BUILDING YOUR PATH TO ESCAPE FROM HOME

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Abstract. Random walks on dynamic graphs have received increasingly more attention over the last decade from different academic communities. Despite the relatively large literature very little is known about random walks that construct the graph where they walk while moving around. In this paper we study one of the simplest conceivable models of this kind, that works as follows: before every walker step, with probability p a new leaf is added to the vertex currently occupied by the walker. The model grows trees and we call it the Bernoulli Growth Random Walk (BGRW). We address the classical dichotomy between recurrence and transience in BGRW and prove that the random walk is transient for any $p \in (0, 1]$.

1. Introduction

Random walks on graphs that change over time have received much attention over the past decades and continues to be a fertile ground for theoretical progress in probability. Within this context, a large body of work assumes only edge (or node) weights change over time while the graph structure (i.e., set of edges) remains constant. Here lies reinforced random walks and random walks in random environments [1, 7, 8, 9, 13]. A much smaller line of work assumes the graph structure (edges and nodes) changes over time. However, such works generally assume graph dynamics to be independent of the walker [4, 11, 14] (an exception is [12]). This work explores a novel model where the random walk constructs its own graph, mutually coupling the walker and graph dynamics.

Models of random walks that construct their graph were first proposed as network growth models that embody *preferential attachment* and *local attachment* principles. To facilitate analysis, such models had a *restart* step where the random walker was periodically placed uniformly at random on the graph constructed [5, 15]. More recently, the following model without restarts was proposed and analyzed [2]:

(0) start with a node with a self-loop;
(1) let the walker take s steps on the current graph;
(2) add and connect a new node to the current location of the walker;
(3) go to step 1.

We refer to this model as NRRW (No Restart Random Walk).

A fundamental question on dynamic graphs is the recurrence or transience nature of the walker. For example and most notably, for edge reinforced random walks on lattices of any dimension there is a phase transition from recurrence to transience with respect to the reinforcement value [3, 9]. A similar dichotomy between transient and recurrent was also more recently established for the once-reinforced random walk [6, 13]. For the NRRW model, the walker is transient for $s = 1$ and recurrent for all s even [2, 10]. This parity dichotomy is related to the fact that for s even the walker must continue to visit the root (traversing the self-loop) in order to continue to extend the tree height. The case for $s > 1$ odd is conjectured to be transient [10].

We consider a model variation that avoids the parity dichotomy of NRRW and works as follows:

(0) start with two nodes connected by an edge with the walker in one of them;
(1) with probability p, add and connect a new node to the current location of the walker;
(2) let the walker take one step on the current graph;
(3) go to step 1.

We refer to this model as BGRW (Bernoulli Growth Random Walk), and a more formal definition is given in Section 2. Note that for large enough p BGRW is expected to be similar to NRRW with $s = 1$. Indeed, for large enough p it can be shown through simpler arguments that the walker is transient, as with the case $s = 1$.

Figure 1 shows simulated sample paths from the BGRW model for different values of p. While all trees have exactly the same number of nodes ($n = 5000$) their structural difference is striking. For larger values of p the generated trees are very slim and long, as p decreases the trees become fatter and shorter. Intuitively, with large p the random walk can escape more easily, while for small p the random walk wonders more. However, our findings show that for any fixed p, the random walk escapes.

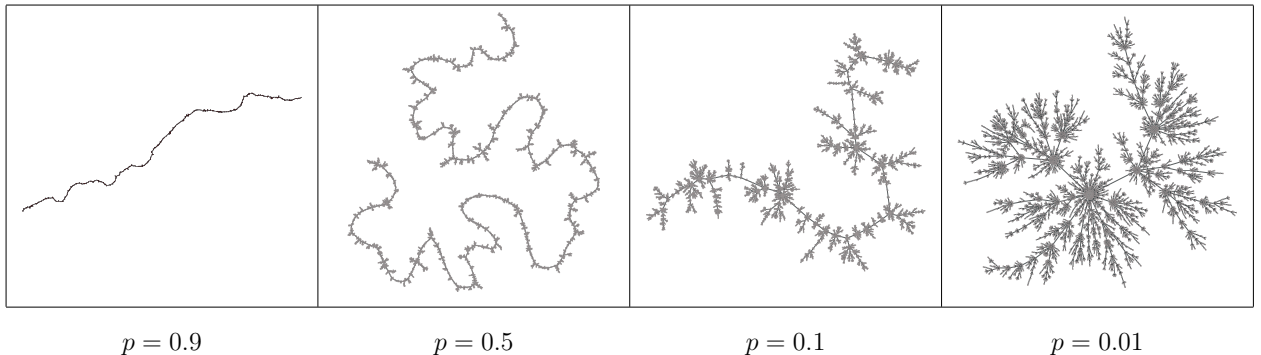$p = 0.9$ $p = 0.5$ $p = 0.1$ $p = 0.01$

FIGURE 1. Trees with $n = 5000$ nodes generated by simulating BGRW using different values for p, illustrating the diversity in the tree structure.

In this paper, we show that walker is transient for any $p > 0$. This indicates that without the parity constraints of NRRW, the random walk always escapes from its current location irrespective of the (expected) number of steps between two node additions. Note that prior models and results constrain the walker to some graph. Our findings suggests that when the walker is *a prori* unconstrained, it constructs a graph that allows it to escape. We believe this finding adds a significant contribution to the ongoing discussion of random walks on dynamic graphs.

1.1. **Organization.** The remainder of the paper is organized as follows. Section 2 provides a formal definition of the model, the statement for our main result and the outline for the proof. In Section 3 we prove that the walker achieves long distances (of order $\log^M n$) from its initial position in less than n steps with high probability. In Section 4 we introduce a new process called *Loop walker* and prove lower bounds for the time this process takes to cross a line of length ℓ. In Section 5 we construct a coupling between the *BGRW* and the *Loop* walker in such way that the latter is always closer to the origin than the former. Finally, in Section 6 we put all the pieces together to prove that the *BGRW* is transient.

