L^p-ESTIMATES AND REGULARITY FOR SPDES WITH MONOTONE SEMILINEARITY

NEELIMA AND DAVID ŠIŠKA

ABSTRACT. Semilinear stochastic partial differential equations on bounded domains $\mathscr{D}$ are considered. The semilinear term may have arbitrary polynomial growth as long as it is continuous and monotone except perhaps near the origin. A typical example is the stochastic Ginzburg–Landau equation. The main result of this article are L^p-estimates for such equations. The L^p-estimates are subsequently employed in obtaining higher regularity. It is shown, under appropriate assumptions, that the solution is continuous in time with values in the Sobolev space $H^2(\mathscr{D}')$ and L^2-integrable with values in $H^3(\mathscr{D}')$, for any compact $\mathscr{D}' \subset \mathscr{D}$. Using results from L^p-theory of SPDEs obtained by Kim [12] we get analogous results in weighted Sobolev spaces on the whole $\mathscr{D}$. Finally it is shown that the solution is Hölder continuous in time of order $\frac{1}{2} - \frac{2}{q}$ as a process with values in a weighted L^q-space, where q arises from the integrability assumptions imposed on the initial condition and forcing terms.

CONTENTS

1. INTRODUCTION

The aim of this article is to obtain L^p-estimates and regularity of solutions to the semilinear stochastic partial differential equation (SPDE)

$$\begin{aligned} du_t &= (L_t u_t + f_t(u_t, \nabla u_t) + f_t^0)dt + \sum_{k\in\mathbb{N}} (M_t^k u_t + g_t^k) dW_t^k \quad \text{on } [0,T]\times\mathscr{D} \\ u_t &= 0 \text{ on } \partial\mathscr{D}, \quad u_0 = \phi \text{ on } \mathscr{D}. \end{aligned} \tag{1}$$

where,

$$L_t u := \sum_{j=1}^d \partial_j \Big(\sum_{i=1}^d a_t^{ij} \partial_i u \Big) + \sum_{i=1}^d b_t^i \partial_i u + c_t u \quad \text{and} \quad M_t^k u := \sum_{i=1}^d \sigma_t^{ik} \partial_i u + \mu_t^k u. \tag{2}$$

Here $\mathscr{D}$ is a bounded domain in $\mathbb{R}^d$ and W^k are independent Wiener processes. The coefficients a and σ are assumed to satisfy stochastic parabolicity condition (and thus our equation is non-degenerate). Moreover all the coefficients a, b, c, σ and μ are assumed to be measurable and bounded, $f = f_t(\omega, x, r, z)$ is measurable, continuous in (r, z), monotone in r except perhaps around the origin, Lipschitz continuous in z, bounded in x and of polynomial growth in r (of arbitrary order). The forcing terms f^0 and g are assumed

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to satisfy appropriate integrability conditions. A typical example of equation fitting this setting is the stochastic Ginzburg–Landau equation. In this case

$$f(r) = -|r|^{\alpha-2}r\,,\;\; \alpha \geq 1.$$

To obtain higher interior regularity we will have to impose further regularity assumptions on the coefficients. To obtain regularity up to the boundary (in weighted Sobolev spaces) we will also need to impose regularity assumptions on the domain. The assumptions will be formulated precisely in further sections.

The main aim of this article is to obtain regularity results for the solutions to the SPDE (1). For a semilinear equation it is natural to consider the term $f := f(u, \nabla u) + f^0$ as a free term in an appropriate linear SPDE and to use established methods and theory to obtain regularity for this linear SPDE. Due to uniqueness of solutions to (1), see Lemma 1, we then get the same regularity for the semilinear equation (1). However, for the theory of regularity of linear SPDEs to apply, we need that the new free term f satisfies appropriate integrability conditions. This would typically mean at least L^2-integrability. Since the semilinear term in (1) is allowed arbitrary polynomial growth, it is clear that we need to obtain L^p-estimates for solution to (1) with $p \geq 2$ sufficiently large. Note that if one attempts to do this using Sobolev embedding theorem then one immediately runs into restrictions on the combination of dimension of $\mathscr{D}$ and the growth of the semilinear term. The main novelty of this article is in allowing arbitrary dimension of $\mathscr{D}$ and growth of the semilinear term. See Theorem 1. This is achieved by using the monotonicity property of the semilinear term and a cutting argument to obtain the required L^p-estimate. Once these have been established we then obtain new spatial regularity results for the SPDE (1), these are both interior regularity and up-to-the-boundary regularity in weighed Sobolev spaces. See Theorems 2 and 5. Finally we have a new time regularity result (in weighted space again), see Theorem 6.

Regularity of solutions to linear SPDEs has been an area of active interest for quite some time and here we point out some of the main results. Regularity of solutions to linear SPDEs on the whole space has been proved in Rozovskii [19]. On domains with a boundary the situation is much more involved and one cannot expect the same regularity up to the boundary as in the interior of the domain. See e.g. Examples 1.1 and 1.2 in Krylov [15]. After this observation two approaches to dealing with boundaries emerge: one is to quantify the loss of regularity near the boundary using weighted Sobolev spaces. These allow oscillations and explosion of the spatial derivatives of the solution near the boundary. The other approach is to side-step the problems created by the boundary by restricting the class of equations under consideration by imposing additional restriction on the noise term near the boundary (effectively disallowing stochastic forcing near the boundary). See Flandoli [3]. Weighted Sobolev spaces have also further employed, in the context of L^p-thoery for linear SPDEs, by Kim [12].

Unsurprisingly, there are fewer results for nonlinear SPDEs. Kim and Kim use the L^p-theory in [10] and [11] to obtain regularity for quasilinear SPDEs where the coefficients are uniformly bounded. Current results in Gerencsér [7] show that for a class of SPDEs, including (1), there exists some Hölder exponent such that the solution is Hölder continuous in space up to the boundary with this exponent. For interior regularity of a class of quasilinear equations associated with the "p-Laplace" operator see Breit [1]. For SPDEs with drift given by the subgradient of a quasi-convex function and with sufficiently regular noise Gess [4] proves higher regularity and existence of (analytically) strong solutions. All the aforementioned work on regularity of nonlinear SPDEs has been done using the variational approach. For results obtained in the semigroup framework we refer the reader to the work of Jentzen and Röckner [5] and references therein.

The article is organised as follows: Section 2 is devoted to the proof of Theorem 1 which gives us the desired L^p-estimates for the solution to semilinear SPDE (1). In Section 3, we first prove interior regularity for the associated linear SPDE, see Theorem 3. We then use the results on interior regularity of the linear SPDE to prove Theorem 2. In Section 4, we prove regularity results up to the boundary and time regularity in weighted Sobolev spaces using L^p-theory from Kim [12]. The main results and required assumptions are stated at the beginning of each section.

2. L^p-ESTIMATES FOR THE SEMILINEAR EQUATION

Let $T > 0$ be given, $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis, $\mathscr{P}$ be the predictable σ-algebra and $W := (W_t)_{t\in[0,T]}$ be an infinite dimensional Wiener martingale with respect to $(\mathscr{F}_t)_{t\in[0,T]}$, i.e. the coordinate processes $(W_t^k)_{t\in[0,T]}$, $k \in \mathbb{N}$ are independent $\mathscr{F}_t$-adapted Wiener processes such that $W_t^k - W_s^k$ is independent of $\mathscr{F}_s$ for $s \le t$. Further, let $\mathscr{D}$ be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. We use standard notation for Lebesgue–Bochner and Sobolev spaces. In general, if X is a normed linear space then we will use $|\cdot|_X$ to denote the norm in this space. There are exceptions: if $x \in \mathbb{R}^d$ then $|x|$ denotes the Euclidean norm. For Lebesgue and Sobolev spaces over the entire domain $\mathscr{D}$ we will omit the dependence on $\mathscr{D}$. So e.g. if $h \in L^p(\mathscr{D})$ then we will write $|h|_{L^p}$ for $|h|_{L^p(\mathscr{D})}$. If $h \in L^p((0,T); L^p(\mathscr{D}))$ then we use $\|h\|_{L^p}$ to denote the norm. Throughout this article N denotes a generic constant that may change from line to line.

Let $n \in \{0\} \cup \mathbb{N}$ and fix constants $K > 0$, $\kappa > 0$, $\alpha \ge 1$ and $p \ge \max(\alpha, 2)$. We assume the following:

A - 1. For any $i, j = 1, \dots, d$, the coefficients a^{ij}, b^i and c are real-valued, $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and are bounded by K. The coefficients $\sigma^i = (\sigma^{ik})_{k=1}^\infty$, $\mu = (\mu^k)_{k=1}^\infty$ are l^2-valued, $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and almost surely

$$\sum_{i=1}^d \sum_{k\in\mathbb{N}} |\sigma_t^{ik}(x)|^2 + \sum_{k\in\mathbb{N}} |\mu_t^k(x)|^2 \le K \quad \forall t \in [0,T], x \in \mathscr{D}.$$

A - 2. Almost surely

$$\sum_{i,j=1}^d \Big(a_t^{ij}(x) - \frac{1}{2}\sum_{k\in\mathbb{N}} \sigma_t^{ik}(x)\sigma_t^{jk}(x)\Big)\xi_i\xi_j \ge \kappa|\xi|^2 \quad \forall t \in [0,T], x \in \mathscr{D}, \xi \in \mathbb{R}^d\,.$$

A - 3. The function $f = f_t(\omega, x, r, z)$ is $\mathscr{P} \times \mathscr{B}(\mathscr{D}) \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^d)$-measurable, it is continuous in (r, z) almost surely for all t and x. Furthermore, almost surely

$$\begin{aligned}(r - r')(f_t(x, r, z) - f_t(x, r', z)) &\le K|r - r'|^2,\\ |f_t(x, r, z) - f_t(x, r, z')| &\le K|z - z'|,\\ |f_t(x, r, z)| &\le K(1 + |r|)^{\alpha-1}\end{aligned}$$

for all t, x, r, r', z, z'.

A - 4. $\phi \in L^p(\Omega, \mathscr{F}_0; L^p(\mathscr{D}))$, $f^0 \in L^p(\Omega \times (0,T), \mathscr{P}; L^p(\mathscr{D}))$ and $g \in L^p(\Omega \times (0,T), \mathscr{P}; L^p(\mathscr{D}; l^2))$.

Remark 1. Without loss of generality, we may assume that almost surely for all t, x and z the function $r \mapsto f_t(x, r, z)$ is decreasing. If not, then (1) can be rewritten by replacing $f_t(x, r, z)$ with $\bar{f}_t(x, r, z) := f_t(x, r, z) - Kr$ and $c_t(x)$ with $\bar{c}_t(x) := c_t(x) + K$, where using Assumption A - 3, $\bar{f}$ is decreasing in r.

Further, we may assume that almost surely for all t and x, $f_t(x, 0, 0) = 0$. Otherwise, we can replace $f_t(x, r, z)$ in (1) by $\tilde{f}_t(x, r, z) := f_t(x, r, z) - f_t(x, 0, 0)$ and f_t^0 by $\tilde{f}_t^0(x) := f_t^0(x) + f_t(x, 0, 0)$.

Definition 1 (L^2-Solution)**.** An adapted, continuous $L^2(\mathscr{D})$-valued process is said to be a solution of stochastic partial differential equation (1) if

(i) $dt \times \mathbb{P}$ almost everywhere $u \in L^\alpha(\mathscr{D}) \cap H^1_0(\mathscr{D})$ and

$$\mathbb{E}\int_0^T (|u_t|^\alpha_{L^\alpha} + |u_t|^2_{H^1_0})\,dt < \infty\,,$$

(ii) almost surely for every $t \in [0,T]$ and $\xi \in C_0^\infty(\mathscr{D})$,

$$(u_t,\xi) = (u_0,\xi) + \int_0^t \langle L_s(u_s) + f_s(u_s,\nabla u_s) + f^0_s, \xi\rangle ds + \sum_{k\in\mathbb{N}}\int_0^t (\xi, M^k_s(u_s) + g^k_s) dW^k_s.$$

The following theorem is the main result of this section.

Theorem 1. *If Assumptions A-1 to A-4 hold, then there exists a unique solution u to* (1) *and*

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}|u_t|^p_{L^p} + \mathbb{E}\int_0^T\int_{\mathscr{D}} |\nabla u_s|^2 |u_s|^{p-2} dx ds \\ \le N\mathbb{E}\Big(|\phi|^p_{L^p} + \|f^0\|^p_{L^p} + \||g|_{l^2}\|^p_{L^p}\Big),\end{aligned} \tag{3}$$

where $N = N(d,p,K,\kappa,T)$.

The rest of Section 2 is devoted to proving Theorem 1 but we give a brief outline of the proof here.

(1) We replace the semilinear term f by truncations f^m, depending on some $m \in \mathbb{N}$, chosen in such a way that that the monotonicity is preserved and f^m are bounded. For standard theory of stochastic evolution equations we obtain u^m which are solutions to the SPDE with f replaced with f^m.
(2) We now wish to get the estimate (3) for these u^m (uniformly in m). If we were allowed to apply Itô's formula directly to $r \mapsto |r|^p$ and the process $u^m_t(x)$ and to integrate over $\mathscr{D}$ then (3) for u^m would follow from A-1, A-2 and A-3.
(3) Since, of course, this is not allowed we instead consider an appropriate bounded smooth approximation ϕ_n to $r \mapsto |r|^p$ and use the Itô formula from Krylov [14]. We then establish an estimate similar to (3) but for $\phi_n(u^m)$ instead of $|u^m|^p$ and with the right-hand-side still depending on m but independent of n. See Lemma 2. This allows us to take the limit $n \to \infty$ and to use the monotonicity of $r \mapsto f^m_t(x,r,z)$ to obtain (3) for u^m. See Lemma 3.
(4) The final step is then to use compactness argument to obtain u as a weak limit of $(u^m)_{m\in\mathbb{N}}$, see Lemma 4, and the usual monotonicity argument to show that u satisfies (1). Fatou's lemma will then yield (3) for u.

