REVISITING THE ATOMIC AND MOLECULAR DECOMPOSITION OF THE WEIGHTED HARDY SPACES

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Abstract. The purpose of this article is to give another molecular decomposition for members of the weighted Hardy spaces different from that given in [9], and review some overlooked details. As an application of this decomposition, we obtain the boundedness on $H^p_w(\mathbb{R}^n)$ of all linear operator T, which is assumed to be bounded on $L^{p_0}(\mathbb{R}^n)$ for some $1 < p_0 < +\infty$, for all weight $w \in \mathcal{A}_\infty$ and all $0 < p \leq 1$ if $1 < \frac{r_w-1}{r_w} p_0$ or all $0 < p < \frac{r_w-1}{r_w} p_0$ if $\frac{r_w-1}{r_w} p_0 \leq 1$, where r_w is the critical index of w for the reverse Hölder condition. In particular, the classical singular integrals result bounded from $H^p_w(\mathbb{R}^n)$ into $L^p_w(\mathbb{R}^n)$ and on $H^p_w(\mathbb{R}^n)$ for all $w \in \mathcal{A}_\infty$ and all $0 < p \leq 1$. We also obtain the $H^p_{w^p}(\mathbb{R}^n) - H^q_{w^q}(\mathbb{R}^n)$ boundedness of the Riesz potential I_α, for $0 < p \leq 1$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and certain weights w.

1. Introduction

The Hardy spaces on $\mathbb{R}^n$ were defined in [5] by C. Fefferman and E. Stein, since then the subject has received considerable attention. One of the most important applications of Hardy spaces is that they are good substitutes of Lebesgue spaces when $p \leq 1$. For example, when $p \leq 1$, it is well known that Riesz transforms are not bounded on $L^p(\mathbb{R}^n)$; however, they are bounded on Hardy spaces $H^p(\mathbb{R}^n)$.

To obtain the boundedness of operators, like singular integrals or fractional type operators, in the Hardy spaces $H^p(\mathbb{R}^n)$, one can appeals to the atomic or molecular characterization of $H^p(\mathbb{R}^n)$,which means that a distribution in H^p can be represented as a sum of atoms or molecules. The atomic decomposition of elements in $H^p(\mathbb{R}^n)$ was obtained by Coifman in [2] ($n = 1$), and by Latter in [8] ($n \geq 1$). In [20], Taibleson and Weiss gave the molecular decomposition of elements in $H^p(\mathbb{R}^n)$. Then the boundedness of linear operators in H^p can be deduced from their behavior on atoms or molecules in principle. However, it must be mentioned that M. Bownik in [1], based on an example of Y. Meyer, constructed a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$, which maps all $(1, \infty, 0)$ atoms into bounded scalars, but yet cannot extended to a bounded linear functional on the whole $H^1(\mathbb{R}^n)$. This implies that it not suffice to check that an operator from a Hardy spaces H^p, $0 < p \leq 1$, into some quasi Banach space X, maps atoms into bounded elements of X to establish that this operator extends to a bounded operator on H^p. Bownik's example is in certain sense pathological. Fortunately, if T is a classical operator, then the uniform boundedness of T on atoms implies the boundedness from H^p into L^p, this follows from the boundedness on L^s, $1 < s < \infty$, of T and since one always can take an atomic decomposition which converges in the norm of L^s, see [21] and [14].

The weighted Lebesge spaces $L^p_w(\mathbb{R}^n)$ are a generalization of the classical Lebesgue spaces $L^p(\mathbb{R}^n)$, replacing the Lebesgue measure dx by the measure $w(x)dx$, where

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w is a non-negative measurable function. Then one can define the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ by generalizing the definition of $H^p(\mathbb{R}^n)$ (see [18]). It is well known that the harmonic analysis on these spaces is relevant if the "weights" w belong to the class $\mathcal{A}_\infty$. The atomic characterization of $H^p_w(\mathbb{R}^n)$ has been given in [4] and [18]. The molecular characterization of $H^p_w(\mathbb{R}^n)$ was developed independently by X. Li and L. Peng in [10] and by M.-Y. Lee and C.-C. Lin in [9]. In both works the authors obtained the boundedness of classical singular integrals in H^p_w for $w \in \mathcal{A}_1$. We extend these results for all $w \in \mathcal{A}_\infty$.

Given $w \in \mathcal{A}_\infty$, for us, a $w-(p, p_0, d)$ atom is a measurable function $a(\cdot)$ with support in a ball B such that

(1) $\|a\|_{L^{p_0}} \leq \dfrac{|B|^{1/p_0}}{w(B)^{1/p}}$, *and*

(2) $\int x^\alpha a(x)\, dx = 0$, *for all multi-index* $|\alpha| \leq d$,

where the parameters p, p_0 and d satisfy certain restrictions. We remark that our definition of atom differs from that given in [4], [18].

Ones of our main results is the theorem 9 of Section 2 below, this assure that:

If $w \in \mathcal{A}_\infty$ and f belongs to a dense subspace of H^p_w , then there exist a sequence of $w-(p, p_0, d)$ atoms $\{a_j\}$ and a sequence of scalars $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq c\|f\|^p_{H^p_w}$ such that $f = \sum_j \lambda_j a_j$, where the series converges to f in $L^s(\mathbb{R}^n)$, for all $s > 1$.

With this result we avoid any problem that could arise with respect to establish the boundedness of classical operators in H^p_w. The verification of the convergence in L^s for the infinite atomic decomposition was sometimes an overlooked detail. As far as the author knows, the above result has been proved for $w-(p, \infty, d)$ atoms in $\mathbb{R}$ by J. García-Cuerva in [4], and for $w-(p, \infty, d)$ atoms in $\mathbb{R}^n$ by D. Cruz-Uribe et al. in [3].

Given $w \in \mathcal{A}_\infty$, we say that a measurable funtion $m(\cdot)$ is a $w-(p, p_0, d)$ molecule centered at a ball $B = B(x_0, r)$ if it satisfies the following conditions:

$(m1)$ $\|m\|_{L^{p_0}(B(x_0,2r))} \leq |B|^{\frac{1}{p_0}} w(B)^{-\frac{1}{p}}$.

$(m2)$ $|m(x)| \leq w(B)^{-\frac{1}{p}} \left(1 + \frac{|x-x_0|}{r}\right)^{-2n-2d-3}$, for all $x \in \mathbb{R}^n \setminus B(x_0, 2r)$.

$(m3)$ $\int_{\mathbb{R}^n} x^\alpha m(x) dx = 0$ for every multi-index α with $|\alpha| \leq d$.

Our definition of molecule is an adaptation from that given in [13] by E. Nakai and Y. Sawano in the setting of the variable Hardy spaces. It is clear that a $w-(p, p_0, d)$ atom is a $w-(p, p_0, d)$ molecule. The pointwise inequality in $(m2)$ seems as a good substitute for "the loss of compactness in the support of an atom".

In Section 3, we obtain the following result (Theorem 13 below):

Let $0 < p \leq 1$, $w \in \mathcal{A}_\infty$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f = \sum_j \lambda_j m_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{\lambda_j\}$ is a sequence of positive numbers belonging to $l^p(\mathbb{N})$ and the functions m_j are (p, p_0, d)-molecules centered at B_j with respect to the weight w. Then $f \in H^p_w(\mathbb{R}^n)$ with

$$\|f\|^p_{H^p_w} \leq C_{w,p,p_0} \sum_j \lambda_j^p.$$

With these results in Section 4 we obtain the boundedness on H^p_w and from H^p_w to L^p_w of certain singular integrals, for all $w \in \mathcal{A}_\infty$ and all $0 < p \leq 1$. We also obtain the $H^p_{w^p} - H^q_{w^q}$ boundedness of the Riesz potential I_α, for $0 < p \leq 1$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and certain weights w.

Notation: The symbol $A \lesssim B$ stands for the inequality $A \leq cB$ for some constant c. We denote by $B(x_0, r)$ the ball centered at $x_0 \in \mathbb{R}^n$ of radius r. Given a ball $B(x_0, r)$ and a constant $c > 0$, we set $cB = B(x_0, cr)$. For a measurable subset $E \subset \mathbb{R}^n$ we denote $|E|$ and χ_E the Lebesgue measure of E and the characteristic function of E respectively. Given a real number $s \geq 0$, we write $\lfloor s \rfloor$ for the integer part of s. As usual we denote with $\mathcal{S}(\mathbb{R}^n)$ the space of smooth and rapidly decreasing functions, with $\mathcal{S}'(\mathbb{R}^n)$ the dual space. If β is the multiindex $\beta = (\beta_1, ..., \beta_n)$, then $|\beta| = \beta_1 + ... + \beta_n$.

