COMMUTING PROBABILITY OF INFINITE GROUPS

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ABSTRACT. Let G be a group, and let $M = (\mu_n)_{n=1}^{\infty}$ be a sequence of finitely supported probability measures on G. Consider the probability that two elements chosen independently according to μ_n commute. Antolín, Martino and Ventura define the *degree of commutativity* $\mathrm{dc}_M(G)$ of G with respect to this sequence to be the lim sup of this probability. The main results of the present paper give quantitative algebraic consequences of the degree of commutativity being above certain thresholds. For example, if μ_n is the distribution of the nth step of a symmetric random walk on G, or if G is amenable and (μ_n) is a sequence of almost-invariant measures on G, we show that if $\mathrm{dc}_M(G) > \frac{5}{8}$ then G is abelian; if $\mathrm{dc}_M(G) \geq \frac{1}{2} + \varepsilon$ then the centre of G has index at most ε^{-1}; and if $\mathrm{dc}_M(G) \geq \alpha > 0$ then G contains a normal subgroup Γ of index at most $\lceil \alpha^{-1} \rceil$ and a normal subgroup H of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and Γ/H is abelian. We also describe some general conditions on (μ_n) under which such theorems hold. These results generalise results for finite groups due to Gustafson and P. M. Neumann, and generalise and quantify a result for certain residually finite groups of subexponential growth due to Antolín, Martino and Ventura. We also present an application to conjugacy ratios, showing that if the sequence of word-metric balls in a finitely generated group is both left- and right-Følner then the conjugacy ratio is equal to the degree of commutativity. Combined with our main results, this generalises a result of Ciobanu, Cox and Martino.

CONTENTS

1. INTRODUCTION

The starting point for this paper is two simple but beautiful results concerning the probability that two randomly chosen elements of a finite group G commute. If two elements $x, y \in G$ are chosen uniformly at random from G and independently of one another, we define the probability that they commute to be the *degree of commutativity* of G, and denote it by $\mathrm{dc}(G)$.

Theorem 1.1 (Gustafson [10, §1]). *Let G be a finite group such that $\mathrm{dc}(G) > \frac{5}{8}$. Then G is abelian.*

Theorem 1.2 (P. M. Neumann [14, Theorem 1]). *Let G be a finite group such that $\mathrm{dc}(G) \geq \alpha > 0$. Then G has a normal subgroup Γ of index at most $\alpha^{-1} + 1$ and a normal subgroup H of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and Γ/H is abelian.*

There are many natural ways in which one might seek to generalise these results. In this paper we are concerned with generalisations to infinite groups. The first question in this setting is how to define the probability that two group elements commute.

Antolín, Martino and Ventura [1] approach this issue by considering sequences of finitely supported probability measures whose supports converge to the whole of G. Given a probability measure μ on G, define the *degree of commutativity* $\mathrm{dc}_\mu(G)$ of G with respect to μ via

$$\mathrm{dc}_\mu(G) = (\mu \times \mu)(\{(x, y) \in G \times G : xy = yx\}).$$

Then, given a sequence $M = (\mu_n)_{n=1}^\infty$ of probability measures on G, define the *degree of commutativity* $\mathrm{dc}_M(G)$ of G with respect to M via

$$\mathrm{dc}_M(G) = \limsup_{n \to \infty} \mathrm{dc}_{\mu_n}(G).$$

Antolín, Martino and Ventura [1, Meta-conjecture 1.3] suggest the following rather natural analogue of Theorem 1.2 for degree of commutativity defined in this way.

Conjecture 1.3 (Antolín, Martino and Ventura). *For any "reasonable" sequence $M = (\mu_n)_{n=1}^\infty$ of probability measures on G we have $\mathrm{dc}_M(G) > 0$ if and only if G is virtually abelian.*

They suggest that "reasonable" might mean that the measures μ_n cover G with "enough homogeneity" as $n \to \infty$. Of course, the terms "reasonable" and "enough homogeneity" are somewhat vague, and so they also mention two explicit sequences of measures for which they believe Conjecture 1.3 should hold. The first such sequence is where S is a fixed finite symmetric generating set for G, and μ_n is the uniform measure on the ball of radius n about the identity in G with respect to S. In this setting we write $\mathrm{dc}_S(G)$ instead of $\mathrm{dc}_M(G)$. The second such sequence is where μ is some finite probability measure on G, and $\mu_n = \mu^{*n}$ is defined by letting $\mu^{*n}(x)$ be the probability that a random walk of length n on G with respect to μ ends at x. Later, in Definition 1.9 and Theorem 1.10, we look in more detail at a possible interpretation of what "enough homogeneity" might mean in Conjecture 1.3. First, however, we consider these two explicit sequences of measures.

There has already been some progress towards Conjecture 1.3 in the case of the sequence of uniform measures on balls with respect to a fixed generating set. Antolín, Martino and Ventura themselves prove the conjecture completely in that setting for hyperbolic groups [1, Theorem 1.9], whilst Valiunas proves it for right-angled Artin groups [18, Theorem 1]. Antolín, Martino and Ventura also prove the following result, which includes a fairly direct analogue of Theorem 1.1.

Theorem 1.4 (Antolín–Martino–Ventura [1, Theorem 1.5]). *Let G be a finitely generated residually finite group and let S be a finite summetric generating set containing the identity and satisfying*

$$\tag{1.1} \frac{|S^{n+1}|}{|S^n|} \to 1$$

as $n \to \infty$. Then $\mathrm{dc}_S(G) > \frac{5}{8}$ if and only if G is abelian, and $\mathrm{dc}_S(G) > 0$ if and only if G is virtually abelian.

In the first main result of the present paper we remove completely from Theorem 1.4 the hypothesis that G is residually finite, and make the second conclusion more quantitative in the spirit of Theorem 1.2, as follows.

Theorem 1.5. *Let G be a finitely generated group, and let S be a finite symmetric generating set containing the identity and satisfying* (1.1). *Then the following hold.*

(1) *If $\mathrm{dc}_S(G) > \frac{5}{8}$ then G is abelian.*

(2) *If $\mathrm{dc}_S(G) \geq \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$ then the centre of G has index at most $\frac{1}{\varepsilon}$ in G.*

(3) *If $\mathrm{dc}_S(G) \geq \alpha > 0$ then G has a normal subgroup Γ of index at most $\lceil \alpha^{-1} \rceil$ and a normal subgroup H of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and Γ/H is abelian.*

We also settle Conjecture 1.3 completely for the case in which the sequence of measures is defined via a symmetric random walk, as follows.

Theorem 1.6. *Let G be a finitely generated group, and let μ be a symmetric, finitely supported generating probability measure on G such that $\mu(e) > 0$. Let $M = (\mu^{*n})_{n=1}^{\infty}$ be the sequence of measures corresponding to the steps of the random walk on G with respect to μ. Then the following hold.*

(1) *If $\mathrm{dc}_M(G) > \frac{5}{8}$ then G is abelian.*

(2) *If $\mathrm{dc}_M(G) \geq \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$ then the centre of G has index at most $\frac{1}{\varepsilon}$ in G.*

(3) *If $\mathrm{dc}_M(G) \geq \alpha > 0$ then G has a normal subgroup Γ of index at most $\lceil \alpha^{-1} \rceil$ and a normal subgroup H of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and Γ/H is abelian.*

Remark. A finitely generated finite-by-abelian group G is virtually abelian, so Theorems 1.5 and 1.6 really do imply the conclusion required by Conjecture 1.3, and in particular recover Theorem 1.4 in full. However, the bound on the index of the abelian subgroup of G obtained in this way depends on the rank of G as well as on α, whereas the bound given by Theorems 1.5 and 1.6 depends only on α, and so the characterisation given by Theorems 1.5 and 1.6 is quantitatively preferable.