Before proceeding with the proof of Theorem 1, we observe the following:

Remark 2. Assumptions A-1 and A-2 imply, after some computations using Hölder's and Young's inequalities, the existence of a constant K' depending on K, d and κ only such that almost surely for all $t \in [0,T]$ and $w, w' \in H^1_0(\mathscr{D})$,

$$2\langle L_t w + f^0_t, w\rangle + \sum_{k\in\mathbb{N}} |M^k_t w + g^k_t|^2_{L^2} + \kappa |w|^2_{H^1_0} \le K'\Big[|f^0_t|^2_{L^2} + \big||g_t|_{l^2}\big|^2_{L^2} + |w|^2_{L^2}\Big]$$

and

$$2\langle L_t w - L_t w', w - w'\rangle + \sum_{k\in\mathbb{N}} |M^k_t w - M^k_t w'|^2_{L^2} + \kappa|w - w'|^2_{H^1_0} \le K'|w-w'|^2_{L^2}\ .$$

Lemma 1 (Uniqueness)**.** *The solution to* (1) *is unique in the sense that if u and $\bar u$ both satisfy* (1) *then*

$$\mathbb{P}\Big(\sup_{t\le T}|u_t - \bar u_t|_{L^2} = 0\Big) = 1.$$

Proof. Let u and $\bar{u}$ be two solutions of (1) in the sense of Definition 1. Then,

$$\begin{aligned} u_t - \bar{u}_t = &\int_0^t \left(L_s(u_s) - L_s(\bar{u}_s) + f_s(u_s, \nabla u_s) - f_s(\bar{u}_s, \nabla \bar{u}_s)\right) ds \\ &+ \sum_{k\in\mathbb{N}} \int_0^t \left(M_s^k(u_s) - M_s^k(\bar{u}_s)\right) dW_s^k \end{aligned} \tag{4}$$

almost surely for all $t \in [0,T]$. Using Remark 1, Assumption A-3 and Young's inequality, we get

$$\begin{aligned} &\langle f_t(u_t, \nabla u_t) - f_t(\bar{u}_t, \nabla \bar{u}_t), u_t - \bar{u}_t\rangle \\ &= \langle f_t(u_t, \nabla u_t) - f_t(\bar{u}_t, \nabla u_t) + f_t(\bar{u}_t, \nabla u_t) - f_t(\bar{u}_t, \nabla \bar{u}_t), u_t - \bar{u}_t\rangle \\ &\le \frac{\kappa}{2}|\nabla(u_t - \bar{u}_t)|_{L^2}^2 + N|u_t - \bar{u}_t|_{L^2}^2 \,. \end{aligned} \tag{5}$$

Using the product rule and applying Itô's formula for the the square of the norm to (4), see Gyöngy and Šiška [9] or Pardoux [18, Chapitre 2, Theoreme 5.2], we obtain

$$\begin{aligned} d\Big(e^{-K''t}|u_t - \bar{u}_t|_{L^2}^2\Big) &= e^{-K''t}\big[d|u_t - \bar{u}_t|_{L^2}^2 - K''|u_t - \bar{u}_t|_{L^2}^2\, dt\big] \\ = e^{-K''t}\Bigg[&\Big(2\langle L_t(u_t) - L_t(\bar{u}_t) + f_t(u_t, \nabla u_t) - f_t(\bar{u}_t, \nabla \bar{u}_t), u_t - \bar{u}_t\rangle \\ &+ \sum_{k\in\mathbb{N}} |M_t^k(u_t) - M_t^k(\bar{u}_t)|_{L^2}^2 - K''|u_t - \bar{u}_t|_{L^2}^2\Big)\, dt \\ &+ \sum_{k\in\mathbb{N}} 2\big(u_t - \bar{u}_t, M_t^k(u_t) - M_t^k(\bar{u}_t)\big) dW_t^k\Bigg] \end{aligned} \tag{6}$$

almost surely for all $t \in [0,T]$. Substituting (5) in (6) and using Remark 2, we get

$$e^{-K''t}|u_t - \bar{u}_t|_{L^2}^2 \le 2\sum_{k\in\mathbb{N}} \int_0^t e^{-K''s}\big(u_s - \bar{u}_s, M_s^k(u_s) - M_s^k(\bar{u}_s)\big) dW_s^k$$

implying that right hand side is a non-negative local martingale (and thus a super-martingale) starting from 0 and hence for all $t \in [0,T]$,

$$\mathbb{E}[e^{-K't}|u_t - \bar{u}_t|_{L^2}^2] \le 0.$$

Thus for all $t \in [0,T]$, we get $\mathbb{P}(|u_t - \bar{u}_t|_{L^2}^2 = 0) = 1$ which, along with the continuity of $u - \bar{u}$ in $L^2(\mathscr{D})$, concludes the proof. □

Having proved uniqueness we start preparing the proof of Theorem 1. For $m \in \mathbb{N}$, consider the truncated function

$$f_t^m(x,r,z) = \begin{cases} f_t(x,-m,z) & \text{if } r < -m \\ f_t(x,r,z) & \text{if } -m \le r \le m \\ f_t(x,m,z) & \text{if } r > m, \end{cases}$$

and the equation

$$\begin{aligned} &du_t^m = (L_t u_t^m + f_t^m(u_t^m, \nabla u_t^m) + f_t^0)dt + \sum_{k\in\mathbb{N}} (M_t^k u_t^m + g_t^k) dW_t^k, \\ &u_t^m = 0 \text{ on } \partial\mathscr{D}, \quad u_0^m = \phi \text{ on } \mathscr{D}. \end{aligned} \tag{7}$$

For each $m \in \mathbb{N}$, using Assumption A-3, $f_t^m(x,r,z)$ is bounded and hence (7) can be viewed as a SPDE on the Gelfand triple $H_0^1(\mathscr{D}) \hookrightarrow L^2(\mathscr{D}) \hookrightarrow H^{-1}(\mathscr{D})$ and all the conditions for existence and uniqueness of solution in [16] are satisfied. Thus (7) has a unique L^2-solution in the sense of [16, Definition 2.2].

We now prove an estimate similar to (3) for the solutions of (7). We will do this by applying the Itô formula from Krylov [14]. To that end we need to consider the functions

$$\phi_n(r) = \begin{cases} |r|^p & \text{if} \quad |r| < n \\ n^{p-2}\frac{p(p-1)}{2}(|r|-n)^2 + pn^{p-1}(|r|-n) + n^p & \text{if} \quad |r| \geq n. \end{cases}$$

We now collect some key properties of these functions. We see that ϕ_n are twice continuously differentiable and

$$|\phi_n(x)| \leq N|x|^2, \; |\phi_n'(x)| \leq N|x|, \; |\phi_n''(x)| \leq N$$

where N depends on p and $n \in \mathbb{N}$ only. Further, for any $r \in \mathbb{R}$,

$$\phi_n(r) \to |r|^p, \; \phi_n'(r) \to p|r|^{p-2}r, \; \phi_n''(r) \to p(p-1)|r|^{p-2} \tag{8}$$

as $n \to \infty$ and

$$\phi_n(r) \leq N|r|^p, \; \phi_n'(r) \leq N|r|^{p-1}, \; \phi_n''(r) \leq N|r|^{p-2}, \tag{9}$$

where N depends on p only.

Remark 3. For any $r \in \mathbb{R}$ we have

(a) $|r\phi_n'(r)| \leq p\phi_n(r)$,
(b) $|r^2\phi_n''(r)| \leq p(p-1)\phi_n(r)$,
(c) $|\phi_n'(r)|^2 \leq 4p\phi_n''(r)\phi_n(r)$,
(d) $|\phi_n''(r)|^{\frac{p}{p-2}} \leq [p(p-1)]^{\frac{p}{p-2}}\phi_n(r)$.

These inequalities along with Young's inequality imply, for any $\epsilon > 0$,

(i) $|u_s^m\phi_n'(u_s^m)| \leq N\phi_n(u_s^m)$,
(ii) $|u_s^m|^2\phi_n''(u_s^m) \leq N\phi_n(u_s^m)$,
(iii) $\sum_{i=1}^d \partial_i u_s^m \phi_n'(u_s^m) \leq \epsilon\phi_n''(u_s^m)|\nabla u_s^m|^2 + N\phi_n(u_s^m)$,
(iv) $|f_s^0\phi_n'(u_s^m)| \leq N|f_s^0|[\phi_n''(u_s^m)]^{\frac{1}{2}}[\phi_n(u_s^m)]^{\frac{1}{2}} \leq N|f_s^0|^p + N\phi_n(u_s^m)$,
(v) $|f_s^m(u_s^m,\nabla u_s^m)\phi_n'(u_s^m)| \leq N|f_s^m(u_s^m,\nabla u_s^m)|[\phi_n''(u_s^m)]^{\frac{1}{2}}[\phi_n(u_s^m)]^{\frac{1}{2}}$
$\leq N|f_s^m(u_s^m,\nabla u_s^m)|^p + N\phi_n(u_s^m) \leq N|f_s(-m,\nabla u_s^m)|^p + N\phi_n(u_s^m)$,
(vi) $|g_s|_{l^2}^2\phi_n''(u_s^m) \leq N\phi_n(u_s^m) + N|g_s|_{l^2}^p$,

where the last inequality is obtained using Hölder's inequality and N depends only on d, p and ϵ.

Using Theorem 3.1 from [14], we get that almost surely

$$\begin{aligned}
&\int_{\mathscr{D}} \phi_n(u_t^m)dx \\
&= \int_{\mathscr{D}} \phi_n(u_0^m)dx + \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathscr{D}} \Big(\sum_{i=1}^d \sigma_s^{ik}\partial_i u_s^m + \mu_s^k u_s^m + g_s^k\Big)\phi_n'(u_s^m)dxdW_s^k \\
&\quad + \int_0^t \int_{\mathscr{D}} \Big(\sum_{i=1}^d b_s^i\partial_i u_s^m + c_s u_s^m + f_s^m(u_s^m,\nabla u_s^m) + f_s^0\Big)\phi_n'(u_s^m)dxds \\
&\quad - \int_0^t \int_{\mathscr{D}} \sum_{i,j=1}^d a_s^{ij}\partial_i u_s^m \phi_n''(u_s^m)\partial_j u_s^m dxds \\
&\quad + \frac{1}{2}\int_0^t \int_{\mathscr{D}} \sum_{k\in\mathbb{N}} \Big|\sum_{i=1}^d \sigma_s^{ik}\partial_i u_s^m + \mu_s^k u_s^m + g_s^k\Big|^2 \phi_n''(u_s^m)dxds,
\end{aligned}$$

for any $t \in [0,T]$ and $n \in \mathbb{N}$. Thus using Assumptions A-1, A-2 and Young's inequality for any $\epsilon > 0$, we obtain almost surely

$$\begin{aligned}\int_{\mathscr{D}} \phi_n(u_t^m)dx \le \int_{\mathscr{D}} \phi_n(u_0^m)dx + \mathscr{M}_t^{n,m} \\ + \int_0^t \int_{\mathscr{D}} \Big(\sum_{i=1}^d b_s^i \partial_i u_s^m + c_s u_s^m + f_s^m(u_s^m, \nabla u_s^m) + f_s^0 \Big) \phi_n'(u_s^m) dxds \\ - \int_0^t \int_{\mathscr{D}} \kappa |\nabla u_s^m|^2 \phi_n''(u_s^m) dxds \\ + \int_0^t \int_{\mathscr{D}} \Big(\epsilon |\nabla u_s^m|^2 + N|u_s|^2 + N|g_s|_{l^2}^2 \Big) \phi_n''(u_s^m)\, dxds,\end{aligned} \tag{10}$$

for any $t \in [0,T]$ and $n \in \mathbb{N}$. Here the generic constant N depends only on d, K and ϵ and

$$\mathscr{M}_t^{n,m} := \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathscr{D}} \Big(\sum_{i=1}^d \sigma_s^{ik} \partial_i u_s^m + \mu_s^k u_s^m + g_s^k \Big) \phi_n'(u_s^m) dx dW_s^k$$

is a martingale.

Further, using Burkholder–Davis–Gundy's inequality, Remark 3(c) and Hölder's inequality, we see that

$$\begin{aligned}&\mathbb{E} \sup_{0\le t\le T} |\mathscr{M}_t^{n,m}| \\ &\le N\mathbb{E}\Bigg(\int_0^T \sum_k \Big(\int_{\mathscr{D}} \Big| \sum_{i=1}^d \sigma_s^{ik}\partial_i u_s^m + \mu_s^k u_s^m + g_s^k \Big| \Big(\phi_n''(u_s^m)\phi_n(u_s^m) \Big)^{\frac12} dx \Big)^2 ds \Bigg)^{\frac12} \\ &\le N\mathbb{E}\Bigg(\int_0^T \sum_k \Big(\int_{\mathscr{D}} \Big| \sum_{i=1}^d \sigma_s^{ik}\partial_i u_s^m + \mu_s^k u_s^m + g_s^k \Big|^2 \phi_n''(u_s^m) dx \int_{\mathscr{D}} \phi_n(u_s^m) dx \Big) ds \Bigg)^{\frac12}\end{aligned}$$

which, using the same steps as before, in particular Remark 3 points (ii) and (iv), gives

$$\begin{aligned}&\mathbb{E} \sup_{0\le t\le T} |\mathscr{M}_t^{n,m}| \\ &\le N\mathbb{E}\Bigg(\int_0^T \Big(\int_{\mathscr{D}} \Big(|\nabla u_s^m|^2 + |u_s^m|^2 + |g_s|_{l^2}^2 \Big) \phi_n''(u_s^m) dx \int_{\mathscr{D}} \phi_n(u_s^m) dx \Big) ds \Bigg)^{\frac12} \\ &\le N\mathbb{E}\Bigg(\sup_{0\le t\le T} \int_{\mathscr{D}} \phi_n(u_t^m) dx \int_0^T \int_{\mathscr{D}} \Big[|\nabla u_s^m|^2 \phi_n''(u_s^m) + \phi_n(u_s^m) + |g_s|_{l^2}^p \Big] dxds \Bigg)^{\frac12} \\ &\le \frac12 \mathbb{E} \sup_{0\le t\le T} \int_{\mathscr{D}} \phi_n(u_t^m)dx + N\mathbb{E} \int_0^T \int_{\mathscr{D}} \Big[|\nabla u_s^m|^2 \phi_n''(u_s^m) + \phi_n(u_s^m) + |g_s|_{l^2}^p \Big] dxds\end{aligned} \tag{11}$$

The next lemma follows from Lemma 3.3 in [6], however we include the proof for convenience of the reader.

Lemma 2. *If u^m is the solution to* (7)*, then*

$$\begin{aligned}\mathbb{E} \sup_{0\le t\le T} |u_t^m|_{L^p}^p + \mathbb{E} \int_0^t \int_{\mathscr{D}} |\nabla u_s^m|^2 |u_s^m|^{p-2} dxds \\ \le N\mathbb{E}\Big(|\phi|_{L^p}^p + C_m + \|f^0\|_{L^p}^p + \| |g|_{l^2} \|_{L^p}^p \Big),\end{aligned} \tag{12}$$

where $N = N(d, K, \kappa, p)$ and $C_m := \mathbb{E} \int_0^T \int_{\mathscr{D}} (1 + |m|)^{\alpha(p-1)} dxds$ are constants.

Proof. From (10) and Remark 3(iv),(v) and Assumption A-3, we get

$$\begin{aligned}
\mathbb{E}\int_{\mathscr{D}}\phi_n(u^m_t)dx+\frac{\kappa}{2}\mathbb{E}\int_0^t\int_{\mathscr{D}}|\nabla u^m_s|^2\phi''_n(u^m_s)dxds\le N\mathbb{E}\int_{\mathscr{D}}\phi_n(u^m_0)dx+C_m\\
+\mathbb{E}\int_0^t\int_{\mathscr{D}}|f^0_s|^p dxds+N\mathbb{E}\int_0^t\int_{\mathscr{D}}|g_s|^p_{l^2}dxds+N\int_0^t\mathbb{E}\int_{\mathscr{D}}\phi_n(u^m_s)dxds\\
\le N\mathbb{E}\mathcal{K}^m_t+N\int_0^t\mathbb{E}\int_{\mathscr{D}}\phi_n(u^m_s)dxds,
\end{aligned}$$

where $N=N(d,p,K,\epsilon)$ and

$$\mathcal{K}^m_t:=\int_{\mathscr{D}}|\phi|^p dx+C_m+\int_0^t\int_{\mathscr{D}}|f^0_s|^p dxds+\int_0^t\int_{\mathscr{D}}|g_s|^p_{l^2}\,dxds.$$

Applying Gronwall's lemma, we obtain for any $t\in[0,T]$

$$\mathbb{E}\int_{\mathscr{D}}\phi_n(u^m_t)dx+\mathbb{E}\int_0^t\int_{\mathscr{D}}|\nabla u^m_s|^2\phi''_n(u^m_s)dxds\le N\mathbb{E}\mathcal{K}^m_t \tag{13}$$

where $N=N(d,p,K,\kappa,T)$.

