UNIFORM DIMENSION RESULTS FOR A FAMILY OF MARKOV PROCESSES

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ABSTRACT. In this paper we prove uniform Hausdorff and packing dimension results for the images of a large family of Markov processes. The main tools are the two covering principles in [59]. As applications, uniform Hausdorff and packing dimension results for self-similar Markov processes, certain classes of Lévy processes, stable jump diffusions and non-symmetric stable-type processes are obtained.

1. PRELIMINARIES

Fractal properties of Brownian motion and more general Lévy processes have been studied extensively. We refer to the recent books of Mörters and Peres [45], Schilling and Partzsch [53], Böttcher, Schilling and Wang [8], the survey papers [55, 59], and more recent articles [28, 31, 37] for further information.

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a stable Lévy process in $\mathbb{R}^d$ of index α ($0 < \alpha \leq 2$). For any Borel set $E \subseteq \mathbb{R}_+ = [0, \infty)$, Blumenthal and Getoor [7] obtained the Hausdorff dimension of the image set $X(E)$, namely,

$$(1.1) \qquad \dim_{\mathrm{H}} X(E) = \min\{d, \alpha \dim_{\mathrm{H}} E\} \quad \text{a.s.,}$$

where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension; see Falconer [13], or [45, 55, 59] for the definitions and properties of Hausdorff measure and Hausdorff dimension.

Result (1.1) has been extended and strengthened in various directions by many authors; see [55, 59] and the references therein for a historical account and information on development on (mostly) Lévy processes. In particular, Hawkes and Pruitt [18, Theorem 4.1] established a uniform version of (1.1): If X is a strictly stable Lévy process of index α in $\mathbb{R}^d$ and $d \geq \alpha$, then there exists a single null probability event outside of which (1.1) holds simultaneously for all Borel sets $E \subseteq \mathbb{R}_+$. The first such uniform dimension result was due to Kaufman [24] for planar Brownian motion (i.e., $\alpha = 2$ and $d = 2$). A short and easily accessible redaction of Kaufman's original argument can be found in [53]. Such uniform dimension results are useful in many situations because it allows E to be a random set (cf. [4, 45] for some applications). Perkins and Taylor [47] further proved uniform Hausdorff and packing measure results for strictly stable Lévy processes. As a consequence, they proved a packing dimension analogue of [18, Theorem 4.1]: If X is a strictly stable Lévy process of index α in $\mathbb{R}^d$ and $d \geq \alpha$, then with probability 1,

$$(1.2) \qquad \dim_{\mathrm{P}} X(E) = \alpha \dim_{\mathrm{P}} E \quad \text{for all Borel sets } E \subseteq \mathbb{R}_+,$$

where $\dim_{\mathrm{P}}$ denotes packing dimension (cf. e.g., [13, 45, 55, 59]). Note that, a strictly α-stable Lévy process is self-similar with index $H = 1/\alpha$, and is an important representative among self-similar processes and random fractals.

In recent years, there has been increasing interest in constructing and studying more general Markov processes related to Lévy processes. A large class of Markov processes are generated by pseudo-differential operators [8, 20, 21, 54], their corresponding transition probabilities and heat kernel estimates have been studied in, for example, [38, 40, 41, 46]. Also, we refer to [3] for the martingale problem of pure jump Markov processes, [9, 10, 11] for stable-like processes related to Dirichlet forms and their heat kernel estimates. Many natural questions regarding sample path and fractal properties arise for such Markov processes. Xiao [59] gives a comprehensive survey on fractal properties of Lévy or more general Markov processes before 2004. A lot of progress has been made since then. See, for example, [26, 27, 28, 29, 30, 31] for various results on Lévy processes, [9, 10, 61] for Hausdorff dimension of the range or graph of stable-like processes, [37, 36] for results on Hausdorff dimensions of the image, level and collision sets of a class of Feller process generated by a pseudo-differential operator. However, many interesting problems described [59] are still open.

The main purpose of this paper is to provide a general method for establishing uniform Hausdorff and packing dimension results for more general Markov processes. In particular, we extend the methods of Hawkes [16] and Hawkes and Pruitt [18] for stable Lévy processes (see also Pruitt [48]) to a large family of Markov processes including more general Lévy processes, the stable-like processes or stable jump diffusions in [10, 11, 38, 46]. The key technical tools are the covering principles in Section 2, which improve Lemmas 8.1 and 8.2 in Xiao [59]. We apply them to show the main result of this paper, Theorem 3.3, in Section 3. In Section 4, we apply Theorem 3.3 to Lévy processes and stable-type processes. We mention that Benjamini, Chen and Rohde [4] studied the normally reflected Brownian motion (RBM) in a class of non-smooth domains and proved their uniform Hausdorff dimension result by using the uniform Hölder continuity of RBM, the aforementioned Kaufman's theorem for Brownian motion and a subordination argument. Moreover, they indicated in Remark 3.10 in [4] that similar result still holds for the stable-like processes in [10] by using the covering principle in Xiao [59]. The scope of the present paper is a lot broader, and we expect that the main result of this paper will also be useful for studying fractal sets related to intersections and multiple points of Markov processes.

In the rest of this paper, we assume that $X = \{X(t),\, t \in \mathbb{R}_+, \mathbb{P}^x\}$ is a time-homogeneous Markov process with state space $\mathbb{R}^d$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and satisfies the strong Markov property. We assume that X is separable and its sample paths are almost surely right continuous and have left limit at every $t \in \mathbb{R}_+$ (such a sample function will be called cadlag).

An unspecified positive and finite constant will be denoted by C, which may be different in each appearance. More specific constants are numbered by $K_1, K_2, \ldots$ and $C_1, C_2, \ldots$.

2. The covering principles

In the review article [59, Lemmas 8.1 and 8.2], Xiao stated without proof two covering principles that extend respectively Lemma 3.1 in [18] and Lemma 3 in [16] for Lévy processes (see also Lemmas 1 and 2 in [48]), and suggested that they are useful for proving uniform Hausdorff and packing dimension results for the images of a general

Markov process. In this section, we weaken the conditions of Lemmas 8.1 and 8.2 in [59] and provide proofs.

The following lemma is useful for proving upper bounds for the Hausdorff and packing dimensions of the image of a Markov process. Its proof is a modification of that of Lemma 3.1 in [18], and we provide it for the sake of completeness. Moreover, in Proposition 3.1 below, we will provide a convenient way to verify condition (2.1).

Lemma 2.1. *Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ be a time homogenous strong Markov process in $\mathbb{R}^d$. Let $\{t_n, n \geq 1\}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} t_n^p < \infty$ for some $p > 0$, and let $\mathcal{C}_n$ be a class of N_n intervals in $\mathbb{R}_+$ of length t_n with $\log N_n = O(1)|\log t_n|$. If there is a sequence $\{\theta_n\}$ of positive numbers such that for all $x \in \mathbb{R}^d$,*

$$(2.1) \qquad \mathbb{P}^x \left\{ \sup_{0 \leq s \leq t_n} |X(s) - x| \geq \theta_n \right\} \leq K_1 t_n^{\delta},$$

where K_1 and δ are some positive constants, then there exists a positive integer K_2, depending on p and δ only, such that, with $\mathbb{P}^x$-probability one, for n large enough, $X(I)$ can be covered by K_2 balls of radius θ_n whenever $I \in \mathcal{C}_n$.

Proof. Let $I \in \mathcal{C}_n$ and write it as $I = [a, a + t_n]$. Let $\tau_0 = a$ and, for all $j \geq 1$, define

$$\tau_j = \inf\{s > \tau_{j-1} : |X_s - X_{\tau_{j-1}}| > \theta_n\},$$

with the convention $\inf \emptyset = \infty$. It is easy to see

$$(2.2) \qquad \{X(I, \omega) \text{ cannot be covered by } k \text{ balls of radius } \theta_n\} \subseteq \{\tau_k - \tau_0 \leq t_n\}.$$

Moreover, by the strong Markov property and (2.1),

$$
\begin{aligned}
\mathbb{P}^x\{\tau_k - \tau_0 \leq t_n\} &\leq \mathbb{E}^x \left\{ \mathbb{E}^x \left[1_{\{\tau_k - \tau_{k-1} \leq t_n\}} 1_{\{\tau_{k-1} - \tau_0 \leq t_n\}} | \mathcal{F}_{\tau_{k-1}} \right] \right\} \\
&= \mathbb{E}^x \left\{ \mathbb{E}^x \left[1_{\{\tau_k - \tau_{k-1} \leq t_n\}} | \mathcal{F}_{\tau_{k-1}} \right] 1_{\{\tau_{k-1} - \tau_0 \leq t_n\}} \right\} \\
&\leq \sup_{y \in \mathbb{R}^d} \mathbb{P}^y \left\{ \sup_{0 \leq s \leq t_n} |X(s) - y| \geq \theta_n \right\} \mathbb{E}^x \left\{ 1_{\{\tau_{k-1} - \tau_0 \leq t_n\}} \right\} \\
&\leq K_1 t_n^{\delta} \mathbb{P}^x\{\tau_{k-1} - \tau_0 \leq t_n\}.
\end{aligned}
$$

Using the above argument recursively, we obtain that for all $n \geq 1$,

$$(2.3) \qquad \mathbb{P}^x\{\tau_k - \tau_0 \leq t_n\} \leq K_1^k t_n^{k\delta}.$$

Define events

$$A_n^k := \{ \exists \, I \in \mathcal{C}_n \text{ such that } X(I, \omega) \text{ cannot be covered by } k \text{ balls of radius } \theta_n \}.$$

Since $\log N_n = O(1)|\log t_n|$, i.e., there exist positive constant C, such that $N_n \leq C t_n^{-C}$, as k is large enough (say, $\delta k - C \geq p$), we obtain from (2.3) that

$$
\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}^x(A_n^k) &\leq \sum_{n=1}^{\infty} N_n K_1^k t_n^{k\delta} \\
&\leq C K_1^k \sum_{n=1}^{\infty} t_n^{-C+\delta k} \leq C K_1^k \sum_{n=1}^{\infty} t_n^p < \infty.
\end{aligned}
$$

Hence, the Borel-Cantelli lemma yields the desired result. The proof is complete. $\qquad \square$

For obtaining lower bounds for the Hausdorff and packing dimensions of the image of a Markov process, one can apply the following lemma. Observe that the condition (2.4) is significantly weaker than (8.7) in Lemma 8.2 in [59] (which is usually satisfied only if X is transient) and is easier to verify (see (3.15) below).

