SIMULTANEOUS MODELS FOR COMMUTING HOLOMORPHIC SELF-MAPS OF THE BALL

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ABSTRACT. We prove that a finite family of commuting holomorphic self-maps of the unit ball $\mathbb{B}^q \subset \mathbb{C}^q$ admits a simultaneous holomorphic conjugacy to a family of commuting automorphisms of a possibly lower dimensional ball, and that such conjugacy satisfies a universal property. As an application we describe when a hyperbolic and a parabolic holomorphic self-map of $\mathbb{B}^q$ can commute.

1. INTRODUCTION

In this paper we study simultaneous linearization of commuting holomorphic self-maps of the unit ball $\mathbb{B}^q \subset \mathbb{C}^q$, and we use such a tool to find obstructions to commutation.

In order to state the results, we need to recall what we mean by linearization of a holomorphic self-map. Even if we are interested in the general case of a holomorphic self-map of $\mathbb{B}^q$, for the sake of clearness we will consider at first the case of a univalent self-map of the unit disc $\mathbb{D} \subset \mathbb{C}$. It follows from the work of Schröder [31], Valiron [33], Pommerenke [30], Baker and Pommerenke [5] (see also Cowen [19] and Bourdon-Shapiro [13]), that given a univalent map $f \colon \mathbb{D} \to \mathbb{D}$ there exists a conjugacy as in the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{D} & \xrightarrow{\ f\ } & \mathbb{D} \\
{\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
\Omega & \xrightarrow{\ \varphi\ } & \Omega,
\end{array}
$$

where Ω is either $\mathbb{D}$ or $\mathbb{C}$, $h \colon \mathbb{D} \to \Omega$ is a univalent map, and φ is an automorphism of Ω, and thus a linear fractional transformation. Let $k \in \mathbb{N}$, $k \geq 2$. Given a k-uple of commuting univalent self-maps $(f_1, \ldots, f_k)$ of $\mathbb{D}$, it is then natural to ask whether such a conjugacy can be performed simultaneously. In other words, the question is whether there exists a triple (Ω, h, Φ), where Ω is either $\mathbb{D}$ or $\mathbb{C}$, $h \colon \mathbb{D} \to \Omega$ is univalent, $\Phi := (\varphi_1, \ldots, \varphi_k)$ is a k-uple of commuting automorphisms of Ω and

$$
h \circ f_j = \varphi_j \circ h, \quad 1 \leq j \leq k.
$$

Applying the previous result to the univalent self map $f_1 \circ \cdots \circ f_k$, Cowen answered positively this question in [20, Theorem 3.1].

Let us now move to higher dimensions. Linearization for holomorphic self-maps of the unit ball has been studied in particular cases and under more or less stringent regularity assumptions by many authors, see, *e.g.*, [6, 7, 8, 10, 12, 16, 21]. For commuting holomorphic self-maps with a common isolated fixed point in $\mathbb{B}^q$, the simultaneous linearization has been investigated by MacCluer in [28] exploiting constructions done by Cowen–MacCluer in [21] (see also [27] and [11]). The case of commuting holomorphic self-maps with no fixed points in $\mathbb{B}^q$ was studied by Gentili and the second named author [16] under suitable regularity assumption at the Denjoy–Wolff point, which allowed linearization by iterative methods. However, without any additional assumption the iterative methods seem to fail (see [18]).

In this paper we propose a new approach to simultaneous linearization in several variables, generalizing the theory of canonical models introduced in [4, 3] from the case of a single holomorphic self-map f of $\mathbb{B}^q$ to the case of a k-uple of commuting holomorphic self maps $(f_1, \ldots, f_k)$ of $\mathbb{B}^q$. If $f : \mathbb{B}^q \to \mathbb{B}^q$ is a univalent self-map, then it is shown in [4] that performing a direct limit one can always find a conjugacy as follows:

$$
\begin{array}{ccc}
\mathbb{B}^q & \xrightarrow{\ f\ } & \mathbb{B}^q \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\
\Omega & \xrightarrow{\ \psi\ } & \Omega,
\end{array}
$$

where Ω is a q-dimensional complex manifold uniquely determined up to biholomorphism, $h : \mathbb{B}^q \to \Omega$ is a univalent map and ψ is an automorphism of Ω. The triple (Ω, h, ψ) is called a *model* for f. Due to the lack of a uniformization theorem in higher dimensions, the complex structure of Ω is not known in general. However, identifying points of Ω which are at zero Kobayashi-pseudodistance, one obtains a conjugacy as follows:

$$
\begin{array}{ccc}
\mathbb{B}^q & \xrightarrow{\ f\ } & \mathbb{B}^q \\
{\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle \ell} \\
\mathbb{B}^d & \xrightarrow{\ \tau\ } & \mathbb{B}^d,
\end{array}
\tag{1.1}
$$

where $0 \leq d \leq q$, $\ell : \mathbb{B}^q \to \mathbb{B}^d$ is a holomorphic map and τ is an automorphism of $\mathbb{B}^d$. The triple $(\mathbb{B}^d, \ell, \tau)$ in (1.1) is called the *canonical Kobayashi hyperbolic semi-model* for f, and is characterized uniquely up to biholomorphisms by the following universal property: every other conjugacy of f with an automorphism of a Kobayashi hyperbolic complex manifold factorizes through the triple $(\mathbb{B}^d, \ell, \tau)$. In particular, one has $d = 0$ if and only if the only possible conjugacy of f with an automorphism of a Kobayashi hyperbolic complex manifold is given by the trivial conjugacy to a point. We call the number d the *type* of f.

Recall that a holomorphic self-map of $\mathbb{B}^q$ is said *elliptic* if it has a fixed point in $\mathbb{B}^q$. A non-elliptic holomorphic self-map f of $\mathbb{B}^q$ has a unique fixed (with respect to K-limits) boundary point $p \in \partial\mathbb{B}^q$ such that the dilation of f at p is less than or equal to 1, called the *Denjoy–Wolff point* of f. A non-elliptic map f is called *hyperbolic* if the dilation at its Denjoy–Wolff point is strictly less than 1, and *parabolic* if the dilation is equal to 1 (see Section 3 for precise definitions).

If f is hyperbolic, then the type d is always strictly greater than 0, and the automorphism τ is hyperbolic with the same dilation at its Denjoy–Wolff point.

While the model (Ω, h, ψ) does not exist in general for holomorphic self-maps of $\mathbb{B}^q$ which are not univalent, it is shown in [3] that the canonical Kobayashi hyperbolic semi-model $(\mathbb{B}^d, \ell, \tau)$ always exists.

Our first result shows that the concept of canonical Kobayashi hyperbolic semi-model can be generalized to a k-uple of commuting holomorphic self-maps $(f_1, \ldots, f_k)$ of $\mathbb{B}^q$.

Theorem 1.1. *Let $(f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of $\mathbb{B}^q$. Then there exist $0 \le d \le q$, a holomorphic map $\ell : \mathbb{B}^q \to \mathbb{B}^d$ and a k-uple of commuting automorphisms $(\tau_1, \ldots, \tau_k)$ of $\mathbb{B}^d$ such that*

$$\ell \circ f_j = \tau_j \circ \ell, \quad 1 \le j \le k.$$

Moreover, if Λ is a Kobayashi hyperbolic complex manifold such that there exists a holomorphic map $g : \mathbb{B}^q \to \Lambda$ and a k-uple of commuting automorphisms $(\varphi_1, \ldots, \varphi_k)$ of Λ such that

$$g \circ f_j = \varphi_j \circ g, \quad 1 \le j \le k,$$

then there exists a unique holomorphic mapping $\eta : \mathbb{B}^d \to \Lambda$ such that $g = \eta \circ \ell$ and $\eta \circ \tau_j = \varphi_j \circ \eta$ for all $1 \le j \le k$.

The triple $(\mathbb{B}^d, \ell, (\tau_1, \ldots, \tau_k))$ is called the *canonical Kobayashi hyperbolic semi-model* for $(f_1, \ldots, f_k)$. Theorem 1.1 actually holds in the more general case of a cocompact Kobayashi hyperbolic complex manifold, *e.g.*, every bounded homogenous domain of $\mathbb{C}^q$ (see Theorem 2.17).

Something more can be said about $(\mathbb{B}^d, \ell, (\tau_1, \ldots, \tau_k))$. It follows from Proposition 2.23 that if there is $1 \le j \le k$ such that f_j is hyperbolic, then the dimension d is strictly greater than zero, and the automorphism τ_j is hyperbolic with the same dilation at its Denjoy–Wolff point. It is also important to notice that for a fixed $1 \le j \le k$ the triple $(\mathbb{B}^d, \ell, \tau_j)$ is in general not the canonical Kobayashi hyperbolic semi-model for f_j. In fact $d \le \mathrm{type}(f_j)$, and strict inequality can occur. On the other hand, the triple $(\mathbb{B}^d, \ell, \tau_1 \circ \cdots \circ \tau_k)$ is the canonical Kobayashi hyperbolic semi-model for $f_1 \circ \cdots \circ f_k$. This shows that the dimension d is related to the dynamics of the map $f_1 \circ \cdots \circ f_k$.

We now turn to the following problem: if $f, g : \mathbb{B}^q \to \mathbb{B}^q$ are two commuting holomorphic self-maps, what can be said about their ellipticity/parabolicity/hyperbolicity? If one of the maps is elliptic, then the answer is trivial. Indeed, if f has a fixed point $z \in \mathbb{B}^q$, then for all $n \ge 0$ the point $g^n(z)$ is fixed by f. Hence, if f has a unique fixed point, then g has to be elliptic too. On the other hand, it is easy to find examples of elliptic mappings (with affine slices of fixed points) commuting with parabolic or hyperbolic mappings (see [14]). Hence in the following discussion we will assume that f and g are not elliptic.

Heins [25] proved that if $q = 1$ and f is a hyperbolic automorphism of $\mathbb{D}$, then g is a hyperbolic automorphism as well. Behan [9, Thm. 6] and Shields [32, Thm. 1] proved that if $q = 1$ and f is not a hyperbolic automorphism of $\mathbb{D}$, then f and g have the same Denjoy–Wolff point. As a consequence of these results, in dimension one f and g may have different Denjoy–Wolff point only if they are both hyperbolic automorphisms. Cowen [20, Cor. 4.1], starting from the previous results and the existence of models, proved the following result.

