Coarse assembly maps

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Abstract

A coarse assembly map relates the coarsification of a generalized homology theory with a coarse version of that homology theory. In the present paper we provide a motivic approach to coarse assembly maps. To every coarse homology theory E we naturally associate a homology theory $E\mathcal{O}^{\infty}$ and construct an assembly map

$$\mu_E : \mathrm{Coarsification}(E\mathcal{O}^{\infty}) \to E \ .$$

For sufficiently nice spaces X we relate the value $E\mathcal{O}^{\infty}(X)$ with the locally finite homology of X with coefficients in $E(*)$. In the example of coarse K-homology we discuss the relation of our motivic constructions with the classical constructions using C^*-algebra techniques.

Contents

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1 Introduction

In the present paper we propose a definition of a coarse assembly map for every strong coarse homology theory and every bornological coarse space. We further study conditions which imply that the coarse assembly map is an equivalence.

Classically, for a separable bornological coarse space X of bounded geometry, the analytic coarse assembly map is a homomorphism of $\mathbb{Z}$-graded groups

$$\mu_X^{an} : QK_*^{an,lf}(X) \to K\mathcal{X}_*(X) \tag{1.1}$$

from the coarsified analytic locally finite K-homology groups $QK_*^{an,lf}(X)$ to the coarse K-homology groups of X; see Roe [Roe96], Higson–Roe [HR95], Yu [Yu95b], Roe–Siegel [RS12] or Definition 16.10. The coarse Baum–Connes conjecture predicts conditions on the space X which imply that the analytic coarse assembly map is an isomorphism; see, e.g., Higson–Roe [HR95], Yu [Yu95a], Skandalis–Tu–Yu [STY02] or Wright [Wri05].

In the present paper we are interested in a refinement of the coarse assembly map from a $\mathbb{Z}$-graded group homomorphism to a morphism between K-theory spectra, see (1.6). Furthermore we ask for generalizations of the coarse assembly map to other coarse homology theories and study its functorial properties, e.g., the compatibility with Mayer–Vietoris sequences.

1.1 Constructing coarse assembly maps

In the following we describe the set-up in which we will construct the coarse assembly map. The basic category is the category **BornCoarse** of bornological coarse spaces introduced in [BE16]. Let $\mathbf{C}$ be a cocomplete stable ∞-category, e.g., the ∞-category of spectra $\mathbf{Sp}$. A $\mathbf{C}$-valued coarse homology theory is a functor

$$E : \mathbf{BornCoarse} \to \mathbf{C}$$

which is coarsely invariant, coarsely excisive, u-continuous, and vanishes on flasques. We refer to [BE16] for a detailed description of these properties. In order to study properties of coarse homology theories in general we constructed in [BE16] a universal coarse homology theory

$$\mathrm{Yo}^s : \mathbf{BornCoarse} \to \mathbf{Sp}\mathcal{X}$$

with values in the stable ∞-category of motivic coarse spectra $\mathbf{Sp}\mathcal{X}$. A $\mathbf{C}$-valued coarse homology theory as above is then equivalently described as a colimit preserving functor

$$E : \mathbf{Sp}\mathcal{X} \to \mathbf{C} \ .$$

Locally finite homology theories are defined on the category **TopBorn** of topological bornological spaces and proper continuous maps [BE16, Sec. 6.5]. A functor

$$H^{lf} : \mathbf{TopBorn} \to \mathbf{C}$$

is a locally finite homology theory if, in addition to the usual homological conditions of excision and homotopy invariance, it satisfies the local finiteness condition that the natural map

$$H^{lf}(X) \to \lim_{B} \mathrm{Cofib}(H^{lf}(X \setminus B) \to H^{lf}(X)) \tag{1.2}$$

is an equivalence for every bornological topological spaces X, where the limit runs over the bounded subsets of X. Every homology theory H has a corresponding locally finite version H^{lf} [BE16, Def. 6.48].

A particular class of coarse homology theories are coarsifications QH^{lf} of locally finite homology theories H^{lf} [BE16, Sec. 6.6]. In contrast to general coarse homology theories, coarsifications of locally finite homology theories seem to be much more tractable because they can be studied by well-established methods of homotopy theory. One could ask whether there are other pairs (besides coarse K-homology $K\mathcal{X}$ and analytic locally finite K-homology $K^{an,lf}$) of a coarse homology theory E and a locally finite homology theory H^{lf} which are related by a coarse assembly map $QH^{lf} \to E$.

Locally finite homology theories are characterized by a limit condition (1.2). It is therefore complicated to construct maps out of locally finite homology theories. The main novelty of the present paper is to introduce the notion of a local homology theory, essentially by replacing the condition of being locally finite by the weaker condition of vanishing on flasques, see Definition 3.10.

In the following we explain this in greater detail. We introduce the category of uniform bornological coarse spaces **UBC**. A **C**-valued local homology theory is then a functor

$$F : \mathbf{UBC} \to \mathbf{C}$$

which is homotopy invariant, excisive, u-continuous, and vanishes on flasques. We will actually construct a universal local homology theory

$$\mathrm{Yo}^s\mathcal{B} : \mathbf{UBC} \to \mathbf{Sp}\mathcal{B}$$

with values in motivic uniform bornological coarse spectra (Corollary 4.16). Note that the nature of the local finiteness condition (1.2) makes it impossible to construct a universal locally finite homology theory in a similar manner. Similarly as in the case of coarse homology theories, a **C**-valued local homology theory is equivalently described as a colimit preserving functor

$$F : \mathbf{Sp}\mathcal{B} \to \mathbf{C} .$$

Any locally finite homology theory H^{lf} gives rise to a local homology theory which in the notation of the present paper appears as $H^{lf} \circ F_{\mathcal{C},\mathcal{U}/2}$ in Lemma 3.13.

A uniform bornological coarse space has an underlying bornological coarse space. But if we simply forget the uniform structure, then we completely lose the local topological structure of the space. A more interesting transition from uniform bornological coarse spaces to bornological coarse spaces keeping the local structure is given by the cone construction. Indeed, with the help of the cone one can encode the uniform structure into a suitable coarse structure.

The cone construction will be investigated in various versions in Section 8; the main version is in Definition 8.1, [BE16, Ex. 5.16] and [BEKWb, Def. 9.24]. It provides a functor

$$\mathcal{O} : \mathbf{UBC} \to \mathbf{BornCoarse} .$$

The cone and the germs at infinity

$$\mathcal{O}^\infty : \mathbf{UBC} \to \mathbf{Sp}\mathcal{X}$$

of the cone (see Definition 8.2, [BE16, Ex. 5.23] and [BEKWb, Sec. 9.5]) can be used to pull-back coarse homology theories to functors defined on **UBC**. If E is a strong (i.e., vanishes on weakly flasques) coarse homology theory, then its pull-backs $E\mathcal{O}^\infty$ and $E\mathcal{O}$ are local homology theories, see Lemma 9.5.

The idea to use some version of cones in order to pull-back coarse homology theories has some history. We refer to Higson–Pederson–Roe [HPR96, Prop. 12.1] (coarse K-homology), Mitchener [Mit10, Thm. 4.9] (coarsely excisive theories), Bartels–Farrell–Jones–Reich [BFJR04, Sec. 5] (equivariant coarse algebraic K-homology), or Weiss [Wei02] (algebraic K-theory of additive categories and retractive spaces) as entry points to the literature.

Given an entourage U of a bornological coarse space X we can form the Rips complex $P_U(X)$ at scale U, see Example 2.6. It is a simplicial complex which will be equipped with the metric induced by the spherical metric on its simplices. The metric induces a coarse and a uniform structure on $P_U(X)$, and the family of subsets $(P_U(B))_{B\in\mathcal{B}}$ (where $\mathcal{B}$ denotes the bornology of X) generates the bornology of the uniform bornological coarse space $P_U(X)$. There is a canonical embedding of sets $X \hookrightarrow P_U(X)$ which induces an equivalence of bornological coarse spaces $X_U \to F_{\mathcal{U}}(P_U(X))$.

On the one hand the family $(F_{\mathcal{U}}(P_U(X)))_{U\in\mathcal{C}}$ (where $\mathcal{C}$ denotes the coarse structure of X) of underlying bornological coarse spaces of the Rips complexes approximates the space X. One the other hand, forming the colimit of the motivic uniform bornological coarse spectra represented by the Rips complexes we obtain a functor

$$\mathbf{P} : \mathbf{BornCoarse} \to \mathbf{Sp}\mathcal{B}$$

called the universal coarsification, see Definition 5.4. In detail,

$$\mathbf{P}(X) \simeq \operatorname{colim}_{U\in\mathcal{C}} \mathrm{Yo}^s\mathcal{B}(P_U(X)) \ .$$

By Proposition 5.2 the functor $\mathbf{P}$ is a $\mathbf{Sp}\mathcal{B}$-valued coarse homology theory and can therefore be interpreted as a colimit preserving functor

$$\mathbf{P} : \mathbf{Sp}\mathcal{X} \to \mathbf{Sp}\mathcal{B} \ .$$

Pull-back along $\mathbf{P}$ associates to every $\mathbf{C}$-valued local homology theory F a $\mathbf{C}$-valued coarse homology theory $F\mathbf{P}$, see Definition 5.5.

For a locally finite homology theory H^{lf} the coarsification of the local homology theory $H^{lf} \circ F_{\mathcal{C},\mathcal{U}/2}$ induced from H^{lf} coincides with the coarsification QH^{lf} from [BE16, Sec. 6.6] which we have discussed earlier, i.e., we have an equivalence

$$QH^{lf} \simeq (H^{lf} \circ \mathcal{F}_{\mathcal{C},\mathcal{U}/2})\mathbf{P} \ .$$

Let us state the main construction of the paper. Let E be a coarse homology theory.

Definition 1.1 (Definition 9.6). *If E is strong, then the coarse assembly map is the natural transformation between coarse homology theories*

$$\mu_E : E\mathcal{O}^\infty\mathbf{P} \to \Sigma E \ ,$$

derived from the boundary map of the cone sequence.

1.2 Isomorphism results and computations

In Section 10 we study various conditions on the coarse homology theory E and the bornological coarse space X which imply that the coarse assembly map

$$\mu_{X,E} : E\mathcal{O}^{\infty}\mathbf{P}(X) \to \Sigma E(X)$$

is an equivalence. Let us mention the following two results which are analogues of instances of the coarse Baum–Connes conjecture.

Let X be a bornological coarse space and E be a strong coarse homology theory.

Theorem 1.2 (Theorem 10.3)**.** *If X admits a cofinal set of entourages U such that X_U has finite asymptotic dimension, then the coarse assembly map $\mu_{E,X}$ is an equivalence.*

Let K be a simplicial complex and K_d be the corresponding uniform bornological coarse space whose structures are induced from the path metric induced by the spherical metric on the simplices. Then $F_{\mathcal{U}}(K_d)$ denotes the underlying bornological coarse space of K_d. Furthermore let E be a $\mathbf{C}$-valued coarse homology theory.

Theorem 1.3 (Corollary 10.21)**.** *Assume:*

1. *E is strong, countably additive, and admits transfers.*
2. *$\mathbf{C}$ is presentable.*
3. *K has bounded geometry.*
4. *K_d is equicontinuously contractible.*
5. *K_d admits a coarse scaling.*

Then the coarse assembly map $\mu_{E,F_{\mathcal{U}}(K_d)}$ is an equivalence.

We refer to Section 10 for a detailed description of the assumptions.

Let $E : \mathbf{BornCoarse} \to \mathbf{C}$ be a coarse homology theory. In general the local homology theory $E\mathcal{O}^{\infty}$ seems to be quite complicated. But if E is additive, then on nice spaces it behaves like a locally finite homology theory. Concretely, we have the following result.

Let X be a uniform bornological coarse space.

Proposition 1.4 (Proposition 12.17)**.** *Assume:*

1. *$\mathbf{C}$ is presentable.*
2. *E is countably additive, see* (12.3)*.*
3. *X is small (Definition 11.1).*
4. *X is homotopy equivalent in* **UBC** *to a countable, locally finite, finite-dimensional simplicial complex.*

Then we have a natural equivalence

$$(\Sigma E(*) \wedge \Sigma^{\infty}_{+})^{lf}(X) \simeq E\mathcal{O}^{\infty}(X) \ . \tag{1.3}$$

The left-hand side of (1.3) is the value on the underlying bornological topological space of X of the locally finite version of the homology represented by the spectrum $\Sigma E(*)$, see Definition 12.1.

The next proposition is a consequence of Proposition 12.17 applied to Rips complexes. It provides, under appropriate conditions, a calculation of the domain of the coarse assembly map.

Let E be a coarse homology theory and X be a bornological coarse space.

Proposition 1.5 (Proposition 13.2)**.** *Assume:*

1. $\mathbf{C}$ *is presentable.*
2. E *is countably additive.*
3. X *is separable and of bounded geometry.*

Then we have a natural equivalence

$$((\Sigma E(*) \wedge \Sigma^{\infty}_{+})^{lf} \circ F_{\mathcal{C},\mathcal{U}/2})\mathbf{P}(X) \simeq E\mathcal{O}^{\infty}\mathbf{P}(X) \ .$$

Assume that $E \to E'$ is a natural transformation between coarse homology theories such that $E(*) \to E'(*)$ is an equivalence. Then we can use Proposition 1.5 in order to show for a bornological coarse space X that $E(X) \to E'(X)$ is an equivalence if the assembly maps $\mu_{E,X}$ and $\mu_{E',X}$ are equivalences, see Theorem 13.3. The precise statement is the following:

Theorem 1.6 (Theorem 13.3)**.** *Assume:*

1. $\mathbf{C}$ *is presentable.*
2. E *and* E' *are strong and countably additive.*
3. $E(*) \to E'(*)$ *is an equivalence.*
4. X *is separable and of bounded geometry.*
5. *The assembly maps* $\mu_{E,X}$ *and* $\mu_{E',X}$ *are equivalences.*

Then $E(X) \to E'(X)$ *is an equivalence.*

It is tempting to apply Theorem 1.6 to the transformation $\mu_E : E\mathcal{O}^{\infty}\mathbf{P} \to \Sigma E$ in order to show that $\mu_{E,X}$ is an equivalence. But in view of Assumption 1.6.5 this would lead to a circular argument.

1.3 The case of coarse K-homology

In the remaining part of this introduction we will discuss the application of our theory to K-homology. Note that coarse K-homology $K\mathcal{X}$ is countably additive and therefore the Proposition 1.5 can be applied. We can choose an identification of spectra

$$K^{an,lf}(*) \simeq K\mathcal{X}(*) \tag{1.4}$$

since both are equivalent to KU. This choice induces an equivalence

$$K^{an,lf}(K) \simeq (K\mathcal{X}(*) \wedge \Sigma^\infty_+)^{lf}(K) \tag{1.5}$$

for every countable and finite-dimensional simplicial complex K, see [BE16, Prop. 6.73] and [BE16, Ex. 6.72]. As a consequence, if X is separable and of bounded geometry, then we get the following formula for the domain of the coarse assembly map

$$K\mathcal{X}\mathcal{O}^\infty\mathbf{P}(X) \overset{\text{Prop. 13.2}}{\simeq} (K\mathcal{X}(*) \wedge \Sigma^\infty_+)^{lf}(F_{\mathcal{C},\mathcal{U}/2}(\mathbf{P}(X))) \overset{(1.5)}{\simeq} K^{an,lf}(F_{\mathcal{C},\mathcal{U}/2}(\mathbf{P}(X))) \simeq QK^{an,lf}(X)\ .$$

The coarse assembly map $\mu_{K\mathcal{X},X}$ can therefore be interpreted as a morphism of spectra

$$\mu^{top}_X : QK^{an,lf}(X) \to K\mathcal{X}(X) \tag{1.6}$$

which induces a map on homotopy groups as in (1.1). This solves one of the problems stated at the beginning of this introduction.

Note that the group of automorphisms of the spectrum KU is huge. Hence there are many choices for the identification (1.4). The equivalence (1.5) and hence the assembly map μ^{top}_X depend non-trivially on this choice. Therefore, in order to fix a canonical identification of the assembly maps μ^{an}_X and μ^{top}_X, we must fix the identification (1.4) appropriately.

An idea in this direction would be to observe that both sides of (1.4) are ring spectra in a natural way. One could then require that (1.4) is an equivalence of ring spectra. We will not discuss this problem further in the present paper.

But we can show that μ^{top}_X induces an equivalence if and only if μ^{an}_X is an isomorphism of $\mathbb{Z}$-graded groups, see Corollary 16.12. Consequently, Theorem 1.2 and Theorem 1.3 imply instances of the coarse Baum-Connes conjecture. But note that these cases were known before by the work of Higson–Roe [HR95], Wright [Wri05] and Yu [Yu98].

The construction of the coarse assembly map as a transformation between coarse homology theories automatically implies its compatibility with the boundary maps in Mayer–Vietoris sequences. We derive the corresponding statement for our version of the assembly map μ^{top}_X in Corollary 14.1. In contrast, for the analytic assembly map μ^{an}_X (1.1) (recall that μ^{an}_X is only defined as a transformation between $\mathbb{Z}$-graded group-valued functors) this compatibility is a non-trivial issue as we explain in the following. For a proper metric space X the coarse analytic assembly map is obtained from the analytic assembly map (which we will recall in Definition 16.7)

$$A_X : K^{an,lf}_*(X) \to K\mathcal{X}_*(X) \tag{1.7}$$

by the process of coarsification (Remark 16.9). The construction of the map A_X requires the choice of an ample Hilbert space on X and uses Paschke duality, see Higson–Roe [HR00] (see also the discussion in [BE16, Sec. 7.10]). There is also the alternative construction in Skandalis–Tu–Yu [STY02, Def. 2.9]. At the moment, Paschke duality or the alternative construcion from [STY02] are understood as maps between K-theory groups, but not as morphisms of spectra. This is the reason that (1.7) is only a map between K-theory groups. The analytic coarse assembly map (1.1) is natural in X, but because of the choices made during the construction this naturality is already difficult to establish.

Siegel [Sie12] has shown, by studying various explicit descriptions of the boundary map in K-theory associated to an exact sequence of C^*-algebras, that A_X is compatible with the boundary map in the Mayer–Vietoris sequence associated to a decomposition of X. For this result he adopts a fixed choice of an ample Hilbert space on X which provides the ample Hilbert spaces on the pieces of X.

In order to show that μ_X^{an} is compatible with the boundary map in the Mayer–Vietoris sequence we would need uniqueness results for the identification of different ample Hilbert spaces up to contractible choice. It seems that such results have not appeared in the literature so far — the currently available uniqueness results are only up to homotopy.

The fact that we can show the compatibility of μ_X^{top} with Mayer–Vietoris sequences does not imply the corresponding compatibility for μ_X^{an}. The problem is that, at the moment, we do not have a *natural* identification between μ_X^{top} and μ_X^{an}, see Remark 16.11.

In Section 15 we will observe that $\mathcal{O}^\infty(X)$ is actually a representable motive. If X is a complete Riemannian manifold, then this will be used in Section 17 in order to provide examples of classes in $K\mathcal{X}\mathcal{O}^\infty_*(X)$ represented by coarse indices of Dirac operators.

We now switch from X to M in order to denote the uniform bornological coarse space associated to a complete Riemannian manifold M. More concretely, given a generalized Dirac operator $\not{D}$ of degree n on M we will construct a class $\sigma(\not{D})$ in $K\mathcal{X}\mathcal{O}^\infty_{n+1}(M)$, see Definition 17.3. This class is an analog of the symbol class of $\not{D}$. For a discussion of the precise relation with the classical notion of a symbol of $\not{D}$ we refer to the end of Section 17 and especially Problem 17.6. The symbol class can further be promoted to a locally finite K-homology class $Q\tilde{\sigma}^{an,lf}(\not{D})$ in $QK^{an,lf}(X)$. We the argue that the coarse assembly map μ_M^{top} sends the symbol class to the coarse index (17.4):

$$\mu_M^{top}(Q\tilde{\sigma}^{an,lf}(\not{D})) = \mathtt{Ind}\mathcal{X}(\not{D}) .$$

This paper is written as an addendum to [BE16] to which we refer for details on coarse and on locally finite homology theories and for more references to the literature.

In the present paper coarse K-homology is considered as a motivating example. But we do not want to put the analytic details to much into the foreground. The index theoretic facts used in the present paper are special cases of results to appear in [BE17] and [Bun]. We also refer to these papers for more references to the previous literature on coarse index theory. For readers interested precisely in coarse index theory it might be unsatisfying that we do not provide answers to the questions raised in Problems 16.6, 16.8 and 17.6. We think that a satisfying solution would require another much more analytic paper.

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2 Uniform bornological coarse spaces

In this section we introduce the category of uniform bornological coarse spaces.

To start with we consider a set equipped with a bornology and a coarse structure. The bornology and the coarse structure are called compatible ([BE16, Def. 2.5]) if the bornology is invariant under thickening with respect to coarse entourages.

A set equipped with compatible bornological and coarse structures is an object of the category **BornCoarse** of bornological coarse spaces. A morphism between bornological coarse spaces is a proper and controlled map. We refer to [BE16, Sec. 2] for a detailed study of the category **BornCoarse**.

We now consider a set with a coarse structure and a uniform structure. The structures are called compatible ([BE16, Def. 5.4]) if there exists an entourage with is both coarse and uniform.

Let $(X,\mathcal{U})$ be a uniform space.

Definition 2.1. *The coarse structure associated to the uniform structure is defined by*

$$\mathcal{C}(\mathcal{U}) := \bigcap_{V\in\mathcal{U}} \mathcal{C}\langle\{V\}\rangle \ .$$

Here $\mathcal{C}\langle\{V\}\rangle$ denotes the coarse structure generated by V ([BE16, Ex. 2.11]).

Example 2.2. The coarse structure $\mathcal{C}(\mathcal{U})$ is not necessarily compatible with the uniform structure $\mathcal{U}$. Let us construct an example of such a space. We let $X := \{0\}\cup\{1/n\,|\,n\in\mathbb{N}\}$ and the uniform structure $\mathcal{U}$ of X is defined to be the one induced from the natural metric on X coming from the canonical inclusion $X \subseteq \mathbb{R}$. This uniform structure is generated by the uniform entourages U_r for all $r > 0$ given by $U_r := \{(x, y) \in X \mid d(x, y) < r\}$. Now a moment of reflection reveals $\mathcal{C}(\mathcal{U}) = \mathcal{C}\langle\{\mathtt{diag}_X\}\rangle$, which is not compatible with $\mathcal{U}$. □

Example 2.3. The notion of a quasi-metric on a set is defined similary as the notion of a metric where one in addition allows that points have infinite distance. For example, a disjoint union of metric spaces is naturally a quasi-metric space. The definition of a coarse structure associated to a metric [BE16, Ex. 2.17] generalizes immediately to the case of quasi-metric spaces. Similarly, a quasi-metric also induces a uniform structure.

We consider a quasi-metric space with the induced coarse and uniform structures $\mathcal{C}$ and $\mathcal{U}$. They are compatible. If the space is in addition a path quasi-metric space, then we have the equality $\mathcal{C} = \mathcal{C}(\mathcal{U})$. In particular, in this case $\mathcal{C}(\mathcal{U})$ is compatible with $\mathcal{U}$. □

A bornological coarse space with a uniform structure $(X, \mathcal{C}, \mathcal{B}, \mathcal{U})$ such that $\mathcal{U}$ and $\mathcal{C}$ are compatible is an object of the category of uniform bornological coarse spaces **UBC**. A morphism between uniform bornological coarse spaces is a morphism between bornological coarse spaces which is in addition uniformly continuous. We refer to [BEKWb, Sec. 9.1] for more details.

Remark 2.4. A map between metric spaces $f : (X, d) \to (X', d')$ is called uniformly continuous if for every δ in $(0, \infty)$ there exists an ϵ in $(0, \infty)$ such that for all pairs of points x, y of X with $d(x, y) \le \epsilon$ we have $d(f(x), f(y)) \le \delta$. A uniformly continuous map between metric spaces in this sense is uniformly continuous as a map between the associated uniform spaces. □

Example 2.5. Let X be a simplicial complex. Then X has a canonical spherical metric which induces a coarse structure $\mathcal{C}$ and a compatible uniform structure $\mathcal{U}$.

A choice of a set A of sub-complexes generates a bornology $\mathcal{B} := \mathcal{B}\langle A\rangle$. It is compatible with the coarse structure if for every entourage U in $\mathcal{C}$ and every sub-complex K in A there exists another sub-complex K' in A with $U[K] \subseteq K'$. The triple $(X, \mathcal{C}, \mathcal{B}, \mathcal{U})$ is a uniform bornological coarse space.

If X' and A' is similar data and $f : X \to X'$ is a simplicial map such that for every Y' in A' we have $f^{-1}(Y') \in A$, then f is a morphism of uniform bornological coarse spaces. □

Example 2.6. If X is a bornological coarse space and U is an entourage of X, then we consider the simplicial complex $P_U(X)$ of probability measures on X which have finite, U-bounded support. For a subset Y of X we let $P_U(Y)$ denote the sub-complex of $P_U(X)$ of measures supported on Y. We let A be the set of sub-complexes $P_U(B)$ for all bounded subsets B of X. The constructions explained in Example 2.5 turn $P_U(X)$ into a uniform bornological coarse space.

Let $f : X \to X'$ be a morphism between bornological coarse spaces and U' be an entourage of X' such that $(f \times f)(U) \subseteq U'$. Then the push-forward of measures provides a morphism $P_U(X) \to P_{U'}(X')$ between uniform bornological coarse spaces in a functorial way. □

Example 2.7. Let X be a uniform bornological coarse space. If Y is a subset of X, then Y has an induced uniform bornological coarse structure. If not said differently, we will always consider subsets with the induced structures. The inclusion $Y \to X$ is then a morphism between uniform bornological coarse spaces. □

3 Local homology theories

In this section we introduce the notion of a local homology theory.

