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A Cheeger-type exponential bound for the number of triangulated manifolds

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Abstract

In terms of the number of triangles, it is known that there are more than exponentially many triangulations of surfaces, but only exponentially many triangulations of surfaces with bounded genus. In this paper we provide a first geometric extension of this result to higher dimensions. We show that in terms of the number of facets, there are only exponentially many geometric triangulations of space forms with bounded geometry in the sense of Cheeger (curvature and volume bounded below, and diameter bounded above). This establishes a combinatorial version of Cheeger's finiteness theorem.

Further consequences of our work are:

(1) There are exponentially many geometric triangulations of S^d.

(2) There are exponentially many convex triangulations of the d-ball.

Contents

1 Introduction

In discrete quantum gravity, one simulates Riemannian structures by considering all possible triangulations of manifolds [Reg61, ADJ97, Wei82]. The metric is introduced a posteriori, by assigning to each edge a certain length (as long as all triangular inequalities are satisfied). For example, in Weingarten's *dynamical triangulations* model, we simply assign to all edges length 1, and view all triangles as equilateral triangles in the plane [ADJ97, Wei82]. The resulting intrinsic metric is sometimes called "equilateral flat metric", cf. [AB17+].

This model gained popularity due to its simplification power. For example, the *partition function* for quantum gravity, a path integral over all Riemannian metrics, becomes a sum over all possible triangulations with N facets [Wei82]. To make sure that this sum converges when N tends to infinity, one needs to establish an exponential bound for the number of triangulated d-manifolds with N facets. However, already for $d = 2$, this dream is simply impossible: There are more than exponentially many surfaces with N triangles (cf. Example 2.0.1). For $d = 2$ the problem can be bypassed by restricting the topology, because for fixed g there are only exponentially many triangulations of the genus-g surface, as explained in [ADJ97, Tut62].

In dimension greater than two, however, it is not clear which geometric tools to use to provide exponential cutoffs for the class of triangulations with N simplices. Among others, Gromov [Gro00, pp. 156–157] asked whether there are more than exponentially many triangulations of S^3; the problem is still open. Part of the difficulty is that when $d \geq 3$ many d-spheres cannot be realized as boundaries of $(d+1)$-polytopes [Kal88, PZ04], and cannot even be shelled [HZ00]. In fact, we know that shellable spheres are only exponentially many [BZ11].

In this paper, we tackle the problem from a new perspective. Cheeger's finiteness theorem states that there are only finitely many diffeomorphism types of space forms with "bounded geometry" — namely, curvature and volume bounded below, and diameter bounded above. What we achieve is a discrete analogue of Cheeger's theorem, which (roughly speaking) shows that *geometric triangulations* of manifolds with bounded geometry are very few, compared to all possible triangulations.

Theorem I (Theorem 3.0.3)**.** *In terms of the number of facets, there are exponentially many* *geometric triangulations of space forms with bounded geometry (and fixed dimension).*

Since every topological triangulation of an orientable surface can be straightened to a geometric one [CdV91, Wag36], this result is a generalization of the classical exponential bound on the number of triangulated surfaces with bounded genus.

Here is the proof idea. Via Cheeger's bounds on the injectivity radius, we chop any manifold of constant curvature into a finite number of convex pieces of small diameter. Up to performing a couple of barycentric subdivisions, we can assume that each piece is a shellable ball [AB17], and in particular endo-collapsible [Ben12]. This implies an upper bound for the number of critical faces that a discrete Morse function on the triangulation can have. From here we are able to conclude, using the second author's result that there are only exponentially many triangulations of manifolds with bounded discrete Morse vector [Ben12].

The outcome of Theorem I suggested us to look at a possible fixed-metric approach. Inspired by Gromov's question on S^3, let us consider the unit sphere S^d with its standard intrinsic metric. Let us agree to call "geometric triangulation" any tiling of S^d into regions that are convex simplices with respect to the given metric and combinatorially form a simplicial complex. For example, all the boundaries of $(d+1)$-polytopes yield geometric triangulations of S^d, but not all geometric triangulations arise this way. How many geometric traingulations are there?

Once again, by proving that the second derived subdivision of every geometric triangulation is endo–collapsible, we obtain

Theorem II (Theorem 2.0.7)**.** *There are at most $2^{d^2 \cdot ((d+1)!)^2 \cdot N}$ distinct combinatorial types of triangulations of the standard S^d with N facets.*

Our methods rely, as explained, on convex and metric geometry. Whether *all* triangulations of S^d are exponentially many, remains open. But even if the answer turned out to be negative, Main Theorems I and II provide some support for the hope of discretizing quantum gravity in all dimensions.

