MULTIGRADED HILBERT SERIES OF NONCOMMUTATIVE MODULES

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Abstract. In this paper, we propose methods for computing the Hilbert series of multigraded right modules over the free associative algebra. In particular, we compute such series for noncommutative multigraded algebras. Using results from the theory of regular languages, we provide conditions when the methods are effective and hence the sum of the Hilbert series is a rational function. Moreover, a characterization of finite-dimensional algebras is obtained in terms of the nilpotency of a key matrix involved in the computations. Using this result, efficient variants of the methods are also developed for the computation of Hilbert series of truncated infinite-dimensional algebras whose (non-truncated) Hilbert series may not be rational functions. We consider some applications of the computation of multigraded Hilbert series to algebras that are invariant under the action of the general linear group. In fact, in this case such series are symmetric functions which can be decomposed in terms of Schur functions. Finally, we present an efficient and complete implementation of (standard) graded and multigraded Hilbert series that has been developed in the kernel of the computer algebra system Singular. A large set of tests provides a comprehensive experimentation for the proposed algorithms and their implementations.

1. Introduction

The concept of what is nowadays known as Hilbert (or Hilbert-Poincaré) series, was first introduced in the 19th century in the context of finitely generated commutative algebras over a field $\mathbb{K}$. This is the generating series of the Hilbert function which relates each nonnegative integer d to the $\mathbb{K}$-linear dimension of the graded or filtered component of degree d of the algebra.

A fundamental property of a finitely generated commutative algebra is that the sum of its Hilbert series is always a rational function allowing a finite description of such an invariant. This was proved by Hilbert himself in 1890 by introducing the other fundamental notion of a free resolution. Indeed, to compute a resolution, that is, a complete chain of syzygies for the generators of the ideal of relations of the algebra is usually much more involved than to determine the Hilbert series. This was essentially observed in 1927 by

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Francis Sowerby Macaulay, whose results imply that the Hilbert series of a commutative algebra is equal to the series of a corresponding monomial algebra, that is, an algebra whose generators are related by monomials. In modern terminology, such a monomial algebra is defined by the leading monomial ideal, with respect to a suitable monomial ordering, of the ideal of relations of the algebra under consideration. In other words, one has to compute a Gröbner basis of such an ideal. In fact, it was only around the mid-1990s when the computer algebra community was able to propose and implement many efficient algorithms for the computation of the Hilbert series of commutative algebras (see, for instance, [1, 4]).

For noncommutative structures, the first result about the rationality of the Hilbert series of a finitely presented monomial algebra is due to Govorov [13] and dates back only to 1972. Another fundamental contribution was given by Ufnarovski [24] who developed a graph theoretic method to study the asymptotic behaviour of the Hilbert functions of finitely presented monomial algebras and also to compute the corresponding rational Hilbert series. We mention that in the noncommutative literature, Hilbert functions and Hilbert series are also called "growth functions and growth series". Clearly, these results of Govorov and Ufnarovski can be immediately extended to (finitely generated) algebras whose ideals of relations admit a finite Gröbner basis. However, the free associative algebra $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ is not Noetherian which implies that noncommutative Gröbner bases are generally infinite sets.

Recently, in [18] it has been proved that the rationality of a noncommutative Hilbert series follows from the condition that the (possibly infinitely generated) leading monomial ideal of relations of the algebra is a "regular language" in the sense of the theory of formal languages. In fact, this result has been stated in the more general context of finitely generated right modules over $\mathbb{K}\langle x_1, \ldots, x_n\rangle$. Moreover, in that paper one finds an iterative method for the computation of the Hilbert series which is proved to be optimal with respect to the number of iterations by means of standard results in automata theory. Since this method is based on the repeated application of the (right) colon ideal operation, from a historical perspective we can say that it generalizes and refines the methods that were developed for commutative Hilbert series, indicating that the worlds of commutative and noncommutative algebra are not far apart.

In [18] the only implementation of the proposed algorithms was an experimental one in Maple. Then, our first goal in the present paper is to develop a fast and well-tested implementation of noncommutative Hilbert series in the kernel of Singular [7] (see Section 8 for details). We remark that this is the only general implementation of these series up to now. In fact, the library fpadim.lib of Singular provides the computation of the Hilbert function of a finite-dimensional algebra by means of the enumeration of normal words. In our tests we propose hence the comparison of the two methods for the finite-dimensional case.

Our second goal is to extend the theory and the methods to multivariate Hilbert series that are defined for multigraded algebras. It is well known that if an algebra with n generators over a field $\mathbb{K}$ of characteristic zero is also a (polynomial) module for the action of the general linear group $\mathrm{GL}_n(\mathbb{K})$, then it is a multigraded algebra. An algebra is a $\mathrm{GL}_n(\mathbb{K})$-module when the ideal of relations of its n generators is invariant under all invertible linear substitutions of them. This is the case for many universal enveloping algebras and also for the algebras that are defined by "T-ideals", that is, ideals that are invariant under all polynomial substitutions of the generators. For $\mathrm{GL}_n(\mathbb{K})$-invariant algebras, the multigraded Hilbert series are in fact symmetric functions. The decomposition of these functions in terms of Schur functions provides all essential information (multiplicities) about the decomposition of the algebra in terms of its simple $\mathrm{GL}_n(\mathbb{K})$-submodules. For the purposes of representation theory, this is of course a very important task. The general theory for the multigraded Hilbert series of finitely generated multigraded right modules over $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ is presented in Section 2 and 3. In particular, we prove Theorem 3.2 which is a noncommutative multigraded version of Macaulay's basis theorem that reduces the computations to the monomial case. In Section 4 and 5 we present the methods by extending the approach that has been introduced in [18] for the graded case. In Section 7 we apply the computation of the multigraded Hilbert series to an interesting $\mathrm{GL}_n(\mathbb{K})$-invariant algebra with the purpose of obtaining the Schur function decomposition of its series. For the reader's convenience, a brief review of the very basic theory of polynomial representations of the general linear group is provided at the beginning of this section.

A last contribution of the present paper consists in developing efficient variants of the proposed methods that can be applied to truncations of infinite-dimensional algebras up to some fixed degree. Note that the truncations yield finite-dimensional algebras and hence their Hilbert series are in fact polynomials. The motivation for developing such variants is two-fold. First, there exist non-regular monomial algebras for which the sum of the corresponding Hilbert series is not available. Thus, polynomial approximations of these functions may be useful to understand them. Moreover, since Schur functions are (symmetric) polynomials, there are fast algorithms to perform the Schur function decomposition for symmetric polynomials but it is much more difficult to decompose even a rational symmetric function as an infinite sum of Schur functions. Again, to have approximations of the latter decomposition may help to understand the complete picture. The truncation methods are presented in Section 6 and applied in Section 7 and Section 8. In Section 6 one also finds a characterization of the finite-dimensionality of a (regular) monomial algebra in terms of the nilpotency of a key matrix that is involved in our method. Finally, in Section 9 we draw some conclusions and suggest further developments of the theory and the methods.

2. Multigraded modules and their Hilbert series

Let $\mathbb{K}$ be any field and let $X = \{x_1, \ldots, x_n\}$ be a finite set. We denote by $W = X^*$ the free monoid that is freely generated by X, that is, the elements of W are words over the alphabet X. Moreover, we denote by $F = \mathbb{K}\langle X\rangle$ the corresponding monoid $\mathbb{K}$-algebra, that is, F is the (finitely generated) free associative algebra freely generated by X. A *monomial* of F is by definition a word of W and an element of F is called a *(noncommutative) polynomial*.

Recall that a standard grading of the algebra F is given by assigning $\mathrm{tdeg}(w)$ as the length of a word $w \in W$. We call also $\mathrm{tdeg}(w)$ the *total degree* of the monomial w. Thus, we have that $F = \bigoplus_{d\in\mathbb{N}} F_d$ where F_d is the span of the words $w \in W$ such that $\mathrm{tdeg}(w) = d$. An element $f \in F_d$ is called a *homogeneous polynomial of total degree d*. A standard multigrading of F is defined in the following way.

Definition 2.1. *For any $w \in W$, we define $\deg(w) = \bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \mathbb{N}$ is the number of times the variable x_i occurs in the word w. Since $\deg(vw) = \deg(v) + \deg(w) \in \mathbb{N}^n$ for all $v, w \in W$, we have an algebra multigrading*

$$F = \bigoplus_{\bar{\alpha}\in\mathbb{N}^n} F_{\bar{\alpha}}$$

where $F_{\bar{\alpha}}$ is the subspace of F that is spanned by the words $w \in W$ such that $\deg(w) = \bar{\alpha}$. Then, an element $f \in F_{\bar{\alpha}}$ is called a multihomogeneous polynomial of multidegree $\bar{\alpha}$.

A variant of the above multigrading can be obtained in the following way. Fix a multidegree $\bar{\delta} = (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n$ and define $\deg'(1) = \bar{\delta}$, that is, $\deg'(w) = \bar{\delta} + \deg(w)$ for each $w \in W$. Then, we denote by $F[-\bar{\delta}]$ the algebra F that is endowed by the multigrading defined by $\deg'$. By denoting $\bar{\alpha} \trianglerighteq \bar{\delta}$ when $\alpha_i \geq \delta_i$ for any $i = 1, 2, \ldots, n$, we have in fact that

$$F[-\bar{\delta}]_{\bar{\alpha}} = \begin{cases} F_{\bar{\alpha}-\bar{\delta}} & \text{if } \bar{\alpha} \trianglerighteq \bar{\delta}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $r > 0$ be any integer and let F^r be the free right F-module of rank r. If $\{e_1, \ldots, e_r\}$ denotes the canonical basis of F^r, then the elements of F^r are right linear combinations $\sum_i e_i f_i$ where $f_i \in F$. A standard right module multigrading of F^r is defined by putting $\deg(e_i) = \deg(1) = (0, \ldots, 0)$, that is, $\deg(e_i w) = \deg(w)$ for all $w \in W$ and $i = 1, 2, \ldots, r$. By using a set of multidegrees $\{\bar{\delta}_1, \ldots, \bar{\delta}_r\}$ we can modify this multigrading of F^r by putting $\deg'(e_i) = \bar{\delta}_i$, for any i. We will denote this multigraded (free) right F-module as $\bigoplus_{1\leq i\leq r} F[-\bar{\delta}_i]$. In fact, an element $f = \sum_i e_i f_i \in \bigoplus_i F[-\bar{\delta}_i]$ is multihomogeneous of multidegree $\bar{\alpha}$ if and only if $f_i \in F[-\bar{\delta}_i]_{\bar{\alpha}}$, for all $i = 1, 2, \ldots, r$.

Definition 2.2. *Consider a right submodule $M \subset \bigoplus_i F[-\bar{\delta}_i]$. We call M a* multigraded submodule *if $M = \sum_{\bar{\alpha}} M_{\bar{\alpha}}$ where $M_{\bar{\alpha}} = M \cap \bigoplus_i F[-\bar{\delta}_i]_{\bar{\alpha}}$. In this case, we consider the quotient multigraded right module $N = \bigoplus_i F[-\bar{\delta}_i]/M$ where the multihomogeneous component $N_{\bar{\alpha}}$ is isomorphic to $\bigoplus_i F[-\bar{\delta}_i]_{\bar{\alpha}}/M_{\bar{\alpha}}$. A finitely generated multigraded right module $N' = \langle g_1, \ldots, g_r\rangle$ where g_i is a multihomogeneous element of multidegree $\bar{\delta}_i$ is clearly isomorphic to N by the multigraded right module homomorphism $\varphi : \bigoplus_i F[-\bar{\delta}_i] \to N', e_i \mapsto g_i$ where $M = \ker\varphi$.*

Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and put $d = |\bar{\alpha}| = \sum_i \alpha_i$. By counting the number of words that have multidegree $\bar{\alpha}$, one has that

$$\dim F_{\bar{\alpha}} = \binom{d}{\alpha_1, \ldots, \alpha_n} = \frac{d!}{\alpha_1! \cdots \alpha_n!}$$

Recall that $(t_1+\cdots+t_n)^d = \sum_{|\bar\alpha|=d}\binom{d}{\alpha_1,\ldots,\alpha_n} t_1^{\alpha_1}\cdots t_n^{\alpha_n}$ by the multinomial theorem, which implies that the multivariate generating series of the function $\bar\alpha\mapsto \dim F_{\bar\alpha}$ satisfies the following formula

$$\sum_{\bar\alpha\in\mathbb{N}^n}\dim F_{\bar\alpha}\, t_1^{\alpha_1}\cdots t_n^{\alpha_n} = \sum_{d\in\mathbb{N}}(t_1+\cdots+t_n)^d = \frac{1}{1-(t_1+\cdots+t_n)}.$$

Moreover, for any fixed multidegree $\bar\delta=(\delta_1,\ldots,\delta_n)$, we have by definition $F[-\bar\delta]_{\bar\alpha} = F_{\bar\alpha-\bar\delta}$ and therefore

$$\sum_{\bar\alpha}\dim F[-\bar\delta]_{\bar\alpha}\, t_1^{\alpha_1}\cdots t_n^{\alpha_n} = \frac{t_1^{\delta_1}\cdots t_n^{\delta_n}}{1-(t_1+\cdots+t_n)}.$$

Next, we will generalize the notions and results above.