This is slightly reminiscent of Gromov's theorem on groups of polynomial growth. Classically, Gromov's theorem says that a group of polynomial growth is virtually nilpotent. However, Breuillard, Green and Tao [2, Corollary 11.5] have given a quantitative refinement of Gromov's theorem showing that a group G of polynomial growth has a finite-index finite-by-nilpotent subgroup. This implies in particular that G is virtually nilpotent, but the bound this gives on the index of a nilpotent subgroup is far worse than the bounds it gives on the index of the finite-by-nilpotent subgroup and the size of the kernel of the projection of that subgroup onto a nilpotent group. As things stand this is a slightly hypothetical remark, since the only known proof of this result renders the bounds in question ineffective, but there are specific classes of group for which effective bounds are known and are much worse for the second characterisation than for the first (see [17, Corollary 1.8], for example). A similar phenomenon occurs in Breuillard and the author's analogous result for finite groups of large diameter [4, Theorem 4.1].

Before we move on to consider the meaning of "enough homogeneity" in Conjecture 1.3, let us note that the condition (1.1) is somewhat suggestive of a particular explicit generalisation, as it implies that the sequence $(S^n)_{n=1}^{\infty}$ is a *Følner sequence* for the group G. In general a sequence $(F_n)_{n=1}^{\infty}$ of finite subsets of G is said to be a *left-Følner sequence*, or simply a *Følner sequence*, if

$$\frac{|xF_n \triangle F_n|}{|F_n|} \to 0$$

for every $x \in G$. Even more generally, for a given Følner sequence $(F_n)_{n=1}^{\infty}$, if μ_n is the uniform probability measure on the set F_n then the sequence $(\mu_n)_{n=1}^{\infty}$ satisfies the condition

(1.2) $$\|x \cdot \mu_n - \mu_n\|_1 \to 0$$

for every $x \in G$ (here $x \cdot \mu$ is defined by setting $x \cdot \mu(A) = \mu(x^{-1}A)$). A sequence $(\mu_n)_{n=1}^{\infty}$ of probability measures satisfying (1.2) is said to be a sequence of *almost-invariant probability measures*. Theorem 1.5 is therefore a special case of the following result.

Theorem 1.7. *Let G be a countable amenable group, and let $M = (\mu_n)_{n=1}^{\infty}$ be a sequence of almost-invariant probability measures on G. Then the following hold.*

    (1) If $\mathrm{dc}_M(G) > \frac{5}{8}$ then G is abelian.

    (2) If $\mathrm{dc}_M(G) \geq \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$ then the centre of G has index at most $\frac{1}{\varepsilon}$ in G.

    (3) If G is finitely generated and $\mathrm{dc}_M(G) \geq \alpha > 0$ then G has a normal subgroup Γ of index at most $\lceil \alpha^{-1} \rceil$ and a normal subgroup H of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and Γ/H is abelian.

It is well known and easy to check that if G is a group of subexponential growth and S is an arbitrary finite symmetric generating set for G containing the identity then there is a sequence $n_1 < n_2 < \ldots$ such that $(S^{n_i})_{i=1}^{\infty}$ is a Følner sequence for G. Theorem 1.7 therefore has the following immediate corollary.

Corollary 1.8. *Let G be a finitely generated group of subexponential growth with finite symmetric generating set S containing the identity, and for each n let μ_n be the uniform probability measure on S^n. Then the following hold.*

    (1) If $\liminf \mathrm{dc}_{\mu_n}(G) > \frac{5}{8}$ then G is abelian.

    (2) If $\liminf \mathrm{dc}_{\mu_n}(G) \geq \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$ then the centre of G has index at most $\frac{1}{\varepsilon}$ in G.

    (3) If $\liminf \mathrm{dc}_{\mu_n}(G) \geq \alpha > 0$ then G has a normal subgroup Γ of index at most $\lceil \alpha^{-1} \rceil$ and a normal subgroup H of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and Γ/H is abelian.

Of course, for Conjecture 1.3 to hold in this case we would need to replace the $\liminf$s with $\limsup$s.

Let us emphasise that there are many groups that do not admit generating sets satisfying the condition (1.1), but that nonetheless admit sequences of almost-invariant measures. Indeed, it is well known that a countable discrete group admits such a sequence if and only if it belongs to the class of *amenable* groups, which includes all soluble groups, for example, whereas a generating set can satisfy (1.1) only in a group of subexponential growth.

Unfortunately we do not prove Conjecture 1.3 for the seqence of uniform probability measures on balls with respect to a fixed generating set that does not satisfy (1.1). As is noted in [1], in order to prove such a result it would be sufficient to prove that if a finite generating set S for a group G did not satisfy (1.1) then $\mathrm{dc}_S(G) = 0$.

A MORE PRECISE GENERAL STATEMENT. We now suggest a precise version of Antolín, Martino and Ventura's idea that Conjecture 1.3 should be satisfied by sequences of measures that cover G with "enough homogeneity" in the limit. Specifically, we show that one possible meaning of homogeneity that works in this context is that the measures should not be too concentrated in any coset of a subgroup, in the following sense.

Definition 1.9 (uniform detection of index). Let $\pi : (0, 1] \to (0, 1]$ be a non-decreasing function such that $\pi(\gamma) \to 0$ as $\gamma \to 0$. We say that a sequence $M = (\mu_n)_{n=1}^{\infty}$ of probability measures on a group G *detects index uniformly at rate π* if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ if $[G : H] \geq m$ then $\mu_n(H) \leq \pi(\frac{1}{m}) + \varepsilon$ for every $n \geq N$. We also say simply that

M *detects index uniformly* to mean that there exists some π such that M detects index uniformly at rate π.

The word "uniform" in Definition 1.9 refers to the requirement that the definition be satisfied by the same $N(\varepsilon)$ for all subgroups H.

Theorem 1.10 (weak Neumann-type theorem for measures that detect index uniformly). *Let G be a finitely generated group, let $M = (\mu_n)_{n=1}^{\infty}$ be a sequence of measures on G that detects index uniformly at rate π, and suppose that $\mathrm{dc}_M(G) \geq \alpha > 0$. Then G has an abelian subgroup of index at most $O_{r,\pi,\alpha}(1)$.*

Remark. In the converse direction, note that if G has an abelian subgroup H such that $\limsup \mu_n(H) \geq \gamma$ then $\mathrm{dc}_M(G) \geq \gamma^2$.

Burillo and Ventura [5, Proposition 2.4] show that if S is a finite symmetric generating set for G containing the identity and satisfying (1.1), and $M = (\mu_n)_{n=1}^{\infty}$ is the sequence of probability measures on G defined by setting μ_n to be the uniform probability measure on S^n, then $\mu_n(xH) \to 1/[G : H]$ for every subgroup H of G and every $x \in G$ (here, and elsewhere, for notational convenience we take $1/[G : H]$ to be zero when H is an infinite-index subgroup of G). This was in turn used in Antolín, Martino and Ventura's proof of Theorem 1.4 [1]. In the present paper we show that this convergence is uniform over H and x, which implies in particular that M detects index uniformly. In order to state this result and others like it more succinctly, we use the following definition.

Definition 1.11 (uniform measurement of index). We say that a sequence $M = (\mu_n)_{n=1}^{\infty}$ of probability measures on a group G *measures index uniformly* if $\mu_n(xH) \to 1/[G : H]$ uniformly over all $x \in G$ and all subgroups H of G.

Remark 1.12. If a sequence of probability measures on a group measures index uniformly then it also detects index uniformly with rate $\iota : (0, 1] \to (0, 1]$ defined by $\iota(x) = x$.

Theorem 1.13 (almost-invariant measures measure index uniformly). *Let G be a finitely generated group. Then every sequence of almost-invariant measures on G measures index uniformly.*

Remark. By contrast with Theorem 1.13, if μ_n is the uniform measure on the ball of radius n in a group of exponential growth then the sequence $(\mu_n)_{n=1}^{\infty}$ may not even *detect* index uniformly. For example, writing $F(x, y)$ for the free group on generators x, y, and taking $G = \mathbb{Z} \times F(x, y)$ with generating set $\{(0, e), (\pm 1, e), (0, x^{\pm 1}), (0, y^{\pm 1})\}$, the subgroup $F(x, y)$ has infinite index but $\mu_n(F(x, y)) \not\to 0$.

