$\mathbb{L}^p$ Solutions of Quadratic BSDEs

Hanlin Yang*

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Abstract

We study a general class of quadratic BSDEs with terminal value in $\mathbb{L}^p$ for $p > 1$. First of all, we give an $\mathbb{L}^p$-type estimate and existence result. Under the additional assumption of monotonicity and convexity, we derive the comparison theorem, uniqueness and stability result via θ-technique (Briand and Hu [7]). The assumptions employed throughout this paper are rather weak and extend the quadratic BSDE literature. Finally, a probabilistic representation for the viscosity solution to the associated quadratic PDEs is given.

Keywords: quadratic BSDEs, Krylov estimate, convexity, FBSDEs, quadratic PDEs

1 Introduction

In this paper, we are concerned with $\mathbb{R}$-valued backward stochastic differential equations (BSDEs)

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{1}$$

where the generator F is continuous and satisfies $\mathbb{P}$-a.s. for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} \operatorname{sgn}(y)F(t, y, z) &\leq \alpha_t + \beta|y| + \gamma|z| + f(|y|)|z|^2, \\ |F(t, y, z)| &\leq \alpha_t + \varphi(|y|) + \gamma|z| + f(|y|)|z|^2, \end{aligned} \tag{2}$$

for an $\mathbb{R}^+$-valued progressively measurable process α, $\beta \in \mathbb{R}, \gamma \geq 0$, a function $f(|\cdot|) : \mathbb{R} \to \mathbb{R}^+$ which is integrable and bounded on any compact subset of $\mathbb{R}$, and a continuous nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. A solution to (1) is a process (Y, Z) adapted to the filtration generated by the Brownian motion W such that (1) holds $\mathbb{P}$-a.s. for all $t \in [0, T]$. We emphasize that, unlike the quadratic BSDEs studied by Briand and Hu [6], [7], the quadratic growth in our study takes the form $f(|y|)|z|^2$. Moreover, we assume that the terminal value ξ and $\int_0^T \alpha_s ds$ belong to $\mathbb{L}^p$ for a certain $p > 1$.

Let us recall that, quadratic BSDEs are first studied by Kobylanski [15], where existence, uniqueness, comparison theorem and monotone stability for bounded solutions are obtained. Proving the existence of a solution consists in constructing a monotone sequence of bounded solutions of better-known BSDEs and then passing the limit. The underlying machinery

*Department of Banking and Finance, Universität Zürich, Plattenstrasse 22, CH-8006 Zürich, Switzerland, e-mail: hanlin.yang@uzh.ch. This paper is part of the author's master thesis.

of this procedure is called the monotone stability of quadratic BSDEs. Later, Briand and Hu [6], [7] extend the existence result by assuming that the terminal value has exponential moments integrability. Recently, Bahlali et al [1] constructs a solution to quadratic BSDEs with its terminal value in $\mathbb{L}^2$ and a generator satisfying

$$|F(t,y,z)| \leq \alpha + \beta|y| + \gamma|z| + f(|y|)|z|^2,$$

for some $\alpha, \beta, \gamma \geq 0$. However, as to the uniqueness of a solution, only purely quadratic BSDEs are studied.

There are two lines of studies on the uniqueness of a solution to quadratic BSDEs. When the terminal value is bounded, one crucial feature is that $\int_0^\cdot Z_s dW_s$ is a BMO martingale. This observation, combined with a local Lipschitz condition, can be used to prove a uniqueness result; see, e.g., [13], [17], [18], [5]. However, $\int_0^\cdot Z_s dW_s$ is in general not a BMO martingale if the terminal value is unbounded. Nevertheless one can also obtain a uniqueness result, by relying on a convexity condition which proves to be convenient to treat the quadratic generators; see [7], [14], [10], etc.

The first contribution of this paper is to study an existence result given (2) and a terminal value in $\mathbb{L}^p$ for a certain $p > 1$. We first briefly present the motivations to assume (2). Among the literature on non-quadratic BSDEs, assumptions of this type are quite convenient to obtain the a priori estimates; see, e.g., [4], [3], [8]. It turns out that the existence and monotone stability of bounded solutions can also be adapted to quadratic BSDEs with a growth of this type. This is stated in Briand and Hu [7], which assumes that

$$\begin{aligned} \operatorname{sgn}(y)F(t,y,z) &\leq \alpha_t + \beta|y| + \eta|z|^2, \\ |F(t,y,z)| &\leq \alpha_t + \varphi(|y|) + \eta|z|^2. \end{aligned}$$

The proof is merely a slight modification of Kobylanski [15]. In parallel with these works, we prove an existence result under (2). In the first step, we derive a $\mathbb{L}^p$-type estimate for quadratic BSDEs, by adapting the method developed by Briand et al [3]. To construct a solution, we use a combination of the localization procedure developed by Briand and Hu [6] and the monotone stability result.

Another contribution is to address the question of uniqueness. In the spirit of Briand and Hu [7], we prove comparison theorem, uniqueness and a stability result via θ-technique under a monotonicity and convexity assumption. It turns out that, our results of existence and uniqueness, not simply provide a broader perspective in quadratic BSDEs, but also, by setting $f(|\cdot|) = 0$, (partially) generalize [19], [4], [3], [8], etc. Hence our approach can be seen as unified to the study of both quadratic BSDEs and non-quadratic BSDEs. Finally, as an application, we prove a probabilistic representation for the viscosity solution of the quadratic PDEs associated with the BSDEs of our study.

This paper is organized as follows. In Section 2, we introduce some functions used to treat the quadratic generator in (2). In Section 3, we prove the Itô-Krylov formula and a generalized Itô formula for $y \mapsto |y|^p (p \geq 1)$. The former one is used to treat discontinuous quadratic generators or discontinuous quadratic growth, and the later one is used to deduce the a priori estimates. Section 4 reviews purely quadratic BSDEs and studies their natural extensions, based on Bahlali et al [2]. Section 5 concerns existence, comparison theorem, uniqueness, etc. Finally, in Section 6, we derive the nonlinear Feynman-Kac formula in our framework.

Let us close this section by introducing all required notations. We fix the time horizon $0 < T < +\infty$ and a d-dimensional Brownian motion $(W_t)_{0 \leq t \leq T}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the filtration generated by W and augmented by

$\mathbb{P}$-null sets of $\mathcal{F}$. Any measurability will refer to this filtration. In particular, Prog denotes the progressive σ-algebra on $\Omega \times [0, T]$.

As mentioned before, we only deal with $\mathbb{R}$-valued BSDEs of type (1). We call the Prog $\otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable random function $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ the *generator* and the $\mathcal{F}_T$-measurable random variable ξ the *terminal value*. The conditions imposed on the generator are called the *structure conditions*. For notational convenience, we sometimes write (F, ξ) instead of (1) to denote the BSDE with generator F and terminal value ξ.

$\int_0^\cdot Z_s dW_s$, sometimes denoted by $Z \cdot W$, refers to the vector stochastic integral; see, e.g., Shiryaev and Cherny [22]. We call a process (Y, Z) valued in $\mathbb{R} \times \mathbb{R}^d$ a *solution* of (1), if Y is a continuous adapted process and Z is a Prog-measurable process such that $\mathbb{P}$-a.s. $\int_0^T |Z_s|^2 ds < +\infty$ and $\int_0^T |F(s, Y_s, Z_s)| ds < +\infty$, and (1) holds $\mathbb{P}$-a.s. for any $t \in [0, T]$. The first inequality above ensures that Z is integrable with respect to W in the sense of vector stochastic integration. As a result, $Z \cdot W$ is a continuous local martingale.

As will be seen later, the BSDEs (1) satisfying (2) is solvable if $f(|\cdot|)$ belongs to $\mathcal{I}$, the set of integrable functions from $\mathbb{R}$ to $\mathbb{R}$ which are bounded on any compact subset of $\mathbb{R}$.

For any random variable or process Y, we say Y has some property if this is true except on a $\mathbb{P}$-null subset of Ω. Hence we omit "$\mathbb{P}$-a.s." in situations without ambiguity. Define $\mathrm{sgn}(x) := \mathbb{I}_{\{x \neq 0\}} \frac{x}{|x|}$. For any càdlàg adapted process Y, set $Y_{s,t} := Y_t - Y_s$ and $Y^* := \sup_{t \in [0,T]} |Y_t|$. For any $\mathbb{R}$-valued Prog-measurable process H, set $|H|_{s,t} := \int_s^t H_u du$ and $|H|_t := |H|_{0,t}$. $\mathcal{T}$ stands for the set of stopping times valued in $[0, T]$ and $\mathcal{S}$ denotes the space of continuous adapted processes. For any local martingale M, we call $\{\sigma_n\}_{n \in \mathbb{N}^+} \subset \mathcal{T}$ a *localizing sequence* if σ_n increases stationarily to T as n goes to $+\infty$ and $M_{\cdot \wedge \sigma_n}$ is a martingale for any $n \in \mathbb{N}^+$. For later use, we specify the following spaces under $\mathbb{P}$.

- $\mathcal{S}^\infty$: the set of bounded processes in $\mathcal{S}$;
- $\mathcal{S}^p (p \geq 1)$: the set of $Y \in \mathcal{S}$ with $Y^* \in \mathbb{L}^p$;
- $\mathcal{D}$: the set of $Y \in \mathcal{S}$ such that $\{Y_\tau | \tau \in \mathcal{T}\}$ is uniformly integrable;
- $\mathcal{M}$: the space of $\mathbb{R}^d$-valued Prog-measurable processes Z such that $\mathbb{P}$-a.s. $\int_0^T |Z_s|^2 ds < +\infty$; for any $Z \in \mathcal{M}$, $Z \cdot W$ is a continuous local martingale;
- $\mathcal{M}^p (p > 0)$: the set of $Z \in \mathcal{M}$ with

$$\|Z\|_{\mathcal{M}^p} := \mathbb{E}\Big[\Big(\int_0^T |Z_s|^2 ds\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p} \wedge 1} < +\infty;$$

- $\mathcal{C}^p(\mathbb{R})$: the space of p times continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$;
- $\mathcal{W}^2_{1,loc}(\mathbb{R})$: the Sobolev space of measurable maps $u : \mathbb{R} \to \mathbb{R}$ such that both u and its generalized derivatives u', u'' belong to $\mathbb{L}^1_{loc}(\mathbb{R})$.

The above spaces are Banach (respectively complete) under suitable norms (respectively metrics); we will not present these facts in more detail since they are not involved in our study. We call (Y, Z) a $\mathbb{L}^p$ *solution* of (1) if (Y, Z) belongs to $\mathcal{S}^p \times \mathcal{M}^p$. This definition simply comes from the fact that the existence holds if $|\xi| + \int_0^T \alpha_s ds$ belongs to $\mathbb{L}^p$. Analogously to most papers on $\mathbb{R}$-valued quadratic BSDEs, our existence result essentially relies on the monotone stability result of quadratic BSDEs; see, e.g., Kobylanski [15] or Briand and Hu [7].

2 Functions of Class $\mathcal{I}$

In this section, we introduce the basic ingredients used to treat the quadratic generator in (2). We recall that $\mathcal{I}$ is the set of integrable functions from $\mathbb{R}$ to $\mathbb{R}$ which are bounded on any compact subset of $\mathbb{R}$.

u^f **Transform.** For any $f \in \mathcal{I}$, define $u^f : \mathbb{R} \to \mathbb{R}$ and M^f by

$$u^f(x) := \int_0^x \exp\Big(2\int_0^y f(u)du\Big)dy,$$
$$M^f := \exp\Big(2\int_{-\infty}^{\infty} |f(u)|du\Big).$$

Obviously, $1 \le M^f < +\infty$. Moreover, the following properties hold by simple computations. Here we set $u := u^f$ for notational convenience.

