A note on stochastic exponential integrators for the finite element discretization of semilinear SPDEs driven by additive noise

Antoine Tambue[a,b], Jean Daniel Mukam[c]

[a]*The African Institute for Mathematical Sciences(AIMS) of South Africa and Stellenbosch University, 6-8 Melrose Road, Muizenberg 7945, South Africa*

[b]*Center for Research in Computational and Applied Mechanics (CERECAM), and Department of Mathematics and Applied Mathematics, University of Cape Town, 7701 Rondebosch, South Africa.*

[c]*Fakultät für Mathematik, Technische Universität Chemnitz, 09126 Chemnitz, Germany*

Abstract

This paper aims to investigate the strong convergence analysis of the exponential Euler method for parabolic stochastic partial differential equation (SPDE) driven by additive noise under more relaxed conditions. The SPDE is discretized in space by the finite element method. In contrast to very restrictive assumptions made in the literature on the drift term to achieve optimal convergence orders with self-adjoint operator, we weaken those assumptions and assume only the standard Lipschitz condition on the drift and deal with not necessarily self-adjoint linear operator. Under this relaxed assumption, we have achieved optimal convergence orders, which depend on the regularity of the noise and the initial data. In particular, for trace class noise we achieve convergence orders of $\mathcal{O}(h^2 + \Delta t)$. These optimal convergence orders are due to a new optimal regularity result we have further derived here, which was not yet proved in the literature, even in the case of self-adjoint operator. Numerical experiments to illustrate our theoretical result are provided.

Keywords: Stochastic parabolic partial differential equations, Stochastic exponential integrators, Additive noise, Finite element method, Errors estimate.

*Corresponding author

Email addresses: `antonio@aims.ac.za` (Antoine Tambue), `jean.d.mukam@aims-senegal.org` (Jean Daniel Mukam)

1. Introduction

We consider the numerical approximation of SPDE defined in $\Lambda \subset \mathbb{R}^d$, $d = 1, 2, 3$, with initial value and boundary conditions of the following type

$$dX(t) = [AX(t) + F(X(t))]dt + dW(t), \quad X(0) = X_0, \quad t \in (0, T] \tag{1}$$

on the Hilbert space $L^2(\Lambda)$, where $T > 0$ is the final time, A is a linear operator which is unbounded and not necessarily self-adjoint. The noise $W(t) = W(x, t)$ is a $Q-$Wiener process defined in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ and $Q : H \longrightarrow H$ is positive and self-adjoint. The filtration is assumed to fulfill the usual conditions. Precise assumptions on F, X_0 and A will be given in the next section to ensure the existence of the unique mild solution X of (1), which has the following representation (see [14, 15])

$$X(t) = S(t)X_0 + \int_0^t S(t - s)F(X(s))ds + \int_0^t S(t - s)dW(s), \quad t \in [0, T]. \tag{2}$$

Equations of type (1) are used to model different real world phenomena in different fields such as biology, chemistry, physics etc [17, 2]. In more cases, explicit solutions of SPDEs are unknown, therefore numerical methods are the only good tools for their approximations. Numerical approximation of SPDE of type (1) is therefore an active research area and have attracted a lot of attentions since two decades (see for example [10, 22, 23, 7, 4, 5] and references therein). For many numerical schemes, the optimal convergence order in time under only standard Lipschitz conditions is $1/2$ (see for example [7, 10]). By incorporating suitable linear functionals of the noise, the accelerated exponential Euler (AEE) methods were proved in [4, 5, 22] to converge strongly to the mild solution of (1). Due to more information incorporated on the noise, the AEE schemes usually achieve convergence orders higher than $1/2$. The convergence analysis is usually done under restrictive assumptions (see for example [4, Assumption 2.4]). The AEE method in [4] was recently analyzed in [22] under more relaxed conditions. These assumptions were also used in [23, 13] for implicit scheme and exponential integrators with standard Brownian increments to obtain optimal convergence orders greater than $1/2$. However assumptions made on the drift function in [22, 23, 13] are still restrictive as they involve first and sometime second order derivatives of the drift function. In many problems the nonlinear function may not be differentiable. An illustrative example is the function $F(u) = |u|$, $u \in H$, which is not differentiable at 0. In

this paper, we investigate the strong convergence analysis of standard stochastic exponential integrators [10] using only standard Lipschitz condition on the drift function. The result indicates how the convergence orders depend on the regularity of the initial data and the noise. In fact, we achieve optimal convergence order $\mathcal{O}(h^\beta + \Delta t^{\beta/2})$, where $\beta \in (0, 2]$ is the regularity's parameter of the noise and initial data (see Assumption 2.1). Let us mention that even under restrictive assumptions (namely [4, Assumption 2.4] and [5, Assumption 2]), the convergence orders in time of the AEE schemes analyzed in [4, 5] are sub-optimal and have a logarithm reduction. It is worth to mention that the convergence order in time of the AEE scheme studied in [22] is optimal and does not have any logarithmic reduction. This is due to the sharp integral estimate [8, Lemma 3.2 (iii)] and the strong regularity assumptions on the drift term, namely [22, Assumption 2.1]. Note that [8, Lemma 3.2 (iii)] is only valid for self-adjoint operator. In this paper, we also extend [8, Lemma 3.2 (iii)] to the case of not necessarily self-adjoint operator in Lemma 2.1 and we further prove a new optimal regularity result, namely (20), which is not yet proved in the literature to the best of our knowledge. Lemma 2.1 and the regularity result (20) of Theorem 2.1 are therefore key ingredients to achieve optimal convergence orders in both space and time in this work.

