ON A CONJECTURE OF KASHIWARA RELATING CHERN AND EULER CLASSES OF $\mathcal{O}$-MODULES

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ABSTRACT. In this note we prove a conjecture of Kashiwara, which states that the Euler class of a coherent analytic sheaf $\mathcal{F}$ on a complex manifold X is the product of the Chern character of $\mathcal{F}$ with the Todd class of X. As a corollary, we obtain a functorial proof of the Grothendieck-Riemann-Roch theorem in Hodge cohomology for complex manifolds.

1. INTRODUCTION

The notation used throughout this article is defined in §2.

Let X be a complex manifold, ω_X be the holomorphic dualizing complex of X, δ_X be the diagonal injection of X in $X \times X$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ be the full subcategory of the bounded derived category of analytic sheaves on X consisting of objects with coherent cohomology. In the letter [7] which is reproduced in Chapter 5 of [6], Kashiwara constructs for every $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ two cohomology classes $\mathrm{hh}_X(\mathcal{F})$ and $\mathrm{thh}_X(\mathcal{F})$ in $\mathrm{H}^0_{\mathrm{supp}(\mathcal{F})}(X, \delta_X^* \delta_{X*} \mathcal{O}_X)$ and $\mathrm{H}^0_{\mathrm{supp}(\mathcal{F})}(X, \delta_X^! \delta_{X!} \omega_X)$; they are the Hochschild and co-Hochschild classes of $\mathcal{F}$.

Let us point out that characteristic classes in Hochschild homology are well-known in homological algebra (see [8, §8]). They have been recently intensively studied in various algebraico-geometric contexts. For further details, we refer the reader to [3], [2], [13] and to the references therein.

If $f : X \longrightarrow Y$ is a holomorphic map, the classes hh_X and thh_X satisfy the following dual functoriality properties:

– for every $\mathcal{G}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$, $\mathrm{hh}_X(f^* \mathcal{G}) = f^* \mathrm{hh}_Y(\mathcal{G})$,

– for every $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that f is proper on $\mathrm{supp}(\mathcal{F})$,

$$\mathrm{thh}_Y(Rf_! \mathcal{F}) = f_! \,\mathrm{thh}_X(\mathcal{F}).$$

The analytic Hochschild-Kostant-Rosenberg isomorphisms constructed in [7] are specific isomorphisms

$$\delta_X^* \delta_{X*} \mathcal{O}_X \simeq \bigoplus_{i \geq 0} \Omega_X^i [i] \qquad \text{and} \qquad \delta_X^! \delta_{X!} \omega_X \simeq \bigoplus_{i \geq 0} \Omega_X^i [i]$$

in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. The Hochschild and co-Hochschild classes of an element $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ are mapped via the above HKR isomorphisms to the so-called Chern and Euler classes of $\mathcal{F}$ in $\bigoplus_{i \geq 0} \mathrm{H}^i_{\mathrm{supp}(\mathcal{F})}(X, \Omega_X^i)$, denoted by $\mathrm{ch}(\mathcal{F})$ and $\mathrm{eu}(\mathcal{F})$.

The natural morphism

$$\bigoplus_{i\geq 0} \mathrm{H}^i_{\mathrm{supp}(\mathcal{F})}(X,\Omega^i_X) \longrightarrow \bigoplus_{i\geq 0} \mathrm{H}^i(X,\Omega^i_X)$$

maps $\mathrm{ch}(\mathcal{F})$ to the usual Chern character of $\mathcal{F}$ in Hodge cohomology, which is obtained by taking the trace of the exponential of the Atiyah class of the tangent bundle TX.[1]

The Chern and Euler classes satisfy the same functoriality properties as the Hochschild and co-Hochschild classes, namely for every holomorphic map f from X to Y:

– for every $\mathcal{G}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$, $\mathrm{ch}(f^*\mathcal{G}) = f^*\,\mathrm{ch}(\mathcal{G})$,