Theorem 1.2 (Cowen)**.** *Let f, g be two nonelliptic commuting holomorphic self-maps of $\mathbb{D}$. If f is hyperbolic, then g is hyperbolic (and thus if g is parabolic, then f is parabolic).*

In higher dimensions, it was proved in [14] that f and g have the same Denjoy–Wolff point unless both are hyperbolic automorphisms when restricted to the one-dimensional slice connecting the two Denjoy–Wolff points. In [23] de Fabritiis–Gentili proved that if f is a hyperbolic automorphism of $\mathbb{B}^q$, then g is hyperbolic and it is a hyperbolic automorphism when restricted to the one-dimensional slice connecting the two fixed points of f. Later on, de Fabritiis [22] completely characterized the mapping g in this case. Bayart [6, Thm. 6.1] proved, under regularity conditions at the Denjoy–Wolff point, that if f is a hyperbolic self-map of $\mathbb{B}^q$, then it cannot commute with certain parabolic maps.

Using the theory developed in the present paper, we are able to describe when a hyperbolic and a parabolic self-map of $\mathbb{B}^q$ can commute in terms of their types (see Theorem 3.14, Proposition 3.15, Proposition 3.16). Recall that the *step* of g at x_0 is defined as the limit $s_1^g(x_0) := \lim_{n \to \infty} k_{\mathbb{B}^q}(g^{n+1}(x_0), g^n(x_0))$, where $k_{\mathbb{B}^q}$ denotes the Kobayashi distance.

Theorem 1.3. *Let f, g be two nonelliptic commuting holomorphic self-maps of $\mathbb{B}^q$.*

    (1) *If f is hyperbolic of type q, then g is hyperbolic.*
    (2) *If g is parabolic of type 0, then f is parabolic.*
    (3) *If g is parabolic of type 1 and there exists x_0 such that $s_1^g(x_0) > 0$, then f is parabolic.*

Moreover, there exist counterexamples in all remaining cases, that is, for all $u, v \in \mathbb{N}$ such that

$$1 \leq u \leq q - 1, \quad 2 \leq v \leq q,$$

there exist commuting holomorphic self-maps $f, g \colon \mathbb{B}^q \to \mathbb{B}^q$ such that f is hyperbolic of type u and g is parabolic of type v.

The assumption on the step in (3) could be redundant, indeed it is actually an open question whether there exists a parabolic self map $g \colon \mathbb{B}^q \to \mathbb{B}^q$ with $\mathrm{type}(g) \geq 1$ which admits a point x_0 such that $s_1^g(x_0) = 0$.

Notice that Theorem 1.3 (1) generalizes the result of de Fabritiis–Gentili [23], and generalizes Cowen's Theorem 1.2 to higher dimensions, since, if $q = 1$, then $\mathrm{type}(f) = 1 = q$. The proof of Theorem 1.3 (1) is based on the following result which is of interest by itself and is proved in Proposition 2.28 in the more general case of a k-uple of commuting holomorphic self-maps.

Proposition 1.4. *Let $f \colon \mathbb{B}^q \to \mathbb{B}^q$ be a holomorphic map with $\mathrm{type}(f) = q$. Let $(\mathbb{B}^q, \ell, \tau)$ be a canonical Kobayashi hyperbolic semi-model for f. Then for every $R > 0$ and every $z \in \mathbb{B}^q$, both ℓ and f are univalent on the Kobayashi ball $B(f^n(z), R)$ for n large enough.*

2. Semi-models for commuting maps

Let X be a complex manifold. We denote by k_X the Kobayashi pseudodistance and by κ_X the Kobayashi pseudometric.

Definition 2.1. We say that X is *cocompact* if $X/\mathrm{aut}(X)$ is compact.

Notice that this implies that X is complete Kobayashi hyperbolic [24, Lemma 2.1].

Definition 2.2. Let $k \in \mathbb{N}$, $k \geq 1$. Let A a set. We call a *k-sequence* in A any mapping $a \colon \mathbb{N}^k \to A$, and we denote it by $(a_N)_{N \in \mathbb{N}^k}$. We consider the following partial ordering on $\mathbb{N}^k$:

$$(m_1, \ldots, m_k) \geq (n_1, \ldots, n_k)$$

if for all $1 \leq j \leq k$ we have $m_j \geq n_j$. We denote by $E_j \in \mathbb{N}^k$ the k-uple with 1 as j-th entry and 0 everywhere else.

Let $N = (n_1, \ldots, n_k) \in \mathbb{N}^k$. If $F = (f_1, \ldots, f_k)$ is a k-uple of commuting holomorphic self-maps of a complex manifold X, we denote by F^N the holomorphic self map of X defined by $f_1^{n_1} \circ \cdots \circ f_k^{n_k}$.

Let $F = (f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X, and let $G = (g_1, \ldots, g_p)$ be a p-uple of commuting holomorphic self-maps of X. We say that F and G commute if

$$f_j \circ g_i = g_i \circ f_j, \quad \forall \, 1 \leq j \leq k, 1 \leq i \leq p.$$

Remark 2.3. Notice that $F^{E_j} = f_j$ for all $1 \leq j \leq k$.

Remark 2.4. We consider the base of neighborhoods for the point ∞ in $\mathbb{N}^k \cup \{\infty\}$ given by $\{N \geq M\}_{M \in \mathbb{N}^k}$. Thus, if $(a_N)_{N \in \mathbb{N}^k}$ is a k-sequence in a Hausdorff space Y, by $\lim_{N \to \infty} a_N = \ell \in Y$ we mean that for any neighborhood W of ℓ in Y, there exists $N_0 \in \mathbb{N}^k$ such that $a_N \in W$ for all $N \geq N_0$.

The following lemmas are classical and the proof is omitted.

Lemma 2.5. *Let $(a_N)_{N \in \mathbb{N}^k}$ be a k-sequence in a Hausdorff space Y. Then $\lim_{N \to \infty} a_N = \ell$ if and only if for every sequence $(N_h)_{h \in \mathbb{N}}$ in $\mathbb{N}^k$ converging to ∞ we have $\lim_{h \to \infty} a_{N_h} = \ell$.*

Lemma 2.6. *Let $(a_N)_{N \in \mathbb{N}^k}$ be a monotone non-increasing k-sequence in $[0, +\infty)$. Then*

$$\lim_{N \to \infty} a_N = \inf_{N \in \mathbb{N}^k} a_N.$$

In what follows, let $k \in \mathbb{N}, k \geq 1$.

Definition 2.7. Let X be a complex manifold. We call *(forward) holomorphic k-dynamical system* on X any family $(F_{N,M} \colon X \to X)_{M \geq N \in \mathbb{N}^k}$ of holomorphic self-maps such that for all $M \geq U \geq N \in \mathbb{N}^k$, we have

$$F_{U,M} \circ F_{N,U} = F_{N,M}.$$

Let $F := (f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps $f_j \colon X \to X$. Then setting $F_{N,M} := F^{M-N}$ for all $M \geq N \in \mathbb{N}^k$, we obtain a holomorphic k-dynamical system naturally associated with F.

Definition 2.8. Let X be a complex manifold and let $(F_{N,M} \colon X \to X)_{M \geq N \in \mathbb{N}^k}$ be a holomorphic k-dynamical system. We call *canonical Kobayashi hyperbolic direct limit* (CDL for short) for $(F_{N,M})$ a pair (Z, α_N) where Z is a Kobayashi hyperbolic complex manifold and $(\alpha_N \colon X \to Z)_{N \in \mathbb{N}^k}$ is a family of holomorphic mappings such that

$$\alpha_M \circ F_{N,M} = \alpha_N, \quad \forall \, M \geq N \in \mathbb{N}^k,$$

which satisfies the following universal property: if Q is a Kobayashi hyperbolic complex manifold and if $(g_N \colon X \to Q)_{N \in \mathbb{N}^k}$ is a family of holomorphic mappings satisfying

$$g_M \circ F_{N,M} = g_N, \quad \forall\, M \geq N \in \mathbb{N}^k,$$

then there exists a unique holomorphic mapping $\Gamma \colon Z \to Q$ such that

$$g_N = \Gamma \circ \alpha_N, \quad \forall\, N \in \mathbb{N}^k.$$

The proof of the following lemma is trivial and is therefore omitted.

Lemma 2.9. *The canonical Kobayashi hyperbolic direct limit of $(F_{N,M})$ is essentially unique, in the following sense. Let (Z, α_N) and (Q, g_N) be two CDL for $(F_{N,M})$. Then there exists a biholomorphism $\Gamma \colon Z \to Q$ such that*

$$g_N = \Gamma \circ \alpha_N, \quad \forall\, N \in \mathbb{N}^k.$$

Definition 2.10. Let X, Y be complex manifolds, and let $f \colon X \to Y$ be a holomorphic map. Then by $f^* k_Y$ and $f^* \kappa_Y$ we denote the pullbacks

$$f^* k_Y(x, y) := k_Y(f(x), f(y)), \quad x, y \in X,$$
$$f^* \kappa_Y(x, v) := \kappa_Y(f(z), d_x f(v)), \quad x \in X, v \in T_x X.$$

Lemma 2.11. *Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $(F_{N,M} \colon X \to X)_{M \geq N \in \mathbb{N}^k}$ be a holomorphic k-dynamical system. There exists a CDL (Z, α_N) for $(F_{N,M})$, where Z is a holomorphic retract of X. Moreover,*

$$Z = \bigcup_{N \in \mathbb{N}^k} \alpha_N(X), \tag{2.1}$$

$$\lim_{M \to \infty} (F_{N,M})^* k_X = \alpha_N^* k_Z, \quad \lim_{M \to \infty} (F_{N,M})^* \kappa_X = \alpha_N^* \kappa_Z, \quad \forall\, N \in \mathbb{N}^k. \tag{2.2}$$

Proof. Let $(N_h)_{h \in \mathbb{N}}$ be a strictly monotone sequence in $\mathbb{N}^k$ converging to ∞. Define a holomorphic dynamical system $(g_{h,j} \colon X \to X)_{j \geq h \in \mathbb{N}}$ by

$$g_{h,j} := F_{N_h, N_j}.$$

Let (Z, α_h) be the CDL given by [3, Theorem 2.11] for $(g_{h,j})$. For every $N \in \mathbb{N}^k$ let $h \in \mathbb{N}$ be such that $N_h \geq N$, and define $\alpha_N \colon X \to Z$ by

$$\alpha_N := \alpha_h \circ F_{N, N_h}. \tag{2.3}$$

It is easy to see that this is well defined and that (Z, α_N) is a CDL for $(F_{N,M})$ satisfying (2.1).

