Local discontinuous Galerkin method for distributed-order time and space-fractional convection-diffusion and Schrödinger type equations

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Abstract

Fractional partial differential equations with distributed-order fractional derivatives describe some important physical phenomena. In this paper, we propose a local discontinuous Galerkin (LDG) method for the distributed-order time and Riesz space fractional convection-diffusion and Schrödinger type equations. We prove stability and optimal order of convergence $\mathcal{O}(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2)$ for the distributed-order time and space-fractional diffusion and Schrödinger type equations, an order of convergence of $\mathcal{O}(h^{N+\frac{1}{2}} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2)$ is established for the distributed-order time and Riesz space fractional convection-diffusion equations where Δt , h and θ are the step sizes in time, space and distributed-order variables, respectively. Finally, the performed numerical experiments confirm the optimal order of convergence.

Keywords: time distributed order and space-fractional convection-diffusion equations, time distributed order and space-fractional Schrödinger type equations, local discontinuous Galerkin method, stability, error estimates.

1. Introduction

The distributed-order differential equation can serve as a natural generalization of the single-order and the multi-term fractional differential equation [1] which arises in many physical and biological applications, for example, the stress behavior of an elastic medium[2], the torsional phenomenon of anelastic or dielectric spherical shells and infinite planes [3], the rheological properties of composite materials [4, 5], dielectric induction and diffusion [6], viscoelastic oscillators [7], distributed order membranes in the ear [8], and anomalous diffusion [9, 10]. The earliest appearance of the idea on the distributed-order equation may date back to the Caputo's work in 1960s [3], which was also stated by Podlubny et al. [11]. Jiao et al. [12] presented a concise and insightful view to understand the usefulness of distributed-order concept in control and signal processing. a more complicated process cannot be described by a single power law and a mixture of power laws leads to a time distributed-order fractional derivative [13]. Chechkin et al. [14] proposed diffusion-like equations with time and space fractional derivatives of the distributed order for the kinetic description of anomalous diffusion and relaxation phenomena

and demonstrated that retarding subdiffusion and accelerating superdiffusion were governed by distributed-order fractional diffusion equation. Luchko [15] investigated some uniqueness and existence results of solutions to boundary value problems of the generalized distributed-order time-fractional diffusion equation by an appropriate maximum principle. Gorenflo et al. [16] obtained a representation of the fundamental solution to the Cauchy problem of a distributed-order time-fractional diffusion-wave equation by employing the technique of the Fourier and Laplace transforms and gave the interpretation of the fundamental solution as a probability density function. Furthermore, they studied waves in a viscoelastic rod of finite length, where viscoelastic material was described by a constitutive equation of fractional distributed-order type (see Atanackovic et al. [17]).

In recent years, developing various numerical algorithms for solving distributed-order and space-fractional equations has received much attention. For the distributed-order time differential equations, Diethelm and Ford [1, 18] presented the numerical methods for solving the distributed-order ordinary differential equations, where the distributed-order integral was rstly approximated using the quadrature formula and then the multi-term fractional differential equations were resulted in, which were nally reduced to a system of single-term equations. The idea was followed by Ford and Morgado [19] still for the distributed-order ordinary differential equations. The matrix approach to the solution of distributed-order differential equations was introduced by Podlubny et al. [11]. For the distributed-order time and space-fractional equations, Ye et al. [20] have treated the time distributedorder and space Riesz fractional diffusion on bounded domains numerically, where the distributed integral was discretized by the mid-point quadrature rule and the time-fractional derivatives in the resultant multi-term fractional diffusion equation were approximated by the classical L1 formula. Hu et al. [21] investigated an implicit numerical method for the time distributed-order and two-sided space-fractional advection-dispersion equation. Jacobi collocation method in two successional steps is developed to numerically solve the multi-dimensional distributed-order generalized Schrödinger equations [22]. To the best of our knowledge, however, the LDG method, which is an important approach to solve partial differential equations and fractional partial differential equations, has not been considered for the distributed-order time and space-fractional partial equations. In this paper, we develop a LDG method to solve the distributed-order time and space-fractional convection-diffusion equations equation

$$\mathcal{D}_{t}^{W(\alpha)}u + \varepsilon(-\Delta)^{\frac{\beta}{2}}u + \frac{\partial}{\partial x}f(u) = 0, \quad x \in \mathbb{R}, \ t \in (0, T],$$

$$u(x, 0) = u_{0}(x), \quad x \in \mathbb{R},$$

$$(1.1)$$

the nonlinear distributed-order time and space-fractional Schrödinger equation

$$i\mathcal{D}_t^{W(\alpha)}u - \varepsilon_1(-\Delta)^{\frac{\beta}{2}}u + \varepsilon_2 f(|u|^2)u = 0, \quad x \in \mathbb{R}, \ t \in (0, T],$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R},$$

$$(1.2)$$

and the coupled nonlinear distributed-order time and space-fractional Schrödinger equations

$$i\mathcal{D}_{t}^{W(\alpha)}u - \varepsilon_{1}(-\Delta)^{\frac{\beta}{2}}u + \varepsilon_{2}f(|u|^{2},|v|^{2})u = 0, \quad x \in \mathbb{R}, \ t \in (0,T],$$

$$i\mathcal{D}_{t}^{W(\alpha)}v - \varepsilon_{3}(-\Delta)^{\frac{\beta}{2}}v + \varepsilon_{4}g(|u|^{2},|v|^{2})v = 0, \quad x \in \mathbb{R}, \ t \in (0,T],$$

$$u(x,0) = u_{0}(x), \quad x \in \mathbb{R},$$

$$v(x,0) = v_{0}(x), \quad x \in \mathbb{R},$$
(1.3)

and homogeneous boundary conditions. f(u) and g(u) are arbitrary (smooth) nonlinear real functions and ε , ε_i , i = 1, 2, 3, 4 are a real constants, and $\mathcal{D}_t^{W(\alpha)}u(x, t)$ denotes the distributed order fractional derivative of u in time t, given by

$$\mathcal{D}_t^{W(\alpha)} u = \int_0^1 W(\alpha) \, {}_0^C \mathcal{D}_t^{\alpha} u(x, t) d\alpha, \tag{1.4}$$

where $W(\alpha)$ is the weight function, ${}^C_0\mathcal{D}^{\alpha}_t u(x,t)$ $(0<\alpha<1)$ is the Caputo fractional derivative of order α with respect to t. The fractional Laplacian $-(-\Delta)^{\frac{\beta}{2}}$, which can be defined using Fourier analysis as [23, 24, 25]

$$-(-\Delta)^{\frac{\beta}{2}}u(x,t) = \mathcal{F}^{-1}(|\xi|^{\beta}\hat{u}(\xi,t)),$$

where \mathcal{F} is the Fourier transform.

The discontinuous Galerkin (DG) method is a class of finite element methods using discontinuous, piecewise polynomials as the solution and the test spaces in the spatial direction. There have been various DG methods suggested in the literature to solve diffusion problem, including the method originally proposed by Bassi and Rebay [26] for compressible Navier-Stokes equations, its generalization called the local discontinuous Galerkin (LDG) methods introduced in [27] by Cockburn and Shu and further studied in [28, 29]. These DG methods have several attractive properties. It can be easily designed for any order of accuracy and it has the advantage of greatly facilitates the handling of complicated geometries and elements of various shapes and types, as well as the treatment of boundary conditions. And the higher-order of convergence can be achieved without over many iterations. For application of the method to fractional problems, Mustapha and McLean [30, 31, 32, 33] have developed and analyzed discontinuous Galerkin methods for time fractional diffusion and wave equations. Xu and Hesthaven [34] proposed a LDG method for fractional convection-diffusion equations. They proved stability and optimal order of convergence N+1 for the fractional diffusion problem when polynomials of degree N, and an order of convergence of $N+\frac{1}{2}$ is established for the general fractional convection-diffusion problem with general monotone flux for the nonlinear term. Aboelenen and El-Hawary [35] proposed a high-order nodal discontinuous Galerkin method for a linearized fractional Cahn-Hilliard equation. They proved stability and optimal order of convergence N+1 for the linearized fractional Cahn-Hilliard problem. A nodal discontinuous Galerkin method was developed to solve the nonlinear Riesz space fractional Schrödinger equation and the strongly coupled nonlinear Riesz space fractional Schrödinger equations [36]. They proved, for both problems, L^2 stability and optimal order of convergence $O(h^{N+1})$. Aboelenen [37] proposed a direct discontinuous Galerkin (DDG) finite element method

for fractional convection-diffusion and Schrödinger type equations. they proved, for both problems, L^2 stability and a priori L^2 error estimates.

This paper is organized as follows. In Section 2, we introduce some basic definitions and recall a few central results. We derive the discontinuous Galerkin formulation for the distributed-order time and Riesz space fractional convection-diffusion equations in Section 3. Then we prove a theoretical result of L^2 stability as well as an error estimate in Section 4. In Section 5, we present and analyze a local discontinuous Galerkin method for the nonlinear distributed-order time and Riesz space fractional Schrödinger type equations. We derive the discontinuous Galerkin formulation for the nonlinear distributed-order time and Riesz space fractional Schrödinger equation in Section 5.1. Moreover, we prove a theoretical result of L^2 stability for the nonlinear case in Section 5.1.1 as well as an error estimate for the linear case in Section 5.1.2. In Section 5.2, we present a local discontinuous Galerkin method for the nonlinear distributed-order time and Riesz space fractional coupled nonlinear Schrödinger equations and give a theoretical result of L^2 stability and error estimates. Section 6 presents some numerical examples to illustrate the efficiency of the scheme. A few concluding remarks are offered in Section 7.

2. Preliminary definitions

We introduce some preliminary definitions of fractional calculus, see, e.g., [38] and associated functional setting for the subsequent numerical schemes and theoretical analysis.

2.1. Liouville-Caputo fractional calculus

The left-sided and right-sided Riemann-Liouville integrals of order μ , when $0 < \mu < 1$, are defined, respectively, as

$$\left(\frac{RL}{-\infty}\mathcal{I}_x^{\mu}f\right)(x) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x \frac{f(s)ds}{(x-s)^{1-\mu}}, \quad x > -\infty, \tag{2.1}$$

and

$${\binom{RL}{x}\mathcal{I}_{\infty}^{\mu}f}(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{\infty} \frac{f(s)ds}{(s-x)^{1-\mu}}, \quad x < \infty,$$
(2.2)

where Γ represents the Euler Gamma function. The corresponding inverse operators, i.e., the left-sided and right-sided fractional derivatives of order μ , are then defined based on (2.1) and (2.2), as

$$\binom{RL}{-\infty} \mathcal{D}_x^{\mu} f(x) = \frac{d}{dx} \binom{RL}{-\infty} \mathcal{I}_x^{1-\mu} f(x) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_{-\infty}^x \frac{f(s)ds}{(x-s)^{\mu}}, \quad x > -\infty, \tag{2.3}$$

and

$${\binom{RL}{x}\mathcal{D}_{\infty}^{\mu}f}(x) = \frac{-d}{dx}{\binom{RL}{x}\mathcal{I}_{\infty}^{1-\mu}f}(x) = \frac{1}{\Gamma(1-\mu)}\left(\frac{-d}{dx}\right)\int_{x}^{\infty} \frac{f(s)ds}{(s-x)^{\mu}}, \quad x < \infty.$$
 (2.4)

This allows for the definition of the left and right Riemann-Liouville fractional derivatives of order μ $(n-1 < \mu < n), n \in \mathbb{N}$ as

$$\binom{RL}{-\infty} \mathcal{D}_x^{\mu} f(x) = \left(\frac{d}{dx}\right)^n \binom{RL}{-\infty} \mathcal{I}_x^{n-\mu} f(x) = \frac{1}{\Gamma(n-\mu)} \left(\frac{d}{dx}\right)^n \int_{-\infty}^x \frac{f(s)ds}{(x-s)^{-n+1+\mu}}, \quad x > -\infty, \tag{2.5}$$

and

$${\binom{RL}{x}\mathcal{D}^{\mu}_{\infty}f}(x) = \left(\frac{-d}{dx}\right)^{n} {\binom{RL}{x}\mathcal{I}^{n-\mu}_{\infty}f}(x) = \frac{1}{\Gamma(n-\mu)} \left(\frac{-d}{dx}\right)^{n} \int_{x}^{\infty} \frac{f(s)ds}{(s-x)^{-n+1+\mu}}, \quad x < \infty.$$
 (2.6)

Furthermore, the corresponding left-sided and right-sided Caputo derivatives of order μ $(n-1 < \mu < n)$ are obtained as

$$\left({_{-\infty}^C \mathcal{D}_x^{\mu} f} \right)(x) = \left({_{-\infty}^{RL} \mathcal{I}_x^{n-\mu} \frac{d^n f}{dx^n}} \right)(x) = \frac{1}{\Gamma(n-\mu)} \int_{-\infty}^x \frac{f^{(n)}(s) ds}{(x-s)^{n-1+\mu}}, \quad x > -\infty, \tag{2.7}$$

and

$${\binom{C}{x}}\mathcal{D}_{\infty}^{\mu}f(x) = (-1)^{n} {\binom{RL}{x}} \mathcal{I}_{\infty}^{n-\mu} \frac{d^{n}f}{dx^{n}}(x) = \frac{1}{\Gamma(n-\mu)} \int_{x}^{\infty} \frac{(-1)^{n}f^{(n)}(s)ds}{(s-x)}^{n-1+\mu}, \quad x < \infty.$$
 (2.8)

To carry out the analysis, we introduce the appropriate fractional spaces.

Definition 2.1. (left fractional space [39]). We define the seminorm

$$|u|_{J_L^{\mu}(\mathbb{R})} = \left\| {_{x_L}^{RL}} \mathcal{D}_x^{\mu} u \right\|_{L^2(\mathbb{R})}. \tag{2.9}$$

and the norm

$$||u||_{J_L^{\mu}(\mathbb{R})} = (|u|_{J_L^{\mu}(\mathbb{R})}^2 + ||u||_{L^2(\mathbb{R})}^2)^{\frac{1}{2}},\tag{2.10}$$

and let $J_L^{\mu}(\mathbb{R})$ denote the closure of $C_0^{\infty}(\mathbb{R})$ with respect to $\|.\|_{J_L^{\mu}(\mathbb{R})}$.

Definition 2.2. (right fractional space [39]). We define the seminorm

$$|u|_{J_R^{\mu}(\mathbb{R})} = \left\| {_x^{RL} \mathcal{D}_{x_R}^{\mu} u} \right\|_{L^2(\mathbb{R})}, \tag{2.11}$$

and the norm

$$||u||_{J_R^{\mu}(\mathbb{R})} = (|u|_{J_R^{\mu}(\mathbb{R})}^2 + ||u||_{L^2(\mathbb{R})}^2)^{\frac{1}{2}},\tag{2.12}$$

and let $J_R^{\mu}(\mathbb{R})$ denote the closure of $C_0^{\infty}(\mathbb{R})$ with respect to $\|.\|_{J_R^{\mu}(\mathbb{R})}$.

Definition 2.3. (symmetric fractional space [39]). We define the seminorm

$$||u||_{J_S^{\mu}(\mathbb{R})} = \left| \left({_{x_L}^{RL} \mathcal{D}_x^{\mu} u, {_{x_L}^{RL} \mathcal{D}_{x_R}^{\mu} u} \right)_{L^2(\mathbb{R})}} \right|^{\frac{1}{2}}, \tag{2.13}$$

and the norm

$$||u||_{J_S^{\mu}(\mathbb{R})} = \left(|u|_{J_S^{\mu}(\mathbb{R})}^2 + ||u||_{L^2(\mathbb{R})}^2\right)^{\frac{1}{2}}.$$
(2.14)

and let $J_S^{\mu}(\mathbb{R})$ denote the closure of $C_0^{\infty}(\mathbb{R})$ with respect to $\|.\|_{J_S^{\mu}(\mathbb{R})}$.

Lemma 2.1. (see [39]). For any 0 < s < 1, the fractional integral satisfies the following property:

$$\binom{RL}{-\infty} \mathcal{I}_x^s u, \binom{RL}{x} \mathcal{I}_\infty^s u)_{\mathbb{R}} = \cos(s\pi) |u|_{J_L^{-s}(\mathbb{R})}^2 = \cos(s\pi) |u|_{J_R^{-s}(\mathbb{R})}^2. \tag{2.15}$$

Generally, we consider the problem in a bounded domain instead of \mathbb{R} . Hence, we restrict the definition to the domain $\Omega = [a, b]$.

