# SEMIORTHOGONAL DECOMPOSITIONS OF EQUIVARIANT DERIVED CATEGORIES OF INVARIANT DIVISORS

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Abstract. Given a smooth variety $Y$ with an action of a finite group $G$, and a semiorthogonal decomposition of the derived category, $\mathcal{D}[Y/G]$, of $G$-equivariant coherent sheaves on $Y$ into subcategories equivalent to derived categories of smooth varieties, we construct a similar semiorthogonal decomposition for a smooth $G$-invariant divisor in $Y$ (under certain technical assumptions). Combining this procedure with the semiorthogonal decompositions constructed in [PV15], we construct semiorthogonal decompositions of some equivariant derived categories of smooth projective hypersurfaces.

## 1. Introduction

1.1. **Semiorthogonal decompositions for $\mathcal{D}[X/G]$.** Let $X$ be a smooth quasiprojective variety over an algebraically closed field $k$ of characteristic zero. Suppose $G$ is a finite group acting on $X$ by automorphisms. Then there is a decomposition of the Hochschild homology of the quotient stack $[X/G]$,

$$HH_*(\mathcal{D}[X/G]) \cong \bigoplus_{\lambda \in G/\sim} HH_*(X_\lambda)^{C(\lambda)}$$

where $\lambda \in G/\sim$ is the set of conjugacy classes of $G$, $C(\lambda)$ is the centralizer of $\lambda$, $X_\lambda \subset X$ is the invariant subvariety of $\lambda$, see [PV15, Lemma 2.1.1]. In [TV16, Theorem 1.1], the authors show that the above decomposition has a motivic origin in an appropriate sense, and that a similar decomposition exists for any additive invariant of dg-categories. In [BGLL17] a related decomposition of the equivariant zeta function is given.

In the case when the geometric quotient $X_\lambda/C(\lambda)$ is smooth one can identify $HH_*(X_\lambda)^{C(\lambda)}$ with $HH_*(X_\lambda/C(\lambda))$ (see [PV15, Proposition 2.1.2]). Thus, it is natural to ask whether in some cases the above decomposition can be realized at the level of derived (or dg) categories.

**Conjecture** ([PV15, Conjecture A]). *Assume a finite group $G$ acts effectively on a smooth variety $X$, and all the geometric quotients $X_\lambda/C(\lambda)$ are smooth for $\lambda \in G/\sim$. Then there is a semiorthogonal decomposition of the derived category $\mathcal{D}[X/G]$ such that the pieces $\mathcal{C}_{[\lambda]}$ of this decomposition are in bijection with conjugacy classes in $G$ and $\mathcal{C}_{[\lambda]} \cong \mathcal{D}(X_\lambda/C(\lambda))$.*

This conjecture was verified in [PV15] in the case where $G$ is a complex reflection group of types $A, B, G_2, F_4$, and $G(m,1,n)$ acting on a vector space $V$, as well as for some actions on $C^n$, where $C$ is a smooth curve. Other global results exist for cyclic quotients, see [KP17, Theorem 4.1] and [Lim16, Theorem 3.3.2], and for

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quotients of curves, see [Pol06, Theorem 1.2]. It is shown in [BGLL17, Theorem D] that the above conjecture fails without the assumption that $G$ acts effectively.

**Definition 1.1.1.** We will refer to a semiorthogonal decomposition of $\mathcal{D}[X/G]$ of the kind appearing in the above conjecture as a *motivic semiorthogonal decomposition.*

1.2. **Statement of the main result.** In this paper, we develop a procedure that starts with a motivic semiorthogonal decomposition of $\mathcal{D}[Y/G]$ and produces a similar decomposition for a smooth $G$-invariant divisor $X$ in $Y$. More precisely, this procedure works under certain assumptions on the kernels giving the semiorthogonal decomposition of $\mathcal{D}[Y/G]$ and on the invariant divisor, that we will now formulate. Some of these assumptions say that the kernels extend to those defining functors from the equivariant derived categories $\mathcal{D}[Y_\lambda/W_\lambda]$; also we need the divisor to be "in generic position" with respect to the kernels in an appropriate sense.

We will work in the following setup. Let $Y$ be a smooth quasiprojective variety equipped with an action of a finite group $G$. For each conjugacy class representative $\lambda \in G/\sim$, we denote by $Y_\lambda$ the fixed locus of $\lambda$ in $Y$. Let $W_\lambda$ be the quotient of the centralizer $C(\lambda)$ by the subgroup fixing $Y_\lambda$ pointwise. We set

$$\overline{Y}_\lambda := Y_\lambda/W_\lambda.$$

Note that there is a natural finite morphism $\overline{Y}_\lambda \to Y/G$.

Let $\overline{\mathcal{L}}$ be a line bundle on the geometric quotient $Y/G$ and let $\overline{s} \in H^0(Y/G, \overline{\mathcal{L}})$ be a nonzero section. We denote by $s$ the induced section of the pull-back $\mathcal{L}$ of $\overline{\mathcal{L}}$ to $Y$. Let $X = V(s) \subset Y$ be the divisor determined by $s$. We assume that $X$ is smooth, and denote by $\iota\colon X \hookrightarrow Y$ the natural closed embedding. The varieties $X_\lambda$ and $\overline{X}_\lambda$ are defined similarly to $Y_\lambda$ and $\overline{Y}_\lambda$, starting from $X$.

Finally, assume that for each representative $\lambda \in G/\sim$, we are given a coherent sheaf $\mathcal{K}_\lambda$ on $\overline{Y}_\lambda \times Y$, equivariant with respect to the natural $G$-action (through the second factor).

We consider the following two conditions on our data.

(SOD) For each $\lambda \in G/\sim$, the support of $\mathcal{K}_\lambda$ is proper over both $\overline{Y}_\lambda$ and $Y$. Also, for each $\lambda$ the Fourier-Mukai functor

$$\mathcal{K}_\lambda\colon \mathcal{D}(\overline{Y}_\lambda) \hookrightarrow \mathcal{D}[Y/G]$$

is fully faithful. Moreover, there is a semiorthogonal decomposition

$$\mathcal{D}[Y/G] = \langle \mathcal{K}_{\lambda_1}(\mathcal{D}(\overline{Y}_{\lambda_1})), \cdots, \mathcal{K}_{\lambda_r}(\mathcal{D}(\overline{Y}_{\lambda_r}))\rangle$$

with respect to some total ordering on the set of conjugacy classes.

(Res) For each $\lambda \in G/\sim$, there is an isomorphism

$$\varphi_\lambda\colon \mathcal{K}_\lambda \otimes \pi_Y^*\mathcal{L} \xrightarrow{\sim} \mathcal{K}_\lambda \otimes \pi_{\overline{Y}_\lambda}^*\overline{\mathcal{L}}_\lambda,$$

where $\overline{\mathcal{L}}_\lambda$ is the pull-back of $\overline{\mathcal{L}}$ to $\overline{Y}_\lambda$. Let $\overline{s}_\lambda \in H^0(\overline{Y}_\lambda, \overline{\mathcal{L}}_\lambda)$ be the pull-back of $\overline{s}$. We require also the morphism

$$[\varphi_\lambda \circ (1 \otimes s) - 1 \otimes \overline{s}_\lambda]\colon \mathcal{K}_\lambda \to \mathcal{K}_\lambda \otimes \pi_{\overline{Y}_\lambda}^*\overline{\mathcal{L}}_\lambda$$

to be zero and the morphism

$$\cdot(1 \otimes \overline{s}_\lambda)\colon \mathcal{K}_\lambda \to \mathcal{K}_\lambda \otimes \pi_{\overline{Y}_\lambda}^*\overline{\mathcal{L}}_\lambda$$

to be injective, where we have abused notation by setting $1 \otimes s = 1 \otimes \pi_Y^*(s)$ and similarly for $1 \otimes s_\lambda$.

*Remarks* 1.2.1. 1. Note that condition (SOD) implies that $\mathcal{D}[Y/G]$ has a motivic semiorthogonal decomposition. In particular, all the quotients $\overline{Y}_\lambda = Y_\lambda/W_\lambda$ are smooth, since the derived categories $\mathcal{D}(\overline{Y}_\lambda)$ are full subcategories of the smooth category $\mathcal{D}[Y/G]$.
2. Condition (Res) implies the nonvanishing of the restriction of $\overline{s}_\lambda$ to each connected component of $\overline{Y}_\lambda$, over which $\mathcal{K}_\lambda$ is nonzero.
3. Let us consider the closed subset

$$Z_\lambda = \cup_{g\in G}(\mathrm{Id}\times g)(\Gamma_\lambda) \subset Y_\lambda \times Y \tag{1.1}$$

where $\Gamma_\lambda \subset Y_\lambda \times Y$ is the graph of the closed embedding $Y_\lambda \hookrightarrow Y$. We equip $Z_\lambda$ with the reduced scheme structure. Note that $Z_\lambda$ is invariant under the action of $W_\lambda \times G$, where $W_\lambda$ acts on the first factor and $G$ acts on the second factor. Let us consider the (reduced) subscheme

$$\overline{Z}_\lambda := Z_\lambda/W_\lambda \subset \overline{Y}_\lambda \times Y.$$

In the examples we will consider in Section 5, for each $\lambda$, the sheaf $\mathcal{K}_\lambda$ will be supported on $\overline{Z}_\lambda$. This implies the condition on support of $\mathcal{K}_\lambda$ imposed in (SOD). Furthermore, if we assume that $\mathcal{K}_\lambda$ is obtained by taking $W_\lambda$-invariants from a $W_\lambda \times G$-equivariant vector bundle over $Z_\lambda$, then the condition (Res) reduces to the requirement that $s$ has nonzero restrictions to all connected components of $Y_\lambda$ for each $\lambda$ (see Lemma 5.1.1).

Now we can state our first main result.

**Theorem 1.2.2.** *Let $Y$ be a smooth $G$-variety, $\overline{s} \in H^0(Y/G, \overline{\mathcal{L}})$ a section, $X \subset Y$ the corresponding divisor, which we assume to be smooth, and for each $\lambda \in G/{\sim}$, let $\mathcal{K}_\lambda$ be a $G$-equivariant coherent sheaf on $\overline{Y}_\lambda \times Y$, such that conditions (SOD, Res) are satisfied. Then there exists a collection of $G$-equivariant coherent sheaves on $\overline{X}_\lambda \times X$, such that condition (SOD) is satisfied. In particular, $\mathcal{D}[X/G]$ admits a motivic semiorthogonal decomposition.*

To get applications of this theorem, one should start with some quotient stacks $[Y/G]$, for which a motivic semiorthogonal decomposition has been constructed. We mostly focus on the case of $[\mathbb{A}^n/S_n]$ (in which case a motivic semiorthogonal decomposition was constructed in [PV15]), and also consider the stacks of the form $[C_1 \times \ldots \times C_n/(G_1 \times \ldots \times G_n)]$, where for each $i$, $G_i$ is a finite abelian group acting effectively on a smooth curve $C_i$.

We combine Theorem 1.2.2 with two simpler procedures: replacing $Y$ by a $G$-invariant open subset and passing to the quotient by a free action of $\mathbb{G}_m$. This leads us in the case of $[\mathbb{A}^n/S_n]$ to the following semiorthogonal decomposition for an $S_n$-invariant projective hypersurface.

**Theorem 1.2.3.** *Let $f$ be an $S_n$-invariant homogeneous polynomial on $V = \mathbb{A}^n$, such that the corresponding projective hypersurface $\mathbb{P}V(f)$ is smooth. Then there exists a semiorthogonal decomposition of $\mathcal{D}[\mathbb{P}V(f)/S_n]$, with the pieces $\mathcal{D}[\mathbb{P}V(\overline{f}_\lambda)]$, where $\mathbb{P}V(\overline{f}_\lambda) \subset \mathbb{P}\overline{V}_\lambda$ is the weighted projective hypersurface stack associated with $\overline{f}_\lambda$. Here $\overline{f}_\lambda$ is the polynomial on $\overline{V}_\lambda = V_\lambda/W_\lambda$ corresponding to $f_\lambda = f|_{V_\lambda}$.*

Note that the decomposition of Theorem 1.2.3 no longer follows the pattern of Conjecture A since some pieces of the decompositions are themselves derived categories of stacks. The only similarity is that in both cases there is a birational morphism of stacks inducing a fully faithful embedding via the pull-back ($[X/G] \to X/G$ in Conjecture A and $[\mathbb{P}V(f)/S_n] \to \mathbb{P}V(\overline{f}_1)$ in Theorem 1.2.3), which is then extended to a semiorthogonal decomposition of the derived category of the source stack.