– for every $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that f is proper on $\mathrm{supp}(\mathcal{F})$,

$$\mathrm{eu}(Rf_!\,\mathcal{F}) = f_!\,\mathrm{eu}(\mathcal{F}).$$

Furthermore, for every $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, $\mathrm{eu}(\mathcal{F}) = \mathrm{ch}(\mathcal{F})\,\mathrm{eu}(\mathcal{O}_X)$. Putting together the previous identity with the functoriality of the Euler class with respect to direct images, Kashiwara obtained the following Grothendieck-Riemann-Roch theorem:

Theorem 1.1. [7] *Let $f : X \longrightarrow Y$ be a holomorphic map and $\mathcal{F}$ be an element of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that f is proper on $\mathrm{supp}(\mathcal{F})$. Then the following identity holds in $\bigoplus_{i\geq 0} \mathrm{H}^i_{f[\mathrm{supp}(\mathcal{F})]}(Y,\Omega^i_Y)$:*

$$\mathrm{ch}(Rf_!\,\mathcal{F})\,\mathrm{eu}(\mathcal{O}_Y) = f_!\left[\mathrm{ch}(\mathcal{F})\,\mathrm{eu}(\mathcal{O}_X)\right].$$

Then Kashiwara stated the following conjecture (see [6, §5.3.4]):

Conjecture. [7] *For any complex manifold X, the class $\mathrm{eu}(\mathcal{O}_X)$ is the Todd class of the tangent bundle TX.*

This conjecture was related to another conjecture of Schapira and Schneiders comparing the Euler class of a $\mathscr{D}_X$-module $\mathfrak{m}$ and the Chern class of the associated $\mathcal{O}_X$-module $\mathrm{Gr}(\mathfrak{m})$ (see [12], [1]).

The aim of this note is to give a simple proof of Kashiwara's conjecture:

Theorem 1.2. *For any complex manifold X, $\mathrm{eu}(\mathcal{O}_X)$ is the Todd class of TX.*

In the algebraic setting, an analogous result is established in [11] (see also [9]).

As a corollary of Theorem 1.2, we obtain the Grothendieck-Riemann-Roch theorem in Hodge cohomology for abstract complex manifolds, which has been already proved by different methods in [10]:

Theorem 1.3. *Let $f : X \longrightarrow Y$ be a holomorphic map between complex manifolds, and let $\mathcal{F}$ be an element of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that f is proper on $\mathrm{supp}(\mathcal{F})$. Then*

$$\mathrm{ch}(Rf_!\,\mathcal{F})\,\mathrm{td}(Y) = f_!\left[\mathrm{ch}(\mathcal{F})\,\mathrm{td}(X)\right].$$

in $\bigoplus_{i\geq 0} \mathrm{H}^i_{f[\mathrm{supp}(\mathcal{F})]}(Y,\Omega^i_Y)$.

[1] This property has been proved in [2] for algebraic varieties using different definitions of the HKR isomorphism and of the Hochschild class. In Kashiwara's setting, this is straightforward.

However, the proof given here is simpler and more conceptual. Besides, we would like to emphasize that it is entirely self-contained and relies only on the results appearing in Chapter 5 of [6].

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2. Notations and basic results

We follow the notation of [6, Ch. 5].

If X is a complex manifold, we denote by $\mathrm{D^b}(X)$ the bounded derived category of sheaves of $\mathcal{O}_X$-modules and by $\mathrm{D^b_{coh}}(X)$ the full subcategory of $\mathrm{D^b}(X)$ consisting of complexes with coherent cohomology.

If $f\colon X\longrightarrow Y$ is a holomorphic map between complex manifolds, the four operations $f^*\colon\mathrm{D^b}(Y)\longrightarrow\mathrm{D^b}(X)$, Rf_*, $Rf_!\colon\mathrm{D^b}(X)\longrightarrow\mathrm{D^b}(Y)$ and $f^!\colon\mathrm{D^b}(Y)\longrightarrow\mathrm{D^b}(X)$ are part of the formalism of Grothendieck's six operations. Let us recall their definitions:

– f^* is the left derived functor of the pullback functor by f, that is $\mathcal{G}\rightarrow\mathcal{G}\otimes_{f^{-1}\mathcal{O}_Y}\mathcal{O}_X$,

– Rf_* is the right derived functor of the direct image functor f_*, it is the left adjoint to the functor f^*;

– $Rf_!$ is the right derived functor of the proper direct image functor $f_!$,

– $f^!$ is the exceptional inverse image, it is the right adjoint to the functor $Rf_!$.

