

COMPLICATED DYNAMICS IN PLANAR POLYNOMIAL EQUATIONS

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ABSTRACT. We deal with a mechanism of generating distributional chaos in planar nonautonomous ODEs and try to measure chaosity in terms of topological entropy. It is based on the interplay between simple periodic solutions. We prove the existence of infinitely many heteroclinic solutions between the periodic ones.

1. INTRODUCTION

The goal of the paper is to investigate the mechanism of generating chaos in the equation

$$(1) \quad \dot{z} = P_n(t, \bar{z}) + f(t, z),$$

where P_n is a polynomial in \bar{z} variable of degree n and it is continuous and periodic in t variable. f is treated as a sufficiently small perturbation.

We deal with $n = 2$ and investigate the model equation

$$(2) \quad \dot{z} = v(t, z) = Re^{it}(\bar{z}^2 - 1) + f(t, z),$$

for degree 2.

Investigated equations are different from the ones described in [9, 6]. We deal with distributional chaos which is not equivalent to the notion of chaos from [9, 6] but in some cases (see [4]) may be implied by it. We also try to set a lower boundary for the topological entropy for the equations.

The main result of the paper in the case of big leading coefficient is the following theorem.

Theorem 1. *Let $f \in \mathcal{C}(\mathbb{R} \times \mathbb{C}, \mathbb{C})$ be 2π -periodic in the first variable i.e. $f(t, z) = f(t + 2\pi, z)$ for every $(t, z) \in \mathbb{R} \times \mathbb{C}$. Moreover, let*

$$(3) \quad R \geq 1$$

and

$$(4) \quad N \leq 0.001R$$

where

$$(5) \quad |f(t, z)| \leq N,$$

$$(6) \quad |f(t, z) - f(t, w)| \leq N|z - w|$$

be satisfied for every $t \in \mathbb{R}$ and $z, w \in Q = \{p \in \mathbb{C} : |p| < 3\}$.

Then

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- (1) the equation (2) is distributionally chaotic,
- (2) both trivial solutions -1 and 1 for the case $f \equiv 0$ continue to 2π -periodic ones and there exist infinitely many heteroclinic solutions between them,
- (3) there exists $\beta > 0$ and $\Lambda \subset Q$ invariant with respect to the Poincaré operator $\varphi_{(-\beta, 2\pi)}$ such that the dynamical system $(\Lambda, \varphi_{(-\beta, 2\pi)}|_{\Lambda})$ is semiconjugated to the dynamical system (Σ, σ^2) , where Σ is the sofic shift given by the Figure 1 and $\sigma : \Sigma \rightarrow \Sigma$ is the shift operator.

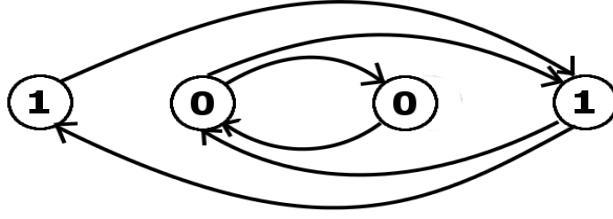


FIGURE 1. Vertex graph of the sofic shift Σ for the equation (2) with big leading coefficient.

Since the topological entropy of the system (Σ, σ^2) is given by $\ln \frac{3+\sqrt{5}}{2} \approx \ln(2.615)$ Theorem 1 gives greater entropy than $\ln(2)$ which could be obtained by semiconjugacy to the (Σ_2, σ) . Although we now get higher lower estimation of topological entropy in equation (2) we do not know whether new semiconjugacy is now conjugacy.

The main result of the paper in the case of small leading coefficient is the following theorem.

Theorem 2. *There exist $L > 0$ and $\varepsilon_0 > 0$ such that the equation*

$$(7) \quad \dot{z} = v_1(t, z) = L [-iz + \bar{z}^2 - \varepsilon e^{it}]$$

is distributionally chaotic provided that

$$(8) \quad 0 < \varepsilon \leq \varepsilon_0.$$

Moreover, there exists $\beta > 0$ and $\Lambda \subset \mathbb{C}$ invariant with respect to the Poincaré operator $\varphi_{(-\beta, 2\pi)}$ such that the dynamical system $(\Lambda, \varphi_{(-\beta, 2\pi)}|_{\Lambda})$ is semiconjugated to the dynamical system $(\hat{\Sigma}, \sigma)$, where $\hat{\Sigma}$ is the sofic shift given by the Figure 2 and $\sigma : \hat{\Sigma} \rightarrow \hat{\Sigma}$ is the shift operator.

The change of variables

$$(9) \quad u(t) = \frac{1}{\varepsilon} e^{\frac{i}{3}t} z \left(\frac{2}{3}t \right)$$

in the equation (7) gives

$$(10) \quad \dot{u} = v_2(t, u) = \frac{2}{3} L \varepsilon e^{it} (\bar{u}^2 - 1) + \left(\frac{1}{2} - L \right) iu.$$

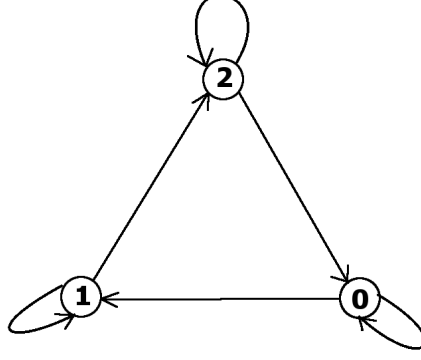


FIGURE 2. Vertex graph of the sofic shift $\hat{\Sigma}$ for the equation (7) with small leading coefficient.

Topological entropy of the system $(\hat{\Sigma}, \sigma)$ equals $\ln 2$. But since the time scaling in (9), topological entropy for discrete system generated by (10) equals $\frac{2}{3} \ln 2$, so it is lower than in the case of (Σ, σ^2) .

The most interesting case seems to be $L = \frac{1}{2}$ when the equation (10) reduces to

$$\dot{u} = v_2(t, u) = \frac{\varepsilon}{3} e^{it} (\bar{u}^2 - 1).$$

But we expect constant L to be much higher in Theorem 2.

2. DEFINITIONS

2.1. Basic notions. Let (X, f) be a dynamical system on a compact metric space such that f is a homeomorphism. By *(full) orbit* of x we mean the set

$$\text{Orb}(x, f) = \{\dots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \dots\}.$$

A point $y \in X$ is an ω -*limit point* (α -*limit point*) of a point x if it is an accumulation point of the sequence $x, f(x), f^2(x), \dots$ (resp. $x, f^{-1}(x), f^{-2}(x), \dots$). The set of all ω -limit points (α -limit points) of x is called ω -*limit set* (resp. α -*limit set*) of x and denoted $\omega_f(x)$ (resp. $\alpha_f(x)$). A point $p \in X$ is said to be *periodic* if $f^n(p) = p$ for some $n \geq 1$. The set of all periodic points for f is denoted by $\text{Per}(f)$.

Let $(X, f), (Y, g)$ be dynamical systems on compact metric spaces. A continuous map $\Phi : X \rightarrow Y$ is called a *semiconjugacy* (or a *factor map*) between f and g if Φ is surjective and $\Phi \circ f = g \circ \Phi$.

Let Y be a topological space. For any set $Z \subset \mathbb{R} \times Y$ and $a, b, t \in \mathbb{R}$, $a < b$ we define

$$\begin{aligned} Z_t &= \{x \in Y : (t, x) \in Z\}, \\ Z_{[a,b]} &= \{(t, x) \in Z : t \in [a, b]\}. \end{aligned}$$

We denote by \mathbb{N} the set of positive integers.

Let $c \in \mathbb{C}$ and $r > 0$. Then $\bar{B}(c, r) \subset \mathbb{C}$ denotes the closed ball centered at c with radius r .

2.2. Shift spaces. Let $\mathcal{A} = \{0, 1, \dots, n-1\}$. We denote $\Sigma_n = \mathcal{A}^{\mathbb{Z}}$.

By a *word*, we mean any element of a free monoid \mathcal{A}^* with the set of generators equal to \mathcal{A} . If $x \in \Sigma_n$ and $i < j$ then by $x_{[i,j]}$ we mean a sequence x_i, x_{i+1}, \dots, x_j . We may naturally identify $x_{[i,j]}$ with the word $x_{[i,j]} = x_i x_{i+1} \dots x_j \in \mathcal{A}^*$. It is also very convenient to denote $x_{[i,j]} = x_{[i,j-1]}$.

We introduce a metric ρ in Σ_n by

$$\rho(x, y) = 2^{-k}, \text{ where } k = \min \{m \geq 0 : x_{[-m,m]} \neq y_{[-m,m]}\}.$$

If $a_{-k} \dots a_{-1} a_0 a_1 \dots a_m \in \mathcal{A}^*$ then we define so called *cylinder set*:

$$[a_{-k} \dots a_m] = \{x \in \Sigma_n : x_{[-k,m]} = a_{-k} \dots a_{-1} a_0 a_1 \dots a_m\}.$$

It is well known that cylinder sets form a neighborhood basis for the space Σ_n .

