Duality for pathwise superhedging in continuous time

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Abstract. We provide a model-free pricing-hedging duality in continuous time. For a frictionless market consisting of d risky assets with continuous price trajectories, we show that the purely analytic problem of finding the minimal superhedging price of an upper semicontinuous path dependent European option has the same value as the purely probabilistic problem of finding the supremum of the expectations of the option over all martingale measures. The superhedging problem is formulated with simple trading strategies and superhedging is required in the pathwise sense on a σ-compact sample space of price trajectories. If the sample space is stable under stopping, the probabilistic problem reduces to finding the supremum over all martingale measures with compact support. As an application of the general results we deduce dualities for Vovk's outer measure and semi-static superhedging with finitely many securities.

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1. Introduction

Given the space $C([0,T],\mathbb{R}^d)$ of all continuous price trajectories, the superhedging problem of a contingent claim $X\colon C([0,T],\mathbb{R}^d)\to\mathbb{R}$ consists of finding the infimum over all $\lambda\in\mathbb{R}$ such that there exists a trading strategy H which satisfies

$$\lambda + (H\cdot S)_T(\omega) \geq X(\omega), \quad \omega\in C([0,T],\mathbb{R}^d), \tag{1.1}$$

where $(H\cdot S)_T(\omega)$ denotes the capital gain by trading according to the strategy H in the underlying assets $S_t(\omega):=\omega(t)$.

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In the classical framework of mathematical finance one commonly postulates a model for the price evolution by fixing a probability measure P such that S is a semimartingale and defines $(H \cdot S)_T$ as the stochastic integral $\int_0^T H_t \, \mathrm{d}S_t$. Then, a consequence of the fundamental theorem of asset pricing states that the infimum over all λ such that there are admissible predictable integrands H fulfilling inequality (1.1) is equal to the supremum of $E_Q[X]$ over all absolutely continuous local martingale measures Q, see Delbaen and Schachermayer [18]. Here, the superhedging (i.e. inequality (1.1)) is assumed to hold P-almost surely and the set of absolutely continuous local martingale measures is non-empty, which is guaranteed by the exclusion of some form of arbitrage, see [18] for the precise formulation.

More recently, there are alternative possibilities to specify the superhedging requirement without referring to a fixed model; for instance, if an investor takes into account a class $\mathcal{P}$ of probabilistic models, then superhedging is naturally required to hold $\mathcal{P}$-quasi surely, i.e. P-almost surely for all considered models $P \in \mathcal{P}$. The pioneering works of Lyons [31] and Avellaneda et al. [4] on Knightian uncertainty in mathematical finance consider models with uncertain volatility in continuous time. The study of the pricing-hedging duality in this setting has given rise to a rich literature starting with the capacity-theoretic approach of Denis and Martini [19]. Further, Peng [37] obtains the duality using stochastic control techniques, whereas Soner et al. [41, 42, 43] rely on supermartingale decomposition results under individual models and eventually build on aggregation results to derive the duality under model uncertainty. This approach has been extended by Neufeld and Nutz [35] to cover measurable claims using the theory of analytic sets, see also Biagini et al. [12] for a robust fundamental theorem under a model ambiguity version of the no-arbitrage of the first kind condition $\mathrm{NA}_1(\mathcal{P})$, and Nutz [36] for the case of jump diffusions.

In the present work we focus on the pathwise/model-free approach and assume that the superhedging requirement (1.1) has to hold pointwise for all price trajectories in a given set $\Omega \subseteq C([0,T], \mathbb{R}^d)$. In this pathwise setting, finding the minimal superhedging price turns out to be a purely analytic problem and its formulation is independent of the probabilistic problem of finding the supremum of the expectation over (a subset of) all martingale measures. This is in contrast to the above mentioned approaches working with a fixed model, under Knightian uncertainty or in a quasi-sure setting. Notice that the pathwise approach corresponds to the quasi-sure approach when $\mathcal{P}$ contains all Dirac measures, which in continuous time is excluded, see e.g. [12, Corollary 3.5].

In the now classical paper [28], Hobson first addressed the problem of pathwise superhedging for the lookback option. His analysis was based on some sharp pathwise martingale inequalities and has motivated Beiglböck et al. [8] to introduce the martingale optimal transport problem in discrete time. Here, the investor takes static positions in some liquidly traded vanilla options and dynamic positions in the stocks. The rationale is that information on the price of options translates into the knowledge of some marginals of the martingale measures; see also [2, 6, 10, 14, 15, 17] for further developments in this direction. In continuous time, the duality for the martingale optimal transport has been obtained by Galichon et al. [24] and Possamaï et al. [39] in the quasi-sure setting. The pathwise formulation was studied by Dolinsky and Soner [20] using a discretization of the sample space. These results have been extended by Hou and Obłój [29], who, in particular, allow incorporation of investor's beliefs (of possible price paths) by relying on the notion of “prediction set” due to Mykland [34].

Following this consideration in our analysis, we also assume that the investor does not deem every continuous paths plausible but focuses on a prediction set $\Omega \subseteq C([0,T],\mathbb{R}^d)$ that is required to be σ-compact (i.e. at most a countable union of compact sets) and define the pathwise superhedging problem on the sample space Ω. Moreover, restricting the set of possible price paths has the financially desirable effect of reducing the superhedging price. See also Aksamit et al. [3] and Acciaio and Larsson [1] for other treatments of belief and information in robust superhedging, and Dolinsky and Soner [21] and Guo et al. [27] for extensions of the pathwise formulation to the Skorokhod space.

In the continuous time setting already the definition of the pathwise “stochastic” integral is a non-trivial issue. We circumvent this problem by working with simple strategies and consider as “stochastic” integral the pointwise limit inferior of pathwise integrals against simple strategies; an approach that was proposed by Perkowski and Prömel [38] to define an outer measure allowing to study stochastic integration under model ambiguity. This outer measure is very similar in spirit to that of Vovk [44] and can be seen as the value of a pathwise superhedging problem, c.f. Section 2.1 for details and Beiglböck et al. [9] and Vovk [45] for existing duality results in this setting.

Formally, for an upper semicontinuous contingent claim $X\colon \Omega \to [0,+\infty]$ we define its superhedging price as the infimum over all $\lambda \in \mathbb{R}$ such that there exists a sequence (H^n) of simple strategies which satisfies

$$\lambda + \liminf_{n\to\infty}(H^n \cdot S)_T(\omega) \geq X(\omega) \quad \text{for all } \omega \in \Omega$$

and the admissibility condition $\lambda + (H^n \cdot S)_t(\omega) \geq 0$ for all $n \in \mathbb{N}$, $\omega \in \Omega$, and $t \in [0,T]$. Under the assumptions that Ω is σ-compact and contains all its stopped paths, we show that the superhedging price coincides with the supremum of $E_Q[X]$ over all martingale measures Q. Furthermore, this duality is generalized to the case when X is unbounded from below and when Ω does not contain all its stopped pahts.

Our main contributions to the pathwise pricing-hedging duality in continuous time and finitely many risky assets are as follows. While in the current literature (see e.g. [20, 27, 29]) pathwise duality results hold for uniformly continuous options, the proposed method allows for upper semicontinuous claims (including for example European options, Spread options, continuously and discretely monitoring Asian options, lookback options and especially certain types of barrier options). In particular, this implies a duality for Vovk’s outer measure on closed sets. A related duality result was given by Vovk [45], however, under an additional closure of the attainable outcomes. Moreover, our pricing-hedging duality holds for every prediction set Ω which is σ-compact. Let us remark that the assumption of σ-compactness is an essential ingredient of the presented method to get the pricing-hedging duality. We will show in Section 3.1 that typical price trajectories for various popular financial models such as local, stochastic or even rough volatility models belong to the σ-compact space of Hölder continuous functions. In the related work [29] the pricing-hedging duality holds for an approximate version of the superhedging price which requires the superhedging on a enlarged prediction set $\Omega^\varepsilon := \{\omega \in C([0,T],\mathbb{R}^d) : \inf_{\omega'\in\Omega} \|\omega-\omega'\|_\infty \leq \varepsilon\} \supset \Omega$ for any given $\varepsilon > 0$.

The article is organized as follows: In Section 2 we present the main results (Theorem 2.3 and Theorem 2.1) and some direct applications. Section 3 contains a detailed discussion of feasible choices for the underlying sample space. The proofs of the main results are carried out in Section 4. A criterion for the sample path regularity of stochastic processes is stated in Appendix A.

2. Main results

Let $\Omega \subset C([0,T],\mathbb{R}^d)$ be a non-empty metric space where $T > 0$ is a finite time horizon and $d \in \mathbb{N}$. The canonical process $S\colon [0,T] \times \Omega \to \mathbb{R}^d$ given by $S_t(\omega) := \omega(t)$ generates the raw filtration $\mathcal{F}_t^0 := \sigma(S_s, s \leq t \wedge T)$, $t \geq 0$. Furthermore, let $(\mathcal{F}_t)$ be the right-continuous version of the raw filtration $(\mathcal{F}_t^0)$, defined by $\mathcal{F}_t := \bigcap_{s>t} \mathcal{F}_s^0$ for all $t \in [0,T]$. Denote by $\mathcal{M} := \mathcal{M}(\Omega)$ the set of all Borel probability measures Q on Ω such that the canonical process S is a Q-martingale, and by $\mathcal{M}_c := \mathcal{M}_c(\Omega) := \{Q \in \mathcal{M} : Q(K) = 1 \text{ for some compact } K \subset \Omega\}$ the subset of all martingale measures with compact support.

A process $H\colon [0,T] \times \Omega \to \mathbb{R}^d$ is called simple predictable if it is of the form

$$H_t(\omega) = \sum_{n=1}^{N} h_n(\omega) \mathbf{1}_{(\tau_n(\omega), \tau_{n+1}(\omega)]}(t), \quad (t,\omega) \in [0,T] \times \Omega,$$

where $N \in \mathbb{N}$, $0 \leq \tau_1 \leq \cdots \leq \tau_{N+1} \leq T$ are stopping times w.r.t. the filtration $(\mathcal{F}_t)$, and $h_n\colon \Omega \to \mathbb{R}^d$ are bounded $\mathcal{F}_{\tau_n}$-measurable functions. The set of all simple predictable processes is denoted by $\mathcal{H}^f := \mathcal{H}^f(\Omega)$. For a simple predictable $H \in \mathcal{H}^f$ the pathwise stochastic integral

$$(H \cdot S)_t(\omega) := \sum_{n=1}^{N} h_n(\omega)(S_{\tau_{n+1}(\omega)\wedge t}(\omega) - S_{\tau_n(\omega)\wedge t}(\omega))$$

is well-defined for all $t \in [0,T]$ and all $\omega \in \Omega$. Similarly, the pathwise stochastic integral $(H \cdot S)$ is also well-defined for each $H\colon [0,T] \times \Omega \to \mathbb{R}^d$ in the set $\mathcal{H} := \mathcal{H}(\Omega)$ of processes of the form

$$H_t(\omega) = \sum_{n=1}^{\infty} h_n(\omega) \mathbf{1}_{(\tau_n(\omega), \tau_{n+1}(\omega)]}(t)$$

where $0 \leq \tau_1 \leq \tau_2 \leq \cdots$ are stopping times such that for each $\omega \in \Omega$ there exists an $N(\omega) \in \mathbb{N}$ with $\tau_k(\omega) = T$ for all $k \geq N(\omega)$, and $h_n\colon \Omega \to \mathbb{R}$ are bounded $\mathcal{F}_{\tau_n}$-measurable functions.

