The Cauchy problem for the Finsler heat equation

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Abstract

Let H be a norm of \mathbf{R}^N and H_0 the dual norm of H. Denote by Δ_H the Finsler-Laplace operator defined by $\Delta_H u := \operatorname{div}(H(\nabla u)\nabla_\xi H(\nabla u))$. In this paper we prove that the Finsler-Laplace operator Δ_H acts as a linear operator to H_0 -radially symmetric smooth functions. Furthermore, we obtain an optimal sufficient condition for the existence of the solution to the Cauchy problem for the Finsler heat equation

$$\partial_t u = \Delta_H u, \quad x \in \mathbf{R}^N, \quad t > 0,$$

where $N \geq 1$ and $\partial_t := \partial/\partial t$.

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1 Introduction

Let $N \geq 1$ and let $H \in C(\mathbf{R}^N) \cap C^1(\mathbf{R}^N \setminus \{0\})$ be a norm of \mathbf{R}^N , that is

$$\begin{cases}
H \ge 0 \text{ in } \mathbf{R}^N \text{ and } H(\xi) = 0 \text{ if and only if } \xi = 0, \\
H \text{ is convex in } \mathbf{R}^N, \\
H(\alpha \xi) = |\alpha| H(\xi) \text{ for } \xi \in \mathbf{R}^N \text{ and } \alpha \in \mathbf{R}.
\end{cases}$$
(1.1)

We denote by H_0 the dual norm of H defined by

$$H_0(x) := \sup_{\xi \in \mathbf{R}^N \setminus \{0\}} \frac{x \cdot \xi}{H(\xi)}.$$

Then

$$|x \cdot \xi| \le H_0(x)H(\xi), \qquad H(\xi) = \sup_{x \in \mathbf{R}^N \setminus \{0\}} \frac{x \cdot \xi}{H_0(x)}.$$
 (1.2)

For any $x \in \mathbf{R}^N$, $\xi \in \mathbf{R}^N$ and R > 0, we set

$$B_H(\xi, R) := \{ \eta \in \mathbf{R}^N : H(\eta - \xi) < R \},$$

$$B_{H_0}(x, R) := \{ y \in \mathbf{R}^N : H_0(y - x) < R \}.$$

Throughout this paper we assume that

$$B_H(0,1)$$
 is strictly convex, (1.3)

which is equivalent to $H_0 \in C^1(\mathbf{R}^N \setminus \{0\})$ (see [33, Corollary 1.7.3]).

Let Δ_H be the Finsler-Laplace operator associated with the norm H, that is

$$\Delta_H u := \operatorname{div}(\nabla_{\xi} V(\nabla u)) = \operatorname{div}(H(\nabla u)\nabla_{\xi} H(\nabla u)),$$

where $V(\xi) := H(\xi)^2/2$. We remark that $\Delta_H = \Delta$ if $H(\xi) = |\xi|$ for $\xi \in \mathbf{R}^N$. The Finsler-Laplace operator has been treated by many mathematicians from various points of view (see e.g., [4], [6], [8], [10], [11], [12], [18], [31], [32], [34], [36] and references therein). This paper is concerned with the Cauchy problem for the Finsler heat equation

$$\partial_t u = \Delta_H u, \qquad x \in \mathbf{R}^N, \quad t > 0,$$
 (1.4)

which is introduced as a gradient flow of the energy

$$I[u] := \frac{1}{2} \int_{\mathbf{R}^N} H(\nabla u)^2 \, dx.$$

Equation (1.4) is a nonlinear parabolic equation with the following nice property:

If
$$u$$
 is a solution of (1.4), then ku and $u(kx, k^2t)$ are also solutions of (1.4) for any $k \in \mathbf{R}$

(see Definition 1.1 and Section 2).

The Finsler-Laplace operator enjoys nice properties for "smooth radial" functions. To be more precise, for any function v in \mathbf{R}^N , we say that v is H_0 -radially symmetric in \mathbf{R}^N if there exists a function v^{\sharp} on $[0,\infty)$ such that

$$v(x) = v^{\sharp}(r)$$
 for $x \in \mathbf{R}^N$ with $r = H_0(x)$.

