

ON GLOBAL OKOUNKOV BODIES OF SPHERICAL VARIETIES

GUIDO PEZZINI AND HENRIK SEPPÄNEN

ABSTRACT. We define and study the *global Okounkov moment cone* of a projective spherical variety X , generalizing both the global Okounkov body and the *moment body* of X defined by Kaveh and Khovanskii. Under mild assumptions on X we show that the global Okounkov moment cone of X is rational polyhedral. As a consequence, also the global Okounkov body of X , with respect to a particular valuation, is rational polyhedral.

INTRODUCTION

Started by the work of A. Okounkov, cf. [Ok96], the theory of Okounkov bodies has been widely developed in recent years, in connection with other research topics such as representation theory, asymptotics of linear series and positivity, and tropical geometry, see e.g. [LM09], [KK09], [KL15], [Se16], [SS17], [PU16].

In this short note, we study a convex cone associated to a projective variety X , spherical under the action of a semisimple and simply connected group G . The convex cone, called *global Okounkov moment cone* and denoted by¹ $\tilde{\Delta}_\nu(X)$, is closely related to the *global Okounkov body*, which we denote here by $\Delta_\nu(X)$. The latter is a “global version” of the Okounkov bodies associated to pseudo effective divisors of projective varieties.

We recall that the construction of $\Delta_\nu(X)$ depends on the choice of a function

$$\nu : \bigsqcup_{D \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D)) \setminus \{0\} \rightarrow \mathbb{N}_0^n,$$

where $n = \dim X$, called a *valuation-like* function.

In the case of a general smooth projective variety X , such function is often constructed using a flag of subvarieties $X = Y_0 \supset Y_1 \supset \dots \supset Y_n = \{\text{pt}\}$, where Y_i is a smooth prime divisor of Y_{i-1} for all $i \in \{1, \dots, n\}$. For D a Cartier divisor and $s \in H^0(X, \mathcal{O}_X(D))$, one defines $\nu(s) = (\nu_1(s), \dots, \nu_n(s))$ setting first $\nu_1(s) = \text{ord}_{Y_1}(s)$. Then, by restriction on Y_1 , the section s determines naturally a section $s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1(s)Y_1))$, and we set $\nu_2(s) = \text{ord}_{Y_2}(s_1)$. Iterating this construction defines $\nu_i(s)$ for all $i \in \{1, \dots, n\}$. Other definitions of valuation-like functions are possible, and sometimes more convenient. This is the case e.g. for flag varieties, where Schubert varieties are natural candidates for the elements of the flag, but the possible

The first author was supported by DFG grant BA 1559/6-1. The second author was partially supported by the DFG Priority Programme 1388 “Representation Theory”.

¹The meaning of ν in this notation is recalled below.

failure of their smoothness leads to alternative constructions. One such, which uses Bott-Samelson varieties, is described in [SS17].

Once the function ν is given, the global Okounkov body is defined as the closed convex cone in $\mathbb{R}^n \times N^1(X)$ generated by all pairs of the form $(\nu(s), [D]) \in \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$ where D is effective and s is as above.

In the setting of spherical varieties it is convenient to use valuation-like functions defined using a mixture of the two constructions mentioned above, see Section 2 below for details.

In addition, given the importance of the structure of G -module of the space of sections $H^0(X, \mathcal{O}_X(D))$, it is natural to add to the points of the global Okounkov body a third component. We assume that s belongs to a simple submodule V of $H^0(X, \mathcal{O}_X(D))$ (we show in Section 2 that this assumption is harmless), and, instead of the pair $(\nu(s), [D])$, we consider the triple $(\nu(s), [D], \lambda)$ where λ is the highest weight of V . The closed convex cone generated by such triples is the global Okounkov moment cone $\tilde{\Delta}_{\nu}(X)$, see Definition 2.7 below. This construction can also be considered a generalization of the *moment body* of X defined by Kaveh-Khovanskii in [KK12].

Our main result is Theorem 2.19 below, where we show under mild hypotheses on X that $\tilde{\Delta}_{\nu}(X)$ is rational polyhedral. This also implies that $\Delta_{\nu}(X)$ is rational polyhedral, answering in our setting a question posed in [LM09].

We end this introduction by remarking that our varieties are Mori dream spaces (see [P14, Theorem 4.1.1]), as defined by Hu and Keel ([HK00]). As a consequence, they also admit valuations defining rational polyhedral Okounkov bodies by the method of Postingshel and Urbinati ([PU16]) that uses tropical compactifications. It is not clear to us, however, whether our valuation can be related to theirs.