By the 0^∞ we denote the element $x \in \Sigma_n$ such that $x_i = 0$ for all $i \in \mathbb{Z}$. The usual map on Σ_n is the shift map σ defined by $\sigma(x)_i = x_{i+1}$ for all i . Dynamical system (Σ_n, σ) is called full two-sided shift over n symbols.

2.3. Dynamical systems and Ważewski method. Let X be a topological space and W be a subset of X . Denote by $\text{cl } W$ the closure of W . The following definitions come from [8].

Let D be an open subset of $\mathbb{R} \times X$. By a *local flow* on X we mean a continuous map $\phi : D \rightarrow X$, such that three conditions are satisfied:

- i) $I_x = \{t \in \mathbb{R} : (t, x) \in D\}$ is an open interval (α_x, ω_x) containing 0, for every $x \in X$,
- ii) $\phi(0, x) = x$, for every $x \in X$,
- iii) $\phi(s+t, x) = \phi(t, \phi(s, x))$, for every $x \in X$ and $s, t \in \mathbb{R}$ such that $s \in I_x$ and $t \in I_{\phi(s, x)}$.

In the sequel we write $\phi_t(x)$ instead of $\phi(t, x)$.

Let ϕ be a local flow on X , $x \in X$ and $W \subset X$. We call the set

$$\phi^+(x) = \phi([0, \omega_x) \times \{x\})$$

the *positive semitrajectory* of $x \in X$.

We distinguish three subsets of W given by

$$\begin{aligned} W^- &= \{x \in W : \phi([0, t] \times \{x\}) \not\subset W, \text{ for every } t > 0\}, \\ W^+ &= \{x \in W : \phi([-t, 0] \times \{x\}) \not\subset W, \text{ for every } t > 0\}, \\ W^* &= \{x \in W : \phi(t, x) \notin W, \text{ for some } t > 0\}. \end{aligned}$$

It is easy to see that $W^- \subset W^*$. We call W^- the *exit set* of W , and W^+ the *entrance set* of W .

We call W a *Ważewski set* provided

- (1) if $x \in W$, $t > 0$, and $\phi([0, t] \times \{x\}) \subset \text{cl } W$ then $\phi([0, t] \times \{x\}) \subset W$,
- (2) W^- is closed relative to W^* .

Proposition 1. *If both W and W^- are closed subsets of X then W is a Ważewski set.*

The function $\sigma : W^* \rightarrow [0, \infty)$

$$\sigma(x) = \sup\{t \in [0, \infty) : \phi([0, t] \times \{x\}) \subset W\}$$

is called the *escape-time function* of W .

The following lemma is called the Ważewski lemma.

Lemma 3 ([8, Lemma 2.1 (iii)]). *Let W be a Ważewski set and σ be its escape-time function. Then σ is continuous.*

Finally, we state one version of the Ważewski theorem.

Theorem 4 ([8, Corollary 2.3]). *Let ϕ be a local flow on X , $W \subset X$ be a Ważewski set and $Z \subset W$. If W^- is not a strong deformation retract of $Z \cup W^-$ in W then there exists an $x_0 \in Z$ such that $\phi^+(x_0) \subset W$.*

(For the definition of the strong deformation retract see e.g. [2].)

2.4. Processes. Let X be a topological space and $\Omega \subset \mathbb{R} \times X \times \mathbb{R}$ be an open set.

By a *local process* on X we mean a continuous map $\varphi : \Omega \rightarrow X$, such that three conditions are satisfied:

- i) $I_{(\sigma, x)} = \{t \in \mathbb{R} : (\sigma, x, t) \in \Omega\}$ is an open interval containing 0, for every $\sigma \in \mathbb{R}$ and $x \in X$,
- ii) $\varphi(\sigma, \cdot, 0) = \text{id}_X$, for every $\sigma \in \mathbb{R}$,
- iii) $\varphi(\sigma, x, s+t) = \varphi(\sigma+s, \varphi(\sigma, x, s), t)$, for every $x \in X$, $\sigma \in \mathbb{R}$ and $s, t \in \mathbb{R}$ such that $s \in I_{(\sigma, x)}$ and $t \in I_{(\sigma+s, \varphi(\sigma, x, s))}$.

For abbreviation, we write $\varphi_{(\sigma, t)}(x)$ instead of $\varphi(\sigma, x, t)$. If $\Omega = \mathbb{R} \times X \times \mathbb{R}$, then the process φ is called *global*.

Local process φ on X generates a local flow ϕ on $\mathbb{R} \times X$ by the formula

$$(11) \quad \phi(t, (\sigma, x)) = (\sigma + t, \varphi(\sigma, x, t)).$$

Let M be a smooth manifold and let $v : \mathbb{R} \times M \rightarrow TM$ be a time-dependent vector field. We assume that v is so regular that for every $(t_0, x_0) \in \mathbb{R} \times M$ the Cauchy problem

$$(12) \quad \dot{x} = v(t, x),$$

$$(13) \quad x(t_0) = x_0$$

has unique solution. Then the equation (12) generates a local process φ on X by $\varphi_{(t_0, t)}(x_0) = x(t_0, x_0, t+t_0)$, where $x(t_0, x_0, \cdot)$ is the solution of the Cauchy problem (12), (13).

Let T be a positive number. In the sequel T denotes the period. We assume that v is T -periodic in t . It follows that the local process φ is T -periodic, i.e.,

$$\varphi_{(\sigma+T, t)} = \varphi_{(\sigma, t)} \text{ for all } \sigma, t \in \mathbb{R},$$

hence there is a one-to-one correspondence between T -periodic solutions of (12) and fixed points of the Poincaré map $\varphi_{(0, T)}$.

2.5. Distributional chaos. Let \mathbb{N} denote the set of positive integers and let f be a continuous self map of a compact metric space (X, ρ) . We define a function $\xi_f : X \times X \times \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{N}$ by:

$$\xi_f(x, y, t, n) = \#\{i : \rho(f^i(x), f^i(y)) < t, 0 \leq i < n\}$$

where $\#A$ denotes the cardinality of the set A . By the means of ξ_f we define the following two functions:

$$F_{xy}(f, t) = \liminf_{n \rightarrow \infty} \frac{1}{n} \xi_f(x, y, t, n), \quad F_{xy}^*(f, t) = \limsup_{n \rightarrow \infty} \frac{1}{n} \xi_f(x, y, t, n).$$

For brevity, we often write ξ , $F_{xy}(t)$, $F_{xy}^*(t)$ instead of ξ_f , $F_{xy}(f, t)$, $F_{xy}^*(f, t)$ respectively.

Both functions F_{xy} and F_{xy}^* are nondecreasing, $F_{xy}(t) = F_{xy}^*(t) = 0$ for $t < 0$ and $F_{xy}(t) = F_{xy}^*(t) = 1$ for $t > \text{diam } X$. Functions F_{xy} and F_{xy}^* are called *lower* and *upper distribution* functions, respectively.

Definition 5. A pair of points $(x, y) \in X \times X$ is called *distributionally chaotic* (of type 1) if

- (1) $F_{xy}(s) = 0$ for some $s > 0$,
- (2) $F_{xy}^*(t) = 1$ for all $t > 0$.

A set containing at least two points is called *distributionally scrambled set of type 1* (or *d-scrambled set for short*) if any pair of its distinct points is *distributionally chaotic*.

A map f is *distributionally chaotic (DC1)* if it has an uncountable *d-scrambled set*. *Distributional chaos* is said to be *uniform* if a constant s from condition (1) may be chosen the same for all the pairs of distinct points of *d-scrambled set*.

We remark here that the definition of distributional chaos was introduced to extend approach proposed by Li and Yorke in their famous paper [3]. Then it is clear why we use the name *d-scrambled set*. Namely each *d-scrambled set* is also *scrambled set* as defined by Li and Yorke.

We also should mention that our notation is a slightly different compared to that introduced by Schweizer and Smítal (founders of distributional chaos) in [7]. It is mainly because the definition of distributional chaos passed a very long journey since its introduction (even its name changed as it was originally called *strong chaos*). The definition we present is one of the strongest possibilities [1] and is usually called *distributional chaos of type 1* to be distinguished from other two weaker definitions - DC2 and DC3.

Definition 6. We say that a T -periodic local process φ on M is (uniform) *distributionally chaotic* if there exists compact set $\Lambda \subset M$ invariant for the Poincaré map $P_T = \varphi_{(0,T)}$ such that $P_T|_\Lambda$ is (uniform) *distributionally chaotic*.

We say that the equation (12) is (uniform) *distributionally chaotic* if it generates a local process which is (uniform) *distributionally chaotic*.

2.6. Useful facts. We recall Theorem 5 from [4] which is crucial in the proof of Theorem 1 (it is also possible to use more general [5, Theorem 11]).

Theorem 7. Let f be a continuous map from a compact metric space (Λ, ρ) into itself and let $\Phi : \Lambda \rightarrow \Sigma_2$ be a semiconjugacy between f and σ .