Further, taking the supremum over $t\in[0,T]$ in (10), using the same estimates as given above and then taking expectation, we get using (11)

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}\phi_n(u^m_t)dx\\
\le& N\mathbb{E}\int_{\mathscr{D}}\phi_n(u^m_0)dx+\mathbb{E}\sup_{0\le t\le T}\int_0^t\int_{\mathscr{D}}f^m_s(u^m_s,\nabla u^m_s)\phi'_n(u^m_s)dxds\\
&+N\mathbb{E}\int_0^T\int_{\mathscr{D}}|f^0_s|^p dxds+N\mathbb{E}\int_0^T\int_{\mathscr{D}}|g_s|^p_{l^2}\,dxds+N\int_0^T\mathbb{E}\int_{\mathscr{D}}\phi_n(u^m_s)dxds\\
&+\frac{1}{2}\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}\phi_n(u^m_t)dx+N\mathbb{E}\int_0^T\int_{\mathscr{D}}\Big[|\nabla u^m_s|^2\phi''_n(u^m_s)+\phi_n(u^m_s)\Big]dxds\\
\le& N\mathbb{E}\int_{\mathscr{D}}\phi_n(u^m_0)dx+NC_m+N\mathbb{E}\int_0^T\int_{\mathscr{D}}|f^0_s|^p dxds\\
&+N\mathbb{E}\int_0^T\int_{\mathscr{D}}\left[|g_s|^p_{l^2}+\phi_n(u^m_s)\right]dxds\\
&+\frac{1}{2}\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}\phi_n(u^m_t)dx+N\mathbb{E}\int_0^T\int_{\mathscr{D}}|\nabla u^m_s|^2\phi''_n(u^m_s)dxds\\
\le& N\mathbb{E}\mathcal{K}^m_T+\frac{1}{2}\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}\phi_n(u^m_t)dx<\infty
\end{aligned}$$

where N does not depend on n and m. Thus, we have

$$\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}\phi_n(u^m_t)dx+\mathbb{E}\int_0^T\int_{\mathscr{D}}|\nabla u^m_s|^2\phi''_n(u^m_s)dxds\le N\mathbb{E}\mathcal{K}^m_T<\infty,$$

where $N=N(d,p,K,\kappa,T)$. Now we let $n\to\infty$ and apply Fatou's lemma to complete the proof. □

We can now use Lemma 2 and the monotonicity of $r\mapsto f^m_t(x,r,z)$ to obtain an estimate for u^m_t, where the right-hand-side no longer depends on m. Let

$$\mathscr{K}_t:=\int_{\mathscr{D}}|\phi|^p dx+\int_0^t\int_{\mathscr{D}}\left[|f^0_s|^p+|g_s|^p_{l^2}\right]dxds.$$

Lemma 3. *If u^m is the solution to* (7) *then there is $N = N(d,p,K,\kappa,T)$ such that*

$$\mathbb{E}\sup_{0\le t\le T}|u_t^m|_{L^p}^p + \mathbb{E}\int_0^T\int_{\mathscr{D}}|\nabla u_s^m|^2|u_s^m|^{p-2}\,dxds \le N\mathbb{E}\mathscr{K}_T. \tag{14}$$

Proof. From (10) and Remark 3(iv), we get

$$\begin{aligned}
&\mathbb{E}\int_{\mathscr{D}}\phi_n(u_t^m)dx + \frac{\kappa}{2}\mathbb{E}\int_0^t\int_{\mathscr{D}}|\nabla u_s^m|^2\phi_n''(u_s^m)dxds \\
&\qquad\le N\mathbb{E}\int_{\mathscr{D}}\phi_n(u_0^m)dx + \mathbb{E}\int_0^t\int_{\mathscr{D}}\left[f_s^m(u_s^m,\nabla u_s^m)\phi_n'(u_s^m) + |f_s^0|^p\right]dxds \\
&\qquad\quad + N\mathbb{E}\int_0^t\int_{\mathscr{D}}\left[|g_s|_{l^2}^p + \phi_n(u_s^m)\right]dxds,
\end{aligned}$$

where $N = N(d,p,K,\kappa)$.

Taking limit $n\to\infty$ and using Lebesgue's dominated convergence theorem in view of (12), (8) and (9), we get

$$\begin{aligned}
&\mathbb{E}\int_{\mathscr{D}}|u_t^m|^pdx + p(p-1)\frac{\kappa}{2}\mathbb{E}\int_0^t\int_{\mathscr{D}}|\nabla u_s^m|^2|u_s^m|^{p-2}dxds \\
&\le N\mathbb{E}\mathscr{K}_t + p\mathbb{E}\int_0^t\int_{\mathscr{D}}|u_s^m|^{p-2}f_s^m(u_s^m,\nabla u_s^m)u_s^mdxds + N\mathbb{E}\int_0^t\int_{\mathscr{D}}|u_s^m|^pdxds.
\end{aligned} \tag{15}$$

Using the fact $rf_t^m(r,0)\le 0$ for any $r\in\mathbb{R}, m\in\mathbb{N}, t\in[0,T]$, Young's inequality and Assumption A-3, we get

$$\begin{aligned}
&p\mathbb{E}\int_0^t\int_{\mathscr{D}}|u_s^m|^{p-2}f_s^m(u_s^m,\nabla u_s^m)u_s^mdxds \\
&= p\mathbb{E}\int_0^t\int_{\mathscr{D}}|u_s^m|^{p-2}\left[f_s^m(u_s^m,\nabla u_s^m) - f_s^m(u_s^m,0) + f_s^m(u_s^m,0)\right]u_s^mdxds \\
&\le \mathbb{E}\int_0^t\int_{\mathscr{D}}|u_s^m|^{p-2}\left[\frac{\kappa}{4}|f_s^m(u_s^m,\nabla u_s^m) - f_s^m(u_s^m,0)|^2 + N|u_s^m|^2\right]dxds \\
&\le \frac{\kappa}{4}\mathbb{E}\int_0^t\int_{\mathscr{D}}|u_s^m|^{p-2}|\nabla u_s^m|^2dxds + N\mathbb{E}\int_0^t\int_{\mathscr{D}}|u_s^m|^pdxds
\end{aligned}$$

Substituting this in (15) and then applying Gronwall's lemma, we obtain for any $t\in[0,T]$

$$\mathbb{E}\int_{\mathscr{D}}|u_t^m|^pdx + \mathbb{E}\int_0^t\int_{\mathscr{D}}|\nabla u_s^m|^2|u_s^m|^{p-2}dxds \le N\mathbb{E}\mathscr{K}_t$$

where $N = N(d,p,K,\kappa,T)$.

Further, taking the supremum over $t\in[0,T]$ in (10), using the same estimates as given above and then taking expectation, we get using (11)

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}\phi_n(u_t^m)dx \\
&\le N\mathbb{E}\int_{\mathscr{D}}\phi_n(u_0^m)dx + \mathbb{E}\sup_{0\le t\le T}\int_0^t\int_{\mathscr{D}}f_s^m(u_s^m,\nabla u_s^m)\phi_n'(u_s^m)dxds \\
&\quad + N\mathbb{E}\int_0^T\int_{\mathscr{D}}\left[|f_s^0|^p + |g_s|_{l^2}^p + \phi_n(u_s^m)\right]dxds \\
&\quad + \frac{1}{2}\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}\phi_n(u_t^m)dx + N\mathbb{E}\int_0^T\int_{\mathscr{D}}|\nabla u_s^m|^2\phi_n''(u_s^m)dxds,
\end{aligned}$$

where N does not depend on n and m. Taking limit $n \to \infty$ using Lebesgue's dominated convergence theorem and using (13) along with the steps as above, we get

$$\mathbb{E} \sup_{0\le t\le T} \int_{\mathscr{D}} |u_t^m|^p dx \le N\mathbb{E}\mathscr{K}_T + \frac{1}{2}\mathbb{E} \sup_{0\le t\le T} \int_{\mathscr{D}} |u_t^m|^p dx$$

and hence the lemma. □

To complete the proof of Theorem 1 we need to take the limit, as $m \to \infty$ in (14) and to show that (1) has a solution. To that end we obtain the following result.

Lemma 4. *There is a subsequence of m (which we denote by m again) and an adapted process u such that $u \in L^\alpha(\Omega\times(0,T),\mathscr{P};L^\alpha(\mathscr{D})) \cap L^2(\Omega\times(0,T),\mathscr{P};H_0^1(\mathscr{D}))$ and almost surely $u \in C([0,T];L^2(\mathscr{D}))$. Moreover, for $\alpha > 1$,*

$$\begin{aligned} u^m &\rightharpoonup v \quad \textit{in} \quad L^\alpha(\Omega\times(0,T),\mathscr{P};L^\alpha(\mathscr{D})),\\ u^m &\rightharpoonup \bar{v} \quad \textit{in} \quad L^2(\Omega\times(0,T),\mathscr{P};H_0^1(\mathscr{D})),\end{aligned}$$

$$\begin{aligned} f^m(u^m,\nabla u^m) &\rightharpoonup f' \quad \textit{in} \quad L^{\frac{\alpha}{\alpha-1}}\big(\Omega\times(0,T),\mathscr{P};L^{\frac{\alpha}{\alpha-1}}(\mathscr{D})\big),\\ L(u^m) &\rightharpoonup L(v) \quad \textit{in} \quad L^{\frac{\alpha}{\alpha-1}}\big(\Omega\times(0,T),\mathscr{P};L^{\frac{\alpha}{\alpha-1}}(\mathscr{D})\big)\\ M(u^m) &\rightharpoonup M(v) \quad \textit{in} \quad L^2\big(\Omega\times(0,T),\mathscr{P};l^2(L^2(\mathscr{D}))\big).\end{aligned}$$

Finally for all $t \in [0,T]$

$$u_t = u_0 + \int_0^t (L_s u_s + f_s' + f_s^0)ds + \sum_{k\in\mathbb{N}} \int_0^t (M_s^k u_s + g_s^k)dW_s^k \ a.s.$$

and

$$\begin{aligned}|u_t|_{L^2}^2 =& |\psi|_{L^2}^2 + 2\int_0^t \langle L_s u_s + f_s^0, u_s\rangle\, ds + 2\int_0^t \langle f_s', u_s\rangle\, ds\\ &+ 2\sum_{k\in\mathbb{N}} \int_0^t (M_s^k u_s + g_s^k, u_s)\, dW_s^k + \sum_{k\in\mathbb{N}} \int_0^t |M_s^k u_s + g_s^k|_{L^2}^2\, ds\,.\end{aligned}$$

Proof. By Lemma 3, we have $u^m \in L^\alpha(\Omega\times(0,T),\mathscr{P};L^\alpha)\cap L^2(\Omega\times(0,T),\mathscr{P};H_0^1(\mathscr{D}))$. Moreover, using Assumption A-3 and (14), we have for $\alpha > 1$,

$$\begin{aligned}\mathbb{E}\int_0^T \int_{\mathscr{D}} |f_t^m(u_t^m(x),\nabla u_t^m(x))|^{\frac{\alpha}{\alpha-1}}\, dxdt &\le K\mathbb{E}\int_0^T \int_{\mathscr{D}} (1+|u_t^m(x)|)^\alpha dxdt\\ &\le N + N\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}} |u_t^m(x)|^\alpha dx < \infty.\end{aligned} \tag{16}$$

Thus, $f^m(u^m,\nabla u^m) \in L^{\frac{\alpha}{\alpha-1}}\big(\Omega\times(0,T),\mathscr{P};L^{\frac{\alpha}{\alpha-1}}(\mathscr{D})\big)$ such that (14) and (16) holds for each $m \in \mathbb{N}$ with a constant independent of m. Since these Banach spaces are reflexive, there exists a subsequence (see, e.g., Theorem 3.18 in [2]), which we denote again by $\{m\}$, such that

$$\begin{aligned} u^m &\rightharpoonup v \quad \text{in} \quad L^\alpha(\Omega\times(0,T),\mathscr{P};L^\alpha(\mathscr{D}))\,,\\ u^m &\rightharpoonup \bar{v} \quad \text{in} \quad L^2(\Omega\times(0,T),\mathscr{P};H_0^1(\mathscr{D})) \text{ and}\\ f^m(u^m,\nabla u^m) &\rightharpoonup f' \quad \text{in} \quad L^{\frac{\alpha}{\alpha-1}}\big(\Omega\times(0,T),\mathscr{P};L^{\frac{\alpha}{\alpha-1}}(\mathscr{D})\big)\,.\end{aligned}$$

Moreover, the operators L and M are bounded and linear and hence map a weakly convergent sequence to a weakly convergent sequence. Thus, we have

$$\begin{aligned} L(u^m) &\rightharpoonup L(v) \quad \text{in} \quad L^{\frac{\alpha}{\alpha-1}}\big(\Omega\times(0,T),\mathscr{P};L^{\frac{\alpha}{\alpha-1}}(\mathscr{D})\big) \text{ and}\\ M(u^m) &\rightharpoonup M(v) \quad \text{in} \quad L^2\big(\Omega\times(0,T),\mathscr{P};l^2(L^2(\mathscr{D}))\big)\,.\end{aligned}$$

Since the Bochner integral and the stochastic integral are bounded linear operators, they are continuous with respect to weak topologies. Now, for any adapted and bounded real valued process η_t and $\xi \in C_0^\infty(\mathscr{D})$ we have

$$\begin{aligned}&\mathbb{E}\int_0^T \eta_t (u_t^m, \xi) dt\\ &= \mathbb{E}\int_0^T \eta_t\Big((u_0^m, \xi) + \int_0^t \langle L_s u_s^m + f_s^m + f_s^0, \xi\rangle ds + \sum_{k\in\mathbb{N}} \int_0^t (\xi, M_s^k u_s^m + g_s^k) dW_s^k\Big) dt.\end{aligned}$$

On taking limit $m \to \infty$, we get

$$\begin{aligned}&\mathbb{E}\int_0^T \eta_t (v_t, \xi) dt\\ &= \mathbb{E}\int_0^T \eta_t\Big((u_0, \xi) + \int_0^t \langle L_s v_s + f'_s + f_s^0, \xi\rangle ds + \sum_{k\in\mathbb{N}} \int_0^t (\xi, M_s^k v_s + g_s^k) dW_s^k\Big) dt\end{aligned}$$

for any adapted and bounded real valued process η_t and $\xi \in C_0^\infty(\mathscr{D})$. Since $C_0^\infty(\mathscr{D})$ is dense in $L^\alpha(\mathscr{D})$ and $H_0^1(\mathscr{D})$, we have

$$v_t = u_0 + \int_0^t (L_s v_s + f'_s + f_s^0) ds + \sum_{k\in\mathbb{N}} \int_0^t (M_s^k v_s + g_s^k) dW_s^k$$

$dt \times \mathbb{P}$ almost everywhere. Similarly, we get

$$\bar{v}_t = u_0 + \int_0^t (L_s \bar{v}_s + f'_s + f_s^0) ds + \sum_{k\in\mathbb{N}} \int_0^t (M_s^k \bar{v}_s + g_s^k) dW_s^k$$

$dt \times \mathbb{P}$ almost everywhere and hence the processes v and $\bar{v}$ are equal $dt \times \mathbb{P}$ almost everywhere. Using Itô formula for processes taking values in intersection of Banach spaces from Gyöngy and Šiška [9], there exists an $L^2(\mathscr{D})$-valued continuous modification u of v and $\bar{v}$ which satisfies above equality almost surely for all $t \in [0, T]$. $\square$

Remark 4. For $\alpha > 1$ and $\psi \in L^\alpha(\Omega \times (0,T), \mathscr{P}; L^\alpha(\mathscr{D})) \cap L^2(\Omega \times (0,T), \mathscr{P}; H_0^1(\mathscr{D}))$, we have

$$f^m(\psi, \nabla\psi) \to f(\psi, \nabla\psi)$$

in $L^{\frac{\alpha}{\alpha-1}}(\Omega \times (0,T), \mathscr{P}; L^{\frac{\alpha}{\alpha-1}}(\mathscr{D}))$. Indeed, by definition of f^m, as $m \to \infty$

$$f_s^m(\psi_s(x), \nabla\psi_s(x)) \to f_s(\psi_s(x), \nabla\psi_s(x)) \;\; \forall \omega, s, x\,.$$

Moreover $|f_s^m(r,z)| \le |f_s(r,z)|$ and due to Assumption A-3,

$$\mathbb{E}\int_0^T |f_s(\psi_s, \nabla\psi_s(x))|_{L^{\frac{\alpha}{\alpha-1}}}^{\frac{\alpha}{\alpha-1}} ds \le N\mathbb{E}\int_0^T \int_{\mathscr{D}} \Big(1 + |\psi_s(x)|^\alpha\Big) dx ds < \infty.$$

Therefore we may use Lebesgue Dominated Convergence Theorem to obtain

$$\begin{aligned}&\lim_{m\to\infty} \mathbb{E}\int_0^T \int_{\mathscr{D}} |f_s^m(\psi_s(x), \nabla\psi_s(x)) - f_s(\psi_s(x), \nabla\psi_s(x))|^{\frac{\alpha}{\alpha-1}} dx ds\\ &\qquad = \mathbb{E}\int_0^T \int_{\mathscr{D}} \lim_{m\to\infty} |f_s^m(\psi_s(x), \nabla\psi_s(x)) - f_s(\psi_s(x), \nabla\psi_s(x))|^{\frac{\alpha}{\alpha-1}} dx ds = 0.\end{aligned}$$

Proof of Theorem 1. In the case when $f_t(r,z)$ is bounded (i.e. the case of $\alpha = 1$ in A-3) the existence of unique L^2-solution follows immediately from Krylov and Rozovskii [16] and the required estimates from Lemma 2.