1.3. **Outline of paper.** In Section 2, we remind the reader about semiorthogonal decompositions and equivariant derived categories. In Section 3, we prove Theorem 1.2.2. In Section 4, we discuss the simpler procedures of inducing semiorthogonal decompositions when passing to an invariant open subset or to the quotient by an action of a reductive algebraic group. In Section 5, we consider applications of Theorem 1.2.2. In particular, in Section 5.3 we prove Theorem 1.2.3. In Section 5.4 we consider applications related to the stacks $[C_1 \times \ldots \times C_n/(G_1 \times \ldots \times G_n)]$, where $G_i$ is a finite abelian group acting on a smooth curve $C_i$.

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1.5. **Conventions.** We work over $\mathbb{C}$. By a *variety* we mean a *quasiprojective variety* over $\mathbb{C}$. All functors are assumed to be derived. We denote by $\mathcal{D}(X)$, for $X$ a variety or a quotient stack, the bounded derived categories of coherent sheaves on $X$. When $G$ is an algebraic group acting on a variety $X$, then we denote by $[X/G]$ the corresponding quotient stack, whereas $X/G$ denotes the geometric quotient (when it exists).

## 2. Semiorthogonal decompositions and equivariant derived categories

In this section, we remind the reader of semiorthogonal decompositions and $G$-equivariant derived categories. For an overview of semiorthogonal decompositions in algebraic geometry, see [BO02, Bri06].

2.1. **Semiorthogonal decompositions.** Recall, a *semiorthogonal decomposition* of a triangulated category $\mathcal{T}$ is a pair $\mathcal{A}, \mathcal{B}$ of full triangulated subcategories of $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{B}, \mathcal{A}) = 0$, and if every object $t \in \mathcal{T}$ fits in an exact triangle

$$b \to t \to a \to b[1]$$

where $a \in \mathcal{A}, b \in \mathcal{B}$. In this case, we write $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$. We can iterate this definition to get semiorthogonal decompositions with any finite number of components $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and we write

$$\mathcal{T} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle.$$

2.2. **Fourier-Mukai functors.** Let $X, Y$ be smooth schemes or DM-stacks. An object $\mathcal{P} \in \mathcal{D}(X \times Y)$, whose support is proper over $Y$, gives rise to an exact functor $\Phi_{\mathcal{P}} \colon \mathcal{D}(X) \to \mathcal{D}(Y)$ defined by

$$\Phi_{\mathcal{P}}(\cdot) = \pi_{Y*}(\pi_X(\cdot) \otimes \mathcal{P}).$$

We will refer to $\mathcal{P}$ as a *Fourier-Mukai kernel* and $\Phi_{\mathcal{P}}$ a *Fourier-Mukai functor*. Starting from Section 3, we will denote the functor $\Phi_{\mathcal{P}}$ simply by $\mathcal{P}$.

Given two Fourier-Mukai kernels $\mathcal{P} \in \mathcal{D}(X \times Y)$ and $\mathcal{Q} \in \mathcal{D}(Y \times Z)$ (where the support of $\mathcal{P}$ is proper over $Y$ and the support of $\mathcal{Q}$ is proper over $Z$), we set

$$\mathcal{P} \circ_Y \mathcal{Q} = \pi_{X,Z*}(\pi_{X,Y}^*\mathcal{P} \otimes \pi_{Y,Z}^*\mathcal{Q}) \in \mathcal{D}(X \times Z).$$

Then the support of $\mathcal{P} \circ_Y \mathcal{Q}$ is proper over $Z$, and we have

$$\Phi_{\mathcal{P}\circ_Y \mathcal{Q}} \cong \Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}}$$

(see [Huy06, Prop. 5.10]).

If $X, Y$ are smooth and the support of $\mathcal{P}$ is proper over both $X$ and $Y$, then the left and right adjoints of $\Phi_{\mathcal{P}}$ are also given by Fourier-Mukai functors. Namely, let us define Fourier-Mukai kernels in $\mathcal{D}(Y \times X)$,

$$\mathcal{P}^R = \sigma^*(\mathcal{P}^\vee \otimes \pi_X^*\omega_X[\dim(X)]) \text{ and } \mathcal{P}^L = \sigma^*(\mathcal{P}^\vee \otimes \pi_Y^*\omega_Y[\dim(Y)]), \tag{2.1}$$

where $\mathcal{P}^\vee$ is the derived dual of $\mathcal{P}$ and $\sigma : Y \times X \to X \times Y$ is the transposition of factors.

**Proposition 2.2.1.** *Assume that $X$ and $Y$ are smooth, and $X$ is a scheme (while $Y$ is possibly a stack). Let $\mathcal{P} \in \mathcal{D}(X \times Y)$ be a kernel whose support is proper over both $X$ and $Y$. Then the Fourier-Mukai functors $\Phi_{\mathcal{P}^L}, \Phi_{\mathcal{P}^R} : \mathcal{D}(Y) \to \mathcal{D}(X)$ are left, respectively right, adjoint to $\Phi_{\mathcal{P}}$.*

*Proof.* In the case when $X$ and $Y$ are projective, this is [Huy06, Proposition 5.9]. The proof in the general case is similar, using the adjunction of the form

$$\mathrm{Hom}(f_*F, G) \simeq \mathrm{Hom}(F, f^!G) \simeq \mathrm{Hom}(F, f^*G \otimes \omega_f[\dim f]),$$

for a smooth morphism $f$ and $F$ with proper support over the target. □

**Lemma 2.2.2.** *Assume that $X$ and $Y$ are smooth, and $X$ is a scheme (while $Y$ is possibly a stack).*
*(i) Let $\mathcal{P} \in \mathcal{D}(X \times Y)$ be a kernel such that $\Phi_{\mathcal{P}} = 0$. Then $\mathcal{P} = 0$.*
*(ii) Assume now that $X$ and $Y$ are smooth, and $\mathcal{P}$ has support which is proper over both $X$ and $Y$. Then the functor $\Phi_{\mathcal{P}}$ is fully faithful if and if the natural map*

$$\mathcal{P} \circ_Y \mathcal{P}^L \to \mathcal{O}_{\Delta_X}$$

*(resp., the natural map*

$$\mathcal{O}_{\Delta_X} \to \mathcal{P} \circ_Y \mathcal{P}^R$$

*is an isomorphism.*
*(iii) In the assumptions of (ii), let $\mathcal{P}' \in \mathcal{D}(X \times Y)$ be another kernel. Then we have the semiorthogonality*

$$\mathrm{Hom}_{\mathcal{D}(Y)}(\Phi_{\mathcal{P}'}(\cdot), \Phi_{\mathcal{P}}(\cdot)) = 0$$

*between the images of the Fourier-Mukai functors if and only if the*

$$\mathcal{P}' \circ_Y \mathcal{P}^L = 0.$$

*Proof.* (i) For every closed point $x \in X$ we have $\mathcal{P}|_{\{x\}\times Y} = \Phi_{\mathcal{P}}(\mathcal{O}_x) = 0$. This easily implies that $\mathcal{P} = 0$.
(ii) Let $C$ be the cone of the morphism $\mathcal{P} \circ_Y \mathcal{P}^L \to \mathcal{O}_{\Delta_X}$. Then for every $F \in \mathcal{D}(X)$, $\Phi_C(F)$ is the cone of the adjunction morphism $\Phi_{\mathcal{P}^L} \circ \Phi_{\mathcal{P}}(F) \to F$. This shows that $\Phi_{\mathcal{P}}$ is fully faithful if and only if $\Phi_C = 0$. By part (i), this is equivalent to $C = 0$. The second case is considered similarly.
(iii) The semiorthogonality in question is equivalent to the vanishing $\Phi_{\mathcal{P}^L} \circ \Phi_{\mathcal{P}} = 0$, so the assertion follows from (i). □

2.3. **Spanning classes.** A subclass of objects $\Omega \subset \mathcal{T}$ of a triangulated category $\mathcal{T}$ is called a *spanning class* if for any object $t \in \mathcal{T}$

$$\mathrm{Hom}_{\mathcal{T}}(t, \omega[i]) = 0 \text{ for all } i \in \mathbb{Z} \text{ and all } \omega \in \Omega \text{ implies } t = 0,$$
$$\mathrm{Hom}_{\mathcal{T}}(\omega[i], t) = 0 \text{ for all } i \in \mathbb{Z} \text{ and all } \omega \in \Omega \text{ implies } t = 0.$$

Spanning classes are useful when checking that an exact functor from $\mathcal{T}$ is fully faithful, due to the following result (see [Bri99, Thm. 2.3]).

**Proposition 2.3.1.** *Let $\Omega \subset \mathcal{T}$ be a spanning class, and let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor with a left and right adjoint. If for every $\omega_1, \omega_2 \in \Omega$, the map*

$$F : \mathrm{Hom}^i(\omega_1, \omega_2) \to \mathrm{Hom}^i(F(\omega_1), F(\omega_2))$$

*is an isomorphism, then $F$ is fully faithful.*

**Proposition 2.3.2.** *Let $X$ be a smooth projective or quasi-projective scheme. Let $\Omega$ be the subclass of $\mathcal{D}(X)$ consisting of structure sheaves of points. Then $\Omega$ is a spanning class.*

*Proof.* In the quasi-projective case, the standard argument works to show if $\mathcal{F}^{\cdot} \neq 0$, then there exists $x \in X$ and $i \in \mathbb{Z}$ so that $\mathbf{R}\mathrm{Hom}(\mathcal{F}^{\cdot}, \mathcal{O}_x[i]) \neq 0$. To get the other direction, we use Serre-duality with proper support see [BKR01, Section 3.1]. $\square$

**Lemma 2.3.3.** *Let $\mathcal{P} \in \mathcal{D}(X \times Y)$, with $X$ a smooth scheme and $Y$ smooth (possibly a stack). Assume that either*

$$\mathcal{P} \circ_Y \mathcal{P}^L \cong \mathcal{O}_{\Delta_X} \quad \text{or} \quad \mathcal{P} \circ_Y \mathcal{P}^R \cong \mathcal{O}_{\Delta_X}$$

*Then $\Phi_{\mathcal{P}}$ is fully-faithful.*

*Proof.* Assume that $\mathcal{P} \circ_Y \mathcal{P}^L \cong \mathcal{O}_{\Delta_X}$. By Propositions 2.3.1 and 2.3.2, it is enough to check that the adjunction morphism

$$\Phi_{\mathcal{P}^L} \circ \Phi_{\mathcal{P}}(\mathcal{O}_x) \to \mathcal{O}_x$$

is an isomorphism for every closed point $x \in X$. But

$$(\Phi_{\mathcal{P}^L} \circ \Phi_{\mathcal{P}})(\mathcal{O}_x) \cong \Phi_{\mathcal{P} \circ_Y \mathcal{P}^L}(\mathcal{O}_x) \cong \mathcal{O}_x.$$

Since the above adjunction morphism is necessarily nonzero, it is an isomorphism. In the case when $\mathcal{P} \circ_Y \mathcal{P}^R \cong \mathcal{O}_{\Delta_X}$ the argument is similar using the adjunction morphisms $\mathcal{O}_x \to \Phi_{\mathcal{P}^R} \circ \Phi_{\mathcal{P}}(\mathcal{O}_x)$. $\square$

2.4. **Equivariant derived categories of coherent sheaves.** Let $G$ be a finite group acting on a smooth variety $X$. A $G$-equivariant sheaf on $X$ is the data of a sheaf $\mathcal{F}$ over $X$ and a collection of isomorphism $\theta_g \colon \mathcal{F} \to g^*\mathcal{F}$ subject to the compatibility condition $g^*\theta_h \circ \theta_g = \theta_{hg}$ for all $g, h \in G$. Morphisms between $G$-equivariant sheaves $(\mathcal{F}, \theta_g)$ and $(\mathcal{G}, \psi_g)$ are morphisms of sheaves $\alpha \colon \mathcal{F} \to \mathcal{G}$, such that $g^*\alpha \circ \theta_g = \psi_g \circ \alpha$, for all $g \in G$.