If there exists $x \in \Sigma_2 \cap \text{Per}(\sigma)$ such that $\#\Phi^{-1}(\{x\}) = 1$ then f is *distributionally chaotic* and *distributional chaos* is *uniform*.

3. MAIN CALCULATIONS

Idea of the proof of Theorem 2. There are many symmetries in the equation (7) e.g. function $z : (a, b) \rightarrow \mathbb{C}$ is a solution of the equation (2) if and only if the function w such that $w(t) = e^{i\frac{2}{3}\pi} z(t + \frac{2}{3}\pi)$ is also a solution. Immediately one gets the third solution $x(t) = e^{i\frac{4}{3}\pi} z(t + \frac{4}{3}\pi)$.

The equation (2) with $\varepsilon = 0$ has three stationary points which are saddle points $z_j = e^{-i\frac{1+4j}{6}\pi}$ for $j \in \{1, 2, 3\}$. So it is enough to investigate what happens between solutions z_1 and z_2 .

It is important to notice that the line $l : \Im(z) = -\frac{1}{2}$ is invariant. There is a heteroclinic solution between z_1 and z_2 contained in this line.

Term $-\varepsilon e^{it}$ makes stationary points continue to simple periodic solutions, but only for small $0 < \varepsilon$. Obviously $\varepsilon < \frac{1}{2}$. So $\varepsilon_0 < \frac{1}{2}$.

The simple periodic solution ξ_1 continued from z_1 lies at both sides of the line l . The same happens for the simple periodic solution ξ_2 continued from z_2 . But there are such time points t that the $\xi_1(t)$ is above the line l , while $\xi_2(t)$ is below it. So starting near ξ_1 and above l , due to L big enough, the solution enters neighbourhood of ξ_2 being above l . But on the other time t the point $\xi_1(t)$ is below l while $\xi_2(t)$ is above it. So starting near ξ_1 and below l , due to L big enough, the solution enters neighbourhood of ξ_2 being below l . This gives intersection of unstable manifold of solution ξ_1 with the stable one of ξ_2 .

To obtain a semiconjugacy to a sofic shift one needs to code the behaviour of solutions. Roughly speaking, being close to ξ_1 for a one period or transferring from the neighbourhood of ξ_1 to the neighbourhood of ξ_2 is coded by one letter 0. Similarly, being close to ξ_2 for a one period or transferring from the neighbourhood of ξ_2 to the neighbourhood of ξ_3 is coded by one letter 1. Of course, periodic solution ξ_j is coded by the sequence $(j-1)^\infty.(j-1)^\infty$ for $j \in \{1, 2, 3\}$. And every sequence $(j-1)^\infty.(j-1)^\infty$ for $j \in \{1, 2, 3\}$ codes exactly one solution (which is ξ_j).

It seems that it is possible to get complicated dynamics for $L = \frac{1}{2}$ but the time of transfer from neighbourhood of ξ_1 to neighbourhood of ξ_2 must be longer than one period. This gives semiconjugacy to shift with lower entropy than $(\hat{\Sigma}, \sigma)$. \square

Sketch of the proof of Theorem 1. Let us fix R satisfying (3). We define $a, \beta, \gamma, \delta, M$ to be such that

$$(14) \quad a = 0.7,$$

$$(15) \quad \beta = \gamma = \delta = 0.01.$$

are satisfied. By (6), the equation (2) generates a local process on Q . We denote it by φ (we do need φ to be a local process on the whole \mathbb{C} because we analyse dynamics only close to the origin). Observe that φ is 2π -periodic and if $f \equiv 0$, then, by Lemma 8, satisfies

$$(16) \quad -\varphi_{(\tau, t)}(z) = \varphi_{(\tau + \pi, t)}(-z) \text{ for every } \tau \in \mathbb{R}, z \in Q \text{ and } t \in I_{(\tau, z)}.$$

Of course, if $f \not\equiv 0$, then (16) is no longer valid, but since the perturbation f is small the equality (16) is still crucial for our calculations.

Write

$$(17) \quad U = \left\{ (t, q) \in \mathbb{R} \times \mathbb{C} : \left| \Re \left[(q-1)e^{-i\frac{t}{2}} \right] \right| \leq a, \left| \Im \left[(q-1)e^{-i\frac{t}{2}} \right] \right| \leq a \right\},$$

$$(18) \quad W = \left\{ (t, q) \in \mathbb{R} \times \mathbb{C} : \left| \Re \left[(q+1)e^{-i\frac{t}{2}} \right] \right| \leq a, \left| \Im \left[(q+1)e^{-i\frac{t}{2}} \right] \right| \leq a \right\},$$

$$(19) \quad Z = \{ (t, q) \in \mathbb{R} \times \mathbb{C} : -0.5 \leq \Re[q] \leq 0.5, |\Im[q]| \leq l((-1)^k \Re[q]), \\ t \in [k\pi - \beta, k\pi + \gamma] \text{ for some } k \in \mathbb{Z} \},$$

where

$$l(x) = \begin{cases} \frac{5}{28} - \frac{9}{14}x, & \text{for } -0.5 \leq x \leq 0.2, \\ 0.05, & \text{for } 0.2 \leq x \leq 1. \end{cases}$$

Moreover we write

$$(20) \quad \Lambda = \{q \in \mathbb{C} : I_{(-\beta, q)} = \mathbb{R} \text{ and for every } k \in \mathbb{Z} \text{ there holds exactly one of the conditions (21), (22), (23), (24)}\}$$

where

$$(21) \quad \varphi_{(-\beta, t)}(q) \in U \text{ for every } t \in [k\pi, (k+1)\pi],$$

$$(22) \quad \varphi_{(-\beta, t)}(q) \in W \text{ for every } t \in [k\pi, (k+1)\pi],$$

$$(23) \quad \begin{cases} \varphi_{(-\beta, k\pi)}(q) \in U, \\ \varphi_{(-\beta, t)}(q) \in Z & \text{for every } t \in [k\pi, k\pi + \beta + \gamma], \\ \varphi_{(-\beta, t)}(q) \in W & \text{for every } t \in [k\pi + \beta + \gamma, (k+1)\pi], \end{cases}$$

$$(24) \quad \begin{cases} \varphi_{(-\beta, k\pi)}(q) \in W, \\ \varphi_{(-\beta, t)}(q) \in Z & \text{for every } t \in [k\pi, k\pi + \beta + \gamma], \\ \varphi_{(-\beta, t)}(q) \in U & \text{for every } t \in [k\pi + \beta + \gamma, (k+1)\pi]. \end{cases}$$

By Lemma 9, the set Λ is compact. Moreover it is invariant with respect to the Poincaré map $\varphi_{(-\beta, 2\pi)}$, i.e. $\varphi_{(-\beta, 2\pi)}(\Lambda) = \Lambda$.

Now we define a map $\Phi : \Lambda \rightarrow \Sigma_2$ by

$$(25) \quad [\Phi(q)]_k = \begin{cases} 0, & \text{if (21) or (24) hold,} \\ 1, & \text{if (22) or (23) hold.} \end{cases}$$

By the definition of Φ , we get immediately

$$\Phi \circ \varphi_{(-\beta, 2\pi)} = \sigma^2 \circ \Phi.$$

Moreover, by the continuous dependence of solutions of (2) on initial conditions, Φ is continuous.

Let $x \in \Sigma \subset \Sigma_2$ be such that there exists $N \in \mathbb{N}$ such that $x_j = 0$ for every $|j| \geq N$, i.e. x is homoclinic to 0^∞ . By Lemma 10, there exists $q_x \in \Lambda$ such that $\Phi(q_x) = x$. Since the set of homoclinic points to 0^∞ is dense in Σ , Φ is continuous and defined on a compact set Λ , we get

$$\Phi(\Lambda) = \Sigma,$$

i.e. surjectivity of Φ . Finally, Φ is a semiconjugacy between Λ and Σ .

By Lemma 15,

$$(26) \quad \#\Phi^{-1}(\{0^\infty\}) = 1$$

holds so the existence of uniform distributional chaos follows by Theorem 7.

By the symmetry (16), we get $\#\Phi^{-1}(\{1^\infty\}) = 1$. The existence of infinitely many heteroclinic solutions between -1 and 1 follows by Lemma 16. Namely, for every $x \in \Sigma$ such that there exists $N \in \mathbb{N}$ such that $x_j = 0$, $x_{-j} = 1$ or $x_j = 1$, $x_{-j} = 0$ for $j > N$ every point from $\Phi^{-1}(\{x\})$ is heteroclinic from 1 to -1 or from -1 to 1 respectively. \square

Now we present lemmas which were used in the proof of Theorem 1. We use the notation introduced in the proof of Theorem 1 unless stated otherwise.

