arXiv:1710.00344v1 [math.PR] 1 Oct 2017

The Edwards-Wilkinson limit of the random heat equation in dimensions three and higher

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October 3, 2017

Abstract

We consider the heat equation with a *multiplicative* Gaussian potential in dimensions $d \geq 3$. We show that the renormalized solution converges to the solution of a deterministic diffusion equation with an effective diffusivity. We also prove that the renormalized large scale random fluctuations are described by the Edwards-Wilkinson model, that is, the stochastic heat equation (SHE) with *additive* white noise, with an effective variance.

1 Introduction

We consider the solutions to the heat equation with a smooth Gaussian random potential:

$$\partial_t u = \frac{1}{2}\Delta u + \lambda V(t,x)u, \quad x \in \mathbb{R}^d, d \geq 3. \tag{1.1}$$

Here, $\lambda > 0$ is a constant, and the random potential $V(t,x)$ is a mean-zero Gaussian field that we assume to be of the form

$$V(t,x) = \int_{\mathbb{R}^{d+1}} \phi(t-s)\psi(x-y)dW(s,y),$$

where $dW(s,y)$ is a space-time white noise built on a probability space $(\Sigma, \mathcal{F}, \mathbb{P})$. We assume that the non-negative functions $\phi, \psi \in \mathcal{C}_c^\infty$, that ϕ is supported on $[0,1]$, and that ψ is even and supported on $\{x : |x| \leq 1/2\}$. The covariance function of V is

$$R(t,x) = \mathbb{E}[V(0,0)V(t,x)] = \int_{\mathbb{R}} \phi(t+s)\phi(s)ds \int_{\mathbb{R}^d} \psi(x+y)\psi(y)dy. \tag{1.2}$$

Here, $\mathbb{E}$ denotes the expectation on Σ. The above assumptions on the correlation function $R(t,x)$ are made mostly to simplify the notation, and the only essential technical assumptions are that $R(t,x)$ is compactly supported in t and is rapidly decaying in x.

As we are interested in the large scale and long time asymptotics of $u(t,x)$, we consider the rescaled function

$$u_\varepsilon(t,x) := u(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}),$$

with $\varepsilon \ll 1$. The function u_ε satisfies

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \frac{\lambda}{\varepsilon^2} V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})u_\varepsilon. \tag{1.3}$$

We assume that the initial condition $u_\varepsilon(0,x) = u_0(x) \in \mathcal{C}_b(\mathbb{R}^d)$. Throughout the paper, we stay in the weak disorder regime and assume that $\lambda \in (0, \lambda_0)$, with a small but fixed constant λ_0 only depending on d, ϕ and ψ. Our main result is as follows.

Theorem 1.1. *There exist c_1, c_2 depending on λ, ϕ, and ψ such that for any $t > 0$ and $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} u_\varepsilon(t,x) \exp\Big\{ -\frac{c_1 t}{\varepsilon^2} - c_2 \Big\} g(x)dx \to \int_{\mathbb{R}^d} \bar{u}(t,x)g(x)dx, \quad \text{as } \varepsilon \to 0, \tag{1.4}$$

in probability, and

$$\frac{1}{\varepsilon^{d/2-1}} \int_{\mathbb{R}^d} (u_\varepsilon(t,x) - \mathbb{E}[u_\varepsilon(t,x)]) \exp\Big\{ -\frac{c_1 t}{\varepsilon^2} - c_2 \Big\} g(x)dx \Rightarrow \int_{\mathbb{R}^d} \mathscr{U}(t,x)g(x)dx \tag{1.5}$$

in distribution. Here, $\bar{u}$ is the solution of the effective heat equation

$$\partial_t \bar{u} = \frac{1}{2} \nabla \cdot \boldsymbol{a}_{\text{eff}} \nabla \bar{u}, \quad \bar{u}(0,x) = u_0(x), \tag{1.6}$$

with the effective diffusion matrix $\boldsymbol{a}_{\text{eff}} \in \mathbb{R}^{d\times d}_{\text{sym}}$ defined in (4.28) below, and $\mathscr{U}$ solves the additive stochastic heat equation

$$\partial_t \mathscr{U} = \frac{1}{2} \nabla \cdot \boldsymbol{a}_{\text{eff}} \nabla \mathscr{U} + \lambda \nu_{\text{eff}} \bar{u} \dot{W}, \quad \mathscr{U}(0,x) = 0, \tag{1.7}$$

with the effective variance $\nu_{\text{eff}}^2 > 0$ defined in (5.6) below.

The renormalization constants c_1 and c_2 are identified in (A.2) below.

1.1 Background and related problems

The study of singular stochastic PDEs has witnessed important progress in recent years, with different approaches developed to make sense of equations which are genuinely ill-posed due to the lack of regularity and the need to make sense of the multiplication of distributions [13, 14, 15, 19, 23]. The existing works typically prove that the solution of the equation with the mollified white noise, after a suitable renormalization, converges to some limit that is independent of the way in which the noise is mollified.

Here, we consider a slightly different situation: the rescaled random field in (1.3) is not a mollification of the white noise, and does not directly converge to the white noise in $d \geq 3$ as $\varepsilon \to 0$. We rather have, formally,

$$\frac{1}{\varepsilon^2} V\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\big) \sim \varepsilon^{d/2-1} \nu_0 \dot{W}(t,x),$$

with

$$\nu_0^2 = \int_{\mathbb{R}^{d+1}} R(s,y)dsdy. \tag{1.8}$$

Hence, one could think that the noise in (1.3) is small and would not produce a non-trivial effect on the solutions, so that the limit would be simply the unperturbed heat equation. This is problematic – if we formally replace the random potential in (1.3) by $\varepsilon^{d/2-1}\dot{W}(t,x)$, we obtain the multiplicative stochastic heat equation. Giving a meaning to its solutions in $d \geq 3$ brings about the aforementioned question of making sense of multiplying two distributions u and $\dot{W}$. Hence, the issue of the limit is much more delicate. Theorem 1.1 shows that even though the random potential in (1.3) formally converges to zero, it still affects the solutions in a non-trivial way: (i) on the level of the law of large numbers, the solution of (1.3) converges to a solution of the deterministic diffusion equation (1.6), with an effective diffusivity that is modified by the presence of the noise, and (ii) on the level of the central limit theorem, the random fluctuations, after a rescaling, fall into the Edwards-Wilkinson universality class in $d \geq 3$, as in (1.7), with an effective (and not a "naive-guess" ν_0) variance. We stress that both the diffusion matrix and the variance of the noise are homogenized in the limit.

We mention two related problems. The weak coupling regime analyzed in [11] concerns the situation when the potential in (1.1) is asymptotically small:

$$\partial_t u = \frac{1}{2}\Delta u + \varepsilon V(t,x)u, \quad x \in \mathbb{R}^d, d \geq 3. \tag{1.9}$$

It was shown that no renormalization is required: the diffusively rescaled solution

$$u_\varepsilon(t,x) = u(t/\varepsilon^2, x/\varepsilon)e^{-V_{\text{eff}}t}$$

converges in probability to the solution of the diffusion equation

$$\partial_t \bar{u} = \frac{1}{2}\Delta \bar{u}, \quad \bar{u}(0,x) = u_0(x), \tag{1.10}$$

with an un-modified diffusivity. The effective potential V_{eff} is explicit:

$$V_{\text{eff}} = \int_0^\infty \mathbb{E}_B[R(t,B_t)]dt.$$

As far as fluctuations are concerned, using a simpler version of what is done in the present paper, one can show that for any $t > 0$ and $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ we have, as $\varepsilon \to 0$:

$$\frac{1}{\varepsilon^{d/2}} \int_{\mathbb{R}^d} (u_\varepsilon(t,x) - \mathbb{E}[u_\varepsilon(t,x)])e^{-V_{\text{eff}}t} g(x)dx \Rightarrow \int_{\mathbb{R}^d} \mathscr{U}(t,x)g(x)dx \tag{1.11}$$

in distribution. Here, $\mathscr{U}$ solves the stochastic heat equation with additive space-time white noise

$$\partial_t \mathscr{U} = \frac{1}{2}\Delta \mathscr{U} + \nu_0 \bar{u}\dot{W}, \quad \mathscr{U}(0,x) = 0. \tag{1.12}$$

Note that neither the diffusivity nor the variance of the noise in (1.12) are homogenized in the weak coupling regime. Indeed, equations (1.10) and (1.12) are precisely the "naive guesses" for the leading order equation and its approximation that fail in our case, when the potential is not weak – it has no pre-factor ε in (1.1) unlike in (1.9).

The case when V is white in time but not in space was considered in [22]:

$$V(t,x) = \dot{W}_\psi(t,x) = \int \psi(x-y)dW(t,y).$$

Equation (1.1) is interpreted in [22] in the Itô sense:

$$\partial_t u = \frac{1}{2}\Delta u + \lambda \dot{W}_\psi(t,x)u, \quad x \in \mathbb{R}^d, d \geq 3. \tag{1.13}$$

It was shown in [22, Theorem 2.1] that there exists $\lambda_1 > 0$ so that if $\lambda \in (0, \lambda_1)$, then the rescaled solution $u_\varepsilon(t,x) = u(t/\varepsilon^2, x/\varepsilon)$ satisfies

$$\int_{\mathbb{R}^d} u_\varepsilon(t,x)g(x)dx \to \int_{\mathbb{R}^d} \bar{u}(t,x)g(x)dx$$

in probability for any $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Here, $\bar{u}$ solves the heat equation

$$\partial_t \bar{u} = \frac{1}{2}\Delta \bar{u}, \quad \bar{u}(0,x) = u_0(x),$$

with an un-modified diffusivity. The same approach as in the present paper gives in that case

Theorem 1.2. *There exists $\lambda_1 = \lambda_1(\psi)$ so that for all $0 < \lambda < \lambda_1$ we have*

$$\frac{1}{\varepsilon^{d/2-1}} \int_{\mathbb{R}^d} (u_\varepsilon(t,x) - \mathbb{E}[u_\varepsilon(t,x)])g(x)dx \Rightarrow \int_{\mathbb{R}^d} \mathscr{U}(t,x)g(x)dx, \tag{1.14}$$

as $\varepsilon \to 0$, with $\mathscr{U}$ solving

$$\partial_t \mathscr{U} = \frac{1}{2}\Delta \mathscr{U} + \lambda \nu_{\text{eff}} \bar{u} \dot{W}, \quad \mathscr{U}(0,x) = 0, \tag{1.15}$$

and

$$\nu_{\text{eff}}^2 = \int_{\mathbb{R}^d} R_\psi(x) \mathbb{E}_B \Big[\exp\Big\{ \frac{1}{2}\lambda^2 \int_0^\infty R_\psi(x+B_s)ds \Big\} \Big] dx.$$

In this case, only the variance of the noise is homogenized but not the diffusivity. Thus, both these regimes also lead to an Edwards-Wilkinson limit, with an un-modified diffusivity, and with either a "naive-guess" noise variance (the weak coupling case), or a homogenized noise variance (in the white in time case), whereas (1.3) leads to both homogenized diffusivity and variance.

We mention the very recent paper [21] that considers essentially the same setup as in the present paper. The main result of [21] implies (1.4) except that the convergence is established for the averages and not in probability, and the renormalization in the exponent is less explicit than in (1.4).

In dimensions $d = 1, 2$, similar problems have been discussed in the literature. For the random PDE (1.3), with $\lambda = \lambda(\varepsilon) \to 0$ chosen appropriately, and after a possible renormalization, the solution u_ε converges to the solution to the stochastic heat equation with multiplicative space-time white noise in $d = 1$ [8, 16, 17], and a Gaussian field in $d = 2$ within the weak-disorder regime [7, 10]. For random polymers and interacting particle systems, the partition function or the height function plays the role of the solution to certain "PDE", and their convergences to the SHE/KPZ equation have been proved in $d = 1$ e.g. in [1, 2, 3].

We comment briefly on the strategy of the proof. The Feynman-Kac representation expresses the solution to the random PDE in the form of a partition function of a directed polymer in a random environment, and the appearance of the effective diffusivity in the limit can be interpreted as the convergence of a diffusively rescaled polymer path converging to a Brownian motion in $d \geq 3$, see the results in [4, 12, 21] for the annealed continuous setting and [6, 18] for the quenched discrete setting. By a construction similar to [21], we utilize the finite range in time correlation of $V(t,x)$ to decompose the polymer path into length-one increments and establish a Markovian dynamics in the space of path increments. The latter Markov chain satisfies the Doeblin condition, greatly simplifying the analysis. The proof of the Edwards-Wilkinson limit for the fluctuations relies on the Clark-Ocone formula which expresses the random fluctuation in terms of a stochastic integral, and the fact that $u_\varepsilon(t,x)$ essentially only depends on $dW(s,x)$ locally around $s = t/\varepsilon^2$.

It may be possible to apply a PDE approach, such as using the correctors in the standard homogenization theory, to identity the limit and prove the convergence. However, the particular scaling considered here requires the construction of infinitely many correctors. Controlling these correctors becomes increasingly more difficult as their order increases. Therefore, we find the probabilistic methods more convenient to use here.

1.2 Connections to the KPZ equation

The recent work [20], which employs completely different methods, is closely related to ours. It considers the KPZ equation, related to (1.3) by a Cole-Hopf transformation. The setup and result are close but not exactly the same as here and we discuss below the connection.

Starting from (1.1), applying the centered Cole-Hopf transformation

$$h(t,x) = \lambda^{-1}\log u(t,x) - c_0 t,$$

one obtains

$$\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}\lambda|\nabla h|^2 + V(t,x) - c_0, \quad x \in \mathbb{R}^d, d \geq 3, \tag{1.16}$$

with a constant c_0. Define

$$h_\varepsilon(t,x) := \varepsilon^{-d/2+1} h(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}), \tag{1.17}$$

which satisfies

$$\partial_t h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \frac{1}{2}\lambda\varepsilon^{d/2-1}|\nabla h_\varepsilon|^2 + \varepsilon^{-d/2-1}\left(V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) - c_0\right). \tag{1.18}$$

The rescaled random potential $\varepsilon^{-d/2-1}V(t/\varepsilon^2, x/\varepsilon)$ converges to the space-time white noise, while the nonlinear term formally disappears as $\varepsilon \to 0$ in $d \geq 3$. The authors in [20, Theorem 0.1] show that if the initial condition $h_0(x)$ for the un-scaled KPZ equation (1.16) is rapidly decaying, then for λ sufficiently small, the Edwards-Wilkinson model shows up in the limit:

$$h_\varepsilon(t,x) - \mathbb{E}[h_\varepsilon(t,x)] \to \mathscr{H}(t,x), \tag{1.19}$$

in the sense of convergence of the corresponding multipoint correlation functions. Here, $\mathscr{H}$ is the solution to

$$\partial_t \mathscr{H} = \frac{1}{2}D_{\text{eff}}\Delta\mathscr{H} + \mu_{\text{eff}}\dot{W} \tag{1.20}$$

with zero initial conditions, for some $D_{\text{eff}}, \mu_{\text{eff}} > 0$. One difference from our setting is that we consider the initial conditions for the un-scaled stochastic heat equation (1.1) that vary on a macroscopic scale: $u(0,x) = u_0(\varepsilon x)$. Disregarding this difference, we try to interpret the convergence in (1.19) on the level of the stochastic heat equation, using the relation

$$u_\varepsilon(t,x) = \exp\left(\lambda\varepsilon^{d/2-1}h_\varepsilon(t,x) + \frac{\lambda c_0 t}{\varepsilon^2}\right).$$

Theorem 1.1 shows that

$$\frac{1}{\varepsilon^{d/2-1}}\int_{\mathbb{R}^d}\left(e^{\lambda\varepsilon^{d/2-1}h_\varepsilon(t,x)} - \mathbb{E}[e^{\lambda\varepsilon^{d/2-1}h_\varepsilon(t,x)}]\right)e^{\lambda c_0 t/\varepsilon^2}e^{-c_1 t/\varepsilon^2 - c_2}g(x)dx \Rightarrow \int_{\mathbb{R}^d}\mathscr{U}(t,x)g(x)dx. \tag{1.21}$$

If we use the approximation

$$e^{\lambda\varepsilon^{d/2-1}h_\varepsilon(t,x)} \approx 1 + \lambda\varepsilon^{d/2-1}h_\varepsilon(t,x),$$

and choose $\lambda c_0 = c_1$, then (1.19) and (1.21) are equivalent.

