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NEW ARTIN-SCHELTER REGULAR AND CALABI-YAU ALGEBRAS VIA NORMAL EXTENSIONS

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Abstract. We develop a method to construct 4-dimensional Artin-Schelter regular algebras as normal extensions of (not necessarily noetherian) 3-dimensional ones. The method produces large classes of new 4-dimensional Artin-Schelter regular algebras. When applied to a 3-Calabi-Yau algebra our method produces a flat family of central extensions of it that are 4-Calabi-Yau, and all 4-Calabi-Yau central extensions having the same generating set as the original 3-Calabi-Yau algebra arise in this way. Some of the 2-generated 4-dimensional Artin-Schelter regular algebras discovered by Lu-Palmieri-Wu-Zhang [17] can be obtained by our method and our results provide a new proof of their regularity. Each normal extension has the same generators as the original 3-dimensional algebra, and its relations consist of all but one of the relations for the original algebra and an equal number of new relations determined by "the missing one" and a tuple of scalars satisfying some numerical conditions. We determine the Nakayama automorphisms of the 4-dimensional algebras obtained by our method and as a consequence show that their homological determinant is 1. This supports the conjecture in [20] that the homological determinant of the Nakayama automorphism is 1 for all Artin-Schelter regular connected graded algebras. Reyes-Rogalski-Zhang proved this is true in the noetherian case [24, Cor. 5.4].

Contents

1. Introduction

1.1. **Notation.** Throughout this paper we fix a field $\Bbbk$. All vector spaces will be $\Bbbk$-vector spaces, all categories will be $\Bbbk$-linear, and all functors will be $\Bbbk$-linear. All algebras will be finitely presented $\mathbb{N}$-graded $\Bbbk$-algebras. Such an algebra, $A = A_0 \oplus A_1 \oplus \cdots$, is connected if $A_0 = \Bbbk$. All graded algebras in this paper are connected and generated by their degree-one components. Thus, we are concerned with algebras $TV/(R)$ where TV denotes the tensor algebra on a finite dimensional vector space V placed in degree 1 and (R) denotes the ideal generated by a finite dimensional graded subspace R of TV whose elements have degrees ≥ 2.

The algebras we study are not required to be noetherian or to have finite Gelfand-Kirillov dimension.

1.2. The assumption that a non-commutative algebra has finite global dimension does not imply that it satisfies good analogues of the Cohen-Macaulay and Gorenstein properties. For example, all free algebras on a finite positive number of generators have global dimension 1; for all $d \geq 0$, there are finite dimensional algebras having global dimension d; for all $d \geq 1$, there are algebras of global dimension d whose only ideals are $\{0\}$ and the rings themselves; and so on. However, for graded algebras there are two ways to strengthen the finite global dimension hypothesis that have proven sufficiently effective that they lead to a good non-commutative analogue of projective algebraic geometry.

The algebras satisfying these stronger definitions are called Artin-Schelter regular and twisted Calabi-Yau algebras, respectively. The definitions vary a little from one paper to another but in this paper they are defined in such a way that they are, in fact, the same (see §1.8). A particularly important subset of these algebras are

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the Calabi-Yau algebras: their place among all twisted Calabi-Yau algebras is analogous to that occupied by the symmetric algebras among all finite dimensional Frobenius algebras.

Many Artin-Schelter regular and Calabi-Yau algebras are known. The 3-dimensional Artin-Schelter regular algebras (meaning those of global dimension 3) that have finite Gelfand-Kirillov dimension were classified by Artin-Schelter [1] and Artin-Tate-Van den Bergh [2]. Using those classifications, the 3-dimensional Calabi-Yau algebras of finite Gelfand-Kirillov dimension were classified by Mori-Smith [21] and Mori-Ueyama [22]. A bewildering variety of 4-dimensional Artin-Schelter regular and Calabi-Yau algebras are known but no classification is in sight. The construction of new examples is an important on-going research goal.

1.3. **The main result.** This paper introduces a new method for producing 4-dimensional Artin-Schelter regular and Calabi-Yau algebras from 3-dimensional ones.

Before stating an approximate version of our main result we need the following terminology. An element Ω in a ring D is normal if $\Omega D = D\Omega$, and is regular if it is not a zero-divisor. Following the terminology in [15], we call D a normal extension by Ω of a ring A if there is a regular normal element Ω in D such that $A \cong D/(\Omega)$. If, in addition, Ω belongs to the center of D we call D a central extension of A.

Now let $A = TV/(R)$ be a 3-dimensional Artin-Schelter regular algebra. Since A is generated by its degree-one component we can, without loss of generality, assume that $R \subseteq V^{\otimes m}$ for some $m \geq 2$ (see §1.10). We do that. Moreover, Dubois-Violette proved (see Theorem 3.1) that there is a twisted superpotential $\mathsf{w} \in V^{\otimes m+1}$ such that A is the algebra $A(\mathsf{w}) := TV/(\partial_1\mathsf{w}, \ldots, \partial_n\mathsf{w})$ where the $\partial_i\mathsf{w}$ are the partial derivatives of w with respect to a basis $x_1, \ldots, x_n$ for V. Such w is uniquely determined up to scalar multiplication and the definition of $A(\mathsf{w})$ does not depend on the choice of basis for V. See §§3.1 and 3.2 for details on twisted superpotentials.

With this notation, our main result is as follows.

Theorem 1.1 (Approximate version of Theorem 3.4)**.** *Let $A = TV/(R)$ be a 3-dimensional Artin-Schelter regular algebra and fix a basis $x_1, \ldots, x_n$ for V. Let $\mathsf{w} \in V^{\otimes(m+1)}$ be a twisted superpotential such that $A = TV/(\partial_1\mathsf{w}, \ldots, \partial_n\mathsf{w})$. Fix an index $1 \leq k \leq n$, let $\mathsf{p} = (p_1, \ldots, p_n)$ be a "good" tuple of non-zero scalars, let $D(\mathsf{w}, \mathsf{p})$ be the free algebra TV modulo the $2(n-1)$ relations*

$$\partial_i\mathsf{w} \;=\; x_i(\partial_k\mathsf{w}) - p_i(\partial_k\mathsf{w})x_i \;=\; 0 \qquad (i = 1, \ldots, n,\; i \neq k),$$

and let Ω denote the image of $\partial_k\mathsf{w}$ in $D(\mathsf{w}, \mathsf{p})$.

1. *$D(\mathsf{w}, \mathsf{p})$ is a 4-dimensional Artin-Schelter regular algebra and is a normal extension of $A = D(\mathsf{w}, \mathsf{p})/(\Omega)$.*
2. *$D(\mathsf{w}, \mathsf{p})$ is noetherian if and only if A is.*
3. *$D(\mathsf{w}, \mathsf{p})$ has finite Gelfand-Kirillov dimension if and only if A does.*

In Theorem 1.1, the algebra A has n generators and n defining relations, all of degree m, and the algebra $D(\mathsf{w}, \mathsf{p})$ has n generators and $2n-2$ relations, $n-1$ of degree m and $n-1$ of degree $m+1$.

The fact that (2) and (3) follow from (1) is well-known so the real content of Theorem 1.1 is (1).

The condition that the tuple p is "good" can be stated in several ways, perhaps the simplest of which is Theorem 3.4(4) (see §3.4).

It is not immediately obvious that there are *any* "good" p. However, as we show in §5, there *are* good p's for almost all the generic 3-dimensional Artin-Schelter regular algebras found by Artin and Schelter. Furthermore, if A is a 3-Calabi-Yau algebra $\mathsf{p} = (1, \ldots, 1)$ is good and in that case $D(\mathsf{w}, \mathsf{p})$ is 4-dimensional Calabi-Yau algebra and is a central extension of A.

Corollary 1.2 (Corollary 3.5)**.** *Let $TV/(R) = A(\mathsf{w})$ be a 3-Calabi-Yau algebra and fix an index $1 \leq k \leq n$. Then TV modulo the relations*

$$\partial_i\mathsf{w} \;=\; [x_i, \partial_k\mathsf{w}] \;=\; 0 \qquad (i = 1, \ldots, n,\; i \neq k)$$

is a 4-Calabi-Yau algebra and is a central extension of $A(\mathsf{w})$ by the image of $\partial_k\mathsf{w}$.

Although the terminology "central extension" and "normal extension" follows that in [15] there is no overlap between this paper and that one. In this paper the natural map $D \to A$ from one of the normal or central extensions in Theorem 1.1 is an isomorphism in degree 1, i.e., A and D have the "same" generating sets, whereas the corresponding map $D \to A$ in [15] has a 1-dimensional kernel in degree 1.

1.4. It is not immediately apparent that Corollary 1.2 produces a flat family of 4-Calabi-Yau algebras over $\mathbb{P}^{n-1}$ that map onto the original 3-Calabi-Yau algebra $A(\mathsf{w})$. To state that result we need to state the corollary in a basis-free way.

To do that, let V be an n-dimensional vector space over $\Bbbk$ and let $m \in \mathbb{Z}_{\geq 1}$. For each $x \in V$, let $x^\perp := \{\psi \in V^* \mid \psi(x) = 0\}$. Define $\langle -, - \rangle : V^* \times V^{\otimes(m+1)} \to V^{\otimes m}$ by $\langle \psi, \mathsf{w}\rangle := (\psi \otimes \mathrm{id}^{\otimes m})(\mathsf{w})$.

Theorem 1.3. *Let $A(\mathsf{w})$ be a 3-Calabi-Yau algebra where $\mathsf{w} \in V^{\otimes(m+1)}$ is a superpotential. For each $x \in V$, let I_x be the ideal in TV generated by $\{\langle \psi, \mathsf{w}\rangle \mid \psi \in x^\perp\}$ and $\{[y, \langle \psi, \mathsf{w}\rangle] \mid y \in V,\ \psi \in V^*\}$. Define $D_x = TV/I_x$. By definition $D_0 = A(\mathsf{w})$.*

1. *$\{D_x \mid x \in \mathbb{P}^{n-1} = \mathbb{P}(V)\}$ is a flat family of 4-Calabi-Yau algebras having degree-m central regular elements Ω_x such that $A(\mathsf{w}) = D_x/(\Omega_x)$.*
2. *All degree-m central extensions of $A(\mathsf{w})$ are obtained in this way (Theorem 3.7).*

In §5, we give some explicit examples to show that not every normal extension of a 3-Calabi-Yau algebra need be a central extension.

1.5. Given part (1) of Theorem 1.3 above, it is natural to consider the algebras D_x as homogeneous coordinate rings of a well-behaved family of non-commutative schemes. In order to make this precise, we need some preparation.

Let $\mathcal{D}$ be the sheaf of algebras on $\mathbb{P}(V)$ defined in Remark 3.8 below. The fiber of $\mathcal{D}$ over a closed point $x \in \mathbb{P}(V)$ is the algebra D_x defined in Theorem 1.3. For each open subscheme $U \subseteq \mathbb{P}(V)$ we denote by $\mathcal{D}_U$ the restriction of $\mathcal{D}$ to U.

Definition 1.4. The category $\mathsf{QGr}(\mathcal{D}_U)$ is the localization of the category $\mathsf{Gr}(\mathcal{D}_U)$ of graded quasi-coherent sheaves of modules over $\mathcal{D}_U$ modulo the subcategory $\mathsf{Tors}(\mathcal{D}_U)$ consisting of sheaves whose fibers are torsion over the fibers D_x, $x \in \mathbb{P}(V)$. ♦

Recall also from [16, Definition 3.2] the notion of flatness over a commutative ring R for an abelian R-linear category. With all of this in place, we can state

Theorem 1.5. *For every open affine $\mathrm{Spec}(R) = U \subseteq \mathbb{P}(V)$, the R-linear abelian category $\mathsf{QGr}(\mathcal{D}_U)$ is flat over R as in [16]. We can express this by saying that $\mathsf{QGr}(\mathcal{D})$ is flat over $\mathbb{P}(V)$.*

The intuition here is that the non-commutative scheme represented by the category $\mathsf{QGr}(\mathcal{D})$ aggregates its fibers $\mathsf{QGr}(D_x)$, $x \in \mathbb{P}(V)$, into a flat family over the parameter space $\mathbb{P}(V)$.

1.6. The 3-dimensional Artin-Schelter regular algebras of finite Gelfand-Kirillov dimension (or, equivalently, the noetherian ones) have been classified. They fall into two broad classes: those on 3 generators subject to 3 quadratic relations and those on 2 generators subject to 2 cubic relations. Applying Theorem 1.1 to these produces 4-dimensional Artin-Schelter regular algebras on 3 generators subject to two quadratic and two cubic relations, and 4-dimensional Artin-Schelter regular algebras on 2 generators subject to one cubic and one quartic relation, respectively.

Following the terminology in [17], we sometimes say that these two classes of 3-dimensional Artin-Schelter regular algebras have types (1331) and (1221), respectively. The terminology comes from the ranks of the free modules that appear in the minimal projective resolution of the trivial module (see §1.7).[1] Extending this terminology to the 4-dimensional case in the obvious way, Theorem 1.1 produces 4-dimensional Artin-Schelter regular algebras of types (13431) and (12221), respectively.

Once one drops the requirement that the 3-dimensional Artin-Schelter regular algebra have finite Gelfand-Kirillov dimension (or, equivalently, be noetherian), other "types" can appear. For example, in [10], Eisenschlos constructs 3-Calabi-Yau algebras of type $(1, n, n, 1)$ for many $n \geq 3$. Theorem 1.1 applies to these algebras and produces 4-Calabi-Yau algebras of type $(1, n, 2n-2, n, 1)$.

Many of the algebras produced by Theorem 1.1 are new. Some are not. For example, two of the four families of 4-dimensional regular algebras introduced by Lu-Palmieri-Wu-Zhang [17] can be obtained by Theorem 1.1. Thus the regularity for these algebras follows from Theorem 1.1; in contrast, the proof of their regularity in [17] proceeded by finding an explicit basis of the defining ideal using Bergman's diamond lemma [5]. The algebras produced by Theorem 1.1 need not have PBW bases.

[1] It is no accident that these numbers form a palindromic sequence. The definition of a d-dimensional Artin-Schelter regular algebra says, in part, that it satisfies a Gorenstein-like property which ensures that $\mathrm{Ext}^i_A(A, \Bbbk) \cong \mathrm{Ext}^{d-i}_A(\Bbbk, A)^*$ and this, in turn, implies that these ranks form a palindromic sequence.

1.7. **Artin-Schelter regular algebras.** When A is connected graded, $\Bbbk = A/A_{\geq 1}$ is a graded left A-module concentrated in degree 0. We call it the trivial module.

Let d be a non-negative integer. A connected graded algebra A is Gorenstein of dimension d if it has left injective dimension d as a module over itself and

$$\operatorname{Ext}^i_A(\Bbbk, A) \cong \begin{cases} \Bbbk(\ell) & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases} \tag{1-1}$$

for some $\ell \in \mathbb{Z}$. The number ℓ in (1-1) is called the Gorenstein parameter of A.

A connected graded algebra A is Artin-Schelter regular (or AS-regular, or simply regular) of dimension d if it

(1) has finite global dimension d and
(2) is Gorenstein of dimension d.

The Gorenstein and AS-regularity properties are known to be left-right symmetric.

Commutative Artin-Schelter regular algebras are polynomial rings.

The notion of regularity we use does not require or imply that the algebra be noetherian or have finite Gelfand-Kirillov dimension. However, by [26, Prop. 3.1(1)] it does imply that ${}_A\Bbbk$ has a finite projective resolution in which every term is a finitely generated free left A-module.

