Set-Valued Tableaux & Generalized Catalan Numbers

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Abstract

Standard set-valued Young tableaux are a generalization of standard Young tableaux in which cells may contain more than one integer, with the added conditions that every integer at position (i, j) must be smaller than every integer at positions $(i, j+1)$ and $(i+1, j)$. This paper explores the combinatorics of standard set-valued Young tableaux with two-rows, and how those tableaux may be used to provide new combinatorial interpretations of generalized Catalan numbers. New combinatorial interpretations are provided for the two-parameter Fuss-Catalan numbers (Raney numbers), the rational Catalan numbers, and the solution to the so-called "generalized tennis ball problem". Methodologies are then introduced for the enumeration of standard set-valued Young tableaux, prompting explicit formulas for the general two-row case. The paper closes by drawing a bijection between arbitrary classes of two-row standard set-valued Young tableaux and collections of two-dimensional lattice paths that lie weakly below a unique maximal path.

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1 Introduction

For a non-increasing integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, a Young diagram Y of shape λ is a left-justified array of cells with exactly λ_i cells in its ith row. If Y is a Young diagram of shape λ with $\sum_i \lambda_i = n$, a Young tableau of shape λ is an assignment of the integers $[n] = \{1, \ldots, n\}$ to the cells of Y such that every integer is used precisely once. A Young tableau in which integers increase from top-to-bottom down every column and increase from left-to-right across every row is said to be a standard Young tableau. We denote the set of all standard Young tableaux of shape λ by $S(\lambda)$. For m-row rectangular shapes $\lambda = (n, \ldots, n)$ we use the abbreviated notation $S(n^m)$. For a comprehensive introduction to Young tableaux, see Fulton [7].

Let Y be a Young diagram of shape λ, and let $\rho = \{\rho_{i,j}\}$ be a collection of positive integers such that $\sum_{i,j} \rho_{i,j} = m$. A **set-valued tableau** of shape λ and density ρ is a function from $[m]$ to the cells of Y such that the cell at position (i, j) receives a set of $\rho_{i,j}$ integers. A set-valued tableau is said to be a **standard set-valued Young tableau** if we additionally require that every integer at position (i, j) is smaller than every integer at positions $(i+1, j)$ and $(i, j+1)$. In analogy with standard Young tableaux, we refer to these added conditions as "column-standardness" and "row-standardness". We denote the set of all standard set-valued Young tableaux of shape λ and density ρ by $\mathbb{S}(\lambda, \rho)$. See Figure 1 for a basic example.

$$\begin{array}{|c|c|} \hline 1\,2 & 3\,4 \\ \hline 5\,6 & 7\,8 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1\,2 & 3\,5 \\ \hline 4\,6 & 7\,8 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1\,2 & 3\,6 \\ \hline 4\,5 & 7\,8 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1\,2 & 4\,5 \\ \hline 3\,6 & 7\,8 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1\,2 & 4\,6 \\ \hline 3\,5 & 7\,8 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1\,2 & 5\,6 \\ \hline 3\,4 & 7\,8 \\ \hline \end{array}$$

Figure 1: The set $\mathbb{S}(\lambda, \rho)$ when $\lambda = (2, 2)$ and $\rho_{i,j} = 2$ for all i, j.

Set-valued tableaux were introduced by Buch [5] in his investigation of the K-theory of Grassmannians. More directly influencing this paper is the work of Heubach, Li and Mansour [11], who argued that the cardinality of

$\mathbb{S}(n^2, \rho)$ with row-constant density $\rho_{1,j} = k - 1$, $\rho_{2,j} = 1$ equalled the k-Catalan number C_n^k. For a more recent appearance of standard set-valued tableaux see Reiner, Tenner and Yong [16], who investigated so-called "barely set-valued tableaux" with a single non-unitary density $\rho_{i,j} = 2$ (not necessarily located at a fixed position i, j).

Currently, the central difficulty in studying standard set-valued Young tableaux is the lack of a closed formula for enumerating general $\mathbb{S}(\lambda, \rho)$: there is no known set-valued analogue of the celebrated hook-length formula for standard Young tableaux. Reiner, Tenner and Yong [16] do utilize a modified insertion algorithm to enumerate their "barely set-valued tableaux", but theirs is an atypically tractable case and cannot be modified to the enumeration of sets $\mathbb{S}(\lambda, \rho)$ with a fixed density at each position.

The purpose of this paper is twofold. In Section 2, we utilize standard set-valued Young tableaux to provide new combinatorial interpretations for various generalizations of the Catalan numbers. In particular, we show how various densities for rectangular set-valued tableaux of shape $\lambda = n^2$ are enumerated by the Raney numbers (two-parameter Fuss Catalan numbers, Theorem 2.2) and the solution to the "(s, t)-tennis ball problem" of Merlini, Sprugnoli, and Verri [13] (Theorem 2.3). See Figure 2 for an overview of the various densities needed to achieve our combinatorial interpretations. In Section 3, we develop methodologies for enumerating two-row standard set-valued tableaux. Concise closed formulas are presented for the number of such tableaux of arbitrary density (Theorem 3.2). In Section 4, we conclude by drawing a bijection between two-row standard set-valued tableaux of arbitrary density and certain classes of two-dimensional integer lattice paths with "East" $E = (1, 0)$ and "North" $N = (0, 1)$. In particular, $\mathbb{S}(n^2, \rho)$ with $\rho_{1,j} = a_j$ and $\rho_{2,j} = b_j$ is placed in bijection with all such lattice paths that lie weakly below the lattice path $P = E^{a_1} N^{b_1} E^{a_2} N^{b_2} \ldots$ (Theorem 4.2).
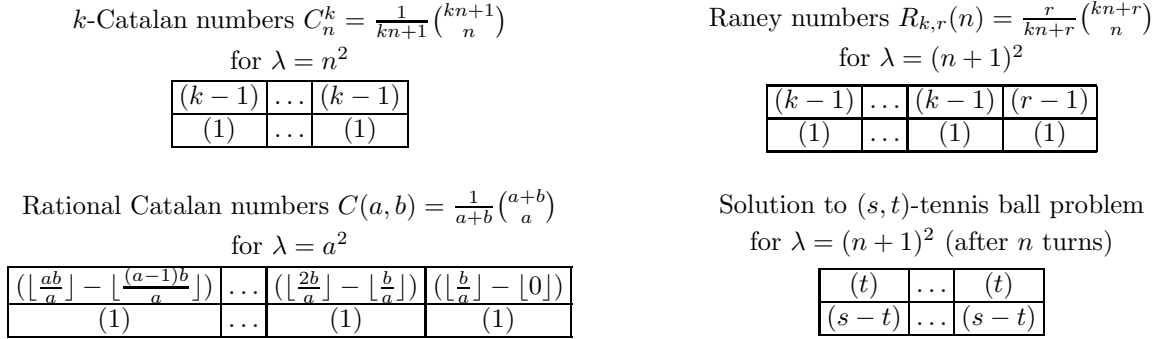
k-Catalan numbers $C_n^k = \frac{1}{kn+1}\binom{kn+1}{n}$
for $\lambda = n^2$

