

WELL-POSEDNESS IN GEVREY FUNCTION SPACE FOR THE THREE-DIMENSIONAL PRANDTL EQUATIONS

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ABSTRACT. In the paper, we study the three-dimensional Prandtl equations, and prove that if one component of the tangential velocity field satisfies the monotonicity assumption in the normal direction, then the system is locally well-posed in the Gevrey function space with Gevrey index in $]1, 2]$. The proof relies on some new cancellation mechanism in the system in addition to those observed in the two-dimensional setting.

1. INTRODUCTION AND MAIN RESULTS

The inviscid limit for the incompressible Navier-Stokes equations is one of the most fundamental problems in fluid mechanics. The justification remains a challenging problem from the mathematical point of view in particular with physical boundary conditions because of the appearance of boundary layer. Under the no-slip boundary condition, the behavior of the boundary layer can be described by the Prandtl system introduced by Prandtl [22] to study the behavior of the incompressible flow near a rigid wall at high Reynolds number. Formally the asymptotic limit of the Navier-Stokes equations is represented by the Euler equations outside away from boundary and by the Prandtl equations within the boundary layer. A mathematically rigorous justification of the vanishing viscosity limit basically remains unsolved up to now even though it has been achieved in some special settings, see for example [3, 7, 8, 17, 23] and the references therein for the recent progress.

The mathematical study on the boundary layer has a very long history, however, so far the theory is only well developed in various function spaces for the 2D Prandtl system. In fact, the 2D Prandtl system can be reduced to a scalar nonlinear and nonlocal degenerate parabolic equation that has loss of derivative in tangential variable. The degeneracy in the viscosity dissipation coupled with the loss of derivative in the nonlocal term is the main difficulty in the well-posedness theories. To overcome the degeneracy, it is natural, in the spirit of abstract Cauchy-Kovalevskaya theorem, to perform estimates within the category of the analytic function space, and in this context the well-posedness was obtained in [23] (see the earlier work [2], and also [12, 20] for further generalization), even with the justification of the vanishing viscosity limit and in 3D. If the initial data have only finite order of regularity, the well-posedness in Sobolev space was first obtained by Oleinik (cf. [21]) under the monotonicity assumption in the normal direction, where the Crocco transformation was used to overcome the loss of the derivative. Recently, an approach based on energy method was developed for the well-posedness in Sobolev space under Oleinik's monotonicity assumption, see [1, 19] where the key observation is some kind of cancellation property in the convection term due to the monotonicity. Furthermore, the well-posedness results in Gevrey space were achieved in two recent works [6, 16] for the

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initial data without analyticity or monotonicity, where some further cancellation properties were observed near the non-degenerate critical points. Different from the analytic context, the Gevrey space with index $\sigma > 1$ contains compactly supported functions. We also refer to [15] for the smoothing effect in Gevrey space under the monotonicity assumption. In the aforementioned works, only local-in-time well-posedness results are obtained. On the other hand, the global weak solution was established by [24] with additional assumption on pressure while the existence of global strong solutions still remains unsolved, although there are several works (see [10, 25, 26] for example) about the almost global solutions or long time behavior.

On the other hand, in general boundary separation happens that implies the Prandtl equations can no longer be a suitable model for describing the behavior of the flow near the boundary. In mathematics, this is related to the fact that the Prandtl equations without the analyticity or monotonicity are in general ill-posed, cf. [4, 5, 9] and the references therein.

Compared to the 2D case, much less is known about the three-dimensional Prandtl equations. As for the well-posedness theories, only partial results have been obtained in some specific settings, cf. [23] in the analytic context and [13] under some constraint on flow structure. In addition to the difficulties for 2D, another major difficulty in 3D arises from the secondary flow. As it will be seen in the later analysis, the cancellation properties observed in 2D case are not enough to overcome the difficulties in the analysis. In addition, we need to use some new cancellations in the 3D setting. We will explain this further in Subsection 2.2 about the new ideas and approach to be used.

In this paper, we will study the well-posedness of 3D Prandtl system without asking the analyticity or structural constraint. For this, let us first mention the paper on the ill-posedness [14] which shows that even for a perturbation of shear flow, without the structural condition, the linearized Prandtl equations are ill-posed. In fact, the ill-posedness estimate on the solution operator implies that the optimal Gevrey index for the well-posedness without any structural condition is 2. Hence, the result of this paper about the well-posedness in Gevrey function space with index in $[1, 2]$ complements the ill-posedness estimate in [14]. Without loss of generality, we will consider the system in a periodic domain in tangential direction, that is, in $\Omega = \mathbb{T}^2 \times \mathbb{R}_+$.

Denote by (u, v) the tangential component and by w the vertical component of the velocity field, then the 3D Prandtl system in Ω reads

$$\begin{cases} \partial_t u + (u\partial_x + v\partial_y + w\partial_z)u - \partial_z^2 u + \partial_x p = 0, & t > 0, (x, y, z) \in \Omega, \\ \partial_t v + (u\partial_x + v\partial_y + w\partial_z)v - \partial_z^2 v + \partial_y p = 0, & t > 0, (x, y, z) \in \Omega, \\ \partial_x u + \partial_y v + \partial_z w = 0, & t > 0, (x, y, z) \in \Omega, \\ u|_{z=0} = v|_{z=0} = w|_{z=0} = 0, & \lim_{z \rightarrow +\infty} (u, v) = (U(t, x, y), V(t, x, y)), \\ u|_{t=0} = u_0, \quad v|_{t=0} = v_0, & (x, y, z) \in \Omega, \end{cases} \quad (1)$$

where $(U(t, x, y), V(t, x, y))$ and $p(t, x, y)$ are the boundary traces of the tangential velocity field and pressure of the outer flow, satisfying

$$\begin{cases} \partial_t U + U\partial_x U + V\partial_y U + \partial_x p = 0, \\ \partial_t V + U\partial_x V + V\partial_y V + \partial_y p = 0. \end{cases}$$

Note p, U, V are given functions determined by the Euler flow, and (1) is a degenerate parabolic system losing one order derivative in the tangential variable. We refer to [18, 21, 22] for the background and mathematical presentation of this fundamental system.

We will consider the Prandtl system (1) under the assumption that one component of the initial tangential velocity component, for example u_0 satisfies the monotonicity condition in normal direction. Under this assumption, take $U > 0$. The precise assumption on the initial data is given as follows.

Assumption 1.1. Let $\delta > 2$ be a given number, and let $u_0 \in C^6(\Omega)$ satisfy that

(i) there exists a constant $0 < c_0 < 1$ such that

$$\forall (x, y, z) \in \Omega, \quad c_0 \langle z \rangle^{-\delta} \leq \partial_z u_0(x, y, z) \leq c_0^{-1} \langle z \rangle^{-\delta},$$

where and throughout the paper $\langle z \rangle = (1 + |z|^2)^{1/2}$.

(ii) there exists a constant $C_0 \geq 1$ such that

$$\forall (x, y, z) \in \Omega, \quad |\partial_z^j u_0(x, y, z)| \leq C_0 \langle z \rangle^{-\delta-1} \text{ for } 2 \leq j \leq 6.$$

Next we introduce the Gevrey function space in the tangential variables x, y . In this paper, we use ℓ, κ to denote two fixed constants satisfying

$$\kappa \geq 1, \quad \ell > 3/2 \text{ and } \ell + 1/2 < \delta \leq \ell + 1, \quad (2)$$

where $\delta > 2$ is given in Assumption 1.1.

Definition 1.2 (Gevrey space in tangential variables x and y). Let U, V be the data given in (1). With each pair (ρ, σ) , $\rho > 0$ and $\sigma \geq 1$, a Banach space $X_{\rho, \sigma}$ consists of all smooth vector functions (u, v) such that $\|(u, v)\|_{\rho, \sigma} < +\infty$, where the Gevrey norm $\|\cdot\|_{\rho, \sigma}$ is defined below. For each multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, denote $\partial^\alpha = \partial_x^{\alpha_1} \partial_y^{\alpha_2}$ and

$$\psi = \partial_z(u - U) = \partial_z u, \quad \eta = \partial_z(v - V) = \partial_z v.$$

Then the Gevrey norm is defined by

$$\begin{aligned} \|(u, v)\|_{\rho, \sigma} &= \sup_{|\alpha| \geq 7} \frac{\rho^{|\alpha|-6}}{[(|\alpha| - 7)!]^\sigma} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha(u - U)\|_{L^2} + \|\langle z \rangle^\kappa \partial^\alpha(v - V)\|_{L^2} \right) \\ &\quad + \sup_{|\alpha| \leq 6} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha(u - U)\|_{L^2} + \|\langle z \rangle^\kappa \partial^\alpha(v - V)\|_{L^2} \right) \\ &\quad + \sup_{|\alpha| \geq 7} \frac{\rho^{|\alpha|-6}}{[(|\alpha| - 7)!]^\sigma} \|\langle z \rangle^\ell \partial^\alpha \psi\|_{L^2} + \sup_{|\alpha| \leq 6} \|\langle z \rangle^\ell \partial^\alpha \psi\|_{L^2} \\ &\quad + \sup_{|\alpha| \geq 7} \frac{\rho^{|\alpha|-5}}{[(|\alpha| - 6)!]^\sigma} |\alpha| \|\langle z \rangle^{\kappa+2} \partial^\alpha \eta\|_{L^2} + \sup_{|\alpha| \leq 6} \|\langle z \rangle^{\kappa+2} \partial^\alpha \eta\|_{L^2} \\ &\quad + \sup_{\substack{1 \leq j \leq 4 \\ |\alpha|+j \geq 7}} \frac{\rho^{|\alpha|+j-6}}{[(|\alpha| + j - 7)!]^\sigma} \|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi\|_{L^2} + \sup_{\substack{1 \leq j \leq 4 \\ |\alpha|+j \leq 6}} \|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi\|_{L^2} \\ &\quad + \sup_{\substack{1 \leq j \leq 4 \\ |\alpha|+j \geq 7}} \frac{\rho^{|\alpha|+j-5}}{[(|\alpha| + j - 6)!]^\sigma} |\alpha| \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta\|_{L^2} + \sup_{\substack{1 \leq j \leq 4 \\ |\alpha|+j \leq 6}} \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta\|_{L^2}, \end{aligned} \quad (3)$$

where and throughout the paper we use L^2 instead of $L^2(\Omega)$ without confusion. Moreover, we define another Gevrey space $Y_{\rho, \sigma}$ consist of smooth functions $F(x, y)$ such that

$\|F\|_{\rho,\sigma} < +\infty$, where

$$\|F\|_{\rho,\sigma} = \sup_{|\alpha| \geq 0} \frac{\rho^{|\alpha|}}{(|\alpha|!)^\sigma} \|\partial^\alpha F\|_{L^2(\mathbb{T}^2)}.$$

Remark 1.3. Note the factors in front of the L^2 -norms of $\partial^\alpha \psi$ and $\partial^\alpha \eta$ are anisotropic, likewise, for the mixed derivatives in the last two lines of (3).

We will look for the solutions to (1) in the Gevrey function space $X_{\rho,\sigma}$. For this, the initial data (u_0, v_0) satisfy the following compatibility conditions

$$\begin{cases} (u_0, v_0)|_{z=0} = (0, 0), \quad \lim_{z \rightarrow +\infty} (u_0, v_0) = (U, V) \text{ and } (\partial_z \psi_0, \partial_z \eta_0)|_{z=0} = (\partial_x p, \partial_y p), \\ \partial_z^3 \psi_0|_{z=0} = \psi_0 (\partial_x \psi_0 - \partial_y \eta_0)|_{z=0} + 2\eta_0 \partial_y \psi_0|_{z=0} + \partial_t \partial_x p, \\ \partial_z^3 \eta_0|_{z=0} = \eta_0 (\partial_y \eta_0 - \partial_x \psi_0)|_{z=0} + 2\psi_0 \partial_x \eta_0|_{z=0} + \partial_t \partial_y p, \end{cases} \quad (4)$$

where $\psi_0 = \partial_z u_0$ and $\eta_0 = \partial_z v_0$,

The main result of this paper can be stated as follows.

Theorem 1.4. *Let $1 < \sigma \leq 2$, under the compatibility conditions (4), suppose $U, V, p \in Y_{2\rho_0, \sigma}$ and $(u_0, v_0) \in X_{2\rho_0, \sigma}$ for some $\rho_0 > 0$. Moreover, suppose $U > 0$ and u_0 satisfies Assumption 1.1. Then the Prandtl system (1) admits a unique solution $(u, v) \in L^\infty([0, T]; X_{\rho, \sigma})$ for some $T > 0$ and some $0 < \rho < 2\rho_0$.*

Remark 1.5. Up to a coordinate transformation, the monotonicity condition on u_0 can be replaced by a monotonicity assumption on the tangential component of the initial velocity field in an arbitrary but fixed direction.