1.8. **Twisted Calabi-Yau algebras.** Let A be a graded algebra. We denote its opposite algebra by A° and its enveloping algebra by $A^e = A \otimes_\Bbbk A^\circ$. An A-bimodule M is regarded as a left A^e-module via $(a \otimes b) \cdot x = axb$ for all $a, b \in A$ and $x \in M$. If ν is an automorphism of A we denote by ${}_\nu A_1$ the left A^e-module that is A as a graded vector space with action $(a \otimes b) \cdot c = \nu(a)cb$. We say that A is twisted Calabi-Yau of dimension d (abbreviated as d-tCY or simply tCY) if A, as left A^e-module, has a finite-length projective resolution in which each term is finitely generated, and there is an isomorphism

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} {}_\nu A_1(\ell) & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases} \tag{1-2}$$

of graded right A^e-modules for some $\ell \in \mathbb{Z}$ and some graded algebra automorphism ν of A. Such a ν is uniquely determined and is called the Nakayama automorphism of A. We also denote it by ν_A to clarify the algebra. (Some authors call ν^{-1} the Nakayama automorphism; see, e.g., [23].) We say that A is d-Calabi-Yau (or simply Calabi-Yau, abbreviated as d-CY or CY) if it is twisted Calabi-Yau of dimension d and $\nu = \mathrm{id}_A$.

Proposition 1.6. [23, Lem. 1.2] *A connected graded algebra is an Artin-Schelter regular algebra of dimension d if and only if it is a twisted Calabi-Yau algebra of dimension d. If this is the case, the numbers ℓ in (1-1) and (1-2) are the same.*

1.9. The next result, which summarizes some of the results in §2, applies to a larger class of algebras than those covered by Theorem 1.1.

Theorem 1.7. *Let D be a connected graded algebra generated as a $\Bbbk$-algebra by D_1. Let $\Omega \in D$ be a homogeneous normal regular element of degree ≥ 2 and let $A = D/(\Omega)$. The algebra D is twisted Calabi-Yau if and only if A is and in that case the following results hold:*

(1) *on the common space of generators $A_1 = D_1$, $\nu_A = \tau^{-1}\nu_D$ where τ is the unique automorphism of D such that $\Omega x = \tau(x)\Omega$ for all $x \in D$;*
(2) *Ω is an eigenvector for the Nakayama automorphism ν_D;*
(3) *if σ is a degree-preserving automorphism of D such that $\sigma(\Omega) = \lambda\Omega$, then $\operatorname{hdet}(\sigma)$, the homological determinant of σ acting on D, is $\lambda \operatorname{hdet}(\sigma|_A)$.*

1.10. m**-Koszul algebras.** Because they are generated by their degree-one components the 3-dimensional Artin-Schelter regular algebras $A(\mathsf{w})$ in this paper are m-Koszul by [1, Thm. 1.5]. A connected graded algebra A is m-Koszul if it is finitely presented and its relations are homogeneous of degree m and $\operatorname{Ext}^i_A(\Bbbk, \Bbbk)$ is concentrated in a single degree for all i. The class of m-Koszul algebras was identified and first studied by R. Berger [4], the justification being that they are a natural generalization of Koszul algebras (the 2-Koszul algebras are exactly the Koszul algebras in the “classical” sense) and that there are many “nice” and “natural” m-Koszul algebras that are not 2-Koszul.

Reyes-Rogalski-Zhang [24, Cor. 5.4] proved that the homological determinant of the Nakayama automorphism of a noetherian Artin-Schelter regular connected graded algebras is 1. Mori-Smith [20] proved a similar result in the non-noetherian case when the algebra is m-Koszul (and Artin-Schelter regular and connected) and conjectured that the equality holds without the m-Koszul assumption. It follows from Theorem 1.7(3) that the conjecture is true for the algebras $D(\mathsf{w},\mathsf{p})$ (which are never m-Koszul and often not noetherian).

1.11. **The structure of the paper.** In §2, we prove some new results about the homological properties of a connected graded algebra and a normal extension of it. The main theorem is stated in §3 and some of its consequences are proved there. Its proof is postponed to §4. In the last section, §5, we determine the good tuples p for all the generic 3-dimensional Artin-Schelter regular algebras of finite Gelfand-Kirillov dimension listed in [1, Tables 3.9 and 3.11]. This provides a host of new noetherian 4-dimensional Artin-Schelter regular algebras on 2 and 3 generators. That section also provides some new normal and central extensions of non-noetherian 3-Calabi-Yau algebras.

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2. Results on normal extensions

2.1. In this section we are concerned about the relation between a ring D having a normal regular element Ω and the quotient $A := D/(\Omega)$.

To do this we make use of the automorphism $\tau : D \to D$ defined by the requirement that

$$\Omega x = \tau(x)\Omega \tag{2-1}$$

for all $x \in D$. Since τ fixes Ω, it descends to an automorphism of A that we also denote by τ.

Let σ be an automorphism of a ring D. If M is a left D-module we write ${}_\sigma M$ for the left D-module that is the abelian group M endowed with the action

$$x \cdot m \;:=\; \sigma(x)m$$

for all $x \in D$ and $m \in M$. There is an autoequivalence $M \rightsquigarrow {}_\sigma M$ of the category $\mathsf{Mod}(D)$ of left D-modules. The inverse $\sigma^{-1} : D \to D$ is an isomorphism ${}_\sigma D \to D$ of left D-modules. Likewise, if σ descends to A, then $\sigma^{-1} : A \to A$ is an isomorphism ${}_\sigma A \to A$ of left D-modules.

We extend the notation in the previous paragraph to D-bimodules: if α and β are automorphisms of D and M is a D-bimodule, we write ${}_\alpha M_\beta$ for the D-bimodule that is M as an abelian group endowed with the action

$$x \cdot m \cdot y \;:=\; \alpha(x)m\beta(y).$$

If $\alpha^{-1}\beta = \gamma^{-1}\delta$, then $\gamma\alpha^{-1} : {}_\alpha D_\beta \to {}_\gamma D_\delta$ is an isomorphism of D-bimodules. For example, ${}_\alpha D_1 \cong {}_1 D_{\alpha^{-1}}$.

2.2. In the situation of §2.1, the projective resolution

$$0 \longrightarrow D \xrightarrow{\cdot\Omega} D \longrightarrow A \longrightarrow 0 \tag{2-2}$$

of A as a left D-module is, in fact, an exact sequence

$$0 \longrightarrow {}_1 D_\tau \xrightarrow{\cdot\Omega} D \longrightarrow A \longrightarrow 0 \tag{2-3}$$

in the category of D-bimodules. The unadorned D and A in this sequence are given their natural D-bimodule structures.

If M is a left D-module, the result of applying $\operatorname{Hom}_D(-,M)$ to (2-3) is the exact sequence

$$0 \longrightarrow \operatorname{Hom}_D(A,M) \longrightarrow \operatorname{Hom}_D({}_1D_1,M) \xrightarrow{\Omega\cdot} \operatorname{Hom}_D({}_1D_\tau,M) \longrightarrow \operatorname{Ext}^1_D(A,M) \longrightarrow 0. \tag{2-4}$$

The map $\Phi : {}_\tau M \to \operatorname{Hom}_D({}_1D_\tau, M)$ given by $\Phi(m)(d) = dm$ is an isomorphism of left D-modules when $\operatorname{Hom}_D({}_1D_\tau, M)$ is given the left D-module structure coming from the right action of D on ${}_1D_\tau$. Thus, $\operatorname{Ext}^1_D(A,M)$ is isomorphic as a left D-module to ${}_\tau(M/\Omega M)$.

Proposition 2.1. *Let D be a connected graded algebra, $\Omega \in D_m$ a normal element of degree m, and let $A = D/(\Omega)$. Consider the following statements.*

1. *Ω is regular.*
2. *The Hilbert series of A and D are related by $h_D(t) = h_A(t)\,(1-t^m)^{-1}$.*
3. *If A is AS-regular of dimension d, then D is AS-regular of dimension $d+1$.*

There are implications (1) ⇔ (2) ⇒ (3)

Proof. (1) $\Leftrightarrow$ (2) This follows from the fact that the complex $0 \longrightarrow D(-m) \xrightarrow{\cdot\Omega} D \longrightarrow A \longrightarrow 0$ is exact if and only if Ω is regular.

(1) $\Rightarrow$ (3) As explained in [2, §2], the global dimensions of each of A and D equal the projective dimension of the trivial module $\Bbbk$ over the respective algebra. Since Ω is regular and $\operatorname{pdim}({}_A\Bbbk) < \infty$, [19, Thm. 7.3.5(i)] implies that $\operatorname{pdim}({}_D\Bbbk)$ equals $\operatorname{pdim}({}_A\Bbbk) + 1 = d + 1$. Thus, $\operatorname{gldim}(D) = d + 1$. The Gorenstein property for D is equivalent to that of A via the isomorphism

$$\operatorname{Ext}^{p+1}_D(\Bbbk, D) \cong \operatorname{Ext}^p_A(\Bbbk, A)$$

which is a consequence of Lemma 2.2 below applied to $N = \Bbbk$ and $M = D$. ■

Lemma 2.2. *Let D be a connected graded algebra, $\Omega \in D_m$ a normal element of degree m, and let $A = D/(\Omega)$. Then, for every left A-module N and all left D-modules M on which Ω acts regularly, there are isomorphisms*

$$\operatorname{Ext}^{p+1}_D(N, M) \cong \operatorname{Ext}^p_A(N, {}_\tau(M/\Omega M))$$

which is functorial on both N and M, for all p.

Proof. The functor $\operatorname{Hom}_D(N, -)$ from left D-modules to vector spaces factors as

$$\begin{tikzcd} \mathsf{Mod}(D) \arrow[rr, bend left, "\operatorname{Hom}_D(N,-)"] \arrow[r, "\operatorname{Hom}_D(A,-)"'] & \mathsf{Mod}(A) \arrow[r, "\operatorname{Hom}_A(N,-)"'] & \mathsf{Vect}. \end{tikzcd}$$

Since the lower left functor preserves injectivity, [12, Thm. III.7.1] gives an isomorphism

$$\operatorname{RHom}_D(N, M) \cong \operatorname{RHom}_A(N, \operatorname{RHom}_D(A, M)) \tag{2-5}$$

in the left-bounded derived category of vector spaces, which is functorial on both M and N.

Since Ω acts faithfully on M, $\operatorname{Hom}_D(A, M) = 0$. As observed above, the canonical map $\operatorname{Hom}_D({}_1D_\tau, M) \to \operatorname{Ext}^1_D(A, M)$ induces an isomorphism ${}_\tau(M/\Omega M) \xrightarrow{\sim} \operatorname{Ext}^1_D(A, M)$ of left D-modules. Since $\operatorname{pdim}({}_DA) = 1$, $\operatorname{Ext}^p_D(A, M) = 0$ for $p \geq 2$. Hence

$$\operatorname{RHom}_D(A, M) \cong {}_\tau(M/\Omega M)[-1]$$

in the left-bounded derived category of left A-modules. Thus,

$$\operatorname{RHom}_D(N, M) \cong \operatorname{RHom}_A(N, {}_\tau(M/\Omega M))[-1],$$

which is nothing but a reformulation of the conclusion of the lemma. ■

On occasion, one might want to impose growth conditions on connected graded algebras that complement the good homological properties in §1.7 (the more prevalent definition of Artin-Schelter regularity, for instance, requires finite Gelfand-Kirillov dimension; see e.g. [1, Introduction] or [2, (2.12)]). The following remark supplements Proposition 2.1 in that direction.

Proposition 2.3. *Let D be a connected graded algebra and $\Omega \in D_m$ a normal element of degree $m \geq 1$, and let $A = D/(\Omega)$. Then*

1. *A has finite Gelfand-Kirillov dimension if and only if D does;*
2. *A is left or right noetherian if and only if D is.*

Proof. (1) The finite Gelfand-Kirillov dimension of D certainly entails that of its quotient $A = D/(\Omega)$. Conversely, it follows from the exact sequence $D(-m) \xrightarrow{\cdot\Omega} D \longrightarrow A \longrightarrow 0$ that

$$\dim(D_k) - \dim(D_{k-m}) \leq \dim(A_k) \tag{2-6}$$

for all k. Thus, if $\dim(A_k)$ has polynomial growth so does $\dim(D_k)$.

(2) Certainly, if D is noetherian so is A. The converse follows from [2, Lem. 8.2]. ■

Later, we determine the relationship between the Nakayama automorphism of A and that of D whenever the equivalent conditions (1) and (2) of Proposition 2.1 are satisfied. As a first step in that direction, we record the following consequence of Lemma 2.2.

Lemma 2.4. *Let D be a connected graded algebra, $\Omega \in D_m$ a normal regular element of degree m, and let $A = D/(\Omega)$. There is an isomorphism*

$$\operatorname{Ext}^{p+2}_{D^e}(A, D^e) \;\cong\; {}_\tau \operatorname{Ext}^p_{A^e}(A, A^e)_{\tau^{-1}}$$

of D-bimodules.

Proof. We write ${}_\tau A^e{}_{\tau^{-1}}$ for A^e endowed with the D-bimodule structure given by

$$c \cdot (a \otimes b) \cdot d \;=\; \tau(\overline{c})a \otimes b\tau^{-1}(\overline{d})$$

where $\overline{c}$ and $\overline{d}$ are the images in A of $c, d \in D$ and $b\tau^{-1}(\overline{d})$ is the product $b \times \tau^{-1}(\overline{d})$ in A viewed as an element of A°. As a left D^e-module, ${}_\tau A^e{}_{\tau^{-1}}$ is annihilated by $1 \otimes \Omega$ and $\Omega \otimes 1$ so ${}_\tau A^e{}_{\tau^{-1}}$ is a left A^e-module (and, equivalently, an A-bimodule) where the subscripts τ and τ^{-1} should now be viewed as the automorphisms of A induced by the automorphisms τ and τ^{-1} of D.

Apply Lemma 2.2 twice, first to the quotient $D \otimes A^\circ = D^e/(1 \otimes \Omega)$, then to $A^e = (D \otimes A^\circ)/(\Omega \otimes 1)$. This results in an isomorphism

$$\operatorname{Ext}^{p+2}_{D^e}(A, D^e) \;\cong\; \operatorname{Ext}^p_{A^e}(A, {}_\tau A^e{}_{\tau^{-1}})$$

that preserves the right D^e-module structures. The result follows from the isomorphisms

$${}_\tau A_1 \cong {}_1 A_{\tau^{-1}}, \quad {}_1 A_{\tau^{-1}} \cong {}_\tau A_1$$

for the left and respectively right-hand tensorands of A^e. ■

2.3. For the rest of this section we assume that D is a connected graded algebra, $\Omega \in D_m$ is a normal regular element of degree m, and $A = D/(\Omega)$ is Artin-Schelter regular of dimension d. Since condition (3) of Proposition 2.1 holds, D is Artin-Schelter regular of dimension $d + 1$.

By Proposition 1.6, A and D are also twisted Calabi-Yau; it is the A^e- and D^e-bimodule structures of A and D that will play the central role.

We denote by $\nu = \nu_{D}$ the Nakayama automorphism of D, defined by

$$\operatorname{Ext}^{d+1}_{D^e}(D, D^e) \cong {}_\nu D_1 \cong {}_1 D_{\nu^{-1}}; \tag{2-7}$$

all other $\operatorname{Ext}^p(D, D^e)$ vanish.

2.4. The next result, together with Lemma 2.4, ensures that $\operatorname{Ext}^{d+2}_{D^e}(A, D^e)$ can function as a bridge between $\operatorname{Ext}^{d+1}_{D^e}(D, D^e)$ and $\operatorname{Ext}^d_{A^e}(A, A^e)$, which in turn are used to define the Nakayama automorphisms of A and D.

Lemma 2.5. *Let D be a connected graded algebra, $\Omega \in D_m$ a normal regular element of degree m, and let $A = D/(\Omega)$. If A is AS-regular of dimension d, then there is an isomorphism*

$$\operatorname{Ext}^{d+2}_{D^e}(A, D^e) \;\cong\; {}_\tau A_{\nu_{D}^{-1}}$$

of D-bimodules.