(i) $u \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$ and $u''(x) = 2f(x)u'(x)$ a.e.; if f is continuous, then $u \in \mathcal{C}^2(\mathbb{R})$;

(ii) u is strictly increasing and bijective from $\mathbb{R}$ to $\mathbb{R}$;

(iii) $u^{-1} \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$; if f is continuous, then $u^{-1} \in \mathcal{C}^2(\mathbb{R})$;

(iv) $\frac{|x|}{M} \le |u(x)| \le M|x|$ and $\frac{1}{M} \le u'(x) \le M$.

v^f **Transform.** For any $f \in \mathcal{I}$, define $v^f : \mathbb{R} \to \mathbb{R}^+$ by

$$v^f(x) := \int_0^{|x|} u^{(-f)}(y) \exp\Big(2\int_0^y f(u)du\Big)dy.$$

Set $v := v^f$. Simple computations give

(i) $v \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$ and $v''(x) - 2f(|x|)|v'(x)| = 1$ a.e.; if f is continuous, then $v \in \mathcal{C}^2(\mathbb{R})$;

(ii) $v(x) \ge 0, \operatorname{sgn}(v'(x)) = \operatorname{sgn}(x)$ and $v''(0) = 1$;

(iii) $\frac{x^2}{2M^2} \le v(x) \le \frac{M^2x^2}{2}$ and $\frac{|x|}{M^2} \le |v'(x)| \le M^2|x|$.

In the sequel of our study, u^f and v^f exclusively stand for the above transforms associated with $f \in \mathcal{I}$. Hence in situations without ambiguity, we denote u^f, v^f, M^f by u, v, M, respectively.

3 Krylov Estimate and the Itô-Krylov Formula

The first auxiliary result is the Krylov estimate. Later, it is used to prove an Itô's-type formula for functions in $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$. This helps to deal with (possibly discontinuous) quadratic generators. As the second application, we derive a generalized Itô formula for $y \mapsto |y|^p (p \ge 1)$ which is not smooth enough for $1 \le p < 2$. This is a basic tool to study $\mathbb{L}^p (p \ge 1)$ solutions.

To allow the existence of a local time in particular situations, we study equations of type

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds + \int_t^T dC_s - \int_t^T Z_s dW_s, \tag{3}$$

where C is a continuous adapted process of finite variation. We denote its total variation process by $V.(C)$. Likewise, sometimes we denote (3) by (F, C, ξ). The solution of (3) is defined analogously to that of (1).

Now we prove the Krylov estimate for (3). A more complicated version not needed for our study can be found in Bahlali et al [1].

Lemma 1 (Krylov Estimate) *Consider* (3). *For any measurable function* $\psi : \mathbb{R} \to \mathbb{R}^+$,

$$\mathbb{E}\Big[\int_0^{\tau_m} \psi(Y_s)|Z_s|^2 ds\Big] \le 6m\|\psi\|_{\mathbb{L}^1([-m,m])}, \tag{4}$$

where τ_m *is a stopping time defined by*

$$\tau_m := \inf\Big\{t \ge 0 : |Y_t| + V_t(C) + \int_0^t |F(s, Y_s, Z_s)|ds \ge m\Big\} \wedge T.$$

Proof. Without loss of generality we assume $\|\psi\|_{\mathbb{L}^1([-m,m])} < +\infty$. For each $n \in \mathbb{N}^+$, set

$$\tau_{m,n} := \tau_m \wedge \inf\Big\{t \ge 0 : \int_0^t |Z_s|^2 ds \ge n\Big\}.$$

Let $a \in [-m, m]$. By Tanaka's formula,

$$\begin{aligned}(Y_{t\wedge\tau_{m,n}} - a)^- &= (Y_0 - a)^- - \int_0^{t\wedge\tau_{m,n}} \mathbb{I}_{\{Y_s<a\}} dY_s + \frac{1}{2} L^a_{t\wedge\tau_{m,n}}(Y) \\ &= (Y_0 - a)^- + \int_0^{t\wedge\tau_{m,n}} \mathbb{I}_{\{Y_s<a\}} F(s, Y_s, Z_s)ds + \int_0^{t\wedge\tau_{m,n}} \mathbb{I}_{\{Y_s<a\}} dC_s \\ &\quad - \int_0^{t\wedge\tau_{m,n}} \mathbb{I}_{\{Y_s<a\}} Z_s dW_s + \frac{1}{2} L^a_{t\wedge\tau_{m,n}}(Y),\end{aligned} \tag{5}$$

where $L^a(Y)$ is the local time of Y at a. To estimate the local time, we put it on the left-hand side and the rest terms on the right-hand side. Since $x \mapsto (x - a)^-$ is Lipschitz-continuous, we deduce from the definition of $\tau_{m,n}$ that

$$(Y_0 - a)^- - (Y_{t\wedge\tau_{m,n}} - a)^- \le |Y_0 - Y_{t\wedge\tau_{m,n}}| \le 2m.$$

Meanwhile, the definition of τ_m also implies that the sum of the ds-integral and dC-integral is bounded by m. Hence, we have

$$\mathbb{E}\big[L^a_{t\wedge\tau_{m,n}}(Y)\big] \le 6m.$$

By Fatou's lemma applied to the sequence indexed by n,

$$\sup_{a\in[-m,m]} \mathbb{E}\big[L^a_{t\wedge\tau_m}(Y)\big] \le 6m.$$

We then use time occupation formula for continuous semimartingales (see Chapter VI., Revuz and Yor [21]) and the above inequality to obtain

$$\begin{aligned}\mathbb{E}\Big[\int_0^{T\wedge\tau_m} \psi(Y_s)|Z_s|^2 ds\Big] &= \mathbb{E}\Big[\int_{-m}^m \psi(x) L^x_{T\wedge\tau_m}(Y)dx\Big] \\ &= \int_{-m}^m \psi(x)\mathbb{E}\big[L^x_{T\wedge\tau_m}(Y)\big]dx \\ &\le 6m\|\psi\|_{\mathbb{L}^1([-m,m])}.\end{aligned}$$

□

As an immediate consequence of Lemma 1, we have $\mathbb{P}$-a.s.

$$\int_0^T \mathbb{I}_{\{Y_s \in A\}} |Z_s|^2 ds = 0, \tag{6}$$

for any $A \subset \mathbb{R}$ with null Lebesgue measure. This will be used later several times.

Given Lemma 1, we turn to the main results of this section. The following generalized Itô formula is proved in Bahlali et al [1].

Theorem 2 (Itô-Krylov Formula) *If (Y, Z) is a solution of (3), then for any $u \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$, we have $\mathbb{P}$-a.s. for all $t \in [0, T]$,*

$$u(Y_t) = u(Y_0) + \int_0^t u'(Y_s) dY_s + \frac{1}{2} \int_0^t u''(Y_s) |Z_s|^2 ds. \tag{7}$$

Proof. We use τ_m defined in Lemma 1 (Krylov estimate). Note that τ_m increases stationarily to T as m goes to $+\infty$. It is therefore sufficient to prove the equality for $u(Y_{t\wedge\tau_m})$. To this end we use an approximation procedure. We consider m such that $\mathbb{P}$-a.s. $m \geq |Y_0|$. Let u_n be a sequence of functions in $\mathcal{C}^2(\mathbb{R})$ satisfying

(i) u_n converges uniformly to u on $[-m, m]$;

(ii) u_n' converges uniformly to u' on $[-m, m]$;

(iii) u_n'' converges in $\mathbb{L}^1([-m, m])$ to u''.

By Itô's formula,

$$u_n(Y_{t\wedge\tau_m}) = u_n(Y_0) + \int_0^{t\wedge\tau_m} u_n'(Y_s) dY_s + \frac{1}{2} \int_0^{t\wedge\tau_m} u_n''(Y_s) |Z_s|^2 ds.$$

Due to (i) and $|Y_{t\wedge\tau_m}| \leq m$, $u_n(Y_{\cdot\wedge\tau_m})$ converges to $u(Y_{\cdot\wedge\tau_m})$ $\mathbb{P}$-a.s. uniformly on $[0, T]$ as n goes to $+\infty$; the second term converges in probability to

$$\int_0^{t\wedge\tau_m} u'(Y_s) dY_s$$

by (ii) and dominated convergence for stochastic integrals; the last term converges in probability to

$$\frac{1}{2} \int_0^{t\wedge\tau_m} u''(Y_s) |Z_s|^2 ds$$

due to (iii) and Lemma 1. Indeed, Lemma 1 implies

$$\mathbb{E}\Big[\int_0^{\tau_m} |u_n'' - u''|(Y_s) |Z_s|^2 ds \Big] \leq 6m \|u_n'' - u''\|_{\mathbb{L}^1([-m,m])}.$$

Hence collecting these convergence results gives (7). By the continuity of both sides of (7), the quality also holds $\mathbb{P}$-a.s. for all $t \in [0, T]$.

□

To study $\mathbb{L}^p (p \geq 1)$ solutions we now prove an Itô's-type formula for $y \mapsto |y|^p (p \geq 1)$ which is not smooth enough for $1 \leq p < 2$. The proof for multidimensional Itô processes can be found, e.g., in Briand et al [3]. In contrast to their approach, we give a novel and simpler proof for BSDE framework but point out that it can be also extended to Itô processes.

Lemma 3 *Let $p \geq 1$. If (Y,Z) is a solution of (3), then we have $\mathbb{P}$-a.s. for all $t \in [0,T]$,*

$$|Y_t|^p + \frac{p(p-1)}{2}\int_t^T \mathbb{I}_{\{Y_s \neq 0\}}|Y_s|^{p-2}|Z_s|^2 ds$$
$$= |\xi|^p - p\int_t^T \mathrm{sgn}(Y_s)|Y_s|^{p-1}dY_s - \mathbb{I}_{\{p=1\}}\int_t^T dL_s^0(Y), \tag{8}$$

where $L^0(Y)$ is the local time of Y at 0.

Proof. (i). $p = 1$. This is immediate from Tanaka's formula.

(ii). $p > 2$. $y \mapsto |y|^p \in \mathcal{C}^2(\mathbb{R})$. Hence this is immediate from Itô's formula.

(iii). $p = 2$. $y \mapsto |y|^p \in \mathcal{C}^2(\mathbb{R})$. Due to (6), $\int_0^\cdot |Y_s|^{p-2}|Z_s|^2 ds$ is indistinguishable from $\int_0^\cdot \mathbb{I}_{\{Y_s \neq 0\}}|Y_s|^{p-2}|Z_s|^2 ds$. By taking this fact into account, this equality is thus immediate from Itô's formula.