Theorem 1.14 (random walks measure index uniformly). *Let G be a finitely generated group, and let μ be a symmetric, finitely supported generating probability measure on G such that $\mu(e) > 0$. Then the sequence $M_\mu = (\mu^{*n})_{n=1}^{\infty}$ measures index uniformly.*

Remark. The well-known example of $G = \mathbb{Z}$ with μ the uniform probability measure supported on ± 1 and $H = 2\mathbb{Z}$ shows that Theorem 1.14 is not necessarily true in the absence of the condition $\mu(e) > 0$ (or some other aperiodicity assumption).

It is this stronger property allows us to give the precise quantitative conclusions of Theorems 1.6 and 1.7, as follows.

Theorem 1.15. *Let G be a countable group, and let $M = (\mu_n)_{n=1}^{\infty}$ be a sequence of measures on G that measures index uniformly. Then the following hold.*

(1) If $\mathrm{dc}_M(G) > \frac{5}{8}$ then G is abelian.
(2) If $\mathrm{dc}_M(G) \geq \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$ then the centre of G has index at most $\frac{1}{\varepsilon}$ in G.
(3) If G is finitely generated and $\mathrm{dc}_M(G) \geq \alpha > 0$ then G has a normal subgroup Γ of index at most $\lceil \alpha^{-1} \rceil$ and a normal subgroup H of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and Γ/H is abelian.

DEGREE OF COMMUTATIVITY WITH RESPECT TO AN INVARIANT MEAN. We now present a variant of Theorem 1.15 for amenable groups, which is both a natural result in its own right (particularly in the context of Theorem 1.7) and, as it turns out, an ingredient in the proof of Theorem 1.15 (see Remark 7.2 for more details).

By definition, a group is amenable if it admits a *finitely additive left-invariant mean*. A *finitely additive mean* on a group G is a linear functional $\int \mathrm{d}\mu : \ell^\infty(G) \to \mathbb{R}$ that is positive in the sense that $\int f \, \mathrm{d}\mu \geq 0$ if $f \geq 0$ pointwise, and normalised in the sense that $\int 1_G \, \mathrm{d}\mu = 1$. Such a mean is said to be *left-invariant* if, defining $g \cdot f$ via $g \cdot f(x) = f(g^{-1}x)$ for $g \in G$, we have $\int g \cdot f \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$ for every $f \in \ell^\infty(G)$ and every $g \in G$. In general we abuse notation slightly and denote the mean $\int \mathrm{d}\mu$ simply by μ. Given a subset $X \subset G$ and a function $f \in \ell^\infty(G)$, we write 1_X for the characteristic function of X, write

$$\int_{x \in X} f(x) \, \mathrm{d}\mu(x) = \int 1_X f \, \mathrm{d}\mu,$$

and abbreviate

$$\mu(X) = \int 1_X \, \mathrm{d}\mu.$$

It is very natural to define the degree of commutativity on an amenable group G via a finitely additive left-invariant mean μ. Given such a mean on G, we define the *product mean* $\mu \times \mu$ on $G \times G$ via

$$\int_{G \times G} f \, \mathrm{d}(\mu \times \mu) = \int_{x \in G} \int_{y \in G} f(x, y) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y).$$

We then define

$$\mathrm{dc}_\mu(G) = (\mu \times \mu)(\{(x, y) \in G \times G : xy = yx\}).$$

Theorem 1.16. *Let G be an amenable group with finitely additive left-invariant mean μ. Then the following hold.*

(1) If $\mathrm{dc}_\mu(G) > \frac{5}{8}$ then G is abelian.
(2) If $\mathrm{dc}_\mu(G) \geq \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$ then the centre of G has index at most $\frac{1}{\varepsilon}$ in G.
(3) If $\mathrm{dc}_\mu(G) \geq \alpha > 0$ then G has a normal subgroup Γ of index at most $\lceil \alpha^{-1} \rceil$ and a normal subgroup H of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and Γ/H is abelian.

Let us emphasise that we do not assume in Theorem 1.16 that G is finitely generated.

Proposition 1.17 (converse to Theorem 1.15 (3) and Theorem 1.16 (3))**.** *Let G be a countable group with a subgroup Γ of index m and a subgroup $H \triangleleft \Gamma$ of cardinality d such that Γ/H is abelian. Then for every finitely additive left-invariant mean μ on G we have $\mathrm{dc}_\mu(G) \geq \frac{1}{m^2 d}$. Moreover, if G is finitely generated and M is a sequence of probability measures on G that measures index uniformly then $\mathrm{dc}_M(G) \geq \frac{1}{m^2 d}$.*

Remark. For non-finitely generated groups it is not the case that finite-by-abelian implies virtually abelian. An example showing this appears on Tao's blog [16]: if V is an infinite vector space over a finite field $\mathbb{F}$ and $b : V \times V \to \mathbb{F}$ is a non-degenerate anti-symmetric bilinear form then the product set $V \times \mathbb{F}$ with group operation defined by $(v, x)(w, y) = (v + w, x + y + b(v, w))$ for $v, w \in V$ and $x, y \in F$ is finite-by-abelian but not virtually abelian. Proposition 1.17 therefore shows that the characterisation proposed by Conjecture 1.3 does not hold for groups that are not finitely generated when degree of commutativity is defined with respect to an invariant mean.

INDEPENDENCE OF DEGREE OF COMMUTATIVITY FROM THE METHOD OF AVERAGING. In a forthcoming paper, Antolín, Martino and Ventura show that in a virtually abelian group G the value of $\mathrm{dc}_S(G)$ is independent of the choice of generating set S, and that the $\limsup$ appearing in the definition of $\mathrm{dc}_S(G)$ is actually a limit. This leads immediately to the following corollary of Theorem 1.4.

Corollary 1.18 (Antolín–Martino–Ventura, unpublished)**.** *Let G be a finitely generated residually finite group and let S, S' be two generating sets, each satisfying (1.1). Then $\mathrm{dc}_S(G) = \mathrm{dc}_{S'}(G)$, and the $\limsup$ appearing in the definition of $\mathrm{dc}_S(G)$ is actually a limit.*

Here we prove a similar result, which appears to be a necessary ingredient in our proof of Theorem 1.15 (see Remark 7.2).

Theorem 1.19. *Let G be a finitely generated group, and let M be a sequence of probability measures on G that measures index uniformly. Then the $\limsup$ in the definition of $\mathrm{dc}_M(G)$ is actually a limit, and $\mathrm{dc}_{M'}(G) = \mathrm{dc}_M(G)$ for every other sequence M' of probability measures on G that measures index uniformly. Moreover, if G is amenable then $\mathrm{dc}_\mu(G) = \mathrm{dc}_M(G)$ for every finitely additive left-invariant mean μ on G.*

As a by-product of the proofs of our theorems we also obtain the following partial version of Theorem 1.19 for groups that are not finitely generated.

Corollary 1.20. *Let G be a countable group. Then the following hold.*
  (1) If μ is a finitely additive left-invariant mean on G such that $\mathrm{dc}_\mu(G) = 0$ then $\mathrm{dc}_{\mu'}(G) = 0$ for every other finitely additive left-invariant mean μ' on G.
  (2) If $\alpha > \frac{1}{2}$, and if there is a finitely additive left-invariant mean μ on G such that $\mathrm{dc}_\mu(G) = \alpha$ or some sequence M of probability measures on G that measures index uniformly such that $\mathrm{dc}_M(G) = \alpha$, then $\mathrm{dc}_{\mu'}(G) = \alpha$ for every finitely additive left-invariant mean μ' on G, and $\mathrm{dc}_{M'}(G) = \alpha$ for every sequence M' of probability measures on G that measures index uniformly. Moreover, the $\limsup$ in the definition of $\mathrm{dc}_{M'}(G)$ is actually a limit.

Remark. It follows from Theorem 1.16 and Proposition 1.17 that in an arbitrary amenable group G the degree of commutativity $\mathrm{dc}_\mu(G)$ with respect to a given finitely additive left-invariant mean μ is bounded from below and above in terms of $\mathrm{dc}_{\mu'}(G)$ for any other finitely additive left-invariant mean μ'. It would be interesting to see whether different invariant means really can give different values of dc in situations not governed by Theorem 1.19 and Corollary 1.20, which is to say when they take values in $(0, \frac{1}{2}]$ in a group that is not finitely generated.

