STABLE RATIONALITY OF BRAUER-SEVERI SURFACE BUNDLES

ANDREW KRESCH AND YURI TSCHINKEL

Abstract. For sufficiently ample linear systems on rational surfaces we show that a very general associated Brauer-Severi surface bundle is not stably rational.

1. Introduction

This paper extends the study of stable rationality of conic bundles over rational surfaces in [10] to the case of Brauer-Severi surface bundles. Our main result is:

Theorem 1. *Let k be an uncountable algebraically closed field of characteristic different from 3, S a rational smooth projective surface over k, and L a very ample line bundle on S such that $H^1(S,L)=0$, and the complete linear system $|L|$ contains a nodal reducible curve $D=D_1\cup D_2$, where D_1 and D_2 are smooth of positive genus, and contains a curve with E_6-singularity. Then the Brauer-Severi surface bundle corresponding to a very general element of $|L|$ with nontrivial unramified cyclic degree 3 cover is not stably rational.*

This is applicable, for instance, to the complete linear system of degree d curves in $\mathbb{P}^2$ for $d\ge 6$.

The proof of Theorem 1 relies on the construction of good models of Brauer-Severi surface bundles in [12]. A new ingredient is a variant of the standard elementary transformation of vector bundles. This is needed to apply the specialization method, which was introduced by Voisin [15] and developed further in [6], [14], [11] and which tells us that in a family where one (mildly singular) member has an obstruction to stable rationality, the very general member fails to be stably rational. In our case, the family is a family of Brauer-Severi surface bundles, where one member has nontrivial 3-torsion in its unramified Brauer group.

In Section 2 we recall some facts on Brauer groups, and in Section 3 we describe the variant of the standard elementary transformation that will be used in the proof of Theorem 1, which occupies Section 4.

Date: September 28, 2017.

Acknowledgments: We are grateful to Asher Auel for helpful comments. The first author is partially supported by the Swiss National Science Foundation. The second author is partially supported by NSF grant 1601912.

2. Basic facts

Recall that the Brauer group of a Noetherian scheme S is defined as the torsion subgroup of the étale cohomology group $H^2(S,\mathbb{G}_m)$ [7]. The same definition extends to Noetherian Deligne-Mumford stacks.

In this section, we work over an algebraically closed field k of characteristic different from 3. We start with two basic facts:

Proposition 2 ([4])**.** *Let S be a smooth surface over k that is* (i) *projective and rational, or* (ii) *quasiprojective. Then there are residue maps fitting in an exact sequence*

$$0\to \mathrm{Br}(K)[3]\to \bigoplus_{\xi\in S^{(1)}} H^1(k(\xi),\mathbb{Z}/3\mathbb{Z})\to \bigoplus_{\xi\in S^{(2)}}\mathbb{Z}/3\mathbb{Z}\quad \textit{in case } \text{(i)},$$

$$0\to \mathrm{Br}(S)[3]\to \mathrm{Br}(K)[3]\to \bigoplus_{\xi\in S^{(1)}} H^1(k(\xi),\mathbb{Z}/3\mathbb{Z})\qquad \textit{in case } \text{(ii)}.$$

Here $K=k(S)$, and $S^{(i)}$ denotes the set of codimension i points of S.

The *root stack* $\sqrt[3]{(S,D)}$ along an effective Cartier divisor D in S is a Deligne-Mumford stack, locally, for D defined by the vanishing of a regular function f on an affine chart $\mathrm{Spec}(A)$ of S, isomorphic to the stack quotient

$$[\mathrm{Spec}(A[t]/(t^3-f))/\mu_3],$$

where the roots of unity μ_3 act by scalar multiplication on t; cf. [5, §2], [1, App. B]. There is a closed substack with morphism to D known as the *gerbe of the root stack* and given locally as

$$[\mathrm{Spec}(A[t]/(t,f))/\mu_3].$$

This is a gerbe since this μ_3 acts trivially, i.e.,

$$[\mathrm{Spec}(A[t]/(t,f))/\mu_3]\cong \mathrm{Spec}(A[t]/(t,f))\times B\mu_3,$$

where $B\mu_3$ denotes the classifying stack of μ_3. The complement of the gerbe of the root stack maps isomorphically to $S\smallsetminus D$.