So we need to consider the case $\alpha > 1$. In order to show the weak limit u obtained in Lemma 4 is indeed the unique solution of SPDE (1), it remains to show that $f' = f(u, \nabla u)$ which can be shown using the monotonicity argument as below.

Define for each $w \in L^\alpha(\mathscr{D}) \cap H_0^1(\mathscr{D})$, $s \in (0,T)$ and $k \in \mathbb{N}$, the operators

$$A_s w := L_s w + f_s^0 \quad \text{and} \quad B_s^k w := M_s^k w + g_s^k.$$

Then for any $w, w' \in L^\alpha(\mathscr{D}) \cap H_0^1(\mathscr{D})$, we have using Remark 2

$$2\langle A_s w - A_s w', w - w'\rangle + \sum_{k\in\mathbb{N}} |B_s^k w - B_s^k w'|_{L^2}^2 \le -\kappa |w - w'|_{H_0^1}^2 + K'|w - w'|_{L^2}^2. \tag{17}$$

Consider $\psi \in L^\alpha(\Omega \times (0,T), \mathscr{P}; L^\alpha(\mathscr{D})) \cap L^2(\Omega \times (0,T), \mathscr{P}; H_0^1(\mathscr{D}))$. Then using Assumption A-3, Remark 1 and definition of f^m, we have

$$\langle f_s^m(u_s^m, \nabla u_s^m) - f_s^m(\psi_s, \nabla u_s^m), u_s^m - \psi_s\rangle \le 0 \tag{18}$$

almost surely for all $s \in [0,T]$. Moreover using Young's inequality and Assumption A-3, we have almost surely for all $s \in [0,T]$

$$2\langle f_s^m(\psi_s, \nabla u_s^m) - f_s^m(\psi_s, \nabla \psi_s), u_s^m - \psi_s\rangle \le \kappa |\nabla(u_s^m - \psi_s)|_{L^2}^2 + N|u_s^m - \psi_s|_{L^2}^2. \tag{19}$$

Define $K'' := K' + N$, where K' and N are as in (17) and (19) above. Then using the product rule and Itô's formula, we obtain

$$\begin{aligned}
&\mathbb{E}\big(e^{-K''t}|u_t|_{L^2}^2\big) - \mathbb{E}(|u_0|_{L^2}^2) \\
&\qquad = \mathbb{E}\Big[\int_0^t e^{-K''s}\Big(2\langle A_s u_s + f_s', u_s\rangle + \sum_{k\in\mathbb{N}} |B_s^k u_s|_{L^2}^2 - K''|u_s|_{L^2}^2\Big) ds\Big]
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\mathbb{E}\big(e^{-K''t}|u_t^m|_{L^2}^2\big) - \mathbb{E}(|u_0^m|_{L^2}^2) = \mathbb{E}\Big[\int_0^t e^{-K''s}\Big(&2\langle A_s u_s^m + f_s^m(u_s^m, \nabla u_s^m), u_s^m\rangle \\
&+ \sum_{k\in\mathbb{N}} |B_s^k u_s^m|_{L^2}^2 - K''|u_s^{m_k}|_{L^2}^2\Big) ds\Big]
\end{aligned} \tag{21}$$

for all $t \in [0,T]$.

We now need to re-arrange the right-hand side of (21) so that we can use the monotonicity assumptions. We have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t e^{-K''s}\Big(2\langle A_s u_s^m + f_s^m(u_s^m, \nabla u_s^m), u_s^m\rangle + \sum_{k\in\mathbb{N}} |B_s^k u_s^m|_{L^2}^2 - K''|u_s^m|_{L^2}^2\Big) ds\Big] \\
=&\mathbb{E}\Big[\int_0^t e^{-K''s}\Big(2\langle A_s u_s^m - A_s\psi_s, u_s^m\rangle + 2\langle A_s\psi_s, u_s^m\rangle + 2\langle A_s u_s^m - A_s\psi_s, \psi_s\rangle \\
&+ 2\langle f_s^m(u_s^m, \nabla u_s^m) - f_s^m(\psi_s, \nabla\psi_s), u_s^m - \psi_s\rangle + 2\langle f_s^m(\psi_s, \nabla\psi_s), u_s^m\rangle \\
&+ 2\langle f_s^m(u_s^m, \nabla u_s^m) - f_s^m(\psi_s, \nabla\psi_s), \psi_s\rangle + \sum_{k\in\mathbb{N}} \big|B_s^k u_s^m - B_s^k\psi_s\big|_{L^2}^2 - \sum_{k\in\mathbb{N}} |B_s^k\psi_s|_{L^2}^2 \\
&+ 2\sum_{k\in\mathbb{N}} \big(B_s^k u_s^m, B_s^k\psi_s\big) - K''\left[|u_s^m - \psi_s|_{L^2}^2 - |\psi_s|_{L^2}^2 + 2(u_s^m, \psi_s)\right]\Big) ds\Big].
\end{aligned} \tag{22}$$

Using (18) and (19), we have

$$\begin{aligned}
&2\langle f_s^m(u_s^m, \nabla u_s^m) - f_s^m(\psi_s, \nabla\psi_s), u_s^m - \psi_s\rangle \\
&= 2\langle f_s^m(u_s^m, \nabla u_s^m) - f_s^m(\psi_s, \nabla u_s^m) + f_s^m(\psi_s, \nabla u_s^m) - f_s^m(\psi_s, \nabla\psi_s), u_s^m - \psi_s\rangle \\
&\le \kappa|\nabla(u_s^m - \psi_s)|_{L^2}^2 + N|u_s^m - \psi_s|_{L^2}^2
\end{aligned}$$

and hence using (17) in (22) together with (21), we obtain for all $t \in [0,T]$

$$
\begin{aligned}
&\mathbb{E}\big(e^{-K''t}|u_t^m|_{L^2}^2\big) - \mathbb{E}(|u_0^m|_{L^2}^2) \\
\le&\mathbb{E}\Big[\int_0^t e^{-K''s}\Big(2\langle A_s\psi_s, u_s^m\rangle + 2\langle A_s u_s^m - A_s\psi_s, \psi_s\rangle \\
&+ 2\langle f_s^m(\psi_s, \nabla\psi_s), u_s^m\rangle + 2\langle f_s^m(u_s^m, \nabla u_s^m) - f_s^m(\psi_s, \nabla\psi_s), \psi_s\rangle \\
&- \sum_{k\in\mathbb{N}} |B_s^k\psi_s|_{L^2}^2 + 2\sum_{k\in\mathbb{N}} \big(B_s^k u_s^m, B_s^k\psi_s\big) + K''\big[|\psi_s|_{L^2}^2 - 2(u_s^m, \psi_s)\big]\Big)ds\Big].
\end{aligned}
$$

Now, integrating over t from 0 to T, letting $m \to \infty$ and using the weak lower semicontinuity of the norm, we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T \big(e^{-K''t}|u_t|_{L^2}^2 - |u_0|_{L^2}^2\big)dt\Big] \\
&\quad\le \liminf_{k\to\infty} \mathbb{E}\Big[\int_0^T \big(e^{-K''t}|u_t^m|_{L^2}^2 - |u_0^m|_{L^2}^2\big)dt\Big] \\
&\quad\le\mathbb{E}\Big[\int_0^T\int_0^t e^{-K''s}\Big(2\langle A_s\psi_s, u_s\rangle + 2\langle A_s u_s - A_s\psi_s, \psi_s\rangle \\
&\qquad + 2\langle f_s(\psi_s, \nabla\psi_s), u_s\rangle + 2\langle f_s' - f_s(\psi_s, \nabla\psi_s), \psi_s\rangle - \sum_{k\in\mathbb{N}} |B_s^k\psi_s|_{L^2}^2 \\
&\qquad + 2\sum_{k\in\mathbb{N}} (B_s^k u_s, B_s^k(\psi_s)) + K''\,\big[|\psi_s|_{L^2}^2 - 2(u_s, \psi_s)\big]\Big)dsdt\Big]
\end{aligned}
\tag{23}
$$

where we have used Remark 4 in last inequality. Again, integrating from 0 to T in (20) and combining this with (23), we get

$$
\begin{aligned}
\mathbb{E}\Big[\int_0^T\int_0^t e^{-K''s}\Big(&2\langle A_s u_s - A_s\psi_s, u_s - \psi_s\rangle + 2\langle f_s' - f_s(\psi_s, \nabla\psi_s), u_s - \psi_s\rangle \\
&+ \sum_{k\in\mathbb{N}} |B_s^k\psi_s - B_s^k u_s|_{L^2}^2 - K''|u_s - \psi_s|_{L^2}^2\Big)dsdt\Big] \le 0
\end{aligned}
$$

which on using (17) gives

$$
\mathbb{E}\Big[\int_0^T\int_0^t e^{-K''s}\Big(2\langle f_s' - f_s(\psi_s, \nabla\psi_s), u_s - \psi_s\rangle\Big)dsdt\Big] \le 0. \tag{24}
$$

Let $\eta \in L^\infty((0,T)\times\Omega;\mathbb{R})$, $\phi \in C_0^\infty(\mathscr{D})$, $\epsilon \in (0,1)$ and let $\psi = u - \epsilon\eta\phi$. Then from (24) one obtains that

$$
\mathbb{E}\Big[\int_0^T\int_0^t 2\epsilon e^{-K''s}\langle f_s' - f_s(u_s - \epsilon\eta_s\phi, \nabla u_s - \epsilon\eta_s\nabla\phi), \eta_s\phi\rangle dsdt\Big] \le 0.
$$

Dividing by ϵ, letting $\epsilon \to 0$, using Lebesgue dominated convergence theorem and Assumption A-3 leads to

$$
\mathbb{E}\Big[\int_0^T\int_0^t 2e^{-K''s}\eta_s\langle f_s' - f_s(u_s, \nabla u_s), \phi\rangle dsdt\Big] \le 0.
$$

Since this holds for any $\eta \in L^\infty((0,T)\times\Omega, \mathscr{P};\mathbb{R})$ and $\phi \in C_0^\infty(\mathscr{D})$, one gets that $f(u, \nabla u) = f'$ which concludes the proof.

Further, taking $m \to \infty$ in (14) and using the weak lower semicontinuity of the norm, we obtain the following estimates for the solution of (1)

$$\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}|u_t|^p_{L^p}+&\mathbb{E}\int_0^T\int_{\mathscr{D}}|\nabla u_s|^2|u_s|^{p-2}dxds\\
&\le \liminf_{m\to\infty}\Big[\mathbb{E}\sup_{0\le t\le T}|u^m_t|^p_{L^p}+\mathbb{E}\int_0^T\int_{\mathscr{D}}|\nabla u^m_s|^2|u^m_s|^{p-2}dxds\Big]\\
&\le N\mathbb{E}\Big(|\phi|^p_{L^p}+\|f^0\|^p_{L^p}+\||g|_{l^2}\|^p_{L^p}\Big).
\end{aligned}$$

□

3. INTERIOR REGULARITY

In this section, we present the results on interior regularity of the solution to SPDE (1). The main result is stated in Theorem 2. The idea is to prove the result for the linear SPDE first and then use it along with the L^p-estimates obtained in Section 2 to prove Theorem 2. We do not claim the result for the linear case to be new, however we could not find such result in literature in sufficient generality.

To raise the regularity of the solution one needs the given data to be sufficiently smooth. Thus, we assume the following condition on the coefficients before stating the main result of this section.

A - 5. For any $i,j=1,\dots,d$, the coefficients a^{ij}, b^i and c and their spatial derivatives up to order n are real-valued, $\mathscr{P}\times\mathscr{B}(\mathscr{D})$-measurable and are bounded by K. The coefficients $\sigma^i=(\sigma^{ik})_{k=1}^\infty$, $\mu=(\mu^k)_{k=1}^\infty$ and their spatial derivatives up to order n are l^2-valued, $\mathscr{P}\times\mathscr{B}(\mathscr{D})$-measurable and almost surely

$$\sum_{i=1}^d\sum_{k\in\mathbb{N}}\sum_{|\gamma|\le n}|D^\gamma\sigma^{ik}_t(x)|^2+\sum_{k\in\mathbb{N}}\sum_{|\gamma|\le n}|D^\gamma\mu^k_t(x)|^2\le K$$

for all t and x.

Theorem 2. *Let Assumptions A-2 to A-4 hold and u be the solution to* (1). *Fix some open $\mathscr{D}'\Subset\mathscr{D}$. If Assumption A-5 holds with $n=1$, and if $\phi\in L^2(\Omega,\mathscr{F}_0;H^1(\mathscr{D}))$ and $g\in L^2(\Omega\times(0,T),\mathscr{P};H^1(\mathscr{D};l^2))$, then*

$$u\in C([0,T],H^1(\mathscr{D}'))\ \text{ a.s. and }\ u\in L^2(\Omega\times(0,T),\mathscr{P};H^2(\mathscr{D}')).$$

Moreover, in case the semilinear term f does not depend on z, if Assumption A-1 holds with $n=2$, if $\phi\in L^2(\Omega,\mathscr{F}_0;H^2(\mathscr{D}))$, $f^0\in L^2(\Omega\times(0,T),\mathscr{P};H^1(\mathscr{D}))$ and $g\in L^2(\Omega\times(0,T),\mathscr{P};H^2(\mathscr{D};l^2))$ and if almost surely

$$|\partial_r f_t(x,r)|\le K(1+|r|)^{\alpha-2}\ \text{ and }\ |\partial_i f_t(x,r)|\le K(1+|r|)^{\alpha-1}\tag{25}$$

for all $i=1,\dots,d$, $t\in[0,T]$, $x\in\mathscr{D}$ and all $r\in\mathbb{R}$, then we have

$$u\in C([0,T],H^2(\mathscr{D}'))\ \text{ a.s. and }\ u\in L^2(\Omega\times(0,T),\mathscr{P};H^3(\mathscr{D}')).$$

One can obtain regularity results up to the boundary in appropriate weighted Sobolev space using results from Krylov [15] along with the L^p-estimates obtained in Theorem 1. However, obtaining the similar results for the linear equations using L^p-theory is more useful . We will discuss this in Section 4.