AN APPLICATION TO CONJUGACY RATIOS OF AMENABLE GROUPS. Given a finitely generated group G with finite symmetric generating set S containing the identity, and writing $\mathcal{C}(G)$ for the set of

conjugacy classes of G, Cox [7] introduces the *conjugacy ratio* $\mathrm{cr}_S(G)$ of G with respect to S via

$$(1.3) \qquad \mathrm{cr}_S(G) = \limsup_{n \to \infty} \frac{|\{C \in \mathcal{C}(G) : C \cap S^n \neq \varnothing\}|}{|S^n|}.$$

For notational convenience, in the present paper we also define a $\liminf$ version

$$\mathrm{cr}_S^*(G) = \liminf_{n \to \infty} \frac{|\{C \in \mathcal{C}(G) : C \cap S^n \neq \varnothing\}|}{|S^n|}.$$

It is well known that in a finite group the conjugacy ratio coincides with the degree of commutativity; Cox asks whether this remains true for infinite groups [7, Question 2], and also whether $\mathrm{cr}_S(G) > 0$ if and only if G is virtually abelian [7, Question 1]. Positive answers to Conjecture 1.3 and the first of these questions would of course imply a postive answer to the second.

Ciobanu, Cox and Martino [6, Conjecture 1.1] conjecture explicitly that $\mathrm{cr}_S(G) > 0$ if and only if G is virtually abelian, and prove it for residually finite groups satisfying (1.1) [6, Theorem 3.7], as well as for hyperbolic groups, lamplighter groups and right-angled Artin groups [6, §4].

In the present paper we answer both of Cox's questions in the affirmative in the case where S satisfies (1.1); in particular, this removes the need for the residually finite hypothesis from the result of Ciobanu, Cox and Martino.

Before we state our results precisely we need some further notation. A sequence $(F_n)_{n=1}^\infty$ of finite subsets of G is said to be a *right-Følner sequence* if

$$\frac{|F_n x \triangle F_n|}{|F_n|} \to 0$$

as $n \to \infty$. We have already noted that if S satisfies (1.1) then $(S^n)_{n=1}^\infty$ is a left-Følner sequence; by symmetry it is also a right-Følner sequence.

By analogy with (1.3), we define the *conjugacy ratio* $\mathrm{cr}_F(G)$ of G with respect to a Følner sequence $F = (F_n)_{n=1}^\infty$ of finite subsets of G via

$$\mathrm{cr}_F(G) = \limsup_{n \to \infty} \frac{|\{C \in \mathcal{C}(G) : C \cap F_n \neq \varnothing\}|}{|F_n|}.$$

We also abuse notation slightly and identify F with the sequence $(\mu_n)_{n=1}^\infty$ of uniform probability measures on the sets F_n, so that the notation $\mathrm{dc}_F(G)$ makes sense.

It is an easy exercise to check that in a finite group the conjugacy ratio is always equal to the degree of commutativity. Here we show that this remains true in an infinite group when both are defined with respect to a simultaneous left- and right-Følner sequence.

Theorem 1.21. *Let G be a finitely generated amenable group, and let $F = (F_n)_{n=1}^\infty$ be both a left- and right-Følner sequence for G. Then $\mathrm{cr}_F(G) = \mathrm{dc}_F(G)$.*

In particular, combined with Theorem 1.7 this implies the following.

Corollary 1.22. *Let G be a finitely generated amenable group, and let $F = (F_n)_{n=1}^\infty$ be both a left- and right-Følner sequence for G. Then the following hold.*

    (1) *If $\mathrm{cr}_F(G) > \frac{5}{8}$ then G is abelian.*

    (2) *If $\mathrm{cr}_F(G) \geq \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$ then the centre of G has index at most $\frac{1}{\varepsilon}$ in G.*

    (3) *If $\mathrm{cr}_F(G) \geq \alpha > 0$ then G has a normal subgroup Γ of index at most $\lceil \alpha^{-1} \rceil$ and a normal subgroup H of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and Γ/H is abelian.*

Remark. It is well known that every finitely generated amenable group admits sequences that are simultaneously left- and right-Følner, and so Theorem 1.21 and Corollary 1.22 have content for every finitely generated amenable group.

If S is a finite symmetric generating set for G containing the identity and satisfying (1.1) then $(S^n)_{n=1}^\infty$ is both a left- and right-Følner sequence, and so we have the following generalisation of Ciobanu, Cox and Martino's result [6, Theorem 3.7].

Corollary 1.23. *Let G be a finitely generated group of subexponential growth, and let S be a finite symmetric generating set containing the identity and satisfying (1.1). Then the following hold.*

  (1) *If $\operatorname{cr}_S(G) > \frac{5}{8}$ then G is abelian.*
  (2) *If $\operatorname{cr}_S(G) \geq \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$ then the centre of G has index at most $\frac{1}{\varepsilon}$ in G.*
  (3) *If $\operatorname{cr}_S(G) \geq \alpha > 0$ then G has a normal subgroup Γ of index at most $\lceil \alpha^{-1} \rceil$ and a normal subgroup H of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and Γ/H is abelian.*

Moreover, if G is a finitely generated group of subexponential growth and S is a finite symmetric generating set containing the identity then it is well known and easy to check that some subsequence of $(S^n)_{n=1}^\infty$ is both a left- and right-Følner sequence, and so Corollary 1.22 also implies the following.

Corollary 1.24. *Let G be a finitely generated group of subexponential growth, and let S be a finite symmetric generating set containing the identity. Then the following hold.*

  (1) *If $\operatorname{cr}_S^*(G) > \frac{5}{8}$ then G is abelian.*
  (2) *If $\operatorname{cr}_S^*(G) \geq \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$ then the centre of G has index at most $\frac{1}{\varepsilon}$ in G.*
  (3) *If $\operatorname{cr}_S^*(G) \geq \alpha > 0$ then G has a normal subgroup Γ of index at most $\lceil \alpha^{-1} \rceil$ and a normal subgroup H of cardinality at most $\exp(O(\alpha^{-O(1)}))$ such that $H \subset \Gamma$ and Γ/H is abelian.*

Proposition 1.17 and Theorem 1.21 combine to give a converse to Corollary 1.22 when F is both a left- and right-Følner sequence. However, it turns out to be extremely straightforward to prove this converse directly for just left-Følner sequences, as follows.

Proposition 1.25 (converse to Corollary 1.22)**.** *Let G be a group with a subgroup Γ of index m and a subgroup $H \lhd \Gamma$ of cardinality d such that Γ/H is abelian, and let $F = (F_n)_{n=1}^\infty$ be a left-Følner sequence for G. Then $\operatorname{cr}_F(G) \geq \frac{1}{m^2 d}$.*

Remark. In particular, since a finitely generated finite-by-abelian group G is virtually abelian, if F is both a left- and right-Følner sequence then $\operatorname{cr}_F(G) > 0$ if and only if G is virtually abelian.

OVERALL STRUCTURE OF THE ARGUMENTS. Theorem 1.5 is a special case of Theorem 1.7. Theorem 1.6 follows from Theorem 1.14 and Theorem 1.15, whilst Theorem 1.7 follows from Theorem 1.13 and Theorem 1.15.

We prove Theorem 1.10 in Section 2, Theorem 1.13 in Section 3, and Theorem 1.14 in Section 4. We prove Theorem 1.19 in Section 6, and Theorems 1.15 and 1.16, Proposition 1.17, and Corollary 1.20 in Section 7. Finally, we prove Theorem 1.21 and Proposition 1.25 in Section 8.

2. A weak Neumann-type theorem

In this section we prove Theorem 1.10. We start with the following result, which is based on a simple technique used by P. M. Neumann [14] that we use repeatedly throughout this paper.