Remark 1.6. The assumption that $U, V, p \in Y_{2\rho_0, \sigma}$ is closely related to the up-to-boundary Gevrey regularity for the Euler equations, see for example [11] and the references therein.

Remark 1.7. It remains unsolved about the well-posedness in the category of Sobolev space with finite regularity that has been well developed in 2D. And a more important open problem is the mathematical justification on the approximation of the solution to Navier-Stokes equation with no-slip boundary condition by the those of Euler and Prandtl equations. We expect the present work may shed some light on the vanishing viscosity limit for Navier-Stokes equations in 3D setting.

To simplify the notations, we will mainly focus only on the constant outer flow when $(U, V) \equiv (1, 0)$, and the argument can be extended, without essential difficulty to general functions U and V as given in the proof of Theorem 1.4 at the end of the paper.

For the constant outer flow, the Prandtl system (1) can be written as

$$\begin{cases} \partial_t u + (u \partial_x + v \partial_y + w \partial_z) u - \partial_z^2 u = 0, & t > 0, (x, y, z) \in \Omega, \\ \partial_t v + (u \partial_x + v \partial_y + w \partial_z) v - \partial_z^2 v = 0, & t > 0, (x, y, z) \in \Omega, \\ (u, v)|_{z=0} = (0, 0), \quad \lim_{z \rightarrow +\infty} (u, v) = (1, 0), \\ (u, v)|_{t=0} = (u_0, v_0), & (x, y, z) \in \Omega, \end{cases} \quad (5)$$

with

$$w(t, x, y, z) = - \int_0^z \partial_x u(t, x, y, \tilde{z}) d\tilde{z} - \int_0^z \partial_y v(t, x, y, \tilde{z}) d\tilde{z}.$$

The existence and uniqueness of solutions to system (5) can be stated as follows.

Theorem 1.8. *Let $1 < \sigma \leq 2$. Suppose the initial datum (u_0, v_0) belongs to $X_{2\rho_0, \sigma}$ for some $\rho_0 > 0$, and satisfies the compatibility condition (4) with constant pressure. Then the system (5) admits a unique solution $(u, v) \in L^\infty([0, T]; X_{\rho, \sigma})$ for some $T > 0$ and some $0 < \rho < 2\rho_0$.*

We will focus on proving Theorem 1.8 and show at the end of the last section about how to extend the argument to the case with general outer flow when $U, V \in Y_{2\rho_0, \sigma}$. Furthermore, we will present in detail the proof of Theorem 1.8 for $\sigma \in [3/2, 2]$. Note that the constraint $\sigma \geq 3/2$ is not essential and indeed it is just a technical assumption for clear presentation. And we will explain at the end of Section 6 about how to modify the proof for the case when $1 < \sigma < 3/2$.

The rest of the paper is organized as follows. In Section 2, we first introduce the notations used in the paper, and then explain the main difficulties and the new ideas. We will prove in Sections 3-6 the a priori estimate given in Section 2. The proof of the well-posedness for the Prandtl system is given in the last section.

2. NOTATIONS AND METHODOLOGY

This section and Sections 3-6 are to derive a priori estimate for the Prandtl system (5), which is crucial for proving Theorem 1.8. In Subsection 2.1, we introduce some functions to be estimated. In Subsection 2.2, we explain the difficulties for the well-posedness of 3D Prandtl system and then present the ideas to overcome them. The a priori estimate is given in Subsection 2.3 with its proof given in Sections 3-6.

2.1. Notations. From now on, we write ∂^α instead of $\partial_x^{\alpha_1} \partial_y^{\alpha_2}$ for $\alpha \in \mathbb{Z}_+^2$. Let (u, v) solve the system (5). Define ψ and η by

$$\psi = \partial_z u \quad \text{and} \quad \eta = \partial_z v.$$

Applying ∂_z to the equations for u and v in (5), we obtain the equations solved by ψ and η , that is,

$$\begin{cases} \partial_t \psi + (u \partial_x + v \partial_y + w \partial_z) \psi - \partial_z^2 \psi = g, \\ \partial_t \eta + (u \partial_x + v \partial_y + w \partial_z) \eta - \partial_z^2 \eta = h, \end{cases} \quad (6)$$

where g, h are given by

$$\begin{cases} g = (\partial_y v) \psi - (\partial_y u) \eta, \\ h = (\partial_x u) \eta - (\partial_x v) \psi. \end{cases} \quad (7)$$

For each multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, we define g_α and h_α by

$$g_\alpha = \partial^\alpha g = \partial_x^{\alpha_1} \partial_y^{\alpha_2} g, \quad \text{and} \quad h_\alpha = \partial^\alpha h = \partial_x^{\alpha_1} \partial_y^{\alpha_2} h. \quad (8)$$

And for each $m \geq 1$, define Γ_m and $\tilde{\Gamma}_m$ by

$$\Gamma_m = (\partial_x^m v) \psi - (\partial_x^m u) \eta, \quad \tilde{\Gamma}_m = (\partial_y^m v) \psi - (\partial_y^m u) \eta. \quad (9)$$

Furthermore, if $\psi \neq 0$ in Ω , then we can define f_m with $m \geq 1$, by

$$f_m = \partial_x^m \psi - \frac{\partial_z \psi}{\psi} \partial_x^m u = \psi \partial_z \left(\frac{\partial_x^m u}{\psi} \right). \quad (10)$$

Likewise, define \tilde{f}_m by

$$\tilde{f}_m = \partial_y^m \psi - \frac{\partial_z \psi}{\psi} \partial_y^m u = \psi \partial_z \left(\frac{\partial_y^m u}{\psi} \right).$$

It follows from (5) that

$$(\partial_z \psi, \partial_z \eta)|_{z=0} = (\partial_z f_m, \Gamma_m)|_{z=0} = (\partial_z \tilde{f}_m, \tilde{\Gamma}_m)|_{z=0} = (g_\alpha, h_\alpha)|_{z=0} = (0, 0).$$

Note the definitions of f_m and \tilde{f}_m are motivated by [19], and the estimates on $\partial_x^m u$ and $\partial_x^m \psi$ can be derived from the weighted L^2 -norm of f_m by Hardy inequality, likewise, for $\partial_y^m u$ and $\partial_y^m \psi$. As a result, we have the upper bounds on $\partial^\alpha u$ and $\partial^\alpha \psi$, by the inequality

$$\forall \alpha \in \mathbb{Z}_+^2, \forall F \in H^\infty, \quad \|\partial^\alpha F\|_{L^2(\mathbb{T}^2)}^2 \leq \|\partial_x^{|\alpha|} F\|_{L^2(\mathbb{T}^2)}^2 + \|\partial_y^{|\alpha|} F\|_{L^2(\mathbb{T}^2)}^2. \quad (11)$$

In order to obtain the upper bound of $\partial^\alpha \eta$, we will apply the energy method to the equation (6) for η . It remains to estimate $\partial^\alpha v$ with $|\alpha| = m$, and this can be deduced by the auxilliary functions Γ_m and $\tilde{\Gamma}_m$ by using $\psi > 0$. Finally, we need to estimate g_α and h_α since they appear in the equations for f_m and $\partial^\alpha \eta$ with the degeneracy in tangential variables. We will explain further in the next subsection about the motivation for introducing the above auxilliary functions.

2.2. Difficulties and methodologies. In this subsection, we will explain the main difficulties and the new ideas introduced in this paper.

When applying ∂^α to the equations in (5), we lose derivatives in the tangential variables x and y in the terms

$$(\partial^\alpha w) \psi, \quad (\partial^\alpha w) \eta. \quad (12)$$

Similar to 2D case, part of these terms can be handled under the Oleinik's monotonicity assumption, by some kind of cancellation properties, cf. [1, 19]. In the 3D setting considered in this paper, under the monotonicity assumption that $\psi > 0$, we can apply the same cancellation as in 2D case, to the equations of $\partial^\alpha u$ and $\partial^\alpha \psi$. This indeed eliminates the first term in (12), but meanwhile a new term

$$g_\alpha = \partial^\alpha g$$

appears when applying ∂^α to the equation (6) for ψ . Note that g also has the loss of one order derivative in y variable. This prevents us to investigate the well-posedness in Sobolev space with finite order of regularity. However, in the setting of Gevrey class, one can apply the following approach on the estimation of the unknown functions.

Estimates on $\partial^\alpha u$ and $\partial^\alpha \psi$. Under the monotonicity assumption, we will make use of the same cancellation as in 2D case to obtain a new equation for the auxilliary function f_m defined by (10). Even though this can not avoid the degeneracy in the tangential variables because of g_α . Our observation is that this kind of cancellation transfers the degeneracy coming from the first non-local term in (12) to a local term g_α . To work with g_α , we have the advantage to use another kind of cancellation. Precisely, multiplying the first equation in (6) by $\partial_y v$, and the second equation by $\partial_y u$, and then their subtraction yields the equation for g . Based on this, we can apply our approach used in 2D [16] to perform energy estimate for g_α and then for f_m , in the context of Gevrey function space rather than the analytic function space, if we ignore at moment the degeneracy caused by the second term in (12). Similar argument applies also to h_α and \tilde{f}_m . As a result, we can obtain the estimates as desired for $\partial^\alpha u$ and $\partial^\alpha \psi$ by Hardy inequality.

Estimates on $\partial^\alpha v$ and $\partial^\alpha \eta$. Now we turn to the second tangential component of velocity field. Firstly, for η , note that if the order of its derivatives in the energy are one order less than the ones of u, v and ψ , then we do not have derivative loss for this term. This is why in the definition of the Gevrey norm, there is an anisotropic term for η (see Remark 1.3).

To handle v and its derivatives, we will not apply the energy estimation directly because it involves the second term in (12). Instead, we observe that the estimate on $\partial_x^m v$ can be derived through $h_{m-1,0} = \partial_x^{m-1} h$ defined in (8), due to the monotonicity assumption that $\psi > 0$. On the other hand, it is not easy to estimate the bounds of the lower order derivatives $\partial_x^j v, j < m$, in the expression of $h_{m-1,0}$. This is why we introduce the auxillary function Γ_m , which contains only the leading term in the representation of $h_{m-1,0}$. It is then clear that we can use the monotonicity assumption $\psi > 0$ to get the upper bound of $\partial_x^m v$ from Γ_m . Furthermore, Γ_m can be handled by a new cancellation as shown in the equation for Γ_m , where the terms in (12) do not appear due to cancellation. Similar argument applies to $\tilde{\Gamma}_m$. And this leads to the desired estimate on $\partial^\alpha v$.

2.3. A priori estimate. Let δ, ℓ and κ be some given numbers satisfying (2), and let (u, v) be a solution to the Prandtl system (5) in $[0, T] \times \Omega$, satisfying the properties stated as follows. There exist two constants \tilde{c} and \tilde{C} such that for any $t \in [0, T]$ and for any $(x, y, z) \in \Omega$

$$\begin{cases} \tilde{c} \langle z \rangle^{-\delta} \leq \psi(t, x, y, z) \leq \langle z \rangle^{-\delta} / \tilde{c}, \\ \sum_{j=1}^5 |\partial_z^j \psi(t, x, y, z)| \leq \langle z \rangle^{-\delta-1} / \tilde{c}, \end{cases} \quad (13)$$

and moreover,

$$\begin{aligned} & \sum_{|\alpha| \leq 3} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha (u-1)\|_{L^\infty} + \|\langle z \rangle^\kappa \partial^\alpha v\|_{L^\infty} + \|\partial^\alpha w\|_{L^\infty} \right) \\ & + \sum_{|\alpha| \leq 4} \|\langle z \rangle^\ell \partial^\alpha \psi\|_{L_{x,y}^\infty(L_z^2)} + \sum_{\substack{|\alpha|+j \leq 4 \\ j \geq 1}} \|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi\|_{L_{x,y}^\infty(L_z^2)} \\ & + \sum_{|\alpha|+j \leq 4} \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta\|_{L_{x,y}^\infty(L_z^2)} \leq \tilde{C}, \end{aligned} \quad (14)$$

where $L_{x,y}^\infty(L_z^2) = L^\infty(\mathbb{T}^2; L^2(\mathbb{R}_+))$ stands for the classical Sobolev space, so does the Sobolev space $L_{x,y}^2(L_z^\infty)$.