The root stack is smooth when D is smooth, and singular when D is singular. For $D=D_1\cup D_2$ as in Theorem 1, however, we may consider the *iterated root stack* [5, Def. 2.2.4]

$$\sqrt[3]{(S,\{D_1,D_2\})}:=\sqrt[3]{(S,D_1)}\times_S\sqrt[3]{(S,D_2)}, \tag{1}$$

which is smooth with stabilizer group μ_3 over the smooth locus of D and $\mu_3 \times \mu_3$ over $D_1 \cap D_2$. Base change by the inclusion of the gerbe of the root stack $\sqrt[3]{(S, D_i)}$ leads to a closed substack of $\sqrt[3]{(S, \{D_1, D_2\})}$ with morphism to the pre-image of D_i in $\sqrt[3]{(S, D_{3-i})}$ which we call the *gerbe over the ith component*, for $i = 1, 2$:

$$\mathfrak{D}_i \to D_i \times_S \sqrt[3]{(S, D_{3-i})}.$$

Proposition 3 ([13]). *Let S be a smooth quasiprojective surface over k, D a curve on S that is either* (i) *smooth or* (ii) *nodal, consisting of two intersecting smooth components, and $U := S \smallsetminus D$. Then the restriction map induces an isomorphism*

$$\begin{aligned} &\mathrm{Br}\left(\sqrt[3]{(S, D)}\right)[3] \to \mathrm{Br}(U)[3] && \textit{in case } \text{(i)}, \\ &\mathrm{Br}\left(\sqrt[3]{(S, \{D_1, D_2\})}\right)[3] \to \mathrm{Br}(U)[3] && \textit{in case } \text{(ii)}. \end{aligned}$$

In each case, nonzero elements of the indicated Brauer groups are represented by sheaves of Azumaya algebras of index 3.

In case (ii) of Proposition 3, we have a morphism

$$\rho\colon \sqrt[3]{(S, \{D_1, D_2\})} \to \sqrt[3]{(S, D)}. \tag{2}$$

Let $\alpha \in \mathrm{Br}\left(\sqrt[3]{(S, \{D_1, D_2\})}\right)$ be the class of a sheaf of Azumaya algebras $\mathcal{A}$ of index 3.

Assumption 4. The restriction of α to $\mathrm{Br}(U)$ does not extend across the generic point of D_1 or of D_2 in S.

Lemma 5. *With notation as above, let $x \in D_1 \cap D_2$ and let*

$$\mu_3 \times \mu_3 \to PGL_3 \tag{3}$$

be the projective representation associated with the restriction of $\mathcal{A}$ to the copy of the classifying stack $B(\mu_3 \times \mu_3)$ in $\sqrt[3]{(S, \{D_1, D_2\})}$ over x, where the factors μ_3 correspond to the stabilizer along D_1 and along D_2. Then the restriction of (3) *to each factor μ_3 is balanced, i.e., is isomorphic to the projectivization of the sum of the three distinct one-dimensional linear representations of μ_3.*

Proof. It suffices to treat just the first factor μ_3. With the fiber product description (1) of the iterated root stack we have the projection morphism

$$p_2\colon \sqrt[3]{(S, \{D_1, D_2\})} \to \sqrt[3]{(S, D_2)}.$$

There is a criterion due to Alper [3, Thm. 10.3] for a vector bundle (e.g., the sheaf of Azumaya algebras $\mathcal{A}$) to descend via a morphism such as p_2. Specifically, Alper considers so-called good moduli spaces, e.g., the coarse moduli space of a finite-type separated Deligne-Mumford stack over k

whose stabilizer groups have order not divisible by the characteristic of k. However, by reasoning étale locally, his criterion applies as well to a relative moduli space as in [2, §3]. Applied to p_2, this reveals that there exists a sheaf of Azumaya algebras $\mathcal{A}'$ on $\sqrt[3]{(S,D_2)}$ and an isomorphism $p_2^*\mathcal{A}' \cong \mathcal{A}$ if and only if the relative stabilizer of p_2 acts trivially on fibers of $\mathcal{A}$.

