Delay embedding with a fixed observation map

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Abstract

Delay coordinates are a widely used technique to pass from observations of a dynamical system to a representation of the dynamical system as an embedding in Euclidean space. Current proofs show that delay coordinates of a given dynamical system result in embeddings generically over a space of observations (Sauer, Yorke, Casdagli, *J. Stat. Phys.*, vol. 65 (1991), p. 579-616). Motivated by applications of the embedding theory, we consider the situation where the observation function is fixed. For example, the observation function may simply be some fixed coordinate of the state vector. For a fixed observation function (any nonzero linear combination of coordinates) and for the special case of periodic solutions, we prove that delay coordinates result in an embedding generically over the space of flows in the C^r topology with $r \geq 2$.

1 Introduction

Suppose a physical system is described by the differential equation $\frac{dx}{dt} = f(x)$, where $f : \mathbb{R}^d \to \mathbb{R}^d$. Often the state vector x is unobservable in its entirety, and that is especially true if d is large. Thus, reconstructing the flow from observations is not straightforward. The technique of delay coordinates makes it possible to look at a single scalar observation and reconstruct the dynamics. We denote the scalar that is observed by πx. The observation function π could be a projection to a single coordinate, for example, when the velocity of a fluid flow is recorded at a single point and in a single direction. It could be some other linear function of x. More generally, the observation function πx could be nonlinear.

If $\phi_t(x)$ is the time-t flow map, the idea behind delay coordinates [11, 16, 18] is to use the delay vector

$$\xi(x; \tau, n) = \left(\pi x, \pi\phi_{-\tau}(x), \ldots, \pi\phi_{-(n-1)\tau}(x)\right),$$

which is observable, as a surrogate for the point x in phase space. For a suitable choice of delay τ and embedding dimension n, delay coordinates yield a faithful representation of the phase space in a sense we will discuss later in this introduction. Delay coordinates have been employed in many applications [2, 17].

Packard et al [11] demonstrated that coordinate vectors such as $(\pi\phi_t(x), \frac{d}{dt}\pi\phi_t(x))$ give good representations of strange attractors. They noted that delay coordinate vectors would be equivalent to coordinate vectors formed using derivatives of the observed quantity.

A mathematical analysis of delay coordinates was undertaken in a famous paper by Takens [18] (and also [1], which we will discuss in a later section). In particular, Takens considered when $x \to \xi(x;\tau,n)$ is an embedding. Suppose M is a manifold of dimension m, $A \subset M$ a submanifold of M of dimension d, and $f : M \to N$ a continuous map from M to the manifold N. The restriction $f\big|_A$ is an embedding of A in N if the tangent map df has full rank at every point of A, $f\big|_A$ is injective, and the $f\big|_A$ has a continuous inverse from its range back to A [6, 8]. For the definition to make sense, the manifolds and f must be at least C^1. More generally, the manifolds M, N and f may be assumed to be C^r with $r \geq 1$ or with $r = \infty$. Takens concluded that delay coordinates yield an embedding of compact manifolds without boundary if $n \geq 2m+1$, for generic observation functions π and generic vector fields f.

The paper by Sauer et al [16] marked a major advance in the theory of delay coordinates. The approach to embedding theorems outlined by Takens relied on parametric transversality. Parametric transversality arguments typically have a local part and a global part, and the transition from local arguments to a global theorem is made using partitions of unity [8].

Sauer et al [16] sidestepped transversality theory almost entirely. Unlike in transversality theory, there is no explicitly local part in the arguments of Sauer et al [16]. The local part of the argument comes down to a verification of Lipshitz continuity. The set being embedded is only assumed to have finite box counting dimension. The arguments are mostly probabilistic and the globalization step relies only on the finiteness of the box counting dimension. The only real analogy to differential topology appears to be to the proof of Sard's theorem [8], which too is proved using probabilistic arguments. Sauer et al prove prevalence [9], which goes beyond genericity. If the consideration is restricted to subsets A of box counting dimension d, Sauer et al only require $n > 2d$. Thus, we could even have $n < m$.

The embedding theorem of Sauer et al [16] fixes the dynamical system and allows only the observation function π to be perturbed. The statements of genericity and prevalence are with regard to π, not the original dynamical system.

In this article, we consider embedding theorems in which the observation map is fixed. For example, π could be fixed as simply a linear projection that extracts some component of the state vector. We allow perturbations only of the dynamical system.

The motivation for considering such embedding theorems is as follows. First, on purely aesthetic grounds, it appears desirable to have an embedding theory that depends upon the dynamics and not the observation function. Second, in many applications the observation function is fixed, whereas the dynamical system itself is parametrized [2, 3, 5, 14, 17]. If π extracts a single component at a single point in the velocity field of a fluid, it is more pertinent to make the embedding theory depend upon the dynamics rather than upon the observation function.

In such a setting, with a fixed observation function, we prove that periodic orbits embed generically in $\mathbb{R}^3$. The techniques we use are those of transversality theory. The theorems that we prove show why an embedding theory that depends only on the dynamics can become more complicated. Although periodic orbits are only a special case, they are an important special case and arise frequently in applications. Thus, periodic orbits are a good place to begin the study of embeddings from more of a dynamical point of view.

To conclude this introduction, we mention some other extensions of delay coordinate embedding theory. Embedding theory has been considered for endomorphisms [19] as well as delay differential equations [4], and in concert with Kalman filtering [7]. The concept of determining modes and points in fluid mechanics and PDE is related to embedding theory [10, 14, 15]. The embedding theory of Sauer et al [16] has been generalized to PDE by Robinson [14, 15]. The embedding theory for PDE also relies on perturbing the observation function.

2 Embedding periodic signals in $\mathbb{R}^3$

Let $\mathcal{O}$ be the set of C^r functions $o : [0, T] \to \mathbb{R}$ with period equal to $T > 0$. Periodicity requires r derivatives of $o(t)$ to match at $t = 0$ and $t = T$. The domain of functions in $\mathcal{O}$, which we will write as $[0, T)$, is compact and homeomorphic to S^1. More precisely, the domain is the identification space obtained by identifying 0 and T in $[0, T]$. For convenience, we shall refer to it as $[0, T)$, with the understanding that when we refer to an interval (α, β) it can wrap around. The elements of $\mathcal{O}$ will be referred to as periodic signals.

