Ordered multiplicity inverse eigenvalue problem for graphs on six vertices

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Abstract

For a graph G, we associate a family of real symmetric matrices, $\mathcal{S}(G)$, where for any $M \in \mathcal{S}(G)$, the location of the nonzero off-diagonal entries of M are governed by the adjacency structure of G. The ordered multiplicity Inverse Eigenvalue Problem of a Graph (IEPG) is concerned with finding all attainable ordered lists of eigenvalue multiplicities for matrices in $\mathcal{S}(G)$.

For connected graphs of order six, we offer significant progress on the IEPG, as well as a complete solution to the ordered multiplicity IEPG. We also show that while $K_{m,n}$ with $\min(m,n) \geq 3$ attains a particular ordered multiplicity list, it cannot do so with arbitrary spectrum.

1 Introduction

A graph G consists of a vertex set V(G) and an edge set E(G). Given G with vertices v_1, \ldots, v_n , a real symmetric matrix M is in S(G) if for all $i \neq j$, $M_{i,j} = 0$ if and only if $v_i v_j \notin E(G)$; there are no restrictions on the diagonal entries.

The spectrum of a matrix M is the set of eigenvalues of M. Let $\lambda_1 < \cdots < \lambda_k$ be the distinct eigenvalues of M in increasing order, and let γ_i be the multiplicity of λ_i as an eigenvalue of M. Then the ordered multiplicity list of M is $(\gamma_1, \ldots, \gamma_k)$. (With this convention, the spectrum of M is $\{\lambda_1^{(\gamma_1)}, \lambda_2^{(\gamma_2)}, \ldots, \lambda_k^{(\gamma_k)}\}$.)

The Inverse Eigenvalue Problem of a Graph (IEPG) is stated as follows: given G and a set of numbers $L = \{\ell_1, \ldots, \ell_n\}$, does there exist a matrix $M \in \mathcal{S}(G)$ with spectrum L?

This problem has been completely resolved through graphs on five vertices (see [5]). More information about the IEPG can be found in the survey of Hogben [12]. Because

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of the difficulty of the IEPG, many relaxations have been considered; previous works have examined inverse inertia (see [7]), minimum rank and maximum nullity (see [9]), and the minimum number of distinct eigenvalues (see [8]).

We consider the ordered multiplicity inverse eigenvalue problem for graphs, a slight relaxation of the IEPG: given a graph G and an ordered list of integers $\Gamma = (\gamma_1, \ldots, \gamma_k)$, does there exist a matrix $M \in \mathcal{S}(G)$ that attains Γ as its ordered multiplicity list?

We label graphs using the Atlas of Graphs [13]; these graphs are also reproduced in the appendix for reference. In this paper, we solve the ordered multiplicity IEPG for all connected graphs of order six. The result is summarized in Figure 1: the graphs are in 26 different equivalence classes based on what ordered multiplicity lists are attainable. To determine what lists are attainable, locate the equivalence class it belongs in and then read off all ordered multiplicity lists (and reversals) on the edges of a directed path from that equivalence class to \emptyset . This diagram also gives all possible relationships between equivalence classes, namely the equivalence class containing G attains all ordered multiplicity lists as the equivalence class containing G to the one containing G.

We say a graph G is spectrally arbitrary for an ordered multiplicity list $(\gamma_1, \ldots, \gamma_k)$ if for any $\lambda_1 < \cdots < \lambda_k$, there is a matrix $M \in \mathcal{S}(G)$ with spectrum $\{\lambda_1^{(\gamma_1)}, \ldots, \lambda_k^{(\gamma_k)}\}$. Many of the techniques we use show that a graph is spectrally arbitrary for an ordered multiplicity list. In the appendix, for each ordered multiplicity list, we give all graphs which can attain that list, indicating those which are not known to be attainable with arbitrary spectrum.

We proceed as follows. In Section 2 we will review what is known for the IEPG for graphs on five or fewer vertices which we will build on for the case of six vertices. In Section 3 we will introduce a technique we call cloning and how it connects with ordered multiplicity lists of eigenvalues. In Section 4 we justify what ordered multiplicity lists are unattainable, while in Section 5 we justify what ordered multiplicity lists are attainable for graphs on six vertices. In Section 6 we show that $K_{m,n}$ with $\min(m,n) \geq 3$ is a graph for which the ordered multiplicity IEPG differs from the IEPG. Finally, in Section 7, we give concluding remarks.

Because we are working on graphs with six or fewer vertices, it will be unambiguous to write the ordered multiplicity list $(\gamma_1, \ldots, \gamma_k)$ as $\gamma_1 \ldots \gamma_k$. We will say a graph G attains $\gamma_1 \ldots \gamma_k$ if there is some $M \in \mathcal{S}(G)$ with multiplicity list $\gamma_1 \ldots \gamma_k$; similarly, G does not attain $\gamma_1 \ldots \gamma_k$ if there is no $M \in \mathcal{S}(G)$ with multiplicity list $\gamma_1 \ldots \gamma_k$. We note that a graph attains $\gamma_1 \ldots \gamma_k$ if and only if it attains $\gamma_k \ldots \gamma_1$, which follows by noting if $M \in \mathcal{S}(G)$ then so also is -M.

2 IEPG for graphs on five or fewer vertices

The IEPG for all graphs of order at most five was recently solved by Barrett et al. [5]. They showed that for graphs with five or fewer vertices, the IEPG is equivalent to the ordered multiplicity IEPG. Thus, a graph on five or fewer vertices attains a given spectrum if and only if the corresponding multiplicity list is attainable. Two of the main tools used to solve this problem were the Strong Spectral Property (SSP) and the Strong Multiplicity Property (SMP), introduced in an earlier paper (see [6]).

Definition 1. An $n \times n$ symmetric matrix A has the Strong Spectral Property (SSP) if the

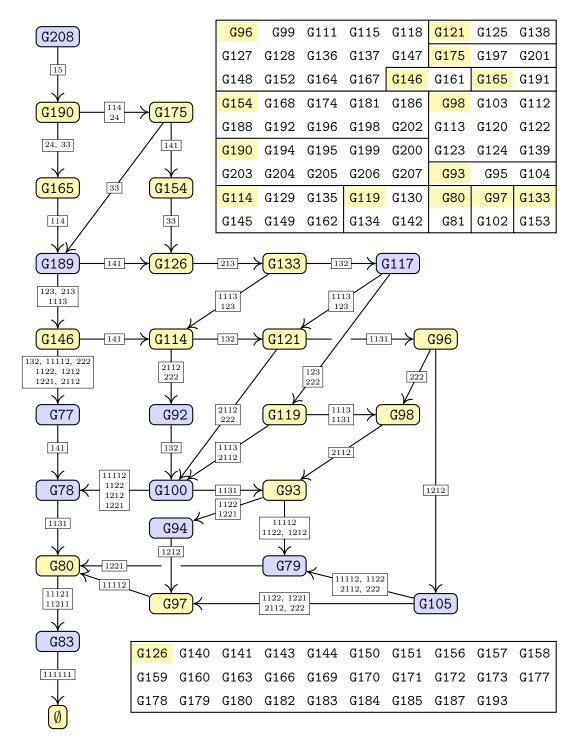


Figure 1: There are 26 equivalence classes; those in blue have one graph while those in yellow have their full membership given in the boxes. To determine attainable ordered multiplicity lists for a graph, find its equivalence class in the diagram and take any path to \emptyset ; the multiplicity lists (and reversals) that occur on the edges of the path are the only ones attainable. The graph G attains all multiplicity lists that H attains if and only if there is a directed path from the class containing G to the class containing H; the difference in what is attainable are the multiplicity lists which occur on any directed path between them.

only symmetric matrix X satisfying $A \circ X = I \circ X = AX - XA = O$ is X = O (where " \circ " indicates the Hadamard, or entry-wise, product of matrices).

Definition 2. An $n \times n$ symmetric matrix A satisfies the *Strong Multiplicity Property (SMP)* if the only symmetric matrix X satisfying $A \circ X = I \circ X = AX - XA = O$ and $\operatorname{tr}(A^iX) = 0$ for $i = 2, \ldots, n-1$ is X = O.

These properties are important for the IEPG because SSP (SMP) allows us to determine the attainability of certain spectra (ordered multiplicity lists) for many graphs simultaneously. It should be noted that testing if a matrix has SSP (SMP) reduces to showing if a large linear system has full rank. This has been implemented and is available online (see [11]); any matrix which is claimed to have SSP follows either from Barrett et al. [5] or by using the online implementation.

Theorem 1 (Barrett et al. [6] Theorems 2.10 and 2.20). If $A \in \mathcal{S}(G)$ has SSP (SMP), then every supergraph of G with the same vertex set has an SSP (SMP) realization with the same spectrum (ordered multiplicity list).

Theorem 2 (Barrett et al. [6], Theorem 3.8). If $A \in \mathcal{S}(G)$ and $B \in \mathcal{S}(H)$ both have SSP (SMP) and $\operatorname{spec}(A) \cap \operatorname{spec}(B) = \emptyset$, then $A \oplus B \in \mathcal{S}(G \cup H)$ has SSP (SMP).

Using the above theorems and several constructions, Barrett et al. [5] determined that the multiplicity lists given in Table 1 are attainable with SSP, and those in Table 2 are attainable but without SMP or SSP. Moreover, they established that the graphs can attain any spectrum compatible with the ordered multiplicity lists it attains. Thus, the IEPG and the ordered multiplicity IEPG are equivalent for graphs on five or fewer vertices.

3 Cloning vertices and ordered multiplicity lists

In this section, we introduce *cloning*, a graph operation that, given G and $v \in V(G)$, constructs a new graph H by adding a new vertex v' which is a twin of v. This operation is sometimes referred to as duplicating or blow-ups.

Definition 3. Two vertices u and w are twins in H if $N_H(u) \setminus \{w\} = N_H(w) \setminus \{u\}$, where $N_H(v)$ is the set of neighbors of v (i.e., vertices which share an edge with v).

Twins do not need to be adjacent. This leads to two variants of cloning: cloning v with an edge requires that $v \sim v'$ (i.e., v and v' are adjacent), while cloning without an edge requires $v \nsim v'$ (i.e., v and v' are not adjacent).

Theorem 3. Let G be a graph with $M \in \mathcal{S}(G)$ having multiplicity list $(\gamma_1, \ldots, \gamma_i, \ldots, \gamma_k)$ where the eigenvalue 0 has multiplicity γ_i . Then the following two cases are possible:

1. If the diagonal entry of M corresponding to v_j is zero, then the graph H attained from G by cloning v_j without an edge has a matrix $N \in \mathcal{S}(H)$ that attains the multiplicity list $(\gamma_1, \ldots, \gamma_i + 1, \ldots, \gamma_k)$.

