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On convergence to equilibrium for one-dimensional chain of harmonic oscillators in the half-line

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Abstract

The mixing boundary-value problem for infinite one-dimensional chain of harmonic oscillators on the half-line is considered. The large time asymptotic behavior of solutions is obtained. The initial data of the system are supposed to be a random function which has some mixing properties. We study the distribution μ_t of the random solution at time moments $t \in \mathbb{R}$. The main result is the convergence of μ_t to a Gaussian probability measure as $t \to \infty$. The mixing properties of the limit measures are studied.

Key words and phrases: one-dimensional system of harmonic oscillators on the half-line; mixing problem; random initial data; mixing condition; Volterra integro-differential equation; compactness of measures; convergence to statistical equilibrium

1 Introduction

We consider the following infinite system of harmonic oscillators on the half–line:

$$\ddot{u}(x,t) = (\Delta_L - m^2)u(x,t), \quad x \in \mathbb{N}, \quad t > 0, \tag{1.1}$$

with the boundary condition as $x = 0$:

$$\ddot{u}(0,t) = F(u(0,t)) - m^2 u(0,t) - \gamma \dot{u}(0,t) + u(1,t) - u(0,t), \quad t > 0, \tag{1.2}$$

and with the initial condition (as $t = 0$)

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), \quad x \geq 0. \tag{1.3}$$

Here $u(x,t) \in \mathbb{R}^1$, $m \geq 0$, $\gamma \geq 0$, Δ_L denotes the second derivative on $\mathbb{Z}^1$:

$$\Delta_L u(x) = u(x+1) - 2u(x) + u(x-1), \quad x \in \mathbb{Z}^1.$$

If $\gamma = 0$, then (1.1)–(1.2) is a Hamiltonian system with the Hamiltonian functional

$$\mathrm{H}(u,\dot{u}) := \frac{1}{2}\sum_{x\geq 0}\Big(|\dot{u}(x)|^2 + |u(x+1) - u(x)|^2 + m^2|u(x)|^2\Big) + P(u(0)), \tag{1.4}$$

where, by definition, $P(q) := -\int F(q)\,dq$, $q \in \mathbb{R}^1$, stands for the potential energy of the external force. To prove the existence of solutions to problem (1.1)–(1.3) we assume that $P \equiv 0$ or

$$P \in C^2(\mathbb{R}), \quad P(q) \to +\infty \quad \text{as } |q| \to \infty, \tag{1.5}$$

so $P(q) \geq P_0$ for all q with some $P_0 \in \mathbb{R}$.

Write $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot,t), \dot{u}(\cdot,t))$, $Y_0 = (Y_0^0, Y_0^1) \equiv (u_0(\cdot), v_0(\cdot))$. We assume that the initial state $Y_0(x)$ belongs to the Hilbert space $\mathcal{H}_{\alpha,+}$, $\alpha \in \mathbb{R}$, consisting of real sequences, see Definition 2.1 below. The existence and uniqueness of the solutions $Y(t)$ is proved in Appendix A.

To prove the main result we assume that $F(q) = -\kappa q$ with $\kappa \geq 0$. Moreover, on the coefficients m, κ, γ of the system we impose some restrictions. Namely, if $\gamma > 0$ we assume that $m > 0$ or $m = 0$ and $\kappa \neq 0$. If $\gamma = 0$, it is assumed that $\kappa \in (0,2)$. The initial state $Y_0(x)$ is supposed to be a random element of the space $\mathcal{H}_{\alpha,+}$, $\alpha < -3/2$, with the distribution μ_0. We assume that μ_0 is a probability measure of mean zero satisfying conditions **S2**–**S4** below. In particular, we assume that the initial measure μ_0 satisfies a mixing condition. Roughly speaking, it means that $Y_0(x)$ and $Y_0(y)$ are asymptotically independent as $|x-y| \to \infty$.

For a given $t \in \mathbb{R}$, denote by μ_t the probability measure on $\mathcal{H}_{\alpha,+}$ giving the random solution $Y(t) = (u(\cdot,t), \dot{u}(\cdot,t))$ to problem (1.1)–(1.3). Our main objective is to prove the weak convergence of the measures μ_t on the space $\mathcal{H}_{\alpha,+}$ with $\alpha < -3/2$ to a limit measure μ_∞, which is an equilibrium Gaussian measure on $\mathcal{H}_{\alpha,+}$,

$$\mu_t \rightharpoonup \mu_\infty \quad \text{as } t \to \infty. \tag{1.6}$$

This means the convergence of the integrals $\displaystyle\int f(Y)\,\mu_t(dY) \to \int f(Y)\,\mu_\infty(dY)$ as $t \to \infty$ for any bounded continuous functional f on $\mathcal{H}_{\alpha,+}$.

For one-dimensional chains of harmonic oscillators in the whole line, the convergence to equilibrium have been established by Boldrighini *et al.* in [1] and by Spohn and Lebowitz in [23]. Ergodic properties of one-dimensional chains of anharmonic oscillators coupled to heat baths were studied by Jakšić, Pillet and others (see, e.g., [12, 13]).

In [2, 3], we studied the convergence to equilibrium for the systems described by partial differential equations. Later on, similar results were obtained in [5] for d-dimensional harmonic crystals with $d \geq 1$, and in [6] for a scalar field coupled to a harmonic crystal. The convergence (1.6) was derived for the harmonic crystals in the whole space $\mathbb{Z}^d$, $d \geq 1$, and in the half-space $\mathbb{Z}^d_+ := \{x = (x_1, \ldots, x_d) \in \mathbb{Z}^d : x_1 \geq 0\}$ with zero boundary condition (see [5] and [7], resp.). In the present paper we prove the convergence (1.6) only in the one-dimensional case but with the boundary condition of the form (1.2).

We outline the strategy of the proof. Using the technique from [4, 7], we derive the convergence (1.6) from the assertions **I** and **II**:
I. The family of measures μ_t, $t \geq 0$, is weakly compact in $\mathcal{H}_{\alpha,+}$ with $\alpha < -3/2$.
II. The characteristic functionals converge to a Gaussian functional,

$$\hat{\mu}_t(\Psi) := \int \exp(i\langle Y, \Psi\rangle_+)\, \mu_t(dY) \to \exp\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi, \Psi)\}, \quad t \to \infty, \tag{1.7}$$

Here $\Psi = (\Psi^0, \Psi^1) \in \mathcal{S} := S \oplus S$, where S denotes a space of real rapidly decreasing sequences, $\langle Y, \Psi\rangle_+ = \sum\limits_{i=0,1}\sum\limits_{x\geq 0} Y^i(x)\Psi^i(x)$ for $Y = (Y^0, Y^1) \in \mathcal{H}_{\alpha,+}$ and $\Psi = (\Psi^0, \Psi^1) \in \mathcal{S}$, $\mathcal{Q}_\infty(\Psi, \Psi)$ is a quadratic form.

To prove **I** we derive the uniform bound (5.1) below for the mean local energy with respect to the measure μ_t, $t \geq 0$, and apply the Prokhorov Theorem.

To prove **II** we study the asymptotic behavior of the solution $Y(t)$ and obtain (see Lemma 6.7) that

$$\langle Y(t), \Psi\rangle_+ \sim \langle U_0(t)Y_0, \Pi_\Psi\rangle, \quad t \to \infty, \quad \text{(in mean)}, \tag{1.8}$$

where $U_0(t)$ is a solving operator to the mixing problem (2.2)–(2.4) with zero boundary condition, the vector–functions Π_Ψ are expressed by $\Psi \in \mathcal{S}$ (see formula (6.19) below).

To prove (1.8) we decompose the solution $u(x,t)$ into two terms: $u(x,t) = z(x,t) + q(x,t)$. Here $z(x,t)$ is a solution of (1.1) with zero boundary condition and satisfying the initial condition (1.3), i.e., $(z(\cdot,t), \dot{z}(\cdot,t)) = U_0(t)Y_0$, $q(x,t)$ is a solution of (1.1) with zero initial condition and some boundary condition (see (2.7) below). The existence and behavior of the solution $z(x,t)$ were studied in [7]. In Section 2.3 we state the basic results on $z(x,t)$ (see also Appendix C). Therefore, the next step is to study the long time behavior of $q(x,t)$, $x \geq 0$. In the case when $x \geq 1$, we obtain the following formula for $q(x,t)$,

$$q(x,t) = \int\limits_0^t K(x, t-s) q(0,s)\, ds, \quad x \in \mathbb{N}, \tag{1.9}$$

where the kernel $K(x,t)$ is defined in (3.5) below. The properties of this kernel are studied in Appendix A. In particular, we prove that $K(x,t)$ satisfies the following bound

$$\sum_{x\in\mathbb{N}} (1+x^2)^\alpha |K(x,t)|^2 \sim C(1+t)^{-3} \quad \text{as } t \to \infty, \quad \text{for any } \alpha < -3/2. \tag{1.10}$$

Therefore, the main step in the proof of (1.8) is to study the behavior of $q(x,t)$ with $x=0$. By (1.2) and (1.9), $q(0,t)$ evolves according to a Volterra integro-differential equation of a form

$$\ddot{q}(0,t)=F(q(0,t))-(1+m^2)q(0,t)-\gamma\dot{q}(0,t)+\int\limits_0^t K(1,t-s)q(0,s)\,ds+h(t),\quad t>0, \quad (1.11)$$

with $h(t)=z(1,t)$. To study the long time behavior for the solutions of Eqn (1.11), we put $F(q)=-\kappa q$ with $\kappa\geq 0$, apply the Fourier-Laplace transform $t\to\omega$ to the solutions $q(0,t)$ and study the analytic properties of $\tilde{q}(0,\omega)$ for complex values $\omega\in\mathbb{C}$ (see Appendix B). These properties allow us to obtain the following bound for the solutions $q(0,t)$ of Eqn (1.11) with $h(t)\equiv 0$:

$$|q(0,t)|\leq C(1+t)^{-3/2},\quad t\geq 0.$$

Applying this estimate together with (1.9) and (1.10), we obtain the asymptotics for $q(x,t)$ in mean (see formula (6.16) below)

$$\langle q(\cdot,t),\psi\rangle_+\sim\langle U_0(t)Y_0,\mathbf{K}^0_\psi\rangle_+,\quad t\to\infty,$$

where $\psi\in S$, the vector valued function $\mathbf{K}^0_\psi$ is defined in (6.15) below. This implies the asymptotics (1.8) which plays the crucial role in our convergence analysis for the statistical solutions of the problem (1.1)–(1.3).

The dynamics of the equations with delay has been extensively investigated by many authors under some restrictions on the kernel $K(1,t)$. For details, we refer to the monograph by Gripenberg, Londen and Staffans [9]. The stability properties of linear Volterra integro-differential equations can be found in the papers by Murakami [19], Nino and Murakami [20], Kordonis and Philos [16]. Note that in the literature frequently the long time asymptotical behavior of solutions is studied assuming that the kernel has the exponential decay or is of one sign. However, in our case, by (1.10), the decay of $K(1,t)$ is like $(1+t)^{-3/2}$.

In recent years the nonlinear equation of a form (1.11) with a stationary Gaussian process $h(t)$ and with a smooth (confining or periodic) potential $P(q)=-\int F(q)dq$ has been investigated also extensively, for example, the ergodic properties of such equations were studied by Jakšić and Pillet in [12], and the qualitative properties of solutions were established by Ottobre and Pavliotis in [21]. In the present paper, we prove the convergence to equilibrium for a linear model. However, we do not assume that the initial distribution of the system is a Gibbs measure or absolutely continuous with respect to a Gibbs measure. Therefore, the force $h(t)=z(1,t)$ in Eqn (1.11) is non-Gaussian, in general.

The paper is organized as follows. In Section 2 we impose the conditions on the model and on the initial measures μ_0 and state the main results. The limit behavior for solutions of Eqn (1.11) is studied in Sections 3 and 4. The compactness of the measures family $\{\mu_t,\,t\in\mathbb{R}\}$ is proved in Section 5. In Section 6 we establish asymptotics (1.8) and complete the proof of the convergence (1.6). In Appendix A we study the properties of $K(x,t)$ and prove the existence of the solutions. The behavior of $q(0,t)$ is studied in Appendix B. Appendix C contains the results on the solutions $z(x,t)$.

2 Main Results

2.1 Phase space

We assume that the initial date Y_0 belongs to the phase space $\mathcal{H}_{\alpha,+}$, $\alpha\in\mathbb{R}$, defined below.

Definition 2.1 *(i)* $\ell^2_{\alpha,+}\equiv\ell^2_{\alpha,+}(\mathbb{Z}^1_+)$, $\alpha\in\mathbb{R}$, *is the Hilbert space of sequences* $u(x)$, $x\geq 0$, *with norm*

$$\|u\|^2_{\alpha,+}=\sum_{x\geq 0}|u(x)|^2\langle x\rangle^{2\alpha}<\infty,\quad \langle x\rangle:=(1+|x|^2)^{1/2}.$$

(ii) $\mathcal{H}_{\alpha,+}=\ell^2_{\alpha,+}\otimes\ell^2_{\alpha,+}$ *is the Hilbert space of pairs* $Y=(u,v)$ *of sequences equipped with norm*

$$\|Y\|^2_{\alpha,+}=\|u\|^2_{\alpha,+}+\|v\|^2_{\alpha,+}<\infty\,.$$

$\mathcal{H}_{\alpha,+}$ *is equipped with the Borel* σ*-algebra* $\mathcal{B}(\mathcal{H}_{\alpha,+})$.
(iii) $\ell^2_\alpha\equiv\ell^2_\alpha(\mathbb{Z}^1)$ *is the Hilbert space of sequences with norm* $\|u\|^2_\alpha=\sum\limits_{x\in\mathbb{Z}}|u(x)|^2\langle x\rangle^{2\alpha}<\infty$. *In particular,* $\ell^2_0\equiv\ell^2$. *Write* $\mathcal{H}_\alpha:=\ell^2_\alpha\otimes\ell^2_\alpha$, $\alpha\in\mathbb{R}$.

Theorem 2.2 *Let* $\gamma,m\geq 0$ *and condition (1.5) hold, and let* $Y_0\in\mathcal{H}_{\alpha,+}$, $\alpha\in\mathbb{R}$. *Then the mixing problem (1.1)–(1.3) has a unique solution* $Y(t)\in C(\mathbb{R},\mathcal{H}_{\alpha,+})$. *The operator* $U(t):Y_0\to Y(t)$ *is continuous on* $\mathcal{H}_{\alpha,+}$. *Moreover, for any* $T>0$, $\sup_{|t|\leq T}\|U(t)Y_0\|_{\alpha,+}\leq B$, *where the constant* B *depends on* $\|Y_0\|_{\alpha,+}$ *and* T.

Theorem 2.2 is proved in Appendix A. The proof is based on the following representation for the solution $u(x,t)$ of the problem (1.1)–(1.3):

$$u(x,t)=z(x,t)+q(x,t),\quad x\geq 0, \tag{2.1}$$

where $z(x,t)$ is a solution of the mixing problem with zero boundary condition,

$$\ddot z(x,t)=(\Delta_L-m^2)z(x,t),\quad x\in\mathbb{N},\quad t>0, \tag{2.2}$$
$$z(0,t)=0,\quad t\geq 0, \tag{2.3}$$
$$z(x,0)=u_0(x),\quad \dot z(x,0)=v_0(x),\quad x\in\mathbb{N}. \tag{2.4}$$

Therefore, $q(x,t)$ is a solution of the mixing problem with zero initial condition,

$$\ddot q(x,t)=(\Delta_L-m^2)q(x,t),\quad x\in\mathbb{N},\quad t>0, \tag{2.5}$$
$$\ddot q(0,t)=F(q(0,t))-m^2q(0,t)-\gamma\dot q(0,t)+q(1,t)-q(0,t)+z(1,t),\ t>0, \tag{2.6}$$
$$q(x,0)=0,\quad \dot q(x,0)=0,\quad x\in\mathbb{N}, \tag{2.7}$$
$$q(0,0)=u_0(0),\quad \dot q(0,0)=v_0(0). \tag{2.8}$$

In Section 2.3 we state the results concerning the solutions $z(x,t)$. In Sections 3 and 4, we study the behavior of the solutions $q(x,t)$.

To prove the main result we assume that $F(q)=-\kappa q$, with $\kappa\geq 0$. Moreover, on the nonnegative constants m,γ,κ of the system we impose the following condition **C**.
C. (i) If $\gamma\neq 0$, then either $m\neq 0$ or $m=0$ and $\kappa\neq 0$;
(ii) if $\gamma=0$, then $\kappa\in(0,2)$.

2.2 Random initial data

We assume that the initial date $Y_0(x) = (Y_0^0(x), Y_0^1(x))$ is a measurable random function with values in $(\mathcal{H}_{\alpha,+}, \mathcal{B}(\mathcal{H}_{\alpha,+}))$. Denote by μ_0 a Borel probability measure on $\mathcal{H}_{\alpha,+}$ giving the distribution of Y_0. The expectation with respect to μ_0 is denoted by $\mathbb{E}$.