Definition 2.4. Define the spaces $J_{R,0}^{\mu}(\Omega), J_{L,0}^{\mu}(\Omega), J_{S,0}^{\mu}(\Omega)$ as the closures of $C_0^{\infty}(\Omega)$ under their respective norms.

Lemma 2.2. (fractional Poincaré-Friedrichs, [39]). For $u \in J_{L,0}^{\mu}(\Omega)$ and $\mu \in \mathbb{R}$, we have

$$||u||_{L^2(\Omega)} \le C|u|_{J^{\mu}_{L,0}(\Omega)},$$
 (2.16)

and for $u \in J_{R,0}^{\mu}(\Omega)$, we have

$$||u||_{L^2(\Omega)} \le C|u|_{J^{\mu}_{B,0}(\Omega)}.$$
 (2.17)

Lemma 2.3. (See [40]) The fractional integration operator \mathcal{I}^s is bounded in $L^2(\Omega)$:

$$\|\mathcal{I}^s u\|_{L^2(\Omega)} \le K \|u\|_{L^2(\Omega)},$$
 (2.18)

where $\mathcal{I}^s = {}^{RL}_{x_L} \mathcal{I}^s_x$ (i.e., right-sided Riemann-Liouville integral of order s).

Lemma 2.4. (See [36]) The fractional integration operator $\Delta_{-\mu}$ is bounded in $L^2(\Omega)$:

$$\|\Delta_{-\mu}u\|_{L^{2}(\Omega)} \le K\|u\|_{L^{2}(\Omega)}.$$
 (2.19)

3. LDG scheme for the time distributed-order and space-fractional convection-diffusion equation

Let us consider the distributed-order time and Riesz space fractional convection-diffusion equation. We first discretize the integral interval [0,1] by the grid $0 = \tau_0 < \tau_1 < ... < \tau_S = 1$ and take $\Delta \tau_j = \tau_j - \tau_{j-1} = \frac{1}{S} = \theta$, $\alpha_j = \frac{\tau_j + \tau_{j-1}}{2} = \frac{2j-1}{2S}$, $j = 1, 2, ..., S, S \in \mathbb{N}$. Then using the mid-point quadrature rule, we obtain

$$\mathcal{D}_t^{W(\alpha)} u(x,t) = \sum_{j=1}^S W(\alpha_j) \, {}_0^C \mathcal{D}_t^{\alpha_j} u(x,t) \Delta \tau_j + \mathcal{O}(\theta^2), \tag{3.1}$$

where θ is the step size of the discretization of the numerical integration. Thus the distributed-order fractional equation (1.1) is now transformed into multi-term fractional equation. An approximation to the time fractional derivative (3.1) can be obtained by simple quadrature formula given as [41]. Let $\Delta t = T/M$ be the time mesh-size, M is a positive integer, $t_n = n\Delta t$, n = 0, 1, ..., M be mesh points.

Lemma 3.1. (See [41]) Suppose $(0 < \alpha < 1)$, $y(t) \in C^2[0, t_n]$. It holds that

$$\left| \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t_{n}} \frac{y'(s)ds}{(t_{n}-s)^{\alpha}} - \frac{1}{\lambda} \left[a_{0}y(t_{n}) - \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l})y(t_{l}) - a_{n-1}y(0) \right] \right| \\
\leq \frac{1}{\Gamma(2-\alpha)} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha}) \right] \max_{0 \leq t \leq t_{n}} |y''(t)| (\Delta t)^{2-\alpha}.$$
(3.2)

For simplicity of the presentation of the proposed method, we introduce the notation

$${}_{0}^{C}\mathcal{D}_{t_{n}}^{\alpha}y \approx \delta_{t}^{\alpha}y^{n} = \frac{1}{\lambda} \left(y^{n} - \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l})y^{l} - a_{n-1}y^{0} \right). \tag{3.3}$$

From (1.4), (3.1) and (3.3) we obtain

$$\mathcal{D}_{t_n}^{W(\alpha)} u \approx \sum_{j=1}^{S} \Delta \tau_j W(\alpha_j) {}_0^C \mathcal{D}_{t_n}^{\alpha_j} u \approx \sum_{j=1}^{S} \Delta \tau_j W(\alpha_j) \delta_t^{\alpha_j} u^n$$

$$= \sum_{j=1}^{S} \frac{W(\alpha_j) \Delta \tau_j}{\lambda_j} \left(u^n - \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_j} - a_{n-l}^{\alpha_j}) u^l - a_{n-1}^{\alpha_j} u^0 \right), \tag{3.4}$$

where
$$\lambda_j = (\Delta t)^{\alpha_j} \Gamma(2 - \alpha_j)$$
 and $a_l^{\alpha_j} = (l+1)^{1-\alpha_j} - l^{1-\alpha_j}, 0 \le l \le M-1$.

To obtain a high order discontinuous Galerkin scheme for the space fractional derivative, we rewrite the fractional derivative as a composite of first order derivatives and a fractional integral to recover the equation to a low order system. However, for the first order system, alternating fluxes are used. We introduce variables p, q and r and set

$$p = \Delta_{(\beta-2)/2}q, \quad q = \frac{\partial}{\partial x}r, \quad r = \frac{\partial}{\partial x}u,$$
 (3.5)

then, the time distributed order and space-fractional convection-diffusion problem can be rewritten as

$$\mathcal{D}_{t}^{W(\alpha)}u + \frac{\partial}{\partial x}f(u) - p = 0,$$

$$p = \Delta_{(\beta-2)/2}q, \quad q = \frac{\partial}{\partial x}r, \quad r = \frac{\partial}{\partial x}u.$$
(3.6)

Now we introduce the broken Sobolev space for any real number r

$$H^{r}(\Omega) = \{ v \in L^{2}(\Omega) : \forall k = 1, 2, ..., K, v|_{D^{k}} \in H^{r}(D^{k}) \}.$$
(3.7)

We define the local inner product and $L^2(D^k)$ norm

$$(u,v)_{D^k} = \int_{D^k} uv dx, \quad ||u||_{D^k}^2 = (u,u)_{D^k}, \tag{3.8}$$

as well as the global broken inner product and norm

$$(u,v) = \sum_{k=1}^{K} (u,v)_{D^k}, \quad ||u||_{L^2(\Omega)}^2 = \sum_{k=1}^{K} (u,u)_{D^k}.$$
(3.9)

We introduce some notation

$$u^{\pm}(x_i) = \lim_{x \to x_i^{\pm}} u(x), \quad \{u\} = \frac{u^+ + u^-}{2}, \quad [u] = u^+ - u^-. \tag{3.10}$$

For simplicity we discretize the computational domain Ω into K non-overlapping elements, $D^k = [x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]$, k = 1, ..., K. Let $u_h^n, p_h^n, q_h^n, r_h^n \in V_k^N$ be the approximation of $u(., t_n), p(., t_n), q(., t_n), r(., t_n)$ respectively, where the approximation space is defined as

$$V_k^N = \{ v : v_k \in \mathbb{P}(D^k), \forall D^k \in \Omega \},\tag{3.11}$$

where $\mathbb{P}(D^k)$ denotes the set of polynomials of degree up to N defined on the element D^k .

We define a fully discrete local discontinuous Galerkin scheme with as follows: find $u_h^n, p_h^n, q_h^n, r_h^n \in V_k^N$, such that for all test functions $v, \psi, \phi, \eta \in V_k^N$,

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} u_{h}^{n}, v\right)_{D^{k}} - \varepsilon(p_{h}^{n}, v)_{D^{k}} - \left(f(u_{h}^{n}), \frac{\partial}{\partial x} v\right)_{D^{k}} + \left((\widehat{f}(u_{h}^{n}) v^{-})_{k+\frac{1}{2}} - (\widehat{f}(u_{h}^{n}) v^{+})_{k-\frac{1}{2}}\right) = 0,
(p_{h}^{n}, \psi)_{D^{k}} = \left(\Delta_{(\beta-2)/2} q_{h}^{n}, \psi\right)_{D^{k}},
(q_{h}^{n}, \phi)_{D^{k}} = -\left(r_{h}^{n}, \frac{\partial \phi}{\partial x}\right)_{D^{k}} + \left((\widehat{r}_{h}^{n} \phi^{-})_{k+\frac{1}{2}} - (\widehat{r}_{h}^{n} \phi^{+})_{k-\frac{1}{2}}\right),
(r_{h}^{n}, \eta)_{D^{k}} = -\left(u_{h}^{n}, \frac{\partial \eta}{\partial x}\right)_{D^{k}} + \left((\widehat{u}_{h}^{n} \eta^{-})_{k+\frac{1}{2}} - (\widehat{u}_{h}^{n} \eta^{+})_{k-\frac{1}{2}}\right).$$
(3.12)

The 'hat' terms in the scheme are the so-called numerical fluxes. In order to ensure the stability, these terms are taken as

$$\widehat{u}_h^n = (u_h^n)^-, \quad \widehat{r}_h^n = (r_h^n)^+, \quad \widehat{f}_h = \widehat{f}((u_h^n)^-, (u_h^n)^+).$$
 (3.13)

Note that we can also choose

$$\widehat{u}_h^n = (u_h^n)^+, \quad \widehat{r}_h^n = (r_h^n)^-, \quad \widehat{f}_h = \widehat{f}((u_h^n)^-, (u_h^n)^+). \tag{3.14}$$

4. Stability and error estimates

In the following we discuss stability and accuracy of the proposed scheme, for time distributed order and space-fractional convection-diffusion problem.

4.1. The analysis of stability for fully discrete scheme

Theorem 4.1. The fully-discrete LDG scheme (3.12) is stable, and

$$||u_h^n||_{L^2(\Omega)} \le C||u_h^0||_{L^2(\Omega)}, \quad n = 1, 2, ..., M.$$
 (4.1)

Proof. Set $(v, \psi, \phi, \eta) = (u_h^n, p_h^n - q_h^n, u_h^n, r_h^n)$ in (3.12), and define $\theta(u_h^n) = \int^{u_h^n} f(s_h^n) ds_h^n$. Then the following result holds:

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} u_{h}^{n}, u_{h}^{n}\right)_{D^{k}} - \varepsilon \left(p_{h}^{n}, u_{h}^{n}\right)_{D^{k}} - \left(\theta(u_{h}^{n})\right)_{k+\frac{1}{2}}^{-} + \left(\theta(u_{h}^{n})\right)_{k-\frac{1}{2}}^{+} + \left(\left(\widehat{f}(u_{h}^{n})u^{-}\right)_{k+\frac{1}{2}}^{-} - \left(\widehat{f}(u_{h}^{n})(u_{h}^{n})^{+}\right)_{k-\frac{1}{2}}\right) \\
- \left(p_{h}^{n}, q_{h}^{n}\right)_{D^{k}} + \left(p_{h}^{n}, p_{h}^{n}\right)_{D^{k}} + \left(\Delta_{(\beta-2)/2}q_{h}^{n}, q_{h}^{n}\right)_{D^{k}} - \left(\Delta_{(\beta-2)/2}q_{h}^{n}, p_{h}^{n}\right)_{D^{k}} + \left(r_{h}^{n}, r_{h}^{n}\right)_{D^{k}} + \left(u_{h}^{n}, \frac{\partial r_{h}^{n}}{\partial x}\right)_{D^{k}} \\
+ \left(q_{h}^{n}, u_{h}^{n}\right)_{D^{k}} + \left(r_{h}^{n}, \frac{\partial u_{h}^{n}}{\partial x}\right)_{D^{k}} - \left(\left(\widehat{u}_{h}^{n}(r_{h}^{n})^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{u}_{h}^{n}(r_{h}^{n})^{+}\right)_{k-\frac{1}{2}}\right) - \left(\left(\widehat{r}_{h}^{n}(u_{h}^{n})^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{r}_{h}^{n}(u_{h}^{n})^{+}\right)_{k-\frac{1}{2}}\right) = 0.$$

$$(4.2)$$

Summing over k, with the definition (3.13) of the numerical fluxes and with simple algebraic manipulations and, we easily obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} u_{h}^{n}, u_{h}^{n}\right) - \varepsilon \left(p_{h}^{n}, u_{h}^{n}\right) - \sum_{k=1}^{K} \left(\left(\theta(u_{h}^{n})\right)_{k+\frac{1}{2}}^{-} - \left(\theta(u_{h}^{n})\right)_{k-\frac{1}{2}}^{+}\right) + \sum_{k=1}^{K} \left(\left(\widehat{f}(u_{h}^{n})u^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{f}(u_{h}^{n})(u_{h}^{n})^{+}\right)_{k-\frac{1}{2}}\right) - \left(p_{h}^{n}, q_{h}^{n}\right) + \left(p_{h}^{n}, p_{h}^{n}\right) + \left(\Delta_{(\beta-2)/2}q_{h}^{n}, q_{h}^{n}\right) - \left(\Delta_{(\beta-2)/2}q_{h}^{n}, p_{h}^{n}\right) + \left(r_{h}^{n}, r_{h}^{n}\right) + \left(q_{h}^{n}, u_{h}^{n}\right) = 0.$$

$$(4.3)$$

From the properties of the monotone flux, we know that $\widehat{f}((u_h^n)^-, (u_h^n)^+)$ nondecreasing function of its first argument and a nonincreasing function of its second argument. Hence, we have

$$\sum_{k=1}^{K} \left((\widehat{f}(u_h^n) u^-)_{k+\frac{1}{2}} - (\widehat{f}(u_h^n) (u_h^n)^+)_{k-\frac{1}{2}} \right) - \sum_{k=1}^{K} \left((\theta(u_h^n))_{k+\frac{1}{2}}^- - (\theta(u_h^n))_{k-\frac{1}{2}}^+ \right) > 0. \tag{4.4}$$

This implies that

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} u_{h}^{n}, u_{h}^{n}\right) + \left(r_{h}^{n}, r_{h}^{n}\right) + \left(p_{h}^{n}, p_{h}^{n}\right) + \left(\Delta_{(\beta-2)/2} q_{h}^{n}, q_{h}^{n}\right)
\leq \left(\Delta_{(\beta-2)/2} q_{h}^{n}, p_{h}^{n}\right) - \left(q_{h}^{n}, u_{h}^{n}\right) + \left(p_{h}^{n}, q_{h}^{n}\right) + \varepsilon\left(p_{h}^{n}, u_{h}^{n}\right).$$
(4.5)

Employing Young's inequality and Lemma 2.4, we obtain

$$\left(\sum_{i=1}^{S} W(\alpha_{i}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} u_{h}^{n}, u_{h}^{n}\right) + \|r_{h}^{n}\|_{L^{2}(\Omega)}^{2} + \left(\Delta_{(\beta-2)/2} q_{h}^{n}, q_{h}^{n}\right) \le c \|u_{h}^{n}\|_{L^{2}(\Omega)}^{2} + c_{1} \|q_{h}^{n}\|_{L^{2}(\Omega)}^{2}.$$

$$(4.6)$$

Recalling Lemma 2.4, we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_j) \Delta \tau_j \delta_t^{\alpha_j} u_h^n, u_h^n\right) + \|r_h^n\|_{L^2(\Omega)}^2 \le c \|u_h^n\|_{L^2(\Omega)}^2. \tag{4.7}$$

It then follows that

$$\left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} u^{n}, u_{h}^{n}\right) \leq \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) u_{h}^{l}, u_{h}^{n}\right) + \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} a_{n-1}^{\alpha_{j}} u_{h}^{0}, u_{h}^{n}\right) + c \|u_{h}^{n}\|_{L^{2}(\Omega)}^{2}.$$
(4.8)

Using Cauchy-Schwarz inequality, we obtain

$$||u_{h}^{n}||_{L^{2}(\Omega)}^{2} \leq c_{1} \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) ||u_{h}^{l}||_{L^{2}(\Omega)} ||u_{h}^{n}||_{L^{2}(\Omega)} + c_{2} \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} ||u_{h}^{0}||_{L^{2}(\Omega)} ||u_{h}^{n}||_{L^{2}(\Omega)} + cQ ||u_{h}^{n}||_{L^{2}(\Omega)}^{2},$$

$$(4.9)$$

where $Q = \frac{1}{\sum_{j=1}^{S} \frac{W(\alpha_j)\Delta\tau_j}{\lambda_j}}$ and provided c is sufficiently small such that 1 - cQ > 0, we obtain that

$$||u_h^n||_{L^2(\Omega)} \le C \left(\sum_{j=1}^S \frac{W(\alpha_j) \Delta \tau_j}{\lambda_j} Q \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_j} - a_{n-l}^{\alpha_j}) ||u_h^l||_{L^2(\Omega)} + \sum_{j=1}^S \frac{W(\alpha_j) \Delta \tau_j}{\lambda_j} Q a_{n-1}^{\alpha_j} ||u_h^0||_{L^2(\Omega)} \right). \tag{4.10}$$

Obviously the theorem holds for n = 0. Assume that it is valid for n = 1, 2, ..., m - 1. Then, by (4.10), we have

$$||u_{h}^{m}||_{L^{2}(\Omega)} \leq C \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{m-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) ||u_{h}^{l}||_{L^{2}(\Omega)} + \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} ||u_{h}^{0}||_{L^{2}(\Omega)} \right)$$

$$\leq C \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{m-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) ||u_{h}^{0}||_{L^{2}(\Omega)} + \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} ||u_{h}^{0}||_{L^{2}(\Omega)} \right)$$

$$= C ||u_{h}^{0}||_{L^{2}(\Omega)}. \quad \Box$$

$$(4.11)$$

4.2. Error estimates

In order to obtain the error estimate to smooth solutions for the considered fully discrete LDG scheme (3.12), we need to first obtain the error equation.

