Variations on known and recent cardinality bounds

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Abstract

Sapirovskii [18] proved that $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$, for a regular space X. We introduce the *θ-pseudocharacter* of a Urysohn space X, denoted by $\psi_\theta(X)$, and prove that the previous inequality holds for Urysohn spaces replacing the bounds on celluarity $c(X) \leq \kappa$ and on pseudocharacter $\psi(X) \leq \kappa$ with a bound on Urysohn cellularity $Uc(X) \leq \kappa$ (which is a weaker conditon because $Uc(X) \leq c(X)$) and on θ-pseudocharacter $\psi_\theta(X) \leq \kappa$ respectivly (Note that in general $\psi(\cdot) \leq \psi_\theta(\cdot)$ and in the class of regular spaces $\psi(\cdot) = \psi_\theta(\cdot)$). Further, in [6] the authors generalized the Dissanayake and Willard's inequality: $|X| \leq 2^{aL_c(X)\chi(X)}$, for Hausdorff spaces X [25], in the class of n-Hausdorff spaces and de Groot's result: $|X| \leq 2^{hL(X)}$, for Hausdorff spaces [11], in the class of T_1 spaces (see Theorems 2.22 and 2.23 in [6]). In this paper we restate Theorem 2.22 in [6] in the class of n-Urysohn spaces and give a variation of Theorem 2.23 in [6] using new cardinal functions, denoted by $UW(X)$, $\psi w_\theta(X)$, θ-$aL(X)$, $h\theta$-$aL(X)$, θ-$aL_c(X)$ and θ-$aL_\theta(X)$. In [5] the authors introduced the *Hausdorff point separating weight of a space* X denoted by $Hpsw(X)$ and proved a Hausdorff version of Charlesworth's inequality $|X| \leq psw(X)^{L(X)\psi(X)}$ [7]. In this paper, we introduce the *Urysohn point separating weight of a space* X, denoted by $Upsw(X)$, and prove that $|X| \leq Upsw(X)^{\theta\text{-}aL_c(X)\psi(X)}$, for a Urysohn space X.

Keywords: Urysohn; θ-closure; pseudocharacter; almost Lindelöf degree; Hausdorff point separating weight.

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1 Introduction

We shall follow notations from [12] and [14]. Recall that a space X is *Urysohn* if for every two distinct points $x,\ y \in X$ there are open sets U and V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.

For a space X, we denote by $\chi(X)$ (resp., $\psi(X)$, $\pi\chi(X), c(X), t(X)$) the *character*, (resp., *pseudocharacter*, *π-character*, *celluarity*, *tightness*) of a space X [12].

The *θ-closure* of a set A in a space X is the set $cl_\theta(A) = \{x \in X :$ for every neighborhood $U \ni x, \overline{U} \cap A \neq \emptyset\}$; A is said to be θ-closed if $A = cl_\theta(A)$ [24]. Considering the fact that the θ-closure operator is not in general idempotent, Bella and Cammaroto defined in [2] the *θ-closed hull* of a subset A of a space X, denoted by $[A]_\theta$, that is the smallest θ-closed subset of X containing A. The *θ-tightness* of X at $x \in X$ is $t_\theta(x, X) = \min\{k :$ for every $A \subseteq X$ with $x \in cl_\theta(A)$ there exists $B \subseteq A$ such that $|B| \leq k$ and $x \in cl_\theta(B)$; the *θ-tightness of* X is $t_\theta(X) = sup\{t_\theta(x, X) :\ x \in X\}$ [8]. We have that tightness and θ-tightness are independent (see Example 11 and Example 12 in [9]), but if X is a regular space then $t(X) = t_\theta(X)$. The θ-density of X is $d_\theta(X) = min\{k :\ A \subseteq X$, A is a dense subset of $X$and $|A| \leq k\}$. We say that a subset A of X is *θ-dense* in X if $cl_\theta(A) = X$.

If X is a Hausdorff space, the *closed pseudocharacter of a point* x in X is $\psi_c(x, X) = \min\{|\mathcal{U}| : \mathcal{U}$ is a family of open neighborhoods of x and $\{x\}$ is the intersection of the closure of $\mathcal{U}\}$; the *closed pseudocharacter of* X is $\psi_c(X) = sup\{\psi_c(x, X) :\ x \in X\}$ (see [19] where it is called $S\psi(X)$). The *Urysohn pseudocharacter of* X, denoted by $U\psi(X)$, is the smallest cardinal k such that for each point $x \in X$ there is a collection $\{V(\alpha, x) :\ \alpha < k\}$ of open neighborhoods of x such that if $x \neq y$, then there exist $\alpha,\ \beta < k$ such that $\overline{V(\alpha, x)} \cap \overline{V(\beta, y)} = \emptyset$ [20]; this cardinal function is defined only for Urysohn spaces. The *Urysohn-cellularity* of a space X is $Uc(X) =\ sup\{|\mathcal{V}| : \mathcal{V}$ is Urysohn-cellular$\}$ (a collection $\mathcal{V}$ of open subsets of X is called *Urysohn-cellular*, if $O_1,\ O_2$ in $\mathcal{V}$ and $O_1 \neq O_2$ implies $\overline{O_1} \cap \overline{O_2} = \emptyset$). Of course, $Uc(X) \leq c(X)$.

