SINGULAR LINES OF TRILINEAR FORMS

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ABSTRACT. We prove that an alternating e-form on a vector space over a quasi-algebraically closed field always has a singular $(e-1)$-dimensional subspace, provided that the dimension of the space is strictly greater than e. Here an $(e-1)$-dimensional subspace is called singular if pairing it with the e-form yields zero. By the theorem of Chevalley and Warning our result applies in particular to finite base fields. Our proof is most interesting in the case where $e=3$ and the space has odd dimension n; then it involves a beautiful equivariant map from alternating trilinear forms to polynomials of degree $\frac{n-1}{2}-1$. We also give a sharp upper bound on the dimension of subspaces all of whose 2-dimensional subspaces are singular for a non-degenerate trilinear form. In certain binomial dimensions the trilinear forms attaining this upper bound turn out to form a single orbit under the general linear group, and we classify their singular lines.

1. Introduction and the main theorem

While alternating bilinear forms on an n-dimensional vector space V are very well understood in terms of their ranks and orbits—the forms of rank at most $2k$ form a Zariski-closed set in which those of rank exactly $2k$ form a single orbit for each $k=0,\ldots,\lfloor n/2\rfloor$—*trilinear* and higher alternating multilinear forms on V are much harder to grasp. For instance, being of rank at most k, that is, being expressible as the sum of at most k decomposable alternating forms, is no longer necessarily a closed condition. Even the generic rank of trilinear forms is not known exactly, although tight asymptotic results have recently been obtained [1]. As for orbits, trilinear forms have been classified on spaces of dimension up to seven over arbitrary fields [6, 11], as well as in dimensions 8 over the complex or real numbers [7, 9]. In dimension 8 there are 23 orbits over the complex numbers, and the Hasse diagram of their orbit closures is known explicitly [8]. For trilinear forms on $\mathbb{C}^9$ the number of orbits is infinite, but the invariant ring of the action of SL_9 on them is well understood—in particular, it is free—and this contributes to the classification in [13]. Beyond that, there seems little hope of a full classification.

This paper settles a question, put forward as conjecture A in [12], about the geometry of trilinear forms in arbitrary dimension. To state our main result we introduce some notation and terminology. Write $\langle .,. \rangle : V\times V^* \to K$ for the natural pairing of V with its dual V^* to the ground field K, and $\bigwedge^d V$ for d-th exterior power of V. Recall that for $e\geq d$ there is a natural bilinear map $\bigwedge^d V\times \bigwedge^e(V^*)\to \bigwedge^{e-d}(V^*)$ determined by

$$(v_1\wedge\cdots\wedge v_d, y_1\wedge\cdots\wedge y_e)\mapsto \sum_{\pi:[d]\to[e]} \mathrm{sgn}\,(\pi)\left(\prod_{i=1}^d \langle v_i, y_{\pi(i)}\rangle\right)\bigwedge\nolimits_{j\notin \mathrm{im}(\pi)} y_j.$$

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Here the sum is taken over all injections $\pi : [d] := \{1, \ldots, d\} \to [e]$, of which the sign $\operatorname{sgn} \pi$ is defined to be the sign of the unique extension of π to a permutation $\pi' : [e] \to [e]$ that is strictly increasing on $\{d+1, \ldots, e\}$. Moreover, the last wedge is taking in order of increasing index j. For $d = e = 1$ this pairing reduces to $\langle ., . \rangle$, and we will use the latter notation for general $d \leq e$, as well. Whenever $d = e$ the pairing $\langle ., . \rangle$ is a non-degenerate K-valued pairing, by which we identify $(\bigwedge^e V)^*$ with $\bigwedge^e (V^*)$. Elements of either of these spaces, or of the space of alternating multilinear forms $V^e \to K$, are called *alternating e-forms on V*.