As mentioned before, we will first get the results for linear equations. So, we consider the following linear stochastic evolution equation:

$$dv_t=(L_tv_t+f_t)dt+\sum_{k\in\mathbb{N}}(M^k_tv_t+g^k_t)dW^k_t\quad\text{on }[0,T]\times\mathscr{D},\tag{26}$$

where the operators L and M^k are defined in (2). As can be seen in what follows, one can raise the regularity to any order for the linear equation by assuming the given data to be sufficiently smooth. Thus we make the following assumption on initial data and the free terms and then state the result in Theorem 3.

Let $n \geq 0$ be an integer.

A - 6. Assume that $v_0 \in L^2(\Omega, \mathscr{F}_0; H^n(\mathscr{D}))$, $g \in L^2(\Omega \times (0,T), \mathscr{P}; H^n(\mathscr{D}; l^2))$ and $f \in L^2(\Omega \times (0,T), \mathscr{P}; H^{n-1}(\mathscr{D}))$.

Theorem 3. *Assume that v is a continuous $L^2(\mathscr{D})$-valued adapted process such that $v \in L^2(\Omega \times (0,T), \mathscr{P}; H^1(\mathscr{D}))$, and it satisfies* (26). *If Assumptions A- 2, A- 5 and A- 6 hold, then for all open $\mathscr{D}' \Subset \mathscr{D}$,*

$$v \in C([0,T], H^n(\mathscr{D}')) \text{ a.s. and } v \in L^2(\Omega \times (0,T), \mathscr{P}; H^{n+1}(\mathscr{D}'))$$

We will prove Theorem 3 via Lemmas 5 and 6. In Lemma 5, we first prove the special case $n = 1$.

Lemma 5. *Assume that $v \in C([0,T]; L^2(\mathscr{D}))$ a.s., v is adapted and satisfies* (26) *and moreover $v \in L^2(\Omega \times (0,T), \mathscr{P}; H^1(\mathscr{D}))$. If Assumptions A-2, A-5 and A-6 hold with $n = 1$, then*

$$\begin{aligned} &\mathbb{E} \sup_{0 \leq t \leq T} |\partial_i v_t|^2_{L^2(\mathscr{D}')} + \mathbb{E} \int_0^T |\partial_i v_t|^2_{H^1(\mathscr{D}')} dt \\ &\leq N \left[\mathbb{E} \int_{\mathscr{D}} |\nabla v_0|^2 dx + \mathbb{E} \int_0^T \int_{\mathscr{D}} \Big[|\nabla v_t|^2 + |f_t|^2 + |v_t|^2 + \sum_{k \in \mathbb{N}} |\nabla g^k_t|^2 \Big] dx dt \right] \end{aligned} \tag{27}$$

for all $i = 1, \dots, d$ and open $\mathscr{D}' \Subset \mathscr{D}$ where $N = N(\mathscr{D}', d, T, K, \kappa)$.

Proof. We consider a cut-off function $\eta \in C_0^\infty(\mathscr{D})$ which is 1 on $\mathscr{D}'$. Define the l^{th}-difference quotient, $l \in \{1, 2, \dots, d\}$, by

$$\delta^h_l u(x) := \frac{1}{h} \big(T^h_l u - u \big)(x), \qquad x \in \mathbb{R}^d$$

where $T^h_l u(x) = u(x + h e_l)$ is the shift operator and the step-size h satisfies $2|h| < \operatorname{dist}(\operatorname{supp} \eta, \partial \mathscr{D})$. From (26), we get

$$d(\eta \delta^h_l v_t) = \eta \delta^h_l (L_t v_t + f_t) dt + \eta \sum_{k \in \mathbb{N}} \delta^h_l (M^k_t v_t + g^k_t) dW^k_t.$$

Applying Itô's formula for the square of L^2-norm, we get

$$\begin{aligned} d|\eta \delta^h_l v_t|^2_{L^2(\mathscr{D})} = 2 \langle \eta \delta^h_l (L_t v_t + f_t), \eta \delta^h_l v_t \rangle dt + 2 \sum_{k \in \mathbb{N}} (\eta \delta^h_l (M^k_t v_t + g^k_t), \eta \delta^h_l v_t) dW^k_t \\ + \sum_{k \in \mathbb{N}} |\eta \delta^h_l (M^k_t v_t + g^k_t)|^2_{L^2(\mathscr{D})} dt. \end{aligned}$$

Note that operators δ^h_l and ∂_j are linear and hence they commute. Thus, using integration by parts and the formula

$$\delta^h_l (vw)(x) = \delta^h_l v(x) T^h_l w(x) + v(x) \delta^h_l w(x)$$

we get,

$$\begin{aligned} \int_{\mathscr{D}} \eta^2 |\delta^h_l v_t|^2 dx = &\int_{\mathscr{D}} \eta^2 |\delta^h_l v_0|^2 dx + 2 \int_0^t \int_{\mathscr{D}} \eta^2 \delta^h_l (L_s v_s + f_s) \delta^h_l v_s dx ds \\ &+ \mathscr{M}^h_t + \sum_{k \in \mathbb{N}} \int_0^t \int_{\mathscr{D}} \eta^2 |\delta^h_l (M^k_s v_s + g^k_s)|^2 dx ds \\ =& I_0 - 2 \int_0^t \int_{\mathscr{D}} \eta^2 \sum_{i,j=1}^d a^{ij}_s \, \partial_i (\delta^h_l v_s) \, \partial_j (\delta^h_l v_s) + I_1 + I_2 + I_3 + \mathscr{M}^h_t + I_4 \end{aligned} \tag{28}$$

where,

$$
\begin{aligned}
I_0 &:= \int_{\mathscr{D}} \eta^2 |\delta_l^h v_0|^2 dx, \\
I_1 &:= -2 \int_0^t \int_{\mathscr{D}} \eta^2 \sum_{i,j=1}^d \delta_l^h a_s^{ij}\, \partial_i(T_l^h v_s) \partial_j(\delta_l^h v_s) dx ds, \\
I_2 &:= -4 \int_0^t \int_{\mathscr{D}} \eta \sum_{i,j=1}^d \left[\delta_l^h a_s^{ij}\, \partial_i(T_l^h v_s) + a_s^{ij}\, \partial_i(\delta_l^h v_s)\right] \partial_j \eta \delta_l^h v_s dx ds
\end{aligned}
$$

$$
\begin{aligned}
I_3 &:= 2 \int_0^t \int_{\mathscr{D}} \eta^2 \Big[\sum_{i=1}^d \{\delta_l^h b_s^i\, \partial_i(T_l^h v_s) + b_s^i\, \delta_l^h(\partial_i v_s)\} \\
&\qquad\qquad + \delta_l^h c_s\, T_l^h v_s + c_s\, \delta_l^h v_s + \delta_l^h f_s \Big] \delta_l^h v_s dx ds, \\
I_4 &:= \sum_{k \in \mathbb{N}} \int_0^t \int_{\mathscr{D}} \eta^2 \Big| \sum_{i=1}^d \delta_l^h \sigma_s^{ik}\, \partial_i(T_l^h v_s) + \delta_l^h \mu_s^k\, T_l^h v_s \\
&\qquad\qquad + \sum_{i=1}^d \sigma_s^{ik}\, \partial_i(\delta_l^h v_s) + \mu_s^k\, \delta_l^h v_s + \delta_l^h g_s^k \Big|^2 dx ds
\end{aligned}
$$

and

$$
\mathscr{M}_t^h := 2 \sum_{k \in \mathbb{N}} \int_0^t \int_{\mathscr{D}} \eta^2 \delta_l^h (M_s^k v_s + g_s^k) \delta_l^h v_s dx dW_s^k.
$$

Now, we see that

$$
\begin{aligned}
I_4 =& \sum_{k \in \mathbb{N}} \int_0^t \int_{\mathscr{D}} \eta^2 \left[\Big| \sum_{i=1}^d \delta_l^h \sigma_s^{ik}\, \partial_i(T_l^h v_s) + \delta_l^h \mu_s^k\, T_l^h v_s \Big|^2 \right. \\
&+ 2 \Big[\sum_{i=1}^d \delta_l^h \sigma_s^{ik}\, \partial_i(T_l^h v_s) + \delta_l^h \mu_s^k\, T_l^h v_s \Big] \Big[\sum_{i=1}^d \sigma_s^{ik}\, \partial_i(\delta_l^h v_s) + \mu_s^k \delta_l^h v_s + \delta_l^h g_s^k \Big] \\
&+ \left. \Big| \sum_{i=1}^d \sigma_s^{ik}\, \partial_i(\delta_l^h v_s) + \mu_s^k\, \delta_l^h v_s + \delta_l^h g_s^k \Big|^2 \right] dx ds \\
\leq& \sum_{i,j=1}^d \sigma_s^{ik}\, \partial_i(\delta_l^h v_s)\, \sigma_s^{jk}\, \partial_j(\delta_l^h v_s) + \bar{I}_4
\end{aligned}
$$

where,

$$
\begin{aligned}
\bar{I}_4 :=& \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathscr{D}} \eta^2 \Bigg[(d+1) \sum_{i=1}^d |\delta_l^h \sigma_s^{ik}|^2 |\partial_i (T_l^h v_s)|^2 + (d+1) |\delta_l^h \mu_s^k \, T_l^h v_s|^2 \\
& + 2 \sum_{i,j=1}^d \delta_l^h \sigma_s^{ik} \, \partial_i (T_l^h v_s) \, \sigma_s^{jk} \, \partial_j (\delta_l^h v_s) + 2 \sum_{i,j=1}^d \delta_l^h \sigma_s^{ik} \, \partial_i (T_l^h v_s) \, \mu_s^k \, \delta_l^h v_s \\
& + 2 \sum_{i,j=1}^d \delta_l^h \sigma_s^{ik} \, \partial_i (T_l^h v_s) \, \delta_l^h g_s^k + 2 \sum_{i=1}^d \sigma_s^{ik} \, \partial_i (\delta_l^h v_s) \, \delta_l^h \mu_s^k \, T_l^h v_s \\
& + 2 \delta_l^h \mu_s^k \, T_l^h v_s \, \mu_s^k \, \delta_l^h v_s + 2 \delta_l^h \mu_s^k \, T_l^h v_s \, \delta_l^h g_s^k \\
& + |\mu_s^k \, \delta_l^h v_s|^2 + |\delta_l^h g_s^k|^2 + 2 \sum_{i=1}^d \sigma_s^{ik} \, \partial_i (\delta_l^h v_s) \, \mu_s^k \, \delta_l^h v_s \\
& + 2 \sum_{i=1}^d \sigma_s^{ik} \, \partial_i (\delta_l^h v_s) \, \delta_l^h g_s^k + 2 \mu_s^k \, \delta_l^h v_s \, \delta_l^h g_s^k \Bigg] dx ds
\end{aligned}
$$

Substituting this in (28), we get

$$
\begin{aligned}
& \int_{\mathscr{D}} \eta^2 |\delta_l^h v_t|^2 dx \\
\leq & I_0 + I_1 - 2 \int_0^t \int_{\mathscr{D}} \eta^2 \sum_{i,j=1}^d \Big[a_s^{ij} - \frac{1}{2} \sum_{k\in\mathbb{N}} \sigma_s^{ik} \sigma_s^{jk} \Big] \partial_i (\delta_l^h v_s) \, \partial_j (\delta_l^h v_s) dx ds \\
& + I_2 + I_3 + \mathscr{M}_t^h + \bar{I}_4.
\end{aligned}
$$

which on using Assumptions A-2, A-5 (with $n = 1$) and Young's inequality for an $\epsilon > 0$ gives

$$
\begin{aligned}
\int_{\mathscr{D}} \eta^2 |\delta_l^h v_t|^2 dx \leq & \int_{\mathscr{D}} \eta^2 |\delta_l^h v_0|^2 dx - 2\kappa \int_0^t \int_{\mathscr{D}} \eta^2 |\nabla (\delta_l^h v_s)|^2 dx ds + \mathscr{M}_t^h \\
& + \int_0^t \int_{\mathscr{D}} \sum_{i,j=1}^d \big[\epsilon K |\partial_i (T_l^h v_s)|^2 + \epsilon K |\partial_i (\delta_l^h v_s)|^2 + C_\epsilon |\delta_l^h v_s|^2 \big] \eta \, \partial_j \eta \, dx ds \\
& + \int_0^t \int_{\mathscr{D}} \eta^2 \Bigg[2 \delta_l^h f_s \, \delta_l^h v_s + C_{K,d,\epsilon} \sum_{i=1}^d |\partial_i (T_l^h v_s)|^2 + C_{K,d,\epsilon} |T_l^h v_s|^2 \\
& + C \sum_{k\in\mathbb{N}} |\delta_l^h g_s^k|^2 + \epsilon C_K \sum_{i=1}^d |\partial_i (\delta_l^h v_s)|^2 + C_{K,\epsilon} |\delta_l^h v_s|^2 \Bigg] dx ds.
\end{aligned}
\tag{29}
$$

Now extending η, f, g and v to $\mathbb{R}^d$ by setting them to 0 on $\mathbb{R}^d \setminus \mathscr{D}$ and using the fact that $\operatorname{supp}\eta \subset \mathscr{D}$ and $\operatorname{supp}(T_l^{-h}\eta) \subset \mathscr{D}$ for our choice of h, we get

$$\begin{aligned}\int_{\mathscr{D}} \eta^2\, \delta_l^h f_s\, \delta_l^h v_s dx &= \int_{\mathbb{R}^d} \eta^2\, \delta_l^h f_s\, \delta_l^h v_s dx \\ &= \int_{\mathbb{R}^d} \eta^2\, \frac{1}{h} T_l^h f_s\, \delta_l^h v_s dx - \int_{\mathbb{R}^d} \eta^2\, \frac{1}{h} f_s\, \delta_l^h v_s dx \\ &= \int_{\mathbb{R}^d} T_l^{-h}(\eta^2) \frac{1}{h} f_s\, T_l^{-h}(\delta_l^h v_s) dx - \int_{\mathbb{R}^d} \eta^2\, \frac{1}{h} f_s\, \delta_l^h v_s dx \\ &= \int_{\mathbb{R}^d} f_s \frac{1}{h}\big[T_l^{-h}(\eta^2 \delta_l^h v_s) - (\eta^2 \delta_l^h v_s)\big] dx \\ &= -\int_{\mathbb{R}^d} f_s\, \delta_l^{-h}(\eta^2\, \delta_l^h v_s) dx = -\int_{\mathscr{D}} f_s\, \delta_l^{-h}(\eta^2\, \delta_l^h v_s) dx \\ &\le \epsilon \int_{\mathscr{D}} |\delta_l^{-h}(\eta^2\, \delta_l^h v_s)|^2 dx + C_\epsilon \int_{\mathscr{D}} |f_s|^2 dx\end{aligned} \tag{30}$$

where last inequality has been obtained using Young's inequality.