We impose the following conditions **S1**–**S4** on μ_0.
S1. The measure μ_0 has zero expectation value, $\mathbb{E}(Y_0(x)) \equiv \int Y_0(x)\mu_0(dY_0) = 0$, $x \in \mathbb{Z}^1_+$.
S2. The measure μ_0 has finite variance, $\sup\limits_{x\geq 0} \mathbb{E}(|Y_0(x)|^2) \leq e_0 < \infty$.

Write $\nu_0 = \mu_0\{Y_0 \in \mathcal{H}_{\alpha,+} : Y_0(0) = 0\}$. The expectation with respect to ν_0 is denoted by $\mathbb{E}_0$. On the measure ν_0 we impose conditions **S3** and **S4**.
S3. The correlation functions of the measure ν_0 are denoted by

$$Q_0^{ij}(x, x') = \mathbb{E}_0(Y_0^i(x)Y_0^j(x')), \quad x, x' \geq 0, \quad i, j = 0, 1.$$

In particular, the matrix $Q_0(x, x') = (Q_0^{ij}(x, x'))_{i,j=0,1}$ vanishes if $x = 0$ or $x' = 0$. Moreover, for every $x \in \mathbb{Z}^1$,

$$\lim_{y\to+\infty} Q_0^{ij}(x + y, y) = q_0^{ij}(x).$$

Here $q_0^{ij}(x)$ are correlation functions of some translation invariant measure ν with zero mean value on the space $\mathcal{H}_\alpha$. By definition, a measure ν is said to be translation invariant if $\nu(T_hB) = \nu(B)$ for $B \in \mathcal{B}(\mathcal{H}_\alpha)$, $h \in \mathbb{Z}^1$, where $T_hY_0(x) = Y_0(x - h)$ for $x \in \mathbb{Z}^1$.

To formulate the last condition on ν_0, denote by $\mathcal{A}$ an open interval of $\mathbb{Z}^1_+$ and by $\sigma(\mathcal{A})$ the σ-algebra on $\mathcal{H}_{\alpha,+}$ generated by $Y_0(x)$ with $x \in \mathcal{A}$. Define the Ibragimov mixing coefficient of a probability measure ν on $\mathcal{H}_{\alpha,+}$ by the rule

$$\varphi(r) \equiv \sup_{\substack{\mathcal{A},\mathcal{B}\subset\mathbb{Z}^1_+ \\ \mathrm{dist}(\mathcal{A},\mathcal{B})\geq r}} \quad \sup_{\substack{A\in\sigma(\mathcal{A}),\, B\in\sigma(\mathcal{B}) \\ \nu(B)>0}} \frac{|\nu(A\cap B) - \nu(A)\nu(B)|}{\nu(B)}. \tag{2.9}$$

Definition 2.3 *A measure ν is said to satisfy the strong uniform Ibragimov mixing condition if $\varphi(r) \to 0$ as $r \to \infty$.*

S4. The measure ν_0 satisfies the strong uniform Ibragimov mixing condition with mixing coefficient φ, and

$$\int_0^{+\infty} \varphi^{1/2}(r)\, dr < \infty. \tag{2.10}$$

The examples of the initial measures ν_0 satisfying **S3** and **S4** are given in [7].

Lemma 2.4 *Let conditions* **S1**–**S4** *hold. Then $q_0^{ij} \in \ell^1$. Moreover, for any $\Phi, \Psi \in \mathcal{H}_{0,+}$, the following bound holds*

$$|\langle Q_0(z, z'), \Phi(z) \otimes \Psi(z')\rangle_+| \leq C\|\Phi\|_{0,+}\|\Psi\|_{0,+}. \tag{2.11}$$

Proof Indeed, by [10, Lemma 17.2.3], conditions **S1**, **S2** and **S4** imply

$$|Q_0^{ij}(z,z')| \le Ce_0\,\varphi^{1/2}(|z-z'|), \quad z,z'\in\mathbb{N}, \quad i,j=0,1. \tag{2.12}$$

Hence, condition (2.10) implies (2.13),

$$\sum_{z'\in\mathbb{N}} |Q_0^{ij}(z,z')| \le Ce_0 \sum_{z\in\mathbb{Z}^1} \varphi^{1/2}(|z|) \le C_1 < \infty \quad \text{for all } z\in\mathbb{N}, \tag{2.13}$$

and

$$\sum_{z\in\mathbb{N}} |Q_0^{ij}(z,z')| \le C_1 < \infty \quad \text{for all } z'\in\mathbb{N}. \tag{2.14}$$

Here the constant C_1 does not depend on $z,z'\in\mathbb{N}$. Estimates (2.13) and (2.14) and the Shur lemma imply the bound (2.11). Moreover, by condition **S3** and the bound (2.12),

$$|q_0^{ij}(z)| \le Ce_0\,\varphi^{1/2}(|z|), \quad z\in\mathbb{N}.$$

Hence, $q_0^{ij}\in\ell^1$. This implies, in particular, that $\hat{q}_0^{ij}\in C(\mathbb{T}^1)$. Here and below $\hat{q}(\theta)\equiv F_{x\to\theta}[q(x)]$ denotes the Fourier transform w.r.t. $x\in\mathbb{Z}^1$,

$$\hat{q}(\theta) = \sum_{x\in\mathbb{Z}^1} e^{ix\theta} q(x), \quad \theta\in\mathbb{T}^1\equiv\mathbb{R}^1/2\pi\mathbb{Z}^1. \;\blacksquare$$

For a probability measure ν on $\mathcal{H}_{\alpha,+}$ denote by $\hat{\nu}$ the characteristic functional (Fourier transform)

$$\hat{\nu}(\Psi) \equiv \int \exp(i\langle Y,\Psi\rangle_+)\,\nu(dY), \;\; \Psi\in\mathcal{S}.$$

A measure ν is called Gaussian (with zero expectation) if its characteristic functional has the form $\hat{\nu}(\Psi)=\exp\{-\frac{1}{2}\mathcal{Q}(\Psi,\Psi)\}$, $\Psi\in\mathcal{S}$, where $\mathcal{Q}$ is a real nonnegative quadratic form in $\mathcal{S}$.

2.3 The mixing problem with zero boundary condition. Convergence to equilibrium for problem (2.2)–(2.4)

In this section we study the problem (2.2)–(2.4). The results of this subsection were proved in [7]. To state these results, write $Z(t)\equiv Z(x,t)=(z(x,t),\dot{z}(x,t))$.

Lemma 2.5 *(see Lemma 2.7 in [7]) Assume that $\alpha\in\mathbb{R}$. Then the following assertions hold.*
(i) For any $Y_0\in\mathcal{H}_{\alpha,+}$, there exists a unique solution $Z(t)\in C(\mathbb{R},\mathcal{H}_{\alpha,+})$ to the mixed problem (2.2)–(2.4).
(ii) For any $t\in\mathbb{R}$, the operator $U_0(t): Y_0\mapsto Z(t)$ is continuous on $\mathcal{H}_{\alpha,+}$, and

$$\sup_{|t|\le T} \|Z(\cdot,t)\|^2_{\alpha,+} \le C(T)\|Y_0\|^2_{\alpha,+}.$$

The proof is based on the following formula for the solution $Z(x,t)$ of problem (2.2)–(2.4):

$$Z^i(x,t) = \sum_{j=0,1}\sum_{x'\ge 1} \mathcal{G}^{ij}_{t,+}(x,x')Y_0^j(x'), \quad x\in\mathbb{Z}^1_+, \tag{2.15}$$

where $Z(x,t)=(Z^0(x,t),Z^1(x,t))\equiv(z(x,t),\dot z(x,t))$, the Green function $\mathcal{G}_{t,+}(x,x')$ is

$$\mathcal{G}_{t,+}(x,x'):=\mathcal{G}_t(x-x')-\mathcal{G}_t(x+x'),\quad \mathcal{G}_t(x)\equiv\frac{1}{2\pi}\int_{\mathbb{T}^1}e^{-ix\theta}\hat{\mathcal{G}}_t(\theta)\,d\theta,\quad x\in\mathbb{Z}^1, \tag{2.16}$$

with

$$\hat{\mathcal{G}}_t(\theta)=(\hat{\mathcal{G}}_t^{ij}(\theta))_{i,j=0}^1=\left(\begin{array}{cc}\cos\phi(\theta)t & \frac{\sin\phi(\theta)t}{\phi(\theta)}\\ -\phi(\theta)\sin\phi(\theta)t & \cos\phi(\theta)t\end{array}\right),\quad \phi(\theta)=\sqrt{2-2\cos\theta+m^2}. \tag{2.17}$$

In particular, $\phi(\theta)=2|\sin(\theta/2)|$ if $m=0$. We see that $Z(0,t)\equiv0\ \ \forall t$, since $\mathcal{G}_t(-x)=\mathcal{G}_t(x)$ (see formulas (2.16) and (2.17)).

Definition 2.6 *We define ν_t as the Borel probability measure on $\mathcal{H}_{\alpha,+}$, which gives the distribution of solution $Z(t)$ to the problem (2.2)–(2.4), i.e.,*

$$\nu_t(B)=\nu_0(U_0(-t)B),\quad where\ B\in\mathcal{B}(\mathcal{H}_{\alpha,+}),\quad t\in\mathbb{R}.$$

Lemma 2.7 *The correlation functions of measures ν_t converge to a limit,*

$$Q_t^{ij}(x,x')=\int Z^i(x)Z^j(x')\,\nu_t(dZ)\to Q_\infty^{ij}(x,x'),\quad t\to\infty,\quad x,x'\in\mathbb{Z}^1_+. \tag{2.18}$$

The correlation matrix $Q_\infty(x,x')=(Q_\infty^{ij}(x,x'))_{i,j=0}^1$ has the form

$$Q_\infty(x,x')=q_\infty(x-x')-q_\infty(x+x')-q_\infty(-x-x')+q_\infty(-x+x'),\quad x,x'\in\mathbb{Z}^1_+. \tag{2.19}$$

Here $q_\infty(x)=q_\infty^+(x)+q_\infty^-(x)$, $x\in\mathbb{Z}^1$, where in the Fourier transform, we have

$$\begin{array}{rcl}\hat q_\infty^+(\theta) &=& \frac12(\hat q_0(\theta)+C(\theta)\hat q_0(\theta)C^T(\theta))=\frac12\left(\begin{array}{cc}\hat q_0^{00}+\hat q_0^{11}\phi^{-2}(\theta) & \hat q_0^{01}-\hat q_0^{10}\\ -\hat q_0^{01}+\hat q_0^{10} & \phi^2(\theta)\hat q_0^{00}+\hat q_0^{11}\end{array}\right),\\ \hat q_\infty^-(\theta) &=& \frac12\,\mathrm{sgn}(\theta)(C(\theta)\hat q_0(\theta)-\hat q_0(\theta)C^T(\theta))\\ &=& \frac12\,\mathrm{sgn}(\theta)\frac{1}{\phi(\theta)}\left(\begin{array}{cc}\hat q_0^{10}-\hat q_0^{01} & \phi^2(\theta)\hat q_0^{00}+\hat q_0^{11}\\ -\phi^2(\theta)\hat q_0^{00}-\hat q_0^{11} & \phi^2(\theta)(\hat q_0^{10}-\hat q_0^{01})\end{array}\right).\end{array} \tag{2.20}$$

Here $\theta\in\mathbb{T}^1$ if $m\ne0$ and $\theta\in\mathbb{T}^1\setminus\{0\}$ if $m=0$, $(\cdot)^T$ denotes a matrix transposition, the functions q_0^{ij}, $i,j=0,1$, are introduced in condition **S3**,

$$C(\theta)=\left(\begin{array}{cc}0 & 1/\phi(\theta)\\ -\phi(\theta) & 0\end{array}\right),\quad \phi(\theta)=\sqrt{2-2\cos\theta+m^2}. \tag{2.21}$$

Denote by $\mathcal{Q}_\infty^\nu(\Psi,\Psi)$ a real quadratic form on $\mathcal{S}$ defined by

$$\mathcal{Q}_\infty^\nu(\Psi,\Psi)=\langle Q_\infty(x,x'),\Psi(x)\otimes\Psi(x')\rangle_+\equiv\sum_{i,j=0,1}\sum_{x,x'\in\mathbb{Z}^1_+}Q_\infty^{ij}(x,x')\Psi^i(x)\Psi^j(x'). \tag{2.22}$$

Remark 2.8 Given $\Psi=(\Psi^0,\Psi^1)\in\ell^2_{0,+}\times\ell^2_{0,+}$, introduce an odd sequence $\Psi_o(x)$, $x\in\mathbb{Z}^1$, such that $\Psi_o(x)=\Psi(x)$ for $x>0$, $\Psi_o(0)=0$ and $\Psi_o(x)=-\Psi(-x)$ for $x<0$. Note that $Q_\infty(x,x')$ is odd w.r.t. x and $x'\in\mathbb{Z}^1$. Moreover,

$$\mathcal{Q}^\nu_\infty(\Psi,\Psi)=\langle q_\infty(x-x'),\Psi_o(x)\otimes\Psi_o(x')\rangle\equiv\sum_{i,j=0}^{1}\sum_{x,x'\in\mathbb{Z}^1}q^{ij}_\infty(x-x')\Psi^i_o(x)\Psi^j_o(x'). \tag{2.23}$$

In the case $m\neq 0$, $q^{ij}_\infty\in\ell^1$, by Lemma 2.4 and formulas (2.20). Then, Young's inequality yields

$$|\mathcal{Q}^\nu_\infty(\Psi,\Psi)|\le C\|q_\infty\|_{\ell^1}\|\Psi_o\|^2_0\le C_1\|q_\infty\|_{\ell^1}\|\Psi\|^2_{0,+},$$

i.e., $\mathcal{Q}^\nu_\infty(\Psi,\Psi)$ is continuous in $\mathcal{H}_{0,+}\equiv\ell^2_{0,+}\times\ell^2_{0,+}$. Here as before $\|\cdot\|_0$ ($\|\cdot\|_{0,+}$) stands for the norm in $\ell^2_0\equiv\ell^2$ or in $\mathcal{H}_0$ (in $\ell^2_{0,+}$ and in $\mathcal{H}_{0,+}$, respectively).

In the case $m=0$, $\phi(\theta)=2|\sin(\theta/2)|$. Note that $|\hat\psi_o(\theta)|\le C(\psi)|\sin\theta|$ for $\psi\in S$. Denote $\ell^2_{1/\phi,+}=\{\psi\in\ell^2_{0,+}:\hat\psi_o/\phi\in L^2(\mathbb{T}^1)\}$ with norm $\|\psi\|^2_{1/\phi,+}:=\|(1+1/\phi(\theta))\hat\psi_o(\theta)\|^2_{L^2(\mathbb{T}^1)}$. Therefore, for any $\Psi=(\Psi^0,\Psi^1)\in\mathcal{H}_{1/\phi,+}:=\ell^2_{1/\phi,+}\times\ell^2_{0,+}$,

$$\mathcal{Q}^\nu_\infty(\Psi,\Psi)\le\|q_0\|_{\ell^1}\left(C_1\|F^{-1}(1/\phi)*\Psi^0_o\|^2_0+C_2\|\Psi_o\|^2_0\right)\le C_3\|\Psi^0\|^2_{1/\phi,+}+C_4\|\Psi^1\|^2_{0,+},$$

by (2.20) and (2.23). Here and below by F^{-1} we denote the inverse Fourier transform.

Theorem 2.9 *Let conditions* **S1**, **S3**, **S4** *hold and let $\alpha<-1/2$ if $m\neq 0$ and $\alpha<-1$ if $m=0$. Then the following assertions are fulfilled. (i) The measures $\nu_t=U_0(t)^*\nu_0$ weakly converge as $t\to\infty$:*

$$\nu_t\to\nu_\infty\quad\text{as } t\to\infty\quad\text{on the space } \mathcal{H}_{\alpha,+}.$$

(ii) The limit measure ν_∞ is Gaussian, with zero mean value and with correlation matrix Q_∞ defined in (2.19).
(iii) The uniform bound holds,

$$\sup_{t\ge 0}\mathbb{E}_0\|U_0(t)Y_0\|^2_{\alpha,+}<\infty. \tag{2.24}$$

Moreover, $\lim_{t\to\infty}\mathbb{E}_0|\langle U_0(t)Y_0,\Psi\rangle_+|^2=\mathcal{Q}^\nu_\infty(\Psi,\Psi)$ for any $\Psi\in\mathcal{S}$.
(iv) The limit measure ν_∞ satisfies a mixing condition w.r.t. the group $U_0(t)$, i.e., for any $f,g\in L^2(\mathcal{H}_{\alpha,+},\nu_\infty)$, $\lim\limits_{t\to\infty}\int f(U_0(t)Y)g(Y)\nu_\infty(dY)=\int f(Y)\nu_\infty(dY)\int g(Y)\nu_\infty(dY)$.

If $m\neq 0$, the results of Lemmas 2.7 and Theorem 2.9 follow directly from [7]. In the case $m=0$, the proof needs in some modification. In Appendix C we prove the bound (2.24) for $m\ge 0$ which implies the compactness of the measures family $\{\nu_t,t\in\mathbb{R}\}$, and outline the proof of Theorem 2.9.