The *almost Lindelöf degree* of a subset Y of a space X is $aL(Y, X) = \min\{k :$ for every cover $\mathcal{V}$ of Y consisting of open subsets of X, there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $|\mathcal{V}'| \leq k$ and $\bigcup\{\overline{V} :\ V \in \mathcal{V}'\} = Y\}$. The function $aL(X, X)$ is called the *almost Lindelöf degree* of X and denoted by $aL(X)$ (see [25] and [15]). The *almost Lindelöf degree of X with respect to closed subsets of* X is $aL_c(X) = \sup\{aL(C, X) :\ C \subseteq X \textit{ is closed}\}$.

For a subset A of a space X we will denote by $[A]^{\leq\lambda}$ the family of all subsets of A of cardinality $\leq \lambda$.

Sapirovskii [18] proved that $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$, for a regular space X.

Later Shu-Hao [19] proved that the previous inequality holds in the class of Hausdorff spaces by replacing the pseudocharacter with the closed pseudocharacter. In Section 2 we introduce the *θ-pseudocharacter* of a Urysohn space X, denoted by $\psi_\theta(X)$ and prove the following result:

- $|X| \leq \pi\chi(X)^{Uc(X)\psi_\theta(X)}$ for a Urysohn space X.

A space X is *n-Urysohn* [4] (resp. *n-Hausdorff* [3]), $n \in \omega$, if for every $x_1, x_2, ..., x_n \in X$ there exist open subsets $U_1, U_2, ..., U_n$ of X such that $x_1 \in U_1$, $x_2 \in U_2, ..., x_n \in U_n$ and $\bigcap_{i=1}^n \overline{U_i} = \emptyset$ (resp, $\bigcap_{i=1}^n U_i = \emptyset$). In [6] the authors generalized the Dissanayake and Willard's inequality: $|X| \leq 2^{aL_c(X)\chi(X)}$, for Hausdorff spaces X [25], in the class of n-Hausdorff spaces and de Groot's result: $|X| \leq 2^{hL(X)}$, for Hausdorff spaces [11], in the class of T_1 spaces. In particular, they used two new cardinal functions, denoted by $HW(X)$, $\psi w(X)$, to obtain the following results:

- If X is a T_1 n-Hausdorff ($n \in \omega$) space, then $|X| \leq HW(X)2^{aL_c(X)\chi(X)}$.
- If X is a T_1 space, then $|X| \leq HW(X)\psi w(X)^{haL(X)}$.

In Section 3 we introduce new cardinal functions, denoted by $UW(X)$, $\psi w_\theta(X)$, θ-$aL(X)$, $h\theta$-$aL(X)$, θ-$aL_c(X)$ and θ-$aL_\theta(X)$ such that $HW(X) \leq UW(X)$, $\psi w(X) \leq \psi w_\theta(X)$ and θ-$aL(X) \leq aL(X)$, restate Theorem 2.22 in [6] in the class of n-Urysohn spaces and give a variation of Theorem 2.23 in [6]. In particular, we prove the following results:

- If X is a T_1 n-Urysohn ($n \in \omega$) space, then $|X| \leq UW(X)2^{\theta\text{-}aL_\theta(X)\chi(X)}$.
- If X is a T_1 space then $|X| \leq UW(X)\psi w_\theta(X)^{h\theta\text{-}aL(X)}$.

In [5] the authors introduced the *Hausdorff point separating weight of a space X* denoted by $Hpsw(X)$ and proved a Hausdorff version of Charlesworth's inequality $|X| \leq psw(X)^{L(X)\psi(X)}$ [7]. In a similar way, in Section 4 we introduce *Urysohn point separating weight of a space X*, denoted by $Hpsw(X)$, and prove the following result:

- If X is a Urysohn space, then $|X| \leq Upsw(X)^{\theta\text{-}aL_c(X)\psi(X)}$.

2 A generalization of Sapirovskii's inequality $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$.

Definition 2.1. If X is a Urysohn space, we define *θ-pseudocharacter of a point* $x \in X$ the smallest cardinal k such that $\{x\}$ is the intersection of the θ-closure of the closure of a family of open neighborhood of x having cardinality less or equal to k; we denote it with $\psi_\theta(x, X)$. The *θ-pseudocharacter of* X is:

$$\psi_\theta(X) = sup\{\psi_\theta(x, X) : x \in X\}.$$

The following result is trivial:

Proposition 2.1. *X is a Urysohn space iff for every $x \in X$, $\{x\}$ is the intersection of the θ-closure of the closure of a family of open neighborood of x.*

Proof. Let X be a Urysohn space and $x \in X$. For every $y \in X \setminus \{x\}$, there exist U_y and V_y open disjoint subsets of X such that $x \in U_y$, $y \in V_y$ and $\overline{U_y} \cap \overline{V_y} = \emptyset$. So, y $\notin cl_\theta(\overline{U_y})$ and $\{x\} = \bigcap_{y \in X \setminus \{x\}} cl_\theta(\overline{U_y})$. Viceversa let x, y be distinct points of X. By hypothesis there exists an open neighbourhood V of x such that $y \notin cl_\theta(\overline{V})$.Then there exists an open subset U of X such that $y \in U$ and $\overline{U} \cap \overline{V} = \emptyset$. So X is Urysohn. $\square$

We have that:

$$\psi(X) \leq \psi_c(X) \leq \psi_\theta(X) \leq U\psi(X) \leq \chi(X).$$

Since for a regular space X, $cl_\theta(A) = \overline{A}$ for every $A \subseteq X$ [13], we have that for a regular space X, $\psi_c(X) = \psi_\theta(X)$. In general this need not be true for non regular spaces. Indeed if we consider $\mathbb{R}$ with the countable complement topology we have that $\overline{\mathbb{Q}} \neq cl_\theta(\mathbb{Q})$.