Now, and several times further below, we use the Kummer sequence

$$0 \to \mu_3 \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

and the corresponding long exact sequence of cohomology groups. We take

$$\alpha_0 \in H^2(\sqrt[3]{(S,\{D_1,D_2\})}, \mu_3)$$

to be a lift of the class

$$\alpha \in \mathrm{Br}\left(\sqrt[3]{(S,\{D_1,D_2\})}\right)[3].$$

To α_0 there is a corresponding gerbe

$$\mathfrak{G} \xrightarrow{\tau} \sqrt[3]{(S,\{D_1,D_2\})}$$

banded by μ_3, meaning that $\mathfrak{G}$ is étale locally over $\sqrt[3]{(S,\{D_1,D_2\})}$ isomorphic to a product with $B\mu_3$, and the automorphism groups of the local sections are equipped with compatible identifications with μ_3. We have $\tau^*\alpha = 0$, hence

$$\tau^*\mathcal{A} \cong \mathrm{End}(\mathcal{E})$$

for some rank 3 vector bundle $\mathcal{E}$ on $\mathfrak{G}$. The stabilizer group of $\mathfrak{G}$ is a central μ_3-extension G of $\mu_3 \times \mu_3$:

$$1 \to \mu_3 \to G \to \mu_3 \times \mu_3 \to 1, \tag{4}$$

and by convention we take $\mathcal{E}$ so that the action of the central μ_3 is by scalar multplication.

The projective representation of the first factor μ_3 is induced by the linear representation of the subgroup of G, pre-image in (4) of $\mu_3 \times \{1\}$ in $\mu_3 \times \mu_3$. We suppose that this is not balanced. If this is trivial then the criterion mentioned above is applicable, and $\mathcal{A} \cong p_2^*\mathcal{A}'$ for some sheaf of Azumaya algebras $\mathcal{A}'$ on $\sqrt[3]{(S,D_2)}$. But then the restriction of α to $\mathrm{Br}(U)$ extends across the generic point of D_1, in contradiction to our assumption. A nontrivial unbalanced representation is the projectivization of a linear representation which is a sum of two copies of one and one copy of another one-dimensional linear representation of μ_3. Then the restriction of $\mathcal{E}$ to

$$\mathfrak{G} \times_{\sqrt[3]{(S,\{D_1,D_2\})}} \mathfrak{D}_1$$

splits canonically according to multiplicity as $\mathcal{E}_1 \oplus \mathcal{E}_2$. Let us denote by h the inclusion in $\mathfrak{G}$ of the above fiber product. Then we may form an exact sequence

$$0 \to \widetilde{\mathcal{E}}^{(j)} \to \mathcal{E} \to h_*\mathcal{E}_j \to 0 \tag{5}$$

for $j = 1, 2$, and consider the respective corresponding sheaf of Azumaya algebras $\widetilde{\mathcal{A}}^{(j)}$ on $\sqrt[3]{(S, \{D_1, D_2\})}$. Reasoning étale locally, we see that for appropriate j the sheaf of Azumaya algebras $\widetilde{\mathcal{A}}^{(j)}$ descends to $\sqrt[3]{(S, D_2)}$, and we have again reached a contradiction to our assumption. $\square$

Assumption 6. The restriction of α to $\mathrm{Br}(K)$ (where $K = k(S)$) is an element whose residue (image under the map to $H^1(k(\xi), \mathbb{Z}/3\mathbb{Z})$ in Proposition 2) at the generic point of D_i is the class of an *unramified* cyclic degree 3 cover $\widetilde{D}_i \to D_i$ for $i = 1, 2$.

We are interested in knowing whether $\mathcal{A}$ descends to $\sqrt[3]{(S, D)}$, i.e., is isomorphic to $\rho^*\mathcal{A}'$ for some sheaf of Azumaya algebras $\mathcal{A}'$ on $\sqrt[3]{(S, D)}$.