Proposition 2.1. *Let $M = (\mu_n)_{n=1}^{\infty}$ be a sequence of measures that detects index uniformly at rate π. Let $\alpha \in (0,1]$, and suppose that $\mathrm{dc}_M(G) \geq \alpha$. Let $\gamma \in (0,1)$ be such that $\pi(\gamma) < \alpha$, and write*

$$X = \{x \in G : [G : C_G(x)] \leq \tfrac{1}{\gamma}\}.$$

Then $\limsup_{n \to \infty} \mu_n(X) \geq \alpha - \pi(\gamma)$.

Proof. By definition of dc_M there exists a sequence $n_1 < n_2 < \ldots$ such that $\mathrm{dc}_{\mu_{n_i}}(G) \geq \alpha - o(1)$. Writing $\mathbb{E}^{(n)}$ for expectation with respect to μ_n, this means precisely that

$$\mathbb{E}_{x \in G}^{(n_i)}(\mu_{n_i}(C_G(x))) \geq \alpha - o(1).$$

Following P. M. Neumann [14], we note that therefore

$$\alpha \leq \mu_{n_i}(X)\mathbb{E}_{x \in X}^{(n_i)}(\mu_{n_i}(C_G(x))) + \mu_{n_i}(G \backslash X)\mathbb{E}_{x \in G \backslash X}^{(n_i)}(\mu_{n_i}(C_G(x))) + o(1)$$
$$\leq \mu_{n_i}(X) + \pi(\gamma) + o(1),$$

the last inequality being by uniform detection of index, and the result follows. $\square$

Lemma 2.2. *Let $m, r \in \mathbb{N}$, and let G be a group generated by r elements. Then G has at most $O_{m,r}(1)$ subgroups of index m.*

Proof. A subgroup H of index m is the stabiliser of an action of G on the set G/H, which has size m. There at most $|\mathrm{Sym}(m)|^r$ possible actions of G on a set of size m, and so there are at most $|\mathrm{Sym}(m)|^r$ possibilities for this stabiliser. $\square$

Proof of Theorem 1.10. Let $\gamma = \frac{1}{2}\inf\{\beta \in (0,1] : \pi(\beta) \geq \frac{\alpha}{2}\}$, noting that therefore $\pi(\gamma) < \frac{\alpha}{2}$, and write $X = \{x \in G : [G : C_G(x)] \leq \frac{1}{\gamma}\}$, so that Proposition 2.1 gives $\limsup_{n \to \infty} \mu_n(X) > \frac{\alpha}{2}$. This implies in particular that the group Γ generated by X satisfies $\limsup_{n \to \infty} \mu_n(\Gamma) > \frac{\alpha}{2}$, and hence $[G : \Gamma] \leq \frac{1}{\gamma}$ by the definitions of γ and uniform detection of index at rate π.

Lemma 2.2 and the definition of γ imply that there are at most $O_{r,\pi,\alpha}(1)$ possibilities for $C_G(x)$ with $x \in X$, and so their intersection has index at most $O_{r,\pi,\alpha}(1)$ in G. However, this intersection is contained in $C_G(\Gamma)$, and so its intersection with Γ, which again has index at most $O_{r,\pi,\alpha}(1)$ in G, is abelian. $\square$

3. Uniform measurement of index by almost-invariant measures

In this section we prove Theorem 1.13. Throughout the section, in order to avoid having to repeat arguments that are essentially identical for finite- and infinite-index subgroups, we adopt the convention that if M is an infinite quantity (such as the index of a subgroup or the number of vertices in a graph) then $1/M$ takes the value zero.

We start with Theorem 1.13, which shows that sequences of almost-invariant measures measure index uniformly. The only tool we really need other than the almost-invariance of the measures under consideration is the following standard fact.

Lemma 3.1. *Let G be a group with finite symmetric generating set S containing the identity, let H be a subgroup of index at least $m \in \mathbb{N}$, and let $x \in G$. Then there exist $y_1, \ldots, y_m \in S^m$ such that the elements $y_i x$ belong to distinct left-cosets of H.*

Proof. We essentially reproduce the proof of the closely related [4, Lemma 2.7]. If $S^{n+1}xH = S^n xH$ for a given n then it follows by induction that $S^r xH = S^n xH$ for every $r \geq n$, and hence that $G = S^n xH$. We may therefore assume that $SxH \subsetneq S^2 xH \subsetneq \ldots \subsetneq S^m xH$. However, this implies that the number of left-cosets of H having non-empty intersection with $S^n x$ is strictly increasing for $n = 1, \ldots, m$, and so the lemma is proved. $\square$

Proof of Theorem 1.13. It follows from the definition of a sequence of almost-invariant measures that for every $y \in G$ we have

$$(3.1) \qquad |\mu_n(yxH) - \mu_n(xH)| \to 0$$

uniformly over all $x \in G$ and all subgroups H of G. Since this convergence is uniform over all $y \in S^m$, Lemma 3.1 implies that

$$\mu_n(xH) \to \frac{1}{[G:H]}$$

uniformly over all $x \in G$ and all subgroups H of G of index at most m. On the other hand, given $\varepsilon > 0$, let m be the smallest integer greater than $\frac{1}{\varepsilon}$. If H has index greater than m then Lemma 3.1 and the fact that the convergence in (3.1) is uniform over $y \in S^{m+1}$ implies that provided n is large enough in terms of ε we have $\mu_n(xH) < \varepsilon$, and hence in particular

$$\left| \mu_n(xH) - \frac{1}{[G:H]} \right| \leq \varepsilon.$$

The convergence is therefore uniform over all $x \in G$ and all subgroups H of G, as required. $\square$

4. Uniform measurement of index by random walks

In this section we prove Theorem 1.14. Our key tools will be results of Morris and Peres [12] concerning mixing time and heat-kernel decay of Markov chains. Following the set-up of their paper, let $\{p(x, y)\}$ be the transition probabilities for an irreducible Markov chain on countable state space V. A measure π on V is said to be *stationary* for p if $\sum_{x \in V} \pi(x)p(x, y) = \pi(y)$ for all $x \in V$; if V is finite we also require π to be a probability measure.

Lemma 4.1. *If p is symmetric and π is uniform then π is stationary for p.*

Proof. We have $\sum_{x \in V} p(x, y) = \sum_{x \in V} p(y, x) = 1$. $\square$

Given a stationary measure π on V, write $\pi_* = \inf_{x \in V} \pi(x)$. For $A \subset V$ write

$$|\partial A| = \sum_{x \in A, y \notin A} \pi(x)p(x, y),$$

and define the *conductance* Φ_A of $A \neq \varnothing$ via

$$\Phi_A = \frac{|\partial A|}{\pi(A)}.$$

Define $\Phi : [\pi_*, \infty) \to \mathbb{R}$ via

$$\Phi(r) = \begin{cases} \inf\{\Phi_A : \pi(A) \leq r\} & \text{if } r \leq \frac{1}{2} \\ \Phi(\frac{1}{2}) & \text{otherwise} \end{cases}$$

if V is finite, or via

$$\Phi(r) = \inf\{\Phi_A : \pi(A) \leq r\}$$

if V is infinite. Note that in each case irreducibility implies that

$$(4.1) \qquad \Phi(r) \geq \frac{\inf\{\pi(x)p(x,y) : p(x,y) > 0\}}{r}$$

for every r.

Theorem 4.2 (Morris–Peres [12]). *Let $\varepsilon > 0$, and let $c \in (0, \frac{1}{2}]$ be such that $p(x,x) \geq c$ for all $x \in V$. Then for every*

$$(4.2) \qquad n \geq 1 + \frac{(1-c)^2}{c^2} \int_{4\pi_*}^{4/\varepsilon} \frac{4\,du}{u\Phi(u)^2}$$

we have

$$\left| \frac{p^n(x,y) - \pi(y)}{\pi(y)} \right| \leq \varepsilon$$

if V is finite [12, Theorem 5], or

$$\left| \frac{p^n(x,y)}{\pi(y)} \right| \leq \varepsilon$$

if V is infinite [12, Theorem 2].

