MODERATE DEVIATIONS FOR A STOCHASTIC WAVE EQUATION IN DIMENSION THREE

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Abstract: In this paper, we prove a central limit theorem and establish a moderate deviation principle for a perturbed stochastic wave equation defined on $[0, T] \times \mathbb{R}^3$. This equation is driven by a Gaussian noise, white in time and correlated in space. The weak convergence approach plays an important role.

Keyword: Stochastic wave equation; Large deviations; Moderate deviations; Central limit theorem.

MSC: 60H15, 60F05, 60F10.

1. Introduction

Since the pioneer work of Freidlin and Wentzell [15], the theory of small perturbation large deviations for stochastic dynamics has been extensively developed, see books [9, 10, 12]. The large deviation principle (LDP for short) for stochastic reaction-diffusion equations driven by the space-time white noise was first obtained by Freidlin [14] and later by Sowers [25], Chenal and Millet [5], Cerrai and Röckner [4] and other authors. Also see [3, 24, 30] and references therein for further development.

Like large deviations, the moderate deviation problems arise in the theory of statistical inference quite naturally. The moderate deviation principle (MDP for short) can provide us with the rate of convergence and a useful method for constructing asymptotic confidence intervals, see [13, 17] and references therein.

Results on the MDP for processes with independent increments were obtained in De Acosta [1], Ledoux [20] and so on. The study of the MDP estimates for other processes has been carried out as well, e.g., Gao [16] for martingales, Wu [29] for Markov processes, Guillin and Liptser [18] for diffusion processes.

The problem of moderate deviations for stochastic partial differential equations has been receiving much attention in very recently years, such as Wang and Zhang [28] for stochastic reaction-diffusion equations, Wang *et al.* [27] for stochastic Navier-Stokes equations, Budhiraja *et al.* [2] and Dong *et al.* [11] for stochastic systems with jumps. Those moderate deviation results are established for the stochastic parabolic equations. However, the hyperbolic case is much more complicated, one difficulty comes from the more complicated stochastic integral, another one comes from the lack of good regularity properties of the Green functions. See [6, 8] for the study of the stochastic wave equation.

Using the weak convergence approach in [3], Ortiz-lópez and Sanz-Solé [23] proved a LDP for a stochastic wave equation defined on $[0,T]\times\mathbb{R}^3$, perturbed by a Gaussian noise which is white in time and correlated in space.

In this paper, we shall study the central limit theorem and moderate deviation principle for the stochastic wave equation in dimension 3.

The rest of this paper is organized as follows. In Section 2, we give the framework of the stochastic wave equation, and state the main results of this paper. In Section 3, we first prove some convergence results and then give the proof of the central limit theorem. In Section 4, we prove the moderate deviation principle by using the weak convergence method.

Throughout the paper, $C(p)$ is a positive constant depending on the parameter p, and C is a positive constant depending on no specific parameter (except T and the Lipschitz constants), whose values may be different from line to line by convention.

We end this section with some notions. For any $T>0$ and $D\subset\mathbb{R}^3$, let $\mathcal{C}([0,T]\times D)$ be the space of all continuous functions from $[0,T]\times D$ to $\mathbb{R}$, and let $\mathcal{C}^\alpha([0,T]\times D)$ be the space of all Hölder continuous functions g of degree α jointly in (t,x), with the Hölder norm

$$\|g\|_\alpha := \sup_{(t,x)\neq(s,y)} \frac{|g(t,x)-g(s,y)|}{(|t-s|+|x-y|)^\alpha},$$

and let

$$\mathcal{C}^{\alpha,0}([0,T]\times D) := \left\{ g\in\mathcal{C}^\alpha([0,T]\times D) : \lim_{\delta\to 0} O_g(\delta)=0\right\},$$

where $O_g(\delta) := \sup_{|t-s|+|x-y|<\delta}\frac{|g(t,x)-g(s,y)|}{(|t-s|+|x-y|)^\alpha}$. Then $\mathcal{C}^{\alpha,0}([0,T]\times D)$ is a Polish space, which is denoted by $\mathcal{E}_\alpha$.

2. Framework and the main results

2.1. **Framework.** Let us give the framework taken from Dalang and Sanz-Solé [8], Ortiz-López and Sanz-Solé [23]. Consider the following stochastic wave equation in spatial dimension $d=3$:

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2}-\Delta\right) u^\varepsilon(t,x) = \sqrt{\varepsilon}\sigma\big(u^\varepsilon(t,x)\big)\dot F(t,x) + b\big(u^\varepsilon(t,x)\big), \\ u^\varepsilon(0,x)=\nu_0(x), \\ \frac{\partial}{\partial t}u^\varepsilon(0,x)=\tilde\nu_0(x) \end{cases} \tag{1}$$

for all $(t,x)\in[0,T]\times\mathbb{R}^3$ ($T>0$ is a fixed constant), where $\varepsilon>0$, the coefficients $\sigma,b:\mathbb{R}\to\mathbb{R}$ are Lipschitz continuous functions, the term Δu^ε denotes the Laplacian of u^ε in the x-variable and the process $\dot F$ is the formal derivative of a Gaussian random field, white in time and correlated in space. Precisely, for any $d\geqslant 1$, let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of Schwartz test functions. $F=(F(\varphi),\varphi\in\mathcal{D}(\mathbb{R}^{d+1}))$ is a Gaussian process defined on some probability space with zero mean and covariance functional

$$E(F(\varphi)F(\psi)) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx) \left(\varphi(s) * \tilde{\psi}(s)\right)(x), \tag{2}$$

where Γ is a non-negative and non-negative definite tempered measure on $\mathbb{R}^d$, $\tilde{\psi}(s)(x) := \psi(s)(-x)$ and the notation " $*$ " means the convolution operator. According to [7], the process F can be extended to a martingale measure

$$M = \left\{M_t(A),\ t \geqslant 0,\ A \in \mathcal{B}_b(\mathbb{R}^d)\right\},$$

where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the collection of all bounded Borel measurable sets in $\mathbb{R}^d$.

Using the tempered measure Γ above, we can define an inner product on $\mathcal{D}(\mathbb{R}^d)$:

$$\langle \varphi, \psi \rangle_{\mathcal{H}} := \int_{\mathbb{R}^d} \Gamma(dx)(\varphi * \tilde{\psi})(x), \quad \forall \varphi, \psi \in \mathcal{D}(\mathbb{R}^d).$$

Let $\mathcal{H}$ be the Hilbert space obtained by the completion of $\mathcal{D}(\mathbb{R}^d)$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, and denote by $\| \cdot \|_{\mathcal{H}}$ the induced norm.

By Walsh's theory of stochastic integration with respect to (w.r.t. for short) martingale measures, for any $t \geqslant 0$ and $h \in \mathcal{H}$, the stochastic integral

$$B_t(h) := \int_0^t \int_{\mathbb{R}^d} h(y) M(ds, dy)$$

is well defined, and

$$\left\{B_t^k := \int_0^t \int_{\mathbb{R}^d} e_k(y) M(ds, dy);\ k \geqslant 1\right\}$$

defines a sequence of independent standard Wiener processes, here $\{e_k\}_{k \geqslant 1}$ is a complete orthonormal system of the Hilbert space $\mathcal{H}$. Thus, $B_t := \sum_{k \geqslant 1} B_t^k e_k$ is a cylindrical Wiener process on $\mathcal{H}$. See [9] or [26].