1. SETTING

Let G be a simply connected semisimple complex algebraic group, let $B \subseteq G$ be a Borel subgroup and $T \subseteq B$ a maximal torus. We denote by W the Weyl group of G . The Lie algebras of G, B , and T will be denoted by the corresponding lower case German letter. Given a dominant weight λ , we denote by $V(\lambda)$ the irreducible G -module of highest weight λ .

We recall that an irreducible normal G -variety is *spherical* if B has an open orbit on X .

Throughout the paper X will be a projective \mathbb{Q} -factorial spherical G -variety of dimension n . We shall further assume that X admits G -invariant Cartier prime divisors E_1, \dots, E_r with $E_i = Z(s_i)$, for a G -invariant section $s_i \in H^0(X, \mathcal{O}_X(E_i))$, with combinatorial normal crossings such that the intersection $E_1 \cap \dots \cap E_r$ is a closed G -orbit Gx_0 . Then $Gx_0 \cong G/P$ for a parabolic subgroup P containing B . We assume that all intersections of the E_i are smooth along the orbit Gx_0 .

An example where these assumptions are satisfied is the case where X is a *wonderful variety*, see e.g. [Pe17].

We also consider a B -equivariant birational morphism

$$(1) \quad \tau : Z \longrightarrow G/P,$$

where Z is a Bott-Samelson variety defined by a reduced sequence $w = (s_{i_1}, \dots, s_{i_m})$, where $s_{i_j} \in W$ are simple reflections (cf. e.g. [J03, Section 13.10]).

We recall that any line bundle on X has a unique G -linearization, by [KKLV89].

2. THE GLOBAL OKOUNKOV MOMENT CONE

We define a partial flag of subvarieties

$$X := Y_0 \supset Y_1 \supset \dots \supset Y_r = Gx_0,$$

of X by

$$Y_i := \bigcap_{j=1}^i E_j,$$

i.e., $Y_i = Z(s_1, \dots, s_i)$, for $i = 1, \dots, r$.

We now define a valuation-like function with values in \mathbb{N}_0^n on the disjoint union $\bigsqcup_{D \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D)) \setminus \{0\}$ of all section spaces of all line bundles by first using the \mathbb{N}_0 -valued function defined by the partial flag above, and then continuing with the valuation-like function defined by the “vertical flag” on a Bott-Samelson variety (cf. [SS17]).

We thus first define

$$\nu' : \bigsqcup_{D \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D)) \setminus \{0\} \rightarrow \mathbb{N}_0^r$$

by $\nu'(s) := (a_1, \dots, a_r)$, for a Cartier divisor D and nonzero section $s \in H^0(X, \mathcal{O}_X(D))$, where

$$(2) \quad \begin{aligned} a_1 &:= \text{ord}_{Y_1}(s), \\ a_2 &:= \text{ord}_{Y_2} \left(\frac{s}{s_1^{a_1}} \mid_{Y_1} \right), \\ &\vdots \\ a_r &:= \text{ord}_{Y_r} \left(\frac{s}{s_1^{a_1} \dots s_{r-1}^{a_{r-1}}} \mid_{Y_{r-1}} \right). \end{aligned}$$

Recall the valuation-like function

$$\widetilde{\nu}'' : \bigsqcup_{D \in \text{Pic}(Z)} H^0(Z, \mathcal{O}_Z(D)) \setminus \{0\} \rightarrow \mathbb{N}_0^m$$

defined by the “vertical flag” of Bott-Samelson-subvarieties of Z with respect to the reduced expression $w = (s_{i_1}, \dots, s_{i_m})$. This is a flag of Bott-Samelson varieties Z_j , where Z_j is defined by the reduced subsequence $(s_{i_{j+1}}, \dots, s_{i_m})$

of w (cf. [SS17]). Using the birational map $\tau : Z \rightarrow G/P$ (eq. (1)), we then define

$$\begin{aligned} \nu'' : \bigsqcup_{D \in \text{Pic}(G/P)} H^0(G/P, \mathcal{O}_{G/P}(D)) \setminus \{0\} &\rightarrow \mathbb{N}_0^m, \\ \nu''(s) &:= \widetilde{\nu''}(\tau^* s), \quad s \in H^0(G/P, \mathcal{O}_{G/P}(D)). \end{aligned}$$

By putting $(a_{r+1}, \dots, a_n) := \nu''(\frac{s}{s_1^{a_1} \dots s_r^{a_r}}|_{Y_r})$ (cf. (2)), and defining $\nu(s) := (a_1, \dots, a_n)$, we obtain a valuation-like function

$$(3) \quad \nu : \bigsqcup_{D \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D)) \setminus \{0\} \rightarrow \mathbb{N}_0^n.$$

We write $\nu(s) = (\nu_1(s), \dots, \nu_n(s))$, for a section s of some line bundle.