Lemma 2.2. *Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ be a time homogenous strong Markov process in $\mathbb{R}^d$. Let $\{r_n, n \geq 1\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} r_n^p < \infty$ for some $p > 0$, and let $\mathcal{D}_n$ be a class of N_n balls (or cubes) of diameter r_n in $\mathbb{R}^d$ with $\log N_n = O(1)|\log r_n|$. If, for every constant $T > 0$, there exists a sequence $\{t_n\}$ of positive numbers and constants K_3 and $\delta > 0$ such that*

$$(2.4) \qquad \mathbb{P}^x \left\{ \inf_{t_n \leq s \leq T} |X(s) - x| \leq r_n \right\} \leq K_3 r_n^{\delta}, \quad \forall x \in \mathbb{R}^d,$$

then there exists a constant K_4, depending on p and δ only, such that, with $\mathbb{P}^x$-probability one, for n large enough, $X^{-1}(B) \cap [0, T]$ can be covered by at most K_4 intervals of length t_n, whenever $B \in \mathcal{D}_n$.

Proof. Let $B \in \mathcal{D}_n$ and assume $B = B(z, \frac{r_n}{2})$ for some $z \in \mathbb{R}^d$ since the diameter of B is r_n. Let $\tau_0 = 0$ and, for any $k \geq 1$, define

$$\tau_k = \inf \left\{ t \geq \tau_{k-1} + t_n, |X_t - z| \leq \frac{r_n}{2} \right\},$$

with the convention $\inf \emptyset = \infty$. It is easy to see

$$\{t : X_t \in B\} \subseteq \bigcup_{i=0}^{\infty} [\tau_i, \tau_i + t_n),$$

which implies

$$\left\{ \tau_k > T \right\} \subseteq \left\{ X^{-1}(B, \omega) \cap [0, T] \text{ can be covered by } k \text{ intervals of length } t_n \right\}.$$

Hence

$$\left\{ X^{-1}(B, \omega) \cap [0, T] \text{ cannot be covered by } k \text{ intervals of length } t_n \right\} \subseteq \left\{ \tau_k \leq T \right\}.$$

By the strong Markov property, (2.4) and the fact that $X(\tau_{k-1}) \in B$ as $\tau_{k-1} \leq T$, we obtain

$$
\begin{aligned}
\mathbb{P}^x\{\tau_k \leq T\} &\leq \mathbb{P}^x\{\tau_k \leq T | \tau_{k-1} \leq T\} \mathbb{P}^x\{\tau_{k-1} \leq T\} \\
&\leq \sup_{y \in B} \mathbb{P}^y \left\{ \inf_{t_n \leq s \leq T} |X(s) - z| \leq \frac{r_n}{2} \right\} \mathbb{P}^x\{\tau_{k-1} \leq T\} \\
&= \sup_{y \in B} \mathbb{P}^y \left\{ \inf_{t_n \leq s < T} |X(s) - y + y - z| \leq \frac{r_n}{2} \right\} \mathbb{P}^x\{\tau_{k-1} \leq T\} \\
&\leq \sup_{y \in B} \mathbb{P}^y \left\{ \inf_{t_n \leq s \leq T} |X(s) - y| \leq r_n \right\} \mathbb{P}^x\{\tau_{k-1} \leq T\} \\
&\leq K_3 r_n^{\delta} \mathbb{P}^x\{\tau_{k-1} \leq T\}.
\end{aligned}
$$

By iterating the above argument, we obtain

$$\mathbb{P}^x\{\tau_k \leq T\} \leq K_3^k r_n^{k\delta}.$$

Define the events

$$A_n^k := \big\{ \omega \in \Omega : \exists B \in \mathcal{D}_n \quad \text{s.t.} \quad X^{-1}(B,\omega) \cap [0,T] \quad \text{cannot be covered by}$$
$$k \text{ intervals of length } t_n \big\}.$$

Since $\log N_n = O(1)|\log r_n|$, i.e., there exists a positive constant C such that $N_n \le r_n^{-C}$ for all integers n, we see that for k large enough (say, $\delta k - C \ge p$),

$$\sum_{n=1}^{\infty} \mathbb{P}^x(A_n^k) \le \sum_{n=1}^{\infty} N_n K_3^k r_n^{k\delta} \le K_3^k \sum_{n=1}^{\infty} r_n^{-C+\delta k} \le K_3^k \sum_{n=1}^{\infty} r_n^p < \infty.$$

Hence, the conclusion of Lemma 2.2 follows from the Borel-Cantelli lemma. $\qquad\square$

3. MAIN RESULT

The objective of this section is to establish uniform Hausdorff and packing dimension results for the images of a time homogeneous Markov process $X = \{X(t), t \ge 0, \mathbb{P}^x\}$ with values in $\mathbb{R}^d$. For any Borel set A in $\mathbb{R}^d$, denote by $P(t, x, A) := \mathbb{P}^x(X_t \in A)$ the transition probability of X. We state the following assumptions, where (A1) will be used for deriving uniform upper bounds, and (A2) or (A3) for uniform lower bounds.

(A1) There is a constant $H > 0$ such that for any $\gamma \in (0, H)$, there exist constants $C > 0$, $\eta > 0$ and $t_0 \in (0,1)$ such that for all $x \in \mathbb{R}^d$ and $0 < t \le t_0$,

$$(3.1) \qquad\qquad \mathbb{P}^x \left\{ \sup_{0 \le s \le t} |X(s) - x| \ge t^\gamma \right\} \le Ct^\eta.$$

(A2) There is a sequence of vectors of non-negative numbers $J = \{(\varepsilon_n, \zeta_n), n \ge 1\}$ such that $\varepsilon_n \to 0$ and $\zeta_n \to 0$ as $n \to \infty$, and has the following property (for simplicity of notation, we omit the subscript n): For any $(\varepsilon, \zeta) \in J$ and constant $T > 0$, there exist positive constants C_1, C_2, and $r_0 \le 1$ such that for all $0 < r \le r_0$, $x, y \in \mathbb{R}^d$ with $|y - x| \le r$, and all $0 < t \le T$,

$$(3.2) \qquad\qquad P(t, y, B(x, r)) \ge C_1 \min \left\{ 1, \left(\frac{r}{t^{H-\zeta}} \right)^{d+\varepsilon} \right\};$$

and

$$(3.3) \qquad\qquad P(t, x, B(x, r)) \le C_2 \min \left\{ 1, \left(\frac{r}{t^{H+\zeta}} \right)^{d-\varepsilon} \right\}.$$

(A3) We strengthen (A2) by further assuming that (3.3) holds for all $t > 0$.

Condition (A2) is quite general due to the flexibility in choosing arbitrarily small constants ε and ζ, in order for (3.2) and (3.3) to hold. This condition can be satisfied by a large class of Markov processes such as those with a bounded transition density and an approximate scaling property; see Section 4 for some interesting examples. (A3) is slightly stronger than (A2), which is needed for our subordination argument in proving Theorem 3.3 below in the critical case of $1 = Hd$.

Condition (A1) is less obvious. In the following, we give a sufficient condition for it to hold. For any $h \geq 0$ and $a > 0$, similar to Manstavičius [44], we consider the function

$$(3.4) \qquad \alpha(h, a) = \sup \left\{ P\big(s, x, B(x, a)^c\big) : x \in \mathbb{R}^d, 0 \leq s \leq h \right\},$$

where $B(x, a)^c = \{ y \in \mathbb{R}^d : |y - x| \geq a \}$. The function $h \mapsto \alpha(h, a)$ carries a lot of information about regularity properties of the sample paths of the Markov process X. For example, Kinney [32] showed that if for each fixed $a > 0$, $\alpha(h, a) \to 0$ as $h \to 0$ then the sample function $X(t)$ is almost surely cadlag; and if $\alpha(h, a) = o(h)$ as $h \to 0$ for every fixed $a > 0$ then the sample function $X(t)$ is almost surely continuous. See Manstavičius [44] and the references therein for further information.

For a given constant $H > 0$, a Markov process X is said to belong to the class $\widetilde{\mathcal{M}}(H)$ if there exist positive and finite constants C, β, h_0 and a_0, depending on d and H only, such that the following property holds: For all $h \in (0, h_0)$ and $a \in (0, a_0)$ such that $ha^{-1/H} < 1$, we have

$$(3.5) \qquad \alpha(h, a) \leq C \left(\frac{h}{a^{1/H}} \right)^{\beta}.$$

Condition (3.5) is the same as (1.1) in Manstavičius [44] for the class $\mathcal{M}(\beta, \gamma)$ with $\gamma = \beta/H$. We mention that [54, Corollary 5.10] proved that the solution of certain SDE driven by a Lévy process belongs to the class $\mathcal{M}(1, \gamma)$ of Manstavičius [44] for suitable choice of γ, thus derived a result on the γ-variation of the solution.

The following sufficient condition for (A1) is often convenient to use (cf. Theorem 4.1 below).

Proposition 3.1. *Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ be a separable, time-homogeneous Markov process taking values in $\mathbb{R}^d$. If X belongs to the class $\widetilde{\mathcal{M}}(H)$, then for any $\varepsilon \in (0, 1)$ and $\gamma \in (0, H)$, X satisfies (3.1) with $\eta = \beta(1 - \frac{\gamma}{H})$.*

Proof. We make use of the following Ottaviani-type inequality (cf. Gikhman and Skorohod [14, page 420], or Manstavičius [44]): For all $x \in \mathbb{R}^d$, all $h > 0$ and $a > 0$ such that $\alpha(h, a/2) < 1$,

$$(3.6) \qquad \mathbb{P}^x \left\{ \sup_{0 \leq s \leq h} |X(s) - X(0)| > a \right\} \leq \frac{\mathbb{P}^x \big\{ |X(h) - X(0)| > a/2 \big\}}{1 - \alpha(h, a/2)}.$$

For any $\gamma \in (0, H)$, it follows from (3.6) with $a = h^{\gamma}$ and (3.5) that for h small enough,

$$(3.7) \quad \mathbb{P}^x \left\{ \sup_{0 \leq s \leq h} |X(s) - X(0)| > h^{\gamma} \right\} \leq C \mathbb{P}^x \left\{ |X(h) - X(0)| > h^{\gamma}/2 \right\} \leq C h^{\beta(1 - \frac{\gamma}{H})}.$$

This proves the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Most examples given in Section 4 are Lévy or Lévy-type processes. For these processes, the maximal tail probability $\mathbb{P}^x \big\{ \sup_{s \in [0,t]} |X_s - x| \geq r \big\}$ in (3.7) has been studied by several authors. Pruitt [49] established an upper bound for the maximal probability for a general Lévy process in terms of its Lévy measure. Schilling [52] and Böttcher et al. [8] extended Pruitt's result to Lévy-type processes and proved an upper bound in terms of the symbol of the process. The following proposition is taken from [8, Corollary 5.2],

which can be applied to verify (A1) for Lévy-type processes. We remark that Kühn [40, Lemma 3.2] has proved recently that the inequality (3.8) still holds if t is a stopping time, with the t on the right-hand side replaced by $\mathbb{E}(t)$. We thank the referee for pointing out these results to us.

