DESTABILISING COMPACT WARPED PRODUCT EINSTEIN MANIFOLDS

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ABSTRACT. The linear stability of warped product Einstein metrics as fixed points of the Ricci flow is investigated. We generalise the results of Gibbons, Hartnoll and Pope and show that in sufficiently low dimensions, all warped product Einstein metrics are unstable. By exploiting the relationship between warped product Einstein metrics, quasi-Einstein metrics and Ricci solitons, we introduce a new destabilising perturbation (the Ricci variation) and show that certain infinite families of warped product Einstein metrics will be unstable in high dimensions.

1. Introduction

1.1. **Main results.** In 2003, Perelman made spectacular use of Hamilton's Ricci flow to prove Thurston's geometrization conjecture [28], [29] and [30]. Put simply, geometrization says that three-dimensional manifolds decompose into pieces that can each be endowed with a canonical geometry. In order to extend geometrization to higher dimensions, a crucial step is finding the right set of canonical geometries in each dimension. One candidate for these geometries are metrics that are the *stable* fixed points (up to diffeomorphism and scaling) of the Ricci flow. Fixed points of the Ricci flow are known as *Ricci solitons* and clearly include Einstein metrics. Roughly speaking, stability can be taken to mean that the Ricci flow starting at small perturbation of a Ricci soliton will return to the soliton.

The study of the linear stability of Ricci solitons was initiated by Cao, Hamilton, and Ilmanen [6] who considered the second variation of Perelman's ν entropy, a monotonic quantity for the flow. In the subsequent years, stability questions for many important classes of metric have been investigated such as: Einstein metrics admitting parallel spinors [11], compact symmetric spaces [7], and Kähler metrics [17].

In this article we take up the study of stability for a class of compact Einstein metrics known as warped products. These are generalisations of ordinary Riemannian products where the underlying manifold M decomposes as $M = B \times F$ for a base B and a fibre F but the metric on the fibre is 'warped' by a factor coming from the base. We refer the reader to Section 2 for the precise definition of these metrics. There are many important examples of such Einstein metrics in low dimensions including the inhomogenous families on $\mathbb{S}^3 \times \mathbb{S}^2$, $\mathbb{S}^3 \times \mathbb{S}^3$, and $\mathbb{S}^4 \times \mathbb{S}^2$ due to Böhm [3], and the warped product Einstein metric on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2 \times \mathbb{S}^2$ due to Lü, Page and Pope [25]. We completely settle the question of stability in low dimensions:

Theorem A. *Let (M^n, g) be a warped product Einstein manifold where $n \leq 6$. Then (M, g) is unstable as a fixed point of the Ricci flow.*

We remark that the proof of Theorem A also shows that warped products with three-dimensional base and a four-dimensional fibre are unstable too. However, this does not account for all possible seven-dimensional products.

Many known examples of Einstein warped products are constructed from one parameter families of Riemannian manifolds (B, g_m) where the metrics g_m (known as *quasi-Einstein metrics*) converge in the C^∞ topology as $m \to \infty$ to a Ricci soliton (B, g_∞). For example, the metrics of Lü–Page–Pope and their generalisation by the second author [16] have this property. We also prove an asymptotic instability result that complements Theorem A:

Theorem B. *Let (B, g_m) be a one parameter family of quasi-Einstein metrics that converge in the C^∞ topology to a non-trivial Ricci soliton g_∞. Then there exists a $K \in \mathbb{N}$ such that the associated warped product Einstein metrics (M^k, g) are unstable for $k \geq K$.*

As discussed in Section 2, the construction of a warped product Einstein metric requires an Einstein metric $\tilde{g}$ on the fibre F. The following theorem makes precise the interaction between the stability of the fibre Einstein metric and that of the warped product.

Theorem C. *Let (M, g) be a warped product Einstein manifold with fibre $(F, \tilde{g})$ satisfying $\mathrm{Ric}(\tilde{g}) = \mu\tilde{g}$. Let σ be a transverse, trace-free eigentensor of the Lichnerowicz Laplacian of $(F, \tilde{g})$ satisfying $\widetilde{\Delta}_L\sigma = -\kappa\sigma$. If $\kappa < \mu$ then the warped product (M, g) is unstable.*

This theorem allows us to find a large class of Ricci flow unstable warped products. We will refer to warped products that are unstable in the manner of Theorem C as fibre-unstable.

Corollary D. *The following fibres yield fibre-unstable warped products:*

- *when $(F, \tilde{g})$ is a Riemannian product $(F_1 \times F_2, g_1 \oplus g_2)$,*
- *when $(F, \tilde{g})$ is a Kähler-Einstein metric with $h^{(1,1)} > 1$,*
- *when $(F, \tilde{g})$ is a fibre-unstable warped product Einstein metric.*

1.2. **The stability of generalised black holes.** A second motivation for studying stability, which historically preceeds the Ricci flow, comes from the role compact Einstein metrics play in the theory of generalised black holes. This was the context of the pioneering study of the stability of Böhm's Einstein metrics on low-dimensional products of spheres conducted by Gibbons, Hartnoll, and Pope [14]. In their study they exploited the fact that the Böhm metrics are invariant under a cohomogeneity one action by a compact Lie group. We generalise their results in low dimensions to arbitrary warped product Einstein metrics without any symmetry assumptions.

Theorem E. *Let M be a warped product Einstein metric with three-dimensional base and two or three-dimensional fibre. Then the associated black hole is unstable.*

1.3. **Conventions.** As the proofs of the theorems rest on certain quantities having a particular sign, it is important to state exactly the conventions used in the paper where there is room for ambiguity. On the product $B \times F$, uppercase letters denote general coordinates, lowercase Roman letters denote the coordinates on B and lowercase Greek letters denote the coordinates on the fibre F.
The divergence of a tensor T is given by

$$\mathrm{div}(T)(\cdot) = g^{AB}(\nabla_A T)(\partial_B, \cdot)$$

and the (connection) Laplacian ΔT is given by

$$\Delta T = -\nabla^*\nabla T = g^{AB}(\nabla^2_{A,B}T) = \mathrm{div}(\nabla T).$$

With this convention, the spectrum of the Laplacian is non-positive. The convention we use for curvature is
$$R(X,Y)Z = \nabla^2_{Y,X}Z - \nabla^2_{X,Y}Z$$
and the curvature operator $\mathrm{Rm} : s^2(TM) \to s^2(TM)$ is given by
$$\mathrm{Rm}(h,\cdot)_{AB} = R_{ACBD}h^{CD}$$
for $h \in s^2(TM)$.