One defines similarly a valuation-like function ν_{Y_i} for sections of line bundles on Y_i , for any variety Y_i belonging to the flag Y_\bullet .

Remark 2.1. We recall that, since X is spherical, for any line bundle \mathcal{L} on X the space of sections $H^0(X, \mathcal{L})$ is a multiplicity-free G -module, that is, any simple submodule appears with multiplicity 1. This fact is used in the proof of the following proposition.

Proposition 2.2. *Let D be an effective divisor on X , and consider the decomposition*

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{\lambda} V(\lambda)$$

of the section space $H^0(X, \mathcal{O}_X(D))$ into a direct sum of irreducible G -modules.

(i) *For any $V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$, and any two nonzero sections $s, t \in V(\lambda)$, we have $\nu'(s) = \nu'(t)$.*

(ii) *If $s \in V(\lambda), t \in V(\mu)$ are nonzero sections for distinct highest weights λ and μ , then $\nu'(s) \neq \nu'(t)$. In particular, $\nu(s) \neq \nu(t)$.*

Proof. For the first part, note that the G -invariance of the Y_i implies that ν' is G -invariant, i.e., $\nu'(gs) = \nu'(s)$, for any nonzero $s \in H^0(X, \mathcal{O}_X(D))$, and $g \in G$. Moreover, $\nu'(s+t) \geq \min\{\nu'(s), \nu'(t)\}$, for any nonzero $s, t \in H^0(X, \mathcal{O}_X(D))$ such that $s+t \neq 0$.

Now, if $s \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$ is a nonzero section, it is a cyclic vector for the G -module. Hence, if $t \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$, we can write t as

$$t = \sum_{i=1}^m c_i g_i s,$$

for some $g_1, \dots, g_m \in G$, and $c_1, \dots, c_m \in \mathbb{C}$. The G -invariance and the valuation-like properties of ν' then yield

$$\nu(t) \geq \min\{\nu(g_i s) \mid i = 1, \dots, m\} = \nu(s).$$

Reversing the roles of s and t , that is, using t as a G -cyclic vector, yields $\nu(s) \geq \nu(t)$. This proves (i).

For (ii), assume that $\nu'(s) = \nu'(t) = (a_1, \dots, a_r)$, for nonzero sections $s \in V(\lambda), t \in V(\mu)$, of $\mathcal{O}_X(D)$. By (i), we may without loss of generality assume that s and t are B -semiinvariants of weight λ and μ , respectively.

The definition of ν' , and the fact that Y_i is the common zero set of the G -invariant sections s_1, \dots, s_i , with $s_i \in H^0(X, \mathcal{O}_X(E_i))$, shows that the restriction $\frac{s}{s_1^{a_1} \dots s_r^{a_r}}|_{G/P}$ defines a regular section of the line bundle $\mathcal{O}_{G/P}(D - (a_1 E_1 + \dots + a_r E_r))$ over G/P . Moreover, since s is B -semiinvariant of weight λ , and the s_i are G -invariant, the restricted section $\frac{s}{s_1^{a_1} \dots s_r^{a_r}}|_{G/P}$ is also B -semiinvariant of weight λ . The same argument, applied to t , shows that $\frac{t}{s_1^{a_1} \dots s_r^{a_r}}|_{G/P}$ is a B -semiinvariant section of $\mathcal{O}_{G/P}(D - (a_1 E_1 + \dots + a_r E_r))$ of weight μ . Since $H^0(G/P, \mathcal{O}_{G/P}(D - (a_1 E_1 + \dots + a_r E_r)))$ defines an irreducible G -module, this is a contradiction unless $\lambda \neq \mu$. \square

Definition 2.3. Let

$$\begin{aligned} \tilde{S}_\nu(X) := \{(\nu(s), [D], \lambda) \in \mathbb{N}_0^n \times N^1(X) \times \Gamma \mid s \in V(\lambda) \setminus \{0\}, \\ V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))\}. \end{aligned}$$

Proposition 2.4. *The set $\tilde{S}_\nu(X)$ is an additive subsemigroup of $\mathbb{N}_0^n \times N^1(X) \times \Gamma$.*

Proof. Let $s \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$ and $t \in V(\mu) \subseteq H^0(X, \mathcal{O}_X(E))$ be nonzero sections. We show that $\nu(s) + \nu(t)$ is the value of a section in the G -submodule $V(\lambda + \mu) \subseteq H^0(X, \mathcal{O}_X(D + E))$.