2. Model and Main Result

The *BGRW* process is a Markov chain $\{(T_n, X_n)\}_{n \in \mathbb{N}}$ where T_n is a rooted tree and X_n is one of its vertices. The model has one parameter $p \in [0, 1]$. In order to describe it, let $\{Z_n\}_{n \in \mathbb{N}}$ be a sequence of *i.i.d* random variables with $Z_1 \sim \text{Ber}(p)$ and fix a rooted locally finite tree T_0 and one of its vertices x_0 to be the initial state. We then define the *BGRW* inductively as follows: start from (T_0, x_0). Then,

(1) Obtain T_{n+1} from T_n by adding a new leaf to X_n whether $Z_{n+1} = 1$;
(2) Obtain X_{n+1} moving X_n to a uniformly chosen neighbor on T_{n+1}.

We stress out the index $n + 1$ of T in (2). This means that X_{n+1} may be a new leaf which was just added to T_n at (1).

From now on, fixed an initial state (T_0, x_0), we denote by $\mathbb{P}_{T_0, x_0}(\cdot)$ the probability measure defined on the space of trajectories of $\{(T_n, X_n)\}_{n \in \mathbb{N}}$ starting from (T_0, x_0). Our main result in this paper is the following theorem.

Theorem 1 (Transience of random walk on growing trees)**.** *For any $p \in (0, 1]$, $\{X_t\}_{t \in \mathbb{N}}$ is transient. That is, for any initial state T_0, X_0 of the process, and any time $m \geq 0$ the set of times n when X_n is in T_m is almost surely finite.*

2.1. **Proof ideas.** The proof of Theorem 1 requires several steps, which we sketch below. For simplicity, we only describe how one would prove that X_0 (which we call "the root") is almost surely visited only a finite number of times. We will derive this from a series of statements.

(1) *Logarithmic distance (trivial).* The simple Lemma 1 shows that, with high probability, X_t will be at distance $\geq \log n$ from X_0 in $t \leq n$ steps. Actually, we simply prove that in n steps the walker sprints from X_0, in the sense that it gives $\log n$ steps in a row in the "opposite direction" of X_0,

(2) *Polylogarithmic distance (less trivial).* We then bootstrap Lemma 1 to obtain a stronger result: in n steps, the walker typically achieves $\log^M n$ distance from X_0, where $M > 0$ can be made arbitrarily large. This is Lemma 2 below.

(3) *Long return time via loop walker.* Lemma 4 implies that, that once the walker is at a distance ℓ from the root, with high probability it typically takes at least $\approx e^{\sqrt{\ell}}$ steps to return there. This is shown via a simplification of our process called the "loop walker". This process is confined to a path connecting X_t to the root. Edges from this path to the rest of the tree are replaced by self-loops. Tree growth corresponds to the number of self-loops increasing over time. This mechanism alone explains why the walker will likely take a long time before it returns to the root. The loop process is explained in section 4;

(4) *End of proof.* With the above tools, we can show that the probability that $X_t = X_0$ for some $2^j < t \leq 2^{j+1}$ is summable in j. An application of the Borel-Cantelli Lemma then finishes the proof.

3. Getting further and further away from home

This section is devoted to prove that the walker X gets distance of order $\log^M n$ in less than n^β for small β. This will be crucial for our argument, since, roughly speaking, we will show that once the walker is sufficiently far away from the root it takes too long to get back.

We start by showing a simpler lemma that assures us distance at least $\log n$.

Lemma 1. *For all $\beta \in (0,1]$ there exists $n_0 = n(\beta, p)$ such that, for all $n \geq n_0$, we have*

$$\inf_{(T_0, x_0)} \mathbb{P}_{T_0, x_0} \left(\exists m \leq n^\beta, d(X_m, root) \geq \beta(4\log(2/p))^{-1} \log n \right) \geq 1 - e^{-n^{\beta/4}}.$$

Proof. Observe that X has probability at least $p/2$ of jumping to a descendant of its current position (the worst case would be if it is on a leaf). Thus, for all $m \in \mathbb{N}$ and all initial state (T_0, x_0), we have

$$\mathbb{P}_{T_0, x_0} \left(X_{m+c\log n} \downarrow, \cdots, X_m \downarrow | \mathcal{F}_{m-1} \right) \geq \left(\frac{p}{2} \right)^{c\log n}.$$

Now, split the interval $[0, n^\beta]$ into subintervals of length $c \log n$. For each subinterval, denote by A_k the following event

(3.1) $$A_k := \left\{ X_{ck\log n} \downarrow, \cdots, X_{c(k-1)\log n} \downarrow \right\}.$$

So, the probability that none of the A_ks occurs is bounded by

$$\mathbb{P}_{T_0, x_0} \left(\bigcap_{k=1}^{n^\beta / c \log n} A_k^c \right) \leq \left[1 - \left(\frac{p}{2} \right)^{c \log n} \right]^{n^\beta / c \log n} \leq \exp\{-n^{\beta/4}\}$$

for sufficiently large n and $c \leq \beta/(4 \log(2/p))$. This proves the Lemma. $\qquad \square$

The next step is to take advantage of distance $\log n$ assured by the previous lemma and bootstrap it to guarantee larger power of $\log n$.