Geometric objects on the base manifold B will usually be denoted using a bar, for example $\bar{g}$ for the metric. Geometric objects on the fibre F are likewise denoted using a tilde, so we have $\tilde{g}$ for the fibre metric. We will set $n = \dim(B)$ and $m = \dim(F)$.

Acknowledgements: The initial work on this project was conducted whilst WB and TM were visiting SH in Dec 2015 and Jan 2016. The visit of WB was funded by an LMS grant under scheme 4. TM was funded by a Cal. State Fullerton Startup Grant.

2. Background

2.1. **Warped Product Einstein metrics.** Let $M = B \times F$ be a product manifold. Equip M with the metric
$$g = \pi_B^*\overline{g} \oplus (f \circ \pi_B)^2 \pi_F^* \tilde{g},$$
where $\overline{g}$ and $\tilde{g}$ are Riemannian metrics on B and F respectively, $\pi_B : M \to B$ and $\pi_F : M \to F$ denote the natural projections, and $f \in C^\infty(B)$. The Riemannian manifold (M,g) is referred to as a *warped product*. We shall henceforth adopt the standard abuse of notation and drop the references to the projections π_B and π_F. The manifold B is referred to as the base F is referred to as the fibre. By taking the function f to be constant, one recovers a Riemannian product.

We will be concerned with the case where the warped product is an Einstein metric with positive scalar curvature. By Myer's theorem this immediately implies the manifolds B and F are compact. If a warped product (M,g) is an Einstein manifold with $\mathrm{Ric}(g) = \lambda g$ for $\lambda > 0$, then the following is well known (e.g. Corollary 9.107 in [2]) :

$(F, \tilde{g})$ is an Einstein manifold with $\mathrm{Ric}(\tilde{g}) = \mu\tilde{g}$ for some $\mu > 0$,
$$f\overline{\Delta}f + (m-1)|\overline{\nabla}f|^2 + \lambda f^2 = \mu, \tag{2.1}$$
$$\mathrm{Ric}(\overline{g}) - mf^{-1}\overline{\nabla}^2 f = \lambda\overline{g}. \tag{2.2}$$
Riemannian manifolds $(B,\overline{g})$ that solve Equation (2.2) for some $f \in C^\infty(B)$ and $m > 0$ are known as *quasi-Einstein* manifolds and are studied in their own right as interesting generalisations of Einstein metrics. A foundational result of Kim and Kim [21] says that if $(B, \overline{g}, f, m)$ solve (2.2), then there exists a $\mu > 0$ such that f solves Equation (2.1). Hence for integral values of m, one can construct warped product Einstein manifolds from a quasi-Einstein metric on the base B.

We also consider Riemannian manifolds (M,g) where the metric g solves
$$\mathrm{Ric}(g) + \nabla^2\phi = \lambda g, \tag{2.3}$$
where $\phi \in C^\infty(M)$ and $\lambda \in \mathbb{R}$. Metrics solving (2.3) are called *gradient Ricci solitons*. One can view Equation (2.3) as the formal limit as $m \to \infty$ of (2.2) by setting $\phi_m = -m\log f$. More detailed results on the sense in which gradient Ricci solitons are the limits of quasi-Einstein metrics can be found in the work of Case [10].

At the time of writing, there are very few general methods for constructing of compact, warped product Einstein metrics. As mentioned in Section 1, the first examples that were found are due to Böhm [3] and occur on the product $\mathbb{S}^n \times F^m$ with $2 \le m \le 6$ and $3 \le n \le 9 - m$. The second family of examples come from a construction due to Lü, Page and Pope [25] of quasi-Einstein metrics on $\mathbb{CP}^1$-bundles over a Fano Kähler–Einstein base. This construction (and its generalisation due to the second author [16]) produce quasi-Einstein metrics for all $m > 1$ and hence infinitely many examples of warped product Einstein manifolds. The lowest dimensional examples of the Lü–Page–Pope construction occur when the base is the non-trivial $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$. In this case one can view the base as $B = \mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$. A very explicit construction of the Lü–Page–Pope metrics on this manifold was given in [1]. The Lü–Page–Pope quasi-Einstein metrics converge as $m \to \infty$ to the Koiso–Cao Kähler–Ricci soliton constructed independently in [4] and [22]. The metrics constructed in [16] should converge to generalisations of the Koiso–Cao soliton due to Dancer and Wang [12] which are known as Dancer–Wang Kähler–Ricci solitons.

2.2. **Linear stability for Ricci flow.** Ricci solitons first arose as the fixed points up to gauge of the Ricci flow

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g).$$

In particular, Einstein metric evolve via homothetic rescaling. Perelman [28] introduced a quantity ν which is monotonically increasing along a Ricci flow and stationary only at Ricci solitons and, in particular, at Einstein metrics. If the second variation of ν at an Einstein metric is positive, then a small perturbation of the metric will increase ν and the Ricci flow cannot flow back to the Einstein metric. Hence the Einstein metric will be unstable. The linear stability of Ricci solitons was considered by Cao, Hamilton and Ilmanen [6] and Cao and Zhu [8]. In order to state their theorem we need to introduce the operators

$$\mathrm{div}_\phi(\cdot) := e^{-\phi}\mathrm{div}(e^{\phi}\cdot) = \mathrm{div}(\cdot) - \iota_{\nabla\phi}(\cdot),$$

and

$$\Delta_\phi(\cdot) := \Delta(\cdot) - \nabla_{\nabla\phi}(\cdot).$$

Theorem 2.1 (Cao–Hamilton–Ilmanen [6], Cao–Zhu [8])**.** *Let (M, g, ϕ) be a gradient Ricci soliton with potential function ϕ and constant λ. Let $h \in s^2(TM)$. Then*