We introduce the following two assumptions, which we shall use frequently.

(A1) Ω is σ-compact, the metric on Ω induces a topology finer than (or equal to) the one induced by the maximum norm $\|\omega\|_\infty := \max_{t\in[0,T]} |\omega(t)|$, and for each Borel probability Q on Ω and every bounded $\mathcal{F}_t^0$-measurable function h there exists a sequence of $\mathcal{F}_t^0$-measurable continuous functions (h_n) which converges Q-almost surely to h.

(A2) For every $\omega \in \Omega$ and each $t \in [0,T]$ the stopped path $\omega^t(\cdot) := \omega(\cdot \wedge t)$ is in Ω and the function $[0,T] \times \Omega \ni (t,\omega) \mapsto \omega^t$ is continuous.

If Ω is a σ-compact space endowed with the topology induced by the maximum norm, then (A1) is always satisfied, see Remark 4.1. Now we are ready to state the main results of this paper. The proofs are given in Section 4.

Theorem 2.1. *Suppose that (A1) and (A2) hold and let $Z\colon \Omega \to [0,+\infty)$ be a continuous function satisfying $Z(\omega^s) \le Z(\omega^t)$ for all $\omega \in \Omega$ and $0 \le s \le t \le T$. Then, for every upper semicontinuous function $X\colon \Omega \to (-\infty,+\infty]$ which satisfies $X(\omega) \ge -Z(\omega)$ for all $\omega \in \Omega$, one has*

$$\inf\left\{\lambda \in \mathbb{R} : \begin{array}{l} \text{there is a sequence } (H^n) \text{ in } \mathcal{H}^f \text{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \ge -Z(\omega^t) \text{ for all } (t,\omega) \in [0,T]\times\Omega \\ \text{and } \lambda + \liminf_n (H^n \cdot S)_T(\omega) \ge X(\omega) \text{ for all } \omega \in \Omega \end{array}\right\} = \sup_{Q \in \mathcal{M}_c} E_Q[X]. \quad (2.1)$$

Moreover, if $\mathcal{H}^f$ is replaced by $\mathcal{H}$, then (2.1) *holds with $\mathcal{M}_c$ replaced by $\mathcal{M}_Z := \{Q \in \mathcal{M} : E_Q[Z] < +\infty\}$.*

Remark 2.2.

1. *By continuity of Z one has $\mathcal{M}_c \subset \mathcal{M}_Z$. In particular, if $X(\omega) \ge Z(\omega)$ for all $\omega \in \Omega$, the expectation $E_Q[X]$ is well-defined under every $Q \in \mathcal{M}_Z$.*

2. *Notice that $Z(\omega) := \max_{t\in[0,T]} |\omega(t)|^p$ for $p \ge 0$ satisfies $Z(\omega^s) \le Z(\omega^t)$ for every $\omega \in \Omega$ and $0 \le s \le t \le T$.*

3. *If $Z \ge \|\cdot\|_\infty$, then $E_Q[\max_{t\in[0,T]} |S_t|] < +\infty$ for every Borel probability measure Q which integrates Z. Hence, the set of all local martingale measures which integrate Z coincides with $\mathcal{M}_Z$.*

If Ω does not contain all its stopped paths, then the following version of Theorem 2.1 holds true.

Theorem 2.3. *Let $Z\colon \Omega \to [1,+\infty)$ be a function with compact sublevel sets $\{Z \le c\}$ for all $c \in \mathbb{R}$ satisfying $Z(\omega) \ge \|\omega\|_\infty$ for all $\omega \in \Omega$. If (A1) holds true and $\mathcal{M}_Z \ne \emptyset$, then*

$$\inf\left\{\lambda \in \mathbb{R} : \begin{array}{l} \text{there is } c \ge 0 \text{ and a sequence } (H^n) \text{ in } \mathcal{H}^f \text{ such that} \\ (H^n \cdot S)_T(\omega) \ge -cZ(\omega) \text{ for all } \omega \in \Omega \text{ and} \\ \lambda + \liminf_n (H^n \cdot S)_T(\omega) \ge X(\omega) \text{ for all } \omega \in \Omega \end{array}\right\} = \sup_{Q \in \mathcal{M}_Z} E_Q[X]$$

for every upper semicontinuous function $X\colon \Omega \to (-\infty,+\infty]$ which is bounded from below.

Finally, using a different set of admissible strategies, one can replace $\mathcal{M}_Z$ by the set of all martingale measures $Q \in \mathcal{M}$ which satisfy $E_Q[\max_{t\in[0,T]} |S_t|] < +\infty$. Indeed, denote by $\mathcal{Z}$ the set of all functions $Z\colon \Omega \to [1,+\infty)$ with compact sublevel sets $\{Z \le c\}$ satisfying $Z \ge \|\cdot\|_\infty$ and define $\mathcal{M}_{\mathcal{Z}} := \bigcup_{Z\in\mathcal{Z}} \mathcal{M}_Z$. For every martingale measure Q on Ω which satisfies $E_Q[\max_{t\in[0,T]} |S_t|] < +\infty$, by σ-compactness of Ω, there exists a function $Z'\colon \Omega \to [1,+\infty)$ with compact sublevel sets $\{Z' \le c\}$ such that $E_Q[Z'] < +\infty$. Then, since $Z := Z' + \|\cdot\| \in \mathcal{Z}$ and $E_Q[Z] < +\infty$, one has $Q \in \mathcal{M}_{\mathcal{Z}}$. The other inclusion is trivial. Therefore, by interchanging two suprema in the previous theorem, one can deduce the following result.

Corollary 2.4. *Assume that (A1) holds true and $\mathcal{M}_{\mathcal{Z}} \neq \emptyset$. Then*

$$\inf \left\{ \lambda \in \mathbb{R} : \begin{array}{l} \text{for every } Z \in \mathcal{Z} \text{ there is } c \geq 0 \text{ and a sequence } (H^n) \\ \text{in } \mathcal{H}^f \text{ such that } (H^n \cdot S)_T(\omega) \geq -cZ(\omega) \text{ for all } \omega \in \Omega \\ \text{and } \lambda + \liminf_n (H^n \cdot S)_T(\omega) \geq X(\omega) \text{ for all } \omega \in \Omega \end{array} \right\} = \sup_{Q \in \mathcal{M}_{\mathcal{Z}}} E_Q[X]$$

for every upper semicontinuous function $X \colon \Omega \to (-\infty, +\infty]$ which is bounded from below.

2.1. Relation to Vovk's outer measure

In recent years (see e.g. [44, 45] and the references therein), Vovk introduced an outer measure on different path spaces, defined as the minimal superhedging price, which allows to quantify the path behavior of "typical price paths" in frictionless financial markets without any reference measure.

In order to recall Vovk's outer measure on a set $\Omega \subset C([0,T], \mathbb{R}^d)$ endowed with the maximum norm, we write $\mathcal{H}_\lambda$ for the set of λ-admissible simple predicable strategies, i.e. the set of all $H \in \mathcal{H}$ such that $(H \cdot S)_t(\omega) \geq -\lambda$ for all $(t, \omega) \in [0,T] \times \Omega$. Furthermore, we define the set of processes

$$\mathcal{V}_\lambda := \left\{ \mathrm{H} := \left(H^k\right)_{k \in \mathbb{N}} : H^k \in \mathcal{H}_{\lambda_k},\, \lambda_k > 0,\, \sum_{k=1}^{\infty} \lambda_k = \lambda \right\}$$

for an initial capital $\lambda \in (0, +\infty)$. Note that for every $\mathrm{H} = \left(H^k\right) \in \mathcal{V}_\lambda$, all $\omega \in \Omega$, and all $t \in [0,T]$, the corresponding capital process

$$(\mathrm{H} \cdot S)_t(\omega) := \sum_{k=1}^{\infty} (H^k \cdot S)_t(\omega) = \sum_{k=1}^{\infty} \left(\lambda_k + (H^k \cdot S)_t(\omega)\right) - \lambda$$

is well-defined and takes values in $[-\lambda, +\infty]$. Then, Vovk's outer measure on Ω is given by

$$\overline{Q}_\Omega(A) := \inf \left\{ \lambda > 0 : \text{there is } \mathrm{H} \in \mathcal{V}_\lambda \text{ such that } \lambda + (\mathrm{H} \cdot S)_T(\omega) \geq \mathbf{1}_A(\omega) \text{ for all } \omega \in \Omega \right\}.$$

A slight modification of the outer measure $\overline{Q}_\Omega$ was introduced in Perkowski and Prömel [38], which is defined as

$$\overline{P}_\Omega(A) := \inf \left\{ \lambda > 0 : \begin{array}{l} \text{there is } (H^n) \text{ in } \mathcal{H}_\lambda \text{ such that} \\ \lambda + \liminf_{n \to \infty} (H^n \cdot S)_T(\omega) \geq \mathbf{1}_A(\omega) \text{ for all } \omega \in \Omega \end{array} \right\}$$

for $A \subseteq \Omega$. The latter definition seems to be more in the spirit of superhedging prices in semimartingale models as discussed in [38, Sections 2.1 and 2.2]. Notice that, even if it would be convenient to just minimize over simple strategies rather than over the limit (inferior) along sequences of simple strategies in both definitions of outer measures, this is essential to obtain the desired countable subadditivity of both outer measures.

Remark 2.5. *In case that $\Omega = C([0,T],\mathbb{R}^d)$ one would expect that the outer measures $\overline{Q}_\Omega$ and $\overline{P}_\Omega$ coincide. However, currently it is only known that*

$$\sup_{Q\in\mathcal{M}} Q(A) \leq \overline{P}_\Omega(A) \leq \overline{Q}_\Omega(A), \tag{2.2}$$

where $A \subset C([0,T],\mathbb{R}^d)$ is a measurable set, see [44, Lemma 6.2] and [38, Lemma 2.9]. In the special case of $\Omega = C([0,+\infty),\mathbb{R})$ and a time-superinvariant set $A \subset C([0,+\infty),\mathbb{R})$ the inequalities in (2.2) *turn out to be true equalities. See Vovk [44, Sections 2 and 3] and Beiglböck et al. [9, Section 2] for the precise definitions and statements in this context.*

By restricting the outer measure $\overline{P}_\Omega$ to a σ-compact space Ω, we get the following duality result for the slightly modified version of Vovk's outer measure as a direct application of Theorem 2.1.

Proposition 2.6. *Under the assumptions on Ω of Theorem 2.1, one has*

$$\overline{P}_\Omega(A) = \sup_{Q\in\mathcal{M}_c} Q(A) = \sup_{Q\in\mathcal{M}} Q(A)$$

for all closed subsets $A \subset \Omega$.