Let ω be an alternating e-form. An element $\lambda \in \bigwedge^d V$ with $d \leq e$ is called *singular for ω* or *ω-singular* if $\langle \lambda, \omega \rangle = 0 \in \bigwedge^{e-d} V^*$. Similarly, a d-dimensional subspace U of V is called singular for ω if $\langle \bigwedge^d U, \omega \rangle = \{0\}$, that is, if the one-dimensional subspace $\bigwedge^d U$ of $\bigwedge^d V$ is spanned by an ω-singular element. More generally, suppose that d, e, f are natural numbers with $f \leq e$. Then a d-dimensional subspace U of V is called *f-singular for ω* if $\bigwedge^f U$ consists entirely of ω-singular elements, or, equivalently, if every f-dimensional subspace of U is ω-singular. For $d < f$ this is automatically true, and for $f = d$ this reduces to the statement that U is ω-singular.

For instance, a vector $v \in V$ is singular for an alternating bilinear form ω if and only if $\omega(v, w) = 0$ for all w, that is, if and only if v lies in the radical of ω. Similarly, a two-dimensional subspace U of V is singular for a trilinear form ω if and only if $\omega(u, u', v) = 0$ for all $u, u' \in U$. In projective terminology, as in [12], such U are called *singular lines*. We will use both projective terminology (point, line) and vector space terminology (one-dimensional subspace, two-dimensional subspace).

Notice that there is some asymmetry in these notions, which we could have avoided by allowing that $e < d$ and by calling a the *pair* $\lambda \in \bigwedge^d V$, $\omega \in \bigwedge^e V^*$ singular. However, in this paper we will be primarily interested in questions of the following flavour: fixing an alternating e-form ω, what can we say about the d-singular subspaces of V for some $d \leq e$? This justifies the present notions.

Theorem 1.1 (Main theorem)**.** *Let K be a quasi-algebraically closed field, that is, every non-constant homogeneous multivariate polynomial of degree less than the number of its variables has a non-zero K-valued root. Let e be an integer with $e \geq 3$, and let V be a vector space over K of dimension at least $e+1$. Then every alternating e-form on V has a singular $(e-1)$-dimensional space.*

The conclusion of the theorem holds in particular for finite fields, which are quasi-algebraically closed by the Theorem of Chevalley and Warning [5, 14]. Note that the statement is false if V has dimension e: an e-form spanning the one-dimensional space $\bigwedge^e V^*$ does not have singular $(e-1)$-spaces. Also, the following construction shows that the statement is, in general, false for trilinear forms over non-quasi-algebraically closed fields.

Example 1.2. Consider a real Euclidean space E of dimension 7 and inner product denoted by $\cdot$. It is known, see [3], that there exist vector cross products $a \times b \in V$ which are bilinear and which satisfy the axioms

$$a \times b \cdot a = 0, \qquad a \times b \cdot b = 0, \tag{1}$$

$$a \times b \cdot a \times b = (a \cdot a)(b \cdot b) - (a \cdot b)^2. \tag{2}$$

It follows that $\omega(a,b,c) := a \times b \cdot c$ defines an alternating trilinear form on E, and (2) implies that $a \times b \neq 0$ for all linearly independent a, b. Hence there are no 2-dimensional ω-singular subspaces.

Such exceptional alternating trilinear forms are of great interest and are well-known, see for example [2], to be related to the composition algebra $\mathbb{O}$ of the real octonions. With respect to an orthonormal basis $\{x_1, \ldots, x_7\}$ of E^* one such $\omega \bigwedge^3 E^*$ is given by

$$\omega := f_{124} + f_{235} + f_{346} + f_{457} + f_{561} + f_{672} + f_{713}, \tag{3}$$

where $f_{ijk} = x_i \wedge x_j \wedge x_k$. It is known, see [4, Theorem 1], that the stabiliser $\mathrm{GL}(E)_\omega$ of ω in $\mathrm{GL}(E)$ is a subgroup of $\mathrm{SO}(E) \cong \mathrm{SO}(7)$ which is isomorphic to the compact exceptional real Lie group G_2, and that $\mathrm{GL}(E)_\omega$ acts transitively on the set of 2-dimensional vector subspaces of E. Now, from (3), the linear form $\omega(e_1, e_2, .)$ is nonzero. Consequently, by the afore-mentioned transitivity, for any 2-dimensional space $\mathbb{R}a \oplus \mathbb{R}b \subseteq E$ the form $\omega(a, b, .)$ is nonzero, thus recovering the fact that ω has no singular lines.