Graphs	Attainable Ordered multiplicity lists
G1	1
G3	11
G6	111
G7	111, 12, 21
G14	1111
G13	1111, 121
G15	1111, 121, 112, 211
G16, G17	1111, 121, 112, 211, 22
G18	1111, 121, 112, 211, 22, 13, 31
G31	11111
G29, G30	11111, 1121, 1211
G35, G36	11111, 1121, 1211, 1112, 2111
G34, G38	11111, 1121, 1211, 1112, 2111, 122, 221
G37, G40, G41	11111, 1121, 1211, 1112, 2111, 122, 221, 212
G42, G43, G47	
G44, G46	11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131
G45	11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311
G48, G49	11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32
G50, G51	
G52	11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32, 14, 41

Table 1: Realizable ordered multiplicity lists for connected graphs with five or fewer vertices and are attainable with SSP.

Graphs	Attainable Ordered multiplicity lists
G29	131
G42	113, 131, 311

Table 2: Realizable ordered multiplicity lists for connected graphs with five or fewer vertices which cannot have SMP or SSP.

2. If the diagonal entry of M corresponding to v_j is nonzero, then the graph H attained from G by cloning v_j with an edge has a matrix $N \in \mathcal{S}(H)$ that attains the multiplicity list $(\gamma_1, \ldots, \gamma_i + 1, \ldots, \gamma_k)$.

Proof. Let $-\lambda_1 \leq \cdots \leq \lambda_{n-\gamma_i}$ be the nonzero eigenvalues of M where a is the last index such that $-\lambda_a < 0$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_{n-\gamma_i}$ be the corresponding orthonormal eigenvectors. Then

$$M = \sum_{k=1}^{n-\gamma_i} \lambda_k \mathbf{x}_k \mathbf{x}_k^T$$

$$= (\mathbf{x}_1 \cdots \mathbf{x}_{n-\gamma_i}) \begin{pmatrix} -\lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n-\gamma_i} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_{n-\gamma_i}^T \end{pmatrix}$$

$$= \left(\sqrt{\lambda_1}\mathbf{x}_1 \quad \cdots \quad \sqrt{\lambda_{n-\gamma_i}}\mathbf{x}_{n-\gamma_i}\right) \underbrace{\begin{pmatrix} -I_a & O \\ O & I_{n-\gamma_i-a} \end{pmatrix}}_{=S} \begin{pmatrix} \sqrt{\lambda_1}\mathbf{x}_1^T \\ \vdots \\ \sqrt{\lambda_{n-\gamma_i}}\mathbf{x}_{n-\gamma_i}^T \end{pmatrix}.$$

The columns of

$$Y = \begin{pmatrix} \sqrt{\lambda_1} \mathbf{x}_1^T \\ \vdots \\ \sqrt{\lambda_{n-\gamma_i}} \mathbf{x}_{n-\gamma_i}^T \end{pmatrix}$$

are an orthogonal representation of G with respect to the indefinite inner product S. That is, if \mathbf{y}_k denotes the k-th column then $\mathbf{y}_k^T S \mathbf{y}_\ell = 0$ if and only if $v_k \nsim v_\ell$ in G.

Let Z be a $(n - \gamma_i) \times (n + 1)$ matrix with columns as follows,

- $\mathbf{z}_k = \mathbf{y}_k$ for $1 \le k < j$;
- $\mathbf{z}_j = \mathbf{z}_{j+1} = \frac{1}{\sqrt{2}} \mathbf{y}_j;$
- $\mathbf{z}_k = \mathbf{y}_{k-1}$ for $j + 1 < k \le n + 1$.

Now consider the matrix $N = Z^T S Z$. Since N is real symmetric, there exists some graph H such that $N \in \mathcal{S}(H)$. The following two observations will now conclude the proof.

First, use the columns of Z as an orthogonal representation for H with respect to S. This corresponds to the graph G with the vertex v_j cloned (i.e., columns still have the same orthogonality relationships as given by Y). This will have cloned with an edge if and only if $\mathbf{y}_j^T S \mathbf{y}_j \neq 0$. The latter holds if and only if the diagonal entry of M corresponding to v_j is nonzero.

Second, the inner product of any two rows of Z agree with the inner product of the corresponding rows of Y. So the nonzero eigenvalues of N are

$$SZZ^{T} = \begin{pmatrix} -I_{a} & O \\ O & I_{n-\gamma_{i}-a} \end{pmatrix} \begin{pmatrix} \lambda_{1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n-\gamma_{i}} \end{pmatrix} = \begin{pmatrix} -\lambda_{1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n-\gamma_{i}} \end{pmatrix},$$

which are the same as those of M. Hence N has the same spectrum of M with the addition of a single eigenvalue of 0, giving us the desired ordered multiplicity list.

Corollary 4. Let G be a graph without isolated vertices, and let $M \in \mathcal{S}(G)$ with multiplicity list (m_1, \ldots, m_k) . If a graph H is attained by cloning $v \in V(G)$ with an edge, then H attains the multiplicity list $(m_1 + 1, \ldots, m_k)$.

Proof. By translation we can assume M is positive semi-definite with nullity m_1 . If any diagonal entry were 0, this would force a row and column of zeroes. However, this implies that G contains an isolated vertex, a contradiction. Thus, the entries of the diagonal are nonzero, and so we apply the previous theorem by cloning with an edge.

4 Unattainable multiplicity lists for graphs

In this section we will determine which ordered multiplicity lists are unattainable for connected graphs on six vertices.

4.1 Using known graph parameters

Since we can assign any particular eigenvalue to 0 by translation, we have the following observations.

Observation 1. If M(G) denotes the maximum nullity of a matrix in S(G), then all entries of a multiplicity list of a matrix in S(G) are bounded above by M(G).

Observation 2. If $M_+(G)$ denotes the maximum nullity of a positive semidefinite matrix in S(G), then the first (and by reversal from negation, the last) entry of a multiplicity list of a matrix in S(G) are bounded above by $M_+(G)$.

In general, the computation of M(G) and $M_+(G)$ is an open problem. However, for graphs on seven or fewer vertices, known techniques can find find these these values (in particular, the inertia tables—see [4]). It suffices to provide an upper bound for these parameters, which can be done through the combinatorial parameters Z(G) and $Z_+(G)$, respectively known as the zero-forcing number and semidefinite zero-forcing number of a graph. Because of their combinatorial nature, Z(G) and $Z_+(G)$ can be easily computed for small graphs through exhaustive analysis. The definition of these parameters, as well as related extensions and results, can be found in the survey of Fallat and Hogben [9]. For our purposes, we will use the following result.

Lemma 5 (AIM [1], Prop. 2.4). For any graph G, we have $M(G) \leq Z(G)$.

Lemma 6 (Barioli et al. [2], Theorem 3.5). For any graph G, we have $M_+(G) \leq Z_+(G)$.

Another useful parameter is q(G), the minimum number of distinct eigenvalues.

Observation 3. The length of any ordered multiplicity list for $M \in \mathcal{S}(G)$ is at least q(G).

This is a harder parameter to compute; for connected graphs of order at most six, q(G) was recently determined (see [8]). For all connected graphs of order six, the parameters Z(G), $Z_{+}(G)$, and q(G) are given in Table 3 and rule out many ordered multiplicity lists.

4.2 Previous results to rule out ordered multiplicity lists

The following two results, both from [5], rule out several cases.

Lemma 7 (Barrett et al. [5], Lemma 3.3). If G is a connected unicyclic graph with odd girth, then at least one of the first or last eigenvalues has multiplicity one.

This rules out 2112 and 222 for G92, G93, G94, G95, G100, and G104.

Lemma 8 (Barrett et al. [5], Lemmas 2.3 and 5.2). A generalized star or a generalized 3-sun does not allow an ordered multiplicity list with consecutive multiple eigenvalues.

This rules out 1122, 1221, and 2211 for G77 and G78 (generalized stars) and G94 (3-sun). The inverse eigenvalue problem for cycles has been determined by Fernandes and Fonseca [10], and in particular it follows that 1212 and 2121 are not attainable for the graph C_6 (G105).

G	Z	Z_{+}	q	G	Z	Z_{+}	q		G	Z	Z_{+}	q	G	Z	Z_{+}	q
G77	4	1	3	G121	3	2	3	•	G151	3	3	3	G181	3	3	2
G78	3	1	4	G122	2	2	4		G152	2	2	3	G182	3	3	3
G79	2	1	4	G123	2	2	4		G153	3	3	3	G183	3	3	3
G80	2	1	5	G124	2	2	4		G154	3	3	2	G184	3	3	3
G81	2	1	5	G125	3	2	3		G156	3	3	3	G185	3	3	3
G83	1	1	6	G126	3	3	3		G157	3	3	3	G186	3	3	2
G92	3	2	3	G127	2	2	3		G158	3	3	3	G187	3	3	3
G93	2	2	4	G128	2	2	3		G159	3	3	3	G188	3	3	2
G94	2	2	4	G129	3	2	3		G160	3	3	3	G189	4	3	3
G95	2	2	4	G130	3	3	4		G161	4	2	3	G190	4	4	2
G96	2	2	3	G133	3	3	3		G162	3	2	3	G191	4	4	3
G97	2	2	5	G134	3	3	4		G163	3	3	3	G192	3	3	2
G98	2	2	4	G135	3	2	3		G164	2	2	3	G193	3	3	3
G99	2	2	3	G136	2	2	3		G165	4	4	3	G194	4	4	2
G100	3	2	4	G137	2	2	3		G166	3	3	3	G195	4	4	2
G102	2	2	5	G138	3	2	3		G167	2	2	3	G196	3	3	2
G103	2	2	4	G139	2	2	4		G168	3	3	2	G197	4	3	2
G104	2	2	4	G140	3	3	3		G169	3	3	3	G198	3	3	2
G105	2	2	3	G141	3	3	3		G170	3	3	3	G199	4	4	2
G111	2	2	3	G142	3	3	4		G171	3	3	3	G200	4	4	2
G112	2	2	4	G143	3	3	3		G172	3	3	3	G201	4	3	2
G113	2	2	4	G144	3	3	3		G173	3	3	3	G202	3	3	2
G114	3	2	3	G145	3	2	3		G174	3	3	2	G203	4	4	2
G115	2	2	3	G146	4	2	3		G175	4	3	2	G204	4	4	2
G117	3	3	3	G147	2	2	3		G177	3	3	3	G205	4	4	2
G118	2	2	3	G148	2	2	3		G178	3	3	3	G206	4	4	2
G119	3	3	4	G149	3	2	3		G179	3	3	3	G207	4	4	2
G120	2	2	4	G150	3	3	3		G180	3	3	3	G208	5	5	2

Table 3: The values for Z(G), $Z_{+}(G)$, and q(G) for connected graphs on six vertices.

