Pseudoholomorphic Maps Relative to Normal Crossings Symplectic Divisors: Compactification

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Abstract

Inspired by the log Gromov-Witten theory of Gross-Siebert/Abramovich-Chen, we introduce a geometric notion of *log pseudoholomorphic map* relative to simple normal crossings symplectic divisors defined in [9]. For certain almost complex structures, we show that the moduli space of stable log pseudoholomorphic maps of any fixed type[1] is compact and metrizable with respect to an enhancement of the Gromov topology. In the case of smooth symplectic divisors, our compactification is often smaller than the relative compactification and there is a projection map from the former onto the latter. The latter is constructed via expanded degenerations of the target. Our construction does not need any modification of (or any extra structure on) the target. Unlike the classical moduli spaces of stable maps, these log moduli spaces are often virtually singular. We describe an explicit toric model for the normal cone[2] to each stratum in terms of the defining combinatorial data of that stratum. In upcoming papers, we will define a natural Fredholm operator which gives us the deformation/obstruction spaces of each stratum and prove a gluing theorem for smoothing log maps in the normal direction to each stratum. With minor modifications to the theory of Kuranishi structures, the latter would allow us to construct a virtual fundamental class for every such log moduli space.

Contents

[1] Every such moduli space is characterized by a second homology class, genus, and contact data.
[2] i.e. the space of gluing parameters.

1 Introduction

Let us begin by setting up the most commonly used notation and recalling some of the known facts about the classical and relative moduli spaces of pseudoholomorphic maps.

1.1 Classical stable maps and GW invariants

For a smooth manifold X, $g,k\in\mathbb{N}$, $A\in H_2(X,\mathbb{Z})$, and an almost complex structure[3] J on X, a (nodal) k-marked genus g degree A J-holomorphic map into X is a tuple $\big(u,\Sigma,\mathfrak{j},z^1,\ldots,z^k\big)$, where

- $(\Sigma,\mathfrak{j})$ is a connected nodal Riemann surface of arithmetic genus g with k distinct ordered marked points $z^1,\ldots,z^k$ away from the nodes,
- $u\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J)$ is a continuous and component-wise smooth map satisfying the Cauchy-Riemann equation
$$\bar\partial u=\frac{1}{2}(\mathrm{d}u+J\mathrm{d}u\circ\mathfrak{j})=0 \tag{1.1}$$
on each smooth component, and
- the map u represents the homology class A.

Two such tuples
$$(u,\Sigma,\mathfrak{j},z^1,\ldots,z^k)\quad\text{and}\quad(u',\Sigma',\mathfrak{j}',w^1,\ldots,w^k)$$
are equivalent if there exists a biholomorphic isomorphism $h\colon(\Sigma,\mathfrak{j})\longrightarrow(\Sigma',\mathfrak{j}')$ such that $h(z^i)=w^i$, for all $i=1,\ldots,k$, and $u=u'\circ h$. Such a tuple is called stable if the group of self-automorphisms is finite. Let $\overline{\mathcal{M}}_{g,k}(X,A,J)$ (or simply $\overline{\mathcal{M}}_{g,k}(X,A)$ when J is fixed in the discussion) denote the space of (equivalence classes of) stable k-marked genus g degree A J-holomorphic maps into X.

By[4] a celebrated theorem of Gromov [15, Theorem 1.5.B], for every smooth closed (i.e. compact and without boundary) symplectic manifold (X,ω), g,k,A as above, and an almost complex structure J compatible[5] with ω (or taming ω), the moduli space $\overline{\mathcal{M}}_{g,k}(X,A,J)$ has a natural sequential convergence topology, called the Gromov topology, which is compact, Hausdorff, and furthermore metrizable. The symplectic structure only gives an energy bound which is needed for establishing

[3] i.e. J is a real-linear endomorphism of TX lifting the identity map satisfying $J^2=-\mathrm{id}_{TX}$.
[4] and its subsequent refinements; see Theorem 4.3.
[5] i.e. $\omega(\cdot,J\cdot)$ is a metric.

the compactness, and the precise choice of that, up to deformation, is not important. If $\overline{\mathcal{M}}_{g,k}(X,A)$ has a "nice" orbifold structure of the expected real dimension

$$2\big(c_1^{TX}(A)+(n-3)(1-g)+k\big),$$

Gromov-Witten (or GW) invariants are obtained by the integration of appropriate cohomology classes against its fundamental class. These numbers are independent of J and only depend on the deformation equivalence class of ω. These allow the formulation of symplectic analogues of enumerative questions from algebraic geometry, as well-defined invariants of symplectic manifolds. However, in general, such moduli spaces can be highly singular. This issue is known as the *transversality problem*. Fortunately, it has been shown (e.g. [22, 23]) that $\overline{\mathcal{M}}_{g,k}(X,A)$ still carries a rational homology class, called virtual fundamental class (VFC); integration of cohomology classes against VFC gives rise to GW-invariants. We denote such an VFC by $[\overline{\mathcal{M}}_{g,k}(X,A)]^{\mathrm{VFC}}$. In [8], we sketch the construction of VFC via the method of Kuranishi structures in [13].

1.2 Relative stable maps

Given a closed symplectic manifold (X,ω) and a submanifold $D\subset X$, we say $D\subset X$ is a symplectic submanifold if $\omega|_D$ is a symplectic structure. A (smooth) symplectic divisor is a symplectic submanifold of real codimension 2. For such D (or a smooth divisor in complex algebraic geometry), relative GW theory (virtually) counts pseudoholomorphic maps in X with a fixed contact order $\mathfrak{s}\equiv(s_1,\cdots,s_k)\in\mathbb{N}^k$ with D. In this theory, we require J to be also compatible with D in the following sense. First, we require D to be J-holomorphic, i.e. $J(TD)=TD$. This implies, for example, that every J-holomorphic map to X from a smooth domain is either mapped into D or intersects D positively in a finite set of points. Furthermore, we require J to be integrable to the first order in the normal direction to D in the sense that

$$N_J(v_1,v_2)\in T_xD \qquad \forall\, x\in D,\ v_1,v_2\in T_xX, \quad \text{where} \tag{1.2}$$

$$N_J\in\Gamma(X,\Omega_X^2\otimes TX),\quad N_J(u,v)\equiv[u,v]+J[u,Jv]+J[Ju,v]-[Ju,Jv] \quad \forall u,v\in TX,$$

is the Nijenhueis tensor of J. This ensures that certain operators are complex linear, see (2.20), and certain sequence of almost complex structures on the normal bundle $\mathcal{N}_XD$ converges to a standard one, see Lemma 4.5. The space $\mathcal{J}(X,D,\omega)$ of (ω,D)-compatible almost complex structures on (X,D,ω) is again non-empty and contractible. For every $J\in\mathcal{J}(X,D,\omega)$ and $\mathfrak{s}\equiv(s_1,\cdots,s_k)\in\mathbb{N}^k$, with

$$\sum_{i=1}^k s_i = A\cdot D, \tag{1.3}$$

let $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ (in the stable range) denote the space of (equivalence classes of) k-marked degree A genus g J-holomorphic maps $(u,\Sigma,\mathfrak{j},z^1,\ldots,z^k)$ such that Σ is smooth and u has a tangency of order s_i at z^i with D. In particular, by (1.3),

$$u^{-1}(D)\subset\{z^1,\ldots,z^k\}.$$

The subset of marked points z^i with $s_i=0$ corresponds to the classical marked points of the classical GW theory with image away from D. The relative compactification $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathfrak{s}}(X,D,A)$ of $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ constructed in [20], in the algebraic case, and in [17, 21], in the symplectic case, includes stable nodal maps with components mapped into X or an expanded degeneration[6] of that so that the contact order $\mathfrak{s}$ still makes sense; we will review this construction in Section 2.

[6] A normal crossing variety made of X and finite copies of some $\mathbb{P}^1$-bundle $\mathbb{P}_XD$ over D.

1.3 Pseudoholomorphic maps relative SC divisors

In [9, 10], we defined topological notions of symplectic normal crossings divisor and variety and showed that they are equivalent, in a suitable sense, to the desired rigid notions. A simple normal crossings (SC) symplectic divisor $D=\bigcup_{i\in S}D_i$ in (X,ω) is a transverse union of smooth symplectic divisors $\{D_i\}_{i\in S}$ in X such that all the strata

$$D_I\equiv\bigcap_{i\in I}D_i \qquad \forall I\subset S$$

are symplectic, and the symplectic orientation of D_I coincides with its "intersection" orientation for all $I\subset S$; see [9, Definition 2.1]. For

$$J\in\mathcal{J}(X,D,\omega)=\bigcap_{j\in S}\mathcal{J}(X,D_j,\omega),$$

we similarly define $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ (in the stable range) to be the space of equivalence classes of degree A J-holomorphic maps from a k-marked genus g connected smooth domain Σ into X of contact order $\mathfrak{s}$ with D, for which

$$\mathfrak{s}\equiv(s_i\equiv(s_{ij})_{j\in S})_{i=1}^k\in(\mathbb{N}^S)^k,$$

each s_i records the intersection numbers of the i-th marked point z^i with the divisors $\{D_j\}_{j\in S}$, and

$$u^{-1}(D)\subset\{z^1,\ldots,z^k\}, \quad \text{or equivalently} \quad A\cdot D_j=\sum_{i=1}^k s_{ij} \quad \forall\, j\in S. \tag{1.4}$$

The expected real dimension of $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ is equal to

$$2\big(c_1^{TX(-\log D)}(A)+(n-3)(1-g)+k\big)=2\big(c_1^{TX}(A)+(n-3)(1-g)+k-A\cdot D\big), \tag{1.5}$$

where $TX(-\log D)$ is the log tangent bundle associated to the deformation equivalence class of (X,D,ω), defined in [11]. The question is then:

($\star$) how to define a "good" compactification $\overline{\mathcal{M}}^{\text{good}}_{g,\mathfrak{s}}(X,D,A)$ of $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ so that the definition of contact order $\mathfrak{s}$ naturally extends to every element of $\overline{\mathcal{M}}^{\text{good}}_{g,\mathfrak{s}}(X,D,A)$, $\overline{\mathcal{M}}^{\text{good}}_{g,\mathfrak{s}}(X,D,A)$ is (virtually) smooth enough to admit a natural class of cobordant (for various choices of J, etc.) Kuranishi structures of the real dimension (1.5) in the sense of [8, Theorem 6.5.1], *and the resulting GW invariants are invariants of the deformation equivalence class of (X,D,ω)?*

For example, if D is smooth, the relative compactification $\overline{\mathcal{M}}^{\text{rel}}_{g,\mathfrak{s}}(X,D,A)$ has (or it is expected[7] to have) these nice properties.

In the algebraic category, every (algebraic) NC variety $D\subset X$ defines a natural "fine saturated log structure" on X. Then the log GW theory of [1, 18] constructs a good compactification with a perfect obstruction theory for every fine saturated log variety X. Unlike in [20], the algebraic log compactification does not require any expanded degeneration of the target. Instead, it uses the extra log structure on X (and various log structures on the domains) to keep track of the contact

[7] cf. [12] for an overview of the analytical approaches of [17, 21].

data for the curves that have image inside the support of the log structure (i.e. D).

On the analytical side, in [28], Brett Parker uses his enriched almost Kähler category of "exploded manifolds", defined in [27], to construct such a compactification relative to an almost Kähler NC divisor. His approach can be considered as a direct translation/generalization of the algebraic log GW theory; however, in addition to some rigidity requirements, his definitions are too complicated than desirable and have so far proved too unwieldy for practical applications. In [16], Eleny Ionel approaches ($\star$), by considering expanded degenerations similar to [17]; we refer to [6, The state of the art] for a list of main issues in her approach. Nevertheless, the main motivation behind the log GW theory of Gross-Siebert-Abramovich-Chen, exploded theory of Parker, and the current paper is that the idea of expanded degenerations is not suitable for addressing ($\star$) in the general case. In particular, it is very unlikely that the idea of expended degenerations can be used to define a good compactification for the moduli space of pseudoholomorphic maps into NC varieties. We discuss this aspect of our approach with details in [6].

1.4 Log compactification and the main results

In this paper, for an arbitrary SC symplectic divisor $D\subset(X,\omega)$ and certain $J\in\mathcal{J}(X,D,\omega)$, we define a "good geometric compactification"

$$\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A) \tag{1.6}$$

that does not require any modification of the target (or the nodal domains). For its connection to the algebraic log maps, and the appearance of various log structures[8] throughout the construction, we call our maps: log pseudoholomorphic maps.

For $J\in\mathcal{J}(X,D,\omega)$, a (nodal) log J-holomorphic map into (X,D) of contact type

$$\mathfrak{s}\equiv(s_i\equiv(s_{ij})_{j\in S})_{i=1}^k\in(\mathbb{Z}^S)^k,$$

and with the marked nodal domain $(\Sigma,\mathfrak{j},\vec{z})=\bigcup_{v\in\mathbb{V}}(\Sigma_v,\mathfrak{j}_v,\vec{z}_v)$, is collection of tuples

$$u_{\log}\equiv\big((u_v\colon\Sigma_v\longrightarrow D_{I_v}),([\zeta_{v,i}])_{i\in I_v}\big)_{v\in\mathbb{V}}$$

over smooth components of Σ such that

- $u\equiv(u_v)_{v\in\mathbb{V}}\colon(\Sigma,\mathfrak{j},\vec{z})\longrightarrow(X,J)$ is a k-marked J-holomorphic nodal map in the classical sense,
- for each $v\in\mathbb{V}$, $I_v\subset S$ is the maximal subset such that $\mathrm{Im}(u_v)\subset D_{I_v}$,
- for each $v\in\mathbb{V}$ and any $i\in I_v$, $[\zeta_{v,i}]$ is a $\mathbb{C}^*$-equivalence class of non-trivial meromorphic sections of the holomorphic[9] line bundle $u^*\mathcal{N}_XD_i$,
- the contact order vectors in $\mathbb{Z}^S$, defined in (3.4) and (3.5), are the reverse of each other at the nodal points,
- every point in Σ with a non-trivial contact vector is either a marked point or a nodal point, and the contact order vector at z^i is $s_i\in\mathbb{Z}^S$,

[8] Such as the log tangent bundle defined in [11].

[9] Since $\dim_{\mathbb{C}}\Sigma_v=1$, the pull-back line bundle $u_v^*\mathcal{N}_XD_i$ is holomorphic.

- there exists a function $s\colon \mathbb{V}\longrightarrow \mathbb{R}^S$ such that $s_v=s(v)\in \mathbb{R}_+^{I_v}\times\{0\}^{S-I_v}$ for all $v\in\mathbb{V}$, and $s_v-s_{v'}$ is a positive multiple of the contact order vector of any nodal point on Σ_v connected to $\Sigma_{v'}$, for all $v,v'\in\mathbb{V}$,

- and, certain group (a complex torus) element associated to $u_{\log}$, defined in (3.24), is equal to 1;

see Definition 3.8 for more details. Two marked log maps are equivalent if one is a "reparametrization" of the other. A marked log map is stable if it has a finite "automorphism group". For $g,k\in\mathbb{N}$, $A\in H_2(X,\mathbb{Z})$, and $\mathfrak{s}\in(\mathbb{Z}^S)^k$, we denote the space of equivalence classes of stable k-marked degree A genus g log maps of contact type $\mathfrak{s}$ by $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$. Given $\mathfrak{s}\in(\mathbb{Z}^S)^k$, it turns out that for every k-marked stable nodal map f in $\overline{\mathcal{M}}_{g,k}(X,A)$, there exists at most finitely many elements $f_{\log}\in\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ (with distinct decorated dual graphs) lifting f; see Remarks 3.2 and 3.13. Furthermore, $f_{\log}$ is stable if and only if f is stable (and the automorphism groups are often the same).

Approaching ($\star$), we face some new problems that are not present in the case of the classical and relative stable maps. Unlike the smooth case, it is not a priori clear whether every SC symplectic divisor $D\subset(X,\omega)$ admits a compatible almost complex structure. Furthermore, even if $\mathcal{J}(X,D,\omega)\neq\emptyset$, it is not clear whether it is contractible (or even connected). In order to address this issue, in [9], we considered the space $\mathrm{Symp}^+(X,D)$ of all symplectic structures ω on X such that D is an SC symplectic divisor in (X,ω). If $\{D_i\}_{i\in S}$ is a transverse union of real codimension 2 submanifolds of X, we say $D=\bigcup_{i\in S}D_i$ is an SC symplectic divisor in X if $\mathrm{Symp}^+(X,D)=\emptyset$. We then defined a space of almost Kähler auxiliary data $\mathrm{AK}(X,D)$ consisting of tuples $(J,\mathcal{R},\omega)$ where $\omega\in\mathrm{Symp}^+(X,D)$, $\mathcal{R}$ is an "ω-regularization" for D in X, and J is an "$(\mathcal{R},\omega)$-compatible" almost complex structure on X; see [9, Page 8]. Roughly speaking, a regularization is a compatible set of symplectic identifications of neighborhoods of $\{D_I\}_{I\subset S}$ in their normal bundle with neighborhoods of them in X; [9, Definitio 2.12]. A regularization serves as a replacement for holomorphic defining equations in holomorphic manifolds. For every $(J,\mathcal{R},\omega)\in\mathrm{AK}(X,D)$, we have $J\in\mathcal{J}(X,D,\omega)$. Let

$$\mathcal{J}(X,D)=\big\{(J,\omega)\colon\,\omega\in\mathrm{Symp}^+(X,D),\;\,J\in\mathcal{J}(X,D,\omega)\big\}\,.$$

Therefore, $\mathrm{AK}(X,D)$ is essentially a nice subset of $\mathcal{J}(X,D)$ consisting of those almost complex structures that are of some specified type in a sufficiently small neighborhood of D. Furthermore, by [9, Theorem 2.13], the forgetful map

$$\mathrm{AK}(X,D)\longrightarrow\mathrm{Symp}^+(X,D),\qquad(J,\mathcal{R},\omega)\longrightarrow\omega,\tag{1.7}$$

is a weak homotopy equivalence. This implies that any invariant of the deformation equivalence classes in $\mathrm{AK}(X,D)$ is an invariant of the symplectic deformation equivalence class of (X,D,ω). In particular, by restricting to the subclass $\mathrm{AK}(X,D)$, the last statement in ($\star$) follows from constructing Kuranishi structures for families. Still, I expect the forgetful map $\mathcal{J}(X,D)\longrightarrow\mathrm{Symp}^+(X,D)$ to be a weak homotopy equivalence as well.

The main goal of the present article is to prove the following compactness result, addressing the first part of ($\star$). We will address the rest, in subsequent papers.

Theorem 1.1. *Assume $D\subset X$ is an SC symplectic divisor and $(J,\omega)\in\mathcal{J}(X,D)$. If further $(J,\mathcal{R},\omega)\in AK(X,D)$ for some regularization $\mathcal{R}$ or if J is integrable, then for every $A\in H_2(X,\mathbb{Z})$,*

$g,k\in\mathbb{N}$, and $\mathfrak{s}\in\big(\mathbb{Z}^S\big)^k$, the Gromov sequential convergence topology on $\overline{\mathcal{M}}_{g,k}(X,A)$ lifts to a compact sequential convergence topology on $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ such that the forgetful map

$$\iota\colon \overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)\longrightarrow \overline{\mathcal{M}}_{g,k}(X,A) \tag{1.8}$$

is a continuous local embedding. In particular, $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ is metrizable. If $g=0$, then (1.8) is a global embedding.

Remark 1.2. Except for the proof of Proposition 4.11, every other statement in the proof of Theorem 1.1 is stated and proved for arbitrary $(J,\omega)\in\mathcal{J}(X,D)$. We expect the local statement of Proposition 4.11, and thus Theorem 1.1, to be true for arbitrary $(J,\omega)\in\mathcal{J}(X,D)$. We hope to prove the stronger result in the near furture. If D is smooth, a significantly simpler version of Proposition 4.11 is sufficient for proving Theorem 4.10, and thus Theorem 1.1; see Remark 4.12. This simpler version can easily be proved for arbitrary $(J,\omega)\in\mathcal{J}(X,D)$. Nevertheless, by the argument around (1.7), the sub-class $\mathrm{AK}(X,D)$ is ideal for defining GW-type invariants and the holomorphic case is sufficient for most of interesting examples/calculations.

While $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ is defined for arbitrary $\mathfrak{s}\in(\mathbb{Z}^S)^k$ satisfying the second identity in (1.4), the resulting moduli spaces do not have some of the nice properties unless $\mathfrak{s}\in(\mathbb{N}^S)^k$; e.g. the (virtually) main stratum $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ would be empty if any of s_{ij} is negative. For $\mathfrak{s}\in(\mathbb{N}^S)^k$, by Lemma 3.20, the expected dimension of $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ is equal to (1.5), and the only stratum with the top expected dimension is $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)$.

If D is smooth, i.e. if $|S|=1$, we show in Proposition 3.24 that there is a surjective projection map

$$\overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathfrak{s}}(X,D,A)\longrightarrow\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A).$$

This is as expected, since our notion of log pseudoholomorphic map involves more $\mathbb{C}^*$-quotients on the set of meromorphic sections than in the relative case. In the algebraic case, [3, Theorem 1.1] shows that an algebraic analogue of this projection map induces an equivalence of the virtual fundamental classes. We expect the same to hold for invariants/VFCs arising from our log compactification.

Approaching the rest of $(\star)$, the transversality issue aside, log moduli spaces constructed in this paper are often virtually singular in the sense that the (virtual) normal cone of each stratum is not necessarily an orbibundle. More precisely, $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ admits a stratification

$$\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)=\bigcup_{\Gamma}\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma$$

where Γ runs over all the possible "decorated dual graphs"; see Definition 3.11. For any f in $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma$, the natural process of describing a neighborhood of f in $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ is by, first, describing a neighborhood U of f in $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ and then extending that by a "gluing" theorem of smoothing the nodes to a neighborhood of the form $U\times N'$ for f in $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$, where N' is a neighborhood of the origin in an affine sub-variety $N\subset\mathbb{C}^m$, for some $m\in\mathbb{N}$. In this situation, we say that N is the normal cone to $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ or it is the space of gluing parameters. In the case of classical stable maps, N is isomorphic to $\mathbb{C}^{\mathbb{E}}$ where $\mathbb{E}$ is the set of the edges of Γ

(or nodes of the nodal domain). Unlike in the classical case, in the case of log compactification, N could be reducible, and the normalization of N might be singular as well; see Example 3.21. Nevertheless, we show that N is (isomorphic to some finite copy of) an affine toric verity that can be explicitly described in terms of Γ. More precisely, in (3.17), associated to every such Γ, we construct a $\mathbb{Z}$-linear homomorphism of free $\mathbb{Z}$-modules

$$\varrho\colon \mathbb{D}(\Gamma) \longrightarrow \mathbb{T}(\Gamma) \tag{1.9}$$

such that N is isomorphic to (some finite copy of) the toric variety associated to a maximal convex rational polyhedral cone in $\mathrm{Ker}(\varrho) \otimes \mathbb{R}$. Moreover, the group element mentioned in the last bullet condition of Page 6 (i.e. in the definition of a log map) is an element of the Lie group $\mathcal{G}(\Gamma)$ with the Lie algebra $\mathrm{Coker}(\varrho) \otimes \mathbb{C}$. In other words, $\mathrm{Ker}(\varrho)$ gives the deformation space in the normal direction and $\mathrm{Coker}(\varrho)$ gives the obstruction for the smoothability of such maps.

1.5 Outline and acknowledgements

In Section 2, we review the construction of the relative compactification. The $\bar{\partial}$-operator $\bar{\partial}_{\mathcal{N}_X D}$ on $\mathcal{N}_X D$ described in Lemma 2.1 plays a key role in defining the basic building blocks of relative and log maps. In Section 3.1, we set up our notation for the decorated dual graph of nodal maps. The $\mathbb{Z}$-linear map (1.9) is defined in terms of such decorated dual graphs. In Section 3.2, we define the moduli spaces of log pseudoholomorphic maps and provide several examples to highlight their features. This is done in two steps: first, in Definition 3.3, we define a straightforward notion of pre-log map, then in Definition 3.8, we impose two non-trivial conditions on a such a pre-log map to define a log map. In the case of smooth divisors, we compare the relative and the log compactifications of the same combinatorial type in Section 3.3. In Section 3.5, we explicitly describe the space of gluing parameters of any fixed type Γ and identify it with an explicit affine toric variety. Proof of Theorem 1.1 relies on Gromov's compactness result for the underlying stable maps. In Section 4.1, we review the Gromov compactness theorem and set up our notation. In Section 4.2, we state a log enhancement of the Gromov compactness theorem. We prove the main result in Section 4.3. The main step of the proof is Proposition 4.11 that compares the limiting behavior of the rescaling and gluing parameters. We finish this paper with the proof of Proposition 4.11 in Section 4.4.

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2 Relative moduli spaces; review

In this section, we review the construction of the relative moduli spaces for smooth symplectic divisors in [17]. With a slight modification of the notation, we follow the description in [12].

Let (X, ω) be a smooth symplectic manifold and J be an ω-compatible almost complex structure on X. Let $\nabla^{\ell c}$ be the Levi-Civita connection of the metric $\omega(\cdot, J\cdot)$ and

$$\nabla_v\zeta=\nabla^{\ell c}_v\zeta-\frac{1}{2}J(\nabla^{\ell c}_v J)\zeta=\frac{1}{2}\big(\nabla^{\ell c}_v\zeta-J\nabla^{\ell c}_v(J\zeta)\big) \qquad \forall\, v\in TX,\ \zeta\in\Gamma(X,TX) \tag{2.1}$$

be the associated complex linear Hermitian connection; see [25, p41]. The Hermitian connection ∇ coincides with $\nabla^{\ell c}$ if and only if (X,ω,J) is Kähler, i.e. $\nabla^{\ell c}J\equiv 0$. The torsion T of ∇ is related to the Nijenhueis tensor by

$$T_\nabla(v,w)=-\frac{1}{4}N_J(v,w) \qquad \forall\, v,w\in TX, \tag{2.2}$$

where N_J is the Nijenhueis tensor of J; see [25, Section 2.1].

2.1 Almost complex structures and $\bar\partial$-operators

Suppose M is a smooth manifold, $\mathfrak{i}_M$ is an almost complex structure on M, and $(L,\mathfrak{i}_L)\longrightarrow M$ is a complex vector bundle. Let

$$\Omega^{1,0}_{M,\mathfrak{i}_M}\equiv\{\eta\in T^*M\otimes_{\mathbb{R}}\mathbb{C}\colon \eta\circ\mathfrak{i}_M=\mathfrak{i}\circ\eta\} \quad\text{and}\quad \Omega^{0,1}_{M,\mathfrak{i}_M}\equiv\{\eta\in T^*M\otimes_{\mathbb{R}}\mathbb{C}\colon \eta\circ\mathfrak{i}_M=-\mathfrak{i}\circ\eta\} \tag{2.3}$$

be the bundles of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear 1-forms on M, where $\mathfrak{i}$ is the unit imaginary number in $\mathbb{C}$. Given a smooth function $f\colon M\longrightarrow\mathbb{C}$, (2.3) gives a decomposition of $\mathrm{d}f$ into $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts ∂f and $\bar\partial f$, respectively. A $\bar\partial$-operator on $(L,\mathfrak{i}_L)$ is a complex linear operator

$$\bar\partial\colon \Gamma(M,L)\longrightarrow\Gamma(M,\Omega^{0,1}_{M,\mathfrak{i}_M}\otimes_{\mathbb{C}}L) \tag{2.4}$$

such that

$$\bar\partial(f\zeta)=\overline{\partial}f\otimes\zeta+f\bar\partial\zeta \qquad \forall f\in C^\infty(M,\mathbb{C}),\ \zeta\in\Gamma(M,L).$$

Given a complex linear connection ∇ on $(L,\mathfrak{i}_L)\longrightarrow(M,\mathfrak{i}_M)$, the $(0,1)$-part

$$\nabla^{(0,1)}=\frac{1}{2}(\nabla+\mathfrak{i}_L\nabla\circ\mathfrak{i}_M) \tag{2.5}$$

of ∇ is a $\bar\partial$-operator which we denote by $\bar\partial_\nabla$. Every $\bar\partial$-operator is the associated $\bar\partial$-operator of some $\mathbb{C}$-linear connection ∇ as above. The connection, however, is not unique. Every two connections ∇ and ∇' differ by a global 1-form α, i.e. $\nabla'=\nabla+\alpha$. If ∇ and ∇' are complex linear connections on $(L,\mathfrak{i}_L)$ with $\nabla'=\nabla+\alpha$, then

$$\bar\partial_{\nabla'}=\bar\partial_\nabla+\alpha^{(0,1)},$$

where $\alpha^{(0,1)}$ is the $(0,1)$-part of α in the decomposition (2.3). In particular, $\bar\partial_{\nabla'}=\bar\partial_\nabla$ whenever α is a $(1,0)$-form.

By [32, Lemma 2.2], corresponding to every $\bar\partial$-operator (2.4), there exists a unique almost complex structure $J=J_{\bar\partial}$ on the total space of L such that

(1) the projection $\pi\colon L\longrightarrow M$ is a $(\mathfrak{i}_M,J)$-holomorphic map (i.e. $\mathrm{d}\pi+\mathfrak{i}\,\mathrm{d}\pi\, J=0$),

(2) the restriction of J to the vertical tangent bundle $TL^{\mathrm{ver}}\cong\pi^*L\subset TL$ agrees with $\mathfrak{i}_L$,

(3) and the map $\zeta\colon M\longrightarrow L$ corresponding to a section $\zeta\in\Gamma(M,L)$ is $(J,\mathfrak{i}_M)$-holomorphic if and only if $\bar\partial\zeta=0$.

Suppose (X,ω) is a symplectic manifold, D is a symplectic submanifold, and J is an almost complex structure on X such that $J(TD)=TD$. The last condition implies that J induces a complex structure $\mathfrak{i}_{\mathcal{N}_XD}$ on (the fibers of) the normal bundle

$$\pi\colon \mathcal{N}_XD \equiv TX|_D/TD \longrightarrow D. \tag{2.6}$$

Under the isomorphism

$$\mathcal{N}_XD \cong TD^{\omega} = \{u\in TX|D\colon\ \omega(u,v)=0 \quad \forall\, v\in TD\},$$

$\mathfrak{i}_{\mathcal{N}_XD}$ is the same as the restriction to TD^{ω} of J. Let J_D denote the restriction of J to TD.