Lemma 7. *With notation and assumption as above, let $x \in D_1 \cap D_2$. Then there exists an étale neighborhood $S' \to S$ of x such that α lies in the kernel of*

$$\mathrm{Br}(U) \to \mathrm{Br}(S' \times_S U).$$

Proof. We take $S' \to S$ trivializing the cyclic covers $\widetilde{D}_i \to D_i$ for $i = 1$, 2. Application of Proposition 2 to S' shows that the pullback of α to $\mathrm{Br}(S' \times_S U)$ is the restriction of an element of $\mathrm{Br}(S')$. This is trivialized upon passage to a suitable further étale neighborhood. $\square$

Proposition 8. *With notation and assumption as above, let $x \in D_1 \cap D_2$. Then the kernel of the projective representation* (3) *is a subgroup, isomorphic to μ_3, embedded either as the diagonal or the antidiagonal in $\mu_3 \times \mu_3$.*

Proof. By Lemma 7, with its notation, the pullback of α to

$$S' \times_S \sqrt[3]{(S, \{D_1, D_2\})}$$

vanishes, and hence the projective representation lifts to a linear representation, which is well-defined up to twist by a character of $\mu_3 \times \mu_3$ and hence may be written as trivial $\oplus\, \chi \oplus \chi'$, for some characters χ and χ' of $\mu_3 \times \mu_3$. By Lemma 5, the restriction of χ and χ' to the first factor μ_3 are nontrivial and opposite, and the same holds for the restrictions to the second factor μ_3.

Let χ_i for $i \in \{0, 1, 2\}$ denote the ith character of μ_3. Swapping χ and χ' if necessary, we may suppose that

$$\chi|_{\mu_3 \times \{1\}} = \chi_1 \quad \text{and} \quad \chi'|_{\mu_3 \times \{1\}} = \chi_2.$$

Now there are two possibilities. If

$$\chi|_{\{1\}\times\mu_3} = \chi_1 \quad \text{and} \quad \chi'|_{\{1\}\times\mu_3} = \chi_2,$$

then the kernel is the antidiagonal copy of μ_3. If

$$\chi|_{\{1\}\times\mu_3} = \chi_2 \quad \text{and} \quad \chi'|_{\{1\}\times\mu_3} = \chi_1,$$

then the kernel is the diagonal copy of μ_3. $\square$

Definition 9. In the two cases in the proof of Proposition 8, leading to antidiagonal μ_3 and diagonal μ_3, we say that the sheaf of Azumaya algebras $\mathcal{A}$ at x is *good*, respectively *bad*.

Proposition 10. *With notation and assumption as above, the sheaf of Azumaya algebras $\mathcal{A}$ descends to $\sqrt[3]{(S,D)}$ if and only if $\mathcal{A}$ is good at every point of $D_1 \cap D_2$.*

Proof. The morphism ρ in (2) is a relative coarse moduli space. Indeed, if near $x \in D_1 \cap D_2$ in S we denote a defining equation of D_i by f_i for $i = 1$, 2, then ρ has the local form

$$[\mathrm{Spec}(A[t_1,t_2]/(t_1^3 - f_1, t_2^3 - f_2))/\mu_3 \times \mu_3] \to [\mathrm{Spec}(A[t]/(t^3 - f_1f_2))/\mu_3]$$

where $t = t_1t_2$ and $\mu_3 \times \mu_3$ maps to μ_3 by multiplication. Letting $\tilde{\mu}_3$ denote the antidiagonal copy of μ_3 in $\mu_3 \times \mu_3$, we obtain

$$[\mathrm{Spec}(A[t_1,t_2]/(t_1^3 - f_1, t_2^3 - f_2))/\tilde{\mu}_3] \to \mathrm{Spec}(A[t]/(t^3 - f_1f_2))$$

upon base change to an étale chart of $\sqrt[3]{(S,D)}$. Triviality of the action of $\tilde{\mu}_3$ is thus necessary and sufficient for the descent of $\mathcal{A}$ to $\sqrt[3]{(S,D)}$. $\square$

3. Elementary transformation

Already the proof of Lemma 5 exhibits the use of an elementary transformation (5) to alter the representation type of fibers of a vector bundle. In this section we use a variant of this to change the type of a sheaf of Azumaya algebras at a point from bad to good (Definition 9).