Organization of the paper. The paper is organized as follows. In Section 2 we introduce a tilted Brownian motion and use the Clark-Ocone formula to establish in Lemma 2.1 a representation for the fluctuation as a stochastic integral, and obtain in Lemma 2.2 an expression for its variance. In Section 3 we prove Theorem 1.1. Assuming the main technical result, Proposition 3.2, we show that the fluctuation does depend only on the "recent past" of the noise, and use this to prove the central limit theorem for the fluctuations. The proof of Proposition 3.2 presented in Section 5 relies on the properties of a Markov chain on the space of path increments that is constructed in Section 4. Finally, some technical results are proved in the Appendix.

Acknowledgments

O.Z. thanks Herbert Spohn for bringing [20] to his attention and motivating this study, and Chiranjib Mukherjee for making [21] available to us and for useful discussions. Y.G. was partially supported by the NSF through DMS-1613301, L.R. was partially supported by the NSF grant DMS-1613603 and ONR grant N00014-17-1-2145, and O.Z. was partially supported by a Poincare visiting professorship at Stanford, and by an Israel Science Foundation grant.

2 Preliminaries: a stochastic integral and variance representation

The goal in this section is to express the deviation of the solution of (1.1) from its mean in terms of a stochastic integral given by the Clark-Ocone formula, and present a convenient formula for its second moment. Let B be a standard Brownian motion starting from the origin that is independent from the random potential V, and let $\mathbb{E}_B$ denote the expectation with respect to B. We define the renormalization constant

$$\zeta_t := \log \mathbb{E}_B\Big[\exp\Big\{\frac{\lambda^2}{2}\int_{[0,t]^2} R(s-u, B_s - B_u)dsdu\Big\}\Big], \tag{2.1}$$

and denote by $\widehat{\mathbb{E}}_{B,t}$ the expectation with respect to a tilted Brownian path on $[0,t]$: for any integrable random variable $f(B)$ depending on $B = \{B_s : s \geq 0\}$, set

$$\widehat{\mathbb{E}}_{B,t}[f(B)] := \mathbb{E}_B\Big[f(B)\exp\Big\{\frac{\lambda^2}{2}\int_{[0,t]^2} R(s-u, B_s - B_u)dsdu - \zeta_t\Big\}\Big]. \tag{2.2}$$

For two independent tilted Brownian motions B^1, B^2 on $[0,t]$, we write

$$\widehat{\mathbb{E}}_{B,t}[f(B^1,B^2)] = \mathbb{E}_B\Big[f(B^1,B^2)\prod_{i=1}^{2}\exp\Big\{\frac{1}{2}\lambda^2\int_{[0,t]^2} R(s-u, B^i_s - B^i_u)dsdu - \zeta_t\Big\}\Big].$$

For $t > 0, x \in \mathbb{R}^d$ and every realization of the Brownian motion, we define

$$\Phi_{t,x,B}(s,y) := \int_0^t \phi(t-r-s)\psi(x + B_r - y)dr, \tag{2.3}$$

and the square-integrable martingale

$$M_{t,x,B}(r) := \int_{-\infty}^r \int_{\mathbb{R}^d} \Phi_{t,x,B}(s,y)dW(s,y), \tag{2.4}$$

with quadratic variation

$$\langle M_{t,x,B}\rangle_r = \int_{-\infty}^r \int_{\mathbb{R}^d} |\Phi_{t,x,B}(s,y)|^2 dsdy. \tag{2.5}$$

Since ϕ is supported on $[0,1]$, $\Phi_{t,x,B}(s,y) \neq 0$ only when $s \in [-1,t]$.

The following lemma expresses the random fluctuations of $u(t,x)$ in terms of a stochastic integral.

Lemma 2.1. *Let $u(t,x)$ be a solution to (1.1), then for any $t > 0$ and $x \in \mathbb{R}^d$, we have*

$$(u(t,x)-\mathbb{E}[u(t,x)])e^{-\zeta_t} = \lambda\int_{-1}^{t}\int_{\mathbb{R}^d} \widehat{\mathbb{E}}_{B,t}\Big[u(0,x+B_t)\Phi_{t,x,B}(r,y)\exp\Big\{\lambda M_{t,x,B}(r) - \frac{\lambda^2}{2}\langle M_{t,x,B}\rangle_r\Big\}\Big]dW(r,y). \tag{2.6}$$

Proof. Since $\phi(s) = 0$ for $s < 0$, $u(t,x)$ is adapted to the filtration generated by dW up to t, denoted by $\mathcal{F}_t$. By the Clark-Ocone formula, we have

$$u(t,x) - \mathbb{E}[u(t,x)] = \int_{-\infty}^{t} \mathbb{E}[D_{r,y}u(t,x)|\mathcal{F}_r]dW(r,y).$$

Here, $D_{r,y}$ denotes the Malliavin derivative. As the function $\phi(s)$ is supported in $[0,1]$, the random potential $V(t,x)$ for $t > 0$ depends only on $\dot{W}(r,y)$ for $r > -1$, and so does $u(t,x)$ for $t > 0$. Therefore, the Malliavin derivative vanishes for $r < -1$, and we have

$$u(t,x) - \mathbb{E}[u(t,x)] = \int_{-1}^{t} \mathbb{E}[D_{r,y}u(t,x)|\mathcal{F}_r]dW(r,y). \tag{2.7}$$

To compute the Malliavin derivative in (2.7), we note that by the Feynman-Kac formula, the solution can be written as

$$u(t,x) = \mathbb{E}_B\Big[u(0,x+B_t)\exp\Big\{\lambda\int_0^t V(t-s,x+B_s)ds\Big\}\Big].$$

Rewriting the exponent above as

$$\begin{aligned}\int_0^t V(t-s,x+B_s)ds &= \int_0^t \Big(\int_{\mathbb{R}^{d+1}} \phi(t-s-s')\psi(x+B_s-y')dW(s',y')\Big)ds\\ &= \int_{\mathbb{R}^{d+1}} \Phi_{t,x,B}(s',y')dW(s',y'),\end{aligned}$$

we see that the Malliavin derivative is given by

$$D_{r,y}u(t,x) = \lambda\mathbb{E}_B\Big[u(0,x+B_t)\Phi_{t,x,B}(r,y)\exp\Big\{\lambda\int_{\mathbb{R}^{d+1}}\Phi_{t,x,B}(s',y')dW(s',y')\Big\}\Big],$$

so that

$$\mathbb{E}[D_{r,y}u(t,x)|\mathcal{F}_r] = \lambda\mathbb{E}_B\Big(u(0,x+B_t)\Phi_{t,x,B}(r,y)\mathbb{E}\Big[\exp\Big\{\lambda\int_{\mathbb{R}^{d+1}}\Phi_{t,x,B}(s',y')dW(s',y')\Big\}\Big|\mathcal{F}_r\Big]\Big). \tag{2.8}$$

For the conditional expectation in the right side, we write

$$\int_{\mathbb{R}^{d+1}}\Phi_{t,x,B}(s',y')dW(s',y') = \Big(\int_{-\infty}^{r} + \int_r^{\infty}\Big)\int_{\mathbb{R}^d}\Phi_{t,x,B}(s',y')dW(s',y'),$$

which gives

$$\begin{aligned}\mathbb{E}\Big[\exp\Big\{\lambda\int_{\mathbb{R}^{d+1}}\Phi_{t,x,B}(s',y')dW(s',y')\Big\}\Big|\mathcal{F}_r\Big] &= \exp\Big\{\lambda M_{t,x,B}(r) - \frac{\lambda^2}{2}\langle M_{t,x,B}\rangle_r\Big\}\\ &\quad\times\exp\Big\{\frac{\lambda^2}{2}\int_{\mathbb{R}^{d+1}}|\Phi_{t,x,B}(s',y')|^2ds'dy'\Big\}.\end{aligned} \tag{2.9}$$

With the help of the definition (2.3) of $\Phi_{t,x,B}$, together with expression (1.2) for $R(t,x)$ and the fact that the function ψ is even, the last integral in (2.9) can be written as

$$\int_{\mathbb{R}^{d+1}}|\Phi_{t,x,B}(s',y')|^2ds'dy' = \int_{[0,t]^2} R(s-u,B_s-B_u)dsdu. \tag{2.10}$$

Finally, using (2.8), (2.9) and (2.10), as well as the definition (2.2) of the tilted measure $\widehat{\mathbb{E}}_{B,t}$, in (2.7), completes the proof of (2.6). $\square$

An expression for the variance

We now use Lemma 2.1 for the re-scaled solution $u_\varepsilon(t,x) = u(t/\varepsilon^2, x/\varepsilon)$, with $u_\varepsilon(0,x) = u_0(x)$. For any test function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} (u_\varepsilon(t,x) - \mathbb{E}[u_\varepsilon(t,x)])e^{-\zeta_{t/\varepsilon^2}} g(x)dx = \lambda \int_{-1}^{t/\varepsilon^2} \int_{\mathbb{R}^d} Z_t^\varepsilon(r,y)dW(r,y), \tag{2.11}$$

with

$$Z_t^\varepsilon(r,y) := \int_{\mathbb{R}^d} g(x)\widehat{\mathbb{E}}_{B,t/\varepsilon^2}\Big[u_0(x+\varepsilon B_{t/\varepsilon^2})\Phi^\varepsilon_{t,x,B}(r,y)\exp\Big\{\lambda M^\varepsilon_{t,x,B}(r) - \frac{\lambda^2}{2}\langle M^\varepsilon_{t,x,B}\rangle_r\Big\}\Big]dx, \tag{2.12}$$

where

$$\Phi^\varepsilon_{t,x,B} := \Phi_{t/\varepsilon^2, x/\varepsilon, B}, \quad M^\varepsilon_{t,x,B} := M_{t/\varepsilon^2, x/\varepsilon, B}.$$

Thus, the proof of the fluctuation convergence (1.5) in Theorem 1.1 reduces to the analysis of the stochastic integral

$$\frac{1}{\varepsilon^{d/2-1}} \int_{-1}^{t/\varepsilon^2} \int_{\mathbb{R}^d} Z_t^\varepsilon(r,y)dW(r,y), \tag{2.13}$$

provided that we can replace $\zeta_{t/\varepsilon^2} \mapsto c_1 t/\varepsilon^2 + c_2$ as $\varepsilon \to 0$.

We express the variance of the stochastic integral in (2.13) in a more explicit form. First, we need to introduce some notation. We define

$$R_\psi(x) = \int_{\mathbb{R}^d} \psi(x-y)\psi(y)dy, \quad R_\phi(t_1,t_2) = \int_0^\infty \phi(s-t_1)\phi(s-t_2)ds. \tag{2.14}$$

Since ψ is supported on $\{x : |x| \le 1/2\}$ and ϕ on $[0,1]$, we know that R_ψ is supported on $\{x : |x| \le 1\}$ and $R_\phi(t_1,t_2) = 0$ if $t_1 < -1$ or $t_2 < -1$. In addition, $R_\phi(t_1,t_2) = 0$ if $|t_1 - t_2| \ge 1$.

From now on, we fix $t > 0$. Given two continuous paths $B^1, B^2 \in \mathcal{C}([0, t/\varepsilon^2])$, we set

$$\Delta B^i_{u,v} = B^i_v - B^i_u.$$

For $x_1, x_2, y \in \mathbb{R}^d, s_1, s_2 \in [0,1], r \in [0,t]$ and $-1 < M_1, M_2 \le r/\varepsilon^2$, we define

$$\mathcal{I}_\varepsilon = \mathcal{I}_\varepsilon(x_1,x_2,y,s_1,s_2,r) = \prod_{i=1}^2 g(\varepsilon x_i + y - \varepsilon B^i_{\frac{t-r}{\varepsilon^2}-s_i})u_0(\varepsilon x_i + y + \varepsilon \Delta B^i_{\frac{t-r}{\varepsilon^2}-s_i, \frac{t}{\varepsilon^2}}), \tag{2.15}$$

and

$$\begin{aligned} &\mathcal{J}_\varepsilon(M_1,M_2) = \mathcal{J}_\varepsilon(M_1,M_2,x_1,x_2,s_1,s_2,r) \\ &= \lambda^2 \int_{-1}^{M_1}\int_{-1}^{M_2} R_\phi(u_1,u_2)R_\psi(x_1 - x_2 + \Delta B^1_{\frac{t-r}{\varepsilon^2}-s_1, \frac{t-r}{\varepsilon^2}+u_1} - \Delta B^2_{\frac{t-r}{\varepsilon^2}-s_2, \frac{t-r}{\varepsilon^2}+u_2})du_1du_2. \end{aligned} \tag{2.16}$$

To simplify the notation, we write $\mathcal{I}_\varepsilon$ and $\mathcal{J}_\varepsilon(M_1,M_2)$ and keep their dependence on B^i, x_i, y, s_i, r implicit.

Lemma 2.2. *For any $-1 \le t_1 < t_2 \le t - \varepsilon^2$, we have, with $d\bar{s} = ds_1ds_2$ and $d\bar{x} = dx_1dx_2$:*

$$\frac{1}{\varepsilon^{d-2}}\mathbb{E}\left[\int_{t_1/\varepsilon^2}^{t_2/\varepsilon^2}\int_{\mathbb{R}^d} |Z_t^\varepsilon(r,y)|^2dydr\right] = \int_{t_1}^{t_2}\int_{\mathbb{R}^{3d}}\int_{[0,1]^2} \widehat{\mathbb{E}}_{B,t/\varepsilon^2}\Big[\mathcal{I}_\varepsilon e^{\mathcal{J}_\varepsilon(\frac{r}{\varepsilon^2},\frac{r}{\varepsilon^2})}\Big]\prod_{i=1}^2 \phi(s_i)\psi(x_i)\ d\bar{s}d\bar{x}dydr. \tag{2.17}$$

Proof. The proof is a straightforward calculation with multiple changes of variables. We first write

$$|Z_t^\varepsilon(r,y)|^2 = \widehat{\mathbb{E}}_{B,t/\varepsilon^2}\int_{\mathbb{R}^{2d}}\prod_{i=1}^2 g(x_i)u_0(x_i+\varepsilon B^i_{t/\varepsilon^2})\Phi^\varepsilon_{t,x_i,B^i}(r,y)\exp\Big\{\lambda M^\varepsilon_{t,x_i,B^i}(r)-\frac{1}{2}\lambda^2\langle M^\varepsilon_{t,x_i,B^i}\rangle_r\Big\}d\bar{x}.$$

Taking the expectation $\mathbb{E}$ above, for each B^1, B^2 fixed we have

$$\mathbb{E}\Big[\prod_{i=1}^2 e^{\lambda M^\varepsilon_{t,x_i,B^i}(r)-\frac{1}{2}\lambda^2\langle M^\varepsilon_{t,x_i,B^i}\rangle_r}\Big] = e^{\lambda^2\langle M^\varepsilon_{t,x_1,B^1},M^\varepsilon_{t,x_2,B^2}\rangle_r},$$

with

$$\langle M^\varepsilon_{t,x_1,B^1},M^\varepsilon_{t,x_2,B^2}\rangle_r = \int_{-\infty}^r\int_{\mathbb{R}^d}\Phi^\varepsilon_{t,x_1,B^1}(s',z)\Phi^\varepsilon_{t,x_2,B^2}(s',z)dzds'.$$

Next, we write

$$\Phi^\varepsilon_{t,x_1,B^1}(r,y)\Phi^\varepsilon_{t,x_2,B^2}(r,y) = \int_{[0,t/\varepsilon^2]^2}\prod_{i=1}^2\phi(\frac{t}{\varepsilon^2}-s_i-r)\psi(\frac{x_i}{\varepsilon}+B^i_{s_i}-y)ds_1ds_2. \tag{2.18}$$

We consider the integral in x,y and change variables $x_i\mapsto \varepsilon x_i-\varepsilon B^i_{s_i}+\varepsilon y$, $y\mapsto y/\varepsilon$ to obtain