Proof. The short exact sequence (2-3) of D-bimodules yields an exact sequence

$$\operatorname{Ext}^{d+1}_{D^e}(D, D^e) \xrightarrow{(\cdot\Omega)^*} \operatorname{Ext}^{d+1}_{D^e}({}_1D_\tau, D^e) \longrightarrow \operatorname{Ext}^{d+2}_{D^e}(A, D^e) \longrightarrow 0$$

of right D^e-modules in which $(\cdot\Omega)^*$ denotes the morphism induced from the morphism $(\cdot\Omega)$ between the first variables. A morphism induced from that between second variables will be denoted by a lower star. We will show that the diagram

$$\begin{tikzcd}
\operatorname{Ext}^{d+1}_{D^e}(D, D^e) \arrow[r, equal] \arrow[d, "(\cdot\Omega)^*"] & \operatorname{Ext}^{d+1}_{D^e}(D, D^e) \arrow[r, equal] \arrow[d, "((1\otimes\Omega)\cdot)_*"] & \operatorname{Ext}^{d+1}_{D^e}(D, D^e) \arrow[r, equal] \arrow[d, "(\cdot(1\otimes\Omega))_*"] & \operatorname{Ext}^{d+1}_{D^e}(D, D^e) \arrow[d, "\Omega\cdot"] \\
\operatorname{Ext}^{d+1}_{D^e}({}_1D_\tau, D^e) \arrow[r, equal] & \operatorname{Ext}^{d+1}_{D^e}(D, {}_{1\otimes\tau^{-1}}D^e) \arrow[r, "\sim", "(1\otimes\tau)_*"'] & \operatorname{Ext}^{d+1}_{D^e}(D, D^e_{1\otimes\tau}) \arrow[r, equal] & {}_\tau \operatorname{Ext}^{d+1}_{D^e}(D, D^e)_1
\end{tikzcd} \tag{2-8}$$

commutes. The commutativity of the middle square follows from the definition of τ. Note that right multiplication by $1 \otimes \Omega$ on $D^e = D \otimes_k D^\circ$ is left multiplication by Ω on the right tensorand. The commutativity of the right-most square is straightforward.

For the left-most square: The left vertical morphism is equal to $((1 \otimes \Omega)\cdot)^*$. For each left D^e-module M, set $FM := {}_1M_{\tau^{-1}}$ and $\varphi_M := ((1 \otimes \Omega)\cdot)\colon M \to FM$. This defines a morphism $\varphi\colon 1 \to F$ of autoequivalences on $\mathsf{Mod}(D^e)$. Since every left D^e-homomorphism $f\colon FM \to N$ satisfies $f\varphi_M = \varphi_N f$, we obtain the following commutative diagram by considering an injective resolution of N:

$$\begin{tikzcd} \operatorname{Ext}^{d+1}_{D^e}(FM,N) \arrow[r, equal] \arrow[d, "(\varphi_M)^*"] & \operatorname{Ext}^{d+1}_{D^e}(FM,N) \arrow[d, "(\varphi_N)_*"] \\ \operatorname{Ext}^{d+1}_{D^e}(M,N) \arrow[r, equal] & \operatorname{Ext}^{d+1}_{D^e}(FM,FN) \end{tikzcd}.$$

This becomes the left-most square of (2-8) after substituting $M := {}_1D_\tau$ and $N := D^e$.

In the following diagram, the right-hand square obviously commutes and the left-hand square commutes since the horizontal arrows are isomorphism of D-bimodules:

$$\begin{tikzcd} \operatorname{Ext}^{d+1}_{D^e}(D,D^e) \arrow[r, "\sim"'] \arrow[d, "\Omega\cdot"] & {}_\nu D_1 \arrow[r, "\nu^{-1}", "\sim"'] \arrow[d, "\Omega\cdot"] & {}_1D_{\nu^{-1}} \arrow[d, "\Omega\cdot"] \\ {}_\tau \operatorname{Ext}^{d+1}_{D^e}(D,D^e)_1 \arrow[r, "\sim"] & {}_\tau({}_\nu D_1)_1 \arrow[r, "\sim", "\nu^{-1}"'] & {}_\tau D_{\nu^{-1}} \end{tikzcd}$$

By adjoining this diagram to the right-hand end of (2-8), we obtain the commutative diagram

$$\begin{tikzcd} \operatorname{Ext}^{d+1}_{D^e}(D,D^e) \arrow[r, "(\cdot\Omega)^*"] \arrow[d, "\wr"] & \operatorname{Ext}^{d+1}_{D^e}({}_1D_\tau,D^e) \arrow[r] \arrow[d, "\wr"] & \operatorname{Ext}^{d+2}_{D^e}(A,D^e) \arrow[r] \arrow[d, "\wr"] & 0 \\ {}_1D_{\nu^{-1}} \arrow[r, "\Omega\cdot"] & {}_\tau D_{\nu^{-1}} \arrow[r] & {}_\tau A_{\nu_D^{-1}} \arrow[r] & 0 \end{tikzcd}$$

which completes the proof of the lemma. ■

We now determine the relation between the Nakayama automorphisms of D and A when D is generated by D_1 as a $\Bbbk$-algebra. The algebras D that we will consider in section 3 *are* generated by D_1.

Theorem 2.6. *Let D be a connected graded $\Bbbk$-algebra that is generated by D_1. Let $\Omega \in D$ be a homogeneous normal regular element of degree ≥ 2 and let $A = D/(\Omega)$. If A is twisted Calabi-Yau so is D and in that case $\nu_A = \tau^{-1}\nu_D$ on the common space of generators $A_1 = D_1$.*

Proof. It follows from (3) of Proposition 2.1 that D is twisted Calabi-Yau of dimension $d+1$. Lemmas 2.4 and 2.5 together with $\operatorname{Ext}^d_{A^e}(A,A^e) \cong {}_{\nu_A}A_1$ imply that

$${}_1A_{\nu_A^{-1}} \;\cong\; \operatorname{Ext}^d_{A^e}(A,A^e) \;\cong\; {}_{\tau^{-1}}\operatorname{Ext}^{d+2}_{D^e}(A,D^e)_\tau \;\cong\; {}_1A_{\nu_D^{-1}\tau};$$

the conclusions follows. ■

We remind the reader that τ and ν_D commute since, by [18, Thm. 4.2], Nakayama automorphisms are central under conditions that our algebras certainly satisfy.

Theorem 2.6 is analogous to [23, Lem. 1.5], but we do not use the theory of dualizing complexes, and do not assume here that Ω is an eigenvector for the Nakayama automorphism ν_D. In fact, the latter condition now follows from the above discussion.

Corollary 2.7. *Let D be a connected graded algebra, $\Omega \in D$ a homogeneous normal regular element of degree ≥ 2. If $D/(\Omega)$ is twisted Calabi-Yau, then Ω is an eigenvector for the Nakayama automorphism ν_D.*

Proof. By Theorem 2.6, ν_D descends to an automorphism of $D/(\Omega)$. Hence $\nu_D(\Omega)$ belongs to the ideal (Ω). Since D is connected and ν_D preserves degree, $\nu_D(\Omega) \in \Bbbk\Omega$. ■

2.5. **Homological determinant.** As before, let D be a connected graded algebra, $\Omega \in D_m$ a normal regular element of degree m, and assume that $A = D/(\Omega)$ is Artin-Schelter regular of dimension d with Gorenstein parameter ℓ. We will determine the relation between the homological determinant of an automorphism of D that preserves the ideal (Ω) with the homological determinant of the induced automorphism of A. The main result in this section, Theorem 2.10, will be used later to show that the homological determinant of the Nakayama automorphism of the algebra $D(\mathsf{w},\mathsf{p})$ is 1.

The homological determinant was introduced by Jørgensen-Zhang [14]. Before defining it we need some notation and other ideas.

Let M be a left A-module. The p^{th} **local cohomology group** of M is $H^p_{\mathfrak{m}}(M) := \varinjlim \operatorname{Ext}^p_A(A/A_{\geq n}, M)$. The right action of A on $A/A_{\geq n}$ determines a left A-module structure on $H^p_{\mathfrak{m}}(M)$, and $H^p_{\mathfrak{m}}$ then becomes an endofunctor on the category of graded left A-modules.

The **Matlis dual** of M is the graded right A-module M' whose degree-i component is $\operatorname{Hom}_{\Bbbk}(M_{-i}, \Bbbk)$, the right action of A coming from the left action of A on M. The operation $M \rightsquigarrow M'$ is a duality between the category of graded left A-modules and the category of graded right A-modules.

Let $\operatorname{Aut}_{\mathsf{gr}}(A)$ denote the group of degree-preserving $\Bbbk$-algebra automorphisms of A and let $\sigma \in \operatorname{Aut}_{\mathsf{gr}}(A)$. Then $\sigma : A \to {}_{\sigma}A_{\sigma}$ is an isomorphism of A-bimodules.

Jørgensen-Zhang [14] defined the homological determinant in the following way. First they show that the top local cohomology group $H^d_{\mathfrak{m}}(A)$ is isomorphic to $A'(\ell)$ and all other $H^p_{\mathfrak{m}}(A)$ vanish. Fix an isomorphism $\psi\colon H^d_{\mathfrak{m}}(A) \to A'(\ell)$. Let $H^d_{\mathfrak{m}}({}_{\sigma}A) \xrightarrow{\sim} {}_{\sigma}H^d_{\mathfrak{m}}(A)$ be the isomorphism

$$H^d_{\mathfrak{m}}({}_{\sigma}A) = \varinjlim \operatorname{Ext}^d_A(A/A_{\geq n}, {}_{\sigma}A) \xrightarrow{\sim} \varinjlim \operatorname{Ext}^d_A({}_{\sigma^{-1}}(A/A_{\geq n}), A) \xrightarrow{\sim} \varinjlim \operatorname{Ext}^d_A((A/A_{\geq m})_{\sigma}, A) = {}_{\sigma}H^d_{\mathfrak{m}}(A).$$

The **homological determinant** of σ is the unique scalar $\operatorname{hdet}(\sigma)$ that makes the diagram

$$\begin{tikzcd} H^d_{\mathfrak{m}}(A) \arrow[r, "H^d_{\mathfrak{m}}(\sigma)", "\sim"'] \arrow[d, "\psi"', "\wr"] & H^d_{\mathfrak{m}}({}_{\sigma}A) \arrow[r, "\sim"'] & {}_{\sigma}H^d_{\mathfrak{m}}(A) \arrow[d, "\psi", "\wr"'] \\ A'(\ell) \arrow[rr, "\sim", "(\operatorname{hdet}(\sigma))^{-1}(\sigma^{-1})'"'] & & {}_{\sigma}A'(\ell) \end{tikzcd} \tag{2-9}$$

commute. When we need to identify the algebra A we write $\operatorname{hdet}_A(\sigma)$.

The degree-$(-\ell)$ components of the diagram (2-9) give another commutative diagram:

$$\begin{tikzcd} \operatorname{Ext}^d_A(\Bbbk, A) \arrow[r, "\sigma_*", "\sim"'] \arrow[d, "\wr"] & \operatorname{Ext}^d_A(\Bbbk, {}_{\sigma}A) \arrow[r, "\sigma^{-1}(-)", "\sim"'] & \operatorname{Ext}^d_A(\Bbbk, A) \arrow[d, "\wr"] \\ \Bbbk(\ell) \arrow[rr, "\sim", "(\operatorname{hdet}(\sigma))^{-1}"'] & & \Bbbk(\ell) \end{tikzcd} \tag{2-10}$$

where the two vertical morphisms are the same. Note that ${}_{\sigma}\operatorname{Ext}^d_A(\Bbbk, A) = \operatorname{Ext}^d_A(\Bbbk, A)$. Because all the objects in (2-10) have dimension 1, the definition of $\operatorname{hdet}(\sigma)$ does not depend on the choice of the isomorphism ψ.

Lemma 2.8. *Let $\sigma \in \operatorname{Aut}_{\mathsf{gr}}(D)$. If Ω is an eigenvector for σ, then there is an isomorphism $\alpha\colon \operatorname{Ext}^d_D(\Bbbk, A) \xrightarrow{\sim} \operatorname{Ext}^d_A(\Bbbk, A)$ that makes the diagram*

$$\begin{tikzcd} \operatorname{Ext}^d_D(\Bbbk, A) \arrow[r, "\sigma_*", "\sim"'] \arrow[d, "\alpha"', "\wr"] & \operatorname{Ext}^d_D(\Bbbk, {}_{\sigma}A) \arrow[r, "\sigma^{-1}(-)", "\sim"'] & \operatorname{Ext}^d_D(\Bbbk, A) \arrow[d, "\alpha", "\wr"'] \\ \operatorname{Ext}^d_A(\Bbbk, A) \arrow[r, "\sim", "\sigma_*"'] & \operatorname{Ext}^d_A(\Bbbk, {}_{\sigma}A) \arrow[r, "\sim", "\sigma^{-1}(-)"'] & \operatorname{Ext}^d_A(\Bbbk, A) \end{tikzcd} \tag{2-11}$$

commutative.

Proof. Since $\sigma(\Bbbk\Omega) = \Bbbk\Omega$, σ descends to an automorphism of A, which is also denoted by σ. By applying $\operatorname{RHom}_D(-, A)$ to the short exact sequence (2-3) of D-bimodules, we obtain a triangle

$$\operatorname{RHom}_D(A, A) \longrightarrow A \xrightarrow{0} {}_{\tau}A \longrightarrow \operatorname{RHom}_D(A, A)[1]$$

in the left-bounded derived category of left D-modules. Hence $\operatorname{RHom}_D(A, A)$ is, in the left-bounded derived category of left A-module, isomorphic to a two-term complex $C^0 \to C^1$ whose cohomologies are A at degree 0 and ${}_{\tau}A$ at degree 1. Since $\tau\colon A \to {}_{\tau}A$ is an isomorphism of left A-modules, ${}_{\tau}A$ is also projective. Therefore the two-term complex splits into $A \oplus {}_{\tau}A[-1]$.

Let $A \to I^{\bullet}$ be an injective resolution of A as a left D-module. Then $\operatorname{Hom}_D(A, I^{\bullet})$ is a complex of injective left A-modules and we have

$$A \xrightarrow{\sim} \operatorname{Hom}_A(A, A) = \operatorname{Hom}_D(A, A) \to \operatorname{Hom}_D(A, I^{\bullet})$$

whose 0^{th} cohomologies are isomorphic. This induces

$$\operatorname{Ext}^d_A(\Bbbk, A) \xrightarrow{\sim} \mathrm{H}^d \operatorname{Hom}_A(\Bbbk, \operatorname{Hom}_D(A, I^{\bullet})) \xrightarrow{\sim} \mathrm{H}^d(\operatorname{Hom}_D(\Bbbk, I^{\bullet})) = \operatorname{Ext}^d_D(\Bbbk, A)$$

where the first isomorphism holds since $\mathrm{H}^d \operatorname{Hom}_A(\Bbbk, {}_\tau A[-1]) \cong \operatorname{Ext}_A^{d-1}(\Bbbk, A) = 0$ by the Gorenstein condition.

