Two applications of polylog functions and Euler sums

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Abstract

Let $I(n) := \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx$. In this paper, we show that $I(n) = \sum_0^\infty \frac{I_i}{n^i}, n \to \infty$ and we compute $I_i, i = 0..5$, obtained by polylog functions and Euler sums. As a corollary, we obtain explicit expressions for some integrals involving functions $u^i, exp(-u), (1+exp(-u))^j, ln(1+exp(-u))^k$. As another asymptotic result, let $S_0(z) := \frac{Li_m(1)}{Li_m(1)-Li_m(z)}$, where $Li_m(z)$ is the polylog function. We provide the asymptotic behaviour of $S_n, n \to \infty$ where $S_n := [z^n]S_0(z)$. This paper fits within the framework of analytic combinatorics.

Keywords: polylog functions, Euler sums, asymptotics, analytic combinatorics

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1 Introduction

Some time ago, the following question was circulating among the Mathematical problems aficionados: let

$$I(n) := \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx,$$

what are $I_0 := \lim I(n), I_2 := \lim n^2(I(n) - I_0), n \to \infty$? I found it interesting to look at a deeper asymptotic analysis of $I(n)$ and found actually that, asymptotically,

$$I(n) = \sum_0^\infty \frac{I_i}{n^i}, n \to \infty,$$

where I_i are curiously obtained by polylog functions and Euler sums. In this paper we compute $I_i, i = 0..5$. As a corollary, we obtain explicit expressions for some integrals involving functions $u^i, exp(-u), (1 + exp(-u))^j, ln(1 + exp(-u))^k$. About polylog functions, see de Doelder, [2], Apostol, [1], Lewin, [6], and about Euler sums, see Flajolet, Salvy, [3], Xu, [7].

Another problem arose in some work in progress on dynamical systems by Gómez-Aiza and Ward [5]. Ward asked the following question: the polylog function is defined as

$$Li_m(z) := \sum_1^\infty \frac{z^n}{n^m}.$$

Set

$$S_0(z) := \frac{Li_m(1)}{Li_m(1) - Li_m(z)}$$

and

$$S_n := [z^n]S_0(z).$$

What is the asymptotic behaviour of $S_n, n \to \infty$? In this paper, we provide the asymptotics of $S_n, m = 3, 4$, up to the $1/n^3$ term. Next terms can be mechanically computed.

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2 A first analysis of I_n

We have

$$I(n) = 2\int_0^{1/2} [x^n + (1-x)^n]^{\frac{1}{n}} dx = 2\int_0^{1/2} (1-x)\left[1 + \left(\frac{x}{1-x}\right)^n\right]^{\frac{1}{n}} dx,$$

and

$$0 \le \frac{x}{1-x} \le 1, \text{ let}$$
$$F(n) := \left[1 + \left(\frac{x}{1-x}\right)^n\right]^{\frac{1}{n}} \sim \exp\left[\left(\frac{x}{1-x}\right)^n \Big/ n\right],$$
$$\left(\frac{x}{1-x}\right)^n \Big/ n \to 0, n \to \infty, \text{ exponentially if } x < 1/2, \text{ as } 1/n, \text{ if } x = 1/2.$$

Hence the asymptotic behaviour of $I(n)$ is related to the behaviour of $F(n)$ in the neighbourhood of $x = 1/2$. We set $x = 1/2 - y$ and get

$$I_0 = 2\int_0^{1/2} (1-x)dx = \frac{3}{4}.$$

We now expand I_n up to the $1/n^5$ term.

$$\begin{aligned}
I(n) &= 2\int_0^{1/2} \left(\frac{1}{2} + y\right)\left[1 + \left(\frac{1-2y}{1+2y}\right)^n\right]^{\frac{1}{n}} dy \\
&= 2\int_0^{1/2} \left(\frac{1}{2} + y\right)\left[1 + \left(1 - 4y + 8y^2 - 16y^3 + 32y^4 + \mathcal{O}(y^5)\right)^n\right]^{\frac{1}{n}} dy, \text{ and with } y = \frac{u}{4n}, \\
&= \frac{2}{4n}\int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right)\left[1 + \left(1 - u/n + 1/2u^2/n^2 - 1/4u^3/n^3 + 1/8u^4/n^4 + \mathcal{O}(u^5/n^5)\right)^n\right]^{\frac{1}{n}} du \\
&= \frac{2}{4n}\int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right)\left[1 + \exp\left(-u - 1/12u^3/n^2 + \mathcal{O}(u^5/n^4)\right)\right]^{\frac{1}{n}} du \\
&= \frac{2}{4n}\int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right)\left[1 + \exp(-u) - 1/12\exp(-u)u^3/n^2 + \exp(-u)\mathcal{O}(u^5/n^4)\right]^{\frac{1}{n}} du \\
&= \frac{2}{4n}\int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right)\exp\left[\ln(1+\exp(-u))/n - 1/12\exp(-u)u^3/((1+\exp(-u))n^3) + \exp(-u)\mathcal{O}(u^5/n^5)\right] du \\
&= \frac{2}{4n}\int_0^{2n} \left(\frac{1}{2} + \frac{u}{4n}\right)\Big[1 + \ln(1+\exp(-u))/n + 1/2\ln(1+\exp(-u))^2/n^2 \\
&+ 1/12\Big[-\exp(-u)u^3/(1+\exp(-u)) + 2\ln(1+\exp(-u))^3\Big]/n^3 \\
&+ 1/24\ln(1+\exp(-u))\Big[-2\exp(-u)u^3/(1+\exp(-u)) + \ln(1+\exp(-u))^3\Big]/n^4 \\
&+ \exp(-u)\mathcal{O}(u^5/n^5)\Big] du
\end{aligned}$$