Our interest later will be in periodic orbits. Suppose that $\frac{dx}{dt} = f(x)$ is a C^r differential equation in $\mathbb{R}^d$ with $r \geq 2$ and that $x_p(t)$, $0 \leq t < T$, is a hyperbolic periodic solution of period T. We will prove later that $t \to (a^T x_p(t), a^T x_p(t - \tau), a^T x_p(t - 2\tau))$ is an embedding of the circle for generic f and fixed a with $a \in \mathbb{R}^d$ and $a \neq 0$.

However, we will begin by limiting ourselves to periodic signals. Given a periodic signal $o \in \mathcal{O}$, the map $t \to (o(t), o(t - \tau), o(t - 2\tau))$ for $0 \leq t < T$ may not be an embedding of the circle. We first focus on perturbing o so that that map becomes an embedding if it is not already an embedding. The embedding result for periodic orbits is proved using the embedding result for periodic signals.

2.1 Local argument for periodic signals

If $r \in \mathbb{Z}^+$ and $o, o' \in \mathcal{O}$ are two periodic signals, define

$$d(o, o') = \sup_{k=0,\ldots r} \sup_{0 \leq s < 1} |o^{(k)}(sT) - o'^{(k)}(sT')| + |T - T'|.$$

The C^r topology on $\mathcal{O}$ is defined by this metric, which may be denoted using d or d_r. The C^∞ topology is the union of C^r topologies over $r \in \mathbb{Z}^+$. The C^∞ topology is also a metric topology. If d_r is the metric for the C^r topology with $r < \infty$, the metric for C^∞ can be taken to be $\sum d_r / 2^r$.

For differential topology, our main reference is [8]; [6] is another reference, and [12, 14] are references from a dynamical point of view.

A periodic signal signal can even be constant for $0 \leq t < T$ implying that any attempt to embed it using delay coordinates will fail. However, if a periodic signal has finitely many local maxima or minima and can be broken up into intervals of strict monotonicity between such critical points, points nearby are mapped injective by delay coordinates. Therefore, we begin by looking at conditions under which the periodic signal can be broken up into intervals of strict monotonicity.

Lemma 1. *Let $o \in \mathcal{O}$ be a periodic signal of period $T > 0$. Assume the C^r topology on $\mathcal{O}$ with $r \geq 2$. If 0 is a regular value of do/dt, then the periodic signal $o(t)$ has finitely many critical points in $[0, T)$.*

Proof. Suppose $do/dt = 0$ at infinitely many points on the compact circle $[0, T)$. Let $p \in [0, T)$ be an accumulation point of the set of zeros. Then $d^2 o(p)/dt^2 = 0$ and $do(p)/dt = 0$ implying that 0 is not a regular value of do/dt. □

Lemma 2. *Given* $(\alpha, \beta) \subset [0, T)$ *and* $\delta > 0$*, for all sufficiently small* ϵ *there exists an infinitely differentiable periodic signal* o *such that* $do(t)/dt = \epsilon$ *for* $t \notin (\alpha, \beta)$ *and* $|do(t)/dt| < \delta$ *for* $t \in (\alpha, \beta)$*. In addition, the* C^r *norm of* o *tends to zero as* $\epsilon \to 0$*.*

Proof. Let $\lambda(x)$ be an infinitely differentiable bump function with $\lambda(x) \in [0, 1]$ for $x \in [0, 1]$, $\lambda(x) = 1$ for $x \in [1/4, 3/4]$, and $\lambda(x) = 0$ for $x \in [0, 1/8]$ and $x \in [7/8, 1]$. If $\int_0^1 \lambda(x)\, dx = c$ then $1/2 < c < 1$.

Define $do(t)/dt = \epsilon$ for $t \notin (\alpha, \beta)$ and more generally

$$\frac{do(t)}{dt} = \epsilon - k\lambda((t - \alpha)/(\beta - \alpha))$$

for $t \in [0, T)$. It follows that $\int_0^T (do(t)/dt)\ dt = \epsilon T - k(\beta - \alpha)c$. The integral is zero if $k = \epsilon T/(\beta - \alpha)c$. For ϵ small, we have k small as well, and we may obtain $o(t)$ by integrating $do(t)/dt$. □

Lemma 3. *If* o' *is a periodic signal, there exists another periodic signal* o *of the same period with* $d(o, o')$ *arbitrarily small in the* C^r *sense,* $2 \leq r \leq \infty$*, such that* o *has only finitely many critical points (including local maxima and minima) and* 0 *is a regular value of* do/dt*.*

Proof. If $o'(t)$ is constant we can perturb to $\epsilon \sin(tT/2\pi)$ for arbitrarily small ϵ and verify the theorem. We will assume that o' is not constant.

Consider $\frac{do'}{dt}(t)$ as a map from the circle $[0, T')$ to $\mathbb{R}$. If 0 is a regular value of this map, we are done by Lemma 1.

If not, there exists a regular value ϵ of do'/dt arbitrarily close to 0 by Sard's theorem (here $r \geq 2$ is needed). Suppose we look at $do'(t)/dt - \epsilon$. This function has a regular value at 0. However, the corresponding perturbation of o' is $o'(t) - t\epsilon$ and is not periodic.

Because $o'(t)$ is not constant, there exists an interval (α, β) in the circle $[0, T)$ over which $do'(t)/dt$ is nonzero. Without loss of generality, we assume $do'(t)/dt > \delta > 0$ in the interval (α, β) (consider $-o'(t)$ for the case where the derivative is negative). Using Lemma 2, we may find a periodic signal $p(t)$ such that $dp/dt = \epsilon$ for $t \notin (\alpha, \beta)$ and $|dp/dt| < \delta$ for $t \in (\alpha, \beta)$. Set $o(t) = o'(t) - p(t)$ to obtain a periodic signal with 0 being a regular value of do/dt to complete the proof. □

Remark. Lemma 1 is evidently true if we only assume second derivative of the periodic signal $o(t)$ to exist and not necessarily continuous. In fact, Lemma 3 is also true under the same weaker assumption because, in one dimension, Sard's theorem requires only the existence of the derivative (see Exercise 1 of Section 3.1 of [8]).

