WEIGHTED SHEAVES AND HOMOLOGY OF ARTIN GROUPS

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ABSTRACT. In this paper we expand the theory of weighted sheaves over posets, and use it to study the local homology of Artin groups. First, we use such theory to relate the homology of classical braid groups with the homology of certain independence complexes of graphs. Then, in the context of discrete Morse theory on weighted sheaves, we introduce a particular class of acyclic matchings. Explicit formulas for the homology of the corresponding Morse complexes are given, in terms of the ranks of the associated incidence matrices. We use such method to perform explicit computations for the new affine case $\tilde{C}_n$, as well as for the cases A_n, B_n and $\tilde{A}_n$ (which were already done before by different methods).

CONTENTS

1. INTRODUCTION

The topological theory of Artin groups started in the 60's with classical braid groups [FN62, Arn69, Arn70], and was broadened to general Artin groups in the 70's [Fuc70, BS72, Del72, Bri73, Coh76, Wei78, Gor78]. It has received big attention in connection with problems in singularity theory, hyperplane arrangements, and combinatorics (see [Par14] for some references), until recent applications of some special cases to conjectures about hyperbolic groups [AGM13]. Therefore any

method to calculate topological invariants of Artin groups (e.g. their homology and cohomology) should be considered interesting and subject to possible applications.

Thanks to some good topological model for the $K(\pi, 1)$ space of Artin groups [Sal87, Sal94], a combinatorial free resolution for these groups was found [DCS96], and several (co)homology computations were carried out [DCPSS99, DCPS01, CS04, Cal05, Cal06, CMS08a, CMS08b, CMS10] (an equivalent resolution for Artin groups of finite type, based on purely algebraic methods, appeared in [Squ94]). The computational methods used in these papers are essentially based on filtering the algebraic complex and using the associated spectral sequence.

In [SV13] a more combinatorial method of calculation was introduced, based on the application of discrete Morse theory to a particular class of sheaves over posets (called *weighted sheaves*). It is interesting to notice that similar ideas were considered later in [CGN16], even if there the authors were mainly interested in computational aspects.

In this paper we expand the theory of weighted sheaves. First, we observe a surprising relationship between some twisted (co)homology of the classical braid groups and the (co)homology of certain *independence complexes*, which in their simplest form have already been studied in a combinatorial context (see for example[Eng09]). This relationship is obtained in an indirect way, after localization. The exact formula we are going to prove is the following (it was announced in [Sal15] without proof).

Theorem 3.4.
$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} \cong \tilde{H}_{*-d+1}\left(\mathrm{Ind}_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)}\right).$$

Here on the left we have the homology with local coefficients of the classical braid group and on the right we have the homology of some independence complex with trivial coefficients (we indicate by φ_d the d-th cyclotomic polynomial). The homology of these independence complexes can be computed separately, so our results can be used in both directions: either in order to compute the homology of these complexes from that of Artin groups, or to compute the homology of these complexes to deduce that of Artin groups.

As said before, our method is based on the notions of weighted sheaves over posets and of *weighted matchings*, which are interesting objects by themselves. In our situation, the d-weight $v_d(\sigma)$ of a simplex σ is the maximal power of the d-th cyclotomic polynomial which divides the Poincaré polynomial of the (finite) parabolic subgroup generated by the vertices of the simplex. We allow matchings between simplices of the same weight. Applying discrete Morse theory, one obtains a Morse complex which still computes the local homology.

Then, by applying these methods to other Coxeter graphs (in particular to finite and affine type cases), we come up with the notion of *precise matching*. In all the cases we consider, it is possible to produce acyclic weighted matchings such that the Morse complex turns out to have a very nice property: the boundary of a critical p-simplex σ has a non-zero coefficient along a critical $(p-1)$-simplex τ if and only if $v_d(\sigma) = v_d(\tau) + 1$. Then the homology of the Morse complex reduces to the computation of the ranks of the incidence matrices among critical simplices (Theorems 4.2 and 4.4). We show that the existence of precise matchings can also be interpreted in terms of second-page collapsing of a spectral sequence which is naturally associated to the weighted sheaf (Proposition 4.3).

As we discuss at the beginning of Section 5, the existence of precise matchings for a certain Artin group has strong implications for its homology. In particular we want to highlight the following non-trivial result.

Theorem 5.2. Let $\mathbf{G_W}$ be an Artin group (corresponding to some finitely generated Coxeter group $\mathbf{W}$) that admits a φ-precise matching for all cyclotomic polynomials $\varphi = \varphi_d$ (with $d \geq 2$). Then the homology $H_*(\mathbf{X_W}; R)$ does not have φ^k-torsion for $k \geq 2$.

Here $\mathbf{X_W}$ (introduced in Section 2.4) is a finite CW-complex which is a deformation retract of the *orbit space* associated to $\mathbf{W}$, and $\pi_1(\mathbf{X_W}) = \mathbf{G_W}$. When $\mathbf{W}$ is finite and in a few other cases $\mathbf{X_W}$ it is known to be a $K(\mathbf{G_W}, 1)$, whereas this property is only conjectured in general [Del72, Bri73, Oko79, VdL83, Hen85, Sal87, Sal94, CD95, CMS10, Par14].

Theorem 5.2 can be seen as a result about the homology of the infinite cyclic covering of $\mathbf{X_W}$ and its monodromy. It is known that the conclusion of such corollary holds for all Artin groups of finite type, due to geometric reasons (see Remark 5.3). Our construction provides a new combinatorial condition that applies to a wider class of Artin groups (including affine groups of type $\tilde{A}_n$ and $\tilde{C}_n$, and also others).

The methods we develop allow explicit and complete computations, in general in a more direct way with respect to other known methods. We use them to compute the twisted homology in the case of affine Artin groups of type $\tilde{C}_n$, obtaining the following result.

Theorem 5.25. Let $\mathbf{G_W}$ be an Artin group of type $\tilde{C}_n$. Then the φ_d-primary component of $H_*(\mathbf{G_W}; R)$ is trivial for d odd, and for d even is as follows (let $n = k\frac{d}{2} + r$):

$$H_m(\mathbf{G_W}; R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus m+k-n+1} & \text{if } n - k \leq m \leq n - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We also carry out complete computations in the cases A_n, B_n and $\tilde{A}_n$: these cases were already done before by different methods [Fre88, DCPSS99, DCPS01, CMS08a], but need them to perform computations of more complicated cases.

In a recent preprint [Pao17], the first author succeeded to construct precise matchings for all remaining Artin groups of finite and affine type. Therefore precise matchings seem to be a suitable and effective tool for studying the homology of Artin groups.

The paper is structured as follows. In Section 2 we recall the most important definitions and results of [SV13]. In particular we introduce the general framework of weighted sheaves over posets, and then explain how it can be used to compute the homology of Artin groups. In Section 3 we study the case of braid groups (i.e. Artin groups of type A_n), and relate their homology with that of certain independence complexes. In Section 4 we introduce precise matchings and derive a general formula for the homology of the Morse complex. In Section 5 we apply the theory developed in Section 4 to the case of Artin groups. We construct precise matchings for Artin groups of type A_n, B_n, $\tilde{A}_n$ and $\tilde{C}_n$, and use such matchings to explicitly compute the homology.

2. Preliminaries

In this section we are going to recall some definitions and constructions from [SV13] (see also [MSV12]).

2.1. **Weighted sheaves over posets.** Let $(P, \preceq)$ be a finite poset.

Definition 2.1. Define a *sheaf of rings* over P (or a *diagram of rings* over P) as a collection

$$\{A_x \mid x \in P\}$$

of commutative rings together with a collection of ring homomorphisms

$$\{\rho_{x,y} \colon A_y \to A_x \mid x \preceq y\}$$

satisfying

$$\rho_{x,x} = \mathrm{id}_{A_x};$$
$$x \preceq y \preceq z \;\Rightarrow\; \rho_{x,z} = \rho_{x,y}\, \rho_{y,z}.$$

In other words, a sheaf of rings over P is a functor from P^{op} to the category of commutative rings.

Fix a PID R; usually we take $R = \mathbb{Q}[q^{\pm 1}]$. The divisibility relation

$$p_1 \mid p_2 \;\Leftrightarrow\; (p_1) \supseteq (p_2)$$

gives R the structure of a small category (and to R/R_* the structure of a poset, where R_* is the group of units in R, with minimum element the class of the units and maximum element 0). Any functor $w \colon (P, \preceq) \to (R, \mid)$, which maps every $x \in P$ to some $w(x) \in R \setminus \{0\}$, defines a sheaf over P by the collection

$$\{R/(w(x)) \mid x \in P\}$$

and

$$\{i_{x,y} \colon R/(w(y)) \to R/(w(x)) \mid x \preceq y\}$$

where $i_{x,y}$ is induced by the identity of R.

Definition 2.2. Given a poset P, a PID R and a morphism $w \colon (P, \preceq) \to (R, \mid)$ as above, we call the triple (P, R, w) a *weighted sheaf* over P and the coefficients $w(x)$ the *weights* of the sheaf.

Remark 2.3. Consider the *poset topology* over P, where a basis for the open sets is given by the upper intervals $\mathcal{B} = \{P_{>p}, \; p \in P\}$. Let (P, R, w) be a weighted sheaf. Then one can see w as a functor from $(\mathcal{B}, \supseteq)$ to $(R, \mid)$ and thus a weighted sheaf defines a sheaf in the usual sense.

From now on our poset will be a simplicial complex K defined over a finite set S, with the partial ordering

$$\sigma \preceq \tau \Leftrightarrow \sigma \subseteq \tau.$$

We adopt the convention that K contains the empty simplex $\varnothing$. A weighted sheaf over K is given by assigning to each simplex $\sigma \in K$ a weight $w(\sigma) \in R$, with

$$\sigma \preceq \tau \;\Rightarrow\; w(\sigma) \mid w(\tau).$$

Let $C^0_*(K; R)$ be the 1-shifted standard algebraic complex computing the simplicial homology of K, i.e.

$$C^0_k(K; R) = \bigoplus_{\substack{\sigma \in K \\ |\sigma| = k}} R\, e^0_\sigma$$

where e_σ^0 is a generator associated to a given orientation of σ ($C_0^0(K; R) = R\, e_\varnothing^0$). The boundary is given by

$$\partial^0(e_\sigma^0) = \sum_{|\tau|=k-1} [\sigma : \tau]\, e_\tau^0,$$

where $[\sigma : \tau]$ is the incidence number (which is equal to ± 1 if $\tau \prec \sigma$, and otherwise is equal to 0).

Definition 2.4. The *weighted complex* associated to the weighted sheaf (K, R, w) is the algebraic complex $L_* = L_*(K)$ defined by

$$L_k = \bigoplus_{|\sigma|=k} \frac{R}{(w(\sigma))}\, \bar{e}_\sigma,$$

with boundary $\partial \colon L_k \to L_{k-1}$ induced by ∂^0:

$$\partial(a_\sigma \bar{e}_\sigma) = \sum_{\tau \prec \sigma} [\sigma : \tau]\, i_{\tau,\sigma}(a_\sigma)\, \bar{e}_\tau.$$

There is a natural projection $\pi \colon C_*^0(K; R) \to L_*$ which sends a generator e_σ in C_*^0 to the generator $\bar{e}_\sigma$ in L_*.

Remark 2.5. The diagram $\{R/(w(\sigma))\}$, for $\sigma \in K$, also defines a sheaf over the poset K in the sense of Remark 2.3. The sheaf cohomology associated to the open covering given by the upper segments coincides with the homology of the weighted sheaf.

2.2. **Decomposition and filtration.** Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf. For any irreducible $\varphi \in R$, we define the φ-primary component $\mathcal{S}_\varphi = (K, R, w_\varphi)$ of the weighted sheaf $\mathcal{S}$ by setting

$$w_\varphi(\sigma) = \varphi^{v_\varphi(\sigma)},$$

where

$$v_\varphi(\sigma) = \text{maximal } r \text{ such that } \varphi^r \text{ divides } w(\sigma).$$

Since R is a PID and $w(\sigma) \neq 0$, such a maximal value exists. Notice that $\mathcal{S}_\varphi$ is a weighted sheaf. The weighted complex $(L_*)_\varphi$ associated to $\mathcal{S}_\varphi$ is called the φ-primary component of the weighted complex L_*. In degree k one has:

$$(L_k)_\varphi = \bigoplus_{|\sigma|=k} \frac{R}{\left(\varphi^{v_\varphi(\sigma)}\right)}\, \bar{e}_\sigma.$$

The complex $(L_*)_\varphi$ has a natural increasing filtration by the following subcomplexes:

$$F^s(L_*)_\varphi = \bigoplus_{v_\varphi(\sigma)\leq s} \frac{R}{\left(\varphi^{v_\varphi(\sigma)}\right)}\, \bar{e}_\sigma.$$

Such filtration is associated to an increasing filtration of the simplicial complex K:

$$K_{\varphi,s} = \{\sigma \in K \mid v_\varphi(\sigma) \leq s\}.$$

Then $F^s(L_*)_\varphi$ is the weighted complex associated to the weighted sheaf

$$(K_{\varphi,s},\, R,\, w_\varphi|_{K_{\varphi,s}}).$$

Theorem 2.6 ([SV13])**.** Let (K, R, w) be a weighted sheaf, with associated weighted complex L_*. For any irreducible $\varphi \in R$, there exists a spectral sequence

$$E_{p,q}^r \;\Rightarrow\; H_*((L_*)_\varphi)$$

that abuts to the homology of the φ-primary component of the associated algebraic complex L_*. Moreover the E^1-term

$$E_{p,q}^1 = H_{p+q}(F^p/F^{p-1}) \cong H_{p+q}(K_{\varphi,p}, K_{\varphi,p-1}; R/(\varphi^p))$$

is isomorphic to the relative homology with *trivial coefficients* of the simplicial pair $(K_{\varphi,p}, K_{\varphi,p-1})$.

2.3. **Discrete Morse theory for weighted complexes.** Given x, y in a poset $(P, \prec)$, the notation $x \lhd y$ means that y *covers* x, i.e. $x \prec y$ and there is no other $z \in P$ such that $x \prec z \prec y$.

Recall the basic facts of discrete Morse theory (see for example [For98, Koz07]). A *matching* in a poset P is a set $\mathcal{M} \subseteq P \times P$ such that $(x, y) \in M \Rightarrow x \lhd y$ and each $x \in P$ belongs to at most one pair of $\mathcal{M}$. An *alternating path* is a sequence

$$y_0 \rhd x_1 \lhd y_1 \rhd x_2 \lhd y_2 \rhd \cdots \rhd x_m \lhd y_m (\rhd x_{m+1})$$

such that each pair $x_i \lhd y_i$ belongs to $\mathcal{M}$ and no pair $x_i \lhd y_{i-1}$ belongs to $\mathcal{M}$. A *cycle* is a closed alternating path with $y_0 = y_m$. An *acyclic matching* over P is a matching with no cycles. We are going to describe a variant of discrete Morse theory which is suitable for our situation.

Definition 2.7. A *weighted acyclic matching* on a weighted sheaf (P, R, w) over P is an acyclic matching $\mathcal{M}$ on P such that

$$(x, y) \in \mathcal{M} \;\Rightarrow\; (w(x)) = (w(y))$$

(in [SV13] we required $w(x) = w(y)$, but the above generalization does not change the results).

The standard theory generalizes as follows. Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf over a finite simplicial complex K, and let $\mathcal{M}$ an acyclic weighted matching. A *critical simplex* of K is a simplex σ which does not belong to any pair of $\mathcal{M}$. The following definition is equivalent to [SV13, Definition 3.3].

Definition 2.8. The *Morse complex* of $\mathcal{S}$ with respect to $\mathcal{M}$ is defined as the torsion complex

$$L_*^{\mathcal{M}} \;=\; \bigoplus_{\sigma \text{ critical}} \frac{R}{(w(\sigma))} \, \bar{e}_\sigma$$

with boundary

$$\partial^{\mathcal{M}}(\bar{e}_\sigma) = \sum_{\substack{\tau \text{ critical} \\ |\tau| = |\sigma| - 1}} [\sigma : \tau]^{\mathcal{M}} \, \bar{e}_\tau,$$

where $[\sigma : \tau]^{\mathcal{M}} \in \mathbb{Z}$ is given by the sum over all alternating paths

$$\sigma \rhd \tau_1 \lhd \sigma_1 \rhd \tau_2 \lhd \sigma_2 \rhd \cdots \rhd \tau_m \lhd \sigma_m \rhd \tau$$

from σ to τ of the quantity

$$(-1)^m [\sigma : \tau_1][\sigma_1 : \tau_1][\sigma_1 : \tau_2][\sigma_2 : \tau_2] \cdots [\sigma_m : \tau_m][\sigma_m : \tau].$$

The boundary map $\partial^{\mathcal{M}}$ is extended by R-linearity, where some care should be taken since each $\bar{e}_\tau$ lives in a component with possibly different torsion ($\partial^{\mathcal{M}}$ is the same as the boundary map of [SV13]).

Theorem 2.9 ([SV13]). Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf over a simplicial complex K. Let $\mathcal{M}$ be an acyclic matching for $\mathcal{S}$. Then there is an isomorphism

$$H_*(L_*, \partial) \cong H_*(L_*^{\mathcal{M}}, \partial^{\mathcal{M}})$$

between the homology of the weighted complex L_* associated to $\mathcal{S}$ and the homology of the Morse complex $L_*^{\mathcal{M}}$ of $\mathcal{S}$.