If W is a closed complex submanifold of Y, the pullback morphism from $f^*\Omega^i_Y[i]$ to $\Omega^i_X[i]$ induces in cohomology a map

$$f^*\colon\bigoplus_{i\geq 0}\mathrm{H}^i_W(Y,\Omega^i_Y)\longrightarrow\bigoplus_{i\geq 0}\mathrm{H}^i_{f^{-1}(W)}(X,\Omega^i_X).$$

If Z is a closed complex submanifold of X and if f is proper on Z, the integration morphism from $\Omega^{i+d_X}_X[i+d_X]$ to $\Omega^{i+d_Y}_Y[i+d_Y]$ induces a Gysin morphism

$$f_!:\bigoplus_{i\geq -d_X}\mathrm{H}^{i+d_X}_Z(X,\Omega^{i+d_X}_X)\longrightarrow\bigoplus_{i\geq -d_Y}\mathrm{H}^{i+d_Y}_{f(Z)}(Y,\Omega^{i+d_Y}_Y).$$

Let X be a complex manifold, ω_X be the holomorphic dualizing complex of X and δ_X be the diagonal injection. If $\mathcal{F}$ belongs to $\mathrm{D^b_{coh}}(X)$, we define the ordinary dual (resp. Verdier dual) of $\mathcal{F}$ by the usual formula $D'\mathcal{F}=\mathcal{RH}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{O}_X)$ (resp. $D\mathcal{F}=\mathcal{RH}om_{\mathcal{O}_X}(\mathcal{F},\omega_X)$).

The Hochschild and co-Hochschild classes of $\mathcal{F}$, denoted by $\mathrm{hh}_X(\mathcal{F})$ and $\mathrm{thh}_X(\mathcal{F})$, lie in $\mathrm{H}^0_{\mathrm{supp}(\mathcal{F})}(X,\delta^*_X\delta_{X*}\mathcal{O}_X)$ and $\mathrm{H}^0_{\mathrm{supp}(\mathcal{F})}(X,\delta^!_X\delta_{X!}\,\omega_X)$ respectively. They are constructed by the chains of maps

$$\mathrm{hh}_X(\mathcal{F}):\ \mathrm{id}\longrightarrow\mathcal{RH}om(\mathcal{F},\mathcal{F})\xrightarrow{\sim}\delta^*_X(D'\mathcal{F}\boxtimes_{\mathcal{O}}\mathcal{F})\longrightarrow\delta^*_X\delta_{X*}\mathcal{O}_X$$
$$\mathrm{thh}_X(\mathcal{F}):\ \mathrm{id}\longrightarrow\mathcal{RH}om(\mathcal{F},\mathcal{F})\xrightarrow{\sim}\delta^!_X(D\mathcal{F}\boxtimes_{\mathcal{O}}\mathcal{F})\longrightarrow\delta^!_X\delta_{X!}\,\omega_X$$

where in both cases the last arrows are obtained from the derived trace maps

$$D'\mathcal{F}\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\mathcal{F}\longrightarrow\mathcal{O}_X \quad\text{and}\quad D\mathcal{F}\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\mathcal{F}\longrightarrow\omega_X$$

by adjunction.

If $f\colon X\longrightarrow Y$ is a holomorphic map between complex manifolds, there are pullback and push-forward morphisms

$$f^*\colon f^*\delta_Y^*\delta_{Y*}\,\mathcal{O}_Y\longrightarrow\delta_X^*\delta_{X*}\,\mathcal{O}_X \quad\text{and}\quad f_!\colon Rf_!\,\delta_X^!\delta_{X!}\omega_X\longrightarrow\delta_Y^!\delta_{Y!}\omega_Y.$$

