Non commutative L^p spaces without the completely bounded approximation property

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Abstract

For any $1 \leq p \leq \infty$ different from 2, we give examples of non-commutative L^p spaces without the completely bounded approximation property. Let F be a non-archimedian local field. If $p > 4$ or $p < 4/3$ and $r \geq 3$ these examples are the non-commutative L^p-spaces of the von Neumann algebra of lattices in $SL_r(F)$ or in $SL_r(\mathbb{R})$. For other values of p the examples are the non-commutative L^p-spaces of the von Neumann algebra of lattices in $SL_r(F)$ for r large enough depending on p.

We also prove that if $r \geq 3$ lattices in $SL_r(F)$ or $SL_r(\mathbb{R})$ do not have the Approximation Property of Haagerup and Kraus. This provides examples of exact C^*-algebras without the operator space approximation property.

Introduction

There are various notions of finite-dimensional approximation properties for C^*-algebras and more generally operator algebras. Among others, we can cite nuclearity, completely bounded approximation property (CBAP), operator space approximation property (OAP), exactness... Although some of these notions will be defined precisely in this paper, the reader is refered to [4] for an exposition of these concepts.

For the reduced C^*-algebra of a discrete group, most of these approximation properties have equivalent reformulations in term of the group : the nuclearity of $C^*_{\text{red}}(G)$ is equivalent to the amenability of G. Haagerup proved in [5] that the CBAP for $C^*_{\text{red}}(G)$ is equivalent to the weak amenability of G, and Haagerup and Kraus [11] proved that the OAP of $C^*_{\text{red}}(G)$ is equivalent to Haagerup's and Kraus' approximation property (AP) of G. For equivalent formulation of exactness for a group, see [4], Chapter 5. For a discrete group, the following implications are known:

$$\text{amenability} \Longrightarrow \text{weak amenability} \Longrightarrow AP \Longrightarrow \text{exactness}. \qquad (1)$$

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It is also known that the first two implications are not equivalences : for the first one, it was proved in [5] that non-abelian free groups are weakly amenable, whereas they are not amenable. For the second implication, a counter-example is given by $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$: since AP is stable by semi-direct product ([11]), this group has the AP. But it was proved in [10] that it does not have the CBAP. In fact Haagerup proved in [10] that the reduced C^*-algebra of any lattice in a locally compact simple lie group of real rank ≥ 2 with finite center does not have the CBAP. To the knowledge of the authors, before the present work there were no counter-example for the implication "exactness $\Longrightarrow OAP$". But it was conjectured by Haagerup and Kraus ([11]) that the (exact) group $SL_3(\mathbb{Z})$ fails AP. We prove this conjecture (Theorem C).

Let us recall some definitions: an operator space E is said to have the completely bounded approximation property (abreviated by CBAP) if there exists a net of finite rank linear maps $T_\alpha : E \to E$, such that $\|T_\alpha x - x\| \to 0$ for any $x \in E$ and such that $\sup_\alpha \|T_\alpha\|_{cb} < \infty$. The infimum over all such T_α of $\sup \|T_\alpha\|_{cb}$ is the CBAP constant of E and is denoted by $\Lambda(E)$. This is the natural analogue for operator spaces of Grothendieck's bounded approximation property (for Banach spaces). The analogue of the metric approximation property is the completely contractive approximation property (CCAP), and corresponds to the case when the maps T_α can be taken as complete contractions. The approximation property has also an analogue: E is said to have the operator space approximation property (OAP) if there exists a net of finite rank linear maps $T_\alpha : E \to E$ such that for all $x \in \mathcal{K}(\ell^2) \otimes_{\min} E$, $\|id \otimes T_\alpha(x) - x\| \to 0$. The CBAP is stronger than OAP. As explained above these notions are of particular interest when E is an operator algebra. They are also interesting for non-commutative L^p-spaces (which have a natural operator space structure, see [26], and subsection 1.1). This has been studied in [15], where the authors discovered some nice phenomena, as a consequence of the unpublished work from [13] : for $1 < p < \infty$, under the assumption that the underlying von Neumann algebra is QWEP (see Remark 1.1), the OAP, the CBAP and the CCAP are equivalent properties for a non-commutative L^p-space.

In this paper we give examples of non-commutative L^p spaces that fail CBAP (and hence OAP by [15]) for any $p \neq 2$. To our knowledge, the only results in this direction for non-commutative L^p spaces ($p \neq 1, \infty$) were consequences of Szankowski's work [27] : he indeed proved that for $p > 80$ (or $p < 80/79$), S^p does not have the uniform approximation property. By an ultrapower argument this implies the existence of non-commutative L^p-spaces without the BAP (and hence without CBAP) for $p > 80$ or $p < 80/79$, see Theorem 2.19 in [14]. Here we get concrete examples for any $p \neq 2$. They are non-commutative L^p-spaces associated to discrete groups, more precisely lattices in $SL_r(F)$ for F a non-archimedian local field (the typical example is to take F as the field of q-adic numbers $\mathbb{Q}_q$ for some prime number q) and r depending on p, or in $SL_r(\mathbb{R})$ with $r \geq 3$ if $p > 4$ or $p < 4/3$. More precisely, we prove the following (in the theorem below and in the rest of the paper by a lattice in a locally compact group G we mean a discrete subgroup with finite covolume) :

Theorem A. *Let F be a non-archimedean local field, $r \in \mathbb{N}$ with $r \geq 3$, and Γ be a lattice in $SL_r(F)$.*

If $1 \leq p < \infty$ and $n \in \mathbb{N}^*$ are such that $r \geq 2n+1$ and $1 \leq p < 2-2/(n+2)$ or $2+\frac{2}{n} < p < \infty$, then the non-commutative L^p space of the von Neumann algebra of Γ does not have the OAP or CBAP.

This theorem is proved at the end of section 4. Taking a direct sum of such discrete groups, we even get a group such that the corresponding non-commutative L^p spaces do not have the CBAP for any $p \neq 2$. In the real case, we prove the following at the end of section 5 :

Theorem B. *Let $r \in \mathbb{N}$ with $r \geq 3$, and Γ be a lattice in $SL_r(F)$ (for example $\Gamma = SL_3(\mathbb{Z})$). Let $1 \leq p < \infty$ with $p > 4$ or $p < 4/3$.*

The non-commutative L^p space of the von Neumann algebra of Γ does not have the OAP or CBAP.

As a consequence of [15], the corresponding discrete groups fail the AP. We also give an elementary proof of this. Since linear groups are exact ([9]), this gives examples of exact groups without the AP.

Theorem C. *Let $\Gamma = SL_3(\mathbb{Z})$ or more generally a lattice in $SL_r(F)$ with $r \geq 3$ and F denoting either $\mathbb{R}$ or a non-archimedean local field. Γ does not have AP; equivalently the reduced C^*-algebra of Γ does not have the OAP.*

To prove Theorem A, B, and C we introduce, for $1 \leq p \leq \infty$, a different approximation property for a group G, (property $\mathrm{AP}_{pcb}^{Schur}$), in terms of completely bounded Schur multipliers on the p-Schatten class on $L^2(G)$. These properties for p and p' coincide if $1/p+1/p' = 1$. When p is 1 or ∞, this property coincides with weak amenability, and when p decreases from ∞ to 2, this property becomes weaker. For discrete groups this property is implied by the completely bounded approximation property of the corresponding non-commutative L^p space, and by Haagerup's and Kraus' AP. As for the weak amenability of a group, we introduce a constant $\Lambda_{pcb}^{Schur}(G)$ of the property $\mathrm{AP}_{pcb}^{Schur}$ for G. We notice however that for discrete groups and $1 < p < \infty$, $\Lambda_{pcb}^{Schur}(G) \in \{1, \infty\}$. We also prove that the property $\mathrm{AP}_{pcb}^{Schur}$ is equivalent for a locally compact group second countable G or for a lattice in G (this was proved by Haagerup in [5] for the weak amenability).

The theorems above are thus consequences of the following results, which are proved in section 4 and 5 using ideas close to [18].

Theorem D. *Let F be a non-archimedean local field, $r \in \mathbb{N}$ with $r \geq 3$.*

If $1 \leq p \leq \infty$ and $n \in \mathbb{N}^*$ are such that $r \geq 2n+1$ and $1 \leq p < 2-2/(n+2)$ or $2+\frac{2}{n} < p < \infty$, then $SL_r(F)$ does not have the property $\mathrm{AP}_{pcb}^{Schur}$.

Theorem E. *Let $r \geq 3$. If $4 < p \leq \infty$ or $1 \leq p < 4/3$ then $SL_r(\mathbb{R})$ does not have the property $\mathrm{AP}_{pcb}^{Schur}$.*

We expect that a result analogous to Theorem D (with $r \to \infty$ as $p \to 2$) holds in the real case, but this would require more work.

Let us mention that there has also been some recent activity in the study of Herz-Schur multipliers (for $p = \infty$) for the groups $PGL_2(\mathbb{Q}_q)$ in [12] (in relation with Schur multipliers on homogeneous trees) and for $SL_2(\mathbb{R})$ in [20].

Let us review the organization of this paper. In a first section, we review some basic notions on completely bounded maps between non-commutative L^p-spaces, and on Schur multipliers. We give definitions and facts on Schur multipliers on the p-Schatten class on $L^2(X, \mu)$ for a general (σ-finite) measure space (X, μ). In a digression (subsection 1.4), we discuss Pisier's conjecture that there exist Schur multipliers that are bounded on $S^p = S^p(\ell^2)$ but not completely bounded. This conjecture is left wide open, but we reformulate it (Proposition 1.15) and we observe that when (X, μ) has no atom, no such phenomenon can occur (Theorem 1.18), *i.e.* the norm and the completely bounded norm of a Schur multiplier coincide. Finally we prove a characterization of Schur multipliers with continuous symbol when μ is a Radon measure on a locally compact space : Theorem 1.19. Apart from the definitions and from this Theorem, this section is quite independent from the rest of the paper.

In section 2, we introduce, for any $1 \leq p \leq \infty$, the property of completely bounded approximation by Schur multipliers on S^p for a group and the corresponding constant $\Lambda_{pcb}^{Schur}(G)$. The main result is Theorem 2.5, which states that the property of completely bounded approximation by Schur multipliers on S^p for a locally compact group is equivalent to the same property for a lattice.

In section 3 we restrict ourselves to discrete groups and investigate the relationship between the property $\mathrm{AP}_{pcb}^{Schur}$ when $1 < p < \infty$ and other approximation properties (AP for the group or OAP for the non-commutative L^p-space). The main results are Corollary 3.11, where we prove that $\Lambda_{pcb}^{Schur}(G)$ can only take the values 1 and ∞, Corollary 3.12, where we prove that AP implies $\mathrm{AP}_{pcb}^{Schur}$, and Corollary 3.13, where we prove that the OAP (and CBAP) of the associated non-commutative L^p-space implies $\mathrm{AP}_{pcb}^{Schur}$. The results in this section are close to [15], but since we are working with Schatten classes S^p instead of general non-commutative L^p-spaces, we are able to give elementary proofs.

In section 4, we prove Theorem D. The method of the proof is similar to the method of the proof of strong property (T) for $SL_3(F)$ in [17]. We also derive Theorem A and the non-archimedean case of Theorem C.

In section 5, we prove the same results for $SL_r(\mathbb{R})$ for $r \geq 3$, using again the methods close to [17].

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1 Schur multipliers on Schatten classes

In this section we fix $1 \leq p \leq \infty$. Given a Hilbert space H, $S^p(H)$ will denote the Schatten class on H: if $p = \infty$ it is the compact operators on H (equipped with the operator norm) and for $p < \infty$ it is the set of operators A on H such that $\|A\|_p := Tr(|A|^p)^{1/p} < \infty$. This quantity is a norm which makes $S^p(H)$ a Banach space. When $H = \ell_n^2$ then $S^p(H)$ is denoted by S_n^p. When no confusion is possible we might denote $S^p(H)$ simply by S^p.

1.1 CB maps on non-commutative L^p spaces

Note that for Hilbert spaces H and K, the algebraic tensor product $S^p(H) \otimes S^p(K)$ is naturally embedded in $S^p(H \otimes_2 K)$ as a dense subspace.

A linear map $T : S^p(H) \to S^p(H)$ is called completely bounded if for any Hilbert space K, the map $T^{(K)} = T \otimes id$ on $S^p(H) \otimes S^p(K)$ extends to a bounded map on $S^p(H \otimes K)$. The completely bounded norm of T is $\|T\|_{cb} = \sup_K \|T^{(K)}\|$. Note that

$$\|T\|_{cb} = \|T^{(\ell^2)}\| = \sup_n \|T^{(\ell_n^2)}\|. \tag{2}$$

The n-norm of T is $\|T^{(\ell_n^2)}\|$.

This definition agrees with the definition by Pisier of the natural operator space structure on $S^p(H)$ ([24]).

More generally (if $p < \infty$) if $\mathcal{M}$ is a von Neumann algebra with a semi-finite trace τ, a linear map T on $L^p(\mathcal{M}, \tau)$ is called completely bounded if $T \otimes id$ extends to a bounded operator on $L^p(\mathcal{M} \bar{\otimes} B(H), \tau \otimes Tr)$ for any Hilbert space H. Again we have that

$$\|T\|_{cb} = \sup_n \|T \otimes id : L^p(\mathcal{M} \otimes M_n) \to L^p(\mathcal{M} \otimes M_n)\|. \tag{3}$$

Remark 1.1. When $1 < p \neq 2 < \infty$, it is not known whether T being completely bounded implies that $T \otimes id$ extends to a bounded map, or completely bounded map, (with norm not greater than $\|T\|_{cb}$) on $L^p(\mathcal{M} \bar{\otimes} \mathcal{N}, \tau \otimes \widetilde{\tau})$ for any von Neumann algebra $\mathcal{N}$ with semi-finite trace $\widetilde{\tau}$. This is related to Connes' embedding problem (which is equivalent to the QWEP conjecture, see [22] for a survey). When $\widetilde{\tau}$ if finite and $(\mathcal{N}, \widetilde{\tau})$ embeds in an ultraproduct of the hyperfinite II_1 factor, then an ultraproduct argument shows that the previous holds. More generally Junge proved [13] that this is the case when $\mathcal{N}$ has QWEP ($\mathcal{N}$ is said to have QWEP if $\mathcal{N}$ is a quotient of a C^*-aglebra with Lance's weak expectation property). For a separable finite von Neumann algebra, Kirchberg [16] proved that QWEP is equivalent to the embedding into an ultraproduct of the hyperfinite II_1 factor.

1.2 Schur multipliers on $S^p(L^2(X, \mu))$.

A Schur multiplier on $M_n(\mathbb{C})$ is a linear map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ of the form $T : (a_{i,j}) \mapsto (\varphi_{i,j} a_{i,j})$ for some family $\varphi = (\varphi_{i,j})_{1 \leq i,j \leq n}$ called the *symbol* of T.

The multiplier T is then also denoted by M_φ. We study Schur multipliers on S^p_n and their continuous generalizations.

We wish to study this notion by replacing $M_n(\mathbb{C}) = B(\ell^2_n)$ by more generally $B(L^2(X,\mu))$ (or $S^p(L^2(X,\mu))$) for a σ-finite measure space (X,μ). Informally for a function $\varphi : X \times X \to \mathbb{C}$ we are interested in the map sending an operator T on $L^2(X,\mu)$ having a representation $(T_{x,y})_{x,y\in X}$ to the operator having $(\varphi(x,y)T_{x,y})_{x,y\in X}$ as representation. But since all the operators cannot be represented in this way we have to be more careful. This is closely related to the notion of double operator integrals.

For $p = 2$ and any Hilbert space H, $S^2(H)$ is identified with the Hilbert space tensor product $H^* \otimes_2 H$ (with the usual identification of $\xi^* \otimes \xi$ with the rank one operator on H, $\eta \mapsto \xi^*(\eta)\xi$). Let us identify (linearly isometrically) the dual of $L^2(X,\mu)$ with $L^2(X,\mu)$ for the duality $\langle f, g\rangle = \int fg d\mu$. We thus can identify $S^2(L^2(X,\mu))$ with $L^2(X,\mu) \otimes_2 L^2(X,\mu) \simeq L^2(X \times X, \mu \otimes \mu)$. We therefore have a good notion of Schur multipliers on $S^2(L^2(X,\mu))$, which coincides with $L^\infty(X \times X, \mu\otimes\mu)$ acting by multiplication on $L^2(X \times X, \mu \otimes \mu)$. Thus for any p and any function $\varphi \in L^\infty(X \times X, \mu \otimes \mu)$ we say that the Schur multiplier with symbol φ is completely bounded on S^p if it maps $S^2 \cap S^p$ into S^p, and if it extends to a completely bounded map from S^p to S^p. This extension is then necessarily unique because $S^2 \cap S^p$ is dense in S^p. We denote by M_φ this map. We will denote by $\|\varphi\|_{MS^p(L^2(X))}$ (resp. $\|\varphi\|_{cbMS^p(L^2(X))}$) its norm (resp. completely bounded norm).