Proof of Theorem 1.14. Write $c = \min_{x \in \operatorname{supp}\mu} \mu(x)$. Let $\varepsilon > 0$ and let H be a subgroup of G. We will prove that for every

$$(4.3) \qquad n \geq 1 + \frac{32(1-c)^2}{c^4\varepsilon^2}$$

we have

$$(4.4) \qquad \left| \mu^{*n}(xH) - \tfrac{1}{[G:H]} \right| \leq \varepsilon$$

for every $x \in G$ if H has finite index, or

$$(4.5) \qquad |\mu^{*n}(xH)| \leq \varepsilon$$

for every $x \in G$ if H has infinite index; in particular, this implies the theorem.

The left random walk on G with respect to μ induces an irreducible Markov chain on the left-cosets xH of H. Let p be the transition matrix of this Markov chain. The symmetry of μ implies that p is symmetric, and hence, by Lemma 4.1, that we may assume that π is the uniform probability measure on G/H if H has finite index, or that π is everywhere equal to 1 if H has infinite index. The inequality (4.4) is therefore equivalent to the inequality

$$\left| \frac{p^n(H, xH) - \pi(xH)}{\pi(xH)} \right| \leq \varepsilon[G : H],$$

whilst the inequality (4.5) is equivalent to the inequality

$$\left| \frac{p^n(H, xH)}{\pi(xH)} \right| \leq \varepsilon.$$

However, since $\inf\{p(xH, yH) : p(xH, yH) > 0\} \geq c$, it follows from (4.1) that the integrand of (4.2) is at most $4u[G : H]^2/c^2$ if H has finite index, or at most $4u/c^2$ if H has infinite index, and so Theorem 4.2 implies that (4.4) is satisfied if

$$n \geq 1 + \frac{4(1-c)^2[G : H]^2}{c^4} \int_0^{4/(\varepsilon[G:H])} u\,du,$$

or that (4.5) is satisfied if

$$n \geq 1 + \frac{4(1-c)^2}{c^4} \int_0^{4/\varepsilon} u \, \mathrm{d}u.$$

In each case this is equivalent to (4.3), as reuqired. □

5. A FINITE-INDEX SUBGROUP WITH UNIFORMLY BOUNDED CONJUGACY CLASSES

At the heart of P. M. Neumann's original proof of Theorem 1.2 lies a result essentially saying that if many elements of G have bounded conjugacy classes then G has a large subgroup in which *every* element has a bounded conjugacy class. In this section we generalise his argument to arbitrary amenable groups as follows.

Proposition 5.1. *Let G be an amenable group with finitely additive left-invariant mean μ, let $\alpha \in (0,1]$, and suppose that $\mathrm{dc}_\mu(G) \geq \alpha$. Given $r \in \mathbb{N}$, write $Y_r = \{x \in G : [G : C_G(x)] \leq r\}$, and write Γ_r for the group generated by Y_r. Then*

(A) $[G : \Gamma_r] \leq \frac{1}{\alpha - 1/r}$, *and*

(B) *if $r \geq \frac{2}{\alpha}$ then $[G : C_G(x)] \leq r^{6/\alpha+2}$ for every $x \in \Gamma_r$.*

We prove this result shortly. First, let us note that the result that allows Neumann to pass from the uniform bound on the size of conjugacy classes given by Proposition 5.1 to the algebraic characterisation given by Theorem 1.2 is the following.

Theorem 5.2 (P. M. Neumann–Vaughan-Lee [15, Theorem 1]). *Let G be a group in which the size of every conjugacy class is at most $k \in \mathbb{N}$. Then*

$$|[G, G]| \leq k^{\frac{1}{2}(3+5 \log k)}.$$

Remarks on the proof. The proof in [15] assumes that G is finite. However, as is remarked there, one can easily reduce to this case using other known results. First, B. H. Neumann [13, Theorem 3.1] shows that if an infinite group G has a uniform bound on the size of its conjugacy classes then $|[G, G]| < \infty$. Then, [11, Lemma 2.1] (which in [11] is attributed to P. Hall) shows that if $|[G, G]| < \infty$ then there exists some finite group G_0 whose largest conjugacy class has the same size as the largest conjugacy class of G, and such that $|[G_0, G_0]| = |[G, G]|$. □

The first step of P. M. Neumann's original proof of Theorem 1.2 is essentially the following observation.

Lemma 5.3. *Let G be an amenable group with finitely additive left-invariant mean μ, let $\alpha \in (0,1]$, and suppose that $\mathrm{dc}_\mu(G) \geq \alpha$. Let $r > \frac{1}{\alpha}$ and write $X = \{x \in G : [G : C_G(x)] \leq r\}$. Then $\mu(X) \geq \alpha - \frac{1}{r}$.*

Proof. We have

$$\mathrm{dc}_\mu(G) = \int_{x \in X} \mu(C_G(x)) \, \mathrm{d}\mu(x) + \int_{x \in G \setminus X} \mu(C_G(x)) \, \mathrm{d}\mu(x)$$
$$\leq \mu(X) + \frac{1}{r}.$$

□

Lemma 5.4. *Let G be an amenable group with finitely additive left-invariant mean μ, let $X \subset G$ be a symmetric subset containing the identity, and let $m \in \mathbb{N}$. Suppose that*

$$(5.1) \qquad\qquad \mu(X) \geq \frac{1}{m}.$$

Then $\langle X \rangle = X^{3m-1}$.

Proof. We follow Eberhard [8, Lemma 2.1], whose proof for the finite case streamlines part of P. M. Neumann's argument [14]. Select for as long as possible a sequence $x_1, x_2, \ldots$ in G such that $x_i \in X^{3i} \backslash X^{3i-1}$. If we have such a sequence $x_1, \ldots, x_j$ then $x_i X \subset X^{3i+1} \backslash X^{3i-2}$, and so the sets X and $x_i X$ are all disjoint. The inequality (5.1) and the left-invariance of μ therefore imply that $j \leq m - 1$. It follows that $X^{3m} = X^{3m-1}$, and hence that $\langle X \rangle = X^{3m-1}$, as required. $\qquad\square$

Proof of Proposition 5.1. We essentially follow P. M. Neumann [14, Theorem 1]. Lemma 5.3 implies that $\mu(Y_r) \geq \alpha - \frac{1}{r}$, which implies (A). Moreover, if $r \geq \frac{2}{\alpha}$ then by Lemma 5.4 it also implies that

$$(5.2) \qquad\qquad \Gamma_r = Y_r^{3\lceil \frac{2}{\alpha} \rceil - 1}.$$

Writing $k_G(x)$ for the size of the conjugacy class in G of $x \in G$, and noting that $k_G(x) = [G : C_G(x)]$ and $k_G(xy) \leq k_G(x)k_G(y)$ for every $x, y \in G$, conclusion (B) then follows from (5.2) and the definition of Y_r. $\qquad\square$

6. Independence of degree of commutativity from the method of averaging

Proof of Theorem 1.19. If $\mathrm{dc}_M(G) > 0$ for some sequence M of probability measures that detects index uniformly then Theorem 1.10 and Remark 1.12 imply that G is virtually abelian, and in particular amenable, and it is then trivial that for every finitely additive left-invariant mean μ we have $\mathrm{dc}_\mu(G) > 0$. We may therefore assume that G is amenable with finitely additive left-invariant mean ν such that $\mathrm{dc}_\nu(G) > 0$, say $\mathrm{dc}_\nu(G) = \alpha$. Let $r \in \mathbb{N}$ large enough to apply Proposition 5.1 (B) with the mean ν, and define Γ_r to be the group generated by $\{x \in G : [G : C_G(x)] \leq r\}$ as in that proposition.