$(k-1)$	...	$(k-1)$
(1)	...	(1)

Raney numbers $R_{k,r}(n) = \frac{r}{kn+r}\binom{kn+r}{n}$
for $\lambda = (n+1)^2$

$(k-1)$	...	$(k-1)$	$(r-1)$
(1)	...	(1)	(1)

Rational Catalan numbers $C(a,b) = \frac{1}{a+b}\binom{a+b}{a}$
for $\lambda = a^2$

$(\lfloor \frac{ab}{a} \rfloor - \lfloor \frac{(a-1)b}{a} \rfloor)$	...	$(\lfloor \frac{2b}{a} \rfloor - \lfloor \frac{b}{a} \rfloor)$	$(\lfloor \frac{b}{a} \rfloor - \lfloor 0 \rfloor)$
(1)	...	(1)	(1)

Solution to (s,t)-tennis ball problem
for $\lambda = (n+1)^2$ (after n turns)

(t)	...	(t)
$(s-t)$	...	$(s-t)$

Figure 2: Densities ρ needed for $|\mathbb{S}(\lambda, \rho)|$ to yield various combinatorial interpretations.

We pause to introduce a foundational result that, in the case of rectangular λ, serves as a set-valued analogue of the Schützeberger involution for standard Young tableaux. In the case of densities ρ that are constant across each row, notice that Proposition 1.1 manifests as invariance under a vertical reflection of those densities.

Proposition 1.1. *For rectangular* $\lambda = n^m$ *and any density* $\rho = \{\rho_{i,j}\}$, *let* $r(\rho) = \{\rho_{n-i+1, m-j+1}\}$. *Then* $|\mathbb{S}(\lambda, \rho)| = |\mathbb{S}(\lambda, r(\rho))|$.

Proof. One may define a bijection $f : S(\lambda, \rho) \to S(\lambda, r(\rho))$ such that $f(T) \in S(\lambda, r(\rho))$ is obtained by reversing the alphabet of $T \in S(\lambda, \rho)$ and rotating the resulting tableau by 180-degrees. $\square$

2 Generalized Catalan Numbers and Set-Valued Tableaux

We begin be briefly recapping established results about the k-Catalan numbers. For any $k \geq 1$, the k-Catalan numbers are given by $C_n^k = \frac{1}{kn+1}\binom{kn+1}{n}$ for all $n \geq 0$. Notice that the k-Catalan numbers specialize to the usual Catalan numbers when $k = 2$.

See Hilton and Pedersen [10] or Heubach, Li and Mansour [11] for various combinatorial interpretations of the k-Catalan numbers. Relevant to our work is the standard result that C_n^k enumerates the set $\mathcal{D}_n^k$ of so-called k-good paths of length kn. These are integer lattice paths from $(0, 0)$ to $(n, (k-1)n)$ that utilize only "East" $E = (1, 0)$ and "North" $N = (0, 1)$ steps and which stay weakly below the line $y = (k-1)x$. The set of k-good paths are obviously in bijection with k-ary paths of length kn: integer lattice paths from $(0, 0)$ to $(nk, 0)$ that use steps $u = (1, \frac{1}{k-1})$, $d = (1, -1)$ and stay weakly above $y = 0$. We prefer working with k-good paths because they are more easily generalized to sets enumerated by the Raney numbers and the rational Catalan numbers.

Heubach, Li and Mansour [11] showed that standard set-valued Young tableaux of shape $\lambda = n^2$ and row-constant density $\rho_{1,j} = k-1$, $\rho_{2,j} = 1$ were counted by C_n^k. This was done by placing such tableaux in bijection with k-ary paths of length kn. Using Proposition 1.1, their map may be modified to give an elementary bijection between $\mathcal{D}_n^k$ and standard set-valued Young tableaux $S(n^2, \rho)$ of row-constant density $\rho_{1,j} = 1$, $\rho_{2,j} = k-1$. As exemplified in Figure 3, this bijection $\phi : \mathcal{D}_n^k \to S(\lambda, w)$ involves associating East steps in $P \in \mathcal{D}_n^k$ with first-row entries in $\phi(P)$ and North steps in P with second-row entries in $\phi(P)$.
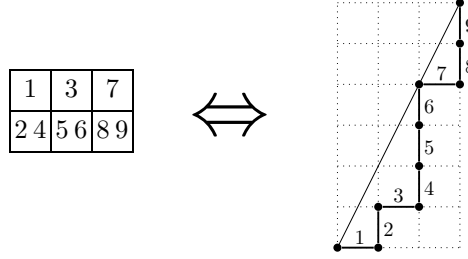
Figure 3: A standard set-valued Young tableau with $\lambda = 3^2$ and $\rho_{1,j} = 1, \rho_{2,j} = k-1$, alongside the corresponding 3-good path in $\mathcal{D}_3^3$.