$$= \frac{3}{4} + \int_0^\infty \Bigg[1/4 \ln(1+\exp(-u))/n^2 + \Big[1/8 u \ln(1+\exp(-u)) + 1/8 \ln(1+\exp(-u))^2 \Big] /n^3$$

$$+ \Bigg[1/16 u \ln(1+\exp(-u))^2 + 1/48 \Big[-\exp(-u)u^3/(1+\exp(-u)) + 2\ln(1+\exp(-u))^3 \Big] \Bigg] /n^4$$

$$+ 1/96 \Bigg[-\exp(-u)u^4/(1+\exp(-u)) + 2u\ln(1+\exp(-u))^3$$

$$- 2\ln(1+\exp(-u))\exp(-u)u^3/(1+\exp(-u)) + \ln(1+\exp(-u))^4 \Bigg] /n^5$$

$$+ \exp(-u)\mathcal{O}(u^6/n^6) \Bigg] du.$$

We immediately recover I_0. The computation of $I_i, i \geq 1$ is detailed in the next sections.

3 Computation of I_1, I_2, I_3

We have

$$I_1 = 0,$$

$$I_2 = \int_0^\infty 1/4 \ln(1+\exp(-u))du = 1/4 \sum_1^\infty \frac{(-1)^{i+1}}{i^2} = \frac{\pi^2}{48},$$

$$I_3 = \int_0^\infty [1/8 u \ln(1+\exp(-u)) + 1/8 \ln(1+\exp(-u))^2] du = 11/32\zeta(3) + 1/8 I \ln(2)^2 \pi + 1/4 \ln(2) Li_2(2) - 1/4 Li_3(2),$$

where the polylog function is defined by

$$Li_n(z) = \sum_1^\infty \frac{z^k}{k^n}.$$

But we know that

$$Li_n(z) = -(-1)^n Li_n(1/z) - \frac{(2\pi I)^n}{n!} B_n\left(\frac{1}{2} + \frac{\ln(-z)}{2\pi I}\right), z \notin [0,1],$$

where $B_n(x)$ is the nth Bernoulli polynomial, and

$Li_2(1/2) = 1/12\pi^2 - 1/2\ln(2)^2,$ hence $Li_2(2) = 1/4\pi^2 - I\pi\ln(2),$
$Li_3(1/2) = 7/8\zeta(3) - 1/12\pi^2\ln(2) + 1/6\ln(2)^3,$ hence $Li_3(2) = 7/8\zeta(3) + 1/4\pi^2\ln(2) - 1/2 I\pi\ln(2)^2.$

The values of $Li_k(1/2), k \geq 4$ are not known to be related to classical constants.

This leads to

$$I_3 = \frac{\zeta(3)}{8}.$$

Another, more elegant, way to compute I_3 is to turn to Euler sums. Following Flajolet, Salvy, [3], we have

$$S^{+-}_{p,q} := \sum_{k=1}^\infty (-1)^{k-1} \frac{H_k^{(p)}}{k^q}, H_n^{(p)} := \sum_{j=1}^n \frac{1}{j^p},$$
$$\bar{\zeta}(s) := (1-2^{1-s})\zeta(s), \bar{\zeta}(1) := \ln(2),$$

$$2S^{+-}_{1,q} = (q+1)\overline{\zeta}(q+1) - \zeta(q+1) - 2\sum_{k=1}^{q/2-1}\overline{\zeta}(k)\zeta(q+1-2k), 1+q \text{ odd },$$

$$2S^{+-}_{1,2} = \frac{5}{4}\zeta(3),$$

$$2S^{+-}_{1,4} = 59/16\zeta(5) - 1/6\pi^2\zeta(3).$$

Hence

$$\int_0^\infty u\ln(1+\exp(-u))du = \sum_1^\infty \frac{(-1)^{i+1}}{i^3} = \frac{3}{4}\zeta(3),$$

$$\int_0^\infty \ln(1+\exp(-u))^2 du = \sum_1^\infty\sum_1^\infty \frac{(-1)^{i+j}}{ij(i+j)} = \sum_{k=2}^\infty\sum_{i=1}^{k-1}\frac{(-1)^k}{ki(k-i)} = \sum_{k=2}^\infty \frac{(-1)^k}{k^2}2H_{k-1}$$
$$= 2\sum_{k=1}^\infty \frac{(-1)^k}{k^2}H_k - 2\sum_{k=1}^\infty \frac{(-1)^k}{k^3} = -2S^{+-}_{1,2} + \frac{3}{2}\zeta(3) = \frac{\zeta(3)}{4}.$$

We immediately recover I_3.

