SIMPLE CURRENT EXTENSIONS OF VERTEX OPERATOR ALGEBRAS BY UNITARY MODULES

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ABSTRACT. In this paper, a condition making vertex operator superalgebras to be unitary is determined and an analogue of conformal spin-statistics theorem in conformal field theory is proved. As an application of these results, it is proved that under some assumptions there exist vertex operator algebra structures on the direct sum of simple current unitary modules of unitary vertex operator algebras.

1. Introduction

Simple current extension is one of important ways to construct new vertex operator algebras. More precisely, let V be a strongly regular vertex operator algebra, then a vertex operator algebra U is called a simple current extension of V if V is a vertex operator subalgebra of U and U viewed as a V-module has the decomposition $U = M^0 \oplus M^1 \oplus \cdots \oplus M^k$ such that M^0, M^1, \cdots, M^k are simple current V-modules [DLM1]. Many important vertex operator algebras can be constructed by simple current extension, for example, the Moonshine vertex operator algebra, code vertex operator algebras and framed vertex operator algebras are simple current extensions of tensor product of Virasoro vertex operator algebras [M], [DGH]. It is also known that simple current extensions of vertex operator algebras have good properties [Y1].

Thus, it is meaningful to determine the conditions such that simple current extensions of vertex operator algebras exist. There have been many works about constructing simple current extensions, for example, simple current extensions of affine vertex operator algebras and Virasoro vertex operator algebras have been constructed in [DLM1], [L1], [LLY]; Lam and Yamauchi have proved under some assumptions that there is a vertex operator algebra structure on the direct sum of simple current modules if the simple current modules form an order two group under tensor product [LY]; Carnahan has proved that there is a vertex operator algebra structure on the direct sum of simple current modules if the evenness conjecture is true [C]; Creutzig, Kanade and Linshaw have proved that there is a vertex operator superalgebra on the direct sum of simple current modules if the simple current modules form a cyclic group under tensor product [CKL]. In this paper, we shall provide such a condition which works in the case that the simple current modules form a general group.

To explain our conditions, we first recall some definitions about unitary vertex operator algebras [DLin]. A vertex operator algebra V is called unitary if there exist an anti-linear involution ϕ of V and a positive definite Hermitian form $(,)_V$ on V such that the following invariant property

$$(Y(u,x)v,w)_V = (v,Y(e^{xL(1)}(-x^{-2})^{L(0)}\phi(u),x^{-1})w)_V$$

holds for any $u, v, w \in V$ (cf. [DLin]). For a unitary vertex operator algebra $(V, \phi, (,)_V)$, a V-module (M, Y_M) is called unitary if there exists a positive definite Hermitian form $(,)_M$ on M such that

$$(Y_M(u,x)w_1,w_2)_M = (w_1,Y_M(e^{xL(1)}(-x^{-2})^{L(0)}\phi(u),x^{-1})w_2)_M$$

holds for any $u \in V, w_1, w_2 \in M$. Then our conditions are as follows: (i) V is a strongly regular, unitary vertex operator algebra; (ii) $M^0 = V, M^1, \dots, M^k$ are unitary, simple current V-modules such that $M^0 = V, M^1, \dots, M^k$ form a group under tensor product and have integral conformal weights. We shall prove under assumptions (i), (ii) that $M^0 \oplus M^1 \oplus \dots \oplus M^k$ is a vertex operator algebra (see Theorem 3.9).

We next explain the main ideas of the proof. Let V be a strongly regular, unitary vertex operator algebra, $M^0 = V, M^1, \dots, M^k$ be unitary, simple current V-modules such that M^0, M^1, \dots, M^k have integral conformal weights and form a group \mathcal{M} under tensor product. The proof is by induction on the group \mathcal{M} . We first consider the case that \mathcal{M} is cyclic, it is known [CKL] that $M^0 \oplus M^1 \oplus \cdots \oplus M^k$ is a vertex operator superalgebra (see Theorem 3.5). We then need to prove that this is indeed a vertex operator algebra. We first prove that this is a unitary vertex operator superalgebra. Actually, this follows from a general result which states that a vertex operator superalgebra V equipped with a positive definite Hermitian form $(,)_V$ satisfying $(L(n)v,w)_V=(v,L(-n)w)_V$ for any $v, w \in V$ and $n \in \mathbb{Z}$ is unitary (see Theorem 2.10). This result has its own interest, it is known [CKLW] that there are close relations between conformal nets and unitary vertex operator algebras. Thus, it is meaningful to determine conditions making vertex operator algebras to be unitary. Our result provides a simple condition making vertex operator algebras to be unitary. To complete the proof, we shall prove an analogue of the conformal spin-statistics theorem, which states that for a unitary vertex operator superalgebra $V=V_{\bar 0}\oplus V_{\bar 1},$ the elements in $V_{\bar 0}$ (resp. $V_{\bar 1})$ must have integral (resp. half-integral) weights (see Theorem 2.11).

The organization of this paper is as follows. In Section 2, we define unitary vertex operator superalgebras and then prove some properties of unitary vertex operator superalgebras. A conditions making vertex operator superalgebras to be unitary is determined and an analogue of conformal spin-statistics theorem in conformal field theory is proved. By applying the results in Section 2, we shall prove in Section 3 that there

exist vertex operator algebra structures on the direct sum of simple current unitary modules of unitary vertex operator algebras.

2. Unitary vertex operator superalgebras

2.1. Basic facts about vertex operator superalgebras. In this subsection, we recall some basic facts about vertex superalgebras from [FHL], [K], [L2]. Let $\mathbb{C}[x_1, x_2]_S$ be the ring of rational functions obtained by inverting the product of elements of the set S of nonzero homogeneous linear polynomials in x_1 and x_2 . We will use ι_{x_1,x_2} to denote the operation of expanding an element of $\mathbb{C}[x_1, x_2]_S$ as a formal series containing at most finitely many negative powers of x_2 ; similarly for ι_{x_2,x_1} . For a \mathbb{Z}_2 -graded vector space $V = V_{\overline{0}} \oplus V_{\overline{1}}$, the elements in $V_{\overline{0}}$ (resp. $V_{\overline{1}}$) are called even (resp. odd). For any $u \in V_{\overline{i}}$, $v \in V_{\overline{j}}$ with i, j = 0, 1, we define $\tilde{v} = j$ and $\epsilon_{u,v} = (-1)^{\tilde{u}\tilde{v}}$.

A vertex superalgebra is a quadruple $(V, Y, \mathbf{1}, \partial_V)$, where $V = V_{\overline{0}} \oplus V_{\overline{1}}$ is a \mathbb{Z}_2 -graded vector space, $\mathbf{1} \in V_{\overline{0}}$ is the vacuum vector of V, ∂_V is a parity preserving endomorphism of V, and Y is a linear map

$$Y: V \to (\operatorname{End} V)[[x, x^{-1}]],$$

$$v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \ v_n \in \operatorname{End} V$$

satisfying the following axioms:

- (i) For any $u \in V_{\overline{i}}, v \in V_{\overline{i}}, Y(u, x)v \in V_{\overline{i+i}}((x))$.
- (ii) $Y(1, x) = id_V$.
- (iii) $Y(v,x)\mathbf{1} \in V[[x]]$ and $Y(v,x)\mathbf{1}|_{x=0} = v$ for any $v \in V$.
- (iv) $[\partial_V, Y(v, x)] = Y(\partial_V v, x) = \frac{d}{dx}Y(v, x).$
- (v) The Jacobi identity for \mathbb{Z}_2 -homogeneous $u, v \in V$ holds,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - \epsilon_{u,v}x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1)$$

$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u, x_0)v, x_2).$$

We will denote the vertex superalgebra briefly by V. It is known [FHL], [LL], [L2] that for any \mathbb{Z}_2 -homogeneous elements $u, v \in V$, there exists a positive integer k such that

$$(x_1 - x_2)^k Y(u, x_1) Y(v, x_2) = \epsilon_{u,v} (x_1 - x_2)^k Y(v, x_2) Y(u, x_1).$$
(2.1)

The operators $Y(u, x_1)$ and $Y(v, x_2)$ satisfying (2.1) are said to be *mutually local*. Conversely, the following theorem allows one to construct vertex superalgebras (see [K]).