Lemma 2. *For all $\beta \in (0, 1]$ and all positive constant M there exist positive constants $c_1 = c(\beta, p, M)$ and $c_2 = c(\beta, p, M)$ and $n_0 = n(\beta, p, M)$ such that, for all $n \geq n_0$*

$$\inf_{(T_0, x_0)} \mathbb{P}_{T_0, x_0} \left(\exists m \leq n^\beta, d(X_m, root) \geq c_1 \log^M n \right) \geq 1 - e^{-c_2 n^{\beta/4}}.$$

Proof. Observe that Lemma 1 implies this Lemma 2 for the specific value $M = 1$. In the remainder of the proof, we prove that if the statement holds for a fixed M, then it holds for $M + 9/10$. So assume that for a fixed M and all $\beta \in (0, 1]$, there exist c_1 and c_2 and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we have

$$(3.2) \qquad \inf_{(T_0, x_0)} \mathbb{P}_{T_0, x_0} \left(\exists m \leq n^\beta, d(X_m, \text{root}) \geq c_1 \log^M n \right) \geq 1 - e^{-c_2 n^{\beta/4}}.$$

Now, for a fixed $\beta \in (0, 1]$, let c_1 and c_2 be the two positive constants given by the inductive hypothesis when we consider $\beta/2$. For any rooted tree T of height at least $c_1 \log^M n$ and any $x_0 \in V(T)$ whose distance from the root is also at least $c_1 \log^M n$, we denote by y_m the ancestor of x_0 at distance $c_1 \log^M(n)/2$ and by $\mathcal{P}$ the path connecting x_0 and y_m. In this setup we define five stopping conditions for the next movements of X. Their usefulness will be explained later.

(1) There exists $m \leq n^{\beta/2}$ such that $d(X_m, y_m) \geq c_1 \log^M n$;
(2) X jumps to a descendant of x_0;
(3) X visits x_0 at least $\log^2 \log n$ times;
(4) X visits y_m in one of its $\log^2 \log n$ excursions from x_0;
(5) X walks $n^{\beta/2}$ steps,

We say $(1) - (5)$ have occurred from X_m whenever one of the five conditions occurred when we set x_0 as X_m.

Let t be $\log^{M+9/10} n$ and split the interval $[0, n^\beta]$ into $n^{\beta/2}/t$ subintervals of length $n^{\beta/2}t$ each. Now, for $i \in \{1, 2, \cdots, n^{\beta/2}/t\}$ and $k \in \{1, 2, \cdots, t\}$ consider the following stopping times

$$(3.3) \qquad \sigma_0^{(i)} := \inf \left\{ (i-1)n^{\beta/2}t \leq m \leq (i-1)n^{\beta/2}t + n^{\beta/2} | d(X_m, \text{root}) \geq c_1 \log^M n \right\},$$

i.e., $\sigma_0^{(i)}$ is the first time in the firsts $n^{\beta/2}$ steps in the i-th subinterval that X is at distance at least $c_1 \log^M n$ from the root. Also define

(3.4)
$$\sigma_1^{(i)} := \inf \left\{ m \geq \sigma_0^{(i)} \wedge ((i-1)n^{\beta/2}t + n^{\beta/2}) | (1) - (5) \text{ occurred from } X_{\sigma_0^{(i)} \wedge ((i-1)n^{\beta/2}t + n^{\beta/2})} \right\},$$

in words, $\sigma_1^{(i)}$ is the first time, after X sees itself at distance at least $c_1 \log^M n$ from the root, one of the conditions $(1) - (5)$ has occurred in the i-th subinterval. In general,

(3.5)
$$\sigma_k^{(i)} := \inf \left\{ m \geq \sigma_{k-1}^{(i)} | (1) - (5) \text{ occurred from } X_{\sigma_{k-1}^{(i)}} \right\}.$$

Finally, we define the two following events

(3.6)
$$B_m := \{(1) \text{ or } (2) \text{ occurred from } X_m\}.$$

(3.7)
$$D[M, m_1, m_2] := \left\{ \exists m \in [m_1, m_2] \text{ such that } d(X_m, \text{root}) \geq c_1 \log^M n \right\}.$$

Observe that whenever one of the stopping times above stops because of conditions (1) or (2) we may assure that X has increased its distance from the root by at least one unit. Moreover, inside each subinterval, X has t attempts to get farther in this way. So, we say that at each attempt, X has got a *strong fail* whenever it stops because one of the $(3) - (5)$ conditions. And we say that X has got a *weak fail* at the i-th subinterval if it has failed to avoid all the t possible strong fails. More specifically, we are interested in the occurrence of the event below

(3.8)
$$A_i := B_{\sigma_t^{(i)}} \cap \cdots B_{\sigma_1^{(i)}} \cap D \left[M, (i-1)n^{\beta/2}t, (i-1)n^{\beta/2}t + n^{\beta/2} \right],$$

in words, A_i means that in the i-th subinterval, X has gone at distance at least $c_1 \log^M n$ from the root and then stopped because of conditions (1) or (2) in a row of t attempts. It is clear that after the occurrence of A_i, X is at distance at least t from the root.

Regarding the probability of A_i we claim that

Claim 1. *For large enough n, there exist positive constants $c_3 = c_3(\beta, M, p)$ and $c_4 = c_4(\beta, M, p)$ such that for all $i \in \{1, 2, \cdots, n^{\beta/2}/t\}$*

$$\inf_{(T_0, x_0)} \mathbb{P}_{T_0, x_0} (A_i) \geq c_3 e^{-c_4 t \log^2 \log n / \log^M n}.$$

Proof of the claim: By our hypothesis (3.2), $D \left[M, (i-1)n^{\beta/2}t, (i-1)n^{\beta/2}t + n^{\beta/2} \right]$ occurs with probability at least $1 - \exp\{-n^{c_2\beta/8}\}$. Thus, by Markov Property, it suffices to bound from above the occurrence of conditions $(3) - (5)$ conditioned on starting from a rooted tree T of height at least $c_1 \log^M n$ and $x_0' \in V(T)$ at distance at least $c_1 \log^M n$ from the root.

