UNBOUNDED MASS RADIAL SOLUTIONS FOR THE KELLER-SEGEL EQUATION IN THE DISK

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Abstract. We construct several families of radial solutions for the stationary Keller-Segel equation in the disk. The first family consists in solutions which blow up at the origin, as a parameter goes to zero, and concentrate on the boundary. The second is made of solutions which blow up at the origin and concentrate on an interior sphere, while the solutions of the third family blow up at the origin and concentrate simultaneously on an interior sphere and on the boundary. Finally, we also show how to construct more families of multi-layered radial solutions provided a suitable non degeneracy assumption is satisfied.

Keywords: Keller-Segel equation, singular solution, internal layer, boundary layer, Lyapunov-Schmidt reduction.
MSC: 35B40, 35B45, 35J55, 92C15, 92C40.

1. Introduction

Chemotaxis is the influence of chemical substances in the environment on the movement of mobile species. It is an important mean for cellular communication by chemical substances, which determines how cells arrange themselves, for instance in living tissues. In 1970, Keller and Segel [KS70] proposed a basic model to describe this phenomenon. They considered an advection-diffusion system consisting of two coupled parabolic equations for the concentration of the species and one for the chemical released, respectively represented by strictly positive quantities $v(x,t)$ and $u(x,t)$ defined on a bounded (smooth) domain $\Omega \subset \mathbb{R}^n$. The system has the form

$$\begin{cases} \dfrac{\partial v}{\partial t} = D_v \Delta v - c\,\mathrm{div}(v \nabla \phi(u)) \\ \dfrac{\partial u}{\partial t} = D_u \Delta u + k(u,v), \\ \partial_\nu u = \partial_\nu v = 0, \ \textit{on } \partial\Omega, \end{cases}$$

where ν denotes the exterior unit normal vector to $\partial\Omega$, D_v, D_u, c are strictly positive constants, the function ϕ, usually called the *sensitive function*, is a smooth function such that $\phi'(r) > 0$ for $r > 0$ and k is a smooth function such that $\frac{\partial k}{\partial v} \geq 0$ and $\frac{\partial k}{\partial u} \leq 0$. The typical choice for k that we adopt from now on is $k(u,v) = -u + v$. The homogeneous Neumann boundary condition for both u and v accounts for the assumption that there is no flux through the boundary, i.e.

$$\nabla v \cdot \nu = \nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

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An important property of this system is the so-called chemotactic collapse. This term refers to the fact that the whole population of organisms concentrate at a single point in finite or infinite time. When $\phi(u) = u$, it is well-known that the chemotactic collapse depends strongly on the dimension of the space. Finite-time blow-up never occurs if $n = 1$, whereas it always occurs if $n \geq 3$. The two-dimensional case is critical: if the initial distribution of organisms exceeds a certain threshold, then the solutions may blow-up in finite time, whereas solutions exist globally in time if the initial mass is below the threshold. We refer the interested reader to the surveys [Hor03, Hor04, BBTW15] and to the references therein for further details about the model and a collection of known results.

Steady states of the Keller-Segel system are of basic importance for the understanding of the global dynamics. They solve the system

$$\begin{cases} -D_v \Delta v + c\,\mathrm{div}(v \nabla \phi(u)) &= 0, \quad v > 0 \quad \text{in } \Omega \\ -D_u \Delta u - u + v &= 0, \quad u > 0 \quad \text{in } \Omega, \end{cases}$$

with homogeneous Neumann boundary conditions on $\partial\Omega$. This system can be reduced to a scalar equation as, indeed, one easily gets

$$\int_\Omega v |\nabla (D_v \log v - c\phi(u))|^2 \, dx = 0,$$

which implies $v = C e^{\frac{c}{D_v}\phi(u)}$ for some constant $C > 0$. In the most common formulation of the Keller-Segel model, one takes $\phi(u) = u$ and this choice yields the so-called Keller-Segel equation

$$\begin{cases} -\sigma^2 \Delta u + u - \lambda e^u &= 0, \ u > 0 \quad \text{in } \Omega \\ \partial_\nu u &= 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the constants σ, λ depend on D_v, D_u, c and C. It is worth to mention that in the case $\phi(u) = \log u$, we get

$$\begin{cases} -\tilde{\sigma}^2 \Delta u + u - u^p &= 0, \ u > 0 \quad \text{in } \Omega \\ \partial_\nu u &= 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

for some constants $\tilde{\sigma}, p > 0$, i.e. we recover the celebrated Lin-Ni-Takagi equation [NT86, LN88, LNT88]. Let us observe that in dimension 2 the Keller-Segel equation is critical, whereas the Lin-Ni-Takagi problem is subcritical. A good account of known results about this equation is given in the book by Wei and Winter [WW14], in the chapter [Ni04], in the recent paper [dPMRW16], and in the references therein.

From now on, we study the Keller-Segel equation (1.1) and we assume without loss of generality that $\sigma = 1$. In the one-dimensional case, Schaaf [Sch85] proved the existence of non-trivial solutions. For a general two-dimensional domain, the first existence results were obtained by Wang and Wei [WW02] and independently by Senba and Suzuki [SS00], when the parameter λ is small enough. Moreover, Senba and Suzuki [SS00, SS02] studied the asymptotic behavior of finite mass solutions when $\lambda \to 0$. These are solutions u_λ to (1.1) such that

$$\lim_{\lambda \to 0} \lambda \int_\Omega e^{u_\lambda} < \infty.$$

Senba and Suzuki showed that there exist points $\xi_i \in \Omega$, with $i \le k$, and points $\eta_i \in \partial\Omega$, with $k < i \le m$, such that

$$u_\lambda(x) \to \sum_{i=1}^{k} 8\pi \mathcal{G}(x,\xi_i) + \sum_{i=k+1}^{m} 4\pi \mathcal{G}(x,\eta_i), \quad \text{as } \lambda \to 0, \tag{1.2}$$

uniformly on compact subsets of $\overline{\Omega}\setminus\{\xi_1,\dots,\xi_k,\eta_{k+1},\dots,\eta_m\}$, where we recall that, given $y \in \overline{\Omega}$, $\mathcal{G}(x,y)$ denotes the Green function associated to homogeneous Neumann boundary condition, namely the unique solution of

$$\begin{cases} -\Delta_x \mathcal{G} + \mathcal{G} &= \delta_y \quad \text{in } \Omega, \\ \nabla \mathcal{G} \cdot \nu &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

The counterpart of this result has been obtained by del Pino and Wei [dPW06]. For any given integers k and m, they constructed a family of solutions to (1.1) satisfying (1.2) for a suitable choice of points $\xi_i \in \Omega$ and $\eta_i \in \partial\Omega$.

Recently, solutions concentrating on higher dimensional sets, with unbounded mass, have been proved to exist. From now on, we denote by B_r the ball of radius r centered at zero. When $\Omega = B_1 \subset \mathbb{R}^n$, with $n \ge 2$, Pistoia and Vaira [PV15] constructed a family u_λ of radial solutions blowing-up on the whole boundary of Ω and such that

$$\lim_{\lambda\to 0} \lambda \int_{B_1} e^{u_\lambda(x)} dx = \infty.$$

More precisely, their solutions satisfy

$$\lim_{\lambda \to 0} \varepsilon_\lambda u_\lambda = \sqrt{2}\,\mathcal{U},$$

C^0-uniformly on compact subsets of B_1, where $\varepsilon_\lambda \approx -\frac{1}{\ln\lambda}$ and $\mathcal{U}$ is the unique (radial) solution to

$$\begin{cases} -\Delta \mathcal{U} + \mathcal{U} &= 0 \quad \text{in } B_1 \\ \mathcal{U} &= 1 \quad \text{on } \partial B_1, \end{cases}$$

Near the boundary, these solutions behave (up to rescaling) like the one-dimensional half standard bubble, namely the solution of

$$-w'' = e^w \quad \text{in } \mathbb{R}, \text{ with } \int_{\mathbb{R}} e^w < \infty.$$

It is worth pointing out that del Pino, Pistoia, and Vaira [dPPV16] generalized this result to general two-dimensional domains. More recently, existence of solutions concentrating on submanifolds of the boundary has also been investigated; see for instance [AP16].

From now on, we suppose that $\Omega = B_1 \subset \mathbb{R}^n$, with $n \ge 2$. In [BCN17b], a bifurcation analysis of radial solutions to (1.1) was performed. Observe that for $\lambda < 1/e$, the equation (1.1) can be rewritten as

$$\begin{cases} -\Delta u + u &= e^{\mu(u-1)}, \ u > 0 \quad \text{in } B_1 \\ \nabla u \cdot \nu &= 0 \quad \text{on } \partial B_1 \end{cases} \tag{1.3}$$

for $\mu > 1$. This equation admits the constant solutions $u \equiv 1$ and $\underline{u}_\mu < 1$. To describe the bifurcation result, we denote by λ_i^{rad} the i-th eigenvalue of the operator $-\Delta + \mathrm{Id}$ in B_1, restricted to the set of radial functions, with homogeneous Neumann boundary conditions.

Theorem 1.1 ([BCN17b])**.** *For every $i \geq 2$, $(\lambda_i^{\mathrm{rad}}, 1)$ is a bifurcation point of (1.3). Denoting by $\mathcal{B}_i$ the continuum that branches out of $(\lambda_i^{\mathrm{rad}}, 1)$, we have*

(i) the branches $\mathcal{B}_i$ are unbounded and do not intersect; close to $(\lambda_i^{\mathrm{rad}}, 1)$, $\mathcal{B}_i$ is a C^1-curve;
(ii) if $u_\mu \in \mathcal{B}_i$ then $u_\mu > 0$;
(iii) each branch consists of two connected components: the component $\mathcal{B}_i^-$, along which $u_\mu(0) < 1$, and the component $\mathcal{B}_i^+$, along which $u_\mu(0) > 1$;
(iv) if $u_\mu \in \mathcal{B}_i$ then $u_\mu - 1$ has exactly $i-1$ zeros, u_μ' has exactly $i-2$ zeros, and each zero of u_μ' lies between two zeros of $u_\mu - 1$;
(v) the functions satisfying $u_\mu(0) < 1$ are uniformly bounded in the C^1-norm.

We conjecture that the solutions constructed by Pistoia and Vaira [PV15] correspond to those on $\mathcal{B}_1^-$, while the solutions constructed by del Pino and Wei [dPW06] (when restricted to the 2-dimensional ball) correspond to the branch $\mathcal{B}_1^+$. In [BCN17b], the authors constructed solutions concentrating on an arbitrary number of internal spheres by combining variational and perturbative methods. Solutions sharing the same qualitative properties were obtained with a different method in [BCN17a] with very precise asymptotics. We conjecture that those solutions are indeed the same and correspond to the solutions on the branches $\mathcal{B}_i^-$.

In this paper, we restrict ourselves to the disk. Our goal is to construct solutions of (1.1) with prescribed asymptotics as $\lambda \to 0$, namely solutions that concentrate at the origin and on spheres belonging to the interior and/or the boundary of B_1. We conjecture that those solutions correspond to the solutions of (1.3) on the branches $\mathcal{B}_i^+$ for $i \geq 2$. We emphasize that only a few results concerning existence of solutions concentrating simultaneously on points and layers are available in the litterature, see for instance [SW13, WW08].

Our first result concerns the existence of solutions concentrating at the origin and on ∂B_1. In the statement of the result, U_λ is defined as the unique (radial) solution to

$$\begin{cases} -\Delta U_\lambda + U_\lambda &= 0 \quad \text{in } B_1 \\ \displaystyle\lim_{r \to 0^+} \frac{U_\lambda(r)}{-\ln r} &= 4 \\ U_\lambda &= \dfrac{\sqrt{2}}{\varepsilon_\lambda} \quad \text{on } \partial B_1 \end{cases}$$

where ε_λ is such that

$$\ln \frac{4}{\varepsilon_\lambda^2} - \ln \lambda = \frac{\sqrt{2}}{\varepsilon_\lambda}. \tag{1.4}$$

Theorem 1.2. *There exists a family of radial solutions $\{u_\lambda \mid \lambda \in (0, \lambda_0)\}$ to (1.1) such that*

$$\lim_{\lambda \to 0} \lambda \int_{B_1} e^{u_\lambda(x)} dx = \infty,$$

$$\lim_{\lambda \to 0} (u_\lambda(x) - U_\lambda(x)) = 0,$$

uniformly on compact subsets of $B_1 \backslash \{0\}$,

$$\lambda e^{u_\lambda} \rightharpoonup 8\pi\delta_0 \quad \text{in } B_{1/2},$$

and

$$\sqrt{2}\lambda e^{u_\lambda} + |\partial_\nu U_\lambda(1)|^{-1}\delta_{\partial B_1} \rightharpoonup 0 \quad \text{in } B_1\backslash B_{1/2}.$$

Next, we state the existence of two families of multi-layered solutions. In order to do so, we first need to introduce some Green's functions which basically give the limit profiles of those multi-layered solutions. Their construction extends the results of [BGNT16].

Theorem 1.3. *Let $k \in \mathbb{N}\backslash\{0\}$. For any constant $b > 0$ small enough,*

(i) there exist a configuration $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ and a continuous radial function $U_{b,k}$ such that

$$\begin{cases} -\Delta U_{b,k} + U_{b,k} &= 0 \quad \text{in } B_1 \setminus (\{0\} \cup_{i=1}^k \partial B_{\alpha_i}) \\ \lim\limits_{r\to 0^+} \dfrac{U_{b,k}(r)}{-\ln r} &= b \\ U_{b,k|\partial B_{\alpha_i}} &= 1 \quad \text{for any } i = 1, \ldots, k, \end{cases}$$

and satisfying, for any $i = 1, \ldots, k-1$, the reflection law

$$\lim_{\varepsilon\to 0^-} \frac{U_{b,k}(\alpha_i+\varepsilon) - U_{b,k}(\alpha_i)}{\varepsilon} = -\lim_{\varepsilon\to 0^+} \frac{U_{b,k}(\alpha_i+\varepsilon) - U_{b,k}(\alpha_i)}{\varepsilon}; \tag{1.5}$$

(ii) there exist a configuration $0 = \tilde\alpha_0 < \tilde\alpha_1 < \ldots < \tilde\alpha_k < \tilde\alpha_{k+1} = 1$ and a continuous radial function $\tilde U_{b,k}$ such that

$$\begin{cases} -\Delta \tilde U_{b,k} + \tilde U_{b,k} &= 0 \quad \text{in } B_1 \setminus (\{0\} \cup_{i=1}^k \partial B_{\tilde\alpha_i}) \\ \lim\limits_{r\to 0^+} \dfrac{\tilde U_{b,k}(r)}{-\ln r} &= b \\ \partial_\nu \tilde U_{b,k} &= 0 \quad \text{on } \partial B_1 \\ \tilde U_{b,k|\partial B_{\tilde\alpha_i}} &= 1 \quad \text{for any } i = 1, \ldots, k \end{cases}$$

and satisfying, for any $i = 1, \ldots, k$, the reflection law

$$\lim_{\varepsilon\to 0^-} \frac{\tilde U_{b,k}(\tilde\alpha_i+\varepsilon) - \tilde U_{b,k}(\tilde\alpha_i)}{\varepsilon} = -\lim_{\varepsilon\to 0^+} \frac{\tilde U_{b,k}(\tilde\alpha_i+\varepsilon) - \tilde U_{b,k}(\tilde\alpha_i)}{\varepsilon}. \tag{1.6}$$

Theorem 1.4 (Singular solution at the origin with an internal layer)**.** *There exists a family of radial solutions $\{\tilde u_\lambda \mid \lambda \in (0, \lambda_1)\}$ to (1.1) such that*

$$\lim_{\lambda\to 0} \left(\tilde u_\lambda - \frac{\sqrt{2}}{\varepsilon_\lambda} \tilde U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},1} \right) = 0,$$

uniformly on compact subsets of $B_1\backslash\{0\}$,

$$\lambda e^{\tilde u_\lambda} \rightharpoonup 8\pi\delta_0 \quad \text{in } B_{\tilde\alpha_1/2},$$

and

$$\varepsilon_\lambda \lambda e^{\tilde u_\lambda} + |\partial_\nu \tilde U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},1}(\tilde\alpha_1)|^{-1}\delta_{\partial B_1} \rightharpoonup 0 \quad \text{in } B_1\backslash\{0\},$$

where ε_λ is given by (1.4).