2.4 Main theorem

Definition 2.10 *(i) We define μ_t as the Borel probability measure on $\mathcal{H}_{\alpha,+}$ given by the rule $\mu_t(B)=\mu_0(U(-t)B)$, where $B\in\mathcal{B}(\mathcal{H}_{\alpha,+})$ and $t\in\mathbb{R}$.*

(ii) The correlation functions of the measure μ_t are defined by

$$\mathbf{Q}^{ij}_t(x,x')\equiv\mathbb{E}\Big(Y^i(x,t)Y^j(x',t)\Big),\quad i,j=0,1,\qquad x,x'\in\mathbb{N}.$$

Here $Y^i(x,t)$ are the components of the random solution $Y(t)=(Y^0(\cdot,t),Y^1(\cdot,t))$ to problem (1.1)–(1.3).

Denote by $\mathcal{Q}_\infty(\Psi,\Psi)$ a real quadratic form in $\mathcal{S}$ of the form

$$\mathcal{Q}_\infty(\Psi,\Psi)=\mathcal{Q}_\infty^\nu(\Pi_\Psi,\Pi_\Psi), \tag{2.25}$$

where $\mathcal{Q}_\infty$ is defined by (2.22), Π_Ψ from (6.19).

Remark 2.11 *For any $\Psi\in\mathcal{S}$, $\Pi_\Psi\in\mathcal{H}_{1/\phi,+}$. This follows from (6.19) and Remark 6.5. Therefore, by Remark 2.8 and formula (2.25), $\mathcal{Q}_\infty(\Psi,\Psi)$ is bounded for all $\Psi\in\mathcal{H}_{0,+}$.*

The main result of the paper is the following theorem.

Theorem 2.12 *Let conditions* **S1**–**S4** *on the initial measure μ_0 hold and the constants m,γ,κ satisfy condition* **C**. *Then the following assertions are fulfilled.*
(i) The measures μ_t weakly converge as $t\to\infty$ to a limit measure μ_∞ on the space $\mathcal{H}_{\alpha,+}$ with $\alpha<-3/2$. Moreover, the limit measure is Gaussian with zero mean and the correlation matrix Q_∞.
(ii) The correlation functions of μ_t converge to a limit, and for $\Psi\in\mathcal{S}$,

$$\lim_{t\to\infty}\mathbb{E}|\langle Y(t),\Psi\rangle_+|^2=\mathcal{Q}_\infty(\Psi,\Psi).$$

3 Large time asymptotic behavior for $q(x,t)$

In Sections 3 and 4, we investigate the behavior of the solutions $q(x,t)$ to the problem (2.5)–(2.8) as $t\to\infty$. At first, we study the properties of $q(x,t)$ with $x\ge 1$ in the Fourier–Laplace transform. In next section, we study the asymptotics of $q(0,t)$.

Definition 3.1 *Let $|q(t)|\le Ce^{Bt}$. The Laplace–Fourier transform of $q(t)$ is given by the formula*

$$\tilde q(\omega)=\int_0^{+\infty}e^{i\omega t}q(t)\,dt,\quad \operatorname{Im}\omega>B. \tag{3.1}$$

The Gronwall inequality and conditions on $F(q)$ imply standard a priori estimates for the solutions $q(x,t)$, $x\ge 1$. In particular, there exist constants $A,B<\infty$ such that

$$\sum_{x\in\mathbb{N}}(|q(x,t)|^2+|\dot q(x,t)|^2)\le Ce^{Bt}\quad \text{as }\ t\to\infty.$$

This bound is proved in Appendix A (see formula (7.19)). Hence the Laplace–Fourier transform with respect to t-variable, $q(x,t)\to\tilde q(x,\omega)$, exists at least for $\operatorname{Im}\omega>B$ and satisfies the following equation

$$(-\Delta_L+m^2-\omega^2)\tilde q(x,\omega)=0,\quad x\in\mathbb{N},\quad \operatorname{Im}\omega>B, \tag{3.2}$$

by (2.7). Now we construct the solution of (3.2). Note first that the Fourier transform of the lattice operator $-\Delta_L$ is the operator of multiplication by $\phi_0(\theta)=2-2\cos\theta$, and $\phi_0(\theta)\in[0,4]$. Second, the following lemma holds (see Lemma 2.1 in [15]).

Denote $\Gamma_m:=[-\sqrt{4+m^2},-m]\cup[m,\sqrt{4+m^2}]$.

Lemma 3.2 *For given $\omega \in C_m = \mathbb{C} \setminus \Gamma_m$, the equation*

$$2 - 2\cos\theta = \omega^2 - m^2 \tag{3.3}$$

has the unique solution (we denote it by $\theta(\omega)$) in the domain $\{\theta \in \mathbb{C} : \operatorname{Im}\theta > 0, -\pi < \operatorname{Re}\theta \le \pi\}$. Moreover, $\theta(\omega)$ is an analytic function in C_m.

Since we seek the solution $q(\cdot, t) \in \ell^2_{\alpha,+}$ with some α, $\tilde{q}(x, \omega)$ has a form

$$\tilde{q}(x, \omega) = \tilde{q}(0, \omega) e^{i\theta(\omega)x}, \quad x \ge 0.$$

Applying the inverse Laplace–Fourier transform with respect to ω-variable, we write the solution $q(x, t)$ in the form

$$q(x,t) = \frac{1}{2\pi} \int\limits_{-\infty+i\mu}^{+\infty+i\mu} e^{-i\omega t} \tilde{q}(0,\omega) e^{i\theta(\omega)x}\, d\omega, \quad x \in \mathbb{N}, \quad t > 0, \quad \mu > \alpha, \tag{3.4}$$

and $q(\cdot, t) \in \ell^2_{\alpha,+}$. Moreover, the left-hand side of (3.4) does not depend on μ. The integral in (3.4) is understood in the sense of principal value. Write $\tilde{K}(x, \omega) = e^{i\theta(\omega)x}$ and put

$$K(x,t) = \frac{1}{2\pi} \int\limits_{-\infty+i\mu}^{+\infty+i\mu} e^{-i\omega t} \tilde{K}(x,\omega)\, d\omega, \quad x \in \mathbb{N}, \quad t > 0, \quad \mu > \alpha. \tag{3.5}$$

Then (3.4) implies the following representation

$$(q(x,t), \dot{q}(x,t)) = \int_0^t K(x, t-s)(q(0,s), \dot{q}(0,s))\, ds, \quad x \in \mathbb{N}. \tag{3.6}$$

To study the large time behavior of $q(x, t)$, we will move down the contour of integration in (3.4) and in (3.5). In Appendix A, we study the analytic properties $\tilde{K}(x, \omega)$ for $\omega \in \mathbb{C}$ and obtain the following result.

Theorem 3.3 *For every $x \in \mathbb{N}$, $K(x, t) = 0$ for $t < 0$. Moreover, for any $\alpha < -3/2$, the following bound holds,*

$$\sum_{x\in\mathbb{N}} \langle x\rangle^{2\alpha} |K(x,t)|^2 \le C(1+t)^{-3} \quad \textit{for} \quad t > 0. \tag{3.7}$$

In particular,

$$|K(1,t)| \le C(1+t)^{-3/2}, \quad t > 0. \tag{3.8}$$

Corollary 3.4 *For any $\alpha < -3/2$, (3.6) and (3.7) imply*

$$\Big(\sum_{x\in\mathbb{N}} \langle x\rangle^{2\alpha} \left(|q(x,t)|^2 + |\dot{q}(x,t)|^2\right)\Big)^{1/2} \le C \int\limits_0^t (1+t-s)^{-3/2} \left(|q(0,s)| + |\dot{q}(0,s)|\right) ds, \;\; t > 0.$$

4 Large time asymptotic behavior of $q(0,t)$

Let $F(q) = -\kappa q$ with $\kappa \geq 0$. Write $q(t) = q(0,t)$. By (3.6), Eqn (2.6) becomes a linear Volterra integro-differential equation with the kernel $K(1,t)$,

$$\ddot{q}(t) = -(\kappa + 1 + m^2)q(t) - \gamma\dot{q}(t) + \int_0^t K(1,t-s)q(s)\,ds + z(1,t). \tag{4.1}$$

At first, we study the solutions of the corresponding homogeneous equation (with $z(1,t) \equiv 0$),

$$\ddot{q}(t) = -(\kappa + 1 + m^2)q(t) - \gamma\dot{q}(t) + \int_0^t K(1,t-s)q(s)\,ds, \tag{4.2}$$

with the initial data

$$q(0) = u_0(0) \equiv q_0, \quad \dot{q}(0) = v_0(0) \equiv p_0. \tag{4.3}$$

Remark 4.1 *For $m \geq 0$, set*

$$\gamma_m := \frac{2}{m + \sqrt{4+m^2}}\,. \tag{4.4}$$

Note that $\gamma_m = \max\limits_{\theta\in[-\pi,\pi]} |\phi'(\theta)|$, where $\phi(\theta) = \sqrt{2 - 2\cos\theta + m^2}$. Moreover, by direct calculation, we obtain

$$\int_0^{+\infty} K(1,s)\,ds = \tilde{K}(1,0) = e^{i\theta(0)} \equiv \gamma_m^2.$$

In particular, if $m \neq 0$, $\gamma_m \in (0,1)$. If $m = 0$, then $\gamma_m = 1$.

Applying the Fourier–Laplace transform (3.1) to the solutions $q(t)$ of Eqn (4.2), we obtain

$$\tilde{D}(\omega)\tilde{q}(\omega) = -i\omega q_0 + q_0\gamma + p_0 \quad \text{for } \operatorname{Im}\omega > B,$$

where, by definition,

$$\tilde{D}(\omega) := -\omega^2 + \kappa + 1 + m^2 - i\omega\gamma - \tilde{K}(1,\omega), \quad \tilde{K}(1,\omega) = e^{i\theta(\omega)}. \tag{4.5}$$

Write $\tilde{N}(\omega) := [\tilde{D}(\omega)]^{-1}$. Then $\tilde{q}(\omega) = \tilde{N}(\omega)\,(-i\omega q_0 + q_0\gamma + p_0)$. The analytic properties of $\tilde{D}(\omega)$ and $\tilde{N}(\omega)$ for $\omega \in \mathbb{C}$ are studied in Appendix B. In particular, $\tilde{N}(\omega)$ is analytic in the upper half-space, i.e., with $\operatorname{Im}\omega > 0$. Denote

$$N(t) = \frac{1}{2\pi}\int_{-\infty+i\mu}^{+\infty+i\mu} e^{-i\omega t}\tilde{N}(\omega)\,d\omega, \quad t \geq 0, \quad \text{with some } \mu > 0. \tag{4.6}$$

The following proposition follows from Theorem 8.5 (see Appendix B).

Proposition 4.2 *Let constants m, γ, κ be nonnegative and satisfy condition* **C**. *Then*

$$|N^{(j)}(t)| \le C(1+t)^{-3/2}, \quad j = 0,1,2. \tag{4.7}$$

Moreover, the solution $q(t)$ of the problem (4.2)–(4.3) has a form

$$q(t) = N(t)(p_0 + \gamma q_0) + \dot N(t) q_0.$$

Hence, the following bound holds, $|q(t)| + |\dot q(t)| \le C(1+|t|)^{-3/2}(|q_0| + |p_0|)$ for any $t \ge 0$.

Remark Let $\gamma = 0$. If $\kappa = 2$ or $m \ne 0$ and $\kappa = 0$, then $|N^{(j)}(t)| \le C(1+t)^{-1/2}$. For $m = \kappa = 0$ with any $\gamma \ge 0$, $N(t) = (\gamma+1)^{-1} + o(1)$ as $t \to \infty$ (see Theorem 8.5 below).

Corollary 4.3 *Let condition* **C** *hold. Denote by $S(t)$ a solving operator of the Cauchy problem (4.2), (4.3). Then the variation constants formula gives the following representation for the solution of problem (4.1), (4.3):*

$$\begin{pmatrix} q(t) \\ \dot q(t) \end{pmatrix} = S(t) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} + \int_0^t S(\tau) \begin{pmatrix} 0 \\ z(1, t-\tau) \end{pmatrix} d\tau, \quad t > 0.$$

Evidently, $S(0) = I$. The matrix $S(t)$, $t > 0$, is called the resolvent or principal matrix solution for Eqn (4.1). Moreover, the matrix $S(t)$ has a form $\begin{pmatrix} \dot N(t) + \gamma N(t) & N(t) \\ \ddot N(t) + \gamma \dot N(t) & \dot N(t) \end{pmatrix}$. Proposition 4.2 implies that $|S(t)| \le C(1+t)^{-3/2}$, and for the solutions of (4.1) the following bound holds:

$$|q(t)| + |\dot q(t)| \le C_1(1+t)^{-3/2}(|q_0| + |p_0|) + C_2 \int_0^t (1+s)^{-3/2} |z(1, t-s)|\, ds \quad \textit{for } t \ge 0. \tag{4.8}$$

5 Compactness of measures family

The compactness of the measures family $\{\mu_t,\, t > 0\}$ follows from the Prokhorov Theorem and the bound 5.1 (see below), since the embedding $\mathcal{H}_{\alpha,+} \subset \mathcal{H}_{\beta,+}$ is compact if $\alpha > \beta$.

Lemma 5.1 *Let $\alpha < -3/2$ and conditions* **S1**–**S3** *and* **C** *hold. Then*

$$\sup_{t>0} \mathbb{E} \|Y(\cdot, t)\|^2_{\alpha,+} \le C < \infty. \tag{5.1}$$

Proof By the representation (2.1), the solution $Y(t)$ has a form $Y(x,t) = Z(x,t) + X(x,t)$, where $X(x,t) = (q(x,t), \dot q(x,t))$. Hence by the bound (2.24), to prove the bound (5.1) it suffices to verify that

$$\sup_{t>0} \mathbb{E} \|X(\cdot, t)\|^2_{\alpha,+} \le C < \infty. \tag{5.2}$$

Applying formula (3.6) and bound (3.7), we have

$$\begin{aligned} \mathbb{E}\|X(\cdot,t)\|^2_{\alpha,+} &= \mathbb{E}|X(0,t)|^2 + \mathbb{E} \sum_{x \in \mathbb{N}} \langle x \rangle^{2\alpha} \left| \int_0^t K(x, t-s) X(0,s)\, ds \right|^2 \\ &\le C \sup_{\tau \in [0,t]} \mathbb{E} |X(0,\tau)|^2. \end{aligned} \tag{5.3}$$

Using the estimate (4.8), we obtain

$$\mathbb{E}|X(0,t)|^2 \le \mathbb{E}(|q_0|^2 + |p_0|^2) + C \sup_{\tau \ge 0} \mathbb{E}|z(1,\tau)|^2, \quad t \ge 0.$$

On the other hand, $\sup_{\tau\in\mathbb{R}} \mathbb{E}|z(1,\tau)|^2 = \sup_{\tau\in\mathbb{R}} Q^{00}_\tau(1,1) \le C < \infty$ by the bound (2.24). Hence,

$$\sup_{\tau \ge 0} \mathbb{E}|X(0,t)|^2 \le C < \infty. \tag{5.4}$$

Estimates (5.3) and (5.4) imply the bound (5.2), since

$$\sup_{t\ge 0} \mathbb{E}\|Y(\cdot,t)\|^2_{\alpha,+} \le \sup_{t\ge 0} \mathbb{E}\|Z(\cdot,t)\|^2_{\alpha,+} + \sup_{t\ge 0} \mathbb{E}\|X(\cdot,t)\|^2_{\alpha,+} \le C < \infty. \blacksquare$$

6 Asymptotic behavior of $Y(t)$ in mean

6.1 The estimates of $q(0,t)$ in mean

Set $q^{(0)}(x,t) = q(x,t)$, $q^{(1)}(x,t) = \dot q(x,t)$, $x \in \mathbb{Z}^1_+$. To formulate the asymptotic behavior for $q^{(j)}(0,t)$ in mean, introduce the following notations. By definition, (see (2.16))

$$\mathbf{G}^i_z(y,t) := \Big(\mathcal{G}^{i0}_{t,+}(z,y), \mathcal{G}^{i1}_{t,+}(z,y)\Big) = \Big(\mathcal{G}^{i0}_t(z-y) - \mathcal{G}^{i0}_t(z+y), \mathcal{G}^{i1}_t(z-y) - \mathcal{G}^{i1}_t(z+y)\Big), \tag{6.1}$$

$y,z \in \mathbb{Z}^1$, $i = 0,1$, $t \in \mathbb{R}$. In particular, $\mathbf{G}^i_z(0,t) = 0$ for any z and t. Let $\mathbf{G}^j(y)$ denote the vector valued functions defined as

$$\mathbf{G}^j(y) = \int_0^{+\infty} N(s)\mathbf{G}^j_1(y,-s)\,ds = \int_0^{+\infty} N^{(j)}(s)\mathbf{G}^0_1(y,-s)\,ds, \quad y \in \mathbb{Z}^1, \quad j = 0,1, \tag{6.2}$$

where the function $N(s)$ is introduced in (4.6), $\mathbf{G}^j_1(y,s)$ is defined in (6.1).