Remark. The main difficulty in proving Lemma 3 is in dealing with repeated oscillations as in $x^2 \sin(1/x)$ near $x = 0$, although that issue does not come up explicitly in the proof. Since the periodic signal is C^2, when the critical points accumulate at a point, the function derivative must become very small to prevent the second derivative from blowing up. The proof indirectly relies on that phenomenon.

If o is a periodic signal with finitely many critical points, then its circular domain $[0, T)$ may be decomposed into finitely many intervals with local minima and maxima at either end. Let μ denote the minimum width among such intervals. Because $o(t)$ is monotonic in each interval, we refer to each such interval as the minimum interval of strict monotonicity. If the delay is τ, we denote the point $(o(t), o(t-\tau), o(t-2\tau))$ by $o(t;\tau)$.

Even if o has only finitely many critical points, $t \to o(t;\tau)$ may not be an embedding of the circle, which is clear by considering $o(t) = \sin 100t$ and $\tau < \pi/100$ for instance. However, if t_1 and t_2 are close enough, we may deduce that $o(t_1;\tau) \neq o(t_2;\tau)$ as the following lemma shows.

Lemma 4. *If $0 < |t_1 - t_2| \leq \mu/3$, where μ is the minimum interval of strict monotonicity, and if the delay τ satisfies $0 < \tau \leq \mu/3$, then $o(t_1;\tau) \neq o(t_2;\tau)$. If 0 is a regular value of $\frac{do(t)}{dt}$, we also have $\frac{do(t;\tau)}{dt} \neq 0$ for all $t \in [0,T)$.*

Proof. Because $|t_1 - t_2| \leq \mu/3$, $t_1$and t_2 lie in either the same interval of strict monotonicity of the periodic signal $o(t)$ or in neighboring intervals. If they lie in the same interval, we must have either $o(t_1) < o(t_2)$ or $o(t_2) < o(t_1)$ proving the lemma.

If t_1 and t_2 lie in neighboring intervals, we may assume $t_1 < t_2$ without loss of generality. If $o(t_1) \neq o(t_2)$, there is nothing to prove. So we assume $o(t_1) = o(t_2)$ in addition. Again without loss of generality, we assume that $o(t)$ first increases and then decreases as t increases from t_1 to t_2.

With these assumptions, t_1 and $t_1 - \tau$ must lie in the same interval of monotonicity because $\tau \leq \mu/3$, and therefore $o(t_1 - \tau) < o(t_1)$. Further $t_2 - \tau \in (t_1 - \tau, t_2)$ and the unique minimum of $o(t)$ for $t \in [t_1 - \tau, t_2]$ is attained when $t = t_1 - \tau$. Therefore $o(t_1 - \tau) < o(t_2 - \tau)$, and we once again have $o(t_1;\tau) \neq o(t_2;\tau)$.

For the claim about $\frac{do(t;\tau)}{dt} \neq 0$, we note that $\frac{do}{dt}$ cannot equal zero at both t and $t-\tau$, because $\tau < \mu$. ☐

With Lemma 4, the local argument for embedding periodic signals is partly complete. Globalizing the argument will involve additional perturbations, which we now define.

Let λ be a C^∞ bump function with $\lambda(x) = 1$ for $|x| \leq 1/2$, $\lambda(x) = 0$ for $|x| \geq 1$, and $\lambda(x) \in [0,1]$ for all $x \in \mathbb{R}$. Let $h = \tau/2$ and $j \in \mathbb{Z}$. Define

$$\lambda_j(t) = \lambda\left(\frac{t - jh}{h}\right)$$

for $j = 0, 1, \ldots, n$ and $n = \lfloor T/h \rfloor$. We interpret t modulo T and regard $\lambda_j(t)$ as a periodic signal with the circular domain $[0,T)$: a pulse of period T and width h centered at jh which is equal to 1 for $|t - jh| \leq h/2$. We now consider the perturbation

$$o_\epsilon(t) = o(t) + \epsilon_1 \lambda_1(t) + \cdots + \epsilon_n \lambda_n(t), \tag{2.1}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n$. For any $t_0 \in [0,T)$, there exists a bump function $\lambda_j(t)$ with $0 \leq j \leq n$ such that $\lambda_j(t_0) = 1$ and therefore $\lambda_j(t) = 0$ if $|t - t_0| \geq \tau = 2h$. In fact the choice of j is unique, except if t_0 is an odd multiple of $h/2$.

Before we turn to the global argument, we must prove that the local structure asserted by Lemma 4 is preserved when o is perturbed to o_ϵ as in (2.1). The lemma below guarantees $o_\epsilon(t_1;\tau) \neq o_\epsilon(t_2;\tau)$ for $|t_1 - t_2| \leq 3\tau$. The bound 3τ ensures that $o_\epsilon(t_1;\tau) = o_\epsilon(t_2;\tau)$ can happen only when the intervals $[t_1 - 2\tau, t_1]$ and $[t_2 - 2\tau, t_2]$ do not overlap.

Lemma 5. *Let o be a periodic signal defined over the domain $[0,T)$ and with minimum interval of strict monotonicity equal to μ. The topology over periodic signals is assumed to be C^r with*

$r \geq 2$. *Assume that* 0 *is a regular value of* do/dt. *There exists* ϵ_0 *such that if* $||\epsilon|| \leq \epsilon_0$, *then for the perturbation defined by (2.1) and delay* τ *satisfying* $0 < \tau < \mu/12$, *we have* $o_\epsilon(t_1;\tau) \neq o_\epsilon(t_2;\tau)$ *for all* (t_1, t_2) *with* $|t_1 - t_2| \leq 3\tau$. *In addition,* 0 *remains a regular value of* $\frac{do_\epsilon}{dt}$.