The rest of the paper is organized as follows. Section 2 deals with the well posedness, the regularity result, the numerical scheme and the main result. In Section 3, we provide the proof of the main result. In Section 4, we have expanded the discussion on the nonlinear term and the associated challenges. We end the paper in Section 5 by giving some numerical experiments to sustain our theoretical result.

2. Mathematical setting and numerical method

2.1. Main assumptions and well posedness problem

Let us define functional spaces, norms and notations that will be used in the rest of the paper. Let $(H, \langle ., . \rangle, \|.\|)$ be a separable Hilbert space, we denote by $L^2(\Omega, U)$ the Banach space of all equivalence classes of squared integrable U-valued random variables, by $L(U, H)$ the space of bounded linear mappings from U to H endowed with the usual operator norm $\|.\|_{L(U,H)}$, by $\mathcal{L}_2(U, H) := HS(U, H)$ the space of Hilbert-Schmidt operators from U to H. We equip $\mathcal{L}_2(U, H)$

with the norm

$$\|l\|_{\mathcal{L}_2(U,H)}^2 := \sum_{i=1}^{\infty} \|l\psi_i\|^2, \quad l \in \mathcal{L}_2(U,H), \tag{3}$$

where $(\psi_i)_{i=1}^{\infty}$ is an orthonormal basis of U. Note that (3) is independent of the orthonormal basis of U. For simplicity, we use the notations $L(U,U) =: L(U)$ and $\mathcal{L}_2(U,U) =: \mathcal{L}_2(U)$. In the sequel, we take $H = L^2(\Lambda)$ and assume A to be second order and is given by

$$Au = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(q_{ij}(x) \frac{\partial u}{\partial x_j} \right) - \sum_{i=1}^{d} q_i(x) \frac{\partial u}{\partial x_i}, \tag{4}$$

where $q_{ij} \in L^{\infty}(\Lambda)$, $q_i \in L^{\infty}(\Lambda)$. We assume that there exists $c_1 > 0$ such that

$$\sum_{i,j=1}^{d} q_{ij}(x)\xi_i\xi_j \geq c_1|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad x \in \overline{\Lambda}.$$

As in [10, 13], we introduce two spaces $\mathbb{H}$ and V, such that $\mathbb{H} \subset V$, that depend on the boundary conditions for the domain of the operator A and the corresponding bilinear form. For example, for Dirichlet boundary conditions, we take

$$V = \mathbb{H} = H_0^1(\Lambda) = \{v \in H^1(\Lambda) : v = 0 \quad \text{on} \quad \partial\Lambda\}.$$

For Robin boundary conditions and Neumann boundary conditions, we take $V = H^1(\Lambda)$

$$\mathbb{H} = \{v \in H^2(\Lambda) : \partial v/\partial v_A + \alpha_0 v = 0, \quad \text{on} \quad \partial\Lambda\}, \quad \alpha_0 \in \mathbb{R}.$$

Let $a : V \times V \longrightarrow \mathbb{R}$ be the bilinear operator associated to $-A$, which depends on the boundary condition, see e.g. [10, 13] for details. Note that using the above assumptions on $q_{i,j}$ and q_j we can prove that A is sectorial and generates an analytic semigroup $S(t) = e^{At}$, the fractional powers of $-A$ are also well defined and characterized [3, 10, 13] for any $\alpha > 0$ by

$$\begin{cases} (-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{tA} dt, \\ (-A)^{\alpha} = ((-A)^{-\alpha})^{-1}, \end{cases} \tag{5}$$

where $\Gamma(\alpha)$ is the Gamma function, see [3]. In addition, the following holds [3, 10, 13]

$$(-A)^{\alpha} S(t) = S(t)(-A)^{\alpha} \quad \text{on} \quad \mathcal{D}((-A)^{\alpha}), \quad \alpha \geq 0. \tag{6}$$

In order to ensure the existence and the uniqueness of solution of (1) and for the purpose of the convergence analysis, we make the following assumption.

Assumption 2.1. *We assume that the initial data $X_0 \in L^2\left(\Omega, \mathcal{D}\left((-A)^{\beta/2}\right)\right)$, $0 < \beta \leq 2$, the nonlinear function $F : H \longrightarrow H$ and the covariance operator Q satisfy the following properties*

$$\|F(0)\| \leq C, \quad \|F(Y) - F(Z)\| \leq C\|Y - Z\|, \quad Y, Z \in H, \tag{7}$$

$$\left\|(-A)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\right\|_{\mathcal{L}_2(H)} < C, \tag{8}$$

where C is a positive constant.