Since $\nu(s) = \nu(s + cv_\lambda)$ for all $c \in \mathbb{C}$, where $v_\lambda \in V(\lambda)$ is a B -semiinvariant section, we may assume that the decomposition of s as a sum of weight vectors has a nonzero component of weight λ . Similarly, we may assume that t has a nonzero weight component of weight μ . The product section $s \cdot t \in H^0(X, \mathcal{O}_X(D + E))$ decomposes as a sum

$$(4) \quad s \cdot t = \sum_{\xi} r_{\xi}, \quad \xi \in V(\xi) \subseteq H^0(X, \mathcal{O}_X(D + E)).$$

Since $\nu(r_{\xi}) \neq \nu(r_{\xi'})$ for $\xi \neq \xi'$, by Proposition 2.2, we have that

$$\nu(s \cdot t) = \min\{\nu(r_{\xi})\},$$

where the minimum is taken over the set of highest weights occurring in the decomposition (4). In fact, this minimum is attained by the highest weight ξ_0 for which $\nu'(s \cdot t) = \nu'(r_{\xi_0})$. Since $\nu'(s \cdot t) = \nu'(s) + \nu'(t) = \nu'(v_{\lambda}) + \nu'(v_{\mu}) = \nu'(v_{\lambda} \cdot v_{\mu}) = \nu'(v_{\lambda + \mu})$, it follows that $\xi_0 = \lambda + \mu$. Hence,

$$(\nu(s), [D], \lambda) + (\nu(t), [E], \mu) = (\nu(r_{\lambda + \mu}), [D] + [E], \lambda + \mu) \in \tilde{S}_\nu(X),$$

where $r_{\lambda + \mu}$ is the component in $V(\lambda + \mu)$ of the product

$$s \cdot t \in H^0(X, \mathcal{O}_X(D + E)).$$

This proves that $\tilde{S}_\nu(X)$ is a semigroup. \square

Definition 2.5. The *global Okounkov semigroup* of X (with respect to ν) is

$$S_\nu(X) := \{(\nu(s), [D]) \in \mathbb{N}_0^n \times N^1(X) \mid s \in H^0(X, \mathcal{O}_X(D)) \setminus \{0\}\}.$$

The *global Okounkov body* of X (with respect to ν) is the closed convex cone $\Delta_\nu(X) = \overline{\text{cone}}(S_\nu(X))$ generated by $S_\nu(X)$.

Proposition 2.6. *The map $p: \tilde{S}_\nu(X) \rightarrow S_\nu(X)$ given by $(\nu(s), [D], \lambda) \mapsto (\nu(s), [D])$ defines an isomorphism of semigroups.*

Proof. Clearly, p is a homomorphism of semigroups. For the surjectivity, let $(\nu(s), [D]) \in S_\nu(X)$, so that $s \in H^0(X, \mathcal{O}_X(D))$. The section s then decomposes as a sum $s = \sum_\xi r_\xi$, where $r_\xi \in V(\xi) \subseteq H^0(X, \mathcal{O}_X(D))$. Since, by Proposition 2.2, the values of the r_ξ are distinct, we have $\nu(s) = \nu(r_\lambda)$, for some $\lambda \in \Gamma$ with $V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$. Then, $(\nu(r_\lambda), [D], \lambda) \in \tilde{S}_\nu(X)$, and

$$p(\nu(r_\lambda), [D], \lambda) = (\nu(r_\lambda), [D]) = (\nu(s), [D]).$$

For the injectivity, let $(\nu(s), [D], \lambda)$ and $(\nu(s'), [D'], \lambda')$ be in $\tilde{S}_\nu(X)$ with $\nu(s) = \nu(s')$ and $[D] = [D']$. Then $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$, and by Proposition 2.2 we have $\lambda = \lambda'$. \square

Definition 2.7. Let $\tilde{\Delta}_\nu(X) := \overline{\text{cone}}(\tilde{S}_\nu(X)) \subseteq \mathbb{R}^n \oplus N^1(X)_\mathbb{R} \oplus \mathfrak{t}^*$ be the closed convex cone generated by the semigroup $\tilde{S}_\nu(X)$. We call $\tilde{\Delta}_\nu(X)$ the *global Okounkov moment cone* of X (with respect to ν).

Similarly, for any variety Y_i in the flag Y_\bullet , we define $\tilde{\Delta}_{\nu_{Y_i}}(Y_i) \subseteq \mathbb{R}^{n-i} \oplus N^1(Y_i)_\mathbb{R} \oplus \mathfrak{t}^*$ to be the closed convex cone generated by the corresponding semigroup $\tilde{S}_{\nu_{Y_i}}(Y_i)$ for the spherical variety Y_i .