Proof. By assumption the periodic signal $o(t)$ has finitely many critical points. Let $t_1 < t_2 < \cdots < t_k$ be the critical points in the circular interval $[0, T)$; at these points and only at these, we have $do/dt = 0$. Since 0 is a regular value of do/dt, we have $o''(t_j) \neq 0$ for $j = 1, \ldots, k$.

In the circle $[0, T)$, choose compact intervals $K_i = [t_i - \delta, t_i + \delta]$, $i = 1, \ldots, k$, such that $\delta < \mu/4$ and $o''(t) \neq 0$ for any $t \in K_i$. By continuity in the perturbing parameters ϵ_i, for sufficiently small $||\epsilon||$ the perturbed periodic signal (2.1) also has nonzero second derivative on $\cup K_i$.

Define the interval K_i' to be $[t_i + \delta/2, t_{i+1} - \delta/2]$ (K_k' wraps around the circle). Each K_k' is an interval of strict monotonicity. By compactness, $|do/dt|$ attains a minimum strictly greater than 0 over $\cup K_i'$. Again by continuity, any perturbation of the form (2.1) with $||\epsilon||$ sufficiently small also has nonzero derivative over $\cup K_i'$.

Thus for $||\epsilon||$ sufficiently small, K_i' remain intervals of strict monotonicity for the perturbed periodic signal, and each K_i can contain at most one critical point of the perturbed periodic signal. The minimum interval of strict monotonicity is at least $\mu - \delta \geq 3\mu/4$. We now apply Lemma 4 to infer that $0 < \tau \leq \mu/4$ implies $o_\epsilon(t_1;\tau) \neq o_\epsilon(t_2;\tau)$ for $0 < |t_1 - t_2| \leq \mu/4$. We limit τ to the interval $(0, \mu/12)$ to complete the proof. □

2.2 Global argument for periodic signals

The global argument relies on the parametric transversality theorem [8, 13].

Lemma 6. *Let* o *be a periodic signal defined over the circle* $[0, T)$. *Assume the* C^r *topology over periodic signals with* $r \geq 2$. *There exists an arbitrarily small perturbation of the periodic signal* o *to* o', *with the same period, and* $\tau_0 > 0$, *such that* $t \to o'(t;\tau)$ *is an embedding for* $0 < \tau < \tau_0$, *with* 0 *a regular value of* do'/dt.

Proof. By Lemma 3, we may make an initial perturbation to o if necessary and assume that o has finitely many critical points, that 0 is a regular value of do/dt, and that $\mu > 0$ is the minimum width of an interval of strict monotonicity.

Now consider perturbations of o to o_ϵ of the form (2.1). By Lemma 5, we may assume $o_\epsilon(t_1;\tau) \neq o_\epsilon(t_2;\tau)$ for $t_1 \neq t_2$ and $|t_1 - t_2| \leq 3\tau$ for $\tau < \mu/18$, *provided* $||\epsilon||$ is sufficiently small.

Consider the set

$$\mathcal{T} = \left\{(t_1, t_2) \Big| |t_1 - t_2| > 3\tau,\ t_1 \in [0, T),\ t_2 \in [0, T)\right\},$$

where $[0, T)$ is interpreted as the circle, as before. For the applicability of the parametric transversality theorem later in the proof, it is important to note that $\mathcal{T}$ is a manifold of dimension 2 *without* a boundary.

Consider $(o_\epsilon(t_1;\tau), o_\epsilon(t_2;\tau))$ as a function from $\{(\epsilon_1, \ldots \epsilon_n)\} \times \mathcal{T}$ to $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$. We will now verify that this function is transverse to the diagonal in $\mathbb{R}^3 \times \mathbb{R}^3$. If $o_\epsilon(t_1;\tau) \neq o_\epsilon(t_2;\tau)$ there is nothing to prove. Suppose $o_\epsilon(t_1;\tau) = o_\epsilon(t_2;\tau)$ and consider the point in $\mathbb{R}^6$ given by

$$(o_\epsilon(t_1), o_\epsilon(t_1 - \tau), o_\epsilon(t_1 - 2\tau), o_\epsilon(t_2), o_\epsilon(t_2 - \tau), o_\epsilon(t_2 - 2\tau))$$

The intervals $[t_1 - 2\tau, t_1]$ and $[t_2 - 2\tau, t_2]$ are disjoint because $|t_1 - t_2| > 3\tau$. By construction, there exist $i_1, i_2, i_3, i_4, i_5, i_6$ such that $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3}, \lambda_{i_4}, \lambda_{i_5}, \lambda_{i_6}$ are each equal to 1 at exactly one of the six points $t_1, t_1 - \tau, t_1 - 2\tau, t_2, t_2 - \tau, t_2 - 2\tau$ and zero at the others. If the tangent direction in the domain is taken to perturb ϵ_{i_j} for $j \in \{1, \ldots, 6\}$, it maps to a perturbation of the j-th coordinate in $\mathbb{R}^6$, more precisely the elementary vector $\boldsymbol{e}_j$. Therefore the tangent map is surjective and transversality is verified.

By the parametric transversality theorem [Hirsch, Chapter 3, Theorem 2.7], we may choose ϵ arbitrarily small such that $(o_\epsilon(t_1;\tau), o_\epsilon(t_2;\tau))$ considered as a function from $\mathcal{T}$ to $\mathbb{R}^6$ is transverse to the diagonal of $\mathbb{R}^3 \times \mathbb{R}^3$. Since $\mathcal{T}$ is of dimension 2, that can only happen if $o_\epsilon(t_1;\tau) \neq o(t_2;\tau)$ for $(t_1, t_2) \in \mathcal{T}$.