(iv). $1 < p < 2$. We use an approximation argument. Define

$$u_\epsilon(y) := \left(y^2 + \epsilon^2\right)^{\frac{1}{2}}.$$

Hence for any $\epsilon > 0$, we have $u_\epsilon^p \in \mathcal{C}^2(\mathbb{R})$. By Itô's formula,

$$u_\epsilon^p(Y_t) = u_\epsilon^p(\xi) - p\int_t^T Y_s u_\epsilon^{p-2}(Y_s)dY_s - \frac{1}{2}\int_t^T \left(p u_\epsilon^{p-2}(Y_s) + p(p-2)|Y_s|^2 u_\epsilon^{p-4}(Y_s)\right)|Z_s|^2 ds. \tag{9}$$

Now we send ϵ to 0. $u_\epsilon(y) \longrightarrow |y|$ pointwise implies $u_\epsilon(Y_t)^p \longrightarrow |Y_t|^p$ and $u_\epsilon(\xi)^p \longrightarrow |\xi|^p$ pointwise on Ω. Secondly, $y u_\epsilon^{p-2}(y) \longrightarrow \mathrm{sgn}(y)|y|^{p-1}$ pointwise implies by dominated convergence for stochastic integrals that

$$\int_t^T Y_s \,\mathrm{sgn}(Y_s) u_\epsilon^{p-2}(Y_s)dY_s \longrightarrow \int_t^T |Y_s|^{p-1}dY_s \text{ in probability.}$$

To prove that the ds-integral in (9) also converges, we split it into two parts and argue their convergence respectively. Note that

$$p u_\epsilon^{p-2}(Y_s) + p(p-2)|Y_s|^2 u_\epsilon^{p-4}(Y_s) = p\epsilon^2 u_\epsilon^{p-4}(Y_s) + p(p-1)|Y_s|^2 u_\epsilon^{p-4}(Y_s). \tag{10}$$

For the second term on the right-hand side of (10), we have

$$|Y_s|^2 u_\epsilon^{p-4}(Y_s) = \mathbb{I}_{\{Y_s \neq 0\}}|Y_s|^{p-2}\left|\frac{|Y_s|}{u_\epsilon(Y_s)}\right|^{4-p}.$$

Since $\frac{|y|}{u_\epsilon(y)} \nearrow \mathbb{I}_{\{y \neq 0\}}$, monotone convergence gives

$$\int_t^T |Y_s|^2 u_\epsilon^{p-4}(Y_s)|Z_s|^2 ds \longrightarrow \int_t^T \mathbb{I}_{\{Y_s \neq 0\}}|Y_s|^{p-2}|Z_s|^2 ds \text{ pointwise on } \Omega.$$

It thus remains to prove the ds-integral concerning the first term on the right-hand side of (10) converges to 0. To this end, we use Lemma 1 (Krylov estimate) and the same localization procedure. This gives

$$\begin{aligned}\mathbb{E}\Big[\int_0^{\tau_m} \epsilon^2 u_\epsilon^{p-4}(Y_s)|Z_s|^2 ds\Big] &\le 6m\epsilon^2 \int_{-m}^{m} (x^2+\epsilon^2)^{\frac{p-4}{2}} dx \\ &\le 12m\epsilon^2 \int_0^m (x^2+\epsilon^2)^{\frac{p-4}{2}} dx \\ &\le 12\cdot 2^{\frac{4-p}{2}} m\epsilon^2 \int_0^m (x+\epsilon)^{p-4} dx \\ &\le 12\cdot 2^{\frac{4-p}{2}} m\epsilon^2 \int_\epsilon^{m+\epsilon} x^{p-4} dx \\ &= \frac{12\cdot 2^{\frac{4-p}{2}} m}{p-3}\big(\epsilon^2(m+\epsilon)^{p-3} - \epsilon^{p-1}\big),\end{aligned}$$

which, due to $1<p$, converges to 0 as ϵ goes to 0. Hence $\int_0^{\cdot} \epsilon^2 u_\epsilon^{p-4}(Y_s)|Z_s|^2 ds$ converges $u.c.p$ to 0. Collecting all convergence results above gives (8). Finally, the continuity of each term in (8) implies that the equality also holds $\mathbb{P}$-a.s. for all $t\in[0,T]$.

□

4 $\mathbb{L}^p(p\ge 1)$ Solutions of Purely Quadratic BSDEs

Before turning to the main results of this paper, we partially extend the existence and uniqueness result for purely quadratic BSDEs studied by Bahlali et al [1]. Later, we present their natural extensions and the motivations of our work. These BSDEs are called purely quadratic, since the generator takes the form $F(t,y,z)=f(y)|z|^2$. The solvability simply comes from the function u^f defined in Section 2 which transforms better known BSDEs to $(f(y)|z|^2,\xi)$ by Itô-Krylov formula.

Theorem 4 *Let $f\in\mathcal{I}$ and $\xi\in\mathbb{L}^p(p\ge 1)$. Then there exists a unique solution of*

$$Y_t = \xi + \int_t^T f(Y_s)|Z_s|^2 ds - \int_t^T Z_s dW_s. \tag{11}$$

Moreover, if $p>1$, the solution belongs to $\mathcal{S}^p\times\mathcal{M}^p$; if $p=1$, the solution belongs to $\mathcal{D}\times\mathcal{M}^q$ for any $q\in(0,1)$.

Proof. Let $u:=u^f$ and $M:=M^f$. Then $u,u^{-1}\in\mathcal{C}^1(\mathbb{R})\cap\mathcal{W}^2_{1,loc}(\mathbb{R})$. The existence and uniqueness result can be seen as a one-on-one correspondence between solutions of BSDEs.

(i). Existence. $|u(x)|\le M|x|$ implies $u(\xi)\in\mathbb{L}^p$. By Itô representation theorem, there exists a unique pair $(\widetilde{Y},\widetilde{Z})$ which solves $(0,u(\xi))$, i.e.,

$$d\widetilde{Y}_t = \widetilde{Z}_t dW_t,\ \widetilde{Y}_T = u(\xi). \tag{12}$$

We aim at proving

$$(Y,Z) := (u^{-1}(\widetilde{Y}), \frac{\widetilde{Z}}{u'(u^{-1}(\widetilde{Y}))}) \tag{13}$$

solves (11). Itô-Krylov formula applied to $Y_t = u^{-1}(\widetilde{Y}_t)$ yields

$$dY_t = \frac{1}{u'(u^{-1}(\widetilde{Y}_t))} d\widetilde{Y}_t - \frac{1}{2}\Big(\frac{1}{u'(u^{-1}(\widetilde{Y}_t))}\Big)^2 \frac{u''(u^{-1}(\widetilde{Y}_t))}{u'(u^{-1}(\widetilde{Y}_t))}|\widetilde{Z}_s|^2 ds. \tag{14}$$

To simplify (14) let us recall that $u''(x) = 2f(x)u'(x)$ a.e. Hence (13), (14) and (6) give

$$dY_t = -f(Y_t)|Z_t|^2 dt + Z_t dW_t, \ Y_T = \xi,$$

i.e., (Y, Z) solves (11).

(ii). Uniqueness. Suppose (Y, Z) and (Y', Z') are solutions of (11). By Itô-Krylov formula applied to $u(Y)$ and $u(Y')$, we deduce that $(u(Y), u'(Y)Z)$ and $(u(Y'), u'(Y')Z')$ solve $(0, u(\xi))$. But from (i) it is known that they coincide. Transforming $u(Y)$ and $u(Y')$ via the bijective function u^{-1} yields the uniqueness result.

(iii). We prove the estimate for the unique solution (Y, Z). For $p > 1$, Doob's $\mathbb{L}^p(p > 1)$ maximal inequality used to (12) implies $(\widetilde{Y}, \widetilde{Z}) \in \mathcal{S}^p \times \mathcal{M}^p$. Hence $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}^p$, due to $|u'(x)| \geq \frac{1}{M}$ and $|u^{-1}(x)| \leq M|x|$. For $p = 1$, $\widetilde{Y} \in \mathcal{D}$ since it is a martingale on $[0, T]$. In view of the above properties of u we have $Y \in \mathcal{D}$. The estimate for Z is immediate from Lemma 6.1, Briand et al [3] which is a version of $\mathbb{L}^p(0 < p < 1)$ maximal inequality for martingales.

$\square$

Remark. If ξ is a general $\mathcal{F}_T$-measurable random variable, Dudley representation theorem (see Dudley [11]) implies that there still exists a solution of (12) and hence a solution of (11). However, the solution in general is not unique.

The proof of Theorem 4 indicates that f being bounded on compact subsets of $\mathbb{R}$ is not needed for the existence and uniqueness result of purely quadratic BSDEs.

Proposition 5 (Comparison) *Let $f, g \in \mathcal{I}$, $\xi, \xi' \in \mathbb{L}^p(p \geq 1)$ and (Y, Z), (Y', Z') be the unique solutions of $(f(y)|z|^2, \xi)$, $(g(y)|z|^2, \xi')$, respectively. If $f \leq g$ a.e. and $\mathbb{P}$-a.s. $\xi \leq \xi'$, then $\mathbb{P}$-a.s. $Y_\cdot \leq Y'_\cdot$.*

Proof. Again we transform so as to compare better known BSDEs. Set $u := u^f$. For any $\tau \in \mathcal{T}$, Itô-Krylov formula yields

$$\begin{aligned} u(Y'_{t\wedge\tau}) &= u(Y'_\tau) + \int_{t\wedge\tau}^{\tau} \Big(u'(Y'_s)g(Y'_s)|Z'_s|^2 - \frac{1}{2}u''(Y'_s)|Z'_s|^2\Big)ds - \int_{t\wedge\tau}^{\tau} u'(Y_s)Z'_s dW_s. \\ &= u(Y'_\tau) + \int_{t\wedge\tau}^{\tau} u'(Y'_s)\big(g(Y'_s) - f(Y'_s)\big)|Z'_s|^2 ds - \int_{t\wedge\tau}^{\tau} u'(Y_s)Z'_s dW_s \\ &\geq u(Y'_\tau) - \int_{t\wedge\tau}^{\tau} u'(Y_s)Z'_s dW_s, \end{aligned}$$

where the last two lines are due to $u''(x) = 2f(x)u'(x)$ a.e., $g \geq f$ a.e. and (6). In the next step, we want to eliminate the local martingale part by a localization procedure. Note that $\int_t^{\cdot} u'(Y_s)Z'_s dW_s$ is a local martingale on $[t, T]$. Set $\{\tau_n\}_{n\in\mathbb{N}^+}$ to be its localizing sequence on $[t, T]$. Replacing τ by τ_n in the above inequality thus gives $\mathbb{P}$-a.s.

$$u(Y'_t) \geq \mathbb{E}\big[u(Y'_{t\wedge\tau_n})\big|\mathcal{F}_t\big].$$

This implies that, for any $A \in \mathcal{F}_t$, we have

$$\mathbb{E}\big[u(Y'_t)\mathbb{I}_A\big] \geq \mathbb{E}\big[u(Y'_{t\wedge\tau_n})\mathbb{I}_A\big].$$

Since $u(Y') \in \mathcal{D}$, we can use Vitali convergence theorem to obtain

$$\mathbb{E}\big[u(Y'_t)\mathbb{I}_A\big] \geq \mathbb{E}\big[u(\xi')\mathbb{I}_A\big] = \mathbb{E}\big[\mathbb{E}\big[u(\xi')\big|\mathcal{F}_t\big]\mathbb{I}_A\big].$$

Note that this inequality holds for any $A \in \mathcal{F}_t$. Hence, by choosing $A = \{u(Y_t') < \mathbb{E}[u(\xi')|\mathcal{F}_t]\}$, we obtain $u(Y_t') \geq \mathbb{E}\big[u(\xi')\big|\mathcal{F}_t\big]$. Since $\xi' \geq \xi$ and u is increasing, we further have $u(Y_t') \geq \mathbb{E}\big[u(\xi)\big|\mathcal{F}_t\big]$. Let us recall that, by Theorem 4, $(u(Y), u'(Y)Z)$ is the unique solution of $(0, u(\xi))$. Hence, $u(Y_t') \geq u(Y_t)$. Transforming both sides via the bijective increasing function u^{-1} yields $\mathbb{P}$-a.s. $Y_t \leq Y_t'$. By the continuity of Y and Y' we have $\mathbb{P}$-a.s. $Y_\cdot \leq Y_\cdot'$.

□

Remark. In Proposition 5, we rely on the fact that $\mathbb{P}$-a.s.

$$\int_0^\cdot \Big(\frac{1}{2}u''(Y_s') - f(Y_s')u'(Y_s')\Big)|Z_s'|^2 ds = 0, \tag{15}$$

even though $u''(x) = 2f(x)u'(x)$ only holds almost everywhere on $\mathbb{R}$. Here we prove it. Let A be the subset of $\mathbb{R}$ on which $u''(x) = 2f(x)u'(x)$ fails. Hence,

$$\int_0^\cdot \mathbb{I}_{\{Y_s' \in \mathbb{R}\setminus A\}}\Big|\frac{1}{2}u''(Y_s') - f(Y_s')u'(Y_s')\Big||Z_s'|^2 ds = 0.$$

Meanwhile, by (6), we have $\mathbb{P}$-a.s.

$$\int_0^\cdot \mathbb{I}_{\{Y_s' \in A\}}\Big|\frac{1}{2}u''(Y_s') - f(Y_s')u'(Y_s')\Big||Z_s'|^2 ds = 0.$$

Hence, (15) holds $\mathbb{P}$-a.s. This fact also applies to Theorem 4 and all results in the sequel of our study.