If we forget about the weights, the matching $\mathcal{M}$ is in particular an acyclic matching for the simplicial complex K. Therefore the algebraic complex $C_*^0(K; A)$, which computes the (shifted and reduced) simplicial homology of K with coefficients in some ring A, admits a "classical" algebraic Morse complex

$$C_*^0(K; A)^{\mathcal{M}} = \bigoplus_{\sigma \text{ critical}} A\, \bar{e}_\sigma$$

with boundary map

$$\delta^{\mathcal{M}}(\bar{e}_\sigma) = \sum_{\substack{\tau \text{ critical} \\ |\tau| = |\sigma| - 1}} [\sigma : \tau]^{\mathcal{M}}\, \bar{e}_\tau.$$

Remark 2.10. Set $C_*^0 = C_*^0(K; R)$ and let $\pi \colon C_*^0 \to L_*$ be the natural projection. The composition with the projection $L_* \to (L_*)_\varphi$, for some fixed irreducible element $\varphi \in R$, yields a natural projection $\pi_\varphi \colon C_*^0 \to (L_*)_\varphi$. Similarly at the level of Morse complexes we have the projection $\bar{\pi}_\varphi \colon (C_*^0)^{\mathcal{M}} \to (L_*)_\varphi^{\mathcal{M}}$ which sends a generator $\bar{e}_\sigma$ in $(C_*^0)^{\mathcal{M}}$ to the generator $\bar{e}_\sigma$ in $(L_*)_\varphi^{\mathcal{M}}$. Then, by construction, the maps induced in homology by the projections π_φ and $\bar{\pi}_\varphi$ commute with the Morse isomorphisms of [Koz07] and [SV13]:

$$
\begin{array}{ccc}
H_*(C_*^0) & \xrightarrow{(\pi_\varphi)_*} & H_*((L_*)_\varphi) \\
\cong \downarrow & & \cong \downarrow \\
H_*((C_*^0)^{\mathcal{M}}) & \xrightarrow{(\bar{\pi}_\varphi)_*} & H_*((L_*)_\varphi^{\mathcal{M}}).
\end{array}
$$

Let $\varphi \in R$ be an irreducible element, and let $\mathcal{M}$ be a weighted acyclic matching on $\mathcal{S}_\varphi$. Then the filtration on the weighted complex induces a filtration on the Morse complex:

$$F^p(L_*)_\varphi^{\mathcal{M}} = \bigoplus_{\substack{\sigma \text{ critical} \\ v_\varphi(\sigma) \leq p}} \frac{R}{(\varphi^{v_\varphi(\sigma)})}\, \bar{e}_\sigma.$$

Consider also the following quotient complex:

$$\mathcal{F}^p(L_*)_\varphi^{\mathcal{M}} = F^p(L_*)_\varphi^{\mathcal{M}} \,/\, F^{p-1}(L_*)_\varphi^{\mathcal{M}} \cong \bigoplus_{\substack{\sigma \text{ critical} \\ v_\varphi(\sigma) = p}} \frac{R}{(\varphi^p)}\, \bar{e}_\sigma.$$

Theorem 2.11 ([SV13]). Let $\mathcal{S}$ and φ be as above and let $\mathcal{M}$ be an acyclic matching for $\mathcal{S}_\varphi$. Then the E^1-page of the spectral sequence of Theorem 2.6 is identified with

$$E_{p,q}^1 \cong H_{p+q}(\mathcal{F}^p(L_*)_\varphi^{\mathcal{M}}),$$

where $(L_*)_\varphi^{\mathcal{M}}$ is the Morse complex of $\mathcal{S}_\varphi$. The differential

$$d_{p,q}^1 \colon E_{p,q}^1 \to E_{p-1,q}^1$$

is induced by the boundary of the Morse complex, and thus it is also computed by using alternating paths.

2.4. **Weighted sheaves for Artin groups.** We use the constructions given above in the context of Artin groups. Let $(\mathbf{W}, S)$ be a Coxeter system, with $|S| = n$ finite, and let Γ be the corresponding Coxeter graph (with S as vertex set). Recall that $\mathbf{W}$ can always be realized as a group of reflections in some $\mathbb{R}^n$ (for example through the *Tits representation*, see [Bou68, Vin71], see also [Par14]), so that it has the following naturally associated objects:

(i) a hyperplane arrangement in $\mathbb{R}^n$

$$\mathcal{A} = \{H \mid H \text{ is the fixed point set of some reflection in } \mathbf{W}\};$$

(ii) configuration spaces

$$\mathbf{Y} = (\text{int}(U) + i\mathbb{R}^n) \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}},$$

$$\mathbf{Y_W} = \mathbf{Y}/\mathbf{W}$$

where $U = \mathbf{W} \cdot \overline{C_0}$ is the *Tits cone* (here C_0 is a fixed chamber of the arrangement, and $\overline{C_0}$ is its topological closure);

(iii) a simplicial complex (defined over the finite set S)

$$K = K_{\mathbf{W}} = \{\sigma \subseteq S \mid \text{the parabolic subgroup } \mathbf{W}_\sigma \text{ generated by } \sigma \text{ is finite}\}.$$

One can define the *Artin group* $\mathbf{G_W}$ of type $\mathbf{W}$ as the fundamental group $\pi_1(\mathbf{Y_W})$. It has a presentation

$$\mathbf{G_W} = \langle\, g_s, s \in S \mid g_s g_{s'} g_s g_{s'} \ldots = g_{s'} g_s g_{s'} g_s \ldots \,\rangle$$

(both the products have $m(s, s')$ factors). Recall also the following results [Sal94, Theorems 1.4 and 1.8].

(i) The orbit space $\mathbf{Y_W}$ deformation retracts onto a *finite CW*-complex $\mathbf{X_W}$ given by a union

$$Q = \bigcup_{\sigma \in K_{\mathbf{W}}} Q_\sigma$$

of convex polyhedra with explicit identifications of their faces.

(ii) Consider the action of the Artin group $\mathbf{G_W}$ on the ring $R = \mathbb{Q}[q^{\pm 1}]$ given by

$$g_s \mapsto [\text{multiplication by } - q] \quad \forall s \in S.$$

Then the homology $H_*(\mathbf{X_W}; R)$ is computed by the algebraic complex

$$C_k = \bigoplus_{\substack{\sigma \in K_{\mathbf{W}} \\ |\sigma| = k}} R\, e_\sigma$$

with boundary

$$\partial(e_\sigma) = \sum_{\tau \lhd \sigma} [\sigma : \tau] \frac{\mathbf{W}_\sigma(q)}{\mathbf{W}_\tau(q)} e_\tau,$$

where $\mathbf{W}_\sigma(q) = \displaystyle\sum_{w \in \mathbf{W}_\sigma} q^{l(w)}$ is the Poincaré polynomial of $\mathbf{W}_\sigma$.

Remark 2.12. The R-module structure on $H_*(\mathbf{X_W}; R)$ is given by the transformation μ_q induced by q-multiplication. If the order of μ_q is n, then the homology groups decompose into cyclic factors which are either free, of the form R^k, or torsion, of the form $R/(\varphi_d)$, where φ_d is the d-th cyclotomic polynomial with $d \mid n$. Therefore we are interested in localizing to cyclotomic polynomials.

Theorem 2.13 ([SV13]). To a Coxeter system $(\mathbf{W}, S)$ we can associate a weighted sheaf (K, R, w) over the simplicial complex $K = K_\mathbf{W}$, by setting $w(\sigma) = \mathbf{W}_\sigma(q)$. Also, for any cyclotomic polynomial $\varphi = \varphi_d$, the map

$$w_\varphi(\sigma) = \text{maximal power of } \varphi \text{ which divides } \mathbf{W}_\sigma(q)$$

defines a weighted sheaf (K, R, w_φ) over $K = K_\mathbf{W}$.

The homology of the associated weighted complex is strictly related to the homology of $\mathbf{X_W}$. Specifically, set $C_*^0 = C_*^0(K; R)$ and consider the diagonal map

$$\Delta \colon C_* \to C_*^0, \quad e_\sigma \mapsto \mathbf{W}_\sigma(q)\, e_\sigma^0.$$

By the formula for the boundary map it follows that Δ is an injective chain-complex homomorphism, so there is an exact sequence of complexes:

$$0 \longrightarrow C_* \xrightarrow{\Delta} C_*^0 \xrightarrow{\pi} L_* \longrightarrow 0,$$

where

$$L_k = \bigoplus_{\substack{\sigma \in K_\mathbf{W} \\ |\sigma| = k}} \frac{R}{(\mathbf{W}_\sigma(q))}\, \bar{e}_\sigma$$

is the quotient complex. Passing to the associated long exact sequence we get:

$$(1) \quad \ldots \xrightarrow{\pi_*} H_{k+1}(L_*) \to H_k(C_*) \xrightarrow{\Delta_*} H_k(C_*^0) \xrightarrow{\pi_*} H_k(L_*) \to H_{k-1}(C_*) \xrightarrow{\Delta_*} \ldots$$

Then the homology of L_* can be used to compute the homology of C_*.

The orbit space $\mathbf{Y_W}$ (and thus the CW-complex $\mathbf{X_W} \simeq \mathbf{Y_W}$) is conjectured to be a classifying space for the Artin group $\mathbf{G_W}$ [Del72, Bri73, VdL83, Par14]. Such conjecture was proved for Artin groups of finite type [Del72], for affine Artin groups of type $\tilde{A}_n$, $\tilde{B}_n$ and $\tilde{C}_n$ [Oko79, CMS10], and for some other families of Artin groups [Hen85, CD95]. Whenever the conjecture holds (in particular this is true for all the cases we consider in this paper), the homology $H_*(\mathbf{X_W}; R)$ coincides with the twisted homology $H_*(\mathbf{G_W}; R)$ of the Artin group $\mathbf{G_W}$.

3. HOMOLOGY OF BRAID GROUPS AND INDEPENDENCE COMPLEXES

In this section we are going to show how the twisted homology of braid groups (i.e. Artin groups of type A_n) is related to the homology of suitable independence complexes. Recall that a Coxeter graph of type A_n is a linear graph with n vertices, usually labeled $1, 2, \ldots, n$ (see Figure 1), and that the corresponding Artin group is the braid group on $n + 1$ strands (which we denote by Br_{n+1}). With a slight abuse of notation, by A_n we will sometimes indicate the graph itself.

FIGURE 1. A Coxeter graph of type A_n.

The homology of Br_{n+1}, with coefficients in the representation $R = \mathbb{Q}[q^{\pm 1}]$ described in the previous section, has been computed in [Fre88, DCPS01]. See also [Cal06] for coefficients in $\mathbb{Z}[q^{\pm 1}]$ (but we will not address such case in this paper).

Theorem 3.1 ([Fre88, DCPS01]). *The φ_d-primary component of the twisted homology of a braid group is given by*

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} = \begin{cases} R/(\varphi_d) & \text{if } n \equiv 0 \text{ or } -1 \pmod{d}, \\ 0 & \text{otherwise,} \end{cases}$$

where the non vanishing term is in degree $(d-2)k$ if $n = dk$ or $n = dk - 1$.

Recall that, if G is a graph with vertex set VG, an *independent set* of G is a subset of VG consisting of pairwise non-adjacent vertices. Also, the *independence complex* $\mathrm{Ind}(G)$ of G is the abstract simplicial complex with VG as set of vertices and whose simplices are all the non-empty independent sets of G. In contrast with the simplicial complex K introduced in Section 2, the simplicial complex $\mathrm{Ind}(G)$ does not contain the empty simplex, and the dimension of a simplex $\sigma \in \mathrm{Ind}(G)$ is given by $|\sigma| - 1$. The homotopy type of $\mathrm{Ind}(A_n)$ has been computed in [Koz07] by means of discrete Morse theory, and the result is the following.

Proposition 3.2 ([Koz07]).

$$\mathrm{Ind}(A_n) \simeq \begin{cases} S^{k-1} & \text{if } n = 3k \text{ or } n = 3k - 1, \\ \{\mathrm{pt}\} & \text{if } n = 3k + 1. \end{cases}$$

By comparing this with Theorem 3.1 we obtain the following relation between the homology of the independence complex of A_n and the φ_3-primary component of the twisted homology of Br_{n+1}.

Corollary 3.3.

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_3} \cong \tilde{H}_{*-2}\left(\mathrm{Ind}(A_{n-3}); \frac{R}{(\varphi_3)}\right),$$

where on the left we have local coefficients and on the right we have trivial coefficients. □

In general, following [Sal15], define the *r-independence complex* of a graph G as

$$\mathrm{Ind}_r(G) = \{\text{full subgraphs } G' \subseteq G \text{ such that each connected component} \\ \text{of } G' \text{ has at most } r \text{ vertices}\}.$$

So $\mathrm{Ind}_r(G)$ is an abstract simplicial complex on the set of vertices VG of the graph G, which coincides with $\mathrm{Ind}(G)$ for $r = 1$. The case $r = 0$ also makes sense: $\mathrm{Ind}_0(G) = \varnothing$ for any graph G. On page 12 we are going to prove the following generalization of Corollary 3.3.

Theorem 3.4.

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} \cong \tilde{H}_{*-d+1}\left(\mathrm{Ind}_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)}\right),$$

where on the left we have local coefficients and on the right we have trivial coefficients.

From the expression of the φ_d-primary component of the local homology of the braid group (Theorem 3.1) we obtain the following consequence.

Corollary 3.5.

$$\tilde{H}_*(\mathrm{Ind}_{d-2}(A_n)) \cong \begin{cases} \tilde{H}_*(S^{dk-2k-1}) & \text{for } n = dk \text{ or } n = dk-1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the knowledge of the twisted homology of the braid group gives the homology (with trivial coefficients) of $\mathrm{Ind}_d(A_n)$. Conversely the knowledge of the homology of $\mathrm{Ind}_d(A_n)$ gives the twisted homology of the braid group.

The Poincaré polynomial of a Coxeter group of type A_k is given by

$$\mathbf{W}_{A_k}(q) = [k+1]_q! \quad \text{where} \quad [k]_q = \frac{q^k - 1}{q - 1} = \prod_{\substack{d|k \\ d \geq 2}} \varphi_d$$

(see for example [BB06]). Consider now a simplex $\sigma \subseteq S \cong \{1, \ldots, n\}$. Denote by $\Gamma(\sigma)$ the subgraph of A_n induced by σ. Denote by $\Gamma_1(\sigma)$, $\Gamma_2(\sigma)$, ..., $\Gamma_m(\sigma)$ the connected components of $\Gamma(\sigma)$, and by $n_1, n_2, \ldots n_m$ their cardinalities (see Figure 2). The i-th connected component is a Coxeter graph of type A_{n_i}, so the entire Coxeter graph induced by σ has Poincaré polynomial

$$\mathbf{W}_\sigma(q) = [n_1 + 1]_q! \cdot [n_2 + 1]_q! \cdots [n_m + 1]_q!$$
$$= \prod_{i=1}^{m} \prod_{d \geq 2} \varphi_d^{\lfloor \frac{n_i+1}{d} \rfloor}$$
$$= \prod_{d \geq 2} \varphi_d^{\sum_{i=1}^{m} \lfloor \frac{n_i+1}{d} \rfloor}.$$

Then we are interested in the homology of the weighted complex $(L_*)_\varphi$ associated to the weighted sheaf (K, R, w_φ), where $\varphi = \varphi_d$ is a cyclotomic polynomial (with $d \geq 2$) and

$$w_\varphi(\sigma) = \varphi^{v_\varphi(\sigma)}, \quad v_\varphi(\sigma) = \sum_{i=1}^{m} \left\lfloor \frac{n_i + 1}{d} \right\rfloor.$$

Notice that only the connected components with at least $d-1$ vertices contribute to the φ-weight. Therefore, the weighted complex $(L_*)_\varphi$ is generated by the subgraphs having at least one component with $\geq d-1$ vertices.
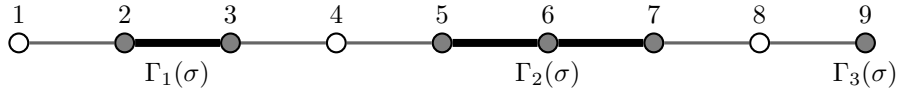

FIGURE 2. An example with $n = 9$ and $\sigma = \{2, 3, 5, 6, 7, 9\} \in \mathrm{Ind}_3(A_9)$. In this case $|\Gamma_1(\sigma)| = 2$, $|\Gamma_2(\sigma)| = 3$, $|\Gamma_3(\sigma)| = 1$.

Theorem 3.6. There is a weighted acyclic matching $\mathcal{M}$ on K such that the set of critical simplices is given by

$$\begin{aligned} \mathrm{Cr}(\mathcal{M}) &= \{\sigma \mid \Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_{m-1} \sqcup A_{d-1}, \ |\Gamma_i| \leq d-2\} \cup \mathrm{Ind}_{d-2}(A_n) \\ &= \{\tau \sqcup A_{d-1} \mid \tau \in \mathrm{Ind}_{d-2}(A_{n-d})\} \cup \mathrm{Ind}_{d-2}(A_n), \end{aligned}$$

where A_{d-1} is the linear graph on the vertices $n-d+2, \ldots, n$.

Proof. First notice that removing the d-th vertex from an A_k component leaves the φ-weight unchanged:

$$v_\varphi(A_k) = \left\lfloor \tfrac{k+1}{d} \right\rfloor = 1 + \left\lfloor \tfrac{k+1-d}{d} \right\rfloor = v_\varphi(A_{d-1} \sqcup A_{k-d})$$
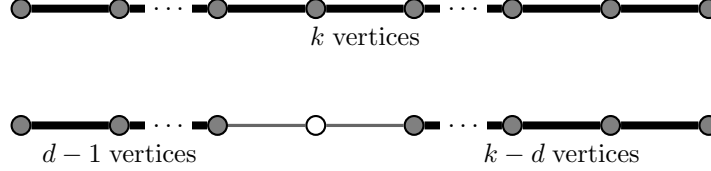
(see Figure 3).

FIGURE 3. The φ-weight of an A_k component remains the same if we remove the d-th vertex, splitting A_k into $A_{d-1} \sqcup A_{k-d}$.

Let $K_0 = \mathrm{Ind}_{d-2}(A_n) \subseteq K$ (this is the set of the simplices $\sigma \in K$ such that all the connected components of the induced subgraph $\Gamma(\sigma)$ have cardinality $< d-1$). Let us define on $K' = K \setminus K_0$ the following matching $\mathcal{M}$. For $\sigma \in K'$, with $\Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_m$, set

$$i(\sigma) = \min\{i \mid |\Gamma_i| \geq d - 1\}.$$

Then match σ with the simplex τ obtained by adding or removing from σ the $(d-1)$-th vertex of the component $\Gamma_{i(\sigma)}$. By the remark at the beginning of the proof, $\mathcal{M}$ is a weighted matching. We prove that it is acyclic. In fact, suppose that an alternating path contains some subpath

$$\tau \lhd \sigma \rhd \tau' \lhd \sigma'$$

with $\tau \neq \tau'$. Then either $i(\tau) < i(\tau')$ or (set $j = i(\tau) = i(\tau')$)

$$\Gamma_j(\tau') = \{a, a+1, \ldots, a+d-2\}, \quad \Gamma_j(\tau') = \{a+1, \ldots, a+d-1\}$$

for some $a \leq n - d + 1$. In this case the first vertex of $\Gamma_j(\tau')$ is greater than the first vertex of $\Gamma_j(\tau)$. Therefore an alternating path in K' cannot be closed.