Let μ be an arbitrary finitely additive left-invariant mean on G, and let $M = (\mu_n)_{n=1}^\infty$ be an arbitrary sequence of probability measures on G that measures index uniformly, and note that

$$(6.1) \qquad \mathrm{dc}_\mu(G) = \int_{x \in \Gamma_r} \mu(C_G(x)) \, \mathrm{d}\mu(x) + \int_{x \in G \backslash \Gamma_r} \mu(C_G(x)) \, \mathrm{d}\mu(x)$$

and

$$(6.2) \qquad \mathrm{dc}_{\mu_n}(G) = \sum_{x \in \Gamma_r} \mu_n(C_G(x))\mu_n(x) + \sum_{x \in G \backslash \Gamma_r} \mu_n(C_G(x))\mu_n(x).$$

The definition of Γ_r implies that the centraliser of any given $x \in G \backslash \Gamma_r$ has index at least r, which implies that

$$(6.3) \qquad \int_{x \in G \backslash \Gamma_r} \mu(C_G(x)) \, \mathrm{d}\mu(x) \leq \frac{1}{r},$$

and also, by definition of uniform measurement of index, that

$$(6.4) \qquad \limsup_{n \to \infty} \sum_{x \in G \backslash \Gamma_r} \mu_n(C_G(x))\mu_n(x) \leq \frac{1}{r}.$$

Proposition 5.1 (B) implies that the centraliser of any given $x \in \Gamma_r$ has index at most $r^{6/\alpha+2}$. Since G is finitely generated, Lemma 2.2 therefore implies that we may enumerate as $C_1, \ldots, C_k$ the finite-index subgroups of G arising as centralisers of elements of Γ_r. We may then partition Γ_r into sets $A_1, \ldots, A_k$ defined by

$$A_i = \{x \in \Gamma_r : C_G(x) = C_i\},$$

giving

$$(6.5) \qquad \int_{x \in \Gamma_r} \mu(C_G(x)) \, \mathrm{d}\mu(x) = \sum_{i=1}^{k} \mu(A_i)\mu(C_i)$$

and

$$(6.6) \qquad \sum_{x \in \Gamma_r} \mu_n(C_G(x))\mu_n(x) = \sum_{i=1}^{k} \mu_n(A_i)\mu_n(C_i)$$

for every n.

However, for each i we have

$$A_i = C_{\Gamma_r}(C_i) \setminus \bigcup_{H \gneqq C_i} C_{\Gamma_r}(H) \,.$$

Writing Z for the set of subgroups $H \gneqq C_i$ such that $C_{\Gamma_r}(H)$ has infinite index in G and P for the set of subgroups $H \gneqq C_i$ such that $C_{\Gamma_r}(H)$ has finite index in G, and noting that each of Z and P is finite by Lemma 2.2, we therefore have

$$A_i = \left(C_{\Gamma_r}(C_i) \setminus \bigcup_{H \in P} C_{\Gamma_r}(H) \right) \setminus \bigcup_{H \in Z} C_{\Gamma_r}(H)$$

However, since P is finite the union $\bigcup_{H \in P} C_{\Gamma_r}(H)$ is also the union of finitely many cosets of the finite-index subgroup $\bigcap_{H \in P} C_{\Gamma_r}(H)$, each of which is contained entirely within $C_{\Gamma_r}(C_i)$. This implies that the right-hand side of (6.5) takes some value λ_r that does not depend on the choice of μ, and that the right-hand side of (6.6) converges to λ_r as $n \to \infty$ for every M that measures index uniformly. Combined with (6.1) and (6.3), this implies that $\lambda_r \to \mathrm{dc}_\mu(G)$ as $r \to \infty$, whilst combined with (6.2) and (6.4) it implies that $\mathrm{dc}_{\mu_n}(G) \to \mathrm{dc}_M(G) = \mathrm{dc}_\mu(G)$ as $n \to \infty$. □

7. Algebraic consequences of a high degree of commutativity

In this section we complete the proofs of our main results. We start with Theorem 1.15 (2) and Theorem 1.16 (2), as they are ingredients in the proofs of Theorem 1.15 (1) and Theorem 1.16 (1), respectively.

Proof of Theorem 1.15 (2) and Theorem 1.16 (2). Given $\delta > 0$ define $X_\delta = \{x \in G : [G : C_G(x)] \leq 2/(1 + 2\delta)\}$, noting that since $2/(1 + 2\delta) < 2$ the set X_δ is central. In the setting of Theorem 1.16 (2), Lemma 5.3 implies that for $\delta < \varepsilon$ we have $\mu(X_\delta) \geq \varepsilon - \delta$, and so letting $\delta \to 0$ we see that $\mu(Z(G)) \geq \varepsilon$, and Theorem 1.16 (2) follows. In the setting of Theorem 1.15 (2), Remark 1.12 and Proposition 2.1 imply that for $\delta < \varepsilon$ we have $\limsup_{n \to \infty} \mu_n(X_\delta) \geq \varepsilon - \delta$, and so letting $\delta \to 0$ we see that $\limsup_{n \to \infty} \mu_n(Z(G)) \geq \varepsilon$. Theorem 1.15 (2) then follows from the fact that the μ_n measure index uniformly. □

The conclusion of Theorem 1.15 (2) and Theorem 1.16 (2) allows us to give an explicit expression for the degree of commutativity, as follows.

Proposition 7.1. *Let G be a countable group with a central subgroup Z of finite index, and let T be a left-transversal for Z in G. Then if μ is a finitely additive left-invariant mean on G then*

$$\mathrm{dc}_\mu(G) = \frac{1}{|G/Z|} \sum_{t \in T} \frac{1}{[G : C_G(t)]}.$$

Moreover, if $M = (\mu_n)_{n=1}^\infty$ is a sequence of probability measures on G that measure index uniformly then

$$\mathrm{dc}_M(G) = \frac{1}{|G/Z|} \sum_{t \in T} \frac{1}{[G : C_G(t)]},$$

and the $\limsup$ *in the definition of* $\mathrm{dc}_M(G)$ *is actually a limit.*

Proof. Elements $x, y \in G$ commute if and only if xz and y commute for every $z \in Z$. This implies that

$$\mathrm{dc}_\mu(G) = \sum_{t \in T} \mu(tZ)\mu(C_G(t))$$

$$= \frac{1}{|G/Z|} \sum_{t \in T} \frac{1}{[G : C_G(t)]},$$

as required. It also implies that

$$\mathrm{dc}_{\mu_n}(G) = \sum_{t \in T} \mu_n(tZ)\mu_n(C_G(t))$$

$$\to \frac{1}{|G/Z|} \sum_{t \in T} \frac{1}{[G : C_G(t)]} \qquad \text{(by uniform measurement of index)},$$

as required. $\square$

Proof of Theorem 1.15 (1) and Theorem 1.16 (1). In this proof we write $\mathrm{dc}(G)$ to mean $\mathrm{dc}_M(G)$ in the context of Theorem 1.15, or $\mathrm{dc}_\mu(G)$ in the context of Theorem 1.16. We may assume that G is not abelian and that $\mathrm{dc}(G) \geq \frac{5}{8}$, and prove that $\mathrm{dc}(G) \leq \frac{5}{8}$.

Let T be a left-transversal in G of the centre Z of G. It follows from Theorem 1.15 (2) or Theorem 1.16 (2), as appropriate, and from Proposition 7.1, that

$$\mathrm{dc}(G) = \frac{1}{|G/Z|} \sum_{t \in T} \frac{1}{[G : C_G(t)]}.$$

Since only one element of T belongs to Z, and every other $t \in T$ has $[G : C_G(t)] \geq 2$, we conclude that

$$\mathrm{dc}(G) \leq \frac{1}{|G/Z|} + \frac{|G/Z| - 1}{2|G/Z|}$$

$$= \frac{1}{2} + \frac{1}{2|G/Z|}.$$

Finally, we observe as Gustafson [10] did that since G is not abelian, the quotient G/Z is neither trivial nor cyclic, and hence $|G/Z| \geq 4$. Substituting this into the last inequality then gives $\mathrm{dc}(G) \leq \frac{5}{8}$, as required. $\square$

Proof of Theorem 1.16 (3). We essentially reproduce P. M. Neumann's proof of Theorem 1.2. Let $r = \frac{1}{\alpha^2} + \frac{1}{\alpha} + 1$, let Γ be the group Γ_r defined in Proposition 5.1 and set $H = [\Gamma, \Gamma]$, so that Γ/H is abelian. Since the set Y_r generating Γ_r in Proposition 5.1 is invariant under conjugation by G, the subgroup Γ is normal in G, and since H is characteristic in Γ it is normal in G.