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\mathbb{R}^d}|Z_t^\varepsilon(r,y)|^2dy\Big] = \widehat{\mathbb{E}}_{B,t/\varepsilon^2}\int_{[0,t/\varepsilon^2]^2}\int_{\mathbb{R}^{3d}}\prod_{i=1}^2 g(x_i)u_0(x_i+\varepsilon B^i_{t/\varepsilon^2})\psi(\frac{x_i}{\varepsilon}+B^i_{s_i}-y)\phi(\frac{t}{\varepsilon^2}-s_i-r)\\
&\times\exp\Big\{\lambda^2\int_{-\infty}^r\int_{\mathbb{R}^d}\Phi^\varepsilon_{t,x_1,B^1}(s',z)\Phi^\varepsilon_{t,x_2,B^2}(s',z)dzds'\Big\}d\bar{x}dyd\bar{s}\\
&=\varepsilon^d\widehat{\mathbb{E}}_{B,t/\varepsilon^2}\int_{[0,t/\varepsilon^2]^2}\int_{\mathbb{R}^{3d}}\prod_{i=1}^2 g(\varepsilon x_i+y-\varepsilon B^i_{s_i})u_0(\varepsilon x_i+y+\varepsilon B^i_{t/\varepsilon^2}-\varepsilon B^i_{s_i})\psi(x_i)\phi(\frac{t}{\varepsilon^2}-s_i-r)\\
&\times\exp\Big\{\lambda^2\int_{-\infty}^r\int_{\mathbb{R}^d}\Phi^\varepsilon_{t,\varepsilon x_1-\varepsilon B^1_{s_1}+y,B^1}(s',z)\Phi^\varepsilon_{t,\varepsilon x_2-\varepsilon B^2_{s_2}+y,B^2}(s',z)dzds'\Big\}d\bar{x}dyd\bar{s}.
\end{aligned} \tag{2.19}$$

The exponent in the last line above can be written as

$$\begin{aligned}
&\int_{-\infty}^r\int_{\mathbb{R}^d}\Phi^\varepsilon_{t,\varepsilon x_1-\varepsilon B^1_{s_1}+y,B^1}(s',z)\Phi^\varepsilon_{t,\varepsilon x_2-\varepsilon B^2_{s_2}+y,B^2}(s',z)dzds'\\
&=\int_{-\infty}^r\int_{\mathbb{R}^d}\int_{[0,t/\varepsilon^2]^2}\phi(\frac{t}{\varepsilon^2}-u_1-s')\phi(\frac{t}{\varepsilon^2}-u_2-s')\psi(x_1-B^1_{s_1}+\frac{y}{\varepsilon}+B^1_{u_1}-z)\\
&\times\psi(x_2-B^2_{s_2}+\frac{y}{\varepsilon}+B^2_{u_2}-z)du_1du_2dzds'\\
&=\int_{[0,t/\varepsilon^2]^2}R_\phi(u_1+r-\frac{t}{\varepsilon^2},u_2+r-\frac{t}{\varepsilon^2})R_\psi(x_1-x_2+\Delta B^1_{s_1,u_1}-\Delta B^2_{s_2,u_2})du_1du_2,
\end{aligned} \tag{2.20}$$

with R_ϕ, R_ψ defined in (2.14). Next, we also integrate in the r-variable, with a change of variable $r\mapsto r/\varepsilon^2$, so that

$$\mathbb{E}\Big[\int_{t_1/\varepsilon^2}^{t_2/\varepsilon^2}\int_{\mathbb{R}^d}|Z_t^\varepsilon(r,y)|^2dydr\Big] = \varepsilon^{d-2}\int_{t_1}^{t_2}\int_{\mathbb{R}^{3d}}\int_{[0,t/\varepsilon^2]^2}\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[Ie^J]\ d\bar{s}d\bar{x}dydr, \tag{2.21}$$

with

$$\begin{aligned}
I &= \prod_{i=1}^2 g(\varepsilon x_i+y-\varepsilon B^i_{s_i})u_0(\varepsilon x_i+y+\varepsilon\Delta B^i_{s_i,\frac{t}{\varepsilon^2}})\phi(\frac{t-r}{\varepsilon^2}-s_i)\psi(x_i),\\
J &= \lambda^2\int_{[0,t/\varepsilon^2]^2}R_\phi(u_1-\frac{t-r}{\varepsilon^2},u_2-\frac{t-r}{\varepsilon^2})R_\psi(x_1-x_2+\Delta B^1_{s_1,u_1}-\Delta B^2_{s_2,u_2})du_1du_2.
\end{aligned} \tag{2.22}$$

As ϕ is supported on $[0,1]$, the integration domain in s_i is is $[\frac{t-r}{\varepsilon^2}-1, \frac{t-r}{\varepsilon^2}]$, because of the corresponding factor in the expression for I in (2.22). A change of variable $s_i \mapsto (t-r)/\varepsilon^2 - s_i$ turns the domain of integration in s_i into $[0,1]$, as in (2.17). It also turns I in (2.22) into expression (2.15) for $\mathcal{I}_\varepsilon$. For the integral in u_i in the expression for J in (2.22), to have $R_\phi \neq 0$, we need $u_i \geq \frac{t-r}{\varepsilon^2} - 1$, so the integration domain for u_i is $[\frac{t-r}{\varepsilon^2} - 1, \frac{t}{\varepsilon^2}]$. The change of variable $u_i \mapsto \frac{t-r}{\varepsilon^2} + u_i$ turns this into $[-1, r/\varepsilon^2]$, and J into $\mathcal{J}_\varepsilon(r/\varepsilon^2, r/\varepsilon^2)$. This completes the proof of (2.17). $\square$

Remark 2.3. The assumption $t_2 \leq t - \varepsilon^2$ in the statement of Lemma 2.2 is only made to simplify the presentation of the result. For any $t_2 \leq t$, a similar result holds – we only need to modify the integration domain for u_1, u_2 in (2.16) to $[-(t-r)/\varepsilon^2, r/\varepsilon^2]^2$ and that of s_1, s_2 in (2.17) to $[0, (t-r)/\varepsilon^2]^2$ – this only makes a difference when $t - r \leq \varepsilon^2$.

3 Proof of Theorem 1.1

We first explain how the renormalization constants c_1 and c_2 are determined.

Lemma 3.1. *There exist c_1, c_2 such that*

$$\zeta_t := \log \mathbb{E}_B \Big[e^{\frac{1}{2}\lambda^2 \int_{[0,t]^2} R(s-u, B_s - B_u) ds du} \Big] = c_1 t + c_2 + o(1), \quad \text{as } t \to \infty. \tag{3.1}$$

Lemma 3.1, which explains the choice of c_1 and c_2, is proved in Appendix A. As a consequence of the lemma, it is sufficient to prove Theorem 1.1 with the term $c_1 t/\varepsilon^2 + c_2$ in the exponent replaced by ζ_{t/ε^2}.

Convergence of the fluctuations: the outline

We first prove the central limit theorem for the centered random fluctuation in (1.5), and then the leading order homogenization result in (1.4). Fix a test function $g(x) \in C_c^\infty(\mathbb{R}^d)$, and go back to (2.11)-(2.13). Our goal will be to show that the integrand $Z_t^\varepsilon(r,y)$ depends mainly on $\dot W(s,\cdot)$ with s close to r, so that the stochastic integral is an approximate linear combination of strongly mixing processes, which should satisfy a central limit theorem. To make the "local dependence" more precise, we decompose the interval $[-1, t/\varepsilon^2]$ of integration in (2.11) into alternating subintervals of size $\varepsilon^{-\alpha}$ and $\varepsilon^{-\beta}$ with $0 < \alpha < \beta < 2$:

$$[-1, \frac{t}{\varepsilon^2}] = [-1, \varepsilon^{-\alpha}] \cup [\varepsilon^{-\alpha}, \varepsilon^{-\beta} + \varepsilon^{-\alpha}] \cup [\varepsilon^{-\beta} + \varepsilon^{-\alpha}, \varepsilon^{-\beta} + 2\varepsilon^{-\alpha}] \cup \ldots \cup [t_\varepsilon, \frac{t}{\varepsilon^2}],$$

with t_ε chosen so that $|t/\varepsilon^2 - t_\varepsilon| = O(\varepsilon^{-\beta})$.

Denote the "short" intervals of length $\varepsilon^{-\alpha}$ by $\{I_{\alpha,j}\}$ and the "long" ones of length $\varepsilon^{-\beta}$ by $\{I_{\beta,j}\}$, and set

$$I_\alpha = \bigcup_j I_{\alpha,j}, \quad I_\beta = \bigcup_j I_{\beta,j}.$$

The last piece $[t_\varepsilon, t/\varepsilon^2]$ is assigned to I_α. We will define a modification $\tilde Z_t^\varepsilon(r,y)$ of $Z_t^\varepsilon(r,y)$ for $r \in I_\beta$, so that $\tilde Z_t^\varepsilon(r,y)$ only depends on $\dot W(s,\cdot)$ with $s \in (r - \varepsilon^{-\alpha}, r]$, and thus the random variables

$$\mathcal{X}_j^\varepsilon := \frac{1}{\varepsilon^{d/2-1}} \int_{I_{\beta,j}} \int_{\mathbb{R}^d} \tilde Z_t^\varepsilon(r,y) dW(r,y) \tag{3.2}$$

are independent. To prove the central limit theorem statement (1.5) in Theorem 1.1, it suffices to show that

(1) The error induced by the modification on the "long" intervals is small (Lemma 3.3):

$$\frac{1}{\varepsilon^{d-2}}\int_{I_\beta}\int_{\mathbb{R}^d}\mathbb{E}[|Z_t^\varepsilon(r,y)-\tilde{Z}_t^\varepsilon(r,y)|^2]dydr\to 0 \text{ as } \varepsilon\to 0. \tag{3.3}$$

(2) The contribution from the "short intervals" I_α is small (Lemma 3.4):

$$\frac{1}{\varepsilon^{d-2}}\int_{I_\alpha}\int_{\mathbb{R}^d}\mathbb{E}[|Z_t^\varepsilon(r,y)|^2]dydr\to 0 \text{ as } \varepsilon\to 0. \tag{3.4}$$

(3) The sum $\sum_j \mathcal{X}_j^\varepsilon$ satisfies a central limit theorem, with variance given by (1.7) (Lemma 3.5).

The modification

We first explain how the modification is done. Recall that

$$Z_t^\varepsilon(r,y)=\int_{\mathbb{R}^d}g(x)\widehat{\mathbb{E}}_{B,t/\varepsilon^2}\Big[u_0(x+\varepsilon B_{t/\varepsilon^2})\Phi_{t,x,B}^\varepsilon(r,y)\exp\Big\{\lambda M_{t,x,B}^\varepsilon(r)-\frac{1}{2}\lambda^2\langle M_{t,x,B}^\varepsilon\rangle_r\Big\}\Big]dx$$

depends on W only through the martingale in the exponent

$$M_{t,x,B}^\varepsilon(r)=\int_{-\infty}^r\int_{\mathbb{R}^d}\left(\int_0^{t/\varepsilon^2}\phi(\frac{t}{\varepsilon^2}-s'-s)\psi(\frac{x}{\varepsilon}+B_{s'}-y)ds'\right)dW(s,y). \tag{3.5}$$

Since ϕ is supported on $[0,1]$, the integration in s' in (3.5) is only over $s'<t/\varepsilon^2-s$ (in fact, over the interval $(t/\varepsilon^2-s-1,t/\varepsilon^2-s)$), so that

$$M_{t,x,B}^\varepsilon(r)=\int_{-\infty}^r\int_{\mathbb{R}^d}\left(\int_0^{t/\varepsilon^2-s}\phi(\frac{t}{\varepsilon^2}-s'-s)\psi(\frac{x}{\varepsilon}+B_{s'}-y)ds'\right)dW(s,y). \tag{3.6}$$

We expect that, because we deal with dimensions $d\geq 3$, and therefore the transience of Brownian motion yields mixing, most of the contributions to $M_{t,x,B}^\varepsilon(r)$ come from s "macroscopically near" r, so that $0<r-s<\varepsilon^{-\alpha}$, with some $\alpha\in(0,2)$. Thus, we set

$$r_\varepsilon:=\frac{t}{\varepsilon^2}-r+\frac{1}{2\varepsilon^\alpha}, \tag{3.7}$$

and define the modification of $M_{t,x,B}^\varepsilon(r)$ on I_β as

$$\tilde{M}_{t,x,B}^\varepsilon(r):=\int_{-\infty}^r\int_{\mathbb{R}^d}\left(\int_0^{r_\varepsilon}\phi(\frac{t}{\varepsilon^2}-s'-s)\psi(\frac{x}{\varepsilon}+B_{s'}-y)ds'\right)dW(s,y),\quad r\in I_\beta. \tag{3.8}$$

Note that for $r\in I_\beta$, we have $r\geq\varepsilon^{-\alpha}$, hence $r_\varepsilon<t/\varepsilon^2$. Due to the dependence of r_ε on r, $\tilde{M}_{t,x,B}^\varepsilon$ is not a martingale. Still, with some abuse of notation, we write

$$\langle\tilde{M}_{t,x,B}^\varepsilon\rangle_r:=\int_{-\infty}^r\int_{\mathbb{R}^d}\left(\int_0^{r_\varepsilon}\phi(\frac{t}{\varepsilon^2}-s'-s)\psi(\frac{x}{\varepsilon}+B_{s'}-y)ds'\right)^2dyds.$$

Note that if $s\leq r-\varepsilon^{-\alpha}$, then

$$\frac{t}{\varepsilon^2}-s'-s\geq\frac{t}{\varepsilon^2}-r_\varepsilon-r+\varepsilon^{-\alpha}=\frac{1}{2\varepsilon^\alpha}>1,$$

so the integrand in (3.8) vanishes. Thus, $\tilde{M}^{\varepsilon}_{t,x,B}(r)$ only depends on $dW(s,\cdot)$ for $s \in (r-\varepsilon^{-\alpha}, r]$. The corresponding modification of $Z^{\varepsilon}_t(r,y)$ is

$$\tilde{Z}^{\varepsilon}_t(r,y) := \int_{\mathbb{R}^d} g(x)\widehat{\mathbb{E}}_{B,t/\varepsilon^2}\left[u_0(x+\varepsilon B_{t/\varepsilon^2})\Phi^{\varepsilon}_{t,x,B}(r,y)\exp\left\{\lambda \tilde{M}^{\varepsilon}_{t,x,B}(r) - \frac{1}{2}\lambda^2\langle \tilde{M}^{\varepsilon}_{t,x,B}\rangle_r\right\}\right]dx,$$

which also depends only on $dW(s,\cdot)$ for $s \in (r-\varepsilon^{-\alpha}, r]$, and the integrals $\{\mathcal{X}^{\varepsilon}_j\}$ defined in (3.2) are independent random variables.

Proof of the central limit theorem (1.5)

Recall (2.17), written as

$$\frac{1}{\varepsilon^{d-2}}\mathbb{E}\left[\int_{t_1/\varepsilon^2}^{t_2/\varepsilon^2}\int_{\mathbb{R}^d}|Z^{\varepsilon}_t(r,y)|^2 dydr\right] = \int_{t_1}^{t_2}\int_{\mathbb{R}^d}\mathscr{F}_{\varepsilon}(r,y,\frac{r}{\varepsilon^2},\frac{r}{\varepsilon^2})dydr. \tag{3.9}$$

Here, for $r \in [0,t]$, $y \in \mathbb{R}^d$ and $M_1, M_2 \le r/\varepsilon^2$, we have set

$$\mathscr{F}_{\varepsilon}(r,y,M_1,M_2) := \int_{\mathbb{R}^{2d}}\int_{[0,1]^2}\widehat{\mathbb{E}}_{B,t/\varepsilon^2}\left[\mathcal{I}_{\varepsilon}e^{\mathcal{J}_{\varepsilon}(M_1,M_2)}\right]\prod_{i=1}^{2}\phi(s_i)\psi(x_i)ds_1ds_1dx_1dx_2, \tag{3.10}$$

with $\mathcal{I}_{\varepsilon}$ and $\mathcal{J}_{\varepsilon}$ defined in (2.15) and (2.16), respectively.

We state the following proposition and postpone its proof to Section 5. The function $\bar{g}(t,x)$ in the proposition is the solution of the effective diffusion equation

$$\partial_t \bar{g} = \frac{1}{2}\nabla\cdot \boldsymbol{a}_{\text{eff}}\nabla\bar{g}, \quad \bar{g}(0,x) = g(x), \tag{3.11}$$

where $\boldsymbol{a}_{\text{eff}}$ is as in (4.28) below.