Theorem 2.1. Let V be a \mathbb{Z}_2 -graded vector space, $\mathbf{1}$ be an even vector of V and ∂_V an even endomorphism of V. Assume that $\{a^{\alpha}(x) = \sum_{n \in \mathbb{Z}} a_n^{\alpha} x^{-n-1}\}_{\alpha \in I}$ is a set of \mathbb{Z}_2 -homogeneous operators on V such that

- (a) $[\partial_V, a^{\alpha}(x)] = \partial_x a^{\alpha}(x)$.
- (b) $\partial_V \mathbf{1} = 0$, $a^{\alpha}(x)\mathbf{1}|_{x=0} = a^{\alpha}$, $\alpha \in I$, where a^{α} are linearly independent in V.
- (c) $a^{\alpha}(x)$ and $a^{\beta}(x)$, $\alpha, \beta \in I$, are mutually local.
- (d) The vectors $a_{j_1}^{\alpha_1}...a_{j_n}^{\alpha_n}\mathbf{1}$ span V.

Then there is a unique vertex superalgebra structure on V such that $\mathbf{1}$ is the vacuum vector, ∂_V is the derivation and $Y(a^{\alpha}, x) = a^{\alpha}(x)$.

A vertex superalgebra $(V, Y, \mathbf{1}, \partial_V)$ is called a *vertex operator superalgebra* if there is an additional element $\omega \in V_{\bar{0}}$, which is called the *conformal vector* of V, such that the following three conditions hold:

- (i) V has the decomposition $V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n$ such that dim $V_n < \infty$ for any n and $V_n = 0$ if n is sufficiently small.
- (ii) The component operators of $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2}$ satisfy the Virasoro algebra relation with central charge $c \in \mathbb{C}$:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c,$$

and

$$L(0)|_{V_n} = n, n \in \frac{1}{2}\mathbb{Z}.$$

(iii) $L(-1) = \partial_V$. For a vector v in V_n , we define the *conformal weight* wtv of v to be n, and v is called a *quasiprimary vector* if L(1)v = 0. If $V = \bigoplus_{n \in \mathbb{Z}} V_n$ and $V_{\overline{1}} = 0$, then V is called a *vertex operator algebra*, and a vertex operator algebra V is called of CFT-type if $V = \bigoplus_{n>0} V_n$ and $\dim V_0 = 1$.

Remark 2.2. Here we adopt the definition of vertex operator superalgebras in [L2], which is slightly different from those in [KW], [DZ1] and [DZ2]. The elements in $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are not demanded to have integral conformal weights (resp. half-integral conformal weights).

We will need the following properties about vertex operator superalgebras, which can be proved by the similar arguments in the proof of formulas (2.3.17), (5.2.37), (5.2.38), (5.3.1) of [FHL].

Theorem 2.3. Let V be a vertex operator superalgebra. Then for any $v \in V$ we have

$$e^{x_0L(-1)}Y(v,x)e^{-x_0L(-1)} = Y(v,x+x_0), (2.2)$$

$$x^{L(0)}Y(v,x_0)x^{-L(0)} = Y(x^{L(0)}v,xx_0), (2.3)$$

$$(\zeta x^2)^{L(0)} e^{xL(1)} (\zeta x^2)^{-L(0)} e^{x^{-1}L(1)} = 1, \tag{2.4}$$

$$e^{xL(1)}Y(u,x_0)e^{-xL(1)} = Y(e^{x(1-xx_0)L(1)}(1-xx_0)^{-2L(0)}u,x_0/(1-xx_0)),$$
(2.5)

where ζ denotes the complex number $e^{i\pi}$ and ι denotes the imaginary unit.

We next consider modules of vertex operator superalgebras. Let V be a vertex operator superalgebra. A weak V-module is a vector space M equipped with a linear map

$$Y_M: V \to (\operatorname{End} M)\{x\},$$

$$v \mapsto Y_M(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \ v_n \in \operatorname{End} M$$

such that the following conditions hold:

- (i) $v_n w = 0$ if n is sufficiently large.
- (ii) $Y_M(\mathbf{1}, x) = \mathrm{id}_M$.
- (iii) The Jacobi identity holds for $u, v \in V$

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_M(u,x_1)Y_M(v,x_2) - \epsilon_{u,v}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_M(v,x_2)Y_M(u,x_1)$$

$$= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_M(Y(u,x_0)v,x_2).$$

A weak V-module M is called a V-module if M has the decomposition $M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}$ such that $M_{\lambda} = \{ w \in M | L(0)w = \lambda w \}$ and is finite dimensional, and for any $\lambda \in \mathbb{C}$, $M_{\lambda+n} = 0$ for sufficiently small integer n.

For a V-module $M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}$, consider the restricted dual $M' = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}^*$. Define a linear map $Y'_M : V \to (\operatorname{End} M')\{x\}$ by

$$\langle Y_M'(v,x)w',w\rangle = \langle w', Y_M(e^{xL(1)}(\zeta x^{-2})^{L(0)}\iota^{\tilde{v}}v,x^{-1})w\rangle$$

for $v \in V, w' \in M'$ and $w \in M$.

Theorem 2.4. Let V be a vertex operator superalgebra, M be a V-module. Then M' equipped with the linear map Y'_M is also a V-module.

Proof: The argument is similar to that in the proof of Theorem 5.2.1 of [FHL]. By the argument in the proof of Lemma 2.2 of [DLM2], we only need to prove that the Jacobi identity holds for any $u, v \in V$ and $w' \in M'$:

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_M'(u,x_1)Y_M'(v,x_2)w' - \epsilon_{u,v}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_M'(v,x_2)Y_M'(u,x_1)w'$$

$$= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_M'(Y(u,x_0)v,x_2)w'.$$

By the definition of Y'_M , we have for any $w \in M$,

$$\begin{split} &\langle x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_M'(u,x_1)Y_M'(v,x_2)w',w\rangle\\ &=\langle w',x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_M(e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)}\iota^{\tilde{v}}v,x_2^{-1})Y_M(e^{x_1L(1)}(\zeta x_1^{-2})^{L(0)}\iota^{\tilde{u}}u,x_1^{-1})w\rangle, \end{split}$$

$$\begin{split} &\langle x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_M'(v, x_2) Y_M'(u, x_1) w', w \rangle \\ &= \langle w', x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_M(e^{x_1 L(1)} (\zeta x_1^{-2})^{L(0)} \iota^{\tilde{u}} u, x_1^{-1}) Y_M(e^{x_2 L(1)} (\zeta x_2^{-2})^{L(0)} \iota^{\tilde{v}} v, x_2^{-1}) w \rangle, \end{split}$$

and

$$\langle x_{2}^{-1}\delta\left(\frac{x_{1}-x_{0}}{x_{2}}\right)Y'_{M}(Y(u,x_{0})v,x_{2})w',w\rangle$$

$$=\langle w',x_{2}^{-1}\delta\left(\frac{x_{1}-x_{0}}{x_{2}}\right)Y_{M}(e^{x_{2}L(1)}(\zeta x_{2}^{-2})^{L(0)}\iota^{\tau(u,v)}Y(u,x_{0})v,x_{2}^{-1})w\rangle$$

$$=\langle w',x_{1}^{-1}\delta\left(\frac{x_{2}+x_{0}}{x_{1}}\right)Y_{M}(e^{x_{2}L(1)}(\zeta x_{2}^{-2})^{L(0)}\iota^{\tau(u,v)}Y(u,x_{0})v,x_{2}^{-1})w\rangle,$$

where $\tau(u,v)=0$ if $u\in V_{\bar{0}},v\in V_{\bar{0}}$ or $u\in V_{\bar{1}},v\in V_{\bar{1}};$ and $\tau(u,v)=1$ if $u\in V_{\bar{0}},v\in V_{\bar{1}}$ or $u\in V_{\bar{1}},v\in V_{\bar{0}}.$ On the other hand, we have

$$\left(-\frac{x_0}{x_1x_2}\right)^{-1} \delta\left(\frac{x_1^{-1} - x_2^{-1}}{-x_0/x_1x_2}\right) Y_M(e^{x_1L(1)}(\zeta x_1^{-2})^{L(0)} \iota^{\tilde{u}} u, x_1^{-1}) Y_M(e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)} \iota^{\tilde{v}} v, x_2^{-1})
- \epsilon_{u,v} \left(-\frac{x_0}{x_1x_2}\right)^{-1} \delta\left(\frac{x_2^{-1} - x_1^{-1}}{x_0/x_1x_2}\right) Y_M(e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)} \iota^{\tilde{v}} v, x_2^{-1})
\cdot Y_M(e^{x_1L(1)}(\zeta x_1^{-2})^{L(0)} \iota^{\tilde{u}} u, x_1^{-1})
= x_2 \delta\left(\frac{x_2 + x_0}{x_1}\right) Y_M(Y(e^{x_1L(1)}(\zeta x_1^{-2})^{L(0)} \iota^{\tilde{u}} u, -x_0/x_1x_2) e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)} \iota^{\tilde{v}} v, x_2^{-1}),$$

which is equivalent to

$$\begin{split} &(-x_0)^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_M(e^{x_1L(1)}(\zeta x_1^{-2})^{L(0)}\iota^{\tilde{u}}u,x_1^{-1})Y_M(e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)}\iota^{\tilde{v}}v,x_2^{-1})\\ &-\epsilon_{u,v}(-x_0)^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_M(e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)}\iota^{\tilde{v}}v,x_2^{-1})Y_M(e^{x_1L(1)}(\zeta x_1^{-2})^{L(0)}\iota^{\tilde{u}}u,x_1^{-1})\\ &=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_M(Y(e^{x_1L(1)}(\zeta x_1^{-2})^{L(0)}\iota^{\tilde{u}}u,-x_0/x_1x_2)e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)}\iota^{\tilde{v}}v,x_2^{-1}). \end{split}$$