Proposition 6.1 *Let conditions* **S1**–**S3** *and* **C** *hold,* $Y_0 \in \mathcal{H}_{\alpha,+}$, *and* $q(0,t)$ *be the solution to problem (4.1). Then*

$$q^{(j)}(0,t) = \langle U_0(t)Y_0, \mathbf{G}^j\rangle_+ + r_j(t), \quad j = 0,1, \quad \textit{where } \mathbb{E}|r_j(t)|^2 \le C(1+t)^{-1}, \tag{6.3}$$

the vector valued functions $\mathbf{G}^j$, $j = 0,1$, *are defined in (6.2).*

Proof First, Corollary 4.3 implies that

$$\mathbb{E}\left|\begin{pmatrix} q(0,t) \\ \dot q(0,t) \end{pmatrix} - \int_0^t S(\tau)\begin{pmatrix} 0 \\ z(1,t-\tau) \end{pmatrix} d\tau\right|^2 \le C(1+t)^{-3}$$

Second, $S(\tau)\begin{pmatrix} 0 \\ z(1,t-\tau) \end{pmatrix} = \begin{pmatrix} N(\tau) \\ \dot N(\tau) \end{pmatrix} z(1,t-\tau)$. Moreover,

$$\mathbb{E}\left|\int_t^{+\infty} N(\tau)z(1,t-\tau)d\tau\right|^2 = \int_t^{+\infty} N(\tau_1)d\tau_1 \int_t^{+\infty} N(\tau_2)\,\mathbb{E}\Big(z(1,t-\tau_1)z(1,t-\tau_2)\Big)d\tau_2.$$

Further, the bound (2.24) gives

$$\Big|\mathbb{E}\left(z(1,t-\tau_1)z(1,t-\tau_2)\right)\Big| \le C\sup_{s\in\mathbb{R}}\mathbb{E}_0|z(1,s)|^2 \le C<\infty.$$

Hence, the bound (4.7) yields

$$\mathbb{E}\left|\int\limits_t^{+\infty} S(\tau)\left(\begin{array}{c}0\\ z(1,t-\tau)\end{array}\right)d\tau\right|^2 \le C\sup_s \mathbb{E}|z(1,s)|^2\left(\int\limits_t^{+\infty}(1+\tau)^{-3/2}d\tau\right)^2 \le C_1(1+t)^{-1}.$$

This implies the representation (6.3) with $j=0$, since by (6.1),

$$z(1,t-\tau)=\langle U_0(t)Y_0(\cdot),\mathbf{G}_1^0(\cdot,-\tau)\rangle_+.$$

The bound (6.3) with $j=1$ is proved by the similar way. ■

Remark 6.2 (i) By (2.16) and (2.17), $\mathbf{G}_z^i(y,t)$ is odd w.r.t. $y\in\mathbb{Z}^1$. Moreover, $\mathbf{G}_z^i(y,t)=0$ for $z=0$. Also, the Parseval identity gives

$$\|\mathbf{G}_z^i(\cdot,t)\|_0^2=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}|\hat{\mathbf{G}}_z^i(\theta,t)|^2\,d\theta=\frac{2}{\pi}\int\limits_{-\pi}^{\pi}\Big(|\hat{\mathcal{G}}_t^{i0}(\theta)|^2+|\hat{\mathcal{G}}_t^{i1}(\theta)|^2\Big)\sin^2(z\theta)\,d\theta,\quad\forall z\in\mathbb{N}.\quad(6.4)$$

Hence, if $m\ne0$, then $\|\mathbf{G}_z^i(\cdot,t)\|_0^2\le C<\infty$ uniformly in $z\in\mathbb{Z}^1$ and $t\in\mathbb{R}$. If $m=0$, $\|\mathbf{G}_z^1(\cdot,t)\|_0^2\le C<\infty$ uniformly on z and t, and

$$\|\mathbf{G}_z^0(\cdot,t)\|_0^2\le C_1+C_2\int_{\pi}^{\pi}\frac{\sin^2(z\theta)}{\sin^2(\theta/2)}\,d\theta\le C_1+C_2|z|.$$

In particular, $\hat{\mathbf{G}}_1^0(\theta,t)=2i\sin\theta\Big(\cos(\phi(\theta)t),\sin(\phi(\theta)t)/\phi(\theta)\Big)$, $\theta\in\mathbb{T}^1$. Then formulas (2.17) and (6.4) imply

$$\|\mathbf{G}_1^0(\cdot,t)\|_0^2=C\int\limits_{-\pi}^{\pi}\Big(\cos^2(\phi(\theta)t)+\frac{\sin^2(\phi(\theta)t)}{\phi^2(\theta)}\Big)\sin^2(\theta)\,d\theta.\quad(6.5)$$

(ii) $\mathbf{G}^j(\cdot)$ is odd. Moreover, by the bounds (4.7) and (6.5), $\mathbf{G}^j\in\mathcal{H}_0$, since

$$\|\mathbf{G}^j(\cdot)\|_0\le\int\limits_0^{+\infty}|N^{(j)}(s)|\|\mathbf{G}_1^0(\cdot,-s)\|_0\,ds\le C\int\limits_0^{+\infty}|N^{(j)}(s)|\,ds<\infty.\quad(6.6)$$

In Fourier transform,

$$\hat{\mathbf{G}}^j(\theta)=2i\sin\theta\int\limits_0^{+\infty}N^{(j)}(s)\Big(\cos(\phi(\theta)s),-\sin(\phi(\theta)s)/\phi(\theta)\Big)ds,\quad\theta\in\mathbb{T}^1.$$

Therefore, $\mathbf{G}^j\in\mathcal{H}_{1/\phi}:=\ell^2_{1/\phi}\times\ell^2$, where $\ell^2_{1/\phi}=\{\psi\in\ell^2:\hat\psi/\phi\in L^2(\mathbb{T}^1)\}$. Note that $\ell^2_{1/\phi}\equiv\ell^2$ if $m\ne0$.

Denote by $U_0'(t)$ the operator adjoint to $U_0(t)$:

$$\langle Y, U_0'(t)\Psi\rangle_+ = \langle U_0(t)Y, \Psi\rangle_+, \quad \text{for } Y \in \mathcal{H}_{\alpha,+}, \quad \Psi \in \mathcal{S}, \quad t \in \mathbb{R}. \tag{6.7}$$

In other words,

$$(U_0'(t)\Psi)^j(y) = \sum_{i=0}^{1} \sum_{x\geq 0} \mathcal{G}_{t,+}^{ij}(x,y)\Psi^i(x), \quad \text{for } \Psi \in \mathcal{S}, \quad t \in \mathbb{R}, \quad y \in \mathbb{Z}_+^1.$$

In particular, (see (6.1)) $\mathbf{G}_1^0(y,t) = (U_0'(t)Y_0)(y)$ with $Y_0(x) = (\delta_{1x}, 0)$, where δ_{1x} denotes the Kronekker symbol.

Corollary 6.3 *(i) For $\mathbf{G}^j$ defined in (6.2), we have*

$$\sup_{t\in\mathbb{R}} \mathbb{E}|\langle U_0(t)Y_0, \mathbf{G}^j\rangle_+|^2 = \sup_{t\in\mathbb{R}} \mathbb{E}|\langle Y_0, U_0'(t)\mathbf{G}^j\rangle_+|^2 \leq C < \infty. \tag{6.8}$$

Indeed, since $U_0'(t)\mathbf{G}_1^0(y,\tau) = \mathbf{G}_1^0(y,\tau+t)$, then

$$\sup_{t\in\mathbb{R}} \|U_0'(t)\mathbf{G}^j\|_0 \leq C < \infty, \tag{6.9}$$

by (6.6). Hence, (6.8) follows from the bounds (2.11) and (6.9).

(ii) The representation (6.3), Lemma 2.7 and the bound (6.8) give the following convergence,

$$\mathbb{E}\Big(q^{(i)}(0,t)q^{(j)}(0,t)\Big) \to \mathcal{Q}_\infty^\nu(\mathbf{G}^i, \mathbf{G}^j) \quad \text{as } t \to \infty, \tag{6.10}$$

where the quadratic form $\mathcal{Q}_\infty^\nu$ is defined by (2.22). The r.h.s. of (6.10) is defined by Remarks 2.8 and 6.2 (ii).

6.2 The large time behavior of $q(x,t)$, $x \in \mathbb{N}$, in mean

Let $\mathbf{K}^j(x,y)$ $j = 0,1$, stand for the vector-valued functions,

$$\mathbf{K}^j(x,y) = \int_0^{+\infty} K(x,s)\Big(U_0'(-s)\mathbf{G}^j\Big)(y)\,ds = \int_0^{+\infty}\int_0^{+\infty} K(x,s)\mathcal{N}^{(j)}(\tau)\mathbf{G}_1^0(y,-s-\tau)\,ds\,d\tau, \tag{6.11}$$

$x \in \mathbb{N}$, $y \in \mathbb{Z}^1$, $K(x,s)$ is defined in (3.5), $\mathbf{G}^j$ is introduced in (6.2). Note that $\mathbf{K}^j(x,y)$ is odd w.r.t. y, $\mathbf{K}^j(\cdot,y) \in \mathcal{H}_{\alpha,+}$ with $\alpha < -3/2$, and $\mathbf{K}^j(x,\cdot) \in \mathcal{H}_{0,+}$ for any x.

The formula (3.6) and Proposition 6.1 imply the following lemma.

Lemma 6.4 *(i) The solution $q(x,t)$, $x \geq 1$, of problem (2.5)–(2.8) admits the following representation*

$$q^{(j)}(x,t) = \langle U_0(t)Y_0, \mathbf{K}^j(x,\cdot)\rangle_+ + r_j(x,t), \quad j = 0,1, \quad t > 0, \tag{6.12}$$

where $\mathbb{E}\|r_j(\cdot,t)\|_{0,+}^2 \leq C(1+t)^{-1}$ as $t \to \infty$, $\mathbf{K}^j$ is introduced in (6.11). Here $\|r\|_{0,+}^2 = \sum_{x\in\mathbb{N}} |r(x)|^2$.

(ii) The correlation functions converge to a limit,

$$\lim_{t\to\infty} \mathbb{E}\Big(q^{(i)}(x,t)q^{(j)}(x',t)\Big) = \mathcal{Q}_\infty^\nu(\mathbf{K}^i(x,\cdot), \mathbf{K}^j(x'\cdot)) \quad \textit{for any } x,x' \in \mathbb{Z}_+^1,\ i,j = 0,1.$$

Proof At first, by (3.6) and (6.3),

$$q^{(j)}(x,t)=\int_0^t K(x,t-s)q^{(j)}(0,s)\,ds=\int_0^t K(x,t-s)\langle U_0(s)Y_0,\mathbf{G}^j(\cdot)\rangle_+\,ds+r'_j(x,t),\quad (6.13)$$

where $x\in\mathbb{N}$, $\mathbb{E}\|r'_j(\cdot,t)\|^2_{0,+}\le C(1+t)^{-1}$. Indeed, by (6.3) and (3.7),

$$\begin{aligned}\mathbb{E}\|r'_j(\cdot,t)\|^2_{0,+} &= \mathbb{E}\Big\|\int_0^t K(\cdot,t-s)r_j(s)\,ds\Big\|^2_{0,+}\le\Big(\int_0^t\|K(\cdot,t-s)\|_{0,+}\sqrt{\mathbb{E}|r_j(s)|^2}\,ds\Big)^2\\ &\le C\Big(\int_0^t(1+t-s)^{-3/2}(1+s)^{-1/2}\,ds\Big)^2\le C_1(1+t)^{-1}.\end{aligned}$$

Second, the first term in the r.h.s. of (6.13) has a form

$$\int_0^t K(x,t-s)\langle U_0(s)Y_0,\mathbf{G}^j(\cdot)\rangle_+\,ds=\langle U_0(t)Y_0,\mathbf{K}^j(x,\cdot)\rangle_+ + r''_j(x,t),$$

where, by (6.11),

$$r''_j(\cdot,t)=\int_t^{+\infty}K(x,s)\langle U_0(t-s)Y_0,\mathbf{G}^j(\cdot)\rangle_+\,ds.$$

Since $\mathbf{G}^j\in\ell^2_{0,+}\times\ell^2_{0,+}$, the bounds (2.24), (3.7) and (6.8) yield

$$\mathbb{E}\|r''_j(\cdot,t)\|^2_{0,+}\le\Big(\int_t^{+\infty}\|K(\cdot,s)\|_{0,+}\Big(\mathbb{E}_0\left|\langle U_0(t-s)Y_0,\mathbf{G}^j(\cdot)\rangle_+\right|^2\Big)^{1/2}ds\Big)^2\le C(1+t)^{-1}.\ (6.14)$$

Hence, (6.13) and (6.14) imply (6.12) with $r_j(x,t)=r'_j(x,t)+r''_j(x,t)$. Finally, the item (ii) of Lemma 6.4 follows from the representation (6.12) and Lemma 2.7. ■

Remark If we set $\tilde K(0,\omega):=e^{i\theta(\omega)x}|_{x=0}=1$, then $K(0,t)=\delta_{0t}$ and $\mathbf{K}^j(0,y)=\mathbf{G}^j(y)$. Hence, we put $\mathbf{K}^j(0,y)=\mathbf{G}^j(y)$, $y\in\mathbb{Z}^1$. Then the representation (6.3) follows from (6.12).

For any $\psi\in S$, denote

$$\mathbf{K}^j_\psi(y)=\langle\mathbf{K}^j(\cdot,y),\psi(\cdot)\rangle_+=\int_0^{+\infty}\langle K(\cdot,s),\psi\rangle_+(U'_0(-s)\mathbf{G}^j)(y)\,ds,\quad y\in\mathbb{Z}^1.\qquad(6.15)$$

Remark 6.5 *Note first that* $\mathbf{G}^j(0)=0$ *(see (6.2)). Hence* $\mathbf{K}^j(x,0)=0$ *for any* $x\in\mathbb{N}$. *Then* $\mathbf{K}^j_\psi(0)=0$. *Also,* $\mathbf{K}^j_\psi$ *is odd. Moreover,* $\mathbf{K}^j_\psi\in\ell^2_{1/\phi}\times\ell^2$ *(see Remark 6.2 (ii)).*

Corollary 6.6 *(i) For any* $\psi\in S$, $j=0,1$,

$$\langle q^{(j)}(\cdot,t),\psi\rangle_+=\langle U_0(t)Y_0,\mathbf{K}^j_\psi\rangle_+ + r_j(t),\quad t>0,\quad \textit{where }\ \mathbb{E}|r_j(t)|^2\le C(1+t)^{-1}.\qquad(6.16)$$

(ii) For any $\psi\in S$,

$$\sup_{t\in\mathbb{R}}\mathbb{E}|\langle U_0(t)Y_0,\mathbf{K}^j_\psi\rangle_+|^2=\sup_{t\in\mathbb{R}}\mathbb{E}|\langle Y_0,U'_0(t)\mathbf{K}^j_\psi\rangle_+|^2\le C<\infty.\qquad(6.17)$$

(iii) For any $\psi,\chi\in S$, $\mathbb{E}\Big(\langle q^{(i)}(\cdot,t),\psi\rangle_+\langle q^{(j)}(\cdot,t),\chi\rangle_+\Big)\to\mathcal{Q}^\nu_\infty(\mathbf{K}^i_\psi,\mathbf{K}^j_\chi)$ *as* $t\to\infty$.

Proof The item (i) follows from (6.12). Now we verify the item (ii).

By definition (6.15), $U_0'(t)\mathbf{K}^j_\psi = \int_0^{+\infty}\langle K(\cdot,s),\psi\rangle_+ U_0'(t-s)\mathbf{G}^j(y)\,ds$. Hence, (6.9) implies the following uniform bound,

$$\sup_{t\in\mathbb{R}}\|U_0'(t)\mathbf{K}^j_\psi\|_0 \le C(\psi)\int_0^{+\infty}\|K(\cdot,s)\|_{0,+}\,ds<\infty. \tag{6.18}$$

Therefore, (6.17) follows from the bounds (2.11) and (6.18). Finally, (6.16) and Lemma 2.7 imply the item (iii) of Corollary 6.6. ■

6.3 Proof of Theorem 2.12

Write $\Pi^j(x,y)=e_j\delta_{xy}+\mathbf{K}^j(x,y)$ with $e_0=(1,0)$, $e_1=(0,1)$, $j=0,1$, $x,y\in\mathbb{Z}^1_+$. Given $\Psi=(\Psi^0,\Psi^1)\in\mathcal{S}$, define the vector valued functions $\Pi_\Psi(y)$, $y\in\mathbb{Z}^1_+$,

$$\Pi_\Psi(y):=\langle\Pi^j(\cdot,y),\Psi^j\rangle_+=\Psi(y)+\sum_{j=0}^{1}\langle\mathbf{K}^j(\cdot,y),\Psi^j(\cdot)\rangle_+=\Psi(y)+\sum_{j=0}^{1}\mathbf{K}^j_{\Psi^j}(y) \tag{6.19}$$

with $\mathbf{K}^j_\psi(y)$ from (6.15). The formula (2.1) and Lemma 6.4 imply the following lemma.

