RETURN- AND HITTING-TIME DISTRIBUTIONS OF SMALL SETS IN INFINITE MEASURE PRESERVING SYSTEMS

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ABSTRACT. We study convergence of return- and hitting-time distributions of small sets E_k with $\mu(E_k) \to 0$ in recurrent ergodic dynamical systems preserving an infinite measure μ. Some properties which are easy in finite measure situations break down in this null-recurrent setup. However, in the presence of a uniform set Y with regularly varying wandering rate there is a scaling function suitable for all subsets of Y. In this case, we show that return distributions for the E_k converge iff the corresponding hitting time distributions do, and we derive an explicit relation between the two limit laws. Some consequences of this result are discussed. In particular, this leads to improved sufficient conditions for convergence to $\mathcal{E}^{\frac{1}{\alpha}} \mathcal{G}_\alpha$, where $\mathcal{E}$ and $\mathcal{G}_\alpha$ are independent random variables, with $\mathcal{E}$ exponentially distributed and $\mathcal{G}_\alpha$ following the one-sided stable law of order α. The same principle also reveals the limit laws (different from the above) which occur at hyperblic periodic ponts of prototypical null-recurrent interval maps

1. Introduction

The asymptotic behaviour of return- and hitting-time distributions of (very) small sets in ergodic probability preserving dynamical systems has been studied in great detail, and there is now a well-developed theory, both for specific types of maps and sets, and for general abstract systems.

For infinite measure preserving situations, however, results are scarce. Only recently some concrete classes of prototypical systems have been studied in [PS1], [PS2], and [PSZ2], where distributional limit theorems for certain natural sequences of sets were established.

The purpose of the present article is to discuss some basic aspects of return- and hitting-time limits for asymptotically rare events in the setup of abstract infinite ergodic theory. After considering questions of scaling, we discuss to what extent the natural relation between return- and hitting-time limits which holds in the finite-measure setup carries over to infinite measures, and how this can be used to prove convergence to specific laws.

General setup. Throughout, *all measures are understood to be σ-finite*. We study *measure preserving transformations* T (not necessarily invertible) on a measure space $(X, \mathcal{A}, \mu)$, i.e. measurable maps $T : X \to X$ for which $\mu \circ T^{-1} = \mu$. Here T will be *ergodic* (i.e. for $A \in \mathcal{A}$ with $T^{-1}A = A$ we have $0 \in \{\mu(A), \mu(A^c)\}$) and *conservative* (meaning that $\mu(A) = 0$ for all wandering sets, that is, $A \in \mathcal{A}$ with

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$T^{-n}A$, $n \geq 0$, pairwise disjoint), and thus *recurrent* (in that $A \subseteq \bigcup_{n\geq 1} T^{-n}A$ mod μ for $A \in \mathcal{A}$). Our emphasis will be on the *infinite measure case*, $\mu(X) = \infty$.

For T such a conservative ergodic measure preserving transformation (*c.e.m.p.t.*) on $(X, \mathcal{A}, \mu)$, and any $Y \in \mathcal{A}$, $\mu(Y) > 0$, we define the *first entrance time* function of Y, $\varphi_Y : X \to \mathbb{N} \cup \{\infty\}$ by $\varphi_Y(x) := \min\{n \geq 1 : T^n x \in Y\}$, $x \in X$, and let $T_Y x := T^{\varphi(x)}x$, $x \in X$. When restricted to Y, φ_Y is called the *first return time* of Y, and $\mu \mid_{Y\cap\mathcal{A}}$ is invariant under the ergodic *first return map*, T_Y restricted to Y. If $\mu(Y) < \infty$, it is natural to regard φ_Y as a random variable on the probability space $(X, \mathcal{A}, \mu_Y)$, where $\mu_Y(E) := \mu(Y)^{-1}\mu(Y \cap E)$. By Kac' formula, $\int \varphi_Y \, d\mu_Y = \mu(X)/\mu(Y)$. It is well known (see [A0]) that, for suitable reference sets Y, the distribution of this variable reflects important features of the system $(X, \mathcal{A}, \mu, T)$.

Return- and hitting-time distributions for small sets. Rather than focusing on a particular set Y, the present article studies the behaviour of such distributions for sequences (E_k) of sets of (strictly) *positive* finite measure with $\mu(E_k) \to 0$, that is, for sequences of *asymptotically rare events*. As return times to small sets will typically be very large, the functions φ_{E_k} need to be normalized, which will be done using a certain scaling function γ.

We will thus study the distributions of random variables of the form $\gamma(\mu(E))\,\varphi_E$ on $(E, E \cap \mathcal{A}, \mu_E)$, with $\mu(E)$ small, and call this the *(normalized) return time distribution of* E,

$$\mathsf{law}_{\mu_E}[\gamma(\mu(E))\,\varphi_E]$$

where, for $\psi : X \to X'$ any $\mathcal{A}$-$\mathcal{A}'$-measurable map and $\nu \ll \mu$ a probability on $(X, \mathcal{A})$, we write $\mathsf{law}_\nu[\psi] := \nu \circ \psi^{-1}$. In fact, we can use any such ν as an initial distribution, in which case we refer to

$$\mathsf{law}_\nu[\gamma(\mu(E))\,\varphi_E]$$

as the *(normalized) hitting time distribution of* E *(under* ν*)*. This leads to two different ways of looking at the φ_{E_k} for a sequence (E_k) as above: *asymptotic return distributions* of (E_k) are limits, as $k \to \infty$, of $(\mathsf{law}_{\mu_{E_k}}[\gamma(\mu(E_k))\,\varphi_{E_k}])_{k\geq 1}$, while *asymptotic hitting distributions* are limits of $(\mathsf{law}_\nu[\gamma(\mu(E_k))\,\varphi_{E_k}])_{k\geq 1}$ for some fixed ν. (The latter limits do not depend on the choice of ν, and we often take $\nu = \mu_Y$ for some nice set Y, see Section 4 below.) *Understanding the relation between these two types of limits will be one central theme of this article.*

It will be convenient to regard the distributions above as measures on $[0, \infty]$. Accordingly, we let $\mathcal{F} := \{F : [0, \infty) \to [0, 1]$, non-decreasing and right-continuous$\}$ be the set of sub-probability distribution functions on $[0, \infty)$. For F, $F_n \in \mathcal{F}$ $(n \geq 1)$ we write $F_n \Rightarrow F$ for *vague convergence*, i.e. $F_n(t) \to F(t)$ at all continuity points of F. For efficiency, we shall also use $F_n(t) \Longrightarrow F(t)$ to express the same thing. (This allows us to use explicit functions of t.) If $\sup F(t) = 1$ this is the usual *weak convergence* of probability distribution functions on $[0, \infty)$.

Pointwise dual ergodicity and uniform sets. Some classes of well-behaved infinite measure preserving systems are characterized by the existence of distinguished *reference sets* Y, $0 < \mu(Y) < \infty$, with special properties. Those are often defined in terms of the *transfer operator* $\widehat{T} : L_1(\mu) \to L_1(\mu)$, with $\int_X u\cdot(v\circ T)\,d\mu = \int_X \widehat{T}u\cdot v\,d\mu$ for all $u \in L_1(\mu)$ and $v \in L_\infty(\mu)$. The operator $\widehat{T}$ naturally extends to $\{u : X \to$

$[0,\infty)$ $\mathcal{A}$-measurable$\}$. It is a linear Markov operator, $\int_X \widehat{T}u\,d\mu = \int_X u\,d\mu$ for $u \geq 0$. The m.p.t. T is conservative and ergodic if and only if $\sum_{k\geq 0} \widehat{T}^k u = \infty$ a.e. for all $u \in L_1^+(\mu) := \{u \in L_1(\mu) : u \geq 0 \text{ and } \mu(u) > 0\}$ or (equivalently) all $u \in \mathcal{D}(\mu) := \{u \in L_1(\mu) : u \geq 0,\, \mu(u) = 1\}$. By invariance of μ we have $\widehat{T}1_X = 1_X$.

A c.e.m.p.t. T on the space $(X, \mathcal{A}, \mu)$ is said to be *pointwise dual ergodic* (cf. [A0], [A1]) if there is some sequence (a_n) in $(0,\infty)$ such that

$$(1.1) \qquad \frac{1}{a_n}\sum_{k=0}^{n-1} \widehat{T}^k u \longrightarrow \mu(u)\cdot 1_X \qquad \begin{array}{c}\text{a.e. on } X \text{ as } n\to\infty, \text{ for every}\\ u\in L_1(\mu) \text{ with } \mu(u)\neq 0.\end{array}$$

In this case, (a_n) (unique up to asymptotic equivalence, with $a_n \to \infty$) is called a *return sequence* of T. W.l.o.g. we will assume throughout that $a_n = a_T(n)$ for some strictly increasing continuous $a_T : [0,\infty) \to [0,\infty)$ with $a_T(0) = 0$. For convenience, we shall call any homeomorphism of $[0,\infty)$ a *scaling function*. Note that in case $\mu(X) = \infty$, we always have $a_T(s) = o(s)$ as $s \to \infty$. Letting b_T denote the inverse function of a_T, we thus see that $s = o(b_T(s))$ as $s \to \infty$. For later use define another scaling function $\gamma_T : [0,\infty) \to [0,\infty)$ with $\gamma_T(s) = o(s)$ as $s \searrow 0$ via

$$(1.2) \qquad \gamma_T(0) := 0 \quad \text{and} \quad \gamma_T(s) := 1/b_T(1/s) \quad \text{for } s > 0.$$

By Egorov's theorem, the convergence in (1.1) is uniform on suitable sets (depending on u) of arbitrarily large measure. It is useful to identify specific pairs (u, Y), with $u \in \mathcal{D}(\mu)$ and $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, such that

$$(1.3) \qquad \left\| 1_Y \cdot \left(\frac{1}{a_n}\sum_{k=0}^{n-1} \widehat{T}^k u - 1_X \right) \right\|_\infty \longrightarrow 0 \quad \text{as } n\to\infty,$$

in which case we shall refer to Y as a *u-uniform set* (compare [A0], [T3]). In [PSZ2] the notion of a *$\mathcal{U}$-uniform set* Y was introduced. This means that $\mathcal{U} \subseteq \mathcal{D}(\mu)$ is a class of densities such that the $L_\infty(\mu)$-convergence asserted in (1.3) holds uniformly in $u \in \mathcal{U}$, that is,

$$(1.4) \qquad \sum_{k=0}^{n-1} \widehat{T}^k u \sim a_n \qquad \begin{array}{c}\text{as } n\to\infty, \text{ uniformly mod } \mu \text{ on } Y,\\ \text{and uniformly in } u\in\mathcal{U}.\end{array}$$

A set Y which is $\mu(Y)^{-1}\cdot 1_Y$-uniform is called a *Darling-Kac (DK) set*, cf. [A0], [A2]. The existence of a uniform set implies pointwise dual ergodicity (as in Proposition 3.7.5 of [A0]), and the a_n in (1.3) then form a return sequence.

Several basic classes of infinite measure preserving systems, including Markov shifts and other Markov maps with good distortion properties (see [A0], [A2], and [T2]), as well as various non-Markovian interval maps (see [Z2], [Z3]), are known to possess DK-sets. Section 6 of [PSZ2] shows that a set Y on which T induces a Gibbs-Markov map T_Y is always $\mathcal{U}$-uniform for a reasonably large family $\mathcal{U}$.

Finer probabilistic statements about pointwise dual ergodic systems usually require a_T to be *regularly varying* with *index* $\alpha \in [0,1]$ (written $a_T \in \mathcal{R}_\alpha$), meaning that for every $c > 0$, $a_T(ct)/a_T(t) \to c^\alpha$ as $t \to \infty$ (see [BGT]). The asymptotics of a_T is intimately related to the return distribution $\mathsf{law}_{\mu_Y}[\varphi_Y]$ of any of its uniform sets Y: Write $q_n(Y) := \mu_Y(\varphi_Y > n)$, $n \geq 0$, for the *tail probabilities* of φ_Y, and define the *wandering rate* $(w_N(Y))_{N\geq 1}$ *of* Y as the sequence of partial sums $w_N(Y) := \mu(Y)\sum_{n=0}^{N-1} q_n(Y)$. By Theorem 5.1 of [AZ], any uniform set Y has *minmal wandering rate* (meaning that $\underline{\lim}_{N\to\infty} w_N(A)/w_N(Y) \geq 1$ whenever

$0 < \mu(A) < \infty$), and satisfies Aaronson's *asymptotic renewal equation* (as in Proposition 3.8.6 of [A0]). Combining the latter with Karamata's Tauberian Theorem (KTT, see Corollary 1.7.3 in [BGT]) shows that for $\alpha \in [0,1]$, one has

$$(w_N) \in \mathcal{R}_{1-\alpha} \quad \text{iff} \quad a_T \in \mathcal{R}_\alpha, \tag{1.5}$$

in which case

$$a_n \sim \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \frac{n}{w_n(Y)} \quad \text{as } n \to \infty. \tag{1.6}$$

The concrete limit theorems of [PS1]**,** [PS2]**, and** [PSZ2]**.** The results of [PS1] and [PS2], were the starting point for the present investigation of return- and hitting-time limits in null-recurrent situations. They apply to certain skew-products which are "barely recurrent" in that $(w_N) \in \mathcal{R}_1$ (corresponding to $\alpha = 0$ above). In that case, only a seriously distorted version of the return-time function can have a nontrivial limit, see the discussion in Section 4 below. For natural sequences (E_k) of sets, those variables were shown to converge to the law with distribution function $t \longmapsto t/(1+t)$, $t \geq 0$.

The skew-product structure was exploited through the use of local limit theorems. That approach has been extended to some (classical probabilistic) $\alpha \in [0, 1/2]$ situations in [PSZ1], and further work on skew-products has been done in [Y]. To go beyond skew-products and the local limit technique, the notion of $\mathcal{U}$-uniform sets was introduced (and shown to work) in [PSZ2], which dealt with $\alpha \in (0,1]$ situations. For certain natural sequences (E_k), suitably normalized return- (and hitting-) times $\gamma(\mu(E_k))\,\varphi_{E_k}$ were shown to converge to a law best expressed as the distribution of the random variable

$$\mathcal{H}_\alpha := \mathcal{E}^{\frac{1}{\alpha}}\,\mathcal{G}_\alpha \ , \quad \alpha \in (0,1], \tag{1.7}$$

where $\mathcal{E}$ and $\mathcal{G}_\alpha$ are independent random variables, with $\mathcal{E}$ exponentially distributed ($\Pr[\mathcal{E} > t] = e^{-t}$ for $t \geq 0$) and $\mathcal{G}_\alpha$, $\alpha \in (0,1)$, following the one-sided stable law of order α (so that $\mathbb{E}[\exp(-s\mathcal{G}_\alpha)] = \exp(-s^\alpha)$ for $s \geq 0$), while $\mathcal{G}_1 = 1$. We use $H_\alpha(t) := \Pr[\mathcal{H}_\alpha \leq t]$, $t \geq 0$, to denote the distribution function of $\mathcal{H}_\alpha$.