Hypothesis (H):

(H.1) The coefficients σ and b are real Lipschitz continuous, i.e., there exists some constant $K > 0$ such that

$$|\sigma(x) - \sigma(y)| \leqslant K|x - y|, \quad |b(x) - b(y)| \leqslant K|x - y|, \quad \forall x, y \in \mathbb{R}. \tag{3}$$

(H.2) The spatial covariance measure Γ is absolutely continuous with respect to the Lebesgue measure, and the density is $f(x) = \varphi(x)|x|^{-\beta}, x \in \mathbb{R}^3 \backslash \{0\}$. Here the function φ is bounded and positive, $\varphi \in \mathcal{C}^1(\mathbb{R}^3)$, $\nabla\varphi \in \mathcal{C}_b^\delta(\mathbb{R}^3)$ with $\delta \in]0, 1]$ and $\beta \in]0, 2[$.

(H.3) The initial values $\nu_0, \tilde{\nu}_0$ are bounded, $\nu_0 \in \mathcal{C}^2(\mathbb{R}^3)$, $\nabla\nu_0$ is bounded, $\triangle\nu_0$ and $\tilde{\nu}_0$ are Hölder continuous with degrees $\gamma_1, \gamma_2 \in]0, 1]$, respectively.

According to Dalang and Sanz-Solé [8], under hypothesis (**H**), Eq.(1) admits a unique solution u^ε:

$$\begin{aligned} u^\varepsilon(t,x) =& w(t,x) + \sqrt{\varepsilon}\sum_{k\geqslant 1}\int_0^t \langle G(t-s,x-\cdot)\sigma(u^\varepsilon(s,\cdot)), e_k(\cdot)\rangle_{\mathcal{H}}\, dB_s^k \\ &+ \int_0^t \big[G(t-s) * b(u^\varepsilon(s,\cdot))\big](x)ds, \end{aligned} \tag{4}$$

where

$$w(t,x) := \left(\frac{d}{dt}G(t) * \nu_0\right)(x) + \left(G(t) * \tilde{\nu}_0\right)(x),$$

and $G(t) = \frac{1}{4\pi t}\sigma_t$, σ_t is the uniform surface measure (with total mass $4\pi t^2$) on the sphere of radius t. Furthermore, for any $p \in [2,\infty[$,

$$\sup_{\varepsilon\in]0,1]}\ \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\left[|u^\varepsilon(t,x)|^p\right] < +\infty, \tag{5}$$

and for any

$$\alpha \in \mathcal{I} := \left]0, \gamma_1\wedge\gamma_2\wedge\frac{2-\beta}{2}\wedge\frac{1+\delta}{2}\right[, \tag{6}$$

there exists $C>0$ such that for any $(t,x),(s,y)\in[0,T]\times D$, it holds that

$$E\left[|u^\varepsilon(t,x) - u^\varepsilon(s,y)|^p\right] \leqslant C(|t-s|+|x-y|)^{\alpha p}.$$

Consequently, almost all the sample paths of the process $\{u^\varepsilon(t,x);(t,x)\in[0,T]\times D\}$ are α-Hölder continuous jointly in (t,x). See Dalang and Sanz-Solé [8] or Hu *et al.* [19] for details.

Intuitively, as the parameter ε tends to zero, the solution u^ε of (4) will tend to the solution of the deterministic equation

$$u^0(t,x) = w(t,x) + \int_0^t [G(t-s) * b(u^0(s,\cdot))](x)ds. \tag{7}$$

In this paper, we shall investigate deviations of u^ε from u^0, as ε decreases to 0. That is, the asymptotic behavior of the trajectories,

$$Z^\varepsilon(t,x) := \frac{1}{\sqrt{\varepsilon}h(\varepsilon)}(u^\varepsilon - u^0)(t,x), \quad (t,x)\in[0,T]\times D. \tag{8}$$

(LDP) The case $h(\varepsilon) = 1/\sqrt{\varepsilon}$ provides some large deviation estimates. Ortiz-López and Sanz-Solé [23] proved that the law of the solution u^ε satisfies a LDP, see Theorem 2.1 below.

(CLT) If $h(\varepsilon)$ is identically equal to 1, we are in the domain of the central limit theorem (CLT for short). We will show that $(u^\varepsilon - u^0)/\sqrt{\varepsilon}$ converges as $\varepsilon\to 0^+$ to a random field, see Theorem 2.2 below.

(MDP) To fill in the gap between the central limit theorem scale and the large deviations scale, we will study moderate deviations, that is when the deviation scale satisfies

$$h(\varepsilon)\to+\infty \quad\text{and}\quad \sqrt{\varepsilon}h(\varepsilon)\to 0, \quad \text{as} \quad \varepsilon\to 0. \tag{9}$$

In this case, we will prove that Z^ε satisfies a LDP, see Theorem 2.3 below. This special type of LDP is called the MDP for u^ε, see [10, Section 3.7].

Throughout this paper, we assume (9) is in place.

2.2. **Main results.** Let $\mathcal{H}_T := L^2([0,T];\mathcal{H})$ and consider the usual L^2-norm $\|\cdot\|_{\mathcal{H}_T}$ on this space. For any $h \in \mathcal{H}_T$, we consider the deterministic evolution equation:

$$V^h(t,x) = w(t,x) + \int_0^t \left\langle G(t-s,x-\cdot)\sigma(V^h(s,\cdot)), h(s,\cdot)\right\rangle_{\mathcal{H}} ds + \int_0^t \left[G(t-s) * b(V^h(s,\cdot))\right](x)ds. \tag{10}$$

By [23, Theorem 2.3], Eq.(10) admits a unique solution $V^h =: \mathcal{G}_1(h) \in \mathcal{E}_\alpha$, where $\mathcal{G}_1$ is the solution functional from $\mathcal{H}_T$ to $\mathcal{E}_\alpha$. For any $f \in \mathcal{E}_\alpha$, define

$$I_1(f) = \inf_{h\in\mathcal{H}_T:\mathcal{G}_1(h)=f} \left\{\frac{1}{2}\|h\|^2_{\mathcal{H}_T}\right\}, \tag{11}$$

with the convention $\inf\varnothing = +\infty$.

Ortiz-López and Sanz-Solé [23] proved the following LDP result for u^ε.

Theorem 2.1 (Ortiz-López and Sanz-Solé [23])**.** Under assumption **(H)**, the family $\{u^\varepsilon;\ \varepsilon \in]0,1]\}$ given by (4) satisfies a large deviation principle on $\mathcal{E}_\alpha$ with the speed function ε^{-1} and with the good rate function I_1 given by (11). More precisely,

(a) for any $L > 0$, the set $\{f \in \mathcal{E}_\alpha; I_1(f) \leqslant L\}$ is compact in $\mathcal{E}_\alpha$;
(b) for any closed subset $F \subset \mathcal{E}_\alpha$,

$$\limsup_{\varepsilon\to 0^+} \varepsilon \log \mathbb{P}(u^\varepsilon \in F) \leqslant -\inf_{f\in F} I_1(f);$$

(c) for any open subset $G \subset \mathcal{E}_\alpha$,

$$\liminf_{\varepsilon\to 0^+} \varepsilon \log \mathbb{P}(u^\varepsilon \in G) \geqslant -\inf_{f\in G} I_1(f).$$

In this paper, we further assume the condition (**D**):

the function b is differentiable and its derivative b' is also Lipschitz.

More precisely, there exists a positive constant K' such that

$$|b'(y) - b'(z)| \leqslant K'|y-z|, \quad \text{for all } y,z \in \mathbb{R}. \tag{12}$$

Combined with the Lipschitz continuity of b, we conclude that

$$|b'(z)| \leqslant K, \quad \text{for all } z \in \mathbb{R}. \tag{13}$$

Our first main result is the following central limit theorem.