Another similar sum will be used in the next section: for $p+q$ odd,

$$S^{+-}_{p,q} := \Bigg[(1-(-1)^p)\zeta(p)\overline{\zeta}(q) + \overline{\zeta}(p+q) + 2\sum_{k=0}^{\lfloor p/2\rfloor}\binom{q+p-2k-1}{q-1}(-1)^{p-2k+1}\overline{\zeta}(q+p-2k)\overline{\zeta}(2k)$$

$$+2(-1)^p\sum_0^{\lfloor q/2\rfloor}\binom{p+q-2k-1}{p-1}\zeta(p+q-2k)\overline{\zeta}(2k)\Bigg]\Bigg/2\ ,$$

$$S^{+-}_{2,3} = -11/32\zeta(5) + 5/48\zeta(3)\pi^2.$$

4 Computation of I_4

Now we have

$$S_1 := \int_0^\infty \frac{u^3e^{-u}}{1+e^{-u}}du = \frac{7\pi^4}{120},$$

$$S_2 := \int_0^\infty \ln(1+\exp(-u))^3 du = \ln(2)^3\pi I + 3\ln(2)^2 Li_2(2) - 6\ln(2)Li_3(2) + 6Li_4(2) - 1/15\pi^4$$
$$= 1/4\pi^2\ln(2)^2 - 21/4\ln(2)\zeta(3) - 6Li_4(1/2) + 1/15\pi^4 - 1/4\ln(2)^4,$$

$$S_3 := \int_0^\infty u\ln(1+\exp(-u))^2du = \sum_{k=2}^\infty\frac{(-1)^k}{k^3}2H_{k-1} = 2\sum_{k=1}^\infty\frac{(-1)^k}{k^3}H_k - 2\sum_{k=1}^\infty\frac{(-1)^k}{k^4} = -2S^{+-}_{1,3} - 2\left(-\frac{7}{720}\pi^4\right),$$

$S^{+-}_{1,3} = -2Li_4(1/2) + 11/4\zeta(4) + 1/2\zeta(2)\ln(2)^2 - 1/12\ln(2)^4 - 7/4\zeta(3)\ln(2)$, this is μ_1 in [3],

$$S_3 = 4Li_4(1/2) - 1/24\pi^4 - 1/6\pi^2\ln(2)^2 + 1/6\ln(2)^4 + 7/2\ln(2)\zeta(3).$$

Hence

$$I_4 := \frac{1}{48}\left[-S_1 + 2S_2 + 3S_3\right] = -\frac{\pi^4}{960}.$$

5 Computation of I_5

We compute

$$S_4 := \int_0^\infty \frac{u^4 e^{-u}}{1+e^{-u}} du = 45/2\zeta(5).$$

Now we turn to S_5, which is the most intricate case of our integral expressions:

$$\begin{aligned}
S_5 := \int_0^\infty u \ln(1+\exp(-u))^3 du &= \sum_{i=1}^\infty \sum_{j=1}^\infty \sum_{\ell=1}^\infty \frac{(-1)^{i+j+\ell+1}}{ij\ell(i+j+\ell)^2} = \sum_{k=3}^\infty \frac{(-1)^{k+1}}{k^2} \sum_{v=2}^{k-1} \frac{1}{k-v} \sum_{i=1}^{v-1} \frac{1}{i(v-i)} \\
&= \sum_{v=2}^\infty 2\frac{H_{v-1}}{v} \sum_{k=v+1}^\infty \frac{(-1)^{k+1}}{k^2(k-v)} \\
&= 2\sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^2} \sum_{j=1}^{k-1} \frac{H_{j-1}}{j(k-j)} \\
&= 2\sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^3} \sum_{j=1}^{k-1} H_{j-1}\Big[\frac{1}{j} + \frac{1}{k-j}\Big] \\
&= \sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^3}\Big[H_{k-1}^2 - H_{k-1}^{(2)} + 2H_{k-1}^2 - 2H_{k-1}^{(2)}\Big] \\
&= 3\sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^3}\Big[H_{k-1}^2 - H_{k-1}^{(2)}\Big].
\end{aligned}$$

But

$$\begin{aligned}
H_{n-1}^2/n^3 &= [H_n^2 - 2H_n/n + 1/n^2]/n^3, \\
H_{n-1}^{(2)}/n^3 &= [H_n^{(2)} - 1/n^2]/n^3.
\end{aligned}$$