Outline of results. In contrast to references [PS1], [PS2], [PSZ2] mentioned before, which study specific classes of systems and particular types of sequences (E_k), the present note discusses the asymptotics of general asymptotically rare sequence (E_k) in an abstract setup.

We first discuss the basic question of how to normalize the functions φ_E, and show that it is impossible to find a scaling function γ such that $\gamma(\mu(E))$ captures the order of magnitude of φ_E for all (small) sets E. However, if T admits a uniform set Y with regularly varying return sequence, then there is some $\gamma = \gamma_T$ which works for every E contained in Y.

In this very setup, we then prove that, for every asymptotically rare sequence (E_k) inside Y, the return-time distributions converge iff the hitting-time distributions converge. We also clarify the relation between the respective limit laws. The latter allows us to characterize convergence to the specific limit laws which occurred in [PS1], [PS2], and [PSZ2]. This also leads to an improved version of the abstract limit theorem of [PSZ2].

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2. How to normalize return-times of small sets

We collect some facts regarding the order of magnitude of a return-time variable φ_E, focusing on its relation to the measure of the set E. We first record some basic observations to point out some of the difficulties which are inevitable when dealing with infinite measures. We then formulate the main results of this section.

Scaling return-times in finite measure systems. As a warm-up, assume first that $(X, \mathcal{A}, \mu, T)$ is ergodic and measure preserving, with $\mu(X) < \infty$. Kac' formula $\int_E \varphi_E \, d\mu_E = \mu(X)/\mu(E)$ for the expectation of the return-time of an arbitrary set $E \in \mathcal{A}$ with $\mu(E) > 0$ not only shows that $\mu(E)\, \varphi_E$ is the canonical choice if we wish to use *normalized return times*, but also yields the simple estimate $\mu_E(\mu(E)\, \varphi_E > t) \leq 1/t$, $t > 0$. The latter can be read as an explicit version of the trivial statement that the family of all normalized return distributions,

$$\{\mathsf{law}_{\mu_E}[\mu(E)\, \varphi_E] : E \in \mathcal{A},\, \mu(E) > 0\}\text{, is tight.} \tag{2.1}$$

We record an obvious consequence of this by also stating that for every $\eta > 0$,

$$\{\mathsf{law}_{\mu_E}[\varphi_E] : E \in \mathcal{A},\, \mu(E) \geq \eta\} \text{ is tight.} \tag{2.2}$$

The relevance of these trivialities for the present paper lies in the fact that they both break down when $\mu(X) = \infty$.

Scaling return-times in infinite measure systems - difficulties. Now let $(X, \mathcal{A}, \mu, T)$ be a c.e.m.p.t. system with $\mu(X) = \infty$. We are interested in the return distributions of sets of positive finite measure. Kac' formula remains valid in that $\int_E \varphi_E \, d\mu_E = \infty$ for every set $E \in \mathcal{A}$ with $\mu(E) > 0$, but it no longer provides us with a canonical normalization for φ_E. Indeed, the situation is much more complicated than it is in the finite measure regime.

Proposition 2.1 (**Basic (non-)tightness properties of return distributions**). *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$ with $\mu(X) = \infty$.*

a) *The family of return distributions of large sets E is not tight: For every $\eta > 0$,*

$$\{\mathsf{law}_{\mu_E}[\varphi_E] : E \in \mathcal{A},\, \mu(E) \geq \eta\} \text{ is not tight.} \tag{2.3}$$

b) *Locally, the family of return distributions of large sets E is tight: Let $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$. Then for every $\eta > 0$,*

$$\{\mathsf{law}_{\mu_E}[\varphi_E] : E \in Y \cap \mathcal{A},\, \mu(E) \geq \eta\} \text{ is tight.} \tag{2.4}$$

c) *Even locally, the family of return distributions of arbitrary sets E with normalization $\mu(E)$ is not tight: Let $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$. Then*

$$\{\mathsf{law}_{\mu_E}[\mu(E)\, \varphi_E] : E \in Y \cap \mathcal{A},\, \mu(E) > 0\} \text{ is not tight.} \tag{2.5}$$

Statement c) of the proposition confirms that $\mu(E)$ is not an appropriate normalizing factor. In Theorem 2.2 below we identify, under additional assumptions, a scaling function $\gamma : [0,\infty) \to [0,\infty)$ for which $\gamma(\mu(E))$ gives a suitable normalization, at least inside certain reference sets Y. Call γ a *tight scale for return times in* Y if

$$\{\mathsf{law}_{\mu_E}[\gamma(\mu(E))\,\varphi_E] : E \in Y \cap \mathcal{A},\ \mu(E) > 0\} \text{ is tight.} \tag{2.6}$$

We will show first that any Y admits a tight scale.

Proposition 2.2 (**Existence of tight scales for arbitrary** Y)**.** *Let T be a c.e.m.p.t. on $(X,\mathcal{A},\mu)$ and $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$. Then there exists a tight scale γ for return times in Y.*

There is a good reason for restricting to subsets of some fixed Y in the definition of a tight scale: there never is a scaling function γ which works for every set Y of positive finite measure. We shall prove

Theorem 2.1 (**No universal tight scale for return times**)**.** *Let T be a c.e.m.p.t. on $(X,\mathcal{A},\mu)$ with $\mu(X) = \infty$, and γ a scaling function. Then there is some $Y \in \mathcal{A}$ with $\mu(Y) = 1$ such that γ is* not *a tight scale for return times in Y.*

Observe next that if γ is such a tight scale for Y, then any scaling function $\widetilde{\gamma}$ with $\widetilde{\gamma}(s) = o(\gamma(s))$ as $s \searrow 0$ kills return time functions of small sets in Y in that $\mu_{E_k}(\gamma(\mu(E_k))\,\varphi_{E_k} \leq t) \Rightarrow 1$ for all $t > 0$ whenever (E_k) is a sequence in $Y \cap \mathcal{A}$ with $\mu(E_k) \to 0$. But, naturally, we will mostly be interested in a scaling function γ which is not only tight, but also a *nontrivial* scale for return times in Y in that

$$\text{there is a sequence } (E_k) \text{ of asymptotically rare events in } Y \text{ s.t.} \\ \underline{\lim}_{k\to\infty}\, \mu_{E_k}(\gamma(\mu(E_k))\,\varphi_{E_k} > t^*) > 0 \quad \text{for some } t^* > 0. \tag{2.7}$$

For the sake of completeness we include the very easy observation that there are always sets with exceptionally short returns, which elude any given scale function:

Proposition 2.3 (**Sets with very short returns**)**.** *Let T be a c.e.m.p.t. on $(X,\mathcal{A},\mu)$, and $\gamma : [0,\infty) \to [0,\infty)$ a scaling function. Assume that $Y \in \mathcal{A}$ satisfies $\mu(Y) > 0$, then there are sets $E_k \in Y \cap \mathcal{A}$, $k \geq 1$, such that $0 < \mu(E_k) \to 0$ and*

$$\mu_{E_k}(\gamma(\mu(E_k))\,\varphi_{E_k} \leq t) \longrightarrow 1 \quad \textit{for } t > 0. \tag{2.8}$$

Identifying nontrivial tight scaling in the presence of regular variation. There are systems which possess distinguished reference sets Y of positive finite measure for which we can explicitly identify a good scaling function. The main positive result of the present section is

Theorem 2.2 (**Nontrivial tight scale in a uniform set,** $\alpha > 0$)**.** *Let T be a c.e.m.p.t. on $(X,\mathcal{A},\mu)$, $\mu(X) = \infty$. Assume that $Y \in \mathcal{A}$ is a uniform set and that $a_T \in \mathcal{R}_\alpha$ for some $\alpha \in (0,1]$. Let b_T be the inverse function of a_T, and*

$$\gamma_T(s) := 1/b_T(1/s), \quad s > 0. \tag{2.9}$$

Then $\gamma_T \in \mathcal{R}_{1/\alpha}(0^+)$ and γ_T is a nontrivial tight scale for return times in Y.

The situation inside a uniform set with regularly varying return sequence therefore is not as wild as it is for arbitrary sets. The function γ_T is the scale which has been used in the concrete limit theorems of [PSZ1], and [PSZ2].

Note that the theorem does not cover the $\alpha = 0$ case. Indeed, for pointwise dual ergodic systems with $a_T \in \mathcal{R}_0$ it is more natural to consider the nonlinear function $a_T(\varphi_E)$ of φ_E rather than just rescaling φ_E by a constant factor. As shown in [DE], [PS1], and [PS2], this often has a natural limit distribution. The next result shows that $\mu(E)$ is once again a natural normalization in this situation.

Theorem 2.3 (**Nontrivial tight distorted times in a uniform set,** $\alpha = 0$**).** *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$. Assume that $Y \in \mathcal{A}$ is a uniform set and that $a_T \in \mathcal{R}_0$. Then,*

$$\{\mathsf{law}_{\mu_E}[\mu(E)\, a_T(\varphi_E)] : E \in Y \cap \mathcal{A},\ \mu(E) > 0\} \text{ is tight.} \tag{2.10}$$

Moreover, there exists a sequence (E_k^*) in $Y \cap \mathcal{A}$ with $0 < \mu(E_k^*) \to 0$ for which

$$\mu_{E_k^*}(\mu(E_k^*)\, a_T(\varphi_{E_k^*}) \le t) \Longrightarrow 1_{[1,\infty)}(t)\left(1 - \tfrac{1}{t}\right) \qquad \text{as } k \to \infty. \tag{2.11}$$

The proofs of Theorems 2.2 and 2.3 will depend on the investigation of the relations between return- and hitting-time limits presented in Section 4.

3. Proofs for some results of Section 2

We begin with the

Proof of Proposition 2.1. **a)** We show that there are $E_k \in \mathcal{A}$ with $\mu(E_k) = \eta$ such that $\varphi_{E_k} \ge k$ on E_k for $k \ge 1$.

Note first that an infinite measure space allowing a c.e.m.p. map T is necessarily nonatomic. Take some $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$, and set $Y_0 := Y$ and $Y_n := Y^c \cap \{\varphi_Y = n\}$, $n \ge 1$. For any $k \ge 1$, the set $A_k := \bigcup_{j \ge 1} Y_{jk}$ satisfies $\varphi_{A_k} \mid_{A_k} \ge k$ and therefore $\varphi_E \mid_E \ge k$ holds for all $E \in A_k \cap \mathcal{A}$. Since $\mu(A_k) = \infty$ and μ is nonatomic, A_k has a subset E_k with $\mu(E_k) = \eta$.

b) We first prove that for every $E \in Y \cap \mathcal{A}$ with $\mu(E) > 0$, and any $m, n \ge 1$,

$$\mu_E(\varphi_E > mn) \le \frac{\mu(Y)}{\mu(E)} \left(\frac{1}{m} + m\, \mu_Y(\varphi_Y > n) \right). \tag{3.1}$$

Note first that decomposing an excursion from E into consecutive excursions from Y, we can represent φ_E as

$$\varphi_E = \sum_{j=0}^{\varphi_E^Y - 1} \varphi_Y \circ T_Y^j \quad \text{on } Y, \tag{3.2}$$

where $\varphi_E^Y(x) := \inf\{i \ge 1 : T_Y^i x \in E\}$ denotes the first entrance time of E under the induced map T_Y. This reveals that

$$E \cap \{\varphi_E > mn\} \subseteq \left(E \cap \{\varphi_E^Y > m\}\right) \cup \left(Y \cap \bigcup_{j=0}^{m-1} T_Y^{-j}\{\varphi_Y > n\} \right). \tag{3.3}$$

Applying Kac' formula to T_Y gives $\mu_E(\varphi_E^Y > m) \le \mu(Y)/(m\mu(E))$, and since T_Y preserves μ_Y, it is clear that $\mu_Y(\bigcup_{j=0}^{m-1} T_Y^{-j}\{\varphi_Y > n\}) \le m\, \mu_Y(\varphi_Y > n)$. Combining these yields (3.1).