The following lemma provides a version of [8, Lemma 3.2 (iii)] for not necessarily self-adjoint operator. Lemma 2.1 is useful to achieve optimal regularity result of the mild solution (2).

Lemma 2.1. *For any $0 \leq \rho \leq 1$ and $0 \leq \gamma \leq 2$, there exists a constant C such that*

$$\int_{t_1}^{t_2} \left\|(-A)^{\rho/2}S(t_2 - r)\right\|_{L(H)}^2 dr \leq C(t_2 - t_1)^{1-\rho}, \quad 0 \leq t_1 \leq t_2 \leq T, \tag{9}$$

$$\int_{t_1}^{t_2} \left\|(-A)^{\gamma/2}S(t_2 - r)\right\|_{L(H)} dr \leq C(t_2 - t_1)^{1-\gamma/2}, \quad 0 \leq t_1 \leq t_2 \leq T. \tag{10}$$

Proof. Let us write $A = A_s + A_n$, where A_s and A_n are respectively the self-adjoint and the non self-adjoint parts of A. As in [21, (147)], we use the Zassenhaus product formula [11, 16] to decompose the semigroup $S(t)$ as follows.

$$S(t) = e^{At} = e^{(A_s+A_n)t} = e^{A_s t}e^{A_n t}\prod_{k=2}^{\infty} e^{C_k}, \tag{11}$$

where $C_k = C_k(t)$ are called Zassenhaus exponents. In (11), let us set

$$S_N(t) := e^{A_n t}\prod_{k=2}^{\infty} e^{C_k}, \quad S(t) = S_s(t)S_N(t), \tag{12}$$

where $S_s(t) := e^{A_s t}$ is the semigroup generated by A_s. Using the Baker-Campbell-Hausdorff representation formula [11, 12, 16], one can prove as in [21] that $S_N(t)$ is a linear bounded operator. Therefore using (12) and the boundedness of $S_N(t)$ yields

$$\int_{t_1}^{t_2} \left\|(-A)^{\rho/2}S(t_2 - r)\right\|_{L(H)}^2 dr = \int_{t_1}^{t_2} \left\|(-A)^{\rho/2}S_s(t_2 - r)S_N(t_2 - r)\right\|_{L(H)}^2 dr$$

$$\leq C \int_{t_1}^{t_2} \left\|(-A)^{\rho/2}S_s(t_2 - r)\right\|_{L(H)}^2 dr. \tag{13}$$

As in [21], note that $\mathcal{D}(-A) = \mathcal{D}(-A_s)$ with equivalent norms (see [1]). Therefore by [9, (3.3)] and by interpolation technique, it holds that $\mathcal{D}((-A)^\alpha) = \mathcal{D}((-A_s)^\alpha)$, $0 \leq \alpha \leq 1$, with equivalent norms. It follows therefore that

$$\left\|(-A)^{\rho/2}S_s(t_2 - r)\right\|_{L(H)} \leq C \left\|(-A_s)^{\rho/2}S_s(t_2 - r)\right\|_{L(H)}. \tag{14}$$

Substituting (14) in (13) yields

$$\int_{t_1}^{t_2} \left\| (-A)^{\rho/2} S(t_2 - r) \right\|_{L(H)}^2 dr \leq C \int_{t_1}^{t_2} \left\| (-A_s)^{\rho/2} S_s(t_2 - r) \right\|_{L(H)}^2 dr. \tag{15}$$

Since A_s is self-adjoint, it follows from [8, Lemma 3.2 (iii)] that

$$\int_{t_1}^{t_2} \left\| (-A_s)^{\rho/2} S_s(t_2 - r) \right\|_{L(H)}^2 dr \leq C(t_2 - t_1)^{1-\rho}. \tag{16}$$

Substituting (16) in (15) complete the proof of (9). To prove (10), we use Cauchy-Schwarz inequality, (6) and (9). This yields

$$
\begin{aligned}
\int_{t_1}^{t_2} \| (-A)^{\gamma/2} S(t_2 - r) \|_{L(H)} dr &= \int_{t_1}^{t_2} \left\| (-A)^{\gamma/4} S\left(\frac{1}{2}(t_2 - r) \right) (-A)^{\gamma/4} S\left(\frac{1}{2}(t_2 - r) \right) \right\|_{L(H)} dr \\
&\leq \int_{t_1}^{t_2} \left\| (-A)^{\gamma/4} S\left(\frac{1}{2}(t_2 - r) \right) \right\|_{L(H)}^2 dr \\
&\leq C(t_2 - t_1)^{1-\gamma/2}.
\end{aligned}
\tag{17}
$$