This paper is organised as follows. In Section 2 we collect some results on divided powers of alternating forms of even degree, which we use in Section 3 to prove our main theorem. It turns out that the proof is most interesting for trilinear forms in odd dimensions n, where we prove that the singular lines either sweep out the entire projective $(n-1)$-space or else a hypersurface of degree $\frac{n-1}{2} - 1$. Finally, in Section 4 we study 2-singular subspaces for a trilinear form on a vector space V. In particular, we give a sharp upper bound in terms of $\dim V$ on the dimension of such subspaces (assuming that ω is non-degenerate), and study the trilinear forms in certain binomial dimensions attaining this bound.

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2. Divided powers in the Grassmann algebra

For an n-dimensional vector space V over a field K let $\bigwedge V = \bigoplus_{d=0}^n \bigwedge^d V$ denote the Grassmann algebra of V. This is an associative K-algebra in which the multiplication, denoted $\wedge$, takes $\bigwedge^d V \times \bigwedge^e V$ into $\bigwedge^{d+e} V$. Let $e_1, \ldots, e_n$ be a basis of V, and for a d-element subset $I = \{i_1 < \ldots < i_d\}$ of $[n]$ write $e_I := e_{i_1} \wedge \ldots \wedge e_{i_d}$. These elements form a basis of $\bigwedge^d V$. Now assume that d is even, and let $\omega \in \bigwedge^d V$. For every natural number k we define an element $\omega^{(k)}$ of $\bigwedge^{kd} V$ as follows. Write $\omega = \sum_{I \subseteq [n], |I| = d} \alpha_I e_I$ and set

$$\omega^{(k)} := \sum_{I \subseteq [n],\ |I| = kd} \left(\sum_{\{I_1, \ldots, I_k\},\ \dot{\bigcup}_j I_j = I,\ |I_j| = d} (\prod_j \alpha_{I_j}) e_{I_1} \wedge \ldots \wedge e_{I_d} \right). \tag{4}$$

The second sum is over all unordered partitions of I into k d-element subsets. It is important that these partitions are taken unordered, so that a permutation of the I_j does not yield further terms in the second sum. Note that the expression being summed is well-defined as interchanging two consecutive factors e_{I_j}s does not change the sign of the wedge-product—here we use that d is even.

Lemma 2.1. *For even d the map $\bigwedge^d V \to \bigwedge^{kd} V$, $\omega \mapsto \omega^{(k)}$ has the following properties:*

1. *$\omega \wedge \omega \wedge \ldots \wedge \omega$, where the number of factors is k, equals $(k!)\omega^{(k)}$;*
2. *the map $\omega \mapsto \omega^{(k)}$ does not depend on the choice of the basis $e_1, \ldots, e_n$;*
3. *for any K-linear map $A : V \to W$ of vector spaces we have $((\bigwedge^d A)\omega)^{(k)} = (\bigwedge^{kd} A)(\omega^{(k)})$; and*
4. *if $d = 2$ and $\dim V = 2k$, then $\omega^{(k)}$ is zero if and only if ω does not have full rank.*

Proof. Property (1) is obvious: multiplying by $k!$ has the same effect as summing, in (4), over all *ordered* partitions.

Property (2) is clear in characteristic zero by property (1). Now if we express $e_1, \ldots, e_n$ by an invertible matrix g in a second basis $e'_1, \ldots, e'_n$, then the fact that $\omega^{(d)}$ does not change when K has characteristic 0 translates into identities among certain polynomial expressions over $\mathbb{Z}$ in $\det(g)^{-1}$ and the g_{ij}. These identities hold over any field, which proves the basis-independence over any K.