Observe that if X stops by the occurrence of condition (3) it means that X has visited x_0' $\log^2 \log n$ times and at none of them has jumped to a descendant of x_0'. But recall that the

probability of jumping down is at least $p/2$, thus we have the following upper bound for the occurrence of condition (3)

$$(3.9) \qquad \mathbb{P}_{T,x_0'} \left(\sigma_1^{(1)} \text{stops because of (3)} \right) \leq (1 - p/2)^{\log^2 \log n} .$$

For the occurrence of condition (4) a simple comparison with the SRW on $\{1, 2, \cdots, c_1/2 \log^M n\}$ tells us that the probability of X leaves x_0' and visit y_m before it returns to x_0' is at most $2/c_1 \log^M n$. Then, by union bound on the number of excursions we obtain

$$\mathbb{P}_{T,x_0'} \left(\sigma_1^{(1)} \text{stops because of (4)} \right) \leq \frac{2 \log^2 \log n}{c_1 \log^M n}.$$

And for (5), observe that if it has occurred then X has walked $n^{\beta/2}$ steps and has not visited y_m neither has increased its distance from y_m to at least $c_1 \log^M n$. This is the same that a process $\bar{X}$ on the subtree T_{y_m} does not go at distance $c_1 \log^M n$ from the root (y_m) in $n^{\beta/2}$ steps. However, by our hypothesis (3.2), this has probability at most $e^{-c_2 n^{\beta/8}}$, which is less than $\frac{2 \log^2 \log n}{c_1 \log^M n}$ for large n. Thus, an union bound yields

$$\mathbb{P}_{T,x_0'} \left(X \text{ has got a strong fail at the first subinterval} \right) \leq \frac{c_4 \log^2 \log n}{\log^M n}.$$

And then Markov Property allows us to conclude that for every initial tree T_0 and initial position x_0

$$\mathbb{P}_{T_0,x_0} (A_i) \geq \left(1 - \frac{c_4 \log^2 \log n}{\log^M n} \right)^t \geq c_3 e^{-c_4 t \log^2 \log n / \log^M n}$$

for large enough n. This proves the claim. ∎

Finally, another use of Markov property gives us that

$$(3.10) \qquad \mathbb{P}_{T_0,x_0} \left(\bigcap_{i=1}^{n^{\beta/2}/t} A_i^c \right) \leq \left(1 - c_3 e^{-c_4 t \log^2 \log n / \log^M n} \right)^{n^{\beta/2}/t}$$

$$\leq \exp\{-c_3 n^{\beta/2} e^{-c_4 t \log^2 \log n / \log^M n} / t\}$$

$$\leq \exp\{-c_3 n^{\beta/4}\},$$

since $t = \log^{M+9/10} n$ implies that

$$n^{\beta/2} e^{-c_4 t \log^2 \log n / \log^M n} / t = \exp\{\beta/2 \log n - \log t - c_4 \log^{9/10}(n) \log^2 \log n\} \geq n^{\beta/4},$$

for large enough n. The upper bound given by (3.10) tells us that with probability close to one, at some subinterval, X succeed to stop by conditions (1) or (2) in a row, which implies that at some time it finds itself at distance at least $\log^{M+9/10} n$ from the root and this proves the Lemma. □

4. The loop walker process

Now we stop the discussion about the $BGRW$ process, to introduce a simpler process which will help us to understand how long the walker X stays on specifics subgraphs of the random trees $\{T_n\}_{n\in\mathbb{N}}$. Roughly speaking, a loop process on an initial graph G is a random walker such that at each step adds a loop to its position according to a coin and then chooses uniformly one edge of its current position to walk on. In other words, the process is quite similar to the $BGRW$ but here the walker adds loops instead of leaves, which makes it possible for it to stand still.

We will be particularly interested in the loop process over specific graphs which we define before the definition of the process itself. We call a finite graph $\mathcal{B}$ a *backbone* of length ℓ if $\mathcal{B}$ is a path of length ℓ having a loop attached to its $\ell+1$-th vertex and possibly to its other vertices, see figure below. The model has one parameter $p \in [0,1]$ which is the parameter of
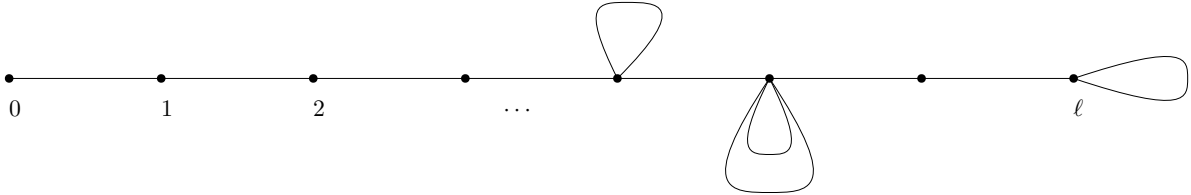

FIGURE 2. A *backbone* of length ℓ

a sequence $\{Z_t\}_{t\in\mathbb{N}}$ of *i.i.d* random variables distributed as a Ber(p). The loop process on a backbone of length ℓ is a stochastic process $\{(\mathcal{B}_t, X_t^{\text{loop}})\}_{t\in\mathbb{N}}$ where $\mathcal{B}_t$ denotes the resultant backbone at time t and X_t^{loop} is one of its $\ell+1$ vertices. Here we will abuse the graph terminology saying *degree* of a vertex even though we do not count loops twice. In this way, we simply write $\deg_t(i)$ to denote the number of edges attached to vertex i at time t. The loop process is also defined inductively according to the update rules below

  (1) Obtain $\mathcal{B}_{t+1}$ from $\mathcal{B}_t$ by adding a new loop to X_t^{loop} whether $Z_{t+1} = 1$;
  (2) Choose uniformly one edge attached to X_t^{loop} in $\mathcal{B}_{t+1}$. Whether the chosen edge is a loop set X_{t+1}^{loop} as X_t^{loop}. Otherwise, X_{t+1}^{loop} becomes the X_t^{loop} neighbor.

As in the definition of the $BGRW$ model, here we also stress out to the index $t+1$ of $\mathcal{B}$ on (2). It means that we may add a new loop at (1) and then choose it at (2).

It is clear that $\{(\mathcal{B}_t, X_t^{\text{loop}})\}_{t\in\mathbb{N}}$ is a Markov chain and for a fixed backbone $\mathcal{B}$ of length ℓ and $i \in \{0, 1, \cdots, \ell\}$ we let $\mathbb{P}_i(\cdot)$ denote the law of $\{(\mathcal{B}_t, X_t^{\text{loop}})\}_{t\in\mathbb{N}}$ when $(\mathcal{B}_0, X_0^{\text{loop}}) \equiv (\mathcal{B}, i)$.