It is easy to verify that the exact solution of (1.1) satisfies

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} u^{n}, v\right)_{D^{k}} - \varepsilon \left(p^{n}, v\right)_{D^{k}} + \left(\gamma(x)^{n}, v\right)_{D^{k}} - \left(f(u^{n}), \frac{\partial}{\partial x} v\right)_{D^{k}} + \left((\widehat{f}(u^{n})v^{-})_{k+\frac{1}{2}} - (\widehat{f}(u^{n})v^{+})_{k-\frac{1}{2}}\right) = 0,
\left(p^{n}, \psi\right)_{D^{k}} = \left(\Delta_{(\beta-2)/2}q^{n}, \psi\right)_{D^{k}},
\left(q^{n}, \phi\right)_{D^{k}} = -\left(r^{n}, \frac{\partial \phi}{\partial x}\right)_{D^{k}} + \left((\widehat{r}^{n}\phi^{-})_{k+\frac{1}{2}} - (\widehat{r}^{n}\phi^{+})_{k-\frac{1}{2}}\right),
\left(r^{n}, \eta\right)_{D^{k}} = -\left(u^{n}, \frac{\partial \eta}{\partial x}\right)_{D^{k}} + \left((\widehat{u}^{n}\eta^{-})_{k+\frac{1}{2}} - (\widehat{u}^{n}\eta^{+})_{k-\frac{1}{2}}\right).$$
(4.12)

Subtracting equation (3.12) from (4.12), we can obtain the error equation

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(u^{n} - u_{h}^{n}), v\right)_{D^{k}} - \varepsilon \left(p^{n} - p_{h}^{n}, v\right)_{D^{k}} + \left(\gamma(x)^{n}, v\right)_{D^{k}} - \left(f(u^{n}) - f(u_{h}^{n}), \frac{\partial}{\partial x}v\right)_{D^{k}} \\
+ \left(\left((\widehat{f}(u^{n}) - \widehat{f}(u_{h}^{n}))v^{-}\right)_{k+\frac{1}{2}} - \left((\widehat{f}(u^{n}) - \widehat{f}(u_{h}^{n}))v^{+}\right)_{k-\frac{1}{2}}\right) + \left(p^{n} - p_{h}^{n}, \psi\right)_{D^{k}} + \left(q^{n} - q_{h}^{n}, \phi\right)_{D^{k}} \\
- \left(\Delta_{(\beta-2)/2}(q^{n} - q_{h}^{n}), \psi\right)_{D^{k}} + \left(r^{n} - r_{h}^{n}, \frac{\partial\phi}{\partial x}\right)_{D^{k}} - \left((\widehat{r}^{n} - \widehat{r}_{h}^{n})\phi^{-}\right)_{k+\frac{1}{2}} + \left((\widehat{r}^{n} - \widehat{r}_{h}^{n})\phi^{+}\right)_{k-\frac{1}{2}} \\
+ \left(r^{n} - r_{h}^{n}, \eta\right)_{D^{k}} + \left(u^{n} - u_{h}^{n}, \frac{\partial\eta}{\partial x}\right)_{D^{k}} - \left((\widehat{u}^{n} - \widehat{u}_{h}^{n})\eta^{-}\right)_{k+\frac{1}{2}} + \left((\widehat{u}^{n} - \widehat{u}_{h}^{n})\eta^{+}\right)_{k-\frac{1}{2}} = 0,$$
(4.13)

where

$$|\gamma(x)^n| = |\mathcal{O}((\Delta t)^{2-\alpha_j} + \theta^2)| \le c((\Delta t)^{1+\frac{\theta}{2}} + \theta^2),$$
 (4.14)

such that

$$1 + \frac{\theta}{2} = 2 - S\theta + \frac{\theta}{2} \le 2 - \alpha_j = 2 - j\theta + \frac{\theta}{2} \le 2 - \theta + \frac{\theta}{2} = 2 - \frac{\theta}{2}.$$
 (4.15)

For the error estimate, we define special projections, \mathcal{P} and \mathcal{P}^{\pm} into V_h^k . For all the elements, D^k , k = 1, 2, ..., K are defined to satisfy

$$(\mathcal{P}u - u, v)_{D^k} = 0, \quad \forall v \in \mathbb{P}_N^k(D^k),$$

$$(\mathcal{P}^{\pm}u - u, v)_{D^k} = 0, \quad \forall v \in \mathbb{P}_N^{k-1}(D^k), \quad \mathcal{P}^{\pm}u_{k+\frac{1}{2}} = u(x_{k+\frac{1}{2}}^{\pm}).$$

$$(4.16)$$

Denoting

$$\pi^{n} = \mathcal{P}^{-}u^{n} - u_{h}^{n}, \quad \pi_{n}^{e} = \mathcal{P}^{-}u^{n} - u^{n}, \quad \sigma^{n} = \mathcal{P}p^{n} - p_{h}^{n}, \quad \sigma_{n}^{e} = \mathcal{P}p^{n} - p^{n}, \quad \varphi^{n} = \mathcal{P}q^{n} - q_{h}^{n},$$

$$\varphi_{n}^{e} = \mathcal{P}q^{n} - q^{n}, \quad \psi^{n} = \mathcal{P}^{+}r^{n} - r_{h}^{n}, \quad \psi_{n}^{e} = \mathcal{P}^{+}r^{n} - r^{n}.$$
(4.17)

For the special projections mentioned above, we have, by the standard approximation theory [42], that

$$\|\pi^e\|_{L^2(\Omega)} + h\|\pi^e\|_{\infty} + h^{\frac{1}{2}}\|\pi\|_{\Gamma_h} \le Ch^{N+1}.$$
(4.18)

where $\pi^e = \mathcal{P}^{\pm}u^n - u_h^n$ or $\pi^e = \mathcal{P}u^n - u_h^n$. The positive constant C, solely depending on u^n , is independent of h. Γ_h denotes the set of boundary points of all elements D^k .

Theorem 4.2. (Diffusion without convection f(u) = 0). Let $u(x, t_n)$ be the exact solution of the problem (1.1), which is sufficiently smooth with bounded derivatives, let u_h^n be the numerical solution of the fully discrete LDG scheme (3.12), then there holds the following error estimates:

$$||u(x,t_n) - u_h^n||_{L^2(\Omega)} \le C(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2).$$
(4.19)

Proof. From (4.13), we can obtain the error equation

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(u^{n} - u_{h}^{n}), v\right)_{D^{k}} - \varepsilon \left(p^{n} - p_{h}^{n}, v\right)_{D^{k}} + \left(\gamma(x)^{n}, v\right)_{D^{k}}
+ \left(p^{n} - p_{h}^{n}, \psi\right)_{D^{k}} - \left(\Delta_{(\beta-2)/2}(q^{n} - q_{h}^{n}), \psi\right)_{D^{k}} + \left(q^{n} - q_{h}^{n}, \phi\right)_{D^{k}} + \left(r^{n} - r_{h}^{n}, \frac{\partial \phi}{\partial x}\right)_{D^{k}}
- \left((\widehat{r}^{n} - \widehat{r}_{h}^{n})\phi^{-}\right)_{k+\frac{1}{2}} + \left((\widehat{r}^{n} - \widehat{r}_{h}^{n})\phi^{+}\right)_{k-\frac{1}{2}} + \left(r^{n} - r_{h}^{n}, \eta\right)_{D^{k}} + \left(u^{n} - u_{h}^{n}, \frac{\partial \eta}{\partial x}\right)_{D^{k}}
- \left((\widehat{u}^{n} - \widehat{u}_{h}^{n})\eta^{-}\right)_{k+\frac{1}{2}} + \left((\widehat{u}^{n} - \widehat{u}_{h}^{n})\eta^{+}\right)_{k-\frac{1}{2}} = 0.$$
(4.20)

Using (4.17), the error equation (4.20) can be written

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(\pi^{n} - \pi_{n}^{e}), v\right)_{D^{k}} - \varepsilon \left(\sigma^{n} - \sigma_{n}^{e}, v\right)_{D^{k}} + \left(\gamma(x)^{n}, v\right)_{D^{k}}
+ \left(\sigma^{n} - \sigma_{n}^{e}, \psi\right)_{D^{k}} - \left(\Delta_{(\beta-2)/2}(\varphi^{n} - \varphi_{n}^{e}), \psi\right)_{D^{k}} + \left(\varphi^{n} - \varphi_{n}^{e}, \phi\right)_{D^{k}} + \left(\psi^{n} - \psi_{n}^{e}, \frac{\partial \phi}{\partial x}\right)_{D^{k}}
- \left((\psi^{n} - \psi_{n}^{e})^{-}\phi^{-}\right)_{k+\frac{1}{2}} + \left((\psi^{n} - \psi_{n}^{e})^{-}\phi^{+}\right)_{k-\frac{1}{2}} + \left(\psi^{n} - \psi_{n}^{e}, \eta\right)_{D^{k}} + \left(\pi^{n} - \pi_{n}^{e}, \frac{\partial \eta}{\partial x}\right)_{D^{k}}
- \left((\pi^{n} - \pi_{n}^{e})^{+}\eta^{-}\right)_{k+\frac{1}{2}} + \left((\pi^{n} - \pi_{n}^{e})^{+}\eta^{+}\right)_{k-\frac{1}{2}} = 0,$$
(4.21)

and taking the test functions

$$v = \pi^n, \quad \psi = \sigma^n - \varphi^n, \quad \phi = \pi^n, \quad \eta = \psi^n,$$
 (4.22)

we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(\pi^{n} - \pi_{n}^{e}), \pi^{n}\right) - \varepsilon \left(\sigma^{n} - \sigma_{n}^{e}, \pi^{n}\right) + \left(\gamma(x)^{n}, \pi^{n}\right)
+ \left(\sigma^{n} - \sigma_{n}^{e}, -\varphi^{n} + \sigma^{n}\right) - \left(\Delta_{(\beta-2)/2}(\varphi^{n} - \varphi_{n}^{e}), -\varphi^{n} + \sigma^{n}\right) + \left(\varphi^{n} - \varphi_{n}^{e}, \pi^{n}\right)
- \left(\psi_{n}^{e}, \frac{\partial \pi^{n}}{\partial x}\right) + \sum_{k=1}^{K} \left(\left((\psi_{n}^{e})^{-}(\pi^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\psi_{n}^{e})^{-}(\pi^{n})^{+}\right)_{k-\frac{1}{2}}\right) + \left(\psi^{n} - \psi_{n}^{e}, \psi^{n}\right)
- \left(\pi_{n}^{e}, \frac{\partial \psi^{n}}{\partial x}\right) + \sum_{k=1}^{K} \left(\left((\pi_{n}^{e})^{+}(\psi^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\pi_{n}^{e})^{+}(\psi^{n})^{+}\right)_{k-\frac{1}{2}}\right) = 0,$$
(4.23)

by the properties of the projection P^+ and P^- we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(\pi^{n} - \pi_{n}^{e}), \pi^{n}\right) - \varepsilon \left(\sigma^{n} - \sigma_{n}^{e}, \pi^{n}\right) + \left(\gamma(x)^{n}, \pi^{n}\right)
+ \left(\sigma^{n} - \sigma_{n}^{e}, -\varphi^{n} + \sigma^{n}\right) - \left(\Delta_{(\beta-2)/2}(\varphi^{n} - \varphi_{n}^{e}), -\varphi^{n} + \sigma^{n}\right) + \left(\varphi^{n} - \varphi_{n}^{e}, \pi^{n}\right)
+ \sum_{k=1}^{K} \left(\left((\psi_{n}^{e})^{-}(\pi^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\psi_{n}^{e})^{-}(\pi^{n})^{+}\right)_{k-\frac{1}{2}}\right) + \left(\psi^{n} - \psi_{n}^{e}, \psi^{n}\right)
+ \sum_{k=1}^{K} \left(\left((\pi_{n}^{e})^{+}(\psi^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\pi_{n}^{e})^{+}(\psi^{n})^{+}\right)_{k-\frac{1}{2}}\right) = 0.$$
(4.24)

Employing Young's inequality and Lemma 2.4 and the interpolation property (4.18) and (4.14), we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} \pi^{n}, \pi^{n}\right) + (\sigma^{n}, \sigma^{n}) + (\Delta_{(\beta-2)/2} \varphi^{n}, \varphi^{n}) + (\psi^{n}, \psi^{n})
\leq C(h^{2N+2} + (\Delta t)^{4+\theta} + \theta^{4}) + \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} \pi_{n}^{e}, \pi^{n}\right) + c_{2} \|\sigma^{n}\|_{L^{2}(\Omega)}^{2} + c_{3} \|\varphi^{n}\|_{L^{2}(\Omega)}^{2}
+ c \|\pi^{n}\|_{L^{2}(\Omega)}^{2} + c_{1} \|\psi^{n}\|_{L^{2}(\Omega)}^{2},$$
(4.25)

by using Lemma 3.1, (4.17) and the interpolation property (4.18), we get

$$\|\delta_t^{\alpha}(\mathcal{P}^+u(x,t_n) - u(x,t_n))\|_{L^2(\Omega)} \le C(h^{N+1} + (\Delta t)^{2-\alpha}). \tag{4.26}$$

From (3.1), (4.15) and (4.26), we obtain

$$\left\| \sum_{j=1}^{S} W(\alpha_j) \Delta \tau_j \delta_t^{\alpha_j} (\mathcal{P}^+ u(x, t_n) - u(x, t_n)) \right\|_{L^2(\Omega)} \le C \left(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2 \right). \tag{4.27}$$

Hence

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} \pi^{n}, \pi^{n}\right) + \left(\sigma^{n}, \sigma^{n}\right) + \left(\Delta_{(\beta-2)/2} \varphi^{n}, \varphi^{n}\right) + \left(\psi^{n}, \psi^{n}\right)
\leq C \left(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}\right) + c_{2} \|\sigma^{n}\|_{L^{2}(\Omega)}^{2} + c_{3} \|\varphi^{n}\|_{L^{2}(\Omega)}^{2} + c \|\pi^{n}\|_{L^{2}(\Omega)}^{2} + c_{1} \|\psi^{n}\|_{L^{2}(\Omega)}^{2}.$$
(4.28)

Recalling Lemma 2.2 and provided c_i , i = 1, 2 are sufficiently small such that $c_i \leq 1$, we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_j) \Delta \tau_j \delta_t^{\alpha_j} \pi^n, \pi^n\right) \le C \left(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^4\right) + c \|\pi^n\|_{L^2(\Omega)}^2. \tag{4.29}$$

It then follows that

$$\left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \pi^{n}, \pi_{h}^{n}\right) \leq \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \pi^{l}, \pi^{n}\right) + \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} a_{n-1}^{\alpha_{j}} \pi^{0}, \pi^{n}\right) + c \|\pi^{n}\|_{L^{2}(\Omega)}^{2} + C(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}).$$
(4.30)

Employing Young's inequality, we obtain

$$\|\pi^{n}\|_{L^{2}(\Omega)}^{2} \leq \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \|\pi^{l}\|_{L^{2}(\Omega)}^{2} + \frac{1}{4} \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q(a_{0}^{\alpha_{j}} - a_{n-1}^{\alpha_{j}}) \|\pi^{n}\|_{L^{2}(\Omega)}^{2}$$

$$+ \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Qa_{n-1}^{\alpha_{j}} \|\pi^{0}\|_{L^{2}(\Omega)}^{2} + \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{4\lambda_{j}} Qa_{n-1}^{\alpha_{j}} \|\pi^{n}\|_{L^{2}(\Omega)}^{2}$$

$$+ cQ\|\pi^{n}\|_{L^{2}(\Omega)}^{2} + CQ(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}).$$

$$(4.31)$$

Notice the facts that

$$\|\pi^0\|_{L^2(\Omega)} \le Ch^{N+1}.$$
 (4.32)

Thus,

$$\|\pi^{n}\|_{L^{2}(\Omega)}^{2} \leq \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \|\pi^{l}\|_{L^{2}(\Omega)}^{2} + (cQ + \frac{1}{4}) \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q \|\pi^{n}\|_{L^{2}(\Omega)}^{2}$$

$$+ C \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} h^{2N+2} + C \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} \left(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}\right),$$

$$(4.33)$$

provided c is sufficiently small such that $\frac{3}{4} - cQ > 0$, we obtain that

$$\|\pi^{n}\|_{L^{2}(\Omega)}^{2} \leq C\left(\sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \|\pi^{l}\|_{L^{2}(\Omega)}^{2} + \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} (h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4})\right). \tag{4.34}$$

Obviously the theorem holds for n = 0. Assume that it is valid for n = 1, 2, ..., m - 1. Then, by (4.34), we have

$$\|\pi^{m}\|_{L^{2}(\Omega)}^{2} \leq C \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{m-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \|\pi^{l}\|_{L^{2}(\Omega)}^{2} + \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} (h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}) \right)$$

$$\leq C \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{m-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) (h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}) + \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} (h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}) \right)$$

$$= C \left(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4} \right). \tag{4.35}$$

Finally, by using triangle inequality and standard approximation theory, we get

$$||u(x,t_m) - u_h^m||_{L^2(\Omega)} \le C(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2). \quad \Box$$
 (4.36)

For the more general fractional convection-diffusion problem, we introduce a few results and then give the error estimate.