Proof. For every closed subset $A \subset \Omega$, it follows from Theorem 2.1 and Remark 2.2 that

$$\begin{aligned}\overline{P}_\Omega(A) &= \inf\left\{\lambda > 0 : \begin{array}{l}\text{there is a sequence } (H^n) \text{ in } \mathcal{H} \text{ such that}\\ \lambda + (H^n\cdot S)_t(\omega) \geq 0 \text{ for all } (t,\omega)\in[0,T]\times\Omega \text{ and}\\ \lambda + \liminf_n (H^n\cdot S)_T(\omega) \geq \mathbf{1}_A(\omega) \text{ for all } \omega\in\Omega\end{array}\right\}\\ &= \sup_{Q\in\mathcal{M}_c} E_Q[\mathbf{1}_A] = \sup_{Q\in\mathcal{M}_c} Q(A)\end{aligned}$$

because $\mathbf{1}_A$ is upper semicontinuous. □

Remark 2.7. *Recently, Vovk [45] obtained a similar duality for open sets by adjusting the definition of the outer measure $\overline{P}_\Omega$. More precisely, his new definition of outer measure allows for superhedging with all processes in the "liminf-closure" of capital processes generated by sequences of λ-admissible simple strategies, see [45, Section 2 and Theorem 2] for more details.*

2.2. Semi-static superhedging

Let us fix a continuous function $Z\colon \Omega \to [1,+\infty)$ and consider a finite number of securities with (discounted) continuous payoffs $G_1,\dots,G_K$ such that $|G_i| \leq cZ$ for $i = 1,\dots,K$ and some $c \geq 0$. We assume that these securities can be bought and sold at prices $g_k \in \mathbb{R}$, and satisfy the no-arbitrage condition

$$(g_1,\dots,g_K) \in \operatorname{ri}\{(E_Q[G_1],\dots,E_Q[G_K]) : Q \in \mathcal{M}_c\} \tag{2.3}$$

where ri denotes the relative interior.

Then the following semi-static hedging duality holds.

Proposition 2.8. *Suppose that the assumptions (A1) and (A2) are satisfied, and the securities with payoffs $G_1, \dots, G_K$ satisfy the static no arbitrage condition* (2.3)*. Then, for every upper semicontinuous function $X\colon \Omega \to \mathbb{R}$ which satisfies $|X| \le cZ$ for some $c \ge 0$, one has*

$$\begin{aligned}
&\inf\left\{ \lambda \in \mathbb{R} : \begin{array}{l} \textit{there is } c \ge 0,\ \alpha \in \mathbb{R}^K, \textit{ and a sequence } (H^n) \textit{ in } \mathcal{H}^f \textit{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \ge -cZ(\omega^t) \textit{ for all } (t,\omega) \in [0,T] \times \Omega \textit{ and} \\ \lambda + \sum_{k=1}^K \alpha_k (G_k(\omega) - g_k) + \liminf_n (H^n \cdot S)_T(\omega) \ge X(\omega) \textit{ for all } \omega \in \Omega \end{array} \right\} \\
&= \sup_{Q \in \mathcal{M}_c(G)} E_Q[X]
\end{aligned} \tag{2.4}$$

where $\mathcal{M}_c(G) := \{Q \in \mathcal{M}_c : E_Q[G_k] = g_k \text{ for all } k = 1, \dots, K\}$.

Proof. For every $Y\colon \Omega \to \mathbb{R}$ which satisfies $|Y| \le cZ$ for some $c \ge 0$ we define

$$\phi(Y) := \inf\left\{ \lambda \in \mathbb{R} : \begin{array}{l} \text{there is } c \ge 0 \text{ and a sequence } (H^n) \text{ in } \mathcal{H}^f \text{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \ge -cZ(\omega^t) \text{ for all } (t,\omega) \in [0,T] \times \Omega \text{ and} \\ \lambda + \liminf_n (H^n \cdot S)_T(\omega) \ge Y(\omega) \text{ for all } \omega \in \Omega \end{array} \right\}$$

and we remark that, by interchanging two infimia, the left hand side of (2.4) can be expressed as $\inf_{\alpha \in \mathbb{R}^K} \phi\big(X - \sum_{k=1}^K \alpha_k (G_k - g_k)\big)$. Further, Theorem 2.1 yields

$$\phi\Big(X - \sum_{k=1}^K \alpha_k (G_k - g_k)\Big) = \sup_{Q \in \mathcal{M}_c} E_Q\Big[X - \sum_{k=1}^K \alpha_k (G_k - g_k)\Big]$$

for every $\alpha \in \mathbb{R}^K$. Now define the function

$$J\colon \mathcal{M}_c \times \mathbb{R}^K \to \mathbb{R}, \quad J(Q,\alpha) := E_Q[X] - \sum_{k=1}^K \alpha_k E_Q[G_k - g_k].$$

It is immediate that $J(Q,\cdot)$ is convex for every $Q \in \mathcal{M}_c$ and that $J(\cdot,\alpha)$ is concave for each $\alpha \in \mathbb{R}^K$ since $\mathcal{M}_c$ is convex. Therefore, it follows exactly as in step (a) of the proof of [5, Theorem 2.1], that the assumption of 0 being in the relative interior of $\{(E_Q[G_1 - g_1], \dots, E_Q[G_K - g_K]) : Q \in \mathcal{M}_c\}$ can be used to show that all requirements of the minimax theorem [40, Theorem 4.1] are satisfied. Hence, one gets

$$\inf_{\alpha \in \mathbb{R}^K} \phi\Big(X - \sum_{k=1}^K \alpha_k (G_k - g_k)\Big) = \inf_{\alpha \in \mathbb{R}^K} \sup_{Q \in \mathcal{M}_c} J(Q,\alpha) = \sup_{Q \in \mathcal{M}_c} \inf_{\alpha \in \mathbb{R}^K} J(Q,\alpha) = \sup_{Q \in \mathcal{M}_c(G)} E_Q[X],$$

where the first equality follows from Theorem 2.1 and the last one by

$$\inf_{\alpha \in \mathbb{R}^K} J(Q,\alpha) = \begin{cases} E_Q[X], & \text{if } Q \in \mathcal{M}_c(G), \\ -\infty, & \text{if } Q \in \mathcal{M}_c \setminus \mathcal{M}_c(G). \end{cases}$$

The proof is complete. □

3. Discussion of σ-compact spaces

By definition, the σ-compactness of the metric space $\Omega \subset C([0,T],\mathbb{R}^d)$ requires to find a covering of Ω by compact sets K^m, $m \in \mathbb{N}$. It is an easy consequence of the Arzelà-Ascoli theorem (see e.g. [23, Theorem 1.4]) that these K^m have to be bounded, closed and equicontinuous.

In the next lemma we provide an easy-to-check criterion for a set Ω of continuous functions to be σ-compact. This leads to many interesting examples of such $\Omega \subset C([0,T],\mathbb{R}^d)$ appearing in the context of (classical) financial modeling, see Subsection 3.1.

Lemma 3.1. *For $n \in \mathbb{N}$ let $c_n\colon [0,T]^2 \to [0,+\infty)$ be a continuous function with $c_n(t,t)=0$ for $t \in [0,T]$ and define the norm*

$$\|\omega\|_{c_n,\alpha} := |\omega(0)| + \sup_{s,t\in[0,T]} \frac{|\omega(t)-\omega(s)|}{c_n(s,t)^\alpha}, \quad \omega \in C([0,T],\mathbb{R}^d),$$

with $\alpha \in (0,1]$ and the convention $\frac{0}{0} := 0$. Then the spaces

$$\Omega_n := \left\{\omega \in C([0,T],\mathbb{R}^d) \,:\, \|\omega\|_{c_n,1} < +\infty\right\}, \quad n \in \mathbb{N},$$

are σ-compact w.r.t. the norm $\|\cdot\|_{c_n,\alpha}$ for $\alpha \in (0,1)$ and in particular w.r.t. the maximum norm $\|\cdot\|_\infty$. Moreover, the set $\Omega := \bigcup_{n\in\mathbb{N}} \Omega_n$ is σ-compact w.r.t. the maximum norm $\|\cdot\|_\infty$.

Proof. For $m,n \in \mathbb{N}$ we observe

$$\Omega_n = \bigcup_{m\in\mathbb{N}} K_n^m \quad \text{with} \quad K_n^m := \left\{\omega \in C([0,T],\mathbb{R}^d) \,:\, \|\omega\|_{c_n,1} \le m\right\}.$$

In order to show the σ-compactness of Ω_n w.r.t. $\|\cdot\|_\infty$, we need to show that each K_n^m is compact. Due to the Arzelà-Ascoli theorem, it is sufficient to show that each K_n^m is bounded, equicontinuous and closed.
Boundedness: For every $\omega \in K_n^m$ we have

$$\|\omega\|_\infty \le |\omega(0)| + \sup_{t\in[0,T]} |\omega(t)-\omega(0)| \le |\omega(0)| + m \sup_{t\in[0,T]} c_n(0,t).$$

Equicontinuity: Because c_n is continuous on a compact set and $c_n(t,t)=0$ for $t \in [0,T]$, for every $\varepsilon > 0$ there exits a $\delta > 0$ such that $|c_n(s,t)| < \varepsilon/m$ for $|t-s| \le \delta$. Hence, for every $\omega \in K_n^m$ and $s,t \in [0,T]$ with $|t-s| \le \delta$ we get $|\omega(t)-\omega(s)| \le \varepsilon$.
Closeness: If $(\omega_k) \subset K_n^m$ converges uniformly to ω, then $\omega \in K_n^m$. Indeed, this can be seen by

$$|\omega(0)| + \frac{|\omega(t)-\omega(s)|}{c_n(s,t)} = \lim_{k\to\infty} \left(|\omega_k(0)| + \frac{|\omega_k(t)-\omega_k(s)|}{c_n(s,t)}\right) \le m.$$

The σ-compactness of Ω_n w.r.t. $\|\cdot\|_{c_n,\alpha}$ for $\alpha \in (0,1)$ follows by the fact that the uniform convergence in each K_n^m implies the convergence w.r.t. $\|\cdot\|_{c_n,\alpha}$, which is a consequence of the following interpolation inequality

$$\frac{|\omega(t)-\omega(s)|}{c_n(s,t)^\alpha} = \left(\frac{|\omega(t)-\omega(s)|}{c_n(s,t)}\right)^\alpha |\omega(t)-\omega(s)|^{1-\alpha} \le 2\|\omega\|_{c_n,1}^\alpha \|\omega\|_\infty^{1-\alpha}, \quad s,t \in [0,T].$$

Finally, Ω is σ-compact (w.r.t. $\|\cdot\|_\infty$) since its is countable union of σ-compact sets. □

From the previous lemma it is easy to deduce that many well-known function spaces $\Omega \subset C([0,T],\mathbb{R}^d)$ are σ-compact spaces. To state the next corollary, we recall the notion of control functions: $c\colon [0,T]^2 \to [0,+\infty)$ is called control function if c is continuous, super-additive, i.e. $c(s,t)+c(t,u) \le c(s,u)$ for $0 \le s \le t \le u \le T$, and $c(t,t)=0$ for every $t \in [0,T]$.