Proposition 3.2. *Let $X = \{X(t), t \geq 0\}$ be a Lévy-type process with a symbol $q(x, \xi) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ given by*

$$q(x, \xi) = -ib(x) \cdot \xi + \frac{1}{2}\xi \cdot Q(x)\xi + \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{iy \cdot \xi} + iy \cdot \xi 1_{(0,1]}(|y|)\right)\nu(x, dy),$$

where for each fixed $x \in \mathbb{R}^d$, $(b(x), Q(x), \nu(x, dy))$ is a Lévy triplet, i.e. $b(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d \times d}$ is a symmetric positive semidefinite matrix and $\nu(x, dy)$ is a measure on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$ such that $\int_{\mathbb{R}^d \setminus \{0\}}(|y|^2 \wedge 1)\nu(x, dy) < \infty$. Then, there exists a constant $C > 0$ such that

$$(3.8) \qquad \mathbb{P}^x\left\{\sup_{s \in [0,t]} |X_s - x| \geq r\right\} \leq Ct \sup_{|y-x| \leq r} \sup_{|\xi| \leq 1/r} |q(y, \xi)|.$$

Our main theorem of this paper is the following uniform Hausdorff and packing dimension result for the images of X.

Theorem 3.3. *Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ be a time homogeneous Markov process in $\mathbb{R}^d$ and satisfies Conditions (A1). Assume either (i) $1 < Hd$ and (A2) hold; or (ii) $1 = Hd$ and (A3) hold. Then for all $x \in \mathbb{R}^d$,*

$$(3.9) \qquad \mathbb{P}^x\left\{\dim_{\mathrm{H}} X(E) = \frac{1}{H}\dim_{\mathrm{H}} E \text{ for all Borel sets } E \subseteq [0, \infty)\right\} = 1$$

and

$$(3.10) \qquad \mathbb{P}^x\left\{\dim_{\mathrm{P}} X(E) = \frac{1}{H}\dim_{\mathrm{P}} E \text{ for all Borel sets } E \subseteq [0, \infty)\right\} = 1.$$

Proof. We will only prove the Hausdorff dimension result (3.9). The proof of (3.10), which is based on the connection between packing dimension and upper box-counting dimension (cf. [13]), is similar and hence omitted.

The proof of is divided into two parts. Namely, we prove the upper and lower bounds for $\dim_{\mathrm{H}} X(E)$, respectively.

Part 1 (Uniform upper bound). By the σ-stability of Hausdorff dimension (cf. [13]), it suffices to consider Borel sets $E \subseteq [0, L]$ for all fixed integers L. For simplicity, we take $L = 1$ in this proof. Let $\gamma \in (0, H)$ be a constant, $t_n = 2^{-n}$ and

$$\mathcal{C}_n = \left\{[(j-1)t_n, jt_n] : j = 1, 2, \ldots, 2^n\right\}.$$

By Condition (A1) we get that for all $x \in \mathbb{R}^d$,

$$\mathbb{P}^x\left\{\sup_{0 \leq s \leq t_n} |X(s) - x| \geq t_n^\gamma\right\} \leq Ct_n^\eta.$$

Thus, by Lemma 2.1, with probability one under $\mathbb{P}^x$, as n is sufficiently large, $X(I)$ can be covered by K_2 balls of radius $\theta_n := t_n^\gamma$ for all intervals $I \in \mathcal{C}_n$.

Let $\chi = \dim_{\mathrm{H}} E$. Then, for any $\delta > 0$, there exists a sequence of intervals $\{F_i \subseteq [0, 1], i \in \mathbb{N}\}$ such that $d_i := \mathrm{diam}(F_i) \leq \varepsilon$,

$$(3.11) \qquad E \subseteq \bigcup_i F_i \quad \text{and} \quad \sum_i d_i^{\chi + \delta} \leq 1.$$

We choose n_i so that $\frac{t_{n_i}}{2} \leq d_i \leq t_{n_i}$, thus F_i is contained in at most two intervals of $\mathcal{C}_{n_i}$. Consequently, $X(F_i)$ can be covered by $2K_2$ balls of radius θ_{n_i}, which are denoted by $B_{i,1}, ..., B_{i,2K_2}$. Hence,

$$X(E) \subseteq \bigcup_i \bigcup_{j=1}^{2K_2} B_{i,j}.$$

Further observe that, by using (3.11), we have

$$\sum_i \sum_{j=1}^{2K_2} [\mathrm{diam}(B_{i,j})]^{(\chi + \delta)/\gamma} \leq 2K_2 \sum_i t_{n_i}^{\chi + \delta}$$

$$\leq 2K_2 \sum_i (2d_i)^{\chi + \delta} \leq 2^{1 + \chi + \delta} K_2.$$

This yields $\dim_{\mathrm{H}} X(E) \leq (\chi + \delta)/\gamma$. Letting $\delta \downarrow 0$ and $\gamma \uparrow H$ yields

$$\dim_{\mathrm{H}} X(E) \leq \frac{1}{H} \dim_{\mathrm{H}} E.$$

Part 2 (Uniform lower bound). In order to prove

$$(3.12) \qquad \mathbb{P}^x \Big\{ \dim_{\mathrm{H}} X(E) \geq \frac{1}{H} \dim_{\mathrm{H}} E \text{ for all Borel sets } E \subseteq [0, \infty) \Big\} = 1,$$

we will treat the two cases (i) $1 < Hd$ and (ii) $1 = Hd$ separately.

We observe that the inequality in (3.12) follows from the following claim: For all $x \in \mathbb{R}^d$,

$$(3.13) \qquad \mathbb{P}^x \big\{ \dim_{\mathrm{H}} X^{-1}(F) \leq H \dim_{\mathrm{H}} F \quad \text{for all } F \subseteq \mathbb{R}^d \big\} = 1,$$

by taking $F = X(E)$. Moreover, by the σ-stability of Hausdorff dimension, (3.13) is equivalent to: For all constants $T > 0$ and all integers $m = 1, 2, ...,$

$$(3.14) \qquad \mathbb{P}^x \Big\{ \dim_{\mathrm{H}} \big(X^{-1}(F) \cap [0, T] \big) \leq H \dim_{\mathrm{H}} F \quad \text{for all } F \subseteq [-m, m]^d \Big\} = 1.$$

Hence, in order to prove (3.12) for the case (i), it suffices to prove (3.14) for all fixed constant $T > 0$ and positive integer m. This will be done by using the covering principle in Lemma 2.2. Recall that in case (i) we assume (A2) holds. Specifically, (3.2) and (3.3) hold for all $0 < t \leq 2T$.

By applying [58, Proposition 2.1], which is an extension of Theorem 1.1 in [25], we get

$$(3.15) \qquad \mathbb{P}^x \Big\{ \inf_{t \leq s \leq T} |X(s) - x| \leq r \Big\} \leq \frac{\int_t^{2T} P(s, x, B(x, r)) ds}{\inf_{|y - x| \leq r} \int_0^{T-t} P(s, y, B(x, r)) ds}.$$

In order to estimate the denominator of (3.15), we assume without loss of generality that $0 < t \le T/2$. For any $y \in \mathbb{R}^d$ with $|y - x| \le r \le r_0$, we use (3.2) in (A2) to get

$$\int_0^{T-t} P(s, y, B(x, r))ds \ge C_1 \int_0^{T/2} \min\left\{1, \left(\frac{r}{s^{H-\zeta}}\right)^{d+\varepsilon}\right\} ds \ge C r^{\frac{1}{H-\zeta}}.$$

On the other hand, since $1 < Hd$, we choose $(\varepsilon, \zeta) \in J$ such that ε and ζ are small enough so that $1 < (H + \zeta)(d - \varepsilon)$. By using (3.3) in (A2) we derive that for any $t > r^{1/(H+\zeta)}$,

$$\int_t^{2T} P(s, x, B(x, r))ds \le C_2 \int_t^{2T} \min\left\{1, \left(\frac{r}{s^{H+\zeta}}\right)^{d-\varepsilon}\right\} ds$$

$$= C_2 \int_t^{2T} \left(\frac{r}{s^{H+\zeta}}\right)^{d-\varepsilon} ds \le C\, r^{d-\varepsilon} t^{1-(H+\zeta)(d-\varepsilon)}.$$

We combine the above with (3.15) to see that for all $x \in \mathbb{R}^d$, $r > 0$ and $t > r^{1/(H+\zeta)}$,

(3.16) $$\mathbb{P}^x\left\{\inf_{t \le s \le T} |X(s) - x| \le r\right\} \le C\, r^{d - \frac{1}{H-\zeta} - \varepsilon} t^{1-(H+\zeta)(d-\varepsilon)}.$$

Now we proceed to prove (3.14) for the case $1 < Hd$. Fix any integer $m \ge 1$ and let $\mathcal{D}_n^m$ be the collection of dyadic cubes in $[-m, m]^d$ of the form $\prod_{i=1}^d [j_i 2^{-n}, (j_i + 1)2^{-n}]$, where each j_i is an integer and $j_i \in [-m2^n, m2^n - 1]$. It is easy to see that $\sharp(\mathcal{D}_n^m) = (m2^{n+1})^d$.

Fix a constant $\gamma \in (0, 1/H)$ and then choose $\varepsilon > 0$ and $\zeta > 0$ sufficiently small such that $(\varepsilon, \zeta) \in J$ and

$$\delta := \left(d - \frac{1}{H-\zeta} - \varepsilon\right)\left(1 - \gamma(H+\zeta)\right) - \frac{2\gamma\zeta}{H-\zeta} > 0.$$

This is possible since $(d - \frac{1}{H})(1 - \gamma H) > 0$.

Let $r_n = 2^{-n}$ and $t_n = 2^{-\gamma n}$, further choose n large enough so that $r_n \le r_0$. Notice that $t_n > r_n^{1/(H+\zeta)}$. By (3.16) we verify that for all $x \in \mathbb{R}^d$,

(3.17) $$\mathbb{P}^x\left\{\inf_{t_n \le s \le T} |X_s - x| \le r_n\right\} \le C r_n^{d - \frac{1}{H-\zeta} - \varepsilon} t_n^{1-(H+\zeta)(d-\varepsilon)} = C\, r_n^{\delta}.$$

Hence, Lemma 2.2 implies that there is an integer K_4 such that with $\mathbb{P}^x$-probability one, for all n large enough (say, $n \ge n_0$) and all dyadic cubes $B \in \mathcal{D}_n^m$, $X^{-1}(B) \cap [0, T]$ can be covered by at most K_4 intervals of length $t_n = 2^{-\gamma n}$.