We are left to prove (2.2). Let $x, y \in X$, let $N \in \mathbb{N}^k$ and let $h \in \mathbb{N}$ be such that $N_h \geq N$. Notice that the k-sequence $(k_X(F_{N,M}(x), F_{N,M}(y)))_{M \geq N \in \mathbb{N}^k}$ is monotone non-increasing, and thus it admits a limit as $M \to \infty$. Then, using [3, Theorem 2.11 (2.8)],

$$\lim_{M \to \infty} k_X(F_{N,M}(x), F_{N,M}(y)) = \lim_{j \to \infty} k_X(F_{N, N_j}(x), F_{N, N_j}(y)) =$$
$$= \lim_{j \to \infty} k_X(g_{h,j}(F_{N, N_h}(x)), g_{h,j}(F_{N, N_h}(y))) =$$
$$= k_Z(\alpha_h(F_{N, N_h}(x)), \alpha_h(F_{N, N_h}(y))) = k_Z(\alpha_N(x), \alpha_N(y)).$$

The proof for the Kobayashi metric κ_X is similar.

$\square$

Definition 2.12. Let X be a complex manifold and let $F := (f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. A *quasi-model* for F is a triple (Λ, h, Φ) where Λ is a complex manifold, $h \colon X \to \Lambda$ is a holomorphic map, and $\Phi := (\varphi_1, \ldots, \varphi_k)$ is a k-uple of commuting automorphisms of Λ such that

$$h \circ f_j = \varphi_j \circ h, \quad \forall\, 1 \le j \le k.$$

Let (Λ, h, Φ) and $(Z, \ell, T := (\tau_1, \ldots, \tau_k))$ be two quasi-models for F. A *morphism of quasi-models* $\hat{\eta} \colon (Z, \ell, T) \to (\Lambda, h, \Phi)$ is given by a holomorphic map $\eta \colon Z \to \Lambda$ such that $\eta \circ \ell = h$ and such that

$$\eta \circ \tau_j = \varphi_j \circ \eta, \quad \forall\, 1 \le j \le k.$$

If the map $\eta \colon Z \to \Lambda$ is a biholomorphism, then we say that $\hat{\eta} \colon (Z, \ell, T) \to (\Lambda, h, \Phi)$ is an *isomorphism of quasi-models*. Notice that then $\eta^{-1} \colon \Lambda \to Z$ induces a morphism $\hat{\eta}^{-1} \colon (\Lambda, h, \Phi) \to (Z, \ell, T)$.

Definition 2.13. Let X be a complex manifold and let $F := (f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. A *semi-model* for F is a quasi-model (Λ, h, Φ) such that

$$\bigcup_{N \in \mathbb{N}^k} \Phi^{-N}(h(X)) = \Lambda. \tag{2.4}$$

If (Z, ℓ, T) and (Λ, h, Φ) are two semi-models for F, then a morphism of quasi-models $\hat{\eta} \colon (Z, \ell, T) \to (\Lambda, h, \Phi)$ is called a *morphism of semi-models*.

Proposition 2.14. *Let X be a complex manifold and let $F := (f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. Let (Z, ℓ, T) be a semi-model for F, and let (Λ, h, Φ) be a quasi-model for F. Then*

(1) *there exists at most one morphism from (Z, ℓ, T) to (Λ, h, Φ), and*
(2) *if $\hat{\eta} \colon (Z, \ell, T) \to (\Lambda, h, \Phi)$ is a morphism, then*

$$\eta(Z) = \bigcup_{N \in \mathbb{N}^k} \Phi^{-N}(h(X)).$$

Proof.

(1) Let $\hat{\eta}, \hat{\beta} \colon (Z, \ell, T) \to (\Lambda, h, \Phi)$ be two morphisms. Then for all $N \in \mathbb{N}^k$,

$$\eta \circ T^{-N} \circ \ell = \Phi^{-N} \circ \eta \circ \ell = \Phi^{-N} \circ h = \Phi^{-N} \circ \beta \circ \ell = \beta \circ T^{-N} \circ \ell.$$

Since (Z, ℓ, T) is a semi-model, $\bigcup_{N \in \mathbb{N}^k} T^{-N}(\ell(X)) = Z$, hence $\eta = \beta$.

(2)

$$\bigcup_{N \in \mathbb{N}^k} \Phi^{-N}(h(X)) = \bigcup_{N \in \mathbb{N}^k} \Phi^{-N}(\eta(\ell(X))) = \eta \left(\bigcup_{N \in \mathbb{N}^k} T^{-N}(\ell(X)) \right).$$

$\square$

Definition 2.15. Let X be a complex manifold and let $F := (f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. Let (Z, ℓ, T) be a semi-model for F whose base space Z is Kobayashi hyperbolic. We say that (Z, ℓ, T) is a *canonical Kobayashi hyperbolic semi-model* for F (for short CSM) if it satisfies the following universal property: for any semi-model (Λ, h, Φ) for

F such that the base space Λ is Kobayashi hyperbolic there exists a morphism of semi-models $\hat{\eta}\colon (Z, \ell, T) \to (\Lambda, h, \Phi)$ (which is necessarily unique by Proposition 2.14).

Remark 2.16. If (Z, ℓ, T) and (Λ, h, Φ) are two CSM for F, then they are isomorphic.

Theorem 2.17. *Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $F :=$ $(f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. Then there exists a CSM (Z, ℓ, T) for F, where Z is a holomorphic retract of X. Moreover, for all $N \in \mathbb{N}^k$,*

$$\lim_{M \to \infty} (F^M)^* k_X = (T^{-N} \circ \ell)^* k_Z, \quad \lim_{M \to \infty} (F^M)^* \kappa_X = (T^{-N} \circ \ell)^* \kappa_Z. \tag{2.5}$$

Proof. Let (Z, α_N) be the CDL given by Lemma 2.11 for the holomorphic k-dynamical system associated with F. Let $1 \le j \le k$. The k-sequence of holomorphic maps $(\beta_N := \alpha_N \circ F^{E_j}\colon X \to Z)_{N \in \mathbb{N}^k}$ satisfies, for all $M \ge N \in \mathbb{N}^k$,

$$\beta_M \circ F^{M-N} = \alpha_M \circ F^{E_j} \circ F^{M-N} = \alpha_M \circ F^{M-N} \circ F^{E_j} = \alpha_N \circ F^{E_j} = \beta_N.$$

By the universal property in Definition 2.8 there exists a unique holomorphic self-map $\tau_j\colon Z \to Z$ such that for all $N \in \mathbb{N}^k$,

$$\tau_j \circ \alpha_N = \alpha_N \circ F^{E_j}.$$

We claim that τ_j is a holomorphic automorphism. For all $N \in \mathbb{N}^k$, set $\gamma_N := \alpha_{N+E_j}$. For all $M \ge N \in \mathbb{N}^k$,

$$\gamma_M \circ F^{M-N} = \alpha_{M+E_j} \circ F^{M-N} = \alpha_{N+E_j} = \gamma_N.$$

Thus there exists a holomorphic self-map $\delta_j\colon Z \to Z$ such that $\delta_j \circ \alpha_N = \alpha_{N+E_j}$ for all $N \in \mathbb{N}^k$. For all $N \in \mathbb{N}^k$ we have

$$\tau_j \circ \delta_j \circ \alpha_N = \tau_j \circ \alpha_{N+E_j} = \alpha_N,$$

and

$$\delta_j \circ \tau_j \circ \alpha_N = \delta_j \circ \alpha_N \circ F^{E_j} = \alpha_{N+E_j} \circ F^{E_j} = \alpha_N.$$

By the universal property in Definition 2.8 we have that τ_j is a holomorphic automorphism and $\delta_j = \tau_j^{-1}$. Set $T := (\tau_1, \ldots, \tau_k)$. Since for all $N \in \mathbb{N}^k$,

$$T^N \circ \alpha_N = \alpha_N \circ F^N = \alpha_0,$$

it follows that $\alpha_N = T^{-N} \circ \alpha_0$.

Set $\ell := \alpha_0$. We claim that the triple (Z, ℓ, T) is a CSM for F. Indeed, let (Λ, h, Φ) be a semi-model for F such that the base space Λ is Kobayashi hyperbolic. For all $N \in \mathbb{N}^k$, let $\lambda_N := \Phi^{-N} \circ h$. Then by the universal property in Definition 2.8 there exists a unique holomorphic map $\eta\colon Z \to \Lambda$ such that for all $N \in \mathbb{N}^k$ we have $\eta \circ \alpha_N = \lambda_N$, that is

$$\eta \circ T^{-N} \circ \ell = \Phi^{-N} \circ h.$$

Setting $N = 0$ this implies $\eta \circ \ell = h$. Now fix $1 \le j \le k$. If $N \in \mathbb{N}^k$,

$$\Phi^{E_j} \circ \eta \circ T^{-E_j} \circ \alpha_N = \Phi^{E_j} \circ \eta \circ T^{-E_j} \circ T^{-N} \circ \ell = \Phi^{E_j} \circ \Phi^{-N-E_j} \circ h = \lambda_N.$$

By the universal property in Definition 2.8, we have

$$\eta = \Phi^{E_j} \circ \eta \circ T^{-E_j}.$$

Hence the map $\eta\colon Z \to \Lambda$ gives a morphism of semi-models $\hat{\eta}\colon (Z, \ell, T) \to (\Lambda, h, \Phi)$. $\square$

Remark 2.18. Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $F :=$ $(f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. Let (Z, ℓ, T) be a CSM for F. The proof of Theorem 2.17 actually shows that (Z, ℓ, T) satisfies the following slightly stronger universal property: for any quasi-model (Λ, h, Φ) for F such that the base space Λ is Kobayashi hyperbolic, there exists a morphism of quasi-models $\hat{\eta} \colon (Z, \ell, T) \to (\Lambda, h, \Phi)$ (which is necessarily unique by Proposition 2.14).