As in the previous section, S is a smooth quasiprojective surface over an algebraically closed field k of characteristic different from 3, and $D = D_1 \cup D_2$ is a nodal divisor with intersecting irreducible smooth components D_1 and D_2. We are given nontrivial unramified cyclic degree 3 covers

$$\widetilde{D}_i \to D_i, \quad \text{for} \quad i = 1, 2,$$

and an element

$$\alpha \in \mathrm{Br}\left(\sqrt[3]{(S,\{D_1,D_2\})}\right)[3],$$

whose residue along D_i is the class of $\widetilde{D}_i \to D_i$, for $i = 1$, 2. Let $\mathcal{A}$ be a sheaf of Azumaya algebras of index 3 on $\sqrt[3]{(S,\{D_1,D_2\})}$ representing α.

At a point $x \in D_1 \cap D_2$, the sheaf of Azumaya algebras $\mathcal{A}$ has a type, good or bad, according to the type of the associated projective representation at the point of $\sqrt[3]{(S, \{D_1, D_2\})}$ with stabilizer $\mu_3 \times \mu_3$ over x.

Let C_0 be a general nonsingular curve in S through x. Specifically, we suppose that C_0 meets D_i transversely, for $i = 1, 2$, and does not pass through any point of $D_1 \cap D_2$ besides x. The pre-image C of C_0 in $\sqrt[3]{(S, \{D_1, D_2\})}$ has a D_4-singularity over x.

Lemma 11. *With the above notation, α restricts to zero in* $\mathrm{Br}(C)$.

Proof. We argue as in [8, Thm. 1.3]. Let $\widehat{C}$ denote the normalization of C, and C' the seminormalization:

$$\widehat{C} \xrightarrow{\sigma} C' \xrightarrow{\nu} C.$$

Then we have an exact sequence

$$0 \to \mathbb{G}_{m,C} \to \nu_* \mathbb{G}_{m,C'} \to i_* \mathcal{L} \to 0,$$

where $\mathcal{L}$ is an invertible sheaf on $B(\mu_3 \times \mu_3)$, identified with the singular substack of C with inclusion map i. So ν induces an isomorphism $\mathrm{Br}(C)[3] \to \mathrm{Br}(C')[3]$, and we are reduced to showing that α restricts to zero in $\mathrm{Br}(C')$.

Identifying as well the singular substack of C' with $B(\mu_3 \times \mu_3)$, with inclusion i', there is an exact sequence

$$0 \to \mathbb{G}_{m,C'} \to \sigma_* \mathbb{G}_{m,\widehat{C}} \to i'_* \mathcal{H} \to 0,$$

for a two-dimensional torus $\mathcal{H}$ over $B(\mu_3 \times \mu_3)$, that appears also in another exact sequence

$$0 \to \mathbb{G}_{m,B(\mu_3 \times \mu_3)} \to j_* \mathbb{G}_{m,B\tilde{\mu}_3} \to \mathcal{H} \to 0$$

that is related to the first by obvious restriction maps. Here we employ the notation $\tilde{\mu}_3$ as in the proof of Proposition 10 and denote by j the morphism $B\tilde{\mu}_3 \to B(\mu_3 \times \mu_3)$. We obtain a commutative diagram of cohomology groups

$$\begin{tikzcd}
\mathrm{Pic}(\widehat{C}) \arrow[r] \arrow[d] & H^1(B(\mu_3 \times \mu_3), \mathcal{H}) \arrow[r] \arrow[d, equal] & \mathrm{Br}(C') \arrow[r] \arrow[d] & 0 \\
\mathbb{Z}/3\mathbb{Z} \arrow[r] & H^1(B(\mu_3 \times \mu_3), \mathcal{H}) \arrow[r] & \mathrm{Br}(B(\mu_3 \times \mu_3)) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since the map on the left is surjective, we have an isomorphism of Brauer groups on the right. So we are further reduced to verifying the triviality of the restriction of α to $B(\mu_3 \times \mu_3)$, which holds by Lemma 7. $\square$