Set $v^*(x) := v^{\sharp}(|x|)$ for $x \in \mathbf{R}^N$. For $k \in \{0, 1, 2, ...\}$, we say that an H_0 -radially symmetric function v is $C_{H_0}^k$ -smooth if $v^* \in C^k(\mathbf{R}^N)$. One of main purposes of this paper is to obtain the following two nice properties of H_0 -radially symmetric $C_{H_0}^2$ -smooth functions.

Theorem 1.1 Assume (1.1) and (1.3). Let v and w be H_0 -radially symmetric $C_{H_0}^2$ -smooth functions in \mathbb{R}^N . Then

- (a) $(\Delta_H v)(x) = (\Delta v^*)(y)$ for $x \in \mathbf{R}^N$ with $y = H_0(x)$;
- (b) $\Delta_H(\alpha v + \beta w) = \alpha \Delta_H v + \beta \Delta_H w$ in \mathbf{R}^N for any $\alpha, \beta \in \mathbf{R}$.

The Finsler-Laplace operator is a nonlinear operator and often has a complicated form. Nevertheless, Theorem 1.1 implies that the Finsler-Laplace operator acts as a linear operator to H_0 -radially symmetric $C_{H_0}^2$ -smooth functions. Furthermore, Theorem 1.1 enables us to bring rich mathematical results for radially symmetric functions to the study of H_0 -radially symmetric functions. See also Section 2.

The other purpose of this paper is to obtain an optimal growth condition on initial data for the existence of solutions to the Cauchy problem for the Finsler heat equation

$$\partial_t u = \Delta_H u, \quad x \in \mathbf{R}^N, \ t > 0, \qquad u(\cdot, 0) = \mu \quad \text{in} \quad \mathbf{R}^N,$$
 (1.6)

where μ is a (signed) Radon measure in \mathbf{R}^N . The study of the growth conditions on initial data for the existence of solutions to parabolic equations is a classical subject. The growth conditions generally depend on the diffusion and the nonlinear terms in the parabolic problems (see e.g., [1]–[3], [5], [7], [15], [16], [19]–[25], [27], [35] and [37]). For the heat equation, the following holds.

(H1) Let u be a nonnegative solution of $\partial_t u = \Delta u$ in $\mathbf{R}^N \times (0,T)$, where T > 0. Then there exists a unique nonnegative Radon measure μ in \mathbf{R}^N such that

$$\lim_{t\to +0} \int_{\mathbf{R}^N} u(y,t)\phi(y)\,dy = \int_{\mathbf{R}^N} \phi(y)\,d\mu(y), \qquad \phi\in C_0(\mathbf{R}^N).$$

Furthermore, μ satisfies

$$\sup_{x \in \mathbf{R}^N} \int_{B(x,1/\sqrt{\Lambda})} e^{-\Lambda |x|^2} d\mu < \infty \quad \text{for some } \Lambda > 0.$$

(H2) Let μ be a (signed) Radon measure in \mathbf{R}^N such that

$$\sup_{x \in \mathbf{R}^N} \int_{B(x, 1/\sqrt{\Lambda})} e^{-\Lambda |x|^2} d|\mu| < \infty \quad \text{for some } \Lambda > 0.$$
 (1.7)

Then

$$u(x,t) := (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} \exp\left(-\frac{|x-y|^2}{4t}\right) d\mu(y)$$

satisfies

$$\partial_t u = \Delta u \quad \text{in} \quad \mathbf{R}^N \times (0, 1/4\Lambda), \qquad u(\cdot, 0) = \mu \quad \text{in} \quad \mathbf{R}^N.$$
 (1.8)

(H3) Let $\Lambda > 0$ and set

$$v(x,t) := (1 - 4\Lambda t)^{-\frac{N}{2}} \exp\left(\frac{\Lambda |x|^2}{1 - 4\Lambda t}\right).$$

Then v is a solution of (1.8) and it satisfies (1.7) and

$$\min_{x \in \mathbf{R}^N} v(x, t) = v(0, t) \to \infty \quad \text{as} \quad t \to \frac{1}{4\Lambda}.$$

See [3] and [35]. (See also [23] and [25]).