Let $(X,\mathcal{U})$ be a uniform space and let A and B be subsets of X with $A \cup B = X$. For an entourage U let $\mathcal{P}(X \times X)_{\subseteq U}$ denote the set of elements of $\mathcal{P}(X \times X)$ (the power set of $X \times X$) which are contained in U. The following is taken from [BE16, Def. 5.18]:

Definition 3.1. *The pair (A, B) is a uniformly excisive decomposition of X if there exists a uniform entourage U and a function $\kappa : \mathcal{P}(X \times X)_{\subseteq U} \to \mathcal{P}(X \times X)$ such that:*

1. *The restriction of κ to $\mathcal{U} \cap \mathcal{P}(X \times X)_{\subseteq U}$ is $\mathcal{U}$-admissible.*
2. *For every W in $\mathcal{P}(X \times X)_{\subseteq U}$ we have $W[A] \cap W[B] \subseteq \kappa(W)(A \cap B)$.*

Remark 3.2. Note that in Definition 3.1 we consider $\mathcal{P}(X \times X)_{\subseteq U}$ and $\mathcal{P}(X \times X)$ as partially ordered sets with the order relation given by the opposite of the inclusion relation. By definition, a function between partially ordered sets is order preserving.

Condition 3.1.1 means that for every V in $\mathcal{U}$ there exists W in $\mathcal{U} \cap \mathcal{P}(X \times X)_{\subseteq U}$ such that $\kappa(W) \subseteq V$. □

For a coarse space $(X,\mathcal{C})$ the notion of a coarsely excisive decomposition [BE16, Def. 3.37] is defined similarly. We again consider two subsets A and B of X such that $X = A \cup B$.

Definition 3.3. *The pair (A, B) is a coarsely excisive decomposition of the space X if for every coarse entourage V of X there exists a coarse entourage W of X such that we have $V[A] \cap V[B] \subseteq W(A \cap B)$.*

Let $(X,\mathcal{U})$ be a uniform space.

Lemma 3.4. *If $\mathcal{C}(\mathcal{U})$ is compatible with $\mathcal{U}$, then any uniformly excisive decomposition (A, B) of X is coarsely excisive for the coarse structure $\mathcal{C}(\mathcal{U})$.*

Proof. Let U and κ be as in the Definition 3.1. Since $\mathcal{C}(\mathcal{U})$ is compatible with $\mathcal{U}$, we can assume that U is also a coarse entourage and $\kappa(W)$ is a coarse entourage for every W in $\mathcal{U} \cap \mathcal{P}(X \times X)_{\subseteq U}$.

Let V be an entourage in $\mathcal{C}(\mathcal{U})$. Then there exists an integer n such that $V \subseteq U^n$. We claim that

$$V[A] \cap V[B] \subseteq (U^{2n+1} \circ \kappa(U))(A \cap B) .$$

Let z be a point in $V[A] \cap V[B]$. Then there exists integers r and s with $r \leq n$ and $s \leq n$ and a sequence of points $(x_0, \ldots, x_{r+s})$ in X such that $x_0 \in A$, $x_{r+s} \in B$, and $(x_i, x_{i+1}) \in U$ for all $i = 0, \ldots, r+s-1$. There exists i_0 in $\{0, \ldots, r+s-1\}$ such that $x_{i_0} \in A$ and $x_{i_0+1} \in B$. But then $x_{i_0+1} \in U[x_i]$, i.e., $x_{i_0+1} \in U[A] \cap U[B]$. Hence there exists a point y in $A \cap B$ such that $(x_{i_0+1}, y) \in \kappa(U)$. This now implies that $z \in (U^{2n+1} \circ \kappa(U))(A \cap B)$ as asserted. □

Example 3.5. On a path quasi-metric space every closed decomposition is coarsely and uniformly excisive. □

Let $\mathbf{C}$ be a cocomplete stable ∞-category and consider a functor $E : \mathbf{UBC} \to \mathbf{C}$.

Definition 3.6. *We say that E satisfies excision if $E(\emptyset) \simeq 0$ and for every uniform bornological coarse space X and uniformly and coarsely excisive closed decomposition (A, B) of X the square*

$$\begin{tikzcd} E(A \cap B) \arrow[r] \arrow[d] & E(A) \arrow[d] \\ E(B) \arrow[r] & E(X) \end{tikzcd}$$

is cocartesian.

We refer to Remark 3.14 for comments on the condition that A and B are closed subsets of X in the above definition.

Let X and Y be two uniform bornological coarse spaces. We define the tensor product $X \otimes Y$ such that the underlying bornological coarse space is the tensor product of the corresponding bornological coarse spaces [BE16, Ex. 2.30] and the uniform structure is the product uniform structure.

The unit interval $[0, 1]$ has a canonical uniform bornological coarse structure (the maximal coarse and bornological structure, and the metric uniform structure). The product $[0, 1] \otimes X$ is now defined, and the projection $[0, 1] \otimes X \to X$ is a morphism of uniform bornological coarse spaces since $[0, 1]$ is bounded.

Let $E : \mathbf{UBC} \to \mathbf{C}$ be a functor.

Definition 3.7. *We say that E is homotopy invariant if for every uniform bornological coarse space X the morphism $E([0, 1] \otimes X) \to E(X)$ induced by the projection is an equivalence.*

A homotopy between morphisms $f_0, f_1 : X \to Y$ of uniform bornological coarse spaces is a morphism $h : [0, 1] \otimes X \to Y$ which restricts to f_i at the endpoints of the interval.

A uniform bornological coarse space X is called flasque with flasqueness implemented by a morphism $f : X \to X$ if f implements flasqueness in the sense of bornological coarse spaces [BE16, Def. 3.21] and f is in addition uniformly homotopic to the identity.

Let $E : \mathbf{UBC} \to \mathbf{C}$ be a functor.

Definition 3.8. *We say that E vanishes on flasques if $E(X) \simeq 0$ for every flasque uniform bornological coarse space X.*

Let X be a uniform bornological coarse space and U be an entourage which is both coarse and uniform. Then for every coarse entourage V such that $U \subseteq V$ we can replace the coarse structure by the coarse structure generated by V and obtain a uniform bornological coarse space X_V. Hence, for a uniform bornological coarse space X the uniform bornological coarse space X_V is well-defined for sufficiently large coarse entourages V. We have a canonical morphism $X_V \to X$ given by the identity of the underlying sets. Hence the colimit and the canonical morphism in the following definition have a well-defined interpretation.

Let $E : \mathbf{UBC} \to \mathbf{C}$ be a functor.

Definition 3.9. *We say that E is u-continuous if for every uniform bornological coarse space X the canonical morphism* $\mathrm{colim}_{V\in\mathcal{C}}E(X_V)\to E(X)$ *is an equivalence.*

Let $\mathbf{C}$ be a cocomplete stable ∞-category and let $E:\mathbf{UBC}\to\mathbf{C}$ be a functor.

Definition 3.10. *E is called a $\mathbf{C}$-valued local homology theory if*

1. *E satisfies excision,*
2. *E is homotopy invariant,*
3. *E vanishes on flasques, and*
4. *E is u-continuous.*

We have a forgetful functor

$$F_{\mathcal{U}}:\mathbf{UBC}\to\mathbf{BornCoarse} \tag{3.1}$$

which forgets the uniform structure.

Lemma 3.11. *If E is a $\mathbf{C}$-valued coarse homology theory, then $E\circ F_{\mathcal{U}}$ is a $\mathbf{C}$-valued local homology theory.*

Proof. It is clear that $E\circ F_{\mathcal{U}}$ is homotopy invariant, u-continuous, and vanishes on flasques. The functor $F_{\mathcal{U}}$ sends uniformly and coarsely excisive closed decompositions to coarsely excisive decompositions. By [BE16, Lem. 3.38] the composition $E\circ F_{\mathcal{U}}$ is excisive. □

Remark 3.12. The reason that the proof of the Lemma 3.11 is not completely trivial is that excision for coarse homology theories was not defined in terms of coarsely excisive decompositions but with complementary pairs. Our main reason for doing this was that the intersection of a coarsely excisive decomposition with a subset need not be coarsely excisive, while intersection with subsets preserves complementary pairs. In fact, the Definition 3.6 suffers from the same defect which causes some work at other points later. □

Let **TopBorn** be the category of topological bornological spaces ([BE16, Sec. 6.5]).

We have a forgetful functor

$$F_{\mathcal{C},\mathcal{U}/2}:\mathbf{UBC}\to\mathbf{TopBorn}$$

which forgets the coarse structure and a part of the uniform structure, i.e., only remembers the bornology and the topology induced from the uniform structure.

A locally finite homology theory in the sense of [BE16, Def. 6.60] will be called closed if it satisfies excision for closed decompositions.

An example of a spectrum-valued closed locally finite homology theory is the analytic K-homology $K^{an,lf}$ constructed in [BE16, Def. 6.92].

We assume that $\mathbf{C}$ is a complete and cocomplete stable ∞-category. Let $E:\mathbf{TopBorn}\to\mathbf{C}$ be a locally finite homology theory.

Lemma 3.13. *If E is closed, then $E \circ F_{\mathcal{C},\mathcal{U}/2}$ is a* $\mathbf{C}$*-valued local homology theory.*

Proof. Homotopy invariance of E implies homotopy invariance of $E \circ F_{\mathcal{C},\mathcal{U}/2}$. The functor sends coarsely and uniformly excisive closed decompositions to closed decompositions. Since we assume that E is closed the composition $E \circ F_{\mathcal{C},\mathcal{U}/2}$ satisfies excison.

The functor $F_{\mathcal{C},\mathcal{U}/2}$ sends a flasque uniform bornological coarse space to a topological bornological space which is flasque in the sense of [BE16, Def. 6.52]. Since E vanishes on flasque topological bornological spaces by [BE16, Lem. 6.54] we conclude that $E \circ F_{\mathcal{C},\mathcal{U}/2}$ vanishes on flasques.

Since $F_{\mathcal{C},\mathcal{U}/2}$ forgets the coarse structure, the composition $E \circ F_{\mathcal{C},\mathcal{U}/2}$ is u-continuous. $\square$

Remark 3.14. Note that in Definition 3.6 we can replace the condition that A and B are closed in X by the condition that these subsets are open in X. Then using the functor $F_{\mathcal{C},\mathcal{U}/2}$ we can pull-back locally finite homology theories satisfying open excision.

A typical example of a locally finite homology theory satisfying open excision is the locally finite version of stable homotopy $\Sigma^{\infty}_{+}(-)^{lf}$, see [BE16, Ex. 6.56].

We choose to work with the condition *closed* since the main example for the present paper is analytic locally finite K-homology which satisfies closed excision. $\square$

4 Motives and the universal local homology theory

In this section we construct the universal local homology theory. The construction here is completely analogous to the construction of the universal coarse homology theory carried out in [BE16, Sec. 3 & 4]. We keep the present section as short as possible and refer to [BE16] for more background and references to the ∞-category literature.

Remark 4.1. For an ∞-category $\mathbf{D}$ we use the standard notation

$$\mathbf{PSh}(\mathbf{D}) := \mathbf{Fun}(\mathbf{D}^{op}, \mathbf{Spc})$$

for the ∞-category space-valued presheaves. If $\mathbf{D}$ is an ordinary category, then we consider it as an ∞-category using the nerve.

In order to perform the localizations below we must assume that $\mathbf{D}$ is small. The category $\mathbf{UBC}$ is not small. Therefore in order to make the theory below precise we must replace $\mathbf{UBC}$ by a small full subcategory which contains all isomorphism classes of uniform bornological coarse spaces we are interested in. This category we choose must be closed under constructions like forming products, taking subspaces, etc. Furthermore, it must contain the Rips complexes $P_U(X)$ for X belonging to a similarly choosen small and full subcategory of $\mathbf{BornCoarse}$. From now on we will drop these set-theoretic issues and pretent that $\mathbf{UBC}$ is small. $\square$

Let E be in $\mathbf{PSh}(\mathbf{UBC})$.

Definition 4.2. *E satisfies descent if for every uniform bornological coarse space X and uniformly and coarsely excisive closed decomposition (A, B) of X the square*

$$\begin{array}{ccc} E(X) & \longrightarrow & E(A) \\ \downarrow & & \downarrow \\ E(B) & \longrightarrow & E(A \cap B) \end{array}$$

is cartesian.

Definition 4.3. *We let* $\mathbf{Sh}(\mathbf{UBC})$ *be the full subcategory of* $\mathbf{PSh}(\mathbf{UBC})$ *of presheaves which satisfy descent. Its objects will be called sheaves.*

Lemma 4.4. *We have a localization*

$$L : \mathbf{PSh}(\mathbf{UBC}) \leftrightarrows \mathbf{Sh}(\mathbf{UBC}) : \mathit{inclusion}\ .$$

Proof. The condition of descent can be written as a locality condition for a set of maps of $\mathbf{PSh}(\mathbf{UBC})$. This implies existence of the localization. □

Remark 4.5. We think that the sheafification adjunction is exact. But since exactness does not play a role in the present paper we refrain from working out the details. □

We let $Y : \mathbf{UBC} \to \mathbf{PSh}(\mathbf{UBC})$ be the Yoneda embedding.

Lemma 4.6. *For every uniform bornological coarse space X the presheaf $Y(X)$ satisfies descent.*

Proof. See [BE16, Lem. 3.12] for a similar argument. □

Let E be in $\mathbf{Sh}(\mathbf{UBC})$.

Definition 4.7. *E is homotopy invariant if for every uniform bornological coarse space X the morphism $E(X) \to E([0,1] \otimes X)$ induced by the projection is an equivalence.*

We let $\mathbf{Sh}^h(\mathbf{UBC})$ denote the full subcategory of $\mathbf{Sh}(\mathbf{UBC})$ of homotopy invariant sheaves.

Lemma 4.8. *We have an adjunction*

$$\mathcal{H} : \mathbf{Sh}(\mathbf{UBC}) \leftrightarrows \mathbf{Sh}^h(\mathbf{UBC}) : \mathit{inclusion}\ .$$

Proof. Homotopy invariance can be written as locality with respect to a set of maps in $\mathbf{UBC}$. This implies the existence of the localization. □

We call $\mathcal{H}$ the homotopification.

Let E be in $\mathbf{Sh}^h(\mathbf{UBC})$.

Definition 4.9. *We say that E vanishes on flasques if for every flasque uniform bornological coarse space X we have $E(X) \simeq \emptyset$.*

We let $\mathbf{Sh}^{h,fl}(\mathbf{UBC})$ denote the full subcategory of $\mathbf{Sh}^{h}(\mathbf{UBC})$ of homotopy invariant sheaves which vanish on flasques.

Lemma 4.10. *We have an adjunction*

$$\mathrm{Fl} : \mathbf{Sh}^{h}(\mathbf{UBC}) \leftrightarrows \mathbf{Sh}^{h,fl}(\mathbf{UBC}) : \textit{inclusion} \ .$$

Proof. The condition of vanishing on flasques can be written as a locality condition with respect to a set of maps in $\mathbf{PSh}(\mathbf{UBC})$. This implies existence of the localization. $\square$

For a uniform bornological coarse space X let $\tilde{\mathcal{C}}$ denote the subset of coarse entourages which are also uniform. Note that this subset is cofinal in $\mathcal{C}$.

Let E be in $\mathbf{Sh}^{h,fl}(\mathbf{UBC})$.

Definition 4.11. *We say that E is u-continuous if for every uniform bornological coarse space X the natural morphism $E(X) \to \mathtt{lim}_{U\in\tilde{\mathcal{C}}} E(X_U)$ is an equivalence.*

We let $\mathbf{Sh}^{h,fl,u}(\mathbf{UBC})$ denote the full subcategory of $\mathbf{Sh}^{h,fl}(\mathbf{UBC})$ of homotopy invariant sheaves which vanish on flasques and are u-continuous.

Lemma 4.12. *We have an adjunction*

$$U : \mathbf{Sh}^{h,fl}(\mathbf{UBC}) \leftrightarrows \mathbf{Sh}^{h,fl,u}(\mathbf{UBC}) : \textit{inclusion} \ .$$

Proof. The condition of being u-continuous can be written as a locality condition with respect to a set of maps in $\mathbf{PSh}(\mathbf{UBC})$. This implies existence of the localization. $\square$

Lemma 4.13. *The category $\mathbf{Sh}^{h,fl,u}(\mathbf{UBC})$ is presentable.*

Proof. The category $\mathbf{Sh}^{h,fl,u}(\mathbf{UBC})$ is a reflective localization of a presheaf category. $\square$

Definition 4.14. *We call $\mathbf{Sh}^{h,fl,u}(\mathbf{UBC})$ the category of motivic uniform bornological coarse spaces and use the notation $\mathbf{Spc}\mathcal{B}$.*

The following is analogous to [BE16, Sec. 4.1].

Definition 4.15. *We define the stable ∞-category of motivic uniform bornological coarse spectra $\mathbf{Sp}\mathcal{B}$ as the stabilization of $\mathbf{Spc}\mathcal{B}$ in the world of presentable stable ∞-categories.*

We have a canonical functor

$$\Sigma^{\infty}_{+} : \mathbf{Spc}\mathcal{B} \to \mathbf{Sp}\mathcal{B} \ .$$

By construction the category $\mathbf{Sp}\mathcal{B}$ is a presentable stable ∞-category.

We have a functor

$$\mathrm{Yo}\mathcal{B}^{s} := \Sigma^{\infty}_{+} \circ U \circ \mathrm{Fl} \circ \mathcal{H} \circ L \circ Y : \mathbf{UBC} \to \mathbf{Sp}\mathcal{B} \ .$$

In view of Lemma 4.6 we could omit L in this composition. For a uniform bornological coarse space X we call $\mathrm{Yo}\mathcal{B}^{s}(X)$ the motive of X.

By construction the functor $\mathrm{Yo}\mathcal{B}^s$ is a $\mathbf{Sp}\mathcal{B}$-valued local homology theory. It is in fact the universal local homology theory.

Corollary 4.16. *If $\mathbf{C}$ is a cocomplete stable ∞-category, then precomposition with $\mathrm{Yo}\mathcal{B}^s$ induces an equivalence between the ∞-categories of $\mathbf{C}$-valued local homology theories and colimit preserving functors $\mathbf{Sp}\mathcal{B} \to \mathbf{C}$.*

For a local homology theory E we will use the notation E also to denote the corresponding colimit preserving functor $\mathbf{Sp}\mathcal{B} \to \mathcal{C}$.

Remark 4.17. The existence of non-trivial local homology theories (see Lemma 3.11 and Lemma 3.13) shows that the category $\mathbf{Sp}\mathcal{B}$ is non-trivial. □

5 Universal coarsification functor $\mathbf{P}$

In this section we extend the construction given in Example 2.6 to a coarse homology theory $\mathbf{P}$ called the universal coarsification.

Let $\mathbf{BornCoarse}^{\mathcal{C}}$ be the category of pairs (X, U) of bornological coarse spaces X and coarse entourages U of X. A morphisms $f : (X, U) \to (X', U')$ is a morphism $f : X \to X'$ of bornological coarse spaces such that $(f \times f)(U) \subseteq U'$. By Example 2.6 we have a functor

$$P : \mathbf{BornCoarse}^{\mathcal{C}} \to \mathbf{UBC}$$

which sends (X, U) to the uniform bornological coarse space $P(X, U)$ associated to the simplical complex $P_U(X)$ and the family A of sub-complexes $P_U(B)$ for all bounded subsets B of X. We furthermore have a forgetful functor

$$F_{\mathcal{C}} : \mathbf{BornCoarse}^{\mathcal{C}} \to \mathbf{BornCoarse} \tag{5.1}$$

which sends the pair (X, U) to X.

Definition 5.1. *We define $\mathbf{P} : \mathbf{BornCoarse} \to \mathbf{Sp}\mathcal{B}$ as the left Kan-extension*

$$\begin{tikzcd} \mathbf{BornCoarse}^{\mathcal{C}} \arrow[r, "\mathrm{Yo}\mathcal{B}^s \circ P"] \arrow[d, "F_{\mathcal{C}}"'] & \mathbf{Sp}\mathcal{B} \\ \mathbf{BornCoarse} \arrow[ur, "\mathbf{P}"'] & \end{tikzcd}$$

Proposition 5.2. *The functor $\mathbf{P}$ is an $\mathbf{Sp}\mathcal{B}$-valued coarse homology theory.*

Proof. If X is a bornological coarse space with coarse structure $\mathcal{C}$, then by the point-wise formula for the left Kan extension

$$\mathbf{P}(X) \simeq \mathrm{colim}_{U \in \mathcal{C}} \mathrm{Yo}\mathcal{B}^s(P(X, U)) \ . \tag{5.2}$$

We have equivalences

$$\begin{aligned}\mathrm{colim}_{V\in\mathcal{C}}\mathbf{P}(X_V) &\simeq \mathrm{colim}_{V\in\mathcal{C}}\mathrm{colim}_{U\in\mathcal{C}\langle V\rangle}\mathrm{Yo}\mathcal{B}^s(P(X,U))\\ &\simeq \mathrm{colim}_{U\in\mathcal{C}}\mathrm{Yo}\mathcal{B}^s(P(X,U))\\ &\simeq \mathbf{P}(X)\ ,\end{aligned}$$

where for the second equivalence we use a cofinality consideration. Hence $\mathbf{P}$ is u-continuous.

Consider two morphisms $f_0, f_1 : (X,U) \to (X',U')$ in $\mathbf{BornCoarse}^{\mathcal{C}}$. If f_0 and f_1 are U'-close, then $P(f_0)$ and $P(f_1)$ are homotopic and $\mathrm{Yo}\mathcal{B}^s(P(f_0)) \simeq \mathrm{Yo}\mathcal{B}^s(P(f_1))$ by the homotopy invariance of $\mathrm{Yo}\mathcal{B}^s$. This implies that $\mathbf{P}$ is coarsely invariant.

Let (X,U) be an object of $\mathbf{BornCoarse}^{\mathcal{C}}$ such that U contains the diagonal of X. For a subset Y of X note that $P_U(Y)$ is a closed subset of $P_U(X)$. If $(Z,\mathcal{Y})$ with $\mathcal{Y} = (Y_i)_{i\in I}$ is a complementary pair, then for every i,j in I such that $Z\cup Y_i = X$ and $U[Y_i]\subseteq Y_j$ the pair $(P_U(Z), P_U(Y_j))$ is a closed decomposition of the path quasi-metric space $P_U(X)$ and hence uniformly and coarsely excisive (see Example 3.5). For sufficiently large j in I and since $\mathrm{Yo}\mathcal{B}^s$ is excisive we get a cocartesian square

$$\begin{tikzcd}\mathrm{Yo}\mathcal{B}^s(P_U(Z)\cap P_U(Y_j)) \arrow[r]\arrow[d] & \mathrm{Yo}\mathcal{B}^s(P_U(Z)) \arrow[d]\\ \mathrm{Yo}\mathcal{B}^s(P_U(Y_j)) \arrow[r] & \mathrm{Yo}\mathcal{B}^s(P_U(X))\end{tikzcd}$$

We form the colimit over j in I and over U in the coarse structure of X. The lower right corner yields $\mathbf{P}(X)$. For the lower left corner we first take the U-colimit and then the j-colimit. Then we obtain the object $\mathbf{P}(\mathcal{Y})$. In the upper right corner we get $\mathbf{P}(Z)$. For the upper left corner we note that $P_U(Z)\cap P_U(Y_j) = P_U(Z\cap Y_j)$ and finally get $\mathbf{P}(Z\cap\mathcal{Y})$. Since we have exhibited the square

$$\begin{tikzcd}\mathbf{P}(Z\cap\mathcal{Y}) \arrow[r]\arrow[d] & \mathbf{P}(Z)\arrow[d]\\ \mathbf{P}(\mathcal{Y}) \arrow[r] & \mathbf{P}(X)\end{tikzcd}$$

as a colimit of cocartesian squares it is cocartesian itself. We conclude that the functor $\mathbf{P}$ satisfies excision.

Finally, assume that a bornological coarse space X is flasque with flasqueness implemented by $f : X \to X$. Let U be an entourage of X such that $\mathtt{id}_X$ and f are U-close to each other. Then $V := \bigcup_{n\in\mathbb{N}}(f^n\times f^n)(U)$ is again an entourage of X which contains U. Now note that $(f\times f)(V)\subseteq V$. Therefore $P(f) : P(X,V)\to P(X,V)$ is defined. This map implements flasqueness of $P(X,V)$, hence $\mathrm{Yo}\mathcal{B}^s(P(X,V))\simeq 0$. In view of (5.2) by cofinality we see that $\mathbf{P}$ vanishes on flasques. □

Remark 5.3. Note that the Proposition 5.2 would also be true (with a slightly different argument for excision) if we would have worked with open instead of closed decompositions in the definition of excision. □

We therefore have defined a colimit preserving functor

$$\mathbf{P} : \mathbf{Sp}\mathcal{X} \to \mathbf{Sp}\mathcal{B} \ .$$

Definition 5.4. *The functor* $\mathbf{P}$ *is the universal coarsification functor.*

Let E be a local homology theory.