Theorem 2.2. Under conditions **(H)** and **(D)**, for any $\alpha \in \mathcal{I}$ and $p \geqslant 2$, the random field $(u^\varepsilon - u^0)/\sqrt{\varepsilon}$ converges in L^p to a random field Y on $\mathcal{E}_\alpha$, determined by

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \Delta\right) Y(t,x) = \sigma(u^0(t,x))\dot{F}(t,x) + b'(u^0(t,x))Y(t,x), \\ Y(0,x) = 0, \\ \frac{\partial}{\partial t} Y(0,x) = 0, \quad t \in [0,T], \quad x \in \mathbb{R}^3. \end{cases} \tag{14}$$

For any $\varepsilon > 0$, let $q_\varepsilon := Y/h(\varepsilon)$. Then q_ε satisfies the following equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) q_\varepsilon(t,x) = \frac{1}{h(\varepsilon)}\sigma(u^0(t,x))\dot{F}(t,x) + b'(u^0(t,x))q_\varepsilon(t,x), \tag{15}$$

with the same initial conditions as those of Y.

Notice that Eq.(15) is a particular case of Eq.(1) if its coefficients σ and b are allowed to depend on (t,x). Now, assume that the coefficients σ and b in Eq.(1) depend on (t,x) and they are Lipschitz continuous in the third variable uniformly over $(t,x) \in [0,T]\times\mathbb{R}^3$, that is, $\sigma, b : [0,T] \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ satisfy that for all $u, v \in \mathbb{R}$,

$$\sup_{t\in[0,T], x\in\mathbb{R}^3} \left(|\sigma(t,x,u) - \sigma(t,x,v)| + |b(t,x,u) - b(t,x,v)|\right) \leqslant K|u-v|.$$

By using the same strategies in [8] and [23], we know that the wave equation under above assumption admits a unique solution and the LDP result in Theorem 2.1 also holds. In fact, their proofs in this generalized case are the same as that in [8] and [23], only the notions are need to be changed. For example, see [21, 30] for other type SPDEs.

Hence, q_ε obeys a LDP on $\mathcal{E}_\alpha$ with the speed $h^2(\varepsilon)$ and with the good rate function

$$I(g) = \begin{cases} \inf\{\frac{1}{2}\|h\|^2_{\mathcal{H}_T}; Z^h = g\}; & \text{if } g \in Im(Z^\cdot); \\ +\infty, & \text{otherwise}, \end{cases} \tag{16}$$

where Z^h is the solution of the following deterministic evolution equation

$$\begin{aligned} Z^h(t,x) = &\int_0^t \langle G(t-s, x-\cdot)\sigma(u^0(s,\cdot)), h(s,\cdot)\rangle_{\mathcal{H}} ds \\ &+ \int_0^t G(t-s) * [b'(u^0(s,\cdot))Z^h(s,\cdot)](x) ds. \end{aligned} \tag{17}$$

Our second main result is that $\{(u^\varepsilon - u^0)/[\sqrt{\varepsilon}h(\varepsilon)]\}$ satisfies the same LDP with q_ε, that is the following theorem.

Theorem 2.3. Under conditions **(H)** and **(D)**, the family $\{(u^\varepsilon - u^0)/[\sqrt{\varepsilon}h(\varepsilon)];\ \varepsilon \in]0,1]\}$ satisfies a large deviation principle on $\mathcal{E}_\alpha$ with the speed function $h^2(\varepsilon)$ and with the good rate function I given by (16).

3. Proof of Theorem 2.2

3.1. **Convergence of solutions.** For any function $\phi : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$, let

$$|\phi|_{t,\infty} := \sup\left\{|\phi(s,x)| : (s,x) \in [0,t] \times \mathbb{R}^3\right\}.$$

The next result is concerned with the convergence of u^ε as $\varepsilon \to 0$.

Proposition 3.1. Under (**H**), for any $p \geqslant 2$, there exists some positive constant $C(p,K,T)$ depending on p,K,T such that

$$\mathbb{E}\left[|u^\varepsilon - u^0|^p_{T,\infty}\right] \leqslant \varepsilon^{\frac{p}{2}} C(p,K,T) \to 0, \quad \text{as } \varepsilon \to 0. \tag{18}$$

Proof. Since for any $0 \leqslant t \leqslant T$,

$$\begin{aligned} u^\varepsilon(t,x) - u^0(t,x) =& \int_0^t \left[G(t-s) * \left(b(u^\varepsilon(s,\cdot)) - b(u^0(s,\cdot))\right)\right](x)ds \\ &+ \sqrt{\varepsilon}\sum_{k\geqslant 1}\int_0^t \langle G(t-s,x-\cdot)\sigma(u^\varepsilon(s,\cdot)), e_k(\cdot)\rangle_{\mathcal{H}} dB_s^k \\ =&: T_1^\varepsilon(t,x) + T_2^\varepsilon(t,x), \end{aligned}$$

we obtain that for any $p \geqslant 2$,

$$|u^\varepsilon - u^0|^p_{t,\infty} \leqslant 2^{p-1}\left(|T_1^\varepsilon|^p_{t,\infty} + |T_2^\varepsilon|^p_{t,\infty}\right). \tag{19}$$

By the Hölder's inequality w.r.t the measure on $[0,T] \times \mathbb{R}^3$ given by $G(t-s,dy)ds$ and the Lipschitz continuity of b, we obtain that

$$\begin{aligned} &\mathbb{E}\left[|T_1^\varepsilon|^p_{t,\infty}\right] \\ \leqslant& C(K)\left(\int_0^t\int_{\mathbb{R}^3} G(t-s,dy)ds\right)^{p-1} \times \int_0^t \mathbb{E}\left[|u^\varepsilon - u^0|^p_{s,\infty}\right]\left(\int_{\mathbb{R}^3} G(t-s,dy)\right)ds \\ \leqslant& C(p,K,T)\int_0^t \mathbb{E}\left[|u^\varepsilon - u^0|^p_{s,\infty}\right]ds. \end{aligned} \tag{20}$$

By Burkholder's inequality, Hölder's inequality, the Lipschitz continuity of σ and (5), we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{k\geqslant 1}\int_0^t \langle G(t-s,x-\cdot)\sigma(u^\varepsilon(s,\cdot)),e_k(\cdot)\rangle_{\mathcal{H}} dB_s^k\right|^p\right]\\
\leqslant & C(p)\mathbb{E}\left[\left|\int_0^t \|G(t-s,x-\cdot)\sigma(u^\varepsilon(s,\cdot))\|_{\mathcal{H}}^2 ds\right|^{\frac{p}{2}}\right]\\
\leqslant & C(p,K)\left(\int_0^t\int_{\mathbb{R}^3}|\mathcal{F}G(t-s)(\xi)|^2\mu(d\xi)ds\right)^{\frac{p}{2}-1}\\
&\times\int_0^t\left[\left(1+\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}\mathbb{E}\left[|u^\varepsilon(r,z)|^p\right]\right)\int_{\mathbb{R}^3}|\mathcal{F}G(t-s)(\xi)|^2\mu(d\xi)\right]ds\\
\leqslant & C(p,K,T)\int_0^t\left(1+\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}\mathbb{E}[|u^\varepsilon(r,z)|^p]\right)ds<\infty,
\end{aligned}$$

where (5) is used in the last inequality. The above estimate yields that

$$\mathbb{E}[|T_2^\varepsilon|_{T,\infty}^p]\leqslant \varepsilon^{\frac{p}{2}}C(p,K,T). \tag{21}$$

Putting (19)-(21) together, and using the Gronwall's inequality, we obtain the desired inequality (18).

The proof is complete. □

3.2. **The proof of CLT.**

The following lemma is a consequence of the Garsia-Rodemich-Rumsey's theorem, see Millet and Sanz-Solé [22, Lemma A2].