2.1 Raney numbers

The first generalization of the Catalan numbers for which we will present a new combinatorial interpretation are the Raney numbers, also known as the two-parameter Fuss-Catalan numbers. For any $k \geq 1$ and $r \geq 1$, the Raney numbers are given by $R_{k,r}(n) = \frac{r}{kn+r}\binom{kn+r}{n}$ for all $n \geq 0$.[1] The Raney numbers specialize to the k-Catalan numbers as $R_{k,1}(n) = C_n^k$.

Hilton and Pedersen [10] showed that the Raney numbers could be calculated from the k-Catalan numbers via Equation 1. This equation may be viewed as a generalization of the standard recurrence for the k-Catalan numbers when one notes that $R_{k,k}(n-1) = C_n^k$, a distinct identity from the "obvious" specialization of Equation 1 to the k-Catalan numbers as $R_{k,1}(n) = C_n^k$.

$$R_{k,r}(n) = \sum_{(i_1,\ldots,i_r)\vdash n} C_{i_1}^k C_{i_2}^k \ldots C_{i_r}^k \tag{1}$$

Equation 1 is useful in that it allows one to define combinatorial interpretations for the Raney numbers as ordered r-tuples of pre-existing interpretations for the k-Catalan numbers:

Proposition 2.1. *Fix $k, r \geq 1$, $n \geq 0$. Then $R_{k,r}(n)$ equals the number of ordered r-tuples $(T_1, \ldots, T_r)$ such that $T_j \in \mathbb{S}(i_j^2, \rho)$ for row-constant weight $\rho_{1,j} = 1$, $\rho_{2,j} = k-1$, where $i_1 + \ldots + i_r = n$.*

Our goal is to replace the ordered r-tuples of Proposition 2.1 with a single set-valued tableau of shape $\lambda = (n+1)^2$. This utilizes a technique that we refer to "horizontal tableaux concatenation", whereby the entries of the ordered r-tuple are continuously reindexed and a new column with density $\rho_{1,1} = 1, \rho_{2,1} = r-1$ is added to the front of the resulting tableau. This additional column carries the information needed to recover the original partition of the tableau into r pieces.

So fix $n \geq 0$ and take any two-row rectangular shape $\lambda = (n+1)^2$. To ease notation, for any $k, r \geq 1$ we temporarily define the density $\rho(k,r) = \{\rho_{i,j}\}$ by $\rho_{1,j} = 1$ for all $1 \leq j \leq n$, $\rho_{2,1} = r-1$, and $\rho_{2,j} = k-1$ for all $2 \leq j \leq n$. Notice that $\rho(k,r)$ is equivalent to the density of Figure 2 via Proposition 1.1.

Theorem 2.2. *Take any $k, r \geq 1$, $n \geq 0$, and define $\rho(k,r)$ as above. Then $R_{k,r}(n) = |\mathbb{S}((n+1)^2, \rho(k,r))|$.*

Proof. Take $(T_1, \ldots, T_r)$, where $T_j \in \mathbb{S}(i_j^2, \rho)$ with $\rho_{1,j} = 1, \rho_{2,j} = k-1$ and $i_1 + \ldots + i_r = n$. Observe that a total of kn integers appear across the $2n$ cells of the T_j. Create a partially-filled Young diagram D of shape $\lambda_D = (n+r)^2$ by adding an empty column c_j in front of each T_j and horizontally concatenating the resulting tableaux in the given order. Notice that this will result in multiple consecutive empty columns if any of the T_j

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[1]Hilton and Pedersen [10] use the alternative notation $d_{qk}(p) = \frac{p-q}{pk-q}\binom{pk-q}{k-1}$ for their two-parameter generalization. Our two notations are related via the change of variables $R_{p,p-q}(k-1) = d_{qk}(p)$.

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are empty. Mark the top cell of column c_1 and the bottom cells of columns $c_2, \ldots, c_r$. This gives r markings in addition to the kn integers of D. Re-index these $kn + r$ items by working through D from left-to-right. Every time a marking is encountered, assign the marked cell the smallest available element of $[kn + r]$. When T_j is encountered, simultaneously replace the ki_j integers of T_j with the ki_j smallest available elements of $[kn + r]$, preserving the relative ordering within T_j.

This gives a partially-filled set-valued Young tableau $\widetilde{D}$ that is row-standard and column-standard if you look past the empty cells. Then "collapse" the entries of $\widetilde{D}$ off the interstitial columns $c_2, \ldots, c_j$ (but not off c_1) by shifting all entries leftward until the cells corresponding to c_1 have density $\rho_{1,1} = 1, \rho_{1,2} = r - 1$, the cells corresponding to the T_j have row-constant density $\rho_{1,j} = 1, \rho_{2,j} = k - 1$, and the cells corresponding to the columns $c_2, \ldots, c_j$ are empty. Deleting the columns corresponding to $c_2, \ldots, c_j$ then produces a set-valued tableau T of shape $\lambda = (n+1)^2$ and density $\rho(k, r)$. T is obviously row-standard. To see that T is also column-standard, notice that first row entries of $\widetilde{D}$ that were originally associated with a particular T_j are shifted leftward by precisely $j - 1$ cells as we pass from $\widetilde{D}$ to T, whereas second row entries in $\widetilde{D}$ that were originally associated with T_j are shifted leftward by at least $j - 1$ cells as we pass from $\widetilde{D}$ to T. The latter observation follows from the fact that $r - 1$ integers must eventually appear in the $(2, 1)$ cell of T, and that there are $j - 1 \leq r - 1$ marked second-row cells to the left of the entries associated with T_j. As second row entries are shifted at least as far left as first row entries, the column-standardness of $\widetilde{D}$ implies that T is also column-standard.