Definition 2.19. Let X be a complex manifold, and let $F := (f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. Let $x \in X$, and let $M \in \mathbb{N}^k$. We define the M-step of F at x as

$$s_M^F(x) := \lim_{N \to \infty} k_X(F^N(x), F^{N+M}(x)).$$

Notice that the limit exists and $s_M^F(x) \in [0, +\infty)$ since $k_X(F^N(x), F^{N+M}(x))_{N \in \mathbb{N}^k}$ is a monotone non increasing k-sequence.

Definition 2.20. Let X be a complex manifold. Let $f \colon X \to X$ be a holomorphic self-map. The *divergence rate* $c(f) \in [0, +\infty)$ of f is defined as

$$c(f) := \lim_{m \to \infty} \frac{k_X(f^m(x), x)}{m},$$

where $x \in X$. It is shown in [4] that such a limit exists and does not depend on $x \in X$, and moreover

$$c(f) = \inf_{m \in \mathbb{N}} \frac{k_X(f^m(x), x)}{m} = \lim_{m \to \infty} \frac{s_m^f(x)}{m} = \inf_{m \in \mathbb{N}} \frac{s_m^f(x)}{m}.$$

Lemma 2.21. *Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $F :=$ $(f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. Let (Z, ℓ, T) be a CSM for F, and let $M \in \mathbb{N}^k$. Then for all $x \in X$,*

$$k_Z(\ell(x), T^M(\ell(x))) = s_M^F(x).$$

In particular, for all $1 \le j \le k$,

$$c(\tau_j) = \lim_{m \to \infty} \frac{s_{mE_j}^F(x)}{m} = \inf_{m \to \infty} \frac{s_{mE_j}^F(x)}{m}. \tag{2.6}$$

Proof. By (2.5),

$$k_Z(\ell(x), T^M(\ell(x))) = k_Z(\ell(x), \ell(F^M(x)) = \lim_{U \to \infty} k_X(F^U(x), F^{U+M}(x)) = s_M^F(x).$$

$\square$

Remark 2.22. Notice that in general, if $1 \le j \le k$,

$$s_{mE_j}^F(x) = \lim_{\mathbb{N}^{k-1} \ni Q \to \infty} s_m^{f_j}(\widehat{F}_j^Q(x)) \le s_m^{f_j}(x),$$

where $\widehat{F}_j$ denotes the $(k-1)$-uple $(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_k)$.

Proposition 2.23. *Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $F :=$ $(f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. Let (Z, ℓ, T) be a CSM for F. Then for all $1 \leq j \leq k$,*

$$c(\tau_j) = c(f_j).$$

Proof. Clearly $c(\tau_j) \leq c(f_j)$. For all $y \in X$ we have that $c(f_j) = \inf_{m \in \mathbb{N}} \frac{s_m^{f_j}(y)}{m}$, in particular, for all $m \in \mathbb{N}$ we have $\frac{s_m^{f_j}(y)}{m} \geq c(f_j)$. Thus, by Remark 2.22,

$$\frac{s_{mE_j}^F(x)}{m} = \lim_{\mathbb{N}^{k-1} \ni Q \to \infty} \frac{s_m^{f_j}(\widehat{F}_j^Q(x))}{m} \geq c(f_j),$$

and the result follows then from (2.6). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 2.24. Let X be a cocompact Kobayashi hyperbolic complex manifold. Let $F :=$ $(f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. The *type* of F (denoted type(F)) is the dimension of a CSM for F. Clearly type(F) $\leq \dim(X)$. If type(F) $= \dim(X)$, then we say that F is of *automorphism type*.

Proposition 2.23 immediately yields the following corollary.

Corollary 2.25. *Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $F :=$ $(f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X and assume that at least one of the f_j's is hyperbolic. Then* type(F) ≥ 1.

Definition 2.26. Let X be a Kobayashi hyperbolic complex manifold, let $p \in X$ and let $r > 0$. We denote by $B(p, r)$ the ball of center p and radius r with respect to the Kobayashi distance k_X, in other words, $B(p, r) := \{z \in X : k_X(z, p) < r\}$.

Proposition 2.27. *Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $F :=$ $(f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. Let (Z, ℓ, T) be a CSM for F, and let $d = $ type(F) be the dimension of Z. Then for all $p \in X$ there exists $r > 0$ and $N_0 \in \mathbb{N}^k$ such that for all $N \geq N_0 \in \mathbb{N}^k$*

(1) $\mathrm{rk}_y \ell = d, \quad \forall\, y \in B(F^N(p), r)$,
(2) $\mathrm{rk}_y F^M \geq d, \quad \forall\, M \in \mathbb{N}^k, y \in B(F^N(p), r)$.

Proof. We first prove (1) by contradiction. Fix $p \in X$. Assume that there exists a strictly increasing sequence $(N_h)_{h \in \mathbb{N}}$ in $\mathbb{N}^k$ with $N_0 = 0$ converging to ∞ such that for all $h \geq 1$ the mapping ℓ has not rank identically d on $B(F^{N_h}(p), \frac{1}{h})$.

Let K be a compact subset in X such that $X = \mathrm{aut}(X) \cdot K$. For all $N \in \mathbb{N}$, let γ_N be an automorphism of X such that $\gamma_N(F^N(p)) \in K$. Define a holomorphic k-dynamical system $(\tilde{F}_{N,M})$ in the following way: for all $M \geq N \in \mathbb{N}^k$,

$$\tilde{F}_{N,M} := \gamma_M \circ F^{M-N} \circ \gamma_N^{-1}.$$

Define a holomorphic dynamical system $(g_{h,j})$ in the following way: for all $j \geq h \in \mathbb{N}$,

$$g_{h,j} := \tilde{F}_{N_h, N_j}.$$

The dynamical system $(g_{h,j} \colon X \to X)$ admits a relatively compact orbit. Indeed $g_{0,j}(\gamma_0(p)) \subset K$ for all $j \in \mathbb{N}$. Let $(A, \tilde{\alpha}_h)$ be the CDL for $(g_{h,j})$ given by [3, Section 2]. This extends, as in Lemma 2.11, to a CDL $(A, \tilde{\alpha}_N)$ for $(\tilde{F}_{N,M})$ satisfying $\tilde{\alpha}_h = \tilde{\alpha}_{N_h}$ for all $h \in \mathbb{N}$. Clearly, $(A, \alpha_N := \tilde{\alpha}_N \circ \gamma_N^{-1})$ is a CDL for (F^{M-N}).

Notice that, as $h \to \infty$, the map $\tilde{\alpha}_{N_h}$ converges uniformly on compacta to a retraction $\tilde{\alpha} \colon X \to X$ of rank d such that $\tilde{\alpha}(X) = A$. Since $\tilde{\alpha}$ has rank d on A, there exists a neighborhood U of A such that $\mathrm{rk}_w(\tilde{\alpha}) = d$ for all $w \in U$. Up to passing to a subsequence, we may assume that the sequence $(\tilde{F}_{0,N_h}(\gamma_0(p)))_{h \geq 0}$ converges to $\tilde{\alpha}_0(\gamma_0(p)) \in A$ as $h \to \infty$. There exists a neighborhood $V \subset\subset U$ of $\tilde{\alpha}_0(\gamma_0(p))$ and $\bar{h} \in \mathbb{N}$ such that $\mathrm{rk}_w(\tilde{\alpha}_{N_h}) = d$ for all $h \geq \bar{h}$ and all $w \in V$.

There exists an index $h_0 \geq \bar{h}$ and $r > 0$ such that for all $h \geq h_0$, the Kobayashi ball $B(\tilde{F}_{0,N_h}(\gamma_0(p)), r)$ is contained in V and thus

$$\mathrm{rk}_w(\tilde{\alpha}_{N_h}) = d, \quad \forall\, w \in B(\tilde{F}_{0,N_h}(\gamma_0(p)), r). \tag{2.7}$$

Since for all $h \geq 1$ the mapping ℓ has not rank identically d on $B(F^{N_h}(p), \frac{1}{h})$, we have that for all $h \geq 1$ the mapping α_{N_h} has not rank identically d on $B(F^{N_h}(p), \frac{1}{h})$, and thus for all $h \geq 1$ $\tilde{\alpha}_{N_h}$ has not rank identically d on $B(\tilde{F}_{0,N_h}(\gamma_0(p)), \frac{1}{h})$, which contradicts (2.7). This proves (1). Statement (2) follows immediately from $\ell \circ F^M = T^M \circ \ell$. $\qquad\square$

Proposition 2.28. *Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $F := (f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. Assume that F is of automorphism type, and let (Z, ℓ, T) be a CSM for F. Then for all $r > 0$ and all $p \in X$ there exists $N_0 \in \mathbb{N}^k$ such that for all $N \geq N_0 \in \mathbb{N}^k$*

(1) *the map ℓ is univalent on $B(F^N(p), r)$,*
(2) *the map F^M is univalent on $B(F^N(p), r)$ for all $M \in \mathbb{N}^k$.*

Proof. The proof is similar to the one of Proposition 2.27. Fix $r > 0$ and $p \in X$. By contradiction, assume that there exists a strictly increasing sequence (N_h) in $\mathbb{N}^k$ with $N_0 = 0$, converging to ∞ and such that for all $h \geq 1$ the mapping ℓ is not univalent on $B(F^{N_h}(p), r)$. Let K be a compact subset in X such that $X = \mathrm{aut}(X) \cdot K$. For all $N \in \mathbb{N}$, let γ_N be an automorphism of X such that $\gamma_N(F^N(p)) \in K$. Define for all $M \geq N \in \mathbb{N}^k$,

$$\tilde{F}_{N,M} := \gamma_M \circ F^{M-N} \circ \gamma_N^{-1}.$$

Define for all $j \geq h \in \mathbb{N}$,

$$g_{h,j} := \tilde{F}_{N_h, N_j}.$$

The dynamical system $(g_{h,j} \colon X \to X)$ admits a relatively compact orbit. Let $(A, \tilde{\alpha}_h)$ be the CDL for $(g_{h,j})$ given by [3, Section 2]. This extends, as in Lemma 2.11, to a CDL $(A, \tilde{\alpha}_N)$ for $(\tilde{F}_{N,M})$ satisfying $\tilde{\alpha}_h = \tilde{\alpha}_{N_h}$ for all $h \in \mathbb{N}$. Clearly, $(A, \alpha_N := \tilde{\alpha}_N \circ \gamma_N^{-1})$ is a CDL for (F^{M-N}).