Since the isomorphism $\operatorname{Ext}_A^d(\Bbbk, A) \xrightarrow{\sim} \operatorname{Ext}_D^d(\Bbbk, A)$ sends each n-extension of A by $\Bbbk$ to itself, the commutativity of the diagram follows. ■

Lemma 2.9. *Let $\sigma \in \operatorname{Aut}_{\mathsf{gr}}(D)$. If Ω is an eigenvector for σ with eigenvalue λ, then there is an isomorphism $\delta\colon \operatorname{Ext}_D^d(\Bbbk, A) \xrightarrow{\sim} \operatorname{Ext}_D^{d+1}(\Bbbk, D)$ that makes the diagram*

(2-12)
$$\begin{tikzcd}
\operatorname{Ext}_D^d(\Bbbk, A) \arrow[r, "\sigma_*", "\sim"'] \arrow[d, "\delta"', "\wr"] & \operatorname{Ext}_D^d(\Bbbk, {}_\sigma A) \arrow[r, "\sigma^{-1}(-)", "\sim"'] & \operatorname{Ext}_D^d(\Bbbk, A) \arrow[d, "\wr"', "\lambda^{-1}\delta"] \\
\operatorname{Ext}_D^{d+1}(\Bbbk, D) \arrow[r, "\sim", "\sigma_*"'] & \operatorname{Ext}_D^{d+1}(\Bbbk, {}_\sigma D) \arrow[r, "\sim", "\sigma^{-1}(-)"'] & \operatorname{Ext}_D^{d+1}(\Bbbk, D)
\end{tikzcd}$$

commutative.

Proof. The short exact sequence (2-2) is extended to the commutative diagram in the derived category

$$\begin{tikzcd}
D \arrow[r, "\cdot\Omega"] \arrow[d, "\wr"', "\sigma"] & D \arrow[r] \arrow[d, "\wr"', "\sigma"] & A \arrow[r, "\varepsilon"] \arrow[d, "\wr"', "\sigma"] & D[1] \arrow[d, "\wr"', "\sigma[1]"] \\
{}_\sigma D \arrow[r, "\cdot\sigma(\Omega)"] \arrow[d, "\wr"', "\lambda"] & {}_\sigma D \arrow[r] \arrow[d, equal] & {}_\sigma A \arrow[r] \arrow[d, equal] & {}_\sigma D[1] \arrow[d, "\wr"', "\lambda"] \\
{}_\sigma D \arrow[r, "(\cdot\Omega)={}_\sigma(\cdot\Omega)"] & {}_\sigma D \arrow[r] & {}_\sigma A \arrow[r, "{}_\sigma\varepsilon"] & {}_\sigma D[1]
\end{tikzcd}$$

where each row is a triangle. The third and fourth columns yield the commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}_D^d(\Bbbk, A) \arrow[r, "\varepsilon_*", "\sim"'] \arrow[d, "\sigma_*"', "\wr"] & \operatorname{Ext}_D^{d+1}(\Bbbk, D) \arrow[d, "\wr"', "\lambda\sigma_*"] \\
\operatorname{Ext}_D^d(\Bbbk, {}_\sigma A) \arrow[r, "({}_\sigma\varepsilon)_*", "\sim"'] \arrow[d, "\sigma^{-1}(-)"', "\wr"] & \operatorname{Ext}_D^{d+1}(\Bbbk, {}_\sigma D) \arrow[d, "\wr"', "\sigma^{-1}(-)"] \\
\operatorname{Ext}_D^d(\Bbbk, A) \arrow[r, "\varepsilon_*", "\sim"'] & \operatorname{Ext}_D^{d+1}(\Bbbk, D)
\end{tikzcd}$$

Letting $\delta := \operatorname{Ext}^d(\Bbbk, \varepsilon)$ completes the proof. ■

Theorem 2.10. *Let D be a connected graded algebra, $\Omega \in D_m$ a normal regular element of degree m, and let $A = D/(\Omega)$. Assume that D is twisted Calabi-Yau of dimension $d+1$. Let $\sigma \in \operatorname{Aut}_{\mathsf{gr}}(D)$. If $\Omega \in D$ is an eigenvector for σ with eigenvalue λ, then*

$$\operatorname{hdet}_D(\sigma) \;=\; \lambda \operatorname{hdet}_A(\sigma|_A).$$

Proof. The concatenation of the diagram (2-10) for D with (2-11) and (2-12) yields the commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}_A^d(\Bbbk, A) \arrow[r, "\sigma_*", "\sim"'] \arrow[d, "\wr"'] & \operatorname{Ext}_A^d(\Bbbk, {}_\sigma A) \arrow[r, "\sigma^{-1}(-)", "\sim"'] & \operatorname{Ext}_A^d(\Bbbk, A) \arrow[d, "\wr"] \\
\Bbbk(\ell) \arrow[rr, "\sim", "(\operatorname{hdet}_D(\sigma))^{-1}"'] & & \Bbbk(\ell)
\end{tikzcd}$$

where the right isomorphism is equal to the left isomorphism multiplied by λ^{-1}. This shows $(\operatorname{hdet}_A(\sigma|_A))^{-1} = (\operatorname{hdet}_D(\sigma))^{-1}\lambda$. ■

2.6. **An equivariant formulation of the homological determinant.** In this section we give a slightly different interpretation of the homological determinant. In this section A denotes a d-dimensional Artin-Schelter regular $\Bbbk$-algebra with Gorenstein parameter ℓ and σ is a graded $\Bbbk$-algebra automorphism of A.

We assume some familiarity with group actions on categories and equivariantizations. A good reference is [11, §2.7]. Additional details can be found in the appendix of [7]. Since categories in this paper are $\Bbbk$-linear, all functors in the following discussion are required to be $\Bbbk$-linear.

Recall [11, Defns. 2.7.1 and 2.7.2]; because of the wide availability of resources on the topic we merely paraphrase the definitions here.

Definition 2.11. An action of a group Γ on a category $\mathcal{C}$ is a collection of autoequivalences $T_g : \mathcal{C} \to \mathcal{C}$, $g \in \Gamma$, and natural isomorphisms $T_g \circ T_h \cong T_{gh}$ satisfying the obvious compatibility conditions.

Given an action of Γ on $\mathcal{C}$, the category $\mathcal{C}^\Gamma$ of Γ-equivariant objects is the category whose objects are collections of isomorphisms

$$(\varphi_g : T_g x \to x \mid g \in \Gamma)$$

that are compatible with the endofunctors T_g in the sense that the diagrams

$$\begin{tikzcd} T_{gh}x \arrow[rrr, bend left, "\varphi_{gh}"] \arrow[r, "\cong"'] & T_gT_hx \arrow[r, "T_g\varphi_h"'] & T_gx \arrow[r, "\varphi_g"'] & x \end{tikzcd}$$

commute for all $g, h \in \Gamma$, where the lower isomorphism is the structural one defining the Γ-action on $\mathcal{C}$. The morphisms in $\mathcal{C}^\Gamma$ are those morphisms $f : x \to y$ in $\mathcal{C}$ for which the diagrams

$$\begin{tikzcd} T_gx \arrow[r, "T_gf"] \arrow[d, "\varphi_g"'] & T_gy \arrow[d, "\varphi_g"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

commute for all $g \in \Gamma$. ♦

Now consider objects $x, y \in \mathcal{C}^\Gamma$ (we are abusing notation by suppressing the isomorphisms $\varphi_g : T_gx \to x$). The action of Γ on $\mathcal{C}$ induces an action $\triangleright$ of Γ on $\mathrm{Hom}_{\mathcal{C}}(x, y)$ defined by the commutativity of the diagram

$$\begin{tikzcd} T_gx \arrow[r, "T_g(g^{-1}\triangleright f)"] \arrow[d, "\varphi_g"'] & T_gy \arrow[d, "\varphi_g"] \\ x \arrow[r, "f"'] & y \end{tikzcd} \tag{2-13}$$

Remark 2.12. When $\mathcal{C}$ is abelian a Γ-action passes over to any of the derived categories associated to $\mathcal{C}$ (the bounded derived category $\mathcal{D}^b(\mathcal{C})$, the left bounded one $\mathcal{D}^+(\mathcal{C})$, etc.). We assume below that $\mathcal{C}$ has enough injectives.

For an equivariant object y in $\mathcal{C}^\Gamma$ as above the isomorphisms $\varphi_g : T_gy \to y$ lift to isomorphisms between the injective resolutions T_gE^* and E^* of T_gy and y respectively, and given another equivariant object $x \in \mathcal{C}$ the action of Γ on

$$\mathrm{Ext}^d(x, y) = \mathrm{Hom}_{\mathcal{D}}(x, y[d])$$

can be computed via the action on $\mathrm{Hom}_{\mathcal{C}}(x, E^d)$ as in (2-13), using the isomorphism $\varphi_g : T_gx \to x$ as the left-hand vertical arrow and the isomorphism $T_gE^d \to E^d$ as the right-hand vertical map.

All group actions on derived categories considered below arise in this fashion, from an action on the original abelian category, and similarly, we only consider equivariant objects in $\mathcal{D}^b(\mathcal{C})$ that are lifts of those in $\mathcal{C}$ in the manner described above. ♦

We now apply the above to the group $\Gamma = \mathbb{Z}$ acting via the twist $M \mapsto {}_\sigma M$ for M in the category of A-modules, graded modules, complexes thereof, derived categories, etc. Since the group Γ is generated by the single element g mapping to $\sigma \in \langle\sigma\rangle$, equivariant structures are determined by single automorphisms $\varphi^x = \varphi_g : T_gx \to x$.

By definition, for (graded) A-modules the twist functor

$$M \;\mapsto\; T_gM \;=\; {}_\sigma M$$

does not change the underlying vector spaces and morphisms, so the diagram (2-13) above, for equivariant objects (M,φ^M) and (N,φ^N), reads

$$\begin{tikzcd} {}_\sigma M \arrow[r, "g^{-1}\triangleright f"] \arrow[d, "\varphi^M"'] & {}_\sigma N \arrow[d, "\varphi^N"] \\ M \arrow[r, "f"'] & N \end{tikzcd} \tag{2-14}$$

We can apply the discussion above to the bounded derived category $\mathcal{D}^b = \mathcal{D}^b(\mathsf{Mod}(A))$ of left A-modules with $\Gamma = \mathbb{Z}$ once more acting by powers by σ.

Now regard all $A/A_{\geq k}$, $k \leq \infty$ (where $A/A_{\geq\infty}$ means A), and their homological shifts $(A/A_{\geq k})[p]$ in $\mathcal{D}^b$ as σ-equivariant A-modules via $\varphi_{(A/A_{\geq k})[p]} := \sigma^{-1}[p]$. This renders the following statement meaningful; it is a repackaging of [14, Defn. 2.3].

Proposition 2.13. *The inverse* $\operatorname{hdet}^{-1}(\sigma) := \operatorname{hdet}(\sigma)^{-1}$ *of the homological determinant is the scalar by which* $g^{-1}\triangleright$ *acts on the degree-*$(-\ell)$ *component of any of the spaces* $H^d_{\mathfrak{m}}(A)$, $\operatorname{Ext}^d_A(\Bbbk, A)$, *or* $\operatorname{Ext}^d_A(\Bbbk,\Bbbk)$.

Proof. First, note that the three scalars in the statement are indeed equal:

As far as the last two are concerned, the long exact sequence attached to the short exact sequence

$$0 \to A_{\geq 1} \to A \to \Bbbk \to 0,$$

the vanishing of $\operatorname{Ext}^{d+1}_A$ and the fact that $\operatorname{Ext}^d_A(\Bbbk, A)$ and $\operatorname{Ext}^d_A(\Bbbk,\Bbbk)$ are both one-dimensional vector spaces jointly ensure that the induced map

$$\operatorname{Ext}^d_A(\Bbbk, A) \cong \operatorname{Ext}^d_A(\Bbbk,\Bbbk)$$

is an isomorphism.

On the other hand, the first two scalars in the statement are equal because the σ-equivariant maps $A/A_{\geq k+1} \to A/A_{\geq k}$ induce a Γ-module morphism

$$\operatorname{Ext}^d_A(\Bbbk, A) \to H^d_{\mathfrak{m}}(A) = \varinjlim \operatorname{Ext}^d_A(A/A_{\geq k}, A)$$

that identifies degree-$(-\ell)$ components.

Thus, it suffices to prove that the scalar hdet^{-1} defined via (2-10) agrees with the eigenvalue of g^{-1} on $\operatorname{Ext}^d_A(\Bbbk, A)$ (note however that for us, here, the top dimension is d rather than the $d+1$ of (2-10)). We proceed to do this.

[14, Defn. 2.3] (which (2-10) follows) dictates that the scalar hdet^{-1} is computed as follows:

Consider the minimal injective resolution

$$0 \to A \to E^0 \to E^1 \to \cdots$$

of A in $\mathsf{Mod}(A)$, and let σ act on it so as to extend the action on A. Then, hdet^{-1} will be the scaling that σ induces on the one-dimensional torsion of the d^{th} term E^d. This is also what we obtain by making g (the generator of $\Gamma = \mathbb{Z}$) act on $\operatorname{Hom}(\Bbbk, E^d)$ by sending a vector space morphism $f : \Bbbk \to E^d$ to $\sigma \circ f \circ \sigma^{-1}$, which is what (2-14) amounts to in the present setting (see Remark 2.12). ∎

3. Normal extensions of 3-dimensional regular algebras

In this section we describe a technique for producing 4-dimensional Artin-Schelter regular algebras by modifying m-Koszul 3-dimensional Artin-Schelter regular algebras. Again, our regular algebras need not be noetherian. We first recall some conventions and terminology from various sources on m-Koszul and superpotential algebras, such as e.g. [8, 9, 6, 20].

3.1. Superpotential algebras.

For our purposes, the relevant setup is as follows.

We fix an n-dimensional vector space V that will be the common space of degree-one generators for the algebras under discussion. We also fix integers $\ell \geq m \geq 2$, $Q \in \mathrm{GL}(V)$, and an element $\mathsf{w} \in V^{\otimes\ell}$ that is invariant under the linear map

$$V^{\otimes\ell} \longrightarrow V^{\otimes\ell}, \quad v_1 \otimes \ldots \otimes v_\ell \mapsto (Q^t v_\ell) \otimes v_1 \otimes \ldots \otimes v_{\ell-1}.$$

We call such a w a Q-twisted superpotential or just a twisted superpotential if we don't want to specify Q. This should be compared to [1, (2.12)] and the terminology in [6, §2.2]. If $Q = \mathrm{id}_V$ we simply call w a superpotential; see also [22, Defn. 2.1].

Frequently, we assume that Q is diagonalizable, but see (2) in §3.4.

Let W be a subspace of some tensor power, $V^{\otimes p}$ say. We introduce the following notation:

$$\begin{aligned} \partial W &:= \big\{(\psi \otimes \mathrm{id}^{\otimes p-1})(\mathsf{w}) \bigm| \psi \in V^* \text{ and } \mathsf{w} \in W\big\}, \\ \partial^{i+1} W &:= \partial(\partial^i W) \quad \text{for all } i \geq 0, \\ A(W,i) &:= TV/(\partial^i W). \end{aligned}$$

The space $\partial^i W$ appears in [9, §4] where it is denoted $W^{(p-i)}$.

Let $\mathsf{w} \in V^{\otimes \ell}$ be a Q-twisted superpotential. We call the algebra

$$A(\mathsf{w},i) \;:=\; A(\Bbbk\mathsf{w},i)$$

a superpotential algebra. In [6], $A(\mathsf{w},i)$ is called a derivation-quotient algebra. We are mostly interested in the case $i=1$ and we then write

$$A(\mathsf{w}) \;:=\; A(\mathsf{w},1).$$

Theorem 3.1 (Dubois-Violette)**.** [9, Thm. 11] [6, Thm. 6.8] *If A is an m-Koszul Artin-Schelter regular algebra of dimension d with Gorenstein parameter ℓ, then there is a unique 1-dimensional subspace $\Bbbk\mathsf{w}$ of $V^{\otimes \ell}$ such that*

$$A \cong A(\mathsf{w}, \ell - m).$$

The global dimension d of $A(\mathsf{w},\ell-m)$ can be recovered from the other numerical data as follows:

- *if $m=2$ then $d=\ell$;*
- *if $m \geq 3$ then d is odd and $\ell = m\frac{d-1}{2}+1$.*

We will use the next lemma in Corollary 3.6.