Lemma 6.7 *For the solution $Y(t)=U(t)Y_0=(u(x,t),\dot u(x,t))$ of the problem (1.1)–(1.3), the following asymptotics holds,*

$$(U(t)Y_0)^j(x)=\langle U_0(t)Y_0,\Pi^j(x,\cdot)\rangle_+ + r_j(x,t),\quad j=0,1, \tag{6.20}$$

where $r_j(x,t)$ from (6.12) and $\mathbb{E}\|r_j(\cdot,t)\|^2_{0,+}\le C(1+t)^{-1}$. Hence, for any $\Psi=(\Psi^0,\Psi^1)\in\mathcal{S}$,

$$\langle Y(t),\Psi\rangle_+=\langle U_0(t)Y_0,\Pi_\Psi\rangle_+ + r(t), \tag{6.21}$$

where $r(t)=\sum_{j=0}^1\langle r_j(\cdot,t),\Psi^j\rangle_+$, and $\mathbb{E}|r(t)|^2\le C(1+t)^{-1}$.

We return to the proof of Theorem 2.12. The compactness of the measures family $\{\mu_t,t\in\mathbb{R}\}$ is proved in Section 5. To end the proof of item (i) of Theorem 2.12, it suffices to check the convergence (1.7) of characteristic functionals for μ_t. By the triangle inequality and the equality (2.25),

$$\begin{aligned}\Big|\mathbb{E}e^{i\langle Y(t),\Psi\rangle_+}-e^{-\frac12\mathcal{Q}_\infty(\Psi,\Psi)}\Big| &\le \Big|\mathbb{E}\Big(e^{i\langle Y(t),\Psi\rangle}-e^{i\langle U_0(t)Y_0,\Pi_\Psi\rangle_+}\Big)\Big|\\ &\quad+\Big|\mathbb{E}_0e^{i\langle U_0(t)Y_0,\Pi_\Psi\rangle_+}-e^{-\frac12\mathcal{Q}^\nu_\infty(\Pi_\Psi,\Pi_\Psi)}\Big|.\end{aligned} \tag{6.22}$$

Applying (6.21), we estimate the first term in the r.h.s. of (6.22) by

$$\mathbb{E}\Big|\langle Y(t),\Psi\rangle_+-\langle U_0(t)Y_0,\Pi_\Psi\rangle_+\Big|\le\mathbb{E}|r(t)|\le\Big(\mathbb{E}|r(t)|^2\Big)^{1/2}\le C(1+t)^{-1/2}\to0\quad\text{as } t\to\infty.$$

The convergence of the characteristic functionals $\hat\nu_t(\Pi_\Psi)=\mathbb{E}_0\left(\exp\{i\langle U_0(t)Y_0,\Pi_\Psi\rangle_+\}\right)$ to a limit as $t\to\infty$ follows from Theorem 2.9 and Remark 2.11. This proves the convergence (1.7). Corollary 6.6 and Lemma 2.7 imply the item (ii) of Theorem 2.12.

7 Appendix A: The behavior of $K(x,t)$ as $t \to \infty$

We first study the analytic properties of $\tilde K(x,\omega) := e^{i\theta(\omega)x}$ (where $\theta(\omega)$ is introduced in Lemma 3.2) applying the technique of Komech, Kopylova and Kunze [15], developed by them for discrete Schrodinger and Klein–Gordon equations in the whole space $\mathbb{Z}^1$.

7.1 Properties of $e^{i\theta(\omega)x}$ for $\omega \in \mathbb{C}$ and $x \in \mathbb{N}$

As before we denote $\Gamma_m := [-\sqrt{4+m^2}, -m] \cup [m, \sqrt{4+m^2}]$. For $x \in \mathbb{N}$, $\omega \in \mathbb{C}$, the function $\tilde K(x,\omega) = e^{i\theta(\omega)x}$ has the following properties **I–III**.

I. Let $\omega \in C_m = \mathbb{C} \setminus \Gamma_m$. Then $\operatorname{Im}\theta(\omega) > 0$. In this case, $\tilde K(x,\omega)$ exponentially decays in x. Hence $\tilde K(x,\omega)$ is an analytic function in the complex ω-plane with the values in the class $\ell^2_+ \equiv \ell^2_{0,+}$. The behavior of $\tilde K(x,\omega)$ for $|\omega| \to \infty$ is following.

Let $\operatorname{Im}\omega = \mu = \text{const} > 0$. Then $\operatorname{Im}\theta(\omega) \to +\infty$ and $\operatorname{Re}\theta(\omega) \to \pm\pi$ as $\operatorname{Re}\omega \to \pm\infty$. (If $\operatorname{Im}\omega = \mu < 0$, then $\operatorname{Im}\theta(\omega) \to +\infty$ and $\operatorname{Re}\theta(\omega) \to \mp\pi$ as $\operatorname{Re}\omega \to \pm\infty$.) Moreover, for any $K > 0$ there exists a positive constant $C = C(K) < \infty$ such that

$$\left|e^{i\theta(\nu+i\mu)}\right| \le \frac{C}{\nu^2} \quad \text{for any } \nu \in \mathbb{R}: \ |\nu| > \sqrt{4+m^2}, \quad \text{and any } \mu \in \mathbb{R}: \ |\mu| \le K. \tag{7.1}$$

Indeed, put $z = 1 - (\omega^2 - m^2)/2$. Then, by (3.3),

$$e^{i\theta(\omega)} = \cos\theta + i\sin\theta = z + \sqrt{z^2-1} = \frac{1}{z - \sqrt{z^2-1}}, \tag{7.2}$$

where $\sqrt{z^2-1}$ is a complex root and its branch is chosen by the condition $\operatorname{Im}\theta(\omega) > 0$. In particular, this condition implies that $\operatorname{sgn}(\operatorname{Im}\sqrt{z^2-1}) = -\operatorname{sgn}(\operatorname{Im} z) = \operatorname{sgn}(\operatorname{Im}\omega^2)$ for $\operatorname{Im}\omega^2 \ne 0$. Moreover, $\sqrt{z^2-1} > 0$ for $\omega \in (-\infty, -\sqrt{4+m^2}) \cup (\sqrt{4+m^2}, +\infty)$. Let us verify (7.1) for $\omega = \nu \in \mathbb{R}$: $|\nu| > \sqrt{4+m^2}$. For such values ω, $\operatorname{Re}(\theta(\nu)) = \pm\pi$ and $\sqrt{z^2-1} = \sqrt{(2-\nu^2+m^2)^2/4 - 1} > 0$. Hence, by (7.2),

$$e^{i\theta(\nu)} = -e^{-\operatorname{Im}\theta(\nu)} = \frac{2}{2 - (\nu^2 - m^2) - \sqrt{(2-\nu^2+m^2)^2 - 4}} \sim -\frac{1}{\nu^2} \quad \text{as } |\nu| \to \infty. \tag{7.3}$$

In the general case, the bound (7.1) can be proved similarly. Moreover, by direct calculation, we have $\left|e^{i\theta(\omega)}\right| \le C|\omega|^{-2}$ as $|\omega| \to \infty$.

Lemma 7.1 *The bound (7.1) implies that there exist constants $C, B < \infty$ such that*

$$\sum_{x\in\mathbb{N}} \langle x\rangle^{2\alpha} |K(x,t)|^2 \le Ce^{Bt} \quad \textit{for any } \alpha \in \mathbb{R}, \quad t > 0. \tag{7.4}$$

Proof Let us apply the inverse Fourier–Laplace transform. Then for some $\mu > 0$, any $x \in \mathbb{N}$ and $t > 0$,

$$K(x,t) = \frac{1}{2\pi} \int\limits_{-\infty+i\mu}^{+\infty+i\mu} e^{-i\omega t} e^{i\theta(\omega)x}\, d\omega = \frac{e^{\mu t}}{2\pi} \int\limits_{-\infty}^{+\infty} e^{-i\nu t} e^{i\theta(\nu+i\mu)x}\, d\nu,$$

where $\omega = \nu + i\mu$. Note that

$$\left|e^{i\theta(\nu+i\mu)x}\right| = e^{-\operatorname{Im}\theta(\nu+i\mu)} =: q_{\nu,\mu} < 1 \quad \text{for any } \nu \in \mathbb{R}, \text{ and } \mu > 0.$$

Hence,

$$\sum_{x\in\mathbb{N}} \langle x\rangle^{2\alpha} |K(x,t)|^2 \le \frac{e^{2\mu t}}{4\pi^2} \int\limits_{-\infty}^{+\infty} \sum_{x\in\mathbb{N}} \langle x\rangle^{2\alpha} \left(q^2_{\nu,\mu}\right)^x \, d\nu.$$

Therefore, (7.4) follows from (7.1), since

$$\begin{aligned} \sum_{x\in\mathbb{N}} \langle x\rangle^{2\alpha} \left(q^2_{\nu,\mu}\right)^x &\le \frac{q^2_{\nu,\mu}}{1-q^2_{\nu,\mu}} \sim \frac{C}{|\nu|^4}, \quad \nu\to\infty, \quad \text{for } \alpha\le 0,\\ \sum_{x\in\mathbb{N}} \langle x\rangle^{2\alpha} \left(q^2_{\nu,\mu}\right)^x &\le C\sum_{x\in\mathbb{N}} (q_{\nu,\mu})^x \le C_1 \frac{q_{\nu,\mu}}{1-q_{\nu,\mu}} \sim \frac{C}{|\nu|^2}, \quad \nu\to\infty, \quad \text{for } \alpha>0. \ \blacksquare \end{aligned}$$

II. Let $\omega\in\mathbb{R}$ and $|\omega|\in(m,\sqrt{m^2+4})$. Then $\omega^2-m^2\in(0,4)$ and $\operatorname{Im}\theta(\omega)=0$. Put $\theta(\omega\pm i0)=\lim_{\varepsilon\to+0}\theta(\omega\pm i\varepsilon)$. For every $x\in\mathbb{N}$, the following pointwise limit exists

$$\tilde K(x,\omega\pm i\varepsilon)\to\tilde K(x,\omega\pm i0), \quad \varepsilon\to+0.$$

Moreover, $|\theta(\omega\pm i\varepsilon)|\le C$ for any ε. Hence, for any $\alpha<-1/2$,

$$\|\tilde K(x,\omega\pm i0)-\tilde K(x,\omega\pm i\varepsilon)\|^2_{\alpha,+} \equiv \sum_{x\in\mathbb{N}} \langle x\rangle^{2\alpha} |\tilde K(x,\omega\pm i0)-\tilde K(x,\omega\pm i\varepsilon)|^2 \to 0 \quad \text{as } \varepsilon\to+0,$$

by the Lebesgue dominated theorem.

Note that $\overline{\theta(\omega)}=-\theta(\bar\omega)$ for $\omega\in C_m\equiv\mathbb{C}\setminus\Gamma_m$. Hence,

$$\tilde K(x,\omega-i0)=\overline{\tilde K(x,\omega+i0)} \quad \text{for } \omega\in(-\sqrt{4+m^2},-m)\cup(m,\sqrt{4+m^2}), \quad x\in\mathbb{N}.$$

III. Now we discuss the behavior of $\tilde K(x,\omega)$ near the points $\pm m$, $\pm\sqrt{4+m^2}$. Eqn (3.3) implies

$$e^{i\theta(\omega)}=\cos\theta(\omega)+i\sin\theta(\omega)=1-\frac12(\omega^2-m^2)+i\sqrt{\omega^2-m^2-\frac14(\omega^2-m^2)^2}, \quad \omega\in C_m.$$

The Taylor expansion implies

$$e^{i\theta(\omega)}=1+i\sqrt{\omega^2-m^2}-\frac12(\omega^2-m^2)-\frac{i}{8}(\omega^2-m^2)^{3/2}+\dots, \quad \omega\to\pm m+i0, \tag{7.5}$$

where $\operatorname{sgn}(\operatorname{Re}\sqrt{\omega^2-m^2})=\operatorname{sgn}(\operatorname{Re}\omega)$ for $\operatorname{Im}\omega>0$. This choice of the branch of the complex root $\sqrt{\omega^2-m^2}$ follows from the condition $\operatorname{Im}\theta(\omega)>0$. Hence, for $x\in\mathbb{N}$,

$$e^{i\theta(\omega)x}=1+\sum_{j=1}^{+\infty}(\omega^2-m^2)^{j/2}R^j(x), \quad \omega\to\pm m+i0. \tag{7.6}$$

Here polynomials $R^j(x)$ are of a form $R^j(x)=\sum\limits_{k=1}^{j} c^j_k x^k$ with some coefficients $c^j_k\in\mathbb{C}$, $j\in\mathbb{N}$. For example, $R^1(x)=ix$, $R^2(x)=-x^2/2$, $R^3(x)=-i(4x^3-x)/24$.

The behavior of $\tilde K(x,\omega)$ near points $\omega=\pm\sqrt{m^2+4}$ is similar. Namely,

$$e^{i\theta(\omega)}=-1+i(m^2+4-\omega^2)^{1/2}+\frac12(m^2+4-\omega^2)-\frac{i}{8}(m^2+4-\omega^2)^{3/2}+\dots, \tag{7.7}$$

where $\sqrt{m^2+4-\omega^2}$ is the complex root and we choose the branch of this root such that $\mathrm{sgn}(\mathrm{Re}\sqrt{m^2+4-\omega^2}) = \mathrm{sgn}(\mathrm{Re}\,\omega)$ by the condition $\mathrm{Im}\,\theta(\omega) > 0$. Hence, for $x \in \mathbb{N}$,

$$e^{i\theta(\omega)x} = (-1)^x(1 - ix\sqrt{m^2+4-\omega^2} - (m^2+4-\omega^2)x^2/2 + \ldots) \quad \text{as } \omega \to \pm\sqrt{m^2+4} + i0. \quad (7.8)$$

If $m = 0$, then the Taylor expansion gives

$$e^{i\theta(\omega)} = 1 - \frac{\omega^2}{2} + i\omega(1 - \frac{\omega^2}{8} - \frac{\omega^4}{128} + \ldots), \quad \omega \to 0, \quad (7.9)$$

and $e^{i\theta(\omega)} = -1 + i\sqrt{4-\omega^2} + \ldots$ as $\omega \to \pm 2 + i0$. Therefore,

$$\left.\begin{array}{lcll} e^{i\theta(\omega)x} & = & 1 + i\omega x - \omega^2 x^2/2 - i\omega^3(4x^3 - x)/24 + \ldots & \text{as } \omega \to 0 \\ e^{i\theta(\omega)x} & = & (-1)^x(1 - ix\sqrt{4-\omega^2} - (4-\omega^2)x^2/2 + \ldots) & \text{as } \omega \to \pm 2 + i0 \end{array}\right| x \in \mathbb{N}. \quad (7.10)$$

The representations (7.6), (7.8) and (7.10) imply the following lemma.

Lemma 7.2 *(cf [15, Lemma 3.2]) Let $m = 0$. Then for every $N \in \mathbb{N}$,*

$$e^{i\theta(\omega)x} = 1 + \sum_{j=1}^{N} \omega^j R^j(x) + r_N(\omega, x), \quad \omega \to 0,$$

where $\|r_N(\omega,\cdot)\|_{\alpha,+} = \mathcal{O}(|\omega|^{N+1})$ for $\alpha < -3/2 - N$.

Let $m \neq 0$. Then for every $N \in \mathbb{N}$,

$$e^{i\theta(\omega)x} = 1 + \sum_{j=1}^{N} (\omega^2 - m^2)^{j/2} R^j(x) + r_N(\omega, x), \quad \omega \to \pm m + i0, \quad (7.11)$$

where $\|r_N(\omega,\cdot)\|_{\alpha,+} = \mathcal{O}((\omega^2 - m^2)^{(N+1)/2})$ for $\alpha < -3/2 - N$. Moreover,

$$D^r_\omega\left(e^{i\theta(\omega)x}\right) = \sum_{j=1}^{N} \frac{d^r}{d\omega^r}(\omega^2 - m^2)^{j/2} R^j(x) + \tilde r_N(\omega, x), \quad \omega \to \pm m + i0, \quad (7.12)$$

where $\|\tilde r_N(\omega,\cdot)\|_{\alpha,+} = \mathcal{O}((\omega^2 - m^2)^{(N+1)/2 - r})$ for $\alpha < -3/2 - N$. The similar representation holds for $\omega \to \pm\sqrt{m^2+4} + i0$.

Indeed, the bound (7.11) follows from the following representation for remainder (see formula (7.6))

$$r_N(\omega, x) = (\omega^2 - m^2)^{(N+1)/2} \sum_{k=1}^{N+1} b_k(\omega) x^k,$$

where $b_k(\omega)$ are uniformly bounded for $\omega \to \pm m$. In particular,

$$e^{i\theta(\omega)x} = 1 + \sqrt{\omega^2 - m^2} R_0(\omega, x) \quad \text{as } \omega \to \pm m,$$

where $\sup \|R_0(\omega,\cdot)\|_{\alpha,+} \le C < \infty$ for $\alpha < -3/2$.