As explained in Subsection 2.2, we introduce the following

Definition 2.1. Denote $\vec{a} = (u, v)$ with (u, v) satisfying the Prandtl system (5) and the conditions (13)-(14). Let $X_{\rho,\sigma}$ be the Gevrey function space given in Definition 1.2, equipped with the norm $\|\cdot\|_{\rho,\sigma}$ defined by (3). Set

$$\begin{aligned} |\vec{a}|_{\rho,\sigma} &= \|\vec{a}\|_{\rho,\sigma} + \sup_{|\alpha| \geq 7} \frac{\rho^{|\alpha|-5}}{[(|\alpha|-6)!]^\sigma} \left(|\alpha| \|\langle z \rangle^{\kappa+\delta} g_\alpha\|_{L^2} + |\alpha| \|\langle z \rangle^{\kappa+\delta} h_\alpha\|_{L^2} \right) \\ &+ \sup_{m \geq 7} \frac{\rho^{m-6}}{[(m-7)!]^\sigma} \left(\|\langle z \rangle^\ell f_m\|_{L^2} + \|\langle z \rangle^\ell \tilde{f}_m\|_{L^2} + \|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2} + \|\langle z \rangle^{\kappa+\delta} \tilde{\Gamma}_m\|_{L^2} \right) \\ &+ \sup_{0 \leq |\alpha| \leq 6} \left(\|\langle z \rangle^{\kappa+\delta} g_\alpha\|_{L^2} + \|\langle z \rangle^{\kappa+\delta} h_\alpha\|_{L^2} \right) \\ &+ \sup_{1 \leq m \leq 6} \left(\|\langle z \rangle^\ell f_m\|_{L^2} + \|\langle z \rangle^\ell \tilde{f}_m\|_{L^2} + \|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2} + \|\langle z \rangle^{\kappa+\delta} \tilde{\Gamma}_m\|_{L^2} \right), \end{aligned}$$

where the functions g_α, h_α, f_m , etc. are defined in Subsection 2.1. Similarly, we define $|\vec{a}_0|_{\rho,\sigma}$ for $\vec{a}_0 = (u_0, v_0)$, the initial datum in (5).

Remark 2.2. It is clear that $|\vec{a}|_{\rho,\sigma} \leq |\vec{a}|_{\tilde{\rho},\sigma}$ for any $\rho \leq \tilde{\rho}$. Moreover, direct calculation gives

$$\|\vec{a}\|_{\rho,\sigma} \leq |\vec{a}|_{\rho,\sigma} \leq C_{\rho,\rho^*} (\|\vec{a}\|_{\rho^*,\sigma} + \|\vec{a}\|_{\rho^*,\sigma}^2)$$

for any $\rho < \rho^*$, with C_{ρ,ρ^*} being a constant depending only on the difference $\rho^* - \rho$.

The main result of this part can be stated as follows.

Theorem 2.3 (A priori estimate in Gevrey space). *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0,\sigma})$ is the solution to the Prandtl system (5) such that the properties listed in (13)-(14) hold. Then there exists a constant $C_* > 1$, such that the estimate*

$$|\vec{a}(t)|_{\rho,\sigma}^2 \leq C_* |\vec{a}_0|_{\rho,\sigma}^2 + C_* \int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + C_* \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \quad (15)$$

holds for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$ and any $t \in [0, T]$. Here, the constant C_* depends only on the constants in (13)-(14) as well as the Sobolev embedding constants.

Remark 2.4. When $1 < \sigma < 3/2$, we can obtain a similar a priori estimate as (15), cf. Theorem 6.1 in Section 6.

In view of Remark 2.2, each term in (15) is well-defined. We will proceed to derive the upper bound of $|\vec{a}|_{\rho,\sigma}$ in the next Sections 3-6.

Before proving Theorem 2.3, we first list some basic inequalities to be used.

Lemma 2.5. *With the notations in Subsection 2.1, the following inequalities hold.*

(i) *For any integer $k \geq 1$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} \leq 1$, we have*

$$k \left(\frac{\rho}{\tilde{\rho}} \right)^k \leq \frac{1}{\tilde{\rho}} k \left(\frac{\rho}{\tilde{\rho}} \right)^k \leq \frac{1}{\tilde{\rho} - \rho}. \quad (16)$$

(ii) *For any suitable function F ,*

$$\|F\|_{L^\infty(\Omega)} \leq \sqrt{2} \left(\|F\|_{L_{x,y}^2(L_z^\infty)} + \|\partial_x F\|_{L_{x,y}^2(L_z^\infty)} + \|\partial_y F\|_{L_{x,y}^2(L_z^\infty)} + \|\partial_x \partial_y F\|_{L_{x,y}^2(L_z^\infty)} \right), \quad (17)$$

and

$$\begin{aligned} \|F\|_{L^\infty(\Omega)} &\leq 2 \left(\|F\|_{L^2} + \|\partial_x F\|_{L^2} + \|\partial_y F\|_{L^2} + \|\partial_z F\|_{L^2} \right) \\ &\quad + 2 \left(\|\partial_x \partial_y F\|_{L^2} + \|\partial_x \partial_z F\|_{L^2} + \|\partial_y \partial_z F\|_{L^2} + \|\partial_x \partial_y \partial_z F\|_{L^2} \right). \end{aligned} \quad (18)$$

(iii) *For any $0 < r \leq 1$ and any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ with $|\alpha| = m$, we have*

$$\begin{aligned} &\| \langle z \rangle^{\ell-1} \partial^\alpha (u-1) \|_{L^2} + \| \langle z \rangle^\ell \partial^\alpha \psi \|_{L^2} + \| \langle z \rangle^\ell f_m \|_{L^2} + \| \langle z \rangle^\kappa \partial^\alpha v \|_{L^2} \\ &\leq \begin{cases} \frac{[(m-7)!]^\sigma}{r^{(m-6)}} |\vec{a}|_{r,\sigma}, & \text{if } m \geq 7, \\ |\vec{a}|_{r,\sigma}, & \text{if } m \leq 6, \end{cases} \end{aligned} \quad (19)$$

and

$$\| \langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi \|_{L^2} \leq \begin{cases} \frac{[(|\alpha|+j-7)!]^\sigma}{r^{(|\alpha|+j-6)}} |\vec{a}|_{r,\sigma}, & \text{if } |\alpha|+j \geq 7 \text{ and } 1 \leq j \leq 4, \\ |\vec{a}|_{r,\sigma}, & \text{if } |\alpha|+j \leq 6 \text{ and } 1 \leq j \leq 4. \end{cases} \quad (20)$$

(iv) For any $0 < r \leq 1$ and any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, we have

$$\begin{aligned} & \|\partial^\alpha w\|_{L_{x,y}^2(L_z^\infty)} + |\alpha| \left(\|\langle z \rangle^{\kappa+\delta} g_\alpha\|_{L^2} + \|\langle z \rangle^{\kappa+\delta} h_\alpha\|_{L^2} + \|\langle z \rangle^{\kappa+2} \partial^\alpha \eta\|_{L^2} \right) \\ & \leq \begin{cases} \frac{[(|\alpha| - 6)!]^\sigma}{r^{(|\alpha| - 5)}} |\vec{a}|_{r,\sigma}, & \text{if } |\alpha| \geq 7, \\ |\vec{a}|_{r,\sigma}, & \text{if } |\alpha| \leq 6, \end{cases} \end{aligned} \quad (21)$$

and

$$|\alpha| \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta\|_{L^2} \leq \begin{cases} \frac{[(|\alpha| + j - 6)!]^\sigma}{r^{(|\alpha| + j - 5)}} |\vec{a}|_{r,\sigma}, & \text{if } |\alpha| + j \geq 7 \text{ and } 1 \leq j \leq 4, \\ |\vec{a}|_{r,\sigma}, & \text{if } |\alpha| + j \leq 6 \text{ and } 1 \leq j \leq 4. \end{cases} \quad (22)$$

Proof. We refer to [16, Lemma 3.2] for the proof of (i), and the proof of (ii) follows from the standard Sobolev inequalities (see for example [16, Lemma A.1]). The other inequalities are direct consequences of the definition of $|\vec{a}|_{r,\sigma}$. \square

3. ESTIMATES ON $f_m, \tilde{f}_m, \Gamma_m$ AND $\tilde{\Gamma}_m$

This section is for deriving the upper bounds of the weighted L^2 -norms of f_m and Γ_m , defined in (9) and (10). And \tilde{f}_m and $\tilde{\Gamma}_m$ can be handled similarly. To estimate f_m , we will use the cancellation introduced in [19] for 2D Prandtl equations. The estimation on Γ_m relies on another kind of cancellation without using any monotonicity.

To simplify the notation, we use from now on the capital letter C to denote some generic constant that may vary from line to line, and it depends only on the constants in (13)-(14) as well as the Sobolev embedding constants, in particular, it is independent of the order of derivatives denoted by m .

The proposition below is the main result in this section.

Proposition 3.1. *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) satisfying the conditions (13)-(14). Then for any $m \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$, we have*

$$\begin{aligned} & \frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} \left(\|\langle z \rangle^\ell f_m(t)\|_{L^2}^2 + \|\langle z \rangle^{\kappa+\delta} \Gamma_m(t)\|_{L^2}^2 + \|\langle z \rangle^\ell \tilde{f}_m(t)\|_{L^2}^2 + \|\langle z \rangle^{\kappa+\delta} \tilde{\Gamma}_m(t)\|_{L^2}^2 \right) \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

We will only estimate f_m and Γ_m because \tilde{f}_m and $\tilde{\Gamma}_m$ can be estimated similarly. To prove the above proposition, we first derive the equations solved by f_m and Γ_m . Let $m \geq 1$. Then we conclude

$$\partial_t f_m + (u \partial_x + v \partial_y + w \partial_z) f_m - \partial_z^2 f_m = \partial_x^m g + \mathcal{J}_m \quad (23)$$

and

$$\partial_t \Gamma_m + (u \partial_x + v \partial_y + w \partial_z) \Gamma_m - \partial_z^2 \Gamma_m = \mathcal{L}_m, \quad (24)$$

where g is defined by (7), and

$$\begin{aligned}\mathcal{J}_m &= \chi \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} u + (\partial_x^j v) \partial_y \partial_x^{m-j} u] + \chi \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_x^{m-j} \psi \\ &\quad - \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} \psi + (\partial_x^j v) \partial_y \partial_x^{m-j} \psi] - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_z \partial_x^{m-j} \psi \\ &\quad - \left[\frac{\partial_z g - \eta \partial_y \psi - \chi g}{\psi} - \partial_x \psi + \chi \partial_x u + \chi \partial_y v + 2\chi \partial_z \chi \right] \partial_x^m u + 2(\partial_z \chi) \partial_x^m \psi\end{aligned}$$

with $\chi \stackrel{\text{def}}{=} \partial_z \psi / \psi$, and

$$\begin{aligned}\mathcal{L}_m &= \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} u + (\partial_x^j v) \partial_y \partial_x^{m-j} u] \eta + \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) (\partial_x^{m-j} \psi) \eta \\ &\quad - \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} v + (\partial_x^j v) \partial_y \partial_x^{m-j} v] \psi - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) (\partial_x^{m-j} \eta) \psi \\ &\quad + g \partial_x^m v - 2(\partial_z \psi) \partial_x^m \eta - \left(h \partial_x^m u - 2(\partial_z \eta) \partial_x^m \psi \right)\end{aligned}$$

with g, h defined by (7). Here and throughout the paper, $\binom{m}{j}$ denotes the binomial coefficient.

We just give a sketch for obtaining (23)-(24). The derivation of (23) relies on the cancellation property observed by [19]. Applying ∂_x^m to the equations (5) and (6) for u and ψ , it follows from Leibniz formula that

$$\begin{aligned}&\partial_t \partial_x^m u + (u \partial_x + v \partial_y + w \partial_z) \partial_x^m u - \partial_z^2 \partial_x^m u + (\partial_x^m w) \psi \\ &= - \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} u + (\partial_x^j v) \partial_y \partial_x^{m-j} u] - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_x^{m-j} \psi\end{aligned}\tag{25}$$

and

$$\begin{aligned}&\partial_t \partial_x^m \psi + (u \partial_x + v \partial_y + w \partial_z) \partial_x^m \psi - \partial_z^2 \partial_x^m \psi + (\partial_x^m w) \partial_z \psi \\ &= \partial_x^m g - \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} \psi + (\partial_x^j v) \partial_y \partial_x^{m-j} \psi] - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_z \partial_x^{m-j} \psi.\end{aligned}$$

Multiplying the first equation above by $\partial_z \psi / \psi$ and then subtracting one by another, we obtain the equation for f_m .