To complete the proof, we only need to check the smoothness/dimension condition in the parametric transversality theorem. The dimension of $\mathcal{T}$ is 2 and the codimension of the diagonal in $\mathbb{R}^6$ is 3. Thus, it is sufficient if the map from $\{(\epsilon_1, \ldots \epsilon_n)\} \times \mathcal{T}$ to $\mathbb{R}^6$ is C^1 which it is. $\square$

Lemma 7. *Let $o \in \mathcal{O}$ be a periodic signal such that $t \to o(t;\tau)$ is an embedding of the circle $[0, T)$ in $\mathbb{R}^3$ for delay $\tau > 0$. There exists $\epsilon_0 > 0$ such that $d(o, o') < \epsilon_0$ in the C^r topology with $r \geq 1$ and $T = T'$ (perturbation has same period) imply that $t \to o'(t;\tau)$ is also an embedding of the circle $[0, T)$.*

Proof. By the inverse function theorem, there exists $\epsilon_0 > 0$ such that for every $\tilde{t} \in [0, T)$ there exists a neighborhood of $\tilde{t}$ over which $t \to o'(t;\tau)$ is an injection if $d(o', o) < \epsilon_0$ and $T = T'$. Using a Lebesgue-δ argument we may assume that $o'(t_1;\tau) \neq o'(t_2;\tau)$ for $0 < |t_1 - t_2| < \epsilon_0$.

Although arguments like the one above are common in differential topology, we state the version of the inverse function theorem invoked for clarity. The version used is as follows. Suppose f is a C^r map from U, an open subset of $\mathbb{R}^m$ to V, an open subset of $\mathbb{R}^n$ with $m < n$. Suppose $f(x) = y$ and that the tangent map $\frac{\partial f}{\partial x}$ is injective at x. Then there exists a neighborhood $\mathcal{N}$ of f in the weak C^r topology ($r \geq 1$), a neighborhood U' of x, V' of y, and W' of $0 \in \mathbb{R}^{n-m}$, such that for every $g \in \mathcal{N}$ there exists a diffeomorphism $G : V' \to U' \times W'$ with G^{-1} restricted to $U' \times 0$ coinciding with g. This theorem is applied with $m = 1$ and $n = 3$.

The rest of the proof is a standard compactness argument. Let

$$\min_{|t_1 - t_2| \geq \epsilon_0} |o(t_1;\tau) - o(t_2;\tau)| = \delta > 0,$$

where the minimum exists because of compactness and is greater than 0 because $t \to o(t;\tau)$ is an embedding. By continuity, the minimum must be positive for o' sufficiently close to o. Thus, $t \to o'(t;\tau)$ is also an embedding. $\square$

Theorem 8. *Let $\mathcal{O}$ be the set of periodic signals with C^r topology and $r \geq 2$. The set of periodic signals o for which there exists a delay $\tau > 0$ such that $t \to o(t;\tau)$ is an embedding of the circle $[0, T)$ in $\mathbb{R}^3$ is open and dense.*

Proof. Let o be any periodic signal in $\mathcal{O}$. By Lemma 6, there exists an arbitrarily small perturbation to o' such that $t \to o'(t;\tau)$ is an embedding for $0 < \tau < \tau_0$ and with 0 a regular value of do'/dt. Thus the set of periodic signals with a delay embedding and with 0 a regular value of do/dt is dense. We only have to prove that the set is open.

Given periodic signal o with $t \to o(t;\tau)$ an embedding, Lemma 7 shows that $t \to o'(t;\tau)$ remains an embedding for $d_1(o,o')$ sufficiently small and with $T = T'$. Here d_1 is the metric for the C^1 topology. If $T \neq T'$, we may still apply Lemma 7, by defining $o''(t) = o'(tT'/T)$ which is a periodic signal of period T. If $d_r(o,o')$ is small enough for $r \geq 2$, we will have $d_1(o,o'')$ small enough. Finally, $t \to o''(t;\tau)$ is an embedding implies that $t \to o'(t;\tilde{\tau})$ is an embedding with $\tilde{\tau} = \tau T'/T$. □

Theorem 9. *Suppose that $o \in \mathcal{O}$ is a C^2 periodic signal and that $t \to o(t;\tau)$ is an embedding of the circle for some delay $\tau > 0$. Then $t \to o(t;\tau')$ remains an embedding if τ' is close enough to τ.*

Proof. The arguments used in Lemma 7 and Theorem 8 apply with little change. □

3 Embedding periodic orbits in $\mathbb{R}^3$

The following proposition proves that the embedding using delay coordinates persists when the vector field is perturbed slightly.

Proposition 10. *Let $\frac{dx}{dt} = f(x)$, where $x \in \mathbb{R}^d$, $f : U \to \mathbb{R}^d$, and U an open subset of $\mathbb{R}^d$, be a C^r, $r \geq 2$, dynamical system. Let $x_p : [0,T) \to U$ be a hyperbolic periodic solution of period $T > 0$. Let $a \in \mathbb{R}^d$ and $a \neq 0$. If $t \to (a^T x_p(t), a^T x_p(t-\tau), a^T x_p(t-2\tau))$ be an embedding of the circle $[0,T)$ in $\mathbb{R}^3$. There exists an open neighborhood of f in the C^r topology such that for each g in that neighborhood, there exists a hyperbolic periodic solution $x_p'(t)$ of period T' and τ' close to τ such that $t \to (a^T x_p'(t), a^T x_p'(t-\tau'), a^T x_p'(t-2\tau'))$ is an embedding of the circle $[0,T')$ in $\mathbb{R}^3$.*

Proof. The fact that a hyperbolic periodic solution such as x_p perturbs to a nearby hyperbolic solution x_p' in a small enough open neighborhood of f is a standard result [14]. If the signal $o(t) = a^T x_p(t)$ is such that $t \to o(t;\tau)$ is an embedding of the circle, then $t \to o'(t;\tau')$ is also an embedding for $o'(t) = a^T x_p'(t;\tau')$ by Theorem 8. In fact the proof of Theorem 8 uses the choice $\tau' = \tau T'/T$. □

Let $\mathcal{P}$ denote the set of periodic orbits $\mathbf{p} : [0,T) \to \mathbb{R}^d$. As before, we assume that $[0,T)$ is a parametrization of S^1 and $T > 0$ for the period. We also assume $\mathbf{p}$ to be C^r with $r \geq 2$. By definition, $\frac{d\mathbf{p}}{dt} \neq 0$ for $t \in [0,T)$.

