

ON THE GENERALIZED NASH PROBLEM FOR SMOOTH GERMS AND ADJACENCIES OF CURVE SINGULARITIES

JAVIER FERNÁNDEZ DE BOBADILLA, MARÍA PE PEREIRA, AND PATRICK POPESCU-PAMPU

ABSTRACT. In this paper we explore the generalized Nash problem for arcs on a germ of smooth surface: given two prime divisors above its special point, to determine whether the arc space of one of them is included in the arc space of the other one. We prove that this problem is combinatorial and we explore its relation with several notions of adjacency of plane curve singularities.

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1. INTRODUCTION

The original motivation, and one of the main purposes of this paper, is to study the *generalized Nash problem* for smooth surface germs. However, working on this problem we have obtained results in the study of adjacencies of plane curve singularities and in the study of divisorial valuations in the plane, as we will explain below.

Let (X, O) be a germ of a normal surface singularity. A *model* of X is a proper birational map $\pi : S \rightarrow X$ for which S is smooth. A *prime divisor over O* is an irreducible divisor E in a model S such that $\pi(E) = O$. If h is a function in the local ring \mathcal{O} of germs of holomorphic functions on (X, O) , consider its order of vanishing $\nu_E(h)$ along E . The function $\nu_E : \mathcal{O} \rightarrow \mathbb{N} \cup \{\infty\}$ is a discrete valuation of

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rank one. We call it the *valuation associated with E* . Two prime divisors over O are *equivalent* if they induce the same valuation. A *divisor over O* is a formal finite combination $\sum_i a_i E_i$ for $a_i \in \mathbb{N}$ and E_i prime over O . We define $\nu_E := \sum_i a_i \nu_{E_i}$, which is a function $\nu_E : \mathcal{O} \rightarrow \mathbb{Z}$, but in general no longer a valuation.

The *maximal divisorial set* \overline{N}_E associated with a prime divisor E over O is the Zariski closure in the space of arcs of (X, O) of the set of arcs whose liftings to S hit E . It is well known that this definition depends only on the equivalence class of E . A *Nash adjacency* is an inclusion $\overline{N}_F \subset \overline{N}_E$.

Given a normal surface singularity (X, O) and given two different prime divisors E, F appearing on its minimal resolution, Nash conjectured in [21] that *none of the spaces $\overline{N}_E, \overline{N}_F$ is included in the other one*. This conjecture, which was proved in [15], can be generalized in the form of the following problem which, as far as we know, was stated first by Ishii [19, 3.10] (here X may denote an irreducible germ of arbitrary dimension):

Let E and F be prime divisors over $O \in X$. Characterize when $\overline{N}_F \subset \overline{N}_E$.

This problem is wide open even for *smooth* germs of surfaces (X, O) . This is the reason why we concentrate here on this case. We will assume without loss of generality that our germ is $(\mathbb{C}^2, 0)$.

The *combinatorial type* of a pair of divisors is the combinatorial type of the minimal sequence of blowing ups needed for making both of them appear. One of the main results in this paper is (see Theorem 4.10):

Theorem 1.1. *Let E and F be two prime divisors over the origin of \mathbb{C}^2 . The Nash-adjacency $\overline{N}_E \supset \overline{N}_F$ only depends on the combinatorial type of the pair (E, F) .*

The proof involves techniques that fall outside Algebraic Geometry, and it is based on the proof of a result of the first author from [14]. However, there are a few improvements and modifications which motivated us to include a detailed version here.

This theorem has a few surprising consequences that we now summarize.

If we have a Nash adjacency $\overline{N}_E \supsetneq \overline{N}_F$, then $\text{codim}(\overline{N}_E) < \text{codim}(\overline{N}_F)$. As a corollary of the last theorem we have the following: if the Nash-adjacency $\overline{N}_E \supset \overline{N}_F$ is true, then $\cap_t \overline{N}_{E_t} \supseteq \cup_s \overline{N}_{F_s}$ for certain infinite families of divisors E_t and F_s such that the pairs (E_t, F_s) have the same combinatorial type as (E, F) for any value of the parameters (t, s) . This immediately improves the codimension bound in an unexpected way: as a special case of a general theorem of Ein, Lazarsfeld and Mustață [10], we know that given a prime divisor E , the codimension $\text{codim}(\overline{N}_E)$ of \overline{N}_E in the arc space of \mathbb{C}^2 equals its *log discrepancy* $a_E(\mathbb{C}^2)$ (its coefficient in the relative canonical divisor $+1$). Using the previous corollary, we improve in most cases the existing bound $a_E(\mathbb{C}^2) < a_F(\mathbb{C}^2)$ for the log discrepancy. Namely, we show that if E is Nash-adjacent to F , then:

$$a_E(\mathbb{C}^2) < a_F(\mathbb{C}^2) - b, \quad (1)$$

where b depends on the combinatorial type of the pair (E, F) and in which often $b > 0$ (see Corollary 4.19).

We also state some conjectures in Section 4.2, relating the generalized Nash problem stated above to the existence of inclusions between maximal divisorial sets in the space of arcs associated with a valuation $q \cdot \nu_E$, as defined in [10] or [19]. We are convinced that the solution of these conjectures is crucial for the final solution of the generalised Nash problem.

A basic fact, proved by Plénat [23] as a generalization of Reguera's [27, Thm. 1.10], which concerned only the case of rational surface singularities, is that one has the implication:

$$\overline{N}_F \subset \overline{N}_E \implies \nu_E \leq \nu_F.$$

This implication, which we call the *valuative criterion* for Nash adjacency, has been used as a main tool for the investigation of Nash's Conjecture until the works [14], [22] and [15]. It has been the essential ingredient in the resolution of some cases (see [20], [24] and [25]). However, Ishii [19] showed with an

example that even in the smooth surface case the converse implication is not true. The second set of results of this paper (which for expository reasons appear in the course of the paper before the results mentioned previously in this introduction) concerns the investigation of the geometric meaning of the inequality $\nu_E \leq \nu_F$. For this purpose there is no need to restrict to prime divisors, although for the case of prime divisors we often get stronger results.

In Proposition 3.8 of Section 3.1, we prove a characterization of the inequality $\nu_E \leq \nu_F$ in terms of finitely many inequalities $\nu_E(h) \leq \nu_F(h)$ with functions h only depending on E . When E and F are prime divisors, we see in Section 3.2 that this is equivalent to an even more reduced number of inequalities, with h varying only among the approximate roots of a function related to E .

Using the previous characterization, we clarify the geometric meaning of the inequality $\nu_E \leq \nu_F$ in the following way. Let $F = \sum_i b_i D_i$ be a divisor in a model S of $(\mathbb{C}^2, 0)$ with exceptional locus $Exc_S = \bigcup_i D_i$. We say that a function f is *weakly associated with F in S* if $b_i = \widetilde{V(f)} \cdot D_i$ where $\widetilde{V(f)}$ is the strict transform in S of $V(f)$, the zero set of f . Furthermore, we say that f is *associated to F* if $\widetilde{V(f)}$ is a disjoint union of smooth irreducible curve germs and if for every i , b_i of them are transversal to the corresponding D_i , at distinct smooth points of Exc_S .

Then, our result is the following (see Theorem 3.25):

Theorem 1.2. *Let E and F be divisors over the origin of \mathbb{C}^2 and let S be the minimal model containing the divisor $E + F$. The following are equivalent:*

- (a) $\nu_E \leq \nu_F$.
- (b) *there exists a deformation $G = (g_s)_s$ with g_0 weakly associated with F in S and g_s weakly associated with E in S , for $s \neq 0$ small enough.*
- (c) *there exists a deformation $G = (g_s)_s$ with g_0 associated with F in S and g_s associated with E in S , for $s \neq 0$ small enough.*
- (d) *there exists a linear deformation $(g_s = g_0 + sg)_s$ with g_0 associated with F in S and g and g_s associated with E in S , for $s \neq 0$ small enough.*

The equivalence (c) \Leftrightarrow (d) was proved in different language by Alberich and Roe in [1]. See also Theorem 3.26 for an improvement when F is a prime divisor.

Let $E = \sum_i a_i E_i$ and $F = \sum_j b_j F_j$ be divisors in a model $S \rightarrow \mathbb{C}^2$. Let $(f_s)_s$ be a curve deformation such that f_0 is associated with F in S and $f_{s \neq 0}$ is associated with E in S . For $s \neq 0$ small enough, the germ $V(f_s)$ is a curve singularity with constant topological type. The topological type only jumps for $s = 0$. Following Arnold's classical terminology, we say that the deformation $(f_s)_s$ exhibits an *adjacency* from the topological type of $V(f_{s \neq 0})$ to the topological type of $V(f_0)$ (note that in this paper, when we speak about the topological type of a curve singularity, we mean its embedded topological type).

As the reader may expect, not all of Arnold's adjacencies of topological types of plane curve singularities can be realized by curve deformations as above. Let us explain why.

In a deformation $(f_s)_s$ giving rise to an Arnold adjacency, the positions of the free infinitely near points of the plane curve singularity $V(f_{s \neq 0})$ move with the parameter s (see Section 2). By contrast, the special property (b) above implies that the free points do not move with s . We say that such deformations *fix the free points*. They correspond to the deformations of Alberich and Roe's paper [1] whose general member is associated with a cluster.

In Section 3.4, we use Theorem 1.2 and its corollaries to show that most of the known Arnold adjacencies (see [2], [5], [31] and [3]) can be realized by deformations fixing the free points. We provide a complete list of adjacencies fixing the free points between all simple, unimodal and bimodal singularities with Milnor number up to 16, and compare it with the list of usual Arnold adjacencies. This improves the study carried in [1] and allows to recover in a fast way many adjacencies from [2], [5], [31].

Using these results, in Remark 3.33 we sketch an algorithm which, starting from a topological type, gives back many topological types Arnold adjacent to the given one. We expect that this algorithm produces a good portion of all the topological types Arnold adjacent to the given one.

In Section 3.5, using the technique developed in this paper and a result of Ishii [19] for the case of deformations whose general member has only one Puiseux pair, we obtain information about δ -constant deformations:

Proposition 1.3. *Let E be a prime divisor which is the last divisor appearing in the minimal embedded resolution of an irreducible plane curve singularity with one Puiseux pair. Let F be any other prime divisor. The following properties are equivalent:*

- (1) *there is a δ -constant deformation $(f_s)_s$ such that f_0 is associated with F and $f_{s \neq 0}$ is associated with E .*
- (2) *the inequality $\nu_E \leq \nu_F$ holds (and hence all the equivalent formulations predicted in Theorem 1.2).*

If the previous properties are satisfied, then there exists a δ -constant deformation whose generic topological type is given by a curve whose minimal embedded resolution hits E and whose special curve lifts transversely to F .

As far as we know, this is the first concrete criterion allowing to find topological types in the δ -constant stratum of a given plane curve singularity.

We are now in a position to return to the case in which E and F are prime and compare the geometric meanings of the inequality $\nu_E \leq \nu_F$ and of the inclusion $N_F \subseteq N_E$.

Using results of Reguera [29] and of the first author [14], the generalized Nash problem can be reformulated as follows in the case of normal surfaces:

Given E and F as before, does there exist a wedge (a germ of a uniparametric family $(\alpha_s)_{s \in (\mathbb{C}, 0)}$ of arcs) such that $\alpha_0 \in \overline{N}_F$ and $\alpha_{s \neq 0} \in \overline{N}_E$ and furthermore the liftings of α_0 and $\alpha_{s \neq 0}$ to the minimal models S_F and S_E of F and E respectively are transversal to the corresponding exceptional divisors?

In the case of $X = \mathbb{C}^2$, with coordinates (x, y) , the image in $\mathbb{C}_{x,y}^2 \times \mathbb{C}_s$ of a wedge $\alpha : \mathbb{C}_t \times \mathbb{C}_s \rightarrow \mathbb{C}_{x,y}^2$ given by $(\alpha(t, s), s)$ has an implicit equation $f(s, x, y) = 0$ which induces a deformation of a plane curve singularity (define $f_s(x, y) := f(s, x, y)$). It is clear that f_0 is associated with F and that $f_{s \neq 0}$ is associated with E . Since the deformation admits a parametrization in family (given by α), it satisfies the additional property of being a δ -constant deformation. In fact, due to a classical theorem of Teissier (see Theorem 3.34), the converse is true: if there is a δ -constant deformation f_s such that f_0 is associated with F and that $f_{s \neq 0}$ is associated with E , then there is a uniparametric family $(\alpha_s)_s$ as above, and by results of the first author [14] we have the inclusion $\overline{N}_F \subseteq \overline{N}_E$.

Thus we find that the geometric difference between the inequality $\nu_E \leq \nu_F$ and the inclusion $N_F \subseteq N_E$ is the δ -constancy of the associated deformations. More precisely:

Proposition 1.4. *Let E, F be prime divisors over O such that the inequality $\nu_E \leq \nu_F$ holds. The inclusion $N_F \subseteq N_E$ is satisfied if and only if there exists a δ -constant deformation $G = (g_s)_s$ with g_0 associated with F in S and g_s associated with E in S , for $s \neq 0$ small enough.*

In Section 4.3 we show an example where condition $\nu_E \leq \nu_F$ holds but the inclusion $\overline{N}_E \supset \overline{N}_F$ does not. Ishii gave already such an example in [19]. But in the case of her example, E is a free divisor and one can disprove the inclusion $\overline{N}_E \supset \overline{N}_F$ by the log discrepancy criterion mentioned above. By contrast, our example concerns two satellite divisors and the inclusion $\overline{N}_E \supset \overline{N}_F$ cannot be discarded even by the enhanced log discrepancy criterion explained above in this introduction.

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2. BACKGROUND AND TERMINOLOGY

In this section we introduce terminology about divisors, curves, infinitely near points and combinatorial types of divisors to be used throughout the paper. The reader can find further details in [35], [8] or [6].

We work with $(\mathbb{C}^2, 0)$, seen as a germ of complex analytic surface, with local \mathbb{C} -algebra \mathcal{O} , whose maximal ideal is denoted \mathfrak{m} . If $h \in \mathfrak{m}$, we denote by $V(h)$ its vanishing locus, seen as a germ of effective divisor on \mathbb{C}^2 . If h is irreducible, then $V(h)$ is called a *branch*. In general we say simply that $V(h)$ is a *curve singularity*, or even a *curve*, if no confusion is possible with a global curve.

We denote by $m_0(h) = m_0(V(h))$ the *multiplicity* at 0 of h or of the curve $V(h)$ it defines.

If C and D are two curves, their *intersection number* at a point p (in general, p will be an infinitely near point of 0, as defined below) is denoted by:

$$I_p(C, D) \in \mathbb{N} \cup \{+\infty\}.$$

One gets $+\infty$ if and only if C and D share irreducible components.

The generalized Nash problem concerns prime divisors above 0. But, since the valuative order and some results concerning it in Section 3 can be extended to divisors which are not necessarily prime, we introduce our terminology in that generality.

A *model* is any smooth surface S obtained from $(\mathbb{C}^2, 0)$ by a finite sequence of blow-ups of points. If $\pi : S \rightarrow \mathbb{C}^2$ is the associated morphism, we denote by $Exc_S := \pi^{-1}(0)$ the corresponding *exceptional divisor*. If S, T are two models and π_S and π_T are the associated morphisms, we say that T *dominates* S if there exists a morphism $\psi : T \rightarrow S$ such that $\pi_S \circ \psi = \pi_T$.

Any point of the exceptional divisor of some model is called an *infinitely near point* of 0. Two such points in different models are considered to be equivalent if the canonical bimeromorphic mapping between the models which contain them restricts to a biholomorphism in their neighborhoods. In the sequel, we will call such equivalent classes of infinitely near points simply *points*.

In a process of blowing up points starting from 0, if a divisor E is created by blowing up a point p , we may also denote that divisor by E_p or that point by p_E . If a point q lies on the strict transform of E_p on some model, we say that q is *proximate* to p .

Recall that in a blowing up process, an infinitely near point of 0 can be either *satellite*, if it is a singular point of the exceptional divisor of the model on which it lies, that is, if it is the intersection point of two components of this divisor, or *free*, if it is a smooth point of it. We say also that a divisor E is *satellite/free* if p_E is so.

In the sequel, a *prime divisor* (above 0, which we won't precise if there is no risk of confusion) will mean any irreducible component of the exceptional divisor of any model, two such divisors being *equivalent* if

there exists a third model on which their strict transforms coincide. In other words, they are equivalent if and only if they define the same divisorial valuation on the local ring \mathcal{O} . We will make an abuse of language and talk about prime divisors also for equivalence classes of divisors according to this relation.

More generally, a *divisor* D will be an element of the free abelian semigroup generated by the prime divisors. That is, $D = \sum_{i=1}^n a_i D_i$, where $a_i \in \mathbb{N}$ and D_i is a prime divisor (that is, our “divisors” will be effective divisors in the usual terminology of algebraic geometry). Whenever $a_i \neq 0$, we say that D_i is a *component* of D .