To show that our map $(T_1, \ldots, T_r) \mapsto T$ is bijective we provide a well-defined inverse. For $T \in \mathbb{S}((n + 1)^2, \rho(k, r))$, collectively shift all entries in the second row rightward so that the $(2, 1)$ cell is empty and all remaining cells in the second row contain precisely $k - 1$ entries (there will be an overflow of $r - 1$ elements at the end of the second row). Then proceed through the second-row from left-to-right and identify the smallest integer c that violates column-standardness. Insert a new, partially-filled column at the position of c whose top cell is empty and whose bottom cell contains c. Then re-allocate the remaining entries of the second row so that $k - 1$ entries appear in each cell to the right of c, and repeat the above procedure until $r - 1$ new columns have been added. The end result of this procedure is identical to the partially-filled tableau $\widetilde{D}$ from above. This is because the second-row entries of the interstitial columns $c_2, \ldots, c_r$ are necessarily smaller than all entries in the "block" corresponding to T_j and hence would violate column-standardness if moved one cell to their right. $\square$
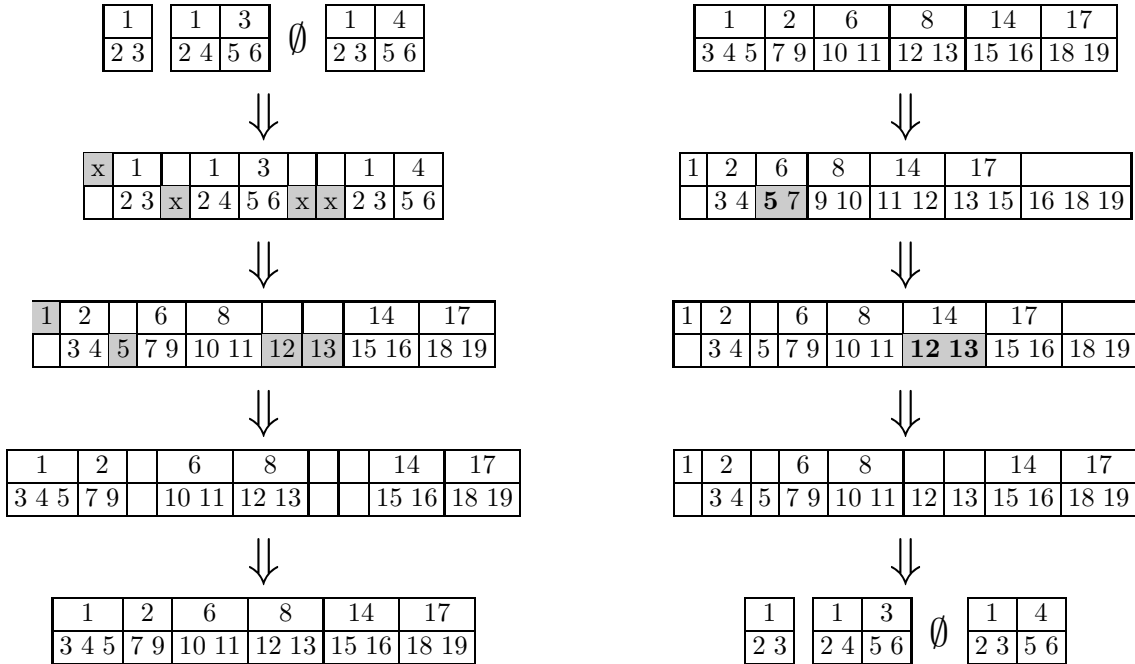

Figure 4: Transforming an r-tuple $(T_1, \ldots, T_r)$ of set-valued tableaux into a single set-valued tableau of density $\rho(k, r)$ via horizontal concatenation, alongside the inverse procedure.

For an example illustrating both directions of the bijection from Theorem 2.2, see Figure 4. Notice that our interpretation of $R_{k,r}(n)$ as the cardinality of $\mathbb{S}((n+1)^2, \rho(k, r))$ immediately recovers the k-Catalan specialization

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$R_{k,k}(n-1) = C_n^k$ of Heubach, Li and Mansour [11] when $r = k$. Also notice the special meaning of Theorem 2.2 as it applies to the extreme case of $r = 1$, as set-valued tableaux of density $\rho(k,1)$ have an empty cell at position $(2,1)$. In this case, one may construct a bijection from $\mathbb{S}((n+1)^2, \rho(k,1))$ to $\mathbb{S}(n^2, \rho(k,k))$ by deleting the first column of $T \in \mathbb{S}((n+1)^2, \rho(k,1))$ and re-indexing the remaining nk entries of T by $x \mapsto (x-1)$. This bijection directly corresponds to the Raney number identity $R_{k,1}(n) = R_{k,k}(n-1) = C_n^k$.

2.2 The (s,t)-Tennis Ball Problem

The so-called "tennis ball problem" was introduced by Tymoczko and Henle in their logic textbook [17], and was subsequently formalized by Mallows and Shapiro [12]. The classic version of the problem begins with $2n$ tennis balls, numbered $1, 2, \ldots, 2n$, and proceeds through n turns. For the first turn, one takes the balls numbers 1 and 2 and randomly throws one of them out of your window onto their lawn. During the ith turn, the balls numbered $2i - 1$ and $2i$ are added to the $i - 1$ balls leftover from previous steps, and one of those $i + 1$ balls is thrown onto your lawn. The problem then asks how many different sets of balls are possible on one's lawn after n steps. Independent from Mallows and Shapiro [12], Grimaldi and Moser [8] proved that the number of such arrangements was the Catalan number C_{n+1}.

Merlini, Sprugnoli, and Verri [13] generalized these phenomena to the (s,t)-tennis ball problem, whereby s new balls are added and t balls are thrown onto the lawn during each turn. If we let $\mathcal{B}_{s,t}(n)$ denote the number of arrangements possible after n turns in the generalized problem, Merlini et al. [13] showed that $\mathcal{B}_{k,1}(n) = C_{n+1}^k$. Generating functions for all $\mathcal{B}_{s,t}(n)$ were later developed by de Mier and Noy [6], who placed the resulting arrangements in bijection with certain classes of N-E lattice paths (see Section 4).