Remark 1.2. If A and B belong to $S^2(L^2(X,\mu))$ and correspond in the identification above to functions f and g in $L^2(X \times X, \mu \otimes \mu)$, then

$$Tr(AB) = \int f(x,y)g(y,x)d\mu(x)d\mu(y). \tag{4}$$

Remark 1.3. By duality, if $1/p + 1/p' = 1$, the norm (resp. completely bounded norm) on $S^p(L^2(X))$ and $S^{p'}(L^2(X))$ of a Schur multiplier are the same.

Remark 1.4. By interpolation, this duality property implies that if $\varphi \in L^\infty(X \times X)$ and $2 \le p \le q \le \infty$, then $\|\varphi\|_{MS^p(L^2(X))} \le \|\varphi\|_{MS^q(L^2(X))}$. This holds because $S^q(H)$ coincides isometrically with the interpolation space (for the complex interpolation method) $[S^p(H), S^{p'}(H)]_\theta$ for $1/q = \theta/p' + (1-\theta)/p$. In particular, for any p,

$$\|\varphi\|_{L^\infty(X\times X)} \le \|\varphi\|_{MS^p(L^2(X))} \le \|\varphi\|_{MS^\infty(L^2(X))}.$$

The same inequalities hold for the cb-norm.

The following is immediate from (2).

Lemma 1.5. *The Schur multiplier corresponding to $\varphi \in L^\infty(X \times X, \mu \otimes \mu)$ is completely bounded on $S^p(L^2(X))$ if and only if the Schur multiplier corresponding to $\widetilde{\varphi}(x,i,y,j) = \varphi(x,y)$ is bounded on $S^p(L^2(X \times \mathbb{N}))$ (where $\widetilde{X} = X \times \mathbb{N}$ is equipped with the product measure of μ and the counting measure on $\mathbb{N}$). More precisely*

$$\|\varphi\|_{cbMS^p(L^2(X))} = \|\widetilde{\varphi}\|_{cbMS^p(L^2(\widetilde{X}))} = \|\widetilde{\varphi}\|_{MS^p(L^2(\widetilde{X}))}.$$

Remark 1.6. In fact we can replace $\mathbb{N}$ by any σ-finite measure space (Ω, ν) : for $\varphi \in L^\infty(X \times X)$, define again $\widetilde{X} = X \times \Omega$ and $\widetilde{\varphi} \in L^\infty(\widetilde{X} \times \widetilde{X})$ by $\widetilde{\varphi}(x, \omega, y, \omega') = \varphi(x, y)$. Then

$$\|\varphi\|_{cbMS^p(L^2(X))} = \|\widetilde{\varphi}\|_{cbMS^p(L^2(\widetilde{X}))},$$

and this is equal to $\|\widetilde{\varphi}\|_{MS^p(L^2(\widetilde{X}))}$ provided that $L^2(\Omega, \nu)$ is infinite dimensional.

When $p = 2$ we obviously have

$$\|\varphi\|_{cbMS^2(L^2(X))} = \|\varphi\|_{MS^2(L^2(X))} = \|\varphi\|_{L^\infty(X \times X)}.$$

For $p = 1, \infty$, the following characterization is well-known, and goes back to Grothendieck (see chapter 5 of [25]). The result is more often expressed when $X = \mathbb{N}$, but the general statement below follows by a martingale/ultraproduct argument. For completeness we include a proof of this generalization, that uses Lemma 1.11 below. This proof was indicated to us by Gilles Pisier.

Theorem 1.7. *Let (X, μ) be a σ-finite measure space. If $p = \infty$ (or $p = 1$) and $\varphi \in L^\infty(X \times X)$, we have that*

$$\|\varphi\|_{MS^p(L^2(X))} = \inf \|f\|_{L^\infty(X,\mu;H)} \|g\|_{L^\infty(X,\mu;H)}$$

where the infimum runs over all separable Hilbert spaces, all measurable functions $f, g : X \to H$ such that $\varphi(x, y) = \langle f(x), g(y)\rangle$ almost everywhere.

For other values of p, there is no known characterization of Schur multipliers. In particular, the following conjecture of Pisier is still open.

Conjecture 1.8 ([24], Conjecture 8.1.12)**.** *For $1 < p < \infty$, $p \neq 2$, there exist Schur multipliers on $S^p = S^p(\ell^2)$ that are bounded but not completely bounded.*

In fact there is not even an example of a Schur multiplier on S^p_n (for $n \in \mathbb{N}^*$ and $1 < p < \infty$, $p \neq 2$) for which the norm and the cb-norm are known to be different.

Proof of Theorem 1.7. First note that by Lemma 1.9 we can assume that μ is a finite measure.

We claim that the Theorem is equivalent to the following fact:

$$\|\varphi\|_{cbMS^1(L^2(X))} = \|\varphi\|_{MS^1(L^2(X))} = \inf \|a\|_{L^1(\mu)\to H} \|b\|_{L^1(\mu)\to H} \tag{5}$$

where the infimum runs over all Hilbert spaces H, all bounded linear maps $a, b : L^1(\mu) \to H$ such that $\int \varphi(x, y)u(x)\overline{v(y)}d\mu(x)d\mu(y) = \langle a(u), b(v)\rangle$.

Indeed since Hilbert spaces have the Radon-Nikodym property, the Riesz representation Theorem ([7], Chapter III) implies that a linear map $a : L^1(\mu) \to H$ takes values in separable subspace of H (hence we can assume that H is separable), and a is of the form $u \mapsto \int uf d\mu$ for some map $f \in L^\infty(X, \mu; H)$ (note that when H is separable Bochner-measurable functions are simply usual measurable functions). Then $\int \varphi(x, y)u(x)\overline{v(y)}d\mu(x)d\mu(y) = \langle a(u), b(v)\rangle$ if and only if $\varphi(x, y) = \langle f(x), g(y)\rangle$ almost everywhere.

Let us now prove (5). As explained before the statement of the Theorem, we only derive the general case from the case when $L^2(X)$ is finite dimensional. Note that the following inequalities are easy:

$$\|\varphi\|_{MS^1(L^2(X))} \leq \|\varphi\|_{cbMS^1(L^2(X))} \leq \inf \|a\|_{L^1(\mu)\to H}\|b\|_{L^1(\mu)\to H}.$$

The first is obvious and the second inequality follows from Lemma (1.5) and from the fact that the unit ball of $S^1(L^2(X\times\mathbb{N}))$ is the closed convex hull of the rank one operators in the unit ball. Let us prove the remaining inequality. For this consider a filtration of finite σ-subalgebras $\mathcal{B}_n$ such that the corresponding martingale $\varphi_n = \mathbb{E}[\varphi|\mathcal{B}_n\otimes\mathcal{B}_n]$ converges almost surely to φ. For any n, (5) gives a Hilbert space H_n and linear map $a_n, b_n : L^1(X,\mathcal{B}_n,\mu) \to H_n$ such that $\int \varphi_n(x,y)u(x)\overline{v(y)}d\mu(y) = \langle a_n(u), b_n(v)\rangle$ and such that $\|a_n\|\|b_n\| \leq \|\varphi_n\|_{MS^1(L^2(\mathcal{B}_n))} + 1/n$ (in fact we can even take $\|a_n\|\|b_n\| = \|\varphi_n\|_{MS^1(L^2(\mathcal{B}_n))}$). We can and will assume that $\|a_n\| = \|b_n\|$. Take $\mathcal{U}$ a non principal ultrafilter on $\mathbb{N}$, and let $H = \prod H_n/\mathcal{U}$ be the ultraproduct. It is a Hilbert space. For $u \in L^1(\mu)$ let $u_n = \mathbb{E}[u|\mathcal{B}_n]$. If $a(u)$ (resp. $b(v)$) denotes the image of $(a_n(u_n))_n$ (resp. $(b_n(v_n))_n$) in the ultraproduct, then a and b are bounded linear maps of norm $\lim_{\mathcal{U}}\|a_n\|$ and $\lim_{\mathcal{U}}\|b_n\|$. In particular by Lemma 1.10 $\|a\|\|b\| \leq \|\varphi\|_{MS^1(L^2(X))}$. Moreover by the dominated convergence Theorem

$$\begin{aligned}
\int \varphi(x,y)u(x)\overline{v(y)}d\mu(x)d\mu(y) &= \lim_{\mathcal{U}} \int \varphi_n(x,y)u(x)\overline{v(y)}d\mu(x)d\mu(y) \\
&= \lim_{\mathcal{U}} \int \varphi_n(x,y)u_n(x)\overline{v_n(y)}d\mu(x)d\mu(y) \\
&= \lim_{\mathcal{U}} \langle a_n(u), b_n(v)\rangle = \langle a(u), b(v)\rangle.
\end{aligned}$$

This concludes the proof. ☐

1.3 Change of measure.

The first obvious remark is that for $\varphi \in L^\infty(X\times X, \mu\otimes\mu)$, the norm (resp. cb-norm) of the corresponding Schur multiplier on $S^p(L^2(X,\mu))$ only depends on the class of the measure μ. More precisely:

Lemma 1.9. *Let $\nu << \mu$ be two σ-finite measures on X and $\varphi \in L^\infty(X\times X, \mu\otimes\mu)$. Then*

$$\|\varphi\|_{MS^p(L^2(X,\nu))} \leq \|\varphi\|_{MS^p(L^2(X,\mu))}.$$

The same holds for the cb-norm.

Proof. If $f = d\nu/d\mu$ is the Radon-Nikodym derivative, and if U denotes the multiplication by $\sqrt{f}$ from $L^2(X,\nu)$ to $L^2(X,\mu)$ (U is an isometry), then $A \mapsto UAU^*$ defines a (completely) isometric embedding of $S^p(L^2(X,\nu))$ into $S^p(L^2(X,\nu))$ such that $M_\varphi(UAU^*) = UM_\varphi(A)U^*$. ☐

1.4 Change of σ-algebra.

We observe basic properties of the Schur multipliers relative to conditional expectations. Except from Lemma 1.10 below, this subsection is independent of the rest of the paper. We will mainly work in the following situation:

$$\begin{array}{c} \mathcal{A} \subset \mathcal{B} \text{ are } \sigma\text{-algebras on } X \\ \mu \text{ is a measure on } (X, \mathcal{B}) \text{ that is } \sigma\text{-finite on } (X, \mathcal{A}) \end{array} \tag{6}$$

Note that this allows us to talk about the conditional expectation from $L^\infty(X, \mathcal{B}, \mu)$ to $L^\infty(X, \mathcal{A}, \mu)$ (resp. from $L^\infty(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$ to $L^\infty(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$). When no confusion is possible we will simply denote $L^2(X, \mathcal{B}, \mu)$ by $L^2(\mathcal{B})$ and $L^2(X, \mathcal{A}, \mu)$ by $L^2(\mathcal{A})$.

The following lemma is essentially obvious:

Lemma 1.10. *In the situation of* (6)*, if* $\varphi \in L^\infty(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$,

$$\| \mathbb{E} \left[\varphi | \mathcal{A} \otimes \mathcal{A} \right] \|_{MS^p(L^2(\mathcal{A}))} \leq \| \varphi \|_{MS^p(L^2(\mathcal{B}))}.$$

The same holds for the cb-norm.

Proof. Let $V : L^2(\mathcal{A}) \to L^2(\mathcal{B})$ be the isometry corresponding to the inclusion map. The map $\iota : B(L^2(\mathcal{A})) \to B(L^2(\mathcal{B}))$ which maps T to VTV^* is a trace preserving $*$-homomorphism (and hence induces a complete isometry $S^p(L^2(\mathcal{A})) \to S^p(L^2(\mathcal{B})))$, and the projection $P : B(L^2(\mathcal{B}))) \to B(L^2(\mathcal{A}))$ mapping T to V^*TV is also completely contractive on S^p. It remains to notice that $V^* : L^2(\mathcal{B}) \to L^2(\mathcal{A})$ corresponds to the conditional expectation on $\mathcal{A}$, which implies that the following diagram commutes:

$$\begin{tikzcd} S^p(L^2(\mathcal{B})) \arrow[r, "M_\varphi"] & S^p(L^2(\mathcal{B})) \arrow[d, "P"] \\ S^p(L^2(\mathcal{A})) \arrow[u, "\iota"] \arrow[r, "M_{\mathbb{E}[\varphi|\mathcal{A}\otimes\mathcal{A}]}"] & S^p(L^2(\mathcal{A})) \end{tikzcd} .$$

□

In the vocabulary of martingales, the previous ideas become:

Lemma 1.11. *Let* $(X, \mathcal{B}, \mu)$ *be a measure space and* $(\mathcal{B}_n)_{n \in \mathbb{N}}$ *be a filtration. Assume that* μ *is* σ*-finite on* $(X, \mathcal{B}_n)$ *for all* n*, and that* $\mathcal{B}$ *is the* σ*-algebra generated by* $\cup_n \mathcal{B}_n$*. For any* $f \in L^\infty(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$ *let* $f_n \in L^\infty(X \times X, \mathcal{B}_n \otimes \mathcal{B}_n, \mu \otimes \mu)$ *be the conditional expectation. Then*

$$\|f\|_{MS^p(L^2(\mathcal{B}))} = \lim_{n \to \infty} \nearrow \|f_n\|_{MS^p(L^2(\mathcal{B}_n))} \tag{7}$$

$$\|f\|_{cbMS^p(L^2(\mathcal{B}))} = \lim_{n \to \infty} \nearrow \|f_n\|_{cbMS^p(L^2(\mathcal{B}_n))} \tag{8}$$

By $l = \lim_{n \to \infty} \nearrow u_n$ we mean that the sequence u_n is non-decreasing and converging to l.

Remark 1.12. This statement remains valid replacing $(\mathcal{B}_n)_{n\in\mathbb{N}}$ by a filtration $(\mathcal{B}_\alpha)_{\alpha\in A}$ with respect to any directed set A.

Proof. The equality for the cb-norm follows from the equality for the norm and Lemma 1.5. So let us focus on the inequality for the norm. The fact that $\|f_n\|_{MS^p(L^2(\mathcal{B}_n))}$ grows with n and stays smaller than $\|f\|_{MS^p(L^2(\mathcal{B}))}$ is Lemma 1.10. Denote by C its limit. We have to prove that for any $A \in S^p \cap S^2(L^2(\mathcal{B}))$ and $B \in S^{p'} \cap S^2(L^2(\mathcal{B}))$, we have that

$$|Tr(M_f(A)B)| \leq C\|A\|_p\|B\|_{p'}. \tag{9}$$

But by the assumption that $\cup_n \mathcal{B}_n$ generate $\mathcal{B}$, $\cup_n S^p(L^2(\mathcal{B}_n))$ is dense (for the norm $\|\cdot\|_p$) in $S^p(L^2(\mathcal{B}))$. We can therefore assume that A (resp. B) belongs to $S^p \cap S^2(L^2(\mathcal{B}_n))$ (resp. $S^{p'} \cap S^2(L^2(\mathcal{B}_n))$). But then (9) follows from the fact $Tr(M_f(A)B) = Tr(M_{f_n}(A)B)$, which can be checked directly: let g_A and $g_B \in L^2(X \times X, \mathcal{B}_n \otimes B_n, \mu \otimes \mu)$ be the functions corresponding to A and B with the identification $S^2(L^2(X)) = L^2(X \otimes X)$. Then

$$\begin{aligned} Tr(M_f(A)B) &= \int f(x,y)g_A(x,y)g_B(y,x)d\mu(x)d\mu(y) \\ &= \int f_n(x,y)g_A(x,y)g_B(y,x)d\mu(x)d\mu(y) \\ &= Tr(M_{f_n}(A)B). \end{aligned}$$

□

For the cb-norm we even have the following generalization of Remark 1.6 :

Lemma 1.13. *In the situation of* (6)*, if* $\varphi \in L^\infty(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \otimes \mu)$*,*

$$\|\varphi\|_{cbMS^p(L^2(X,\mathcal{A}))} = \|\varphi\|_{cbMS^p(L^2(X,\mathcal{B}))}.$$

Therefore, if $\varphi \in L^\infty(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$*,*

$$\|\mathbb{E}\left[\varphi|\mathcal{A} \otimes \mathcal{A}\right]\|_{cbMS^p(L^2(X,\mathcal{B},\mu))} \leq \|\varphi\|_{cbMS^p(L^2(X,\mathcal{B},\mu))}.$$

Proof. The second statement is the combination of the first statement and of Lemma 1.10 for the cb-norm. So let us focus on the first statement. It is immediate when $\mathcal{A}$ is finite.