Definition 5.5. *The coarse homology theory* $E\mathbf{P} := E \circ \mathbf{P}$ *will be called the coarsification of the theory* E*.*

Example 5.6. In Example 3.13 we have seen that for any closed locally finite homology theory $E : \mathbf{TopBorn} \to \mathbf{C}$ we can define a local homology theory $E \circ F_{\mathcal{C},\mathcal{U}/2} : \mathbf{UBC} \to \mathbf{C}$. The coarsification $(E \circ F_{\mathcal{C},\mathcal{U}/2})\mathbf{P}$ is equivalent to the coarse homology theory QE, which is the coarsification of E from [BE16, Defn. 6.74]. □

6 From coarse to local homology theories via $\mathbf{F}$

In this section we refine the forgetful functor $F_{\mathcal{U}} : \mathbf{UBC} \to \mathbf{BornCoarse}$ from (3.1) to a local homology theory. We define

$$\mathbf{F} := \mathrm{Yo}^s \circ F_{\mathcal{U}} : \mathbf{UBC} \to \mathbf{Sp}\mathcal{X} \ .$$

Lemma 6.1. $\mathbf{F}$ *is a local homology theory.*

Proof. The proof is straightforward and similar to the one of Lemma 3.11. □

We therefore get a colimit-preserving functor

$$\mathbf{F} : \mathbf{Sp}\mathcal{B} \to \mathbf{Sp}\mathcal{X} \tag{6.1}$$

For a $\mathbf{C}$-valued coarse homology theory E we write

$$E\mathbf{F} := E \circ \mathbf{F} : \mathbf{Sp}\mathcal{B} \to \mathbf{C}$$

for the associated local homology theory (compare with Lemma 3.11 where the notation $E \circ F_{\mathcal{U}}$ was used).

Proposition 6.2. *We have a canonical equivalence*

$$\mathtt{id} \xrightarrow{\simeq} \mathbf{F} \circ \mathbf{P} \ . \tag{6.2}$$

Proof. We have a functor

$$I : \mathbf{BornCoarse}^{\mathcal{C}} \to \mathbf{BornCoarse}$$

which is defined on objects by $I(X, U) := X_U$. By u-continuity of Yo^s the left Kan extension of $\mathrm{Yo}^s \circ I$ along $F_{\mathcal{C}}$ (see (5.1) for the definition of $F_{\mathcal{C}}$) is equivalent to Yo^s. Let

(X, U) be in $\mathbf{BornCoarse}^{\mathcal{C}}$. Dirac measures provide a canonical inclusion $X \to P_U(X)$ of sets. This map is an equivalence

$$X_U \xrightarrow{\simeq} F_{\mathcal{U}}(P(X, U)) \tag{6.3}$$

of bornological coarse spaces. Hence we get an equivalence of functors from $\mathbf{BornCoarse}^{\mathcal{C}}$ to $\mathbf{Sp}\mathcal{X}$

$$\mathrm{Yo}^s \circ I \xrightarrow{\simeq} \mathrm{Yo}^s \circ F_{\mathcal{U}} \circ P \simeq \mathbf{F} \circ \mathrm{Yo}\mathcal{B}^s \circ P\ . \tag{6.4}$$

Since $\mathbf{F}$ is colimit-preserving the equivalence (6.4) induces an equivalence of left Kan extensions along $F_{\mathcal{C}}$:

$$\mathrm{Yo}^s \to \mathbf{F} \circ \mathbf{P} : \mathbf{BornCoarse} \to \mathbf{Sp}\mathcal{X}\ .$$

We finally interpret $\mathbf{P}$ as a colimit-preserving functor $\mathbf{Sp}\mathcal{X} \to \mathbf{Sp}\mathcal{B}$ to get the desired equivalence. □

Corollary 6.3. *Every coarse homology theory E is equivalent to the coarsification of the local homology theory $E\mathbf{F}$. Similarly, every morphism between coarse homology theories is induced by coarsification from a morphism between the associated local homology theories.*

7 Coarsifying spaces

Under certain finiteness conditions on the uniform bornological coarse space X we can construct a morphism

$$c_X : \mathrm{Yo}\mathcal{B}^s(X) \to \mathbf{P}(\mathbf{F}(X))$$

called the comparison morphism. We will furthermore show that it is an equivalence for simplicial complexes of bounded geometry which are uniformly contractible. Part of the material here is inspired by Roe [Roe96, Ch. 2, Part “Coarse algebraic topology”].

Let X be a coarse space with a uniform structure.

Definition 7.1. *We say that the uniform structure is numerable if there exists an entourage U which is both coarse and uniform, and an equicontinuous, uniformly point-wise locally finite partition of unity $(\chi_\alpha)_{\alpha \in A}$ such that $\mathtt{supp}(\chi_\alpha)$ is U-bounded for all α in A.*

Remark 7.2. Here uniform point-wise local finiteness means

$$\sup_{x \in X} |\{\alpha \in A \mid \chi_\alpha(x) \neq 0\}| < \infty\ .$$

Let us spell out the meaning of *equicontinuous* explicitly: For every positive real number ϵ there exists a uniform entourage V of X such that for all α in A and (x, x') in V we have the inequality $|\chi_\alpha(x) - \chi_\alpha(x')| \leq \epsilon$. □

Let X be a simplicial complex with the coarse and uniform structures both induced from the spherical path metric.

Lemma 7.3. *If X is finite-dimensional, then X is numerable.*

Proof. We consider the entourage U_2 of width 2. We define the equicontinuous partition of unity $(\chi_v)_{v\in X^{(0)}}$ using the baricentric coordinates of the simplices, where $X^{(0)}$ is the set of vertices of X. If σ is a simplex in X and x is a point in σ, then $\chi_v(x) \neq 0$ exactly if v is a vertex of σ. Hence for every point x the number of vertices v of X with $\chi_v(x) \neq 0$ is bounded by $\dim(X)+1$.

The support of χ_v is U_2-bounded for every vertex v of X. □

Let X be a numerable uniform bornological coarse space. By numerability of the uniform structure we can choose an entourage U which is coarse and uniform such that there exists an equicontinuous, uniformly point-wise locally finite partition of unity $(\chi_\alpha)_{\alpha\in A}$ on X such that $\mathtt{supp}(\chi_\alpha)$ is U-bounded for every α in A. We choose a family of points $(x_\alpha)_{\alpha\in A}$ in X such that $x_\alpha \in \mathtt{supp}(\chi_\alpha)$ for all α in A. We can then define a map

$$X \to P_{U^2}(F_{\mathcal{U}}(X)) \ , \quad x \mapsto \sum_{\alpha\in A} \chi_\alpha(x)\delta_{x_\alpha} \ . \tag{7.1}$$

This map is uniform. Note that at this point we use the uniformity of the point-wise locally finiteness condition since we measure distances in the simplices of $P_{U^2}(F_{\mathcal{U}}(X))$ in the spherical metric and not in the maximum metric with respect to baricentric coordinates, cf. [BE16, Ex. 5.37].

The map defined in (7.1) can also be regarded as a morphism of uniform bornological coarse spaces $\tilde{c} : X_U \to P_{U^2}(F_{\mathcal{U}}(X))$. It induces a morphism

$$\mathrm{Yo}\mathcal{B}^s(X_U) \to \mathrm{Yo}\mathcal{B}^s(P_{U^2}(F_{\mathcal{U}}(X))) \to \mathbf{P}(\mathbf{F}(X))$$

for every sufficiently large entourage U of X, and by u-continuity of $\mathrm{Yo}\mathcal{B}^s$, a morphism

$$c_X : \mathrm{Yo}\mathcal{B}^s(X) \to \mathbf{P}(\mathbf{F}(X)) \ .$$

Definition 7.4. *For a numerable uniform bornological coarse space the transformation c_X is called the comparison map.* □

Remark 7.5. We must assume that X is numerable in order to produce a uniform map $X \to P_{U^2}(F_{\mathcal{U}}(X))$ by (7.1).

In the classical approach to the coarsification of locally finite homology theories (see, e.g., Higson–Roe [HR95, Sec. 3]) one only needs a coarse and continuous map. In this case the same formula works, and we only have to assume that the members of the partition of unity have uniformly controlled support. The existence of such a partition of unity follows from the compatibility of the uniform and the coarse structure if we in addition assume that the underlying topological space of X is paracompact.

In our approach we must work with uniform maps since this is required by functoriality of the cone functor $\mathcal{O}$ which we employ below in order to construct the assembly map. □

Lemma 7.6. *Up to equivalence the comparison map does not depend on the choice of the partition of unity.*

Proof. We consider a second choice of partition of unity (without loss of generality for the same entourage U) and denote the associated morphism by $\tilde{c}' : X_U \to P_{U^2}(F_{\mathcal{U}}(X))$. Then $s \mapsto (1-s)c + s\tilde{c}'$ is a homotopy between $\tilde{c}$ and $\tilde{c}'$. Moreover $\tilde{c}$ and $\tilde{c}'$ are U^2-close to each other. □

Let $f : X \to X'$ be a morphism of uniform bornological coarse spaces which are assumed to be numerable.

Lemma 7.7. *We have an equivalence*

$$c_{X'} \circ \mathrm{Yo}\mathcal{B}^s(f) \simeq (\mathbf{P} \circ \mathbf{F})(f) \circ c_X \ .$$

Proof. After choosing partitions of unity for X and X' with bounds U and U' such that $(f \times f)(U) \subseteq U'$ we have a square (not necessarily commuting) of morphisms of uniform bornological coarse spaces

$$\begin{tikzcd} X_U \arrow[r, "\tilde{c}_X"] \arrow[d, "f"] & P_{U^2}(F_{\mathcal{U}}(X)) \arrow[d, "P(F_{\mathcal{U}}(f))"] \\ X'_{U'} \arrow[r, "\tilde{c}_{X'}"] & P_{U',2}(F_{\mathcal{U}}(X')) \end{tikzcd}$$

We now observe that the compositions $P(F_{\mathcal{U}}(f)) \circ \tilde{c}_X$ and $\tilde{c}_{X'} \circ f$ are close and (linearly) homotopic to each other. □

Let Y be a uniform bornological coarse space .

Definition 7.8. *We say that Y is coarsifying if it is numerable and the comparison map c_Y is an equivalence.*

Let E be a local homology theory. If Y is coarsifying, then the comparison map induces an equivalence

$$E(c_X) : E(Y) \xrightarrow{\simeq} E\mathbf{P}(F_{\mathcal{U}}(Y)) \ .$$

Let X be a numerable uniform bornological coarse space .

Definition 7.9. *A morphism $f : X \to Y$ in* **UBC** *is called a coarsifying approximation if Y is coarsifying and $(\mathbf{P} \circ \mathbf{F})(f)$ is an equivalence.*

Let E be a local homology theory. If $X \to Y$ is a coarsifying approximation, then by construction we have an equivalence

$$E\mathbf{P}(F_{\mathcal{U}}(X)) \simeq E(Y) \ .$$

We refer to [BE16, Sec. 6.8] for more information.

Let us discuss now an important class of examples of coarsifying spaces.

Let K be a simplicial complex. We get a uniform bornological coarse space K_d by equipping K with the bornology of bounded subsets and the metric coarse and uniform structures.

Below B^{q+1} is the unit ball in $\mathbb{R}^{q+1}$ and S^q its boundary.

Definition 7.10. 1. *K has bounded geometry if the number of vertices in the stars of its vertices is uniformly bounded.*

2. *K is equicontinuously contractible, if for every q in $\mathbb{N}$ and for every equicontinuous family of maps $\{\varphi_i : S^q \to K\}_{i\in I}$ there exists an equicontinuous family of maps $\{\Phi_i : B^{q+1} \to K\}_{i\in I}$ with $\Phi_i|_{\partial B^{q+1}} = \varphi_i$.*[1]

Let A be a subcomplex of K, X be a metric space and $f : K_d \to X_d$ be a morphism of bornological coarse spaces such that $f|_A$ is uniformly continuous.

Lemma 7.11. *If K is finite-dimensional and X is equicontinuously contractible, then f is close to a morphism of uniform bornological coarse spaces which extends $f|_A$ and is in addition uniformly continuous.*

Proof. The proof given in [BE16, Lem. 6.97] (which covers the non-uniform version of this lemma) also works word-for-word here. □

Proposition 7.12. *If K is a simplicial complex of bounded geometry which is equicontinuously contractible, then K_d is coarsifying.*

Proof. Note that K is finite-dimensional and hence K_d is numerable by Lemma 7.3. The verification that the comparison map for K_d is an equivalence is the core of the argument of [BE16, Prop. 6.105], which is itself taken from Nowak–Yu [NY12, Proof of Thm. 7.6.2].

As in the beginning of the proof of [BE16, Prop. 6.105], by extensive use of Lemma 7.11, we construct maps

$$\begin{tikzcd}
K_d \arrow[r, bend left, "f"] & P_U(F_{\mathcal{U}}(K_d)) \arrow[l, "g"] \arrow[r, bend left, "f_0"] & P_{U_1}(F_{\mathcal{U}}(K_d)) \arrow[ll, bend left, "g_0"] \arrow[r, bend left, "f_1"] & P_{U_2}(F_{\mathcal{U}}(K_d)) \arrow[lll, bend left, "g_1"] \arrow[r, bend left, "f_2"] & \cdots
\end{tikzcd}$$

with $g_n \circ (f_n \circ \cdots \circ f_0 \circ f)$ being homotopic in $\mathbf{UBC}$ to $\mathtt{id}_{K_d}$, and $(f_n \circ \cdots \circ f_0 \circ f) \circ g_{n-1}$ being homotopic to f_n in $\mathbf{UBC}$. We claim that the induced comparison map

$$\mathrm{Yo}^s\mathcal{B}(K_d) \to \mathrm{colim}_{n\in\mathbb{N}}\mathrm{Yo}^s\mathcal{B}(P_{U_n}(F_{\mathcal{U}}(K_d))) \simeq \mathbf{P}(\mathbf{F}(K_d))$$

is an equivalence.

Let T be any object of $\mathbf{Sp}\mathcal{B}$. Then we must check that

$$\lim_{n\in\mathbb{N}} \mathtt{Map}(\mathrm{Yo}^s\mathcal{B}(P_{U_n}(F_{\mathcal{U}}(K_d))), T) \to \mathtt{Map}(\mathrm{Yo}^s\mathcal{B}(K_d), T)$$

is an equivalence. We first check that

$$\lim_{n\in\mathbb{N}} \pi_*(\mathtt{Map}(\mathrm{Yo}^s\mathcal{B}(P_{U_n}(F_{\mathcal{U}}(K_d))), T)) \to \pi_*(\mathtt{Map}(\mathrm{Yo}^s\mathcal{B}(K_d), T)) \tag{7.2}$$

[1] This is a slight strengthening of the notion of uniform contractibility which is commonly used in the coarse geometry literature.

is an isomorphism.

Let us first check that (7.2) is surjective: Let $t \in \pi_*(\mathtt{Map}(\mathrm{Yo}^s\mathcal{B}(K_d), T))$. Then we consider the family $(g_n^* t)_{n\in\mathbb{N}}$. We observe that

$$f_n^* g_n^* t = (g_{n-1}^* f^* f_0^* \cdots f_n^*)(g_n^* t) = g_{n-1}^* t \ . \tag{7.3}$$

Therefore our family belongs to the limit. Furthermore $f^* f_0^* g_0^* t = t$, hence the family is a preimage of t.

Let now (t_n) be any preimage of t. Then we have

$$t_n - g_n^* f^* f_0^* \cdots f_n^* t_n = f_{n+1}^* t_{n+1} - g_n^* f^* f_0^* \cdots f_n^* f_{n+1}^* t_{n+1} = f_{n+1}^* t_{n+1} - f_{n+1}^* t_{n+1} = 0 \ ,$$

which shows injectivity.

We verify now the Mittag-Leffler condition. We claim $\mathrm{Im}(f_{n+1}^*) = \mathrm{Im}(f_{n+1}^* \cdots f_{n+k}^*)$ for all k, which follows from the next calculation. Let t_{n+1} be in $\pi_*(\mathtt{Map}(\mathrm{Yo}^s\mathcal{B}(P_{U_{n+1}}(F_{\mathcal{U}}(K_d))), T))$. Then using the identities show above we get

$$\begin{aligned} f_{n+1}^* \cdots f_{n+k}^* g_{n+k}^* f^* f_0^* \cdots f_{n+1}^* t_{n+1} &\overset{(7.3)}{=} f_{n+1}^* \cdots f_{n+k-1}^* g_{n+k-1}^* f^* f_0^* \ldots f_{n+1}^* t_{n+1} \\ &\ \vdots \\ &= g_n^* f^* f_0^* \ldots f_{n+1}^* t_{n+1} \\ &= f_{n+1}^* t_{n+1} \end{aligned}$$

finishing this proof. $\square$

Example 7.13. The following is taken from [BE16, Ex. 6.96] and originally goes back to Gromov [Gro93, Ex. 1.D_1]:

Let G be a finitely generated group admitting a model for its classifying space BG which is a finite simplicial complex. Then the universal cover EG of BG is a simplicial complex of bounded geometry which is equicontinuously contractible, i.e., EG_d is coarsifying by the above Proposition 7.12.

The group G quipped with a word-metric becomes a metric spaces and hence a uniform bornological coarse G_d. The action of G on EG provides a morphism $f : G_d \to EG_d$ which depends on the choice of a base-point in EG. The morphism $(\mathbf{P} \circ \mathbf{F})(f)$ is an equivalence.

Therefore we have shown that $f : G_d \to EG_d$ is a coarsifying approximation. $\square$

8 Cone functors

In this section we describe the cone functor $\mathcal{O} : \mathbf{UBC} \to \mathbf{BornCoarse}$ and its germs at infinity $\mathcal{O}^\infty : \mathbf{UBC} \to \mathbf{Sp}\mathcal{X}$. These functors play a crucial role in the construction of the coarse assembly map. After the introduction of the cone functor, we compare it with variants which occur in the literature on coarse geometry and which are useful in certain arguments.

In short, the cone of a uniform bornological coarse space X is the bornological coarse space $\mathcal{O}(X)$ obtained from the bornological coarse space $F_{\mathcal{U}}([0,\infty)\otimes X)$ by replacing the coarse structure by the hybrid structure (cf. [BE16, Sec. 5.1]) associated to the family of subsets $\mathcal{Y} := ([0,n]\times X)_{n\in\mathbb{N}}$ and the uniform structure on $[0,\infty)\otimes X$.

In the following we spell out the definition of the cone explicitly. Let $\mathcal{T}$ denote the uniform structure of X. Recall that a function $\phi : [0,\infty)\to\mathcal{T}$ is cofinal if for every entourage U in $\mathcal{T}$ there exists an element t in $[0,\infty)$ such that $\phi(s)\subseteq U$ for all s in $[t,\infty)$.

Definition 8.1. *We let $\mathcal{O}(X)$ be the bornological coarse space defined as follows:*

1. *The underlying set of $\mathcal{O}(X)$ is $[0,\infty)\times X$.*
2. *The bornology of $\mathcal{O}(X)$ is generated by the subsets $[0,n]\times B$ for n in $\mathbb{N}$ and bounded subsets B of X*
3. *The coarse structure of $\mathcal{O}(X)$ is generated by the entourages of the form $V\cap U_{(\kappa,\phi)}$, where V is a coarse entourage of $[0,\infty)\otimes X$ and*

$$U_{(\kappa,\phi)} := \{((s,x),(t,y))\in([0,\infty)\times X)^2 \mid |s-t|\le\kappa(\max\{s,t\})\ \&\ (x,y)\in\phi(\max\{s,t\})\}\ .$$

for all cofinal functions $\phi : [0,\infty)\to\mathcal{T}$ and functions $\kappa : [0,\infty)\to[0,\infty)$ satisfying $\mathtt{lim}_{t\to\infty}\kappa(t)=0$.

If $f : X\to X'$ is a morphism of uniform bornological coarse spaces, then the map

$$\mathtt{id}_{[0,\infty)}\times f : [0,\infty)\times X\to[0,\infty)\times X$$

is a morphism of bornological coarse spaces

$$\mathcal{O}(f) : \mathcal{O}(X)\to\mathcal{O}(X')\ .$$

We thus have described the cone functor

$$\mathcal{O} : \mathbf{UBC}\to\mathbf{BornCoarse}\ .$$

Let X be a uniform bornological coarse space. Then $\mathcal{Y}(X) := ([0,n]\times X)_{n\in\mathbb{N}}$ is a big family in $\mathcal{O}(X)$. The inclusion $X\to\{0\}\times X\to[0,n]\times X$ induces a coarse equivalence and hence induces an equivalence $\mathrm{Yo}^s(F_{\mathcal{U}}(X))\to\mathrm{Yo}^s(([0,n]\times X)_{\mathcal{O}(X)})$ for every n in $\mathbb{N}$. The collection of these equivalences for all n in $\mathbb{N}$ induces an equivalence

$$\mathrm{Yo}^s(F_{\mathcal{U}}(X))\simeq\mathrm{Yo}^s(\mathcal{Y}(X))\ .$$

The pair sequence of $(\mathcal{O}(X),\mathcal{Y}(X))$ therefore gives rise to the cone sequence of motivic coarse spectra

$$F_{\mathcal{U}}(X)\to\mathrm{Yo}^s(\mathcal{O}(X))\to\mathcal{O}^\infty(X)\to\Sigma\mathrm{Yo}^s(F_{\mathcal{U}}(X))\ , \tag{8.1}$$

where, by definiton,

$$\mathcal{O}^\infty(X) := \mathrm{Cofib}\big(\mathrm{Yo}^s(\mathcal{Y}(X))\to\mathrm{Yo}^s(\mathcal{O}(X))\big)\ .$$

This construction is functorial for X in $\mathbf{UBC}$.

Definition 8.2. *We call the resulting functor $\mathcal{O}^\infty : \mathbf{UBC} \to \mathbf{Sp}\mathcal{X}$ the germs at infinity of the cone.*

We refer to [BE16, Ex. 5.16] and [BEKWb, Sec. 9] for more details.

In the proof of Proposition 10.12 below it is useful to use a modified version of the cone over a uniform bornological coarse space X which we will denote by $\tilde{\mathcal{O}}(X)$.

Definition 8.3. *We let $\tilde{\mathcal{O}}(X)$ be the bornological coarse space defined as follows:*

1. *The underlying set of $\tilde{\mathcal{O}}(X)$ is $[0,\infty) \times X$.*
2. *The bornology of $\tilde{\mathcal{O}}(X)$ is generated by the subsets $[0,n] \times B$ for n in $\mathbb{N}$ and bounded subsets B of X*
3. *The coarse structure of $\tilde{\mathcal{O}}(X)$ is generated by the entourages of the form $V \cap U_\phi$, where V is a coarse entourage of $[0,\infty) \otimes X$ and*

$$U_\phi := \{((s,x),(t,y)) \in ([0,\infty) \times X) \times ([0,\infty) \times X) \mid (x,y) \in \phi(\max\{s,t\})\}$$

for all cofinal functions $\phi : [0,\infty) \to \mathcal{T}$.

Note that the underlying bornological spaces of $\mathcal{O}(X)$, $\tilde{\mathcal{O}}(X)$ and $[0,\infty) \otimes X$ coincide. The identity map of the underlying sets induces a morphism

$$i : \mathcal{O}(X) \to \tilde{\mathcal{O}}(X) \ . \tag{8.2}$$

Lemma 8.4. *The morphism* (8.2) *induces an equivalence*

$$\mathrm{Yo}^s(i) : \mathrm{Yo}^s(\mathcal{O}(X)) \to \mathrm{Yo}^s(\tilde{\mathcal{O}}(X)) \ .$$

Proof. We define a map of sets

$$q : [0,\infty) \times X \to [0,\infty) \times X \ , \quad q(t,x) := (\sqrt{1+t}, x) \ .$$

The map q induces a morphism of bornological coarse spaces $j : \tilde{\mathcal{O}}(X) \to \mathcal{O}(X)$. Note that the compositions $i \circ j$ and $j \circ i$ are both given on the level of sets by the map q. It suffices to show that the morphisms on $\mathrm{Yo}^s(\mathcal{O}(X))$ or $\mathrm{Yo}^s(\tilde{\mathcal{O}}(X))$, respectively, induced by q are equivalent to the respective identities.

We first consider the case of the modified cone $\tilde{\mathcal{O}}(X)$. In this case we shall see that q is coarsely homotopic to the identity (see [BE16, Defn. 4.17]). In order to define the homotopy we let the map $p_+ : \tilde{\mathcal{O}}(X) \to [0,\infty)$ be given by

$$p_+(t,x) := t + 1 - \sqrt{t+1}$$

and set $p := (p_+, 0)$. Note that p_+ is bornological and controlled. Then we define the coarse homotopy

$$I_p\tilde{\mathcal{O}}(X) \to \tilde{\mathcal{O}}(X) \ , \quad (u,t,x) \mapsto \left(\left(1 - \frac{u}{p_+(t)}\right)t + \frac{u}{p_+(t)}\sqrt{1+t}, x\right) \tag{8.3}$$

(see [BE16, Defn. 4.14] for notation of coarse cylinders). One easily checks that this map is proper and controlled. Since Yo^s is invariant under coarse homotopies (in particular using [BE16, Cor. 4.18]) we conclude that

$$\mathrm{Yo}^s(q) : \mathrm{Yo}^s(\tilde{\mathcal{O}}(X)) \to \mathrm{Yo}^s(\tilde{\mathcal{O}}(X))$$

is equivalent to the identity.