Theorem 4 and Proposition 5 are based on a one-on-one correspondence between solutions (respectively the unique solution) of BSDEs. Hence it is natural to generalize as follows. Set $f \in \mathcal{I}, u := u^f$, $F(t,y,z) := G(t,y,z) + f(y)|z|^2$ and

$$\widetilde{F}(t,y,z) := u'(u^{-1}(y))G(t, u^{-1}(y), \frac{z}{u'(u^{-1}(y))}). \tag{16}$$

If G ensures the existence of a solution of $(\widetilde{F}, u(\xi))$, we can transform it via u^{-1} to a solution of (F, ξ). An example is that G is of continuous linear growth in (y,z) where the existence of a maximal (respectively minimal) solution of $(\widetilde{F}, u(\xi))$ can be proved in the spirit of Lepeltier and San Martin [16].

When the generator is continuous in (y,z), a more general situation is linear-quadratic growth, i.e.,

$$|H(t,y,z)| \leq \alpha + \beta|y| + \gamma|z| + f(|y|)|z|^2 := F(t,y,z), \tag{17}$$

for some $\alpha, \beta, \gamma \geq 0$. The existence result then consists of viewing the maximal (respectively minimal) solution of (F, ξ^+) (respectively $(-F, -\xi^-)$) as the a priori bounds for solutions of (H, ξ), and using a combination of a localization procedure and the monotone stability result developed by Briand and Hu [6], [7]. For details the reader shall refer to Bahlali et al [1].

However, either an additive structure in (16) or a linear-quadratic growth (17) is too restrictive and uniqueness is not available in general. Considering this limitation, we devote Section 5 to the solvability under milder structure conditions.

5 $\mathbb{L}^p(p>1)$ Solutions of Quadratic BSDEs

With the preparatory work in Section 2, 3, 4, we study $\mathbb{L}^p(p>1)$ solutions of quadratic BSDEs under general assumptions. We deal with the quadratic generators in the spirit of Bahlali et al [1], derive the a priori estimates in the spirit of Briand et al [3] and prove the existence and uniqueness result in the spirit of Briand et al [6], [7], [8]. This section can also be seen as a generalization of these works. The following assumptions on (F,ξ) ensure the estimates and an existence result.

Assumption (A.1) Let $p\geq 1$. There exist $\beta\in\mathbb{R},\gamma\geq 0$, an $\mathbb{R}^+$-valued Prog-measurable process α, $f(|\cdot|)\in\mathcal{I}$ and a continuous nondecreasing function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ with $\varphi(0)=0$ such that $|\xi|+|\alpha|_T\in\mathbb{L}^p$ and $\mathbb{P}$-a.s.

(i) for any $t\in[0,T]$, $(y,z)\longmapsto F(t,y,z)$ is continuous;

(ii) F is "monotonic" at $y=0$, i.e., for any $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$,

$$\operatorname{sgn}(y)F(t,y,z)\leq \alpha_t+\beta|y|+\gamma|z|+f(|y|)|z|^2;$$

(iii) for any $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$,

$$|F(t,y,z)|\leq \alpha_t+\varphi(|y|)+\gamma|z|+f(|y|)|z|^2.$$

It is worth noticing that, given (A.1)(iii) and $f(|\cdot|)=0$, (A.1)(ii) is a consequence of F being monotonic at $y=0$. Indeed,

$$\operatorname{sgn}(y-0)\big(F(t,y,z)-F(t,0,z)\big)\leq\beta|y|$$

implies

$$\begin{aligned}\operatorname{sgn}(y)F(t,y,z)&\leq F(t,0,z)+\beta|y|\\&\leq \alpha_t+\beta|y|+\gamma|z|.\end{aligned}$$

This explains why we keep saying that F is monotonic at $y=0$, even though y also appears in the quadratic term. Secondly, our results don't rely on the specific choice of φ. Hence the growth condition in y can be arbitrary as long as (A.1)(i)(ii) hold. Assumptions of this type for different settings can also be found in, e.g., [4], [3], [7]. Finally, $f(|\cdot|)$ can be discontinuous; $f(|\cdot|)$ being $\mathbb{R}^+$-valued appears more naturally in the growth condition.

Lemma 6 (A Priori Estimate (i)) *Let $p\geq 1$ and (A.1) hold for (F,ξ). If $(Y,Z)\in\mathcal{S}^p\times\mathcal{M}$ is a solution of (F,ξ), then*

$$\mathbb{E}\Big[\Big(\int_0^T|Z_s|^2ds\Big)^{\frac{p}{2}}\Big]+\mathbb{E}\Big[\Big(\int_0^T f(|Y_s|)|Z_s|^2ds\Big)^{p}\Big]\leq c\Big(\mathbb{E}\big[(Y^*)^p+|\alpha|_T^p\big]\Big),$$

where c is a constant only depending on $T,M^{f(|\cdot|)},\beta,\gamma,p$.

Proof. Set $v:=v^{f(|\cdot|)}$ and $M:=M^{f(|\cdot|)}$. For any $\tau\in\mathcal{T}$, Itô-Krylov formula yields

$$\begin{aligned}v(Y_0)=v(Y_\tau)&+\int_0^\tau v'(Y_s)F(s,Y_s,Z_s)ds\\&-\frac{1}{2}\int_0^\tau v''(Y_s)|Z_s|^2ds-\int_0^\tau v'(Y_s)Z_sdW_s.\end{aligned}\tag{18}$$

Due to $\mathrm{sgn}(v'(x)) = \mathrm{sgn}(x)$ and (A.1)(ii), we have

$$v'(Y_s)F(s,Y_s,Z_s) \le |v'(Y_s)|\big(\alpha_t + \beta|Y_s| + \gamma|Z_s| + f(|Y_s|)|Z_s|^2\big). \tag{19}$$

Recall that $v''(x) - 2f(|x|)|v'(x)| = 1$ a.e. Hence (18) and (19) give

$$\frac{1}{2}\int_0^\tau |Z_s|^2 ds \le v(Y_\tau) + \int_0^\tau |v'(Y_s)|\big(\alpha_s + \beta|Y_s| + \gamma|Z_s|\big)ds - \int_0^\tau v'(Y_s)Z_s dW_s.$$

Moreover, since $v(x) \le \frac{M^2x^2}{2}$ and $|v'(x)| \le M^2|x|$, this inequality gives

$$\int_0^\tau |Z_s|^2 ds \le c_1(Y^*)^2 + c_1\int_0^\tau |Y_s|\big(\alpha_s + |Y_s| + |Z_s|\big)ds - 2\int_0^\tau v'(Y_s)Z_s dW_s, \tag{20}$$

where $c_1 := 2M^2(1\vee\beta\vee\gamma)$. Note that in (20),

$$\int_0^\tau |Y_s|\alpha_s ds \le \frac{1}{2}(Y^*)^2 + \frac{1}{2}|\alpha|_T^2,$$
$$c_1\int_0^\tau |Y_s||Z_s| ds \le \frac{1}{2}c_1^2T\cdot(Y^*)^2 + \frac{1}{2}\int_0^\tau |Z_s|^2 ds.$$

Hence (20) yields

$$\int_0^\tau |Z_s|^2 ds \le (3c_1 + c_1^2T)(Y^*)^2 + c_1|\alpha|_T^2 - 4\int_0^\tau v'(Y_s)Z_s dW_s.$$

This estimate implies that for any $p \ge 1$,

$$\mathbb{E}\Big[\Big(\int_0^\tau |Z_s|^2 ds\Big)^{\frac{p}{2}}\Big] \le c_2\mathbb{E}\Big[(Y^*)^p + |\alpha|_T^p + \Big|\int_0^\tau v'(Y_s)Z_s dW_s\Big|^{\frac{p}{2}}\Big], \tag{21}$$

where $c_2 := 3^{\frac{p}{2}}\big((3c_1 + c_1^2T)\vee 4\big)^{\frac{p}{2}}$. Define for each $n \in \mathbb{N}^+$, $\tau_n := \inf\big\{t \ge 0 : \int_0^t |Z_s|^2 ds \ge n\big\}\wedge T$. We then replace τ by τ_n and use Davis-Burkholder-Gundy inequality to obtain

$$\begin{aligned}c_2\mathbb{E}\Big[\Big(\int_0^{\tau_n} v'(Y_s)Z_s dW_s\Big)^{\frac{p}{2}}\Big] &\le c_2c(p)M^p\mathbb{E}\Big[\Big(\int_0^{\tau_n} |Y_s|^2|Z_s|^2 ds\Big)^{\frac{p}{4}}\Big]\\ &\le \frac{1}{2}c_2^2c(p)^2M^{2p}\mathbb{E}\big[(Y^*)^p\big] + \frac{1}{2}\mathbb{E}\Big[\Big(\int_0^{\tau_n} |Z_s|^2 ds\Big)^{\frac{p}{2}}\Big]\\ &< +\infty.\end{aligned}$$

We explain that in this inequality, $c(p)$ comes from Davis-Burkholder-Gundy inequality and only depends on p. With this estimate, we come back to (21). Transferring the quadratic term to the left-hand side of (21) and using Fatou's lemma, we obtain

$$\mathbb{E}\Big[\Big(\int_0^T |Z_s|^2 ds\Big)^{\frac{p}{2}}\Big] \le c\Big(\mathbb{E}\big[(Y^*)^p + |\alpha|_T^p\big]\Big),$$

where $c := c_2^2c(p)^2M^{2p} + 2c_2$.

To estimate $\int_0^T f(|Y_s|)|Z_s|^2 ds$ we use $u := u^{2f(|\cdot|)}$. This helps to transfer $\int_0^T f(|Y_s|)|Z_s|^2 ds$ to the left-hand side so that standard estimates can be used. The proof is omitted since it is not relevant to our study.

$\square$

We continue our study by sharpening Lemma 6 for $p>1$. We follow Proposition 3.2, Briand et al [3] and extend it to quadratic BSDEs. As an important byproduct, we obtain the a priori bound for solutions which is crucial to the construction of a solution.

Lemma 7 (A Priori Estimate (ii)) *Let $p>1$ and (A.1) hold for (F,ξ). If $(Y,Z)\in\mathcal{S}^p\times\mathcal{M}$ is a solution to (F,ξ), then*

$$\mathbb{E}\big[(Y^*)^p\big]+\mathbb{E}\Big[\Big(\int_0^T|Z_s|^2ds\Big)^{\frac{p}{2}}\Big]+\mathbb{E}\Big[\Big(\int_0^T f(|Y_s|)|Z_s|^2ds\Big)^p\Big]\le c\Big(\mathbb{E}\big[|\xi|^p+|\alpha|_T^p\big]\Big).$$

In particular,

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|Y_s|^p\Big|\mathcal{F}_t\Big]\le c\mathbb{E}\big[|\xi|^p+|\alpha|_{t,T}^p\big|\mathcal{F}_t\big].$$

In both cases, c is a constant only depending on $T,M^{f(|\cdot|)},\beta,\gamma,p$.

