma 6. *Let $w \in T(v_1)$, with $d(v_1, w) = l$. Suppose that $w \in T(u_1)$, with $d(u_1, w) = m$. Then either $m \leq l$ or $w \in N^{k-1}(u_1)$. A similar result holds for $w \in T(u_1)$.*

Proof. Let w be as described and suppose that $m > l$. Consider the set $N^{k-m}(w)$. By construction, $N^{k-m}(w) \subseteq N^k(u_1)$, so by k-geodecity $N^{k-m}(w) \cap T(u_1) = \varnothing$. At the same time, we have $l+k-m \leq k-1$, so $N^{k-m}(w) \subseteq T(v_1)$. This implies that $N^{k-m}(w) \cap T(v_2) = N^{k-m}(w) \cap T(u_2) = \varnothing$. As $V(G) = \{u\} \cup T(u_1) \cup T(u_2) \cup O(u)$, it follows that $N^{k-m}(w) \subseteq \{u\} \cup O(u)$. Therefore $|N^{k-m}(w)| = 2^{k-m} \leq 3$. By assumption $0 \leq m \leq k-1$, so it follows that $m = k-1$. □

Corollary 3. *If $w \in T(v_1)$, then either $w \in \{u\} \cup O(u)$ or $w \in T(u_1)$ with $d(u_1, w) = k-1$ or $d(u_1, w) \leq d(v_1, w)$.*

Proof. By k-geodecity and Lemma 6. □

Corollary 4. *$v_1 \in N^{k-1}(u_1) \cup O(u)$ and $u_1 \in N^{k-1}(v_1) \cup O(v)$.*

Proof. We prove the first inclusion. By Corollary 3, $v_1 \in \{u\} \cup O(u) \cup \{u_1\} \cup N^{k-1}(u_1)$. By k-geodecity, $v_1 \neq u$ and by construction, $v_1 \neq u_1$. □

We now have enough information to identify one member of $O(u)$ and $O(v)$.

Lemma 7. *$v_1 \in O(u)$ and $u_1 \in O(v)$.*

Proof. We prove that $v_1 \in O(u)$. Suppose that neither v_1 nor v lies in $O(u)$. Then by Lemma 5 and Corollary 4 we have $v, v_1 \in N^{k-1}(u_1)$. As v_1 is an out-neighbour of v, it follows that v_1 appears twice in $T_k(u_1)$, violating k-geodecity. Therefore $O(u) \cap \{v, v_1\} \neq \varnothing$.

Now assume that $v_1, v_3 \in T_k(u)$. Again by Corollary 4, $v_1 \in N^{k-1}(u_1)$. By k-geodecity we also have $v_3 \in T(u_1)$. However, $v_3 \in N^+(v_1)$, so v_3 appears twice in $T_k(u_1)$, which is impossible. Hence $O(u) \cap \{v_1, v_3\} \neq \varnothing$. Similarly, $O(u) \cap \{v_1, v_4\} \neq \varnothing$. In the terminology of the previous section, G contains no bad pairs. Therefore, if $v_1 \notin O(u)$, then $\{v, v_3, v_4\} \subseteq O(u)$. Since these vertices are distinct, this is a contradiction and the result follows.

$\square$

Lemma 7 allows us to conclude that for vertices sufficiently close to v_1 one of the potential situations mentioned in Corollary 3 cannot occur.

Lemma 8. *$T_{k-3}(v_1) \cap N^{k-1}(u_1) = T_{k-3}(u_1) \cap N^{k-1}(v_1) = \varnothing$.*

Proof. Let $w \in T_{k-3}(v_1) \cap N^{k-1}(u_1)$. Consider the position of the vertices of $N^+(w)$ in $T_k(u) \cup O(u)$. As $v_1 \notin N^+(w)$, it follows from Lemma 7 that at most one of the vertices of $N^+(w)$ can be an outlier of u, so let us write $w_1 \in N^+(w) - O(u)$. By k-geodecity, $w_1 \notin T(u_1) \cup \{u\}$. Hence $w_1 \in T(u_2) = T(v_2)$. However, w_1 also lies in $T(v_1)$, so this violates k-geodecity. $\square$

Corollary 5. *There is at most one vertex in $T_{k-3}(v_1) - \{v_1\}$ that does not lie in $T(u_1)$; for all other vertices $w \in T_{k-3}(v_1) - \{v_1\}$, $d(u_1, w) = d(v_1, w)$. A similar result for $T_{k-3}(u_1) - \{u_1\}$ also holds.*

Lemma 9. *For $k = 3$, $N^+(u_1) \cap N^2(v_1) = N^+(v_1) \cap N^2(u_1) = \varnothing$.*