For Γ_m , to use another kind of cancellation, applying ∂_x^m to the equation (5) for v gives

$$\begin{aligned}&\partial_t \partial_x^m v + (u \partial_x + v \partial_y + w \partial_z) \partial_x^m v - \partial_z^2 \partial_x^m v + (\partial_x^m w) \eta \\ &= - \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} v + (\partial_x^j v) \partial_y \partial_x^{m-j} v] - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_x^{m-j} \eta.\end{aligned}$$

We multiply the above equation by ψ and multiply the equation (25) by η , and then subtract one by another to have

$$\begin{aligned} & \partial_t \Gamma_m + (u \partial_x + v \partial_y + w \partial_z) \Gamma_m - \partial_z^2 \Gamma_m \\ = & \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} u + (\partial_x^j v) \partial_y \partial_x^{m-j} u] \eta + \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) (\partial_x^{m-j} \psi) \eta \\ & - \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} v + (\partial_x^j v) \partial_y \partial_x^{m-j} v] \psi - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) (\partial_x^{m-j} \eta) \psi \\ & + g \partial_x^m v - 2 (\partial_z \psi) \partial_x^m \eta - (h \partial_x^m u - 2 (\partial_z \eta) \partial_x^m \psi). \end{aligned}$$

Note the terms in the last line arise from the commutators between the functions ψ, η and the differential operators, where we have used the equations (6) for ψ and η .

The following three lemmas are for the estimates on the terms involving $\partial_x^m g, \mathcal{J}_m$ and \mathcal{L}_m appearing on the right sides of (23) and (24).

Lemma 3.2. *Let $\sigma \leq 2$. Then for any $m \geq 7$, and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, we have*

$$\left(\langle z \rangle^\ell \partial_x^m g, \langle z \rangle^\ell f_m \right)_{L^2} \leq \frac{C[(m-7)!]^{2\sigma} |\vec{a}|_{\tilde{\rho}, \sigma}^2}{\rho^{2(m-6)} \tilde{\rho} - \rho}.$$

Proof. Note that $\partial_x^m g = g_\alpha$ with $\alpha = (m, 0) \in \mathbb{Z}_+^2$, and that $\ell \leq \kappa + \delta$ in view of (2). Then we use (21) and (19) to get

$$\begin{aligned} \left(\langle z \rangle^\ell \partial_x^m g, \langle z \rangle^\ell f_m \right)_{L^2} & \leq m^{-1} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-5}} |\vec{a}|_{\tilde{\rho}, \sigma} \frac{[(m-7)!]^\sigma}{\tilde{\rho}^{m-6}} |\vec{a}|_{\tilde{\rho}, \sigma} \\ & \leq \tilde{\rho}^{-1} m^{\sigma-1} \frac{\rho^{2(m-6)}}{\tilde{\rho}^{2(m-6)}} \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\tilde{\rho}, \sigma}^2. \end{aligned}$$

Moreover, it follows from the fact $\sigma \leq 2$ and (16) that

$$\tilde{\rho}^{-1} m^{\sigma-1} \frac{\rho^{2(m-6)}}{\tilde{\rho}^{2(m-6)}} \leq \tilde{\rho}^{-1} m \frac{\rho^{m-6}}{\tilde{\rho}^{m-6}} \leq \frac{C}{\tilde{\rho} - \rho}.$$

Then combining these estimates completes the proof. \square

Lemma 3.3. *Let $\sigma \in [3/2, 2]$. Then for any $m \geq 7$, and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, we have*

$$\left(\langle z \rangle^\ell \mathcal{J}_m, \langle z \rangle^\ell f_m \right)_{L^2} \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\vec{a}|_{\rho, \sigma}^2 + |\vec{a}|_{\rho, \sigma}^3) + \frac{C[(m-7)!]^{2\sigma} |\vec{a}|_{\tilde{\rho}, \sigma}^2}{\rho^{2(m-6)} \tilde{\rho} - \rho}.$$

Proof. We divide the term into

$$\left(\langle z \rangle^\ell \mathcal{J}_m, \langle z \rangle^\ell f_m \right)_{L^2} = A_1 + A_2 + A_3$$

with

$$\begin{aligned}
A_1 &= \sum_{j=1}^m \binom{m}{j} \left(\chi \langle z \rangle^\ell [(\partial_x^j u) \partial_x \partial_x^{m-j} u + (\partial_x^j v) \partial_y \partial_x^{m-j} u], \langle z \rangle^\ell f_m \right)_{L^2} \\
&\quad - \sum_{j=1}^m \binom{m}{j} \left(\langle z \rangle^\ell [(\partial_x^j u) \partial_x \partial_x^{m-j} \psi + (\partial_x^j v) \partial_y \partial_x^{m-j} \psi], \langle z \rangle^\ell f_m \right)_{L^2} \\
A_2 &= \sum_{j=1}^{m-1} \binom{m}{j} \left(\chi \langle z \rangle^\ell (\partial_x^j w) \partial_x^{m-j} \psi, \langle z \rangle^\ell f_m \right)_{L^2} \\
&\quad - \sum_{j=1}^{m-1} \binom{m}{j} \left(\langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi, \langle z \rangle^\ell f_m \right)_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
A_3 &= 2 \left(\langle z \rangle^\ell (\partial_z \chi) \partial_x^m \psi, \langle z \rangle^\ell f_m \right)_{L^2} \\
&\quad - \left(\langle z \rangle^\ell \left[\frac{\partial_z g - \eta \partial_y \psi - \chi g}{\psi} - \partial_x \psi + \chi \partial_x u + \chi \partial_y v + 2\chi \partial_z \chi \right] \partial_x^m u, \langle z \rangle^\ell f_m \right)_{L^2}.
\end{aligned}$$

We now derive the upper bounds for $A_j, 1 \leq j \leq 3$, term by term.

Upper bound for A_3 . In view of the conditions (13)-(14) and the fact that $|\chi| \leq C \langle z \rangle^{-1}$, by observing (2), we have

$$\|\partial_z \chi\|_{L^\infty} + \|\langle z \rangle \left[\frac{-\eta \partial_y \psi}{\psi} - \partial_x \psi + \chi \partial_x u + \chi \partial_y v + 2\chi \partial_z \chi \right]\|_{L^\infty} \leq C.$$

And using (2) and the representation (7) of g gives

$$\|\langle z \rangle \frac{\partial_z g - \chi g}{\psi}\|_{L^\infty} \leq C.$$

Thus,

$$A_3 \leq C \left(\|\langle z \rangle^{\ell-1} \partial_x^m u\|_{L^2} + \|\langle z \rangle^\ell \partial_x^m \psi\|_{L^2} \right) \|\langle z \rangle^\ell f_m\|_{L^2} \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\rho, \sigma}^2, \quad (26)$$

the last inequality following from (19).

Upper bound for A_2 . We will show that A_2 satisfies

$$A_2 \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\rho, \sigma}^3 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho}. \quad (27)$$

For this, firstly, we have

$$\begin{aligned}
A_2 &\leq \|\langle z \rangle^\ell f_m\|_{L^2} \sum_{1 \leq j \leq m-1} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi\|_{L^2} \\
&\quad + \|\langle z \rangle^\ell f_m\|_{L^2} \sum_{1 \leq j \leq m-1} \binom{m}{j} \|\chi \langle z \rangle^\ell (\partial_x^j w) \partial_x^{m-j} \psi\|_{L^2} \\
&\stackrel{\text{def}}{=} A_{2,1} + A_{2,2}.
\end{aligned}$$

We will only estimate $A_{2,1}$ because $A_{2,2}$ can be handled similarly. By (14), (20) and (21), we have

$$\begin{aligned} & \left[\sum_{1 \leq j \leq 2} + \sum_{m-2 \leq j \leq m-1} \right] \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi\|_{L^2} \\ & \leq C m \frac{[(m-7)!]^\sigma}{\tilde{\rho}^{m-6}} |\vec{a}|_{\tilde{\rho}, \sigma} \leq \frac{C[(m-7)!]^\sigma}{\rho^{m-6}} \frac{|\vec{a}|_{\tilde{\rho}, \sigma}}{\tilde{\rho} - \rho}, \end{aligned} \quad (28)$$

where in the last inequality we have used (16). Next, we estimate the remaining terms in the summation. Note that

$$\sum_{3 \leq j \leq m-3} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi\|_{L^2} \leq T_1 + T_2$$

with

$$\begin{aligned} T_1 &= \sum_{j=3}^{[(m-3)/2]} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi\|_{L^2}, \\ T_2 &= \sum_{j=[(m-3)/2]+1}^{m-3} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi\|_{L^2}, \end{aligned}$$

where as standard, $[p]$ denotes the largest integer less than or equal to p . By using Sobolev inequality (17) and (19)-(21), we have

$$\begin{aligned} T_1 &\leq \sum_{j=3}^{[(m-3)/2]} \binom{m}{j} \|(\partial_x^j w)\|_{L^\infty} \|\langle z \rangle^\ell \partial_z \partial_x^{m-j} \psi\|_{L^2} \\ &\leq C \frac{m!}{3!(m-3)!} |\vec{a}|_{\rho, \sigma} \frac{[(m-9)!]^\sigma}{\rho^{m-8}} |\vec{a}|_{\rho, \sigma} \\ &\quad + C \sum_{j=4}^{[(m-3)/2]} \frac{m!}{j!(m-j)!} \frac{[(j-4)!]^\sigma}{\rho^{j-3}} |\vec{a}|_{\rho, \sigma} \frac{[(m-j-6)!]^\sigma}{\rho^{m-j-5}} |\vec{a}|_{\rho, \sigma} \\ &\leq \frac{C |\vec{a}|_{\rho, \sigma}^2}{\rho^{m-8}} m^3 [(m-9)!]^\sigma + \frac{C |\vec{a}|_{\rho, \sigma}^2}{\rho^{m-8}} \sum_{j=4}^{[(m-3)/2]} \frac{m! [(j-4)!]^\sigma [(m-j-6)!]^\sigma}{j^4 (m-j)^6} \\ &\leq \frac{C |\vec{a}|_{\rho, \sigma}^2}{\rho^{m-6}} [(m-7)!]^\sigma (m^3 m^{-2\sigma}) + \frac{C |\vec{a}|_{\rho, \sigma}^2}{\rho^{m-6}} \sum_{j=4}^{[(m-3)/2]} \frac{(m-7)! m^7}{j^4 m^6} [(m-10)!]^\sigma \\ &\leq \frac{C [(m-7)!]^\sigma |\vec{a}|_{\rho, \sigma}^2}{\rho^{m-6}} \left(m^3 m^{-2\sigma} + \sum_{j=4}^{[(m-3)/2]} \frac{m}{j^4 m^{3(\sigma-1)}} \right) \\ &\leq \frac{C [(m-7)!]^\sigma}{\rho^{m-6}} |\vec{a}|_{\rho, \sigma}^2, \end{aligned}$$

where in the last inequality we have used the fact that $\sigma \geq 3/2$. Similarly,

$$\begin{aligned}
T_2 &\leq \sum_{j=[(m-3)/2]+1}^{m-3} \binom{m}{j} \|(\partial_x^j w)\|_{L_{x,y}^2(L_z^\infty)} \|\langle z \rangle^\ell \partial_z \partial_x^{m-j} \psi\|_{L_{x,y}^\infty(L_z^2)} \\
&\leq \sum_{j=[(m-3)/2]+1}^{m-4} \frac{m!}{j!(m-j)!} \frac{[(j-6)!]^\sigma}{\rho^{j-5}} |\vec{a}|_{\rho,\sigma} \frac{[(m-j-4)!]^\sigma}{\rho^{m-j-3}} |\vec{a}|_{\rho,\sigma} \\
&\quad + \frac{C |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-8}} m^3 [(m-9)!]^\sigma \\
&\leq \frac{C |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-8}} \sum_{j=[(m-3)/2]+1}^{m-4} \frac{m![(j-6)!]^\sigma [(m-j-4)!]^\sigma}{j^6 (m-j)^4} + \frac{C |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-8}} m^3 [(m-9)!]^\sigma \\
&\leq \frac{C |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-6}} \sum_{j=[(m-3)/2]+1}^{m-4} \frac{(m-7)! m^7}{m^6 (m-j)^4} [(m-10)!]^\sigma + \frac{C |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-6}} m^3 [(m-9)!]^\sigma \\
&\leq \frac{C[(m-7)!]^\sigma |\vec{a}|_{\rho,\sigma}^2}{\rho^{m-6}} \left(m^3 m^{-2\sigma} + \sum_{j=[(m-3)/2]+1}^{m-4} \frac{m}{(m-j)^4 m^{3(\sigma-1)}} \right) \\
&\leq \frac{C[(m-7)!]^\sigma}{\rho^{m-6}} |\vec{a}|_{\rho,\sigma}^2.
\end{aligned}$$

Combining these inequalities yields

$$\sum_{3 \leq j \leq m-3} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi\|_{L^2} \leq \frac{C[(m-7)!]^\sigma}{\rho^{m-6}} |\vec{a}|_{\rho,\sigma}^2,$$

which along with (28) gives

$$\sum_{1 \leq j \leq m-1} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j w) \partial_z \partial_x^{m-j} \psi\|_{L^2} \leq \frac{C[(m-7)!]^\sigma}{\rho^{m-6}} |\vec{a}|_{\rho,\sigma}^2 + \frac{C[(m-7)!]^\sigma}{\rho^{m-6}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho},$$

and thus,

$$A_{2,1} \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\rho,\sigma}^3 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}.$$

The upper bound for $A_{2,2}$ is similar and thus (27) follows.