The basis independence implies property (3): choose a basis $e_1, \ldots, e_m, \ldots, e_n$ of V such that $e_{m+1}, \ldots, e_n$ span $\ker(A)$, and extend $Ae_1, \ldots, Ae_m$ to a basis of W. In these bases it is trivial to verify that $((\bigwedge^d A)\omega)^{(k)} = (\bigwedge^{kd} A)\omega^{(k)}$.

Property (4) also follows from basis independence. Indeed, one can choose a basis $e_1, \ldots, e_{2m}, \ldots, e_{2k}$ of V with $m \leq k$ such that $\omega = \sum_{i=1}^m e_{2i-1} \wedge e_{2i}$. If $m < d$ then all terms in (4) are zero. If $m = d$ then the expression equals $e_1 \wedge \ldots \wedge e_{2d} \neq 0$. □

Remark 2.2.
1. We call $\omega^{(k)}$ the k-th *divided power* of ω.
2. If $d = 2$ and $n = 2k$, then the k-th divided power of ω is known as its *Pfaffian*.
3. In our application below, this lemma will be applied to V^*.

3. Proof of the main theorem

We first prove our main theorem for trilinear forms. Here we distinguish two cases, according to the parity of $\dim V$.

Proposition 3.1. *Let V be a vector space of* even *dimension over any field and let $\omega \in \bigwedge^3 V^*$. Then every one-dimensional subspace of V is contained in an ω-singular two-dimensional subspace of V.*

Proof. For any one-dimensional subspace $\langle u \rangle$ of V the alternating bilinear form $\langle u, \omega \rangle \in \bigwedge^2(V^*)$ has rank at most $\dim V - 1$, as u is in its radical. But the rank of an alternating bilinear form is even, so the rank of $\langle u, \omega \rangle$ is at most $\dim V - 2$. Hence there exists a u', linearly independent of u, such that $\langle u \wedge u', \omega \rangle = 0$. □

Theorem 3.2. *Let V be a vector space of* odd *dimension $n \geq 5$ over a field K and let $\omega \in \bigwedge^3 V^*$. Then the union of all ω-singular lines is either all of V or a hypersurface defined by a homogeneous polynomial in $K[V]$ of degree $(n-1)/2 - 1$.*

In particular, if K is quasi-algebraically closed, then this hypersurface contains K-rational points, since $(n-1)/2 - 1$ is greater than zero and less than n, the number of variables.

Proof. For any non-zero $u \in V$ consider the alternating bilinear form $\omega_u := \langle u, \omega \rangle \in \bigwedge^2 V^*$. This is an element of $\bigwedge^2(u^0) \subseteq \bigwedge^2(V^*)$, where u^0 is the annihilator of u

in V^*. Setting $k := (n-1)/2$, the k-th divided power $\omega_u^{(k)}$ of ω_u lies in the one-dimensional subspace $\bigwedge^{n-1}(u^0)$ of the n-dimensional space $\bigwedge^{n-1}(V^*)$. By choosing a basis in the one-dimensional space $\bigwedge^n(V^*)$ the space $\bigwedge^{n-1}(V^*)$ can be identified with $(V^*)^* = V$. Under this identification the one-dimensional subspace $\bigwedge^{n-1}(u^0)$ corresponds to the one-dimensional subspace Ku, and hence $\omega_u^{(k)}$ corresponds to a multiple $f_\omega(u)u$ of u. Now $f_\omega(u)$ is either zero or a homogeneous polynomial in u of degree $k-1 = (n-1)/2 - 1$, which as $n > 3$ is strictly positive. Its non-zero roots are precisely the vectors $u \neq 0$ for which ω_u does not have full rank, by property (4) in Lemma 2.1 applied to the even-dimensional space u^0. These, in turn, are precisely the vectors $u \neq 0$ for which there exists a $u' \in V$, linearly independent of u, for which $\langle u \wedge u', \omega\rangle = 0$—that is, the vectors $u \neq 0$ lying in some two-dimensional ω-singular space. □

Now we can prove the main theorem in full generality.