All of this is relevant in that the $\mathcal{B}_{s,t}(n)$ admits a straightforward combinatorial interpretation in terms of Young tableaux, an interpretation that has yet to appear anywhere in the literature. As seen with the row-constant densities of Theorem 2.3, the $\mathcal{B}_{s,t}(n)$ represent a one-parameter generalization of the k-Catalan numbers that are distinct from the Raney numbers.

Theorem 2.3. *Fix $s, t \geq 1$ such that $s \geq t$. The solution to the (s,t)-tennis ball problem after n turns is $\mathcal{B}_{s,t}(n) = |\mathbb{S}((n+1)^2, \rho)|$, where ρ is the row-constant density $\rho_{1,j} = t$, $\rho_{2,j} = s - t$.*

Proof. We define a bijection from the set of arrangements after n turns to $\mathbb{S}((n+1)^2, \rho)$. Place the nt numbers corresponding to balls on the lawn in increasing order across the first row of the Young diagram of shape $\lambda = (n+1)^2$, beginning with the cell at $(1,2)$ and ensuring each cell receives t integers. Then place the remaining $n(s-t)$ integers in increasing order across the first n second row cells of that same Young diagram, ensuring each cell receives $s - t$ integers. After reindexing entries by $x \mapsto x + t$, place the integers $1, \ldots, t$ in the cell at position $(1,1)$, and place the integers $ns - s + t + 1, \ldots, ns$ at position $(2, n+1)$. The resulting set-valued tableau is row-standard by construction. To see that it is column-standard, notice that the first i turns of the procedure collectively involve throwing ti balls with labels at most equal to si. This means that, before reindexing, the largest entry at position $(1, i+1)$ is at most si. It also implies that, before reindexing, the largest entry at position $(2,i)$ is at least si. It follows that the smallest entry at $(2, i+1)$, before reindexing, is at least $si + 1$. $\square$

Comparing Theorem 2.3 with the results of Subsection 2.1, in the case of $t = 1$ we directly recover the result of Merlini, Sprugnoli, and Verri [13] that $\mathcal{B}_{s,t}(n) = C_{n+1}^s$. Also note that the result of Theorem 2.3 may be further generalized to the "non-constant" tennis ball problem of de Mier and Noy [6]. If $\vec{s} = \{s_i\}$ and $\vec{t} = \{t_i\}$ are sequences of positive integers such that $t_i < s_i$ for all i, the $(\vec{s}, \vec{t})$-tennis ball problem is the generalization of the tennis ball problem wherein s_i new balls are added and t_i balls are thrown out the window during the ith turn. Equivalent reasoning to Theorem 2.3, yields the following combinatorial interpretation of the solution $\mathcal{B}_{\vec{s}, \vec{t}}(n)$ to this fully-generalized problem.

Theorem 2.4. *Let $\vec{s} = \{s_i\}$ and $\vec{t} = \{t_i\}$ be sequences of positive integers such that $t_i < s_i$ for all i. Then the solution to the $(\vec{s}, \vec{t})$-tennis ball problem after n turns is $\mathcal{B}_{\vec{s}, \vec{t}}(n) = |\mathbb{S}((n+1)^2, \rho)|$, where ρ is shown below.*

(1)	(t_1)	$\ldots$	(t_{n-1})	(t_n)
$(s_1 - t_1)$	$(s_2 - t_2)$	$\ldots$	$(s_n - t_n)$	(1)

3 Enumeration of Two-Row Set-Valued Tableaux

Although an enumeration of $\mathbb{S}(\lambda, \rho)$ for general λ and ρ isn't currently tractable, the two-row case of $\lambda = (n_1, n_2)$ is sufficiently simple that methodologies may be developed for arbitrary ρ. In this section, we present a technique for such enumerations that we refer to as "density shifting". This procedure sets up a bijection between $\mathbb{S}(\lambda, \rho)$ and a collection of sets $\mathbb{S}(\lambda', \rho_i')$, where $\lambda' = (n_1 - 1, n_2 - 1)$ and the varying densities ρ_i' are determined by ρ.

To define our procedure of **density shifting**, fix $\lambda = (n_1, n_2)$ and a density ρ with $\rho_{1,j} = a_j$, $\rho_{2,j} = b_j$. We focus on the first two columns of arbitrary $T \in \mathbb{S}(\lambda, \rho)$, and consider the relationship of the integers $\beta_1 < \ldots < \beta_{b_1}$ at position $(2,1)$ to the integers $\alpha_1 < \ldots < \alpha_{a_2}$ at position $(1,2)$. In particular, we identify the smallest integer β_m such that $\beta_m > \alpha_{a_2}$. The integers $\beta_m, \beta_{i+1}, \ldots, \beta_{b_1}$ may then be moved to the cell at $(2,2)$ without violating column-standardness, where they are necessarily the $b_1 - m + 1$ smallest integers at $(2,2)$. The remaining integers $\beta_1, \ldots, \beta_{i-1}$ at position $(2,1)$ are then moved to the cell at $(1,2)$, where they are smaller than α_{a_2} but their relationship to $\alpha_1, \ldots, \alpha_{a_2-1}$ depends upon the choice of T. With the cell at $(2,1)$ empty, the entire first row of the tableau is deleted and the remaining entries are re-indexed according to $x \mapsto x - a_1$. This produces a tableau $d(T) \in \mathbb{S}(\lambda', \rho')$ for $\lambda' = (n_1 - 1, n_2 - 1)$ and some ρ' with $\rho_{i,j}' = \rho_{i+1,j+1}$ for $j > 1$ and first row densities $\rho_{1,j}'$ determined by T. We refer to this new tableau $d(T)$ as the **density shift** of T. See Figure 5 for an example.

$$
\boxed{\begin{array}{|c|c|c|} \hline 1\,2 & 4\,5\,7 & \ldots \\ \hline 3\,6\,8\,9 & 10\,11 & \ldots \\ \hline \end{array}} \implies \boxed{\begin{array}{|c|c|c|} \hline 1\,2 & \mathbf{3}\,4\,5\,\mathbf{6}\,\mathbf{7} & \ldots \\ \hline & \mathbf{8}\,\mathbf{9}\,10\,11 & \ldots \\ \hline \end{array}} \implies \boxed{\begin{array}{|c|c|} \hline 1\,2\,3\,4\,5 & \ldots \\ \hline 6\,7\,8\,9 & \ldots \\ \hline \end{array}}
$$

Figure 5: A two-row set-valued tableau T and its density shift $d(T)$.