Throughout this paper, C will denote a positive constant, not necessarily the same at each occurrence.

2. Preliminaries

2.1. **Weighted Theory.** A weight is a non-negative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere, i.e. : the weights are allowed to be zero or infinity only on a set of Lebesgue measure zero.

Given a weight w and $0 < p < \infty$, we denote by $L^p_w(\mathbb{R}^n)$ the spaces of all functions f satisfying $\|f\|^p_{L^p_w} := \int_{\mathbb{R}^n} |f(x)|^p w(x)dx < \infty$. When $p = \infty$, we have that $L^\infty_w(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ with $\|f\|_{L^\infty_w} = \|f\|_{L^\infty}$. If E is a measurable set, we use the notation $w(E) = \int_E w(x)dx$.

Let f be a locally integrable function on $\mathbb{R}^n$. The function

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|dy,$$

where the supremum is taken over all balls B containing x, is called the uncentered Hardy-Littlewood maximal function of f.

We say that a weight $w \in \mathcal{A}_1$ if there exists $C > 0$ such that

$$M(w)(x) \leq Cw(x), \quad a.e.\, x \in \mathbb{R}^n, \tag{1}$$

the best possible constant is denoted by $[w]_{\mathcal{A}_1}$. Equivalently, a weight $w \in \mathcal{A}_1$ if there exists $C > 0$ such that for every ball B

$$\frac{1}{|B|} \int_B w(x)dx \leq C \, ess \inf_{x \in B} w(x). \tag{2}$$

Remark 1. *If $w \in \mathcal{A}_1$ and $0 < r < 1$, then by Hölder inequality we have that $w^r \in \mathcal{A}_1$.*

For $1 < p < \infty$, we say that a weight $w \in \mathcal{A}_p$ if there exists $C > 0$ such that for every ball B

$$\left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B [w(x)]^{-\frac{1}{p-1}} dx \right)^{p-1} \leq C.$$

It is well known that $\mathcal{A}_{p_1} \subset \mathcal{A}_{p_2}$ for all $1 \leq p_1 < p_2 < \infty$. Also, if $w \in \mathcal{A}_p$ with $1 < p < \infty$, then there exists $1 < q < p$ such that $w \in \mathcal{A}_q$. We denote by $\widetilde{q}_w = \inf\{q > 1 : w \in \mathcal{A}_q\}$ *the critical index of w.*

Lemma 2. *If $w \in \mathcal{A}_p$ for some $1 \leq p < \infty$, then the measure $w(x)dx$ is doubling: precisely, for all $\lambda > 1$ an all ball B we have*

$$w(\lambda B) \leq \lambda^{np} [w]_{\mathcal{A}_p} w(B),$$

where λB denotes the ball with the same center as B and radius λ times the radius of B.

Theorem 3. (*Theorem 9 in* [11]) *Let $1 < p < \infty$. Then*

$$\int_{\mathbb{R}^n} [Mf(x)]^p w(x) dx \leq C_{w,p,n} \int_{\mathbb{R}^n} |f(x)|^p w(x) dx,$$

for all $f \in L^p_w(\mathbb{R}^n)$ if and only if $w \in \mathcal{A}_p$.

Given $1 < p \leq q < \infty$, we say that a weight $w \in \mathcal{A}_{p,q}$ if there exists $C > 0$ such that for every ball B

$$\left(\frac{1}{|B|} \int_B [w(x)]^q dx \right)^{1/q} \left(\frac{1}{|B|} \int_B [w(x)]^{-p'} dx \right)^{1/p'} \leq C < \infty.$$

For $p = 1$, we say that a weight $w \in \mathcal{A}_{1,q}$ if there exists $C > 0$ such that for every ball B

$$\left(\frac{1}{|B|} \int_B [w(x)]^q dx \right)^{1/q} \leq C \, ess \inf_{x \in B} w(x).$$

When $p = q$, this definition is equivalent to $w^p \in \mathcal{A}_p$.

Remark 4. *From the inequality in (2) it follows that if a weight $w \in \mathcal{A}_1$, then $0 < ess \inf_{x \in B} w(x) < \infty$ for each ball B. Thus $w \in \mathcal{A}_1$ implies that $w^{\frac{1}{q}} \in \mathcal{A}_{p,q}$, for each $1 \leq p \leq q < \infty$.*

Given $0 < \alpha < n$, we define the fractional maximal operator M_α by

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)| dy,$$

where f is a locally integrable function and the supremum is taken over all balls B containing x

The fractional maximal operators satisfies the following weighted inequality.

Theorem 5. (*Theorem 3 in* [12]) *If $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $w \in \mathcal{A}_{p,q}$, then*

$$\left(\int_{\mathbb{R}^n} [M_\alpha f(x)]^q w^q(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p w^p(x) dx \right)^{1/p},$$

for all $f \in L^p_{w^p}(\mathbb{R}^n)$.

A weight w satisfies the reverse Hölder inequality with exponent $s > 1$, denoted by $w \in RH_s$, if there exists $C > 0$ such that for every ball B,

$$\left(\frac{1}{|B|} \int_B [w(x)]^s dx \right)^{\frac{1}{s}} \leq C \frac{1}{|B|} \int_B w(x) dx;$$

the best possible constant is denoted by $[w]_{RH_s}$. We observe that if $w \in RH_s$, then by Hölder's inequality, $w \in RH_t$ for all $1 < t < s$, and $[w]_{RH_t} \leq [w]_{RH_s}$. Moreover, if $w \in RH_s$, $s > 1$, then $w \in RH_{s+\epsilon}$ for some $\epsilon > 0$. We denote by $r_w = \sup\{r > 1 : w \in RH_r\}$ *the critical index of w for the reverse Hölder condition.*

It is well known that a weight w satisfies the condition $\mathcal{A}_\infty$ if and only if $w \in \mathcal{A}_p$ for some $p \geq 1$ (see corollary 7.3.4 in [6]). So $\mathcal{A}_\infty = \cup_{1 \leq p < \infty} \mathcal{A}_p$. Also, $w \in \mathcal{A}_\infty$ if and only if $w \in RH_s$ for some $s > 1$ (see Theorem 7.3.3 in [6]). Thus $1 < r_w \leq +\infty$ for all $w \in \mathcal{A}_\infty$.

Other remarkable result about the reverse Hölder classes was discovered by Stromberg and Wheeden, they proved in [19] that $w \in RH_s$, $1 < s < +\infty$, if and only if $w^s \in \mathcal{A}_\infty$.

Given a weight w, $0 < p < \infty$ and a measurable set E we set $w^p(E) = \int_E [w(x)]^p dx$. The following result is a immediate consequence of the reverse Hölder condition.

Lemma 6. *For $0 < \alpha < n$, let $0 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $w^p \in RH_{\frac{q}{p}}$ then*

$$[w^p(B)]^{-\frac{1}{p}}[w^q(B)]^{\frac{1}{q}} \leq [w^p]^{1/p}_{RH_{q/p}}|B|^{-\frac{\alpha}{n}},$$

for each ball B in $\mathbb{R}^n$.

2.2. **Weighted Hardy Spaces.** Topologize $\mathcal{S}(\mathbb{R}^n)$ by the collection of semi-norms $\|\cdot\|_{\alpha,\beta}$, with α and β multi-indices, given by

$$\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)|.$$

For each $N \in \mathbb{N}$, we set $\mathcal{S}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \|\varphi\|_{\alpha,\beta} \leq 1, |\alpha|, |\beta| \leq N\}$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\mathcal{M}_N$ the grand maximal operator given by

$$\mathcal{M}_N f(x) = \sup_{t>0} \sup_{\varphi\in\mathcal{S}_N} \left| \left(t^{-n}\varphi(t^{-1}\cdot) * f\right)(x) \right|.$$

Given a weight $w \in \mathcal{A}_\infty$ and $p > 0$, the weighted Hardy space $H^p_w(\mathbb{R}^n)$ consists of all tempered distributions f such that

$$\|f\|_{H^p_w(\mathbb{R}^n)} = \|\mathcal{M}_N f\|_{L^p_w(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} [\mathcal{M}_N f(x)]^p w(x) dx\right)^{1/p} < \infty.$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a function such that $\int \phi(x)dx \neq 0$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we define the maximal function $M_\phi f$ by

$$M_\phi f(x) = \sup_{t>0} \left| \left(t^{-n}\phi(t^{-1}\cdot) * f\right)(x) \right|.$$

For N sufficiently large, we have $\|M_\phi f\|_{L^p_w} \simeq \|\mathcal{M}_N f\|_{L^p_w}$, (see [18]).