Lemma 2.1. *Suppose (X,ω) is a symplectic manifold, D is a symplectic submanifold, J is an almost complex structure on X such that $J(TD)=TD$, and ∇ is the complex linear connection associated to (ω,J) in (2.1). Then the $\bar\partial$-operator*

$$\bar\partial_\nabla \equiv \nabla^{(0,1)}\colon \Gamma\big(X,TX\big) \longrightarrow \Gamma\big(X,\Omega^{0,1}_{X,J}\otimes_{\mathbb{C}}TX\big)$$

in (2.5) descends to a $\bar\partial$-operator

$$\bar\partial_{\mathcal{N}_XD}\colon \Gamma\big(D,\mathcal{N}_XD\big) \longrightarrow \Gamma\big(D,\Omega^{0,1}_{D,J_D}\otimes_{\mathbb{C}}\mathcal{N}_XD\big) \tag{2.7}$$

on $(\mathcal{N}_XD,\mathfrak{i}_{\mathcal{N}_XD})\longrightarrow(D,J_D)$.

Proof. We need to show that $\bar\partial_\nabla$ maps $\Gamma\big(D,TD\big)$ to $\Gamma\big(D,\Omega^{0,1}_{D,J_D}\otimes_{\mathbb{C}}TD\big)$. Let $\nabla^{\ell c}$ and $\widetilde\nabla^{\ell c}$ be the Levi-Civita connections associated to the metrics $\omega(\cdot,J\cdot)$ and $\omega|_D(\cdot,J_D\cdot)$ on X and D, respectively. Then

$$\nabla^{\ell c}\zeta = \widetilde\nabla^{\ell c}\zeta + \widehat\nabla^{\ell c}\zeta \qquad \forall\,\zeta\in\Gamma(D,TD),$$

such that

$$\widehat\nabla^{\ell c}\zeta\in\Gamma\big(D,\Omega^1_D\otimes TD^{\omega}\big) \qquad \forall\,\zeta\in\Gamma(D,TD).$$

Similarly, let ∇ and $\widetilde\nabla$ be the complex linear connections on TX and TD associated to $\nabla^{\ell c}$ and $\widetilde\nabla^{\ell c}$ as in (2.1), respectively. It follows from (2.1) that

$$\nabla\zeta = \widetilde\nabla\zeta + \widehat\nabla\zeta \qquad \forall\,\zeta\in\Gamma(D,TD), \tag{2.8}$$

where

$$\widehat\nabla\zeta=\frac{1}{2}\big(\widehat\nabla^{\ell c}\zeta - J\widehat\nabla^{\ell c}(J\zeta)\big)\in\Gamma\big(D,\Omega^1_D\otimes_{\mathbb{C}}TD^{\omega}\big) \qquad \forall\,\zeta\in\Gamma(D,TD).$$

From (2.2), (2.8), and

$$N_J(u,v)=N_{J_D}(u,v)\in TD \qquad \forall\,x\in D,\ u,v\in T_xD,$$

we conclude that

$$\begin{aligned}\widehat\nabla_\xi\zeta-\widehat\nabla_\zeta\xi &= \big(\nabla_\xi\zeta-\nabla_\zeta\xi\big)-\big(\widetilde\nabla_\xi\zeta-\widetilde\nabla_\zeta\xi\big) = \\ &\big(\nabla_\xi\zeta-\nabla_\zeta\xi-[\zeta,\xi]\big)-\big(\widetilde\nabla_\xi\zeta-\widetilde\nabla_\zeta\xi-[\zeta,\xi]\big) = T_\nabla(\zeta,\xi)-T_{\widetilde\nabla}(\zeta,\xi)=0;\end{aligned}$$

i.e.

$$\widehat\nabla_\xi\zeta = \widehat\nabla_\zeta\xi \qquad \forall\zeta,\xi\in\Gamma(D,TD).$$

From the last identity we get

$$\widehat{\nabla}_\xi\zeta + J\widehat{\nabla}_{J\xi}\zeta = \widehat{\nabla}_\zeta\xi + J\widehat{\nabla}_\zeta J\xi = \widehat{\nabla}_\zeta\xi - \widehat{\nabla}_\zeta\xi = 0 \qquad \forall\, \zeta,\xi\in\Gamma(D,TD).$$

Therefore,

$$\begin{aligned}\nabla^{(0,1)}_\xi\zeta =& \frac{1}{2}\big(\nabla_\xi\zeta + J\nabla_{J\xi}\zeta\big) = \frac{1}{2}\big(\widehat{\nabla}_\xi\zeta + J\widehat{\nabla}_{J\xi}\zeta\big) + \frac{1}{2}\big(\widehat{\nabla}_\xi\zeta + J\widehat{\nabla}_{J\xi}\zeta\big) = \\ & \frac{1}{2}\big(\widehat{\nabla}_\xi\zeta + J\widehat{\nabla}_{J\xi}\zeta\big)\in\Gamma(D,TD) \qquad \forall\, \zeta,\xi\in\Gamma(D,TD).\end{aligned}$$

□

Suppose $D\subset(X,\omega)$ is a smooth symplectic divisor, J is an almost complex structure on X such that $J(TD)=TD$, and $\bar{\partial}_{\mathcal{N}_XD}$ is the $\bar{\partial}$-operator in Lemma 2.1. Choose a Hermitian connection $\widetilde{\nabla}$ on $(\mathcal{N}_XD, \mathfrak{i}_{\mathcal{N}_XD})$ such that $\bar{\partial}_{\mathcal{N}_XD}=\bar{\partial}_{\widetilde{\nabla}}$. The connection $\widetilde{\nabla}$ gives a splitting of the exact sequence

$$0\longrightarrow \pi^*\mathcal{N}_XD\longrightarrow T(\mathcal{N}_XD)\xrightarrow{\mathrm{d}\pi}\pi^*TD\longrightarrow 0 \tag{2.9}$$

of vector bundles over $\mathcal{N}_XD$ which restricts to the canonical splitting over the zero section and is preserved by the multiplication by $\mathbb{C}^*$; see [12, Section 4.1]. Let

$$\mathbb{P}_XD=\mathbb{P}(\mathcal{N}_XD\oplus D\times\mathbb{C}), \qquad D_0=\mathbb{P}(0\oplus D\times\mathbb{C}) \;\text{ and }\; D_\infty=\mathbb{P}(\mathcal{N}_XD\oplus 0)\subset\mathbb{P}_XD. \tag{2.10}$$

The splitting of (2.9) extends to a splitting of the exact sequence

$$0\longrightarrow T^{\mathrm{vrt}}(\mathbb{P}_XD)\longrightarrow T(\mathbb{P}_XD)\xrightarrow{\mathrm{d}\pi}\pi^*TD\longrightarrow 0,$$

where $\pi:\mathbb{P}_XD\longrightarrow D$ is the bundle projection map induced by (2.6); this splitting restricts to the canonical splittings over $D_0\cong D_\infty\cong D$ and is preserved by the multiplication by $\mathbb{C}^*$. Via this splitting, the almost complex structure J_D and the complex structure $\mathfrak{i}_{\mathcal{N}_XD}$ in the fibers of π induce an almost complex structure $J_{X,D}$ on $\mathbb{P}_XD$ which restricts to J_D on D_0 and D_∞ and is preserved by the $\mathbb{C}^*$-action. In fact, $J_{X,D}|_{\mathcal{N}_XD}$ is the almost complex structure $J_{\bar{\partial}_{\mathcal{N}_XD}}$ associated to $\bar{\partial}_{\mathcal{N}_XD}$ described in (1)-(3) above and is independent of choice of $\widetilde{\nabla}$. By (1), the projection $\pi:\mathbb{P}_XD\longrightarrow D$ is $(J_D,J_{X,D})$-holomorphic. By (3), there is a one-to-one correspondence between the space of $J_{X,D}$-holomorphic maps $u:(\Sigma,\mathfrak{j})\longrightarrow(\mathbb{P}_XD,J_{X,D})$ (not mapped into $D_{X,0}$ and $D_{X,\infty}$) and tuples (u_D,ζ) where $u_D:(\Sigma,\mathfrak{j})\longrightarrow(D,J_D)$ is a J_D-holomorphic map into D and ζ is a non-trivial meromorphic section of $u_D^*\mathcal{N}_XD$ with respect to the holomorphic structure defined by $u^*\bar{\partial}_{\mathcal{N}_XD}$.

2.2 Relative compactification

Let (X,ω) be a smooth symplectic manifold, $D\subset X$ be a smooth symplectic divisor, and $J\in\mathcal{J}(X,D,\omega)$. With notation as in (2.10), for each $m\in\mathbb{N}$, let

$$\begin{aligned} X[m]=\big(X\sqcup\{1\}\times\mathbb{P}_XD\sqcup\ldots\sqcup\{m\}\times\mathbb{P}_XD\big)/\!\sim, \qquad \text{where} \\ D\sim\{1\}\times D_\infty, \quad \{r\}\times D_0\sim\{r+1\}\times\mathbb{P}_\infty D \quad \forall\, r=1,\ldots,m-1;\end{aligned} \tag{2.11}$$

see Figure 1. This is a basic (i.e. there are no triple or higher intersections) smoothable SC variety which is smoothable to (a symplectic manifold deformation equivalent to) X itself. There exists a

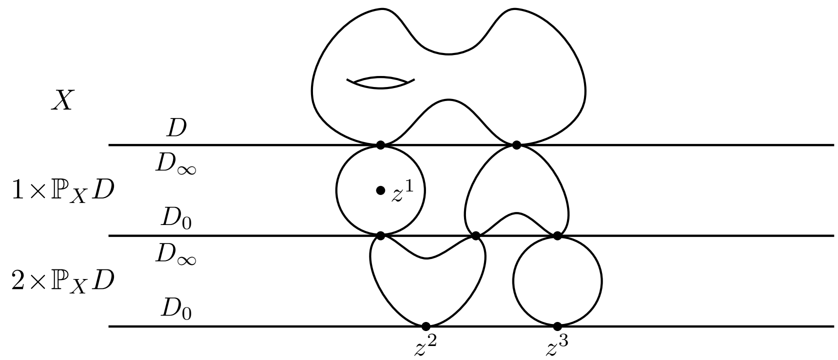

Figure 1: A relative map with $k=3$ and $\mathfrak{s}=(0,2,2)$ into the expanded degeneration $X[2]$.

continuous projection map $\pi_m\colon X[m]\longrightarrow X$ which is identity on X and π on each $\mathbb{P}_X D$. We denote by J_m the almost complex structure on $X[m]$ so that

$$J_m|_X = J_X \qquad \text{and} \qquad J_m|_{\{r\}\times\mathbb{P}_X D} = J_{X,D} \quad \forall\, r=1,\ldots,m.$$

For each $(c_1,\ldots,c_m)\in\mathbb{C}^*$, define

$$\Theta_{c_1,\ldots,c_m}\colon X[m]\longrightarrow X[m] \qquad \text{by} \quad \Theta_{c_1,\ldots,c_m}(x)=\begin{cases}(r,[c_r v,w]), & \text{if } x=(r,[v,w])\in\{r\}\times\mathbb{P}_X D;\\ x, & \text{if } x\in X.\end{cases} \tag{2.12}$$

This diffeomorphism is biholomorphic with respect to J_m and preserves the fibers of the projection $\mathbb{P}_X D\longrightarrow D$ and the sections D_0 and D_∞.

The moduli space of relative stable maps into (X,D) in [17, Section 7] is defined in the following way. With slight modifications, we follow the description in [12]. Suppose $k\in\mathbb{N}$, $A\in H_2(X,\mathbb{Z})$, and $\mathfrak{s}=(s_1,\ldots,s_k)\in\mathbb{N}^k$ is a tuple satisfying

$$\sum_{i=1}^{k} s_i = A\cdot D. \tag{2.13}$$

A level zero genus g k-marked degree A relative J-holomorphic map into X of contact type $\mathfrak{s}$ with D is simply a stable J-holomorphic map in $\overline{\mathcal{M}}_{g,k}(X,A)$ such that

$$u^{-1}(D)\subset\{z^1,\ldots,z^k\}, \quad \mathrm{ord}_{z^i}\big(u,D\big)=s_i \quad \forall i=1,\ldots,k. \tag{2.14}$$

For $m\in\mathbb{Z}_+$, a level m k-marked relative J-holomorphic map of contact type $\mathfrak{s}$ is a continuous map $u\colon\Sigma\longrightarrow X[m]$ from a marked connected nodal curve $\big(\Sigma,\mathfrak{j},\vec{z}=(z^1,\ldots,z^k)\big)$ such that

$$u^{-1}\big(\{m\}\times D_0\big)\subset\{z^1,\ldots,z^k\}, \quad \mathrm{ord}_{z^i}(u,\{m\}\times D_0)=s_i \quad \forall\, z^i\in u^{-1}\big(\{m\}\times D_0\big), \tag{2.15}$$

$s_i=0$ if and only if $z^i\notin u^{-1}\big(\{m\}\times D_0\big)$, and the restriction of u to each irreducible component Σ_j of Σ is either

(1) a J-holomorphic map to X such that the set $u|_{\Sigma_j}^{-1}(D)$ consists of the nodes joining Σ_j to irreducible components of Σ mapped to $\{1\}\times\mathbb{P}_X D$, or

(2) a $J_{X,D}$-holomorphic map to $\{r\}\times\mathbb{P}_X D$ for some $r=1,\ldots,m$ such that

(a) the set $u|_{\Sigma_j}^{-1}(\{r\}\times D_\infty)$ consists of the nodes $q_{j,i}$ joining Σ_j to irreducible components of Σ mapped to $\{r-1\}\times\mathbb{P}_X D$ if $r>1$ and to X if $r=1$ and

$$\mathrm{ord}_{q_{j,i}}(u,D_\infty)=\begin{cases}\mathrm{ord}_{q_{i,j}}(u,D_0), & \text{if } r>1,\\ \mathrm{ord}_{q_{i,j}}(u,D), & \text{if } r=1,\end{cases}$$

where $q_{i,j}\in\Sigma_{i,j}$ is the point identified with $q_{j,i}$,

(b) if $r<m$, the set $u|_{\Sigma_j}^{-1}(\{r\}\times D_0)$ consists of the nodes joining Σ_j to irreducible components of Σ mapped to $\{r+1\}\times\mathbb{P}_X D$;

see Figure 1. The genus and the degree of such a map $u\colon\Sigma\longrightarrow X[m]$ are the arithmetic genus of Σ and the homology class

$$A=\big[\pi_m\circ u\big]\in H_2(X,\mathbb{Z}). \tag{2.16}$$

Two tuples $(\Sigma_\alpha,\mathfrak{j}_\alpha,\vec{z}_\alpha,u_\alpha)$ and $(\Sigma_\beta,\mathfrak{j}_\beta,\vec{z}_\beta,u_\beta)$ as above are equivalent if there exist a biholomorphic map $\varphi\colon(\Sigma_\alpha,\mathfrak{j}_\alpha)\longrightarrow(\Sigma_\beta,\mathfrak{j}_\beta)$ and $c_1,\ldots,c_m\in\mathbb{C}^*$ so that

$$\varphi(z_\alpha^i)=z_\beta^i\quad\forall\, i=1,\ldots,k\quad\text{and}\quad u_\beta=\Theta_{c_1,\ldots,c_m}\circ u_\alpha\circ\varphi.$$

A tuple as above is stable if it has finitely many automorphisms (self-equivalences).

If $A\in H_2(X,\mathbb{Z})$, $g,k\in\mathbb{N}$, and $\mathfrak{s}=(s_1,\ldots,s_k)\in\mathbb{N}^k$ is a tuple satisfying (2.13), then the relative moduli space

$$\overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathfrak{s}}(X,D,A) \tag{2.17}$$

is the set of equivalence classes of such connected stable k-marked genus g degree A J-holomorphic maps into into $X[m]$ for any $m\in\mathbb{N}$. The latter space has a natural compact Hausdorff topology[10].

Remark 2.2. In (2.13), we are allowing s_i to be zero for some $i=1,\ldots,k$. A marked point z with contact order 0 has image away from D (or D_0, D_∞). Therefore, such points are ordinary marked points as in the classical moduli spaces of J-holomorphic maps. In the literature, marked points are usually divided into the classical part $(z^1,\ldots,z^k)$ and the relative part $(z^{k+1},\ldots,z^{k+\ell})$ such that $s_i=\mathrm{ord}_{z^{k+i}}(u,D)>0$ and $\sum_{i=1}^{\ell}s_i=A\cdot D$. Then the moduli space (2.17) is denoted by $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,k,\mathfrak{s}}(X,D,A)$ with $\mathfrak{s}\in(\mathbb{Z}_+)^\ell$. This sort of separation works fine in the relative case, because there are only two types of points: in D or away from D. In the general SC case $D=\bigcup_{i\in S}D_i$, however, there are 2^S types of points and it is notationally cumbersome (and useless) to divide points into separate groups based on their type.

Remark 2.3. Let (X,ω) be a smooth symplectic manifold, $D\subset X$ be a smooth symplectic divisor, and J be an ω-compatible almost complex structure on X such that $J(TD)=TD$. If $u\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J)$ is J-holomorphic, the linearization of the Cauchy-Riemann operator (1.1) at u is given by

$$\begin{aligned}D_u\bar\partial\colon\Gamma\big(\Sigma;u^*TX\big)&\longrightarrow\Gamma\big(\Sigma,\Omega^{0,1}_{\Sigma,\mathfrak{j}}\otimes_{\mathbb{C}}u^*TX\big),\\ D_u\bar\partial(\xi)&=u^*\bar\partial_\nabla+\frac{1}{4}N_J(\xi,\mathrm{d}u),\end{aligned} \tag{2.18}$$

where ∇ is the complex linear connection in (2.1) and $\bar\partial_\nabla$ is the associated $\bar\partial$-operator on $\Gamma(X,TX)$ given by (2.4); see [25, Chapter 3.1]. The kernel of $D_u\bar\partial$ corresponds to infinitesimal deformations

[10] The proof of which is not complete in [17].

of u (over the fixed domain $(\Sigma,\mathfrak{j})$) and the cokernel of that is the obstruction space for integrating infinitesimal deformations to actual deformations.

If furthermore $\mathrm{Im}(u)\subset D$, then the linearization map $D_u\bar\partial$, defined in (2.18), satisfies

$$D_u\bar\partial\big(\Gamma(\Sigma,u^*TD)\big)\subset\Gamma\big(\Sigma,\Omega^{0,1}_{\Sigma,\mathfrak{j}}\otimes_{\mathbb{C}}u^*TD\big), \tag{2.19}$$

because the restriction of $D_u\bar\partial$ to $\Gamma(\Sigma,u^*TD)$ is the linearization[11] of the $\bar\partial$-operator at u for the space of maps into D. Thus, $D_u\bar\partial$ descends to a first-order differential operator

$$D_u^{\mathcal{N}_XD}\bar\partial\colon\Gamma(\Sigma,u^*\mathcal{N}_XD)\longrightarrow\Gamma\big(\Sigma,\Omega^{0,1}_{\Sigma,\mathfrak{j}}\otimes_{\mathbb{C}}u^*\mathcal{N}_XD\big). \tag{2.20}$$

If $J\in\mathcal{J}(X,D,\omega)$, i.e. (1.2) holds, then the normal part of $N_J(*,\mathrm{d}u)$ vanishes. From (2.18) and Lemma 2.1 we conclude that

$$D_u^{\mathcal{N}_XD}\bar\partial=u^*\bar\partial_{\mathcal{N}_XD} \tag{2.21}$$

is a complex linear operator. A priori, the Nijenhueis condition (1.2), and thus the equality (2.21), is not need for defining relative/log maps. The $\bar\partial$-operator $\bar\partial_{\mathcal{N}_XD}$ is defined for any J that preserves TD and is complex linear. However, we use (1.2) in the proof of compactness; this condition implies that certain sequence of almost complex structures on the normal bundle $\mathcal{N}_XD$ converges to $J_{X,D}$, see Lemma 4.5. One may alternatively show that the limiting sections are in the kernel of $D_u^{\mathcal{N}_XD}\bar\partial$ and use (2.21) to show that kernel of $D_u^{\mathcal{N}_XD}\bar\partial$ is complex-linear.

3 Log pseudoholomorphic maps

In this section, we introduce the moduli spaces of log pseudoholomorphic maps relative to SC symplectic divisors defined in [9]. This is done by first introducing a notion of pre-log pseudoholomorphic map which only involves a matching condition of contact orders at the nodes. We then define a $\mathbb{Z}$-linear map between certain $\mathbb{Z}$-modules associated to the dual graph of such a pre-log map which encodes the essential deformation/obstruction data for defining and studying log maps.

3.1 Decorated dual graphs

Let $\Gamma=\Gamma(\mathbb{V},\mathbb{E},\mathbb{L})$ be a graph with the set of vertices $\mathbb{V}$, edges $\mathbb{E}$, and legs $\mathbb{L}$; the latter, also called flags or roots, are half edges that have a vertex at one end and are open at the other end. Let $\underrightarrow{\mathbb{E}}$ be the set of edges with an orientation. Given $\underrightarrow{e}\in\underrightarrow{\mathbb{E}}$, let $\underleftarrow{e}$ denote the same edge with the opposite orientation. For each $\underrightarrow{e}\in\underrightarrow{\mathbb{E}}$, let $v_1(\underrightarrow{e})$ and $v_2(\underrightarrow{e})$ in $\mathbb{V}$ denote the starting and ending points of the arrow, respectively. For $v,v'\in\mathbb{V}$, let $\mathbb{E}_{v,v'}$ denote the subset of edges between the two vertex and $\underrightarrow{\mathbb{E}}_{v,v'}$ denote the subset of oriented edges from v to v'. For every $v\in\mathbb{V}$, let $\underrightarrow{\mathbb{E}}_v$ denote the subset of oriented edges starting from v.

A genus labeling of Γ is a function $g\colon\mathbb{V}\longrightarrow\mathbb{N}$. An ordering of the legs of Γ is a bijection $o\colon\mathbb{L}\longrightarrow\{1,\ldots,|\mathbb{L}|\}$. If a decorated graph Γ is connected, the arithmetic genus of Γ is

$$g=g_\Gamma=\sum_{v\in\mathbb{V}}g_v+\mathrm{rank}\ H_1(\Gamma,\mathbb{Z}), \tag{3.1}$$

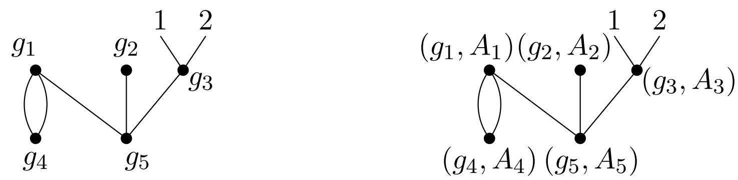

Figure 2: On left, a labeled graph Γ representing elements of $\overline{\mathcal{M}}_{g,2}$. On right, a labeled graph Γ representing elements of $\overline{\mathcal{M}}_{g,2}(X,A)$.

where $H_1(\Gamma,\mathbb{Z})$ is the first homology group of the underlying topological space of Γ. Figure 2-left illustrates a labeled graph with 2 legs.

Such decorated graphs Γ characterize different topological types of nodal marked surfaces

$$(\Sigma, \vec{z}=(z^1,\ldots,z^k))$$

in the following way. Each vertex $v\in\mathbb{V}$ corresponds to a smooth[12] component Σ_v of Σ with genus g_v. Each edge $e\in\mathbb{E}$ corresponds to a node q_e obtained by connecting Σ_v and $\Sigma_{v'}$ at the points $q_{\underset{\rightarrow}{e}}\in\Sigma_v$ and $q_{\underset{\leftarrow}{e}}\in\Sigma_{v'}$, where $e\in\mathbb{E}_{v,v'}$ and $\underset{\rightarrow}{e}$ is an orientation on e with $v_1(\underset{\rightarrow}{e})=v$. The last condition uniquely specifies $\underset{\rightarrow}{e}$ unless e is a loop connecting v to itself. Finally, each leg $l\in\mathbb{L}$ connected to the vertex $v(l)$ corresponds to a marked point $z^{o(l)}\in\Sigma_{v(l)}$ disjoint from the connecting nodes. If Σ is connected, then g_Γ is the arithmetic genus of Σ. Thus we have

$$(\Sigma,\vec{z})=\coprod_{v\in\mathbb{V}}(\Sigma_v,\vec{z}_v,q_v)/\sim, \quad q_{\underset{\rightarrow}{e}}\sim q_{\underset{\leftarrow}{e}} \quad \forall\, e\in\mathbb{E}, \tag{3.2}$$

where

$$\vec{z}_v=\vec{z}\cap\Sigma_v \quad \text{and} \qquad q_v=\{q_{\underset{\rightarrow}{e}}\colon \underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}_v\} \qquad \forall\, v\in\mathbb{V}.$$

We treat q_v as an un-ordered set of marked points on Σ_v. If we fix an ordering on the set q_v, we denote the ordered set by $\vec{q}_v$. In this situation, we say Γ is the dual graph of $(\Sigma,\vec{z})$.

A complex structure $\mathfrak{j}$ on Σ is a set of complex structures $(\mathfrak{j}_v)_{v\in\mathbb{V}}$ on its components. By a (complex) marked nodal curve, we mean a marked nodal real surface together with a complex structure $(\Sigma,\mathfrak{j},\vec{z})$. Figure 3 illustrates a nodal curve with $(g_1,g_2,g_3,g_4,g_5)=(0,2,0,1,0)$ corresponding to Figure 2-left.

Similarly, for nodal marked surfaces mapping into a topological space X, we consider similar decorated graphs where the vertices carry an additional degree labeling

$$A\colon \mathbb{V}\longrightarrow H_2(X,\mathbb{Z}), \quad v\longrightarrow A_v,$$

recording the homology class of the image of the corresponding component. Figure 2-right illustrates a dual graph associated to a marked nodal map over the graph on the left.

[11] The linearization of (1.1) is independent of the choice of the connection at every J-holomorphic map.
[12] We mean a smooth closed oriented surface.

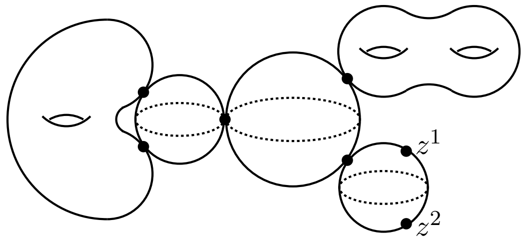

Figure 3: A nodal curve in $\overline{\mathcal{M}}_{4,2}$.

Assume $D=\bigcup_{i\in S} D_i\subset X$ is an SC symplectic divisor, $(J,\omega)\in\mathcal{J}(X,D)$, and $(\Sigma,\mathfrak{j})$ is a connected smooth complex curve. Then every J-holomorphic map $u\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J)$ has a well-defined depth $I\subset S$, which is the maximal subset such that $\mathrm{Image}(u)\subset D_I$. In particular, any map u intersecting D in a discrete set is of depth $I=\emptyset$. We say a point $w\in\Sigma$ is of depth I, if I is the maximal subset where $u(w)\in D_I$. Let $\mathcal{P}(S)$ be the set of subsets of S. In this situation, the dual graph of (u,Σ) carries additional labelings

$$I\colon \mathbb{V},\mathbb{E}\longrightarrow\mathcal{P}(S), \qquad v\longrightarrow I_v \quad \forall v\in\mathbb{V}, \qquad e\longrightarrow I_e \quad \forall e\in\mathbb{E} \tag{3.3}$$

recording the depths of smooth components and nodes of Σ.

3.2 Log moduli spaces

Assume $D=\bigcup_{i\in S} D_i\subset X$ is an SC symplectic divisor, $(J,\omega)\in\mathcal{J}(X,D)$, and $u\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J)$ is a J-holomorphic map of depth $I\subset S$ with smooth domain. Then, for every $i\in S-I$, the function

$$\mathrm{ord}_u^i\colon \Sigma\longrightarrow\mathbb{N}, \quad \mathrm{ord}_u^i(x)=\mathrm{ord}_x(u,D_i), \tag{3.4}$$

recording the contact order of u with D_i at x is well-defined. For every $i\in I$, let $u^*\bar\partial_{\mathcal{N}_X D_i}$ be the pull-back of $\bar\partial$-operator $\bar\partial_{\mathcal{N}_X D_i}$ associated (J,D_i) in (2.7). Since every $\bar\partial$-operator over a complex curve is integrable, $u^*\bar\partial_{\mathcal{N}_X D_i}$ defines a holomorphic structure on $u^*\mathcal{N}_X D_i$; see [25, Remark C.1.1]. The holomorphic line bundles

$$\big(u^*\mathcal{N}_X D_i, u^*\bar\partial_{\mathcal{N}_X D_i}\big) \qquad \forall\ i\in I$$

play a key role in definition of the log moduli space below. Let $\Gamma_{\mathrm{mero}}(\Sigma,u^*\mathcal{N}_X D_i)$ be the space of non-trivial meromorphic sections of $u^*\mathcal{N}_X D_i$ with respect to $u^*\bar\partial_{\mathcal{N}_X D_i}$. For every

$$\zeta\in\Gamma_{\mathrm{mero}}(\Sigma,u^*\mathcal{N}_X D_i)$$

the function

$$\mathrm{ord}_\zeta\colon\Sigma\longrightarrow\mathbb{Z}, \quad \mathrm{ord}_\zeta(x)=\mathrm{ord}_x(\zeta), \tag{3.5}$$

recording the vanishing order of ζ at x (which is negative if ζ has a pole at x) is well-defined and does not change if we replace ζ with a non-zero constant multiple of that.

Definition 3.1. Let $D=\bigcup_{i\in S} D_i\subset X$ be an SC symplectic divisor and $(\omega,J)\in\mathcal{J}(X,D)$. A log J-holomorphic tuple $(u,[\zeta],\Sigma,\mathfrak{j},w)$ consists of a smooth (closed) connected curve $(\Sigma,\mathfrak{j})$, ℓ distinct points $w=\{w^1,\ldots,w^\ell\}$ on Σ, a $(J,\mathfrak{j})$-holomorphic map $u\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J)$ of depth $I\subset S$, and

$$[\zeta]\equiv\big([\zeta_i]\big)_{i\in I}\in\bigoplus_{i\in I}\big(\Gamma_{\mathrm{mero}}(\Sigma,u^*\mathcal{N}_X D_i)/\mathbb{C}^*\big) \tag{3.6}$$

such that

$$\mathrm{ord}_{u,\zeta}(x)\neq 0 \quad\Rightarrow\quad x\in w \tag{3.7}$$

where the vector-valued order function

$$\mathrm{ord}_{u,\zeta}(x)=\big((\mathrm{ord}^i_u(x))_{i\in S-I},(\mathrm{ord}_{\zeta_i}(x))_{i\in I}\big)\in\mathbb{Z}^S \qquad \forall\; x\in\Sigma$$

is defined via (3.4) and (3.5).