The case of the cone $\mathcal{O}(X)$ is more involved. By Definition 8.1 the hybrid structure on $\mathcal{O}(X)$ is generated by entourages of the form $V \cap U_{(\kappa,\phi)}$. We fix the pair (κ, ϕ) and V. We can now choose a differentiable function $\sigma : [0,\infty) \to [0,\infty)$ such that $\mathtt{lim}_{t\to\infty}\sigma(t) = \infty$ and $p_+ : \mathcal{O}(X)_{V\cap U_{(\kappa,\phi)}} \to [0,\infty)$ given by

$$p_+(t) := \sigma(t)(t+1-\sqrt{t+1})$$

is controlled. To this end we must make sure that $(1+t)\sigma'(t)$ and $\sigma\kappa$ are both uniformly bounded. Note that p_+ is also bornological. We then define the coarse homotopy

$$I_p\mathcal{O}(X)_{V\cap U_{(\kappa,\phi)}} \to \mathcal{O}(X)$$

between the maps induced by $\mathtt{id}_{[0,\infty)\times X}$ and q by the same formula as in (8.3) as above. Indeed one checks that this map is proper and controlled. Hence we have an equivalence of morphisms

$$\mathrm{Yo}^s(q) \simeq \mathrm{Yo}^s(\mathtt{id}) : \mathrm{Yo}^s(\mathcal{O}(X)_{V\cap U_{(\kappa,\phi)}}) \to \mathrm{Yo}^s(\mathcal{O}(X)) \ .$$

We now perform the colimit of these equivalences over the poset of data $(V, (\kappa, \phi))$. By u-continuity we get the desired equivalence of

$$\mathrm{Yo}^s(q) : \mathrm{Yo}^s(\mathcal{O}(X)) \to \mathrm{Yo}^s(\mathcal{O}(X))$$

with the identity. □

Note that in the definition of the modified cone $\tilde{\mathcal{O}}(X)$ we have not fixed the decay rate (encoded in the function ϕ in Definition 8.3.3) of the entourages in the X-direction as t and s tend to ∞. Let us fix such a function ϕ which we assume to be monotoneous and such that $\phi(0) = X \times X$.

Definition 8.5. *We let $\tilde{\mathcal{O}}_\phi(X)$ be the bornological coarse space defined as follows:*

1. *The underlying set of $\tilde{\mathcal{O}}_\phi(X)$ is $[0,\infty) \times X$.*
2. *The bornology of $\tilde{\mathcal{O}}_\phi(X)$ is generated by the subsets $[0,n] \times B$ for n in $\mathbb{N}$ and bounded subsets B of X.*
3. *The coarse structure of $\tilde{\mathcal{O}}_\phi(X)$ is generated by entourages of the form $V \cap U_\phi$, where V is a coarse entourage of $[0,\infty) \otimes X$.*

Example 8.6. Let X be a metric space. Recall that its coarse structure is generated by the collection of entourages $W_r := \{(x,y) \in X \times X \mid d(x,y) \le r\}$ for all $r > 0$. If we set $\phi(t) = W_{1/t}$, then $\tilde{\mathcal{O}}_\phi(X)$ is the open cone over X as considered at many places in the coarse geometry literature and usually called the Euclidean cone over X. $\square$

We have a canonical morphism

$$k_\phi : \tilde{\mathcal{O}}_\phi(X) \to \tilde{\mathcal{O}}(X) \tag{8.4}$$

given by the identity of the underlying sets.

Lemma 8.7. *If ϕ is monotoneous and satisfies $\phi(0) = X \times X$, then the map* (8.4) *induces an equivalence*

$$\mathrm{Yo}^s(k_\phi) : \mathrm{Yo}^s(\tilde{\mathcal{O}}_\phi(X)) \to \mathrm{Yo}^s(\tilde{\mathcal{O}}(X)) \ .$$

Proof. If ϕ' is a second monotoneous function as in 8.3.3 such that $\phi(t) \subseteq \phi'(t)$ for all t in $[0,\infty)$, then $U_\phi \subseteq U_{\phi'}$. Therefore the identity of the underlying maps induces a morphism

$$k_\phi^{\phi'} : \tilde{\mathcal{O}}_\phi(X) \to \tilde{\mathcal{O}}_{\phi'}(X) \ .$$

By u-continuity we have an equivalence

$$\mathrm{Yo}^s(\tilde{\mathcal{O}}(X)) \simeq \mathrm{colim}_{\phi' \ge \phi} \mathrm{Yo}^s(\tilde{\mathcal{O}}_{\phi'}(X)) \ .$$

It therefore suffices to show that

$$\mathrm{Yo}^s(k_\phi^{\phi'}) : \mathrm{Yo}^s(\tilde{\mathcal{O}}_\phi(X)) \to \mathrm{Yo}^s(\tilde{\mathcal{O}}_{\phi'}(X))$$

is an equivalence for all pairs ϕ, ϕ' such that $\phi(t) \subseteq \phi'(t)$ for all t in $[0,\infty)$.

We will show now that there exists a controlled function $\sigma : [0,\infty) \to [0,\infty)$ such that $\phi'(t) \subseteq \phi(\sigma(t))$ and $\mathtt{lim}_{t\to\infty} \sigma(t) = \infty$. To this end set

$$\delta : [0,\infty) \to [0,\infty) \ , \quad \delta(s) := \sup\{t \in [0,\infty) \mid \phi'(s) \subseteq \phi(t)\} \ .$$

This function is monotoneously increasing and satisfies $\mathtt{lim}_{s\to\infty} \delta(s) = \infty$. The idea is now to define σ to be δ. But to ensure that σ is controlled, we have to modify this idea slightly. We choose $t_0 \in [0,\infty)$ such that $\delta(t_0) \ge 2$. We can find $\sigma(t)$ for all t in $[t_0,\infty)$ by solving the equation

$$t = \int_0^{\sigma(t)} h(s) ds \ ,$$

where h is a function with $h \ge 1$ and

$$t \le \int_0^{\delta(t)} h(s) ds \ .$$

More concretely, we can take

$$h(t) := \max\left\{1, \sup_{s\in[1,t]} \delta^{-1}(2s)\right\} \ ,$$

where we set $\delta^{-1}(u) := \sup\{r \in [0,\infty) \mid \delta(r) \le u\}$. Note that if $t \in [t_0,\infty)$, then the interval $[\delta(t)/2, \delta(t)]$ in the domain of integration yields the estimate

$$\int_0^{\delta(t)} h(s)ds \ge \frac{\delta(t)}{2}\delta^{-1}(2\delta(t)/2) \ge t \ .$$

For $t \in [0,t_0]$ we set $\sigma(t) = 0$. The Lipschitz constant of σ on $[t_0,\infty)$ is bounded by 1. It follows that σ is controlled.

We consider the map of sets

$$q : [0,\infty) \times X \to [0,\infty) \times X \ , \quad q(t,x) := (\sigma(t), x) \ .$$

By construction it induces a morphism

$$j : \tilde{\mathcal{O}}_{\phi'}(X) \to \tilde{\mathcal{O}}_{\phi}(X) \ .$$

We now note that the compositions

$$j \circ k_{\phi}^{\phi'} : \tilde{\mathcal{O}}_{\phi}(X) \to \tilde{\mathcal{O}}_{\phi}(X) \ , \quad k_{\phi}^{\phi'} \circ j : \tilde{\mathcal{O}}_{\phi'}(X) \to \tilde{\mathcal{O}}_{\phi'}(X)$$

are both induced by q.

It suffices to show that these morphisms are both coarsely homotopic to the identity.

We set $p := (\sigma + 1, 0)$ and observe that the map

$$I_p\tilde{\mathcal{O}}_{\phi}(X) \to \tilde{\mathcal{O}}_{\phi}(X) \ , \quad (u,t,x) \mapsto \Big(\Big(1 - \frac{u}{\sigma(t)+1}\Big)t + \frac{u}{\sigma(t)+1}\sigma(t), x\Big)$$

is a suitable homotopy (i.e., proper and controlled) that does the job. The same construction also works in the case of ϕ'. □

Remark 8.8. The cone $\tilde{\mathcal{O}}(X)$ has a big family $\mathcal{Y}(X) := ([0,n] \otimes X)_{n\in\mathbb{N}}$ and we can define a modified version of the germs at infinity

$$\tilde{\mathcal{O}}^{\infty}(X) := \mathrm{Cofib}\big(\mathrm{Yo}^s(\mathcal{Y}(X)) \to \mathrm{Yo}^s(\tilde{\mathcal{O}}(X))\big) \ .$$

Similarly we can define

$$\tilde{\mathcal{O}}_{\phi}^{\infty}(X) := \mathrm{Cofib}\big(\mathrm{Yo}^s(\mathcal{Y}(X)) \to \mathrm{Yo}^s(\tilde{\mathcal{O}}_{\phi}(X))\big) \ .$$

The inclusion $X \to [0,n] \times X$ is a coarse equivalence for every n in $\mathbb{N}$ and the structure induced by $\tilde{\mathcal{O}}(X)$ or $\tilde{\mathcal{O}}_{\phi}(X)$, respectively. In the latter case this is granted by the condition that $\phi(0) = X \times X$. Therefore we get fibre sequences

$$\mathrm{Yo}^s(F_{\mathcal{U}}(X)) \to \mathrm{Yo}^s(\tilde{\mathcal{O}}(X)) \to \tilde{\mathcal{O}}^{\infty}(X) \to \Sigma\mathrm{Yo}^s(F_{\mathcal{U}}(X)) \ ,$$

and

$$\mathrm{Yo}^s(F_{\mathcal{U}}(X)) \to \mathrm{Yo}^s(\tilde{\mathcal{O}}_{\phi}(X)) \to \tilde{\mathcal{O}}_{\phi}^{\infty}(X) \to \Sigma\mathrm{Yo}^s(F_{\mathcal{U}}(X)) \ ,$$

respectively. By a comparison with the cone sequence (8.1) and by Lemmas 8.4 and 8.7 we get induced equivalences

$$\mathcal{O}^{\infty}(X) \simeq \tilde{\mathcal{O}}^{\infty}(X) \simeq \tilde{\mathcal{O}}^{\infty}_{\phi}(X) \ .$$

So we could have defined the germs at infinity of the cone using a modified version of the cone. But since the modified cones do not come from a hybrid structure construction we can not apply the general theorems (Homotopy Theorem and Decomposition Theorem) for hybrid spaces shown in [BE16, Sec. 5.2 & 5.3] in order to deduce the properties of this functor, see e.g. Lemma 9.1 below. For this reason we prefer to work with $\mathcal{O}(X)$ instead of $\tilde{\mathcal{O}}(X)$ or $\tilde{\mathcal{O}}_{\phi}(X)$. □

Example 8.9. Let X be a geodesic, locally compact hyperbolic metric space. One can construct a nice compactification of X by attaching the Gromov boundary ∂X. Note that ∂X is a compact metric space. Higson–Roe [HR95] showed that X is coarsely homotopy equivalent to the Euclidean cone $\tilde{\mathcal{O}}_{\phi}(\partial X)$ over its Gromov boundary ∂X. Together with the results of the present section we therefore get the equivalence

$$\mathrm{Yo}^{s}(X) \simeq \mathrm{Yo}^{s}(\mathcal{O}(\partial X)) \ . \tag{8.5}$$

Fukaya–Oguni [FO17] generalized the result of Higson–Roe to all proper coarsely convex spaces (examples are hyperbolic spaces, CAT(0) spaces and systolic complexes). Especially, we have the equivalence (8.5) where ∂X is a suitable version of Gromov's boundary. □

9 The coarse assembly map

In this section we define the coarse assembly map.

Taking the functoriality of cone sequence (8.1) into account we get a fibre sequence of functors from **UBC** to $\mathbf{Sp}\mathcal{X}$

$$\mathbf{F} \to \mathrm{Yo}^{s} \circ \mathcal{O} \to \mathcal{O}^{\infty} \xrightarrow{\partial} \Sigma \mathbf{F} \tag{9.1}$$

which we call the cone sequence.

Lemma 9.1. *The functors* $\mathrm{Yo}^{s} \circ \mathcal{O}, \mathcal{O}^{\infty} : \mathbf{UBC} \to \mathbf{Sp}\mathcal{X}$ *satisfy excision for uniformly and coarsely excisive decompositions, and they are homotopy invariant.*

Proof. This is shown in [BEKWb, Sec. 9.4 & 9.5]. □

Remark 9.2. Since we consider excision for decompositions which are uniform and coarse at the same time it is not necessary to assume that our uniform spaces are Hausdorff, see [BEKWb, Rem. 9.26]. □

By Lemma 6.1 functor $\mathbf{F}$ vanishes on flasque spaces, but we do not expect that $\mathcal{O}$ vanishes on flasque spaces. Assume that X is a flasque uniform bornological coarse space with the flasqueness witnessed by the self-map f. Then in general $\mathcal{O}(f)$ is not close to the identity, but it is equivalent to it [BE16, Cor. 5.31]. In fact, the map $\mathcal{O}(f)$ exhibits the cone $\mathcal{O}(X)$ as a weakly flasque bornological coarse space in the sense of [BEKWb, Def. 4.17], see [BEKWb, Proof of Prop. 11.20].

Definition 9.3 ([BEKWb, Def. 4.18]). *A coarse homology theory is called strong if it vanishes on weakly flasque bornological coarse spaces.*

Example 9.4. Ordinary coarse homology $H\mathcal{X}$, algebraic K-theory $K\mathcal{A}\mathcal{X}$ of an additive category $\mathcal{A}$ and coarse K-homology $K\mathcal{X}$ are strong coarse homology theories. □

Let $\mathbf{C}$ be a cocomplete stable ∞-category and $E : \mathbf{BornCoarse} \to \mathbf{C}$ be a coarse homology theory. We set

$$E\mathcal{O}^\infty := E \circ \mathcal{O}^\infty, \quad E\mathcal{O} := E \circ \mathcal{O} .$$

Lemma 9.5. *If E is strong, then both*

$$E\mathcal{O}^\infty, E\mathcal{O} : \mathbf{UBC} \to \mathbf{C}$$

are local homology theories.

Proof. For a uniform bornological coarse space X we have a natural fibre sequence

$$E(F_{\mathcal{U}}(X)) \to E\mathcal{O}(X) \to E\mathcal{O}^\infty(X) \xrightarrow{E(\partial)} \Sigma E(F_{\mathcal{U}}(X)) . \tag{9.2}$$

By Lemma 9.1 both functors $E\mathcal{O}$ and $E\mathcal{O}^\infty$ are homotopy invariant and satisfy excision, and by Lemma 3.11 the functor $E \circ F_{\mathcal{U}}$ also has these properties.

Lemma 3.11 shows that the functor $E \circ \mathcal{F}_{\mathcal{U}}$ is u-continuous. The functor $\mathcal{O}^\infty$ is invariant under coarsenings ([BEKWb, Prop. 9.31] or Definition 12.8) which implies the equivalence $\mathcal{O}^\infty(X_U) \xrightarrow{\simeq} \mathcal{O}^\infty(X)$ for sufficiently large entourages U of X. In particular, the functor $E\mathcal{O}^\infty$ is u-continuous. It follows from the fibre sequence (9.2) that $E\mathcal{O}$ is u-continuous.

If X is flasque, then $F_{\mathcal{U}}(X)$ is flasque and $\mathcal{O}(X)$ is weakly flasque. Since $E(F_{\mathcal{U}}(X)) \simeq 0$ and also $E\mathcal{O}(X) \simeq 0$ due to strongness of E, we conclude that $E\mathcal{O}^\infty(X) \simeq 0$. □

Let E be a strong coarse homology theory.

Definition 9.6. *The coarse assembly map is the natural transformation between coarse homology theories*

$$\mu_E : E\mathcal{O}^\infty\mathbf{P} \to \Sigma E$$

defined as the composition of $E(\partial \circ \mathbf{P})$ with the identification $(E\mathbf{F})\mathbf{P} \simeq E$ from (6.2).

Remark 9.7. It follows from the above fibre sequence (9.1) that for a bornological coarse space X the coarse assembly map

$$\mu_{E,X} : E\mathcal{O}^\infty\mathbf{P}(X) \to \Sigma E(X) \tag{9.3}$$

is an equivalence if and only if $E\mathcal{O}\mathbf{P}(X) \simeq 0$. Therefore we have identified $E\mathcal{O}\mathbf{P}$ as the coarse homology theory which detects the obstructions to μ_E being an equivalence. □

Remark 9.8. At the moment the local homology theory $E\mathcal{O}^\infty$ appearing in the domain of the coarse assembly map might appear mysterious. In Proposition 12.17 we calculate the evaluation of this homology theory on countable, finite-dimensional, locally finite simplicial complexes under the assumption that E is countably additive.

In general, using the fact that $\mathcal{O}^\infty(X)$ is representable (see Section 15) one can express the value of $E\mathcal{O}^\infty$ on a uniform bornological coarse space by the value of E on an explicitly given bornological coarse space. □

10 Isomorphism results

In this section we discuss conditions which imply that the coarse assembly map $\mu_{E,X}$ (Equation 9.3) is an equivalence. We will discuss the cases of finite asymptotic dimension, finite decomposition complexity, and scaleable spaces. Our goal is to show that in many cases the reasons for the validity of the coarse Baum–Connes conjecture for X in fact imply in greater generality that the coarse assembly map $\mu_{E,X}$ is an equivalence for suitable coarse homology theories E.

Note that the coarse assembly map $\mu_E : E\mathcal{O}^\infty\mathbf{P} \to \Sigma E$ is a morphism between coarse homology theories. So it is clear from the outset that the property of $\mu_{E,X}$ of being an equivalence only depends on the coarse motivic spectrum $\mathrm{Yo}^s(X)$.

10.1 Finite asymptotic dimension

Let $(X, \mathcal{C}, \mathcal{B})$ be a bornological coarse space. Recall that X is called discrete as a coarse space, if $\mathcal{C} = \mathcal{C}\langle\{\mathtt{diag}_X\}\rangle$.

Let X be a bornological coarse space and E be a strong coarse homology theory.

Proposition 10.1. *If X is discrete as a coarse space, then the coarse assembly map $\mu_{E,X}$ is an equivalence.*

Proof. Assume that X is a discrete bornological coarse space. Then we have $X \cong P_{\mathtt{diag}}(X)$ as uniform bornological coarse spaces if we equip X with the discrete uniform structure. By [BEKWb, Prop. 9.35] the boundary map of the cone sequence (9.1) induces an equivalence

$$\mathcal{O}^\infty(X) \xrightarrow{\simeq} \Sigma\mathrm{Yo}^s(F_{\mathcal{U}}(X)) .$$

This implies the result immediately. □

Corollary 10.2. *If E is strong, then the coarse assembly map $\mu_{X,E}$ is an equivalence for all bornological coarse spaces X such that $\mathrm{Yo}^s(X)$ belongs to the subcategory $\mathbf{Sp}\mathcal{X}\langle \mathrm{disc}\rangle$ generated under colimits by discrete bornological coarse spaces.*

Let X be a bornological coarse space with coarse structure $\mathcal{C}$ and E be a strong coarse homology theory.

Theorem 10.3. *Assume that there exists a cofinal set of entourages U in $\mathcal{C}$ such that X_U has finite asymptotic dimension.*

Then the coarse assembly map $\mu_{E,X} : E\mathcal{O}^\infty\mathbf{P}(X) \to \Sigma E(X)$ is an equivalence.

Proof. The assumptions on the space X imply by [BE16, Thm. 6.114] that the motive $\mathrm{Yo}^s(X)$ belongs to $\mathbf{Sp}\mathcal{X}\langle \mathrm{disc}\rangle$. We now apply the above Corollary 10.2. □

Note that the condition of having finite asymptotic dimension only depends on the coarse structure of X.

10.2 Finite decomposition complexity

Guentner–Tessera–Yu [GTY12] introduced a weaker condition than finite asymptotic dimension called finite decomposition complexity FDC. In [BEKWa] we investigated under which assumptions on E the condition that a bornological coarse space X has FDC implies that $E\mathcal{O}\mathbf{P}(X) \simeq 0$ (even in the equivariant case).

On the side of the coarse homology theory E we need weak additivity and transfers. Let us define and discuss these notions now.

The notion of a coarse homology theory with transfers was introduced in [BEKWa]. In [BEKWa, Sec. 2.2] we introduced the category $\mathbf{BornCoarse}_{tr}$ of bornological coarse spaces with transfers. It has the same objects as $\mathbf{BornCoarse}$, but its morphisms $X \to X'$ are compositions $f \circ \mathtt{tr}_{X,I}$ of a transfer morphisms

$$\mathtt{tr}_{X,I} : X \to I_{min,min} \otimes X$$

for some well-ordered set I and a morphism

$$f : I_{min,min} \otimes X \to X'$$

of bornological coarse spaces. In particular we have a canonically given inclusion functor $\mathbf{BornCoarse} \to \mathbf{BornCoarse}_{tr}$.

A $\mathbf{C}$-valued coarse homology theory with transfers is a functor $E_{tr} : \mathbf{BornCoarse}_{tr} \to \mathbf{C}$ such that the restriction $E : \mathbf{BornCoarse} \to \mathbf{C}$ of E_{tr} along the canonical inclusion is a coarse homology theory and for every i in I the composition

$$E_{tr}(X) \xrightarrow{E_{tr}(\mathtt{tr}_{X,I})} E_{tr}(I_{min,min} \otimes X) \stackrel{\text{excision}}{\simeq} E_{tr}(X) \oplus E_{tr}((I \setminus \{i\})_{min,min} \otimes X) \qquad (10.1)$$

is equivalent to

$$\mathtt{id}_{E_{tr}(X)} \oplus E_{tr}(\mathtt{tr}_{X,I\setminus\{i\}}) \ .$$

We say that a coarse homology theory E admits transfers if it has an extension to a coarse cohomology theory with transfers.

For the definition of the property of E being weakly additive we refer to [BEKWa]. But note that E is weakly additive if it is strongly additive. Recall that E is strongly additive if it sends free unions to products, i.e,

$$E(\bigsqcup_{i\in I}^{free} X_i) \simeq \prod_{i\in I} E(X_i)$$

for every family $(X_i)_{i\in I}$ of bornological coarse spaces, where the map is induced by the family of projections $(E(\bigsqcup_{i\in I}^{free} X_i) \to E(X_i))_{i\in I}$ given by excision.

Example 10.4. Examples of coarse homology theories which admit transfers are ordinary coarse homology $H\mathcal{X}$, algebraic K-homology $K\mathcal{A}\mathcal{X}$ with coefficients in an additive category $\mathcal{A}$, and coarse K-homology $K\mathcal{X}$. We refer to [BEKWa] for the first two cases and [BE] for the last case. In these references we actually considered the equivariant case for a group G. For the present application just need the case of a trivial group $G = \{1\}$.

In [BEKWa] we show that the cohomology theories $H\mathcal{X}$ and $K\mathcal{A}\mathcal{X}$ are strongly additive. At the moment we do not know whether coarse K-homology $K\mathcal{X}$ is strongly additive, see the discussion in [BE16, Rem. 7.76]. □

Let E be a $\mathbf{C}$-valued coarse homology theory.

Assumption 10.5. *Assume:*

1. $\mathbf{C}$ *is compactly generated.*
2. E *is strong.*
3. E *is weakly additive.*
4. E *admits transfers.*

Remark 10.6. The category of spectra $\mathbf{Sp}$ is compactly generated. □

Let X be a bornological coarse space and E be a coarse homology theory.

Theorem 10.7. *If E satisfies the Assumption 10.5 and X_U has FDC for a cofinal set of entourages U of X, then the coarse assembly map $\mu_{E,X}$ is an equivalence.*

Proof. This follows from [BEKWa, Thm. 1.6] and Remark 9.7. □

Remark 10.8. Note that finite asymptotic dimension implies FDC. Hence in the case that the coarse homology theory E satisfies Assumption 10.5 the above Theorem 10.7 generalizes Theorem 10.3. □

Remark 10.9. In [BE16, Lem. 7.74] we show that coarse K-homology $K\mathcal{X}$ is strongly additive on the subcategory of **BornCoarse** of locally countable bornological coarse spaces. This should suffices to conclude the statement of Theorem 10.7 under the assumption that X is locally countable. But we have not checked this in detail. □

10.3 Scaleable spaces

In the literature on the coarse Baum–Connes conjecture it is an important observation that the existence of a suitable scaling implies that the analytic coarse assembly map in coarse K-homology is an isomorphism [HR95]. In the following we show analogous results for general coarse homology theories.

Let X be a uniform bornological coarse space and $s : X \to X$ be a morphism of uniform bornological coarse spaces. We assume that the uniform structure of X is induced by a metric.

Definition 10.10. *The morphism s is a scaling if it satisfies the following conditions:*

1. *s is 1-Lipschitz.*
2. *For every coarse entourage W and uniform entourage V of X there exists k in $\mathbb{N}$ such that $(s^k \times s^k)(W) \subseteq V$.*
3. *For every coarse entourage U of X the union $\bigcup_{k\in\mathbb{N}}(s^k \times s^k)(U)$ is also a coarse entourage of X.*

Example 10.11. Assume that X is a proper metric space whose structures are induced from the metric. If $s : X \to X$ is a map which is 1/2-Lipschitz and proper, then s is a scaling in the sense of Definition 10.10. Note that in order to be a scaling in the sense of [HR95, Def. 7.1] one must in addition assume that s is coarsely and properly homotopic to the identity. These conditions will be added in Definition 10.15 which characterizes coarse scalings. □

Using the existence of a scaling for X we want to deduce that $E\mathcal{O}(X) \simeq 0$ for suitable coarse homology theories E. Similarly as in the proof of [HR95, Thm. 7.2] the argument is based on an Eilenberg swindle. In order to make this work in our abstract setting we need to assume that the homology theory admits transfers.

Let X be a uniform bornological coarse space and $s : X \to X$ be a morphism. Furthermore let E be a coarse homology theory.

Proposition 10.12. *Assume:*

1. *$s : X \to X$ is a scaling.*
2. *$E(F_{\mathcal{U}}(s)) \simeq \mathtt{id}_{E(F_{\mathcal{U}}(X))}$.*
3. *E admits transfers.*
4. *$E\mathcal{O}^\infty(s) \simeq \mathtt{id}_{E\mathcal{O}^\infty(X)}$.*

Then $E\mathcal{O}(X) \simeq 0$.