Hence

$$\begin{aligned}
S_5 &= -3(T_1+T_2), \text{ with} \\
T_2 &= -\Big[-S_{2,3}^{+-} + 15/16\zeta(5)\Big] = 5/48\zeta(3)\pi^2 - \frac{41}{32}\zeta(5), \\
T_1 &= T_3 + 2S_{1,4}^{+-} - 15/16\zeta(5), \\
T_3 &= \sum_{k=1}^\infty (-1)^k \frac{H_k^2}{k^3} \\
&= -(4Li_5(1/2) + 4\ln(2)Li_4(1/2) + 2/15\ln(2)^5 + 7/4\zeta(3)\ln(2)^2 \\
&- 19/32\zeta(5) - 2/3\zeta(2)\ln(2)^3 - 11/8\zeta(2)\zeta(3)), \text{ see [7], where many recent references can be found, so} \\
T_1 &= -4Li_5(1/2) - 4\ln(2)Li_4(1/2) - 2/15\ln(2)^5 - 7/4\zeta(3)\ln(2)^2 + \frac{107}{32}\zeta(5) + 1/9\pi^2\ln(2)^3 + 1/16\zeta(3)\pi^2,
\end{aligned}$$

and finally

$$\begin{aligned}
S_5 &= 12Li_5 1/2) + 12\ln(2)Li_4(1/2) + 2/5\ln(2)^5 + 21/4\zeta(3)\ln(2)^2 - \frac{99}{16}\zeta(5) - 1/3\pi^2 ln(2)^3 - 1/2\zeta(3)\pi^2, \\
S_6 &:= \int_0^\infty \frac{u^3 e^{-u}\ln(1+\exp(-u))}{1+e^{-u}} du = \int_0^\infty u^3 \sum_{k=2}^\infty e^{-uk}(-1)^k \sum_{i=1}^{k-1}\frac{1}{i} du = 3!\sum_{k=2}^\infty \frac{1}{k^4}(-1)^k H_{k-1} \\
&= 3!\left[\sum_{k=1}^\infty \frac{1}{k^4}(-1)^k H_k - \sum_{k=1}^\infty \frac{1}{k^5}(-1)^k\right] = 3!\left[-S_{1,4}^{+-} + \frac{15}{16}\zeta(5)\right] = -87/16\zeta(5) + 1/2\pi^2\zeta(3),
\end{aligned}$$

$$\begin{aligned}S_7 &:= \int_0^\infty \ln(1+\exp(-u))^4 du \\ &= 2/3\pi^2\ln(2)^3 - 21/2\ln(2)^2\zeta(3) - 24\ln(2)Li_4(1/2) - 4/5\ln(2)^5 - 24Li_5(1/2) + 24\zeta(5).\end{aligned}$$

Hence

$$I_5 = \frac{1}{96}\left[-S_4 + 2S_5 - 2S_6 + S_7\right] = -1/48\zeta(3)\pi^2.$$

Let us summarize our results in the following theorem:

Theorem 5.1 *Let*

$$I(n) := \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx.$$

We have

$$I(n) = \sum_0^\infty \frac{I_i}{n^i}, n \to \infty, I_0 = \frac{3}{4}, I_1 = 0, I_2 = \frac{\pi^2}{48}, I_3 = \frac{\zeta(3)}{8}, I_4 = -\frac{\pi^4}{960}, I_5 = -1/48\zeta(3)\pi^2.$$

We also have the following corollary (many other similar results can be found in Xu [7])

Corollary 5.2

$$\int_0^\infty 1/4\ln(1+\exp(-u))du = \frac{\pi^2}{48},$$

$$\int_0^\infty u\ln(1+\exp(-u))du = \frac{3}{4}\zeta(3),$$

$$\int_0^\infty \ln(1+\exp(-u))^2 du = \frac{\zeta(3)}{4},$$

$$\int_0^\infty \frac{u^3e^{-u}}{1+e^{-u}}du = \frac{7\pi^4}{120},$$

$$\int_0^\infty \ln(1+\exp(-u))^3 du = 1/4\pi^2\ln(2)^2 - 21/4\ln(2)\zeta(3) - 6Li_4(1/2) + 1/15\pi^4 - 1/4\ln(2)^4,$$

$$\int_0^\infty u\ln(1+\exp(-u))^2 du = 4Li_4(1/2) - 1/24\pi^4 - 1/6\pi^2\ln(2)^2 + 1/6\ln(2)^4 + 7/2\ln(2)\zeta(3),$$

$$\int_0^\infty \frac{u^4e^{-u}}{1+e^{-u}}du = 45/2\zeta(5),$$

$$\begin{aligned}&\int_0^\infty u\ln(1+\exp(-u))^3 du \\ &= 12Li_51/2) + 12\ln(2)Li_4(1/2) + 2/5\ln(2)^5 + 21/4\zeta(3)\ln(2)^2 - \frac{99}{16}\zeta(5) - 1/3\pi^2 ln(2)^3 - 1/2\zeta(3)\pi^2,\end{aligned}$$