Proposition 3.2. *For any $r \in (0,t), y \in \mathbb{R}^d$, as $\varepsilon \to 0$ and $M_1, M_2 \to \infty$,*

$$\mathscr{F}_{\varepsilon}(r,y,M_1,M_2) \to \nu^2_{\text{eff}}|\bar{g}(t-r,y)\bar{u}(r,y)|^2, \tag{3.12}$$

where $\bar{u}, \bar{g}$ solve (1.6) and (3.11), and ν_{eff} is defined in (5.6). In addition, for any $k > 0$,

$$|\mathscr{F}_{\varepsilon}(r,y,M_1,M_2)| \le C(1\wedge |y|^{-k}) \tag{3.13}$$

for some constant $C > 0$ independent of ε, r, M_1, M_2.

Proposition 3.2 is instrumental in completing the proof of Lemmas 3.3, 3.4 and 3.5, which in turn finish the proof of (1.5). First, we show that (3.3) holds: the total error induced by the modification on the "long" intervals is small.

Lemma 3.3. *We have*

$$\frac{1}{\varepsilon^{d-2}}\int_{I_\beta}\int_{\mathbb{R}^d}\mathbb{E}\left[|Z^{\varepsilon}_t(r,y) - \tilde{Z}^{\varepsilon}_t(r,y)|^2\right]dydr \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.14}$$

Proof. By Lemma 2.2, we have

$$\frac{1}{\varepsilon^{d-2}}\int_{I_\beta}\int_{\mathbb{R}^d}\mathbb{E}[|Z^{\varepsilon}_t(r,y)|^2]dydr = \int_0^t\int_{\mathbb{R}^d}1_{\{r/\varepsilon^2\in I_\beta\}}\mathscr{F}_{\varepsilon}(r,y,\frac{r}{\varepsilon^2},\frac{r}{\varepsilon^2})dydr.$$

The same calculation as in the proof of that lemma gives

$$\frac{1}{\varepsilon^{d-2}}\int_{I_\beta}\int_{\mathbb{R}^d}\mathbb{E}[|\tilde{Z}_t^\varepsilon(r,y)|^2]dydr=\int_0^t\int_{\mathbb{R}^d}1_{\{r/\varepsilon^2\in I_\beta\}}\mathscr{F}_\varepsilon(r,y,\frac{1}{2\varepsilon^\alpha},\frac{1}{2\varepsilon^\alpha})dydr, \tag{3.15}$$

and

$$\frac{1}{\varepsilon^{d-2}}\int_{I_\beta}\int_{\mathbb{R}^d}\mathbb{E}[Z_t^\varepsilon(r,y)\tilde{Z}_t^\varepsilon(r,y)]dydr=\int_0^t\int_{\mathbb{R}^d}1_{\{r/\varepsilon^2\in I_\beta\}}\mathscr{F}_\varepsilon(r,y,\frac{r}{\varepsilon^2},\frac{1}{2\varepsilon^\alpha})dydr. \tag{3.16}$$

Indeed, the only required modification in replacing M^ε by $\tilde{M}^\varepsilon$ is to replace the upper limit t/ε^2 of integration in s in (2.18) by r_ε. This leads to the same change of the upper limit of integration in u in (2.20), and in the expression for J in (2.22). The changes of variables described below (2.22) then bring about (3.15) and (3.16). By Proposition 3.2, the proof is complete, as (3.13) allows us to apply the Lebesgue dominated convergence theorem. □

The next step is to establish (3.4): the contribution of the "short" intervals is small.

Lemma 3.4. *We have*

$$\frac{1}{\varepsilon^{d-2}}\int_{I_\alpha}\int_{\mathbb{R}^d}\mathbb{E}[|Z_t^\varepsilon(r,y)|^2]dydr\to 0\quad \text{as } \varepsilon\to 0. \tag{3.17}$$

Proof. By Lemma 2.2, we have

$$\frac{1}{\varepsilon^{d-2}}\int_{I_\alpha}\int_{\mathbb{R}^d}\mathbb{E}[|Z_t^\varepsilon(r,y)|^2]dydr=\int_0^t\int_{\mathbb{R}^d}1_{\{r/\varepsilon^2\in I_\alpha\}}\mathscr{F}_\varepsilon(r,y,\frac{r}{\varepsilon^2},\frac{r}{\varepsilon^2})dydr.$$

Note that when $t-r\le\varepsilon^2$, the expressions for $\mathcal{I}_\varepsilon,\mathcal{J}_\varepsilon$, as well as $\mathscr{F}_\varepsilon$ are slightly different, see Remark 2.3. In this case, it is easy to check that Proposition 3.2 still holds. The uniform bound (3.13), as well as the fact that

$$|\{r\in[0,t]:\ r/\varepsilon^2\in I_\alpha\}|\to 0 \text{ as } \varepsilon\to 0,$$

complete the proof. □

The last step in the proof of (1.5) is to establish the central limit theorem for the sums over the variables $\mathcal{X}_j^\varepsilon$ defined by (3.2).

Lemma 3.5. *We have*

$$\lambda\sum_j\mathcal{X}_j^\varepsilon\Rightarrow\int_{\mathbb{R}^d}\mathscr{U}(t,x)g(x)dx\quad \text{in distribution as } \varepsilon\to 0.$$

Here, $\mathscr{U}(t,x)$ is the solution of (1.7).

Proof. First, it is easy to check that the solution of (1.7) satisfies

$$\mathrm{Var}\Big[\int_{\mathbb{R}^d}\mathscr{U}(t,x)g(x)dx\Big]=\lambda^2\nu_{\mathrm{eff}}^2\int_0^t\int_{\mathbb{R}^d}|\bar{g}(t-s,x)\bar{u}(s,x)|^2dxds. \tag{3.18}$$

Let

$$s_{n,\varepsilon}^2=\lambda^2\sum_j\mathrm{Var}[\mathcal{X}_j^\varepsilon],$$

then, by the same calculation as in the proofs of Lemma 2.2 and 3.3, we have

$$s_{n,\varepsilon}^2=\lambda^2\sum_j\int_0^t\int_{\mathbb{R}^d}1_{\{r/\varepsilon^2\in I_{\beta,j}\}}\mathscr{F}_\varepsilon(r,y,\frac{1}{2\varepsilon^\alpha},\frac{1}{2\varepsilon^\alpha})dydr\to\mathrm{Var}\Big[\int_{\mathbb{R}^d}\mathscr{U}(t,x)g(x)dx\Big].$$

The last step comes from Proposition 3.2 and (3.18).

Since $\mathcal{X}_j^\varepsilon$ are independent random variables, it remains to check the Lindeberg condition which reduces in our case to: for any $\delta > 0$,

$$\sum_j \mathbb{E}[|\mathcal{X}_j^\varepsilon|^2 1_{\{|\mathcal{X}_j^\varepsilon|>\delta\}}] \to 0 \tag{3.19}$$

as $\varepsilon \to 0$. By the Cauchy-Schwarz and Chebyshev inequality, we have

$$\sum_j \mathbb{E}[|\mathcal{X}_j^\varepsilon|^2 1_{\{|\mathcal{X}_j^\varepsilon|>\delta\}}] \le \sum_j \sqrt{\mathbb{E}[|\mathcal{X}_j^\varepsilon|^4]}\sqrt{\mathbb{E}[|\mathcal{X}_j^\varepsilon|^2]/\delta^2} \le \frac{1}{\delta}\Big(\sum_j \sqrt{\mathbb{E}[|\mathcal{X}_j^\varepsilon|^4]}\Big)\Big(\sup_j \sqrt{\mathbb{E}[|\mathcal{X}_j^\varepsilon|^2]}\Big).$$

Proposition 3.2 implies that for all j we have

$$\mathbb{E}[|\mathcal{X}_j^\varepsilon|^2] = \int_0^t \int_{\mathbb{R}^d} 1_{\{r/\varepsilon^2 \in I_{\beta,j}\}} \mathscr{F}_\varepsilon(r, y, \frac{1}{2\varepsilon^\alpha}, \frac{1}{2\varepsilon^\alpha}) dy dr \lesssim \varepsilon^{2-\beta}.$$

Lemma A.3 proved in Appendix A shows that

$$\sum_j \sqrt{\mathbb{E}[|\mathcal{X}_j^\varepsilon|^4]} \lesssim 1,$$

and (3.19) follows. □

Proof of the homogenization limit (1.4)

The proof of (1.4) is now straightforward. We write

$$\int_{\mathbb{R}^d} u_\varepsilon(t,x) e^{-\zeta_{t/\varepsilon^2}} g(x) dx = \int_{\mathbb{R}^d} (u_\varepsilon(t,x) - \mathbb{E}[u_\varepsilon(t,x)]) e^{-\zeta_{t/\varepsilon^2}} g(x) dx + \int_{\mathbb{R}^d} \mathbb{E}[u_\varepsilon(t,x)] e^{-\zeta_{t/\varepsilon^2}} g(x) dx.$$

The first term goes to zero in probability by (1.5). For the second term, by Lemma 4.2 below, we have

$$\mathbb{E}[u_\varepsilon(t,x)] e^{-\zeta_{t/\varepsilon^2}} = \widehat{\mathbb{E}}_{B,t/\varepsilon^2}[u_0(x + \varepsilon B_{t/\varepsilon^2})] \to \bar{u}(t,x),$$

finishing the proof. □

The rest of the paper is devoted to the proof of Proposition 3.2, as well as the other auxiliary statements used in this section, such as the technical lemmas in Appendix A.

4 The tilted Brownian motion

The previous section relies on analyzing the expectations under the tilted measure

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[f(B)] = \mathbb{E}_B\Big[f(B) \exp\Big\{\frac{1}{2}\lambda^2 \int_{[0,t/\varepsilon^2]} R(s-u, B_s - B_u) ds du\Big\}\Big] e^{-\zeta_{t/\varepsilon^2}}.$$

The goal of this section is to construct a Markov chain taking values in $\mathcal{C}([0,1])$ so that the tilted Brownian path on $\mathcal{C}([0,t/\varepsilon^2])$ can be represented by the chain, and satisfies an invariance principle. We also analyze the intersection of two independent paths and show that the total "intersection time" has exponential tails.

4.1 Construction of the Markov chain on $\mathcal{C}([0,1])$

For any $T>0$, let

$$\Omega_T = \{\omega : \omega \in \mathcal{C}([0,T]), \omega(0)=0\}$$

be the configuration space. Denoting the tilted measure by $\widehat{\mathbb{P}}_T$, and the Wiener measure by $\mathbb{P}_T$, we have

$$\frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}_T}(\omega) = \exp\Big\{\frac{1}{2}\lambda^2 \int_{[0,T]^2} R(s-u,\omega(s)-\omega(u))dsdu - \zeta_T\Big\}. \tag{4.1}$$

Define the probability space $(\Omega, \mathcal{A}, \pi)$ with $\Omega = \Omega_1$, $\mathcal{A}$ the sigma-algebra on Ω_1, $\pi = \widehat{\mathbb{P}}_1$, and denote the expectation by $\mathbb{E}_\pi$. We will decompose the path of length T into increments of length 1 which take values in Ω. In order to consider the distribution of the path on $[t,t+1]$ for any $t>0$, we introduce a parameter $\tau \in (0,1]$, set $N=[T-\tau]$, and divide the interval $[0,T]$ into $N+2$ subintervals (τ_k,τ_{k+1}), $k=0,\dots,N+1$, with $\tau_0=0$, $\tau_1=\tau$, $\tau_{k+1}=\tau_k+1$ for $k=1,\dots,N$, and $\tau_{N+2}=T$. The increments of the path on (τ_k,τ_{k+1}) are denoted by $\{x_k\}$, with $x_0\in\Omega_\tau$, $x_k\subset\Omega$ for $k=1,\dots,N+1$, and $x_{N+1}\in\Omega_{T-\tau-N}$. Given x_k, we define ω_s, $0\le s\le T$, as

$$\omega_s = \begin{cases} x_0(s) & s\in[0,\tau], \\ \omega_{\tau+k-1} + x_k(s-\tau-k+1) & s\in[\tau+k-1,\tau+k], k=1,\dots,N, \\ \omega_{\tau+N} + x_{N+1}(s-\tau-N) & s\in[\tau+N,T], \end{cases} \tag{4.2}$$

and write

$$\{\omega_s\} = (x_0,\dots,x_{N+1}).$$

For $t>0$ and $t\notin\mathbb{Z}_{\ge1}$, we only need to choose $\tau=t-[t]$ so that $\{\omega_s : s\in[t,t+1]\}=x_k$ for some k. Write

$$\int_{[0,T]^2} R(s-u,\omega(s)-\omega(u))dsdu = \sum_{k,m=0}^{N+1} Q_{km}, \quad Q_{km} = \int_{\tau_k}^{\tau_{k+1}}\int_{\tau_m}^{\tau_{m+1}} R(s-u,\omega(s)-\omega(u))dsdu. \tag{4.3}$$

Since $R(s,\cdot)=0$ when $|s|\ge1$, we only have nearest-neighbor interactions of $(x_0,\dots,x_{N+1})$ in (4.3): $Q_{km}=0$ unless $|m-k|\le1$, and

$$\int_{[0,T]^2} R(s-u,\omega(s)-\omega(u))dsdu = \sum_{k=0}^{N+1} Q_{kk} + 2\sum_{k=0}^{N} Q_{k,k+1}. \tag{4.4}$$

For $k=1,\dots,N$ and $0\le s\le1$, we can write

$$\omega(\tau_k+s)=\omega(\tau_k)+x_k(s), \quad \omega(\tau_k+1+s)=\omega(\tau_k)+x_k(1)+x_{k+1}(s),$$

so that

$$\begin{aligned} Q_{k,k+1} &= \int_{\tau_k}^{\tau_k+1}\int_{\tau_k+1}^{\tau_k+2} R(s-u,\omega(s)-\omega(u))dsdu \\ &= \int_0^1\int_0^1 R(s+1-u,\omega(\tau_k+1+s)-\omega(\tau_k+u))dsdu \\ &= \int_0^1\int_0^1 R(s+1-u,x_k(1)+x_{k+1}(s)-x_k(u))dsdu. \end{aligned} \tag{4.5}$$

Thus, for $x,y\in\Omega$, we define the interaction term

$$I(x,y) = \lambda^2\int_0^1\int_0^1 R(s+1-u,y(s)+x(1)-x(u))dsdu. \tag{4.6}$$

The interactions between x_0, x_1 and that of x_N, x_{N+1} are defined slightly differently as

$$I_{0,1}(x_0,x_1) = \lambda^2 \int_0^\tau du \int_0^1 ds\ R(s+\tau-u, x_1(s)+x_0(\tau)-x_0(u)),$$
$$I_{N,N+1}(x_N,x_{N+1}) = \lambda^2 \int_0^1 du \int_0^{T-\tau-N} ds\ R(s+1-u, x_{N+1}(s)+x_N(1)-x_N(u)).$$

It is now straightforward to check that

$$\widehat{\mathbb{P}}_T(d\omega) \propto\ \widehat{\mathbb{P}}_\tau(dx_0)e^{I_{0,1}(x_0,x_1)}\prod_{k=1}^{N-1}\pi(dx_k)e^{I(x_k,x_{k+1})}\pi(dx_N)e^{I_{N,N+1}(x_N,x_{N+1})}\widehat{\mathbb{P}}_{T-\tau-N}(dx_{N+1}). \quad (4.7)$$