This completes the proof of Lemma 2.1. ∎

Theorem 2.1. *Let Assumption 2.1 be satisfied. If X_0 is an $\mathcal{F}_0$-measurable H-valued random variable, then there exists a unique mild solution X of problem (1) represented by (2) and satisfying*

$$\mathbb{P} \left[\int_0^T \|X(s)\|^2 ds < \infty \right] = 1, \tag{18}$$

and for any $p \geq 2$, there exists a constant $C = C(p, T) > 0$ such that

$$\sup_{t \in [0,T]} \mathbb{E}\|X(t)\|^p \leq C(1 + \mathbb{E}\|X_0\|^p). \tag{19}$$

Moreover, the following optimal regularity estimate holds

$$\left\| (-A)^{\beta/2} X(t) \right\|_{L^2(\Omega, H)} \leq C \left(1 + \| (-A)^{\beta/2} X_0 \|_{L^2(\Omega, H)} \right), \quad t \in [0, T]. \tag{20}$$

Proof. The proof of the existence, the uniqueness and (19) can be found in [14, Theorem 7.2]. The proof of (20) for $\beta \in (0, 2)$ is similar to that of [6, Theorem 1] for multiplicative noise and can also be found in [13, Lemma 3]. Note that the case $\beta = 2$ is of great importance in numerical analysis, it allows to avoid reduction of convergence order. It corresponds to [6, Theorem 1] with $\gamma = 1$ and [8, Theorem 3.1] with $r = 1$. To the best of our knowledge the case $\beta = 2$ is not treated

so far in the scientific literature, even for self-adjoint operators. We fill that gap with the help of Lemma 2.1. Indeed, from the mild solution (2), it follows by using triangle inequality that

$$
\begin{aligned}
\left\|(-A)^{\beta/2} X(t)\right\|_{L^2(\Omega,H)} &\leq \left\|(-A)^{\beta/2} X_0\right\|_{L^2(\Omega,H)} + \int_0^t \left\|(-A)^{\beta/2} S(t-s)\right\|_{L(H)} \left\|F(X(s))\right\| ds \\
&+ \left\|\int_0^t (-A)^{\beta/2} S(t-s) dW(s)\right\|_{L(H)} := I_0 + I_1 + I_2.
\end{aligned}
\tag{21}
$$

Using Lemma 2.1 we can easily prove that $I_1 \leq C$. Using Itô-isometry property, Lemma 2.1 and Assumption 2.1 yields

$$
\begin{aligned}
I_2^2 &= \int_0^t \left\|(-A)^{\beta/2} S(t-s) Q^{1/2}\right\|_{\mathcal{L}_2(H)}^2 ds \\
&\leq \int_0^t \left\|(-A)^{1/2} S(t-s)\right\|_{L(H)}^2 \left\|(-A)^{\frac{\beta-1}{2}} Q^{1/2}\right\|_{\mathcal{L}_2(H)}^2 ds \\
&\leq C \int_0^t \left\|(-A)^{1/2} S(t-s)\right\|_{L(H)}^2 ds \\
&\leq C.
\end{aligned}
\tag{22}
$$

Substituting (22) and the estimate of I_1 in (21) complete the proof of the regularity result in (20). ∎

2.2. Numerical scheme and main result

Let us first perform the space approximation of problem (1). We start by discretizing our domain Ω by a finite triangulation. Let $\mathcal{T}_h$ be a triangulation with maximal length h. Let $V_h \subset V$ denote the space of continuous functions that are piecewise linear over the triangulation $\mathcal{T}_h$. We consider the projection P_h and the discrete operator A_h defined respectively from $L^2(\Omega)$ to V_h and from V_h to V_h by

$$
(P_h u, \chi) = (u, \chi), \quad \forall \chi \in V_h, u \in H, \quad \text{and} \quad (A_h \phi, \chi) = -a(\phi, \chi), \quad \forall \phi, \chi \in V_h.
\tag{23}
$$

The discrete operator A_h is also a generator of an analytic semigroup $S_h(t) := e^{tA_h}$. The semi-discrete in space version of problem (1) consists of finding $X^h \in V_h$ such that

$$
dX^h(t) = [A_h X^h(t) + P_h F(X^h(t))] dt + P_h dW(t), \quad X^h(0) = P_h X_0, \quad t \in (0, T].
\tag{24}
$$

The mild solution $X^h(t)$ of (24) has the following integral form

$$
X^h(t) = S_h(t) X^h(0) + \int_0^t S_h(t-s) P_h F(X^h(s)) ds + \int_0^t S_h(t-s) P_h dW(s), \quad t \in [0, T].
\tag{25}
$$

Let $t_m = m\Delta t \in [0, T]$, where $M \in \mathbb{N}$ and $\Delta t = T/M$. The mild solution of (1) at t_m, $m = 1, \cdots, M$ can be also written as follows.

$$X(t_m) = S(\Delta t)X(t_{m-1}) + \int_{t_{m-1}}^{t_m} S(t_m - s)F(X(s))ds + \int_{t_{m-1}}^{t_m} S(t_m - s)dW(s). \tag{26}$$