The map $T \mapsto d(T)$ is well-defined into $\bigcup_i \mathbb{S}(\lambda', \rho_i')$, assuming that one appropriately determines the collection of shifted densities ρ_i'. However, the map is far from injective, as $d(T)$ does not remember which of its (non-maximal) entries at position $(1,1)$ were shifted to that position. Relating $|\mathbb{S}(\lambda, \rho)|$ to the $|\mathbb{S}(\lambda', \rho_i')|$ requires that we account for all possible positioning of shifted entries at $(2,1)$.

In the statement of Theorem 3.1 and all that follows, we use a Young diagram of shape λ labelled with cell densities $\rho_{i,j}$ from ρ to denote the cardinality $\mathbb{S}(\lambda, \rho)$.

Theorem 3.1. *For any two-row shape $\lambda = (n_1, n_2)$ and density ρ as shown,*

$$
|\mathbb{S}(\lambda, \rho)| = \boxed{\begin{array}{|c|c|c|c|} \hline (a_1) & (a_2) & (a_3) & \ldots \\ \hline (b_1) & (b_2) & (b_3) & \ldots \\ \hline \end{array}} = \sum_{i=0}^{b_1} \binom{a_2 + i - 1}{i} \boxed{\begin{array}{|c|c|c|} \hline (a_2) & (a_3) & \ldots \\ \hline (b_1 + b_2 - i) & (b_3) & \ldots \\ \hline \end{array}}
$$

where the sets $\mathbb{S}(\lambda', \rho_i')$ inside the summation on the right are all of shape $\lambda = (n_1 - 1, n_2 - 1)$.

Proof. For fixed $\lambda = (n_1, n_2)$ and ρ as shown, partition $\mathbb{S}(\lambda, \rho)$ into subsets $S_1, \ldots, S_{b_1}$ where $S_i = \{T \in \mathbb{S}(\lambda, \rho) \mid$ precisely i entries at $(2,1)$ smaller than largest entry at $(1,2)\}$. When restricted to a specific S_i, $T \mapsto d(T)$ defines a function $d_i : \mathbb{S}(\lambda, \rho) \to \mathbb{S}(\lambda', \rho')$ with $\lambda' = (n_1 - 1, n_2 - 1)$ and $\rho_{1,1}' = a_2 - i$, $\rho_{2,1}' = b_2 + b_1 - i$. For any $0 \le i \le b_1$, we claim d_i is onto and is exactly m-to-1, where $m = \binom{a_2 + i - 1}{i}$.

So consider $\mathbb{S}(\lambda', \rho')$, and notice that the upper-leftmost cell of any $T' \in \mathbb{S}(\lambda', \rho')$ is filled with the integers $\{1, 2, \ldots, a_2 + i\}$. For any choice $\vec{u}$ of i integers from $[a_2 + i - 1]$, define a map $d_i^{-1} : \mathbb{S}(\lambda', \rho') \to \mathbb{S}(\lambda, \rho)$ as follows:

1. For any $T' \in \mathbb{S}(\lambda', \rho')$, remove the i integers at position $(1,1)$ corresponding to $\vec{u}$ as well as the $b_1 - i$ smallest integers at position $(2,1)$.

2. Append a new column to the left of the tableau and populate the bottom cell of that new column with the b_1 integers removed during Step #1.

3. Reindex all entries in the resulting (partially-filled) tableau by $x \mapsto x + a_1$ and add the set $[a_1]$ to the top cell in the new leftmost column, resulting in $d_i^{-1}(T) \in \mathbb{S}(\lambda, \rho)$.

The map d_i^{-1} has been defined so that $d \circ d_i^{-1}(T') = T'$ for every $T' \in \mathbb{S}(\lambda', \rho')$. If we once again let $\beta_1 < \ldots < \beta_{b_1}$ denote the integers at $(2,1)$ of some $T \in S_i$, we have $d_i^{-1} \circ d_i(T) = T$ for precisely those T where $\beta_1, \ldots, \beta_i$ correspond to $\vec{u}$ in $d_i(T)$. If we let $S_i|_{\vec{u}}$ denote the subset of S_i with this restriction upon the $\beta_1, \ldots, \beta_i$, it follows that $S_i|_{\vec{u}}$ is in bijection with $\mathbb{S}(\lambda', \rho')$. Ranging over all choices of $\vec{u}$ then allows us to conclude that $|S_i| = \binom{a_2 + i - 1}{i} |\mathbb{S}(\lambda', \rho')|$.

To obtain the summation from the theorem, note that changing the density $\rho'_{1,1}$ at position $(1,1)$ has no effect on the size of the sets $\mathbb{S}(\lambda, \rho)$. We are then allowed to assume that $\rho'_{1,1} = a_2 = \rho_{1,2}$ is unchanged as we pass from ρ to ρ'. As the S_i partition $\mathbb{S}(\lambda, \rho)$, varying $1 \le i \le b_1$ then yields the required summation. $\quad\square$

Now consider a pair of n-tuples of non-negative integers $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$. One may define a dominance ordering on these tuples whereby $\vec{x} \preceq \vec{y}$ if $x_1 + \ldots + x_i \le y_1 + \ldots y_i$ for every $1 \le i \le n$. Using this notation, Theorem 3.1 may be repeatedly applied to derive the following.