7.2 Proof of Theorem 3.3

By the properties **I–III**, $K(x,t)=0$ for $t<0$. To prove (3.7), we use the properties **I–III** and apply the technique from [15]. At first, we rewrite $K(x,t)$ in the form

$$K(x,t)=\frac{1}{2\pi}\int\limits_{\mathrm{Im}\,\omega=\mu>0} e^{-i\omega t}\tilde K(x,\omega)\,d\omega=-\frac{1}{2\pi}\int\limits_{\Gamma} e^{-i\omega t}\tilde K(x,\omega)\,d\omega,\quad x\in\mathbb{N},$$

where $\Gamma=\{|\omega|=R:\ R>\sqrt{4+m^2}\}$ and Γ is oriented anticlockwise. Since $\tilde K(x,\omega)$ is analytic in $\mathbb{C}\setminus\Gamma_m$, we can change the contour of integration into $\Gamma_{m,\varepsilon}$, where the contour $\Gamma_{m,\varepsilon}$ surrounds of Γ_m and belongs to the ε-neighborhood of Γ_m ($\Gamma_{m,\varepsilon}$ is oriented anticlockwise). Taking $\varepsilon\to 0$, we obtain

$$\begin{aligned}K(x,t)&=\frac{1}{2\pi}\int\limits_{\Gamma_m} e^{-i\omega t}\Big(\tilde K(x,\omega+i0)-\tilde K(x,\omega-i0)\Big)\,d\omega\\&=\frac{i}{\pi}\int\limits_{\Gamma_m} e^{-i\omega t}\mathrm{Im}\,\tilde K(x,\omega+i0)\,d\omega=\sum_{\pm}\sum_{j=1}^{2}\frac{i}{\pi}\int\limits_{\Gamma_m} e^{-i\omega t}P_j^{\pm}(x,\omega)\,d\omega.\end{aligned}$$

Here $P_j^{\pm}(x,\omega):=\zeta_j^{\pm}(\omega)\mathrm{Im}\,\tilde K(x,\omega+i0)$, where $\zeta_j^{\pm}(\omega)$, $j=1,2$, are smooth functions such that $\sum\limits_{\pm,j}\zeta_j^{\pm}(\omega)=1$, $\omega\in\mathbb{R}$, $\mathrm{supp}\,\zeta_1^{\pm}\subset\mathcal{O}(\pm m)$, $\mathrm{supp}\,\zeta_2^{\pm}\subset\mathcal{O}(\pm\sqrt{4+m^2})$ (Here $\mathcal{O}(a)$ denotes a neighborhood of a point $\omega=a$). If $m=0$, we introduce ζ_1 (P_1) instead of $\zeta_1^{\pm}$ ($P_1^{\pm}$, resp.) with $\mathrm{supp}\,\zeta_1\subset\mathcal{O}(0)$. By the property **III**,

$$\begin{aligned}&\|P_1^{\pm}(\cdot,\omega)\|_{\alpha,+}=O(|\omega\mp m|^{1/2})\quad\text{if}\ \ m\neq 0,\quad \|P_1(\cdot,\omega)\|_{\alpha,+}=O(|\omega|)\quad\text{if}\ \ m=0,\\&\|P_2^{\pm}(\cdot,\omega)\|_{\alpha,+}=O(|\omega\mp\sqrt{4+m^2}|)\end{aligned}$$

for any $\alpha<-3/2$. Therefore, using the lemma of Jensen and Kato [14, Lemma 10.2] or the Vainberg lemma [25, Lemma 2], we obtain

$$\Big\|\int\limits_{\Gamma_m} e^{-i\omega t}P_j^{\pm}(x,\omega)\,d\omega\Big\|_{\alpha,+}=O(t^{-3/2}),\quad t\to\infty,\quad j=1,2,\quad\text{for any}\ \alpha<-3/2.\tag{7.13}$$

The bound (3.7) is proved. ■

7.3 Existence of solutions

Lemma 7.3 *Let $\alpha\in\mathbb{R}$, $m,\gamma\geq 0$, and let P satisfy the condition (1.5). Then the following assertions hold. (i) For every $Y_0\in\mathcal{H}_{\alpha,+}$, the mixing problem (1.1)–(1.3) has a unique solution $Y(t)\in C(\mathbb{R},\mathcal{H}_{\alpha,+})$. Moreover, the operator $U(t):Y_0\mapsto Y(t)$, $t\in\mathbb{R}$, is continuous on $\mathcal{H}_{\alpha,+}$, and there exist constants $C,B<\infty$ such that*

$$\|Y(t)\|_{\alpha,+}\leq Ce^{Bt}\|Y_0\|_{\alpha,+}\quad\textit{for}\ \ t\in\mathbb{R}.\tag{7.14}$$

(ii) For $Y_0\in\mathcal{H}_{0,+}$,

$$H(Y(t))+\gamma\int_0^t|\dot u(0,s)|^2\,ds=H(Y_0),\quad t\in\mathbb{R},\tag{7.15}$$

where $H(Y(t))$ is defined in (1.4). In particular, if $\gamma=0$, the energy $H(Y(t))$ is conserved and finite.

Proof of Lemma 7.3 We use the representation (2.1), $u(x,t) = z(x,t) + q(x,t)$. By Lemma 2.5 and formula (3.6), to prove the existence of $u(x,t)$, it suffices to prove the existence of the solutions $q(t) \equiv q(0,t)$ to the problem (4.2)–(4.3). Indeed, first, rewrite (4.2) in the equivalent integral form

$$q(t) = \int_0^t \Big(\int_0^s \mathcal{F}(\tau, q(\tau))\, d\tau \Big)\, ds + \int_0^t \Big(\int_0^s z(1,\tau)\, d\tau - \gamma q(s) \Big)\, ds + C_0 + C_1 t, \quad t > 0. \quad (7.16)$$

with $\mathcal{F}(\tau, q) := -(1+m^2)q + F(q) + \int_0^t K(1, t-s) q(s)\, ds$, $C_0 = q(0) = q_0$, $C_1 = \dot q(0) = p_0$. By the contraction mapping principle, the solution $q(t)$ of Eqn (7.16) is unique on a certain interval $t \in [0, \varepsilon)$ with some $\varepsilon > 0$ depending on the initial data q_0, p_0. Hence, by (3.6), the solution $q(x,t)$ of problem (2.5)–(2.8) is unique on a certain interval $t \in [0,\varepsilon)$ with some $\varepsilon > 0$ depending on Y_0. The existence of $z(x,t)$ is stated in Appendix C. This implies the existence of the local solution $u(x,t) = z(x,t) + q(x,t)$ for sufficiently small t. This local solution can be extended to the global solution using the a priori estimate (7.14). Now we verify (7.14). Indeed, by (4.2),

$$\begin{aligned} &\frac{1}{2}\Big(|\dot q(t)|^2 + (m^2+1)|q(t)|^2 \Big) + P(q(t)) + \gamma \int_0^t |\dot q(s)|^2\, ds \\ = \ &\frac{1}{2}\Big(|p_0|^2 + (m^2+1)|q_0|^2 \Big) + P(q_0) + \int_0^t \dot q(s) \Big(z(1,s) - \int_0^s K(1, s-\tau) q(\tau) d\tau \Big)\, ds. \quad (7.17) \end{aligned}$$

Define $M(t) = \sup\limits_{0 \le s \le t} (|q(s)|^2 + |\dot q(s)|^2)$. Then the equality (7.17) and the bound (3.8) for $K(1,t)$ yield

$$M(t) \le C_1 + \int_0^t \sqrt{M(s)}\, |z(1,s)|\, ds + C_2 \int_0^t M(s)\, ds, \quad t > 0.$$

Applying the Gronwall–Bellman integral type inequality [22] we have

$$M(t) \le e^{C_2 t} \Big(\sqrt{C_1} + \frac{1}{2} \int_0^t |z(1,s)| e^{C_2 s/2}\, ds \Big)^2, \quad t > 0.$$

Since $|z(1,t)| \le C(1+|t|)^N \|Y_0\|_{\alpha,+}$ (see Appendix C), we obtain the a priori bound

$$|q(t)| + |\dot q(t)| \le C e^{Bt} \|Y_0\|_{\alpha,+} \quad (7.18)$$

with some constants $C, B < \infty$. By Lemma 7.1 and formula (3.6),

$$\Big(\sum_{x \in \mathbb{N}} \langle x \rangle^{2\alpha} (|q(x,t)|^2 + |\dot q(x,t)|^2) \Big)^{1/2} \le C_1 e^{Bt} \|Y_0\|_{\alpha,+}. \quad (7.19)$$

Since $\|Y(t)\|^2_{\alpha,+} \le \|q(\cdot,t)\|^2_{\alpha,+} + \|\dot q(\cdot,t)\|^2_{\alpha,+} + \|z(\cdot,t)\|^2_{\alpha,+} + \|\dot z(\cdot,t)\|^2_{\alpha,+}$, the a priori bound (7.14) follows from (7.19) and (9.5). ■

Remark 7.4 *Let* $F(q) = -\kappa q$ *with* $\kappa \ge 0$, $Y_0 \in \mathcal{H}_{0,+}$. *Then the energy* $H(Y(t))$ *is nonnegative and finite,* $H(Y(t)) \le H(Y_0)$ *by (7.15).*

8 Appendix B: The analytic properties of $\tilde{D}(\omega)$ for $\omega \in \mathbb{C}$

In this section, we study the properties of $\tilde{D}(\omega)$ and $\tilde{N}(\omega) = (\tilde{D}(\omega))^{-1}$ for $\omega \in \mathbb{C}$.

Denote by $\mathbb{C}_+$ ($\mathbb{C}_-$) the upper (lower, resp.) half-plane, $\mathbb{C}_\pm = \{\omega \in \mathbb{C} : \pm \mathrm{Im}\, \omega > 0\}$.

Lemma 8.1 *(i)* $\tilde{N}(\omega)$ *is meromorphic for* $\omega \in C_m = \mathbb{C} \setminus \Gamma_m$.

(ii) For any $K > 0$, *there exists a constant* $C(K) > 0$ *such that for any* $\mu \in \mathbb{R}$ *such that* $|\mu| \le K$, *we have*

$$|\tilde{D}(\nu + i\mu)| \ge C(K)\nu^2, \quad \nu \to \infty. \tag{8.1}$$

(iii) $(\tilde{D}(\omega))^{-1}$ *is analytic for* $\omega \in \mathbb{C}_+$.

Proof (i) It follows from the formula (4.5) and from the property **I** of $\tilde{K}(x, \omega)$ (see Appendix A) that $\tilde{D}(\omega)$ is analytic for $\omega \in C_m$.

(ii) For any $\beta > 0$, there exists a constant $K_\beta > \sqrt{4 + m^2}$ such that $|\tilde{D}(\omega)| \ge C|\omega|^2$ for any $\omega \in \mathbb{C}$ with $|\omega| \ge K_\beta$. Hence, $(\tilde{D}(\omega))^{-1}$ exists for large $|\omega|$. Moreover, item (ii) follows from (4.5) and (7.1). Hence, for any $\mu \in \mathbb{R}$ such that $|\mu| \le K$ there exists $C_1 = C_1(K) > 0$ such that $|\tilde{N}(\nu + i\mu)| \le C_1 \nu^{-2}$ as $\nu \to \infty$.

(iii) Assume that $\tilde{D}(\omega_0) = 0$ for some $\omega_0 \in \mathbb{C}_+$. Introduce a function $q_*(t)$, $t \in \mathbb{R}$, such that $q_*(t) = e^{-i\omega_0 t}$ for $t \ge 0$, and $q_*(t)$ vanishes for $t < 0$. Then $q_*(t)$ is a solution of Eqn (4.2) with the initial data $(1, -i\omega_0)$. Denote by $Y_*(x) := (u_0(x), v_0(x))$ the initial data for the problem (1.1)–(1.2) such that $Y_*(x) = 0$ for $x \in \mathbb{N}$ and $Y_*(0) = (1, -i\omega_0)$. Then the solution of the system (2.2)–(2.3) with the initial data Y_* vanishes, i.e.,

$$U_0(t)Y_* = (z(\cdot, t), \dot{z}(\cdot, t)) = 0 \quad \text{for any } t > 0 \quad \text{and } x \in \mathbb{N}.$$

In particular, $z(1, t) \equiv 0$. Hence, $q_*(t)$ is the solution of (4.1) for $t > 0$. Then $\tilde{q}_*(x, \omega) := e^{i\theta(\omega)x}\tilde{q}_*(0, \omega) = ie^{i\theta(\omega)x}/(\omega - \omega_0)$. Therefore, the solution of the problem (1.1)–(1.2) with the initial data Y_* is of the form $u_*(x, t) = q_*(x, t) = e^{i\theta(\omega_0)x}e^{-i\omega_0 t}$, $x \in \mathbb{N}$, $t > 0$. On the other hand, the Hamiltonian $H(Y(t))$ (see (1.4)) becomes

$$H(u_*(\cdot, t), \dot{u}_*(\cdot, t)) = e^{2t\,\mathrm{Im}\,\omega_0} H(Y_*) \quad \text{for any } t > 0, \quad \text{where } H(Y_*) > 0.$$

Since $\mathrm{Im}\, \omega_0 > 0$ and $Y_* \in \mathcal{H}_{0,+}$, this exponential growth contradicts the energy estimate (7.15). Hence, $\tilde{D}(\omega) \ne 0$ for any $\omega \in \mathbb{C}_+$. ■

Corollary 8.2 *If* $\gamma = 0$, *then* $\tilde{D}(\omega) \ne 0$ *for any* $\omega \in \mathbb{C}_-$.

Indeed, if $\gamma = 0$, then $\overline{\tilde{D}(x, \omega)} = \tilde{D}(x, \bar{\omega})$, because $\overline{\theta(\omega)} = -\theta(\bar{\omega})$ for $\omega \in C_m$. Then the assertion of the corollary follows from item (iii) of Lemma 8.1.

Now we study the invertibility of $\tilde{D}(\omega + i0)$ for $\omega \in \mathbb{R}$.

Lemma 8.3 *Let* $\omega \in \mathbb{R}$, *and the constants* γ, m, κ *satisfy condition* **C**. *Then* $\tilde{D}(\omega + i0) \ne 0$ *for any* $\omega \in \mathbb{R}$.

Proof *Step 1*: Let $\omega \in \mathbb{R}$ and $|\omega| > \sqrt{4 + m^2}$. Then $\mathrm{Re}\, \theta(\omega + i0) = \pm\pi$. Therefore,

$$\tilde{D}(\omega + i0) = -\omega^2 + \kappa + 1 + m^2 - i\omega\gamma + e^{-\mathrm{Im}\,\theta(\omega+i0)}, \quad \text{with } \mathrm{Im}\,\theta(\omega + i0) > 0.$$

Hence, $\operatorname{Im}\tilde D(\omega+i0)\neq 0$ iff $\gamma\neq 0$. On the other hand,

$$\operatorname{Re}\tilde D(\omega+i0)=-\omega^2+\kappa+1+m^2+e^{-\operatorname{Im}\theta(\omega+i0)}<\kappa-2\quad\text{for}\quad|\omega|>\sqrt{4+m^2},$$

and $\operatorname{Re}\tilde D(\omega+i0)=\kappa-2$ for $\omega=\pm\sqrt{4+m^2}$. Moreover, $\operatorname{Re}\tilde D(\omega+i0)\to-\infty$ as $|\omega|\to\infty$. Hence, for $|\omega|>\sqrt{4+m^2}$, $\operatorname{Re}\tilde D(\omega+i0)\neq 0$ iff $\kappa\le 2$. Therefore, for such values ω, $\tilde D(\omega+i0)\neq 0$ iff $\gamma\neq 0$ or $\gamma=0$ and $\kappa\le 2$. If $\gamma=0$ and $\kappa>2$, then there exist two points $\pm\omega_0$ ($\omega_0>\sqrt{4+m^2}$) such that $\tilde D(\pm\omega_0+i0)=0$.

Step 2: Let $m\neq 0$ and $\omega\in(-m,0)\cup(0,m)$. For such values ω, $\operatorname{Re}\theta(\omega+i0)=0$ and $e^{i\theta(\omega+i0)}\in(e^{i\theta(0)},e^{i\theta(\pm m+i0)})=(\gamma_m^2,1)$ with γ_m defined in (4.4). Hence,

$$\operatorname{Re}\tilde D(\omega+i0)=-\omega^2+\kappa+1+m^2-e^{i\theta(\omega+i0)}>\kappa\quad\text{for}\quad|\omega|<m,$$

and $\operatorname{Re}\tilde D(\omega+i0)=\kappa$ if $\omega=\pm m$. Therefore, $\tilde D(\omega+i0)\neq 0$ for any $|\omega|<m$, since $\kappa\ge 0$.

Step 3: Let $\omega\in(-\sqrt{4+m^2},-m)\cup(m,\sqrt{4+m^2})$. Then $\operatorname{Im}\theta(\omega+i0)=0$ and $\operatorname{Re}\theta(\omega+i0)\in(-\pi,0)\cup(0,\pi)$. Moreover, $\operatorname{sign}(\sin\theta(\omega))=\operatorname{sign}\omega$. Hence, for such values ω and for $m\neq 0$,

$$\operatorname{Im}\tilde D(\omega+i0)=-\omega\gamma-\sin\theta(\omega)=-\operatorname{sgn}(\omega)\Big(|\omega|\gamma+\sqrt{\omega^2-m^2}\sqrt{1-(\omega^2-m^2)/4}\Big).$$

If $m=0$, $\tilde D(\omega+i0)=\kappa-\omega^2/2-i\omega\left(\gamma+\sqrt{1-\omega^2/4}\right)$. Therefore, for such values ω, $\operatorname{Im}\tilde D(\omega+i0)\neq 0$ for any $\kappa,\gamma\ge 0$.