Thus, we only need to prove the following identity holds for any $u, v \in V$:

$$\begin{split} &\epsilon_{u,v}Y_{M}(e^{x_{2}L(1)}(\zeta x_{2}^{-2})^{L(0)}\iota^{\tau(u,v)}Y(u,x_{0})v,x_{2}^{-1})\\ &=Y_{M}(Y(e^{(x_{2}+x_{0})L(1)}(\zeta(x_{2}+x_{0})^{-2})^{L(0)}\iota^{\tilde{u}}u,-x_{0}/(x_{2}+x_{0})x_{2})e^{x_{2}L(1)}(\zeta x_{2}^{-2})^{L(0)}\iota^{\tilde{v}}v,x_{2}^{-1}). \end{split}$$

Note that for any $u, v \in V$, we have $\epsilon_{u,v} \iota^{\tau(u,v)} = \iota^{\tilde{u}} \iota^{\tilde{v}}$. Then we only need to prove the following identity holds for any $u \in V$:

$$e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)}Y(u,x_0)(\zeta x_2^{-2})^{-L(0)}e^{-x_2L(1)}$$

$$=Y(e^{(x_2+x_0)L(1)}(\zeta(x_2+x_0)^{-2})^{L(0)}u,-x_0/(x_2+x_0)x_2).$$

By the formulas (2.3) and (2.5), we have

$$\begin{split} &e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)}Y(u,x_0)(\zeta x_2^{-2})^{-L(0)}e^{-x_2L(1)}\\ &=e^{x_2L(1)}Y((\zeta x_2^{-2})^{L(0)}u,\zeta x_2^{-2}x_0)e^{-x_2L(1)}\\ &=Y(e^{x_2(1-x_2\zeta x_2^{-2}x_0)L(1)}(1-x_2\zeta x_2^{-2}x_0)^{-2L(0)}(\zeta x_2^{-2})^{L(0)}u,\zeta x_2^{-2}x_0/(1-x_2\zeta x_2^{-2}x_0))\\ &=Y(e^{x_2(1+x_2^{-1}x_0)L(1)}(1+x_2^{-1}x_0)^{-2L(0)}(\zeta x_2^{-2})^{L(0)}u,\zeta x_2^{-2}x_0/(1+x_2^{-1}x_0))\\ &=Y(e^{(x_2+x_0)L(1)}(\zeta(x_2+x_0)^{-2})^{L(0)}u,-x_0/(x_2+x_0)x_2). \end{split}$$

This completes the proof.

Remark 2.5. If V is a vertex operator superalgebra defined in [KW], then Theorem 2.4 has been proved in [Y2].

For a vertex operator algebra V, a V-module M is called *self dual* if M' viewed as a V-module is isomorphic to M.

2.2. Properties of unitary vertex operator superalgebras. In this subsection, we shall consider unitary structures of vertex operator superalgebras. Let V be a vertex operator superalgebra. An anti-linear map $\phi: V \to V$ is called an anti-linear involution of V if $\phi^2 = \mathrm{id}$, $\phi(\mathbf{1}) = \mathbf{1}$, $\phi(\omega) = \omega$ and $\phi(u_n v) = \phi(u)_n \phi(v)$ for any $u, v \in V$. A vertex operator superalgebra V is called unitary if there exist an anti-linear involution ϕ of V and a positive definite Hermitian form $(,)_V$ on V such that the following invariant property:

$$(Y(u,x)v,w)_V = (v,Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\iota^{\tilde{u}}\phi(u),x^{-1})w)_V$$

holds for any $u, v, w \in V$.

Remark 2.6. The definition of unitary vertex operator superalgebras is a slight generalization of that in [AL].

Proposition 2.7. Let $(V, \phi, (,)_V)$ be a unitary vertex operator superalgebra. Then $(Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\iota^{\tilde{u}}\phi(u), x^{-1})v, w)_V = (v, Y(u, x)w)_V$ holds for any $u, v, w \in V$.

Proof: This follows immediately from the following property of Hermitian form $(,)_V$: $(u,v)_V = \overline{(v,u)_V}$ for any $u,v \in V$.

As a corollary, we have

Corollary 2.8. Let $(V, \phi, (,)_V)$ be a unitary vertex operator superalgebra. Then ϕ is an even map, that is, $\widetilde{\phi(u)} = \widetilde{u}$ for any \mathbb{Z}_2 -homogeneous vector $u \in V$.

Proof: By the definition of unitary vertex operator superalgebras and the formula (2.4), we have

$$(Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\iota^{\tilde{u}}\phi(u), x^{-1})v, w)_{V}$$

$$= (v, Y(e^{x^{-1}L(1)}(\zeta x^{2})^{L(0)}\iota^{\widetilde{\phi(u)}}e^{xL(1)}x^{-2L(0)}\overline{\zeta^{L(0)}\iota^{\tilde{u}}}u, x)w)_{V}$$

$$= \overline{\iota^{\widetilde{\phi(u)}}}\iota^{\tilde{u}}(v, Y(u, x)w)_{V}.$$

This implies that $\widetilde{\iota^{\phi(u)}}\iota^{\tilde{u}}=1$ for any \mathbb{Z}_2 -homogeneous vector $u\in V$. Hence, $\widetilde{\phi(u)}=\tilde{u}$ for any \mathbb{Z}_2 -homogeneous vector $u\in V$.

We will need the following property of unitary vertex operator superalgebras.

Proposition 2.9. Let $(V, \phi, (,)_V)$ be a unitary vertex operator superalgebra. Then for any $w \in V$ we have $(\phi(w), \phi(w))_V = (w, w)_V$.

Proof: We first prove that for any $u \in V$, $(\zeta x^2)^{-L(0)} e^{xL(-1)} (\zeta x^2)^{L(0)} = e^{-x^{-1}L(-1)}$. We only need to prove that $(\zeta x^2)^{-L(0)} x L(-1) (\zeta x^2)^{L(0)} = \zeta x^{-1} L(-1)$. This immediately follows from the fact $x_0^{-L(0)} L(-1) x_0^{L(0)} = x_0^{-1} L(-1)$. By the formulas (2.3), (2.4),

$$\begin{split} &(\phi(v),\phi(w))_{V}\\ &=(e^{x^{-1}L(-1)}Y(\phi(v),-x^{-1})\mathbf{1},\phi(w))_{V}\\ &=(e^{x^{-1}L(-1)}Y(\phi(v),-x^{-1})(\zeta x^{2})^{-L(0)}e^{-x^{-1}L(1)}\mathbf{1},\phi(w))_{V}\\ &=(e^{x^{-1}L(-1)}(\zeta x^{2})^{-L(0)}Y((\zeta x^{2})^{L(0)}\phi(v),-\zeta x^{2}x^{-1})e^{-x^{-1}L(1)}\mathbf{1},\phi(w))_{V}\\ &=((\zeta x^{2})^{-L(0)}e^{-xL(-1)}Y((\zeta x^{2})^{L(0)}\phi(v),x)e^{-x^{-1}L(1)}\mathbf{1},\phi(w))_{V}\\ &=(Y((\zeta x^{2})^{L(0)}\phi(v),x)e^{-x^{-1}L(1)}\mathbf{1},e^{-xL(1)}\overline{\zeta^{-L(0)}}x^{-2L(0)}\phi(w))_{V}\\ &=(Y(e^{x^{-1}L(1)}(\zeta x^{2})^{L(0)}e^{xL(1)}\phi(v),x)e^{-x^{-1}L(1)}\mathbf{1},e^{-xL(1)}\overline{\zeta^{-L(0)}}x^{-2L(0)}\phi(w))_{V}\\ &=\iota^{-\tilde{v}}(e^{-x^{-1}L(1)}\mathbf{1},Y(e^{xL(1)}v,x^{-1})e^{-xL(1)}(\zeta x^{-2})^{L(0)}\phi(w))_{V}\\ &=\iota^{-\tilde{v}}\epsilon_{w,v}\overline{\iota^{-\tilde{w}}}(Y(\overline{(-1)^{2L(0)}}w,-x)\mathbf{1},e^{xL(1)}v)_{V}\\ &=\iota^{-\tilde{v}}\epsilon_{w,v}\overline{\iota^{-\tilde{w}}}(Y(\overline{(-1)^{2L(0)}}w,-x)\mathbf{1},e^{xL(1)}v)_{V}\\ &=\iota^{-\tilde{v}}\epsilon_{w,v}\overline{\iota^{-\tilde{w}}}(\overline{(-1)^{2L(0)}}w,v)_{V}. \end{split}$$