Corollary 3.2.

1. *The space $C^\alpha([0,T],\mathbb{R}^d)$ of α-Hölder continuous functions, i.e.*

$$C^\alpha([0,T],\mathbb{R}^d) := \left\{ \omega \in C([0,T],\mathbb{R}^d) \,:\, \sup_{s,t\in[0,T]} \frac{|\omega(t)-\omega(s)|}{|t-s|^\alpha} < +\infty \right\}, \quad \alpha \in (0,1],$$

is σ-compact w.r.t. $\|\cdot\|_\infty$ and w.r.t. the Hölder norm $\|\cdot\|_\beta$ for $\beta \in (0,\alpha)$ defined by

$$\|\omega\|_\beta := |\omega(0)| + \sup_{s,t\in[0,T]} \frac{|\omega(t)-\omega(s)|}{|t-s|^\beta} \quad \text{for} \quad \omega \in C^\alpha([0,T],\mathbb{R}^d).$$

2. *The space $C^{\text{Hölder}}([0,T],\mathbb{R}^d) := \bigcup_{\alpha\in(0,1]} C^\alpha([0,T],\mathbb{R}^d)$ of all Hölder continuous functions is σ-compact w.r.t. the maximum norm $\|\cdot\|_\infty$.*

3. *The fractional Sobolev space $W^{\delta,p}([0,T],\mathbb{R}^d)$ with $\delta - 1/p > 0$, given by*

$$W^{\delta,p}([0,T],\mathbb{R}^d) := \left\{ \omega \in C([0,T],\mathbb{R}^d) \,:\, \int_{[0,T]^2} \frac{|\omega(t)-\omega(s)|^p}{|t-s|^{\delta p+1}} \,\mathrm{d}s\,\mathrm{d}t < +\infty \right\}$$

for $\delta \in (0,1)$ and $p \in [1,+\infty)$, is σ-compact w.r.t. maximum norm $\|\cdot\|_\infty$.

4. *The space $C^{p\text{-var},c}([0,T],\mathbb{R}^d)$, which is a subspace of continuous functions with finite p-variation, given by*

$$C^{p\text{-var},c}([0,T],\mathbb{R}^d) := \left\{ \omega \in C([0,T],\mathbb{R}^d) \,:\, \sup_{s,t\in[0,T]} \frac{|\omega(t)-\omega(s)|}{c(s,t)^{1/p}} < +\infty \right\}$$

for $p \in [1,+\infty)$ and a control function c, is σ-compact w.r.t. the maximum norm $\|\cdot\|_\infty$ and w.r.t. the p'-variation norm $\|\cdot\|_{p'\text{-var}}$ for $p' \in (p,+\infty)$ defined by

$$\|\omega\|_{p'\text{-var}} := |\omega(0)| + \sup_{0\le t_0\le\cdots\le t_n\le T,\, n\in\mathbb{N}} \left(\sum_{i=0}^{n-1} |\omega(t_{i+1})-\omega(t_i)|^{p'} \right)^{1/p'}.$$

Proof. 1. and 2. follow directly by Lemma 3.1 and the fact that

$$C^\alpha([0,T],\mathbb{R}^d) \subset C^{\frac{1}{n}}([0,T],\mathbb{R}^d) \quad \text{for} \quad \alpha \in [n^{-1},(n-1)^{-1}], \quad n \in \mathbb{N}.$$

3. Classical Sobolev embedding results, see e.g. [23, Corollary A.2], imply

$$W^{\delta,p}([0,T],\mathbb{R}^d) \subset C^{\delta-1/p}([0,T],\mathbb{R}^d) \quad \text{and} \quad \|\omega\|_{\delta-1/p} \le C(\delta,p)\|\omega\|_{W^{\delta,p}}$$

for $\omega \in W^{\delta,p}([0,T],\mathbb{R}^d)$ with $\delta - 1/p > 0$ and for a constant $C(\delta,p) > 0$ depending only on δ and p. Here $\|\cdot\|_{W^{\delta,p}}$ denotes the fractional Sobolev semi-norm, see (A.1) below. Hence, to obtain the stated σ-compactness from Lemma 3.1, it remains to show that, if a sequence $(\omega_k) \subset W^{\delta,p}([0,T],\mathbb{R}^d)$ with $\|\omega\|_{W^{\delta,p}} \le K$ for some constant $K > 0$ converges uniformly to a function ω, then $\|\omega\|_{W^{\delta,p}} \le K$. However, this is a simple consequence of Fatou's lemma.

4. The σ-compactness w.r.t. $\|\cdot\|_\infty$ and $\|\cdot\|_{c,\alpha}$ for $\alpha \in (0,1)$ follows again by Lemma 3.1. The σ-compactness w.r.t. $\|\cdot\|_{p'\text{-var}}$ can be deduced from the inequality $\|\omega\|_{p'\text{-var}} \le \|\omega\|_{c,\frac{1}{p}} c(0,T)^{1/p}$ for $\omega \in C^{p\text{-var},c}([0,T],\mathbb{R}^d)$ and for $p' \in (p,+\infty)$. □

Remark 3.3.

1. *The function spaces stated in Corollary 3.2 satisfy also the first part of assumption (A2): for every $\omega \in \Omega$ and $t \in [0,T]$ the stopped path $\omega^t(\cdot) := \omega(\cdot \wedge t)$ is in Ω; in the case of the Hölder-type spaces this fairly easy to verify and for the Sobolev space we refer to [26, Lemma 1.5.1.8]. Hence, all these function spaces equipped with the maximum norm satisfy the assumptions (A1) and (A2), see also Remark 4.1.*

2. *From the perspective of (completely) model-free financial mathematics it might be desirable to consider the space $C^{p\text{-var}}([0,T],\mathbb{R}^d)$ of all continuous functions possessing finite p-variation for $p > 2$ since this space includes the support of all martingale measures. Unfortunately, the elementary covering as used in the proof of Lemma 3.1 cannot work as the unit ball in $C^{p\text{-var}}([0,T],\mathbb{R}^d)$ is not compact, see e.g. [32, Example 3.4].*

3.1. Examples from mathematical finance

As mentioned in the introduction, the prediction set Ω can be interpreted to contain all the price paths that an investor believes could possibly appear on a financial market. Hence, it is natural to choose Ω in a way that it includes those price processes coming from financial models which have been proven to provide fairly reasonable underlying price processes.

One way to proceed this idea goes as follows: Choose a set of financial models given by the set $\mathcal{P}$ of probability measures describing the law of the underlying price process S on $C([0,T],\mathbb{R}^d)$. In the present pathwise setting, that means, one wants to consider only those paths $\omega \in C([0,T],\mathbb{R}^d)$ which are "visited" by some model $P \in \mathcal{P}$. In order to do so, let us assume that there exits a σ-compact (w.r.t. the maximum norm) function space $\hat{\Omega}$ such that $P(\hat{\Omega}) = 1$ for all $P \in \mathcal{P}$. Notice that $\hat{\Omega}$ is separable since it is the union of compact sets in a topology induced by a metric. In this case the quasi-sure support of $\mathcal{P}$,

$$\operatorname{supp}\mathcal{P} := \bigcap \left\{ C \subset \hat{\Omega} \,:\, P(C) = 1 \text{ for all } P \in \mathcal{P} \text{ and } C \text{ is closed} \right\},$$

is well-defined, closed, and satisfies $P(\operatorname{supp}\mathcal{P}) = 1$ for all $P \in \mathcal{P}$, which follows exactly as in [13, Lemma 4.2]. Therefore, a natural choice of the prediction set Ω, in order to include beliefs coming from the models $\mathcal{P}$, is to set $\Omega := \operatorname{supp}\mathcal{P}$, which satisfies assumption (A1) due to Remark 4.1.

In the following we present several examples coming from the modeling of financial market and guarantee the existence of σ-compact metric spaces $\hat{\Omega} \subset C([0,T],\mathbb{R}^d)$, which include all the possible price trajectories produced by these models. For simplicity we consider one-dimensional processes and denote by W a one-dimensional Brownian motion on a probability space $(\tilde{\Omega}, P, \mathcal{F})$. However, all arguments extend straightforward to multi-dimensional settings.

1. Classical Black-Scholes model: A classical example from mathematical finance is the famous Black-Scholes model, which is given by

$$\mathrm{d}S_t = \sigma S_t \,\mathrm{d}W_t + \mu S_t \,\mathrm{d}t, \quad t \in [0, T],$$

for $\mu \in \mathbb{R}$ and $\sigma > 0$. In this case the price process S is a so-called geometric Brownian motion, which possesses the same sample path regularity as a Brownian motion. Hence, one has almost surely $S \in C^{\alpha}([0, T], \mathbb{R})$ and $S \in W^{\alpha-\frac{1}{q},q}([0, T], \mathbb{R})$ for every $\alpha \in (0, 1/2)$ and $q > 2$, cf. Corollary A.2, that is, we can take $\hat{\Omega}$ as one of the aforementioned function spaces.

2. Local volatility models: Other examples are local volatility models

$$\mathrm{d}S_t = \sigma(t, S_t) \,\mathrm{d}W_t, \quad S_0 = s_0, \quad t \in [0, T],$$

for a volatility function $\sigma\colon [0, T] \times \mathbb{R} \to \mathbb{R}$. For these classes of models one again has $S \in \hat{\Omega} := C^{\alpha}([0, T], \mathbb{R})$ a.s. for every $\alpha < 1/2$ if $s_0 \in \mathbb{R}$ and σ is Lipschitz continuous and satisfies the linear growth condition $|\sigma(t, x)|^2 \leq K(1 + |x|^2)$ for $(t, x) \in [0, T] \times \mathbb{R}$ and positive constant $K > 0$. Indeed, the Hölder regularity of S can be deduced from Corollary A.2 combined with the estimate

$$E_P\left[\int_0^T |\sigma(s, S_s)|^q \,\mathrm{d}s\right] \leq \tilde{C} E_P\left[\int_0^T (1 + |S_s|)^q \,\mathrm{d}s\right] \leq \tilde{C}'\left(1 + \int_0^T E_P\big[|S_s|^q\big] \,\mathrm{d}s\right) \leq C,$$

for constants $\tilde{C}, \tilde{C}' > 0$ and $C = C(q, K, T, S_0) > 0$, and for every $q \geq 2$, where the last inequality follows by the L^q-estimate in [33, Theorem 4.1].