For any Borel set $F \subseteq [-m, m]^d$, let $\theta > \dim_{\mathrm{H}} F$. Then there exists a sequence of dyadic cubes $\{B_i, i \in \mathbb{N}\}$ in $[-m, m]^d$ of sides $r_{n_i} = 2^{-n_i}$ such that $n_i \ge n_0$,

(3.18) $$F \subseteq \bigcup_{i=1}^\infty B_i \quad \text{and} \quad \sum_{i=1}^\infty r_{n_i}^\theta \le 1.$$

Since, for every i, $X^{-1}(B_i) \cap [0, T]$ can be covered by K_4 intervals I_{ij} of length $t_{n_i} = 2^{-\gamma n_i}$, we see that

$$X^{-1}(F) \cap [0, T] \subseteq \bigcup_{i=1}^\infty \bigcup_{j=1}^{K_4} I_{ij}.$$

Moreover,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{K_4} \left[\operatorname{diam}(I_{ij})\right]^{\theta/\gamma} \leq K_4 \sum_{i=1}^{\infty} r_{n_i}^{\theta} \leq K_4,$$

this implies $\dim_{\mathrm{H}}\left(X^{-1}(F) \cap [0, T]\right) \leq \frac{\theta}{\gamma}$. Letting $\gamma \uparrow 1/H$ and $\theta \downarrow \dim_{\mathrm{H}} F$ yields (3.14) and, thus, (3.12) for the case $1 < Hd$.

Finally we prove (3.12) for the case $1 = Hd$ by making use of a "subordination argument" that is similar to that in Hawkes [16] (see also Pruitt [48]).

Let $\tau = \{\tau_t, t \geq 0\}$ be a stable subordinator with stability index $\rho \in (0, 1)$, and independent of the process X. Consider the Markov process $Y = \{Y_t, t \geq 0\}$ defined by $Y_t = X(\tau_t)$. It is easy to see $\{Y_t, t \geq 0\}$ is a time-homogeneous strong Markov process. Denote the transition probability of Y by $\tilde{P}(t, x, A) := \mathbb{P}^x(Y_t \in A)$. We claim that, if (A3) holds, then $\tilde{P}(t, x, A)$ satisfies Condition (A2) with H replaced by H/ρ. Consequently, because $1 < Hd/\rho$, we can apply the conclusion of the first part to the process Y.

More specifically, we now verify the following claim under assumption (A3):

(A2′) We can find a sequence $J' = \{(\varepsilon', \zeta')\}$ of arbitrarily small numbers with the following property: For any $(\varepsilon', \zeta') \in J'$ and $A > 0$, there exist positive constants C_3 and C_4 such that for all $0 < r \leq r_0$ (as in (A2)), $x, y \in \mathbb{R}^d$ with $|y - x| \leq r$ and $0 < t \leq A$,

$$(3.19) \qquad \tilde{P}(t, y, B(x, r)) \geq C_3 \min\left\{1, \left(\frac{r}{t^{(H-\zeta')/\rho}}\right)^{d+\varepsilon'}\right\};$$

and

$$(3.20) \qquad \tilde{P}(t, x, B(x, r)) \leq C_4 \min\left\{1, \left(\frac{r}{t^{(H+\zeta')/\rho}}\right)^{d-\varepsilon'}\right\}.$$

To verify (A2′), we denote, for all $t > 0$, the density function of τ_t by p_{τ_t}. Then the self-similarity of τ_t implies $p_{\tau_t}(s) = t^{-\frac{1}{\rho}} p_{\tau_1}(t^{-\frac{1}{\rho}} s)$. Hence

$$
\begin{aligned}
\tilde{P}\big(t, y, B(x, r)\big) &= \mathbb{P}^y(|Y_t - x| \leq r) \\
&= \int_0^{\infty} \mathbb{P}^y(|X_{\tau_t} - x| \leq r | \tau_t = s) p_{\tau_t}(s) ds \\
(3.21) \qquad &= \int_0^{\infty} P(s, y, B(x, r)) p_{\tau_t}(s) ds \\
&= \int_0^{\infty} P(s, y, B(x, r)) t^{-\frac{1}{\rho}} p_{\tau_1}(t^{-\frac{1}{\rho}} s) ds \\
&= \int_0^{\infty} P\left(t^{\frac{1}{\rho}} s, y, B(x, r)\right) p_{\tau_1}(s) ds.
\end{aligned}
$$

Consequently, we can make use of Condition (A3) to estimate $\tilde{P}\big(t, y, B(x, r)\big)$. Recall that $J = \{(\varepsilon, \zeta)\}$ is the sequence in (A2).

On one hand, for any constant $A > 0$, we apply (3.2) with $T \geq 2A^{1/\rho}$ to derive that for any $x, y \in \mathbb{R}^d$ with $|y - x| \leq r$, $0 < r \leq r_0$ and $0 < t \leq A$, we have

$$\tilde{P}(t, y, B(x, r)) \geq C_1 \int_0^{Tt^{-1/\rho}} \min\left\{1, \left(\frac{r}{(t^{1/\rho}s)^{H-\zeta}}\right)^{d+\varepsilon}\right\} p_{\tau_1}(s) ds$$

$$= C_1 \int_0^{r^{1/(H-\zeta)}t^{-1/\rho}} p_{\tau_1}(s)\, ds$$

$$+ C_1 \int_{r^{1/(H-\zeta)}t^{-1/\rho}}^{Tt^{-1/\rho}} \left(\frac{r}{t^{(H-\zeta)/\rho}}\right)^{d+\varepsilon} \frac{p_{\tau_1}(s)}{s^{(H-\zeta)(d+\varepsilon)}}\, ds.$$

It is easy to see that when $r^{1/(H-\zeta)} > t^{1/\rho}$,

$$(3.22) \qquad \tilde{P}(t, y, B(x, r)) \geq C_1 \int_0^1 p_{\tau_1}(s) ds$$

and when $r^{1/(H-\zeta)} \leq t^{1/\rho}$,

$$(3.23) \qquad \tilde{P}(t, y, B(x, r)) \geq C_1 \left(\frac{r}{t^{(H-\zeta)/\rho}}\right)^{d+\varepsilon} \int_1^2 \frac{1}{s^{(H-\zeta)(d+\varepsilon)}} p_{\tau_1}(s) ds,$$

where we have used that fact that $Tt^{-1/\rho} \geq TA^{-1/\rho} \geq 2$ for all $0 < t \leq A$ and the last integral is positive since the density $p_{\tau_1}(s)$ is positive for $s > 0$. Then, (3.22) and (3.23) imply (3.19) with $\varepsilon' = \varepsilon$ and $\zeta' = \zeta$.

On the other hand, similarly to (3.21), we use (3.3), which is now assumed to hold for all $t > 0$, to derive that for any $x \in \mathbb{R}^d$, $0 < r \leq r_0$ and $t > 0$,

$$\tilde{P}(t, x, B(x, r)) \leq C_2 \int_0^\infty \min\left\{1, \left(\frac{r}{(t^{1/\rho}s)^{H+\zeta}}\right)^{d-\varepsilon}\right\} p_{\tau_1}(s) ds$$

$$(3.24) \qquad = C_2 \int_0^{r^{1/(H+\zeta)}t^{-1/\rho}} p_{\tau_1}(s) ds$$

$$+ C_2 \int_{r^{1/(H+\zeta)}t^{-1/\rho}}^\infty \left(\frac{r}{t^{(H+\zeta)/\rho}}\right)^{d-\varepsilon} \frac{p_{\tau_1}(s)}{s^{(H+\zeta)(d-\varepsilon)}} ds.$$

Since $\tilde{P}(t, y, B(x, r)) \leq 1$, in order to verify (3.20), we only need to consider the case when $r^{1/(H+\zeta)} \leq t^{1/\rho}$. If $(H + \zeta)(d - \varepsilon) < 1$, then by the boundedness of the density function $p_{\tau_1}(\cdot)$, i.e., $\sup_{0 < s \leq 1} p_{\tau_1}(s) \leq M$ for some $M > 0$, we obtain that

$$(3.25) \qquad \tilde{P}(t, x, B(x, r)) \leq C_2 M \frac{r^{1/(H+\zeta)}}{t^{1/\rho}} + C_2 \left(\frac{r}{t^{(H+\zeta)/\rho}}\right)^{d-\varepsilon} \int_0^\infty \frac{p_{\tau_1}(s)}{s^{(H+\zeta)(d-\varepsilon)}}\, ds.$$

The last integral is convergent at 0 because $(H + \zeta)(d - \varepsilon) < 1$ and at infinity because $p_{\tau_1}(s) \sim \frac{C_\rho}{s^{1+\rho}}$ as $s \to +\infty$.

If $(H+\zeta)(d-\varepsilon) > 1$, then we further split the last integral in (3.24) over $[r^{1/(H+\zeta)}t^{-1/\rho}, 1]$ and $[1, \infty)$, and to derive

$$(3.26) \qquad \tilde{P}(t, x, B(x, r)) \leq C_2' M \frac{r^{1/(H+\zeta)}}{t^{1/\rho}} + C_2 \left(\frac{r}{t^{(H+\zeta)/\rho}}\right)^{d-\varepsilon} \int_1^\infty \frac{p_{\tau_1}(s)}{s^{(H+\zeta)(d-\varepsilon)}}\, ds,$$

where C_2' is a finite constant and, again, the last integral is convergent. The case of $(H+\zeta)(d-\varepsilon) = 1$ can be treated in the same way, and (3.26) still holds with an extra factor of $\log(\frac{r^{1/(H+\zeta)}}{t^{1/\rho}})$ in the first term on the right hand side, which can be absorbed by choosing $\tilde{\varepsilon}$ below slightly bigger. However, for simplicity, we ignore this case because one may choose $J = \{(\varepsilon, \zeta)\}$ so that $(H+\zeta)(d-\varepsilon) \neq 1$ for all $(\varepsilon, \zeta) \in J$.