Now take any $\varepsilon > 0$. First choose $m \geq 1$ so large that $\frac{\mu(Y)}{\eta m} < \frac{\varepsilon}{2}$, then pick $n \geq 1$ so large that $\frac{\mu(Y)}{\eta} m\, \mu_Y(\varphi_Y > n) < \frac{\varepsilon}{2}$ as well. Now (3.1) shows that $\mu_E(\varphi_E > mn) < \varepsilon$ whenever $E \in Y \cap \mathcal{A}$ satisfies $\mu(E) \geq \eta$.

c) Assume w.l.o.g. that $\mu(Y) = 1$. We prove that there are sets $E_k \in Y \cap \mathcal{A}$ with $0 < \mu(E_k) < 1/k$ and

$$\mu_{E_k}(\mu(E_k)\, \varphi_{E_k} > k) > (k-1)/k \quad \text{for } k \geq 1. \tag{3.4}$$

Fix any $k \geq 1$. Since, according to Kac' formula, $\int_Y \varphi_Y \, d\mu = \infty$, we have $m^{-1} \sum_{j=0}^{m-1} \varphi_Y \circ T_Y^j \to \infty$ a.e. on Y by (an obvious extension of) the ergodic theorem. An Egorov-type argument shows that there is some $Z \in Y \cap \mathcal{A}$ with $\mu(Y \setminus Z) < 1/(2k)$ and an integer $M > k$ such that

$$\sum_{j=0}^{m-1} \varphi_Y \circ T_Y^j > 2mk \quad \text{on } Z \text{ for } m \geq M. \tag{3.5}$$

Recalling the representation (3.2), we conclude that for every $E \in Y \cap \mathcal{A}$,

$$\mu(E)\, \varphi_E > 2M\mu(E)k \quad \text{on } Z \cap \{\varphi_E^Y \geq M\}. \tag{3.6}$$

Now appeal to the Rokhlin lemma to obtain some $F \in Y \cap \mathcal{A}$ for which the sets $F_i := T_Y^{-i} F$, $i \in \{0, \ldots, M-1\}$ are pairwise disjoint, and $\mu(Y \setminus \bigcup_{i=0}^{M-1} F_i) < 1/(M+1)$. In particular, $1/(M+1) < \mu(F_i) \leq 1/M$ for each i. Observing that $\mu(\bigcup_{i=0}^{M-1} F_i \setminus Z) < \mu(\bigcup_{i=0}^{M-1} F_i)/k$, we see that there is some $i_0 \in \{0, \ldots, M-1\}$ for which

$$\mu_{F_{i_0}}(F_{i_0} \setminus Z) < 1/k. \tag{3.7}$$

Let $E_k := F_{i_0}$, then $E_k \in Y \cap \mathcal{A}$ with $1/(M+1) < \mu(E_k) < 1/k$. On the other hand, by the Rokhlin tower structure, we have $\varphi_{E_k}^Y \geq M$ on E_k, and can thus employ the estimate (3.6) to see that

$$\mu(E_k)\, \varphi_{E_k} > k \quad \text{on } Z \cap E_k. \tag{3.8}$$

In view of (3.7), this implies our claim (3.4). □

Given Proposition 2.1, it is easy to proceed to the

Proof of Proposition 2.2. By part b) of Proposition 2.1 there is some non-decreasing sequence $(\vartheta_m)_{m\geq 1}$ in $(0, \infty)$ such that $\mu_E(\varphi_E > \vartheta_m) < 1/m$ for all $E \in Y \cap \mathcal{A}$ with $\mu(E) > 1/m$. Choose some scaling function γ for which $\gamma(1/m) = 1/(m\vartheta_{m+1})$, $m \geq 1$. Then, for $E \in Y \cap \mathcal{A}$ with $\mu(E) \in (\frac{1}{m+1}, \frac{1}{m}]$,

$$\begin{aligned} \mu_E(\gamma(\mu(E))\, \varphi_E > 1/m) &\leq \mu_E(\gamma(1/m)\, \varphi_E > 1/m) \\ &= \mu_E(\varphi_E > \vartheta_{m+1}) < 1/m. \end{aligned}$$

This easily implies that γ is tight for Y: Fix any $\varepsilon > 0$ and choose $M := \lfloor 1/\varepsilon \rfloor + 1$. For any $E \in Y \cap \mathcal{A}$ with $\mu(E) < 1/M$ there is some $m \geq M$ with $\mu(E) \in (\frac{1}{m+1}, \frac{1}{m}]$, and hence $\mu_E(\gamma(\mu(E))\, \varphi_E > 1) < \varepsilon$ by the above. On the other hand, part b) of Proposition 2.1 provides us with some K such that $\mu_E(\gamma(\mu(E))\, \varphi_E > K) < \varepsilon$ whenever $\mu(E) \geq 1/M$. □

Proof of Proposition 2.3. Assume w.l.o.g. that $\mu(Y)=1$. We construct $E_k \in Y\cap\mathcal{A}$, $k\geq 1$, s.t. $\mu(E_k)=1/k$ and $\mu_{E_k}(\varphi_{E_k}>1)\leq 1/k$. Then (E_k) satisfies (2.8).

Fix any $k\geq 1$. By the Rokhlin lemma, there is some $F\in Y\cap\mathcal{A}$, $\mu(F)>0$, such that the sets $F_i := T_Y^{-i}F$, $i\in\{0,\ldots,k-1\}$ are pairwise disjoint. Since μ is nonatomic, there is some $F'\in F\cap\mathcal{A}$ such that $\mu(F')=1/k^2$. Set $E_k := \bigcup_{i=0}^{k-1} T_Y^{-i}F'$ (disjoint), then $\mu(E_k)=1/k$, and $\varphi_{E_k}=1$ on $E_k\setminus F'$. □

We prepare the proof of Theorem 2.1 by recording two easy lemmas.

Lemma 3.1 (**Sufficient conditions for non-trivial and non-tight scales**). *Let T be a c.e.m.p.t. on $(X,\mathcal{A},\mu)$ with $\mu(X)=\infty$, $Y\in\mathcal{A}$ a set with $0<\mu(Y)<\infty$, and let $\gamma:[0,\infty)\to[0,\infty)$ be a scaling function.*

$$\text{(3.9)}\qquad \textit{If } \overline{\lim_{n\to\infty}}\, n\,\gamma(\mu(Y)q_n(Y))>0, \textit{ then } \gamma \textit{ is a non-trivial scale for } Y$$

in the sense of (2.7), and

$$\text{(3.10)}\qquad \textit{if } \overline{\lim_{n\to\infty}}\, n\,\gamma(\mu(Y)q_n(Y))=\infty, \textit{ then } \gamma \textit{ is not a tight scale for } Y.$$

Proof. For $n\geq 1$ set $E_n^* := Y\cap\{\varphi_Y>n\}\subseteq Y$, then $\mu(E_n^*)=\mu(Y)q_n(Y)$, while $\varphi_{E_n^*}\geq\varphi_Y>n$ on E_n^*. Hence,

$$\gamma(\mu(E_n^*))\,\varphi_{E_n^*}>n\,\gamma(\mu(Y)q_n(Y)) \quad \text{ on } E_n^*,$$

and the implications (3.9) and (3.10) follow at once. □

Remark 3.1 (**Non-triviality of γ_T**). Condition of (3.9) immediately shows that *under the assumptions of Theorem 2.2, if $\alpha\in(0,1)$, then γ_T is non-trivial.*

Indeed, the asymptotic renewal equation (1.6) together with the Monotone Density Theorem (Theorem 1.7.2 of [BGT], this uses $\alpha<1$) shows that in this case $q_n(Y)\sim\kappa_\alpha/a_T(n)$ as $n\to\infty$ with $\kappa_\alpha := (1-\alpha)/(\Gamma(2-\alpha)\Gamma(1+\alpha))\in(0,\infty)$. Therefore, $n\,\gamma_T(\mu(Y)q_n(Y))\to(\mu(Y)\kappa_\alpha)^{1/\alpha}\in(0,\infty)$ as $n\to\infty$.

Lemma 3.2 (**Prescribing the measure of a set with given wandering rate**). *Let T be a c.e.m.p.t. on $(X,\mathcal{A},\mu)$ with $\mu(X)=\infty$, $Z\in\mathcal{A}$ a set with $0<\mu(Z)<\infty$.*

a) *For every $m\geq 1$, we have $w_N(Z^c\cap\{\varphi_Z=m\})\sim w_N(Z)$ as $N\to\infty$.*

b) *For every $\eta\in(0,\infty)$ there is some $Y\in\mathcal{A}$ with $\mu(Y)=\eta$ and $w_N(Y)\sim w_N(Z)$.*

Proof. **a)** Fix $m\geq 1$ and note that $Z' := Z^c\cap\{\varphi_Z=m\}\subseteq\bigcup_{k=0}^m T^{-n}Z$, so that

$$w_N(Z')\leq w_N(\textstyle\bigcup_{k=0}^m T^{-n}Z)\sim w_N(Z).$$

On the other hand, for $N>m$ we have

$$\bigcup_{k=0}^{N-1}T^{-n}Z\subseteq\left(\bigcup_{k=0}^{m-1}T^{-n}Z\right)\cup\left(\bigcup_{k=0}^{N-m}T^{-k}(Z')\right),$$

and hence $w_N(Z)\leq\mu(\bigcup_{k=0}^{m-1}T^{-n}Z)+w_{N-m}(Z')\sim w_N(Z')$, as required.

b) Since $\mu(Z^c\cap\{\varphi_Z=m\})=\mu(Z\cap\{\varphi_Z>m\})\to 0$ as $m\to\infty$, statement a) shows that there are arbitrarily small sets which asymptotically have the same wandering rate as Z. We can therefore assume w.l.o.g. that $\mu(Z)\leq\eta$.

Now, as $\bigcup_{k=0}^{L-1} T^{-k}Z \nearrow X \pmod{\mu}$, there is some integer $L \geq 1$ for which

$$\mu(\bigcup_{k=0}^{L-1} T^{-k}Z) \leq \eta < \mu(\bigcup_{k=0}^{L} T^{-k}Z).$$

Since μ is necessarily non-atomic, there is some measurable $W \subseteq T^{-L}Z \setminus \bigcup_{k=0}^{L-1} T^{-k}Z$ for which $\mu(W) = \eta - \mu(\bigcup_{k=0}^{L-1} T^{-k}Z)$. Set $Y := \bigcup_{k=0}^{L-1} T^{-k}Z \cup W$, then $\mu(Y) = \eta$ and since $\bigcup_{k=0}^{L-1} T^{-k}Z \subseteq Y \subseteq \bigcup_{k=0}^{L} T^{-k}Z$ we have $w_N(Y) \sim w_N(Z)$. □

Proof of Theorem 2.1. In view of Lemma 3.1 it suffices to show that there is some $Y \in \mathcal{A}$ with $\mu(Y) = 1$ such that

$$\overline{\lim}_{n\to\infty} n\,\gamma(q_n(Y)) = \infty. \tag{3.11}$$

Observe first that we can choose a sequence $(M_n)_{n\geq 1}$ in $(0,\infty)$ such that $M_n \nearrow \infty$ while $M_n/n \searrow 0$ so slowly that $a_N^\circ := \sum_{n=1}^{N} \gamma^{-1}(M_n/n) \nearrow \infty$ as $N \to \infty$. Since $\gamma^{-1}(M_n/n) \searrow 0$, we also have $a_N^\circ/N \searrow 0$ in this case.

By Proposition 3.8.2 of [A0], there is some $Z \in \mathcal{A}$, $0 < \mu(Z) < \infty$, such that $w_N(Z) \geq 2a_N$ for $N \geq 1$. According to Lemma 3.2, there is some $Y \in \mathcal{A}$, $\mu(Y) = 1$, with $w_N(Y) \sim w_N(Z)$. In particular, there is some N° such that $w_N(Y) > 3a_N/2$ for $N \geq N^\circ$, that is,

$$\sum_{n=0}^{N} q_n(Y) > \frac{3}{2} \sum_{n=1}^{N} \gamma^{-1}(M_n/n) \quad \text{for } N \geq N^\circ.$$

But since $a_N^\circ \nearrow \infty$ this implies that there are indices $n_k \nearrow \infty$ for which

$$q_{n_k}(Y) > \gamma^{-1}(M_{n_k}/n_k) \quad \text{whenever } k \geq 1,$$

and hence $n_k\,\gamma(q_{n_k}(Y)) > M_{n_k}$ for $k \geq 1$. As $M_{n_k} \nearrow \infty$, this proves (3.11). □

4. Return-time limits versus hitting-time limits

This section contains the central results of the present paper. Our proofs of several other results (including Theorem 2.2 above) rely on them.

We clarify the relation between asymptotic return-time distributions and asymptotic hitting-time distributions in the situation of Theorem 2.2, where a suitable scaling function has been identified.

Strong distributional convergence of hitting times. In contrast to asymptotic return distributions, asymptotic hitting distributions are robust under any change of (absolutely continuous) initial measure. This property was mentioned before. Since it plays a crucial role in the following, we provide a formal statement (taken from [Z4], Corollary 5).

Lemma 4.1 (**Strong distributional convergence of hitting times**). *Let T be a c.e.m.p.t. on $(X,\mathcal{A},\mu)$, γ a scaling function, and (E_k) a sequence in $\mathcal{A}$ with $0 < \mu(E_k) \to 0$. If there are some $F \in \mathcal{F}$ and some probability $\nu^* \ll \mu$ such that*

$$\nu^*(\gamma(\mu(E_k))\,\varphi_{E_k} \leq t) \Longrightarrow F(t) \quad \textit{as } k \to \infty, \tag{4.1}$$

then

$$\nu(\gamma(\mu(E_k))\,\varphi_{E_k} \leq t) \Longrightarrow F(t) \quad \textit{as } k \to \infty \quad \textit{for all } \nu \ll \mu. \tag{4.2}$$

The reason behind this principle is *asymptotic T-invariance in measure* of the variables $\gamma(\mu(E_k))\,\varphi_{E_k}$, which means that

$$\gamma(\mu(E_k))\,\varphi_{E_k} - \gamma(\mu(E_k))\,\varphi_{E_k} \circ T \xrightarrow{\nu} 0 \quad \text{as } k \to \infty \tag{4.3}$$

for all probabilities $\nu \ll \mu$. The latter is immediate from the fact that

$$\varphi_E = \varphi_E \circ T + 1 \quad \text{on } \{\varphi_E > 1\} = T^{-1}E. \tag{4.4}$$

Arguments related to those responsible for this lemma will be used in Section 6.

Hitting versus returning in finite measure systems. Recall that in the probability-preserving setup, a simple general principle relates limit laws for normalized hitting-times and for normalized return-times to each other (see [HLV]):

Theorem HLV (Return and hitting-time limits for finite measure). *Let T be an ergodic m.p.t. on (X, A, μ), $\mu(X) = 1$. Suppose that E_k, $k \geq 1$, are sets of positive measure with $\mu(E_k) \to 0$.*

Then the normalized return-time distributions of the E_k converge in that

$$\mu_{E_k}(\mu(E_k)\,\varphi_{E_k} \leq t) \Longrightarrow \widetilde{F}(t) \quad \text{as } k \to \infty \tag{4.5}$$

for some $\widetilde{F} \in \mathcal{F}$, if and only if the normalized hitting-time distributions converge,

$$\mu(\mu(E_k)\,\varphi_{E_k} \leq t) \Longrightarrow F(t) \quad \text{as } k \to \infty \tag{4.6}$$

for some $F \in \mathcal{F}$. In this case the limit laws satisfy

$$F(t) = \int_0^t [1 - \widetilde{F}(s)]\, ds \quad \text{for } t \geq 0. \tag{4.7}$$

If (4.7) is satisfied, F is sometimes called the *integrated tail distribution* of $\widetilde{F}$.

Remark 4.1. In view of Kac' formula, it is a priori clear that any limit law $\widetilde{F}$ of the normalized return times has expectation not exceeding 1. On the other hand, it can (and often does) happen that the limit law is concentrated at $t = 0$, so that $\widetilde{F} = 1_{[0,\infty)}$. According to (4.7), this happens iff the scaled hitting-times $\mu(E_k)\,\varphi_{E_k}$ diverge to infinity, $F = 0$.