4.3 Remaining Cases

After applying the graph parameters, Lemma 7, and Lemma 8, the remaining unattainable cases are shown in Figure 2.

Proving that these cases are unattainable will be done by contradiction; namely, by assuming the existence of a matrix that achieves the specified ordered multiplicity list on a graph. An examination of the corresponding orthogonal representation allows us to argue that two of the vectors are scalar multiples. This pair of vectors allows us to "declone" the original graph to a smaller graph with a corresponding matrix that has an impossible ordered multiplicity list.

Recall that cloning works by taking a single vertex and creating a pair of vertices where the vector corresponding with each vertex is a scalar multiple of the original. Decloning does

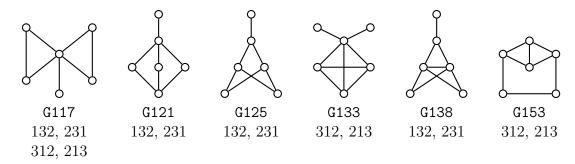


Figure 2: The remaining unattainable ordered multiplicity lists.

the opposite: if we can show that the vectors corresponding to a pair of vertices must be scalar multiples, then the two vertices must be twins. We can then delete one of the twins and decrease an entry in the ordered multiplicity list.

Lemma 9 (Decloning). Let G be a graph with $M = Q^T SQ \in \mathcal{S}(G)$ having multiplicity list $(\gamma_1, \ldots, \gamma_k)$ where the eigenvalue 0 has multiplicity γ_i . Further assume that S is symmetric and has dimension $n - \gamma_i$, where n is the order of G. If the columns of Q corresponding to vertices a and b are scalar multiples, then there exists $N \in \mathcal{S}(H)$ where H is attained from G by deleting vertex a and N has ordered multiplicity list $(\gamma_1, \ldots, \gamma_i - 1, \ldots, \gamma_k)$.

Proof. Let $Q = (\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_n)$, and without loss of generality assume that x_1 and x_2 are scalar multiples, i.e., $\mathbf{x}_2 = \alpha \mathbf{x}_1$. Let $\widehat{Q} = (\sqrt{1 + \alpha^2} \mathbf{x}_2 \cdots \mathbf{x}_n)$; we claim $N = \widehat{Q}^T S \widehat{Q}$.

Note that two vertices u and v in the graph are not adjacent if and only if $\mathbf{x}_u^T S \mathbf{x}_v = 0$. Since scaling by a nonzero value does not change this, we have that the adjacencies are preserved and $N \in \mathcal{S}(H)$. It remains to check the spectrum of N, but for this we note that $QQ^T = \widehat{Q}\widehat{Q}^T$ since the dot products of any pair of corresponding rows are identical. Since the nonzero portion of the spectrum comes from $SQQ^T = S\widehat{Q}\widehat{Q}^T$, the result follows. \square

Proposition 10. If a symmetric matrix M has nullity three and ordered multiplicity 312, then $M = Q^T I Q$ where I is the 3×3 identity matrix.

Proof. The matrix M has spectrum $\{0^{(3)}, \lambda^{(1)}, \mu^{(2)}\}$. Let \mathbf{x} be a unit eigenvector for λ and \mathbf{y}, \mathbf{z} be orthonormal eigenvectors for μ . We have that

$$M = \begin{pmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{x}^T \\ \mathbf{y}^T \\ \mathbf{z}^T \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda} \mathbf{x} & \sqrt{\mu} \mathbf{y} & \sqrt{\mu} \mathbf{z} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \sqrt{\lambda} \mathbf{x}^T \\ \sqrt{\mu} \mathbf{y}^T \\ \sqrt{\mu} \mathbf{z}^T \end{pmatrix}}_{=O}. \quad \Box$$

Theorem 11. The graphs G117 and G133 cannot attain ordered multiplicity lists 312 or 213.

Proof. Label the vertices of G117 as shown in Figure 3. Suppose $M \in \mathcal{S}(G117)$ attains ordered multiplicity list 312 with nullity three. Applying Proposition 10, we can write $M = Q^T I Q$, where Q is a matrix which forms an orthogonal representation for G117.

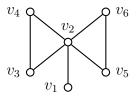


Figure 3: The graph G117.

Since v_1 and v_3 are not adjacent, the corresponding vectors \mathbf{v}_1 and \mathbf{v}_3 are orthogonal and thus form a plane in \mathbb{R}^3 . Both v_5 and v_6 are not adjacent to v_1 and v_3 , so \mathbf{v}_5 and \mathbf{v}_6 are orthogonal to this plane and must be scalar multiples.

The decloning lemma implies that the graph with vertex e deleted has a matrix that attains 212. However, this graph is odd unicyclic and thus cannot attain 212 by Lemma 7.

A similar argument establishes the result for G133.

Theorem 12. The graph G153 cannot attain ordered multiplicity lists 213 or 312.

Proof. Label the vertices of G153 as shown in Figure 4. Suppose that $M \in \mathcal{S}(G153)$ attains ordered multiplicity list 312 with nullity three. Applying Proposition 10 we can write $M = Q^T I Q$, where Q forms an orthogonal representation for G153.

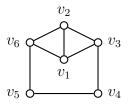


Figure 4: The graph G153.

If \mathbf{v}_1 and \mathbf{v}_2 are scalar multiples, the decloning lemma implies a matrix for C_5 with multiplicity list 212, which is impossible by Lemma 7.

If \mathbf{v}_1 and \mathbf{v}_2 are not scalar multiples, then \mathbf{v}_1 and \mathbf{v}_2 form a plane in \mathbb{R}^3 . Since vertices v_4 and v_5 are not adjacent to v_1 and v_2 , the vectors \mathbf{v}_4 and \mathbf{v}_5 are orthogonal to the aforementioned plane. Thus, \mathbf{v}_4 and \mathbf{v}_5 are scalar multiples, an impossibility given that v_4 and v_5 have distinct neighbors.

In either case, we get a contradiction; thus, G153 cannot attain 312.

For the remaining cases, we first establish an analogous result to Proposition 10.

Proposition 13. If a symmetric matrix M has nullity three and ordered multiplicity 132, then $M = Q^T S Q$ where S = diag(-1, 1, 1).

Proof. The matrix M has spectrum $\{-\lambda^{(1)}, 0^{(3)}, \mu^{(2)}\}$. Let \mathbf{x} be a unit eigenvector for $-\lambda$ and \mathbf{y}, \mathbf{z} be orthonormal eigenvectors for μ . We have

$$M = \begin{pmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{pmatrix} \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{x}^T \\ \mathbf{y}^T \\ \mathbf{z}^T \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda} \mathbf{x} & \sqrt{\mu} \mathbf{y} & \sqrt{\mu} \mathbf{z} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \sqrt{\lambda} \mathbf{x}^T \\ \sqrt{\mu} \mathbf{y}^T \\ \sqrt{\mu} \mathbf{z}^T \end{pmatrix}}_{=Q}. \quad \Box$$

Theorem 14. The graph G117 cannot attain ordered multiplicity lists 132 or 231.

Proof. Label the vertices of G117 as shown in Figure 3. Suppose that $M \in \mathcal{S}(G117)$ attains ordered multiplicity list 132 with nullity three. Applying Proposition 13, $M = Q^T S Q$ where S = diag(-1, 1, 1).

Since v_1 , v_3 , and v_5 have distinct sets of neighbors, no two of \mathbf{v}_1 , \mathbf{v}_3 and \mathbf{v}_5 are scalar multiples of each other.

Because S is invertible, $R = \begin{pmatrix} \mathbf{v}_1^T S \\ \mathbf{v}_3^T S \end{pmatrix}$ has rank two and nullity one. Since $v_1, v_3 \notin N(v_5)$, we have $R\mathbf{v}_5 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; similarly, $v_1, v_3 \notin N(v_6)$, so $R\mathbf{v}_6 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus, \mathbf{v}_5 and \mathbf{v}_6 are scalar multiples. A similar argument allows us to conclude that \mathbf{v}_3 and \mathbf{v}_4 are scalar multiples.

Now we can apply the decloning lemma *twice* (i.e., once for each scalar multiple pair) to produce a matrix for the graph $K_{1,3}$ which attains 112. But $Z_+(K_{1,3}) = 1$, so the end terms of any multiplicity list of $K_{1,3}$ must be 1, a contradiction.

In the following proposition, we introduce a tool that shows two vectors are scalar multiples, a technique similar to the one used in the previous results.

Proposition 15. If S is a 3×3 symmetric invertible matrix and \mathbf{x}, \mathbf{y} are vectors that satisfy $\mathbf{x}^T S \mathbf{x} = \mathbf{y}^T S \mathbf{y} = \mathbf{x}^T S \mathbf{y} = 0$, then \mathbf{x} and \mathbf{y} are scalar multiples of each other.

Proof. Assume \mathbf{x} and \mathbf{y} are not scalar multiples; then $S\mathbf{x}$ and $S\mathbf{y}$ are also not scalar multiples. This implies the matrix

$$R = \begin{pmatrix} \mathbf{x}^T S \\ \mathbf{y}^T S \end{pmatrix}$$

has rank two and nullity one. On the other hand, by our hypothesis we have

$$R\mathbf{x} = \begin{pmatrix} \mathbf{x}^T S \mathbf{x} \\ \mathbf{y}^T S \mathbf{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{x}^T S \mathbf{y} \\ \mathbf{y}^T S \mathbf{y} \end{pmatrix} = R\mathbf{y}$$

which shows that R has nullity at least two, a contradiction.

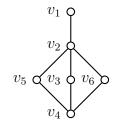
Theorem 16. The graph G121 cannot attain ordered multiplicity lists 132 or 231.

Proof. Label the vertices of G121 as shown in Figure 5(a). Suppose that $M \in \mathcal{S}(G121)$ has an ordered multiplicity list 132 with nullity three.

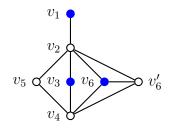
We claim that $M_{3,3} = M_{5,5} = M_{6,6} = 0$. To see this, suppose that $M_{6,6} \neq 0$; cloning v_6 with an edge produces the graph shown in Figure 5(b) which attains an ordered multiplicity list 142. The set marked in Figure 5(b) is a zero forcing set of order three. Thus, the cloned graph cannot attain 142, a contradiction. Hence, $M_{6,6} = 0$, and $M_{3,3} = M_{5,5} = 0$ by symmetry.