Since for all $h \geq 1$ the mapping ℓ is not univalent on $B(F^{N_h}(p), r)$, we have that for all $h \geq 1$ the mapping α_{N_h} is not univalent on $B(F^{N_h}(p), r)$, and thus $\tilde{\alpha}_{N_h}$ is not univalent on $B(K, r)$. Since by assumption F is of automorphism type, it follows that $\tilde{\alpha}_{N_h} \to \mathrm{id}_X$ uniformly on compact subsets as $h \to \infty$, which gives a contradiction. This proves (1). Statement (2) follows immediately from $\ell \circ F^M = T^M \circ \ell$. $\qquad\square$

Corollary 2.29. *Let X be a cocompact Kobayashi hyperbolic complex manifold. Let*

$$E = (f_1, \ldots f_k, g_1, \ldots g_p)$$

be a $(k+p)$-uple of commuting holomorphic self-maps of X. Let $F := (f_1, \ldots, f_k)$. Then

$$\mathrm{type}(E) \leq \mathrm{type}(F).$$

Proof. Let $G := (g_1, \ldots, g_p)$. Let $(Z, \ell, (T_F, T_G))$ be a CSM for E and let (Λ, h, Φ) be a CSM for F. Since (Z, ℓ, T_F) is a quasi-model for F, by Remark 2.18 there exists a morphism $\hat{\eta} \colon (\Lambda, h, \Phi) \to (Z, \ell, T_F)$. In particular we have $\ell = \eta \circ h$. We claim that there exists a point $x \in X$ such that $\mathrm{rk}_x \ell = \mathrm{type}(E)$ and $\mathrm{rk}_x h = \mathrm{type}(F)$, and this clearly yields the result. Fix $z \in X$. By Proposition 2.27, there exists $U_0 := (N_0, M_0) \in \mathbb{N}^{k+p}$ such that for all $U \geq U_0$, we have $\mathrm{rk}_{E^U(z)} \ell = \mathrm{type}(E)$. Define

$$y := G^{M_0}(z).$$

By Proposition 2.27, there exists $N_1 \in \mathbb{N}^k$ such that for all $N \geq N_1$, we have $\mathrm{rk}_{F^N(y)} h = \mathrm{type}(F)$. Let $N_2 := \max(N_0, N_1)$. Setting

$$x := F^{N_2}(G^{M_0}(z)),$$

the claim is proved. $\square$

The remaining of the section is devoted to a useful procedure to construct a CSM for a finite family of commuting holomorphic self-maps.

Let X be a cocompact Kobayashi hyperbolic complex manifold, and let $F := (f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. Let (Λ, h, Φ) be a CSM for F. Let g be a holomorphic self map of X commuting with F. Since

$$\varphi_j \circ (h \circ g) = (h \circ g) \circ f_j \quad \text{for all } 1 \leq j \leq k,$$

the triple $(\Lambda, h \circ g, \Phi)$ is a quasi-model for F with Kobayashi hyperbolic base space.

Definition 2.30. We denote by $\Gamma_F(g)$ the holomorphic self-map of Λ which commutes with φ_j for all $1 \leq j \leq k$ and which satisfies

$$\Gamma_F(g) \circ h = h \circ g,$$

whose existence and uniqueness follows from Remark 2.18.

Remark 2.31. Notice that $\Gamma_F(\mathrm{id}_X) = \mathrm{id}_\Lambda$.

Lemma 2.32. *If $g_1, g_2 \colon X \to X$ are two holomorphic self-maps commuting with F, then*

$$\Gamma_F(g_1 \circ g_2) = \Gamma_F(g_1) \circ \Gamma_F(g_2).$$

Proof. We have that

$$\Gamma_F(g_1) \circ \Gamma_F(g_2) \circ h = \Gamma_F(g_1) \circ h \circ g_2 = h \circ g_1 \circ g_2,$$

and the result follows from the uniqueness of $\Gamma_F(g_1 \circ g_2)$. $\square$

Corollary 2.33. *If g is an automorphism of X, then $\Gamma_F(g)$ is an automorphism of Λ.*

Proof. Clearly $\Gamma_F(g^{-1})$ is an holomorphic inverse for $\Gamma_F(g)$. $\square$

Corollary 2.34. *Let $g_1, g_2 \colon X \to X$ be two holomorphic self-maps commuting with F. If g_1 commutes with g_2, then $\Gamma_F(g_1)$ commutes with $\Gamma_F(g_2)$.*
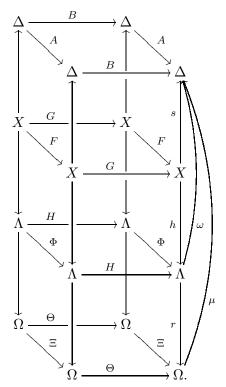
Proof. By Lemma 2.32,

$$\Gamma_F(g_1) \circ \Gamma_F(g_2) = \Gamma_F(g_1 \circ g_2) = \Gamma_F(g_2 \circ g_1) = \Gamma_F(g_2) \circ \Gamma_F(g_1).$$

$\square$

Theorem 2.35. *Let X be a cocompact Kobayashi hyperbolic complex manifold. Let*

$$E = (f_1, \ldots f_k, g_1, \ldots g_p)$$

be a $(k+p)$-uple of commuting holomorphic self-maps of X. Let $F := (f_1, \ldots, f_k)$ and let $G := (g_1, \ldots, g_p)$. Let (Λ, h, Φ) be a CSM for F and assume that Λ is cocompact. Let (Ω, r, Θ) be a CSM for $\Gamma_F(G)$. Denote $H := \Gamma_F(G)$ and $\Xi := \Gamma_H(\Phi)$. Then a CSM for E is given by the triple $(\Omega, r \circ h, (\Xi, \Theta))$.

Proof. Denote $\Theta = (\vartheta_1, \ldots, \vartheta_p), H = (\eta_1, \ldots \eta_p), \Xi = (\xi_1, \ldots, \xi_k), \Phi = (\varphi_1, \ldots \varphi_k)$. First of all, notice that by Corollary 2.33, Ξ is a k-uple of automorphisms of Ω. Let $(\Delta, s, (A, B))$ be a semi-model for E such that Δ is Kobayashi hyperbolic, where $A = (\alpha_1, \ldots \alpha_k)$ and $B = (\beta_1, \ldots, \beta_p)$.

The triple (Δ, s, A) is a quasi-model for F. By the universal property stated in Remark 2.18, there exists a morphism $\hat{\omega} \colon (\Lambda, h, \Phi) \to (\Delta, s, A)$.

We claim that (Δ, ω, B) is a quasi-model for H, and it suffices to show that

$$\omega \circ \eta_j = \beta_j \circ \omega, \quad \forall 1 \leq j \leq p.$$

Fix $1 \leq j \leq p$. Then for all $N \in \mathbb{N}^k$,

$$\omega \circ \eta_j \circ \Phi^{-N} \circ h = \omega \circ \Phi^{-N} \circ \eta_j \circ h = A^{-N} \circ \omega \circ h \circ g_j = A^{-N} \circ s \circ g_j =$$
$$= A^{-N} \circ \beta_j \circ s = \beta_j \circ A^{-N} \circ \omega \circ h = \beta_j \circ \omega \circ \Phi^{-N} \circ h.$$

Hence $\omega \circ \eta_j = \beta_j \circ \omega$ on $\bigcup_{N \in \mathbb{N}^k} \Phi^{-N} \circ h = \Lambda$.

By the universal property stated in Remark 2.18, there exists a morphism $\hat{\mu} \colon (\Omega, r, \Theta) \to (\Delta, \omega, B)$. We claim that the holomorphic map $\mu \colon \Omega \to \Delta$ also defines a morphism $\hat{\mu} \colon (\Omega, r \circ h, (\Xi, \Theta)) \to (\Delta, s, (A, B))$, and it suffices to show that

$$\mu \circ \xi_j = \alpha_j \circ \mu, \quad \forall 1 \leq j \leq k.$$

Fix $1 \leq j \leq k$. For all $N \in \mathbb{N}^p$,

$$\mu \circ \xi_j \circ \Theta^{-N} \circ r = \mu \circ \Theta^{-N} \circ \xi_j \circ r = \mu \circ \Theta^{-N} \circ r \circ \varphi_j = B^{-N} \circ \omega \circ \varphi_j = \alpha_j \circ B^{-N} \circ \omega = \alpha_j \circ \mu \circ \Theta^{-N} \circ r.$$

Hence $\mu \circ \xi_j = \alpha_j \circ \mu$ on $\bigcup_{N \in \mathbb{N}^p} \Theta^{-N}(r(\Lambda)) = \Omega$. □

Proposition 2.36. *Let X be a cocompact Kobayashi hyperbolic complex manifold. Let $F := (f_1, \ldots, f_k)$ be a k-uple of commuting holomorphic self-maps of X. Then for all holomorphic self-map $g \colon X \to X$ commuting with F we have $c(\Gamma_F(g)) = c(g)$.*

Proof. Let (Λ, h, Φ) be a CSM for F, and set $E := (f_1, \ldots, f_k, g)$. Let $(Z, \ell, (T_F, T_g))$ be a CSM for E. Let

$$\hat{t} \colon (\Lambda, h, \Phi) \to (Z, \ell, T_F)$$

be the morphism given by the universal property stated in Remark 2.18. Then, arguing as in the proof of Theorem 2.35, we see that the triple (Z, t, T_g) is a quasi-model for $\Gamma_F(g) \colon \Lambda \to \Lambda$. But then

$$c(g) \geq c(\Gamma_F(g)) \geq c(T_g) = c(g),$$

where the last equality follows from Proposition 2.23. □

3. APPLICATIONS TO THE UNIT BALL

We begin this section recalling some definitions and results for the unit ball $\mathbb{B}^q \subset \mathbb{C}^q$.