Lemma 3.2. *Let $k \in \mathbb{Z}$. If $\mathsf{w} \in V^{\otimes \ell}$ is a Q-twisted superpotential such that $A(\mathsf{w},\ell-m)$ is an m-Koszul twisted Calabi-Yau algebra, then*

$$(Q^k)^{\otimes \ell}\mathsf{w} = \mathsf{w}.$$

Proof. This follows from the conjunction of [20, Thms. 1.2, 1.6 and 1.8]. ■

3.2. **Notation.** It will be useful in what follows to use a fixed basis $x_1,\ldots,x_n$ for V.

For each $i=1,\ldots,n$, let $\partial_i : TV \to TV$ be the linear map that acts on words u on the letters x_j as follows:

$$\partial_i \mathsf{u} \;=\; \begin{cases} \mathsf{v} & \text{if } \mathsf{u} = x_i\mathsf{v}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $A(\mathsf{w},\ell-m)$ is equal to the tensor algebra TV modulo the ideal generated by

$$\{\partial_{i_1}\ldots\partial_{i_{\ell-m}}(\mathsf{w}) \mid 1 \leq i_1,\ldots,i_{\ell-m} \leq n\}.$$

From now on A will denote a 3-dimensional m-Koszul Artin-Schelter regular algebra so, by Theorem 3.1, $\ell = m+1$ and there is a unique subspace $\Bbbk\mathsf{w} \subseteq V^{\otimes(m+1)}$ such that

$$A \;\cong\; A(\mathsf{w}) \;=\; \frac{TV}{(\partial_1\mathsf{w},\ldots,\partial_n\mathsf{w})}.$$

The minimal projective resolution of ${}_A\Bbbk$ is of the form $0 \to A(-\ell) \to A \otimes R \to A \otimes V \to A \to \Bbbk \to 0$ where R is the linear span of a "minimal" set of relations. The Gorenstein condition implies that $\dim(V) = \dim(R)$, so $\{\partial_1\mathsf{w},\ldots,\partial_n\mathsf{w}\}$ is linearly independent in TV. This follows also from the "pre-regularity" of w in [9, Thm. 11].

If $p \in \Bbbk$ and x and y are elements in a ring, we use the notation

$$[x,y]_p \;:=\; xy - pyx.$$

3.3. **The main theorem.** Let $\mathsf{w} \in V^{\otimes(m+1)}$ be a Q-twisted superpotential and let $\mathsf{p} = (p_1, \ldots, p_n)$ be a tuple of non-zero scalars. The algebra $D(\mathsf{w}, \mathsf{p})$ is the tensor algebra TV modulo the relations

$$\partial_i \mathsf{w} \;=\; [x_i, \partial_1 \mathsf{w}]_{p_i} \;=\; 0 \qquad \text{for } i = 2, \ldots, n. \tag{3-1}$$

Thus $D(\mathsf{w}, \mathsf{p})$ has $n-1$ relations in degree m and $n-1$ relations of degree $m+1$. The scalar p_1 is not used in the relation and will later be fixed to be a particular entry in Q.

We will frequently write A for $A(\mathsf{w})$ and $D(\mathsf{p})$, or just D, for $D(\mathsf{w}, \mathsf{p})$.

We will write Ω for $\partial_1 \mathsf{w}$ viewed as an element in $D(\mathsf{w}, \mathsf{p})$. Clearly, $A(\mathsf{w}) = D(\mathsf{w}, \mathsf{p})/(\Omega)$.

The following notation will be used in what follows. First, we write

$$\mathsf{f} \;=\; \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \;=\; \begin{pmatrix} \partial_1 \mathsf{w} \\ \vdots \\ \partial_n \mathsf{w} \end{pmatrix} \qquad \text{and} \qquad \mathsf{x} \;=\; \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Since $\mathsf{w} \in V^{\otimes(m+1)}$, there is a unique $n \times n$ matrix M with entries in $V^{\otimes(m-1)}$ such that

$$\mathsf{w} \;=\; \mathsf{x}^t \mathsf{M} \mathsf{x}.$$

It follows from the definition of the cyclic partial derivatives $\partial_j \mathsf{w}$ that

$$\mathsf{M}\mathsf{x} = \mathsf{f}.$$

Because w is a Q-twisted superpotential

$$\mathsf{x}^t \mathsf{M} = (Q\mathsf{f})^t.$$

The last two equalities take place in TV. We are following the notation in [1]; see, for example, [1, (2.1)].

Lemma 3.3. *If Q is diagonal with entries $q_1, \ldots, q_n$ with respect to the basis $x_1, \ldots, x_n$ for V, then Ω is a normal element and $A(\mathsf{w}) = D(\mathsf{w}, \mathsf{p})/(\Omega)$.*

Proof. Since $[x_j, \partial_1 \mathsf{w}]_{p_j} = 0$ in $D(\mathsf{w}, \mathsf{p})$, to prove Ω is normal it only remains to show that $x_1 \Omega$ is a non-zero scalar multiple of Ωx_1. It follows from the remarks prior to this lemma that there are equalities

$$\sum_{i=1}^{n} x_i f_i \;=\; \mathsf{x}^t \mathsf{f} \;=\; \mathsf{x}^t \mathsf{M} \mathsf{x} \;=\; (Q\mathsf{f})^t \mathsf{x} \;=\; \mathsf{f}^t Q^t \mathsf{x} \;=\; \mathsf{f}^t Q \mathsf{x} \;=\; \sum_{i=1}^{n} q_i f_i x_i,$$

in TV. But $f_i = 0$ in $D(\mathsf{w}, \mathsf{p})$ for all $i \geq 2$, so $x_1 \Omega = q_1 \Omega x_1$. Thus, Ω is normal as claimed. ■

Now we state our main theorem.

Theorem 3.4. *Let $A(\mathsf{w})$ be an m-Koszul 3-dimensional Artin-Schelter regular algebra generated by $x_1, \ldots, x_n$. Let $\mathsf{p} = (p_1, \ldots, p_n)$ be a tuple of non-zero scalars and define $D(\mathsf{w}, \mathsf{p})$ as above. If Q is a diagonal matrix* $\operatorname{diag}(q_1, \ldots, q_n)$ *with respect to $x_1, \ldots, x_n$ and $p_1 = q_1$, then the following conditions are equivalent:*

1. *$D(\mathsf{w}, \mathsf{p})$ is a 4-dimensional Artin-Schelter regular algebra.*
2. *$[\Omega, \mathsf{M}_{ij}]_{q_1^{-1} p_i p_j} = 0$ in $D(\mathsf{w}, \mathsf{p})$ for all $1 \leq i, j \leq n$;*
3. *for all $1 \leq i, j \leq d$ and all words $x_{l_1} \ldots x_{l_{m-1}}$ appearing in M_{ij}, $q_1 = p_i p_j p_{l_1} \ldots p_{l_{m-1}}$;*
4. *the automorphism $\varphi\colon V \to V$ defined by $\varphi(x_i) = p_i x_i$ has the property*

$$\varphi^{\otimes(m+1)}(\mathsf{w}) = q_1 \mathsf{w}.$$

Furthermore, if one of these conditions holds, then Ω is a regular element of $D(\mathsf{w}, \mathsf{p})$.

We will prove the theorem in section 4.

3.4. **Remarks.** (1) We say the tuple p is good if it satisfies one of the conditions (2)-(4) in Theorem 3.4.

(2) Theorem 3.4 can be generalized and extended in a number of ways that we will take for granted below. For example, instead of replacing the relation $\partial_1 \mathsf{w} = 0$ we could replace the relation $\partial_k \mathsf{w} = 0$ for *any* $k \in \{1, \ldots, n\}$; the proof proceeds just as for $k = 1$ after making the obvious changes to the indexing. Furthermore, it is not necessary that Q be diagonal; as we note below in Remark 4.3, it is enough to assume that the entry q_k relevant to the relation $\partial_k \mathsf{w}$ being omitted splits off as a block diagonal entry.

3.5. **Consequences of the main theorem.** Theorem 3.4 is already useful when $Q = \mathrm{id}_V$, i.e., when w is a superpotential. In our setting, it follows from [20, Cor. 4.5] that w is a superpotential if and only if $A(\mathsf{w})$ is a 3-Calabi-Yau algebra. In that case, $\mathsf{p} = (1, \dots, 1)$ is good because it obviously satisfies condition (3), and equally obviously satisfies condition (4), in Theorem 3.4. We record this observation since we will use it later.

Corollary 3.5. *Let $A = A(\mathsf{w})$ be a 3-Calabi-Yau algebra generated by $x_1, \dots, x_n$. Then the algebra D generated by $x_1, \dots, x_n$ subject to relations*

$$\partial_i \mathsf{w} \;=\; [x_i, \partial_1 \mathsf{w}] \;=\; 0 \qquad (i = 2, \dots, n)$$

is a 4-dimensional Artin-Schelter regular algebra and is a central extension of A by $\Omega = \partial_1 \mathsf{w}$.

There is a more general version of Corollary 3.5.

Corollary 3.6. *Let $k \in \mathbb{Z}$. Let $A(\mathsf{w})$ be 3-dimensional Artin-Schelter regular (= twisted Calabi-Yau) algebra generated by $x_1, \dots, x_n$. If $Q = \operatorname{diag}(1, q_2, \dots, q_n)$ with respect to the basis $x_1, \dots, x_n$, then the algebra D generated by $x_1, \dots, x_n$ subject to the relations*

$$\partial_i \mathsf{w} \;=\; [x_i, \partial_1 \mathsf{w}]_{q_i^k} \;=\; 0 \qquad (i = 2, \dots, n)$$

is a 4-dimensional Artin-Schelter regular algebra and is a normal extension of $A(\mathsf{w})$ by $\Omega = \partial_1 \mathsf{w}$.

Proof. It suffices to show that $\mathsf{p} = (1, q_2^k, \dots, q_n^k)$ is good. By Lemma 3.2, for every monomial $x_{l_1} \cdots x_{l_{m+1}}$ appearing in w the product $\prod_{i=1}^{m+1} q_{l_i}^k$ equals 1. Hence p satisfies the criterion for goodness in Theorem 3.4(4). ■

In Corollary 3.12 we will show that the algebra D in Corollary 3.5 is 4-Calabi-Yau. The next result shows that every 4-Calabi-Yau central extension of $A(\mathsf{w})$ can be obtained in this way.

Theorem 3.7. *Let $A(\mathsf{w})$ be a m-Koszul 3-Calabi-Yau algebra. Then every 4-Calabi-Yau algebra that is a central extension of $A(\mathsf{w})$ by a degree-m element can be obtained by the construction in Corollary 3.5.*

Proof. Let D' be a central extension of $A(\mathsf{w})$. Explicitly, let $\Omega \in D'$ be a degree-m central regular element such that $D'/(\Omega) \cong A(\mathsf{w})$. Note that $D'_1 = A_1 = V$. Since the relations for A is concentrated in degree m and span an n-dimensional subspace of TV (see §3.2), there exist degree-m elements $f_1, \dots, f_n$ of TV such that $A = TV/(f_1, \dots, f_n)$, $f_2 = \cdots = f_n = 0$ in D', and Ω is the image of f_1 in D'.

Let $y_1, \dots, y_n$ be a basis of V and write $\mathsf{w} = y_1 h_1 + \cdots + y_n h_n$. Since $A(\mathsf{w})$ is defined by the relations $h_1, \dots, h_n$, there exists $P \in \mathsf{GL}(n)$ such that $(h_1, \dots, h_n)^t = P \cdot (f_1, \dots, f_n)^t$. Define the basis $x_1, \dots, x_n$ of V by $(x_1, \dots, x_n) = (y_1, \dots, y_n)P$. Then

$$\mathsf{w} = (y_1, \dots, y_n) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = (y_1, \dots, y_n) P \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = (x_1, \dots, x_n) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

Hence $f_i = \partial_i \mathsf{w}$ where the partial derivatives are taken with respect to the basis $x_1, \dots, x_n$. Since $\Omega \in D'$ is central, the defining relations for the algebra D in Corollary 3.5 are satisfied by D'. Hence there is a surjective homomorphism $D \to D'$. But D and D' have the same Hilbert series because A is a quotient of each of them by a central regular element of degree m. Thus $D' = D$. ■

The other consequence of Theorem 3.4 that we wish to highlight is Theorem 1.3 in the introduction. That result says that if $A(\mathsf{w})$ is a 3-Calabi-Yau algebra on n generators, then the recipe in Corollary 3.5 produces a flat family over $\mathbb{P}^{n-1}$ of 4-Calabi-Yau algebras that are central extensions of $A(\mathsf{w})$.

Proof of Theorem 1.3 (1). Let $k = 1$ and let D be the algebra in Corollary 3.5. We first show that $D = D_{x_1}$. The hypothesis that w is a superpotential says there are elements $\mathsf{w}_1, \dots, \mathsf{w}_n \in V^{\otimes m}$ such that

$$\mathsf{w} \;=\; x_1 \mathsf{w}_1 + \cdots + x_n \mathsf{w}_n \;=\; \mathsf{w}_1 x_1 + \cdots + \mathsf{w}_n x_n.$$

By definition, $D = TV/I$ where I is the ideal generated by $\{\mathsf{w}_2, \dots, \mathsf{w}_n\}$ and $\{[x_i, \mathsf{w}_1] \mid i = 2, \dots, n\}$. This is clearly the same as the ideal generated by $\{\mathsf{w}_2, \dots, \mathsf{w}_n\}$ and $\{[x_i, \mathsf{w}_j] \mid 1 \le i, j \le n\}$. It is also clear that $\operatorname{span}\{\mathsf{w}_1, \dots, \mathsf{w}_n\} = \{\langle \psi, \mathsf{w} \rangle \mid \psi \in V^*\}$. We denote this subspace of $V^{\otimes m}$ by $\langle V^*, \mathsf{w} \rangle$. In a similar way, we write $\langle x_1^\perp, \mathsf{w} \rangle$ for $\{\langle \psi, \mathsf{w} \rangle \mid \psi \in x_1^\perp\}$ which is equal to $\operatorname{span}\{\mathsf{w}_2, \dots, \mathsf{w}_n\}$. It is now apparent that $I = I_{x_1}$, i.e., $D = D_{x_1}$.

Now let $x \in V - \{0\}$. Clearly, $\langle x^\perp, \mathsf{w} \rangle \subseteq \langle V^*, \mathsf{w} \rangle$.

Since $\{\mathsf{w}_1, \ldots, \mathsf{w}_n\}$ is linearly independent (see §3.2), $\dim_{\Bbbk}(\langle V^*, \mathsf{w}\rangle) = n$. The linear map $\langle -, \mathsf{w}\rangle : V^* \to V^{\otimes m}$ is therefore injective. The codimension of $\langle x^\perp, \mathsf{w}\rangle$ in $\langle V^*, \mathsf{w}\rangle$ is therefore 1 for all non-zero $x \in V$.

The image of $\langle V^*, \mathsf{w}\rangle$ in D_x is a 1-dimensional subspace of the degree-m component of D_x. If Ω_x is a basis for this subspace then it follows from the relations $[y, \langle \psi, \mathsf{w}\rangle] = 0$ that Ω_x is in the center of D_x. It is clear that $D_x/(\Omega_x) = A(\mathsf{w})$.

Since the Hilbert series of D_x is the same for all non-zero x, the D_x's form a flat family of algebras parametrized by $V - \{0\}$. Since $D_x = D_{\lambda x}$ for all $\lambda \in \Bbbk - \{0\}$, the family is parametrized by $\mathbb{P}(V) = \mathbb{P}^{n-1}$. ■

It is reasonable to ask whether as x varies over $V - \{0\}$ every codimension-1 subspace of the degree-m relations for $A(\mathsf{w})$ arises as the degree-m relations for D_x. The answer is "yes" and the proof is as follows. Let $R' = \operatorname{span}\{\mathsf{w}'_2, \ldots, \mathsf{w}'_n\}$ be a codimension-1 subspace of $\langle V^*, \mathsf{w}\rangle$. Let $\mathsf{w}'_1 \in \langle V^*, \mathsf{w}\rangle$ be an element that is not in R'. Since the linear map $\langle -, \mathsf{w}\rangle : V^* \to V^{\otimes m}$ has rank n, there is a basis $\{x, x'_2, \ldots, x'_n\}$ for V such that $\mathsf{w} = x\mathsf{w}'_1 + x'_2\mathsf{w}'_2 + \cdots + x'_n\mathsf{w}'_n$. It is now clear that $R' = \langle x^\perp, \mathsf{w}\rangle$.