$$\frac{d^2}{ds^2}\nu(g+sh)|_{s=0} = \frac{(2\lambda)^{-1}}{(8\pi\lambda)^{n/2}} \int_M \langle N(h), h\rangle e^{-\phi} dV_g,$$

where

$$N(h) = \frac{1}{2}\Delta_\phi(h) + \mathrm{Rm}(h,\cdot) + \mathrm{div}^*\mathrm{div}_\phi h + \frac{1}{2}\nabla^2 v_h - \mathrm{Ric}\frac{\int_M \langle \mathrm{Ric}, h\rangle e^{-\phi} dV_g}{\int_M R e^{-\phi} dV_g}, \tag{2.4}$$

and v_h is the unique solution to

$$\Delta_\phi v_h + \lambda v_h = \mathrm{div}_\phi \mathrm{div}_\phi h.$$

As Perelman's ν-entropy is invariant under homothetic rescaling and diffeomorphisms of the metric, we restrict to perturbations given by tensors h satisfying the gauge-fixing conditions

$$\mathrm{div}_\phi(h) = 0 \text{ and } \int_M \langle \mathrm{Ric}(g), h\rangle e^{-\phi} dV_g = 0.$$

In the Einstein case (where the soliton potential ϕ is constant), the stability operator (2.4) restricted to these tensors is given by

$$N(h) = \frac{1}{2}(\Delta_L h + 2\lambda h),$$

where Δ_L is the Lichnerowicz Laplacian

$$\Delta_L h = \Delta h + 2\mathrm{Rm}(h, \cdot) - \mathrm{Ric} \cdot h - h \cdot \mathrm{Ric}. \tag{2.5}$$

Hence we can state a stability criterion for Einstein metrics in terms of the spectrum of the Lichnerowicz Laplacian.

Proposition 2.2 (Ricci flow linear stability [6])**.** *Let (M, g) be a compact Einstein manifold satisfying $\mathrm{Ric}(g) = \lambda g$. If the Lichnerowicz Laplacian has a divergence free, g-orthogonal eigentensor with eigenvalue $-\kappa$ satisfying*

$$\kappa < 2\lambda,$$

then g is unstable as a fixed point of the Ricci flow.

It is expected that Einstein metrics which are stable under the Ricci flow are quite special. In dimension four Richard Hamilton has conjectured that the only linearly stable examples of positively curved Einstein metrics (or Ricci solitons) are $\mathbb{S}^4$ and $\mathbb{CP}^2$ with their standard metrics. The Fubini–Study on $\mathbb{CP}^n$ metric is *neutrally* linearly stable as the eigentensors of the Lichnerowicz Laplacian achieve the bound -2λ. The relationship between dynamic and linear (in)stability was established by Sesum [31]. In particular, she showed that a linearly unstable metric is always dynamically unstable. Recent work by Kröncke [23] showed the surprising result that the Fubini–Study metric on $\mathbb{CP}^n$ is not dynamically stable. Various works [11], [15], [17], [18], [19] have provided case-by-case evidence for Hamilton's conjecture but as yet, very little general theory exists. Recent work by Pali has addressed this in the case of Kähler–Ricci solitons and Kähler–Ricci flow [26], [27] . By looking at compact symmetric spaces, Cao and He showed that there do exist a wider variety of stable Einstein metrics in higher dimensions [7]. The linear stability of noncompact warped product constructions has also been investigated in [15] and [24].

In the case that function f in Equations (2.1) and (2.2) is constant, we recover the usual notion of a product Einstein metric. It is well-known that ordinary products can be destabilised by 'inflating' one of the factors. The destabilising perturbations we use to prove the main theorems follow a similar idea but of course the presence of the non-constant warping factor f complicates this procedure.

We end this section with a lemma that will prove useful in subsequent calculations.

Lemma 2.3. *Let (M^n, g) be an Einstein manifold with Einstein constant $\lambda > 0$ and let h be a divergence free tensor. Then:*

(1) The tensor $h + cg$ satisfies

$$\int_M \langle h + cg, g\rangle \; dV_g = 0,$$

where

$$c = \frac{-\int_M \mathrm{tr}(h) dV_g}{\|g\|^2}.$$

(2) The stability integral for $h+cg$ is given by

$$\langle N(h+cg), h+cg\rangle_{L^2(g)} = \langle \frac{1}{2}\Delta h + \mathrm{Rm}(h,\cdot), h\rangle_{L^2(g)} - \lambda \frac{\left(\int_M \mathrm{tr}(h) dV_g\right)^2}{\|g\|^2}.$$

Proof. (1) is a trivial calculation. To see (2) note that by Equation 2.4, the stability operator N is given by

$$N(h+cg) = \frac{1}{2}\Delta(h+cg) + \mathrm{Rm}(h+cg,\cdot).$$

Hence we see

$$\langle N(h+cg), h+cg\rangle_{L^2(g)} = \langle \frac{1}{2}\Delta(h+cg) + \mathrm{Rm}(h+cg,\cdot), h+cg\rangle_{L^2(g)}.$$

Using the fact that $\Delta(cg) = 0$ and, as g is an Einstein metric,

$$\langle \mathrm{Rm}(h+cg,\cdot), cg\rangle_{L^2(g)} = c\lambda \int_M \mathrm{tr}(h+cg) dV_g = 0,$$

we obtain

$$\langle N(h+cg), h+cg\rangle_{L^2(g)} = \langle \frac{1}{2}\Delta h + \mathrm{Rm}(h,\cdot), h\rangle_{L^2(g)} + \langle \mathrm{Rm}(cg,\cdot), h\rangle_{L^2(g)}.$$

The claim follows by noting $\mathrm{Rm}(cg,\cdot) = c\mathrm{Ric}(g) = c\lambda g$ and the value of c from part (1). □

2.3. **Black hole stability.** In [13] the authors developed the stability theory of generalised Schwarzschild-Tangherlini black holes. These are metrics of the form

$$d\hat{s}^2 = -\left[1 - \left(\frac{l}{r}\right)^{d-1}\right] dt^2 + \frac{dr^2}{\left[1 - \left(\frac{l}{r}\right)^{d-1}\right]} + r^2 ds_d^2,$$

where l is a constant and ds_d^2 is the metric on a d dimensional compact Einstein manifold normalised so that its Einstein constant is $d-1$. They found a stability criterion involving the spectrum of the Lichnerowicz Laplacian restricted to the divergence free (called transverse in the physics literature), trace-free tensors on the Einstein manifold. We state their stability criterion with respect to our convention that the ordinary Laplacian has a non-positive spectrum (this is the opposite convention to that taken in [13]).