Theorem 1.5 (Singular solution at the origin with an internal layer and a boundary layer)**.** *There exists a family of radial solutions* $\{\bar{u}_\lambda \mid \lambda \in (0, \lambda_2)\}$ *to* (1.1) *such that*

$$\lim_{\lambda \to 0} \left(\bar{u}_\lambda - \frac{\sqrt{2}}{\varepsilon_\lambda} U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},2} \right) = 0,$$

uniformly on compact subsets of $B_1 \backslash \{0\}$,

$$\lambda e^{\bar{u}_\lambda} \rightharpoonup 8\pi\delta_0 \quad \text{in } B_{\alpha_1/2},$$

and

$$\varepsilon_\lambda \lambda e^{\bar{u}_\lambda} + |\partial_\nu U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},2}(\alpha_1)|^{-1}\delta_{\partial B_{\alpha_1}} + |\partial_\nu U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},2}(1)|^{-1}\delta_{\partial B_1} \rightharpoonup 0 \quad \text{in} \quad B_1 \backslash \{0\},$$

where ε_λ *is given by* (1.4).

Theorem 1.4 and Theorem 1.5 will be proven using perturbation arguments following a Lyapunov-Schmidt scheme. In such a construction, it is well-known that the localization of the layers is driven by some Green's functions which give the asymptotic profile. It is essential that the derivative of those Green's functions satisfy the weak reflexion law (1.5) (or (1.6)) at each of their maxima which eventually give the asymptotic positions of the layers. As usual, the success of the construction also relies on a non degeneracy condition that we now explain briefly regarding Theorem 1.5. Fix $\alpha = (\alpha_1, \ldots, \alpha_k)$ as in Theorem 1.3 (i). For $a = (a_1, \ldots, a_k)$, $\sigma = (\sigma_1, \ldots, \sigma_k)$ satisfying, for $i = 2, \ldots, k-1$,

$$\sigma_1 \in \left(\frac{\alpha_1}{4}, \frac{\alpha_2 - \alpha_1}{4} \right), \quad \sigma_i \in \left(\frac{\alpha_i - \alpha_{i-1}}{4}, \frac{\alpha_{i+1} - \alpha_i}{4} \right), \quad \sigma_k = 0, \tag{1.7}$$

and $b, \varepsilon > 0$, we denote by $U_{\varepsilon,a,b,\sigma}$ the (radial) solution of

$$\begin{cases} -\Delta U_{\varepsilon,a,b,\sigma} + U_{\varepsilon,a,b,\sigma} &= 0 \qquad \text{in } B_1 \setminus (\{0\} \cup_{i=1}^k \partial B_{\alpha_i}) \\ \lim_{r \to 0^+} -\dfrac{U_{\varepsilon,a,b,\sigma}(r)}{\ln r} &= b \\ U_{\varepsilon,a,b,\sigma|\partial B_{\alpha_i + \sigma_i}} &= 1 + \varepsilon a_i \quad \text{for any } i \in \{1, \ldots, k\}. \end{cases} \tag{1.8}$$

Let

$$U'^{\pm}_{\sigma_i}(\alpha_j) = \frac{\partial}{\partial \sigma_i} (U'^{\pm}_{\varepsilon,a,b,\sigma}(\alpha_j + \sigma_j))|_{\varepsilon,a,b,\sigma=0},$$

where $'$ is used to denote radial derivatives and $+$, resp. $-$, stands for the right, resp. left, derivative. Define the $k \times k$ tridiagonal square matrix A_k as follows :

– the elements of the diagonal are given by

$$A_{i,i} = (U'^{+}_{\sigma_i} + U'^{-}_{\sigma_i})(\alpha_i) \quad \text{for } i = 1, \ldots, k;$$

– the elements of the subdiagonal of A_k are given by

$$A_{i+1,i} = U'^{-}_{\sigma_i}(\alpha_{i+1}) \quad \text{for } i = 1, \ldots, k-1;$$

– the elements of the superdiagonal of A_k are given by

$$A_{i,i+1} = U'^{+}_{\sigma_{i+1}}(\alpha_i) \quad \text{for } i = 1, \ldots, k-1.$$

We can now state the non degeneracy condition

$$M_k := \det(A_k) \neq 0. \tag{1.9}$$

Actually, we are able to construct a family of k-layers solutions (which are singular at the origin and have a boundary layer) as soon as (1.9) holds. The limit profile is then given by $U_{b,k+1}$. The same non degeneracy condition also allows to build a family of k-layers whose limit profile is given by $\tilde{U}_{b,k+1}$. We mention that a condition of the same kind was used in [BCN17a]. Numerical simulations suggest that $M_k \neq 0$ for any $k > 0$.

However since we are only able to prove theoretically (1.9) for $k = 1$, we stated only the existence of 1-layer solutions in Theorem 1.5 and Theorem 1.4, leaving the general case as a conjecture.

Conjecture 1.1. *Let ε_λ be defined by (2.4) and $U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}$ (resp. $\tilde{U}_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}$) be defined as in Theorem 1.3 (i) (resp. (ii)), with $b = 4\frac{\varepsilon_\lambda}{\sqrt{2}}$ and $k \in \mathbb{N}\backslash\{0\}$. Then*

(i) there exists a family of radial solutions $\{u_{\lambda,k} \mid \lambda \in (0, \lambda_k)\}$ to (1.1) in B_1 such that

$$\lim_{\lambda\to 0}\left(\varepsilon_\lambda u_{\lambda,k} - \sqrt{2}U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}\right) = 0,$$

uniformly on compact subsets of $B_{\alpha_1} \cup_{i=1}^{k-1} B_{\alpha_{i+1}}\backslash B_{\alpha_i}$,

$$\lambda e^{u_{\lambda,k}} \rightharpoonup 8\pi\delta_0 \quad \text{in } B_{\alpha_1/2},$$

and

$$\varepsilon_\lambda \lambda e^{u_{\lambda,k}} + \sum_{i=1}^{k}(|\partial_\nu U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}(\alpha_i)|)^{-1}\delta_{\partial B_{\alpha_i}} \rightharpoonup 0 \quad \text{in } B_1\backslash\{0\};$$

(ii) there exists a family of radial solutions $\{\tilde{u}_{\lambda,k} \mid \lambda \in (0, \tilde{\lambda}_k)\}$ to (1.1) in B_1 such that

$$\lim_{\lambda\to 0}\left(\varepsilon_\lambda \tilde{u}_{\lambda,k} - \sqrt{2}\tilde{U}_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}\right) = 0,$$

uniformly on compact subsets of $B_{\tilde{\alpha}_1} \cup_{i=1}^{k-1} B_{\tilde{\alpha}_{i+1}}\backslash B_{\tilde{\alpha}_i}$,

$$\lambda e^{\tilde{u}_{\lambda,k}} \rightharpoonup 8\pi\delta_0 \quad \text{in } B_{\tilde{\alpha}_1/2},$$

and

$$\varepsilon_\lambda \lambda e^{\tilde{u}_{\lambda,k}} + \sum_{i=1}^{k}(|\partial_\nu \tilde{U}_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}(\tilde{\alpha}_i)|)^{-1}\delta_{\partial B_{\tilde{\alpha}_i}} \rightharpoonup 0 \quad \text{in } B_1\backslash\{0\}.$$

With the additional assumption that $M_{k-1} \neq 0$ for the assertion (i) and $M_k \neq 0$ for the assertion (ii), the conjecture holds true and is stated as Theorem 5.1. The proofs of Theorem 1.4 and Theorem 1.5 are then a consequence from the fact that $M_1 \neq 0$, see Remark 6.1.

The paper is organized as follows. In Section 2, we provide the ansatz of solution that will be used to prove Theorem 1.2. We then estimate the error introduced by our ansatz in Section 3. In Section 4, we prove the solvability of the linearized equation around our ansatz. This allows us to use a fixed point argument to prove Theorem 1.2. In Section 5, we give the proof of Theorem 5.1. Finally, we prove Theorem 1.3 and the invertibility of the matrix A_1 in Section 6 completing the proofs of Theorem 1.4 and Theorem 1.5.

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2. The approximate solution

We look for a radial solution to (1.1) concentrating at 0 and on ∂B_1. To do so, we take an ansatz of solution of the form

$$U = \begin{cases} u_0 & \text{in } [0,\delta) \\ u_1 & \text{in } [\delta, 2\delta) \\ u_2 & \text{in } [2\delta, 1-2\delta_1) \\ u_3 & \text{in } [1-2\delta_1, 1-\delta_1) \\ u_4 & \text{in } [1-\delta_1, 1]. \end{cases}$$

In a first time, let us describe intuitively our ansatz. In the previous definition, δ and δ_1 are suitable constants depending on λ. Near the origin, we want $U = u_0$ to behave approximately like U_0, the two dimensional standard bubble given by

$$U_0(r) = \ln \frac{8\mu^2}{(\mu^2\lambda + r^2)^2}, \tag{2.1}$$

for some constant $\mu > 0$. Let us recall that these functions correspond to all solutions of the problem

$$-\Delta U_0 = \lambda e^{U_0} \quad \text{in } \mathbb{R}^2, \text{ with } \lambda \int_{\mathbb{R}^2} e^{U_0} dx < \infty.$$

Near the unit sphere ∂B_1, we want that $U = u_4$ behaves up to rescaling like $W_{\tilde{\mu}} - \ln \lambda$ where $W_{\tilde{\mu}}$ is the one dimensional standard bubble solving $-w'' = e^w$ in $\mathbb{R}$, which is given by

$$W_{\tilde{\mu}}(r) = \ln \left(\frac{4}{\tilde{\mu}^2} \frac{e^{-\frac{\sqrt{2}(r-1)}{\tilde{\mu}}}}{\left(1 + e^{-\frac{\sqrt{2}(r-1)}{\tilde{\mu}}}\right)^2} \right), \tag{2.2}$$

for some $\tilde{\mu}$ depending on λ to be determined later. Far from the origin and ∂B_1, we choose $U = G$ where G is the singular at the origin Green's function given in Lemma 6.2 for some suitable constant $\tilde{b}$ depending on λ. Finally, we choose u_1 and u_3 to be linear interpolations between u_{i-1} and u_{i+1}, for $i = 1, 3$, namely

$$u_i(r) = \chi_i(r) u_{i-1}(r) + (1 - \chi_i(r)) u_{i+1}(r), \tag{2.3}$$

where $\chi_i \in C^2((0,1))$ are cut-off functions such that

$$\chi_1(r) \equiv 1 \text{ in } (0,\delta),\ \chi_1(r) \equiv 0 \text{ in } (2\delta, 1),\ |\chi_1(r)| \leq 1,\ |\chi_1'(r)| \leq c, |\chi_1''(r)| \leq c,$$

and

$$\chi_3 \equiv 1 \text{ in } (0, 1-2\delta_1),\ \chi_3 \equiv 0 \text{ in } (1-\delta_1, 1),\ |\chi_3(r)| \le 1,\ |\chi_3'(r)| \le c, |\chi_3''(r)| \le c.$$

2.1. **Construction of u_4.** First, we set ε such that

$$\ln \frac{4}{\varepsilon^2} - \ln \lambda = \frac{\sqrt{2}}{\varepsilon}, \tag{2.4}$$

and choose

$$\delta_1 = \varepsilon^\eta,\ \text{for some } \eta \in \left(\frac{2}{3}, 1\right). \tag{2.5}$$

We define u_4 in the same way as the function "u_1" of [PV15] (or [BCN17a]) with $r_0 = 1$. The construction of this function is quite lenghty so we only briefly recall it and refer to the above two papers for more details. We take u_4 as follows

$$u_4 = \underbrace{W_{\tilde\mu} - \ln\lambda + \alpha_\varepsilon}_{1^{st}\ order\ approx.} + \underbrace{v_\varepsilon + \beta_\varepsilon}_{2^{nd}\ order} + \underbrace{z_\varepsilon}_{3^{rd}\ order}.$$

The function $W_{\tilde\mu}$ has been defined in (2.2) for some $\tilde\mu = O(\varepsilon)$. We refer to Subsection 2.2 for the precise definition. We also set

$$W\left(\frac{r-1}{\tilde\mu}\right) + \ln\frac{4}{\tilde\mu^2} - \ln 4 = W_{\tilde\mu}(r).$$

The function α_ε satisfies

$$\begin{cases} -(\alpha_\varepsilon)'' - \dfrac{n-1}{r}(\alpha_\varepsilon)' &= \dfrac{n-1}{r}(w_\varepsilon^i)' - w_\varepsilon^i + \ln\lambda \quad \text{in } (0,1) \\ \alpha_\varepsilon(1) &= 0 \\ (\alpha_\varepsilon)'(1) &= 0 \end{cases}$$

and the following estimate holds, for $s \le 0$,

$$\alpha_\varepsilon(\tilde\mu s + 1) = \tilde\mu(\alpha_\varepsilon)_1(s) + \tilde\mu^2(\alpha_\varepsilon)_2(s) + O(\tilde\mu^3 s^4), \tag{2.6}$$

where

$$(\alpha_\varepsilon)_1(s) = -(n-1)\int_0^s W(\sigma)d\sigma + \frac{\tilde\mu}{\sqrt{2}\varepsilon}s^2,$$

and

$$(\alpha_\varepsilon)_2(s) = \int_0^s\int_0^\sigma (W(\rho) - \ln 4)d\rho d\sigma + (n-1)(n-2)\int_0^s\int_0^\sigma W(\rho)d\rho d\sigma$$
$$+ (n-1)\int_0^s \sigma W(\sigma)d\sigma - s^2\ln\left(\frac{\tilde\mu}{\varepsilon}\right).$$