We denote the projection from $\mathbb{R}^d$ to the first coordinate by π_1. If $\mathbf{p}$ is a solution of the dynamical system $\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$, we wish to show that either $o(t) = \pi_1 \mathbf{p}(t)$ is such that $t \to o(t;\tau)$ is an embedding of the circle $[0,T)$ for some delay $\tau > 0$, or that there exists an arbitrarily close perturbed dynamical system $\frac{d\mathbf{x}}{dt} = f'(\mathbf{x})$ with a nearby periodic orbit $\mathbf{p}'$ such that $t \to o'(t;\tau)$ is an embedding of the circle, if $o' = \pi_1 \circ \mathbf{p}'$.

To begin with, the signal $o(t)$ may even be identically zero. In our proof, we use the results of the previous section to perturb it to $o'(t)$ such that $t \to o'(t;\tau)$ is an embedding and then show how to perturb the flow to realize $o'(t)$ as $\pi_1 \circ \mathbf{p}'$.

The first set of lemmas we prove construct a tube around the periodic orbit $\mathbf{p}$ in $\mathbb{R}^d$. That tube will be used to perturb f to f'.

Lemma 11. *Let* $\mathbf{p}$ *be a periodic orbit of period* T*. There exist* $q_i : [0, T) \to \mathbb{R}^d$*, with each* q_i *a periodic* C^r *function for* $i = 1, \ldots, d-1$*, such that*

$$\frac{d\mathbf{p}}{dt}\Big/\left|\left|\frac{d\mathbf{p}}{dt}\right|\right|, q_1, \ldots, q_{d-1}$$

is an orthonormal basis of $\mathbb{R}^d$ *for each* $t \in [0, T)$*. In addition, the* q_i *can be chosen such that the map* $\mathbf{p} \to q_i$ *is continuous.*

Proof. The plan of the proof is to construct q_i for a specific periodic orbit and then use orthogonal transformations to transport it to all other periodic orbits $\mathbf{p}$.

Begin with the periodic orbit $\mathbf{P}$ given by $\mathbf{P}(t) = (\cos t, \sin t, 0, \ldots, 0)^T$. The unit tangent vector is given by $(-\sin t, \cos t, 0, \ldots, 0)$. For this orbit we may take $Q_1(t) = (\cos t, \sin t, 0, \ldots, 0)^T$, $Q_2(t) = (0, 0, 1, 0, \ldots, 0)$, and finally, $Q_{d-1}(t) = (0, \ldots, 0, 1)^T$.

Given $\mathbf{p}(t)$ of period T, denote its unit tangent vector by $\mathbf{s}(t)$. Let $M(t)$ be a $d \times d$ orthogonal matrix that rotates $Q_1(2\pi t/T)$ to $\mathbf{s}(t)$ as follows. The matrix $M(t)$ will leave the space orthogonal to the span of $\mathbf{s}(t)$ and $Q_1(2\pi t/T)$ invariant. In the plane spanned by $\mathbf{s}(t)$ and $Q_1(2\pi t/T)$, it rotates $\mathbf{s}(t)$ to $Q_1(2\pi t/T)$. The matrix $M(t)$ is given by

$$M(t) = (I - Q_1 Q_1^T - \mathbf{s}\mathbf{s}^T) + \begin{pmatrix} Q_1 & \mathbf{s} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} Q_1^T \\ \mathbf{s}^T \end{pmatrix}.$$

The angle of rotation θ is uniquely determined module 2π and, therefore, the rotation matrix is uniquely determined. The vectors $q_i(t)$ may be obtained as $M(t)Q_i(2\pi t/T)$. ☐

Lemma 12. *Let the periodic orbit* $\mathbf{p}$ *and* $q_1, \ldots, q_{d-1}$ *be as in Lemma 11. Then there exists a* $\delta > 0$ *such that the* C^r *map from* $(\alpha_1, \ldots, \alpha_{d-1}, t) \in \mathbb{B}^{d-1} \times S^1$ *given by*

$$\mathbf{p}(t) + \delta \sum_{j=0}^{d-1} \alpha_j q_j$$

is diffeomorphic to its image. Here, the circle S^1 *is parametrized by* $[0, T)$ *and* $\mathbb{B}^{d-1}$ *is the solid unit sphere in* $\mathbb{R}^{d-1}$*. (In effect, the diffeomorphism is a parametrization of a tube around the periodic orbit). In addition, "the tube radius"* $\delta > 0$ *may be chosen to hold for all periodic orbits in some* C^r *neighborhood of* $\mathbf{p}$*.*

Proof. Let $\mathbf{s}(t)$ denote the tangent vector to the periodic orbit at t. Then $\mathbf{s}.\frac{d\mathbf{p}}{dt} > 2m$ for some $m > 0$ and all $t \in [0, T)$. There exists $r > 0$ such that $|t_1 - t| < r$ and $|t_2 - t| < r$ and $t_1 < t_2$ implies that

$$\mathbf{s}(t).(\mathbf{p}(t_2) - \mathbf{p}(t_1)) > m(t_2 - t_1) > 0. \tag{3.1}$$

The short interval from t_1, t_2 may cross the boundary point of the parametrization $[0, T)$ of the circle. The difference $t_2 - t_1$, in that case, must be taken to be small and positive modulo T.

We also have $\mathbf{s}(t).q_j(t) = 0$. Thus, we may assume that

$$|\mathbf{s}(t).q_j(t_1)| < M|t - t_1| \tag{3.2}$$

for $|t-t_1|<r$, by reducing r if necessary. Once again the difference $t-t_1$ must be interpreted appropriately if the t, t_1 interval wraps around $[0, T)$. From now on, we ignore the wrap around case in the proof.

Suppose that

$$\mathbf{p}(t_2)+\sum b_j q_j(t_2)=\mathbf{p}(t_1)+\sum a_j q_j(t_1) \tag{3.3}$$

for some numbers a_j, b_j and with $t_1<t_2$ and $|t_1-t_2|<2r$. Take $t=(t_1+t_2)/2$ and write

$$\mathbf{s}(t).(\mathbf{p}(t_2)-\mathbf{p}(t_1))=\sum a_j\mathbf{s}(t).q_j(t_1)-\sum b_j\mathbf{s}(t).q_j(t_2).$$

Cauchy's inequality along with (3.1) and (3.1) give

$$m|t_2-t_1|<M|t_2-t_1|\sqrt{2d}\left(\sum a_j^2+b_j^2\right)^{1/2}.$$

Thus, in the event of an "intersection" as in (3.3), we must have either $\sum a_j^2$ or $\sum b_j^2$ greater than $m/(2Md^{1/2})$.