Definition 2.3. *Let $N=\bigoplus_{\bar\alpha} N_{\bar\alpha}$ be a finitely generated multigraded right module over F. We define the function $\mathrm{HF}(N)(\bar\alpha)=\dim N_{\bar\alpha}$, for any $\bar\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n$ and the corresponding multivariate generating series*

$$\mathrm{HS}(N)=\sum_{\bar\alpha\in\mathbb{N}^n}\dim N_{\bar\alpha}\, t_1^{\alpha_1}\cdots t_n^{\alpha_n}$$

We call HF, HS *respectively the* multigraded Hilbert function and multigraded Hilbert series of the module N.

Observe that the *(graded) Hilbert function of* N is defined for any total degree $d\in\mathbb{N}$ as

$$\mathrm{HF}'(N)(d)=\sum_{|\bar\alpha|=d}\mathrm{HF}(N)(\bar\alpha)$$

In other words, the corresponding *(graded) Hilbert series* $\mathrm{HS}'(N)$ is obtained by identifying in the series $\mathrm{HS}(N)$ all variables t_i $(1\le i\le n)$ with a single variable t.

We have already obtained explicit formulas for multigraded Hilbert function and series in the case of free algebras. Of course, these formulas immediately extend to finitely generated free right modules. Note now that an important property of the finitely generated free associative algebra $F=\mathbb{K}\langle X\rangle$ is that it is a *free right ideal ring* [6], that is, each right submodule $M\subset F^r$ is in fact a free one of unique rank. Unfortunately, owing to non-Noetherianity of the algebra F, we cannot always assume that M is also finitely generated. Nevertheless, for this case one has the following formula.

Theorem 2.4. *Let $N=\bigoplus_{1\le i\le r}F[-\bar\delta_i]/M$ be a finitely presented multigraded right module where $\bar\delta_i=(\delta_{i1},\ldots,\delta_{in})$, for all i. Consider $\{g_1,\ldots,g_s\}$ a multihomogeneous free basis of the right submodule $M\subset\bigoplus_i F[-\bar\delta_i]$. If $\bar\Delta_j=(\Delta_{j1},\ldots,\Delta_{jn})=\deg(g_j)$ then*

$$\mathrm{HS}(N)=\frac{\sum_{1\le i\le r}t_1^{\delta_{i1}}\cdots t_n^{\delta_{in}}-\sum_{1\le j\le s}t_1^{\Delta_{j1}}\cdots t_n^{\Delta_{jn}}}{1-(t_1+\cdots+t_n)}$$

In particular, we have that $\mathrm{HS}(N)$ is a rational function with integer coefficients.

Proof. Consider $\{\epsilon_1, \ldots, \epsilon_s\}$ the canonical basis of the free right module $\bigoplus_{1\leq j\leq s} F[-\bar{\Delta}_j]$ and define the multigraded right module homomorphism

$$\bigoplus_j F[-\bar{\Delta}_j] \to \bigoplus_i F[-\bar{\delta}_i], \epsilon_j \mapsto g_j.$$

Since $\{g_j\}$ is a free basis of M, the map above implies a short exact sequence

$$0 \to \bigoplus_j F[-\bar{\Delta}_j] \to \bigoplus_i F[-\bar{\delta}_i] \to N \to 0.$$

Because all the homomorphisms are multigraded ones, we obtain that

$$\mathrm{HS}(\bigoplus_j F[-\bar{\Delta}_j]) - \mathrm{HS}(\bigoplus_i F[-\bar{\delta}_i]) + \mathrm{HS}(N) = 0$$

which implies the stated formula. □

It is clear that for finitely generated but infinitely presented right modules the above result does not apply and hence we have to follow a different path. In the next section we begin by reducing the problem of determining multigraded Hilbert series to the case of monomial cyclic right modules.

3. Monomial cyclic right modules

Let F^r be a free right module and consider $\{e_1, \ldots, e_r\}$ its canonical basis. We denote $W(r) = \cup_{i=1}^r e_i W = \{e_i w \mid 1 \leq i \leq r, w \in W\}$ which is a canonical $\mathbb{K}$-linear basis of F^r. The elements of $W(r)$ are called the *monomials of* F^r.

Definition 3.1. *Let $\prec$ be a well-ordering of $W(r)$. We call $\prec$ a* monomial ordering *of F^r if $\prec$ is compatible with the right module structure of F^r, that is, for all $e_i u, e_j v \in W(r)$ and $w \in W$, one has that*

$$e_i u \prec e_j v \Rightarrow e_i u w \prec e_j v w.$$

There are well-known examples of monomial orderings of F^r. For instance, for any $e_i v, e_j w \in W(r)$ we can define that $e_i v \prec e_j w$ if and only if either $v < w$ in some graded lexicographic ordering or $v = w$ and $i < j$. Assume now that F^r is endowed with a monomial ordering $\prec$. For any element $f \in F^r$ we denote by $\mathrm{lm}(f)$ the greatest, with respect to $\prec$, among the monomials occurring in f. The element $\mathrm{lm}(f) \in W(r)$ is called the *leading monomial of f*. We denote in addition by $\mathrm{lc}(f) \in \mathbb{K}$ the coefficient that the monomial $\mathrm{lm}(f)$ has in f and we call it the *leading coefficient of f*. If $M \subset F^r$ is a right submodule, then we denote by $\mathrm{LM}(M)$ the right submodule of F^r that is generated by the set $\mathrm{lm}(M) = \{\mathrm{lm}(f) \mid f \in M, f \neq 0\} \subset W(r)$. Note that $\mathrm{LM}(M)$ is a *monomial right submodule* which means that it is generated by monomials of F^r. Then, we call $\mathrm{LM}(M)$ the *leading monomial module of M*. It is well known that to compute a minimal set of monomials generating $\mathrm{LM}(M)$ one uses the notion of (minimal) Gröbner basis [15, 17]. Although these bases are generally infinite because of the non-Noetherianity of the free associative algebra F, in many cases they can be described in closed forms by the aid of partial computations and formal arguments (see, for instance, [10]).

We now state a generalization of Macaulay's basis theorem for commutative modules in the noncommutative setting.

Theorem 3.2. *Let $N = \bigoplus_{1\leq i\leq r} F[-\bar{\delta}_i]/M$ be a finitely generated multigraded right module. Fix any monomial ordering for the free right module F^r and consider $N' = \bigoplus_{1\leq i\leq r} F[-\bar{\delta}_i]/\mathrm{LM}(M)$. Then, we have that* $\mathrm{HF}(N) = \mathrm{HF}(N')$ *and hence* $\mathrm{HS}(N) = \mathrm{HS}(N')$.

Proof. Put $M' = \mathrm{LM}(M)$ and denote $W(r)_{\bar{\alpha}} = W(r) \cap \bigoplus_i F[-\bar{\delta}_i]_{\bar{\alpha}}$, for any multidegree $\bar{\alpha} \in \mathbb{N}^n$. Since $M' \subset \bigoplus_i F[-\bar{\delta}_i]$ is a monomial module, a linear basis of $\bigoplus_i F[-\bar{\delta}_i]_{\bar{\alpha}}/M'_{\bar{\alpha}}$ is clearly given by the set $\{e_i w + M'_{\bar{\alpha}} \mid e_i w \in W(r)_{\bar{\alpha}} \setminus M'\}$. We have to prove that the corresponding set $\{e_i w + M_{\bar{\alpha}} \mid e_i w \in W(r)_{\bar{\alpha}} \setminus M'\}$ is a linear basis of $\bigoplus_i F[-\bar{\delta}_i]_{\bar{\alpha}}/M_{\bar{\alpha}}$. Consider any multihomogeneous element $f \in \bigoplus_i F[-\bar{\delta}_i]_{\bar{\alpha}}, f \neq 0$. If $\mathrm{lm}(f) \in M'$ then there exists a multihomogeneous element $g \in M$ such that $\mathrm{lm}(f) = \mathrm{lm}(g)v$, for some $v \in W$. By putting $f_1 = f - gv\frac{\mathrm{lc}(f)}{\mathrm{lc}(g)}$ we obtain that either $f \equiv f_1 \mod M_{\bar{\alpha}}$ with $f_1 = 0$ or $\mathrm{lm}(f) \succ \mathrm{lm}(f_1)$. In the latter case, we can repeat this division step for the multihomogeneous element $f_1 \in \bigoplus_i F[-\bar{\delta}_i]_{\bar{\alpha}}, f_1 \neq 0$. Since $W(r)_{\bar{\alpha}}$ is a finite set, we conclude that either $f \equiv f_2 \mod M_{\bar{\alpha}}$, for some multihomogeneous element $f_2 \in \bigoplus_i F[-\bar{\delta}_i]_{\bar{\alpha}}$ such that $f_2 = 0$ or $\mathrm{lm}(f) \succ \mathrm{lm}(f_2) \notin M'$. If $f_2 \neq 0$, then we consider the multihomogeneous element $f_3 = f_2 - \mathrm{lm}(f_2)\mathrm{lc}(f_2)$ so that $f - \mathrm{lm}(f_2)\mathrm{lc}(f_2) \equiv f_3 \mod M_{\bar{\alpha}}$. Note that one has either $f_3 = 0$ or $\mathrm{lm}(f_2) \succ \mathrm{lm}(f_3)$. By iterating the division process, we finally obtain that $f \equiv f' \mod M_{\bar{\alpha}}$ where $f' = \sum_k e_{i_k} w_k c_k$ with $e_{i_k} w_k \in W(r)_{\bar{\alpha}} \setminus M'$ and $c_k \in \mathbb{K}$. Moreover, it is clear that $f' \in M_{\bar{\alpha}}$ if and only if $f' = 0$. $\square$

The result above implies that the problem of computing a multigraded Hilbert series can be reduced to the case of a finitely generated multigraded right module $N = \bigoplus_i F[-\bar{\delta}_i]/M$ where $M \subset \bigoplus_i F[-\bar{\delta}_i]$ is a monomial right submodule. In this case, we call N a (finitely generated) *monomial right module*. For these modules we are immediately reduced to the cyclic case by means of the following result.

Proposition 3.3. *Let $N = \bigoplus_{1\leq i\leq r} F[-\bar{\delta}_i]/M$ be a finitely generated monomial right module and denote by $\{e_{i_k} w_k\} \subset W(r)$ a monomial generating set of M. For each index $i = 1, 2, \ldots, r$, let $I_i \subset F[-\bar{\delta}_i]$ be the monomial right ideal that is generated by set $\{w_k \mid e_{i_k} = e_i\}$. Moreover, let $C_i = F[-\bar{\delta}_i]/I_i$ be the corresponding monomial cyclic right module. Thus, one has that $M = \bigoplus_{i=1}^r e_i I_i$ and hence N is isomorphic to $\bigoplus_{i=1}^r C_i$. For the multigraded Hilbert series this implies that*

$$\mathrm{HS}(N) = \sum_{i=1}^{r} \mathrm{HS}(C_i).$$

We recall that F is a free right ideal ring which implies that a minimal basis of a monomial right ideal I is in fact a free one. If this basis is finite, then one can immediately obtain the multigraded Hilbert series of the cyclic module $C = F[-\bar{\delta}]/I$ by means of the formula in Theorem 2.4. Unfortunately, this happens very seldom. For instance, even if I is a finitely generated two-sided ideal, it may be infinitely generated as a right ideal. In the next section, to solve the problem of computing $\mathrm{HS}(C)$ in the general case, we develop an iterative method which relates this series to the Hilbert series of monomial cyclic right modules that are obtained from C.