Conclusion (A) of Proposition 5.1 implies that the index of Γ in G is less than $\alpha^{-1} + 1$, and hence at most $\lceil \alpha^{-1} \rceil$, as required. On the other hand, conclusion (B) implies that the maximum size of a conjugacy class in Γ is at most $\exp(O(\alpha^{-O(1)}))$, and so Theorem 5.2 implies that the cardinality of H is at most $\exp(O(\alpha^{-O(1)}))$, as required. □

Proof of Theorem 1.15 (3). Since G is finitely generated, Theorem 1.10 and Remark 1.12 imply that G is virtually abelian, and in particular amenable. It follows that G has a finitely additive left-invariant measure μ, and then Theorem 1.19 implies that $\mathrm{dc}_\mu(G) = \alpha$. The desired result therefore follows from Theorem 1.16 (3). □

Remark 7.2. In the specific case where M is a sequence of almost-invariant measures, one can prove Theorem 1.15 (3) directly by a similar proof to Theorem 1.16 (3), without using Theorem 1.19. However, it does not appear that such an argument is possible when M is assumed more generally to be a sequence of measures that measures index uniformly. This is due to the use of Proposition 5.1 in the proof of Theorem 1.16 (3), and in particular the use of Lemma 5.4 in the proof of Proposition 5.1. More precisely, Lemma 5.4 uses the invariance of the invariant mean μ, and whilst the almost-invariance of a sequence of almost-invariant means is enough to prove a version of it, for more general sequences of measures that measure index uniformly there does not appear to be any reason to expect it to hold.

One reason one might be interested in proving Theorem 1.15 (3) directly for almost-invariant measures is that the existence of an invariant mean on a virtually abelian group requires the axiom of choice; one could avoid this in the proof of Theorem 1.15 (3) by replacing Theorem 1.16 (3) with the special case of Theorem 1.15 (3) for almost-invariant measures. We leave it to the interested reader to prove Theorem 1.15 (3) in this way.

Proof of Proposition 1.17. Since G is amenable, Theorem 1.19 implies that it is sufficient to prove that $\mathrm{dc}_\mu(G) \geq \frac{1}{m^2 d}$ for every finitely additive left-invariant mean μ.

Fix $x \in \Gamma$. The fact that Γ/H is abelian implies that $[x, y] \subset H$ for every $y \in \Gamma$. Enumerating the elements of H as $c_1, \ldots, c_d$, we may therefore partition Γ into sets $\Gamma_1, \ldots, \Gamma_d$ defined by

$$\Gamma_i = \{ y \in \Gamma : [x, y^{-1}] = c_i \}.$$

Applying a technique that was used in [3], for each i with $\Gamma_i \neq \varnothing$ we fix some arbitrary $y_i \in \Gamma_i$ and note that $y_i^{-1} y \in C_G(x)$ for every $y \in \Gamma_i$. In particular,

(7.1) $$y_i^{-1} \Gamma_i \subset C_G(x)$$

for every such i. Finite additivity and the pigeonhole principle imply that there exists some i with

$$\mu(\Gamma_i) \geq \frac{1}{d} \mu(\Gamma)$$
$$= \frac{1}{md},$$

and then (7.1) and left invariance imply that $\mu(C_G(x)) \geq \frac{1}{md}$. Since this holds for every $x \in \Gamma$, the proposition is proved. □

Proof of Corollary 1.20. Case (1) follows from Theorem 1.16 (3) and Proposition 1.17. Case (2) follows from Theorem 1.15 (2), Theorem 1.16 (2) and Proposition 7.1. □

8. Conjugacy ratios

In this section we prove Theorem 1.21 and Proposition 1.25. An important ingredient in the proof of Theorem 1.21 is the following reformulation of the definition of the conjugacy ratio with respect to a left- and right-Følner sequence.

Proposition 8.1. *If F is both left- and right-Følner then*

$$\frac{|\{C \in \mathcal{C}(G) : C \cap F_n \neq \varnothing \text{ and } C \not\subset F_n\}|}{|F_n|} \to 0.$$

In particular,

$$\mathrm{cr}_F(G) = \limsup_{n \to \infty} \frac{|\{C \in \mathcal{C}(G) : C \subset F_n\}|}{|F_n|}.$$

Proof. Let S be a finite generating set for G. The fact that F is both left- and right-Følner implies that

$$\frac{|S^{-1}F_n S \backslash F_n|}{|F_n|} \to 0,$$

and hence in particular that

$$\frac{|\{C \in \mathcal{C}(G) : C \cap (S^{-1}F_n S \backslash F_n) \neq \varnothing\}|}{|F_n|} \to 0.$$

Note, however, that if X is any subset of a conjugacy class C such that $S^{-1}XS \cap C = X$ then in fact $X = C$, which implies in particular that

$$\{C \in \mathcal{C}(G) : C \cap F_n \neq \varnothing \text{ and } C \not\subset F_n\} \subset \{C \in \mathcal{C}(G) : C \cap (S^{-1}F_n S \backslash F_n) \neq \varnothing\},$$

and so the result follows. □

Proof of Theorem 1.21. Writing

$$\varepsilon_n(x) = \frac{|C_G(x) \cap F_n|}{|F_n|} - \frac{1}{[G : C_G(x)]},$$

we have

$$\mathrm{dc}(F_n) = \frac{1}{|F_n|} \sum_{C \in \mathcal{C}(G) : C \cap F_n \neq \varnothing} \left(\sum_{x \in C \cap F_n} \frac{1}{[G : C_G(x)]} \right) + \frac{1}{|F_n|} \sum_{x \in F_n} \varepsilon_n(x),$$

Theorem 1.13 implies that $\varepsilon_n(x) \to 0$ uniformly in x as $n \to \infty$, and so this implies that

$$(8.1) \qquad \mathrm{dc}(F_n) = \frac{1}{|F_n|} \sum_{C \in \mathcal{C}(G) : C \cap F_n \neq \varnothing} \left(\sum_{x \in C \cap F_n} \frac{1}{[G : C_G(x)]} \right) + o(1).$$

The index of $C_G(x)$ is equal to the cardinality of the conjugacy class of x, and so

$$\sum_{x \in C \cap F_n} \frac{1}{[G : C_G(x)]} \leq 1$$

for every $C \in \mathcal{C}(G)$, which with (8.1) implies that $\mathrm{dc}_F(G) \leq \mathrm{cr}_F(G)$.

On the other hand, (8.1) also implies that

$$
\begin{aligned}
\mathrm{dc}_F(G) &\geq \limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{C\in\mathcal{C}(G):C\subset F_n} \left(\sum_{x\in C} \frac{1}{[G:C_G(x)]} \right) \\
&= \limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{C\in\mathcal{C}(G):C\subset F_n} \left(\sum_{x\in C} \frac{1}{|C|} \right) \\
&= \limsup_{n\to\infty} \frac{|\{C\in\mathcal{C}(G):C\subset F_n\}|}{|F_n|} \\
&= \mathrm{cr}_F(G),
\end{aligned}
$$

the last equality being by Proposition 8.1. $\qquad\square$

Proof of Proposition 1.25. Let $x\in\Gamma$. Since $[\Gamma,\Gamma]\subset H$, for every $y\in\Gamma$ we have $y^{-1}xy\in xH$. Fixing a right-transversal T for Γ in G, for every $t\in T$ and every $y\in\Gamma$ we therefore have $t^{-1}y^{-1}xyt\in t^{-1}xHt$. The conjugacy class of x in G therefore has size at most $|T||H|=md$. Since this is true for every $x\in H$, we conclude that

$$
|\{C\in\mathcal{C}(G):C\cap F_n\neq\varnothing\}| \geq \frac{|\Gamma\cap F_n|}{md}.
$$

Since Theorem 1.13 implies that

$$
\frac{|\Gamma\cap F_n|}{|F_n|} \to \frac{1}{m},
$$

the result follows. $\qquad\square$

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