Proof. Suppose that $v_3 = u_7$. By the reasoning of Lemma 8 we can set $u = v_7$ and $O(u) = \{v_1, v_8\}$. $v \notin O(u)$ and by 3-geodecity $v \notin N^+(u_3)$, so we can assume that $v = u_9$. $u_3 \to v_3$ implies that $u_3 \notin T(v_1)$, so $O(v) = \{u_1, u_3\}$. We must have $\{u_4, u_8, u_{10}\} = \{v_4, v_9, v_{10}\}$. As $u_4 \to v$, it follows that $v_4 = u_8$ and hence $\{u_4, u_{10}\} = \{v_9, v_{10}\}$, which is impossible. $\square$

As u_1 is an outlier of v, neither v_3 nor v_4 can be equal to u_1. It follows from Corollary 5 and Lemma 9 that either $N^+(u_1) = N^+(v_1)$ or u_1 and v_1 have a single common out-neighbour, with one vertex of $N^+(v_1)$ being an outlier of u.

Lemma 10. $N^2(u) \neq N^2(v)$

Proof. Let $N^2(u) = N^2(v)$, with $N^+(u_1) = N^+(v_1) = \{u_3, u_4\}$. Suppose that $v \notin O(u)$. By Lemma 5, $v \in N^{k-2}(u_3) \cup N^{k-2}(u_4)$. But then there is a k-cycle through v. It follows that $O(u) = \{v, v_1\}, O(v) = \{u, u_1\}$. By Lemma 5, $u_2 \in O(u_1) \cap O(v_1)$. Therefore by Lemma 1 $O(u_1) = \{u_2, v_1\}, O(v_1) = \{u_2, u_1\}$.

Consider the in-neighbour u' of u_1 that is distinct from u. We have either $|N^+(u') \cap N^+(u)| = 1$ or $|N^+(u') \cap N^+(u)| = 2$. In the first case, it follows from Lemma 7 that $u_2 \in O(u')$. Every vertex of G is an outlier of exactly two vertices, so $u' = u_1$ or v_1. In either case, we have a contradiction. Therefore $N^+(u') = N^+(u)$. It now follows from Lemma 1 that $u' \in O(u) = \{v, v_1\}$, which is impossible.

$\square$

Noticing that u_1 and v_1 also have a unique common out-neighbour, we have the following corollary.

Corollary 6. *Without loss of generality,* $u_3 = v_3, u_9 = v_9, O(u) = \{v_1, v_4\}, O(v) = \{u_1, u_4\}, O(u_1) = \{v_4, v_{10}\}$ *and* $O(v_1) = \{u_4, u_{10}\}$.

We are now in a position to complete the proof by deriving a contradiction.

Theorem 2. *There are no diregular* $(2, k, +2)$*-digraphs for* $k \geq 3$.

Proof. $u, v \notin \{u_1, u_4, v_1, v_4\}$, so by Lemma 5 $d(u, v) = d(v, u) = k$. In fact, $u_3 = v_3$ implies that $v \in N^{k-2}(u_4)$ and $u \in N^{k-2}(v_4)$. Let $k \geq 4$. Then $u, v \notin \{u_{10}, v_{10}\}$, so $u, v \in T_k(u_1) \cap T_k(v_1)$. If $u \in T(u_3) = T(v_3)$, then u would appear twice in $T_k(v_1)$, so $u \in N^{k-1}(u_4)$. However, as u and v have a common out-neighbour, this violates k-geodecity.

Finally, suppose that $k = 3$. The above analysis will hold unless $u = v_{10}$ and $v = u_{10}$. Let $N^-(u_1) = \{u, u'\}, N^-(v_1) = \{v, v'\}$. It is evident that $v' \notin \{v_1, v_4\}$, so that $v' \in T_3(u)$. As $v \in N^+(u_4)$, we must have $v' \in N^2(u_2)$. Similarly $u' \in N^2(u_2)$. Since u_1 and v_1 have a common out-neighbour, we can assume that $u' \in N^+(u_5)$ and $v' \in N^+(u_6)$. v_4 can be the outlier of only two vertices, namely u and u_1, so $v_4 \in N^3(u_2)$ and likewise $u_4 \in N^3(u_2)$. By 3-geodecity $v_4 \in N^2(u_5)$ and $u_4 \in N^2(u_6)$. It follows that $u, v \notin N^3(u_2)$, so $u \notin T_3(u_1) \cup T_3(u_2)$. Hence $O(u) = N^-(u) = \{v_1, v_4\}$, which again is impossible.

□

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of [6].

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