Upper bound for A_1 . Just following the argument used above for A_2 with slight modification, we can obtain

$$A_1 \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} |\vec{a}|_{\rho,\sigma}^3 + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}.$$

This completes the proof of Lemma 3.3. \square

Lemma 3.4. *Let $\sigma \in [3/2, 2]$. Then for any $m \geq 7$, and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, we have*

$$\left(\langle z \rangle^{\kappa+\delta} \mathcal{L}_m, \langle z \rangle^{\kappa+\delta} \Gamma_m \right)_{L^2} \leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} (|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^3) + \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}.$$

Proof. The proof is similar to those for Lemmas 3.2-3.3. For instance, following the argument in Lemma 3.2, it yields

$$-2 \left(\langle z \rangle^{\kappa+\delta} (\partial_z \psi) \partial_x^m \eta, \langle z \rangle^{\kappa+\delta} \Gamma_m \right)_{L^2} \leq \frac{C[(m-7)!]^{2\sigma} |\vec{a}|_{\tilde{\rho},\sigma}^2}{\rho^{2(m-6)} \tilde{\rho} - \rho}.$$

Meanwhile the other terms involving in \mathcal{L}_m can be handled as in Lemma 3.3. We omit the detail here for brevity. \square

Proof of Proposition 3.1. We first estimate f_m . Multiplying both sides of equation (23) by $\langle z \rangle^{2\ell} f_m$ and then integrating over Ω , by integration by parts and observing $\partial_z f_m|_{z=0} = 0$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\langle z \rangle^\ell f_m\|_{L^2}^2 + \|\langle z \rangle^\ell \partial_z f_m\|_{L^2}^2 \\ &= \left(\langle z \rangle^\ell (\partial_x^m g + \mathcal{J}_m), \langle z \rangle^\ell f_m \right)_{L^2} + \left(w(\partial_z \langle z \rangle^\ell) f_m, \langle z \rangle^\ell f_m \right)_{L^2} + \frac{1}{2} \left((\partial_z^2 \langle z \rangle^{2\ell}) f_m, f_m \right)_{L^2} \\ &\leq \left(\langle z \rangle^\ell \partial_x^m g, \langle z \rangle^\ell f_m \right)_{L^2} + \left(\langle z \rangle^\ell \mathcal{J}_m, \langle z \rangle^\ell f_m \right)_{L^2} + C \|\langle z \rangle^\ell f_m\|_{L^2}^2 \\ &\leq \frac{C[(m-7)!]^{2\sigma}}{\rho^{2(m-6)}} \left(|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^3 \right) + \frac{C[(m-7)!]^{2\sigma} |\vec{a}|_{\tilde{\rho},\sigma}^2}{\rho^{2(m-6)} \tilde{\rho} - \rho}, \end{aligned}$$

where the last line follows from Lemmas 3.2-3.3 as well as (19). Then integration over $[0, t]$ gives the upper bound as desired for the term $\|\langle z \rangle^\ell f_m\|_{L^2}$. The estimate on $\|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2}$ is similar by using the equation (24), Lemma 3.4 and the boundary condition that $\Gamma_m|_{z=0} = 0$. Then the upper bound for $\|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2}$ follows. The estimate on \tilde{f}_m and $\tilde{\Gamma}_m$ can be obtained similarly. Then we complete the proof of Proposition 3.1. \square

4. ESTIMATES ON g_α AND h_α

In this section, we estimate g_α and h_α which are defined by (8), and the main result can be stated as follows.

Proposition 4.1. *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) satisfying the conditions (13)-(14). Then for any $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$, we have*

$$\begin{aligned} & \frac{\rho^{2(|\alpha|-5)}}{[(|\alpha|-6)!]^{2\sigma}} \left(|\alpha|^2 \|\langle z \rangle^{\kappa+\delta} g_\alpha(t)\|_{L^2}^2 + |\alpha|^2 \|\langle z \rangle^{\kappa+\delta} h_\alpha(t)\|_{L^2}^2 \right) \\ &\leq C |\vec{a}|_{\rho,\sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

We will only estimate g_α because the estimation on h_α is similar. By (11), it suffices to estimate g_m and \tilde{g}_m with

$$g_m \stackrel{\text{def}}{=} \partial_x^m g, \quad \tilde{g}_m \stackrel{\text{def}}{=} \partial_y^m g. \quad (29)$$

For the same reason, we handle only g_m . And we begin with the equation solved by g_m . Let $m \geq 1$ and let g_m be given by (29). Then

$$\partial_t g_m + (u \partial_x + v \partial_y + w \partial_z) g_m - \partial_z^2 g_m = \mathcal{K}_m, \quad (30)$$

where

$$\begin{aligned} \mathcal{K}_m &= - \sum_{1 \leq j \leq m} \binom{m}{j} [(\partial_x^j u) \partial_x g_{m-j} + (\partial_x^j v) \partial_y g_{m-j} + (\partial_x^j w) \partial_z g_{m-j}] \\ &\quad + 2 \partial_x^m [(\partial_y \psi) \partial_z \eta] - 2 \partial_x^m [(\partial_y \eta) \partial_z \psi]. \end{aligned}$$

To see this, multiplying the first equation in (6) by $\partial_y v$ and the second equation by $\partial_y u$, and then subtracting one by another, we get the equation solved by g , i.e.,

$$\partial_t g + (u\partial_x + v\partial_y + w\partial_z)g - \partial_z^2 g = 2(\partial_y \psi)\partial_z \eta - 2(\partial_y \eta)\partial_z \psi. \quad (31)$$

Taking ∂_x^m on both sides and then using Leibniz formula, we obtain the equation (30).

The following three lemmas are for the estimates on the terms in \mathcal{K}_m .

Lemma 4.2. *Let $\sigma \in [3/2, 2]$. Then for any $m \geq 7$, any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, and any $\varepsilon > 0$, we have*

$$\begin{aligned} & 2m^2 \left(\langle z \rangle^{\kappa+\delta} \partial_x^m [(\partial_y \psi)\partial_z \eta], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho, \sigma}^3 + |\vec{a}|_{\rho, \sigma}^4 + \frac{|\vec{a}|_{\rho, \sigma}^2}{\tilde{\rho} - \rho} \right), \end{aligned} \quad (32)$$

where C_ε is a constant depending on ε .

Proof. We use Leibniz formula to get

$$\begin{aligned} & m^2 \left(\langle z \rangle^{\kappa+\delta} \partial_x^m [(\partial_y \psi)\partial_z \eta], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & = m^2 \left(\sum_{0 \leq j \leq [m/2]} + \sum_{[m/2]+1 \leq j \leq m} \right) \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_z \partial_x^{m-j} \eta, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}. \end{aligned}$$

Furthermore, using integration by parts for the first term on the right side gives

$$m^2 \left(\langle z \rangle^{\kappa+\delta} \partial_x^m [(\partial_y \psi)\partial_z \eta], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \leq B_1 + B_2 + B_3 + B_4$$

with

$$\begin{aligned} B_1 &= m^2 \sum_{0 \leq j \leq [m/2]} \binom{m}{j} \|\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_x^{m-j} \eta\|_{L^2} \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}, \\ B_2 &= m^2 \sum_{0 \leq j \leq [m/2]} \binom{m}{j} \|\langle z \rangle^{\kappa+\delta} (\partial_y \partial_z \partial_x^j \psi) \partial_x^{m-j} \eta\|_{L^2} \|\langle z \rangle^{\kappa+\delta} g_m\|_{L^2}, \\ B_3 &= m^2 \sum_{0 \leq j \leq [m/2]} \binom{m}{j} \|\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_x^{m-j} \eta\|_{L^2} \|\langle z \rangle^{\kappa+\delta} g_m\|_{L^2}, \\ B_4 &= m^2 \sum_{[m/2]+1 \leq j \leq m} \binom{m}{j} \|\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_z \partial_x^{m-j} \eta\|_{L^2} \|\langle z \rangle^{\kappa+\delta} g_m\|_{L^2}. \end{aligned}$$

By (19)-(21) and the Sobolev inequality (18), we have

$$\begin{aligned} & m \sum_{0 \leq j \leq [m/2]} \binom{m}{j} \|\langle z \rangle^{\kappa+\delta} (\partial_y \partial_z \partial_x^j \psi) \partial_x^{m-j} \eta\|_{L^2} \\ & \leq C m \sum_{j=2}^{[m/2]} \frac{m!}{j!(m-j)!} \frac{[(j-2)!]^\sigma}{\rho^{j-1}} |\vec{a}|_{\rho, \sigma} \frac{[(m-j-6)!]^\sigma}{\rho^{m-j-5}(m-j)} |\vec{a}|_{\rho, \sigma} + \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} |\vec{a}|_{\rho, \sigma}^2 \\ & \leq \frac{C |\vec{a}|_{\rho, \sigma}^2}{\rho^{m-5}} \sum_{j=2}^{[m/2]} \frac{m![(j-2)!]^{\sigma-1} [(m-j-6)!]^{\sigma-1} m}{j^2 (m-j)^7} + \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} |\vec{a}|_{\rho, \sigma}^2 \\ & \leq \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} |\vec{a}|_{\rho, \sigma}^2. \end{aligned}$$

Similarly,

$$m \sum_{0 \leq j \leq [m/2]} \binom{m}{j} \|\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_x^{m-j} \eta\|_{L^2} \leq \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} |\vec{a}|_{\rho,\sigma}^2.$$

Thus, we combine the above estimates to have for any $\varepsilon > 0$,

$$B_1 + B_2 + B_3 \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^3 + |\vec{a}|_{\rho,\sigma}^4 \right).$$

Applying a similar argument as for (27), we have

$$\begin{aligned} & m \left(\sum_{[m/2]+1 \leq j \leq m-2} + \sum_{m-1 \leq j \leq m} \right) \binom{m}{j} \|\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \psi) \partial_z \partial_x^{m-j} \eta\|_{L^2} \\ & \leq \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} |\vec{a}|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^\sigma}{\rho^{m-5}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}. \end{aligned}$$

Thus,

$$B_4 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |\vec{a}|_{\rho,\sigma}^3 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}.$$

Combining the estimates on B_1 - B_4 , we obtain the desired inequality and complete the proof. \square

Lemma 4.3. *Let $\sigma \in [3/2, 2]$. Then for any $m \geq 7$, any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, and any $\varepsilon > 0$, we have*

$$\begin{aligned} & -2m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} [(\partial_x^j u) \partial_x g_{m-j} + (\partial_x^j v) \partial_y g_{m-j} + (\partial_x^j w) \partial_z g_{m-j}], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^3 + |\vec{a}|_{\rho,\sigma}^4 + \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right), \end{aligned}$$

where C_ε is a constant depending on ε .