$$\int_0^\infty \frac{u^3e^{-u}\ln(1+\exp(-u))}{1+e^{-u}}du = -87/16\zeta(5) + 1/2\pi^2\zeta(3),$$

$$\begin{aligned}&\int_0^\infty \ln(1+\exp(-u))^4 du \\ &= 2/3\pi^2\ln(2)^3 - 21/2\ln(2)^2\zeta(3) - 24\ln(2)Li_4(1/2) - 4/5\ln(2)^5 - 24Li_5(1/2) + 24\zeta(5).\end{aligned}$$

We have two Open problem:

Open problem 1: how to explain the relatively simple I_i expressions?

Open problem 2: can we find 'easily' similar computations for $I_i, i \geq 6$?

Let

$$\begin{aligned}I_{n,2} &:= I_0 + I_2/n^2, \\ I_{n,3} &:= I_0 + I_2/n^2 + I_3/n^3, \\ I_{n,4} &:= I_0 + I_2/n^2 + I_3/n^3 + I_4/n^4.\end{aligned}$$

To check the quality of our asymptotics, we display, in Figure 1, $I_0, I(n), I_{n,2}, I_{n,3}, I_{n,4}$.

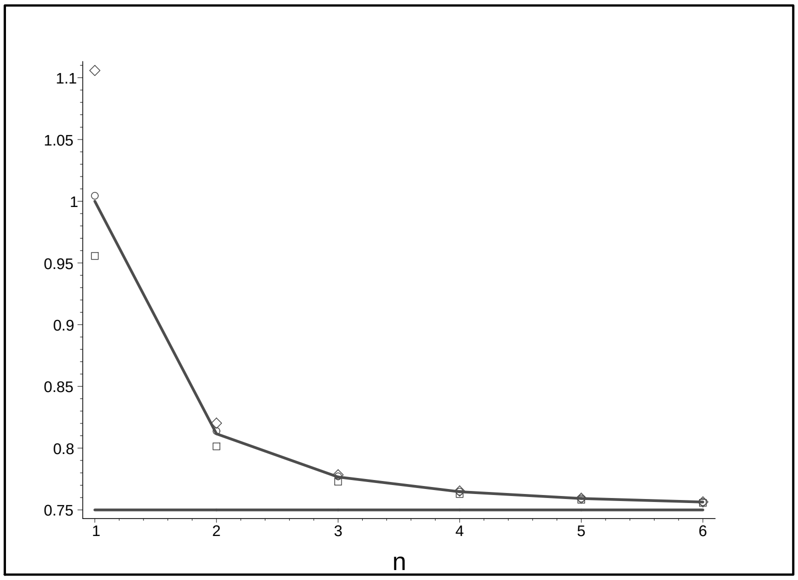

Figure 1: $I(n)$(line),I_0(line),$I_{n,2}$(box),$I_{n,3}$(diamond),$I_{n,4}$(circle)

6 A first analysis of S_n

We know that

$$Li_m(z) = \frac{(-1)^m}{(m-1)!} w^{m-1}(\ln(w) - H_{m-1}) + \sum_{j \neq m-1, j \geq 0} \frac{(-1)^j}{j!} \zeta(m-j) w^j, \text{ see [4],VI.20, with} \tag{1}$$

$$w := -\ln(z).$$

The singularity $z = 1$ in $S_0(z)$ leads to the desired expansion of S_n. Set

$$\begin{aligned}
L &:= \ln\left(\frac{1}{1-z}\right) = Li_1(z), \\
L_{k,n} &:= [z^n]L^k, \text{ with} \\
L_{1,n} &= \frac{1}{n}. \\
&\text{Let} \\
\varepsilon &:= 1 - z, \\
S(\varepsilon, L) &:= \text{ expansion of } S_0(1-\varepsilon) \text{ w.r.t. } \varepsilon, \\
&\quad \text{we compute successively} \\
D_{i,j} &:= [\varepsilon^i L^j] S(\varepsilon, L), \text{ which depends on } m, \\
G_{i,j,n} &:= [z^n]\varepsilon^i L^j, \text{ which is independent of } m, \\
T_{i,j,n} &= D_{i,j} G_{i,j,n} = [z^n \varepsilon^i L^j] S(\varepsilon, L), \\
S_n &:= [z^n] S_0(z),
\end{aligned}$$

$$T_n := \text{ asymptotics of } S_n = \sum_i \sum_j T_{i,j,n}, n \to \infty.$$

We also define

$$C_{n,k} := \text{ asymptotics of } S_n \text{ up to the } 1/n^k \text{ term.}$$

In this paper, we will compute $C_{n,k}, k = 0..3, m = 3, 4$, but as we will see, more terms can be mechanically obtained. We will also show some graphs of $C_{n,k}$.