In the sequel we consider the following set

$$\widehat{\mathcal{D}}_0 = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \widehat{\phi} \in C^\infty_c(\mathbb{R}^n), \textit{and } \operatorname{supp}(\hat{\phi}) \subset B(0,\delta) \textit{ for some } \delta > 0\}.$$

The following theorem is crucial to get the main results.

Theorem 7. (*Theorem 1 in* [18] *p. 103*) *Let w be a doubling weight on $\mathbb{R}^n$. Then $\widehat{\mathcal{D}}_0$ is dense in $H^p_w(\mathbb{R}^n)$, $0 < p < \infty$.*

2.2.1. *Weighted atom Theory.* Let $w \in \mathcal{A}_\infty$ with critical index $\widetilde{q}_w$ and critical index r_w for reverse Hölder condition. Let $0 < p \leq 1$, $\max\{1, p(\frac{r_w}{r_w-1})\} < p_0 \leq +\infty$, and $d \in \mathbb{Z}$ such that $d \geq \lfloor n(\frac{\widetilde{q}_w}{p} - 1)\rfloor$, we say that a function $a(\cdot)$ is a $w-(p, p_0, d)$ atom centered in $x_0 \in \mathbb{R}^n$ if

($a1$) $a \in L^{p_0}(\mathbb{R}^n)$ with support in the ball $B = B(x_0, r)$.

($a2$) $\|a\|_{L^{p_0}(\mathbb{R}^n)} \leq |B|^{\frac{1}{p_0}} w(B)^{-\frac{1}{p}}$.

($a3$) $\int x^\alpha a(x)dx = 0$ for all multi-index α such that $|\alpha| \leq d$.

We observe that the condition $\max\{1, p(\frac{r_w}{r_w-1})\} < p_0 < +\infty$ implies that $w \in RH_{(\frac{p_0}{p})'}$. If $r_w = +\infty$, then $w \in RH_t$ for each $1 < t < +\infty$. So, if $r_w = +\infty$ and since $\displaystyle\lim_{t\to+\infty} \frac{t}{t-1} = 1$ we put $\frac{r_w}{r_w-1} = 1$. For example, if $w \equiv 1$, then $\widetilde{q}_w = 1$ and $r_w = +\infty$ and our definition of atom in this case coincide with the definition of atom in the classical Hardy spaces.

Lemma 8. *Let $w \in \mathcal{A}_\infty$ with critical index q_w and critical index r_w. If $a(\cdot)$ is a $w-(p,p_0,d)$ atom, then $a(\cdot) \in H^p_w(\mathbb{R}^n)$. Moreover, there exists a positive constant c independent of the atom a such that $\|a\|_{H^p_w} \leq c$.*

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \phi(x)dx \neq 0$. Since ϕ has a radial majorant that is non-incresing, bounded and integrable we have that

$$M_\phi a(x) \leq cMa(x), \quad for\ all\ \ x \in \mathbb{R}^n.$$

In view of the moment condition of a we have

$$(a * \phi_t)(x) = \int [\phi_t(x-y) - q_{x,t}(y)]a(y)dy, \quad if\ \ x \in \mathbb{R}^n \setminus B(x_0, 4r)$$

where $q_{x,t}$ is the degree d Taylor polynomial of the function $y \to \phi_t(x-y)$ expanded around x_0. By the standard estimate of the remainded term of the Taylor expansion, the condition $(a2)$ and Hölder's inequality, we obtain that

$$\begin{aligned} M_\phi a(x) &\leq c\|a\|_1 r^{d+1}|x-x_0|^{-n-d-1} \\ &\leq cw(B)^{-1/p} r^{n+d+1}|x-x_0|^{-n-d-1} \\ &\leq cw(B)^{-1/p}[M(\chi_B)(x)]^{\frac{n+d+1}{n}}, \ \ if\ x \in \mathbb{R}^n \setminus B(x_0,4r). \end{aligned}$$

Therefore

$$\int [M_\phi a(x)]^p w(x)dx \leq c \int \left(\chi_{B(x_0,4r)}[Ma(x)]^p + \frac{[M(\chi_B)(x)]^{\frac{(n+d+1)p}{n}}}{w(B)} \right) w(x)dx.$$

In the right-side of this inequality, we apply Hölder's inequality with p_0/p and use that $w \in RH_{(\frac{p_0}{p})'}$ $(p_0 > p(\frac{r_w}{r_w-1}))$ and Lemma 2 for the first term, for the second term we have that $\frac{(n+d+1)p}{n} > \widetilde{q}_w$, so $w \in \mathcal{A}_{\frac{(n+d+1)p}{n}}$, then to invoke Theorem 3 we obtain

$$\|M_\phi a\|^p_{L^p_w} = \int_{\mathbb{R}^n} [M_\phi a(x)]^p w(x)dx \leq c,$$

where the constant c is independent of the $w-(p,p_0,d)$ atom a. Thus $a \in H^p_w(\mathbb{R}^n)$. □

Theorem 9. *Let $f \in \hat{\mathcal{D}}_0$, and $0 < p \leq 1$. If $w \in \mathcal{A}_\infty$, then there exist a sequence of $w-(p,p_0,d)$ atoms $\{a_j\}$ and a sequence of scalars $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq c\|f\|^p_{H^p_w}$ such that $f = \sum_j \lambda_j a_j$, where the series converges to f in $L^s(\mathbb{R}^n)$, for each $1 < s < \infty$.*

Proof. Given $f \in \hat{\mathcal{D}}_0$, let $\mathcal{O}_j = \{x : \mathcal{M}_N f(x) > 2^j\}$ and let $\mathcal{F}_j = \{Q^j_k\}_k$ be the Whitney decomposition associated to $\mathcal{O}_j$ such that $\bigcup_k Q^{j*}_k = \mathcal{O}_j$. Fixed $j \in \mathbb{Z}$, we define the following set

$$E^j = \{(i,k) \in \mathbb{Z} \times \mathbb{Z} : Q^{j+1*}_i \cap Q^{j*}_k \neq \emptyset\}$$

and let $E^j_k = \{i : (i,k) \in E^j\}$ and $E^j_i = \{k : (i,k) \in E^j\}$. Following the proof 2.3 in [16] (p. 107-109), we have a sequence of functions A^j_k such that

(i) $supp(A^j_k) \subset Q^{j*}_k \cup \bigcup_{i \in E^j_k} Q^{j+1*}_i$ and $|A^j_k(x)| \leq c2^j$ for all $k, j \in \mathbb{Z}$.

(ii) $\int x^\alpha A^j_k(x)dx = 0$ for all α with $|\alpha| \leq n(\frac{\widetilde{q}_w}{p} - 1)$ and all $k, j \in \mathbb{Z}$.

(iii) The sum $\sum_{j,k} A^j_k$ converges to f in the sense of distributions.