To prove the general case we can first assume that μ is a finite measure (replacing μ by $f\mu$ for some $\mathcal{A}$-measurable almost everywhere positive function $f \in L^1(X, \mathcal{A}, \mu)$). Then consider a filtration $(\mathcal{B}_n)_{n\geq 0}$ of finite σ-subalgebras of $\mathcal{A}$, such that the corresponding martingale $(\varphi_n)_{n\geq 0}$ converges almost surely. Since $\mathcal{B}_n$ is finite, we get, using that $\mathcal{B}_n \subset \mathcal{A}$ (resp. $\mathcal{B}_n \subset \mathcal{B}$) that

$$\|\varphi_n\|_{cbMS^p(L^2(X,\mathcal{A}))} = \|\varphi_n\|_{cbMS^p(L^2(X,\mathcal{B}_n))} = \|\varphi_n\|_{cbMS^p(L^2(X,\mathcal{B}))}. \tag{10}$$

We claim that $\|\varphi_n\|_{cbMS^p(L^2(X,\mathcal{C}))} \to \|\varphi\|_{cbMS^p(L^2(X,\mathcal{C}))}$ for $\mathcal{C} = \mathcal{A}$ or $\mathcal{B}$. This would conclude the proof. By Lemma 1.10 for the cb-norm and (10), it is enough to prove that $\|\varphi\|_{cbMS^p(L^2(X,\mathcal{C}))} \leq \limsup_n \|\varphi_n\|_{cbMS^p(L^2(X,\mathcal{C}))}$. Take $A \in S^2 \cap S^p(L^2(X \times \mathbb{N}, \mathcal{C} \otimes \mathcal{P}(\mathbb{N}))$ and $B \in S^2 \cap S^{p'}(L^2(X \times \mathbb{N}, \mathcal{C} \otimes \mathcal{P}(\mathbb{N}))$. Let $\widetilde{X} = X \times \mathbb{N}$, and consider $\widetilde{\varphi}_n \in L^\infty(\widetilde{X} \times \widetilde{X})$ as in Lemma 1.5. Since $\widetilde{\varphi}_n$ converges almost surely to $\widetilde{\varphi}$ and $\sup_n \|\widetilde{\varphi}_n\|_{L^\infty} \leq \|\varphi\|_\infty < \infty$, the dominated convergence Theorem and (4) imply that $\lim_n Tr(M_{\widetilde{\varphi}_n}(A)B) = Tr(M_\varphi(A)B)$. Hence,

$$\left|Tr(M_{\widetilde{\varphi}}(A)B)\right| \leq \limsup_n \|\widetilde{\varphi}_n\|_{MS^p(L^2(\widetilde{X},\mathcal{C}))} \|A\|_p \|B\| p'.$$

By Lemma 1.5, this proves the claim because $S^2 \cap S^p$ (resp. $S^2 \cap S^{p'}$) is dense in S^p (resp. $S^{p'}$). □

We do not know the answer to the following question for $1 < p \neq 2 < \infty$, although we suspect that the answer should be negative :

Question 1.14. *With the same assumptions as in Lemma 1.10, is it true that*

$$\|\mathbb{E}\left[\varphi|\mathcal{A} \otimes \mathcal{A}\right]\|_{MS^p(L^2(X,\mathcal{B},\mu))} \leq \|\varphi\|_{MS^p(L^2(X,\mathcal{B},\mu)}?$$

But we can prove that this question is related to Pisier's conjecture 1.8 :

Proposition 1.15. *Fix $1 \leq p \leq \infty$ and $K \geq 1$. Then the following are equivalent:*

(i) For all $n \in \mathbb{N}^*$, the norm and the cb-norm of a Schur multiplier on S^p_n are equal.

(ii) For all σ-finite measure space $(X, \mathcal{B}, \mu)$ and $\varphi \in L^\infty(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$,

$$\|\varphi\|_{MS^p(L^2(X,\mathcal{B},\mu))} = \|\varphi\|_{cbMS^p(L^2(X,\mathcal{B},\mu))}.$$

(iii) For all measure spaces $(X, \mathcal{B}, \mu)$, all σ-subalgebras $\mathcal{A} \subset B$ such that μ is σ-finite on $(X, \mathcal{A})$, and all $\varphi \in L^\infty(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$,

$$\|\mathbb{E}[\varphi|\mathcal{A} \otimes \mathcal{A}]\|_{MS^p(L^2(X,\mathcal{B},\mu))} \leq \|\varphi\|_{MS^p(L^2(X,\mathcal{B},\mu))}.$$

Remark 1.16. In fact the proof shows more generally that Pisier's conjecture 1.8 is equivalent to the fact that there exists (X, μ), $\mathcal{A}$, $\mathcal{B}$ and φ as in (iii) such that $\|\varphi\|_{MS^p(L^2(X,\mathcal{B},\mu))} < \infty$ but the Schur multiplier with symbol $\mathbb{E}\left[\varphi|\mathcal{A} \otimes \mathcal{A}\right]$ is not bounded on $MS^p(L^2(X, \mathcal{B}, \mu))$.

Proof. First remark that since any σ-finite measure is equivalent to a probability measure, both assertions (ii) and (iii) are equivalent to the same assertions with μ being a probability measure.

The assertion (i) is just (ii) restricted to the case when $\mathcal{B}$ is finite. Thus (ii) implies (i) and the other direction follows by Lemma 1.11 (or rather the remark following, applied to the filtration of all finite σ-subalgebras of $\mathcal{B}$, provided that μ is finite).

(ii) ⇒ (iii) follows from Lemma 1.13.

Let us prove now that (iii) $\Rightarrow$ (ii). Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $\varphi \in L^\infty(X \times X)$. Let $\widetilde{X} = X \times \mathbb{N}$ and define $\widetilde{\varphi}$ on $\widetilde{X} \times \widetilde{X}$ by

$$\widetilde{\varphi}(x, i, y, j) = \begin{cases} \varphi(x, y) & \text{if } i = j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Fix $\varepsilon > 0$ and consider the probability measure P_ε on $\mathbb{N}$ such that $P_\varepsilon(0) = 1 - \varepsilon$ and $P_\varepsilon(i) = \varepsilon 2^{-i}$ if $i > 0$. Let $\mathcal{B}_1 = \mathcal{B} \otimes \mathcal{P}(\mathbb{N})$ and $\mathcal{A}_1 = \mathcal{B} \otimes \{\emptyset; \mathbb{N}\}$. Then the conditional expectation of $\widetilde{\varphi}$ with respect to P_ε on $\mathcal{A}_1 \otimes \mathcal{A}_1$ is $\mathbb{E}[\widetilde{\varphi}|\mathcal{A}_1 \otimes \mathcal{A}_1](x, i, y, j) = (1 - \varepsilon)\varphi(x, y)$. But the equality

$$\|\widetilde{\varphi}\|_{MS^p(L^2(\widetilde{X}, \mathcal{B}_1, \mu \otimes P_\varepsilon))} = \|\varphi\|_{MS^p(L^2(X, \mu))}$$

is obvious, whereas the equality

$$\|\mathbb{E}[\widetilde{\varphi}|\mathcal{A}_1 \otimes \mathcal{A}_1]\|_{MS^p(L^2(\widetilde{X}, \mathcal{B}_1 \mu \otimes P_\varepsilon))} = (1 - \varepsilon)\|\varphi\|_{cbMS^p(L^2(X, \mathcal{B}, \mu))}$$

follows from the fact that P_ε is equivalent to the counting measure on $\mathbb{N}$ and from Lemma 1.5. The assumption (iii) thus implies that

$$(1 - \varepsilon)\|\varphi\|_{cbMS^p(L^2(X, \mathcal{B}, \mu))} \leq \|\varphi\|_{cbMS^p(L^2(X, \mathcal{B}, \mu))}.$$

Making $\varepsilon \to 0$ we get (ii). $\square$

The following Lemma gives a positive answer to question 1.14, in the setting when “the conditional expectation is implemented by random permutations”. By an atom in a measure space $(X, \mathcal{B}, \mu)$, we mean a measurable subset that cannot be partitioned into two subsets of positive measure.

Lemma 1.17. *Let $\mathcal{A} \subset \mathcal{B}$ be two* finite *σ-algebras on X, μ a finite measure on $(X, \mathcal{B})$ such that every atom of $\mathcal{A}$ is partitioned into atoms of $\mathcal{B}$* of same measure*. Then for any $\mathcal{B} \otimes \mathcal{B}$-measurable $\varphi : X \times X \to \mathbb{C}$,*

$$\|\mathbb{E}\left[\varphi|\mathcal{A} \otimes \mathcal{A}\right]\|_{MS^p(L^2(X, \mathcal{B}))} \leq \|\varphi\|_{MS^p(L^2(X, \mathcal{B}))}.$$

Proof. We can as well assume that X is a finite set and $\mathcal{B} = \mathcal{P}(X)$. If σ and σ' are permutations of X, denote by $\varphi^{\sigma,\sigma'}(x, y) = \varphi(\sigma(x), \sigma'(y))$. Note that by invariance of the norm on $S^p(\ell^2(X))$ by permutation of rows and columns

$$\|\varphi\|_{MS^p(L^2(X, \mathcal{B}))} = \|\varphi^{\sigma,\sigma'}\|_{MS^p(L^2(X, \mathcal{B}))}. \tag{11}$$

Let now σ be a random permutation of X satisfying the following: for any atom A of $\mathcal{A}$, $\sigma(A) = A$ and for any $x, y \in A$, the probability that $\sigma(x) = y$ is $1/|A|$. Let σ' be an independent copy of σ. Then for any $x, y \in X$, $\mathbb{E}\left[\varphi|\mathcal{A} \otimes \mathcal{A}\right](x, y)$ is the expected value of $\varphi^{\sigma,\sigma'}(x, y)$, and the triangle inequality and (11) conclude the proof. $\square$

We can thus conclude by the following result:

Theorem 1.18. *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space with no atom. Then for any $\varphi \in L^\infty(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$ and any $1 \leq p \leq \infty$,*

$$\|\varphi\|_{MS^p(L^2(X,\mathcal{B}))} = \|\varphi\|_{cbMS^p(L^2(X,\mathcal{B}))}.$$

Proof. Replacing μ by a probability measure which is equivalent, we can assume that μ is a probability measure.

By Lemma 1.11 it is enough to prove that for any finite σ-subalgebra $\mathcal{A} \subset \mathcal{B}$, if $\varphi_{\mathcal{A}} = \mathbb{E}[\varphi|\mathcal{A} \otimes \mathcal{A}]$, then

$$\|\mathbb{E}\left[\varphi|\mathcal{A} \otimes \mathcal{A}\right]\|_{cbMS^p(L^2(\mathcal{A}))} \leq \|\varphi\|_{MS^p(L^2(\mathcal{B}))}.$$

Fix such $\mathcal{A}$, and some integer n. Use the assumption that $\mathcal{B}$ has no atom: every atom A of $\mathcal{A}$ can be partitioned into n $\mathcal{B}$-measurable subsets $A^1, \ldots, A^n$ of same measure $\mu(A)/n$. Let $\mathcal{B}'$ be the σ-algebra generated by the set A^i for $1 \leq i \leq n$ and A atom of $\mathcal{A}$. Then by Lemma 1.17,

$$\|\mathbb{E}\left[\varphi|\mathcal{A} \otimes \mathcal{A}\right]\|_{MS^p(L^2(\mathcal{B}'))} \leq \|\mathbb{E}\left[\varphi|\mathcal{B}' \otimes \mathcal{B}'\right]\|_{MS^p(L^2(\mathcal{B}'))}.$$

But the left-hand side is equal to the norm of $\mathbb{E}\left[\varphi|\mathcal{A} \otimes \mathcal{A}\right] \otimes id$ acting on $S^p(L^2(\mathcal{A}) \otimes \ell^2_n)$, and the right-hand side is by Lemma 1.10 not greater than $\|\varphi\|_{MS^p(L^2(X,\mathcal{B}))}$. Since n was arbitrary, this concludes the proof. $\square$

1.5 Multipliers with continuous symbol.

We now study Schur multipliers in the setting when X is a locally compact space, μ is a σ-finite Radon measure, and the symbol φ is continuous.

Theorem 1.19. *Let μ be a σ-finite Radon measure on a locally compact space X, and $\varphi : X \times X \to \mathbb{C}$ a continuous function. Let $1 \leq p \leq \infty$ and $C > 0$. The following are equivalent:*

(i) φ defines a bounded multiplier on $S^p(L^2(X,\mu))$ with norm less than C.

(ii) For any finite subset $F = \{x_1, \ldots, x_N\}$ in X belonging to the support of μ, the multiplier $(\varphi(x_i, x_j))$ is bounded on $S^p(\ell^2(F))$ with norm less than C.

The same equivalence is true for the cb-norms.

In particular, the norm and cb-norm on S^p of the multiplier with symbol φ only depends on the support of μ, and if this support has no isolated point, its norm and cb-norm coincide.

Proof. Since any σ-finite Radon measure is equivalent to a finite measure, we can assume that μ is a probability measure.

Let us first prove that (i)$\Rightarrow$(ii). Assume (i) and fix a finite subset $F = \{x_1, \ldots, x_N\}$ of the support of μ. Then for any family $V_1, \ldots, V_N$ of disjoint

Borel subsets such that $x_i \in V_i$ and $\mu(V_i) > 0$, we can consider $\mathcal{A}$ the σ-subalgebra of $\mathcal{B}$ generated by the V_i's. By Lemma 1.10, we get that the norm on S^p_n of the Schur multiplier with symbol given by

$$(i,j) \mapsto \text{average value of } \varphi \text{ on } V_i \times V_j$$

is not greater than the norm on $S^p(L^2(X))$ of M_φ, *i.e.* is not greater than C. But if the V_i's are chosen to be contained in arbitrary small neighbourhouds of x_i (which is possible because x_i belongs to the support of μ), we get at the limit that the average value of φ on $V_i \times V_j$ tends to $\varphi(x_i, x_j)$. This proves (ii).

For the converse, assume (ii). By a density argument it is enough to prove that

$$|Tr(M_\varphi(A)B)| \leq C\|A\|_p\|B\|_{p'}$$

for finite rank operators on A and B on $L^2(X,\mu)$ that correspond to elements g_A, g_B of $C_c(X) \otimes C_c(X)$ in the identification $S^2(L^2(X,\mu)) = L^2(X \times X, \mu \otimes \mu)$ (here $C_c(X)$ denotes the continuous functions from X to $\mathbb{C}$ with compact support). Find (μ_α) a net a probability measures on X with finite support contained in the support of μ converging vaguely to μ (*i.e.* such that $\int f d\mu_\alpha \to \int f d\mu$ for all $f \in C_c(X)$). For the existence of such a net, see [3], Chap. IV, §2, 4, Corollaire 2. Then for any α denote by A_α and B_α the operators on $L^2(X, \mu_\alpha)$ corresponding to g_A and g_B viewed in $L^2(X,\mu_\alpha) \otimes L^2(X,\mu_\alpha)$. We claim that $\lim_\alpha \|A_\alpha\|_p = \|A\|_p$ and $\lim_\alpha \|B_\alpha\|_{p'} = \|B\|_{p'}$. This would conclude the proof of (ii)⇒(i) since by (4) and the vague convergence of μ_α to μ, we have that

$$Tr(M_f(A_\alpha)B_\alpha) \xrightarrow{\alpha} Tr(M_f(A)B).$$

To prove the claim (say for A), write (using the Gram-Schmidt orthonormalization process) $g_A = \sum_{i,j=1}^N a_{i,j} f_i \otimes f_j$ for a family $f_i \in C_c(X)$ which is orthonormal in $L^2(\mu)$. Thus $\|A\|_p = \|(a_{i,j})_{i,j\leq N}\|_{S^p_N}$. But by the vague convergence of μ_α to μ, the family $f_1, \ldots, f_N$ is almost orthonormal in $L^2(X,\mu_\alpha)$, and thus it is close to an orthonormal family f_i^α, and thus we can write $g_A = \sum_{i,j=1}^N a_{i,j}^\alpha f_i^\alpha \otimes f_j^\alpha$ with $a_{i,j}^\alpha$ converging to $a_{i,j}$. This indeed implies that

$$\|A_\alpha\|_p = \|(a_{i,j}^\alpha)_{i,j\leq N}\|_{S^p_N} \xrightarrow{\alpha} \|(a_{i,j})_{i,j\leq N}\|_{S^p_N} = \|A\|_p.$$

This proves (i)⇔(ii). For the cb-norm, apply this equivalence with X replaced by $X \times \mathbb{N}$ and use Lemma 1.5.

It remains to note that when the support of μ has no isolated point, the norm and cb-norm of a Schur multiplier coincide. We show that the best C such that (ii) holds is equal to the best C such that (ii) holds for the cb-norm. For this, fix a finite subset $F = \{x_1, \ldots, x_N\}$ in the support of μ and an integer n. For any $1 \leq i \leq N$, find n nets $(y_\alpha^{i,j})_\alpha$ for $j = 1, \ldots, n$ of elements of the support of μ such that $y_\alpha^{i,j} \xrightarrow{\alpha} x_i$ and such that for fixed α, the $y_\alpha^{i,j}$ for $1 \leq i \leq N$ and $j = 1, \ldots, n$ are all disctinct. This is possible because the support of μ has no isolated point. Note that $\varphi(y_\alpha^{i,j}, y_\alpha^{i',j'}) \xrightarrow{\alpha} \varphi(x_i, x_{i'})$. Expressing, for any α, (ii) with the finite set $\{y_\alpha^{i,j}, 1 \leq i \leq N, 1 \leq j \leq n\}$, one gets at the limit that the

n-norm of the multiplier with symbol $\varphi(x_i, x_{i'})$ is bounded by C. But n was arbitrary. $\square$

2 Approximation by Schur multipliers

In this section locally compact groups will always be assumed to be second countable. The reason is that we want to deal with σ-finite measure spaces, and a Haar measure on a locally compact group is σ-finite if and only if the group is second countable.