Remark 3.8. Next, as was discussed in §1.5, we define a sheaf of algebras $\mathcal{D}$ on the projective space $\mathbb{P}(V)$ whose fibers at the $\Bbbk$-points $x \in \mathbb{P}(V)$ are the algebras D_x.

Having fixed the basis $\{x_1, \ldots, x_n\}$, we write $\mathsf{w} = x_1\mathsf{w}_1 + \cdots + x_n\mathsf{w}_n$. Each $x \in V - \{0\}$ can be written as $x = c_1x_1 + \cdots + c_nx_n$ where $c_1, \ldots, c_n \in \Bbbk$ and at least one of the scalars, say c_1, is non-zero. Since

$$c_1\mathsf{w} \;=\; (c_1x_1 + \cdots + c_nx_n)\mathsf{w}_1 + \sum_{j=2}^{n}(c_1x_j\mathsf{w}_j - c_jx_j\mathsf{w}_1) \;=\; x\mathsf{w}_1 + \sum_{j=2}^{n} x_j(c_1\mathsf{w}_j - c_j\mathsf{w}_1),$$

$$\langle x^\perp, \mathsf{w}\rangle \;=\; \langle x^\perp, c_1\mathsf{w}\rangle \;=\; \operatorname{span}\{c_1\mathsf{w}_j - c_j\mathsf{w}_1 \mid 2 \le j \le n\} \;=\; \operatorname{span}\{c_i\mathsf{w}_j - c_j\mathsf{w}_i \mid 1 \le i, j \le n\}.$$

Now let $R = \Bbbk[c_1, \ldots, c_n]$ be the polynomial ring on indeterminates $c_1, \ldots, c_n$ placed in degree-one and identify $\mathbb{P}(V)$ with $\operatorname{Proj}(R)$. Let A be the R-algebra generated by $x_1, \ldots, x_n$ modulo the relations

$$c_i\mathsf{w}_j \;-\; c_j\mathsf{w}_i \;=\; [x_i, \mathsf{w}_j] \;=\; 0, \qquad 1 \le i, j \le n.$$

Viewing A as a graded R-module with respect to $c_1, \ldots, c_n$, we define $\mathcal{D}$ to be the quasi-coherent sheaf of algebras on $\mathbb{P}(V)$ whose sections over a basic open set $U_f = \operatorname{Spec}(R[f^{-1}]_0)$ is $A[f^{-1}]_0$. Since A has an algebra structure and also has a grading with respect to $x_1, \ldots, x_n$, $\mathcal{D}$ is a sheaf of graded $\mathcal{O}_{\mathbb{P}(V)}$-algebras on $\mathbb{P}(V)$. This definition of $\mathcal{D}$ does not depend on the basis $x_1, \ldots, x_n$ for V. It follows from the observation in the previous paragraph that for each $\Bbbk$-point $x \in \mathbb{P}(V)$ the algebra D_x is $\mathcal{D}/\mathfrak{m}_x\mathcal{D}$ where $\mathfrak{m}_x$ is the maximal ideal in $\mathcal{O}_{\mathbb{P}(V)}$ vanishing at x.

Applying Theorem 3.4 over the base field that is the function field of $\mathbb{P}(V)$, we conclude that the dimension of the homogeneous component $\mathcal{D}_n$ at the generic point of $\mathbb{P}(V)$ is equal to $\dim_{\Bbbk}(D_x)_n$. The flatness of the sheaf $\mathcal{D}$ (in fact, the local freeness of the sheaves $\mathcal{D}_n$) then follows from [13, Lemma II.8.9]. ♦

Proof of Theorem 1.5. This follows from Theorem 1.3 (1), proven above, together with the general remark that flatness of families of connected graded algebras transports over to their QGr categories, recorded in Lemma 3.9 below. ■

Lemma 3.9. *Let R be a commutative ring and D a non-negatively graded R-algebra with $D_0 = R$. If the homogeneous components D_n are flat over R then so is the abelian category $\mathsf{QGr}(D)$ where $\mathsf{QGr}(D)$ is the category defined as in Definition 1.4.*

Proof. By [16, Proposition 3.7], the R-linear category $\mathsf{Gr}(D)$ of graded D-modules is flat over R because it is, in fact, the category of modules over the small R-linear category $\mathfrak{d}$ with objects $i \in \mathbb{Z}$ and $\operatorname{Hom}_{\mathfrak{d}}(i, j) = D_{j-i}$ (the hypothesis implies that $\mathfrak{d}$ is R-flat as a *linear* category).

Since $\mathsf{QGr}(D)$ is a quotient of $\mathsf{Gr}(D)$, its flatness over R follows from the flatness of $\mathsf{Gr}(D)$ over R by [16, Lemma 8.13]. ■

3.6. **Automorphisms.** Let $A = A(\mathsf{w})$ and $D = D(\mathsf{w}, \mathsf{p})$ be algebras satisfying the assumptions of Theorem 3.4. We now consider the relation between $\operatorname{Aut}_{\mathsf{gr}}(A)$ and $\operatorname{Aut}_{\mathsf{gr}}(D)$.

Proposition 3.10 identifies a subgroup of $\operatorname{Aut}_{\mathsf{gr}}(A)$ that consists of automorphisms that lift to D.

This result does not depend on Theorem 3.4 and its proof will be used in the proof of Theorem 3.4.

Proposition 3.10. *Let $A = A(\mathsf{w})$ and $D = D(\mathsf{w}, \mathsf{p})$ be algebras satisfying the hypotheses of Theorem 3.4. Let $\sigma \in \operatorname{Aut}_{\mathsf{gr}}(A)$. If every x_i is an eigenvector for σ, then σ lifts to an automorphism of D.*

Proof. By definition, the relations $f_i = \partial_i \mathsf{w}$ are the respective images of $x_i^* \otimes \mathsf{w}$ through the map

$$V^* \otimes V^{\otimes(m+1)} \xrightarrow{\mathrm{ev}\otimes \mathrm{id}_{V^{\otimes m}}} V^{\otimes m} \tag{3-2}$$

that evaluates the two leftmost tensorands against each other, where $\{x_i^*\} \subset V^*$ is the basis dual to $\{x_i\}$.

By [20, Thm. 1.1], the automorphisms of A are (identified in the obvious manner with) precisely those linear automorphisms of $V = A_1$ which scale the superpotential $\mathsf{w} \in V^{\otimes(m+1)}$. In other words, the element w is an eigenvector for the action of $\Gamma = \langle\sigma\rangle$ on $V^{\otimes(m+1)}$ (with the tensor product being taken in the monoidal category of Γ-modules).

Now suppose that the generators x_i are eigenvectors for $\sigma \in GL(V)$. Then each $\Bbbk f_i$ is the image of the Γ-submodule $\Bbbk x_i^* \otimes \Bbbk\mathsf{w}$ of the left-hand side of the Γ-module morphism (3-2), so they are Γ-submodules of $V^{\otimes m}$. Thus, each f_i is an eigenvector for $\sigma^{\otimes m}$. Since x_1 is, by hypothesis, also an eigenvector for σ, it acts as a scalar on each of the defining relations (3-1) for D. ∎

In what follows we assume that the four conditions in Theorem 3.4 are satisfied.

We have the following consequence of Theorem 2.6 on how the respective Nakayama automorphisms interact.

Proposition 3.11. *Let $A(\mathsf{w})$ and $D(\mathsf{w},\mathsf{p})$ be Artin-Schelter regular algebras satisfying the hypotheses, and hence the conclusions, of Theorem 3.4. If $Q = \operatorname{diag}(q_1,\dots,q_n)$, then the Nakayama automorphism of $D(\mathsf{w},\mathsf{p})$ scales x_i by $p_i^{-1}q_i^{-1}$, i.e., $\nu_D(x_i) = (p_iq_i)^{-1}x_i$ for all i.*

Proof. This follows from Theorem 2.6, together with the fact that τ ("conjugation" by Ω) acts by

$$x_i \mapsto p_i^{-1}x_i$$

and [20, Thm. 1.8] which, in our setting, says that ν_A scales x_i by q_i^{-1}. ∎

As an immediate consequence, in the cases treated in Corollary 3.6

Corollary 3.12. *Suppose $q_1 = 1$. If $\mathsf{p} = (1, q_2^{-1},\dots,q_n^{-1})$, then $D(\mathsf{w},\mathsf{p})$ is 4-Calabi-Yau.*

Proof. Proposition 3.11 implies that $\nu_D = \mathrm{id}_D$ in this case. ∎

By [24, Cor. 5.4], the homological determinant of the Nakayama automorphism is 1 for all noetherian Artin-Schelter regular algebras. By [20, Thm. 1.6], the same result holds in the non-noetherian case provided the algebra is m-Koszul and Artin-Schelter regular. It is conjectured at [20, Conj. 4.12] that this holds for *all* Artin-Schelter regular algebras. Theorem 3.14 below shows that the normal extensions $D(\mathsf{w},\mathsf{p})$ produced by Theorem 3.4 have this property.

Before stating that result, we give a relatively self-contained proof of the following auxiliary result. It is essentially [20, Thm. 1.2], with the minor caveat that the homological determinant in [20] is inverse to that of [14]. Nevertheless, [20, Thm. 1.2] appeals to [28], which agrees with [14]. For this reason, [20, Thm. 1.2] is correct for the version of the homological determinant used here and in [14, 28], but not precisely as stated, for the version of hdet defined in [20, §2.2].

Throughout the statement and the proof of Lemma 3.13, and in later sections, we identify each graded algebra automorphism of $A = TV/(R)$ (when the relations R have degree ≥ 2) with the automorphism of TV that agrees on the degree-1 component V.

Lemma 3.13. *Let A be an m-Koszul Artin-Schelter regular algebra and let $\mathsf{w} \in V^{\otimes\ell}$ be a Q-twisted superpotential such that $A \cong A(\mathsf{w}, \ell - m)$ as in Theorem 3.1. If $\sigma \in \operatorname{Aut}_{\mathsf{gr}}(A)$, then w is an eigenvector for $\sigma^{\otimes\ell}$ with eigenvalue $\operatorname{hdet}(\sigma)$.*

Proof. By Proposition 2.13, $\operatorname{hdet}(\sigma)^{-1}$ is the scalar by which $\sigma\triangleright$ acts on the degree-$(-\ell)$ component of $\operatorname{Ext}_A^d(\Bbbk,\Bbbk)$. Resolving ${}_A\Bbbk$ by projective modules $P_\bullet$, this amounts to the action obtained by pre-composing morphisms in $\operatorname{Hom}_A(P_d,\Bbbk)$ with the action of $\varphi_{P_d}^{-1}$.

Now, according to [9, equation (4.2)], the minimal resolution $P_\bullet$ consists of free left A-modules generated by the partial derivatives of w, and the left-most projective P_d is precisely the free A-module with basis w. It follows from that equation that $\varphi_{P_d}^{-1}$ is the application of $(\sigma^{-1})^{\otimes(\ell+1)}$ to

$$A \otimes \Bbbk\mathsf{w} \subseteq A^{\otimes(\ell+1)}.$$

Proposition 2.13 now implies that $\operatorname{hdet}(\sigma)^{-1}$ is the eigenvalue associated to the eigenvector w of $(\sigma^{-1})^{\otimes\ell}$, hence the conclusion. ∎

Theorem 3.14. *Let $A = A(\mathsf{w})$ and $D = D(\mathsf{w}, \mathsf{p})$ be Artin-Schelter regular connected graded algebras satisfying all conditions in Theorem 3.4. Then* $\operatorname{hdet}(\nu_D) = 1$.

Proof. Theorem 2.6 shows that the restriction $\nu_D|_A$ is $\tau|_A \circ \nu_A$, where τ is defined on D by

$$\Omega x = \tau(x)\Omega$$

and $\tau|_A$ is its descent to A modulo Ω. Theorem 2.10 shows that

$$\operatorname{hdet}(\nu_D) = \lambda \operatorname{hdet}(\nu_D|_A), \tag{3-3}$$

where λ is defined by $\nu_D(\Omega) = \lambda\Omega$. Hence, by Theorem 2.6 and the multiplicativity of hdet ([14, Prop. 2.5]),

$$\operatorname{hdet}(\nu_D) = \lambda \cdot \operatorname{hdet}(\tau|_A) \cdot \operatorname{hdet}(\nu_A). \tag{3-4}$$

But $\operatorname{hdet}(\nu_A) = 1$ by [20, Thm. 1.6], so the rightmost factor in (3-4) is 1.

On the other hand, since τ fixes Ω, the equation $\nu_D|_A = \tau \circ \nu_A$ shows that λ is defined by

$$\nu_A(\Omega) = \lambda\Omega, \tag{3-5}$$

where we have abused notation by using ν_A to denote the automorphism of the free algebra TV that agrees with ν_A on $A_1 = V$ (see the discussion preceding Lemma 3.13).

By [20, Thm. 1.6] and Lemma 3.13, ν_A fixes w, and [20, Thm. 1.8] shows that the same automorphism scales x_1 by q_1^{-1}. It now follows from the fact that $x_1\Omega$ is a sum of monomials appearing in w that ν_A scales Ω by q_1. In other words, (3-5) implies $\lambda = q_1$.

Finally, [20, Thm. 1.2] shows that $\operatorname{hdet}(\tau|_A)$ is equal to the eigenvalue of the $\tau|_A$-eigenvector w which condition (3) in Theorem 3.4 shows is equal to q_1^{-1}.

To summarize:

- the leftmost factor in the right-hand side of (3-4) is q_1;
- the middle factor is q_1^{-1};
- the rightmost factor is 1.

The desired conclusion that $\operatorname{hdet}(\nu_D) = 1$ now follows from (3-4). ■

3.7. **Zhang twists.** We will observe some relationship between our construction of normal extensions and Zhang twists. Zhang twist is a method to transform a graded algebra using a family of graded $\Bbbk$-linear automorphisms called a twisting system. It was introduced by Zhang [29] by generalizing the idea in [3, section 8]. Here we only consider a Zhang twist defined by a graded algebra automorphism on A.

Definition 3.15. Let A be a graded algebra and σ an automorphism on A. The Zhang twist ${}^{\sigma}A$ of A by σ is defined by ${}^{\sigma}A := A$ as an graded vector space and a new multiplication $a * b := \sigma^{\deg b}(a)b$ where $a, b \in A$ are homogeneous elements.[2] ♦

For each $f \in V^{\otimes r}$, set ${}^{\sigma}f := (\mathrm{id}_V \otimes \sigma \otimes \cdots \otimes \sigma^{r-1})(f)$. Then the relations in ${}^{\sigma}A$ are given by ${}^{\sigma}f$ where f runs over all relations in A.

Proposition 3.16. *Let $A(\mathsf{w})$ and $D(\mathsf{w}, \mathsf{p})$ be Artin-Schelter regular algebras as in Theorem 3.4. Let σ be an automorphism of $A(\mathsf{w})$ such that $\sigma(x_i) = s_i x_i$ for some $s_i \in \Bbbk^{\times}$. Then σ lifts to an automorphism of $D(\mathsf{w}, \mathsf{p})$, and*

$${}^{\sigma}D(\mathsf{w}, \mathsf{p}) \cong D({}^{\sigma}\mathsf{w}, \mathsf{p}')$$

where $\mathsf{p}' = (p_1', \dots, p_n')$ is defined by $p_i' = p_i s_1 s_i^m (\operatorname{hdet}_A(\sigma))^{-1}$.