In particular, if u is of degree $A\in H_2(X,\mathbb{Z})$, then (3.7) implies

$$(A\cdot D_i)_{i\in S}=\sum_{w^i\in w}\mathrm{ord}_{u,\zeta}(w^i)\in\mathbb{Z}^S. \tag{3.8}$$

Remark 3.2. For every J-holomorphic map $u\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J)$ with smooth domain, ℓ distinct points $w^1,\ldots,w^\ell$ in Σ, and $s_1,\ldots,s_\ell\in\mathbb{Z}$, if $\mathrm{Im}(u)\subset D_i$, up to $\mathbb{C}^*$-action there exists at most one meromorphic section $\zeta_i\in\Gamma_{\mathrm{mero}}(\Sigma,u^*\mathcal{N}_XD_i)$ with zeros/poles of order s_i at w^i (and nowhere else).

Definition 3.3. Let $D=\bigcup_{i\in S}D_i\subset X$ be an SC symplectic divisor, $(J,\omega)\in\mathcal{J}(X,D)$, and

$$C\equiv(\Sigma,\mathfrak{j},\vec{z})=\coprod_{v\in\mathbb{V}}C_v\equiv(\Sigma_v,\mathfrak{j}_v,\vec{z}_v,q_v)/\sim,\quad q_{\underset{\rightarrow}{e}}\sim q_{\underset{\leftarrow}{e}}\quad\forall\;\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}},$$

be a k-marked connected nodal curve with smooth components C_v and dual graph $\Gamma=\Gamma(\mathbb{V},\mathbb{E},\mathbb{L})$ as in (3.2). A pre-log J-holomorphic map of contact type $\mathfrak{s}=(s_i)_{i=1}^k\in\big(\mathbb{Z}^S\big)^k$ from C to X is a collection

$$f\equiv\big(f_v\equiv(u_v,[\zeta_v],C_v)\big)_{v\in\mathbb{V}} \tag{3.9}$$

such that

(1) for each $v\in\mathbb{V}$, $(u_v,[\zeta_v]=([\zeta_{v,i}])_{i\in I_v},\Sigma_v,\mathfrak{j}_v,z_v\cup q_v)$ is a log J-holomorphic tuple as in Definition 3.1,

(2) $u_v(q_{\underset{\rightarrow}{e}})=u_{v'}(q_{\underset{\leftarrow}{e}})\in X$ for all $\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}_{v,v'}$;

(3) $s_{\underset{\rightarrow}{e}}\equiv\mathrm{ord}_{u_v,\zeta_v}(q_{\underset{\rightarrow}{e}})=-\mathrm{ord}_{u_{v'},\zeta_{v'}}(q_{\underset{\leftarrow}{e}})\equiv-s_{\underset{\leftarrow}{e}}$ for all $v,v'\in\mathbb{V}$ and $\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}_{v,v'}$;

(4) $\mathrm{ord}_{u_v,\zeta_v}(z^i)=s_i$ for all $v\in\mathbb{V}$ and $z^i\in z_v$.

Remark 3.4. For every $v\in\mathbb{V}$ and $\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}_v$, let

$$s_{\underset{\rightarrow}{e}}=(s_{\underset{\rightarrow}{e},i})_{i\in S}=\big((\mathrm{ord}^i_{u_v}(q_{\underset{\rightarrow}{e}}))_{i\in S-I_v},(\mathrm{ord}_{\zeta_{v,i}}(q_{\underset{\rightarrow}{e}}))_{i\in I_v}\big)\in\mathbb{Z}^S. \tag{3.10}$$

For $\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}_{v,v'}$, if u_v and $u_{v'}$ have image in D_{I_v} and $D_{I_{v'}}$, respectively, by Condition (2), we have

$$u(q_e)=u_v(q_{\underset{\rightarrow}{e}})=u_{v'}(q_{\underset{\leftarrow}{e}})\in D_{I_v}\cap D_{I_{v'}}=D_{I_v\cup I_{v'}};$$

i.e. $I_e\supset I_v\cup I_{v'}$. If $i\in S\backslash I_v\cup I_{v'}$, by (3.4), we have

$$s_{\underset{\rightarrow}{e},i},\,s_{\underset{\leftarrow}{e},i}\geq 0.$$

Therefore, by Condition (3), they are both zero, i.e.

$$I_e=I_v\cup I_{v'}\quad\text{and}\quad s_{\underset{\rightarrow}{e}}\in\mathbb{Z}^{I_e}\times\{0\}^{S-I_e}\subset\mathbb{Z}^S\qquad\forall\;\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}_{v,v'}. \tag{3.11}$$

The dual graph Γ of every pre-log map in Definition 3.8 carries an additional decoration $s_{\underset{\rightarrow}{e}} \in \mathbb{Z}^S$, for all $\underset{\rightarrow}{e} \in \underset{\rightarrow}{\mathbb{E}}$, which records the contact order of $(u_v, [\zeta_v])$ at $q_{\underset{\rightarrow}{e}}$ for every $\underset{\rightarrow}{e} \in \underset{\rightarrow}{\mathbb{E}}_v$; see Figure 4. The set $\mathbb{L}$ of legs of Γ is also decorated with the contact order function

$$\mathrm{ord}\colon \mathbb{L} \longrightarrow \mathbb{Z}^S, \quad l \longrightarrow s_l,$$

recording the contact vector at the marked point $z^{o(l)}$ corresponding to l.

Two pre-log maps $(u, [\zeta], C) \equiv \big(u_v, [\zeta_v], C_v\big)_{v \in \mathbb{V}}$ and $(\widetilde{u}, [\widetilde{\zeta}], \widetilde{C}) \equiv \big(\widetilde{u}_v, [\widetilde{\zeta}_v], \widetilde{C}_v\big)_{v \in \mathbb{V}}$ with isomorphic decorated dual graphs Γ as in Definition 3.3 are equivalent if there exists a biholomorphic identification

$$(h\colon C \longrightarrow \widetilde{C}) \equiv \big(h_v\colon (\widetilde{\Sigma}_v, \widetilde{\mathfrak{j}}_v) \longrightarrow (\Sigma_{h(v)}, \mathfrak{j}_{h(v)})\big)_{v \in \mathbb{V}} \tag{3.12}$$

such that

$$h(\widetilde{z}^i) = z^i \quad \forall i = 1, \ldots, k, \quad u \circ h = \widetilde{u}, \quad [h_v^* \zeta_{h(v),i}] = [\widetilde{\zeta}_{v,i}] \quad \forall\, v \in \mathbb{V},\ i \in I_v.$$

A pre-log map f is stable if the group of self-equivalences $\mathfrak{Aut}(f)$ is finite. By Remark 3.2, a pre-log map is stable if and only if the underlying nodal marked J-holomorphic map is stable. Clearly, the automorphism group of a pre-log map is a subgroup of the automorphism group of the underlying nodal marked J-holomorphic map. Example 3.16 below illustrates some rare cases when the two groups are different. For every such Γ, we denote the space of equivalence classes of k-marked degree A pre-log J-holomorphic maps with dual graph Γ and contact pattern $\mathfrak{s}$ by

$$\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X, D, A)_\Gamma. \tag{3.13}$$

If Γ has only one vertex v with $I = I_v$ then

$$\mathcal{M}^{\log}_{g,\mathfrak{s}}(X, D, A)_I \equiv \mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X, D, A)_\Gamma$$

is the space of equivalence classes of genus g degree A k-marked log J-holomorphic tuples with an ordering on the marked points and contact pattern $\mathfrak{s}$.

In $g = 0$, the forgetful map

$$\mathcal{M}^{\log}_{0,\mathfrak{s}}(X, D, A)_I \longrightarrow \mathcal{M}_{0,k}(D_I, A), \quad [u, [\zeta], \Sigma, \mathfrak{j}, \vec{z}] \longrightarrow [u, \Sigma, \mathfrak{j}, \vec{z}] \tag{3.14}$$

into the (virtually) main stratum of moduli space of k-marked degree A J-holomorphic maps into D_I gives an identification of two sets. In the higher genus case, however, the (virtual) normal bundle of this embedding is the direct sum of I copies of dual of Hodge bundle (i.e. tangent space of $\mathrm{Pic}^0(\Sigma)$ at the trivial line bundle). In [7], we will use this description to define a natural class of Kuranishi charts around log maps with smooth domain.

In [7], we will study the deformation theory of $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X, D, A)_I$ and show that it admits a class of natural Kuranishi charts of the expected real dimension

$$\begin{aligned} \dim_{\mathrm{vir}} \mathcal{M}^{\log}_{g,\mathfrak{s}}(X, D, A)_I =& 2\big(c_1^{TX}(A) + (\dim_{\mathbb{C}} X - 3)(1-g) + k - |I| - A \cdot D\big) = \\ & 2\big(c_1^{TX(-\log D)}(A) + (\dim_{\mathbb{C}} X - 3)(1-g) + k - |I|\big), \end{aligned} \tag{3.15}$$

where $TX(-\log D)$ is the logarithmic tangent bundle defined in [11].

Example 3.5. If D is smooth, i.e. $|S|=1$, a (pre-)log map with smooth domain of depth $\emptyset$ is just a J-holomorphic map u with image not into D, $u^{-1}(D)\subset\vec{z}$, and

$$\mathfrak{s}=\big(\mathrm{ord}_{z^i}(u,D)\big)_{i=1}^k\in\mathbb{N}^k$$

as in the definition of the relative moduli spaces in (2.14). Thus there exists a one-to-one correspondence between the virtually main stratum of the moduli space of relative J-holomorphic maps of contact order $\mathfrak{s}$, and the space of depth $\emptyset$ (pre-)log maps of the same contact pattern. Also, a depth $\{1\}$ (pre-)log J-holomorphic map with smooth domain is made of a J-holomorphic map $u\colon(\Sigma,\mathfrak{j})\longrightarrow(D,J|_{TD})$ and a meromorphic section ζ of $u^*\mathcal{N}_XD$ such that $\vec{z}$ includes the set of zeros and poles of ζ and

$$\mathfrak{s}=\big(\mathrm{ord}_{z^i}(\zeta)\big)_{i=1}^k\in\mathbb{Z}^k$$

as in the definition of the relative moduli spaces. The definitions, however, become different if we consider maps with nodal domain.

For some decorated dual graphs Γ, the expected dimension of $\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma$, calculated via (3.15) and the matching conditions at the nodes, could be bigger than or equal to the expected dimension of the (virtually) main stratum $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)=\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\emptyset$ (something that we do not want to happen); see the following example. In order for a nodal pre-log map to be in the limit of the (virtually) main stratum, there are other global combinatorial and non-combinatorial obstructions that we are going to describe next.

Example 3.6. Let $X=\mathbb{P}^2$ with projective coordinates $[x_1,x_2,x_3]$ and $D=D_1\cup D_2$ be a transverse union of two hyperplanes (lines). For

$$g=0,\quad \mathfrak{s}=((3,2),(0,1))\in(\mathbb{N}^{\{1,2\}})^2,\quad\text{and}\quad A=[3]\in H_2(X,\mathbb{Z})\cong\mathbb{Z},$$

$\mathcal{M}_{0,\mathfrak{s}}(X,D,[3])$ is a manifold of complex dimension 4. If $D_1=(x_1=0)$ and $D_2=(x_2=0)$, every element in $\mathcal{M}_{0,\mathfrak{s}}(X,D,[3])$ is equivalent to a holomorphic map of the form

$$[z,w]\longrightarrow[z^3,z^2w,a_3z^3+a_2z^2w+a_1zw^2+a_0w^3].\tag{3.16}$$

Let Γ be the dual graph with three vertices v_1,v_2,v_3, and two edges e_1,e_2 connecting v_1 to v_3 and v_2 to v_3, respectively. Furthermore, choose the orientations $\underrightarrow{e}_1$ and $\underrightarrow{e}_2$ to end at v_3 and assume

$$I_{v_1}=I_{v_2}=\emptyset,\quad I_{v_3}=\{1,2\},\quad s_{\underrightarrow{e}_1}=(2,1),\quad s_{\underrightarrow{e}_2}=(1,1),\quad A_{v_1}=[2],\quad A_{v_2}=[1];$$

see Figure 4. Note that u_{v_3} is map of degree 0 from a sphere with three special points, two of which are the nodes connecting Σ_{v_3} to Σ_{v_1} and Σ_{v_2} and the other one is the first marked point z^1 with contact order $(3,2)$. The second marked point with contact order $(0,1)$ lies on Σ_{v_1}. A simple calculation shows that $\mathcal{M}^{\mathrm{plog}}_{0,\mathfrak{s}}(X,D,[3])_\Gamma$ is also a manifold of complex dimension 4. Image of u_2 could be any line different from D_1 and D_2 passing through D_{12}, and every such u_1 is equivalent to a holomorphic map of the form

$$[z,w]\longrightarrow[z^2,zw,a_2z^2+a_1zw+a_0w^2].$$

$\square$

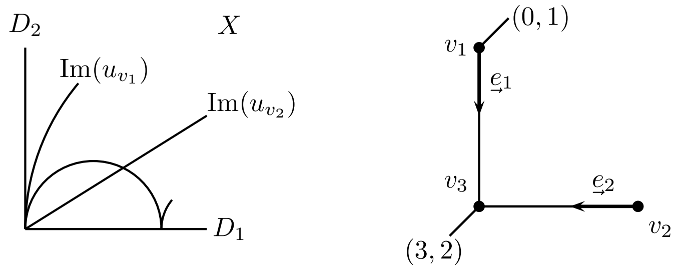

Figure 4: A 2-marked genus 0 degree 3 pre-log map in $\mathbb{P}^3$ relative to two lines.

Corresponding to the decorated dual graph $\Gamma=\Gamma(\mathbb{V},\mathbb{E},\mathbb{L})$ of a pre-log map as in Definition 3.3 and an arbitrary orientation $O\equiv\{\underset{\to}{e}\}_{e\in\mathbb{E}}\subset\underset{\to}{\mathbb{E}}$ on the edges, we define a homomorphism of $\mathbb{Z}$-modules

$$\mathbb{D}=\mathbb{D}(\Gamma)\equiv\mathbb{Z}^{\mathbb{E}}\oplus\bigoplus_{v\in\mathbb{V}}\mathbb{Z}^{I_v}\xrightarrow{\varrho=\varrho_O}\mathbb{T}=\mathbb{T}(\Gamma)\equiv\bigoplus_{e\in\mathbb{E}}\mathbb{Z}^{I_e}\tag{3.17}$$

in the following way. For every $e\in\mathbb{E}$, let

$$\varrho(1_e)=s_{\underset{\to}{e}}\in\mathbb{Z}^{I_e},\tag{3.18}$$

where 1_e is the generator of $\mathbb{Z}^e$ in $\mathbb{Z}^{\mathbb{E}}$ and $\underset{\to}{e}$ is the chosen orientation on e in O. In particular, $\varrho(1_e)=0$ for any e with $I_e=\emptyset$. Similarly, for every $v\in\mathbb{V}$ and $i\in I_v$, let $1_{v,i}$ be the generator of the i-th factor in $\mathbb{Z}^{I_v}$, and define

$$\varrho(1_{v,i})=\xi_{v,i}\in\bigoplus_{e\in\mathbb{E}}\mathbb{Z}^{I_e}\tag{3.19}$$

to be the vector which has $1_{e,i}\in\mathbb{Z}^{I_e}\subset\mathbb{Z}^S$ in the e-th factor, if $v=v_1(\underset{\to}{e})$ and e is not a loop, it has $-1_{e;i}\in\mathbb{Z}^{I_e}$ in the e-th factor, if $v=v_2(\underset{\to}{e})$ and e is not a loop, and is zero otherwise. This is well-defined by the first equality in (3.11). Let

$$\Lambda=\Lambda(\Gamma)=\mathrm{image}(\varrho),\quad \mathbb{K}=\mathbb{K}(\Gamma)=\mathrm{Ker}(\varrho)\quad\text{and}\quad \mathbb{CK}=\mathbb{CK}(\Gamma)=\mathbb{T}/\Lambda=\mathrm{coker}(\varrho).\tag{3.20}$$

By Definition 3.3.(3), the $\mathbb{Z}$-modules Λ, $\mathbb{K}$, and $\mathbb{CK}$ are independent of choice of the orientation O on $\mathbb{E}$ and are invariants of the decorated graph Γ. In particular,

$$\mathbb{K}=\Big\{\big((\lambda_e)_{e\in\mathbb{E}},(s_v)_{v\in\mathbb{V}}\big)\in\mathbb{Z}^{\mathbb{E}}\oplus\bigoplus_{v\in\mathbb{V}}\mathbb{Z}^{I_v}:\ s_v-s_{v'}=\lambda_e s_{\underset{\to}{e}}\quad\forall\ v,v'\in\mathbb{V},\ \underset{\to}{e}\in\underset{\to}{\mathbb{E}}_{v',v}\Big\}.\tag{3.21}$$

In this equation, via the first identity in (3.11) and the inclusion $\mathbb{Z}^{I_v}\cong\mathbb{Z}^{I_v}\times\{0\}^{I_e-I_v}\subset\mathbb{Z}^{I_e}$, we think of s_v as a vector also in $\mathbb{Z}^{I_e}$, for all $e\in\mathbb{E}_v$. For any field of characteristic zero F, let

$$\mathbb{D}_F=\mathbb{D}\otimes_{\mathbb{Z}}F,\quad \mathbb{T}_F=\mathbb{T}\otimes_{\mathbb{Z}}F,\quad \Lambda_F=\Lambda\otimes_{\mathbb{Z}}F,\quad \mathbb{K}_F=\mathbb{K}\otimes_{\mathbb{Z}}F,\quad\text{and}\quad \mathbb{CK}_F=\mathbb{CK}\otimes_{\mathbb{Z}}F\tag{3.22}$$

be the corresponding F-vector spaces and $\varrho_F:\mathbb{D}_F\longrightarrow\mathbb{T}_F$ be the corresponding F-linear map. Via the exponentiation map, let

$$\exp(\Lambda_{\mathbb{C}})\subset\prod_{e\in\mathbb{E}}(\mathbb{C}^*)^{I_e}$$

be the subgroup corresponding to the sub-Lie algebra $\Lambda_{\mathbb{C}}\subset\mathbb{T}_{\mathbb{C}}$, and denote the quotient group by $\mathcal{G}=\mathcal{G}(\Lambda)=\exp(\mathbb{CK}_{\mathbb{C}})$. Therefore,

$$\dim_{\mathbb{C}}\mathcal{G}=\dim\mathbb{CK}_{\mathbb{C}}=\sum_{e\in\mathbb{E}}(|I_e|-1)-\sum_{v\in\mathbb{V}}|I_v|+\dim\mathbb{K}_{\mathbb{C}}.\tag{3.23}$$

Lemma 3.7. *Let $D=\bigcup_{i\in S}D_i\subset X$ be an SC symplectic divisor, $(J,\omega)\in\mathcal{J}(X,D)$, and $\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ be as in (3.13). Then there exists a natural map*

$$\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma \xrightarrow{\mathrm{ob}_\Gamma} \mathcal{G}(\Gamma). \tag{3.24}$$

Proof. Given a pre-log map $f\equiv\big(f_v\equiv(u_v,[\zeta_v],C_v)\big)_{v\in\mathbb{V}}$ as in Definition 3.3, fix an arbitrary set of representatives

$$\zeta_v=(\zeta_{v,i})_{i\in I_v}\in\Gamma_{\mathrm{mero}}(\Sigma_v,u_v^*\mathcal{N}_XD_{I_v})\qquad\forall\ v\in\mathbb{V}. \tag{3.25}$$

For each $v\in\mathbb{V}$ and $\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}_v$, let $z_{\underset{\rightarrow}{e}}$ be an arbitrary holomorphic coordinate in a sufficiently small disk $\Delta_{\underset{\rightarrow}{e}}$ around the nodal point $(z_{\underset{\rightarrow}{e}}=0)=q_{\underset{\rightarrow}{e}}\in\Sigma_v$. By (3.5), for every $v\in\mathbb{V}$, $\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}_v$, and $i\in I_v$, in a local holomorphic trivialization

$$u^*\mathcal{N}_XD_i|_{\Delta_{\underset{\rightarrow}{e}}}\approx\mathcal{N}_XD_i|_{u(q_e)}\times\Delta_{\underset{\rightarrow}{e}},$$

we have

$$\zeta_{v,i}(z_{\underset{\rightarrow}{e}})=z_{\underset{\rightarrow}{e}}^{s_{\underset{\rightarrow}{e},i}}\widetilde{\zeta}_{v,i}(z_{\underset{\rightarrow}{e}}) \tag{3.26}$$

such that

$$0\neq\widetilde{\zeta}_{v,i}(0)\equiv\eta_{\underset{\rightarrow}{e},i}\in\mathcal{N}_XD_i|_{u(q_e)}$$

is independent of the choice of the trivialization. Similarly, by [12, (6.1)], for every $v\in\mathbb{V}$, $\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}_v$, and $i\in I_e-I_v$, the map u_v has a well-defined $s_{\underset{\rightarrow}{e},i}$-th derivative

$$\eta_{\underset{\rightarrow}{e},i}\in\mathcal{N}_XD_i|_{u(q_e)} \tag{3.27}$$

(with respect to the coordinate $z_{\underset{\rightarrow}{e}}$) in the normal direction to D_i at the nodal marked point $q_{\underset{\rightarrow}{e}}$.

With the choice of orientation $O\equiv\{\underset{\rightarrow}{e}\}_{e\in\mathbb{E}}\subset\underset{\rightarrow}{\mathbb{E}}$ on the edges as before, since $\eta_{\underset{\rightarrow}{e},i}\neq0$ for all $\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}$ and $i\in I_e$, the tuples

$$\eta_e=\big(\eta_{\underset{\rightarrow}{e},i}/\eta_{\underset{\leftarrow}{e},i}\big)_{i\in I_e}\in(\mathbb{C}^*)^{I_e}\qquad\forall\ \underset{\rightarrow}{e}\in O \tag{3.28}$$

give rise to an element

$$\eta\equiv(\eta_e)_{e\in\mathbb{E}}\in\prod_{e\in\mathbb{E}}(\mathbb{C}^*)^{I_e}.$$

The action of the subgroup $\exp(\Lambda_{\mathbb{C}})$ on η corresponds to rescalings of (3.25) and change of coordinates in (3.26); i.e the class $\mathrm{ob}_\Gamma(f)=[\eta]$ of η in $\mathcal{G}=\prod_{e\in\mathbb{E}}(\mathbb{C}^*)^{I_e}/\exp(\Lambda_{\mathbb{C}})$ is independent of the choice of representatives in (3.25) and local coordinates in (3.26). If f and f' are equivalent with respect to a reparametrization $h\colon\Sigma'\longrightarrow\Sigma$ as in (3.12), the associated group elements η and η', respectively, would be the same with respect to any h-symmetric choice of holomorphic coordinates $\{z_{\underset{\rightarrow}{e}}\}_{\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}}$. Therefore, $\mathrm{ob}_\Gamma([f])=[\eta]\in\mathcal{G}$ is well-defined. □

Definition 3.8. Let $D=\bigcup_{i\in S}D_i\subset X$ be an SC symplectic divisor and $(J,\omega)\in\mathcal{J}(X,D)$. A log J-holomorphic is a pre-log J-holomorphic map f with the decorated dual graph Γ such that

(1) there exist functions

$$s\colon\mathbb{V}\longrightarrow\mathbb{R}^S,\quad v\longrightarrow s_v,\qquad\text{and}\qquad\lambda\colon\mathbb{E}\longrightarrow\mathbb{R}_+,\quad e\longrightarrow\lambda_e,$$

such that

(*a*) $s_v\in\mathbb{R}_+^{I_v}\times\{0\}^{S-I_v}$ for all $v\in\mathbb{V}$,

(b) $s_{v_2(\underrightarrow{e})} - s_{v_1(\underrightarrow{e})} = \lambda_e s_{\underrightarrow{e}}$ for every $\underrightarrow{e} \in \underrightarrow{\mathbb{E}}$;

(2) and $\mathrm{ob}_\Gamma(f) = 1 \in \mathcal{G}(\Gamma)$.

Condition (1)(b) is well-defined because of Definition 3.3.(3). If (2) holds, we say that the pre-log map f is $\mathcal{G}$-unobstructed. Condition (2) is independent of the choice of orientation O on $\mathbb{E}$. Note that the functions s and λ are not part of the defining data of a log map.

Remark 3.9. A nodal map in the relative compactification (when D is smooth) with image in an expanded degeneration $X[m]$ comes with a partial ordering of the smooth components of the domain, such that the components mapped into X have order 0 and those mapped into the r-th copy of $\mathbb{P}_X D$ are of order r. In the compactification process, a component sinking faster into D gives a component with higher order. From our perspective, the function s in Condition (1) is a generalization of this partial ordering to the SC case with $\mathbb{Z}^S$ instead of $\mathbb{Z}$; see Lemma 3.22. From the tropical perspective of [2, Definition 2.5.3], Condition (1) is equal to the existence of a tropical map from a tropical curve associated to Γ into $\mathbb{R}^S_{\geq 0}$. This condition puts a big restriction on the set of contact vectors $s_{\underrightarrow{e}}$. For example if $I_v, I_{v'} = \emptyset$, then for any other $v'' \in \mathbb{V}$ and oriented edges $\underrightarrow{e} \in \underrightarrow{\mathbb{E}}_{v,v''}$ and $\underrightarrow{e}' \in \underrightarrow{\mathbb{E}}_{v',v''}$, the contact vectors $s_{\underrightarrow{e}}$ and $s_{\underrightarrow{e}'}$ should be positively proportional.

Example 3.10. Example 3.6 does not satisfy Definition 3.8.(1). Since $I_{v_1} = I_{v_2} = \emptyset$, we should have $s_{v_1} = s_{v_2} = (0,0)$. Then Condition (1)($b$) requires $s_{\underrightarrow{e}_1} = (2,1)$ and $s_{\underrightarrow{e}_1} = (1,1)$ to be positive multiples of s_{v_3}, which is impossible. A straightforward calculation shows that the line component u_{v_2} in any limit of (3.16) with a component u_{v_1} as in Figure 4 should lie in D_1. Then the function $s\colon \mathbb{V} \longrightarrow \mathbb{R}^2$ given by $s_{v_1} = (0,0)$, $s_{v_2} = (1,0)$, and $s_{v_3} = (2,1)$ satisfies Definition 3.8.(1).

Definition 3.11. For a fixed SC symplectic divisor $D = \bigcup_{i\in S} D_i$ in X, given $g, k \in \mathbb{N}$, $A \in H_2(X,\mathbb{Z})$, and $\mathfrak{s} \in (\mathbb{Z}^S)^k$, we denote by $\mathrm{DG}(g,\mathfrak{s},A)$ to be the set of (stable) dual graphs $\Gamma = \Gamma(\mathbb{V},\mathbb{E},\mathbb{L})$ with k legs and

(a) a genus decoration of total genus g,

(b) a degree decoration of total degree A,

(c) an ordering $\mathbb{L} \longrightarrow \{1,\ldots,k\}$,

(d) set decorations $I\colon \mathbb{V},\mathbb{E} \longrightarrow \mathbb{R}^S$ satisfying $I_e = I_v \cup I_{v'}$ for all $v, v' \in \mathbb{V}$ and $e \in \mathbb{E}_{v,v'}$,

(e) and a vector decoration on the set of oriented edges $\underrightarrow{\mathbb{E}}$, $\underrightarrow{e} \to s_{\underrightarrow{e}} \in \mathbb{Z}^{I_e} \subset \mathbb{Z}^S$, with $s_{\underrightarrow{e}} + s_{\underleftarrow{e}} = 0$ for all $\underrightarrow{e} \in \underrightarrow{\mathbb{E}}$,

such that Condition (1) of Definition 3.8 holds and

$$(A_v \cdot D_i)_{i\in S} = \sum_{\underrightarrow{e} \in \underrightarrow{\mathbb{E}}_v} s_{\underrightarrow{e}} + \sum_{\substack{l\in\mathbb{L} \\ v(l)=v}} s_l \qquad \forall\, v \in \mathbb{V}. \tag{3.29}$$

$\mathrm{DG}(g,\mathfrak{s},A)$ is the set of possible combinatorial types of stable genus g k-marked degree A log maps of contact order $\mathfrak{s}$. Note that the defining conditions of $\mathrm{DG}(g,\mathfrak{s},A)$ do not capture Definition 3.8.(2); the latter is a non-combinatorial condition. Example 3.12 below illustrates a legitimate Γ such that the space of pre-log maps has an expected dimension larger than the expected dimension of the (virtually) main stratum. Then, imposing Condition (2) of Definition 3.8 would reduce

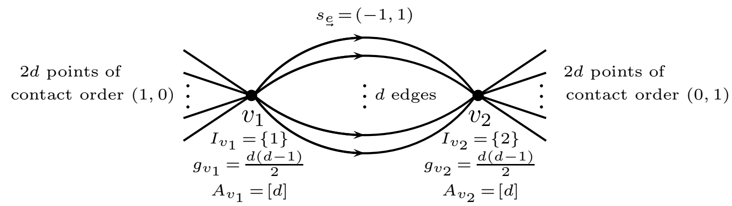

Figure 5: A decorated graph in $\mathrm{DG}(g=(d-1)^2,\mathfrak{s},A=[2d])$, corresponding to two generic degree d curves in D_1 and D_2 intersecting at d points along D_{12}.

the dimension to less than the expected dimension of the (virtually) main stratum.

For every $\Gamma\in\mathrm{DG}(g,\mathfrak{s},A)$ define

$$\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma\subset\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma \tag{3.30}$$

to be the stratum of log J-holomorphic maps of type Γ. We then define the moduli space of genus g degree A stable nodal log pseudoholomorphic maps of contact type $\mathfrak{s}$ to be the union

$$\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)\equiv\bigcup_{\Gamma\in\mathrm{DG}(g,\mathfrak{s},A)}\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma, \tag{3.31}$$

Example 3.12. Let

$$X=\mathbb{P}^3,\quad D_1\cup D_2=\mathbb{P}^2\cup\mathbb{P}^2,\quad A=[2d]\in H_2(X,\mathbb{Z})\cong\mathbb{Z},\quad g=(d-1)^2,$$
$$\mathfrak{s}=\big((1,0),\ldots,(1,0),(0,1),\ldots,(0,1)\big)\in\big(\mathbb{Z}^{\{1,2\}}\big)^{4d},$$

and $\Gamma\in\mathrm{DG}(g,\mathfrak{s},A)$ be the decorated dual graph illustrated in Figure 5. Note that the function $s\colon\mathbb{V}\longrightarrow\mathbb{R}^2$ given by $s_{v_1}=(1,0)$ and $s_{v_2}=(0,1)$ satisfies Definition 3.8.(1). Every element of $\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ is supported on two generic degree d plane curves in D_1 and D_2 intersecting at d points along D_{12}. By (3.15) and Definition 3.3.(2), the expected $\mathbb{C}$-dimension of $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\emptyset$ and $\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ are $8d$ and $9d-2$, respectively.