7 Some asymptotics for $L_{k,n}$

Set

$$H := \Gamma(n+\alpha)/(\Gamma(\alpha)\Gamma(n+1)), \text{ see [1],VI.7,}$$

$$\frac{\partial^2 H(\alpha)}{\partial \alpha^2} = [\psi(1, n+\alpha) + \psi(n+\alpha)^2 - 2\psi(n+\alpha)\psi(\alpha) + \psi(\alpha)^2 - \psi(1,\alpha)]\Gamma(n+\alpha)/(\Gamma(\alpha)\Gamma(n+1)),$$

where $\psi(n, x)$ is the nth polygamma function, which is the nth derivative of the digamma function, we have

$$L_{2,n} = \lim_{\alpha \to 0} \frac{\partial^2 H(\alpha)}{\partial \alpha^2} = (2\psi(n) + 2\gamma)/n,$$

$$L_{2,n} = (2\ln(n) + 2\gamma)/n - 1/n^2 - 1/(6n^3) + 1/(60n^5) + \mathcal{O}(1/n^6), \text{ see [1], Figure VI.5 for the first terms,}$$

$$\begin{aligned}\frac{\partial^3 H(\alpha)}{\partial \alpha^3} &= [\psi(2, n+\alpha) + 3\psi(1, n+\alpha)\psi(n+\alpha) - 3\psi(1, n+\alpha)\psi(\alpha) \\ &+ \psi(n+\alpha)^3 - 3\psi(n+\alpha)^2\psi(\alpha) + 3\psi(n+\alpha)\psi(\alpha)^2 \\ &- 3\psi(n+\alpha)\psi(1,\alpha) - \psi(\alpha)^3 + 3\psi(\alpha)\psi(1,\alpha) - \psi(2,\alpha)]\Gamma(n+\alpha)/(\Gamma(\alpha)\Gamma(n+1)),\end{aligned}$$

$$L_{3,n} = \lim_{\alpha \to 0} \frac{\partial^3 H(\alpha)}{\partial \alpha^3} = 1/2(6\psi(1,n) + 12\psi(n)\gamma - \pi^2 + 6\gamma^2 + 6\psi(n)^2)/n,$$

$$\begin{aligned}L_{3,n} &= (3\ln(n)^2 + 6\ln(n)\gamma - 1/2\pi^2 + 3\gamma^2)/n + (3 - 3\ln(n) - 3\gamma)/n^2 + (-1/2\gamma + 9/4 - 1/2\ln(n))/n^3 \\ &+ 3/(4n^4) + (1/20\gamma + 1/48 + 1/20\ln(n))/n^5 + \mathcal{O}(1/n^6).\end{aligned}$$

8 The case $m = 3, 4$

We have, for $m = 3$, by (1),

$$Li_3(z) = -1/2w^2(\ln(w) - 3/2) + \zeta(3) - 1/6\pi^2 w + 1/12w^3 - 1/288w^4 + 1/86400w^6 - 1/10160640w^8 + \mathcal{O}(w^9),$$

$$S_0(z) = \zeta(3)/(1/2w^2(\ln(w) - 3/2) + 1/6\pi^2 w - 1/12w^3 + 1/288w^4 - 1/86400w^6 + 1/10160640w^8 + \mathcal{O}(w^9)),$$

we have the expansions

$$w = \varepsilon + 1/2\varepsilon^2 + 1/3\varepsilon^3 + 1/4\varepsilon^4 + \mathcal{O}(\varepsilon^5),$$

$$\ln(w) = -L + 1/2\varepsilon + 5/24\varepsilon^2 + 1/8\varepsilon^3 + \mathcal{O}(\varepsilon^4).$$

Hence

$$\begin{aligned}S(\varepsilon, L) &= 6\zeta(3)/(\pi^2\varepsilon) + 1/8\zeta(3)(-24\pi^6 + 144\pi^4 L + 216\pi^4)/\pi^8 \\ &+ 1/8\zeta(3)(-4\pi^6 - 48\pi^4 + 432\pi^2 L^2 + 1296\pi^2 L + 972\pi^2)/\pi^8\varepsilon \\ &+ 1/8\zeta(3)(-2\pi^6 - 19\pi^4 + 360\pi^2 L + 216\pi^2 L^2 + 54\pi^2 + 1296L^3 + 5832L^2 + 8748L + 4374)/\pi^8\varepsilon^2 + \mathcal{O}(\varepsilon^3).\end{aligned}$$

This leads successively to

$$D_{-1,0} = 6\zeta(3)/\pi^2,$$

$$D_{0,1} = 18\zeta(3)/\pi^4,$$

$$D_{1,1} = 162\zeta(3)/\pi^6,$$
$$D_{1,2} = 54\zeta(3)/\pi^6,$$
$$T_{-1,0,n} = 6\zeta(3)/\pi^2, T_{-1,k,n} = 0, k > 0,$$
$$T_{0,1,n} = 18\zeta(3)/(\pi^4 n), T_{0,k,n} = 0, k > 1,$$