Lemma 4.1. (see [43]). For any piecewise smooth function $\pi \in L^2(\Omega)$, on each cell boundary point we define

$$\kappa(\widehat{f}; \pi) \equiv \kappa(\widehat{f}; \pi^{-}, \pi^{+}) = \begin{cases} [w]^{-1} (f(\pi) - \widehat{f}(\pi)), & \text{if } [\pi] \neq 0; \\ \frac{1}{2} |f'(\overline{\pi})|, & \text{if } [\pi] = 0, \end{cases}$$

$$(4.37)$$

where $\widehat{f}(\pi) \equiv \widehat{f}(\pi, \pi^+)$ is a monotone numerical flux consistent with the given flux f. Then $\kappa(\widehat{f}, \pi)$ is nonnegative and bounded for any $(\pi, \pi^+) \in \mathbb{R}$.

To estimate the nonlinear part, we can write it into the following form

$$\sum_{k=1}^{K} \mathcal{H}_{k}(f; u, u_{h}; \pi) = \sum_{k=1}^{K} \left(f(u) - f(u_{h}), \frac{\partial}{\partial x} \pi \right)_{D^{k}} + \sum_{k=1}^{K} \left((f(u) - f(u_{h})[\pi])_{k+\frac{1}{2}} + \sum_{k=1}^{K} \left((f(u_{h}) - \widehat{f})[\pi] \right)_{k+\frac{1}{2}}.$$
(4.38)

We can rewrite (4.38) as:

$$\sum_{k=1}^{K} \mathcal{H}_{k}(f; u, u_{h}; \pi) = \sum_{k=1}^{K} \left(f(u) - f(u_{h}), \frac{\partial}{\partial x} \pi \right)_{D^{k}} + \sum_{k=1}^{K} ((f(u) - f(\{u_{h}\})[\pi])_{k + \frac{1}{2}} + \sum_{k=1}^{K} ((f(\{u_{h}\}) - \widehat{f})[\pi])_{k + \frac{1}{2}}.$$

$$(4.39)$$

Lemma 4.2. (see [43]) For $\mathcal{H}_k(f; u, u_h; \pi)$ defined above, we have the following estimate:

$$\sum_{k=1}^{K} \mathcal{H}_{k}(f; u, u_{h}; v) \leq -\frac{1}{4} \kappa(\hat{f}; u_{h}) + (C + C_{*}(\|v\|_{\infty} + h^{-1}\|e_{u}\|_{\infty}^{2}))\|v\|^{2} + (C + C_{*}h^{-1}\|e_{u}\|_{\infty}^{2})h^{2N+1}. \tag{4.40}$$

To deal with the nonlinearity of the flux f(u), we make the following assumption for h small enough and $k \ge 1$, which can be verified [44]:

$$||e_u|| = ||u - u_h|| < h. (4.41)$$

Theorem 4.3. Let $u(x, t_n)$ be the exact solution of the problem (1.1), which is sufficiently smooth with bounded derivatives, let u_h^n be the numerical solution of the fully discrete LDG scheme (3.12), then there holds the following error estimates:

$$||u(x,t_n) - u_h^n||_{L^2(\Omega)} \le C(h^{N + \frac{1}{2}} + (\Delta t)^{1 + \frac{\theta}{2}} + \theta^2). \tag{4.42}$$

Proof. Using (4.17), the error equation (4.13) can be written

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(\pi^{n} - \pi_{n}^{e}), v\right) - \varepsilon \left(\sigma^{n} - \sigma_{n}^{e}, v\right) + \left(\gamma(x)^{n}, v\right) - \sum_{k=1}^{K} \mathcal{H}_{k}(f; u, u_{h}; v)
+ \left(\sigma^{n} - \sigma_{n}^{e}, \psi\right) - \left(\Delta_{(\beta-2)/2}(\varphi^{n} - \varphi_{n}^{e}), \psi\right) + \left(\varphi^{n} - \varphi_{n}^{e}, \phi\right) + \left(\psi^{n} - \psi_{n}^{e}, \frac{\partial \phi}{\partial x}\right)
- \sum_{k=1}^{K} \left(\left((\psi^{n} - \psi_{n}^{e})^{-} \phi^{-}\right)_{k+\frac{1}{2}} - \left((\psi^{n} - \psi_{n}^{e})^{-} \phi^{+}\right)_{k-\frac{1}{2}}\right) + \left(\psi^{n} - \psi_{n}^{e}, \eta\right) + \left(\pi^{n} - \pi_{n}^{e}, \frac{\partial \eta}{\partial x}\right)
- \sum_{k=1}^{K} \left(\left((\pi^{n} - \pi_{n}^{e})^{+} \eta^{-}\right)_{k+\frac{1}{2}} - \left((\pi^{n} - \pi_{n}^{e})^{+} \eta^{+}\right)_{k-\frac{1}{2}}\right) = 0.$$
(4.43)

Following the proof of Theorem 4.2, we take the test functions

$$v = \pi^n, \quad \psi = -\varphi^n + \sigma^n, \quad \phi = \pi^n, \quad \eta = \psi^n, \tag{4.44}$$

we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(\pi^{n} - \pi_{n}^{e}), \pi^{n}\right) - \varepsilon \left(\sigma^{n} - \sigma_{n}^{e}, \pi^{n}\right) + \left(\gamma(x)^{n}, \pi^{n}\right) - \sum_{k=1}^{K} \mathcal{H}_{k}(f; u, u_{h}; \pi^{n}) \\
+ \left(\sigma^{n} - \sigma_{n}^{e}, -\varphi^{n} + \sigma^{n}\right) - \left(\Delta_{(\beta-2)/2}(\varphi^{n} - \varphi_{n}^{e}), -\varphi^{n} + \sigma^{n}\right) + \left(\varphi^{n} - \varphi_{n}^{e}, \pi^{n}\right) + \left(\psi^{n} - \psi_{n}^{e}, \frac{\partial \pi^{n}}{\partial x}\right) \\
- \sum_{k=1}^{K} \left(\left((\psi^{n} - \psi_{n}^{e})^{-}(\pi^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\psi^{n} - \psi_{n}^{e})^{-}(\pi^{n})^{+}\right)_{k-\frac{1}{2}}\right) + \left(\psi^{n} - \psi_{n}^{e}, \psi^{n}\right) + \left(\pi^{n} - \pi_{n}^{e}, \frac{\partial \psi^{n}}{\partial x}\right) \\
- \sum_{k=1}^{K} \left(\left((\pi^{n} - \pi_{n}^{e})^{+}(\psi^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\pi^{n} - \pi_{n}^{e})^{+}(\psi^{n})^{+}\right)_{k-\frac{1}{2}}\right) = 0,$$
(4.45)

by the properties of the projection P^+ and P^- , we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(\pi^{n} - \pi_{n}^{e}), \pi^{n}\right) - \varepsilon \left(\sigma^{n} - \sigma_{n}^{e}, \pi^{n}\right) + \left(\gamma(x)^{n}, \pi^{n}\right) - \sum_{k=1}^{K} \mathcal{H}_{k}(f; u, u_{h}; \pi^{n})
+ \left(\sigma^{n} - \sigma_{n}^{e}, -\varphi^{n} + \sigma^{n}\right) - \left(\Delta_{(\beta-2)/2}(\varphi^{n} - \varphi_{n}^{e}), -\varphi^{n} + \sigma^{n}\right) + \left(\varphi^{n} - \varphi_{n}^{e}, \pi^{n}\right)
+ \sum_{k=1}^{K} \left(\left((\psi_{n}^{e})^{-}(\pi^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\psi_{n}^{e})^{-}(\pi^{n})^{+}\right)_{k-\frac{1}{2}}\right) + \left(\psi^{n} - \psi_{n}^{e}, \psi^{n}\right)
+ \sum_{k=1}^{K} \left(\left((\pi_{n}^{e})^{+}(\psi^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\pi_{n}^{e})^{+}(\psi^{n})^{+}\right)_{k-\frac{1}{2}}\right) = 0.$$
(4.46)

Employing Young's inequality and Lemma 2.4 and the interpolation property (4.18), (4.26) and (4.27), we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} \pi^{n}, \pi^{n}\right) + \left(\sigma^{n}, \sigma^{n}\right) + \left(\Delta_{(\beta-2)/2} \varphi^{n}, \varphi^{n}\right) + \left(\psi^{n}, \psi^{n}\right)
\leq C \left(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}\right) + c_{2} \|\sigma^{n}\|_{L^{2}(\Omega)}^{2} + c_{3} \|\varphi^{n}\|_{L^{2}(\Omega)}^{2} + c \|\pi^{n}\|_{L^{2}(\Omega)}^{2} + c_{1} \|\psi^{n}\|_{L^{2}(\Omega)}^{2}
- \frac{1}{4} \kappa (f^{*}; u_{h}^{n}) + \left(C + C_{*}(\|\pi^{n}\|_{\infty} + h^{-1} \|e_{u}\|_{\infty}^{2})) \|\pi^{n}\|^{2} + \left(C + C_{*}h^{-1} \|e_{u}\|_{\infty}^{2}\right) h^{2N+1}.$$
(4.47)

Recalling Lemma 2.2 and provided c_i , i = 1, 2 are sufficiently small such that $c_i \leq 1$, we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_j) \Delta \tau_j \delta_t^{\alpha_j} \pi^n, \pi^n\right) \le C \left(h^{2N+1} + (\Delta t)^{2+\theta} + \theta^4\right) + c_4 \|\pi^n\|_{L^2(\Omega)}^2. \tag{4.48}$$

It then follows that

$$\left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \pi^{n}, \pi_{h}^{n}\right) \leq \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \pi^{l}, \pi^{n}\right) + \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} a_{n-1}^{\alpha_{j}} \pi^{0}, \pi^{n}\right) + c \|\pi^{n}\|_{L^{2}(\Omega)}^{2} + C\left(h^{2N+1} + (\Delta t)^{2+\theta} + \theta^{4}\right). \tag{4.49}$$

Employing Young's inequality, we obtain

$$\|\pi^{n}\|_{L^{2}(\Omega_{h})}^{2} \leq \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \|\pi^{l}\|_{L^{2}(\Omega_{h})}^{2} + (cQ + \frac{1}{4}) \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q \|\pi^{n}\|_{L^{2}(\Omega_{h})}^{2} + C \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} h^{2N+2} + C \sum_{j=1}^{S} \frac{W(\alpha_{j})\Delta\tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} (h^{2N+1} + (\Delta t)^{2+\theta} + \theta^{4}),$$

$$(4.50)$$

provided c is sufficiently small such that $\frac{3}{4} - cQ > 0$, we obtain that

$$\|\pi^{n}\|_{L^{2}(\Omega)}^{2} \leq C \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \|\pi^{l}\|_{L^{2}(\Omega)}^{2} + \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} (h^{2N+1} + (\Delta t)^{2+\theta} + \theta^{4}) \right).$$

$$(4.51)$$

Obviously the theorem holds for n = 0. Assume that it is valid for n = 1, 2, ..., m - 1. Then, by (4.51), we have

$$\|\pi^{m}\|_{L^{2}(\Omega)}^{2} \leq C \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{m-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \|\pi^{l}\|_{L^{2}(\Omega)}^{2} + \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} \left(h^{2N+1} + (\Delta t)^{2+\theta} + \theta^{4} \right) \right)$$

$$\leq C \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{m-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \left(h^{2N+1} + (\Delta t)^{2+\theta} + \theta^{4} \right) + \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} \left(h^{2N+1} + (\Delta t)^{4+\theta} + \theta^{4} \right) \right)$$

$$= C \left(h^{2N+1} + (\Delta t)^{2+\theta} + \theta^{4} \right). \tag{4.52}$$

Finally, by using triangle inequality and standard approximation theory, we can get (4.42).