For the time discretization, we consider the exponential Euler method proposed in [10], which gives the numerical approximation X_m^h of $X^h(t_m)$ through the following recurrence

$$X_m^h = e^{A_h \Delta t} X_{m-1}^h + A_h^{-1} \left(e^{A_h \Delta t} - \mathbf{I} \right) P_h F(X_{m-1}^h) + e^{A_h \Delta t} P_h \left(W_{t_{m+1}} - W_{t_m} \right). \tag{27}$$

with $X_0^h := P_h X_0$. The scheme (27) can be writen in the following integral form, useful for the error estimate

$$X_m^h = S_h(\Delta t)X_{m-1}^h + \int_{t_{m-1}}^{t_m} S_h(t_m - s)P_h F(X^h(s))ds + \int_{t_{m-1}}^{t_m} S_h(\Delta t)P_h dW(s). \tag{28}$$

An equivalent formulation of (27) easy for simulation is given by

$$X_m^h = X_{m-1}^h + P_h \Delta W_{m-1} + \Delta t \varphi_1(\Delta t A_h) \left[A_h(X_{m-1}^h + P_h \Delta W_{m-1}) + P_h F(X_{m-1}^h) \right], \tag{29}$$

where

$$\varphi_1(\Delta t A_h) = (\Delta t A_h)^{-1} \left(e^{\Delta t A_h} - \mathbf{I} \right) = \frac{1}{\Delta t} \int_0^{\Delta t} e^{(\Delta t - s)A_h} ds. \tag{30}$$

With the numerical method in hand, we can state our strong convergence result.

Throughout this paper, C is a positive constant independent of h, m, M and Δt and that may change from one place to another.

Theorem 2.2 (Main Result). *Let X be the mild solution of problem (1) and X_m^h the approximated solution at time t_m with the scheme (27). If Assumption 2.1 is fulfilled, then the following error estimate holds*

$$\|X(t_m) - X_m^h\|_{L^2(\Omega, H)} \leq C \left(h^\beta + \Delta t^{\beta/2} \right), \quad m = 0, 1, \cdots, M. \tag{31}$$

Remark 2.1. *Theorem 2.2 also holds if we replace the numerical approximation X_m^h by the stochastic exponential Rosenbrock-Euler scheme proposed in [13]. Note that the arguments in this paper can be applied to the numerical schemes in [22, 23] with the drift term satisfying only the global Lipschitz condition with the linear self-adjoint operator A. In this case, the time error*

estimate in [22] will have the convergence rate $\mathcal{O}\left(\Delta t^{\min(\beta,1)}\right)$ [1] and that in [23] will have the same convergence rate as in (31). Note that in the case of not necessarily self-adjoint operator A, the arguments in this paper are not enough to applied to the numerical scheme in [23]. In fact one need in addition some error estimates of the associated deterministic problem, which do not rely on the spectral decomposition of A. This can be found in our accompanied paper [20].

3. Preliminary results and proof of the main result

3.1. Preliminary results

The proof of the main result requires three important lemmas. Let us start by considering the following deterministic problem: find $u \in V$ such that $u'(t) = Au(t)$, $t \in (0, T]$, $u(0) = v$. The corresponding semi-discrete problem in space consists of finding $u_h \in V_h$ such that $u'_h(t) = A_h u_h(t)$, $t \in (0, T]$, $u_h(0) = P_h v$. The following lemma can be found in [10, Lemma 3.1] or [13, Lemma 7].

Lemma 3.1. *Let $r \in [0, 2]$ and $\gamma \in [0, r]$. Then the following estimate holds*

$$\|u(t) - u_h(t)\| = \|(e^{At} - e^{A_h t} P_h)v\| \le Ch^r t^{-(r-\gamma)/2}\|v\|_\gamma, \quad v \in \mathcal{D}\left((-A)^{\gamma/2}\right). \tag{32}$$

Lemma 3.2. *Let Assumption 2.1 be fulfilled. Then the following error estimate holds*

$$\left\|(-A_h)^{\frac{\beta-1}{2}} P_h Q^{\frac{1}{2}}\right\|_{\mathcal{L}_2(H)} \le C, \tag{33}$$

$$\int_0^t \|(S(s) - S_h(s)P_h)v\|^2 ds \le Ch^\gamma \|v\|_{\gamma-1}, \quad v \in \mathcal{D}\left((-A)^{(\gamma-1)/2}\right), \quad 0 \le \gamma \le 2. \tag{34}$$

Proof. The proof of (33) can be found in [13, Lemma 11] and the proof of (34) can be found in [21, Lemma 6.1 (ii)]. ∎

Lemma 3.3. *(Discrete Gronwall lemma) Let (x_k) and (z_k) be sequences of non negatives numbers. Let c be a non negative constant. If*

$$x_k \le c + \sum_{0 \le i < k} z_i x_i, \quad k \ge 0, \tag{35}$$

---

[1]Note that in our error estimates the deterministic part is of order $\mathcal{O}(h^\beta + \Delta t)$. To update the proof in [22], we just need to follow the deterministic part of the current work as the noise term is already high order in [22].