From (i) we obtain

$$\sum_k |A^j_k| \leq c2^j \left(\sum_k \chi_{Q^{j*}_k} + \sum_k \chi_{\bigcup_{i \in E^j_k} Q^{j+1*}_i} \right)$$

following the proof of Theorem 5 in [14] we obtain that

$$\leq c2^j \left(\chi_{\mathcal{O}_j} + \sum_k \sum_{i \in E_k^j} \chi_{Q_i^{j+1*}} \right) = c2^j \left(\chi_{\mathcal{O}_j} + \sum_i \sum_{k \in E_i^j} \chi_{Q_i^{j+1*}} \right)$$

$$\leq c2^j \left(\chi_{\mathcal{O}_j} + 84^n \sum_i \chi_{Q_i^{j+1*}} \right) \leq c2^j \left(\chi_{\mathcal{O}_j} + \chi_{\mathcal{O}_{j+1}} \right) \leq c2^j \chi_{\mathcal{O}_j},$$

by Lemma 4 in [14] we have that

$$\sum_{j,k} |A_k^j(x)| \leq c \sum_j 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}}(x), \quad a.e. x \in \mathbb{R}^n.$$

Thus, for $1 < s < \infty$ fixed

$$\int \left(\sum_{j,k} |A_k^j(x)| \right)^s dx \leq c \sum_j \int_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} 2^{js} dx \leq c \sum_j \int_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} (\mathcal{M}f(x))^s dx$$

$$\leq c \int_{\mathbb{R}^n} (\mathcal{M}f(x))^s dx < \infty,$$

since $f \in \hat{\mathcal{D}}_0 \subset L^s(\mathbb{R}^n)$. From (iii) we obtain that the sum $\sum_{j,k} A_k^j$ converges to f in $L^s(\mathbb{R}^n)$, for each $1 < s < \infty$.

Now, we set $a_{j,k} = \lambda_{j,k}^{-1} A_k^j$ with $\lambda_{j,k} = c2^j w(B_k^j)^{1/p}$, where B_k^j is the smallest ball containing Q_k^{j*} as well as all the Q_i^{j+1*} that intersect Q_k^{j*}. Then we have a sequence $\{a_{j,k}\}$ of $w-(p,p_0,d)$ atoms and a sequence of scalars $\{\lambda_{j,k}\}$ such that the sum $\sum_{j,k} \lambda_{j,k} a_{j,k}$ converges to f in $L^s(\mathbb{R}^n)$. On the other hand there exists an universal constant c_1 such that $B_k^j \subset c_1 Q_k^j$ so

$$\sum_{j,k} |\lambda_{j,k}|^p \lesssim \sum_{j,k} 2^{jp} w(B_k^j) \lesssim \sum_{j,k} 2^{jp} w(c_1 Q_k^j) \lesssim c_1^{np} \sum_{j,k} 2^{jp} w(Q_k^j) = c \sum_j 2^{jp} w(\mathcal{O}_j).$$

If $x \in \mathbb{R}^n$, there exists a unique $j_0 \in \mathbb{Z}$ such that $2^{j_0 p} < \mathcal{M}_N f(x)^p \leq 2^{(j_0+1)p}$. So

$$\sum_j 2^{jp} \chi_{\mathcal{O}_j}(x) \leq \sum_{j \leq j_0} 2^{jp} = \frac{2^{(j_0+1)p}}{2^p - 1} \leq \frac{2^p}{2^p - 1} \mathcal{M}_N f(x)^p.$$

From this it follows that

$$\sum_{j,k} |\lambda_{j,k}|^p \leq c \sum_j 2^{jp} w(\mathcal{O}_j) \leq c \frac{2^p}{2^p - 1} \|\mathcal{M}_N f\|_{L_w^p}^p = c \frac{2^p}{2^p - 1} \|f\|_{H_w^p}^p,$$

which proves the theorem. □

Theorem 10. *Let T be a bounded linear operator from $L^{p_0}(\mathbb{R}^n)$ into $L^{p_0}(\mathbb{R}^n)$ for some $1 < p_0 < +\infty$. If $w \in \mathcal{A}_\infty$ with critical index r_w, $0 < p \leq 1 < \frac{r_w - 1}{r_w} p_0$ or $0 < p < \frac{r_w - 1}{r_w} p_0 \leq 1$, then T can be extended to an $H_w^p(\mathbb{R}^n) - L_w^p(\mathbb{R}^n)$ bounded linear operator if and only if T is bounded uniformly in L_w^p norm on all $w-(p,p_0,d)$ atom a.*

Proof. Since T is a bounded linear operator on $L^{p_0}(\mathbb{R}^n)$, T is well defined on $H_w^p(\mathbb{R}^n) \cap L^{p_0}(\mathbb{R}^n)$. If T can be extended to a bounded operator from $H_w^p(\mathbb{R}^n)$ into $L_w^p(\mathbb{R}^n)$, then $\|Ta\|_{L_w^p} \leq c_p \|a\|_{H_w^p}$ for all w-atom a. By Lemma 8, there exists a universal constant C such that $\|a\|_{H_w^p} \leq C < \infty$ for all w-atom a; so $\|Ta\|_{L_w^p} \leq C_p$ for all w-atom a.

Reciprocally, taking into account the assumptions on p and p_0, given $f \in \widehat{\mathcal{D}}_0$, by Theorem 9, there exists a $w-(p,p_0,d)$ atomic decomposition such that $\sum_j |\lambda_j|^p \lesssim \|f\|_{H_w^p}$ and $\sum_j \lambda_j a_j = f$ in $L^{p_0}(\mathbb{R}^n)$. Since T is bounded on $L^{p_0}(\mathbb{R}^n)$ we have that the sum $\sum_j \lambda_j T a_j$ converges a Tf in $L^{p_0}(\mathbb{R}^n)$, thus there exists a subsequence of natural numbers $\{k_N\}_{N\in\mathbb{N}}$ such that $\lim_{N\to+\infty}\sum_{j=-k_N}^{k_N}\lambda_j T a_j(x) = Tf(x)$ a.e.$x \in \mathbb{R}^n$, this implies

$$|Tf(x)| \leq \sum_j |\lambda_j T a_j(x)|, \quad a.e. x \in \mathbb{R}^n.$$

If $\|Ta\|_{L^p_w} \leq C_p < \infty$ for all $w-(p,p_0,d)$ atom a, and since $0 < p \leq 1$ we get

$$\|Tf\|^p_{L^p_w} \leq \sum_j |\lambda_j|^p \|Ta_j\|^p_{L^p_w} \leq C^p_p \sum_j |\lambda_j|^p \leq C^p_p \|f\|^p_{H^p_w}$$

for all $f \in \widehat{\mathcal{D}}_0$. By theorem 7, we have that $\widehat{\mathcal{D}}_0$ is a dense subspace of $H^p_w(\mathbb{R}^n)$, so the theorem follows by a density argument. $\square$

3. Molecular decomposition

Our definition of molecule is an adaptation from that given in [13] by E. Nakai and Y. Sawano in the setting of the variable Hardy spaces.

Definition 11. *Let $w \in \mathcal{A}_\infty$ with critical index q_w and critical index r_w for reverse Hölder condition. Let $0 < p \leq 1$, $\max\{1, p(\frac{r_w}{r_w-1})\} < p_0 \leq +\infty$ and $d \in \mathbb{Z}$ such that $d \geq \lfloor n(\frac{\widetilde{q}_w}{p} - 1)\rfloor$. We say that a function $m(\cdot)$ is a $w-(p,p_0,d)$ molecule centered at a ball $B = B(x_0, r)$ if it satisfies the following conditions:*

$(m1)$ $\|m\|_{L^{p_0}(B(x_0,2r))} \leq |B|^{\frac{1}{p_0}} w(B)^{-\frac{1}{p}}$.

$(m2)$ $|m(x)| \leq w(B)^{-\frac{1}{p}}\left(1 + \frac{|x-x_0|}{r}\right)^{-2n-2d-3}$ for all $x \in \mathbb{R}^n \setminus B(x_0, 2r)$.

$(m3)$ $\int_{\mathbb{R}^n} x^\alpha m(x)dx = 0$ for every multi-index α with $|\alpha| \leq d$.

Remark 12. *The conditions $(m1)$ and $(m2)$ imply that $\|m\|_{L^{p_0}(\mathbb{R}^n)} \leq c\frac{|B|^{\frac{1}{p_0}}}{w(B)^{\frac{1}{p}}}$, where c is a positive constant independent of the molecule m.*

From the definition of molecule is clear that a $w-(p,p_0,d)$ atom is a $w-(p,p_0,d)$ molecule.

In view of Lemma 8, the following theorem assure, among other things, that the pointwise inequality in $(m2)$ is a good substitute for "the loss of compactness in the support of an atom".