The set of critical elements of $\mathcal{M}$ in K' is given by

$$\mathrm{Cr}(\mathcal{M}) = \{\sigma \mid \Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_{m-1} \sqcup A_{d-1}, \ |\Gamma_i| \leq d - 2\},$$

where A_{d-1} is as in the statement of the theorem. $\square$

Proof of Theorem 3.4. Consider the matching $\mathcal{M}$ of Theorem 3.6. Since $K_0 = \mathrm{Ind}_{d-2}(A_n)$ does not contribute to the weighted complex $(L_*)_\varphi$, we concentrate on the poset $K' = K \setminus K_0$. Notice that there are no non-trivial alternating paths between critical elements of K'. Therefore the boundaries between simplices in

$$\mathrm{Cr}'(\mathcal{M}) = \{\tau \sqcup A_{d-1} \mid \tau \in \mathrm{Ind}_{d-2}(A_{n-d})\}$$

are the same as the boundaries of $\mathrm{Ind}_{d-2}(A_{n-d})$. In the long exact sequence (1) the algebraic complex C_*^0 has vanishing homology in all degrees (because K is the full simplicial complex on n vertices), so we have an isomorphism $H_{*+1}(L_*) \cong H_*(C_*)$. Therefore

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} \cong H_*(C_*)_{\varphi_d}$$
$$\cong H_{*+1}(L_*)_{\varphi_d}$$

$$\cong \tilde{H}_{*-d+1}\left(\mathrm{Ind}_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)}\right).$$

In the last isomorphism there is a $(d-1)$-shift in degree due to the loss of $d-1$ vertices when passing from $\mathrm{Cr}'(\mathcal{M})$ to $\mathrm{Ind}_{d-2}(A_{n-d})$; there is then a further 1-shift in degree due to the fact that in L_* a simplex σ has dimension $|\sigma|$, and in $\mathrm{Ind}_{d-2}(A_{n-d})$ it has dimension $|\sigma| - 1$; finally, it is necessary to pass to reduced homology because the empty simplex is missing in $\mathrm{Ind}_{d-2}(A_{n-d})$. $\qquad \square$

For the sake of completeness we also give an independent and purely topological proof of Corollary 3.5, which is a straightforward generalization of the one in [Koz07] for the case $d = 3$.

Proof of Corollary 3.5. We denote here for brevity $K_0 = \mathrm{Ind}_{d-2}(A_n)$. Let $n = qd + r$ be the euclidean division of n by d.

Let $Q = \{c_d > c_{2d} > c_{3d} > \cdots > c_{qd} > c_*\}$ be a linearly ordered set, where c_* is the minimum element. Define a map $f\colon K_0 \to Q$ as follows. If $\sigma \in K_0$ does not contain any multiples of d, then set $f(\sigma) = c_*$. Otherwise, set $f(\sigma) = c_{jd}$ if $jd \in \sigma$ but $j'd \notin \sigma$ for $j' < j$. Clearly f is a poset map (if we remove a vertex from σ, j increases or remains the same).

In $f^{-1}(c_d)$ consider the matching $\mathcal{M}_1 = \{(\sigma \setminus \{1\}, \sigma \cup \{1\}) \mid \sigma \in f^{-1}(c_d)\}$, which is justified by:

$$\sigma \in f^{-1}(c_d) \;\Rightarrow\; \{2, \ldots, d-1\} \nsubseteq \sigma.$$

Similarly, in $f^{-1}(c_{jd})$ for $j \leq q$, consider the matching

$$\mathcal{M}_j = \left\{\left(\sigma \setminus \{(j-1)d+1\}, \; \sigma \cup \{(j-1)d+1\}\right) \mid \sigma \in f^{-1}(c_{jd})\right\},$$

justified by:

$$\sigma \in f^{-1}(c_{jd}) \;\Rightarrow\; \{(j-1)d+2, \ldots, jd-1\} \nsubseteq \sigma.$$

Each $\mathcal{M}_j$ is acyclic in $f^{-1}(c_{jd})$, thus

$$\mathcal{M} = \bigcup_{j=1}^{q} \mathcal{M}_j$$

is an acyclic matching on K_0 by the Patchwork Theorem (see for example [Jon08, Lemma 4.2], [Koz07, Theorem 11.10]). Since each $\mathcal{M}_j$ is a perfect matching on $f^{-1}(c_{jd})$ for $j = 1, \ldots, q$, and $K_1 = f^{-1}(c_*)$ is a subcomplex of K_0, it follows that K_0 deformation retracts over K_1.

Notice now that K_1 is the join of q copies of $\mathrm{Ind}_{d-2}(A_{d-1}) \cong S^{d-3}$ and one copy of $\mathrm{Ind}_{d-2}(A_r)$. For $r = 0$ we get

$$K_0 \cong (S^{d-3})^{*q} \cong S^{q(d-3)+q-1} = S^{qd-2q-1}$$

and for $r = d-1$ we get

$$K_0 \cong (S^{d-3})^{*(q+1)} \cong S^{(q+1)(d-3)+q} = S^{(q+1)d-2(q+1)-1},$$

which give the first case in the corollary. If r is not 0 or $d-1$ then $\mathrm{Ind}_{d-2}(A_r)$ is a full simplex Δ^{r-1} on r vertices, therefore K_0 is contractible. $\qquad \square$

4. Precise matchings

In this section we are going to introduce the notion of *precise matching* on a weighted sheaf. The motivation comes from the study of the twisted homology of some families of Artin groups, as we will show in the subsequent sections. For now we go back to the general framework of weighted sheaves over simplicial complexes (later we will specialize in the case of Artin groups).

Assume from now on that the PID R contains some field $\mathbb{K}$. Our main case of interest is $\mathbb{K} = \mathbb{Q}$ and $R = \mathbb{Q}[q^{\pm 1}]$. Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf over the finite simplicial complex K, with associated weighted complex L_*. Given a fixed irreducible element φ of R, let $\mathcal{S}_\varphi$ be the φ-primary component of $\mathcal{S}$ and let $(L_*)_\varphi$ be its associated weighted complex. Let $\mathcal{M}$ be a weighted acyclic matching for $\mathcal{S}_\varphi$.

Let $G^{\mathcal{M}}$ be the incidence graph of the corresponding Morse complex: the vertices of $G^{\mathcal{M}}$ are the critical simplices of K, and there is an (oriented) edge $\sigma \to \tau$ whenever $[\sigma : \tau]^{\mathcal{M}}$ is not 0 in R (or equivalently in $\mathbb{K}$), where $[\sigma : \tau]^{\mathcal{M}} \in \mathbb{Z}$ is the incidence number between σ and τ in the Morse complex of K. In other words, there is an edge $\sigma \to \tau$ if $[\sigma : \tau]^{\mathcal{M}}$ is not multiple of $\operatorname{char} \mathbb{K} = \operatorname{char} R$. When $\mathbb{K} = \mathbb{Q}$, this simply means that $[\sigma : \tau]^{\mathcal{M}} \neq 0$.

Let $\mathcal{I}$ be the set of connected components of $G^{\mathcal{M}}$ (computed ignoring the orientation of the edges). Recall that $v_\varphi(\sigma)$ is the maximal $k \in \mathbb{N}$ such that φ^k divides $w(\sigma)$.

Definition 4.1. The matching $\mathcal{M}$ is *φ-precise* (or simply *precise*) if, for any edge $\sigma \to \tau$ of $G^{\mathcal{M}}$, we have that $v_\varphi(\sigma) = v_\varphi(\tau) + 1$.

In other words $\mathcal{M}$ is precise if, for any two simplices σ and τ lying in the same connected component $i \in \mathcal{I}$, the following relation holds:

$$v_\varphi(\sigma) - v_\varphi(\tau) = |\sigma| - |\tau|.$$

Equivalently the quantity $|\sigma| - v_\varphi(\sigma)$, as a function of σ, is constant within a fixed connected component of $G^{\mathcal{M}}$. This definition is motivated by the fact that precise matchings exist in many cases of interest (we will see this in Section 5), and that the homology of the Morse complex is much simpler to compute (and takes a particularly nice form) when the matching is precise. The name "precise" has been chosen because for a generic matching one only has $v_\varphi(\sigma) \geq v_\varphi(\tau)$ (when $\sigma \to \tau$ is an edge of $G^{\mathcal{M}}$), and we require $v_\varphi(\sigma)$ to be precisely $v_\varphi(\tau) + 1$.

Assume from now on in this section that $\mathcal{M}$ is a φ-precise matching. To simplify the notation, set $(A_*, \partial) = ((L_*)_\varphi^{\mathcal{M}}, \partial^{\mathcal{M}})$ and $(V_*, \delta) = (C_*^0(K, \mathbb{K})^{\mathcal{M}}, \delta^{\mathcal{M}})$. Our aim is to derive a formula for the homology of the Morse complex A_*. Since the differential δ vanishes between simplices in different connected components of $G^{\mathcal{M}}$, the complex (V_*, δ) splits as follows:

$$(V_*, \delta) = \bigoplus_{i \in \mathcal{I}} (V_*^i, \delta^i),$$

where

$$V_*^i = \bigoplus_{\substack{\sigma \text{ critical} \\ \sigma \in i}} \mathbb{K}\, \bar{e}_\sigma$$

and the boundary map $\delta^i \colon V_*^i \to V_*^i$ is the restriction of δ to V_*^i. The differential ∂ of A_* is induced by δ, and thus it also vanishes between simplices in different

connected components. Therefore we have an analogous splitting for (A_*, ∂):

$$(A_*, \partial) = \bigoplus_{i \in \mathcal{I}} (A_*^i, \partial^i),$$

where

$$A_*^i = \bigoplus_{\substack{\sigma \text{ critical} \\ \sigma \in i}} \frac{R}{(w_\varphi(\sigma))} \bar{e}_\sigma$$

and the boundary map $\partial^i \colon A_*^i \to A_*^i$ is simply the restriction of ∂ to A_*^i.

Fix now a connected component $i \in \mathcal{I}$. Since $\mathcal{M}$ is precise, there exists some $k \in \mathbb{Z}$ (which depends on i) such that $v_\varphi(\sigma) = |\sigma| + k$ for all $\sigma \in i$. Therefore in degree m we have

$$A_m^i = \bigoplus_{\sigma \in \mathrm{Cr}_m^i} \frac{R}{(\varphi^{m+k})} \bar{e}_\sigma = \left(\bigoplus_{\sigma \in \mathrm{Cr}_m^i} \mathbb{K} \bar{e}_\sigma \right) \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k})} = V_m^i \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k})},$$

where Cr_m^i is the set of critical simplices σ such that $\sigma \in i$ and $|\sigma| = m$. By construction the boundary $\partial_m^i \colon A_m^i \to A_{m-1}^i$ factors accordingly:

$$\partial_m^i = \delta_m^i \otimes_\mathbb{K} \pi_m,$$

where

$$\pi_m \colon \frac{R}{(\varphi^{m+k})} \to \frac{R}{(\varphi^{m+k-1})}$$

is the projection induced by the identity $R \to R$. Since $\mathrm{im}\, \delta_{m+1}^i \subseteq \ker \delta_m^i$, each V_m^i splits (as a vector space over $\mathbb{K}$) as a direct sum of linear subspaces:

$$V_m^i = W_{m,1}^i \oplus W_{m,2}^i \oplus W_{m,3}^i,$$

where $W_{m,1}^i = \mathrm{im}\, \delta_{m+1}^i$ and $W_{m,1}^i \oplus W_{m,2}^i = \ker \delta_m^i$. Then

$$\ker(\partial_m^i) = \ker(\delta_m^i \otimes_\mathbb{K} \pi_m)$$

$$= \left((W_{m,1}^i \oplus W_{m,2}^i) \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k})} \right) \oplus \left(W_{m,3}^i \otimes_\mathbb{K} \frac{(\varphi^{m+k-1})}{(\varphi^{m+k})} \right);$$

$$\mathrm{im}\, (\partial_{m+1}^i) = \mathrm{im}\, (\delta_{m+1}^i \otimes_\mathbb{K} \pi_{m+1})$$

$$= W_{m,1}^i \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k})}.$$

Therefore the homology of (A_*^i, δ^i) is given, as an R-module, by

$$H_m(A_*^i) = \frac{\ker(\partial_m^i)}{\mathrm{im}\, (\partial_{m+1}^i)}$$

$$= \left(W_{m,2}^i \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k})} \right) \oplus \left(W_{m,3}^i \otimes_\mathbb{K} \frac{(\varphi^{m+k-1})}{(\varphi^{m+k})} \right)$$

$$\cong \left(H_m(V_*^i, \delta^i) \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k})} \right) \oplus \left(\mathbb{K}^{\mathrm{rk}\, \delta_m^i} \otimes_\mathbb{K} \frac{R}{(\varphi)} \right).$$

In the last isomorphism we used the fact that $\dim W_{m,3}^i = \dim V_m^i - \dim \ker \delta_m^i = \mathrm{rk}\, \delta_m^i$.

Recall that the previous formula holds for a fixed connected component $i \in \mathcal{I}$, and k depends on i. Since we now need to take the direct sum over the connected components, let k_i be the value of k for the component i.

Theorem 4.2. The homology of $(L_*)_\varphi$ is given, as an R-module, by

$$H_m((L_*)_\varphi, \partial) \cong \left(\bigoplus_{i \in \mathcal{I}} H_m(V_*^i) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})} \right) \oplus \left(\mathbb{K}^{\mathrm{rk}\, \delta_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right).$$

Proof. By Theorem 2.9, $H_m((L_*)_\varphi, \partial) \cong H_m((L_*)_\varphi^{\mathcal{M}}, \partial^{\mathcal{M}})$. Using what we have done in this section, we have that

$$
\begin{aligned}
H_m((L_*)_\varphi^{\mathcal{M}}, \partial^{\mathcal{M}}) &= H_m(A_*, \partial) \\
&= \bigoplus_{i \in \mathcal{I}} H_m(A_*^i, \partial) \\
&\cong \left(\bigoplus_{i \in \mathcal{I}} H_m(V_*^i) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})} \right) \oplus \left(\left(\bigoplus_{i \in \mathcal{I}} \mathbb{K}^{\mathrm{rk}\, \delta_m^i} \right) \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right) \\
&\cong \left(\bigoplus_{i \in \mathcal{I}} H_m(V_*^i) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})} \right) \oplus \left(\mathbb{K}^{\mathrm{rk}\, \delta_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right). \quad \square
\end{aligned}
$$

Let us see how the existence of a precise matching can be interpreted in terms of the spectral sequence associated to the weighted sheaf (see Theorem 2.6).

Proposition 4.3. If a φ-precise matching $\mathcal{M}$ exists, then the spectral sequence $E_{p,q}^r$ associated to the weighted sheaf $\mathcal{S}_\varphi$ collapses at the E^2-page.

Proof. By Theorem 2.11, the E^1-page can be computed through the Morse complex of our matching $\mathcal{M}$:

$$E_{p,q}^1 \cong H_{p+q}(\mathcal{F}^p(L_*)_\varphi^{\mathcal{M}}),$$

and the differential $d_{p,q}^1$ is induced by the boundary of the Morse complex. The spectral sequence then splits as a direct sum over the connected components of $G^{\mathcal{M}}$:

$$E_{p,q}^r = \bigoplus_{i \in \mathcal{I}} E_{p,q}^{r,i},$$

where $E_{p,q}^{1,i} \cong H_{p+q}(\mathcal{F}^p A_*^i)$. Since the matching is precise, for $m \neq p - k_i$ we have

$$\mathcal{F}^p A_m^i = F^p A_m^i / F^{p-1} A_m^i = 0.$$

This means that the page $E_{p,q}^{0,i} \cong E_{p,q}^{1,i}$ is non-trivial only in the row $q = -k_i$, and the entire spectral sequence $E_{p,q}^r$ collapses at the E^2-page. $\square$

What we have done so far in this section assumed φ to be some fixed irreducible element of the PID R. In order to recover the full homology of L_* we need to make φ vary among all equivalence classes of irreducible elements of R modulo the units. Suppose from now on to have a φ-precise matching $\mathcal{M}_\varphi$ on $\mathcal{S}_\varphi$, for each φ. The following results follows immediately from Theorem 4.2, provided that we add a "φ" subscript (or superscript) to all the quantities that depend on the matching $\mathcal{M}_\varphi$.

Theorem 4.4. The homology of L_* is given, as an R-module, by

$$H_m(L_*, \partial) \cong \bigoplus_\varphi \left(\bigoplus_{i \in \mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \right) \oplus \left(\mathbb{K}^{\mathrm{rk}\, \delta_m^\varphi} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right). \quad \square$$

For later applications, we finally need to study how the isomorphism of Theorem 4.4 behaves with respect to the projection $\pi\colon C_*^0 \to L_*$ of Remark 2.10. Such projection is the direct sum over φ of the projections

$$\pi_\varphi\colon C_*^0 \to (L_*)_\varphi.$$

Instead of studying the induced map $(\pi_\varphi)_*\colon H_m(C_*^0) \to H_m((L_*)_\varphi)$, we study the map

$$(\bar\pi_\varphi)_*\colon H_m((C_*^0)^{\mathcal{M}}) \to H_m((L_*)_\varphi^{\mathcal{M}})$$

between the Morse complexes (here $\mathcal{M} = \mathcal{M}_\varphi$ is a precise matching which depends on φ). For $i \in \mathcal{I}_\varphi$, let $\pi_i\colon (C_*^0)^{\mathcal{M}} \to V_*^{\varphi,i} \otimes_{\mathbb{K}} R \subseteq (C_*^0)^{\mathcal{M}}$ be the projection on the subcomplex corresponding to the connected component i, and let $(\pi_i)_*$ be the map induced in homology. Let $[c] \in H_m((C_*^0)^{\mathcal{M}})$, for some cycle $c \in \ker \delta^{\mathcal{M}} \subseteq (C_m^0)^{\mathcal{M}}$. Applying the map $(\bar\pi_\varphi)_*\colon H_m((C_*^0)^{\mathcal{M}}) \to H_m((L_*)_\varphi^{\mathcal{M}})$ we obtain

$$(\bar\pi_\varphi)_*([c]) = (\bar\pi_\varphi)_* \left(\sum_{i \in \mathcal{I}_\varphi} (\pi_i)_*([c]) \right)$$

$$= \sum_{i \in \mathcal{I}_\varphi} (\bar\pi_\varphi)_* \Big((\pi_i)_*([c]) \Big).$$

Applying the isomorphism of Theorem 4.2, this element is sent to

$$\sum_{i \in \mathcal{I}_\varphi} \Big((\pi_i)_*([c]) \Big) \otimes_{\mathbb{K}} [1] \ \in \ \bigoplus_{i \in \mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}.$$

We are going to use these computations to prove the following result which describes the kernel and the cokernel of π_*.

Proposition 4.5. The cokernel of $\pi_*\colon H_m(C_*^0) \to H_m(L_*)$ is given by

$$\operatorname{coker} \pi_* \cong \bigoplus_\varphi \left(\frac{R}{(\varphi)} \right)^{\oplus \operatorname{rk} \delta_m^\varphi}.$$

In addition, the kernel of π_* is a free R-module isomorphic to $H_m(C_*^0)$.

Proof. Throughout the proof, consider the following R-modules identified one with each other, without explicitly mentioning the isomorphisms between them:

$$H_m(L_*) \cong \bigoplus_\varphi H_m((L_*)_\varphi)$$

$$\cong \bigoplus_\varphi H_m((L_*)_\varphi^{\mathcal{M}})$$

$$\cong \bigoplus_\varphi \left(\bigoplus_{i \in \mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \right) \oplus \left(\mathbb{K}^{\operatorname{rk} \delta_m^\varphi} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right).$$

Recall that the matching $\mathcal{M}$ depends on φ, although we write $\mathcal{M}$ instead of $\mathcal{M}_\varphi$ in order to make the notations more readable. Also recall that the isomorphisms $H_m((L_*)_\varphi) \cong H_m((L_*)_\varphi^{\mathcal{M}})$ occur in the commutative diagram of Remark 2.10.