Proof. Applying the argument used in the proof of Lemma 3.3, we have

$$\begin{aligned} & -2m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} [(\partial_x^j u) \partial_x g_{m-j} + (\partial_x^j v) \partial_y g_{m-j}], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^3 + \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right). \end{aligned}$$

Moreover, we write

$$\begin{aligned} & -2m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_x^j w) \partial_z g_{m-j}, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & = -2m^2 \left[\sum_{1 \leq j \leq m-2} + \sum_{m-1 \leq j \leq m} \right] \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_x^j w) \partial_z g_{m-j}, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}. \end{aligned}$$

Direct calculation shows that, by (13)-(14) and the representation (7) of g ,

$$-2m^2 \sum_{m-1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_x^j w) \partial_z g_{m-j}, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho}.$$

Furthermore, following the approach used to estimate B_1 - B_3 in Lemma 4.2 as well as the argument in Lemma 3.3, we can obtain

$$\begin{aligned} & -2m^2 \sum_{1 \leq j \leq m-2} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_x^j w) \partial_z g_{m-j}, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^3 + |\vec{a}|_{\rho,\sigma}^4 \right). \end{aligned}$$

Thus, combining these estimates yields

$$\begin{aligned} & -2m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_x^j w) \partial_z g_{m-j}, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^3 + |\vec{a}|_{\rho,\sigma}^4 + \frac{|\vec{a}|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} \right). \end{aligned}$$

Then the desired upper bound estimate follows, and it completes the proof. \square

Lemma 4.4. *Let $\sigma \in [3/2, 2]$. Then for any $m \geq 7$, any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, and any $\varepsilon > 0$, we have*

$$\begin{aligned} & -2m^2 \left(\langle z \rangle^{\kappa+\delta} \partial_x^m [(\partial_y \eta) \partial_z \psi], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4 + \frac{|\vec{a}|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} \right), \end{aligned} \quad (33)$$

where C_ε is a constant depending on ε .

Proof. By using (14) and (19)-(22), direct calculation yields

$$\begin{aligned} & -2m^2 \left(\sum_{0 \leq j \leq 1} + \sum_{m-2 \leq j \leq m-1} \right) \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \eta) \partial_z \partial_x^{m-j} \psi, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \frac{C[(m-6)!]^{2\sigma} |\vec{a}|_{\tilde{\rho},\sigma}^2}{\rho^{2(m-5)} \tilde{\rho} - \rho}. \end{aligned}$$

Moreover, following the argument used in Lemma 4.2, we have

$$-2m^2 \sum_{2 \leq j \leq m-3} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^j \eta) \partial_z \partial_x^{m-j} \psi, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |\vec{a}|_{\rho,\sigma}^3.$$

Then it remains to show that, for any $\varepsilon > 0$,

$$\begin{aligned} & -2m^2 \left(\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^m \eta) \partial_z \psi, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4 + \frac{|\vec{a}|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} \right). \end{aligned} \quad (34)$$

For this, by the representation of g , applying $\partial_z \partial_x^m$ to the first equation in (7) gives

$$\begin{aligned} \partial_z g_m &= (\partial_y \partial_x^m \eta) \psi + \sum_{0 \leq j \leq m-1} \binom{m}{j} (\partial_y \partial_x^j \eta) \partial_x^{m-j} \psi \\ &\quad + \partial_x^m [(\partial_y v) \partial_z \psi] - \partial_x^m [(\partial_y \psi) \eta] - \partial_x^m [(\partial_y u) \partial_z \eta]. \end{aligned}$$

Thus

$$-2m^2 \left(\langle z \rangle^{\kappa+\delta} (\partial_y \partial_x^m \eta) \partial_z \psi, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} = \sum_{1 \leq j \leq 5} S_j,$$

where S_j are given by, using the notation $\chi = \partial_z \psi / \psi$,

$$\begin{aligned} S_1 &= -2m^2 \left(\langle z \rangle^{\kappa+\delta} \chi \partial_z g_m, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}, \\ S_2 &= 2m^2 \sum_{0 \leq j \leq m-1} \binom{m}{j} \left(\langle z \rangle^{\kappa+\delta} \chi (\partial_y \partial_x^j \eta) \partial_x^{m-j} \psi, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}, \\ S_3 &= 2m^2 \left(\langle z \rangle^{\kappa+\delta} \chi \partial_x^m [(\partial_y v) \partial_z \psi], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}, \\ S_4 &= -2m^2 \left(\langle z \rangle^{\kappa+\delta} \chi \partial_x^m [(\partial_y \psi) \eta], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}, \\ S_5 &= -2m^2 \left(\langle z \rangle^{\kappa+\delta} \chi \partial_x^m [(\partial_y u) \partial_z \eta], \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2}. \end{aligned}$$

The estimation on S_3 - S_5 is similar to that for (32), in fact, we have

$$S_3 + S_4 + S_5 \leq \varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^3 + |\vec{a}|_{\rho,\sigma}^4 + \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right).$$

And following the argument in Lemma 3.3, it gives

$$S_2 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^3 + \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right).$$

Finally,

$$S_1 = m^2 \left([\partial_z (\langle z \rangle^{2(\kappa+\delta)} \chi)] g_m, g_m \right)_{L^2} \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |\vec{a}|_{\rho,\sigma}^2.$$

Combining the upper bounds for S_1 - S_5 gives (34), and then (33) follows. The proof is completed. \square

We are now ready to give

Proof of Proposition 4.1. Observe $g_m|_{z=0} = 0$. We multiply both sides of the equation (30) by $m^2 \langle z \rangle^{2(\kappa+\delta)} g_m$, then integrate over Ω , to have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} m^2 \|\langle z \rangle^{\kappa+\delta} g_m\|_{L^2}^2 + m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 \\ &= m^2 \left(\langle z \rangle^{\kappa+\delta} \mathcal{K}_m, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} + m^2 \left(w(\partial_z \langle z \rangle^{\kappa+\delta}) g_m, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} \\ & \quad + \frac{1}{2} m^2 \left((\partial_z^2 \langle z \rangle^{2(\kappa+\delta)}) g_m, g_m \right)_{L^2} \\ &\leq m^2 \left(\langle z \rangle^{\kappa+\delta} \mathcal{K}_m, \langle z \rangle^{\kappa+\delta} g_m \right)_{L^2} + C m^2 \|\langle z \rangle^{\kappa+\delta} g_m\|_{L^2}^2. \end{aligned}$$

Furthermore, by Lemmas 4.2-4.4 and (21) we see the terms in the last line are bounded from above by

$$\varepsilon m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 + \frac{C_\varepsilon [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4 + \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right)$$

for any $\varepsilon > 0$. Then choosing ε small enough gives

$$\frac{d}{dt} m^2 \|\langle z \rangle^{\kappa+\delta} g_m\|_{L^2}^2 + m^2 \|\langle z \rangle^{\kappa+\delta} \partial_z g_m\|_{L^2}^2 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4 + \frac{|\vec{a}|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} \right),$$

and thus, integrating over $[0, t]$ yields

$$\begin{aligned} & \frac{\rho^{2(m-5)}}{[(m-6)!]^{2\sigma}} m^2 \|\langle z \rangle^{\kappa+\delta} g_m(t)\|_{L^2}^2 \\ & \leq C |\vec{a}_0|_{\rho,\sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

A similar argument applies to the estimation on the upper bound for \tilde{g}_m . As a result, it follows from (11) that

$$\begin{aligned} & \frac{\rho^{2(|\alpha|-5)}}{[(|\alpha|-6)!]^{2\sigma}} |\alpha|^2 \|\langle z \rangle^{\kappa+\delta} g_\alpha(t)\|_{L^2}^2 \\ & \leq C |\vec{a}_0|_{\rho,\sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Note that similar to (31), we have

$$\partial_t h + (u \partial_x + v \partial_y + w \partial_z) h - \partial_z^2 h = 2 (\partial_x \eta) \partial_z \psi - 2 (\partial_x \psi) \partial_z \eta.$$

This enables us to estimate h_α in the same way as g_α . Then we complete the proof of Proposition 4.1. \square

5. ESTIMATE ON $\|\vec{a}\|_{\rho,\sigma}$

This section is about the estimate on $\|\vec{a}\|_{\rho,\sigma}$ with $\vec{a} = (u, v)$. We will handle the tangential derivatives and mixed derivatives in Subsection 5.1 and Subsection 5.2 respectively.

5.1. Upper bound for the tangential derivatives. For the derivatives in x, y variables, we have the following

Proposition 5.1. *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0,\sigma})$ is the solution to the Prandtl system (5) satisfying the conditions (13)-(14). Then for any $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$, we have*

$$\begin{aligned} & \frac{\rho^{2(|\alpha|-6)}}{[(|\alpha|-7)!]^{2\sigma}} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha u(t)\|_{L^2}^2 + \|\langle z \rangle^\ell \partial^\alpha \psi(t)\|_{L^2}^2 + \|\langle z \rangle^\kappa \partial^\alpha v(t)\|_{L^2}^2 \right) \\ & + \frac{\rho^{2(|\alpha|-5)}}{[(|\alpha|-6)!]^{2\sigma}} |\alpha|^2 \|\langle z \rangle^{\kappa+2} \partial^\alpha \eta(t)\|_{L^2}^2 \\ & \leq C |\vec{a}_0|_{\rho,\sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

The proof of this proposition is based on the following two lemmas.

Lemma 5.2 (Estimates on $\partial^\alpha u, \partial^\alpha \psi$ and $\partial^\alpha v$). *Under the same assumption as in Proposition 5.1, for any $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, we have*

$$\begin{aligned} & \frac{\rho^{2(|\alpha|-6)}}{[(|\alpha|-7)!]^{2\sigma}} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha u(t)\|_{L^2}^2 + \|\langle z \rangle^\ell \partial^\alpha \psi(t)\|_{L^2}^2 + \|\langle z \rangle^\kappa \partial^\alpha v(t)\|_{L^2}^2 \right) \\ & \leq C |\vec{a}_0|_{\rho,\sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Proof. The estimates on $\partial^\alpha u$ and $\partial^\alpha \psi$ rely on the following Hardy inequality

$$\|\langle z \rangle^{\ell-1} \partial_x^m u\|_{L^2} + \|\langle z \rangle^\ell \partial_x^m \psi\|_{L^2} \leq C \|\langle z \rangle^\ell f_m\|_{L^2},$$

whose proof is standard, cf. [16, Lemma 6.2] for example. As a result, we use the above Hardy inequality and Proposition 3.1 to obtain

$$\begin{aligned} & \frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} (\|\langle z \rangle^{\ell-1} \partial_x^m u(t)\|_{L^2}^2 + \|\langle z \rangle^\ell \partial_x^m \psi(t)\|_{L^2}^2) \\ & \leq C |\vec{a}_0|_{\rho,\sigma}^2 + C \left(\int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned} \quad (35)$$

Similar estimates hold for the weighted L^2 -norms of $\partial_y^m u$ and $\partial_y^m \psi$. Then we use (11) with $|\alpha| = m$ for the desired estimates on $\partial^\alpha u$ and $\partial^\alpha \psi$.

It remains to handle $\partial^\alpha v$. And again it suffices to consider $\partial_x^m v$ and $\partial_y^m v$ with $m = |\alpha|$, due to (11). In view of (9), we have

$$\partial_x^m v = (\Gamma_m + (\partial_x^m u)\eta)/\psi,$$

and thus, recalling $\psi \sim \langle z \rangle^{-\delta}$, and using (14) and (2),

$$\begin{aligned} \|\langle z \rangle^\kappa \partial_x^m v\|_{L^2} & \leq C \|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2} + C \|\langle z \rangle^{\kappa+\delta} \eta \partial_x^m u\|_{L^2} \\ & \leq C \|\langle z \rangle^{\kappa+\delta} \Gamma_m\|_{L^2} + C \|\langle z \rangle^{\ell-1} \partial_x^m u\|_{L^2}. \end{aligned}$$

This along with Proposition 3.1 and (35), yields the upper bound for $\partial_x^m v$, i.e.,

$$\begin{aligned} & \frac{\rho^{2(m-6)}}{[(m-7)!]^{2\sigma}} \|\langle z \rangle^\kappa \partial_x^m v(t)\|_{L^2}^2 \\ & \leq C |\vec{a}_0|_{\rho,\sigma}^2 + C \left(\int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

The same argument applies to $\partial_y^m v$. Thus, the desired bound for $\partial^\alpha v$ follows and the proof is completed. \square

Lemma 5.3 (Estimate on $\partial^\alpha \eta$). *Under the same assumption as in Proposition 5.1, for any $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \geq 7$, any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, we have*

$$\begin{aligned} & \frac{\rho^{2(|\alpha|-5)}}{[(|\alpha|-6)!]^{2\sigma}} |\alpha|^2 \|\langle z \rangle^{\kappa+2} \partial^\alpha \eta(t)\|_{L^2}^2 \\ & \leq C |\vec{a}_0|_{\rho,\sigma}^2 + C \left(\int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Proof. As before, it suffices to consider only $\partial_x^m \eta$ and $\partial_y^m \eta$. We apply ∂_x^m to the equation for η in (6) to obtain

$$\begin{aligned} & \partial_t \partial_x^m \eta + (u \partial_x + v \partial_y + w \partial_z) \partial_x^m \eta - \partial_z^2 \partial_x^m \eta \\ & = \partial_x^m h - \sum_{1 \leq j \leq m} \binom{m}{j} [(\partial_x^j u) \partial_x \partial_x^{m-j} \eta + (\partial_x^j v) \partial_y \partial_x^{m-j} \eta + (\partial_x^j w) \partial_z \partial_x^{m-j} \eta]. \end{aligned}$$