Hitting versus returning in infinite measure systems - difficulties. We address the obvious question of how the two different types of limit theorems are related in null-recurrent situations. As a first complication, meaningful limits must involve some nontrivial scaling function γ, and we have seen above that there is no natural choice which works for all small sets. But even if we can find γ such that a specific sequence (E_k) has a nondegenerate return-time limit on this scale, this does not imply that it also has a hitting-time limit different from ∞ if we use γ (in contrast to the finite measure case, compare Remark 4.1.)

Theorem 4.1 (**All return time limits are compatible with exploding hitting times**). *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$ with $\mu(X) = \infty$, and take any $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$. Let γ be any scaling function.*

Suppose that $E_k \in \mathcal{A}$, $k \geq 1$, are sets of positive measure with $\mu(E_k) \to 0$. If, for some $\widetilde{F} \in \mathcal{F}$,

$$\mu_{E_k}(\gamma(\mu(E_k))\,\varphi_{E_k} \leq t) \Longrightarrow \widetilde{F}(t) \quad \textit{as } k \to \infty, \tag{4.8}$$

then there exist $E'_k \in \mathcal{A}$ *with* $\mu(E'_k) = \mu(E_k)$ *and*

$$\mu_{E'_k}(\gamma(\mu(E'_k))\,\varphi_{E'_k} \leq t) \Longrightarrow \widetilde{F}(t) \quad \text{as } k \to \infty, \tag{4.9}$$

while

$$\mu_Y(\gamma(\mu(E'_k))\,\varphi_{E'_k} \leq t) \Longrightarrow 0 \quad \text{as } k \to \infty. \tag{4.10}$$

(Recall that by Helly's selection theorem, every asymptotivally rare sequence (E_k) has a subsequence along which return distibutions converge as in (4.8).)

The (simple) principle behind this theorem is that we can move copies E'_k of the E_k, which have the same return distributions, to places which are far away from any given Y. Therefore we can only hope for a result parallel to Theorem HLV if we focus on sequences (E_k) inside some reference set.

Hitting versus returning inside a uniform set. We prove an abstract result in the spirit of Theorem HLV which applies to infinite measure preserving maps possessing a uniform set with regularly varying return sequence. We use the normalization discussed in the preceding section. The following result confirms once again that the latter is a sensible choice.

Theorem 4.2 (**Return- versus hitting-time limits in uniform sets**). *Let* T *be a c.e.m.p.t. on* $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$. *Assume that* $Y \in \mathcal{A}$ *is a uniform set and that* $a_T \in \mathcal{R}_\alpha$ *for some* $\alpha \in (0, 1]$. *Define* γ_T *by* $\gamma_T(s) := 1/b_T(1/s)$, $s > 0$.

Suppose that $E_k \subseteq Y$, $k \geq 1$, *are sets of positive measure with* $\mu(E_k) \to 0$. *Then the normalized return-time distributions of the* E_k *converge in that*

$$\mu_{E_k}(\gamma_T(\mu(E_k))\,\varphi_{E_k} \leq t) \Longrightarrow \widetilde{F}(t) \quad \text{as } k \to \infty \tag{4.11}$$

for some $\widetilde{F} \in \mathcal{F}$, *if and only if the normalized hitting-time distributions converge,*

$$\mu_Y(\gamma_T(\mu(E_k))\,\varphi_{E_k} \leq t) \Longrightarrow F(t) \quad \text{as } k \to \infty \tag{4.12}$$

for some $F \in \mathcal{F}$. *In this case the limit laws satisfy*

$$F(t) = \int_0^t [1 - \widetilde{F}(s)]\,\alpha\,(t-s)^{\alpha-1}\,ds \quad \text{for } t \geq 0. \tag{4.13}$$

Remark 4.2 (**Some comments and consequences**). We record the following:
a) In the $\alpha = 1$ case of a "barely infinite" measure, (4.13) reduces to (4.7).
b) In (4.13), the function $\widetilde{F}$ clearly determines F, and vice versa.
c) Relation (4.13) shows that F is necessarily continuous on $[0, \infty)$ with $F(0) = 0$. Moreover, since $\int_0^t \alpha\,(t-s)^{\alpha-1}\,ds = t^\alpha \to \infty$ as $t \to \infty$, it is immediate that $\widetilde{F}(s) \to 1$ as $s \to \infty$, so that $\widetilde{F}$ is a *probability* distribution function on $[0, \infty)$.
d) By the previous remark and Theorem 2.1 we see that under the assumptions of Theorem 4.2 there are always sets $Y' \in \mathcal{A}$, $0 < \mu(Y') < \infty$, inside which the conclusion of Theorem 4.2 fails.
e) If (E_k) is a sequence such that (4.11) takes place with $\widetilde{F} = 1_{[0,\infty)}$, that is $\mathsf{law}_{\mu_{E_k}}[\gamma_T(\mu(E_k))\,\varphi_{E_k}] \Longrightarrow \delta_0$ (by Proposition 2.3 the set Y always contains such a sequence), then (4.12) holds with $F = 0$, so that $\mathsf{law}_{\mu_Y}[\gamma_T(\mu(E_k))\,\varphi_{E_k}] \Longrightarrow \delta_\infty$.
f) In (4.12) the measure μ_Y can be replaced by any fixed probability measure $\nu \ll \mu$, see Lemma 4.1.

Theorem 4.2 is a consequence of the following result where return- and hitting times are distorted by the nonlinear function a_T. Theorem 4.3 is more general in that it also gives nontrivial information about the "barely recurrent" $\alpha = 0$ case, which is excluded in Theorem 4.2. This case is of interest, because of very natural examples (recurrent random walks on $\mathbb{Z}^2$ and recurrent $\mathbb{Z}^2$-extensions including the Lorentz process, see [PS1] and [PS2], and slowly recurrent random walks on $\mathbb{R}$, [PSZ1]) in which (for typical sequences (E_k)) the $\mu(E_k)\, a_T(\varphi_{E_k})$ have been shown to converge to the limit law with distribution function $t \mapsto t/(1+t)$, $t \geq 0$.

Theorem 4.3 (**Distorted return- and hitting-time limits in uniform sets**). *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$. Assume that $Y \in \mathcal{A}$ is a uniform set and that $a_T \in \mathcal{R}_\alpha$ for some $\alpha \in [0, 1]$.*

Suppose that $E_k \subseteq Y$, $k \geq 1$, are sets of positive measure with $\mu(E_k) \to 0$. Then the distorted return-time distributions of the E_k converge,

$$\mu_{E_k}(\mu(E_k)\, a_T(\varphi_{E_k}) \leq t) \Longrightarrow \widetilde{G}(t) \quad \textit{as } k \to \infty \tag{4.14}$$

for some $\widetilde{G} \in \mathcal{F}$, if and only if the distorted hitting-time distributions converge,

$$\mu_Y(\mu(E_k)\, a_T(\varphi_{E_k}) \leq t) \Longrightarrow G(t) \quad \textit{as } k \to \infty \tag{4.15}$$

for some $G \in \mathcal{F}$. In this case the limit laws satisfy, for $t \geq 0$,

$$G(t) = \begin{cases} t \displaystyle\int_0^1 [1 - \widetilde{G}\,(t\,(1-r)^\alpha)]\, \alpha r^{\alpha-1}\, dr & \textit{if } \alpha \in (0,1], \\ t\,[1 - \widetilde{G}(t)] & \textit{if } \alpha = 0, \end{cases} \tag{4.16}$$

and G is Lipschitz continuous on $[0, \infty)$.

The proofs of the two theorems are given in the next section.

Remark 4.3. For $\alpha \in (0, 1]$, the statement of Theorem 4.3 is equivalent to Theorem 4.2, with $(\widetilde{G}(t), G(t)) = (\widetilde{F}(t^{1/\alpha}), F(t^{1/\alpha}))$, see Lemma 5.2 below. Remark 4.2 translates accordingly.

Regarding the $\alpha = 0$ case, some related facts are recorded in

Remark 4.4 (**More comments**). **a)** If $G \in \mathcal{F}$ satisfies

$$G(t) = t[1 - G(t)] \quad \text{for } t > 0, \tag{4.17}$$

then $G(t) = t/(1+t)$, which is the $\alpha = 0$ limit law from [PS1], [PS2], and [PSZ1].
b) If $G, \widetilde{G} \in \mathcal{F}$ satisfy $G(t) = t\,[1 - \widetilde{G}(t)]$ for $t \geq 0$, as in the $\alpha = 0$ case of (4.16), then $\widetilde{G}$ is necessarily a probability distribution function on $[0, \infty)$, because $G \leq 1$. Moreover, in the $\alpha = 0$ case, $\widetilde{G}$ is continuous on $[0, \infty)$ since G is.
c) Regarding the right-hand side of (4.16), note that for every $\widetilde{G} \in \mathcal{F}$ and every continuity point $t > 0$ of $\widetilde{G}$ we have

$$\int_0^1 [1 - \widetilde{G}\,(t\,(1-r)^\alpha)]\, \alpha r^{\alpha-1}\, dr \longrightarrow 1 - \widetilde{G}(t) \quad \text{as } \alpha \searrow 0.$$

While the $a_T(\varphi_{E_k})$ do exhibit nontrivial asymptotic distributional behaviour in the interesting $\alpha = 0$ situations mentioned above, the original φ_{E_k} do not. (Hence, studying the $a_T(\varphi_{E_k})$ is the right thing to do.) A formal version of this statement is immediate from the following general fact.

Proposition 4.1 (**No way back from $\ell(R_n)$ to R_n**). *Let $\ell : [0,\infty) \to [0,\infty)$ be a slowly varying homeomorphism. Assume that (R_n) is a sequence of $[0,\infty]$-valued random variables with $R_n \Longrightarrow \infty$ for which $(\ell(R_n))$ has a continuous limit distribution on $(0,\infty)$, that is, there are (η_n) in $(0,\infty)$ and a continuous random variable L with $0 < L < \infty$ a.s., such that*

$$\eta_n\, \ell(R_n) \Longrightarrow L \quad \text{as } n \to \infty. \tag{4.18}$$

Then any limit distribution of (R_n) is concentrated on $\{0,\infty\}$, that is, if

$$\gamma_n\, R_n \Longrightarrow R \quad \text{as } n \to \infty, \tag{4.19}$$

for some (γ_n) in $(0,\infty)$ and some random variable R, then $\Pr[R \in \{0,\infty\}] = 1$.

5. Proofs for Sections 2 and 4

General setup. As a warm-up we provide the

Proof of Proposition 4.1. Assume, for a contradiction, that (4.19) holds and $\Pr[e^{-c} < R \le e^{c}] > 0$ for some $c > 1$. Then there are $\kappa > 0$ and $n_0 \ge 1$ such that the events $A_n := \{e^{-2c} < \gamma_n\, R_n \le e^{2c}\}$ satisfy $\Pr[A_n] \ge \kappa$ for $n \ge n_0$. Set $\overline{\eta}_n := 1/\ell(1/\gamma_n)$ and $t_n^{\pm} := \overline{\eta}_n \ell(e^{\pm 2c}/\gamma_n)$, so that the above becomes

$$\Pr[t_n^- < \overline{\eta}_n\, \ell(R_n) \le t_n^+] \ge \kappa \quad \text{for } n \ge n_0. \tag{5.1}$$

Since $R_n \Longrightarrow \infty$, we have $\gamma_n \to 0$, and slow variation of ℓ yields $t_n^{\pm} \to 1$.

Passing to a subsequence if necessary, we may assume that there is some $[0,\infty]$-valued random variable $\overline{L}$ such that

$$\overline{\eta}_n\, \ell(R_n) \Longrightarrow \overline{L} \quad \text{as } n \to \infty. \tag{5.2}$$

Now (5.1) and $t_n^{\pm} \to 1$ clearly imply $\Pr[\overline{L} = 1] \ge \kappa > 0$. In view of the standard *convergence of types* theorem, applied to $(\ell(R_n))$, this contradicts our assumption (4.18), where L is a *continuous* variable in $(0,\infty)$. $\square$

Throughout the rest of this section, assume (as in the theorems) that T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$, and that $0 < \mu(Y) < \infty$. We begin with the

Proof of Theorem 4.1. **(i)** We first recall that for any $B \in \mathcal{A}$ of finite measure,

$$\frac{1}{n} \sum_{k=0}^{n-1} \int_B \widehat{T}^k 1_Y \, d\mu \longrightarrow 0 \quad \text{as } n \to \infty. \tag{5.3}$$

Observe next that for every sequence (m_k) of integers $m_k \ge 0$ the sets $E_k' := T^{-m_k} E_k$, $k \ge 1$, satisfy

$$\mu_{E_k'}(\gamma_T(\mu(E_k'))\, \varphi_{E_k'} \le t) = \mu_{E_k}(\gamma_T(\mu(E_k))\, \varphi_{E_k} \le t) \quad \text{for } t \ge 0,\ k \ge 1.$$

Indeed, for $E \in \mathcal{A}$ with $0 < \mu(E) < \infty$, and integers $m \ge 0$, it is immediate that

$$\varphi_{T^{-m}E} = \varphi_E \circ T^m \quad \text{and} \quad \mu_{T^{-m}E} \circ T^{-m} = \mu_E, \tag{5.4}$$

and thus

$$\mathsf{law}_{\mu_{T^{-m}E}}[\gamma(\mu(T^{-m}E))\,\varphi_{T^{-m}E}] = \mathsf{law}_{\mu_E}[\gamma(\mu(E))\,\varphi_E]. \tag{5.5}$$

(ii) Fix any $k \geq 1$ and set $B_k := \{\varphi_{E_k} \leq k/\gamma(\mu(E_k))\} = \bigcup_{j=1}^{\lfloor k/\gamma(\mu(E_k))\rfloor} T^{-j}E_k \in \mathcal{A}$, which has finite measure. If $m \geq 0$ and $E' := T^{-m}E_k$, then $\{\gamma(\mu(E'))\varphi_{E'} \leq k\} = T^{-m}B_k$ and hence

$$\mu_Y(\gamma(\mu(E'))\,\varphi_{E'} \leq k) = \mu_Y(T^{-m}B_k) = \mu(Y)^{-1}\int_{B_k} \widehat{T}^m 1_Y\,d\mu.$$

But in view of (5.3) there is some $m =: m_k$ for which $\int_{B_k} \widehat{T}^m 1_Y\,d\mu < \mu(Y)/k$, and we take $E'_k := T^{-m_k}E_k$. □

Arguments involving uniform sets. Assume now that the system is pointwise dual ergodic with $a_T \in \mathcal{R}_\alpha$ for some $\alpha \in [0,1]$, and let b_T be asymptotically inverse to a_T. Suppose that Y is a u-uniform set, w.l.o.g. with $\mu(Y) = 1$, and that $E_k \subseteq Y$, $k \geq 1$, are sets of positive finite measure with $\mu(E_k) \to 0$.