The function v_ε solves

$$\begin{cases} -(v_\varepsilon)'' - e^{W_{\tilde\mu}}v_\varepsilon &= \tilde\mu e^{W_{\tilde\mu}}(\alpha_\varepsilon)_1\left(\dfrac{r-1}{\tilde\mu}\right) \quad \text{in } \mathbb{R} \\ v_\varepsilon(1) &= 0 \\ (v_\varepsilon)'(1) &= 0 \end{cases},$$

where $(\alpha_\varepsilon)_1$ is defined in (2.6). Moreover, we have

$$v_\varepsilon(r) = \nu_1(r-1) + \nu_2\tilde\mu + O(\tilde\mu e^{-\frac{|r-1|}{\tilde\mu}}), \tag{2.7}$$

where

$$\nu_2 \in \mathbb{R} \quad \text{and} \quad \nu_1 = -2(n-1)(1-\ln 2) + 2\ln 2\frac{\tilde{\mu}}{\varepsilon}.$$

We also set

$$v_\varepsilon(r) = \tilde{\mu} v\left(\frac{r-1}{\tilde{\mu}}\right).$$

The function β_ε is the solution of

$$\begin{cases} -(\beta_\varepsilon)'' - \dfrac{n-1}{r}(\beta_\varepsilon)' &= \dfrac{n-1}{r}(v_\varepsilon)' \quad \text{in } (0,1), \\ \beta_\varepsilon(1) &= 0, \\ (\beta_\varepsilon)'(1) &= 0, \end{cases}$$

and the following estimate holds, for $s \le 0$,

$$\beta_\varepsilon(\tilde{\mu}s + 1) = \tilde{\mu}^2(\beta_\varepsilon)_1(s) + O(\tilde{\mu}^3 s^3),$$

where

$$(\beta_\varepsilon)_1(s) = -(n-1)\int_0^s \int_0^\sigma v'(\rho)d\rho d\sigma.$$

Finally, the function z_ε satisfies

$$\begin{cases} -(z_\varepsilon)'' - e^{W_{\tilde{\mu}}} z_\varepsilon &= \tilde{\mu}^2 e^{W_{\tilde{\mu}}}\Bigg[(\alpha_\varepsilon)_2\left(\dfrac{r-1}{\tilde{\mu}}\right) + (\beta_\varepsilon)_1\left(\dfrac{r-1}{\tilde{\mu}}\right) \\ &\qquad + \dfrac{1}{2}\left((\alpha_\varepsilon)_1\left(\dfrac{r-1}{\tilde{\mu}}\right) + v\left(\dfrac{r-1}{\tilde{\mu}}\right)\right)^2\Bigg] \quad \text{in } (0,1) \\ z_\varepsilon(1) &= 0 \\ (z_\varepsilon)'(1) &= 0 \end{cases}$$

and there holds

$$z_\varepsilon(r) = \tilde{\mu}\zeta_1(r-1) + \zeta_2\tilde{\mu}^2 + O(\tilde{\mu}^2 e^{-\frac{|r-1|}{\tilde{\mu}}}), \tag{2.8}$$

for some $\zeta_j \in \mathbb{R}$, $j = 1, 2$.

2.2. **Construction of u_2.** Thanks to Lemma 6.3, we know that, for any b small enough, there exists a function G_b satisfying

$$\begin{cases} -G_b'' - \dfrac{1}{r}G_b' + G_b &= 0 \quad \text{in } (0,1) \\ \lim_{r\to 0^+} \dfrac{G_b(r)}{-\ln r} &= b \\ G_b(1) &= 1. \end{cases}$$

Arguing as in [dPPV16, Lemma 2.8] (see also Lemma 5.1), we perturb the function G_b in the following way. There exists ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, there exist $\gamma_\varepsilon \in \mathbb{R}$ and a radial function U_ε solution to

$$\begin{cases} -\Delta U_\varepsilon + U_\varepsilon &= 0 \qquad \text{in } (0,1) \\ \lim\limits_{r\to 0^+} \dfrac{U_\varepsilon(r)}{-\ln r} &= \dfrac{4}{\sqrt{2}}\varepsilon \\ U_\varepsilon(1) &= 1 + \dfrac{\sqrt{2}}{\varepsilon}(-\ln(\gamma_\varepsilon)^2 + \varepsilon\gamma_\varepsilon\nu_2) \\ U_\varepsilon'(1) &= \dfrac{1}{\gamma_\varepsilon} + \dfrac{\varepsilon}{\sqrt{2}}(-2 + 2\gamma_\varepsilon \ln 2 + \varepsilon\gamma_\varepsilon\zeta_1), \end{cases}$$

where ν_2 and ζ_1 are respectively defined in (2.7) and (2.8). We then define u_2 as

$$u_2(r) = \frac{\sqrt{2}}{\varepsilon} U_\varepsilon(r). \tag{2.9}$$

Observe that there exists $\tilde r = O(\sqrt{\varepsilon})$ such that $u_2'(\tilde r) = 0$. We denote by H the regular part of u_2, namely

$$H(r) = u_2(r) + 4\ln r. \tag{2.10}$$

Observe that thanks to (6.4) and (6.5), we have, for some constant $C > 0$,

$$H(0) < 0, \quad |H(0)| \le \frac{C}{\varepsilon}, \text{ and } \quad \lim_{r\to 0^+} H'(r) = 0. \tag{2.11}$$

We choose $\tilde\mu$ in (2.2) as $\tilde\mu = \varepsilon\gamma_\varepsilon$. Thanks to our choices of u_2 and u_4, one can show, arguing as in [BCN17a], the following estimate.

Lemma 2.1. *For any $\delta_1 < |r-1| < 2\delta_1$, we have*

$$u_4(r) - u_2(r) = O\left(\varepsilon^2 + \varepsilon|r-1|^2 + |r-1|^3 + \frac{|r-1|^4}{\varepsilon} + \exp\left(-\frac{|r-1|}{\varepsilon}\right)\right)$$

and

$$u_4'(r) - u_2'(r) = O\left(\varepsilon|r-1| + |r-1|^2 + \frac{|r-1|^3}{\varepsilon} + \frac{1}{\varepsilon}\exp\left(-\frac{|r-1|}{\varepsilon}\right)\right).$$

2.3. **Construction of u_0.** We define $u_0 = U_0 + H_0$ where U_0 is the function defined in (2.1) and H_0 is the solution to

$$\begin{cases} -\Delta H_0 + H_0 &= -U_0 \qquad \text{in } (0,\tilde r) \\ H_0'(\tilde r) &= -U_0'(\tilde r). \end{cases} \tag{2.12}$$

We introduced the function H_0 in order to get a better matching between u_0 and u_2. We choose δ such that

$$2\delta < \tilde r \text{ and } \delta = O(\sqrt{\varepsilon}). \tag{2.13}$$

Arguing in a similar way to the proof of [dPW06, Lemma 2.1], we obtain the following estimates.

Lemma 2.2. *For any $\alpha \in \left(0, \frac{1}{2}\right)$, we have, for $r \in (0,\tilde r)$,*

$$H_0(r) = H(r) - \ln(8\mu^2) + O(\lambda^\alpha), \tag{2.14}$$

$C^{0,\gamma}(B_{\tilde r})$–uniformly, for $\gamma \in [0,1)$, where $H(r)$ is defined in (2.10). Moreover, (2.14) holds uniformly in $C^1(B_{2\delta}\backslash B_\delta)$. Finally, by choosing $\mu^2 = \dfrac{e^{H(0)}}{8}$ and recalling (2.11), the estimate

$$H_0(r) = O\left(\lambda^\alpha + \frac{r^2}{\varepsilon}\right), \tag{2.15}$$

holds true for $r \in (0, \tilde r)$.

Proof. Let us consider the function $z = H_0 - H + \ln 8\mu^2$, which satisfies

$$\begin{cases} -\Delta z + z &= -\ln \dfrac{1}{(\mu^2\lambda + r^2)^2} + \ln \dfrac{1}{r^4} \quad \text{in } (0,\tilde r) \\ z'(\tilde r) &= \dfrac{4\tilde r}{\mu^2\lambda + \tilde r^2} - \dfrac{4}{\tilde r}. \end{cases}$$

By recalling (2.4) and that $\tilde r = O(\sqrt{\varepsilon})$, we deduce that

$$z'(\tilde r) = \frac{4\mu^2\lambda}{\tilde r(\mu^2\lambda + \tilde r^2)} = O(\lambda^\alpha),$$

for any $\alpha \in \left(0, \frac{1}{2}\right)$. We set $f = -\ln \dfrac{1}{(\mu^2\lambda + r^2)^2} + \ln \dfrac{1}{r^4}$ and let $p > 2$. We have

$$\int_{B_{\tilde r}} |f|^p dx = \int_{B_{\tilde r}\backslash B_{\mu\sqrt{\lambda}}} |f|^p dx + \int_{B_{\mu\sqrt{\lambda}}} |f|^p dx.$$

It is easy to see that

$$\int_{B_{\mu\sqrt{\lambda}}} |f|^p dx \le C\lambda |\ln \lambda|^p,$$

and, using the fact that $|f(r)| \le \dfrac{C\sqrt{\lambda}}{r}$, one gets

$$\int_{B_{\tilde r}\backslash B_{\mu\sqrt{\lambda}}} |f|^p dx \le C\lambda^{p/2}\tilde r^{2-p} \le \lambda^{p/2}.$$

Using elliptic regularity theory (see Lemma 6.5), we deduce that

$$\|z\|_{C^{0,\gamma}(B_{\tilde r})} \le C\lambda^\alpha$$

for any $\gamma \in (0,1)$ and $\alpha \in \left(0, \frac{1}{2}\right)$.

On the other hand, for any $q \ge 2$, since $\delta = O(\sqrt{\varepsilon})$, we have

$$\int_{B_{2\delta}\backslash B_\delta} |f|^q dx \le C\lambda^{q/2}\delta^{2-q} \le C\lambda^{q/2}\varepsilon^{\frac{1}{2}(2-q)} \le C\lambda^{\alpha q},$$

for any $\alpha \in \left(0, \frac{1}{2}\right)$. We deduce that

$$\|z\|_{C^1(B_{2\delta}\backslash B_\delta)} \le C\lambda^\alpha.$$

Finally, (2.15) is a direct consequence of the fact that $H \in C^{1,\beta}(B_{\tilde r})$, $\beta \in (0,1)$. ☐

Thanks to the previous lemma, we are able to show that u_0 and u_2 are very close for the C^1–norm in the interval $[\delta, 2\delta]$.

Lemma 2.3. *For $\delta \leq r \leq 2\delta$, we have*

$$|u_0(r) - u_2(r)| = O(\lambda^\alpha) \quad \text{and} \quad |u_0'(r) - u_2'(r)| = O(\lambda^\alpha)$$

for any $\alpha \in \left(0, \frac{1}{2}\right)$.

Proof. The proof is a direct consequence of Lemma 2.2. Indeed, by definition we have, for $r \in [\delta, 2\delta]$,

$$u_0(r) = U_0(r) + H_0(r) = \ln \frac{8\mu^2}{(\mu^2\lambda + r^2)^2} + H(r) - \ln 8\mu^2 + O(\lambda^\alpha)$$

and

$$u_2(r) = -4\ln r + H(r).$$

It follows that

$$\begin{aligned} u_0(r) - u_2(r) &= -2\ln\left(1 + \frac{\mu^2\lambda}{r^2}\right) + O(\lambda^\alpha) \\ &= O(\lambda^\alpha). \end{aligned}$$

Arguing in a similar way, one shows that

$$u_0'(r) - u_2'(r) = O\left(\frac{\mu^2\lambda}{\delta^3}\right) + O(\lambda^\alpha) = O(\lambda^\alpha).$$

□

We now look for a solution of (1.1) of the form $U + \phi$. Let us observe that $U + \phi$ is a solution to (1.1) if and only if ϕ solves

$$\left\{ \begin{array}{rcll} L(\phi) &=& N(\phi) + R(U) & \text{in } (0,1), \\ \phi'(0) &=& 0, & \\ \phi'(1) &=& 0, & \end{array} \right. \tag{2.16}$$

where

$$L(\phi) = -\Delta\phi + \phi - \lambda e^U \phi, \tag{2.17}$$

$$N(\phi) = \lambda(e^{U+\phi} - e^U - e^U\phi), \tag{2.18}$$

and

$$R(U) = -\Delta U + U - \lambda e^U. \tag{2.19}$$

3. The error estimate

In this section we estimate the terms $R(U)$ and $N(\phi)$. In order to take benefit of the estimates in [PV15], we are going to work with the norm $\|\cdot\|_*$ (see (4.1)) which is a weighted L^∞ norm on $B_{\frac{1}{2}}$ and a L^1-norm elsewhere. We begin by estimating $N(\phi)$.

Lemma 3.1. *We have, for any $\beta > 0$,*

$$N(\phi) \leq C|\phi|^2 \left\{ \begin{array}{ll} \dfrac{8\mu^2}{\lambda\left(\mu^2 + \left(\dfrac{r}{\sqrt{\lambda}}\right)^2\right)^2} & \text{if } r \leq 2\delta \\ \varepsilon^\beta & \text{if } 2\delta \leq r \leq 1 - 2\delta_1 \end{array} \right.$$

and

$$\|N(\phi)\|_{L^1\left(B_1\setminus B_{\frac{1}{2}}\right)} \le C\varepsilon^{-1}\|\phi\|^2_{L^\infty\left(B_1\setminus B_{\frac{1}{2}}\right)}. \tag{3.1}$$

Proof. First, using a Taylor's expansion, it is immediate to see that

$$N(\phi) \le C\lambda e^U|\phi|^2.$$

Therefore, the proof reduces to estimate e^U. First, we consider the case $r \in [0, 2\delta]$. In this range, using (2.15) and a Taylor's expansion, we see that

$$e^{u_0} = e^{U_0+H_0} = \frac{8\mu^2}{(\mu^2\lambda + r^2)^2} e^{O(\lambda^\alpha + \frac{r^2}{\varepsilon})} = O(\frac{8\mu^2}{(\mu^2\lambda+r^2)^2}).$$

Next, we consider $r \in [\delta, 1-2\delta_1]$. By definition of u_2, we know that it is decreasing in $r \in (0, \tilde r)$ and increasing elsewhere. Then, for $r \in [\delta, 1-2\delta_1]$, we have

$$e^{u_2(r)} \le e^{u_2(\delta)} + e^{u_2(1-2\delta_1)}.$$

Making a Taylor's expansion and using (2.9), we obtain, for some $\theta \in (1-2\delta_1, 1)$,

$$u_2(1-2\delta_1) = u_2(1) - 2\delta_1 u_2'(1) + 2\delta_1^2 u_2''(\theta) \le \frac{\sqrt{2}}{\varepsilon} - \delta_1 u_2'(1).$$

Thus, recalling the relation (2.4) and the definition of δ_1, we deduce that

$$\lambda e^{u_2(1-2\delta_1)} \le C\varepsilon^{-2} e^{\frac{\sqrt{2}}{\varepsilon}(-\delta_1 u_2'(1))} \le C\varepsilon^\beta,$$

for any $\beta > 0$. On the other hand, using (2.9), we see that $e^{u_2(\delta)} \le \dfrac{C}{\delta^4} \le C\varepsilon^{-8}$. The estimate then follows by noticing that $\lambda\varepsilon^{-8} \le \varepsilon^\beta$ for any $\beta > 0$. Finally, we refer to [PV15, Lemma 4.3] for the proof of (3.1). $\square$

Next, we estimate $R(U)$.