Multiplying both sides above by $m^2 \langle z \rangle^{2(\kappa+2)} \partial_x^m \eta$, and then integrating over Ω , we have by noting $\partial_z \eta|_{z=0} = 0$,

$$\frac{1}{2} \frac{d}{dt} m^2 \|\langle z \rangle^{\kappa+2} \partial_x^m \eta\|_{L^2}^2 + m^2 \|\langle z \rangle^{\kappa+2} \partial_z \partial_x^m \eta\|_{L^2}^2 = P_1 + P_2 + P_3$$

with

$$\begin{aligned}
P_1 &= m^2 \left(\langle z \rangle^{\kappa+2} \partial_x^m h, \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2} + m^2 \left(w(\partial_z \langle z \rangle^{\kappa+2}) \partial_x^m \eta, \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2} \\
&\quad + \frac{1}{2} m^2 \left((\partial_z^2 \langle z \rangle^{2(\kappa+2)}) \partial_x^m \eta, \partial_x^m \eta \right)_{L^2}, \\
P_2 &= -m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+2} [(\partial_x^j u) \partial_x \partial_x^{m-j} \eta + (\partial_x^j v) \partial_y \partial_x^{m-j} \eta], \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2}, \\
P_3 &= -m^2 \sum_{1 \leq j \leq m} \binom{m}{j} \left(\langle z \rangle^{\kappa+2} (\partial_x^j w) \partial_z \partial_x^{m-j} \eta, \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2}.
\end{aligned}$$

Furthermore, using (21)-(22) we obtain by noting $\kappa + 2 \leq \kappa + \delta$ due to (2) and $\partial_x^m h = h_\gamma$ with $\gamma = (m, 0)$,

$$P_1 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |\vec{a}|_{\rho, \sigma}^2.$$

We now estimate P_3 . By (21), (22) and (14), we compute

$$\begin{aligned}
&-m^2 \left(\sum_{1 \leq j \leq 2} + \sum_{m-1 \leq j \leq m} \right) \binom{m}{j} \left(\langle z \rangle^{\kappa+2} (\partial_x^j w) \partial_z \partial_x^{m-j} \eta, \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2} \\
&\leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \frac{|\vec{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho},
\end{aligned}$$

moreover, for the terms in between in the summation, it follows from the argument used in Lemma 3.3 that

$$-m^2 \sum_{3 \leq j \leq m-2} \binom{m}{j} \left(\langle z \rangle^{\kappa+2} (\partial_x^j w) \partial_z \partial_x^{m-j} \eta, \langle z \rangle^{\kappa+2} \partial_x^m \eta \right)_{L^2} \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |\vec{a}|_{\rho, \sigma}^3.$$

Thus, we obtain

$$P_3 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho, \sigma}^3 + \frac{|\vec{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} \right).$$

Similarly, we have

$$P_2 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho, \sigma}^3 + \frac{|\vec{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} \right).$$

Combining these estimates gives

$$\frac{1}{2} \frac{d}{dt} m^2 \|\langle z \rangle^{\kappa+2} \partial_x^m \eta\|_{L^2}^2 + m^2 \|\langle z \rangle^{\kappa+2} \partial_z \partial_x^m \eta\|_{L^2}^2 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|\vec{a}|_{\rho, \sigma}^2 + |\vec{a}|_{\rho, \sigma}^3 + \frac{|\vec{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} \right).$$

Integrating over $[0, t]$ yields

$$\begin{aligned}
&\frac{\rho^{2(m-5)}}{[(m-6)!]^{2\sigma}} m^2 \|\langle z \rangle^{\kappa+2} \partial_x^m \eta(t)\|_{L^2}^2 \\
&\leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right).
\end{aligned}$$

The upper bound of $\|\langle z \rangle^{\kappa+2} \partial_y^m \eta(t)\|_{L^2}$ can be obtained similarly. Thus, the estimate on $\partial^\alpha \eta$ follows and this completes the proof. \square

As an immediate consequence of Lemmas 5.2-5.3, we have Proposition 5.1.

5.2. Upper bound for the mixed derivatives. Now we estimate the mixed derivatives that is stated in

Proposition 5.4. *Let $3/2 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) satisfying the conditions (13)-(14). Then we have, for any pair $(\alpha, j) \in \mathbb{Z}_+^2 \times \mathbb{Z}_+$ with $1 \leq j \leq 4$ and $|\alpha| + j \geq 7$, any $t \in [0, T]$, and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$,*

$$\begin{aligned} & \frac{\rho^{2(|\alpha|+j-6)}}{[(|\alpha|+j-7)!]^{2\sigma}} \|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi(t)\|_{L^2}^2 + \frac{\rho^{2(|\alpha|+j-5)}}{[(|\alpha|+j-6)!]^{2\sigma}} |\alpha|^2 \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta(t)\|_{L^2}^2 \\ & \leq C |\vec{a}_0|_{\rho, \sigma}^2 + C \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\rho, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Sketch of the proof. Based on the above discussion, we will only sketch the proof for brevity. First of all, we need to consider here the boundary conditions since the normal derivatives are involved when we use integration by parts. The situation is exactly the same as in 2D, where we need to use the equation (6) and the boundary conditions

$$\partial_z \psi|_{z=0} = \partial_z \eta|_{z=0} = 0,$$

to reduce the order of normal derivatives in the boundary terms. Precisely, we use the equation for ψ in (6) to obtain

$$\partial_z^3 \psi|_{z=0} = \partial_z (\partial_t \psi + (u \partial_x + v \partial_y + w \partial_z) \psi - g)|_{z=0} = \psi (\partial_x \psi - \partial_y \eta)|_{z=0} + 2\eta \partial_y \psi|_{z=0},$$

and moreover, direct computation yields

$$\partial_z^5 \psi|_{z=0} = -(\partial_z^2 \psi) (\partial_x \psi + \partial_y \eta)|_{z=0} + 4\psi \partial_x \partial_z^2 \psi|_{z=0} + 4\eta \partial_y \partial_z^2 \psi|_{z=0}.$$

Similar equalities hold for $\partial_z^3 \eta|_{z=0}$ and $\partial_z^5 \eta|_{z=0}$. Then, we can follow the argument in [16] to deduce the energy estimates on $\partial^\alpha \partial_z^j \psi$ and $\partial^\alpha \partial_z^j \eta$, with only difference coming from the appearance of two additional terms

$$\left(\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j g, \langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi \right)_{L^2} \text{ and } |\alpha|^2 \left(\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j h, \langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta \right)_{L^2}. \quad (36)$$

However, there will be no additional difficulty to control the above two terms because we can use (7) to write, by noting $1 \leq j \leq 4$,

$$\partial^\alpha \partial_z^j g = \partial^\alpha \partial_z^{j-1} \left[(\partial_y \eta) \psi + (\partial_y v) \partial_z \psi - (\partial_y \psi) \eta - (\partial_x u) \partial_z \eta \right].$$

Then following the argument in the proof of Lemma 3.3 yields that

$$\|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^{j-1} g\|_{L^2} \leq \frac{C[(|\alpha|+j-7)!]^\sigma}{\rho^{|\alpha|+j-6}} |\vec{a}|_{\rho, \sigma}^2 + \frac{C[(|\alpha|+j-7)!]^\sigma}{\rho^{|\alpha|+j-6}} \frac{|\vec{a}|_{\tilde{\rho}, \sigma}}{\tilde{\rho} - \rho}.$$

Hence, we have

$$\left(\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j g, \langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi \right)_{L^2} \leq \frac{C[(|\alpha|+j-7)!]^{2\sigma}}{\rho^{2(|\alpha|+j-6)}} \left(|\vec{a}|_{\rho, \sigma}^3 + \frac{|\vec{a}|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} \right).$$

Similarly, we can show that the second term in (36) is bounded from above by

$$\frac{C[(|\alpha|+j-6)!]^{2\sigma}}{\rho^{2(|\alpha|+j-5)}} |\vec{a}|_{\rho, \sigma}^3.$$

And the other terms can be estimated in the same way as [16], so that we omit the detail for brevity, cf. Subsection 6.2 in [16]. Then the estimate given in Proposition 5.4 follows. \square

6. PROOF OF A PRIORI ESTIMATES

In this section, we will prove the a priori estimate, in the two cases of $\sigma \in [3/2, 2]$ and $1 < \sigma < 3/2$ separately. For this, we will first prove Theorem 2.3 for the case when $\sigma \in [3/2, 2]$.

The case when $\sigma \in [3/2, 2]$. By Propositions 3.1, 4.1, 5.1 and 5.4, we have the upper bound on the terms in $|\vec{a}|_{\rho, \sigma}$ when the order of derivatives is greater than or equal to 7, that is, these terms are bounded by

$$C |\vec{a}_0|_{\rho, \sigma}^2 + C \int_0^t \left(|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4 \right) ds + C \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds.$$

For the derivatives with order less than 7, it can be checked straightforwardly that the same upper bound holds. Hence, we have the a priori estimate (15) and then complete the proof of Theorem 2.3.

The case when $1 < \sigma < 3/2$. The proof for this case is similar to the one when $\sigma \in [3/2, 2]$, just replacing the norm $\|\cdot\|_{\rho, \sigma}$ by an equivalent norm. Precisely, if $1 < \sigma < 3/2$, then we can find an integer $N \geq 2$ such that

$$(N+1)/N \leq \sigma. \quad (37)$$

Define another Gevrey norm $\|\cdot\|_{\rho, \sigma, N}$ by replacing respectively $\sup_{|\alpha| \geq 7}$ and $\sup_{|\alpha| \leq 6}$ in (3) by $\sup_{|\alpha| \geq N+5}$ and $\sup_{|\alpha| \leq N+4}$. Do the same for $\sup_{|\alpha|+j \geq 7}$ and $\sup_{|\alpha|+j \leq 6}$. This new norm $\|\cdot\|_{\rho, \sigma, N}$ is equivalent to $\|\cdot\|_{\rho, \sigma}$ in the sense that

$$\|\cdot\|_{\rho, \sigma} \leq \|\cdot\|_{\rho, \sigma, N} \leq C_N \rho^{2-N} \|\cdot\|_{\rho, \sigma}$$

for $\rho \leq 1$, where C_N is a constant depending only on N . Moreover, following the same argument as above, we replace $|\cdot|_{\rho, \sigma}$ given in Definition 2.1 by $|\cdot|_{\rho, \sigma, N}$, and replace (14) by

$$\begin{aligned} & \sum_{|\alpha| \leq N+1} \left(\|\langle z \rangle^{\ell-1} \partial^\alpha (u-1)\|_{L^\infty} + \|\langle z \rangle^\kappa \partial^\alpha v\|_{L^\infty} + \|\partial^\alpha w\|_{L^\infty} \right) \\ & + \sum_{|\alpha| \leq N+2} \|\langle z \rangle^\ell \partial^\alpha \psi\|_{L_{x,y}^\infty(L_z^2)} + \sum_{\substack{|\alpha|+j \leq N+2 \\ j \geq 1}} \|\langle z \rangle^{\ell+1} \partial^\alpha \partial_z^j \psi\|_{L_{x,y}^\infty(L_z^2)} \\ & + \sum_{|\alpha|+j \leq N+2} \|\langle z \rangle^{\kappa+2} \partial^\alpha \partial_z^j \eta\|_{L_{x,y}^\infty(L_z^2)} \leq \tilde{C}. \end{aligned} \quad (38)$$

And then we have

Theorem 6.1. *Let $1 < \sigma < 3/2$ and $0 < \rho_0 \leq 1$. Suppose $(u, v) \in L^\infty([0, T]; X_{\rho_0, \sigma})$ is the solution to the Prandtl system (5) such that the properties in (13) and (38) hold. Then there exists a constant $C_* > 1$, such that*

$$|\vec{a}(t)|_{\rho, \sigma, N}^2 \leq C_* |\vec{a}_0|_{\rho, \sigma, N}^2 + C_* \int_0^t \left(|\vec{a}(s)|_{\rho, \sigma, N}^2 + |\vec{a}(s)|_{\rho, \sigma, N}^4 \right) ds + C_* \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma, N}^2}{\tilde{\rho} - \rho} ds \quad (39)$$

holds for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$ and any $t \in [0, T]$.

Sketch of the proof. For brevity, we only give a sketch of the proof. In fact, it is similar to that for Theorem 2.3. Here we will show how to modify the argument used there.