Question 2.1. Is there a Urysohn space such that $\psi_c(X) < \psi_\theta(X)$?

It was proved in [2] that for Urysohn spaces, $|cl_\theta(A)| \leq |A|^{\chi(X)}$ for every $A \subseteq X$ and further this inequality was used for the estimation of cardinality of Lindelöf spaces. Since $t_\theta(X)\psi_\theta(X) \leq \chi(X)$, the following proposition improves the result in [2]. (Note that if $X = \omega \cup \{p\}$, with $p \in \omega^*$, we have that $\aleph_0 = t_\theta(X)\psi_\theta(X) < \chi(X)$.)

Proposition 2.2. *Let X be a Urysohn space such that $t_\theta(X)\psi_\theta(X) \leq k$. Then for every $A \subseteq X$ we have that $|cl_\theta(A)| \leq |A|^k$.*

Proof. Let $x \in cl_\theta(A)$, since $\psi_\theta(X) \leq k$ there exist a family $\{U_\alpha(x)\}_{\alpha<k}$ of neighborhood of x such that $\{x\} = \bigcap_{\alpha<k} cl_\theta(\overline{U_\alpha(x)})$. We want to prove that $x \in cl_\theta(\overline{U_\alpha(x)} \cap A)$, $\forall \alpha < k$. Let U be a neighborhood of x and $\alpha < k$. Then $\emptyset \neq \overline{U \cap U_\alpha(x)} \cap A \subseteq \overline{U} \cap \overline{U_\alpha(x)} \cap A$. This shows that $x \in cl_\theta(\overline{U_\alpha(x)} \cap A)$. Since $t_\theta(X) \leq k$, there exists $A_\alpha \subset \overline{U_\alpha(x)} \cap A$ such that $|A_\alpha| \leq k$ and $x \in cl_\theta(A_\alpha)$. Then $\{x\} = \bigcap_{\alpha<k} cl_\theta(A_\alpha)$ and $\{A_\alpha\}_{\alpha<k} \in [[A]^{\leq k}]^{\leq k}$, so $|cl_\theta(A)| \leq |[[A]^{\leq k}]^{\leq k}| = |A|^k$. $\square$

Corollary 2.1. *[2] If X is a Urysohn space then for every $A \subseteq X$ we have that $|cl_\theta(A)| \leq |A|^{\chi(X)}$.*

The following result is the analogue of 2.20 in [16] in the case of Urysohn spaces.

Corollary 2.2. *If X is a Urysohn space then $|X| \leq d_\theta(X)^{t_\theta(X)\psi_\theta(X)}$.*

Proof. If A is θ-dense subset of X, i.e. $cl_\theta(A) = X$, we have that $|A| \leq d_\theta(X)$ and from the above theorem we have that $|cl_\theta(A)| \leq |A|^{t_\theta(X)\psi_\theta(X)}$, so $|X| \leq d_\theta(X)^{t_\theta(X)\psi_\theta(X)}$. □

The authors know that I. Gotchev obtained independently the results given in Proposition 2.2 and Corollary 2.2.

Now we prove the following result:

Lemma 2.1. *Let X be a topological space, $\mathcal{B}$ a π-base for X and $\mathcal{W}$ a family of open sets. Let $\mathcal{M}$ be a maximal Urysohn cellular subfamily of $\{U \in \mathcal{B} : U \subseteq W \text{ for some } W \in \mathcal{W}\}$. Then $cl_\theta\left(\bigcup \overline{\mathcal{M}}\right) \supseteq \bigcup \mathcal{W}$.*

Proof. Using Zorn's Lemma we can say that there exists a maximal Urysohn-cellular subfamily $\mathcal{M}$ of $\{U \in \mathcal{B} : U \subseteq W \text{ for some } W \in \mathcal{W}\}$. We want to prove that $cl_\theta\left(\bigcup \overline{\mathcal{M}}\right) \supseteq \bigcup \mathcal{W}$. Assume, by the way of contradiction, that $cl_\theta\left(\bigcup \overline{\mathcal{M}}\right) \not\supseteq \bigcup \mathcal{W}$. Let $x \in \bigcup \mathcal{W}$ such that $x \notin cl_\theta(\bigcup \overline{\mathcal{M}})$. Then there exists an open set U such that $x \in U$ such that $\overline{U} \cap \overline{M} = \emptyset$, $\forall M \in \mathcal{M}$. So $x \notin M$, $\forall M \in \mathcal{M}$. Let $W \in \mathcal{W}$ such that $x \in W$. $\mathcal{M} \cup \{U \cap W\}$ is a Urysohn cellular family. Since $\mathcal{B}$ is a π-base for X and $U \cap W$ is an open set containing x, there exists $B \in \mathcal{B}$ such that $B \subseteq U \cap W$, so $\mathcal{M}' = \mathcal{M} \cup \{B\}$ is a Urysohn cellular subfamily of $\{U \in \mathcal{B} : U \subseteq W \text{ for some } W \in \mathcal{W}\}$ containing $\mathcal{M}$; a contradiction. □

Theorem 2.1. *Let X be a Urysohn space. Then $|X| \leq \pi\chi(X)^{Uc(X)\psi_\theta(X)}$.*

Proof. Let $\pi\chi(X) = \lambda$ and $Uc(X)\psi_\theta(X) = k$; for each $p \in X$, let $\mathcal{U}_p$ be a local π-base at p such that $|\mathcal{U}_p| \leq \lambda$.