3. Stochastic volatility models (with uncertainty): A frequently used generalization of the Black-Scholes model is given by stochastic volatility models

$$\mathrm{d}S_t = \sigma_t S_t \,\mathrm{d}W_t + \mu_t S_t \,\mathrm{d}t, \quad S_0 = s_0, \quad t \in [0, T], \tag{3.1}$$

for $s_0 \in \mathbb{R}$ and predictable real-valued processes μ and σ. This type of linear stochastic differential equations can be explicitly solved by

$$S_t := s_0 \exp\left(\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) \mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}W_s\right), \quad t \in [0, T].$$

Based on Corollary A.2, one can easily deduce the sample path regularity of the price process S: For $q \in (2, +\infty)$, $\alpha \in (0, 1/2 - 1/(2q))$ and $\delta := \alpha - 1/q$, if $E_P\big[\int_0^T |\mu_s|^q \,\mathrm{d}s\big] < +\infty$ and $E_P\big[\int_0^T |\sigma_s|^{2q} \,\mathrm{d}s\big] < +\infty$, then

$$S \in C^{\alpha}([0, T], \mathbb{R}) \quad \text{and} \quad S \in W^{\delta,q}([0, T], \mathbb{R}), \quad \text{a.s.} \tag{3.2}$$

For example the Heston model is a stochastic volatility model, in which the volatility process σ satisfies such a bound.

In the context of stochastic volatility modeling with Knightian uncertainty, one usually replaces the fixed volatility process σ by a class of volatility processes. For example the seminal works on volatility uncertainty [4] and [31] require the volatility processes σ to be such that $\sigma_t \in [\sigma_{\min}, \sigma_{\max}]$ for all $t \in [0,T]$ and some constants $\sigma_{\min}, \sigma_{\max} > 0$ with $\sigma_{\min} < \sigma_{\max}$. Therefore, due to the bounds on the volatility, all possible price paths considered in [4] and [31] belong to the function spaces as stated in (3.2).

4. Rough volatility models: Recently, analyzing time series of volatility using high frequency data, Gatheral, Jaisson and Rosenbaum [25] showed that the log-volatility behaves essentially like a fractional Brownian motion with Hurst exponent H close to 0.1. This new insight has led to various fractional extensions of classical volatility models (see e.g. [7, 11, 22, 25]), which nicely lead to price paths belonging to the σ-compact metric space of Hölder continuous functions. Indeed, if the stochastic volatility σ fulfills for some $M > 0$ and $q > r \geq 1$ the bound

$$E_P[|\sigma_t - \sigma_s|^q] \leq |t-s|^{\frac{q}{r}} \quad \text{for } s,t \in [0,T] \quad \text{and} \quad \sigma_0 \in \mathbb{R}, \tag{3.3}$$

then we observe that

$$E_P\bigg[\int_0^T |\sigma_s|^q \,\mathrm{d}s\bigg] \leq C\big(|\sigma_0|^q + E_P\big[\|\sigma\|_\beta^q\big]\big) < +\infty,$$

for some constant $C = C(q,M,T) > 0$ and $\beta \in (0, 1/r - 1/q)$. Note, that condition (3.3) is exactly the condition usually required by the Kolmogorov continuity criterion (cf. Theorem A.1), which is frequently used to verify the Hölder regularity of a stochastic process. In particular, every rough volatility model satisfying (3.3) with associated price process given by (3.1) generates price paths possessing Hölder regularity as provided in (3.2). For example, a simple fractional Brownian motion with Hurst index H fulfills the bound (3.3) with $q \in [2,+\infty)$ and $r = H$ and the rough Heston model as introduced by El Euch and Rosenbaum [22, (3)] fulfills the bound (3.3) with $q \in [2, +\infty)$ and $1/r = \alpha - 1/2$ for $\alpha \in (1/2, 1)$, where α denotes the parameter specified in the rough Heston model [22, (3)].

5. Volatility uncertainty: The most general case of volatility uncertainty is usually provided by simultaneously considering all processes of the type

$$S_t = \int_0^t \sqrt{\sigma_s}\,\mathrm{d}W_s, \quad t \in [0,T],$$

for strictly positive and predictable processes σ, see [35, 39]. While they can deal with all σ such that $\int_0^T \sigma_s \,\mathrm{d}s < +\infty$ a.s., we have seen in 3. that we can deal with all volatility processes σ such that $E_P[\int_0^T \sigma_s^q \,\mathrm{d}s] < +\infty$ for $q \in (1, +\infty)$.

Another sub-class of these price processes S leading to σ-compact sets of price paths is given by all processes S with corresponding volatility process σ such that $\sigma \leq f$ for some deterministic integrable function $f\colon [0,T] \to (0,+\infty)$. Indeed, defining the quadratic variation of S by $\langle S\rangle_t = \int_0^t \sigma_s \,\mathrm{d}s$ for $t \in [0,T]$ and using Dambis Dubin-Schwarz theorem, one has $S_t = B_{\langle S\rangle_t}$ for a suitable Brownian motion B. Based on this observation, it is easy to derive that

$$S \in C^{p\text{-var},c}([0,T],\mathbb{R}), \quad \text{a.s.,} \quad \text{with} \quad c(s,t) := \int_s^t f(r)\,\mathrm{d}r, \quad s,t \in [0,T],$$

and $p > 2$. Recall that $C^{p\text{-var},c}([0,T],\mathbb{R})$ is σ-compact by Corollary 3.2.

4. Proofs of the main results

Denote by C_b the set of all bounded continuous functions $X\colon \Omega \to \mathbb{R}$.

Remark 4.1. *If Ω is a σ-compact space endowed with the maximum norm, then (A1) is always satisfied.*

Proof. Fix $t \in [0,T]$, a bounded $\mathcal{F}_t^0$-measurable function h, and a Borel probability Q. Define $\pi\colon \Omega \to C([0,t],\mathbb{R}^d)$, $\pi(\omega)(s) := \omega(s)$, and set $\Omega_t := \pi(\Omega)$ endowed with the maximum norm $\|\omega\|_\infty := \max_{s\in[0,t]} |\omega(s)|$. By σ-compactness there exist compact sets K_n, $n \in \mathbb{N}$, such that $\Omega = \bigcup_n K_n$. Further, since $\Omega_t = \bigcup_n \pi(K_n)$, and $\pi(K_n)$ is compact by continuity of π, it follows that Ω_t is σ-compact and therefore separable. Standard arguments show that $\mathcal{F}_t^0 = \{\pi^{-1}(B) : B \in \mathcal{B}(\Omega_t)\}$, where $\mathcal{B}(\Omega_t)$ denotes the Borel sets of Ω_t. Hence, $h = \tilde{h} \circ \pi$ for some Borel function $\tilde{h}\colon \Omega_t \to \mathbb{R}$. Again by σ-compactness of Ω_t, the probability measure $\tilde{Q} := Q \circ \pi^{-1}$ is tight and thus regular, i.e. Borel sets can be approximated by compact subsets in measure. In particular, there exists a sequence of continuous functions $\tilde{h}_n\colon \Omega_t \to \mathbb{R}$ such that $\tilde{h}_n \to \tilde{h}$ $\tilde{Q}$-almost surely, which in turn implies $h_n := \tilde{h}_n \circ \pi \to \tilde{h} \circ \pi = h$ Q-almost surely. □

Lemma 4.2. *If $Q \in \mathcal{M}$ and $H \in \mathcal{H}^f$ such that $E_Q[(H\cdot S)_T^-] < +\infty$, then $(H\cdot S)_T$ is Q-integrable and $E_Q[(H\cdot S)_T] = 0$.*

Proof. This follows from a fact on discrete-time local martingales. Indeed, by definition $H = \sum_{n=1}^N h_n \mathbf{1}_{(\tau_n,\tau_{n+1}]}$, so that $(H \cdot S)_T$ can be viewed as the integral $\sum_{n=1}^N h_n(S_{\tau_{n+1}} - S_{\tau_n})$ of $(h_n)_{n=1,\dots,N}$ with respect to the discrete time martingale $(S_{\tau_n})_{n=0,\dots,N+1}$ adapted to the filtration $(\mathcal{F}_{\tau_n})_{n=0,\dots,N+1}$, where we set $\tau_0 := 0$ and $h_0 \equiv 0$. Thus, [30, Theorems 1 and 2] yields the claim. □

Lemma 4.3. *Let $d = 1$, $0 \le s < t \le T$, $m > 0$, and define*

$$\tau := \inf\{r \ge s : S_r > m \text{ or } S_r \le -m\} \wedge T.$$

Then the function $\omega \mapsto S_{\tau(\omega)\wedge t}(\omega)$ is lower semicontinuous w.r.t. the maximum norm.

Proof. Define $\tau_+ := \inf\{r \ge s : S_r > m\} \wedge T$ and $\tau_- := \inf\{r \ge s : S_r \le -m\} \wedge T$, and note that $\tau = \tau_+ \wedge \tau_-$. Moreover, fix ω and a sequence (ω_n) such that $\|\omega_n - \omega\|_\infty \to 0$. We claim that

$$\limsup_n \tau_+(\omega_n) \le \tau_+(\omega) \quad \text{and} \quad \liminf_n \tau_-(\omega_n) \ge \tau_-(\omega).$$

Indeed, assume without loss of generality that $r := \tau_+(\omega) < T$. Then, by defintion, for every $\varepsilon > 0$ there is $\delta \in (0,\varepsilon)$ such that $\omega(r+\delta) > m$. Therefore $\omega_n(r+\delta) > m$ for eventually all n, showing that $\tau_+(\omega_n) \le r + \varepsilon$ for eventually all n. As ε was arbitrary, the first part of the claim follows. Next, we may assume without loss of generality that $r := \tau_-(\omega) > s$. Then necessarily $\omega(u) > -m$ for $u \in [s,r)$. By continuity of ω and since $\|\omega_n - \omega\|_\infty \to 0$, for every $\varepsilon > 0$, it holds

$\omega_n(u) > -m$ for all $u \in [s, r-\varepsilon]$ and therefore $\tau_-(\omega_n) \geq r - \varepsilon$ for eventually all n. As ε was arbitrary, the second part of the claim follows. In the following we prove the lower semicontinuity of S_t^τ.

(a) If $S_t^\tau(\omega) > m$, then $\tau(\omega) = \tau_+(\omega) = s$ and $\omega(s) > m$. In particular $\omega_n(s) > m$ and $\tau_+(\omega_n) = s$ for eventually all n, hence $\lim_n S_t^\tau(\omega_n) = \lim_n \omega_n(s) = \omega(s) = S_t^\tau(\omega)$.

(b) If $S_t^\tau(\omega) = m$, then either $\tau_+(\omega) < t$ or $\tau_+(\omega) \geq t$. In the first case it follows that $\tau_+(\omega) < \tau_-(\omega)$ so that $\tau_+(\omega_n) < \tau_-(\omega_n)$ and $\tau_+(\omega_n) < t$ for all but finely many n by the first part of the proof and therefore

$$\liminf_n S_t^\tau(\omega_n) = \liminf_n \omega_n(\tau_+(\omega_n)) = m = S_t^\tau(\omega).$$

On the other hand, if $\tau_+(\omega) \geq t$, then $\omega(t) = m$ and $\omega(r) > -m$ for $r \in [s,t]$. This implies that $\tau_-(\omega_n) \geq t$ for eventually all n and therefore

$$\liminf_n S_t^\tau(\omega_n) = \liminf_n \omega_n(t \wedge \tau_+(\omega_n)) = m = S_t^\tau(\omega).$$

(c) If $S_t^\tau(\omega) \in (-m, m)$, then either $\tau(\omega) > t$ or $\tau(\omega) = T$ (in which case necessarily $t = T$). In the latter case it follows that $\omega(r) > -m$ for $r \in [s, T]$, hence $\tau_-(\omega_n) = T$ for eventually all n and thus

$$\liminf_n S_t^\tau(\omega_n) = \liminf_n \omega_n(t \wedge \tau_+(\omega_n)) \geq \omega(t) = S_t^\tau(\omega).$$

If $\tau(\omega) > t$, then again $\tau_-(\omega_n) > t$ for eventually all n so that the same argument shows that $\liminf_n S_t^\tau(\omega_n) \geq S_t^\tau(\omega)$.