Preliminaries

By $\mathbb{R}^d$ and S^d we denote the euclidean d-space and the unit sphere in $\mathbb{R}^{d+1}$, respectively. A *(euclidean) polytope* in $\mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$. A *spherical polytope* in S^d is the convex hull of a finite number of points that all belong to some open hemisphere of S^d. Spherical polytopes are in natural one-to-one correspondence with euclidean polytopes, just by taking radial projections. A *geometric polytopal complex* in $\mathbb{R}^d$ (resp. in S^d) is a finite collection of polytopes in $\mathbb{R}^d$ (resp. S^d) such that the intersection of any two polytopes is a face of both. An *intrinsic polytopal complex* is a collection of polytopes that are attached along isometries of their faces, so that the intersection of any two polytopes is a face of both.

The *face poset* $(C, \subseteq)$ of a polytopal complex C is the set of nonempty faces of C, ordered with respect to inclusion. Two polytopal complexes C, D are *combinatorially equivalent*, denoted by $C \cong D$, if their face posets are isomorphic. Any polytope combinatorially equivalent to the d-simplex, or to the regular unit cube $[0,1]^d$, shall simply be called a *d-simplex* or a *d-cube*, respectively. A polytopal complex is *simplicial* (resp. *cubical*) if all its faces are simplices (resp. cubes).

The *underlying space* $|C|$ of a polytopal complex C is the topological space obtained by taking the union of its faces. If two complexes are combinatorially equivalent, their underlying spaces are homeomorphic. If C is simplicial, C is sometimes called a *triangulation* of $|C|$ (and of any topological space homeomorphic to $|C|$). If $|C|$ is isometric to some metric space M, then C is called a *geometric triangulation* of M.

A *subdivision* of a polytopal complex C is a polytopal complex C' with the same underlying space of C, such that for every face F' of C' there is some face F of C for which $F' \subset F$. A *derived subdivision* $\operatorname{sd} C$ of a polytopal complex C is any subdivision of C obtained by stellarly subdividing at all faces in order of decreasing dimension of the faces of C, cf. [Hud69]. An example of a derived subdivision is the *barycentric subdivision*, which uses as vertices the barycenters of all faces of C.

If C is a polytopal complex, and A is some set, we define the *restriction* $\mathrm{R}\,(C, A)$ *of C to A* as the inclusion-maximal subcomplex D of C such that D lies in A. The *star* of σ in C, denoted by $\mathrm{St}\,(\sigma, C)$, is the minimal subcomplex of C that contains all faces of C containing σ. The *deletion* $C - D$ of a subcomplex D from C is the subcomplex of

C given by $\mathrm{R}\,(C, C\backslash \mathrm{relint} D)$. The *(first) derived neighborhood* $N(D, C)$ of D in C is the simplicial complex

$$N(D, C) := \bigcup_{\sigma \in \mathrm{sd}\, D} \mathrm{St}\,(\sigma, \mathrm{sd}\, C).$$

For the geometric definition of *link*, we refer the reader to [AB17+]. For the purpose of the present paper, it suffices to know that $\mathrm{Lk}\,(\sigma, C)$ is a spherical complex whose face poset is the upper order ideal of σ in the face poset of C. In particular, $\mathrm{Lk}\,(\emptyset, C) = C$.

If C is a simplicial complex, and σ, τ are faces of C, we denote by $\sigma * \tau$ the minimal face of C containing both σ and τ (if there is one). If σ is a face of C, and τ is a face of $\mathrm{Lk}\,(\sigma, C)$, then $\sigma * \tau$ is defined as the face of C with $\mathrm{Lk}\,(\sigma, \sigma * \tau) = \tau$. In both cases, the operation $*$ is called the *join*.

Inside a polytopal complex C, a *free* face σ is a face strictly contained in only one other face of C. An *elementary collapse* is the deletion of a free face σ from a polytopal complex C. We say that C *(elementarily) collapses* onto $C - \sigma$, and write $C \searrow_e C - \sigma$. We also say that the complex C *collapses* to a subcomplex C', and write $C \searrow C'$, if C can be reduced to C' by a sequence of elementary collapses. A *collapsible* complex is a complex that collapses onto a single vertex. Collapsibility is, clearly, a combinatorial property (i.e. it only depends on the combinatorial type), and does not depend on the geometric realization of a polytopal complex. We have however the following results:

Theorem 1.0.1 (Adiprasito–Benedetti [AB17+]). *Let C be a simplicial complex. If the underlying space of C in $\mathbb{R}^d$ is convex, then the (first) derived subdivision of C is collapsible.*

We are also going to use the following "spherical version" of the statement above.