Our argument exploits the Ansatz of [PSZ2] (which goes back to [DE]), and uses several auxiliary facts already mentioned or established there. The following (which slightly generalizes Lemma 5.3 of [PSZ2]) is our starting point.

Lemma 5.1 (**Decomposition according to the last visit before time** n)**.** *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, and $B \in \mathcal{A}$ while $u \in \mathcal{D}(\mu)$. Then*

$$\int_{\{\varphi_B \leq n\}} u\,d\mu = \sum_{l=1}^{n} \int_{B\cap\{\varphi_B > n-l\}} \widehat{T}^l u\,d\mu \quad \textit{for } n \geq 0. \tag{5.6}$$

Proof. Fix $n \geq 0$ and decompose $\{\varphi_B \leq n\}$ according to the last instant $l \in \{1,\dots,n\}$ at which the orbit visits B to get

$$\{\varphi_B \leq n\} = \textstyle\bigcup_{l=1}^{n} T^{-l}(B \cap \{\varphi_B > n-l\}) \quad \text{(disjoint)}.$$

Now measure these sets using the probability with density u. □

As our goal is to prove (4.14) and (4.15), we define

$$\mathsf{R}_k := \mu(E_k)\,a_T(\varphi_{E_k}), \quad \text{for } k \geq 1, \tag{5.7}$$

and denote the relevant distribution functions by $\widetilde{G}_k$ and G_k,

$$\widetilde{G}_k(t) := \mu_{E_k}(\mathsf{R}_k \leq t), \quad G_k(t) := \mu_Y(\mathsf{R}_k \leq t) \quad \text{for } k \geq 1,\ t \in [0,\infty). \tag{5.8}$$

It is convenient to set, for $t \in [0,\infty)$ and $k \geq 1$,

$$n_k^{[t]} := b_T\left(t/\mu(E_k)\right), \tag{5.9}$$

and, for $l \in \{0,\dots,n_k^{[t]}\}$, $\vartheta_{k,l}^{[t]} := \mu(E_k)\cdot a_T(n_k^{[t]} - l)$. These allow us to represent some important events, as

$$\{\mathsf{R}_k > t\} = \{\varphi_{E_k} > n_k^{[t]}\}, \quad \text{while} \quad \{\mathsf{R}_k > \vartheta_{k,l}^{[t]}\} = \{\varphi_{E_k} > n_k^{[t]} - l\}. \tag{5.10}$$

Note that, for any fixed t and k, the function $l \mapsto \vartheta_{k,l}^{[t]}$ is non-increasing.

We are now ready for the

Proof. **Proof of Theorem 4.3, case** $\alpha \in (0,1]$**. (i)** note first that given $\rho \in [0,1)$ and a sequence $(l_k)_{k\geq 1}$ in $(0,\infty)$,

$$\text{if} \quad l_k \sim \rho \cdot n_k^{[t]}, \quad \text{then} \quad \vartheta_{k,l_k}^{[t]} \sim t \cdot (1-\rho)^\alpha \quad \text{as } k \to \infty. \tag{5.11}$$

Moreover, by the uniform set property of Y, if $0 \leq c_1 < c_2$, then

$$\sum_{j=c_1 n}^{c_2 n - 1} \widehat{T}^j u \sim (c_2^\alpha - c_1^\alpha) \cdot a_n \qquad \text{as } n \to \infty \text{, uniformly mod } \mu \text{ on } Y. \tag{5.12}$$

(ii) To exploit the fact that Y is a u-uniform set, we consider hitting-time distributions with respect to the probability with density u, and set

$$G_k^u(t) := \int_{\{\mathsf{R}_k \leq t\}} u \, d\mu, \quad \text{for } k \geq 1,\ t \in [0,\infty). \tag{5.13}$$

In view of Remark 4.2 g), we see that (4.15) is equivalent to

$$G_k^u(t) \Longrightarrow G(t) \quad \text{as } k \to \infty, \tag{5.14}$$

and for the rest of the proof we work with the latter condition.
(iii) Assume that $\widetilde{G}_k \Rightarrow \widetilde{G}$, and define G via

$$G(t) = \alpha t \int_0^1 [1 - \widetilde{G}\,(t\,(1-r)^\alpha)]\, r^{\alpha-1}\, dr \quad \text{for } t \in [0,\infty). \tag{5.15}$$

We are going to prove that there is a dense subset $\mathcal{T}$ of $[0,\infty)$ such that

$$G_k^u(t) \longrightarrow G(t) \quad \text{for } t \in \mathcal{T}, \tag{5.16}$$

It is then immediate that G is a sub-probability distribution function on $[0,\infty)$ for which $G_k^u \Longrightarrow G$.

To this end, let $\mathcal{T}$ be the set of those continuity points $t \in (0,\infty)$ of $\widetilde{G}$ with the property that for all integers $0 \leq m \leq M$, the $t(1-\frac{m}{M})^\alpha$ also are continuity points of $\widetilde{G}$. The complement of this set is only countable.

Henceforth, we fix some $t \in \mathcal{T}$, and abbreviate $n_k := n_k^{[t]}$ and $\vartheta_{k,i} := \vartheta_{k,i}^{[t]}$.

Lemma 5.1, with $B := E_k$, and $n := n_k$, gives

$$G_k^u(t) = \int_{\{\varphi_{E_k} \leq n_k\}} u\, d\mu = \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l u \, d\mu, \tag{5.17}$$

and we are going to prove, for $k \to \infty$, that

$$\sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l u \, d\mu \longrightarrow \alpha t \int_0^1 [1 - \widetilde{G}\,(t\,(1-r)^\alpha)]\, r^{\alpha-1}\, dr. \tag{5.18}$$

(iv) Since $b_T \in \mathcal{R}_{1/\alpha}$, we have $n_k \sim t^{1/\alpha} b_T(1/\mu(E_k))$ as $k \to \infty$. Fix some $M \geq 1$, and take any $\varepsilon \in (0,1)$. Decomposing the sum in (5.18) into M sections and recalling (5.10), we find that

$$\begin{aligned}\sum_{l=1}^{n_k}\int_{E_k\cap\{\varphi_{E_k}>n_k-l\}}\widehat{T}^l u\,d\mu &= \sum_{m=0}^{M-1}\sum_{l=\lfloor\frac{m}{M}n_k\rfloor+1}^{\lfloor\frac{m+1}{M}n_k\rfloor}\int_{E_k\cap\{\mathsf{R}_k>\vartheta_{k,l}\}}\widehat{T}^l u\,d\mu\\ &\le \sum_{m=0}^{M-1}\int_{E_k\cap\{\mathsf{R}_k>\vartheta_{k,\lfloor\frac{m+1}{M}n_k\rfloor}\}}\sum_{l=\lfloor\frac{m}{M}n_k\rfloor+1}^{\lfloor\frac{m+1}{M}n_k\rfloor}\widehat{T}^l u\,d\mu,\end{aligned}$$

where the second step uses that, by monotonicity of $l\mapsto\vartheta_{k,l}$, $\{\mathsf{R}_k>\vartheta_{k,l}\}\subseteq\{\mathsf{R}_k>\vartheta_{k,\lfloor(m+1)n_k/M\rfloor}\}$ for $l\le(m+1)n_k/M$. In view of (5.12) and $a_T(n_k)=t/\mu(E_k)$ we have, for $m\ge 0$,

$$\sum_{l=\lfloor\frac{m}{M}n_k\rfloor+1}^{\lfloor\frac{m+1}{M}n_k\rfloor}\widehat{T}^l u\sim\left(\left(\frac{m+1}{M}\right)^\alpha-\left(\frac{m}{M}\right)^\alpha\right)\frac{t}{\mu(E_k)}\qquad\begin{array}{c}\text{as } k\to\infty,\\ \text{uniformly mod } \mu \text{ on } Y.\end{array}$$

Since $(\frac{m+1}{M})^\alpha-(\frac{m}{M})^\alpha\le\alpha\frac{1}{M}(\frac{m}{M})^{\alpha-1}$ by the mean-value theorem, we thus get

$$\sum_{l=1}^{n_k}\int_{E_k\cap\{\varphi_{E_k}>n_k-l\}}\widehat{T}^l u\,d\mu\le e^\varepsilon\alpha\,t\sum_{m=1}^{M-1}\mu_{E_k}\left(\mathsf{R}_k>\vartheta_{k,\lfloor\frac{m+1}{M}n_k\rfloor}\right)\left(\frac{m}{M}\right)^{\alpha-1}\frac{1}{M}$$

for $k\ge K=K(M,\varepsilon)$. By our choice of t, $\widetilde{G}$ is continuous at each $t(1-\frac{m+1}{M})^\alpha$, so that (5.11) ensures, as $k\to\infty$,

$$\begin{aligned}\mu_{E_k}\left(\mathsf{R}_k>\vartheta_{k,\lfloor\frac{m+1}{M}n_k\rfloor}\right)&=1-\widetilde{G}_k\left(\vartheta_{k,\lfloor\frac{m+1}{M}n_k\rfloor}\right)\\ &\longrightarrow 1-\widetilde{G}\left(t\left(1-\frac{m+1}{M}\right)^\alpha\right).\end{aligned}$$

Combining this with the above, and letting $\varepsilon\searrow 0$, we obtain

$$\varlimsup_{k\to\infty}\sum_{l=1}^{n_k}\int_{E_k\cap\{\varphi_{E_k}>n_k-l\}}\widehat{T}^j u\,d\mu\le\alpha\,t\sum_{m=1}^{M-1}\left[1-\widetilde{G}\left(t\left(1-\frac{m+1}{M}\right)^\alpha\right)\right]\left(\frac{m}{M}\right)^{\alpha-1}\frac{1}{M}.$$

Now $M\to\infty$ yields

$$\varlimsup_{k\to\infty}\sum_{l=1}^{n_k}\int_{E_k\cap\{\varphi_{E_k}>n_k-l\}}\widehat{T}^j u\,d\mu\le\alpha t\int_0^1[1-\widetilde{G}\left(t\,(1-r)^\alpha\right)]\,r^{\alpha-1}\,dr.$$

A parallel argument proves the corresponding lower estimate,

$$\varliminf_{k\to\infty}\sum_{l=1}^{n_k}\int_{E_k\cap\{\varphi_{E_k}>n_k-l\}}\widehat{T}^j u\,d\mu\ge\alpha t\int_0^1[1-\widetilde{G}\left(t\,(1-r)^\alpha\right)]\,r^{\alpha-1}\,dr,$$

and hence our claim (5.18).

(v) Now assume that $G_k^u\Rightarrow G$. Our goal is to show that $\widetilde{G}_k\Rightarrow\widetilde{G}$ with $\widetilde{G}$ satisfying (5.15). In view of the Helly selection theorem we need only check that whenever $\widetilde{G}_{k_i}\Rightarrow\widetilde{G}_*$ for some subsequence $k_i\nearrow\infty$ of indices, this limit point $\widetilde{G}_*$ is indeed the unique sub-distribution function satisfying (5.15).

However, if we apply the conclusion of step (i) above, we see that $\widetilde{G}_{k_i}\Rightarrow\widetilde{G}_*$ entails $G_{k_i}^u\Rightarrow G_*$ with the pair $(G_*,\widetilde{G}_*)$ satisfying the desired integral equation.

Since $G_k^u \Rightarrow G$ it is clear that $G_* = G$, so that in fact $(G, \widetilde{G}_*)$ satisfying the integral equation. $\square$

A slight modification of the argument gives the

Proof. **Proof of Theorem 4.3, case** $\alpha = 0$**. (i)** As in the previous proof, we will work with the G_k^u from (5.13) rather than the G_k, and replace (4.15) by (5.14).

(ii) Assume that $\widetilde{G}_k \Rightarrow \widetilde{G}$. By a subsequence-in-subsequence argument, we may assume that also $G_k^u \Rightarrow G$ for some $G \in \mathcal{F}$. We are going to prove that for every point $t > 0$ (henceforth fixed) at which both G and $\widetilde{G}$ are continuous, we have

$$G_k^u(t) \longrightarrow t\,[1 - \widetilde{G}\,(t)] \quad \text{as } k \to \infty, \tag{5.19}$$

whence $G(s) = s\,[1 - \widetilde{G}\,(s)]$, $s > 0$. Abbreviate $n_k := n_k^{[t]}$.