The Krein-Rutman and Doob-Krein-Rutman theorems (see Appendix to Chapter VIII of [9]) imply that there exist $\rho > 0$ and $\Psi(y)$ solving the eigenvalue problem

$$\int_\Omega e^{I(x,y)}\Psi(y)\pi(dy) = \rho\Psi(x), \quad (4.8)$$

such that ρ is the largest possible eigenvalue,

$$0 < c_1 \le \Psi(y) \le c_2 < +\infty \quad \text{for all } y \in \Omega, \quad (4.9)$$

and Ψ is the unique eigenvector associated with ρ, normalized so that

$$\int_\Omega \Psi(y)\pi(dy) = 1. \quad (4.10)$$

Such an argument was also used in [21]. The bounds on ρ and Ψ only depend on $\|I\|_{L^\infty}$. Indeed, (4.10) implies that

$$\rho = \int_{\Omega\times\Omega} e^{I(x,y)}\Psi(y)\pi(dx)\pi(dy),$$

so we have

$$e^{-\|I\|_\infty} \le \rho \le e^{\|I\|_\infty}. \quad (4.11)$$

Since

$$\Psi(x) = \frac{1}{\rho}\int_\Omega e^{I(x,y)}\Psi(y)\pi(dy),$$

we also have

$$e^{-2\|I\|_\infty} \le \Psi(x) \le e^{2\|I\|_\infty}. \quad (4.12)$$

Now we can re-write (4.7) as

$$\widehat{\mathbb{P}}_T(d\omega) \propto \widehat{\mathbb{P}}_\tau(dx_0)e^{I_{0,1}(x_0,x_1)}\Psi(x_1)\pi(dx_1)\prod_{k=1}^{N-1}\hat{\pi}(x_k,dx_{k+1})\frac{e^{I_{N,N+1}(x_N,x_{N+1})}}{\Psi(x_N)}\widehat{\mathbb{P}}_{T-\tau-N}(dx_{N+1}), \quad (4.13)$$

with the transition probability density

$$\hat{\pi}(x,dy) = \frac{e^{I(x,y)}\Psi(y)\pi(dy)}{\rho\Psi(x)}. \quad (4.14)$$

Setting

$$f_{0,1}(x_0) = \int_\Omega e^{I_{0,1}(x_0,x_1)}\Psi(x_1)\pi(dx_1), \quad f_{N,N+1}(x_N) = \int_{\Omega_{T-\tau-N}} e^{I_{N,N+1}(x_N,x_{N+1})}\widehat{\mathbb{P}}_{T-\tau-N}(dx_{N+1}),$$

and

$$\hat{\pi}_{0,1}(x_0, dx_1) = \frac{e^{I_{0,1}(x_0,x_1)}\Psi(x_1)\pi(dx_1)}{f_{0,1}(x_0)}, \quad \hat{\pi}_{N,N+1}(x_N, dx_{N+1}) = \frac{e^{I_{N,N+1}(x_N,x_{N+1})}\widehat{\mathbb{P}}_{T-\tau-N}(dx_{N+1})}{f_{N,N+1}(x_N)},$$

we obtain

$$\widehat{\mathbb{P}}_T(d\omega) \propto f_{0,1}(x_0)\widehat{\mathbb{P}}_\tau(dx_0)\left(\hat{\pi}_{0,1}(x_0, dx_1)\prod_{k=1}^{N-1}\hat{\pi}(x_k, dx_{k+1})\hat{\pi}_{N,N+1}(x_N, dx_{N+1})\right)\frac{f_{N,N+1}(x_N)}{\Psi(x_N)}. \tag{4.15}$$

Now, we construct the Markov chain X_k, with $X_0 \in \Omega_\tau$, $\{X_k\}_{k=1}^N \subset \Omega$, and $X_{N+1} \in \Omega_{T-\tau-N}$, as follows:

(1) X_0 is sampled from the (normalized) distribution $f_{0,1}(x_0)\widehat{\mathbb{P}}_\tau(dx_0)$,

(2) $(X_1, \ldots, X_{N+1})$ are sampled according to

$$\hat{\pi}_{0,1}(X_0, dx_1)\left(\prod_{k=1}^{N-1}\hat{\pi}(x_k, dx_{k+1})\right)\hat{\pi}_{N,N+1}(x_N, dx_{N+1}).$$

We construct the path B by stitching together all increments as in (4.2):

$$B = \{B_s : s \in [0, T]\} = (X_0, \ldots, X_{N+1}). \tag{4.16}$$

We use $\mathbb{E}_\pi$ to denote the expectation with respect to this Markov chain. In light of (4.15), for any $F : \Omega_T \to \mathbb{R}$, we have the relation

$$\widehat{\mathbb{E}}_{B,T}[F(B)] := \int_{\Omega_T} F(\omega)\widehat{\mathbb{P}}_T(d\omega) = \mathbb{E}_\pi\left[F(B)c_{\tau,T}\frac{f_{N,N+1}(X_N)}{\Psi(X_N)}\right]. \tag{4.17}$$

Here, $c_{\tau,T}$ is the normalization constant:

$$\frac{1}{c_{\tau,T}} := \mathbb{E}_\pi\left[\frac{f_{N,N+1}(X_N)}{\Psi(X_N)}\right].$$

Using (4.11) and (4.12), we see that the Doeblin condition is satisfied:

$$\sup_{x\in\Omega, A\subset\Omega} \hat{\pi}(x, A) \geq \gamma\pi(A) \tag{4.18}$$

for some $\gamma \in (0, 1)$ that depends only on $\|I\|_{L^\infty}$. Therefore, there exists a unique invariant measure for $\hat{\pi}$, and

$$d_{\mathrm{TV}}(X_k, \tilde{X}) \lesssim (1-\gamma)^k, \tag{4.19}$$

where $\tilde{X}$ is sampled from the invariant measure.

The two-component chain

To consider the interaction between two independent paths B^1, B^2, we construct a two component Markov chain $Z_k = (X_k, Y_k) \in \Omega^2$ by sampling X_k, Y_k independently. By the same discussion we have

$$B^1 = \{B^1_s : s \in [0, t/\varepsilon^2]\} = (X_0, \ldots, X_{N_\varepsilon+1}),$$
$$B^2 = \{B^2_s : s \in [0, t/\varepsilon^2]\} = (Y_0, \ldots, Y_{N_\varepsilon+1}),$$

where $T = t/\varepsilon^2$ and $N_\varepsilon = [t/\varepsilon^2 - \tau]$. For any $F : \Omega_{t/\varepsilon^2} \times \Omega_{t/\varepsilon^2} \to \mathbb{R}$, we have

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[F(B^1, B^2)] = \mathbb{E}_\pi[F(B^1, B^2)\mathscr{G}_\varepsilon(X_{N_\varepsilon})\mathscr{G}_\varepsilon(Y_{N_\varepsilon})]. \tag{4.20}$$

For $k = 2, \ldots, N_\varepsilon$, Z_k is sampled from $\hat{\boldsymbol{\pi}}(Z_{k-1}, dz_k)$ with

$$\hat{\boldsymbol{\pi}}(z_1, dz_2) := \hat{\pi}(x_1, dx_2)\hat{\pi}(y_1, dy_2), \quad z_i = (x_i, y_i). \tag{4.21}$$

As for the single-component chain, the Doeblin condition is satisfied for Z_k as well:

$$\sup_{z \in \Omega^2, B \subset \Omega^2} \hat{\boldsymbol{\pi}}(z, B) \ge \gamma(\pi \times \pi)(B). \tag{4.22}$$

After possibly decreasing the parameter γ, we can ensure that (4.18) and (4.22) hold with the same $\gamma \in (0, 1)$.

Writing

$$\hat{\boldsymbol{\pi}}(z_1, dz_2) = \gamma(\pi \times \pi)(dz_2) + (1 - \gamma)\frac{\hat{\boldsymbol{\pi}}(z_1, dz_2) - \gamma(\pi \times \pi)(dz_2)}{1 - \gamma}, \tag{4.23}$$

we couple the two-component chain with a sequence of i.i.d. Bernoulli random variables η_k, $k \in \mathbb{N}$, with the parameter γ: for $k = 2, \ldots, N_\varepsilon$, if $\eta_k = 1$, we sample Z_k from $(\pi \times \pi)(dz)$, and if $\eta_k = 0$, we sample Z_k from

$$\frac{\hat{\boldsymbol{\pi}}(Z_{k-1}, dz) - \gamma(\pi \times \pi)(dz)}{1 - \gamma},$$

which is possible because of the Doeblin condition (4.22). The same coupling works for the one-component chain, of course, with the help of (4.18). We enlarge the probability space so that η_k are also defined on $(\Omega, \mathcal{A}, \pi)$.

4.2 The invariance principle for the tilted Brownian path

We will use here the re-scaled version of (4.17): set $T = t/\varepsilon^2$, $N_\varepsilon = [t/\varepsilon^2 - \tau]$, and for any $F : \Omega_{t/\varepsilon^2} \to \mathbb{R}$ write

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[F(B)] = \mathbb{E}_\pi\left[F(B)\mathscr{G}_\varepsilon(X_{N_\varepsilon})\right], \tag{4.24}$$

with

$$\mathscr{G}_\varepsilon(X_{N_\varepsilon}) := c_{\tau,t/\varepsilon^2}\frac{f_{N_\varepsilon,N_\varepsilon+1}(X_{N_\varepsilon})}{\Psi(X_{N_\varepsilon})}. \tag{4.25}$$

To simplify the notation, we kept the dependence on τ implicit in (4.24). Since both $f_{N_\varepsilon,N_\varepsilon+1}$ and Ψ are bounded from above and below, and $\mathbb{E}_\pi[\mathscr{G}_\varepsilon(X_{N_\varepsilon})] = 1$, we know that $\mathscr{G}_\varepsilon$ is uniformly bounded in ε.

We fix $\tau = 1$ in this section, so that $N_\varepsilon = [t/\varepsilon^2] - 1$,

$$B = \{B_s : s \in [0, t/\varepsilon^2]\} = (X_0, \ldots, X_{N_\varepsilon+1}). \tag{4.26}$$

Here, X_k is the increment of B on $[k, k+1]$ for $k = 0, \ldots, N_\varepsilon$, and $X_{N_\varepsilon+1}$ is the increment on the last interval $[[T], T]$. In this case, X_0 is sampled from $\Psi(x_0)\pi(x_0)$, X_k is sampled from $\hat{\pi}(X_{k-1}, dx_k)$ for $k = 1, \ldots, N_\varepsilon$, and $X_{N_\varepsilon+1}$ is sampled from $\hat{\pi}_{N_\varepsilon,N_\varepsilon+1}(X_{N_\varepsilon}, dx_{N_\varepsilon+1})$.

For $k = 1, \ldots, N_\varepsilon$, we take independent Bernoulli random variables η_k with parameter $\gamma \in (0, 1)$ as in the Doeblin condition, and consider the regeneration times

$$T_0 = 0, \quad T_i = \inf\{j > T_{i-1} : \eta_j = 1\}, \quad i \ge 1. \tag{4.27}$$

We define the path increment in each regeneration block as

$$\mathbf{X}_j := \sum_{k=T_j}^{T_{j+1}-1} X_k(1), \quad j = 0, 1, \ldots$$

Proposition 4.1. *For any $t > 0$,*

$$\varepsilon B_{s/\varepsilon^2} \Rightarrow W_s$$

in $\mathcal{C}([0,t])$ in $(\Omega, \mathcal{A}, \pi)$, where W_s is a Brownian motion with the covariance matrix $\boldsymbol{a}_{\text{eff}}$:

$$\boldsymbol{a}_{\text{eff}} := \gamma \mathbb{E}_\pi[\mathbf{X}_1 \mathbf{X}_1^t]. \tag{4.28}$$

It is a straightforward computation to check that $\boldsymbol{a}_{\text{eff}}$ in (4.28) does not depend on γ. In fact, the right side of (4.28) can be written as

$$\boldsymbol{a}_{\text{eff}} = \lim_{n \to +\infty} \frac{1}{n} \mathbb{E}_\pi \left[(X_1(1) + \cdots + X_n(1))(X_1(1) + \cdots + X_n(1))^t \right]. \tag{4.29}$$

Proof. We show in Lemma A.1 that $\mathbf{X}_1$ has zero mean and exponential tails, and further that the random variables

$$Z_i = \max_{s \in [T_i, T_{i+1}]} |B_s - B_{T_i}|, \quad i \geq 0,$$

have exponential tails, and for $i \geq 1$ they are i.i.d. From the first fact, one obtains by Donsker's invariance principle that

$$Y_n(t) := n^{-1/2} \sum_{j=1}^{[nt]} \mathbf{X}_i$$

converges weakly to a Brownian motion with diffusivity $\mathbb{E}_\pi[\mathbf{X}_1 \mathbf{X}_1^t]$. On the other hand, T_n/n converges a.s. to $1/\gamma$, on account of the independence of the increments $T_i - T_{i-1}$ and the fact that they have mean $1/\gamma$ and are geometrically distributed. Setting

$$N_t^\varepsilon = \max\{i : T_i < t/\varepsilon^2\} - 1,$$

we deduce from [5, Theorem 14.4] that the process

$$\varepsilon \sum_{i=1}^{N_t^\varepsilon} \mathbf{X}_i$$

converges in distribution to a Brownian motion with the diffusivity $\boldsymbol{a}_{\text{eff}}$ given by (4.28). On the other hand, we have

$$\max_{s \leq t} |\varepsilon B_{s/\varepsilon^2} - \varepsilon \sum_{i=1}^{N_s^\varepsilon} \mathbf{X}_i| \lesssim \varepsilon \max_{i=1}^{N_t^\varepsilon + 1} |Z_i| \to_{\varepsilon \to 0} 0, \quad a.s.,$$

because of the exponential tails of the Z_i. This completes the proof. $\square$

With the invariance principle, we can show the convergence of the average of the solution.

Lemma 4.2. *We have $\mathbb{E}[u_\varepsilon(t,x)] e^{-\zeta_{t/\varepsilon^2}} \to \bar{u}(t,x)$ as $\varepsilon \to 0$.*

Proof. We first show that

$$\mathbb{E}_\pi[|\varepsilon B_{t_2/\varepsilon^2} - \varepsilon B_{t_1/\varepsilon^2}|^2] \le C(t_2 - t_1) \tag{4.30}$$

with a constant $C > 0$ independent of $0 \le t_1 < t_2 \le t$ and $\varepsilon > 0$. Define

$$K_{1,\varepsilon} = \min\{i : \frac{t_1}{\varepsilon^2} < T_i < \frac{t_2}{\varepsilon^2}\}, \quad K_{2,\varepsilon} = \max\{i : \frac{t_1}{\varepsilon^2} < T_i < \frac{t_2}{\varepsilon^2}\},$$

and if there is no regeneration time in $(t_1/\varepsilon^2, t_2/\varepsilon^2)$, we define $T_{K_{1,\varepsilon}} = t_1/\varepsilon^2$ and $T_{K_{2,\varepsilon}} = t_2/\varepsilon^2$. We decompose

$$\varepsilon B_{t_2/\varepsilon^2} - \varepsilon B_{t_1/\varepsilon^2} = (\varepsilon B_{T_{K_{1,\varepsilon}}} - \varepsilon B_{t_1/\varepsilon^2}) + (\varepsilon B_{T_{K_{2,\varepsilon}}} - \varepsilon B_{T_{K_{1,\varepsilon}}}) + (\varepsilon B_{t_2/\varepsilon^2} - \varepsilon B_{T_{K_{2,\varepsilon}}}) := I_1 + I_2 + I_3.$$

For I_2, we write

$$I_2 = \varepsilon \sum_{j=K_{1,\varepsilon}}^{K_{2,\varepsilon}-1} \mathbf{X}_j,$$

and, conditioning on all the regeneration times, denoted by $\{T_i\}$, we obtain

$$\mathbb{E}_\pi[|I_2|^2 \mid \{T_i\}] = \varepsilon^2 \sum_{j=K_{1,\varepsilon}}^{K_{2,\varepsilon}-1} \mathbb{E}_\pi[\mathbf{X}_j^2|\{T_i\}].$$

Here, we used the fact that $\mathbf{X}_j$ are independent with zero mean conditioning on $\{T_i\}$. By Lemma A.1, we have

$$\mathbb{E}_\pi[\mathbf{X}_j^2|\{T_i\}] \lesssim (T_{j+1} - T_j)^2.$$

As $K_{2,\varepsilon} - K_{1,\varepsilon} \le \frac{t_2 - t_1}{\varepsilon^2}$, it follows that

$$\mathbb{E}_\pi[|I_2|^2] \lesssim \varepsilon^2 \mathbb{E}_\pi \sum_{j=K_{1,\varepsilon}}^{K_{2,\varepsilon}-1} (T_{j+1} - T_j)^2 \le C\varepsilon^2 \frac{t_2 - t_1}{\varepsilon^2} = C(t_2 - t_1).$$

Estimating the terms I_1 and I_3 is also straightforward using Lemma A.1, finishing the proof of (4.30).