Lemma 8. *If $f \equiv 0$ then the condition (16) holds.*

Proof. Let $\Delta \subset \mathbb{R}$ be an interval and $z : \Delta \rightarrow \mathbb{C}$ be a solution of (2). It is enough to show that ξ given by $\xi(t) = -z(t + \pi)$ is also a solution of (2). To see this let us calculate

$$\dot{\xi}(t) = -\dot{z}(t + \pi) = -Re^{i(t+\pi)} [\bar{z}^2(t + \pi) - 1] = Re^{it} [\bar{\xi}^2(t) - 1].$$

□

Lemma 9. *The set Λ given by (20) is compact.*

Proof. It is enough to prove that Λ is a closed subset of \mathbb{C} . Let $\{z_n\}_{n \in \mathbb{N}} \subset \Lambda$ be such that $\lim_{n \rightarrow \infty} z_n$ exists. We denote this limit by z .

Let us fix $t \in \mathbb{R}$. Then $\varphi_{(-\beta, t)}(z_n) \in \overline{B}(0, 2)$ holds for every $n \in \mathbb{N}$, so $\lim_{n \rightarrow \infty} \varphi_{(-\beta, t)}(z_n)$ exists and it must be equal to $\varphi_{(-\beta, t)}(z)$, so $\varphi_{(-\beta, t)}(z)$ exists. By the arbitrariness of the choice of t , we get $I_{(-\beta, z)} = \mathbb{R}$.

Now we fix $k \in \mathbb{Z}$.

For every $z \in \Lambda$ let us write $\eta_z : \mathbb{R} \rightarrow \mathbb{C}$ where $\eta_z(t) = \varphi_{(-\beta, z, t)}$. By the continuous dependence on initial conditions,

$$\lim_{n \rightarrow \infty} \eta_{z_n} = \eta_z \text{ uniformly on the interval } [k\pi, (k+1)\pi].$$

Thus exactly one condition among (21), (22), (23), (24) is satisfied by almost all η_{z_n} . Since for every $t \in \mathbb{R}$ sets U_t , W_t and Z_t are compact (or empty), η_z satisfies this condition.

Finally, $z \in \Lambda$. □

Lemma 10. *Let $x \in \Sigma$ be such that there exists $N \in \mathbb{N}$ such that $x_j = 0$ for every $|j| \geq N$. Then there exists $q_x \in \Lambda$ such that $\Phi(q_x) = x$.*

Proof. First of all we assume that $f \equiv 0$.

In the sequel we investigate (2) (especially in a neighbourhood of 1) in the coordinates

$$(27) \quad w = w(q, t) = (q - 1)e^{-i\frac{t}{2}}$$

which has the form

$$(28) \quad \dot{w} = u(t, w) = 2R\bar{w} + Re^{-i\frac{t}{2}}\bar{w}^2 - \frac{i}{2}w + e^{-i\frac{t}{2}}f\left(t, we^{i\frac{t}{2}} + 1\right).$$

We also investigate (2) (especially in a neighbourhood of -1) in the coordinates

$$(29) \quad p = p(q, t) = (q + 1)e^{-i\frac{t}{2}}$$

which has the form

$$(30) \quad \dot{p} = \tilde{u}(t, p) = -2R\bar{p} + Re^{-i\frac{t}{2}}\bar{p}^2 - \frac{i}{2}p + e^{-i\frac{t}{2}}f\left(t, pe^{i\frac{t}{2}} - 1\right).$$

We denote by ψ and $\tilde{\psi}$ the local processes generated by (28) and (30) respectively. Let us stress that

$$\begin{aligned} q &= \Upsilon(w, t) = e^{i\frac{t}{2}}w + 1, \\ q &= \Xi(p, t) = e^{i\frac{t}{2}}p - 1 \end{aligned}$$

hold. Thus the following equalities

$$\begin{aligned}\varphi(\tau, q, t) &= \Upsilon(\psi(\tau, w(q, \tau), t), \tau + t), \\ \psi(\tau, w, t) &= w(\varphi(\tau, \Upsilon(w, \tau), t), \tau + t), \\ \varphi(\tau, q, t) &= \Xi(\tilde{\psi}(\tau, p(q, \tau), t), \tau + t), \\ \tilde{\psi}(\tau, p, t) &= p(\varphi(\tau, \Xi(p, \tau), t), \tau + t)\end{aligned}$$

are satisfied wherever they have sense.

It is easy to see, that

$$\begin{aligned}w(U) &= \{w \in \mathbb{C} : |\Re[w]| \leq a, |\Im[w]| \leq a\}, \\ p(W) &= \{p \in \mathbb{C} : |\Re[p]| \leq a, |\Im[p]| \leq a\}.\end{aligned}$$

Let us notice that inside $w(U)$ the vector field u is close to $2R\overline{w}$. The other terms are treated as perturbation so the qualitative behaviour of u inside $\mathbb{R} \times w(U)$ is just as the term $2R\overline{w}$. Similarly, the qualitative behaviour of \tilde{u} inside $\mathbb{R} \times p(W)$ is just as the term $-2R\overline{p}$.

Let

$$\begin{aligned}(31) \quad K &= \left\{ w \in \mathbb{C} : |\Re[w]| \leq \frac{11}{10}\beta, |\Im[w]| \leq 2\beta^2 \right\}, \\ L &= \left\{ p \in \mathbb{C} : |\Re[p]| \leq 2\beta^2, |\Im[p]| \leq \frac{11}{10}\beta \right\}.\end{aligned}$$

By Lemma 11, there exists a continuous function

$$\xi : \left[-\frac{11}{10}\beta, \frac{11}{10}\beta \right] \ni o \mapsto \xi(o) \in [-2\beta^2, 2\beta^2]$$

such that for every $o \in [-\frac{11}{10}\beta, \frac{11}{10}\beta]$ we get

$$\begin{aligned}\lim_{t \rightarrow -\infty} \psi(\delta, o + i\xi(o), t) &= 0, \\ \psi(\delta, o + i\xi(o), t) &\in K \text{ for every } t \leq 0\end{aligned}$$

where $\delta = -\beta - 2(N-1)\pi$.

It immediately follows by Lemma 12, that there exists an interval $[\mu, \nu] \subset [-\frac{11}{10}\beta, \frac{11}{10}\beta]$ such that the following conditions hold: (32), exactly one out of (33) and (34), for every $l \in \{1, 2, \dots, 2N\}$ exactly one out of (35), (36), (37) and (38) where

$$\begin{aligned}
 (32) \quad & \psi(\delta, o + i\xi(o), 4N\pi) \in K \text{ for every } o \in [\mu, \nu], \\
 (33) \quad & \Re[\psi(\delta, \mu + i\xi(\mu), 4N\pi)] = -\frac{11}{10}\beta, \quad \Re[\psi(\delta, \nu + i\xi(\nu), 4N\pi)] = \frac{11}{10}\beta, \\
 (34) \quad & \Re[\psi(\delta, \mu + i\xi(\mu), 4N\pi)] = \frac{11}{10}\beta, \quad \Re[\psi(\delta, \nu + i\xi(\nu), 4N\pi)] = -\frac{11}{10}\beta, \\
 (35) \quad & \begin{cases} \text{if } x_{-N+l-1} = x_{-N+l} = 0, \text{ then for every } o \in [\mu, \nu] \\ \psi(\delta, o + i\xi(o), t) \in K \text{ for every } t \in [(l-1)\pi, l\pi], \end{cases} \\
 (36) \quad & \begin{cases} \text{if } x_{-N+l-1} = x_{-N+l} = 1, \text{ then for every } o \in [\mu, \nu] \\ \tilde{\psi}(\delta, p(\Upsilon(o + i\xi(o), \delta), \delta), t) \in L \text{ for every } t \in [(l-1)\pi, l\pi], \end{cases} \\
 (37) \quad & \begin{cases} \text{if } x_{-N+l-1} = 0, x_{-N+l} = 1, \text{ then for every } o \in [\mu, \nu] \\ \psi(\delta, o + i\xi(o), (l-1)\pi) \in K, \\ \varphi(\delta, \Upsilon(o + i\xi(o), \delta), t) \in Z \text{ for every } t \in [(l-1)\pi, (l-1)\pi + \beta + \gamma], \\ \tilde{\psi}(\delta, p(\Upsilon(o + i\xi(o), \delta), \delta), t) \in p(W) \text{ for every } t \in [(l-1)\pi + \beta + \gamma, l\pi], \\ \tilde{\psi}(\delta, p(\Upsilon(o + i\xi(o), \delta), \delta), l\pi) \in L, \end{cases} \\
 (38) \quad & \begin{cases} \text{if } x_{-N+l-1} = 1, x_{-N+l} = 0, \text{ then for every } o \in [\mu, \nu] \\ \tilde{\psi}(\delta, p(\Upsilon(o + i\xi(o), \delta), \delta), (l-1)\pi) \in L, \\ \varphi(\delta, \Upsilon(o + i\xi(o), \delta), t) \in Z \text{ for every } t \in [(l-1)\pi, (l-1)\pi + \beta + \gamma], \\ \psi(\delta, o + i\xi(o), t) \in w(U) \text{ for every } t \in [(l-1)\pi + \beta + \gamma, l\pi], \\ \psi(\delta, o + i\xi(o), l\pi) \in K. \end{cases}
 \end{aligned}$$

Reversing time in (28) and applying Lemma 11, we get the existence of a continuous function

$$\tilde{\xi} : \left[-\frac{11}{10}\beta, \frac{11}{10}\beta\right] \ni o \mapsto \tilde{\xi}(o) \in [-2\beta^2, 2\beta^2]$$

such that for every $o \in [-\frac{11}{10}\beta, \frac{11}{10}\beta]$ we get

$$\begin{aligned}
 \lim_{t \rightarrow +\infty} \psi(-\beta + 2(N+1)\pi, \tilde{\xi}(o) + io, t) &= 0, \\
 \psi(-\beta + 2(N+1)\pi, \tilde{\xi}(o) + io, t) &\in w(U) \text{ for every } t \geq 0.
 \end{aligned}$$

By the continuity of ψ and (32), (33), (34), there exist $\hat{o} \in [\mu, \nu]$, $\hat{y} \in [-\frac{11}{10}\beta, \frac{11}{10}\beta]$ such that

$$\psi(\delta, \hat{o} + i\xi(\hat{o}), 4N\pi) = \tilde{\xi}(\hat{y}) + i\hat{y}$$

holds.