Proposition 2.4 (Black hole linear stability [13])**.** *Let (M^n, g) be a compact Einstein manifold satisfying $\mathrm{Ric}(g) = \lambda g$. Then the associated Schwarzschild-Tangherlini black hole is linearly unstable if the Lichnerowicz Laplacian has a divergence free, trace-free eigentensor with eigenvalue $-\kappa$ satisfying*

$$\kappa < \frac{\lambda}{n-1}\left(4 - \frac{(5-n)^2}{4}\right) = \frac{(9-n)\lambda}{4}.$$

Note that if an Einstein metric is unstable in the black hole sense, then it is unstable as a fixed point of the Ricci flow. We also note that we require genuinely trace-free perturbations in this definition rather than perturbations which are L^2-orthogonal to the metric (i.e. the integral of the trace is zero).

In [14], Gibbons, Hartnoll and Pope investigated the linear stability of the Böhm warped product metrics. They proved that the Böhm metrics (or rather the associated black holes) on $\mathbb{S}^3 \times \mathbb{S}^m$ for $m = 2, 3$ are unstable. Their proof used the cohomogeneity one symmetry that the Böhm metrics exhibit. Our Theorem E is a generalisation of this result to an arbitrary warped product metric on these spaces.

2.4. **A heuristic for destabilising one parameter families of warped products.** The methods for proving the Theorems A and B are inspired by considering how to destabilise a product gradient Ricci soliton. Given a gradient Ricci soliton $(B, \bar{g}_\infty, \phi_\infty)$ satisfying

$$\mathrm{Ric}(\bar{g}_\infty) + \overline{\nabla}^2 \phi_\infty = \lambda \bar{g}_\infty,$$

and an Einstein manifold $(F, \tilde{g})$ satisfying

$$\mathrm{Ric}(\tilde{g}) = \mu \tilde{g},$$

where $\mu = f_\infty \lambda$ for some constant f_∞. Thus the metric $g = \bar{g}_\infty \oplus f_\infty^2 \tilde{g}$ is a gradient Ricci soliton on $B \times F$ with potential function $\phi_\infty \circ \pi_B$ (as usual, we will drop the reference to the projection from now on and also denote this function ϕ_∞).

As mentioned previously, it is natural to use gauge-fixed tensors to destabilise. For a Ricci soliton, this means choosing tensors which are $\mathrm{div}_{\phi_\infty}$-free and L^2-orthogonal with the weighted volume form $e^{-\phi_\infty} dV_g$ to the Ricci tensor. On a product soliton there are two natural tensors satisfying the gauge-fixing conditions. The first is the tensor

$$h_1 = e^{\phi_\infty} \left(\frac{\bar{g}_\infty}{n} \oplus -\frac{f_\infty^2}{m} \tilde{g} \right).$$

The second is the tensor

$$h_2 = \mathrm{Ric}(\bar{g}_\infty) \oplus c f_\infty^2 \tilde{g},$$

with the constant c chosen so that

$$\int_M \langle \mathrm{Ric}(g), h_2 \rangle e^{-\phi_\infty} dV_g = 0.$$

One can compute the stability integral in Theorem 2.1 for each of the perturbations (the h_2 calculation uses a result of of Cao and Zhu in [8] that $N(\mathrm{Ric}_\infty) = 0$ as well as the fact that one can compute the stability operator on both factors of a product separately).

Proposition 2.5. *Let $(B \times F, g = \bar{g}_\infty \oplus f_\infty^2 \tilde{g}, \phi_\infty)$ be a gradient product Ricci soliton and let h_1 and h_2 be as above. Then*

$$\langle N(h_1), h_1 \rangle = \mathrm{Vol}(F) \int_B \left(-\left(\frac{1}{2n} + \frac{1}{2m} + \frac{1}{n^2} \right) |\overline{\nabla} e^{\phi_\infty}|^2 + (\frac{1}{n} + \frac{1}{m}) \lambda e^{2\phi_\infty} \right) e^{-\phi_\infty} dV_{\bar{g}_\infty},$$

and

$$\langle N(h_2), h_2 \rangle = \lambda \|h_2\|^2.$$

While it is clear from the Proposition that the perturbation h_2 always destabilises, it is not clear that this is true for h_1. Noting that

$$\int_B |\overline{\nabla} e^{\phi_\infty}|^2 e^{-\phi_\infty} dV_{\bar{g}_\infty} = \frac{1}{2} \int_B \langle \overline{\nabla} \phi_\infty, \overline{\nabla} e^{2\phi_\infty} \rangle e^{-\phi_\infty} dV_{\bar{g}_\infty}.$$

It is well-known [5] that it is also possible to normalise ϕ_∞ so that

$$\overline{\Delta}_{\phi_\infty} \phi_\infty = -2\lambda \phi_\infty.$$

In this case

$$\int_B |\overline{\nabla} e^{\phi_\infty}|^2 e^{-\phi_\infty} dV_{\bar{g}_\infty} = \lambda \int_B \phi_\infty e^{\phi_\infty} dV_{\bar{g}_\infty},$$

and the perturbation h_1 is destabilising if

$$\int_B \phi_\infty e^{\phi_\infty} dV_{\bar{g}_\infty} < \frac{2n(m+n)}{n^2 + mn + 2m} \int_B e^{\phi_\infty} dV_{\bar{g}_\infty}.$$

Such an inequality does not in general hold for functions ψ satisfying

$$\int_B \psi e^{-\psi} dV_{\bar{g}_\infty} = 0,$$

and so not possible to conclude that h_1 is a destabilising perturbation of a product Ricci soliton.