then the following estimate holds

$$x_k \leq c \prod_{0 \leq j < k} (1 + z_j) \leq c \exp \left(\sum_{0 \leq j < k} z_j \right), \quad k \geq 0. \tag{36}$$

Proof. As in [18], we use the convention that an empty product is 1. Applying [18, Lemma 2] with $f_k \equiv c$ and using the telescopic identity yields

$$
\begin{aligned}
x_k &\leq c + \sum_{0 \leq i < k} c z_i \prod_{i < j < k} (1 + z_j) = c + c \sum_{0 \leq i < k} \left[\prod_{i \leq j < k} (1 + z_j) - \prod_{i+1 \leq j < k} (1 + z_j) \right] \\
&= c + c \left[\prod_{0 \leq j < k} (1 + z_j) - \prod_{k \leq j < k} (1 + z_j) \right] = c \prod_{0 \leq j < k} (1 + z_j) \leq c \exp \left(\sum_{0 \leq j < k} z_j \right), \tag{37}
\end{aligned}
$$

where at the last step we used the inequality $1 + z_j \leq e^{z_j}$. $\blacksquare$

3.2. Proof of the main result

Let us now turn to the the proof of Theorem 2.2. Subtracting (28) from (26) and taking the norm in both sides yields

$$
\begin{aligned}
\|X(t_m) - X_m^h\|_{L^2(\Omega, H)} &\leq \|S(\Delta t) X(t_{m-1}) - S_h(\Delta t) X_{m-1}^h\|_{L^2(\Omega, H)} \\
&+ \int_{t_{m-1}}^{t_m} \|S(t_m - s) F(X(s)) - S_h(t_m - s) P_h F(X_{m-1}^h)\|_{L^2(\Omega, H)} ds \\
&+ \left\| \int_{t_{m-1}}^{t_m} (S(t_m - s) - S_h(\Delta t) P_h) \, dW(s) \right\|_{L^2(\Omega, H)} \\
&:= II_1 + II_2 + II_3. \tag{38}
\end{aligned}
$$

Using triangle inequality, (20) and Lemma 3.1 with $r = \gamma = \beta$, it holds that

$$
\begin{aligned}
II_1 &\leq \|(S(\Delta t) - S_h(\Delta t) P_h) X(t_{m-1})\|_{L^2(\Omega, H)} + \|S_h(\Delta t)(X(t_{m-1}) - X_{m-1}^h)\|_{L^2(\Omega, H)} \\
&\leq Ch^\beta + C\|X(t_{m-1}) - X_{m-1}^h\|_{L^2(\Omega, H)}. \tag{39}
\end{aligned}
$$

Using Assumption 2.1, the boundness of $X(s)$ and X_{m-1}^h ([21, Lemma 4.2]), it holds that

$$II_2 \leq C \int_{t_{m-1}}^{t_m} \|F(X(s))\|_{L^2(\Omega, H)} ds + C \int_{t_{m-1}}^{t_m} \|P_h F(X_{m-1}^h))\|_{L^2(\Omega, H)} ds \leq C\Delta t. \tag{40}$$

Using the Itô-isometry property and triangle inequality, we split II_3 in two terms.

$$
\begin{aligned}
II_3 &\leq \int_{t_{m-1}}^{t_m} \left\| (S(t_m - s) - S_h(t_m - s) P_h) Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(H)}^2 ds \\
&+ \int_{t_{m-1}}^{t_m} \left\| (S_h(t_m - s) - S_h(\Delta t)) P_h Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(H)}^2 ds \\
&:= II_{31} + II_{32}. \tag{41}
\end{aligned}
$$

10

Using Lemma 3.2 and Assumption 2.1, it holds that

$$
\begin{aligned}
II_{31} &\leq \int_{t_{m-1}}^{t_m} \sum_{i=1}^{\infty} \|(S(t_m - s) - S_h(t_m - s)P_h)Q^{\frac{1}{2}}\psi_i\|^2 ds \\
&= \sum_{i=1}^{\infty} \int_{t_{m-1}}^{t_m} \|(S(t_m - s) - S_h(t_m - s)P_h)Q^{\frac{1}{2}}\psi_i\|^2 ds \\
&\leq \sum_{i=1}^{\infty} Ch^{2\beta}\|Q^{\frac{1}{2}}\psi_i\|_{\beta-1}^2 = Ch^{2\beta}\left\|(-A)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\right\|_{\mathcal{L}_2(H)} \leq Ch^{2\beta}.
\end{aligned}
\tag{42}
$$