Remark 2.8. Since the semigroup $S_\nu(X)$ generating $\Delta_\nu(X)$ also keeps track of the G -modules from which sections with given values come, the cone $\Delta_\nu(X)$ can also be seen as a global version of the moment polytope of X (cf. [Bri87]), with the additional information of the values of ν ; hence the name.

Let $Y := Y_1$ be the divisor of the flag Y_\bullet . The goal now is to prove that the convex cone $\tilde{\Delta}_\nu(X)$ is rational polyhedral if $\tilde{\Delta}_{\nu_Y}(Y)$ is rational polyhedral.

We first observe that, for an effective divisor D on X and a G -submodule $V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$, because of the G -equivariance, the restriction map

$$(5) \quad V(\lambda) \longrightarrow H^0(Y, \mathcal{O}_Y(D)), \quad s \mapsto s|_Y,$$

is either zero or injective. This motivates the following definition.

Definition 2.9. (i) Let $J \subseteq \overline{\text{Eff}}(X) \times \mathfrak{t}^*$ be the subset of all pairs (D, λ) with D and λ as above, and such that the restriction map (5) is injective, and define $K \subseteq \overline{\text{Eff}}(X) \times \mathfrak{t}^*$ to be the closed convex cone generated by J .

(ii) Let $J_Y \subseteq \overline{\text{Eff}}(Y) \times \mathfrak{t}^*$ be the subset of all pairs $(D|_Y, \lambda)$ with $(D, \lambda) \in J$, and define K_Y to be the closed convex cone generated by J_Y .

Lemma 2.10. *The convex cone K is rational polyhedral.*

Proof. Since X is a Mori dream space, the Cox ring, $\text{Cox}(X)$, of X is a finitely generated \mathbb{C} -algebra. Hence, the subalgebra $\text{Cox}(X)^U$ of unipotent invariants is also finitely generated over \mathbb{C} (cf. [G97, Theorem 16.2]). We

can then choose a set of generators ζ_1, \dots, ζ_N for $\text{Cox}(X)^U$ which are homogeneous with respect to the grading by divisors as well as with respect to the grading of the U -invariants by the weights, i.e., we can assume that $\zeta_i \in V(\lambda_i)^U \subseteq H^0(X, \mathcal{O}_X(D_i))^U$, for an effective divisor D_i and a weight λ_i . Moreover, we can assume that the restriction maps (5) are injective for $\lambda_1, \dots, \lambda_\ell$, and zero for $\lambda_{\ell+1}, \dots, \lambda_N$.

Now, let (D, λ) be a pair for which the map (5) is injective, and let $s \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$ be a highest weight section, so $s \in V(\lambda)^U$. Writing

$$s = \sum_{\underline{m}=(m_1, \dots, m_N) \in \mathbb{N}_0^N} c_{\underline{m}} \zeta_1^{m_1} \cdots \zeta_N^{m_N}, \quad c_{\underline{m}} \in \mathbb{C},$$

we see that the sum of the terms for (m_1, \dots, m_N) not satisfying $D = m_1 D_1 + \cdots + m_N D_N$ is zero. The same holds for the sum of the terms for (m_1, \dots, m_N) not satisfying $\lambda = m_1 \lambda_1 + \cdots + m_N \lambda_N$. Hence, we can write s as a linear combination of monomials in the s_i for which both conditions are satisfied.

We now claim that the sum $s = \sum_{\underline{m} \in \mathbb{N}_0^N} c_{\underline{m}} \zeta_1^{m_1} \cdots \zeta_N^{m_N}$ is even a linear combination of monomials only in $\zeta_1, \dots, \zeta_\ell$, that is, we have $m_{\ell+1} = \cdots = m_N = 0$ for all \underline{m} occurring in the sum. Indeed, if we had $m_i > 0$ for some $i \in \{\ell+1, \dots, N\}$, the monomial $\zeta_1^{m_1} \cdots \zeta_N^{m_N}$ would restrict to zero on Y , so that the restriction map (5) would be zero for the highest weight module $V(m_1 \lambda_1 + \cdots + m_N \lambda_N) \subseteq H^0(X, \mathcal{O}_X(m_1 D_1 + \cdots + m_N D_N)) = H^0(X, \mathcal{O}_X(D))$. In particular, this would contradict the fact that $m_1 \lambda_1 + \cdots + m_N \lambda_N = \lambda$.

It follows that K is the closed convex cone generated by the points $(D_1, \lambda_1), \dots, (D_\ell, \lambda_\ell)$. \square

Definition 2.11. Let $\varphi: N^1(X)_{\mathbb{R}} \oplus \mathfrak{t}^* \longrightarrow N^1(Y)_{\mathbb{R}} \oplus \mathfrak{t}^*$ be the unique \mathbb{R} -linear map satisfying $\varphi((D, \lambda)) = (D|_Y, \lambda)$ for all pairs (D, λ) where D is a divisor and $\lambda \in \mathfrak{t}^*$.