5. LDG method for the nonlinear distributed-order time and space-fractional Schrödinger type equations

5.1. LDG method for the nonlinear distributed-order time and space-fractional Schrödinger equation

We rewrite the fractional derivative as a composite of first order derivatives and a fractional integral to recover the equation to a low order system. However, for the first order system, alternating fluxes are used. We introduce three variables e, r, s and set

$$e = \Delta_{(\beta-2)/2}r, \quad r = \frac{\partial}{\partial x}s, \quad s = \frac{\partial}{\partial x}u,$$
 (5.1)

then, the nonlinear distributed-order time and space-fractional Schrödinger problem can be rewritten as

$$i\mathcal{D}_{t}^{W(\alpha)}u + \varepsilon_{1}e + \varepsilon_{2}f(|u|^{2})u = 0,$$

$$e = \Delta_{(\beta-2)/2}r, \quad r = \frac{\partial}{\partial x}s, \quad s = \frac{\partial}{\partial x}u.$$
(5.2)

For actual numerical implementation, it might be more efficient if we decompose the complex function u(x,t) into its real and imaginary parts by writing

$$u(x,t) = p(x,t) + iq(x,t), \tag{5.3}$$

where p, q are real functions. Under the new notation, the problem (5.2) can be written as

$$\mathcal{D}_{t}^{W(\alpha)} p + \varepsilon_{1} e + \varepsilon_{2} f(p^{2} + q^{2}) q = 0,$$

$$e = \Delta_{(\beta-2)/2} r, \quad r = \frac{\partial}{\partial x} s, \quad s = \frac{\partial}{\partial x} q,$$

$$\mathcal{D}_{t}^{W(\alpha)} q - \varepsilon_{1} l - \varepsilon_{2} f(p^{2} + q^{2}) p = 0,$$

$$l = \Delta_{(\alpha-2)/2} w, \quad w = \frac{\partial}{\partial x} z, \quad z = \frac{\partial}{\partial x} p.$$

$$(5.4)$$

Let $p_h^n, q_h^n, e_h^n, l_h^n, r_h^n, s_h^n, w_h^n, z_h^n \in V_k^N$ be the approximation of $p(., t_n), q(., t_n), e(., t_n), l(., t_n)r(., t_n), s(., t_n),$ $w(., t_n), z(., t_n)$ respectively. We a fully discrete local discontinuous Galerkin scheme as follows: find $p_h^n, q_h^n, e_h^n, l_h^n, r_h^n, s_h^n, w_h^n, z_h^n \in V_k^N$, such that for all test functions $\vartheta_1, \rho, \phi, \varphi, \chi, \varrho, \psi, \zeta \in V_k^N$,

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} p_{h}^{n}, \vartheta_{1}\right)_{D^{k}} + \varepsilon_{1} \left(e_{h}^{n}, \vartheta_{1}\right)_{D^{k}} + \varepsilon_{2} \left(f\left((p_{h}^{n})^{2} + (q_{h}^{n})^{2}\right) q_{h}^{n}, \vartheta_{1}\right)_{D^{k}} = 0,$$

$$\left(e_{h}^{n}, \rho\right)_{D^{k}} = \left(\Delta_{(\beta-2)/2} r_{h}^{n}, \rho\right)_{D^{k}},$$

$$\left(r_{h}^{n}, \phi\right)_{D^{k}} = \left(\frac{\partial}{\partial x} s_{h}^{n}, \phi\right)_{D^{k}},$$

$$\left(s_{h}^{n}, \varphi\right)_{D^{k}} = \left(\frac{\partial}{\partial x} q_{h}^{n}, \varphi\right)_{D^{k}},$$

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} q_{h}^{n}, \chi\right)_{D^{k}} - \varepsilon_{1} \left(l_{h}^{n}, \chi\right)_{D^{k}} - \varepsilon_{2} \left(f\left((p_{h}^{n})^{2} + (q_{h}^{n})^{2}\right)) p_{h}^{n}, \chi\right)_{D^{k}} = 0,$$

$$\left(l_{h}^{n}, \varrho\right)_{D^{k}} = \left(\Delta_{(\beta-2)/2} w_{h}^{n}, \varrho\right)_{D^{k}},$$

$$\left(w_{h}^{n}, \psi\right)_{D^{k}} = \left(\frac{\partial}{\partial x} z_{h}^{n}, \psi\right)_{D^{k}},$$

$$\left(z_{h}^{n}, \zeta\right)_{D^{k}} = \left(\frac{\partial}{\partial x} p_{h}^{n}, \zeta\right)_{D^{k}}.$$
(5.5)

Applying integration by parts to (5.5), and replacing the fluxes at the interfaces by the corresponding numerical fluxes, we obtain

$$\begin{split} & \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} p_{h}^{n}, \vartheta_{1}\right)_{D^{k}} + \varepsilon_{1} \left(e_{h}^{n}, \vartheta_{1}\right)_{D^{k}} + \varepsilon_{2} \left(\left(f((p_{h}^{n})^{2} + (q_{h}^{n})^{2}))q_{h}^{n}, \vartheta_{1}\right)_{D^{k}} = 0, \\ & \left(e_{h}^{n}, \rho\right)_{D^{k}} = \left(\Delta_{(\beta-2)/2} r_{h}^{n}, \rho\right)_{D^{k}}, \\ & \left(r_{h}^{n}, \phi\right)_{D^{k}} = -\left(s_{h}^{n}, \phi_{x}\right)_{D^{k}} + \left(\left(\widehat{s}_{h}^{n} \phi^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{s}_{h}^{n} \phi^{+}\right)_{k-\frac{1}{2}}\right), \\ & \left(s_{h}^{n}, \varphi\right)_{D^{k}} = -\left(q_{h}^{n}, \varphi_{x}\right)_{D^{k}} + \left(\left(\widehat{q}_{h}^{n} \varphi^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{q}_{h}^{n} \varphi^{+}\right)_{k-\frac{1}{2}}\right), \\ & \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} q_{h}^{n}, \chi\right)_{D^{k}} - \varepsilon_{1} \left(l_{h}^{n}, \chi\right)_{D^{k}} - \varepsilon_{2} \left(\left(f\left((p_{h}^{n})^{2} + (q_{h}^{n})^{2}\right))p_{h}^{n}, \chi\right)_{D^{k}} = 0, \\ & \left(l_{h}^{n}, \varrho\right)_{D^{k}} = \left(\Delta_{(\beta-2)/2} w_{h}^{n}, \varrho\right)_{D^{k}}, \\ & \left(w_{h}^{n}, \psi\right)_{D^{k}} = - \left(z_{h}^{n}, \psi_{x}\right)_{D^{k}} + \left(\left(\widehat{z}_{h}^{n} \psi^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{z}_{h}^{n} \psi^{+}\right)_{k-\frac{1}{2}}\right), \\ & \left(z_{h}^{n}, \zeta\right)_{D^{k}} = -\left(p_{h}^{n}, \zeta_{x}\right)_{D^{k}} + \left(\left(\widehat{p}_{h}^{n} \zeta^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{p}_{h}^{n} \zeta^{+}\right)_{k-\frac{1}{2}}\right). \end{split}$$

The numerical traces $(\hat{p}_h^n, \hat{q}_h^n, \hat{s}_h^n, \hat{z}_h^n)$ are defined on interelement faces as the alternating fluxes [45, 46]:

$$\widehat{p}_h^n = (p_h^n)^-, \ \widehat{s}_h^n = (s_h^n)^+, \ \widehat{q}_h^n = (q_h^n)^-, \ \widehat{z}_h^n = (z_h^n)^+. \tag{5.7}$$

Note that we can also choose

$$\widehat{p}_h^n = (p_h^n)^+, \ \widehat{s}_h^n = (s_h^n)^-, \ \widehat{q}_h^n = (q_h^n)^+, \ \widehat{z}_h^n = (z_h^n)^-. \tag{5.8}$$

5.1.1. The analysis of stability for fully discrete scheme

In order to carry out the analysis of the LDG scheme, we have the following results.

Theorem 5.1. (L^2 stability). The semidiscrete scheme (5.6) is stable, and

$$||p_h^n||_{L^2(\Omega)}^2 + ||q_h^n||_{L^2(\Omega)}^2 \le C(||p_h^0||_{L^2(\Omega)}^2 + ||q_h^0||_{L^2(\Omega)}^2).$$

$$(5.9)$$

Proof. Set $(\vartheta_1, \rho, \phi, \varphi, \chi, \varrho, \psi, \zeta) = (p_h^n, -r_h^n + e_h^n, p_h^n, -z_h^n, q_h^n, l_h^n - w_h^n, -q_h^n, s_h^n)$ in (5.6), we get

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} p_{h}^{n}, p_{h}^{n}\right)_{D^{k}} + \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} q_{h}^{n}, q_{h}^{n}\right)_{D^{k}} + \left(e_{h}^{n}, e_{h}^{n}\right)_{D^{k}} + \left(l_{h}^{n}, l_{h}^{n}\right)_{D^{k}} + \left(\Delta_{(\beta-2)/2} w_{h}^{n}, w_{h}^{n}\right)_{D^{k}} + \left(\Delta_{(\beta-2)/2} w_{h}^{n}, w_{h}^{n}\right)_{D^{k}} + \left(\Delta_{(\beta-2)/2} v_{h}^{n}, e_{h}^{n}\right)_{D^{k}} - \left(r_{h}^{n}, p_{h}^{n}\right)_{D^{k}} + \left(w_{h}^{n}, q_{h}^{n}\right)_{D^{k}} + \left(e_{h}^{n}, r_{h}^{n}\right)_{D^{k}} + \left(e_{h}^{n}, r_{h}^{n}\right)_{D^{k}} - \left(s_{h}^{n}, (p_{h}^{n})_{x}\right)_{D^{k}} + \left(r_{h}^{n}, (q_{h}^{n})_{x}\right)_{D^{k}} - \left(r_{h}^{n}, (s_{h}^{n})_{x}\right)_{D^{k}} + \left(l_{h}^{n}, w_{h}^{n}\right)_{D^{k}} - \varepsilon_{1}\left(e_{h}^{n}, p_{h}^{n}\right)_{D^{k}} + \varepsilon_{1}\left(l_{h}^{n}, q_{h}^{n}\right)_{D^{k}} + \left(\left(\widehat{s}_{h}^{n}(p_{h}^{n})^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{s}_{h}^{n}(p_{h}^{n})^{+}\right)_{k-\frac{1}{2}}\right) - \left(\left(\widehat{q}_{h}^{n}(z_{h}^{n})^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{q}_{h}^{n}(z_{h}^{n})^{+}\right)_{k-\frac{1}{2}}\right) - \left(\left(\widehat{s}_{h}^{n}(q_{h}^{n})^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{s}_{h}^{n}(q_{h}^{n})^{+}\right)_{k-\frac{1}{2}}\right) + \left(\left(\widehat{p}_{h}^{n}(s_{h}^{n})^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{p}_{h}^{n}(s_{h}^{n})^{+}\right)_{k-\frac{1}{2}}\right). \tag{5.10}$$

Summing over k, with the definition (5.7) of the numerical fluxes and with simple algebraic manipulations, we easily obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} p_{h}^{n}, p_{h}^{n}\right) + \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} q_{h}^{n}, q_{h}^{n}\right) + \left(e_{h}^{n}, e_{h}^{n}\right) + \left(l_{h}^{n}, l_{h}^{n}\right) + \left(\Delta_{(\beta-2)/2} w_{h}^{n}, w_{h}^{n}\right) \\
+ \left(\Delta_{(\beta-2)/2} r_{h}^{n}, r_{h}^{n}\right) = \left(\Delta_{(\beta-2)/2} w_{h}^{n}, l_{h}^{n}\right) + \left(\Delta_{(\beta-2)/2} r_{h}^{n}, e_{h}^{n}\right) - \left(r_{h}^{n}, p_{h}^{n}\right) + \left(w_{h}^{n}, q_{h}^{n}\right) + \left(e_{h}^{n}, r_{h}^{n}\right).$$
(5.11)

Employing Young's inequality and Lemma 2.4, we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} p_{h}^{n}, p_{h}^{n}\right) + \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} q_{h}^{n}, q_{h}^{n}\right) + \left(e_{h}^{n}, e_{h}^{n}\right)_{D^{k}} + \left(l_{h}^{n}, l_{h}^{n}\right) + \left(\Delta_{(\beta-2)/2} w_{h}^{n}, w_{h}^{n}\right) + \left(\Delta_{(\beta-2)/2} v_{h}^{n}, r_{h}^{n}\right)_{\Omega_{h}} \leq c \|p_{h}^{n}\|_{L^{2}(\Omega)}^{2} + c \|q_{h}^{n}\|_{L^{2}(\Omega)}^{2} + c_{6} \|w_{h}^{n}\|_{L^{2}(\Omega)}^{2} + c_{5} \|r_{h}^{n}\|_{L^{2}(\Omega)}^{2} + c_{1} \|e_{h}^{n}\|_{L^{2}(\Omega)}^{2} + c_{1} \|e_{h}^{n}\|_{L^{2}(\Omega)}^{2} + c_{2} \|l_{h}^{n}\|_{L^{2}(\Omega)}^{2}.$$

$$(5.12)$$

Recalling Lemma 2.2 and provided c_i , i = 1, 2 are sufficiently small such that $c_i \leq 1$, we obtain that

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} p_{h}^{n}, p_{h}^{n}\right) + \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} q_{h}^{n}, q_{h}^{n}\right) \leq c(\|p_{h}^{n}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{n}\|_{L^{2}(\Omega)}^{2}). \tag{5.13}$$

It then follows that

$$\begin{split} &\left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} p^{n}, p_{h}^{n}\right) + \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} q^{n}, q_{h}^{n}\right) \\ &\leq \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) p_{h}^{l}, p_{h}^{n}\right) + \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} a_{n-1}^{\alpha_{j}} p_{h}^{0}, p_{h}^{n}\right) \\ &+ \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) q_{h}^{l}, q_{h}^{n}\right) + \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} a_{n-1}^{\alpha_{j}} q_{h}^{0}, q_{h}^{n}\right) + c(\|p_{h}^{n}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{n}\|_{L^{2}(\Omega)}^{2}). \end{split}$$

Employing Young's inequality, we obtain

$$\begin{split} &\|p_{h}^{n}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{n}\|_{L^{2}(\Omega)}^{2} \\ &\leq \frac{1}{2} \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \bigg(\sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \big(\|p_{h}^{l}\|_{L^{2}(\Omega)}^{2} + \|p_{h}^{n}\|_{L^{2}(\Omega)}^{2} \big) \bigg) + \frac{1}{2} \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} (\|p_{h}^{0}\|_{L^{2}(\Omega)}^{2} + \|p_{h}^{n}\|_{L^{2}(\Omega)}^{2}) \bigg) \\ &+ \|p_{h}^{n}\|_{L^{2}(\Omega)}^{2} \big) + \frac{1}{2} \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \bigg(\sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \big(\|q_{h}^{l}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{n}\|_{L^{2}(\Omega)}^{2} \big) \bigg) \\ &+ \frac{1}{2} \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} \big(\|q_{h}^{0}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{n}\|_{L^{2}(\Omega)}^{2} \big) + cQ \big(\|p_{h}^{n}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{n}\|_{L^{2}(\Omega)}^{2} \big), \end{split}$$

$$(5.15)$$

provided c is sufficiently small such that $\frac{1}{2} - cQ > 0$, we obtain that

$$\|p_{h}^{n}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{n}\|_{L^{2}(\Omega)}^{2}$$

$$\leq C \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \left(\sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \left(\|p_{h}^{l}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{l}\|_{L^{2}(\Omega)}^{2} \right) \right) + C \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} \left(\|p_{h}^{0}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{0}\|_{L^{2}(\Omega)}^{2} \right)$$

$$+ \|q_{h}^{0}\|_{L^{2}(\Omega)}^{2} \right).$$

$$(5.16)$$

Obviously the theorem holds for n = 0. Assume that it is valid for n = 1, 2, ..., m - 1. Then, by (5.16), we have

$$\begin{split} \|p_{h}^{m}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{m}\|_{L^{2}(\Omega)}^{2} \\ &\leq C \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \bigg(\sum_{l=1}^{m-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \big(\|p_{h}^{l}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{l}\|_{L^{2}(\Omega)}^{2} \big) \bigg) + C \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} \big(\|p_{h}^{0}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{0}\|_{L^{2}(\Omega)}^{2} \big) \bigg) \\ &+ \|q_{h}^{0}\|_{L^{2}(\Omega)}^{2} \big). \\ &\leq C \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \bigg(\sum_{l=1}^{m-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \big(\|p_{h}^{0}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{0}\|_{L^{2}(\Omega)}^{2} \big) \bigg) + C \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} \big(\|p_{h}^{0}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{0}\|_{L^{2}(\Omega)}^{2} \big) \bigg) \\ &+ \|q_{h}^{0}\|_{L^{2}(\Omega)}^{2} \big). \\ &= C \Big(\|p_{h}^{0}\|_{L^{2}(\Omega)}^{2} + \|q_{h}^{0}\|_{L^{2}(\Omega)}^{2} \big). \quad \Box \end{split} \tag{5.17}$$

5.1.2. Error estimates

We consider the linear distributed-order time and space-fractional Schrödinger equation

$$i\mathcal{D}_t^{W(\alpha)}u - \varepsilon_1(-\Delta)^{\frac{\alpha}{2}}u + \varepsilon_2 u = 0. \tag{5.18}$$

It is easy to verify that the exact solution of the above (5.18) satisfies

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} p^{n}, \vartheta_{1}\right)_{D^{k}} + \varepsilon_{1} \left(e^{n}, \vartheta_{1}\right)_{D^{k}} + \varepsilon_{2} \left(q^{n}, \vartheta_{1}\right)_{D^{k}} + \left(\gamma(x)^{n}, \vartheta_{1}\right)_{D^{k}} = 0,$$

$$\left(e^{n}, \rho\right)_{D^{k}} = \left(\Delta_{(\beta-2)/2} r^{n}, \rho\right)_{D^{k}},$$

$$\left(r^{n}, \phi\right)_{D^{k}} = -\left(s^{n}, \phi_{x}\right)_{D^{k}} + \left(\left(\widehat{s}^{n} \phi^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{s}^{n} \phi^{+}\right)_{k-\frac{1}{2}}\right),$$

$$\left(s^{n}, \varphi\right)_{D^{k}} = -\left(q^{n}, \varphi_{x}\right)_{D^{k}} + \left(\left(\widehat{q}^{n} \varphi^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{q}^{n} \varphi^{+}\right)_{k-\frac{1}{2}}\right),$$