Notice that, when $1 = Hd$, we can write $\frac{1}{H+\zeta} = d - \tilde{\varepsilon}$, where $\tilde{\varepsilon} = \frac{\zeta}{H(H+\zeta)}$. It follows from (3.25) and (3.26) that for $r^{1/(H+\zeta)} \leq t^{1/\rho}$,

$$(3.27) \qquad \tilde{P}(t, x, B(x, r)) \leq C \left(\frac{r}{t^{(H+\zeta)/\rho}} \right)^{d-\varepsilon'}$$

for some finite constant C, where $\varepsilon' = \max\{\varepsilon, \tilde{\varepsilon}\}$. Thus we have verified (3.20) with $\varepsilon' = \max\{\varepsilon, \tilde{\varepsilon}\}$ and $\zeta' = \zeta$. Moreover, because $0 < t \leq A$, the inequality in (3.19) remains valid (with a modified constant C_3) if we take $\varepsilon' = \max\{\varepsilon, \tilde{\varepsilon}\}$. Thus, Condition (A2') has been verified.

Because of Condition (A2') and the fact that $\frac{Hd}{\rho} > 1$, we can apply the above uniform lower bound result in the case of $Hd > 1$ to Markov process Y to obtain

$$\mathbb{P}\left\{ \dim_{\mathrm{H}} Y(E) \geq \frac{\rho}{H} \dim_{\mathrm{H}} E \text{ for all Borel sets } E \subseteq [0, \infty) \right\} = 1.$$

It follows that, with probability 1, for all Borel sets $E \subseteq [0, \infty)$,

$$(3.28) \qquad \dim_{\mathrm{H}} X(E) \geq \dim_{\mathrm{H}} X(\tau(B)) = \dim_{\mathrm{H}} Y(B) \geq \frac{\rho}{H} \dim_{\mathrm{H}} B,$$

where $B = \{t : \tau_t \in E\} = \tau^{-1}(E)$. Even though both $\dim_{\mathrm{H}} X(E)$ and $\dim_{\mathrm{H}} B$ in (3.28) are random, they are determined by two independent processes X and τ, respectively. Hence we have

$$(3.29) \qquad \mathbb{P}\left\{ \dim_{\mathrm{H}} X(E) \geq \frac{\rho}{H} \|\dim_{\mathrm{H}} \tau^{-1}(E)\|_\infty \text{ for all Borel sets } E \subseteq [0, \infty) \right\} = 1,$$

where $\|\cdot\|_\infty$ is the $L^\infty(\mathbb{P})$-norm in the underlying probability space. (A more illustrative way for deriving (3.29) is to use the setting of product probability space. Namely, we assume that X is defined on Ω, τ is defined on Ω', then Y is defined on $\Omega \times \Omega'$ and (3.28) holds for almost all (ω, ω'). One can see that (3.29) follows from (3.28) and Fubini's Theorem.)

Recalling from Hawkes [17] that

$$(3.30) \qquad \|\dim_{\mathrm{H}} \tau^{-1}(E)\|_\infty = \frac{\rho + \dim_{\mathrm{H}} E - 1}{\rho},$$

we derive that

$$\mathbb{P}\left\{ \dim_{\mathrm{H}} X(E) \geq \frac{\rho + \dim_{\mathrm{H}} E - 1}{H} \text{ for all Borel sets } E \subseteq [0, \infty) \right\} = 1.$$

Letting $\rho \uparrow 1$ yields (3.12) for the case $1 = Hd$. This completes the proof of Theorem 3.3. □

4. EXAMPLES

Theorem 3.3 is applicable to a wide class of Markov processes. In this section, we provide some examples which include self-similar Markov processes, Lévy processes, stable jump diffusion processes and non-symmetric stable-type processes.

4.1. Self-similar Markov processes.
The class of H-self-similar (H-$s.s.$) Markov processes with values in $[0, \infty)$ was introduced and studied by Lamperti [42], who used the term "semi-stable" instead of "self-similar". The H-$s.s.$ Markov processes on $\mathbb{R}^d$ or $\mathbb{R}^d \backslash \{0\}$ were investigated by, in chronicle order, Kiu [33], Graversen and Vuolle-Apiala [15], Vuolle-Apiala and Graversen [57], Vuolle-Apiala [56], Liu and Xiao [43], Xiao [58], Bertoin and Yor [6], Chaumont, et al [12], Alili, et al [1], among others.

We recall the definition of H-self-similar processes. Let $(E, \mathcal{B})$ denotes $\mathbb{R}^d$, $\mathbb{R}^d \setminus \{0\}$ or $\mathbb{R}^d_+$ with the usual Borel σ- algebra, $\{e\}$ is a point attached to E as an isolated point. Ω denotes the space of all functions ω from $[0, \infty)$ to $E \cup \{e\}$ having the following properties:

   (i) $\omega(t) = e$ for $t \geq \tau$, where $\tau = \inf\{t \geq 0 : \omega(t) = e\}$;
   (ii) ω is right continuous and has a left limit at every $t \in [0, \infty)$.

Let $H > 0$ be a fixed constant. A time homogeneous Markov process $X = \{X(t), t \geq 0, \mathbb{P}^x\}$ with state space $E \cup \{e\}$ is called H-self-similar (H-s.s.) if its transition probability function $P(t, x, A)$ satisfies

$$(4.1) \qquad P(t, x, A) = P(rt, r^H x, r^H A), \quad \text{for all } t > 0, r > 0, x \in E, A \in \mathcal{B}.$$

The constant H is called the self-similarity index of X. Condition (4.1) is equivalent to the statement that for every constant $r > 0$ the $\mathbb{P}^x$-distribution of $\{X(t), t \geq 0\}$ is equal to the $\mathbb{P}^{r^H x}$-distribution of $\{r^{-H} X(rt), t \geq 0\}$. Important examples of self-similar Markov processes include strictly α-stable Lévy processes which are $1/\alpha$-s.s., the Bessel processes which form exactly the class of $1/2$-s.s. diffusions on $(0, \infty)$ (see [50]). More examples of H-s.s. Markov processes can be found in [42, 15, 33, 57, 58].

In this section, we take $E = \mathbb{R}^d$ and assume the following two conditions:

(B1) There exist positive constants β, C and a_1 such that

$$(4.2) \qquad P(1, x, B(x, a)^c) \leq Ca^{-\beta} \quad \text{for all } x \in \mathbb{R}^d \text{ and } a > a_1.$$

(B2) For any $\varepsilon > 0$ small, there exist positive constants C_1 and C_2 such that for all $r > 0$ and $x, y \in \mathbb{R}^d$ with $|x - y| \leq r$, we have

$$(4.3) \qquad C_1 \min\{1, r^{d+\varepsilon}\} \leq P(1, x, B(y, r)) \leq C_2 \min\{1, r^{d-\varepsilon}\}.$$

The following theorem provides a uniform version for the Hausdorff dimension result in [43].

Theorem 4.1. *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an H-s.s. Markov process in $\mathbb{R}^d$ that satisfies conditions (B1) and (B2). If $1 \leq Hd$, then for all $x \in \mathbb{R}^d$, with $\mathbb{P}^x$-probability one,*

$$\dim_{\mathrm{H}} X(E) = \frac{1}{H} \dim_{\mathrm{H}} E \quad and \quad \dim_{\mathrm{P}} X(E) = \frac{1}{H} \dim_{\mathrm{P}} E$$

for all Borel sets $E \subseteq \mathbb{R}_+$.

Proof. It is sufficient to verify that Conditions (A1) and (A3) are satisfied. It follows from (4.1) and (4.2) that for any $t > 0$, $x \in \mathbb{R}^d$ and $a > 0$,

$$(4.4) \qquad P\big(t, x, B(x, a)^c\big) = P\big(1, xt^{-H}, B(xt^{-H}, at^{-H})^c\big) \leq C(at^{-H})^{-\beta}$$

provided $at^{-H} \geq a_1$. This implies that $\alpha(h, a) \leq C(ha^{-1/H})^{H\beta}$ for all $h, a > 0$ satisfying $ha^{-1/H} \leq a_1^{-1/H}$. Hence X belongs to the class $\widetilde{\mathcal{M}}(H)$. It follows from Proposition 3.1 that Condition (A1) is satisfied. For verifying (A3), we apply (4.1) again, together with the first inequality in (4.3), to see that for all $t, r > 0$ and $x, y \in \mathbb{R}^d$ with $|x - y| \leq r$,

$$(4.5) \qquad P\big(t, x, B(y, r)\big) = P\big(1, xt^{-H}, B(yt^{-H}, rt^{-H})\big) \geq C_1 \min\left\{1, \left(\frac{r}{t^H}\right)^{d+\varepsilon}\right\}.$$

Similarly, we have

$$P\big(t, x, B(x, r)\big) = P\big(1, xt^{-H}, B(xt^{-H}, rt^{-H})\big) \leq C_2 \min\left\{1, \left(\frac{r}{t^H}\right)^{d-\varepsilon}\right\}.$$

Thus, Condition (A3) is satisfied with $J = \{(\varepsilon_n, 0)\}$, where $\varepsilon_n \downarrow 0$ can be taken arbitrarily. Therefore the conclusion of Theorem 4.1 follows from Theorem 3.3. $\qquad\square$

4.2. **Lévy processes.** A stochastic process $X = \{X(t), \ t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{R}^d$, is called a *Lévy process*, if X has stationary and independent increments, $t \mapsto X(t)$ is continuous in probability and $\mathbb{P}\{X(0) = 0\} = 1$. It is well known that for $t \geq 0$, the characteristic function of $X(t)$ is given by

$$\mathbb{E}\big[e^{i\langle \xi, X(t) \rangle}\big] = e^{-t\psi(\xi)},$$

where, by the Lévy-Khintchine formula,

$$(4.6) \qquad \psi(\xi) = i\langle \mathsf{a}, \xi \rangle + \frac{1}{2}\langle \xi, \Sigma\xi' \rangle + \int_{\mathbb{R}^d}\left[1 - e^{i\langle x, \xi \rangle} + \frac{i\langle x, \xi \rangle}{1 + |x|^2}\right]\mathsf{L}(dx), \qquad \forall \xi \in \mathbb{R}^d,$$

and $\mathsf{a} \in \mathbb{R}^d$ is fixed, Σ is a non-negative definite, symmetric, $(d \times d)$ matrix, and L is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}^d}\frac{|x|^2}{1 + |x|^2}\mathsf{L}(dx) < \infty.$$

The function ψ is called the *characteristic or Lévy exponent* of X, and L is the corresponding *Lévy measure*. The characteristic exponent ψ plays very important roles in studying the Lévy process X and many sample path properties of X can be described in terms of ψ. We also note that

$$\Re\,\psi(\xi) \geq 0, \ \text{ and } \ \Re\,\psi(-\xi) = \Re\,\psi(\xi), \qquad \forall \xi \in \mathbb{R}^d.$$

Notice that, if X is symmetric (i.e., X and $-X$ have the same law), then its Lévy exponent $\psi(\xi)$ is a nonnegative function.