Theorem 13. *Let $0 < p \leq 1$, $w \in \mathcal{A}_\infty$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f = \sum_j \lambda_j m_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{\lambda_j\}$ is a sequence of positive numbers belonging to $l^p(\mathbb{N})$ and the functions m_j are (p,p_0,d)-molecules centered at B_j with respect to the weight w. Then $f \in H^p_w(\mathbb{R}^n)$ with*

$$\|f\|^p_{H^p_w} \leq C_{w,p,p_0}\sum_j \lambda^p_j.$$

Proof. Let $\phi \in C^\infty_c(\mathbb{R}^n)$ such that $\chi_{B(0,1)} \leq \phi \leq \chi_{B(0,2)}$, we set $\phi_{2^k}(x) = 2^{kn}\phi(2^k x)$ where $k \in \mathbb{Z}$. Since $f = \sum_j \lambda_j m_j$ in the sense of the distributions we have that

$$|(\phi_{2^k} * f)(x)| \leq \sum_{j=1}^\infty \lambda_j |(\phi_{2^k} * m_j)(x)|,$$

for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{Z}$. We observe that the argument used in the proof of Theorem 5.2 in [13] to obtain the pointwise inequality (5.2) works in this setting, but considering now the conditions $(m1)$, $(m2)$ and $(m3)$. Therefore we get

$$M_\phi(f)(x) \lesssim \sum_j \lambda_j \chi_{2B_j}(x) M(m_j)(x) + \sum_j \lambda_j \frac{\left[M(\chi_{B_j})(x)\right]^{\frac{n+d_w+1}{n}}}{w(B_j)^{\frac{1}{p}}}, \quad (x \in \mathbb{R}^n)$$

where M is the Hardy-Littlewood maximal operator.

Since $0 < p \leq 1$, it follows that

$$[M_\phi(f)(x)]^p \lesssim \sum_j \lambda_j^p \chi_{2B_j}(x) [M(m_j)(x)]^p + \sum_j \lambda_j^p \frac{\left[M(\chi_{B_j})(x)\right]^{p\frac{n+d_w+1}{n}}}{w(B_j)}, \quad (x \in \mathbb{R}^n)$$

by integrating with respect to w we get

$$\int [M_\phi(f)(x)]^p w(x)dx \lesssim \sum_j \lambda_j^p \int \chi_{2B_j}(x)[M(m_j)(x)]^p w(x)dx$$

$$+ \sum_j \lambda_j^p \int \frac{\left[M(\chi_{B_j})(x)\right]^{p\frac{n+d_w+1}{n}}}{w(B_j)} w(x)dx.$$

In the right-side of this inequality, we apply Hölder's inequality with p_0/p, remark 12, Lemma 2 and use that $w \in RH_{(\frac{p_0}{p})'}$ ($p_0 > p(\frac{r_w}{r_w-1})$) for the first term, for the second term we have that $\frac{(n+d+1)p}{n} > \widetilde{q}_w$, so $w \in \mathcal{A}_{\frac{(n+d+1)p}{n}}$, then to invoke Theorem 3 we obtain

$$\|f\|^p_{H^p_w} \leq C_{w,p,p_0} \sum_j \lambda_j^p.$$

This completes the proof. □

Theorem 14. *Let T be a bounded linear operator from $L^{p_0}(\mathbb{R}^n)$ into $L^{p_0}(\mathbb{R}^n)$ for some $1 < p_0 < +\infty$. If $w \in \mathcal{A}_\infty$ with critical index r_w, $0 < p \leq 1 < \frac{r_w-1}{r_w}p_0$ or $0 < p < \frac{r_w-1}{r_w}p_0 \leq 1$ and Ta is a $w-(p,p_0,d)$ molecule for each $w-(p,p_0,d)$ atom a, then T can be extended to an $H^p_w(\mathbb{R}^n) - H^p_w(\mathbb{R}^n)$ bounded linear operator.*

Proof. Taking into account the assumptions on p and p_0, given $f \in \widehat{\mathcal{D}}_0$, from Theorem 9 it follows that there exists a sequence of $w-(p,p_0,d)$ atoms $\{a_j\}$ and a sequence of scalars $\{\lambda_j\}$ with

$$\sum_j |\lambda_j|^p \lesssim \|f\|^p_{H^p_w}, \tag{3}$$

such that $f = \sum_j \lambda_j a_j$ in $L^{p_0}(\mathbb{R}^n)$. Since T is a bounded linear operator on $L^{p_0}(\mathbb{R}^n)$ we have that $Tf = \sum_j \lambda_j T a_j$ in $L^{p_0}(\mathbb{R}^n)$ and therefore in $\mathcal{S}'(\mathbb{R}^n)$. Theorem 13 and the inequality in (3) allow us to obtain

$$\|Tf\|^p_{H^p_w} \lesssim \sum_j |\lambda_j|^p \lesssim \|f\|^p_{H^p_w},$$

for all $f \in \widehat{\mathcal{D}}_0$, so the theorem follows from the density of $\widehat{\mathcal{D}}_0$ in $H^p_w(\mathbb{R}^n)$. □

4. Applications

4.1. **Singular integrals.** Let $\Omega \in C^\infty(S^{n-1})$ with $\int_{S^{n-1}} \Omega(u)d\sigma(u) = 0$. We define the operator T by

$$Tf(x) = \lim_{\epsilon \to 0^+} \int_{|y|>\epsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy, \quad x \in \mathbb{R}^n. \tag{4}$$

It is well known that $\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi)$, where the multiplier m is homogeneous of degree 0 and is indefinitely diferentiable on $\mathbb{R}^n \setminus \{0\}$. Moreover, if $k(y) = \frac{\Omega(y/|y|)}{|y|^n}$ we have

$$|\partial_y^\alpha k(y)| \le C|y|^{-n-|\alpha|}, \quad \textit{for all } \; y \neq 0 \textit{ and all multi-index } \alpha. \tag{5}$$

The operator T results bounded on $L^s(\mathbb{R}^n)$ for all $1 < s < +\infty$ and of weak-type $(1,1)$ (see [15]).

Let $0 < p \le 1$. Given a $w-(p,p_0,d)$ atom $a(\cdot)$ with support in the ball $B(x_0,r)$, since T is bounded on $L^{p_0}(\mathbb{R}^n)$, we have that

$$\|Ta\|_{L^{p_0}(B(x_0,2r))} \le C\|a\|_{p_0} \le C|B|^{1/p_0} w(B)^{-1/p}. \tag{6}$$

In view of the moment condition of $a(\cdot)$ we obtain

$$Ta(x) = \int_{B(x_0,r)} k(x-y)a(y)dy = \int_{B(x_0,r)} [k(x-y) - q_d(x,y)]a(y)dy, \quad x \notin B(x_0,2r)$$

where q_d is the degree d Taylor polynomial of the function $y \to k(x-y)$ expanded around x_0. From the estimate in (5), and the standard estimate of the remainder term of the Taylor expansion, there exists ξ between y and x_0 such that

$$|Ta(x)| \le C\|a\|_1 \frac{|y-x_0|^{d+1}}{|x-\xi|^{n+d+1}} \le C\frac{r^{n+d+1}}{w(B)^{1/p}}|x-x_0|^{-n-d-1}, \quad x \notin B(x_0,2r), \tag{7}$$

this inequality and a simple computation allows us to obtain

$$|Ta(x)| \le Cw(B)^{-\frac{1}{p}}\left(1 + \frac{|x-x_0|}{r}\right)^{-2n-2d-3}, \quad \textit{for all } x \notin B(x_0,2r). \tag{8}$$

From the estimate in (7) we obtain that the function $x \to x^\alpha Ta(x)$ belongs to $L^1(\mathbb{R}^n)$ for each $|\alpha| \le d$, so

$$|((-2\pi i x)^\alpha Ta)\widehat{\ }(\xi)| = |\partial_\xi^\alpha (m(\xi)\widehat{a}(\xi))| = \left|\sum_{\beta \le \alpha} c_{\alpha,\beta}\, (\partial_\xi^{\alpha-\beta} m)(\xi)\, (\partial_\xi^\beta \widehat{a})(\xi)\right|$$

$$= \left|\sum_{\beta \le \alpha} c_{\alpha,\beta}\, (\partial_\xi^{\alpha-\beta} m)(\xi)\, ((-2\pi i x)^\beta a)\widehat{\ }(\xi)\right|,$$

from the homogeneity of the function $\partial_\xi^{\alpha-\beta} m$ we obtain that

$$|((-2\pi i x)^\alpha Ta)\widehat{\ }(\xi)| \le C\sum_{\beta \le \alpha} |c_{\alpha,\beta}| \frac{\left|((-2\pi i x)^\beta a)\widehat{\ }(\xi)\right|}{|\xi|^{|\alpha|-|\beta|}}, \quad \xi \neq 0. \tag{9}$$