If $|t_1-t_2|\geq r$, there exists $\Delta>0$ such that $||\mathbf{p}(t_2)-\mathbf{p}(t_1)||>\Delta$. Here too we may reduce r if necessary. We may thus choose δ to be smaller than both $\Delta/2$ and $m/(2Md^{1/2})$.

The last part of the lemma, about some $\delta>0$ that holds uniformly for all periodic orbits in a C^r neighborhood, $r\geq 2$, follows because r, m, M, and Δ depend only on bounds for derivatives of $\mathbf{p}(t)$. In greater detail, the argument for (3.1) is as follows:

$$\mathbf{p}(t_2)-\mathbf{p}(t_1)=(t_2-t_1)\int_0^1 \left.\frac{d\mathbf{p}}{dt}\right|_{t=t_1+u(t_2-t_1)} u\,du.$$

To get (3.1) with an r and an m that is uniformly valid, we first require that $\mathbf{s}'.\frac{d\mathbf{p}'}{dt}>2m$ for all periodic orbits $\mathbf{p}'$ in some neighborhood of $\mathbf{p}$. Such a requirement can be met because $\mathbf{s}'.\frac{d\mathbf{p}'}{dt}$ takes a positive minimum for each periodic orbit $\mathbf{p}'$, and that minimum is continuous with respect to $\mathbf{p}'$ in the C^r topology. Next we assume $|t-t_1|<r$ and consider

$$\begin{aligned}\mathbf{s}'.\left.\frac{d\mathbf{p}'}{dt}\right|_{t_1} &= \mathbf{s}'.\left.\frac{d\mathbf{p}'}{dt}\right|_{t}+\left(\mathbf{s}'.\left.\frac{d\mathbf{p}'}{dt}\right|_{t_1}-\mathbf{s}'.\left.\frac{d\mathbf{p}'}{dt}\right|_{t}\right)\\ &= \mathbf{s}'.\left.\frac{d\mathbf{p}'}{dt}\right|_{t}+(t-t_1)\mathbf{s}'.\int_0^1\left.\frac{d^2\mathbf{p}'}{dt^2}\right|_{t+s(t_1-t)}ds.\end{aligned}$$

The first requirement ensures that $\mathbf{s}'.\left.\frac{d\mathbf{p}'}{dt}\right|_{t}>2m$. In the C^r topology with $r\geq 2$, we may assume a uniform upper bound on the norm of $d^2\mathbf{p}'/dt^2$ in some suitably small neighborhood of $\mathbf{p}$. Thus we may make the second term smaller than m in magnitude by restricting r to be small enough, thus obtaining (3.1) in a neighborhood of $\mathbf{p}$.

The details for obtaining a uniform M in (3.2) are similar. Here we begin with

$$\mathbf{s}(t).q(t_1)=\mathbf{s}(t_1).q(t_1)+(t-t_1)q(t_1).\int_{s=0}^1\left.\frac{d\mathbf{s}}{dt}\right|_{t=t_1+s(t-t_1)}ds.$$

A uniform bound can be obtained as in the previous argument by considering $d^2\mathbf{s}'/dt^2$ for $\mathbf{p}'$ in some C^r neighborhood of $\mathbf{p}$, with $r\geq 2$. The argument for obtaining a uniform Δ too is quite similar and we therefore omit it. ∎

The following theorem asserts that a vector field may be perturbed such that the delay coordinates of the perturbed system imply an embedding of the perturbed periodic orbit.

Theorem 13. *Let $\mathbf{p}(t)$ be a periodic solution of the C^r dynamical system $d\mathbf{x}/dt = f(\mathbf{x})$. If $o(t) = \pi_1\mathbf{p}(t)$ is a periodic signal, there exists either a delay $\tau > 0$ such that $t \to o(t;\tau)$, $0 \le t < T$, is an embedding of the circle $[0,T)$ or another vector field f', arbitrarily close to f in the C^{r-1} topology, with a periodic solution $\mathbf{p}'(t)$ arbitrarily close to $\mathbf{p}(t)$ and of the same period such that $t \to \pi_1\mathbf{p}'(t;\tau)$ is an embedding of the circle $[0,T)$.*

Proof. Let $o(t) = \pi_1\mathbf{p}(t)$ and assume that there is no delay $\tau > 0$ such that $t \to o(t;\tau)$ is an embedding. By Lemma 6, we can find a periodic signal $o'(t)$ of period T, and arbitrarily close to $o(t)$ in the C^r norm, such that $t \to o'(t;\tau)$ for some $\tau > 0$. Define

$$\mathbf{p}'(t) = \mathbf{p}(t) + \begin{pmatrix} o'(t) - o(t) \\ 0 \\ \vdots \end{pmatrix}. \tag{3.4}$$

It suffices to construct a vector field f' such that $\mathbf{p}'(t)$ is a periodic solution of $\frac{d\mathbf{x}}{dt} = f'(\mathbf{x})$ and $f' \to f$ as $\mathbf{p}' \to \mathbf{p}$.

Using Lemma 12, find an open neighborhood U of $\mathbf{p}$ and a $\delta > 0$, such that a δ-tube may be constructed as in the lemma for all periodic orbits in U.

Choose $o'(t)$ of Lemma 6 such that $\mathbf{p}'(t)$, defined by (3.4), lies within the neighborhood U of $\mathbf{p}$. Now

$$\begin{aligned} \frac{d\mathbf{p}'(t)}{dt} &= \frac{d\mathbf{p}(t)}{dt} + \epsilon_1(t) \\ &= f(\mathbf{p}(t)) + \epsilon_1(t) \\ &= f(\mathbf{p}'(t)) + \epsilon_1(t) + \epsilon_2(t), \end{aligned}$$

where

$$\epsilon_1(t) = \begin{pmatrix} \frac{d(o'(t)-o(t))}{dt} \\ 0 \\ \vdots \end{pmatrix}$$

and $\epsilon_2(t) = f(\mathbf{p}(t)) - f(\mathbf{p}'(t))$. Evidently, as $o' \to o$ in the C^r topology over periodic signals, the periodic signals $\epsilon_1(t)$ and $\epsilon_2(t)$ go to 0 in the C^{r-1} topology over periodic orbits.

Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a C^∞ bump function with $\lambda(x) = 1$ for $|x| \le 1/2$ and $\lambda(x) = 0$ for $|x| \ge 3/4$. Following Lemma 12, consider the C^r diffeomorphism from the δ-tube around $\mathbf{p}'(t)$ to $(\alpha_1, \dots, \alpha_{d-1}, t)$, and denote it by Φ. If $\mathbf{x}$ is a point in the δ-tube around $\mathbf{p}'(t)$, we may parameterize the neighborhood as $(\alpha_1(\mathbf{x}), \dots, \alpha_{d-1}(\mathbf{x}), t(\mathbf{x}))$. The perturbation $\delta f : \mathbb{R}^d \to \mathbb{R}^d$ is defined as

$$\delta f(\mathbf{x}) = (\epsilon_1(t(\mathbf{x})) + \epsilon_2(t(\mathbf{x})))\lambda\left(\frac{(\alpha_1^2 + \cdots + \alpha_{d-1}^2)^{1/2}}{\delta}\right)$$

for $\mathbf{x}$ in the δ-tube around $\mathbf{p}'$, and zero otherwise. Evidently, $\delta f \to 0$ in the C^{r-1} sense as $o' \to o$ because δ holds uniformly for all $\mathbf{p}'$ in U.

By construction, $\mathbf{p}'(t)$ is a periodic solution of the dynamical system $d\mathbf{x}/dt = f'(\mathbf{x})$, with $f' = f + \delta f$. $\square$

Finally, as a consequence of Proposition 10 and Theorem 13, we have the following theorem.

Theorem 14. *Let $\frac{dx}{dt} = f(x)$, where $x \in \mathbb{R}^d$, $f : U \to \mathbb{R}^d$, and U an open subset of $\mathbb{R}^d$, be a C^r, $r \geq 2$, dynamical system. Let $x_p : [0, T) \to U$ be a hyperbolic periodic solution of period $T > 0$. There exists an open neighborhood of f in the C^r topology such that for an open and dense set of g in that neighborhood and a nearby hyperbolic periodic solution $x_p'(t)$ of $dx'/dt = g(x')$ of period T' and a delay $\tau' > 0$ such that $t \to (a^T x_p'(t), a^T x_p'(t-\tau'), a^T x_p'(t-2\tau'))$ is an embedding of the circle $[0, T')$ in $\mathbb{R}^3$.*

The above theorem does not assert that periodic orbits can be embedded in $\mathbb{R}^3$ for an open and dense set of C^r vector fields g. Instead, the theorem limits itself to a neighborhood of a vector field f which is known to admit a hyperbolic periodic orbit. Such a restriction is essential because there exist open sets of vector fields none which admits any periodic solution.

Theorem 2 of [1] states the embedding theorem for a fixed observation function π and for a generic vector field f. First, we note that that theorem does not imply our Theorem 14, which embeds periodic orbits in $\mathbb{R}^3$ independently of the dimension of the phase space of the dynamical system.

Second, the proof of Theorem 2 of [1] is not complete for reasons we will now explain. To explain that, we will restrict ourselves to Theorem 1 of [1] which fixes the vector field f and perturbs only the observation map π. That is because a complete proof in that setting is available in [16]. The statement made on p. 598 that if x^* is not an equilibrium point then it is possible to pick an observation function g such that

$$\left\{ \begin{array}{c} g \circ \phi_{t_1}(x^*) \\ \vdots \\ g \circ \phi_{t_{2n+1}}(x^*) \end{array} \right\}$$

equals any vector is correct. However, a problem arises with

$$\left\{ \begin{array}{c} g \circ \phi_{t_1}(x^*) \\ \vdots \\ g \circ \phi_{t_{2n+1}}(x^*) \end{array} \right\} \left\{ \begin{array}{c} g \circ \phi_{t_1}(y^*) \\ \vdots \\ g \circ \phi_{t_{2n+1}}(y^*) \end{array} \right\}$$

if the delay coordinates of x^* and y^* have components that overlap, for example when $\phi_{t_3}(x^*) = \phi_{t_2}(y^*)$.

That difficulty is unfortunately not easy to resolve. The difficulty is recognized explicitly in the second paragraph on p. 370 of [18], which deals with maps and flows. There it is asserted (without a complete proof) that the map to delay coordinates is an immersion for generic observation functions.

That the proof cannot be trivial may be seen by examining one of the proofs of the Whitney embedding theorem in [8]. Theorem 2.13 of [8] is the Whitney embedding theorem, whose proof is left as an exercise. However, the proof of the immersive counterpart, which is Theorem 2.12, is given in full. That proof is quite nontrivial. There are yet other proofs of the Whitney embedding theorem in [8] but none of them extend to delay coordinates easily.

4 Discussion

In this paper, we have considered an extension of the delay coordinate embedding theory. The current embedding theory of Sauer et al [16] is based on fixing the dynamical system and perturbing the observation function. Our intention is to derive an embedding theory that fixes the observation function and perturbs the dynamical system.

In this article, we proved a special case that applies to hyperbolic periodic orbits and observation functions that are linear in the coordinates. In future work, we will remove the restriction to periodic orbits and allow the observation function to be any nonlinear function with a gradient whose norm is strictly greater than a positive quantity. Removing the restriction to periodic orbits would be more consequential in the context of applications but also more difficult.

The present work indicates the nature of the difficulty. Suppose we want to produce an embedding of a periodic orbit. Given any map into $\mathbb{R}^3$ it is sufficient to perturb the observation function by linear functions or polynomials to produce an embedding [16]. However, when we allow only the flow or dynamical system to be perturbed, the manner which perturbations to the flow propagate to the delay coordinates becomes more complicated. The present work handles the important special case of periodic orbits and offers evidence that those difficulties can be overcome.

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