Given a divisor D , we say that it *appears* on a model S if its components are equivalent to irreducible components of the exceptional divisor Exc_S of S . Among all the models on which D appears, there is a minimal one under the relation of domination of models. We call it the *minimal model* of D .

Definition 2.1. Let E and F be two prime divisors. We say that F *dominates* E , and we write $F \geq_d E$, if and only if the minimal model of F dominates the minimal model of E .

Definition 2.2. Given a divisor $D = \sum_{i=1}^n a_i D_i$ with D_i irreducible and $a_i \geq 0$ that appears in a model S , then we say that $h \in \mathcal{O}$ is *associated with D in S* if:

$$h = \prod_j h^{(j)}$$

with $h^{(j)}$ irreducible factors and for every $i \in \{1, \dots, n\}$, we have that a_i of them are functions defining pairwise disjoint *curvetas* of D_i in S (that is, branches whose strict transforms on S are smooth and hit D_i transversely in different smooth points of Exc_S). We say that h and $V(h)$ are, respectively, a *function* and a *curve associated with D in S* . We will denote a function associated with D by h_D .

Recall that the final prime divisors of the minimal embedded resolutions of branches are satellite. Conversely, given any satellite prime divisor D , it is the final divisor of the minimal embedded resolution of any branch associated with it.

The *combinatorial type* of a (not necessarily satellite) prime divisor D may be encoded in the following equivalent ways (see more details in [8, Chapter 5] and [35, Section 3.6]):

- (1) by the dual graph, which we denote by \mathcal{G}_D , of the exceptional divisor of the minimal model of D , each vertex being weighted by the self-intersection of the corresponding component in this minimal model;
- (2) by the character free/satellite of the finite set of points which have to be blown up in order to reach the minimal model of D , enriched with the information about the proximity binary relation; this information can be represented for example by the Enriques diagram (see [6, Chapter 3]);
- (3) the *multiplicity sequence*. This sequence associates to the height k above O of each infinitely near point p_k which is blown up in order to reach D , the multiplicity at p_k of the strict transform of $V(h_D)$ for some associated function h_D .

The *combinatorial type* of a not necessarily prime divisor D can be encoded in analogous ways, by keeping also the coefficients in D of each component of the minimal model of D . For example, in the dual graph \mathcal{G}_D we mark the vertices which correspond to the components of D and we add their coefficients in D as a second decoration of those vertices of \mathcal{G}_D .

We write $D \equiv D'$ to say that D and D' have the same combinatorial type.

Observe that a divisor D is not determined by its combinatorial type, even up to isomorphisms above $(\mathbb{C}^2, 0)$. The existence of free points in the resolution process (and the points to be blown up immediately after 0 are always free points) gives moduli in the family of divisors with the same combinatorial type. In [13], [30] moduli spaces of divisors with the same combinatorial type are constructed.

Note that for a free prime divisor E , the embedded topology of a curve $V(h)$ associated with it (in some model) contains less information than the combinatorial type of E . Indeed, the combinatorial type of a prime divisor E is codified by the minimal embedded resolution of any associated curve $V(h)$ and the

number of free points we have to blow up after resolving such a curve $V(h)$ in order to get the minimal model of E . That is, it is codified adding to the multiplicity sequence of the associated curve $V(h)$ as many 1's as free points we have to blow up after resolving it, in order to get E .

Given a pair of prime divisors E and F , we define the *contact order* $\text{Cont}(E, F)$ between them as the number of common points p_i they share in the process of finding their minimal models. Said slightly differently, it is the number of blow ups that we perform before the strict transforms of associated curves $V(h_E)$ and $V(h_F)$ to E and F respectively in a model of $E + F$ separate. For instance, assume that both E and F are different from E_0 . Then their contact order is 1 if and only if the branches $V(h_E)$ and $V(h_F)$ have different tangent lines at 0.

Using this notion, we can say that the combinatorial type of a non-prime divisor is given by the combinatorial types of its prime components, their multiplicities in the divisors and the contact orders between them.

Given two divisors E and F , we will talk about the *combinatorial type* of the pair (E, F) as the combinatorial type of both E and F taken separately, to which we add the information of all the contact orders between all their prime components. We write $(E, F) \equiv (E', F')$ to say that these two pairs have the same combinatorial type. Equivalently, we can consider the dual graph of the exceptional divisor of the minimal model where all the components of E and F appear and mark differently the components of E and F and their corresponding multiplicities.

3. THE VALUATIVE PARTIAL ORDER

In this section we introduce the *valuative domination* partial order \leq_ν on the set of non necessarily prime divisors above O and explore several properties of it.

In the first subsection we show that the valuative inequality $E \leq_\nu F$ is equivalent to a finite number of inequalities $\nu_E(h) \leq \nu_F(h)$, for well-chosen functions h depending only on E . In particular, we see in this way that the valuative domination between two divisors E and F only depends on the combinatorics of the pair (E, F) .

In Subsection 3.2 we prove that when E and F are *prime*, then in order to show the valuative inequality $E \leq_\nu F$, it is enough to check those inequalities when h varies among the approximate roots of a function which defines a curvetta of E .

In Subsection 3.3 we prove a characterization of the pairs of divisors (E, F) such that $E \leq_\nu F$ in terms of the existence of certain types of deformations, which we call *deformations fixing the free points* (see Definition 3.29 and Theorems 3.25 and 3.26). This recovers and generalizes in a different language results of Alberich and Roe [1].

In Subsection 3.4 we see that these deformations realize many classical adjacencies of topological types, testing the previous criterion on Arnold's list of adjacencies. In Remark 3.33 we sketch an algorithm which, given the embedded topological type of a branch, finds all the Arnold adjacent topological types obtained by deformations fixing the free points.

3.1. Definition and first properties of the valuative partial order.

Given a prime divisor E , we denote by ν_E the divisorial valuation associated with it. Let $\pi_S : S \rightarrow \mathbb{C}^2$ be a model in which E appears. Given $h \in \mathcal{O}$, the value $\nu_E(h)$ is the vanishing order along E of the total transform $\pi_S^*(h)$ of h on S . If the divisor E is not prime, then it may be written as a sum $\sum_i a_i E_i$ for some prime E_i 's and non-negative integers a_i 's. We define then:

$$\nu_E := \sum_i a_i \nu_{E_i}.$$

Note that, whenever E has at least two components, ν_E is no longer a valuation.

The value $\nu_E(h)$ can be also computed as the intersection multiplicity of $V(h)$ with a curve $V(h_E)$ associated with E in S (see 2.2), whenever h_E is chosen such that the strict transforms of $V(h_E)$ and $V(h)$ in the minimal model of E do not meet. An immediate consequence of this fact is:

Lemma 3.1. *For any two divisors E, F above 0, one has $\nu_E(h_F) = \nu_F(h_E)$ for h_E and h_F generic associated functions to E and F in the minimal model of $E + F$.*

Definition 3.2. We define the *valuative domination partial order* \leq_ν among divisors, by saying that $E \leq_\nu F$ if and only if $\nu_E(h) \leq \nu_F(h)$ for all $h \in \mathcal{O}$. We also write $\nu_E \leq \nu_F$, where ν_E and ν_F are seen as functions defined on the set \mathcal{O} with values in \mathbb{N} . We say then that F *valuatively dominates* E or that E *is valuably dominated by* F and we call $E \leq_\nu F$ the *valuative inequality*.

Remark 3.3. *Let E and F be two prime divisors. If F dominates E in the sense of Definition 2.1, then F valuably dominates E .*

The only non-trivial axiom of partial order is the antisymmetry, which is verified in Lemma 3.7.

Consider two curves $V(f)$ and $V(g)$. Let $\{x_i\}_{i \in I}$ be the infinitely near points which we blow up to obtain an embedded resolution of $V(fg)$. Let \tilde{f}_i and \tilde{g}_i be the local equations of the strict transforms of $V(f)$ and $V(g)$ at x_i . Denote $m_i^f = m_{x_i} \tilde{f}_i$ and $m_i^g = m_{x_i} \tilde{g}_i$. Then, we have (see for example [8]):

$$I_O(V(f), V(g)) = \sum_{i \in I} m_i^f \cdot m_i^g. \quad (2)$$

The following two lemmas are immediate consequences of this equality:

Lemma 3.4. *Let E be a divisor above 0 and h_E a function associated with E in the minimal model S_E of E . Then for any $h \in \mathfrak{m}$, one has:*

$$\nu_E(h) \leq I_0(V(h_E), V(h))$$

with equality if and only if the strict transforms of $V(h_E)$ and $V(h)$ do not meet in S_E .

Lemma 3.5. *The value of $\nu_E(h)$ only depends on the combinatorial type of the exceptional divisor of the minimal model which both gives an embedded resolution of $V(h)$ and makes E appear, endowed with the marking of the divisor E and of the divisor met by the strict transform of $V(h)$ in this model.*

Proof. The combinatorial type we refer to may be described by the following decorated graph, associated to the minimal embedded resolution in which E appears:

- let $\mathcal{G}(E, h)$ be the dual graph of the exceptional divisor;
- weight each vertex by the self-intersection of the corresponding divisor;
- if an irreducible component of the strict transform of $V(h)$ meets an irreducible component of the exceptional divisor, attach an arrow to the vertex which corresponds to the strict transform;
- mark the vertices corresponding to the components of E with the label “ E ”.

Choose a function h_E such that the strict transforms of h_E and of h do not meet in the minimal resolution considered in the statement. Then by Lemma 3.4 we have the equality $\nu_E(h) = I_0(V(h_E), V(h))$. By the usual formulas controlling the behaviour of intersection multiplicity of plane branches under blow up, it is clear that the combinatorics of the decorated graph $\mathcal{G}(E, h)$ determine $I_0(V(h_E), V(h))$. \square

Lemma 3.6. *If $(E, F) \equiv (E', F')$, that is, if the two pairs have the same combinatorial type, then:*

$$E \leq_\nu F \Leftrightarrow E' \leq_\nu F'. \quad (3)$$

Proof. Consider the minimal blowing up sequence $\pi_{E,F} : X_{E,F} \rightarrow \mathbb{C}^2$ such that every component of E and F appear in the exceptional divisor. Consider the graph $\mathcal{H}(E, F)$ to be the dual graph of the exceptional divisor; weight each vertex by the self-intersection of the corresponding prime divisor; mark

the vertices corresponding to prime components in E by a label “ E ” and the vertices corresponding to prime components in F by a label “ F ” (observe that a vertex can carry the 2 labels).

Given any function $h \in \mathcal{O}$, we consider the strict transform $\widetilde{V(h)}$ to $X_{E,F}$ of the curve defined by it. To any vertex of $\mathcal{H}(E, F)$ we give a secondary weight given by the intersection number of the corresponding prime divisor and the strict transform $\widetilde{V(h)}$. Observe that the graph $\mathcal{H}(E, F)$ together with the second weighting determine $\nu_E(h)$ and $\nu_F(h)$.

Moreover, given any non-negative secondary system of weights of the vertices, there exists a function h which gives rise to those weights by the procedure above. It can be constructed as follows: in the model $X_{E,F}$ we consider a curve that intersects each divisor E with the intersection multiplicity prescribed by the weight (this can be done because the exceptional divisor has only normal crossings). Blow down the curve and consider an equation h for it.

The lemma holds because we have the combinatorial equivalence $(E, F) \equiv (E', F')$ if and only if the 2 decorated graphs $\mathcal{H}(E, F)$ and $\mathcal{H}(E', F')$ are isomorphic, and because the possible pairs $(\nu_E(h), \nu_F(h))$ are determined from the non-negative weights explained above. \square

The following lemma is obvious for prime divisors, but for the general case we include a short proof.

Lemma 3.7. *If E and F are divisors above 0 and $E \leq_\nu F \leq_\nu E$, then $E = F$, that is \leq_ν is a partial order relation.*

Proof. Given E and F with $E \leq_\nu F \leq_\nu E$, we have to check that the multiplicity sequences of E and F are the same and that the same sequence of infinitely near points is blown up to obtain their minimal models.

Take a branch $V(h)$ with contact order 1 with both $V(h_E)$ and $V(h_F)$, for h_E and h_F associated functions to E and F . Then, $\nu_E(h) = \nu_F(h)$ implies that the multiplicities of h_E and h_F are the same. To see this implication we use Lemma 3.4 and (2).

Take another branch $V(h)$ with contact order 2 with some branch of $V(h_E)$ and no higher contact order with $V(h_F)$. We impose $\nu_E(h) = \nu_F(h)$. Using $m_0(h_E) = m_0(h_F)$, we conclude that $V(h)$ has also contact order 2 with some branch of $V(h_F)$ and that the strict transforms of $V(h_E)$ and $V(h_F)$ at that second contact point have the same multiplicity.

Using functions h with increasingly contact order with the different branches of $V(h_E)$, we inductively conclude that $V(h_E)$ and $V(h_F)$ share all the infinitely near points above the origin 0 and that their multiplicities are the same at all of them. \square

To check whether two divisors are comparable for the valuative order, we can use the following criterion:

Proposition 3.8. *Let $E \neq F$ be divisors above 0. Denote by S the minimal model of $E + F$. Given a divisor D above 0 in the model S , h_D denotes an associated function in S . The following are equivalent:*

- (1) $E \leq_\nu F$, that is $\nu_E(h) \leq \nu_F(h)$ for all $h \in \mathcal{O}$.
- (2) $\nu_H(h_E) \leq \nu_H(h_F)$ for all prime divisors H .
- (3) $\nu_H(h_E) \leq \nu_H(h_F)$ for all prime divisors H appearing in S .
- (4) $\nu_E(h_H) \leq \nu_F(h_H)$ for all prime divisors H appearing in S .
- (5) $\nu_E(h_H) \leq \nu_F(h_H)$ for all prime divisors H appearing in the minimal model of E .
- (6) $\nu_H(h_E) \leq \nu_H(h_F)$ for all prime divisors H appearing in the minimal model of E .

Proof. The equivalences (1) \Leftrightarrow (2), (3) \Leftrightarrow (4) and (5) \Leftrightarrow (6) are immediate consequences of Lemma 3.1. The implications (2) \Rightarrow (3) \Rightarrow (6) are immediate.

Let us prove that (6) \Rightarrow (2). Consider any prime divisor H . Let $\{p_j\}_{j \in J}$ be the infinitely near points that we blow up to obtain the minimal model of H . The index set J is ordered by saying that $j > i$ if p_j is infinitely near p_i . This order is total since H is prime. Let H_j be the divisor which appears after

blowing up p_j . Let $J' \subset J$ be the subset consisting of the indices $j' \in J$ such that $H_{j'}$ appears in the minimal model of E . It is enough to check that $\nu_{H_j}(h_E) \leq \nu_{H_j}(h_F)$ for all $j \in J$. The inequality is true by hypothesis for $j \in J'$. Assuming it is true up to j , we check it for $j+1$, the immediate successor of j for the previous total order (provided it does not belong to J'):

- if p_{j+1} is a satellite point, then it is the intersection point of H_j with H_k for some $k < j$ and then:

$$\begin{aligned}\nu_{H_{j+1}}(h_E) &= \nu_{H_j}(h_E) + \nu_{H_k}(h_E), \\ \nu_{H_{j+1}}(h_F) &= \nu_{H_j}(h_F) + \nu_{H_k}(h_F) + m_j^F,\end{aligned}$$

where $m_j^F \geq 0$,

- if p_{j+1} is a free point, then:

$$\nu_{H_{j+1}}(h_E) = \nu_{H_j}(h_E) \leq \nu_{H_j}(h_F) + m_j^F = \nu_{H_{j+1}}(h_F) \quad (4)$$

where m_{j+1}^F denotes the multiplicity at p_{j+1} of the strict transform of $V(h_F)$ (which could be 0).

In both cases we get the desired inequality $\nu_{H_{j+1}}(h_E) \leq \nu_{H_{j+1}}(h_F)$. \square

Note that each of the conditions (3) – (6) gives a *finite* number of inequalities to be checked.

Corollary 3.9. *Assume $\nu_E \leq \nu_F$ with E and F prime divisors. If the prime divisor F' is such that $F' \equiv F$ and $\text{Cont}(E, F') \geq \text{Cont}(E, F)$, then we have $\nu_E \leq \nu_{F'}$.*

Proof. By hypothesis and using the equivalence (1) \Leftrightarrow (6) of Proposition 3.8, we have that $\nu_H(h_E) \leq \nu_H(h_F)$ for all prime divisors H in the minimal model of E . Using formula (2) and the hypothesis, it is clear that $\nu_H(h_F) \leq \nu_H(h_{F'})$. Using the equivalence (6) \Leftrightarrow (1) of Proposition 3.8, we conclude. \square

Using the same type of arguments, Lemma 3.5 and Lemma 3.6, we get also the following:

Corollary 3.10. *Let $E = \sum_i a_i E_i$, $F = \sum_j b_j F_j$ be two divisors above $O \in \mathbb{C}^2$ such that $E \leq_\nu F$. Assume that $E' = \sum_i a_i E'_i$ and $F' = \sum_j b_j F'_j$ are such that $E_i \equiv E'_i$ for all i , $F_j \equiv F'_j$ for all j and moreover $E \equiv E'$, $F \equiv F'$. If $\text{Cont}(E'_i, F'_j) \geq \text{Cont}(E_i, F_j)$ for all i and j , then $E' \leq_\nu F'$.*

Observing the rules of computation of the divisorial valuations, we get the following corollary:

Corollary 3.11. *Given a divisor F , there exists a finite number of combinatorial types of pairs (E, F) such that $E \leq_\nu F$.*

Proof. If $E \leq_\nu F$, then each of the prime components E_i of E satisfies the analogous inequality $E_i \leq_\nu F$. Thus it is enough to assume that E is prime. In that case the sequence of blowing up centers of the minimal model where E appears is totally ordered.