Proof of the main theorem. Let ω be an alternating e-form on a space of dimension larger than e, and assume that $e \geq 3$. We have to prove that there exist $(e-1)$-dimensional ω-singular spaces. Choose an $(e-3)$-dimensional subspace U of V, let $\lambda \in \bigwedge^{e-3} V$ span $\bigwedge^{e-3} U$, and consider $\omega' := \langle \lambda, \omega\rangle \in \bigwedge^3(V/U)^*$. By Proposition 3.1 and Theorem 3.2 the space V/U, which is of dimension greater than 3, contains an ω'-singular two-dimensional space V'. The pre-image of V' in V is an $(e-1)$-dimensional ω-singular space. □

Remark 3.3. The following remarks all concern trilinear forms.

(1) The map $\bigwedge^3 V^* \to S^{(n-1)/2-1}V^*$ sending ω to f_ω is $\mathrm{GL}(V)$-equivariant by construction. This map may prove useful in the further study of alternating trilinear forms.
(2) If K is finite and n is odd, the theorem of Chevalley and Warning allows one to add another $(n+1)/2$ linear equations, which then still have a non-zero common root with f. Hence every space of vector dimension $(n-1)/2$ intersects some singular line.
(3) Suppose that K is algebraically closed. Then every line intersects some singular lines. If f is non-zero, then a general line has $(n-1)/2-1$ intersections with singular lines.
(4) From the classification in [6] one can deduce that for trilinear forms on spaces of dimensions 5 and 7 the polynomial f is identically zero if and only if ω has a singular one-dimensional space, that is, if and only if $\omega \in \wedge^3 U^*$ for some proper subspace U^* of V^*. The implication $\Rightarrow$ clearly always holds, but the converse does not. Indeed, consider the form

$$\omega = x_1 \wedge x_2 \wedge x_3 + x_4 \wedge x_5 \wedge x_6 + x_7 \wedge x_8 \wedge x_9,$$

where $x_1, \ldots, x_9$ are a basis of a 9-dimensional space V^*. For general v the radical of ω_v is three-dimensional, hence f_ω is identically zero, but ω does not have a singular point.
(5) In the previous example ω equals $\omega_1 + \omega_2 + \omega_3$ for a suitable decomposition $V = V_1 \oplus V_2 \oplus V_3$ and $\omega_i \in V_i^* = (V_j \oplus V_k)^0$ for all distinct i, j, k. One may be led to think that f_ω is identically zero if and only if ω is the sum of forms ω_i, where each $\omega_i \in V_i^* = (\bigoplus_{j \neq i} V_j)^*$ for some non-trivial vector space decomposition $V = \bigoplus_i V_i$. This is, however, not true: take

V equal to a simple Lie algebra of odd dimension n and rank l, say in characteristic zero. For instance, one may take $V = \mathfrak{sl}_m$ with m even, so that $n = m^2 - 1$ and $l = m - 1$ are odd. Let ω be the trilinear form on V defined by $\omega(u, v, w) = \kappa([u, v], w)$, where $[.,.]$ is the Lie bracket and κ is the Killing form. This form is alternating as the Killing form is invariant $(\kappa([u, v], w) + \kappa(v, [u, w]) = 0)$ and the Lie bracket is alternating. Now for all u the space of elements v having zero Lie bracket with u has dimension at least l. Hence if $l > 1$, then the alternating bilinear form ω_u has a radical. We conclude that $f_\omega = 0$. On the other hand, ω cannot be split as a sum of ω_is as above. Indeed, ω does not have singular one-dimensional spaces, as κ is a non-degenerate symmetric bilinear form and the centre of V is trivial. Hence ω is non-degenerate in the sense of [10], and by the results of that paper the finest decomposition of V and of ω as above would be unique. Then, since ω is V-invariant, the V_i would have to be ideals in V, which would contradict the fact that V is simple. Concluding, at present we have no better geometric description for $f_\omega \equiv 0$ than "the union of all singular lines is $\mathbb{P}V$".