Write $q_x = \Upsilon(\psi(\delta, \hat{o} + i\xi(\hat{o}), 2(N-1)\pi), -\beta)$. It is easy to see that $\Phi(q_x) = x$ holds.

The proof for the case $f \not\equiv 0$ is similar. The role of trivial solutions -1 and 1 plays periodic ones which are contained from them. \square

In the following lemma we use the notation introduced in the proofs of Theorem 1 and Lemma 10.

Lemma 11. *For every $\tau \in \mathbb{R}$ there exists a continuous function*

$$\xi : \left[-\frac{11}{10}\beta, \frac{11}{10}\beta\right] \ni o \mapsto \xi(o) \in [-2\beta^2, 2\beta^2]$$

such that for every $o \in [-\frac{11}{10}\beta, \frac{11}{10}\beta]$ we get

$$\lim_{t \rightarrow -\infty} \psi(\tau, o + i\xi(o), t) = 0,$$

$$\psi(\tau, o + i\xi(o), t) \in K \text{ for every } t \leq 0.$$

Proof. To use a Ważewski method, let us reverse the time in (28) by setting $a(t) = w(-t)$. We get

$$(39) \quad \dot{a} = \widehat{u}(t, a) = -2R\bar{a} - Re^{i\frac{t}{2}}\bar{a}^2 + \frac{1}{2}a.$$

Let $\widehat{\psi}$ and $\widehat{\phi}$ denote the local process on $\mathbb{C} \setminus \{0\}$ and the local flow on $\mathbb{R} \times (\mathbb{C} \setminus \{0\})$, respectively, generated by (39). Of course, the relation between $\widehat{\psi}$ and $\widehat{\phi}$ is given by (11).

Let us fix $\tau \in \mathbb{R}$. To finish the proof it is enough to show that there exists a continuous function

$$\xi : \left[-\frac{11}{10}\beta, \frac{11}{10}\beta\right] \ni o \mapsto \xi(o) \in [-2\beta^2, 2\beta^2]$$

such that $\xi(0) = 0$ and for every $o \in [-\frac{11}{10}\beta, \frac{11}{10}\beta] \setminus \{0\}$ we get

$$(40) \quad \lim_{t \rightarrow +\infty} \widehat{\psi}(\tau, o + i\xi(o), t) = 0,$$

$$(41) \quad \widehat{\psi}(\tau, o + i\xi(o), t) \in K \text{ for every } t \geq 0.$$

Let us fix $o \in (0, \frac{11}{10}\beta]$. We define

$$\Gamma = \left\{ (t, a) \in \mathbb{R} \times \mathbb{C} : t \in \mathbb{R}, \Re[a] \in \left(0, \frac{11}{10}\beta\right], |\Arg[a]| \leq \beta \right\}.$$

We show that

$$(42) \quad \Gamma^- = \left\{ (t, a) \in \mathbb{R} \times \mathbb{C} : t \in \mathbb{R}, \Re[a] \in \left(0, \frac{11}{10}\beta\right], |\Arg[a]| = \beta \right\}.$$

We parameterize part of $\partial\Gamma$ by

$$s_1 : \mathbb{R} \times \left(0, \frac{11\beta}{10 \cos(\beta)}\right) \ni (t, \theta) \mapsto (t, \theta e^{i\beta}).$$

An outward orthonormal vector to this part of $\partial\Gamma$ is given by

$$n_1 : \mathbb{R} \times \left(0, \frac{11\beta}{10 \cos(\beta)}\right) \ni (t, \theta) \mapsto (0, ie^{i\beta}).$$

The inner product of the outward orthonormal vector and the vector field \widehat{u} has the form

$$\begin{aligned} \langle n_1(t, \theta), \widehat{u}(s_1(t, \theta)) \rangle &= \Re \left[-ie^{-i\beta}(-2)R\theta e^{-i\beta} - ie^{-i\beta}(-R)e^{i\frac{t}{2}}\theta^2 e^{-2i\beta} \right. \\ &\quad \left. - ie^{-i\beta}\theta \frac{1}{2}e^{i\beta} \right] \\ &\geq 2R \sin(2\beta) - R\theta^2 - \frac{1}{2}\theta \\ &> 0. \end{aligned}$$

Another part of $\partial\Gamma$ can be parameterized by

$$s_2 : \mathbb{R} \times \left(0, \frac{11\beta}{10 \cos(\beta)}\right) \ni (t, \theta) \mapsto (t, \theta e^{-i\beta}).$$

An outward orthonormal vector to this part of $\partial\Gamma$ is given by

$$n_2 : \mathbb{R} \times \left(0, \frac{11\beta}{10 \cos(\beta)}\right) \ni (t, \theta) \mapsto (0, -ie^{-i\beta}).$$

By calculations similar to the above, one can see that $\langle n_1(t, \theta), \hat{u}(s_1(t, \theta)) \rangle > 0$ holds for every $(t, \theta) \in \left(0, \frac{11\beta}{10 \cos(\beta)}\right)$.

Now let $(t, a) \in \Gamma$ i.e. $t \in \mathbb{R}$, $\Re[a] \in (0, \frac{11}{10}\beta]$ and

$$(43) \quad |\Im[a]| \leq \tan(\beta)\Re[a]$$

hold. We calculate

$$\begin{aligned} \Re[\hat{u}(t, a)] &= \Re \left[-2R\bar{a} - Re^{i\frac{\pi}{2}}\bar{a}^2 + \frac{1}{2}a \right] \\ (44) \quad &\leq -2R\Re[a] + R|a|^2 + \frac{1}{2}|a| \\ &\leq \Re[a] \left(-2R + R\Re[a](1 + \tan^2(\beta)) + \frac{1}{2}\sqrt{1 + \tan^2(\beta)} \right) \\ &< 0. \end{aligned}$$

Finally, the vector field \hat{u} points outwards Γ on some part of $\partial\Gamma$ and points inwards on the other part of $\partial\Gamma$. Thus (42) holds.

Let us notice, that since both Γ and Γ^- are closed in $\mathbb{R} \times (\mathbb{C} \setminus \{0\})$, by Proposition 1, Γ is a Ważewski set.

Write

$$\Theta = \{(t, a) \in \Gamma : t = \tau, \Re[a] = o\}.$$

Since Γ^- , as a not connected set, is not a strong deformation retract of a connected $\Gamma^- \cup \Theta$ in Γ , by Theorem 4, there exists $a_0 \in \Theta_\tau$ such that

$$(45) \quad \hat{\phi}^+(\tau, a_0) \subset \Gamma$$

holds.

We set $\xi(o) = \Im[a_0]$ (if there are more than one a_0 's, we choose one of them). To define ξ for negative o 's we repeat the above construction with $\hat{\Gamma}$ instead of Γ where $\hat{\Gamma} = \{(t, a) \in \mathbb{R} \times \mathbb{C} : (t, -a) \in \Gamma\}$. The same relation holds between $\hat{\Gamma}^-$ and Γ^- . All calculations are similar to the above due to the symmetries of $-2R\bar{a}$ which is the leading term of \hat{u} close to the origin.

Let us notice, that by (45), we immediately get (41). Moreover, by (44), we get (40).

To finish the proof it is enough to show that ξ is continuous. By (43), ξ is continuous at 0.

To obtain a contradiction, let us assume that there exists $o_1 \in [-\frac{11}{10}\beta, \frac{11}{10}\beta] \setminus \{0\}$ such that

$$(46) \quad \xi(o_1) \neq \lim_{o \rightarrow o_1} \xi(o)$$

holds.

Let χ be a solution of (39) satisfying $\chi(\tau) = o_1 + i\xi(o_1)$. Let us notice that

$$(47) \quad |\chi(t)| \leq \frac{12}{10}\beta$$

for every $t \geq \tau$.