We apply Proposition 13 to write $M = Q^T S Q$ where S = diag(-1, 1, 1). Since v_3 , v_5 , and v_6 form an independent set, we have $\mathbf{v}_3^T S \mathbf{v}_6 = \mathbf{v}_3^T S \mathbf{v}_5 = \mathbf{v}_5^T S \mathbf{v}_6 = 0$. Moreover, because $M_{3,3} = M_{5,5} = M_{6,6} = 0$, we have $\mathbf{v}_3^T S \mathbf{v}_3 = \mathbf{v}_5^T S \mathbf{v}_5 = \mathbf{v}_6^T S \mathbf{v}_6 = 0$. From Proposition 15, it follows that \mathbf{v}_3 , \mathbf{v}_5 , and \mathbf{v}_6 are pairwise scalar multiples.

Applying the decloning lemma *twice* produces a matrix for P_4 which attains 112. However P_4 can only attain 1111, a contradiction.



(a) Labeled graph.

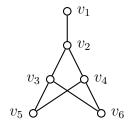


(b) A clone of the graph with an edge and marked zero forcing set.

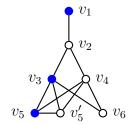
Figure 5: The graph G121.

Theorem 17. The graphs G125 and G138 cannot attain ordered multiplicity lists 132 or 231.

Proof. Label the vertices of G125 as shown in Figure 6(a). Suppose $M \in \mathcal{S}(G125)$ has spectrum $\{-\lambda, 0^{(3)}, 2^{(2)}\}$ with $\lambda > 0$ (by scale and shift, this holds without loss of generality).



(a) Labeled graph.



(b) A clone of the graph with an edge and marked zero forcing set.

Figure 6: The graph G125.

We claim that $M_{5,5} = M_{6,6} = 0$. To see this, suppose that $M_{5,5} \neq 0$; cloning v_5 with an edge produces the graph shown in Figure 6(b) which attains an ordered multiplicity list 142. The set marked in Figure 6(b) is a zero forcing set of order three. Thus, the cloned graph cannot attain 142, a contradiction. Hence, $M_{5,5} = 0$, and $M_{6,6} = 0$ by symmetry.

Applying Proposition 13 gives $M = Q^T S Q$, where S = diag(-1, 1, 1). Note $v_5 \nsim v_6$, so $\mathbf{v}_5^T S \mathbf{v}_6 = 0$. Moreover, $M_{5,5} = M_{6,6} = 0$, so $\mathbf{v}_5^T S \mathbf{v}_5 = \mathbf{v}_6^T S \mathbf{v}_6 = 0$. From Proposition 15, it follows that \mathbf{v}_5 and \mathbf{v}_6 are scalar multiples.

Applying the decloning lemma, we attain a matrix N for the banner graph as labeled in Figure 7, where N has eigenvalues $\{-\lambda, 0^{(2)}, 2^{(1)}\}$ and $N_{5,5} = 0$; particularly, N has the following form, where c_i are nonzero and d_i are arbitrary:

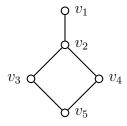


Figure 7: The banner graph.

$$N = \begin{pmatrix} d_1 & c_1 & 0 & 0 & 0 \\ c_1 & d_2 & c_2 & c_3 & 0 \\ 0 & c_2 & d_3 & 0 & c_4 \\ 0 & c_3 & 0 & d_4 & c_5 \\ 0 & 0 & c_4 & c_5 & 0 \end{pmatrix}.$$

The matrix N has rank three; moreover, the first three rows are linearly independent. Therefore the fifth row is a linear combination of the first three rows, which implies that $c_4 = \alpha c_2$ and $c_5 = \alpha c_3$ for some $\alpha \neq 0$.

Let $R = (N - I)^2$ which has spectrum $\{1^{(4)}, (1 + \lambda)^2\}$. The only graphs that attain the ordered multiplicity list 41 are unions of complete graphs with isolated vertices. Since $R_{1,3} = c_1 c_2 \neq 0$ and $R_{1,5} = 0$, we must have that v_1 is in the clique and v_5 is isolated. In particular

$$0 = R_{2,5} = c_2 c_4 + c_3 c_5 = \alpha \left(c_2^2 + c_3^2\right) \neq 0,$$

which is impossible.

A similar argument establishes the result for G138.

5 Attainable multiplicity lists for graphs

In this section we establish which ordered multiplicity lists are attainable for connected graphs on six vertices. We first utilize techniques which will also achieve arbitrary spectrum, and then describe those that fail to do so.

Before we begin, we note that a few special cases have already been done in the literature; we refer the reader elsewhere for details on the following cases.

• The inverse eigenvalue problem for cycles has been determined by Fernandes and Fonseca.

Theorem 18 (Fernandes and Fonseca [10], Theorem 3.3). The numbers $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ are the spectrum for some matrix $M \in \mathcal{S}(C_n)$ if and only if

$$\lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \lambda_5 \le \lambda_6 < \cdots$$

or

$$\lambda_1 < \lambda_2 \le \lambda_3 < \lambda_4 \le \lambda_5 < \lambda_6 < \cdots$$

(In particular G105 attains 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1221, and 2112 spectrally arbitrary.)

- All trees on six vertices are generalized stars or double stars for which the IEPG has been solved (see [3]). (In particular G77 attains 1131 and G79 attains 1221 with arbitrary spectra.)
- The graph G129 attains 222; a construction can be found in [8].

5.1 Using SSP for connected graphs of order at most five

Let G be a disconnected graph on six vertices. From each of its connected components, select an attainable ordered multiplicity list (see Table 1). Let $\gamma_1 \dots \gamma_k$ be the ordered multiplicity list built by interlacing in some way these lists. By Theorem 2, if each attains its multiplicity list with SSP, G attains $\gamma_1 \dots \gamma_k$ with SSP. By Theorem 1, any supergraph of G—namely, any connected supergraph H of order six—attains $\gamma_1 \dots \gamma_k$ (with SSP as well).

Since the components are graphs on at most five vertices, all of the attainable multiplicity lists are spectrally arbitrary. Since SSP preserves spectrum, any ordered multiplicity list constructed in this fashion must be spectrally arbitrary as well. Exhaustively performing the process detailed above gives the results listed in Table 4.

5.2 Using cloning for connected graphs of order five

Let G be a connected graph on five vertices, and let H be the graph constructed by cloning $v \in v(G)$ with an edge. If G attains the ordered multiplicity list $(\gamma_1, \ldots, \gamma_k)$, by Corollary 4 it follows that H attains both $(\gamma_1 + 1, \ldots, \gamma_k)$ and $(\gamma_1, \ldots, \gamma_k + 1)$.

The method above produces the results listed in Table 5 (note we exclude information that follows from previous results).

The full application of Theorem 3 generally requires a constructed matrix; then, confirming the value of the appropriate diagonal entry allows for the manipulation of the interior entries of the corresponding ordered multiplicity list. However, an explicit construction is not always necessary. Given a prescribed multiplicity list, for some graphs G we can sometimes guarantee that a particular diagonal entry of any $M \in \mathcal{S}(G)$ must be zero (or nonzero), as we now demonstrate.

Proposition 19. Let $M \in \mathcal{S}(G40)$ have nullity two, then the diagonal entry of M corresponding to the leaf is nonzero. In particular, for any ordered multiplicity list of G40 with a 2 we can clone the leaf vertex with an edge to get G144 and change the 2 to a 3.

Proof. In Figure 8 we have G40 and the two possible graphs that result from cloning without an edge (G111) and with an edge (G144). In addition we have marked minimal zero forcing sets for the two clones.

The graph G111 has a zero forcing number of two, which would imply that the maximum nullity (and hence also maximum multiplicity of an eigenvalue) is at most two. Therefore no matrix associated with G111 can have an ordered multiplicity list with an entry which is ≥ 3 . On the other hand that is possible for G144.

Graph(s)	Supergraphs	Multiplicity lists
G54 ∪ G1	G191, G200, G205, G207, G208	111111, 11112, 11121, 11211, 12111, 21111,
		1122, 1212, 1221, 2112, 2121, 2211,
		1113, 1131, 1311, 3111, 123, 132,
		213, 231, 312, 321, 114, 141, 411
G48 ∪ G1	G140, G141, G143, G156, G157,	111111, 11112, 11121, 11211, 12111, 21111,
	G158, G159, G160, G166, G168,	1122, 1212, 1221, 2112, 2121, 2211,
	G170, G172, G173, G177, G178,	1113, 1131, 1311, 3111, 123, 132,
	G179, G180, G181, G182, G183,	213, 231, 312, 321
	G184, G185, G186, G188, G189,	
	G190, G192, G193, G194, G195,	
	G196, G197, G198, G199, G201,	
	G202, G203, G204, G206	
G18 ∪ 2G1	G133, G134, G142, G165, G169	111111, 11112, 11121, 11211, 12111, 21111,
		1122, 1212, 1221, 2112, 2121, 2211,
		1113, 1131, 1311, 3111
G44 ∪ G1	G121, G125, G135, G138, G146,	111111, 11112, 11121, 11211, 12111, 21111,
	G149, G154, G161, G162, G175	1122, 1212, 1221, 2112, 2121, 2211,
		1131, 1311
G7 ∪ G7	G96, G98, G99, G103, G111,	111111, 11112, 11121, 11211, 12111, 21111,
G42 ∪ G1	G112, G113, G114, G115, G117,	1122, 1212, 1221, 2112, 2121, 2211
G16 ∪ 2G1	G118, G119, G120, G122, G123,	
	G124, G126, G128, G129, G130,	
	G136, G137, G139, G144, G145,	
	G147, G148, G150, G151, G152,	
	G153, G163, G164, G167, G171,	
70.1 71	G174, G187	11111 1110 11101 11011 10111 01111
G34 ∪ G1	G92, G93, G95, G104, G127	111111, 11112, 11121, 11211, 12111, 21111,
G38 ∪ G1	go4 go7 gu65 gu65	1122, 1212, 1221, 2121, 2211
G7 ∪ 3G1	G94, G97, G100, G102	111111, 11112, 11121, 11211, 12111, 21111
G13 ∪ 2G1	G77, G78, G79, G80, G81	111111, 11121, 11211, 12111
6 G 1	G83, G105	111111

Table 4: Using SSP properties for graphs of order at most five to attain (spectrally arbitrary) ordered multiplicity lists for graphs of order six.