Orient each edge such that $v_1(\underset{\rightarrow}{e}_i)=v_1$ for all $i=1,\ldots,d$. Then $\Lambda=\varrho(\mathbb{D})$ in (3.20) is generated by the vectors $s_{\underset{\rightarrow}{e}_1},\ldots,s_{\underset{\rightarrow}{e}_d}$, $\xi_{v_1}=\xi_{v_1,1}$, and $\xi_{v_2}=\xi_{v_2,2}$, such that the only relation is

$$\xi_{v_1}+\xi_{v_2}+(s_{\underset{\rightarrow}{e}_1}+\ldots+s_{\underset{\rightarrow}{e}_d})=0.$$

We conclude that the group $\mathcal{G}(\Gamma)$ is complex $(d-1)$-dimensional. Therefore, the subset of log maps

$$\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma\subset\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma$$

is of the expected $\mathbb{C}$-dimension $(9d-2)-(d-1)=8d-1<8d$. $\square$

Remark 3.13. By Remark 3.2, for every k-marked stable nodal map f in $\overline{\mathcal{M}}_{g,k}(X,A)$ with dual graph Γ, fixing $\mathfrak{s}\in(\mathbb{Z}^S)^k$ and the vector decoration $\{s_{\underset{\rightarrow}{e}}\}_{\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}}$ in Definition 3.11.(e), there exists at most one element $f_{\log}\in\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ with orders s_i at z^i and $s_{\underset{\rightarrow}{e}}$ at $q_{\underset{\rightarrow}{e}}$ lifting f. Furthermore, $f_{\log}$ is stable if and only if f is stable.

Lemma 3.14. *For every genus 0 k-marked stable nodal map f in $\overline{\mathcal{M}}_{0,k}(X,A)$ with dual graph Γ and a fixed $\mathfrak{s}$, there exists at most one vector decoration $\{s_{\underset{\rightarrow}{e}}\}_{\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}}$ as in Definition 3.11.(e) satisfying (3.29). In particular, the forgetful map*

$$\overline{\mathcal{M}}^{\log}_{0,\mathfrak{s}}(X,D,A)\longrightarrow\overline{\mathcal{M}}_{0,k}(X,D,A)$$

is an embedding (of sets).

Proof. Without loose of generality, we may assume D is smooth, i.e $|S|=1$. Assume there are two different decorations $\{s_{\underset{\rightarrow}{e}}\}_{\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}}$ and $\{s'_{\underset{\rightarrow}{e}}\}_{\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}}$ as in Definition 3.11 satisfying (3.29). Since $g=0$, Γ is a tree and the subset of edges $\Omega\subset\mathbb{E}$ where $s_{\underset{\rightarrow}{e}}\neq s'_{\underset{\rightarrow}{e}}$ determines a sub-tree of that. In particular, there exists a vertex $v\in\mathbb{V}$ that is connected to only one edge $e'\in\Omega$. Orient e' so that v is the starting point. Then, by (3.29),

$$A_v\cdot D=s_{\underset{\rightarrow}{e'}}+\sum_{\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}_v-\{\underset{\rightarrow}{e'}\}}s_{\underset{\rightarrow}{e}}+\sum_{\substack{l\in\mathbb{L}\\ v(l)=v}}s_l\neq s'_{\underset{\rightarrow}{e'}}+\sum_{\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}_v-\{\underset{\rightarrow}{e'}\}}s'_{\underset{\rightarrow}{e}}+\sum_{\substack{l\in\mathbb{L}\\ v(l)=v}}s'_l=A_v\cdot D\,;$$

that is a contradiction. $\square$

For $g\neq 0$, Example 3.15 describes a situation where f has different lifts but the automorphism groups of f and its lifts are the same. Example 3.16 describes a situation where f has different lifts and some of them have smaller automorphism groups. For a given stable map f with the dual graph Γ and a given $\mathfrak{s}$, an argument similar to the proof of Lemma 3.14 shows that the set of possible vector decorations $\{s_{\underset{\rightarrow}{e}}\}_{\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}}$ as in Definition 3.11 satisfying (3.29), and thus the set of possible log lifts of f, is finite.

Example 3.15. Let $X=\mathbb{P}^2$, D be a hyperplane (line), and p_1,p_2,p_3,p_4 be four distinct points in D. Let $u_{v_1}:\Sigma_{v_1}\xrightarrow{\approx}D$ be a degree one map and ζ_{v_1} be a meromorphic section of $u_{v_1}^*\mathcal{N}_XD$ with two poles of order 1 and 2 at $q_{\underset{\rightarrow}{e}_1}=u_{v_1}^{-1}(p_1)$ and $q_{\underset{\rightarrow}{e}_2}=u_{v_1}^{-1}(p_2)$, respectively, and a zero of order 4 at $z^1=u_{v_1}^{-1}(p_3)$. Similarly, let $u_{v_2}:\Sigma_{v_2}\xrightarrow{\approx}D$ be a degree one map and ζ_{v_2} be a meromorphic section of $u_{v_2}^*\mathcal{N}_XD$ with two zeros of order 1 and 2 at $q_{\underset{\leftarrow}{e}_1}=u_{v_2}^{-1}(p_1)$ and $q_{\underset{\leftarrow}{e}_2}=u_{v_2}^{-1}(p_2)$, respectively, and a pole of order 2 at $q_{\underset{\rightarrow}{e}_3}=u_{v_2}^{-1}(p_4)$. Finally, let $u_{v_3}:\Sigma_{v_3}\longrightarrow X$ be a smooth conic with a tangency of order 2 with D at $q_{\underset{\leftarrow}{e}_3}=u_{v_3}^{-1}(p_4)$. The tuple

$$u_{\log}\equiv(u_{v_3},(u_{v_2},\zeta_{v_2}),(u_{v_1},\zeta_{v_1}))$$

with the nodal 1-marked domain

$$(\Sigma,z^1)=(\Sigma_{v_1},z^1,q_{\underset{\rightarrow}{e}_1},q_{\underset{\rightarrow}{e}_2})\sqcup(\Sigma_{v_2},q_{\underset{\leftarrow}{e}_1},q_{\underset{\leftarrow}{e}_2},q_{\underset{\rightarrow}{e}_3})\sqcup(\Sigma_{v_3},q_{\underset{\leftarrow}{e}_3})/q_{\underset{\rightarrow}{e}}\sim q_{\underset{\leftarrow}{e}}\quad\forall\ e\in\{e_1,e_2,e_3\}$$

defines an element of $\overline{\mathcal{M}}^{\log}_{1,(4)}(X,D,[4])$. Let $u'_{\log}$ be a similar tuple with the roles of p_1 and p_2 reversed, i.e. $u=(u_{v_1},u_{v_2},u_{v_3})$ remains the same but ζ_{v_1} and ζ_{v_2} exchange their orders at the pre-images of p_1 and p_2. Therefore, $(u_{\log},\Sigma,z^1)$ and $(u'_{\log},\Sigma,z^1)$ are different lifts of the same 1-marked stable map (u,Σ,z^1) in $\overline{\mathcal{M}}_{1,1}(X,[4])$. Note that e_1 and e_2 form a loop in Γ. In this example, the two vector decorations corresponding to $(u_{\log},\Sigma,z^1)$ and $(u'_{\log},\Sigma,z^1)$ yield isomorphic decorated dual graphs Γ. In other words, the forgetful map

$$\mathcal{M}^{\log}_{1,(4)}(X,D,[4])_\Gamma\longrightarrow\overline{\mathcal{M}}_{1,1}(X,[4])$$

is a double-covering of its image. $\square$

Example 3.16. Assume $u\colon\Sigma\longrightarrow D\subset X$ is a stable map, where Σ is the genus 1 nodal curve made of two copies of $\mathbb{P}^1$, say $\mathbb{P}^1_1$ and $\mathbb{P}^1_2$, attached at 0 and ∞, and $u_i=u|_{\mathbb{P}^1_i}\colon\mathbb{P}^1_i\longrightarrow D$, for $i=1,2$, is a double-covering of some rational curve $C_i\subset D$, with $u_i(z^{-1})=u_i(z)$; i.e.

$$u_i(0)=u_i(\infty)=x\in C_1\cap C_2\subset D.$$

Furthermore, assume $\mathcal{N}_X D|_{C_1}=\mathcal{O}(2)$ and $\mathcal{N}_X D|_{C_2}=\mathcal{O}(-2)$. The automorphism group of the stable map $f=(u,\Sigma)$ is $\mathbb{Z}_2$. Since $u_1^*\mathcal{N}_X D=\mathcal{O}(4)$ and $u_2^*\mathcal{N}_X D=\mathcal{O}(-4)$, there are 2 possible ways to lift f to a log map $f_{\log}\in\overline{\mathcal{M}}^{\log}_{1,\emptyset}(X,D,2(C_1+C_2))$. The holomorphic section ζ_1 of $u_1^*\mathcal{N}_X D$ can be chosen to have zeros of orders $(3,1)$, $(2,2)$, or $(1,3)$ at $(0,\infty)$. In the middle case, the automorphism group of $f_{\log}$ is $\mathbb{Z}_2$. In the remaining two cases, the two lifts are equivalent with respect to the reparametrization map

$$h\colon\Sigma\longrightarrow\Sigma,\quad h|_{\mathbb{P}^1_i}(z)=z^{-1},\quad i=1,2,$$

and their equivalence class defines a single element of $\overline{\mathcal{M}}^{\log}_{1,\emptyset}(X,D,2(C_1+C_2))$ with the trivial automorphism group.

In Section 4, for J as in the statement of Theorem 1.1, we will lift the Gromov convergence topology to a compact sequential convergence topology on (3.31) such that the forgetful map (1.8) is a continuous local embedding. It follows that the lifted topology is also metrizable. If $g>0$, globally, (1.8) behaves like an immersion. If $\mathfrak{s}\in(\mathbb{N}^S)^k$, by Lemma 3.20 bellow, $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ is a compact space of the expected real dimension

$$2\big(c_1^{TX(-\log D)}(A)+(\dim_{\mathbb{C}}X-3)(1-g)+k\big). \tag{3.32}$$

In subsequent papers we will construct Kuranishi-type charts of dimension (3.32) around every point of $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$.

The following example describes the log compactification of the moduli space of lines in $\mathbb{P}^2$ relative to a transverse union of two hyperplanes (lines). Even in this simple example, E. Ionel was not[13] able to precisely describe her compactification and verify that it is compact/Hausdorff. In [29], Brett Parker gives a tropical description of "a compactification" which is different than the natural outcome of Example 3.17.

Example 3.17. Let $X=\mathbb{P}^2$ with projective coordinates $[x_1,x_2,x_3]$, $D_1=(x_1=0)$, $D_2=(x_2=0)$, $D=D_1\cup D_2$, $A=[1]\in H_2(\mathbb{P}^2,\mathbb{Z})\cong\mathbb{Z}$, and $\mathfrak{s}=((1,0),(0,1))$. Then, as we show below, the moduli space

$$\overline{\mathcal{M}}^{\log}_{0,\mathfrak{s}}(X,D,[1]) \tag{3.33}$$

can be identified[14] with $B_{\mathrm{pt}_1,\mathrm{pt}_2}\mathbb{P}^2_{\mathrm{dual}}$ (two point blowup of $\mathbb{P}^2$), where $\mathbb{P}^2_{\mathrm{dual}}$ is the dual space of lines in $X=\mathbb{P}^2$, pt_1 is the point corresponding to the line D_1, and pt_2 is the point corresponding to the line D_2. Let E_1 and E_2 be the exceptional curves of $B_{\mathrm{pt}_1,\mathrm{pt}_2}\mathbb{P}^2_{\mathrm{dual}}$ and L be the proper transform of the line connecting pt_1 and pt_2. Any line in X not passing through D_{12} intersects D_1 and D_2 at two disjoint points z^1 and z^2, respectively. By (3.4),

$$\mathrm{ord}(z^1)=(1,0)\qquad\text{and}\qquad\mathrm{ord}(z^2)=(0,1).$$

[13] In a discussion that happened at Simons Center in Spring 2014 and a series of follow-up emails.

[14] The identification is homeomorphic with respect to the topology that we describe in Section 4.

This gives an identification of

$$\mathcal{M}^{\log}_{0,\mathfrak{s}}(X,D,[1])_{\emptyset} \subset \overline{\mathcal{M}}^{\log}_{0,\mathfrak{s}}(X,D,[1])$$

with $B_{\mathrm{pt}_1,\mathrm{pt}_2}\mathbb{P}^2-(E_1\cup E_2\cup L)$. Every other log map $(u,[\zeta])$ with smooth domain in (3.33) is either of depth $\{1\}$ or of depth $\{2\}$ with two marked points z^1 and z^2 of the corresponding orders. Those of depth $\{1\}$ are given by an isomorphism $u\colon \mathbb{P}^1 \xrightarrow{\cong} D_1$ and a holomorphic section ζ of $\mathcal{N}_X D_1 \cong \mathcal{O}_{\mathbb{P}^1}(1)$, such that ζ has a simple zero at the marked point z^1 and $z^1\neq z^2=u^{-1}(D_2)$. Such $[\zeta]$ is uniquely determined by $u(z^1)\in D_1\cong\mathbb{P}^1$. Therefore, via the identification

$$E_1 \cong \mathbb{P}\big(H_0(\mathcal{N}_X D_1)\big) \cong \mathbb{P}^1,$$

such maps correspond to $E_1-\{E_1\cdot L\}\cong\mathbb{C}$. Similarly, the maps of depth $\{2\}$ with smooth domain correspond to $E_2-\{E_2\cdot L\}\cong\mathbb{C}$.

For other log maps f in (3.33), z^1 and z^2 are mapped to the point D_{12} and thus live on a "ghost bubble" $u_2\colon\mathbb{P}^1\longrightarrow X$, with $u_2\equiv D_{12}$. This ghost bubble and the non-trivial map $u_1\colon\mathbb{P}^1\longrightarrow X$ are attached to each other at nodal points $z^3\in\mathrm{Dom}(u_2)$ and $z'\in\mathrm{Dom}(u_1)$. By definition, the meromorphic section $\zeta=(\zeta_1,\zeta_2)$ defining the log map $(u_2,[\zeta]\equiv([\zeta_1],[\zeta_2]))$ is a meromorphic section of the trivial bundle $u_2^*\mathcal{N}_X D_{12}\cong\mathbb{P}^1\times\mathbb{C}^2$, such that

$$\mathrm{ord}_{z^1}(\zeta)=(1,0) \qquad \text{and} \quad \mathrm{ord}_{z^2}(\zeta)=(0,1).$$

Since $u_2^*\mathcal{N}_X D_{12}$ is trivial, we should have $\mathrm{ord}_{z^3}(\zeta)=(-1,-1)$ and these restrictions specify a unique $(\mathbb{C}^*)^2$-class $[\zeta]$. The are thus three possibilities for f.

(1) u_1 is of depth $\emptyset$: in this case, by Definition 3.3.(3), u_1 specifies an element of $\mathcal{M}^{\log}_{0,((1,1))}(X,D,[1])_{\emptyset}$ and we get an identification of such maps $f=[u_1,(u_2,[\zeta])]$ in (3.33) with the points of $L-\{L\cdot E_1, L\cdot E_2\}$. The associated decorated dual graph Γ is made of two vertices v_1 and v_2 corresponding to u_1 and u_2, with $I_{v_1}=\emptyset$ and $I_{v_2}=\{1,2\}$, connected by an edge e with $I_e=\{1,2\}$ and $s_{\vec{e}}=\pm(1,1)$ (depending on the choice of the orientation). The group $\mathcal{G}(\Gamma)$ in this case is trivial and the function s in Definition 3.8.(1) can be taken to be $s_{v_1}=(0,0)$ and $s_{v_2}=(1,1)$.

(2) u_1 is of depth $\{1\}$: in this case u_1 comes with a holomorphic section ζ' of $\mathcal{O}_{\mathbb{P}^1}(1)$ as before. Since $\mathrm{ord}(z')=(1,1)$, by Definition 3.3.(3), ζ' should be zero at z' and this uniquely determines $[\zeta']$. This unique element $f=[(u_1,[\zeta']),(u_2,[\zeta])]$ corresponds to the point $E_1\cdot L$. The associated decorated dual graph Γ is made of two vertices v_1 and v_2 corresponding to u_1 and u_2, with $I_{v_1}=\{1\}$ and $I_{v_2}=\{1,2\}$, connected by an edge e with $I_e=\{1,2\}$ and $s_{\vec{e}}=\pm(1,1)$ (depending on the choice of orientation). The group $\mathcal{G}(\Gamma)$ in this case is trivial and the function s in Definition 3.8.(1) can be taken to be $s_{v_1}=(1,0)$ and $s_{v_2}=(2,1)$.

(3) u_1 is of depth $\{2\}$: similarly, there is a unique such map which corresponds to the point $E_2\cdot L$.

□

In the case of the classical moduli space of stable J-holomorphic maps $\overline{\mathcal{M}}_{g,k}(X,A)$, for a J-holomorphic map $u\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J)$ with smooth domain, the linearization $D_u\bar\partial$ of the Cauchy-Riemann equation in (2.18) is Fredholm. Therefore, the real vector spaces

$$\mathrm{Def}(u)=\ker(\mathrm{D}_u\bar\partial) \quad \text{and} \quad \mathrm{Obs}(u)=\mathrm{coker}(\mathrm{D}_u\bar\partial)$$

are finite dimensional. The first space corresponds to infinitesimal deformations of u (over the fixed domain C) and the second one is the obstruction space for integrating the elements of $\mathrm{Def}(u)$ to actual deformations. In the nodal case, the kernel $\mathrm{Def}(u)$ of the similarly defined linearization map in [8, Section 6.3] corresponds to infinitesimal deformations of u in the stratum $\overline{\mathcal{M}}_{g,k}(X,A)_\Gamma$. Deformations into $\overline{\mathcal{M}}_{g,k}(X,A)$ correspond to gluing the nodes of the domain with gluing parameters from $\mathbb{C}^{\mathbb{E}}$ and the gluing is virtually un-obstructed, i.e. if $\mathrm{Obs}(u)=0$, for every sufficiently small smoothing $(\Sigma',\mathfrak{j}')$ of the nodes of the domain $(\Sigma,\mathfrak{j})$, there exists a pseudoholomorphic map $u'\colon(\Sigma',\mathfrak{j}')\longrightarrow(X,J)$ close u; see [8, Theorem 6.3.5] for $\mathrm{Obs}(u)\neq 0$. In other words, moduli spaces $\overline{\mathcal{M}}_{g,k}(X,A)$ are virtually smooth (orbifolds) and the "virtual normal cone" of the stratum $\overline{\mathcal{M}}_{g,k}(X,A)_\Gamma$ is an (orbi-) bundle of rank $|\mathbb{E}|$. For the log moduli spaces defined in this paper, as (3.30) indicates, there are new obstructions for smoothing of nodal pre-log maps. The claim is that, in addition to a logarithmic version of $D_u\bar\partial$, the gluing part of the deformation-obstruction is encoded in the combinatorial linear map (3.17) in the following sense.

With notation as in (3.22), let

$$\sigma=\sigma(\Gamma)=\mathbb{K}_{\mathbb{R}}\cap\big(\mathbb{R}^{\mathbb{E}}_{\geq 0}\oplus\bigoplus_{v\in\mathbb{V}}\mathbb{R}^{I_v}_{\geq 0}\big)\subset\mathbb{K}_{\mathbb{R}} \tag{3.34}$$

be the cone of non-negative elements in the kernel of $\varrho_{\mathbb{R}}\colon\mathbb{D}_{\mathbb{R}}\longrightarrow\mathbb{T}_{\mathbb{R}}$. This cone is independent of the choice of the orientation O used to define (3.17); in fact, by (3.21),

$$\sigma=\big\{\big((\lambda_e)_{e\in\mathbb{E}},(s_v)_{v\in\mathbb{V}}\big)\in\mathbb{R}^{\mathbb{E}}_{\geq 0}\oplus\bigoplus_{v\in\mathbb{V}}\mathbb{R}^{I_v}_{\geq 0}\colon\quad s_v-s_{v'}=\lambda_e s_{\underrightarrow{e}}\quad\forall\, v,v'\in\mathbb{V},\ \underrightarrow{e}\in\underrightarrow{\mathbb{E}}_{v',v}\big\}. \tag{3.35}$$

The integral part of σ coincides with the monoid $Q^\vee$ in [2, Section 2.3.9].

Lemma 3.18. *For every $\Gamma\in\mathrm{DG}(g,\mathfrak{s},A)$, $\sigma(\Gamma)$ is a top-dimensional strictly convex rational polyhedral cone in $\mathbb{K}_{\mathbb{R}}(\Gamma)$.*

Proof. The functions s and λ in Definition 3.8.(1) define an element m_+ of

$$\mathbb{K}_{\mathbb{R}}\cap\big(\mathbb{R}^{\mathbb{E}}_{+}\oplus\bigoplus_{v\in\mathbb{V}}\mathbb{R}^{I_v}_{+}\big). \tag{3.36}$$

Since all of the coefficients in m_+ are positive, for any arbitrary $m\in\mathbb{K}_{\mathbb{R}}$ there exists a sufficiently large $r>0$ such that $m+rm_+\in\sigma$. We conclude that σ is top-dimensional. Since $\mathbb{R}^{\mathbb{E}}_{\geq 0}\oplus\bigoplus_{v\in\mathbb{V}}\mathbb{R}^{I_v}_{\geq 0}$ is a strictly convex rational polyhedral cone and $\mathbb{K}_{\mathbb{R}}$ is an integrally defined sub-vector space, the intersection (3.36) is a strictly convex rational polyhedral cone. $\square$

Corollary 3.19. *By Lemma 3.18, the functions s and λ in Definition 3.8.*(1) *can be chosen to be integral-valued*

Lemma 3.20. *For any decorated dual graph $\Gamma\in\mathrm{DG}(g,\mathfrak{s},A)$, the expected complex dimension of $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ is*

$$c_1^{TX(-\log D)}(A)+(n-3)(1-g)+k-\dim_{\mathbb{R}}\mathbb{K}_{\mathbb{R}}(\Gamma) \tag{3.37}$$

The only stratum with $\dim\mathbb{K}_{\mathbb{R}}(\Gamma)=0$ *is* $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\emptyset$.

Proof. Equation 3.37 follows from (3.15), Definition 3.3.(2), (3.1), (3.23), and (3.24). By Definition 3.8.(1) and (3.21), a function (s,λ) as in Definition 3.8.(1) gives us an element of $\mathbb{K}_{\mathbb{R}}(\Gamma)$. This element is trivial only if $\Gamma=\{v\}$ is a one-vertex graph with no edge and $I_v=\emptyset$. This establishes the last claim. □

The log moduli spaces (3.31) are not always virtually smooth. For example, the log moduli space of Example 3.21 below has an A_1-singularity along some stratum. For the log moduli spaces, as we will explain in Section 3.5, the the space of gluing parameters along $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ belongs to (a neighborhood of the origin in finitely many copies of) the affine toric variety $Y_{\sigma(\Gamma)}$ constructed from the toric fan $\sigma(\Gamma)\subset\mathbb{K}_{\mathbb{R}}$. In other words, the kernel of (3.17) gives the gluing deformation and, by (3.30), the cokernel of that gives the obstruction space for smoothability of such pre-log maps.

In the following example, we describe a tuple $(X,D,g,\mathfrak{s},A,\Gamma)$ where $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ is a point and Y_σ has an A_1-singularity at its center. In this example, the relative moduli space $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathfrak{s}}(X,D,A)$ replaces the A_1-singularity with a small resolution of that.

Example 3.21. Suppose $X=\mathbb{P}^3$, $D\cong\mathbb{P}^1\times\mathbb{P}^1$ is a smooth degree 2 hypersurface,

$$g=1,\qquad A=[2]\in H_2(\mathbb{P}^3,\mathbb{Z})\cong\mathbb{Z},\qquad \text{and}\quad \mathfrak{s}=(0,0,4).$$

By [17, Lemma 4.2] and (3.37), both $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathfrak{s}}(X,D,A)$ and $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ are of the expected complex dimension 7. Let $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ be the stratum of maps in the expanded degeneration $X[2]$ with connected components:

- a degree 1 map $u_0\colon\mathbb{P}^1\to X$ (a line) that intersects D at two distinct points (with multiplicity 1),
- a map $u_3\colon\mathbb{P}^1\to\mathbb{P}_XD$ in the second layer $\{2\}\times\mathbb{P}_XD$ of $X[2]$ which is made of a degree 1 map $\overline{u}_3\colon\mathbb{P}^1\to D$ and a meromorphic section ζ of $\overline{u}_3^*\mathcal{N}_XD\cong\mathcal{O}_{\mathbb{P}^1}(2)$ with a zero of order 4 and 2 poles of order one, and
- two maps $u_1,u_2\colon\mathbb{P}^1\to\mathbb{P}_XD$ in the first layer $\{1\}\times\mathbb{P}_XD$ of $X[2]$ carrying the first and the second marked point, respectively, which are degree 1 covers of fibers of $\mathbb{P}_XD$ connecting u_0 and u_4;

see Figure 6-Left. While the stratum $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ is of virtual $\mathbb{C}$-codimension 2, by (3.37), its image

$$\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma=\pi(\overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathfrak{s}}(X,D,A)_\Gamma)$$

in the log moduli space, given by the projection map π of Proposition 3.24 below, is of virtual $\mathbb{C}$-codimension 3. In fact, with the labeling and the choice of orientation on the edges of the associated decorated dual graph Γ in Figure 6-Right, we have

$$\varrho\colon\mathbb{Z}^{\mathbb{E}}\oplus\bigoplus_{i\in[3]}\mathbb{Z}^{I_{v_i}}\cong\mathbb{Z}^{\{e_1,e_2,e_3,e_4\}}\oplus\mathbb{Z}^{\{v_1,v_2,v_3\}}\longrightarrow\bigoplus_{i\in[4]}\mathbb{Z}^{I_{e_i}}\cong\mathbb{Z}^{\{e_1,e_2,e_3,e_4\}}$$

$$\varrho(1_{e_i})=1_{e_i}\quad\forall\, i\in[4],\quad \varrho(1_{v_1})=-1_{e_1}+1_{e_3},\quad \varrho(1_{v_2})=-1_{e_2}+1_{e_4},\quad\text{and}\quad \varrho(1_{v_3})=-1_{e_3}-1_{e_4}.$$

Therefore,

$$\sigma=\ker(\varrho_{\mathbb{R}})\cap\big(\mathbb{R}_{\geq0}^{\{e_1,e_2,e_3,e_4\}}\oplus\mathbb{R}_{\geq0}^{\{v_1,v_2,v_3\}}\big)$$

is the cone generated by the set of 4 vectors

$$\alpha_1=1_{v_3}+1_{e_3}+1_{e_4},\quad \alpha_2=1_{v_1}+1_{v_3}+1_{e_1}+1_{e_4},\quad \alpha_3=1_{v_2}+1_{v_3}+1_{e_2}+1_{e_3},\quad \alpha_4=1_{v_1}+1_{v_2}+1_{v_3}+1_{e_1}+1_{e_2}.$$

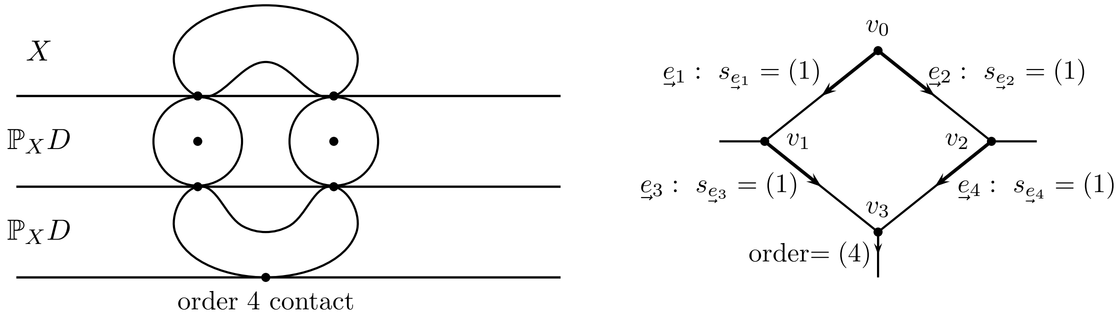

Figure 6: On left, a nodal 2-marked $g=1$ relative map in $X[2]$. On right, the decorated dual graph of the image log map.

Since the only relation among α_i is $\alpha_1+\alpha_4=\alpha_2+\alpha_3$, the associated toric variety Y_σ is isomorphic to the 3-dimensional affine sub-variety

$$(x_1x_4-x_2x_3=0)\subset\mathbb{C}^4.$$

3.3 Relative vs. Log compactification

In this section, for the case where D is smooth, i.e. $|S|=1$ in Definition 3.8, we compare $\overline{\mathcal{M}}^{\text{rel}}_{g,\mathfrak{s}}(X,D,A)$ and $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$. Proposition 3.24 shows that the latter is smaller and there is a projection map from the relative compactification onto the log compactification. This is expected, since the notion of nodal log map involves more $\mathbb{C}^*$-quotients on the set of meromorphic sections. In the algebraic case, [3, Theorem 1.1.] shows that an algebraic analogue of the projection map (3.43) induces an equivalence of virtual fundamental classes. We expect the same to hold for the invariants/VFC arising from our log compactification.

We first start with a simple lemma that highlights the relation between Definition 3.8.(1) and the layer structure in the relative compactification.

Lemma 3.22. *Let $D\subset(X,\omega)$ be a smooth symplectic divisor, $J\in\mathcal{J}(X,D,\omega)$, and*

$$\big[f\equiv\big((u_v,[\zeta_v],C_v)_{\substack{v\in\mathbb{V}\\ I_v=\{1\}}},(u_v,C_v)_{\substack{v\in\mathbb{V}\\ I_v=\emptyset}}\big)\big]\in\mathcal{M}^{\text{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma \tag{3.38}$$

be a pre-log J-holomorphic map with dual graph $\Gamma(\mathbb{V},\mathbb{E},\mathbb{L})$. The there exists a function $s\colon\mathbb{V}\longrightarrow\mathbb{R}_{\geq0}$ satisfying Definition 3.8.(1), if and only if the relations

(a) *$v_1\approx_\Gamma v_2$ if v_1 and v_2 are connected and $s_{\underset{\to}{e}}=0$ for any $\underset{\to}{e}\in\underset{\to}{\mathbb{E}}_{v_1,v_2}$, and*

(b) *$v_1\prec_\Gamma v_2$ if v_1 and v_2 are connected and $s_{\underset{\to}{e}}>0$ for any $\underset{\to}{e}\in\underset{\to}{\mathbb{E}}_{v_1,v_2}$,*

are independent of the choice of $\underset{\to}{e}\in\underset{\to}{\mathbb{E}}_{v_1,v_2}$ (i.e. they are well-defined), and generate a partial order $\preceq_\Gamma$ on $\mathbb{V}$.