4. Two-singular subspaces for alternating trilinear forms

Recall that a subspace U of a vector space V is called 2-singular for an alternating trilinear form ω if all 2-dimensional subspaces of U are ω-singular; in particular, we consider to be 2-singular all subspaces of dimension at most one, as well as all ω-singular 2-dimensional subspaces. Here we present a result on the possible dimensions of such a space U. The kernel of $V \mapsto \bigwedge^2 V^*,\ v \mapsto \langle v, \omega\rangle$ is called the *radical* of ω; and ω is called *non-degenerate* if its radical is trivial.

Theorem 4.1. *Assume that* $\dim V \geq 3$ *and let* $s \geq 2$ *be the natural number for which* $\binom{s}{2} < n := \dim V \leq \binom{s+1}{2}$. *Then no non-degenerate trilinear form on* V *can have a* 2*-singular space of codimension strictly smaller than* s*; but there exist non-degenerate trilinear forms on* V *having* 2*-singular spaces of codimension exactly* s*. Moreover, if* $n = \binom{s+1}{2}$*, then the non-degenerate trilinear forms having a* 2*-singular space of codimension* s *form a single* $\mathrm{GL}(V)$*-orbit.*

Note that if V is three-dimensional this theorem reduces to the known fact that there exist non-degenerate trilinear forms on V, and that these form a single orbit. For the next interesting case $n = \binom{4}{2} = 6$ see Example 4.2 below.

Proof. Suppose that U is a 2-singular subspace for the non-degenerate trilinear form ω on V. Then we have a linear map $U \to \bigwedge^2 (V/U)^*,\ u \mapsto \langle u, \omega\rangle$, whose kernel is contained in the radical of ω, hence zero by assumption. Hence we find that $r := \dim U \leq \binom{n-r}{2} = \dim \bigwedge^2 (V/U)^*$, or $s' \geq n - \binom{s'}{2}$ where $s' := n - r$, or $\binom{s'+1}{2} \geq n$, so that the codimension s' of U is at least s, as claimed.

Now let U be a subspace of V of codimension s. For the remainder of this proof it is convenient to choose a vector space complement W of U in V. We may then identify W^* with the annihilator of U in V^*, and vice versa. Since $\dim U \leq \dim \bigwedge^2 W^*$, there exist injective linear maps $L : U \to \bigwedge^2 W^*$. In fact we may chose such an injection L to have the property that the intersection of the radicals of all images $L(u)$ is trivial. For if s is even, then we may take L such that some $L(u)$ is a non-degenerate alternating 2-form, while if s is odd, then

s, $\binom{s}{2} \geq 3$ by the dimension restriction on V and we can ensure that $\mathrm{im}(L)$ contains two alternating forms of rank $s-1$ whose radicals are distinct.

We also view L as an element of $U^* \otimes \bigwedge^2 W^*$ and hence as an element $\omega = \omega_L$ of $\bigwedge^3 V^*$ by means of the (injective) linear map $U^* \otimes \bigwedge^2 W^* \to \bigwedge^3 V^*$ determined by $\xi \otimes \zeta \mapsto \xi \wedge \zeta$. Then ω has U as a 2-singular subspace, and we claim that ω is non-degenerate. For this we have to prove that the linear map $H : V \to \bigwedge^2 V^*,\ v \mapsto \langle v, \omega \rangle$ is injective. This H maps U into $\bigwedge^2 W^*$ and W into $U^* \otimes W^*$, considered as a subspace of $\bigwedge^2 V^*$ by the injective linear map determined by $\xi \otimes \zeta \mapsto \xi \wedge \zeta$. Since the two subspaces $\bigwedge^2 W^*$ and $U^* \otimes W^*$ of $\bigwedge^2 V^*$ intersect trivially, the injectivity of H is equivalent to the joint injectivity of $H|_U$ and of $H|_W$. Now $H|_U = L$ is injective by assumption, and $H(w) = 0$ implies that w lies in the radical of $L(u)$ for all $u \in U$, a contradiction to the choice of L. This proves that ω is non-degenerate.