Lemma 3.2. *Let $\alpha \in \left(0, \frac{1}{2}\right)$. We have*

$$R(U) \le C \begin{cases} \dfrac{8\mu^2}{\lambda\left(\mu^2 + \left(\frac{r}{\sqrt{\lambda}}\right)^2\right)^2}(\lambda^\alpha + \dfrac{r^2}{\varepsilon}) & \text{if } r \le \delta \\ \varepsilon^\beta & \text{if } \delta \le r \le 1-2\delta_1 \end{cases}$$

for any $\beta > 0$, and

$$\|R(U)\|_{L^1\left(B_1\setminus B_{\frac{1}{2}}\right)} \le C\varepsilon^{1+\sigma}$$

for some $\sigma > 0$.

Proof. First, we consider the case $r \le \delta$ so that $U(r) = u_0(r) = U_0(r) + H_0(r)$. Combining (2.1), (2.12), and (2.15), we infer that

$$\begin{aligned} R(u_0) &= -\Delta(U_0 + H_0) + U_0 + H_0 - \lambda e^{U_0+H_0} \\ &= \lambda e^{U_0}\left(1 - e^{H_0}\right) \\ &\le C\frac{8\mu^2}{\lambda\left(\mu^2 + \left(\frac{r}{\sqrt{\lambda}}\right)^2\right)^2}(\lambda^\alpha + \frac{r^2}{\varepsilon}). \end{aligned} \tag{3.2}$$

Next, when $2\delta \le r \le 1-2\delta_1$, we have $U(r) = u_2(r)$. Arguing as in the previous lemma, we obtain

$$R(u_2(r)) = \lambda e^{u_2(r)} \le C\varepsilon^{\beta} \tag{3.3}$$

for any $\beta > 0$.

On the other hand, it is proven in [PV15, Lemma 4.2] that

$$\|R(u_4)\|_{L^1\left(B_1\setminus B_{1-\delta_1}\right)} = O(\varepsilon^{1+\sigma}) \quad \text{for some } \sigma > 0. \tag{3.4}$$

Finally, we consider the two intermediate regimes. First, let us consider the case $\delta \le r \le 2\delta$. In this interval, we have $U(r) = u_1(r)$. Using (2.3), we get

$$\begin{aligned} R(u_1) &= \chi_1 R(u_0) + (1-\chi_1)R(u_2) - 2\chi_1'(u_0' - u_2') + (-\Delta\chi_1 + \chi_1)(u_0 - u_2) \\ &\quad + \lambda\chi_1 e^{u_0} + \lambda(1-\chi_1)e^{u_2} - \lambda e^{\chi_1 u_0 + (1-\chi_1)u_2} \\ &\le R(u_0) + R(u_2) + C\left(\frac{|u_0' - u_2'|}{\delta} + \frac{|u_0 - u_2|}{\delta^2}\right) \\ &\quad + \lambda e^{u_2} + \lambda e^{u_0}\left(e^{(\chi_1 - 1)(u_2 - u_0)} - 1\right). \end{aligned} \tag{3.5}$$

First, using a Taylor's expansion and Lemma 2.3, we have

$$\lambda e^{u_0}\left(e^{(\chi_1-1)(u_2-u_0)} - 1\right) \le \lambda e^{u_0}|u_0 - u_2| \le \lambda^{1+\alpha} e^{u_0}.$$

From Lemma 2.3 again, we get

$$\frac{|u_0' - u_2'|}{\delta} + \frac{|u_0 - u_2|}{\delta^2} \le C\lambda^{\alpha}\delta^{-2}.$$

Plugging these two last estimates into (3.5) and using (3.2), (3.3), we obtain

$$\begin{aligned} R(u_1) &= O\left(\sup_{\delta\le r\le 2\delta} \frac{8\mu^2}{\lambda\left(\mu^2 + \left(\frac{r}{\sqrt{\lambda}}\right)^2\right)^2} + \varepsilon^{\beta} + \frac{\lambda^{\alpha}}{\delta^2}\right) \\ &= O\left(\frac{\lambda}{\delta^3} + \varepsilon^{\beta} + \frac{\lambda^{\alpha}}{\delta^2}\right) = O\left(\varepsilon^{\beta}\right). \end{aligned}$$

Finally, when $1-2\delta_1 \le r \le 1-\delta_1$, arguing as above, we have

$$\begin{aligned} R(u_3) &= \chi_3 R(u_2) + (1-\chi_3)R(u_4) - 2\chi_3'(u_4' - u_2') + (-\Delta\chi_3 + \chi_3)(u_4 - u_2) \\ &\quad + \lambda\chi_3 e^{u_4} + \lambda(1-\chi_3)e^{u_2} - \lambda e^{\chi_3 u_4 + (1-\chi_3)u_2} \\ &\le R(u_2) + R(u_4) + C\left(\frac{|u_4' - u_2'|}{\delta} + \frac{|u_4 - u_2|}{\delta^2}\right) \\ &\quad + \lambda e^{u_2} + \lambda e^{u_4}\left|u_4 - u_2\right|. \end{aligned} \tag{3.6}$$

Using Lemma 2.1, we obtain

$$\int_{1-2\delta_1}^{1-\delta_1} \left(\frac{|u_4' - u_2'|}{\delta} + \frac{|u_4 - u_2|}{\delta^2}\right) r\,dr = O(\delta_1^2) = O(\varepsilon^{1+\sigma})$$

and

$$\int_{1-2\delta_1}^{1-\delta_1} \lambda e^{u_4} |u_4 - u_2| \, r dr = O(\lambda\varepsilon^2).$$

Thanks to (3.3) and (3.4), we see that

$$\int_{1-2\delta_1}^{1-\delta_1} (R(u_2) + R(u_4) + \lambda e^{u_2}) \, r dr = O(\varepsilon^{1+\sigma}).$$

Plugging the three previous estimates into (3.6), we obtain

$$\|R(u_3)\|_{L^1\left(B_1 \setminus B_{\frac{1}{2}}\right)} = O(\varepsilon^{1+\sigma}).$$

This concludes the proof of the lemma.

□

4. Inversibility of the linearized operator

In this section we develop an inversibility theory for the operator L defined in (2.17). To do so, we utilize ideas used in [dPKM05, dPR15, dPW06, PV15]. First, we define the norms

$$\|u\|_* = \max\{|\log\lambda| \|\tilde{\chi}_1 u\|_\star, \|\tilde{\chi}_2 u\|_{L^1}\} \tag{4.1}$$

and

$$\|u\|_{**} = \max\{\|\tilde{\chi}_1 u\|_\star, \|\tilde{\chi}_2 u\|_{L^1}\},$$

where

$$\tilde{\chi}_1(r) = \begin{cases} 1 & \text{if } r \le \frac{1}{2} \\ 0 & \text{if } r \ge \frac{3}{4} \end{cases}, \quad \tilde{\chi}_2(r) = \begin{cases} 1 & \text{if } r \ge \frac{1}{2} \\ 0 & \text{if } r \le \frac{1}{4} \end{cases},$$

and

$$\|u\|_\star = \sup \frac{\lambda |u(r)|}{\lambda + \left(1 + \frac{r}{\sqrt{\lambda}}\right)^{-2-\nu}} = \sup f_\lambda(r)|u(r)|$$

for some $\nu \in (0,1)$. The following proposition is the main result of the section.

Proposition 4.1. *There exist positive constants λ_0 and C such that for any $\lambda \in (0, \lambda_0)$ and for any $h \in L^\infty(B_1)$, there exists a unique radial function $\phi \in W^{2,2}(B_1)$ solution of the problem*

$$\begin{cases} L(\phi) = h & \text{in } B_1 \\ \phi'(1) = 0, \end{cases} \tag{4.2}$$

which satisfies

$$\|\phi\|_{L^\infty(B_1)} \le C \|h\|_*. \tag{4.3}$$

Rather than proving directly this statement, we first prove a priori estimates for the solution of (4.2) when ϕ is orthogonal to

$$z_0(r) = \frac{r^2 - \lambda\mu^2}{r^2 + \lambda\mu^2}.$$

It is important to notice that z_0 solves

$$-\Delta z_0 = \frac{8\lambda\mu^2}{(\lambda\mu^2+r^2)^2} z_0, \tag{4.4}$$

which is the linearization of the equation $-\Delta v = e^v$ around the radial solution $v(r) = U_0(r) + \log\lambda = \log\frac{8\lambda\mu^2}{(\lambda\mu^2+|r|^2)^2}$. It is well-known that the only bounded radial solutions of (4.4) are multiples of z_0 (see [CL02, Lemma 2.1]).

Consider a large but fixed number $R_0 > 0$ and a radial smooth cut-off function $\chi(r)$ such that $\chi(r) = 1$ if $r \leq R_0\sqrt{\lambda}$ and $\chi(r) = 0$ if $r > (R_0+1)\sqrt{\lambda}$.

Lemma 4.1. *There exist positive constants λ_0 and C such that for, any $\lambda \in (0, \lambda_0)$, any radial solution $\phi \in W^{2,2}(B_1)$ to*

$$\begin{cases} L(\phi) &= h \quad \text{in } B_1 \\ \phi'(1) &= 0 \\ \displaystyle\int_{B_1} \chi z_0 \phi \, dx &= 0 \end{cases} \tag{4.5}$$

satisfies

$$\|\phi\|_{L^\infty(B_1)} \leq C \|h\|_{**}.$$

Proof. Assume by contradiction that there exist a sequence of positive numbers $\lambda_n \to 0$ and a sequence of solutions ϕ_n to (4.5) such that

$$\|\phi_n\|_{L^\infty(B_1)} = 1 \quad \text{and} \quad \|h_n\|_{**} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.6}$$

We denote by ε_n the sequence defined by the relation

$$\ln\frac{4}{\varepsilon_n^2} - \ln\lambda_n = \frac{\sqrt{2}}{\varepsilon_n}.$$

We also use the notation $o_n(1)$ to denote functions $f_n(r)$ such that $\lim_{n\to\infty} f_n(r) = 0$ uniformly in r. Our goal is to prove that $\phi_n(r) = o_n(1)$, for any $r \in [0,1]$, which yields to a contradiction with (4.6). We split the proof into 4 steps. In the first one we prove that $\phi_n(r) = C\phi_n(1/2) + o_n(1)$ for $r \in \left[2\delta, \frac{1}{2}\right]$ and some constant $C \in \mathbb{R}$. In the second step we show that $\phi_n = o_n(1)$ when $r \in \left[\frac{1}{2}, 1\right]$ and finally in the last two steps we consider the case $r \in [0, 2\delta]$.

Step 1. There holds $\phi_n(r) = o_n(1)$ for $r \in \left[2\delta, \frac{1}{2}\right]$.

First, we recall that for $r \in \left[2\delta, \frac{1}{2}\right]$, $U(r) = u_2(r)$. Observe that thanks to (3.3), we have $\lambda_n e^{u_2} = O(\varepsilon_n^{1+\sigma})$ for any $\sigma > 0$. Since by assumption $\|h_n\|_\infty \to 0$, it is easy to see that, up to subsequence, ϕ_n converges uniformly on compact subsets of $B_{\frac{1}{2}} \setminus \{0\}$ to a function $\hat{\phi} \in H^1\left(B_{\frac{1}{2}}\right) \cap L^\infty\left(B_{\frac{1}{2}}\right)$ solution to

$$-\Delta\hat{\phi} + \hat{\phi} = 0 \quad \text{in } B_{\frac{1}{2}} \setminus \{0\}. \tag{4.7}$$

We claim that $\hat{\phi} \equiv 0$. To prove this, let us consider the unique radial solution Φ of the problem

$$\begin{cases} -\Delta \Phi + \Phi &= \delta_0 \quad \text{in } B_{\frac{1}{2}} \\ \Phi\left(\frac{1}{2}\right) &= 0. \end{cases}$$

It is well-known that

$$\Phi(x) = -\frac{1}{2\pi} \log|x| + H(x)$$

for some smooth function H. Since $\hat{\phi} \in L^\infty\left(B_{\frac{1}{2}}\right)$, for any sufficiently small $\tilde{\varepsilon}$ and τ, we have

$$|\hat{\phi}(\tau) - \hat{\phi}(1/2)| \le \tilde{\varepsilon}\Phi(\tau).$$

By testing (4.7) against $\varphi = \max(\hat{\phi} - \hat{\phi}(1/2) - \tilde{\varepsilon}\Phi, 0)$, integrating by parts over $B_{\frac{1}{2}} \setminus B_\tau$, and using that $\varphi = 0$ on $\partial(B_{\frac{1}{2}} \setminus B_\tau)$, we deduce that $\varphi \equiv 0$, i.e. $\hat{\phi} - \hat{\phi}(1/2) \le \tilde{\varepsilon}\Phi$ in $B_{\frac{1}{2}} \setminus B_\tau$. Arguing as above with $\varphi = \min(\hat{\phi} - \hat{\phi}(1/2) + \tilde{\varepsilon}\Phi, 0)$, we conclude that $|\hat{\phi} - \hat{\phi}(1/2)| \le \tilde{\varepsilon}\Phi$ in $B_{\frac{1}{2}} \setminus B_\tau$. Passing to the limit $\tilde{\varepsilon} \to 0$ and then $\tau \to 0$, we deduce that $\hat{\phi} \equiv \hat{\phi}(1/2)$. Since the only constant solution to (4.7) is zero, we deduce $\hat{\phi}(1/2) \equiv \hat{\phi} \equiv 0$. This implies that $\phi_n(r) = o_n(1)$ for $r \in \left[2\delta, \frac{1}{2}\right]$.