Note that for a classical edges e connecting $v_1,v_2\in\mathbb{V}_0$, since $I_e=\emptyset$ by (3.11), we always have

$$s_{\underset{\to}{e}}=0\in0=\mathbb{R}^\emptyset\subset\mathbb{R}^S=\mathbb{R}.$$

Proof. If (a) and (b) define a partial order $(\mathbb{V},\preceq_\Gamma)$, we construct $s\colon\mathbb{V}\longrightarrow\mathbb{R}$ satisfying Definition 3.8.(1) in the following way. For every $v\in\mathbb{V}_0$ define $s_v=0$. Let $\mathbb{V}_{\rm min}^{(1)}$ be the subset of minimal vertices in $\mathbb{V}_1$. For every $v\in\mathbb{V}_{\rm min}^{(1)}$ define $s_v=1$. Having constructed $\mathbb{V}_{\rm min}^{(1)},\ldots,\mathbb{V}_{\rm min}^{(k)}$, let $\mathbb{V}_{\rm min}^{(k+1)}$ be the subset of minimal vertices in

$$\mathbb{V}_1-\big(\mathbb{V}_{\rm min}^{(1)}\cup\cdots\mathbb{V}_{\rm min}^{(k)}\big).$$

For every $v\in\mathbb{V}_{\rm min}^{(k+1)}$ define $s_v=k+1$. This function clearly satisfies Definition 3.8.(1). Conversely, given such a function $s\colon\mathbb{V}\longrightarrow\mathbb{R}$ satisfying Definition 3.8.(1), define $v_1\approx_\Gamma v_2$ (resp. $v_1\prec_\Gamma v_2$) if they are connected by a path and $s_{v_1}=s_{v_2}$ (resp. $s_{v_1}<s_{v_2}$). This is a partial order whose defining conditions match with (a) and (b). $\square$

Lemma 3.23. *With notation as in Lemma 3.22, the pre-log map f satisfies Definition 3.8.*(2) *if and only if there exists a set of representatives $\{\zeta_v\}_{v\in\mathbb{V},I_v=\{1\}}$ such that*

$$\zeta_v(q_{\underset{\rightarrow}{e}})=\zeta_{v'}(q_{\underset{\leftarrow}{e}})\qquad\forall\ v,v'\in\mathbb{V},\ e\in\mathbb{E}_{v,v'}\quad\text{s.t.}\quad I_v,I_{v'}=\{1\},\ s_{\underset{\rightarrow}{e}}=0.\tag{3.39}$$

Proof. For $S=\{1\}$, define

$$\begin{aligned}&\mathbb{V}_i=\{v\in\mathbb{V}\colon |I_v|=i\},\quad \mathbb{E}_i=\{e\in\mathbb{E}\colon |I_e|=i\},\qquad\text{with } i=0,1,\\&\mathbb{E}_{1,0}=\{e\in\mathbb{E}\colon |I_e|=1,\ s_{\underset{\rightarrow}{e}}=0\},\quad \mathbb{E}_{1,\star}=\{e\in\mathbb{E}\colon |I_e|=1,\ s_{\underset{\rightarrow}{e}}\neq 0\}.\end{aligned}\tag{3.40}$$

The last equation is well-defined by Definition 3.3.(3). Then the homomorphism (3.17) (corresponding to some fixed orientation O on $\mathbb{E}$) takes the form

$$\mathbb{Z}^{\mathbb{E}_0}\oplus\mathbb{Z}^{\mathbb{E}_1}\oplus\mathbb{Z}^{\mathbb{V}_1}\xrightarrow{\ \ \varrho\ \ }\mathbb{Z}^{\mathbb{E}_1}\tag{3.41}$$

where $\varrho|_{\mathbb{Z}^{\mathbb{E}_0}}\equiv 0$, $\varrho(1_e)=s_{\underset{\rightarrow}{e}}\in\mathbb{Z}$ for all $e\in\mathbb{E}_1$, and

$$\varrho(1_v\equiv 1_{v,1})_e=\begin{cases}1_e & \text{if } v_1(\underset{\rightarrow}{e})=v;\\ -1_e & \text{if } v_2(\underset{\rightarrow}{e})=v;\\ 0 & \text{if } e \text{ is a loop or otherwise.}\end{cases}$$

Therefore, tensoring (3.41) with $\mathbb{C}$, the cokernel $\mathbb{CK}_{\mathbb{C}}$ of $\varrho_{\mathbb{C}}$ is equal to the cokernel of the induced map

$$\mathbb{C}^{\mathbb{V}_1}\xrightarrow{\ \ \overline{\varrho}_{\mathbb{C}}\ \ }\mathbb{C}^{\mathbb{E}_{1,0}}.$$

Fix an arbitrary set of representatives

$$(\zeta_v\in\Gamma_{\rm mero}(\Sigma_v,u_v^*\mathcal{N}_XD))_{v\in\mathbb{V}_1}.\tag{3.42}$$

By (3.26) and (3.28), for every $e\in\mathbb{E}_{1,0}$ with $v=v_1(\underset{\rightarrow}{e})$ and $v'=v_2(\underset{\rightarrow}{e})$, we have

$$\eta_e=\zeta_v(q_{\underset{\rightarrow}{e}})/\zeta_{v'}(q_{\underset{\leftarrow}{e}})\in\mathbb{C}^*.$$

Therefore,

$$\overline{\eta}\equiv(\eta_e)_{e\in\mathbb{E}_{1,0}}\in\prod_{e\in\mathbb{E}_{1,0}}(\mathbb{C}^*)^{\mathbb{E}_{1,0}}$$

is equal to $(1)_{e\in\mathbb{E}_{1,0}}$ if and only if (3.39) holds. Since cokernel of $\overline{\varrho}_{\mathbb{C}}$ coincides with cokernel of $\varrho_{\mathbb{C}}$, the element

$$[\eta]\in(\mathbb{C}^*)^{\mathbb{E}}/\exp(\mathrm{im}(\varrho_{\mathbb{C}}))$$

in the proof of Lemma 3.7 is the identity element if and only if

$$[\overline{\eta}]\in(\mathbb{C}^*)^{\mathbb{E}_{1,0}}/\exp(\overline{\varrho}_{\mathbb{C}}(\mathbb{C}^{\mathbb{V}_1}))$$

is the identity element. The latter holds if and only if is there exists a rescaling of the sections $(\zeta_v)_{v\in\mathbb{V}_1}$ for which (3.39) holds. $\square$

Proposition 3.24. *Let $D\subset(X,\omega)$ be a smooth symplectic divisor, $J\in\mathcal{J}(X,D)$, and $\mathfrak{s}\in\mathbb{N}^k$. Then there exists a natural surjective map*

$$\pi\colon \overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathfrak{s}}(X,D,A)\longrightarrow\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A). \tag{3.43}$$

Proof. For each relative map f, $\pi(f)$ is the log map obtained by forgetting those unstable $\mathbb{P}^1$-components of the domain which are isomorphically mapped to the trivial fibers of $\mathbb{P}_XD$, and restricting the equivalence class of each section defining a map into a $\mathbb{P}_XD$ to the equivalence classes of its restrictions to each connected component. The required function $s\colon\mathbb{V}(\Gamma)\longrightarrow\mathbb{R}_{\geq 0}$ in Definition 3.8(1) can be taken to be the one given by the layer structure of the relative moduli space. Moreover, by Lemma 3.23, $\pi(f)$ satisfies (3.30) because a set of sections representing f have equal values at the nodes q_e with $I_e=\{1\}$ and $s_{\vec{e}}=0$.

Conversely, let f be any log map with dual graph Γ. By Corollary 3.19, we can assume that the function $s\colon\mathbb{V}(\Gamma)\longrightarrow\mathbb{R}_{\geq 0}$ in Definition 3.8.(1) is integral. We take one such s such that $\max(s)$ is the smallest among all such s. For each connected component Σ_v of Σ in f with $I_v=\{1\}$, choose an arbitrary section ζ_v representing the equivalence class $[\zeta_v]$ in f. By Lemma 3.23, we can choose these sections to have equal values at the nodes q_e with $I_e=\{1\}$ and $s_{\vec{e}}=0$. Define a relative map $\widetilde{f}$ whose restriction to Σ_v is the map corresponding to ζ_v into the s_v-th $\mathbb{P}_XD$ and such that disconnected nodes are connected by adding extra $\mathbb{P}^1$-components to the domain and by mapping them bijectively to the $\mathbb{P}^1$-fibers of $\mathbb{P}_XD$. Since $\max(s)$ is the smallest among all such s, there is at least one non-trivial component in each $\mathbb{P}_XD$ of the expanded degeneration $X[\max(s)]$; i.e. $\widetilde{f}$ defines a stable map into $X[\max(s)]$. It is clear from the construction that $\pi(\widetilde{f})=f$. $\square$

Next, we give an example where the projection map (3.43) is non-trivial and both the relative and the log moduli spaces are smooth. The relative moduli space in this example is some blowup of the log moduli space.

Example 3.25. Let $X=\mathbb{P}^1$, $D=D_1=\mathrm{pt}_1\sqcup\mathrm{pt}_2$ be the disjoint union of two points, $g=0$, $k=4$, and $A=[1]\in H_2(\mathbb{P}^1,\mathbb{Z})\cong\mathbb{Z}$. Therefore $\mathfrak{s}=(0,0,1,1)\in\mathbb{Z}^2$ (or a permutation of that) is the only option for the contact pattern. Then the relative moduli space $\overline{\mathcal{M}}^{\mathrm{rel}}_{0,\mathfrak{s}}(X,D,[1])$ can be identified with a blowup of $\mathbb{P}^1\times\mathbb{P}^1$ at 4 points, while $\overline{\mathcal{M}}^{\log}_{0,\mathfrak{s}}(X,D,[1])$ can be identified with a blowup of $\mathbb{P}^1\times\mathbb{P}^1$ at 2 (of those) points. The projection map in (3.43) corresponds to the blowdown of the two extra exceptional curves.

3.4 Forgetful maps

In this section, we show that the process of forgetting some of the smooth components of an SC divisor $D=\bigcup_{i\in S}D_i$ gives us a forgetful map between the corresponding log moduli spaces. The results are not used in the rest of the paper. While (1.8) is not always an embedding, the map (3.47) below is an embedding. This embedding can be used to reduce certain arguments to the case of smooth divisors.

Let $D=\bigcup_{i\in S}D_i\subset X$ be an SC symplectic divisor, $(J,\omega)\in\mathcal{J}(X,D)$, $g,k\in\mathbb{N}$,

$$\mathfrak{s}=(s_i=(s_{ij})_{j\in S})_{i=1}^k\in(\mathbb{Z}^S)^k, \tag{3.44}$$

and $\Gamma\in\mathrm{DG}(g,\mathfrak{s},A)$. Given $S'\subset S$, let

$$\mathfrak{s}|_{S'}=(s_i=(s_{ij})_{j\in S'})_{i=1}^k\in(\mathbb{Z}^{S'})^k, \quad D|_{S'}=\bigcup_{i\in S'}D_i,$$

and $\Gamma|_{S'}\in\mathrm{DG}(g,\mathfrak{s}|_{S'},A)$ be the decorated dual graph with the same set of vertices and edges, but with the reduced set of decorations

$$I'_v=I_v\cap S' \quad \forall\, v\in\mathbb{V}, \quad I'_e=I_v\cap S' \quad \forall\, e\in\mathbb{E}, \quad s'_{\underset{\rightarrow}{e}}=(s_{\underset{\rightarrow}{e},j})_{j\in S'}\in\mathbb{Z}^{S'} \quad \forall\, \underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}.$$

Define

$$\iota_{S,S'}\colon\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma\longrightarrow\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}|_{S'}}(X,D|_{S'},A)_{\Gamma|_{S'}} \tag{3.45}$$

to be the (well-defined) forgetful map obtained by removing the meromorphic sections

$$(\zeta_{v,i})_{i\in I_v-I'_v\subset S-S'}$$

in (3.9) for all $v\in\mathbb{V}$.

Lemma 3.26. *With notation as above, the map $\iota_{S,S'}$ in (3.45) sends $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma\subset\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ to $\mathcal{M}^{\log}_{g,\mathfrak{s}|_{S'}}(X,D|_{S'},A)_{\Gamma|_{S'}}\subset\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}|_{S'}}(X,D|_{S'},A)_{\Gamma|_{S'}}$.*

Proof. Fix an orientation O on $\mathbb{E}$. With notation as in (3.17), the commutative diagram

$$\begin{array}{ccc}\mathbb{Z}^{\mathbb{E}}\oplus\bigoplus_{v\in\mathbb{V}}\mathbb{Z}^{I_v} & \xrightarrow{\ \varrho\ } & \bigoplus_{e\in\mathbb{E}}\mathbb{Z}^{I_e}\\ \Big\downarrow{\scriptstyle \mathrm{pr}_{\mathbb{D}}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_{\mathbb{T}}}\\ \mathbb{Z}^{\mathbb{E}}\oplus\bigoplus_{v\in\mathbb{V}}\mathbb{Z}^{I'_v} & \xrightarrow{\ \varrho'\ } & \bigoplus_{e\in\mathbb{E}}\mathbb{Z}^{I'_e}\end{array},$$

where $\mathrm{pr}_{\mathbb{D}}$ and $\mathrm{pr}_{\mathbb{T}}$ are the obvious projection maps and ϱ and ϱ' are defined via O, induces a group homomorphism $\mathrm{pr}_{S,S'}\colon\mathcal{G}(\Gamma)\longrightarrow\mathcal{G}(\Gamma|_{S'})$ such that

$$\mathrm{pr}_{S,S'}\big(\mathrm{ob}_\Gamma(f)\big)=\mathrm{ob}_{\Gamma|_{S'}}\big(\iota_{S,S'}(f)\big) \qquad \forall\, f\in\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma.$$

Therefore, $\mathrm{ob}_\Gamma(f)=1$ implies $\mathrm{ob}_{\Gamma|_{S'}}\big(\iota_{S,S'}(f)\big)=1$. $\square$

Taking union over all Γ, we obtain the stratified forgetful map

$$\iota_{S,S'}\colon\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)\longrightarrow\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}|_{S'}}(X,D|_{S'},A).$$

For example, the $S'=\emptyset$ case of (3.45) is the map (1.8) into the underlying moduli space of stable maps; moreover,

$$\iota_{S,S''}=\iota_{S',S''}\circ\iota_{S,S'}\colon\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)\longrightarrow\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}|_{S''}}(X,D|_{S''},A) \qquad \forall S''\subset S'\subset S. \tag{3.46}$$

For $\mathfrak{s}$ as in (3.44), let $\mathfrak{s}_j=\mathfrak{s}|_{\{j\}}=(s_{ij})_{i=1}^k\in(\mathbb{Z})^k$, for all $j\in S$, and define

$$\iota_{S,1}=\prod_{j\in S}\iota_{S,\{j\}}:\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)\longrightarrow\times_{j\in S}\,\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}_j}(X,D_j,A),\tag{3.47}$$

where the right-hand side is the fiber product of

$$\big\{\iota_{\{j\},\emptyset}:\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}_j}(X,D_j,A)\longrightarrow\overline{\mathcal{M}}_{g,k}(X,A)\big\}_{j\in S}.$$

The map ι is well-defined by (3.46) and it is an embedding[15] by Remark 3.13. As the following example shows, this embedding can be proper (i.e. not an equality).

Example 3.27. In Example 3.12, the obstruction groups $\mathcal{G}(\Gamma|_{\{1\}})$ and $\mathcal{G}(\Gamma|_{\{2\}})$ associated to $\Gamma|_{\{1\}}$ and $\Gamma|_{\{2\}}$ are trivial. Therefore, for an element of the right-hand side in (3.47), the corresponding sections $\zeta_{v_1,1}$ and $\zeta_{v_2,2}$ can be arbitrary (modulo the combinatorial conditions imposed by Definitions 3.3 and 3.8). On the otherhand, for such a pair $(\zeta_{v_1,1},\zeta_{v_2,2})$ to define an element of the left-hand side, the corresponding group element in the non-trivial group $\mathcal{G}$ has be the identity. Therefore, the restriction

$$\iota_{\{1,2\},1}:\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma\longrightarrow\times_{j=1,2}\,\mathcal{M}^{\log}_{g,\mathfrak{s}_j}(X,D_j,A)_{\Gamma|_{\{j\}}}$$

of (3.47) to $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ is not an isomorphism.

3.5 Gluing parameters

In this section, we describe the space of gluing parameters for each $\Gamma\in\mathrm{DG}(g,\mathfrak{s},A)$ and show that it is essentially an affine toric variety.

For every log map $f\in\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\Gamma$ choose a representative

$$\big(u_v,\{\zeta_{v,i}\}_{i\in I_v},C_v\equiv(\Sigma_v,\mathfrak{j}_v,\vec{z}_v)\big)_{v\in\mathbb{V}}\tag{3.48}$$

and a set of local coordinates $\{z_{\underset{\rightarrow}{e}}\}_{\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}}$ around the nodes. Since f is unobstructed, by the proof of Lemma 3.7 we can choose $\zeta_{v,i}$ and $z_{\underset{\rightarrow}{e}}$ such that the leading coefficient vectors $\eta_{\underset{\rightarrow}{e}}$ in (3.28) satisfy

$$\eta_{\underset{\rightarrow}{e}}=\eta_{\underset{\leftarrow}{e}}\qquad\forall\ e\in\mathbb{E}.\tag{3.49}$$

For every $v\in\mathbb{V}$ and $i\in S-I_v$, let $t_{v,i}=1$ in (3.50). Then the space of gluing parameters for f is a sufficiently small neighborhood of the origin in

$$\begin{split}\mathfrak{Gl}_\Gamma=\Big\{\big((\varepsilon_e)_{e\in\mathbb{E}},(t_{v,i})_{v\in\mathbb{V},i\in I_v}\big)\in\mathbb{C}^{\mathbb{E}}\times\prod_{v\in\mathbb{V}}\mathbb{C}^{I_v}:\quad\varepsilon_e^{s_{\underset{\rightarrow}{e},i}}t_{v,i}=t_{v',i}\\ \forall\ v,v'\in\mathbb{V},\ e\in\mathbb{E}_{v,v'},i\in I_e,\ \underset{\rightarrow}{e}\ \text{ s.t }\ s_{\underset{\rightarrow}{e},i}\geq0\Big\}\subset\mathbb{C}^{\mathbb{E}}\times\prod_{v\in\mathbb{V}}\mathbb{C}^{I_v}.\end{split}\tag{3.50}$$

The complex numbers ε_e are the gluing parameters for the nodes of Σ and $t_{v,i}$ are the parameters for pushing u_v out in the direction of $\zeta_{v,i}$. In the proof of gluing construction in [7], given a set of representatives $\big(\{z_{\underset{\rightarrow}{e}}\}_{\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}},\{\zeta_{v,i}\}_{v\in\mathbb{V},i\in I_v}\big)$ satisfying (3.49) and a sufficiently small

$$(\varepsilon,t)\equiv\big((\varepsilon_e)_{e\in\mathbb{E}},(t_{v,i})_{v\in\mathbb{V},i\in I_v}\big)\in\mathfrak{Gl}_\Gamma,$$

[15]By the results of Section 4, for appropriate choice of J, all maps $\iota_{S,S'}$ and thus $\iota_{S,1}$ are continuous.

we will construct a pre-gluing almost log map $\widetilde{f}_{\varepsilon,t}$ and show that there is an actual log pseudoholomorphic map "close" to it.

Let

$$\mathbb{T}^\vee \cong \bigoplus_{e\in\mathbb{E}} \mathbb{Z}^{I_e} \xrightarrow{\quad \varrho^\vee \quad} \mathbb{D}^\vee \cong \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v\in\mathbb{V}} \mathbb{Z}^{I_v}$$

be the dual of $\mathbb{Z}$-linear map ϱ associated to Γ in (3.17) (for a fixed choice of orientation O on $\mathbb{E}$). For the kernel subspace $\mathbb{K}=\ker(\varrho)\subset\mathbb{D}$ as in (3.20), let

$$\mathbb{K}^\perp = \{m \in \mathbb{D}^\vee\colon\ \langle m, \alpha\rangle = 0 \quad \forall\ \alpha\in\mathbb{K}\} \subset \mathbb{D}^\vee.$$

Then $\mathrm{Im}(\varrho^\vee)\subset\mathbb{K}^\perp$ with the finite qoutient

$$\mathbb{K}^\perp/\mathrm{image}(\varrho^\vee).$$

Proposition 3.28. *The space of gluing parameters $\mathfrak{Gl}_\Gamma$ in (3.50) is a possibly non-irreducible and non-reduced affine toric sub-variety of $\mathbb{C}^{\mathbb{E}}\times\prod_{v\in\mathbb{V}}\mathbb{C}^{I_v}$ that is isomorphic to $\left|\mathbb{K}^\perp/\mathrm{Im}(\varrho^\vee)\right|$ copies of the irreducible reduced affine toric variety $Y_{\sigma(\Gamma)}$ (with toric fan σ), counting with multiplicities*[16]*. Replacing $\{z_{\underset{\rightarrow}{e}}\}_{\underset{\rightarrow}{e}\in\underset{\rightarrow}{\mathbb{E}}}$ and $\{\zeta_{v,i}\}_{v\in\mathbb{V},i\in I_v}$ with another choice satisfying (3.49) corresponds to a torus action on $\mathfrak{Gl}_\Gamma$.*

Proof. Let us start with some general facts about toric varieties. For $n\in\mathbb{Z}_+$, every vector $m\in\mathbb{Z}^n$ has a unique presentation $m=m_+-m_-$ such that $m_+,m_-\in(\mathbb{Z}_{\geq 0})^n$. Every $m=(a_1,\ldots,a_n)\in(\mathbb{Z}_{\geq 0})^n$ corresponds to the monomial

$$x^m \equiv x_1^{a_1}\cdots x_n^{a_n} \in\mathbb{C}[x_1,\ldots,x_n].$$

For every arbitrary $m\in\mathbb{Z}^n$, the binomial corresponding to m is the expression

$$x^{m,\pm} \equiv x^{m_+} - x^{m_-} \in\mathbb{C}[x_1,\ldots,x_n].$$

For example, if $m=0$, then $x^{m,\pm}=1-1=0$. A binomial ideal[17] I in $\mathbb{C}[x_1,\ldots,x_n]$ is an ideal generated by a finite set of binomials $x^{m_1,\pm},\ldots,x^{m_\ell,\pm}$.

Suppose $\mathbb{K}^\vee\cong\mathbb{Z}^\ell$ is a lattice and $\mathbb{Z}^n\longrightarrow\mathbb{K}^\vee$ is a surjective $\mathbb{Z}$-linear map. Let $\mathbb{R}^n\longrightarrow\mathbb{K}^\vee_{\mathbb{R}}$ be the corresponding $\mathbb{R}$-linear projection map and $\sigma^\vee$ be the image of the cone $\mathbb{R}^n_{\geq 0}$ in $\mathbb{K}^\vee_{\mathbb{R}}$. Then the dual map $\iota\colon\mathbb{K}\hookrightarrow\mathbb{Z}^n$ is an embedding and the dual of $\sigma^\vee$ is the toric fan

$$\sigma=\mathbb{K}_{\mathbb{R}}\cap\iota^{-1}(\mathbb{R}^n_{\geq 0}).$$

In this situation, by [4, Proposition 1.1.9], the toric variety Y_σ associated to the toric fan σ is the zero set of the binomial ideal

$$I = \{x^{m,\pm}\colon\quad m\in\mathbb{K}^\perp\subset\mathbb{Z}^n\}. \tag{3.51}$$

With $\mathbb{Z}^n=\mathbb{Z}^{\mathbb{E}}\oplus\bigoplus_{v\in\mathbb{V}}\mathbb{Z}^{I_v}$, $\mathbb{K}$ in (3.21), and $\sigma=\sigma(\Gamma)$ as in (3.34), the previous argument implies that $Y_{\sigma(\Gamma)}$ is the zero set of the binomial ideal (3.51).

[16]We dont know of any example, arising from such dual graphs, such that the multiplicities are bigger than 1.
[17]For more general binomial ideals see [5].

Let $I' \subset I$ be the binomial sub-ideal generated by the elements of $\mathrm{Im}(\varrho^\vee) \subset \mathbb{K}^\perp$. By definition of ϱ and (3.50), the space of gluing parameters $\mathfrak{Gl}_\Gamma$ is the zero set (scheme) of I'. Therefore $Y_{\sigma(\Gamma)} \subset \mathfrak{Gl}_\Gamma$. Note that $Y_{\sigma(\Gamma)}$ is the Zariski closure of the irreducible subgroup

$$\{t \in (\mathbb{C}^*)^n : t^m = 1 \qquad \forall\, m \in \mathbb{K}^\perp\} \subset (\mathbb{C}^*)^n$$

and $\mathfrak{Gl}_\Gamma$ is the Zariski closure of possibly non-irreducible subgroup

$$\{t \in (\mathbb{C}^*)^n : t^m = 1 \qquad \forall\, m \in \mathrm{Im}(\varrho^\vee)\} \subset (\mathbb{C}^*)^n; \tag{3.52}$$

see [4, Definition 1.1.7]. Therefore, all the irreducible components of $\mathfrak{Gl}_\Gamma$ are isomorphic to $Y_{\sigma(\Gamma)}$. Since

$$\big|I/I'\big| = \Big|\mathbb{K}^\perp/\mathrm{Im}(\varrho^\vee)\Big|,$$

$\mathfrak{Gl}_\Gamma$ is isomorphic to $|\mathbb{K}^\perp/\mathrm{Im}(\varrho^\vee)|$ copies of $Y_{\sigma(\Gamma)}$, counting with multiplicities. The last statement in Proposition 3.28 follows how the subgroup (3.52) acts on (3.50). $\square$

Example 3.29. Suppose $S = \{1, 2\}$ and Γ is the decorated dual graph with two vertices $\mathbb{V} = \{v_1, v_2\}$ and two edges e_1 and e_2 connecting them. Choose $\underrightarrow{e}_1$ and $\underrightarrow{e}_2$ to be the orientations starting at v_1. Suppose

$$I_{v_1} = \{1\}, \quad I_{v_2} = \{2\}, \quad s_{\underrightarrow{e}_1} = s_{\underrightarrow{e}_2} = (-2, 2).$$

Then the linear map

$$\varrho\colon \mathbb{Z}^{\mathbb{E}} \oplus \mathbb{Z}^{I_{v_1}} \oplus \mathbb{Z}^{I_{v_2}} \equiv \mathbb{Z}_{e_1} \oplus \mathbb{Z}_{e_2} \oplus \mathbb{Z}_{v_1} \oplus \mathbb{Z}_{v_2} \longrightarrow \mathbb{Z}^{\{1,2\}}_{e_1} \oplus \mathbb{Z}^{\{1,2\}}_{e_2}$$

is given by

$$\begin{aligned} \varrho(1_{e_1}) &= ((-2,2)_{e_1}, (0,0)_{e_2}), & \varrho(1_{e_2}) &= ((0,0)_{e_1}, (-2,2)_{e_2}), \\ \varrho(1_{v_1}) &= ((1,0)_{e_1}, (1,0)_{e_2}), & \varrho(1_{v_2}) &= ((0,-1)_{e_1}, (0,-1)_{e_2}). \end{aligned}$$

It is straightforward to check that $\mathrm{Ker}(\varrho)$ is one-dimensional and is generated by

$$1_{e_1} + 1_{e_2} + 2 \cdot 1_{v_1} + 2 \cdot 1_{v_2}\,;$$

i.e. $Y_{\sigma(\Gamma)} \cong \mathbb{C}$. On the other hand, $\mathfrak{Gl}_\Gamma$ is the sub-variety cut-out by

$$\varepsilon_1^2 = t_{v_2}, \quad \varepsilon_2^2 = t_{v_2}, \quad \varepsilon_1^2 = t_{v_1}, \quad \varepsilon_2^2 = t_{v_1}.$$

This is isomorphic to 2 copies of $\mathbb{C}$, the component $Y_{\sigma(\Gamma)}$ is the image of $t \longrightarrow (t, t, t^2, t^2)$ and the other one is the image of $t \longrightarrow (t, -t, t^2, t^2)$. It is straightforward to see that

$$\mathrm{Ker}(\varrho)^\perp/\mathrm{Im}(\varrho^\vee)$$

is isomorphic to $\mathbb{Z}_2$ and is generated by the class of $[1^\vee_{e_1} - 1^\vee_{e_2}]$.

4 Compactness

In this section, after a quick review of the convergence problem for the Deligne-Mumford space and for the classical moduli spaces of pseudoholomorphic maps, we prove Theorem 1.1 in several steps. The main step of the proof is Proposition 4.11, that relates the sequence of "gluing" and "rescaling" parameters, when a sequence of J-holomorphic maps breaks into two pieces with at least one of them mapped into D. We are currently working to generalize the proof of Proposition 4.11 to arbitrary $(J, \omega) \in \mathcal{J}(X, D)$.

4.1 Classical Gromov convergence

Definition 4.1. Given a k-marked genus g (possibly not stable) nodal surface $C \equiv (\Sigma, \vec{z})$ with dual graph Γ, a cutting configuration with dual graph Γ' is a set of disjoint embedded circles

$$\gamma \equiv \{\gamma_e\}_{e\in\mathbb{E}(\Gamma'/\Gamma)} \subset \Sigma,$$

away from the nodes and marked points, such that the nodal marked surface $(\Sigma', \vec{z}')$ obtained by pinching every γ_e into a node q_e has dual graph Γ'.