Definition 3.1. The Siegel upper half-space $\mathbb{H}^q$ is defined by

$$\mathbb{H}^q = \left\{ (z, w) \in \mathbb{C} \times \mathbb{C}^{q-1}, \mathsf{Im}\,(z) > \|w\|^2 \right\}.$$

Recall that $\mathbb{H}^q$ is biholomorphic to the ball $\mathbb{B}^q$ via the *Cayley transform* $\Psi \colon \mathbb{B}^q \to \mathbb{H}^q$ defined as

$$\Psi(z, w) = \left(i\frac{1+z}{1-z}, \frac{iw}{1-z} \right), \quad (z, w) \in \mathbb{C} \times \mathbb{C}^{q-1}.$$

Let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian product in $\mathbb{C}^q$. In several complex variables, the natural generalization of the non-tangential limit at the boundary is the following. If $\zeta \in \partial \mathbb{B}^q$, then the set

$$K(\zeta, R) := \{ z \in \mathbb{B}^q : |1 - \langle z, \zeta \rangle| < R(1 - \|z\|) \}$$

is a *Korányi region of vertex ζ and amplitude $R > 1$*. If $f \colon \mathbb{B}^q \to \mathbb{C}^m$ is a holomorphic map, then we say that f has *K-limit $L \in \mathbb{C}^m$ at ζ* (we write $K\text{-}\lim_{z \to \zeta} f(z) = L$) if for each sequence (z_n) converging to ζ such that (z_n) belongs eventually to some Korányi region of vertex ζ, we have that $f(z_n) \to L$.

Definition 3.2. A point $\zeta \in \partial\mathbb{B}^q$ such that $K\text{-}\lim_{z\to\zeta} f(z) = \zeta$ and

$$\liminf_{z\to\zeta} \frac{1 - \|f(z)\|}{1 - \|z\|} = \lambda < +\infty$$

is called a *boundary regular fixed point* (BRFP for short), and λ is called its *dilation*.

The following result by Hervé [26] generalizes the classical Denjoy–Wolff theorem in the unit disc.

Theorem 3.3. *Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be holomorphic. Assume that f admits no fixed points in $\mathbb{B}^q$. Then there exists a point $p \in \partial\mathbb{B}^q$, called the Denjoy–Wolff point of f, such that (f^n) converges uniformly on compact subsets to the constant map $z \mapsto p$. The Denjoy–Wolff point of f is a BRFP and its dilation $\lambda \in (0,1]$.*

Remark 3.4. Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a holomorphic self-map without fixed points, and let λ be the dilation at its Denjoy–Wolff point. Then by [4, Proposition 5.8] the divergence rate of f satisfies

$$c(f) = -\log \lambda.$$

Definition 3.5. A holomorphic self-map $f\colon \mathbb{B}^q \to \mathbb{B}^q$ is called

(1) *elliptic* if it admits a fixed point $z \in \mathbb{B}^q$,
(2) *parabolic* if it admits no fixed points $z \in \mathbb{B}^q$, and its dilation at the Denjoy–Wolff point is equal to 1,
(3) *hyperbolic* if it admits no fixed points $z \in \mathbb{B}^q$, and its dilation at the Denjoy–Wolff point is strictly less than 1.

For a proof of the following result, see, *e.g.*, [15, Proposition 2.9]

Proposition 3.6. *Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be holomorphic. Assume that f admits a BRFP $p \in \partial\mathbb{B}^q$ with dilation $0 < \lambda < 1$. Then f is hyperbolic with Denjoy–Wolff point p.*

In the unit ball $\mathbb{B}^q$ every holomorphic retract is an affine slice, that is, the intersection of $\mathbb{B}^q$ with an affine complex space (see *e.g.*, [1, Corollary 2.2.16]). In particular, every holomorphic retract is (affine) biholomorphic to a (possibly lower dimensional) ball. Also, recall that a *complex geodesic* for $\mathbb{B}^q$ is a holomorphic map $\varphi : \mathbb{D} \to \mathbb{B}^q$ such that for all $\zeta, \xi \in \mathbb{D}$ it holds $k_{\mathbb{D}}(\zeta, \xi) = k_{\mathbb{B}^q}(\varphi(\zeta), \varphi(\xi))$. The image of each complex geodesic for $\mathbb{B}^q$ is a one-dimensional affine slice of $\mathbb{B}^q$, and, conversely, every one-dimensional affine slice of $\mathbb{B}^q$ is the image of a complex geodesic for $\mathbb{B}^q$ (see *e.g.* [1, Corollary 2.6.9]).

By Theorem 2.17, Proposition 2.23 and Remark 3.4 we have the following.

Theorem 3.7. *Le $F = (f_1, \ldots, f_k)$ be a family of commuting holomorphic self-maps of $\mathbb{B}^q$. Then there exists a CSM $(\mathbb{B}^d, \ell, T := (\tau_1, \ldots, \tau_k))$ for F, where $0 \le d \le q$. Moreover, if for some $1 \le j \le k$ the map f_j is hyperbolic with dilation λ_j at its Denjoy–Wolff point, then $d \ge 1$ and τ_j is hyperbolic with dilation λ_j at its Denjoy–Wolff point.*

The following theorem is proved in [14, Thm. 3.3].

Theorem 3.8. *Let $f, g\colon \mathbb{B}^q \to \mathbb{B}^q$ be commuting holomorphic self-maps without inner fixed points. Let $p_f, p_g \in \partial\mathbb{B}^q$ be the respective Denjoy–Wolff points. Assume that $p_f \ne p_g$. Let Δ be the complex geodesic whose closure contains p_f and p_g. Then Δ is invariant for f and g and $f|_\Delta, g|_\Delta$ are commuting hyperbolic automorphisms of Δ.*

Remark 3.9. It follows from Theorem 3.8, that the map f is hyperbolic with a BRFP in p_g and that the dilation of f at p_g is reciprocal to its dilation at p_f. A symmetric statement holds for g.

The next results generalize Theorem 3.8 to a family of commuting holomorphic maps.

Proposition 3.10. *Let A be a set of indices. Let $\{f_i\}_{i \in A}$ be a family of commuting holomorphic self-maps of $\mathbb{B}^q$ without inner fixed points. For all $i \in A$ let $p_i \in \partial \mathbb{B}^q$ be the Denjoy–Wolff point of f_i.*

(1) *If there exists $i \in A$ such that f_i is parabolic, then there exists $p \in \partial \mathbb{B}^q$ such that $p_i = p$ for all $i \in A$.*

(2) *If for all $i \in A$ the map f_i is hyperbolic then*
   i) *either there exists $p \in \partial \mathbb{B}^q$ such that $p_i = p$ for all $i \in A$,*
   ii) *or there exist $p, p' \in \partial \mathbb{B}^q$, $p \neq p'$ such that $p_i \in \{p, p'\}$ for all $i \in A$, and there exist $j, k \in A$ such that $p_j = p$ and $p_k = p'$. Moreover, for all $i \in A$, the complex geodesic Δ whose closure contains p and p' is f_i-invariant, and $f_i|_\Delta : \Delta \to \Delta$ is a hyperbolic automorphism of Δ with p and p' BRFP. In particular, both p and p' are BRFP's for f_i for all $i \in A$.*

Proof. If f_i is parabolic and $p_k \neq p_i$ for some $k \in A$ we immediately reach a contradiction by Remark 3.9.

Suppose now that for all $i \in A$ the map f_i is hyperbolic. Suppose $p_j \neq p_k$ for some $j \neq k$. If for all $m \in A$ we have that $p_m \in \{p_j, p_k\}$, then the result follows applying Theorem 3.8 to the couple of commuting holomorphic self-maps f_j, f_m if $p_j \neq p_m$ or to the couple f_k, f_m if $p_k \neq p_m$.

Thus, in order to conclude the proof, we need to show that if $p_j \neq p_k$ for some $j \neq k$ then for all $m \in A$ it holds $p_m \in \{p_j, p_k\}$. Assume by contradiction that this is not the case, and let $m \in A$ be such that $p_m \notin \{p_j, p_k\}$. For all $t \in A$, let $\lambda_t \in (0, 1)$ be the dilation of f_t at p_t.

By Theorem 3.8, both p_k and p_m are BRFP's for f_j with dilation $1/\lambda_j$. Similarly, p_j and p_m are BRFP's for f_k with dilation $1/\lambda_k$ and p_j and p_k are BRFP's for f_m with dilation $1/\lambda_m$.

We first assume that $\lambda_j < \lambda_k$. Let $g := f_j \circ f_k$. Since p_j is the Denjoy–Wolff point of f_j and a BRFP for f_k, it follows by the chain rule for dilations (see *e.g.* [2, Lemma 2.6]) that p_j is a BRFP for g with dilation given by the product of the dilations of f_k and of f_j at p_j. Therefore, the dilation of g at p_j is $\mu := \frac{\lambda_j}{\lambda_k} < 1$. By Theorem 3.6, p_j is the Denjoy–Wolff point of g. Now, g commutes with f_m, and by Theorem 3.8, it has a BRFP at p_m with dilation $1/\mu$. However, since p_m is a BRFP for both f_j and f_k, the dilation of $g = f_j \circ f_k$ at p_m is given by $\frac{1}{\lambda_j \lambda_k}$. This implies

$$\frac{\lambda_k}{\lambda_j} = \frac{1}{\lambda_j \lambda_k},$$

which gives a contradiction since $\lambda_k < 1$.

A similar argument works in case $\lambda_k < \lambda_j$. If $\lambda_j = \lambda_k$, one can argue as before replacing g with $g = f_j^2 \circ f_k$.   □

Theorem 3.11. *Let $F := (f_1, \ldots, f_k)$ be a k-uple of commuting hyperbolic holomorphic self-maps of $\mathbb{B}^q$. For all $1 \leq j \leq k$ let $0 < \lambda_j < 1$ be the dilation of f_j at its Denjoy–Wolff point. Then there exists a CSM $(\mathbb{B}^d, h, \Theta := (\tau_1, \ldots, \tau_k))$ for F, where $1 \leq d \leq q$, satisfying the following*

properties. For all $1 \leq j \leq k$ the automorphism τ_j is hyperbolic with fixed points $-e_1, e_1$ and the dilation at its Denjoy–Wolff point is λ_j. Moreover, one of the following two cases occurs.

(1) *If the f_j's have the same Denjoy–Wolff point $p \in \partial\mathbb{B}^q$, then $K\text{-}\lim_{z\to p} h(z) = e_1$ and e_1 is the common Denjoy–Wolff point of the τ_j's.*

(2) *If the set of Denjoy–Wolff points of the f_j's is formed by two points $p, p' \in \partial\mathbb{B}^q$, then $K\text{-}\lim_{z\to p} h(z) = e_1$, and $K\text{-}\lim_{z\to p'} h(z) = -e_1$. The point e_1 (respectively $-e_1$) is the Denjoy–Wolff point of τ_j if and only if p (respect. p') is the Denjoy–Wolff point of f_j. Let Δ be the complex geodesic whose closure contains p and p'. Then $h|_\Delta : \Delta \to \mathbb{C}e_1 \cap \mathbb{B}^d$ is an affine bijective transformation.*

Proof. Let $(\mathbb{B}^d, h, \Theta = (\tau_1, \ldots \tau_k))$ be the CSM given by Theorem 3.7. By [23, Proposition 1.8], the mappings $(\tau_1, \ldots \tau_k)$ have their two boundary repelling fixed points in common. Up to conjugacy with an automorphism of $\mathbb{B}^d$ we can assume them to be $-e_1, e_1$.