Let $\{m_i^E\}_{i \in I_E}$ be the multiplicity sequence of a curve associated with E in the minimal model of E . Let g, h be functions associated with E in its minimal model, so that the strict transforms in this minimal model of the curves defined by them are disjoint. In that case, by Lemma 3.4 and Formula (2) we have:

$$\nu_E(h) = I_0(V(g), V(h)) = \sum_{i \in I_E} (m_i^E)^2.$$

The set $\{x_i\}_{i \in J}$ of infinitely near points that we need to blow up in order to obtain the minimal model S where all components of F appear is finite. Let f be a function associated with F in S . Denote by m_i^F the multiplicity of the strict transform of $V(f)$ at the point x_i . Choosing f so that the strict transforms of $V(f)$ and $V(h)$ in the minimal model of $E \cup F$ are disjoint, by Lemma 3.4 and formula (2) we get:

$$\nu_F(h) = I_0(V(f), V(h)) = \sum_{i \in I_E \cap J} m_i^F \cdot m_i^E.$$

Hence we have

$$\nu_F(h) - \nu_E(h) = \sum_{i \in I_E \cap J} (m_i^F - m_i^E) \cdot m_i^E - \sum_{i \in I_E \setminus J} (m_i^E)^2. \quad (5)$$

The inequality $\nu_E(h) \leq \nu_F(h)$ implies at once the inequalities $m_i^E \leq m_i^F$ for any $i \in J \cap I_E$ and that the cardinality of $I_E \setminus J$ is bounded by $\sum_{i \in J} (m_i^F)^2$. Hence the cardinality of I_E for any E satisfying $\nu_E \leq \nu_F$ is bounded by a universal bound only depending on the combinatorial type of F .

Moreover, the multiplicities m_i^E are also universally bounded by bounds only depending on the combinatorial type of F : otherwise, Formula (5) would contradict the inequality $\nu_E(h) \leq \nu_F(h)$.

Since the multiplicity sequences of the prime components of a given divisor and the orders of contact of its pairs of components determine the combinatorial type of the divisor, the corollary is proven. \square

3.2. The special case of prime divisors.

In this subsection we will show that Proposition 3.8 may be improved when both E and F are prime divisors (see Proposition 3.17 below). We start by recalling several facts about the approximate roots of polynomials, whose proofs may be found in [26]. In that reference one may also find explanations about the papers of Abhyankar and Moh in which this theory was developed.

In the sequel, if C is a branch on $(\mathbb{C}^2, 0)$, we say that $f \in \mathbb{C}\{x\}[y]$ is a *Weierstrass polynomial associated with C* if:

- it is monic and irreducible;
- it defines C in some local coordinates (x, y) on $(\mathbb{C}^2, 0)$;
- the y -degree $d_y(f)$ is equal to the multiplicity of C at the origin.

Whenever the coordinate system (x, y) is chosen such that the y -axis is transversal to C , it determines a unique Weierstrass polynomial associated with C . In the sequel we assume that such a coordinate system is fixed and that f is the associated polynomial.

If one computes the roots of f as Newton-Puiseux series in x , the sequence $\alpha_1 < \dots < \alpha_g$ of their rational *characteristic exponents* is a topological invariant of C . One has $g \geq 1$ if and only if C is singular. Denoting by $\beta_0 \in \mathbb{N}^*$ the least common denominator of those numbers, let us introduce Zariski's notation (see [36]):

$$\beta_i := \beta_0 \cdot \alpha_i, \forall i \in \{1, \dots, g\}.$$

We say that $(\beta_0, \beta_1, \dots, \beta_g)$ is the *characteristic sequence* of C or of f . One has $d_y(f) = \beta_0 < \dots < \beta_g$. Introduce also the integers:

$$e_i := \gcd(\beta_0, \dots, \beta_i), \forall i \in \{0, \dots, g\}$$

and their successive quotients:

$$n_i := \frac{e_{i-1}}{e_i}, \forall i \in \{1, \dots, g\}.$$

Those quotients are integers ≥ 2 , because one has the following sequence of strict divisibilities: $1 = e_g | \dots | e_1 | e_0 = \beta_0 = d_y(f)$. Finally, define another sequence $(\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_g)$ (which is the minimal generating sequence of the semigroup of C , see [26, Prop. 4.2]):

$$\bar{\beta}_0 := \beta_0, \quad \bar{\beta}_1 := \beta_1, \quad \bar{\beta}_{i+1} := n_i \bar{\beta}_i + \beta_{i+1} - \beta_i, \forall i \in \{1, \dots, g-1\}.$$

In general, when one has a divisor of the degree of a monic polynomial which is invertible in the ring of coefficients of the polynomial, it is possible to associate to it canonically a new polynomial (see [26, Prop. 3.1]):

Proposition 3.12. *Let A be a commutative ring with unit and $P \in A[y]$ a monic polynomial of degree $d > 0$. Let $e > 0$ be a divisor of d , which is moreover invertible in A . Then there exists a unique monic polynomial $Q \in A[y]$ such that $d_y(P - Q^e) < d - \frac{d}{e}$.*

This polynomial is called the e -th approximate root of P . It is of degree $\frac{d}{e}$.

In particular, one may consider the e_0, \dots, e_g -th approximate roots of f . Denote them by f_0, \dots, f_g respectively. Therefore $f_g = f$. Moreover (see [26, Thm. 5.1]):

Proposition 3.13. *The approximate roots $(f_k)_{0 \leq k \leq g}$ have the following properties:*

- (1) $d_y(f_k) = \frac{\beta_0}{e_k}$ and $I_0(V(f), V(f_k)) = \bar{\beta}_{k+1}$ (where, by convention, $\bar{\beta}_{g+1} = +\infty$);
- (2) the polynomial f_k is a Weierstrass polynomial for $V(f_k)$ and its characteristic sequence is $(\frac{\beta_0}{e_k}, \dots, \frac{\beta_k}{e_k})$.

For us, the most important property of the previous sequence of approximate roots of f is the possibility to express canonically any other polynomial (f_0, \dots, f_g) -adically, similarly to the expansion of integers in a basis of numeration (see [26, Cor. 5.4]):

Proposition 3.14. *Every (not necessarily monic) polynomial $h \in \mathbb{C}\{x\}[y]$ may be expressed uniquely as a finite sum of the form:*

$$h = \sum_{(i_0, \dots, i_g) \in J(h)} a_{i_0, \dots, i_g} f_0^{i_0} \cdots f_g^{i_g}$$

where:

- the coefficients a_{i_0, \dots, i_g} are elements of the ring $\mathbb{C}\{x\}$;
- the indices satisfy the inequalities $0 \leq i_k \leq n_{k+1} - 1$ for any $k \in \{0, \dots, g-1\}$ and i_g is any non-negative integer;
- $J(h)$ is the set of indices with non-vanishing coefficients, that is, the support of h for this expansion.

Moreover, the intersection numbers $I_0(V(f), V(a_{i_0, \dots, i_{g-1}, 0} f_0^{i_0} \cdots f_{g-1}^{i_{g-1}}))$ of C with the curves defined by the various terms of the sum which do not vanish on C are pairwise distinct.

Let us recall now some facts about the minimal embedded resolution $\pi_S : S \rightarrow \mathbb{C}^2$ of $C = V(f)$. Extend first the notation f_k to the index $k = -1$ by:

$$f_{-1} := x.$$

Then (see [26, Cor. 5.6]):

Proposition 3.15. *The components of Exc_S which intersect exactly one other component of Exc_S are precisely those which are intersected by the strict transforms of the branches $V(f_k)$, for $k \in \{-1, \dots, g-1\}$. Those components are pairwise distinct, and if we denote by $E_C^{(k)}$ the one intersected by $V(f_k)$, then f_k is a curve associated with $E_C^{(k)}$, for all $k \in \{-1, \dots, g-1\}$ (see Figure 1).*

More generally, if a prime divisor E is given and S denotes its minimal model, then S is an embedded resolution of the branch $V(h_E)$, where h_E is any function associated with E . Apply the previous proposition to $f = h_E$. It may be shown that the components $(E_{V(h_E)}^{(k)})_{-1 \leq k \leq g-1}$ are independent of the choice of function h_E . We will denote them simply by $(E^{(k)})_{-1 \leq k \leq g-1}$. Extend this notation by saying that $E^{(g)} := E$. Therefore, the strict transform of $V(f_k)$ on S is a curvetta for $E^{(k)}$, for all $k \in \{-1, \dots, g\}$.

Using the two previous propositions, one may reprove the following description of the valuation ν_E in terms of the approximate roots f_k , which is also a consequence of Spivakovsky's [32, Thm. 8.6]:

Lemma 3.16. *For any $h \in \mathcal{O}$, if one expands it as in Proposition 3.14, then one has:*

$$\nu_E(h) = \min\{\nu_E(a_{i_0, \dots, i_g} f_0^{i_0} \cdots f_g^{i_g}) \mid (i_0, \dots, i_g) \in J(h)\}. \quad (6)$$

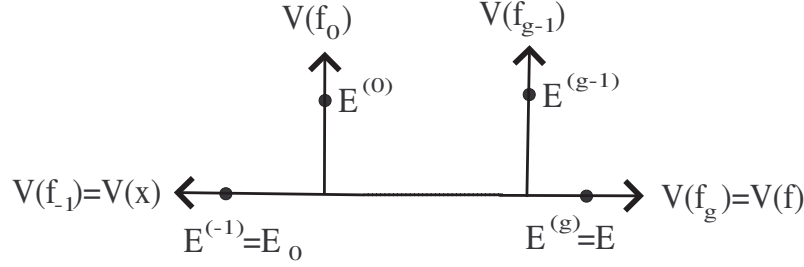


FIGURE 1. The strict transform of the branch $V(f_k)$ defined by the approximate root f_k is represented by an arrow labeled “ $V(f_k)$ ”

Proof. We reason by contradiction, assuming that:

$$\nu_E(h) > \eta \quad (7)$$

where η denotes the right-hand-side of (6). As ν_E is a valuation, this implies that there are at least two terms which realize that minimum, and that the sum of such terms has ν_E -value strictly bigger than η . Replacing h by this sum, we may therefore assume that:

$$\nu_E(a_{i_0, \dots, i_g} f_0^{i_0} \cdots f_g^{i_g}) = \eta, \quad \forall (i_0, \dots, i_g) \in J(h). \quad (8)$$

Dividing h by the highest power of $f = f_g$ appearing in the expansion, we may assume that there are terms in it with $i_g = 0$. For such terms, Lemma 3.4 implies that:

$$\eta = \nu_E(a_{i_0, \dots, i_{g-1}, 0} f_0^{i_0} \cdots f_{g-1}^{i_{g-1}}) = I_0(V(f), V(a_{i_0, \dots, i_{g-1}, 0} f_0^{i_0} \cdots f_{g-1}^{i_{g-1}})). \quad (9)$$

Indeed, Proposition 3.15 shows that the strict transforms on S of the two curves $V(f)$ and $V(a_{i_0, \dots, i_{g-1}, 0} f_0^{i_0} \cdots f_{g-1}^{i_{g-1}})$ do not intersect. Proposition 3.14 implies that there is only one such term $a_{i_0, \dots, i_{g-1}, 0} f_0^{i_0} \cdots f_{g-1}^{i_{g-1}}$ in the expansion of h , therefore:

$$I_0(V(f), V(h)) = I_0(V(f), V(a_{i_0, \dots, i_{g-1}, 0} f_0^{i_0} \cdots f_{g-1}^{i_{g-1}})). \quad (10)$$

Combining now, in order, formula (7), Lemma 3.4, formula (10) and formula (9), we get the contradiction:

$$\eta < \nu_E(h) \leq I_0(V(f), V(h)) = I_0(V(f), V(a_{i_0, \dots, i_{g-1}, 0} f_0^{i_0} \cdots f_{g-1}^{i_{g-1}})) = \eta.$$

The lemma is proved. \square

We are ready to prove our improvement of Proposition 3.8 in the case when both divisors E and F are prime (recall from Definition 2.2 that h_D denotes a function associated with the divisor D):

Proposition 3.17. *Let $E \neq F$ be different prime divisors above O . Denote by S the minimal model of E . The following statements are equivalent:*

- (1) $E \leq_\nu F$;
- (2) $\nu_E(h_{E^{(k)}}) \leq \nu_F(h_{E^{(k)}})$ for all the prime divisors $(E^{(k)})_{-1 \leq k \leq g}$.

Proof. The implication (1) \Rightarrow (2) is an immediate consequence of the definition of the valuative partial order.

Let us prove that (2) \Rightarrow (1). A function associated with E can be chosen so that it is a Weierstrass polynomial f . Let $(f_k)_{0 \leq k \leq g}$ be its approximate roots. By Proposition 3.15, each f_k is a function associated with $E^{(k)}$. Our hypothesis may therefore be rewritten as:

$$\nu_E(f_k) \leq \nu_F(f_k), \quad \forall k \in \{-1 \leq k \leq g\}. \quad (11)$$

Take any function $h \in \mathcal{O}$, which is identified to $\mathbb{C}\{x, y\}$ by the choice of local coordinates (x, y) . By Weierstrass' preparation theorem, it is equal to a polynomial in $\mathbb{C}\{x\}[y]$ multiplied by a unit. Therefore we may assume that $h \in \mathbb{C}\{x\}[y]$. Expand it (f_0, \dots, f_g) -adically (see Proposition 3.14):

$$h = \sum_{(i_0, \dots, i_g) \in J(h)} a_{i_0, \dots, i_g} f_0^{i_0} \cdots f_g^{i_g}.$$

As F is prime, $\nu_F : \mathcal{O} \rightarrow \mathbb{N} \cup \{+\infty\}$ is a valuation, therefore:

$$\nu_F(h) \geq \min\{\nu_F(a_{i_0, \dots, i_g} f_0^{i_0} \cdots f_g^{i_g}) \mid (i_0, \dots, i_g) \in J(h)\}. \quad (12)$$

The coefficients a_{i_0, \dots, i_g} being moreover series in the variable x alone, one deduces also the following equality from the fact that ν_F is a valuation:

$$\nu_F(a_{i_0, \dots, i_g} f_0^{i_0} \cdots f_g^{i_g}) = \text{ord}_x(a_{i_0, \dots, i_g}) \cdot \nu_F(x) + \sum_{k=0}^g i_k \cdot \nu_F(f_k).$$

As E is also prime, one has the analogous equality obtained by replacing F by E . Therefore, our hypothesis (11) implies that (recall that $f_{-1} = x$):

$$\nu_F(a_{i_0, \dots, i_g} f_0^{i_0} \cdots f_g^{i_g}) \geq \nu_E(a_{i_0, \dots, i_g} f_0^{i_0} \cdots f_g^{i_g}), \quad \forall (i_0, \dots, i_g) \in J(h),$$

from which we get:

$$\min\{\nu_F(a_{i_0, \dots, i_g} f_0^{i_0} \cdots f_g^{i_g}) \mid (i_0, \dots, i_g) \in J(h)\} \geq \min\{\nu_E(a_{i_0, \dots, i_g} f_0^{i_0} \cdots f_g^{i_g}) \mid (i_0, \dots, i_g) \in J(h)\}.$$

By combining Lemma 3.16 with the inequality (12), we would deduce that $\nu_F(h) \geq \nu_E(h)$. This inequality being true for any $h \in \mathcal{O}_0$, we would get the desired conclusion $\nu_F \geq_\nu \nu_E$. \square

Let $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ be a sequence of key polynomials (a SKP as defined in [12]) of length k with $1 \leq k < \infty$. By [12, Chap. 2, Thm. 2.8], since the sequence is finite, there exists a prime divisor E such that it is the sequence of key polynomials associated with the valuation ν_E . The polynomials of the sequence coincide with the approximate roots of the last polynomial of the sequence. Interpreted in this language, the previous result complements [12, Chap. 2, Thm. 2.8] in the following sense:

Corollary 3.18. *Let $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ be a sequence of key polynomials (a SKP as defined in [12]) of length k with $1 \leq k < \infty$ and $\tilde{\beta}_k \in \mathbb{Q}$. Then there exists a unique divisorial centered valuation ν_k satisfying*

$$(Q1) \quad \nu_k(U_j) = \tilde{\beta}_j \text{ for } 0 \leq j \leq k;$$

$$(Q2') \quad \nu_k \leq \nu \text{ for any divisorial valuation } \nu \text{ satisfying } \nu(U_j) \geq \tilde{\beta}_j \text{ for } 0 \leq j \leq k.$$

3.3. The valuative criterion and deformations.

Some of the results in this section reinterpret and generalize results of [1] in terms of the valuative criterion.