Besides, there is a natural pairing

$$\delta_X^*\delta_{X*}\,\mathcal{O}_X\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\delta_X^!\delta_{X!}\,\omega_X \;\rightarrow\; \delta_X^!\delta_{X!}\,\omega_X \tag{1}$$

given by the chain

$$\delta_X^*\delta_{X*}\,\mathcal{O}_X\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\delta_X^!\delta_{X!}\,\omega_X\simeq\delta_X^!(\delta_{X*}\,\mathcal{O}_X\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{X\times X}}\delta_{X!}\,\omega_X)\;\rightarrow\;\delta_X^!\delta_{X!}\,\omega_X.$$

Theorem 2.1. [7]

(i) *For all elements $\mathcal{F}$ and $\mathcal{G}$ in $D^{\mathrm{b}}_{\mathrm{coh}}(X)$ and $D^{\mathrm{b}}_{\mathrm{coh}}(Y)$ respectively,* $\mathrm{hh}_X(f^*\mathcal{G})=f^*\,\mathrm{hh}_Y(\mathcal{G})$ *and* $f_!\,\mathrm{thh}_X(\mathcal{F})=\mathrm{thh}_Y(Rf_!\mathcal{F})$.

(ii) *Via the pairing (1), for every $\mathcal{F}$ in $D^{\mathrm{b}}_{\mathrm{coh}}(X)$,*

$$\mathrm{hh}_X(\mathcal{F})\,\mathrm{thh}(\mathcal{O}_X)=\mathrm{thh}_X(\mathcal{F}).$$

The Hochschild and co-Hochschild classes are translated into Hodge cohomology classes by Kashiwara's analytic Hochschild-Kostant-Rosenberg isomorphisms[2]

$$\delta_X^*\delta_{X*}\mathcal{O}_X\simeq\bigoplus_{i\geq 0}\Omega_X^i[i] \qquad\text{and}\qquad \delta_X^!\delta_{X!}\,\omega_X\simeq\bigoplus_{i\geq 0}\Omega_X^i[i], \tag{2}$$

the resulting classes are called Chern and Euler classes. If $\mathcal{F}$ is an element of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, then $\mathrm{ch}(\mathcal{F})$ and $\mathrm{eu}(\mathcal{F})$ lie in $\bigoplus_{i\geq 0}\mathrm{H}^i_{\mathrm{supp}(\mathcal{F})}(X,\Omega_X^i)$.

The first HKR isomorphism commutes with pullback and the second one with push forward. Besides, the pairing (1) between $\delta_X^*\delta_{X*}\,\mathcal{O}_X$ and $\delta_X^!\delta_{X!}\,\omega_X$ is exactly the cup-product on holomorphic differential forms after applying the HKR isomorphisms (2).

Theorem 2.2. [7]

(i) *For every $\mathcal{F}$ in $D^{\mathrm{b}}_{\mathrm{coh}}(X)$,* $\mathrm{ch}(\mathcal{F})$ *is the usual Chern character obtained by the Atiyah exact sequence.*

(ii) *For all elements $\mathcal{F}$ and $\mathcal{G}$ in $D^{\mathrm{b}}_{\mathrm{coh}}(X)$ and $D^{\mathrm{b}}_{\mathrm{coh}}(X)$ respectively,* $\mathrm{ch}(f^*\mathcal{G})=f^*\,\mathrm{ch}(\mathcal{G})$ *and* $f_!\,\mathrm{eu}(\mathcal{F})=\mathrm{eu}(Rf_!\mathcal{F})$.

(iii) *For every $\mathcal{F}$ in $D^{\mathrm{b}}_{\mathrm{coh}}(X)$,* $\mathrm{eu}(\mathcal{F})=\mathrm{ch}(\mathcal{F})\,\mathrm{eu}(\mathcal{O}_X)$.

For the proofs of Theorems 2.1 and 2.2, we refer to [6, Ch. 5].

For any complex manifold X, we denote by $\mathrm{td}(X)$ the Todd class of the holomorphic tangent bundle TX in $\bigoplus_{i\geq 0}\mathrm{H}^i(X,\Omega_X^i)$.

[2] For a detailed account of the HKR isomorphisms, we refer to the introduction of [5] and to the references therein.

3. Proof of Theorem 1.2

We proceed in several steps.