Recall that the Fourier algebra $A(G)$ of a locally compact group G is the set of coefficients of the left regular representation of G and is naturally identified with the predual of the von Neumann algebra of G.

Notation 2.1. For a locally compact and second countable group G (say equipped with a left Haar measure) and a function $\varphi \in L^\infty(G)$ we will denote by $\check{\varphi} \in L^\infty(G \times G)$ the function defined by $\check{\varphi}(g, h) = \varphi(g^{-1}h)$. The corresponding Schur multiplier is sometimes called Toeplitz-Schur multiplier, or Herz-Schur multiplier.

Bożejko-Fendler's characterization [1] (see also [2]) states that for $\varphi : G \to \mathbb{C}$, the completely bounded norm on $VN(G)$ of the Fourier multiplier $\lambda(g) \mapsto \varphi(g)\lambda(g)$, denoted by $\|\varphi\|_{M_0A(G)}$ (by duality it is the cb-norm of the multiplication by φ on $A(G)$) is equal to the norm of the Schur multiplier $\check{\varphi}$:

$$\|\check{\varphi}\|_{cbMS^\infty(L^2(\Gamma))} = \|\varphi(g)\|_{M_0A(G)}.$$

As defined in [6], G is said to be weakly amenable if there exists a constant C and a net $\varphi_\alpha \in A(G)$ that converges uniformly on compact subsets to 1 and such that $\|\varphi_\alpha(g)\|_{M_0A(G)} \leq C$. The infimum of such C is denoted by Λ_G.

We generalize this notion as follows :

Definition 2.2. If G is a locally compact second countable group and $1 \leq p \leq \infty$, we say that G has the *property of completely bounded approximation by Schur multipliers on* S^p ($\mathrm{AP}_{pcb}^{Schur}$) if there is a constant C, a net of functions $\varphi_\alpha \in A(G)$ such that $\varphi_\alpha \to 1$ uniformly on compact subsets of G and such that $\|\check{\varphi}_\alpha\|_{cbMS^p(L^2(G))} \leq C$. The infimum of such C is denoted by $\Lambda_{pcb}^{Schur}(G)$.

Note that if G is not discrete, Theorem 1.19 shows that the condition $\|\check{\varphi}_\alpha\|_{cbMS^p(L^2(G))} \leq C$ is equivalent to $\|\check{\varphi}_\alpha\|_{MS^p(L^2(G))} \leq C$.

Here are some basic properties of $\Lambda_{pcb}^{Schur}(G)$:

Proposition 2.3. *For a locally compact second countable group G:*

- *For $p = \infty$, G has the property of completely bounded approximation by Schur multipliers on S^p if and only if it is weakly amenable, and*

$$\Lambda_{\infty cb}^{Schur}(G) = \Lambda_G.$$

- $\Lambda^{Schur}_{2cb}(G) = 1$.
- *If* $1 \leq p \leq \infty$*, and* $1/p + 1/p' = 1$ *then* $\Lambda^{Schur}_{pcb}(G) = \Lambda^{Schur}_{p'cb}(G)$.
- *If* $2 \leq p \leq q \leq \infty$*, then* $\Lambda^{Schur}_{pcb}(G) \leq \Lambda^{Schur}_{qcb}(G)$.
- *If* H *is a closed subgroup of* G *and* $1 \leq p \leq \infty$*,* $\Lambda^{Schur}_{pcb}(H) \leq \Lambda^{Schur}_{pcb}(G)$.

Proof. The first point is by definition of weak amenability and of Λ_G.

The second assertion is obvious because for any $\varphi \in L^\infty(G)$, $\|\check{\varphi}\|_{cbMS^2(L^2(G))} = \|\varphi\|_\infty$. The next two assertions are consequences of Remarks 1.3 and 1.4. The last assertion is a consequence of Theorem 1.19 (remember that $A(G) \subset C(G)$). □

It is also natural to study the approximation by continuous functions with compact support. This yields to a property which might be weaker in general but which is equivalent when the group is discrete (by the proof of Theorem 2.5, we also get the same notion when G contains a lattice).

Lemma 2.4. *Let* G *be a locally compact second countable group, and* $1 \leq p \leq \infty$*. In the definition of* $\Lambda^{Schur}_{pcb}(G)$ *the functions* φ_α *can be taken in* $A(G) \cap C_c(G)$*.*

In particular when G *is discrete* $\Lambda^{Schur}_{pcb}(G)$ *is the smallest* C *such that there exists a net of functions with finite support* $\varphi_\alpha : G \to \mathbb{C}$ *such that* $\varphi_\alpha(g) \to 1$ *for all* $g \in G$ *and such that* $\|\check{\varphi}_\alpha\|_{cbMS^p(L^2(G))} \leq C$*.*

Proof. The first point is because $C_c(G)$ is dense in $A(G)$ and because, by Remark 1.4 and the inequality $\|\cdot\|_{M_0A(G)} \leq \|\cdot\|_{A(G)}$, for any $\varphi \in A(G)$,

$$\|\varphi\|_\infty \leq \|\check{\varphi}\|_{cbMS^p(L^2(G))} \leq \|\varphi\|_{M_0A(G)} \leq \|\varphi\|_{A(G)}.$$

The second statement is because $A(G)$ contains all functions with finite support when G is discrete. □

2.1 From a lattice to the whole group

In this subsection we prove that the property of completely bounded approximation by Schur multipliers on S^p for a group is equivalent to the same property for a lattice in this group. This was proved in [10] for $p = \infty$. With the tools developped in section 1, the proof is the very close to Haagerup's proof. The main result is :

Theorem 2.5. *Let* G *be a locally compact second countable group and* Γ *a lattice in* G*. Then for* $1 \leq p \leq \infty$

$$\Lambda^{Schur}_{pcb}(G) = \Lambda^{Schur}_{pcb}(\Gamma).$$

We now fix p, and G, Γ as in Theorem 2.5. We denote by μ a Haar measure on G (μ is a left and right Haar measure because a group containing a lattice is unimodular). Let Ω be a Borel fundamental domain of the action of Γ by right-multiplication on G, *i.e.* Ω is a Borel subset of G such that the restriction of the quotient map $G \to G/\Gamma$ is bijective. Since Γ is a lattice, Ω has finite Haar measure, and we can assume that it has measure 1. For $g \in G$ denote by $g = \omega(g)\gamma(g)$ the unique decomposition of g with $\omega(g) \in \Omega$ and $\gamma(g) \in \Gamma$.

For any bounded function $\psi : \Gamma \to \mathbb{C}$ we define $\varphi : G \to \mathbb{C}$ by

$$\varphi = \chi_\Omega \star \psi\mu_\Gamma \star \widetilde{\chi}_\Omega,$$

where μ_Γ is the counting measure on Γ, and χ_Ω (resp. $\widetilde{\chi}_\Omega$) is the characteristic function of Ω (resp. Ω^{-1}). Equivalently,

$$\varphi(g) = \int_\Omega \psi(\gamma(g\omega))d\mu(\omega).$$

Lemma 2.6. *For* $\psi : \Gamma \to \mathbb{C}$*,*

$$\|\check{\varphi}\|_{cbMS^p(L^2(G))} \leq \|\check{\psi}\|_{cbMS^p(L^2(\Gamma))}.$$

Proof. Since for any $h \in G$, the measure $\mu|_\Omega$ is invariant under $\omega' \mapsto \omega(h\omega')$, and since $gh^{-1}\omega(h\omega') = \omega(g\omega')\gamma(g\omega')\gamma(h\omega')^{-1}$, we get that

$$\varphi(gh^{-1}) = \int_\Omega \psi(\gamma(g\omega')\gamma(h\omega')^{-1})d\mu(\omega').$$

By Fubini's theorem it is enough to prove that for any $\omega' \in \Omega$ the Schur multiplier with symbol $(g,h) \mapsto \psi(\gamma(g\omega')\gamma(h\omega')^{-1})$ has cb-norm on $S^p(L^2(G))$ not larger than the cb-norm on $S^p(\ell^2(\Gamma))$ of the Schur multiplier with symbol $(\gamma,\gamma') \mapsto \psi(\gamma\gamma'^{-1})$. But since measure-theoretically, we have $G = \Gamma \times \Omega$ for the identification of g with $(\gamma(g\omega'), \omega(g\omega'))$ these Schur multipliers have in fact the same cb-norm, by Remark 1.6. □

We will also use the following Lemma from [10]. Since [10] is not easily available we reproduce a proof.

Lemma 2.7. $\|\varphi\|_{A(G)} \leq \|\psi\|_{A(\Gamma)}$.

Proof. If $\psi \in A(\Gamma)$ there exist $f, g \in \ell^2(\Gamma)$ such that $\|f\|_2\|g\|_2 = \|\psi\|_{A(\Gamma)}$ and $\varphi = f \star \widetilde{g}$ where $\widetilde{g}(\gamma) = g(\gamma^{-1})$. Put $f_1 = f\mu_\Gamma \star \chi_\Omega$ and $g_1 = g\mu_\Gamma \star \chi_\Omega$. Then $f_1 \star \widetilde{g}_1 = \varphi$ and hence

$$\|\psi\|_{A(G)} \leq \|f_1\|_{L^2(G)}\|g_1\|_{L^2(G)} = \|f\|_{\ell^2(\Gamma)}\|g\|_{\ell^2(\Gamma)} = \|\psi\|_{A(\Gamma)}.$$

□

Proof of Theorem 2.5. The inequality $\Lambda^{Schur}_{pcb}(G) \geq \Lambda^{Schur}_{pcb}(\Gamma)$ holds for any closed subgroup of G by Proposition 2.3.

For the other inequality, let $\psi_\alpha \in A(\Gamma)$ converging pointwise to 1 and such that $\sup_\alpha \|\check{\psi}_\alpha\|_{cbMS^p(\ell^2(\Gamma))} = \Lambda^{Schur}_{pcb}(\Gamma)$. Use Lemma 2.6 and define $\varphi^0_\alpha = \chi_\Omega \star \psi_\alpha \mu_\Gamma \star \widetilde{\chi}_\Omega$. Lemma 2.6 implies that $\|\check{\varphi}^0_\alpha\|_{cbMS^p(L^2(G))} \leq \|\check{\psi}_\alpha\|_{cbMS^p(\ell^2(\Gamma))}$. Also, by Lemma 2.7, $\varphi^0_\alpha \in A(G)$. However φ^0_α may not converge to 1 uniformly on compact subsets. Take $h \in C_c(G)^+$ (a continuous nonnegative function with compact support) such that $\int h d\mu = 1$, and define $\varphi_\alpha = h \star \varphi^0_\alpha$. Then $\check{\varphi}_\alpha$ is the average with respect to the probability measure $h(x)d\mu(x)$ of $(s,t) \mapsto \check{\varphi}_\alpha(sx,t)$. But for any x, the Schur multiplier with symbol $(s,t) \mapsto \check{\varphi}_\alpha(sx,t)$ has same norm as the multiplier with symbol $\check{\varphi}_\alpha$. This implies that $\|\check{\varphi}_\alpha\|_{cbMS^p(L^2(G))} \leq \|\check{\psi}_\alpha\|_{cbMS^p(\ell^2(\Gamma))}$. In the same way, since left translations by G act on $A(G)$ isometrically, $\varphi_\alpha \in A(G)$. The fact that $\lim_\alpha \varphi_\alpha(g) = 1$ follows from the dominated convergence Theorem in

$$\varphi_\alpha(g) = \int_G \int_\Omega h(gs^{-1}) \psi_\alpha(\gamma(s\omega'))d\mu(\omega')d\mu(s)$$

The convergence is uniform in compact subsets of G because the family $h(g\cdot)$, when g belong to a compact subset of G, is relatively compact in $L^1(G)$. □

3 The case of discrete groups

In this section we restrict ourselves to discrete groups and we study the relation between the property of completely bounded approximation by Schur multipliers on S^p and various other approximation properties. We prove that the AP of Haagerup and Kraus (see definition 3.6) implies $\mathrm{AP}^{Schur}_{pcb}$ for any $1 < p < \infty$. We also prove that for such p, if the non-commutative L^p-space associated to a discrete group has the OAP (or the stronger property CBAP), then this group has the property $\mathrm{AP}^{Schur}_{pcb}$. When G is hyperlinear, these results are consequences of [15]. Here we prove these results without the hypothesis of hyperlinearity. Since we are working in S^p instead of general non-commutative L^p-spaces, we are able to adapt the argument of [15] and give elementary proofs that avoid some technicalities (in particular we avoid the use of the results from the unpublished work [13]). The results in this section are however certainly well-known to experts, and the proofs standard. We also prove that, for discrete groups and $1 < p < \infty$, $\Lambda^{Schur}_{pcb}(G)$ can only take the two values 1 or ∞. All the aforementioned results are corollaries of a same result (Theorem 3.10) on the approximation, in the stable point-norm topology (see below for definitions), of the identity on a Schatten class.

For a discrete group G, we denote by τ_G the usual tracial state on the von Neumann algebra of G, and by $L^p(\tau_G)$ the corresponding non-commutative L^p space (for $1 \leq p \leq \infty$).

Before we give precise statements and proofs we have to recall some basic facts on the stable point-norm topology.

3.1 The stable point-norm topology

For an operator space V, we recall the definition of the *stable point-norm* topology $\mathcal{T}_{\mathrm{n}}$ on $CB(V,V)$: $\mathcal{T}_{\mathrm{n}}$ is the weakest topology making the seminorms $T \mapsto \|id \otimes T(x)\|$ for $x \in \mathcal{K}(\ell^2) \otimes_{\min} V = S^\infty[V]$ continuous. In this section we use the notation $S^\infty[V]$ for $\mathcal{K}(\ell^2) \otimes_{\min} V$.

We recall the definition of OAP, which was given in the introduction :

Definition 3.1. An operator space V has the operator space approximation property (OAP) if the identity on V belongs to the $\mathcal{T}_{\mathrm{n}}$-closure of the space $F(V,V)$ of finite rank operators on V.

We wish to study this notion when V is a non-commutative L^p-space $L^p(\mathcal{M},\tau)$. Non-commutative L^p spaces indeed have a natural operator space structure but, as explained in subsection 1.1, this structure is more simply described in terms of $L^p(B(\ell^2)\overline{\otimes}\mathcal{M}, Tr \otimes \tau)$ (Tr denotes the usual semi-finite trace on $B(\ell^2)$). Lemma 3.2 below will allow us to give a simpler equivalent definition of the topology $\mathcal{T}_{\mathrm{n}}$ in Definition 3.4.

Lemma 3.2 is a characterization of the topology $\mathcal{T}_{\mathrm{n}}$, in terms of vector-valued Schatten classes $S^p[V]$ defined in [24]. Except in the following two lemmas, in the remaining of the paper the notation $S^p[V]$ will only be used when $V = S^p(H)$ or $V = L^p(\tau_G)$ for a discrete group G. In this case the space $S^p[V]$ coincides with $S^p(\ell^2 \otimes H)$ or (if $p < \infty$) $L^p(Tr \otimes \tau)$.

Lemma 3.2. *Let $1 \le p \le \infty$. The topology $\mathcal{T}_{\mathrm{n}}$ on $CB(V,V)$ coincides with the topology defined by the family of seminorms $T \mapsto \sup_i \|id_{S^p} \otimes T(x_i)\|_{S^p[V]}$, for all $(x_i)_{i\ge 0} \in c_0(S^p[V])$.*

Remark 3.3. We view S^p as the increasing union of S^p_n, $n \ge 1$.

Let us denote by $\mathcal{T}^p_{\mathrm{n}}$ the topology described in this lemma. Since $\cup_n S^p_n[V]$ is dense in $S^p[V]$, this topology $\mathcal{T}^p_{\mathrm{n}}$ coincides with the topology defined by the seminorms $T \mapsto \sup_i \|id_{S^p} \otimes T(x_i)\|_{S^p[V]}$ for $(x_i)_{i\ge 0} \in c_0(\cup_n S^p_n[V])$. We will use this elementary fact in the proof below.

Proof. We first consider the case $p = \infty$ (note that by definition, $S^\infty[V] = \mathcal{K}(\ell^2) \otimes_{\min} V$). The inclusion $\mathcal{T}_{\mathrm{n}} \subset \mathcal{T}^\infty_{\mathrm{n}}$ is obvious. The other direction is classical and follows very easily from the fact that $\mathcal{K}(\ell^2) \otimes_{\min} \mathcal{K}(\ell^2) \otimes_{\min} V = \mathcal{K}(\ell^2 \otimes_2 \ell^2) \otimes_{\min} V$. Indeed if $x_i \in \mathcal{K}(\ell^2) \otimes V$ converges to 0, then $x = \oplus_i x_i$ belongs to $\mathcal{K}(\ell^2) \otimes_{\min} \mathcal{K}(\ell^2) \otimes_{\min} V$, and for any $T \in CB(V,V)$, $\|id \otimes T(x)\| = \sup_i \|id \otimes T(x_i)\|$.