We begin by explaining some notations used in the sequel.

Notation 3.19. *A germ of holomorphic mapping:*

$$H : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$$

$$(x, y, s) \rightarrow H(x, y, s)$$

such that $H(s, 0, 0) = 0$ is said to be a deformation of the plane curve $H(x, y, 0) = 0$, which is called the special curve of the deformation. We denote $h_s(x, y) = H(s, x, y)$. It is known that there exists a representative Λ of $(\mathbb{C}, 0)$ such that for $s \in \Lambda \setminus \{0\}$, the topological type of the curves $(V(h_s), 0)$ is constant. We call them the generic curves of the deformation. We denote the deformation by $(h_s)_{s \in \Lambda}$ or simply by $(h_s)_s$. We say that Λ is the space of parameters and in this context, we will always assume that $s \in \Lambda$.

Notation 3.20. Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ and let S be a model of $(\mathbb{C}^2, 0)$. Let $\text{Exc}_S = \bigcup_i D_i$ be the decomposition into irreducible components of its exceptional set. We say that $H_f = \sum c_i D_i$ is the divisor associated with f in the model S if $c_i = \widetilde{V(f)} \cdot D_i$ for all i , where $\widetilde{V(f)}$ denotes the strict transform of $V(f)$ on S .

In particular, given a deformation $(g_s)_s$, we can define $E := H_{g_s}$ for $s \neq 0$ small enough, since it does not depend on this choice.

As a first result relating deformations and the valuative criterion, we have the following:

Proposition 3.21. Let $G = (g_s)_s$ be a deformation. Let S be any model and let $F := H_{g_0}$ and $E := H_{g_s}$ (with $s \neq 0$) as in Notation 3.20. Then $\nu_E \leq \nu_F$.

Proof. We proceed as in [15, Section 3.2]. We use Notation 3.20.

Consider the mapping:

$$\sigma := \pi \times \text{Id}_\Lambda : S \times \Lambda \rightarrow \mathbb{C}^2 \times \Lambda$$

where $(\Lambda, 0) \subsetneq (\mathbb{C}, 0)$ is the space of parameters s of the deformation.

Let Y be the strict transform of $V(G)$ in $S \times \Lambda$ by σ . Denote $Y_s := Y \cap (S \times \{s\})$. Observe that:

$$\begin{aligned} Y_s &= \widetilde{V(g_s)} \quad \text{for } s \neq 0 \\ Y_0 &= \widetilde{V(g_0)} + \sum_k d_k D_k, \quad \text{with } d_k \geq 0. \end{aligned}$$

Due to the invariance of the intersection number under deformation, we have the system of equations:

$$Y_0 \cdot D_i = Y_s \cdot D_i \quad (13)$$

for any i . If we define $b_i := \widetilde{V(g_0)} \cdot D_i$ and $a_i := Y_s \cdot D_i$, then the system (13) can be rewritten as follows using the intersection matrix $M = (D_i \cdot D_j)_{i,j}$:

$$(b_0, b_1, \dots, b_n)^t + M(d_1, \dots, d_n)^t = (a_0, \dots, a_n)^t \quad (14)$$

or equivalently:

$$-M^{-1}(b_0 - a_0, \dots, b_n - a_n)^t = (d_1, \dots, d_n)^t \geq 0. \quad (15)$$

The entries of the matrix $-M^{-1}$ are exactly $\nu_{D_i}(h_{D_j}) = I_0(V(h_{D_i}), V(h_{D_j}))$ (see Lemma 3.4), and since $d_i \geq 0$ for all i , the inequality (15) says exactly $\nu_E(h_{D_i}) \leq \nu_F(h_{D_i})$ for all i . Then, using the implication (4) \Rightarrow (1) of Proposition 3.8, we conclude that $\nu_E \leq \nu_F$. \square

Proposition 3.24 below is a kind of reciprocal of the previous one. In particular, it shows that with the hypothesis of the valuative inequality we can control the generic member of a pencil of curves. For the case of prime divisors and $V(\tilde{g})$ a curvetta for E , this was proved in [1].

To state Proposition 3.24 in the maximal generality, we need the following definition, which weakens Definition 2.2, in the sense that one does not impose now any transversality hypothesis:

Definition 3.22. Let $F = \sum_i b_i D_i$ be a divisor in a model S of $(\mathbb{C}^2, 0)$ with $\text{Exc}_S = \bigcup_i D_i$. We say that a function f is *weakly associated with F in S* if $b_i = \nu_{D_i}(f) = \widetilde{V(f)} \cdot D_i$ for all i .

We need also the following lemma:

Lemma 3.23. Let h be a function weakly associated with a divisor $F = \sum b_i D_i$ in a model S . Then $\nu_{D_i}(h)$ does not depend on the choice of h , for any prime component D_i of Exc_S .

As a consequence, let S be obtained by blowing up points $\{x_i\}_{i \in I}$ and \tilde{h}_i denote the local equation of the strict transform of h at x_i . If h is weakly associated with F , then $m_{x_i}(V(\tilde{h}_i))$ does not depend on the choice of h for all $i \in I$.

Proof. Let h be a holomorphic function weakly associated with F in S . We know that $\nu_D(h) = I_0(h, h_D)$ for some associated function h_D to D whose strict transform in S does not meet the strict transform of h . But we have the equality $I_0(h, h_D) = \pi^*(h_D) \cdot \tilde{h}$, which only depends on the intersection products $b_i := \widetilde{V(h)} \cdot D_i$.

Let us prove the consequence. Let $\{D_j\}_{j \in J}$ be the prime divisors meeting x_i at the minimal model where x_i appears. We just observe the equality:

$$m_{x_i}(V(\tilde{h}_i)) = \nu_{D_i}(h) - \sum_{j \in J} \nu_{D_j}(h).$$

□

Proposition 3.24. *Let $E = \sum_i a_i D_i$, $F = \sum_i b_i D_i$ be divisors such that $\nu_E \leq \nu_F$. Let $V(g)$ and $V(f)$ be curves weakly associated with E and F in the minimal model S of $E + F$. Then $V(f + sg)$ are weakly associated with E in S for all $s \neq 0$ small enough.*

Moreover, if g is associated with E then so is $V(f + sg)$ for $s \neq 0$ small enough.

Proof. Let $\{x_i\}_{i \in I}$ be the infinitely near points that we blow up in order to obtain the minimal model of E . Here I is partially ordered by saying that $i < j$ precisely when x_j is infinitely near x_i . We denote $x_0 := 0$ and for any $i > 0$ we will denote by x_{i-1} the point which is blown up in order to create the minimal model of x_i . Let E_i be the divisor obtained after blowing up x_i . Let f_i be a local defining function at x_i of the total transform of f and \tilde{f}_i a local defining function at x_i of the strict transform of f . We use the same notations for g and for $f + sg$. Let $\{m_i\}_{i \in I}$ be the multiplicity sequence of g , that is, $m_i = m_{x_i}(\tilde{g}_i)$.

Note that by the choice of f and g and the inequality $\nu_E \leq \nu_F$ we have:

$$\nu_{E_i}(g) = I_O(h_{E_i}, g) = I_O(h_{E_i}, h_E) = \nu_E(h_{E_i}) \leq \nu_F(h_{E_i}) = I_O(h_{E_i}, h_F) = I_O(h_{E_i}, f) = \nu_{E_i}(f) \quad (16)$$

for any E_i in Exc_S and for h_D denoting functions associated with the divisor D , chosen generically relative to f and g .

1. We prove that for $s \neq 0$ the strict transform of $V(f + sg)$ after the blow-up of x_{i-1} passes through x_i and has the same multiplicity as $V(g)$, that is, $m_{x_i}(\widetilde{(f + sg)_i}) = m_i$.

We proceed by induction on the partial order of I .

Initial step at x_0 . Both f and g go through the origin x_0 and $m_{x_0}(f + sg) = m_{x_0}(g) = m_0$ for $s \neq 0$ small enough because $m_{x_0}(f) \geq m_{x_0}(g)$: the multiplicity of $f + sg$ is different from that of g only for certain s , so it is equal to it if $s \neq 0$ is small enough.

Let's do in detail the case where x_i is a satellite point and $x_i = E_{i-1} \cap E_j$. Assume that $E_{i-1} = V(y)$ and $E_j = V(x)$ for local coordinates (x, y) around x_i . Then we may write:

$$\begin{aligned} f_i &= x^{\nu_{E_j}(f)} y^{\nu_{E_{i-1}}(f)} \tilde{f}_i, \\ g_i &= x^{\nu_{E_j}(g)} y^{\nu_{E_{i-1}}(g)} \tilde{g}_i \end{aligned}$$

and so the total transform of $f + sg$ around x_i is:

$$\begin{aligned} (f + sg)_i &= x^{\nu_{E_j}(f)} y^{\nu_{E_{i-1}}(f)} \tilde{f}_i + s \cdot x^{\nu_{E_j}(g)} y^{\nu_{E_{i-1}}(g)} \tilde{g}_i = \\ &= x^{\nu_{E_j}(g)} y^{\nu_{E_{i-1}}(g)} (x^{\nu_{E_j}(f) - \nu_{E_j}(g)} y^{\nu_{E_{i-1}}(f) - \nu_{E_{i-1}}(g)} \tilde{f}_i + s \cdot \tilde{g}_i) \end{aligned}$$

where what is inside the parenthesis is $\widetilde{(f + sg)_i}$. If the two strict transforms $V(\tilde{f}_i)$ and $V(\tilde{g}_i)$ pass through x_i , then it is also the case for $V(\widetilde{(f + sg)_i})$. If only $V(\tilde{g}_i)$ passes through x_i , in order to have that $\widetilde{(f + sg)_i}$ is zero in x_i , we need to guarantee that $\nu_{E_{i-1}}(f) - \nu_{E_{i-1}}(g)$ and $\nu_{E_j}(f) - \nu_{E_j}(g)$ are not

both zero. But if they were both zero, then $\nu_{E_{i-1}}(f) + \nu_{E_j}(f) = \nu_{E_{i-1}}(g) + \nu_{E_j}(g)$ and since $m_i > 0$ we would have that:

$$\nu_{E_i}(f) = \nu_{E_{i-1}}(f) + \nu_{E_j}(f) < \nu_{E_{i-1}}(g) + \nu_{E_j}(g) + m_i = \nu_{E_i}(g)$$

which cannot be the case by inequality (16).

Let us show now that the multiplicity of $\widetilde{(f + sg)_i}$ at x_i is $m_i = m_{x_i}(\widetilde{g}_i)$. One has $m_{x_i} \widetilde{f}_i = \nu_{E_i}(f) - \nu_{E_{i-1}}(f) - \nu_{E_j}(f)$ and $m_{x_i} \widetilde{g}_i = \nu_{E_i}(g) - \nu_{E_{i-1}}(g) - \nu_{E_j}(g)$, therefore the inequality:

$$m_{x_i}(x^{\nu_{E_j}(f) - \nu_{E_j}(g)} y^{\nu_{E_{i-1}}(f) - \nu_{E_{i-1}}(g)} \widetilde{f}_i) \geq m_{x_i}(\widetilde{g}_i) \quad (17)$$

is equivalent to the inequality $\nu_{E_i}(f) \leq \nu_{E_i}(g)$, which we have proved before. Moreover, the equality in (17) (and the possible jump of multiplicity of $\widetilde{(f + sg)_i}$) may only occur for certain isolated values of s (and only when we have the equality $\nu_{E_i}(f) = \nu_{E_i}(g)$), therefore not for $s \neq 0$ small enough.

The case where x_i is a free point is analogous.

2. We claim that the fact that \widetilde{g}_i and $\widetilde{(f + sg)_i}$ have the same multiplicity at x_i for any infinitely near point x_i which is blowing up center for the model S implies the equality:

$$V(\widetilde{(f + sg)_i}) \cdot D_i = \widetilde{V(g)_i} \cdot D_i$$

for any exceptional divisor D_i in S and $s \neq 0$ small enough. This exactly means that $f + sg$ is weakly associated with E for $s \neq 0$ small enough.

Let us prove this claim. Give a total order to the infinitely near points involved in the blowing up process leading to S , which is compatible with the natural partial order given by domination. Let x_i be the blowing up center giving rise to D_i in its minimal model. Denote by $\widetilde{\widetilde{V(g)_i}}$ the strict transform of $V(g)$ at the result of blowing up the point x_i . In a neighbourhood of D_i it coincides with the strict transform of $V(\widetilde{g}_i)$ by the blowing up of x_i . Therefore we have the equality:

$$\widetilde{\widetilde{V(g)_i}} \cdot D_i = m_{x_i}(\widetilde{g}_i) = m_{x_i}(\widetilde{(f + sg)_i}) = \widetilde{\widetilde{V(f + sg)_i}} \cdot D_i.$$

Let x_j be the blowing up center appearing right after x_i . If x_j is not located in D_i , then we have the equalities:

$$\widetilde{\widetilde{V(g)_j}} \cdot D_i = \widetilde{\widetilde{V(g)_i}} \cdot D_i$$

and:

$$\widetilde{\widetilde{V(f + sg)_j}} \cdot D_i = \widetilde{\widetilde{V(f + sg)_i}} \cdot D_i.$$

If x_j is on D_i , then we have the equalities:

$$\begin{aligned} \widetilde{\widetilde{V(g)_j}} \cdot (D_i + D_j) &= \widetilde{\widetilde{V(g)_i}} \cdot D_i = m_{x_i}(\widetilde{g}_i), \\ \widetilde{\widetilde{V(f + sg)_j}} \cdot (D_i + D_j) &= \widetilde{\widetilde{V(f + sg)_i}} \cdot D_i = m_{x_i}(\widetilde{(f + sg)_i}), \\ \widetilde{\widetilde{V(g)_j}} \cdot D_j &= m_{x_j}(\widetilde{g}_j), \\ \widetilde{\widetilde{V(f + sg)_j}} \cdot D_j &= m_{x_j}(\widetilde{(f + sg)_j}). \end{aligned}$$

Since we have shown the equality $m_{x_i}(\widetilde{g}_i) = m_{x_j}(\widetilde{(f + sg)_i})$ for any i , we deduce the equality:

$$\widetilde{\widetilde{V(g)_j}} \cdot D_i = \widetilde{\widetilde{V(f + sg)_j}} \cdot D_i.$$

Iterating this procedure for all blowing up points, the claim is proven.

3. Let us show now the last assertion of the proposition. Consider the pencil $\lambda f + \mu g$ for $(\lambda : \mu) \in \mathbb{P}^1$. The fact that g is associated with E and $f + sg$ is weakly associated with E for almost all s implies that the generic member of the pencil is associated with E : the intersection number $V(\widetilde{f + sg}) \cdot D_i$ equals b_i and there is a member of the pencil (the one corresponding to the parameter value $(0 : 1)$, which is g) for which the set $V(\widetilde{f + sg}) \cap D_i$ consists of b_i distinct points. Since this is the most generic behaviour we have that for generic s the set $V(\widetilde{f + sg}) \cap D_i$ consists of b_i distinct points. \square

Putting the last two propositions together we have the following theorem:

Theorem 3.25. *Let E and F be divisors over the origin of \mathbb{C}^2 and let S be the minimal model containing the divisor $E + F$. The following are equivalent:*

- (a) $\nu_E \leq \nu_F$;
- (b) *there exists a deformation $G = (g_s)_s$ with g_0 weakly associated with F in S and g_s weakly associated with E in S , for $s \neq 0$ small enough;*
- (c) *there exists a deformation $G = (g_s)_s$ with g_0 associated with F in S and g_s associated with E in S , for $s \neq 0$ small enough;*
- (d) *there exists a linear deformation $(g_0 + sg)_s$ with g_0 associated with F in S and g and g_s associated with E in S , for $s \neq 0$ small enough.*

For the case that F is prime, we have the following finer theorem, of which Theorem 1.2 stated in the introduction is the special case where E is also prime:

Theorem 3.26. *Let F be a prime divisor, and E be any divisor. Then the following are equivalent:*

- (a) $\nu_E \leq \nu_F$;
- (b) *there exists a deformation $G = (g_s)_s$ with g_0 associated with F in the minimal model of F and g_s associated with E in the minimal model of E for $s \neq 0$ small enough.*

Proof. The implication $(a) \Rightarrow (b)$ follows from $(a) \Rightarrow (d)$ of Theorem 3.25.