Since $(,)_V$ is positive definite, it follows immediately that $(\phi(w), \phi(w))_V = (w, w)_V$ for any $w \in V$. This completes the proof.

We next provide a condition making vertex operator superalgebras to be unitary.

Theorem 2.10. Let V be a vertex operator superalgebra. Assume that there exists a positive definite Hermitian form $(,)_V$ on V such that (u,v)=0 if $\tilde{u}\neq\tilde{v}$ and that $(L(n)u,v)_V=(u,L(-n)v)_V$ for any $u,v\in V$. Then V is a unitary vertex operator superalgebra.

Proof: Define a map $\phi: V \to V$ by

$$(Y(\phi(u), x)v, w)_V = (v, Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\iota^{\tilde{u}}u, x^{-1})w)_V$$

for any $u, v, w \in V$. It is sufficient to prove that ϕ is an anti-linear involution of V. We first prove that ϕ is an anti-linear even map. By definition, we have

$$(Y(\phi(\lambda u), x)v, w)_{V}$$

$$= (v, Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\iota^{\tilde{u}}\lambda u, x^{-1})w)_{V}$$

$$= \bar{\lambda}(v, Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\iota^{\tilde{u}}u, x^{-1})w)_{V}$$

$$= (Y(\bar{\lambda}\phi(u), x)v, w)_{V}.$$

This implies that $\phi(\lambda u) = \bar{\lambda}\phi(u)$ for any $u \in V$, as desired. We now prove that ϕ is an even map. Otherwise, assume that there exists an element $u \in V$ such that $\phi(u) \neq \tilde{u}$. Then by assumption we have

$$(Y(\phi(u), x)\mathbf{1}, \phi(u))_V = (\mathbf{1}, Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\iota^{\tilde{u}}u, x^{-1})\phi(u))_V = 0.$$

This is a contradiction.

We next prove that $\phi(\mathbf{1}) = \mathbf{1}$ and $\phi(\omega) = \omega$. By definition, we have

$$(Y(\phi(\mathbf{1}), x)v, w)_{V}$$

$$= (v, Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\mathbf{1}, x^{-1})w)_{V}$$

$$= (v, Y(\mathbf{1}, x^{-1})w)_{V} = (v, w)_{V},$$

and

$$(Y(\phi(\omega), x)v, w)_{V}$$

$$= (v, Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\omega, x^{-1})w)_{V}$$

$$= (v, Y(x^{-4}\omega, x^{-1})w)_{V}$$

$$= \sum_{n \in \mathbb{Z}} x^{n-2}(v, L(n)w)_{V}$$

$$= \sum_{n \in \mathbb{Z}} x^{n-2}(L(-n)v, w)_{V}$$

$$= (Y(\omega, x)v, w)_{V}.$$

These imply that $\phi(\mathbf{1}) = \mathbf{1}$ and $\phi(\omega) = \omega$.

We now prove that $\phi(u_n v) = \phi(u)_n \phi(v)$ for any $u, v \in V$ and $n \in \mathbb{Z}$. By definition,

$$(Y(\phi(u_nv), x)w_1, w_2)_V = (w_1, Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\iota^{\widetilde{u_nv}}u_nv, x^{-1})w_2)_V$$

On the other hand, from the proof of Theorem 2.4, we have the following identity

$$\begin{split} &\epsilon_{u,v}(-x_0)^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(e^{x_1L(1)}(\zeta x_1^{-2})^{L(0)}\iota^{\tilde{u}}u,x_1^{-1})Y(e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)}\iota^{\tilde{v}}v,x_2^{-1})\\ &+x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)}\iota^{\tilde{v}}v,x_2^{-1})Y(e^{x_1L(1)}(\zeta x_1^{-2})^{L(0)}\iota^{\tilde{u}}u,x_1^{-1})\\ &=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y(e^{x_2L(1)}(\zeta x_2^{-2})^{L(0)}\iota^{\tau(u,v)}Y(u,x_0)v,x_2^{-1}). \end{split}$$

Thus, we have

$$(Y(\phi(u)_{n}\phi(v), x)w_{1}, w_{2})_{V}$$

$$= \operatorname{Res}_{x_{1}} \operatorname{Res}_{x_{0}} x_{0}^{n} (x_{0}^{-1}\delta\left(\frac{x_{1}-x}{x_{0}}\right) Y(\phi(u), x_{1}) Y(\phi(v), x)w_{1}, w_{2})_{V}$$

$$- \epsilon_{u,v} \operatorname{Res}_{x_{1}} \operatorname{Res}_{x_{0}} x_{0}^{n} (x_{0}^{-1}\delta\left(\frac{x-x_{1}}{-x_{0}}\right) Y(\phi(v), x) Y(\phi(u), x_{1})w_{1}, w_{2})_{V}$$

$$= \operatorname{Res}_{x_{1}} \operatorname{Res}_{x_{0}} x_{0}^{n} (w_{1}, x_{0}^{-1}\delta\left(\frac{x_{1}-x}{x_{0}}\right) Y(e^{xL(1)}(\zeta x^{-2})^{L(0)} \iota^{\tilde{v}}v, x^{-1})$$

$$\cdot Y(e^{x_{1}L(1)}(\zeta x_{1}^{-2})^{L(0)} \iota^{\tilde{u}}u, x_{1}^{-1})w_{2})_{V} - \epsilon_{u,v} \operatorname{Res}_{x_{1}} \operatorname{Res}_{x_{0}} x_{0}^{n} (w_{1}, x_{0}^{-1}\delta\left(\frac{x-x_{1}}{-x_{0}}\right)$$

$$\cdot Y(e^{x_{1}L(1)}(\zeta x_{1}^{-2})^{L(0)} \iota^{\tilde{u}}u, x_{1}^{-1}) Y(e^{xL(1)}(\zeta x^{-2})^{L(0)} \iota^{\tilde{v}}v, x^{-1})w_{2})_{V}$$

$$= \operatorname{Res}_{x_{1}} \operatorname{Res}_{x_{0}} x_{0}^{n} (w_{1}, x_{1}^{-1}\delta\left(\frac{x+x_{0}}{x_{1}}\right) Y(e^{xL(1)}(\zeta x^{-2})^{L(0)} \iota^{\tau(u,v)} Y(u, x_{0})v, x^{-1})w_{2})_{V}$$

$$= (w_{1}, Y(e^{xL(1)}(\zeta x^{-2})^{L(0)} \iota^{\tilde{u}_{n}\tilde{v}}u_{n}v, x^{-1})w_{2})_{V}$$

$$= (Y(\phi(u_{n}v), x)w_{1}, w_{2})_{V}.$$

This implies that $\phi(u_n v) = \phi(u)_n \phi(v)$, as desired.

Finally, we prove that ϕ is an anti-linear involution. By definition,

$$(Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\iota^{\tilde{u}}\phi(u), x^{-1})v, w)_{V}$$

$$= (v, Y(e^{x^{-1}L(1)}(\zeta x^{2})^{L(0)}\iota^{\tilde{u}}e^{xL(1)}x^{-2L(0)}\overline{\zeta^{L(0)}\iota^{\tilde{u}}}u, x)w)_{V}$$

$$= (v, Y(u, x)w)_{V}.$$

This implies that for any $u, v, w \in V$

$$(Y(u, x)w, v)_{V}$$

$$= (w, Y(e^{xL(1)}(\zeta x^{-2})^{L(0)}\iota^{\tilde{u}}\phi(u), x^{-1})v)_{V}$$

$$= (Y(\phi^{2}(u), x)w, v)_{V}.$$

It follows that $\phi^2 = id$. This completes the proof.