The category of $G$-equivariant sheaves on $X$ is Abelian and there is an exact equivalence

$$\mathcal{D}[X/G] \simeq \mathcal{D}(X)^G$$

between the bounded derived category of coherent sheaves on the quotient stack $[X/G]$ and the bounded derived category of $G$-equivariant sheaves on $X$, [Vis05, Section 3.8].

## 3. Construction of restricted kernels and proof of Theorem 1.2.2

Throughout this section we fix a smooth variety $Y$ with an action of a finite group $G$, and a smooth $G$-invariant divisor $X \subset Y$, which is the zero locus of a $G$-invariant section $s$ of a $G$-equivariant line bundle $\mathcal{L}$ such that the conditions (SOD, Res) are satisfied. We then proceed to construct a motivic semiorthogonal decomposition of $\mathcal{D}[X/G]$.

3.1. **Smoothness of the quotients.** Recall that $X_\lambda \subset X$ is the $\lambda$-invariant locus in $X$ and

$$\overline{X}_\lambda := X/W_\lambda.$$

We also have $\overline{X}_\lambda = V(\overline{s}_\lambda) \subset \overline{Y}_\lambda$.

We are going to show that the quotients $\overline{X}_\lambda$ are smooth. We start by observing that the smoothness of the geometric quotient is preserved upon passing to a smooth $G$-invariant divisor.

**Proposition 3.1.1.** *Suppose $Y$ is smooth and $G$ is a finite group acting by automorphisms on $Y$ such that the geometric quotient $Y/G$ is smooth. Let $X$ be a smooth $G$-invariant divisor. Then the geometric quotient $X/G$ is smooth.*

*Proof.* Let $x \in X$ be a $G$-invariant point. Formally, near $x$ in $Y$, the divisor $X$ is an invariant hyperplane. Since $Y/G$ is smooth at $x$, $G$ is generated by pseudo-reflections and hence the quotient $X/G$ is smooth at $x$.

Now let $x \in X$ be arbitrary. By Luna's étale slice theorem, [Lun73], the map $Y/\mathrm{St}_x \to Y/G$ is étale near the image of $x$. Since $Y/G$ is smooth, this implies $Y/\mathrm{St}_x$ is also smooth. Thus $X/\mathrm{St}_x$ is smooth by the previous argument. Since the mapping $X/\mathrm{St}_x \to X/G$ is étale, we conclude $X/G$ is smooth at the image of $x$, [Gro67, Theorem 17.11.1]. □

**Corollary 3.1.2.** *Each of the geometric quotients $\overline{X}_\lambda = X_\lambda/W_\lambda$ are smooth.*

*Proof.* The scheme $Y_\lambda$ is smooth as the fixed locus of a finite order automorphism in $Y$. Similarly, $X_\lambda$ is smooth as $X$ is smooth and $X_\lambda$ is the fixed locus of $\lambda$. We know that the quotients $\overline{Y}_\lambda = Y_\lambda/W_\lambda$ are smooth (see Remark 1.2.1.2) and so, by Proposition 3.1.1, the quotients $\overline{X}_\lambda = X_\lambda/W_\lambda$ are also smooth. □

3.2. **The Fourier-Mukai kernels.** We will use Condition (Res) to construct Fourier-Mukai kernels $\mathcal{F}_\lambda$ which give fully-faithful embeddings $\mathcal{D}(\overline{X}_\lambda) \hookrightarrow \mathcal{D}[X/G]$. For each $\lambda \in G/\!\sim$ we denote by $\iota_\lambda : \overline{X}_\lambda \to \overline{Y}_\lambda$ the natural closed embedding.

**Lemma 3.2.1.** *(i) Let us consider the $G$-equivariant coherent sheaf*

$$\mathcal{F}_\lambda := \underline{H}^0(\iota_\lambda \times \iota)^*\mathcal{K}_\lambda$$

*on $\mathcal{D}(\overline{X}_\lambda \times X)$. Then we have isomorphisms of $G$-equivariant sheaves,*

$$(\mathrm{Id}_{\overline{X}_\lambda} \times \iota)_*\mathcal{F}_\lambda \simeq (\iota_\lambda \times \mathrm{Id}_Y)^*\mathcal{K}_\lambda$$

*and*

$$(\iota_\lambda \times \mathrm{Id}_X)_*\mathcal{F}_\lambda \simeq (\mathrm{Id}_{\overline{Y}_\lambda} \times \iota)^*\mathcal{K}_\lambda.$$

*(ii) For $\mathcal{G} \in \mathcal{D}(\overline{X}_\lambda)$ and $\mathcal{G}' \in \mathcal{D}(\overline{Y}_\lambda)$ there are natural isomorphisms*

$$\iota_*\mathcal{F}_\lambda(\mathcal{G}) \simeq \mathcal{K}_\lambda(\iota_{\lambda*}\mathcal{G}) \quad and$$

$$\mathcal{F}_\lambda(\iota_\lambda^*\mathcal{G}') \simeq \iota^*\mathcal{K}_\lambda(\mathcal{G}')$$

*in $\mathcal{D}[Y/G]$ and $\mathcal{D}[X/G]$, respectively.*

*Proof.* (i) We can rewrite Condition (Res) as stating that the composition

$$\mathcal{K}_\lambda \otimes \pi^*_{\overline{Y}_\lambda} \overline{\mathcal{L}}_\lambda^{-1} \xrightarrow{\varphi_\lambda} \mathcal{K}_\lambda \otimes \pi^*_Y \mathcal{L}^{-1} \xrightarrow{1\otimes s} \mathcal{K}_\lambda$$

is equal to $1 \otimes s_\lambda$ and is injective. It follows that we have an isomorphism (recall all functors are derived)

$$\begin{aligned}(\mathrm{Id}_{\overline{Y}_\lambda} \times \iota)_*(\mathrm{Id}_{\overline{Y}_\lambda} \times \iota)^* \mathcal{K}_\lambda &\cong \mathrm{coker}(1 \otimes s : \mathcal{K}_\lambda \otimes \pi^*_Y \mathcal{L}^{-1} \to \mathcal{K}_\lambda) \\ &\cong \mathrm{coker}(1 \otimes s_\lambda : \mathcal{K}_\lambda \otimes \pi^*_{\overline{Y}_\lambda} \overline{\mathcal{L}}_\lambda^{-1} \to \mathcal{K}_\lambda) \\ &\cong (\iota_\lambda \times \mathrm{Id}_Y)_*(\iota_\lambda \times \mathrm{Id}_Y)^* \mathcal{K}_\lambda.\end{aligned}$$

Note that both $1 \otimes s$ and $1 \otimes s_\lambda$ act by zero on the above coherent sheaf, so we can view it as a coherent sheaf $\mathcal{F}_\lambda$ on $\overline{X}_\lambda \times X$. Now the above isomorphisms immediately give the required properties of $\mathcal{F}_\lambda$.
(ii) These isomorphisms are immediate consequences of isomorphisms of kernels in (i). □

We will need the following compatibility between adjoint kernels.

**Lemma 3.2.2.** *There is an isomorphism*

$$(\iota \times \mathrm{Id}_{\overline{X}_\lambda})_* \mathcal{F}^L_\lambda \cong \mathcal{K}^L_\lambda|_{Y \times \overline{X}_\lambda},$$

*where we use the notation* (2.1).

*Proof.* We have an isomorphism

$$\mathcal{K}^L_\lambda|_{Y \times \overline{X}_\lambda} \cong (\mathcal{K}_\lambda|_{\overline{X}_\lambda \times Y})^L \cong ((\mathrm{Id}_{\overline{X}_\lambda} \times \iota)_* \mathcal{F}_\lambda)^L.$$

Now we use Grothendieck duality to compute $((\mathrm{Id}_{\overline{X}_\lambda} \times \iota)_* \mathcal{F}_\lambda)^L$. Namely, we have

$$\begin{aligned}&((\mathrm{Id}_{\overline{X}_\lambda} \times \iota)_* \mathcal{F}_\lambda)^L \cong \sigma^* \underline{\mathbf{R}\mathrm{Hom}}((\mathrm{Id}_{\overline{X}_\lambda} \times \iota)_* \mathcal{F}_\lambda, \mathcal{O}_{\overline{X}_\lambda \times Y}) \otimes \pi^*_Y \omega_Y[\dim(Y)] \\ &\cong \sigma^*(\mathrm{Id}_{\overline{X}_\lambda} \times \iota)_*(\underline{\mathbf{R}\mathrm{Hom}}(\mathcal{F}_\lambda, (\mathrm{Id}_{\overline{X}_\lambda} \times \iota)^! \mathcal{O}_{\overline{X}_\lambda \times Y} \otimes (\mathrm{Id}_{\overline{X}_\lambda} \times \iota)^*(\pi^*_Y \omega_Y)[\dim(\overline{Y}_\lambda)]).\end{aligned}$$

Next, we observe that

$$\begin{aligned}(\mathrm{Id}_{\overline{X}_\lambda} \times \iota)^! \mathcal{O}_{\overline{X}_\lambda \times Y} \otimes (\mathrm{Id}_{\overline{X}_\lambda} \times \iota)^*(\pi^*_Y \omega_Y)[\dim(\overline{Y}_\lambda)] &\cong \\ (\mathrm{Id}_{\overline{X}_\lambda} \times \iota)^!(\pi^*_Y \omega_Y)[\dim(\overline{Y}_\lambda)] &\cong \\ \pi^*_X \omega_X[\dim(\overline{X}\lambda)].&\end{aligned}$$

Thus, we deduce an isomorphism

$$((\mathrm{Id}_{\overline{X}_\lambda} \times \iota)_* \mathcal{F}_\lambda)^L \cong (\iota \times \mathrm{Id}_{\overline{X}_\lambda})_* \sigma^* \underline{\mathbf{R}\mathrm{Hom}}(\mathcal{F}_\lambda, \pi^*_X \omega_X)[\dim(\overline{X}\lambda)] \cong (\iota \times \mathrm{Id}_{\overline{X}_\lambda})_* \mathcal{F}^L_\lambda.$$

□

3.3. **Proof of Theorem 1.2.2.** We will check that the collection of coherent sheaves $(\mathcal{F}_\lambda)$ satisfies condition (SOD).

**Step 1**. For each $\lambda$, the functor

$$\mathcal{F}_\lambda \colon \mathcal{D}(\overline{X}_\lambda) \to \mathcal{D}[X/G]$$

fully-faithful.

By Lemma 2.3.3, we just need an isomorphism of sheaves

$$\pi_{13*}(\pi^*_{12} \mathcal{F}_\lambda \otimes \pi^*_{23} \mathcal{F}^L_\lambda)^G \cong \mathcal{O}_{\Delta_{\overline{X}_\lambda}}.$$

The following diagram is helpful for following the computations below.

$$\begin{tikzcd}
 & \overline{Y}_\lambda \times Y \times \overline{Y}_\lambda \arrow[r] & \overline{Y}_\lambda \times \overline{Y}_\lambda \\
\overline{Y}_\lambda \times X \times \overline{X}_\lambda \arrow[r, hook] & \overline{Y}_\lambda \times Y \times \overline{X}_\lambda \arrow[r] \arrow[u] & \overline{Y}_\lambda \times \overline{X}_\lambda \arrow[u] \\
\overline{X}_\lambda \times X \times \overline{X}_\lambda \arrow[u, hook] \arrow[rr] & & \overline{X}_\lambda \times \overline{X}_\lambda \arrow[u, hook]
\end{tikzcd}$$

The maps are the obvious embeddings and projections, and the two rectangles are Cartesian.