It only remains to prove that $(b) \Rightarrow (a)$. Assume that we have a deformation as in the statement. Consider the minimal model S for E . We claim that the divisor F does not appear in S unless we have the equality $E = F$, case in which the implication is obvious.

Let $E = \sum a_i E_i$. If F appears in the minimal model of E then there exists E_i in the support of E such that either it is equal to F or it dominates it. But then, by Remark 3.3, we have the valuative domination $\nu_{E_i} \geq \nu_F$. Since by hypothesis $\nu_E \leq \nu_F$ and obviously $\nu_{E_i} \leq \nu_E$, we have $\nu_{E_i} = \nu_F$. We deduce the equality $E_i = E = F$.

Now, we keep blowing up until we get a model where F also appears. It may happen that the strict transform of $V(g_s)$ for small enough $s \neq 0$ meet now another divisor $E' > E$. We can apply now Lemma 3.21 to obtain $E' \leq_\nu F$. Since we have that $E \leq_\nu E'$ we get $E \leq_\nu F$ as desired. \square

Remark 3.27. *Given a deformation as in (b) in Theorem 3.26 we can consider the minimal model S of $E + F$ and the divisors H_{g_0} and H_{g_s} as in Notation 3.20. A quick reflection will convince the reader that the equalities $F = H_{g_0}$ or $E = H_{g_s}$ do not necessarily hold, for example if g_0 is also associated with a prime divisor $F' >_d F$ which appears in the minimal model of E .*

3.4. Classical adjacency of functions and deformations fixing the free points.

The classical *adjacency problem for plane curves* asks whether, given any two different embedded topological types encoded, for instance, by the two dual graphs \mathcal{G}_0 and \mathcal{G}_1 of the minimal embedded resolution of two curves germs, there exists a deformation $(h_s)_s$ where $V(h_0)$ and $V(h_{s \neq 0})$ have the embedded topological types given by \mathcal{G}_0 and \mathcal{G}_1 respectively. We say in this case that the topological type given by \mathcal{G}_0 is *upper-adjacent* to the one given by \mathcal{G}_1 .

Let $(h_s)_s$ be a curve deformation and C any plane curve germ at the origin. It is classical that:

$$I_0(V(h_0), C) \geq I_0(V(h_s), C), \quad (18)$$

for $s \neq 0$ small enough.

Remark 3.28. *Gorsky and Némethi proved in [16, Proposition 4.5.1] a refined criterion, consisting on an upper semi-continuity of the Hilbert function associated with a (multi)-branch, which obstructs the existence of Arnold adjacencies in the case in which the special curve is irreducible. Their proof boils down to the inequality (18).*

Given a deformation as in Notation 3.19, we know that the family of curves $V(h_s)$ for $s \neq 0$ have the same embedded topological type, that is, the final divisors of their minimal embedded resolutions have the same combinatorial type. These divisors can change with $s \neq 0$ because the free infinitely near points appearing as blowing up centers in the minimal embedded resolutions of the curves $V(h_{s \neq 0})$ can move continuously with s without changing the combinatorial type. In the deformations in item (c) of Theorem 3.25 this does not happen and the generic curves for small value of the parameter $s \neq 0$ can be resolved by a single sequence of point blow ups, and can sometimes even lift together after further blowups. We codify this phenomenon in a definition (in [1], this corresponds to a deformation associated with a cluster of infinitely near points):

Definition 3.29. A deformation $(h_s)_s$ is said to *fix the free points for the divisor E* if, for $s \neq 0$ small enough, the function h_s is associated with E in the minimal model where E appears.

A deformation $(h_s)_s$ is said to *fix the free points*, if there exists a divisor E such that it fixes the free points for E .

A deformation $(h_s)_s$ is called *dicritical* along a prime divisor E if the strict transforms of $V(h_s)$ for $s \neq 0$ small enough move along E . Otherwise we call it *non-dicritical* along E .

Remark 3.30. (1) *Let be given a deformation $(h_s)_s$ fixing the free points for E . If the family of strict transforms of $V(h_s)$ for all $s \neq 0$ meets some component E_i of E in a non-dicritical way, then we can keep blowing up until each of the branches of the family of strict transforms meets a component of the exceptional divisor of the new model in a dicritical way.*
 (2) *The divisor for which the family is dicritical along every components is the maximal divisor with respect to domination for which the deformation fixes the free points.*

Observe that two curves have the same topological type if and only if they are respectively associated with two divisors D and D' which are combinatorially equivalent.

Remark 3.31. *Let $D = \sum_i E_i$ and $D' = \sum_i E'_i$ be two divisors satisfying the condition that E'_i dominates E_i in such a way that it appears after further blowing ups only in free points. Then the curve associated with D' has the same topological type as the curve associated with D .*

The previous remark and Theorem 3.25 give a necessary and sufficient criterion for the existence of deformations fixing the free points with prescribed embedded topological types. Let us explain this criterion.

A topological type is determined by a class of combinatorially equivalent divisors $[E]$, but due to the previous remark there are further divisors whose associated curves have the same topological type. Denote by $[E]_{top}$ the collection of divisors whose associated curves have the given topological type.

Consider another topological type $[F]$ and let $[F]_{top}$ be the associated set of divisors.

By Theorem 3.25, there is an adjacency fixing the free points between the two topological types, if and only if there exist divisors $E' \in [E]_{top}$ and $F' \in [F]_{top}$ which appear in a common model, such that we have the inequality:

$$\nu_{E'} \leq \nu_{F'}.$$

Then, given $[F]_{top}$, our interest is to identify all the topological types $[E]_{top}$ such that the previous inequality is satisfied for some $E' \in [E]_{top}$ and $F' \in [F]_{top}$.

Let us call *the complexity of E* the length of the longest totally ordered chain of infinitely near points appearing as blowing up centers leading to the minimal model of E .

The validity of the previous inequality only depends on the combinatorial type of the pair (E', F') . The next lemma implies that, in order to check the existence of the adjacency fixing the free points between the topological types, we only have to look at pairs (E', F') with E' and F' of bounded complexity, and therefore at finitely many combinatorial types of the pair (E', F') . Being finite in number they can be enumerated, and this gives a method to check the existence of the adjacency in finite time.

Given a topological type, there is a combinatorial class of divisors $[E]$ of minimal complexity included in $[E]_{top}$. It corresponds to the divisors associated with the minimal embedded resolutions of the curves with the given topological type. Observe that any other divisor in $[E]_{top}$ dominates one in this class by a blowing up process involving only free points. The *complexity of a topological type* is, by definition, the complexity of any divisor associated to the minimal resolution of a curve with the given topological type.

Lemma 3.32. *Denote by M the maximum of the complexities of the two topological types we are considering. If there exist $E' \in [E]_{top}$ and $F' \in [F]_{top}$ appearing in a common model such that the inequality $\nu_{E'} \leq \nu_{F'}$ holds, then there exist $E'' \in [E]_{top}$ and $F'' \in [F]_{top}$ such that the inequality $\nu_{E''} \leq \nu_{F''}$ holds and the complexities of both E'' and F'' are bounded by M .*

Proof. Let S be the minimal model in which E' and F' appear. By Theorem 3.25, there exists a deformation $(f_s)_s$ such that the divisor associated with f_0 in S equals F' and the divisor associated with f_s ($s \neq 0$) in S equals E' .

Let $\{p_1, \dots, p_r\}$ be a maximal linearly ordered chain of infinitely near points appearing in the blowing up process leading to S . Let S' be the model obtained after blowing up the union of $\{p_1, \dots, p_{\min\{M, r\}}\}$ over all maximal linearly ordered chains leading to S .

Let F'' be the divisor weakly associated with f_0 in S' . Since M is at least the complexity of the topological type given by F' , we have that F'' belongs to $[F]_{top}$ (and that f_0 is associated to F'').

Let E'' be the divisor weakly associated with $f_{s \neq 0}$ in S'' ; for the same reason we have $E'' \in [E]_{top}$ (and that $f_{s \neq 0}$ is associated to E'').

Since E'' and F'' are in a common model S' , Theorem 3.25 implies the desired inequality. \square

Moreover, we have the following reductions which help avoiding unnecessary computations:

- using Corollary 3.10 we are reduced to check some “maximal” contact orders;
- when F is a prime divisor, Theorem 3.26 allows us to avoid checking for the divisors F' of a combinatorial type different from that of F .

Using the previous procedure, let us study now the adjacencies which can be realized fixing the free points among simple, unimodal and bimodal singularities with Milnor number less or equal 16, that is, those classified in [2].

In Figure 2 one can see all adjacencies between simple and parabolic plane curve singularities. The complete list of adjacencies was originally given in [31]. Here an arrow directed from a singularity type X to a singularity type Y means that X is *upper adjacent* to Y , that is, there is a deformation of a singularity of type X with generic singularities of type Y . **A dotted arrow means that the adjacency is not realizable by deformations fixing the free points.**

Among the 93 (classical) adjacencies between simple singularities with Milnor number less or equal 8, only 7 are not realized by deformations fixing the free points. Moreover, between A_* and D_* type simple singularities, only the adjacencies from D_{2n+1} to A_{2n} are not realizable in that way for any n .

Example 1. Consider the deformation:

$$y^3 + x^4 + s^2 y^2 + 2s x^2 y$$

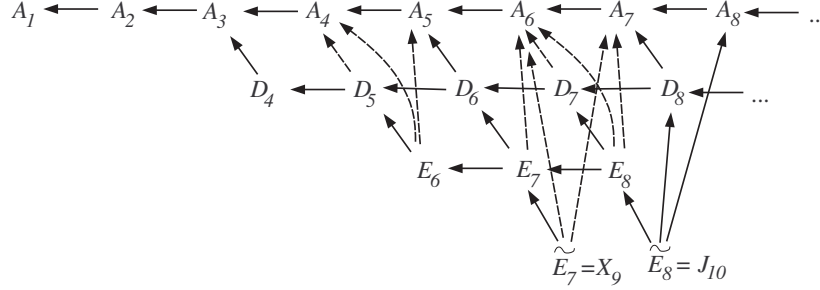


FIGURE 2. Adjacencies realizable by deformations fixing the free points among simple and parabolic singularities. Any adjacency of this type follows by transitivity from the arrows in the picture. Dotted arrows indicate the other classical adjacencies.

of the singularity E_6 . For $s \neq 0$ we get the singularity A_4 . It is easy to check that the valuative inequality is not satisfied for the corresponding divisors for any contact order between them, so it is not possible to realize the adjacency with a deformation fixing the free points.

All adjacencies between unimodal singularities¹ were described in [5]. All adjacencies between hyperbolic singularities $T_{a,b,c}$ are realizable by linear adjacencies as it was pointed out in [5] (in this case $T_{a,b,c} \rightarrow T_{\alpha,\beta,\gamma}$ if and only if $\alpha \leq a$, $\beta \leq b$, $\gamma \leq c$). On the contrary, not all adjacencies from exceptional to hyperbolic singularities are realizable fixing the free points as it is shown in figure 4 and 3. In these two diagrams one can find all the adjacencies between unimodal singularities realised fixing the free points apart from the ones between hyperbolic singularities. The adjacency from $X_{1,1}$ to E_8 seems not to appear in the literature but it follows from the comparison of the Newton diagrams of their normal equations, criterion used in [2].

The adjacencies between bimodal singularities, between unimodal singularities and simple and between unimodal and bimodal are mainly described in Arnold where it is stated that the list is not complete. The list given in [3] of adjacencies between bimodal singularities with Milnor number less or equal than 16 is assumed to be more complete than the one in [2]. We find that again almost all of these adjacencies can be realized by deformations fixing the free points. Those that are not, appear with dotted lines in Figure 4. The adjacencies $Z_{17} \rightarrow E_{14}$, $W_{17} \rightarrow Z_{13}$, $E_{19} \rightarrow J_{3,2}$ and $E_{20} \rightarrow J_{3,2}$ are in [3] but not in Arnold's list. The one $Z_{11} \rightarrow E_8$ does not appear even in [3] but it follows also from the comparison of the Newton diagrams of their normal equations, a criterium used in [2]. They are all realizable by families fixing the free points.

Observe that the singularity E_8 is simple, $Z_{11} = S_{2,4,5}$, $Z_{13} = S_{2,4,7}$, $E_{14} = S_{2,3,9}$ and $J_{3,2}$ are unimodal and Z_{17} , E_{19} , W_{17} and Z_{17} are bimodal.

To check these diagrams, we have considered the equations for the singularities of plane curves given in [2] and we computed their embedded resolutions and multiplicity sequences using Singular (these singularities are resolved in at most 5 blow ups).

Remark 3.33. *It is easy to build up an algorithm which, given a prime divisor F , finds all those combinatorial types of divisors E such that $\nu_{E'} \leq \nu_F$ for some E' with the same combinatorics as E and with certain contact order $\text{Cont}(E', F)$. By Corollary 3.11 there are only finitely many of them. In particular, this list of divisors would encounter all the embedded topological types \mathcal{G}_E that are adjacent*

¹For the convenience of the reader, we recall the different names of the unimodal singularities. The *parabolic* ones are $X_9 = T_{2,4,4} = \tilde{E}_7 = X_{1,0}$ and $J_{10} = T_{2,3,6} = \tilde{E}_8 = J_{2,0}$. For the *hyperbolic* singularities we use Arnold's notation (not the one in [5]) with the following translation: $J_{2,i} = T_{2,3,6+i}$. We have also $X_{1,p} = T_{2,4,4+p}$ and $Y_{r,s}^1 = T_{2,4+r,4+s}$. We use also Arnold's notations for the exceptional singularities. For the correspondence with the other name $S_{a,b,c}$, see [5].

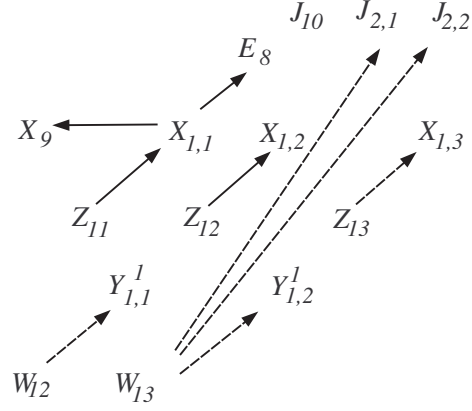


FIGURE 3. Adjacencies realizable by deformations fixing the free points among unimodal singularities, from exceptional singularities to hyperbolic singularities. Any adjacency of this type between the singularities in the diagram follows by transitivity from the arrows in the picture. Dotted arrows indicate the other classical adjacencies.

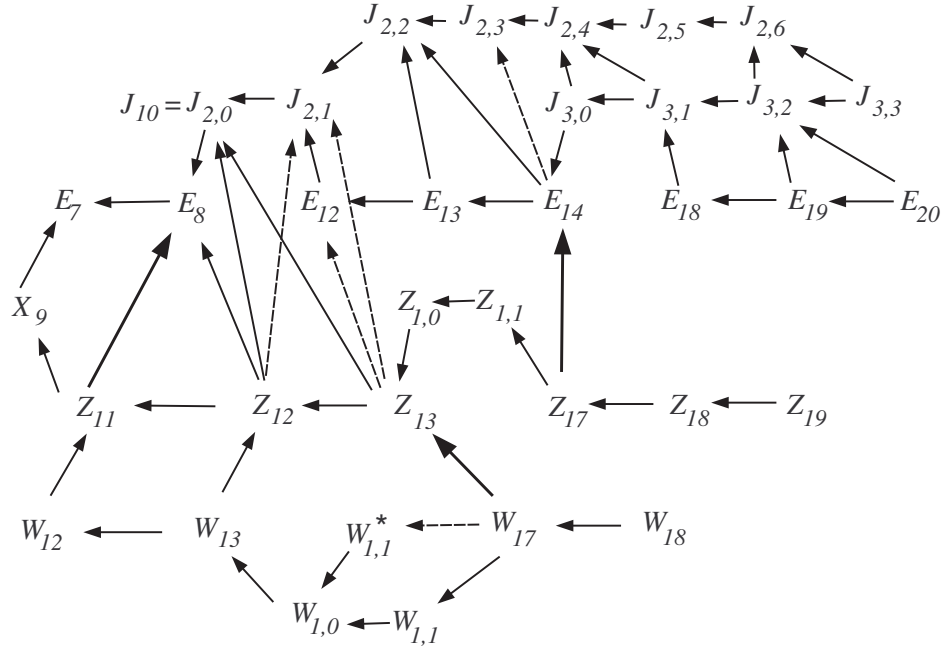


FIGURE 4. Adjacencies realizable by deformations fixing the free points among some simple, unimodal and bimodal plane curve singularities. Any adjacency of this type between the singularities in the diagram follows by transitivity from the arrows in the picture. Dotted arrows indicate the other classical adjacencies.

to \mathcal{G}_F as in Definition 3.29, by deformations fixing the free points. The second author is planning to program it in *Singular*.