Now, by Proposition 3.10, there are two cases: either the f_j's have a common Denjoy–Wolff point $p \in \partial\mathbb{B}^q$ or the set of Denjoy–Wolff points if formed by two points $p, p' \in \partial\mathbb{B}^q$.

In the first case, up to conjugating (if necessary) the family Θ with the automorphism $-\mathrm{id}_{\mathbb{B}^d}$, we can assume that e_1 is the Denjoy–Wolff point of τ_1. Now, let $x \in \mathbb{B}^q$. Note that the orbit $(f_1^n(x))$ converges to p and

$$\lim_{n\to\infty} h(f_1^n(x)) = \lim_{n\to\infty} \tau_1^n(h(x)) = e_1. \tag{3.1}$$

Since f_1 is hyperbolic, by [17, Sect. 3.5] the orbit $(f_1^n(x))$ is eventually contained in a Korányi region with vertex in its Denjoy–Wolff point and by [4, Lemma 5.2] it satisfies therefore the hypotheses of [4, Theorem 5.2], hence

$$K\text{-}\lim_{z\to p} h(z) = e_1.$$

Moreover, by the same token, for all $2 \leq j \leq k$ the orbit $(f_j^n(x))$ lies eventually in a Korányi region with vertex p, therefore $h(f_j^n(x)) = \tau_j^n(h(x)) \to e_1$, proving that e_1 is the Denjoy–Wolff point of τ_j for all $1 \leq j \leq k$.

In the second case, we conjugate the family F with an automorphism of $\mathbb{B}^q$ in such a way that $p = e_1$ and $p' = -e_1$. Then $\Delta = \mathbb{D} \times \{0\} \subset \mathbb{C} \times \mathbb{C}^{q-1}$. Let $1 \leq m \leq k$ such that the Denjoy–Wolff point of f_m is e_1. Then up to conjugating (if necessary) the family Θ with the automorphism $-\mathrm{id}_{\mathbb{B}^d}$, we can assume that the Denjoy–Wolff point of τ_m is e_1. Denote

$$h(\zeta, 0, \ldots, 0) := (h_1(\zeta), \ldots, h_d(\zeta)), \quad \zeta \in \mathbb{D}.$$

By the proofs of [23, Proposition 1.3 and 2.7] it follows that $h_1 \colon \mathbb{D} \to \mathbb{D}$ is a hyperbolic automorphism with fixed points $\{1, -1\}$, while $h_i(\zeta) \equiv 0$ for all $2 \leq i \leq d$. This implies that $K\text{-}\lim_{z\to e_1} h(z) = e_1$ and $K\text{-}\lim_{z\to -e_1} h(z) = -e_1$. Arguing as before it is easy to see that for all $1 \leq j \leq k$ such that f_j has Denjoy–Wolff point e_1 (resp. $-e_1$), the automorphism τ_j has Denjoy–Wolff point e_1 (resp $-e_1$). $\qquad\square$

Corollary 3.12. *Let $F := (f_1, \ldots, f_k)$ be a k-uple of commuting hyperbolic holomorphic self-maps of $\mathbb{B}^q$. For all $1 \leq j \leq k$ let $0 < \lambda_j < 1$ be the dilation of f_j at its Denjoy–Wolff point. Then there exists a CSM $(\mathbb{H}^d, \ell, \Phi := (\varphi_1, \ldots, \varphi_k))$ for F with $1 \leq d \leq q$, such that one of the following two cases occurs.*

(1) *If the f_j's share the same Denjoy–Wolff point $p \in \partial \mathbb{B}^q$, then $K\text{-}\lim_{z \to p} \ell(z) = \infty$, and each φ_j has the form*

$$(z, w) \mapsto \left(\frac{1}{\lambda_j} z, \frac{e^{it_1^j}}{\sqrt{\lambda_j}} w_1, \ldots, \frac{e^{it_{d-1}^j}}{\sqrt{\lambda_j}} w_{d-1} \right), \tag{3.2}$$

where $t_a^j \in \mathbb{R}$ for $1 \le a \le d - 1$.

(2) *If the set of Denjoy–Wolff points of the f_j's consists of two points $p, p' \in \partial \mathbb{B}^q$, then*

   i) *if p is the Denjoy–Wolff point of f_j, then $K\text{-}\lim_{z \to p} \ell(z) = \infty$ and φ_j has the form (3.2)*

   ii) *if p' is the Denjoy–Wolff point of f_j, then $K\text{-}\lim_{z \to p'} \ell(z) = 0$ and φ_j has the form*

$$(z, w) \mapsto \left(\lambda_j z, e^{it_1^j} \sqrt{\lambda_j} w_1, \ldots, e^{it_{d-1}^j} \sqrt{\lambda_j} w_{d-1} \right), \tag{3.3}$$

   where $t_a^j \in \mathbb{R}$ for $1 \le a \le d - 1$. Let Δ be the complex geodesic whose closure contains p and p'. Then $\ell|_\Delta : \Delta \to \mathbb{H}$ is a Möbius transformation.

Proof. Let $(\mathbb{B}^d, h, \Theta)$ be the CSM for F given by Theorem 3.11. Let $C : \mathbb{B}^d \to \mathbb{H}^d$ be the Cayley transform from $\mathbb{B}^d$ to $\mathbb{H}^d$ which maps e_1 to ∞ and $-e_1$ to 0. Setting $\tilde{\ell} := C \circ h$ and $\tilde{\varphi}_j := C \circ \tau_j \circ C^{-1}$, it follows that $(\mathbb{H}^d, \tilde{\ell}, (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_k))$ is a CSM for F.

In the first case, note that for each j, the point $\infty \in \partial \mathbb{H}^d$ is the Denjoy–Wolff point of $\tilde{\varphi}_j$, and the point 0 is a BRFP. By [1, Proposition 2.2.11], for all $1 \le j \le k$ there exists a unitary $(d-1) \times (d-1)$-matrix U_j such that $\tilde{\varphi}_j$ has the form

$$(z, w) \mapsto \left(\frac{1}{\lambda_j} z, \frac{1}{\sqrt{\lambda_j}} U_j w \right).$$

Since the unitary matrices $U_1, \ldots, U_k$ commute, there exists a unitary matrix Q such that for all $1 \le j \le k$ we have

$$Q U_j Q^{-1} = D_j,$$

where D_j is a diagonal unitary matrix

$$D_j := \mathsf{diag}(e^{it_1^j}, \ldots, e^{it_{d-1}^j}),$$

with $t_a^j \in \mathbb{R}$ for all $1 \le a \le d - 1$. Define an automorphism of $\mathbb{H}^d$ by setting $\Psi(z, w) = (z, Q w)$. For all $1 \le j \le k$ set $\varphi_j := \Psi \circ \tilde{\varphi}_j \circ \Psi^{-1}$, and set $\Phi := (\varphi_1, \ldots \varphi_k)$, $\ell := \Psi \circ \tilde{\ell}$. Then $(\mathbb{H}^d, \ell, \Phi)$ is a CSM for F satisfying the required properties. The second case is similar. $\qquad \square$

In the remaining of the section we study obstructions to commutation in the unit ball. Cowen in [20, Corollary 4.1] proved that if $f, g : \mathbb{D} \to \mathbb{D}$ are commuting holomorphic maps different from the identity, then f and g are either both elliptic, or both hyperbolic, or both parabolic. In higher dimensions the situation is far more complicated. Examples of elliptic maps different from the identity commuting with hyperbolic or parabolic maps can be easily constructed in $\mathbb{B}^q$ for $q \ge 2$. On the other hand we can say something about such elliptic self-maps.

Remark 3.13. Let $q \geq 2$. Let $f \colon \mathbb{B}^q \to \mathbb{B}^q$ be an elliptic self-map and assume that it commutes with a parabolic or hyperbolic self-map $g \colon \mathbb{B}^q \to \mathbb{B}^q$. Then $\mathsf{Fix}(f)$ is an affine slice of $\mathbb{B}^q$ of nonzero dimension, and the Denjoy–Wolff point of g is in $\overline{\mathsf{Fix}(g)}$.

The next natural question is if a hyperbolic mapping can commute with a parabolic mapping in $\mathbb{B}^q$. If $q = 1$ this is not possible by Cowen's Theorem. Notice that in dimension one every hyperbolic map is of automorphism type. The following result generalizes Cowen's theorem to several variables assuming that the hyperbolic map is of automorphism type.