We want to show that the image of $\pi_* \colon H_m(C^0_*) \to H_m(L_*)$ is given by

$$\operatorname{im} \pi_* = \bigoplus_\varphi \left(\bigoplus_{i \in \mathcal{I}_\varphi} H_m(V^{\varphi,i}_*) \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \right) \subseteq H_m(L_*).$$

Let $\psi_\varphi \colon H_*(C^0_*) \to H_*((L_*)^\mathcal{M}_\varphi)$ be the map defined as the composition

$$H_*(C^0_*) \xrightarrow{\cong} H_*((C^0_*)^\mathcal{M}) \xrightarrow{(\bar{\pi}_\varphi)_*} H_*((L_*)^\mathcal{M}_\varphi).$$

By commutativity of the diagram of Remark 2.10, the image of $\pi_* = \bigoplus_\varphi (\pi_\varphi)_*$ is the same as the image of

$$\bigoplus_\varphi \psi_\varphi \colon H_*(C^0_*) \to \bigoplus_\varphi H_*((L_*)^\mathcal{M}_\varphi).$$

We have already proved that, for any $[c] \in H_m((C^0_*)^\mathcal{M})$,

$$(\bar{\pi}_\varphi)_*([c]) = \sum_{i \in \mathcal{I}_\varphi} \left((\pi_i)_*([c]) \right) \otimes_R [1] \ \in \ \bigoplus_{i \in \mathcal{I}_\varphi} H_m(V^{\varphi,i}_*) \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k_{\varphi,i}})},$$

which means in particular that

$$\operatorname{im} (\bar{\pi}_\varphi)_* \subseteq \bigoplus_{i \in \mathcal{I}_\varphi} H_m(V^{\varphi,i}_*) \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k_{\varphi,i}})}.$$

Therefore we immediately have the inclusion

$$\operatorname{im} (\pi_*) \subseteq \sum_\varphi \operatorname{im} \psi_\varphi = \sum_\varphi \operatorname{im} (\bar{\pi}_\varphi)_* \subseteq \bigoplus_\varphi \left(\bigoplus_{i \in \mathcal{I}_\varphi} H_m(V^{\varphi,i}_*) \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \right).$$

To prove the opposite inclusion, we show that any element of the form

$$[c] \otimes_\mathbb{K} [1] \in H_m(V^{\varphi,i}_*) \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k_{\varphi,i}})}$$

is in the image of π_* (for any fixed φ and i). To do so, choose $\alpha \in R$ such that $\alpha \equiv 1 \pmod{\varphi^{m+k_{\varphi,i}}}$ and $\alpha \equiv 0 \pmod{\eta^{m+k_{\eta,j}}}$ for any irreducible element $\eta \neq \varphi$ which divides some weight $w(\sigma)$, and for any connected component $j \in \mathcal{I}_\eta$ (there is only a finite number of such η up to multiplication by units, because K is finite). The element $c \otimes_\mathbb{K} \alpha$ is a cycle in $V^{\varphi,i}_m \otimes_\mathbb{K} R \subseteq C^0_m(K, \mathbb{K})^\mathcal{M} \otimes_\mathbb{K} R \cong (C^0_m)^\mathcal{M}$. Then, if $[\tilde{c}]$ is the preimage of $[c \otimes_\mathbb{K} \alpha]$ under the isomorphism $H_*(C^0_*) \xrightarrow{\cong} H_*((C^0_*)^\mathcal{M})$, we have that:

$$\psi_\varphi([\tilde{c}]) = (\bar{\pi}_\varphi)_*([c \otimes_\mathbb{K} \alpha]) = [c] \otimes_\mathbb{K} [1] \in H_m(V^{\varphi,i}_*) \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k_{\varphi,i}})};$$

$$\psi_\eta([\tilde{c}]) = (\bar{\pi}_\eta)_*([c' \otimes_\mathbb{K} \alpha]) = 0 \quad \text{for any } \eta \neq \varphi,$$

where $[c']$ is the image of $[c]$ under the isomorphism

$$H_*((C^0_*)^{\mathcal{M}_\varphi}) \xrightarrow{\cong} H_*(C^0_*) \xrightarrow{\cong} H_*((C^0_*)^{\mathcal{M}_\eta}).$$

Therefore $[c] \otimes_\mathbb{K} [1]$ is in the image of π_*. We have thus proved that

$$\operatorname{im} \pi_* = \bigoplus_\varphi \left(\bigoplus_{i \in \mathcal{I}_\varphi} H_m(V^{\varphi,i}_*) \otimes_\mathbb{K} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \right).$$

Then the cokernel of π_* can be easily computed:

$$\operatorname{coker}\pi_* = \frac{H_m(L_*)}{\operatorname{im}\pi_*} = \bigoplus_\varphi \left(\mathbb{K}^{\operatorname{rk}\delta_m^\varphi} \otimes_\mathbb{K} \frac{R}{(\varphi)} \right) = \bigoplus_\varphi \left(\frac{R}{(\varphi)} \right)^{\oplus \operatorname{rk}\delta_m^\varphi}.$$

The R-module $H_m(C_*^0) \cong H_m(C_*^0(K;\mathbb{K})) \otimes_\mathbb{K} R$ is free and finitely generated, because $H_m(C_*^0(K;\mathbb{K}))$ is a finite-dimensional vector space over $\mathbb{K}$ (recall that K is a finite simplicial complex). The kernel of π_* is a submodule of $H_m(C_*^0)$, so it is itself a free R-module with lower or equal rank. Let $[c_1],\ldots,[c_k]$ be an R-base of $H_m(C_*^0)$. Consider the non-zero ideal

$$I = \bigcap_\varphi \bigcap_{i\in\mathcal{I}_\varphi} (\varphi^{m+k_i}) \subseteq R,$$

where φ varies among the (finitely many) irreducible elements which divide some weight $w(\sigma)$ (for $\sigma \in K$). Fix any non-zero element $\alpha \in I$. Then the elements $\alpha[c_1],\ldots,\alpha[c_k]$ generate a free submodule of $\ker\pi_*$ of rank $k = \operatorname{rk}H_m(C_*^0)$. Therefore $\ker\pi_*$ and $H_m(C_*^0)$ have the same rank, so they are isomorphic as R-modules. $\qquad\square$

5. Precise matchings for Artin groups

Consider now the case of Artin groups, as in Section 2.4. For a Coxeter system $(\mathbf{W},S)$ we have constructed a weighted sheaf $\mathcal{S} = (K,R,w)$ with

$$K = \{\sigma \subset S \mid \text{the parabolic subgroup } \mathbf{W}_\sigma \text{ is finite}\}$$

and $R = \mathbb{Q}[q^{\pm 1}]$. The associated weighted complex L_* fits into the short exact sequence

$$0 \to C_* \xrightarrow{\Delta} C_*^0 \xrightarrow{\pi} L_* \to 0,$$

which gives rise to the long exact sequence (1). In order to compute $H_*(C_*) = H_*(\mathbf{X_W};R)$, we split such long exact sequence into the short exact sequences

$$0 \to \operatorname{coker}\pi_* \to H_m(C_*) \xrightarrow{\Delta_*} \ker\pi_* \to 0,$$

where on the left we have the cokernel of $\pi_*\colon H_{m+1}(C_*^0) \to H_{m+1}(L_*)$ and on the right we have the kernel of $\pi_*\colon H_m(C_*^0) \to H_m(L_*)$. Since $\ker\pi_*$ is a free R-module, these short exact sequences split:

$$H_m(C_*) \cong \operatorname{coker}\pi_* \oplus \ker\pi_*.$$

Recall that the only irreducible elements of R that occur in the factorization of the weights are the cyclotomic polynomials φ_d for $d \geq 2$. As in Section 4, suppose from now on that we have constructed a φ-precise matching $\mathcal{M}_\varphi$ for each cyclotomic polynomial $\varphi = \varphi_d$ (with $d \geq 2$). Then we have an explicit description of $\operatorname{coker}\pi_*$ and $\ker\pi_*$ thanks to Proposition 4.5, and we obtain the following result.

Theorem 5.1. Under the above hypothesis, the homology of $\mathbf{X_W}$ with coefficients in the representation $R = \mathbb{Q}[q^{\pm 1}]$ is given by

$$H_m(\mathbf{X_W};R) \cong \left(\bigoplus_\varphi \left(\frac{R}{(\varphi)} \right)^{\oplus \operatorname{rk}\delta_{m+1}^\varphi} \right) \oplus H_m(C_*^0). \qquad\qquad \square$$

In particular the term $H_m(C_*^0)$ gives the free part of the homology, and the other direct summands give the torsion part. The torsion part actually takes a very particular form, and we are going to highlight this in the following corollary.

Theorem 5.2. Let $\mathbf{G_W}$ be an Artin group that admits a φ-precise matching for all cyclotomic polynomials $\varphi = \varphi_d$ (with $d \geq 2$). Then the homology $H_*(\mathbf{X_W}; R)$ does not have φ^k-torsion for $k \geq 2$. □

We are particularly interested in Artin groups of finite and affine type. When $\mathbf{G_W}$ is an Artin group of finite type with n generators, K is the full simplicial complex on $S \cong \{1, \ldots, n\}$ and therefore C_*^0 has trivial homology in every dimension. Thus the formula of Theorem 5.1 reduces to

$$ H_m(\mathbf{X_W}; R) \cong \bigoplus_\varphi \left(\frac{R}{(\varphi)} \right)^{\oplus \operatorname{rk} \delta_{m+1}^\varphi}. $$

When $\mathbf{G_W}$ is an Artin group of affine type with $n+1$ generators, K is the simplicial complex on $S \cong \{0, \ldots, n\}$ which misses only the full simplex. Then we have

$$ H_m(\mathbf{X_W}; R) \cong \begin{cases} R & \text{for } m = n, \\ \bigoplus_\varphi \left(\frac{R}{(\varphi)} \right)^{\oplus \operatorname{rk} \delta_{m+1}^\varphi} & \text{for } m < n. \end{cases} $$

Remark 5.3. When $\mathbf{G_W}$ is an Artin group of finite type, the corresponding reflection arrangement of hyperplanes $\mathcal{A}$ is finite. In this case it is well known that there is an R-module isomorphism between the twisted homology $H_*(\mathbf{X_W}; R)$ and the homology with constant coefficients $H_*(F; \mathbb{Q})$ of the Milnor fiber F of $\mathcal{A}$ [Cal05]. The q-multiplication on the homology of $\mathbf{X_W}$ corresponds to the action of the monodromy operator on the homology of F. If $N = |\mathcal{A}|$, the square of the defining polynomial of the arrangement is $\mathbf{W}$-invariant, thus the order of the monodromy of the Milnor fibration divides $2N$. It follows that the polynomial $q^{2N} - 1$ must annihilate the homology. Since $q^{2N} - 1$ is square-free in characteristic 0, the homology cannot have φ^k-torsion for $k \geq 2$. So the conclusion of Theorem 5.2 is not surprising in the case of Artin groups of finite type.

In the rest of this paper we are going to construct precise matchings for Artin groups of type A_n, B_n, $\tilde{A}_n$ and $\tilde{C}_n$. For each of these cases we are then going to use Theorem 5.1 to compute the homology $H_*(\mathbf{X_W}; R)$.

Let us first introduce some notations. We will have $S = \{1, 2, \ldots, n\}$ for the finite cases (A_n and B_n), and $S = \{0, 1, 2, \ldots, n\}$ for the affine cases ($\tilde{A}_n$ and $\tilde{C}_n$). A simplex $\sigma \subseteq S$ will be also represented as a string of bits $\epsilon_i \in \{0, 1\}$ (for $i \in S$), where $\epsilon_i = 1$ if $i \in S$ and $\epsilon_i = 0$ if $i \notin S$. For example, if $S = \{1, 2, 3, 4\}$, the string representation of $\sigma = \{1, 2, 4\}$ is 1101. Also, for $\sigma \subseteq S$ and $v \in S$, let

$$ \sigma \veebar v = \begin{cases} \sigma \cup \{v\} & \text{if } v \notin \sigma, \\ \sigma \setminus \{v\} & \text{if } v \in \sigma. \end{cases} $$

(it can be regarded as the bitwise xor between the string representation of σ and the string with $\epsilon_v = 1$ and $\epsilon_i = 0$ for $i \neq v$).

5.1. **Case A_n.** Many properties of the homology $H_*(\mathbf{X_W}; R)$ in the case A_n have been thoroughly discussed in Section 3. Using precise matchings we are going to obtain a new proof of Theorem 3.1.

Let $S = \{1, 2, \ldots, n\}$ be the vertices of the Coxeter graph of type A_n, as in Section 3, Figure 1. In this case K is the full simplicial complex on S. Fix now an integer $d \geq 2$ and set $\varphi = \varphi_d$. The φ-weight of a simplex $\sigma \in K$ has been computed in Section 3 and is as follows:

$$v_\varphi(\sigma) = \sum_{i=1}^{m} \omega_\varphi(A_{n_i}),$$

where n_i is the size of the i-th (linear) connected component of the subgraph induced by σ, and $\omega_\varphi(A_k)$ stands for the φ-weight of a connected component of type A_k, given by

$$\omega_\varphi(A_k) = \left\lfloor \frac{k+1}{d} \right\rfloor.$$

Fix also an integer f with $0 \leq f \leq d - 1$. Let $K_{n,f} \subseteq K$ be as follows:

$$K_{n,f} = \{\sigma \in K \mid 1, 2, \ldots, f \in \sigma\}.$$

Notice that $K_{n,0} = K$, and $K_{n,f}$ is not a subcomplex of K for $f \geq 1$ (but it is still a subposet of K, so it makes sense to define a matching on it). We are going to construct a φ-precise matching on $K_{n,f}$. In particular, for $f = 0$, we will get a φ-precise matching for K. The precise matchings on $K_{n,f}$ for $f \geq 1$ will become useful when treating the cases B_n, $\tilde{A}_n$ and $\tilde{C}_n$.

For a fixed d, the matching will be constructed recursively in n and f for $n \geq 0$ and $0 \leq f \leq d - 1$. In the subsequent sections we will also write $K_{n,f}^A$ instead of $K_{n,f}$ to avoid confusion with other K's. The matching is as follows.

(a) If $\{1, \ldots, d-1\} \subseteq \sigma$ then match σ with $\sigma \searrow d$ (unless $n = d - 1$, in which case σ is critical). Notice that for $f = d - 1$ this is always the case, thus in the subsequent steps we can assume $f \leq d - 2$. Here σ is matched with a simplex which also occurs in step (a).
(b) Otherwise, if $n = f$ then σ is critical.
(c) Otherwise, if $f + 1 \in \sigma$ then match σ with $\sigma \setminus \{f + 1\}$. Notice that $\sigma \setminus \{f + 1\}$ occurs in step (d).
(d) Otherwise, if $\{f + 2, \ldots, d - 1\} \not\subseteq \sigma$ then match σ with $\sigma \cup \{f + 1\}$. Notice that $\sigma \cup \{f + 1\}$ occurs in step (c).
(e) We are left with the simplices σ such that $\{1, \ldots, f, f + 2, \ldots, d - 1\} \subseteq \sigma$ and $f + 1 \notin \sigma$. If we ignore the vertices $1, \ldots, f + 1$ we are left with the simplices on the vertex set $\{f + 2, \ldots, n\}$ which contain $f + 2, \ldots, d - 1$; relabeling the vertices, these are the same as the simplices on the vertex set $\{1, \ldots, n - f - 1\}$ which contain $1, \ldots, d - 2 - f$. Then construct the matching recursively as in $K_{n-f-1,\, d-2-f}$.

Example 5.4. For $n = 5$, $d = 3$ and $f = 1$, $K_{n,f}$ contains $2^4 = 16$ simplices of which 14 are matched and 2 are critical. For instance, consider the simplex $\sigma = \{1, 4, 5\}$. Step (e) applies because $1 \in \sigma$ and $2 \notin \sigma$; the recursion requires to consider the new simplex $\sigma' = \{2, 3\} \in K_{3,0}$. Again step (e) applies because $1 \notin \sigma'$ and $2 \in \sigma'$; the recursion requires to consider the new simplex $\sigma'' = \{1, 2\} \in K_{2,1}$. Finally step (a) applies because $\{1, 2\} \subseteq \sigma''$, and σ'' is critical because $\sigma'' = \{1, 2\}$. Therefore $\sigma = \{1, 4, 5\} \in K_{5,1}$ is also critical. See Table 1 for an explicit description of the matching for $K_{5,1}$, $d = 3$.
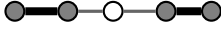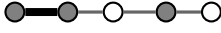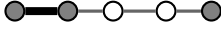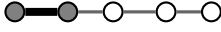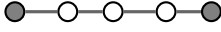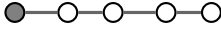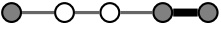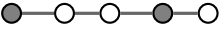
Simplices	$v_\varphi(\sigma)$	Step
(diagram) $\longrightarrow$ (diagram)	2	(a)
(diagram) $\longrightarrow$ (diagram)	1	(a)
(diagram) $\longrightarrow$ (diagram)	1	(a)
(diagram) $\longrightarrow$ (diagram)	1	(a)
(diagram) $\longrightarrow$ (diagram)	1	(e) $\to$ (a)
(diagram) $\longrightarrow$ (diagram)	0	(e) $\to$ (c)/(d)
(diagram) $\longrightarrow$ (diagram)	0	(e) $\to$ (c)/(d)
(diagram) *(critical)*	1	(e) $\to$ (e) $\to$ (a)
(diagram) *(critical)*	0	(e) $\to$ (e) $\to$ (e) $\to$ (b)

TABLE 1. Matching in the case A_n with $n = 5$, $d = 3$, $f = 1$. The last columns indicates the step where simplices occur. When step (e) is reached, an arrow indicates how the recursion continues (after $K_{5,1}$, the recursion involves $K_{3,0}$, $K_{2,1}$ and $K_{0,0}$).

Remark 5.5. A peculiarity of this matching is that, if $\sigma' \to \tau'$ is in the matching, then $\sigma' = \tau' \cup \{v\}$ with $v \equiv f+1$ or $v \equiv 0 \pmod{d}$. This can be easily checked by induction.

Lemma 5.6. The matching described above is an acyclic matching on $K_{n,f}$.

Proof. The proof is by induction on n, the case $n = 0$ being trivial. Assume that $n > f$ since, for $n \le f$, in $K_{n,f}$ there are either 1 or 0 simplices. Consider the map $\eta \colon K_{n,f} \to \{p_\mathrm{a} > p_\mathrm{e} > p_\mathrm{c,d}\}$ which sends $\sigma \in K_{n,f}$ to the p_x such that σ occurs in step (x) (here steps (c) and (d) are united). For instance, $\eta(\sigma) = p_\mathrm{a}$ if and only if $\{1, \ldots, d-1\} \subseteq \sigma$. Notice that the map η is compatible with the matching, i.e. two matched simplices lie in the same fiber of η.