We first recall how the assumption $\sigma \in [3/2, 2]$ is used when proving Theorem 2.3. For $\sigma \geq 3/2$, we use the following type of splitting in the summation

$$\left[\sum_{0 \leq j \leq 1} + \sum_{m-1 \leq j \leq m} \right] \binom{m}{j} \cdots + \sum_{2 \leq j \leq m-2} \binom{m}{j} \cdots. \quad (40)$$

By direct computation, we see that the factors m and $m^{2-\sigma}$ appear in the first two summations in (40) respectively. Then we can use (16) and (14) to conclude that the summation

$$\left[\sum_{0 \leq j \leq 1} + \sum_{m-1 \leq j \leq m} \right] \binom{m}{j} \cdots$$

is basically bounded from above by

$$\frac{C[(m-7)!]^\sigma |\vec{a}|_{\tilde{\rho}, \sigma}}{\rho^{m-6} \tilde{\rho} - \rho} \quad \text{or} \quad \frac{C[(m-6)!]^\sigma |\vec{a}|_{\tilde{\rho}, \sigma}}{\rho^{m-5} \tilde{\rho} - \rho}.$$

Meanwhile, for the last summation in (40) we have factors like

$$m^{3-2\sigma}, m^{4-3\sigma}, \dots,$$

which are less than 1 when $\sigma \geq 3/2$. Thus, the last summation in (40) is basically bounded from above by

$$\frac{C[(m-7)!]^\sigma |\vec{a}|_{\rho, \sigma}^2}{\rho^{m-6}} \quad \text{or} \quad \frac{C[(m-6)!]^\sigma |\vec{a}|_{\rho, \sigma}^2}{\rho^{m-5}}.$$

Now we turn to the case when $1 < \sigma < 3/2$, and instead of (40), we can use a new splitting

$$\left[\sum_{0 \leq j \leq N-1} + \sum_{m-N+1 \leq j \leq m} \right] \binom{m}{j} \cdots + \sum_{N \leq j \leq m-N} \binom{m}{j} \cdots. \quad (41)$$

Then we have factors

$$m, m^{2-\sigma}, m^{3-2\sigma}, m^{4-3\sigma}, m^{N-(N-1)\sigma},$$

appearing in the first two summations. Note that these factors are bounded by m , and then we can use again (16) and (38) to estimate the first two summations. Meanwhile, factors like

$$m^{N+1-N\sigma}, m^{N+2-(N+1)\sigma}, \dots,$$

appear in the last summation in (41), and these factors are less than 1 because of (37). So the situation is similar to the case of $\sigma \in [3/2, 2]$. Then one can apply the argument used in Sections 3-5 to the case when $1 < \sigma < 3/2$, with $|\cdot|_{\rho, \sigma}$ and (14) therein replaced by $|\cdot|_{\rho, \sigma, N}$ and (38). Then the desired a priori estimate (39) follows, and it completes the proof. \square

7. PROOF OF THE MAIN RESULTS

We will prove in this section the main results on the existence and uniqueness for Prandtl system. We first prove Theorem 1.8 that corresponds to the constant outer flow when $(U, V) = (1, 0)$, and then explain how to extend the argument to the general outer flow. Since the proof is similar as in 2D case after we have the a priori estimate, we will only give a sketch.

Sketch of the proof of Theorem 1.8. The proof relies on the a priori estimates given in Theorems 2.3 and 6.1.

In order to obtain the existence of solutions to the Prandtl equations (5), there are two main ingredients, one of which is to investigate the existence of approximate solutions to the regularized Prandtl system

$$\begin{cases} \partial_t u_\varepsilon + (u_\varepsilon \partial_x + v_\varepsilon \partial_y + w_\varepsilon \partial_z) u_\varepsilon - \partial_z^2 u_\varepsilon - \varepsilon \partial_x^2 u_\varepsilon - \varepsilon \partial_y^2 u_\varepsilon = 0, \\ \partial_t v_\varepsilon + (u_\varepsilon \partial_x + v_\varepsilon \partial_y + w_\varepsilon \partial_z) v_\varepsilon - \partial_z^2 v_\varepsilon - \varepsilon \partial_x^2 v_\varepsilon - \varepsilon \partial_y^2 v_\varepsilon = 0, \\ (u_\varepsilon, v_\varepsilon)|_{z=0} = (0, 0), \quad \lim_{z \rightarrow +\infty} (u_\varepsilon, v_\varepsilon) = (1, 0), \\ (u_\varepsilon, v_\varepsilon)|_{t=0} = (u_0, v_0), \end{cases} \quad (42)$$

with $w_\varepsilon = -\int_0^z (\partial_x u_\varepsilon + \partial_y v_\varepsilon) d\tilde{z}$. We remark that the regularized equations above share the same compatibility condition (4) as the original system (5). Another ingredient is to derive a uniform estimate with respect to ε for the approximate solutions $(u_\varepsilon, v_\varepsilon)$.

The existence for the parabolic system (42) is standard. Indeed, suppose that $(u_0, v_0) \in X_{2\rho_0, \sigma}$. Then we can construct, following the similar scheme as that in [16, Section 7], a solution $(u_\varepsilon, v_\varepsilon) \in L^\infty([0, T_\varepsilon^*]; X_{3\rho_0/2, \sigma})$ to (42) such that

$$\sup_{t \in [0, T_\varepsilon^*]} \|(u_\varepsilon(t), v_\varepsilon(t))\|_{3\rho_0/2, \sigma} \leq C \|(u_0, v_0)\|_{2\rho_0, \sigma}, \quad (43)$$

where T_ε^* may depend on ε .

It remains to derive a uniform estimate for the approximate solutions $(u_\varepsilon, v_\varepsilon)$, so that we can remove the ε -dependence of the lifespan T_ε^* . For this, we need to verify that $(u_\varepsilon, v_\varepsilon)$ satisfies the condition (13), and the condition (14) if $\sigma \in [3/2, 2]$ and (38) if $1 < \sigma < 3/2$ respectively.

In view of (43) and by Sobolev inequalities and the definition of $\|\cdot\|_{\rho, \sigma}$, we know that u_ε and v_ε satisfy the condition (14) if $\sigma \in [3/2, 2]$, and the condition (38) in the case of $1 < \sigma < 3/2$. In order to verify (13), we suppose u_0 satisfies Assumption 1.1 additionally and show that these properties therein preserve in time. This is clear when we consider a small perturbation around a shear flow. For the general initial data, it was shown in [19] by using the classical maximum principle for the parabolic equation, that such properties listed in Assumption 1.1 also preserve in time. Here, we can adopt the argument in [19, Subsection 5.2], and apply the maximum principle to the first equation in (42). Precisely, applying ∂_z to the first equation in (42) and using the notations $\psi_\varepsilon = \partial_z u_\varepsilon$ and $\eta_\varepsilon = \partial_z v_\varepsilon$, we have

$$\partial_t \psi_\varepsilon + (u_\varepsilon \partial_x + v_\varepsilon \partial_y + w_\varepsilon \partial_z) \psi_\varepsilon - \partial_z^2 \psi_\varepsilon - \varepsilon \partial_x^2 \psi_\varepsilon - \varepsilon \partial_y^2 \psi_\varepsilon = (\partial_y v_\varepsilon) \psi_\varepsilon - (\partial_y u_\varepsilon) \eta_\varepsilon,$$

that is,

$$\begin{aligned} & \left[\partial_t + \left(u_\varepsilon \partial_x + v_\varepsilon \partial_y + (w_\varepsilon + 2\delta(1+z)^{-1}) \partial_z \right) - \partial_z^2 - \varepsilon \partial_x^2 - \varepsilon \partial_y^2 \right] ((1+z)^\delta \psi_\varepsilon) \\ &= \left(\partial_y v_\varepsilon - (\partial_y u_\varepsilon) \eta_\varepsilon / \psi_\varepsilon + \delta w_\varepsilon / (1+z) + \delta(\delta+1)/(1+z)^2 \right) (1+z)^\delta \psi_\varepsilon. \end{aligned}$$

By using (43) and the maximal principle for parabolic equations (see [19, Lemma E.2]), one can show that there exists $c_* > 0$ independent of ε such that for any $(t, x, y, z) \in [0, T_\varepsilon^*] \times \Omega$,

$$\psi_\varepsilon(t, x, y, z) \geq c_*(1+z)^{-\delta}.$$

This gives the lower bound on $\partial_z u_\varepsilon$ given in (13). Similarly, we can derive also the upper bounds on $\partial_z u_\varepsilon$ and its normal derivatives given in (13).

Consequently, following the argument in Sections 3-6, the estimates (15) and (39) also hold, with \vec{a} there replaced by $(u_\varepsilon, v_\varepsilon)$. Finally, by the uniform estimate on $(u_\varepsilon, v_\varepsilon)$, we can repeat the argument in [16, Section 8] to conclude the existence and uniqueness of solutions to the Prandtl system (5). For brevity, we omit the detail here and refer it to [19, Section 5.2] and [16, Section 8] for the comprehensive discussion. Thus, the proof of Theorem 1.8 is completed. \square

Sketch of the proof of Theorem 1.4. Now we consider the general outer flow $U, V, p \in Y_{2\rho_0, \sigma}$. The argument is similar to the case with the constant outer flow discussed above. The main difference comes from the appearance of extra source term and boundary term. Since the extra source terms only involve U, V and p , they are bounded by the Gevrey norms of U, V and p .

In addition, the boundary terms do not create additional difficulty in the estimation. To see this, we consider for instance \hat{f}_m which is defined in the same way as f_m , that is,

$$\hat{f}_m = \partial_x^m \psi - \frac{\partial_z \psi}{\psi} \partial_x^m (u - U) = \psi \partial_z \left(\frac{\partial_x^m (u - U)}{\psi} \right).$$

Different from the case with constant data U , since $\hat{f}_m \partial_z \hat{f}_m|_{z=0} \neq 0$, then we have boundary terms like

$$\int_{\mathbb{T}^2} \hat{f}_m(x, y, 0) \partial_z \hat{f}_m(x, y, 0) dx dy,$$

when applying integration by parts. Furthermore, note that, denoting $\chi = \partial_z \psi / \psi$,

$$\hat{f}_m(x, y, 0) = \partial_x^m \psi(x, y, 0) + \chi(x, y, 0) \partial_x^m U(x, y),$$

by using the fact that $\partial_z \psi|_{z=0} = \partial_x p$, we have

$$\partial_z \hat{f}_m(x, y, 0) = \partial_x^{m+1} p(x, y) + (\partial_z \chi)(x, y, 0) \partial_x^m U(x, y) - \chi(x, y, 0) \partial_x^m \psi(x, y, 0).$$

Then

$$\left| \int_{\mathbb{T}^2} \hat{f}_m(x, y, 0) \partial_z \hat{f}_m(x, y, 0) dx dy \right| \leq C \|\partial_x^m \psi(\cdot, 0)\|_{L^2(\mathbb{T}^2)}^2 + \text{terms involving } U, V, p.$$

As for the first term on the right side, we have

$$\begin{aligned} \|\partial_x^m \psi(\cdot, 0)\|_{L^2(\mathbb{T}^2)}^2 &\leq \varepsilon \|\partial_z \partial_x^m \psi\|_{L^2(\Omega)}^2 + C_\varepsilon \|\partial_x^m \psi\|_{L^2(\Omega)}^2 \\ &\leq \varepsilon \|\partial_z \hat{f}_m\|_{L^2(\Omega)}^2 + C_\varepsilon \left(\|\partial_x^m \psi\|_{L^2(\Omega)}^2 + \|\partial_x^m (u - U)\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

where in the last inequality we have used the representation of \hat{f}_m . We can choose ε small enough to absorb the first term $\|\partial_z \hat{f}_m\|_{L^2(\Omega)}^2$. Then the quantity

$$\left| \int_{\mathbb{T}^2} \hat{f}_m(x, y, 0) \partial_z \hat{f}_m(x, y, 0) dx dy \right|$$

has a suitable upper bound, just as $\|\partial_x^m \psi\|_{L^2}^2$ and $\|\partial_x^m (u - U)\|_{L^2}^2$. We can apply a similar argument as above to the other boundary terms when estimating $g_\alpha, h_\alpha, \Gamma_m$, etc.

In summary, we can extend the result to general outer flow (U, V) , with (U, V) in a Gevrey space with the same index σ as the initial data. For this, we replace $(u - 1, v)$ by $(u - U, v - V)$ in the estimation, and perform the estimates on $(u - U, v - V)$ following the discussions in Sections 3-6. Since it does not involve extra essential difficulty, we omit the detail for brevity. \square

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