Suppose that there is a one parameter family of quasi-Einstein metrics $(M, \bar{g}_m, f_m, m)$ solving Equations (2.2) and (2.1) for fixed λ and μ. Setting $\phi_m = -m \log f_m$, equation (2.2) becomes

$$\mathrm{Ric}(\bar{g}_m) + \overline{\nabla}^2 \phi_m - \frac{d\phi_m \otimes d\phi_m}{m} = \lambda \bar{g}_m.$$

If as $m \to \infty$ the metrics $\bar{g}_m$ and functions ϕ_m converge (we do not make the sense of this convergence precise at this stage), then the limiting metric $\bar{g}_\infty$ and function ϕ_∞ solve the gradient Ricci soliton equation

$$\mathrm{Ric}(\bar{g}_\infty) + \overline{\nabla}^2 \phi_\infty = \lambda \bar{g}_\infty.$$

For large values of m, we can make the following approximations

$$\bar{g}_m \approx \bar{g}_\infty, \quad f_m \approx f_\infty, \text{ and } f_m^m \approx e^{-\phi_\infty},$$

where f_∞ is a non-zero constant that is the limit of the f_m. Hence the warped product Einstein metric $\bar{g}_m \oplus f_m^2 \tilde{g}$ can be approximated by the metric $\bar{g}_\infty \oplus f_\infty^2 \tilde{g}$. As Equation 2.1 ensures $\mu = f_\infty^2 \lambda$, the metric $\bar{g}_\infty \oplus f_\infty^2 \tilde{g}$ is a product Ricci soliton.

In Sections 4 and 5 we define two different tensors. The first is the GHP variation (Definition 4.1) which is the analogue of the tensor h_1. The fact that h_1 is not obviously universally destabilising goes some way to explain why the GHP variation fails to destabilise warped products in all but the lowest dimensions. The Ricci variation (Definition 5.1) is the analogue of h_2 and Theorem B could be paraphrased as saying that, providing the metrics $\bar{g}_m$ and $\bar{g}_\infty$ are close to each other, the fact that h_2 is universally destabilising means the Ricci variation also destabilises the warped product for large values of m.

3. Geometric operators for warped products

In this section we collect some useful identities that are used in the proof of the main theorems. The calculations are entirely straightforward so we omit the proofs. All of the theorems involve choosing a potentially destabilising tensor $h \in s^2(TM)$ and then computing the Rayleigh quotient

$$\frac{\int_M \langle \Delta_L h, h \rangle dV_g}{\int_M |h|^2 dV_g},$$

which provides a lower bound for the least negative eigentensor of Δ_L. The class of destabilising tensors that we consider can be written in the form

$$h = \overline{h} \oplus \psi \tilde{h}, \tag{3.1}$$

where $\overline{h} \in s^2(TB)$, $\tilde{h} \in s^2(TF)$ and $\psi \in C^\infty(B)$.

One fundamental calculation is of the Christoffel symbols for the Levi-Civita connection of a warped product metric.

Lemma 3.1 (Christoffel symbols of g)**.** *Let $M = B \times F$ be a product manifold and let $g = \overline{g} \oplus f^2\tilde{g}$ be a warped product metric on M. Then the Christoffel symbols for the Levi-Civita connection of g are given by:*

$$\begin{aligned}
\Gamma^c_{ab} &= \overline{\Gamma}^c_{ab},\\
\Gamma^c_{\alpha\beta} &= -\tilde{g}_{\alpha\beta} f \overline{g}^{cd}(\overline{\nabla}_d f),\\
\Gamma^\gamma_{a\beta} &= (\overline{\nabla}_a \log f)\delta^\gamma_\beta,\\
\Gamma^\gamma_{\alpha\beta} &= \tilde{\Gamma}^\gamma_{\alpha\beta}.
\end{aligned}$$

All other symbols are zero.

As mentioned in Section 2, one need only check stability on divergence free tensors. The next lemma computes the divergence of tensors of the form (3.1).

Lemma 3.2 (Divergence of h)**.** *Let $(B \times F^m, \overline{g} \oplus f^2\tilde{g})$ be a warped product manifold and let h be of the form (3.1). Then*

$$\begin{aligned}
\operatorname{div}(h)(\cdot) =& \overline{\operatorname{div}}(\overline{h})(\cdot) + m\overline{h}(\overline{\nabla}\log f, \cdot)\\
&- f^{-2}\psi(\tilde{\operatorname{tr}}(\tilde{h}))d\log f(\cdot) + f^{-2}\psi\widetilde{\operatorname{div}}(\tilde{h})(\cdot).
\end{aligned}$$

To break down the calculation of the Lichnerowicz Laplacian, we first compute the connection Laplacian of the tensors h.

Lemma 3.3 (Connection Laplacian)**.** *Let $(B \times F^m, \overline{g} \oplus f^2\tilde{g})$ be a warped product manifold. and let h be of the form (3.1). Then*

$$\begin{aligned}
\Delta h =& \overline{\Delta}\overline{h} + 2f^{-2}\psi\tilde{\operatorname{tr}}(\tilde{h})(d\log f \otimes d\log f)\\
&- m(d\log f \otimes \iota_{\overline{\nabla}\log f}\overline{h} + \iota_{\overline{\nabla}\log f}\overline{h} \otimes d\log f - (\overline{\nabla}_{\overline{\nabla}\log f}\overline{h}))\\
&+ (\overline{\Delta}\psi - 2\psi\overline{\Delta}\log f + (m-4)\overline{g}(\overline{\nabla}\psi, \overline{\nabla}\log f) + 2(1-m)\psi|\overline{\nabla}\log f|^2)\tilde{h}\\
&+ f^{-2}\psi\tilde{\Delta}\tilde{h} + 2\overline{h}(\overline{\nabla}f, \overline{\nabla}f)\tilde{g}\\
&- 2\psi f^{-2}(d\log f \otimes \widetilde{\operatorname{div}}(\tilde{h}) + \widetilde{\operatorname{div}}(\tilde{h}) \otimes d\log f).
\end{aligned}$$

The other component of the Lichnerowicz Laplacian is the curvature operator.