Proof. As observed in Proposition 3.10 and its proof, σ lifts to D, and $\sigma^{\otimes m}$ scales each f_i in $V^{\otimes m}$. ${}^{\sigma}D(\mathsf{w}, \mathsf{p})$ is defined by the relations ${}^{\sigma}f_i$ and ${}^{\sigma}[x_i, \partial_1\mathsf{w}]_{p_i}$ $(i = 2, \dots, n)$. By interpreting [20, Thm. 5.5] to our convention, ${}^{\sigma}A(\mathsf{w}) = A({}^{\sigma}\mathsf{w})$. Note that $\partial_i {}^{\sigma}\mathsf{w} = s_i^{-1}(\operatorname{hdet}_A(\sigma))({}^{\sigma}f_i)$. By Lemma 3.13,

$$(\operatorname{hdet}_A(\sigma))\mathsf{w} = \sigma^{\otimes(m+1)}(\mathsf{w}) = \sigma^{\otimes(m+1)}(x_1 f_1 + \cdots + x_n f_n) = s_1 x_1 \sigma^{\otimes m}(f_1) + \cdots + s_n x_n \sigma^{\otimes m}(f_n)$$

and hence $\sigma^{\otimes m}(f_i) = s_i^{-1}(\operatorname{hdet}_A(\sigma)) f_i$. Therefore

$${}^{\sigma}[x_i, \partial_1\mathsf{w}]_{p_i} = x_i {}^{\sigma}(\sigma^{\otimes m}(f_1)) - p_i {}^{\sigma}f_1 \sigma^m(x_i) = s_i^{-1}(\operatorname{hdet}_A(\sigma)) x_i {}^{\sigma}(f_1) - p_i {}^{\sigma}f_1 \sigma^m(x_i) = [x_i, \partial_1 {}^{\sigma}\mathsf{w}]_{p_i'}.$$

The defining ideal of ${}^{\sigma}D(\mathsf{w}, \mathsf{p})$ is generated by $\partial_i {}^{\sigma}\mathsf{w}$ and $[x_i, \partial_1 {}^{\sigma}\mathsf{w}]_{p_i'}$ $(i = 2, \dots, n)$. ■

[2] Our definition of Zhang twist agrees with that of [23] but differs from [20].

4. Proof of the main theorem

In this section we prove Theorem 3.4. Let $A = A(\mathsf{w})$ and $D = D(\mathsf{w}, \mathsf{p})$ be algebras as in the theorem.

4.1. In this subsection we show that (1) $\Rightarrow$ (4) $\Leftrightarrow$ (3) $\Leftrightarrow$ (2).

The proof that (1) $\Rightarrow$ (4) also shows that Ω is a regular element in $D(\mathsf{w}, \mathsf{p})$.

(1) $\Rightarrow$ (4) We first show that the relations (3-1) form a minimal set of relations of $D(\mathsf{w}, \mathsf{p})$. Assume they are not. Since $\partial_1 \mathsf{w}, \ldots, \partial_n \mathsf{w}$ are linearly independent (see §3.2), there is a linear equation

$$\sum_{i=2}^{n} a_i [x_i, \partial_1 \mathsf{w}]_{p_i} + \sum_{i=2}^{n}\sum_{j=1}^{n} (b_{ij} x_j (\partial_i \mathsf{w}) + c_{ij} (\partial_i \mathsf{w}) x_j) = 0$$

in TV, where $a_i, b_{ij}, c_{ij} \in \Bbbk$ and at least one of these scalars is non-zero. This can be written as

$$\sum_{i=2}^{n} a_i x_i (\partial_1 \mathsf{w}) + \sum_{i=2}^{n}\sum_{j=1}^{n} b_{ij} x_j (\partial_i \mathsf{w}) = \sum_{i=2}^{n} a_i p_i (\partial_1 \mathsf{w}) x_i - \sum_{i=2}^{n}\sum_{j=1}^{n} c_{ij} (\partial_i \mathsf{w}) x_j. \tag{4-1}$$

Let R be the subspace of TV spanned by $\partial_1 \mathsf{w}, \ldots, \partial_n \mathsf{w}$. The left-hand side of (4-1) belongs to $V \otimes R$ and the right-hand side of (4-1) belongs to $R \otimes V$. Since $(V \otimes R) \cap (R \otimes V) = \Bbbk \mathsf{w}$ by [20, Prop. 2.12], the left-side of (4-1) is a scalar multiple of $\mathsf{w} = x_1(\partial_1 \mathsf{w}) + \cdots + x_n(\partial_n \mathsf{w})$. This is a contradiction. Hence (3-1) is a minimal set of relations.

Thus the minimal free resolution of ${}_D\Bbbk$ is of the form

$$0 \to D(-2m-1) \to D(-2m)^n \to D(-m)^{n-1} \oplus D(-m-1)^{n-1} \to D(-1)^n \to D \to {}_D\Bbbk \to 0.$$

That of ${}_A\Bbbk$ is of the form

$$0 \to A(-m-1) \to A(-m)^n \to A(-1)^n \to A \to {}_A\Bbbk \to 0.$$

These resolutions determine the Hilbert series of D and A and it is easy to verify that $h_D(t) = h_A(t)(1-t^m)^{-1}$. Therefore by Proposition 2.1, Ω is a regular element of $D(\mathsf{w}, \mathsf{p})$.

Since Ω is regular, we can define an automorphism $\tau \in \operatorname{Aut}_{\mathsf{gr}}(D(\mathsf{w}, \mathsf{p}))$ by the formula in (2-1). The argument in the proof of Lemma 3.3 shows that $\tau(x_i) = p_i^{-1} x_i$ for each $i = 1, \ldots, n$. Of course, τ can also be viewed as an element in $\mathsf{GL}(V)$ and, as observed in the proof of Proposition 3.10, $\mathsf{w} \in V^{\otimes(m+1)}$ and $f_1 \in V^{\otimes m}$ are eigenvectors for $\tau^{\otimes(m+1)}$ and $\tau^{\otimes m}$, respectively. Since $\tau(\Omega) = \Omega$, $\tau^{\otimes m}(f_1) = f_1$. Since $\mathsf{w} = x_1 f_1 + \cdots + x_n f_n$, the $\tau^{\otimes(m+1)}$-eigenvalue of w is the same as the τ-eigenvalue of x_1. Therefore $\tau^{\otimes(m+1)}(\mathsf{w}) = p_1^{-1}\mathsf{w} = q_1^{-1}\mathsf{w}$. Since the automorphism φ defined in Theorem 3.4(4) is the inverse of τ, (4) holds.

(3) $\Leftrightarrow$ (4) Since $\mathsf{w} = \sum_{i,j} x_i \mathsf{M}_{ij} x_j$, $\varphi^{\otimes(m+1)}$ scales w by q_1 if and only if it scales $x_i x_{l_1} \cdots x_{l_{m-1}} x_j$ by q_1 for all i, j and all words $x_{l_1} \cdots x_{l_{m-1}}$ appearing in M_{ij}. But $\varphi^{\otimes(m+1)}(x_i x_{l_1} \cdots x_{l_{m-1}} x_j) = p_i p_j p_{l_1} \cdots p_{l_{m-1}} x_i x_{l_1} \cdots x_{l_{m-1}} x_j$ by definition of φ.

(2) $\Rightarrow$ (3) Fix $1 \le i, j \le d$ and write $\mathsf{M}_{ij} = \sum_l a_l x_{l_1} \ldots x_{l_{m-1}}$ where every $a_l \neq 0$ and $l = (l_1, \ldots, l_{m-1})$. As described in the proof of Lemma 3.3, $[x_j, \Omega]_{p_j} = 0$ for each j. Hence condition (2) can be written as

$$[\Omega, \mathsf{M}_{ij}]_{q_1^{-1}p_i p_j} \;=\; \sum_l a_l [\Omega, x_{l_1} \ldots x_{l_{m-1}}]_{q_1^{-1}p_i p_j} \;=\; \Omega \sum_l a_l (1 - q_1^{-1} p_i p_j p_{l_1} \ldots p_{l_{m-1}}) x_{l_1} \ldots x_{l_{m-1}}. \tag{4-2}$$

If this is zero, then Proposition 2.1 tells us that $D(\mathsf{w}, \mathsf{p})$ is Artin-Schelter regular and that Ω is a regular element. Hence $(1 - q_1^{-1} p_i p_j p_{l_1} \ldots p_{l_{m-1}}) x_{l_1} \ldots x_{l_{m-1}} = 0$ in $D(\mathsf{w}, \mathsf{p})$. Since all defining relations of $D(\mathsf{w}, \mathsf{p})$ have degree at least m, it follows that $1 - q_1^{-1} p_i p_j p_{l_1} \ldots p_{l_{m-1}} = 0$ for all l.

(3) $\Rightarrow$ (2) This is clear.

4.2. To complete the proof of Theorem 3.4 we show that (2) $\Rightarrow$ (1). The proof of this occupies the rest of §4.

First we identify a candidate for the minimal projective resolution of the trivial module ${}_D\Bbbk$. The minimal resolution of $\Bbbk$ as a left A-module is

$$0 \longrightarrow A(-m-1) \xrightarrow{\cdot \mathsf{x}^t} A(-m)^n \xrightarrow{\cdot \mathsf{M}} A(-1)^n \xrightarrow{\cdot \mathsf{x}} A \longrightarrow \Bbbk \longrightarrow 0. \tag{4-3}$$

By definition, the relations defining D are f_i and $g_i := [x_i, \partial_1 \mathsf{w}]_{p_i}$ for $2 \le i \le n$. We aggregate them into two column vectors of size $2(n-1) \times 1$, namely

$$\mathsf{g}_l := (q_1 p_2^{-1} g_2, \ldots, q_1 p_n^{-1} g_n \mid q_2 f_2, \ldots, q_n f_n)^t \qquad \text{and} \qquad \mathsf{g}_r := (f_2, \ldots, f_n \mid g_2, \ldots, g_n)^t.$$

The subscripts l and r denote "left" and "right" respectively. There are unique matrices M_l and M_r such that

$$\mathsf{x}^t\mathsf{M}_l \;=\; \mathsf{g}_l^t \qquad \text{and} \qquad \mathsf{M}_r\mathsf{x} \;=\; \mathsf{g}_r. \tag{4-4}$$

Let $P_\bullet$ denote the sequence

$$D(-2m-1) \xrightarrow{\cdot\mathsf{x}^t} D(-2m)^n \xrightarrow{\cdot\mathsf{M}_l} D(-m)^{n-1} \oplus D(-m-1)^{n-1} \xrightarrow{\cdot\mathsf{M}_r} D(-1)^n \xrightarrow{\cdot\mathsf{x}} D \tag{4-5}$$

of left D-modules and D-module homomorphisms. Eventually, we will show that

$$0 \to P_\bullet \to \Bbbk \to 0 \tag{4-6}$$

is a minimal resolution of the trivial D-module ${}_D\Bbbk$.

As usual, when working with resolutions, we place $\Bbbk$ in homological degree -1 in (4-3), (4-5), and (4-6).

We denote the homology of the complex (4-6) by H_i, $0 \le i \le 4$. Each H_i is a graded vector space; its homogeneous components will be denoted by $H_{i\ell}$. The dimension of $H_{i\ell}$ will be denoted by $h_{i\ell}$ or $(h_i)_\ell$.

4.3. **Notation.** Let X be a matrix. For subsets S and T of the set of rows and vectors respectively, we denote by $X_{S,T}$ (or X_{ST} if there is no danger of ambiguity) the matrix consisting of only those entries of X whose row and column indices belong to S and T. Moreover, if the sets S and T are one-sided intervals, we simply substitute the corresponding inequality for the set.

For example, $X_{\ge 2,\le 6}$ is the sub-matrix of X consisting of entries x_{ij} for $i \ge 2$ and $j \le 6$.

We apply the same convention to row or column vectors, except that we adorn these with a single symbol as in $\mathsf{v}_{\ge 5}$ denoting the vector consisting of the entries of v with index ≥ 5.

We use the symbol $\bullet$ for no condition at all, e.g., $X_{\bullet\bullet} = X$, $\mathsf{v}_\bullet = \mathsf{v}$ and $X_{\ge 8,\bullet}$ is the matrix obtained from X by deleting the first 7 rows.

The hat indicates the negation of the condition in question (or the complement of the set). For instance, $X_{\bullet\widehat{j}}$ is the matrix obtained from X by discarding its j^{th} column.

We define

$$\begin{aligned} J_h &:= \text{ the matrix obtained by adding a row of 0's on top of } q_1\operatorname{diag}(p_2^{-1},\dots,p_n^{-1}) \quad \textit{and} \\ J_v &:= \text{ the matrix obtained by adding a column of 0's to the left of } \operatorname{diag}(p_2\dots p_n). \end{aligned}$$

4.4. The next lemma follows immediately from the definitions of M_l and M_r in (4-4). The horizontal and vertical bars in its statement split the respective matrices into blocks of equal size.

Lemma 4.1. *The $2(n-1)\times n$ matrix M_r is*

$$\left(\begin{array}{c} \mathsf{M}_{\widehat{1}\bullet} \\ \hline \mathsf{x}_{\ge 2}\mathsf{M}_{1\bullet} - f_1 J_v \end{array}\right)$$

The $n \times 2(n-1)$ matrix M_l is

$$\left(-\mathsf{M}_{\bullet 1}(\mathsf{x}_{\ge 2})^t + f_1 J_h \;\middle|\; \mathsf{M}_{\bullet\widehat{1}}\right).$$

Lemma 4.2. *If condition (2) in Theorem 3.4 holds, then the diagram $0 \to P_\bullet \to \Bbbk \to 0$ in (4-5) is a complex.*

Proof. The composition $P_1 \to P_0 \to \Bbbk$ is zero because, the image of the map $P_1 \to P_0 = D$ is the maximal ideal $(x_1,\dots,x_n)$. The definitions of M_l and M_r, respectively, imply that the compositions $P_4 \to P_3 \to P_2$ and $P_2 \to P_1 \to P_0$ are zero. It remains to show that the composition

$$D(-2m)^n \xrightarrow{\cdot\mathsf{M}_l} D(-m)^{n-1} \oplus D(-m-1)^{n-1} \xrightarrow{\cdot\mathsf{M}_r} D(-1)^n$$

is zero, i.e., that the entries of the $n\times n$ matrix $\mathsf{M}_l\mathsf{M}_r \in M_n\big(V^{\otimes(2m-1)}\big)$ are zero in D.

To this end, we consider the product of the i^{th} row of M_l by the j^{th} column of M_r. The i^{th} row of M_l is

$$\left(-\mathsf{M}_{i1}(\mathsf{x}_{\ge 2})^t + f_1(J_h)_{i\bullet} \;\middle|\; \mathsf{M}_{i\widehat{1}}\right), \tag{4-7}$$

and the j^{th} column of M_r is

$$\left(\begin{array}{c} \mathsf{M}_{\widehat{1}j} \\ \hline \mathsf{x}_{\ge 2}\mathsf{M}_{1j} - f_1(J_v)_{\bullet j} \end{array}\right). \tag{4-8}$$

Since $\mathsf{x}^t\mathsf{M} = (Q\mathsf{f})^t$, the product of the left-hand side of (4-7) and the top half of (4-8) is

$$-\,\mathsf{M}_{i1}(q_jf_j - x_1\mathsf{M}_{1j}) + (1-\delta_{i1})q_1p_i^{-1}f_1\mathsf{M}_{ij} \tag{4-9}$$

Since $\mathsf{M}\mathsf{x} = \mathsf{f}$, the product of the right-hand half of (4-7) by the bottom half of (4-8) is

$$(f_i - \mathsf{M}_{i1}x_1)\mathsf{M}_{1j} - (1-\delta_{1j})p_j\mathsf{M}_{ij}f_1. \tag{4-10}$$

Upon adding (4-9) and (4-10) the terms containing x_1 cancel out, and the remaining sums, in the four cases

(1) $i = 1 = j$;
(2) $i = 1 \neq j$;
(3) $i \neq 1 = j$;
(4) $i \neq 1 \neq j$

precisely coincide with the respective expressions in the statement of Theorem 3.4. The conclusion then follows from our assumption that these vanish in D. ■

Remark 4.3. A simultaneous change of basis for the spaces spanned by x_i and f_i for $i \geq 2$ induces, as explained in [1, §2], a conjugation of the diagonal matrix Q by a block diagonal scalar matrix of the form $T = \mathrm{diag}(1, T')$, where T' is an $(n-1)\times(n-1)$ matrix. The resulting matrix TQT^{-1} is then of the form $\mathrm{diag}(q_1, Q')$ for some $Q' \in M_{n-1}(\Bbbk)$.