First, one can proceed by looking for all the combinatorial types of prime divisors E recursively on the number of points one has to blow up to obtain their minimal model. The equivalence (1) \Leftrightarrow (6) from Proposition 3.8 allows to do the computations in every step of the recursion in a fast way. Then, one can proceed recursively in the number of components for the non-prime case. Note also that according to Corollaries 3.9 and 3.10, in order to check if a divisor is adjacent to F it is sufficient to check the cases of “maximal” contact order.

The case where F is not prime can be treated with the same ideas but one has to carefully enumerate all the possibilities of contact orders between the branches of E and F .

3.5. A result on the δ -constant stratum through the study of arc spaces.

Another classical problem related to the classical study of adjacencies (see section 3.4) is the study of the δ -constant stratum of a topological type. Using the following result of Teissier ([34], see also [7]), it consists in studying, among the possible adjacencies, the ones that are realizable by deformations of parametrizations:

Theorem 3.34. *A family of curves on $(\mathbb{C}^2, 0)$ admits a parametrization in family if and only if it is δ -constant.*

A family of parametrizations is a convergent wedge realising the corresponding adjacency. Observe that a representative of the wedge may have reducible generic curves at the origin. This means that a wedge can have *returns* (see [22]):

Definition 3.35. The *returns* of a wedge $\alpha : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ are the branches of $(\alpha^{-1}(0), 0)$ different from $\Lambda \times \{0\}$ where Λ is the parameter space of the family.

In the general surface case, the existence of returns can be essential and unavoidable, see [22]. This is not the case in \mathbb{C}^2 :

Lemma 3.36. *Given a wedge α strictly realizing a Nash-adjacency $\overline{N}_F \subsetneq \overline{N}_E$, there exists a wedge strictly realizing it and without returns.*

Proof. Write $\alpha(t, s) = (\alpha_1(t, s), \alpha_2(t, s)) = \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Picking appropriate coordinates of the source \mathbb{C}^2 , we can get parametrizations of the branches of the zero set of $\alpha_1 : \mathbb{C}^2 \rightarrow \mathbb{C}$ in the form $(q_i(s), s^{k_i})$, for i varying in some index set I . We can always redefine $\alpha(t, s) := \alpha(t, s^k)$ with $k = \text{lcm}\{k_i, i \in I\}$, which is a wedge with the same properties as the original one. So, we can assume that the parametrizations of the zero set of α_1 are now of the form $(q_i(s), s)$.

A branch $(q_i(s), s)$ gives a return of the wedge if and only if it is contained in the zero set of α_2 , and if $q_i(s)$ is not identically zero.

Let us consider a number $M \in \mathbb{N}^*$ such that $\alpha_2(q_i(s), s)$ is different from $s \cdot q_i(s)^M$ for all i .

Then, the wedge given by $\beta(t, s) = (\alpha_1(t, s), \alpha_2(t, s) + st^M)$ does not have returns.

Besides, by taking M big enough, we can guarantee that the generic arc of β is also transversal to the divisor E , provided the generic arc of α is transversal to E . \square

Now, we can state our main result about δ -constant deformations:

Proposition 3.37. *Let E and F be satellite prime divisors (that means that give topological types). Let $V(h_E)$ be a curve associated to E and assume it has only one Puiseux pair (which means that E is toric). If the inequality $\nu_E \leq \nu_F$ holds, then there is a δ -constant deformation with generic topological type a curvetta for E and special curve a curvetta for F .*

Proof. Lemma 3.11 in [19] says that in our case $\nu_E \leq \nu_F$ implies $\overline{N}_F \subset \overline{N}_E$. Then, by Theorem 4.7 we have a convergent wedge realizing the adjacency and by the previous proposition we can suppose that

the wedge doesn't have returns and then gives a deformation of parametrizations whose generic type is the one associated to E . \square

We have also this easy consequence:

Corollary 3.38. *Let E and F be satellite prime divisors such that $\nu_E \leq \nu_F$ holds. Then $\delta(h_E) \leq \delta(h_F)$ for h_E and h_F functions associated to E and F respectively.*

Proposition 3.37 can be seen as a generalization of Alberich and Roe's [1, Corollary 2.4], which only deals with classical adjacencies.

4. INCLUSIONS OF MAXIMAL DIVISORIAL SETS IN THE ARC SPACE OF \mathbb{C}^2

In this section we prove Theorem 1.1 of the introduction and we derive some consequences of it. Before entering in the core of the proof, we need to recall a few known results about wedges and adjacencies and to introduce *dicritical* wedges.

4.1. Nash adjacency for (E, F) is a combinatorial property of the pair (E, F) .

Let (X, O) be any normal surface singularity. In this paper we talk only about the case $X = \mathbb{C}^2$, but some of the results we are using were proved in the possibly singular setting, therefore we state them in that generality.

Definition 4.1. An *arc through the origin* $O \in X$ is a morphism

$$\text{Spec}(\mathbb{C}[[t]]) \rightarrow X$$

sending the special point to the origin O . We denote by \mathcal{X}_∞ the space of arcs through the origin of X . An arc is said to be *convergent* if the power series that define it are convergent.

The space of arcs \mathcal{X}_∞ may be endowed canonically with a structure of affine scheme over X , as inverse limit of jet spaces at all truncation orders.

We will only consider arcs through the origin, so an *arc* will be always through the origin.

As when (X, O) is smooth, we say that S is a *model* of (X, O) if:

$$\pi : (S, E) \rightarrow (X, O)$$

is a proper birational map and S is smooth. The divisor E is the preimage of O and is called the *exceptional divisor*. Let $E = \cup_{k=1}^r E_k$ be the decomposition of E into irreducible components. As in Section 2, we can talk about the *minimal model* in which a divisor D appears.

Given any model S of X , by the valuative criterion of properness, any non-constant arc γ admits a unique lifting:

$$\tilde{\gamma} : \text{Spec}(\mathbb{C}[[t]]) \rightarrow S.$$

Of course, if γ is convergent, so is $\tilde{\gamma}$.

Definition 4.2. An arc is said *to have a transverse lifting* to S if its lifting $\tilde{\gamma}$ to S meets only one irreducible component E_k transversely at a smooth point of E . An arc is said *to have a transverse lifting by a divisor F* if its lifting to the minimal model of F meets F transversally at a smooth point of the exceptional divisor.

Given a prime divisor E_k and any model $(\tilde{X}, E) \rightarrow (X, O)$ where it appears, we can consider the following subsets of the space of arcs:

$$N_{E_k} = \{\gamma \in \mathcal{X}_\infty : \tilde{\gamma}(0) \in E_k\},$$

$$\dot{N}_{E_k} = \{\gamma \in N_{E_k} : \tilde{\gamma} \text{ is transverse to } E_k\}.$$

The Zariski closure \overline{N}_{E_k} of N_{E_k} in \mathcal{X}_∞ is called the *maximal divisorial set associated with E_k* and does not depend on the model containing E_k which is considered.

These sets were introduced by Nash in [21]. We collect their basic properties in the following lemma:

Lemma 4.3. (see [21] and [14] for more details) *We have the following statements:*

- (1) *The sets N_E and \overline{N}_E are irreducible sets of \mathcal{X}_∞ .*
- (2) *The subset \dot{N}_E is open and dense in N_E , therefore \overline{N}_E is also the closure of \dot{N}_E .*
- (3) *The sets \dot{N}_E are cylindrical (there exists an $m \in \mathbb{N}$ such that they are the preimages of constructible sets of m -jets by the m -truncation map).*
- (4) *The sets \overline{N}_E are of finite codimension.*
- (5) *If (X, O) is smooth, then the sets \overline{N}_E are cylindrical.*
- (6) *If $E \neq F$ then $\overline{N}_E \neq \overline{N}_F$.*

Proof. Only (5) is not directly contained in the literature. The sets \overline{N}_E are closures of cylindrical sets, and since (X, O) is smooth they are also cylindrical. \square

Definition 4.4. Given two prime divisors E and F , a *Nash-adjacency* is an inclusion $\overline{N}_F \subset \overline{N}_E$. This defines a partial order $E \leq_N F$ among prime divisors.

Note that in previous articles (see [14], [22]), what we denote here as Nash-adjacencies were denoted there by *adjacencies*.

Observe that a Nash-adjacency $\overline{N}_F \subsetneq \overline{N}_E$ implies that the topological type of curves associated to F is adjacent (in the sense of section 3.4) to the one of curves associated to E (in particular, $E \leq_\nu F$).

We will study inclusions of maximal divisorial sets via a characterization using *wedges*, which are uni-parametric families of arcs:

Definition 4.5. A *wedge* is a morphism:

$$\alpha : \text{Spec}(\mathbb{C}[[t, s]]) \rightarrow X$$

such that the closed subset $V(t)$ of $\text{Spec}(\mathbb{C}[[t, s]])$ is sent to the origin $O \in X$.

A wedge is said to be *convergent* if the power series that define it are convergent. In this case we define $\alpha_s := \alpha|_{\mathbb{C} \times \{s\}}$. It is known that we can choose a small enough representative Λ of $(\mathbb{C}, 0)$ such that for $s \in \Lambda \setminus \{0\}$, the topological type of the curve parametrized by α_s is constant. We say that α_0 and α_s with $s \in \Lambda \setminus \{0\}$ are respectively *the special arc* and *the generic arcs* of the wedge.

A wedge is said to be *algebraic* if for every i , the series $\alpha_i(s, t)$ that define it are algebraic power series, that is, for any i there exists a polynomial $G_i(t, s, x)$ such that $G_i(t, s, \alpha_i(s, t)) = 0$.

More precisely, we are interested in the following wedges (compare with the last paragraph of Definition 3.29):

Definition 4.6. A *wedge realizing a Nash-adjacency* $\overline{N}_F \subsetneq \overline{N}_E$ is a wedge α such that α_0 is in \dot{N}_F and α_s is in N_E for $s \neq 0$. Moreover:

- (1) we say that α is *dicritical* if and only if E is dicritical for the wedge, that is, if and only if:

$$\alpha_s : \text{Spec}(\mathbb{C}((s))[[t]]) \rightarrow X$$

sends the special point to the generic point of E . In the convergent case, this may be interpreted as the fact that $\tilde{\alpha}_s(0)$ moves along the divisor E , when the parameter s varies.

- (2) we say that α *strictly realises the Nash-adjacency* if and only if α_s lifts transversally to E for all $s \neq 0$ small enough.

Observe that a wedge realising a Nash-adjacency neither has to be dicritical nor has to strictly realize an adjacency. There is also no implication between being dicritical and strictly realizing an adjacency.

The following theorem was fundamental to prove the Nash Conjecture in [15] and will be used in the proof of Theorem 1.1. It was originally stated for any 2 divisors appearing in the minimal resolution of a singular surface germ but it works in general for any two divisors appearing in a model (which is always

the case of a pair of divisors) It follows from the results in [29] and [14] and can be found in [14, Thm. A, Prop. 17]:

Theorem 4.7. (see [14]) *Let E and F be two prime divisors over $O \in X$. Then, the following assertions are equivalent:*

- (1) *one has the Nash adjacency $\overline{N}_F \subsetneq \overline{N}_E$;*
- (2) *there exists a wedge α realising the Nash-adjacency $\overline{N}_F \subsetneq \overline{N}_E$;*
- (3) *for any convergent $\gamma \in \dot{N}_F$, there exists an algebraic wedge α with $\alpha_0 = \gamma$, strictly realizing the Nash-adjacency $\overline{N}_F \subsetneq \overline{N}_E$.*

Remark 4.8. *In the proof of (1) \Rightarrow (2), a version of the Curve Selection Lemma from [29] was used. For the case of \mathbb{C}^2 we are interested in, we can substitute it by a simpler proof since the arc space of \mathbb{C}^2 is an infinite dimensional affine space.*

The following new lemma will be used:

Lemma 4.9. *Let α be a wedge realising a Nash adjacency $\overline{N}_F \subsetneq \overline{N}_E$. Then there exists $E' \geq_d E$ such that α realizes the Nash-adjacency $\overline{N}_F \subsetneq \overline{N}_{E'}$ and is dicritical for E' .*

Besides, if α strictly realizes the Nash-adjacency $\overline{N}_F \subsetneq \overline{N}_E$, then it also strictly realizes the Nash-adjacency $\overline{N}_F \subsetneq \overline{N}_{E'}$.

Proof. Consider the minimal model of E . If the wedge is not dicritical we can assume that there is a point $p \in E$ where all the generic arcs $(\alpha_s)_{s \neq 0}$ meet. We blow-up p and keep blowing up points $p_1 \in E_p$, p_2, \dots, p_n if the liftings of the generic arcs still meet at $p_i \in E_{p_{i-1}}$. After a finite number of blow ups, the intersection multiplicity $I_{p_i}(\tilde{\alpha}_{s_1}, \tilde{\alpha}_{s_2})$ goes down and then after another finite number of blow ups we make the wedge dicritical. \square

We prove now Theorem 1.1 of the introduction, that is, that the existence of an inclusion $\overline{N}_F \subsetneq \overline{N}_E$ only depends on the combinatorics of the pair (E, F) . It is based on a similar result in [14] (Theorem D) and we quote [14] for technical aspects that are fully explained there. Using Theorem 4.7, we see that it is equivalent to the following:

Theorem 4.10. *Let (X, O) be smooth. Assume that there exists a wedge α realizing the Nash-adjacency $\overline{N}_F \subsetneq \overline{N}_E$. If $(E', F') \equiv (E, F)$, then there exists a wedge β realizing the Nash-adjacency $\overline{N}_{F'} \subsetneq \overline{N}_{E'}$.*

Proof. Identify (X, O) with $(\mathbb{C}^2, 0)$ by a choice of local coordinates.

The theorem is trivial if $E \leq_d F$, because in this case F' also dominates E' , which implies that $\overline{N}_{F'} \subsetneq \overline{N}_{E'}$. Therefore, we can assume that this is not the case.

Using point (3) of Theorem 4.7, we can assume that $\alpha : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ is an algebraic wedge that strictly realizes the adjacency for (E, F) .

We can reduce it to the case of α a dicritical wedge as follows. If α is not dicritical for (E, F) , following Lemma 4.9, we can take $E_2 >_d E$ such that it is dicritical for (E_2, F) . Now, we take $E'_2 > E'$ such that $(E'_2, F') \equiv (E_2, F)$. If we can find β realizing the adjacency for (E'_2, F') , then it results from the domination $E'_2 >_d E'$ that β also realizes the adjacency for (E', F') .

So, let us assume that α is dicritical.

Let:

$$\pi_{E,F} : S_{E,F} \rightarrow (\mathbb{C}^2, 0)$$

be the minimal model of $E + F$. Then, since $E \not\leq_d F$, we have that the liftings to $S_{E,F}$ of α_0 and α_s for small enough $s \neq 0$ hit F and E respectively.

By Proposition 22 in [14] there is a smooth algebraic surface Σ , a proper morphism:

$$\Phi : \Sigma \rightarrow \mathbb{C}^2$$

and an embedding:

$$i : (\mathbb{C}^2, 0) \rightarrow \Sigma$$

such that the following diagram commutes:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\Phi} & (\mathbb{C}^2, 0) \\ \uparrow i & \nearrow \alpha & \\ (\mathbb{C}^2, 0) & & \end{array} \quad (19)$$

In other words, the wedge α is the germ of the mapping Φ around a point $p := i(0) \in \Sigma$ and we can think about the coordinates (s, t) as local coordinates around p . Note that the map Φ is not necessarily a model above $(\mathbb{C}^2, 0)$, as it is not always bimeromorphic (it may have generic degree ≥ 2). We denote by L_s and L_t the images by i of the s -axis (defined by $t = 0$) and the t -axis of $(\mathbb{C}^2)_{(s,t)}$ respectively. We denote by Ω a small ball around p . The arc α_0 is identified with $\Phi|_{L_t}$. Its lifting to $S_{E,F}$ is denoted by $\tilde{\alpha}_0$.

Let U be a Milnor ball for the image of α_0 around the origin of \mathbb{C}^2 . Consider $\tilde{U} := \pi_{E,F}^{-1}(U)$, which is a small tubular neighborhood at the exceptional divisor of $\pi_{E,F}$.

Define $V := \Phi^{-1}(U)$. We assume that Ω was chosen such that $\Omega \subset V$. Consider the fiber product $\bar{V} := V \times_U \tilde{U}$. Denote by σ_1, σ_2 its projections to the each of the factors. Let $\rho : \tilde{V} \rightarrow \bar{V}$ be the minimal resolution of \bar{V} . Then $\tilde{\Phi} := \sigma_2 \circ \rho$ is a proper map:

$$\begin{array}{ccccc} \tilde{V} & & & & \\ \rho \searrow & & \tilde{\Phi} \searrow & & \\ & \bar{V} & \xrightarrow{\sigma_2} & \tilde{U} \subset S_{E,F} & \\ \kappa \searrow & \downarrow \sigma_1 & & \downarrow \pi_{E,F} & \\ & V & \xrightarrow{\Phi} & U \subset \mathbb{C}^2 & \\ & \uparrow i & \nearrow \alpha & & \\ & (\mathbb{C}^2, 0) & & & \end{array} \quad (20)$$

We denote by \tilde{L}_s and \tilde{L}_t the strict transform by κ of L_s and L_t in \tilde{V} .