Lemma 2.10 has the following immediate corollary.

Corollary 2.12. *The convex cone K_Y is given by $K_Y = \varphi(K)$. In particular, K_Y is rational polyhedral.*

Definition 2.13. We define the semigroups

$$(6) \quad \tilde{J} := \{(\nu(s), D, \lambda) \mid s \in V(\lambda) \setminus \{0\} \subseteq H^0(X, \mathcal{O}_X(D)), (D, \lambda) \in J\},$$

$$(7) \quad \tilde{J}_Y := \{(\nu_Y(t), E, \lambda) \mid t \in V(\lambda) \setminus \{0\} \subseteq H^0(Y, \mathcal{O}_Y(E)), (E, \lambda) \in J_Y\},$$

and the homomorphism of semigroups

$$\tilde{\varphi}: \tilde{J} \longrightarrow \tilde{J}_Y, \quad (\nu(s), D, \lambda) \mapsto (\nu_Y(s|_Y), D|_Y, \lambda).$$

We also denote by $\tilde{K} = \overline{\text{cone}}(\tilde{J})$ and $\tilde{K}_Y = \overline{\text{cone}}(\tilde{J}_Y)$ the closed convex cones generated by \tilde{J} and \tilde{J}_Y , respectively. The map $\tilde{\varphi}$ extends uniquely to a continuous map $\tilde{K} \longrightarrow \tilde{K}_Y$, also denoted by $\tilde{\varphi}$, for simplicity, between the respective closed convex cones.

Since the convex cone K is generated by pairs (D, λ) for which the restriction map (5) is injective, the map $\tilde{\varphi}$ between the convex cones is surjective,

i.e.,

$$\tilde{K}_Y = \tilde{\varphi}(\tilde{K}).$$

Lemma 2.14. *For any big divisor D on Y there exist a rational weight λ and $a \in \mathbb{Q}^{n-1}$ such that (a, D, λ) is in the relative interior of $\tilde{\Delta}_{\nu_Y}(Y)$.*

Proof. By the inductive assumption, the convex cone $\tilde{\Delta}_{\nu_Y}(Y)$ is rational polyhedral. Let $\{(a_i, D_i, \lambda_i) \mid i \in \{1, \dots, m\}\}$, for some $m \in \mathbb{N}$, be a set of rational generators. We may assume that the rational divisors D_1, \dots, D_m generate the pseudoeffective cone $\overline{\text{Eff}}(Y)$ of Y . Note that we do not require the λ_i nor the D_i to be distinct here.

Since D is in the interior of $\overline{\text{Eff}}(Y)$, we can write D as a linear combination $D = \xi_1 D_1 + \dots + \xi_m D_m$ with rational coefficients $\xi_1, \dots, \xi_m > 0$.

Put

$$a = \sum_{i=1}^m \xi_i a_i, \quad \lambda = \sum_{i=1}^m \xi_i \lambda_i.$$

Then

$$(a, D, \lambda) = \sum_{i=1}^m \xi_i (a_i, D_i, \lambda_i)$$

is in the relative interior of $\tilde{\Delta}_{\nu_Y}(Y)$. □

Definition 2.15. Let $\pi: \mathbb{R}^{n-1} \oplus N^1(Y)_{\mathbb{R}} \oplus \mathfrak{t}^* \longrightarrow N^1(Y)_{\mathbb{R}} \oplus \mathfrak{t}^*$ denote the projection onto the direct summand $N^1(Y)_{\mathbb{R}} \oplus \mathfrak{t}^*$.

Lemma 2.16. *If the convex cone $\tilde{\Delta}_{\nu_Y}(Y)$ is rational polyhedral, then*

$$\tilde{K}_Y = \tilde{\Delta}_{\nu_Y}(Y) \cap \pi^{-1}(K_Y).$$

In particular, the convex cone \tilde{K}_Y is rational polyhedral, being the intersection of two rational polyhedral convex cones.

Proof. Notice that $\tilde{J}_Y = \tilde{\Delta}_{\nu_Y}(Y) \cap \pi^{-1}(J_Y)$ and that the convex cone $\pi^{-1}(K_Y)$ is rational polyhedral and generated by $\pi^{-1}(J_Y)$.

We claim that $\pi^{-1}(J_Y)$ intersects the relative interior of $\tilde{\Delta}_{\nu_Y}(Y)$. Given this claim, the lemma is a consequence of [LM09, Prop. A.1.] (and its proof).