2.3. Conformal spin-statistics theorem. In this subsection, we shall prove an analogue of conformal spin-statistics theorem in conformal field theory (cf. Page 619 of [G]). This result will play a key role in the later section.

Theorem 2.11. Let $(V, \phi, (,)_V)$ be a unitary vertex operator superalgebra. Then for any element v in $V_{\bar{0}}$ we have $\operatorname{wt} v \in \mathbb{Z}$ and for any element v in $V_{\bar{1}}$ we have $\operatorname{wt} v \in 1/2 + \mathbb{Z}$.

Proof: For any homogenous quasiprimary vectors $u, v \in V$, consider the formal power series

$$(Y(u, x_1)Y(v, x_2)\mathbf{1}, \mathbf{1})_V$$
.

Since $L(1)\mathbf{1} = 0$, by the formula (2.2) we have

$$(Y(u, x_1)Y(v, x_2)\mathbf{1}, \mathbf{1})_V$$

$$= (e^{-x_2L(-1)}Y(u, x_1)e^{x_2L(-1)}v, \mathbf{1})_V$$

$$= (Y(u, x_1 - x_2)v, \mathbf{1})_V.$$

Thus, we have

$$(Y(\phi(u), x_1)Y(u, x_2)\mathbf{1}, \mathbf{1})_{V}$$

$$= (Y(\phi(u), x_1 - x_2)u, \mathbf{1})_{V}$$

$$= (u, Y(e^{(x_1 - x_2)L(1)}(\zeta(x_1 - x_2)^{-2})^{L(0)}\iota^{\tilde{u}}u, (x_1 - x_2)^{-1})\mathbf{1})_{V}$$

$$= \overline{\zeta^{\text{wt}u}\iota^{\tilde{u}}}(x_1 - x_2)^{-2\text{wt}u}(u, u)_{V}.$$

Similarly, $(Y(u, x_2)Y(\phi(u), x_1)\mathbf{1}, \mathbf{1}) = \overline{\zeta^{\text{wt}u}\iota^{\tilde{u}}}(\phi(u), \phi(u))_V(x_2 - x_1)^{-2\text{wt}u}$. Hence, from the locality of vertex operator superalgebras, we have

$$\overline{\zeta^{\operatorname{wt} u}\iota^{\overline{u}}}(\phi(u),\phi(u))_{V} = \epsilon_{u,u}(-1)^{2\operatorname{wt} u}\overline{\zeta^{\operatorname{wt} u}\iota^{\overline{u}}}(u,u)_{V}.$$

It follows from Proposition 2.9 that

$$\epsilon_{u,u}(-1)^{-2\mathrm{wt}u} = 1.$$

In particular, $\epsilon_{u,u} = 1$ if and only if wt $u \in \mathbb{Z}$, and $\epsilon_{u,u} = -1$ if and only if wt $u \in 1/2 + \mathbb{Z}$. Finally, since V viewed as a module for the Virasoro algebra is unitary, it follows that V is a direct sum of highest weight modules of the Virasoro algebra. It follows that for any homogeneous vector $u \in V$, $\epsilon_{u,u} = 1$ if and only if wt $u \in \mathbb{Z}$, and $\epsilon_{u,u} = -1$ if and only if wt $u \in 1/2 + \mathbb{Z}$. This completes the proof.

Remark 2.12. If V is not a unitary vertex operator algebra, then the conformal spin-statistics theorem is not true. The counter example can be found in [DZ1].

3. Simple current extensions of vertex operator algebras

As an application of results in the last section, we shall construct simple current extensions of vertex operator algebras in this section.

3.1. Associativity and commutativity. In this subsection, we recall some facts about intertwining operators from [H1]-[H4]. Let V be a vertex operator algebra and M^1 , M^2 , M^3 be weak V-modules. An intertwining operator $\mathcal Y$ of type $\begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix}$ is a linear map

$$\mathcal{Y}: M^1 \to \operatorname{Hom}(M^2, M^3)\{x\}$$
$$w^1 \mapsto \mathcal{Y}(w^1, x) = \sum_{n \in \mathbb{C}} w_n^1 x^{-n-1}$$

satisfying a number of conditions (cf. [FHL]). Denote the vector space of intertwining operators of type $\begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix}$ by $\mathcal{V}_{M^1,M^2;V}^{M^3}$. The dimension of $\mathcal{V}_{M^1,M^2;V}^{M^3}$ is called the fusion rule for M^1 , M^2 and M^3 , and is denoted by $N_{M^1,M^2}^{M^3}$.

Assume that V is a vertex operator algebra satisfying the following conditions:

- (1) V is regular, i.e., any weak V-module is a direct sum of irreducible V-modules.
- (2) V is simple, self dual and of CFT-type.

Then we have the following results which were established in [H1]-[H4].

Theorem 3.1. Let V be a vertex operator algebra satisfying conditions (1), (2). Let M^i (i = 1, 2, 3, 4, 5) be weak V-modules. Then for any intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2$ of $type\begin{pmatrix} M^4 \\ M^1 M^5 \end{pmatrix}$ and $\begin{pmatrix} M^5 \\ M^2 M^3 \end{pmatrix}$, respectively,

$$\langle w_4', \mathcal{Y}_1(w^1, x_1)\mathcal{Y}_2(w^2, x_2)w^3\rangle|_{x_1=z_1, x_2=z_2}$$

is absolutely convergent when $|z_1| > |z_2| > 0$. And for any intertwining operators $\mathcal{Y}_3, \mathcal{Y}_4$ of type $\begin{pmatrix} M^5 \\ M^1 M^2 \end{pmatrix}$ and $\begin{pmatrix} M^4 \\ M^5 M^3 \end{pmatrix}$, respectively,

$$\langle w_4', \mathcal{Y}_4(\mathcal{Y}_3(w^1, x_0)w^2, x_2)w^3 \rangle |_{x_0=z_1-z_2, x_2=z_2}$$

is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$.

Theorem 3.2. Let V be a vertex operator algebra satisfying conditions (1), (2). Then we have

Associativity: For any weak V-modules M^i (i = 1, 2, 3, 4, 5) and intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2$ of type $\begin{pmatrix} M^4 \\ M^1 M^5 \end{pmatrix}$ and $\begin{pmatrix} M^5 \\ M^2 M^3 \end{pmatrix}$, respectively. There exist a module

 M^6 and intertwining operators \mathcal{Y}_3 and \mathcal{Y}_4 of type $\binom{M^6}{M^1 M^2}$ and $\binom{M^4}{M^6 M^3}$, respectively, such that for any $w_4' \in (M^4)'$, $w^i \in M^i$ (i = 1, 2, 3) the multivalued analytic function

$$\langle w_4', \mathcal{Y}_1(w^1, x_1)\mathcal{Y}_2(w^2, x_2)w^3 \rangle |_{x_1=z_1, x_2=z_2}$$
 (3.1)

on $\{(z_1, z_2) \in \mathbb{C}^2 | |z_1| > |z_2| > 0\}$ and the multivalued analytic function

$$\langle w_4', \mathcal{Y}_4(\mathcal{Y}_3(w^1, x_0)w^2, x_2)w^3 \rangle |_{x_0 = z_1 - z_2, x_2 = z_2}$$
 (3.2)

on $\{(z_1, z_2) \in \mathbb{C}^2 | |z_2| > |z_1 - z_2| > 0\}$ are equal on $\{(z_1, z_2) \in \mathbb{C}^2 | |z_1| > |z_2| > |z_1 - z_2| > 0\}$. In addition,

$$\langle w_4', \mathcal{Y}_1(w^1, x_1) \mathcal{Y}_2(w^2, x_2) w^3 \rangle |_{x_1^n = e^{n \log z_1}, \ x_2^n = e^{n \log z_2}}$$

$$= \langle w_4', \mathcal{Y}_4(\mathcal{Y}_3(w^1, x_0) w^2, x_2) w^3 \rangle |_{x_0^n = e^{n \log(z_1 - z_2)}, \ x_2^n = e^{n \log z_2}}$$

for any $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$.