We can run cloning with the matrix M which has nullity two and produce a matrix that has nullity three. This can only be possible if we cloned to $\tt G144$ and not $\tt G111$ which means that the diagonal entry corresponding to the leaf vertex is nonzero.

Finally, since we can always translate any particular eigenvalue to 0 then for any matrix in $\mathcal{S}(G40)$ with an entry of 2 first we translate so that the entry corresponds to an eigenvalue of 0, apply the preceding argument, and then translate back.

A similar argument works for several other cases which are listed in Table 6. Because we know that graphs on five vertices attain their ordered multiplicity lists spec-

Graph	Cloned graph(s)	Multiplicity lists
G29	G92, G161	132, 231
G30	G100	1122, 1212, 2121, 2211
G34	G117, G133, G179	1113, 123, 222, 3111, 321
G35	G119	1113, 3111
G36	G130, G150	1113, 3111
G37	G126, G141, G143, G168	1113, 123, 213, 222, 3111, 312, 321
G38	G153	1113, 123, 222, 3111, 321
G40	G144, G156, G177, G192	1113, 123, 213, 222, 3111, 312, 321
G41	G150, G160, G178, G181	1113, 123, 213, 222, 3111, 312, 321
G42	G165, G195	114, 123, 132, 213, 222, 231, 312, 321, 411
G43	G169, G172, G185	123, 213, 222, 312, 321
G44	G170, G189	222
G45	G165, G191, G200	114, 123, 132, 213, 222, 231, 312, 321, 411
G46	G179, G201	222
G47	G183, G193, G202	222
G48	G190, G194, G199	114, 222, 24, 33, 411, 42
G49	G195, G200, G205	114, 222, 24, 33, 411, 42
G50	G203, G206	114, 222, 24, 33, 411, 42
G51	G205, G207	222, 24, 33, 42
G52	G208	15, 222, 24, 33, 42, 51

Table 5: Using cloning for graphs on five vertices to attain ordered multiplicity lists for graphs on six vertices.

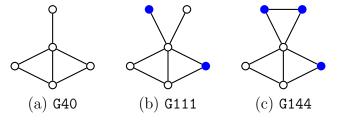


Figure 8: G40 and its two clones via the leaf. Minimal zero forcing sets have been marked for the clones.

trally arbitrarily and cloning preserves the eigenvalues, we can conclude that these lists found by cloning are also spectrally arbitrary.

5.3 Constructions of graphs which are spectrally arbitrary

Several cases were handled by finding matrices, usually through the aid of orthogonal representations. Constructions with SSP allowed multiple cases to be handled simultaneously.

Proposition 20. For G96 we can attain 222 spectrally arbitrary using SSP matrices.

Graph	Cloned graph	Multiplicity lists
G29	G77	141
G30	G78	1131, 1311
G30	G100	1131, 1311
G30	G114	1131, 1311
G34	G133	132, 231
G35	G119	1131, 1311
G37	G126	1131, 1311, 132, 231
G40	G144	1131, 1311, 132, 231
G42	G165	141
G44	G146	141
G46	G161	141
G48	G190	141
G48	G194	141
G49	G195	141

Table 6: Using cloning for graphs on five vertices to attain ordered multiplicity lists for graphs on six vertices where a middle entry is increased.

Proof. For b > 0, the following matrix for G96 attains $\{0^{(2)}, 1^{(2)}, (1+4b^2)^{(2)}\}$ with SSP.

$$\begin{pmatrix}
4b^2 & \sqrt{2}b & -\sqrt{2}b & 0 & 0 & 0 \\
\sqrt{2}b & 1 & 0 & b & 0 & 0 \\
-\sqrt{2}b & 0 & 1 & b & 0 & 0 \\
0 & b & b & 4b^2 & b & b \\
0 & 0 & 0 & b & 1 & 0 \\
0 & 0 & 0 & b & 0 & 1
\end{pmatrix}$$

By choosing b, we can make the ratio of the two gaps between the eigenvalues arbitrarily large. By scaling and shifting, this establishes attainability with arbitrary spectrum. \Box

By Theorem 1, we have that all supergraphs which contain G96 can attain the ordered multiplicity list 222 with arbitrary spectrum. Thus, the graphs G111, G114, G118, G121, G135, G136, G137, G140, G145, G146, G147, G148, G149, G157, G158, G159, G161, G162, G163, G164, G166, G167, G171, G173, G180, G182, G184, G186, G187, G188, G196, G197, G198, and G204 are spectrally arbitrary for 222.

Proposition 21. For G105 we can attain 222 spectrally arbitrary using SSP matrices.

Proof. The matrix

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & a \\ -1 & 0 & a & 0 & 0 & 0 \\ 0 & a & a^2 & a & 0 & 0 \\ 0 & 0 & a & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & a \\ a & 0 & 0 & 0 & a & a^2 \end{pmatrix} \in \mathcal{S}(\texttt{G105})$$

has SSP and has spectrum

$$\left\{ (\frac{1}{2}a^2 - \frac{1}{2}\sqrt{a^4 + 10a^2 + 5} - \frac{1}{2})^{(2)}, 0^{(2)}, (\frac{1}{2}a^2 + \frac{1}{2}\sqrt{a^4 + 10a^2 + 5} - \frac{1}{2})^{(2)} \right\}.$$

Since we can scale the spectrum, it suffices to show that the following ratio of the absolute value of the two nonzero eigenvalues contains the interval $[1, \infty)$:

$$\frac{a^2 + \sqrt{a^4 + 10 * a^2 + 5} - 1}{-a^2 + \sqrt{a^4 + 10 * a^2 + 5} + 1}.$$

This is continuous for $a \ge 1$ and if a = 1 we get 1. Furthermore, the numerator has growth a^2 and the denominator approaches 6 so the ratio is unbounded.

By Theorem 1, we have that all supergraphs which contain G105 can attain the ordered multiplicity list 222 with arbitrary spectrum. Thus, the graphs G127, G128, G151, G152, G154, G174, and G175 are spectrally arbitrary for 222.

Proposition 22. For G99 we can attain 222 spectrally arbitrary using non-SSP matrices.

Proof. For a > 0, the following matrix for G99 attains $\{0^{(2)}, 1^{(2)}, (1+3a^2)^{(2)}\}$.

$$\begin{pmatrix}
1 & a & 0 & 0 & 0 & 0 \\
a & 3 a^2 & a & a & 0 & 0 \\
0 & a & 1 & 0 & -a & 0 \\
0 & a & 0 & 1 & a & 0 \\
0 & 0 & -a & a & 3 a^2 & a \\
0 & 0 & 0 & 0 & a & 1
\end{pmatrix}$$

By choosing a we can make the ratios of the two gaps arbitrarily big. By scaling and shifting, this establishes attainability with arbitrary spectrum.

It is impossible for any matrix to have 222 and SSP for this graph as G112 is a supergraph which cannot attain 222 because q(G112) = 4.

Proposition 23. For G189 we can attain 141 spectrally arbitrary using SSP matrices.

Proof. The matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & a \\ 0 & 0 & 0 & 1 & 1 & a \\ 0 & 0 & 0 & 1 & 1 & a \\ 1 & 1 & 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & -1 & -1 & 0 \\ a & a & a & 0 & 0 & a^2 \end{pmatrix} \in \mathcal{S}(G189)$$

has SSP and has characteristic polynomial $p(x) = x^4(x^2 - (a^2 - 2)x - (5a^2 + 6))$. The spectrum is

$$\left\{\frac{1}{2}\left((a^2-2)-\sqrt{a^4+16a^2+28}\right),0^{(4)},\frac{1}{2}\left((a^2-2)+\sqrt{a^4+16a^2+28}\right)\right\}$$

Since we can scale the spectrum, it suffices to show that the following ratio of the absolute value of the two nonzero eigenvalues contains the interval $[1, \infty)$:

$$\frac{(a^2-2)+\sqrt{a^4+16a^2+28}}{-(a^2-2)+\sqrt{a^4+16a^2+28}}.$$

This is continuous for $a \ge \sqrt{2}$ and if $a = \sqrt{2}$ we get 1. Furthermore, the numerator has growth a^2 and the denominator approaches 10 so the ratio is unbounded.

By Theorem 1, all supergraphs which contain G189 can attain the ordered multiplicity list 141 with arbitrary spectrum. Thus, the graphs G197, G199, G201, G203, and G206 are spectrally arbitrary for 141.

The remaining case for 141 is G204. Consider

$$\begin{pmatrix} a^2 & 0 & a & a & a & a \\ 0 & -1 & -1 & -1 & 1 & 1 \\ a & -1 & 0 & 0 & 2 & 2 \\ a & -1 & 0 & 0 & 2 & 2 \\ a & 1 & 2 & 2 & 0 & 0 \\ a & 1 & 2 & 2 & 0 & 0 \end{pmatrix} \in \mathcal{S}(G204),$$

which has eigenvalues $\{-5,0^{(4)},4+a^2\}$. Arbitrary spectrums are attained through appropriate choices of $a \ge 1$, scaling, and shifting.

Proposition 24. For G151 we can attain 213, 312, 1113, and 3111 spectrally arbitrary using SSP matrices.

Proof. For a, b > 0, the following matrix for G151 has spectrum $\{0^{(3)}, 2a^2, 2a^2 + 2, a^2 + b^2 + 2\}$ and satisfies SSP:

$$\begin{pmatrix} a^2 & a & 0 & 0 & a^2 & 0 \\ a & a^2 + 2 & a & 0 & 0 & ab \\ 0 & a & a^2 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & a^2 + 1 & 1 & -b \\ a^2 & 0 & 0 & 1 & a^2 + 1 & -b \\ 0 & ab & 0 & -b & -b & b^2 \end{pmatrix}.$$

Setting a = b gives 312 with a fixed gap between the last two eigenvalues and an arbitrary gap between the first two; the attainability of an arbitrary spectrum follows.

For 3111, by scale and shift we can assume a non-negative spectrum such that 0 is the eigenvalue of multiplicity 3, and the gap between the first and second positive eigenvalues is 2. Set $2a^2$ and $a^2 + b^2 + 2$ to the first and third positive eigenvalues, respectively, and solve for a and b. Thus, an arbitrary spectrum is attainable.

By Theorem 1 we have that all supergraphs of G151 can attain the ordered multiplicity lists 213, 312, 1113, and 3111 with arbitrary spectrum. Thus, the graphs G171 and G187 are spectrally arbitrary for 213, 312, 1113, and 3111.

Proposition 25. For G163 we can attain 213, 312, 1113, and 3111 spectrally arbitrary using SSP matrices.