Lemma 3.4 (Curvature operator)**.** *Let $(B \times F^m, \overline{g} \oplus f^2\tilde{g})$ be a warped product manifold. and let h be of the form (3.1). Then*

$$\begin{aligned}
\operatorname{Rm}(h, \cdot) =& \overline{\operatorname{Rm}}(\overline{h}, \cdot) - f^{-3}\psi\tilde{\operatorname{tr}}(\tilde{h})(\overline{\nabla}^2 f) + f^{-2}\psi\widetilde{\operatorname{Rm}}(\tilde{h}, \cdot)\\
&- \psi|\overline{\nabla}\log f|^2(\tilde{\operatorname{tr}}(\tilde{h})\tilde{g} - \tilde{h}) - f\langle\overline{\nabla}^2 f, \overline{h}\rangle_{\overline{g}}\tilde{g}.
\end{aligned}$$

4. The proof of Theorems A, C and E

In order to prove Theorems A and E, we consider a generalisation of the variation considered by Gibbons, Hartnoll, and Pope in [14]. Using the terminology of [14], the perturbation is a 'balloning mode' which generalises the manner one destabilises an ordinary Riemannian product by changing the relative volumes of the base and fibre.

Definition 4.1 (The GHP variation)**.** Let $(B \times F^m, \bar{g} \oplus f^2\tilde{g})$ be a warped product manifold. The GHP variation is the tensor

$$h = \left(\frac{f^k}{n}\bar{g}\right) \oplus \left(\frac{(m+k)f^{k+2}}{mn}\tilde{g}\right). \tag{4.1}$$

Proposition 4.2. *Let $(B^n \times F^m, \bar{g} \oplus f^2\tilde{g})$ be a warped product Einstein manifold. Then the GHP variation h is divergence free. Furthermore if c is the constant defined in Lemma 2.3, then*

$$\langle N(h+cg), h+cg\rangle_{L^2(g)} = C^1_{n,m,k} \int_B f^{2k+m-2}|\overline{\nabla} f|^2 dV_{\bar{g}} + \lambda \left(\|h\|^2 - \frac{\left(\int_M \operatorname{tr}(h) dV_g\right)^2}{\|g\|^2} \right),$$

where

$$C^1_{n,m,k} = -\frac{\operatorname{Vol}(F)}{2} \left(\frac{k^2(4k+2m+mn+(m+k)^2)}{n^2 m} \right).$$

Proof. Consider a variation

$$h = Af^k \bar{g} \oplus Bf^{k+2}\tilde{g}.$$

By Lemma 3.2

$$\operatorname{div}(h) = kAf^{k-1}df + mAf^{k-1}df - mBf^{k-1}df.$$

Hence if the constants A and B are chosen so that

$$A = \frac{m}{m+k} B,$$

then h is divergence free. The GHP variation (4.1) has $A = 1/n$ and $B = (m+k)/mn$ and so is divergence free. Using Lemma 3.3 we see

$$\langle \Delta h, h\rangle = \frac{f^k \overline{\Delta} f^k}{n} + \frac{(m+k)^2}{mn^2} f^{k-2}\overline{\Delta} f^{k+2} - 2\frac{(m+k)^2}{mn^2} f^{2k}\overline{\Delta}\log f + C^2_{n,m,k} f^{2k-2}|\overline{\nabla} f|^2,$$

where

$$C^2_{n,m,k} = \frac{km^3 - 4km^2 - 8k^2m - 8km + k^3m - 6k^2 - 4k^3 - 4m^2 + 2k^2m^2 + km^2n}{mn^2}.$$

Integrating by parts yields

$$\int_M \langle \Delta h, h\rangle dV_g = -\operatorname{Vol}(F) \left(\frac{k^2(k^2+2km+m^2+nm+2)}{mn^2} \right) \int_B f^{2k+m-2} |\overline{\nabla} f|^2 dV_{\bar{g}}.$$

Using Lemma 3.4 (and simplifying using Equations (2.1) and (2.2)) we obtain

$$\langle \operatorname{Rm}(h,\cdot), h\rangle = \left(\frac{(m+k)^2}{mn^2} - \frac{m+2k}{n^2} \right) f^{2k-1}\overline{\Delta} f + \lambda f^{2k} \left(\frac{1}{n} + \frac{(m+k)^2}{mn^2} \right).$$

Integrating by parts yields

$$\int_M \langle \operatorname{Rm}(h,\cdot), h\rangle dV_g = \operatorname{Vol}(F) \left(\frac{-k^2(2k+m-1)}{mn^2} \right) \int_B f^{2k+m-2}|\overline{\nabla} f|^2 dV_{\bar{g}} + \lambda\|h\|^2.$$

The result follows immediately using the simplication given in Lemma 2.3. □

The proofs of Theorems A and E now follow by taking special values of the parameter k.

Proof of Theorem A. We take $k = -(2+m)$ in Proposition 4.2. (We note that for fixed values of n and m, this value of k minimises the coefficient in front of the gradient terms.) In this case the stability integral becomes

$$\langle N(h+cg), h+cg\rangle_{L^2(g)} = C^3_{n,m} \int_B f^{-m-6}|\overline{\nabla} f|^2 dV_{\bar{g}} + \lambda \left(\|h\|^2 - \frac{\left(\int_M \operatorname{tr}(h) dV_g\right)^2}{\|g\|^2} \right),$$

where

$$C^3_{n,m} = -\frac{\operatorname{Vol}(F)}{2} \left(\frac{(m+2)^2((n-2)m-4)}{n^2 m} \right).$$

The Cauchy-Schwarz inequality applied to $\langle h, g\rangle_{L^2(g)}$ implies that

$$\|h\|^2 - \frac{\left(\int_M \mathrm{tr}(h) dV_g\right)^2}{\|g\|^2} > 0,$$

and the remaining term is non-negative when $(n-2)m \leq 4$. The result follows noting that either $n = 3$ and $m \in \{2, 3, 4\}$ or $n = 4$ and $m = 2$. This covers all possible six-dimensional products as if the base has dimension 2 then, by the rigidity theorem of Case–Shu–Wei [9], the function f is constant and the product is trivially unstable. The fibre of a compact warped product Einstein manifold must be at least two-dimensional by Myer's theorem.