Construct an increasing chain $\{A_\alpha : \alpha < k^+\}$ of subsets of X and a sequence $\{\mathcal{U}_\alpha : 0 < \alpha < k^+\}$ of open collections in X such that:

1. $|A_\alpha| \leq \lambda^k$, $0 \leq \alpha < k^+$;
2. $\mathcal{U}_\alpha = \{V \in \mathcal{U}_p : p \in \bigcup_{\beta<\alpha} A_\beta\}$, $0 < \alpha < k^+$;
3. for each $\gamma < k$, if $\mathcal{V}_\gamma \in [\mathcal{U}_\alpha]^{\leq k}$ and $W = \bigcup_{\gamma<k} cl_\theta(\bigcup \overline{\mathcal{V}_\gamma}) \neq X$, then $A_\alpha \setminus W \neq \emptyset$.

The construction is by trasfinite induction. Let $0 < \alpha < k^+$ and assume that $\{A_\beta : \beta < \alpha\}$ has already been constructed. Then $\mathcal{U}_\alpha$ is defined by 2., i.e., we put $\mathcal{U}_\alpha = \{V : \exists p \in \bigcup_{\beta<\alpha} A_\beta,\ V \in \mathcal{U}_p\}$. It follows that $|\mathcal{U}_\alpha| \leq \lambda^k$. If $\{\mathcal{V}_\gamma\}_{\gamma<k} \in [[\mathcal{U}_\alpha]^{\leq k}]^{\leq k}$ and $W = \bigcup_{\gamma<k} cl_\theta(\bigcup \overline{\mathcal{V}_\gamma}) \neq X$, then we can choose one point of $X \setminus W$. Let S_α be the set of points chosen in this way. Note that $|[[\mathcal{U}_\alpha]^{\leq k}]^{\leq k}| \leq \lambda^k$. Define A_α to be the set $S_\alpha \cup (\bigcup_{\beta<\alpha} A_\beta)$. Then A_α satisfies 1., and 3. is also satisfied if $\beta \leq \alpha$. This completes the construction.

Now let $S = \bigcup_{\alpha<k^+} A_\alpha$; then $|S| \leq k^+\lambda^k = \lambda^k$. The proof is complete if $S = X$. Suppose not and let $p \in X \setminus S$; since $\psi_\theta(X) \leq k$, there exist open neighbourhoods $\{U_\alpha\}_{\alpha<k}$ of p such that $\{p\} = \bigcap_{\alpha<k} cl_\theta(\overline{U_\alpha})$. For each $\alpha < k$, let $V_\alpha = X \setminus cl_\theta(\overline{U_\alpha})$. Then $S = \bigcup_{\alpha<k} V_\alpha \cap S$. Fix $\alpha < k$. For each $q \in V_\alpha \cap S$, there exists $V_q \in \mathcal{U}_q$ such that $\overline{V_q} \cap \overline{U_\alpha} = \emptyset$ (from the definition of V_α). We have that $\{V \in \mathcal{U}_q : V \subseteq V_q\}$ is a local π-base at q. Since $q \in \overline{\bigcup\{V \in \mathcal{U}_q : V \subseteq V_q\}}$, we have that $S \cap V_\alpha \subseteq \bigcup_{q\in S\cap V_\alpha} \overline{\bigcup\{V \in \mathcal{U}_q : V \subseteq V_q\}} \subseteq \overline{\bigcup\{V : V \in \mathcal{U}_q, V \subseteq V_q, q \in S \cap V_\alpha\}}$. We put $\mathcal{W}_\alpha = \{V : V \in \mathcal{U}_q, V \subseteq V_q, q \in S \cap V_\alpha\}$. Since $Uc(X) \leq k$, by Lemma 2.1 we have that $\forall\alpha < k$ there exists a maximal Urysohn cellular family $\mathcal{W}'_\alpha \in [\mathcal{W}_\alpha]^{\leq k}$ such that $cl_\theta(\bigcup \overline{\mathcal{W}'_\alpha}) \supseteq \bigcup \mathcal{W}_\alpha$. Since $cl_\theta(\bigcup \overline{\mathcal{W}'_\alpha})$ is closed, it follows that $S \cap V_\alpha \subseteq \overline{\bigcup \mathcal{W}_\alpha} \subseteq cl_\theta(\bigcup \overline{\mathcal{W}'_\alpha}) \subseteq cl_\theta(\bigcup_{q\in S\cap V_q} \overline{V_q})$. Then, since $(\bigcup_{q\in S\cap V_\alpha} \overline{V_q}) \cap \overline{U_\alpha} = \emptyset$ and $p \notin cl_\theta(\bigcup_{q\in S\cap V_\alpha} \overline{V_q})$, we have that $p \notin cl_\theta(\bigcup \overline{\mathcal{W}'_\alpha})$. Put $W = \bigcup_{\alpha<k} cl_\theta(\bigcup \overline{\mathcal{W}'_\alpha})$. Since $|\{V : V \in \mathcal{W}'_\alpha \text{ for some } \alpha < k\}| \leq kk = k < k^+$, there is an $\alpha_0 < k^+$ such that $\mathcal{W}'_\alpha \in [\mathcal{U}_{\alpha_0}]^{\leq k}$ for each $\alpha < k$. Hence, by 3., one has $A_{\alpha_0} \setminus W \neq \emptyset$. But $W \supseteq \bigcup_{\alpha<k}(V_\alpha \cap S) = S$ and $A_{\alpha_0} \setminus W \subseteq S \setminus W = \emptyset$; a contradiction. $\square$

Corollary 2.3. *[18] Let X be a regular space. Then $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$.*

3 Variations of the Dissanayake and Willard's inequality $|X| \leq 2^{aL_c(X)\chi(X)}$ and of the de Groot's inequality $|X| \leq 2^{hL(X)}$ in the class of T_1 spaces.