Step 2. We have that $\phi_n(r) = o_n(1)$ for $r \in \left[\frac{1}{2}, 1\right]$.

We set $\psi_n(s) = \phi_n(\varepsilon_n s + 1)$ for $s \in [-\varepsilon_n^{-1}, 0]$. Then, since ψ_n is bounded, by arguing as in [PV15, Proposition 5.1] it is possible to show that $\psi_n \to \psi$ C^2–uniformly on compact subsets of $(-\infty, 0]$ where ψ satisfies

$$\begin{cases} -\psi'' &= e^\psi \quad \text{in } \mathbb{R}^- \\ \psi'(0) &= 0 \\ \|\psi\|_{L^\infty} &\le 1. \end{cases}$$

We know (see [Gro06]) that any solution ψ to $-\psi'' = e^\psi$ is of the form

$$\psi(s) = a\frac{e^{\sqrt{2}s} - 1}{e^{\sqrt{2}s} + 1} + b\left(-2 + \sqrt{2}s\frac{e^{\sqrt{2}s} - 1}{e^{\sqrt{2}s} + 1}\right)$$

for some $a, b \in \mathbb{R}$. Since $\|\psi\|_\infty \le 1$, we deduce that $a = b = 0$.

Next, we denote by $G(r,t)$ the radial Green's function associated to the operator $(-\Delta \cdot + \cdot)$ satisfying $G\left(r, \frac{1}{2}\right) = G'(r,1) = 0$ and singular at the point $r \in \left(\frac{1}{2}, 1\right)$. Now, using Green's formula, we have, for $\frac{1}{2} \le r \le 1$,

$$\begin{aligned} \phi_n(r) - G'\left(r, \frac{1}{2}\right)\phi_n\left(\frac{1}{2}\right) &= \int_{\frac{1}{2}}^1 G(r,t)h_n(t)dt + \lambda_n \int_{\frac{1}{2}}^1 G(r,t)e^{U_{\lambda_n}}\phi_n(t)dt \\ &= \int_{\frac{1}{2}}^1 G(r,t)h_n(t)dt + G(r,1)\varepsilon_n\lambda_n \int_{-\frac{1}{2\varepsilon_n}}^0 e^{U_{\lambda_n}(\varepsilon_n s+1)}\psi_n(s)ds \\ &\quad + \varepsilon_n\lambda_n \int_{-\frac{1}{2\varepsilon_n}}^0 (G(r,\varepsilon_n s+1) - G(r,1))e^{U_{\lambda_n}(\varepsilon_n s+1)}\psi_n(s)ds. \end{aligned}$$

From Step 1, we infer that $\|h_n\|_{**} \to 0$ as $n \to \infty$, and therefore

$$G'\left(r, \frac{1}{2}\right) \phi_n \left(\frac{1}{2}\right) + \int_{\frac{1}{2}}^{1} G(r,t) h_n(t) dt = o_n(1),$$

since G is bounded in C^1. Arguing as in [PV15], one shows that

$$\varepsilon_n \lambda_n \int_{-\frac{1}{2\varepsilon_n}}^{0} (G(r, \varepsilon_n s + 1) - G(r,1)) e^{u_{\lambda_n}(\varepsilon_n s+1)} \psi_n(s) ds = o_n(1).$$

From this, we get

$$\phi_n(r) = C_n G(r,1) + o_n(1),$$

where $C_n = \varepsilon_n \lambda_n \int_{-\frac{1}{2\varepsilon_n}}^{0} e^{U_{\lambda_n}(\varepsilon_n s+1)} \psi_n(s) ds$. Evaluating the previous expression at $r = 1$, we obtain

$$\phi_n(1) = \psi_n(0) = o_n(1) = C_n G(1,1) + o_n(1).$$

Since $G(1,1) \neq 0$, we deduce $C_n = 0$ and therefore $\phi_n = o_n(1)$ for $r \in \left[\frac{1}{2}, 1\right]$.

In the following steps it is convenient to work with rescaled variables. We set $s = \frac{r}{\sqrt{\lambda_n}}$, $\tilde{\phi}_n(s) = \phi_n(\sqrt{\lambda_n} s)$, $\tilde{U}(s) = U(\sqrt{\lambda_n} s) + 2 \ln \lambda_n$, $\tilde{h}_n(s) = \lambda_n h_n(\sqrt{\lambda_n} s)$, and $\tilde{L} = -\Delta + \lambda_n - e^{\tilde{U}}$. We also define (with some abuse of notation)

$$\|\tilde{h}\|_\star := \sup_{s \in [0, \lambda_n^{-1/2}/4]} \frac{\tilde{h}(s)}{\lambda_n + (1+s)^{-2-\nu}} = \|h\|_\star, \tag{4.8}$$

for functions $\tilde{h}$ defined in the rescaled variable.

Step 3. Up to subsequence, we have that $\tilde{\phi}_n \to 0$ as $n \to \infty$ uniformly over compact sets of $\mathbb{R}^2$.

It is easy to see that $\tilde{\phi}_n$ satisfies

$$\tilde{L}(\tilde{\phi}_n(s)) = \tilde{h}_n(s).$$

Elliptic estimates imply that, up to subsequence, $\tilde{\phi}_n$ converges uniformly over compact sets of $\mathbb{R}^2$ to a bounded solution $\tilde{\phi}$ of

$$-\Delta \tilde{\phi} = e^{\hat{U}} \tilde{\phi} \quad \text{in } \mathbb{R}^2.$$

This implies that there exists a constant C_0 such that $\tilde{\phi} = C_0 \tilde{Z}_0(s)$, where

$$\tilde{Z}_0(s) = z_{0,n}(\sqrt{\lambda_n} s), \quad \text{with } z_{0,n} = \frac{r^2 - \lambda_n \mu^2}{r^2 + \lambda_n \mu^2}.$$

From the orthogonality condition on ϕ_n we have

$$\int_{B_1} \chi z_{0,n} \phi_n dx = \lambda_n \int_{B_{\lambda_n^{-1/2}}} \tilde{\chi} \tilde{Z}_0 \tilde{\phi}_n dx = 0,$$

where $\tilde{\chi}(s) = \chi(\sqrt{\lambda_n} s)$. Passing to the limit yields

$$\int_{\mathbb{R}^2} \tilde{\chi} \tilde{Z}_0 \tilde{\phi} dx = 0,$$

which implies $C_0 = 0$. This gives the result.

The final step is based on the maximum principle.

Step 4. We have that $\phi_n(r) = o_n(1)$ for $r \leq 2\delta$.

Let $\tilde{\delta} > 0$ be a fixed constant such that $2\delta < 2\tilde{\delta} < 1/4$. Next, we show that there exists a constant $C > 0$, independent of n, such that

$$\|\tilde{\phi}_n\|_{L^\infty\left(B_{2\tilde{\delta}\lambda_n^{-1/2}}\right)} \leq C\left[\sup_{s\leq R}|\tilde{\phi}_n(s)| + \|\tilde{h}_n\|_\star\right], \tag{4.9}$$

where $R > 0$ is a large but fixed real number. To prove this, we need the following version of the maximum principle. We claim that there exists a fixed number $R_1 > 0$ such that for all $R > R_1$ if $\tilde{L}(Z) > 0$ in $A_{\tilde{\delta}} := B_{2\tilde{\delta}\lambda_n^{-1/2}} \setminus B_R$ and $Z \geq 0$ on $\partial A_{\tilde{\delta}}$ then $Z \geq 0$ in $A_{\tilde{\delta}}$. To prove this claim, we consider the function $Z_0(s) = \frac{s^2-1}{s^2+1}$. Observe that it satisfies

$$-\Delta Z_0 = \frac{8}{(1+s^2)^2}Z_0 \quad s \in \mathbb{R}^2.$$

We define the function $Z(s) = Z_0(\alpha s)$ for some constant α that we will fix afterwards. Observe that

$$-\Delta Z = \frac{8\alpha^2}{(\alpha^2 s^2+1)^2}\frac{\alpha^2 s^2-1}{\alpha^2 s^2+1}.$$

In particular, if $\alpha^2 s^2 > 100$ then $-\Delta Z \geq \frac{2}{\alpha^2 s^4}$. On the other hand, we have

$$e^{\tilde{U}}Z = O\left(\frac{8\mu^2}{(\mu^2+s^2)^2}\right)\frac{\alpha^2 s^2-1}{\alpha^2 s^2+1} \leq \frac{C}{s^4},$$

where C is a constant independent of α. We get

$$\tilde{L}(Z) = -\Delta Z + \lambda_n Z - e^{\tilde{U}}Z \geq \frac{1}{s^4}\left(\frac{2}{\alpha^2} - C\right).$$

Hence if α is chosen small and fixed, and $R > 0$ is sufficiently large depending on α, then $\tilde{L}(Z) > 0$ and $Z > 0$ in $A_{\tilde{\delta}}$, which gives the result.

Thanks to this maximum principle, we are in position to prove (4.9). Let $R_2 > \max\{R_1, R_0\}$. Consider the unique solution ψ_n^0 to

$$\begin{cases} -\Delta\psi_n^0 + \lambda_n\psi_n^0 - \lambda_n = 0 & \text{in } B_{2\tilde{\delta}\lambda_n^{-1/2}} \setminus B_{R_2} \\ \psi_n^0 = 0 & \text{on } \partial B_{R_2} \\ \psi_n^0 = |\tilde{\phi}_n| & \text{on } \partial B_{2\tilde{\delta}\lambda_n^{-1/2}}, \end{cases}$$

and let $\psi^1 = 1 - s^{-\nu}$. We set $\psi_n = \psi_n^0 + \psi^1$. For $s > R_2$, we have

$$\tilde{L}(\psi_n) \geq \lambda_n + \nu^2 s^{-2-\nu} - O(e^{\tilde{U}}) \geq \frac{\nu^2}{2}s^{-2-\nu} + \lambda_n,$$

since

$$e^{\tilde{U}} = O(s^{-4}).$$

We let $\bar{\phi}_n = C_1\left[\max_{s\in(0,R_2)}|\tilde{\phi}_n(s)| + \|\tilde{h}_n\|_\star\right]\psi_n$ for a constant C_1 independent of n. Observe that if $C_1 \geq \dfrac{4}{\nu^2}$ then

$$\tilde{L}(\bar{\phi}_n) \geq 2\|\tilde{h}_n\|_\star(s^{-2-\nu}+\lambda_n) \geq |\tilde{h}_n|\frac{2(s^{-2-\nu}+\lambda_n)}{((1+s)^{-2-\nu}+\lambda_n)} \geq |\tilde{h}_n| = |\tilde{L}(\tilde{\phi}_n)|$$

in $B_{2\tilde{\delta}\lambda_n^{-1/2}} \setminus B_{R_2}$, since $\dfrac{2(s^{-2-\nu}+\lambda_n)}{((1+s)^{-2-\nu}+\lambda_n)} \geq 1$ for $s \in [R_2,+\infty)$ (taking R_2 larger if necessary). On the other hand, for $C_1 \geq (1-R_2^{-\nu})^{-1}$ we have

$$\bar{\phi}_n \geq |\tilde{\phi}_n| \quad \text{on } \partial B_{2\tilde{\delta}\lambda_n^{-1/2}} \setminus B_{R_2}.$$

Applying the maximum principle and taking into account that ψ_n is uniformly bounded (since $|\tilde{\phi}_n| \leq 1$ for all n), we get

$$|\tilde{\phi}_n(s)| \leq C\left[\max_{s\in(0,R_2)}|\tilde{\phi}_n(s)| + \|\tilde{h}_n\|_\star\right]$$

for every $s \in B_{2\tilde{\delta}\lambda_n^{-1/2}} \setminus B_{R_2}$. From this, we deduce (4.9).

Noting that $\|\tilde{h}_n\|_\star \leq \|h_n\|_{**}$, we conclude from the previous steps that $\|\phi_n\|_{L^\infty(B_1)} = o_n(1)$ which contradicts the fact that $\|\phi_n\|_{L^\infty(B_1)} = 1$. This completes the proof. □

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. We reuse the notation introduced in the proof of the previous lemma. For a scaled function $\tilde{g}(s) = \lambda g(\sqrt{\lambda}s)$, with $s = r/\sqrt{\lambda}$, we define

$$\|\tilde{g}\|_{**} := \|g\|_{**}. \tag{4.10}$$

Let $R > R_2 + 1$ be a large fixed number, $\delta < 1/4$, and $\hat{z}_0$ be the solution of the problem

$$\begin{cases} -\Delta\hat{z}_0 &= \frac{8\mu^2}{(\mu^2+s^2)^2}\hat{z}_0 \quad \text{in} B_{\delta\lambda^{-1/2}}\backslash B_R \\ \hat{z}_0(R) &= \tilde{Z}_0(R) \\ \hat{z}_0\left(\frac{\delta}{\sqrt{\lambda}}\right) &= 0, \end{cases}$$

where $\tilde{Z}_0$ is defined in Step 3 of the proof of Lemma 4.1. A direct computation shows that

$$\hat{z}_0(s) = \tilde{Z}_0(s)\left[1 - \frac{\displaystyle\int_R^s \frac{dt}{t\tilde{Z}_0^2(t)}}{\displaystyle\int_R^{\frac{\delta}{\sqrt{\lambda}}} \frac{dt}{t\tilde{Z}_0^2(t)}}\right].$$

We consider smooth cut-off functions $\eta_1(s)$ and $\eta_2(s)$ with the following properties: $\eta_1(s) = 1$ for $s < R$, $\eta_1(s) = 0$ for $s > R+1$, $|\eta_1'(s)| \leq 2$, $\eta_2(s) = 1$ for $s < \frac{\delta}{2\sqrt{\lambda}}$, $\eta_2(s) = 0$ for $s > \frac{\delta}{\sqrt{\lambda}}$, $|\eta_2'(s)| \leq C\sqrt{\lambda}$, and $|\eta_2''(s)| \leq C\lambda$. We then define the test function

$$\tilde{z}_0 = \eta_1\tilde{Z}_0 + (1-\eta_1)\eta_2\hat{z}_0.$$

Let ϕ be a solution to (4.2). As previously, we denote $\tilde{\phi}(s) = \phi(\sqrt{\lambda}s)$ and we let $\tilde{\chi}(s) = \chi(\sqrt{\lambda}s)$. Next, we modify $\tilde{\phi}$ so that the orthogonality condition with respect to $\tilde{z}_0$ is satisfied. We let

$$\hat{\phi} = \tilde{\phi} + A\tilde{z}_0,$$

where the number A is such that

$$A \int_{B_{\lambda^{-1/2}}} \tilde{\chi}|\tilde{z}_0|^2 dx + \int_{B_{\lambda^{-1/2}}} \tilde{\chi}\tilde{z}_0\tilde{\phi} dx = 0.$$