Commutativity: For any weak V-modules M^i (i = 1, 2, 3, 4, 5) and intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2$ of type $\begin{pmatrix} M^4 \\ M^1 M^5 \end{pmatrix}$ and $\begin{pmatrix} M^5 \\ M^2 M^3 \end{pmatrix}$, respectively. There exist a module

 M^7 and intertwining operators \mathcal{Y}_5 and \mathcal{Y}_6 of type $\binom{M^4}{M^2 M^7}$ and $\binom{M^7}{M^1 M^3}$, respectively, such that for any $w_4' \in (M^4)'$, $w^i \in M^i$ (i = 1, 2, 3) the multivalued analytic function

$$\langle w_4', \mathcal{Y}_1(w^1, x_1)\mathcal{Y}_2(w^2, x_2)w^3\rangle|_{x_1=z_1, x_2=z_2}$$

of z_1 and z_2 in the region $|z_1|>|z_2|>0$ and the multivalued analytic function

$$\langle w_4', \mathcal{Y}_5(w^2, x_2)\mathcal{Y}_6(w^1, x_1)w^3 \rangle |_{x_1=z_1, x_2=z_2}$$
 (3.3)

of z_1 and z_2 in the region $|z_2| > |z_1| > 0$, are analytic extensions of each other.

We also need the following result which was obtained in [H5].

Proposition 3.3. There exists a multivalued analytic function $E(z_1, z_2)$ defined on $\mathcal{M}^2 = \{(z_1, z_2) \in \mathbb{C}^2 | z_1, z_2 \neq 0, z_1 \neq z_2\}$ such that the functions (3.1), (3.2) and (3.3) are restrictions of $E(z_1, z_2)$ to their domains.

3.2. Simple current extensions of vertex operator algebras. Throughout this subsection, we shall fix a vertex operator algebra V which satisfies the conditions (1) and (2). Let M be a weak V-module, then M has the following decomposition,

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}$$

where $M_{\lambda} = \{w \in M | L(0)w = \lambda w\}$ (see [DLM3], [Z]). If $w \in M_{\lambda}$, we define the conformal weight wtw of w to be λ .

For weak V-modules M^1 , M^2 , a tensor product for the ordered pair (M^1, M^2) is a pair $(M, \mathcal{Y}_{M^1, M^2})$ consisting of a weak V-module M and an intertwining operator \mathcal{Y}_{M^1, M^2} of type $\begin{pmatrix} M \\ M^1 & M^2 \end{pmatrix}$ satisfying the following universal property: For any weak V-module

W and any intertwining operator I of type $\binom{W}{M^1 M^2}$, there exists a unique Vhomomorphism ψ from M to W such that $I = \psi \circ \mathcal{Y}_{M^1,M^2}$. We shall denote the tensor
product M by $M^1 \boxtimes M^2$. Recall [DLM1] that a V-module M is called a *simple current*if for any irreducible V-module X, the tensor product $M \boxtimes X$ is also irreducible. Denote
the set of simple current V-modules by \mathcal{S}_V , it is known that the tensor product operation
induces an abelian group structure on \mathcal{S}_V such that the identity element is V (see [LY]).

In the following we assume that D is a cyclic subgroup of \mathcal{S}_V and each $M \in D$ has integral conformal weights. Set $U_D = \bigoplus_{M \in D} M$, the main goal of this subsection is to show that there exists a simple vertex operator superalgebra structure on U_D . By assumption, D is isomorphic to the cyclic group \mathbb{Z}_k for some $k \in \mathbb{N}$. Let $M^{\overline{i}} \in D$ be the irreducible V-module corresponding to $\overline{i} \in \mathbb{Z}_k$. Then we have $M^{\overline{0}} = V$ and $M^{\overline{i}} \boxtimes M^{\overline{j}} = M^{\overline{i+j}}$. Therefore, we can fix an intertwining operator $\mathcal{Y}_{\overline{i}}$ of type $\binom{M^{\overline{i+1}}}{M^{\overline{1}}M^{\overline{i}}}$, $0 \le i \le k-1$, such that $\mathcal{Y}_{\overline{0}} = Y_{M^{\overline{1}}}^*$, where $Y_{M^{\overline{1}}}^*$ is the intertwining operator defined by

$$Y_{M^{\overline{1}}}^{*}(w,x)v = e^{L(-1)x}Y_{M^{\overline{1}}}(v,-x)w$$

for any $v \in M^{\overline{0}} = V$, $w \in M^{\overline{1}}$. Consider the intertwining operator $\mathcal{Y}_{\overline{1}}^*$ of type $\begin{pmatrix} M^{\overline{2}} \\ M^{\overline{1}} M^{\overline{1}} \end{pmatrix}$ defined by

$$\mathcal{Y}_{\bar{1}}^*(w^1, x)w^2 = e^{L(-1)x}\mathcal{Y}_{\bar{1}}(w^2, -x)w^1$$

for any $w^1, w^2 \in M^{\overline{1}}$. Since $M^{\overline{1}}$ is a simple current module, it follows immediately that $\mathcal{Y}_{\overline{1}}^* = \mathcal{Y}_{\overline{1}}$ or $\mathcal{Y}_{\overline{1}}^* = -\mathcal{Y}_{\overline{1}}$. If $\mathcal{Y}_{\overline{1}}^* = \mathcal{Y}_{\overline{1}}$, set $(U_D)_{\overline{0}} = U_D$ and $(U_D)_{\overline{1}} = 0$; If $\mathcal{Y}_{\overline{1}}^* = -\mathcal{Y}_{\overline{1}}$, set $(U_D)_{\overline{0}} = \{w \in M^{\overline{i}} | i \in 2\mathbb{Z}\}$ and $(U_D)_{\overline{1}} = \{w \in M^{\overline{i}} | i \in 2\mathbb{Z} + 1\}$. It is clear that $U_D = (U_D)_{\overline{0}} \oplus (U_D)_{\overline{1}}$ is a \mathbb{Z}_2 -graded vector space and L(-1) is an even endomorphism of U_D .

For $v \in V$, $w \in M^{\overline{1}}$, define the operators $Y_D(v,x)$, $Y_D(w,x)$ by setting $Y_D(v,x)|_{M^{\overline{i}}} = Y_{M^{\overline{i}}}(v,x)$, $Y_D(w,x)|_{M^{\overline{i}}} = \mathcal{Y}_{\overline{i}}(w,x)$, respectively. It is clear that $\{Y_D(v,x)|v \in V\} \cup \{Y_D(w,x)|w \in M^{\overline{1}}\}$ is a set of \mathbb{Z}_2 -homogeneous operators on U_D .

Lemma 3.4. The operators $\{Y_D(v,x)|v\in V\}\cup\{Y_D(w,x)|w\in M^{\overline{1}}\}$ are mutually local.

Proof: It is sufficient to prove that $Y_D(w^1,x)$ and $Y_D(w^2,x)$ with $w^1,w^2 \in M^{\overline{1}}$ are mutually local. For any $M^{\overline{i}}$, it is known from Theorem 3.2 that there exist an intertwining operator $\mathcal{Y}^{\overline{2}}_{\overline{i}}$ of type $\begin{pmatrix} M^{\overline{2+i}} \\ M^{\overline{2}} M^{\overline{i}} \end{pmatrix}$ and a complex number $\lambda_{\overline{i}}$ such that for any $w^1,w^2 \in M^{\overline{1}}$, $w^3 \in M^{\overline{i}}$ and $w'_4 \in (M^{\overline{i+2}})'$, the multivalued analytic function

$$\langle w_4', Y_D(w^1, x_1)Y_D(w^2, x_2)w^3 \rangle |_{x_1=z_1, x_2=z_2}$$
 (3.1)

on $\{(z_1,z_2)\in\mathbb{C}^2||z_1|>|z_2|>0\}$ and the multivalued analytic function

$$\langle w_4', \mathcal{Y}_{\bar{i}}^{\bar{2}}(Y_D(w^1, x_0)w^2, x_2)w^3 \rangle |_{x_0 = z_1 - z_2, x_2 = z_2}$$
 (3.2)

on $\{(z_1, z_2) \in \mathbb{C}^2 | |z_2| > |z_1 - z_2| > 0\}$ are equal on $\{(z_1, z_2) \in \mathbb{C}^2 | |z_1| > |z_2| > |z_1 - z_2| > 0\}$, and that the multivalued analytic function

$$\langle w_4', Y_D(w^1, x_1)Y_D(w^2, x_2)w^3 \rangle |_{x_1=z_1, x_2=z_2}$$

of z_1 and z_2 in the region $|z_1| > |z_2| > 0$ and the multivalued analytic function

$$\lambda_{\bar{i}}\langle w_4', Y_D(w^2, x_2)Y_D(w^1, x_1)w^3\rangle|_{x_1=z_1, x_2=z_2}$$
(3.3)

of z_1 and z_2 in the region $|z_2| > |z_1| > 0$, are analytic extensions of each other.