Proposition 3.1. *Let Y and Z be complex manifolds such that Z is a closed complex submanifold of Y, and let i_Z be the corresponding inclusion. Then, for every coherent sheaf $\mathcal{F}$ on Z, we have*

$$i_{Z!}\,[\mathrm{ch}(\mathcal{F})\,\mathrm{td}(Z)] = \mathrm{ch}(i_{Z*}\mathcal{F})\,\mathrm{td}(Y)$$

in $\bigoplus_{i\geq 0} \mathrm{H}^i_Z(Y, \Omega^i_Y)$.

Proof. This is proved in the classical way using the deformation to the normal cone as in [4, §15.2], except that we use local cohomology. For the sake of completeness, we provide a detailed proof.

We start by a particular case:

– $\mathcal{N}$ is a holomorphic vector bundle on Z, and $Y = \mathbb{P}\,(\mathcal{N} \oplus \mathcal{O}_Z)$,

– Z embeds in Y by identifying Z with the zero section of $\mathcal{N}$.

Let d be the rank of $\mathcal{N}$, π be the projection of the projective bundle $\mathbb{P}\,(\mathcal{N} \oplus \mathcal{O}_Z)$ and $\mathcal{Q}$ be the universal quotient bundle on Y; $\mathcal{Q}$ is the quotient of $\pi^*(\mathcal{N} \oplus \mathcal{O}_Z)$ by the tautological line bundle $\mathcal{O}_{\mathcal{N}\oplus\mathcal{O}_Z}(-1)$. Then $\mathcal{Q}$ has a canonical holomorphic section s which is obtained by the composition

$$s\colon \mathcal{O}_Y \simeq \pi^*\mathcal{O}_Z \longrightarrow \pi^*(\mathcal{N} \oplus \mathcal{O}_Z) \longrightarrow \mathcal{Q}.$$

This section vanishes transversally along its zero locus, which is exactly Z. Therefore, we have a natural locally free resolution of $i_{Z!}\mathcal{O}_Z$ given by the Koszul complex associated with the pair $(\mathcal{Q}^*, s^*)$:

$$0 \longrightarrow \wedge^d\mathcal{Q}^* \longrightarrow \wedge^{d-1}\mathcal{Q}^* \longrightarrow \cdots \longrightarrow \mathcal{O}_Y \longrightarrow i_{Z!}\mathcal{O}_Z \longrightarrow 0.$$

This gives the equality

$$\mathrm{ch}(i_{Z!}\mathcal{O}_Z) = \sum_{k=0}^{d}(-1)^k\mathrm{ch}(\wedge^k\mathcal{Q}^*) = \mathrm{c}_d(\mathcal{Q})\,\mathrm{td}(\mathcal{Q})^{-1}$$

in $\bigoplus_{i\geq 0} \mathrm{H}^i(Y, \Omega^i_Y)$, where $\mathrm{c}_d(\mathcal{Q})$ denotes the d-th Chern class of $\mathcal{Q}$ (for the last equality, see [4, § 3.2.5]). Since $\mathrm{c}_d(\mathcal{Q})$ is the image of the constant class 1 by $i_{Z!}$ and since $i_Z^*\mathcal{Q} = \mathcal{N}$, we get

$$\mathrm{ch}(i_{Z!}\mathcal{O}_Z) = i_{Z!}(i_Z^*\mathrm{td}(\mathcal{Q})^{-1}) = i_{Z!}(\mathrm{td}(\mathcal{N})^{-1}).$$

For any coherent sheaf $\mathcal{F}$ on Z, we have $i_{Z!}\mathcal{F} = i_{Z!}\mathcal{O}_Z \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \pi^*\mathcal{F}$ so that we obtain by the projection formula

$$\mathrm{ch}(i_{Z!}\mathcal{F}) = i_{Z!}(\mathrm{ch}(\mathcal{F})\mathrm{td}(\mathcal{N})^{-1}) \tag{3}$$

in $\bigoplus_{i\geq 0} \mathrm{H}^i(Y, \Omega^i_Y)$. Remark now that by Theorem 2.2 (ii) and (iii), we have