Lemma 3.2. Let $\{V^\varepsilon(t,x);\ (t,x)\in[0,T]\times D\}$ be a family of real-valued stochastic processes. Assume that there exists $p\in]1,\infty[$ such that

(A1). for any $(t,x)\in[0,T]\times D$,
$$\lim_{\varepsilon\to 0}\mathbb{E}\left[|V^\varepsilon(t,x)|^p\right]=0;$$

(A2). there exists $\gamma_0>0$ such that for any $(t,x),(s,y)\in[0,T]\times D$,
$$\mathbb{E}\left[|V^\varepsilon(t,x)-V^\varepsilon(s,y)|^p\right]\leqslant C(|t-s|+|x-y|)^{\gamma_0+4},$$
where C is a positive constant independent of ε.

Then for any $\alpha\in]0,\gamma_0/p[$, $r\in[1,p[$, we have
$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\|V^\varepsilon\|_\alpha^r\right]=0.$$

Proof of Theorem 2.2. Denote $Y^\varepsilon:=(u^\varepsilon-u^0)/\sqrt{\varepsilon}$. We will prove that for any $\alpha\in\mathcal{I},\ p\geqslant 2$,
$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\|Y^\varepsilon-Y\|_\alpha^p\right]=0. \tag{22}$$

To this end, we only need to verify (A1) and (A2) in Lemma 3.2 for $V^\varepsilon := Y^\varepsilon - Y$. Notice that

$$\begin{aligned}
&Y^\varepsilon(t,x) - Y(t,x)\\
=&\sum_{k\geqslant 1}\int_0^t \left\langle G(t-s,x-\cdot)\left(\sigma(u^\varepsilon(s,\cdot))-\sigma(u^0(s,\cdot))\right), e_k(\cdot)\right\rangle_{\mathcal{H}} dB_s^k\\
&+\int_0^t G(t-s)*\left(\frac{b(u^\varepsilon(s,\cdot))-b(u^0(s,\cdot))}{\sqrt{\varepsilon}} - b'(u^0(s,\cdot))Y(s,\cdot)\right)(x)ds\\
=:&I_1^\varepsilon(t,x)+I_2^\varepsilon(t,x)+I_3^\varepsilon(t,x),
\end{aligned}\tag{23}$$

where

$$\begin{aligned}
I_1^\varepsilon(t,x) &:= \sum_{k\geqslant 1}\int_0^t \left\langle G(t-s,x-\cdot)\left(\sigma(u^\varepsilon(s,\cdot))-\sigma(u^0(s,\cdot))\right), e_k(\cdot)\right\rangle_{\mathcal{H}} dB_s^k,\\
I_2^\varepsilon(t,x) &:= \int_0^t G(t-s)*\left(\frac{b(u^\varepsilon(s,\cdot))-b(u^0(s,\cdot))}{\sqrt{\varepsilon}} - b'(u^0(s,\cdot))Y^\varepsilon(s,\cdot)\right)(x)ds,\\
I_3^\varepsilon(t,x) &:= \int_0^t G(t-s)*\left[b'(u^0(s,\cdot)\big(Y^\varepsilon(s,\cdot)-Y(s,\cdot)\big)\right](x)ds.
\end{aligned}$$

Next, we shall verify (A1) and (A2) for $I_i^\varepsilon, i=1,2,3$.

Step 1. Following the similar calculation in the proof of (21) and using the Lipschitz continuity of σ, we can deduce that for any $p\geqslant 2$,

$$\mathbb{E}\left[|I_1^\varepsilon|_{T,\infty}^p\right] \leqslant C(p,K,T)\mathbb{E}\left[|u^\varepsilon-u^0|_{T,\infty}^p\right] \leqslant \varepsilon^{\frac{p}{2}}C(p,K,T),\tag{24}$$

where we have used Proposition 3.1 in the last inequality.

Notice that $u^\varepsilon = u^0+\sqrt{\varepsilon}Y^\varepsilon$. By the mean theorem for derivatives, there exists a random field $v^\varepsilon(t,x)$ taking values in $(0,1)$ such that

$$\frac{1}{\sqrt{\varepsilon}}\left[b(u^\varepsilon)-b(u^0)\right] = b'\big(u^0+\sqrt{\varepsilon}v^\varepsilon Y^\varepsilon\big)Y^\varepsilon.$$

By the Lipschitz continuity of b', we have

$$\frac{1}{\sqrt{\varepsilon}}\left[b(u^\varepsilon)-b(u^0)\right] - b'(u^0)Y^\varepsilon = \left[b'\big(u^0+\sqrt{\varepsilon}v^\varepsilon Y^\varepsilon\big)-b'(u^0)\right]Y^\varepsilon \leqslant \sqrt{\varepsilon}K'|Y^\varepsilon|^2.\tag{25}$$

Hence

$$|I_2^\varepsilon(t,x)| \leqslant \sqrt{\varepsilon}K'\int_0^t G(t-s)*|Y^\varepsilon(s,\cdot)|^2(x)ds.$$

By Hölder's inequality and Proposition 3.1, we obtain that for any $p\geqslant 2$,

$$\begin{aligned}
&\mathbb{E}\left[|I_2^\varepsilon|_{t,\infty}^p\right]\\
\leqslant&\varepsilon^{\frac{p}{2}}K'^p\left(\int_0^t\int_{\mathbb{R}^3}G(t-s,dy)ds\right)^{p-1}\times\int_0^t \mathbb{E}\left[|Y^\varepsilon|_{s,\infty}^{2p}\right]\left(\int_{\mathbb{R}^3}G(t-s,dy)\right)ds\\
\leqslant&\varepsilon^{\frac{p}{2}}C(p,K,K',T).
\end{aligned}\tag{26}$$

By Hölder's inequality and (13), we deduce that for any $p \geqslant 2$,

$$\begin{aligned}&\mathbb{E}\left[|I_3^\varepsilon|_{t,\infty}^p\right]\\&\leqslant K^p\left(\int_0^t\int_{\mathbb{R}^3}G(t-s,dy)\right)^{p-1}\times\int_0^t\mathbb{E}\left[|Y^\varepsilon-Y|_{s,\infty}^p\right]\left(\int_{\mathbb{R}^3}G(t-s,dy)\right)ds\\&\leqslant C(p,K,T)\int_0^t\mathbb{E}\left[|Y^\varepsilon-Y|_{s,\infty}^p\right]ds.\end{aligned}\tag{27}$$

Putting (23), (24), (26) and (27) together, we have

$$\mathbb{E}\left[|Y^\varepsilon-Y|_{t,\infty}\right]^p\leqslant C(p,K,K',T)\left(\varepsilon^{\frac{p}{2}}+\int_0^t\mathbb{E}\left[|Y^\varepsilon-Y|_{s,\infty}^p\right]ds\right).$$

By Gronwall's inequality, we have

$$\mathbb{E}\left[|Y^\varepsilon-Y|_{T,\infty}^p\right]\leqslant\varepsilon^{\frac{p}{2}}C(p,K,K',T)\to0,\quad\text{as }\varepsilon\to0,\tag{28}$$

which, in particular, implies (A1) in Lemma 3.2.