Before starting the proof of the above proposition let us first prove the following statement. Recall Definition 8.3 of the modified cone $\tilde{\mathcal{O}}(X)$. We define the map of sets

$$\Phi : \mathbb{N} \times [0,\infty) \times X \to [0,\infty) \times X \ , \quad \Phi(n,t,x) := (n+t, s^n(x)) \ .$$

Lemma 10.13. *The map* Φ *is a morphism of bornological coarse spaces*

$$\Phi : \mathbb{N}_{min,min} \otimes \tilde{\mathcal{O}}(X) \to \tilde{\mathcal{O}}(X) \ .$$

Proof. First we show that Φ is proper. Let B be a bounded subset in X and u be in $\mathbb{N}$ and consider the bounded subset $[0,u] \times B$ in $\tilde{\mathcal{O}}(X)$. Then $\Phi^{-1}([0,u] \times B)$ is contained in $[0,u] \times [0,\infty) \times X$. The restriction of Φ to $\{n\} \times \tilde{\mathcal{O}}(X)$ is proper for every n in $\mathbb{N}$ since the maps $s^n : X \to X$ and $t \mapsto n+t : [0,\infty) \to [0,\infty)$ are proper. Therefore we can conclude that $\Phi^{-1}([0,u] \times B)$ is bounded.

We now show that Φ is controlled. It is easy to check using 10.10.3 and the fact that $t \mapsto n+t$ is 1-Lipschitz that Φ is a morphism of bornological coarse spaces

$$\mathbb{N}_{min,min} \otimes F_{\mathcal{U}}([0,\infty) \otimes X) \to F_{\mathcal{U}}([0,\infty) \otimes X) \ .$$

Let $\psi : [0,\infty) \to [0,\infty)$ be a function such that $\mathtt{lim}_{t\to\infty} \psi(t) = 0$. For simplicity we can assume that ψ is monotoneously decreasing. It determines a function $\phi : [0,\infty) \to \mathcal{T}$ by $\phi(t) := U_{\psi(t)}$ as used in Definition 8.3.3. Let W be a coarse entourage of X and $V := U_r \times W$ be a coarse entourage of $[0,\infty) \otimes X$ for r in $(0,\infty)$. Then we must show that

$$(\Phi \times \Phi)(\mathtt{diag}(\mathbb{N}) \times V \cap U_\phi) \subseteq U_{\phi'}$$

for $\phi'(t) = U_{\psi'(t)}$ with ψ' having the same properties as ψ. This boils down to the assertion that for all t in $[0,\infty)$ we have $d(s^n(x), s^n(y)) \le \psi'(t)$ for all $n \in \mathbb{N}$ with $t \ge n$ and $(x,y) \in W$ with $d(x,y) \le \psi(t-n-r)$ (here we use the monotonicity of ψ). Here we set $\psi(t) := \psi(0)$ for negative t.

We define the monotonously decreasing function

$$e : \mathbb{N} \to [0,\infty] \ , \quad e(n) := \sup\{d(s^n(x), s^n(y)) \mid (x,y) \in W\} \ .$$

By 10.10.2 we have $\mathtt{lim}_{n\to\infty} e(n) = 0$. We define

$$\psi'(t) := \max\{\min\{\psi(t-n-r), e(n)\} \mid n \in \mathbb{N} \,\&\, t \ge n\} \ .$$

In view of 10.10.1 this function would do the job if $\mathtt{lim}_{t\to\infty} \psi'(t) = 0$. Let ϵ in $(0,\infty)$ be given. Then we choose n_0 in $\mathbb{N}$ so large that $e(n) \le \epsilon$ for all n in $\mathbb{N}$ with $n \ge n_0$. Let furthermore t_0 in $[0,\infty)$ be so large that $\psi(t) \le \epsilon$ for all t in $[t_0,\infty)$. If t in $[0,\infty)$ satisfies $t \ge n_0 + t_0 + r$, then $\psi'(t) \le \epsilon$. $\square$

Proof of Proposition 10.12. Let E_{tr} be an extension of E to a coarse homology theory with transfers. An application of the relation (10.1) yields a decomposition

$$E_{tr}(\Phi \circ \mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}}) \simeq E_{tr}(\Phi' \circ \mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}^{\geq 1}}) + \mathtt{id}_{E(\tilde{\mathcal{O}}(X))} \ , \tag{10.2}$$

where Φ' is the restriction of Φ to $\mathbb{N}^{\geq 1}_{min,min} \otimes \tilde{\mathcal{O}}(X)$. We consider the following commuting diagram in $\mathbf{BornCoarse}_{tr}$:

$$\begin{tikzcd}
\tilde{\mathcal{O}}(X) \arrow[r, "\mathtt{tr}_{X,\mathbb{N}^{\geq 1}}"] \arrow[d, "\tilde{\mathcal{O}}(s)"'] & \mathbb{N}^{\geq 1}_{min,min} \otimes \tilde{\mathcal{O}}(X) \arrow[r, "\Phi'"] \arrow[d, "(n\mapsto n-1)\otimes\tilde{\mathcal{O}}(s)"] & \tilde{\mathcal{O}}(X) \\
\tilde{\mathcal{O}}(X) \arrow[r, "\mathtt{tr}_{X,\mathbb{N}}"] & \mathbb{N}_{min,min} \otimes \tilde{\mathcal{O}}(X) \arrow[r, "\Phi"] & \tilde{\mathcal{O}}(X) \arrow[u, "T"']
\end{tikzcd}$$

where $T : \tilde{\mathcal{O}}(X) \to \tilde{\mathcal{O}}(X)$ is given by $T(t,x) := (t+1,x)$. Note that the morphism T is close to the identity.

Since T is close to the identity the commutativity of the above diagram implies

$$\begin{aligned}
E_{tr}(\Phi' \circ \mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}^{\geq 1}}) &\simeq E_{tr}(T) \circ E_{tr}(\Phi) \circ E_{tr}(\mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}}) \circ E_{tr}\tilde{\mathcal{O}}(s) \\
&\simeq E_{tr}(\Phi \circ \mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}}) \circ E\tilde{\mathcal{O}}(s) \ ,
\end{aligned}$$

from which we get, using (10.2),

$$E_{tr}(\Phi \circ \mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}}) \simeq E_{tr}(\Phi \circ \mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}}) \circ E\tilde{\mathcal{O}}(s) + \mathtt{id}_{E\tilde{\mathcal{O}}(X)} \ . \tag{10.3}$$

We now consider the diagram (note that we are now using the cone instead of the modified cone as above)

$$\begin{tikzcd}
E(F_{\mathcal{U}}(X)) \arrow[r, "\iota"] \arrow[d, shift right=3, bend right] \arrow[d, shift left=3, bend left, "E(F_{\mathcal{U}}(s))"] & E\mathcal{O}(X) \arrow[r] \arrow[d, shift right=3, bend right] \arrow[d, shift left=3, bend left, "E\mathcal{O}(s)"] \arrow[dl, dotted, "\delta"'] & E\mathcal{O}^{\infty}(X) \arrow[r] \arrow[d, shift right=3, bend right] \arrow[d, shift left=3, bend left, "E\mathcal{O}^{\infty}(s)"] & \Sigma E(F_{\mathcal{U}}(X)) \arrow[d, shift right=3, bend right] \arrow[d, shift left=3, bend left, "\Sigma E(F_{\mathcal{U}}(s))"] \\
E(F_{\mathcal{U}}(X)) \arrow[r, "\iota"] & E\mathcal{O}(X) \arrow[r] & E\mathcal{O}^{\infty}(X) \arrow[r] & \Sigma E(F_{\mathcal{U}}(X))
\end{tikzcd}$$

whose horizontal sequences are two copies of the cone sequence and the non-labeled vertical maps are induced by the identity. The diagram is a picture of two morphisms between fibre sequences (one is the identity) which we want to compare. The Condition 4 yields a morphism $\delta : E\mathcal{O}(X) \to E(F_{\mathcal{U}}(X))$ such that

$$E\mathcal{O}(s) - \mathtt{id}_{E\mathcal{O}(X)} \simeq \iota \circ \delta \ . \tag{10.4}$$

Condition 2 then implies that

$$\iota \circ \delta \circ \iota \simeq 0 \ . \tag{10.5}$$

In view of Lemma 8.4 we get the same relations if we replace the cone by the modified cone. Equivalence (10.3) now implies that (using in the second line (10.4) for the modified cone)

$$\begin{aligned}
\mathtt{id}_{E\tilde{\mathcal{O}}(X)} &\simeq E_{tr}(\Phi \circ \mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}}) - E_{tr}(\Phi \circ \mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}}) \circ E\tilde{\mathcal{O}}(s) \\
&\simeq E_{tr}(\Phi \circ \mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}}) - E_{tr}(\Phi \circ \mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}}) \circ (\iota \circ \delta + \mathtt{id}_{E\tilde{\mathcal{O}}(X)}) \\
&\simeq -E_{tr}(\Phi \circ \mathtt{tr}_{\tilde{\mathcal{O}}(X),\mathbb{N}}) \circ \iota \circ \delta
\end{aligned}$$

If we compose this equivalence from the right with ι and use (10.5), then we get

$$\iota \simeq -E_{tr}(\Phi \circ \mathtt{tr}_{\mathbb{N},\tilde{\mathcal{O}}(X)}) \circ \iota \circ \delta \circ \iota \simeq 0 \ .$$

Hence we get

$$\mathtt{id}_{E\tilde{\mathcal{O}}(X)} \simeq 0 \ ,$$

which in view of Lemma 8.4 implies $E\mathcal{O}(X) \simeq 0$. □

Our next concern are the conditions 10.12.2 and 10.12.4. Condition 10.12.2 is satisfied, e.g., if $F_{\mathcal{U}}(s)$ is coarsely homotopic to the identity map. In the literature this is a standard assumption on a scaling; see, e.g., Higson–Roe [HR95].

Condition 10.12.4 is more problematic. If s is homotopic to the identity in the sense of **UBC**, then 10.12.4 is satisfied by the homotopy invariance of the functor $E\mathcal{O}^\infty : \mathbf{UBC} \to \mathbf{C}$, see Lemma 9.1. Unfortunately, in applications s is rarely homotopic to the identity in the sense of **UBC**. The standard assumption made in, e.g., Higson–Roe [HR95] is that $F_{\mathcal{C},\mathcal{U}/2}(s)$ is homotopic to the identity map, i.e., that s is homotopic to the identity in the sense of **TopBorn** (i.e., after forgetting the coarse and the uniform structures, but the homotopies are still required to be proper). If E is countably additive, then $E\mathcal{O}^\infty$ has better homotopy invariance properties on nice spaces which we will use in the following to make the standard assumption of Higson–Roe also work in our situation.

Let X be a uniform bornological coarse space and let E be a **C**-valued coarse homology theory.

Lemma 10.14. *Assume:*

1. *X is homotopy equivalent (in* **UBC***) to a countable, locally finite, finite-dimensional simplicial complex.*
2. *E is countably additive.*
3. ***C** is presentable.*
4. *$F_{\mathcal{C},\mathcal{U}/2}(s)$ is homotopic to $\mathtt{id}_{F_{\mathcal{C},\mathcal{U}/2}(X)}$.*

Then $E\mathcal{O}^\infty(s) \simeq \mathtt{id}_{E\mathcal{O}^\infty(X)}$.

Proof. This is an immediate consequence of Corollary 12.18. □

In the following definition of a coarse scaling we introduce a class of scalings with additional properties ensuring that Proposition 10.12 is applicable.

Let X be a uniform bornological coarse space whose uniform structure is induced by a metric and let $s : X \to X$ be a scaling.

Definition 10.15. *The scaling s is a coarse scaling if it satisfies in addition:*

1. *$F_{\mathcal{U}}(s)$ is coarsely homotopic to the identity.*
2. *$F_{\mathcal{C},\mathcal{U}/2}(s)$ is properly homotopic to the identity.*

Remark 10.16. A scaling in the sense of [HR95, Def. 7.1] is a coarse scaling; see also Example 10.11. $\square$

The following corollary is an analog of Higson-Roe [HR95, Thm. 7.2]. Assumption 10.17.4 does not occur in [HR95] because the analogue of our $E\mathcal{O}^\infty$ is the functor $X \mapsto K_*(D^*(X))$ in the notation of [HR95] which has good homotopy invariance properties replacing the application of our Lemma 10.14.

Corollary 10.17. *Assume:*

1. *E is countably additive and admits transfers.*
2. *$\mathbf{C}$ is presentable.*
3. *The uniform structure of X is induced by a metric.*
4. *X is homotopy equivalent (in **UBC**) to a countable, locally finite, finite-dimensional simplicial complex.*
5. *X admits a coarse scaling (see Definition 10.15).*

Then $E\mathcal{O}(X) \simeq 0$ and the cone boundary $E\mathcal{O}^\infty(X) \to \Sigma E(X)$ is an equivalence.

Proof. This follows from Proposition 10.12. Lemma 10.14 verifies Assumption 10.12.4. $\square$

Example 10.18. A typical example of a uniform bornological coarse space which admits a coarse scaling is a Euclidean cone. Let Y be a subset of the unit sphere in a Hilbert space and let X be the cone over Y with the metric induced from the Hilbert space. We consider X as a uniform bornological coarse space with all structures induced from the metric. Then the map

$$s : X \to X\ , \quad s(x) := x/2$$

is a coarse scaling.

If Y has a finite-dimensional, locally finite triangulation with a uniform bound on the size of its simplices, then so does X. In this case Corollary 10.17 can be applied to X. $\square$

Let X be a uniform bornological coarse space and E a $\mathbf{C}$-valued coarse homology theory.

Theorem 10.19. *Assume:*

1. *E is strong, countably additive, and admits transfers.*
2. *$\mathbf{C}$ is presentable.*
3. *The uniform structure of X is induced by a metric.*
4. *X is homotopy equivalent (in **UBC**) to a countable, locally finite, finite-dimensional simplicial complex.*
5. *X admits a coarse scaling (see Definition 10.15).*
6. *X is coarsifying (Definition 7.8).*

Then $E\mathcal{O}\mathbf{P}(F_{\mathcal{U}}(X)) \simeq 0$ and therefore the coarse assembly map $\mu_{E,F_{\mathcal{U}}(X)}$ is an equivalence.

Proof. Since X is coarsifying and $E\mathcal{O}$ is a local homology theory (Lemma 9.5) we have an equivalence $E\mathcal{O}\mathbf{P}(F_{\mathcal{U}}(X)) \simeq E\mathcal{O}(X)$. We now apply Corollary 10.17 in order to conclude that $E\mathcal{O}(X) \simeq 0$. □

Example 10.20. Let Y and X be as in Example 10.18. In general we can not expect X to be coarsifying even if Y is compact and the Hilbert space is finite-dimensional. Especially, we do not expect that the analogue of [HR95, Prop. 4.3] is true in our generality. By using Proposition 7.12 one can prove that X is coarsifying if Y is a finite simplicial complex. Hence one can apply Theorem 10.19 to Euclidean cones over finite complexes.

Therefore we get the analogue of [HR95, Cor. 7.3] under the additional assumption of Y being a finite simplicial complex (instead of a finite-dimensional compact metric space).

Every complete, simply-connected, non-positively curved Riemannian manifold is coarsely homotopy equivalent to the Euclidean cone over a finite-dimensional sphere. Because a finite-dimensional sphere has a finite triangulation, Theorem 10.19 provides a generalization of [HR95, Cor. 7.4].

Because of the Assumptions 10.19.4 and 10.19.6 we are not able to apply Theorem 10.19 to cones over arbitrary compact metric spaces. In particular, we do not obtain the analogue of [HR95, Cor. 8.2] asserting the coarse Baum–Connes conjecture for all hyperbolic (proper) metric spaces.

We do not know whether we should expect that the assmbly map $\mu_{E,F_{\mathcal{U}}(X)}$ is an equivalence for all hyperbolic (proper) metric spaces or Euclidean cones over finite-dimensional compact metric spaces and arbitrary coarse homology theories E satisfying the Assumptions 10.19.1 and 10.19.2. □

The next corollary specializes Theorem 10.19 by utilizing a convenient condition on the space X to be coarsifying.

Let K be a simplicial complex and K_d be the associated uniform bornological coarse space.

Corollary 10.21. *Assume:*

1. *E is strong, countably additive, and admits transfers.*
2. *$\mathbf{C}$ is presentable.*
3. *K has bounded geometry.*
4. *K_d is equicontinuously contractible.*
5. *K_d admits a coarse scaling.*

Then the coarse assembly map $\mu_{E,F_{\mathcal{U}}(K_d)}$ is an equivalence.

Proof. Combine Proposition 7.12 with Proposition 10.19. □

Example 10.22. If X is a tree or an affine Bruhat–Tits building of bounded geometry, then Corollary 10.21 applies to X. Hence we obtain the analogue of [HR95, Cor. 7.5]. □

Example 10.23. Recall that the stable ∞-category $\mathbf{Sp}$ of spectra is presentable. Examples of $\mathbf{Sp}$-valued coarse homology theories which are strong, countably additive and admit transfers are ordinary coarse homology $H\mathcal{X}$, algebraic K-homology $K\mathcal{A}\mathcal{X}$ with coefficients in an additive category $\mathcal{A}$, and coarse K-homology $K\mathcal{X}$.

Examples of spaces admitting coarse scalings and which are homotopy equivalent (in $\mathbf{UBC}$) to uniformly contractible simplicial complexes of bounded geometry are simply-connected complete Riemannian manifolds M with sectional curvatures satisfying $-C \leq \sec \leq 0$ for a positive constant C. The coarse scaling s is in this case given by, e.g., $s(x) := \exp(\log(x)/2)$, where we have fixed a base point x_0 in M, $\exp : T_{x_0}M \to M$ is the Riemannian exponential map and $\log : M \to T_{x_0}M$ is its inverse. □

Remark 10.24. If E is the coarse K-homology $K\mathcal{X}$, then using the comparison between the analytic and the topological assembly maps obtained in Section 16 below one can in fact deduce the isomorphism statements of Higson–Roe [HR95, Sec. 7] (under additional assumptions on X related to Conditions 10.19.4 and 10.19.6; see the discussion at the end of Example 10.20) formally as special cases of Proposition 10.19 or Corollary 10.21. But note that specialized to these cases our proof is not really different from the proof given by Higson–Roe [HR95]. In fact, in our approach we have just separated the geometric and homological arguments from the analysis which is hidden in the verification that $K\mathcal{X}$ is a coarse homology theory, the existence of transfers, and the comparison between the two assembly maps. □

11 Extension from locally compact, separable spaces

In this section we will describe an extension process from locally compact, separable spaces to uniform bornological coarse spaces. It will be used to construct analytic local K-homology.

Definition 11.1. *A uniform bornological coarse space X is called small if the underlying topological space of X is separable and locally compact, and the bornology consists of the relatively compact subsets.*

We let $\mathbf{UBC}^{small}$ denote the full subcategory of $\mathbf{UBC}$ consisting of small uniform bornological coarse spaces. Adapting Definition 3.10 we can talk about local homology theories $E : \mathbf{UBC}^{small} \to \mathbf{C}$. We call them restricted.

Definition 11.2. *E is called a closed restricted local homology theory, if E is a restricted local homology theory which satisfies in addition excision for closed decompositions.*

Remark 11.3. Note that the closed decompositions considered in Definition 11.2 need not be uniformly or coarsely excisive. □

A subset Y of a uniform bornological coarse space X is called small if Y with the induced uniform bornological coarse structure is small. Note that a small subset is closed (because of the requirement on the bornology), and a closed subset of a small uniform bornological coarse space is small.

We let $\mathbf{UBC}^{loc}$ be the category of pairs (X, Y), where X is a uniform bornological coarse space and Y is a small subset of X. A morphism $f : (X, Y) \to (X', Y')$ is a morphism $f : X \to X'$ in $\mathbf{UBC}$ such that $f(Y) \subseteq Y'$. We have functors

$$\ell : \mathbf{UBC}^{loc} \to \mathbf{UBC}^{small} \ , \quad (X, Y) \mapsto Y$$

and

$$F_{loc} : \mathbf{UBC}^{loc} \to \mathbf{UBC} \ , \quad (X, Y) \mapsto X \ .$$

Let $E : \mathbf{UBC}^{small} \to \mathbf{C}$ be a functor whose target is a cocomplete stable ∞-category. We define the functor $\mathrm{Ex}(E) : \mathbf{UBC} \to \mathbf{C}$ as the left Kan-extension

$$\begin{tikzcd} \mathbf{UBC}^{loc} \arrow[r, "E\circ\ell"] \arrow[d, "F_{loc}"'] & \mathbf{C} \\ \mathbf{UBC} \arrow[ur, "\mathrm{Ex}(E)"'] & \end{tikzcd}$$

Let X be a uniform bornological coarse space.

Lemma 11.4. *If X is small, then* $\mathrm{Ex}(E(X)) \simeq E(X)$.

Proof. Let X be a uniform bornological coarse space. By the pointwise formula for the left Kan-extension we have

$$\mathrm{Ex}(E)(X) \simeq \mathrm{colim}_Y E(Y) \ , \tag{11.1}$$

where Y runs over all small subsets of X. If X itself is small, then X is the final element of the index set of the colimit. $\square$

Proposition 11.5. *If E is a closed restricted local homology theory, then* $\mathrm{Ex}(E)$ *is a local homology theory.*

Proof. We will use the point-wise formula (11.1) for the left Kan-extension. The subspaces of the form $[0,1] \otimes Y$ of $[0,1] \otimes X$ for small subspaces Y are cofinal in all small subspaces of $[0,1] \otimes X$. Hence homotopy invariance of the restricted homology theory E implies homotopy invariance of $\mathrm{Ex}(E)$.

If Y is a small subset on X, then a coarsely and uniformly excisive closed decomposition (A, B) induces a closed decomposition $(Y \cap A, Y \cap B)$. Note that we do not expect that the latter is coarsely or uniformly excisive.

The analog of (11.1) for closed pairs expresses the excision square for $\mathrm{Ex}(E)$ as a colimit over the small subsets Y of X of the corresponding squares for E. Since E satisfies closed excision, the colimit square is a colimit of cocartesian squares and hence cocartesian.

Assume that the uniform bornological coarse space X is flasque with flasqueness implemented by $f : X \to X$. Let Y be a small subset. Let $h : [0,1] \otimes X \to X$ be the homotopy from $\mathtt{id}$ to f. We define inductively $\tilde{Y}_0 := Y$ and $\tilde{Y}_n := \tilde{Y}_{n-1} \cup h([0,1] \otimes \tilde{Y}_{n-1})$. Note that $\tilde{Y}_0$ small. We show now inductively that $\tilde{Y}_n$ is small. Note that a morphism from a small uniform bornological coarse space to a uniform bornological coarse space has the property that every point in the target has a neighbourhood (take a bounded one) whose preimage is relatively compact. Using the induction hypothesis that $\tilde{Y}_{n-1}$ is small this implies that $h([0,1] \otimes \tilde{Y}_{n-1})$ is closed and locally compact. Moreover, it is separable.

We set $\tilde{Y} := \bigcup_{n \geq 0} \tilde{Y}_n$. This union is locally finite. Hence $\tilde{Y}$ is still locally compact and has the induced bornology of relatively compact subsets. Furthermore it is separable. The morphism f restricts to $\tilde{Y}$ and is homotopic to $\mathtt{id}_{\tilde{Y}}$ by restriction of the homotopy h. We conclude that $\tilde{Y}$ is flasque and $Y \subseteq \tilde{Y}$.

Hence, if X is a flasque uniform bornological coarse space, then the index set of the colimit in (11.1) contains a cofinal subset of flasque small subsets. Since E vanishes on flasques, it follows that $\mathrm{Ex}(E)$ vanishes on X.

Finally, u-continuity of E implies u-continuity of $\mathrm{Ex}(E)$. □

Our main example of a closed restricted local homology theory is the functor

$$K^{an} : \mathbf{UBC}^{small} \to \mathbf{Sp}$$

defined by (see [BE16, Sec. 6.7])

$$K^{an}(X) := KK(C_0(X), \mathbb{C}) \ .$$

By [BE16, Lem. 6.89] the functor K^{an} satisfies excision for closed decompositions and is homotopy invariant. By [BE16, Prop. 6.91] it is locally finite and therefore in particular vanishes on flasques by [BE16, Lem. 6.54]. Finally, $K^{an}(X)$ does not depend on the coarse structure of X. Hence K^{an} is u-continuous.

Definition 11.6. *We define the analytic local K-homology by*

$$K^{an,loc} := \mathrm{Ex}(K^{an}) \ .$$

Remark 11.7. The analytic local K-homology is the analogue of the functor

$$L(K^{an}) : \mathbf{TopBorn} \to \mathbf{Sp}$$

appearing in [BE16, Def. 6.92] of analytic locally finite K-homology. In particular, we do not expect that $K^{an,loc}$ is locally finite on all of $\mathbf{UBC}$. □

12 Calculation of $E\mathcal{O}^\infty$

The goal of this section is to provide a computation of $E\mathcal{O}^\infty(X)$ in terms of the value $E(*)$ of E at the one-point space, see Proposition 12.17. For this calculation we must adopt some finiteness assumptions on X and require that E is countably additive.

In this section we assume that $\mathbf{C}$ is a presentable stable ∞-category.

Let $F : \mathbf{UBC}^{small} \to \mathbf{C}$ be a functor and let X be a small (Definition 11.1) uniform bornological coarse space.

Definition 12.1. *We define the locally finite evaluation of F at X by*

$$F^{lf}(X) := \mathtt{lim}_W \ \mathrm{Cofib}(F(X \setminus W) \to F(X)) \ , \qquad (12.1)$$

where W runs over all open subsets of X with compact closure.