Since $\eta^2\, \delta_l^h v_s \in H^1(\mathscr{D})$, using and using the relation between difference quotients and weak derivatives (see e.g. [8, Ch. 5, Sec. 8, Theorem 3]), we have

$$\int_{\mathscr{D}} |\delta_l^{-h}(\eta^2\, \delta_l^h v_s)|^2 dx = \int_{\mathscr{D}_l^h(\eta)} |\delta_l^{-h}(\eta^2\, \delta_l^h v_s)|^2 dx \le C \int_{\mathscr{D}} |\nabla(\eta^2\, \delta_l^h v_s)|^2 dx$$

for some constant C and $\mathscr{D}_l^h(\eta) := \operatorname{supp}\eta \cup \operatorname{supp}(T_l^h \eta) \cup \operatorname{supp}(T_l^{-h}\eta) \Subset \mathscr{D}$. Substituting this in (30), we get

$$\begin{aligned}\int_{\mathscr{D}} \eta^2\, \delta_l^h f_s\, \delta_l^h v_s dx &\le \epsilon C \int_{\mathscr{D}} |\nabla(\eta^2\, \delta_l^h v_s)|^2 dx + C_\epsilon \int_{\mathscr{D}} |f_s|^2 dx \\ &= \epsilon C \int_{\mathscr{D}} |\eta^2\, \nabla(\delta_l^h v_s) + 2\eta\, \nabla\eta\, \delta_l^h v_s|^2 dx + C_\epsilon \int_{\mathscr{D}} |f_s|^2 dx \\ &\le \epsilon C_\eta \int_{\mathscr{D}} |\eta\, \nabla(\delta_l^h v_s)|^2 dx + \epsilon C_\eta \int_{\mathscr{D}} |(\eta\, \delta_l^h v_s)|^2 dx + C_\epsilon \int_{\mathscr{D}} |f_s|^2 dx.\end{aligned} \tag{31}$$

Similarly,

$$\int_{\mathscr{D}} \eta^2 |T_l^h v_s|^2 dx = \int_{\mathscr{D}_l^h(\eta)} \eta^2 |T_l^h v_s|^2 dx = \int_{\mathscr{D}_l^h(\eta)} |T_l^{-h}\eta|^2 |v_s|^2 dx \le C_\eta \int_{\mathscr{D}} |v_s|^2 dx$$

and

$$\begin{aligned}\sum_{i=1}^d \int_{\mathscr{D}} \eta^2 |\partial_i(T_l^h v_s)|^2 dx &= \sum_{i=1}^d \int_{\mathscr{D}_l^h(\eta)} \eta^2 |T_l^h(\partial_i v_s)|^2 dx \\ &\le C_\eta \sum_{i=1}^d \int_{\mathscr{D}} |\partial_i v_s|^2 dx = C_\eta \int_{\mathscr{D}} |\nabla v_s|^2 dx.\end{aligned}$$

Using the assumption $g \in L^2(\Omega \times (0,T), \mathscr{P}; H^1(\mathscr{D}; l^2))$ and the property of difference quotients mentioned above,

$$\sum_{k\in\mathbb{N}} \int_{\mathscr{D}} \eta^2 |\delta_l^h\, g_s^k|^2 dx = \sum_{k\in\mathbb{N}} \int_{\mathscr{D}_l^h(\eta)} \eta^2 |\delta_l^h g_s^k|^2 dx \le C_\eta \sum_{k\in\mathbb{N}} \int_{\mathscr{D}} |\nabla g_s^k|^2 dx.$$

Similarly, $v \in L^2(\Omega \times (0,T), \mathscr{P}; H^1(\mathscr{D}))$ and the property of difference quotients imply

$$\int_{\mathscr{D}} \eta^2 |\delta_l^h v_s|^2 dx \le C_\eta \int_{\mathscr{D}} |\nabla v_s|^2 dx. \tag{32}$$

Substituting (31)-(32) in (29), we get

$$\begin{aligned}\int_{\mathscr{D}} \eta^2|\delta_l^h v_t|^2dx \leq C_\eta \int_{\mathscr{D}} |\nabla v_0|^2 dx - 2\kappa \int_0^t \int_{\mathscr{D}} \eta^2 |\nabla(\delta_l^h v_s)|^2 dxds \\ + \mathscr{M}_t^h + \int_0^t \int_{\mathscr{D}} \Big[C_{K,\epsilon,\eta,d}|\nabla v_s|^2 + \epsilon C_{K,\eta}|\eta\nabla(\delta_l^h v_s)|^2 + C_\epsilon |f_s|^2 \\ + C_{K,\epsilon,\eta,d}|v_s|^2 + C_\eta \sum_{k\in\mathbb{N}} |\nabla g_s^k|^2\Big] dxds.\end{aligned} \tag{33}$$

Further, it can be seen that the process $\mathscr{M}_t^h$ defined in (28) is a local martingale where a localizing sequence of stopping times converging to T as $n \to \infty$ is given by

$$\tau_n := \inf\{t \in [0,T] : |\eta\delta_l^h v_s|_{L^2(\mathscr{D})}| > n\} \wedge T. \tag{34}$$

Thus, replacing t by $t \wedge \tau_n$ in (33), then taking expectation and choosing $\epsilon > 0$ small enough such that $2\kappa - \epsilon C_{K,\eta} = C_\kappa > 0$ and finally using Fatou's lemma, we get

$$\begin{aligned}\mathbb{E}\int_{\mathscr{D}} \eta^2|\delta_l^h v_t|^2 dx + C_\kappa \mathbb{E}\int_0^t \int_{\mathscr{D}} \eta^2|\nabla(\delta_l^h v_s)|^2 dxds \leq C_\eta \mathbb{E}\int_{\mathscr{D}} |\nabla v_0|^2 dx \\ + \mathbb{E}\int_0^t \int_{\mathscr{D}} \Big[C_{K,\epsilon,\eta,d}|\nabla v_s|^2 + C_\epsilon|f_s|^2 + C_{K,\epsilon,\eta,d}|v_s|^2 + C_\eta \sum_{k\in\mathbb{N}} |\nabla g_s^k|^2\Big] dxds.\end{aligned} \tag{35}$$

Using the inequalities of Burkholder–Davis–Gundy, Hölder and Young together with the estimates above we get that

$$\begin{aligned}&\mathbb{E}\sup_{0\leq t\leq T} |\mathscr{M}_{t\wedge\tau_n}^h| = \mathbb{E}\sup_{0\leq t\leq T}\Big|2\sum_{k\in\mathbb{N}}\int_0^{t\wedge\tau_n}\int_{\mathscr{D}} \eta^2 \delta_l^h(M_s^k v_s + g_s^k)\delta_l^h v_s dx dW_s^k\Big| \\ &\leq 4\mathbb{E}\Big(\sum_{k\in\mathbb{N}}\int_0^{\tau_n}\Big|2\int_{\mathscr{D}} \eta^2\delta_l^h(M_s^k v_s + g_s^k)\delta_l^h v_s dx\Big|^2 ds\Big)^{\frac{1}{2}} \\ &\leq 8\mathbb{E}\Big(\sum_{k\in\mathbb{N}}\int_0^{\tau_n} |\eta\,\delta_l^h(M_s^k v_s + g_s^k)|_{L^2(\mathscr{D})}^2 |\eta\,\delta_l^h v_s|_{L^2(\mathscr{D})}^2 ds\Big)^{\frac{1}{2}} \\ &\leq \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T} |\eta\,\delta_l^h v_t|_{L^2(\mathscr{D})}^2 + N\sum_{k\in\mathbb{N}}\mathbb{E}\int_0^{\tau_n} |\eta\,\delta_l^h(M_s^k v_s + g_s^k)|_{L^2(\mathscr{D})}^2 ds \\ &\leq \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T} |\eta\,\delta_l^h v_t|_{L^2(\mathscr{D})}^2 + N\mathbb{E}\int_0^{\tau_n}\int_{\mathscr{D}} \big[|\nabla v_s|^2 + |f_s|^2 + |v_s|^2 + |\nabla g_s|_{l^2}^2\big] dxds.\end{aligned} \tag{36}$$

Replacing t by $t\wedge\tau_n$ in (33), taking the supremum over $t \in [0,T]$ and using (36) we obtain

$$\begin{aligned}&\mathbb{E}\sup_{0\leq t\leq T}\int_{\mathscr{D}} \eta^2|\delta_l^h v_{t\wedge\tau_n}|^2 dx \\ &\leq N\left[\mathbb{E}\int_{\mathscr{D}} |\nabla v_0|^2 dx + \mathbb{E}\int_0^T \int_{\mathscr{D}} \Big[|\nabla v_s|^2 + |f_s|^2 + |v_s|^2 + |\nabla g_s|_{l^2}^2\Big] dxds\right],\end{aligned}$$

which, on applying Fatou's lemma, yields

$$\begin{aligned}&\mathbb{E}\sup_{0\leq t\leq T}\int_{\mathscr{D}} \eta^2|\delta_l^h v_t|^2 dx \\ &\leq N\left[\mathbb{E}\int_{\mathscr{D}} |\nabla v_0|^2 dx + \mathbb{E}\int_0^T \int_{\mathscr{D}} \Big[|\nabla v_s|^2 + |f_s|^2 + |v_s|^2 + |\nabla g_s|_{l^2}^2\Big] dxds\right],\end{aligned}$$

where $N = N(K,d,\eta,\epsilon)$. Now note that the right hand side of above equation and (35) are independent of h and are finite and hence using e.g. [8, Ch. 5, Sec. 8, Theorem 3]), we get (27). □

We now extend the result to the case $n = 2$ as follows. From Lemma 5 we have that v is a continuous $H^1(\mathscr{D}')$-valued adapted process such that $v \in L^2(\Omega \times (0,T), \mathscr{P}; H^2(\mathscr{D}'))$, and it satisfies (26). If Assumptions A-5 and A-6 hold for $n = 2$, then from (26), we get

$$\begin{aligned} d(\partial_l v_t) &= \partial_l (L_t v_t + f_t) dt + \sum_{k \in \mathbb{N}} \partial_l (M_t^k v_t + g_t^k) dW_t^k \\ &= \big(L_t(\partial_l v_t) + \bar{f}_t\big) dt + \sum_{k \in \mathbb{N}} \big(M_t^k(\partial_l v_t) + \bar{g}_t^k\big) dW_t^k \end{aligned} \tag{37}$$

on $[0,T] \times \mathscr{D}$, where

$$\bar{f}_t := \sum_{j=1}^d \partial_j \Big(\sum_{i=1}^d \partial_l a_t^{ij}\, \partial_i v_t \Big) + \sum_{i=1}^d \partial_l b_t^i\, \partial_i v_t + \partial_l c_t\, v_t + \partial_l f_t$$

and

$$\bar{g}_t^k := \sum_{i=1}^d \partial_l \sigma_t^{ik}\, \partial_i v_t + \partial_l \mu_t^k\, v_t + \partial_l g_t^k.$$

Using Assumptions A-5 , A-6 with $n = 2$ we get that $\bar{f} \in L^2(\Omega \times (0,T), \mathscr{P}; L^2(\mathscr{D}'))$ and $\bar{g} \in L^2(\Omega \times (0,T), \mathscr{P}; H^1(\mathscr{D}'; l^2))$.

Thus replacing $f, g^k, \mathscr{D}$ in (26) by $\bar{f}, \bar{g}^k$ and $\mathscr{D}'$ respectively, we see that $z = \partial_l v$ satisfies (26). Clearly $z \in C([0,T]; L^2(\mathscr{D}'))$ almost surely and $z \in L^2(\Omega \times (0,T); H^1(\mathscr{D}'))$ and hence all the assumptions of Lemma 5 are satisfied for the new linear equation (37). Therefore for all open $\mathscr{D}'' \Subset \mathscr{D}'$, we have

$$\begin{aligned} &\mathbb{E} \sup_{0 \le t \le T} |\partial_i z_t|^2_{L^2(\mathscr{D}'')} + \mathbb{E} \int_0^T |\partial_i z_t|^2_{H^1(\mathscr{D}'')} dt \\ &\le N \left[\mathbb{E} \int_{\mathscr{D}'} |\nabla z_0|^2 dx + \mathbb{E} \int_0^T \int_{\mathscr{D}'} \Big[|\nabla z_t|^2 + |\bar{f}_t|^2 + |z_t|^2 + |\nabla \bar{g}_t|^2_{l^2} \Big] dx dt \right] . \end{aligned}$$

which, substituting back the values of $\bar{f}, \bar{g}^k$ and $z = \partial_l v$ and then using Assumption A-5 with $n = 2$ and (27), gives

$$\begin{aligned} &\mathbb{E} \sup_{0 \le t \le T} |\partial_i \partial_l v_t|^2_{L^2(\mathscr{D}'')} + \mathbb{E} \int_0^T |\partial_i \partial_l v_t|^2_{H^1(\mathscr{D}'')} dt \\ &\le N \Bigg[\mathbb{E} \int_{\mathscr{D}'} \sum_{|\gamma| \le 2} |D^\gamma v_0|^2 dx + \mathbb{E} \int_0^T \int_{\mathscr{D}'} \Big[|\nabla v_t|^2 + \sum_{|\gamma| \le 1} |D^\gamma f_t|^2 + |v_t|^2 \\ &\qquad + \sum_{|\gamma| \le 2} |D^\gamma g_t|^2_{l^2} \Big] dx dt \Bigg] \end{aligned}$$

for all $i = 1, \dots, d$ and open $\mathscr{D}'' \Subset \mathscr{D}'$ where $N = N(\mathscr{D}'', d, T, K, \kappa)$. Repeating the above procedure k times, we have the following result.

Lemma 6. *Assume that v is a continuous $L^2(\mathscr{D})$-valued adapted process satisfying* (26) *and such that $v \in L^2(\Omega \times (0,T), \mathscr{P}; H^1(\mathscr{D}))$. If Assumptions A-2, A-5 and A-6 hold for*

$n = k$, *then*

$$\mathbb{E}\sup_{0\le t\le T}|\partial_{i_k}\ldots\partial_{i_1}v_t|^2_{L^2(\mathscr{D}^k)}+\mathbb{E}\int_0^T|\partial_{i_k}\ldots\partial_{i_1}v_t|^2_{H^1(\mathscr{D}^k)}dt$$
$$\le N\Bigg[\mathbb{E}\int_{\mathscr{D}^{k-1}}\sum_{|\gamma|\le k}|D^\gamma v_0|^2dx+\mathbb{E}\int_0^T\int_{\mathscr{D}^{k-1}}\Big[|\nabla v_t|^2+\sum_{|\gamma|\le k-1}|D^\gamma f_t|^2$$
$$+|v_t|^2+\sum_{|\gamma|\le k}|D^\gamma g_t|^2_{l^2}\Big]dxdt\Bigg]$$

for all $i_k = 1,\ldots,d$ *and open* $\mathscr{D}^k \Subset \mathscr{D}^{k-1}$ *where* $N = N(\mathscr{D}^k, d, T, K, \kappa)$.

We immediately see that Theorem 3 follows from Lemma 6. Using Theorems 1 and 3, we can now prove Theorem 2.

Proof of Theorem 2. Let u be the solution to (1) given by Theorem 1. Then considering $f_t(u_t,\nabla u_t)+f_t^0$ as a new free term f_t, we observe that u satisfies (26) with such free term.

Now under the Assumptions A-3, A-4 and due to Theorem 1, applied with $p \ge 2\alpha-2$, we get the estimate (3) and hence

$$\begin{aligned}\mathbb{E}\int_0^T|f_t|^2_{L^2(\mathscr{D})}dt=&\mathbb{E}\int_0^T\int_{\mathscr{D}}|f(u_t,\nabla u_t)+f_t^0|^2dxdt\\ \le&2\Big[\mathbb{E}\int_0^T\int_{\mathscr{D}}K^2(1+|u_t|)^{2\alpha-2}dxdt+\mathbb{E}\int_0^T\int_{\mathscr{D}}|f_t^0|^2dxdt\Big]\\ \le&N\Big[1+\mathbb{E}\sup_{0\le t\le T}\int_{\mathscr{D}}|u_t|^{2\alpha-2}dx\Big]+2\mathbb{E}\int_0^T\int_{\mathscr{D}}|f_t^0|^2dxdt<\infty.\end{aligned}$$

Hence we can apply Theorem 3 with $n = 1$ thus proving the first claim.