Theorem 3.2. *For any two-row shape $\lambda = (n_1, n_2)$ and density ρ as shown,*

$$|\mathbb{S}(\lambda, \rho)| = \boxed{\begin{array}{|c|c|c|c|} \hline (a_1) & (a_2) & (a_3) & \ldots \\ \hline (b_1) & (b_2) & (b_3) & \ldots \\ \hline \end{array}} = \sum_{(i_1, \ldots, i_{n_1-1}) \preceq (b_1, \ldots, b_{n_1-1})} \prod_{j=1}^{n_1-1} \binom{a_{j+1} + i_j - 1}{i_j}$$

Proof. Repeated application of Theorem 3.1 immediately yields

$$\begin{array}{|c|c|c|c|} \hline (a_1) & (a_2) & (a_3) & \ldots \\ \hline (b_1) & (b_2) & (b_3) & \ldots \\ \hline \end{array} = \sum_{i_1=0}^{b_1} \binom{a_2 + i_1 - 1}{i_1} \sum_{i_2=0}^{b_1+b_2-i_1} \binom{a_3 + i_2 - 1}{i_2} \sum_{i_3=0}^{b_1+b_2+b_3-i_1-i_2} \binom{a_4 + i_3 - 1}{i_3} \ldots$$

If we assume that $i_1, i_2, \ldots$ must be positive integers, the later summations on the right side are equivalent to $i_1 + i_2 \le b_1 + b_2$, $i_1 + i_2 + i_3 \le b_1 + b_2 + b_3$, etc. $\quad\square$

Observe that the equation of Theorem 3.2 involves all cell densities apart from a_1 and b_n, aligning with our intuition that changing a_1 or b_n does not effect $|\mathbb{S}(\lambda, \rho)|$. Also note that applying Theorems 3.1 and 3.2 to non-rectangular shapes $\lambda = (n_1, n_2)$ merely requires that we set $b_j = 0$ for every $j > n_2$.

Example 3.3. *For $\lambda = (n, n)$ and $a_j = 1$, $b_j = k - 1$ for all j, the product of Theorem 3.2 becomes*

$$\prod_{j=1}^{n_1-1} \binom{a_{j+1} + i_j - 1}{i_j} = \prod_{j=1}^{n_1-1} \binom{i_j}{i_j} = 1$$

Thus $|\mathbb{S}(\lambda, \rho)|$ is the number of $(n-1)$-tuples of non-negative integers $(i_1, \ldots, i_{n-1}) \preceq (k-1, \ldots, k-1)$. If we let $y = kn - i_1 - \ldots i_{n-1}$, these tuples may be placed in bijection with the set $\mathcal{D}_n^k$ of k-good paths by $(i_1, \ldots, i_{n-1}) \mapsto EN^{i_1} EN^{i_2} \ldots EN^{i_{n-1}} EN^y$. Hence $|\mathbb{S}(\lambda, \rho)| = C_n^k$, as expected from Section 2.

Example 3.4. *More generally, for $\lambda = (n, n)$ and any density with $a_j = 1$ for all j, Theorem 3.2 shows that $|\mathbb{S}(\lambda, \rho)|$ equals the number of $(n-1)$-tuples of non-negative integers $(i_1, \ldots, i_{n-1}) \preceq (b_1, \ldots, b_{n-1})$. These tuples may be placed in bijection with the set of N-E lattice paths from $(0,0)$ to $(n, b_1 + \ldots + b_n)$ that lie weakly below the path $P = EN^{b_1} EN^{b_2} \ldots EN^{b_n}$ via the same map as the previous example. See Section 4 for a further generalization of this result.*

4   Set-Valued Tableaux & Two-Dimensional Lattice Paths

We close by drawing a bijection between arbitrary $\mathbb{S}(\lambda, \rho)$ with $\lambda = n^2$ and various classes of two-dimensional lattice paths, generalizing the phenomenon exemplified in Figure 3. This requires a consideration of all integer lattice paths from $(0,0)$ to (a, b) that use only East $E = (1, 0)$ and North $N = (0, 1)$ steps, which we refer to as N-E lattice paths of shape (a, b). To avoid ambiguities in the definition of our lattice paths, for the rest of this section we restrict our attention to densities ρ that lack cells of density zero. It is straightforward to extend all of the results below to ρ with zero density cells, so long as there does not exist j where $\rho_{2,j} = \rho_{1,j+1} = 0$.

So fix $\lambda = n^2$ and consider the density ρ where $\rho_{1,j} = a_j, \rho_{2,j} = b_j$. If $\sum_j a_j = a$ and $\sum_j b_j = b$, there exists a map $\psi_\rho : \mathbb{S}(\lambda, \rho) \to \mathcal{P}$ into the set $\mathcal{P}$ of lattice paths of shape (a, b) such that first row entries of $T \in \mathbb{S}(\lambda, \rho)$ correspond to East steps in $\psi_\rho(T)$ and second row entries of T correspond to North steps in $\psi_\rho(T)$. The map ψ_ρ is always an injection, but its image is dependent upon the choice of ρ.

To characterize $\mathrm{im}(\psi_\rho)$, we introduce a partial order on $\mathcal{P}$. For $P_1, P_2 \in \mathcal{P}$, define $P_1 \ge P_2$ if P_1 lies weakly above P_2 across $0 \le x \le a$.[2] Our map ψ_ρ respects this partial order in the following sense.

---

[2]This poset is isomorphic to Young's lattice via the map that takes a path to the Young diagram lying above its conjugate.