Proof. For a, b > 0, the following matrix for G163 has spectrum $\{0^{(3)}, 1, 1 + 3a^2, 1 + a^2 + 2b^2\}$ and satisfies SSP:

$$\begin{pmatrix} a^2 + b^2 & b^2 & b & 0 & a^2 & a \\ b^2 & a^2 + b^2 & b & a & -a^2 & 0 \\ b & b & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & -a & 0 \\ a^2 & -a^2 & 0 & -a & 2 a^2 & a \\ a & 0 & 0 & 0 & a & 1 \end{pmatrix}.$$

By appropriate choices of a, b, scaling, and translating, this attains 312, 213, 3111, and 1113 spectrally arbitrary.

5.4 Two distinct eigenvalues

For the graphs G154, G168, G174, G175, G181, G186, G188, G192, G196, G197, G198, G201, and G202, q(G) = 2 and $Z_{+}(G) = 3$ (see Table 3), which implies that all these graphs attain 33. (For more on matrix realizations for these graphs, see [8].) For the graph G204, this attains 33 by the matrix below on the left; and 42 (24) by the matrix below on the right.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 & -1 & 2 & -1 \\ 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 3 & 1 \\ -1 & 1 & 0 & 2 & -1 & 3 \\ 2 & 1 & 3 & -1 & 5 & 0 \\ -1 & 2 & 1 & 3 & 0 & 5 \end{pmatrix}$$

When a matrix for a graph has two distinct eigenvalues, we can modify the matrix to get additional attainable ordered multiplicity lists. The following will suffice for our purposes (generalizations are possible for graphs with larger order).

Lemma 26. Let $G \neq K_6$ be a connected graph on six vertices. If G attains ordered multiplicity list 33, then with arbitrary spectrum it attains multiplicity lists 1113, 123, 213, 312, 321, and 3111. Similarly, if G attains 42 or 24, then with arbitrary spectrum it attains 411 and 114.

Proof. Since G is not the complete graph, there are two non-adjacent vertices, which we assume to be v_1 and v_2 .

Let $M \in \mathcal{S}(G)$ attain the ordered multiplicity list 33 with spectrum $\{0^{(3)}, 1^{(3)}\}$. We can write $M = Q^T Q$, where Q is a 3×6 matrix whose rows are any orthonormal basis of the eigenspace of 1 (in particular, M is the projection matrix onto the eigenspace associated with eigenvalue 1).

We claim we can choose our orthonormal basis so that for some $x, y \neq 0$,

$$Q = \left(\begin{array}{cccc} x & 0 & * & * & * & * \\ 0 & y & * & * & * & * \\ 0 & 0 & * & * & * & * \end{array}\right).$$

To see this, first consider the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, which form a basis for our eigenspace.

- Note that no fixed entry can be 0 for all three of **a**, **b**, **c**; this would imply that the corresponding vertex is isolated, a contradiction given that our graph is connected. Thus, at least one vector has a nonzero first entry. By taking linear combinations, we can assume that the first entry is 0 for **a** and **b** and nonzero for **c**.
- Run Gram-Schmidt on \mathbf{a} , \mathbf{b} , \mathbf{c} (in this order) to get an orthonormal set \mathbf{a}' , \mathbf{b}' , \mathbf{c}' , where the first entries of \mathbf{a}' , \mathbf{b}' are 0 and the first entry of \mathbf{c}' is nonzero. Note the second entry of \mathbf{c}' must be zero; otherwise, using this set as an orthonormal basis for Q would force $M_{1,2} \neq 0$, a contradiction given that $v_1 \nsim v_2$.
- Now repeat the argument for \mathbf{a}', \mathbf{b}' by taking a linear combination so that the second entry of \mathbf{a}' is zero. Run Gram-Schmidt again to produce $\mathbf{a}'', \mathbf{b}''$.
- Thus, the rows of Q are (from top to bottom) $\mathbf{c}', \mathbf{b}'', \mathbf{a}''$.

Note we can now introduce parameters $\lambda, \mu > 0$ to give

$$\widehat{Q} = \left(\begin{array}{cccc} \lambda x & 0 & * & * & * & * \\ 0 & \mu y & * & * & * & * \\ 0 & 0 & * & * & * & * \end{array}\right).$$

Since the orthogonality of the columns of \widehat{Q} agree with the orthogonality of the columns of Q, the matrix $\widehat{M} = \widehat{Q}^T \widehat{Q} \in \mathcal{S}(G)$. Because the nonzero eigenvalues of \widehat{M} are the norms of the rows, \widehat{M} has spectrum $\{0^{(3)}, 1, 1 + (\lambda^2 - 1)x^2, 1 + (\mu^2 - 1)y^2\}$. Appropriate choices of λ and μ , combined with scaling and translation, arbitrarily attain any spectrum that starts or ends with an eigenvalue of multiplicity 3.

A similar argument handles the 42 case.

This lemma establishes that 1113, 123, 213, 312, 321, 3111 are all attainable with arbitrary spectrum for graphs G154, G174, and G175; similarly, 411 and 114 are attainable with arbitrary spectrum for G204.

5.5 Graph minor results

We now turn to results for graphs which attain certain ordered multiplicity lists, but which are not enough to prove we do so arbitrarily. We start with the following result which connects SSP and graph minors.

Theorem 27 (Barrett et al. [5], Theorem 6.12). Let G be attained from H by contraction of a single edge, and let $M \in \mathcal{S}(G)$ have SSP and ordered multiplicity list $(\gamma_1, \ldots, \gamma_k)$. Then there is $N \in \mathcal{S}(H)$ with ordered multiplicity list $(\gamma_1, \ldots, \gamma_k, 1)$.

Applying Theorem 27 by looking for minors on graphs of order six gives the results listed in Table 7 (we exclude information that follows from previous results).

Note Theorem 27 does not guarantee spectrally arbitrary results; the newly appended one on the ordered multiplicity list might need to be large (see [5] for more information).

Graph(s)	Minor	Multiplicity lists
G100	G34	1221
G129, G145, G151, G153,	G44	1311, 1131
G171, G174, G187		
G151, G153, G154, G169,	G48	321, 123, 231, 132
G171, G174, G175, G187		

Table 7: Using graph minors to attain ordered multiplicity lists for graphs of order six.

5.6 Graphs with SSP/SMP

Table 8 lists matrices that attain the corresponding ordered multiplicity list for that graph. All these matrices have either SSP or SMP, which gives the results listed in Table 7.

Graph	Multiplicity list(s)		Matrix					
			0	1	0	0	0	-2
			1	0	2	0	0	0
G105	2112	İ	0	2	0	1	0	0
	2112		0	0	1	0	2	0
			0	0	0	2	0	1
			-2	0	0	0	1	0 /
			-1	0	1	1	1	0 \
	222		0	-1	1	1	-1	0
G125		Ī	1	1	0	0	0	0
G125		-	1	1	0	0	0	0
			1	-1	0	0	-1	1
			0	0	0	0	1	1 /
			2	2	1	1	0	0 \
			2	1	0	0	0	-1
G129	139 931		1	0	0	0	1	0
0123	132, 231		1	0	0	0	1	0
			0	0	1	1	2	2
			0	-1	0	0	2	1 /

Table 8: Matrices with SSP or SMP

- Because G127 is a supergraph of G105, G127 attains 2112.
- Because G138 is a supergraph of G125, G138 attains 222.
- Because G145, G149, and G162 are supergraphs of G129, they attain 132 and 231.

5.7 Remaining cases

Table 9 gives the remaining attainable cases. The construction of these matrices included exhaustive searches for matrices with simple entries (i.e., $0, \pm 1$), as well as the use of orthogonal representations with respect to some (possibly indefinite) inner product.

$ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{pmatrix} \in \mathcal{S}(G92) $ 1131 1311	$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}(G94)$
$ \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix} $ $ \in \mathcal{S}(G114) $ $ 132 $ $ 231 $	$ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{S}(\texttt{G115}) $
$ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \in \mathcal{S}(G117) $	$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \in \mathcal{S}(\texttt{G130})$
$ \begin{pmatrix} 0 & 0 & 0 & \sqrt{3} & \sqrt{3} & 0 \\ 0 & 0 & 0 & \sqrt{3} & \sqrt{3} & 0 \\ 0 & 0 & 0 & \sqrt{3} & \sqrt{3} & 0 \\ \sqrt{3} & \sqrt{3} & \sqrt{3} & 0 & 3 & 0 \\ \sqrt{3} & \sqrt{3} & \sqrt{3} & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix} $ $ \in \mathcal{S}(G135) $ $ 132 $ $ 231 $	$ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & -1 & 0 & -1 \end{pmatrix} \in \mathcal{S}(G146) $
$ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \in \mathcal{S}(G150) $	$ \begin{pmatrix} \gamma^2 & \gamma & \gamma & 0 & 0 & 0 \\ \gamma & 0 & 2 & -1 & 0 & 0 \\ \gamma & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & \gamma & \gamma \\ 0 & 0 & 0 & \gamma & \gamma^2 & \gamma^2 \\ 0 & 0 & 0 & \gamma & \gamma^2 & \gamma^2 \end{pmatrix} \begin{pmatrix} \gamma^2 = \sqrt{10} - 2 \\ \in \mathcal{S}(G150) \\ 132 \\ 231 \end{pmatrix} $
$ \begin{pmatrix} 1 & 3 & 1 & 0 & 1 & 2 \\ 3 & 5 & 2 & 1 & 0 & 3 \\ 1 & 2 & 7 & 4 & -1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 2 \end{pmatrix} \in \mathcal{S}(G163) $ 1131 1311	$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \in \mathcal{S}(G163)$
$ \begin{pmatrix} 5 & \sqrt{5} & 5 & 0 & 0 & 0 \\ \sqrt{5} & 2 & \sqrt{5} & \sqrt{5} & \sqrt{5} & 0 \\ 5 & \sqrt{5} & 0 & 0 & -5 & -5 \\ 0 & \sqrt{5} & 0 & 5 & 5 & 0 \\ 0 & \sqrt{5} & -5 & 5 & 0 & -5 \\ 0 & 0 & -5 & 0 & -5 & -5 \end{pmatrix} $ $ \in \mathcal{S}(G163) $ $ 132 $ $ 231 $	

Table 9: Remaining cases.

6 Ordered Multiplicity IEPG differs from IEPG

Through five vertices, the ordered multiplicity IEPG and the IEPG are equivalent: a graph attains an ordered multiplicity list if and only if it attains that multiplicity list with arbitrary spectrum. However, for graphs of order at least six, this relationship no longer holds.