In Proposition 2.1 it was shown that Urysohn axiom is equivalent to $\{x\} = \bigcap\{cl_\theta(\overline{U}) : U \text{ open}, x \in U\}$, for every point x of the space. The following example shows that in spaces which are not Urysohn the previous intersection can be large.

Example 3.1. Any infinite space X with the cofinite topology is a T_1, not Hausdorff space for which there is a point x such that $\bigcap\{cl_\theta(\overline{U}) : x \in U\}$ has large cardinality.

The example above gives a motivation to introduce the following definition:

Definition 3.1. Let X be a T_1 topological space and for all $x \in X$, let

$$Uw(x) = \bigcap\{cl_\theta(\overline{U}) : x \in U, U \textit{ open}\}.$$

The *Urysohn width* is:

$$UW(X) = sup\{|Uw(x)| : \ x \in X\}.$$

It is clear that if X is a Urysohn space then $UW(X) = 1$.

Recall that $HW(X) = sup\{|Hw(x)| : \ x \in X\}$ is the *Hausdorff width*, where $Hw(x) = \bigcap\{\overline{U} : \ x \in U, \ U \ open\}$ [6]. Since the θ-closure of a set contains its closure we have that $HW(X) \leq UW(X)$.

Question 3.1. Is $HW(X) = UW(X)$ in some class of non regular spaces?

Definition 3.2. [6] Let X be a space and $x \in X$.

$$\psi w(x) = min\{|\mathcal{U}_x| : \ \bigcap\{\overline{U} : \ U \in \mathcal{U}_x\} = Hw(x), \ \mathcal{U}_x \ is \ a$$

$$family \ of \ open \ neighborhood \ of \ x\};$$

and

$$\psi w(X) = sup\{\psi w(x) : \ x \in X\}.$$

Similarly, we introduce the following definition.

Definition 3.3. Let X be a space and $x \in X$.

$$\psi w_\theta(x) = min\{|\mathcal{U}_x| : \ \bigcap\{cl_\theta(\overline{U}) : \ U \in \mathcal{U}_x\} = Uw(x), \ \mathcal{U}_x \ is \ a$$

$$family \ of \ open \ neighborhood \ of \ x\};$$

and

$$\psi w_\theta(X) = sup\{\psi w_\theta(x) : \ x \in X\}.$$

Of course, if X is a T_1 space then $\psi w(X) \leq \psi w_\theta(X) \leq \chi(X)$; further if X is a Urysohn space then we have that $\psi w_\theta(X) = \psi_\theta(X)$.

We introduce the following definition:

Definition 3.4. Let Y be a subset of a space X.

The *θ-almost Lindelöf degree* of a subset Y of a space X is

θ-$aL(Y, X) = \min\{k$: for every cover $\mathcal{V}$ of Y consisting of open subsets of X, there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $|\mathcal{V}'| \leq k$ and $\bigcup\{cl_\theta(\overline{V}) : \ V \in \mathcal{V}'\} = Y\}$.

The function θ-$aL(X, X)$ is called *θ-almost Lindelöf degree of the space* X and denoted by θ-$aL(X)$.

The *θ-almost Lindelöf degree with respect to closed subsets of* X, denoted by θ-$aL_c(X)$, is the cardinal $sup\{\theta\text{-}aL(C, X) : \ C \subseteq X \ is \ closed\}$.

The *θ-almost Lindelöf degree with respect to θ-closed subsets of* X, denoted by θ-$aL_\theta(X)$, is the cardinal $sup\{\theta\text{-}aL(B, X) : \ B \subseteq X \ is \ \theta\text{-}closed\}$.

Of course θ-$aL(X) \leq aL(X)$, for every space X. Using a slight modification of Example 2.3 in [1] we prove that the previous inequality can be strict.

Example 3.2. A space X such that θ-$aL(X) < aL(X)$.

Let k be any uncountable cardinal, let $\mathbb{Q}$ be the set of all the rationals and let $\mathbb{P}$ be the set of the irrationals. Put $X = (\mathbb{Q} \times k) \cup \mathbb{P}$. We topologized X as follows. If $q \in \mathbb{Q}$ and $\alpha < k$ then a neighborhood base at (q, α) is $\mathcal{U}(q, \alpha) = \{U_n(q, \alpha) : n \in \omega\}$ where

$$U_n(q, \alpha) = \{(r, \alpha) : r \in \mathbb{Q} \text{ and } |r\text{-}q| < \frac{1}{n}\}.$$

If $p \in \mathbb{P}$ a neighborhood base at p takes the form:

$$\{\{b \in \mathbb{P} : |b\text{-}p| < \frac{1}{n}\} \cup \{(q, \alpha) : \alpha < k \text{ and } |q\text{-}p| < \frac{1}{n}\} : n \in \omega\}.$$

For every $q \in \mathbb{Q}$, $\alpha < k$ and $n \in \omega$ we have that:

$$\overline{U_n(q, \alpha)} = U_n(q, \alpha) \bigcup \{(r, \alpha) : r \in \mathbb{Q},\ |r\text{-}q| < \frac{1}{n}\} \bigcup \{p \in \mathbb{P} : |q\text{-}p| < \frac{1}{n}\};$$

and:

$$cl_\theta(\overline{U_n(q, \alpha)}) = \overline{U_n(q, \alpha)} \bigcup \{(r, \beta) : |r - q| < \frac{1}{n},\ \beta < k \text{ and } \beta \neq \alpha\}.$$

Let $\alpha < k$, we have that $X = \bigcup_{q \in \mathbb{Q}} cl_\theta(\overline{\mathcal{U}(q, \alpha)})$ and so θ-$aL(X) = \aleph_0$ but we have that $aL(X) = 2^{\aleph_0}$.