4. A key linear equation

Let $C = F[-\bar\delta]/I$ be a monomial cyclic right module. For any $i = 1,2,\ldots,n$, denote by $\bar\theta_i$ the multidegree that the variable x_i has in F, that is, $\bar\theta_i = (0,\ldots,0,1,0,\ldots,0)$ where 1 occurs in the i-th position. Put $\bar\delta_i = \bar\delta + \bar\theta_i$ and observe that x_i has multidegree $\bar\delta_i$ in $F[-\bar\delta]$. Let $\{e_1,\ldots,e_n\}$ be the canonical basis of the multigraded free right module $\bigoplus_{1\leq i\leq n} F[-\bar\delta_i]$ and denote $\bar x_i = x_i + I \in C$. We define the multigraded right module homomorphism

$$\varphi : \bigoplus_i F[-\bar\delta_i] \to C, e_i \mapsto \bar x_i.$$

By definition, the image of this map is the multigraded right submodule $B = \langle \bar x_1,\ldots,\bar x_n\rangle \subset C$. Thus, the cokernel C/B is either zero if $C = 0$ or it is isomorphic to the base field $\mathbb{K}$ otherwise. To compute the kernel of φ one has to consider the colon right ideal

$$(I :_R x_i) = \{f \in F \mid x_i f \in I\}$$

which is also a monomial right ideal of F (see [18] for more details). By putting $I_{x_i} = (I :_R x_i)$, we consider the monomial cyclic right module $C_{x_i} = F[-\bar\delta]/I_{x_i}$ and we denote $C_{x_i}[-\bar\theta_i] = F[-\bar\delta_i]/I_{x_i}$. Since I is a monomial ideal, one has immediately that

$$\ker\varphi = \bigoplus_i e_i I_{x_i} \subset \bigoplus_i F[-\bar\delta_i].$$

Therefore, we obtain the following short exact sequence of multigraded right module homomorphisms

$$0 \to \bigoplus_{i=1}^{n} C_{x_i}[-\bar\theta_i] \to C \to C/B \to 0.$$

For the corresponding multigraded Hilbert series, since one has clearly that $\mathrm{HS}(C_{x_i}[-\bar\theta_i]) = t_i \cdot \mathrm{HS}(C_{x_i})$, we obtain the following key linear equation

$$\mathrm{HS}(C) = \sum_{i=1}^{n} t_i \cdot \mathrm{HS}(C_{x_i}) + c(I) \tag{1}$$

where, by definition, $c(I)$ is the dimension of the cokernel C/B, that is

$$c(I) = \begin{cases} 0 & \text{if } I = \langle 1\rangle, \\ 1 & \text{otherwise.} \end{cases}$$

In view of the equation (1), one may think to reduce the computation of the multigraded Hilbert series $\mathrm{HS}(C)$ to the one of the series $\mathrm{HS}(C_{x_i})$ $(1 \leq i \leq n)$ and iteratively to the computation of $\mathrm{HS}(C_{x_i x_j})$ $(1 \leq i,j \leq n)$ and so on. It may happen that this process terminates in a finite number of steps because many of these monomial cyclic right modules coincide with each other. In the next section, we explain why and how, in this case, the series $\mathrm{HS}(C)$ can be immediately obtained.

5. Orbits of monomial right ideals

In this section, it is essential to recall the following two definitions that have been used in [18].

Definition 5.1. *Let $\mathcal{MI}$ denote the set of all monomial right ideals of F. For all variables x_i ($1 \leq i \leq n$) we let T_{x_i} be the colon right ideal operator on $\mathcal{MI}$ defined by x_i, namely $T_{x_i}(I) = (I :_R x_i)$ for any $I \in \mathcal{MI}$. Moreover, we denote by $\mathcal{O}_I$ the minimal subset of $\mathcal{MI}$ containing I such that $T_{x_i}(\mathcal{O}_I) \subset \mathcal{O}_I$ for any variable x_i. The set $\mathcal{O}_I$ is called the* orbit *of $I \in \mathcal{MI}$. A monomial right ideal I is called* regular *if its orbit $\mathcal{O}_I$ is a finite set.*

Definition 5.2. *Consider $I \subset F$ a regular (monomial) right ideal and let its orbit $\mathcal{O}_I = \{I_1, \ldots, I_r\}$ be an ordered set where $I_1 = I$. Define a square matrix $A_I = (a_{kl}) \in \mathbb{Z}^{r \times r}$ such that*

$$a_{kl} = \#\{1 \leq i \leq n \mid T_{x_i}(I_k) = I_l\}.$$

Let $E_r \in \mathbb{Z}^{r\times r}$ be the identity matrix and consider the field of rational functions $\mathbb{Q}(t)$ in the variable t and with coefficients in $\mathbb{Q}$ (in fact in $\mathbb{Z}$). We denote $p_I(t) = \det(t \cdot E_r - A_I) \in \mathbb{Q}(t)$ the characteristic polynomial of A_I. Finally, define the column vector $\mathbf{C}_I = (c(I_1), \ldots, c(I_r))^t$. We call $A_I, p_I(t)$ and $\mathbf{C}_I$ respectively, the adjacency matrix, characteristic polynomial and constant vector of (the orbit of) I.

Observe that $\det(E_r - t \cdot A_I) = t^r p_I(1/t) \neq 0$, since $E_r - t \cdot A_I = t(\frac{1}{t} \cdot E_r - A_I) \in \mathbb{Q}(t)^{r\times r}$. We now introduce a new set of matrices.

Definition 5.3. *Let I be a regular right ideal. For each index $i = 1, 2, \ldots, n$, we denote by $A_I^{(i)} \in \mathbb{Z}^{r\times r}$ the square matrix that is defined as follows*

$$A_I^{(i)} = (a_{kl}^{(i)})\,,\; a_{kl}^{(i)} = \begin{cases} 1 & \text{if } T_{x_i}(I_k) = I_l, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, one has that $A_I = A_I^{(1)} + \cdots + A_I^{(n)}$ and therefore we call $A_I^{(i)}$ the i-th component of the adjacency matrix A_I.

Let $\mathbb{Q}(t_1, \ldots, t_n)$ be the field of rational functions in the variables $t_1, \ldots, t_n$ and with coefficients in $\mathbb{Q}$. We may consider the square matrix $E_r - \sum_i t_i \cdot A_I^{(i)}$ as an element of the matrix algebra $\mathbb{Q}(t_1, \ldots, t_n)^{r\times r}$.

Lemma 5.4. *$E_r - \sum_i t_i \cdot A_I^{(i)}$ is an invertible matrix.*

Proof. Consider the algebra homomorphism $\mathbb{Q}[t_1, \ldots, t_n] \to \mathbb{Q}[t]$ such that $t_i \mapsto t$, for any $i = 1, 2, \ldots, n$. Clearly, this homomorphism maps $\det(E_r - \sum_i t_i \cdot A_I^{(i)})$ into $\det(E_r - t \cdot A_I)$. Since we have already observed that the latter determinant is different from zero, we conclude that the same holds for the former one. □

We now show that, in the regular case, the multigraded Hilbert series of a monomial cyclic right module can be obtained by solving a linear system corresponding to the non-singular matrix $E_r - \sum_i t_i \cdot A_I^{(i)}$.

Theorem 5.5. *Let $C = F[-\delta]/I$ be a* regular monomial cyclic right module*, that is, I is a regular ideal. Then, the multigraded Hilbert series* $\mathrm{HS}(C)$ *belongs to the rational function field $\mathbb{Q}(t_1, \ldots, t_n)$.*

Proof. Let $\mathcal{O}_I = \{I_1, \ldots, I_r\}$ be the finite orbit of I and consider the monomial cyclic right module $C_k = F[-\delta]/I_k$, for all $k = 1, 2, \ldots, r$. Note that all I_k are in fact regular ideals, because by definition $\mathcal{O}_{I_k} \subset \mathcal{O}_I$. For each module C_k, the linear equation (1) becomes

$$\mathrm{HS}(C_k) = \sum_{i=1}^{n} t_i \cdot \mathrm{HS}(C_{l_{ki}}) + c(I_k) \tag{2}$$

where the index $l_{ki} \in \{1, 2, \ldots, r\}$ is defined as $T_{x_i}(I_k) = I_{l_{ki}}$. The matrix form of this linear system with coefficients in $\mathbb{Q}(t_1, \ldots, t_n)$ is clearly

$$(E_r - \sum_{i=1}^{n} t_i \cdot A_I^{(i)})\mathbf{H} = \mathbf{C}_I$$

where $\mathbf{H} = (\mathrm{HS}(C_1), \ldots, \mathrm{HS}(C_r))^t$ is a column vector of unknown multigraded Hilbert series. From Lemma 5.4 it follows that the linear system (2) has a unique solution, that is, all series $\mathrm{HS}(C_k)$ ($C = C_1$) are rational functions with integer coefficients. $\square$

On account of Theorem 3.2 and Proposition 3.3, the result above can be immediately generalized in the following way.

Theorem 5.6. *Let $N = \bigoplus_{1 \leq i \leq r} F[-\bar{\delta}_i]/M$ be a finitely generated multigraded right module. Fix any monomial ordering for the free right module F^r and consider $N' = \bigoplus_{1 \leq i \leq r} F[-\bar{\delta}_i]/M'$ where $M' = \mathrm{LM}(M)$ is the leading monomial module of M. According to Proposition 3.3, the finitely generated monomial right module N' is isomorphic to a direct sum $\bigoplus_i C_i$ where $C_i = F[-\bar{\delta}_i]/I_i$ and I_i $(1 \leq i \leq r)$ is a monomial right ideal such that M' is isomorphic to $\bigoplus_i I_i$. If each I_i is a regular ideal, then the multigraded Hilbert series $\mathrm{HS}(N) = \sum_{1 \leq i \leq r} \mathrm{HS}(C_i)$ is a rational function with integer coefficients.*

Observe now that two problems need to be solved in order to use the arguments of Theorem 5.5 as an effective method to compute multigraded Hilbert series. One question is how to compute the colon right ideal operators T_{x_i} and the second consists in providing an "internal characterization" of the property that a monomial right ideal I is regular, that is, the orbit $\mathcal{O}_I$ is a finite set. Both these problems have been solved in [18] and, for the sake of completeness, we recall briefly here the corresponding results. Recall that we denote by W the set of monomials of the free associative algebra $F = \mathbb{K}\langle X \rangle$, that is, W is the set of all words over the alphabet $X = \{x_1, \ldots, x_n\}$. Note that for computing the orbit $\mathcal{O}_I$ one needs to iteratively apply the operators T_{x_i} on the monomial right ideal I until one obtains a stable set. In other words, for any monomial $w = x_{i_1} \cdots x_{i_d} \in W$, we define

$$T_w(I) = (T_{x_{i_d}} \cdots T_{x_{i_1}})(I) = (((I :_R x_{i_1}) \cdots) :_R x_{i_d}) = (I :_R w),$$

where $(I :_R w) = \{f \in F \mid wf \in I\}$. Note that $w \in I$ if and only if $(I :_R w) = \langle 1 \rangle$.

Proposition 5.7 ([18]). *Consider a monomial right basis* $\{w_j\} \subset W$ *of a monomial right ideal* $I \subset F$ *and let* $w \in W$. *For all* j, *we denote*

$$w'_j = \begin{cases} 1 & \text{if } w = w_j v_j \ (v_j \in W), \\ v_j & \text{if } w_j = w v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the subset $\{w'_j\} \subset W$ *is a monomial right basis of* $T_w(I) = (I :_R w)$.

An important case is when I is a monomial two-sided ideal, that is, $A = F/I$ is a monomial algebra. It is easy to prove that a two-sided ideal is always contained in the colon right ideals it defines.

Proposition 5.8 ([18]). *Consider a monomial two-sided basis* $\{w_j\} \subset W$ *of a monomial two-sided ideal* $I \subset F$ *and let* $w \in W, w \notin I$. *For all* j, *we define the monomial right ideal*

$$R(w, w_j) = \langle v_{jk} \mid u_{jk} w_j = w v_{jk}, u_{jk}, v_{jk} \in W, \operatorname{tdeg}(v_{jk}) < \operatorname{tdeg}(w_j) \rangle.$$

Then, one has that $(I :_R w) = \sum_j R(w, w_j) + I$.

We now recall some notions from the theory of formal languages (see, for instance, [8]).

Definition 5.9. *Any subset* $L \subset W$ *is called a* (formal) language. *Given two languages* $L, L' \subset W$, *we consider their set-theoretic union* $L \cup L'$ *and their product* $L \cdot L' = \{ww' \mid w \in L, w' \in L'\}$. *Moreover, one defines the* star operation $L^* = \bigcup_{d \geq 0} L^d$, *where* $L^0 = \{1\}$ *and* $L^d = L \cdot L^{d-1}$ *for* $d \geq 1$. *The union, the product and the star operation are called the* rational operations *over the languages. A language* $L \subset W$ *is called* regular *if it can be obtained from finite languages by applying a finite number of rational operations.*

A characterization of the finiteness of the orbit of a monomial right ideal is provided by the following key result.

Theorem 5.10 ([18]). *The monomial right ideal* $I \subset F$ *is regular if and only if the corresponding language* $L = I \cap W$ *is regular.*

By using standard results of the theory of finite-state automata, which is strictly related to the theory of regular languages, in [18] it was proved that the number of variables in the key linear system (2) is minimal to obtain the Hilbert series of the monomial cyclic right module under consideration. In other words, the proposed method for the computation of such a series (both graded and multigraded) is optimal with respect to the required number of iterations of the colon right ideal operators. This is essentially a promise of efficiency that we are able to verify in practice in Section 8.