Once we have defined the loop process, we are interested on the time it takes to go from one end of the backbone to the other. More precisely, we would like to obtain bounds on the

stopping time bellow

(4.1) $$\eta_0^{\text{loop}} := \inf \left\{ t \geq 1 \middle| X_t^{\text{loop}} = 0 \right\}$$

when the process starts from ℓ. The next Lemma gives us some estimates.

Lemma 3. *There exist positive constants c_1 and c_2 depending on p only, such that, for all integer K*

$$\mathbb{P}_\ell \left(\eta_0^{\text{loop}} \leq e^K \right) \leq 1 - \left(1 - \frac{1}{2\ell} \right)^{c_2 K} + e^{-c_1 K}$$

Proof. As usual, we begin introducing a few notation and definitions. Consider the following sequence of stopping times

(4.2)
$$\tau_0 \equiv 0,$$
$$\tau_k := \inf \left\{ t > \tau_{k-1} | X_t^{\text{loop}} \neq X_{\tau_{k-1}}^{\text{loop}} \right\}.$$

Observe that the probability of X_t^{loop} leaving its current position is at least $1/t$ since the degree of a vertex at time t is at most t. This implies that τ_k is finite a.s. for all k. This implies us to define the process

(4.3) $$Y_k := X_{\tau_k}^{\text{loop}}.$$

Note that by strong Markov Property, $\{Y_k\}_k$ is a simple RW on $\{0, 1, \cdots, l\}$ with reflecting barriers. Regarding the process $\{Y_k\}_k$ we let σ be the following stopping time

(4.4) $$\sigma := \inf \left\{ k > 0 | Y_k = 0 \right\}.$$

Observe that $\eta_0^{\text{loop}} = \tau_\sigma$.

We prove the Lemma by showing that X^{loop} spends at least $\exp\{K\}$ steps on the vertex ℓ. More precisely, we prove that the degree of ℓ at time τ_σ is at least $\exp\{K\}$, *w.h.p.* To do this, first observe that the degree of a vertex may be written in terms of Y_k and $\Delta \tau_k$ as follows

(4.5) $$deg_{\tau_n}(l) = 2 + \sum_{k=0}^{n} \mathbb{1}\{Y_k = l\} \left(\sum_{m=\tau_k}^{\tau_{k+1}-1} Z_m \right) = 2 + \sum_{k=0}^{n} \mathbb{1}\{Y_k = l\} Bin\left(\Delta \tau_k, p\right).$$

Also notice that whether $Y_k = l$, then the number of steps X^{loop} spends on l is exactly $\Delta \tau_k$, which satisfies

$$\Delta \tau_k \geq Geo\left(1/deg_{\tau_k}(l)\right).$$

To see the bound above, consider the random variable which counts the number of steps X^{loop} spends on l by choosing only the loops which were already attached on l when X^{loop} arrived at l. This random variable is clearly smaller than $\Delta \tau_k$ and follows a geometric distribution of parameter $1/deg_{\tau_k}(l)$. Thus, we have

(4.6) $$deg_{\tau_n}(l) \geq \sum_{k=0}^{n} \mathbb{1}\{Y_k = l\} Bin\left(Geo\left(1/deg_{\tau_k}(l)\right), p\right).$$

Regarding the random variables $Bin\left(Geo\left(1/deg_{\tau_k}(l)\right),p\right)$, we claim that

Claim 2. *For all $\varepsilon \in (0,1)$, there exists a positive constant $q = q(\varepsilon, p)$ such that*

$$(4.7) \qquad \mathbb{P}_l\left(Y_k = l, Bin\left(Geo\left(1/deg_{\tau_k}(l)\right),p\right) \geq \varepsilon p deg_{\tau_k}(l) \middle| \mathcal{F}_{\tau_k}\right) \geq q\mathbb{1}\{Y_k = l\}.$$

Proof of the claim: To simplify our writing, write

$$G_k := Geo\left(1/deg_{\tau_k}(l)\right); d_k := deg_{\tau_k}(l)$$

and let $\tilde{\mathcal{F}}_{\tau_k}$ be the σ-algebra generated by G_k and $\mathcal{F}_{\tau_k}$. Now, by Chernoff bounds, we have that

$$(4.8) \quad \begin{aligned} \mathbb{P}_l\left(Y_k = l, Bin\left(G_k,p\right) \geq \varepsilon p d_k, G_k \geq d_k \middle| \tilde{\mathcal{F}}_{\tau_k}\right) &\geq \left(1 - e^{-(1-\varepsilon)^2 p G_k}\right)\mathbb{1}\{Y_k = l, G_k \geq d_k\} \\ &\geq \left(1 - e^{-(1-\varepsilon)^2 p d_k}\right)\mathbb{1}\{Y_k = l, G_k \geq d_k\}. \end{aligned}$$

Recall that d_k is greater than 2 for all k. So, taking the conditional expectation *wrt* $\mathcal{F}_{\tau_k}$ on the above inequality yields

$$(4.9)$$

$$\begin{aligned} \mathbb{P}_l\left(Y_k = l, Bin\left(G_k,p\right) \geq \varepsilon p d_k, G_k \geq d_k \middle| \tilde{\mathcal{F}}_{\tau_k}\right) &\geq \left(1 - e^{-2(1-\varepsilon)^2 p}\right)\mathbb{P}_l\left(Y_k = l, G_k \geq d_{\tau_k} \middle| \mathcal{F}_{\tau_k}\right) \\ &\geq \left(1 - e^{-2(1-\varepsilon)^2 p}\right)\left(1 - e^{-1}\right)\mathbb{1}\{Y_k = l\}, \end{aligned}$$

which proves the claim. ∎

The above claim tells us that conditionally on the past, every time $\{Y_k\}_k$ visits l it has probability at least q of increasing the degree of l by a factor of at least $1 + \varepsilon p$. This points out that the degree of l must be at least exponential of the number of visits l receives from $\{Y_k\}_k$. So, let $N_\sigma(l)$ the number of visits made by Y to l before it reaches vertex 0. Since Y is a simple random walk on $\{0, 1, \cdots, l\}$, $N_\sigma(l)$ follows a geometric distribution of parameter $1/2l$. Moreover, the random variable that counts how many times we have successfully multiplied the degree of l by $1 + \varepsilon p$, which may be written as follows