Similary as in [BE16, Rem. 6.49] one can turn the above definition into a construction of a functor $F^{lf} : \mathbf{UBC}^{small} \to \mathbf{C}$.

Remark 12.2. Here are the details. We consider the category $\mathbf{UBC}^{small,\mathcal{B}}$ of pairs (X,W), where X is in $\mathbf{UBC}^{small}$ and W is an open subset of X with compact closure. A morphism $f : (X,W) \to (X',W')$ is a morphism $f : X \to X'$ in $\mathbf{UBC}^{small}$ with $f(W) \subseteq W'$. We have the functors

$$p : \mathbf{UBC}^{small,\mathcal{B}} \to \mathbf{UBC}^{small} \ , \quad p(X,W) := X$$

and

$$\tilde{F} : \mathbf{UBC}^{small,\mathcal{B}} \to \mathbf{C} \ , \quad \tilde{F}(X,W) := \mathrm{Cofib}(F(X \setminus W) \to F(X)) \ .$$

We then define the functor F^{lf} as the right Kan extension of $\tilde{F}$ along p:

$$\begin{tikzcd} \mathbf{UBC}^{small,\mathcal{B}} \arrow[r, "\tilde{F}"] \arrow[d, "p"'] & \mathbf{C} \\ \mathbf{UBC}^{small} \arrow[ur, "F^{lf}"'] & \end{tikzcd}$$

The formula (12.1) now follows from the pointwise formula for the evaluation of the right Kan extension. □

Remark 12.3. If F is induced from a functor $F' : \mathbf{TopBorn} \to \mathbf{C}$ by $F = F' \circ F_{\mathcal{C},\mathcal{U}/2}$, then we have an equivalence

$$F^{lf} \simeq F'^{,lf} \circ F_{\mathcal{C},\mathcal{U}/2} \ ,$$

where $F'^{,lf}$ is exactly the locally finite evaluation as defined in [BE16, Def. 6.48]. □

We have a natural morphism $F(X) \to F^{lf}(X)$.

Lemma 12.4. *If F is homotopy invariant, then so is F^{lf}.*

Proof. The proof if the same as the one of [BE16, Lem. 6.67]. One must observe that the subsets of the form $[0,1] \times W$ of $[0,1] \otimes X$ are cofinal in the open subsets with compact closure. $\square$

Lemma 12.5. *If F satisfies closed or open excision, then so does F^{lf}*

Proof. The argument is the same as for [BE16, Lem. 6.68]. $\square$

Remark 12.6. If F satisfies excision in the sense of Definition 3.6, then it is not clear what kind of excison properties F^{lf} has. The problem is that the intersection with $X \setminus W$ does not necessarily preserve coarsely or uniformly excisive pairs. $\square$

Let X be a small uniform bornological coarse space.

Definition 12.7. *A coarsening X' of X is a small uniform bornological coarse space obtained from X by replacing the coarse structure by a larger one which is still compatible with the bornology.*

Note that the identity of the underlying sets is a morphism $X \to X'$ of uniform bornological coarse spaces.

Let $F : \mathbf{UBC}^{small} \to \mathbf{C}$ be a functor.

Definition 12.8. *We say that F is invariant under coarsening if for every small uniform bornological coarse space and coarsening $X \to X'$ the induced morphism $F(X) \to F(X')$ is an equivalence.*

Example 12.9. The functor $\mathcal{O}^\infty : \mathbf{UBC}^{small} \to \mathbf{Sp}\mathcal{X}$ is invariant under coarsening, see [BEKWb, Prop. 9.31].

The functor $K^{an} : \mathbf{UBC}^{small} \to \mathbf{Sp}$ is invariant under coarsening, since $K^{an}(X)$ does not depend on the coarse structure of X at all. $\square$

Note that countable, locally finite simplicial complexes naturally provide small uniform bornological coarse spaces.

Let X be a countable, locally finite simplicial complex with a decomposition (A, B) into sub-complexes.

Lemma 12.10. *If F satisfies excision in the sense of Definition 3.6 and is invariant under coarsening, then we have a push-out square*

$$\begin{tikzcd} F^{lf}(A \cap B) \arrow[r] \arrow[d] & F^{lf}(A) \arrow[d] \\ F^{lf}(B) \arrow[r] & F^{lf}(X) \end{tikzcd} \tag{12.2}$$

Proof. We use that the cofibre of a map of cocartesian squares is a cocartesian square.

In the limit (12.1) we can restrict W to run only over the interiors of finite sub-complexes. Then $X \setminus W$ is again a simplicial complex and $(A \setminus W, B \setminus W)$ is a decomposition of it into closed sub-complexes.

Note that in the terms $F(X \setminus W)$ in (12.1) we must equip the set $X \setminus W$ with the uniform bornological coarse structures induced from X, and not with the structures coming from a path-metric on $X \setminus W$. Although the uniform and bornological structures on $X \setminus W$ are also induced from the path-metric of $X \setminus W$, this will be in general not true for the coarse structure. But since F is invariant under coarsening, we can, without changing the value of F on the spaces $X \setminus W$, equip these spaces with the coarse structures associated to the intrinsic path-metrics.

Using Example 3.5 we now see that excisiveness of F in the sense of Definition 3.6 can be applied to the decompositions $(A \setminus W, B \setminus W)$ of the complexes $X \setminus W$ occuring in the limit (12.1). We therefore have expressed the square (12.2) as a limit of cofibres of maps of cocartesian squares, i.e., as a limit of cocartesian squares. Since **C** is stable, cartesian and cocartesian squares in **C** are the same. Hence (12.2) itself is a cocartesian square. □

Let $F : \mathbf{UBC}^{small} \to \mathbf{C}$ be a functor and assume that F is excisive (in any of the senses discussed above). If X is a small uniform bornological coarse space with the discrete uniform and coarse structures and x is a point in X, then we have a natural projection morphism $F(X) \to F(\{x\})$, see [BE16, Ex. 4.11].

Definition 12.11. *F is called additive if for every small uniform bornological coarse space X with the discrete uniform and coarse structures and the minimal bornology the natural morphism*

$$F(X) \to \prod_{x \in X} F(\{x\})$$

induced by the projections is an equivalence.

Let us underline that in Definition 12.11 we really mean the product and not the sum.

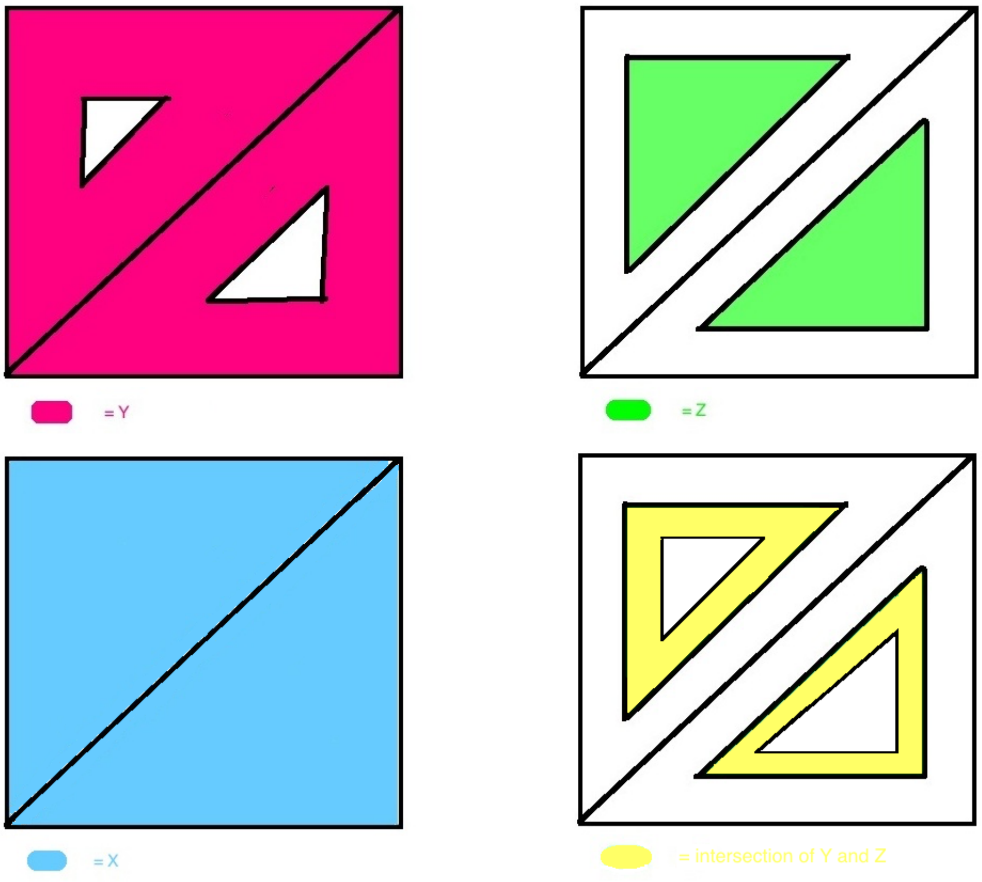
Let $F : \mathbf{UBC}^{small} \to \mathbf{C}$ be a functor.

Lemma 12.12. *Assume:*

1. *F satisfies excision in the sense of Definition 3.6.*
2. *F is homotopy invariant.*
3. *F invariant under coarsening.*
4. *F is additive.*

Then for every countable, locally finite, finite-dimensional simplicial complex X the natural morphism $F(X) \to F^{lf}(X)$ is an equivalence.

Proof. We argue by a finite induction over the dimension. The assertion is true for zero dimensional complexes since they are discrete and F, F^{lf} are additive (see [BE16, Lem. 6.63] for the latter). Assume now that the assertion is true for complexes of dimension $n-1$. If X is n-dimensional, then we can decompose X into a closed tubular neighbourhood Y of thickness $1/3$ of its $n-1$-skeleton and a disjoint union Z of n-simplices of size $2/3$ (see the picture). The intersection is then a disjoint union of tubular neighbourhoods of thickness $1/3$ of the boundaries of simplices of size $2/3$. See the picture on Page 48.



Decomposition $X = Y \cup Z$ used in the proof of Lemma 12.12

This closed decomposition is coarsely and uniformly excisive. Hence we can apply excision for F in the sense of Definition 3.6. For F^{lf} we use Lemma 12.10.

We use homotopy invariance in order to replace the evaluation on Y by the evaluation on the $n-1$-skeleton X^{n-1} itself. Furthermore, we can contract the n-simplices of size $2/3$ in Z to the set C of their centers. Finally, we contract $Y \cap Z$ to the set W of the boundaries of these simplices of size $2/3$.

We use invariance under coarsening (note that F^{lf} is also invariant under coarsening) in order to replace the induced coarse structures by the coarse structures induced by the intrinsic path-quasi-metric on the $n-1$-skeleton X^{n-1} and on W and the discrete coarse structure on the set C of centers of n-simplices. Then we can apply the induction assumption to X^{n-1}, W (which is also $n-1$-dimensional) and C. $\square$

Example 12.13. Analytic local K-homology K^{an} satisfies the assumptions of the above Lemma 12.12. Excision for closed decompositions and homotopy invariance is shown in [BE16, Lem. 6.89] and additivity in [BE16, Lem. 6.90]. It is invariant under coarsenings since $K^{an}(X)$ does not depend on the coarse structure of X. $\square$

Recall that a coarse homology theory E is countably additive if we have an equivalence

$$E(\mathbb{N}_{min,min}) \simeq \prod_{\mathbb{N}} E(*) \tag{12.3}$$

induced by the canonical projections, where $\mathbb{N}_{min,min}$ denotes the set $\mathbb{N}$ equipped with the minimal coarse and bornological structures.

Let E be a coarse homology theory.

Proposition 12.14. *If E is countably additive and X is a countable, locally finite, finite-dimensional simplicial complex, then the natural morphism*

$$E\mathcal{O}^\infty(X) \to (E\mathcal{O}^\infty)^{lf}(X)$$

is an equivalence.

Proof. We will check that the assumptions of Lemma 12.12 are satisfied. By Lemma 9.1 the functor $\mathcal{O}^\infty_{|\mathbf{UBC}^{small}}$ satisfies excision in the sense of Definition 3.6 and is homotopy invariant. Therefore $E\mathcal{O}^\infty$ has these properties. Furthermore, by Example 12.9 the functor $E\mathcal{O}^\infty$ is invariant under coarsening.

Let X be a uniform bornological coarse space which is discrete both as a uniform and as a coarse space. Then

$$\mathcal{O}^\infty(X) \simeq \Sigma \mathrm{Yo}^s(F_{\mathcal{U}}(X))$$

by [BEKWb, Prop. 9.33]. Using that E is countably additive at the marked equivalence in the following chain of equivalences, we have for a small uniform bornological coarse space X with the discrete uniform and coarse structures and the minimal bornology (note that the space X is countable under these assumptions)

$$\begin{aligned} E\mathcal{O}^\infty(X) &\simeq E(\Sigma F_{\mathcal{U}}(X)) \\ &\simeq \Sigma E(F_{\mathcal{U}}(X)) \\ &\overset{!}{\simeq} \Sigma\Big(\prod_{x\in X} E(\{x\})\Big) \\ &\simeq \prod_{x\in X} \Sigma E(\{x\}) \\ &\simeq \prod_{x\in X} E\mathcal{O}^\infty(\{x\}) \end{aligned}$$

showing that $E\mathcal{O}^\infty$ is additive, and therefore finishing this proof. $\square$

Following Weiss–Williams [WW95], for a homotopy invariant functor $F : \mathbf{UBC}^{small} \to \mathbf{C}$ we can construct a best approximation of F by a homology theory. It is given by the Kan extension procedure described in the proof of [BE16, Prop. 6.73] which produces a functor and a natural transformation

$$F^{\%} : \mathbf{UBC}^{small} \to \mathbf{C} \ , \quad F^{\%} \to F \ .$$

The objectwise formula for $F^{\%}$ is

$$F^{\%}(X) := \mathrm{colim}_{(\Delta^n \to X)} F(\Delta^n) \ ,$$

where the colimit runs over the category of simplices of X.

Since $\mathbf{C}$ is a presentable stable ∞-category, it is tensored over $\mathbf{Sp}$. Since F is homotopy invariant, the projection $\Delta^n \to *$ induces an equivalence $F(\Delta^n) \to F(*) \simeq \Sigma^\infty_+(*) \wedge F(*)$. Using the equivalence $\mathrm{colim}_{(\Delta^n \to X)} \Sigma^\infty_+(*) \simeq \Sigma^\infty_+(X)$ and the fact that $\wedge$ commutes with colimits we therefore have an equivalence

$$F^{\%}(X) \simeq \Sigma^\infty_+(X) \wedge F(*) \ . \tag{12.4}$$

This implies that the functor $F^{\%}$ is homotopy invariant and satisfies open excision. Hence its locally finite evaluation $(F^{\%})^{lf} : \mathbf{UBC}^{small} \to \mathbf{C}$ satisfies open excision, is homotopy invariant and is countably additive [BE16, Lem. 6.63]. Alternatively one can use the fact that Remark 12.3 applies to $F^{\%}$ by (12.4). Note that in the argument of Σ^∞_+ we dropped the obvious forgetful functor from $\mathbf{UBC}^{small}$ to topological spaces.

Remark 12.15. If F is induced from a functor $F' : \mathbf{TopBorn} \to \mathbf{C}$ by $F = F' \circ F_{\mathcal{C},\mathcal{U}/2}$, then

$$F^{\%} \simeq F'^{,\%} \circ F_{\mathcal{C},\mathcal{U}/2} \ ,$$

where $F'^{,\%}$ is precisely the functor defined in the proof of [BE16, Prop. 6.73]. □

Lemma 12.16. *If F satisfies the assumptions stated in Lemma 12.12, and X is a countable, locally finite, finite-dimensional simplicial complex, then the natural morphism*

$$(F^{\%})^{lf}(X) \to F^{lf}(X)$$

is an equivalence.

Proof. The argument is the same as for Lemma 12.12. We just observe that $(F^{\%})^{lf}$ is also excisive for decompositions of simplicial complexes into closed sub-complexes, and that it is also invariant under coarsening. □

Let $E : \mathbf{BornCoarse} \to \mathbf{C}$ be a coarse homology theory and X be a uniform bornological coarse space.

Proposition 12.17. *Assume:*

1. $\mathbf{C}$ *is presentable.*

2. *E is countably additive, see* (12.3).

3. *X is homotopy equivalent in* **UBC** *to a countable, locally finite, finite-dimensional simplicial complex.*

Then we have a natural equivalence

$$(\Sigma E(*) \wedge \Sigma^{\infty}_{+})^{lf}(X) \simeq E\mathcal{O}^{\infty}(X) \ .$$

Proof. We first observe that $\Sigma E(*) \simeq E\mathcal{O}^{\infty}(*)$. Then we just combine Proposition 12.14 and Lemma 12.16 with $F := E\mathcal{O}^{\infty}$. □

Note that the functor $X \mapsto (\Sigma E(*) \wedge \Sigma^{\infty}_{+})^{lf}(X)$ is naturally defined on **TopBorn** and is locally finite, homotopy invariant (in the sense of **TopBorn**, i.e., for proper homotopies which are not necessarily uniform), and satisfies open excision. Therefore the functor $X \mapsto E\mathcal{O}^{\infty}(X)$ for spaces X in **UBC** (which are homotopy equivalent in the sense of **UBC** to a countable, locally finite, finite-dimensional simplicial complexes), also has these stronger homological properties. In particular:

Let $E : \mathbf{BornCoarse} \to \mathbf{C}$ be a coarse homology theory. Furthermore, let X, X' be in **UBC** and $f, g : X \to X'$ be morphisms in **UBC**.

Corollary 12.18. *Assume:*

1. **C** *is presentable.*

2. *E is countably additive.*

3. *X and X' are homotopy equivalent in* **UBC** *to countable, locally finite and finite-dimensional simplicial complexes.*

4. *$F_{\mathcal{C},\mathcal{U}/2}(f)$ and $F_{\mathcal{C},\mathcal{U}/2}(g)$ are properly homotopic (there is a homotopy $[0,1] \times X \to X'$ which is continuous and proper after forgetting the coarse and uniform structures).*

Then $E\mathcal{O}^{\infty}(f)$ is equivalent to $E\mathcal{O}^{\infty}(g)$.

13 Comparison of coarse homology theories

In ordinary homotopy theory a transformation between spectrum-valued homology theories which induces an equivalence on a point is an equivalence at least on all CW-complexes. In the present section we consider an analogous statement for coarse homology theories.

Let $\mathbf{C}$ be a presentable stable ∞-category. Assume that we have a transformation $E \to E'$ of $\mathbf{C}$-valued coarse homology theories which induces an equivalence $E(*) \to E'(*)$. In this section we provide sufficient conditions on a space X and on the theories E and E' which imply that $E(X) \to E'(X)$ is an equivalence. The main result is formulated in Corollary 13.4.

Let X be a bornological coarse space. The following definitions are from [BE16, Def. 6.100], [BE16, Def. 6.102] and [BE16, Def. 7.32].

Definition 13.1. *1. X has strongly bounded geometry if it has the minimal bornology compatible with the coarse structure and for every coarse entourage U of X the number of points in U-bounded subsets of X is uniformly bounded.*

2. *X has bounded geometry if it is equivalent to a bornological coarse space with strongly bounded geometry.*

3. *X is called separable it admits a coarse entourage U and a countable family of points $(x_i)_{i\in I}$ that $\bigcup_{i\in I} U[x_i] = X$.*

Let X be a bornological coarse space and E be a $\mathbf{C}$-valued coarse homology theory.

Proposition 13.2. *Assume:*

1. *$\mathbf{C}$ is presentable.*
2. *E is countably additive.*
3. *X is separable and of bounded geometry.*

Then we have a natural equivalence

$$((\Sigma E(*) \wedge \Sigma^\infty_+)^{lf} \circ F_{\mathcal{C},\mathcal{U}/2})\mathbf{P}(X) \simeq E\mathcal{O}^\infty\mathbf{P}(X) \ .$$

Proof. Since both sides of the equivalence are coarsely invariant we can assume that X is a countable bornological coarse space of strongly bounded geometry. Then for every entourage U of X the complex $P_U(X)$ is a countable, locally finite, finite-dimensional simplicial complex. Hence by Proposition 12.17 we get an equivalence

$$(\Sigma E(*) \wedge \Sigma^\infty_+)^{lf}(F_{\mathcal{C},\mathcal{U}/2}(P_U(X))) \simeq E\mathcal{O}^\infty(P_U(X)) \ .$$

Forming the colimit over the entourages U of X and using (5.2) we get the equivalence

$$((\Sigma E(*) \wedge \Sigma^\infty_+)^{lf} \circ F_{\mathcal{C},\mathcal{U}/2})\mathbf{P}(X) \simeq E\mathcal{O}^\infty\mathbf{P}(X)$$

as claimed. □

Let $E \to E'$ be a transformation between $\mathbf{C}$-valued coarse homology theories and let X be a bornological coarse space.

Theorem 13.3. *Assume:*

1. *$\mathbf{C}$ is presentable.*
2. *E and E' are strong and countably additive.*
3. *$E(*) \to E'(*)$ is an equivalence.*
4. *X is separable and of bounded geometry.*

5. *The assembly maps $\mu_{E,X}$ and $\mu_{E',X}$ are equivalences (Definition 9.6).*

Then $E(X) \to E'(X)$ is an equivalence.

Proof. We have a commuting diagram

$$\begin{tikzcd}
((\Sigma E(*) \wedge \Sigma^\infty_+)^{lf} \circ F_{\mathcal{C},\mathcal{U}/2})\mathbf{P}(X) \arrow[r, "\text{Prop. 13.2}", "\simeq"'] \arrow[d, "\simeq"] & E\mathcal{O}^\infty\mathbf{P}(X) \arrow[r, "\mu_{E,X}", "\simeq"'] \arrow[d] & \Sigma E(X) \arrow[d] \\
((\Sigma E'(*) \wedge \Sigma^\infty_+)^{lf} \circ F_{\mathcal{C},\mathcal{U}/2})\mathbf{P}(X) \arrow[r, "\text{Prop. 13.2}", "\simeq"'] & E'\mathcal{O}^\infty\mathbf{P}(X) \arrow[r, "\mu_{E',X}", "\simeq"'] & \Sigma E'(X)
\end{tikzcd}$$

The left vertical morphism is an equivalence by Condition 3. We conclude that the right vertical morphism is an equivalence, too. □

We can use Theorems 10.3, 10.7 and 10.19 in order to check Condition 5 in the statement of Theorem 13.3.

Let $E \to E'$ be a transformation between **C**-valued coarse homology theories and X be a bornological coarse space.

Corollary 13.4. *Assume:*

1. **C** *is presentable.*
2. *E and E' are strong and countably additive.*
3. *$E(*) \to E'(*)$ is an equivalence.*
4. *X is separable and of bounded geometry.*

Furthermore assume one of the following three conditions:

1. *There is a cofinal set of coarse entourages U of X such that X_U has finite asymptotic dimension.*
2. a) **C** *is compactly generated.*
   b) *E and E' are weakly additive and admit transfers.*
   c) *There is a cofinal set of entourages U of X such that X_U has finite decomposition complexity.*
3. a) *E and E' are strong, countably additive, and admit transfers.*
   b) *There exists a uniform bornological coarse space Y with $\mathrm{Yo}^s(X) \simeq \mathrm{Yo}^s(F_{\mathcal{U}}(Y))$ and the following holds true:*
      i. *The uniform structure of Y is induced by a metric.*
      ii. *Y is homotopy equivalent (in* **UBC***) to a countable, locally finite, finite-dimensional simplicial complex.*
      iii. *Y admits a coarse scaling (see Definition 10.15).*

iv. Y is coarsifying (Definition 7.8).

Then $E(X) \to E'(X)$ is an equivalence.

Remark 13.5. The case of finite asymptotic dimension in the above corollary has been shown in [BE16, Thm. 6.116] in the slightly more general form that the coarse homology theories are not assumed to be strong. □

14 Coarse assembly map for coarse K-homology

Let us apply the theory developed so far to coarse K-homology to interpret the coarse assembly map as a morphism from the coarsification of the locally finite K-homology to the coarse K-homology, see (14.5).

Note that the spectra $KK(\mathbb{C},\mathbb{C})$ and $K\mathcal{X}(*)$ are both equivalent to the complex K-theory spectrum KU. We can fix once and for all an identification of spectra

$$KK(\mathbb{C},\mathbb{C}) \simeq K\mathcal{X}(*) \ . \tag{14.1}$$

Since KU has many self-equivalences this choice is not unique.

Since

$$(K^{an,lf} \circ F_{\mathcal{C},\mathcal{U}/2}) : \mathbf{UBC}^{small} \to \mathbf{Sp}$$

is a locally finite homology theory, we have by [BE16, Prop. 6.73] the first equivalence in the following chain of equivalences of functors

$$(K^{an,lf} \circ F_{\mathcal{C},\mathcal{U}/2}) \simeq (K^{an,lf}(*) \wedge \Sigma^{\infty}_{+})^{lf} \simeq (K(\mathbb{C},\mathbb{C}) \wedge \Sigma^{\infty}_{+})^{lf} \simeq (K\mathcal{X}(*) \wedge \Sigma^{\infty}_{+})^{lf} \tag{14.2}$$

on $\mathbf{UBC}^{small}$. The second equivalence uses the definition of the analytic locally finite K-homology $K^{an,lf}(*) \simeq KK(C_0(*),\mathbb{C}) \simeq KK(\mathbb{C},\mathbb{C})$, and the third equivalence involves the choice (14.1).