Observe that $\{\varphi_{E_k} > n_k/2\} = \{\mathsf{R}_k > \mu(E_k)\,a(n_k/2)\}$, where, due to slow variation of a_T, $\mu(E_k)\,a(n_k/2) \sim \mu(E_k)\,a(n_k) = t$ as $k \to \infty$. Since $\widetilde{G}$ is continuous at t, we thus see that

$$\begin{aligned} \mu_{E_k}(\varphi_{E_k} > n_k/2) &= 1 - \widetilde{G}_k(\mu(E_k)\,a(n_k/2)) \\ &\longrightarrow 1 - \widetilde{G}(t) \quad \text{as } k \to \infty. \end{aligned} \tag{5.20}$$

(iii) By the uniform set property and $a(n_k/2) \sim a(n_k)$ we have

$$\sum_{l=n_k/2+1}^{n_k} \widehat{T}^l u = o\,(a(n_k)) \qquad \text{as } k \to \infty, \text{ uniformly mod } \mu \text{ on } Y,$$

and hence, since $a(n_k) = t/\mu(E_k)$ and $E_k \subseteq Y$, we find that

$$\begin{aligned} \sum_{l=n_k/2+1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l u \, d\mu &\leq \int_{E_k} \sum_{l=n_k/2+1}^{n_k} \widehat{T}^l u \, d\mu \\ &= \mu(E_k) o\,(a(n_k)) \\ &\longrightarrow 0 \qquad \text{as } k \to \infty. \end{aligned} \tag{5.21}$$

On the other hand, since $\{\varphi_{E_k} > n_k - l\} \subseteq \{\varphi_{E_k} > n_k/2\}$ for $l \leq n_k/2$, we can appeal to the uniform set property and to (5.20) to conclude that

$$\begin{aligned} \sum_{l=1}^{n_k/2} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l u \, d\mu &\leq \int_{E_k \cap \{\varphi_{E_k} > n_k/2\}} \sum_{l=1}^{n_k/2} \widehat{T}^l u \, d\mu \\ &\sim a(n_k/2)\, \mu(E_k \cap \{\varphi_{E_k} > n_k/2\}) \\ &\sim a(n_k)\, \mu(E_k \cap \{\varphi_{E_k} > n_k/2\}) \\ &\sim t\, \mu_{E_k}(\varphi_{E_k} > n_k/2) \\ &\longrightarrow t\,[1 - \widetilde{G}(t)] \quad \text{as } k \to \infty. \end{aligned} \tag{5.22}$$

In view of (5.17), (5.21) and (5.22) together give

$$\varlimsup_{k \to \infty} G_k^u(t) \leq t\,[1 - \widetilde{G}(t)].$$

(iv) To also prove $\underline{\lim}_{k\to\infty} G_k^u(t) \geq t\,[1-\widetilde{G}(t)]$, and hence (5.19), we need only observe that by arguments similar to the above,

$$\begin{aligned} G_k^u(t) = \sum_{l=1}^{n_k} \int_{E_k \cap \{\varphi_{E_k} > n_k - l\}} \widehat{T}^l u \, d\mu &\geq \int_{E_k \cap \{\varphi_{E_k} > n_k\}} \sum_{l=1}^{n_k} \widehat{T}^l u \, d\mu \\ &\sim a(n_k)\, \mu(E_k \cap \{\varphi_{E_k} > n_k\}) \\ &\sim t\, \mu_{E_k}(\varphi_{E_k} > n_k) \\ &\longrightarrow t\,[1-\widetilde{G}(t)] \quad \text{as } k \to \infty. \end{aligned} \tag{5.23}$$

(v) Conversely, if we start from the assumption that $G_k^u \Rightarrow G$, and want to show that $\widetilde{G}_k \Rightarrow \widetilde{G}$ with $\widetilde{G}$ satisfying $G(s) = s\,[1-\widetilde{G}\,(s)]$, $s>0$, we can use the above by arguing as in the last step of the proof for the $\alpha \in (0,1]$ case.

(vi) Finally, we check that $G(t') - G(t) \leq t' - t$ for $t' > t > 0$. Let $n_k' := n_k^{[t']}$ and use (5.17) twice to see that

$$G_k^u(t') \leq G_k^u(t) + \int_{E_k} \sum_{l=n_k+1}^{n_k'} \widehat{T}^l u \, d\mu.$$

But

$$\int_{E_k} \sum_{l=n_k+1}^{n_k'} \widehat{T}^l u \, d\mu \sim \mu(E_k)\,(a_T(n_k') - a_T(n_k)) = t' - t \quad \text{ as } k \to \infty,$$

and the desired estimate follows. This proves Lipschitz continuity of G. □

Now recall the following folklore principle (see e.g. Lemma 1 of [BZ]).

Lemma 5.2 (**Regular variation preserves distributional convergence**). *Assume that R_n and R are random variables taking values in $(0,\infty)$, and that $\rho_n^{-1} R_n \Longrightarrow R$ for constants $\rho_n \to \infty$. If B is regularly varying of index $\beta \neq 0$, then*

$$\frac{B(R_n)}{B(\rho_n)} \Longrightarrow R^\beta. \tag{5.24}$$

This easily leads to

Proof of Theorem 4.2. Fix $\alpha \in (0,1]$. Applying Lemma 5.2 proves that (4.11) is equivalent to (4.14) with $\widetilde{G}(t) = \widetilde{F}(t^{1/\alpha})$, while (4.12) is equivalent to (4.15) with $G(t) = F(t^{1/\alpha})$. This relation between $(\widetilde{G}, G)$ and $(\widetilde{F}, F)$ turns (4.16) into

$$F(t) = t^\alpha \int_0^1 [1 - \widetilde{F}\,(t\,(1-r))]\, \alpha r^{\alpha-1}\, dr$$

which, after an obvious change of variables, becomes (4.13). □

We can now establish the main positive result of Section 2.

Proof of Theorem 2.2. **(i)** We first show that γ_T is a tight scale for return times in Y. Assume otherwise, then there is some $\delta > 0$ and a sequence (E_k) in $Y \cap \mathcal{A}$, $\mu(E_k) > 0$ such that $\mu_{E_k}(\gamma_T(\mu(E_k))\cdot\varphi_{E_k} > k) > \delta$ for $k \geq 1$. In view of Proposition 2.1 b), this sequence must satisfy $\mu(E_k) \to 0$.

Now Helly's selection theorem guarantees $k_j \nearrow \infty$ such that

$$\mu_{E_{k_j}}(\gamma_T(\mu(E_{k_j}))\,\varphi_{E_{k_j}} \le t) \Longrightarrow \widetilde{F}(t),$$

for some $\widetilde{F} \in \mathcal{F}$. By our choice of (E_k), the latter satisfies $\sup_{t\in[0,\infty)} \widetilde{F}(t) < 1-\delta$. But this contradicts Remark 4.2 c).

(ii) To prove that γ_T is a nontrivial scale for return times in Y, we will construct a sequence (E_k) in $Y \cap \mathcal{A}$ with $0 < \mu(E_k) \to 0$ for which there are $t^*, k^* > 0$ s.t.

$$\mu_Y(\gamma_T(\mu(E_k))\,\varphi_{E_k} \le t^*) > 1/3 \quad \text{for } k \ge k^*. \tag{5.25}$$

Helly's selection theorem then provides us with $k_j \nearrow \infty$ and $F, \widetilde{F} \in \mathcal{F}$ such that $\mu_Y(\gamma_T(\mu(E_{k_j}))\,\varphi_{E_{k_j}} \le t) \Longrightarrow F(t)$ and $\mu_{E_{k_j}}(\gamma_T(\mu(E_{k_j}))\,\varphi_{E_{k_j}} \le t) \Longrightarrow \widetilde{F}(t)$ as $j \to \infty$. Due to (5.25), it is clear that $F(t^*) \ge 1/3$ (use the Portmanteau Theorem). In view of (4.13) from Theorem 4.2, this implies that there is some $s^* \in (0, t^*)$ for which $1 - \widetilde{F}(s^*) > 0$. But then

$$\underline{\lim}_{j\to\infty}\, \mu_{E_{k_j}}(\gamma_T(\mu(E_{k_j}))\,\varphi_{E_{k_j}} > s^*) > 0, \tag{5.26}$$

confirming that γ_T is a nontrivial scale for return times in Y.

(iii) To find (E_k) satisfying (5.25), we combine two results regarding the first-return map $T_Y : Y \to Y$. The first of these only uses the fact that T_Y is an ergodic m.p.t. on the probability space $(Y, Y \cap \mathcal{A}, \mu_Y)$, and that this space has no atoms. To check the latter property, note that a space with atoms supporting an ergodic m.p.t. must be purely atomic with a finite number of atoms. But if the measure is supported on finitely many atoms, the space cannot support a non-integrable almost surely finite function φ_Y.

Due to these basic properties, T_Y admits a sequence (E_k) in $Y \cap \mathcal{A}$ for which $\mu_Y(E_k) \to 0$ and

$$\mu_Y(\mu_Y(E_k)\,\varphi^Y_{E_k} \le t) \Longrightarrow 1 - e^{-t} \quad \text{as } k \to \infty, \tag{5.27}$$

see [KL]. In particular, fixing some $\theta > 0$ so large that $1 - e^{-\theta} > 2/3$, we see that the sets $V_k := Y \cap \{\varphi^Y_{E_k} \le \theta/\mu_Y(E_k)\}$, $k \ge 1$, satisfy $\mu_Y(V_k) > 2/3$ for $k \ge k'$.

(iv) The second ingredient is Aaronson's Darling-Kac limit theorem (Corollary 3.7.3 of [A0], see also [A1]), which applies under the present assumptions. This result is equivalent to a stable limit theorem for the return time function φ_Y under T_Y. The latter result asserts that

$$\mu_Y\left(\frac{1}{b_T(m)}\sum_{j=0}^{m-1}\varphi_Y \circ T_Y^j \le t\right) \Longrightarrow \Pr[\Gamma(1+\alpha)^{-1/\alpha}\,\mathcal{G}_\alpha \le t] \quad \text{as } k \to \infty. \tag{5.28}$$

Pick $\rho > 0$ so large that $\Pr[\Gamma(1+\alpha)^{-1/\alpha}\,\mathcal{G}_\alpha \le \rho] > 2/3$. Then there is some m' such that the sets $W_m := Y \cap \{b_T(m)^{-1}\sum_{j=0}^{m-1}\varphi_Y \circ T_Y^j \le \rho\}$ satisfy $\mu_Y(W_m) > 2/3$ for $m \ge m'$.

(v) Since $b_T \in \mathcal{R}_{1/\alpha}$ and $\mu(E_k) \to 0$, we have $b_T(\lfloor \theta/\mu_Y(E_k) \rfloor)/b_T(1/\mu_Y(E_k)) < 2\theta^{1/\alpha}$ for $k \geq k''$. Recalling (3.2) we therefore see that

$$\begin{aligned}\gamma_T(\mu(E_k))\,\varphi_{E_k} &\leq \gamma_T(\mu(E_k)) \sum_{j=0}^{\lfloor \theta/\mu_Y(E_k) \rfloor - 1} \varphi_Y \circ T_Y^j \quad \text{on } V_k \\ &= \frac{b_T(\lfloor \theta/\mu_Y(E_k) \rfloor)}{b_T(1/\mu_Y(E_k))} \frac{1}{b_T(\lfloor \theta/\mu_Y(E_k) \rfloor)} \sum_{j=0}^{\lfloor \theta/\mu_Y(E_k) \rfloor - 1} \varphi_Y \circ T_Y^j \\ &< 2\theta^{1/\alpha}\rho =: t^* \qquad \text{on } W_{\lfloor \theta/\mu_Y(E_k) \rfloor} \text{ for } k \geq k''.\end{aligned}$$

Hence,

$$\mu_Y(\gamma_T(\mu(E_k))\,\varphi_{E_k} \leq t^*) \geq \mu_Y(V_k \cap W_{\lfloor \theta/\mu_Y(E_k) \rfloor}) \quad \text{for } k \geq k''.$$

But there is some k''' such that $\lfloor \theta/\mu_Y(E_k) \rfloor \geq m'$ for $k \geq k'''$, and then

$$\mu_Y(V_k \cap W_{\lfloor \theta/\mu_Y(E_k) \rfloor}) > 1/3 \quad \text{for } k \geq k^* := k' \vee k'' \vee k''',$$

which shows that (E_k) satisfies (5.25). □

Finally, we turn to the

Proof of Theorem 2.3. **(i)** As a consequence of Proposition 2.1 b) we see that whenever a is a scaling function, then for every $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$ and any $\eta > 0$, $\{\mathsf{law}_{\mu_E}[a(\varphi_E)] : E \in Y \cap \mathcal{A},\, \mu(E) \geq \eta\}$ is tight.

(ii) Now take Y and $a = a_T$ as in the statement of the proposition. Assume for a contradiction that (2.10) fails. Then there is some $\delta > 0$ and a sequence (E_k) in $Y \cap \mathcal{A}$ such that $\mu_{E_k}[\mu(E_k)\,a_T(\varphi_{E_k}) > k] > \delta$ for $k \geq 1$. By step (i) we necessarily have $\mu(E_k) \to 0$. The Helly selection theorem provides us with $k_j \nearrow \infty$ and $\widetilde{G} \in \mathcal{F}$ for which

$$\mu_{E_{k_j}}(\mu(E_{k_j})\,a_T(\varphi_{E_{k_j}}) \leq t) \Longrightarrow \widetilde{G}(t) \quad \text{as } j \to \infty. \tag{5.29}$$

According to Theorem 4.3 and Remark 4.4 b), however, $\widetilde{G}$ has to be a probability distribution function on $[0, \infty)$. This contradicts our choice of (E_k).

(iii) To show that this nonlinear scaling is nontrivial, consider $E_k^* := Y \cap \{\varphi_Y > k\}$, $k \geq 1$. Then, $\mu(E_k^*) = \mu(Y)q_k(Y)$, and $\varphi_{E_k^*} \geq \varphi_Y > k$ on E_k^*. Consequently,

$$\mu(E_k^*)\,a_T(\varphi_{E_k^*}) > \mu(Y)q_k(Y)a_T(k) \quad \text{on } E_k^*. \tag{5.30}$$

But in the $\alpha = 0$ case, the asymptotic renewal equation (1.6) and the (monotone density part of) KTT yield

$$a_T(k) \sim \frac{k}{w_k(Y)} \sim \frac{1}{\mu(Y)q_k(Y)} \quad \text{as } k \to \infty. \tag{5.31}$$

Together with (5.30) this shows that any limit point $\widetilde{G}$ as in (5.29) of the (nonlinearly rescaled) return-time distributons, vanishes on $[0, 1)$. In view of Theorem 4.3, this means that $\mu_Y(\mu(E_{k_j}^*)\,a_T(\varphi_{E_{k_j}^*}) \leq t) \Longrightarrow G(t)$ with $G(t) = t$ for $t \in [0, 1]$, and hence (since $G \in \mathcal{F}$) that $G(t) = t \wedge 1$ for $t \geq 0$. Appealing to (4.16) once again, we obtain the explicit form (2.11) of $\widetilde{G}$.

This uniqueness of the limit point $\widetilde{G}$, together with Helly's theorem implies convergence along the full sequence. $\square$

6. Proving convergence to $\mathcal{H}_\alpha$ in uniform sets

Characterizing convergence to $\mathcal{H}_\alpha$. In the case of finite measure preserving systems, Theorem HLV is not only of interest in its own right, but it is also the basis of a method for proving convergence to the exponential distribution (see [HSV]). Indeed, it is clear that the only $F \in \mathcal{F}$ which satisfies

$$F(t) = \int_0^t [1 - F(s)]\, ds \quad \text{for } t \geq 0,$$

is $F = H_1$, where $H_1(t) := 1 - e^{-t}$, $t > 0$. The most prominent limit law is thus characterized as the unique distribution which can appear both as return- and as hitting-time limit for the same sequence of sets. In view of this and the Helly selection principle, one can prove convergence to $\mathcal{E}$ of both return- and hitting-time distributions by showing that the two types of distributions are asymptotically the same.