With the notation of the proof of Proposition 10 we have

$$R_0 := A[t_1, t_2]/(t_1^3 - f_1, t_2^3 - f_2),$$

with $\mu_3 \times \mu_3$-action, as well as twists by characters $\chi_{i,j}$ of $\mu_3 \times \mu_3$ defined by

$$\chi_{i,j}(\lambda, \lambda') := \lambda^i \lambda'^j.$$

We introduce the following notation:

$$\begin{aligned} R_1 &:= R_0 \otimes \chi_{1,1}, & R_2 &:= R_0 \otimes \chi_{2,2}, \\ R' &:= R_0 \otimes \chi_{1,2}, & R'' &:= R_0 \otimes \chi_{2,1}. \end{aligned}$$

We let I_0 denote the ideal sheaf of $B(\mu_3 \times \mu_3)$ in C, with twists $I_i := I_0 \otimes \chi_{i,i}$. Then there is an exact sequence of coherent sheaves on a Zariski neighborhood of the point of $\sqrt[3]{(S, \{D_1, D_2\})}$ over x, given algebraically by

$$0 \to R_1 \oplus R_2 \oplus R_0 \xrightarrow{\begin{pmatrix} -t_2^2 & t_1 & 0 \\ -t_1^2 & t_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}} R' \oplus R'' \oplus R_0 \xrightarrow{\begin{pmatrix} -t_2 & t_1 & 0 \end{pmatrix}} I_1 \to 0.$$

We view this as an analytic local model of an elementary transformation.

Proposition 12. *With notation as above, we suppose that $\mathcal{A}$ is bad at x. Let $\alpha_0 \in H^2(\sqrt[3]{(S, \{D_1, D_2\})}, \mu_3)$ be a lift of α,*

$$\mathfrak{G} \xrightarrow{\tau} \sqrt[3]{(S, \{D_1, D_2\})}$$

a corresponding gerbe banded by μ_3, and $\mathcal{E}$ a rank 3 vector bundle on $\mathfrak{G}$ such that $\tau^*\mathcal{A} \cong \mathrm{End}(\mathcal{E})$. Then there exist a line bundle $\mathcal{L}$ on $\tau^{-1}(C)$ and an exact sequence

$$0 \to \widetilde{\mathcal{E}} \to \mathcal{E} \to h_*(\mathcal{I} \otimes \mathcal{L}) \to 0,$$

where $\mathcal{I}$ denotes the ideal sheaf in $\tau^{-1}(C)$ of its singular locus, as a reduced substack, and h denotes the inclusion $\tau^{-1}(C) \to \mathfrak{G}$. Furthermore, the sheaf $\widetilde{\mathcal{E}}$ on the left is locally free and determines a sheaf of Azumaya algebras $\widetilde{\mathcal{A}}$ on $\sqrt[3]{(S, \{D_1, D_2\})}$ that is good at x.

Proof. Lemma 11 tells us that there is a line bundle $\mathcal{T}$ on

$$\mathfrak{G} \times_{\sqrt[3]{(S, \{D_1, D_2\})}} C$$

for which the induced character of the constant μ_3 stabilizer is χ_1. Consequently, the restriction of $\mathcal{E}$, tensored with $\mathcal{T}^\vee$, descends to a vector bundle E on C. Since we are free to twist $\mathcal{T}$ by the pullback of any line bundle from C, there is no loss of generality in supposing that the isomorphism type of E over x is $\chi_{1,2} \oplus \chi_{2,1} \oplus \chi_{0,0}$.

Let L be a line bundle on C whose isomorphism type over x is $\chi_{1,1}$. We let I denote the ideal sheaf in C of its singular locus (as a reduced substack); the fiber of I at the point over x is a two-dimensional vector space with representation $\chi_{1,0} \oplus \chi_{0,1}$. So there exists an equivariant surjective linear map from the fiber of E to the fiber of $I \otimes L$. This extends to a morphism of modules (first non-equivariantly, then equivariantly by averaging), which we may view as a surjective morphism of sheaves

$$E|_{V\times_S C} \to (I \otimes L)|_{V\times_S C},$$

for some affine neighborhood $V \subset S$ of x. As explained in [12, §4.3] this extends, after possibly modifying L away from x, to a surjective morphism of sheaves on C. Pulling back to the gerbe and tensoring with $\mathcal{T}$ determines a surjective morphism of sheaves on $\mathfrak{G}$ and hence an exact sequence as in the statement.