It is easy to show that the almost Lindelöf degree is hereditary with respect to θ-closed subsets. It is natural to ask:

Question 3.2. Is the θ-almost Lindelöf degree hereditary with respect to θ-closed subsets?

We find out (Proposition 3.1) that the θ-almost Lindelöf degree is hereditary with respect to a new class of spaces that we call γ-*closed*.

Definition 3.5. Let X be a topological space and $A \subseteq X$. The γ-*closure* of the set A is

$cl_\gamma(A) = \{x : \text{for every open neighborhood of } X,\ cl_\theta(\overline{U}) \cap A \neq \emptyset\}$.

A is said to be γ-closed if $A = cl_\gamma(A)$.

The following example shows that the γ-closure and the θ-closure of a subset of a topological space can be different.

Example 3.3. A Urysohn space X having a subset Y such that $cl_\gamma(Y) \neq cl_\theta(Y)$.

Proof. Let $\mathbb{R} = A \cup B \cup C \cup D$ where A, B, C, D are pairwise disjoint and each is dense in $\mathbb{R}$. Let A' be a topological copy of A; points in A' are denoted as a' where $a \in A$.

Let $a, b \in \mathbb{R}$. A base for X is generated by these families of open sets:
(1)$\{(a,b) \cap A : a, b \in \mathbb{R}, a < b\}$
(2)$\{(a,b) \cap C : a, b \in \mathbb{R}, a < b\}$,
(3)$\{(a,b) \cap A' : a, b \in \mathbb{R}, a < b\}$,
(4)$\{(a,b) \cap (A \cup B \cup C) : a, b \in \mathbb{R}, a < b\}$, and
(5)$\{(a,b) \cap (C \cup D \cup A') : a, b \in \mathbb{R}, a < b\}$.

Note that for every $a,\ b \in \mathbb{R}$, $\overline{(a,b) \cap A} = [a,b] \cap (A \cup B)$, $\overline{(a,b) \cap A'} = [a,b] \cap (A' \cup D)$, $\overline{(a,b) \cap C} = [a,b] \cap (B \cup C \cup D)$, $cl_\theta(\overline{(a,b) \cap A}) = [a,b] \cap (A \cup B \cup C)$ and $cl_\theta(\overline{(a,b) \cap A'}) = [a,b] \cap (A' \cup D \cup C)$. For these reasons we can say that if $a,\ b \in \mathbb{R}$ and if we put $Y = (a,b) \cap C$, we have that $cl_\theta(Y) = [a,b] \cap (B \cup C \cup D)$ and $cl_\gamma(Y) = [a,b] \cap (A \cup B \cup C \cup D \cup A')$. □

We have the following:

Proposition 3.1. *The θ-almost Lindelöf degree is hereditary with respect to γ-closed subsets.*

Proof. Let X be a topological space such that θ-$aL(X) \leq k$ and let $C \subseteq X$ be γ-closed set. $\forall x \in X \setminus C$ we have that there exists an open neighborhood U_x of x such that $cl_\theta(\overline{U}) \subseteq X \setminus C$. Let $\mathcal{U}$ be a cover of C consisting of open subsets of X. Then $\mathcal{V} = \mathcal{U} \bigcup \{U_x : \ x \in X \setminus C\}$ is an open cover of X and since θ-$aL(X) \leq k$, there exists $\mathcal{V}' \in [\mathcal{V}]^{\leq k}$ such that $X = \bigcup\{cl_\theta(\overline{V}) : V \in \mathcal{V}'\}$. Then there exists $\mathcal{V}'' \in [\mathcal{U}]^{\leq k}$ such that $C \subseteq \bigcup\{cl_\theta(\overline{V}) : V \in \mathcal{V}''\}$; this proves that θ-$aL(C) \leq k$. □

Now we use $UW(X)$ and θ-$aL_\theta(X)$ to restate Theorem 2.22 in [6] in the class of n-Uysohn spaces. The proof follows step by step the proof of Theorem 2.22 in [6].

Theorem 3.1. *If X is a T_1 n-Urysohn ($n \in \omega$) space, then $|X| \leq UW(X)2^{\theta\text{-}aL_\theta(X)\chi(X)}$.*

Proof. Let $UW(X) \leq k$, θ-$aL_\theta(X)\chi(X) \leq \tau$. For all $x \in X$, let $\mathcal{U}_x$ be a local base and $|\mathcal{U}_x| \leq \tau$. Note that for all $x \in X$, $Uw(x) = \bigcap\{cl_\theta(\overline{U}) : \ U \in \mathcal{U}_x\}$. Construct $\{H_\alpha : \ \alpha \in \tau^+\}$ and $\{\mathcal{B}_\alpha : \ \alpha \in \tau^+\}$ such that:

1. $H_\alpha \subset H_\beta \subset X$, for all $\alpha,\ \beta \in \tau^+$;

2. H_α is θ-closed for all $\alpha \in \tau^+$;

3. $|H_\alpha| \leq 2^\tau$ for all $\alpha \in \tau^+$;

4. if $\{H_\beta : \beta \in \alpha\}$ are defined for some $\alpha \in \tau^+$, then $\mathcal{B}_\alpha = \bigcup\{\mathcal{U}_x : x \in \bigcup\{H_\beta : \beta \in \alpha\}\}$;

5. if $\alpha \in \tau^+$ and $\mathcal{W} \in [\mathcal{B}_\alpha]^{\leq\tau}$ is such that $X \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\}) \neq \emptyset$ then $H_\alpha \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\}) \neq \emptyset$.