Then

$$\tilde{L}(\hat{\phi}) = \tilde{h} + A\tilde{L}(\tilde{z}_0), \tag{4.11}$$

and $\int_{B_{\lambda^{-1/2}}} \tilde{\chi}\tilde{z}_0\hat{\phi} dx = 0$. Recalling (4.10), the previous lemma thus allows us to estimate

$$\|\phi\|_{L^\infty(B_1)} = \|\hat{\phi}\|_{L^\infty(B_{\lambda^{-1/2}})} \leq C\left[\|\tilde{h}\|_{**} + |A|\|\tilde{L}(\tilde{z}_0)\|_{**}\right]. \tag{4.12}$$

Observe that $\tilde{z}_0 = 0$ for $s > \lambda^{-1/2}/4$. Thus, remembering (4.8), we have

$$\|\tilde{L}(\tilde{z}_0)\|_{**} = \|\tilde{L}(\tilde{z}_0)\|_{\star}.$$

Now, let us estimate the size of $|A|\|\tilde{L}(\tilde{z}_0)\|_{\star}$. Testing equation (4.11) with $\tilde{z}_0$ and integrating by parts, we find

$$\langle\hat{\phi}, \tilde{L}(\tilde{z}_0)\rangle = \langle\tilde{h}, \tilde{z}_0\rangle + A\langle\tilde{L}(\tilde{z}_0), \tilde{z}_0\rangle,$$

where $\langle f, g\rangle = \int_{B_{\lambda^{-1/2}}} fg dx$. Combining this with (4.12),

$$\int_{B_{\lambda^{-1/2}}} |\hat{\phi}||\tilde{L}(\tilde{z}_0)| dx \leq C\|\hat{\phi}\|_\infty\|\tilde{L}(\tilde{z}_0)\|_{\star}, \quad \text{and} \quad \int_{B_{\lambda^{-1/2}}} |\tilde{h}||\tilde{z}_0| dx \leq C\|\tilde{h}\|_{\star},$$

yield

$$A\langle\tilde{L}(\tilde{z}_0), \tilde{z}_0\rangle \leq C\|\tilde{h}\|_{\star}\left[1 + \|\tilde{L}(\tilde{z}_0)\|_{\star}\right] + C|A|\|\tilde{L}(\tilde{z}_0)\|_{\star}^2. \tag{4.13}$$

We next measure the size of $\|\tilde{L}(\tilde{z}_0)\|_{\star}$. We have

$$\tilde{L}(\tilde{z}_0) = \lambda\tilde{z}_0 + 2\nabla\eta_1\nabla(\hat{z}_0 - \tilde{Z}_0) + \Delta\eta_1(\hat{z}_0 - \tilde{Z}_0) - 2\nabla\eta_2\nabla\hat{z}_0 - \Delta\eta_2\hat{z}_0. \tag{4.14}$$

It is easy to see that, for $s \in (R, R+1)$,

$$|\tilde{Z}_0 - \hat{z}_0| = |\tilde{Z}_0 \frac{\int_R^r \frac{dt}{t\tilde{Z}_0^2(t)}}{\int_R^{\frac{\delta}{\sqrt{\lambda}}} \frac{dt}{t\tilde{Z}_0^2(t)}}| \leq C|\log\lambda|^{-1} \quad \text{and} \quad |\tilde{Z}_0' - \hat{z}_0'| \leq C|\log\lambda|^{-1}.$$

On the other hand, for $s \in \left(\frac{\delta}{2\sqrt{\lambda}}, \frac{\delta}{\sqrt{\lambda}}\right)$, we have

$$|\hat{z}_0| \leq C|\log\lambda|^{-1} \text{ and } \quad |\hat{z}_0'| \leq C\sqrt{\lambda}|\log\lambda|^{-1}. \tag{4.15}$$

We conclude that

$$\|\tilde{L}(\tilde{z}_0)\|_{\star} \leq C|\log\lambda|^{-1}. \tag{4.16}$$

Finally, we estimate $\langle \tilde{L}(\tilde{z}_0), \tilde{z}_0\rangle$. We decompose

$$\langle \tilde{L}(\tilde{z}_0), \tilde{z}_0\rangle = \int_{B_{R+1}\setminus B_R} \tilde{L}(\tilde{z}_0)\tilde{z}_0 dx + \int_{B_{\frac{\delta}{\sqrt{\lambda}}}\setminus B_{\frac{\delta}{2\sqrt{\lambda}}}} \tilde{L}(\tilde{z}_0)\tilde{z}_0 dx + O(\sqrt{\lambda}).$$

Using (4.14) and (4.15), we get

$$\begin{aligned}\left|\int_{B_{\frac{\delta}{\sqrt{\lambda}}}\setminus B_{\frac{\delta}{2\sqrt{\lambda}}}} \tilde{L}(\tilde{z}_0)\tilde{z}_0 dx\right| &\leq C\int_{B_{\frac{\delta}{\sqrt{\lambda}}}\setminus B_{\frac{\delta}{2\sqrt{\lambda}}}} |\nabla \eta_2||\nabla \hat{z}_0||\hat{z}_0|dx + C\int_{B_{\frac{\delta}{\sqrt{\lambda}}}\setminus B_{\frac{\delta}{2\sqrt{\lambda}}}} |\Delta\eta_2||\hat{z}_0|^2 x \\ &\quad + \lambda \int_{B_{\frac{\delta}{\sqrt{\lambda}}}\setminus B_{\frac{\delta}{2\sqrt{\lambda}}}} |\hat{z}_0|^2 dx \\ &\leq C|\log\lambda|^{-2}. \end{aligned} \tag{4.17}$$

On the other hand, we have

$$\begin{aligned} I := \int_{B_{R+1}\setminus B_R} \tilde{L}(\tilde{z}_0)\tilde{z}_0 dx = \\ 2\int_{B_{R+1}\setminus B_R} \nabla\eta_1 \nabla(\hat{z}_0 - \tilde{Z}_0)\tilde{z}_0 dx + \int_{B_{R+1}\setminus B_R} \Delta\eta_1(\hat{z}_0 - \tilde{Z}_0)\tilde{z}_0 dx + O(\sqrt{\lambda}).\end{aligned}$$

Thus, integrating by parts, we find

$$I = \int_{B_{R+1}\setminus B_R} \nabla\eta_1 \nabla(\hat{z}_0 - \tilde{Z}_0)\tilde{z}_0 dx - \int_{B_{R+1}\setminus B_R} \nabla\eta_1(\hat{z}_0 - \tilde{Z}_0)\nabla\tilde{z}_0 dx + O(\sqrt{\lambda}).$$

Now, we observe that, for $s \in (R, R+1)$, $|\tilde{Z}_0(s) - \hat{z}_0(s)| \leq C|\log\lambda|^{-1}$, while $|\tilde{z}_0'(s)| \leq \frac{1}{R^3} + \frac{1}{R}|\log\lambda|^{-1}$. Thus

$$\left|\int_{B_{R+1}\setminus B_R} \nabla\eta_1(\hat{z}_0 - \tilde{Z}_0)\nabla\tilde{z}_0 dx\right| \leq \frac{D}{R^3}|\log\lambda|^{-1},$$

where D is a constant that does not depend on R. Note that

$$\begin{aligned}\int_{B_{R+1}\setminus B_R} \nabla\eta_1\nabla(\hat{z}_0 - \tilde{Z}_0)\tilde{z}_0 dx &= 2\pi\int_R^{R+1} \eta_1'(\hat{z}_0 - \tilde{Z}_0)'\tilde{Z}_0 t dt + + O(|\log\lambda|^{-2}) \\ &= -\frac{2\pi}{\int_R^{\frac{1}{\delta\sqrt{\lambda}}} \frac{dt}{t\tilde{Z}_0^2(t)}} \int_R^{R+1} \eta_1'\left[1 - 4\frac{(\mu t)^2 \tilde{Z}_0 \int_R^t \frac{ds}{s\tilde{Z}_0^2(s)}}{(\mu^2+t^2)^2}\right] dt \\ &\quad + O(|\log\lambda|^{-2}) \\ &= E|\log\lambda|^{-1}\left[1 + O(|\log\lambda|^{-1})\right],\end{aligned}$$

where E is a positive constant independent of λ. We thus conclude, choosing R large enough, that $I \sim -E|\log\lambda|^{-1}$. Combining this with (4.17), we find

$$\langle \tilde{L}(\tilde{z}_0), \tilde{z}_0\rangle = -\frac{E}{|\log\lambda|}\left[1 + O(R^{-3} + O(|\log\lambda|^{-1}))\right].$$

This, together with (4.13) and (4.16), gives

$$|A| \le C|\log\lambda|\|\tilde{h}\|_\star.$$

Using the definition of $\hat{\phi}$ and (4.12), we then deduce that

$$\|\tilde{\phi}\|_{L^\infty(B_{\lambda^{-1/2}})} \le C(\|\tilde{h}\|_{**} + |\log\lambda|\|\tilde{h}\|_\star).$$

Observe that

$$\|\tilde{h}\|_\star = \sup_{s\in[0,\lambda^{-1/2}/4]} \frac{\tilde{h}(s)}{\lambda + (1+s)^{-2-\nu})} \le \sup_{r\in[0,1/4]} \frac{\lambda|h(r)|}{\lambda + \left(1+\frac{r}{\sqrt{\lambda}}\right)^{-2-\nu}} \le \|\tilde{\chi}_1 h\|_\star.$$

The previous two inequalities yield

$$\|\phi\|_{L^\infty(B_1)} \le C(\|h\|_{**} + |\log\lambda|\|\chi_1 h\|_\star).$$

Recalling the definition of the norm $\|\cdot\|_*$, we conclude that

$$\|\phi\|_{L^\infty(B_1)} \le C\|h\|_*.$$

It only remains to prove the existence part of the statement. For this purpose, we consider the space

$$H = \left\{\phi \in H^1(B_1) \mid \phi \text{ is radial}\right\},$$

endowed with the inner product $\langle\phi,\psi\rangle_{H^1} = \int_{B_1}\nabla\phi\nabla\psi dx + \int_{B_1}\phi\psi dx$. Problem (4.2) expressed in weak form is

$$\langle\phi,\psi\rangle_{H^1} = \int_{B_1}[\lambda e^U\phi + h]\psi dx \quad \text{for all } \psi \in H.$$

By Fredholm's alternative the existence of at least one solution is equivalent to its uniqueness, which is guaranteed by (4.3). □

We are now in position to prove Theorem 1.2.

Proof of Theorem 1.2. Thanks to the previous proposition, we know that the operator L is invertible. Therefore, we can rewrite (2.16) as

$$\phi = T(\phi) = L^{-1}[R(U) + N(\phi)].$$

Let ρ be a fixed number. We define

$$A_\rho = \left\{\phi \in L^\infty(B_1):\ \|\phi\|_{L^\infty(B_1)} \le \rho\varepsilon^{1+\sigma}\right\},$$

where σ is the constant defined in Lemma 3.2. We will show that the map $T: A_\rho \to A_\rho$ is a contraction. Using Lemma 3.1, recalling the definition of $\|\cdot\|_*$ given in (4.1), and since $|\log\lambda| = O\left(\varepsilon^{-1}\right)$, we see that

$$\begin{aligned}\left\|\lambda e^U\right\|_* &\le C\max\left(|\log\lambda|\sup_{r\le 2\delta} f_\lambda(r)\frac{\mu^2}{\lambda\left(\mu^2 + \left(\frac{r}{\sqrt{\lambda}}\right)^2\right)^2}, |\log\lambda|\sup_{\delta\le r\le 1-\delta_1} f_\lambda(r)\varepsilon^\beta, \varepsilon^{-1}\right)\\ &\le C\varepsilon^{-1}.\end{aligned}$$

From this and recalling the definition of $N(\cdot)$ (see (2.18)), we deduce that, for $\phi, \psi \in A_\rho$,

$$\|N(\phi)\|_* \le \left\|\lambda e^U\right\|_* \|\phi\|^2_{L^\infty(B_{R_1}\setminus B_{R_0})} \le C\varepsilon^{-1} \|\phi\|^2_{L^\infty(B_1)} \tag{4.18}$$

and

$$\|N_\lambda(\phi) - N_\lambda(\psi)\|_* \le C\varepsilon^{-1} \max\left\{\|\phi\|_{L^\infty(B_1)}, \|\psi\|_{L^\infty(B_1)}\right\} \|\phi-\psi\|_{L^\infty(B_1)}.$$

Next, using Lemma 3.2, we obtain

$$\begin{aligned}\|R(U)\|_* &\le C\max(|\log\lambda| \sup_{r\le 2\delta} f_\lambda(r)\frac{\mu^2(\lambda^\alpha + \frac{r^2}{\varepsilon})}{\lambda(\mu^2 + (\frac{r}{\sqrt{\lambda}})^2)^2}, |\log\lambda| \sup_{\delta\le r\le 1-\delta_1} f_\lambda(r)\varepsilon^\alpha, \varepsilon^{1+\sigma}) \\ &\le C\varepsilon^{1+\sigma}. \end{aligned} \tag{4.19}$$

Thus, combining (4.18) and (4.19), we get that, for $\phi, \psi \in A_\rho$ and some $\rho > 0$,

$$\|T(\phi)\|_{L^\infty(B_1)} \le C(\|N(\phi)\|_* + \|R(U)\|_*) \le \rho\varepsilon^{1+\sigma}$$

and

$$\|T_\lambda(\phi) - T_\lambda(\psi)\|_{L^\infty(B_1)} \le C\|N_\lambda(\phi) - N_\lambda(\psi)\|_* \le C\varepsilon^\sigma \|\phi-\psi\|_{L^\infty(B_1)}.$$

This implies that T is a contraction mapping in A_ρ for a suitable ρ. Therefore, we conclude that T has a unique fixed point in A_ρ. This establishes the theorem. □

5. Multi-layered solutions

In this section, we prove our main result.