Thus, a cutting configuration corresponds to a continuous map

$$\varphi_\gamma \colon C \longrightarrow C',$$

denoted by a γ-degeneration[18] in what follows, onto a k-marked genus g nodal surface C' with dual graph Γ' such that $\vec{z}' = \varphi_\gamma(\vec{z})$, the preimage of every node of Σ is either a node in Σ' or a circle in γ, and the restriction

$$\varphi_\gamma \colon \Sigma\backslash\gamma \longrightarrow \Sigma'\backslash\big(\varphi_\gamma(\gamma) \equiv \{q_e\}_{e\in\mathbb{E}(\Gamma'/\Gamma)}\big)$$

is a diffeomorphism. Let

$$\gamma^* \colon \Gamma' \longrightarrow \Gamma \tag{4.1}$$

be the map corresponding to φ_γ between the dual graphs. We have

$$\mathbb{E}(\Gamma') \approx \mathbb{E}(\Gamma) \cup \mathbb{E}(\Gamma'/\Gamma), \qquad \mathbb{L}(\Gamma') \approx \mathbb{L}(\Gamma),$$

such that $\gamma^*|_{\mathbb{E}(\Gamma)\subset\mathbb{E}(\Gamma')}$ and $\gamma^*|_{\mathbb{L}(\Gamma')}$ are isomorphisms and

$$\gamma^* \colon \mathbb{E}(\Gamma'/\Gamma) \longrightarrow \mathbb{V}(\Gamma)$$

sends the edge e corresponding to γ_e to v, if $\gamma_e \subset \Sigma_v$. For every $v' \in \mathbb{V}(\Gamma')$ there exists a unique $v \in \mathbb{V}(\Gamma)$ and a connected component $U_{v'}$ of $\Sigma_v \setminus \{\gamma_e\}_{e\in\mathbb{E}(\Gamma'/\Gamma)}$ such that $\Sigma'_{v'} \subset \Sigma'$ is obtained by collapsing the boundaries of $\mathrm{cl}(U_{v'})$ (cl means closure). This identification determines the surjective map

$$\gamma^* \colon \mathbb{V}(\Gamma') \longrightarrow \mathbb{V}(\Gamma), \quad v' \longrightarrow v. \tag{4.2}$$

From another perspective, a cutting configuration corresponds to expanding each vertex $v \in \mathbb{V}(\Gamma)$ into a sub-graph $\Gamma'_v \subset \Gamma'$ (some times, this is just adding more loops to the existing graph) with the set of vertices and edges

$$\mathbb{V}(\Gamma'_v) = (\gamma^*)^{-1}(v) \quad \text{and} \quad \mathbb{E}(\Gamma'_v) = (\gamma^*)^{-1}(v) \cap \mathbb{E}(\Gamma'/\Gamma).$$

Moreover, $g_v = g_{\Gamma'_v}$, the ordering of marked points are as before, and

$$A_v = \sum_{v'\in\mathbb{V}(\Gamma'_v)} A_{v'}. \tag{4.3}$$

Figure 7 illustrates a cutting configuration over a 1-nodal curve of genus 3 and the corresponding dual graphs.

[18] It is called "deformation" in [30].

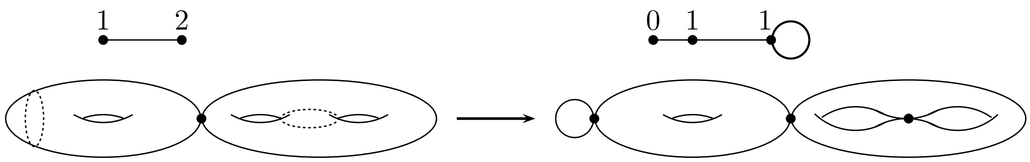

Figure 7: On left, a 1-nodal curve of genus 3 and a cutting set made of two circles. On right, the resulting pinched curve.

A sequence $\{\varphi_{\gamma_a}\colon C_a \longrightarrow C'\}_{a\in\mathbb{N}}$ of degenerations of marked nodal curves is called monotonic if $\Gamma(C_a)\cong\Gamma$ for some fixed Γ and the induced maps $\gamma_a^*\colon \Gamma\longrightarrow\Gamma'$ are all the same. In this situation, the underlying marked nodal surfaces are isomorphic; i.e.,

$$(C_a,\gamma_a)\cong\big((\Sigma,\mathfrak{j}_a,\vec{z}),\gamma\big)\quad\forall\, a\in\mathbb{N}, \tag{4.4}$$

for some fixed marked surface $(\Sigma,\vec{z})$ with dual graph Γ and cutting configuration γ. In the following, we denote the complement of the set of nodes

$$\{q_e\}_{e\in\mathbb{E}(\Gamma'/\Gamma)}\subset\Sigma'$$

by Σ'_*.

Definition 4.2 ([30, Definition 13.3])**.** A sequence $\{C_a\equiv(\Sigma_a,\mathfrak{j}_a,\vec{z}_a)\}_{a\in\mathbb{N}}$ of genus g k-marked nodal curves monotonically converges to $C'\equiv(\Sigma',\mathfrak{j}',\vec{z}')$, if there exist a sequence of cutting configurations γ_a on C_a of type Γ' and a monotonic sequence $\varphi_{\gamma_a}\colon C_a\longrightarrow C'$ of γ_a-degenerations such that the sequence $(\varphi_{\gamma_a}|_{\Sigma_a\setminus\gamma_a})_*\mathfrak{j}_a$ converges to $\mathfrak{j}'|_{\Sigma'_*}$ in the C^∞-topology[19].

By [30, Section 13], the topology underlying the holomorphic orbifold structure of $\overline{\mathcal{M}}_{g,k}$ is equivalent to the sequential DM-convergence topology: a sequence $\{C_a\}_{a\in\mathbb{N}}$ of genus g k-marked stable nodal curves DM-converges to C' if a subsequence of that monotonically converges to C'. The following result, known as Gromov's Compactness Theorem [15, Theorem 1.5.B], describes a convergence topology on $\overline{\mathcal{M}}_{g,k}(X,A,J)$ which is compact and metrizable; see [26], [19, Theorem 1.2], [31, Theorem 0.1], and [25, Section 5] for further details. In the special case of Deligne-Mumford space, Gromov convergence is equal to the DM-convergence discussed above.

Theorem 4.3. *Let (X,ω) be a compact symplectic manifold, $\{J_a\}_{a\in\mathbb{N}}$ be a sequence of ω-compatible*[20] *almost complex structures on X converging in C^∞-topology to J, and*

$$\{f_a\equiv(u_a,C_a\equiv(\Sigma_a,\mathfrak{j}_a,\vec{z}_a))\}_{a\in\mathbb{N}}$$

be a sequence of stable J_a-holomorphic maps of bounded (symplectic) area into X. After passing to a subsequence, still denoted by $\{f_a\}_{a\in\mathbb{N}}$, there exists a unique (up to automorphism) stable J-holomorphic map

$$f'\equiv(u',C'\equiv(\Sigma',\mathfrak{j}',\vec{z}'))$$

such that $\{C_a\}_{a\in\mathbb{N}}$ monotonically converges to C', and

(1) *we can choose the γ_a-degeneration maps $\varphi_{\gamma_a}:\Sigma_a\longrightarrow\Sigma'$ of the monotonic convergence such that the restriction*

$$u_a|_{\Sigma_a\setminus\gamma_a}\circ\varphi_{\gamma_a}^{-1}|_{\Sigma'_*}$$

converges uniformly with all the derivatives to $u|_{\Sigma'_*}$ over compact sets;

[19] uniform convergence on compact sets with all derivatives.

[20] or tame.

(2) *with the dual graphs* $\Gamma \cong \Gamma(C_a)$ *and* $\Gamma' = \Gamma(C')$ *as in the definition of monotonic sequences,*

$$\lim_{a\longrightarrow\infty} u_a(\gamma_{a,e}) = u'(q_e) \qquad \forall\ e\in\mathbb{E}(\Gamma'/\Gamma);$$

(3) *symplectic area of* f' *coincides with the symplectic area of* f_a*, for all* $a\in\mathbb{N}$*.*

It follows from the properties (1) and (3) that for every $v'\in\Gamma'$, with $U_{a,v'}\subset\Sigma_a$ as in the definition of $\gamma_a^*(v')$,

$$\lim_{a\longrightarrow\infty}\int_{\mathrm{cl}(U_{a;v'})} {u_a}^*\omega = \int_{\Sigma'_{v'}} (u')^*\omega.$$

Moreover, the stronger identity (4.3) holds. With respect to the identification of the domains and degeneration maps

$$(\varphi_{\gamma_a}\colon \Sigma_a \longrightarrow \Sigma') \cong (\varphi_\gamma\colon \Sigma \longrightarrow \Sigma')$$

as in (4.4), the second property implies that the sequence $(u_a\colon\Sigma\longrightarrow X)_{a\in\mathbb{N}}$ C^0-converge to $u\circ\varphi_\gamma$.

Assume $D\subset X$ is an SC symplectic divisor, $(J,\omega)\in\mathcal{J}(X,D)$, and

$$\{f_{\log,a} \equiv (u_a,[\zeta_a],C_a\equiv(\Sigma_a,\mathfrak{j}_a,\vec{z}_a))\}_{a\in\mathbb{N}} \tag{4.5}$$

is a sequence of stable log maps in $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$. After passing to a subsequence, we may assume that all the maps in (4.5) have the same decorated dual graph $\Gamma(\mathbb{V},\mathbb{E},\mathbb{L})$, and that the underlying sequence of stable maps

$$\{f_a \equiv (u_a,C_a\equiv(\Sigma_a,\mathfrak{j}_a,\vec{z}_a))\}_{a\in\mathbb{N}} \tag{4.6}$$

in $\overline{\mathcal{M}}_{g,k}(X,A)$ (with the same domain) Gromov convergences to the stable map

$$f\equiv(u,C\equiv(\Sigma,\mathfrak{j},\vec{z}))\in\overline{\mathcal{M}}_{g,k}(X,A)$$

as in Theorem 4.3. Then, in order to prove Theorem 1.1, (for J as in the statement of the theorem) after passing to a further subsequence, we prove that f lifts to a unique log map $f_{\log}\in\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$. The (meromorphic sections that enhance f to a) log map $f_{\log}$ is constructed in Section 4.2. We first prove that $f_{\log}$ is a pre-log map in Lemma 4.9; the proof works for arbitrary $(J,\omega)\in\mathcal{J}(X,D)$. Then, in Theorem 4.10, we prove that $f_{\log}$ satisfies the conditions of Definition 3.8. Since there are only finitely many possible log lifts of a stable map f, it follows that (1.8) is a continuous local embedding.

Remark 4.4. A log map should be thought of as an stable map plus some partial tangent vector (to the moduli space) data. A sequence of stable maps $\{f_a\}_{a\in\mathbb{N}}$ converging to f may approach it from various directions. Therefore, we need to further pass to a subsequence to pick one particular direction.

4.2 Log-Gromov convergence

We start by recalling some basic structures associated to smooth symplectic divisors.

Let $D\subset(X,\omega)$ be a smooth symplectic divisor, $J\in\mathcal{J}(X,D,\omega)$, and $\mathfrak{i}_{\mathcal{N}_XD}$ be the induced complex structure on $\mathcal{N}_XD$. Let $J_{X,D}$ be the almost complex structure on $\mathcal{N}_XD$ induced by the $\bar\partial$-operator $\bar\partial_{\mathcal{N}_XD}$ associated to $(\mathcal{N}_XD,\mathfrak{i}_{\mathcal{N}_XD})$ as in the end of Section 2.1. Fix a compatible pair of a Hermitian metric ρ and a Hermitian connection ∇ on $\mathcal{N}_X D$. Such a connection ∇ defines a 1-form α_∇ on $\mathcal{N}_X D - D$ whose restriction to each fiber $\mathcal{N}_X D|_p - \{p\} \cong \mathbb{C}^*$ is the 1-form $\mathrm{d}\theta$ with respect to the polar coordinates (r,θ) determined by $\rho \approx r^2$ and the complex structure $\mathfrak{i}_{\mathcal{N}_X D}$. Recall from Section 2.1 that the connection ∇ gives a splitting

$$T\mathcal{N}_X D \cong \pi^* TD \oplus \pi^* \mathcal{N}_X D,$$

such that $J_{X,D}$ is equal to $\pi^* J_D$ on the first summand and $\pi^* \mathfrak{i}_{\mathcal{N}_X D}$ on the second one. By the Symplectic Neighborhood Theorem [24, Theorem 3.30], for $\mathcal{N}'_X D$ sufficiently small, there exists a diffeomorphism

$$\Psi : \mathcal{N}'_X D \longrightarrow X \tag{4.7}$$

from a neighborhood of D in $\mathcal{N}_X D$ onto a neighborhood of D in X such that $\Psi(x) = x$, the isomorphism

$$\mathcal{N}_X D|_x = T_x^{\mathrm{ver}} \mathcal{N}_X D \hookrightarrow T_x \mathcal{N}_X D \stackrel{\mathrm{d}_x \Psi}{\longrightarrow} T_x X \longrightarrow \frac{T_x X}{T_x D} \equiv \mathcal{N}_X D|x$$

is the identity map for every $x \in D$, and

$$\Psi^* \omega = \omega_{X,D} = \pi^*(\omega|_D) + \frac{1}{2}\mathrm{d}(\rho \alpha_\nabla). \tag{4.8}$$

The last property is not needed for most of the following arguments. In the language of [9, Definition 2.9], the tuple $\mathcal{R} = (\rho, \nabla, \Psi)$ is called an ω-regularization. If $\Psi^* J = J_{X,D}$, then the tuple $(J, \mathcal{R}, \omega)$ is an element of $\mathrm{AK}(X,D)$ mentioned in (1.7).

For any $c \in \mathbb{R}_{>0}$, define

$$\mathcal{N}_X D(c) = \{v \in \mathcal{N}_X D \colon \rho(v) < c\}.$$

For any $t \in \mathbb{C}^*$, define

$$\begin{aligned} &R_t \colon \mathcal{N}_X D \longrightarrow \mathcal{N}_X D, && R_t(v) = tv \quad \forall v \in \mathcal{N}_X D, \\ &\Psi_t = \Psi \circ R_t \colon R_t^{-1}(\mathcal{N}'_X D) \longrightarrow X, && J_t = \Psi_t^* J. \end{aligned} \tag{4.9}$$

Lemma 4.5. *We have*

$$\lim_{t \longrightarrow 0} J_t|_{\overline{\mathcal{N}_X D(c)}} = J_0 := J_{X,D}|_{\overline{\mathcal{N}_X D(c)}} \qquad \forall\ c \in \mathbb{R}_{>0} \tag{4.10}$$

uniformly with all derivatives.

Proof. In order to simplify the notation, let us forget about Ψ and think of J as an almost complex structure on $\mathcal{N}'_X D$ itself; then $J|_D = J_{X,D}|_D$ and $J_t = R_t^* J$, for every $t \in \mathbb{C}^*$. We decompose J into various compoenents,

$$\begin{gathered} J_v(\alpha) = \Big(J_v^{hh}(\alpha^h) + J_v^{\perp h}(\alpha^\perp)\Big) \oplus \Big(J_v^{h\perp}(\alpha^h) + J_v^{\perp\perp}(\alpha^\perp)\Big), \\ \forall\ x \in D,\ v \in \mathcal{N}_X D|_x,\ \alpha = \alpha^h \oplus \alpha^\perp \in \big(\pi^* TD \oplus \pi^* \mathcal{N}_X D\big)|_v, \end{gathered}$$

where, for example, J^{hh} is the component which maps the horizontal subspace $\pi^* TD$ to $\pi^* TD$. Identifying α^h and $\alpha^\perp$ with the corresponding vectors in $T_x D$ and $\mathcal{N}_X D|_x$, respectively, we get

$$(J_t)_v(\alpha) = \Big(J_{tv}^{hh}(\alpha^h) + J_{tv}^{\perp h}(t\alpha^\perp)\Big) \oplus \Big(\frac{1}{t} J_{tv}^{h\perp}(\alpha^h) + J_{tv}^{\perp\perp}(\alpha^\perp)\Big).$$

On each compact set $\overline{\mathcal{N}_X D(c)}$, the first summand uniformly converges to $J_D(\alpha^h)$, and $J^{\perp\perp}_{tv}(\alpha^\perp)$ uniformly converges to $\mathfrak{i}_{\mathcal{N}_X D}(\alpha^\perp)$ (with all the derivatives). Finally, the term

$$\frac{1}{t} J^{h\perp}_{tv}(\alpha^h)$$

C^∞-converges to the normal part of $N_J(v,\alpha^h)$, which is zero by (1.2). $\square$

For any (continuous) map $u\colon\Sigma\longrightarrow\mathcal{N}_X D$, let

$$\overline{u}=\pi\circ u\colon\,\Sigma\longrightarrow D$$

denote its projection to D. Then u is equivalent to a section $\zeta\in\Gamma(\Sigma,\overline{u}^*\mathcal{N}_X D)$ in the sense that $u(x)=\zeta(x)\in\mathcal{N}_X D|_{\overline{u}(x)}$ for all $x\in\Sigma$. We will use this correspondence repeatedly in the following arguments. In particular, by (1)-(3) in Page 9, u is $J_{X,D}$-holomorphic if and only if $\overline{u}$ is J_D-holomorphic and $\bar{\partial}_{\mathcal{N}_X D}\zeta=0$.

For two sequences of non-zero complex numbers $(t_a)_{a\in\mathbb{N}}$ and $(t'_a)_{a\in\mathbb{N}}$, we write

$$(t_a)_{a\in\mathbb{N}}\sim(t'_a)_{a\in\mathbb{N}}\qquad\text{if}\qquad\lim_{a\longrightarrow\infty}t_a/t'_a=1. \tag{4.11}$$

The right-hand side of (4.11) defines an equivalence relation on the set of such sequences and we denote the equivalence class of a sequence $(t_a)_{a\in\mathbb{N}}$ by $[(t_a)_{a\in\mathbb{N}}]$. For an equivalence class $[(t_a)_{a\in\mathbb{N}}]$ and $t\in\mathbb{C}^*$, the equation

$$t[(t_a)_{a\in\mathbb{N}}]:=[(tt_a)_{a\in\mathbb{N}}]$$

is well-defined and defines an action of $\mathbb{C}^*$ on the set of equivalence classes. Moreover, the operation of point-wise multiplication/divison between such sequences

$$(t_a)_{a\in\mathbb{N}}\cdot(t'_a)_{a\in\mathbb{N}}=(t_at'_a)_{a\in\mathbb{N}}$$

descends to a well-defined multiplication/division operation between the equivalence classes.

Proposition 4.6. *With (X,D,ω,J,Ψ) as above (i.e. D is smooth), let*

$$\{f_a\equiv(u_a,C_a\equiv(\Sigma_a,\mathfrak{j}_a,\vec{z}_a))\}_{a\in\mathbb{N}} \tag{4.12}$$

be a sequence of stable maps with smooth domain in $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_\emptyset$ that Gromov converges (considered as a sequence in $\mathcal{M}_{g,k}(X,A)$) to the marked nodal map

$$f\equiv(u_v,C_v\equiv(\Sigma_v,\mathfrak{j}_v,\vec{z}_v))_{v\in\mathbb{V}}\in\overline{\mathcal{M}}_{g,k}(X,A)$$

with dual graph $\Gamma=\Gamma(\mathbb{V},\mathbb{E},\mathbb{L})$, nodal domain $\Sigma=\bigcup_{v\in\mathbb{V}}\Sigma_v$, and nodal map $u\equiv(u_v)_{v\in\mathbb{V}}\colon\Sigma\longrightarrow X$. With notation as in (3.2), (3.40), and Theorem 4.3, after passing to a subsequence of $\mathbb{N}$ which we still denote by $\mathbb{N}$, for every $v\in\mathbb{V}_1$, there exists a unique $\mathbb{C}^*$-equivalence class of meromorphic sections

$$[\zeta_v]\in\Gamma_{\rm mero}(\Sigma_v,u_v^*\mathcal{N}_X D)/\mathbb{C}^*$$

such that for any representative ζ_v, there exists a unique equivalence class of sequences of non-zero complex numbers $[(t_{\zeta_v,a})_{a\in\mathbb{N}}]$ satsfying

$$\lim_{a\longrightarrow\infty}\Psi^{-1}_{t_{\zeta_v,a}}\circ u_a\circ\varphi^{-1}_{\gamma_a}|_K=\zeta_v|_K \tag{4.13}$$

for every compact set $K\subset\Sigma_v-q_v$. Furthermore, $[\zeta_v]$ has no pole/zero in $\Sigma_v-(q_v\cup z_v)$, and it has a zero of order s_i at z^i, for all $z^i\in\vec{z}_v$.

Proof. For every fixed such K, by Theorem 4.3, the sequence

$$\overline{u}_{a,K}=\pi\circ u_{a,K}\colon K\longrightarrow D, \quad \text{with} \quad u_{a,K}\equiv\Psi^{-1}\circ u_a\circ\varphi_{\gamma_a}^{-1}|_K\colon K\longrightarrow\mathcal{N}'_XD \qquad \forall\ a\gg1,$$

converges uniformly with all the derivative to $u_v|_K$, and

$$u_{a,K}(z)=\zeta_{a,K}(z)$$

for some non-trivial smooth section $\zeta_{a,K}\in\Gamma(K,\overline{u}_{a,K}^*\mathcal{N}_XD)$, such that the sequence $\zeta_{a,K}$ converges uniformly with all the derivatives (with respect to a connection ∇) to 0. Choose $(t_{v,K,a})_{a\gg1}$ so that

$$||t_{v,K,a}^{-1}\zeta_{a,K}||_{L^\infty(K)}=c \qquad \forall\ a\gg1 \tag{4.14}$$

for some arbitrary non-zero constant c. Then, by [25, Theorem 4.1.1] (after passing to a subsequence), the sequence $(\Psi_{t_{v,K,a}}^{-1}\circ u_v\circ\varphi_{\gamma_a}^{-1}|_K)_{a\gg1}$ of $J_{t_{v,K,a}}$-holomorphic maps in $\mathcal{N}_XD(c)$ converges uniformly with all the derivatives to a $J_{X,D}$-holomorphic map

$$u_{\infty,K}\colon K\longrightarrow\mathcal{N}_XD(c).$$

By (4.14), (3) in Page 9, and since $\overline{u}_{a,v}$ converges to $u_v|K$, we have

$$\overline{u}_{\infty,K}=u_v|K \qquad \text{and} \qquad u_{\infty,K}=\zeta_{v,K}$$

for some non-trivial $\bar\partial_{\mathcal{N}_XD}$-holomorphic section $\zeta_{v,K}$ of $u_v^*\mathcal{N}_XD|_K$.

Since $\zeta_{a,K}$ is non-zero away from $\vec{z}_a\cap\varphi_{\gamma_a}^{-1}(K)$, $\zeta_{v,K}$ is non-zero away from $\vec{z}_v\cap K$. Choose a reference point $p\in\Sigma_v-(q_v\cup z_v)$. We may assume $p\in K$ and uniformly rescale c_K and $(t_{v,K,a})_{a\gg1}$ so that $\zeta_{v,K}|_p$ is some fixed non-zero vector $v_p\in\mathcal{N}_XD|_{u_v(p)}$. Let

$$K_1\subset K_2\subset\cdots$$

be a sequence exhausting Σ_v-q_v. By the normalization assumption (and the uniqueness of the limiting sections), $\zeta_{v,K_i}=\zeta_{v,K_{i+1}}|_{K_i}$ for all $i\in\mathbb{N}$ and the equation

$$\zeta_v(x):=\zeta_{v,K_i}(x) \qquad \forall\ x\in\Sigma_v-q_v,\ i\in\mathbb{N}\ \text{ s.t. }\ x\in K_i,$$

defines a holomorphic section of $u_v^*\mathcal{N}_XD|_{\Sigma_v-q_v}$ such that (4.13) holds. Moreover,

$$(t_{v,K_i,a})_{a\in\mathbb{N}}\sim(t_{v,K_j,a})_{a\in\mathbb{N}} \qquad \forall i,j\in\mathbb{N}.$$

It remains to show that ζ_v has at most finite order poles at the nodes and $\mathrm{ord}_{z^i}(\zeta_v)=s_i$ for all $z^i\in\vec{z}_v$.

For any marked point $z^i\in\vec{z}_v$, let $\Delta_i\subset\Sigma_v$ be a sufficiently small disk around z^i that contains no other marked point or nodal point. For a sufficiently large, the order of vanishing of u_a at z_a^i is equal to the winding number of

$$\Psi_{t_{v,\partial\overline{\Delta}_i,a}}^{-1}\circ u_a\circ\varphi_{\gamma_a}^{-1}|_{\partial\overline{\Delta}_i} \qquad \forall\ a\gg1$$

around D. With $K=\overline{\Delta}_i$ in (4.13), these numbers are the same for $a\gg1$ and they are equal to the winding number of $u_{\infty,\overline{\Delta}_i}|_{\partial\overline{\Delta}_i}$ around D. The latter is equal to the order of ζ_v at z^i. We conclude that the contact orders stay the same at the marked points.

Similarly, for any nodal point $q_{\underrightarrow{e}}\in\Sigma_v$, with $\underrightarrow{e}\in\overrightarrow{\mathbb{E}}$ and $v_1(\underrightarrow{e})=v$, let $\Delta\subset\Sigma_v$ be a sufficiently small disk around $q_{\underrightarrow{e}}$ that contains no other marked point or nodal point. Choose a compact set $K\subset\Sigma_v-q_v$ so that one of whose boundary circles coincides with $\partial\overline{\Delta}$. The winding numbers of

$$\Psi^{-1}_{t_{v,K,a}}\circ u_a\circ\varphi^{-1}_{\gamma_a}|_{\partial\overline{\Delta}}\qquad\forall\, a\gg 1$$

around D are finite and, by (4.13), they are the same as the winding number of $u_{\infty,K}|_{\partial\overline{\Delta}}$ around D. The latter is equal to the order of ζ_v at $q_{\underrightarrow{e}}$. We conclude that ζ_v extends to a meromorphic section at $q_{\underrightarrow{e}}$. ◻

Remark 4.7. Note that that the sections ζ_v and the equivalence class of the rescaling sequence $[(t_{\zeta_v,a})_{a\in\mathbb{N}}]$ are independent of the choice of Ψ. It is also clear from (4.13) that

$$[(t_{c\zeta_v,a})]_{a\in\mathbb{N}}=c^{-1}[(t_{\zeta_v,a})_{a\in\mathbb{N}}]\qquad\forall c\in\mathbb{C}^*.\tag{4.15}$$

The following is the analogue of Proposition 4.6 for a sequence of stable log maps with smooth domain in $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_{\{1\}}$.

Corollary 4.8. *Suppose*

$$\{f_a\equiv(u_a,\zeta_a,C_a\equiv(\Sigma_a,\mathfrak{j}_a,\vec{z}_a))\}_{a\in\mathbb{N}}\tag{4.16}$$

is a sequence of representatives of stable log maps with smooth domain in $\mathcal{M}^{\log}_{g,\mathfrak{s}}(X,D,A)_{\{1\}}$ such that the underlying sequence of stable J_D-holomorphic maps

$$\{f_a\equiv(u_a,C_a\equiv(\Sigma_a,\mathfrak{j}_a,\vec{z}_a))\}_{a\in\mathbb{N}}\tag{4.17}$$

Gromov converges, as a sequence in $\mathcal{M}_{g,k}(D,A)$, to the nodal map

$$f\equiv(u_v,C_v\equiv(\Sigma_v,\mathfrak{j}_v,\vec{z}_v))_{v\in\mathbb{V}}\in\overline{\mathcal{M}}_{g,k}(D,A)$$

with the dual graph $\Gamma=\Gamma(\mathbb{V},\mathbb{E},\mathbb{L})$, nodal domain $\Sigma=\bigcup_{v\in\mathbb{V}}\Sigma_v$, and nodal map $u\equiv(u_v)_{v\in\mathbb{V}}:\Sigma\longrightarrow X$. With notation as in (3.2), (3.40), and Theorem 4.3, (after passing to a subsequence) for every $v\in\mathbb{V}$, there exists a unique $\mathbb{C}^*$-equivalence class of meromorphic sections

$$[\zeta_v]\in\Gamma_{\mathrm{mero}}(\Sigma_v,u_v^*\mathcal{N}_XD)/\mathbb{C}^*$$

such that for every representative ζ_v, there exists a unique equivalence class of sequences of non-zero complex numbers $[(t_{\zeta_v,a})_{a\in\mathbb{N}}]$ such that

$$\lim_{a\longrightarrow\infty}t^{-1}_{\zeta_v,a}\;\zeta_a\circ\varphi^{-1}_{\gamma_a}|_K=\zeta_v|_K,\tag{4.18}$$

for any compact set $K\subset\Sigma_v-q_v$. Furthermore, $[\zeta_v]$ only depends on the sequence of equivalence classes $\big([\zeta_a]\big)_{a\in\mathbb{N}}$, it has no pole/zero in $\Sigma_v-(q_v\cup z_v)$, and it has a zero/pole of the same order s_i at z^i, for all $z^i\in\vec{z}_v$.

Proof. If (4.18) holds for a sequence $(\zeta_a,t_{\zeta_v,a})_{a\in\mathbb{N}}$, then it also holds for any other simultaneous reparametrization $(t_a\zeta_a,t_at_{\zeta_v,a})_{a\in\mathbb{N}}$. Therefore, (4.18) only depends on the sequence of equivalence classes $\big([\zeta_a]\big)_{a\in\mathbb{N}}$. Every map in the sequence (4.16) corresponds to a $J_{X,D}$-holomorphic map in

$$\mathcal{M}^{\log}_{g,\mathfrak{s}}(\mathcal{N}_XD,D,A)_{\emptyset}.$$

We can choose the representatives ζ_a so that their image in $\mathcal{N}_X D$ lie in an arbitrary small compact neighborhood[21] of D. Replacing (X, D, ω, J) with $(\mathcal{N}_X D, D, \omega_{X,D}, J_{X,D})$ and Ψ with the identity map in Proposition 4.6, we get the desired result. $\square$

We are now ready to state the main results which equip $\overline{\mathcal{M}}_{g,\mathfrak{s}}(X, D, A)$ with a convergence topology such that the map (1.8) is locally a continuous embedding. We do this in two steps.