Assume now $p < \infty$. We prove first that $\mathcal{T}^p_{\mathrm{n}} \subset \mathcal{T}^\infty_{\mathrm{n}}$. Take $(x_i)_{i\ge 0} \in c_0(S^p[V])$. By the properties of $S^p[V]$ (Theorem 1.5 in [24]), x_i can be written as $x_i = a_i \cdot v_i \cdot b_i$ with a_i, b_i in the unit ball of S^{2p} and $\|v_i\|_{S^\infty[V]} \le 2\|x_i\|_{S^p[V]}$ ($\|x_i\|_{S^p[V]}$ is in fact equal to the infimum of $\|v_i\|_{S^\infty[V]}$ over all such decompositions). In particular, $\lim_i \|v_i\|_{S^\infty[V]} = 0$, and moreover for any $T \in CB(V,V)$ and $n \in \mathbb{N}$, $\|id \otimes T(x_i)\|_{S^p[V]} \le \|id \otimes T(v_i)\|_{S^\infty[V]}$.

For the reverse inclusion $\mathcal{T}^\infty_{\mathrm{n}} \subset \mathcal{T}^p_{\mathrm{n}}$, we use the above remark for $p = \infty$. Let us consider $x_i \in M_{n_i}(V)$ such that $\|x_i\|_{M_{n_i}(V)} \to 0$. By Lemma 1.7 in [24], we have that

$$\|x_i\|_{M_{n_i}(V)} = \sup\left\{\|ax_ib\|_{S^p_{n_i}[V]}, a, b \text{ in the unit ball of } S^{2p}_{n_i}\right\}.$$

Consider a sequence $(y_{i,j})_{j\geq 0}$ in the ball of radius $\|x_i\|_{M_{n_i}(V)}$ in $S^p_{n_i}[V]$ converging to 0 and such that for any a, b in the unit ball of $S^{2p}_{n_i}$, $\|ax_ib\|_{S^p_{n_i}[V]}$ belongs to the closed convex hull of $\{y_{i,j}, j \geq 0\}$. Then $\lim_{|i|+|j|\to\infty} \|y_{i,j}\|_{S^p[V]} = 0$ and for any $T \in CB(V, V)$, and a, b in the unit ball of $S^{2p}_{n_i}$, we have that

$$\|(id \otimes T)(ax_ib)\|_{S^p_{n_i}[V]} \leq \sup_j \|(id \otimes T)(y_{i,j})\|_{S^p_{n_i}[V]}.$$

Hence,

$$\sup_i \|(id \otimes T)(x_i)\|_{M_{n_i}(V)} \quad \leq \quad \sup_i \sup_j \|(id \otimes T)(y_{i,j})\|_{S^p_{n_i}[V]}.$$

This concludes the proof of $\mathcal{T}^\infty_{\mathrm{n}} \subset \mathcal{T}^p_{\mathrm{n}}$ and of the Lemma. □

When $V = S^p$ or $V = L^p(\tau_G)$ (or more generally $V = L^p(\mathcal{M}, \tau)$ for a semi-finite normal faithful trace τ on $\mathcal{M}$), Lemma 3.2 shows that the definition of the topology $\mathcal{T}_{\mathrm{n}}$ and of the property OAP is equivalent to the following definition, which has the advantage not to rely on the precise definition of the operator space structure on V. In this definition G is a discrete group, and H a Hilbert space.

Definition 3.4. Let $1 \leq p \leq \infty$.

The topology $\mathcal{T}_{\mathrm{n}}$ on $CB(S^p(H), S^p(H))$ is the weakest topology making the seminorms $T \mapsto \sup_i \|id \otimes T(x_i)\|_{S^p(\ell^2 \otimes H)}$ for $(x_i)_{i\geq 0} \in c_0(S^p(\ell^2 \otimes H))$ continuous.

If $p < \infty$ the topology $\mathcal{T}_{\mathrm{n}}$ on $CB(L^p(\tau_G), L^p(\tau_G))$ is the weakest topology making the seminorms $T \mapsto \sup_i \|id \otimes T(x_i)\|_{L^p(Tr\otimes\tau_G)}$ for $(x_i)_{i\geq 0} \in c_0(L^p(Tr \otimes \tau_G))$ continuous.

$L^p(\tau_G)$ has OAP if the identity on $L^p(\tau_G)$ is in the $\mathcal{T}_{\mathrm{n}}$-closure of the space of finite rank operators on $L^p(\tau_G)$.

The reader unfamiliar with the notions of vector-valued S^p can start with this definition, forget Lemma 3.2 which will not be used later, and take in Lemma 3.5, $V = S^p(H)$ of $L^p(\tau_G)$, so that $S^p[V]$ is elementary.

Since the weak closure and the norm closure of a convex set coincide, we even get :

Lemma 3.5. *Let C be a convex subset of $CB(V, V)$, $u \in CB(V, V)$ and $1 \leq p \leq \infty$. Then u belongs to the $\mathcal{T}_{\mathrm{n}}$-closure of C if and only if for any $a \in S^p[V]$ and $b \in S^p[V]^*$, $\langle b, id \otimes u(a)\rangle$ belongs to the closure of*

$$\{\langle b, id \otimes T(a)\rangle, T \in C\}.$$

Proof. By Lemma 3.2, u belongs to the $\mathcal{T}_{\mathrm{n}}$-closure of C if and only if for any $(x_i)_{i\geq 0} \in c_0(S^p[V])$, $(id \otimes u(x_i))_{i\geq 0}$ belongs to the norm closure in $c_0(S^p[V])$ of

$$\left\{ (id \otimes T)(x_i))_{i\geq 0}\, , T \in C \right\}.$$

Since this latter set is convex, this is equivalent to saying that $((id \otimes u)(x_i))_{i\geq 0}$ belongs to its weak closure, *i.e.* that $\sum_i \langle b_i, (id \otimes u)(x_i)\rangle$ belongs to the closure of

$$\left\{ \sum_i \langle b_i, (id \otimes T)(x_i)\rangle, T \in C \right\}$$

for every $b_i \in (S^p[V])^*$ such that $\sum_i \|b_i\|_{(S^p[V])^*} < \infty$. Fix such $(x_i)_i \in c_0(S^p[V])$ and $(b_i)_i \in \ell^1(S^p[V]^*)$. We now construct $\widetilde{b} \in S^p[V]^*$ and $\widetilde{x} \in S^p[V]$ such that for any $T \in CB(V, V)$,

$$\sum_i \langle b_i, (id \otimes T)(x_i)\rangle = \langle \widetilde{b}, (id \otimes T)(\widetilde{x})\rangle \tag{12}$$

This will conclude the proof. Let $\lambda_i = \|b_i\|_{(S^{p'}[V])^*}$, and $\widetilde{b}_i = \lambda_i^{-1/p} b_i$ (with $0^{-1/p}0 = 0$) and $\widetilde{x}_i = \lambda_i^{1/p} x_i$. If X denotes the space $\ell^p(S^p[V])$ (if $p < \infty$) or $c_0(S^\infty[V])$ (if $p = \infty$), we therefore have that $\widetilde{b} = (\widetilde{b}_i)_{i\geq 0} \in \ell^{p'}(S^p[V]^*) \simeq X^*$ and $\widetilde{x} = (\widetilde{x}_i)_{i\geq 0} \in X$. Note that the space X is naturally contained in $S^p(\ell^2 \otimes \ell^2)[V]$ as a complemented subspace. Indeed, if $p < \infty$, $\ell^p(S^p)$ is naturally embedded in $S^p(\ell^2 \otimes \ell^2)$, and there is a completely positive projection $P : S^p(\ell^2 \otimes \ell^2) \to \ell^p(S^p)$ (the conditional expectation). By [23], Theorem 0.1, $P \otimes id_V$ extends to a bounded map on the vector-valued spaces. The same proof holds for $p = \infty$. The element $\widetilde{a} \in X^*$ therefore defines an element in the dual of $S^p(\ell^2 \otimes \ell^2)[V]$ (by $x \mapsto \widetilde{a}(Px)$), and with these identifications, (12) is easy to check. □

3.2 AP for groups and approximation on S^p

For facts on AP (Haagerup's and Kraus' approximation property) for discrete groups, see [4], Appendix D. For a discrete group G and a function $\varphi : G \to \mathbb{C}$ we denote by m_φ the corresponding Fourier multiplier on $C^*_{\mathrm{red}}(G)$ defined by $m_\varphi \lambda(s) = \varphi(s)\lambda(s)$. Recall that we denote also by $M_{\check{\varphi}}$ the corresponding Schur multiplier.

Definition 3.6. A discrete group G is said to have the approximation property (AP) if there is a net φ_α of functions from G to $\mathbb{C}$ with finite support and such that for any $a \in \mathcal{K}(\ell^2) \otimes_{\min} C^*_{\mathrm{red}}(G)$ and $f \in L^1(Tr \otimes \tau_G)$,

$$\lim_\alpha \langle f, id \otimes m_{\varphi_\alpha}(a)\rangle = \langle a, f\rangle.$$

Remark 3.7. The AP for a discrete group G implies that $id_{\mathcal{K}(\ell^2 G)}$ belongs to the $\mathcal{T}_{\mathrm{n}}$-closure in $CB(\mathcal{K}(\ell^2 G), \mathcal{K}(\ell^2 G))$ of

$$\{M_{\check{\varphi}}, \varphi : G \to \mathbb{C} \text{ of finite support.}\}.$$

Proof. By Lemma 3.5, we have to prove that for any $a \in \mathcal{K}(\ell^2) \otimes_{\min} \mathcal{K}(\ell^2 G)] = \mathcal{K}(\ell^2 \otimes \ell^2 G)$, and $b \in \mathcal{K}(\ell^2 \otimes \ell^2 G)^* = S^1(\ell^2 \otimes \ell^2 G)$, $\langle a, b\rangle$ belongs to the closure of

$$\{\langle b, (id \otimes M_{\check{\varphi}})(a)\rangle, \varphi \text{ of finite support}\}.$$

(we choose to denote by $\langle a, b\rangle$ the duality bracket $Tr(ab)$).

To do this consider the trace-preserving embedding $i : \mathcal{K}(\ell^2 G) \to \mathcal{K}(\ell^2 G) \otimes_{\min} C^*_{\text{red}}(G)$ defined on the dense subspace spanned by the elementary matrices $e_{s,t}$ for $s, t \in G$ by $i(e_{s,t}) = e_{s,t} \otimes \lambda(s^{-1}t)$. Let E be the conditional expectation. Then $E \circ id \otimes m_\varphi \circ i$ corresponds to the Schur multiplier with symbol $\check{\varphi}$. Hence for $a \in \mathcal{K}(\ell^2 \otimes \ell^2 G)$ and $b \in S^1(\ell^2 \otimes \ell^2 G)$,

$$\langle b, id \otimes M_{\check{\varphi}}(a)\rangle = \langle (id \otimes i)(b), (id \otimes m_\varphi) \circ (id \otimes i)(a)\rangle.$$

Since $id \otimes i(a)$ (resp. $id \otimes i(b)$) belongs to $\mathcal{K}(\ell^2 \otimes \ell^2(G)) \otimes_{\min} C^*_{red}(G)$ (resp. $L^1(Tr \otimes \tau_G)$, where Tr denotes the usual trace on $B(\ell^2 \otimes \ell^2 G)$), this proves the claim. □

Combining the above proof and the proof in [11] that the OAP for $C^*_{\text{red}}(G)$ implies the AP for G (the same idea was already used in [10], Theorem 2.6, to prove that the CBAP for $C^*_{\text{red}}(G)$ implies the weak amenability for G), we get the following Proposition :

Proposition 3.8. *Let G be a discrete group and $1 \leq p < \infty$. If $L^p(\tau_G)$ has the OAP, then the identity on $S^p(\ell^2 G)$ belongs to the $\mathcal{T}_{\text{n}}$-closure of the space*

$$\{M_{\check{\varphi}}, \varphi : G \to \mathbb{C} \textit{ of finite support}\}.$$

Proof. We use again Lemma 3.5. Since (see the proof of Lemma 2.4) the space $\{M_{\check{\varphi}}, \varphi : G \to \mathbb{C}$ of finite support$\}$ is norm-dense (for the cb-norm of linear maps on $S^p(\ell^2 G)$) in $\{M_{\check{\varphi}}, \varphi \in A(G)\}$, it is in fact enough to prove that for any $a \in S^p[S^p(\ell^2 G)] = S^p(\ell^2 \otimes \ell^2 G)$, and $b \in S^p(\ell^2 \otimes \ell^2 G)^*$, $\langle a, b\rangle$ belongs to the closure of

$$\{\langle b, (id \otimes M_{\check{\varphi}})(a)\rangle, \varphi \in A(G)\}.$$

For any finite rank map $T : L^p(\tau_G) \to L^p(\tau_G)$ define $\varphi_T : G \to \mathbb{C}$ by

$$\varphi_T(g) = \tau\left(T(\lambda(g))\lambda(g)^*\right). \tag{13}$$

We claim that $\varphi_T \in A(G)$. We even prove that $\varphi_T \in \ell^2(G)$. To prove this we can assume that T has rank one, *i.e.* is of the form $x \mapsto \xi(x)a$ for some $\xi \in L^p(\tau_G)^*$ and $a \in L^p(\tau_G)$. Then $\varphi_T(g) = \xi(\lambda(g))\tau_G(a\lambda(g)^*)$. If $p \geq 2$, then $|\xi(\lambda(g))| \leq \|\xi\|$ and

$$\|\left(\tau_G(a\lambda(g)^*)\right)_g\|_{\ell^2(G)} = \|a\|_{L^2(\tau_G)} \leq \|a\|_{L^p(\tau_G)}.$$

If $p < 2$ then since $L^p(\tau_G)^* \sim L^{p'}(\tau_G)$ with $1/p' + 1/p = 1$ (note $p' > 2$), the previous computation implies that $(\xi(\lambda(g)))_g$ belongs to $\ell^2(G)$ and $\tau_G(a\lambda(g)^*)$ is bounded.

We now prove that for any $a \in S^p[S^p(\ell^2G)] = S^p(\ell^2 \otimes \ell^2G)$, and $b \in S^p(\ell^2 \otimes \ell^2G)^*$, $\langle a, b\rangle$ belongs to the closure of

$$\{\langle b, (id \otimes M_{\check{\varphi}_T})(a)\rangle, T \in F(L^p(\tau_G), L^p(\tau_G))\}.$$

For simplicity of notation we prove the case $p > 1$. Then $S^p(\ell^2 \otimes \ell^2G)^* = S^{p'}(\ell^2 \otimes \ell^2G)$. The proof for $p = 1$ is the same, except that $S^{p'}(\ell^2 \otimes \ell^2G)$ has to be replaced by $B(\ell^2 \otimes \ell^2G)$. The inclusion i in the proof of Remark 3.7 induces a completely contractive map (that we still denote by the same letter) $i : S^p(\ell^2G) \to L^p(Tr \otimes \tau_G)$. Here Tr denotes the usual semi-finite trace on $S^p(\ell^2G)$. The same holds for p'. Moreover we have, for $a \in S^p(\ell^2 \otimes \ell^2G)$ and $b \in S^{p'}(\ell^2 \otimes \ell^2G)$,

$$\langle b, id \otimes M_{\check{\varphi}_T}(a)\rangle = \langle (id \otimes i)(b), (id \otimes T) \circ (id \otimes i)(a)\rangle.$$

But $(id \otimes i)(a)$ belongs to $S^p(\ell^2 \otimes \ell^2G)[L^p(\tau_G)]$ and $(id \otimes i)(b)$ belongs to its dual space $S^{p'}(\ell^2 \otimes \ell^2G)[L^{p'}(\tau_G)]$. Therefore, by the assumption that $L^p(\tau_G)$ has the OAP and by Lemma 3.5, $\langle b, a\rangle = \langle (id \otimes i)(b), (id \otimes T) \circ (id \otimes i)(a)\rangle$ belongs to the closure of

$$\{\langle (id \otimes i)(b), (id \otimes T) \circ (id \otimes i)(a)\rangle, T \in F(L^p(\tau_G), L^p(\tau_G))\}.$$

This proves the Proposition. ∎

The proof of the following Proposition is very close to the proof of Theorem 1.1 in [15]. In fact this Proposition also follows from Theorem 1.1 in [15] and from Proposition 3.8.