Next, note that by (4.24), we have

$$\begin{aligned}\mathbb{E}[u_\varepsilon(t,x)]e^{-\zeta_{t/\varepsilon^2}} =&\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[u_0(x+\varepsilon B_{t/\varepsilon^2})] = \mathbb{E}_\pi[u_0(x+\varepsilon B_{t/\varepsilon^2})\mathscr{G}_\varepsilon(X_{N_\varepsilon})] \\ =&\mathbb{E}_\pi[u_0((x+\varepsilon B_{t/\varepsilon^2-1/\varepsilon}) + \varepsilon(B_{t/\varepsilon^2} - B_{t/\varepsilon^2-1/\varepsilon}))\mathscr{G}_\varepsilon(X_{N_\varepsilon})].\end{aligned}$$

Using (4.30), it suffices to consider

$$\mathbb{E}_\pi[u_0(x+\varepsilon B_{t/\varepsilon^2-1/\varepsilon})\mathscr{G}_\varepsilon(X_{N_\varepsilon})].$$

We apply Lemma A.2 and Proposition 4.1 to see that

$$\mathbb{E}_\pi[u_0(x+\varepsilon B_{t/\varepsilon^2-1/\varepsilon})\mathscr{G}_\varepsilon(X_{N_\varepsilon})] - \mathbb{E}_\pi[u_0(x+\varepsilon B_{t/\varepsilon^2-1/\varepsilon})] \to 0,$$

and

$$\mathbb{E}_\pi[u_0(x+\varepsilon B_{t/\varepsilon^2-1/\varepsilon})] \to \bar{u}(t,x),$$

which completes the proof. □

4.3 Intersection of independent paths

The previous section shows that the tilted Brownian path behaves like a Brownian motion with an effective diffusivity, and this has been used to prove the convergence of

$$\int_{\mathbb{R}^d} \mathbb{E}[u_\varepsilon(t,x)]e^{-\zeta_{t/\varepsilon^2}} g(x)dx.$$

To control the variance of

$$\int_{\mathbb{R}^d} u_\varepsilon(t,x)e^{-\zeta_{t/\varepsilon^2}} g(x)dx,$$

it is necessary to consider two independent tilted Brownian paths. We will show that the two paths can not intersect too much – this is the goal of this section and is only true in dimensions $d \geq 3$.

For the sake of simplicity of presentation, we consider here only a homogeneous chain, assuming that t/ε^2 is an integer, to avoid dealing with the last step of the chain that has a different law. A modification for a general t is straightforward. Given any $Z_0 = (X_0, Y_0) \in \Omega^2$, we generate the chain $Z_k = (X_k, Y_k)$ according to the transition kernel $\hat{\boldsymbol{\pi}}$ defined in (4.21). The two components X_k and Y_k generate two paths, that we denote by $\omega_{X_0}, \omega_{Y_0} \in \mathcal{C}([0,\infty))$, via (4.2). We recall that the regeneration times are defined as

$$T_0 = 0, \quad T_i = \inf\{j > T_{i-1} : \eta_j = 1\}, \quad i \geq 1, \tag{4.31}$$

where η_j are i.i.d Bernoulli random variables with parameter $\gamma \in (0,1)$.

Throughout the section, X_0, Y_0 are fixed, so we simply write $\pi[\cdot|X_0,Y_0] = \pi[\cdot]$. Define

$$\ell(x,y,X_0,Y_0) = \int_0^\infty 1_{\{|x+\omega_{X_0}(s)-y-\omega_{Y_0}(s)|\leq 1\}} ds$$

as the total "nearby time" of ω_{X_0} and ω_{Y_0}. We have the following result.

Proposition 4.3. *In $d \geq 3$, there exist constants $C_1, C_2 > 0$ such that*

$$\sup_{x,y\in\mathbb{R}^d} \sup_{X_0,Y_0\in\Omega} \pi[\ell(x,y,X_0,Y_0) > t] \leq C_1 e^{-C_2 t}. \tag{4.32}$$

As a consequence, if $\lambda < C_2$, then

$$\sup_{x,y\in\mathbb{R}^d} \sup_{X_0,Y_0\in\Omega} \mathbb{E}_\pi[e^{\lambda \ell(x,y,X_0,Y_0)}] < \infty.$$

Proof. The proof is divided into two steps.

Step 1. We show that there exists $K > 0$ such that

$$\pi[\ell(x,y,X_0,Y_0) > K] < \frac{1}{2} \tag{4.33}$$

for all x, y, X_0, Y_0. Since

$$\pi[\ell(x,y,X_0,Y_0) > K] \leq \pi[\ell(x,y,X_0,Y_0) > T_N] + \pi[T_N > K],$$

with T_N the $N-$th regeneration time, we only need to show

$$\pi[\ell(x,y,X_0,Y_0) > T_N] < \frac{1}{4} \tag{4.34}$$

for some N independent of x, y, X_0, Y_0, and choose K so large that $\pi[T_N > K] < 1/4$. To this end, it suffices to show that

$$\pi[E_N] < \frac{1}{4}, \quad \text{where} \quad E_N = \{ \min_{s \geq T_N} |x + \omega_{X_0}(s) - y - \omega_{Y_0}(s)| \leq 1 \}. \tag{4.35}$$

Recall that

$$\omega_{X_0}(T_k) - \omega_{Y_0}(T_k) = \sum_{j=0}^{k-1} (\mathbf{X}_j - \mathbf{Y}_j), \quad \mathbf{X}_j - \mathbf{Y}_j = \sum_{i=T_j}^{T_{j+1}-1} [X_i(1) - Y_i(1)], \qquad k \geq 1.$$

By the regeneration structure, $\mathbf{X}_j - \mathbf{Y}_j$ are i.i.d. random variables and are also independent of $\mathbf{X}_0 - \mathbf{Y}_0$. For any $\alpha > 0$, define

$$A_k = \{ |x + \omega_{X_0}(T_k) - y - \omega_{Y_0}(T_k)| \leq k^\alpha \}, \quad A(N) = \bigcup_{k \geq N} A_k,$$

and write

$$\pi[E_N] \leq \pi[A(N)] + \pi[E_N \cap A(N)^c] \leq \sum_{k=N}^{\infty} \pi[A_k] + \pi[E_N \cap A(N)^c].$$

By the local limit theorem in [24, Theorem on p. 1], we have

$$\pi[A_k] \leq C \frac{k^{\alpha d}}{k^{d/2}} = \frac{C}{k^{(\frac{1}{2} - \alpha) d}} \tag{4.36}$$

for some constant C independent of x, y, X_0, Y_0. Thus, we can choose $\alpha < 1/2 - 1/d$ (in $d \geq 3$) and N so large that

$$\sum_{k=N}^{\infty} \pi[A_k] < \frac{1}{8}.$$

On the other hand, we have

$$\pi[E_N \cap A(N)^c] \leq \sum_{k \geq N} \pi[B_k],$$

with

$$B_k := A_k^c \cap \{ \min_{s \in [T_k, T_{k+1}]} |x + \omega_{X_0}(s) - y - \omega_{Y_0}(s)| \leq 1 \},$$

and $B_k \subset B_{k,X} \cup B_{k,Y}$ with

$$B_{k,X} = \left\{ \sum_{i=T_k}^{T_{k+1}-1} \max_{s \in [0,1]} |X_i(s)| > \frac{k^\alpha}{3} \right\}, \quad B_{k,Y} = \left\{ \sum_{i=T_k}^{T_{k+1}-1} \max_{s \in [0,1]} |Y_i(s)| > \frac{k^\alpha}{3} \right\}.$$

By Lemma A.1, the random variable

$$\sum_{i=T_k}^{T_{k+1}-1} \max_{s \in [0,1]} |X_i(s)|$$

has an exponential tail, which implies that

$$\pi[E_N \cap A(N)^c] \leq \sum_{k \geq N} e^{-Ck^\alpha} < \frac{1}{8}$$

when N is large. The proof of (4.34) is complete.

Step 2. We define a sequence of stopping times as follows: $\tau_0 = 0$ and

$$\tau_k = \min\Big\{n > \tau_{k-1} : \int_{\tau_{k-1}}^{n+1} 1_{\{|x+\omega_{X_0}(s)-y-\omega_{Y_0}(s)|\le 1\}} ds > K\Big\}, \quad k \ge 1,$$

with K chosen as in step 1. Let $n = [t/K]$, and apply (4.33) to obtain

$$\pi[\ell(x,y,X_0,Y_0) > t] \le \pi[\tau_n < \infty] = \pi[\tau_n < \infty | \tau_1 < \infty]\pi[\tau_1 < \infty] \le \frac{1}{2}\pi[\tau_n < \infty|\tau_1 < \infty].$$

We consider

$$\pi[\tau_2 < \infty | X_{\tau_1}, Y_{\tau_1}] = \pi\Big[\int_{\tau_1}^{\infty} 1_{\{|x+\omega_{X_0}(s)-y-\omega_{Y_0}(s)|\le 1\}} ds > K | X_{\tau_1}, Y_{\tau_1}\Big],$$

and write for $s \ge \tau_1$:

$$x + \omega_{X_0}(s) - y - \omega_{Y_0}(s) = x + \omega_{X_0}(\tau_1) + [\omega_{X_0}(s) - \omega_{X_0}(\tau_1)] - y - \omega_{Y_0}(\tau_1) - [\omega_{Y_0}(s) - \omega_{Y_0}(\tau_1)].$$

Conditioning on X_{τ_1}, Y_{τ_1} gives

$$(\omega_{X_0}(\tau_1 + \cdot) - \omega_{X_0}(\tau_1), \omega_{Y_0}(\tau_1 + \cdot) - \omega_{Y_0}(\tau_1)) \overset{\text{law}}{=} (\tilde\omega_{X_{\tau_1}}(\cdot), \tilde\omega_{Y_{\tau_1}}(\cdot)),$$

where $\tilde\omega$ is independent of ω. Hence, we may apply (4.33) again to get

$$\begin{aligned}
&\pi\Big[\int_{\tau_1}^{\infty} 1_{\{|x+\omega_{X_0}(s)-y-\omega_{Y_0}(s)|\le 1\}} ds > K | X_{\tau_1}, Y_{\tau_1}\Big] \\
&= \pi\Big[\int_{0}^{\infty} 1_{\{x+\omega_{X_0}(\tau_1)+\tilde\omega_{X_{\tau_1}}(s)-y-\omega_{Y_0}(\tau_1)-\tilde\omega_{Y_{\tau_1}}(s)\}} ds > K | X_{\tau_1}, Y_{\tau_1}\Big] < \frac{1}{2}
\end{aligned}$$

uniformly in $x, y, X_{\tau_1}, Y_{\tau_1}$. Iterating the same argument gives

$$\pi[\ell(x,y,X_0,Y_0) > t] \le \Big(\frac{1}{2}\Big)^2 \pi[\tau_n < \infty | \tau_2 < \infty] \le \ldots \le \Big(\frac{1}{2}\Big)^n,$$

which completes the proof. $\square$

Corollary 4.4. *In $d \ge 3$, there exists λ_0 only depending on ϕ, ψ such that for $\lambda < \lambda_0$, we have*

$$\sup_{x,y\in\mathbb{R}^d} \sup_{(X_0,Y_0)\in\Omega^2} \mathbb{E}_\pi\Big[\exp\Big\{\lambda \int_0^\infty \int_0^\infty R_\phi(u_1,u_2) R_\psi(x - y + \omega_{X_0}(u_1) - \omega_{Y_0}(u_2)) du_1 du_2\Big\}\Big] < \infty.$$

Proof. As $R_\phi(u_1,u_2) = 0$ if $|u_1 - u_2| > 1$ and R_ψ is supported on $\{x : |x| \le 1\}$, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty R_\phi(u_1,u_2) R_\psi(x - y + \omega_{X_0}(u_1) - \omega_{Y_0}(u_2)) du_1 du_2 \\
&\lesssim \int_0^\infty \int_0^\infty 1_{\{|u_1-u_2|\le 1\}} 1_{\{|x-y+\omega_{X_0}(u_1)-\omega_{Y_0}(u_2)|\le 1\}} du_1 du_2.
\end{aligned}$$

Consider the region $u_2 > u_1$. After a change of variable and an application of Jensen's inequality, we have

$$\begin{aligned}
&\exp\Big\{\int_0^1 \Big(\int_0^\infty 1_{\{|x-y+\omega_{X_0}(u_1)-\omega_{Y_0}(u_1+u_2)|\le 1\}} du_1\Big) du_2\Big\} \\
&\le \int_0^1 \Big(\exp\Big\{\int_0^\infty 1_{\{|x-y+\omega_{X_0}(u_1)-\omega_{Y_0}(u_1+u_2)|\le 1\}} du_1\Big\}\Big) du_2.
\end{aligned}$$

It suffices to show that there exists $\lambda_0 > 0$ so that for $\lambda \in (0, \lambda_0)$ we have

$$\mathbb{E}_\pi[e^{\lambda \ell(u_2,x,y,X_0,Y_0)}] \text{ is bounded uniformly in } u_2 \in [0,1], x, y \in \mathbb{R}^d, X_0, Y_0 \in \Omega, \tag{4.37}$$

where

$$\ell(u_2, x, y, X_0, Y_0) = \int_0^\infty 1_{\{|x-y+\omega_{X_0}(u_1)-\omega_{Y_0}(u_1+u_2)|\le 1\}} du_1$$

is the total "nearby" time of ω_{X_0} and the "shifted" ω_{Y_0}. We can repeat the proof of (4.32) verbatim to establish an identical estimate for $\ell(u_2, x, y, X_0, Y_0)$, from which (4.37) follows immediately, for $0 < \lambda < C_2$. This completes the proof. $\square$

5 Proof of Proposition 3.2

Before proving Proposition 3.2, we discuss some heuristics of the convergence of $\mathscr{F}_\varepsilon(r, y, M_1, M_2)$ as $\varepsilon \to 0$ and $M_1, M_2 \to \infty$. Recall that

$$\mathscr{F}_\varepsilon(r, y, M_1, M_2) = \int_{\mathbb{R}^{2d}} \int_{[0,1]^2} \widehat{\mathbb{E}}_{B,t/\varepsilon^2}\Big[\mathcal{I}_\varepsilon e^{\mathcal{J}_\varepsilon(M_1,M_2)}\Big] \prod_{i=1}^2 \phi(s_i)\psi(x_i) ds_1 ds_2 dx_1 dx_2, \tag{5.1}$$

with

$$\mathcal{I}_\varepsilon = \mathcal{I}_\varepsilon(x_1, x_2, y, s_1, s_2, r) = \prod_{i=1}^2 g(\varepsilon x_i + y - \varepsilon B^i_{(t-r)/\varepsilon^2 - s_i}) u_0(\varepsilon x_i + y + \varepsilon B^i_{t/\varepsilon^2} - \varepsilon B^i_{(t-r)/\varepsilon^2 - s_i}). \tag{5.2}$$

As shown in Proposition 4.1, the diffusively rescaled Brownian path $\varepsilon B^i_{s/\varepsilon^2}$ behaves like W^i_s, so we expect that

$$\mathcal{I}_\varepsilon \Rightarrow \prod_{i=1}^2 g(y - W^i_{t-r}) u_0(y + W^i_t - W^i_{t-r}). \tag{5.3}$$

in distribution. The exponential factor in (5.1) is

$$\begin{aligned} &\mathcal{J}_\varepsilon(M_1, M_2) = \mathcal{J}_\varepsilon(M_1, M_2, x_1, x_2, s_1, s_2, r) \\ &= \lambda^2 \int_{-1}^{M_1} \int_{-1}^{M_2} R_\phi(u_1, u_2) R_\psi(x_1 - x_2 + B^1_{\frac{t-r}{\varepsilon^2}+u_1} - B^1_{\frac{t-r}{\varepsilon^2}-s_1} - B^2_{\frac{t-r}{\varepsilon^2}+u_2} + B^2_{\frac{t-r}{\varepsilon^2}-s_2}) du_1 du_2, \end{aligned} \tag{5.4}$$

and measures the "nearby" time of two independent paths. Since R_ψ is compactly supported, most of the contribution in (5.4) comes from $u_1, u_2 \in [-1, M]$, with some large M fixed, as indicated by Corollary 4.4. Thus, $\mathcal{J}_\varepsilon$ depends only on the microscopic increments of $B^{1,2}$ around $(t-r)/\varepsilon^2$ that are asymptotically decorrelated from both $W^{1,2}_{t-r}$ and $W^{1,2}_t$. Thus, $\mathcal{J}_\varepsilon$ should be asymptotically independent from $\mathcal{I}_\varepsilon$, and the limit of $\mathcal{J}_\varepsilon$ determines the effective variance ν^2_{eff} in (3.12).

The goal of this section is to make the above heuristics precise. The proof is in two steps. We first show the convergence of $\mathscr{F}_\varepsilon$ for a fixed $r \in (0, t), y \in \mathbb{R}^d$. Then, we prove a uniform bound on $\mathscr{F}_\varepsilon$.