Step 4: For $\omega=\pm\sqrt{4+m^2}$, $e^{i\theta(\omega+i0)}=-1$, and $\tilde D(\omega+i0)=\kappa-2\mp i\gamma\sqrt{4+m^2}$. Hence,

$$\tilde D(\pm\sqrt{4+m^2}+i0)\neq 0\quad\text{iff}\ \gamma\neq 0\quad\text{or}\ \gamma=0\quad\text{and}\ \kappa\neq 2.$$

For $\omega=\pm m$, $e^{i\theta(\omega+i0)}=1$, and $\tilde D(\omega+i0)=\kappa\mp i\gamma m$. Hence, for $m\neq 0$, $\tilde D(\pm m+i0)\neq 0$ iff $\gamma\neq 0$ or $\gamma=0$ and $\kappa\neq 0$. For $m=0$, $\tilde D(0+i0)=\kappa\neq 0$ iff $\kappa\neq 0$. Lemma 8.3 is proved. ■

Corollary 8.4 *Let $m,\gamma,\kappa\ge 0$. If constants m,γ,κ do not satisfy condition* **C**, *then there exist points $\omega\in\mathbb{R}$ in which $\tilde D(\omega+i0)$ vanishes. Indeed, if $m=\kappa=0$, then $\tilde D(0+i0)=0$ for any $\gamma\ge 0$. If $\gamma=\kappa=0$, then $\tilde D(\pm m+i0)=0$ for any $m\ge 0$. If $\gamma=0$ and $\kappa=2$, then $\tilde D(\pm\sqrt{m^2+4}+i0)=0$ for any m. If $\gamma=0$ and $\kappa>2$, then there exists a point $\omega_0>\sqrt{4+m^2}$ such that $\tilde D(\pm\omega_0+i0)=0$.*

Now we study the behavior of $\tilde D(\omega)$ in the neighborhood of the special points $\omega=\pm m$ and $\omega=\pm\sqrt{4+m^2}$.

For $\omega\to\pm\sqrt{4+m^2}+i0$, the representation (7.7) holds. Hence, in the neighborhood of the points $\omega=\pm\sqrt{4+m^2}$ we have

$$\begin{aligned}\tilde D(\omega)\ \ &\sim\ \ \kappa-2\mp i\sqrt{4+m^2}\gamma-i(4+m^2-\omega^2)^{1/2}+\frac{1}{2}(4+m^2-\omega^2)-i(\omega\mp\sqrt{4+m^2})\gamma\\ &\quad+\frac{i}{8}(4+m^2-\omega^2)^{3/2}+...,\end{aligned}$$

where $\operatorname{sgn}(\operatorname{Re}\sqrt{m^2+4-\omega^2})=\operatorname{sgn}(\operatorname{Re}\omega)$. Therefore, if $\gamma\neq 0$ or $\gamma=0$ and $\kappa\neq 2$,

$$(\tilde D(\omega))^{-1}\sim C_0+C_1(4+m^2-\omega^2)^{1/2}+\ldots,\quad\omega\to\pm\sqrt{4+m^2}+i0,\tag{8.2}$$

with $C_0=(\kappa-2\mp i\sqrt{4+m^2}\gamma)^{-1}$ and $C_1=i(\kappa-2\mp i\sqrt{4+m^2}\gamma)^{-2}$.

If $\gamma=0$ and $\kappa=2$, then in the neighborhood of the points $\omega=\pm\sqrt{4+m^2}$,

$$(\tilde{D}(\omega))^{-1}\sim i(4+m^2-\omega^2)^{-1/2}+\frac{1}{2}-\frac{i}{8}(4+m^2-\omega^2)^{1/2}+\dots. \tag{8.3}$$

For $\omega\to\pm m+i0$, the representation (7.5) holds (if $m\neq 0$). Hence, in the neighborhood of the points $\omega=\pm m$,

$$\tilde{D}(\omega)\sim\kappa\mp im\gamma-i(\omega^2-m^2)^{1/2}-i(\omega\mp m)\gamma-\frac{1}{2}(\omega^2-m^2)+\frac{i}{8}(\omega^2-m^2)^{3/2}+\dots,$$

where $\mathrm{sgn}(\mathrm{Re}\sqrt{\omega^2-m^2})=\mathrm{sgn}(\mathrm{Re}\,\omega)$. If $m=0$, then (7.9) holds. Hence, in the neighborhood of the point $\omega=0$,

$$\tilde{D}(\omega)\sim\kappa-i\omega(\gamma+1)-\frac{1}{2}\omega^2+\frac{i}{8}\omega^3+..,\quad\omega\to 0.$$

Hence, if (i) $\gamma\neq 0$ and $m\neq 0$ either (ii) $\gamma\neq 0$, $m=0$ and $\kappa\neq 0$, or (iii) $\gamma=0$ and $\kappa\neq 0$, then

$$(\tilde{D}(\omega))^{-1}\sim\begin{cases}1/\kappa+i\omega(\gamma+1)/\kappa^2+\dots, & \omega\to 0, \quad \text{if } m=0,\\ C_0+C_1(\omega^2-m^2)^{1/2}+\dots, & \omega\to\pm m+i0,\quad \text{if } m\neq 0,\end{cases} \tag{8.4}$$

with $C_0=(\kappa\mp im\gamma)^{-1}$ and $C_1=i(\kappa\mp im\gamma)^{-2}$. If $\gamma=0$, $m\neq 0$ and $\kappa=0$, then

$$(\tilde{D}(\omega))^{-1}\sim i(\omega^2-m^2)^{-1/2}-\frac{1}{2}-\frac{i}{8}(\omega^2-m^2)^{1/2}...,\quad\omega\to\pm m+i0. \tag{8.5}$$

If $m=\kappa=0$ and $\gamma\geq 0$, then

$$(\tilde{D}(\omega))^{-1}\sim\frac{i}{\omega(\gamma+1)}-\frac{1}{2(\gamma+1)^2}-\frac{i\omega(\gamma-1)}{8(\gamma+1)^3}+...,\quad\omega\to 0. \tag{8.6}$$

Applying the inverse Fourier–Laplace formula, we have

$$N(t)=\frac{1}{2\pi}\int\limits_{\mathrm{Im}\,\omega=\mu>0}e^{-i\omega t}\tilde{N}(\omega)\,d\omega,\quad t\geq 0,\quad\text{with some }\mu>0. \tag{8.7}$$

Note that $N(t)=0$ for $t<0$.

Theorem 8.5 *Let $\gamma,m,\kappa\geq 0$. (i) If condition* **C** *is fulfilled, then*

$$|N(t)|\leq C(1+t)^{-3/2}. \tag{8.8}$$

(ii) If $\gamma=0$, $m\neq 0$ and $\kappa=0$ or $\gamma=0$ and $\kappa=2$, then

$$|N(t)|\leq C(1+t)^{-1/2}. \tag{8.9}$$

(iii) If $\gamma=0$ and $\kappa>2$, then

$$N(t)\sim C_1\sin\omega_0 t+O(t^{-3/2}),\quad t\to\infty. \tag{8.10}$$

(iv) If $m=\kappa=0$, then

$$N(t)=\frac{1}{\gamma+1}+O(t^{-3/2}),\quad t\to\infty. \tag{8.11}$$

Proof (i) The proof of the item (i) is similar to the proof of Theorem 3.3 (see subsection 7.2). Indeed, using Lemma 8.1, we change the contour of integration in (8.7) into $\{\omega : |\omega| = R\}$,

$$N(t) = -\frac{1}{2\pi} \int\limits_{|\omega|=R} e^{-i\omega t} \tilde{N}(\omega)\, d\omega, \quad t \geq 0. \tag{8.12}$$

Here R is chosen enough large such that $\tilde{N}(\omega)$ has no poles in the region $\mathbb{C}_- \cap \{|\omega| \geq R\}$. Note that if $\gamma = 0$, then $\tilde{N}(\omega)$ has no poles in $\mathbb{C}_-$ by Corollary 8.2. Denote by σ_j the poles of $\tilde{N}(\omega)$ in $\mathbb{C}_-$ (if they exist). By Lemmas 8.1 and 8.3, there exists a $\delta > 0$ such that $\tilde{N}(\omega)$ has no poles in the region $\operatorname{Im}\omega \in [-\delta, 0]$. Hence, we can rewrite $N(t)$ as

$$N(t) = -i \sum_{j=1}^{K} \operatorname{Res}_{\omega=\sigma_j} \left[e^{-i\omega t} \tilde{N}(\omega) \right] - \frac{1}{2\pi} \int\limits_{\Gamma_{m,\varepsilon}} e^{-i\omega t} \tilde{N}(\omega)\, d\omega, \quad t \geq 0,$$

where $\varepsilon \in (0, \delta)$, the contour $\Gamma_{m,\varepsilon}$ surrounds of Γ_m and belongs to the ε-neighborhood of Γ_m ($\Gamma_{m,\varepsilon}$ is oriented anticlockwise). Taking $\varepsilon \to 0$, we obtain

$$\begin{aligned} N(t) &= \frac{1}{2\pi} \int\limits_{\Gamma_m} e^{-i\omega t} \left(\tilde{N}(\omega + i0) - \tilde{N}(\omega - i0) \right) d\omega + o(t^{-N}) \\ &= \sum_{\pm} \sum_{j=1}^{2} \frac{1}{2\pi} \int\limits_{\Gamma_m} e^{-i\omega t} P_j^{\pm}(\omega)\, d\omega + o(t^{-N}), \quad \text{with any } N > 0. \end{aligned}$$

Here $P_j^{\pm}(\omega) := \zeta_j^{\pm}(\omega)(\tilde{N}(\omega + i0) - \tilde{N}(\omega - i0))$, where $\zeta_j^{\pm}(\omega)$, $j = 1, 2$, are smooth functions such that $\sum\limits_{\pm,j} \zeta_j^{\pm}(\omega) = 1$, $\omega \in \mathbb{R}$, $\operatorname{supp} \zeta_1^{\pm} \subset \mathcal{O}(\pm m)$, $\operatorname{supp} \zeta_2^{\pm} \subset \mathcal{O}(\pm\sqrt{4 + m^2})$ (Here as before $\mathcal{O}(a)$ denotes a neighborhood of a point $\omega = a$). If $m = 0$, we introduce ζ_1 (P_1) instead of $\zeta_1^{\pm}$ ($P_1^{\pm}$, resp.) with $\operatorname{supp} \zeta_1 \subset \mathcal{O}(0)$. Then (8.2) and (8.4) imply the bound (8.8). Here we use the following estimate with $j = 1$,

$$\Big| \int_{\mathbb{R}^1} \zeta(\omega) e^{-i\omega t} \sqrt{a^2 - \omega^2}\, d\omega \Big| \leq C(1 + t)^{-3/2} \quad \text{as } t \to \infty, \tag{8.13}$$

$\zeta(\omega)$ is a smooth function, and $\zeta(\omega) = 1$ for $|\omega - a| \leq \delta$ with some $\delta > 0$ (see, for example, [27] or [25]).

(ii) Let $\gamma = 0$, $m \neq 0$, $\kappa = 0$. Then $\tilde{N}(\omega + i0) - \tilde{N}(\omega - i0) = 2i \operatorname{Im} \tilde{N}(\omega + i0)$ and $\tilde{N}(\omega)$ has no poles in $\mathbb{C}_-$. Introduce the circles $c_{\pm}$, $c_{\pm} = \{|\omega \mp m| = \varepsilon\}$, with some $\varepsilon \in (0, m)$. We change the contour of the integration in (8.12) into $\Gamma_\varepsilon := \cup_{\pm} c_{\pm} \cup_j \gamma_j$, where γ_j, $j = 1, 2, 3$, stand for the segments of the real axis connecting the circles $c_{\pm}$ and passing in two directions, $\gamma_1 = [-\sqrt{m^2 + 4}, -m - \varepsilon]$, $\gamma_2 = [-m + \varepsilon, m - \varepsilon]$, $\gamma_3 = [m + \varepsilon, \sqrt{m^2 + 4}]$. Using the Cauchy theorem and Lemma 8.1, we have

$$N(t) = -\frac{1}{2\pi} \int\limits_{c_- \cup c_+} e^{-i\omega t} \tilde{N}(\omega)\, d\omega + \sum_{j=1}^{3} \frac{i}{\pi} \int\limits_{\gamma_j} e^{-i\omega t} \operatorname{Im} \tilde{N}(\omega + i0)\, d\omega.$$

Applying representations (8.2) and (8.5) and the well-known estimate (see, for example, [25])

$$-\frac{1}{2\pi}\int\limits_{|\omega|=m+1} e^{-i\omega t}\sqrt{\omega^2-m^2}\,d\omega=\sqrt{\frac{2}{\pi mt}}i\cos(mt-\pi/4)+O(t^{-3/2}),\quad t\to\infty,$$

we obtain $N(t)=-(2/(\pi m))^{1/2}t^{-1/2}\cos(mt-\pi/4)+O(t^{-3/2})$ as $t\to\infty$, and the bound (8.9) follows.

Similarly, if $\gamma=0$ and $\kappa=2$, the representations (8.3) and (8.4) imply the bound (8.9), $N(t)=(2/(\pi\sqrt{m^2+4}))^{1/2}t^{-1/2}\cos(t\sqrt{m^2+4}-\pi/4)+O(t^{-3/2})$ as $t\to\infty$.

(iii) If $\kappa>2$, $\tilde N(\omega)$ has two simple poles in the points $\pm\omega_0$, $\omega_0>\sqrt{\omega^2+m^2}$. Then calculating the rescue of $e^{-i\omega t}\tilde N(\omega)$ in these points we have (8.10).

(iv) Using formulas (8.6) and (8.2) and calculating the rescue of $\tilde N(\omega)$ in the point $\omega=0$, we obtain (8.11). ■

9 Appendix C: Zero boundary condition

Consider the following mixed initial-boundary problem on the half-line:

$$\begin{cases} \ddot u(x,t)=(\Delta_L-m^2)u(x,t), \quad x\in\mathbb{N},\quad t\in\mathbb{R},\\ u(0,t)=0,\\ u(x,0)=u_0(x),\quad \dot u(x,0)=v_0(x),\quad x\in\mathbb{N}. \end{cases}\tag{9.1}$$

In [7], we studied the convergence to equilibrium for the harmonic crystals in the half-space in any dimension with zero boundary condition. However, the one-dimensional case with $m=0$ was not considered in [7]. Therefore, we outline the proof using the methods from [7].

We rewrite the problem (9.1) in the more general form:

$$\begin{cases} \ddot u(x,t)=-\sum\limits_{x'\ge 0}\left(V(x-x')-V(x+x')\right)u(x',t), \quad x\in\mathbb{N},\quad t\in\mathbb{R},\\ u(0,t)=0,\\ u(x,0)=u_0(x),\quad \dot u(x,0)=v_0(x),\quad x\in\mathbb{N}. \end{cases}\tag{9.2}$$

We assume that $u_0(0)=v_0(0)=0$. We impose the following conditions **V1**–**V5** on the interaction function V.

V1. There exist positive constants C and β such that $|V(x)|\le Ce^{-\beta|x|}$ for $x\in\mathbb{Z}^1$.

V2. V is real and even, i.e., $V(-x)=V(x)\in\mathbb{R}$, $x\in\mathbb{Z}^1$.

Conditions **V1** and **V2** imply that $\hat V(\theta)$ is a real-analytic function of $\theta\in\mathbb{T}^1$.

V3. $\hat V(\theta)\ge 0$ for every $\theta\in\mathbb{T}^1$.

Let us define the real-valued nonnegative function, $\phi(\theta)=(\hat V(\theta))^{1/2}\ge 0$. $\phi(\theta)$ can be chosen as the real-analytic function in $\mathbb{T}^1\setminus\mathcal{C}_*$, where $\mathcal{C}_*$ is a closed subset of $\mathbb{T}^1$ such that the Lebesgue measure of $\mathcal{C}_*$ is zero (see Lemma 2.2 in [4]).

V4. $\phi''(\theta)$ does not vanish identically on $\theta\in\mathbb{T}^1\setminus\mathcal{C}_*$.

V5. (i) $\hat V(\theta)\ne 0$ for all $\theta\in\mathbb{T}^1$ or (ii) $\hat V(\theta)\ne 0$ for all $\theta\ne 0$, and $\hat V(0)=0$.

Since $\hat V(\theta)$ is even, then in the second case $\hat V(\theta)$ is of order θ^2 as $\theta\to 0$.

Conditions **V1**–**V5** are fulfilled, for example, in the case when $V(x)$ has a form

$$V(\pm 1) = -1, \quad V(x) = 0 \text{ for } |x| \geq 2, \quad \text{and} \qquad V(0) = 2 + m^2 \text{ with } m \geq 0. \tag{9.3}$$

Then $\phi(\theta) = \sqrt{2 - 2\cos\theta + m^2}$. If $m \neq 0$, $\mathcal{C}_* = \emptyset$, and condition **V5** (i) holds. If $m = 0$, $\phi(\theta) = 2|\sin(\theta/2)|$. Then $\mathcal{C}_* = \{0\}$, and condition **V5** (ii) holds. Moreover, in the case when $V(x)$ is of the form (9.3), then the problem (9.2) becomes (9.1).