Proposition 3.9. *Let G be a discrete group with AP and $1 < p < \infty$. Then the identity on $S^p(\ell^2G)$ belongs to the $\mathcal{T}_{\mathrm{n}}$-closure of the space*

$$\{M_{\check{\varphi}}, \varphi : G \to \mathbb{C} \text{ of finite support}\}.$$

Proof. Denote $H = \ell^2 \otimes \ell^2G$. By Lemma 3.5, it is enough to prove that for any $a \in S^p(H)$ and $b \in S^{p'}(H)$, $Tr(ab) = \langle a, b\rangle$ belongs to the closure of

$$\{\langle a, (id \otimes M_{\check{\varphi}})(b)\rangle, \varphi : G \to \mathbb{C} \text{ with finite support}\}.$$

We prove this using the complex variable. We use the notation $S^\infty(H) = \mathcal{K}(H)$. Let S be the strip $\{z \in \mathbb{C}, 0 < Re(z) < 1\}$ and consider maps f, g in $C_0(\overline{S}; S^\infty(H))$ that are holomorphic on S, such that $f(1/p) = a$, $g(1/p) = b$ and such that $t \mapsto f(1 + it)$ belongs to $C_0(\mathbb{R}; S^1(H))$ and $t \mapsto g(it)$ belongs to $C_0(\mathbb{R}; S^1(H))$. Such maps exist because $S^p(H)$ coincides with the complex interpolation space $[S^\infty(H), S^1(H)]_{1/p}$, but they can be constructed explicitely. To construct f, write $a = a_0a_1$ with $a_0 \in S^\infty(H)$ and a_1 a positive element in $S^p(H)$, and take $f(z) = e^{(z-1/p)^2}a_0a_1^{pz}$. In the same way, write $b = b_0b_1$ with $b_0 \in S^\infty(H)$ and b_1 a positive element in $S^{p'}(H)$, and take $g(z) = e^{(z-1/p)^2}b_0b_1^{p'(1-z)}$.

Then the set $K = 0 \cup \{g(1+it)^T, t \in \mathbb{R}\} \cup \{f(it), t \in \mathbb{R}\}$ is a compact subset of $S^\infty(H)$ ($\cdot^T$ denotes the transpose map). It is classical that any compact subset containing 0 in a Banach space is contained in the closed convex hull of a sequence converging to 0. By the assumption that G has AP and by Remark 3.7, for any $\varepsilon > 0$, there is a $\varphi : G \to \mathbb{C}$ of finite support such that for any $x \in K$

$$\|(id \otimes M_{\check{\varphi}})x - x\|_{S^\infty(H)} < \varepsilon.$$

In particular, if $Re(z) = 0$

$$|\langle g(z), (id \otimes M_{\check{\varphi}})f(z) - f(z)\rangle| < \varepsilon \|g(z)\|_{S^1(H)}.$$

In the same way, since $Tr(x(id \otimes M_\varphi)(y)) = Tr(y^T(id \otimes M_\varphi)(x^T))$, we get that for $Re(z) = 1$

$$|\langle g(z), (id \otimes M_{\check{\varphi}})f(z) - f(z)\rangle| < \varepsilon \|f(z)\|_{S^1(H)}.$$

By the maximum principle, if $C = \max(\sup_{t \in R} \|g(it)\|_{S^1(H)}, \sup_{t \in R} \|f(1+it)\|_{S^1(H)})$, we get that for $z = 1/p$,

$$|\langle b, (id \otimes M_{\check{\varphi}})a\rangle - \langle b, a\rangle| < \varepsilon C.$$

Since ε is arbitrary, this concludes the proof. ☐

3.3 Different approximation properties on S^p

The main result of this section is the following Theorem (and its corollaries). This is in the same spirit as the theorem of Grothendieck which states that for a separable dual Banach space, the approximation property implies the metric approximation property. Its proof is an adaptation of Grothendieck's argument to the stable topology.

Theorem 3.10. *Let H be a Hilbert space and let F_0 be a subspace of the space $F(S^p(H), S^p(H))$ of bounded finite rank operators on $S^p(H)$, such that id_{S^p} belongs to the $\mathcal{T}_{\mathrm{n}}$-closure of F_0. Then id_{S^p} belongs to the $\mathcal{T}_{\mathrm{n}}$-closure of $\{T \in F_0, \|T\|_{cb} \leq 1\}$.*

Before we give the proof of this Theorem, let us state three corollaries.

Corollary 3.11. *If G is a discrete group and $1 < p < \infty$, then $\Lambda_{pcb}^{Schur}(G) = 1$ or $\Lambda_{pcb}^{Schur}(G) = \infty$.*

Proof. Note that $\Lambda_{pcb}^{Schur}(G) \leq c$ if and only if $id_{S^p(\ell^2 G)}$ belongs to the $\mathcal{T}_{\mathrm{n}}$-closure in $CB(S^p(\ell^2 G), S^p(\ell^2 G))$ of $\{M_{\check{\varphi}}, \varphi : G \to \mathbb{C}$ with finite support$\} \cap \{T, \|T\|_{cb} \leq c\}$. This Corollary therefore follows from Theorem 3.10 applied for the space F_0 consisting of the $M_{\check{\varphi}}$ for all $\varphi : G \to \mathbb{C}$ with finite support. ☐

Corollary 3.12. *If G is a discrete group with AP and $1 < p < \infty$, then $\Lambda_{pcb}^{Schur}(G) = 1$.*

Proof. This follows from Proposition 3.9 and from Theorem 3.10 applied for the space F_0 consisting of the $M_{\check{\varphi}}$ for all $\varphi : G \to \mathbb{C}$ with finite support. $\square$

Corollary 3.13. *If $1 < p < \infty$ and G is a discrete group such that $L^p(\tau_G)$ has the OAP (or the CBAP), then $\Lambda_{pcb}^{Schur}(G) = 1$.*

Proof. The CBAP is stronger than OAP. If $L^p(\tau_G)$ has the OAP, then by Proposition 3.8, the hypothesis in Theorem 3.10 holds with the space F_0 consisting of the $M_{\check{\varphi}}$ for all $\varphi : G \to \mathbb{C}$ with finite support. This implies that $\Lambda_{pcb}^{Schur}(G) = 1$. $\square$

The main tool in the proof of Theorem 3.10 will be the following Lemma, which expresses (in the vocabulary of [8], chapter 12) that a completely integral map on S^p is completely nuclear. Junge proved in the unpublished paper [13] that this holds for any non-commutative L^p-space (with $1 < p < \infty$) of a QWEP von Neumann algebra. We give an elementary statement and an elementary proof, due to Gilles Pisier :

Lemma 3.14. *Let $1 < p < \infty$ and let H_1, H_2 be Hilbert spaces and Ψ a linear map of norm less than 1 on $F(S^p(H_1), S^p(H_2))$ equipped with the completely bounded norm. Then there exist $x \in S^p(\ell^2 \otimes H_1)$ and $y \in S^{p'}(\ell^2 \otimes H_2)$ satisfying $\|x\|_p \|y\|_{p'} < 1$ and such that*

$$\Psi(T) = \langle y, (id \otimes T)(x) \rangle \text{ for any } T \in F(S^p(H_1), S^p(H_2)).$$

In particular, Ψ extends to a $\mathcal{T}_{\mathrm{n}}$-continuous linear map on $CB(S^p(H_1), S^p(H_2))$ of norm less than 1.

Proof. For any linear map $\Psi : F(S^p(H_1), S^p(H_2)) \to \mathbb{C}$, denote

$$N_1(\Psi) = \sup_{T \in F(S^p(H_1), S^p(H_2)), \|T\|_{cb} \le 1} |\Psi(T)|$$
$$N_2(\Psi) = \inf_{x \in S^p(\ell^2 \otimes H_1), y \in S^{p'}(\ell^2 \otimes H_2)} \|x\|_p \|y\|_{p'},$$

where the infimum is taken over all x, y satisfying $\Psi(T) = \langle y, id \otimes T(x) \rangle$ for all $T \in F(S^p(H_1), S^p(H_2))$. For $i = 1, 2$, N_i is a norm which makes $\{\Psi, N_i(\Psi) < \infty\}$ a Banach space, and obviously $N_1 \le N_2$. We prove that $N_1 = N_2$. When H_1 or H_2 is finite dimensional, this is classical and very easy : namely, for $i = 1$ or 2, the space $\{\Psi : F(S^p(H_1), S^p(H_2)) = CB(S^p(H_2), S^p(H_2)) \to \mathbb{C}$ linear bounded$\}$ coincides (as a vector space) with $S^p(H_1) \otimes S^p(H_2)^*$, and when equipped with the norm N_i, its dual space is naturally $CB(S^p(H_1), S^p(H_2))$ with the norm $\| \cdot \|_{cb}$.

If K is a closed subspace of H_1, denote by $e_K \in B(H_1)$ the orthogonal projection on K and $P_K : A \in S^p(H_1) \mapsto e_K A e_K \in S^p(H_1)$. Denote also by Ψ_K the map $T \in F(S^p(H_1), S^p(H_2)) \to \Psi(T P_K)$. By the case $\dim(H_1) < \infty$, we have that $N_1(\Psi_K) = N_2(\Psi_K)$ for any finite dimensional subspace K of H_1.

If $\{0\} = K_0 \subset K_1 \subset K_2 \subset \dots K_N$ is an increasing family of orthogonal finite dimensional subspaces of H_1 and if $q = \max(p, 2)$, we claim that

$$\frac{1}{2}\left(\sum_{n=1}^{N} N_2(\Psi_{K_n} - \Psi_{K_{n-1}})^q\right)^{1/q} \leq N_2(\Psi_{K_N}) = N_1(\Psi_{K_N}) \leq N_1(\Psi). \qquad (14)$$

The middle equality has already been proved, and the second inequality is obvious. The first inequality follows from the following inequality valid for any $x \in S^p(\ell^2 \otimes H_1)$:

$$\left(\sum_{n=1}^{N} \|(id \otimes P_n - id \otimes P_{n-1})(x)\|_p^q\right)^{1/q} \leq 2\|x\|_p,$$

which follows from the inequalities, valid for any family $(q_n)_{n\geq 1}$ of orthogonal projections on $\ell^2 \otimes H_1$

$$\left(\sum_{n=1}^{N} \|q_n x\|_p^q\right)^{1/q} \leq \|x\|_p$$

$$\left(\sum_{n=1}^{N} \|x q_n\|_p^q\right)^{1/q} \leq \|x\|_p.$$

When $p \geq 2$ this can be proved using the triangle inequality in $S^{p/2}$. When $p = 1$, this can be proved using the fact that the unit ball in S^1 is the closed convex hull of rank one operators, and for $p < 2$, this follows by interpolation between $p = 1$ and $p = 2$.

(14) then implies that the net (Ψ_K) (for K a finite dimensional subspace of H_1) is Cauchy for N_2, *i.e.* for any ε there exists a finite dimensional subspace K_ε such that for any finite dimensional K containing K_ε, $N_2(\Psi_K - \Psi_{K_\varepsilon}) < \varepsilon$. This implies that it converges for the norm N_2 to an element of N_2-norm not greater than $N_1(\Psi)$. This limit is Ψ, which shows that $N_2(\Psi) \leq N_1(\Psi)$ and which concludes the proof of $N_2 = N_1$.

The second statement of the Lemma is then immediate, because for $x \in S^p(\ell_2 \otimes H_1)$ and $y \in S^{p'}(\ell^2 \otimes H_2)$, the formula $T \mapsto \langle y, (id \otimes T)(x)\rangle$ defines a $\mathcal{T}_{\rm n}$-continuous map on $CB(S^p(H_1), S^p(H_2))$. $\square$

Proof of Theorem 3.10. This proof relies on the Hahn-Banach Theorem. For convenience we denote $S^p(H)$ simply by S^p. Let $\Phi : CB(S^p, S^p) \to \mathbb{C}$ be a $\mathcal{T}_{\rm n}$-continuous linear form such that $|\Phi(T)| \leq 1$ for all $T \in F_0$ with $\|T\|_{cb} \leq 1$ (equivalently $|\Phi(T)| \leq \|T\|_{cb}$ for all $T \in F_0$). The aim is to prove that $|\Phi(id_V)| \leq 1$. For this we show that for any $\varepsilon > 0$, Φ coincides on the space F_0 with a linear map Ψ on $CB(S^p, S^p)$, which is also $\mathcal{T}_{\rm n}$-continuous and for which $\|\Psi\| \leq 1 + \varepsilon$. This would conclude the proof because then $\Psi = \Phi$ on the $\mathcal{T}_{\rm n}$-closure of F_0, and in particular $\Phi(id_{S^p}) = \Psi(id_{S^p})$ is less than $1 + \varepsilon$.

The restriction of Φ to F_0 is of norm 1. By Hahn-Banach it extends to a norm 1 functional Φ_1 on $F(S^p, S^p)$. By Lemma 3.14, for any $\varepsilon > 0$, Φ_1 extends to a $\mathcal{T}_{\rm n}$-continuous map Ψ on $CB(S^p, S^p)$ of norm less than $1 + \varepsilon$. $\square$

4 Case of $SL_{r+1}(F)$

The aim of this section is to prove Theorem D. This is done at the end of this section, as a consequence of Proposition 4.1.

Let $p > 2$. Let $n \in \mathbb{N}^*$ such that $p > 2 + \frac{2}{n}$. Set

$$\varepsilon = n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{p} = \frac{n}{2p}\big(p - (2 + \frac{2}{n})\big) \in \mathbb{R}_+^*.$$

Let $r \in \mathbb{N}^*$ such that $r \geq 2n$, F be a non-archimedian local field and $\mathcal{O}$ its ring of integers. Let $G = SL_{r+1}(F)$ and $K = SL_{r+1}(\mathcal{O})$ which is a maximal compact subgroup of G.

Proposition 4.1. *The constant function* 1 *on* G *cannot be approximated (for the topology of uniform convergence on compact subsets) by functions* f *in* $C_0(G)$ *such that* $\|\check{f}\|_{MS^p(L^2(G))}$ *is bounded uniformly. In particular,*

$$\Lambda_{pcb}^{Schur}(SL_{r+1}(F)) = \infty.$$

This proposition follows from

Proposition 4.2. *The constant function* 1 *on* G *cannot be approximated (for the topology of uniform convergence on compact subsets) by* K*-biinvariant functions* f *in* $C_0(G)$ *such that* $\|\check{f}\|_{MS^p(L^2(G))}$ *is bounded uniformly.*

Proof of Proposition 4.2 using Proposition 4.1. Averaging on the left and on the right by K one sees that it is enough to show that one cannot approximate 1 by K-biinvariant functions in $C_0(G)$ uniformly bounded for $\|\check{f}\|_{MS^p(L^2(G))}$. $\square$

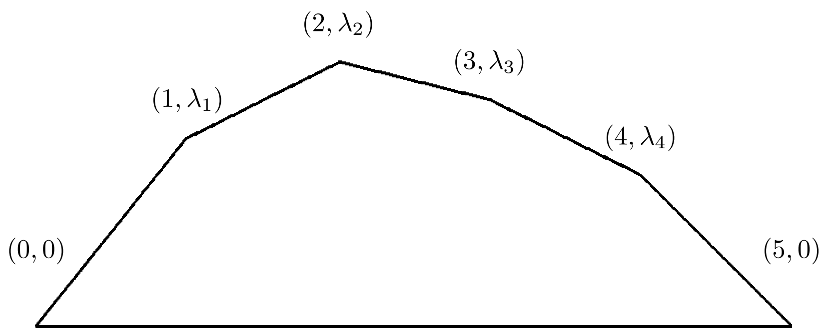
Let π be a uniformizer of $\mathcal{O}$, and let $\mathcal{O}^\times$ denote the units (or invertibles) of $\mathcal{O}$. Denote by $\mathbb{F} = \mathcal{O}/\pi\mathcal{O}$ the residue field of F. To define an absolute value $|\cdot|$ on F we have to choose $|\pi| \in (0,1)$. Then $|\cdot|$ is defined in the following way : $|x| = |\pi|^\lambda$ if $x \in \pi^\lambda\mathcal{O}^\times$ for $\lambda \in \mathbb{Z}$ and $|x| = 0$ if $x = 0$. The standard choice is to take $|\pi| = q^{-1}$, because with this choice $d(xa) = |x|da$ for any $x \in F$, where da denotes a Haar measure on F. Since we do not use this property, we prefer to keep the choice of $|\pi| \in (0,1)$ arbitrary. The coefficients of the matrices below are easier to understand if they are written as powers of π^{-1} instead of powers of π. To keep the size of matrices reasonnable we introdude the notation $e = \pi^{-1}$, so that $|e| = |\pi|^{-1}$ is an arbitrary number in $(1, \infty)$. The important property of $|\cdot|$ is that it is non-archimedian, *i.e.* the triangle inequality has the stronger form $|x + y| \leq \max(|x|, |y|)$ for any $x, y \in F$.

Remark 4.3. The reader unfamiliar with these notions can consider the special case where q is a prime number and $F = \mathbb{Q}_q$ (we avoid the usual notation $\mathbb{Q}_p$ because the letter p is already used). Note that $\mathbb{Q}_q$ is the field obtained by completion of $\mathbb{Q}$ for the distance given by the absolute value on $\mathbb{Q}$, $|a/b| = |q|^{v_q(a) - v_q(b)}$, where $|q| \in (0,1)$ is arbitrary and $v_q(a)$ is the greatest k such that q^k divides a (the resulting field does not depend on the choice of $|q| \in (0,1)$). In the special case where $F = \mathbb{Q}_q$, $\mathcal{O}$ is $\mathbb{Z}_q$, the unit ball in $\mathbb{Q}_q$ (or equivalently the closure of $\mathbb{Z}$), a convenient choice for π is to simply take $\pi = q$ and the residue field is $\mathbb{Z}/q\mathbb{Z}$.