Step 2. Notice that Y^ε satisfies that

$$\begin{aligned}Y^\varepsilon(t,x)=&\sum_{k\geqslant1}\int\left\langle G(t-s,x-\cdot)\sigma\left(u^0(s,\cdot)+\sqrt{\varepsilon}Y^\varepsilon(s,\cdot)\right),e_k(\cdot)\right\rangle_{\mathcal{H}}dB_s^k\\&+\int_0^tG(t-s)*\frac{b\big(u^0(s,\cdot)+\sqrt{\varepsilon}Y^\varepsilon(s,\cdot)\big)-b(u^0(s,\cdot))}{\sqrt{\varepsilon}}(x)ds.\end{aligned}\tag{29}$$

For any $\varepsilon\in]0,1]$, set the mapping $\tilde{\sigma}_\varepsilon,\tilde{b}_\varepsilon:[0,T]\times\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}\tilde{\sigma}_\varepsilon(t,x,r):=&\sigma(u^0(t,x)+\sqrt{\varepsilon}r),\\\tilde{b}_\varepsilon(t,x,r):=&\frac{1}{\sqrt{\varepsilon}}\left[b(u^0(t,x)+\sqrt{\varepsilon}r)-b(u^0(t,x))\right].\end{aligned}$$

By the Lipschitz continuity of σ and b, we know that $\tilde{\sigma}_\varepsilon$ and $\tilde{b}_\varepsilon$ are Lipschitz continuous in the third variable uniformly over $(t,x)\in[0,T]\times\mathbb{R}^3$ and $\varepsilon\in]0,1]$. Using the same strategy of the proof for Theorem 2.3 in [23], one can obtain that for any $\alpha\in\mathcal{I}$, $p>4/\alpha$,

$$\sup_{\varepsilon\in]0,1]}\mathbb{E}\left[|Y^\varepsilon(t,x)-Y^\varepsilon(s,y)|^p\right]\leqslant C(|t-s|+|x-y|)^{\alpha p}\tag{30}$$

and

$$\mathbb{E}\left[|Y(t,x)-Y(s,y)|^p\right]\leqslant C(|t-s|+|x-y|)^{\alpha p}.\tag{31}$$

Putting (30) and (31) together, we obtain the Hölder continuity of V^ε, that is for any $\alpha\in\mathcal{I}$, $p>4/\alpha$, there exists a constant $C>0$ such that

$$\sup_{\varepsilon\in]0,1]}\mathbb{E}\left[|(Y^\varepsilon(t,x)-Y(t,x))-(Y^\varepsilon(s,y)-Y(s,y))|^p\right]\leqslant C(|t-s|+|x-y|)^{\alpha p}.\tag{32}$$

By Lemma 3.2, (28) and (32), we obtain that for any $\tilde{\alpha}\in]0,\alpha-4/p[$, $r\in[1,p[$,

$$\lim_{\varepsilon\to0}\mathbb{E}[\|Y^\varepsilon-Y\|_{\tilde{\alpha}}^r]=0.$$

By the arbitrariness of $p > 4/\alpha, r \in [1, p[$, we get the desired result in Theorem 2.2 for any $\alpha \in \mathcal{I}, p \geqslant 2$.

The proof is complete. $\square$

4. Proof of Theorem 2.3

Let $\mathcal{P}$ denote the set of predictable processes belonging to $L^2(\Omega \times [0,T];\ \mathcal{H})$. For any $N > 0$, we define

$$\mathcal{H}_T^N := \{h \in \mathcal{H}_T : \|h\|_{\mathcal{H}_T} \leqslant N\},$$
$$\mathcal{P}_T^N := \{v \in \mathcal{P} : v \in \mathcal{H}_T^N, a.s.\},$$

and we endow $\mathcal{H}_T^N$ with the weak topology of $\mathcal{H}_T$.

Let $Z^\varepsilon := (u^\varepsilon - u^0)/(\sqrt{\varepsilon}h(\varepsilon))$. Then

$$\begin{aligned} Z^\varepsilon(t,x) =& \frac{1}{h(\varepsilon)} \sum_{k \geqslant 1} \int_0^t \left\langle G(t-s, x-\cdot)\sigma\big(u^0(s,\cdot) + \sqrt{\varepsilon}h(\varepsilon)Z^\varepsilon(s,\cdot)\big), e_k(\cdot)\right\rangle_{\mathcal{H}} dB_s^k \\ &+ \int_0^t G(t-s) * \left(\frac{b\big(u^0(s,\cdot) + \sqrt{\varepsilon}h(\varepsilon)Z^\varepsilon(s,\cdot)\big) - b(u^0(s,\cdot))}{\sqrt{\varepsilon}h(\varepsilon)} \right)(x)ds. \end{aligned} \tag{33}$$

For any $\varepsilon \in]0,1]$ and $v \in \mathcal{P}_T^N$, consider the controlled equation $Z^{\varepsilon,v}$ defined by

$$\begin{aligned} &Z^{\varepsilon,v}(t,x) \\ =& \frac{1}{h(\varepsilon)} \sum_{k \geqslant 1} \int_0^t \left\langle G(t-s, x-\cdot)\sigma\big(u^0(s,\cdot) + \sqrt{\varepsilon}h(\varepsilon)Z^{\varepsilon,v}(s,\cdot)\big), e_k(\cdot)\right\rangle_{\mathcal{H}} dB_s^k \\ &+ \int_0^t \left\langle G(t-s, x-\cdot)\sigma\left(u^0(s,\cdot) + \sqrt{\varepsilon}h(\varepsilon)Z^{\varepsilon,v}(s,\cdot)\right), v(s,\cdot)\right\rangle_{\mathcal{H}} ds \\ &+ \int_0^t G(t-s) * \left(\frac{b\big(u^0(s,\cdot) + \sqrt{\varepsilon}h(\varepsilon)Z^{\varepsilon,v}(s,\cdot)\big) - b(u^0(s,\cdot))}{\sqrt{\varepsilon}h(\varepsilon)} \right)(x)ds. \end{aligned} \tag{34}$$

Following the proof of Theorem 2.3 in [23], similarly to (29), one can prove that Eq.(34) admits a unique solution $\{Z^{\varepsilon,v}(t,x);\ (t,x) \in [0,T] \times \mathbb{R}^3\}$ satisfying that for any $p \in [2, \infty[$,

$$\sup_{\varepsilon \in]0,1]} \sup_{v \in \mathcal{P}_T^N} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}\left[|Z^{\varepsilon,v}(t,x)|^p\right] < \infty, \tag{35}$$

and there exists $C > 0$ such that for $(t,x),(s,y) \in [0,T] \times D$ and $\alpha \in \mathcal{I}$,

$$\sup_{\varepsilon \in]0,1]} \sup_{v \in \mathcal{P}_T^N} \mathbb{E}\left[|Z^{\varepsilon,v}(t,x) - Z^{\varepsilon,v}(s,y)|^p\right] \leqslant C(|t-s| + |x-y|)^{\alpha p}. \tag{36}$$

Particularly, taking $v \equiv 0$, we know that for any $p \in [2, \infty[$,

$$\sup_{\varepsilon \in]0,1]} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}\left[|Z^\varepsilon(t,x)|^p\right] < \infty, \tag{37}$$

and

$$\sup_{\varepsilon \in]0,1]} \mathbb{E}\left[|Z^\varepsilon(t,x) - Z^\varepsilon(s,y)|^p\right] \leqslant C(|t-s| + |x-y|)^{\alpha p}. \tag{38}$$

Recall Z^h defined in Eq.(17). Consider the following conditions:

(a) For any family $\{v^\varepsilon;\ \varepsilon>0\}\subset\mathcal{P}_T^N$ which converges in distribution as $\varepsilon\to 0$ to $v\in\mathcal{P}_T^N$, as $\mathcal{H}_T^N$-valued random variables,
$$\lim_{\varepsilon\to 0} Z^{\varepsilon,v^\varepsilon}=Z^v \quad \text{in distribution},$$
as $\mathcal{E}_\alpha$-valued random variables, where Z^v denotes the solution of Eq.(17) corresponding to the $\mathcal{H}_T^N$-valued random variable v (instead of a deterministic function h);

(b) The set $\{Z^h;\ h\in\mathcal{H}_T^N\}$ is a compact set of $\mathcal{E}_\alpha$, where Z^h is the solution of Eq.(17).