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} q^{n}, \chi\right)_{D^{k}} - \varepsilon_{1} \left(l^{n}, \chi\right)_{D^{k}} - \varepsilon_{2} \left(p^{n}, \chi\right)_{D^{k}} = 0,$$

$$\left(l^{n}, \varrho\right)_{D^{k}} = \left(\Delta_{(\beta-2)/2} w^{n}, \varrho\right)_{D^{k}},$$

$$\left(w^{n}, \psi\right)_{D^{k}} = -\left(z^{n}, \psi_{x}\right)_{D^{k}} + \left(\left(\widehat{z}^{n} \psi^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{z}^{n} \psi^{+}\right)_{k-\frac{1}{2}}\right),$$

$$\left(z^{n}, \zeta\right)_{D^{k}} = -\left(p^{n}, \zeta_{x}\right)_{D^{k}} + \left(\left(\widehat{p}^{n} \psi^{-}\right)_{k+\frac{1}{2}} - \left(\widehat{p}^{n} \zeta^{+}\right)_{k-\frac{1}{2}}\right).$$

Subtracting (5.6) from (5.19), we can obtain the error equation

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(p^{n} - p_{h}^{n}), \vartheta_{1}\right)_{D^{k}} + \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(q^{n} - q_{h}^{n}), \chi\right)_{D^{k}} - \left(\Delta_{(\beta-2)/2}(r^{n} - r_{h}^{n}), \rho\right)_{D^{k}} - \left(\Delta_{(\beta-2)/2}(w^{n} - w_{h}^{n}), \varrho\right)_{D^{k}} + \left(s^{n} - s_{h}^{n}, \phi_{x}\right)_{D^{k}} + \left(q^{n} - q_{h}^{n}, \varphi_{x}\right)_{D^{k}} + \left(z^{n} - z_{h}^{n}, \psi_{x}\right)_{D^{k}} + \left(p^{n} - p_{h}^{n}, \zeta_{x}\right)_{D^{k}} + \left(\gamma(x)^{n}, \vartheta_{1}\right)_{D^{k}} + \varepsilon_{2}\left(q^{n} - q_{h}^{n}, \vartheta_{1}\right)_{D^{k}} - \varepsilon_{2}\left(p^{n} - p_{h}^{n}, \chi\right)_{D^{k}} + \left(r^{n} - r_{h}^{n}, \phi\right)_{D^{k}} + \left(s^{n} - s_{h}^{n}, \varphi\right)_{D^{k}} + \left(l^{n} - l_{h}^{n}, \varrho\right)_{D^{k}} + \left(e^{n} - e_{h}^{n}, \rho\right)_{D^{k}} + \left(w^{n} - w_{h}^{n}, \psi\right)_{D^{k}} + \left(z^{n} - z_{h}^{n}, \zeta\right)_{D^{k}} - \varepsilon_{1}\left(l^{n} - l_{h}^{n}, \chi\right)_{D^{k}} + \varepsilon_{1}\left(e^{n} - e_{h}^{n}, \vartheta_{1}\right)_{D^{k}} - \left(\left((\widehat{s}_{h}^{n} - \widehat{s}^{n})\phi^{-}\right)_{k+\frac{1}{2}} - \left((\widehat{s}_{h}^{n} - \widehat{s}^{n})\phi^{+}\right)_{k-\frac{1}{2}}\right) - \left(\left((\widehat{q}_{h}^{n} - \widehat{q}^{n})\varphi^{-}\right)_{k+\frac{1}{2}} - \left((\widehat{q}_{h}^{n} - \widehat{q}^{n})\varphi^{+}\right)_{k-\frac{1}{2}}\right) - \left(\left((\widehat{s}_{h}^{n} - \widehat{s}^{n})\psi^{-}\right)_{k+\frac{1}{2}} - \left((\widehat{s}_{h}^{n} - \widehat{s}^{n})\psi^{+}\right)_{k-\frac{1}{2}}\right) - \left(\left((\widehat{p}_{h}^{n} - \widehat{p}^{n})\zeta^{-}\right)_{k+\frac{1}{2}} - \left((\widehat{p}_{h}^{n} - \widehat{p}^{n})\zeta^{+}\right)_{k-\frac{1}{2}}\right) = 0. \tag{5.20}$$

Denoting

$$\pi^{n} = \mathcal{P}^{-}p^{n} - p_{h}^{n}, \quad \pi_{n}^{e} = \mathcal{P}^{-}p^{n} - p^{n}, \quad \epsilon^{n} = \mathcal{P}r^{n} - r_{h}^{n}, \quad \epsilon_{n}^{e} = \mathcal{P}r^{n} - r^{n}, \quad \phi^{n} = \mathcal{P}e^{n} - e_{h}^{n}, \quad \phi_{n}^{e} = \mathcal{P}e^{n} - e^{n},$$

$$\tau^{n} = \mathcal{P}^{+}s^{n} - s_{h}^{n}, \quad \tau_{n}^{e} = \mathcal{P}^{+}s^{n} - s^{n}, \quad \sigma^{n} = \mathcal{P}^{-}q^{n} - q_{h}^{n}, \quad \sigma_{n}^{e} = \mathcal{P}^{-}q^{n} - q^{n}, \quad \varpi^{n} = \mathcal{P}l^{n} - l_{h}^{n}, \quad \varpi_{n}^{e} = \mathcal{P}l^{n} - l^{n},$$

$$\varphi^{n} = \mathcal{P}w^{n} - w_{h}^{n}, \quad \varphi_{n}^{e} = \mathcal{P}w^{n} - w^{n}, \quad \vartheta^{n} = \mathcal{P}^{+}z^{n} - z_{h}^{n}, \quad \vartheta_{n}^{e} = \mathcal{P}^{+}z^{n} - z^{n}.$$

$$(5.21)$$

Lemma 5.1.

$$\left(\sum_{j=1}^{S} W(\alpha_j) \Delta \tau_j \delta_t^{\alpha_j} \pi^n, \pi^n\right) + \left(\sum_{j=1}^{S} W(\alpha_j) \Delta \tau_j \delta_t^{\alpha_j} \sigma^n, \sigma^n\right) + \left(\Delta_{(\beta-2)/2} \epsilon^n, \epsilon^n\right) + \left(\Delta_{(\beta-2)/2} \varphi^n, \varphi^n\right) + \left(\varphi^n, \varphi^n\right) + \left(\varpi^n, \varpi^n\right) = Q_1 + Q_2 + Q_3 + Q_4,$$
(5.22)

where

$$Q_1 = -(\epsilon^n, \pi^n) + (\varphi^n, \sigma^n) + (\Delta_{(\beta-2)/2}\epsilon^n, \phi^n) + (\Delta_{(\beta-2)/2}\varphi^n, \varpi^n)$$
(5.23a)

$$-\varepsilon_1(\phi^n, \pi^n) + \varepsilon_1(\varpi^n, \sigma^n) + (\varpi^n, \varphi^n) + (\phi^n, \epsilon^n), \tag{5.23b}$$

$$Q_2 = (\tau_n^e, \pi_n^n) - (\sigma_n^e, \vartheta_n^n) - (\vartheta_n^e, \sigma_n^n) + (\pi_n^e, \tau_n^n) + (\vartheta_n^e, \tau^n) - (\tau_n^e, \vartheta^n), \tag{5.23c}$$

$$Q_{3} = \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} \pi_{n}^{e}, \pi^{n}\right) + \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} \sigma_{n}^{e}, \sigma^{n}\right) + \left(\varpi^{e}, \varpi - \varphi\right)$$

$$(5.23d)$$

$$+\left(\phi_n^e,\phi^n-\epsilon^n\right)+\varepsilon_2\left(\sigma_n^e,\pi^n\right)-\varepsilon_2\left(\pi_n^e,\sigma^n\right)+\left(\epsilon_n^e,\pi^n\right)-\left(\varphi_n^e,\sigma^n\right) \tag{5.23e}$$

$$-\left(\Delta_{(\beta-2)/2}\epsilon_n^e, \phi^n - \epsilon^n\right) - \left(\Delta_{(\beta-2)/2}\varphi_n^e, \varpi^n - \varphi^n\right) + \varepsilon_1\left(\phi_n^e, \pi^n\right) - \varepsilon_1\left(\varpi_n^e, \sigma^n\right) \tag{5.23f}$$

$$+\left(\gamma(x)^n,\pi^n\right),$$
 (5.23g)

$$Q_{4} = -\sum_{k=1}^{K} \left(\left((\tau_{n}^{e})^{+}(\pi^{n})^{-} \right)_{k+\frac{1}{2}} - \left((\tau_{n}^{e})^{+}(\pi^{n})^{+} \right)_{k-\frac{1}{2}} \right) + \sum_{k=1}^{K} \left(\left((\sigma_{n}^{e})^{-}(\vartheta^{n})^{-} \right)_{k+\frac{1}{2}} - \left((\sigma_{n}^{e})^{-}(\vartheta^{n})^{+} \right)_{k-\frac{1}{2}} \right)$$

$$+ \sum_{k=1}^{K} \left(\left((\vartheta_{n}^{e})^{+}(\sigma^{n})^{-} \right)_{k+\frac{1}{2}} - \left((\vartheta_{n}^{e})^{+}(\sigma^{n})^{+} \right)_{k-\frac{1}{2}} \right) - \sum_{k=1}^{K} \left((\pi_{n}^{e})^{-}(\tau^{n})^{-} \right)_{k+\frac{1}{2}} - \left((\pi_{n}^{e})^{-}(\tau^{n})^{+} \right)_{k-\frac{1}{2}} \right).$$

$$(5.23i)$$

Proof. From the Galerkin orthogonality (5.20), we get

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(\pi^{n} - \pi_{n}^{e}), \vartheta_{1}\right)_{D^{k}} + \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(\sigma - \sigma^{e}), \chi\right)_{D^{k}} - \left(\Delta_{(\beta-2)/2}(\epsilon^{n} - \epsilon_{n}^{e}), \rho\right)_{D^{k}} \\
- \left(\Delta_{(\beta-2)/2}(\varphi^{n} - \varphi_{n}^{e}), \varrho\right)_{D^{k}} + \left(\tau^{n} - \tau_{n}^{e}, \phi_{x}\right)_{D^{k}} + \left(\sigma^{n} - \sigma_{n}^{e}, \varphi_{x}\right)_{D^{k}} + \left(\vartheta^{n} - \vartheta_{n}^{e}, \psi_{x}\right)_{D^{k}} + \left(\pi^{n} - \pi_{n}^{e}, \zeta_{x}\right)_{D^{k}} \\
+ \varepsilon_{2}\left(\sigma^{n} - \sigma_{n}^{e}, \vartheta_{1}\right)_{D^{k}} - \varepsilon_{2}\left(\pi^{n} - \pi_{n}^{e}, \chi\right)_{D^{k}} + \left(\epsilon^{n} - \epsilon_{n}^{e}, \phi\right)_{D^{k}} + \left(\tau^{n} - \tau_{n}^{e}, \varphi\right)_{D^{k}} + \left(\varpi^{n} - \varpi_{n}^{e}, \varrho\right)_{D^{k}} \\
+ \left(\phi^{n} - \phi_{n}^{e}, \rho\right)_{D^{k}} + \left(\varphi^{n} - \varphi_{n}^{e}, \psi\right)_{D^{k}} + \left(\vartheta^{n} - \vartheta_{n}^{e}, \zeta\right)_{D^{k}} + \varepsilon_{1}\left(\phi^{n} - \phi_{n}^{e}, \vartheta_{1}\right)_{D^{k}} - \varepsilon_{1}\left(\varpi^{n} - \varpi_{n}^{e}, \chi\right)_{D^{k}} \\
- \sum_{k=1}^{K} \left(\left((\tau^{n} - \tau_{n}^{e})^{+}(\phi)^{-}\right)_{k+\frac{1}{2}} - \left((\tau^{n} - \tau_{n}^{e})^{+}(\phi)^{+}\right)_{k-\frac{1}{2}}\right) - \sum_{k=1}^{K} \left(\left((\sigma^{n} - \sigma_{n}^{e})^{-}(\varphi)^{-}\right)_{k+\frac{1}{2}} - \left((\sigma^{n} - \sigma_{n}^{e})^{-}(\varphi)^{+}\right)_{k-\frac{1}{2}}\right) \\
- \sum_{k=1}^{K} \left(\left((\vartheta^{n} - \vartheta_{n}^{e})^{+}(\psi)^{-}\right)_{k+\frac{1}{2}} - \left((\vartheta^{n} - \vartheta_{n}^{e})^{+}(\psi)^{+}\right)_{k-\frac{1}{2}}\right) - \sum_{k=1}^{K} \left(\left(\pi^{n} - \pi_{n}^{e}\right)^{-}(\zeta)^{-}\right)_{k+\frac{1}{2}} \\
- \left((\pi^{n} - \pi_{n}^{e})^{-}(\zeta)^{+}\right)_{k-\frac{1}{2}}\right) = 0. \tag{5.24}$$

We take the test functions

$$\theta_1 = \pi^n, \quad \rho = \phi^n - \epsilon^n, \quad \phi = \pi^n, \quad \varphi = -\theta^n, \quad \chi = \sigma^n, \quad \rho = \varpi^n - \varphi^n, \quad \psi = -\sigma^n, \quad \zeta = \tau^n, \quad (5.25)$$

we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(\pi^{n} - \pi_{n}^{e}), \pi^{n}\right)_{D^{k}} + \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}}(\sigma^{n} - \sigma_{n}^{e}), \sigma^{n}\right) - \left(\Delta_{(\beta-2)/2}(\epsilon^{n} - \epsilon_{n}^{e}), \phi^{n} - \epsilon^{n}\right)_{D^{k}} \\
- \left(\Delta_{(\beta-2)/2}(\varphi^{n} - \varphi_{n}^{e}), \varpi^{n} - \varphi^{n}\right)_{D^{k}} + \left(\tau^{n} - \tau_{n}^{e}, \pi_{x}^{n}\right)_{D^{k}} - \left(\sigma^{n} - \sigma_{n}^{e}, \vartheta_{x}^{n}\right)_{D^{k}} - \left(\vartheta^{n} - \vartheta_{n}^{e}, \sigma_{x}^{n}\right)_{D^{k}} + \left(\pi^{n} - \pi_{n}^{e}, \tau_{x}^{n}\right)_{D^{k}} \\
+ \varepsilon_{2}\left(\sigma^{n} - \sigma_{n}^{e}, \pi\right)_{D^{k}} - \varepsilon_{2}\left(\pi_{n} - \pi_{n}^{e}, \sigma\right)_{D^{k}} + \left(\epsilon^{n} - \epsilon_{n}^{e}, \pi\right)_{D^{k}} - \left(\tau^{n} - \tau_{n}^{e}, \vartheta\right)_{D^{k}} + \left(\varpi^{n} - \varpi_{n}^{e}, \varpi^{n} - \varphi^{n}\right)_{D^{k}} \\
+ \left(\phi^{n} - \phi_{n}^{e}, \phi^{n} - \epsilon^{n}\right)_{D^{k}} - \left(\varphi^{n} - \varphi_{n}^{e}, \sigma^{n}\right)_{D^{k}} + \left(\vartheta^{n} - \vartheta_{n}^{e}, \tau^{n}\right)_{D^{k}} + \varepsilon_{1}\left(\phi^{n} - \phi_{n}^{e}, \pi^{n}\right)_{D^{k}} - \varepsilon_{1}\left(\varpi^{n} - \varpi_{n}^{e}, \sigma^{n}\right)_{D^{k}} \\
- \left(\left((\tau^{n} - \tau_{n}^{e})^{+}(\pi^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\tau^{n} - \tau_{n}^{e})^{+}(\pi^{n})^{+}\right)_{k-\frac{1}{2}}\right) + \left(\left((\sigma^{n} - \sigma_{n}^{e})^{-}(\vartheta^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\sigma^{n} - \sigma_{n}^{e})^{-}(\vartheta^{n})^{+}\right)_{k-\frac{1}{2}}\right) \\
+ \left(\left((\vartheta^{n} - \vartheta_{n}^{e})^{+}(\sigma^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\vartheta^{n} - \vartheta_{n}^{e})^{+}(\sigma^{n})^{+}\right)_{k-\frac{1}{2}}\right) - \sum_{k=1}^{K} \left((\pi^{n} - \pi_{n}^{e})^{-}(\tau^{n})^{-}\right)_{k+\frac{1}{2}} - \left((\pi^{n} - \pi_{n}^{e})^{-}(\tau^{n})^{+}\right)_{k-\frac{1}{2}}\right) = 0.$$
(5.26)