We make the change of variables $\zeta = a - \chi$ and get

$$(48) \quad \dot{\zeta} = \dot{u}(t, \zeta) = -2R\bar{\zeta} - Re^{i\frac{t}{2}}\bar{\zeta}(\bar{\zeta} + 2\bar{\chi}) + \frac{1}{2}\zeta.$$

We define

$$\begin{aligned} \tilde{K} &= \left\{ \zeta \in \mathbb{C} : |\Im[\zeta]| \leq \frac{11}{10}\beta, |\Im[\zeta]| \geq |\Re[\zeta]| \right\}, \\ \hat{K} &= \left\{ \zeta \in \tilde{K} : |\Im[\zeta]| = \frac{11}{10}\beta \right\}. \end{aligned}$$

We show that every solution ς of (48) such that $\varsigma(\tau) \in \tilde{K}$ leaves \tilde{K} through \hat{K} and it happens for some $t \geq \tau$. Keeping in mind (47), for any $\zeta \in \tilde{K}$ let us estimate

$$\begin{aligned} |\Im[\dot{\zeta}]| &\geq |\Im[-2R\bar{\zeta}]| - \left| \Im \left[Re^{i\frac{t}{2}}\bar{\zeta}(\bar{\zeta} + 2\bar{\chi}) \right] \right| - \frac{1}{2}|\Im[\zeta]| \\ &\geq |\Im[\zeta]| \left(2R - \frac{1}{2} \right) - R|\zeta|(|\zeta| + 2|\chi|) \\ &\geq |\Im[\zeta]| \left(2R - \frac{1}{2} - R\sqrt{2}\frac{11\sqrt{2} + 24}{10}\beta \right) \\ &\geq R|\Im[\zeta]|. \end{aligned}$$

Let us investigate the behaviour of \dot{u} on $\partial\tilde{K}$. We parameterize a part of $\partial\tilde{K}$ by

$$s_3 : \left(0, \frac{11}{10}\beta \right] \ni \theta \mapsto \theta(1 + i).$$

An outward orthogonal vector is given by $n_3 = 1 - i$. The inner product of the outward orthogonal vector and the vector field \dot{u} has the form

$$\begin{aligned} \langle n_3(\theta), \dot{u}(t, s_3(\theta)) \rangle &= \Re \left[(1 + i)(-2)R\theta(1 - i) \right. \\ &\quad \left. - (1 + i)(-R)e^{i\frac{t}{2}}\theta(1 - i)(\theta(1 - i) + 2\bar{\chi}(t)) + (1 + i)^2\theta\frac{1}{2} \right] \\ &\leq -4R\theta + 2R\theta \left(\sqrt{2}\theta + \frac{24}{10}\beta \right) + \theta \\ &< 0. \end{aligned}$$

So on this side of boundary of \tilde{K} the vector field points inwards \tilde{K} . By the symmetries of \tilde{K} and the dominating term $-2R\bar{\zeta}$ of \dot{u} close to the origin, the vector field \dot{u} points inwards \tilde{K} on the other sides of the boundary, except \hat{K} . Finally, every nonzero ς leaves \tilde{K} for some time $t \geq \tau$ through \hat{K} .

So if (46) holds, then there exists a solution $\tilde{\varsigma}$ of (39) satisfying $\tilde{\varsigma}(\tau) = o + i\xi(o)$ for some o close to o_1 such that $\varsigma_1(\tau) = \tilde{\varsigma}(\tau) - \chi(\tau) \in \tilde{K}$. But, as we know, ς_1 must leave \tilde{K} for some time $t \geq \tau$ through \hat{K} . It means that $\tilde{\varsigma}$ leaves K for some time $t \geq \tau$ which contradicts (45). \square

In the following lemma we use the notation introduced in the proofs of Theorem 1 and Lemma 10.

Lemma 12. *There exists an interval $[\mu, \nu] \subset [-\frac{11}{10}\beta, \frac{11}{10}\beta]$ such that the following conditions hold: (32), exactly one out of (33) and (34), for every $l \in \{1, 2, \dots, 2N\}$ exactly one out of (35), (36), (37) and (38).*

Proof. Let

$$\begin{aligned}
K_1 &= \left\{ w \in K : \Re w = -\frac{11}{10}\beta \right\}, \\
K_2 &= \left\{ w \in K : \Re w = \frac{11}{10}\beta \right\}, \\
K_3 &= \{ w \in K : \Im w = -2\beta^2 \}, \\
K_4 &= \{ w \in K : \Im w = 2\beta^2 \}, \\
L_1 &= \left\{ p \in L : \Im p = -\frac{11}{10}\beta \right\}, \\
L_2 &= \left\{ p \in L : \Im p = \frac{11}{10}\beta \right\}.
\end{aligned}$$

We prove more than (35), (36), (37) and (38). Namely, we prove that there exist sequences $\{\mu_j\}_{j=0}^{2N}$, $\{\nu_j\}_{j=0}^{2N}$ such that $\mu_0 = -\frac{11}{10}\beta$, $\nu_0 = \frac{11}{10}\beta$, $\mu_j < \mu_{j+1}$, $\nu_j > \nu_{j+1}$ for $j \in \{0, 1, \dots, 2N-1\}$, $\mu_{2N} < \nu_{2N}$ and

$$\left\{ \begin{array}{l} \text{in the case (35)} \\ \psi(\delta, o + i\xi(o), t) \in K \text{ for every } t \in [(l-1)\pi, l\pi], o \in [\mu_l, \nu_l] \\ \text{and either } \psi(\delta, \mu_l + i\xi(\mu_l), l\pi) \in K_1, \psi(\delta, \nu_l + i\xi(\nu_l), l\pi) \in K_2 \\ \text{or } \psi(\delta, \mu_l + i\xi(\mu_l), l\pi) \in K_2, \psi(\delta, \nu_l + i\xi(\nu_l), l\pi) \in K_1, \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{in the case (36)} \\ \tilde{\psi}(\delta, p(\Upsilon(o + i\xi(o), \delta), \delta), t) \in L \text{ for every } t \in [(l-1)\pi, l\pi], o \in [\mu_l, \nu_l] \\ \text{and either} \\ \tilde{\psi}(\delta, p(\Upsilon(\mu_l + i\xi(\mu_l), \delta), \delta), l\pi) \in L_1, \tilde{\psi}(\delta, p(\Upsilon(\nu_l + i\xi(\nu_l), \delta), \delta), l\pi) \in L_2 \\ \text{or} \\ \tilde{\psi}(\delta, p(\Upsilon(\mu_l + i\xi(\mu_l), \delta), \delta), l\pi) \in L_2, \tilde{\psi}(\delta, p(\Upsilon(\nu_l + i\xi(\nu_l), \delta), \delta), l\pi) \in L_1, \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{in the case (37) for every } o \in [\mu_l, \nu_l] \\ \psi(\delta, o + i\xi(o), (l-1)\pi) \in K, \\ \varphi(\delta, \Upsilon(o + i\xi(o), \delta), t) \in Z \text{ for every } t \in [(l-1)\pi, (l-1)\pi + \beta + \gamma], \\ \tilde{\psi}(\delta, p(\Upsilon(o + i\xi(o), \delta), \delta), t) \in p(W) \text{ for every } t \in [(l-1)\pi + \beta + \gamma, l\pi], \\ \tilde{\psi}(\delta, p(\Upsilon(o + i\xi(o), \delta), \delta), l\pi) \in L, \\ \text{and either} \\ \tilde{\psi}(\delta, p(\Upsilon(\mu_l + i\xi(\mu_l), \delta), \delta), l\pi) \in L_1, \tilde{\psi}(\delta, p(\Upsilon(\nu_l + i\xi(\nu_l), \delta), \delta), l\pi) \in L_2 \\ \text{or} \\ \tilde{\psi}(\delta, p(\Upsilon(\mu_l + i\xi(\mu_l), \delta), \delta), l\pi) \in L_2, \tilde{\psi}(\delta, p(\Upsilon(\nu_l + i\xi(\nu_l), \delta), \delta), l\pi) \in L_1, \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{in the case (38) for every } o \in [\mu_l, \nu_l] \\ \tilde{\psi}(\delta, p(\Upsilon(o + i\xi(o), \delta), \delta), (l-1)\pi) \in L, \\ \varphi(\delta, \Upsilon(o + i\xi(o), \delta), t) \in Z \text{ for every } t \in [(l-1)\pi, (l-1)\pi + \beta + \gamma], \\ \psi(\delta, o + i\xi(o), t) \in w(U) \text{ for every } t \in [(l-1)\pi + \beta + \gamma, l\pi], \\ \psi(\delta, o + i\xi(o), l\pi) \in K \\ \text{and either } \psi(\delta, \mu_l + i\xi(\mu_l), l\pi) \in K_1, \psi(\delta, \nu_l + i\xi(\nu_l), l\pi) \in K_2 \\ \text{or } \psi(\delta, \mu_l + i\xi(\mu_l), l\pi) \in K_2, \psi(\delta, \nu_l + i\xi(\nu_l), l\pi) \in K_1 \end{array} \right.$$

hold.