Finally suppose that $n = \binom{s+1}{2}$, so that $\dim U = \binom{s}{2}$. Then we need to show that all non-degenerate trilinear forms ω' on V having a 2-singular subspace of codimension s are in the $\mathrm{GL}(V)$-orbit of the form ω constructed above. First we move a 2-singular codimension-s subspace for ω' to U by an element of $\mathrm{GL}(V)$. Then ω' determines a linear isomorphism $L' : U \to \bigwedge^2 W^*$, and we still have the group of upper triangular linear maps

$$g = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in \mathrm{GL}(V) = \mathrm{GL}(U \oplus W)$$

with $A \in \mathrm{GL}(U)$, $B \in \mathrm{Hom}(W, U)$, and $C \in \mathrm{GL}(W)$ to move ω' to ω. First we take $B = 0$ and $C = I$ and observe that acting with g on ω' corresponds to replacing L' by $L' \circ A^{-1}$. Hence by taking $A = L^{-1}L'$ we move ω' such that L' becomes equal to L.

Now $\omega, \omega' \in (U^* \otimes \bigwedge^2 W^*) \oplus \bigwedge^3 W^*$ have the same component L in the first summand, but ω' may have a non-zero component μ' in the second summand while ω does not. Take $A = C = I$ in the element g and verify that g then acts trivially on U and on W^*, while it sends an element ξ of U^* to $\xi - \xi \circ B \in U^* \oplus W^* = V^*$. Hence g fixes $\mu' \in \bigwedge^3 W^*$ and maps L to $L - L \circ B$, with the slight abuse of notation that the latter expression stands for the *image* of $L \circ B$ under the projection $W^* \otimes \bigwedge^2 W^* \to \bigwedge^3 W^*$. By surjectivity of L we may choose B such that this image coincides with μ', so that g maps ω' to ω. This completes the proof that ω' lies in the orbit of ω. $\square$

We conclude by determining the singular lines of ω in the orbit described above. We think of U as equal to $\bigwedge^2 W^*$, and then the alternating trilinear form ω is determined by

$$\begin{aligned} &\omega(\mu_1, \mu_2, .) = 0 \text{ for } \mu_1, \mu_2 \in \bigwedge^2 W^*, \\ &\omega(\mu, w_1, w_2) = \mu(w_1, w_2) \text{ for } \mu \in \bigwedge^2 W^*,\ w_1, w_2 \in W, \text{ and} \\ &\omega(w_1, w_2, w_3) = 0 \text{ for } w_1, w_2, w_3 \in W. \end{aligned}$$

In addition to the 2-dimensional subspaces of $U = \bigwedge^2 W^*$ also the 2-dimensional subspaces of the form $K\mu_1 \oplus K(\mu_2 + w_2)$ with $\mu_1, \mu_2 \in \bigwedge^2 W^*$ and w_2 in the radical of μ_1 are singular. We claim that these are the only singular lines. Indeed, consider a 2-dimensional subspace of the form $K(\mu_1 + w_1) \oplus K(\mu_2 + w_2)$ with w_1, w_2 linearly independent. Then choose any alternating bilinear form μ_3 on W such that

$\mu_3(w_1, w_2) \neq 0$. Then we have $\omega(\mu_1 + w_1, \mu_2 + w_2, \mu_3) = \mu_3(w_1, w_2) \neq 0$, so the line is non-singular. This argument also implies that U is the only codimension-s subspace that is 2-singular: any other subspace U' with this property cannot have a projection along U onto W that is more than 1-dimensional, and hence U' must intersect U in a codimension-1 subspace. But if $\mu + w \in U'$ with $w \neq 0$, then the elements of $U \cap U'$ must all have w in their radicals. The space of alternating bilinear forms on W having w in their radicals is $\bigwedge^2 (W/Kw)^*$ and has dimension $\binom{s-1}{2}$. Hence this space cannot contain a codimension-1 subspace of $\bigwedge^2 W^*$.