Moreover if f is a function of t,ω,x and r only such that (25) holds, then taking $f_t(u_t)+ f_t^0$ as a new free term f_t, similarly as above, we get

$$\mathbb{E}\int_0^T|\partial_if_t|^2_{L^2(\mathscr{D})}dt=\mathbb{E}\int_0^T\int_{\mathscr{D}}|\partial_iu_t\,\partial_rf_t(u_t)+\partial_if_t(u_t)+\partial_if_t^0|^2dxdt$$
$$\le N\mathbb{E}\int_0^T\int_{\mathscr{D}}\big[|\nabla u_t|^2(1+|u_t|)^{2\alpha-4}+(1+|u_t|)^{2\alpha-2}+|\partial_if_t^0|^2\big]dxdt$$
$$\le N\mathbb{E}\int_0^T\int_{\mathscr{D}}\big[1+|\nabla u_t|^2+|\nabla u_t|^2|u_t|^{2\alpha-4}+|u_t|^{2\alpha-2}+|\partial_if_t^0|^2\big]dxdt<\infty$$

for any $i\in\{1,\ldots,d\}$. Hence $f(u)+f^0$ is in $L^2(\Omega\times(0,T),\mathscr{P},H^1(\mathscr{D}))$. Thus all the conditions of Theorem 3 are satisfied for $n = 2$. This yields the second claim. □

4. Regularity in Weighted Spaces using L^p-theory & Time Regularity

In this section, we raise the regularity of the solution to the SPDE (1) using L^p-theory from Kim [12]. The reason for using L^p-theory is that one gets better estimates for the solution of the corresponding linear equation, see Theorem 4, given below, which follows immediately from Kim [12, Theorem 2.9].

We will use this together with the L^p-estimates we proved in Theorem 1 to obtain regularity results (both space and time) for the solution of the semilinear equation (1), see Theorems 5 and 6 below. In particular we obtain Hölder continuity in time of order $\frac{1}{2}-\frac{2}{q}$ for the solution to (1) as a process in weighted L^q-space, where q comes from the integrability assumptions imposed on the data.

First, we introduce some notations, concepts and assumptions from Kim [12]. For $r_0 > 0$ and $x\in\mathbb{R}^d$, let $B_{r_0}(x) := \{y\in\mathbb{R}^d : |x-y| < r_0\}$.

Definition 2 (Domain of class C^1_u)**.** The domain $\mathscr{D} \subset \mathbb{R}^d$ is said to be of class C^1_u if for any $x_0 \in \partial\mathscr{D}$, there exist $r_0, K_0, L_0 > 0$ and a one-one, onto continuously differentiable map $\Psi : B_{r_0}(x_0) \to G$, for a domain $G \subset \mathbb{R}^d$, satisfying the following:

(i) $\Psi(x_0) = 0$ and $\Psi\big(B_{r_0}(x_0) \cap \mathscr{D}\big) \subset \{y \in \mathbb{R}^d : y^1 > 0\}$,

(ii) $\Psi\big(B_{r_0}(x_0) \cap \partial\mathscr{D}\big) = G \cap \{y \in \mathbb{R}^d : y^1 = 0\}$,

(iii) $|\Psi|_{C^1(B_{r_0}(x_0))} \leq K_0$ and $|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \leq K_0|y_1 - y_2|$ for any $y_1, y_2 \in G$,

(iv) $|\Psi_x(x_1) - \Psi_x(x_2)| \leq L_0|x_1 - x_2|$ for any $x_1, x_2 \in B_{r_0}(x_0)$.

Let $\mathscr{D}$ be of class C^1_u and $\rho(x) := \mathrm{dist}(x, \partial\mathscr{D})$. Then, by [12, Lemma 2.5], there exists a bounded real valued function ψ defined on $\bar{\mathscr{D}}$ satisfying

$$\sup_{x\in\mathscr{D}} \rho^{|\gamma|}(x)|D^\gamma \partial_i \psi(x)| < \infty \tag{38}$$

for any $i = 1, \ldots, d$ and any multi-index γ. Further it follows from Remark 2.7 in [13] and from the boundedness of $\mathscr{D}$ that for some constant N

$$\frac{1}{N}\rho \leq \psi \leq N\rho \;\text{ in }\; \mathcal{D}.$$

In other words, ψ and ρ are comparable in $\mathscr{D}$, and in estimates they can be used interchangeably (up to multiplication by a constant). Moreover this implies $\psi \geq 0$.

For $1 \leq q < \infty$, $\theta \in \mathbb{R}$ and a non-negative integer n, define the weighted Sobolev space $H^{n,q}_\theta(\mathscr{D})$ by

$$H^{n,q}_\theta(\mathscr{D}) := \{u : \rho^{|\gamma|+(\theta-d)/q} D^\gamma u \in L^q(\mathscr{D}) \;\text{ for any }\; |\gamma| \leq n\}$$

where the norm for $u \in H^{n,q}_\theta(\mathscr{D})$ is given by

$$|u|^q_{H^{n,q}_\theta} := \sum_{i=0}^{n} \sum_{|\gamma|=i} \int_{\mathscr{D}} |D^\gamma u(x)|^q \rho^{\theta-d+iq}(x)dx.$$

For functions $u : \mathbb{R}^d \to \mathbb{R}^{d'}$, we define the norm analogously and use the same notation. The following result from Lototsky [17] plays an important role in proving our results.

Remark 5. The following are equivalent:

(i) $u \in H^{n,q}_\theta(\mathscr{D})$,

(ii) $u \in H^{n-1,q}_\theta(\mathscr{D})$ and $\psi\partial_i u \in H^{n-1,q}_\theta(\mathscr{D})$ for all $i = 1, 2, \ldots d$,

(iii) $u \in H^{n-1,q}_\theta(\mathscr{D})$ and $\partial_i(\psi u) \in H^{n-1,q}_\theta(\mathscr{D})$ for all $i = 1, 2, \ldots d$.

Further, let

$$\mathbb{H}^{n,q}_\theta(\mathscr{D}) := L^q(\Omega \times (0, T), \mathscr{P}, H^{n,q}_\theta(\mathscr{D})).$$

In the rest of the article, we assume that

$$q \geq 2 \quad \text{and} \quad d - 2 + q < \theta < d - 1 + q \tag{39}$$

so that in view of [12, Remark 2.7], the assumption regarding existence of an $\mathscr{A}_{p,\theta}$-type set (see [12, Assumption 2.8]), is satisfied. Finally, we need the following assumption on the coefficients:

A - 7. For any $i, j = 1, \ldots, d$,

(i) the real valued coefficients a^{ij} and their spatial derivatives up to order $n + 1$ are $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and bounded by K,

(ii) the real-valued coefficients b^i, c and their spatial derivatives up to order n are $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and are bounded by K,

(iii) the coefficients $\sigma^i = (\sigma^{ik})_{k=1}^\infty$, $\mu = (\mu^k)_{k=1}^\infty$ and their spatial derivatives up to order $n + 1$ are l^2-valued $\mathscr{P} \times \mathscr{B}(\mathscr{D})$-measurable and almost surely

$$\sum_{i=1}^{d} \sum_{k\in\mathbb{N}} \sum_{|\gamma|\leq n+1} |D^\gamma \sigma^{ik}_t(x)|^2 + \sum_{k\in\mathbb{N}} \sum_{|\gamma|\leq n+1} |D^\gamma \mu^k_t(x)|^2 \leq K$$

for all t and x,

(iv) and for almost every (t,ω), the coefficients $a^{ij}(t,x)$ and $\sigma^i(t,x)$ are uniformly continuous in $x \in \mathscr{D}$.

Note that, the operator L given by (2) is in divergence form but the results from [12] are for operators in non-divergence form. One knows that (1) can be expressed in non-divergence form if the coefficients a^{ij} are differentiable. Thus Assumption A-7 implies Assumptions 2.2 and 2.3 in [12]. Hence the following theorem follows from Theorem 2.9 of Kim [12].

Theorem 4. *Assume $\mathscr{D}$ is of class C^1_u. Further, let Assumptions A-2 and A-7 hold with some $n \geq 0$. If $\psi f \in \mathbb{H}^{n,q}_\theta(\mathscr{D})$, $g \in \mathbb{H}^{n+1,q}_\theta(\mathscr{D}; l^2)$ and $\psi^{\frac{2}{q}-1}\phi \in \mathbb{H}^{n+2,q}_\theta(\mathscr{D})$, then*

$$\begin{cases} dv_t = (L_t v_t + f_t)dt + \sum_{k\in\mathbb{N}} (M^k_t v_t + g^k_t) dW^k_t \quad \text{on } [0,T]\times\mathscr{D}, \\ v_t = 0 \ \text{ on } \partial\mathscr{D}, \quad v_0 = \phi \ \text{ on } \mathscr{D} \end{cases} \tag{40}$$

has a unique solution v such that $\psi^{-1}v \in \mathbb{H}^{n+2,q}_\theta(\mathscr{D})$.

In fact Theorem 2.9 in Kim [12] is proved even for fractional weighted Sobolev spaces and under somewhat weaker assumptions. We do not use fractional spaces here to keep the presentation simpler. As to being able to use weaker assumptions: to obtain results for the semilinear equation (1) we will need to apply our results from Section 2, in particular Theorem 1 and thus we cannot substantially weaken our assumptions here. Finally, we can state the main results on regularity for the solution to semilinear SPDE (1).

Theorem 5. *Assume $\mathscr{D}$ is of class C^1_u and u is the solution to* (1)*. Further, let Assumptions A-2 to A-4 hold with $p \geq \max(q\alpha - q, 2)$ and Assumption A-7 holds with $n = 0$. If for some q satisfying* (39)*, $\psi^{\frac{2}{q}-1}\phi \in \mathbb{H}^{2,q}_\theta(\mathscr{D}), g \in \mathbb{H}^{1,q}_\theta(\mathscr{D}; l^2)$ and $f^0 \in \mathbb{H}^{0,q}_\theta(\mathscr{D})$, then $\psi^{-1}u \in \mathbb{H}^{2,q}_\theta(\mathscr{D})$.*

Moreover, in the case Assumption A-7 holds with $n = 1$ and almost surely

$$\begin{aligned} &|\partial_i f_t(x,r,z)| \leq K(1+|r|)^{\alpha-1}, \ \ |\partial_r f_t(x,r,z)| \leq K(1+|r|)^{\alpha-2} \\ &\text{and } \ |\partial_z f_t(x,r,z)| \leq K(1+|r|)^{\alpha-1} \end{aligned} \tag{41}$$

for all $i = 1,\ldots,d$, $t \in [0,T], x \in \mathscr{D}, r \in \mathbb{R}$ and all $z \in \mathbb{R}^d$, if for some q satisfying (39)*, $\psi^{\frac{2}{q}-1}\phi \in \mathbb{H}^{3,q}_\theta(\mathscr{D})$, $g \in \mathbb{H}^{2,q}_\theta(\mathscr{D}; l^2)$ and $f^0 \in \mathbb{H}^{1,q}_\theta(\mathscr{D})$, then $\psi^{-1}u \in \mathbb{H}^{3,\frac{q}{2}}_\theta(\mathscr{D})$.*

Remark 6. Note that if $\psi^{-1}u \in \mathbb{H}^{2,q}_\theta(\mathscr{D})$, then by using Remark 5, we get

$$\psi^{-1}u \in \mathbb{H}^{1,q}_\theta(\mathscr{D}) \ \text{ and } \ \partial_i u \in \mathbb{H}^{1,q}_\theta(\mathscr{D}) \ \ \forall i = 1,2,\ldots d.$$

Invoking Remark 5 again, we have

$$\psi^{-1}u \in \mathbb{H}^{0,q}_\theta(\mathscr{D}), \ \ \partial_i u \in \mathbb{H}^{0,q}_\theta(\mathscr{D}) \ \text{ and } \ \psi\partial_i\partial_j u \in \mathbb{H}^{0,q}_\theta(\mathscr{D}) \ \ \forall i,j = 1,2,\ldots d.$$

Finally, we present the result on time regularity of the solution of (1).

Theorem 6. *Under the assumptions of Theorems 1 and 5,*

$$u \in C^{\frac{1}{2}-\frac{2}{q}}\big([0,T]; H^{0,q}_{\theta+q}(\mathscr{D})\big) \quad a.s.$$

i.e., the solution u to SPDE (1)*, as a $H^{0,q}_{\theta+q}(\mathscr{D})$- valued process, is Hölder continuous of order $\frac{1}{2} - \frac{2}{q}$ for every q satisfying* (39)*.*

Before proving these theorems, we first prove the following lemma:

Lemma 7. *Let $\tilde\theta > d$ and $\tilde q \geq 1$. Further, let assumptions of Theorem 1 hold with $p \geq \max(\tilde q\alpha - \tilde q, 2)$ and $f^0 \in \mathbb{H}^{0,\tilde q}_{\tilde\theta}(\mathscr{D})$. If u is the solution to* (1) *and $f_t := f_t(u_t, \nabla u_t) + f^0_t$, then $f \in \mathbb{H}^{0,\tilde q}_{\tilde\theta}(\mathscr{D})$ and thus $\psi f \in \mathbb{H}^{0,\tilde q}_{\tilde\theta}(\mathscr{D})$.*

Proof. First we note that $\tilde\theta > d$ and $\mathscr{D}$ is bounded, therefore $\sup_{x\in\mathscr{D}} \rho^{\tilde\theta-d}(x) < \infty$. Using this along with Assumption A-3 implies

$$\begin{aligned}\mathbb{E}\int_0^T\int_{\mathscr{D}} |f_t|^{\tilde q}\rho^{\tilde\theta-d}dxdt &= \mathbb{E}\int_0^T\int_{\mathscr{D}} |f_t(u_t,\nabla u_t)+f_t^0|^{\tilde q}\rho^{\tilde\theta-d}dxdt \\ &\le N\Big[\mathbb{E}\int_0^T\int_{\mathscr{D}}(1+|u_t|)^{\tilde q\alpha-\tilde q}dxdt + \mathbb{E}\int_0^T\int_{\mathscr{D}}|f_t^0|^q\rho^{\tilde\theta-d}dxdt\Big] \\ &\le N\Big[1+\mathbb{E}\sup_{0\le t\le T}|u_t|^{\tilde q\alpha-\tilde q}_{L^{\tilde q\alpha-\tilde q}}\Big] + N\mathbb{E}\int_0^T\int_{\mathscr{D}}|f_t^0|^{\tilde q}\rho^{\tilde\theta-d}dxdt\end{aligned} \tag{42}$$

which is finite in view of Theorem 1 and the fact $f^0 \in \mathbb{H}^{0,\tilde q}_{\tilde\theta}(\mathscr{D})$. Now note that ψ is bounded on $\bar{\mathscr{D}}$ and hence

$$\mathbb{E}\int_0^T\int_{\mathscr{D}}|\psi f_t|^q\rho^{\theta-d}\,dxdt \le N\mathbb{E}\int_0^T\int_{\mathscr{D}}|f_t|^q\rho^{\theta-d}\,dxdt < \infty\,.$$

□

Proof of Theorem 5. Let u be the solution to (1) given by Theorem 1. Then considering $f_t(u_t,\nabla u_t)+f_t^0$ as a new free term f_t, the solution u satisfies (40). We wish to apply Theorem 4 with $n=0$ and in order to do so we need to show that $\psi f \in \mathbb{H}^{0,q}_\theta(\mathscr{D})$. Indeed this follows immediately by using Lemma 7 with $\tilde\theta=\theta$ and $\tilde q = q$. Hence applying Theorem 4 with $n=0$ we obtain $\psi^{-1}u \in \mathbb{H}^{2,q}_\theta(\mathscr{D})$. This completes the proof of the first statement of the theorem.