If a right or two-sided ideal is finitely generated, then this ideal is clearly regular by means of Theorem 5.10. For instance, if $\{w_1, \ldots, w_k\} \subset W$ is a finite basis of the monomial two-sided ideal I, then

$$I \cap W = W \cdot \{w_1, \ldots, w_k\} \cdot W$$

where clearly $W = X^*$. In other words, the language $I \cap W$ is obtained by applying a finite number of rational operations over the finite languages $X = \{x_1, \ldots, x_n\}$ and $\{w_1, \ldots, w_k\}$. By Theorem 5.5, this implies that a finitely presented monomial algebra $A = F/I$ has always a rational multigraded

Hilbert series. Note that for the graded case this agrees with the classical results in [13] and [24]. Since Theorem 5.5 and the corresponding method are more general than the finitely presented case, in Section 7 we will illustrate the computation of the multigraded Hilbert series of an infinitely presented but regular monomial algebra.

6. Finite-dimensional case and truncation

Let C be a regular monomial cyclic right module. In this section we aim to characterize the case in which C is finite-dimensional. Moreover, for an infinite-dimensional C, we want to develop methods for the polynomial approximation of the multigraded Hilbert series of C, which is possibly not rational and unknown. To develop the theory for the finite-dimensional case, it is convenient to switch to the (graded) Hilbert series $\mathrm{HS}'(C)$ that is obtained from $\mathrm{HS}(C)$ by identifying all variables t_i $(1 \leq i \leq n)$ with a single variable t. Let $C = F/I$ and consider the orbit $\mathcal{O}_I = \{I_1, \ldots, I_r\}$ $(I_1 = I)$. Moreover, let A_I and $\mathbf{C}_I$ be the corresponding adjacency matrix and constant vector. As in Theorem 5.5, we have that the Hilbert series $\mathrm{HS}'(C_k)$ $(C_k = F/I_k)$ are obtained by solving the matrix equation

$$(E_r - t \cdot A_I)\mathbf{H'} = \mathbf{C}_I$$

where $\mathbf{H'} = (\mathrm{HS}'(C_1), \ldots, \mathrm{HS}'(C_r))^t$. We want to understand when the (regular monomial cyclic) right modules C_k have finite dimensions, that is, the solution vector $\mathbf{H'}$ has all entries in the polynomial algebra $\mathbb{Q}[t]$. For any monomial $w \in W$, we have already observed that $w \in I$ if and only if $T_w(I) = \langle 1 \rangle$ which implies that $\langle 1 \rangle \in \mathcal{O}_I$. Assume now that $C \neq 0$, that is, $I \neq \langle 1 \rangle$ and hence $r > 1$. Assume that $I_r = \langle 1 \rangle$ and define the *reduced orbit of I* as

$$\bar{\mathcal{O}}_I = \mathcal{O}_I \setminus \{\langle 1 \rangle\} = \{I_1, \ldots, I_{r-1}\}.$$

Clearly, $\mathbf{C}_I = (1, \ldots, 1, 0)^t$ and $\mathrm{HS}'(C_r) = 0$ and therefore we consider $\bar{\mathbf{H}}' = (\mathrm{HS}'(C_1), \ldots, \mathrm{HS}'(C_{r-1}))^t$ and $\bar{\mathbf{C}}_I = (1, \ldots, 1)^t$. Moreover, we denote by $\bar{A}_I$ the square matrix that is obtained from A_I by deleting the r-th row and the r-th column. Finally, we put $\bar{p}_I(t) = \det(t \cdot E_{r-1} - \bar{A}_I)$, that is, $\bar{p}_I(t)$ is the characteristic polynomial of $\bar{A}_I$. We call $\bar{A}_I, \bar{p}(t)$ and $\bar{\mathbf{C}}_I$ respectively, the *reduced adjacency matrix, reduced characteristic polynomial and reduced constant vector of I*.

Since $\mathrm{HS}'(C_r) = 0$, it is clear that one can obtain the Hilbert series $\mathrm{HS}'(C_k)$ $(1 \leq k \leq r-1)$ by solving the reduced matrix equation

$$(E_{r-1} - t \cdot \bar{A}_I)\bar{\mathbf{H}}' = \bar{\mathbf{C}}_I. \tag{3}$$

Note that if we assume $\det(E_{r-1} - t \cdot \bar{A}_I) \in \mathbb{Q} \setminus \{0\}$, then the matrix inverse $(E_{r-1} - t \cdot \bar{A}_I)^{-1}$ belongs to $\mathbb{Q}[t]^{(r-1)\times(r-1)}$ and hence all solutions of equation (3) are in fact in the polynomial algebra $\mathbb{Q}[t]$. Moreover, since $\det(E_{r-1} - t \cdot \bar{A}_I) = t^{r-1}\bar{p}_I(1/t) \neq 0$, we have in particular that $\det(E_{r-1} - t \cdot \bar{A}_I) = 1$ if and only if the matrix $\bar{A}_I$ is nilpotent, that is, $\bar{p}_I(t) = t^{r-1}$. In other words, the latter condition implies that the right module C (in fact each C_k) is finite-dimensional. We now show that the nilpotency of $\bar{A}_I$ is also a necessary condition.

Theorem 6.1. *Let $C = F/I$ be a finite-dimensional monomial cyclic right module. It holds that C is a regular module. Moreover, if $C \neq 0$ and $\bar{\mathcal{O}}_I = \{I_1, \ldots, I_{r-1}\}$ is the reduced orbit of I, then $T_{x_i}(I_k) = I_l$ $(1 \leq i \leq n, 1 \leq k, l \leq r-1)$ implies that $\dim C_k > \dim C_l$.*

Proof. Consider the key short exact sequence of Theorem 5.5, namely

$$0 \to \bigoplus_{i=1}^{n} C_{x_i} \to C \to C/B \to 0$$

where $C_{x_i} = F/I_{x_i}$ and $I_{x_i} = T_{x_i}(I) = (I :_R x_i)$. Since $\dim C < \infty$, one has that all (monomial cyclic) right modules C_{x_i} are also finite-dimensional and

$$\dim C = \sum_{i=1}^{n} \dim C_{x_i} + 1.$$

We conclude that $\dim C > \dim C_{x_i} \geq 0$, for all $i = 1, 2, \ldots, n$. By iterating the above argument along the orbit of I, we obtain that this is finite. In fact, at each iteration, for a non-zero right module $C_k = F/I_k$ one has that the condition $T_{x_i}(I_k) = I_l$ implies that $\dim C_k > \dim C_l$. □

Theorem 6.2. *Let $C = F/I$ be a non-zero regular monomial cyclic right module and consider the corresponding reduced adjacency matrix $\bar{A}_I$. Then, C is finite-dimensional if and only if $\bar{A}_I$ is a nilpotent matrix.*

Proof. By the arguments at the beginning of this section, it remains to prove the necessary condition and hence let us assume that $\dim C < \infty$. By Theorem 6.1, we have that $\dim C_k < \infty$ where $C_k = F/I_k$ and $\bar{\mathcal{O}}_I = \{I_1, \ldots, I_{r-1}\}$. Assume now that the reduced orbit $\bar{\mathcal{O}}_I$ is ordered according to the dimensions, namely $\dim C_k > \dim C_l$ implies that $k < l$. Again by Theorem 6.1, we obtain that the matrix $\bar{A}_I$ is strictly upper triangular and hence nilpotent. □

For the finite-dimensional case, we remark that the strictly upper triangular structure of the reduced adjacency matrix makes the computation of the Hilbert series (actually, polynomial) a fast one. In fact, in Section 8 we will present some tests where this computation performs better than normal words enumeration. This is of course true also for multigraded Hilbert series since $\bar{A}_I$ is a strictly upper triangular matrix if and only if so are the matrices $\bar{A}_I^{(i)}$ $(1 \leq i \leq n)$, where by definition $\bar{A}_I = \bar{A}_I^{(1)} + \cdots + \bar{A}_I^{(n)}$. Finally, it is important to mention that Ufnarovski [24] also provided a graph theoretic characterization of the finite-dimensionality of a finitely presented monomial algebra.

We now consider the problem of computing some truncation of a (multigraded) Hilbert series. In fact, solving this problem may be of interest if the sum of the series is difficult to determine because, for instance, it is not a rational function. Moreover, if an algebra $A = F/I$ is invariant under the action of the general linear group $\mathrm{GL}_n(\mathbb{K})$, the multigraded Hilbert series of A is a symmetric function. Hence, one may want to decompose this function as a sum of Schur (polynomial) functions because this provides the $\mathrm{GL}_n(\mathbb{K})$-module structure of A. We will give more details about this application of the truncation by studying a concrete example in Section 7.

For the computation of Hilbert series, on account of Theorem 3.2 and Proposition 3.3, we are reduced to consider the monomial cyclic case. Since the main applications concern algebras, let us consider a monomial algebra $A = F/I$, that is, $I \subset F$ is a monomial two-sided ideal of F. We assume here the standard multigrading for F and A, in order to simplify the notation. Consider the (monomial) two-sided ideal $B = \langle x_1, \ldots, x_n \rangle$ and its power B^{d+1} ($d \geq 0$) that is generated by all monomials $w \in W$ such that $\mathrm{tdeg}(w) = d + 1$. In other words, one has that $B^{d+1} = \sum_{k \geq d+1} F_k$. We consider the finite-dimensional monomial algebra $A^{(d)} = F/I^{(d)}$ where $I^{(d)} = I + B^{d+1}$ and we call this algebra the *d-th truncation of* A. In fact, it is clear that $A^{(d)}$ is isomorphic to the vector space $\bigoplus_{k \leq d} A_k$ and hence the multigraded Hilbert series (in fact, polynomial) $\mathrm{HS}(A^{(d)})$ is the truncation at total degree d of the multigraded Hilbert series $\mathrm{HS}(A)$, that is

$$\mathrm{HS}(A^{(d)}) = \sum_{|\bar{\alpha}| \leq d} \dim A_{\bar{\alpha}}\, t_1^{\alpha_1} \cdots t_n^{\alpha_n}.$$

Since the computation of the Hilbert series is based on the computation of the colon right ideals, we now analize this operation for the two-sided ideals $I^{(d)}$. We show that one can compute $T_w(I^{(d)}) = (I^{(d)} :_R w)$ ($w \in W$) without having to deal with the n^{d+1} monomial generators of the ideal B^{d+1}.

Lemma 6.3. *Let I, J be monomial two-sided ideals and $w \in W$. Then $(I + J :_R w) = (I :_R w) + (J :_R w)$.*

Proof. Consider the monomial basis $\{v_i\} \cup \{w_j\}$ of the ideal $I + J$, where $\{v_i\}$ is a monomial basis of I and $\{w_j\}$ is a monomial basis of J. With the notation of Proposition 5.8, we immediately have that

$$\begin{aligned}(I + J :_R w) &= \textstyle\sum_i R(w, v_i) + \sum_j R(w, w_j) + I + J \\ &= (I :_R w) + (J :_R w).\end{aligned}$$

□

Lemma 6.4. *If $d' = \mathrm{tdeg}(w)$ ($w \in W$) then*

$$(B^d :_R w) = \begin{cases} \langle 1 \rangle & \text{if } d' \geq d, \\ B^{d-d'} & \text{otherwise.} \end{cases}$$

Proof. If $d' \geq d$ then clearly $w \in B^d$ and therefore $(B^d :_R w) = \langle 1 \rangle$. Assume now $d' < d$. For any $v \in W$ such that $v \in B^{d-d'}$, that is, $\mathrm{tdeg}(v) \geq d - d'$ one has that $wv \in B^d$ and hence $v \in (B^d :_R w)$. Conversely, if $v \in (B^d :_R w)$ ($v \in W$) then $wv \in B^d$ and therefore $d' + \mathrm{tdeg}(v) \geq d$, that is, $v \in B^{d-d'}$. □

Proposition 6.5. *Let $I \subset F$ be a two-sided monomial ideal and $w \in W$. Put $d' = \mathrm{tdeg}(w)$. It holds that*

$$(I^{(d)} :_R w) = \begin{cases} \langle 1 \rangle & \text{if } d' > d, \\ (I :_R w)^{(d-d')} & \text{otherwise.} \end{cases}$$

Proof. By assuming $d' > d$ we have that $w \in B^{d+1}$ and hence $w \in I^{(d)} = I + B^{d+1}$, that is, $(I^{(d)} :_R w) = \langle 1 \rangle$. Otherwise, if $d' \leq d$ then from Lemmas 6.3 and 6.4 it follows that

$$\begin{aligned}(I^{(d)} :_R w) &= (I :_R w) + (B^{d+1} :_R w) = (I :_R w) + B^{d-d'+1} \\ &= (I :_R w)^{(d-d')}.\end{aligned}$$

□

The results above show that, to compute the finite reduced orbit $\bar{\mathcal{O}}_{I^{(d)}}$, one has simply to compute (according to Proposition 5.8) the monomial generators of the colon right ideals $(I :_R w)$ $(\mathrm{tdeg}(w) = d' \leq d)$ up to the total degree $d - d'$. In other words, the n^{d+1} monomial generators of B^{d+1} are not involved at all in these computations. Moreover, solving the corresponding matrix equation to obtain the Hilbert series is very efficient, because we are in the finite-dimensional (strictly upper triangular) case.