Proof. Let $u:=u^{f(|\cdot|)}$ and $M:=M^{f(|\cdot|)}$, and denote $u(|Y_t|),u'(|Y_t|),u''(|Y_t|)$ by u_t,u'_t,u''_t, respectively. By Tanaka's formula applied to $|Y_t|$ and Itô-Krylov formula applied to u_t,

$$\begin{aligned}u_t&=u_T+\int_t^T\operatorname{sgn}(Y_s)u'_sF(s,Y_s,Z_s)ds-\frac{1}{2}\int_t^T\mathbb{I}_{\{Y_s\neq0\}}u''_s|Z_s|^2ds\\&-\int_t^T\operatorname{sgn}(Y_s)u'_sZ_sdW_s-\int_t^Tu'_sdL_s^0(Y),\end{aligned}$$

where $L^0(Y)$ is the local time of Y at 0. Lemma 3 applied to u_t then gives

$$\begin{aligned}&|u_t|^p+\frac{p(p-1)}{2}\int_t^T\mathbb{I}_{\{u_s\neq0\}}\mathbb{I}_{\{Y_s\neq0\}}|u_s|^{p-2}|u'_s|^2|Z_s|^2ds\\&=|u_T|^p+p\int_t^T\operatorname{sgn}(u_s)|u_s|^{p-1}\Big(\operatorname{sgn}(Y_s)u'_sF(s,Y_s,Z_s)-\frac{1}{2}\mathbb{I}_{\{Y_s\neq0\}}u''_s|Z_s|^2\Big)ds\\&-p\int_t^T\operatorname{sgn}(u_s)|u_s|^{p-1}u'_sdL_s^0(Y)-p\int_t^T\operatorname{sgn}(u_s)\operatorname{sgn}(Y_s)|u_s|^{p-1}u'_sZ_sdW_s.\end{aligned}$$

To simplify this equality, we recall that $\operatorname{sgn}(u_s)=\mathbb{I}_{\{u_s\neq0\}}=\mathbb{I}_{\{Y_s\neq0\}}$ and $u''(x)=2f(x)u'(x)$ a.e. Hence

$$\begin{aligned}&|u_t|^p+\frac{p(p-1)}{2}\int_t^T\mathbb{I}_{\{Y_s\neq0\}}|u_s|^{p-2}|u'_s|^2|Z_s|^2ds\\&\le|u_T|^p+p\int_t^T\mathbb{I}_{\{Y_s\neq0\}}|u_s|^{p-1}u'_s\big(\alpha_s+\beta|Y_s|+\gamma|Z_s|\big)ds\\&-p\int_t^T\operatorname{sgn}(Y_s)|u_s|^{p-1}u'_sZ_sdW_s.\end{aligned}$$

Let $\{c_n\}_{n\in\mathbb{N}^+}$ be constants to be determined. Since $\frac{|x|}{M}\le u(|x|)\le M|x|$ and $\frac{1}{M}\le u'(|x|)\le M$, this inequality yields

$$\begin{aligned}&|Y_t|^p+c_1\int_t^T\mathbb{I}_{\{Y_s\neq0\}}|Y_s|^{p-2}|Z_s|^2ds\\&\le M^p|\xi|^p+M^p\int_t^T\mathbb{I}_{\{Y_s\neq0\}}|Y_s|^{p-1}\big(\alpha_s+|\beta||Y_s|+\gamma|Z_s|\big)ds\\&-p\int_t^T\operatorname{sgn}(Y_s)|u_s|^{p-1}u'_sZ_sdW_s,\end{aligned}\tag{22}$$

where $c_1 := \frac{p(p-1)}{2M^p} > 0$. Observe that in (22),

$$M^p\gamma \mathbb{I}_{\{Y_s\neq 0\}}|Y_s|^{p-1}|Z_s| \le \frac{M^{2p}\gamma^2}{2c_1}|Y_s|^p + \frac{c_1}{2}\mathbb{I}_{\{Y_s\neq 0\}}|Y_s|^{p-2}|Z_s|^2.$$

We then use this inequality to (22). Set $c_2 := M^p \vee \big(M^p|\beta| + \frac{M^{2p}\gamma^2}{2c_1}\big)$,

$$X := c_2\Big(|\xi|^p + \int_0^T |Y_s|^{p-1}\big(\alpha_s + |Y_s|\big)ds\Big),$$

and N to be the local martingale part of (22). Hence (22) gives

$$|Y_t|^p + \frac{c_1}{2}\int_t^T \mathbb{I}_{\{Y_s\neq 0\}}|Y_s|^{p-2}|Z_s|^2 ds \le X - N_T + N_t. \tag{23}$$

We claim that N is a martingale. Let $c(1)$ be the constant in Davis-Burkholder-Gundy inequality for $p = 1$. We have

$$\begin{aligned}
\mathbb{E}\big[N^*\big] &\le c(1)\mathbb{E}\big[\langle N\rangle_T^{\frac{1}{2}}\big]\\
&\le c(1)M^p\mathbb{E}\Big[\Big(\int_0^T |Y_s|^{2p-2}|Z_s|^2 ds\Big)^{\frac{1}{2}}\Big]\\
&\le \frac{c(1)M^p}{p}\Big((p-1)\mathbb{E}\big[(Y^*)^p\big] + \mathbb{E}\Big[\Big(\int_0^T |Z_s|^2 ds\Big)^{\frac{p}{2}}\Big]\Big)\\
&< +\infty,
\end{aligned}$$

where the last two lines come from Young's inequality and Lemma 6 (a priori estimate (i)). Hence N is a martingale. Coming back to (23), we deduce that

$$\mathbb{E}\Big[\int_0^T \mathbb{I}_{\{Y_s\neq 0\}}|Y_s|^{p-2}|Z_s|^2 ds\Big] \le \frac{2}{c_1}\mathbb{E}[X]. \tag{24}$$

Now we estimate Y via X. To this end, taking supremum over $t \in [0,T]$ and using Davis-Burkholder-Gundy inequality to (23) give

$$\mathbb{E}\big[(Y^*)^p\big] \le \mathbb{E}[X] + c(1)\mathbb{E}\big[\langle N\rangle_T^{\frac{1}{2}}\big]. \tag{25}$$

The second term in (25) yields by Cauchy-Schwartz inequality that

$$\begin{aligned}
c(1)\mathbb{E}[\langle N\rangle_T^{\frac{1}{2}}] &\le c(1)M^p\mathbb{E}\Big[(Y^*)^{\frac{p}{2}}\Big(\int_0^T \mathbb{I}_{\{Y_s\neq 0\}}|Y_s|^{p-2}|Z_s|^2 ds\Big)^{\frac{1}{2}}\Big]\\
&\le \frac{1}{2}\mathbb{E}\big[(Y^*)^p\big] + \frac{c(1)^2M^{2p}}{2}\mathbb{E}\Big[\int_0^T \mathbb{I}_{\{Y_s\neq 0\}}|Y_s|^{p-2}|Z_s|^2 ds\Big].
\end{aligned}$$

Using (24) to this inequality gives the estimate of $\langle N\rangle^{\frac{1}{2}}$ via Y and X. With this estimate we come back to (25) and obtain

$$\mathbb{E}[(Y^*)^p] \le 2\Big(1 + \frac{2c(1)^2M^{2p}}{c_1}\Big)\mathbb{E}[X].$$

Set $c_3 := 2c_2\big(1 + \frac{c(1)^2M^{2p}}{2}\big)$. This inequality yields

$$\mathbb{E}[(Y^*)^p] \le c_3\Big(\mathbb{E}\big[|\xi|^p\big] + \mathbb{E}\Big[\int_0^T |Y_s|^{p-1}\alpha_s ds\Big] + \mathbb{E}\Big[\int_0^T |Y_s|^p ds\Big]\Big). \tag{26}$$

Young's inequality used to the second term on the right-hand side of this inequality gives

$$c_3 \int_0^T |Y_s|^{p-1} \alpha_s ds \leq \frac{1}{2}(Y^*)^p + \frac{c_3}{p}\Big(\frac{2}{c_3 q}\Big)^{\frac{p}{q}} |\alpha|_T^p,$$

where q is the conjugate index of p. Set $c_4 := 2\Big(c_3 \vee \frac{c_3}{p}\big(\frac{2}{c_3 q}\big)^{\frac{p}{q}}\Big)$. (26) and the above inequality yield

$$\mathbb{E}\big[(Y^*)^p\big] \leq c_4\Big(\mathbb{E}\big[|\xi|^p + |\alpha|_T^p\big] + \mathbb{E}\Big[\int_0^T \sup_{u\in[0,s]} |Y_u|^p ds\Big]\Big),$$

By Gronwall's lemma,

$$\mathbb{E}\big[(Y^*)^p\big] \leq c_4 \exp(c_4 T)\mathbb{E}\big[|\xi|^p + |\alpha|_T^p\big].$$

Finally, by Lemma 6 we conclude that there exists a constant c only depending on T, M, β, γ, p such that

$$\mathbb{E}\big[(Y^*)^p\big] + \mathbb{E}\Big[\Big(\int_0^T |Z_s|^2 ds\Big)^{\frac{p}{2}}\Big] + \mathbb{E}\Big[\Big(\int_0^T f(|Y_s|)|Z_s|^2 ds\Big)^p\Big] \leq c\mathbb{E}\big[|\xi|^p + |\alpha|_T^p\big].$$

To prove the remaining statement, we view any fixed $t \in [0,T]$ as the initial time, reset

$$X := c_2\Big(|\xi|^p + \int_t^T |Y_s|^{p-1}\big(\alpha_s + |Y_s|\big)ds\Big)$$

and replace all estimates by conditional estimates.

□

An immediate consequence of Lemma 7 is that

$$|Y_t| \leq \Big(c\mathbb{E}\big[|\xi|^p + |\alpha|_T^p\big|\mathcal{F}_t\big]\Big)^{\frac{1}{p}},$$

i.e., Y has an a priori bound which is a continuous supermartingale.

With this estimate we are ready to construct a $\mathbb{L}^p(p > 1)$ solution via inf-(sup-)convolution as in Briand et al [6], [7], [8]. A localization procedure where the a priori bound plays a crucial role is used and the monotone stability result takes the limit.

Theorem 8 (Existence) *Let $p > 1$ and (A.1) hold for (F, ξ). Then there exists a solution of (F, ξ) in $\mathcal{S}^p \times \mathcal{M}^p$.*

Proof. We introduce the notations used throughout the proof. Define the process

$$X_t := \Big(c\mathbb{E}\big[|\xi|^p + |\alpha|_T^p\big|\mathcal{F}_t\big]\Big)^{\frac{1}{p}},$$

where c is the constant defined in Lemma 7. Obviously X is continuous by Itô representation theorem. Moreover, for each $m, n \in \mathbb{N}^+$, set

$$\tau_m := \inf\big\{t \geq 0 : |\alpha|_t + X_t \geq m\big\} \wedge T,$$
$$\sigma_n := \inf\big\{t \geq 0 : |\alpha|_t \geq n\big\} \wedge T.$$

It then follows from the continuity of $|\alpha|_\cdot$ and X that τ_m and σ_n increase stationarily to T as m, n goes to $+\infty$, respectively. To apply a double approximation procedure, we define

$$\begin{aligned} F^{n,k}(t,y,z) : &= \mathbb{I}_{\{t\le\sigma_n\}} \inf_{y',z'} \left\{ F^+(t,y',z') + n|y-y'| + n|z-z'| \right\} \\ &\quad - \mathbb{I}_{\{t\le\sigma_k\}} \inf_{y',z'} \left\{ F^-(t,y',z') + k|y-y'| + k|z-z'| \right\}, \end{aligned}$$

and $\xi^{n,k} := \xi^+ \wedge n - \xi^- \wedge k$.

Before proceeding to the proof we give some useful facts. By Lepeltier and San Martin [16], $F^{n,k}$ is Lipschitz-continuous in (y,z); as k goes to $+\infty$, $F^{n,k}$ converges decreasingly uniformly on compact sets to a limit denoted by $F^{n,\infty}$; as n goes to $+\infty$, $F^{n,\infty}$ converges increasingly uniformly on compact sets to F. Moreover, $\big||F^{n,k}(\cdot,0,0)|\big|_T$ and $\xi^{n,k}$ are bounded.

Hence, by Briand et al [3], there exists a unique solution $(Y^{n,k}, Z^{n,k}) \in \mathcal{S}^p \times \mathcal{M}^p$ of $(F^{n,k}, \xi^{n,k})$; by comparison theorem, $Y^{n,k}$ is increasing in n and decreasing in k. We are about to take the limit by the monotone stability result.

However, $\big||F^{n,k}(\cdot,0,0)|\big|_T$ and $Y^{n,k}$ are not uniformly bounded in general. To overcome this difficulty, we use Lemma 7 and work on random time interval where $Y^{n,k}$ and $\big||F^{n,k}(\cdot,0,0)|\big|_\cdot$ are uniformly bounded. This is the motivation to introduce X and τ_m. To be more precise, the localization procedure is as follows.