Let us prove that η is a poset map. Given two simplices $\sigma \ge \tau$, we want to prove that $\eta(\sigma) \ge \eta(\tau)$. If $\eta(\tau) = p_\mathrm{a}$ then $\{1, \ldots, d-1\} \subseteq \tau \subseteq \sigma$, so also $\eta(\sigma) = p_\mathrm{a}$. If $\eta(\tau) = p_\mathrm{e}$ then $\{1, \ldots, f, f+2, \ldots, d-1\} \subseteq \tau \subseteq \sigma$, thus $\eta(\sigma) \in \{p_\mathrm{a}, p_\mathrm{e}\}$. Finally, if $\eta(\tau) = p_\mathrm{c,d}$ there is nothing to prove.

Since η is a poset map, our matching is acyclic if and only if it is acyclic on each fiber of η. The restriction of the matching to $\eta^{-1}(p_\mathrm{a})$ consists of edges $\sigma \to \tau$ with $\sigma = \tau \cup \{d\}$, so there is no alternating path of length ≥ 4 and the matching is acyclic. The restriction to $\eta^{-1}(p_\mathrm{c,d})$ is acyclic for the same reason (it is always the vertex $f+1$ which is added or removed). The restriction to $\eta^{-1}(p_\mathrm{e})$ is acyclic by induction. $\square$

Lemma 5.7. The matching described above is a weighted matching on $K_{n,f}$.

Proof. The proof is by induction on n, the case $n = 0$ being trivial.

In step (a), $v_\varphi(\sigma) = v_\varphi(\sigma \veebar d)$ by what we have already said in Section 3 (see the proof of Theorem 3.6). In steps (c) and (d), both the simplices σ and $\sigma \veebar (f+1)$ do not contain $\{1, \ldots, d-1\}$. Thus the (possibly empty) connected component of 1 has size $\leq d - 2$ and does not contribute to the weight. Therefore $v_\varphi(\sigma) = v_\varphi(\sigma \veebar (f+1))$. Finally, suppose that σ and τ are simplices that occur in step (e). Let $\hat\sigma, \hat\tau \in K_{n-f-1,\, d-2-f}$ be the simplices obtained ignoring the vertices 1, ..., $f + 1$, as described above. Since $f \leq d - 2$ we have that

$$v_\varphi(\sigma) = v_\varphi(\tau) \quad \text{if and only if} \quad v_\varphi(\hat\sigma) = v_\varphi(\hat\tau),$$

so we are done by induction on n. $\qquad\square$

The critical simplices of the matching on $K_{n,f}$ are quite simple to describe. If $f \leq d - 2$, $n > f$ and $n \equiv f$ or $-1 \pmod d$, there are 2 critical simplices, one of weight 1 and one of weight 0. If $n = f$ there is 1 critical simplex (in fact there is only one simplex in $K_{n,f}$). In all the other cases the matching is perfect. We are going to prove this in the following theorem. See Table 2 and Figures 4 and 5 for an explicit description of the critical simplices. Notice in particular that, when there are 2 critical simplices, one is a face of the other in $K_{n,f}$.

Case		Simplices	$|\sigma|$	$v_\varphi(\sigma)$
$n > f$ $f \leq d - 2$	$n = kd + f$ (Fig. 4)	$(1^f 0 1^{d-2-f} 0)^{k-1} 1^f 0 1^{d-1}$	$n - 2k + 1$	1
		$(1^f 0 1^{d-2-f} 0)^k 1^f$	$n - 2k$	0
	$n = kd - 1$ (Fig. 5)	$(1^f 0 1^{d-2-f} 0)^{k-1} 1^{d-1}$	$n - 2k + 2$	1
		$(1^f 0 1^{d-2-f} 0)^{k-1} 1^f 0 1^{d-2-f}$	$n - 2k + 1$	0
$n = f$		1^f	n	1 or 0

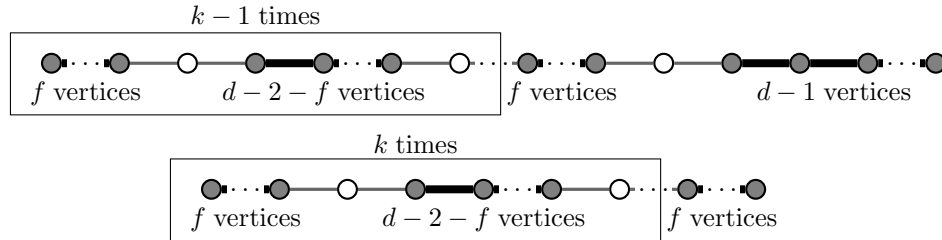TABLE 2. Description of the critical simplices for A_n.

FIGURE 4. Critical simplices for $f \leq d - 2$ and $n = kd + f$ ($k \geq 1$).

Theorem 5.8. The critical simplices for the matching on $K_{n,f}$ are those listed in Table 2. In particular the matching is always precise, and is perfect if $n \not\equiv f, -1 \pmod d$ or $n > f = d - 1$. In addition, when there are two critical simplices there is only one alternating path between them (the trivial one).
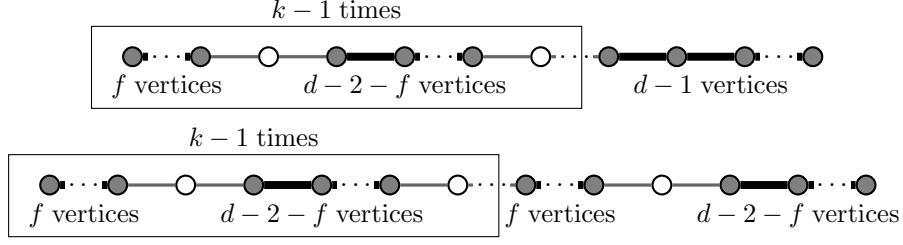
FIGURE 5. Critical simplices for $f \leq d - 2$ and $n = kd - 1$ ($k \geq 1$).

Proof. We prove the first part by induction on n, the case $n = 0$ being trivial (there is one critical cell for $f = 0$, namely the empty simplex $\varnothing$, and no critical cells for $f > 0$). Let $\sigma \in K_{n,f}$ be a simplex. Let us consider each of the five cases that can occur in the construction of the matching.

(a) We have $\{1, \ldots, d-1\} \subseteq \sigma$. Then σ is critical if and only if $n = d - 1$. If $n = d - 1$ then σ is indeed listed in Table 2, as the first of the two critical simplices in the case $n = kd - 1$ (here $k = 1$). Conversely, the only simplex of Table 2 which contains $\{1, \ldots, d-1\}$ is the first one of the case $n = kd - 1$ when $k = 1$.

(b) We have $n = f$ and $\sigma = \{1, \ldots, f\}$. Then σ is critical, and it is indeed listed as the second of the two critical simplices in the case $n = kd + f$ (here $k = 0$).

(c) We have $\{1, \ldots, f+1\} \subseteq \sigma$. Then σ is not critical. The only listed simplex which contains $\{1, \ldots, f+1\}$ is the first one of the case $n = kd - 1$ for $k = 1$, but it is equal to $\{1, \ldots, d-1\}$ and so it must be different from σ.

(d) We have $f + 1 \notin \sigma$ and $\{f+2, \ldots, d-1\} \nsubseteq \sigma$. Then σ is not critical. It is easy to check that all the listed simplices τ with $n > f$ and $f + 1 \notin \tau$ satisfy $\{f+2, \ldots, d-1\} \subseteq \tau$, so none of them can be equal to σ.

(e) We have $\{1, \ldots, f, f+2, \ldots, d-1\} \subseteq \sigma$ and $f + 1 \notin \sigma$. Then σ is critical if and only if the simplex $\hat{\sigma} \in K_{n-f-1, \, d-2-f}$ is critical, where $\hat{\sigma}$ is constructed from σ ignoring the first $f + 1$ vertices. By induction, $\hat{\sigma}$ is critical if and only if it is listed in Table 2. By taking the simplices of Table 2 for $K_{n-f-1, \, d-2-f}$ and adding $1^f 0$ at the beginning, one obtains exactly the simplices of Table 2 for $K_{n,f}$ (but the two cases are exchanged). This concludes the induction argument.

For fixed n and f, the quantity $|\sigma| - v_\varphi(\sigma)$ is constant among the critical simplices σ. More precisely, it is equal to $n - 2k$ for $n = kd + f$ (except for $f = d - 1$) and to $n - 2k + 1$ for $n = kd - 1$. Therefore the matching is precise.

Finally we need to prove that the only alternating path between critical simplices is the trivial one. Consider the case $n = kd + f$ for $k \geq 1$ (the case $n = kd - 1$ is analogous). Let σ and τ be the two critical simplices (the ones of Figure 4). Notice that $\sigma = \tau \cup \{n - f\}$. Suppose by contradiction to have a non-trivial alternating path

$$\sigma \vartriangleright \tau_1 \vartriangleleft \sigma_1 \vartriangleright \tau_2 \vartriangleleft \sigma_2 \vartriangleright \cdots \vartriangleright \tau_m \vartriangleleft \sigma_m \vartriangleright \tau, \quad m \geq 1.$$

Since $\tau_1 \neq \tau$, we must have $\sigma = \tau_1 \cup \{v_1\}$ with $v_1 \neq n - f$. Notice that $n - f$ is the only vertex $v \in \sigma$ which satisfies $v \equiv f + 1$ or $v \equiv 0 \pmod{d}$. Therefore, by Remark 5.5, the vertex v_1 will never be recovered throughout the alternating path. This is a contradiction since $v_1 \in \tau$. $\qquad\square$

As a consequence we can compute the homology $H_*(\mathbf{G_W}; R)$ for $\mathbf{G_W}$ of type A_n, and obtain a proof of Theorem 3.1.

Proof of Theorem 3.1. We simply need to apply the formula given by Theorem 5.1 using our precise matching on $K = K_{n,0}$. Since $f = 0$, there are critical cells for $n = kd$ or $n = kd - 1$. The boundary δ_{m+1}^φ is non-trivial only for

$$\begin{cases} m = n - 2k = k(d-2) & \text{if } n = kd, \\ m = n - 2k + 1 = k(d-2) & \text{if } n = kd - 1. \end{cases}$$

In both cases for $m = k(d-2)$ we have $\mathrm{rk}\, \delta_{m+1}^\varphi = 1$, and all the other boundaries are trivial. Theorem 3.1 follows. $\qquad\square$

Remark 5.9. A relation with independence complexes can be found for $K_{n,f}$ also when $1 \leq f \leq d - 2$. Indeed, choosing a suitable weighted matching (similar to the one of Theorem 3.6), the set of critical simplices of positive weight is

$$\{\sigma \sqcup A_{d-1} \mid \sigma \in \mathrm{Ind}_{d-2}(A_{n-d}) \cap K_{n-d,\,f}\}.$$

Remark 5.10. Throughout this section, the condition $f \leq d - 1$ is not very important. We chose to restrict to this case in order to have simpler results and proofs, but everything can be easily extended to the cases in which $f \geq d$ (ignoring the first sd vertices, where sd is the biggest multiple of d which is $\leq f$). We will apply this argument in Section 5.3, when constructing the matching for the case $\tilde{A}_n$.

5.2. **Case B_n.** Consider now a Coxeter graph of type B_n, as in Figure 6. In this case K is again the full simplex on vertices $\{1, 2, \ldots, n\}$. The Poincaré polynomial of a Coxeter group of type B_k is given by

$$\mathbf{W}_{B_k}(q) = [2k]_q!! = \prod_{i=1}^n [2i]_q = \prod_\varphi \varphi^{\omega_\varphi(B_k)},$$

where, for a given cyclotomic polynomial $\varphi = \varphi_d$, we have

$$\omega_\varphi(B_k) = \begin{cases} \left\lfloor \frac{k}{d} \right\rfloor & \text{if } d \text{ is odd}, \\ \left\lfloor \frac{k}{d/2} \right\rfloor & \text{if } d \text{ is even} \end{cases}$$

(see for example [BB06]). Then we can compute the φ-weight of any simplex $\sigma \in K$ by looking at the subgraph $\Gamma(\sigma)$ induced by σ. Let $\Gamma_1(\sigma)$, ..., $\Gamma_m(\sigma)$ be the connected components of $\Gamma(\sigma)$, with cardinality n_1, ..., n_m respectively, where $\Gamma_1(\sigma)$ is the (possibly empty) component that contains the vertex $1 \in S$. Then

$$v_\varphi(\sigma) = \omega_\varphi(B_{n_1}) + \sum_{i=2}^m \omega_\varphi(A_{n_i}).$$

FIGURE 6. A Coxeter graph of type B_n.

The situation is quite different depending on the parity of d. If d is odd, the φ-weight of a B_{k+1} component is equal to the φ-weight of an A_k component:

$$\omega_\varphi(B_{k+1}) = \left\lfloor \frac{k+1}{d} \right\rfloor = \omega_\varphi(A_k) \quad (d \text{ odd}).$$

For this reason it is possible to construct a very simple perfect matching on K: match each simplex $\sigma \in K$ with $\sigma \veebar 1$. Such matching is acyclic and perfect, but we also want it to be a weighted matching.

Lemma 5.11. The matching constructed above for d odd is a weighted matching on K.

Proof. Let $\sigma \in K$. The only difference between $\Gamma(\sigma)$ and $\Gamma(\sigma \veebar 1)$ is the leftmost connected component, which in one case is of type B_{k+1} and in the other case is of type A_k. Therefore $v_\varphi(\sigma) = v_\varphi(\sigma \veebar 1)$. $\qquad\square$

Suppose from now on that d is even. The simplicial complex K is partitioned as

$$K = \bigsqcup_{q \geq 0} K_q, \quad \text{where} \quad K_q = \left\{ \sigma \in K \ \middle| \ \left\lfloor \frac{|\Gamma_1(\sigma)|}{d/2} \right\rfloor = q \right\}.$$

Here $\Gamma_1(\sigma)$ is the (possibly empty) connected component of $\Gamma(\sigma)$ which contains the vertex 1. Notice that each K_q is a subposet of K, but not a subcomplex in general. For a given simplex $\sigma \in K_q$, let $|\Gamma_1(\sigma)| = q\frac{d}{2} + r$ with $0 \leq r < \frac{d}{2}$. The matching on K_q is as follows.

(a) If $r \geq 1$ (i.e. $q\frac{d}{2} + 1 \in \sigma$) then match σ with $\sigma \setminus \{q\frac{d}{2} + 1\}$.
(b) If $r = 0$ (i.e. $q\frac{d}{2} + 1 \notin \sigma$) and $\{q\frac{d}{2} + 2, \ldots, (q+1)\frac{d}{2}\} \not\subseteq \sigma$, then match σ with $\sigma \cup \{q\frac{d}{2} + 1\}$ (unless $n = q\frac{d}{2}$, in which case σ is critical).
(c) We are left with the simplices σ for which neither (a) nor (b) apply, i.e. with $r = 0$ and $\{q\frac{d}{2} + 2, \ldots, (q+1)\frac{d}{2}\} \subseteq \sigma$. Ignore the first $q\frac{d}{2} + 1$ vertices and relabel the remaining ones from 1 to $n - q\frac{d}{2} - 1$, so that we are left exactly with the simplices of $K^A_{n-q\frac{d}{2}-1, \frac{d}{2}-1}$. Then construct the matching on $K^A_{n-q\frac{d}{2}-1, \frac{d}{2}-1}$ as in Section 5.1.

Putting together the matchings on each K_q we obtain a matching on the full simplicial complex K.

Example 5.12. For $n = 4$ and $d = 4$, the simplicial complex K contains $2^4 = 16$ simplices of which 12 are matched and 4 are critical. For instance, consider $\sigma = \{1, 2\} \in K$. Then $q = 1$ and $r = 0$. Since $4 \notin \sigma$, step (b) occurs. Therefore σ is matched with $\sigma \cup \{3\} = \{1, 2, 3\}$. See Table 3 for an explicit description of the matching in this case.

Lemma 5.13. The matching constructed above for d even is an acyclic matching on K.

Proof. The map $K \to (\mathbb{N}, \leq)$ which sends σ to $q = \left\lfloor \frac{|\Gamma_1(\sigma)|}{d/2} \right\rfloor$ is a poset map compatible with the matching, and its fibers are exactly the subsets K_q for $q \in \mathbb{N}$. Therefore we only need to prove that the matching on each fiber K_q is acyclic.

For a fixed $q \in \mathbb{N}$, consider the map $\eta \colon K_q \to \{p_c > p_{a,b}\}$ which sends σ to the p_x such that σ occurs in step (x) (here steps (a) and (b) are united). Clearly η is compatible with the matching. We want to prove that it is a poset map, and for this we only need to show that if $\eta(\tau) = p_c$ and $\tau \leq \sigma$ then also $\eta(\sigma) = p_c$. We have
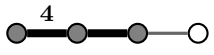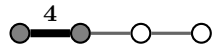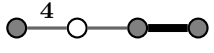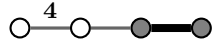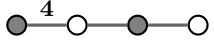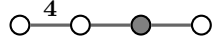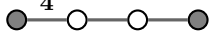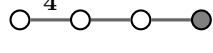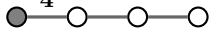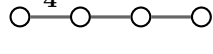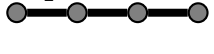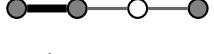
Simplices		$v_\varphi(\sigma)$	q	Step
(diagram)	$\longrightarrow$ (diagram)	1	1	(a)/(b)
(diagram)	$\longrightarrow$ (diagram)	0	0	(a)/(b)
(diagram)	$\longrightarrow$ (diagram)	0	0	(a)/(b)
(diagram)	$\longrightarrow$ (diagram)	0	0	(a)/(b)
(diagram)	$\longrightarrow$ (diagram)	0	0	(a)/(b)
(diagram)	*(critical)*	2	2	(b)
(diagram)	*(critical)*	1	1	(c)
(diagram)	*(critical)*	1	0	(c)
(diagram)	$\longrightarrow$ (diagram)	0	0	(c)
(diagram)	*(critical)*	0	0	(c)

TABLE 3. Matching in the case B_n with $n = 4$ and $d = 4$.

that $\left\{1, \ldots, q\frac{d}{2}, q\frac{d}{2} + 2, \ldots, (q+1)\frac{d}{2}\right\} \subseteq \tau \subseteq \sigma$. The simplex σ cannot contain the vertex $q\frac{d}{2} + 1$, because otherwise we would have $\sigma \in K_{q+1}$. Therefore $\eta(\sigma) = p_c$.

On the fiber $\eta^{-1}(p_{a,b})$ the matching is acyclic because the same vertex $q\frac{d}{2} + 1$ is always added or removed. On the fiber $\eta^{-1}(p_c)$ the matching is acyclic by Lemma 5.6. Therefore the entire matching on K_q is acyclic. $\qquad\square$

Lemma 5.14. The matching constructed above for d even is a weighted matching on K.