Denote $A := \tilde{\Phi}^{-1}(\pi_{E,F}^{-1}(0))$ and consider its decomposition into irreducible components:

$$A = \cup_i A_i.$$

Consider the total transform of $L_s + L_t$ in \tilde{V} by κ , and its decomposition into irreducible components. Then:

- (1) there is a unique non-compact irreducible component that contains \tilde{L}_t ;
- (2) we have that $\tilde{\alpha}_0 \circ \kappa|_{\tilde{L}_t} = \tilde{\Phi}|_{\tilde{L}_t}$;
- (3) the rest of components are contained in A (and are compact);
- (4) \tilde{L}_t is transverse to a compact component;
- (5) there is only one compact component that contains \tilde{L}_s ; we denote it by A_s ;
- (6) since α is a dicritical wedge, A_s is dicritical for $\tilde{\Phi}$ and $\tilde{\Phi}(A_s) = E$.

Take the Stein factorization of $\tilde{\Phi}$:

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\theta} & W \\ & \searrow \tilde{\Phi} & \downarrow \Psi \\ & & \tilde{U} \end{array} \quad (21)$$

where θ is proper and birational, W normal and $\Psi : W \rightarrow \tilde{U}$ finite. Then Ψ is a branched cover.

Let $R \subset W$ be the *ramification locus* of Ψ .

We have the following:

- (1) a divisor A_i is dicritical for $\tilde{\Phi}$ if and only if A_i is not collapsed by θ ;
- (2) the divisor A_s (that contains $\tilde{L}_s \subset \tilde{V}$) is dicritical for $\tilde{\Phi}$, thus, it is not collapsed by θ and $\theta(A_s)$ is a divisor in W ;
- (3) since the lifting $\tilde{\alpha}_s$ of the arc α_s is transverse to E , the curve $\theta(A_s)$ is not contained in the ramification locus, that is $\theta(A_s) \not\subseteq R$.

Define the *branching locus* $\Delta := \Psi(R)$. We decompose $\Delta = \Delta^{exc} \cup \Delta'$, where Δ^{exc} is the part collapsed by $\pi_{E,F}$, that is *the exceptional part*.

Let $\pi_{E',F'} : S_{E',F'} \rightarrow \mathbb{C}^2$ be the minimal model of $E' + F'$. Then, $\pi_{E',F'}^{-1}(0)$ has the same dual graph as $\pi_{E,F}^{-1}(0)$. Let \tilde{U}' be a tubular neighbourhood of $\pi_{E',F'}^{-1}(0)$.

As in [14, Section 6, Proof of Prop. 25 Part I or Proof of Prop. 32], we construct a diffeomorphism:

$$\tau : \tilde{U} \rightarrow \tilde{U}' \subset S_{E',F'}$$

such that:

- (i) $\tau(E) = E'$, $\tau(F) = F'$, $\tau(\pi_{E,F}^{-1}(0)) = \pi_{E',F'}^{-1}(0)$
- (ii) τ is a biholomorphism in neighbourhoods of the singularities of $\pi_{E,F}^{-1}(0)$, the points where Δ' meet $\pi_{E,F}^{-1}(0)$ and the point where $\tilde{\alpha}_0$ meets F .

Note that (ii) implies that $\beta_0 := \pi_{E',F'} \circ \tau \circ \tilde{\alpha}_0$ is a holomorphic arc whose lifting to \tilde{U}' is transversal to F' and equals $\tau \circ \tilde{\alpha}_0$. We denote the lifting by $\tilde{\beta}_0$ (i.e. $\tilde{\beta}_0 = \tau \circ \tilde{\alpha}_0$).

Define the composition:

$$\lambda := \tau \circ \Psi.$$

The morphism λ is unramified over $\tilde{U}' \setminus \tau(\Delta)$ where $\tau(\Delta)$ is a complex analytic curve in \tilde{U}' , since τ satisfies (ii).

Then, using a theorem of Grauert and Remmert (see [17], [18] or [4]), as in the proof of [14, Lemma 28], we deduce the existence of a unique normal holomorphic structure in W which makes λ a complex analytic morphism. We denote by W' the variety W with this new complex structure, by $\bar{\tau}$ the identity diffeomorphism (as topological spaces) from W to W' and by Ψ' the complex analytic morphism from W' to \tilde{U}' given by λ (or more precisely $\lambda \circ \bar{\tau}^{-1}$):

$$\begin{array}{ccc} W & \xrightarrow{\bar{\tau}} & W' \\ \downarrow \Psi & \searrow \lambda & \downarrow \Psi' \\ \tilde{U} & \xrightarrow{\tau} & \tilde{U}' \end{array} \quad (22)$$

In the left handside of diagram (24), one can find the diagrams (20)-(21) that we have associated with the wedge α . Diagram (22) appears in the center. In the rest of the proof we will complete the right handside of diagram (24).

Denote $U' := \pi_{E',F'}(\tilde{U}')$.

First observe that since \tilde{V} is smooth, the mapping $\theta : \tilde{V} \rightarrow W$ can be factorised as:

$$\begin{array}{ccc} & & \tilde{W} \\ & \nearrow \nu & \downarrow \mu \\ \tilde{V} & \xrightarrow{\theta} & W \end{array} \quad (23)$$

where μ is the minimal resolution of W and ν is a sequence of blow ups of points.

Let $\mu' : \tilde{W}' \rightarrow W'$ be the minimal resolution of W' . Since τ is a homeomorphism and the combinatorics of the minimal resolution of a surface singularity only depends on the topology of the link, the dual graphs of the divisors $\mu^{-1}(\pi_{E,F}^{-1}(0))$ and $\mu'^{-1}(\pi_{E',F'}^{-1}(0))$ are the same.

We define \tilde{V}' as the space obtained from \tilde{W}' by a sequence of blow ups $\nu' : \tilde{V}' \rightarrow \tilde{W}'$ such that the dual graph of $B' := \nu'^{-1}(\mu'^{-1}(\pi_{E',F'}^{-1}(0)))$ is isomorphic to the dual graph of $B := \nu^{-1}(\mu^{-1}(\pi_{E,F}^{-1}(0)))$.

Define $\theta' := \nu' \circ \mu'$ and $\tilde{\Phi}' := \Psi' \circ \theta'$.

$$\begin{array}{ccccccc} & & \bar{\tau} & & & & \\ & \nearrow \theta & & \xrightarrow{\tau} & & \nwarrow \theta' & \\ \tilde{V} & & W & & W' & & \tilde{V}' \\ & \searrow \rho & \downarrow \Psi & \searrow \lambda & \downarrow \Psi' & \nearrow \tilde{\Phi}' & \\ & \tilde{V} & \tilde{U} \subset S_{E,F} & \xrightarrow{\tau} & \tilde{U}' & & \\ & \downarrow \kappa & \downarrow \pi_{E,F} & & \downarrow \pi_{E',F'} & \nearrow \xi_1 & \\ & V & U \subset \mathbb{C}^2 & & U' & & V' \\ & & \xrightarrow{\Phi} & & \xleftarrow{\Phi'} & & \\ & & & & v & & \end{array} \quad (24)$$

Notice that the pairs of complex spaces:

$$(W, \Psi^{-1}(\pi_{E,F}^{-1}(0) \cup \tilde{\alpha}_0))$$

$$(W', \Psi'^{-1}(\pi_{E',F'}^{-1}(0) \cup \tilde{\beta}_0))$$

are diffeomorphic because τ sends $\tilde{\alpha}_0$ to $\tilde{\beta}_0$. Hence the pairs:

$$(\tilde{V}, \tilde{\Phi}^{-1}(\pi_{E,F}^{-1}(0) \cup \tilde{\alpha}_0)) \quad (25)$$

$$(\tilde{V}', \tilde{\Phi}'^{-1}(\pi_{E',F'}^{-1}(0) \cup \tilde{\beta}_0)) \quad (26)$$

are diffeomorphic. To construct a diffeomorphism $\bar{\tau}$ from \tilde{V} to \tilde{V}' that makes the diagram (24) commutative, we first lift τ to a diffeomorphism of the minimal resolutions of W and W' obtaining $\bar{\tau}' : \tilde{W} \rightarrow \tilde{W}'$. Now, recall that \tilde{V} and \tilde{V}' are obtained by a sequence of blow ups with the same combinatorics from \tilde{W} and \tilde{W}' . Then, we lift $\bar{\tau}'$ step by step following this sequence of the blow ups, in both origin and target, starting from \tilde{W} and \tilde{W}' respectively. It is clear we can lift it for a single blow up, so finally we can find a diffeomorphism $\bar{\tau} : \tilde{V} \rightarrow \tilde{V}'$ such that the diagram commutes.

In particular, a divisor is collapsed by $\tilde{\Phi}$ if and only if its image by $\bar{\tau}$ is collapsed by $\tilde{\Phi}'$. Also a divisor D_1 in \tilde{V} is dicritical onto some divisor D_2 in $\pi_{E,F}^{-1}(0)$ if and only if $\bar{\tau}(D_1)$ is dicritical onto $\tau(D_2)$ in $\pi_{E',F'}^{-1}(0)$.

Let C be the divisor that is collapsed by $\kappa : \tilde{V} \rightarrow V$ and let C' be its image by $\bar{\tau}$. Then, since $\bar{\tau}$ is a diffeomorphism, we can collapse $C' \subset \tilde{V}'$ and obtain a smooth surface V' . We call this map $\kappa' : \tilde{V}' \rightarrow V'$.

Observe that a divisor in \tilde{V}' is collapsed by κ' if and only if it is the image by $\bar{\tau}$ of a divisor in \tilde{V} that is collapsed by κ respectively.

Let L'_t and L'_s be the images of \tilde{L}_s and \tilde{L}_t by $\kappa' \circ \bar{\tau}$. The union meets in a normal crossing in V' because they are not contracted by κ' , $\bar{\tau}$ is a diffeomorphism and the smoothness and transversality depend only on the combinatorics of the divisor C' that we collapse (that is the same as the one of C).

Applying Stein factorization to the composition $\pi_{E',F'} \circ \tilde{\phi}' : \tilde{V}' \rightarrow U'$ we obtain a normal space T , a proper and birational mapping $\xi_1 : \tilde{V}' \rightarrow T$ and a finite mapping $\xi_2 : T \rightarrow U'$.

The mapping $\tilde{V}' \rightarrow T$ is the result of collapsing the divisor $\tilde{\Phi}'^{-1}(\pi_{E',F'}^{-1}(0))$. Since $\kappa' : \tilde{V}' \rightarrow V'$ collapses C' which is a union of components of $\tilde{\Phi}'^{-1}(\pi_{E',F'}^{-1}(0))$, then there is a morphism $v : V' \rightarrow T$.

Define $\Phi' = \xi_2 \circ v$ and let β be the germ of Φ' at the point $p' := L'_s \cap L'_t$. Choose coordinates (t, s) around p' so that the t -axis is L'_t and the s -axis is L'_s .

We see that β is a dicritical wedge realizing the Nash-adjacency for (E', F') . We think about β as being defined in $\Omega' := \kappa'(\bar{\tau}(\kappa^{-1}(\Omega)))$.

Indeed, by construction we have that:

- The curve L'_t lifts to \tilde{U}' to the lifting of β_0 , that is, to $\tau \circ \tilde{\alpha}_0$ which is transversal to F' at a smooth point of the exceptional divisor.
- The curve L'_s lifts to \tilde{V}' by κ' to $\bar{\tau}(A_s)$, which is transformed by $\tilde{\Phi}'$ to E' in a dicritical way. Thus the special point of the lifting of the generic arc moves along E' when s moves.
- The lifting $\tilde{\beta}_s$ meets E' transversely because $\tilde{\Phi}'$ is unramified at \tilde{L}'_s (this is because $\tilde{\Phi}$ is unramified at $L_s \subset A_s$ and $\tilde{\Phi}'$ is obtained from $\tilde{\Phi}$ by composition with a diffeomorphism).

□

4.2. Consequences of Theorem 4.10 and some conjectures.

In this section, we use Theorem 4.10 to look for necessary or sufficient conditions for the existence of Nash-adjacencies.

Let E, E' be prime divisors. We write $E \equiv_{\geq i} E'$ to say that E and E' have the same combinatorics and that their contact order is greater or equal than i .

We denote by $\overline{\text{Cont}^q(E)}$ the closure of the set of arcs whose lifting to the minimal model of E meet E with intersection multiplicity q . Following the terminology of [10] or [19], it is the *maximal divisorial set* associated with the valuation $q \cdot \nu_E$.

Lemma 4.11. *Given a prime divisor E , for any i , the intersection:*

$$\bigcap_{E' \equiv_{\geq i} E} \overline{N}_{E'}$$

is a closed cylindrical set. Moreover, there exists a finite set of divisors $\{E_j\}_{j \in J}$ satisfying $E_j \equiv_{\geq i} E$ for $j \in J$ such that:

$$\bigcap_{E' \equiv_{\geq i} E} \overline{N}_{E'} = \bigcap_{j \in J} \overline{N}_{E_j}.$$

Proof. The sets $\overline{N}_{E'}$ are closed and cylindrical for the same order of truncation. Then, the intersection occurs at the level of truncation, that is, in a finite dimensional set, where the intersection of an infinite number of algebraic sets is the intersection of only a finite number of them. □

Corollary 4.12. *Let (E, F) be a pair of prime divisors such that we have the Nash adjacency $N_F \subseteq N_E$. If (E', F') is a pair of divisors with E' and F' combinatorially equivalent to E and F respectively and $\text{Cont}(E', F') \geq \text{Cont}(E, F)$, then we have the Nash adjacency $N_{F'} \subseteq N_{E'}$.*

Proof. Let $i + 1$ be the contact order between E and F . Let p_i be the last point in common on the process of finding the minimal model for E and F . The corollary does not say anything unless the next

points in both resolutions are free, because only in this case there exist other divisors E' and F' with the same combinatorics as E and F respectively, but with strictly higher contact than E and F .

By Lemma 4.11, there is a finite set J such that:

$$\bigcap_{j \in J} \overline{N}_{E_j} = \bigcap_{E'' \equiv_{\geq i} E} \overline{N}_{E''}.$$

By Theorem 4.10, if a divisor F'' is such that (E_j, F'') is combinatorially equivalent to (E, F) for any $j \in J$, then we have the inclusion:

$$\overline{N}_{F''} \subset \bigcap_{j \in J} \overline{N}_{E_j}.$$

We deduce the inclusion:

$$\bigcup_{(E_j, F'') \equiv (E, F) \text{ for all } j \in J} \overline{N}_{F''} \subset \bigcap_{E'' \equiv_{\geq i} E} \overline{N}_{E''}. \quad (27)$$

Now, take F'' with $F'' \equiv_{\geq i} F$ and $(E_j, F'') \equiv (E, F)$ for all $j \in J$. Choose E'' with $E'' \equiv_{\geq i} E$ and such that $(E'', F'') \equiv (E', F')$. By the previous inclusion, we have that F'' is upper Nash-adjacent to E'' . Therefore, by Theorem 4.10, F' is upper Nash-adjacent to E' . \square

Corollary 4.13. *If E and F have contact order i , and $\overline{N}_F \subsetneq \overline{N}_E$, then we have the following inclusions:*

$$\overline{N}_F \subsetneq \bigcup_{F' \equiv_{\geq i} F} \overline{N}_{F'} \subset \bigcap_{E' \equiv_{\geq i} E} \overline{N}_{E'} \subsetneq \overline{N}_E$$

Proof. We have to show that $\overline{N}_{F'} \subsetneq \overline{N}_{E'}$ for any $F' \equiv_{\geq i} F$ and $E' \equiv_{\geq i} E$. Consider such an F' . Choose E'' such that $E'' \equiv_{\geq i} E$ and that (E'', F') is combinatorially equivalent to (E, F) . By Theorem 4.10, we have $\overline{N}_{F'} \subsetneq \overline{N}_{E''}$. Since the contact order of E' and F' is at least i when $E' \equiv_{\geq i} E$, by the previous corollary we have $\overline{N}_{F'} \subsetneq \overline{N}_{E'}$ for any $E' \equiv_{\geq i} E$. \square

We state now two conjectures concerning maximal divisorial sets.

Conjecture 1. Let E be a prime divisor. Consider the infinitely near points $\{p_i\}_{i \in I}$ which are blown-up in order to obtain the minimal model of E . Let p_{i_0} be any free point among them. We conjecture that the set $\bigcap_{E' \equiv_{i_0} E} \overline{N}_{E'}$ coincides with $\overline{\text{Cont}^q(E_{p_{i_0}})}$ for certain q . In case the conjecture is true, it would be interesting to compute q .