Now we compute:

$$\begin{aligned}
&(\iota_\lambda \times \mathrm{Id}_{\overline{X}_\lambda})_*(\pi_{13*}(\pi_{12}^*\mathcal{F}_\lambda \otimes \pi_{23}^*\mathcal{F}_\lambda^L)) \\
&\cong \pi_{13*}(((\mathrm{Id}_{\overline{Y}_\lambda} \times \iota \times \mathrm{Id}_{\overline{X}_\lambda})_*(\iota_\lambda \times \mathrm{Id}_{X \times \overline{X}_\lambda})_*(\pi_{12}^*\mathcal{F}_\lambda \otimes \pi_{23}^*\mathcal{F}_\lambda^L))) \\
&\cong \pi_{13*}(((\mathrm{Id}_{\overline{Y}_\lambda} \times \iota \times \mathrm{Id}_{\overline{X}_\lambda})_*(\pi_{12}^*(\iota_\lambda \times \mathrm{Id}_X)_*\mathcal{F}_\lambda \otimes \pi_{23}^*\mathcal{F}_\lambda^L))) \\
&\cong \pi_{13*}((\mathrm{Id}_{\overline{Y}_\lambda} \times \iota \times \mathrm{Id}_{\overline{X}_\lambda})_*(\pi_{12}^*\mathcal{K}_\lambda|_{\overline{Y}_\lambda \times X} \otimes \pi_{23}^*\mathcal{F}_\lambda^L)) \\
&\cong \pi_{13*}(\pi_{12}^*\mathcal{K}_\lambda \otimes (\mathrm{Id}_{\overline{Y}_\lambda} \times \iota \times \mathrm{Id}_{\overline{X}_\lambda})_*\pi_{23}^*\mathcal{F}_\lambda^L) \\
&\cong \pi_{13*}(\pi_{12}^*\mathcal{K}_\lambda \otimes \pi_{23}^*(\iota \times \mathrm{Id}_{\overline{X}_\lambda})_*\mathcal{F}_\lambda^L),
\end{aligned}$$

where we used commutativity of the bottom rectangle, the projection formula, and one of the defining isomorphisms for $\mathcal{F}_\lambda$ (see Lemma 3.2.1). Now using Lemma 3.2.2, we can rewrite this as

$$\pi_{13*}(\pi_{12}^*\mathcal{K}_\lambda \otimes \pi_{23}^*\mathcal{K}_\lambda^L|_{Y \times \overline{X}_\lambda}) \cong (\pi_{13*}(\pi_{12}^*\mathcal{K}_\lambda \otimes \pi_{23}^*\mathcal{K}_\lambda^L))|_{\overline{Y}_\lambda \times \overline{X}_\lambda}$$

Now by taking $G$-invariants, we get

$$\begin{aligned}
(\bar{\iota}_\lambda \times \mathrm{Id})_*(\pi_{13*}(\pi_{12}^*\mathcal{F}_\lambda \otimes \pi_{23}^*\mathcal{F}_\lambda^L))^G &\cong (\pi_{13*}(\pi_{12}^*\mathcal{N}_\lambda \otimes \pi_{23}^*\mathcal{N}_\lambda^L))^G|_{\overline{Y}_\lambda \times \overline{X}_\lambda} \\
&\cong \mathcal{O}_{\Delta_{\overline{Y}_\lambda}}|_{\overline{Y}_\lambda \times \overline{X}_\lambda} \\
&\cong (\bar{\iota}_\lambda \times \mathrm{Id})_*\mathcal{O}_{\Delta_{\overline{X}_\lambda}}
\end{aligned}$$

The only isomorphism that needs explaining is the second which is true by Condition (SOD) and by Lemma 2.2.2(ii). This completes the proof.

**Step 2**. Let $\leq$ be the total order given by Condition (SOD). Then if $i > j$, one has

$$\mathbf{R}\mathrm{Hom}(\mathcal{F}_{\lambda_i}(\cdot), \mathcal{F}_{\lambda_j}(\cdot)) = 0.$$

By Lemma 2.2.2(iii), we need to prove that

$$\pi_{13*}(\pi_{12}^*\mathcal{F}_{\lambda_i} \otimes \pi_{23}^*\mathcal{F}_{\lambda_j}^L)^G = 0.$$

The proof is analogous to the one in Step 1, where instead we use the fact that for $\lambda > \mu$,

$$\pi_{13*}(\pi_{12}^*\mathcal{N}_\lambda \otimes \pi_{23}^*\mathcal{N}_\mu^L)^{W_\lambda \times G \times W_\mu} = 0.$$

**Step 3**. Let $\leq$ be the total order given by Condition (SOD). Then there is a semiorthogonal decomposition

$$\mathcal{D}[X/G] = \langle \mathcal{F}_{\lambda_1}(\mathcal{D}(\overline{X}_{\lambda_1})), \ldots, \mathcal{F}_{\lambda_r}(\mathcal{D}(\overline{X}_{\lambda_r}))\rangle$$

We already know the needed semiorthogonalities. Suppose $\mathcal{H} \in \mathcal{D}[X/G]$ is right-orthogonal to the images of $\mathcal{F}_\lambda$. By adjunction, we have for any $\lambda$ and any $\mathcal{G} \in \mathcal{D}(\overline{Y}_\lambda)$,

$$\mathbf{R}\mathrm{Hom}(\mathcal{K}_\lambda(\mathcal{G}), \iota_*\mathcal{H}) \cong \mathbf{R}\mathrm{Hom}(\iota^*\mathcal{K}_\lambda(\mathcal{G}), \mathcal{H}) \cong \mathbf{R}\mathrm{Hom}(\mathcal{F}_\lambda(\iota_\lambda^*\mathcal{G}), \mathcal{H}) = 0,$$

where we used Lemma 3.2.1(ii). Thus, $\iota_*\mathcal{H}$ is in the right-orthogonal to the images of $\mathcal{K}_\lambda$. By Condition (SOD), we know that this collection of subcategories is full. Hence, we get $\iota_*\mathcal{H} = 0$ and so $\mathcal{H} = 0$.

## 4. Restriction to an invariant open subset and passing to quotient stacks

4.1. **Restriction to an invariant open subset.** Let $Y$ be a fixed smooth variety with an action of a finite group $G$, and assume that $(\mathcal{K}_\lambda)$ are $G$-equivariant coherent sheaves on $\overline{Y}_\lambda \times Y$ satisfying the condition (SOD).

**Lemma 4.1.1.** *Let $\overline{U} \subset Y/G$ be an open subset, and let $U \subset Y$ and $\overline{U}_\lambda \subset \overline{Y}_\lambda$ be its preimages in $Y$ and $\overline{Y}_\lambda$. Assume that for each $\lambda$, we have*

$$\mathrm{supp}(\mathcal{K}_\lambda) \cap (\overline{U}_\lambda \times Y) \subset \overline{U}_\lambda \times U \quad \textit{and} \quad \mathrm{supp}(\mathcal{K}_\lambda) \cap (\overline{Y}_\lambda \times U) \subset \overline{U}_\lambda \times U.$$

*Then the $G$-equivariant coherent sheaves $(\mathcal{K}_\lambda^U := \mathcal{K}_\lambda|_{\overline{U}_\lambda \times U})$ still satisfy condition (SOD) with respect to the action of $G$ on $U$. Furthermore, we have a commutative diagram*

$$\begin{tikzcd} \mathcal{D}(\overline{Y}_\lambda) \arrow[r, "\mathcal{K}_\lambda"] \arrow[d, "j_\lambda^*"'] & \mathcal{D}[Y/G] \arrow[d, "j^*"'] \\ \mathcal{D}(\overline{U}_\lambda) \arrow[r, "\mathcal{K}_\lambda^U"] & \mathcal{D}[U/G] \end{tikzcd} \tag{4.1}$$

*where the vertical arrows are the restriction functors with respect to the open embeddings $j : U \hookrightarrow Y$ and $j_\lambda : \overline{U}_\lambda \to Y$.*

*Proof.* First, we observe that

$$\mathrm{supp}(\mathcal{K}_\lambda) = \mathrm{supp}(\mathcal{K}_\lambda) \cap (\overline{U}_\lambda \times Y) = \mathrm{supp}(\mathcal{K}_\lambda) \cap (Y_\lambda \times U).$$

This implies that $\mathrm{supp}(\mathcal{N}_\lambda)$ is proper over both $\overline{U}_\lambda$ and $U$. We claim that

$$\mathcal{K}_\lambda|_{\overline{U}_\lambda \times Y} \simeq (\mathrm{Id} \times j)_*\mathcal{K}_\lambda^U, \tag{4.2}$$

$$\mathcal{K}_\lambda|_{\overline{Y}_\lambda \times U} \simeq (j_\lambda \times \mathrm{Id})_*\mathcal{K}_\lambda^U. \tag{4.3}$$

Indeed, let $i : S \hookrightarrow \overline{Y}_\lambda \times Y$ be the embedding of a closed subscheme with the underlying closed subset $\mathrm{supp}(\mathcal{K}_\lambda)$, such that $\mathcal{K}_\lambda = i_*\mathcal{G}$, for some coherent sheaf $\mathcal{G}$ on $S$. Let $S' = S \cap (\overline{U}_\lambda \times Y)$. Then we have

$$\mathcal{K}_\lambda|_{\overline{U}_\lambda \times Y} \simeq i'_*\mathcal{G}',$$

where $i' : S' \hookrightarrow \overline{U}_\lambda \times Y$ is the natural closed embedding, and $\mathcal{G}' = \mathcal{G}|_{S'}$. Since $S'$ is contained in $\overline{U}_\lambda \times U$, the embedding $i'$ factors as the composition of a closed embedding $S' \hookrightarrow \overline{U}_\lambda \times U$ followed by $\mathrm{Id} \times j$. This easily implies the claimed isomorphism (4.2). The proof of (4.3) is the same, exchanging the roles of the two factors.

Next, let us check that $\mathcal{K}_\lambda^U$ defines a fully faithful functor

$$\mathcal{D}(\overline{U}_\lambda) \to \mathcal{D}[U/G].$$

By Lemma, 2.2.2(i), we have

$$\pi_{13*}(\pi_{12}^*\mathcal{K}_\lambda \otimes \pi_{23}^*\mathcal{K}_\lambda^L)^G \simeq \mathcal{O}_{\Delta_{\overline{Y}_\lambda}}.$$

Restricting this isomorphism to $\overline{U}_\lambda \times \overline{U}_\lambda$ we get

$$[\pi_{13*}(\pi_{12}^*\mathcal{K}_\lambda \otimes \pi_{23}^*\mathcal{K}_\lambda^L)|_{\overline{U}_\lambda \times Y \times \overline{U}_\lambda}]^G \simeq \mathcal{O}_{\Delta_{\overline{U}_\lambda}}.$$

Now using (4.2), we obtain

$$\begin{aligned}(\pi_{12}^*\mathcal{K}_\lambda \otimes \pi_{23}^*\mathcal{K}_\lambda^L)|_{\overline{U}_\lambda \times Y \times \overline{U}_\lambda} &\simeq \pi_{12}^*\mathcal{K}_\lambda|_{\overline{U}_\lambda \times Y} \otimes \pi_{23}^*\mathcal{K}_\lambda^L|_{Y \times \overline{U}_\lambda} \\ &\simeq (\mathrm{Id}_{\overline{U}_\lambda} \times j \times \mathrm{Id}_{\overline{U}_\lambda})_*(\pi_{12}^*\mathcal{K}_\lambda^U) \otimes \pi_{23}^*\mathcal{K}_\lambda^L|_{Y \times \overline{U}_\lambda} \\ &\simeq (\mathrm{Id}_{\overline{U}_\lambda} \times j \times \mathrm{Id}_{\overline{U}_\lambda})_*(\pi_{12}^*\mathcal{K}_\lambda^U \otimes \pi_{23}^*\mathcal{K}_\lambda^L|_{U \times \overline{Y}_\lambda}).\end{aligned}$$

Finally, we observe that $\mathcal{K}_\lambda^L|_{U \times \overline{Y}_\lambda} \simeq (\mathcal{K}_\lambda^U)^L$, so the above calculation gives an isomorphism

$$(\mathcal{K}_\lambda^U \circ_U (\mathcal{K}_\lambda^U)^L)^G \simeq \mathcal{O}_{\Delta_{\overline{U}_\lambda}},$$

Thus, by Lemma 2.2.2(ii), we deduce that the functors defined by $(\mathcal{K}_\lambda^U)$ are fully faithful.