Note that $(F^{n,k}, \xi^{n,k})$ satisfies (A.1) associated with $(\alpha, \beta, \gamma, \varphi, f)$. Hence by Lemma 7 (a priori estimate (ii)),

$$\begin{aligned} |Y_t^{n,k}| &\le \left(c\mathbb{E}\big[|\xi^{n,k}|^p + |\mathbb{I}_{[0,\sigma_n\vee\sigma_k]}\alpha|_T^p \big| \mathcal{F}_t \big] \right)^{\frac{1}{p}} \\ &\le X_t. \end{aligned} \tag{27}$$

In view of the definition of τ_m, we deduce that

$$|Y_{t\wedge\tau_m}^{n,k}| \le X_{t\wedge\tau_m} \le m. \tag{28}$$

Hence $Y^{n,k}$ is uniformly bounded on $[0,\tau_m]$. Secondly, given $(Y^{n,k}, Z^{n,k})$ which solves $(F^{n,k}, \xi^{n,k})$, it is immediate that $(Y_{\cdot\wedge\tau_m}^{n,k}, \mathbb{I}_{[0,\tau_m]}Z^{n,k})$ solves $(\mathbb{I}_{[0,\tau_m]}F^{n,k}, Y_{\tau_m}^{n,k})$. To make the monotone stability result adaptable, we use a truncation procedure. Define

$$\rho(y) := -\mathbb{I}_{\{y<-m\}}m + \mathbb{I}_{\{|y|\le m\}}y + \mathbb{I}_{\{y>m\}}m.$$

Hence from (28) $(Y_{\cdot\wedge\tau_m}^{n,k}, \mathbb{I}_{[0,\tau_m]}Z^{n,k})$ meanwhile solves $(\mathbb{I}_{[0,\tau_m]}(t)F^{n,k}(t,\rho(y),z), Y_{\tau_m}^{n,k})$. Secondly, we have

$$\begin{aligned} |\mathbb{I}_{[0,\tau_m]}(t)F^{n,k}(t,\rho(y),z)| &\le \mathbb{I}_{\{t\le\tau_m\}} \big(\alpha_t + \varphi(|\rho(y)|) + \gamma|z| + f(|\rho(y)|)|z|^2 \big) \\ &\le \mathbb{I}_{\{t\le\tau_m\}} \Big(\alpha_t + \varphi(m) + \gamma|z| + \sup_{|y|\le m} f(|\rho(y)|)|z|^2 \Big) \\ &\le \mathbb{I}_{\{t\le\tau_m\}} \Big(\alpha_t + \varphi(m) + \frac{\gamma^2}{4} + \big(\sup_{|y|\le m} f(|\rho(y)|) + 1 \big)|z|^2 \Big), \end{aligned}$$

where $\sup_{|y|\le m} f(|\rho(y)|)$ is bounded for each m due to $f(|\cdot|) \in \mathcal{I}$. Moreover, the definition of τ_m implies $|\alpha|_{\tau_m} \le m$. Hence we can use the monotone stability result (see Kobylanksi [15] or Briand and Hu [7]) to obtain $(Y^{m,n,\infty}, Z^{m,n,\infty}) \in \mathcal{S}^\infty \times \mathcal{M}^2$ which solves $(\mathbb{I}_{[0,\tau_m]}(t)F^{n,\infty}(t,\rho(y),z), \inf_k Y_{\tau_m}^{n,k})$. Moreover, $Y_{\cdot\wedge\tau_m}^{m,n,\infty}$ is the $\mathbb{P}$-a.s. uniform limit of $Y_{\cdot\wedge\tau_m}^{n,k}$ as k goes to $+\infty$. These arguments hold for any $m, n \in \mathbb{N}^+$.

Due to this convergence result we can pass the comparison property to $Y^{m,n,\infty}$. We use the monotone stability result again to the sequence indexed by n to obtain $(\widetilde{Y}^m, \widetilde{Z}^m) \in \mathcal{S}^\infty \times \mathcal{M}^2$ which solves $(\mathbb{I}_{[0,\tau_m]}(t)F(t,\rho(y),z),\ \sup_n \inf_k Y^{n,k}_{\tau_m})$. Likewise, $\widetilde{Y}^m_\cdot$ is the $\mathbb{P}$-a.s. uniform limit of $Y^{m,n,\infty}_\cdot$ as n goes to $+\infty$. Hence we obtain from (28) that $|\widetilde{Y}^m_t| \le X_{t\wedge\tau_m} \le m$. Therefore, $(\widetilde{Y}^m, \widetilde{Z}^m)$ solves $(\mathbb{I}_{[0,\tau_m]}F, \sup_n \inf_k Y^{n,k}_{\tau_m})$, i.e.,

$$\widetilde{Y}^m_{t\wedge\tau_m} = \sup_n \inf_k Y^{n,k}_{\tau_m} + \int_{t\wedge\tau_m}^{\tau_m} F(s,\widetilde{Y}^m_s,\widetilde{Z}^m_s)ds - \int_{t\wedge\tau_m}^{\tau_m} \widetilde{Z}^m_s dW_s. \tag{29}$$

We recall that the monotone stability result also implies that $\widetilde{Z}^m$ is the $\mathcal{M}^2$-limit of $\mathbb{I}_{[0,\tau_m]}Z^{n,k}$ as k,n goes to $+\infty$. This fact and previous convergence results give

$$\begin{aligned} \widetilde{Y}^{m+1}_{\cdot\wedge\tau_m} &= \widetilde{Y}^m_{\cdot\wedge\tau_m}\ \mathbb{P}\text{-a.s.}, \\ \mathbb{I}_{\{t\le\tau_m\}}\widetilde{Z}^{m+1}_t &= \mathbb{I}_{\{t\le\tau_m\}}\widetilde{Z}^m_t\ dt\otimes d\mathbb{P}\text{-a.e.} \end{aligned} \tag{30}$$

Define (Y,Z) on $[0,T]$ by

$$\begin{aligned} Y_t &:= \mathbb{I}_{\{t\le\tau_1\}}\widetilde{Y}^1_t + \sum_{m\ge2} \mathbb{I}_{]\tau_{m-1},\tau_m]}\widetilde{Y}^m_t, \\ Z_t &:= \mathbb{I}_{\{t\le\tau_1\}}\widetilde{Z}^1_t + \sum_{m\ge2} \mathbb{I}_{]\tau_{m-1},\tau_m]}\widetilde{Z}^m_t. \end{aligned}$$

By (30), we have $Y_{\cdot\wedge\tau_m} = \widetilde{Y}^m_{\cdot\wedge\tau_m}$ and $\mathbb{I}_{\{t\le\tau_m\}}Z_t = \mathbb{I}_{\{t\le\tau_m\}}\widetilde{Z}^m_t$. Hence we can rewrite (29) as

$$Y_{t\wedge\tau_m} = \sup_n \inf_k Y^{n,k}_{\tau_m} + \int_{t\wedge\tau_m}^{\tau_m} F(s,Y_s,Z_s)ds - \int_{t\wedge\tau_m}^{\tau_m} Z_s dW_s.$$

By sending m to $+\infty$, we deduce that (Y,Z) solves (F,ξ). Since $(Y^{n,k},Z^{n,k})$ verifies Lemma 7, we can use Fatou's lemma to prove that $(Y,Z)\in\mathcal{S}^p\times\mathcal{M}^p$.

□

Theorem 8 proves the existence of a $\mathbb{L}^p(p>1)$ solution under (A.1) which to our knowledge the most general asssumption. For example, (A.1)(ii) allows one to get rid of monotonicity in y which is required by, e.g., Pardoux [19] and Briand et al [4], [3], [8]. Meanwhile, in contrast to these works, the generator can also be quadratic by setting $f(|\cdot|)\in\mathcal{I}$. Hence Theorem 8 provides a unified way to construct solutions of both non-quadratic and quadratic BSDEs via the monotone stability result.

On the other hand, Theorem 8 is an extension of Bahlali et al [1] which only studies BSDEs with $\mathbb{L}^2$ integrability and linear-quadratic growth. However, in contrast to their work, (A.1) is not sufficient in our setting to ensure the existence of a maximal or minimal solution, since the double approximation procedure makes the comparison between solutions impossible.

However, to prove the existence of a maximal or minimal solution is no way impossible. Since we have X as the a priori bound for solutions, we can convert the question of existence into the question of existence for quadratic BSDEs with double barriers. This problem has been solved by introducing the notion of generalized BSDEs; see Essaky and Hassani [12].

Let us turn to the uniqueness result. Motivated by Briand and Hu [7] or Da Lio and Ley [9] from the point of view of PDEs, we impose a convexity condition so as to use θ-techinique which proves to be convenient to treat quadratic generators. We start from comparison theorem and then move to uniqueness and stability result. To this end, the following assumptions on (F,ξ) are needed.

Assumption (A.2) Let $p > 1$. There exist $\beta_1, \beta_2 \in \mathbb{R}$, $\gamma_1, \gamma_2 \geq 0$, an $\mathbb{R}^+$-valued Prog-measurable process α, a continuous nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$, $f(|\cdot|) \in \mathcal{I}$ and $F_1, F_2 : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ which are $\mathrm{Prog} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable such that $F = F_1 + F_2$, $|\xi| + |\alpha|_T \in \mathbb{L}^p$ and $\mathbb{P}$-a.s.

(i) for any $t \in [0,T]$, $(y,z) \longmapsto F(t,y,z)$ is continuous;

(ii) $F_1(t,y,z)$ is monotonic in y and Lipschitz-continuous in z, and $F_2(t,y,z)$ is monotonic at $y = 0$ and of linear-quadratic growth in z, i.e., for any $t \in [0,T]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$,

$$\begin{aligned} \operatorname{sgn}(y-y')\big(F_1(t,y,z) - F_1(t,y',z)\big) &\leq \beta_1 |y-y'|, \\ \big|F_1(t,y,z) - F_1(t,y,z')\big| &\leq \gamma_1 |z-z'|, \\ \operatorname{sgn}(y) F_2(t,y,z) &\leq \beta_2 |y| + \gamma_2 |z| + f(|y|)|z|^2; \end{aligned}$$

(iii) for any $t \in [0,T]$, $(y,z) \longmapsto F_2(t,y,z)$ is convex;

(iv) for any $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|F(t,y,z)| \leq \alpha_t + \varphi(|y|) + (\gamma_1 + \gamma_2)|z| + f(|y|)|z|^2.$$

Intuitively, (A.2) specifies an additive structure consisting of two classes of BSDEs. The cases where $F_2 = 0$ coincide with classic existence and uniqueness results for $\mathbb{R}$-valued BSDEs; see, e.g., Pardoux [20] or Briand et al [4], [3]. When $F_1 = 0$, the BSDEs concern and generalize those studied by Bahlali et al [2]. Given convexity as an additional requirement, we can prove an existence and uniqueness result in the presence of both components. This can be seen as a general version of the additive structure discussed in Section 4 and a complement to the quadratic BSDEs studied by Bahlali et al [2] and Briand and Hu [7].

We start our proof of comparison theorem by observing that (A.2) implies (A.1). Hence the existence of a $\mathbb{L}^p (p > 1)$ solution is ensured.