Consider the Cauchy problem for the porous medium equation

$$\partial_t u = \Delta(|u|^{m-1}u), \quad x \in \mathbf{R}^N, \ t > 0, \qquad u(\cdot, 0) = \mu \quad \text{in} \quad \mathbf{R}^N,$$
 (1.9)

where m > 1. Bénilan, Crandall and Pierre [7] proved that problem (1.9) possesses a solution provided that

$$\limsup_{\rho \to \infty} \rho^{-N - \frac{2}{m-1}} \int_{B(0,\rho)} d|\mu| < \infty.$$

On the other hand, in the case of $(N-2)_+/2 < m < 1$, Herrero and Pierre [20] proved that problem (1.9) possesses a solution for any $L^1_{loc}(\mathbf{R}^N)$ initial data. For similar results to parabolic p-Laplace equations and more general nonlinear parabolic equations, see e.g., [1], [15], [16], [22], [23] and [24].

We formulate a definition of solutions to (1.6).

Definition 1.1 Let u be a measurable function u in $\mathbb{R}^N \times (0,T)$, where T > 0.

(i) We say that u is a solution of (1.4) in $\mathbf{R}^N \times (0,T)$ if

$$u \in C((0,T): L^1(B_{H_0}(0,R))) \cap L^1((0,T-\epsilon): W^{1,1}(B_{H_0}(0,R)))$$
 (1.10)

for any R > 0 and $0 < \epsilon < T$ and u satisfies

$$\int_{\mathbf{R}^{N}} u(y,t)\varphi(y,t) \, dy + \int_{\tau}^{t} \int_{\mathbf{R}^{N}} \left[-u\partial_{t}\varphi + H(\nabla u)\nabla_{\xi}H(\nabla u)\nabla\varphi \right] \, dy \, ds$$

$$= \int_{\mathbf{R}^{N}} u(y,\tau)\varphi(y,\tau) \, dy$$

 $\label{eq:total_state} \textit{for all } 0 < \tau < t < T \ \textit{ and } \varphi \in C_0^\infty(\mathbf{R}^N \times (0,T)).$

(ii) Let μ be a (signed) Radon measure in \mathbb{R}^N . Then we say that u is a solution of (1.6) in $\mathbb{R}^N \times [0,T)$ if u satisfies (1.10) and

$$\int_{\mathbf{R}^{N}} u(y,t)\varphi(y,t) \, dy + \int_{0}^{t} \int_{\mathbf{R}^{N}} \left[-u\partial_{t}\varphi + H(\nabla u)\nabla_{\xi}H(\nabla u)\nabla\varphi \right] \, dy \, ds$$

$$= \int_{\mathbf{R}^{N}} \varphi(y,0) \, d\mu(y)$$

for all 0 < t < T and $\varphi \in C_0^{\infty}(\mathbf{R}^N \times [0, T))$.

Now we are ready to state the second theorem of this paper. In the following Theorem 1.2 we obtain similar results as in statements (H1), (H2) and (H3) for the Cauchy problem to the Finsler heat equation.

Theorem 1.2 Assume (1.1) and (1.3).

(i) Let u be a nonnegative solution of (1.4) in $\mathbf{R}^N \times (0,T)$ for some T > 0. Then there exists a unique nonnegative Radon measure μ in \mathbf{R}^N such that

$$\lim_{t \to +0} \int_{\mathbf{R}^N} u(y,t)\phi(y) \, dy = \int_{\mathbf{R}^N} \phi(y) \, d\mu(y), \qquad \phi \in C_0(\mathbf{R}^N). \tag{1.11}$$

Furthermore,

$$\sup_{x \in \mathbf{R}^N} \int_{B_{H_0}(x,1/\sqrt{\Lambda})} e^{-\Lambda H_0(y)^2} d\mu(y) < \infty \quad \textit{for some } \Lambda > 0.$$

(ii) Let μ be a (signed) Radon measure in \mathbf{R}^N such that

$$\sup_{x \in \mathbf{R}^N} \int_{B_{H_0}(x, 1/\sqrt{\Lambda})} e^{-\Lambda H_0(y)^2} d|\mu|(y) < \infty \quad \text{for some } \Lambda > 0.$$
 (1.12)