7. An illustrative example

In this section, by means of a concrete example, we show how the proposed method for multigraded Hilbert series can be applied to study in an effective way, finitely generated algebras that are invariant under the action of the general linear group. To begin with, we introduce some general notions and results about the action of $G = \mathrm{GL}_n(\mathbb{K})$ on the free associative algebra $F = \mathbb{K}\langle X\rangle$ where $X = \{x_1, \ldots, x_n\}$. For a complete reference we refer to the monographs [9, 11]. Let $g = (g_{ij})$ be any matrix of the group G and define the algebra automorphism $\rho_g : F \to F$ such that $x_i \mapsto \sum_j g_{ij} x_j$. Clearly, $\rho_{gh} = \rho_h \rho_g$ for all $g, h \in G$, that is, one has a right action of G on F. A subspace $V \subset F$ such that $\rho_g(V) \subset V$ for all $g \in G$ is called a *G-submodule of F*. The corresponding (anti)homomorphism $G \to \mathrm{End}_{\mathbb{K}}(V)$ is called a *polynomial representation of G*. If $F = \bigoplus_d F_d$ is the decomposition of the algebra F in its homogeneous components, it is clear that each F_d is a G-submodule of F. By definition, a subspace $V \subset F$ is *graded* if $V = \sum_d V_d$ where $V_d = V \cap F_d$. In a similar way, one defines also *multigraded* subspaces. Clearly, for graded and multigraded subspaces we can consider the corresponding Hilbert series as the generating series of the dimensions of their homomogeneous and multihomogeneous components, respectively. A G-submodule $V \subset F$ is said *simple* if there is no G-submodule of V other than 0 and V.

Assume now that $\mathrm{char}(\mathbb{K}) = 0$. By the "Vandermonde argument" one has that all G-submodules $V \subset F$ are in fact graded subspaces. Note that each homogeneous component $V_d = V \cap F_d$ is clearly a G-submodule. By Schur's theory on polynomial representations, all G-submodules of F are *semisimple*, that is, they are direct sum of simple G-submodules. Moreover, a complete set of (non-isomorphic) simple G-submodules $\{W^\lambda\}_\lambda$ is parametrized by integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $k \leq n$. By denoting $|\lambda| = \sum_i \lambda_i$, one has in particular that the set $\{W^\lambda\}_{|\lambda|=d}$ occurs in the decomposition of F_d. Each W^λ with $|\lambda| = d$ is actually a multigraded subspace of F_d and one defines its multigraded Hilbert series

$$\mathrm{S}_\lambda = \mathrm{HS}(W^\lambda) = \sum_{|\bar{\alpha}|=d} \dim W^\lambda_{\bar{\alpha}}\, t_1^{\alpha_1} \cdots t_n^{\alpha_n}.$$

These polynomials are called *Schur functions*. Every S_λ is a symmetric polynomial which can be computed by well-known formulas (see, for instance, [11]). It follows that any G-submodule $V \subset F$ is in fact multigraded and its multigraded Hilbert series $\mathrm{HS}(V) = \sum_{\bar{\alpha}\in\mathbb{N}^n} \dim V_{\bar{\alpha}}\, t_1^{\alpha_1} \cdots t_n^{\alpha_n}$

is also a symmetric function which is a (possibly infinite) linear combination of Schur functions. Precisely, let $V = \bigoplus_\lambda m_\lambda W^\lambda$ be the decomposition of V in its simple G-submodules, where the integer $m_\lambda \geq 0$ denotes the number of times (multiplicity) that a simple G-submodule isomorphic to W^λ occurs in V. Thus, for the multigraded Hilbert series we clearly have that $\mathrm{HS}(V) = \sum_\lambda m_\lambda \mathrm{S}_\lambda$. In other words, it is sufficient to decompose $\mathrm{HS}(V)$ in terms of Schur functions to have a complete description of how the general linear group G acts on V. When the G-submodule $V \subset F$ is infinite-dimensional, the symmetric function $\mathrm{HS}(V)$ may not be rational and even in the rational case one has the problem that the Schur function decomposition is indeed a series. There are some "nice rational symmetric functions" that allow the computation of this decomposition in the infinite-dimensional case. We refer to [2] for details about such methods. When V is finite-dimensional, and hence $\mathrm{HS}(V)$ is simply a symmetric polynomial, there are fast algorithms that compute the Schur function decomposition. These procedures are implemented, for instance, in the Maple package SF or in the C library SYMMETRICA which are freely distributed over the Internet [21, 23]. We will make use of these algorithms to perform the Schur function decomposition in our example.

Consider now a finitely generated algebra $A = F/J$, where the two-sided ideal $J \subset F$ is also a G-submodule. Clearly, the group G acts on A as well and by the semisemplicity of F we have that A is isomorphic to a G-submodule of F. Then, A is multigraded and semisimple and the multigraded Hilbert series $\mathrm{HS}(A)$ is a linear combination of Schur functions according to the G-module structure of A. In particular, the symmetric function $\mathrm{HS}(A)$ is a polynomial when A is finite-dimensional.

An important class of algebras that are invariant under the action of the general linear group are the following ones. Let R be an associative algebra and consider the two-sided ideal

$$T(R) = \{f \in F \mid f(r_1, \ldots, r_n) = 0, \text{ for all } r_1, \ldots, r_n \in R\}.$$

It is clear that $T(R) \subset F$ is a *T-ideal*, that is, it is invariant under all algebra endomorphisms of F and in particular $T(R)$ is a G-submodule. The finitely generated algebra $A = F/T(R)$ is called the *relatively free algebra in n variables that is defined by R*. It is known that the multigraded Hilbert series of these algebras are rational symmetric functions and the computation of their Schur function decomposition is very important in the study of a *PI-algebra*, that is, an algebra R such that $T(R) \neq 0$. As a reference for PI-theory, we suggest the books [9, 12].

Let $E = E(V) = \bigwedge(V)$ be the Grassmann (or exterior) algebra over a vector space V of countable dimension. Moreover, denote $[f, g] = fg - gf$, for any $f, g \in F$. For $\mathrm{char}(\mathbb{K}) = 0$, Latyshev first proved in [20] that the two-sided ideal $T(E) \subset F$ is generated by the following set of polynomials

$$[[x_i, x_j], x_k]\,,\ [x_i, x_j][x_k, x_l] + [x_i, x_k][x_j, x_l]$$

for all $x_i, x_j, x_k, x_l \in X$. In our example, we will consider the relatively free algebra $A = F/T(E)$. Note that the G-module structure of A was essentially obtained by Krakowski and Regev in [14]. However, we want to show here how such a structure can be studied in an algorithmic way by combining our method for multigraded Hilbert series with algorithms for computing the Schur function decomposition. Our aim is both to illustrate our method and to suggest that other interesting G-invariant algebras may be investigated in a similar way. In fact, in Section 8 we will present more computations of this kind as a test set.

Fix $n = 3$ and denote $X = \{x, y, z\}$. By Theorem 3.2, a first step to obtain the multigraded Hilbert series $\mathrm{HS}(A)$ consists in computing the leading monomial ideal $I = \mathrm{LM}(T(E))$. With respect to the graded left lexicographic ordering with $x \succ y \succ z$, the two-sided ideal I is minimally generated by the following infinite set of monomials

$$x^2y, x^2z, xy^2, xyz, xzy, xz^2, y^2z, yz^2,$$
$$xyxy, xyxz, xzxy, xzxz, yzyz,$$
$$yzy^dxy, yzy^dxz \ (d \geq 0).$$

This generating set was found in [10], where in fact a minimal Gröbner basis of $T(E)$ was given for any number of variables. Clearly, the set of monomials $I \cap W$ ($W = X^*$) can be obtained by applying a finite number of rational operations over finite languages, namely

$$I \cap W = W \cdot (\{x^2y, x^2z, xy^2, xyz, xzy, xz^2, y^2z, yz^2, xyxy, xyxz,$$
$$xzxy, xzxz, yzyz\} \cup \{yz\} \cdot \{y\}^* \cdot \{xy, xz\}) \cdot W.$$

By Definition 5.9, we have that $I \cap W$ is a regular language. By Theorem 5.10 we conclude that I is a regular ideal, that is, the orbit $\mathcal{O}_I$ is a finite set and therefore the multigraded Hilbert series $\mathrm{HS}(A)$ is a rational function. Note that this function is also a symmetric one because $T(E)$ is a G-submodule of F. Recall that to describe the orbit $\mathcal{O}_I$ and the linear equations relating all the series corresponding to the monomial right ideals in $\mathcal{O}_I$, one has to compute the colon right ideals $I_w = (I :_R w)$ ($w \in W$). Since I is a two-sided ideal, one may use Proposition 5.8 for that purpose. For instance, to compute $I_x = (I :_R x)$ one considers the following right ideals

$$R(x, x^2y) = \langle xy \rangle, R(x, x^2z) = \langle xz \rangle, R(x, xy^2) = \langle y^2 \rangle, R(x, xyz) = \langle yz \rangle,$$
$$R(x, xzy) = \langle zy \rangle, R(x, xz^2) = \langle z^2 \rangle, R(x, y^2z) = 0, R(x, yz^2) = 0,$$
$$R(x, xyxy) = \langle yxy \rangle, R(x, xyxz) = \langle yxz \rangle, R(x, xzxy) = \langle zxy \rangle,$$
$$R(x, xzxz) = \langle zxz \rangle, R(x, yzyz) = 0,$$
$$R(x, yzy^dxy) = 0, R(x, yzy^dxz) = 0 \ (d \geq 0).$$

We conclude that $I_x = \langle xy, xz, y^2, yz, zy, z^2, yxy, yxz, zxy, zxz \rangle + I$. In a similar way, one obtains

$$R(y, x^2y) = 0, R(y, x^2z) = 0, R(y, xy^2) = 0, R(y, xyz) = 0,$$
$$R(y, xzy) = 0, R(y, xz^2) = 0, R(y, y^2z) = \langle yz \rangle, R(y, yz^2) = \langle z^2 \rangle,$$
$$R(y, xyxy) = 0, R(y, xyxz) = 0, R(y, xzxy) = 0,$$
$$R(y, xzxz) = 0, R(y, yzyz) = \langle zyz \rangle,$$
$$R(y, yzy^dxy) = \langle zy^dxy \rangle, R(y, yzy^dxz) = \langle zy^dxz \rangle \ (d \geq 0)$$

and hence $I_y = \langle yz, z^2, zyz\rangle + \langle zy^d xy, zy^d xz \mid d \geq 0\rangle + I$. Moreover, it holds immediately that $I_z = I$. By denoting $C_x = F/I_x, C_y = F/I_y$ the monomial cyclic right modules corresponding to the monomial right ideals I_x, I_y, we obtain the first linear equation

$$\mathrm{HS}(A) = t_1\mathrm{HS}(C_x) + t_2\mathrm{HS}(C_y) + t_3\mathrm{HS}(A) + 1. \tag{4}$$

Since $I_x, I_y \in \mathcal{O}_I$, we now have to compute the right colon ideals I_{x^2}, I_{xy}, I_{xz} and I_{yx}, I_{y^2}, I_{yz}. The following equalities hold

$$\begin{gathered}
R(x^2, x^2y) = \langle y, xy\rangle, R(x^2, x^2z) = \langle z, xz\rangle, R(x^2, xy^2) = \langle y^2\rangle, R(x^2, xyz) = \langle yz\rangle, \\
R(x^2, xzy) = \langle zy\rangle, R(x^2, xz^2) = \langle z^2\rangle, R(x^2, y^2z) = 0, R(x^2, yz^2) = 0, \\
R(x^2, xyxy) = \langle yxy\rangle, R(x^2, xyxz) = \langle yxz\rangle, R(x^2, xzxy) = \langle zxy\rangle, \\
R(x^2, xzxz) = \langle zxz\rangle, R(x^2, yzyz) = 0, \\
R(x^2, yzy^d xy) = 0, R(x^2, yzy^d xz) = 0 \ (d \geq 0).
\end{gathered}$$