Proof. Let $\sigma \in K_q$ be a simplex which occurs either in step (a) or step (b). We want to show that $v_\varphi(\sigma) = v_\varphi\left(\sigma \overset{\vee}{} \left(q\frac{d}{2} + 1\right)\right)$. Suppose without loss of generality that σ occurs in step (a), i.e. $r \geq 1$ and $q\frac{d}{2} + 1 \in \sigma$. Let $\tau = \sigma \setminus \left\{q\frac{d}{2} + 1\right\}$. The only difference between $\Gamma(\sigma)$ and $\Gamma(\tau)$ is that $\Gamma(\sigma)$ has a $B_{q\frac{d}{2}+r}$ component, whereas $\Gamma(\tau)$ has a $B_{q\frac{d}{2}}$ component and an A_{r-1} component instead. Since $1 \leq r < \frac{d}{2}$, we have that

$$\omega_\varphi(B_{q\frac{d}{2}+r}) = \omega_\varphi(B_{q\frac{d}{2}}) = q$$
$$\text{and} \quad \omega_\varphi(A_{r-1}) = 0$$

(the second equality holds because $r - 1 \leq \frac{d}{2} - 1 \leq d - 2$). Therefore $v_\varphi(\sigma) = v_\varphi(\tau)$.

For simplices occurring in step (c), the matching only involves changes in the connected components not containing the first vertex (i.e. connected components of type A_k). Therefore Lemma 5.7 applies. $\qquad\square$

Now we are going to describe the critical simplices. The matching is perfect when n is not a multiple of $\frac{d}{2}$. On the other hand, if $n = k\frac{d}{2}$, we have two families of critical simplices: σ_q (for $0 \leq q \leq k-2$) and σ'_q (for $0 \leq q \leq k$). See Table 4 and Figure 7 for the definition of such simplices. For instance, in Example 5.12, the critical simplices are: $\sigma_0 = \{2,3,4\}$, $\sigma'_2 = \{1,2,3,4\}$, $\sigma'_1 = \{1,2,4\}$ and $\sigma'_0 = \{1,3\}$. See also Table 5, where the critical simplices are listed by dimension.

Simplices	$|\sigma|$	$v_\varphi(\sigma)$	
$\sigma_q = 1^q 1^{\frac{d}{2}}(01^{\frac{d}{2}-1})^{k-q-2}01^{d-1}$	$n-k+q+1$	$q+1$	$0 \leq q \leq k-2$
$\sigma'_q = 1^q 1^{\frac{d}{2}}(01^{\frac{d}{2}-1})^{k-q}$	$n-k+q$	q	$0 \leq q \leq k$

TABLE 4. Description of the critical simplices for B_n, where d is even and $n = k\frac{d}{2}$.
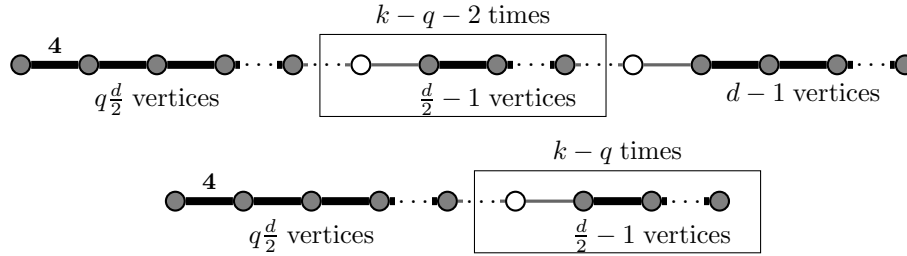
FIGURE 7. Critical simplices for B_n, where d is even and $n = k\frac{d}{2}$. Above is the simplex σ_q ($0 \leq q \leq k-2$) and below is the simplex σ'_q ($0 \leq q \leq k$).

$|\sigma|$	$v_\varphi(\sigma)$	Simplices
n	k	σ'_k
$n-1$	$k-1$	$\sigma_{k-2}, \sigma'_{k-1}$
$n-2$	$k-2$	$\sigma_{k-3}, \sigma'_{k-2}$
$\vdots$	$\vdots$	$\vdots$
$n-k+1$	1	σ_0, σ'_1
$n-k$	0	σ'_0

TABLE 5. Critical simplices for B_n by dimension, where d is even and $n = k\frac{d}{2}$.

Theorem 5.15. The critical simplices for the matching on K are those defined in Table 4. In particular the matching is always precise, and is perfect if d is odd or if d is even and $n \not\equiv 0 \pmod{\frac{d}{2}}$. In addition, if d is even and $n \equiv 0 \pmod{\frac{d}{2}}$, the incidence numbers between the critical simplices in the Morse complex are as follows:

$$[\sigma_q : \sigma_{q-1}]^{\mathcal{M}} = (-1)^{(q-1)\frac{d}{2}},$$

$$[\sigma_q : \sigma_q']^{\mathcal{M}} = (-1)^{(k-1)(\frac{d}{2}-1)+q},$$

$$[\sigma_q' : \sigma_{q-2}]^{\mathcal{M}} = (-1)^{k(\frac{d}{2}-1)+q},$$

$$[\sigma_q' : \sigma_{q-1}']^{\mathcal{M}} = (-1)^{(q-1)\frac{d}{2}}.$$

Proof. For d odd there is nothing to prove. Suppose from now on that d is even. For a given simplex $\sigma \in K$, let us consider each of the three cases that can occur in the construction of the matching.

(a) We have $q\frac{d}{2} + 1 \in \sigma$, and σ is not critical. Indeed, none of the simplices of Table 4 contains the vertex $q\frac{d}{2} + 1$.

(b) We have $q\frac{d}{2} + 1 \notin \sigma$ and $\{q\frac{d}{2}+2, \ldots, (q+1)\frac{d}{2}\} \not\subseteq \sigma$. In this case σ is critical if and only if $n = q\frac{d}{2}$. Indeed, the only simplex of this type in Table 4 is σ_k' (which occurs for $q = k$ i.e. $n = q\frac{d}{2}$).

(c) In the remaining case, we apply the matching on $K^A_{n-q\frac{d}{2}-1,\, \frac{d}{2}-1}$. By Theorem 5.8, such matching admits critical simplices if and only if

$$n - q\frac{d}{2} - 1 \equiv \tfrac{d}{2} - 1 \pmod{d} \quad \text{or} \quad n - q\frac{d}{2} - 1 \equiv -1 \pmod{d},$$

i.e. for $n \equiv 0 \pmod{\frac{d}{2}}$. Notice that if $f = \frac{d}{2} - 1$ then also $d - 2 - f = \frac{d}{2} - 1$. Therefore, again by Theorem 5.8, for $n \equiv 0 \pmod{\frac{d}{2}}$ the critical simplices are exactly the ones listed in Table 4.

For a fixed $n = k\frac{d}{2}$, the quantity $|\sigma| - v_\varphi(\sigma)$ is constant among the critical simplices and is equal to $n - k$. Thus the matching is precise.

Finally we need to prove the formulas for the incidence numbers. We are going to show that there is exactly one alternating path for each of the four pairs. Notice that, if $\sigma \to \tau$ is in the matching of K_q, then $\sigma = \tau \cup \{v\}$ with $v \equiv 1 \pmod{\frac{d}{2}}$ and $v \geq q\frac{d}{2} + 1$. In particular, if at a certain point of an alternating path one removes a vertex $v \not\equiv 1 \pmod{\frac{d}{2}}$, then that vertex will never be added again. But all the critical simplices contain every vertex $\not\equiv 1 \pmod{\frac{d}{2}}$, so any alternating path between critical simplices cannot ever drop a vertex $\not\equiv 1 \pmod{\frac{d}{2}}$. Let us now consider each of the four pairs.

- (σ_q, σ_{q-1}). We have that $\sigma_q = \sigma_{q-1} \cup \{v\}$ where $v = (q-1)\frac{d}{2} + 1$, so there is the trivial alternating path $\sigma_q \to \sigma_{q-1}$ which contributes to the incidence number $[\sigma_q : \sigma_{q-1}]^{\mathcal{M}}$ by

$$[\sigma_q : \sigma_{q-1}] = (-1)^{|\{w \in \sigma_q \mid w < v\}|} = (-1)^{(q-1)\frac{d}{2}}.$$

  Suppose by contradiction that there exists some other (non-trivial) alternating path

$$\sigma_q \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma_{q-1}, \quad m \geq 1.$$

  Let $\sigma_q = \tau_1 \cup \{v\}$. By the previous considerations, we must have $v \equiv 1 \pmod{\frac{d}{2}}$. If $v = (k-1)\frac{d}{2} + 1$ then $\tau_1 = \sigma_q'$, but this is not possible since

alternating paths stop at critical simplices. Similarly, if $v = (q-1)\frac{d}{2} + 1$ then we would stop at $\tau_1 = \sigma_{q-1}$. Therefore we must have $v \leq (q-2)\frac{d}{2} + 1$. But then $\tau_1 \in K_{q'}$ with $q' \leq q-2$, and by induction all the subsequent simplices in the alternating path must lie in

$$\bigcup_{q'' \leq q-2} K_{q''}.$$

In particular $\sigma_{q-1} \in K_{q''}$ for some $q'' \leq q-2$, but this is a contradiction since $\sigma_{q-1} \in K_{q-1}$.

- (σ_q, σ_q'). This case is similar to the previous one, except for the fact that $\sigma_q = \sigma_q' \cup \{v\}$ for $v = (k-1)\frac{d}{2} + 1$. So the only alternating path is the trivial one, which contributes to the incidence number by

$$\begin{aligned}[\sigma_q : \sigma_q'] &= (-1)^{|\{w \in \sigma_q \,|\, w < v\}|} \\ &= (-1)^{q\frac{d}{2} + (k-q-1)(\frac{d}{2}-1)} \\ &= (-1)^{(k-1)(\frac{d}{2}-1)+q}.\end{aligned}$$

- $(\sigma_q', \sigma_{q-1}')$. This case is also similar to the previous ones. Here we have $\sigma_q' = \sigma_{q-1}' \cup \{v\}$ with $v = (q-1)\frac{d}{2} + 1$, so the contribution to the incidence number due to the trivial alternating path is

$$[\sigma_q' : \sigma_{q-1}'] = (-1)^{|\{w \in \sigma_q' \,|\, w < v\}|} = (-1)^{(q-1)\frac{d}{2}}.$$

- $(\sigma_q', \sigma_{q-2})$. This case is more complicated because the only alternating path is non-trivial. Suppose to have an alternating path

$$\sigma_q' \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma_{q-2}, \quad m \geq 1$$

(m must be at least 1 because σ_{q-2} is not a face of σ_q' in K). Let $\sigma_q' = \tau_1 \cup \{v\}$, with $v \equiv 1 \pmod{\frac{d}{2}}$. If $v = (q-1)\frac{d}{2} + 1$ then $\tau_1 = \sigma_{q-1}'$ and we must stop. If $v \leq (q-3)\frac{d}{2} + 1$ we fall into some $K_{q'}$ with $q' \leq q-3$ and it is not possible to reach $\sigma_{q-2} \in K_{q-2}$. Therefore necessarily $v = (q-2)\frac{d}{2} + 1$. Then τ_1 is matched with

$$\rho_1 = \tau_1 \cup \left\{ q\frac{d}{2} + 1 \right\}.$$

Then again, $\tau_2 = \rho_1 \setminus \{v_2\}$ and the only possibility for v_2 (in order to have $\tau_2 \neq \tau_1$ and not to fall into some $K_{q'}$ with $q' \leq q-3$) is $v_2 = (q-1)\frac{d}{2} + 1$. Proceeding by induction we obtain that

$$\begin{aligned}\tau_i &= \rho_{i-1} \setminus \left\{ (q-3+i)\frac{d}{2} + 1 \right\}, \\ \rho_i &= \tau_i \cup \left\{ (q-1+i)\frac{d}{2} + 1 \right\}.\end{aligned}$$

The path stops at $\tau_{k-q+1} = \sigma_{q-2}$, and its length is $m = k - q$. See Table 6 for an example. The contribution of this path to the incidence number is

$$(-1)^{k-q}[\sigma_q' : \tau_1][\rho_1 : \tau_1] \cdots [\rho_{k-q} : \tau_{k-q}][\rho_{k-q} : \sigma_{q-2}]$$

where

$$\begin{aligned}[\rho_{i-1} : \tau_i] &= (-1)^{(q-2)\frac{d}{2} + (i-1)(\frac{d}{2}-1)}, \\ [\rho_i : \tau_i] &= (-1)^{(q-2)\frac{d}{2} + (i-1)(\frac{d}{2}-1)+d-1} \\ &= (-1)^{(q-2)\frac{d}{2} + (i-1)(\frac{d}{2}-1)+1}\end{aligned}$$

(these formulas also hold for $\rho_0 = \sigma'_q$ and $\tau_{k-q+1} = \sigma_{q-2}$). Then

$$[\sigma'_q : \sigma_{q-2}]^{\mathcal{M}} = (-1)^{k-q} \left(\prod_{i=1}^{k-q} [\rho_{i-1} : \tau_i][\rho_i : \tau_i] \right) [\rho_{k-q} : \sigma_{q-2}]$$

$$= (-1)^{k-q}(-1)^{k-q}(-1)^{(q-2)\frac{d}{2}+(k-q)(\frac{d}{2}-1)}$$

$$= (-1)^{k(\frac{d}{2}-1)+q}. \qquad \square$$

	Simplices	$v_\varphi(\sigma)$
$\sigma'_2 \quad =$		2
$\tau_1 \quad =$		1
$\rho_1 \quad =$		1
$\tau_2 \quad =$		1
$\rho_2 \quad =$		1
$\tau_3 \quad =$		1
$\rho_3 \quad =$		1
$\sigma_0 \quad =$		1

TABLE 6. The only alternating path from σ'_q to σ_{q-2} for $n = 10$, $d = 4$, $k = 5$ and $q = 2$.

Having a complete description of a precise matching on K, we can now compute the homology $H_*(\mathbf{G_W}; R)$ for $\mathbf{G_W}$ of type B_n. We recover the result of [DCPSS99].

Theorem 5.16 ([DCPSS99]). For an Artin group $\mathbf{G_W}$ of type B_n, we have

$$H_m(\mathbf{G_W}; R)_{\varphi_d} \cong \begin{cases} R/(\varphi_d) & \text{if } d \text{ is even, } n = k\frac{d}{2} \text{ and } n - k \leq m \leq n - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For a fixed $\varphi = \varphi_d$, we need to compute the boundary δ^φ_{m+1}. Assume that $n = k\frac{d}{2}$, otherwise there are no critical simplices. In top dimension $(m + 1 = n)$ we have $\mathrm{rk}\,\delta^\varphi_n = 1$, because $[\sigma'_k : \sigma'_{k-1}] \neq 0$. For $m \leq n-k-1$ the boundary δ^φ_n vanishes, because there are no critical simplices in dimension $\leq n - k - 1$. For $m = n - k$ we have $\mathrm{rk}\,\delta^\varphi_{m+1} = 1$, because $[\sigma'_1 : \sigma'_0] \neq 0$. Finally, for $n - k + 1 \leq m \leq n - 2$, we have (set $q = m - n + k$):

$$\mathrm{rk}\,\delta^\varphi_{m+1} = \mathrm{rk} \begin{pmatrix} [\sigma_q : \sigma_{q-1}] & [\sigma'_{q+1} : \sigma_{q-1}] \\ [\sigma_q : \sigma'_q] & [\sigma'_{q+1} : \sigma'_q] \end{pmatrix}$$

$$= \mathrm{rk} \begin{pmatrix} (-1)^{(q-1)\frac{d}{2}} & (-1)^{k(\frac{d}{2}-1)+q+1} \\ (-1)^{(k-1)(\frac{d}{2}-1)+q} & (-1)^{q\frac{d}{2}} \end{pmatrix} = 1,$$

because

$$\det \begin{pmatrix} (-1)^{(q-1)\frac{d}{2}} & (-1)^{k(\frac{d}{2}-1)+q+1} \\ (-1)^{(k-1)(\frac{d}{2}-1)+q} & (-1)^{q\frac{d}{2}} \end{pmatrix} = (-1)^{\frac{d}{2}} - (-1)^{\frac{d}{2}} = 0.$$

To summarize, we have: $\mathrm{rk}\,\delta^{\varphi}_{m+1} = 1$ for $n - k \leq m \leq n - 1$, and $\mathrm{rk}\,\delta^{\varphi}_{m+1} = 0$ otherwise. We conclude applying Theorem 5.1. □

5.3. **Case** $\tilde{A}_n$. Consider now the case of affine Artin groups of type $\tilde{A}_n$ (see Figure 8 for a picture of the corresponding Coxeter graph). Recall that, for affine Artin groups, the simplicial complex K consists of all the simplices $\sigma \subseteq S = \{0, \ldots, n\}$ except for the full simplex $\sigma = \{0, \ldots, n\}$. For any $\sigma \in K$, the induced subgraph $\Gamma(\sigma)$ consists only of connected components of type A_k. The weight $v_\varphi(\sigma)$ is then computed as in Section 5.1.
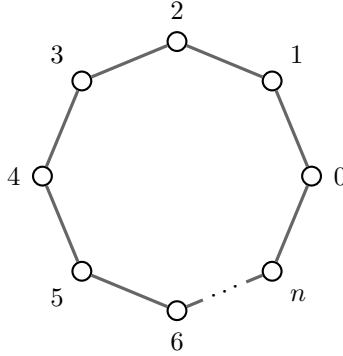
FIGURE 8. A Coxeter graph of type $\tilde{A}_n$.

For $h \in \{0, \ldots, n\}$ set

$$K_h = \{\sigma \in K \mid h \notin \sigma \text{ and } \{0, \ldots, h-1\} \subseteq \sigma\}.$$

Since every simplex $\sigma \in K$ misses at least one vertex, the subsets K_h form a partition of K:

$$K = \bigsqcup_{h=0}^{n} K_h.$$

Construct a matching on a fixed K_h as follows. Let $h = qd + r$ with $0 \leq r \leq d - 1$, and let $m = n - qd$. Consider the following clockwise relabeling of the vertices: $r - 1$ becomes 1, $r - 2$ becomes 2, ..., 0 becomes r, n becomes $r + 1$, ..., $h + 2$ becomes $m - 1$, $h + 1$ becomes m (the vertices $r, r + 1, \ldots, h$ are forgotten). Such relabeling induces a poset isomorphism

$$K_h \xrightarrow{\cong} K^A_{m,r}.$$

Then put on K_h the pull-back of the matching on $K^A_{m,r}$ defined in Section 5.1 (for $h = n$ the only simplex in K_h is critical).

Lemma 5.17. The matching defined above is an acyclic matching on K.

Proof. Consider the map $\eta\colon K \to (\mathbb{N}, \leq)$ which sends $\sigma \in K$ to $\min\{h \in \mathbb{N} \mid h \notin \sigma\}$. Then η is a poset map with fibers $\eta^{-1}(h) = K_h$. The matching on each $K_h \cong K_{m,r}^A$ is acyclic by Lemma 5.6, therefore the whole matching on K is acyclic.   □

Lemma 5.18. The matching defined above is a weighted matching on K.