A similar argument shows that the semiorthogonalities are still satisfied for the functors given by $(\mathcal{K}_\lambda^U)$. The commutativity of diagram (4.1) follows easily from (4.3).

Finally, to check that the images of these functors generate $\mathcal{D}[U/G]$, we observe that for any object $\mathcal{G} \in \mathcal{D}[U/G]$, there exists $\mathcal{F}$ in $\mathcal{D}[Y/G]$ such that $j^*\mathcal{F} \cong \mathcal{G}$. Since the images of the functors $\mathcal{K}_\lambda$ generate $\mathcal{D}[Y/G]$, there exists a sequence of cones that builds $\mathcal{F}$ from the subcategories given by the images of $(\mathcal{K}_\lambda)$. Using commutative diagrams (4.1), we can build $j^*\mathcal{F}$ with a sequence of cones from the images of $(\mathcal{K}_\lambda^U)$.

□

4.2. **Passing to the quotient stacks.** Now assume again that we have a smooth variety $Y$ with an action of a finite group $G$, and $G$-equivariant coherent sheaves $(\mathcal{K}_\lambda)$ on $\overline{Y}_\lambda \times Y$, satisfying condition (SOD). Assume in addition that there is an reductive algebraic group $\mathbb{G}$ acting on $Y$, such that the actions of $G$ and $\mathbb{G}$ commute. In particular, the subvarieties $Y_\lambda$ acquire the action of $W_\lambda \times \mathbb{G}$ and there is an induced action of $\mathbb{G}$ on $\overline{Y}_\lambda = Y_\lambda/W_\lambda$. Assume also that each sheaf $\mathcal{K}_\lambda$ is equipped with a $(G \times \mathbb{G}$-equivariant structure (where $\mathbb{G}$ acts diagonally on $\overline{Y}_\lambda \times Y$). In this case each sheaf $\mathcal{K}_\lambda$ defines the Fourier-Mukai functor

$$\Phi_\lambda^{\mathbb{G}} = \Phi_{\mathcal{K}_\lambda}^{\mathbb{G}} : \mathcal{D}[\overline{Y}_\lambda/\mathbb{G}] \to \mathcal{D}[Y/(G \times \mathbb{G})]$$

that fits into a commutative square

$$\begin{tikzcd} \mathcal{D}[\overline{Y}_\lambda/\mathbb{G}] \arrow[r, "\Phi_\lambda^{\mathbb{G}}"] \arrow[d] & \mathcal{D}[Y/(G \times \mathbb{G})] \arrow[d] \\ \mathcal{D}(\overline{Y}_\lambda) \arrow[r, "\Phi_\lambda"] & \mathcal{D}[Y/G] \end{tikzcd}$$

where $\Phi_\lambda = \Phi_{\mathcal{K}_\lambda}$ and the vertical arrows are given by forgetting the $\mathbb{G}$-action.

**Lemma 4.2.1.** *Assume in addition that there exists a $\mathbb{G}$-equivariant ample line bundle on $Y$. Then the functors $\Phi_\lambda^{\mathbb{G}}$ are fully faithful and their images form a semiorthogonal decomposition of $\mathcal{D}[Y/(G\times\mathbb{G})]$ (ordered in the same way as for the semiorthogonal decomposition of $\mathcal{D}[Y/G]$).*

*Proof.* For a pair of objects $\mathcal{F},\mathcal{G}\in\mathcal{D}[\overline{Y}_\lambda/\mathbb{G}]$, we have a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}[\overline{Y}_\lambda/\mathbb{G}]}(\mathcal{F},\mathcal{G}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}[Y/(G\times\mathbb{G})]}(\Phi_\lambda^{\mathbb{G}}(\mathcal{F}),\Phi_\lambda^{\mathbb{G}}(\mathcal{G})) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}(\overline{Y}_\lambda)}(\mathcal{F},\mathcal{G})^{\mathbb{G}} \arrow[r] & \mathrm{Hom}_{\mathcal{D}[Y/G]}(\Phi_\lambda(\mathcal{F}),\Phi_\lambda(\mathcal{G}))^{\mathbb{G}}
\end{tikzcd}$$

in which the vertical arrows are isomorphisms. Furthermore, since $\Phi_\lambda$ is fully faithful, the bottom horizontal arrow is an isomorphism. Hence, the top horizontal arrow is also an isomorphism, i.e., $\Phi_\lambda^{\mathbb{G}}$ is fully faithful.

Similarly, if $\mathrm{Hom}(\Phi_\lambda(\cdot),\Phi_\mu(\cdot))=0$ then by passing to $\mathbb{G}$-invariants, we deduce that $\mathrm{Hom}(\Phi_\lambda^{\mathbb{G}}(\cdot),\Phi_\mu^{\mathbb{G}}(\cdot))=0$. Hence, the semiorthogonality still holds for the images of $(\Phi_\lambda^{\mathbb{G}})$.

Finally, to see that the images of $(\Phi_\lambda^{\mathbb{G}})$ generate everything, we observe that each functor $\Phi_\lambda^{\mathbb{G}}$ commutes with tensoring with any finite-dimensional $\mathbb{G}$-representation. Thus, if for some $\mathcal{F}\in\mathcal{D}[Y/(G\times\mathbb{G})]$ we have

$$\mathrm{Hom}_{\mathcal{D}[Y/(G\times\mathbb{G})]}(\Phi_\lambda^{\mathbb{G}}(\cdot),\mathcal{F})=0 \text{ for all } \lambda,$$

then this implies that

$$\mathrm{Hom}_{\mathcal{D}[Y/G]}(\Phi_\lambda(\mathcal{G})\otimes V,\mathcal{F})^{\mathbb{G}} \simeq (\mathrm{Hom}_{\mathcal{D}[Y/G]}(\Phi_\lambda(\mathcal{G}),\mathcal{F})\otimes V^\vee)^{\mathbb{G}}=0$$

for all $\lambda$ and all $\mathcal{G}\in\mathcal{D}[\overline{Y}_\lambda/\mathbb{G}]$. Since, $\mathrm{Hom}_{\mathcal{D}[Y/G]}(\Phi_\lambda(\mathcal{G}),\mathcal{F})$ is the union of finite-dimensional $\mathbb{G}$-representations, we deduce the vanishing of this space for any $\lambda$ and any $\mathcal{G}\in\mathcal{D}[\overline{Y}_\lambda/\mathbb{G}]$.

We will be able to conclude that $\mathcal{F}=0$ once we know that for each $\lambda$, the image of the forgetful functor $\mathcal{D}[\overline{Y}_\lambda/\mathbb{G}]\to\mathcal{D}(\overline{Y}_\lambda)$ generates $\mathcal{D}(\overline{Y}_\lambda)$. To this end, we use the fact that $\mathcal{D}(\overline{Y}_\lambda)$ is generated by tensor powers of an ample line bundle. Thus, it is enough to construct a $\mathbb{G}$-equivariant ample line bundle on $Y/G$ (which by the pull-back would give a $\mathbb{G}$-equivariant ample line bundle on each $\overline{Y}_\lambda$). We know that there exists a $\mathbb{G}$-equivariant ample line bundle $L$ on $Y$. By taking tensor products of $g^*L$ over all $g\in G$, we can assume that $L$ is $G\times\mathbb{G}$-equivariant. Then there exists $N>0$ such that $L^N$ descends to a (necessarily ample) $\mathbb{G}$-equivariant line bundle on $Y/G$.

$\square$

## 5. Examples of semiorthogonal decompositions obtained from Theorem 1.2.2

5.1. **Case when $\mathcal{K}_\lambda$ comes from a vector bundle over $Z_\lambda$.** Recall (see (1.1)) that for each $\lambda\in G/\sim$ we define the subscheme $Z_\lambda\subset Y_\lambda\times Y$ as the union of the graphs of the embeddings $g: Y_\lambda\to Y$, over $g\in G$, and we set $\overline{Z}_\lambda = Z_\lambda/W_\lambda$.

In the examples we will consider, the kernels $(\mathcal{K}_\lambda)$, giving the semiorthogonal decomposition of $\mathcal{D}[Y/G]$, will actually be $G$-equivariant sheaves on $\overline{Z}_\lambda$ of the form

$$\mathcal{K}_\lambda = \overline{\mathcal{N}}_\lambda := (p_{\lambda *}\mathcal{N}_\lambda)^{W_\lambda}$$

where $p_\lambda : Z_\lambda \to \overline{Z}_\lambda$ is the natural projection, and $\mathcal{N}_\lambda$ is a $W_\lambda \times G$-equivariant vector bundles over $Z_\lambda$. Note that the sheaf $\overline{\mathcal{N}}_\lambda$ has proper support over both $Y_\lambda$ and $Y$. Furthermore, as the following lemma shows, checking condition (Res) for them also becomes quite easy.

**Lemma 5.1.1.** *Assume that for each $\lambda$ we have a $W_\lambda \times G$-equivariant vector bundle $\mathcal{N}_\lambda$ over $Z_\lambda$. Assume that the section $s$ of $\mathcal{L}$ (coming from a section $\overline{s} \in H^0(Y/G, \overline{\mathcal{L}})$ has nonzero restrictions to all connected components of $Y_\lambda$ for each $\lambda$. Then condition (Res) is satisfied for $(\overline{\mathcal{N}}_\lambda)$.*

*Proof.* Let us define the closed subscheme $Z \subset Y \times Y$ from the commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "p"'] & Y \times Y \arrow[d] \\ Y/G \arrow[r, "\Delta"] & Y/G \times Y/G \end{tikzcd}$$

Let $p_1, p_2 : Z \to Y$ be two natural projections. Then we have a natural isomorphism

$$p_1^*\mathcal{L} \simeq p_2^*\mathcal{L} \simeq p^*\overline{\mathcal{L}}.$$

Furthermore, if $\overline{s}$ is a section of $\overline{\mathcal{L}}$ on $Y/G$ then its pull-backs $p_1^*(\overline{s})$ and $p_2^*(\overline{s})$ get identified by this isomorphism.

Now it is easy to see that for each $\lambda$, we have $Z_\lambda \subset Z$. Hence, we deduce isomorphisms

$$\mathcal{N}_\lambda \otimes \pi_Y^*\mathcal{L} \simeq \mathcal{N}_\lambda \otimes \pi_{Y_\lambda}^*\mathcal{L}|_{Y_\lambda}$$

such that the morphisms given by $\pi_Y^* s$ and $\pi_{Y_\lambda}^* s_\lambda$ get identified (where $s_\lambda$ is the section of the pull-back $\mathcal{L}_\lambda$ of $\mathcal{L}$ to $Y_\lambda$, induced by $s$).

Next, we claim that the section $\pi_{Y_\lambda}^* s_\lambda|_{Z_\lambda}$ of $\pi_{Y_\lambda}^*\mathcal{L}|_{Z_\lambda}$ is not a zero-divisor. Indeed, since $Z_\lambda$ is reduced, local functions on $Z_\lambda$ are determined by their restrictions to the irreducible components $(\mathrm{Id} \times g)(\Gamma_\lambda)$. Thus, we need to check that $s_\lambda$ is not a zero-divisor on $Y_\lambda$. Since $Y_\lambda$ is smooth, this is equivalent to $s_\lambda$ being nonzero on each connected component, which is true by assumption.

Finally, taking the push-forward to $\overline{Y}_\lambda \times Y$ and passing to $W_\lambda$-invariants, we deduce the similar assertions for $\overline{\mathcal{N}}_\lambda$. □

Using the above lemma we can restate a special case of Theorem 1.2.2, combined with Lemma 4.1.1, in the following way. Let us consider the following condition (SOD+) on a collection $(\mathcal{N}_\lambda)$, $\lambda \in G/\sim$, where $\mathcal{N}_\lambda$ is a $W_\lambda \times G$-equivariant vector bundle on $Z_\lambda$:

(SOD+) For each $\lambda \in G/\sim$, set $\overline{\mathcal{N}}_\lambda = (p_{\lambda *}\mathcal{N}_\lambda)^{W_\lambda}$. Then the functors

$$\overline{\mathcal{N}}_\lambda : \mathcal{D}(Y_\lambda/W_\lambda) \to \mathcal{D}[Y/G]$$

are fully faithful, and the corresponding subcategories form a semiorthogonal decomposition of $\mathcal{D}[Y/G]$ with respect to some total ordering on $G/\sim$.