□

Proof of Theorem E. We take $k = -(m+n)$ in Proposition 4.2. In this case the variation h is transverse trace-free and the stability condition becomes

$$\langle \Delta_L h, h\rangle_{L^2(g)} = \mathrm{Vol}(F)\left(\frac{(m+n)^2(4n+2m-n^2-mn)}{n^2 m}\right)\int_B f^{-2n-m-2}|\overline{\nabla} f|^2 dV_{\bar{g}}.$$

This is manifestly non-negative when $n = 3$ and $m = 2$ or $m = 3$. □

We use a similar method to prove Theorem C.

Proof of Theorem C. We consider the perturbation

$$h = 0 \oplus \sigma.$$

Decomposing the variation h in the manner of (3.1) we see that

$$\bar{h} = 0, \quad \tilde{h} = \sigma \text{ and } \psi = 1.$$

This immediately yields $\mathrm{div}(h) = 0$ by Lemma (3.2) and it is clear h is trace-free as σ is trace-free. Using Lemma 3.3 and Lemma 3.4 we obtain

$$\Delta h = f^{-2}\widetilde{\Delta}\sigma + (2(1-m)|\overline{\nabla}\log f|^2 - 2\overline{\Delta}\log f)\sigma,$$

and

$$\mathrm{Rm}(h, \cdot) = f^{-2}\widetilde{\mathrm{Rm}}(\sigma, \cdot) + |\overline{\nabla}\log f|^2\sigma.$$

Hence

$$\langle \Delta_L h, h\rangle_g = f^{-6}\langle \widetilde{\Delta}_L \sigma, \sigma\rangle_{\tilde{g}} + 2(\mu f^{-6} - \lambda f^{-4} + (2-m) f^{-6}|\overline{\nabla} f|^2 - f^{-4}\overline{\Delta}\log f)|\sigma|^2_{\tilde{g}}.$$

Integrating by parts and using the fact $\widetilde{\Delta}_L \sigma = -\kappa\sigma$ we obtain

$$\int_M \langle \Delta_L h, h\rangle_g dV_g = 2\|\sigma\|^2_{\tilde{g}} \int_B \left(\mu - \frac{\kappa}{2}\right) f^{m-6} - \lambda f^{m-4} - 2f^{m-6}|\overline{\nabla} f|^2 dV_{\bar{g}}. \tag{4.2}$$

Multiplying Equation (2.1) by f^{m-6} and integrating we see that

$$\int_B \frac{\mu}{2} f^{m-6} - 2f^{m-6}|\overline{\nabla} f|^2 dV_{\bar{g}} = \frac{\lambda}{2}\int_B f^{m-4} dV_{\bar{g}}.$$

Hence if $\kappa \leq \mu$, we can substitute into Equation (4.2) and get the inequality

$$\int_M \langle \Delta_L h, h\rangle dV_g \geq -\lambda\|\sigma\|^2_{\tilde{g}} \int_B f^{m-4} dV_{\bar{g}} = -\lambda\|h\|^2_g. \tag{4.3}$$

Hence the result follows. □

We can now prove Corollary D.

Proof of Corollary D. It is well known that Einstein products and Kähler-Einstein metrics with $h^{1,1} > 1$ (all with Einstein constant μ) admit transverse, trace-free eigentensors of the Lichnerowicz Laplacian with eigenvalue 0 [6], [8], and [17]. Equation (4.3) shows that a fibre-unstable warped product must also have a transverse, trace-free eigentensor satisfying the destabilising conditions and so must be unstable as a fibre. □

5. The proof of Theorem B

We begin with the definition of the Ricci-variation.

Definition 5.1 (Ricci variation)**.** Let $(B^n \times F^m, \bar{g} \oplus f^2\tilde{g})$ be a warped product Einstein manifold with Einstein constant λ. Then the Ricci variation is the tensor

$$h = \left(m\overline{\nabla}^2 f + \frac{m}{m-1}(\overline{\Delta} f + \lambda f)\bar{g} \right) \oplus 0. \tag{5.1}$$

We remark that this tensor is really fP, where P is the tensor defined by He, Petersen and Wylie in [20] and the following lemma was essentially first proved in this paper. We give a short self-contained proof.

Lemma 5.2 (cf. He–Petersen–Wylie Prop 5.6 in [20])**.** *The Ricci variation (5.1) is divergence free.*

Proof. We use Lemma 3.2, noting the formulae (valid for any metric g and function Φ)

$$\mathrm{div}(\nabla^2\Phi) = d\Delta\Phi + \mathrm{Ric}(\nabla\Phi, \cdot),$$

and

$$\mathrm{div}(\Phi g) = d\Phi.$$

So

$$\mathrm{div}(m\overline{\nabla}^2 f \oplus 0) = md\overline{\Delta} f + \frac{2m^2\overline{\nabla}^2 f(\overline{\nabla} f, \cdot)}{f} + m\lambda df,$$

and

$$\mathrm{div}(\frac{m}{m-1}(\overline{\Delta} f + \lambda f)\bar{g} \oplus 0) = \frac{m}{m-1}\left(d\overline{\Delta} f + \lambda df + m((\overline{\Delta} f + \lambda f)\frac{df}{f}) \right).$$

Hence

$$\mathrm{div}(h) = \frac{m^2}{m-1}\left(d\overline{\Delta} f + \overline{\Delta} f\frac{df}{f} + (m-1)\frac{d|\overline{\nabla} f|^2}{f} + 2\lambda df \right),$$

and so using Equation (2.1)

$$\mathrm{div}(h) = \frac{m^2}{f(m-1)} d\left((f\overline{\Delta} f) + (m-1)|\overline{\nabla} f|^2 + \lambda f^2\right) = \frac{m^2}{f(m-1)} d\mu = 0.$$