Remark 4.14. *For the case $i_0 = 1$, the previous conjecture becomes:*

$$\bigcap_{E' \equiv_{\geq 1} E} \overline{N}_{E'} = \overline{\text{Cont}^M(E_0)}$$

for a certain M , where E_0 is the divisor that appears after blowing up the origin of \mathbb{C}^2 . By the combinatorial nature of Nash adjacency, M must depend only on the Puiseux exponents of a curve $V(h_E)$ associated with E . In the case of one Puiseux pair, Ishii's Lemma 3.11 [19] implies the equality $M = \beta_1$. This comes from the fact that the divisorial valuation associated to the last divisor of the minimal embedded resolution of a branch with 1 Puiseux pair is a toric valuation relative to conveniently chosen coordinates. But this equality is false in general. Until now, we only have been able to prove that M is bounded from above by $\beta_k - 1$.

Conjecture 2. Assume $\text{Cont}(E, F) = i$ and that p_{i+1}^E and p_{i+1}^F are both free points. Consider:

$$k_F := \max \left\{ m : \bigcup_{F' \equiv_i F} \overline{N}_{F'} \subsetneq \overline{\text{Cont}^m E_{p_i}} \right\},$$

$$k_E := \min \left\{ m : \overline{\text{Cont}^m E_{p_i}} \subsetneq \bigcap_{E' \equiv_{\geq i} E} \overline{N}_{E'} \right\}.$$

We conjecture that $\overline{N}_F \subsetneq \overline{N}_E$ if and only if $k_F \geq k_E$. It would also be interesting to compute k_E and k_F in terms of combinatorial invariants of E and F .

The previous conjectures are motivated by our feeling that solving the generalized Nash problem as stated in the introduction seems very close to describing the inclusions among the maximal divisorial sets $\overline{\text{Cont}^q(E)}$.

Definition 4.15. Given a prime divisor E , its *log-discrepancy* $a_E(\mathbb{C}^2)$ is the positive integer

$$1 + \nu_E(dx \wedge dy).$$

Here $\nu_E(dx \wedge dy)$ denotes the order of vanishing along E of the pull-back of the holomorphic 2-form $dx \wedge dy$ to a model in which E appears.

One has the following well-known rules of computation of the log-discrepancy, whose proofs are direct:

- If E_0 is the divisor that appears after the blow up of the origin of \mathbb{C}^2 , then $a_{E_0}(\mathbb{C}^2) = 2$.
- If E is *satellite* and appears after blowing up $p \in E_1 \cap E_2$, then $a_E(\mathbb{C}^2) = a_{E_1}(\mathbb{C}^2) + a_{E_2}(\mathbb{C}^2)$.
- If E is *free* right after E_1 then $a_E(\mathbb{C}^2) = a_{E_1}(\mathbb{C}^2) + 1$.

As a special case of the result [10, Theorem 2.1] of Ein, Lazarsfeld and Mustață (see also [11, Theorem 6.2]), we have:

Theorem 4.16. *The codimension $\text{codim}(\overline{N}_E)$ of \overline{N}_E in the arc space of \mathbb{C}^2 is equal to the log-discrepancy $a_E(\mathbb{C}^2)$.*

Observe that our subsets \overline{N}_E are always contained in the set of arcs centered at the origin, which is a set of codimension 2 in the whole arc space of \mathbb{C}^2 . We will use now the previous theorem in combination with Theorem 4.10 in order to improve the inequalities between log-discrepancies (see Corollary 4.19 below).

As an immediate consequence of Corollary 4.13, we get:

Corollary 4.17. *If $\text{Cont}(E, F) = i$, and $\overline{N}_F \subsetneq \overline{N}_E$, then:*

$$\text{codim} \left(\bigcup_{F' \equiv_{\geq i} F} \overline{N}_{F'} \right) \geq \text{codim} \left(\bigcap_{E' \equiv_{\geq i} E} \overline{N}_{E'} \right).$$

Lemma 4.18. *Let D be a prime divisor that appears after blowing up $n + 1$ points $\{x_i\}_{i=0}^n$. Fix $i \in \{0, \dots, n-1\}$. Let k be the number of free points x_j with $i \leq j \leq n$. Then:*

(1) *if $k \geq 1$, then we have:*

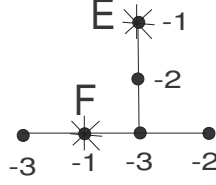
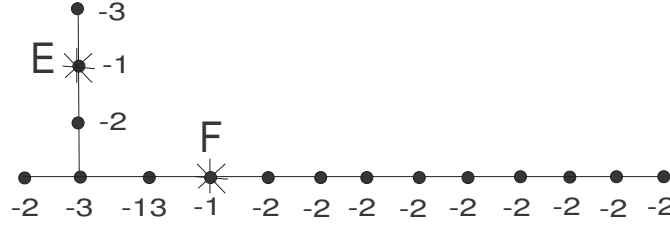
$$\text{codim} \left(\bigcap_{D' \equiv_{\geq i} D} \overline{N}_{D'} \right) \geq \text{codim}(\overline{N}_D) + 1.$$

(2) *we have:*

$$\text{codim} \left(\bigcup_{D' \equiv_{\geq i} D} \overline{N}_{D'} \right) = \text{codim}(\overline{N}_D) - k,$$

Proof. Part (1) is obvious since $k \geq 1$ implies that there are at least two different subsets $\overline{N}_{D'}$ intersecting.

For part (2), notice that $\bigcup_{D' \equiv_{\geq i} D} \overline{N}_{D'}$ fibers over the variety parametrising the possible positions of the free points x_j with $i \leq j \leq n$. This variety has dimension k and its fibre over a point is $\overline{N}_{D'}$ for a certain D' satisfying $D' \equiv_{\geq i} D$. \square

FIGURE 5. Dual graph of the exceptional set of the minimal model of $E + F$ in Ishii's example.FIGURE 6. Dual graph of the exceptional set of the minimal model of $E + F$ in our example.

Corollary 4.19. *Assume that $\overline{N}_F \subsetneq \overline{N}_E$. Let $i = \text{Cont}(E, F)$. Let k be the number of free points after x_i (and including it) that we blow up to obtain the minimal model of F . Let h be equal to 1 if there is at least one free point after x_i in the blowing up process leading to the minimal model of E and be equal to 0 otherwise. Then the following inequality holds:*

$$a_E(\mathbb{C}^2) < a_F(\mathbb{C}^2) - k - h. \quad (28)$$

Proof. This is a direct application of the previous lemma. \square

4.3. An Example.

Plénat showed in [23, 2.2] that a Nash adjacency $\overline{N}_F \subseteq \overline{N}_E$ implies the valuative inequality $\nu_E \leq \nu_F$.

In [19], Ishii gave an example of a pair (E, F) where the valuative inequality $\nu_E \leq \nu_F$ holds, but $\overline{N}_F \not\subseteq \overline{N}_E$. The example was for E and F as in Figure 5. Since $a_E(\mathbb{C}^2) = 9$ and $a_F(\mathbb{C}^2) = 8$, we can use instead Corollary 4.19 to see directly that the Nash-adjacency $\overline{N}_F \subseteq \overline{N}_E$ does not hold.

Observe that in Ishii's example, the divisor E is not a satellite divisor (that is, it is not the final divisor of the minimal embedded resolution of a plane branch). One could still ask oneself whether the valuative inequality implies the Nash-adjacency for satellite divisors. In the following example we will see that this is not true.

Namely, we exhibit two satellite divisors (E, F) with contact order 1 for which we have $\nu_E \leq \nu_F$, $a_E(\mathbb{C}^2) < a_F(\mathbb{C}^2) - k - h$ as in Corollary 4.19, but $E \not\leq_N F$. That is, $\overline{N}_F \not\subseteq \overline{N}_E$.

The example is the following. Take the curve $V(h_E)$ parametrized by $(t^9, t^6 + t^8)$. The curve has multiplicity sequence $(6, 3, 3, 2, 1, 1)$ and characteristic exponents $(6, 9, 11)$. Consider E with the combinatorial type of the final divisor of the minimal embedded resolution of $V(h_E)$. Take F a divisor whose associated curves have multiplicity sequence $(10, 1, 1, \dots, 1)$. We consider divisors E and F of these combinatorial types and contact order 1 as in Figure 6.

It is easy to check using Proposition 3.8 (4) that $\nu_E \leq \nu_F$. Moreover we have that $a_F(\mathbb{C}^2) = 21$ and $a_E(\mathbb{C}^2) = 17$.

We will obtain $\overline{N}_F \not\subseteq \overline{N}_E$ as a consequence of Proposition 4.22. To prove it we will need Lemma 4.20 below, based on the following observation:

Fact 1. Given a satellite divisor E , all the curves $V(h)$ associated with E have the same Newton-Puiseux expansion with exactly the same coefficients up to the coefficient corresponding to the greatest exponent which varies as the strict transform moves along E .

Lemma 4.20. *Let $\alpha(t, s) = (\alpha_1(s, t), \alpha_2(s, t))$ be a wedge realising a Nash-adjacency $\overline{N}_F \subsetneq \overline{N}_E$ whose generic arc lifts transversely by E . Then there exists $t(u, s) \in \mathbb{C}((s))[[u]]$ such that:*

$$\alpha(t(u, s), s) = (u^{\beta_0}, \sum_{i=\beta_1}^{\beta_k-1} c_i u^i + \lambda(s) u^{\beta_k} + \sum_{i>\beta_k} c_i(s) u^i) \quad (29)$$

where $c_i \in \mathbb{C}$ for $i = \beta_1, \beta_1 + 1, \dots, \beta_k - 1$ and $\lambda(s)$ and $c_i(s)$ for $i > \beta_k$ are meromorphic in s and such that (29) is the Newton-Puiseux expansion for any generic arc $\alpha_{s \neq 0}$.

Proof. We just perform in family the usual procedure to get the Newton-Puiseux expansion. Note that the tangent line of the generic arcs is constant unless the arcs are smooth and $E = E_O$. This case is easy so we assume that the tangent of the generic arc is constantly $y = 0$.

Let $\alpha_1(s, t)$ be equal to $\sum_j a_j(s) t^j$ for some holomorphic germs $a_j(s)$. If s^{β_0} does not divide $a_{\beta_0}(s)$, we do the substitution $s = s^{\beta_0}$. Then, we can write α_1 as follows:

$$\alpha_1(s, t) = s^{\beta_0 R} v(s) t^{\beta_0} \left(1 + \frac{a_{\beta_0+1}(s)}{a_{\beta_0}(s)} t^1 + \dots \right)$$

with $R > 0$, $v(0) \neq 0$. Now we can take the β_0 -root of $\alpha_1(t, s)$ for $s \neq 0$ which will be a function:

$$U(t, s) = t s^R \sqrt[\beta_0]{v(s)} \left(1 + \sum_{i>1} b_i(s) t^i \right)$$

for some $b_i(s)$ possibly meromorphic in s . Observe that since $v(s)$ is holomorphic, so is $\sqrt[\beta_0]{v(s)}$.

Then, we have that:

$$U(t, s)^{\beta_0} = \alpha_1(t, s). \quad (30)$$

Consider the inverse function $T(u, s)$ of $U(t, s)$ in $\mathbb{C}((s))[[t]]$, that is, the unique element in $\mathbb{C}((s))[[t]]$ such that $U(T(u, s), s) = u$. Substituting $t = T(u, s)$ in 30, we have that $\alpha_1(T(u, s), s) = u^{\beta_0}$ for all $s \neq 0$. Then, $\alpha(T(u, s), s)$ is in the Newton-Puiseux form for $s \neq 0$, so it is constant in s up to order $\beta_k - 1$ by the previous fact. \square

Making the change of variables $u = u(t)$ (where $u(t)$ is the inverse function of $t(u)$) we get the equivalent formulation:

Corollary 4.21. *Let E be any prime divisor and let $(u^{\beta_0}, \sum_{i \geq \beta_1}^{\beta_k} c_i u^i + \lambda u^{\beta_k})$, with $c_i, \lambda \in \mathbb{C}$, be the Newton-Puiseux expansion of an associated curve $V(h_E)$. Any wedge α with generic arc in \dot{N}_E is obtained from an expression of the form:*

$$\left(u^{\beta_0}, \sum_{i \geq \beta_1}^{\beta_k-1} c_i u^i + \lambda(s) u^{\beta_k} + \sum_{i>\beta_k} c_i(s) u^i \right)$$

for some $\lambda(s)$ and $c_i(s)$ for $i > \beta_k$ meromorphic in s , by performing a meromorphic change of coordinates $u = u(t)$ with:

$$u(t) = s^R t w(s) \left(1 + \sum_{i \geq 1} b_i(s) t^i \right)$$

for some $R > 0$, some holomorphic function $w(s)$ and some meromorphic functions $b_i(s)$.

As a consequence:

Proposition 4.22. *Let E be a prime divisor such that the characteristic sequence of its associated curves is $(\beta_0, \beta_1, \dots, \beta_k)$ (see Subsection 3.2). Let F be a prime divisor with contact order 1 with E . If $m_O(h_F) = \beta_1 + 1$ for an associated curve $V(h_F)$, then $\overline{N}_F \not\subseteq \overline{N}_E$.*

Proof. Assume we have $\overline{N}_F \subsetneq \overline{N}_E$. Using Theorem 4.10, we know that $\overline{N}_{F'} \subsetneq \overline{N}_{E'}$ for any $(E', F') \equiv (E, F)$. Observe that the characteristic exponents and the multiplicities of curves associated to E and F do not change if we replace (E, F) by a combinatorially equivalent pair (E', F') . Therefore, in order to prove the statement we can assume that the tangent lines to the curves associated with F and E are respectively $x = 0$ and $y = 0$ and that the curves associated with E have Newton-Puiseux expansion:

$$(u^{\beta_0}, u^{\beta_1} + \dots + u^{\beta_{k-1}} + \lambda u^{\beta_k})$$

for some $\lambda \in \mathbb{C}$. Then, if $\overline{N}_F \subsetneq \overline{N}_{E'}$, then there exists a (holomorphic) wedge $\alpha(s, t)$ realising this Nash-adjacency. By the previous corollary, the wedge α can be obtained by a change of variables $u = u(t, s)$ with $u(t, s) = ts^R v(s)(1 + \sum c_i(s)t^i)$ for some $R > 0$, $v(s) \in \mathbb{C}[[s]]$ with $v(0) \neq 0$ and $c_i(s) \in \mathbb{C}((s))$, in:

$$\beta(u, s) = (\beta_1(u, s), \beta_2(u, s)) = (u^{\beta_0}, u^{\beta_1} + \dots + u^{\beta_{k-1}} + \lambda(s)u^{\beta_k} + \dots)$$

for some $\lambda(s) \in \mathbb{C}((s))$.

In particular, this means that $\alpha(t, s) = (\alpha_1(t, s), \alpha_2(t, s)) := \beta(u(t, s), s)$ is a holomorphic function and that $\alpha_2(t, 0)$ equals $\mu t^{m(E)} + \dots$ for some $\mu \in \mathbb{C}^*$.

Now, on the one hand, the coefficient of t^{β_1+1} in $\alpha_2(t, s)$ is:

$$\beta_1 s^{R\beta_1} v(s)^{\beta_1} c_1(s).$$

We know that this expression does not have either poles or zeros at $s = 0$ (recall that $m_O(h_F)$ equals $\beta_1 + 1$). This only happens when $c_1(s)$ has exactly a pole of order $R\beta_1$ at $s = 0$.

On the other hand, the coefficient of t^{β_0+1} in $\alpha_1(t, s)$ is $\beta_0 s^{R\beta_0} v(s)^{\beta_0} c_1(s)$. We want it to be holomorphic, so c_1 cannot have a pole at $s = 0$ of order greater than $R\beta_0$. But this is not compatible with the previous requirement. We can conclude that such a wedge does not exist. \square

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JAVIER FERNÁNDEZ DE BOBADILLA: (1) IKERBASQUE, BASQUE FOUNDATION FOR SCIENCE, MARIA DIAZ DE HARO 3, 48013, BILBAO, BIZKAIA, SPAIN (2) BCAM BASQUE CENTER FOR APPLIED MATHEMATICS, MAZARREDO 14, E48009 BILBAO, BASQUE COUNTRY, SPAIN

E-mail address: javierbobadilla73@gmail.com

MARÍA PE PEREIRA: INSTITUTO DE CIENCIAS MATEMÁTICAS (ICMAT), NICOLÁS CABRERA, 13-15, CAMPUS DE CANTO-BLANCO, UAM, 28049 MADRID, SPAIN.

E-mail address: maria.pe@icmat.es

PATRICK POPESCU-PAMPU: UNIVERSITÉ LILLE 1, UFR DE MATHS., BÂTIMENT M2, CITÉ SCIENTIFIQUE, 59655, VILLENEUVE D'ASCQ CEDEX, FRANCE.

E-mail address: patrick.popescu@math.univ-lille1.fr