Let $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ be already defined. For all $\mathcal{W}$ as in 5., choose a point $x(\mathcal{W}) \in X \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\})$ and let C_α be the set of these points. Let $H_\alpha = [\bigcup\{H_\beta : \beta \in \alpha\} \cup C_\alpha]_\theta$. Considering the fact that if X is a n-Urysohn space we have that for every $A \subseteq X$, $|[A]_\theta| \leq |A|^{\chi(X)}$ [4] we have that $|H_\alpha| \leq 2^\tau$. Let $H = \bigcup\{H_\beta : \beta \in \tau^+\}$. Since $t_\theta(X) \leq \chi(X) \leq \tau$, τ^+ is regular and $\{H_\alpha : \alpha \in \tau^+\}$ is an increasing family of myθ-closed sets of lenght τ^+, we have that H is θ-closed. Also $|H| \leq 2^\tau$. Let $H^* = \bigcup\{Uw(x) : x \in H\} \supseteq H$. Then $|H^*| \leq k2^\tau$.

We want to prove that $X = H^*$. Suppose that there exists a point $q \in X \setminus H^* \subset X \setminus H$. Then for all $x \in H$ there is $U(x) \in \mathcal{U}_x$ such that $q \notin cl_\theta(\overline{U(x)})$. From θ-$aL_\theta(X) \leq \tau$ choose $H' \in [H]^{\leq\tau}$ such that $H \subseteq \bigcup\{cl_\theta(\overline{U(x)}) : x \in H'\}$. Then $H' \subseteq H_\alpha$ for some $\alpha \in \tau^+$ and hence $\mathcal{W} = \{cl_\theta(\overline{U(x)}) : x \in H'\} \in [\mathcal{B}_{\alpha+1}]^{\leq\tau}$ and $q \in X\setminus(\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\}) \neq \emptyset$. Hence we have already chosen $x(\mathcal{W}) \in H_\alpha+\} \cap (H \setminus \bigcup\{cl_\theta(\overline{U(x)}) : x \in H'\}) \subseteq H \cap (X \setminus H)$ a contradiction. Hence $X = H^*$ and $|X| \leq k2^\tau$. □

Now we use $UW(X)$,$\psi w_\theta(X)$ and $h\theta$-$aL(X)$ to present a variation of the Theorem 2.23 in [6]. The proof of Theorem 3.2 follows step by step the proof of Theorem 2.23 in [6].

Theorem 3.2. *If X is a T_1 space then* $|X| \leq UW(X)\psi w_\theta(X)^{h\theta\text{-}aL(X)}$.

Proof. Let $UW(X) \leq k$, $h\theta$-$aL(X) \leq \tau$ and $\psi w_\theta(X) \leq \lambda$. For all $x \in X$, let $\mathcal{U}_x$ be a family of open neighborhood of x such that $|\mathcal{U}_x| \leq \lambda$ and $Uw(x) = \bigcap\{cl_\theta(\overline{U}) : U \in \mathcal{U}_x\}$. By trasfinite induction we construct two families $\{H_\alpha : \alpha \in \tau^+\}$ and $\{\mathcal{B}_\alpha : \alpha \in \tau^+\}$ such that:

1. $\{H_\alpha : \alpha \in \tau^+\}$ is an increasing sequence of subsets of X;

2. $|H_\alpha| \leq k\lambda^\tau$ for all $\alpha \in \tau^+$;

3. if $\{H_\beta : \beta \in \alpha\}$ are defined for some $\alpha \in \tau^+$, then $\mathcal{B}_\alpha = \bigcup\{\mathcal{U}_x : x \in \bigcup\{Uw(y) : y \in \bigcup\{H_\beta : \beta \in \alpha\}\}\}$;

4. if $\alpha \in \tau^+$ and $\mathcal{W} \in [\mathcal{B}_\alpha]^{\leq\tau}$ is such that $X \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\}) \neq \emptyset$ then $H_\alpha - (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\}) \neq \emptyset$.

Let $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ be already defined. For all $\mathcal{W}$ as in 4., choose a point $x(\mathcal{W}) \in X \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\})$ and let C_α be the set of these points.

Let $H_\alpha = \bigcup\{H_\beta : \beta \in \alpha\} \cup C_\alpha$. Then $|H_\alpha \leq k\lambda^\tau$.

Let $H = \bigcup\{H_\alpha : \alpha \in \tau^+\}$ and $H^* = \bigcup\{Uw(x) : x \in H\} \supseteq H$. Then $|H^*| \leq k\lambda^\tau$.