Theorem 9 (Comparison) *Let $p > 1$, and $(Y,Z), (Y',Z') \in \mathcal{S}^p \times \mathcal{M}$ be solutions of (F,ξ), (F',ξ'), respectively. If $\mathbb{P}$-a.s. for any $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, $F(t,y,z) \leq F'(t,y,z)$ and $\xi \leq \xi'$, and F verifies (A.2), then $\mathbb{P}$-a.s. $Y_\cdot \leq Y'_\cdot$.*

Proof. We introduce the notations used throughout the proof. For any $\theta \in (0,1)$, define

$$\begin{aligned} \delta F_t &:= F(t,Y'_t,Z'_t) - F'(t,Y'_t,Z'_t), \\ \delta_\theta Y &:= Y - \theta Y', \\ \delta Y &:= Y - Y', \end{aligned}$$

and $\delta_\theta Z, \delta Z$, etc. analogously. θ-technique applied to the generators yields

$$\begin{aligned} &F(t,Y_t,Z_t) - \theta F'(t,Y'_t,Z'_t) \\ &= \big(F(t,Y_t,Z_t) - \theta F(t,Y'_t,Z'_t)\big) + \theta\big(F(t,Y'_t,Z'_t) - F'(t,Y'_t,Z'_t)\big) \\ &= \theta \delta F_t + \big(F(t,Y_t,Z_t) - \theta F(t,Y'_t,Z'_t)\big) \\ &= \theta \delta F_t + \big(F_1(t,Y_t,Z_t) - \theta F_1(t,Y'_t,Z'_t)\big) + \big(F_2(t,Y_t,Z_t) - \theta F_2(t,Y'_t,Z'_t)\big). \end{aligned} \tag{31}$$

By (A.2)(iii),

$$\begin{aligned} F_2(t,Y_t,Z_t) &= F_2\Big(t, \theta Y'_t + (1-\theta)\frac{\delta_\theta Y_t}{1-\theta}, \theta Z'_t + (1-\theta)\frac{\delta_\theta Z_t}{1-\theta}\Big) \\ &\leq \theta F_2(t,Y'_t,Z'_t) + (1-\theta) F_2\Big(t, \frac{\delta_\theta Y_t}{1-\theta}, \frac{\delta_\theta Z_t}{1-\theta}\Big). \end{aligned}$$

Hence we have

$$F_2(t, Y_t, Z_t) - \theta F_2(t, Y_t', Z_t') \leq (1-\theta) F_2(t, \frac{\delta_\theta Y_t}{1-\theta}, \frac{\delta_\theta Z_t}{1-\theta}). \tag{32}$$

Let u be the function defined in Section 2 associated with a function of class $\mathcal{I}$ to be determined later. Denote $u((\delta_\theta Y_t)^+), u'((\delta_\theta Y_t)^+), u''((\delta_\theta Y_t)^+)$ by u_t, u_t', u_t'', respectively. It is then known from Section 2 that $u_t \geq 0$ and $u_t' > 0$. For any $\tau \in \mathcal{T}$, Tanaka's formula applied to $(\delta_\theta Y)^+$, Itô-Krylov formula applied to $u((\delta_\theta Y_t)^+)$ and Lemma 3 give

$$\begin{aligned}
&|u_{t\wedge\tau}|^p + \frac{p(p-1)}{2}\int_{t\wedge\tau}^{\tau} \mathbb{I}_{\{\delta_\theta Y_s>0\}} |u_s|^{p-2}|u_s'|^2|\delta_\theta Z_s|^2 ds \\
&\quad \leq |u_\tau|^p + p\int_{t\wedge\tau}^{\tau} \mathbb{I}_{\{\delta_\theta Y_s>0\}} |u_s|^{p-1} \underbrace{\Big(u_s'\big(F(s,Y_s,Z_s) - \theta F'(s,Y_s',Z_s')\big) - \frac{1}{2}u_s''|\delta_\theta Z_s|^2\Big)}_{:=\Delta_s} ds \\
&\quad - p\int_{t\wedge\tau}^{\tau} \mathbb{I}_{\{\delta_\theta Y_s>0\}} |u_s|^{p-1} u_s' \delta_\theta Z_s dW_s.
\end{aligned} \tag{33}$$

By (31), (32), (A.2)(ii) and $\delta F \leq 0$, we deduce that, on $\{\delta_\theta Y_s > 0\}$,

$$\Delta_s \leq u_s'\Big(F_1(s,Y_s,Z_s) - \theta F_1(s,Y_s',Z_s') + \beta_2(\delta_\theta Y_s)^+ + \gamma_2|\delta_\theta Z_s| + \frac{f(\frac{|\delta_\theta Y_s|}{1-\theta})}{1-\theta}|\delta_\theta Z_s|^2\Big) - \frac{1}{2}u_s''|\delta_\theta Z_s|^2.$$

To eliminate the quadratic term, we associate u with $\frac{f(\frac{|\cdot|}{1-\theta})}{1-\theta}$, i.e.,

$$\begin{aligned}
u(x) :&= \int_0^x \exp\Big(2\int_0^y \frac{f(\frac{|u|}{1-\theta})}{1-\theta} du\Big) dy \\
&= \int_0^x \exp\Big(2\int_0^{\frac{y}{1-\theta}} f(|u|) du\Big) dy.
\end{aligned}$$

Hence, on $\{\delta_\theta Y_s > 0\}$, the above inequality gives

$$\Delta_s \leq u_s'\big(F_1(s,Y_s,Z_s) - \theta F_1(s,Y_s',Z_s') + \beta_2(\delta_\theta Y_s)^+ + \gamma_2|\delta_\theta Z_s|\big). \tag{34}$$

We are about to send θ to 1, and to this end we give some auxiliary facts. Reset $M := \exp\big(2\int_0^\infty f(u)du\big)$. Obviously $1 \leq M < +\infty$. By dominated convergence, for $x \geq 0$, we have

$$\begin{aligned}
&\lim_{\theta\to 1} u(x) = Mx, \\
&\lim_{\theta\to 1} u'(x) = M\mathbb{I}_{\{x>0\}} + \mathbb{I}_{\{x=0\}}.
\end{aligned} \tag{35}$$

Taking (34) and (35) into account, we come back to (33) and send θ to 1. Fatou's lemma used to the ds-integral on the left-hand side of (33) and dominated convergence used to the rest integrals give

$$\begin{aligned}
&((\delta Y_{t\wedge\tau})^+)^p + \frac{p(p-1)}{2}\int_{t\wedge\tau}^{\tau} \mathbb{I}_{\{\delta Y_s>0\}}((\delta Y_s)^+)^{p-2}|\delta Z_s|^2 ds \\
&\leq ((\delta Y_\tau)^+)^p + p\int_{t\wedge\tau}^{\tau} \mathbb{I}_{\{\delta Y_s>0\}}((\delta Y_s)^+)^{p-1}(F_1(s,Y_s,Z_s) - F_1(s,Y_s',Z_s') + \beta_2(\delta Y_s)^+ + \gamma_2|\delta Z_s|)ds \\
&- p\int_{t\wedge\tau}^{\tau} \mathbb{I}_{\{\delta Y_s>0\}}((\delta Y_s)^+)^{p-1}\delta Z_s dW_s.
\end{aligned} \tag{36}$$

Moreover, (A.2)(ii) implies

$$\mathbb{I}_{\{\delta Y_s>0\}}\big(F_1(s,Y_s,Z_s)-F_1(s,Y'_s,Z'_s)\big)\leq \mathbb{I}_{\{\delta Y_s>0\}}\big(\beta_1(\delta Y_s)^+ +\gamma_1|\delta Z_s|\big).$$

We then use this inequality to (36). To eliminate the local martingale, we replace τ by its localization sequence $\{\tau_n\}_{n\in\mathbb{N}^+}$. By the same way of estimation as in Lemma 7 (a priori estimate (ii)), we obtain

$$((\delta Y_{t\wedge\tau_n})^+)^p\leq c\mathbb{E}\big[((\delta Y_{\tau_n})^+)^p\big|\mathcal{F}_t\big],$$

where c is a constant only depending on $T,\beta_1,\beta_2,\gamma_1,\gamma_2,p$. Since $Y,Y'\in\mathcal{S}^p$ and $\mathbb{P}$-a.s. $\xi\leq\xi'$, dominated convergence yields $\mathbb{P}$-a.s. $Y_t\leq Y'_t$. Finally, by the continuity of Y and Y' we conclude that $\mathbb{P}$-a.s. $Y_\cdot\leq Y'_\cdot$.

□

As a byproduct, we obtain the following existence and uniqueness result.

Corollary 10 (Uniqueness) *Let (A.2) hold for (F,ξ). Then there exists a unique solution in $\mathcal{S}^p\times\mathcal{M}^p$.*

Proof. (A.2) implies (A.1). Hence existence result holds. The uniqueness is immediate from Theorem 9 (comparison theorem).

□

It turns out that a stability result also holds given the convexity condition. We denote (F,ξ) satisfying (A.2) by (F,F_1,F_2,ξ). We set $\mathbb{N}^0:=\mathbb{N}^+\cup\{0\}$.

Proposition 11 (Stability) *Let $p>1$. Let $(F^n,F_1^n,F_2^n,\xi^n)_{n\in\mathbb{N}^0}$ satisfy (A.2) associated with $(\alpha^n,\beta_1,\beta_2,\gamma_1,\gamma_2,\varphi,f)$, and (Y^n,Z^n) be their unique solutions in $\mathcal{S}^p\times\mathcal{M}^p$, respectively. If $\xi^n-\xi\longrightarrow 0$ and $\int_0^T|F^n-F^0|(s,Y_s^0,Z_s^0)ds\longrightarrow 0$ in $\mathbb{L}^p$ as n goes to $+\infty$, then (Y^n,Z^n) converges to (Y,Z) in $\mathcal{S}^p\times\mathcal{M}^p$.*

Proof. We prove the stability result in the spirit of Theorem 9 (comparison theorem). For any $\theta\in(0,1)$, define

$$\begin{aligned}\delta F_t^n&:=F^0(t,Y_t^0,Z_t^0)-F^n(t,Y_t^0,Z_t^0),\\ \delta_\theta Y^n&:=Y^0-\theta Y^n,\\ \delta Y^n&:=Y^0-Y^n,\end{aligned}$$

and $\delta_\theta Z^n,\delta Z^n$, etc. analogously. We observe the θ-difference of the generators. Likewise, (A.2)(iii) implies that

$$\begin{aligned}&F^0(t,Y_t^0,Z_t^0)-\theta F^n(t,Y_t^n,Z_t^n)\\ &=\delta F_t^n+\big(F^n(t,Y_t^0,Z_t^0)-\theta F^n(t,Y_t^n,Z_t^n)\big)\\ &\leq\delta F_t^n+\big(F_1^n(t,Y_t^0,Z_t^0)-\theta F_1^n(t,Y_t^n,Z_t^n)\big)+(1-\theta)F_2^n(t,\frac{\delta_\theta Y_s^n}{1-\theta},\frac{\delta_\theta Z_s^n}{1-\theta}).\end{aligned}$$

We first prove convergence of Y^n and later use it to show that Z^n also converges.

(i). By exactly the same arguments as in Theorem 9 but keeping δF^n_t along the deductions, we obtain

$$
\begin{aligned}
&((\delta Y^n_t)^+)^p + \frac{p(p-1)}{2}\int_t^T \mathbb{I}_{\{\delta Y^n_s>0\}}((\delta Y^n_s)^+)^{p-2}|\delta Z^n_s|^2 ds \\
&\le ((\delta \xi^n)^+)^p + p\int_t^T \mathbb{I}_{\{\delta Y^n_s>0\}}((\delta Y^n_s)^+)^{p-1}\big(|\delta F^n_s| + (\beta_1+\beta_2)(\delta Y^n_s)^+ + (\gamma_1+\gamma_2)|\delta Z^n_s|\big)ds \\
&- p\int_t^T \mathbb{I}_{\{\delta Y^n_s>0\}}((\delta Y^n_s)^+)^{p-1}\delta Z^n_s dW_s. \qquad (37)
\end{aligned}
$$

By the same way of estimation as in Lemma 7 (a priori estimate (ii)), we obtain

$$
\mathbb{E}\big[(((\delta Y^n)^+)^*)^p\big] \le c\Big(\mathbb{E}\big[((\delta\xi^n)^+)^p\big] + \mathbb{E}\big[||\delta F^n_\cdot||^p_T\big]\Big),
$$

where c is a constant only depending on $T,\beta_1,\beta_2,\gamma_1,\gamma_2,p$. Interchanging Y^0 and Y^n and analogous deductions then yield

$$
\mathbb{E}\big[(((-\delta Y^n)^+)^*)^p\big] \le c\Big(\mathbb{E}\big[((-\delta\xi^n)^+)^p\big] + \mathbb{E}\big[||\delta F^n_\cdot||^p_T\big]\Big).
$$

Hence a combination of the two inequalities implies the convergence of Y^n.