Using (33) it holds for $\epsilon > 0$ small enough that

$$
\begin{aligned}
II_{32} &\leq \int_{t_{m-1}}^{t_m} \left\|S_h(t_m - s)(\mathbf{I} - S_h(s - t_{m-1}))(-A_h)^{\frac{1-\beta}{2}}\right\|_{L(H)}^2 \left\|(-A_h)^{\frac{\beta-1}{2}}P_hQ^{\frac{1}{2}}\right\|_{\mathcal{L}_2(H)}^2 ds \\
&\leq C \int_{t_{m-1}}^{t_m} \left\|S_h(t_m - s)(-A_h)^{\frac{1-\epsilon}{2}}\right\|_{L(H)}^2 \left\|(\mathbf{I} - S_h(s - t_{m-1}))(-A_h)^{\frac{-\beta+\epsilon}{2}}\right\|_{L(H)}^2 ds \\
&\leq C \int_{t_{m-1}}^{t_m} (t_m - s)^{-1+\epsilon}(s - t_{m-1})^{\beta-\epsilon} ds \\
&\leq C\Delta t^{\beta-\epsilon} \int_{t_{m-1}}^{t_m} (t_m - s)^{-1+\epsilon} ds \leq C\Delta t^{\beta}.
\end{aligned}
\tag{43}
$$

Substituting (43), (42), (40) and (39) in (38) yields

$$
\begin{aligned}
\|X(t_m) - X_m^h\|_{L^2(\Omega,H)} &\leq Ch^{\beta} + C\Delta t^{\beta/2} + C\|X(t_{m-1}) - X_{m-1}^h\|_{L^2(\Omega,H)} \\
&\leq C\left(h^{\beta} + \Delta t^{\beta/2}\right) + C\|X(t_{m-1}) - X_{m-1}^h\|_{L^2(\Omega,H)} \\
&+ C\Delta t \sum_{k=0}^{m-2} \|X(t_k) - X_k^k\|_{L^2(\Omega,H)}.
\end{aligned}
\tag{44}
$$

Applying the discrete Gronwall's lemma (Lemma 3.3) to (44) yields

$$
\|X(t_m) - X_m^h\|_{L^2(\Omega,H)} \leq C\left(h^{\beta} + \Delta t^{\beta/2}\right)\exp\left(C + \sum_{k=0}^{m-2}\Delta t\right) \leq C\left(h^{\beta} + \Delta t^{\beta/2}\right).
\tag{45}
$$

This completes the proof of Theorem 2.2.

4. Remark on current error estimates in the literature

In the literature, to obtain the error estimates many authors [22, 23, 10, 13, 4, 5, 7] usually iterate the numerical scheme, the mild solution of the problem (1) and the mild solution of the semi-discrete problem in space (24). Indeed we usually have

$$
X_m^h = S_h(t_m)P_hX_0 + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} S_h(t_m - s)P_hF(X_k^h)ds + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} S_h(\Delta t)P_h dW(s)
\tag{46}
$$

and

$$X^h(t_m) = S_h(t_m)P_hX_0 + \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m - s)P_hF(X^h(s))ds + \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S_h(t_m - s)P_hdW(s). \quad (47)$$

Subtracting (46) from (47), taking the L^2 norm and using triangle inequality yields

$$
\begin{aligned}
\|X^h(t_m) - X_m^h\|_{L^2(\Omega,H)} &\leq \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left\|S_h(t_m - s)(P_hF(X^h(s)) - P_hF(X_k^h))\right\|_{L^2(\Omega,H)}ds \\
&+ \sum_{k=0}^{m-1}\left\|\int_{t_k}^{t_{k+1}} (S_h(t_m - s) - S_h(\Delta t))\, dW(s)\right\|_{L^2(\Omega,H)} \\
&=: \quad III_1 + III_2. \quad (48)
\end{aligned}
$$

Note that compare to (38), the summation appears in the right hand side of (48). This usually leads to an optimal convergence order $1/2$ in time if the drift term F is only Lipschitz continuous (see e.g. [10]). In fact, the term involving the drift function (deterministic term) is usually estimated as follows.

$$
\begin{aligned}
III_1 &\leq \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|S(t_m - s)(P_hF(X^h(s)) - P_hF(X^h(t_k)))\|_{L^2(\Omega,H)} \quad (49) \\
&+ \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|S(t_m - s)(P_hF(X^h(t_k)) - P_hF(X_k^h))\|_{L^2(\Omega,H)}ds \\
&\leq C\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|X^h(s) - X^h(t_k)\|_{L^2(\Omega,H)}ds + C\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|X^h(t_k) - X_k^h\|_{L^2(\Omega,H)}ds \\
&\leq C\Delta t^{\min(\beta,1)/2} + C\Delta t\sum_{k=0}^{m-1} \|X^h(t_k) - X_k^h\|_{L^2(\Omega,H)}, \quad (50)
\end{aligned}
$$

where we have used the triangle inequality, (7) and [13, Lemma 4].

To the best of our knowledge, to obtain convergence rate in time greater more than $1/2$, almost all authors [22, 23, 13, 4, 5] require F to be twice differentiable with derivatives satisfying certain assumptions (e.g. [4, Assumption 2.4] or [22, Assumption 2.1]), so that they can apply the Taylor expansion to (49). But note that in the case where F is not differentiable, this is not feasible. We fill that gap in this paper by a mean of a sharp integral estimate, a new regularity result (namely Lemma 2.1 and Theorem 2.1 and (20)). Note also that, in our convergence proof, we do not iterate the numerical scheme and the mild solution from time $t_0 = 0$ to t_m.