Theorem 5.1. *Let ε_λ be defined by (2.4) and $U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}$ (resp. $\tilde{U}_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}$) be defined as in Theorem 1.3 (i) (resp. (ii)), with $b = 4\frac{\varepsilon_\lambda}{\sqrt{2}}$ and $k \in \mathbb{N}\setminus\{0\}$. The following holds:*

(i) suppose that $M_{k-1} \neq 0$. There exists $\lambda_k > 0$ such that for all $\lambda \in (0, \lambda_k)$ there exists a family of radial solutions u_λ to (1.1) in B_1 such that

$$\lim_{\lambda\to 0}\left(\varepsilon_\lambda u_\lambda - \sqrt{2}U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}\right) = 0,$$

uniformly on compact subsets of $B_{\alpha_1} \cup_{i=1}^{k-1} B_{\alpha_{i+1}}\setminus B_{\alpha_i}$,

$$\lambda e^{u_\lambda} \rightharpoonup 8\pi\delta_0 \quad \text{in } B_{\alpha_1/2},$$

and

$$\varepsilon_\lambda \lambda e^{u_\lambda} + \sum_{i=1}^{k}(|\partial_\nu U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}(\alpha_i)|)^{-1}\delta_{\partial B_{\alpha_i}} \rightharpoonup 0 \quad \text{in } B_1\setminus\{0\};$$

(ii) suppose that $M_k \neq 0$. There exists $\tilde{\lambda}_k > 0$ such that for all $\lambda \in (0, \tilde{\lambda}_k)$ there exists a family of radial solutions $\tilde{u}_\lambda$ to (1.1) in B_1 such that

$$\lim_{\lambda\to 0}\left(\varepsilon_\lambda \tilde{u}_\lambda - \sqrt{2}\tilde{U}_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}\right) = 0,$$

uniformly on compact subsets of $B_{\tilde{\alpha}_1} \cup_{i=1}^{k-1} B_{\tilde{\alpha}_{i+1}}\setminus B_{\tilde{\alpha}_i}$,

$$\lambda e^{\tilde{u}_\lambda} \rightharpoonup 8\pi\delta_0 \quad \text{in } B_{\tilde{\alpha}_1/2},$$

and

$$\varepsilon_\lambda \lambda e^{\tilde{u}_\lambda} + \sum_{i=1}^{k} (|\partial_\nu \tilde{U}_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}(\tilde{\alpha}_i)|)^{-1}\delta_{\partial B_{\tilde{\alpha}_i}} \rightharpoonup 0 \quad \text{in } B_1\backslash\{0\}.$$

We next give the proof of assertion (i) whereas the proof of (ii) can be done arguing in a similar way.

We define ε by (2.4) and let δ and δ_1 be defined as in Section 2 (see (2.13) and (2.5)). Let us consider constants $(R_i)_{i=1}^k$ depending on ε and to be determined below, such that $0 < R_1 < \ldots < R_k = 1$. We look for a solution of the form

$$\bar{u}_\lambda(r) = \begin{cases} u_0 & \text{in } (0,\delta) \\ (u^0_{\text{trans}})^0 & \text{in } (\delta, 2\delta) \\ u^1_{\text{int}} & \text{in } (2\delta, R_1 - 2\delta_1) \\ (u^1_{\text{trans}})^1 & \text{in } (R_1 - 2\delta_1, R_1 - \delta_1) \\ u^1_{\text{peak}} & \text{in } (R_1 - \delta_1, \tilde{R}_1) \end{cases}$$

and, for any $i = 2, \ldots, k$,

$$\bar{u}_\lambda(r) = \begin{cases} u^{i-1}_{\text{peak}} & \text{in } (R_{i-1}, R_{i-1} + \delta_1) \\ (u^0_{\text{trans}})^i & \text{in } (R_{i-1} + \delta_1, R_{i-1} + 2\delta_1) \\ u^i_{\text{int}} & \text{in } (R_{i-1} + 2\delta_1, R_i - 2\delta_1) \\ (u^1_{\text{trans}})^i & \text{in } (R_i - 2\delta_1, R_i - \delta_1) \\ u^i_{\text{peak}} & \text{in } (R_i - \delta_1, R_i). \end{cases}$$

The functions u^i_{peak} are defined as in [BCN17a, Section 3], for some μ_i's such that $\mu_i = O(\varepsilon)$. The function $(u^0_{\text{trans}})^0$ is a linear interpolation between u_0 and u^1_{int}. The functions $(u^0_{\text{trans}})^i$'s (resp. $(u^1_{\text{trans}})^i$'s) are linear interpolations between u^i_{int} (resp. u^{i-1}_{peak}) and u^i_{peak} (resp. u^i_{int}) for $i = 1, \ldots, k$ (resp. $i = 2, \ldots, k$). The functions u^i_{int} are shaped on the function $\frac{\sqrt{2}}{\varepsilon} U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},k}$ and the precise definitions are given below.

Fix now $\alpha = (\alpha_1, \ldots, \alpha_k)$ as in Theorem 1.3 (i). For $a = (a_1, \ldots, a_k)$, $\sigma = (\sigma_1, \ldots, \sigma_k)$ satisfying (1.7) and $b, \varepsilon > 0$, recall that $U_{\varepsilon,a,b,\sigma}$ is defined by (1.8). Note that $U_{0,a,b,0} = U_{b,k}$, according to the definition given in Theorem 1.3 (i). Observe that to prove Theorem 5.1 (ii), one defines $\tilde{U}_{\varepsilon,a,b,\sigma}$ as an analogous perturbation of $\tilde{U}_{b,k}$.

We then define the operator

$$F(1+\varepsilon a, \alpha + \sigma) = \begin{pmatrix} (U'_{\varepsilon,a,b,\sigma})^-(\alpha_1 + \sigma_1) \\ (U'_{\varepsilon,a,b,\sigma})^+(\alpha_1 + \sigma_1) \\ \vdots \\ (U'_{\varepsilon,a,b,\sigma})^-(\alpha_{k-1} + \sigma_{k-1}) \\ (U'_{\varepsilon,a,b,\sigma})^+(\alpha_{k-1} + \sigma_{k-1}) \\ (U'_{\varepsilon,a,b,\sigma})^-(1) \end{pmatrix},$$

where

$$(U'_{\varepsilon,a,b,\sigma})^\pm(\alpha_i + \sigma_i) = \lim_{\varepsilon \to 0^\pm} \frac{U_{\varepsilon,a,b,\sigma}(\alpha_i + \sigma_i + \varepsilon) - U_{\varepsilon,a,b,\sigma}(\alpha_i + \sigma_i)}{\varepsilon}.$$

Notice that the reflexion law (1.5) implies, for any $i \in \{1, \ldots, k-1\}$, that

$$(U'_{b,k})^+(\alpha_i) + (U'_{b,k})^-(\alpha_i) = 0. \tag{5.1}$$

Define also $\varphi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^k$ and $\tilde{\varphi}_\varepsilon : \mathbb{R} \to \mathbb{R}^k$ by

$$\varphi_\varepsilon(x,t) = (\varphi_\varepsilon^1(x,t), \dots, \varphi_\varepsilon^k(x,t)) \qquad \text{with } \varphi_\varepsilon^i(x,t) = \frac{1}{\sqrt{2}}\left(\frac{2(n-1)}{t} - 2x\ln 2 - \varepsilon x \zeta_1^i\right),$$

$$\tilde{\varphi}_\varepsilon(x) = (\tilde{\varphi}_\varepsilon^1(x), \dots, \tilde{\varphi}_\varepsilon^k(x)) \qquad \text{with } \tilde{\varphi}_\varepsilon^i(x) = \frac{1}{\sqrt{2}}(-\ln x^2 + \varepsilon x \nu_2^i),$$

where ζ_1^i and ν_2^i are some constants (see [BCN17a, Section 2] for more details). We have the following result.

Lemma 5.1. *Let b sufficiently small and define $\alpha = (\alpha_1, \dots, \alpha_k)$ as in Theorem 1.3 (i). There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a solution $(\gamma_\varepsilon, \sigma_\varepsilon) \in \mathbb{R}^k \times \mathbb{R}^k$, with σ_ε satisfying (1.7), to the equation*

$$F(1+\varepsilon\tilde{\varphi}_\varepsilon(\gamma_\varepsilon), \alpha + \sigma_\varepsilon) = \begin{pmatrix} -\frac{1}{\gamma_\varepsilon^1} + \varepsilon\varphi_\varepsilon^1(\gamma_\varepsilon^1, \alpha_1 + (\sigma_\varepsilon)_1) \\ \frac{1}{\gamma_\varepsilon^1} + \varepsilon\varphi_\varepsilon^1(\gamma_\varepsilon^1, \alpha_1 + (\sigma_\varepsilon)_1) \\ \vdots \\ -\frac{1}{\gamma_\varepsilon^k} + \varepsilon\varphi_\varepsilon^k(\gamma_\varepsilon^k, 1) \end{pmatrix}.$$

In addition, defining $U_{b,k}$ as in Theorem 1.3 (i), we have, for $i = 1, \dots, k$, that

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon^i = -\frac{1}{|U'_{b,k}(\alpha_i)|}.$$

Proof. We define, for $x \in \mathbb{R}^k$ and $\sigma \in (0,1)^k$ such satisfying (1.7),

$$H(\varepsilon; x; \sigma) = F(1+\varepsilon\tilde{\varphi}_\varepsilon(x), \alpha+\sigma) - \begin{pmatrix} -\frac{1}{x_1} + \varepsilon\varphi_\varepsilon^1(x_1, \alpha_1+\sigma_1) \\ \frac{1}{x_1} + \varepsilon\varphi_\varepsilon^1(x_1, \alpha_1+\sigma_1) \\ \vdots \\ -\frac{1}{x_k} + \varepsilon\varphi_\varepsilon^k(x_k, 1) \end{pmatrix}.$$

Evaluating H at $\varepsilon = 0$, $x_i = -\frac{1}{(U'_{b,k})^-(\alpha_i)}$, $i \in \{1, \dots, k\}$, $\sigma = 0$, we find, using (5.1), that

$$H\left(0; -\frac{1}{(U'_{b,k})^-(\alpha_1)}, -\frac{1}{(U'_{b,k})^-(\alpha_2)}, \dots, -\frac{1}{(U'_{b,k})^-(1)}; 0\right) = \begin{pmatrix} (U'_{b,k})^-(\alpha_1) \\ (U'_{b,k})^+(\alpha_1) \\ \vdots \\ (U'_{b,k})^-(1) \end{pmatrix} - \begin{pmatrix} (U'_{b,k})^-(\alpha_1) \\ -(U'_{b,k})^-(\alpha_1) \\ \vdots \\ (U'_{b,k})^-(1) \end{pmatrix} = 0.$$

Moreover, we have

$$\frac{\partial}{\partial \xi_i} H\left(0; -\frac{1}{(U'_{b,k})^-(\alpha_1)}, -\frac{1}{(U'_{b,k})^-(\alpha_2)}, \dots, -\frac{1}{(U'_{b,k})^-(1)}; 0\right)$$
$$= \begin{pmatrix} -|U'_{b,k}(\alpha_1)|^2 & 0 & \dots & & 0 & \partial_{k+1}F_1(b,1,\alpha) & \partial_{k+2}F_1(b,1,\alpha) & \dots\partial_{2k-1}F_1(b,1,\alpha) \\ |U'_{b,k}(\alpha_1)|^2 & 0 & \dots & & 0 & \partial_{k+1}F_2(b,1,\alpha) & \partial_{k+2}F_2(b,1,\alpha) & \dots\partial_{2k-1}F_2(b,1,\alpha) \\ 0 & -|U'_{b,k}(\alpha_2)|^2 & 0 & \dots & 0 & \partial_{k+1}F_3(b,1,\alpha) & \partial_{k+2}F_3(b,1,\alpha) & \dots\partial_{2k-1}F_3(b,1,\alpha) \\ 0 & |U'_{b,k}(\alpha_2)|^2 & 0 & \dots & 0 & \partial_{k+1}F_4(b,1,\alpha) & \partial_{k+2}F_4(b,1,\alpha) & \dots\partial_{2k-1}F_4(b,1,\alpha) \\ \dots & & & & & & & \\ 0 & & \dots & & -|U'_{b,k}(1)|^2 & \partial_{k+1}F_{2k}(b,1,\alpha) & \partial_{k+2}F_{2k}(b,1,\alpha) & \dots\partial_{2k-1}F_{2k}(b,1,\alpha) \end{pmatrix}$$
$$= N_k,$$

where $\xi_i = x_i$ and $\xi_{k+i} = \sigma_i$, for $i \in \{1, \dots, k\}$. It is shown in [BCN17a, Appendix] that $\det N_k = M_{k-1}$. Therefore, by assumption, we have that $\det N_k \neq 0$. The proof thus follows from the Implicit Function Theorem. $\square$

Thanks to the previous lemma, we can make explicit our choice of μ_i and R_i as

$$\mu_i = \varepsilon \gamma_\varepsilon^i \quad \text{and} \quad R_i = \alpha_i + (\sigma_i)_\varepsilon.$$

Next, we define the function u_{int}^i by

$$u_{int}^i = \frac{\sqrt{2}}{\varepsilon} U_{\varepsilon, \tilde{\varphi}_\varepsilon^i(\gamma_\varepsilon^i), 4\varepsilon/\sqrt{2}, (\sigma_\varepsilon)_i}.$$

The end of the proof follows along the same lines as the proof of Theorem 1.2.

6. Green's functions

This section is devoted to the study of Green's functions. In particular, we will prove Theorem 1.3. First, let us recall the following lemma from [BCN17b].

Lemma 6.1. *There exist two positive linearly independent solutions $\zeta \in C^2((0,1])$ and $\xi \in C^2([0,1])$ of the equation*

$$-u'' - \frac{1}{r}u' + u = 0 \quad \text{in } (0,1),$$

satisfying

$$\xi'(0) = \zeta'(1) = 0 \quad \text{and} \quad r(\xi'(r)\zeta(r) - \xi(r)\zeta'(r)) = 1 \text{ for any } r \in (0,1].$$

We have that ξ is bounded and increasing in $[0,1]$, ζ is decreasing in $(0,1]$,

$$\xi(0) = 1, \quad \lim_{r \to 0^+} \frac{\zeta(r)}{-\ln r} = 1, \quad \text{and} \quad \lim_{r \to 0^+} (-r\zeta'(r)) = 1.$$

Moreover, as r goes to 0, we have (see [Wei])

$$-\frac{2}{\pi}\zeta(r) = \frac{2}{\pi}(\log r - \ln 2 + \gamma) - \frac{r^2}{2\pi}(\ln r - \ln 2 + \gamma - 1) + O(r^3) \tag{6.1}$$

and

$$-\frac{2}{\pi}\zeta'(r) = \frac{2}{\pi r} - \frac{r(-2\ln r - 2\gamma + 1 + \log 4)}{2\pi} + O(r^2), \tag{6.2}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant.

Using the previous lemma, we are able to construct a radial Green's function on the unit ball B_1 blowing up at 0 and equal to 1 on ∂B_1.