$$(4.10) \qquad W = \sum_{k=0}^{\sigma} \mathbb{1}\{Y_k = l\}\mathbb{1}\{Bin\left(Geo\left(1/deg_{\tau_k}(l)\right),p\right) \geq \varepsilon p deg_{\tau_k}(l)\}$$

dominates a random variable distributed as $Bin(N_\sigma(l), q)$. Consequently, by Chernoff bounds

$$(4.11) \quad \begin{aligned} \mathbb{P}_\ell\left(W \leq \frac{K}{\log(1+\varepsilon p)}\right) &\leq \mathbb{P}_\ell\left(Bin(N_\sigma(l), q) \leq \frac{K}{\log(1+\varepsilon p)}\right) \\ &\leq \mathbb{P}_\ell\left(Bin(N_\sigma(l), q) \leq \frac{K}{\log(1+\varepsilon p)}\middle| N_\sigma(l) \geq \frac{2K}{q\log(1+\varepsilon p)}\right) \\ &\quad + \mathbb{P}_\ell\left(N_\sigma(l) < 2q^{-1}K/\log(1+\varepsilon p)\right) \\ &\leq \exp\{-c_1 K\} + 1 - \left(1 - \frac{1}{2l}\right)^{c_2 K}. \end{aligned}$$

Finally, observe that if $W \geq \frac{K}{\log(1+\varepsilon p)}$, then $deg_{\tau_\sigma}(\ell) \geq 2e^K$ which implies that τ_σ is at least this amount, finishing the proof. □

The following special case of the above Lemma will be particularly useful to our proposes.

Corollary 4.12. *There exists a positive constant C depending on p only, such that*

$$\mathbb{P}_\ell \left(\eta_0^{loop} \leq e^{\sqrt{\ell}} \right) \leq \frac{C}{\sqrt{\ell}}.$$

Proof. Letting $K = \sqrt{\ell}$ on (4.11) of Lemma 3 we obtain

$$
\mathbb{P}_\ell \left(W \leq \frac{\sqrt{\ell}}{\log(1 + \varepsilon p)} \right) \leq \mathbb{P}_\ell \left(\mathrm{Bin}(N_\sigma(\ell), q) \leq \frac{\sqrt{\ell}}{\log(1 + \varepsilon p)} \right)
$$

(4.13)
$$
\leq \mathbb{P}_\ell \left(Bin(N_\sigma(\ell), q) \leq \frac{\sqrt{\ell}}{\log(1 + \varepsilon p)} \middle| N_\sigma(\ell) \geq \frac{2\sqrt{\ell}}{q \log(1 + \varepsilon p)} \right)
$$

$$
+ \mathbb{P}_\ell \left(N_\sigma(\ell) \leq 2q^{-1} \sqrt{\ell}/\log(1 + \varepsilon p) \right)
$$

$$
\leq \exp\{-c_1\sqrt{\ell}\} + 1 - \left(1 - \frac{1}{2\ell} \right)^{c_2\sqrt{\ell}}.
$$

Since $\left(1 - \frac{1}{2\ell} \right)^{c_2\sqrt{\ell}} \approx e^{-c_3/\sqrt{\ell}}$ and $1 - e^{-x} \leq x$, choosing properly the constants we obtain the desired result. □

5. Coupling the BGRW and the loop process

In this section we construct a coupling of the processes $BGRW$ and loop in such way that the loop process is always closer to the root than the walker X. For this, let T be a rooted locally finite tree of height at least $2(\ell + 1)$, x a vertex such that $d(x, \mathrm{root}) \geq l$ and $\deg(x) \geq 2$ and y its ancestor at distance ℓ. Since T is a tree, there exists only one path $\mathcal{P}$ connecting x to the y. With this in mind, we define a graph operation $\mathcal{B}$ which associates to each pair (T, x) and ancestor y satisfying the aforementioned conditions a backbone $\mathcal{B}(T, y, x)$ of length ℓ as follows:

(1) delete all vertices of T whose distance from $\mathcal{P}$ is at least 2;
(2) identify each vertex of distance 1 from $\mathcal{P}$ to its neighbor on $\mathcal{P}$ in such way that the edge outside the path becomes a loop;
(3) label the vertices according to its distance from y.

The figure below gives a concrete example of $\mathcal{B}$ in action when y is taken as the root:

So far, we have shown that the $BGRW$ is capable of reaching long distances – powers of $\log n$ – away from the root. Now, we would like to argue that once it has gone so far, it takes
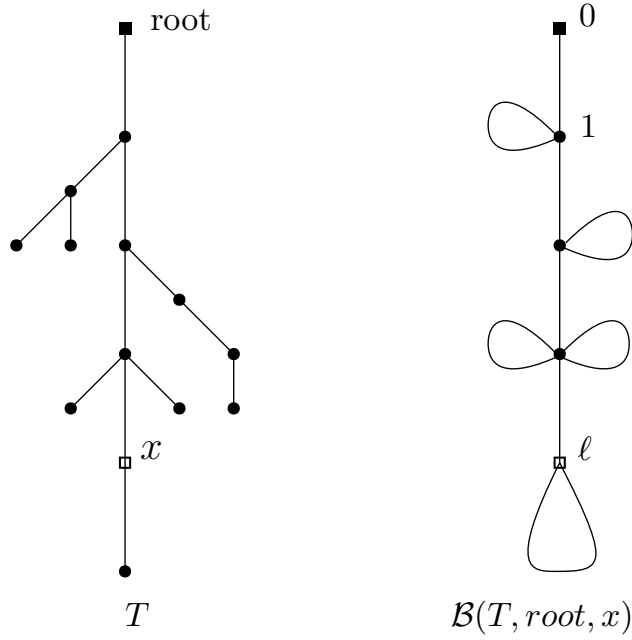
FIGURE 3. Example of a pair (T, x) transformed into a backbone

too long to return. More generally, if the *BGRW* starts on x_0 and y is one of its ancestors, we would like to obtain lower bounds on the following stopping time

$$(5.1) \qquad \eta_y := \inf \{n \geq 1 | X_n = y\}.$$

The way we bound η_y from below is comparing it with η_0^{loop}, which we recall its definition

$$\eta_0^{\mathrm{loop}} := \inf \left\{ t \geq 1 \,\middle|\, X_t^{\mathrm{loop}} = 0 \right\}.$$

The next proposition tells us that η_y is greater than η_0^{loop} almost surely.