Claim: For any $w^1, w^2 \in M^{\overline{1}}$ and $w^3 \in M^{\overline{i}}$, there is a positive integer k such that

$$(x_0 + x_2)^k Y_D(w^1, x_0 + x_2) Y_D(w^2, x_2) w^3 = (x_0 + x_2)^k \mathcal{Y}_{\bar{i}}^{\bar{2}} (Y_D(w^1, x_0) w^2, x_2) w^3.$$
(3.4)

From Proposition 3.3, there is a multivalued analytic function $E(z_1, z_2)$ defined on \mathcal{M}^2 such that the multivalued analytic functions (3.1), (3.2) and (3.3) are the restriction of $E(z_1, z_2)$ in the domains $|z_1| > |z_2| > 0$, $|z_2| > |z_1 - z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively. Since each $M^{\bar{j}} \in D$ has integral conformal weights, it follows from the proof of Lemma 4.1 of [H5] that $E(z_1, z_2)$ is single-valued. Hence, by the Cauchy theorem for contour integrals, we have for any $m, n, l \in \mathbb{Z}$,

$$\begin{split} &\langle v', \operatorname{Res}_{x_2} \operatorname{Res}_{x_1 - x_2} \mathcal{Y}_{\overline{i}}^{\overline{2}} (Y_D(w^1, x_1 - x_2) w^2, x_2) w^3 (x_1 - x_2)^m \iota_{x_2, x_1 - x_2} (x_2 + (x_1 - x_2))^n x_2^l \rangle \\ &= \oint_{C_2^{\rho}(0)} \oint_{C_1^{\epsilon}(z_2)} E(z_1, z_2) (z_1 - z_2)^m z_1^n z_2^l \mathrm{d}z_1 \mathrm{d}z_2 \\ &= \oint_{C_2^{\rho}(0)} \oint_{C_1^{R}(0)} E(z_1, z_2) (z_1 - z_2)^m z_1^n z_2^l \mathrm{d}z_1 \mathrm{d}z_2 \\ &- \oint_{C_2^{\rho}(0)} \oint_{C_1^{\tau}(0)} E(z_1, z_2) (z_1 - z_2)^m z_1^n z_2^l \mathrm{d}z_1 \mathrm{d}z_2 \\ &= \langle v', \operatorname{Res}_{x_2} \operatorname{Res}_{x_1} Y_D(w^1, x_1) Y_D(w^2, x_2) w^3 \iota_{x_1, x_2} (x_1 - x_2)^m x_1^n x_2^l \rangle \\ &- \langle v', \operatorname{Res}_{x_2} \operatorname{Res}_{x_1} \lambda_{\overline{i}} Y_D(w^2, x_2) Y_D(w^1, x_1) w^3 \iota_{x_2, x_1} (x_1 - x_2)^m x_1^n x_2^l \rangle, \end{split}$$

where contour $C_i^r(z)$ denotes the circular contour of radius r around the point $z \in \mathbb{C}$ in the variable z_i and $R > \rho > r$, $\epsilon < \min(R - \rho, \rho - r)$. Thus,

$$\operatorname{Res}_{x_{2}} \operatorname{Res}_{x_{1}-x_{2}} (\mathcal{Y}_{\overline{i}}^{\overline{2}} (Y_{D}(w^{1}, x_{1}-x_{2})w^{2}, x_{2})w^{3} (x_{1}-x_{2})^{m} \iota_{x_{2},x_{1}-x_{2}} (x_{2}+(x_{1}-x_{2}))^{n}) x_{2}^{l}$$

$$= \operatorname{Res}_{x_{2}} \operatorname{Res}_{x_{1}} Y_{D}(w^{1}, x_{1}) Y_{D}(w^{2}, x_{2})w^{3} \iota_{x_{1},x_{2}} (x_{1}-x_{2})^{m} x_{1}^{n} x_{2}^{l}$$

$$- \operatorname{Res}_{x_{2}} \operatorname{Res}_{x_{1}} \lambda_{\overline{i}} Y_{D}(w^{2}, x_{2}) Y_{D}(w^{1}, x_{1})w^{3} \iota_{x_{2},x_{1}} (x_{1}-x_{2})^{m} x_{1}^{n} x_{2}^{l},$$

which implies that

$$\begin{split} & \sum_{j \geq 0} \binom{n}{j} (w_{m+j}^1 w^2)_{n-j+l} w^3 \\ & = \sum_{j \geq 0} \binom{m}{j} (-1)^j w_{m+n-j}^1 w_{l+j}^2 w^3 - \lambda_{\bar{i}} \sum_{j \geq 0} \binom{m}{j} (-1)^{m-j} w_{m+l-j}^2 w_{n+j}^1 w^3 \end{split}$$

holds for any $m, n, l \in \mathbb{Z}$. Hence, we have

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_D(w^1,x_1)Y_D(w^2,x_2)w^3 - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\lambda_{\bar{i}}Y_D(w^2,x_2)Y_D(w^1,x_1)w^3$$

$$= x_2\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}_{\bar{i}}^{\bar{2}}(Y_D(w^1,x_0)w^2,x_2)w^3.$$

Therefore, we can prove the claim by the similar argument in the proof of Proposition 3.3.1 in [LL].

We now prove that the formula (3.4) is good enough to imply the locality. The proof is similar to that of Proposition 3 in [LY]. Let N be a positive integer such that $x^N Y_D(w^2, x) w^1 \in U_D[[x]]$. Take sufficiently large $s, t \in \mathbb{Z}_+$ such that $x^t Y_D(w^1, x) w^3 \in U_D[[x]]$ and (3.4) holds for (w^1, w^2, w^3) and (w^2, w^1, w^3) with k = t, s, respectively. Then

$$\begin{split} &x_1^t x_2^s (x_1 - x_2)^N Y_D(w^1, x_1) Y_D(w^2, x_2) w^3 \\ &= e^{-x_2 \partial_{x_1}} ((x_1 + x_2)^t x_2^s x_1^N Y_D(w^1, x_1 + x_2) Y_D(w^2, x_2) w^3) \\ &= e^{-x_2 \partial_{x_1}} ((x_1 + x_2)^t x_2^s x_1^N \mathcal{Y}_{\overline{i}}^{\overline{2}} (Y_D(w^1, x_1) w^2, x_2) w^3) \\ &= e^{-x_2 \partial_{x_1}} ((x_1 + x_2)^t x_2^s x_1^N \epsilon_{w^1, w^2} \mathcal{Y}_{\overline{i}}^{\overline{2}} (e^{L(-1)x_1} Y_D(w^2, -x_1) w^1, x_2) w^3) \\ &= e^{-x_2 \partial_{x_1}} e^{x_1 \partial_{x_2}} (x_2^t (x_2 - x_1)^s x_1^N \epsilon_{w^1, w^2} \mathcal{Y}_{\overline{i}}^{\overline{2}} (Y_D(w^2, -x_1) w^1, x_2) w^3). \end{split}$$

Define $p(x_1, x_2) := x_2^t(x_2 - x_1)^s x_1^N \mathcal{Y}_{\bar{i}}^{\bar{2}}(Y_D(w^2, -x_1)w^1, x_2)w^3$. In particular, by assumptions, $p(x_1, x_2) \in U_D[[x_1]]((x_2))$. On the other hand, we have

$$x_1^t x_2^s (-x_2 + x_1)^N Y_D(w^2, x_2) Y_D(w^1, x_1) w^3$$

$$= e^{-x_1 \partial_{x_2}} (x_1^t (x_2 + x_1)^s (-x_2)^N Y_D(w^2, x_2 + x_1) Y_D(w^1, x_1) w^3)$$

$$= e^{-x_1 \partial_{x_2}} (x_1^t (x_1 + x_2)^s (-x_2)^N \mathcal{Y}_{\bar{i}}^{\bar{2}} (Y_D(w^2, x_2) w^1, x_1) w^3)$$

$$= e^{-x_1 \partial_{x_2}} p(-x_2, x_1)$$

$$= p(-x_2 + x_1, x_1)$$

$$= p(x_1 - x_2, x_1).$$

Note that by assumptions and the equations above, we have $p(-x_2, x_1) \in U_D[[x_1]]((x_2))$. Therefore, $p(x_1, x_2) \in U_D[[x_1, x_2]]$. Then we have

$$\begin{split} &(x_1-x_2)^N Y_D(w^1,x_1) Y_D(w^2,x_2) w^3 \\ &= x_1^{-t} x_2^{-s} \epsilon_{w^1,w^2} e^{-x_2 \partial_{x_1}} e^{x_1 \partial_{x_2}} p(x_1,x_2) \\ &= x_1^{-t} x_2^{-s} \epsilon_{w^1,w^2} e^{-x_2 \partial_{x_1}} p(x_1,x_2+x_1) \\ &= x_1^{-t} x_2^{-s} \epsilon_{w^1,w^2} e^{-x_2 \partial_{x_1}} p(x_1,x_1+x_2) \\ &= x_1^{-t} x_2^{-s} \epsilon_{w^1,w^2} p(x_1-x_2,x_1) \\ &= \epsilon_{w^1,w^2} (x_1-x_2)^N Y_D(w^2,x_2) Y_D(w^1,x_1) w^3. \end{split}$$

This completes the proof.