Let

$$\Lambda = \{(\lambda_1, ..., \lambda_r) \in \mathbb{N}^r, \lambda_1 \geq \lambda_2 - \lambda_1 \geq \lambda_3 - \lambda_2 \geq ... \geq \lambda_r - \lambda_{r-1} \geq -\lambda_r\}.$$

For $(\lambda_1, ..., \lambda_r) \in \mathbb{N}^r$ denote by $P(\lambda_1, ..., \lambda_r)$ the polygon whose vertices are the points (i, λ_i) for $i \in \{0, ..., r+1\}$, setting $\lambda_0 = 0$ and $\lambda_{r+1} = 0$. Then Λ is the set of $(\lambda_1, ..., \lambda_r) \in \mathbb{N}^r$ such that $P(\lambda_1, ..., \lambda_r)$ is convex (or equivalently such that the piecewise affine map on $[0, r+1]$ taking values λ_i on i is concave). The $\lambda_{i+1} - \lambda_i$ for $i \in \{0, ..., r\}$ are the slopes of the polygon and $2\lambda_i - \lambda_{i-1} - \lambda_{i+1}$ is called break at vertex i, for $i \in \{1, ..., r\}$. A polygon is convex if all its breaks are nonnegative. The picture below gives an example for $r = 4$.



For $(\lambda_1, ..., \lambda_r) \in \Lambda$ denote

$$D(\lambda_1, ..., \lambda_r) = \begin{pmatrix} e^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2 - \lambda_1} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & e^{\lambda_r - \lambda_{r-1}} & 0 \\ 0 & \dots & 0 & 0 & e^{-\lambda_r} \end{pmatrix} \in G,$$

where the exponants of e are the slopes of the polygon $P(\lambda_1, ..., \lambda_r)$.

The map associating $KD(\lambda_1, ..., \lambda_r)K$ to $(\lambda_1, ..., \lambda_r) \in \Lambda$ induces a bijection between Λ et $K \backslash G / K$.

For a matrix $A = (a_{kl})$ denote $\|A\| = \max(|a_{kl}|)$. Then for $A \in G$,

$$A \in KD(\lambda_1, ..., \lambda_r)K$$
$$\text{if and only if } \|\Lambda^i A\| = |e|^{\lambda_i} \text{ for all } i \in \{1, ..., r\}. \tag{15}$$

More concretely $\|\Lambda^i A\|$ is the maximum of the norms of all $i \times i$-minors of A. When $A \in KD(\lambda_1, ..., \lambda_r)K$ one says that $P(\lambda_1, ..., \lambda_r)$ is the polygon of A. The reason why we introduce these polygons is that the λ_i are more convenient parameters than the slopes $\lambda_{i+1} - \lambda_i$ (see (15) above and lemma 4.6 below) and that the convexity condition satisfied by the λ_i is best seen by drawing the polygon.

Denote by B the Borel subgroup of G (formed of upper-triangular matrices).

Proposition 4.4. *For any function $f \in C_c(G)$, let $g = f|_B \in C_c(B)$ be the restriction of f to B. Then*

$$\|\check{g}\|_{MS^p(L^2(B))} \leq \|\check{f}\|_{MS^p(L^2(G))}.$$

If f is K-biinvariant it is an equality: $\|\check{g}\|_{MS^p(L^2(B))} = \|\check{f}\|_{MS^p(L^2(G))}$.

Remark 4.5. The notation $\check{f}$ was introduced at the beginning of section 2. Note that by Theorem 1.19, the norms of all the multipliers appearing in this proposition are equal to their cb-norms.

Proof. For $p = \infty$ this is proved in proposition 1.6 of [6].

For general p it is a consequence of the results in section 1. The first point follows from Theorem 1.19. Moreover since B and G are both without isolated points (and the Haar measure has full support) Theorem 1.19 implies that $\|\check{g}\|_{MS^p(L^2(B))} = \|\check{g}\|_{cbMS^p(L^2(B))}$ and $\|\check{f}\|_{MS^p(L^2(G))} = \|\check{f}\|_{cbMS^p(L^2(G))}$. But by Lemma 1.13, since $G/K = B/(B \cap K)$, both terms $\|\check{g}\|_{cbMS^p(L^2(B))}$ and $\|\check{f}\|_{cbMS^p(L^2(G))}$ are equal to $\|\check{f}\|_{cbMS^p(L^2(G/K))}$. $\square$

Lemma 4.6. *There is a constant C such that for all K-biinvariant $f \in C_c(G)$, for $(\lambda_1, ..., \lambda_r) \in \Lambda$ and $i \in \{1, ..., r\}$ such that*

$$(\lambda_1, ..., \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, ..., \lambda_r) \in \Lambda,$$

one has

$$\begin{aligned} &\left|f(D(\lambda_1, ..., \lambda_r)) - f(D(\lambda_1, ..., \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, ..., \lambda_r))\right| \\ &\quad \leq Cq^{-\varepsilon(2\lambda_{i+1} - \lambda_i - \lambda_{i+2})} \|\check{f}\|_{MS_p(L^2(G))} \text{ if } \quad r - i \geq n \qquad (16) \\ \text{and } &\left|f(D(\lambda_1, ..., \lambda_r)) - f(D(\lambda_1, ..., \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, ..., \lambda_r))\right| \\ &\quad \leq Cq^{-\varepsilon(2\lambda_{i-1} - \lambda_{i-2} - \lambda_i)} \|\check{f}\|_{MS_p(L^2(G))} \text{ for } \ i - 1 \geq n. \qquad (17) \end{aligned}$$

The following lemma is very close to Lemma 5.5 in [18] (and of the estimates following).

Let $m \in \mathbb{N}^*$. For $k \in \{0, ..., m\}$ let us denote by

$$T_k = \left((T_k)_{(a_1,...,a_n,b),(x_1,...,x_n,y)}\right)_{(a_1,...,a_n,b) \in (\mathcal{O}/\pi^m\mathcal{O})^{n+1}, (x_1,...,x_n,y) \in (\mathcal{O}/\pi^m\mathcal{O})^{n+1}}$$

the matrix defined by

$$\begin{aligned} (T_k)_{(a_1,...,a_n,b),(x_1,...,x_n,y)} &= q^{-mn} \text{ if } y = \sum_{i=1}^{n} a_i x_i + b + \pi^k \text{ in } \mathcal{O}/\pi^m\mathcal{O}, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Lemma 4.7. *One has*

$$\|T_m - T_{m-1}\|_{S_p} \leq 2q^{-\varepsilon m} \tag{18}$$

and for $u, v \in \mathbb{C}$ one has

$$\|uT_m - vT_{m-1}\|_{S_p} \geq |u - v|. \tag{19}$$

Proof. Since

$$(T_m)_{(a_1,...,a_n,b),(x_1,...,x_n,y)} \text{ and } (T_{m-1})_{(a_1,...,a_n,b),(x_1,...,x_n,y)}$$

only depend on $y - b$ one has

$$\|T_m - T_{m-1}\|_{S_p}^p$$
$$= \sum_{\eta \in \widehat{\mathcal{O}/\pi^m\mathcal{O}}} \left|1 - \eta(\pi^{m-1})\right|^p \left\| q^{-mn}(\eta(\sum_{i=1}^n a_i x_i))_{(a_1,...,a_n),(x_1,...,x_n)\in(\mathcal{O}/\pi^m\mathcal{O})^n} \right\|_{S_p}^p$$
$$= \sum_{\eta \in \widehat{\mathcal{O}/\pi^m\mathcal{O}}} \left|1 - \eta(\pi^{m-1})\right|^p \left\| q^{-m}(\eta(ax))_{a,x\in\mathcal{O}/\pi^m\mathcal{O}} \right\|_{S_p}^{pn}.$$

If $1-\eta(\pi^{m-1}) \neq 0$ one has $|1-\eta(\pi^{m-1})| \leq 2$ and η is a nondegenerated character of $\mathcal{O}/\pi^m\mathcal{O}$. But for such a character

$$\left\| q^{-m}(\eta(ax))_{a,x\in\mathcal{O}/\pi^m\mathcal{O}} \right\|_{S_p} = q^{-\frac{m}{2}+\frac{m}{p}}$$

because the matrix

$$q^{-\frac{m}{2}}(\eta(ax))_{a,x\in\mathcal{O}/\pi^m\mathcal{O}}$$

is unitary (as a matrix of a Fourier transform). But there are exactly $(1-\frac{1}{q})q^m$ non degenerated characters of $\mathcal{O}/\pi^m\mathcal{O}$. One thus has

$$\|T_m - T_{m-1}\|_{S_p}^p \leq (1-\frac{1}{q})q^m 2^p q^{-\frac{npm}{2}+nm} \leq 2^p q^{(1-\frac{np}{2}+n)m} = 2^p q^{-p\varepsilon m},$$

which proves (18).

The inequality (19) holds because the vector in $\ell^2((\mathcal{O}/\pi^m\mathcal{O})^{n+1})$ with coordinates all equal to 1 is an eigenvector for T_m and T_{m-1} with eigenvalue 1. Hence it is an eigenvector of $uT_m - vT_{m-1}$ with eigenvalue $u - v$. □

The following Lemma is a rephrasing of Theorem 1.19.

Lemma 4.8. *Let $k \in \mathbb{N}$, $A \in M_k(\mathbb{C})$, H a locally compact group, $f \in C_c(H)$ and $\alpha, \beta : \{1, ..., k\} \to H$ two injective maps. Then*

$$\left\| \big(f(\alpha(i)\beta(j))A_{ij}\big)_{i,j\in\{1,...,k\}} \right\|_{S^p} \leq \|\check{f}\|_{MS^p(L^2(H))} \|A\|_{S^p}.$$

Proof. Theorem 1.19 implies this with $f(\alpha(i)^{-1}\beta(j))$ instead of $f(\alpha(i)\beta(j))$, but the two versions are equivalent. □

We use a combination of the two preceding lemmas.

Lemma 4.9. *Let $m \in \mathbb{N}^*$. Let H be a locally compact group and $f \in C_c(H)$. Let $\alpha, \beta : (\mathcal{O}/\pi^m\mathcal{O})^{n+1} \to H$ be two injective applications and $u, v \in \mathbb{C}$ such that*

$$f(\alpha(a_1,...,a_n,b)\beta(x_1,...,x_n,y)) = u \text{ if } y = \sum_{i=1}^{n} a_i x_i + b \text{ in } \mathcal{O}/\pi^m\mathcal{O} \tag{20}$$

$$f(\alpha(a_1,...,a_n,b)\beta(x_1,...,x_n,y)) = v \text{ if } y = \sum_{i=1}^{n} a_i x_i + b + \pi^{m-1} \text{ in } \mathcal{O}/\pi^m\mathcal{O}. \tag{21}$$

Then $|u-v| \leq 2q^{-\varepsilon m}\|\check{f}\|_{MS_p(L^2(H))}$.

Proof. By Lemma 4.8 applied to $A = T_m - T_{m-1}$, one has $\|uT_m - vT_{m-1}\|_{S_p} \leq \|\check{f}\|_{MS_p(L^2(H))}\|T_m - T_{m-1}\|_{S_p}$. One then applies the inequalities (18) and (19) of Lemma 4.7. □

Proof of Lemma 4.6. The estimate (17) can be deduced from the estimate (16) by the automorphism

$$\theta : A \mapsto \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & \diagup & \diagup & 0 \\ 0 & \diagup & \diagup & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} {}^tA^{-1} \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & \diagup & \diagup & 0 \\ 0 & \diagup & \diagup & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

of G, which preserves K and B. Indeed $\theta(D(\lambda_1,...,\lambda_r)) = D(\lambda_r,...,\lambda_1)$. It is thus enough to prove (16).

Let $(\lambda_1,...,\lambda_r) \in \Lambda$ and $i \in \{1,...,r-n\}$ such that

$$(\lambda_1,...,\lambda_{i-1},\lambda_i+1,\lambda_{i+1},...,\lambda_r) \in \Lambda. \tag{22}$$

Set $\lambda_0 = 0$ and $\lambda_{r+1} = 0$. Denote by $\mu_1,...,\mu_{r+1}$ the slopes of the polygon $P(\lambda_1,...,\lambda_r)$, *i.e.* $\mu_i = \lambda_i - \lambda_{i-1}$. Since $(\lambda_1,...,\lambda_r) \in \Lambda$ one has $\mu_1 \geq \mu_2 \geq ... \geq \mu_{r+1}$ and moreover $\sum_{i=1}^{r+1}\mu_i = 0$. The condition (22) is equivalent to

$$\mu_{i-1} > \mu_i \quad \text{and} \quad \mu_{i+1} > \mu_{i+2} \tag{23}$$

because the slopes of the polygon

$$P(\lambda_1,...,\lambda_{i-1},\lambda_i+1,\lambda_{i+1},...,\lambda_r)$$

are

$$(\mu_1,...,\mu_{i-1},\mu_i+1,\mu_{i+1}-1,\mu_{i+2}...,\mu_{r+1}).$$

We are going to apply Lemma 4.9 with

$$H = B \text{ and } m = \mu_{i+1} - \mu_{i+2} = 2\lambda_{i+1} - \lambda_i - \lambda_{i+2} \in \mathbb{N}^*. \tag{24}$$

In other words, m is the break of $P(\lambda_1, ..., \lambda_r)$ at vertex $i+1$.

Let us fix a section $\sigma : \mathcal{O}/\pi^m\mathcal{O} \to \mathcal{O}$ of the projection $\mathcal{O} \to \mathcal{O}/\pi^m\mathcal{O}$. The choice of this section has no importance.

Let us define two maps $\alpha, \beta : (\mathcal{O}/\pi^m\mathcal{O})^{n+1} \to B$ (where B is the subgroup of upper-triangular matrices in SL_{r+1}) in the following way :

$$\alpha(a_1, ..., a_n, b) =$$

$$\begin{pmatrix} e^{\mu_1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & e^{\mu_{i-1}} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha'(a_1, ..., a_n, b) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & e^{\mu_{i+n+2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & e^{\mu_{r+1}} \end{pmatrix}$$

$$\text{and} \quad \beta(x_1, ..., x_n, y) =$$

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \beta'(x_1, ..., x_n, y) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

where the matrices are block-diagonal with all blocks of size 1 except the blocks $\alpha'(a_1, ..., a_n, b)$ and $\beta'(x_1, ..., x_n, y)$ which are square matrices of size $n+2$. The position of the block $\beta'(x_1, ..., x_n, y)$ is the same as the position of the block $\alpha'(a_1, ..., a_n, b)$, so that

$$\alpha(a_1, ..., a_n, b)\beta(x_1, ..., x_n, y) =$$

$$\begin{pmatrix} e^{\mu_1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & e^{\mu_{i-1}} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha'(a_1, ..., a_n, b)\beta'(x_1, ..., x_n, y) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & e^{\mu_{i+n+2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & e^{\mu_{r+1}} \end{pmatrix}.$$

The matrices $\alpha'(a_1, ..., a_n, b)$ and $\beta'(x_1, ..., x_n, y)$ are defined by

$$\alpha'(a_1, ..., a_n, b) =$$

$$\begin{pmatrix} e^{\mu_i} & -e^{\mu_i}\sigma(a_1) & -e^{\mu_i}\sigma(a_2) & \dots & -e^{\mu_i}\sigma(a_n) & -e^{\mu_i+\mu_{i+1}-\mu_{i+2}}\sigma(b) \\ 0 & e^{\mu_{i+2}} & 0 & \dots & \dots & 0 \\ \vdots & \ddots & e^{\mu_{i+3}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & e^{\mu_{i+n+1}} & 0 \\ 0 & \dots & \dots & \dots & 0 & e^{\mu_{i+1}} \end{pmatrix}$$

$$\text{and} \quad \beta'(x_1, ..., x_n, y) =$$

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 & e^{\mu_{i+1}-\mu_{i+2}}\sigma(y) \\ 0 & 1 & \ddots & \ddots & \vdots & e^{\mu_{i+1}-\mu_{i+2}}\sigma(x_1) \\ \vdots & \ddots & 1 & \ddots & \vdots & e^{\mu_{i+1}-\mu_{i+2}}\sigma(x_2) \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & e^{\mu_{i+1}-\mu_{i+2}}\sigma(x_n) \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$$

Let us compute

$$\alpha'(a_1, ..., a_n, b)\beta'(x_1, ..., x_n, y) =$$

$$\begin{pmatrix} e^{\mu_i} & -e^{\mu_i}\sigma(a_1) & -e^{\mu_i}\sigma(a_2) & \dots & -e^{\mu_i}\sigma(a_n) & e^{\mu_i+\mu_{i+1}-\mu_{i+2}}w \\ 0 & e^{\mu_{i+2}} & 0 & \dots & \dots & e^{\mu_{i+1}}\sigma(x_1) \\ \vdots & \ddots & e^{\mu_{i+3}} & \ddots & \ddots & e^{\mu_{i+1}-\mu_{i+2}+\mu_{i+3}}\sigma(x_2) \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & e^{\mu_{i+n+1}} & e^{\mu_{i+1}-\mu_{i+2}+\mu_{i+n+1}}\sigma(x_n) \\ 0 & \dots & \dots & \dots & 0 & e^{\mu_{i+1}} \end{pmatrix}$$

with $w = \sigma(y) - \left(\sum_{i=1}^{n} \sigma(a_i)\sigma(x_i) + \sigma(b)\right) \in \mathcal{O}$.