Lemma 4.1. *For fixed $\lambda = n^2$ and ρ, take $P_1, P_2 \in \mathcal{P}$ such that $P_1 \geq P_2$. If $P_1 \in \text{im}(\psi_\rho)$, then $P_2 \in \text{im}(\psi_\rho)$.*

Proof. We prove the statement for when P_1 directly covers P_2. This corresponds to the situation where P_2 may be obtained from P_1 by replacing a single NE subsequence with an EN subsequence at the same position. Assume that this $NE \mapsto EN$ replacement occurs at the i and $i+1$ steps of both P_1 and P_2. For $T_1 \in \mathbb{S}(\lambda, \rho)$ with $\psi_\rho(T_1) = P_1$, the integer i must appear in the second row of T_1 and $i+1$ must appear in the first row of T_1. Column-standardness of T_1 implies that i must appear in a more leftward column of T_1 than does $i+1$. Then define $T_2 \in \mathbb{S}(\lambda, \rho)$ to be the tableau obtained by flipping the positions of i and $i+1$ in T_1. As i and $i+1$ are consecutive integers and since i originally appeared left of $i+1$ in T_1, T_2 is row- and column-standard. By construction, $\psi_\rho(T_2) = P_2$. □

For any two-row density ρ with non-zero cell densities, there exists unique $T_{max} \in \mathbb{S}(\lambda, \rho)$ such that, for all j, every integer in the jth column of T_{max} is smaller than every integer in the $(j+1)^{st}$ column of T_{max}. This is precisely the tableau such that $\psi_\rho(T_{max}) = E^{a_1} N^{b_1} \ldots E^{a_n} N^{b_n}$. For any such ρ, the order ideal generated by $\psi_\rho(T_{max})$ will precisely correspond to $\text{im}(\psi_\rho)$.

Theorem 4.2. *Fix $\lambda = n^2$ and density ρ with $\rho_{1_j} = a_j > 0$ $\rho_{2,j} = b_j > 0$ for all j. If we define $P_{max} \in \mathcal{P}$ by $P_{max} = E^{a_1} N^{b_1} \ldots E^{a_n} N^{b_n}$, then $\mathbb{S}(\lambda, \rho)$ is in bijection with $I = \{P \in \mathcal{P} \mid P \leq P_{max}\}$.*

Proof. As $P_{max} = \psi_\rho(T_{max})$, $P_{max} \in \text{im}(\psi_\rho)$ and Lemma 4.1 immediately gives $I \subseteq \text{im}(\psi_\rho)$. Since ψ_ρ is known to be injective, it is only left to show that $\text{im}(\psi_\rho) \subseteq I$.

So assume by contradiction there exists $T \in \mathbb{S}(\lambda, \rho)$ with $\psi_\rho(T) \not\leq P_{max}$. There exists a smallest index i such that the ith steps of both $\psi_\rho(T)$ and P_{max} begin at the same point, the ith step of $\psi_\rho(T)$ is a N step, and the ith step of P_{max} is an E step. This means that the subtableaux of T and T_{max} containing only $\{1, \ldots, i-1\}$ must have the same shape and density, while the integer i lies in the first row of T_{max} but in the second row of T. If i lies at position $(1, j)$ of T, the construction of T_{max} then implies that every integer in the $(j-1)^{st}$ columns of both T and T_{max} is smaller than i. It follows that i must lie in the $(2, j)$ cell of T. Yet then there must exist an entry at position $(1, j)$ of T that is larger than i, implying that T is not column-standard. □

Example 4.3. *For $\lambda = n^2$ and row-constant density ρ with $\rho_{1,j} = 1, \rho_{1,j} = k-1$, $\psi_\rho(T_{max}) = (EN^{k-1})^n$. N-E lattice paths lying weakly below $\psi_\rho(T_{max})$ are in bijection with N-E lattice paths lying weakly below the line $y = (k-1)x$, recovering the bijection of Section 2 between $\mathcal{D}_n^k$ and $\mathbb{S}(n^2, \rho)$.*

Example 4.4. *For $\lambda = (n+1)^2$ and row-constant density ρ with $\rho_{1,j} = t, \rho_{2,j} = s-t$, $\psi_\rho(T_{max}) = (E^t N^{s-t})^{n-1}$. This recovers the bijection between $\mathcal{B}_{s,t}(n)$ and N-E lattice paths utilized by de Mier and Noy [6].*

As a more involved example, we use Theorem 4.2 to derive a new combinatorial interpretation of the rational Catalan numbers in terms of standard set-valued Young tableaux. For relatively prime positive integers a and b, there exists a rational Catalan number $C(a, b) = \frac{1}{a+b}\binom{a+b}{a}$. As originally shown by Bizley [2] and extended by Grossman [9], the rational Catalan numbers equal the number of rational Dyck paths of shape (a, b), by which we mean N-E lattice paths from $(0,0)$ to (a, b) that lie weakly below the line of rational slope $y = \frac{b}{a}x$.

Applying Theorem 4.2 merely requires the identification of a unique maximal lattice path P_{max} among the set of all rational Dyck paths of shape (a, b). The path $P_{(a,b)} = E^1 N^{c_1} \ldots E^1 N^{c_a}$ with $c_i = \lfloor \frac{bi}{a} \rfloor - \lfloor \frac{b(i-1)}{a} \rfloor$ satisfies this condition, as $\sum_{i=1}^k c_i = \lfloor bk \rfloor a$ for all $1 \leq k \leq a$ and $P_{(a,b)}$ has a Northwest corner at the first integer lattice point below the intersection of $y = \frac{b}{a}x$ with $x = k$ for every $1 \leq k \leq a$.

Corollary 4.5. *Take positive integers a, b such that $\gcd(a, b) = 1$. Then $|\mathbb{S}(a^2, \rho)| = C(a, b) = \frac{1}{a+b}\binom{a+b}{a}$ for the density ρ with $\rho_{1,j} = 1$, $\rho_{2,j} = \lfloor \frac{bj}{a} \rfloor - \lfloor \frac{b(j-1)}{a} \rfloor$.*

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