We now consider the case when Assumption A-7 holds with $n=1$. Again we will apply Theorem 4 (but now with $n=1$ and $\frac{q}{2}$ in place of q) and so we need to show that $\psi f\in\mathbb{H}^{1,\bar q}_\theta(\mathscr{D})$ with $\bar q := \frac{q}{2}$. Taking $\tilde\theta=\theta$ and $\tilde q=\bar q$ in Lemma 7, we get $\psi f\in\mathbb{H}^{0,\bar q}_\theta(\mathscr{D})$. Thus we consider

$$\mathbb{E}\int_0^T\int_{\mathscr{D}}|\partial_i(\psi f_t)|^{\bar q}\rho^{\theta-d+\bar q}dxdt = I_1+I_2\,,$$

where,

$$I_1 := \mathbb{E}\int_0^T\int_{\mathscr{D}}|f_t|^{\bar q}|\partial_i\psi|^{\bar q}\rho^{\theta-d+\bar q}dxdt \;\text{ and }\; I_2 := \mathbb{E}\int_0^T\int_{\mathscr{D}}|\partial_i f_t|^{\bar q}\psi^{\bar q}\rho^{\theta-d+\bar q}dxdt\,.$$

Clearly $I_1<\infty$ using (38), the fact ρ is bounded on $\mathscr{D}$ and Lemma 7 (with $\tilde\theta=\theta$ and $\tilde q=\bar q$). Further observe that

$$\begin{aligned}\partial_i f_t &= \partial_i(f_t(u_t,\nabla u_t)+f_t^0)\\ &= \partial_i f_t(u_t,\nabla u_t) + \partial_i u_t\,\partial_r f_t(u_t,\nabla u_t) + \partial_i(\nabla u_t)\,\nabla_z f_t(u_t,\nabla u_t) + \partial_i f_t^0,\end{aligned}$$

where $\nabla_z f_t$ is the gradient with respect to z of $f_t = f_t(x,r,z)$. Thus, we have

$$I_2 \le N(I_3+I_4+I_5+I_6) \tag{43}$$

where,

$$\begin{aligned}I_3 &:= \mathbb{E}\int_0^T\int_{\mathscr{D}}|\partial_i f_t(u_t,\nabla u_t)|^{\bar q}\psi^{\bar q}\rho^{\theta-d+\bar q}\,dxdt\,,\\ I_4 &:= \mathbb{E}\int_0^T\int_{\mathscr{D}}|\partial_i u_t\,\partial_r f_t(u_t,\nabla u_t)|^{\bar q}\psi^{\bar q}\rho^{\theta-d+\bar q}\,dxdt\,,\\ I_5 &:= \mathbb{E}\int_0^T\int_{\mathscr{D}}|\partial_i(\nabla u_t)\,\nabla_z f_t(u_t,\nabla u_t)|^{\bar q}\psi^{\bar q}\rho^{\theta-d+\bar q}\,dxdt\,,\end{aligned}$$

and

$$I_6 := \mathbb{E}\int_0^T\int_{\mathscr{D}}|\partial_i f_t^0|^{\bar q}\psi^{\bar q}\rho^{\theta-d+\bar q}\,dxdt\,.$$

Now, using the fact that ψ and ρ are bounded on $\mathscr{D}$ and the assumption on growth of derivatives of the semilinear term, see (41), we observe that

$$I_3 \leq N\mathbb{E}\int_0^T \int_{\mathscr{D}} \big(1 + |\partial_i f_t(u_t, \nabla u_t)|\big)^q dxdt \leq N\Big[1 + \mathbb{E}\int_0^T \int_{\mathscr{D}} (1+|u_t|)^{q\alpha - q} dxdt\Big] .$$

This is finite in view of Theorem 1, see the estimate (42) for details. Further, using Young's inequality and the fact that ψ and ρ are bounded on $\mathscr{D}$ along with growth assumption (41), we get

$$\begin{aligned} I_4 &\leq N\mathbb{E}\int_0^T \int_{\mathscr{D}} \Big[|\partial_i u_t|^q + |\partial_r f_t(u_t, \nabla u_t)|^q\Big]\rho^{\theta-d} dxdt \\ &\leq N\Big[|\partial_i u|^q_{\mathbb{H}^{0,q}_\theta} + \mathbb{E}\int_0^T \int_{\mathscr{D}} (1+|u_t|)^{q\alpha-2q} dxdt\Big] . \end{aligned}$$

We see that this is finite using Remark 6 and Theorem 1 again. Furthermore, using Young's inequality, growth assumption (41) and the fact that ψ and ρ are comparable, we obtain

$$\begin{aligned} I_5 &\leq N\mathbb{E}\int_0^T \int_{\mathscr{D}} \Big[|\partial_i(\nabla u_t)|^q + |\nabla_z f_t(u_t, \nabla u_t)|^q\Big]\psi^q\rho^{\theta-d} dxdt \\ &\leq N\Big[|\psi\partial_i(\nabla u)|^q_{\mathbb{H}^{0,q}_\theta} + \mathbb{E}\int_0^T \int_{\mathscr{D}} (1+|u_t|)^{q\alpha-q} dxdt\Big]. \end{aligned}$$

Thus, applying Remark 6 and Theorem 1 as before, we obtain $I_5 < \infty$. Finally, the fact that ψ and ρ are comparable and bounded on $\mathscr{D}$ implies

$$I_6 \leq N\mathbb{E}\int_0^T \int_{\mathscr{D}} \big(1 + |\partial_i f^0_t|\big)^q \rho^{\theta-d+q} dxdt \leq N\Big[1 + \mathbb{E}\int_0^T \int_{\mathscr{D}} |\partial_i f^0_t|^q \rho^{\theta-d+q} dxdt\Big]$$

which is finite since $f^0 \in \mathbb{H}^{1,q}_\theta(\mathscr{D})$. Thus $\psi f \in \mathbb{H}^{1,\bar{q}}_\theta(\mathscr{D})$ and we can apply Theorem 4 with $n = 1$ and $\bar{q}$ in place of q to complete the proof. □

Proof of Theorem 6. We will prove the result using Kolmogorov continuity theorem. To ease the notation we let $f_t := f_t(u_t, \nabla u_t) + f^0_t$. Then from (1) we see that

$$\mathbb{E}|u_t - u_s|^q_{H^{0,q}_{\theta+q}} \leq 2^{q-1}(I_1(s,t) + I_2(s,t)), \tag{44}$$

where

$$I_1(s,t) := \mathbb{E}\Big|\int_s^t (L_r u_r + f_r) dr\Big|^q_{H^{0,q}_{\theta+q}} \text{ and } I_2(s,t) := \Big|\sum_{k\in\mathbb{N}}\int_s^t (M^k_r u_r + g^k_r) dW^k_r\Big|^q_{H^{0,q}_{\theta+q}} .$$

We note that $f^0 \in \mathbb{H}^{0,q}_\theta(\mathscr{D})$ implies $f^0 \in \mathbb{H}^{0,q}_{\theta+q}(\mathscr{D})$ because ρ is bounded on $\mathscr{D}$. Now using Hölder's inequality, we get

$$\begin{aligned} I_1(s,t) &\leq (t-s)^{q-1}\mathbb{E}\int_s^t |L_r u_r + f_r|^q_{H^{0,q}_{\theta+q}} dr \\ &\leq N(t-s)^{q-1}\Big[\mathbb{E}\int_s^t |L_r u_r|^q_{H^{0,q}_{\theta+q}} dr + \mathbb{E}\int_s^t |f_r|^q_{H^{0,q}_{\theta+q}} dr\Big] . \end{aligned} \tag{45}$$

Using Assumption A-7 with $n = 0$, we get

$$\begin{aligned} |L_r u_r|^q_{H^{0,q}_{\theta+q}} &= \int_{\mathscr{D}} \Big|\sum_{j=1}^d \partial_j\Big(\sum_{i=1}^d a^{ij}_t \partial_i u_r\Big) + \sum_{i=1}^d b^i_t \partial_i u_r + c_t u_r\Big|^q \rho^{\theta+q-d} dx \\ &\leq N\int_{\mathscr{D}} \Big(\sum_{i,j=1}^d |\partial_i\partial_j u_r|^q + \sum_{i=1}^d |\partial_i u_r|^q + |u_r|^q\Big)\rho^{\theta+q-d} dx \\ &\leq N\left(\sum_{i,j=1}^d |\psi\partial_i\partial_j u_r|^q_{H^{0,q}_\theta} + |\psi|^q_{C(\bar{\mathscr{D}})}\sum_{i=1}^d |\partial_i u_r|^q_{H^{0,q}_\theta} + |\psi|^{2q}_{C(\bar{\mathscr{D}})}|\psi^{-1}u_r|^q_{H^{0,q}_\theta}\right). \end{aligned}$$

Substituting this in (45) and using the fact that ψ is bounded on $\bar{\mathscr{D}}$, we obtain

$$\begin{aligned} &I_1(s,t) \\ &\le N(t-s)^{q-1}\Bigg(\sum_{i,j=1}^d |\psi\partial_i\partial_j u|^q_{\mathbb{H}^{0,q}_\theta} + \sum_{i=1}^d |\partial_i u|^q_{\mathbb{H}^{0,q}_\theta} + |\psi^{-1}u|^q_{\mathbb{H}^{0,q}_\theta} + |f|^q_{\mathbb{H}^{0,q}_{\theta+q}}\Bigg) \\ &\le N(t-s)^{q-1}, \end{aligned} \tag{46}$$

where last statement follows using Remark 6 and Lemma 7 with $\tilde{\theta} = \theta + q$ and $\tilde{q} = q$.

Furthermore using Burkholder–Davis–Gundy's inequality, Assumption A-7 with $n = 0$, Hölder's inequality and the fact that ρ is bounded on $\mathscr{D}$, we see that

$$\begin{aligned} I_2(s,t) &= \mathbb{E}\int_{\mathscr{D}} \Big| \sum_{k\in\mathbb{N}} \int_s^t (M_r^k u_r + g_r^k) dW_r^k \Big|^q \rho^{\theta+q-d} dx \\ &\le \int_{\mathscr{D}} \mathbb{E}\Big[\int_s^t \sum_{k\in\mathbb{N}} |M_r^k u_r + g_r^k|^2 dr\Big]^{\frac{q}{2}} \rho^{\theta+q-d} dx \\ &= \int_{\mathscr{D}} \mathbb{E}\Big[\int_s^t \sum_{k\in\mathbb{N}} \Big| \sum_{i=1}^d \sigma_r^{ik}\partial_i u_r + \mu_r^k u_r + g_r^k\Big|^2 dr\Big]^{\frac{q}{2}} \rho^{\theta+q-d} dx \\ &\le N\int_{\mathscr{D}} \mathbb{E}\Big[\int_s^t \Big(\sum_{i=1}^d |\partial_i u_r|^2 + |u_r|^2 + \sum_{k\in\mathbb{N}} |g_r^k|^2\Big) dr\Big]^{\frac{q}{2}} \rho^{\theta+q-d} dx \\ &\le N\int_{\mathscr{D}} (t-s)^{\frac{q}{2}-1}\mathbb{E}\Big[\int_s^t \Big(\sum_{i=1}^d |\partial_i u_r|^q + |u_r|^q + |g_r|^q_{l^2}\Big) dr\Big] \rho^{\theta+q-d} dx \\ &\le N(t-s)^{\frac{q}{2}-1}\Bigg(\sum_{i=1}^d |\partial_i u|^q_{\mathbb{H}^{0,q}_\theta} + |\psi^{-1}u|^q_{\mathbb{H}^{0,q}_\theta} + |g|^q_{\mathbb{H}^{0,q}_\theta}\Bigg) \le N(t-s)^{\frac{q}{2}-1}. \end{aligned} \tag{47}$$

Here, the last inequality is obtained using Remark 6 as before and the assumption that $g \in \mathbb{H}^{1,q}_\theta(\mathscr{D}; l^2)$. Using (46) and (47) in (44), we obtain

$$\mathbb{E}|u_t - u_s|^q_{H^{0,q}_\theta} \le N|t-s|^{\frac{q}{2}-1}$$

which on using Kolmogorov Continuity theorem concludes the result. □

REFERENCES

[1] D. Breit, Regularity theory for nonlinear systems of SPDEs, *arXiv:1311.2001v4* (2016).

[2] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2010.

[3] F. Flandoli, Dirichlet boundary value problem for stochastic parabolic equations: compatibility relations and regularity of solutions, *Stochastics* (1990), 29(3), 331–357.

[4] B. Gess, Strong solutions for stochastic partial differential equations of gradient type, *J. Funct. Anal.* (2012), 263(8), 2355–2383.

[5] A. Jentzen, M. Röckner, Regularity analysis for stochastic partial differential equations with nonlinear multiplicative trace class noise *J. Differ. Equations* (2012), 252(1), 114–136.

[6] K. Dareiotis, M. Gerencsér, On the boundedness of solutions of SPDEs, *Stoch. Partial Differ. Equ. Anal. Comput.* (2015), 3(1), 84–102.

[7] M. Gerencsér, Boundary regularity of stochastic PDEs, *arXiv:1705.05364* (2017).

[8] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island, 1998.

[9] I. Gyöngy, D.Šiška, Itô Formula for Processes Taking Values in Intersection of Finitely Many Banach Spaces, *Stoch. Partial Differ. Equ. Anal. Comput.* (2017).

[10] I. Kim, K.-H. Kim, Some L_p and Hölder Estimates for Divergence Type Nonlinear SPDEs on C^1-Domains, *Potential Anal.* (2014) 41, 583–612.

[11] I. Kim, K.-H. Kim, A Regularity Theory for Quasi-Linear Stochastic Partial Differential Equations in Weighted Sobolev Spaces, *arXiv:1705.01717* (2017).

[12] K.-H. Kim, On stochastic partial differential equations with variable coefficients in C^1 domains, *Stoch. Proc. Appl.* (2004) 112, 261–283.
[13] K.-H. Kim, N. V. Krylov, On the Sobolev space theory of parabolic and elliptic equations in C^1 domains, *SIAM J. Math. Anal.* (2004) 36(2), 618–642.
[14] N. V. Krylov, A relatively short proof of Itô's formula for SPDEs and its applications, *Stoch. Partial Differ. Equ. Anal. Comput.* (2013) 1(1), 152–174.
[15] N. V. Krylov, A W_2^n-theory of the Dirichlet problem for SPDEs in general smooth domains, *Probab. Theory Relat. Fields* (1994) 98, 389–421.
[16] N. V. Krylov, B. L. Rozovskii, Stochastic Evolution Equations, *Journal of Soviet Mathematics*, (1981) 14, 1233-1277.
[17] S. V. Lototsky, Sobolev spaces with weights in domains and boundary value problems for degenerate elliptic equations, *Methods and Applications of Analysis* (2000) 7(1), 195–204.
[18] E. Pardoux, Equations aux dérivées partielles stochastiques non lineaires monotones. Etude des solutions forte de type Ito, Thése Doct. Sci. Math. Univ. Paris Sud., 1975.
[19] B.L. Rozovskii, *Stochastic Evolution Systems. Linear Theory and Applications to Nonlinear Filtering*, Kluwer, Dordrecht, 1990.

SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, UNITED KINGDOM AND RAMJAS COLLEGE, UNIVERSITY OF DELHI, DELHI, INDIA
E-mail address: N.Neelima@sms.ed.ac.uk

SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, UNITED KINGDOM
E-mail address: D.Siska@ed.ac.uk