$$\mathrm{ch}(i_{Z!}\,\mathcal{F}) = i_{Z!}(\mathrm{ch}(\mathcal{F})\mathrm{eu}(\mathcal{O}_Z)\,i_Z^*\,\mathrm{eu}(\mathcal{O}_Y)^{-1})$$

in $\bigoplus_{i\geq 0} \mathrm{H}^i_Z(Y, \Omega^i_Y)$. This proves that $\mathrm{ch}(i_{Z!}\,\mathcal{F})$ lies in the image of

$$i_{Z!}\colon \bigoplus_{i\geq 0} \mathrm{H}^i(Z, \Omega^i_Z) \longrightarrow \bigoplus_{i\geq 0} \mathrm{H}^{i+d}_Z(Y, \Omega^{i+d}_Y).$$

Let us denote this image by W. The map

$$\iota\colon W \longrightarrow \bigoplus_{i\geq 0} \mathrm{H}^{i+d}(Y, \Omega^{i+d}_Y)$$

obtained by forgetting the support is injective. Indeed, for every class $i_{Z!}\alpha$ in W, $\pi_![\iota(i_{Z!}\alpha)] = \alpha$. This implies that (3) holds in $\bigoplus_{i\geq 0} \mathrm{H}^i_Z(Y, \Omega^i_Y)$.

We now turn to the general case, using deformation to the normal cone. Let us introduce some notation:

– M is the blowup of $Z \times \{0\}$ in $Y \times \mathbb{P}^1$, σ is the blowup map and $q = \mathrm{pr}_1 \circ \sigma$,

– for any divisor D on M, $[D]$ denotes its cohomological cycle class in $\mathrm{H}^1(M, \Omega^1_M)$,

– $E = \mathbb{P}(N_{Z/Y} \oplus \mathcal{O}_Z)$ is the exceptional divisor of the blowup and $\widetilde{Y}$ is the strict transform of $Y \times \{0\}$ in M,

– μ is the embedding of Z in E, where Z is identified with the zero section of $N_{Z/Y}$,

– F is the embedding of (the strict transform of) $Z \times \mathbb{P}^1$ in M and for any t in $\mathbb{P}^1$, j_t is the embedding of M_t in M,

– k is the embedding of E in M.

Then M is flat over $\mathbb{P}^1$, M_0 is a Cartier divisor with two smooth components E and $\widetilde{Y}$ intersecting transversally along $\mathbb{P}(N_{Z/Y})$, and M_t is isomorphic to Y if t is nonzero.

Let $\mathcal{G} = F_!(\mathrm{pr}_1^*\,\mathcal{F})$. Since M is flat over $\mathbb{P}^1$, for any t in $\mathbb{P}^1 \setminus \{0\}$,

$$j_t^*\mathcal{G} = i_{Z!}\mathcal{F} \quad \text{and} \quad k^*\mathcal{G} = \mu_!\mathcal{F}.$$

If $\mathrm{ch}(\mathcal{G})$ is the Chern character of $\mathcal{G}$ in $\bigoplus_{i\geq 0} \mathrm{H}^i_{Z\times\mathbb{P}^1}(M, \Omega^i_M)$, using the identity (3) in $\bigoplus_{i\geq 0} \mathrm{H}^i_Z(E, \Omega^i_E)$, we get

$$\begin{aligned}
j_{t!}\,\mathrm{ch}(i_{Z!}\mathcal{F}) &= j_{t!}\,j_t^*\,\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{G})\,[M_t] \\
&= \mathrm{ch}(\mathcal{G})\,[M_0] = \mathrm{ch}(\mathcal{G})\,[E] + \mathrm{ch}(\mathcal{G})\,[\widetilde{Y}] \\
&= \mathrm{ch}(\mathcal{G})\,[E] = k_!\,k^*\,\mathrm{ch}(\mathcal{G}) = k_!\,\mathrm{ch}(\mu_!\mathcal{F}) \\
&= k_!\,\mu_!(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(N_{Z/E})^{-1}) \\
&= k_!\,\mu_!(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(N_{Z/X})^{-1})
\end{aligned}$$

in $\bigoplus_{i\geq 0} \mathrm{H}^i_{Z\times\mathbb{P}^1}(M, \Omega^i_M)$.