(d) If $S_t^\tau(\omega) = -m$, then $\omega(s) \geq -m$. Assume that $\liminf_n S_t^\tau(\omega_n) < -m$. Then there is a subsequence still denoted By (ω_n) such that $\tau(\omega_n) = \tau_-(\omega_n) = s$. However, this contradicts $\liminf_n S_t^\tau(\omega_n) = \lim_n \omega_n(s) = \omega(s) \geq -m$.

(e) If $S_t^\tau(\omega) < -m$, then $\tau_-(\omega) = s$ and $\omega(s) < -m$. This implies $\omega_n(s) < -m$ and therefore $\tau_-(\omega_n) = s$ for eventually all n, so that $\lim_n S_t^\tau(\omega_n) = \lim_n \omega_n(s) = \omega(s) = S_t^\tau(\omega)$. □

Proposition 4.4. *Assume that (A1) holds true. Then, for any Borel probability measure Q on Ω which is not a local martingale measure, there exist $X \in C_b$ and $H \in \mathcal{H}^f$ such that $X \leq (H \cdot S)_T$ and $E_Q[X] > 0$.*

Proof. Notice that S is a local martingale if and only if each component is a local martingale, which means we may assume without loss of generality that $d = 1$.

We prove that if $E_Q[X] \leq 0$ for all $X \in G := \{X \in C_b : X \leq (H \cdot S)_T \text{ for some } H \in \mathcal{H}^f\}$, then Q is a local martingale measure, i.e. for every $m \in \mathbb{N}$, the stopped process

$$S_t^\tau := S_{t \wedge \tau} \quad \text{where} \quad \tau := \inf\{t \geq 0 : |S_t| \geq m\} \wedge T$$

is a martingale. Fix $m \in \mathbb{N}$, $0 \leq s < t \leq T$, and define the stopping times

$$\begin{aligned} \sigma &:= \inf\{r \geq s : |S_r| \geq m\} \wedge T, \\ \sigma_\varepsilon &:= \inf\{r \geq s : S_r > m - \varepsilon \text{ or } S_r \leq \varepsilon - m\} \wedge T \end{aligned}$$

for $0 < \varepsilon \leq 1$. First note that, by continuity of S and right-continuity of $(\mathcal{F}_t)$, one has that σ_ε, σ, and τ are in fact stopping times. By Lemma 4.3 the function $\omega \mapsto S_{t\wedge\sigma_\varepsilon(\omega)}(\omega)$ is lower semicontinuous w.r.t. $\|\cdot\|_\infty$ for every ε. In particular, for every continuous $\mathcal{F}_s^0$-measurable function $h\colon \Omega \to [0,1]$, it holds that

$$(H \cdot S)_T \text{ is lower semicontinuous, where } H := h\mathbf{1}_{(s,\sigma_\varepsilon \wedge t]} \in \mathcal{H}^f.$$

Since additionally $|S_t^{\sigma_\varepsilon} - S_s| \leq 2m$, there exists a sequence of continuous functions $X_n\colon \Omega \to [-2m, 2m]$ such that $X_n \leq (H \cdot S)_T$ which increases pointwise to $(H \cdot S)_T$. Since $X_n \in G$ for all n, it follows that

$$E_Q[h(S_t^{\sigma_\varepsilon} - S_s)] = E_Q[(H \cdot S)_T] = \sup_n E_Q[X_n] \leq 0.$$

By assumption (A1), for every bounded and $\mathcal{F}_s^0$-measurable function h, there exists a sequence of continuous $\mathcal{F}_s^0$-measurable functions $h_n\colon \Omega \to [0,1]$ which converges Q-almost surely to h, in particular

$$E_Q[h(S_t^{\sigma_\varepsilon} - S_s)] = \lim_n E_Q[h_n(S_t^{\sigma_\varepsilon} - S_s)] \leq 0.$$

The fact that σ_ε increases to σ as ε tends to 0 (and therefore $S_t^{\sigma_\varepsilon} \to S_t^\sigma$ by continuity of S), shows that

$$E_Q[h(S_t^\sigma - S_s)] = \lim_{\varepsilon\to 0} E_Q[h(S_t^{\sigma_\varepsilon} - S_s)] \leq 0.$$

Furthermore, notice that $\sigma = \tau$ on $\{\tau \geq s\}$, so that $\mathbf{1}_{\{\tau\geq s\}}(S_t^\sigma - S_s) = S_t^\tau - S_s^\tau$. Since τ is the hitting time of a closed set, it is also a stopping time w.r.t. the raw filtration $(\mathcal{F}_t^0)$, so that $h\mathbf{1}_{\{\tau\geq s\}}\colon \Omega \to [0,1]$ is $\mathcal{F}_s^0$-measurable. This shows that

$$E_Q[h(S_t^\tau - S_s^\tau)] = E_Q[(h\mathbf{1}_{\{\tau\geq s\}})(S_t^\sigma - S_s)] \leq 0,$$

which implies $E_Q[S_t^\tau|\mathcal{F}_s^0] \leq S_s^\tau$, i.e. S^τ is a supermartingale w.r.t. the raw filtration $(\mathcal{F}_t^0)$. Finally, using that S^τ is bounded and $\mathcal{F}_s \subseteq \mathcal{F}_{s+\varepsilon}^0$ yields

$$E_Q[S_t^\tau - S_s^\tau|\mathcal{F}_s] = \lim_{\varepsilon\to 0} E_Q[S_t^\tau - S_{s+\varepsilon}^\tau|\mathcal{F}_s] = \lim_{\varepsilon\to 0} E_Q\big[E_Q[S_t^\tau - S_{s+\varepsilon}^\tau|\mathcal{F}_{s+\varepsilon}^0]|\mathcal{F}_s\big] \leq 0$$

which shows that S^τ is a supermartingale.

By similar arguments one can also show that S^τ is a submartingale (and thus a martingale). Indeed, replace h by a continuous $\mathcal{F}_s^0$-measurable function $\tilde{h}\colon \Omega \to [-1,0]$, and the stopping times σ_ε by the stopping times $\tilde{\sigma}_\varepsilon := \inf\{r \geq s : S_r \geq m - \varepsilon \text{ or } S_r < \varepsilon - m\} \wedge T$ for $\varepsilon > 0$. The same arguments as in Lemma 4.3 show that $\omega \mapsto S_{t\wedge\tilde{\sigma}_\varepsilon(\omega)}(\omega)$ is upper semicontinuous, which implies that $(H \cdot S)_T$ is lower semicontinuous for $H := \tilde{h}\mathbf{1}_{(s,\tilde{\sigma}_\varepsilon\wedge t]} \in \mathcal{H}^f$. The rest follows the same way as before. □

Lemma 4.5. *Assume that (A1) and (A2) hold true. Then there exists an increasing sequence of non-empty compacts (K_n) such that $\Omega = \bigcup_n K_n$, and $\omega^t \in K_n$ for every $(t, \omega) \in [0,T] \times K_n$.*

Proof. By assumption $\Omega = \bigcup_n K_n'$ for some non-empty compacts (K_n'), where we assume without loss of generality that $K_n' \subset K_{n+1}'$ for every n. Define the function $\rho\colon [0,T] \times \Omega \to \Omega$, $(t,\omega) \mapsto \omega^t$ which, again by assumption, is continuous. Therefore $K_n := \{\omega^t : t \in [0,T], \omega \in K_n'\} = \rho([0,T], K_n')$ has the desired properties. □

Lemma 4.6. *Assume that (A1) and (A2) hold true and fix a sequence of compacts (K_j) as in Lemma 4.5. Further fix a continuous function $Z\colon \Omega \to [1,+\infty)$, $H \in \mathcal{H}^f$, and $n \in \mathbb{N}$. If $(H\cdot S)_T(\omega) \geq -Z(\omega)$ for all $\omega \in K_j$, then $(H\cdot S)_t(\omega) \geq -Z(\omega^t)$ for all $(t,\omega) \in [0,T]\times K_j$.*

Proof. Fix $H = \sum_{n=1}^N h_n \mathbf{1}_{(\tau_n,\tau_{n+1}]} \in \mathcal{H}^f$, $\omega \in K_j$, and $t \in [0,T)$ (for $t = T$ the statement holds by assumption). We may assume that $\tau_{N+1} = T$ by adding an additional stopping time and setting $h_N \equiv 0$. Further, fix $\varepsilon > 0$ with $t + \varepsilon \leq T$, and $m \in \mathbb{N}$ such that $\tau_m(\omega^{t+\varepsilon}) \leq t \leq \tau_{m+1}(\omega^{t+\varepsilon})$. Then

$$\begin{aligned}(H\cdot S)_t(\omega^{t+\varepsilon}) - (H\cdot S)_T(\omega^{t+\varepsilon}) =& h_m(\omega^{t+\varepsilon})(S_t(\omega^{t+\varepsilon}) - S_{\tau_{m+1}(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon})) \\ & - \sum_{n=m+1}^{N} h_m(\omega^{t+\varepsilon})(S_{\tau_{n+1}(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon}) - S_{\tau_n(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon}))\end{aligned}$$

and

$$|S_t(\omega^{t+\varepsilon}) - S_{\tau_{m+1}(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon})| \leq \delta(\varepsilon) \quad \text{and} \quad |S_{\tau_{n+1}(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon}) - S_{\tau_n(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon})| \leq \delta(\varepsilon)$$

for all $n \geq m+1$, where $\delta(\varepsilon) := \max_{r,s\in[t,t+\varepsilon]} |\omega(r) - \omega(s)|$. Let C be a constant such that $|h_n| \leq C$. Then, since $\lim_{\varepsilon\downarrow 0} \delta(\varepsilon) = 0$, it holds

$$|(H\cdot S)_t(\omega^{t+\varepsilon}) - (H\cdot S)_T(\omega^{t+\varepsilon})| \leq NC\delta(\varepsilon) \to 0$$

as $\varepsilon \downarrow 0$. Since $\mathcal{F}_t \subset \mathcal{F}^0_{t+\varepsilon}$, it follows that $(H\cdot S)_t(\omega) = (H\cdot S)_t(\omega^{t+\varepsilon})$ for all $\varepsilon > 0$, so that

$$(H\cdot S)_t(\omega) = \lim_{\varepsilon\downarrow 0}(H\cdot S)_T(\omega^{t+\varepsilon}) \geq \liminf_{\varepsilon\downarrow 0} -Z(\omega^{t+\varepsilon}) = -Z(\omega^t)$$

since $\omega^{t+\varepsilon} \in K_j$ for all $\varepsilon > 0$, and $\varepsilon \mapsto Z(\omega^{t+\varepsilon})$ is continuous by assumption. □

We have now all ingredients at hand to prove the main results of the present paper.