5. Numerical simulations

We consider the stochastic advection diffusion reaction SPDE (1) with constant diagonal difussion tensor $\mathbf{Q} = 5\mathbf{I}_2 = (q_{i,j})$ in (4), and mixed Neumann-Dirichlet boundary conditions on $\Lambda = [0, L_1] \times [0, L_2]$. The Dirichlet boundary condition is $X = 1$ at $\Gamma = \{(x, y) : \ x = 0\}$ and we use the homogeneous Neumann boundary conditions elsewhere. The eigenfunctions $\{e_{i,j}\} = \{e_i^{(1)} \otimes e_j^{(2)}\}_{i,j \geq 0}$ of the covariance operator Q are the same as for Laplace operator $-\Delta$ with homogeneous boundary condition given by

$$e_0^{(l)}(x) = \sqrt{\frac{1}{L_l}}, \qquad e_i^{(l)}(x) = \sqrt{\frac{2}{L_l}} \cos\left(\frac{i\pi}{L_l} x\right), \, i = 1, 2, 3, \cdots$$

where $l \in \{1, 2\}$, $x \in \Lambda$. We assume that the noise can be represented as

$$W(x, t) = \sum_{i \in \mathbb{N}^2} \sqrt{\lambda_{i,j}} e_{i,j}(x) \beta_{i,j}(t), \tag{51}$$

where $\beta_{i,j}(t)$ are independent and identically distributed standard Brownian motions, $\lambda_{i,j}$, $(i, j) \in \mathbb{N}^2$ are the eigenvalues of Q, with

$$\lambda_{i,j} = \left(i^2 + j^2\right)^{-(\beta+\epsilon)}, \, \beta > 0, \tag{52}$$

in the representation (51) for some small $\epsilon > 0$. Assumption 2.1 on the noise term is obviously satisfied for $\beta = (0, 2]$. We obtain the Darcy velocity field $\mathbf{q} = (q_i)$ by solving the following system

$$\nabla \cdot \mathbf{q} = 0, \qquad \mathbf{q} = -\mathbf{k}\nabla p, \tag{53}$$

with Dirichlet boundary conditions on $\Gamma_D^1 = \{0, 2\} \times [0, 2]$ and Neumann boundary conditions on $\Gamma_N^1 = (0, 2) \times \{0, 2\}$ such that

$$p = \begin{cases} 1 & \text{in} \quad \{0\} \times [0, 2] \\ 0 & \text{in} \quad \{2\} \times [0, 2] \end{cases}$$

and $-\mathbf{k}\nabla p(\mathbf{x}, t) \cdot \mathbf{n} = 0$ in Γ_N^1. Note that $\mathbf{k}$ is the permeability tensor. We use a heterogeneous medium with three parallel high permeability streaks, 1000 times higher compared to the other part of the medium. This could represent for example a highly idealized fracture pattern. To deal with high Péclet flows we discretise in space using finite volume method, viewed as a finite element method (see [19]). The nonlinear term is $F(X) = -\max(0, 1 - 2X)$. Note that F is not

13

differentiable at $X = 1/2$ and satisfies the gobal Lipschitz condition in Assumption 2.1. We take $L_1 = L_2 = 2$ and our reference solutions samples are numerical solutions using a time step of $\Delta t = 1/1024$. The errors are computed at the final time $T = 2$. The initial solution is $X_0 = 0$, so we can therefore expect high order convergence, which depends only on the noise term. In Figure 1, the order of convergence is 0.71 for $\beta = 1.5$ and 0.97 for $\beta = 2$, which are close to 0.75 and 1 in our theoretical result in Theorem 2.2.
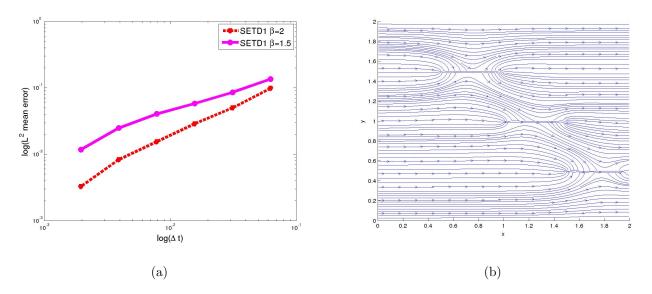

| (a) | (b) |

Figure 1: Convergence in the root mean square L^2 norm at $T = 2$ as a function of Δt. We show convergence for noise where $\beta \in \{1.5, 2\}$ and $\epsilon = 0.001$ in relation (52). We have used here 30 realizations. The order of convergence is 0.71 for $\beta = 1.5$ and 0.97 for $\beta = 2$.

Acknowledgements

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