The map q is proper on $Z \times \mathbb{P}^1$, $q \circ j_t = \mathrm{id}$ and $q \circ k \circ \mu = i_Z$. Applying $q_!$, we get

$$\mathrm{ch}(i_{Z!}\mathcal{F}) = i_{Z!}(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(N_{Z/X})^{-1})$$

in $\bigoplus_{i\geq 0} \mathrm{H}^i_Z(Y, \Omega^i_Y)$. □

Definition 3.2. For any complex manifold X, let $\alpha(X)$ be the cohomology class in $\bigoplus_{i\geq 0} \mathrm{H}^i(X, \Omega_X^i)$ defined by $\alpha(X) = \mathrm{eu}(\mathcal{O}_X)\,\mathrm{td}(X)^{-1}$.

Lemma 3.3. *Let Y and Z be complex manifolds such that Z is a closed complex submanifold of Y, and let i_Z be the corresponding injection. Assume that there exists a holomorphic retraction R of i_Z. Then we have $\alpha(Z) = i_Z^* \,\alpha(Y)$.*

Proof. By Theorem 2.2 (ii), $\mathrm{eu}(i_{Z*}\mathcal{O}_Z) = i_{Z!}\,\mathrm{eu}(\mathcal{O}_Z)$. By Proposition 3.1 and Theorem 2.2 (iii),

$$\mathrm{eu}(i_{Z*}\mathcal{O}_Z) = \mathrm{ch}(i_{Z*}\mathcal{O}_Z)\mathrm{eu}(\mathcal{O}_Y) = (i_{Z!}\,\mathrm{td}(Z))\,\mathrm{td}(Y)^{-1}\mathrm{eu}(\mathcal{O}_Y),$$

so that we obtain in $\bigoplus_{i\geq 0} \mathrm{H}_Z^i(Y, \Omega_Y^i)$ the formula

$$i_{Z!}\big[\mathrm{eu}(\mathcal{O}_Z) - \mathrm{td}(Z)\, i_Z^*(\mathrm{eu}(\mathcal{O}_Y)\,\mathrm{td}(Y)^{-1})\big] = 0.$$

Since R is proper on Z, we can apply $R_!$ and we get the result.

□

Lemma 3.4. *The class $\alpha(X)$ satisfies $\alpha(X)^2 = \alpha(X)$.*

Proof. We apply Lemma 3.3 with $Z = X$ and $Y = X \times X$, where X is diagonally embedded in $X \times X$. Then $\alpha(X) = i_\Delta^* \,\alpha(X \times X)$. The Euler class commutes with external products so that

$$\mathrm{eu}(\mathcal{O}_{X\times X}) = \mathrm{eu}(\mathcal{O}_X) \boxtimes \mathrm{eu}(\mathcal{O}_X)$$

Thus $\alpha(X \times X) = \alpha(X) \boxtimes \alpha(X)$ and we obtain

$$\alpha(X) = i_\Delta^*[\,\alpha(X) \boxtimes \alpha(X)\,] = \alpha(X)^2.$$

□

Proof of Theorem 1.2. There is a natural isomorphism ϕ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ between $\delta^*\delta_*\mathcal{O}_X$ and $\delta^!\delta_!\omega_X$ given by the chain

$$\delta^*\delta_*\mathcal{O}_X \simeq \mathcal{O}_X \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \delta^*\delta_*\mathcal{O}_X \simeq \delta^!(\omega_X \boxtimes \mathcal{O}_X) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \delta^*\delta_*\mathcal{O}_X \simeq \delta^!\delta_!\omega_X.$$

Besides, after applying the two HKR isomorphisms (2), ϕ is given by derived cup-product with the Euler class of $\mathcal{O}_X$ (see [6]). Therefore, the class $\mathrm{eu}(\mathcal{O}_X)$ is invertible in the Hodge cohomology ring of X, and so is $\alpha(X)$. Lemma 3.4 implies that $\alpha(X) = 1$.

q.e.d.

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