Theorem 1.0.2 (Adiprasito–Benedetti [AB17+]). *Let C be a convex polytopal d-complex in S^d and let $\overline{H}_+$ be a closed hemisphere of S^d in general position with respect to C. Then we have the following:*

(A) *If $\partial C \cap \overline{H}_+ = \emptyset$, then $N(\mathrm{R}\,(C, \overline{H}_+), C)$ is collapsible.*
(B) *If $\partial C \cap \overline{H}_+$ is nonempty, and C does not lie in $\overline{H}_+$, then $N(\mathrm{R}\,(C, \overline{H}_+), C)$ collapses to the subcomplex $N(\mathrm{R}\,(\partial C, \overline{H}_+), \partial C)$.*
(C) *If C lies in $\overline{H}_+$, then there exists some facet F of $\mathrm{sd}\,\partial C$ such that $\mathrm{sd}\, C$ collapses to $C_F := \mathrm{sd}\,\partial C - F$.*

2 Geometric triangulations

To make the reader more familiar with the techniques of counting combinatorial types of manifolds asymptotically, let us start with a couple of examples. The first one shows that if we want to reach exponential bounds for triangulations of d-manifolds, and d is at least two, we *must* add some geometric or topological assumption.

Example 2.0.1. Starting with a $1 \times 4g$ grid of squares, let us triangulate the first $2g$ squares (left to right) by inserting "backslash" diagonals, and the other $2g$ squares by "slash" diagonals. If we cut away the last triangle, we obtain a triangulated disc B with $8g-1$ triangles. This B contains a collection of $2g$ pairwise disjoint triangles. In fact, if we set

$$a_j := \begin{cases} 4j-2 & \text{if } j \in \{1, \ldots, g\} \\ 4j-1 & \text{if } j \in \{g+1, \ldots, 2g\} \end{cases}$$

the triangles in position a_1, ..., a_{2g} (left to right) are disjoint. For simplicity, we relabel these $2g$ disjoint triangles by $1, \ldots, g, 1', \ldots, g'$.

Consider the complex S_B that is obtained as boundary of the cone over base B and remove from it the interiors of the triangles $1, \ldots, g, 1', \ldots, g'$. The resulting "sphere with $2g$ holes" can be completed to a closed surface by attaching g handles. More precisely, let us fix a bijection $\pi : \{1, \ldots, g\} \longrightarrow \{1', \ldots, g'\}$. In the triangle i, let us denote by x_i resp. u_i the leftmost resp. the uppermost vertex; symmetrically, in the triangle $\pi(i)$ let us call $x_{\pi(i)}$ resp. $u_{\pi(i)}$ the rightmost resp. the upper vertex. For each $i \in \{1, \ldots, g\}$, we can attach a (non-twisted) triangular prism onto the holes i and $\pi(i)$, so that x_i resp. u_i gets connected via an edge to $x_{\pi(i)}$ resp. $u_{\pi(i)}$. Each prism can be triangulated with six facets by subdividing each lateral rectangle into two; as a result, we obtain a simplicial closed 2-manifold $M_g(\pi)$.

The number of triangles of $M_g(\pi)$ equals $16g$ (the number of triangles of S_B) minus $2g$ (the holes) plus $6g$ (the handles). Therefore, $M_g(\pi)$ has genus g and $20g$ facets. Any two different permutations π and ρ give rise to two combinatorially different surfaces $M_g(\pi)$ and $M_g(\rho)$. Thus, there are at least $g!$ surfaces with genus g and $20g$ triangles. Obviously, $\lfloor N/20 \rfloor !$ grows faster that any exponential function. So, surfaces are more than exponentially many.

Example 2.0.2. Let us subdivide both the top and bottom edge of a unit square S into

r segments of equal length. By triangulating S linearly without adding further vertices into $N = 2r$ triangles, we obtain roughly

$$C_{N+2} = \frac{1}{N+3}\binom{2N+4}{N+2}$$

different triangulations (without accounting for symmetry) of S, where C_{N+2} is the $(N+2)$-nd Catalan Number. For N large, this number grows exponentially in N: In fact, Stirling's formula yields the asymptotics

$$C_n \approx \frac{4^n}{n^{1.5}\sqrt{\pi}}.$$

By identifying the sides of the square to a torus, we get roughly 4^N combinatorially distinct triangulations of the torus with N facets.

We leave it to the reader to conclude the following Lemma:

Lemma 2.0.3. *Let M be a space form of dimension $d \geq 2$. There are* at least *exponentially many triangulations of M.*

This motivates the search for an upper bound to the number of such geometric triangulations. We will show that Lemma 2.0.3 is best possible, in the sense that these triangulations are also *at most* exponentially many (Theorem 3.0.3).