(ii). To prove the convergence of Z^n, we combine the arguments in Lemma 6 (a priori estimate (i)) and Theorem 9. To this end, we introduce the function v defined in Section 2 associated with a function of class $\mathcal{I}$ to be determined later. By Itô-Krylov formula,

$$
\begin{aligned}
v(\delta_\theta Y^n_0) &= v(\delta_\theta\xi^n) + \int_0^T v'(\delta_\theta Y^n_s)\big(F^0(s,Y^0_s,Z^0_s) - \theta F^n(s,Y^n_s,Z^n_s)\big)ds \\
&\quad - \frac{1}{2}\int_0^T v''(\delta_\theta Y^n_s)|\delta_\theta Z^n_s|^2 ds - \int_0^T v'(\delta_\theta Y^n_s)\delta_\theta Z^n_s dW_s. \qquad (38)
\end{aligned}
$$

Note that (A.2)(ii)(iii) and $v'(\delta_\theta Y^n_s) = \operatorname{sgn}(\delta_\theta Y^n_s)|v'(\delta_\theta Y^n_s)|$ give

$$
\begin{aligned}
v'(\delta_\theta Y^n_s)&\big(F^0(s,Y^0_s,Z^0_s) - \theta F^n(s,Y^n_s,Z^n_s)\big) \\
&\le |v'(\delta_\theta Y^n_s)||\delta F^n_s| \\
&+ |v'(\delta_\theta Y^n_s)|\operatorname{sgn}(\delta_\theta Y^n_s)\big(F^n_1(s,Y^0_s,Z^0_s) - \theta F^n_1(s,Y^n_s,Z^n_s)\big) \\
&+ |v'(\delta_\theta Y^n_s)|\Big(\beta_2|\delta_\theta Y^n_s| + \gamma_2|\delta_\theta Z^n_s| + \frac{f(\frac{|\delta_\theta Y^n_s|}{1-\theta})}{1-\theta}|\delta_\theta Z^n_s|^2\Big). \qquad (39)
\end{aligned}
$$

We associate v with $\frac{f(\frac{|\cdot|}{1-\theta})}{1-\theta}$ so as to eliminate the quadratic term. Note that

$$
\begin{aligned}
&\lim_{\theta\to 1} v(x) = \frac{1}{2}|x|^2, \\
&\lim_{\theta\to 1} v'(x) = x. \qquad (40)
\end{aligned}
$$

With (39), (40) and (A.2)(ii), we come back to (38) and send θ to 1. This gives

$$
\begin{aligned}
\frac{1}{2}\int_0^T |\delta Z^n_s|^2 ds &\le \frac{1}{2}|\delta\xi^n|^2 + \int_0^T |\delta Y^n_s|\big(|\delta F^n_s| + (|\beta_1|+|\beta_2|)|\delta Y^n_s| + (\gamma_1+\gamma_2)|\delta Z^n_s|\big)ds \\
&\quad - \int_0^T \delta Y^n_s\delta Z^n_s dW_s.
\end{aligned}
$$

Now we use the same way of estimation as in Lemma 6 to obtain

$$\mathbb{E}\Big[\Big(\int_0^T |\delta Z_s^n|^2 ds\Big)^{\frac{p}{2}}\Big] \leq c\mathbb{E}\big[((\delta Y^n)^*)^p + \big||\delta F_\cdot^n|\big|_T^p\big],$$

where c is a constant only depending on $T, \beta_1, \beta_2, \gamma_1, \gamma_2, p$. The convergence of Z^n is then immediate from (i).

□

Remark. So far we have obtained the existence and uniqueness of a $\mathbb{L}^p(p > 1)$ solution. The solvability for $p = 1$ is not included due to the failure of Lemma 7 (a priori estimate (ii)). One may overcome this difficulty by imposing additional structure conditions as in Briand et al [3], [6]. To save pages the analysis of $\mathbb{L}^1$ solutions is hence omitted.

6 Applications to Quadratic PDEs

In this section, we give an application of our results to quadratic PDEs. More precisely, we prove the probablistic representation for the nonlinear Feymann-Kac formula associated with the BSDEs in our study. Let us consider the following semilinear PDE

$$\begin{aligned} &\partial_t u(t,x) + \mathcal{L}u(t,x) + F(t,x,u(t,x),\sigma^\top \nabla_x u(t,x)) = 0, \\ &u(T,\cdot) = g, \end{aligned} \tag{41}$$

where $\mathcal{L}$ is the infinitesimal generator of the solution X^{t_0,x_0} to the Markovian SDE

$$X_t = x_0 + \int_{t_0}^t b(s,X_s)ds + \int_{t_0}^t \sigma(s,X_s)dB_s, \tag{42}$$

for any $(t_0,x_0) \in [0,T] \times \mathbb{R}^n$, $t \in [t_0,T]$. Denote a solution to the BSDE

$$Y_t = g(X_T^{t_0,x_0}) + \int_t^T F(s,X_s^{t_0,x_0},Y_s,Z_s)ds - \int_t^T Z_s dW_s,\ t \in [t_0,T], \tag{43}$$

by $(Y^{t_0,x_0}, Z^{t_0,x_0})$ or (Y,Z) when there is no ambiguity. The probablistic representation for nonlinear Feymann-Kac formula consists of proving that, in Markovian setting, $u(t,x) := Y_t^{t,x}$ is a solution at least in the viscosity sense to (41) when the source of nonlinearity F is quadratic in $\nabla_x u(t,x)$ and g is an unbounded function. To put it more precisely, let us introduce the FBSDEs.

The Forward Markovian SDEs. Let $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{n\times d}$ be continuous functions and assume there exists $\beta \geq 0$ such that $\mathbb{P}$-a.s. for any $t \in [0,T]$, $|b(t,0)| + |\sigma(t,0)| \leq \beta$ and $b(t,x), \sigma(t,x)$ are Lipschitz-continuous in x, i.e., $\mathbb{P}$-a.s. for any $t \in [0,T]$, $x, x' \in \mathbb{R}^n$,

$$|b(t,x) - b(t,x')| + |\sigma(t,x) - \sigma(t,x')| \leq \beta|x - x'|.$$

Then for any $(t_0,x_0) \in [0,T] \times \mathbb{R}^n$, (42) has a unique solution X^{t_0,x_0} in $\mathcal{S}^p$ for any $p \geq 1$.

The Markovian BSDE. We continue with the setting of the forward equations above. Set $q \geq 1$. Let $F_1, F_2 : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ be continuous functions, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a continuous nondecreasing function with $\varphi(0) = 0$ and $f(|\cdot|) \in \mathcal{I}$, and assume moreover $F = F_1 + F_2$ such that

(i) $F_1(t,x,y,z)$ is monotonic in y and Lipschitz-continuous in z, and $F_2(t,x,y,z)$ is monotonic at $y=0$ and of linear-quadratic growth in z, i.e., for any $(t,x) \in [0,T] \times \mathbb{R}^n$, $y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d$,

$$\begin{aligned}\operatorname{sgn}(y-y')\big(F_1(t,x,y,z) - F_1(t,x,y',z)\big) &\leq \beta|y-y'|,\\ \big|F_1(t,x,y,z) - F_1(t,x,y,z')\big| &\leq \beta|z-z'|,\\ \operatorname{sgn}(y)F_2(t,x,y,z) &\leq \beta|y| + \beta|z| + f(|y|)|z|^2;\end{aligned}$$

(ii) $(y,z) \longmapsto F_2(t,x,y,z)$ is convex ;

(iii) for any $(t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned}|F(t,x,y,z)| &\leq \beta\big(1+|x|^q+2|z|\big) + \varphi(|y|) + f(|y|)|z|^2,\\ |g(x)| &\leq \beta\big(1+|x|^q\big).\end{aligned}$$

Since $X^{t_0,x_0} \in \mathcal{S}^p$ for any $p \geq 1$, the above structure conditions on F and g allow one to use Corollary 10 to construct a unique solution $(Y^{t_0,x_0}, Z^{t_0,x_0})$ in $\mathcal{S}^p \times \mathcal{M}^p$ of (43) for any $p > 1$. Moreover, by standard arguments, $Y_{t_0}^{t_0,x_0}$ is deterministic for any $(t_0,x_0) \in [0,T] \times \mathbb{R}^n$. Hence $u(t,x)$ defined as $Y_t^{t,x}$ is a deterministic function. With this fact we now turn to the main result of this section: u is a viscosity solution of (41). Before our proof let us recall the definition of a viscosity solution.

Viscosity Solution. A continuous function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is called a viscosity subsolution (respectively supersolution) to (41) if $u(T,x) \leq g(x)$ (respectively $u(T,x) \geq g(x)$) and for any smooth function ϕ such that $u - \phi$ reaches the local maximum (respectively local minimum) at (t_0,x_0), we have

$$\partial_t\phi(t_0,x_0) + \mathcal{L}\phi(t_0,x_0) + F(t_0,x_0,u(t_0,x_0),\sigma^\top\nabla_x\phi(t_0,x_0)) \geq 0 \text{ (respectively } \leq 0).$$

A function u is called a viscosity solution to (41) if it is both a viscosity subsolution and supersolution.

Proposition 12 *Given the above assumptions, $u(t,x)$ is continuous with*

$$|u(t,x)| \leq c\big(1+|x|^q\big),$$

where c is a constant. Moreover, u is a viscosity solution to (41).

Proof. Due to the Lipschitz-continuity of b and σ, $X^{t,x}$ is continuous in (t,x), e.g., in mean square sense. The continuity of u is then an immediate consequence of Theorem 11 (stability). The proof relies on standard arguments and hence is omitted. By Lemma 7 (a priori estimate (ii)), we prove that u satisfies the above polynomial growth. It thus remains to prove that u is a viscosity solution to (41).

Let ϕ be a smooth function such that $u - \phi$ reaches local maximum at (t_0,x_0). Without loss of generality we assume that the local maximum is global and $u(t_0,x_0) = \phi(t_0,x_0)$. We aim at proving

$$\partial_t\phi(t_0,x_0) + \mathcal{L}\phi(t_0,x_0) + F(t_0,x_0,u(t_0,x_0),\sigma^\top\nabla_x\phi(t_0,x_0)) \geq 0.$$

From (43) we obtain

$$Y_t = Y_{t_0} - \int_{t_0}^t F(s, X_s^{t_0,x_0}, Y_s, Z_s)ds + \int_{t_0}^t Z_s dW_s.$$

By Itô's formula,

$$\phi(t, X_t^{t_0,x_0}) = \phi(t_0, x_0) + \int_{t_0}^{t} \big\{\partial_s \phi + \mathcal{L}\phi\big\}(s, X_s^{t_0,x_0}) ds + \int_{t_0}^{t} \sigma^\top \nabla_x \phi(s, X_s^{t_0,x_0}) dW_s.$$

Now we take any $t \in [t_0, T]$. Note that the existence of a unique solution of (42) and (43) implies by Markov property that $Y_t = u(t, X_t^{t_0,x_0})$. Hence, $\phi(t, X_t^{t_0,x_0}) \geq u(t, X_t^{t_0,x_0}) = Y_t$. By touching property, on the set $\big\{\phi(t, X_t^{t_0,x_0}) = Y_t\big\}$ we have

$$\begin{aligned}&\partial_t \phi(t, X_t^{t_0,x_0}) + \mathcal{L}\phi(t, X_t^{t_0,x_0}) + F(t, X_t^{t_0,x_0}, Y_t, Z_t) \geq 0 \quad \mathbb{P}\text{-a.s.},\\ &\sigma^\top \nabla_x \phi(t, X_t^{t_0,x_0}) - Z_t = 0 \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

Now we set $t = t_0$. We have $\phi(t_0, X_{t_0}^{t_0,x_0}) = \phi(t_0, x_0) = u(t_0, x_0) = Y_{t_0}$. Moreover, the above equality implies $Z_{t_0} = \sigma^\top \nabla_x \phi(t_0, x_0)$. Plugging the two equalities into the above inequality gives

$$\partial_t \phi(t_0, x_0) + \mathcal{L}\phi(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), \sigma^\top \nabla_x \phi(t_0, x_0)) \geq 0.$$

Hence u is a viscosity subsolution of (41). u being a viscosity supersolution and thus a viscosity solution can be proved analogously.

□

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