The ideal sheaf $\mathcal{I}$ is Cohen-Macaulay of depth 1, so by the Auslander-Buchsbaum formula has projective dimension 1, and $\widetilde{\mathcal{E}}$ is locally free.

For the analysis of the type of the sheaf of Azumaya algebras $\widetilde{\mathcal{A}}$ at x, which is sensitive only to the projective representation of the $\mu_3 \times \mu_3$ stabilizer over x, we may pass to an étale neighborhood of $x \in S$ as in Lemma 7 and thus assume that we have an exact sequence as in the statement of the proposition on $\sqrt[3]{(S, \{D_1, D_2\})}$, rather than on a gerbe. As before, $\mathcal{E}$ is only determined up to twisting by a line bundle. Since the map from the Picard group of $\sqrt[3]{(S, \{D_1, D_2\})}$ to the character group of $\mu_3 \times \mu_3$ (given by restriction to the copy of $B(\mu_3 \times \mu_3)$ over x) is surjective, there is no loss of generality in supposing as before that the isomorphism type of $\mathcal{E}$ over x is $\chi_{1,2} \oplus \chi_{2,1} \oplus \chi_{0,0}$, and of the coherent sheaf on the right is $\chi_{1,2} \oplus \chi_{2,1}$. Restriction to the copy of $B(\mu_3 \times \mu_3)$ over x determines a four-term exact sequence with a Tor sheaf on the left

$$0 \to \mathrm{Tor} \to \widetilde{\mathcal{E}}|_{B(\mu_3\times\mu_3)} \to \chi_{1,2} \oplus \chi_{2,1} \oplus \chi_{0,0} \to \chi_{1,2} \oplus \chi_{2,1} \to 0.$$

Since the configuration of D_1, D_2, and C in S at x has a unique analytic isomorphism type, the model computation just before the statement of the proposition may be used to see that

$$\mathrm{Tor} \cong \chi_{1,1} \oplus \chi_{2,2}.$$

It follows that $\widetilde{\mathcal{A}}$ is good at x. □

4. Proof of the main theorem

The argument begins as in the proof of the main theorem of [10]. The hypotheses guarantee that the monodromy action on nontrivial unramified cyclic degree 3 covers of a nonsingular member of $|L|$ is transitive; cf. the proof of [9, Lem. 3.1]. We take the space of reduced nodal curves in $|L|$ with nontrivial degree 3 cyclic étale covering, and the member $D = D_1 \cup D_2$ with degree 3 cyclic étale cover, nontrivial over each component, as pointed variety (B, b_0). There is an associated element

$$\alpha \in \mathrm{Br}\left(\sqrt[3]{(S,\{D_1,D_2\})}\right),$$

by Propositions 2 and 3, represented by a sheaf of Azumaya algebras $\mathcal{A}$ of index 3. By repeated application of Proposition 12, we may suppose that $\mathcal{A}$ is good at all nodes of D. By Proposition 10, $\mathcal{A}$ descends to the (singular) root stack $\sqrt[3]{(S,D)}$; we let

$$\beta \in \mathrm{Br}\left(\sqrt[3]{(S,D)}\right)$$

denote its Brauer class, and

$$\gamma \in H^2\big(\sqrt[3]{(S,D)}, \mu_3\big)$$

a choice of lift, with gerbe $\mathfrak{G}_0$ associated with γ and locally free coherent sheaf $\mathcal{E}_0$ of rank 3 associated with the sheaf of Azumaya algebras.