We are going to study the following cases :

- i) $w = 0 \mod \pi^m \mathcal{O}$
- ii) $w = \pi^{m-1} \mod \pi^m \mathcal{O}$.

Since $m = \mu_{i+1} - \mu_{i+2}$ by (24) and $e = \pi^{-1}$, one has

$$e^{\mu_i+\mu_{i+1}-\mu_{i+2}}w \in \pi^{-\mu_i}\mathcal{O} \quad \text{in case i)}$$

$$\text{and} \quad e^{\mu_i+\mu_{i+1}-\mu_{i+2}}w \in \pi^{-\mu_i-1} + \pi^{-\mu_i}\mathcal{O} \quad \text{in case ii).}$$

Since $\mu_i \geq \mu_{i+1} \geq ... \geq \mu_{i+n+1}$, it follows that in case i)

$$\|\alpha'(a_1, ..., a_n, b)\beta'(x_1, ..., x_n, y)\| = |e|^{\mu_i}$$

whereas in case ii)

$$\|\alpha'(a_1,...,a_n,b)\beta'(x_1,...,x_n,y)\| = |e|^{\mu_i+1}.$$

Thanks to the second inequality in (23) one checks that in both cases, for all $j \in \{2,...,n+2\}$,

$$\left\|\Lambda^j\left(\alpha'(a_1,...,a_n,b)\beta'(x_1,...,x_n,y)\right)\right\| = |e|^{\mu_i+\mu_{i+1}+...+\mu_{i+j-1}}.$$

As a consequence, $\alpha'(a_1,...,a_n,b)\beta'(x_1,...,x_n,y)$ belongs to

$$GL_{n+2}(\mathcal{O})\begin{pmatrix} e^{\mu_i} & 0 & \dots & \dots & 0 \\ 0 & e^{\mu_{i+1}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & e^{\mu_{i+2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & e^{\mu_{i+n+1}} \end{pmatrix} GL_{n+2}(\mathcal{O})$$

in case i) and to

$$GL_{n+2}(\mathcal{O})\begin{pmatrix} e^{\mu_i+1} & 0 & \dots & \dots & 0 \\ 0 & e^{\mu_{i+1}-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & e^{\mu_{i+2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & e^{\mu_{i+n+1}} \end{pmatrix} GL_{n+2}(\mathcal{O})$$

in case ii).

Thanks to condition (23), it follows that $\alpha(a_1,...,a_n,b)\beta(x_1,...,x_n,y)$ belongs to

$$KD(\lambda_1,...,\lambda_r)K \text{ in case i)}$$
$$\text{and to } KD(\lambda_1,...,\lambda_{i-1},\lambda_i+1,\lambda_{i+1},...,\lambda_r)K \text{ in case ii).}$$

The hypotheses of Lemma 20 are therefore satisfied with

$$H = B, u = f(D(\lambda_1,...,\lambda_r)) \text{ and } v = f(D(\lambda_1,...,\lambda_{i-1},\lambda_i+1,\lambda_{i+1},...,\lambda_r)).$$

This concludes the proof of Lemma 4.6. □

For all $m \in \mathbb{N}^*$ denote $\lambda^m = (\lambda_1^m,...,\lambda_r^m) \in \Lambda$ the element defined by $\lambda_i^m = mi(r+1-i)$. Note that all the breaks of the associated polygon are equal to $2m$. One has

$$D(\lambda^m) = \begin{pmatrix} e^{mr} & 0 & \dots & \dots & 0 \\ 0 & e^{m(r-2)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & e^{m(2-r)} & 0 \\ 0 & \dots & \dots & 0 & e^{m(-r)} \end{pmatrix}$$

Lemma 4.10. *There is a constant C such that for all K-biinvariant function $f \in C_c(G)$, for all $m \in \mathbb{N}^*$ one has*

$$\left|f(D(\lambda^m))\right| \leq Cq^{-2\varepsilon m}\|\check{f}\|_{MS_p(L^2(G))}. \tag{25}$$

Proof of Lemma 4.10. It is enough to prove that there exists C such that for all K-biinvariant function $f \in C_c(G)$, for all $m \in \mathbb{N}^*$ one has

$$\left|f(D(\lambda^m)) - f(D(\lambda^{m+1}))\right| \leq Cq^{-2\varepsilon m}\|\check{f}\|_{MS_p(L^2(G))}.$$

This inequality follows from Lemma 4.6. One can indeed pass from λ^m to λ^{m+1} by $\sum_{i=1}^r i(r+1-i)$ successive transformations consisting in increasing by 1 the i^{th} coefficient and letting the others fixed. One applies
(16) if $i \leq \frac{r+1}{2}$ (which implies that $i \leq r-n$ thanks to the hypothesis $r \geq 2n+1$) and (17) if $i \geq \frac{r+1}{2}$ (which implies that $i-1 \geq n$ thanks to the hypothesis $r \geq 2n+1$).

Moreover one can manage the keep all the breaks $\geq 2m-2$. If C is the constant in Lemma 4.6 one thus gets

$$\left|f(D(\lambda^m)) - f(D(\lambda^{m+1}))\right| \leq C\big(\sum_{i=1}^r i(r+1-i)\big)q^{-\varepsilon(2m-2)}\|\check{f}\|_{MS_p(L^2(G))}$$

and this concludes the proof of Lemma 4.10. □

Proof of Proposition 4.2. If is an immediate consequence of Lemma 4.10. □

Remark 4.11. In (25), the function $m \mapsto f(D(\lambda^m))$ is exponentially small when $m \to \infty$ whereas the proof of Haagerup in [10] (in the case $G = SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ and $p = \infty$) does not imply such a result. For more on this, see [19].

We are now able to prove the main results of the introduction in the non-archimedian case.

Proof of Theorem D. The statement for $p > 2 + 2/n$ is an immediate consequence of Proposition 4.1. If $p < 2 - 2/(n+2)$, notice that $p' > 2 + 2/n$ if p' is the conjugate exponent of p : $1/p + 1/p' = 1$. Proposition 4.1 and hence Theorem D therefore also hold, by Remark 1.3. □

Proof of Theorem A. By Theorem D and Theorem 2.5, Γ does not have $\mathrm{AP}_{pcb}^{Schur}$. The statement follows from Corollary 3.13. □

Proof of Theorem C (non-archimedian case). If $4 < p < \infty$, as in the proof above, Γ does not have $\mathrm{AP}_{pcb}^{Schur}$. The theorem thus from Corollary 3.12. □

5 Case of $SL_r(\mathbb{R})$

This section is devoted to the proof of Theorem E and its consequences. This will be deduced at the end of this section from the following Proposition.

Proposition 5.1. *Let $r \geq 3$ and $G = SL_r(\mathbb{R})$. Let $1 \leq p \leq \infty$ such that $p > 4$ or $p < 4/3$. The constant function 1 on G cannot be approximated (for the topology of uniform convergence on compact subsets) by functions f in $C_0(G)$ such that $\|\check{f}\|_{MS^p(L^2(G))}$ is bounded uniformly :*

$$\Lambda_{pcb}^{Schur}(SL_r(\mathbb{R})) = \infty.$$

This main tool to prove the Proposition is

Lemma 5.2. *Let $G = SL_3(\mathbb{R})$, $K = SO_3(\mathbb{R})$, and $4 < p \leq \infty$. Let $0 < \varepsilon < 1/2 - 2/p$. There is a constant $C > 0$ such that for any K-biinvariant function $\varphi \in C_0(G)$, and any $t > 0$*

$$\left| \varphi \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \right| \leq Ce^{-\varepsilon t} \|\check{\varphi}\|_{MS^p(L^2(G))}.$$

We first deduce Proposition 5.1 from this Lemma.

Proof of Proposition 5.1. Lemma 5.2 implies that, if $4 < p \leq \infty$, the function 1 on $SL_3(\mathbb{R})$ cannot be approximated (pointwise) by $SO_3(\mathbb{R})$-biinvariant functions such f in $C_0(G)$ such that $\|\check{f}\|_{MS^p(L^2(G))}$ is bounded uniformly. By the same averaging argument as in the proof of Proposition 4.1, we deduce Proposition 5.1 in the case $r = 3$ and $p > 4$. The case $r = 3$ and $p < 4/3$ follows from Remark 1.3.

For $r > 3$, the map

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1_{r-3} \end{pmatrix}$$

realizes $SL_3(\mathbb{R})$ as a closed subgroup of $SL_r(\mathbb{R})$. Theorem 1.19 implies that Proposition 5.1 holds also for $r > 3$. ☐

Lemma 5.2 is proved as in section 4, using the same techniques as in the proof of strong property (T) for $SL_3(\mathbb{R})$ in [17]. From now on we fix G, K, $p > 4$ and $\varepsilon > 0$ as in Lemma 5.2. We use some notation and facts from [17], section 2. We denote by $\mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$, equipped with its usual probability measure denoted by dx. For any $\delta \in [-1, 1]$, we denote by T_δ the operator on $L^2(\mathbb{S}^2)$ defined, for a continuous function $f : \mathbb{S}^2 \to \mathbb{C}$ in the following way (and extended by continuity to a norm 1 operator). If $x \in \mathbb{S}^2$, $T_\delta f(x)$ is the average of f on the circle $\{y \in \mathbb{S}^2, \langle x, y\rangle = \delta\}$. We first state the analogue of Lemma 4.7 of this paper.

Lemma 5.3. *There is a constant C_1 such that for $\delta \in [-1/2, 1/2]$*

$$\|T_0 - T_\delta\|_{S^p(L^2(\mathbb{S}^2))} \leq C_1 |\delta|^{1/2-2/p}.$$

Moreover for any $a, b \in \mathbb{C}$,

$$\|aT_0 - bT_\delta\|_{S^p(L^2(\mathbb{S}^2))} \geq |a-b|.$$

Sketch of proof. Let P_n be the n-th Legendre polynomial normalized by $P_n(1) = 1$. It follows the proof of Lemma 2.2 in [17], that

$$\|T_0 - T_\delta\|_{S^p(L^2(\mathbb{S}^2))} = \left(\sum_{n \geq 0} (2n+1) |P_n(0) - P_n(\delta)|^p \right)^{1/p}.$$

Here $2n+1$ appears as the dimension of the space H_n of restrictions to $\mathbb{S}^2$ of the harmonic homogeneous polynomials of degree n on $\mathbb{R}^3$ (more precisely $L^2(\mathbb{S}^2)$ decomposes as $\oplus_{n\geq 0} H_n$, and T_δ acts as the multiplication by $P_n(\delta)$ on H_n).

If $|\delta| \leq 1/2$, the estimate $|P_n(0) - P_n(\delta)| \leq C \min(n|\delta|, 1)/\sqrt{n+1}$ for some constant C was proved in the proof of Lemma 2.2 in [17] and implies the first inequality of Lemma 5.3.

The second inequality holds because the function 1 on $\mathbb{S}^2$ is an eigenvector with eigenvalue 1 for all the T_δ's. □

For any $s, t \in \mathbb{R}_+$ (the non-negative real numbers) , we denote

$$D(s,t) = e^{-\frac{s+2t}{3}} \begin{pmatrix} e^{s+t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 5.4. *Let $\varphi \in C(G)$ be a K-biinvariant function, $s, t, s', t' \in \mathbb{R}_+$, and C_1 the constant in Lemma 5.3.*

- *If $s + 2t = s' + 2t'$ and $0 \leq t' \leq t \leq s+t \leq s'+t' \leq s+2t$, then*

$$|\varphi(D(s,t)) - \varphi(D(s',t'))| \leq C_1 e^{-(1/2-2/p)t'} \|\check{\varphi}\|_{MS^p(L^2(G))}.$$

- *If $2s + t = 2s' + t'$ and $0 \leq s' \leq s \leq s+t \leq s'+t' \leq 2s+t$, then*

$$|\varphi(D(s,t)) - \varphi(D(s',t'))| \leq C_1 e^{-(1/2-2/p)s'} \|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Sketch of proof. As in the proof of Lemma 4.6, the second inequality follows from the first by inversing the role of s, s' and t, t'.

Let us now fix s, t, s', t' as in the first inequality. We can assume that $e^{-t'} \leq 1/2$ because otherwise the inequality $\|\varphi\|_\infty \leq \|\check{\varphi}\|_{MS^p(L^2(G))}$ implies that the desired inequality holds with $C_1 = 2$. In [17] the first author constructed two continuous injective maps $\alpha, \beta : \mathbb{S}^2 \to G/K$ such that there is some for some $0 \leq \delta \leq e^{-t'}$ satisfying :

$$\alpha(x)^{-1}\beta(y) = KD(s,t)K \text{ if } \langle x, y\rangle = 0 \tag{26}$$

$$\alpha(x)^{-1}\beta(y) = KD(s',t')K \text{ if } \langle x, y\rangle = \delta \tag{27}$$

This is contained in Lemma 2.7 in [17], with $\alpha(\cdot) = q_{-(s+t)}(\cdot)$ and $\beta(\cdot) = q_t(\cdot)$.

Let μ be some Radon measure on G/K with full support such that the image measures of the measure dx on S^2 by α and β are absolutely continuous with respect to μ. By Theorem 1.19 we have that

$$\|\check{\varphi}\|_{cbMS^p(L^2(G/K,\mu))} \leq \|\check{\varphi}\|_{cbMS^p(L^2(G))} \leq \|\check{\varphi}\|_{MS^p(L^2(G))}.$$

The image measures of dx by α and β are absolutely continuous with respect to μ, and since $\mathbb{S}^2$ is compact α and β are homeomorphisms onto their images. Therefore, as in Lemma 1.9, α and β induce isometries $U_\alpha, U_\beta : L^2(\mathbb{S}^2) \to L^2(G/K,\mu)$ and hence an isometric embedding $i : S^p(L^2(\mathbb{S}^2)) \to S^p(L^2(G/K),\mu)$ given by $i(T) = U_\alpha T U_\beta^*$. It is straightforward to see that (26) (resp. (27)) implies $M_{\check{\varphi}}(i(T_0)) = \varphi(D(s,t))i(T_0)$ (resp. $M_{\check{\varphi}}(i(T_\delta)) = \varphi(D(s',t'))i(T_\delta)$). We thus get

$$\|\varphi(D(s,t))T_0 - \varphi(D(s',t'))T_\delta\|_p \leq \|\check{\varphi}\|_{MS^p(L^2(G))}\|T_0 - T_\delta\|_p.$$

Lemma 5.3 and the inequality $|\delta| \leq e^{-t'}$ allows to conclude the proof. ☐

Proof of Lemma 5.2. We copy the proof of [17], Proposition 2.3. Take $\varphi \in C_0(G)$. Assume for simplicity $\|\check{\varphi}\|_{MS^p(L^2(G))} = 1$. Let $u, v \in \mathbb{R}_+$ such that $u/v \in]1,2[$. Apply the first part of Lemma 5.4 to $(s,t) = (2v-u, 2u-v)$ and $(s',t') = (u,u)$ and get

$$|\varphi(D(u,u)) - \varphi(D(2v-u, 2u-v))| \leq C_1 e^{-(1/2-2/p)u}.$$

Apply the second part of Lemma 5.4 to $(s,t) = (v,v)$ and $(s',t') = (2v-u, 2u-v)$ and get

$$|\varphi(D(v,v)) - \varphi(D(2v-u, 2u-v))| \leq C_1 e^{-(1/2-2/p)(2v-u)}.$$

Hence,

$$|\varphi(D(v,v)) - \varphi(D(u,u))| \leq C_1 \left(e^{-(1/2-2/p)u} + e^{-(1/2-2/p)(2v-u)}\right).$$

Taking u/v close enough to 1, we can have $(1/2-2/p)(2v-u) \geq \varepsilon v$ and we deduce easily Lemma 5.2. ☐

We are now able to prove the main results of the introduction in the real case.

Proof of Theorem E. This is immediate from Proposition 5.1. ☐

Proof of Theorem B. Theorem E and Theorem 2.5 imply that Γ does not have $\mathrm{AP}_{pcb}^{Schur}$. We conclude using Corollary 3.13. $\square$

Proof of Theorem C (real case). If $4 < p < \infty$, as in the proof above, Γ does not have $\mathrm{AP}_{pcb}^{Schur}$. The theorem thus from Corollary 3.12. $\square$

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