We want to prove that $X = H^*$. Suppose that there exists a point $q \in X \setminus H^*$. Then $q \notin Uw(x), \forall x \in H$. Hence for all $x \in H$ there is $U(x) \in \mathcal{U}_x$ such that $q \notin cl_\theta(\overline{U(x)})$. From $h\theta\text{-}aL(X) \leq \tau$ choose $H' \in [H]^{\leq\tau}$ such that $H \subseteq \bigcup\{cl_\theta(\overline{U(x)}) : x \in H'\}$. Let $\mathcal{W} = \{\overline{U(x)} : x \in H'\}$. We have that $H' \subseteq H_\alpha$ for some $\alpha \in \tau^+$ and $\mathcal{W} \in [\mathcal{B}_{\alpha+1}]^{\leq\tau}$ and $X \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\}) \neq \emptyset$. Hence we have already chosen $x(\mathcal{W}) \in X \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\} \subseteq X \setminus H$ and $x(\mathcal{W}) \in H$ a contradiction. Hence $X = H^*$ and $|X| \leq k\lambda^\tau$. □

Corollary 3.1. *If X is a Urysohn space then $|X| \leq \psi_\theta(X)^{h\theta\text{-}aL(X)}$.*

4 The Urysohn point separating weight

Definition 4.1. [5] A *Hausdorff point separating open cover* $\mathcal{S}$ for a space X is an open cover of X having the property that for each distinct points $x,\ y \in X$ there exists $S \in \mathcal{S}$ such that $x \in S$ and $y \notin \overline{S}$.

The *Hausdorff point separating weight of a space* X is

$Hpsw(X) = min\{\tau :$ X has a Hausdorff point separating open cover $\mathcal{S}$ such that each point of X is contained in at most τ elements of $\mathcal{S}\}$.

Following the same idea as in [5] we introduce the following definition:

Definition 4.2. A *Urysohn point separating open cover* $\mathcal{S}$ for a space X is an open cover of X having the property that for each distinct points $x,\ y \in X$ there exists $S \in \mathcal{S}$ such that $x \in S$ and $y \notin cl_\theta(\overline{S})$.

Definition 4.3. The *Urysohn point separating weight* of a Urysohn space X is the cardinal:

$$Upsw(X) = min\{\tau : \ X \text{ has a Urysohn point separating open cover } \mathcal{S}$$

$$\text{such that each point of X is contained in at most } \tau \text{elements of } \mathcal{S}\} + \aleph_0.$$

Note that $Hpsw(X) \leq Upsw(X)$, for every Urysohn space X.

The proof of the following theorem follows step by step the proof of Theorem 20 in [5].

Theorem 4.1. *If X is a Urysohn space then $nw(X) \leq Upsw(X)^{\theta\text{-}aL_c(X)}$.*

Proof. Let θ-$aL_c(X) = k$ and $\mathcal{S}$ a Urysohn point separating open cover for X such that for each $x \in X$, $|\mathcal{S}_x| \leq \lambda$, where $\mathcal{S}_x$ is the collection of members of $\mathcal{S}$ containing x.

We first show that $d(X) \leq \lambda^k$. $\forall\ \alpha < k$ construct a subset D_α of X such that:

1. $|D_\alpha| \leq \lambda^k$;
2. if $\mathcal{U}$ is a subcollection of $\bigcup\{\mathcal{S}_x :\ x \in \bigcup_{\beta<\alpha} D_\beta\}$ such that $|\mathcal{U}| \leq k$ and if $X \setminus \bigcup cl_\theta(\overline{\mathcal{U}}) \neq \emptyset$ we have that $D_\alpha \setminus \bigcup cl_\theta(\overline{\mathcal{U}}) \neq \emptyset$.

Such a D_α can be constructed since the member of possible $\mathcal{U}'s$ at the αth stage of construction is $\leq (\lambda^k k\lambda)^k = \lambda^k$.

Let $D = \bigcup_{\alpha<k^+} D_\alpha$. We have that $|D| \leq \lambda^k$. We want to prove that $\overline{D} = X$. Suppose that there exists $p \in X \setminus \overline{D}$, since $Upsw(X) \leq \lambda$, $\forall x \in \overline{D}$, there exists $V_x \in \mathcal{S}_x$: $p \notin cl_\theta(\overline{V_x})$. Since $x \in \overline{D}$, $V_x \cap D \neq \emptyset$. Let $y \in V_x \cap D$, so $V_x \in \bigcup\{\mathcal{S}_y :\ y \in D\}$. Put $\mathcal{W} = \{V_x :\ x \in \overline{D}\} \subseteq \bigcup\{\mathcal{S}_y :\ y \in D\}$. $\mathcal{W}$ is an open cover of $\overline{D}$ and since θ-$aL_c(X) \leq k$, there exists $\mathcal{W}' \subseteq \mathcal{W}$ with $|\mathcal{W}'| \leq k$ such that $\overline{D} \subseteq \bigcup\{cl_\theta(\overline{V}) :\ V \in \mathcal{W}'\}$ and $p \notin \bigcup\{cl_\theta(\overline{V}) :\ V \in \mathcal{W}'\}$ and this contradicts 2..

Since $d(X) \leq \lambda^k$ we have that $|\mathcal{S}| \leq \lambda^k$.

Let $\mathcal{N} = \{X \setminus S :\ S \textit{ is the union of at most } k \textit{ members of } \mathcal{S}\}$. $|\mathcal{N}| \leq \lambda^k$ and $\mathcal{N}$ is a network for X.

□

Theorem 4.2. *If X is a Urysohn space then $|X| \leq Upsw(X)^{\theta\text{-}aL_c(X)\psi(X)}$.*

Proof. If X is a T_1 space then $|X| \leq nw(X)^{\psi(X)}$ and using the theorem above we have that $|X| \leq nw(X)^{\psi(X)} \leq Upsw(X)^{\theta\text{-}aL_c(X)\psi(X)}$. □

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