A Lévy process X in $\mathbb{R}^d$ is called a *stable* Lévy process with index $\alpha \in (0, 2]$ if its Lévy measure L is of the form

$$(4.7) \qquad \mathsf{L}(dx) = \frac{dr}{r^{1+\alpha}}\,\nu(dy), \quad \forall\, x = ry, \ \ (r, y) \in \mathbb{R}_+ \times \mathbb{S}_d,$$

where $\mathbb{S}_d = \{y \in \mathbb{R}^d : |y| = 1\}$ is the unit sphere in $\mathbb{R}^d$ and $\nu(dy)$ is an arbitrary finite Borel measure on $\mathbb{S}_d$. Stable Lévy processes in $\mathbb{R}^d$ of index $\alpha = 1$ are also called *Cauchy*

processes. It follows from (4.6) and (4.7) that the Lévy exponent ψ_α of a stable Lévy process of index $\alpha \in (0, 2]$ can be written as

$$\psi_\alpha(\xi) = \int_{\mathbb{S}_d} |\langle \xi, y \rangle|^\alpha \Big[1 - i \, \text{sgn}(\langle \xi, y \rangle) \tan \Big(\frac{\pi\alpha}{2} \Big) \Big] \mathsf{M}(dy) + i\langle \xi, A_0 \rangle \quad \text{if} \ \ \alpha \neq 1,$$

$$\psi_1(\xi) = \int_{\mathbb{S}_d} |\langle \xi, y \rangle| \Big[1 + i \, \frac{\pi}{2} \text{sgn}(\langle \xi, y \rangle) \log |\langle \xi, y \rangle| \Big] \mathsf{M}(dy) + i\langle \xi, A_0 \rangle,$$

where the pair (M, A_0) is unique, and the measure M, which depends on ν in (4.7), is called the *spectral measure* of X. See Samorodnitsky and Taqqu [51, pp.65–66].

Remark 4.2. Due to the space-homogeneity of Lévy processes, Theorem 3.3 and its proof can be simplified. Firstly, we assume the following simpler conditions:

$(\text{A2}'')$ For any $\zeta > 0$ and $T > 0$, there exist positive constants C_1, C_2, $r_0 \leq 1$ such that for all $0 < t \leq T$ and $0 < r \leq r_0$,

$$(4.8) \qquad \qquad \mathbb{P}(|X(t)| \leq r) \geq C_1 \min \Big\{ 1, \Big(\frac{r}{t^{H-\zeta}} \Big)^d \Big\};$$

and

$$(4.9) \qquad \qquad \mathbb{P}(|X(t)| \leq r) \leq C_2 \min \Big\{ 1, \Big(\frac{r}{t^{H+\zeta}} \Big)^d \Big\}.$$

$(\text{A3}'')$ We strengthen $(\text{A2}'')$ by assuming additionally that (4.9) holds for all $t > 0$.

It is clear that the inequalities in (4.8) and (4.9) imply Condition (A2) with $J = \{(0, \zeta_n)\}$, where ζ_n is an arbitrary sequence with $\zeta_n \downarrow 0$ as $n \to \infty$. Secondly, instead of (3.15), we use directly Theorem 1.1 in [25] to get

$$(4.10) \qquad \qquad \mathbb{P}^x \Big\{ \inf_{t \leq s \leq T} |X(s) - x| \leq r \Big\} \leq \frac{\int_t^{2T} \mathbb{P}(|X(s)| \leq 2r) ds}{\int_0^{T-t} \mathbb{P}(|X(s)| \leq r) ds}.$$

Note that, due to the space-homogeneity of X, the denominator in the right hand side of (4.10) is simpler that that in (3.15). One can check that a slightly modified version of (3.16) holds under condition (A2''). Hence, for a space-homogeneous Markov process that satisfies (A1), the conclusion of Theorem 3.3 still holds under either $1 < Hd$ and (A2''); or $1 = Hd$ and (A3''). We will use this modified version of Theorem 3.3 to Lévy processes.

As we mentioned earlier, for a Lévy process $X = \{X(t), t \in \mathbb{R}_+\}$, many of its sample path properties are characterized by the analytic or asymptotic properties of its characteristic exponent $\psi(\xi)$. In order to determine the Hausdorff and packing dimension of $X(E)$, we will make use of the following conditions:

(B3) There is a constant $\alpha \in (0, 2]$ such that the following hold:

(i) If $0 < \alpha < 2$, then for every $\zeta' \in (0, 2 - \alpha)$ we have

$$(4.11) \qquad K_5^{-1} |\xi|^{\alpha - \zeta'} \leq \psi(\xi) \leq K_5 |\xi|^{\alpha + \zeta'}, \quad \forall \xi \in \mathbb{R}^d \text{ with } |\xi| \geq \tau,$$

where $K_5 \geq 1$ and τ are positive and finite constants.

(ii) If $\alpha = 2$, then for any $\zeta' \in (0, 2)$,

$$(4.12) \qquad K_5^{-1} |\xi|^{2-\zeta'} \leq \psi(\xi) \leq K_5 |\xi|^2, \quad \forall \xi \in \mathbb{R}^d \text{ with } |\xi| \geq \tau.$$

(B4) In addition to (B3), we assume that the left inequalities in (4.11) and (4.12) hold for all $\xi \in \mathbb{R}^d$.

Remark 4.3. The following are some remarks about Conditions (B3) and (B4).

(i) Since $\psi(\xi)$ is a negative definite function, the right inequality in (4.12) always holds for $|\xi| \geq 1$. (cf. [5, p.46]).
(ii) Conditions (4.11) and (4.12) are satisfied by a large class of symmetric Lévy processes whose Lévy measures have certain (approximate) regularly varying properties at the origin. This can be explicitly formulated by modifying Condition (2.17) (use $|\lambda| \to 0$ instead of $|\lambda| \to \infty$) and the proof of Theorem 2.5 in Xiao [60].

Now we are ready to state and prove the following theorem, which extends the uniform Hausdorff and packing dimension results of Hawkes [16], Hawkes and Pruitt [18], Perkins and Taylor [47] for stable Lévy processes to a class of symmetric Lévy processes.

Theorem 4.4. *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a symmetric Lévy process in $\mathbb{R}^d$ with exponent $\psi(\xi)$. We assume either (i) $1 < \alpha d$ and (B3) hold; or (ii) $1 = \alpha d$ and (B4) hold. Then with probability one,*

$$\dim_{\mathrm{H}} X(E) = \alpha \dim_{\mathrm{H}} E \quad and \quad \dim_{\mathrm{P}} X(E) = \alpha \dim_{\mathrm{P}} E$$

for all Borel sets $E \subseteq \mathbb{R}_+$.

Proof. This theorem is a consequence of Theorem 3.3 and Remark 4.2. It is sufficient to verify that

(a) When $1 < \alpha d$, (B3) implies Conditions (A1) and (A2″) with $H = \frac{1}{\alpha}$.
(b) When $1 = \alpha d$, (B4) implies Conditions (A1) and (A3″) hold with $H = \frac{1}{\alpha}$.

It will be clear that (a) and (b) can be verified by the same method. For simplicity, we only show (a) in case (i) where $0 < \alpha < 2$ and leave the rest of the verification to an interested reader.

In order to verify Conditions (A1), we apply Proposition 3.2. For any fixed $\gamma \in (0, 1/\alpha)$, there exists $\epsilon_0 \in (0, 2 - \alpha)$ such that $\gamma(\alpha + \epsilon_0) < 1$. By (3.8), we know that for $t \in (0, 1]$ small enough,

$$
\begin{aligned}
\sup_{x \in \mathbb{R}^d} \mathbb{P}^x \left\{ \sup_{s \in [0,t]} |X(s) - x| \geq t^\gamma \right\} &\leq Ct \sup_{|\xi| \leq t^{-\gamma}} |\psi(\xi)| \\
&\leq Ct \left(\sup_{|\xi| \leq \tau} |\psi(\xi)| + K_5 t^{-\gamma(\alpha+\epsilon_0)} \right) \\
&\leq Ct^{1-\gamma(\alpha+\epsilon_0)}.
\end{aligned}
$$

Since $1 - \gamma(\alpha + \epsilon_0) > 0$, we see that Conditions (A1) holds with $\eta = 1 - \gamma(\alpha + \epsilon_0)$.

In order to verify Condition (A2″) under (4.11), we use an argument from Khosh-nevisan and Xiao [27, 29]. For any $r > 0$, consider the nonnegative function

$$\varphi_r(y) = \prod_{j=1}^{d} \frac{1 - \cos(2ry_j)}{2\pi r y_j^2}, \qquad \forall y \in \mathbb{R}^d.$$

Its Fourier transform is given by

$$(4.13) \qquad \widehat{\varphi_r}(\xi) = \prod_{j=1}^{d} \left(1 - \frac{|\xi_j|}{2r}\right)^+, \qquad \forall \xi \in \mathbb{R}^d,$$

where $a^+ = \max(a, 0)$.