**Proposition 5.1.2.** *Assume that $Y$ is smooth and condition (SOD+) is satisfied for a collection $(\mathcal{N}_\lambda)_{\lambda\in G/\sim}$ of $W_\lambda \times G$-equivariant bundles. Then*
*(i) Condition (SOD+) still holds for the restrictions $(\mathcal{N}_\lambda|_{U_\lambda\times U})$, where $U \subset Y$ is any $G$-invariant open subset, $U_\lambda = U \cap Y_\lambda$.*
*(ii) Let $X \subset Y$ be a divisor given by the pull-back $s$ of a global section $\overline{s}$ of a line bundle on $Y/G$. Let $T \subset X$ be a $G$-invariant closed subset containing the singular locus of $X$. Assume that for each $\lambda$, the restriction of $s$ to every connected component of $Y_\lambda \setminus T$ is nonzero. Then condition (SOD) is satisfied for some collection of $G$-equivariant coherent sheaves on $(X_\lambda \setminus T) \times X \setminus T$.*

*Proof.* (i) This follows from Lemma 4.1.1 since

$$Z_\lambda \cap (U_\lambda \times Y) \subset U_\lambda \times U \quad\text{and}\quad Z_\lambda \cap (Y_\lambda \times U) \subset U_\lambda \times U.$$

(ii) First, using part (i), we replace $Y$ by the $G$-invariant open subset $U = Y \setminus T$ and $X$ by $X \cap U = X \setminus T$, which is smooth. Now Lemma 5.1.1 shows that (SOD) and (Res) hold for this situation, so it remains to apply Theorem 1.2.2. □

For applications it is useful to know that the pull-back functor $\mathcal{D}(Y/G) \to \mathcal{D}[Y/G]$ always corresponds to a kernel of the type required in (SOD+). In this case we take $\lambda = 1$, the trivial conjugacy class, so that $Z_1 \subset Y \times Y$ is the union of the graphs of elements $g \in G$ acting on $Y$, with the reduced subscheme structure. The following lemma shows that if we take $\mathcal{N}_\lambda = \mathcal{O}_{Z_1}$ then the corresponding kernel $\overline{\mathcal{N}}_\lambda = \mathcal{O}_{\overline{Z}_1}$ is exactly the structure sheaf of the graph of the projection $Y \to Y/G = \overline{Y}_1$.

**Lemma 5.1.3.** *Let $G$ act on $Z_1$ via the first component. Then the closed subscheme $Z_1/G \subset Y/G \times Y$ coincides with the graph of the projection $Y \to Y/G$.*

*Proof.* We can assume $Y$ to be affine, $Y = \mathrm{Spec}(A)$. Then $Z_1$ is the closed subscheme of $Y \times Y$ corresponding to the image of the algebra homomorphism

$$\alpha : A \otimes A \to \prod_g A : a_1 \otimes a_2 \mapsto (a_1 \cdot g(a_2)).$$

One immediately checks that $\alpha$ is $G$-equivariant, where $G$ acts on the first component of $A \otimes A$ and acts on $\prod_g A$ as follows:

$$g_1 \cdot (a_g) = (b_g), \quad \text{where } b_g = g_1(a_{g_1^{-1}g}).$$

Note that $G$-invariants in $\prod_g A$ are identified with $A$, via the projection $(a_g) \mapsto a_1$. Thus, the morphism of $G$-invariant subalgebras induced by $\alpha$ can be identified with the surjective map

$$\overline{\alpha} : A^G \otimes A \to A : a_0 \otimes a \mapsto a_0 \cdot a.$$

This easily implies the statement. □

5.2. **Motivic decomposition for $\mathcal{D}[\mathbb{A}^n/S_n]$.** Now we will focus on the case of the standard action of the symmetric group $S_n$ on the affine $n$-space, $V = \mathbb{A}^n$. The results of [PV15, Section 6.1] imply that condition (SOD+) is satisfied for this action, as well as for its restriction to invariant open subsets. In particular, there is a semiorthogonal decomposition for the category $\mathcal{D}[\mathbb{P}^{n-1}/S_n]$ (see Corollary 5.2.3 below).

The conjugacy classes of $S_n$ are labelled by partitions $\lambda$ of $n$. Recall that the *dominance partial ordering* $\le$ on partitions of $n$ is defined by $\lambda \ge \mu$ if $\lambda_1 + \cdots + \lambda_i \ge \mu_1 + \cdots + \mu_i$ for all $i \ge 1$. Then $(n)$ is the biggest partition and $(1)^n$ is the smallest.

For each partition $\lambda$ of $n$, define $V_\lambda \subset V$ to be the fixed locus of a permutation with the cycle type $\lambda$ (this is well-defined up to the $S_n$-action). The group $W_\lambda = \prod_i S_{r_i}$, where the $r_i$ are the multiplicity of the part $i$ in $\lambda$, acts on $V_\lambda$ and we set $\overline{V}_\lambda = V_\lambda / W_\lambda$.

Recall that we have the reduced subscheme $Z_\lambda \subset V_\lambda \times V$, invariant under the action of $W_\lambda \times S_n$ (see (1.1)). We set

$$\mathcal{N}_\lambda = \mathcal{O}_{Z_\lambda},$$

so that $\overline{\mathcal{N}}_\lambda$ is the structure sheaf of $\overline{Z}_\lambda = Z_\lambda / W_\lambda$.

**Theorem 5.2.1** ([PV15, Theorem 6.3.1])**.** *For each $\lambda$, $|\lambda| = n$, the functor $\overline{\mathcal{N}}_\lambda : \mathcal{D}(\overline{V}_\lambda) \to \mathcal{D}[V/S_n]$ is fully-faithful. We have*

$$\mathbf{R}\mathrm{Hom}(\overline{\mathcal{N}}_\lambda(\cdot), \overline{\mathcal{N}}_\mu(\cdot)) = 0$$

*for $\lambda \not\leq \mu$. For any total ordering $\lambda_1 < \cdots < \lambda_p$ of partitions of $n$, refining the dominance order, we have a semiorthogonal decomposition*

$$\mathcal{D}[\mathbb{A}^n/S_n] = \langle \overline{\mathcal{N}}_{\lambda_1}(\mathcal{D}(\overline{V}_{\lambda_1})), \ldots, \overline{\mathcal{N}}_{\lambda_p}(\mathcal{D}(\overline{V}_{\lambda_p})) \rangle.$$

Thus, condition (SOD+) holds for the action of $S_n$ on $\mathbb{A}^n$ and the collection $(\mathcal{O}_{Z_\lambda})$. By Proposition 5.1.2, we derive the following

**Corollary 5.2.2.** *For any $S_n$-invariant open subset $U \subset \mathbb{A}^n$, condition (SOD+) holds for the $S_n$-action on $U$ and the structure sheaves of $Z_\lambda \cap (U_\lambda \times U)$.*

For each $\lambda$, the natural $\mathbb{G}_m$-action on $V_\lambda$ induces a $\mathbb{G}_m$-action on $\overline{V}_\lambda$. We denote by

$$\mathbb{P}\overline{V}_\lambda = [(\bar{V}_\lambda \setminus \{0\})/\mathbb{G}_m]$$

the corresponding weighted projective stack. Note that the induced weights on $\bar{V}_\lambda$ are not all 1. For example, for $\lambda = (1)^n$, we have $\overline{V}_\lambda = V/S_n$, and we can take the elementary symmetric functions as coordinates on $V/S_n$, which have weights $1, 2, \ldots, n$, so in this case we get the weighted projective stack $\mathbb{P}(1, 2, \ldots, n)$.

**Corollary 5.2.3.** *There is a semiorthogonal decomposition*

$$\mathcal{D}[\mathbb{P}^{n-1}/S_n] \cong \langle \mathcal{D}(\mathbb{P}\overline{V}_{\lambda_1}), \ldots, \mathcal{D}(\mathbb{P}\overline{V}_{\lambda_p}) \rangle.$$

*Proof.* This follows from Corollary 5.2.2 applied to the open subset $\mathbb{A}^n \setminus \{0\} \subset \mathbb{A}^n$ and from Lemma 4.2.1 applied to the natural $\mathbb{G}_m$-equivariant structures on the sheaves $\mathcal{N}_\lambda$. □

5.3. **$S_n$-invariant hypersurfaces.** Let $f \in \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ be an $S_n$-invariant polynomial, and let $V(f) \subset \mathbb{A}^n$ be the corresponding hypersurface. Using Theorem 5.2.1 and Proposition 5.1.2, we get the motivic decomposition for the action of $S_n$ on any $S_n$-invariant smooth open part of $V(f)$.

**Corollary 5.3.1.** *Let $T \subset \mathbb{A}^n$ be an $S_n$-invariant closed subset containing the singular locus of $V(f)$. Assume that for every partition $\lambda$, such that $V_\lambda \setminus T \neq \emptyset$, the restriction $f|_{V_\lambda}$ is not identicially zero. Then there exists a collection of $W_\lambda \times G$-equivariant sheaves for the $G$-action on $V(f) \setminus T$ such that conditon (SOD) is satisfied. In particular, $\mathcal{D}[V(f) \setminus T/S_n]$ admits a motivic decomposition.*

Now we are ready to prove Theorem 1.2.3. Recall that in this theorem we assume that $f$ is a *homogeneous* $S_n$-invariant polynomial such that $\mathbb{P}V(f)$ is smooth.

*Proof of Theorem 1.2.3.* First, we claim that

$$f(1,1,\ldots,1)\neq 0. \tag{5.1}$$

Indeed, we have the identity $d\cdot f=\sum_{i=1}^n x_i f_{x_i}$, where $(f_{x_i})$ are the partial derivatives of $f$ and $d$ is the degree of $f$. Evaluating this at $(1,\ldots,1)$ we get

$$d\cdot f(1,\ldots,1)=\sum_{i=1}^n f_{x_i}(1,\ldots,1).$$

But by $S_n$-invariance, we have $f_{x_i}(1,\ldots,1)=f_{x_j}(1,\ldots,1)$ for all $i,j$. Thus, the above identity implies that

$$\frac{d}{n}f(1,\ldots,1)=f_{x_i}(1,\ldots,1)$$

for all $i=1,\ldots,n$. Thus, if $f(1,\ldots,1)=0$ then $\mathbb{P}V(f)$ would be singular.

Note that condition (5.1) implies that for every partition $\lambda$, the restriction $f_\lambda := f|_{V_\lambda}$ is not identically zero. It remains to apply Corollary 5.3.1 to get a semiorthogonal decomposition of $\mathcal{D}[V(f)\setminus\{0\}/S_n]$, and then use Lemma 4.2.1 to pass to quotients by $\mathbb{G}_m$. □

The semiorthogonal decomposition given by Theorem 1.2.3 is usually not motivic, since its pieces are derived categories of some quotient stacks. The biggest piece of the semiorthogonal decomposition of $\mathcal{D}[\mathbb{P}V(f)/S_n]$ corresponds to the partition $\lambda=(1)^n$ and is the image of the pull-back functor with respect to the natural morphism of stacks

$$\pi:[\mathbb{P}V(f)/S_n]\to\mathbb{P}V(\overline{f})$$

where $\overline{f}$ is $f$ viewed as a quasihomogeneous polynomial on $\mathbb{A}^n/S_n$ (and the target is the weighted projective stacky hypersurface). The morphism $\pi$ fits into a Cartesian diagram

$$\begin{tikzcd}
{[V(f)\setminus\{0\}/S_n]} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\mathbb{G}_m"'] & V(\overline{f})\setminus\{0\} \arrow[d, "\mathbb{G}_m"'] \\
{[\mathbb{P}V(f)/S_n]} \arrow[r, "\pi"] & \mathbb{P}V(\overline{f})
\end{tikzcd}$$

in which the vertical arrows are $\mathbb{G}_m$-torsors and the top horizontal arrow is the coarse moduli map for the action of $S_n$ on $V(f)\setminus\{0\}$. Note that the fact that the pull-back functor under $\pi$ is fully faithful can be directly deduced from the above diagram. Indeed, by the projection formula, it is enough to check that $R\pi_*\mathcal{O}\simeq\mathcal{O}$. By the base change formula, this reduces to a similar assertion for the morphism $\widetilde{\pi}$, so it boils down to the same fact about the projection $[\mathbb{A}^n/S_n]\to\mathbb{A}^n/S_n$.