We have therefore that $I_{x^2} = \langle y, z, xy, xz\rangle + I$. In a similar way, we compute that $I_{x^2} = I_{xy} = I_{xz}$ and one obtains the equation

$$\mathrm{HS}(C_x) = t_1\mathrm{HS}(C_{x^2}) + t_2\mathrm{HS}(C_{x^2}) + t_3\mathrm{HS}(C_{x^2}) + 1. \tag{5}$$

It is easy to check that $I_{yx} = I_x$ and $I_{y^2} = \langle z, yz\rangle + I$. Moreover, we compute that $I_{yz} = \langle z, yz\rangle + \langle y^d xy, y^d xz \mid d \geq 0\rangle + I$ and one obtains the equation

$$\mathrm{HS}(C_y) = t_1\mathrm{HS}(C_x) + t_2\mathrm{HS}(C_{y^2}) + t_3\mathrm{HS}(C_{yz}) + 1. \tag{6}$$

Then, we have that $I_{x^2}, I_{y^2}, I_{yz} \in \mathcal{O}_I$ and one has to compute the corresponding colon right ideals $I_{x^3}, I_{x^2y}, I_{x^2z}, I_{y^2x}, I_{y^3}, I_{y^2z}$ and $I_{yzx}, I_{yzy}, I_{yz^2}$. By similar computations, one obtains $I_{x^3} = I_{x^2}$. Moreover, from $x^2y, x^2z \in I$ it follows immediately that $I_{x^2y}, I_{x^2z} = \langle 1\rangle$. A new equation is hence the following one

$$\mathrm{HS}(C_{x^2}) = t_1\mathrm{HS}(C_{x^2}) + t_2\mathrm{HS}(C_{x^2y}) + t_3\mathrm{HS}(C_{x^2y}) + 1. \tag{7}$$

We have the following identities $I_{y^2x} = I_x, I_{y^3} = I_{y^2}, I_{y^2z} = I_{x^2y}$ which imply the equation

$$\mathrm{HS}(C_{y^2}) = t_1\mathrm{HS}(C_x) + t_2\mathrm{HS}(C_{y^2}) + t_3\mathrm{HS}(C_{x^2y}) + 1. \tag{8}$$

One has also the identities $I_{yzx} = I_{x^2}, I_{yzy} = I_{yz}, I_{yz^2} = I_{x^2y}$ and the equation

$$\mathrm{HS}(C_{yz}) = t_1\mathrm{HS}(C_{x^2}) + t_2\mathrm{HS}(C_{yz}) + t_3\mathrm{HS}(C_{x^2y}) + 1. \tag{9}$$

Finally, we have that $I_{x^2y} = \langle 1\rangle \in \mathcal{O}_I$ where clearly $\mathrm{HS}(C_{x^2y}) = 0$. This can be also obtained by the obvious identities $I_{x^2yx} = I_{x^2y^2} = I_{x^2yz} = I_{x^2y}$ and by the corresponding linear equation (with $c(I_{x^2y}) = 0$)

$$\mathrm{HS}(C_{x^2y}) = t_1\mathrm{HS}(C_{x^2y}) + t_2\mathrm{HS}(C_{x^2y}) + t_3\mathrm{HS}(C_{x^2y}). \tag{10}$$

We conclude that $\mathcal{O}_I = \{I, I_x, I_y, I_{x^2}, I_{y^2}, I_{yz}, I_{x^2y}\}$. By solving the (nonsingular) system of the obtained linear equations, one computes the multigraded Hilbert series of all monomial cyclic right modules corresponding to the elements of the orbit $\mathcal{O}_I$. In particular, we obtain that

$$\mathrm{HS}(A) = \frac{t_1t_2 + t_1t_3 + t_2t_3 + 1}{(1 - t_1)(1 - t_2)(1 - t_3)}.$$

Note that this formula is a special case, for $n = 3$, of the general formula (see [3]) for the multigraded Hilbert series of the relatively free algebra in n variables of the Grassmann algebra, which is

$$\frac{\prod_{i=1}^{n}(1+t_i) + \prod_{i=1}^{n}(1-t_i)}{2 \cdot \prod_{i=1}^{n}(1-t_i)}.$$

We now consider the problem of understanding the Schur function decomposition of $\mathrm{HS}(A)$. Recall that $A = F/T(E)$ and $I = \mathrm{LM}(T(E))$ and consider the corresponding monomial algebra $A' = F/I$. Our previous computations have been made according to Theorem 3.2 which yields that $\mathrm{HS}(A') = \mathrm{HS}(A)$. To obtain information about the Schur function decomposition of $\mathrm{HS}(A)$, one possible approach consists in truncating A' at a sufficiently high total degree d to see which decomposition holds up to that degree. Precisely, with the notation of Section 6, we consider the finite-dimensional monomial algebra $A'^{(d)} = F/I^{(d)}$ which is the d-th truncation of A'. We obtain that $\mathrm{HS}(A'^{(d)})$ is the truncation at the total degree d of the multigraded Hilbert series $\mathrm{HS}(A') = \mathrm{HS}(A)$ which implies that $\mathrm{HS}(A'^{(d)})$ is a symmetric polynomial. Then, the Schur function decomposition of $\mathrm{HS}(A'^{(d)})$ can be obtained by means of efficient algorithms and this decomposition corresponds to the G-module structure of A up to the fixed degree d.

To show how feasible this method could be for obtaining a sufficiently large truncation of the series $\mathrm{HS}(A) = \sum_\lambda m_\lambda \mathrm{S}_\lambda$, in our example we fix $d = 10$. In this case, the orbit $\mathcal{O}_{I^{(10)}}$ consists of 51 ideals and the corresponding polynomial $\mathrm{HS}(A'^{(10)})$ has 286 monomials in the variables t_1, t_2, t_3. This symmetric function is computed by our methods in 33 milliseconds on a Linux server (see Section 8 for the specification of the machine). Moreover, the Schur function decomposition of $\mathrm{HS}(A'^{(10)})$ takes 0.38 seconds in the Magma interface of the library SYMMETRICA. These computations return the decomposition

$$\mathrm{HS}(A'^{(10)}) = \sum_{0\leq k\leq 10} \sum_{\substack{p+q=k,\\ 0\leq q\leq 2}} \mathrm{S}_{(p,1^q)}$$

where $(p, 1^q)$ reads $(p, 1, \ldots, 1)$ and we agree that $S_{(0,0)} = 1$. This suggests the following formula, which was obtained in [14] for an arbitrary number n of variables

$$\mathrm{HS}(A) = \sum_{k\geq 0} \sum_{\substack{p+q=k,\\ 0\leq q\leq n-1}} \mathrm{S}_{(p,1^q)}.$$

Observe that, in this example, we have computed the rational form of the complete Hilbert series $\mathrm{HS}(A)$ and therefore $\mathrm{HS}(A'^{(10)})$ could also be obtained by means of a Taylor expansion. Nevertheless, since we are in the multivariate case, even this task may be a non-trivial one (recall that $\mathrm{HS}(A'^{(10)})$

has 286 monomials). Moreover, we remark that for a non-regular monomial algebra, the sum of the Hilbert series may be completely unknown.

8. Implementation and Timings

In this section, we provide the practical performance of the proposed algorithms for the computation of graded and multigraded Hilbert series, both in the complete and in the truncated case. All tests are performed by means of an implementation that we have developed in the kernel of the computer algebra system Singular [7]. Recall that the Hilbert series of any graded or multigraded algebra $A = F/J$ ($F = \mathbb{K}\langle x_1, \ldots, x_n \rangle$) is the same as the series of the corresponding monomial algebra $A' = F/I$ where $I = \mathrm{LM}(J)$ is the leading monomial ideal of J. Then, for computing the Hilbert series, our kernel implementation requires as input a set of monomial generators of I. Precisely, user has to compute a Gröbner basis of J with respect to some monomial ordering and to input this basis into the procedure nchilb which has been implemented in Singular's interpreted language. This function collects the leading monomials of the Gröbner basis and converts them in a suitable format for the kernel's code. See our Singular library ncHilb.lib for detailed instructions to use it. Note that internally to the kernel, the implementation uses indeed commutative analogues of noncommutative monomial ideals according to the "letterplace correspondence". For more details about the wide scope of letterplace methods we refer to [15, 16, 17, 19].

We display here the pseudo code of the proposed algorithm for multigraded algebras that runs in the kernel.

Algorithm 1 Multigraded Hilbert series algorithm

Require: A basis of a monomial two-sided ideal $I \subset F$.
Ensure: The multigraded Hilbert series of the monomial algebra $A = F/I$.

$\mathcal{O}_I := \{I\}$, $\mathcal{N} := \{I\}$
matrix $P = (p_{ki}) := 0$, column vector $\mathbf{C}_I = (c_k)^t := 0$
while $\mathcal{N} \neq \emptyset$ **do**
  choose $J \in \mathcal{N}$, $\mathcal{N} := \mathcal{N} \setminus \{J\}$
  $k :=$ position of J in $\mathcal{O}_I$
  if $J \neq \langle 1 \rangle$ **then**
    $c_k = 1$
  for $1 \leq i \leq n$ **do**
    compute the colon right ideal $J_{x_i} := T_{x_i}(J)$
    if $J_{x_i} \notin \mathcal{O}_I$ **then**
      $\mathcal{N} := \mathcal{N} \cup \{J_{x_i}\}$, $\mathcal{O}_I := \mathcal{O}_I \cup \{J_{x_i}\}$
    $p_{ki} :=$ position of J_{x_i} in $\mathcal{O}_I$
$(r \times r)$-matrix $M := 0$ ($r :=$ size of $\mathcal{O}_I$)
unit matrix E_r
for $1 \leq k \leq r$ **do**
  for $1 \leq i \leq n$ **do**
    $M[k, p_{ki}] := M[k, p_{ki}] + t_i$
return H_1, where $\mathbf{H} = (\mathrm{H}_1, \ldots, \mathrm{H}_r)^t$ is a solution of the matrix equation $(E_r - M)\mathbf{H} = \mathbf{C}_I$ over the field $\mathbb{Q}(t_1, \ldots, t_n)$.

If all variables t_i are identified in Step 17 with a single variable t, then the algorithm above computes the graded Hilbert series. User needs to provide the optional parameter "2" for the multigraded Hilbert series since by default the implementation returns the graded one.

8.1. **Computation of Hilbert series of infinitely generated ideals.** If I is a regular but infinitely generated monomial ideal, it is clear that one can provide only the finite set of monomial generators of I up to some fixed total degree d. We remark that this situation is different from the notion of truncation of Section 6 where, at least formally (see Proposition 6.5), one inputs also all monomials of F of degree $d+1$. Thus, for infinitely generated regular ideals, Algorithm 1 essentially guesses the sum of the Hilbert series whose correctness has to be proved by the handling of regular expressions, as we have done for the example in Section 7. In fact, one has a strong indication of the correctness of the computer calculation when this rational function stabilizes as the degree bound d increases. To obtain correct guesses, a modification of the Algorithm 1 is actually required. Observe that each time a colon right ideal operator T_{x_i} $(1 \leq i \leq n)$ is applied to I, one has a complete set of monomial generators of $I_{x_i} = T_{x_i}(I)$ just up to the total degree $d-1$. This is because a generator of I_{x_i} of degree d may arise from a generator of I of degree $d+1$ which has not been included in the input. By iterating the operators T_{x_i}, one has that two right colon ideals $I_w = T_w(I), I_{w'} = T_{w'}(I)$ $(w, w' \in W)$ in the orbit of I can be only compared by means of their monomial generators up to degree $d - d'$, where $d' = \max(\mathrm{tdeg}(w), \mathrm{tdeg}(w'))$. If d is a suitable large bound, then this trick usually provides correct comparisons and hence correct Hilbert series. To access the variant of the Algorithm 1 for infinitely generated ideals, user has to provide an optional parameter "1" along with the input.

8.2. **Computation of affine Hilbert series.** There are many interesting (noncommutative) algebras $A = F/J$ which are not graded ones. Consider, for instance, the group algebras of finitely generated groups. For these algebras, one still has the notion of affine Hilbert function and affine Hilbert series which are also called "growth function and growth series". Precisely, consider $F_{\leq d} = \sum_{0\leq k\leq d} F_d$ the subspace of F of all polynomials of total degree $\leq d$. Moreover, denote $I_{\leq d} = I \cap F_{\leq d}$ and $A_{\leq d} = F_{\leq d}/I_{\leq d}$. The *affine Hilbert function* $\mathrm{HF}'_a(A)$ is defined by putting, for all $d \geq 0$,

$$\mathrm{HF}'_a(A)(d) = \dim A_{\leq d}.$$

The corresponding generating series $\mathrm{HS}'_a(A)$ is called the *affine Hilbert series*. Let $\prec$ be a graded monomial ordering of F, that is, $\mathrm{tdeg}(w) < \mathrm{tdeg}(w')$ $(w, w' \in W)$ implies that $w \prec w'$. As usual, one considers the corresponding monomial algebra $A' = A/I$ where $I = \mathrm{LM}(J)$. By similar arguments to the ones of Theorem 3.2, one proves that the algebras A and A' share the same affine Hilbert function and series (see [18]). Moreover, we immediately have that

$$\mathrm{HS}'_a(A') = \sum_{d\geq 0}\left(\sum_{k=0}^{d} \mathrm{HF}'(A')(k)\right) t^d = \left(\sum_{k\geq 0} t^k\right)\left(\sum_{d\geq 0} \mathrm{HF}'(A')(d) t^d\right)$$

and hence $\mathrm{HS}'_a(A) = \mathrm{HS}'(A')/(1-t)$. Thus, one can easily obtain the affine Hilbert series of A by computing the graded Hilbert series of A'. For the non-homogeneous test cases, we provide the computational timings of $\mathrm{HS}'(A')$.