First we recall the notion of *endo-collapsibility*, introduced in [Ben12]. A triangulation C of a d-manifold with non-empty (resp. empty) boundary is called *endo-collapsible* if C minus a d-face collapses onto ∂C (resp. onto a point). This notion is of interest to us for the following exponential upper bound:

Theorem 2.0.4 (Benedetti–Ziegler [BZ11])**.** *For fixed d, in terms of the number N of facets, there are at most $2^{d^2 \cdot N}$ triangulations of endo-collapsible d-manifolds with N facets.*

Not all triangulations of simplicial balls and spheres are endo-collapsible, cf. [BZ11, Thm. 3]. Our goal is to show that all geometric triangulations of spheres become endo-collapsible after at most two derived subdivisions.

Lemma 2.0.5 (Benedetti [Ben12, Corollary 3.21])**.** *Let B be a collapsible PL d-ball. If* $\mathrm{sd}\,\mathrm{Lk}\,(\sigma, B)$ *is endo-collapsible for every face σ, then* $\mathrm{sd}\, B$ *is endo-collapsible.*

Proposition 2.0.6. *Let C denote a geometric triangulation of a convex subset of S^d. Then $\mathrm{sd}^2 C$ is endo-collapsible.*

Proof. Let H denote a general position hemisphere of S^d, and let H^c denote the complement. Then by Theorem 1.0.2, $N^1(\mathrm{R}\,(C,H),C)$ and $N^1(\mathrm{R}\,(C,H^c),C)$ are both collapsible. By Lemma 2.0.5, $\mathrm{sd}\,N^1(\mathrm{R}\,(C,H),C)$ and $\mathrm{sd}\,N^1(\mathrm{R}\,(C,H^c),C)$ are endo-collapsible. Thus $\mathrm{sd}^2 C$ is endo-collapsible. □

Using the bound from Theorem 2.0.4, we can achieve three non-trivial exponential bounds:

Theorem 2.0.7. *In terms of the number N of facets, there are:*

- *less than $2^{d^2 \cdot (d+1)! \cdot N}$ geometric triangulations of convex balls in $\mathbb{R}^d$,*
- *less than $2^{d^2 \cdot ((d+1)!)^2 \cdot N}$ geometric triangulations of S^d,*
- *less than $2^{d^2 \cdot ((d+1)!)^{(d-2)} \cdot N}$ star-shaped balls in $\mathbb{R}^d$.*

Proof. We only prove the first item here, the other ones can be proven analogously. Let C be any simplicial subdivision of a convex d-polytope. By Theorem 1.0.2(C), the derived subdivision $\mathrm{sd}\,C$ is endo-collapsible. Furthermore, if C has N facets, then $\mathrm{sd}\,C$ has $(d+1)! \cdot N$ facets. Hence, by Theorem 2.0.4, $\mathrm{sd}\,C$ is one of at most $2^{d^2 \cdot (d+1)! \cdot N}$ combinatorial types. Since simplicial complexes with isomorphic derived subdivisions are isomorphic, C is one of at most $2^{d^2 \cdot (d+1)! \cdot N}$ combinatorial types. □

3 Triangulated space forms with bounded geometry

In this section, we wish to study *space forms*, which are Riemannian manifolds of uniform sectional curvature [CE75]. We focus on space forms with "bounded geometry", with the goal of establishing an exponential upper bound for the number of triangulations. Our idea is to chop a geometric triangulation of a space form with bounded geometry into a bounded number of endo-collapsible balls. The key for this is given by the following two lemmas: One is Cheeger's bound on the injectivity radius, the other a direct consequence of Topogonov's theorem.

Lemma 3.0.1 (Cheeger [Che91]). *Let $-\infty < k < \infty$ and $D, V > 0$. There exists a positive number $\mathcal{C}(k, D, V) > 0$ such that every Riemannian d-manifold with Cheeger constraints (k, D, V) has no closed geodesic of length less than $\mathcal{C}(k, D, V)$.*

Lemma 3.0.2. *Let k, D be real numbers, with $D > 0$. Let d be a positive integer. For every $\varepsilon > 0$, there exists a positive integer N_ε such that every Riemannian d-manifold with curvature bounded below by k and diameter at most D can be covered with at most N_ε balls of radius ε.*