Write $Z(x,t) = (Z^0(x,t), Z^1(x,t)) \equiv (u(x,t), \dot{u}(x,t))$, $Z_0(x) = (u_0(x), v_0(x))$. The existence of dynamics is stated by the following lemma which is proved as in [7].

Lemma 9.1 *Let* $\alpha \in \mathbb{R}$ *and conditions* **V1** *and* **V2** *hold. For any* $Z_0 \in \mathcal{H}_{\alpha,+}$, *there exists a unique solution* $Z(\cdot,t) \in \mathcal{H}_{\alpha,+}$ *to problem (9.2). Moreover, the operator* $U_0(t) : Z_0 \to Z(\cdot,t)$ *is continuous in* $\mathcal{H}_{\alpha,+}$.

Indeed, since $\mathcal{G}_t(x)$ is even by condition **V2**, the solution of the problem (9.2) can be represented as the restriction of the solution to the Cauchy problem with odd initial data on the half-line,

$$Z^i(x,t) = \sum_{y \in \mathbb{Z}} \mathcal{G}_t^{ij}(x-y) Z^j_{odd}(y), \quad x \in \mathbb{N}, \quad i = 0, 1.$$

Here $\mathcal{G}_t(x) = F^{-1}_{\theta \to x}[e^{\hat{\mathcal{A}}(\theta)t}]$ with $\hat{\mathcal{A}}(\theta) = \begin{pmatrix} 0 & 1 \\ \hat{V}(\theta) & 0 \end{pmatrix}$ (see also formulas (2.16), (2.17) with $\phi(\theta) = (\hat{V}(\theta))^{1/2}$), and by definition, $Z_{odd}(x) = Z_0(x)$ for $x > 0$, $Z_{odd}(0) = 0$, and $Z_{odd}(x) = -Z_0(-x)$ for $x < 0$. Then the solution $Z(x,t)$ of the problem (9.2) is of a form

$$Z^i(x,t) = \sum_{y \geq 1} \Big(\mathcal{G}_t^{ij}(x-y) - \mathcal{G}_t^{ij}(x+y) \Big) Z_0^j(y), \quad x \in \mathbb{N}, \quad i = 0, 1. \tag{9.4}$$

Moreover, the following bound holds,

$$\|U_0(t) Z_0\|_{\alpha,+} \leq C(1 + |t|)^N \|Z_0\|_{\alpha,+}, \tag{9.5}$$

with some constants $C = C(\alpha), N = N(\alpha) < \infty$.

Denote by ν_0 a Borel probability measure on $\mathcal{H}_{\alpha,+}$ giving the distribution of Z_0. We impose conditions **S1**–**S4** on ν_0.
S1. ν_0 has zero mean value.
S2. The mean energy density of ν_0 is finite, $\mathbb{E}_0(|Z_0(x)|^2) \leq C < \infty$.
Finally, it is assumed that ν_0 satisfies conditions **S3** and **S4** from Section 2.2.

By ν_t, $t \in \mathbb{R}$, we denote the distribution of the solution $Z(x,t) = U_0(t) Z_0$ (see Definition 2.6). Then the following theorem holds.

Theorem 9.2 *Let conditions* **S1**–**S4** *and* **V1**–**V5** *hold, and let* $\alpha < -1/2$ *in the case* **V5** *(i), and* $\alpha < -1$ *in the case* **V5** *(ii). Then the following assertions are true.*
(i) The measures ν_t *weakly converge on the space* $\mathcal{H}_{\alpha,+}$ *as* $t \to \infty$. *Moreover, the limit measure* ν_∞ *is Gaussian on* $\mathcal{H}_{\alpha,+}$ *with correlation matrix of the form (2.19)–(2.20) with* $\phi(\theta) = (\hat{V}(\theta))^{1/2}$.
(ii) The correlation functions of ν_t *converge to a limit as* $t \to \infty$, *i.e., (2.18) holds.*

The derivation of this theorem is based on the proof of the compactness of the measures family $\{\nu_t, t\in\mathbb{R}\}$ and the convergence of the characteristic functionals (cf [7]). Below we prove the compactness and outline the proof of the convergence for correlation functions.

The compactness follows from the bound (9.6) below by the Prokhorov compactness theorem [26, Lemma II.3.1] by a method used in [26, Theorem XII.5.2], since the embedding $\mathcal{H}_{\alpha,+}\subset\mathcal{H}_{\beta,+}$ is compact if $\alpha>\beta$.

Lemma 9.3 *Let conditions* **S1**, **S2**, **S4** *hold, and* $\alpha<-1/2$ *if condition* **V5** *(i) holds, and* $\alpha<-1$ *if condition* **V5** *(ii) holds. Then the following bound holds:*

$$\sup_{t\ge 0}\mathbb{E}_0\|U_0(t)Z_0\|^2_{\alpha,+}<\infty. \tag{9.6}$$

Proof. By Definition 2.1 (ii),

$$\mathbb{E}_0\|Z(\cdot,t)\|^2_{\alpha,+}=\sum_{z\ge 0}(1+|z|^2)^\alpha\Big(Q^{00}_t(z,z)+Q^{11}_t(z,z)\Big). \tag{9.7}$$

The representation (9.4) gives

$$Q^{ij}_t(z,z')=\mathbb{E}_0\Big(Z^i(z,t)\otimes Z^j(z',t)\Big)=\langle Q_0(y,y'),\mathbf{G}^i_z(y,t)\otimes\mathbf{G}^j_{z'}(y',t)\rangle_+, \tag{9.8}$$

where $\mathbf{G}^i_z(y,t)$ is defined in (6.1). If condition **V5** (i) holds, then the Parseval identity and formula (2.17) with $\phi(\theta)=(\hat V(\theta))^{1/2}$ yield

$$\|\mathbf{G}^i_z(\cdot,t)\|^2_{l^2}=\frac{1}{2\pi}\int\limits_{\mathbb{T}^1}|\hat{\mathbf{G}}^i_z(\theta,t)|^2\,d\theta\le C\int\limits_{\mathbb{T}^1}\Big(|\hat{\mathcal{G}}^{i0}_t(\theta)|^2+|\hat{\mathcal{G}}^{i1}_t(\theta)|^2\Big)\sin^2(z\theta)\,d\theta\le C_0<\infty. \tag{9.9}$$

By conditions **S1**, **S2** and **S4**, the bound (2.11) holds. Therefore,

$$\begin{aligned}|Q^{ij}_t(z,z')| &= \big|\langle Q_0(y,y'),\mathbf{G}^i_z(y,t)\otimes\mathbf{G}^j_{z'}(y',t)\rangle_+\big|\\ &\le C\|\mathbf{G}^i_z(\cdot,t)\|_{0,+}\|\mathbf{G}^j_{z'}(\cdot,t)\|_{0,+}\le C_1<\infty,\end{aligned} \tag{9.10}$$

where the constant C_1 does not depend on $z,z'\ge 0$ and $t\in\mathbb{R}$. Therefore, (9.6) follows from (9.7) and (9.10), since $\alpha<-1/2$.

If condition **V5** (ii) holds, then $\hat V(\theta)\sim\theta^2$ as $\theta\to 0$. The estimate (9.9) with $i=1$ remains true. Then $|Q^{11}_t(z,z')|\le C<\infty$ by (9.10). However, since $\hat{\mathcal{G}}^{01}_t(\theta)=\sin(\phi(\theta)t)/\phi(\theta)$, we have

$$\|\mathbf{G}^0_z(\cdot,t)\|^2_{l^2}\le C+C_1\int\limits_{\mathbb{T}^1}\frac{\sin^2(\phi(\theta)t)}{\hat V(\theta)}\sin^2(z\theta)\,d\theta\le C+C_2\int\limits_{\mathcal{O}(0)}\frac{\sin^2(z\theta)}{\sin^2(\theta)}\,d\theta\le C+C_3|z|,$$

uniformly on $t\in\mathbb{R}$. Hence, $|Q^{00}_t(z,z)|\le C\|\mathbf{G}^0_z(\cdot,t)\|^2_{0,+}\le C_1+C_2|z|$, by (9.10). Therefore, for any $\alpha<-1$,

$$\mathbb{E}_0\|Z(\cdot,t)\|^2_{\alpha,+}\le\sum_{z\ge 0}(1+|z|^2)^\alpha(C_1+C_2|z|)<\infty.\ \blacksquare$$

Now we outline the proof of item (ii) of Theorem 9.2. At first, introduce quadratic forms

$$\mathcal{Q}_t(\Psi,\Psi)=\langle Q_t(x,x'),\Psi(x)\otimes\Psi(x')\rangle_+,\quad t\in\mathbb{R},\quad \Psi\in\mathcal{S},$$

and let $\mathcal{Q}_\infty(\Psi,\Psi)$ be the quadratic form with the kernel $Q_\infty(x,x')$. Then the assertion (2.18) is equivalent to the next convergence,

$$\lim_{t\to\infty}\mathcal{Q}_t(\Psi,\Psi)=\mathcal{Q}_\infty(\Psi,\Psi)\quad\text{for any}\quad\Psi\in\mathcal{S}. \tag{9.11}$$

Using the operator $U_0'(t)$ (see formula (6.7)), we rewrite $\mathcal{Q}_t(\Psi,\Psi)$ as

$$\mathcal{Q}_t(\Psi,\Psi)=\mathcal{Q}_0(\Phi(\cdot,t),\Phi(\cdot,t)),\quad \Phi(x,t):=U_0'(t)\Psi=\sum_{z\in\mathbb{Z}^1_+}\mathcal{G}^T_{t,+}(z,x)\Psi(z). \tag{9.12}$$

Since $\mathcal{G}_{t,+}(z,0)=\mathcal{G}_{t,+}(0,x)=0$, we put $\Psi(0)=0$, without loss of generality. Given $\Psi\in\mathcal{S}$, introduce an odd function Ψ_o as in Remark 2.8. Hence,

$$\Phi(x,t):=U_0'(t)\Psi=\sum_{z\in\mathbb{Z}^1}\mathcal{G}^T_t(z-x)\Psi_o(z). \tag{9.13}$$

Further, (see [7, p.468]) define $Q_*(z,z')$ to be equal to $Q_0(z,z')$ and to 0 otherwise. We split the function $Q_*(z,z')$ into the following three matrices (see condition **S3**)

$$\begin{aligned} Q^+(z,z') &:= \frac{1}{2}\,q_0(z-z'),\quad Q^-(z,z')=\frac{1}{2}\,q_0(z-z')\,\mathrm{sign}(z'), \\ Q^r(z,z') &:= Q_*(z,z')-Q^+(z,z')-Q^-(z,z'),\quad z,z'\in\mathbb{Z}^1. \end{aligned} \tag{9.14}$$

Then $\mathcal{Q}_t(\Psi,\Psi)=\langle Q_*(z,z'),\Phi(z,t)\otimes\Phi(z',t)\rangle=\sum\limits_{a=\pm,r}\langle Q^a(z,z'),\Phi(z,t)\otimes\Phi(z',t)\rangle$. Hence, the convergence (9.11) follows from the following lemma.

Lemma 9.4 *(i)* $\lim\limits_{t\to\infty}\langle Q^\pm(z,z'),\Phi(z,t)\otimes\Phi(z',t)\rangle=\langle q^\pm_\infty(z-z'),\Psi_o(z)\otimes\Psi_o(z')\rangle$, *with the matrices* $q^\pm_\infty$ *defined in (2.20). (ii)* $\lim\limits_{t\to\infty}\langle Q^r(z,z'),\Phi(z,t)\otimes\Phi(z',t)\rangle=0$.

This lemma can be proved using the technique of [5, Proposition 7.1]. To justify the main idea of the proof of Lemma 9.4 (i), we first introduce a set $\mathcal{C}\in\mathbb{T}^1$,

$$\mathcal{C}:=\mathcal{C}_*\cup\big\{\theta\in\mathbb{T}^1\setminus\mathcal{C}_*:\phi'(\theta)=0\big\}\cup\{\theta\in\mathbb{T}^1:\hat V(\theta)=0\},$$

and a space $\mathcal{S}_0:=\{\Psi\in\mathcal{S}:\hat\Psi_o(\theta)=0\text{ in a neighborhood of }\mathcal{C}\}$. Note that mes $\mathcal{C}=0$ (see [5, lemma 7.3]) Next, we define a norm $\|\cdot\|_V$ in the space $\mathcal{S}$ such that i) $\mathcal{S}_0$ is dense in $\mathcal{S}$ in this norm, while ii) the quadratic forms $\mathcal{Q}_t(\Psi,\Psi)$, $t\in\mathbb{R}$, are equicontinuous in this norm. By definition, $\mathcal{S}_V$ is the space $\mathcal{S}$ endowed with the norm

$$\|\Psi\|^2_V:=\int_{\mathbb{T}^1}\Big(|\hat\Psi^1_o(\theta)|^2+(1+|\hat V(\theta)|^{-1})|\hat\Psi^0_o(\theta)|^2\Big)\,d\theta,\quad \Psi=(\Psi^0,\Psi^1)\in\mathcal{S}.$$

Note that the set $\mathcal{S}_0$ is dense in $\mathcal{S}_V$, and the quadratic forms $\mathcal{Q}_t(\Psi,\Psi)$, $t\in\mathbb{R}$, are equicontinuous in $\mathcal{S}_V$. Indeed, by (9.12) and (2.11), $\sup\limits_{t\in\mathbb{R}}|\mathcal{Q}_t(\Psi,\Psi)|\le C\sup\limits_{t\in\mathbb{R}}\|\Phi(\cdot,t)\|^2_{l^2}$. On the other hand, by the Parseval identity and (9.12), (2.17), we have

$$\|\Phi(\cdot,t)\|^2_{l^2}=\frac{1}{2\pi}\int_{\mathbb{T}^1}\|\hat{\mathcal{G}}^*_t(\theta)\|^2|\hat\Psi_o(\theta)|^2d\theta\le C\|\Psi\|^2_V,$$

because $|\hat{\Psi}_o(\theta)| \le C|\sin\theta| \sum_{x\in\mathbb{N}} |x||\Psi(x)| \le C(\Psi)|\sin\theta|$ for $\Psi \in \mathcal{S}$. Therefore,

$$\sup_{t\in\mathbb{R}} |\mathcal{Q}_t(\Psi,\Psi)| \le C\|\Psi\|_V^2, \quad \Psi \in \mathcal{S}.$$

Hence, it suffices to prove Lemma 9.4 for $\Psi \in \mathcal{S}_0$ only.

By Parseval identity and (9.14),

$$\langle Q^+(z,z'), \Phi(z,t)\otimes\Phi(z',t)\rangle = \frac{1}{4\pi}\langle \hat{q}_0(\theta), \hat{\mathcal{G}}_t^T(\theta)\hat{\Psi}_o(\theta) \otimes \overline{\hat{\mathcal{G}}_t^T(\theta)\hat{\Psi}_o(\theta)}\rangle.$$

Note that $\hat{\mathcal{G}}_t(\theta) = \cos\phi(\theta)t\, I + \sin\phi(\theta)t\, C(\theta)$. Therefore,

$$\langle Q^+(z,z'), \Phi(z,t)\otimes\Phi(z',t)\rangle = \frac{1}{4\pi}\int_{\mathbb{T}^1} \Big(R_t(\theta), \hat{\Psi}_o(\theta) \otimes \overline{\hat{\Psi}_o}(\theta)\Big)\, d\theta,$$

where $R_t(\theta)$ stands for the 2×2 matrix,

$$\begin{aligned} R_t(\theta) \quad = \quad & \frac{1}{2}\Big[\hat{q}_0(\theta) + C(\theta)\hat{q}_0(\theta)C(\theta)^*\Big] + \frac{1}{2}\cos\left(2\phi(\theta)t\right)\Big[\hat{q}_0(\theta) - C(\theta)\hat{q}_0(\theta)C(\theta)^*\Big] \\ & + \frac{1}{2}\sin\left(2\phi(\theta)t\right)\Big[C(\theta)\hat{q}_0(\theta) + \hat{q}_0(\theta)C(\theta)^*\Big]. \end{aligned}$$

By formulas (2.20), one obtains

$$\langle Q^+(z,z'), \Phi(z,t)\otimes\Phi(z',t)\rangle = \frac{1}{2\pi}\int_{\mathbb{T}^1} \Big(\hat{q}_\infty^+(\theta), \hat{\Psi}_o(\theta) \otimes \overline{\hat{\Psi}_o}(\theta)\Big)\, d\theta + \ldots,$$

where "$\ldots$" stands for the oscillatory integrals which contain $\cos(2\phi(\theta))t$ and $\sin(2\phi(\theta))t$ with $\phi(\theta) \not\equiv \text{const}$. The oscillatory integrals converge to zero by the Lebesgue–Riemann theorem since all the integrands in '...' are summable by Lemma 2.4 and condition **V5**. The convergence in Lemma 9.4 (i) with sign "+" is proved. The another assertions of Lemma 9.4 can be proved using methods of [7]. ■

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