Theorem 28. The complete bipartite graph $K_{m,n}$ where $\min(m,n) \geq 3$ attains the ordered multiplicity list (1, m + n - 2, 1), but not spectrally arbitrary.

Proof. The spectrum of the adjacency matrix of $K_{m,n}$ is $\{-\sqrt{mn}, 0^{(m+n-2)}, \sqrt{mn}\}$. Thus, $K_{m,n}$ attains (1, m+n-2, 1). Note the gaps between consecutive eigenvalues are equal. We claim that any $M \in \mathcal{S}(K_{m,n})$ that attains (1, m+n-2, 1) preserves this relationship. Thus, $K_{m,n}$ cannot attain (1, m+n-2, 1) with arbitrary spectrum.

Let $M \in \mathcal{S}(K_{m,n})$ attain multiplicity list (1, m + n - 2, 1), where after translation we may assume 0 is the eigenvalue of multiplicity m + n - 2. Let $-\lambda_1 < \lambda_2$ be the nonzero eigenvalues of M with the corresponding orthogonal eigenvectors \mathbf{x}_1 and \mathbf{x}_2 . Note,

$$M = -\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \begin{pmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{pmatrix}$$
$$= \begin{pmatrix} \sqrt{\lambda_1} \mathbf{x}_1 & \sqrt{\lambda_2} \mathbf{x}_2 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{=S} \underbrace{\begin{pmatrix} \sqrt{\lambda_1} \mathbf{x}_1^T \\ \sqrt{\lambda_2} \mathbf{x}_2^T \end{pmatrix}}_{=Y}.$$

Let \mathbf{y}_i be the i^{th} column of Y; thus, $y_i \in \mathbb{R}^2$. Note the association between y_i and v_i of $K_{m,n}$: two vertices $v_i \nsim v_j$ if and only if $\mathbf{y}_i^T S \mathbf{y}_j = 0$. Moreover, $K_{m,n}$ is connected, so $y_i \neq 0$. Let $a, b \in K_{m,n}$ be nonadjacent. Since $\min(m, n) \geq 3$, there exists c such that a, b, and c are pairwise nonadjacent. If the corresponding vectors \mathbf{a} and \mathbf{b} are not scalar multiples, then the matrix

$$\begin{pmatrix} -a_1 & a_2 \\ -b_1 & b_2 \end{pmatrix}$$

has rank two. However, the adjacency structure of $K_{m,n}$ requires that

$$\begin{pmatrix} -a_1 & a_2 \\ -b_1 & b_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which shows the matrix lacks full rank, a contradiction. Thus, **a** and **b** are scalar multiples. By symmetry, the vectors associated with pairwise nonadjacent vertices must be scalar multiples. These vectors must also satisfy $\mathbf{a}^T S \mathbf{a} = 0$, and thus must have the form $\begin{pmatrix} x \\ \pm x \end{pmatrix}$. Hence, Y is of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m & \beta_1 & \beta_2 & \cdots & \beta_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_m & -\beta_1 & -\beta_2 & \cdots & -\beta_n \end{pmatrix},$$

for appropriate choice of α_i and β_j , so M has spectrum $\{0^{(m+n-2)}, \pm 2\sqrt{(\sum \alpha_i^2)(\sum \beta_j^2)}\}$. Therefore $K_{m,n}$ cannot attain (1, m+n-2, 1) with arbitrary spectrum.

Corollary 29. The graph $K_{3,3}$ (G175) attains 141, but not spectrally arbitrary.

7 Conclusion

We have given a complete solution for the ordered multiplicity IEPG for connected graphs on six vertices. Moreover, many of the techniques used for attainability also allow for an arbitrary spectrum, allowing for significant progress on the IEPG for connected graphs on six vertices. In particular, there are 1326 cases of attainability. Among these, 1285 are known to be done with arbitrary spectrum, one does so without, and 40 remain undetermined (see Table 10). Finishing these cases, and thus solving the IEPG for graphs on six vertices, is an open problem.

Multiplicity list(s)	Remaining cases
1212, 2121	G94
1221	G100
2112	G127
222	G115, G125, G129, G138
1131, 1311	G92, G117, G129, G130, G145, G150, G151, G153, G163, G171,
	G174, G187
123, 321	G151, G163, G171, G187
132, 231	G114, G129, G135, G145, G146, G149, G150, G151, G153, G154,
	G162, G163, G169, G171, G174, G175, G187

Table 10: The remaining cases for the IEPG for connected graphs on six vertices.

We also showed that $K_{m,n}$ with $\min(m,n) \geq 3$ has at least one attainable multiplicity list which cannot be attained spectrally arbitrary. This shows that the ordered multiplicity IEPG and the IEPG differ for graphs on six or more vertices.

A natural next problem to consider is the ordered multiplicity IEPG for connected graphs on seven or more vertices. While many of the techniques implemented in this work can be utilized in that setting (and indeed solves "most" of the cases), the IEPG becomes dramatically harder, Thus, new tools will likely be need to continue to make progress.

The difficulty lies in part with the number of cases involved, both in terms of the number of graphs and the number of potential ordered multiplicity lists. In addition, one of the most useful tools we had was SMP and SSP which allowed for simultaneous handling of many cases by establishing a result for a graph and all its supergraphs. This does have some limitations.

Observation 4. If $M \in \mathcal{S}(G)$ attains $\gamma_1 \dots \gamma_k$ and H is a supergraph which does not attain $\gamma_1 \dots \gamma_k$, then M does not have SSP or SMP.

This can be used to explain why the cases given in Table 2 do not have SSP, and for graphs on six vertices can be used to show that there are over forty occurrences where a graph attains an ordered multiplicity list but does so without SSP or SMP. These cases required either cloning or finding some appropriate orthogonal constructions. As the number of vertices increases the number of cases needing individual attention (and hence difficulty) will rise as well.

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A Connected graphs on six vertices realizing given ordered multiplicities

For the following ordered multiplicities we list all connected graphs which can achieve a given multiplicity list. Any graph which is underlined is a graph which has not yet been determined to be spectrally arbitrary for that ordered multiplicity list. Any graph which is boxed is a graph which has been shown to *not* be spectrally arbitrary for that multiplicity list.

111111											
G77	G78	G79	G80	G81	G83	G92	G93	G94	G95	G96	G97
G98	G99	G100	G102	G103	G104	G105	G111	G112	G113	G114	G115
G117	G118	G119	G120	G121	G122	G123	G124	G125	G126	G127	G128
G129	G130	G133	G134	G135	G136	G137	G138	G139	G140	G141	G142
G143	G144	G145	G146	G147	G148	G149	G150	G151	G152	G153	G154
G156	G157	G158	G159	G160	G161	G162	G163	G164	G165	G166	G167
G168	G169	G170	G171	G172	G173	G174	G175	G177	G178	G179	G180
G181	G182	G183	G184	G185	G186	G187	G188	G189	G190	G191	G192
G193	G194	G195	G196	G197	G198	G199	G200	G201	G202	G203	G204
G205	G206	G207	G208								
11112 an	d 2111	.1									
G92	G93	G94	G95	G96	G97	G98	G99	G100	G102	G103	G104
G105	G111	G112	G113	G114	G115	G117	G118	G119	G120	G121	G122
G123	G124	G125	G126	G127	G128	G129	G130	G133	G134	G135	G136
G137	G138	G139	G140	G141	G142	G143	G144	G145	G146	G147	G148
G149	G150	G151	G152	G153	G154	G156	G157	G158	G159	G160	G161
G162	G163	G164	G165	G166	G167	G168	G169	G170	G171	G172	G173
G174	G175	G177	G178	G179	G180	G181	G182	G183	G184	G185	G186
G187	G188	G189	G190	G191	G192	G193	G194	G195	G196	G197	G198
G199	G200	G201	G202	G203	G204	G205	G206	G207	G208		
11121 an	d 1211	.1									
G77	G78	G79	G80	G81	G92	G93	G94	G95	G96	G97	G98
G99	G100	G102	G103	G104	G105	G111	G112	G113	G114	G115	G117
G118	G119	G120	G121	G122	G123	G124	G125	G126	G127	G128	G129
G130	G133	G134	G135	G136	G137	G138	G139	G140	G141	G142	G143
G144	G145	G146	G147	G148	G149	G150	G151	G152	G153	G154	G156
G157	G158	G159	G160	G161	G162	G163	G164	G165	G166	G167	G168
G169	G170	G171	G172	G173	G174	G175	G177	G178	G179	G180	G181
G182	G183	G184	G185	G186	G187	G188	G189	G190	G191	G192	G193
G194	G195	G196	G197	G198	G199	G200	G201	G202	G203	G204	G205
G206	G207	G208									
_											

11211												
G77	G78	G79	G80	G81	G92	G93	G94	G95	G96	G97	G98	
G99	G100	G102	G103	G104	G105	G111	G112	G113	G114	G115	G117	
G118	G119	G120	G121	G122	G123	G124	G125	G126	G127	G128	G129	
G130	G133	G134	G135	G136	G137	G138	G139	G140	G141	G142	G143	
G144	G145	G146	G147	G148	G149	G150	G151	G152	G153	G154	G156	
G157	G158	G159	G160	G161	G162	G163	G164	G165	G166	G167	G168	
G169	G170	G171	G172	G173	G174	G175	G177	G178	G179	G180	G181	
G182	G183	G184	G185	G186	G187	G188	G189	G190	G191	G192	G193	
G194	G195	G196	G197	G198	G199	G200	G201	G202	G203	G204	G205	
G206	G207	G208										
1122 and	2211											_
G92	G93	G95	G96	G98	G99	G100	G103	G104	G105	G111	G112	
G113	G114	G115	G117	G118	G119	G120	G121	G122	G123	G124	G125	
G126	G127	G128	G129	G130	G133	G134	G135	G136	G137	G138	G139	
G140	G141	G142	G143	G144	G145	G146	G147	G148	G149	G150	G151	
G152	G153	G154	G156	G157	G158	G159	G160	G161	G162	G163	G164	
G165	G166	G167	G168	G169	G170	G171	G172	G173	G174	G175	G177	
G178	G179	G180	G181	G182	G183	G184	G185	G186	G187	G188	G189	
G190	G191	G192	G193	G194	G195	G196	G197	G198	G199	G200	G201	
G202	G203	G204	G205	G206	G207	G208	420.	4200	4-00	0.200		
$\overline{1221}$												_
G79	G92	G93	G95	G96	G98	G99	G100	G103	G104	G105	G111	
G112	G113	G114	G115	G117	G118	G119	G120	G121	G122	G123	G124	
G125	G126	G127	G128	G129	G130	G133	G134	G135	G136	G137	G138	
G139	G140	G141	G142	G143	G144	G145	G146	G147	G148	G149	G150	
G151	G152	G153	G154	G156	G157	G158	G159	G160	G161	G162	G163	
G164	G165	G166	G167	G168	G169	G170	G171	G172	G173	G174	G175	
G177	G178	G179	G180	G181	G182	G183	G184		G186	G187	G188	
G189	G190	G191	G192	G193	G194	G195	G196	G197	G198	G199	G200	
G201	G202	G203	G204	G205	G206	G207	G208					
1212 and	2121											_
G92	G93	G94	G95	G96	G98	G99	G100	G103	G104	G111	G112	
G113	G114	G115	G117	G118	G119	G120	G121	G122	G123	G124	G125	
G126	G127	G128	G129	G130	G133	G134	G135	G136	G137	G138	G139	
G140	G141	G142	G143	G144	G145	G146	G147	G148	G149	G150	G151	
G152	G153	G154	G156	G157	G158	G159	G160	G161	G162	G163	G164	
G165	G166	G167	G168	G169	G170	G171	G172	G173	G174	G175	G177	
G178	G179	G180	G181	G182	G183	G184	G185	G186	G187	G188	G189	
G190	G191	G192	G193	G194	G195	G196	G197	G198	G199	G200	G201	
G202	G203	G204	G205	G206	G207	G208	. = 2 ,	3 3	- - -			