The proof of Theorem 2.3. Applying [3, Theorem 6] to the solution functional $\mathcal{G}^\varepsilon$: $\mathcal{C}([0,T];\mathbb{R}^\infty)\to\mathcal{E}_\alpha : \mathcal{G}^\varepsilon(\sqrt{\varepsilon}B):=Z^\varepsilon$, the solution of Eq.(33), and the solution functional $\mathcal{G}^0:\mathcal{H}_T\to\mathcal{E}^\alpha$, $\mathcal{G}^0(h):=Z^h$, the solution of Eq.(17), conditions (a) and (b) above imply the MDP result in Theorem 2.3. The verification of condition (a) will be given in Proposition 4.1. Since $\mathcal{H}_T^N$ is compact in the weak topology of $\mathcal{H}$, condition (b) follows from the continuity of the mapping $\mathcal{H}_T^N\ni h\to Z^h\in\mathcal{E}_\alpha$ which will be proved in Proposition 4.2. The proof is complete. □

Proposition 4.1. Under conditions **(H)** and **(D)**, for any family $\{v^\varepsilon;\ \varepsilon>0\}\subset\mathcal{P}_T^N$ which converges in distribution as $\varepsilon\to 0$ to $v\in\mathcal{P}_T^N$, as $\mathcal{H}_T^N$-valued random variables, it holds that
$$\lim_{\varepsilon\to 0} Z^{\varepsilon,v^\varepsilon}=Z^v, \quad \text{in distribution},$$
as $\mathcal{E}_\alpha$-valued random variables.

Proof. By the Skorokhod representation theorem, there exist a probability space $(\bar\Omega,\bar{\mathcal{F}},(\bar{\mathcal{F}}_t),\bar{\mathbb{P}})$, and, on this basis, a sequence of independent Brownian motions $\bar B=(\bar B^k)_{k\geqslant 1}$ and also a family of $\bar{\mathcal{F}}_t$-predictable processes $\{\bar v^\varepsilon;\ \varepsilon>0\}$, $\bar v$ taking values on $\mathcal{H}_T^N$, $\bar{\mathbb{P}}$-a.s., such that the joint law of (v^ε,v,B) under $\mathbb{P}$ coincides with that of $(\bar v^\varepsilon,\bar v,\bar B)$ under $\bar{\mathbb{P}}$ and
$$\lim_{\varepsilon\to 0}\langle\bar v^\varepsilon-\bar v,g\rangle_{\mathcal{H}_T}=0,\quad \forall g\in\mathcal{H}_T,\bar{\mathbb{P}}\text{-a.s.}. \tag{39}$$
Let $\bar Z^{\varepsilon,\bar v^\varepsilon}$ be the solution to a similar equation as (34) replacing v by $\bar v^\varepsilon$ and B by $\bar B$, and let $\bar Z^{\bar v}$ be the solution of Eq.(17) corresponding to the $\mathcal{H}_T^N$-valued random variable $\bar v$ (instead of a deterministic function h).

Now, we shall prove that for any $p\geqslant 2,\alpha\in\mathcal{I}$,
$$\lim_{\varepsilon\to 0}\bar{\mathbb{E}}\left[\|\bar Z^{\varepsilon,\bar v^\varepsilon}-\bar Z^{\bar v}\|_\alpha^p\right]=0, \tag{40}$$
which implies the validity of Proposition 4.1. Here the expectation in (40) refers to the probability $\bar P$.

From now on, we drop the bars in the notation for the sake of simplicity, and we denote
$$X^{\varepsilon,v^\varepsilon,v}:=Z^{\varepsilon,v^\varepsilon}-Z^v.$$
By (35) and (37), we know that for any $p\geqslant 2$,
$$\sup_{\varepsilon\in]0,1]}\sup_{v\in\mathcal{H}_T^N}\sup_{(t,x)\in[0,T]\times\mathbb{R}^3}\mathbb{E}\left[|X^{\varepsilon,v^\varepsilon,v}(t,x)|^p\right]<\infty.$$

According to Lemma 3.2, to prove (40), it is sufficient to prove that for any $(t,x),(s,y)\in[0,T]\times D$ and $p\geqslant 2$, the following conditions hold:

(1) Pointwise convergence:

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[|X^{\varepsilon,v^\varepsilon,v}(t,x)|^p\right]=0. \tag{41}$$

(2) Estimation of the increments: there exists a positive constant C satisfied that

$$\sup_{\varepsilon\in]0,1]}\mathbb{E}\left[\left|X^{\varepsilon,v^\varepsilon,v}(t,x)-X^{\varepsilon,v^\varepsilon,v}(s,y)\right|^p\right]\leqslant C(|t-s|+|x-y|)^{\alpha p}. \tag{42}$$

By (36) and (38), it is easy to obtain (42). Now, it remains to prove (41).

Notice that for any $(t,x)\in[0,T]\times\mathbb{R}^3$,

$$\begin{aligned}
&Z^{\varepsilon,v^\varepsilon}(t,x)-Z^v(t,x)\\
=&\frac{1}{h(\varepsilon)}\sum_{k\geqslant 1}\int_0^t\left\langle G(t-s,x-\cdot)\sigma\big(u^0(s,\cdot)+\sqrt{\varepsilon}h(\varepsilon)Z^{\varepsilon,v^\varepsilon}(s,\cdot)\big),e_k(\cdot)\right\rangle_{\mathcal{H}}dB_s^k\\
&+\left\{\int_0^t\left\langle G(t-s,x-\cdot)\sigma\big(u^0(s,\cdot)+\sqrt{\varepsilon}h(\varepsilon)Z^{\varepsilon,v^\varepsilon}(s,\cdot)\big),v^\varepsilon(s,\cdot)\right\rangle_{\mathcal{H}}ds\right.\\
&\qquad\left.-\int_0^t\left\langle G(t-s,x-\cdot)\sigma\big(u^0(s,\cdot)\big),v(s,\cdot)\right\rangle_{\mathcal{H}}ds\right\}\\
&+\left\{\int_0^t G(t-s)*\left[\frac{b\big(u^0(s,\cdot)+\sqrt{\varepsilon}h(\varepsilon)Z^{\varepsilon,v^\varepsilon}(s,\cdot)\big)-b(u^0(s,\cdot))}{\sqrt{\varepsilon}h(\varepsilon)}\right](x)ds\right.\\
&\qquad\left.-\int_0^t\left\{G(t-s)*\left[b'(u^0(s,\cdot))Z^v(s,\cdot)\right](x)\right\}ds\right\}\\
=:&A_1^\varepsilon(t,x)+A_2^\varepsilon(t,x)+A_3^\varepsilon(t,x).
\end{aligned} \tag{43}$$

Step 1. For the first term $A_1^\varepsilon(t,x)$, noticing that u^0 is bounded, by Burkholder's inequality, Hölder's inequality and the linear growth property of σ, we have

$$\begin{aligned}
&\sup_{(t,x)\in[0,T]\times\mathbb{R}^3}\mathbb{E}\left[|A_1^\varepsilon(t,x)|^p\right]\\
\leqslant&\frac{1}{h^p(\varepsilon)}\sup_{(t,x)\in[0,T]\times\mathbb{R}^3}\mathbb{E}\left(\int_0^t\left\|G(t-s,x-\cdot)\sigma\big(u^0(s,\cdot)+\sqrt{\varepsilon}h(\varepsilon)Z^{\varepsilon,v^\varepsilon}(s,\cdot)\big)\right\|_{\mathcal{H}}^2ds\right)^{\frac{p}{2}}\\
\leqslant&\frac{C(p,u^0,K)}{h^p(\varepsilon)}\left(\int_0^t\int_{\mathbb{R}^3}|\mathcal{F}G(t-s)(\xi)|^2\mu(d\xi)ds\right)^{\frac{p}{2}-1}\\
&\times\int_0^t\left(1+\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}\mathbb{E}\left[|Z^{\varepsilon,v^\varepsilon}(r,z)|^p\right]\right)\int_{\mathbb{R}^3}|\mathcal{F}G(t-s)(\xi)|^2\mu(d\xi)ds\\
\leqslant&\frac{C(p,u^0,K,T)}{h^p(\varepsilon)},
\end{aligned} \tag{44}$$

where (35) is used in the last inequality.