Since $\lim_{\xi \to 0} \frac{\left|((-2\pi i x)^\beta a)\widehat{\ }(\xi)\right|}{|\xi|^{|\alpha|-|\beta|}} = 0$ for each $\beta \le \alpha$ (see 5.4, p. 128, in [16]), taking the limit as $\xi \to 0$ at (9), we get

$$\int_{\mathbb{R}^n} (-2\pi i x)^\alpha Ta(x)\, dx = ((-2\pi i x)^\alpha Ta)\widehat{\ }(0) = 0, \quad \textit{for all } |\alpha| \le d. \tag{10}$$

From (6), (8) and (10) it follows that there exists an universal constant $C > 0$ such that $CTa(\cdot)$ is a $w-(p,p_0,d)$ molecule if $a(\cdot)$ is a $w-(p,p_0,d)$ atom. Taking $p_0 \in (1,+\infty)$ such that $1 < \frac{r_w-1}{r_w}p_0$ and since T is bounded on $L^{p_0}(\mathbb{R}^n)$, by Theorem 14, we get the following result.

Theorem 15. *Let T be the operator defined in (4). If $w \in \mathcal{A}_\infty$ and $0 < p \le 1$, then T can be extended to an $H^p_w(\mathbb{R}^n) - H^p_w(\mathbb{R}^n)$ bounded operator.*

In particular, the Hilbert transform and the Riesz transform admit a continuous extension on $H^p_w(\mathbb{R})$ and $H^p_w(\mathbb{R}^n)$, for each $w \in \mathcal{A}_\infty$ and $0 < p \le 1$, respectively.

Remark 16. *If $a(\cdot)$ is a $w-(p,p_0,d)$ atom with $1 < \frac{r_w-1}{r_w}p_0$, the estimation in (7) allow us to obtain*

$$|Ta(x)| \leq C\frac{[M(\chi_B)(x)]^{\frac{n+d+1}{n}}}{w(B)^{1/p}}, \ \ x \notin B(x_0, 2r),$$

where M is the Hardy-Littlewood maximal operator.

Lemma 17. *Let $p_0 \in (1,+\infty)$ such that $1 < \frac{r_w-1}{r_w}p_0$. If T is the operator defined in (4) and $0 < p \leq 1$, then there exists an universal constant $C > 0$ such that $\|Ta\|_{L^p_w} \leq C$ for all $w-(p,p_0,d)$ atom $a(\cdot)$.*

Proof. Given a $w-(p,p_0,d)$ atom $a(\cdot)$, let $2B = B(x_0, 2r)$, where $B = B(x_0, r)$ is the ball containing the support of $a(\cdot)$. We write

$$\int_{\mathbb{R}^n} |Ta(x)|^p w(x)dx = \int_{2B} |Ta(x)|^p w(x)dx + \int_{\mathbb{R}^n \setminus 2B} |Ta(x)|^p w(x)dx = I + II.$$

Since T is bounded on $L^{p_0}(\mathbb{R}^n)$ and $w \in RH_{(\frac{p_0}{p})'}$ $(p \leq 1 < \frac{r_w-1}{r_w}p_0)$, the Hölder's inequality applied with $\frac{p_0}{p}$, and the condition $(a2)$ give

$$I \leq C\|a\|_{p_0}^p |B|^{-p/p_0} w(B) = C.$$

From remark 16 and since that $w \in \mathcal{A}_{p\frac{n+d+1}{n}}$, $(p\frac{n+d+1}{n} > \widetilde{q}_w)$, we get

$$II \leq w(B)^{-1} \int_{\mathbb{R}^n} [M(\chi_B)(x)]^{\frac{n+d+1}{n}} w(x)dx \leq Cw^{-1}(B) \int_B w(x)dx = C,$$

where the second inequality follows from Theorem 3. This complete the proof. □

Theorem 18. *Let T be the operator defined in (4). If $w \in \mathcal{A}_\infty$ and $0 < p \leq 1$, then T can be extended to an $H^p_w(\mathbb{R}^n) - L^p_w(\mathbb{R}^n)$ bounded operator.*

Proof. The theorem follows from Lemma 17 and Theorem 10. □

4.2. **The Riesz potential.** For $0 < \alpha < n$, let I_α be the Riesz potential defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y)dy, \tag{11}$$

$f \in L^s(\mathbb{R}^n)$, $1 \leq s < \frac{n}{\alpha}$. A well known result of Sobolev gives the boundedness of I_α from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. In [17] E. Stein and G. Weiss used the theory of harmonic functions of several variables to prove that these operators are bounded from $H^1(\mathbb{R}^n)$ into $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$. In [20], M. Taibleson and G. Weiss obtained, using the molecular decomposition, the boundedness of the Riesz potential I_α from $H^p(\mathbb{R}^n)$ into $H^q(\mathbb{R}^n)$, for $0 < p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, independently S. Krantz obtained the same result in [7]. We extend these results to the context of the weighted Hardy spaces using the weighted molecular theory developed in Section 3.

First we recall the definition of the critical indices for a weight w.

Definition 19. *Given a weight w, we denote by $\widetilde{q}_w = \inf\{q > 1 : w \in \mathcal{A}_q\}$ the critical index of w, and we denote by $r_w = \sup\{r > 1 : w \in RH_r\}$ the critical index of w for the reverse Hölder condition.*

Lemma 20. *Let $0 < p < 1$. If $w^{1/p} \in \mathcal{A}_1$, then $p \cdot r_{w^p} \leq r_w \leq r_{w^p}$.*

Proof. The condition $w^{1/p} \in \mathcal{A}_1$, with $0 < p < 1$, implies that $w^p \in RH_{1/p}$. It is well known that if $w \in RH_r$, then $w \in RH_{r+\epsilon}$ for some $\epsilon > 0$, thus $1/p < r_{w^p}$. Taking $1/p < t < r_{w^p}$ we have that $1 < pt < t$ and $w^p \in RH_t$, so

$$\left(\frac{1}{|B|}\int_B [w(x)]^{pt} dx\right)^{1/pt} = \left(\frac{1}{|B|}\int_B [w^p(x)]^{t} dx\right)^{1/pt} \leq C\left(\frac{1}{|B|}\int_B w^p(x) dx\right)^{1/p}$$
$$\leq C\frac{1}{|B|}\int_B w(x)dx,$$

where the last inequality follows from the Jensen's inequality. This implies that $p\,t < r_w$ for all $t < r_{w^p}$, thus $p \cdot r_{w^p} \leq r_w$.

By the other hand, since $0 < p < 1$ and $w^{1/p} \in \mathcal{A}_1$ we have that $w \in RH_{1/p}$. So $1/p < r_w$, taking $1/p < t < r_w$ it follows that $1 < pt < t$, and therefore $w \in RH_{pt}$. Then

$$\left(\frac{1}{|B|}\int_B [w^p(x)]^{t} dx\right)^{1/t} = \left(\frac{1}{|B|}\int_B [w(x)]^{tp} dx\right)^{p/pt} \leq C\left(\frac{1}{|B|}\int_B w(x) dx\right)^{p}$$
$$= C\left(\frac{1}{|B|}\int_B [w^p(x)]^{1/p} dx\right)^{p} \leq C\frac{1}{|B|}\int_B [w^p(x)]dx,$$

where the last inequality it follows from that $w^p \in RH_{1/p}$. So $t < r_{w^p}$ for all $t < r_w$, this gives $r_w \leq r_{w^p}$. $\square$

Lemma 21. *Let $0 < p < q$. If $w^q \in \mathcal{A}_1$, then $p \cdot r_{w^p} \leq q \cdot r_{w^q}$.*