Applying the deformation-theoretic machinery of [10, §4.3], we obtain by (usual) elementary transformation a subsheaf $\widetilde{\mathcal{E}}_0$, also locally free of rank 3, for which the space of obstructions vanishes. Upon replacing B by a suitable étale neighborhood of b_0, we obtain the root stack $\sqrt[3]{(B\times S,\mathcal{D})}$, where $\mathcal{D}$ denotes the corresponding family of divisors in $B \times S$, class

$$\Gamma \in H^2\big(\sqrt[3]{(B\times S,\mathcal{D})}, \mu_3\big)$$

restricting to γ, gerbe

$$\mathfrak{G} \to \sqrt[3]{(B\times S,\mathcal{D})}$$

restricting to $\mathfrak{G}_0$, and locally free sheaf $\widetilde{\mathcal{E}}$ on $\mathfrak{G}$ restricting to $\widetilde{\mathcal{E}}_0$. The locally free sheaf $\widetilde{\mathcal{E}}$ determines a smooth $\mathbb{P}^2$-bundle

$$\mathcal{P} \to \sqrt[3]{(B\times S,\mathcal{D})}.$$

We now apply the final step in the proof of [12, Thm. 1.4] to the $\mathbb{P}^2$-bundle $\mathcal{P}$. The construction of good models of Brauer-Severi surface bundles from op. cit., applied to $\mathcal{P}$ produces a Brauer-Severi surface bundle

$$\mathcal{X} \to B \times S.$$

Over B, this is a flat family of Brauer-Severi surface bundles over S. Since the discriminant curve of the fiber over b_0 has two components, and the Brauer class is given by nontrivial étale cyclic covers, this fiber has nontrivial unramified Brauer group [4]. Such a Brauer-Severi surface bundle has singularities of toric type, and these are mild enough for the specialization method to be applicable. We conclude that the very general Brauer-Severi surface bundle in this family is not stably rational.

References

[1] D. Abramovich, T. Graber, and A. Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *Amer. J. Math.*, 130(5):1337–1398, 2008.

[2] D. Abramovich, M. Olsson, and A. Vistoli. Twisted stable maps to tame Artin stacks. *J. Algebraic Geom.*, 20(3):399–477, 2011.

[3] J. Alper. Good moduli spaces for Artin stacks. *Ann. Inst. Fourier (Grenoble)*, 63(6):2349–2402, 2013.

[4] M. Artin and D. Mumford. Some elementary examples of unirational varieties which are not rational. *Proc. London Math. Soc. (3)*, 25:75–95, 1972.

[5] C. Cadman. Using stacks to impose tangency conditions on curves. *Amer. J. Math.*, 129(2):405–427, 2007.

[6] J.-L. Colliot-Thélène and A. Pirutka. Hypersurfaces quartiques de dimension 3: non-rationalité stable. *Ann. Sci. École Norm. Sup. (4)*, 49(2):371–397, 2016.

[7] A. Grothendieck. Le groupe de Brauer. I–III. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 46–188. North-Holland, Amsterdam, 1968.

[8] Y. Harpaz and A. N. Skorobogatov. Singular curves and the étale Brauer-Manin obstruction for surfaces. *Ann. Sci. École Norm. Sup. (4)*, 47(4):765–778, 2014.

[9] B. Hassett, A. Kresch, and Yu. Tschinkel. On the moduli of degree 4 Del Pezzo surfaces. In *Development of moduli theory (Kyoto 2013)*, volume 69 of *Adv. Stud. Pure Math.*, pages 349–386. Math. Soc. Japan, Tokyo, 2016.

[10] B. Hassett, A. Kresch, and Yu. Tschinkel. Stable rationality and conic bundles. *Math. Ann.*, 365(3-4):1201–1217, 2016.

[11] M. Kontsevich and Yu. Tschinkel. Specialization of birational types, 2017. `arXiv:1708.05699`.

[12] A. Kresch and Yu. Tschinkel. Models of Brauer-Severi surface bundles, 2017. `arXiv:1708.06277`.

[13] M. Lieblich. Period and index in the Brauer group of an arithmetic surface. *J. Reine Angew. Math.*, 659:1–41, 2011. With an appendix by Daniel Krashen.

[14] J. Nicaise and E. Shinder. The motivic nearby fiber and degeneration of stable rationality, 2017. `arXiv:1708.02790`.

[15] C. Voisin. Unirational threefolds with no universal codimension 2 cycle. *Invent. Math.*, 201(1):207–237, 2015.

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: `andrew.kresch@math.uzh.ch`

Courant Institute, 251 Mercer Street, New York, NY 10012, USA

Simons Foundation, 160 Fifth Av., New York, NY 10010, USA

E-mail address: `tschinkel@cims.nyu.edu`