Summing over k, simplify by integration by parts and (5.7). This completes the proof. \Box

Theorem 5.2. Let u be the exact solution of the problem (5.18), and let u_h be the numerical solution of the fully discrete LDG scheme (5.6). Then for small enough h, we have the following error estimates:

$$||u(x,t_n) - u_h^n||_{L^2(\Omega)} \le C(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2).$$
(5.27)

Proof. We estimate the term Q_i , i = 1, ..., 4. So we employ Young's inequality, Lemma 2.4 and the approximation results (4.18), we obtain

$$Q_1 \le c_5 \|\epsilon^n\|_{L^2(\Omega)}^2 + c_6 \|\varphi^n\|_{L^2(\Omega)}^2 + c_1 \|\pi^n\|_{L^2(\Omega)}^2 + c_2 \|\sigma^n\|_{L^2(\Omega)}^2 + c_3 \|\phi^n\|_{L^2(\Omega)}^2 + c_4 \|\varpi^n\|_{L^2(\Omega)}^2. \tag{5.28}$$

Using the definition of the numerical traces, (5.7), and the definitions of the projections $\mathcal{P}^+, \mathcal{P}^-$ (4.16), we get

$$Q_2 = Q_4 = 0. (5.29)$$

From the approximation results (4.18), (4.26) and (4.27) and Young's inequality, we obtain

$$Q_{3} \leq c_{5} \|\epsilon^{n}\|_{L^{2}(\Omega)}^{2} + c_{6} \|\varphi^{n}\|_{L^{2}(\Omega)}^{2} + c_{1} \|\pi^{n}\|_{L^{2}(\Omega)}^{2} + c_{2} \|\sigma^{n}\|_{L^{2}(\Omega)}^{2} + c_{3} \|\phi^{n}\|_{L^{2}(\Omega)}^{2} + c_{4} \|\varpi^{n}\|_{L^{2}(\Omega)}^{2} + C(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}).$$

$$(5.30)$$

Combining (5.28), (5.29), (5.30) and (5.22), we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} \pi^{n}, \pi^{n}\right) + \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} \sigma^{n}, \sigma^{n}\right) + \left(\Delta_{(\beta-2)/2} \epsilon^{n}, \epsilon^{n}\right) + \left(\Delta_{(\beta-2)/2} \varphi^{n}, \varphi^{n}\right) + \left(\varphi^{n}, \phi^{n}\right) + \left(\varpi^{n}, \varpi^{n}\right) \leq c_{5} \|\epsilon^{n}\|_{L^{2}(\Omega)}^{2} + c_{6} \|\varphi^{n}\|_{L^{2}(\Omega)}^{2} + c_{1} \|\pi^{n}\|_{L^{2}(\Omega)}^{2} + c_{2} \|\sigma^{n}\|_{L^{2}(\Omega)}^{2} + c_{3} \|\phi^{n}\|_{L^{2}(\Omega)}^{2} + c_{4} \|\varpi^{n}\|_{L^{2}(\Omega)}^{2} + C\left(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}\right).$$
(5.31)

Recalling Lemmas 2.2 and provided c_3, c_4 are sufficiently small such that $c_3, c_4 \leq 1$, we obtain

$$\left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} \pi^{n}, \pi^{n}\right) + \left(\sum_{j=1}^{S} W(\alpha_{j}) \Delta \tau_{j} \delta_{t}^{\alpha_{j}} \sigma^{n}, \sigma^{n}\right) \leq c_{1} \|\pi^{n}\|_{L^{2}(\Omega)}^{2} + c_{2} \|\sigma^{n}\|_{L^{2}(\Omega)}^{2} + c_{2} \|\sigma^{n}\|_{L^{2}(\Omega)}^{2} + C(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}).$$
(5.32)

It then follows that

$$\left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \pi^{n}, \pi_{h}^{n}\right) + \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \sigma^{n}, \sigma_{h}^{n}\right) \\
\leq \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \pi^{l}, \pi^{n}\right) + \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \sigma^{l}, \sigma^{n}\right) \\
+ \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} a_{n-1}^{\alpha_{j}} \sigma^{0}, \sigma^{n}\right) + \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} a_{n-1}^{\alpha_{j}} \pi^{0}, \pi^{n}\right) + c_{1} \|\pi^{n}\|_{L^{2}(\Omega)}^{2} + c_{2} \|\sigma^{n}\|_{L^{2}(\Omega)}^{2} \\
+ C(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}).$$
(5.33)

Employing Young's inequality, we obtain

$$\|\pi^n\|_{L^2(\Omega)}^2 + \|\sigma^n\|_{L^2(\Omega)}^2$$

$$\leq \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q\left(\sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) \left(\|\pi^{l}\|_{L^{2}(\Omega)}^{2} + \|\sigma^{l}\|_{L^{2}(\Omega)}^{2} \right) \right) + (cQ + \frac{1}{4}) \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} \left(\|\pi^{n}\|_{L^{2}(\Omega)}^{2} \right) \\
+ \|\sigma^{n}\|_{L^{2}(\Omega)}^{2} + c \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} h^{2N+2} + C \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} \left(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4} \right), \tag{5.34}$$

provided c is sufficiently small such that $\frac{3}{4} - cQ > 0$, we obtain that

$$\|\pi^{n}\|_{L^{2}(\Omega)}^{2} + \|\sigma^{n}\|_{L^{2}(\Omega)}^{2} \leq C \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \left(\sum_{l=1}^{n-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) (\|\pi^{l}\|_{L^{2}(\Omega)}^{2} + \|\sigma^{l}\|_{L^{2}(\Omega)}^{2}) \right) + \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} (h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}) \right).$$

$$(5.35)$$

Obviously the theorem holds for n = 0. Assume that it is valid for n = 1, 2, ..., m - 1. Then, by (5.35), we have

$$\|\pi^{m}\|_{L^{2}(\Omega)}^{2} + \|\sigma^{m}\|_{L^{2}(\Omega)}^{2} \leq C \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \left(\sum_{l=1}^{m-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) (\|\pi^{l}\|_{L^{2}(\Omega)}^{2} + \|\sigma^{l}\|_{L^{2}(\Omega)}^{2}) \right) \right)$$

$$+ \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} (h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4}) \right)$$

$$\leq C \left(\sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q \sum_{l=1}^{m-1} (a_{n-l-1}^{\alpha_{j}} - a_{n-l}^{\alpha_{j}}) (h^{2N+2} + (\Delta t)^{2+\sigma} + \sigma^{4}) \right)$$

$$+ \sum_{j=1}^{S} \frac{W(\alpha_{j}) \Delta \tau_{j}}{\lambda_{j}} Q a_{n-1}^{\alpha_{j}} (h^{2N+1} + (\Delta t)^{4+\theta} + \theta^{4}) \right)$$

$$= C \left(h^{2N+2} + (\Delta t)^{2+\theta} + \theta^{4} \right).$$

$$(5.36)$$

Finally, by using triangle inequality and standard approximation theory, we can get (5.27). \Box

5.2. LDG method for the coupled nonlinear distributed-order time and space-fractional Schrödinger equations

In this section, we present and analyze the LDG method for the coupled nonlinear distributed-order time and space- fractional Schrödinger equations

$$i\mathcal{D}_{t}^{W(\alpha)}u_{1} - \varepsilon_{1}(-\Delta)^{\frac{\beta}{2}}u_{1} + \varepsilon_{2}f(|u_{1}|^{2}, |u_{2}|^{2})u_{1} = 0,$$

$$i\mathcal{D}_{t}^{W(\alpha)}u_{2} - \varepsilon_{3}(-\Delta)^{\frac{\beta}{2}}u_{2} + \varepsilon_{4}g(|u_{1}|^{2}, |u_{2}|^{2})u_{2} = 0.$$
(5.37)

To define the local discontinuous Galerkin method, we rewrite (5.37) as a first-order system:

$$i\mathcal{D}_{t}^{W(\alpha)}u_{1} + \varepsilon_{1}e + \varepsilon_{2}f(|u_{1}|^{2}, |u_{2}|^{2})u_{1} = 0,$$

$$e = \Delta_{(\beta-2)/2}r, \quad r = \frac{\partial}{\partial x}s, \quad s = \frac{\partial}{\partial x}u_{1},$$

$$i\mathcal{D}_{t}^{W(\alpha)}u_{2} + \varepsilon_{3}l + \varepsilon_{4}g(|u_{1}|^{2}, |u_{2}|^{2})u_{2} = 0,$$

$$l = \Delta_{(\beta-2)/2}w, \quad w = \frac{\partial}{\partial x}z, \quad z = \frac{\partial}{\partial x}u_{2}.$$

$$(5.38)$$

We decompose the complex functions u(x,t) and v(x,t) into their real and imaginary parts. Setting $u_1(x,t) = p(x,t) + iq(x,t)$ and $u_2(x,t) = v(x,t) + i\vartheta(x,t)$ in system (5.37), we can obtain the following coupled system

$$\mathcal{D}_{t}^{W(\alpha)}p + \varepsilon_{1}Q + \varepsilon_{2}f(|u_{1}|^{2}, |u_{2}|^{2})q = 0,$$

$$Q = \Delta_{(\beta-2)/2}r, \quad r = \frac{\partial}{\partial x}s, \quad s = \frac{\partial}{\partial x}q,$$

$$\mathcal{D}_{t}^{W(\alpha)}q - \varepsilon_{1}H - \varepsilon_{2}f(|u_{1}|^{2}, |u_{2}|^{2})p = 0,$$

$$H = \Delta_{(\beta-2)/2}w, \quad w = \frac{\partial}{\partial x}z, \quad z = \frac{\partial}{\partial x}p,$$

$$\mathcal{D}_{t}^{W(\alpha)}v + \varepsilon_{3}L + \varepsilon_{4}g(|u_{1}|^{2}, |u_{2}|^{2})\vartheta = 0,$$

$$L = \Delta_{(\beta-2)/2}\rho, \quad \rho = \frac{\partial}{\partial x}\varpi, \quad \varpi = \frac{\partial}{\partial x}\vartheta,$$

$$\mathcal{D}_{t}^{W(\alpha)}\vartheta - \varepsilon_{3}E - \varepsilon_{4}g(|u_{1}|^{2}, |u_{2}|^{2})v = 0,$$

$$E = \Delta_{(\beta-2)/2}\xi, \quad \xi = \frac{\partial}{\partial x}\varrho, \quad \varrho = \frac{\partial}{\partial x}v.$$

$$(5.39)$$

We define a fully discrete local discontinuous Galerkin scheme with as follows: find $p_h^n, q_h^n, Q^n, r_h^n, s_h^n, H_h^n, w_h^n, z_h^n, v_h^n, \vartheta_h^n, L_h^n, \rho_h^n, \varpi_h^n, E_h^n, \xi_h^n, \varrho_h^n \in V_k^N$, such that for all test functions $\vartheta_1, \beta_1, \phi, \varphi, \chi, \beta_2, \psi$,

 $\zeta, \gamma, \beta_3, \delta, \varsigma, o, \beta_4, \omega, \kappa \in V_k^N$

$$\begin{split} &\left(\sum_{j=1}^{S}W(\alpha_{j})\Delta\tau_{j}\delta_{t}^{\alpha_{j}}p_{h}^{n},\vartheta_{1}\right)_{D^{k}}+\varepsilon_{1}\left(Q_{h}^{n},\vartheta_{1}\right)_{D^{k}}+\varepsilon_{2}\left(f(|u_{1h}^{n}|^{2},|u_{2h}^{n}|^{2})q_{h}^{n},\vartheta_{1}\right)_{D^{k}}=0,\\ &\left(Q_{h}^{n},\beta_{1}\right)_{D^{k}}=\left(\Delta_{(\beta-2)/2}r_{h}^{n},\beta_{1}\right)_{D^{k}},\\ &\left(r_{h}^{n},\varphi_{0}\right)_{D^{k}}=-\left(s_{h}^{n},\varphi_{x}\right)_{D^{k}}+\left(\left(\tilde{s}_{h}^{n}\varphi^{-}\right)_{k+\frac{1}{2}}-\left(\tilde{s}_{h}^{n}\varphi^{+}\right)_{k-\frac{1}{2}}\right),\\ &\left(s_{h}^{n},\varphi_{0}\right)_{D^{k}}=-\left(q_{h}^{n},\varphi_{x}\right)_{D^{k}}+\left(\left(\tilde{q}_{h}^{n}\varphi^{-}\right)_{k+\frac{1}{2}}-\left(\tilde{r}_{h}^{n}\varphi^{+}\right)_{k-\frac{1}{2}}\right),\\ &\left(\sum_{j=1}^{S}W(\alpha_{j})\Delta\tau_{j}\delta_{t}^{\alpha_{j}}q_{h}^{n},\chi\right)_{D^{k}}-\varepsilon_{1}(H_{h}^{n},\chi)_{D^{k}}-\varepsilon_{2}\left(f(|u_{1h}^{n}|^{2},|u_{2h}^{n}|^{2})p_{h}^{n},\chi\right)_{D^{k}}=0,\\ &\left(H_{h}^{n},\beta_{2}\right)_{D^{k}}=\left(\Delta_{(\beta-2)/2}w_{h}^{n},\beta_{2}\right)_{D^{k}},\\ &\left(w_{h}^{n},\psi\right)_{D^{k}}=\left(Z_{h}^{n},\psi_{x}\right)_{D^{k}}+\left(\left(\tilde{s}_{h}^{n}\psi^{-}\right)_{k+\frac{1}{2}}-\left(\tilde{s}_{h}^{n}\psi^{+}\right)_{k-\frac{1}{2}}\right),\\ &\left(z_{h}^{n},\zeta\right)_{D^{k}}=-\left(p_{h}^{n},\zeta_{x}\right)_{D^{k}}+\left(\left(\tilde{p}_{h}^{n}\zeta^{-}\right)_{k+\frac{1}{2}}-\left(\tilde{p}_{h}^{n}\zeta^{+}\right)_{k-\frac{1}{2}}\right),\\ &\left(\sum_{j=1}^{S}W(\alpha_{j})\Delta\tau_{j}\delta_{t}^{\alpha_{j}}v_{h}^{n},\gamma\right)_{D^{k}}+\varepsilon_{3}(L_{h}^{n},\gamma)_{D^{k}}+\varepsilon_{4}\left(g(|u_{1h}^{n}|^{2},|u_{2h}^{n}|^{2})\vartheta_{h}^{n},\gamma\right)_{D^{k}}=0,\\ &\left(L_{h}^{n},\delta_{3}\right)_{D^{k}}=\left(\Delta_{(\beta-2)/2}\rho_{h}^{n},\beta_{3}\right)_{D^{k}},\\ &\left(\rho_{h}^{n},\delta\right)_{D^{k}}=-\left(\vartheta_{h}^{n},\delta_{x}\right)_{D^{k}}+\left(\left(\widehat{\varphi}_{h}^{n}\delta^{-}\right)_{k+\frac{1}{2}}-\left(\widehat{\varphi}_{h}^{n}\delta^{+}\right)_{k-\frac{1}{2}}\right),\\ &\left(\sum_{j=1}^{S}W(\alpha_{j})\Delta\tau_{j}\delta_{t}^{\alpha_{j}}\vartheta_{h}^{n},o\right)_{D^{k}}-\varepsilon_{3}(E_{h}^{n},o)_{D^{k}}-\varepsilon_{4}\left(g(|u_{1h}^{n}|^{2},|u_{2h}^{n}|^{2})v_{h}^{n},o\right)_{D^{k}}=0,\\ &\left(E_{h}^{n},\beta_{4}\right)_{D^{k}}=\left(\Delta_{(\beta-2)/2}\xi_{h}^{n},\beta_{4}\right)_{D^{k}},\\ &\left(\xi_{h}^{n},\omega\right)_{D^{k}}-\left(\varrho_{h}^{n},\omega\right)_{D^{k}}+\left(\left(\widetilde{\varrho_{h}^{n}}\omega^{-}\right)_{k+\frac{1}{2}}-\left(\widetilde{\varrho_{h}^{n}}\omega^{+}\right)_{k-\frac{1}{2}}\right),\\ &\left(\xi_{h}^{n},\omega\right)_{D^{k}}=\left(\Delta_{(\beta-2)/2}\xi_{h}^{n},\beta_{4}\right)_{D^{k}},\\ &\left(\xi_{h}^{n},\omega\right)_{D^{k}}-\left(\varrho_{h}^{n},\omega\right)_{D^{k}}+\left(\left(\widetilde{\varrho_{h}^{n}}\omega^{-}\right)_{k+\frac{1}{2}}-\left(\widetilde{\varrho_{h}^{n}}\omega^{+}\right)_{k-\frac{1}{2}}\right),\\ &\left(\xi_{h}^{n},\omega\right)_{D^{k}}=\left(\Delta_{(\beta-2)/2}\xi_{h}^{n},\beta_{4}\right)_{D^{k}},\\ &\left(\xi_{h}^{n},\omega\right)_{D^{k}}-\left(\xi_{h}^{n},\omega\right)_{D^{k}}+\left(\left(\widetilde{\varrho_{h}^{n}}\omega^{-}\right)_{k+\frac{1}{2}}-\left($$