Proof. Since $w^q \in \mathcal{A}_1$ and $0 < p < q$ we have that $w^p \in \mathcal{A}_1$ and $w^p \in RH_{q/p}$. thus $q/p < r_{w^p}$. Taking $q/p < s < r_{w^p}$ we have that $w^p \in RH_s$ and $1 < ps/q < s$, so

$$\left(\frac{1}{|B|}\int_B [w^q(x)]^{ps/q} dx\right)^{q/ps} = \left(\frac{1}{|B|}\int_B [w^p(x)]^{s} dx\right)^{q/ps} \leq C\left(\frac{1}{|B|}\int_B w^p(x) dx\right)^{q/p}$$
$$\leq C\frac{1}{|B|}\int_B w^q(x)dx,$$

where the last inequality follows from the Jensen's inequality. This implies that $\frac{p}{q}\,s < r_{w^q}$ for all $s < r_{w^p}$, thus $p \cdot r_{w^p} \leq q \cdot r_{w^q}$. $\square$

Proposition 22. *For $0 < \alpha < n$, let I_α be the Riesz potential defined in (11) and let $w^{1/s} \in \mathcal{A}_1$, $0 < s < \frac{n}{n+\alpha}$, with $\frac{r_w}{r_w-1} < \frac{n}{\alpha}$. If $s \leq p \leq \frac{n}{n+\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then $I_\alpha a(\cdot)$ is a $w^q - (q, q_0, \lfloor n(\frac{1}{q} - 1)\rfloor)$ molecule for each $w^p - (p, p_0, 2\lfloor n(\frac{1}{q}-1)\rfloor + 3 + \lfloor \alpha \rfloor + n)$ atom $a(\cdot)$, where $\frac{r_w}{r_w-1} < p_0 < \frac{n}{\alpha}$ and $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$.*

Proof. The condition $w^{1/s} \in \mathcal{A}_1$ implies that w^p and w^q belong to $\mathcal{A}_1$, so $\widetilde{q}_{w^p} = \widetilde{q}_{w^q} = 1$. We observe that $2\lfloor n(\frac{1}{q}-1)\rfloor + 3 + \lfloor \alpha \rfloor + n > \lfloor n(\frac{1}{p}-1)\rfloor$, thus $a(\cdot)$ is an atom with additional vanishing moments.

By the lemma 20 and the lemma 21 we have that $p\frac{r_{w^p}}{r_{w^p}-1} \leq \frac{r_w}{r_w-1} < p_0$ and $\frac{r_w^q}{r_{w^q}-\frac{p}{q}} \leq \frac{r_{w^p}}{r_{w^p}-1}$, respectively. If $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and since the function $t \to \frac{t}{t-1}$ is decreasing on the region $(1, +\infty)$, a computation gives $q\frac{r_{w^q}}{r_{w^q}-1} < q_0$. The condition $p\frac{r_{w^p}}{r_{w^p}-1} < p_0$ is required in the definion of atom and $q\frac{r_{w^q}}{r_{w^q}-1} < q_0$ in the definition of molecule.

Now we will show that $I_\alpha a(\cdot)$ satisfies the conditions $(m1)$, $(m2)$ and $(m3)$ in the definition of molecule, if $a(\cdot)$ is a $w^p - (p, p_0, 2\lfloor n(\frac{1}{q}-1)\rfloor + 3 + \lfloor \alpha \rfloor + n)$ atom.

Since I_α is bounded from $L^{p_0}(\mathbb{R}^n)$ into $L^{q_0}(\mathbb{R}^n)$ and $w^p \in RH_{q/p}$, by Lemma 6, we get

$$\|I_\alpha a\|_{L^{q_0}(B(x_0,2r))} \leq C\|a\|_{L^{p_0}(\mathbb{R}^n)} \leq C|B|^{1/p_0}(w^p(B))^{-1/p} \leq C|B|^{1/q_0}(w^q(B))^{-1/q},$$

so $I_\alpha a(\cdot)$ satisfies $(m1)$.

Let $d = 2\lfloor n(\frac{1}{q}-1)\rfloor + 3 + \lfloor \alpha \rfloor + n$, and let $a(\cdot)$ be a $w^p - (p, p_0, d)$ atom supported on the ball $B(x_0, r)$. In view of the moment condition of $a(\cdot)$ we obtain

$$I_\alpha a(x) = \int_{B(x_0,r)} \left(|x-y|^{\alpha-n} - q_d(x,y)\right) a(y)dy, \quad \textit{for all } x \notin B(x_0, 2r),$$

where q_d is the degree d Taylor polynomial of the function $y \to |x-y|^{\alpha-n}$ expanded around x_0. By the standard estimate of the remainder term of the Taylor expansion, there exists ξ between y and x_0 such that

$$\left||x-y|^{\alpha-n} - q_d(x,y)\right| \leq C|y-x_0|^{d+1}|x-\xi|^{-n+\alpha-d-1},$$

for any $y \in B(x_0, r)$ and any $x \notin B(x_0, 2r)$, since $|x - \xi| \geq \frac{|x-x_0|}{2}$, we get

$$\left||x-y|^{\alpha-n} - q_d(x,y)\right| \leq Cr^{d+1}|x-x_0|^{-n+\alpha-d-1},$$

this inequality and the condition $(a2)$ of our w^p - atom $a(\cdot)$ allow us to conclude that

$$|I_\alpha a(x)| \leq C\frac{r^{n+d+1}}{(w^p(B))^{1/p}}|x-x_0|^{-n+\alpha-d-1}, \quad \textit{for all } \; x \notin B(x_0, 2r), \tag{12}$$

Lemma 6 and a simple computation gives

$$|I_\alpha a(x)| \leq C(w^q(B))^{-1/q}\left(1 + \frac{|x-x_0|}{r}\right)^{-2n-d_q-3}, \quad \textit{for all } \; x \notin B(x_0, 2r),$$

where $d_q = \lfloor n(\frac{1}{q}-1)\rfloor$. So $I_\alpha a(\cdot)$ satisfies $(m2)$.

Finally, in [20] Taibleson and Weiss proved that

$$\int_{\mathbb{R}^n} x^\beta I_\alpha a(x)dx = 0,$$

for all $0 \leq |\beta| \leq \lfloor n(\frac{1}{q}-1)\rfloor$. This shows that $I_\alpha a(\cdot)$ is a w^q-molecule. The proof of the proposition is therefore concluded. $\square$

Theorem 23. *For $0 < \alpha < n$, let I_α be the Riesz potential defined in (11). If $w^{1/s} \in \mathcal{A}_1$ with $0 < s < \frac{n}{n+\alpha}$ and $\frac{r_w}{r_w-1} < \frac{n}{\alpha}$, then I_α can be extended to an $H^p_{w^p}(\mathbb{R}^n) - H^q_{w^q}(\mathbb{R}^n)$ bounded operator for each $s \leq p \leq \frac{n}{n+\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.*

Proof. Let $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. For the range $p \leq \frac{n}{n+\alpha}$ we have that $p < q \leq 1$. If $p \in [s, \frac{n}{n+\alpha}]$, the condition $w^{1/s} \in \mathcal{A}_1$, $0 < s < \frac{n}{n+\alpha}$, implies that w, $w^{1/p}$ and w^p belong to $\mathcal{A}_1$, so $w^q \in \mathcal{A}_1$. Then $\widetilde{q}_{w^p} = \widetilde{q}_{w^q} = 1$. We put $d_p = \lfloor n(\frac{1}{p}-1)\rfloor$ and $d_q = \lfloor n(\frac{1}{q}-1)\rfloor$. We recall that in the atomic decomposition, we can always choose atoms with additional vanishing moments (see Corollary 2.1.5 p. 105 in [16]). This is, if l is any fixed integer with $l > d_p$, then we have an atomic decomposition such that all moments up to order l of our atoms are zero.