Note that, for $\xi \in B(0, r)$, we have $1 - (2r)^{-1}|\xi_j| \geq \frac{1}{2}$. In light of (4.13), this implies $\mathbb{1}_{B(0,r)}(\xi) \leq 2^d \widehat{\varphi_r}(\xi)$, where $\mathbb{1}_A$ denotes the indicator function of the set A. On the other hand, if $|\xi_j| \geq 2r$ for some $j \in \{1, \ldots, d\}$, by (4.13), then $\widehat{\varphi_r}(\xi) = 0$. Hence we have shown that for all $\forall \xi \in \mathbb{R}^d$,

$$(4.14) \qquad \mathbb{1}_{B(0,r)}(\xi) \leq 2^d \widehat{\varphi_r}(\xi) \leq 2^d \, \mathbb{1}_{B(0,2\sqrt{d}\,r)}(\xi).$$

Integrating the first inequality in (4.14) with respect to ν_t, the distribution of $X(t)$, and using Parseval's formula yield

$$(4.15) \qquad \begin{aligned} \mathbb{P}\{|X(t)| \leq r\} &\leq 2^d \int_{\mathbb{R}^d} \widehat{\varphi_r}(\xi)\, \nu_t(d\xi) \\ &= 2^d \int_{\mathbb{R}^d} \varphi_r(\xi)\, \widehat{\nu_t}(\xi)\, d\xi \\ &= 2^d \int_{\mathbb{R}^d} e^{-t\psi(\xi)} \prod_{j=1}^{d} \frac{1 - \cos(2r\,\xi_j)}{2\pi r \xi_j^2}\, d\xi. \end{aligned}$$

We split the last integral in (4.15) over $B(0, \tau) = \{\xi \in \mathbb{R}^d : |\xi| < \tau\}$ and its complement, respectively. For the first integral, we use the elementary inequality $1 - \cos x \leq x^2$ ($\forall x \in \mathbb{R}$), to derive

$$(4.16) \qquad \int_{|\xi| \leq \tau} e^{-t\,\psi(\xi)} \prod_{j=1}^{d} \frac{1 - \cos(2r\,\xi_j)}{2\pi r \xi_j^2}\, d\xi \leq K\, r^d.$$

For the second integral, we use (4.11), Parseval's formula and (4.14) to derive

$$(4.17) \qquad \begin{aligned} \int_{|\xi| > \tau} e^{-t\,\psi(\xi)} \prod_{j=1}^{d} \frac{1 - \cos(2r\,\xi_j)}{2\pi r \xi_j^2}\, d\xi &\leq \int_{\mathbb{R}^d} e^{-K_5^{-1} t\, \|\xi\|^{\alpha - \zeta'}} \prod_{j=1}^{d} \frac{1 - \cos(2r\,\xi_j)}{2\pi r \xi_j^2}\, d\xi \\ &= \int_{\mathbb{R}^d} \widehat{\varphi_r}(\xi)\, \mu_t(d\xi) \\ &\leq \int_{\mathbb{R}^d} \mathbb{1}_{B(0,2\sqrt{d}\,r)}(\xi)\, \mu_t(d\xi) \\ &\leq C \min\left\{1, \left(\frac{r}{t^{1/(\alpha - \zeta')}}\right)^d\right\}. \end{aligned}$$

In the above, μ_t denotes the distribution of the isotropic stable law with characteristic function $\widehat{\mu}_t(\xi) = e^{-K_5^{-1}t\,|\xi|^{\alpha-\zeta'}}$ and the last inequality follows from the boundedness and scaling property of the density function of μ_t. Combining (4.15)–(4.17) we derive that, for all $0 < t \le T$, (4.9) holds with $H = 1/\alpha$ and $\zeta = \frac{\zeta'}{\alpha(\alpha-\zeta')}$.

Next we verify the lower bound in (4.8) . Let $\tilde{r} = r/(2\sqrt{d})$. It follows from (4.14) that

$$
\begin{aligned}
(4.18) \qquad \mathbb{P}\{|X(t)| \le r\} &\ge \int_{\mathbb{R}^d} \widehat{\varphi}_{\tilde{r}}(\xi)\, \nu_t(d\xi) \\
&= \int_{\mathbb{R}^d} \varphi_{\tilde{r}}(\xi)\, \widehat{\nu}_t(\xi)\, d\xi \\
&= \int_{\mathbb{R}^d} e^{-t\psi(\xi)} \prod_{j=1}^{d} \frac{1 - \cos(2\tilde{r}\,\xi_j)}{2\pi\tilde{r}\xi_j^2}\, d\xi.
\end{aligned}
$$

Again, we split the last integral over $B(0,\tau) = \{\xi \in \mathbb{R}^d : |\xi| < \tau\}$ and its complement, and use a similar argument as in (4.17) to obtain

$$
\begin{aligned}
(4.19) \qquad \mathbb{P}\{|X(t)| \le r\} &\ge \int_{|\xi| \ge \tau} e^{-K_5 t|\xi|^{\alpha+\zeta'}} \prod_{j=1}^{d} \frac{1 - \cos(2\tilde{r}\,\xi_j)}{2\pi\tilde{r}\xi_j^2}\, d\xi \\
&= \int_{\mathbb{R}^d} e^{-K_5 t|\xi|^{\alpha+\zeta'}} \prod_{j=1}^{d} \frac{1 - \cos(2\tilde{r}\,\xi_j)}{2\pi\tilde{r}\xi_j^2}\, d\xi \\
&\quad - \int_{|\xi| \le \tau} e^{-K_5 t|\xi|^{\alpha+\zeta'}} \prod_{j=1}^{d} \frac{1 - \cos(2\tilde{r}\,\xi_j)}{2\pi\tilde{r}\xi_j^2}\, d\xi \\
&\ge C \min\left\{1, \left(\frac{r}{t^{1/(\alpha+\zeta')}}\right)^d\right\}.
\end{aligned}
$$

Note that $\frac{1}{\alpha+\zeta'} = \frac{1}{\alpha} - \frac{\zeta'}{\alpha(\alpha+\zeta')} > \frac{1}{\alpha} - \zeta$. It follows from (4.19) that we can choose a constant C_1 such that for all $0 < t \le T$, the lower bound in (4.8) holds with $H = 1/\alpha$ and $\zeta = \frac{\zeta'}{\alpha(\alpha-\zeta')}$. Hence, we have shown that (4.11) implies Condition (A2″) with $H = 1/\alpha$ and $\zeta = \frac{\zeta'}{\alpha(\alpha-\zeta')}$. This completes the proof of Theorem 4.4. □

Remark 4.5. Our method for verifying of Condition (A2″) provides a comparison theorem for the transition probabilities of Lévy processes in terms of their Lévy exponents. This may be of independent interest. Several authors have established estimates on the transition density functions of Lévy processes based on information on their Lévy measures or Lévy exponents; see Kaleta and Sztonyk [23] and the references therein for further information.

4.3. Stable jump-diffusions.
Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x, x \in \mathbb{R}^d\}$ be a Feller process with values in $\mathbb{R}^d$ corresponding to the Feller semigroup defined by the following equation:

$$
(4.20)\quad \frac{\partial u}{\partial t} = \langle A(x), \frac{\partial u}{\partial x}\rangle + \int_0^\infty \int_{\mathbb{S}_d} \left(u(x + |\xi|s) - u(x) - \frac{\langle|\xi|s, \frac{\partial u}{\partial x}(x)\rangle}{1 + |\xi|^2}\right) \frac{d|\xi|}{|\xi|^{1+\alpha}}\, \widetilde{\mathsf{M}}(x, ds),
$$

where the drift A and the spectral measure $\widetilde{\mathsf{M}}$ on $\mathbb{S}_d$ depend smoothly on x, see Theorem 3.1 of Kolokoltsov [38] for the precise conditions on A and $\widetilde{\mathsf{M}}$.

Following Kolokoltsov [38], we call X a stable jump-diffusion. Roughly speaking, these are the processes corresponding to stable Lévy processes in the same way as the ordinary diffusions corresponding to Brownian motion.

Locally, the stable jump-diffusion X resembles a stable Lévy process, hence it is expected that a stable jump-diffusion has sample path properties similar to those of a stable Lévy process. Some of these properties such as the limsup behavior of $X(t)$ as $t \to 0$ have been established by Kolokoltsov [38, section 6]. Moreover, for every fixed Borel set $E \subseteq \mathbb{R}_+$, the Hausdorff dimension of the image set $X(E)$ can be derived from Theorem 4.14 in Xiao [59].

The following theorem proves a uniform Hausdorff and packing dimension result for stable jump diffusions.

Theorem 4.6. *Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ be a stable jump-diffusion in $\mathbb{R}^d$ with index $\alpha \in (0, 2]$ as described above. If $\alpha \le d$, then for every $x \in \mathbb{R}^d$, $\mathbb{P}^x$-almost surely*

$$(4.21) \qquad \dim_{\mathrm{H}} X(E) = \alpha \dim_{\mathrm{H}} E \quad and \quad \dim_{\mathrm{P}} X(E) = \alpha \dim_{\mathrm{P}} E$$

hold for all Borel sets $E \subseteq \mathbb{R}_+$.

Proof. It follows from Theorem 6.1 of Kolokoltsov [38] that Condition (A1) holds for X with $H = 1/\alpha$. On the other hand, Theorem 3.1 of [38] implies that (A2) with $H = 1/\alpha$, $\varepsilon = 0$ and $\zeta = 0$. Hence (4.21) follows immediately from Theorem 3.3. $\square$

Remark 4.7. We remark that one can also apply the inequality (3.8) to verify Condition (A1) for this case. By (1.9) of [38], the symbol of the stable jump diffusion defined by (4.20) has the form:

$$q(x, \xi) = i(A(x), \xi) - \int_{\mathbb{S}_d} |(\xi, s)|^\alpha M(x, ds).$$

Moreover, we assume that same conditions on A and M as in Theorem 3.1 in [38], which contain

- there exist positive constants C_1 and C_2 such that

$$C_1 |\xi|^\alpha \le \int_{\mathbb{S}_d} |(\xi, s)|^\alpha M(x, ds) \le C_2 |\xi|^\alpha,$$

- A is uniformly bounded in x, i.e. $\sup_{x \in \mathbb{R}^d} |A(x)| < \infty$,
- $A(x) \equiv 0$ for $\alpha \le 1$.

Then, by (3.8), for any $\gamma \in (0, 1/\alpha)$, we have, for all $x \in \mathbb{R}^d$ and $t \in [0, 1]$,

$$\mathbb{P}^x \left\{ \sup_{s \in [0,t]} |X_s - x| \ge t^\gamma \right\} \le Ct \sup_{|y-x| \le t^\gamma} \sup_{|\xi| \le t^{-\gamma}} |q(y, \xi)|$$

$$\le Ct \sup_{|\xi| \le t^{-\gamma}} \left(\sup_{x \in \mathbb{R}^d} |A(x)||\xi| + C_2 |\xi|^\alpha \right)$$

$$\le Ct^{1-\gamma\alpha} \left(\sup_{x \in \mathbb{R}^d} |A(x)| + C_2 \right).$$

This verifies Condition (A1) with $\eta = 1 - \gamma\alpha$.

4.4. Non-symmetric stable-type processes. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a pure jump process such that its infinitesimal generator has the following form:

$$\mathcal{L}_\alpha^\kappa f(x) := \lim_{\varepsilon \to 0} \int_{\{z \in \mathbb{R}^d : |z| \geq \varepsilon\}} (f(x+z) - f(x)) \frac{\kappa(x,z)}{|z|^{d+\alpha}} dz,$$

where $d \geq 1, 0 < \alpha < 2$, and $\kappa(x, z)$ is a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$0 < \kappa_0 \leq \kappa(x, z) \leq \kappa_1, \quad \kappa(x, z) = \kappa(x, -z),$$

and for some $\beta \in (0, 1)$

$$|\kappa(x, z) - \kappa(y, z)| \leq \kappa_2 |x - y|^\beta.$$

This class of Markov processes has been studied by [10, 11], among others.