Proof. Fix $h \in \{0, \ldots, n\}$. The bijection $K_h \xrightarrow{\cong} K_{m,r}^A$ is such that, if $\sigma \mapsto \hat{\sigma}$, then $v_\varphi(\sigma) = v_\varphi(\hat{\sigma}) + q$. Indeed, $\Gamma(\sigma)$ is obtained from $\Gamma(\hat{\sigma})$ by adding qd vertices to one (possibly empty) connected component, and this increases the weight by q. Then, since the matching on $K_{m,r}^A$ is weighted, its pull-back also is.   □

Case	Simplices	$|\sigma|$	$v_\varphi(\sigma)$	
$n = kd + r$	τ_q	$n - 2(k - q) + 1$	$q + 1$	$0 \leq q \leq k - 1$
$\not\equiv -1 \pmod{d}$	τ_q'	$n - 2(k - q)$	q	$0 \leq q \leq k$
	$\sigma_{q,r}$	$n - 2(k - q) + 2$	$q + 1$	$0 \leq q \leq k - 2,\, 0 \leq r \leq d - 2$
$n = kd - 1$	$\sigma_{q,r}'$	$n - 2(k - q) + 1$	q	$0 \leq q \leq k - 1,\, 0 \leq r \leq d - 2$
	$\bar{\sigma} = 1^n 0$	n	k	

Simplices
$\tau_q = 1^{qd+r} 0 1^{d-1} 0 (1^r 0 1^{d-2-r} 0)^{k-q-1}$
$\tau_q' = 1^{qd+r} 0 (1^r 0 1^{d-2-r} 0)^{k-q}$
$\sigma_{q,r} = 1^{qd+r} 0 1^{d-1} 0 (1^{d-2-r} 0 1^r 0)^{k-q-2} 1^{d-2-r} 0$
$\sigma_{q,r}' = 1^{qd+r} 0 (1^{d-2-r} 0 1^r 0)^{k-q-1} 1^{d-2-r} 0$

TABLE 7. Description of the critical simplices for $\tilde{A}_n$.

Theorem 5.19. The critical simplices for the matching on K are those listed in Table 7. The only non-trivial incidence numbers between critical cells are:

$$[\tau_q : \tau_q']^{\mathcal{M}} = \pm 1 \quad (\text{for } n = kd + r \text{ with } 0 \leq r \leq d - 2);$$
$$[\sigma_{q,r} : \sigma_{q,r}']^{\mathcal{M}} = \pm 1,$$
$$[\bar{\sigma} : \sigma_{k-1,r}']^{\mathcal{M}} = \pm 1 \quad (\text{for } n = kd - 1).$$

In particular the matching is precise.

Proof. For $n \not\equiv -1 \pmod{d}$, the matching on K_h has critical simplices only for $h \equiv n \pmod{d}$. If we write $n = kd + r$ with $0 \leq r \leq d - 2$, then $K_{qd+r} \cong K_{n-qd,\,r}^A$ has two critical simplices for $0 \leq q \leq k - 1$ and one critical simplex for $q = k$. Such simplices are those listed in Table 7.

Suppose now that $n = kd - 1$. For any $h = qd + r$, with $0 \leq r \leq d - 2$, the complex $K_{qd+r} \cong K_{n-qd,\,r}^A$ has two critical simplices because $n - qd \equiv -1 \pmod{d}$. Moreover, for $h = n$ the complex $K_n \cong K_{d-1,\,d-1}^A$ has one critical simplex. In the remaining cases the matching on K_h is perfect. Again, the critical simplices are those listed in Table 7.

Let us go back to the case $n = kd + r$, with $0 \leq r \leq d - 2$. We want to find the incidence numbers between critical simplices of consecutive dimensions. First, let us look for alternating paths from τ_q to τ_q' (for $0 \leq q \leq k - 1$). Set $h = qd + r$. Suppose to have one such path:

$$\tau_q \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \tau_q'.$$

If at some point a vertex $v \in \{0, ..., h-1\}$ is removed, then the path falls into some $K_{h'}$ with $h' < h$ and can never return in K_h. Therefore the path must be entirely contained in $K_h \cong K_{n-qd,\,r}^A$, and by Theorem 5.8 it must be the trivial path $\tau_q \rhd \tau_q'$. Thus $[\tau_q : \tau_q'] = \pm 1$.

The other pairs of critical simplices in consecutive dimensions are (τ_{q+1}', τ_q) for $0 \leq q \leq k - 1$. There is a trivial path $\tau_{q+1}' \rhd \tau_q$ which consists in removing the vertex $h = qd + r$, and contributes to the incidence number by

$$[\tau_{q+1}' : \tau_q] = (-1)^h.$$

Suppose to have some other (non-trivial) alternating path:

$$\tau_{q+1}' \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \tau_q, \quad m \geq 1.$$

Let $\rho_m = \tau_q \cup \{v\}$. If $v = h$ then $\rho_m = \tau_{q+1}'$, which is excluded. Then v must be one of the other vertices that don't belong to τ_q. They are of the form $v = n - sd$ (for $0 \leq s \leq k - q - 1$) or $v = n - (d - 1 - r) - sd$ (for $0 \leq s \leq k - q - 2$). If $v = n - (d - 1 - r) - sd$, then ρ_m is matched with $\rho_m \cup \{n - (s+1)d\}$, impossible because ρ_m must be matched with some $\zeta_m \lhd \rho_m$. Similarly, if $v = n - sd$ with $0 \leq s \leq k - q - 2$, then ρ_m is matched with $\rho_m \cup \{n - (d - 1 - r) - sd\}$ and not with some $\zeta_m \lhd \rho_m$. The only remaining possibility is $v = n - sd$ with $s = k - q - 1$, i.e. $v = n - (k - q - 1)d = (q + 1)d + r$. In this case ρ_m is matched with $\zeta_m = \rho_m \setminus \{qd + 2r + 1\}$. Going on with the same argument, we have exactly one way to continue the alternating path (from right to left), and we eventually end up with τ_{q+1}'. From left to right, the obtained alternating path is the following:

$$\zeta_1 = \tau_{q+1}' \setminus \{qd + 2r + 1\},$$
$$\rho_1 = \zeta_1 \cup \{n\},$$
$$\zeta_2 = \rho_1 \setminus \{qd + r\},$$
$$\rho_2 = \zeta_2 \cup \{n - (d - 1 - r)\},$$
$$\zeta_3 = \rho_2 \setminus \{n\},$$
$$\rho_3 = \zeta_3 \cup \{n - d\},$$
$$\zeta_4 = \rho_3 \setminus \{n - (d - 1 - r)\},$$
$$\vdots$$
$$\zeta_m = \rho_{m-1} \setminus \{(q + 1)d + 2r + 1\},$$
$$\rho_m = \zeta_m \cup \{qd + 2r + 1\},$$
$$\tau_q = \rho_m \setminus \{(q + 1)d + r\}.$$

The length of the path is $m = 2(k - q)$. Apart from $qd + r$, which is the vertex in $\tau_{q+1}' \setminus \tau_q$, the other vertices are added and removed exactly once during the path. If a certain v is added in some ρ_i and removed in some ζ_j (the removal might possibly

come before the addition), then

$$[\rho_i : \zeta_i][\rho_{j-1} : \zeta_j] = \begin{cases} 1 & \text{if } v = n, \\ -1 & \text{otherwise.} \end{cases}$$

This is true because, except for $v = n$, between the addition and the removal of v an odd number of vertices $< v$ has been added/removed. For instance, between the addition of $v = n - (d - 1 - r)$ (which occurs for $i = 2$) and its removal (which occurs for $j = 4$), one vertex $< d$ has been added (namely $n - d$) and none has been removed. Therefore this alternating path contributes to the incidence number by

$$(-1)^m \cdot (-1)^{m-1} \cdot [\zeta_2 : \rho_1] = (-1)^{qd+r+1} = (-1)^{h+1}.$$

Finally, the incidence number is given by

$$[\tau'_{q+1} : \tau_q]^{\mathcal{M}} = (-1)^h + (-1)^{h+1} = 0.$$

Consider now the case $n = kd - 1$. Consider a generic alternating path starting from $\bar{\sigma} = \{0, \ldots, n-1\}$:

$$\bar{\sigma} \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \zeta_{m+1}.$$

Let $\bar{\sigma} = \zeta_1 \cup \{v\}$. If $n - d + 1 \leq v \leq n - 1$ then $\zeta_1 = \sigma'_{k-1,r}$ for $r = n - 1 - v$. Therefore we have trivial alternating paths from $\bar{\sigma}$ to any of the $\sigma'_{k-1,r}$. If $v \leq n - d$ then $\zeta_1 \in K_h$ with $h \leq n - d = (k-1)d - 1$. None of such K_h contains critical simplices ζ_{m+1} with $|\zeta_{m+1}| = n - 1$, and the alternating path cannot return in any $K_{h'}$ with $h' \geq (k-1)d$. Thus there no other alternating paths from $\bar{\sigma}$ to critical simplices of K. Then the non-trivial incidence numbers involving $\bar{\sigma}$ are:

$$[\bar{\sigma} : \sigma'_{k-1,r}]^{\mathcal{M}} = \pm 1.$$

Let us check the alternating paths from σ_{q,r_1} to σ'_{q,r_2}. Suppose to have one:

$$\sigma_{q,r_1} \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \sigma'_{q,r_2}.$$

Let $\rho_m = \sigma'_{q,r_2} \cup \{v\}$. Adding v to σ'_{q,r_2} causes the creation of a connected component Γ_i of size $\equiv -1 \pmod{d}$. This means that ρ_m is matched with a simplex of higher dimension, obtained by adding the vertex immediately after Γ_i (clockwise), or is not matched at all (this happens for $v = (q+1)d - 1$). Therefore the alternating path must be trivial, and it occurs only for $r_1 = r_2$. Then the non-trivial incidence numbers of the form $[\sigma_{q,r_1} : \sigma'_{q,r_2}]^{\mathcal{M}}$ are:

$$[\sigma_{q,r} : \sigma'_{q,r}]^{\mathcal{M}} = \pm 1.$$

Finally we need to check the alternating paths from σ'_{q+1,r_1} to σ_{q,r_2}. Suppose to have one:

$$\sigma'_{q+1,r_1} \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \sigma_{q,r_2}.$$

As before, we work backwards. Let $\rho_m = \sigma_{q,r_2} \cup \{v\}$. Apart from the choices $v = qd + r$ and $v = (q+1)d + r$, in all other cases ρ_m has a connected component of size $\equiv -1 \pmod{d}$ and this prevents the continuation of the alternating path. For $v = qd + r$ we obtain $\rho_m = \sigma'_{q+1,r_2}$, so we have a trivial alternating path from σ'_{q+1,r_2} to σ_{q,r_2}. For $v = (q+1)d + r$, iterating the same argument, we have exactly one way to continue the alternating path (from right to left) and we end up with σ'_{q+1,r_2}. From left to right, the path is as follows (set $r = r_2$):

$$\zeta_1 = \sigma'_{q+1,r} \setminus \{(q+1)d - 1\},$$
$$\rho_1 = \zeta_1 \cup \{n\},$$

$$\zeta_2 = \rho_1 \setminus \{qd + r\},$$
$$\rho_2 = \zeta_2 \cup \{n - (d - 1 - r)\},$$
$$\zeta_3 = \rho_2 \setminus \{n\},$$
$$\rho_3 = \zeta_3 \cup \{n - d\},$$
$$\zeta_4 = \rho_3 \setminus \{n - (d - 1 - r)\},$$
$$\vdots$$
$$\zeta_m = \rho_{m-1} \setminus \{(q + 2)d - 1\},$$
$$\rho_m = \zeta_m \cup \{(q + 1)d - 1\},$$
$$\sigma_{q,r} = \rho_m \setminus \{(q + 1)d + r\}.$$

The length of the path is $m = 2(k - q) - 1$. Like what happened in the case $n = kd + r$, the contribution of this alternating path to the incidence number is given by

$$(-1)^m \cdot (-1)^{m-1} \cdot [\zeta_2 : \rho_1] = (-1)^{qd+r+1}.$$

The contribution of the trivial path $\sigma'_{q+1,r} \rhd \sigma_{q,r}$ is given by $(-1)^{qd+r}$. Therefore

$$[\sigma'_{q+1,r} : \sigma_{q,r}]^{\mathcal{M}} = (-1)^{qd+r} + (-1)^{qd+r+1} = 0. \qquad \square$$

We are now able to recover the result of [CMS08b] about the homology $H_*(\mathbf{G_W}; R)$ when $\mathbf{G_W}$ is an Artin group of type $\tilde{A}_n$.

Theorem 5.20 ([CMS08b]). For an Artin group $\mathbf{G_W}$ of type $\tilde{A}_n$, we have

$$H_m(\mathbf{G_W}; R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus d-1} & \text{if } n = kd - 1 \text{ and } m = n - 2i + 1 \ (1 \leq i \leq k), \\ R/(\varphi_d) & \text{if } n = kd + r \text{ and } m = n - 2i \ (1 \leq i \leq k), \\ 0 & \text{otherwise.} \end{cases}$$

In the second case, the condition on r is $0 \leq r \leq d - 2$.

Proof. It is a matter of applying Theorems 5.1 and 5.19. For $n = kd + r$ ($0 \leq r \leq d - 2$), the boundary map δ^φ_{m+1} has rank 1 when $m + 1 = n - 2(k - q) + 1$ (for $0 \leq q \leq k - 1$, due to τ_q); it has rank 0 otherwise. For $n = kd - 1$, the boundary map δ^φ_{m+1} has rank $d - 1$ if $m + 1 = n - 2(k - q) + 2$ (for $0 \leq q \leq k - 2$, due to the simplices $\sigma_{q,r}$) or if $m + 1 = n$ (due to $\bar{\sigma}$); it has rank 0 otherwise. $\qquad \square$

5.4. **Case $\tilde{C}_n$.** The last case we are going to consider is that of Artin groups of type $\tilde{C}_n$. The corresponding Coxeter graph Γ is shown in Figure 9. For any $\sigma \in K$, the subgraph $\Gamma(\sigma)$ of Γ splits as a union of connected components of type B_k (those containing the first or the last vertex) and of type A_k (the remaining ones).

FIGURE 9. A Coxeter graph of type $\tilde{C}_n$.

Let K_m^B be the full simplicial complex on $\{1, \ldots, m\}$, endowed with the weight function of B_m (see Section 5.2). We are going to construct a precise matching for the case $\tilde{C}_n$ using the precise matching on K_m^B.

For $h \in \{0, \ldots, n\}$ set

$$K_h = \{\sigma \in K \mid h \notin \sigma \text{ and } \{0, \ldots, h-1\} \subseteq \sigma\}.$$

Like in the $\tilde{A}_n$ case, since every simplex $\sigma \in K$ misses at least one vertex, the subsets K_h form a partition of K:

$$K = \bigsqcup_{h=0}^{n} K_h.$$

Construct a matching on a fixed K_h as follows. Ignore the vertices $0, \ldots, h$ and relabel the remaining ones from right to left: n becomes 1, $n-1$ becomes 2, $\ldots$, $h+1$ becomes $n-h$. This induces a poset isomorphism

$$K_h \xrightarrow{\cong} K_{n-h}^B.$$

Then put on K_h the pull-back of the matching on K_{n-h}^B constructed in Section 5.2 (for $h = n$ the only simplex in K_h is critical).

Remark 5.21. For d odd we are simply matching σ with $\sigma \vee n$ for all σ except $\sigma = \{0, \ldots, n-1\}$.

Lemma 5.22. The matching defined above is an acyclic matching on K.

Proof. As in the proof of Lemma 5.17, the subsets K_h are the fibers of a poset map and the matching is acyclic on each $K_h \cong K_{n-h}^B$. $\square$

Lemma 5.23. The matching defined above is a weighted matching on K.

Proof. If a simplex $\sigma \in K_h$ is sent to $\hat{\sigma}$ by the isomorphism $K_h \xrightarrow{\cong} K_{n-h}^B$, then $v_\varphi(\sigma) = v_\varphi(\hat{\sigma}) + \omega_\varphi(B_h)$. Since the matching on K_{n-h}^B is weighted, the matching on K_h also is. $\square$

As we have already said, for d odd there is exactly one critical simplex. Suppose now that d is even. Let $n = k\frac{d}{2} + r$, with $0 \le r \le \frac{d}{2} - 1$. In K_h there are critical simplices if and only if $n - h \equiv 0 \pmod{\frac{d}{2}}$, i.e. when $h = q_1\frac{d}{2} + r$ for some q_1 (with $0 \le q_1 \le k$). By Theorem 5.15 there are two families of critical simplices: σ_{q_1,q_2} (for $0 \le q_1 + q_2 \le k - 2$) and σ'_{q_1,q_2} (for $0 \le q_1 + q_2 \le k$), as shown in Table 8 and Figure 10. In a fixed dimension $m = n - k + l$ ($0 \le l \le k$) there are: $2l + 1$ critical simplices if $l \le k - 1$; l critical simplices if $l = k$. See Table 9.
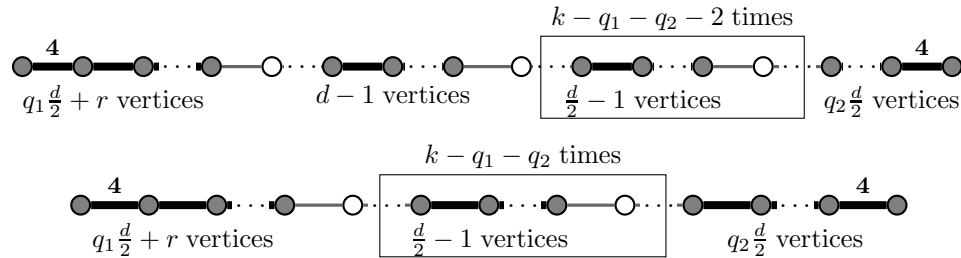

FIGURE 10. Critical simplices for $\tilde{C}_n$, with d even and $n = k\frac{d}{2} + r$. Above is the simplex σ_{q_1,q_2} and below is the simplex σ'_{q_1,q_2}.

Case	Simplices	$|\sigma|$	$v_\varphi(\sigma)$	
d odd	$\bar\sigma = 1^n 0$	n	1 or 0	
d even	σ_{q_1,q_2}	$n - k + q_1 + q_2 + 1$	$q_1 + q_2 + 1$	$0 \le q_1 + q_2 \le k - 2$
	σ'_{q_1,q_2}	$n - k + q_1 + q_2$	$q_1 + q_2$	$0 \le q_1 + q_2 \le k$

Simplices
$\sigma_{q_1,q_2} = 1^{q_1\frac{d}{2}+r} 0 1^{d-1} 0 (1^{\frac{d}{2}-1} 0)^{k-q_1-q_2-2} 1^{q_2\frac{d}{2}}$
$\sigma'_{q_1,q_2} = 1^{q_1\frac{d}{2}+r} 0 (1^{\frac{d}{2}-1} 0)^{k-q_1-q_2} 1^{q_2\frac{d}{2}}$

TABLE 8. Description of the critical simplices for $\tilde{C}_n$. When d is even, set $n = k\frac{d}{2} + r$.