Example 4.2. In the last part of Theorem 4.1 the smallest dimension of interest is $n = 6$, a representative of the single $\mathrm{GL}(V)$-orbit being the form

$$\omega = x_2 \wedge x_3 \wedge x_4 + x_1 \wedge x_3 \wedge x_5 + x_1 \wedge x_2 \wedge x_6,$$

for which the 3-dimensional subspace $U := \prec e_4, e_5, e_6 \succ$ is the unique 2-singular subspace of codimension $s = 3$. In this example the map $L : U \to \bigwedge^2 W^*$ in the preceding proof is chosen to be that which sends e_4, e_5, e_6 to $x_2 \wedge x_3, x_1 \wedge x_3, x_1 \wedge x_2$, respectively. As pointed out in [12, Section 3], in the case $K = \mathrm{GF}(2)$ a trilinear form belonging to the same orbit as ω arises from the cubic equation of the 35-set $\psi \subset \mathrm{PG}(5, 2)$ supporting a non-maximal partial spread Σ_5 of five planes in $\mathrm{PG}(5, 2)$. The unique projective plane U singled out as being 2-singular for ω is in fact one of the planes of Σ_5, and can also be picked out geometrically by the property that each of the seven planes $\notin \Sigma_5$ which lie in ψ meets U in a line and meets each of the four other planes $\in \Sigma_5$ in a point.

References

[1] Hirotachi Abo, Giorgio Ottaviani, and Chris Peterson. Non-defectivity of Grassmannians of planes. 2009. Preprint, available from `http://arxiv.org/abs/0901.2601`.

[2] J.C. Baez. The octonions. *Bull. Amer. Math. Soc. (N.S.)*, 39:145–205, 2002.

[3] R.B. Brown and A. Gray. Vector cross products. *Comment. Math. Helv.*, 42:222–236, 1967.

[4] R.L. Bryant. Metrics with exceptional holonomy. *Ann. of Math.*, 126:525–576, 1987.

[5] Claude Chevalley. Démonstration d'une hypothèse de M. Artin. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 11:73–75, 1936.

[6] Arjeh M. Cohen and Aloysius G. Helminck. Trilinear alternating forms on a vector space of dimension 7. *Commun. Algebra*, 16(1):1–25, 1988.

[7] Dragomir Ž. Djoković. Classification of trivectors of an eight-dimensional real vector space. *Linear Multilinear Algebra*, 13:3–39, 1983.

[8] Dragomir Ž. Djoković. Closures of equivalence classes of trivectors of an eight-dimensional complex vector space. *Can. Math. Bull.*, 26:92–100, 1983.

[9] G.B. Gurevich. Classification des trivecteurs ayant le rang huit. *C. R. (Dokl.) Acad. Sci. URSS*, 2:353–356, 1935. Russian; French text 355–356.

[10] Jan Hora. Orthogonal decompositions and canonical embeddings of multilinear alternating forms. *Linear Multilinear Algebra*, 52(2):121–132, 2004.

[11] Jan A. Schouten. Klassifizierung der alternierenden Grössen dritten Grades in sieben Dimensionen. *Rend. Circ. Mat. Palermo*, 55:137–156, 1931.

[12] Ron Shaw. Trivectors and cubics: $\mathrm{PG}(5, 2)$ aspects. 2008. Preprint, available from `http://www.hull.ac.uk/php/masrs/recentpublications.html`.

[13] Ernest B. Vinberg and Alexander G. Elashvili. Classification of trivectors of a 9-dimensional space. *Sel. Math. Sov.*, 7(1):63–98, 1988. Translation from *Tr. Semin. Vektorn. Tensorn. Anal. Prilozh. Geom. Mekh. Fiz.* **18**, 197–233.

[14] Ewald Warning. Bemerkung zur vorstehenden Arbeit von Herrn Chevalley. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 11:76–83, 1936.

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