Here, we obtain (with hardly any effort) a result which allows a parallel approach to proving convergence to $\mathcal{H}_\alpha$ inside uniform sets in case $a_T \in \mathcal{R}_\alpha$, $\alpha \in (0, 1]$. The following is contained in Lemma 7 of [PSZ1][1].

Lemma 6.1 (**Characterization of** H_α)**.** *For every $\alpha \in (0, 1]$ the distribution function $F = H_\alpha$ of $\mathcal{H}_\alpha$ is the unique element of $\mathcal{F}$ which satisfies*

$$F(t) = \int_0^t [1 - F(s)]\, \alpha\, (t - s)^{\alpha - 1}\, ds \quad \textit{for } t \geq 0. \tag{6.1}$$

As a consequence, we obtain

Theorem 6.1 (**Characterizing convergence to $\mathcal{H}_\alpha$ in uniform sets**)**.** *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$. Assume that $Y \in \mathcal{A}$ is a uniform set and that $a_T \in \mathcal{R}_\alpha$ for some $\alpha \in (0, 1]$.*

Suppose that $E_k \subseteq Y$, $k \geq 1$, are sets of positive measure with $\mu(E_k) \to 0$. Then the normalized return-time distributions of the E_k converge to $\mathcal{H}_\alpha$,

$$\mu_{E_k}(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t) \Longrightarrow H_\alpha(t) \quad \textit{as } k \to \infty, \tag{6.2}$$

if and only if the normalized hitting-time distributions converge to $\mathcal{H}_\alpha$,

$$\mu_Y(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t) \Longrightarrow H_\alpha(t) \quad \textit{as } k \to \infty, \tag{6.3}$$

if and only if for a dense set of points t in $(0, \infty)$,

$$\mu_{E_k}(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t) - \mu_Y(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t) \longrightarrow 0 \quad \textit{as } k \to \infty. \tag{6.4}$$

Proof. By Theorem 4.2 it is clear that (6.2) is equivalent to (6.3). Trivially, either of these statements therefore implies (6.4).

To prove the converse, start from (6.4), and assume for a contradiction that, say, (6.2) fails, so that by Helly's selection principle there is a subsequence $k_j \nearrow \infty$ of

[1]Note that the parameter α appearing in Lemma 7 of [PSZ1] is not the same as our α. In the notation of the present paper it equals $1/(1 - \alpha)$.

indices and some $\widetilde{F} \in \mathcal{F}$, $\widetilde{F} \neq H_\alpha$, such that (4.11) holds along that subsequence. By Theorem 4.2, so does (4.12), where F and $\widetilde{F}$ are related by (4.13). But (6.4) ensures that $\widetilde{F} = F$, which in view of Lemma 6.1 contradicts $\widetilde{F} \neq H_\alpha$.

Exactly the same argument works if we assume that (6.3) fails. □

Sufficient conditions for convergence to $\mathcal{H}_\alpha$. It is natural to review the abstract distributional limit theorem of [PSZ2], which gives sufficient conditions for convergence to $\mathcal{H}_\alpha$, in the light of the preceding result. We establish an improved result which is similar in spirit, but uses easier assumptions.

The key idea goes back to [Z4], [Z5]: Ergodicity ensures that the asymptotics of distributions of variables R_k which are asymptotically invariant in measure does not depend on the choice of the initial density, see Lemma 4.1. In fact, there is uniform control over important quantities as long as all densities involved belong to a sufficiently small family. We shall use the following (see statement (3.1) of [Z5]).

Remark 6.1 (**Dual ergodic sums for compact sets of densities**). Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, and $\mathcal{U} \subseteq L_1(\mu)$ a (strongly) compact set of probablity densities. Then

$$\left\| \frac{1}{M} \sum_{j=0}^{M-1} \widehat{T}^j u - \frac{1}{M} \sum_{j=0}^{M-1} \widehat{T}^j u^* \right\|_1 \longrightarrow 0 \qquad \begin{array}{c} \text{as } M \to \infty, \\ \text{uniformly in } u, u^* \in \mathcal{U}. \end{array} \tag{6.5}$$

This will entail a uniform (in the initial density) version of Lemma 4.1. We can use this to replace the fixed measure μ_Y in the criterion (6.4) of the previous theorem by any sequence of probabilities Q_k as long as their respective densities v_k stay in some definite compact set $\mathcal{U}$. Suppose this holds for the particular case where each Q_k is just a push-forward of the conditional measure μ_{E_k} by a comparatively small (condition (6.8) below) number τ of steps. Then, by virtue of (6.5) those will exhibit the same asymptotics as the μ_{E_k} provided that the exceptional set $\{\varphi_{E_k} < \tau\}$ is not too large (condition (6.9)).

This is the core of the argument to follow. To get a flexible result, we allow for delays τ which depend on the point. Given a measurable $\tau : E \to \mathbb{N}_0$ we can define $T^\tau : E \to X$ by $T^\tau x := T^{\tau(x)} x$, which gives a null-preserving map. For $u \in L_1(\mu)$ supported on E, let

$$\widehat{T^\tau} u := \sum_{j \geq 0} \widehat{T^j} (1_{E \cap \{\tau = j\}} u), \tag{6.6}$$

then $\widehat{T^\tau}$ is the transfer operator of T^τ, describing the push-forward of measures by T^τ on the level of densities,

$$\int_X f \cdot \widehat{T^\tau} u \, d\mu = \int_E (f \circ T^\tau) \cdot u \, d\mu \quad \text{ for } u \in L_1(\mu) \text{ and } f \in L_\infty(\mu). \tag{6.7}$$

Using such an auxiliary map T^τ, we can formulate the advertised result.

Theorem 6.2 (**Sufficient conditions for convergence to $\mathcal{H}_\alpha$**). *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$, pointwise dual ergodic with $a_T \in \mathcal{R}_\alpha$ for some $\alpha \in (0, 1]$. Suppose that Y is a uniform set, and that $E_k \subseteq Y$, $k \geq 1$, are sets of positive measure with $\mu(E_k) \to 0$.*

Assume that there are measurable functions $\tau_k : E_k \to \mathbb{N}_0$ and an $L_1(\mu)$-compact set $\mathcal{U} \subseteq \mathcal{D}(\mu)$ of probability densities for which

$$\gamma_T(\mu(E_k)) \, \tau_k \overset{\mu_{E_k}}{\longrightarrow} 0 \quad \text{as } k \to \infty, \tag{6.8}$$

(in that $\mu_{E_k}[\gamma_T(\mu(E_k))\,\tau_k > \varepsilon] \to 0$ *for every* $\varepsilon > 0$*), and*

$$\mu_{E_k}(\varphi_{E_k} < \tau_k) \longrightarrow 0 \quad \text{as } k \to \infty, \tag{6.9}$$

while

$$\widehat{T^{\tau_k}}(\mu(E_k)^{-1} 1_{E_k}) \in \mathcal{U} \quad \text{for } k \geq 1. \tag{6.10}$$

Then the return-time distributions of the E_k *converge to* $\mathcal{H}_\alpha$*,*

$$\mu_{E_k}(\gamma_T(\mu(E_k))\,\varphi_{E_k} \leq t) \longrightarrow H_\alpha(t) \quad \text{as } k \to \infty, \tag{6.11}$$

and so do the hitting-time distributions,

$$\mu_Y(\gamma_T(\mu(E_k))\,\varphi_{E_k} \leq t) \longrightarrow H_\alpha(t) \quad \text{as } k \to \infty. \tag{6.12}$$

Remark 6.2. This result improves Theorem 4.1 of [PSZ2] in that the conditions on the delay times τ_k are simpler and more general. Moreover, to control the densities in (6.10), we no longer require $\mathcal{U}$ to be such that Y is $\mathcal{U}$-uniform (a dynamical condition), but merely assume that $\mathcal{U}$ is compact in $L_1(\mu)$ (a property of $\mathcal{U}$ which does not involve the dynamics). This immediately shortens the proof of Theorem 3.1 of [PSZ2] by the pages devoted to checking the original dynamical condition.

The proof of the theorem above also seems more transparent than that of Theorem 4.1 of [PSZ2], its basic strategy being parallel to a standard argument used in the finite-measure case.

Proof. **(i)** Equivalence of (6.11) and (6.12) is clear from Theorem 6.1, and we prove them by validating condition (6.4) of that result. Let $R_k := \gamma_T(\mu(E_k))\,\varphi_{E_k}$ and $F_k(t) := \mu_Y(R_k \leq t)$, $\widetilde{F}_k(t) := \mu_{E_k}(R_k \leq t)$, $t \geq 0$. We prove (6.4) by showing that

$$F_k - \widehat{F}_k \to 0 \quad \text{and} \quad \widehat{F}_k - \widetilde{F}_k \to 0 \quad \lambda\text{-a.e. on } [0,\infty) \text{ as } k \to \infty, \tag{6.13}$$

where $\widehat{F}_k$ denotes the distribution function of R_k with respect to the measure with density $v_k := \widehat{T^{\tau_k}}(\mu(E_k)^{-1} 1_{E_k})$, so that $\widehat{F}_k(t) := \int_{\{R_k \leq t\}} v_k \, d\mu$.

(ii) Recall that by definition (6.6) of $\widehat{T^{\tau}}$ we have $\widehat{F}_k(t) = \mu_{E_k}(R_k \circ T^{\tau_k} \leq t)$. To first check that for λ-a.e. $t \in [0,\infty)$,

$$\mu_{E_k}(R_k \leq t) - \mu_{E_k}(R_k \circ T^{\tau_k} \leq t) \longrightarrow 0 \quad \text{as } k \to \infty. \tag{6.14}$$

we need only observe that (generalizing (4.4)), for any measurable E and $\tau \geq 0$,

$$\varphi_E = \varphi_E \circ T^\tau + \tau \quad \text{on } \{\varphi_E > \tau\}, \tag{6.15}$$

and hence

$$R_k = R_k \circ T^{\tau_k} + \gamma_T(\mu(E_k))\,\tau_k \quad \text{on } \{\varphi_{E_k} > \tau_k\}. \tag{6.16}$$

Given this, (6.14) is an easy consequence of (6.8) and (6.9). This shows that $\widehat{F}_k - \widetilde{F}_k \to 0$ holds λ-a.e. on $[0,\infty)$.

(iii) To also prove that $F_k - \widehat{F}_k \to 0$ a.e. on $[0,\infty)$, note first that (by the general definition of distributional convergence) this statement is equivalent to saying that for every bounded Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ one has

$$\int (\psi \circ R_k)\, 1_Y \, d\mu - \int (\psi \circ R_k)\, v_k \, d\mu \longrightarrow 0 \quad \text{as } k \to \infty, \tag{6.17}$$

which, by (6.10), will follow once we show that for any such ψ,

$$\int (\psi \circ R_k)\, u \, d\mu - \int (\psi \circ R_k)\, u^* \, d\mu \longrightarrow 0 \qquad \begin{array}{c}\text{as } k \to \infty,\\ \text{uniformly in } u, u^* \in \mathcal{U}\end{array} \tag{6.18}$$

(assume w.l.o.g. that $1_Y \in \mathcal{U}$). To this end, we now fix ψ and any $\varepsilon > 0$.

(iv) In view of Remark 6.1 there is some $M \geq 1$ such that

$$\begin{aligned} \left| \int (\psi \circ R_k)\, \frac{1}{M} \sum_{j=0}^{M-1} \widehat{T}^j (u - u^*)\, d\mu \right| &\leq \sup|\psi| \, \left\| \frac{1}{M} \sum_{j=0}^{M-1} \widehat{T}^j (u - u^*) \right\|_1 \\ &< \frac{\varepsilon}{3} \quad \text{for all } u, u^* \in \mathcal{U}. \end{aligned} \tag{6.19}$$

On the other hand, for any $j \in \{0, \ldots, M-1\}$ and $u \in \mathcal{U}$, we find recalling (6.15),

$$\begin{aligned} \left| \int (\psi \circ R_k)\, (u - \widehat{T}^j u)\, d\mu \right| &\leq \int \left| \psi \circ R_k - \psi \circ R_k \circ T^j \right| \, u \, d\mu \\ &\leq 2 \sup|\psi| \int_{\{\varphi_{E_k} \leq j\}} u \, d\mu + \mathrm{Lip}(\psi)\, \gamma_T(\mu(E_k))\, j, \end{aligned}$$

so that

$$\begin{aligned} \left| \int (\psi \circ R_k) \left(u - \frac{1}{M} \sum_{j=0}^{M-1} \widehat{T}^j u \right) d\mu \right| &\leq 2 \sup|\psi| \int_{\{\varphi_{E_k} \leq M\}} u \, d\mu \\ &\quad + \mathrm{Lip}(\psi)\, \gamma_T(\mu(E_k))\, M. \end{aligned}$$

Since $\mathcal{U}$ is, in particular, uniformly integrable, there is some $\delta > 0$ such that

$$2 \sup|\psi| \int_A u \, d\mu < \frac{\varepsilon}{6} \quad \text{for all } u \in \mathcal{U} \text{ and } A \in \mathcal{A} \text{ with } \mu(A) < \delta. \tag{6.20}$$

But as $\mu(E_k) \to 0$, this shows there is some k_0 such that

$$\left| \int (\psi \circ R_k) \left(u - \frac{1}{M} \sum_{j=0}^{M-1} \widehat{T}^j u \right) d\mu \right| < \frac{\varepsilon}{3} \quad \text{for } k \geq k_0 \text{ and } u \in \mathcal{U}. \tag{6.21}$$

But combining (6.19) with two applications of (6.21) yields

$$\left| \int (\psi \circ R_k)\, u \, d\mu - \int (\psi \circ R_k)\, u^* \, d\mu \right| < \varepsilon \quad \text{for } k \geq k_0 \text{ and } u, u^* \in \mathcal{U},$$

which proves our earlier claim (6.18). □

7. The limit variables $\widetilde{\mathcal{H}}_{\alpha,\theta}$ and $\mathcal{H}_{\alpha,\theta}$

Further natural limit laws. We continue our discussion of rare events in the setup of the preceding positive results: Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$, $Y \in \mathcal{A}$ a uniform set and $a_T \in \mathcal{R}_\alpha$ for some $\alpha \in (0, 1]$. Consider $\gamma_T(\mu(E_k))\, \varphi_{E_k}$ for sequences (E_k) of asymptotically rare events in $Y \cap \mathcal{A}$.