The case (35) comes directly from Lemma 13, because for every $t \in \mathbb{R}$ the vector field $u(\cdot, t)$ from (28) points outward K on $K_1 \cup K_2$ and points inward K on $K_3 \cup K_4$. Indeed, to see this let us fix $w \in K_1$. Then

$$\begin{aligned} \Re[w] &= \Re \left[2R\overline{w} + Re^{-i\frac{t}{2}}\overline{w}^2 - \frac{i}{2}w \right] \\ &\leq -2R\frac{11}{10}\beta + R \left| e^{-i\frac{t}{2}}\overline{w}^2 \right| + \left| \frac{i}{2}w \right| \\ (49) \quad &\leq -R \left(\frac{11}{5}\beta - \left(\frac{11}{10}\beta \right)^2 - 4\beta^4 \right) + \frac{11}{10}\beta + 2\beta^2 \\ &< 0. \end{aligned}$$

Similar calculations show that $\Re[w] > 0$ for every $w \in K_2$. Let us fix now $w \in K_3$. Then

$$\begin{aligned} \Im[w] &= \Im \left[2R\overline{w} + Re^{-i\frac{t}{2}}\overline{w}^2 - \frac{i}{2}w \right] \\ &\geq 4R\beta^2 - R \left| e^{-i\frac{t}{2}}\overline{w}^2 \right| - \frac{1}{2}\Re[w] \\ (50) \quad &\geq R \left(4\beta^2 - \frac{121}{100}\beta^2 - 4\beta^4 \right) - \frac{11}{5}\beta \\ &\geq R \left(\frac{279}{100}\beta^2 - 4\beta^4 - \frac{22}{5}\beta^3 \right) \\ &> 0. \end{aligned}$$

Similar calculations show that $\Im[w] < 0$ for every $w \in K_4$.

The case (36) is very similar to the case (35) (we only need to interchange the horizontal and vertical directions). By the 2π -periodicity of φ and symmetry (16), the case (38) is equivalent to (37), so it is enough to prove the statement in the case (37).

By the 2π -periodicity of φ , there is no loss of generality in assuming that $\delta = -\beta$ and $l = 1$.

So at the beginnig $\psi(\delta, o + i\xi(o), (l-1)\pi) = \psi(-\beta, o + i\xi(o), 0) \in K$ holds for every $o \in [\mu_{l-1}, \nu_{l-1}] = [\mu_0, \nu_0]$. Thus $\varphi(-\beta, \Upsilon(o + i\xi(o), -\beta), 0) \in Z$ for every $o \in [\mu_0, \nu_0]$.

Let

$$\begin{aligned}
 (51) \quad \tilde{\zeta} &= \beta \frac{(1 - e^{-4R\beta})(2 - \beta)}{2 - \beta + e^{-4R\beta}}, \\
 K_{\tilde{\zeta}} &= \left\{ w \in K : -\beta \leq \Re[w] \leq -\beta + \tilde{\zeta} \right\}, \\
 \tilde{K}_{\tilde{\zeta}} &= \left\{ w \in K : \Re[w] = -\beta + \tilde{\zeta} \right\}, \\
 \tilde{K}_{\beta} &= \{ w \in K : \Re[w] = -\beta \}, \\
 \tilde{L} &= \left\{ p \in \mathbb{C} : |\Re[p]| \leq \frac{11}{10}\beta, |\Im[p]| \leq 3\beta \right\}, \\
 \tilde{L}_1 &= \left\{ p \in \tilde{L} : \Im[p] = -3\beta \right\}, \\
 \tilde{L}_2 &= \left\{ p \in \tilde{L} : \Im[p] = 3\beta \right\}, \\
 \tilde{L}^u &= \left\{ p \in \tilde{L} : \Im[p] > 4\beta^2 \right\}, \\
 \tilde{L}^l &= \left\{ p \in \tilde{L} : \Im[p] < -4\beta^2 \right\}.
 \end{aligned}$$

Obviously, $\tilde{\zeta} < \beta$.

We finish the proof in the following steps:

- (1) showing that for every $w \in K_{\tilde{\zeta}}$ there exists $t_w \in (0, \beta + \gamma]$ such that

$$(52) \quad p(\varphi(-\beta, \Upsilon(w, -\beta), t_w), -\beta + t_w) \in \tilde{L}$$

and

$$(53) \quad \varphi(-\beta, \Upsilon(w, -\beta), t) \in Z_{-\beta+t} \text{ for every } t \in [0, t_w]$$

holds,

- (2) showing that for every $w \in K_{\beta}$ the inclusion

$$p(\varphi(-\beta, \Upsilon(w, -\beta), \gamma - \beta), -2\beta + \gamma) \in \tilde{L}^u$$

holds,

- (3) showing that for every $w \in \tilde{K}_{\tilde{\zeta}}$ the inclusion

$$p(\varphi(-\beta, \Upsilon(w, -\beta), \gamma + \beta), \gamma) \in \tilde{L}^l$$

holds,

- (4) showing that for every $\sigma \in \mathbb{R}$ and $p \in \tilde{L}^u$ there exists $t \in [0, \beta]$ such that $\tilde{\psi}(\sigma, p, t) \in \tilde{L}_2$,
 (5) showing that for every $\sigma \in \mathbb{R}$ and $p \in \tilde{L}^l$ there exists $t \in [0, \beta]$ such that $\tilde{\psi}(\sigma, p, t) \in \tilde{L}_1$,
 (6) showing that for every $\sigma \in \mathbb{R}$ and $z \in p(W_{\sigma} \times \{\sigma\})$ there exists $t \in [0, \beta]$ such that $\tilde{\psi}(\sigma, p, t) \in L$ or $|\Im[\tilde{\psi}(\sigma, p, t)]| = \frac{11}{10}\beta$,
 (7) observing that, by above steps and Lemma 13, for any curve contained in K which connects K_1 and K_2 there must exist its connected part which is after transfer via flow contained in L and connects L_1 and L_2 .

To follow the steps let us observe that, by (4) and (5), one gets the inequalities

$$\Re[v(t, z)] \leq N + R \left[-\cos \beta + \frac{(\Re z)^2}{\cos \beta} \right] < 0$$

for every $(t, z) \in [-\beta, \beta] \times \{z \in \mathbb{C} : |\Re z| \leq 0.98, |\Im z| \leq 1\}$. It means that the vector field points to the left in the whole set. This information combined with Lemma 14 gives some estimates how quickly one can move from set U to set W through set Z . \square

Lemma 13. *Let $\alpha, \beta, \gamma, \delta, \zeta, \eta, a, b \in \mathbb{R}$, $\alpha < \beta$, $\gamma < \delta$, $\zeta < \eta$, $a < b$, $K = [\alpha, \beta] \times [\gamma, \delta] \subset \mathbb{R}^2$ and the time dependent vector field $v : \mathbb{R} \times \mathbb{R}^2 \ni (t, x, y) \mapsto v(t, x, y) = (v_1(t, x, y), v_2(t, x, y)) \in \mathbb{R}^2$ be continuous and so regular that the equation*

$$\begin{cases} \dot{x} = v_1(t, x, y), \\ \dot{y} = v_2(t, x, y) \end{cases}$$

generates a local process φ on \mathbb{R}^2 . Write

$$\begin{aligned} K_1 &= \{(x, y) \in K : x = \alpha\}, \\ K_2 &= \{(x, y) \in K : x = \beta\}, \\ K_3 &= \{(x, y) \in K : y = \gamma\}, \\ K_4 &= \{(x, y) \in K : y = \delta\}. \end{aligned}$$

Let the conditions

$$(54) \quad v_1(t, x, y) < 0 \text{ for every } t \in \mathbb{R}, (x, y) \in K_1,$$

$$(55) \quad v_1(t, x, y) > 0 \text{ for every } t \in \mathbb{R}, (x, y) \in K_2,$$

$$(56) \quad v_2(t, x, y) > 0 \text{ for every } t \in \mathbb{R}, (x, y) \in K_3,$$

$$(57) \quad v_2(t, x, y) < 0 \text{ for every } t \in \mathbb{R}, (x, y) \in K_4$$

hold. Let $\xi \in \mathcal{C}([\zeta, \eta], K)$ be such that

$$(58) \quad \xi(\zeta) \in K_1,$$

$$(59) \quad \xi(\eta) \in K_2$$

hold. Then there exist $\mu, \nu \in \mathbb{R}$, $\zeta < \mu < \nu < \eta$ such that

$$\varphi_{(a, t)}(\xi(p)) \in K \text{ for every } t \in [0, b - a], p \in [\mu, \nu],$$

$$\varphi_{(a, b-a)}(\xi(\mu)) \in K_1,$$

$$\varphi_{(a, b-a)}(\xi(\nu)) \in K_2$$

hold.