Then there exists a nonnegative solution u of (1.6) in $\mathbf{R}^N \times [0, S_{\Lambda})$, where $S_{\Lambda} := 1/4\Lambda$, such that

- (a) $u \in C^{1,\alpha;0,\alpha/2}(\mathbf{R}^N \times (0, S_\Lambda))$ for some $\alpha \in (0,1)$;
- (b) For any $\lambda > \Lambda$, there exists a constant C such that

$$\sup_{0 < t < S_{\lambda}} \int_{\mathbf{R}^{N}} e^{-g_{\lambda}(x,t)} |u(y,t)| \, dy \le C \sup_{x \in \mathbf{R}^{N}} \int_{B_{H_{0}}(x,1/\sqrt{\Lambda})} e^{-\Lambda H_{0}(y)^{2}} \, d|\mu|(y). \tag{1.13}$$

Here $g_{\lambda}(x,t) := \lambda H_0(x)^2/(1 - 4\lambda t)$.

Theorem 1.3 assures that S_{Λ} is the optimal maximal existence time of the solution of (1.6) under assumption (1.12).

Theorem 1.3 Let $\Lambda > 0$ and set

$$v(x,t) := (1 - 4\Lambda t)^{-\frac{N}{2}} \exp\left(\frac{\Lambda H_0(x)^2}{1 - 4\Lambda t}\right).$$

Then v is a solution of (1.4) in $\mathbf{R}^N \times [0, S_{\Lambda})$ and it satisfies (1.12) and

$$\min_{x \in \mathbf{R}^N} v(x,t) = v(0,t) \to \infty \quad as \quad t \to S_{\Lambda}.$$

We explain an idea of the proof of Theorem 1.2. Since Δ_H is a nonlinear operator, we cannot apply the standard theory for linear parabolic equations to the Finsler heat equation. On the other hand, setting

$$A(\xi) := H(\xi)\nabla_{\xi}H(\xi)$$
 for $\xi \neq 0$, $A(\xi) := 0$ for $\xi = 0$,

we have

$$A \in C(\mathbf{R}^N; \mathbf{R}^N), \quad \Delta_H u = \operatorname{div} A(\nabla u) \text{ in } \mathscr{D}',$$

 $A(\xi) \cdot \xi = H(\xi)^2 \text{ and } H_0(A(\xi)) = H(\xi) \text{ for } \xi \in \mathbf{R}^N.$ (1.14)

Since H and H_0 are equivalent to the Euclidean norm of \mathbb{R}^N , it follows from (1.14) that

$$A(\xi) \cdot \xi \ge C_1 |\xi|^2$$
 and $|A(\xi)| \le C_2 |\xi|$ for $\xi \in \mathbf{R}^N$, (1.15)

where C_1 and C_2 are positive constants. Then we can apply the arguments in [25, Theorem 1.8] to obtain assertion (i).

For the proof of assertion (ii), by property (1.5) it suffices to consider the case of $\Lambda < 1$. We introduce some new techniques to obtain uniform estimates of approximate solutions. Firstly, we construct approximate solutions $\{u_n\}$ by using a subdifferential formulation, in order to preserve structure of the Finsler-Laplace operator (see, e.g., (1.5)). Secondly, for any sufficiently small $\delta > 0$, we obtain uniformly local $L^{1+\delta}$ estimates of $\{u_n\}$ with the aid of the Besicovitch covering theorem (see Lemma 3.2). Then we modify the arguments used in [22]–[24] to obtain uniformly local L^1 estimates of the functions

$$U_n := F\left(e^{-H_0(y)^2(1+s^\ell)}|u_n(y,s)|\right),$$

where $0 < \ell < 1$ and

$$F(v) := \int_0^v \min\{\tau^{\delta}, 1\} d\tau = \begin{cases} v^{1+\delta}/(1+\delta) & \text{for } 0 \le v \le 1, \\ v - \delta/(1+\delta) & \text{for } v > 1. \end{cases}$$

Furthermore, employing property (1.5), we improve uniformly local L^1 estimates for U_n , which also asserts those for the functions $e^{-H_0(y)^2(1+s^\ell)}u_n$. Finally, carrying out a similar argument as in [22]–[24], we show the existence of solutions to (1.6) and complete the proof of assertion (ii).