Among all possible limits for return- and hitting time distributions of sequences in Y, the variables $\mathcal{H}_\alpha$, $\alpha \in (0, 1]$, stand out as the only limits which can occur simultaneously as asymptotic hitting distribution and as asymptotic return distribution. This property leads to the strategy for proving convergence to these particular laws developed above, and via Theorem 6.2 or its predecessor in [PSZ2] one finds that the $\mathcal{H}_\alpha$ occur at almost every point of prototypical examples.

In the present section we discuss a larger family $(\widetilde{H}_{\alpha,\theta}, H_{\alpha,\theta})$ with $\alpha, \theta \in (0,1]$ of pairs $(\widetilde{F}, F)$ in $\mathcal{F}$ related as in (4.13), which still appear in a very natural way. Let $\mathcal{E}$, $\mathcal{G}_\alpha$ and Θ_θ be independent random variables, with $\mathcal{E}$ and $\mathcal{G}_\alpha$ as before, and $\Pr[\Theta_\theta = 1] = 1 - \Pr[\Theta_\theta = 0] = \theta$. Define random variables

$$\mathcal{H}_{\alpha,\theta} := \theta^{1/\alpha}\mathcal{H}_\alpha \quad \text{and} \quad \widetilde{\mathcal{H}}_{\alpha,\theta} := \Theta_\theta \cdot \theta^{1/\alpha}\mathcal{H}_\alpha, \tag{7.1}$$

which have distribution functions $H_{\alpha,\theta}$ and $\widetilde{H}_{\alpha,\theta}$ respectively, given by

$$H_{\alpha,\theta}(t) := H_\alpha(\theta^{1/\alpha}t) \quad \text{and} \quad \widetilde{H}_{\alpha,\theta}(t) := (1-\theta) + \theta\, H_\alpha(\theta^{1/\alpha}t), \quad t \geq 0. \tag{7.2}$$

Lemma 7.1 (**Characterization of** $\widetilde{H}_{\alpha,\theta}$ **and** $H_{\alpha,\theta}$)**.** *For $\alpha, \theta \in (0,1]$ the pair $(\widetilde{F}, F) := (\widetilde{H}_{\alpha,\theta}, H_{\alpha,\theta})$ satisfies (4.13). The distribution function $F = \widetilde{H}_{\alpha,\theta}$ is the unique element of $\mathcal{F}$ which satisfies*

$$\widetilde{F}(t) = (1-\theta) + \theta \int_0^t [1 - \widetilde{F}(s)]\, \alpha\, (t-s)^{\alpha-1}\, ds \quad \text{for } t \geq 0. \tag{7.3}$$

Proof. Straightforward from the corresponding properties of H_α, Lemma 6.1. □

Convergence to $(\widetilde{\mathcal{H}}_{\alpha,\theta}, \mathcal{H}_{\alpha,\theta})$ **in uniform sets.** This characterization of $\widetilde{H}_{\alpha,\theta}$ leads to an easy characterization (generalizing Theorem 6.1) of those situations in which the pair $(\widetilde{H}_{\alpha,\theta}, H_{\alpha,\theta})$ occurs in the limit.

Theorem 7.1 (**Characterizing convergence to** $(\widetilde{\mathcal{H}}_{\alpha,\theta}, \mathcal{H}_{\alpha,\theta})$ **in uniform sets**)**.** *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$. Assume that $Y \in \mathcal{A}$ is a uniform set and that $a_T \in \mathcal{R}_\alpha$ for some $\alpha \in (0,1]$. Suppose that $\theta \in (0,1]$, and that $E_k \subseteq Y$, $k \geq 1$, are sets of positive measure with $\mu(E_k) \to 0$.*

Then the normalized return-time distributions of the E_k converge to $\widetilde{\mathcal{H}}_{\alpha,\theta}$,

$$\mu_{E_k}(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t) \Longrightarrow \widetilde{H}_{\alpha,\theta}(t) \quad \text{as } k \to \infty, \tag{7.4}$$

if and only if the normalized hitting-time distributions converge to $\mathcal{H}_{\alpha,\theta}$,

$$\mu_Y(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t) \Longrightarrow H_{\alpha,\theta}(t) \quad \text{as } k \to \infty, \tag{7.5}$$

if and only if for a dense set of points t in $(0, \infty)$,

$$\mu_{E_k}(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t) - \theta\mu_Y(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t) \longrightarrow 1 - \theta \quad \text{as } k \to \infty. \tag{7.6}$$

Proof. By Theorem 4.2 and the first assertion of Lemma 7.1, (7.4) is equivalent to (7.5). Via (4.13) and (7.3) either of these statements therefore implies (7.6).

For the converse, start from (7.6), and assume for a contradiction that, say, (7.4) fails, so that by Helly's selection principle there is a subsequence $k_j \nearrow \infty$ of indices and some $\widetilde{F} \in \mathcal{F}$, $\widetilde{F} \neq \widetilde{H}_{\alpha,\theta}$, such that (4.11) holds along that subsequence. By Theorem 4.2, so does (4.12), where F and $\widetilde{F}$ are related by (4.13). But then (7.6) ensures that $\widetilde{F} = \widetilde{H}_{\alpha,\theta}$, see Lemma 7.1.

Exactly the same argument works if we assume that (7.5) fails. □

Obviously, $(\widetilde{H}_{\alpha,1}, H_{\alpha,1}) = (\widetilde{H}_\alpha, H_\alpha)$. The pair $(\widetilde{H}_{\alpha,\theta}, H_{\alpha,\theta})$ with $\theta \in (0,1)$ occurs in situations where a proportion $1-\theta$ of E_k returns very quickly (at a rate smaller than $\gamma_T(\mu(E_k))$) to this very set, while the remaining part of relative measure θ does not, and instead becomes spread out macroscopically, as in the hitting-time statistics. This is made precise in the following result which extends Theorem 6.2

to situations with $\theta \neq 1$. In concrete maps, this is what happens at (hyperbolic) periodic points, see below.

Theorem 7.2 (Sufficient conditions for convergence to $(\widetilde{\mathcal{H}}_{\alpha,\theta}, \mathcal{H}_{\alpha,\theta})$). *Let T be a c.e.m.p.t. on $(X, \mathcal{A}, \mu)$, $\mu(X) = \infty$, pointwise dual ergodic with $a_T \in \mathcal{R}_\alpha$ for some $\alpha \in (0, 1]$. Suppose that Y is a uniform set, and that $E_k \subseteq Y$, $k \geq 1$, are sets of positive measure with $\mu(E_k) \to 0$.*

Suppose that there are measurable functions $\tau_k : E_k \to \mathbb{N}_0$ for which

$$\gamma_T(\mu(E_k))\, \tau_k \xrightarrow{\mu_{E_k}} 0 \quad \text{as } k \to \infty. \tag{7.7}$$

Assume further that there is some $\theta \in (0, 1)$ such that for each $k \geq 1$,

$$E_k = E_k^\bullet \cup E_k^\circ \text{ (disjoint)} \quad \text{with} \quad \mu_{E_k}(E_k^\circ) \longrightarrow \theta \text{ as } k \to \infty, \tag{7.8}$$

and

$$\mu_{E_k^\bullet}(\varphi_{E_k} > \tau_k) \longrightarrow 0 \quad \text{as } k \to \infty, \tag{7.9}$$

while

$$\mu_{E_k^\circ}(\varphi_{E_k} < \tau_k) \longrightarrow 0 \quad \text{as } k \to \infty, \tag{7.10}$$

and there is some $L_1(\mu)$-compact set $\mathcal{U} \subseteq \mathcal{D}(\mu)$ for which

$$\widehat{T^{\tau_k}}(\mu(E_k^\circ)^{-1} 1_{E_k^\circ}) \in \mathcal{U} \quad \text{for } k \geq 1. \tag{7.11}$$

Then the return-time distributions of the E_k converge to $\widetilde{\mathcal{H}}_{\alpha,\theta}$,

$$\mu_{E_k}(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t) \longrightarrow \widetilde{H}_{\alpha,\theta}(t) \quad \text{as } k \to \infty, \tag{7.12}$$

and the hitting-time distributions converge to $\mathcal{H}_{\alpha,\theta}$,

$$\mu_Y(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t) \longrightarrow H_{\alpha,\theta}(t) \quad \text{as } k \to \infty. \tag{7.13}$$

Proof. We use Theorem 7.1. Represent $\widetilde{F}_k(t) := \mu_{E_k}(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t)$ as

$$\widetilde{F}_k(t) = \mu_{E_k}(E_k^\bullet)\, \widetilde{F}_k^\bullet(t) + \mu_{E_k}(E_k^\circ)\, \widetilde{F}_k^\circ(t)$$

with $\widetilde{F}_k^\bullet(t) := \mu_{E_k^\bullet}(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t)$ and $\widetilde{F}_k^\circ(t) := \mu_{E_k^\circ}(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t)$. By assumptions (7.8) and (7.9),

$$\mu_{E_k}(E_k^\bullet) \longrightarrow 1 - \theta \quad \text{and} \quad \widetilde{F}_k^\bullet(t) \longrightarrow 1 \quad \text{as } k \to \infty.$$

To validate (7.6) it therefore suffices to show that for a dense set of points t in $(0, \infty)$, the normalized hitting-time laws $F_k(t) := \mu_Y(\gamma_T(\mu(E_k))\, \varphi_{E_k} \leq t)$ satisfy

$$\widetilde{F}_k^\circ(t) - F_k(t) \longrightarrow 0 \quad \text{as } k \to \infty. \tag{7.14}$$

But the proof of (7.14) works exactly like that of Theorem 6.2: Simply replace $\widetilde{F}_k$ by $\widetilde{F}_k^\circ$ and v_k by $v_k^\circ := \widehat{T^{\tau_k}}(\mu(E_k^\circ)^{-1} 1_{E_k^\circ})$ etc. $\square$

Returning to a hyperbolic fixed point of a null-recurrent map. We illustrate our abstract Theorem 7.2 in the setup of simple prototypical interval maps with an indifferent fixed point and full branches. Example 7.1 below extends to hyperbolic fixed points (or periodic points) of more general maps (like those of [Z1], [Z2]) by routine arguments.

Consider maps T on $X = [0, 1]$ which satisfy, for some $c \in (0, 1)$, the following:

(a) The restrictions of T to $(0,c)$ and $(c,1)$ map onto $(0,1)$, and possess $\mathcal{C}^2$-extensions to $[0,c]$ and $[c,1]$ respectively.
(b) $T'(0) = 1$ while $T' > 1$ on $(0,c] \cup [c,1]$.
(c) T' is increasing in some neighbourhood of 0.

According to [T1], any such T has an infinite invariant measure μ with a density h relative to Lebesgue measure λ which is strictly positive and continuous on $(0,1]$, and T is conservative ergodic for μ. Due to [A2] (or [T2]), $Y := (c,1)$ is a uniform set. If, in addition, there are $\alpha \in (0,1]$ and $a > 0$ such that

(d) $Tx = x + ax^{1+1/\alpha} + o(ax^{1+1/\alpha})$ as $x \searrow 0$,

then $a_T \in \mathcal{R}_\alpha$ where (for a constant κ_α, explicit in terms of T)

$$a_T(n) \sim \kappa_\alpha \cdot \begin{cases} n/\log n & \text{if } \alpha = 1, \\ n^\alpha & \text{if } \alpha < 1, \end{cases} \qquad \text{as } n \to \infty. \tag{7.15}$$

For such maps there is an easy concrete scenario in which $\widetilde{\mathcal{H}}_{\alpha,\theta}$ and $\mathcal{H}_{\alpha,\theta}$ occur. The following is parallel to well-known results about return- and hitting-time statistics at hyperbolic periodic points in a finite measure setup (see e.g. [K], [F]).

Example 7.1 (**Hyperbolic fixed point of a null-recurrent interval map**). *Let T be a map satisfying (a)-(d). Consider $E_k := \bigcap_{j=0}^{k-1} T^{-j}Y$, $k \geq 1$, the cylinders containing the hyperbolic fixed point $x^* = 1$, and set $\theta := 1 - 1/T'x^*$. Then*

$$\mu_{E_k}(\gamma(\mu(E_k))\,\varphi_{E_k} \leq t) \Longrightarrow \widetilde{H}_{\alpha,\theta}(t) \quad \text{as } k \to \infty, \tag{7.16}$$

while

$$\mu_Y(\gamma(\mu(E_k))\,\varphi_{E_k} \leq t) \Longrightarrow H_{\alpha,\theta}(t) \quad \text{as } k \to \infty, \tag{7.17}$$

where

$$\gamma(s) \sim \kappa_\alpha^\alpha \cdot \begin{cases} s/(-\log s) & \text{if } \alpha = 1, \\ s^{1/\alpha} & \text{if } \alpha < 1, \end{cases} \qquad \text{as } s \searrow 0. \tag{7.18}$$

Proof. We set $F := (0,c)$. Take $E_k^\bullet := E_{k+1}$, $E_k^\circ := E_k \setminus E_{k+1}$ and define $\tau_k := k = \varphi_F$ on E_k° while $\tau_k := 1$ on $E_k^\bullet$. Then it is clear that $\varphi_{E_k} = 1 = \tau_k$ on $E_k^\bullet$ and $\varphi_{E_k} \geq k = \tau_k$ on E_k°, so that (7.9) and (7.10) are satisfied. Since $E_k = TE_k^\bullet$ we obtain $\lambda_{E_k}(E_k^\bullet) \longrightarrow 1 - \theta$ and since $h = d\mu/d\lambda$ is continuous at x^*, (7.8) follows. To check (7.7), note that $\tau_k \leq k$ while $\mu(E_k)$ is exponentially small, $\mu(E_k) \leq \kappa q^k$ for $k \geq 1$ with suitable constants $\kappa \in (0,\infty)$ and $q \in (1/T'x^*, 1)$.

It remains to check (7.11). But $v_k^\circ := \widehat{T^{\tau_k}}(\mu(E_k^\circ)^{-1} 1_{E_k^\circ}) = \mu(E_k^\circ)^{-1}(f^k)'$ on F, where $f := (T\mid_Y)^{-1}$ is the inverse of the uniformly expanding branch of T. By the standard bounded distortion estimate for uniformly expanding $\mathcal{C}^2$-maps (e.g. §4.3 of [A0]), we see that $v_k^\circ \in \mathcal{U}$ for all $k \geq 1$, where $\mathcal{U}$ contains all probability densities on F which respect a Lipschitz constant which can be computed explicitly from T.

Finally, (7.15) easily translates into the asymptotic expression (7.18) for γ_T. □

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