Proof of Theorem 2.1. Fix a continuous function $Z\colon \Omega \to [0,+\infty)$, a sequence of compact sets (K_n) as in Lemma 4.5, and define $\mathcal{M}_n := \{Q \in \mathcal{M} : Q(K_n) = 1\} \subset \mathcal{M}_c$ for every $n \in \mathbb{N}$.

Step (a): Fix $n \in \mathbb{N}$ and define

$$\phi_n(X) := \inf\left\{\lambda \in \mathbb{R} : \begin{array}{l} \text{there is } H \in \mathcal{H}^f \text{ and } c \in \mathbb{R} \text{ such that} \\ (H\cdot S)_T \geq c \text{ on } \Omega \text{ and } \lambda + (H\cdot S)_T \geq X \text{ on } K_n \end{array}\right\}$$

for $X\colon \Omega \to \mathbb{R}$. By Lemma 4.2 it follows that

$$\phi_n(X) \geq \sup_{Q\in\mathcal{M}_n} E_Q[X] \tag{4.1}$$

for every Borel measurable X which is bounded from below on K_n. Let $\bar\omega \in K_n$ be the constant path $t \mapsto \bar\omega(t) := \omega(0)$ for some $\omega \in K_n$. Since the Dirac measure $\delta_{\bar\omega}$ assigning probability 1 to $\bar\omega$ belongs to $\mathcal{M}$ and satisfies $\delta_{\bar\omega}(K_n) = 1$, it follows that ϕ_n is real-valued on C_b and $\phi_n(m) = m$ for every $m \in \mathbb{R}$.

Further, it is straightforward to check that ϕ_n is convex and increasing in the sense that $\phi_n(X) \leq \phi_n(Y)$ whenever $X \leq Y$. Moreover, ϕ_n is continuous from above on C_b, i.e. $\phi_n(X_k) \downarrow \phi_n(0)$ for every sequence (X_k) in C_b such that $X_k \downarrow 0$. To that end, fix such a sequence (X_k) and let $\varepsilon > 0$ be arbitrary. By Dini's lemma one has $X_k \leq \varepsilon$ on K_n for all k large enough, so that $\phi_n(X_k) \leq \varepsilon$ for all such k, which shows that $\phi_n(X_k) \downarrow 0$. It follows from [16, Proposition 1.1] that

$$\phi_n(X) = \max_{Q \in ca^+} \left(E_Q[X] - \phi_n^*(Q)\right) \tag{4.2}$$

for all $X \in C_b$, where $\phi_n^*(Q) := \sup_{X \in C_b}(E_Q[X] - \phi_n(X))$ and ca^+ denotes the set of non-negative countably additive Borel measures on Ω. We claim that

$$\phi_n^*(Q) = \begin{cases} 0, & \text{if } Q \in \mathcal{M}_n, \\ +\infty, & \text{else,} \end{cases} \tag{4.3}$$

for all $Q \in ca^+$. First notice that (4.1) implies $\phi_n^*(Q) \leq 0$ whenever $Q \in \mathcal{M}_n$. Since in addition $\phi_n(0) = 0$, it follows that $\phi_n^*(Q) = 0$. On the other hand, if $Q \notin \mathcal{M}_n$, then $\phi_n^*(Q) = +\infty$. Indeed, if Q is not a probability, then $\phi_n(m) = m$ implies that $\phi_n^*(Q) \geq \sup_{m \in \mathbb{R}}(mQ(\Omega) - m) = +\infty$. Similarly, since K_n^c is open, there exists a sequence of bounded continuous functions (X_k) such that $X_k \uparrow +\infty \mathbf{1}_{K_n^c}$ with the convention $0 \cdot (+\infty) := 0$. By definition $\phi_n(X_k) \leq 0$ for all k, from which it follows that

$$\phi_n^*(Q) \geq \sup_k E_Q[X_k] = +\infty E_Q[\mathbf{1}_{K_n^c}].$$

It remains to show that if Q is a probability with $Q(K_n) = 1$ but not a martingale measure, then $\phi_n^*(Q) = +\infty$. Note that compactness of K_n implies boundedness of K_n w.r.t. $\|\cdot\|_\infty$, and therefore Q is also not a local martingale measure. Thus Proposition 4.4 yields the existence of $X \in C_b$ and $H \in \mathcal{H}^f$ such that $X \leq (H \cdot S)_T$ and $E_Q[X] > 0$. Since $\phi_n(mX) \leq 0$ for all $m > 0$, it follows that $\phi_n^*(Q) \geq \sup_{m>0}(E_Q[mX] - \phi_n(mX)) = +\infty$.

Next, fix some upper semicontinuous X which is bounded from above (i.e. $X = X \wedge m$ for some $m > 0$) and satisfies $X \geq -Z$. We claim that

$$\phi_n(X) = \max_{Q \in \mathcal{M}_n} E_Q[X]. \tag{4.4}$$

To that end, let (X_k) be a sequence in C_b such that $X_k \downarrow X$. By (4.2) and (4.3) there exist $Q_k \in \mathcal{M}_n$ such that $\phi_n(X_k) = E_{Q_k}[X_k]$. Since $\mathcal{M}_n$ is (sequentially) compact in the weak topology induced by the continuous bounded functions, possibly after passing to a subsequence, we may assume that $Q_k \to Q$ for some $Q \in \mathcal{M}_n$. For every $\varepsilon > 0$ there exists k' such that $E_Q[X_{k'}] \leq E_Q[X] + \varepsilon$. Choose $k \geq k'$ such that $E_{Q_k}[X_{k'}] \leq E_Q[X_{k'}] + \varepsilon$. Then

$$E_{Q_k}[X_k] \leq E_{Q_k}[X_{k'}] \leq E_Q[X_{k'}] + \varepsilon \leq E_Q[X] + 2\varepsilon$$

so that

$$\phi_n(X) \leq \lim_k \phi_n(X_k) = \lim_k E_{Q_k}[X_k] \leq E_Q[X] + 2\varepsilon \leq \sup_{R \in \mathcal{M}_n} E_R[X] + 2\varepsilon \leq \phi_n(X) + 2\varepsilon,$$

where the last inequality follows from (4.1). This shows (4.4).

Step (b): We claim that $\sup_n \phi_n(X \wedge n) = \phi(X)$ for all upper semicontinuous functions $X \colon \Omega \to (-\infty, +\infty]$ which satisfies $X \geq -Z$, where

$$\phi(X) := \inf \left\{ \lambda \in \mathbb{R} : \begin{array}{l} \text{there is } (H^n) \text{ in } \mathcal{H}^f \text{ such that } \lambda + (H^n \cdot S)_t(\omega) \geq -Z(\omega^t) \\ \text{for all } (t,\omega) \in [0,T] \times \Omega \text{ and } \lambda + \liminf_n (H^n \cdot S)_T \geq X \end{array} \right\}.$$

Indeed, fix such X and first notice that for every $Q \in \mathcal{M}_Z$ Fatou's lemma and Lemma 4.2 imply

$$\lambda = \lambda + \liminf_n E_Q[(H^n \cdot S)_T] \geq E_Q[\lambda + \liminf_n (H^n \cdot S)_T] \geq E_Q[X]$$

for every $\lambda \in \mathbb{R}$ and (H^n) in $\mathcal{H}^f$ such that $\lambda + \liminf_n (H^n \cdot S)_T \geq X$ and $\lambda + (H^n \cdot S)_T \geq -Z$ for all n. Hence, one gets

$$\phi(X) \geq \sup_{Q \in \mathcal{M}_Z} E_Q[X] \geq \sup_{Q \in \mathcal{M}_c} E_Q[X] \geq \sup_n \sup_{Q \in \mathcal{M}_n} E_Q[X \wedge n] = \sup_n \phi_n(X \wedge n), \tag{4.5}$$

where the last equality follows from (4.4).

On the other hand, let $m > \sup_n \phi_n(X \wedge n)$ so that, by definition, for each n there exists $H^n \in \mathcal{H}^f$ such that $m + (H^n \cdot S)_T \geq X \wedge n \geq -Z$ on K_n. Thus, it follows from Lemma 4.6 that

$$m + (H^n \cdot S)_t(\omega) \geq -Z(\omega^t) \quad \text{for all } (t,\omega) \in [0,T] \times K_n. \tag{4.6}$$

Fix $\varepsilon > 0$. Define the stopping times

$$\sigma_n(\omega) := \inf\{t \in [0,T] : m + \varepsilon + (H^n \cdot S)_t(\omega) + Z(\omega^t) = 0\} \wedge T$$

and notice that

$$(H^n \cdot S)_{\sigma_n} = (\tilde{H}^n \cdot S)_T \quad \text{for } \tilde{H}^n := \sum_{i=1}^N h_i^n \mathbf{1}_{\{\sigma_n \geq \tau_i\}} \mathbf{1}_{(\tau_i \wedge \sigma_n, \tau_{i+1} \wedge \sigma_n]} \in \mathcal{H}^f, \tag{4.7}$$

where $H^n = \sum_{i=1}^N h_i^n \mathbf{1}_{(\tau_i, \tau_{i+1}]}$. Fix $\omega \in \Omega$. Then $\omega \in K_j$ for some $j \in \mathbb{N}$ and therefore, by (4.6) it follows that $\sigma_n(\omega) = T$ whenever $n \geq j$. Hence, we have

$$m + \varepsilon + (\tilde{H}^n \cdot S)_T(\omega) = m + \varepsilon + (H^n \cdot S)_T(\omega) \geq X(\omega) \wedge n \quad \text{for } n \geq j.$$

As ω was arbitrary, it follows that $\liminf_n (m + \varepsilon + (\tilde{H}^n \cdot S)_T) \geq X$. Moreover, it follows from (4.7) that

$$m + \varepsilon + (\tilde{H}^n \cdot S)_t(\omega) \geq -Z(\omega^{t \wedge \sigma_n(\omega)}) \geq -Z(\omega^t) \quad \text{for all } (t,\omega) \in [0,T] \times \Omega,$$

which shows that $\phi(X) \leq m + \varepsilon$. Finally, since $m > \sup_n \phi_n(X)$ and $\varepsilon > 0$ were arbitrary, we conclude that $\phi(X) \leq \sup_n \phi_n(X \wedge n)$, which shows that all inequalities in (4.5) are equalities. In particular, $\phi(X) = \sup_{Q \in \mathcal{M}_c} E_Q[X]$, which shows (2.1).