In order to prove the claim, let D be an ample divisor on X , and let $E := D|_Y$. By Lemma 2.14, there is a point of the form (a, E, λ) in the relative interior of $\tilde{\Delta}_{\nu_Y}(Y)$. By replacing this triple by a positive integer multiple, if necessary, we may assume that (a, E, λ) is an integral point in $\tilde{\Delta}_{\nu_Y}(Y)$, and, by Serre's vanishing theorem, that the restriction map $R_D: H^0(X, \mathcal{O}_X(D)) \longrightarrow H^0(Y, \mathcal{O}_Y(E))$ of sections is surjective.

By the G -equivariance of R_D , the weight λ occurs as the highest weight of an irreducible G -submodule $V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$, and the restriction $R_D|_{V(\lambda)}: V(\lambda) \longrightarrow H^0(Y, \mathcal{O}_Y(E))$ is injective, i.e., $(E, \lambda) \in J_Y$. This finishes the proof. □

The above lemma enables us to deduce the rational polyhedrality of the convex cone \tilde{K} .

Lemma 2.17. *If the convex cone $\tilde{\Delta}_{\nu_Y}(Y)$ is rational polyhedral, then*

$$\tilde{K} = ((\{0\} \times \mathbb{R}_{\geq 0}^{n-1}) \times K) \cap \tilde{\varphi}^{-1}(\tilde{K}_Y)$$

In particular, the convex cone \tilde{K} is rational polyhedral.

Proof. We use the same strategy as in Lemma 2.16. If a point of the form $(\nu(s), D, \lambda)$ is in \tilde{J} for some $s \in V(\lambda) \setminus \{0\} \subseteq H^0(X, \mathcal{O}_X(D))$ and $(D, \lambda) \in J$, then $\nu(s) \in \{0\} \times \mathbb{R}^{n-1}$ by the definition of J .

This yields

$$\tilde{J} = ((\{0\} \times \mathbb{R}_{\geq 0}^{n-1}) \times J) \cap \tilde{\varphi}^{-1}(\tilde{J}_Y).$$

We claim that $((\{0\} \times \mathbb{R}_{\geq 0}^{n-1}) \times J)$ intersects the relative interior of $\tilde{\varphi}^{-1}(\tilde{K}_Y)$. Given this claim, the lemma follows from [LM09, Prop. A.1.].

To prove the claim, let $(a, E, \lambda) \in \tilde{J}_Y$ be in the relative interior of \tilde{K}_Y , choose D such that $(D, \lambda) \in J$ and $E = D|_Y$, and choose $t \in V(\lambda) \setminus \{0\} \subseteq H^0(Y, \mathcal{O}_Y(E))$ such that $\nu_Y(t) = a$. Up to replacing $(a, E, \lambda) \in \tilde{J}_Y$ by a positive integer multiple, we may assume that the restriction map $R_D: H^0(X, \mathcal{O}_X(D)) \rightarrow H^0(Y, \mathcal{O}_Y(E))$ is surjective. Then we have $V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$, and the map R_D sends $V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$ isomorphically to $V(\lambda) \subseteq H^0(Y, \mathcal{O}_Y(E))$. In other words, the section t extends to a section $s \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$.

It follows that $\nu(s) = (0, a)$, so $(\nu(s), D, \lambda) \in (\{0\} \times \mathbb{R}_{\geq 0}^{n-1}) \times J$ and $\tilde{\varphi}(\nu(s), D, \lambda) = (t, E, \lambda)$, which finishes the proof. \square

We can now prove the main result, deducing the rational polyhedrality of $\tilde{\Delta}_\nu(X)$ from that of $\tilde{\Delta}_{\nu_Y}(Y)$.

Lemma 2.18. *If the convex cone $\tilde{\Delta}_{\nu_Y}(Y)$ is rational polyhedral, then so is $\tilde{\Delta}_\nu(X)$.*

Proof. Let

$$((0, \eta_1), D_1, \lambda_1), \dots, ((0, \eta_m), D_m, \lambda_m) \in \{0\} \times \mathbb{R}_{\geq 0}^{n-1} \times K \subseteq \mathbb{R}^n \oplus N^1(X)_{\mathbb{R}} \oplus \mathfrak{t}^*$$

be integral generators of the convex cone \tilde{K} .