As a consequence, we have the following result which has been proved in Theorem 3.12 of [CKL].

Theorem 3.5. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra satisfying conditions (1), (2) and D be a cyclic subgroup of S_V such that each $M \in D$ has integral conformal weights. Then there exists a unique vertex operator superalgebra structure on U_D such that U_D is an extension of V. Moreover, U_D is a simple vertex operator superalgebra.

Proof: Combining Lemma 3.4, Theorem 2.1, we know that U_D is a vertex operator superalgebra. Moreover, recall [LY] that simple current modules are irreducible, by the definition of vertex operators $\{Y_D(w,x)|w\in M^{\overline{1}}\}$, we know that U_D is a simple vertex operator superalgebra. The uniqueness can be proved by the similar argument in the proof Proposition 5.3 of [DM].

3.3. Simple current extensions of vertex operator algebras by unitary modules. In this subsection, we consider simple current extensions of vertex operator algebras by unitary modules. First, we recall from [LY], [Y1] some facts about modules of simple current extensions. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra satisfying the

conditions (1), (2) and D be a subgroup of S_V such that $U_D = \bigoplus_{M \in D} M$ is a vertex operator algebra satisfying the conditions: (a) U_D is simple, self dual and of CFT-type, (b) U_D is an extension of V. Then it is known that U_D is a vertex operator algebra satisfying the conditions (1), (2) (see Proposition 1 of [LY] and Theorem 4.5 of [ABD]). For any V-module W, define $D_W = \{M \in D | M \boxtimes W = W\}$. Then we have the following result about modules of U_D (see Theorem 3.2 of [Y1] and Theorem 1 of [LY]).

Theorem 3.6. Let V and D be as above. Assume that W is a V-module such that for any $M \in D$, $M \boxtimes W$ has integral conformal weights and that $D_W = 0$. Then there exists an irreducible U_D -module structure on $\operatorname{Ind}_V^{U_D}W = \bigoplus_{M \in D} M \boxtimes W$ such that W is a V-submodule of $\operatorname{Ind}_V^{U_D}W$.

We also need the following result (see Lemmas 3.10, 3.12 of [SY] and Lemma 2.16 of [Y1]).

Theorem 3.7. Let V and D be as above. Let W^1, W^2, W^3 be irreducible V-modules such that $D_{W^i} = 0$ and that $\operatorname{Ind}_{V}^{U_D} W^i$ is a U_D -module for i = 1, 2, 3. Then there is a linear isomorphism between $\mathcal{V}_{\operatorname{Ind}_{V}^{U_D} W^1, \operatorname{Ind}_{V}^{U_D} W^2; U_D}^{\operatorname{Ind}_{V}^{U_D} W^3}$ and $\mathcal{V}_{W^1, W^2; V}^{\operatorname{Ind}_{V}^{U_D} W^3}$.

As a consequence, we have the following

Lemma 3.8. Let V and D be as above. Assume that W is a simple current V-module such that for any $M \in D$, $M \boxtimes W$ and $M \boxtimes W'$ have integral conformal weights and that $D_W = 0 = D_{W'}$. Then $\operatorname{Ind}_V^{U_D} W$ is a simple current module of U_D .

Proof: By Theorem 3.6, we know that $\operatorname{Ind}_{V}^{U_{D}}W$ and $\operatorname{Ind}_{V}^{U_{D}}W'$ are irreducible U_{D} -modules. By assumptions, W is a simple current V-module, then we know that $W \boxtimes W' = V$ (see Corollary 1 of [LY]). It follows from Theorem 3.7 that $N_{\operatorname{Ind}_{V}^{U_{D}}W,\operatorname{Ind}_{V}^{U_{D}}W';U_{D}}^{U_{D}} = 1$. On the other hand, for any irreducible U_{D} -module \tilde{W} , it is known that there is an injective morphism from $\mathcal{V}_{\operatorname{Ind}_{V}^{U_{D}}W,\operatorname{Ind}_{V}^{U_{D}}W';U_{D}}^{\tilde{W}}$ to $\mathcal{V}_{W,W';V}^{\tilde{W}}$ (see Lemma 3.10 of [SY]). This implies that the tensor product of $\operatorname{Ind}_{V}^{U_{D}}W$ and $\operatorname{Ind}_{V}^{U_{D}}W'$ is U_{D} . Thus, $\operatorname{Ind}_{V}^{U_{D}}W$ is a simple current module of U_{D} (see Lemma 1 of [LY]). This completes the proof.

Theorem 3.9. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra satisfying conditions (1), (2) and D be a subgroup of S_V such that each $M \in D$ has integral conformal weights. Assume further that for each $M \in D$ there exists a positive definite Hermitian form $(,)_M$ on M such that

$$(L(n)w^1, w^2)_M = (w^1, L(-n)w^2)_M$$

holds for any $w^1, w^2 \in M$ and $n \in \mathbb{Z}$. Then there exists a unique vertex operator algebra structure on U_D such that:

- (a) U_D is an extension of V.
- (b) U_D is a simple vertex operator algebra.

Proof: If D a cyclic subgroup of S_V . This follows immediately from Theorems 2.10, 2.11, 3.5. We now prove the general case by induction. It is known that there exist subgroups D_1 , D_2 of D such that D_1 is cyclic and that $D = D_1 \oplus D_2$. We now further assume that there exists a vertex operator algebra structure on U_{D_2} such that U_{D_2} is an extension of V and simple. By assumptions and Theorem 2.10, we know that U_{D_2} is a unitary vertex operator algebra. Hence, by Proposition 2.3 and Theorem 3.2 of [AL], U_{D_2} is self dual and of CFT-type. It follows that U_{D_2} is a vertex operator algebra satisfying the conditions (1), (2) (see Proposition 1 of [LY] and Theorem 4.5 of [ABD]). Moreover, by assumptions and Theorem 3.6, it is known that for any element $M^{\alpha} \in D_1$, $\operatorname{Ind}_V^{U_{D_2}} M^{\alpha}$ is an irreducible U_{D_2} -module. It is also known from Lemma 3.8 that $\operatorname{Ind}_V^{U_{D_2}} M^{\alpha}$ is a simple current U_{D_2} -module. Furthermore, by Theorems 2.10, $\operatorname{Ind}_V^{U_{D_2}} M^{\alpha} | \alpha \in D_1$ is a subgroup of $S_{U_{D_2}}$ isomorphic to D_1 . Thus, by Theorems 2.10, 2.11, 3.5, there exists a vertex operator algebra structure on U_D such that U_D is an extension of V and simple. This completes the proof.

Recall [DLin] that for a unitary vertex operator algebra $(V, \phi, (,)_V)$, a V-module (M, Y_M) is called *unitary* if there exists a positive definite Hermitian form $(,)_M$ on M such that

$$(Y_M(u,x)w_1,w_2)_M = (w_1,Y_M(e^{xL(1)}(-x^{-2})^{L(0)}\phi(u),x^{-1})w_2)_M$$

holds for any $u \in V$, $w_1, w_2 \in M$. In particular, we have $(L(n)w_1, w_2)_M = (w_1, L(-n)w_2)_M$ holds for any $n \in \mathbb{Z}$, $w_1, w_2 \in M$. Hence, we have

Corollary 3.10. Let $(V, Y, \mathbf{1}, \omega)$ be a unitary vertex operator algebra satisfying conditions (1), (2) and D be a subgroup of S_V such that each $M \in D$ is a unitary V-module and has integral conformal weights. Then there exists a unique simple vertex operator algebra structure on U_D such that U_D is an extension of V.

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