Simplices	
$\sigma_{l-1,0}$	$\sigma'_{l,0}$
$\sigma_{l-2,1}$	$\sigma'_{l-1,1}$
$\vdots$	$\vdots$
$\sigma_{1,l-2}$	$\sigma'_{2,l-2}$
$\sigma_{0,l-1}$	$\sigma'_{1,l-1}$
	$\sigma'_{0,l}$

TABLE 9. Critical simplices for $\tilde{C}_n$ in dimension $m = n - k + l$ ($0 \le l \le k$), where d is even and $n = k\frac{d}{2} + r$. For $l = k$ only the second column occurs.

Theorem 5.24. The critical simplices for the matching on K are those defined in Table 8. In particular the matching is precise. Moreover the only non-trivial incidence numbers between critical simplices in the Morse complex are as follows (for d even and $n = k\frac{d}{2} + r$):

$$[\sigma_{q_1,q_2} : \sigma_{q_1, q_2-1}]^{\mathcal{M}} = (-1)^\alpha,$$
$$[\sigma_{q_1,q_2} : \sigma_{q_1-1, q_2}]^{\mathcal{M}} = (-1)^{\beta+1},$$
$$[\sigma_{q_1,q_2} : \sigma'_{q_1,q_2}]^{\mathcal{M}} = (-1)^{\beta+\frac{d}{2}+1},$$
$$[\sigma'_{q_1,q_2} : \sigma_{q_1, q_2-2}]^{\mathcal{M}} = (-1)^{\beta+1},$$
$$[\sigma'_{q_1,q_2} : \sigma_{q_1-2, q_2}]^{\mathcal{M}} = (-1)^{\beta},$$
$$[\sigma'_{q_1,q_2} : \sigma'_{q_1, q_2-1}]^{\mathcal{M}} = (-1)^{\alpha+1},$$
$$[\sigma'_{q_1,q_2} : \sigma'_{q_1-1, q_2}]^{\mathcal{M}} = (-1)^{\beta+\frac{d}{2}},$$

where $\alpha = (k - q_2)\left(\frac{d}{2} - 1\right) + q_1 + r + \frac{d}{2}$ and $\beta = q_1\frac{d}{2} + r$.

Proof. We have already proved that the critical simplices are those listed in Table 8. For d odd there is nothing more to prove, so let d be even and $n = k\frac{d}{2} + r$. Consider a generic alternating path starting from any critical cell ρ_0 and ending in a critical cell of the form σ_{q_1,q_2}:

$$\rho_0 \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma_{q_1,q_2}.$$

Let $\rho_m = \sigma_{q_1,q_2} \cup \{v\}$. If $v = q_1\frac{d}{2} + r$ then $\rho_m = \sigma'_{q_1+2,\,q_2}$ and the alternating path stops. If $v = n - q_2\frac{d}{2}$ then the alternating path stops at $\rho_m = \sigma_{q_1,\,q_2+1}$. Suppose $q_1 + q_2 \le k - 3$, otherwise there are no more cases. If $q_1\frac{d}{2} + r + d < v < n - q_2\frac{d}{2}$, then $\Gamma(\rho_m)$ has at least one connected component of size $d - 1$ and therefore ρ_m is matched with a simplex of higher dimension; thus the path stops without having reached a critical simplex. If $v = q_1\frac{d}{2} + r + d$ then ρ_m is matched with

$$\tau_m = \rho_m \setminus \left\{q_1\tfrac{d}{2} + r + \tfrac{d}{2}\right\} = 1^{q_1\frac{d}{2}+r}01^{\frac{d}{2}-1}01^{d-1}0(1^{\frac{d}{2}-1}0)^{k-q_1-q_2-3}01^{q_2\frac{d}{2}}.$$

From here we can continue in many ways. Let $\rho_{m-1} = \tau_m \cup \{w\}$.

- If $w = q_1\frac{d}{2} + r$, we end up with $\rho_{m-1} = \sigma_{q_1+1,\,q_2}$.
- If $w = q_1\frac{d}{2} + r + \frac{d}{2}$ we would be going back to ρ_m.
- If $w > q_1\frac{d}{2} + r + \frac{3}{2}d$, then $\Gamma(\rho_{m-1})$ has at least one connected component of size $d-1$ and therefore ρ_{m-1} is matched with a simplex of higher dimension.
- If $w = q_1\frac{d}{2} + r + \frac{3}{2}d$, then ρ_{m-1} is matched with

$$\tau_{m-1} = \rho_{m-1} \setminus \left\{q_1\tfrac{d}{2} + r + d\right\}$$
$$= 1^{q_1\frac{d}{2}+r}01^{\frac{d}{2}-1}01^{\frac{d}{2}-1}01^{d-1}0(1^{\frac{d}{2}-1}0)^{k-q_1-q_2-4}01^{q_2\frac{d}{2}}.$$

By induction, repeating the same argument as above, this path can be continued in exactly one way and it eventually arrives at the critical simplex $\sigma'_{q_1,\,q_2+2}$. The path has length $m = k - q_1 - q_2 - 2$ and is as follows.

$$\tau_1 = \sigma'_{q_1,\,q_2+2} \setminus \left\{n - q_2\tfrac{d}{2}\right\},$$
$$\rho_1 = \tau_1 \cup \left\{n - q_2\tfrac{d}{2} - d\right\},$$
$$\tau_2 = \rho_1 \setminus \left\{n - q_2\tfrac{d}{2} - \tfrac{d}{2}\right\},$$
$$\rho_2 = \tau_2 \cup \left\{n - q_2\tfrac{d}{2} - \tfrac{3}{2}d\right\},$$
$$\tau_3 = \rho_2 \setminus \left\{n - q_2\tfrac{d}{2} - d\right\},$$
$$\vdots$$
$$\rho_{m-1} = \tau_{m-1} \cup \left\{q_1\tfrac{d}{2} + r + d\right\},$$
$$\tau_m = \rho_{m-1} \setminus \left\{q_1\tfrac{d}{2} + r + \tfrac{3}{2}d\right\},$$
$$\rho_m = \tau_m \cup \left\{q_1\tfrac{d}{2} + r + \tfrac{d}{2}\right\},$$
$$\sigma_{q_1,q_2} = \rho_m \setminus \left\{q_1\tfrac{d}{2} + r + d\right\}.$$

Consider now a generic alternating path starting from any critical cell ρ_0 and ending in a critical cell of the form σ'_{q_1,q_2}:

$$\rho_0 \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma_{q_1,q_2}.$$

As usual, let $\rho_m = \sigma'_{q_1,q_2} \cup \{v\}$. For the same reasons as above, there are only three possibilities: $v = q_1\frac{d}{2} + r$, $v = q_1\frac{d}{2} + r + \frac{d}{2}$, $v = n - q_2\frac{d}{2}$; in all the other cases, ρ_m is matched with a simplex of higher dimension. If $v = q_1\frac{d}{2} + r$ then the path ends (to

the left) at $\rho_m = \sigma'_{q_1+1, q_2}$. If $v = q_1\frac{d}{2} + r + \frac{d}{2}$ then the path ends at σ_{q_1,q_2}. Finally, if $v = n - q_2\frac{d}{2}$ then the path ends at σ'_{q_1, q_2+1}.

We have seven families of incidence numbers to compute, each coming from one of the alternating paths we have found.

- From σ'_{q_1+2, q_2} to σ_{q_1,q_2}. The alternating path is trivial and consists in removing the vertex $v = q_1\frac{d}{2} + r$, so

$$[\sigma'_{q_1+2, q_2} : \sigma_{q_1,q_2}]^{\mathcal{M}} = (-1)^{|\{w \in \sigma_{q_1,q_2} | w < v\}|} = (-1)^{q_1\frac{d}{2}+r},$$
$$\implies [\sigma'_{q_1,q_2} : \sigma_{q_1-2, q_2}]^{\mathcal{M}} = (-1)^{q_1\frac{d}{2}+r} = (-1)^{\beta}.$$

- From σ_{q_1, q_2+1} to σ_{q_1,q_2}. Again the path is trivial, and it consists in removing $v = n - q_2\frac{d}{2}$. Therefore

$$[\sigma_{q_1, q_2+1} : \sigma_{q_1,q_2}]^{\mathcal{M}} = (-1)^{|\{w \in \sigma_{q_1,q_2} | w < v\}|}$$
$$= (-1)^{n - q_2\frac{d}{2} - (k - q_1 - q_2 - 1)}$$
$$= (-1)^{k\frac{d}{2}+r - q_2\frac{d}{2} - k + q_1 + q_2 + 1}$$
$$= (-1)^{(k-q_2)\left(\frac{d}{2}-1\right) + r + q_1 + 1},$$
$$\implies [\sigma_{q_1,q_2} : \sigma_{q_1, q_2-1}]^{\mathcal{M}} = (-1)^{(k-q_2+1)\left(\frac{d}{2}-1\right) + r + q_1 + 1}$$
$$= (-1)^{\alpha}.$$

- From σ_{q_1+1, q_2} to σ_{q_1,q_2}. The path consists in removing $q_1\frac{d}{2} + r$, adding $q_1\frac{d}{2} + r + \frac{d}{2}$, and removing $q_1\frac{d}{2} + r + d$. Therefore

$$[\sigma_{q_1+1, q_2} : \sigma_{q_1,q_2}]^{\mathcal{M}} = (-1)(-1)^{q_1\frac{d}{2}+r}(-1)^{q_1\frac{d}{2}+r+\frac{d}{2}-1}(-1)^{q_1\frac{d}{2}+r+d-1}$$
$$= (-1)^{q_1\frac{d}{2}+r+\frac{d}{2}+1},$$
$$\implies [\sigma_{q_1,q_2} : \sigma_{q_1-1, q_2}]^{\mathcal{M}} = (-1)^{(q_1-1)\frac{d}{2}+r+\frac{d}{2}+1}$$
$$= (-1)^{\beta+1}.$$

- From σ'_{q_1, q_2+2} to σ_{q_1,q_2}. The path is the one we explicitly wrote before. Notice that from ρ_{i-1} to τ_i one removes the vertex $v_i = n - (q_2 - i + 1)\frac{d}{2}$, and from τ_i to ρ_i one adds the vertex $v'_i = n - (q_2 - i - 1)\frac{d}{2}$. Therefore

$$[\rho_{i-1} : \tau_i][\rho_i : \tau_i] = (-1)^{|\{w \in \tau_i | v'_i < w < v_i\}|} = (-1)^{d-1} = -1.$$

Then we can compute the incidence number in the Morse complex:

$$[\sigma'_{q_1, q_2+2} : \sigma_{q_1,q_2}]^{\mathcal{M}} = (-1)^m \left(\prod_{i=1}^{m} [\rho_{i-1} : \tau_i][\rho_i : \tau_i]\right) [\sigma_{q_1,q_2} : \rho_m]$$
$$= (-1)^m (-1)^m (-1)^{q_1\frac{d}{2}+r+(d-1)}$$
$$= (-1)^{q_1\frac{d}{2}+r+1},$$
$$\implies [\sigma'_{q_1,q_2} : \sigma_{q_1, q_2-2}]^{\mathcal{M}} = (-1)^{q_1\frac{d}{2}+r+1}$$
$$= (-1)^{\beta+1}.$$

- From $\sigma'_{q_1+1,\,q_2}$ to σ'_{q_1,q_2}. The path is trivial and consists in removing $v = q_1\frac{d}{2} + r$. Then

$$[\sigma'_{q_1+1,\,q_2} : \sigma'_{q_1,q_2}]^{\mathcal{M}} = (-1)^{q_1\frac{d}{2}+r},$$
$$\implies [\sigma'_{q_1,q_2} : \sigma'_{q_1-1,\,q_2}]^{\mathcal{M}} = (-1)^{(q_1-1)\frac{d}{2}+r} = (-1)^{\beta+\frac{d}{2}}.$$

- From σ_{q_1,q_2} to σ'_{q_1,q_2}. The path is trivial and consists in removing $v = q_1\frac{d}{2} + r + \frac{d}{2}$. Then

$$[\sigma_{q_1,\,q_2} : \sigma'_{q_1,q_2}]^{\mathcal{M}} = (-1)^{q_1\frac{d}{2}+r+\frac{d}{2}-1} = (-1)^{\beta+\frac{d}{2}+1}.$$

- From $\sigma'_{q_1,\,q_2+1}$ to σ'_{q_1,q_2}. The path is trivial and consists in removing $v = n - q_2\frac{d}{2}$. Then

$$[\sigma'_{q_1,\,q_2+1} : \sigma'_{q_1,q_2}]^{\mathcal{M}} = (-1)^{n-q_2\frac{d}{2}-(k-q_1-q_2)}$$
$$= (-1)^{k\frac{d}{2}+r-q_2\frac{d}{2}-k+q_1+q_2}$$
$$= (-1)^{(k-q_2)\left(\frac{d}{2}-1\right)+r+q_1},$$
$$\implies [\sigma'_{q_1,q_2} : \sigma'_{q_1,\,q_2-1}]^{\mathcal{M}} = (-1)^{(k-q_2+1)\left(\frac{d}{2}-1\right)+r+q_1}$$
$$= (-1)^{\alpha+1}. \qquad \square$$

We can finally compute the homology $H_*(\mathbf{G_W}; R)$ for Artin groups of type $\tilde{C}_n$.

Theorem 5.25. Let $\mathbf{G_W}$ be an Artin group of type $\tilde{C}_n$. Then the φ_d-primary component of $H_*(\mathbf{G_W}; R)$ is trivial for d odd, and for d even is as follows (let $n = k\frac{d}{2} + r$):

$$H_m(\mathbf{G_W}; R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus m+k-n+1} & \text{if } n-k \leq m \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In order to apply Theorem 5.1 we need to find the rank the boundary maps δ^{φ}_{m+1} of the Morse complex. For d odd there is only one critical cell, thus all the boundaries vanish. Suppose from now on that d is even, and let $n = k\frac{d}{2} + r$. In order to have a non-trivial boundary δ^{φ}_{m+1} we must have at least one critical simplex both in dimension m and in dimension $m+1$, thus $m = n-k+l$ with $0 \leq l \leq k-1$ (see Table 9).

We are going to prove that, for $l \leq k-2$, a basis for the image of $\delta = \delta^{\varphi}_{m+1}$ is given by

$$\mathcal{B} = \{\delta\sigma_{q_1,q_2} \mid q_1 + q_2 = l\}.$$

By Theorem 5.24 we obtain the following formula for $\delta\sigma_{q_1,q_2}$ (whenever a simplex does not exist, the corresponding term simply vanishes):

$$\delta\sigma_{q_1,q_2} = (-1)^{\alpha}\,\sigma_{q_1,\,q_2-1} + (-1)^{\beta+1}\,\sigma_{q_1-1,\,q_2} + (-1)^{\beta+\frac{d}{2}+1}\,\sigma'_{q_1,q_2},$$

where $\alpha = (k-q_2)\left(\frac{d}{2}-1\right) + q_1 + r + \frac{d}{2}$ and $\beta = q_1\frac{d}{2} + r$. The term σ'_{q_1,q_2} appears in $\delta\sigma_{q_1,q_2}$ but not in any other element of $\mathcal{B}$, thus $\mathcal{B}$ is made of linearly independent vectors. Moreover, if $q_1 + q_2 = l+1$, we have that

$$\delta\sigma'_{q_1,q_2} = (-1)^{\beta+1}\sigma_{q_1,\,q_2-2} + (-1)^{\beta}\sigma_{q_1-2,\,q_2} + (-1)^{\alpha+1}\sigma'_{q_1,\,q_2-1} + (-1)^{\beta+\frac{d}{2}}\sigma'_{q_1-1,\,q_2}$$
$$= (-1)^{\alpha+\beta+\frac{d}{2}}\left((-1)^{\alpha+\frac{d}{2}+1}\,\sigma_{q_1,\,q_2-2} + (-1)^{\beta+1}\,\sigma_{q_1-1,\,q_2-1} + (-1)^{\beta+\frac{d}{2}+1}\,\sigma'_{q_1,\,q_2-1}\right)$$

$$+ (-1)^{\frac{d}{2}+1} \left((-1)^{\alpha+1} \, \sigma_{q_1-1, \, q_2-1} + (-1)^{\beta+\frac{d}{2}+1} \, \sigma_{q_1-2, \, q_2} + (-1)^{\beta+1} \, \sigma'_{q_1-1, \, q_2} \right)$$

$$= (-1)^{\alpha+\beta+\frac{d}{2}} \, \delta\sigma_{q_1, \, q_2-1} + (-1)^{\frac{d}{2}+1} \, \delta\sigma_{q_1-1, \, q_2}.$$

Therefore $\mathcal{B}$ generates the image of δ^{φ}_{m+1}. Thus $\operatorname{rk} \delta^{\varphi}_{m+1} = |\mathcal{B}| = l+1$ for $l \leq k-2$.

For $l = k-1$, i.e. $m = n-1$, the situation is a bit different because there are no critical simplices of the form σ_{q_1, q_2} with $q_1 + q_2 = l$. However we can still define

$$\epsilon_{q_1, q_2} = (-1)^{\alpha} \, \sigma_{q_1, \, q_2-1} + (-1)^{\beta+1} \, \sigma_{q_1-1, \, q_2} + (-1)^{\beta+\frac{d}{2}+1} \, \sigma'_{q_1, q_2}$$

for $q_1 + q_2 = l$, and

$$\mathcal{B} = \left\{ \epsilon_{q_1, q_2} \mid q_1 + q_2 = l \right\}.$$

As above we have that, for $q_1 + q_2 = l+1$,

$$\delta\sigma'_{q_1, q_2} = (-1)^{\alpha+\beta+\frac{d}{2}} \, \epsilon_{q_1, \, q_2-1} + (-1)^{\frac{d}{2}+1} \, \epsilon_{q_1-1, \, q_2}.$$

Then, for the same reasons as above, $\mathcal{B}$ is a basis for the image of δ^{φ}_n.

We have proved that, for $0 \leq l \leq k-1$, the rank of δ^{φ}_{m+1} is equal to $l+1 = m+k-n+1$. Then we conclude applying Theorem 5.1. $\qquad\square$

Remark 5.26. Future works will focus on other families of Artin groups. In particular it seems that precise matchings can be constructed in all finite and affine cases (see [Pao17]), possibly allowing explicit homology computations.

The methods developed in this paper are particularly powerful when the coefficients are over a PID, but Artin groups with roots of different lengths (e.g. B_n, $\tilde{B}_n$ and $\tilde{C}_n$) also admit natural representations over polynomial rings with more than one variable. We believe that some of our theory can be extended to such cases, and this can also be the aim of future works.

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