Now, let $s \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$ be a nonzero section. If s vanishes to order a along Y , we have that $s' := \frac{s}{s_1^a} \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D - aY))$ is a section with nonzero restriction to Y , so that $(\nu(s'), D - aY, \lambda) \in \tilde{J}$. Then

$$\begin{aligned} (\nu(s), D, \lambda) &= (\nu(s'), D - aY, \lambda) + ((a, 0, \dots, 0), aY, 0) \\ &\in \text{cone}\{((0, \eta_1), D_1, \lambda_1), \dots, ((0, \eta_m), D_m, \lambda_m), ((1, 0, \dots, 0), Y, 0)\}. \end{aligned}$$

It follows that

$$\tilde{\Delta}_\nu(X) = \text{cone}\{((0, \eta_1), D_1, \lambda_1), \dots, ((0, \eta_m), D_m, \lambda_m), ((1, 0, \dots, 0), Y, 0)\}.$$

\square

Theorem 2.19. *The convex cones $\tilde{\Delta}_\nu(X)$ and $\Delta_\nu(X)$ are rational polyhedral.*

Proof. We proceed by induction on r (the number of the G -invariant prime Cartier divisors E_1, \dots, E_r , cf. Section 1). If $r = 0$ then $X = G/P$ is G -homogeneous, and $\Delta_\nu(X)$ is rational polyhedral by [SS17, Corollary 4.11].

The inductive step follows from the fact that $Y = Y_1$ admits the partial flag of subvarieties $Y = Y_1 \supset Y_2 \supset \dots \supset Y_r$, from Lemma 2.18, and the fact that $\Delta_\nu(X)$ is the image of $\tilde{\Delta}_\nu(X)$ under the projection $\mathbb{R}^n \oplus N^1(X)_\mathbb{R} \oplus \mathfrak{t}^* \rightarrow \mathbb{R}^n \oplus N^1(X)_\mathbb{R}$. \square

REFERENCES

- [Bri87] Brion, M., *Sur l'image de l'application moment*, Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin (Paris, 1986), Lecture Notes in Math., vol. 1296, Springer, Berlin, 1987, pp. 177–192 MR MR932055 (89i:32062)
- [G97] Grosshans, F.D., *Algebraic homogeneous spaces and invariant theory*, Lecture Notes in Mathematics, vol. 1673, Springer-Verlag, Berlin, 1997
- [HK00] Hu, Y., Keel, S., *Mori Dream Spaces and GIT*, Michigan Math. J. **48** (2000), 331–348
- [J03] Jantzen, J.C., *Representations of algebraic groups*, Mathematical Surveys and Monographs **107**, American Mathematical Society, Providence, RI, 2003
- [KK09] Kaveh, K., Khovanskii, A. G., *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. Math. **176** (2012), 925–978
- [KK12] Kaveh, K., Khovanskii, A. G., *Convex bodies associated to actions of reductive groups*, Mosc. Math. J. **12** (2012), no. 2, 369–396
- [KKLV89] Knop, F., Kraft, H., Luna, D., Vust, T., *Local properties of algebraic group actions*, Algebraische Transformationsgruppen und Invariantentheorie, 63–76, DMV Sem., 13, Birkhäuser, Basel, 1989
- [KL15] Küronya, A., Lozovanu, V., *Positivity of line bundles and Newton-Okounkov bodies*, preprint 2015, arXiv:1506.06525
- [LM09] Lazarsfeld, R., Mustață, M., *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. **42** (2009), 783–835
- [Ok96] Okounkov, A., *Brunn-Minkowski inequality for multiplicities*, Invent. Math. **125** (1996), 405–411
- [P14] Perrin, N., *On the geometry of spherical varieties*, Transform. Groups **19** (2014), no.1, 171–223.
- [Pe17] Pezzini, G., *Lectures on wonderful varieties*, preprint 2017, to appear in Acta Math. Sinica
- [PU16] Postinghel, E., Urbinati, M., *Newton-Okounkov bodies and Toric Degenerations of Mori dream spaces via Tropical compactifications*, preprint 2016, arXiv:1612.03861
- [SS17] Schmitz, D., Seppänen, H., *Global Okounkov bodies for Bott-Samelson varieties*, Journal of Algebra **490** (2017), 518–554
- [Se16] Seppänen, H., *Okounkov bodies for ample line bundles with applications to multiplicities for group representations*, Beitr. Algebra Geom. **57** (2016), no. 4, 735–749

GUIDO PEZZINI, DIPARTIMENTO DI MATEMATICA “GUIDO CASTELNUOVO”, “SAPIENZA” UNIVERSITÀ DI ROMA, PIAZZALE A. MORO 5, I-00185 ROMA, ITALY

E-mail address: pezzini@mat.uniroma1.it

HENRIK SEPPÄNEN, MATHEMATISCHES INSTITUT, GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN, BUNSENSTRASSE 3-5, D-37073 GÖTTINGEN, GERMANY

E-mail address: henrik.seppaenen@mathematik.uni-goettingen.de