□

We now consider some of the consequences of the convergence of a one parameter family of quasi-Einstein metrics to a Ricci soliton. We really have a one parameter family of pairs $(\bar{g}_m, f_m)$ solving Equations (2.1) and (2.2). We fix scale and normalise f by requiring that λ and μ do not depend upon the parameter m. As the metrics $\bar{g}_m \to \bar{g}_\infty$ in the C^∞ topology we have

$$\mathrm{Ric}(\bar{g}_m) \to \mathrm{Ric}(\bar{g}_\infty),$$

and so

$$\frac{m}{f_m}\overline{\nabla}^2_m f_m \to -\overline{\nabla}^2_\infty \phi,$$

where ϕ is the soliton potential for $\bar{g}_\infty$. We have the standard identity for any metric g and function Φ

$$\nabla^2 \log \Phi = \frac{1}{\Phi}\nabla^2\Phi - \frac{1}{\Phi^2}d\Phi \otimes d\Phi,$$

and this yields

$$\overline{\nabla}^2_m \log f_m^m + \frac{1}{m} d\log f_m^m \otimes d\log f_m^m \to -\overline{\nabla}^2_\infty \phi. \tag{5.2}$$

Taking the trace of equation (5.2) and integrating we see

$$m\int_B |\overline{\nabla}_m \log f_m|^2 dV_{\bar{g}_m} = \int_B |\overline{\nabla}_m \log f_m^{(m^{1/2})}|^2 dV_{\bar{g}_m} \to 0.$$

Thus $d\log f_m^{m^{1/2}} \to 0$ and so (5.2) yields

$$\overline{\nabla}^2_m \log f_m^m \to -\overline{\nabla}^2_\infty \phi.$$

Hence as B is compact (so the only harmonic functions are constant), we must have in C^∞

$$f_m^m \to e^{-\phi},$$

where ϕ is normalised by the requirement that λ and μ remain fixed. We can now see that f_m must converge to a constant and Equation (2.1) gives this as $\sqrt{\mu/\lambda}$. We can now prove Theorem B.

Proof of Theorem B. For any appropriate fibre metric $\tilde{g}$ we can write the tensor h in the following manner,

$$h = m\nabla^2 f + \frac{m}{m-1}(\overline{\Delta} f + \lambda f)g - \frac{\mu m f}{m-1}\tilde{g}.$$

We now compute $\langle \frac{1}{2}\Delta h + \mathrm{Rm}(h,\cdot), h\rangle_{L^2(g)}$ term-by-term. In the proof of their Lemma 3.5 in [7], Cao–He show that for an Einstein metric

$$\frac{1}{2}\Delta\nabla^2\Phi + \mathrm{Rm}(\nabla^2\Phi,\cdot) = \nabla^2\Delta\Phi + \lambda\nabla^2\Phi,$$

for any function Φ. So, as h is divergence-free by Lemma 5.2, we obtain

$$\langle\frac{1}{2}\Delta(\nabla^2 f) + \mathrm{Rm}(\nabla^2 f,\cdot), h\rangle_{L^2(g)} = \lambda\langle\nabla^2 f, h\rangle_{L^2(g)}.$$

It is straightforward to see that

$$\langle\frac{1}{2}\Delta((\overline{\Delta} f+\lambda f)g)+\mathrm{Rm}((\overline{\Delta} f+\lambda f)g,\cdot),h\rangle_{L^2(g)} = \langle\frac{1}{2}\Delta(\overline{\Delta} f+\lambda f)g+\lambda(\overline{\Delta} f+\lambda f)g,h\rangle_{L^2(g)}.$$

In dealing with the final term we note that $\langle\tilde{g},h\rangle = 0$ and using Lemma 3.3 and Lemma 3.4 we obtain

$$\langle\frac{1}{2}\Delta(f\tilde{g}) + \mathrm{Rm}(f\tilde{g},\cdot), h\rangle_{L^2(g)} = \langle m(f^{-3}df\otimes df - f^{-2}\overline{\nabla}^2 f), h\rangle_{L^2(g)}.$$

Hence we see that

$$\langle\frac{1}{2}\Delta h+\mathrm{Rm}(h,\cdot),h\rangle_{L^2(g)} = \lambda\|h\|^2+\frac{m}{m-1}\langle(\Delta(\overline{\Delta} f+\lambda f)\bar{g}-m\mu(f^{-3}df\otimes df-f^{-2}\overline{\nabla}^2 f),h\rangle_{L^2(g)}. \tag{5.3}$$

We now observe two facts, firstly the general identity for a function Φ

$$\nabla^2\Phi^{-1} = -\frac{1}{\Phi^2}\nabla^2\Phi + \frac{2}{\Phi^3}d\Phi\otimes d\Phi.$$

The second is that as h is both divergence free and pointwise orthogonal to tensors of the form $0 \oplus \tilde{h}$ we have

$$\langle \overline{\nabla}^2 \Phi, h \rangle = 0.$$

Hence equation (5.3) becomes

$$\langle \frac{1}{2}\Delta h + \mathrm{Rm}(h, \cdot), h\rangle_{L^2(g)} = \lambda \|h\|^2 + \frac{m}{m-1}\langle (\Delta(\overline{\Delta} f + \lambda f)\bar{g} + m\mu(f^{-3} df \otimes df, h\rangle_{L^2(g)}.$$

From the discussion preceeding the proof, the term

$$\mathrm{Vol}(F)^{-1}\frac{m}{m-1}\langle (\Delta(\overline{\Delta} f + \lambda f)\bar{g} + m\mu(f^{-3} df \otimes df, h\rangle_{L^2(g)},$$

does not depend upon the fibre geometry and converges to 0. It is straightforward to see that

$$\mathrm{Vol}(F)^{-1}\left(\|h\|^2 - \frac{\left(\int_M \mathrm{tr}(h) dV_g\right)^2}{\|g\|^2} \right),$$

does not depend upon the fibre geometry and must be bounded away from 0 if the limiting soliton is non-trivial (i.e. ϕ is non-constant). Hence using Lemma 2.3, we see for large enough m,

$$\int_M \langle N(h + cg), h + cg\rangle \; dV_g > 0,$$

and the result follows. □

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