**Example 5.3.2** ($S_3$-invariant plane curves)**.** Let $C=\mathbb{P}V(f)\subset\mathbb{P}^2$ be an $S_3$-invariant degree $d$ plane curve. We assume that $C$ is smooth. This implies that for $\lambda=(3)$, we get $V(\bar{f}_\lambda)=\{0\}$. Hence, the corresponding piece in the semiorthogonal decomposition of $\mathcal{D}[\mathbb{P}V(f)/S_3]$ is empty. Let us consider the contributions of the two remaining partitions: $\lambda_1=(1)^3$ and $\lambda_2=(2,1)$.

$\lambda_1$: We have identifications $V_{\lambda_1} = V$, $X_{\lambda_1} = X = V(f)$, $\bar{V}_{\lambda_1} \cong \mathbb{A}^3_{1,2,3}$, where the subscripts indicate the $\mathbb{G}_m$-weights. The vanishing locus of $\bar{f}_{\lambda_1}$, $V(\bar{f}_{\lambda_1})$ will give a smooth stacky curve in $\mathbb{P}(1,2,3)$.

$\lambda_2$: We have identifications $V_{\lambda_2} = \{y = z\} \subset V$, and $f_{\lambda_2}$ is the restriction of $f$ to this plane. Since $V(f_{\lambda_2})$ is smooth away from the origin, it is the union of $d$ lines through the origin, say $l_1, \ldots, l_d$. The projectivization is the union of $d$ distinct (non-stacky) points $p_1, \ldots, p_d$.

In the case $d = 3$, i.e., when $C$ is an elliptic curve, we can be even more precise about the piece corresponding to $\lambda_1$. Namely, in this case

$$f(x, y, z) = \alpha e_1^3 + \beta e_1 e_2 + \gamma e_3,$$

where $e_1, e_2, e_3$ are elementary symmetric functions in $x, y, z$. Furthermore, we have $\gamma \neq 0$ (otherwise, $C$ would contain the line $e_1 = 0$). Thus, the equation $f = 0$ gives a way to express $e_3$ in terms of $e_1$ and $e_2$ (recall that $\alpha \neq -3$). Hence, $\mathbb{P}V(\bar{f}_{\lambda_1})$ is the weighted projective line $\mathbb{P}(1,2)$.

In general, the derived category of $\mathbb{P}V(\bar{f}_{\lambda_1})$ has a semiorthogonal decomposition with the main piece given by the derived category of the coarse moduli, which is $C/S_3$, and some exceptional objects. The obtained semiorthogonal decomposition of $\mathcal{D}[C/S_3]$ matches the one constructed in [Pol06] since the special fibers of the projection $C \to C/S_3$ are either orbits of the points $p_1, \ldots, p_d$, corresponding to $\lambda_2$, or the points of $C$ mapping to the two stacky points of $\mathbb{P}(1,2,3)$.

Note that if $d < 6$ then the coarse quotient of $\mathbb{P}V(\bar{f}_{\lambda_1})$ is rational, so in this case the category $\mathcal{D}[C/S_3]$ has a full exceptional collection.

Some features of the above example occur in a more general situation.

**Proposition 5.3.3.** *(i) Let $\lambda$ be a partition of $n$ such that all parts of $\lambda$ are distinct. Then for a generic $S_n$-invariant homogeneous polynomial $f(x_1, \ldots, x_n)$ of degree $d > 0$, the stack $[\mathbb{P}V(\overline{f}_\lambda)]$ is actually a smooth projective variety.*
*(ii) Now assume that $\lambda$ has one part of multiplicity 2 and all the other parts have multiplicity 1. Then the same conclusion holds for a generic $S_n$-invariant polynomial $f$, provided its degree $d$ is even.*

*Proof.* (i) In this case $\mathbb{P}\overline{V}_\lambda$ is the usual projective space, so the assertion is clear.
(ii) We have coordinates $(x, y; z_1, \ldots, z_p)$ on $V_\lambda$, so that the embedding $V_\lambda \hookrightarrow V$ has form

$$(x, y; z_1, \ldots, z_p) \mapsto (x, y, \ldots, x, y; z_1, \ldots, z_1, \ldots, z_p, \ldots, z_p),$$

where $(x, y)$ is repeated $l$ times, each $z_j$ is repeated $m_j$ times, so that $(l, m_1, \ldots, m_p)$ are all the distinct parts of $\lambda$, and $l$ (resp., $m_j$) occur with multiplicity 2 (resp., 1) in $\lambda$. Set $p_1 = x + y$, $p_2 = x^2 + y^2$, so that $(p_1, p_2), (z_j)$ are the coordinates on $\overline{V}_\lambda$. Now it is enough to check that $\mathbb{P}V(\overline{f}_\lambda)$ does not contain stacky points of $\mathbb{P}\overline{V}_\lambda$, i.e., the points with $p_1 = 0$ and all $z_j = 0$. Thus, it is enough that $f_\lambda$ does not vanish at the point of $V_\lambda$ with $x = -y = 1$ and $z_j = 0$. Note that $p_2(1, -1, \ldots, 1, -1) \neq 0$. Therefore, the same is true for any power of $p_2$, and hence, for a generic $S_n$-invariant polynomial of even degree. □

In the next two propositions we consider the case of $S_n$-invariant homogeneous cubics.

**Proposition 5.3.4.** *Let $f(x_1,\dots,x_n)$ be an $S_n$-invariant homogeneous cubic polynomial such that $\mathbb{P}V(f)$ is smooth. Then for each partition $\lambda$, which has a part of multiplicity $\geq 3$, the stack $[\mathbb{P}V(\overline{f}_\lambda)]$ is isomorphic to a weighted projective stack.*

*Proof.* We can write $f$ in the form

$$f=\alpha p_1^3+\beta p_1p_2+\gamma p_3,$$

where $p_i=x_1^i+\ldots+x_n^i$. Note that $\gamma\neq 0$, since otherwise $f$ would be reducible. Let $x_1,\dots,x_m,z_1,\dots,z_p$ be the coordinates on $V_\lambda$, where $x_1,\dots,x_k$ correspond to the part $l$ of multiplicity $m\geq 3$ in $\lambda$, so that in the embedding $V_\lambda\hookrightarrow V$ the group of variables $(x_1,\dots,x_m)$ is repeated $l$ times. It is sufficient to make sure that $p_3(x_1,\dots,x_m)$ occurs with nonzero coefficient in $f_\lambda$. Note that the same coefficient occurs as the coefficient of the restriction $f_\lambda|_{z_1=\ldots=z_p=0}$. Since $\gamma\neq 0$, this follows from the equality

$$p_3(x_1,\dots,x_m,\dots,x_1,\dots,x_m)=l\cdot p_3(x_1,\dots,x_m),$$

where the group $(x_1,\dots,x_m)$ is repeated $l$ times. □

In the case of cubic forms in $\leq 6$ variables, we obtain from Theorem 1.2.3 the following decompositions of $S_n$-equivariant derived categories.

**Proposition 5.3.5.** *Let $f(x_1,\dots,x_n)$ be a generic $S_n$-invariant homogeneous cubic polynomial, where $n\leq 5$. Then $\mathcal{D}[\mathbb{P}V(f)/S_n]$ has a full exceptional collection. For $n=6$, there is an exceptional collection in $\mathcal{D}[\mathbb{P}V(f)/S_6]$ such that its right orthogonal is equivalent to $\mathcal{D}(E)$, where $E$ is the elliptic curve given by the cubic $f_{(3,2,1)}$ in $\mathbb{P}V_{(3,2,1)}\simeq\mathbb{P}^2$.*

*Proof.* By Proposition 5.3.4, if $\lambda$ has a part of multiplicity $\geq 3$ then the corresponding piece in the semiorthogonal decomposition of Theorem 1.2.3 is the derived category of the weighted projective stack, so it has a full exceptional collection.

On the other hand, if $\lambda=(l_1,l_2)$, where $l_1>l_2$, then $f_\lambda$ is a cubic on the 2-dimensional space $V_\lambda$, with isolated singularity at the origin, so $\mathbb{P}V(f_\lambda)$ is the union of three distinct points.

If $\lambda=(l,l)$ then $V_\lambda$ has coordinates $x,y$ and $\overline{V}_\lambda$ has coordinates $p_1=x+y$, $p_2=x^2+y^2$, and the line $p_1=0$ corresponds to the unique stacky point of the weighted projective line $\mathbb{P}\overline{V}_\lambda$. Note that $p_3=x^3+y^3$ is divisible by $p_1$, so $\overline{f}_\lambda$ vanishes at this point. It follows that $\mathbb{P}V(\overline{f}_\lambda)$ is the union of two points and of one stacky point with the automorphism group $\mathbb{Z}/2$. The derived category of such stacky point splits as the direct sum of two derived categories of the usual point.

Next, let us consider the case $\lambda=(l_1,l_2,l_2)$ where $l_1\neq l_2$. Then $V_\lambda$ has coordinates $x,y,z$, where $W_\lambda=S_2$ swaps $x$ and $y$, so that $\overline{V}_\lambda$ has coordinates $p_1=x+y$, $p_2=x^2+y^2$ and $z$. The cubic $\overline{f}_\lambda$ should have form

$$\overline{f}_\lambda=p_2(\alpha z+\beta p_1)+C(p_1,z),$$

where $C(p_1,z)$ is a binary cubic form. It is easy to see that for generic $S_n$-invariant $f$, one has $\alpha\neq 0$, so we can make the change of variables $z_1=\alpha z+\beta p_1$. Furthermore, $C(p_1,z)$ is not divisible by $z_1$, since $f_\lambda$ has an isolated singularity at 0. Thus, rescaling the variables, we can bring $f$ to the form

$$\overline{f}_\lambda=p_2z_1+z_1Q(p_1,z_1)+p_1^3,$$

where $Q$ is a binary quadratic form. Now taking $u = p_2 + Q(p_1, z_1)$ as a new variable of weight 2, we get

$$\overline{f}_\lambda = uz_1 + p_1^3.$$

It is easy to see that $\mathbb{P}V(\overline{f}_\lambda)$ is isomorphic to the weighted projective line $\mathbb{P}(1,2)$. Namely, there is an isomorphism given by

$$\mathbb{P}(1,2) \to \mathbb{P}V(\overline{f}_\lambda) : (t : v) \mapsto (u = v^3, z_1 = -t^3, p_1 = vt).$$

Next, assume that $\lambda = (l_1, l_1, l_2, l_2)$, where $l_1 > l_2$. Then we have coordinates $x_1, y_1, x_2, y_2$ on $V_\lambda$, and $W_\lambda = S_2 \times S_2$ permutes $x_1$ with $y_1$ and $x_2$ with $y_2$. Set $p_1(i) = x_i + y_i$, $p_2(i) = x_i^2 + y_i^2$. Then the cubic $\overline{f}_\lambda$ has form

$$\overline{f}_\lambda = p_2(1)z_1 + p_2(2)z_2 + C(p_1(1), p_1(2)),$$

where $z_1$ and $z_2$ are some linear forms in $p_1(1), p_1(2)$. It is easy to see that for generic $f$, the linear forms $z_1$ and $z_2$ will be linearly independent, so we can view $p_2(1), p_2(2), z_1, z_2$ as independent variables. Now adding to $p_2(i)$ appropriate quadratic expressions of $z_1, z_2$, we can rewrite $\overline{f}_\lambda$ as

$$\overline{f}_\lambda = u_1z_1 + u_2z_2,$$

where $u_1, u_2, z_1, z_2$ are independent variables ($\deg(u_i) = 2$, $\deg(z_i) = 1$). Thus, we can identify $\mathbb{P}V(\overline{f}_\lambda)$ with $\mathbb{P}(1,2) \times \mathbb{P}^1$ via the isomorphism $\mathbb{P}(1,2) \times \mathbb{P}^1 \to \mathbb{P}V(\overline{f}_\lambda)$ sending

$$(t : v), (s_1 : s_2) \mapsto (u_1 = vs_1, u_2 = vs_2, z_1 = ts_2, z_2 = -ts_1).$$

All of the above pieces admit full exceptional collections. The remaining piece in the case $n = 6$ corresponds to $\lambda = (3,2,1)$ and is equivalent to $\mathcal{D}(E)$, where $E$ is the elliptic curve given by $f_{(3,2,1)}$. $\square$

5.4. **Products of curves.** First, let us consider the case of an action of a finite group on a curve.