Lemma 4.9. *Let $D \subset X$ be an SC symplectic divisor, $(J,\omega) \in \mathcal{J}(X, D)$, and*

$$\big(f_a \equiv \big(u_{a,v}, [\zeta_{a,v}] = ([\zeta_{a,v,i}])_{i\in I_v}, C_{a,v} = (\Sigma_v, \mathfrak{j}_{a,v}, \vec{z}_v)\big)_{v\in\mathbb{V}}\big)_{a\in\mathbb{N}} \in \overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X, D, A) \tag{4.19}$$

be a sequence of stable log maps in $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X, D, A)$ with a fixed decorated dual graph $\Gamma = \Gamma(\mathbb{V}, \mathbb{E}, \mathbb{L})$ such that the underlying sequence of stable maps Gromov converges to the marked nodal map

$$f \equiv \big(u_{v'}, C_{v'}\big)_{v'\in\mathbb{V}'} \in \overline{\mathcal{M}}_{g,k}(X, A) \tag{4.20}$$

with the dual graph $\Gamma' = \Gamma(\mathbb{V}', \mathbb{E}', \mathbb{L}')$, nodal domain $\Sigma' = \bigcup_{v'\in\mathbb{V}'} \Sigma_{v'}$, and nodal map

$$u' \equiv (u_{v'})_{v'\in\mathbb{V}'} : \Sigma' \longrightarrow X.$$

With $\gamma^* : \mathbb{V}' \longrightarrow \mathbb{V}$ as in (4.2) and notation as in Theorem 4.3, (after passing to a subsequence) for each $v \in \mathbb{V}$ and $v' \in \mathbb{V}'$, with $\gamma^*(v') = v$, and $i \in I_{v'} - I_v$, let

$$[\zeta_{v',i}] \in \Gamma_{\mathrm{mero}}(\Sigma_{v'}, u^*_{v'}\mathcal{N}_X D_i)/\mathbb{C}^*$$

be the unique $\mathbb{C}^*$-equivalence class of the meromorphic section obtained from applying Proposition 4.6 to $(u_{a,v}, C_{a,v})_{a\in\mathbb{N}}$ and (X, D_i). Similarly, for each $i \in I_v$, let

$$[\zeta_{v',i}] \in \Gamma_{\mathrm{mero}}(\Sigma_{v'}, u^*_{v'}\mathcal{N}_X D_i)/\mathbb{C}^*$$

be the unique $\mathbb{C}^*$-equivalence class of the meomorphic section obtained from applying Corollary 4.8 to

$$(u_{a,v}, [\zeta_{a,v,i}], C_{a,v})_{a\in\mathbb{N}} \qquad \text{and} \qquad (X, D_i).$$

Then

$$f_{\log} = \big(u_{v'}, [\zeta_{v'}] = ([\zeta_{v',i}])_{i\in I_{v'}}, C_{v'}\big)_{v'\in\mathbb{V}'} \tag{4.21}$$

is a pre-log map.

Proof. (1) **Continuity.** The matching condition (2) of Definition 3.3 is about the continuity of the underlying stable map f and already holds by the Gromov compactness.

In order to show that the condition (3) of Definition 3.3 is satisfied, let us first fix some notation. Since

$$s_{\underset{\rightarrow}{e}} = -s_{\underset{\leftarrow}{e}} \quad \Leftrightarrow \quad s_{\underset{\rightarrow}{e},i} = -s_{\underset{\leftarrow}{e},i} \in \mathbb{Z} \qquad \forall i \in S,$$

[21] So that we can still apply the Gromov convergence theorem. We can also use the compact manifold $\mathbb{P}_X D$ in (2.10) instead of $\mathcal{N}_X D$ with the symplectic form $\omega_{X,D} = \pi^*(\omega|D) + \epsilon \mathrm{d}\big(\frac{\rho\alpha_\nabla}{1+\rho}\big)$, where $\epsilon > 0$ is a sufficiently small constant. Then, for t sufficiently small, by interpolating between

$$J_t|_{R_t^{-1}(\mathcal{N}'_X D)} \quad \text{and} \quad J_{X,D}|_{\mathbb{P}_X D},$$

we can construct a family of almost complex structures $\widetilde{J}_t$ on $\mathbb{P}_X D$ such that $\widetilde{J}_t$ converges to $J_{X,D}$; see [17, Proposition 6.6].

it is enough to show that the condition (3) is satisfied relative to each smooth component D_i; i.e. we may assume D is smooth. In the context/notation of Proposition 4.6, for every $v,v'\in\mathbb{V}$ and any node $q_e=(q_{\underrightarrow{e}}\sim q_{\underleftarrow{e}})$, with $\underrightarrow{e}\in\underrightarrow{\mathbb{E}}_{v,v'}$, connecting Σ_v and $\Sigma_{v'}$, let $\Delta_{\underrightarrow{e}}\subset\Sigma_v$ be a sufficiently small disk around $q_{\underrightarrow{e}}$ (not containing any other marked point or nodal point), $\Delta_{\underleftarrow{e}}\subset\Sigma_{v'}$ be a sufficiently small disk around $q_{\underleftarrow{e}}$, and $A_e=\Delta_{\underrightarrow{e}}\cup\Delta_{\underleftarrow{e}}$ be a the resulting neighborhood of q_e in Σ. We orient each circle $\partial\Delta_{\underrightarrow{e}}$ in the direction of the counter-clock wise rotation in $\Delta_{\underrightarrow{e}}\subset\mathbb{C}$. For each $e\in\mathbb{E}$, $A_{a,e}=\varphi_{\gamma_a}^{-1}(A_e)$ is a cylinder in Σ_a with two (oppositely oriented) boundaries

$$\varphi_{\gamma_a}^{-1}(\partial\Delta_{\underrightarrow{e}})\qquad\text{and}\quad\varphi_{\gamma_a}^{-1}(\partial\Delta_{\underleftarrow{e}})$$

such that $u_a|_{A_{a,e}}$ does not intersect D for $a\gg1$. Therefore, the winding numbers of

$$u_a|_{\varphi_{\gamma_a}^{-1}(\partial\Delta_{\underrightarrow{e}})}\quad\text{and}\quad u_a|_{\varphi_{\gamma_a}^{-1}(\partial\Delta_{\underleftarrow{e}})}$$

are opposite of each other. If $v\in\mathbb{V}_1$, by the proof of Proposition 4.6,

$$s_{\underrightarrow{e}}:=\mathrm{ord}_{q_{\underrightarrow{e}}}\zeta_v=\text{winding number of }(u_a|_{\varphi_{\gamma_a}^{-1}(\partial\Delta_{\underrightarrow{e}})})\qquad\forall\ a\gg1.$$

Similarly, if $v\in\mathbb{V}_0$, then

$$s_{\underrightarrow{e}}:=\mathrm{ord}_{q_{\underrightarrow{e}}}(u_v,D)=\text{winding number of }(u_a|_{\varphi_{\gamma_a}^{-1}(\partial\Delta_{\underrightarrow{e}})})\qquad\forall\ a\gg1.$$

Therefore,

$$s_{\underrightarrow{e}}=-s_{\underleftarrow{e}}\qquad\forall\ v,v'\in\mathbb{V},\ e\in\mathbb{E}_{v,v'}.\tag{4.22}$$

The same conclusion holds in the case of Corollary 4.8 (since it is a corollary of Proposition 4.6).

(2) **Contact order at nodes.** The contact order condition (3) in Definition 3.3, for every $e\in\mathbb{E}(\Gamma'/\Gamma)$, follows from (4.22). For each $\underrightarrow{e}\in\underrightarrow{\mathbb{E}}=\underrightarrow{\mathbb{E}}(\Gamma)\subset\underrightarrow{\mathbb{E}}'=\underrightarrow{\mathbb{E}}(\Gamma')$, with $v=v_1(\underrightarrow{e})\in\mathbb{V}$, the nodal point $q_{\underrightarrow{e}}$ is a marked point for (u_v,C_v). For such e, by the last statements in Proposition 4.6 and Corollary 4.8, the contact order $s_{\underrightarrow{e}}$ remains unchanged in the limiting process. Therefore, the contact order condition (3) in Definition 3.3, for every $e\in\mathbb{E}\subset\mathbb{E}'$, follows from the corresponding condition on $(f_a)_{a\in\mathbb{N}}$. ☐

In order to prove Theorem 1.1, it just remains to prove the following theorem.

Theorem 4.10. *If further $(J,\mathcal{R},\omega)\in AK(X,D)$ for some regularization $\mathcal{R}$ or if J is integrable, then the pre-log J-holomorphic map $f_{\log}$ in (4.21) satisfies Conditions* (1) *and* (2) *of Definition 3.8.*

We prove Theorem 4.10 in Section 4.3. The proof uses a fine comparison result between the rescaling parameters $(t_{\zeta_{v'},i,a})_{a\in\mathbb{N}}$ corresponding to the sections $\zeta_{v',i}$, for all $v'\in\mathbb{V}'$ and $i\in I_{v'}$, and the "gluing parameters" of the nodes. We expect Proposition 4.11 and thus Theorem 4.10 to be true for arbitrary $(J,\omega)\in\mathcal{J}(X,D)$.

4.3 Proof of Theorem 4.10

Theorem 4.10 is essentially a consequence of Proposition 4.11 below that relates the sequence of rescaling parameters $(t_{\zeta_{v'},i,a})_{a\in\mathbb{N}}$ corresponding to the sections $\zeta_{v',i}$ in Lemma 4.9, for all $v'\in\mathbb{V}'$ and $i\in I_{v'}$, to the "gluing parameters" at the nodes and the leading order coefficients $0\neq\eta_{\underrightarrow{e}',i}\in\mathcal{N}_XD_i|_{u'(q_{e'})}$ in (3.28), for all $\underrightarrow{e}'\in\underrightarrow{\mathbb{E}}'$ and $i\in I_{e'}$.

Let us start with a local picture of what is happening in Theorem 4.9 with respect to any smooth component of D. Suppose D is a smooth symplectic divisor in (X,ω) and $J\in\mathcal{J}(X,D,\omega)$. Fix a regularization $\Psi\colon\mathcal{N}'_X D\longrightarrow X$ as in (4.7). Let Δ_1 and Δ_2 be compact discs of some fixed sufficiently small radius δ around $0\in\mathbb{C}$ with coordinates z_1 and z_2. For $i=1,2$, let $\{z_{i,a}\}_{a\in\mathbb{N}}$ be a sequence of complex-coordinates[22] on Δ_i converging to z_i uniformly with all the derivatives.

Local case 1. For a sequence of complex numbers $(\varepsilon_a)_{a\in\mathbb{N}}$ converging to zero, suppose $u_a\colon A_a\longrightarrow \mathrm{Im}(\Psi)\subset X$, where

$$A_a=\{(z_{1,a},z_{2,a})\colon z_{1,a}z_{2,a}=\varepsilon_a\colon z_{1,a}\in\Delta_1, z_{2,a}\in\Delta_2\}\subset\Delta_1\times\Delta_2,\quad\forall\, a\in\mathbb{N},\tag{4.23}$$

is a sequence of J-holomorphic maps that Gromov converges to the nodal map

$$\big(u_1(z_1)\colon\Delta_1\longrightarrow X,\ u_2(z_2)\colon\Delta_2\longrightarrow D\big),\quad x=u_1(0)=u_2(0)\in D.$$

In other words, for any $\epsilon>0$,

(a) the sequence of J-holomorphic maps

$$u_a(z_{1,a})\equiv u_a(z_{1,a},\varepsilon_a/z_{1,a})\colon A_a\approx\{z_{1,a}\in\mathbb{C}\colon \varepsilon_a/\delta\le|z_{1,a}|\le\delta\}\longrightarrow X$$

converges uniformly with all the derivatives on the compact set

$$\{(z_{1,a},z_{2,a})\in A_a\colon \epsilon\le|z_{1,a}|\}\approx\{z_{1,a}\in\mathbb{C}\colon\epsilon\le|z_{1,a}|\le\delta\}$$

to $u_1|_{\{z_1\in\mathbb{C}\colon\,\epsilon\le|z_1|\le\delta\}}$,

(b) the reparametrization

$$u_a(z_{2,a})\equiv u_a(\varepsilon_a/z_{2,a},z_{2,a})\colon A_a\approx\{z_{2,a}\in\mathbb{C}\colon \varepsilon_a/\delta\le|z_{2,a}|\le\delta\}\longrightarrow X$$

converges uniformly with all the derivatives on the compact set

$$\{(z_{1,a},z_{2,a})\in A_a\colon \epsilon\le|z_{2,a}|\}\approx\{z_{2,a}\in\mathbb{C}\colon\epsilon\le|z_{2,a}|\le\delta\}$$

to $u_2|_{\{z_2\in\mathbb{C}\colon\,\epsilon\le|z_2|\le\delta\}}$, and

(c) we do not get any bubbling in between the two maps (i.e. the energy in between shrinks to zero with ϵ).

Furthermore, suppose

(1) u_1 has a tangency order of $s>0$ with D at 0, and

(2) there exists a meromorphic section ζ of $u_2^*\mathcal{N}_X D$ with a pole of order s at the origin and a sequence of complex numbers $(t_a)_{a\in\mathbb{N}}$ converging to zero such that $t_a^{-1}\Psi^{-1}(u_a(z_{2,a}))$ converges to $\zeta(z_2)$ uniformly with all the derivatives on any compact set $\{z_2\in\mathbb{C}\colon\epsilon\le|z_2|\le\delta\}\subset\Delta_2$.

[22] More precisely, $z_{i,a}\colon\Delta_i\longrightarrow\mathbb{C}$ is a sequence of smooth functions converging to the function $z_i\colon\Delta_i\longrightarrow\mathbb{C}$ in C^∞-topology.

Let $0 \neq \eta_2 \in \mathcal{N}_X D|_x$ be the leading coefficient of ζ with respect to the coordinate z_2 as in (3.26), and $0 \neq \eta_1 \in \mathcal{N}_X D|_x$ be the s-th derivative of u_1 in the normal direction to D at 0 with respect to the coordinate z_1 as in (3.27). Proposition 4.11 below shows that there is an explicit relation between the sequence of gluing parameters $(\varepsilon_a)_{a\in N}$, the sequence of rescaling parameters $(t_a)_{a\in\mathbb{N}}$, and the ratio $\eta_2/\eta_1 \in \mathbb{C}^*$.

Local case 2. Similarly, consider the situation where the sequence of J-holomorphic maps $\{u_a\}_{a\in\mathbb{N}}$ in (4.23) Gromov converges to the nodal map

$$\big(u_1\colon \Delta_1 \longrightarrow D, u_2\colon \Delta_2 \longrightarrow D\big), \quad x = u_1(0) = u_2(0) \in D,$$

with the following property: there exist meromorphic sections $\zeta_1(z_1)$ and $\zeta_2(z_2)$ of $u_1^*\mathcal{N}_X D$ and $u_2^*\mathcal{N}_X D$, respectively, such that

$$\mathrm{ord}_0(\zeta_1) = s, \qquad \mathrm{ord}_0(\zeta_2) = -s,$$

and, for $i = 1, 2$, there exists a sequence of complex numbers $(t_{i,a})_{a\in\mathbb{N}}$ converging to zero such that $t_{i,a}^{-1}\Psi^{-1}(u_a(z_{i,a}))$ converges to $\zeta_i(z_i)$ uniformly with all the derivatives on any compact set $\{z_i\colon \epsilon \leq |z_i| \leq \delta\} \subset \Delta_i$. With η_1 and η_2 as before, the following theorem also shows that there is a similar relation between the sequence of gluing parameters $(\varepsilon_a)_{a\in\mathbb{N}}$, rescaling parameters $(t_{i,a})_{a\in\mathbb{N}}$, and the ratio $\eta_2/\eta_1 \in \mathbb{C}^*$.

Proposition 4.11. *With notation as above, if further $(J, \mathcal{R}, \omega) \in AK(X, D)$ for some regularization $\mathcal{R}$ or if J is integrable, in the local case 1, we have*

$$\lim_{a\longrightarrow\infty} \frac{\varepsilon_a^s}{t_a} = \frac{\eta_2}{\eta_1}. \tag{4.24}$$

In the local case 2, we have

$$\lim_{a\longrightarrow\infty} \frac{t_{1,a}\ \varepsilon_a^s}{t_{2,a}} = \frac{\eta_2}{\eta_1}. \tag{4.25}$$

Note that the situation in (4.25) reduces to the situation in (4.24) after a rescaling of the sequence $\{u_a\}_{a\in\mathbb{N}}$ via $(t_{1,a})_{a\in\mathbb{N}}$. For the rescaled sequence we will have $\big(t_a = \frac{t_{2,a}}{t_{1,a}}\big)_{a\in\mathbb{N}}$. We prove Proposition 4.11 in the next section. The proof uses the extra assumption on J in the statement of Theorem 1.1, but we expect this proposition, and thus Theorem 1.1, to be true for arbitrary $(J, \omega) \in \mathcal{J}(X, D)$.

Remark 4.12. It is easy to see that the limit conditions in (4.24) and (4.25) are independent of Ψ, the representatives ζ_1, ζ_2, and the local coordinates z_1 and z_2. For example, in (4.25), changing z_2 with αz_2 and ζ_2 with $\beta\zeta_2$, for some $\alpha, \beta \in \mathbb{C}^*$, changes η_2 on the right-hand side of (4.25) to $\alpha^s\beta\eta_2$, changes ε_a and $t_{2,a}$ on the left-hand side of (4.25) to $\alpha\varepsilon_a$ and $\beta^{-1}t_{2,a}$, respectively, and has no effect on the other terms. Thus it affects both sides of (4.25) equally. It is also clear that (4.24) and (4.25) only depend on the equivalence classes $[(\varepsilon_a)_{a\in\mathbb{N}}]$, $[(t_a)_{a\in\mathbb{N}}]$, $[(t_{1,a})_{a\in\mathbb{N}}]$, and $[(t_{2,a})_{a\in\mathbb{N}}]$.

Remark 4.13. In the case of smooth divisors, a significantly simpler version of (4.25) suffices for proving Theorem 4.10. Instead of (4.25), we only need to prove that

$$\lim_{a\longrightarrow\infty} \frac{t_{1,a}}{t_{2,a}} = \frac{\eta_2}{\eta_1} \qquad \text{if } \ s = 0, \quad \text{and} \quad \lim_{a\longrightarrow\infty} \frac{t_{1,a}}{t_{2,a}} = \infty \qquad \text{if } \ s > 0. \tag{4.26}$$

The equalities in (4.26) can be proved without the extra restriction on J. Thus, if D is smooth, Theorem 1.1 holds for arbitrary $(J, \omega) \in \mathcal{J}(X, D)$.

Proof of Proposition 4.16 uses the following lemma with the linear map

$$\varrho_{\mathbb{C}}\colon \mathbb{C}^{\mathbb{E}'} \oplus \bigoplus_{v'\in\mathbb{V}'} \mathbb{C}^{I_{v'}} \longrightarrow \bigoplus_{e'\in\mathbb{E}'} \mathbb{C}^{I_{e'}}$$

defined in (3.17).

Lemma 4.14. *Assume $f\colon\mathbb{C}^n\longrightarrow\mathbb{C}^m$ is a complex-linear map and $(\xi_a)_{a\in\mathbb{N}}\subset\mathbb{C}^n$ is a sequence such that*

$$\lim_{a\longrightarrow\infty} f(\xi_a)=\eta. \tag{4.27}$$

Then, there exists a convergent sequence $(\xi'_a)_{a\in\mathbb{N}}\subset\mathbb{C}^n$ (i.e. $\exists\ \xi'\in\mathbb{C}^n$ such that $\lim_{a\longrightarrow\infty}\xi'_a=\xi'$) such that $f(\xi_a-\xi'_a)=0$ for all $a\in\mathbb{N}$.

Proof. Since $\mathrm{Im}(f)\subset\mathbb{C}^m$ is closed, (4.27) implies that $\eta\in\mathrm{Im}(f)$. Let $\eta=f(\xi)$. Fix an affine subspace[23] $H\subset\mathbb{C}^n$ passing through ξ and transverse[24] to the hyperplane $f^{-1}(\eta)\subset\mathbb{C}^n$. By (4.27), there exists $M\in\mathbb{N}$ such that H is transverse to $f^{-1}(f(\xi_a))$ for all $a>M$. Then the sequence $\xi'_a=f^{-1}(f(\xi_a))\cap H$, if $a>M$, and $\xi'_a=\xi_a$, if $a\le M$, has the desired properties. □

Going back to the set up of Theorem 4.10, assume that the dual graph Γ of $f_{\log}$ in (4.21) is made of only one vertex $\mathbb{V}=\{v\}$ and fix a set of representatives

$$(\zeta_{a,v,i})_{i\in I_v}$$

for $[\zeta_{a,v}]$ (in other words, restrict to the v-th component of the sequence $(f_a)_{a\in\mathbb{N}}$ in (4.19)). For each $v'\in\mathbb{V}'$ and $i\in I_{v'}$ fix a representative $\zeta_{v',i}$ of the $\mathbb{C}^*$-equivalence class $[\zeta_{v',i}]$ in Lemma 4.9, and a sequence of rescaling parameters $(t_{\zeta_{v',i},a})_{a\in\mathbb{N}}$ satisfying Proposition 4.6 or Corollary 4.8, depending on whether $i\notin I_v$ or $i\in I_v$, respectively.

By the surjectivity of the classical gluing theorem of J-holomorphic maps (e.g. [14, Section 7]), for a sufficiently large, the domain Σ_a of f_a can be obtained from the nodal domain Σ' of the stable nodal map f in (4.20) in the following way. There exist

- a sequence of complex structure $\mathfrak{j}'_a=(\mathfrak{j}_{v',a})_{v'\in\mathbb{V}'}$ on the nodal domain $\Sigma'=(\Sigma_{v'})_{v'\in\mathbb{V}'}$ of the stable nodal map f in (4.20),
- a sequence of local $\mathfrak{j}_{v',a}$-holomorphic coordinates $z_{\underrightarrow{e}',a}\colon\Delta_{\underrightarrow{e}'}\longrightarrow\mathbb{C}$ around $q_{\underrightarrow{e}'}\in\Sigma_{v'}$, for all $v'\in\mathbb{V}'$ and $\underrightarrow{e}'\in\underrightarrow{\mathbb{E}}'_{v'}$, and
- a sequence of non-zero complex numbers $(\varepsilon_{e',a})_{e'\in\mathbb{E}'}$ converging to zero,

such that

(1) $(\Sigma_a,\mathfrak{j}_a,\vec{z_a})$ is isomorphic to the smoothing of $(\Sigma',\mathfrak{j}'_a=(\mathfrak{j}_{v',a})_{v'\in\mathbb{V}'})$ defined by

$$z_{\underrightarrow{e}',a}z_{\underleftarrow{e}',a}=\varepsilon_{e',a}\qquad\forall\ e'\in\mathbb{E}',\quad\text{and} \tag{4.28}$$

(2) the sequence $(\mathfrak{j}_{v',a})_{a\in\mathbb{N}}$ C^∞-converges to $\mathfrak{j}_{v'}$ for all $v'\in\mathbb{V}'$,

[23] i.e. a shifted linear subspace.
[24] assuming f in not trivial; otherwise, the Lemma is obvious.

(3) the sequence $(z_{\underset{\rightarrow}{e}',a})_{a\in\mathbb{N}}$ C^∞-converges to $z_{\underset{\rightarrow}{e}'}$, where $z_{\underset{\rightarrow}{e}'}\colon \Delta_{\underset{\rightarrow}{e}'} \longrightarrow \mathbb{C}$ is some fixed local $\mathfrak{j}_{v'}$-holomorphic coordinate around $q_{\underset{\rightarrow}{e}'}\in\Sigma_{v'}$, for all $v'\in\mathbb{V}'$ and $\underset{\rightarrow}{e}'\in\underset{\rightarrow}{\mathbb{E}}'_{v'}$.

We use this standard presentation of Σ_a in the proof of Theorem 4.10.

Remark 4.15. For $\delta>0$ sufficiently small, let

$$\begin{aligned}
&\Delta_{\underset{\rightarrow}{e}',a}(\delta) = \{x\in\Delta_{\underset{\rightarrow}{e}'}\colon\ |z_{\underset{\rightarrow}{e}',a}(x)|<\delta\} \qquad \forall\ \underset{\rightarrow}{e}'\in\mathbb{E}',\ a\gg 1, \quad \text{and}\\
&A_{e',a} = \{z_{\underset{\rightarrow}{e}',a}z_{\underset{\leftarrow}{e}',a}=\varepsilon_{e',a}\colon z_{\underset{\rightarrow}{e}',a}\in\Delta_{\underset{\rightarrow}{e}'}(2\varepsilon_{e',a}), z_{\underset{\leftarrow}{e}',a}\in\Delta_{\underset{\leftarrow}{e}'}(2\varepsilon_{e',a})\}\subset\Sigma_a \quad \forall\ e'\in\mathbb{E}',\ a\gg 1.
\end{aligned}$$

Then, with respect to the identification of the domains in (1), the γ_a-degeneration maps

$$\varphi_{\gamma_a}\colon\Sigma_a\longrightarrow\Sigma',$$

can be taken to be identity on the complement of $\cup_{e'\in\mathbb{E}'}A_{e',a}$ and some "nice" degeneration map

$$A_{e',a}\longrightarrow \Delta_{\underset{\rightarrow}{e}'}(2\varepsilon_{e',a})\cup\Delta_{\underset{\leftarrow}{e}'}(2\varepsilon_{e',a})$$

on the neck region.

For each $\underset{\rightarrow}{e}'\in\underset{\rightarrow}{\mathbb{E}}'$ and $i\in I_{e'}$, let

$$0\neq\eta_{\underset{\rightarrow}{e}',i}\in\mathcal{N}_X D_i|_{u'(q_{e'})}$$

be the leading coefficient term in (3.28) with respect to $z_{\underset{\rightarrow}{e}'}$ (and $\zeta_{v',i}$, if $i\in I_{v'}$). By Proposition 4.11, for every $v_1',v_2'\in\mathbb{V}'$ and $\underset{\rightarrow}{e}'\in\underset{\rightarrow}{\mathbb{E}}'_{v_1',v_2'}$ we have

$$\lim_{a\longrightarrow\infty}\frac{t_{\zeta_{v_1'},i,a}\ \varepsilon_{e',a}^{s_{\underset{\rightarrow}{e}',i}}}{t_{\zeta_{v_2'},i,a}} = \eta_{\underset{\leftarrow}{e}',i}/\eta_{\underset{\rightarrow}{e}',i} \qquad \forall\ i\in I_{v_1'}\cap I_{v_2'}, \tag{4.29}$$

$$\lim_{a\longrightarrow\infty} t_{\zeta_{v_1'},i,a}\ \varepsilon_{e',a}^{s_{\underset{\rightarrow}{e}',i}} = \eta_{\underset{\leftarrow}{e}',i}/\eta_{\underset{\rightarrow}{e}',i} \qquad \forall\ i\in I_{v_1'}-I_{v_2'}. \tag{4.30}$$

Proposition 4.16. *There exists a choice of the coordinates $\{z_{\underset{\rightarrow}{e}'}\}_{\underset{\rightarrow}{e}'\in\underset{\rightarrow}{\mathbb{E}}'}$ and $\{z_{\underset{\rightarrow}{e}',a}\}_{\underset{\rightarrow}{e}'\in\underset{\rightarrow}{\mathbb{E}}',a\in\mathbb{N}}$ satisfying (4.28) and item (3) after that, and the representatives $\zeta_{v',i}$ and $(t_{\zeta_{v'},i,a})_{a\in\mathbb{N}}$ for $[\zeta_{v',i}]$ and $[(t_{\zeta_{v'},i,a})_{a\in\mathbb{N}}]$, respectively, such that*

$$t_{\zeta_{v_1'},i,a}\ \varepsilon_{e',a}^{s_{\underset{\rightarrow}{e}',i}} = t_{\zeta_{v_2'},i,a} \qquad \forall\ i\in I_{v_1'}\cap I_{v_2'},\ a\gg 1, \tag{4.31}$$

$$t_{\zeta_{v_1'},i,a}\ \varepsilon_{e',a}^{s_{\underset{\rightarrow}{e}',i}} = 1 \qquad \forall\ i\in I_{v_1'}-I_{v_2'},\ a\gg 1. \tag{4.32}$$

Proof. Throughout the proof we assume $I_v=\emptyset$; for $I_v\neq\emptyset$, the argument reduces to $I_v=\emptyset$ by considering the associated sequence of maps in $\mathcal{N}_X D_{I_v}$. We modify a given set of representatives to another set satisfying (4.31) and (4.32). Assuming $I_v=\emptyset$, fix an orientation O on $\mathbb{E}'$, and choose some branch

$$\eta=\bigoplus_{\underset{\rightarrow}{e}'\in O}\eta_{e'}\in\bigoplus_{e'\in\mathbb{E}'}\mathbb{C}^{I_{e'}}, \qquad \eta_{\underset{\rightarrow}{e}'}=\big(-\log(\eta_{\underset{\leftarrow}{e}',i}/\eta_{\underset{\rightarrow}{e}',i})\big)_{i\in I_{e'}}\in\mathbb{C}^{I_{e'}} \quad \forall\ \underset{\rightarrow}{e}'\in O,$$

of the multi-valued function log. By (4.29) and (4.30) and definition of $\varrho_{\mathbb{C}}$ in (3.17) (via the chosen orientation O), we can choose the branches

$$\xi_a=\big((-\log(\varepsilon_{e',a}))_{e'\in\mathbb{E}'},(-\log(t_{\zeta_{v'},i,a}))_{v'\in\mathbb{V}',i\in I_{v'}}\big)\in\mathbb{C}^{\mathbb{E}'}\oplus\bigoplus_{v'\in\mathbb{V}'}\mathbb{C}^{I_{v'}} \qquad \forall\ a\in\mathbb{N}.$$

so that

$$\lim_{a\longrightarrow\infty} \varrho_{\mathbb{C}}(\xi_a) = \eta.$$

By Lemma 4.14 applied to $\varrho_{\mathbb{C}}$, there exists a sequence

$$(\xi'_a)_{a\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{E}'} \oplus \bigoplus_{v'\in\mathbb{V}'} \mathbb{C}^{I_{v'}}$$

such that $\varrho_{\mathbb{C}}(\xi_a - \xi'_a) = 0$ for all $a \in \mathbb{N}$ and $\lim_{a\longrightarrow\infty} \xi'_a = \xi'$. Taking the exponential of ξ'_a and ξ', we conclude that there exist

$$\big((\alpha_{e'})_{e'\in\mathbb{E}'}, (\alpha_{v',i})_{v'\in\mathbb{V}', i\in I_{v'}}\big), \big((\alpha_{e',a})_{e'\in\mathbb{E}'}, (\alpha_{v',i,a})_{v'\in\mathbb{V}', i\in I_{v'}}\big)_{a\in\mathbb{N}} \in (\mathbb{C}^*)^{\mathbb{E}'} \times \prod_{v'\in\mathbb{V}'} (\mathbb{C}^*)^{I_{v'}}$$

such that

$$\lim_{a\longrightarrow\infty} \big((\alpha_{e',a})_{e'\in\mathbb{E}'}, (\alpha_{v',i,a})_{v'\in\mathbb{V}', i\in I_{v'}}\big) = \big((\alpha_{e'})_{e'\in\mathbb{E}'}, (\alpha_{v',i})_{v'\in\mathbb{V}', i\in I_{v'}}\big)$$

and

$$\frac{\big(\alpha^{-1}_{v'_1,i} t_{\zeta_{v'_1,i},a}\big)\ \big(\alpha^{-1}_{e',a}\varepsilon_{e',a}\big)^{s_{\underset{\rightarrow}{e'},i}}}{\big(\alpha^{-1}_{v'_2,i} t_{\zeta_{v'_2,i},a}\big)} = 1 \qquad \forall\ i\in I_{v'_1}\cap I_{v'_2},\ a\in\mathbb{N}, \tag{4.33}$$

$$\big(\alpha^{-1}_{v'_1,i} t_{\zeta_{v'_1,i},a}\big)\ \big(\alpha^{-1}_{e',a}\varepsilon_{e',a}\big)^{s_{\underset{\rightarrow}{e'},i}} = 1 \qquad \forall\ i\in I_{v'_1} - I_{v'_2},\ a\in\mathbb{N}. \tag{4.34}$$

By (4.33) and (4.34), for a sufficiently large, replacing

- $\{z_{\underset{\rightarrow}{e'}}\}_{\underset{\rightarrow}{e'}\in O}$ with $\{\alpha^{-1}_{e'} z_{\underset{\rightarrow}{e'}}\}_{\underset{\rightarrow}{e'}\in O}$,
- $\{z_{\underset{\rightarrow}{e'},a}\}_{\underset{\rightarrow}{e'}\in O}$ with $\{\alpha^{-1}_{e',a} z_{\underset{\rightarrow}{e'},a}\}_{\underset{\rightarrow}{e'}\in O}$,
- $\{\varepsilon_{e',a}\}_{e'\in\mathbb{E}'}$ with $\{\alpha^{-1}_{e',a}\varepsilon_{e',a}\}_{e'\in\mathbb{E}'}$,
- $(t_{\zeta_{v',i},a})_{v'\in\mathbb{V}', i\in I_{v'}}$ with $(\alpha^{-1}_{v',i,a} t_{\zeta_{v',i},a})_{v'\in\mathbb{V}', i\in I_{v'}}$, and
- $(\zeta_{v',i})_{v'\in\mathbb{V}', i\in I_{v'}}$ with $(\alpha_{v',i}\zeta_{v',i})_{v'\in\mathbb{V}', i\in I_{v'}}$,

we get a new set of representatives satisfying (4.31) and (4.32). In particular, the limits in (4.29) and (4.30) can be set to be equal to 1.