Step (c): We finally show that $\mathcal{M}_c$ can be replaced by the set $\mathcal{M}_Z$, and $\mathcal{H}^f$ by $\mathcal{H}$. To that end, fix $X\colon \Omega \to (-\infty,+\infty]$ satisfying $X \geq -Z$ for some $\lambda \in \mathbb{R}$, $Q \in \mathcal{M}_Z$, and (H^n) in $\mathcal{H}$ such that $\lambda + (H^n \cdot S)_t(\omega) \geq -Z(\omega^t)$ for all $(t,\omega) \in [0,T]\times\Omega$ and $\lambda + \liminf_n (H^n \cdot S)_T \geq X$. Define

$$H^{n,K} := \sum_{k=1}^{K} h_k^n \mathbf{1}_{(\tau_k^n, \tau_{k+1}^n]} \in \mathcal{H}^f \quad \text{and} \quad H^n = \sum_{k=1}^{\infty} h_k^n \mathbf{1}_{(\tau_k^n, \tau_{k+1}^n]}.$$

Therefore, one gets

$$\lambda + (H^{n,K} \cdot S)_T(\omega) = \lambda + (H^n \cdot S)_{\tau_{K+1}^n(\omega)}(\omega) \geq -Z(\omega^{\tau_{K+1}^n(\omega)}) \geq -Z(\omega),$$

where the last inequality holds by assumption. Hence, by Lemma 4.2 and Fatou's lemma, it follows that

$$\begin{aligned}\lambda &= \lambda + \liminf_n \liminf_K E_Q[(H^{n,K} \cdot S)_T] \geq \liminf_n E_Q[\lambda + \liminf_K (H^{n,K} \cdot S)_T] \\ &= \liminf_n E_Q[\lambda + (H^n \cdot S)_T] \geq E_Q[\lambda + \liminf_n (H^n \cdot S)_T] \geq E_Q[X].\end{aligned}$$

This shows

$$\begin{aligned}&\inf\left\{\lambda \in \mathbb{R} : \begin{array}{l}\text{there is } c \geq 0 \text{ and a sequence } (H^n) \text{ in } \mathcal{H}^f \text{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \geq -Z(\omega^t) \text{ for all } (t,\omega) \in [0,T]\times\Omega \text{ and} \\ \lambda + \liminf_n (H^n \cdot S)_T(\omega) \geq X(\omega) \ \text{ for all } \omega \in \Omega\end{array}\right\} \\ &\geq \inf\left\{\lambda \in \mathbb{R} : \begin{array}{l}\text{there is } c \geq 0 \text{ and a sequence } (H^n) \text{ in } \mathcal{H} \text{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \geq -Z(\omega^t) \text{ for all } (t,\omega) \in [0,T]\times\Omega \text{ and} \\ \lambda + \liminf_n (H^n \cdot S)_T(\omega) \geq X(\omega) \ \text{ for all } \omega \in \Omega\end{array}\right\} \\ &\geq \sup_{Q \in \mathcal{M}_Z} E_Q[X] \geq \sup_{Q \in \mathcal{M}_c} E_Q[X],\end{aligned}$$

where the first and last terms coincide by the previous steps (a) and (b). The proof is complete. □

Proof of Theorem 2.3. Step (a): For $n \in \mathbb{N}$ and every function $X\colon \Omega \to \mathbb{R}$ define

$$\phi_n(X) := \inf\left\{\lambda \in \mathbb{R} : \begin{array}{l}\text{there is } H \in \mathcal{H}^f \text{ and } c > 0 \text{ such that} \\ (H \cdot S)_T \geq -c \text{ and } \lambda + (H \cdot S)_T \geq X - Z/n\end{array}\right\}.$$

It follows from Lemma 4.2 that $\phi_n(X) \geq \sup_{Q \in \mathcal{M}_Z} \big(E_Q[X] - E_Q[Z]/n\big)$ for every Borel function X which is bounded from below. Moreover, if (X_k) is a sequence in C_b decreasing pointwise to 0, then $\phi(X_n) \downarrow \phi(0)$. Indeed, fix $\varepsilon > 0$ arbitrary and $H \in \mathcal{H}^f$ with $(H \cdot S)_T \geq -c$ for some $c \geq 0$ such that

$$\varepsilon + \phi_n(0) + (H \cdot S)_T + Z/n \geq 0.$$

Now define $\tilde{c} := \|X_1\|_\infty - \varepsilon - \phi_n(0) + c$ so that $\tilde{c} + \varepsilon + \phi_n(0) + (H \cdot S)_T \geq X_1$. Since $\{Z \leq \tilde{c}n\}$ is compact, it follows from Dini's lemma that $X_k \mathbf{1}_{\{Z \leq \tilde{c}n\}} \leq \varepsilon$ for k large enough. Hence

$$\begin{aligned}X_k \leq X_k \mathbf{1}_{\{Z \leq \tilde{c}n\}} + X_1 \mathbf{1}_{\{Z > \tilde{c}n\}} &\leq \varepsilon + (\varepsilon + \phi_n(0) + (H \cdot S)_T + Z/n)\mathbf{1}_{\{Z > \tilde{c}n\}} \\ &\leq 2\varepsilon + \phi_n(0) + (H \cdot S)_T + Z/n\end{aligned}$$

so that $\phi_n(X_k) \le \phi_n(0) + 2\varepsilon$ for k large enough which shows that $\phi_n(X_k) \downarrow \phi_n(0)$. Now, a computation similar to the one in the proof of Theorem 2.1 shows that

$$\phi_n(X) = \max_{Q \in \mathcal{M}_Z} \big(E_Q[X] - E_Q[Z]/n \big) \tag{4.8}$$

for every bounded upper semicontinuous function $X \colon \Omega \to \mathbb{R}$. Indeed, first notice that since by assumption $Z \ge \|\cdot\|_\infty$, the set $\mathcal{M}_Z$ coincides with the set of all local martingale measures which integrate Z. Therefore, the same arguments as in the proof of Theorem 2.1 show that

$$\phi_n^*(Q) := \sup_{X \in C_b} (E_Q[X] - \phi_n(X)) = \begin{cases} E_Q[Z]/n, & \text{if } Q \in \mathcal{M}_Z, \\ +\infty, & \text{else,} \end{cases}$$

and thus that (4.8) is true, at least whenever $X \in C_b$. As for the extension to upper semicontinuous functions, notice that $\phi(X) = \max_{Q \in \Lambda_{2c}} (E_Q[X] - E_Q[Z]/n)$ for every $X \in C_b$ satisfying $|X| \le c$ where $\Lambda_{2c} := \{\phi_n^* \le 2c\}$. Using the fact that Z has compact sublevel sets and Proposition 4.4, it follows that Λ_c is (sequentially) compact. The rest follows analog.

Step (b): For the function ϕ defined by

$$\phi(X) := \inf \left\{ \lambda \in \mathbb{R} : \begin{array}{l} \text{there is } (H^n) \text{ in } \mathcal{H}^f \text{ and } c \ge 0 \text{ such that } (H^n \cdot S)_T \ge -cZ \\ \text{for all } n \text{ and } \lambda + \liminf_n (H^n \cdot S)_T \ge X \end{array} \right\},$$

it follows from Fatou's lemma and Lemma 4.2 that

$$\begin{aligned} \phi(X) &\ge \sup_{Q \in \mathcal{M}_Z} E_Q[X] = \sup_{Q \in \mathcal{M}_Z} \big(\sup_n E_Q[X \wedge n] - E_Q[Z]/n \big) \\ &= \sup_n \sup_{Q \in \mathcal{M}_Z} \big(E_Q[X \wedge n] - E_Q[Z]/n \big) = \sup_n \phi_n(X \wedge n), \end{aligned}$$

for every upper semicontinuous function $X \colon \Omega \to (-\infty, +\infty]$ which is bounded from below. On the other hand, if $m > \sup_n \phi_n(X \wedge n)$, then for every n there exists $H^n \in \mathcal{H}^f$ such that $m + (H^n \cdot S)_T \ge X \wedge n - Z/n$. Hence, $(H^n \cdot S)_T \ge -cZ$ for $c := \|X \wedge 0\|_\infty + m + 1$ and $m + \liminf_n (H^n \cdot S)_T \ge \liminf (X \wedge n - Z/n) = X$, which completes the proof. □

A. Kolmogorov continuity criterion

In this section we briefly recall a version of the so-called Kolmogorov continuity criterion, which provides a sufficient condition for Hölder and Sobolev regularity of stochastic processes. The presented version is a slight reformulation of [23, Theorem A.10].

Let $(\tilde{\Omega}, \mathcal{F}, P)$ be a probability space, $X \colon [0,T] \times \tilde{\Omega} \to \mathbb{R}^d$ be a stochastic process, $T \in (0, +\infty)$, $(\mathbb{R}^d, |\cdot|)$ be the Euclidean space and W be a d-dimensional Brownian motion.

Theorem A.1. *Let $q > r \ge 1$ and suppose that there exists a constant $M > 0$ such that*

$$E_P\big[|X_t - X_s|^q\big] \le M|t-s|^{\frac{q}{r}} \quad \textit{for all} \quad s,t \in [0,T].$$

Then, for any $\alpha \in [0, 1/r - 1/q)$ and $\delta := \alpha + 1/q$ there exists a constant $C = C(r, q, \alpha, T)$ such that

$$E_P\big[\|X\|_\alpha^q\big] \leq CM \quad \text{and} \quad E_P\big[\|X\|_{W^{\delta,q}}^q\big] \leq CM,$$

where we recall the semi-norms

$$\|X\|_\alpha := \sup_{s,t\in[0,T]} \frac{|X_t - X_s|}{|t-s|^\alpha} \quad \text{and} \quad \|X\|_{W^{\delta,q}} := \bigg(\int_{[0,T]^2} \frac{|X_t - X_s|^q}{|t-s|^{\delta q+1}} \,\mathrm{d}s \,\mathrm{d}t \bigg)^{\frac{1}{q}}. \tag{A.1}$$

Applying Theorem A.1 to Itô processes reveals the following regularity criterion.

Corollary A.2. *Let X be a d-dimensional Itô process of the form*

$$X_t = x_0 + \int_0^t a_s \,\mathrm{d}W_s, \quad t \in [0, T],$$

for a predicable process $a\colon [0,T] \times \tilde{\Omega} \to \mathbb{R}^{d\times d}$ and $x_0 \in \mathbb{R}^d$. Suppose $q \in (2, +\infty)$, $\alpha \in (0, 1/2 - 1/(2q))$ and $\delta = \alpha - 1/q$. If $E_P\big[\int_0^T |a_s|^q \,\mathrm{d}s\big] < +\infty$, then

$$X \in C^\alpha([0,T], \mathbb{R}^d) \quad \text{and} \quad X \in W^{\delta,q}([0,T], \mathbb{R}^d), \quad P\text{-a.s.}$$

Proof. Using the Burkholder-Davis-Gundy inequality and Jensen's inequality, one has

$$E_P\big[|X_t - X_s|^q\big] \leq E_P\bigg[\bigg(\int_s^t |a_r|^2 \,\mathrm{d}r\bigg)^{q/2}\bigg] \leq E_P\bigg[\int_0^T |a_r|^q \,\mathrm{d}r\bigg] |t-s|^{q(\frac{1}{2}-\frac{1}{q})}.$$

Therefore, Theorem A.1 implies the assertion. □

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