Proof. Let $\lambda > 0$. Write $K^\lambda = [\alpha - \lambda, \beta + \lambda] \times [\gamma, \delta] \subset \mathbb{R}^2$ and

$$K_1^\lambda = \{(x, y) \in K^\lambda : x = \alpha - \lambda\},$$

$$K_2^\lambda = \{(x, y) \in K^\lambda : x = \beta + \lambda\},$$

$$K_3^\lambda = \{(x, y) \in K^\lambda : y = \gamma\},$$

$$K_4^\lambda = \{(x, y) \in K^\lambda : y = \delta\}.$$

By the continuity of v and compactness of $[a, b] \times K$, there exists $\lambda > 0$ such that the qualitative behaviour of v on the ∂K^λ is the same as on the ∂K i.e. the vector field points inwards on K_3^λ and K_4^λ and points outwards on K_1^λ and K_2^λ (the inequalities analogous to (54)–(57) hold).

By the continuity of φ , compactness of K and the fact that the vector field points on K_3^λ and K_4^λ inward K^λ , there exists $\rho > 0$ such that for every $\tau \in [a, b]$ the condition

$$(60) \quad \varphi_{(\tau, t)}(K) \subset K^\lambda$$

holds for every $t \in [0, \rho]$.

Let us fix $\tau \in [a, b]$. Since the interval $[a, b]$ can be divided into finitely many intervals which lengths are not greater than ρ , it is enough to prove that there exist $\mu, \nu \in \mathbb{R}$, $\zeta < \mu < \nu < \eta$ such that

$$(61) \quad \varphi_{(\tau, t)}(\xi(p)) \in K \text{ for every } t \in [0, \rho], p \in [\mu, \nu],$$

$$(62) \quad \varphi_{(\tau, \rho)}(\xi(\mu)) \in K_1,$$

$$(63) \quad \varphi_{(\tau, \rho)}(\xi(\nu)) \in K_2$$

hold.

Write

$$J_1 = \{(x, y) \in \mathbb{R}^2 : x \in [\alpha - \lambda, \alpha]\},$$

$$J_2 = \{(x, y) \in \mathbb{R}^2 : x \in [\beta, \beta + \lambda]\}.$$

Since, by (60), (58) and (59), $\varphi_{(\tau, \rho)}(\xi(\zeta)) \in J_1$ and $\varphi_{(\tau, \rho)}(\xi(\eta)) \in J_2$, there are points $p_1, p_2 \in (\zeta, \eta)$ such that $\varphi_{(\tau, \rho)}(\xi(p_1)) \in K_1$ and $\varphi_{(\tau, \rho)}(\xi(p_2)) \in K_2$.

We set

$$(64) \quad \mu = \sup \{p \in [\zeta, \eta] : \varphi_{(\tau, \rho)}(\xi(p)) \in K_1\},$$

$$(65) \quad \nu = \inf \{p \in [\mu, \eta] : \varphi_{(\tau, \rho)}(\xi(p)) \in K_2\}.$$

We claim that conditions (61)–(63) hold.

Indeed, (62) and (63) hold by the continuity of φ , ξ , compactness of K_1 and K_2 , respectively.

To obtain a contradiction, we assume that (61) does not hold. Then there exists $p \in (\mu, \nu)$ such that either $\varphi_{(\tau, t)}(\xi(p)) \in J_1 \setminus K$ or $\varphi_{(\tau, t)}(\xi(p)) \in J_2 \setminus K$ for some $t \in (0, \rho)$.

Without losing of generality we may assume that $\varphi_{(\tau, t)}(\xi(p)) \in J_1 \setminus K$ holds. Since the part of trajectory of $\xi(p)$ cannot enter K and must stay in K^ε for times from interval $[t, \rho]$, it must stay in J_1 . So $\varphi_{(\tau, \rho)}(\xi(p)) \in J_1 \setminus K_1$. The first coordinate of $\varphi_{(\tau, \rho)}(\xi(p))$ is lower than α and the first coordinate of $\varphi_{(\tau, \rho)}(\xi(\nu))$ is equal to β , so there exists $q \in (p, \nu)$ such that the first coordinate of $\varphi_{(\tau, \rho)}(q)$ is equal to α . But it means that $\varphi_{(\tau, \rho)}(q) \in K_1$. Since $q > \mu$ we obtain the desired contradiction. \square

Lemma 14. *Let (3), (4) and (15) be satisfied. The local flow ϕ on \mathbb{R} generated by the equation*

$$(66) \quad \dot{x} = N - R \cos \beta + \frac{R}{\cos \beta} x^2$$

is given by

$$(67) \quad \phi(t, x, y) = \sqrt{\cos^2 \beta - \frac{N}{R} \cos \beta} \cdot \left(\frac{2}{1 - \frac{x\sqrt{R} - \sqrt{R \cos^2 \beta - N \cos \beta}}{x\sqrt{R} + \sqrt{R \cos^2 \beta - N \cos \beta}} \exp\left(2t\sqrt{R^2 - \frac{NR}{\cos \beta}}\right)} - 1 \right).$$

Proof. The proof is a matter of straightforward computation and is left to the reader. \square

Lemma 15. *The equation (26) holds.*

Proof. We use the notation from Lemma 10.

Let us assume that $f \equiv 0$.

Since the vector field u has a dominating term $2R\overline{w}$ inside the set $\mathbb{R} \times w(U)$ the qualitative behaviour of u is the same as $\mathbb{R} \times w(U)$ (saddle node). So the only solution staying in $w(U)$ for all times is the trivial one.

When $f \equiv 0$ is no longer valid then the trivial solution of (28) continues to the periodic one κ . By (4) and (5), it can be shown that $|\Re[\kappa]| < 0.0051$ and $|\Im[\kappa]| < 0.0051$.

Now by the change of variables $y = q - \kappa(t)$ the equation (28) has the form

$$(68) \quad \dot{y} = m(t, y) = 2R\overline{y} + Re^{-i\frac{t}{2}}\overline{y}^2 + 2Re^{-i\frac{t}{2}}\overline{y}\kappa - \frac{i}{2}y + e^{-i\frac{t}{2}} \left[f(t, (y + \kappa)e^{i\frac{t}{2}} + 1) - f(t, \kappa e^{i\frac{t}{2}} + 1) \right].$$

The dominating term of the vector field m is $2R\overline{y}$ so, by (6), situation is qualitatively the same as in the case of $f \equiv 0$. \square

Lemma 16. *Let (X, d) , (Y, ρ) be compact metric spaces, $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$ be homeomorphisms and $\Phi \in \mathcal{C}(X, Y)$ be a semiconjugacy between f and g . Let $y_1, y_2 \in \text{Per}(g)$ be such that*

$$(69) \quad \Phi^{-1}(\{y_i\}) = \{o_i\}$$

holds for $i \in \{1, 2\}$ where $\{o_1, o_2\} \subset \text{Per}(f)$. Let $y \in Y$ be such that $\alpha_g(y) = \text{Orb}(y_1, g)$ and $\omega_g(y) = \text{Orb}(y_2, g)$ hold. Then every point $o \in \Phi^{-1}(\{y\})$ satisfies $\alpha_f(o) = \text{Orb}(o_1, f)$ and $\omega_f(o) = \text{Orb}(o_2, f)$.

Proof. I. Let us find $y \in Y$ such that $\alpha_g(y) = \text{Orb}(y_1, g)$ holds and $o \in \Phi^{-1}(\{y\})$. We show that $\alpha_f(o) = \text{Orb}(o_1, f)$.

Let $n \in \mathbb{N}$ be period of y_1 . It is easy to see, that by (69), point o_1 is also n -periodic and

$$(70) \quad \Phi^{-1}(\text{Orb}(y_1, g)) = \text{Orb}(o_1, f)$$

holds.

To obtain a contradiction, let us assume that there exists $p \in \alpha_f(o)$ such that $p \notin \text{Orb}(o_1, f)$. It means that there exists sequence $\{k_j\}_{j \in \mathbb{N}} \subset \mathbb{Z} \setminus \mathbb{N}$ such that

$\lim_{j \rightarrow \infty} f^{k_j}(o) = p$ holds. But

$$\begin{aligned} \Phi(p) &= \Phi\left(\lim_{j \rightarrow \infty} f^{k_j}(o)\right) = \lim_{j \rightarrow \infty} \Phi(f^{k_j}(o)) = \lim_{j \rightarrow \infty} (g^{k_j} \circ \Phi)(o) \\ &= \lim_{j \rightarrow \infty} g^{k_j}(y) \in \alpha_g(y) = \text{Orb}(y_1, g) \end{aligned}$$

which contradict (70). Finally, $\alpha_f(o) \subset \text{Orb}(o_1, f)$, which immediately gives $\alpha_f(o) = \text{Orb}(o_1, f)$.

II. Let us now find $y \in Y$ such that $\omega_g(y) = \text{Orb}(y_2, g)$ holds and $o \in \Phi^{-1}(\{y\})$. To finish the proof it is enough to show that $\omega_f(o) = \text{Orb}(o_1, f)$. The proof is similar to the one from part I. \square

4. FURTHER REMARKS

Symmetry (16) is not essential in our investigations. It only simplifies calculations.

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