Theorem 3.14. *Let $f \colon \mathbb{B}^q \to \mathbb{B}^q$ be a hyperbolic holomorphic self-map of automorphism type. If $g \colon \mathbb{B}^q \to \mathbb{B}^q$ is a holomorphic self-map commuting with f, then g is not parabolic.*

Proof. By [3, Theorem 4.6], f admits a CSM $(\mathbb{H}^q, h, \varphi)$, where $\varphi \colon \mathbb{H}^q \to \mathbb{H}^q$ is a hyperbolic automorphism of the form

$$\varphi(z, w) = \left(\frac{1}{\lambda} z, \frac{e^{it_1}}{\sqrt{\lambda}} w_1, \ldots, \frac{e^{it_{q-1}}}{\sqrt{\lambda}} w_{q-1} \right). \tag{3.4}$$

The map g induces a holomorphic self-map $\Gamma_f(g) \colon \mathbb{H}^q \to \mathbb{H}^q$ commuting with φ and satisfying

$$h \circ g = \Gamma_f(g) \circ h. \tag{3.5}$$

By [23, Propositions 1.3 and 1.7], the mapping $\Gamma_f(g)$ leaves the one-dimensional slice $\Delta := \{w = 0\} \subset \mathbb{H}^q$ invariant and there exists $a > 0$ such that

$$\Gamma_f(g)(z, 0) = (az, 0), \quad \forall z \in \mathbb{H}.$$

If $a \neq 1$, then $\Gamma_f(g)$ is hyperbolic, and hence g is also hyperbolic. Assume now that $a = 1$, that is, $\Gamma_f(g)$ fixes every point of the slice Δ. Since $\mathbb{H}^q = \bigcup_{n \geq 0} \varphi^{-n} h(\mathbb{B}^q)$, it follows that $h^{-1}(\Delta) \neq \varnothing$. Notice that $h^{-1}(\Delta)$ is f-invariant. Let $x \in h^{-1}(\Delta)$. Let $r > k_{\mathbb{B}^q}(x, g(x))$. By Proposition 2.28, there exists $n_0 = n_0(x, r)$ such that for all $n \geq n_0$ the mapping h is univalent on the Kobayashi ball $B(f^n(x), r)$. We have that

$$k_{\mathbb{B}^q}(g(f^n(x)), f^n(x)) = k_{\mathbb{B}^q}(f^n(g(x)), f^n(x)) \leq k_{\mathbb{B}^q}(g(x), x) < r,$$

thus $g(f^n(x)) \in B(f^n(x), r)$. Since $f^n(x) \in h^{-1}(\Delta)$, it follows by (3.5) that $h(g(f^n(x))) = h(f^n(x))$, which implies that $g(f^n(x)) = f^n(x)$. Hence g is elliptic. $\square$

Hence if the type of the hyperbolic mapping is maximal, then it cannot commute with a parabolic mapping. Other obstructions to commutation can be found if the type of the parabolic mapping is too small, as the following results show.

Proposition 3.15. *Let $f \colon \mathbb{B}^q \to \mathbb{B}^q$ be a hyperbolic holomorphic self-map and let $g \colon \mathbb{B}^q \to \mathbb{B}^q$ be a parabolic holomorphic self-map of type 0. Then f and g do not commute.*

Proof. Assume by contradiction that f and g commute. Let (Z, ℓ, T) be a CSM for the pair (f, g). Since f is hyperbolic we have by Corollary 2.25 that $\dim Z \geq 1$. Since g has type 0, it follows from Corollary 2.29 that $\dim Z = 0$, contradiction. $\square$

Proposition 3.16. *Let $f \colon \mathbb{B}^q \to \mathbb{B}^q$ be a hyperbolic holomorphic self-map and let $g \colon \mathbb{B}^q \to \mathbb{B}^q$ be a parabolic holomorphic self-map of type 1. If f and g commute, then g admits a CSM of the form $(\mathbb{D}, h, \mathsf{id})$.*

Proof. Since g is of type 1, it admits a CSM of the form $(\mathbb{D}, h, \varphi)$, where φ is not hyperbolic. Let $\Gamma_g(f) \colon \mathbb{D} \to \mathbb{D}$ be the holomorphic self-map induced by f. By Proposition 2.36 the mapping $\Gamma_g(f)$ is hyperbolic. Since $\Gamma_g(f)$ commutes with φ, it follows from Cowen's theorem [20, Corollary 4.1] (or Theorem 3.14 for $q = 1$) that $\varphi = \mathsf{id}_{\mathbb{H}}$. $\square$

Corollary 3.17. *Let $f : \mathbb{B}^q \to \mathbb{B}^q$ be a hyperbolic holomorphic self-map and let $g \colon \mathbb{B}^q \to \mathbb{B}^q$ be a parabolic holomorphic self-map of type 1 such that there exists $x \in \mathbb{B}^q$ with $s_1^g(x) > 0$. Then f and g do not commute.*

Remark 3.18. It is actually an open question whether there exists a parabolic self map $g \colon \mathbb{B}^q \to \mathbb{B}^q$ with type $g \geq 1$ such that there exists $x \in \mathbb{B}^q$ with $s_1^g(x) = 0$. In other words, it is an open question whether there exists a parabolic self map $g \colon \mathbb{B}^q \to \mathbb{B}^q$ whose CSM $(\mathbb{B}^k, h, \varphi)$ is such that $k \geq 1$ and φ is elliptic.

The following proposition shows that there cannot be more type-related obstructions to the commutation of a hyperbolic and a parabolic mapping.

Proposition 3.19. *Let $m \in \mathbb{N}$, $m \geq 2$. Let $1 \leq q \leq m - 1$, and let $2 \leq p \leq m$. Then there exist commuting holomorphic self-maps $f, g \colon \mathbb{H}^m \to \mathbb{H}^m$ such that f is hyperbolic of type q and g is parabolic of type p.*

Proof. Let $r \in \mathbb{R}$, $r \neq 0$. Denote a point in $\mathbb{C}^{m+2}$ by (z, w, y), where $z, w \in \mathbb{C}$ and $y \in \mathbb{C}^m$. With a slight abuse of notation, if $m = 2$, the triple (z, w, y) simply denotes the point (z, w). Consider the holomorphic self maps f, g of $\mathbb{H}^m$ defined by

$$f(z, w, y) := (2z + iw^2, w, \sqrt{2}y_1, \dots, \sqrt{2}y_{q-1}, 0, \dots, 0)$$

$$g(z, w, y) := (z + ir^2 - 2rw, w - ir, y_1, \dots, y_{p-2}, 0, \dots, 0).$$

It is easy to see that f is hyperbolic and g is parabolic and that they commute. We claim that the type of f is q, the type of g is p, and that the type of the pair (f, g) is $\min(p - 1, q)$.

To prove that the type of f is q, we will study the univalent self-map $\varphi \colon \mathbb{H}^{q+1} \to \mathbb{H}^{q+1}$ defined by

$$\varphi(z, w, y_1, \dots, y_{q-1}) := (2z + iw^2, w, \sqrt{2}y_1, \dots, \sqrt{2}y_{q-1}),$$

which clearly has the same type as f. Notice that φ actually defines a univalent mapping on the whole $\mathbb{C}^{q+1}$. Set

$$\Omega := \{x \in \mathbb{C}^{q+1} \colon \varphi^n(x) \in \mathbb{H}^{q+1} \text{ eventually}\},$$

then $(\Omega, \iota \colon \mathbb{H}^{q+1} \to \Omega, \varphi)$ is a model for φ. For all $n \geq 1$ and $(z, w, y_1, \dots, y_{q-1}) \in \mathbb{C}^{q+1}$,

$$\varphi^n(z, w, y_1, \dots, y_{q-1}) = (2^n z + (2^n - 1)iw^2, w, 2^{\frac{n}{2}}y_1, \dots, 2^{\frac{n}{2}}y_{q-1})$$
$$= (2^n(z + iw^2) - iw^2, w, 2^{\frac{n}{2}}y_1, \dots, 2^{\frac{n}{2}}y_{q-1}).$$

The point $\varphi^n(z, w, y_1, \dots, y_{q-1})$ is in $\mathbb{H}^{q+1}$ if and only if

$$2^n \left(\mathsf{Im}\,(z + iw^2) - \sum_{j=1}^{q-1} |y_j|^2 \right) > |w|^2 + \mathsf{Im}\,(iw^2),$$

and thus

$$\Omega = \{(z, w, y_1, \ldots, y_{q-1}) \in \mathbb{C}^{q+1} \colon \mathsf{Im}\,(z + iw^2) > \sum_{j=1}^{q-1} |y_j|^2\}.$$

The domain Ω is biholomorphic to $\mathbb{H}^q \times \mathbb{C}$. Indeed the mapping $\eta \colon \Omega \to \mathbb{H}^q \times \mathbb{C}$ defined by

$$\eta(z, w, y_1, \ldots, y_{q-1}) := ((z + iw^2, y_1, \ldots, y_{q-1}), w),$$

is a biholomorphism with inverse

$$\eta^{-1}((\zeta, \xi_1, \ldots, \xi_{q-1}), \gamma) = (\zeta - i\gamma^2, \gamma, \xi_1, \ldots, \xi_{q-1}).$$

Define $\ell \colon \mathbb{H}^{q+1} \to \mathbb{H}^q$ by $\ell(z, w, y) := (z + iw^2, y_1, \ldots, y_{q-1})$. Denote by $\Phi \colon \mathbb{H}^q \to \mathbb{H}^q$ the hyperbolic automorphism defined by $\Phi(\zeta, \xi) = (2\zeta, \sqrt{2}\xi)$, where $\zeta \in \mathbb{C}$ and $\xi \in \mathbb{C}^{q-1}$. Notice that

$$\ell(\varphi(z, w, y_1, \ldots, y_{q-1})) = (2z + 2iw^2, \sqrt{2}y_1, \ldots, \sqrt{2}y_{q-1}) = \Phi(\ell(z, w, y_1, \ldots, y_{q-1})).$$

Then a CSM for φ is given by $(\mathbb{H}^q, \ell, \Phi)$, and this proves that type $f = $ type $\varphi = q$.

Let now $\pi \colon \mathbb{H}^m \to \mathbb{H}^p$ be given by $\pi(z, w, y) = (z, w, y_1, \ldots, y_{p-2})$, and let $\Psi \colon \mathbb{H}^p \to \mathbb{H}^p$ be the parabolic automorphism defined by $\Psi(a, b, c) := (a + ir^2 - 2rb, b - ir, c)$, where $a, b \in \mathbb{C}$ and $c \in \mathbb{C}^{p-2}$. A CSM for g is easily seen to be $(\mathbb{H}^p, \pi, \Psi)$, and thus type $g = p$. Set $u := \min(p-2, q-1)$. Then

$$\Gamma_g(f)(a, b, c) = (2a + ib^2, b, \sqrt{2}c_1, \ldots, \sqrt{2}c_u, 0, \ldots, 0).$$

Applying the previous argument we see that the type $\Gamma_g(f) = u + 1$ and thus Theorem 2.35 yields type $(f, g) = u + 1$.

$\square$

Remark 3.20. Let $r \in \mathbb{R}$, $r \neq 0$. The commuting holomorphic self-maps of $\mathbb{H}^2$ defined by

$$f(z, w) := (2z + iw^2, w), \quad g(z, w) := (z + ir^2 - 2rw, w - ir),$$

(which correspond to the minimal dimension case $m = 2$ in Proposition 3.19) were studied in [29] to show the existence of a holomorphic self-map of $\mathbb{B}^2$ with a non-isolated boundary repelling fixed point.

Remark 3.21. It follows from Proposition 3.19 that type (f, g) can be strictly smaller than $\min\{\text{type}(f), \text{type}(g)\}$.

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