211	12											
	G96	G98	G99	G103	G105	G111	G112	G113	G114	G115	G117	G118
	G119	G120	G121	G122	G123	G124	G125	G126	<u>G127</u>	G128	G129	G130
	G133	G134	G135	G136	G137	G138	G139	G140	G141	G142	G143	G144
	G145	G146	G147	G148	G149	G150	G151	G152	G153	G154	G156	G157
	G158	G159	G160	G161	G162	G163	G164	G165	G166	G167	G168	G169
	G170	G171	G172	G173	G174	G175	G177	G178	G179	G180	G181	G182
	G183	G184	G185	G186	G187	G188	G189	G190	G191	G192	G193	G194
	G195	G196	G197	G198	G199	G200	G201	G202	G203	G204	G205	G206
	G207	G208										
222	2											
	G96	G99	G105	G111	G114	G115	G117	G118	G121	G125	G126	G127
	G128	G129	G133	G135	G136	G137	<u>G138</u>	G140	G141	G143	G144	G145
	G146	G147	G148	G149	G150	G151	G152	G153	G154	G156	G157	G158
	G159	G160	G161	G162	G163	G164	G165	G166	G167	G168	G169	G170
	G171	G172	G173	G174	G175	G177	G178	G179	G180	G181	G182	G183
	G184	G185	G186	G187	G188	G189	G190	G191	G192	G193	G194	G195
	G196	G197	G198	G199	G200	G201	G202	G203	G204	G205	G206	G207
	G208											
111	13 and	3111										
	G117	G119	G126	G130	G133	G134	G140	G141	G142	G143	G144	G150
	G151	G153	G154	G156	G157	G158	G159	G160	G163	G165	G166	G168
	G169	G170	G171	G172	G173	G174	G175	G177	G178	G179	G180	G181
	G182	G183	G184	G185	G186	G187	G188	G189	G190	G191	G192	G193
	G194	G195	G196	G197	G198	G199	G200	G201	G202	G203	G204	G205
	G206	G207	G208									
113	31 and	1311										
	G77	G78	<u>G92</u>	G100	G114	G117	G119	G121	G125	G126	G129	G130
	G133	G134	G135	G138	G140	G141	G142	G143	G144	G145	G146	G149
	G150	G151	G153	G154	G156	G157	G158	G159	G160	G161	G162	G163
	G165	G166	G168	G169	G170	G171	G172	G173	G174	G175	G177	G178
	G179	G180	G181	G182	G183	G184	G185	G186	G187	G188	G189	G190
	G191	G192	G193	G194	G195	G196		G198	G199	G200	G201	G202
	G203	G204	G205	G206	G207	G208						
$\overline{123}$	3 and 3	321										
	G117	G126	G133	G140	G141	G143	G144	G150	G151	G153	G154	G156
	G157	G158	G159	G160	G163	G165	G166	G168	G169	G170	G171	G172
	G173	G174	G175	G177	G178	G179	G180	G181	G182	G183	G184	G185
	G186	G187	G188	G189	G190	G191	G192	G193	G194	G195	G196	G197
	G198	G199	G200	G201	G202	G203	G204	G205	G206	G207	G208	

132 and	231										
G92	<u>G114</u>	G126	G129	G133	<u>G135</u>	G140	G141	G143	G144	<u>G145</u>	<u>G146</u>
<u>G149</u>	<u>G150</u>	<u>G151</u>	<u>G153</u>	<u>G154</u>	G156	G157	G158	G159	G160	G161	<u>G162</u>
<u>G163</u>	G165	G166	G168	<u>G169</u>	G170	<u>G171</u>	G172	G173	<u>G174</u>	<u>G175</u>	G177
G178	G179	G180	G181	G182	G183	G184	G185	G186	<u>G187</u>	G188	G189
G190	G191	G192	G193	G194	G195	G196	G197	G198	G199	G200	G201
G202	G203	G204	G205	G206	G207	G208					
213 and	312										
G126	G140	G141	G143	G144	G150	G151	G154	G156	G157	G158	G159
G160	G163	G165	G166	G168	G169	G170	G171	G172	G173	G174	G175
G177	G178	G179	G180	G181	G182	G183	G184	G185	G186	G187	G188
G189	G190	G191	G192	G193	G194	G195	G196	G197	G198	G199	G200
G201	G202	G203	G204	G205	G206	G207	G208				
33											
G154	G168	G174	G175	G181	G186	G188	G190	G192	G194	G195	G196
G197	G198	G199	G200	G201	G202	G203	G204	G205	G206	G207	G208
114 and	411										
G165	G190	G191	G194	G195	G199	G200	G203	G204	G205	G206	G207
G208											
141											
G77	G146	G161	G165	G175	G189	G190	G191	G194	G195	G197	G199
G200	G201	G203	G204	G205	G206	G207	G208				
24 and 4	2										
G190	G194	G195	G199	G200	G203	G204	G205	G206	G207	G208	
-											

15 and 51

G208

B Attainable ordered multiplicitie lists

For each connected graph on six or fewer vertices we give the Atlas of Graph numbering, draw the graph, and list all attainable ordered multiplicity lists.

diaw one g	grapii, and nst	an attainable ordered multiplicity lists.
G1	0	1.
G3	o	11
G6		111
G 7		111, 12, 21
G13		1111, 121
G14		1111
G15		1111, 121, 112, 211
G16		1111, 121, 112, 211, 22
G17		1111, 121, 112, 211, 22
G18		1111, 121, 112, 211, 22, 13, 31
G29		11111, 1121, 1211, 131

G30	11111, 1121, 1211
G31	11111
G34	11111, 1121, 1211, 1112, 2111, 122, 221
G35	11111, 1121, 1211, 1112, 2111
G36	11111, 1121, 1211, 1112, 2111
G37	11111, 1121, 1211, 1112, 2111, 122, 221, 212
G38	11111, 1121, 1211, 1112, 2111, 122, 221
G40	11111, 1121, 1211, 1112, 2111, 122, 221, 212
G41	11111, 1121, 1211, 1112, 2111, 122, 221, 212
G42	11111, 1121, 1211, 1112, 2111, 122, 221, 212, 113, 131, 311
G43	11111, 1121, 1211, 1112, 2111, 122, 221, 212
G44	11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131

G45		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311
G46		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131
G47		11111, 1121, 1211, 1112, 2111, 122, 221, 212
G48		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32
G49		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32
G50		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32
G51		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32
G52		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32, 14, 41
G77		111111, 11121, 12111, 11211, 1131, 1311, 141
G78		111111, 11121, 12111, 11211, 1131, 1311
G79	000	111111, 11121, 12111, 11211, 1221
G80	○ ○ ○ ○ ○ ○ ○ ○ ○ ○	111111, 11121, 12111, 11211

G81	111111, 11121, 12111, 11211
G83	111111
G92	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 1211, 1311, 132, 231
G93	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221
G94	111111, 11112, 21111, 11121, 12111, 11211, <u>1212</u> , <u>2121</u>
G95	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221
G96	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G97	111111, 11112, 21111, 11121, 12111, 11211
G98	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G99	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G99 G100	

G103	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G104	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221
G105	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1221, 2112, 222
G111	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G112	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G113	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G114	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, <u>132</u> , <u>231</u>
G115	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, <u>222</u>
G117	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, <u>1131</u> , <u>1311</u> , 123, 321
G118	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G119	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 1113, 3111, 1131, 1311
G120	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112

G121	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311
G122	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G123	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G124	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G125	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, <u>222</u> , 1131, 1311
G126	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G127	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, <u>2112</u> , 222
G128	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G129	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, <u>222</u> , <u>1131</u> , <u>1311</u> , <u>132</u> , <u>231</u>
G130	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 1113, 3111, <u>1131</u> , <u>1311</u>
G133	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231
G134	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 1113, 3111, 1131, 1311

G135	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, <u>132</u> , <u>231</u>
G136	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G137	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G138	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, <u>222</u> , 1131, 1311
G139	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G140	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G141	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G142	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 1113, 3111, 1131, 1311
G143	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G144	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G145	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, <u>1131</u> , <u>1311</u> , <u>132</u> , <u>231</u>
G146	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, <u>132</u> , <u>231</u> , 141

G147	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G148	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G149	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, <u>132</u> , <u>231</u>
G150	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G151	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G152	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G153	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, <u>1131</u> , <u>1311</u> , 123, 321, <u>132</u> , <u>231</u>
G154	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, <u>132</u> , <u>231</u> , 213, 312, 33
G156	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G157	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G158	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G159	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312

G160	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G161	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231, 141
G162	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, <u>132</u> , <u>231</u>
G163	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G164	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G165	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2121, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 114, 411, 141
G166	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G167	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G168	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G169	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, <u>132</u> , <u>231</u> , 213, 312
G170	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G171	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312

G172	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G173	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G174	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G175	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, <u>132</u> , <u>231</u> , 213, 312, 33, <u>141</u>
G177	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G178	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G179	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G180	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G181	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G182	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G183	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G184	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312

G185	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G186	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G187	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G188	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G189	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 141
G190	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42
G191	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 114, 411, 141
G192	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
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G194	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42
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G198	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
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G201	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 141
G202	111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
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	 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212,
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