The general form of $G_{i,j}$ is given in [4], Equ. (27). The detailed computation goes as follows,

$$G_{1,1,n} = L_{1,n} - L_{1,n-1} = -1/n^2 - 1/n^3 - 1/n^4 - 1/n^5 + \mathcal{O}(1/n^6),$$
$$T_{1,1,n} = D_{1,1}G_{1,1,n} = -162\zeta(3)/(\pi^6 n^2) - 162\zeta(3)/(\pi^6 n^3) + \mathcal{O}(1/n^4),$$
$$G_{1,2,n} = L_{2,n} - L_{2,n-1} = (2 - 2\ln(n) - 2\gamma)/n^2 + (5 - 2\ln(n) - 2\gamma)/n^3$$
$$+ (43/6 - 2\ln(n) - 2\gamma)/n^4 + (55/6 - 2\ln(n) - 2\gamma)/n^5 + \mathcal{O}(1/n^6),$$
$$T_{1,2,n} = D_{1,2}G_{1,2,n} = 54(2 - 2\ln(n) - 2\gamma)\zeta(3)/(\pi^6 n^2) + 54(5 - 2\ln(n) - 2\gamma)\zeta(3)/(\pi^6 n^3) + \mathcal{O}(1/n^4),$$
$$D_{2,3} = 162\zeta(3)/\pi^8,$$
$$D_{2,2} = 27\zeta(3)(27 + \pi^2)/\pi^8,$$
$$D_{2,1} = 9/2\zeta(3)(10\pi^2 + 243)/\pi^8,$$
$$G_{2,3,n} = L_{3,n} - 2L_{3,n-1} + L_{3,n-2} = (6 - 18\gamma - \pi^2 - 18\ln(n) + 6\gamma^2 + 6\ln(n)^2 + 12\ln(n)\gamma)/n^3 + \mathcal{O}(1/n^4),$$
$$G_{2,2,n} = L_{2,n} - 2L_{2,n-1} + L_{2,n-2} = (-6 + 4\ln(n) + 4\gamma)/n^3 + \mathcal{O}(1/n^4),$$
$$G_{2,1,n} = L_{1,n} - 2L_{1,n-1} + L_{1,n-2} = 2/n^3 + \mathcal{O}(1/n^4),$$
$$T_{2,3,n} = D_{2,3}G_{2,3,n} = 162(6 - 18\gamma - \pi^2 - 18\ln(n) + 6\gamma^2 + 6\ln(n)^2 + 12\ln(n)\gamma)\zeta(3)/(n^3\pi^8) + \mathcal{O}(1/n^4),$$
$$T_{2,2,n} = D_{2,2,n}G_{2,2,n} = 27(-6 + 4\ln(n) + 4\gamma)\zeta(3)(27 + \pi^2)/(n^3\pi^8) + \mathcal{O}(1/n^4),$$
$$T_{2,1,n} = D_{2,1}G_{2,1,n} = 9\zeta(3)(10\pi^2 + 243)/(n^3\pi^8) + \mathcal{O}(1/n^4).$$

Finally

$$T_n = T_{-1,0,n} + T_{0,1,n} + \sum_i \sum_j T_{i,j,n}.$$

This leads to the following theorem:

Theorem 8.1

Let

$$S_0(z) := \frac{Li_3(1)}{Li_3(1) - Li_3(z)}, \text{ then}$$
$$S_n := [z^n]S_0(z) = 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n) + 3\zeta(3)(-18\pi^2 - 36\pi^2\ln(n) - 36\pi^2\gamma)/(\pi^8 n^2)$$
$$+ 3\zeta(3)(-42\pi^2 - 405 + 324\gamma^2 + 324\ln(n)^2 + 648\ln(n)\gamma)/(\pi^8 n^3) + \mathcal{O}(1/n^4).$$

This gives

$$C_{n,0} = 6\zeta(3)/\pi^2,$$
$$C_{n,1} = 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n),$$
$$C_{n,2} = 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n) + 3\zeta(3)(-18\pi^2 - 36\pi^2\ln(n) - 36\pi^2\gamma)/(\pi^8 n^2),$$
$$C_{n,3} = 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n) + 3\zeta(3)(-18\pi^2 - 36\pi^2\ln(n) - 36\pi^2\gamma)/(\pi^8 n^2)$$
$$+ 3\zeta(3)(-42\pi^2 - 405 + 324\gamma^2 + 324\ln(n)^2 + 648\ln(n)\gamma)/(\pi^8 n^3).$$

To check the quality of our asymptotics, we display, in Figure 2, $S_n, C_{n,0}, C_{n,1}, C_{n,2}, C_{n,3}$.