The expression (5.4) shows that $\mathcal{J}_\varepsilon$ depends on the trajectories of B^1, B^2 starting from $(t-r)/\varepsilon^2 - 1$, and for a fixed $r \in (0, t), \varepsilon > 0$, we choose

$$\tau = \frac{t-r}{\varepsilon^2} - \Big[\frac{t-r}{\varepsilon^2}\Big].$$

Recall that $T = t/\varepsilon^2$, $N_\varepsilon = [t/\varepsilon^2 - \tau]$, and

$$B^1 = \{B^1_s : s \in [0, t/\varepsilon^2]\} = (X_0, \dots, X_{N_\varepsilon+1}),$$
$$B^2 = \{B^2_s : s \in [0, t/\varepsilon^2]\} = (Y_0, \dots, Y_{N_\varepsilon+1}).$$

It is clear that $\mathscr{J}_\varepsilon$ is determined by the increments of B^1 and B^2 for times larger than $(t-r)/\varepsilon^2 - 2$, that is, for $n > N_{\varepsilon,r}$, with

$$N_{\varepsilon,r} = \Big[\frac{t-r}{\varepsilon^2}\Big] - 1.$$

To simplify the notation, we define

$$\tilde{X}_\varepsilon = X_{N_{\varepsilon,r}}, \quad \tilde{Y}_\varepsilon = Y_{N_{\varepsilon,r}}.$$

We also note that by (4.20), we have

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[\mathcal{I}_\varepsilon e^{\mathscr{J}_\varepsilon}] = \mathbb{E}_\pi[\mathcal{I}_\varepsilon e^{\mathscr{J}_\varepsilon} \mathscr{G}_\varepsilon(X_{N_\varepsilon}) \mathscr{G}_\varepsilon(Y_{N_\varepsilon})].$$

5.1 Pointwise convergence

We first explain how the effective variance ν_{eff} is defined. For any "starting pieces" $X_0, Y_0 \in \Omega$, and starting points $x_1, x_2 \in \mathbb{R}^d$, as well as $M_1, M_2 > 0$, and $s_1, s_2 \in [0,1]$, we define

$$\begin{aligned}
&\mathscr{H}_{M_1,M_2}(X_0, Y_0, x_1, x_2, s_1, s_2) = \mathbb{E}_\pi\Big[\exp\Big\{\lambda^2 \int_{-1}^{M_1}\int_{-1}^{M_2} R_\phi(u_1, u_2) \\
&\times R_\psi(x_1 - x_2 + \omega_{X_0}(2+u_1) - \omega_{X_0}(2-s_1) - \omega_{Y_0}(2+u_2) + \omega_{Y_0}(2-s_2)) du_1 du_2\Big\} \mid X_0, Y_0\Big].
\end{aligned} \tag{5.5}$$

The effective variance is then

$$\nu_{\text{eff}}^2 = \int_{\mathbb{R}^{2d}}\int_{[0,1]^2} \mathbb{E}_\pi[\mathscr{H}_{\infty,\infty}(\tilde{X}, \tilde{Y}, x_1, x_2, s_1, s_2)] \prod_{i=1}^2 \phi(s_i)\psi(x_i) ds_1 ds_2 dx_1 dx_2, \tag{5.6}$$

with $\tilde{X}$ and $\tilde{Y}$ sampled, independently, from the invariant measure of $\hat{\pi}$.

In the following, we fix $x_1, x_2 \in \mathbb{R}^d$ and $s_1, s_2 \in [0,1]$, and simply write $\mathscr{H}_{M_1,M_2}(X_0, Y_0)$. The next two lemmas show the convergence

$$\mathscr{F}_\varepsilon(r, y, M_1, M_2) \to \nu_{\text{eff}}^2 |\bar{g}(t-r, y)\bar{u}(r, y)|^2, \quad \text{as } \varepsilon \to 0 \text{ and } M_1, M_2 \to \infty, \tag{5.7}$$

for fixed $r \in (0,t), y \in \mathbb{R}^d$.

Lemma 5.1. *There exists $C > 0$ independent of $\varepsilon, M_1, M_2, x_1, x_2, s_1, s_2$ such that*

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[\mathcal{I}_\varepsilon e^{\mathscr{J}_\varepsilon(M_1,M_2)}] \le C. \tag{5.8}$$

Lemma 5.2. *As $\varepsilon \to 0$ and $M_1, M_2 \to \infty$, we have*

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[\mathcal{I}_\varepsilon e^{\mathscr{J}_\varepsilon(M_1,M_2)}] \to \mathbb{E}_\pi[\mathscr{H}_{\infty,\infty}(\tilde{X}, \tilde{Y})] |\bar{g}(t-r, y)\bar{u}(r, y)|^2.$$

Proof of Lemma 5.1. Since $\mathcal{I}_\varepsilon$ and $\mathscr{G}_\varepsilon$ are both bounded, we have

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[\mathcal{I}_\varepsilon e^{\mathscr{J}_\varepsilon}] \lesssim \mathbb{E}_\pi[e^{\mathscr{J}_\varepsilon}].$$

We first condition on $\tilde{X}_\varepsilon, \tilde{Y}_\varepsilon$ and assume that

$$\frac{t-r}{\varepsilon^2} + M_i \le \tau + N_\varepsilon.$$

In this case, $\mathcal{J}_\varepsilon$ is not related to $X_{N_\varepsilon+1}, Y_{N_\varepsilon+1}$ (which are sampled differently), and we can replace B^1, B^2 with $\omega_{\tilde{X}_\varepsilon}, \omega_{\tilde{Y}_\varepsilon}$, that is, the homogeneous chains started from $\tilde{X}_\varepsilon, \tilde{Y}_\varepsilon$, respectively, with the transition kernel $\hat{\pi}$. It is easy to check that in this case

$$\mathbb{E}_\pi[e^{\mathcal{J}_\varepsilon(M_1,M_2)}|\tilde{X}_\varepsilon, \tilde{Y}_\varepsilon] = \mathscr{H}_{M_1,M_2}(\tilde{X}_\varepsilon, \tilde{Y}_\varepsilon). \tag{5.9}$$

In the case when $\mathcal{J}_\varepsilon$ involves the last increment $X_{N_\varepsilon+1}, Y_{N_\varepsilon+1}$, it is clear that we still have (5.9), with equality replaced by $\lesssim$.

By Corollary 4.4, we have

$$\mathscr{H}_{M_1,M_2}(\tilde{X}_\varepsilon, \tilde{Y}_\varepsilon) \lesssim 1,$$

uniformly in $x_1, x_2 \in \mathbb{R}^d$, $s_1, s_2 \in [0,1]$, $M_1, M_2 > 0$ and $\tilde{X}_\varepsilon, \tilde{Y}_\varepsilon \in \Omega$, and (5.8) follows. $\square$

Proof of Lemma 5.2. We divide the proof into three steps.

Step 1. We claim that for any $\delta > 0$, there exists a universal $M > 0$ such that if $\min(M_1, M_2) > M$, we have

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}\Big[|e^{\mathcal{J}_\varepsilon(M_1,M_2)} - e^{\mathcal{J}_\varepsilon(M,M)}|\Big] < \delta. \tag{5.10}$$

First, since $R_\phi, R_\psi \ge 0$, $\mathscr{G}_\varepsilon$ is bounded and $R_\phi(u_1,u_2)$ is supported on $|u_1 - u_2| \le 1$, we have

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}\Big[|e^{\mathcal{J}_\varepsilon(M_1,M_2)} - e^{\mathcal{J}_\varepsilon(M,M)}|\Big] \lesssim \mathbb{E}_\pi\Big[e^{\mathcal{J}_\varepsilon(M_1,M_2)}1_{\{\mathcal{E}_1(M)>0\}}\Big], \tag{5.11}$$

with

$$\mathcal{E}_1(M) = \sup_{M_1,M_2>M}\int_{M-1}^{M_1}\int_{M-1}^{M_2} R_\phi(u_1,u_2)R_\psi(x_1-x_2+B^1_{\frac{t-r}{\varepsilon^2}+u_1}-B^1_{\frac{t-r}{\varepsilon^2}-s_1}-B^2_{\frac{t-r}{\varepsilon^2}+u_2}+B^2_{\frac{t-r}{\varepsilon^2}-s_2})du_1du_2.$$

After applying the Cauchy-Schwarz inequality to the r.h.s. of (5.11) and using Lemma 5.1, we only need to consider $\pi[\mathcal{E}_1(M) > 0]$, which is essentially the same as the probability of the "nearby time" of B^1, B^2 being greater than M. By the same argument as in the proof of Lemma 5.1, Proposition 4.3 and Corollary 4.4, we have $\pi[\mathcal{E}_1] \to 0$ as $M \to \infty$, which proves (5.10).

Step 2. We show that

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[(\mathcal{I}_\varepsilon - \tilde{\mathcal{I}}_\varepsilon)e^{\mathcal{J}_\varepsilon(M,M)}] \to 0, \quad \text{as } \varepsilon \to 0, \tag{5.12}$$

where

$$\tilde{\mathcal{I}}_\varepsilon = \prod_{i=1}^2 g(y - \varepsilon B^i_{(t-r)/\varepsilon^2-\varepsilon^{-\alpha}})u_0(y + \varepsilon B^i_{t/\varepsilon^2-\varepsilon^{-\alpha}} - \varepsilon B^i_{T_M^\varepsilon}),$$

with

$$T_M^\varepsilon = \min\{T_i : \frac{t-r}{\varepsilon^2} + \varepsilon^{-\alpha} \le T_i \le \frac{t}{\varepsilon^2} - \varepsilon^{-\alpha}\},$$

and the convention that $T_M^\varepsilon = t/\varepsilon^2 - \varepsilon^{-\alpha}$ if there is no regeneration time in the interval $[(t-r)/\varepsilon^2 + \varepsilon^{-\alpha}, t/\varepsilon^2 - \varepsilon^{-\alpha}]$. As $\mathscr{G}_\varepsilon$ and $\mathcal{J}_\varepsilon(M,M)$ are bounded, we have

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[|\mathcal{I}_\varepsilon - \tilde{\mathcal{I}}_\varepsilon|e^{\mathcal{J}_\varepsilon(M,M)}] \lesssim \mathbb{E}_\pi[|\mathcal{I}_\varepsilon - \tilde{\mathcal{I}}_\varepsilon|]. \tag{5.13}$$

By Proposition 4.1, we have the convergence in distribution of

$$(\varepsilon B^i_{\frac{t-r}{\varepsilon^2}-\varepsilon^{-\alpha}}, \varepsilon B^i_{\frac{t-r}{\varepsilon^2}-s_i}, \varepsilon B^i_{T^\varepsilon_M}, \varepsilon B^i_{\frac{t}{\varepsilon^2}-\varepsilon^{-\alpha}}, \varepsilon B^i_{\frac{t}{\varepsilon^2}}) \Rightarrow (W^i_{t-r}, W^i_{t-r}, W^i_{t-r}, W^i_t, W^i_t), \tag{5.14}$$

which implies that the r.h.s. of (5.13) goes to zero as $\varepsilon \to 0$.

Step 3. We prove the convergence of

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[\tilde{\mathcal{I}}_\varepsilon e^{\mathcal{J}_\varepsilon(M,M)}] \to \mathbb{E}_\pi[\mathscr{H}_{M,M}(\tilde{X},\tilde{Y})]|\bar{g}(t-r,y)\bar{u}(r,y)|^2, \tag{5.15}$$

where $\tilde{X},\tilde{Y}$ are sampled independently from the invariant measure of $\hat{\pi}$. First, we have

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[\tilde{\mathcal{I}}_\varepsilon e^{\mathcal{J}_\varepsilon(M,M)}] = \mathbb{E}_\pi[\tilde{\mathcal{I}}_\varepsilon e^{\mathcal{J}_\varepsilon(M,M)}\mathscr{G}_\varepsilon(X_{N_\varepsilon})\mathscr{G}_\varepsilon(Y_{N_\varepsilon})].$$

Note that, for ε sufficiently small (depending on M and r), both $\tilde{\mathcal{I}}_\varepsilon$ and $\mathcal{J}_\varepsilon(M,M)$ depend only on $\{B^i_s : s \le t/\varepsilon^2 - \varepsilon^{-\alpha}\}$. Lemma A.2 implies that it suffices to prove the convergence of $\mathbb{E}_\pi[\tilde{\mathcal{I}}_\varepsilon \exp\{\mathcal{J}_\varepsilon(M,M)\}]$. We write

$$\mathbb{E}_\pi[\tilde{\mathcal{I}}_\varepsilon e^{\mathcal{J}_\varepsilon(M,M)}] = \mathbb{E}_\pi\Big[\prod_{i=1}^2 g(y-\varepsilon B^i_{(t-r)/\varepsilon^2-\varepsilon^{-\alpha}})\mathscr{H}_{M,M}(\tilde{X}_\varepsilon,\tilde{Y}_\varepsilon)\Big]\mathbb{E}_\pi\Big[\prod_{i=1}^2 u_0(y+\varepsilon B^i_{t/\varepsilon^2-\varepsilon^{-\alpha}} - \varepsilon B^i_{T^\varepsilon_M})\Big]. \tag{5.16}$$

Here, we used the independence of the increments after the regeneration time T^ε_M to split off the second factor, and the separation between the time $(t-r)/\varepsilon^2 - \varepsilon^{-\alpha}$ and the times appearing in the integration in $\mathcal{J}_\varepsilon(M,M)$ in the first factor. By the weak convergence in (5.14), we have

$$\mathbb{E}_\pi\Big[\prod_{i=1}^2 u_0(y+\varepsilon B^i_{t/\varepsilon^2-\varepsilon^{-\alpha}} - \varepsilon B^i_{T^\varepsilon_M})\Big] \to |\bar{u}(r,y)|^2.$$

It remains to consider the first factor in the right side of (5.16). We claim that as $\varepsilon \to 0$

$$\mathbb{E}_\pi\Big[\prod_{i=1}^2 g(y-\varepsilon B^i_{\frac{t-r}{\varepsilon^2}-\varepsilon^{-\alpha}})\mathscr{H}_{M,M}(\tilde{X}_\varepsilon,\tilde{Y}_\varepsilon)\Big] - \mathbb{E}_\pi\Big[\prod_{i=1}^2 g(y-\varepsilon B^i_{\frac{t-r}{\varepsilon^2}-\varepsilon^{-\alpha}})\Big]\mathbb{E}_\pi\Big[\mathscr{H}_{M,M}(\tilde{X},\tilde{Y})\Big] \to 0. \tag{5.17}$$

The proof of (5.17) is the same as the proof of Lemma A.2, as $\mathscr{H}_{M,M}$ is bounded and $\tilde{X}_\varepsilon, \tilde{Y}_\varepsilon$ are the increments of B^1, B^2 on the interval $[(t-r)/\varepsilon^2 - 2, (t-r)/\varepsilon^2 - 1]$. We apply the weak convergence (5.14) again to get

$$\mathbb{E}_\pi\Big[\prod_{i=1}^2 g(y-\varepsilon B^i_{(t-r)/\varepsilon^2-\varepsilon^{-\alpha}})\Big] \to |\bar{g}(t-r,y)|^2$$

and complete the proof of (5.15).

Combining steps 1-3 and sending $\delta \to 0$, completes the proof. □

5.2 Proof of the uniform bound (3.13)

We now prove the uniform bound (3.13) in Proposition 3.2. By Lemma 5.1, we have

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[|e^{\mathcal{J}_\varepsilon(M_1,M_2)}|^2] \lesssim 1,$$

so by the Cauchy-Schwarz inequality,

$$|\mathscr{F}_\varepsilon(r,y,M_1,M_2)| \lesssim \int_{\mathbb{R}^d}\int_{[0,1]} \widehat{\mathbb{E}}_{B,t/\varepsilon^2}[|g(\varepsilon x + y - \varepsilon B_{(t-r)/\varepsilon^2-s})|]\phi(s)\psi(x)dsdx.$$

Lemma 5.3. *For any $k \in \mathbb{Z}_{\geq 1}$, there exists C_k such that*

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[1_{\{|\varepsilon B_{(t-r)/\varepsilon^2-s}|>M\}}] \leq \frac{C_k}{M^{2k}} \tag{5.18}$$

for all $M > 0$.

By the above lemma and the fact that $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $|x| \leq 1$, we have

$$\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[|g(\varepsilon x + y - \varepsilon B_{(t-r)/\varepsilon^2-s})|] \lesssim 1 \wedge \frac{1}{|y|^k},$$

which implies (3.13) and finishes the proof of Proposition 3.2.