The following uniform dimension result is a consequence of Theorem 3.3.

Theorem 4.8. *Let* $X = \{X(t), t \in \mathbb{R}_+\}$ *be an non-symmetric α-stable-type Markov process defined above. If $1 < \alpha d$, then, for every $x \in \mathbb{R}^d$, $\mathbb{P}^x$-almost surely*

$$(4.22) \qquad \dim_{\mathrm{H}} X(E) = \alpha \dim_{\mathrm{H}} E \quad and \quad \dim_{\mathrm{P}} X(E) = \alpha \dim_{\mathrm{P}} E$$

hold for all Borel sets $E \subseteq \mathbb{R}_+$.

Proof. By [11, 22], X_t has a Hölder continuous transition density function $p(t, x, y)$. Furthermore, for any $T > 0$, there exists a constant $C > 1$ depending on $d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2, T$ such that for all $t \in (0, T], x, y \in \mathbb{R}^d$,

$$(4.23) \quad C^{-1} \min\left\{ t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}} \right\} \leq p(t, x, y) \leq C \min\left\{ t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}} \right\}.$$

Define

$$P_1(t, x, A) := C^{-1} \int_A \min\left\{ t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}} \right\} dy$$

and

$$P_2(t, x, A) := C \int_A \min\left\{ t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}} \right\} dy.$$

Both P_1 and P_2 have $1/\alpha$-self-similar property, i.e., for any $r > 0$

$$P_i(t, x, A) = P_i(rt, r^{1/\alpha}x, r^{1/\alpha}A), i = 1, 2.$$

Indeed, for all $A \in \mathcal{B}(\mathbb{R}^d)$ and $r > 0$, we have

$$\int_{r^{1/\alpha}A} \min\left\{ (rt)^{-d/\alpha}, \frac{rt}{|r^{1/\alpha}x - y|^{d+\alpha}} \right\} dy$$

$$= \int_A \min\left\{ (rt)^{-d/\alpha}, \frac{rt}{r^{(d+\alpha)/\alpha}|x - z|^{d+\alpha}} \right\} r^{d/\alpha} dz$$

$$= \int_A \min\left\{ t^{-d/\alpha}, \frac{t}{|x - z|^{d+\alpha}} \right\} dz.$$

By a straightforward computation, there exists constant $C > 1$ such that

$$P_1(1, y, B(x, r)) \geq C^{-1} \min\{1, r^d\}, \quad \forall |y - x| \leq r, \quad r > 0.$$

and

$$P_2(1, x, B(x, r)) \leq C \min\{1, r^d\}, \quad \forall x \in \mathbb{R}^d, \quad r > 0.$$

Then for all $t \in (0,T]$ and $|y - x| \leq r$,

$$
\begin{aligned}
P(t, y, B(x, r)) &\geq P_1(t, y, B(x, r)) \\
&= P_1(1, t^{-\frac{1}{\alpha}} y, B(t^{-\frac{1}{\alpha}} x, t^{-\frac{1}{\alpha}} r)) \\
&\geq C^{-1} \min \left\{ 1, \left(\frac{r}{t^{\frac{1}{\alpha}}} \right)^d \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
P(t, x, B(x, r)) &\leq P_2(t, x, B(x, r)) \\
&= P_2(1, t^{-\frac{1}{\alpha}} x, B(t^{-\frac{1}{\alpha}} x, t^{-\frac{1}{\alpha}} r)) \\
&\leq C \min \left\{ 1, \left(\frac{r}{t^{\frac{1}{\alpha}}} \right)^d \right\}.
\end{aligned}
$$

Thus Condition (A2) holds.

Next, we verify the condition (A1). As in Remark 4.7 , this can be done by applying Proposition 3.2 and the fact that the symbol q of X has the form

$$
q(x, \xi) = \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{iz \cdot \xi} + iz \cdot \xi 1_{(0,1]}(|z|) \right) \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz,
$$

which can be bounded from above by $C|\xi|^\alpha$. Here we provide a different proof.

By the Lévy-Itô decomposition, one has

$$
X_t = X_0 + \int_0^t \int_{|y| \leq l} y \tilde{N}(X_{s-}, dy, ds) + \int_0^t \int_{|y| > l} y N(X_{s-}, dy, ds),
$$

where l can be any positive constant, $N(x, dy, ds)$ is the Poisson random measure with the intensity measure $\nu(x, dy)ds := \frac{\kappa(x,y)}{|y|^{d+\alpha}} dy ds$ and $\tilde{N}(X_{s-}, dy, ds) = N(X_{s-}, dy, ds) - \nu(X_{s-}, dy)ds$ is the compensated Poisson random measure.

Since $\alpha \in (0, 2)$, we can find some p satisfying $0 < p \leq 1$ and $p < \alpha < 2p$. For some $l > 0$ and $x \in \mathbb{R}^d$, we define a smooth function f on $\mathbb{R}^d$ by

$$
f(y) = (|y - x|^2 + l^2)^{p/2}, \quad y \in \mathbb{R}^d,
$$

it is easy to check that for all $x, y \in \mathbb{R}^d$,

(4.24) $$|f(y_1) - f(y_2)| \leq |y_1 - y_2|^p.$$

Now let $X_0 = x$, by Itô's formula [2, Sect. 4.4.2], we get, for any $l > 0$,

$$
\begin{aligned}
f(X_t) =\ & f(x) + \int_0^t \int_{|y| \leq l} [f(X_{s-} + y) - f(X_{s-})] \tilde{N}(X_{s-}, dy, ds) \\
& + \int_0^t \int_{|y| \leq l} [f(X_{s-} + y) - f(X_{s-}) - \langle \nabla f(X_{s-}), y \rangle] \nu(X_{s-}, dy) ds \\
& + \int_0^t \int_{|y| > l} [f(X_{s-} + y) - f(X_{s-})] N(X_{s-}, dy, ds) \\
=:\ & l^p + I_1(t) + I_2(t) + I_3(t).
\end{aligned}
$$

Take $l = T^{1/\alpha}$, for any $T > 0$, by Burkholder's inequality and (4.24), we have

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0 \le t \le T} |I_1(t)|\right) &\le \mathbb{E}\left[\int_0^T \int_{|y| \le T^{1/\alpha}} |y|^{2p} N(X_{s-}, dy, ds)\right]^{1/2} \\
&\le \left[\int_0^T \int_{|y| \le T^{1/\alpha}} |y|^{2p} \frac{\kappa_1}{|y|^{d+\alpha}} dy ds\right]^{1/2} \\
&\le CT^{p/\alpha}.
\end{aligned}
$$
(4.25)

For $I_2(t)$, by Taylor's expansion, we have

$$
f(X_{s-} + y) - f(X_{s-}) - \langle \nabla f(X_{s-}), y \rangle = \langle y, \nabla^2 f(X_{s-} + \theta y) y \rangle,
$$
(4.26)

where $\theta \in [0, 1]$ depending on X_{s-} and y, one can verify that

$$
\begin{aligned}
&\left|\langle y, \nabla^2 f(X_{s-} + \theta y) y \rangle\right| \\
&= \left|\frac{p|y|^2}{\left(T^{2/\alpha} + |X_{s-} + \theta y - x|^2\right)^{1-\frac{p}{2}}} + \frac{p(p-2)|\langle y, X_{s-} + \theta y - x \rangle|^2}{\left(T^{2/\alpha} + |X_{s-} + \theta y - x|^2\right)^{2-\frac{p}{2}}}\right| \\
&\le \frac{p|y|^2}{T^{\frac{2}{\alpha}(1-\frac{p}{2})}} + \frac{p(2-p)|y|^2 |X_{s-} + \theta y - x|^2}{\left(T^{2/\alpha} + |X_{s-} + \theta y - x|^2\right)^{2-\frac{p}{2}}} \\
&\le \frac{(3p - p^2)|y|^2}{T^{\frac{2}{\alpha}(1-\frac{p}{2})}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0 \le t \le T} |I_2(t)|\right) &\le CT^{\frac{2}{\alpha}(\frac{p}{2}-1)} \int_0^T \mathbb{E} \int_{|y| \le T^{1/\alpha}} |y|^2 \nu(X_{s-}, dy) ds \\
&\le CT^{\frac{2}{\alpha}(\frac{p}{2}-1)} \int_0^T \int_{|y| \le T^{1/\alpha}} \frac{\kappa_1 |y|^2}{|y|^{d+\alpha}} dy ds \\
&\le CT^{p/\alpha}.
\end{aligned}
$$

For $I_3(t)$, by (4.24) again, we get

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0 \le t \le T} |I_3(t)|\right) &\le \mathbb{E} \int_0^T \int_{|y| > T^{1/\alpha}} |y|^p \nu(X_{s-}, dy) ds \\
&\le \int_0^T \int_{|y| > T^{1/\alpha}} \frac{\kappa_1 |y|^p}{|y|^{d+\alpha}} dy ds \\
&\le CT^{p/\alpha}.
\end{aligned}
$$
(4.27)

Combining (4.25)-(4.27) yields

$$
\mathbb{E}^x\left(\sup_{t \in [0,T]} \left|(|X_t - x|^2 + T^{\frac{2}{\alpha}})^{p/2} - T^{\frac{p}{\alpha}}\right|\right) \le CT^{p/\alpha}.
$$

Hence, for any $r > \alpha$,

$$
\begin{aligned}
\mathbb{P}^x \left\{ \sup_{0 \leq s \leq t} |X_s - x| \geq t^{1/r} \right\} \quad &\leq \quad \frac{\mathbb{E}^x \left[\sup_{s \in [0,t]} |X_s - x|^p \right]}{t^{p/r}} \\
&\leq \quad \frac{\mathbb{E}^x \left[\sup_{s \in [0,t]} |(|X_s - x|^2 + t^{\frac{2}{\alpha}})^{p/2} - t^{\frac{p}{\alpha}} \right] + t^{\frac{p}{\alpha}}}{t^{p/r}} \\
&\leq \quad C t^{p/\alpha - p/r}.
\end{aligned}
$$

Thus Condition (A1) also holds. Hence, the conclusion of Theorem 4.8 follows from Theorem 3.3. □

Remark 4.9. Notice that the estimate of transition density $p(t, x, y)$ in (4.23) holds for any $t \in (0, T]$, which is sufficient to prove condition (A2). But for condition (A3), we need the estimate (4.23) holds for any $t > 0$. Hence the above theorem holds only for $1 < \alpha d$.

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