To show the performance of our implementation, we have carried out the computations on a Dell PowerEdge R720 with two Intel(R) Xeon(R) CPU E5-2690 @ 2.90GHz, 20 MB Cache, 16 Cores, 32 Threads, 192 GB RAM with a Linux operating system (Gentoo). Besides the experimental implementation in [18], this is the first implementation in the kernel of a computer algebra system performing the computation of Hilbert series of noncommutative algebras in general. To test the performance of our algorithms and their implementations, we provide data for graded and multigraded Hilbert series, together with their truncations, for various examples.

In the tables below, we abbreviate milliseconds, seconds, minutes as ms, s, m, respectively. The symbol ∞ indicates that the computation has not been finished within 1 hour. The computing times for graded and multigraded Hilbert series are indicated by HS and mHS, respectively. We denote by $\#\mathcal{O}_I$ the cardinality of the orbit $\mathcal{O}_I$. Moreover, we let $\mathcal{O}_I = \{T_{w_1}(I), \dots, T_{w_r}(I)\}$ ($w_i \in W$), where we assume that if $T_{w_i}(I) = T_{w'_i}(I)$ then $\mathrm{tdeg}(w_i) \leq \mathrm{tdeg}(w'_i)$. We indicate by $\max\{|w|\}$ the maximal total degree of the words $w_1, \dots, w_r$ and by Sol and mSol the cpu timings for solving the linear systems over the rational functions fields involved in the computation of graded and multigraded series, respectively. Since the complexity of a computation depends also on the cardinality of a (minimal) monomial basis of I and the maximum total degree in it, we provide these details, as well, for some examples and we denote them by $\#I$ and $deg(I)$, respectively. The base field $\mathbb{K}$ is always assumed to be the field of rational numbers.

8.3. **Tests of affine Hilbert series.**

Example 8.1. *Here we give the computational details for some classes of non-graded algebras.*

- *Consider the following Coxeter matrices*

$$C_1 = \begin{bmatrix} 1 & 3 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 \\ 3 & 3 & 1 & 3 & 3 \\ 2 & 2 & 3 & 1 & 3 \\ 2 & 2 & 3 & 3 & 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 3 & 2 & 2 \\ 3 & 3 & 1 & 3 & 3 \\ 2 & 2 & 3 & 1 & 3 \\ 3 & 2 & 3 & 3 & 1 \end{bmatrix}$$

For a parameter $\delta \neq 0$, we consider the following two-sided ideals of the free associative algebra $F = \mathbb{K}\langle x_1, \dots, x_5\rangle$

$$\begin{aligned} J_1 = \langle & (x_j - \delta)(x_j + 1)\,(1 \leq j \leq 5), x_1x_2x_1 - x_2x_1x_2, x_1x_3x_1 - x_3x_1x_3, \\ & x_1x_4 - x_4x_1, x_1x_5 - x_5x_1, x_2x_3x_2 - x_3x_2x_3, x_2x_4 - x_4x_2, x_2x_5 \\ & - x_5x_2, x_3x_4x_3 - x_4x_3x_4, x_3x_5x_3 - x_5x_3x_5, x_4x_5x_4 - x_5x_4x_5\rangle, \end{aligned}$$

$$\begin{aligned} J_2 = \langle & (x_j - \delta)(x_j + 1)\,(1 \leq j \leq 5), x_1x_2 - x_2x_1, x_1x_3x_1 - x_3x_1x_3, x_1x_4 \\ & - x_4x_1, x_1x_5 - x_5x_1, x_2x_3x_2 - x_3x_2x_3, x_2x_4 - x_4x_2, x_2x_5 - x_5x_2, \\ & x_3x_4x_3 - x_4x_3x_4, x_3x_5x_3 - x_5x_3x_5, x_4x_5x_4 - x_5x_4x_5\rangle. \end{aligned}$$

Then, the quotient algebras $HA_1 = F/J_1, HA_2 = F/J_2$ are by definition the Hecke algebras corresponding to the matrices C_1, C_2, respectively.

- *Again, let us consider the following Coxeter matrices*

$$C_3 = \begin{bmatrix} 1 & 2 & 2 & 3 & 3 \\ 2 & 1 & 3 & 2 & 2 \\ 2 & 3 & 1 & 3 & 3 \\ 3 & 2 & 3 & 1 & 3 \\ 3 & 2 & 3 & 3 & 1 \end{bmatrix}, C_4 = \begin{bmatrix} 1 & 2 & 2 & 3 & 3 \\ 2 & 1 & 2 & 3 & 2 \\ 2 & 2 & 1 & 3 & 3 \\ 3 & 3 & 3 & 1 & 3 \\ 3 & 2 & 3 & 3 & 1 \end{bmatrix}$$

and the following two-sided ideals of F

$$J_3 = \langle (x_i x_j)^{C_3[i,j]} - 1\,(1 \leq i, j \leq 5)\rangle,$$
$$J_4 = \langle (x_i x_j)^{C_4[i,j]} - 1\,(1 \leq i, j \leq 5)\rangle.$$

The quotient algebras $CA_3 = F/J_3, CA_4 = F/J_4$ are the group algebras of the Coxeter groups defined by the matrices C_3, C_4, respectively.

- *Let $F_n = \mathbb{K}\langle x_1, \ldots, x_{n-1}\rangle$ be the free associative algebra in $n-1$ variables. For a scalar $\lambda \neq 0$, consider the two-sided ideal $L_n \subset F_n$ generated by the following relations*

$$x_i^2 = \lambda x_i \ \forall i,\ x_i x_j = x_j x_i \text{ if } |j-i| > 1,\ x_i x_j x_i = \lambda x_i \text{ if } |j-i| = 1.$$

Then, the quotient algebra $TL_n = F_n/L_n$ is by definition a Temperley-Lieb algebra. We consider TL_{11} and TL_{12} for the computation. These are finite dimensional algebras.

Tests	HS	$\#\mathcal{O}_I$	$\max\{\lvert w\rvert\}$	Sol	#I	deg(I)
HA_1	263 ms	66	10	20 ms	80	14
HA_2	696 ms	87	13	69 ms	121	17
CA_3	1.05 s	88	12	64 ms	155	19
CA_4	1.10 s	92	13	116 ms	147	21
TL_{11}	3.04 s	213	9	25 ms	136	11
TL_{12}	7.03 s	278	10	47 ms	166	12

8.4. **Tests of multigraded Hilbert series.**

Example 8.2 (Relative free algebras)**.** *In these examples, we provide the data for the graded and multigraded Hilbert series of the following* $\mathrm{GL}_n(\mathbb{K})$*-invariant algebras.*

- *The relatively free algebra in n variables that is defined by the Grassmann (or exterior) algebra E. This example has been described in full details in Section 7. We denote by* rf_ext*n this relatively free algebra.*
- *The relatively free algebra in n variables corresponding to the algebra $UT_2(\mathbb{K})$ of 2 by 2 upper triangular matrices. General results for $UT_m(\mathbb{K})$, which were obtained by Maltsev [22], imply that the T-ideal of the polynomial identities satified by $UT_2(\mathbb{K})$ is generated, as a two-sided ideal of $F = \mathbb{K}\langle x_1, \ldots, x_n\rangle$, by the following infinite basis*

$$[x_i, x_j]w[x_k, x_l]$$

where x_i, x_j, x_k, x_l are any variables and w is any word. In the examples under consideration, we assume that w is also a variable, that is, we input the generators of $T(UT_2(\mathbb{K}))$ up to the total degree 5. In fact, this *is enough to obtain the correct multigraded Hilbert series by means of the variant of the Algorithm 1 described in Subsection 8.1. We denote this test set as* rf_tri$n$. *For the reader's convenience, we recall that the formula for the multigraded Hilbert series of the relatively free algebra in n variables of $UT_m(\mathbb{K})$ is the following one (see* [5]*)*

$$\sum_{j=1}^{m} \binom{m}{j} \left(\prod_{i=1}^{n} \frac{1}{1-t_i} \right)^{j} (t_1 + \cdots + t_n - 1)^{j-1}.$$

Tests	HS	mHS	$\#\mathcal{O}_I$	$\max\{\lvert w\rvert\}$	mSol	#I	deg(I)
rf_ext6	441 ms	654 ms	28	4	215 ms	715	7
rf_ext7	2.03 s	4.36 s	39	4	2.33 s	1675	7
rf_tri6	278 ms	1.39 s	13	4	1.11 s	981	5
rf_tri7	903 ms	18.17 s	15	4	17.26 s	2079	5

We remark that the obtained multigraded Hilbert series are rational symmetric functions.

Example 8.3 (Universal enveloping algebras)**.** *We consider the following algebras.*

- *Let $I \subset F = \mathbb{K}\langle x_1, \ldots, x_n \rangle$ be the two-sided ideal that is generated by all the commutators $[x_{i_1}, \ldots, x_{i_d}]$ of length d. We have that F/I is the universal enveloping algebra of the free nilpotent of class $d-1$ Lie algebra with n generators. This is clearly a $\mathrm{GL}_n(\mathbb{K})$-invariant algebra and hence its multigraded Hilbert series is a symmetric function. We denote this algebra by* $n$unil$d$ *when F is generated by n variables and I is generated by d-length commutators.*
- *Another $\mathrm{GL}_n(\mathbb{K})$-invariant two-sided ideal $J \subset F$ is generated by all commutators $[[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]]$ where $1 \leq i_1, i_2, i_3, i_4 \leq n$. This defines F/J as the universal enveloping algebra of the free metabelian Lie algebra with n generators. We denote this example as* umeta$n$ *for the n variables.*

Tests	HS	mHS	$\#\mathcal{O}_I$	$\max\{\lvert w\rvert\}$	mSol	#I	deg(I)
4unil4	345 ms	∞	77	5	∞	110	6
5unil3	20 ms	1.84 m	26	3	1.84 m	50	4
umeta3	< 1 ms	19 ms	7	4	18 ms	3	4
umeta4	4 ms	3.49 s	15	4	3.49 s	15	4

It can be observed from the two tables above that for multigraded Hilbert series most of the computing time is spent in solving the linear system. Maybe this is a side effect of the implementation of multivariate rational function fields in Singular.

8.5. **Tests of truncated Hilbert series.** As we have explained in Section 7, the motivation to develop an algorithm to compute truncated Hilbert series begins with $\mathrm{GL}_n(\mathbb{K})$-invariant algebras and the necessity to obtain the Schur function decomposition of the symmetric function which is the sum of the multigraded series. Since it is difficult to obtain this decomposition when this symmetric function is not a polynomial, we may decide to truncate the Hilbert series at a sufficiently high degree and compute the corresponding partial decomposition in order to guess the complete one. Of course, one may obtain the truncation of a rational Hilbert series as its Taylor approximation up to some fixed degree but sometimes computing such a series may not be feasible, mainly because of two reasons:

- Fairly often, computing a Gröbner basis up to an appropriate high degree required to obtain the correct Hilbert series when the corresponding leading monomial ideal is an infinitely generated regular ideal, is either too costly or not feasible. Even in the case of finitely generated leading monomial ideals, computing Gröbner bases may not be efficient.
- Solving a linear system in a multivariate rational function field (with rational coefficients) may also be unfeasible.

In these cases, our implementation of the algorithm for truncated algebras presented in Section 6, offers a feasible computation of the truncated Hilbert series. We remark that it computes also such a truncation for non-regular monomial algebras, where the sum of the complete Hilbert series cannot be obtained, up to now, by general methods. User needs to provide an optional parameter "$d+1$" to the implementation for obtaining the truncated Hilbert series up to the total degree d. If no such parameter is given to nchilb, it computes the complete Hilbert series.

Furthermore, we notice that the main difference of timings between the graded and multigraded computation is due to the complexity of solving a linear system over multivariate versus univariate function fields. The timings do not differ much for finite dimensional algebras and the truncated case. In fact, for these cases the reduced adjacency matrix is strictly upper triangular and the corresponding linear system is therefore easy to solve.