Step 2. The second term is further divided into two terms:

$$\begin{aligned}A_2^\varepsilon(t,x) =& \int_0^t \left\langle G(t-s,x-\cdot)\left[\sigma\left(u^0(s,\cdot)+\sqrt{\varepsilon}h(\varepsilon)Z^{\varepsilon,v^\varepsilon}(s,\cdot)\right)-\sigma(u^0(s,\cdot))\right], v^\varepsilon(s,\cdot)\right\rangle_{\mathcal{H}} ds \\ &+\int_0^t \left\langle G(t-s,x-\cdot)\sigma(u^0(s,\cdot)), v^\varepsilon(s,\cdot)-v(s,\cdot)\right\rangle_{\mathcal{H}} ds \\ =:& A_{2,1}^\varepsilon(t,x)+A_{2,2}^\varepsilon(t,x).\end{aligned} \tag{45}$$

By Cauchy-Schwarz's inequality, Hölder's inequality, the Lipschitz continuity of σ and the fact that $v^\varepsilon \in \mathcal{P}_T^N$, we obtain that

$$\begin{aligned}&\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}[|A_{2,1}^\varepsilon(t,x)|^p] \\ \leqslant& \varepsilon^{\frac{p}{2}} h^p(\varepsilon) K^p \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\left[\left(\int_0^t \left\|G(t-s,x-\cdot)Z^{\varepsilon,v^\varepsilon}(s,\cdot)\right\|_{\mathcal{H}}^2 ds\right)^{\frac{p}{2}} \cdot \left(\int_0^t \|v^\varepsilon(s,\cdot)\|_{\mathcal{H}}^2 ds\right)^{\frac{p}{2}}\right] \\ \leqslant& \varepsilon^{\frac{p}{2}} h^p(\varepsilon) N^p K^p \left(\int_0^t \int_{\mathbb{R}^3} |\mathcal{F}G(t-s)(\xi)|^2 \mu(d\xi) ds\right)^{\frac{p}{2}-1} \\ &\times \int_0^t \left(\sup_{(r,z)\in[0,s]\times\mathbb{R}^3} \mathbb{E}\left[|Z^{\varepsilon,v^\varepsilon}(r,z)|^p\right]\right) \int_{\mathbb{R}^3} |\mathcal{F}G(t-s)(\xi)|^2 \mu(d\xi) ds \\ \leqslant& \varepsilon^{\frac{p}{2}} h^p(\varepsilon) C(N,p,K,T),\end{aligned} \tag{46}$$

where (35) is also used in the last inequality.

Next, we will show that

$$\lim_{\varepsilon\to 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}\left[\left|A_{2,2}^\varepsilon(t,x)\right|^p\right] = 0. \tag{47}$$

By Hölder's inequality with respect to the measure on $\mathbb{R}^3$ given by $|\mathcal{F}G(t-s)(\xi)|^2\mu(d\xi)$, we obtain that for any $(t,x) \in [0,T]\times\mathbb{R}^3$,

$$\begin{aligned}&\int_0^t \|G(t-s,x-\cdot)\sigma(u^0(s,\cdot))\|_{\mathcal{H}}^2 ds \\ \leqslant& C\int_0^t ds \left(\int_{\mathbb{R}^3} |\mathcal{F}G(t-s)(\xi)|^2 \mu(d\xi)\right) \times \left(1+\sup_{(s,y)\in[0,T]\times\mathbb{R}^3} |u^0(s,y)|^2\right) \\ <& +\infty.\end{aligned}$$

This implies that for any $(t,x) \in [0,T]\times\mathbb{R}^3$, the function $\{G(t-s,x-y)\sigma(u^0(s,y));\ (s,y) \in [0,T]\times\mathbb{R}^3\}$ takes its values in $\mathcal{H}_T$. Since $v^\varepsilon \to v$ weakly in $\mathcal{H}_T^N$, we know that

$$\lim_{\varepsilon\to 0} A_{2,2}^\varepsilon(t,x) = 0, \quad \text{a.s.}. \tag{48}$$

By Cauchy-Schwarz's inequality on the Hilbert space $\mathcal{H}_T$ and the facts that $\|v^\varepsilon\|_{\mathcal{H}_T} \leqslant N, \|v\|_{\mathcal{H}_T} \leqslant N$, we obtain that

$$\begin{aligned}
|A^\varepsilon_{2,2}(t,x)| \leqslant & \left(\int_0^t \|G(t-s,x-\cdot)\sigma(u^0(s,\cdot))\|^2_{\mathcal{H}} ds\right)^{\frac{1}{2}} \cdot \left(\int_0^t \|v^\varepsilon(s,\cdot)-v(s,\cdot)\|^2_{\mathcal{H}} ds\right)^{\frac{1}{2}} \\
\leqslant & C(N,T)\left(\int_0^t \|G(t-s,x-\cdot)\sigma(u^0(s,\cdot))\|^2_{\mathcal{H}} ds\right)^{\frac{1}{2}} \\
\leqslant & C(N,K,T) < +\infty, \qquad (49)
\end{aligned}$$

here $C(N,K,T)$ is independent of (ε,t,x). By the Hölder regularity of the path-wise integral

$$\int_0^t \left\langle G(t-s,x-\cdot)\sigma(u^0(s,\cdot)), v^\varepsilon(s,\cdot)-v(s,\cdot)\right\rangle_{\mathcal{H}} ds,$$

(see [23, Section 2]), we know that a.s., $\{A^\varepsilon_{2,2}(t,x),(t,x)\in[0,T]\times D\}$ has Hölder continuous sample paths of degree $\alpha\in\mathcal{I}$ jointly in (t,x), and

$$\sup_{\varepsilon\in]0,1]} \|A^\varepsilon_{2,2}\|_\alpha < \infty, \quad \text{a.s..}$$

This, in particular, implies that $\{A^\varepsilon_{2,2}(t,x);\ (t,x)\in[0,T]\times D\}$ is equicontinuous. By the arbitrariness of $D\subset\mathbb{R}^3$, we known that $\{A^\varepsilon_{2,2}(t,x);\ (t,x)\in[0,T]\times\mathbb{R}^3\}$ is equiv-continuous. Thus, by (48), (49) and Arzelà-Ascoli Theorem, we know that $A^\varepsilon_{2,2}$ converges to 0 in the space $\mathcal{C}([0,T]\times\mathbb{R}^3;\mathbb{R})$, a.s. as $\varepsilon\to0$. This implies that

$$\lim_{\varepsilon\to0}\sup_{(t,x)\in[0,T]\times\mathbb{R}^3}|A^\varepsilon_{2,2}(t,x)| = 0, \qquad \text{a.s..} \qquad (50)$$

By the dominated convergence theorem, (49) and (50), we obtain that

$$\lim_{\varepsilon\to0}\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times\mathbb{R}^3}|A^\varepsilon_{2,2}(t,x)|^p\right] = 0,$$

which is stronger than (47).