We recall that $K\mathcal{X}$ is additive [BE16, Prop. 7.77], so in particular it is countably additive. If X is homotopy equivalent to a countable, locally finite, finite-dimensional simplicial complex, then using Proposition 12.17 and (14.2) we get an equivalence

$$(K^{an,lf} \circ F_{\mathcal{C},\mathcal{U}/2})(X) \simeq \Sigma^{-1} K\mathcal{X}\mathcal{O}^{\infty}(X) \ . \tag{14.3}$$

If X is a separable bornological coarse space of bounded geometry, then we have a natural equivalence (see Proposition 13.2)

$$(K^{an,lf} \circ F_{\mathcal{C},\mathcal{U}/2})\mathbf{P}(X) \simeq \Sigma^{-1} K\mathcal{X}\mathcal{O}^{\infty}\mathbf{P}(X) \ . \tag{14.4}$$

We can now interpret the coarse assembly map $\mu_{K\mathcal{X},X}$ as a morphism

$$\mu^{top}_{X} : (K^{an,lf} \circ F_{\mathcal{C},\mathcal{U}/2})\mathbf{P}(X) \to K\mathcal{X}(X) \tag{14.5}$$

from the coarsification of the locally finite K-homology to the coarse K-homology of X.

Observe that this construction produces a morphism of spectra. It is natural in X. It does not involve Paschke duality or similar results from functional analysis. On the other hand we must assume that X has bounded geometry and is separable.

In the following we spell out explicitly the statement about the compatibility of the coarse assembly map with the coarse Mayer-Vietoris sequences. We use the notation introduced in [BE16, Sec. 6.6] for the coarsification

$$(K^{an,lf} \circ F_{\mathcal{C},\mathcal{U}/2})\mathbf{P}_*(X) = QK_*^{an,lf}(X) \tag{14.6}$$

of the locally finite analytic K-homology.

Corollary 14.1. *Let X be a separable bornological coarse space of bounded geometry and let (A, B) be a coarsely excisive decomposition of X. Then the following square commutes:*

$$\begin{tikzcd} QK_{*+1}^{an,lf}(X) \arrow[r, "\partial_{MV}^{K^{an,lf}}"] \arrow[d, "\mu_X^{top}"] & QK_*^{an,lf}(A\cap B) \arrow[d, "\mu_{A\cap B}^{top}"] \\ K\mathcal{X}_{*+1}(X) \arrow[r, "\partial_{MV}^{K\mathcal{X}}"] & K\mathcal{X}_*(A\cap B) \end{tikzcd}$$

15 $\mathcal{O}^\infty(X)$ is representable

In this section we show that the motivic coarse spectrum $\mathcal{O}^\infty(X)$ and the boundary map of the cone sequence are representable. We will use these facts in Section 17 in order to provide examples of local homology classes for the theory $K\mathcal{X}\mathcal{O}^\infty$.

Let X be a uniform bornological coarse space. Then we consider the bornological coarse space $\mathcal{O}(X)_-$ obtained from the uniform bornological coarse space $\mathbb{R}\otimes X$ by taking the hybrid coarse structure [BE16, Def. 5.10] associated to the big family $((-\infty, n]\times X)_{n\in\mathbb{N}}$. Note that the subset $[0,\infty)\times X$ of $\mathcal{O}(X)_-$ with the induced structures is the cone $\mathcal{O}(X)$. We then have maps of bornological coarse spaces

$$F_{\mathcal{U}}(X) \xrightarrow{i} \mathcal{O}(X) \xrightarrow{j} \mathcal{O}(X)_- \xrightarrow{d} F_{\mathcal{U}}(\mathbb{R}\otimes X)\ .$$

The first two maps i and j are the inclusions, and the last map d is given by the identity of the underlying sets.

The following proposition identifies a segment of the cone sequence (9.1) with a sequence represented by maps between bornological coarse spaces. It in particular shows that the cone $\mathcal{O}^\infty(X)$ is represented by the bornological coarse space $\mathcal{O}(X)_-$.

Proposition 15.1. *We have a commutative diagram in* $\mathbf{Sp}\mathcal{X}$

$$\begin{tikzcd} \mathrm{Yo}^s(\mathcal{O}(X)) \arrow[r, "j"] \arrow[d, equal] & \mathrm{Yo}^s(\mathcal{O}(X)_-) \arrow[r, "d"] \arrow[d, "\iota"', "\simeq"] & \mathrm{Yo}^s(F_{\mathcal{U}}(\mathbb{R}\otimes X)) \arrow[d, "s"', "\simeq"] \\ \mathrm{Yo}^s(\mathcal{O}(X)) \arrow[r] & \mathcal{O}^\infty(X) \arrow[r, "\partial"] & \Sigma\mathrm{Yo}^s(F_{\mathcal{U}}(X)) \end{tikzcd} \tag{15.1}$$

Proof. We consider the diagram of motivic coarse spectra

$$\begin{tikzcd} \mathrm{Yo}^s(F_{\mathcal{U}}(X)) \arrow[r, "i"] \arrow[d] & \mathrm{Yo}^s(\mathcal{O}(X)) \arrow[r] \arrow[d, "j"] & \mathrm{Yo}^s(F_{\mathcal{U}}([0,\infty)\otimes X)) \arrow[d] \\ \mathrm{Yo}^s(F_{\mathcal{U}}((-\infty,0]\otimes X)) \arrow[r] & \mathrm{Yo}^s(\mathcal{O}(X)_-) \arrow[r, "d"] & \mathrm{Yo}^s(F_{\mathcal{U}}(\mathbb{R}\otimes X)) \end{tikzcd} \tag{15.2}$$

The left and right vertical and the lower left horizontal map are given by the canonical inclusions. The upper right horizontal map is induced from the identity of the underlying sets. This diagram commutes since it is obtained by applying Yo^s to a commuting diagram of bornological coarse spaces.

The left square in (15.2) is cocartesian since the pair $((-\infty,0]\times X, \mathcal{O}(X))$ in $\mathcal{O}(X)_-$ is coarsely excisive. Furthermore, because $((-\infty,0]\times X, [0,\infty)\times X)$ is coarsely excisive in $F_{\mathcal{U}}(\mathbb{R}\otimes X)$ the outer square is cocartesian. It follows that the right square is cocartesian.

Since the upper right and the lower left corners in (15.2) are trivial by flasqueness of the spaces the diagram is equivalent to

$$\begin{tikzcd} \mathrm{Yo}^s(F_{\mathcal{U}}(X)) \arrow[r, "i"] \arrow[d] & \mathrm{Yo}^s(\mathcal{O}(X)) \arrow[r] \arrow[d, "j"] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{Yo}^s(\mathcal{O}(X)_-) \arrow[r, "d"] & \mathrm{Yo}^s(F_{\mathcal{U}}(\mathbb{R}\otimes X)) \end{tikzcd}$$

Note that $\mathcal{O}^\infty(X)$ is defined as the cofibre of the upper left horizontal map i. Hence the left square yields the middle vertical equivalence ι in (15.1).

The outer square yields the equivalence

$$s : \mathrm{Yo}^s(F_{\mathcal{U}}(\mathbb{R}\otimes X)) \xrightarrow{\simeq} \Sigma\mathrm{Yo}^s(F_{\mathcal{U}}(X)) \ .$$

Then the right square identifies d with the boundary map ∂ of the cone sequence. $\square$

16 Comparison with the analytic assembly map

The classical instance of the coarse assembly map is the coarse analytic assembly map for coarse K-homology which is constructed using C^*-algebra theory. The main goal of the present section is the comparison of the coarse analytic assembly map defined in Definition 16.10 and the coarse assembly map defined in Definiton 9.6. Our main result is a non-canonical identification of these two maps in Corollary 16.12.

We start with Proposition 16.4 comparing the analytic assembly map (Definition 16.7) and the cone boundary. To this end we must identify the domains of these maps appropriately.

Let X be a proper metric space. We consider X with the bornology of metrically bounded subsets. We will also use the notation X for the uniform bornological coarse space with the coarse and uniform structures induced from the metric.

In addition to the metric coarse structure on X we will need other coarse structures $\mathcal{C}$ on X which are compatible with the bornology, but not necessarily associated to any other metric on X. We will assume that such a coarse structure $\mathcal{C}$ contains open entourages. For the moment we write $X_{\mathcal{C}}$ for the corresponding bornological coarse space. In our application $\mathcal{C}$ will be a hybrid coarse structure.

An ample continuously X-controlled Hilbert space is a pair (H,ρ), where H is a separable Hilbert space and ρ is a non-degenerate representation of the C^*-algebra $C_0(X)$ on H such that no non-trivial element of $C_0(X)$ acts by a compact operator. Note that this definition does not involve the coarse structure and also applies to $X_{\mathcal{C}}$ in place of X.

Using ρ we can talk about local compactness, propagation and pseudolocality of operators T in $B(H)$.

1. T is locally compact if $\rho(f)T$ and $T\rho(f)$ are compact operators for all f in $C_0(X)$.
2. T has propagation controlled by the entourage U if $\rho(f)T\rho(g)=0$ for all functions f and g in $C_0(X)$ with $U[\mathtt{supp}(g)]\cap \mathtt{supp}(f)=\emptyset$.
3. T is pseudolocal if $[T,\rho(f)]$ is a compact operator for every function f in $C_0(X)$.

A continuously X-controlled Hilbert space is called very ample, if it is unitarily equivalent to a direct sum of countably many copies of some ample continuously X-controlled Hilbert space.

Remark 16.1. An $X_{\mathcal{C}}$-controlled Hilbert space in the sense of [BE16, Def. 7.1] is a pair (H,ϕ), where H is a Hilbert space and ϕ is a unital representation of the algebra of all bounded $\mathbb{C}$-valued functions on $X_{\mathcal{C}}$ such that $\phi(\chi_B)H$ is separable for every bounded subset B of X. We write $H(Y):=\phi(\chi_Y)H$ for the image of the projection $\phi(\chi_Y)$ associated to a subset Y of X. By definition, the $X_{\mathcal{C}}$-controlled Hilbert space (H,ϕ) is determined on points [BE16, Def. 7.3] if the natural inclusions induce an isomorphism $H\cong\bigoplus_{x\in X}H(\{x\})$. The $X_{\mathcal{C}}$-controlled Hilbert space is called ample [BE16, Def. 7.12] if it is determined on points and there exists an entourage U of $X_{\mathcal{C}}$ such that $H(U[x])$ is infinite-dimensional for every point x in X. In contrast to the continuously controlled case this definition of ampleness depends on the coarse structure.

We now explain a construction which associates to an ample continuously X-controlled Hilbert space (H,ρ) an ample $X_{\mathcal{C}}$-controlled Hilbert space (H,ϕ).

We first observe that the representation ρ of $C_0(X)$ on H naturally extends to a representation of the algebra $L^\infty(X)$ of bounded Borel-measurable functions on X. We can choose an open entourage U of $X_{\mathcal{C}}$ and a partition of X into U-bounded subsets with non-empty interior $(B_\alpha)_{i\in I}$. For every i in I we let $H_i:=\rho(\chi_{B_i})$ and $P_i:H\to H_i$ be the orthogonal projection. Note that H_i is ∞-dimensional. For every i in I we choose a point b_i in B_i. Then we set $\phi:=\sum_{i\in I}\delta_{b_i}P_i$. We get the $X_{\mathcal{C}}$-controlled Hilbert space (H,ϕ).

If U is also an entourage of X, then (H,ϕ) is an ample X-controlled Hilbert space. $\square$

Let X be a proper metric space and (H, ϕ) be a continuously X-controlled Hilbert space. Let us fix an open entourage U of X. If (H, ρ) is very ample, then every other continuously X-controlled Hilbert space (H', ρ') with ρ' non-degenerate admits a pseudolocal isometric inclusion into (H, ρ) of U-controlled propagation [HR00, Lem. 12.4.6]. In the reference the coarse structures are assumed to come from a metric, but this is not necessary. Since we assume that the entourage U is open and X (as a proper metric space) is locally compact we can find a locally finite open covering of X by U-bounded open subsets. Now the proof of [HR00, Lem. 12.4.6] carries over verbatim.

We have an exact sequence of C^*-algebras

$$0 \to C^*(X_{\mathcal{C}}, H, \rho) \to D^*(X_{\mathcal{C}}, H, \rho) \to Q^*(X_{\mathcal{C}}, H, \rho) \to 0 \ , \tag{16.1}$$

where the entries have the following description.

1. $D^*(X_{\mathcal{C}}, H, \rho)$ is the sub-C^*-algebra of $B(H)$ generated by the pseudolocal operators of U-controlled propagation for some U in $\mathcal{C}$.
2. $C^*(X_{\mathcal{C}}, H, \rho)$ is the Roe algebra, which is generated by locally compact operators of U-controlled propagation for some U in $\mathcal{C}$.
3. $Q^*(X_{\mathcal{C}}, H, \rho)$ is defined as the quotient.

Remark 16.2. Let $X_{\mathcal{C}}$ be a bornological coarse space and let (H, ϕ) be an $X_{\mathcal{C}}$-controlled Hilbert space. By [BE16, Def. 7.29] we have a Roe algebra $C^*(X_{\mathcal{C}}, H, \phi)$. If X is a proper metric space and (H, ϕ) is obtained from an ample continuously X-controlled Hilbert space (H, ρ) by the construction described in Remark 16.1, then we have the equality

$$C^*(X_{\mathcal{C}}, H, \phi) = C^*(X_{\mathcal{C}}, H, \rho) \tag{16.2}$$

as subalgebras of $B(H)$. Indeed, the local compactness conditions and the propagation conditions defined in the continuously controlled and controlled contexts are equivalent.

Note that the algebras $D^*(X_{\mathcal{C}}, H, \rho)$ and $Q^*(X_{\mathcal{C}}, H, \rho)$ can not be defined in the controlled context. Their definition requires continuous control. $\square$

Let X be a proper metric space and we choose a very ample continuously X-controlled Hilbert space (H, ρ). Then for every integer n we have an isomorphism of groups

$$K_n^{an,lf}(X) \overset{\text{def}}{\cong} KK_n(C_0(X), \mathbb{C}) \overset{\text{Paschke}}{\cong} K_{n+1}(Q^*(X, H, \rho)) \ , \tag{16.3}$$

given by the Paschke duality isomorphism, see e.g. Paschke [Pas81], Higson [Hig95] or also Higson–Roe [HR95, Prop. 5.2]. Furthermore, by [BE16, Thm. 7.64] we have a canonical isomorphism of groups

$$K_n(C^*(X_{\mathcal{C}}, H, \rho)) \xrightarrow{\cong} K\mathcal{X}_n(X_{\mathcal{C}}) \ . \tag{16.4}$$

By using appropriate product metrics we consider $\mathbb{R} \times X$ and its subspace $[0, \infty) \times X$ as proper metric spaces. We note that the hybrid coarse structure on $\mathbb{R} \times X$ contains open entourages. Hence the definitions above apply to $\mathcal{O}(X)_-$ and $\mathcal{O}(X)$.

Let $i : X \to [0, \infty) \times X$ and $j : [0, \infty) \times X \to \mathbb{R} \times X$ denote the inclusions.

Data 16.3. We now make the following choices.

1. We choose a very ample continuously X-controlled Hilbert space (H, ρ).
2. We choose a very ample continuously $[0, \infty) \otimes X$-controlled Hilbert space (H_+, ρ_+).
3. We choose a very ample continuously $\mathbb{R} \otimes X$-controlled Hilbert space $(\tilde{H}, \tilde{\rho})$.
4. We choose a pseudolocal unitary embedding $u : H \to H_+$ which is U-controlled as a morphism $i_*(H, \rho) \to (H_+, \rho_+)$, where U is an open coarse entourage of the hybrid structure of $\mathcal{O}(X)$.
5. We choose a pseudolocal unitary embedding $v : H_+ \to \tilde{H}$ which is $\tilde{U}$-controlled as a morphism $j_*(H_+, \rho_+) \to (\tilde{H}, \tilde{\rho})$, where $\tilde{U}$ is an open coarse entourage of the hybrid structure of $\mathcal{O}(X)_-$.

The embedding u induces an embedding u_* of algebras by $A \mapsto uAu^*$, and similarly for v.

Proposition 16.4. *We assume that X is isomorphic in* **UBC** *to a countable, locally finite and finite-dimensional simplicial complex. The choices made above then determine naturally an equivalence of fibre sequences of spectra*

$$\begin{tikzcd} K(C^*(X,H,\rho)) \arrow[r] & K(D^*(X,H,\rho)) \arrow[r] & K(Q^*(X,H,\rho)) \arrow[r, "\partial^{C^*}"] & \Sigma K(C^*(X,H,\rho)) \\ K\mathcal{X}(X) \arrow[r] \arrow[u, "\simeq"] & K\mathcal{X}(\mathcal{O}(X)) \arrow[r] \arrow[u, "\simeq"] & K\mathcal{X}(\mathcal{O}^\infty(X)) \arrow[r, "\partial"] \arrow[u, "\simeq", "b"'] & \Sigma K\mathcal{X}(X) \arrow[u, "\simeq"] \end{tikzcd} \tag{16.5}$$

Proof. We have the following commuting diagram of spectra:

$$\begin{tikzcd} \Sigma K(C^*(X,H,\rho)) \arrow[r, "u_*"] & \Sigma K(C^*(\mathcal{O}(X),H_+,\rho_+)) \arrow[r, "v_*"] & \Sigma K(C^*(\mathcal{O}(X)_-,\tilde{H},\tilde{\rho})) \arrow[r, "d"] & \Sigma K(C^*(\mathbb{R}\otimes X,\tilde{H},\tilde{\rho})) \\ K(Q^*(X,H,\rho)) \arrow[r, "u_*"] \arrow[u, "\partial^{C^*}"] & K(Q^*(\mathcal{O}(X),H_+,\rho_+)) \arrow[r, "v_*"] \arrow[u] & K(Q^*(\mathcal{O}(X)_-,\tilde{H},\tilde{\rho})) \arrow[r, "d"] \arrow[u, "\delta"] & K(Q^*(\mathbb{R}\otimes X,\tilde{H},\tilde{\rho})) \arrow[u] \\ K(D^*(X,H,\rho)) \arrow[r, "u_*"] \arrow[u] & K(D^*(\mathcal{O}(X),H_+,\rho_+)) \arrow[r, "v_*"] \arrow[u] & K(D^*(\mathcal{O}(X)_-,\tilde{H},\tilde{\rho})) \arrow[r, "d"] \arrow[u] & K(D^*(\mathbb{R}\otimes X,\tilde{H},\tilde{\rho})) \arrow[u] \\ K(C^*(X,H,\rho)) \arrow[r, "u_*"] \arrow[u] & K(C^*(\mathcal{O}(X),H_+,\rho_+)) \arrow[r, "v_*"] \arrow[u] & K(C^*(\mathcal{O}(X)_-,\tilde{H},\tilde{\rho})) \arrow[r, "d"] \arrow[u] & K(C^*(\mathbb{R}\otimes X,\tilde{H},\tilde{\rho})) \arrow[u] \end{tikzcd} \tag{16.6}$$

The vertical sequences are fibre sequences of K-theory spectra associated to versions of the short exact sequence (16.1) of C^*-algebras. The horizontal maps d in the third column are induced by the identity of $\tilde{H}$. By the comparison theorem [BE16, Thm. 7.70] and the equality (16.2) (for the transition from the controlled to the continuously controlled situation) we get the vertical equivalences in the following diagram

$$\begin{tikzcd}
K(C^*(X,H,\rho)) \arrow[r, "u_*"] & K(C^*(\mathcal{O}(X),H_+,\rho_+)) \arrow[r, "v_*"] & K(C^*(\mathcal{O}(X)_-,\tilde{H},\tilde{\rho})) \arrow[r, "d"] & K(C^*(\mathbb{R}\otimes X,\tilde{H},\tilde{\rho})) \\
K\mathcal{X}(X) \arrow[u, "\simeq"] \arrow[r, "i_*"] & K\mathcal{X}(\mathcal{O}(X)) \arrow[u, "\simeq"] \arrow[r, "j_*"] & K\mathcal{X}(\mathcal{O}(X)_-) \arrow[u, "\simeq"] \arrow[r, "\partial'"] & \Sigma K\mathcal{X}(X) \arrow[u, "\simeq"]
\end{tikzcd}$$

where $\partial' := \partial \circ \iota$ with ι as in (15.1) and we use Proposition 15.1 for the commutativity of the right square. In particular, the lowest and the highest row in (16.6) are equivalent to segments of a fibre sequence of spectra.

We now use the following facts:

1. The algebra $Q^*(X_{\mathcal{C}},H,\rho)$ does not depend on the coarse structure of $X_{\mathcal{C}}$, Higson–Roe [HR95, Lem. 6.2]. In the reference it is assumed that the coarse structure comes from a metric. But the argument only uses that $\mathcal{C}$ has open entourages U and that we can find locally finite, open and U-bounded coverings of X.

   Applied to $[0,\infty)\otimes X$ and $\mathbb{R}\otimes X$ we conclude that the canonical maps

$$\begin{aligned} Q^*(\mathcal{O}(X),H_+,\rho_+) &\to Q^*([0,\infty)\otimes X,H_+,\rho_+) \text{ and} \\ Q^*(\mathcal{O}(X)_-,\tilde{H},\tilde{\rho}) &\to Q^*(\mathbb{R}\otimes X,\tilde{H},\tilde{\rho}) \end{aligned} \tag{16.7}$$

   are isomorphisms of C^*-algebras.

2. By Paschke duality we have an isomorphism of groups

$$K_*(Q^*([0,\infty)\otimes X,H_+,\rho_+)) \cong K_*^{an,lf}([0,\infty)\times X)\ .$$

   Since $[0,\infty)\times X$ is flasque we have the marked isomorphism in the chain

$$K_*(Q^*(\mathcal{O}(X),H_+,\rho_+)) \cong K_*(Q^*([0,\infty)\otimes X,H_+,\rho_+)) \cong K_*^{an,lf}([0,\infty)\otimes X) \overset{!}{\cong} 0\ .$$

3. A sequence of spectra of the form

$$\cdots \to 0 \to A \xrightarrow{\sim} B \to 0 \to \dots$$

   is clearly a segment of a fibre sequence. Hence the above Points 1 and 2 together imply that the second row in (16.6) is a fibre sequence. It then follows that the third row in (16.6) is a fibre sequence, too.

4. We want to show that δ in (16.6) is an equivalence. We have a map

$$\begin{aligned} K_*^{an,lf}(X) \cong K_{*+1}^{an,lf}(\mathbb{R}\otimes X) &\cong K_{*+1}(Q^*(\mathcal{O}(X)_-,\tilde{H},\tilde{\rho})) \\ &\xrightarrow{\delta} K_*(C^*(\mathcal{O}(X)_-,\tilde{H},\tilde{\rho})) \cong K\mathcal{X}_*(\mathcal{O}(X)_-)\ . \end{aligned} \tag{16.8}$$

   The first isomorphism is the suspension isomorphism for locally finite homology theories and the second isomorphism is given by Paschke duality together with the isomorphism (16.7). And finally, the last isomorphism comes from the comparison theorem [BE16, Thm. 7.70].

If assume that X is isomorphic in the category **UBC** to a countable, locally finite and finite-dimensional simplicial complex, then we have equivalences

$$K\mathcal{X}(\mathcal{O}(X)_-) \overset{\text{Prop. 15.1}}{\simeq} K\mathcal{X}\mathcal{O}^\infty(X) \overset{\text{Prop. 12.14}}{\simeq} (K\mathcal{X}\mathcal{O}^\infty)^{lf}(X)\ .$$

The transformation (16.8) is natural with respect to restrictions to subspaces of X. By the result of Siegel [Sie12] this transformation is also compatible with the bounday maps of the Mayer–Vietoris sequences associated to open coverings. Since the target (by Proposition 12.14) and the domain of it both behave like locally finite homology theories and (16.8) induces an isomorphism on bounded contractible subsets, (16.8) is an isomorphism.

The fact that δ is an equivalence implies that $K(D^*(\mathcal{O}(X)_-, \tilde{H}, \tilde{\rho})) \simeq 0$.

Remark 16.5. An alternative option to show that $K(D^*(\mathcal{O}(X)_-, \tilde{H}, \tilde{\rho})) \simeq 0$ is to show the equivalence $K(D^*(X, H, \rho)) \simeq K(D^*(\mathcal{O}(X), H_+, \rho_+)$ directly using the invariance of the functor $K_* \circ D^*$ under coarse homotopies [HR95, Lem. 7.8]. $\square$

Putting all these facts together we get the diagram of vertical and horizontal fibre sequences

$$\begin{tikzcd}
\Sigma K(C^*(X,H,\rho)) \arrow[r] & \Sigma K(C^*(\mathcal{O}(X),H_+,\rho_+)) \arrow[r] & \Sigma K(C^*(\mathcal{O}(X)_-,\tilde{H},\tilde{\rho})) \arrow[r] & \Sigma K(C^*(\mathbb{R}\otimes X,\tilde{H},\tilde{\rho})) \\
K(Q^*(X,H,\rho)) \arrow[r] \arrow[u, "\partial^{C^*}"] & 0 \arrow[r] \arrow[u] & K(Q^*(\mathcal{O}(X)_-,\tilde{H},\tilde{\rho})) \arrow[r, "\simeq"] \arrow[u, "\simeq"'] & K(Q^*(\mathbb{R}\otimes X,\tilde{H},\tilde{\rho})) \arrow[u] \\
K(D^*(X,H,\rho)) \arrow[r, "\simeq"] \arrow[u] & K(D^*(\mathcal{O}(X),H_+,\rho_+)) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & K(D^*(\mathbb{R}\otimes X,\tilde{H},\tilde{\rho})) \arrow[u] \\
K(C^*(X,H,\rho)) \arrow[r, "u_*"] \arrow[u] & K(C^*(\mathcal{O}(X),H_+,\rho_+)) \arrow[r, "v_*"] \arrow[u, "\simeq"'] & K(C^*(\mathcal{O}(X)_-,\tilde{H},\tilde{\rho})) \arrow[r, "d"] \arrow[u] & K(C^*(\mathbb{R}\otimes X,\tilde{H},\tilde{\rho})) \arrow[u]
\end{tikzcd}$$

It provides the asserted morphism of fibre sequences. $\square$

The vertical morphisms and the fillers of the squares in (16.5) may depend non-trivially on the choice of the Data 16.3. In particular we ask:

Problem 16.6. *Does the map b in* (16.5) *depend non-trivially on the choice of Data 16.3 made in addition to 16.3.1?*

Assume that we have just chosen a very ample continuously X-controlled Hilbert space (H, ρ), i.e., the Datum 16.3.1.