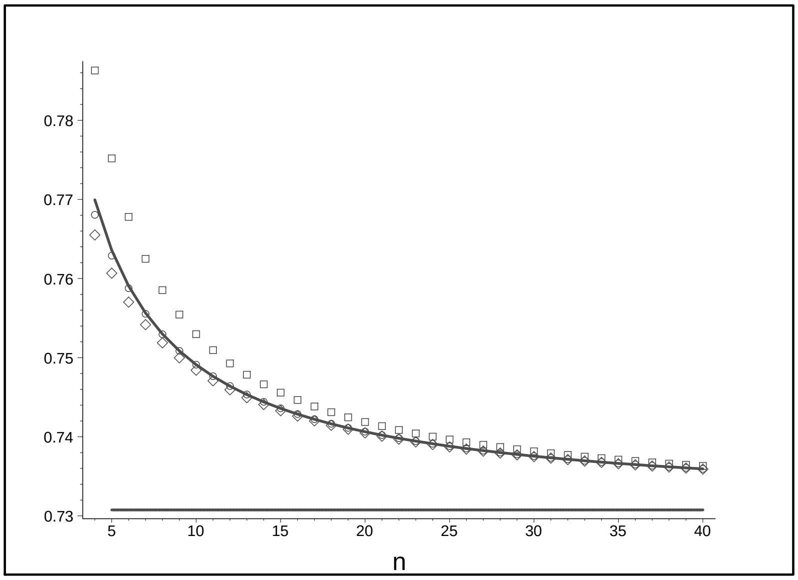

Figure 2: $m = 3$, S_n(line),$C_{n,0}$(line),$C_{n,1}$(box),$C_{n,2}$(diamond),$C_{n,3}$(circle)

The convergence of S_n to $C_{n,0}$ is rather slow: we have $C_{n,0} = 0.7307629692\ldots$ and $S_{100} = 0.7329\ldots$.

The case $m = 4$ is mechanically treated like the case $m = 3$. We obtain

$$\begin{aligned}
C_{n,0} &= 1/90\pi^4/\zeta(3),\\
T_{0,1,n} &= 0,\\
C_{n,2} &= 1/90\pi^4/\zeta(3) + 1/540\pi^4/(\zeta(3)^2n^2),\\
C_{n,3} &= 1/90\pi^4/\zeta(3) + 1/540\pi^4(\zeta(3)^2n^2) - 1/1620\pi^6/(\zeta(3)^3n^3),
\end{aligned}$$

we display, in Figure 3, $S_n, C_{n,0}, C_{n,2}, C_{n,3}$. The convergence of S_n to $C_{n,0}$ is faster.

Remarks

1. in order to expand $S_0(1-\varepsilon)$, we first expand w.r.t ε as $\varepsilon = o(1/L^k), k > 0$
2. $G_{k,j}$ starts with a $1/n^{k+1}$ term, which allows an easy expansion
3. more and more terms in the expansion of $S(\varepsilon, L)$ are needed when m increases: the first terms don't contain any L terms. For instance, for $m = 6$, only the ε^3 contains a linear L contribution, and the asymptotics of S_n starts as $1/945\pi^6/\zeta(5) + 1/18900\pi^6/(\zeta(5)^2n^4)$. More terms can be mechanically computed.

9 Acknowledgements

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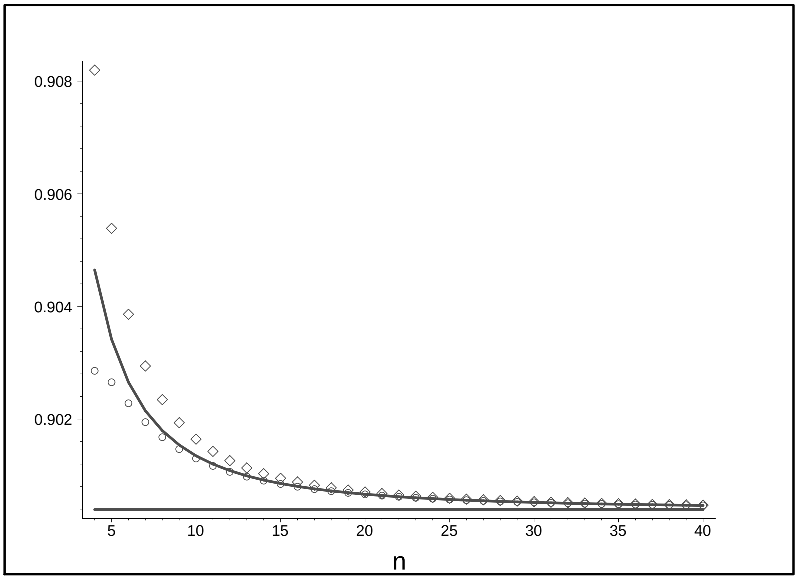

Figure 3: $m = 4$, S_n(line),$C_{n,0}$(line),$C_{n,2}$(diamond),$C_{n,3}$(circle)

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