Proposition 5.2 (Coupling). *Let T_0 be a rooted locally finite tree, x_0 one of its vertices different from the root and y an ancestor of x_0. Then, there exists a coupling of $\{(T_n, X_n)\}_{n \in \mathbb{N}}$ starting from (T_0, x_0) and a loop process $\{(\mathcal{B}_n, X_n^{loop})\}_{n \in \mathbb{N}}$ starting from $(\mathcal{B}(T_0, y, x_0), x_0)$ such that*

$$\mathbb{P}\left(\eta_y \geq \eta_0^{loop} \right) = 1.$$

Proof. Let $\mathcal{P}$ denote the path connecting x_0 to its ancestor y on T_0 and ℓ its length. Also consider the following sequence of stopping times,

$$(5.3) \qquad \zeta_0 \equiv 0; \quad \zeta_k := \inf \{m > \zeta_{k-1} | X_m \in \mathcal{P}\},$$

and let $\{W_k\}_{k \in \mathbb{N}}$ be a sequence of *i.i.d.* r.v's independent of the process *BGRW*, such that $W_k \sim \mathrm{Ber}(p)$. We couple a loop process $\{(\mathcal{B}_k, X_k^{\mathrm{loop}})\}_{k \in \mathbb{N}}$ to the *BGRW* inductively as follows. Start by defining

$$(\mathcal{B}_0, X_0^{\mathrm{loop}}) := (\mathcal{B}(T_0, y, x_0), x_0)$$

and assume we have defined $\{(\mathcal{B}_i, X_i^{\text{loop}})\}_{i=0}^{k-1}$ in such way that this random vector is distributed as $k - 1$ steps of a loop process starting from $(\mathcal{B}_0, \ell)$. Now, we define $(\mathcal{B}_k, X_k^{\text{loop}})$

- If $\zeta_{k-1} < \infty$, we set $\mathcal{B}_k = \mathcal{B}(T_{\zeta_{k-1}+1}, y, x_0)$;
- If $\zeta_{k-1} = \infty$, we make use of our independent sequence $\{W_k\}_{k \in \mathbb{N}}$. We modify $\mathcal{B}_{k-1}$ adding a loop to X_{k-1}^{loop} whether $W_k = 1$;

Observe that if ζ_{k-1} is finite, then $X_{k-1}^{\text{loop}} = X_{\zeta_{k-1}}$. In fact, when $X_{k-1}^{\text{loop}} = X_{\zeta_{k-2}}$ we know that $X_{\zeta_{k-2}}$ has jumped outside $\mathcal{P}$. Thus, the only way of X coming back to $\mathcal{P}$ is through $X_{\zeta_{k-2}}$, which implies $X_{\zeta_{k-1}} = X_{\zeta_{k-2}}$. On the other hand, when $X_{k-1}^{\text{loop}} = X_{\zeta_{k-2}+1}$ we know that $X_{\zeta_{k-2}}$ has moved on $\mathcal{P}$ implying that $\zeta_{k-1} = \zeta_{k-2} + 1$. Consequently, regardless the finiteness of ζ_{k-1} – since W_k is independent of the *BGRW*, $\mathcal{B}_k$ is obtained by adding a loop on X_{k-1}^{loop} independently of the whole past and with probability p.

To define X_k^{loop} we proceed in the following way

- If $\zeta_{k-1} < \infty$,
  - we set $X_k^{\text{loop}} = X_{\zeta_{k-1}+1}$, if $X_{\zeta_{k-1}}$ moves on $\mathbb{P}$;
  - otherwise, when $X_{\zeta_{k-1}}$ jumps outside $\mathcal{P}$, we let X_k^{loop} be $X_{\zeta_{k-1}}$.
- In case $\zeta_{k-1} = \infty$, we select uniformly an edge of X_{k-1}^{loop} on $\mathcal{B}_k$ to walk X_{k-1}^{loop} through.

By definition of the above coupling, we obtain that $\eta_y \geq \eta_0^{\text{loop}}$. The best scenario would be that in which the *BGRW* only walks over $\mathcal{P}$, in this case the stopping times are equals. This concludes the proof. $\qquad\square$

6. Not coming back

We start this section exploring the estimates for the return time of the loop walker. The next lemma is an important consequence of the coupling constructed in Proposition 5.2. It tells us that if X is at distance at least $2 \log^{15/7} n$ from the root, then it takes more than $4n$ steps to climb half way. So, if (T_0, x_0) is an initial state for the *BGRW*, we recall from the proof of Lemma 2 the notation y_m for the ancestor of x_0 half way to the root. Regarding X visiting y_m we have the following

Lemma 4. *There exist a positive constant c_5 and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,*

$$\sup_{T_0, x_0} \mathbb{P}_{T_0, x_0} \left(X \text{ visits } y_m \text{ in less than } 4n \text{ steps} \right) \leq \frac{c_5}{\log^{15/14} n},$$

where the supremum is taken over all possible trees T_0 and initial vertex x_0 such that

$$d(x_0, root) \geq 2 \log^{15/7} n.$$

Proof. We may assume x_0 has degree at least two. Otherwise, we wait for X landing on such vertex. Let $\mathcal{P}$ be the path connecting x_0 and its ancestor y_m at distance $\log^{15/7} n$ and ordinate

the vertices of $\mathcal{P}$ by its distance from y_m. Now, by Proposition 5.2 there exists a coupling of $\{(T_n, X_n)\}_{n\in\mathbb{N}}$ starting from (T_0, x_0) with a loop process on the backbone $\mathcal{B}(T_0, y_m, x_0)$, which has length $\log^{15/7} n$, such that

$$\text{(6.1)} \qquad \mathbb{P}\left(\eta_{y_m} \geq \eta_0^{\text{loop}}\right) = 1.$$