Definition 16.7. *The map*

$$A_X : K_n^{an,lf}(X) \overset{(16.3)}{\cong} K_{n+1}(Q^*(X,H,\rho)) \xrightarrow{\partial^{C^*}} K_n(C^*(X,H,\rho)) \overset{(16.4)}{\cong} K\mathcal{X}_n(X)$$

is called the analytic assembly map.

If we now choose the full Data 16.3, then we get the following commuting diagram:

$$\begin{tikzcd} K_n^{an,lf}(X) \arrow[r, "\cong", "(16.3)"'] \arrow[d, "\cong"', "\mathtt{comp}"] \arrow[rr, bend left, "A_X"] & K_{n+1}(Q^*(X,H,\rho)) \arrow[r, "\tilde{\partial}^{C^*}"] & K\mathcal{X}_n(X) \arrow[d, equal] \\ K_n^{an,lf}(X) \arrow[r, "\cong", "(14.3)"'] & K\mathcal{X}_{n+1}(\mathcal{O}^\infty(X)) \arrow[r, "\partial"] \arrow[u, "\cong"', "b"] & K\mathcal{X}_n(X) \end{tikzcd} \tag{16.9}$$

where we are using the notation $\tilde{\partial}^{C^*} := (16.4) \circ \partial^{C^*}$. We note that the isomorphism `comp` is determined by the condition that the left square commutes. It involves the choice of a spectrum equivalence (14.1) via the equivalence (14.3) and also the Data 16.3 via the isomorphism b.

Problem 16.8. *Can one canonically choose the spectrum equivalence* (14.1) *and the Data 16.3 such that the isomorphism* `comp` *becomes the identity?*

Remark 16.9. In this remark we explain the process of coarsification involved in the transition from the analytic assembly map A_X to the analytic coarse assembly map μ_X^{an}.

We assume that X is a separable bornological coarse space of bounded geometry. If U is an entourage of X, then $P(X,U)$ is a countable, finite-dimensional, locally finite simplicial complex and hence a proper metric space. Consequently, we can apply the above theory to the metric space $P(X,U)$. After choosing a very ample continuously $P(X,U)$-controlled Hilbert space (H_U, ρ_U) we get the analytic assembly map

$$A_{P(X,U)} : K_*^{an,lf}(P(X,U)) \to K\mathcal{X}_*(P(X,U))$$

by Definition 16.7.

We have an Equivalence (6.3) of bornological coarse spaces (recall that in the present section we drop the forgetful functor $F_{\mathcal{U}}$ from the notation)

$$X_U \to P(X,U) \tag{16.10}$$

For every two coarse entourages U, U' of X with $U \subseteq U'$ we choose an isometry $i_{U,U'} : H_U \to H_{U'}$ which induces a pseudolocal morphism $f_{U,U',*}(H_U, \rho_U) \to (H_{U'}, \rho_{U'})$, where $f_{U,U'} : P(X,U) \to P(X,U')$ is the natural embedding. We then have a commuting diagram

$$\begin{tikzcd} K_*^{an,lf}(P(X,U)) \arrow[r, "\cong"] \arrow[d, "f_{U,U',*}"] \arrow[rr, bend left, "A_{P(X,U)}"] & K_{*+1}(Q^*(P(X,U),H_U,\rho_U)) \arrow[r, "\partial^{C^*}"] \arrow[d, "i_{U,U',*}"] & K\mathcal{X}_*(P(X,U)) \arrow[d] & K\mathcal{X}_*(X_U) \arrow[l, "\cong", "(16.10)"'] \arrow[d, equal] \\ K_*^{an,lf}(P(X,U')) \arrow[r, "\cong"] \arrow[rr, bend right, "A_{P(X,U')}"'] & K_{*+1}(Q^*(P(X,U'),H_{U'},\rho_{U'})) \arrow[r, "\partial^{C^*}"] & K\mathcal{X}_*(P(X,U')) & K\mathcal{X}_*(X_{U'}) \arrow[l, "\cong", "(16.10)"'] \end{tikzcd}$$

We now form the colimit of the horizontal maps over the entourages U of X. In view of u-continuity of $K\mathcal{X}_*$ we get the homomorphism

$$\mu_X^{an} : QK_*^{an,lf}(X) \to K\mathcal{X}_*(X) \tag{16.11}$$

finishing the construction of the coarse analytic assembly map. □

Definition 16.10. *The homorphism μ_X^{an} is called the coarse analytic assembly map.*

Remark 16.11. In this remark we compare the coarse analytic assembly map (16.11) with the assembly map (14.5). We assume that X is a separable bornological coarse space of bounded geometry. We assume that the coarse structure admits a countable cofinal monotoneously increasing family.

We must choose the Data 16.3 for $P(X,U)$ in place of X compatibly with the inclusions $f_{U,U'} : P(X,U) \to P(X,U')$. In order to simplify matters and to avoid a discussion of relations in the index poset for the colimit over the coarse entourages of X, we reduce this construction to a cofinal monotoneously increasing family $(U_n)_{n\in\mathbb{N}}$ of entourages. We then have diagrams

$$\begin{tikzcd}
K_n^{an,lf}(P(X,U)) \arrow[r, "\cong", "(16.3)"'] \arrow[d, "\cong"', "\mathtt{comp}"] \arrow[rr, bend left, "A_X"] & K_{n+1}(Q^*(P(X,U),H_U,\rho_U)) \arrow[r, "\tilde\partial^{C^*}"] & K\mathcal{X}_n(P(X,U)) \arrow[d, equal] & K\mathcal{X}_n(X) \arrow[l, "\cong"', "(16.10)"] \arrow[d, equal] \\
K_n^{an,lf}(P(X,U)) \arrow[r, "\cong", "(14.3)"'] & K\mathcal{X}_{n+1}(\mathcal{O}^\infty(P(X,U))) \arrow[r, "\partial"] \arrow[u, "\cong", "b"'] & K\mathcal{X}_n(P(X,U)) & K\mathcal{X}_n(X) \arrow[l, "\cong"', "(16.10)"]
\end{tikzcd} \tag{16.12}$$

for all coarse entourages U of X and connecting maps between such diagrams for inclusions $U \to U'$. If we form the colimit over the coarse entourages of X, then the colimits of the outer squares yield the diagram

$$\begin{tikzcd}
QK_n^{an,lf}(X) \arrow[r, "\mu_X^{an}"] \arrow[d, "\cong"', "Q\mathtt{comp}"] & K\mathcal{X}_n(X) \arrow[d, equal] \\
QK_n^{an,lf}(X) \arrow[r, "\mu_X^{top}"] & K\mathcal{X}_n(X)
\end{tikzcd} \tag{16.13}$$

where we use (14.5) and (14.6) for the identification of the lower horizontal map called μ_X^{top}, and Remark 16.9 for the upper horizontal map. The isomorphism $Q\mathtt{comp}$ possibly depends on the choices of the ample Hilbert space data, the various embeddings, and a spectrum equivalence (14.1). □

The upshot of the above discussion is the following statement:

Corollary 16.12. *There is an equivalence between the coarse analytic assembly map μ_X^{an} and coarse assembly map μ_X^{top}. In particular, if one of these maps is an isomorphism then so is the other.*

The equivalence is canonical up to an automorphism of $QK_n^{an,lf}(X)$. At the moment we are not able to make the comparison more canonical.

17 Index theory

Let M be a complete Riemannian manifold and $\not{D}$ be a generalized Dirac operator on M of degree n. The Dirac operator acts on sections of a $\mathbb{Z}/2\mathbb{Z}$-graded Dirac bundle $E \to M$ with a right action by the Clifford algebra $\mathtt{Cl}^n$. In this situation one can define the coarse index class

$$\mathtt{Ind}(\not{D}) \in K\mathcal{X}_n(M)$$

of the Dirac operator, see Higson–Roe [HR00] or Zeidler [Zei16]. In the original construction of the coarse index class by Roe [Roe96] the degree was incorporated differently. Moreover, the index class arises as a K-theory class of the Roe algebra associated to the Dirac bundle. It requires some argument in order to interpret the index class as a coarse K-homology class in the theory $K\mathcal{X}$ as indicated above, see [BE16, Sec. 7.9]. A detailed construction of the index class, even in the equivariant case and with support conditions, will be given in [BE17, Def. 9.6].

The goal of this section is to construct a K-homology class $\sigma(\not{D})$ in $K\mathcal{X}_{n+1}(\mathcal{O}^\infty(M))$ such that

$$\mu_{K\mathcal{X},M}(\sigma(\not{D})) = \mathtt{Ind}(\not{D})$$

in $K\mathcal{X}_n(M)$. The class $\sigma(\not{D})$ is the analogue of the symbol class of $\not{D}$ (this motivates the notation).

Let (M,g) be a complete Riemannian manifold and $\not{D}$ be a generalized Dirac operator on M of degree n. We will need the following two operations with Dirac operators.

1. ([Bun, Ex. 4.3]) Assume that g' is a second complete Riemannian metric on M. In [Bun] we explain a construction which starts from $\not{D}$ and produces a canonical Dirac operator $\not{D}'$ associated to the metric g'. The idea is to write $\not{D}$ locally as a twisted *spin*-Dirac operator. If we change the metric, then we change the *spin*-Dirac operator correspondingly and keep the twisting fixed.

2. ([Bun, Ex. 4.4]) There is a natural way to extend the Dirac operator $\not{D}$ of degree n to a Dirac operator $\tilde{\not{D}}$ of degree $n+1$ on the Riemannian product $\tilde{M} := \mathbb{R} \times M$. We denote by $\tilde{E}' \to \tilde{M}$ the pull-back of the bundle $E \to M$ with the induced metric and connection, and then form the graded bundle $\tilde{E} := \tilde{E}' \otimes \mathtt{Cl}^1$. Under the identification $\mathtt{Cl}^{n+1} \cong \mathtt{Cl}^n \otimes \mathtt{Cl}^1$ it has a right action of the Clifford algebra $\mathtt{Cl}^{n+1}$, where $\mathtt{Cl}^n$ acts on $\tilde{E}'$ and $\mathtt{Cl}^1$ acts on the $\mathtt{Cl}^1$-factor of $\tilde{E}$ by right-multiplication. The Clifford action $TM \otimes E \to E$ extends to a Clifford action $T\tilde{M} \otimes \tilde{E} \to \tilde{E}$ such that ∂_t acts by left-multiplication by the generator of $\mathtt{Cl}^1$ on the $\mathtt{Cl}^1$-factor of $\tilde{E}$, where t is the coordinate of the $\mathbb{R}$-factor in $\tilde{M} = \mathbb{R} \times M$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function

Assumption 17.1. *We assume that f is smooth, positive, and that $f(t) = 1$ for t in $(-\infty, 0]$ and $\mathtt{lim}_{t\to\infty} f(t) = \infty$.*

We form the new complete Riemannian metric

$$g_f := dt^2 + f(t) \cdot \mathtt{pr}^* g$$

on $\mathbb{R} \times M$, where $\mathtt{pr} : \mathbb{R} \times M \to M$ is the projection. We denote the resulting Riemannian manifold by $\tilde{M}_f$. Then we let $\tilde{\not{D}}_f$ denote the Dirac operator associated to the metric g_f obtained from $\tilde{\not{D}}$ by the Construction 2 mentioned above. We then have a class

$$\mathtt{Ind}(\tilde{\not{D}}_f) \text{ in } K\mathcal{X}_{n+1}(\tilde{M}_f) .$$

The Riemannian manifold $\tilde{M}_f$ can be considered as a bornological coarse space with the structures induced by the metric. We now observe that the identity map of underlying sets induces a morphism of bornological coarse spaces

$$p_f : \tilde{M}_f \to \mathcal{O}(M)_- .$$

Proposition 17.2. *The class $p_{f,*}(\mathtt{Ind}(\tilde{\not{D}}_f))$ in $K\mathcal{X}_{n+1}(\mathcal{O}(M)_-)$ is independent of the choice of f as long as f satisfies Assumption 17.1.*

Proof. This is shown in [Bun, Prop. 4.11]. □

The motivic equivalence $\mathrm{Yo}^s(\mathcal{O}(M)_-) \simeq \mathcal{O}^\infty(M)$ obtained in Proposition 15.1 induces an equivalence of coarse K-homology spectra $K\mathcal{X}(\mathcal{O}(M)_-) \simeq K\mathcal{X}\mathcal{O}^\infty(M)$. Hence we can consider $p_{f,*}(\mathtt{Ind}(\tilde{\not{D}}_f))$ as a class in $K\mathcal{X}_{n+1}(\mathcal{O}^\infty(M))$.

Definition 17.3. *We define the class*

$$\sigma(\not{D}) \text{ in } K\mathcal{X}\mathcal{O}^\infty_{n+1}(M)$$

to be the class $p_{f,*}(\mathtt{Ind}(\tilde{\not{D}}_f))$ for some choice of function f satisfying Assumption 17.1.

Recall the boundary map $\partial : K\mathcal{X}\mathcal{O}^\infty_{*+1}(M) \to K\mathcal{X}_*(M)$ of the cone sequence from (9.1).

Proposition 17.4. *We have the equality*

$$\partial(\sigma(\not{D})) = \mathtt{Ind}(\not{D}) \tag{17.1}$$

in $K\mathcal{X}_*(M)$.

Proof. This is shown in detail in [Bun, Lem. 4.14]. But for completeness we recall the idea. We have a commuting diagram of bornological coarse spaces

$$\begin{tikzcd} \tilde{M}_f \arrow[r, "i"] \arrow[d, "p"] & \tilde{M} \arrow[d, equal] \\ \mathcal{O}(M)_- \arrow[r, "d"] & \mathbb{R} \otimes M \end{tikzcd}$$

all induced by the identity of the underlying maps. We observe that the image $i_*(\mathtt{Ind}(\tilde{\not{D}}_f)$ is independent of f as long as $f \geq 1$ (the argument is similar as for Proposition 17.2). This shows that $d_*(p_*(\mathtt{Ind}(\tilde{\not{D}}_f))) = \mathtt{Ind}(\tilde{\not{D}})$ in $K\mathcal{X}_{n+1}(\mathbb{R} \times M)$. We now use the square

$$\begin{array}{ccc} K\mathcal{X}(\mathcal{O}(M)_-) & \xrightarrow{d} & K\mathcal{X}(\mathbb{R} \otimes M) \\ \downarrow{\simeq} & & {\scriptstyle s}\downarrow{\simeq} \\ K\mathcal{X}\mathcal{O}^\infty(M) & \xrightarrow{\partial} & \Sigma K\mathcal{X}(M) \end{array}$$

obtained from Proposition 15.1 and the compatibility of the coarse index class with the suspension equivalence s expressed by the equality $s_*(\mathtt{Ind}(\tilde{\not{D}})) = \mathtt{Ind}(\not{D})$ (for the latter see Zeidler [Zei16, Thm. 5.5] or [BE17, Thm. 11.1]). □

Let us discuss now the problem of identifying the class $\sigma(\not{D})$ defined in Definition 17.3 with the classical symbol class $\sigma^{an,lf}(\not{D})$ whose definition will be recalled below.

We consider a function $\chi : \mathbb{R} \to [-1, 1]$ which is smooth, odd, and satisfies $\chi(t) > 0$ for all t in $(0, \infty)$ and $\mathtt{lim}_{t\to\pm\infty} \chi(t) = \pm 1$. Then we get a $(C_0(M), \mathtt{Cl}^n)$-Kasparov module $(H, \rho, \chi(\not{D}))$, see Higson–Roe [HR00, Sec. 10.6].

Definition 17.5. *The classical symbol class $\sigma^{an,lf}(\not{D})$ of $\not{D}$ is defined to be the class of the $(C_0(M), \mathtt{Cl}^n)$-Kasparov module $(H, \rho, \chi(\not{D}))$ in $K^{an,lf}_n(M) \stackrel{def}{=} KK(C_0(M), \mathtt{Cl}^n)$.*

It follows from the details of the construction of the Paschke duality isomorphism that

$$A_M(\sigma^{an,lf}(\not{D})) = \mathtt{Ind}(\not{D}) \ . \tag{17.2}$$

If the Riemannian manifold M has bounded geometry, then M is isomorphic (in the category $\mathbf{UBC}^{small}$) to a countable, locally finite, finite-dimensional simplicial complex. Indeed, there exists a triangulation of M as a simplicial complex K of bounded geometry such that M is bi-Lipschitz equivalent to K, Attie [Att04, Thm. 2.1]. Here K is equipped with the spherical metric.

In this case, after choosing all the Data 16.3 and a spectrum equivalence (14.1) we can consider the Diagram (16.9) (with X replaced by M).

The class $\sigma(\not{D})$ corresponds under the isomorphism induced by the Equivalence (14.3) to a class $\tilde{\sigma}^{an,lf}(\not{D})$ in $K^{an,lf}_n(M)$. We note that the class $\tilde{\sigma}^{an,lf}(\not{D})$ depends on the choice of the spectrum equivalence (14.1).

Problem 17.6. *Do we have the equality*

$$\mathtt{comp}(\sigma^{an,lf}(\not{D})) = \tilde{\sigma}^{an,lf}(\not{D}) \ ? \tag{17.3}$$

It is clear that the dependence on the choice of the spectrum equivalence (14.1) on both sides of the Equation (17.3) cancels out because of the way we define the map `comp`. The answer to the Question (17.3) can only be positive if the answer to the question formulated in Problem 16.6 is *no*.

We can map the classes $\tilde{\sigma}^{an,lf}(\not{D})$ and $\sigma^{an,lf}(\not{D})$ into the group $QK_n^{an,lf}(M)$ by combining the procedures explained in Remark 16.11 with the map (7.1) to get classes that we denote by $Q\tilde{\sigma}^{an,lf}(\not{D})$ and $Q\sigma^{an,lf}(\not{D})$. From (17.1) we get that $\tilde{\sigma}^{an,lf}(\not{D})$ is mapped to $\mathtt{Ind}(\not{D})$ by the boundary map of the cone sequence, and by (17.2) we know that $\sigma^{an,lf}(\not{D})$ is mapped to $\mathtt{Ind}(\not{D})$ by A_M. We conclude that we have

$$\mu_M^{an}\big(Q\sigma^{an,lf}(\not{D})\big) = \mathtt{Ind}(\not{D}) \quad \text{and} \quad \mu_M^{top}\big(Q\tilde{\sigma}^{an,lf}(\not{D})\big) = \mathtt{Ind}(\not{D}). \tag{17.4}$$

From this together with the Diagram (16.13) we get the following result:

Lemma 17.7. *If the coarse assembly maps μ_M^{an} and μ_M^{top} are injective, then we have*

$$Q\mathtt{comp}\big(Q\sigma^{an,lf}(\not{D})\big) = Q\tilde{\sigma}^{an,lf}(\not{D}) \ . \tag{17.5}$$

Note that by Theorem 10.7 the coarse assembly map μ_M^{top} is an isomorphism if M has finite decomposition complexity. Yu [Yu00] proved that μ_M^{an} is an isomorphism is M has bounded geometry and is coarsely embeddable into a Hilbert space. More generally, Kasparov–Yu [KY06] proved injectivity of μ_M^{an} if M has bounded geometry and is coarsely embeddable into a uniformly convex Banach space and Chen–Wang–Yu [CWY15] proved injectivity of μ_M^{an} if M has bounded geometry and is coarsely embeddable into a Banach space with Property (H).

The Equality (17.5) should be independent of results on the coarse Novikov conjecture (i.e., independent of injectivity of the coarse assembly map). So let us phrase this as a separate question, which is a weakening of the Question (17.3):

Problem 17.8. *Do we always have the equality*

$$Q\mathtt{comp}(Q\sigma^{an,lf}(\not{D})) = Q\tilde{\sigma}^{an,lf}(\not{D}) \ ? \tag{17.6}$$

References

[Att04] O. Attie. A Surgery Theory for Manifolds of Bounded Geometry, 2004. Online available at `http://arxiv.org/abs/math/0312017/`.

[BE] U. Bunke and A. Engel. Equivariant coarse K-homology. In preparation.

[BE16] U. Bunke and A. Engel. Homotopy theory with bornological coarse spaces. arXiv:1607.03657v3, 2016.

[BE17] U. Bunke and A. Engel. The coarse index class with support. arXiv:1706.06959, 2017.

[BEKWa] U. Bunke, A. Engel, D. Kasprowski, and Ch. Winges. Coarse motives with transfer and finite decomposition complexity. In preparation.

[BEKWb] U. Bunke, A. Engel, D. Kasprowski, and Ch. Winges. Equivariant coarse homotopy theory and coarse algebraic K-homology. In preparation.

[BFJR04] A. Bartels, T. Farrell, L. Jones, and H. Reich. On the isomorphism conjecture in algebraic K-theory. *Topology*, 43(1):157–213, 2004.

[Bun] U. Bunke. Abstract boundary value problems. In preparation.

[CWY15] X. Chen, Q. Wang, and G. Yu. The coarse Novikov conjecture and Banach spaces with Property (H). *J. Funct. Anal.*, 268:2754–2786, 2015.

[FO17] T. Fukaya and Sh.-I. Oguni. A coarse Cartan–Hadamard theorem with applications to the coarse Baum–Connes conjecture. arXiv:1705.05588, 2017.

[Gro93] M. Gromov. Asymptotic Invariants of Infinite Groups. In G. A. Niblo and M. A. Roller, editors, *Geometric Group Theory*, volume 182 of *London Mathematical Society Lecture Note Series*, 1993.

[GTY12] E. Guentner, R. Tessera, and G. Yu. A notion of geometric complexity and its application to topological rigidity. *Invent. math.*, 189(2):315–357, 2012.

[Hig95] N. Higson. C^*-Algebra Extension Theory and Duality. *J. Funct. Anal.*, 129:349–363, 1995.

[HPR96] N. Higson, E. K. Pedersen, and J. Roe. C^*-algebras and controlled topology. *K-Theory*, 11:209–239, 1996.

[HR95] N. Higson and J. Roe. On the coarse Baum–Connes conjecture. In S. C. Ferry, A. Ranicki, and J. Rosenberg, editors, *Novikov conjectures, index theorems and rigidity, Vol. 2*, London Mathematical Society Lecture Notes 227. Cambridge University Press, 1995.

[HR00] N. Higson and J. Roe. *Analytic K-Homology.* Oxford University Press, 2000.

[KY06] G. Kasparov and G. Yu. The coarse geometric Novikov conjecture and uniform convexity. *Adv. Math.*, 206:1–56, 2006.

[Mit10] P. D. Mitchener. The general notion of descent in coarse geometry. *Alg. Geom. Topol.*, 10:2419–2450, 2010.

[NY12] P. W. Nowak and G. Yu. *Large Scale Geometry.* European Mathematical Society, 2012.

[Pas81] W.L. Paschke. K-Theory for Commutants in the Calkin Algebra. *Pacific J. Math.*, 95(2):427–434, 1981.

[Roe96] J. Roe. *Index Theory, Coarse Geometry, and Topology of Manifolds*, volume 90 of *CBMS Regional Conference Series in Mathematics.* American Mathematical Society, 1996.

[RS12] J. Roe and P. Siegel. Sheaf theory and Paschke duality. arXiv:math/1210.6420, 2012.

[Sie12] P. Siegel. The Mayer–Vietoris Sequence for the Analytic Structure Group. arXiv:math/1212.0241, 2012.

[STY02] G. Skandalis, J.L. Tu, and G. Yu. The coarse Baum–Connes conjecture and groupoids. *Topology*, 41:807–834, 2002.

[Wei02] M. Weiss. Excision and restriction in controlled K-theory. *Forum Math.*, 14(1):85–119, 2002.

[Wri05] N. J. Wright. The coarse Baum-Connes conjecture via C_0 coarse geometry. *J. Funct. Anal.*, 220(2):265–303, 2005.

[WW95] M. Weiss and B. Williams. Pro-excisive functors. In S. C. Ferry, A. Ranicki, and J. Rosenberg, editors, *Novikov conjectures, index theorems and rigidity, Vol. 2*, London Mathematical Society Lecture Notes 227. Cambridge University Press, 1995. www.math.uni-muenster.de/u/michael.weiss/preprints/proexci.pdf.

[Yu95a] G. Yu. Baum–Connes Conjecture and Coarse Geometry. *K-Theory*, 9:223–231, 1995.

[Yu95b] G. Yu. Coarse Baum–Connes Conjecture. *K-Theory*, 9:199–221, 1995.

[Yu98] G. Yu. The Novikov conjecture for groups with finite asymptotic dimension. *Ann. Math.*, 147:325–355, 1998.

[Yu00] G. Yu. The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. math.*, 139(1):201–240, 2000.

[Zei16] R. Zeidler. Positive scalar curvature and product formulas for secondary index invariants. *J. Topol.*, 9(3):687–724, 2016.