□

Proof of Theorem 4.10. First, assume that the dual graph Γ of $f_{\log}$ in (4.21) is made of only one vertex $\mathbb{V} = \{v\}$ and fix a set of representatives

$$(\zeta_{a,v,i})_{i\in I_v}$$

for $[\zeta_{a,v}]$. By Proposition 4.11, we can choose the coordinates $\{z_{\underset{\rightarrow}{e'}}\}_{\underset{\rightarrow}{e'}\in\underset{\rightarrow}{\mathbb{E}'}}$ and $\{z_{\underset{\rightarrow}{e'},a}\}_{\underset{\rightarrow}{e'}\in\underset{\rightarrow}{\mathbb{E}'}, a\in\mathbb{N}}$, and the representatives $\zeta_{v',i}$ and $(t_{\zeta_{v',i},a})_{a\in\mathbb{N}}$ so that (4.31) and (4.32) hold. For each $v'\in\mathbb{V}'$ and $i\in I_{v'} - I_v$, note that $(t_{\zeta_{v',i},a})_{a\in\mathbb{N}}$ converges to 0; therefore,

$$-\log(|t_{\zeta_{v',i},a}|) > 0 \qquad \forall\ v'\in\mathbb{V},\ i\in I_{v'} - I_v,\ a \gg 1,$$

and it converges to infinity. Choose a sequence of positive vectors $s_v^a=(s_{v,i}^a)_{i\in I_v}\in\mathbb{R}_+^{I_v}$ such that

$$s_{v,i}^a-\log(|t_{\zeta_{v',i},a}|)>1 \qquad \forall\, v'\in\mathbb{V}',\ i\in I_v. \tag{4.35}$$

With these choices, for $a\gg 1$, the functions $s^a\colon \mathbb{V}'\longrightarrow\mathbb{R}^S$ defined by

$$s_{v'}^a=\Big(\big(s_{v,i}^a-\log(|t_{\zeta_{v',i},a}|)\big)_{i\in I_v},\big(-\log(|t_{\zeta_{v',i},a}|)\big)_{i\in I_{v'}-I_v}\Big)\in\mathbb{R}_+^{I_{v'}} \qquad \forall\, v'\in\mathbb{V}', \tag{4.36}$$

and $\lambda^a\colon \mathbb{E}'\longrightarrow\mathbb{R}_+$ defined by

$$\lambda_{e'}^a=-\log(|\varepsilon_{e',a}|) \qquad \forall\, e'\in\mathbb{E}'$$

satisfy Condition (1) of Definition 3.8. By (4.29) and (4.30), $f_{\log}$ also satisfies Condition (2) of Definition 3.8.

For general Γ, by the proof of Lemma 3.7, we can choose a set of representatives

$$(\zeta_{a,v,i})_{a\in\mathbb{N},v\in\mathbb{V},i\in I_v}$$

and coordinates $(z_{\underrightarrow{e},a}=\delta_{a,\underrightarrow{e}}z_{\underrightarrow{e}})_{a\in\mathbb{N},\underrightarrow{e}\in\underrightarrow{\mathbb{E}}}$ such that the leading coefficients $\eta_{\underrightarrow{e},i,a}$ in (3.28) satisfy

$$\eta_{\underrightarrow{e},i,a}=\eta_{\underleftarrow{e},i,a} \qquad \forall\, e\in\mathbb{E},\ i\in I_e,\ a\in\mathbb{N}. \tag{4.37}$$

Let $\delta_{a,e}=\delta_{a,\underrightarrow{e}}\delta_{a,\underleftarrow{e}}$, for all $e\in\mathbb{E}$. For each $v\in\mathbb{V}$, choose the representatives

$$(\zeta_{v',i})_{v'\in\gamma^*(v),i\in I_{v'}} \quad \text{and} \quad (t_{\zeta_{v',i},a})_{v'\in\gamma^*(v),i\in I_{v'},a\in\mathbb{N}}$$

so that (4.31) and (4.32) hold. By (4.37), we have

$$\lim_{a\longrightarrow\infty}\frac{t_{\zeta_{v_1',i},a}\ \delta_{e,a}^{s_{\underrightarrow{e},i}}}{t_{\zeta_{v_2',i},a}}=1 \qquad \forall\, i\in I_{v_1'}\cap I_{v_2'}, \tag{4.38}$$

$$\lim_{a\longrightarrow\infty} t_{\zeta_{v_1',i},a}\ \delta_{e,a}^{s_{\underrightarrow{e},i}}=1 \qquad \forall\, i\in I_{v_1'}-I_{v_2'}. \tag{4.39}$$

With an argument similar to the proof of Proposition 4.11, we can choose these representatives so that further

$$t_{\zeta_{v_1',i},a}\ \delta_{e,a}^{s_{\underrightarrow{e},i}}=t_{\zeta_{v_2',i},a} \qquad \forall\, i\in I_{v_1'}\cap I_{v_2'},\ a\in\mathbb{N}, \tag{4.40}$$

$$t_{\zeta_{v_1',i},a}\ \delta_{e,a}^{s_{\underrightarrow{e},i}}=1 \qquad \forall\, i\in I_{v_1'}-I_{v_2'},\ a\in\mathbb{N}. \tag{4.41}$$

Also choose the functions $s^a\colon \mathbb{V}\longrightarrow\mathbb{R}^S$ and $\lambda^a\colon \mathbb{V}\longrightarrow\mathbb{R}_+$ satisfying Definition 3.8.(1) so that (4.35) holds and

$$\lambda_e^a-\log(\delta_{e,a})>1 \qquad \forall\, e\in\mathbb{E},\ a\gg 1.$$

Then, similarly to (4.36), for $a\gg 1$, the extended functions $s_{\text{new}}^a\colon \mathbb{V}'\longrightarrow\mathbb{R}^S$ given by

$$s_{\text{new},v'}^a=\Big(\big(s_{v,i}^a-\log(|t_{\zeta_{v',i},a}|)\big)_{i\in I_v},\big(-\log(|t_{\zeta_{v',i},a}|)\big)_{i\in I_{v'}-I_v}\Big)\in\mathbb{R}_+^{I_{v'}} \qquad \forall\, v'\in\mathbb{V}',\ v=\gamma^*(v'), \tag{4.42}$$

and $\lambda_{\text{new}}^a\colon \mathbb{E}'\longrightarrow\mathbb{R}_+$ given by $\lambda_e^a-\log(\delta_{e,a})$, if $e\in\mathbb{E}\subset\mathbb{E}'$, and

$$\lambda_{e'}^a=-\log(|\varepsilon_{e',a}|) \qquad \text{if } e'\in\mathbb{E}'/\mathbb{E},$$

satisfy Condition (1) of Definition 3.8. By (4.29) and (4.30) applied to $\mathbb{E}'/\mathbb{E}$, the assumption (4.37), and (4.40) and (4.41), $f_{\log}$ also satisfies Condition (2) of Definition 3.8. $\square$

4.4 Proof of Proposition 4.11

In this section, we prove Proposition 4.11. This is done by constructing a modified sequence of pseudoholomorphic maps in $\mathcal{N}_X D$.

Let $(\mathcal{R}=(\rho,\nabla,\Psi),\mathfrak{i}_{\mathcal{N}_X D},\bar\partial_{\mathcal{N}_X D},J_{X,D})$ be as in the beginning of Section 4.2. If $(J,\mathcal{R},\omega)\in\mathrm{AK}(X,D)$, then $\Psi^*J = J_{X,D}$. If J is holomorphic, we consider a holomorphic chart $(z_1,\ldots,z_n)$ around $x=u_1(0)=u_2(0)\in D$ such that $D=(z_1=0)$. Then, replacing the rescaling procedure in the proof below with holomorphic rescaling of z_1, the same proof works for the holomorphic case.

Assume $\Psi^*J=J_{X,D}$. Note that $J_{X,D}$ is $\mathbb{C}^*$-invariant. Since the argument is local, in order to simplify the notation, let us forget about Ψ and think of $\{u_a\}_{a\in\mathbb{N}}$ as a sequence of $J_{X,D}$-holomorphic maps into $\mathcal{N}'_X D$ itself.

Assume that we are in the situation of local case 1. For each $a\in\mathbb{N}$, let

$$\lambda_a=\frac{\varepsilon_a^s}{t_a}.$$

(**Claim 1**) First, we show that there is NO subsequence $(a_1,a_2,\ldots)$ of $\mathbb{N}$ such that

$$\lim_{i\longrightarrow\infty}\lambda_a = 0 \text{ or } \infty.$$

Thus, we conclude that there is $M>0$ such that $M^{-1}<|\lambda_a|<M$ for all $a\in\mathbb{N}$.

(**Claim 2**) Then, for any subsequence $(a_1,a_2,\ldots)$ such that the limit

$$\lim_{i\longrightarrow\infty}\lambda_a=\lambda$$

exists, we show that $\lambda=\eta_2/\eta_1$. This implies that (4.24) holds over entire $\mathbb{N}$.

For any $c>0$, there exists a sufficiently small $\epsilon_c>0$ such that

$$\omega_{\epsilon_c}=\pi^*(\omega|_D)+\frac{\epsilon_c}{2}\mathrm{d}(\rho\alpha_\nabla)$$

tames $J_{X,D}$ on $\overline{\mathcal{N}_X D(c)}$. For any compact 2-dimensional domain Σ and a smooth map $u\colon\Sigma\longrightarrow\overline{\mathcal{N}_X D(c)}$, let

$$\omega_{\epsilon_c}(u)=\int_\Sigma u^*\omega_{\epsilon_c}$$

denote the symplectic area of u.

For $a\in\mathbb{N}$, define

$$u_{1,a}\colon A_a\longrightarrow\mathcal{N}_X D,\qquad u_{1,a}(z_{1,a},z_{2,a})=z_{1,a}^{-s}\,u_a(z_{1,a},z_{2,a}),\tag{4.43}$$

$$u_{2,a}\colon A_a\longrightarrow\mathcal{N}_X D,\qquad u_{2,a}(z_{1,a},z_{2,a})=z_{1,a}^{-s}\,\lambda_a\,u_a(z_{1,a},z_{2,a}),\tag{4.44}$$

where the multiplications on the right-hand sides are with respect to the complex structure $\mathfrak{i}_{\mathcal{N}_X D}$ on $\mathcal{N}_X D$. By (1)-(3) in Page 9, both (4.43) and (4.44) are sequences of $J_{X,D}$-holomorphic maps in

$\mathcal{N}_X D$.

Proof of Claim 1 part 1. After passing to a subsequence, suppose

$$\lim_{a\longrightarrow\infty} \lambda_a = \infty. \tag{4.45}$$

By assumption (a) in Page 45 and the previous paragraph, for any $0<r<\delta$, restricted to $r\leq|z_1|\leq\delta$ (and its pre-images in A_a), the sequence $\{u_{1,a}(z_{1,a})\}_{a\in\mathbb{N}}$ converges uniformly with all the derivatives to the $J_{X,D}$-holomorphic map

$$u_{1,\infty,1}(z_1) = z_1^{-s}\, u_1(z_1).$$

By definition of η_1, the function $u_{1,\infty,1}(z_1)$ extends to $z_1=0$ with $u_{1,\infty,1}(0)=\eta_1\in\mathcal{N}_X D|_x$, where $x=u_1(0)=u_2(0)\in D$. By assumptions (b) and (2) in Page 45, (4.45), and since

$$z_{1,a}^{-s} = \varepsilon_a^s\, z_{2,a}^s,$$

restricted to $r\leq|z_2|\leq\delta$ (and its pre-images in A_a), the sequence $\{u_{1,a}(z_{2,a})\}_{a\in\mathbb{N}}$ converges uniformly with all the derivatives to the $J_{X,D}$-holomorphic map

$$u_{1,\infty,2}(z_2) = u_2(z_2)\subset D.$$

This obviously extends to entire Δ_2 with $u_{1,\infty,2}(0)=x$. The following sub-claim shows that the sequence $\{u_{1,a}\}_{a\in\mathbb{N}}$ is bounded in between, so that Gromov's convergence applies.

Sub-Claim. There exists a sufficiently large $c>0$ such that

$$\mathrm{Im}\big(u_{1,a}\big)\subset\overline{\mathcal{N}_X D(c)} \quad\text{and}\quad \omega_{c_\epsilon}\big(u_{1,a}\big)\leq c \qquad \forall\, a\in\mathbb{N}. \tag{4.46}$$

Proof of Sub-Claim. Suppose (4.46) does not hold. Then (after passing to a subsequence), by assumptions (a)-(c) in Page 45, for any $c>1$, there exists a sequence $\{r_a\}_{a\in\mathbb{N}}$, with

$$\lim_{a\longrightarrow\infty} r_a=\infty \qquad \text{and} \tag{4.47}$$

$$\max_{(z_{1,a},z_{2,a})\in A_a}\bigg(\Big|r_a^{-1}z_{1,a}^{-s}\,u_a(z_{1,a},z_{2,a})\Big|,\ \omega_{\epsilon_c}\Big(\big(z_{1,a},r_a^{-1}z_{1,a}^{-s}\,u_a(z_{1,a},z_{2,a})\big)\Big)\bigg)=c \quad \forall\, a\gg 1. \tag{4.48}$$

Let

$$\widetilde{u}_{1,a}\colon A_a\longrightarrow\mathcal{Z}, \qquad u_{1,a}(z_{1,a},z_{2,a}) = r_a^{-1}z_{1,a}^{-s}\,u_a(z_{1,a},z_{2,a}).$$

Then

- by assumption (a) in Page 45 and (4.47), for any $0<r>\delta$, restricted to $r\leq|z_1|\leq\delta$, the rescaled sequence $\{\widetilde{u}_{1,a}(z_{1,a})\}_{a\in\mathbb{N}}$ converges uniformly with all the derivatives to the $J_{X,D}$-holomorphic map

  $$\widetilde{u}_{1,\infty,1}(z_1) = \overline{u}_1(z_1)\subset D,$$

  where $\overline{u}_1$ is the image of u_1 in D;

- by assumptions (b) and (2) in Page 45, restricted to $r\leq|z_2|\leq\delta$, the sequence $\{\widetilde{u}_{1,a}(z_{2,a})\}_{a\in\mathbb{N}}$ still converges uniformly with all the derivatives to the $J_{X,D}$-holomorphic map

  $$\widetilde{u}_{1,\infty,2}(z_2) = u_2(z_2)\subset D;$$

- and, by (4.48), (the proof of) Gromov convergence theorem in [25] applies[25] to the sequence $\{\widetilde{u}_{1,a}\}_{a\in\mathbb{N}}$. In the limit we get a bubble domain Σ_∞ with Δ_1 and Δ_2 at the two ends and at least one closed bubble in between (because of (4.48)), and a continuous $J_{X,D}$-holomorphic map

$$\widetilde{u}_{1,\infty}\colon \Sigma_\infty \longrightarrow \mathcal{Z}$$

  such that

$$\widetilde{u}_{1,\infty}|_{\Delta_1} = \widetilde{u}_{1,\infty,1} \qquad \text{and} \qquad \widetilde{u}_{1,\infty}|_{\Delta_2} = \widetilde{u}_{1,\infty,2}.$$

Any non-trivial bubble would have trivial image in D, thus its image lives in $\overline{\mathcal{N}_X D(c)}|_x$. This is impossible since the latter is open and there are no-marked points to stabilize such a bubble. □

Going back to the proof of Claim 1-Part 1, by (4.46), (the proof of) Gromov convergence theorem in [25] applies[26] to the sequence $\{u_{1,a}\}_{a\in\mathbb{N}}$. In the limit we get a bubble domain Σ_∞ with Δ_1 and Δ_2 at the two ends and possibly some closed bubbles in between, and a continuous $J_{X,D}$-holomorphic map

$$u_{1,\infty}\colon \Sigma_\infty \longrightarrow \mathcal{Z}$$

such that

$$u_{1,\infty}|_{\Delta_1} = u_{1,\infty,1} \qquad \text{and} \qquad u_{1,\infty}|_{\Delta_2} = u_{1,\infty,2}.$$

Since

$$u_{1,\infty,1}(0) \neq u_{1,\infty,2}(0),$$

Σ_∞ should include at least one non-trivial bubble. Such a non-trivial bubble would have trivial image in D, thus its image lives in $\overline{\mathcal{N}_X D(c)}|_x$. This is impossible since the latter is a domain in $\mathbb{C}$ and there are no marked points to stabilize such a bubble.

□

Proof of Claim 1 part 2. After passing to a subsequence, suppose

$$\lim_{a\longrightarrow\infty} \lambda_a = 0. \tag{4.49}$$

By assumptions (b) and (2) in Page 45, since

$$u_{2,a}(z_{1,a}, z_{2,a}) = z_{1,a}^{-s}\, \lambda_a\, u_a(z_{1,a}, z_{2,a}) = z_{2,a}^{s}\, t_a^{-1}\, u_a(z_{1,a}, z_{2,a}),$$

for any $0<r<\delta$, restricted to $r\leq |z_2| \leq \delta$ (and its pre-images in A_a), the sequence $\{u_{2,a}(z_{2,a})\}_{a\in\mathbb{N}}$ converges uniformly with all the derivatives to the $J_{X,D}$-holomorphic map

$$u_{2,\infty,2}(z_2) = z_2^s\, \zeta(z_2).$$

By definition of η_2, the function $u_{2,\infty,2}(z_2)$ extends to $z_2=0$ with $u_{2,\infty,2}(0)=\eta_2$. On the other hand, by assumption (a) in Page 45 and (4.49), restricted to $r\leq|z_1|\leq\delta$ (and its pre-images in A_a), the sequence $\{u_{2,a}(z_{1,a})\}_{a\in\mathbb{N}}$ converges uniformly with all the derivatives to the $J_{X,D}$-holomorphic map

$$u_{2,\infty,1}(z_1) = \overline{u_1}(z_1) \subset D.$$

[25]The Gromov convergence applies, because on the open ends of A_a we already know that $\{\widetilde{u}_{1,a}\}_{a\in\mathbb{N}}$ uniformly converges to $\widetilde{u}_{1,\infty,1}$ and $\widetilde{u}_{1,\infty,2}$, and in the middle the sequence is bounded with bounded energy.

[26]The Gromov convergence applies, because on the open ends of A_a we already know that $\{u_{1,a}\}_{a\in\mathbb{N}}$ uniformly converges to $u_{1,\infty,1}$ and $u_{1,\infty,2}$, and in the middle the sequence is bounded with bounded energy.

This obviously extends to the entire Δ_1 with $u_{2,\infty,1}(0)=x$. By a similar argument as in the previous case, the inequality

$$u_{2,\infty,1}(0)\neq u_{2,\infty,2}(0),$$

leads to a contradiction. This finishes the proof of Claim 1. □

Proof of Claim 2. After passing to a subsequence, suppose

$$\lim_{a\longrightarrow\infty}\lambda_a=\lambda\neq 0.$$

Then, going back to the proof of Claim 1 part 1, since

$$z_{1,a}^{-s}=\varepsilon_a^s\, z_{2,a}^s \qquad \forall\ a\in\mathbb{N},$$

restricted to $r\le|z_2|\le\delta$, the sequence $\{u_{1,a}(z_{2,a})\}_{a\in\mathbb{N}}$ converges uniformly with all the derivatives to the $J_{X,D}$-holomorphic map

$$u_{1,\infty,2}(z_2)=\lambda^{-1}\, z_2^s\, \zeta_2(z_2).$$

This extends to the entire Δ_2 with $u_{1,\infty,2}(0)=\lambda\eta_2$. By a similar argument as in the proof of Claim 1-part 1, if

$$u_{1,\infty,1}(0)\neq u_{1,\infty,2}(0),$$

we get a contradiction. Therefore,

$$\eta_1=u_{1,\infty,1}(0)=u_{1,\infty,2}(0)=\lambda^{-1}\eta_2;$$

in other words, $\lambda=\eta_2/\eta_1$. This finishes the proof of Proposition 4.11 in the local case 1.

For the local case 2, repeat the exact same proof with

$$\begin{aligned} &u_{1,a}\colon A_a\longrightarrow\mathcal{Z}, \qquad u_{1,a}(z_{1,a},z_{2,a})=z_{1,a}^{-s}\,t_{1,a}^{-1}\,u_a(z_{1,a},z_{2,a}) \quad \text{and}\\ &u_{2,a}\colon A_a\longrightarrow\mathcal{Z}, \qquad u_{2,a}(z_{1,a},z_{2,a})=z_{1,a}^{-s}\,\lambda_a\,u_a(z_{1,a}), \end{aligned}$$

in place of (4.43) and (4.43), respectively, where

$$\lambda_a=\frac{t_{1,a}\,\varepsilon_a^s}{t_{2,a}} \qquad \forall\ a\in\mathbb{N}.$$

This finishes the proof of Proposition 4.11 under the assumption that $\Psi^*J=J_{X,D}$.

Remark 4.17. For arbitrary J on $\mathcal{N}'_X D$, define

$$\mathcal{Z}=\left\{(t,v)\in\mathbb{C}\times\mathcal{N}_X D:\ t^s v\in\mathcal{N}'_X D\right\}, \quad \mathcal{Z}_*=\left\{(t,v)\in\mathcal{Z}:\ t\in\mathbb{C}^*\right\},$$

and

$$F\colon\mathcal{Z}_*\longrightarrow\mathbb{C}\times\mathcal{N}'_X D, \qquad F(t,v)=(t,t^s v).$$

Let $J_{\mathcal{Z}}=F^*(\mathfrak{i}\times J)$, where $\mathfrak{i}$ is the standard almost complex structure on $\mathbb{C}$ and $\mathfrak{i}\times J$ is the product almost complex structure on the target. By an argument similar to Lemma 4.5, the almost complex structure $J_{\mathcal{Z}}$ on $\mathcal{Z}_*$ extends to a (similarly denoted) almost complex structure on the entire $\mathcal{Z}$ satisfying

$$J_{\mathcal{Z}}|_{\{0\}\times\mathcal{N}_X D\,\cup\,\mathbb{C}\times D}\cong\mathfrak{i}\times J_{X,D}. \tag{4.50}$$

Similarly, for every $a\in\mathbb{N}$, let

$$\widetilde{\mathcal{Z}}_a=\big\{(t,v)\in\mathbb{C}\times\mathcal{N}_X D:\ t^s\lambda_a^{-1}v\in\mathcal{N}'_X D\big\}\,,\quad \widetilde{\mathcal{Z}}_{a,*}=\{(t,v)\in\widetilde{\mathcal{Z}}_a\!:t\in\mathbb{C}^*\},$$

and define

$$F_a\colon\widetilde{\mathcal{Z}}_{a,*}\longrightarrow\mathbb{C}\times\mathcal{N}'_X D,\qquad F_a(t,v)=(t,\lambda_a^{-1}t^s v). \tag{4.51}$$

For each $a\in\mathbb{N}$, let $J_a=F_a^*(\mathfrak{i}\times J)$. By Lemma 4.5 and the previous paragraph, for each $a\in\mathbb{N}$, the almost complex structure J_a on $\widetilde{\mathcal{Z}}_{a,*}$ extends to a (similarly denoted) almost complex structure on the entire $\widetilde{\mathcal{Z}}_a$ satisfying (4.50).

For $a\in\mathbb{N}$, define

$$u_{1,a}\colon A_a\longrightarrow\mathcal{Z},\qquad u_{1,a}(z_{1,a},z_{2,a})=\big(z_{1,a},z_{1,a}^{-s}\,u_a(z_{1,a},z_{2,a})\big), \tag{4.52}$$

$$u_{2,a}\colon A_a\longrightarrow\mathcal{Z}_a,\qquad u_{2,a}(z_{1,a},z_{2,a})=\big(z_{1,a},z_{1,a}^{-s}\,\lambda_a\,u_a(z_{1,a},z_{2,a})\big). \tag{4.53}$$

By definition, (4.43) is a sequence of $J_{\mathcal{Z}}$-holomorphic maps in $\mathcal{Z}$ and (4.44) is a sequence of J_a-holomorphic maps in $\mathcal{Z}_a$. In principle, one may try the proof above by replacing (4.43) and (4.43) with (4.52) and (4.52), respectively. However, multiplication by λ_a^{-1} in (4.51) and by r_a^{-1} in (4.48) have adverse effects on the almost complex structure, making it hard to apply the Gromov convergence.

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References

[1] D. Abramovich and Q. Chen, *Stable logarithmic maps to Deligne-Faltings pairs II*, Asian J. Math. 18 (2014), no. 3, 465–488.

[2] D. Abramovich, Q. Chen, M. Gross and B. Siebert, *Decomposition of degenerate Gromov-Witten Invariants*, math/1709.09864.

[3] D. Abramovich, S. Marcus, and J. Wise, *Comparison theorems for Gromov-Witten invariants of smooth pairs and of degenerations*, Annales de Institut Fourier, Vol. 64 no. 4 (2014), 1611–1667.

[4] D. A. Cox, J. B. Little, H. K. Schenck, *Toric varieties*, AMS Graduate Studies in Mathematics,Volume: 124, 2011.

[5] D. Eisenbud and B. Sturmfels, *Binomial ideals*, Duke Math. J. Volume 84, Number 1 (1996), 1–45.

[6] M. Farajzadeh-Tehrani, *Towards a degeneration formula for the Gromov-Witten invariants of symplectic manifolds.*

[7] M. Farajzadeh-Tehrani, *Pseudoholomorphic maps relative to normal crossings symplectic divisors: deformation theory*, in progress.

[8] M. Farajzadeh-Tehrani and K. Fukaya, *Gromov-Witten theory via Kuranishi structures*, math/1701.07821; to appear in the vol. 1 of AMS-Simons Center series.

[9] M. Farajzadeh-Tehrani, M. McLean, and A. Zinger, *Normal crossings singularities for symplectic topology*, math/1410.0609.

[10] M. Farajzadeh-Tehrani, M. McLean, and A. Zinger, *Normal crossings singularities for symplectic topology II*, in preparation.

[11] M. Farajzadeh-Tehrani, M. McLean, and A. Zinger, *Normal crossings singularities for symplectic topology: Structures*, in preparation.

[12] M. Farajzadeh-Tehrani and A. Zinger, *On symplectic sum formulas in Gromov-Witten theory*, math/1404.1898.

[13] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology 38 (1999), no. 5, 933–1048.

[14] K. Fukaya, Y. H. Oh, H. Ohta, and K. Ono, *Exponential decay estimates and smoothness of the moduli space of pseudoholomorphic curves*, math/1603.07026.

[15] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math., 82(2):307–347, 1985.

[16] E. Ionel, *GW invariants relative to normal crossing divisors*, Advances in Math. 281 (2015), 40–141.

[17] E. Ionel and T. Parker, *Relative Gromov-Witten invariants*, Ann. of Math. 157 (2003), no. 1, 45–96.

[18] M. Gross and B. Siebert, *Logarithmic Gromov-Witten invariants*, J. Amer. Math. Soc. 26 (2013), no. 2, 451–510

[19] C. Hummel, *Gromov's compactness theorem for pseudo-holomorphic curves*, volume 151 of Progress in Mathematics. Birkhäuser Verlag, Basel, 1997.

[20] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Diff. Geom. 57 (2001), no. 3, 509–578.

[21] A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. 145 (2001), no. 1, 151–218.

[22] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. 11 (1998), no. 1, 119–174.

[23] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 47–83, First Int. Press Lect. Ser., I, Int. Press, Cambridge, MA, 1998.

[24] D. McDuff and D. Salamon, *Symplectic Topology*, 2nd Ed., Oxford University Press, 1998.

[25] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, volume 52 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004.

[26] P. Pansu, *Pseudo-holomorphic curves in symplectic manifolds*, Ecole Polytechnique, Palaiseau, 1986.

[27] B. Parker, *Exploded manifolds*, Advances in Mathematics, v. 229, Issue 6, 1 April 2012, pp. 3256–3319.

[28] B. Parker, *Gromov-Witten invariants of exploded manifolds*, math/1102.0158.

[29] B. Parker, *On the value of thinking tropically to understand Ionel's GW invariants relative normal crossing divisors*, math/1407.3020.

[30] W. Robbin and D.A. Salamon, *A construction of the Deligne-Mumford orbifold*, J. Eur. Math. Soc. (JEMS), 8(4):611–699, 2006.

[31] R. Ye, *Gromov's compactness theorem for pseudo holomorphic curves*, Trans. Amer. Math. Soc., 342(2):671–694, 1994.

[32] A. Zinger, *Basic Riemannian geometry and Sobolev estimates used in symplectic topology*, math/1012.3980