Proof of Lemma 5.3. Since $\mathscr{G}_\varepsilon$ is bounded, it suffices to prove the same estimate for

$$\pi[|\varepsilon B_{(t-r)/\varepsilon^2-s}| > M].$$

We will assume $r = 0, s = 0$ to simplify the notation and the proof of the general case is the same. First, we write B_{t/ε^2} as a sum of independent zero-mean random variables using the regeneration structure. Let $\tau = 1$ and $N_\varepsilon = [t/\varepsilon^2] - 1$ and set

$$B_{t/\varepsilon^2} = \sum_{k=0}^{N_\varepsilon} X_k(1) + B_{t/\varepsilon^2} - B_{[t/\varepsilon^2]}.$$

We also write

$$B_{t/\varepsilon^2} = \sum_{j=0}^{K_\varepsilon} \mathbf{X}_j,$$

where

$$\mathbf{X}_j = \begin{cases} \sum_{k=T_j}^{T_{j+1}-1} X_k(1), & j = 0, \ldots, K_\varepsilon - 1, \\ B_{t/\varepsilon^2} - B_{T_{K_\varepsilon}}, & j = K_\varepsilon, \end{cases}$$

and $K_\varepsilon = \max\{j : T_j \leq N_\varepsilon\}$. Since $\mathbf{X}_j$ are independent random variables with zero mean conditioning on $\{T_j\}_{j=0}^{K_\varepsilon}$, the sum

$$\mathbf{M}_k = \sum_{j=0}^{k} \mathbf{X}_j, \quad k = 0, \ldots, K_\varepsilon,$$

is a martingale. By the Chebyshev and martingale inequalities, we have

$$\pi\Big[|\varepsilon \mathbf{M}_{K_\varepsilon}| > M \mid \{T_j\}_{j=0}^{K_\varepsilon}\Big] \leq \frac{1}{M^{2k}} \mathbb{E}_\pi\Big[|\varepsilon \mathbf{M}_{K_\varepsilon}|^{2k} \mid \{T_j\}_{j=0}^{K_\varepsilon}\Big] \lesssim \frac{1}{M^{2k}} \mathbb{E}_\pi\Big[\Big|\varepsilon^2 \sum_{j=0}^{K_\varepsilon} \mathbf{X}_j^2\Big|^k \mid \{T_j\}_{j=0}^{K_\varepsilon}\Big].$$

Since $K_\varepsilon \leq N_\varepsilon$, we only need to show that

$$\varepsilon^{2k} \mathbb{E}_\pi\Big[\Big|\sum_{j=0}^{N_\varepsilon} \mathbf{X}_j^2\Big|^k\Big] \lesssim 1. \tag{5.19}$$

If we expand $|\sum_{j=0}^{N_\varepsilon} \mathbf{X}_j^2|^k$, the number of terms is smaller than $(t/\varepsilon^2)^k$, and each term is of the form $\prod_{l=1}^k \mathbf{X}_{j_l}^2$ for some $j_l = 0, \ldots, N_\varepsilon$, whose expectation is uniformly bounded, in light of Lemma A.1. Thus, (5.19) holds and the proof is complete. □

A Some technical lemmas

A.1 Proof of Lemma 3.1

Proof. Recall that we need to prove that

$$\zeta_T = c_1 T + c_2 + o(1), \text{ as } T \to +\infty. \tag{A.1}$$

We employ the setup of Section 4. The proof is divided into three steps, in which we prove that (A.1) holds for $T \in \mathbb{N}, \mathbb{Q}, \mathbb{R}$.

Step 1, $T \in \mathbb{N}$. In the construction of the chain, set $\tau = 1$. As in Section 4.1, we have

$$\widehat{\mathbb{P}}_T(d\omega) = \Psi(x_0)\pi(dx_0)\left(\prod_{k=0}^{T-2} \hat{\pi}(x_k, x_{k+1})\right)\Psi^{-1}(x_{T-1})\rho^{T-1}e^{T\zeta_1 - \zeta_T}.$$

Using the normalization (4.10) gives,

$$\mathbb{E}_\pi[\Psi^{-1}(X_{T-1})] = e^{\zeta_T - T\zeta_1}\rho^{1-T}.$$

By (4.19), we have

$$e^{\zeta_T - T\zeta_1}\rho^{1-T} = e^{\zeta_T - T\zeta_1 - (T-1)\log\rho} \to \mathbb{E}_\pi[\Psi^{-1}(\tilde{X})]$$

exponentially fast as $T \to \infty$, where $\tilde{X}$ is sampled from the invariant measure of $\hat{\pi}$. This proves (A.1) for integer T with

$$c_1 = \zeta_1 + \log\rho, \quad c_2 = \log\rho^{-1} + \log\mathbb{E}_\pi[\Psi^{-1}(\tilde{X})]. \tag{A.2}$$

We note that the convergence rate of the remainder $o(1) \to 0$ as $T \to \infty$ only depends on the estimates on Ψ and γ which are determined by $\|I\|_{L^\infty}$.

Step 2, $T \in \mathbb{Q}$. In the construction of the chain, the choice of the length-one increment is arbitrary – we can take any length that is greater than one and follow the same construction. Take the increment of length $r \in \mathbb{Q}$ such that $r \in (1,2)$ (then the corresponding $I(x,y)$ is uniformly bounded), so there exist $m_1, m_2 \in \mathbb{N}$ such that $rm_1 = m_2$. For any $k \in \mathbb{N}$, the same proof as in Step 1 shows that

$$\zeta_{rm_1k} = c_{1,r}m_1k + c_{2,r} + o(1)$$

for some $c_{1,r}, c_{2,r}$. Since

$$\zeta_{m_2k} = c_1m_2k + c_2 + o(1)$$

from step 1, we conclude that $c_{1,r} = c_1r$ and $c_{2,r} = c_2$ by sending $k \to \infty$. Thus, for any $r \in \mathbb{Q}$, we have

$$\zeta_{rk} = c_1rk + c_2 + o(1),$$

with $o(1) \to 0$ as $k \to \infty$, uniformly in $r \in (1,2)$. Choosing $r = T/[T]$, we see that (A.1) holds for $T \in \mathbb{Q}$.

Step 3, $T \in \mathbb{R}$. As ζ_T is continuous in T, we simply take $T_n \in \mathbb{Q}$ so that $T_n \to T$ and $\zeta_{T_n} \to \zeta_T$. Since

$$\zeta_{T_n} = c_1T_n + c_2 + o(1)$$

with $o(1) \to 0$ as $T_n \to \infty$, the proof is complete. □

Lemma A.1. *Assuming $X_0 \sim \pi(dx_0)$, $X_{k+1} \sim (1-\gamma)^{-1}(\hat{\pi}(X_k, dx_{k+1}) - \gamma\pi(dx_{k+1}))$ for $k \geq 0$, and θ is an independent geometric random variable with parameter γ. Then, for all $k \geq 0$, $\mathbb{E}_\pi[X_k(1)] = 0$ and there exists $c > 0$ such that*

$$\pi\big[\max_{s\in[0,1]} |X_k(s)| \geq t\big] \lesssim e^{-ct^2}, \tag{A.3}$$

$$\pi\Big[\sum_{k=0}^{\theta} \max_{s\in[0,1]} |X_k(s)| > t\Big] \lesssim e^{-ct}. \tag{A.4}$$

Proof. For any measure ν_0 on Ω that is symmetric, so that $\nu_0(A) = \nu_0(-A)$ with $-A := \{f : -f \in A\}$, set

$$\nu_1(A) = \int_\Omega \nu_0(dx)\hat{\pi}(x, A).$$

Recall that

$$\int_\Omega e^{I(x,y)}\Psi(y)\pi(dy) = \rho\Psi(x).$$

Since $I(x, y) = I(-x, -y)$, π is symmetric, and Ψ is the unique eigenvector corresponding to ρ satisfying (4.10), we have that $\Psi(-x) = \Psi(x)$, hence $\hat{\pi}(x, A) = \hat{\pi}(-x, -A)$ and ν_1 is symmetric. Thus, the distribution of X_k is symmetric, and

$$\mathbb{E}_\pi[X_k(1)] = -\mathbb{E}_\pi[X_k(1)] = 0.$$

For the Gaussian tail in (A.3), we note that

$$\sup_{x\in\Omega} \frac{\hat{\pi}(x, dy) - \gamma\pi(dy)}{1-\gamma} \lesssim \sup_{x\in\Omega} \hat{\pi}(x, dy) \lesssim \pi(dy). \tag{A.5}$$

As π is the Wiener measure on $\mathcal{C}([0, 1])$ tilted by the bounded factor

$$\exp\Big\{\frac{1}{2}\lambda^2 \int_{[0,1]^2} R(s-u, \omega(s) - \omega(u))dsdu - \zeta_1\Big\},$$

there exists $c > 0$ such that

$$\pi[\max_{s\in[0,1]} |X_{k+1}(s)| \geq t \mid X_k] \lesssim \pi[\max_{s\in[0,1]} |X_0(s)| \geq t] \lesssim e^{-ct^2} \tag{A.6}$$

uniformly in X_k. After averaging with respect to X_k, we obtain (A.3).

To prove (A.4), we note that

$$\pi[\theta > [\alpha t]] \lesssim (1-\gamma)^{\alpha t}$$

for any $\alpha > 0$. By the Chebyshev inequality, we have

$$\pi\Big[\sum_{k=0}^{[\alpha t]} \max_{s\in[0,1]} |X_k(s)| > t\Big] \leq e^{-C_1 t}\mathbb{E}_\pi\Big[\exp\Big\{C_1 \sum_{k=0}^{[\alpha t]} \max_{s\in[0,1]} |X_k(s)|\Big\}\Big]$$

for any $C_1 > 0$. Using (A.5) again, we have

$$e^{-C_1 t}\mathbb{E}\Big[\exp\Big\{C_1 \sum_{k=0}^{[\alpha t]} \max_{s\in[0,1]} |X_k(s)|\Big\}\Big] \lesssim e^{-C_1 t} C_2^{[\alpha t]}$$

for some constant $C_2 > 0$ independent of α. Taking $\alpha < C_1/\log C_2$ finishes the proof. □

Lemma A.2. *If $F:\Omega_{t/\varepsilon^2}\to\mathbb{R}$ is bounded and only depends on $X_0,\ldots,X_{M_\varepsilon}$, with $N_\varepsilon-M_\varepsilon\to\infty$, then*

$$|\mathbb{E}_\pi[F(B)\mathscr{G}_\varepsilon(X_{N_\varepsilon})]-\mathbb{E}_\pi[F(B)]|\to 0 \tag{A.7}$$

as $\varepsilon\to 0$.

Proof. First, we have

$$|\mathbb{E}_\pi[F(B)\mathscr{G}_\varepsilon(X_{N_\varepsilon})]-\mathbb{E}_\pi[F(B)]|=|\mathbb{E}_\pi[F(B)\mathbb{E}_\pi[\mathscr{G}_\varepsilon(X_{N_\varepsilon})-1|X_{M_\varepsilon}]]|\lesssim\mathbb{E}_\pi[|\mathbb{E}_\pi[\mathscr{G}_\varepsilon(X_{N_\varepsilon})|X_{M_\varepsilon}]-1|].$$

Since $\mathscr{G}_\varepsilon$ is bounded, by (4.19), we have

$$|\mathbb{E}_\pi[\mathscr{G}_\varepsilon(X_{N_\varepsilon})|X_{M_\varepsilon}]-\mathbb{E}_\pi[\mathscr{G}_\varepsilon(\tilde{X})]|\to 0 \text{ as } \varepsilon\to 0,$$

uniformly in X_{M_ε}. Here, $\tilde{X}$ is sampled from the invariant measure of $\hat{\pi}$. Since $\mathbb{E}_\pi[\mathscr{G}_\varepsilon(X_{N_\varepsilon})]=1$, we know that $\mathbb{E}_\pi[\mathscr{G}_\varepsilon(\tilde{X})]\to 1$ as $\varepsilon\to 0$. Hence,

$$\mathbb{E}_\pi[\mathscr{G}_\varepsilon(X_{N_\varepsilon})|X_{M_\varepsilon}]-1\to 0,\ \text{ as } \varepsilon\to 0,$$

which completes the proof. □

Lemma A.3. *There exists $C>0$ independent of ε such that $\sum_j\sqrt{\mathbb{E}[|\mathcal{X}_j^\varepsilon|^4]}\le C$.*

Proof. Recall that

$$\mathcal{X}_j^\varepsilon=\frac{1}{\varepsilon^{d/2-1}}\int_{I_{\beta,j}}\int_{\mathbb{R}^d}\tilde{Z}_t^\varepsilon(r,y)dW(r,y),$$

and by the martingale inequality, we have

$$\mathbb{E}[|\mathcal{X}_j^\varepsilon|^4]\lesssim\frac{1}{\varepsilon^{2d-4}}\int_{I_{\beta,j}^2}\int_{\mathbb{R}^{2d}}\mathbb{E}[|\tilde{Z}_t^\varepsilon(r,y)\tilde{Z}_t^\varepsilon(r',y')|^2]dydy'drdr'.$$

For $\mathbb{E}[|\tilde{Z}_t^\varepsilon(r,y)\tilde{Z}_t^\varepsilon(r',y')|^2]$, we repeat the calculation in the proof of Lemma 2.2. To simplify the notation, we let $r=r_1=r_2, r'=r_3=r_4$ and $y=y_1=y_2, y'=y_3=y_4$ and consider

$$\mathbb{E}\Big[\prod_{i=1}^4\tilde{Z}_t^\varepsilon(r_i,y_i)\Big]=\int_{\mathbb{R}^{4d}}\mathbb{E}\widehat{\mathbb{E}}_{B,t/\varepsilon^2}\Big[\prod_{i=1}^4 g(x_i)u_0(x_i+\varepsilon B^i_{t/\varepsilon^2})\Phi^\varepsilon_{t,x_i,B^i}(r_i,y_i)e^{\lambda\tilde{M}^\varepsilon_{t,x_i,B^i}(r_i)-\frac12\lambda^2\langle\tilde{M}^\varepsilon_{t,x_i,B^i}\rangle_{r_i}}\Big]d\boldsymbol{x}.$$

As in the proof of Lemma 2.2, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon^{2d-4}}\int_{I_{\beta,j}^2}\int_{\mathbb{R}^{2d}}\mathbb{E}[|\tilde{Z}_t^\varepsilon(r,y)\tilde{Z}_t^\varepsilon(r',y')|^2]dydy'drdr'\\
&\le\int_{[0,t]^2}\int_{\mathbb{R}^{6d}}\int_{[0,1]^4}1_{\{r/\varepsilon^2\in I_{\beta,j}\}}1_{\{r'/\varepsilon^2\in I_{\beta,j}\}}\widehat{\mathbb{E}}_{B,t/\varepsilon^2}[Ie^J]\prod_{i=1}^4\phi(s_i)\psi(x_i)d\boldsymbol{s}d\boldsymbol{x}dydy'drdr',
\end{aligned}$$

where

$$\begin{aligned}
I&=\prod_{i=1}^4|g(\varepsilon x_i+y_i-\varepsilon B^i_{(t-r_i)/\varepsilon^2-s_i})u_0(\varepsilon x_i+y_i+\varepsilon B^i_{t/\varepsilon^2}-\varepsilon B^i_{(t-r_i)/\varepsilon^2-s_i})|,\\
J=&\lambda^2\sum_{1\le i<l\le 4}\int_{-1}^{1/2\varepsilon^\alpha}\int_{-1}^{1/2\varepsilon^\alpha}R_\phi(u_i,u_l)\\
&\times R_\psi(x_i-x_l+\frac{y_i-y_l}{\varepsilon}+B^i_{\frac{t-r_i\wedge r_l}{\varepsilon^2}+u_i}-B^i_{\frac{t-r_i}{\varepsilon^2}-s_i}-B^l_{\frac{t-r_i\wedge r_l}{\varepsilon^2}+u_l}+B^l_{\frac{t-r_l}{\varepsilon^2}-s_l})du_idu_l.
\end{aligned}$$

By the same proof as that of (3.13), we have

$$\frac{1}{\varepsilon^{2d-4}}\int_{I_{\beta,j}^2}\int_{\mathbb{R}^{2d}}\mathbb{E}[|\tilde{Z}_t^\varepsilon(r,y)\tilde{Z}_t^\varepsilon(r',y')|^2]dydy'drdr'\lesssim\Big(\int_0^t 1_{\{r/\varepsilon^2\in I_{\beta,j}\}}dr\Big)^2.$$

The proof is complete. □

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