The numerical traces $(\widehat{p}_h^n, \widehat{q}_h^n, \widehat{s}_h^n, \widehat{z}_h^n, \widehat{v}_h^n, \widehat{v}_h^n, \widehat{\phi}_h^n, \widehat{\varphi}_h^n, \widehat{\varphi}_h^n)$ are defined on interelement faces as the alternating fluxes

$$\widehat{p}_{h}^{n} = (p_{h}^{n})^{-}, \ \widehat{s}_{h}^{n} = (s_{h}^{n})^{+}, \ \widehat{q}_{h}^{n} = (q_{h}^{n})^{-}, \ \widehat{z}_{h}^{n} = (z_{h}^{n})^{+},
\widehat{v}_{h}^{n} = (v_{h}^{n})^{-}, \ \widehat{\varpi}_{h}^{n} = (\varpi_{h}^{n})^{+}, \ \widehat{\varrho}_{h}^{n} = (\varrho_{h}^{n})^{+}, \ \widehat{\vartheta}_{h}^{n} = (\vartheta_{h}^{n})^{-}.$$
(5.41)

Theorem 5.3. (L^2 stability). Suppose $u_1(x,t) = p(x,t) + iq(x,t)$ and $u_2(x,t) = v(x,t) + i\vartheta(x,t)$ and let $u_{1h}^n, u_{2h}^n \in V_k^N$ be the approximation of $u_1(x,t_n), u_2(x,t_n)$ then the solution to the scheme (5.40) and (5.41) satisfies the L^2 stability

$$||u_{1h}^n||_{L^2(\Omega)}^2 + ||u_{2h}^n||_{L^2(\Omega)}^2 \le C(||u_{1h}^0||_{L^2(\Omega)}^2 + ||u_{2h}^0||_{L^2(\Omega)}^2).$$

Theorem 5.4. Let $u_1(x, t_n)$ and $u_2(x, t_n)$ be the exact solutions of the linear coupled fractional Schrödinger equations (5.37), and let u_{1h}^n and u_{2h}^n be the numerical solutions of the fully discrete LDG scheme (5.40). Then for small enough h, we have the following error estimates:

$$||u_1(x,t_n) - u_{1h}^n||_{L^2(\Omega)} + ||u_2(x,t_n) - u_{2h}^n||_{L^2(\Omega)} \le C(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2).$$
(5.42)

Theorem 5.4 and 5.3 can be proven by similar techniques as that in the proof of Theorem 5.1 and 5.2. We will thus not give the details here.

6. Numerical examples

In the following, we present some numerical experiments to show the accuracy and the performance of the present LDG method for the distributed-order time and space-fractional convection-diffusion and Schrödinger type equations.

Example 6.1. Consider the distributed-order time and space-fractional diffusion equation

$$\mathcal{D}_{t}^{W(\alpha)}u(x,t) + \varepsilon(-\Delta)^{\frac{\beta}{2}}u(x,t) = g(x,t), \quad x \in [-1,1], \quad t \in (0,0.5],$$

$$u(x,0) = 0,$$
(6.1)

and the corresponding forcing term g(x,t) is of the form

$$g(x,t) = \left((x^2 - 1)^4 \mathcal{D}_t^{W(\alpha)} t^2 + \varepsilon t^2 (-\Delta)^{\frac{\beta}{2}} (x^2 - 1)^4 \right), \tag{6.2}$$

then the exact solution is $u(x,t) = t^2(x^2-1)^4$ with $\varepsilon = \frac{\Gamma(8-\beta)}{\Gamma(8)}$.

The problem is solved for several different values of β , polynomial orders (N), and numbers of elements (K). The errors and spatial convergence orders are listed in Table 1 and show that the LDG method can achieve the accuracy of order N+1.

Example 6.2. We consider the distributed-order time and space-fractional Burgers' equation

$$\mathcal{D}_{t}^{W(\alpha)}u(x,t) + \varepsilon(-\Delta)^{\frac{\beta}{2}}u(x,t) + \frac{\partial}{\partial x}\left(\frac{u^{2}(x,t)}{2}\right) = g(x,t), \quad x \in [-1,1], \quad t \in (0,0.5],$$

$$u(x,0) = 0,$$
(6.3)

and the corresponding forcing term g(x,t) is of the form

$$g(x,t) = \left((x^2 - 1)^4 \mathcal{D}_t^{W(\alpha)} t^2 + 8t^4 x (x^2 - 1)^7 + \varepsilon t^2 (-\Delta)^{\frac{\beta}{2}} (x^2 - 1)^4 \right). \tag{6.4}$$

In this case, the exact solution will be $u(x,t)=t^2(x^2-1)^4$ with $\varepsilon=\frac{\Gamma(8-\beta)}{\Gamma(8)}$.

To complete the scheme, we choose a Lax-Friedrichs flux for the nonlinear term. We take $\Delta t = T/500$, $\theta =$

	N = 1									
K	5	10)	1	15		20			
β	L^2 -Error	L^2 -Error	order	L^2 -Error	order	L^2 -Error	order			
1.2	5.97e-02	8.6e-03	2.8	3.4e-03	2.29	1.8e-03	2.21			
1.4	2.84e-02	5.8e-03	2.28	2.5e-03	2.08	1.3e-03	2.27			
1.8	1.91e-02	4.5e-03	2.09	1.9e-03	2.13	9.9e-04	2.27			
				N=2						
K	5	10)	15		2	20			
β	L^2 -Error	L^2 -Error	order	L^2 -Error	order	L^2 -Error	order			
1.2	3.52e-02	4.3e-03	3.03	1.2e-03	3.15	4.8e-04	3.19			
1.4	1.57e-02	2.1e-03	2.9	5.9e-04	3.13	2.6e-04	2.85			
1.8	1.45e-02	1.8e-03	3.01	5.5e-04	2.92	2.2e-04	3.19			

Table 1: L^2 -Error and order of convergence for Example 6.1 with K elements and polynomial order N.

	N = 1									
K	10	2	20		30		40			
β	L^2 -Error	L^2 -Error	order	L^2 -Error	order	L^2 -Error	order			
1.2	7.8e-03	1.9e-03	2.04	8.5e-04	1.98	4.6e-04	2.13			
1.4	4.9e-03	1.1e-03	2.16	4.6e-04	2.15	2.5e-04	2.12			
1.8	1.9e-03	5.1e-04	1.9	2.2e-04	2.07	1.2e-04	2.11			
				N=2	2					
K	10	2	20		30		40			
β	L^2 -Error	L^2 -Error	order	L^2 -Error	order	L^2 -Error	order			
1.2	3.4e-03	4.2e-04	3.02	1.3e-04	2.89	5.2e-05	3.19			
1.4	1.3e-03	1.8e-04	2.85	5.6e-05	2.88	2.5 e-05	2.8			
1.8	8.2e-04	1.1e-04	2.9	3.1e-05	3.12	1.3e-08	3.02			

Table 2: L^2 -Error and order of convergence for Example 6.2 with K elements and polynomial order N.

1/50. The errors and spatial convergence orders are listed in Table 2. Table 3 provides some numerical results of the errors and the temporal convergence orders with $\beta=1.2,1.6$ respectively at T=0.5 with N=1, K=30. Numerical results of the errors and the numerical integration convergence orders in Table 4 with $\beta=1.2,1.6$ respectively at T=0.5. From these tables, we can see that the convergence order of the scheme is

 $\mathcal{O}(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2)$, which matches the theoretical convergence order when θ is small enough.

β		β =1.2	β =1.6		
Δt	L^2 -Error	order	Δt	L^2 -Error	order
T/100	4.38e-03	-	T/100	1.6e-03	-
T/200	2.21e-03	0.99	T/200	7.82e-04	1.03
T/400	1.1e-03	1.01	T/400	4.1e-04	0.93

Table 3: L^2 -Error and temporal convergence orders for u with $\beta = 1.2, 1.8$ at T = 0.5.

β	β	=1.2		β =1.6			
θ	L^2 -Error	order	θ	L^2 -Error	order		
1/10	2.14e-02	-	1/10	7.37e-03	-		
1/20	4.84e-03	2.15	1/20	2.01e-03	1.88		
1/40	1.3e-03	1.9	1/40	4.6e-04	2.13		

Table 4: L^2 -Error and numerical integration convergence orders for u with $\beta = 1.2, 1.8$ at T = 0.5.

Example 6.3. Consider the following nonlinear distributed-order time and space-fractional Schrödinger equation

$$i\mathcal{D}_{t}^{W(\alpha)}u(x,t) - \varepsilon(-\Delta)^{\frac{\beta}{2}}u + |u|^{2}u = g(x,t), \quad x \in [-1,1], \quad t \in (0,0.5],$$

$$u(x,0) = 0,$$
(6.5)

and the corresponding forcing term g(x,t) is of the form

$$g(x,t) = (1+i)\left(i(x^2-1)^5 \mathcal{D}_t^{W(\alpha)} t^2 - \varepsilon t^2 (-\Delta)^{\frac{\beta}{2}} (x^2-1)^5 + 2t^6 (x^2-1)^{15}\right). \tag{6.6}$$

The exact solution $u(x,t) = (1+i)t^2(x^2-1)^5$ with $\varepsilon = \frac{\Gamma(10-\beta)}{\Gamma(10)}$. The errors and spatial convergence orders are listed in Table 5. Table 6 provides some numerical results of the errors and the temporal convergence orders with $\beta = 1.2, 1.6$ respectively at T = 0.5. Numerical results of the errors and the numerical integration convergence orders in Table 7 with $\beta = 1.2, 1.6$ respectively at T = 0.5. From these tables, we can see that the convergence order of the scheme is $\mathcal{O}(h^{N+1} + (\Delta t)^{1+\frac{\theta}{2}} + \theta^2)$, which matches the theoretical convergence order when θ is small enough.

Example 6.4. We consider the coupled nonlinear distributed-order time and space-fractional Schrödinger equations

$$i\mathcal{D}_{t}^{W(\alpha)}u_{1}(x,t) - \varepsilon_{1}(-\Delta)^{\frac{\beta}{2}}u_{1}(x,t) + 2(|u_{1}(x,t)|^{2} + |u_{2}(x,t)|^{2})u_{1}(x,t) = g_{1}(x,t), \ x \in [-1,1], \ t \in (0,0.5],$$

$$i\mathcal{D}_{t}^{W(\alpha)}u_{2}(x,t) - \varepsilon_{2}(-\Delta)^{\frac{\beta}{2}}u_{2}(x,t) + 4(|u_{1}(x,t)|^{2} + |u_{2}(x,t)|^{2})u_{2}(x,t) = g_{2}(x,t), \ x \in [-1,1], \ t \in (0,0.5],$$

$$(6.7)$$

	N = 1									
K	10	2	20		30		40			
β	L^2 -Error	L^2 -Error	order	L^2 -Error	order	L^2 -Error	order			
1.2	1.23e-02	4.61e-03	1.42	1.97e-03	2.1	1.1e-03	2.03			
1.4	1.01e-02	2.51e-03	2.01	1.11e-03	2.01	6.31e-04	1.96			
1.8	7.31e-03	1.91e-03	1.94	8.35e-04	2.04	4.71e-04	1.99			
				N=2	2					
K	10	2	20		30		40			
β	L^2 -Error	L^2 -Error	order	L^2 -Error	order	L^2 -Error	order			
1.2	8.35e-03	1.21e-03	2.79	3.55e-04	3.02	1.41e-04	3.21			
1.4	6.24e-03	9.23 e-04	2.76	2.79e-04	2.95	1.13e-04	3.14			
1.8	2.62e-03	3.54 e- 04	2.89	1.13e-04	2.82	4.66e-05	3.08			

Table 5: L^2 -Error and order of convergence for Example 6.3 with K elements and polynomial order N.

β	β:	=1.2	β =1.6		
Δt	L^2 -Error	order	Δt	L^2 -Error	order
T/100	6.25e-03	-	T/100	5.64e-03	-
T/200	3.12e-03	1.00	T/200	2.81e-03	1.01
T/400	1.58e-03	0.98	T/400	1.25e-03	1.17

Table 6: L^2 -Error and temporal convergence orders for u with $\beta=1.2,1.8$ at T=0.5.

β	β	=1.2		β =1.6			
θ	L^2 -Error	order	θ	L^2 -Error	order		
1/10	5.28e-02	-	1/10	2.25e-02	-		
1/20	1.25e-02	2.08	1/20	5.4e-03	2.06		
1/40	2.98e-03	2.07	1/40	1.55e-03	1.80		

Table 7: L^2 -Error and numerical integration convergence orders for u with $\beta = 1.2, 1.8$ at T = 0.5.

and the corresponding forcing terms $g_1(x,t)$ and $g_2(x,t)$ are of the form

$$g_1(x,t) = (1+i) \left(i(x^2 - 1)^6 \mathcal{D}_t^{W(\alpha)} t^2 - \varepsilon_1 t^2 (-\Delta)^{\frac{\beta}{2}} (x^2 - 1)^6 + 8t^6 (x^2 - 1)^{18} \right),$$

$$g_2(x,t) = (1+i) \left(i(x^2 - 1)^6 \mathcal{D}_t^{W(\alpha)} t^2 - \varepsilon_2 (-\Delta)^{\frac{\beta}{2}} (x^2 - 1)^6 + 16t^6 (x^2 - 1)^{18} \right),$$

$$(6.8)$$

to obtain an exact solutions $u_1(x,t) = (1+i)t^2(x^2-1)^6$ and $u_2(x,t) = (1+i)t^2(x^2-1)^6$ with $\beta = 1.3$, $\varepsilon_1 = \frac{\Gamma(13-\beta)}{2\Gamma(13)}$, $\varepsilon_2 = \frac{\Gamma(13-\beta)}{2\Gamma(13)}$. The errors and spatial convergence orders are listed in Tables 8 and 9, confirming optimal $\mathcal{O}(h^{N+1})$ order of convergence across.

N	N=1		N=2				N	=3
K	L^2 -Error	order	К	L^2 -Error	order	К	L^2 -Error	order
10	4.23e-02	-	10	1.45e-02	-	10	7.87e-03	-
20	9.98e-03	2.08	20	1.87e-03	2.96	20	4.65e-04	4.08
40	2.54e-03	1.97	40	2.12e-04	3.14	40	2.59e-05	4.17
80	6.48e-04	1.97	80	2.68e-05	2.98	80	1.63e-06	3.99

Table 8: L^2 -Error and order of convergence for u_1 with K elements and polynomial order N.

N	N=1		N=2				N=3		
K	L^2 -Error	order	К	L^2 -Error	order	К	L^2 -Error	order	
10	3.98e-02	-	10	1.26e-02	-	10	6.89e-03	-	
20	9.19e-03	2.12	20	1.54e-03	3.03	20	3.84e-04	4.17	
40	2.23e-03	2.04	40	1.82e-04	3.08	40	2.39e-05	4.01	
80	5.54e-04	2.01	80	1.94e-05	3.23	80	1.47e-06	4.02	

Table 9: L^2 -Error and order of convergence for u_2 with K elements and polynomial order N.

7. Conclusions

In this work, we developed and analyzed a local discontinuous Galerkin method for solving the distributedorder time and space-fractional convection-diffusion and Schrödinger type equations, and have proven the stability and error estimates of these methods. Numerical experiments confirm that the optimal order of convergence is recovered. Future work will include the analysis of LDG method for two-dimensional fractional problems.

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