Proof. Let X be a Riemannian manifold satisfying the assumptions. Let x be a point of X. Let B^d_D be a ball of radius D in the d-dimensional space of constant curvature $m = \min\{k, 0\}$. The ball B^d_D has a cover with N_ε balls of diameter ε. Consider the map $\exp_x \widehat{\exp}^{-1}$, where $\widehat{\exp}$ is the exponential map in B^d_D with respect to the center and $\exp_x$ is the exponential map in X with respect to x, cf. [CE75, Ch. 1, Sec. 2]. By Topogonov's Theorem, X has curvature bounded below by $\geq k$. Hence, for all $a, b \subset B^d_D$, we have

$$|(\exp_x \widehat{\exp}^{-1})(a)(\exp_x \widehat{\exp}^{-1})(b)| \leq |ab|;$$

in other words, $\exp_x \widehat{\exp}^{-1}$ is a non-expansive map. Thus, the images of the N_ε balls that cover B^d_D are contained in N_ε balls of radius at most ε.

□

Theorem 3.0.3. *For fixed k, D, d and V, in terms of the number N of facets, there are only exponentially many intrinsic simplicial complexes whose underlying spaces are d-dimensional space forms of curvature bounded below by k, of diameter $\leq D$ and of volume $\geq V$.*

Proof. Our proof has three parts:

- **(I)** we cover a space form X satisfying the constraints above with convex open balls;
- **(II)** we count the number of geometric triangulations restricted to each ball;
- **(III)** we assemble the triangulated balls together, thus estimating the number of triangulations of X.

Part (I). Let X be a space form of dimension d satisfying the Cheeger constraints (k, D, V). By Lemma 3.0.2, there exists a set S of $s = s(k, D, V)$ points in such that every point in X lies in distance less than $\varepsilon := \frac{\mathcal{C}(k,D,V)}{4}$ of S. With this choice, any ball of radius ε in X is isometric to a ball of radius ε in the unique simply-connected space form of

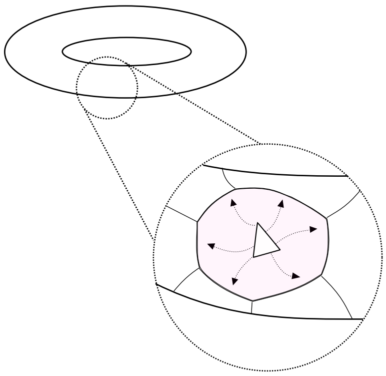

Figure 1: The restriction of the triangulation to each of the B_i ($i \in S$) is endo-collapsible. By counting the number of ways in which two of the B_i's can be glued to one another, we determine an upper bound on the number of triangulations.

curvature equal to the curvature of X. Let T be a triangulation of X into N simplices. Let $(B_i)_{i\in\{1,\cdots,s\}}$ be the family of open convex balls with radius ε, centered at the points of S.

Part (II). For any subset $A \subset X$, let V_A denote the vertices of $\mathrm{sd}\, T$ corresponding to faces of T intersecting A. Define T_A to be the subcomplex of $\mathrm{sd}\, T$ induced by V_A.

Let now B_i be one of the convex balls as above. Then $N(T_{B_i}, \mathrm{sd}\, T)$ is collapsible by Theorem 1.0.2(A). Also, for every face σ of T_{B_i}, $\mathrm{sd}^2 \mathrm{Lk}\,(\sigma, T_{B_i})$ is endo-collapsible by Proposition 2.0.6. Thus, by Lemma 2.0.5, $\mathrm{sd}^3 T_{B_i}$ is endo-collapsible. Since every facet of T intersects B_i at most once, Theorem 2.0.4 provides a constant κ such that the number of combinatorial types of T_{B_i} is bounded above by $e^{\kappa N}$.

Part (III). The triangulation $\mathrm{sd}\, T$ of X is completely determined by

(i) the triangulation of each T_{B_i},

(ii) the triangulation $T_{B_i \cap B_j}$ and its position in T_{B_i} and T_{B_j}. (This means we have to specify which of the faces of T_{B_i} are faces of $T_{B_i \cap B_j}$, too.)

As we saw in Part II, we have $e^{\kappa N}$ choices for triangulating each T_{B_i}. Since $T_{B_i \cap B_j}$ is strongly connected, it suffices to determine the location of one facet Δ of $T_{B_i \cap B_j}$ in T_{B_j} and T_{B_i} (including its orientation) to determine the position of $T_{B_i \cap B_j}$ in T_{B_i} and T_{B_j}. For this, we have at most $(d+1)!^2 N^2$ possibilities. Thus the number of geometric triangulations T of d-dimensional space forms with N facets, diameter $\leq D$, volume

$\geq v$, and curvature bounded below by k is bounded above by

$$e^{\kappa s N}\ \left((d+1)!\,N\right)^{2s(s-1)}. \qquad \square$$

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