But, by Lemma 3 we have that there exists a positive constant C and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have

$$\text{(6.2)} \qquad \mathbb{P}\left(\eta_0^{\text{loop}} \leq e^{\log^{15/14} n}\right) \leq \frac{C}{\log^{15/14} n}.$$

Combining (6.1) with (6.2) we prove the Lemma. $\square$

Finally we have the results needed to prove our main result

Theorem 2 (The walker is transient)**.** *For all initial tree T, we have*

$$\mathbb{P}_{T,root}\left(X \text{ returns to the root i.o}\right) = 0$$

Proof. We prove first that, for sufficiently large n, there exists a positive constant c_6, such that

$$\text{(6.3)} \qquad \sup_{T_0, x_0} \mathbb{P}_{T_0, x_0}\left(X \text{ visits the root between time } n \text{ and } 2n\right) \leq \frac{c_6}{\log^{15/14} n}.$$

To see why (6.3) is true, start by fixing a initial state (T_0, x_0) and for each $m \in [n/2, n]$ let (T_m, x_m) be a state which may be achieved from (T_0, x_0) in exactly m steps and having the special property that $d(x_m, \text{root}) \geq \log^{15/7} n$. By Lemma 2, in at most n steps X achieves distance at least $\log^{15/7} n$, *w.h.p.* Combining this with Markov property we obtain

$$\text{(6.4)} \qquad \mathbb{P}_{T_0, x_0}\left(\nexists m \in [n/2, n], d(X_m, \text{root}) \geq \log^{15/7} n\right) \leq e^{-c_2 n}.$$

And by Lemma 4, X takes at least $4n$ steps to visit its ancestor half way from the *root*, *w.h.p*, thus we have that

$$\text{(6.5)} \qquad \mathbb{P}_{T_m, x_m}\left(X \text{ visits the root between time } n - m \text{ and } 2n - m\right) \leq \frac{c_5}{\log^{15/14} n}.$$

Combining (6.4) with (6.5) we are able to get (6.3). Now, consider the events

$$\text{(6.6)} \qquad A_j = \left\{X \text{ visits the root between time } 2^j \text{ and } 2^{j+1}\right\}.$$

By (6.3) we have that, for large j,

$$\mathbb{P}_{T_0, x_0}\left(A_j\right) \leq \frac{c_6}{j^{15/14}}.$$

Then, by the first Borel-Cantelli Lemma, we obtain that the occurrence of infinitely many A_j's is zero which implies that the *BGRW* visits the root only a finite number of times with probability 1. $\square$

References

[1] G. Amir, I. Benjamini, O. Gurel-Gurevich, and G. Kozma. Random Walk in Changing Environment. *ArXiv e-prints*, April 2015.

[2] Bernardo Amorim, Daniel R. Figueiredo, Giulio Iacobelli, and Giovanni Neglia. Growing networks through random walks without restarts. In *Proc 7th Workshop on Complex Networks (CompleNet'16)*, pages 199–211, 2016.

[3] Omer Angel, Nicholas Crawford, Gady Kozma, et al. Localization for linearly edge reinforced random walks. *Duke Mathematical Journal*, 163(5):889–921, 2014.

[4] Luca Avena, Hakan Guldas, Remco van der Hofstad, and Frank den Hollander. Mixing times of random walks on dynamic configuration models. *The Annals of Probability*, 2017. (to appear).

[5] Chris Cannings and Jonathan Jordan. Random walk attachment graphs. *Electronic Communications in Probability*, 18:1–5, 2013.

[6] Andrea Collevecchio. On the transience of processes defined on galton-watson trees. *The Annals of Probability*, pages 870–878, 2006.

[7] Codina Cotar, Debleena Thacker, et al. Edge-and vertex-reinforced random walks with super-linear reinforcement on infinite graphs. *The Annals of Probability*, 45(4):2655–2706, 2017.

[8] Amir Dembo, Ruojun Huang, Ben Morris, and Yuval Peres. Transience in growing subgraphs via evolving sets. *Ann. Inst. H. Poincar Probab. Statist.*, 53(3):1164–1180, 08 2017.

[9] Margherita Disertori, Christophe Sabot, and Pierre Tarrès. Transience of edge-reinforced random walk. *Communications in Mathematical Physics*, 339(1):121–148, 2015.

[10] Daniel Figueiredo, Giulio Iacobelli, and Giovanni Neglia. Graph builder random walk model. *arXiv preprint arXiv:xxx*, 2017.

[11] Daniel Figueiredo, Philippe Nain, Bruno Ribeiro, Edmundo de Souza e Silva, and Don Towsley. Characterizing continuous time random walks on time varying graphs. In *ACM SIGMETRICS Performance Evaluation Review*, volume 40, pages 307–318. ACM, 2012.

[12] Giulio Iacobelli and Daniel Ratton Figueiredo. Edge-attractor random walks on dynamic networks. *Journal of Complex Networks*, 5(1):84–110, 2016.

[13] Daniel Kious and Vladas Sidoravicius. Phase transition for the once-reinforced random walk on $\{Z\}$-like trees. *arXiv preprint arXiv:1604.07631*, 2017.

[14] Yuval Peres, Alexandre Stauffer, and Jeffrey E Steif. Random walks on dynamical percolation: mixing times, mean squared displacement and hitting times. *Probability Theory and Related Fields*, 162(3-4):487–530, 2015.

[15] Jari Saramäki and Kimmo Kaski. Scale-free networks generated by random walkers. *Physica A: Statistical Mechanics and its Applications*, 341:80–86, 2004.