For a finite group $G$ acting effectively on a smooth curve $C$, let $D_1, \dots, D_r$ be all special fibers of the morphism $C \to C/G$ and let $m_i$ be the order of the stablizer of a point in $D_i$. Then the proof of [Pol06, Thm. 1.2] implies that for each $i$, there is an exceptional collection of $G$-equivariant sheaves on $C$,

$$(\mathcal{O}_{D_i}, \omega_C|_{D_i}, \dots, \omega_C^{\otimes m_i - 2}|_{D_i}),$$

and if $\mathcal{B}_i \subset \mathcal{D}[C/G]$ is the subcategory generated by this collection, then there is a semiorthogonal decomposition

$$\mathcal{D}[C/Y] = \langle \pi^*\mathcal{D}(C/G), \mathcal{B}_1, \dots, \mathcal{B}_r \rangle, \tag{5.2}$$

where $\pi^* : \mathcal{D}(C/G) \to \mathcal{D}[C/G]$ is the pull-back functor.

In the case when $G$ is abelian this leads to the following result.

**Proposition 5.4.1.** *Let $G$ be a finite abelian group acting effectively on a smooth curve $C$. Then condition (SOD+) is satisfied for some collection $(\mathcal{N}_\lambda)$ such that $\mathcal{N}_1 = \mathcal{O}_{Z_1}$.*

*Proof.* By Lemma 5.1.3, we already know that $\overline{\mathcal{N}}_1$ is the kernel for the pull-back functor $\pi^* : \mathcal{D}(C/G) \to \mathcal{D}[C/G]$.

Next, we are going to construct functors $\mathcal{D}(C^g/G) \to \mathcal{D}[C/G]$ for $g \neq 1$, where $C^g \subset C$ is the set of $g$-invariant points, coming from some $G\times G$-equivariant vector bundles $\mathcal{N}_g$ on $Z_g \subset C^g \times C$. The set $C^g$ is a disjoint union of $G$-orbits, so we have

$$Z_g = \sqcup_{D\subset C^g} Z_g(D),$$

where the union is over $G$-orbits $D \subset C^g$, and $Z_g(D) \subset D \times C$ is the preimage of $D$ in $Z_g$ (in fact, $Z_g(D) \subset D \times D$). Thus, we have to define a $G \times G$-equivariant vector bundle $\mathcal{N}_{g,D}$ on each $Z_g(D)$.

Now let us fix a bijection

$$\tau : G \setminus \{1\} \xrightarrow{\sim} \{0, \ldots, |G| - 1\}.$$

This is equivalent to choosing a total order on $G \setminus \{1\}$. For each cyclic subgroup $H \subset G$, $H \neq 1$, we have the induced total order on $H \setminus \{1\}$. The corresponding bijection

$$\tau_H : H \setminus \{1\} \xrightarrow{\sim} \{0, \ldots, |H| - 1\}$$

has the property that for every $h, h' \in H$, one has $\tau_H(h) < \tau_H(h')$ if and only if $\tau(h) < \tau(h')$. Let $D(H) \subset C$ denote the set of points with the stabilizer subgroup $H$. Note that for every $g \in G \setminus \{1\}$ and a $G$-orbit $D \subset C^g$ we have $D \subset D(H)$, where $H$ is the stabilizer subgroup of any point in $D$. Now we set

$$\mathcal{N}_{g,D} := (\mathcal{O}_D \boxtimes \omega_C^{\otimes \tau_H(g)})|_{Z_g(D)}.$$

Note that $D/G$ is a point, so we can view $\overline{\mathcal{N}}_{g,D}$ as a $G$-equivariant sheaf on $C$. By definition, we have

$$\overline{\mathcal{N}}_{g,D} = \omega_C^{\otimes \tau_H(g)}|_D.$$

Now we claim that the subcategory $\pi^*\mathcal{D}(C/G)$, followed by the exceptional collection of sheaves $(\overline{\mathcal{N}}_{g,D})$, ordered in any way compatible with the total order on $G \setminus \{1\}$, gives a semiorthogonal decomposition of $\mathcal{D}[C/G]$. Indeed, we observe that for a given $G$-orbit $D$ in $C$ with a nontrivial stabilizer $H$ of order $k > 1$, the $G$-equivariant sheaves

$$\mathcal{O}_D, \omega_C|_D, \ldots, \omega_C^{\otimes k-2}|_D$$

appear in our collection in exactly this order. Hence, our assertion follows from the decomposition (5.2). □

Next, we make a simple observation that condition (SOD+) is compatible with products.

**Lemma 5.4.2.** *Let $G$ (resp., $G'$) be a finite group acting on a smooth variety $Y$ (resp., $Y'$), and assume that condition (SOD+) is satisfied for some collection $(\mathcal{N}_\lambda)$ (resp., $(\mathcal{N}_{\lambda'})$). Then condition (SOD+) is also satisfied for the action of $G \times G'$ on $Y \times Y'$ and for some collection $(\mathcal{N}_{\lambda,\lambda'})$.*

*Proof.* Let $(\lambda, \lambda')$ be a conjugacy class in $G\times G'$ then we have a natural identification of

$$Z_{\lambda,\lambda'} \subset (Y \times Y') \times (Y \times Y') \simeq (Y \times Y) \times (Y' \times Y')$$

with $Z_\lambda \times Z_{\lambda'}$ so we can define $\mathcal{N}_{\lambda,\lambda'}$ to correspond to the exterior tensor product $\mathcal{N}_\lambda \boxtimes \mathcal{N}_{\lambda'}$. It is easy to check that the corresponding kernels $\overline{\mathcal{N}}_{\lambda,\lambda'}$ define a semiorthogonal decomposition of $\mathcal{D}[Y \times Y'/(G \times G')]$ with respect to any total ordering of conjugacy classes in $G \times G'$ compatible with the partial order

$(\lambda_1, \lambda'_1) \le (\lambda_2, \lambda'_2)$ if $\lambda_1 \le \lambda_2$ and $\lambda'_1 \le \lambda'_2$ (where we use the total orders on conjugacy classes in $G$ and $G'$). $\square$

Thus, we get the following corollary from Proposition 5.4.1.

**Corollary 5.4.3.** *Let $C_1, \dots, C_n$ be smooth curves, and for each $i$, let $G_i$ be a finite abelian group acting on $C_i$. Then condition (SOD+) holds for the action of $G_1 \times \cdots \times G_n$ on $C_1 \times \cdots \times C_n$.*

**Example 5.4.4.** For the standard action of the cyclic group $\mu_d$ on $\mathbb{A}^1$ the coarse quotient is isomorphic to $\mathbb{A}^1_d$, where the $d$ indicates the $\mathbb{G}_m$-weight, so that the quotient map $\pi\colon \mathbb{A}^1 \to \mathbb{A}^1_d$ is given by $x \mapsto x^d$. There is a semiorthogonal decomposition (obtained from (5.2) by mutations)

$$\mathcal{D}[\mathbb{A}^1/\mu_d] = \langle \mathcal{O}_p \otimes \chi^{d-1}, \dots, \mathcal{O}_p \otimes \chi, \pi^*\mathcal{D}(\mathbb{A}^1_d)\rangle.$$

where $\mathcal{O}_p$ denotes the structure sheaf of the origin, and $\chi : \mu_d \to \mathbb{G}_m$ is the character given by the natural embedding.

Now, for positive integers $d_1, \dots, d_k$, let us consider the natural action of $G = \mu_{d_1} \times \cdots \times \mu_{d_k}$ on $\mathbb{A}^k$ (where the $i$th factor acts on the $i$th coordinate). By Corollary 5.4.3, we have a motivic semiorthogonal decomposition of $\mathcal{D}[\mathbb{A}^k/G]$. We can describe explicitly the pieces of this decomposition as follows. The fixed locus of an element of $g = (z_1, \dots, z_k) \in G$ is isomorphic to the affine space $\mathbb{A}^{n_g}$, where $n_g$ is the number of trivial components of $g$ . The coarse quotient by $G$ is $\pi_g\colon \mathbb{A}^{n_g} \to \mathbb{A}^{n_g}_{\mathbf{d}_g}$, where $\mathbf{d}_g$ is a multi-index giving weights for the $\mathbb{G}_m$-action ($\mathbf{d}_g$ is the set of $d_i$ for which $z_i = 1$ ). Let $\iota_g\colon \mathbb{A}^{n_g} \hookrightarrow \mathbb{A}^k$ denote the closed embedding. Then the composite functor

$$\iota_{g*} \circ \pi_g^*\colon \mathcal{D}(\mathbb{A}^{n_g}_{\mathbf{d}_g}) \to \mathcal{D}[\mathbb{A}^k/G]$$

is fully faithful.

For each $i$, let $\zeta_{d_i}$ be a $d_i$th primitive root of unity. For $g = (\zeta_1^{n_1}, \dots, \zeta_k^{n_k}) \in G$, where $0 \le n_i < d_i$, we define the character $\chi_g$ of $G$ by setting $\chi_g = \chi_1^{n_1} \cdots \chi_k^{n_k}$, where $\chi_i\colon G \to \mathbb{G}_m$ is given by the $i$th projection.

Then the functors giving the semiorhogonal decomposition of $\mathcal{D}[\mathbb{A}^k/G]$ (numbered by $g \in G$) are

$$(\iota_{g*} \circ \pi_g^*) \otimes \chi_g\colon \mathcal{D}(\mathbb{A}^{n_g}_{\mathbf{d}_g}) \to \mathcal{D}[\mathbb{A}^k/G],$$

ordered lexicographically with respect to the reverse order on each set $\{0, \dots, d_i - 1\}$.

As before, we can delete the origin in all the affine spaces and pass to $\mathbb{G}_m$-equivariant categories. In this way we get a semiorthogonal decomposition of $\mathcal{D}[\mathbb{P}^{k-1}/G]$ indexed by the elements of $G$. The pieces of this semiorthogonal decomposition will be the weighted projective stacks $\mathbb{P}(\mathbf{d}_g)$.

We can also apply Proposition 5.1.2 to get, as in Section 5.3, a semiorthogonal decomposition of $\mathcal{D}[\mathbb{P}V(f)/G]$, where $f$ is a $G$-invariant homogeneous polynomial on $\mathbb{A}^k$. More precisely, we have to assume that $\mathbb{P}V(f)$ is smooth and that restrictions of $f$ to certain coordinate subspaces are nonzero. Namely, in the case when there is a trivial factor in $G$ (i.e., some $d_i = 1$), we have to assume the nonvanishing of the restriction of $f$ to the subspace where all coordinates with $d_i > 1$ are set to zero. In the case when all $d_i > 1$, we have to assume that the restriction of $f$ to each coordinate line is nonzero.

For example, if $d_1 > 1$, $d_2 = \ldots = d_k = 1$, and $f = x_1^{d_1} - g(x_2, \ldots, x_k)$, then $\mathbb{P}V(f)$ is a cyclic cover of $\mathbb{P}^{k-2}$ and our decomposition of $\mathcal{D}[\mathbb{P}V(f)/\mu_{d_1}]$ matches the one given by Kuznetsov-Perry in [KP17, Theorem 4.1].

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