Step 3. For the third term A^ε_3, using the same argument in the proof of (25), we have

$$\begin{aligned}
&|A^\varepsilon_3(t,x)| \\
\leqslant & \left|\int_0^t \left\{G(t-s) * \left[\frac{b\big(u^0(s,\cdot)+\sqrt{\varepsilon}h(\varepsilon)Z^{\varepsilon,v^\varepsilon}(s,\cdot)\big)-b(u^0(s,\cdot))}{\sqrt{\varepsilon}h(\varepsilon)} - b'(u^0(s,\cdot))Z^{\varepsilon,v^\varepsilon}(s,\cdot)\right](x)\right\} ds\right| \\
& + \left|\int_0^t \left\{G(t-s) * \left[b'(u^0(s,\cdot))\big(Z^{\varepsilon,v^\varepsilon}(s,\cdot)-Z^v(s,\cdot)\big)\right](x)\right\} ds\right| \\
\leqslant & C(K')\sqrt{\varepsilon}h(\varepsilon)\int_0^t G(t-s) * \left|Z^{\varepsilon,v^\varepsilon}(s,\cdot)\right|^2(x)ds \\
& + C(K)\int_0^t G(t-s) * \left|Z^{\varepsilon,v^\varepsilon}(s,\cdot)-Z^v(s,\cdot)\right|(x)ds.
\end{aligned}$$

By Hölder's inequality with respect to the Lebesgue measure on $[0,t]\times\mathbb{R}^3$ and (35), we have

$$\begin{aligned}
&\sup_{(t,x)\in[0,T]\times\mathbb{R}^3}\mathbb{E}\left[|A_3^\varepsilon(t,x)|^p\right]\\
\leqslant&\varepsilon^{\frac{p}{2}}h^p(\varepsilon)C(p,K')\sup_{(t,x)\in[0,T]\times\mathbb{R}^3}\left(\int_0^t\int_{\mathbb{R}^3}G(t-s,x-y)dsdy\right)^{p-1}\\
&\quad\times\int_0^t\int_{\mathbb{R}^3}G(t-s,x-y)\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}\mathbb{E}\left[\left|Z^{\varepsilon,v^\varepsilon}(r,z)\right|^{2p}\right]dsdy\\
&+C(p,K)\sup_{(t,x)\in[0,T]\times\mathbb{R}^3}\left(\int_0^t\int_{\mathbb{R}^3}G(t-s,x-y)dsdy\right)^{p-1}\\
&\quad\times\int_0^t\int_{\mathbb{R}^3}G(t-s,x-y)\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}\mathbb{E}\left[\left|Z^{\varepsilon,v^\varepsilon}(r,z)-Z^v(r,z)\right|^{p}\right]dsdy\\
\leqslant&\varepsilon^{\frac{p}{2}}h^p(\varepsilon)C(p,K',T)+C(p,K,T)\int_0^t\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}\mathbb{E}\left[\left|X^{\varepsilon,v^\varepsilon,v}(r,z)\right|^p\right]ds.
\end{aligned}\tag{51}$$

Putting (43), (44), (46), (50) and (51) together, we have

$$\begin{aligned}
&\sup_{(s,x)\in[0,t]\times\mathbb{R}^3}\mathbb{E}\left[|X^{\varepsilon,v^\varepsilon,v}(s,x)|^p\right]\\
\leqslant&C(N,p,u^0,K,K',T)\Bigg(h^{-p}(\varepsilon)+\varepsilon^{\frac{p}{2}}h^p(\varepsilon)+\sup_{(s,x)\in[0,t]\times\mathbb{R}^3}\mathbb{E}\left[|A_{2,2}^\varepsilon(s,x)|^p\right]\\
&+\int_0^t\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}\mathbb{E}\left[\left|X^{\varepsilon,v^\varepsilon,v}(r,z)\right|^p\right]ds\Bigg).
\end{aligned}$$

By Gronwall's inequality and (47), we obtain that

$$\begin{aligned}
&\sup_{(s,x)\in[0,T]\times\mathbb{R}^3}\mathbb{E}\left[|X^{\varepsilon,v^\varepsilon,v}(s,x)|^p\right]\\
\leqslant&C(N,p,u^0,K,K',T)\left(h^{-p}(\varepsilon)+\varepsilon^{\frac{p}{2}}h^p(\varepsilon)+\sup_{(s,x)\in[0,t]\times\mathbb{R}^3}\mathbb{E}\left[|A_{2,2}^\varepsilon(s,x)|^p\right]\right)\\
&\longrightarrow 0,\quad\text{as }\varepsilon\to 0.
\end{aligned}$$

The proof is complete. □

Proposition 4.2. Under conditions **(H)** and **(D)**, for any $\alpha\in\mathcal{I}$, the mapping $\mathcal{H}_T^N\ni h\to Z^h\in\mathcal{E}_\alpha$ is continuous with respect to the weak topology.

Proof. Let $\{h,(h_n)_{n\geqslant1}\}\subset\mathcal{H}_T^N$ such that for any $g\in\mathcal{H}_T$,

$$\lim_{n\to\infty}\langle h_n-h,g\rangle_{\mathcal{H}_T}=0.$$

We need to prove that

$$\lim_{n\to\infty}\|Z^{h_n}-Z^h\|_\alpha=0.\tag{52}$$

Applying the deterministic version of Lemma 3.2 to Z^{h_n} and Z^h, the proof of (52) can be divided into two steps :

(1) Pointwise convergence : for any $(t,x) \in [0,T] \times D$,

$$\lim_{n\to\infty} \left| Z^{h_n}(t,x) - Z^h(t,x) \right| = 0. \tag{53}$$

(2) Estimation of the increments : for any $(t,x),(s,y) \in [0,T] \times D$, $\alpha \in \mathcal{I}$,

$$\begin{aligned}&\sup_{n\geqslant 1} \left| (Z^{h_n}(t,x) - Z^h(t,x)) - (Z^{h_n}(s,y) - Z^h(s,y)) \right| \\ \leqslant & C\left(|t-s| + |x-y|\right)^\alpha .\end{aligned} \tag{54}$$

By using the similar (but more easier) strategy in the proof of [23, Theorem 2.3], one can prove that the solution Z^h of (17) satisfies that for any $\alpha \in \mathcal{I}$, there exists $C > 0$ such that for any $(t,x),(s,y) \in [0,T] \times D$,

$$\sup_{h\in\mathcal{H}_T^N} |Z^h(t,x) - Z^h(s,y)| \leqslant C(|t-s| + |x-y|)^\alpha. \tag{55}$$

Thus, (54) holds. Next, it remains to prove (53).

Notice that for any $(t,x) \in [0,T] \times \mathbb{R}^3$,

$$\begin{aligned}& Z^{h_n}(t,x) - Z^h(t,x) \\ =& \int_0^t \left\langle G(t-s, x-\cdot)\sigma(u^0(s,\cdot)), h_n(s,\cdot) - h(s,\cdot) \right\rangle_{\mathcal{H}} ds \\ &+ \int_0^t G(t-s) * \left\{ b'(u^0(s,\cdot)) \left[Z^{h_n}(s,\cdot) - Z^h(s,\cdot) \right] \right\} (x) ds \\ =:& I_1^n(t,x) + I_2^n(t,x).\end{aligned} \tag{56}$$

Using the similar arguments as in the proof (50), we can obtain that

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} |I_1^n(t,x)| = 0. \tag{57}$$

Set $\zeta^n(t) := \sup_{(s,x)\in[0,t]\times\mathbb{R}^3} |Z^{h_n}(s,x) - Z^h(s,x)|$. By (13), we have

$$\begin{aligned}|I_2(t,x)| \leqslant & \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) \left| b'(u^0(s,y)) \left[Z^{h_n}(s,y) - Z^h(s,y) \right] \right| dsdy \\ \leqslant & K \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) \sup_{(u,z)\in[0,s]\times\mathbb{R}^3} \left| Z^{h_n}(u,z) - Z^h(u,z) \right| dsdy \\ \leqslant & C(K,T) \int_0^t \zeta^n(s) ds.\end{aligned} \tag{58}$$

By (56) and (58), we have

$$\zeta^n(t) \leqslant C(K,T) \int_0^t \zeta^n(s) ds + \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} |I_1^n(t,x)|.$$

Hence, by Gronwall's lemma and (57), we obtain that

$$\zeta^n(T) \leqslant e^{C(K,T)T} \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} |I_1^n(t,x)| \longrightarrow 0, \quad \text{as } n \to \infty.$$

The proof is complete. □

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