Now, we provide data of the computation of truncated Hilbert series of the $\mathrm{GL}_n(\mathbb{K})$-invariant algebras that we have considered in Subsection 8.4. We compare our truncation algorithm with an available implementation lpHilbert in SINGULAR (see library fpadim.lib) that computes truncated graded Hilbert series by means of normal words enumeration. In the following tables, tdeg denotes the truncation degree and #mHS indicates the number of monomials in the corresponding truncated multigraded Hilbert series.

Tests	tdeg	lpHilbert	HS	mHS	$\#\mathcal{O}_I$	mSol	#mHS
rf_ext6	7	56.19 s	3.04 s	3.05 s	108	7 ms	1716
rf_ext7	7	5.08 m	11.79 s	11.81 s	143	15 ms	3432
rf_tri6	7	58.13 m	21.19 s	21.20 s	53	3 ms	1716
rf_tri7	7	∞	1.53 m	1.53 m	61	6 ms	3432
5unil5	9	∞	1.62 m	1.62 m	1270	511 ms	2002
6unil6	7	∞	12.27 m	12.07 m	492	14 ms	1716
umeta6	10	∞	13.41 s	13.49 s	365	90 ms	8008
umeta7	10	∞	2.93 m	2.94 m	610	340 ms	19448

By comparing the table above with the tables of Subsection 8.4, one can observe that the computation of the complete Hilbert series may sometimes be faster than computing a truncated one, as for the examples rf_ext$n$ and rf_tri$n$. In many other cases, one has the opposite situation, as for the examples $n$unil$d$ and umeta$n$, where many complete multigraded Hilbert series cannot be computed in a reasonable time.

We conclude with the beatiful picture of the Schur function decomposition of one of the truncated (multigraded) Hilbert series that we have computed. The following decomposition of $\mathrm{HS}(A^{(10)})$ for $A = umeta6$ takes 5 minutes using MAGMA

$1+ \mathrm{S}[1]+ \mathrm{S}[1,1]+ \mathrm{S}[2]+ \mathrm{S}[1,1,1]+2\,\mathrm{S}[2,1]+ \mathrm{S}[3]+ \mathrm{S}[1,1,1,1]+2\,\mathrm{S}[2,1,1]+ 2\,\mathrm{S}[2,2]+3\,\mathrm{S}[3,1]+ \mathrm{S}[4]+ \mathrm{S}[1,1,1,1,1]+2\,\mathrm{S}[2,1,1,1]+3\,\mathrm{S}[2,2,1]+4\,\mathrm{S}[3,1,1]+ 5\,\mathrm{S}[3,2]+4\,\mathrm{S}[4,1]+ \mathrm{S}[5]+ \mathrm{S}[1,1,1,1,1,1]+2\,\mathrm{S}[2,1,1,1,1]+3\,\mathrm{S}[2,2,1,1]+3\,\mathrm{S}[2,2,2]+ 5\,\mathrm{S}[3,1,1,1]+10\,\mathrm{S}[3,2,1]+5\,\mathrm{S}[3,3]+7\,\mathrm{S}[4,1,1]+9\,\mathrm{S}[4,2]+5\,\mathrm{S}[5,1]+ \mathrm{S}[6]+ 2\,\mathrm{S}[2,1,1,1,1,1]+3\,\mathrm{S}[2,2,1,1,1]+4\,\mathrm{S}[2,2,2,1]+5\,\mathrm{S}[3,1,1,1,1]+13\,\mathrm{S}[3,2,1,1]+ 11\,\mathrm{S}[3,2,2]+13\,\mathrm{S}[3,3,1]+10\,\mathrm{S}[4,1,1,1]+23\,\mathrm{S}[4,2,1]+14\,\mathrm{S}[4,3]+11\,\mathrm{S}[5,1,1]+ 14\,\mathrm{S}[5,2]+6\,\mathrm{S}[6,1]+ \mathrm{S}[7]+3\,\mathrm{S}[2,2,1,1,1,1]+4\,\mathrm{S}[2,2,2,1,1]+3\,\mathrm{S}[2,2,2,2]+ 5\,\mathrm{S}[3,1,1,1,1,1]+14\,\mathrm{S}[3,2,1,1,1]+20\,\mathrm{S}[3,2,2,1]+20\,\mathrm{S}[3,3,1,1]+21\,\mathrm{S}[3,3,2]+ 11\,\mathrm{S}[4,1,1,1,1]+37\,\mathrm{S}[4,2,1,1]+30\,\mathrm{S}[4,2,2]+45\,\mathrm{S}[4,3,1]+14\,\mathrm{S}[4,4]+18\,\mathrm{S}[5,1,1,1]+ 44\,\mathrm{S}[5,2,1]+28\,\mathrm{S}[5,3]+16\,\mathrm{S}[6,1,1]+20\,\mathrm{S}[6,2]+7\,\mathrm{S}[7,1]+ \mathrm{S}[8]+4\,\mathrm{S}[2,2,2,1,1,1]+ 4\,\mathrm{S}[2,2,2,2,1]+14\,\mathrm{S}[3,2,1,1,1,1]+25\,\mathrm{S}[3,2,2,1,1]+17\,\mathrm{S}[3,2,2,2]+24\,\mathrm{S}[3,3,1,1,1]+ 48\,\mathrm{S}[3,3,2,1]+19\,\mathrm{S}[3,3,3]+12\,\mathrm{S}[4,1,1,1,1,1]+46\,\mathrm{S}[4,2,1,1,1]+70\,\mathrm{S}[4,2,2,1]+ 84\,\mathrm{S}[4,3,1,1]+86\,\mathrm{S}[4,3,2]+54\,\mathrm{S}[4,4,1]+23\,\mathrm{S}[5,1,1,1,1]+84\,\mathrm{S}[5,2,1,1]+ 67\,\mathrm{S}[5,2,2]+108\,\mathrm{S}[5,3,1]+42\,\mathrm{S}[5,4]+30\,\mathrm{S}[6,1,1,1]+75\,\mathrm{S}[6,2,1]+48\,\mathrm{S}[6,3]+ 22\,\mathrm{S}[7,1,1]+27\,\mathrm{S}[7,2]+8\,\mathrm{S}[8,1]+ \mathrm{S}[9]+4\,\mathrm{S}[2,2,2,2,1,1]+3\,\mathrm{S}[2,2,2,2,2]+ 26\,\mathrm{S}[3,2,2,1,1,1]+30\,\mathrm{S}[3,2,2,2,1]+25\,\mathrm{S}[3,3,1,1,1,1]+69\,\mathrm{S}[3,3,2,1,1]+ 50\,\mathrm{S}[3,3,2,2]+55\,\mathrm{S}[3,3,3,1]+50\,\mathrm{S}[4,2,1,1,1,1]+103\,\mathrm{S}[4,2,2,1,1]+69\,\mathrm{S}[4,2,2,2]+ 117\,\mathrm{S}[4,3,1,1,1]+238\,\mathrm{S}[4,3,2,1]+94\,\mathrm{S}[4,3,3]+117\,\mathrm{S}[4,4,1,1]+127\,\mathrm{S}[4,4,2]+ 27\,\mathrm{S}[5,1,1,1,1,1]+120\,\mathrm{S}[5,2,1,1,1]+186\,\mathrm{S}[5,2,2,1]+237\,\mathrm{S}[5,3,1,1]+238\,\mathrm{S}[5,3,2]+ 190\,\mathrm{S}[5,4,1]+42\,\mathrm{S}[5,5]+44\,\mathrm{S}[6,1,1,1,1]+166\,\mathrm{S}[6,2,1,1]+131\,\mathrm{S}[6,2,2]+ 217\,\mathrm{S}[6,3,1]+90\,\mathrm{S}[6,4]+47\,\mathrm{S}[7,1,1,1]+118\,\mathrm{S}[7,2,1]+75\,\mathrm{S}[7,3]+29\,\mathrm{S}[8,1,1]+ 35\,\mathrm{S}[8,2]+9\,\mathrm{S}[9,1]+ \mathrm{S}[10].$

Remark 8.4. *Note that all timings that have been presented so far, display the computing time for the system command* nc_hilb *of* SINGULAR. *This can be used directly (without* nchilb*) if a finite set of monomials is provided in the letterplace format as an input.*

9. Conclusions and further directions

We believe that the two previous sections clearly show the power and flexibility of the proposed approach to the computation of Hilbert series for noncommutative structures. We plan to further extend our implementation from the case of algebras to the case of (finitely generated) right modules according to Theorem 3.2 and Proposition 3.3. Researchers who have interests also in representation theory may obtain essential information about the decomposition of a $\mathrm{GL}_n(\mathbb{K})$-invariant algebra in terms of its simple $\mathrm{GL}_n(\mathbb{K})$-submodules by combining our algorithms with procedures performing the Schur function decomposition. Moreover, with fast approximations of non-rational Hilbert series at hand, one can venture to enter the intriguing realm of such functions. Finally, note that the iterative design of our algorithms is immediately applicable also to the commutative case and automata theory provides optimality in the number of iterations. Therefore, we suggest to develop commutative variants of the proposed methods and to compare them with the many existing implementations of commutative Hilbert series.

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References

[1] Bayer D.; Stillman M., Computation of Hilbert functions. J. Symbol. Comput., 14 (1992), 31–50.

[2] Benanti, F.; Boumova, S.; Drensky, V.; Genov, G.K.; Koev, P., Computing with rational symmetric functions and applications to invariant theory and PI-algebras. Serdica Math. J., 38 (2012), 137–188.

[3] Berele, A.; Regev, A., Applications of Hook Young Diagrams to P.I. algebras. J. Algebra, 82 (1983), 559–567.

[4] Bigatti, A.M., Computation of Hilbert-Poincaré series. J. Pure Appl. Algebra, 119 (1997), 237–253.

[5] Boumova, S.; Drensky, V., Cocharacters of polynomial identities of upper triangular matrices. J. Algebra Appl., 11 (2012), 1250018, 24 pages.

[6] Cohn, P.M., Free Ideal Rings and Localization in General Rings, New Mathematical Monographs, 3. Cambridge University Press, Cambridge, 2006.

[7] Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H.: Singular 4-1-0 — A computer algebra system for polynomial computations (2016). http://www.singular.uni-kl.de.

[8] de Luca, A., Varricchio, S., Finiteness and Regularity in Semigroups and Formal Languages, Monographs in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 1999.

[9] Drensky, V., Free algebras and PI-algebras. Graduate course in algebra. Springer-Verlag Singapore, Singapore, 2000.

[10] Drensky, V.; La Scala, R., Gröbner bases of ideals invariant under endomorphisms. J. Symbolic Comput., 41 (2006), 835–846.

[11] Fulton, W., Young tableaux. With applications to representation theory and geometry. London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.

[12] Giambruno, A.; Zaicev, M., Polynomial identities and asymptotic methods. Mathematical Surveys and Monographs, 122. American Mathematical Society, Providence, RI, 2005.

[13] Govorov, V.E.; Graded algebras (Russian). Mat. Zametki, 12 (1972), 197–204.

[14] Krakowski, D.; Regev, A., The polynomial identities of the Grassmann algebra. Trans. Amer. Math. Soc., 181 (1973), 429–438.

[15] La Scala, R.; Levandovskyy, V., Letterplace ideals and non-commutative Gröbner bases. J. Symbolic Comput., 44 (2009), 1374–1393.

[16] La Scala, R.; Levandovskyy, V., Skew polynomial rings, Gröbner bases and the letterplace embedding of the free associative algebra. J. Symbolic Comput., 48 (2013), 110–131.

[17] La Scala, R.; Extended letterplace correspondence for nongraded noncommutative ideals and related algorithms. Internat. J. Algebra Comput., 24 (2014), 1157–1182.

[18] La Scala R., Monomial Right Ideal and the Hilbert Series of Noncommutative Modules. J. Symb. Comput., 80 (2017), 403–415.

[19] La Scala, R., Computing minimal free resolutions of right modules over noncommutative algebras. J. Algebra, 478 (2017), 458–483.

[20] Latyshev, V.N., On the choice of basis in a T-ideal (Russian). Sib. Mat. Zh., 4 (1963), 1122–1127.

[21] Laue, R., SYMMETRICA.
http://www.algorithm.uni-bayreuth.de/en/research/SYMMETRICA.

[22] Maltsev, Yu.N., A basis for the identities of the algebra of upper triangular matrices (Russian). Algebra i Logika, 10, (1971), 393–400. Translation: Algebra and Logic, 10 (1971), 242–247.

[23] Stembridge, J., John Stembridge's Maple packages for symmetric functions, posets, root systems, and finite Coxeter groups.
http://www.math.lsa.umich.edu/~jrs/maple.html.

[24] Ufnarovski, V.A, A growth criterion for graphs and algebras defined by words. Math. Notes, 31 (1982), 238–241.

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