For $\frac{r_w}{r_w-1} < p_0 < \frac{n}{\alpha}$ we consider $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$. We observe that $2\lfloor n(\frac{1}{q} - 1)\rfloor + 3 + \lfloor \alpha \rfloor + n > \lfloor n(\frac{1}{p}-1)\rfloor$. Since $w^{1/p} \in \mathcal{A}_1$, from Lemma 20, we have $p\frac{r_{w^p}}{r_{w^p}-1} \leq \frac{r_w}{r_w-1} < p_0$. Thus, given $f \in \widehat{\mathcal{D}}_0$ we can write $f = \sum_j \lambda_j a_j$, where a_j are $w^p - (p, p_0, 2\lfloor n(\frac{1}{q_0}-1)\rfloor + 3 + \lfloor\alpha\rfloor + n)$ atoms, $\sum_j |\lambda_j|^p \lesssim \|f\|^p_{H^p_{w^p}}$ and the series converges in $L^{p_0}(\mathbb{R}^n)$. Since I_α is a $L^{p_0}(\mathbb{R}^n) - L^{q_0}(\mathbb{R}^n)$ bounded operator it follows that $I_\alpha f = \sum_j \lambda_j I_\alpha a_j$ in $L^{q_0}(\mathbb{R}^n)$ and therefore in $\mathcal{S}'(\mathbb{R}^n)$. By Proposition 22, we have that the operator I_α maps $w^p - (p, p_0, 2\lfloor n(\frac{1}{q_0}-1)\rfloor + 3 + \lfloor\alpha\rfloor + n))$ atoms $a(\cdot)$ to $w^q - (q, q_0, d_q)$ molecules $I_\alpha a(\cdot)$, applying Theorem 13 we get

$$\|I_\alpha f\|^q_{H_{w^q}} \lesssim \sum_j |\lambda_j|^q \lesssim \left(\sum_j |\lambda_j|^p\right)^{q/p} \lesssim \|f\|^q_{H^p_{w^p}},$$

for all $f \in \widehat{\mathcal{D}}_0$, so the theorem follows from the density of $\widehat{\mathcal{D}}_0$ in $H^p_{w^p}(\mathbb{R}^n)$. $\square$

For $\frac{n}{n+\alpha} < p \leq 1$, we have that $1 < q \leq \frac{n}{n-\alpha}$. For this range of q's the space H^q_w can be identify with the space L^q_w.

Theorem 24. *For $0 < \alpha < n$, let I_α be the Riesz potential defined in (11). If $w^{\frac{n}{(n-\alpha)s}} \in \mathcal{A}_1$ with $0 < s < 1$ and $\frac{r_w}{r_w - 1} < \frac{n}{\alpha}$, then I_α can be extended to an $H^p_{w^p}(\mathbb{R}^n) - L^q_{w^q}(\mathbb{R}^n)$ bounded operator for each $s \leq p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.*

Proof. The condition $w^{n/(n-\alpha)s} \in \mathcal{A}_1$, $0 < s < 1 < \frac{n}{n-\alpha}$, implies that w, $w^{1/p}$, w^p and w^q belong to $\mathcal{A}_1$, for all $s \leq p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

We take p_0 such that $\frac{r_w}{r_w-1} < p_0 < \frac{n}{\alpha}$, from lemma 20, we have that $p\frac{r_{w^p}}{r_{w^p}-1} \leq \frac{r_w}{r_w-1} < p_0$. Given $f \in \widehat{\mathcal{D}}_0$ we can write $f = \sum \lambda_j a_j$ where the a_j's are $w^p - (p, p_0, d)$ atoms, the scalars λ_j satisfies $\sum_j |\lambda_j|^p \lesssim \|f\|^p_{H^p_{w^p}}$ and the series converges in $L^{p_0}(\mathbb{R}^n)$. For $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$, I_α is a bounded operator from $L^{p_0}(\mathbb{R}^n)$ into $L^{q_0}(\mathbb{R}^n)$, since $f = \sum_j \lambda_j a_j$ in $L^{p_0}(\mathbb{R}^n)$, we have that

$$|I_\alpha f(x)| \leq \sum_j |\lambda_j||I_\alpha a_j(x)|, \quad a.e. x \in \mathbb{R}^n. \tag{13}$$

If $\|I_\alpha a_j\|_{L^q_{w^q}} \leq C$, with C independent of the $w^p - (p, p_0, d)$ atom $a_j(\cdot)$, then (13) allows us to obtain

$$\|I_\alpha f\|_{L^q_{w^q}} \leq C \left(\sum_j |\lambda_j|^{\min\{1,q\}} \right)^{\frac{1}{\min\{1,q\}}} \leq C \left(\sum_j |\lambda_j|^p \right)^{1/p} \lesssim \|f\|_{H^p_{w^p}},$$

for all $f \in \widehat{\mathcal{D}}_0$, so the theorem follows from the density of $\widehat{\mathcal{D}}_0$ in $H^p_{w^p}(\mathbb{R}^n)$.

To conclude the proof we will prove that there exists $C > 0$ such that

$$\|I_\alpha a\|_{L^q_{w^q}} \leq C, \quad \text{for all } w^p - (p, p_0, d) \text{ atom } a(\cdot). \tag{14}$$

To prove (14), let $2B = B(x_0, 2r)$, where $B = B(x_0, r)$ is the ball containing the support of the atom $a(\cdot)$. So

$$\int_{\mathbb{R}^n} |I_\alpha a(x)|^q w^q(x) dx = \int_{2B} |I_\alpha a(x)|^q w^q(x) dx + \int_{\mathbb{R}^n \setminus 2B} |I_\alpha a(x)|^q w^q(x) dx$$

To estimate the first term in the right-side of this equality, we apply Hölder's inequality with $\frac{q_0}{q}$ and use that $w^q \in RH_{(\frac{q_0}{q})'}$ ($q_0 > q\frac{r_{w^q}}{r_{w^q}-1}$), thus

$$\int_{2B} |I_\alpha a(x)|^q w^q(x) dx \leq \|I_\alpha a\|^q_{L^{q_0}} \left(\int_{2B} [w^q(x)]^{(\frac{q_0}{q})'} dx \right)^{1/(\frac{q_0}{q})'}$$
$$\leq C|B|^{q/p_0} (w^p(B))^{-q/p} |2B|^{1/(\frac{q_0}{q})'} \left(\frac{1}{|2B|} \int_{2B} w^q(x) dx \right)$$
$$\leq C|B|^{q\alpha/n} (w^p(B))^{-q/p} w^q(B).$$

Lemma 6 gives

$$\int_{2B} |I_\alpha a(x)|^q w^q(x) dx \leq C. \tag{15}$$

From (12), taking there $d = \lfloor n(\frac{1}{p} - 1) \rfloor$, we obtain

$$|I_\alpha a(x)| \leq C(w^p(B))^{-1/p} \left[M_{\frac{\alpha n}{n+d+1}}(\chi_B)(x) \right]^{\frac{n+d+1}{n}}, \quad \text{for all } x \notin B(x_0, 2r).$$

So

(16)
$$\int_{\mathbb{R}^n\setminus 2B} |I_\alpha a(x)|^q w^q(x)dx \leq C(w^p(B))^{-q/p} \int_{\mathbb{R}^n} \left[M_{\frac{\alpha n}{n+d+1}}(\chi_B)(x)\right]^{q\frac{n+d+1}{n}} w^q(x)dx,$$

Since $d = \lfloor n(\frac{1}{p}-1)\rfloor$, we have $q\frac{n+d+1}{n} > 1$. We write $\widetilde{q} = q\frac{n+d+1}{n}$ and let $\frac{1}{\widetilde{p}} = \frac{1}{\widetilde{q}} + \frac{\alpha}{n+d+1}$, so $\frac{\widetilde{p}}{\widetilde{q}} = \frac{p}{q}$ and $w^{q/\widetilde{q}} \in \mathcal{A}_{\widetilde{p},\widetilde{q}}$ (see remark 4). From Theorem 5 we obtain

$$\int_{\mathbb{R}^n} \left[M_{\frac{\alpha n}{n+d+1}}(\chi_B)(x)\right]^{q\frac{n+d+1}{n}} w^q(x)dx \leq C\left(\int_{\mathbb{R}^n} \chi_B(x)w^p(x)dx\right)^{q/p} = C(w^p(B))^{q/p}.$$

This inequality and (16) give

$$\int_{\mathbb{R}^n\setminus 2B} |I_\alpha a(x)|^q w^q(x)dx \leq C. \tag{17}$$

Finally, (15) and (17) allow us to obtain (14). This completes the proof. □

To finish, we recover the classical result obtained by Taibleson and Weiss in [20].

Corollary 25. *For $0 < \alpha < n$, let I_α be the Riesz potential defined in (11). If $0 < p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then I_α can be extended to an $H^p(\mathbb{R}^n) - H^q(\mathbb{R}^n)$ bounded operator.*

Proof. If $w(x) \equiv 1$, then $r_w = +\infty$ and therefore $\frac{r_w}{r_w - 1} = 1$. Applying the theorems 23 and 24, with $w \equiv 1$, the corollary follows. □

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