Under this basis change the above proof of Lemma 4.2 consists of an identity involving the entries of the lower right-hand corner Q' of Q, and hence functions equally well for any diagonalizable Q'. But then, by Zariski continuity, it is valid for any Q' whatsoever. This justifies the remark in §3.4 that the proof of Theorem 3.4 extends to the case $Q = \mathrm{diag}(1, Q')$. ♦

4.5. **Inductive argument.** Define $Z := \{a \in D \mid a\Omega = 0\}$ and $z_i := \dim_{\Bbbk}(Z_i)$.

Lemma 4.4. *Suppose condition (2) in Theorem 3.4 holds. If $(h_3)_{k-m} = z_{k-2m-1} = z_{k-2m} = 0$, then $(h_3)_k = 0$.*

Proof. Let $\varphi^t = (\varphi_1, \ldots, \varphi_n)$ be an element in D^d_{k-2m} representing a class in $(H_3)_k$. Thus, $\varphi^t\mathsf{M}_l = 0$ in D.

Let $\eta := \varphi^t\mathsf{M}$. Because $f_1 = \Omega = 0$ in A, the expression for M_l in Lemma 4.1 implies that

$$\eta \begin{pmatrix} -(\mathsf{x}_{\geq 2})^t & \\ & 1_{n-1} \end{pmatrix} = 0$$

in A. Hence $\eta_1(\mathsf{x}_{\geq 2})^t = 0$ and $\eta_{\geq 2} = 0$ in A. It also follows that

$$\eta_1 x_1 \;=\; \eta_1 x_1 + \eta_{\geq 2}\mathsf{x}_{\geq 2} \;=\; \varphi^t\mathsf{M}\mathsf{x} \;=\; 0$$

in A. We have obtained $\eta_1\mathsf{x}^t = 0$. The exact sequence (4-3) ensures $\eta_1 = 0$, and hence $\varphi^t\mathsf{M} = 0$ in A.

Again by the exactness of (4-3), φ^t is in the image of $\cdot\mathsf{x}^t$ modulo Ω, i.e.,

$$\varphi^t \;=\; u\mathsf{x}^t + \mathsf{y}^t\Omega \tag{4-11}$$

for some $u \in D_{k-2m-1}$ and a column vector y with entries in D_{k-3m}. The equations (4-4) show that the first summand of (4-11) is annihilated by right multiplication by M_l, and hence $\mathsf{y}^t\Omega\mathsf{M}_l = 0$ in D.

Since $[\Omega, \mathsf{M}_{ij}]_{q_1^{-1}p_ip_j} = 0$, Lemma 4.1 implies that $\Omega\mathsf{M}_l$ equals $\mathsf{M}_l\Omega$ up to multiplication on the left by an invertible $n\times n$ diagonal matrix and on the right by an invertible diagonal matrix of size $2(n-1)$, namely

$$\Omega\mathsf{M}_l \;=\; \mathrm{diag}(q_1, p_2, \ldots, p_n)\,\mathsf{M}_l\Omega\;\mathrm{diag}(p_2^{-1}, \ldots, p_n^{-1}, q_1^{-1}p_2, \ldots, q_1^{-1}p_n).$$

It follows that $\mathsf{y}^t\,\mathrm{diag}(q_1, p_2, \ldots, p_n)\mathsf{M}_l$ belongs to $Z^{d-1}_{k-2m} \oplus Z^{d-1}_{k-2m-1}$ which is assumed to be zero, and hence $\mathsf{y}^t\,\mathrm{diag}(q_1, p_2, \ldots, p_n)$ represents a class in $(H_3)_{k-m}$ which is again assumed zero. In conclusion we have $\mathsf{y}^t\,\mathrm{diag}(q_1, p_2, \ldots, p_n) = v\mathsf{x}^t$ for some $v \in D(-2m-1)$ and hence, by (4-11),

$$\varphi^t \;=\; u\mathsf{x}^t + v\,\mathrm{diag}(q_1^{-1}, p_2^{-1}, \ldots, p_n^{-1})\mathsf{x}^t\Omega \;=\; (u + v\Omega)\mathsf{x}^t.$$

In other words φ^t is a boundary, so the homology $(H_3)_k$ vanishes, as desired. ■

Now define

$$d_k \;:=\; \dim_{\Bbbk}(D_k), \qquad d'_k \;:=\; \text{the coefficient of } t^k \text{ in } h_A(t)\,(1-t^m)^{-1}, \qquad \text{and} \qquad e_k := d'_k - d_k.$$

It follows from the exact sequence $0 \to Z(-m) \to D(-m) \xrightarrow{\cdot\Omega} D \to A \to 0$ that $e_k \geq 0$ (as noted in (2-6)) and

$$e_k \;=\; e_{k-m} + z_{k-m}. \tag{4-12}$$

for all k. Moreover, subtracting

$$-\delta_{k,0} + d_k - nd_{k-1} + (n-1)d_{k-m-1} + (n-1)d_{k-m} - nd_{k-2m} + d_{k-2m-1} \;=\; (h_2)_k - (h_3)_k + (h_4)_k$$

from the same expression with d' in place of d (in which case the right-hand side $= 0$) we get

$$e_k - ne_{k-1} + (n-1)e_{k-m-1} + (n-1)e_{k-m} - ne_{k-2m} + e_{k-2m-1} \;=\; -(h_2)_k + (h_3)_k - (h_4)_k. \tag{4-13}$$

This is now sufficient preparation for

Lemma 4.5. *Suppose condition (2) in Theorem 3.4 holds. Let k be an integer. If $e_i = 0$ for all $i \leq k-1$ and $z_j = 0$ for all $j \leq k-2m$, then $e_k = z_{k+1-2m} = 0$.*

Proof. By Lemma 4.4, the vanishing of the z_j's implies that $(h_3)_k = 0$ (by an induction argument since $(h_3)_\ell = 0$ when $\ell < 0$). Equation (4-13) then gives

$$0 \;\leq\; e_k \;=\; -(h_2)_k - (h_4)_k \;\leq\; 0$$

so $e_k = 0$ and $(h_2)_k + (h_4)_k = 0$.

Finally, $z_{k+1-2m} = 0$ because the e_i's vanish and $e_{k+1-m} = e_{k+1-2m} + z_{k+1-2m}$ (by (4-12)). ∎

To complete the proof of Theorem 3.4 we must show that D is Artin-Schelter regular of dimension 4. By Proposition 2.1, D is Artin-Schelter regular of dimension 4 if $h_D(t) = h_A(t)\,(1-t^m)^{-1}$, i.e., if $e_k = 0$ for all k. Certainly, $e_k = z_k = 0$ if $k < 0$; Lemma 4.5 then provides the induction step thereby showing that the vanishing of e_k and z_k perpetuates for all $k \geq 0$.

The proof of Theorem 3.4 is now complete.

5. Examples

In this section we apply Theorem 3.4 to some important classes of 3-dimensional Artin-Schelter regular algebras thereby producing some new 4-dimensional Artin-Schelter regular algebras.

5.1. **The Calabi-Yau case.** As observed in Corollary 3.5, if $A(\mathsf{w})$ is 3-Calabi-Yau, then $\mathsf{p} = (1, \ldots, 1)$ is good so the procedure in Theorem 3.4 produces at least one 4-Calabi-Yau algebra that is a normal, in fact central when $\mathsf{p} = (1, \ldots, 1)$, extension of $A(\mathsf{w})$. The 3-Calabi-Yau algebras of finite Gelfand-Kirillov dimension are noetherian domains and are classified by Mori-Smith [21] and Mori-Ueyama [22].

Proposition 5.1. *For all the 3-Calabi-Yau algebras A classified in* [21, Table 2] *and* [22, Table 2] *and all numberings of the generators, the procedure described in Corollary 3.5 produces a noetherian 4-dimensional Artin-Schelter regular central extension of A.*

Non-noetherian Calabi-Yau algebras are less well understood, but they do exist and a number of them fit into our framework. For example, [10] constructs, usually non-noetherian, 3-Calabi-Yau algebras from superpotentials w defined in terms of combinatorial objects known as Steiner systems. The idea for using Steiner (triple) systems for constructing 3-Calabi-Yau algebras is due to Mariano Suárez-Alvarez [27]. The smallest Steiner triple system is provided by the points and lines in the Fano plane, $\mathbb{P}^2_{\mathbb{F}_2}$, and the 3-Calabi-Yau algebra in that case is the one studied by Smith in [25].

Proposition 5.2. *The 3-Calabi-Yau algebras described in the theorem in the introduction of* [10] *satisfy the hypotheses of Corollary 3.5 and therefore admit 4-dimensional Artin-Schelter regular central extensions.*

5.2. **Generic cases with finite Gelfand-Kirillov dimension.** The generic 3-dimensional Artin-Schelter regular algebras of finite Gelfand-Kirillov dimension are listed in [1, Tables 3.9 and 3.11]. In the terminology of §1.6, they have type (1221) or (1331).

As remarked after Theorem 3.4, there is a version of Corollary 3.5 in which the relations for $D(\mathsf{w}, \mathsf{p})$ are defined by using f_k in place of f_1. If we do that we set $p_k := q_k$ instead of $p_1 := q_1$.

Let ζ_n be a primitive n^{th} root of unity.

Proposition 5.3. *For each 3-dimensional regular algebra A in Tables 3.9 and 3.11 of* [1] *and each index $k = 1, \ldots, n$ of the relation $\partial_k \mathsf{w}$ for $A(\mathsf{w}) = A$ that we omit when defining $D(\mathsf{w}, \mathsf{p})$, Tables 1 and 2 list all tuples $\mathsf{p} = (p_1, \ldots, p_n)$ that satisfy condition (4) in Theorem 3.4. For each such p, $D(\mathsf{w}, \mathsf{p})$ is a 4-dimensional Artin-Schelter regular normal extension of A. (In Tables 1 and 2, λ is an arbitrary non-zero scalar.)*

Type	(q_1, q_2)	$k=1$	$k=2$
A	$(1,1)$	$(1,\pm1)$	$(\pm1,1)$
E	$(1,\zeta_3)$	$(1,\zeta_3^r),\ r=0,1,2$	(ζ_3,ζ_3)
H	$(\zeta_8,-\zeta_8)$	none	none
S_1	(α,α^{-1})	$(\alpha,\alpha^{-1/2})$	$(\alpha^{1/2},\alpha^{-1})$
S_2	$(\alpha,-\alpha^{-1})$	$(\alpha,\alpha^{-1/2})$	$((-\alpha)^{1/2},-\alpha^{-1})$
S_2'	$(1,-1)$	$(1,\pm1)$	none

TABLE 1. Good (p_1, p_2) for the generic cubic AS-regular algebras in [1, Table (3.9)]

Type	(q_1, q_2, q_3)	$k=1$	$k=2$	$k=3$
A	$(1,1,1)$	$(1,1,1)$, $(1,\zeta_3,\zeta_3^2)$	$(1,1,1)$, $(\zeta_3^2,1,\zeta_3)$	$(1,1,1)$, $(\zeta_3,\zeta_3^2,1)$
B	$(1,1,-1)$	$(1,1,\pm1)$	$(1,1,\pm1)$	$(-1,-1,-1)$
E	$(\zeta_9,\zeta_9^4,\zeta_9^7)$	none	none	none
H	$(1,-1,\zeta_4)$	$(1,1,\pm1)$, $(1,-1,\pm\zeta_4)$	$(-1,-1,\pm1)$	$(-\zeta_4,-\zeta_4,\zeta_4)$
S_1	$(\alpha,\beta,(\alpha\beta)^{-1})$	$(\alpha,\lambda,\lambda^{-1})$	$(\lambda,\beta,\lambda^{-1})$	$(\lambda,\lambda^{-1},(\alpha\beta)^{-1})$
S_1'	$(\alpha,\alpha^{-1},1)$	$(\alpha,\alpha^{-1/3},\alpha^{1/3})$	$(\alpha^{1/3},\alpha^{-1},\alpha^{-1/3})$	$(\lambda,\lambda^{-1},1)$
S_2	$(\alpha,-\alpha,\alpha^{-2})$	$(\alpha,\pm\alpha,\alpha^{-1})$	$(\pm\alpha,-\alpha,-\alpha^{-1})$	$(1,\pm1,\alpha^{-2})$, $(-1,\pm1,\alpha^{-2})$

TABLE 2. Good (p_1, p_2, p_3) for the generic quadratic AS-regular algebras in [1, Table (3.11)]

5.3. In [17], four classes of 4-dimensional Artin-Schelter regular algebras of type (12221) were discovered. They are labeled by $A(p)$, $B(p)$, $C(p)$, and $D(v,p)$, with parameters $0 \neq p \in \Bbbk$ and $v \in \Bbbk$. These algebras are *all* the noetherian regular algebras of type (12221) that satisfy the (m_2, m_3)-*generic* condition in [17]; roughly speaking, these algebras are generic in terms of the A_∞-structure on their Yoneda Ext-algebras. In [17], the regularity of these algebras was proved in a computational way using Bergman's diamond lemma [5]. Among these, all $A(p)$ and $D(v,p)$ are among the algebras $D(\mathsf{w},\mathsf{p})$ in Theorem 3.4 so their regularity may also be proved as a consequence Proposition 5.3.

Proposition 5.4.

(1) *Consider the 3-dimensional cubic regular algebra of type S_2 in Table (3.9) of* [1]*. Then the normal extension at $k=2$ with $p_1 = (-\alpha)^{1/2}$ is the algebra $A(-p_1)$ defined in* [17, Thm. A]*. The normal extension at $k=1$ with $p_2 = \alpha^{-1/2}$ is isomorphic to $A(-p_2)$ after interchanging the variables x and y.*

(2) *Consider the 3-dimensional cubic regular algebra of type S_1 in Table (3.9) of* [1]*. Then the normal extension at $k=2$ with $p_1 = \alpha^{1/2}$ is the algebra $D(a,-p_1)$ defined in* [17, Thm. A]*. The normal extension at $k=1$ with $p_2 = \alpha^{-1/2}$ is isomorphic to $D(-ap_2^2,-p_2)$ after interchanging the variables x and y.*

Proof. Let A be the algebra of type S_2, $k=2$, and $p := p_1 = (-\alpha)^{1/2}$. "The" relations for A are

$$\begin{cases} f_1 = xy^2 + \alpha y^2 x = xy^2 + (-p)^2 y^2 x, \\ f_2 = \alpha^2 yx^2 - \alpha x^2 y. \end{cases}$$

Thus, "the" relations for $D(\mathsf{w},\mathsf{p})$ are f_1 and

$$\begin{aligned} g_2 &= x(\alpha^2 yx^2 - \alpha x^2 y) - p(\alpha^2 yx^2 - \alpha x^2 y)x \\ &= p^2(x^3 y + (-p)x^2 yx + (-p)^2 xyx^2 + (-p)^3 yx^3). \end{aligned}$$

This description agrees with the defining relation of $A(-p)$ up to scalar multiplication. The other cases are also computed straightforwardly. ∎

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