Lemma 6.2. *For any $\tilde{b} > 0$ small enough, there exists a positive radial function G solution to*

$$-G'' - \frac{1}{r}G' + G = 0 \quad \text{in } (0,1), \tag{6.3}$$

such that

$$\lim_{r \to 0^+} \frac{G(r)}{-\ln r} = \tilde{b}, \quad \lim_{r \to 0^+} rG'(r) = \tilde{b}, \quad G(1) = 1.$$

Moreover, there exists $\tilde r \in (0,1)$, *with* $\tilde r = O(\sqrt{\tilde b})$, *such that* $G'(\tilde r) = 0$, *and, as* r *goes to zero, we have*

$$G(r) + \tilde b \ln r = \frac{\tilde b \pi}{2}(\gamma - \ln 2) + o(r) \tag{6.4}$$

and

$$G'(r) + \frac{\tilde b}{r} = O(r \ln r). \tag{6.5}$$

Proof. Using the properties of the functions ξ and ζ (defined in Lemma 6.1), it is immediate to see that, for any $b \in (0,1)$,

$$u_b(r) = \frac{\xi'(b)\zeta(r) - \xi(r)\zeta'(b)}{\xi'(b)\zeta(1) - \xi(1)\zeta'(b)}$$

is a solution to (6.3) such that

$$u_b(1) = 1 \quad \text{and} \quad \lim_{r \to 0^+} \frac{u_b(r)}{-\ln r} = \frac{\xi'(b)}{\xi'(b)\zeta(1) - \xi(1)\zeta'(b)}.$$

Moreover, for b small enough, we have

$$\begin{array}{rcl} \xi'(b)\zeta(1) - \xi(1)\zeta'(b) & = & \xi(1) b^{-1} + o(b^{-1}), \\ \xi'(b) & = & \delta b + o(b), \end{array}$$

for some positive constant δ which does not depend on b. Therefore, for b small enough, we have, for some constant C_0 independent of b,

$$\lim_{r \to 0^+} \frac{u_b(r)}{-\ln r} = C_0 b^2 + o(b^2).$$

Multiplying u_b by a suitable constant, we get the result. The estimates (6.4) and (6.5) follow from (6.1) and (6.2) and the fact that $\xi(0) = \xi'(0) = 0$. □

Next, we are going to construct two Green's functions, the first one is singular at the origin and on an interior sphere and the second one is singular at the origin and on ∂B_1.

Before proceeding, it is useful to recall the following result.

Lemma 6.3. *Let* $0 \le a < b \le 1$. *The following holds:*

(i) denote by u_x, $x \in (a,b)$, *the function satisfying*

$$\left\{ \begin{array}{rcll} -u_x''(r) - \frac{1}{r} u_x'(r) + u_x(r) & = & 0 & r \in (a,x) \\ u_x'(a) & = & 0 & \\ u_x(x) & = & 1. & \end{array} \right.$$

Then, the function $x \to u_x'(x)$ *is strictly increasing;*

(ii) denote by v_x, $x \in (a,b)$, *the function satisfying*

$$\left\{ \begin{array}{rcll} -v_x''(r) - \frac{1}{r} v_x'(r) + v_x(r) & = & 0 & r \in (x,b) \\ v_x(x) & = & 1 & \\ v_x(b) & = & 1. & \end{array} \right.$$

Then, the function $x \to v_x'(x)$ *is strictly increasing.*

Proof. We refer to [BGNT16, Lemma 2.4] for a proof of *(i)* and to [BCN17a, Proposition A.1] for a proof of *(ii)*. □

Thanks to the previous lemma, we are able to prove the existence and uniqueness of the two Green's functions mentioned above.

Lemma 6.4. *Let $0 < \beta \le 1$. Then, for any $\tilde{b} > 0$ small enough, there exist a unique $\alpha \in (0,\beta)$ and a unique continuous function U solution to*

$$\begin{cases} -U'' - \dfrac{1}{r}U' + U &= 0 \quad \text{in } (0,\alpha)\cup(\alpha,\beta) \\ \lim\limits_{r\to 0^+} -\dfrac{U(r)}{\ln r} &= \tilde{b} \\ U'(\beta) &= 0 \\ U(\alpha) &= 1 \end{cases}$$

satisfying the reflection law

$$\lim_{\varepsilon\to 0^-}\frac{U(\alpha+\varepsilon)-U(\alpha)}{\varepsilon} = -\lim_{\varepsilon\to 0^+}\frac{U(\alpha+\varepsilon)-U(\alpha)}{\varepsilon}.$$

We also have that, for any $\tilde{b} > 0$ small enough, there exist a unique $\alpha \in (0,\beta)$ and a unique continuous function V solution to

$$\begin{cases} -V'' - \dfrac{1}{r}V' + V &= 0 \quad \text{in } (0,\alpha)\cup(\alpha,\beta) \\ \lim\limits_{r\to 0^+} -\dfrac{V(r)}{\ln r} &= \tilde{b} \\ V(\alpha) &= 1 \\ V(\beta) &= 1 \end{cases}$$

satisfying the reflection law

$$\lim_{\varepsilon\to 0^-}\frac{V(\alpha+\varepsilon)-V(\alpha)}{\varepsilon} = -\lim_{\varepsilon\to 0^+}\frac{V(\alpha+\varepsilon)-V(\alpha)}{\varepsilon}.$$

Proof. We only prove the second part. The proof of the first part is analogous.

Let $b < \beta$ be a small constant to be fixed afterwards. We consider the function u : $(0,\beta)\times(b,\beta)$, defined for $\alpha \in (b,\beta)$ as

$$u(r,\alpha) = \begin{cases} \dfrac{\xi'(b)\zeta(r)-\xi(r)\zeta'(b)}{\xi'(b)\zeta(\alpha)-\xi(\alpha)\zeta'(b)} & r \in (0,\alpha) \\ \dfrac{\xi'(\beta)\zeta(r)-\xi(r)\zeta'(\beta)}{\xi'(\beta)\zeta(\alpha)-\xi(\alpha)\zeta'(\beta)} & r \in (\alpha,\beta), \end{cases}$$

where the functions ξ and ζ are the ones defined in Lemma 6.1. Notice that $u(r,\alpha)$ satisfies the equation

$$-u'' - \frac{1}{r}u' + u = 0 \quad \text{ in } (0,\alpha)\cup(\alpha,\beta),$$

together with the boundary conditions $u(\alpha,\alpha) = u(\beta,\alpha) = 1$. Proceeding as in Lemma 6.2, one checks that

$$\lim_{r\to 0^+} -\frac{u(r,\alpha)}{\ln r} = \frac{\xi''(0)}{\xi(\alpha)}b^2 + o(b^2).$$

Thus, for any $\tilde{b}$ sufficiently small, choosing $b = \left(\frac{\xi(\alpha)}{\xi''(0)}\tilde{b}\right)^{1/2}$, we have $\lim_{r\to 0^+} -\frac{u(r,\alpha)}{\ln r} = \tilde{b}$. It remains to prove that there exists a unique $\alpha_1 \in (b,\beta)$ such that

$$F(\alpha_1) = (u'(\alpha_1,\alpha_1))^+ + (u'(\alpha_1,\alpha_1))^- = 0,$$

where

$$(u'(\alpha,\alpha))^{\pm} = \lim_{\varepsilon\to 0^{\pm}} \frac{u(\alpha+\varepsilon,\alpha) - u(\alpha,\alpha)}{\varepsilon}.$$

Observe that F can be rewritten as

$$F(\alpha) = \frac{\xi'(b)\zeta'(\alpha) - \xi'(\alpha)\zeta'(b)}{\xi'(b)\zeta(\alpha) - \xi(\alpha)\zeta'(b)} + \frac{\xi'(\beta)\zeta'(\alpha) - \xi'(\alpha)\zeta'(\beta)}{\xi'(\beta)\zeta(\alpha) - \xi(\alpha)\zeta'(\beta)}.$$

Thanks to Lemma 6.3, we already know that the function $\alpha \to (u'(\alpha,\alpha))^+$ is strictly increasing. We are going to prove that $\alpha \to (u'(\alpha,\alpha))^-$ is also strictly increasing. Indeed, recalling that $b = \left(\frac{\xi(\alpha)}{\xi''(0)}\tilde{b}\right)^{1/2}$, provided $\tilde{b}$ small enough, we see that

$$\frac{\partial}{\partial\alpha}\left(\frac{\xi'(b)\zeta'(\alpha) - \xi'(\alpha)\zeta'(b)}{\xi'(b)\zeta(\alpha) - \xi(\alpha)\zeta'(b)}\right) = \frac{\zeta'(b)^2(\xi(\alpha)\xi''(\alpha) - \xi'(\alpha)^2)}{(\xi'(b)\zeta(\alpha) - \xi(\alpha)\zeta'(b))^2} + o(1) > 0.$$

So, in order to prove the existence of α_1, since F is continuous, it is sufficient to show that $\lim_{\alpha\to b^+} F(\alpha) < 0$ and $\lim_{\alpha\to\beta^-} F(\alpha) > 0$. First, thanks to Lemma 6.1, we notice that, when $\alpha \to b^+$, we have

$$\xi'(b)\zeta(\alpha) - \xi(\alpha)\zeta'(b) = 1/b + o(1/b)$$

and

$$\xi'(b)\zeta'(\alpha) - \xi'(\alpha)\zeta'(b) = -\frac{b}{\alpha}\xi''(0) + \xi''(0)\frac{\alpha}{b} + o\left(\frac{\alpha}{b}\right) > 0$$

when $\alpha \to b^+$, we also have

$$\xi'(\beta)\zeta(\alpha) - \xi(\alpha)\zeta'(\beta) = -\xi'(\beta)\ln\alpha + o(\ln\alpha),$$

and

$$\xi'(\beta)\zeta'(\alpha) - \xi'(\alpha)\zeta'(\beta) = -\xi'(\beta)1/\alpha + o(1/\alpha).$$

Combining the previous estimates, we deduce that, when $\alpha \to b^+$,

$$F(\alpha) = -\frac{b^2}{\alpha}\xi''(0) + \xi''(0)\alpha + \frac{1}{\alpha\ln\alpha} + o\left(\frac{1}{\alpha\ln\alpha}\right) < 0.$$

On the other hand, when $\alpha \to \beta^-$, we have

$$\xi'(b)\zeta(\alpha) - \xi(\alpha)\zeta'(b) = \xi(\beta)(1/b) + o(1/b)$$

and

$$\xi'(b)\zeta'(\alpha) - \xi'(\alpha)\zeta'(b) = b\xi''(0)\zeta'(\beta) + \xi'(\beta)(1/b) = \frac{\xi'(\beta)}{b} + o(\frac{1}{b}).$$

Since $\lim_{\alpha\to\beta^-} \xi'(\beta)\zeta(\alpha) - \xi(\alpha)\zeta'(\beta) = 1/\beta$ and $\xi'(\beta)\zeta'(\alpha) - \xi'(\alpha)\zeta'(\beta) = O(\alpha-\beta)$, we get that, as $\alpha \to \beta^-$,

$$F(\alpha) = \frac{\xi'(\beta)}{\xi(\beta)} + o(1) > 0.$$

This concludes the proof. □

Remark 6.1. *Observe that along the proof we showed that* $M_1 = -|U'_{\sigma_1}(\alpha_1)|^2(U'^{+}_{\sigma_1} + U'^{-}_{\sigma_1})(\alpha_1) \neq 0$.

Proof of Theorem 1.3. The proof can be done as the one of [BGNT16, Theorem 2.14], by substituting $u_{\infty,1-\text{layer}}(\beta_1; 0, \beta_1)$ by the function U defined in the previous lemma with $\beta = \beta_1$. □

Appendix

We show a very rough elliptic estimate which is needed in the proof of Lemma 2.2.

Lemma 6.5. *Let* $R > 0$ *and* $u \in H^1(B_R(0))$ *be a radial solution to*

$$\begin{cases} -\Delta u + u &= f \quad \text{in } B_R(0) \\ u'(R) &= g, \end{cases}$$

for some $f \in L^q(B_R(0))$, *with* $q > 2$. *Then, we have*

$$\|u\|_{L^\infty(B_R(0))} \leq C\left[\left(\frac{1}{R} + |\ln R| + R\right)R^{1-2/q}\|f\|_{L^q(B_R(0))} + R(1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R(0))}\right]$$

and

$$\|u'\|_{L^\infty(B_R(0))} \leq C\left[R^{1-2/q}\|f\|_{L^q(B_R(0))} + (1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R(0))}\right]$$

for some constant C *not depending on* R.

Proof. Multiplying the equation by u and integrating by parts, we get

$$\|u\|^2_{H^1(B_R)} \leq \|f\|_{L^2(B_R)}\|u\|_{H^1(B_R)} + R|u'(R)||u(R)|. \tag{6.6}$$

Since $u(R) - u(r) = \int_r^R u'(s)ds$, one deduces that

$$|u(R)|^2 \leq C\left[|u(r)|^2 + \|u'\|^2_{L^2(B_R)} \ln\frac{R}{r}\right],$$

where throughout the proof C denotes a constant not depending on R. Multiplying the previous inequality by r and integrating, we find

$$R^2|u(R)|^2 \leq C[\|u\|^2_{L^2(B_R)} + \|u'\|^2_{L^2(B_R)}R^2|\ln R|].$$

This implies that

$$|u(R)| \leq C\left(\frac{1}{R} + |\ln R|\right)\|u\|_{H_1(B_R)}. \tag{6.7}$$

From (6.6), (6.7), and $u'(R) = g$, we obtain that

$$\|u\|^2_{H^1(B_R)} \leq \|f\|_{L^2(B_R)}\|u\|_{H^1(B_R)} + C(1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R)}\|u\|_{H^1(B_R)}.$$

Thanks to Hölder inequality, we find that

$$\|u\|_{H^1(B_R)} \leq C[R^{1-2/q}\|f\|_{L^q(B_R)} + (1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R)}]. \tag{6.8}$$

Next, observe that for any $s \in (0, R)$ we can rewrite the equation as

$$u'(s)s = \int_0^s (u - f)rdr.$$

From Hölder inequality, we obtain that

$$|u'(s)| \leq C\|u - f\|_{L^2(B_R)} \leq C(\|u\|_{L^2(B_R)} + R^{1-2/q}\|f\|_{L^q(B_R)}).$$

From (6.8), we deduce that

$$\|u'\|_{L^\infty(B_R)} \leq C(R^{1-2/q}\|f\|_{L^q(B_R)} + (1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R)}).$$

By noting that

$$u(R) - u(\tilde{s}) = \int_{\tilde{s}}^R u'(r)dr,$$

we get from (6.7) that

$$\begin{aligned}
\|u\|_{L^\infty(B_R)} &\leq C\left[\left(\frac{1}{R} + |\ln R|\right)\|u\|_{H^1(B_R)} + R\|u'\|_{L^\infty(B_R)}\right] \\
&\leq C\left[\left(\frac{1}{R} + |\ln R| + R\right)R^{1-2/q}\|f\|_{L^q(B_R)} + R(1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R)}\right].
\end{aligned}$$

This concludes the proof. □

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