The rest of this paper is organized as follows. In Section 2 we recall some basic properties of H and H_0 and prove Theorems 1.1 and 1.3. In Section 3 we obtain a priori estimates of the solutions of (1.6) by improving the arguments in [23]. In Section 4 we complete the proof of Theorem 1.2.

2 Proof of Theorems 1.1 and 1.3

In this section we recall some basic properties of H and H_0 and prove Theorems 1.1 and 1.3. Assume (1.1) and (1.3). Then H, $H_0 \in C^1(\mathbf{R}^N \setminus \{0\})$. Furthermore, we have

$$\begin{cases}
\xi \cdot \nabla_{\xi} H(\xi) = H(\xi), & \xi \in \mathbf{R}^{N}, \\
\nabla_{\xi} H(t\xi) = \operatorname{sign}(t) \nabla_{\xi} H(\xi), & \xi \in \mathbf{R}^{N} \setminus \{0\}, & t \neq 0, \\
H_{0}(\nabla_{\xi} H(\xi)) = 1, & \xi \in \mathbf{R}^{N} \setminus \{0\}, \\
H(\xi) \nabla_{x} H_{0}(\nabla_{\xi} H(\xi)) = \xi, & \xi \in \mathbf{R}^{N},
\end{cases} (2.1)$$

$$\begin{cases}
 x \cdot \nabla H_0(x) = H_0(x), & x \in \mathbf{R}^N, \\
 \nabla H_0(tx) = \operatorname{sign}(t) \nabla H_0(x), & x \in \mathbf{R}^N \setminus \{0\}, & t \neq 0, \\
 H(\nabla H_0(x)) = 1, & x \in \mathbf{R}^N \setminus \{0\}, \\
 H_0(x) \nabla_{\xi} H(\nabla H_0(x)) = x, & x \in \mathbf{R}^N.
\end{cases}$$
(2.2)

Here $\xi \cdot \nabla_{\xi} H(\xi)$, $H(\xi) \nabla_{\xi} H(\xi)$ and $x \cdot \nabla H_0(x)$, $H_0(x) \nabla H_0(x)$ are taken to be 0 at $\xi = 0$ and x = 0, respectively. See [10] and [18].

Proof of Theorem 1.1. Let $v(x) = v^{\sharp}(H_0(x))$, $v^*(x) := v^{\sharp}(|x|)$, $w(x) = w^{\sharp}(H_0(x))$ and $w^*(x) := w^{\sharp}(|x|)$ for $x \in \mathbf{R}^N$. Assume that v^* , $w^* \in C^2(\mathbf{R}^N)$. It follows that

$$(\nabla v)(x) = (\partial_r v^{\sharp})(H_0(x))\nabla H_0(x),$$

$$H((\nabla v)(x))\nabla_{\xi}H((\nabla v)(x)) = (\partial_r v^{\sharp})(H_0(x))\nabla_{\xi}H(\nabla H_0(x)) = \frac{(\partial_r v^{\sharp})(H_0(x))x}{H_0(x)},$$
(2.3)

for $x \in \mathbf{R}^N \setminus \{0\}$. Then

$$(\Delta_{H}v)(x) = \operatorname{div}\left(\frac{(\partial_{r}v^{\sharp})(H_{0}(x))}{H_{0}(x)}x\right)$$

$$= \frac{\left[(\partial_{r}^{2}v^{\sharp})(H_{0}(x))\nabla H_{0}(x) \cdot x + (\partial_{r}v^{\sharp})(H_{0}(x))N\right]H_{0}(x) - (\partial_{r}v^{\sharp})(H_{0}(x))x\nabla H_{0}(x)}{H_{0}(x)^{2}}$$

$$= \frac{(\partial_{r}^{2}v^{\sharp})(H_{0}(x))H_{0}(x)^{2} + (N-1)(\partial_{r}v^{\sharp})(H_{0}(x))H_{0}(x)}{H_{0}(x)^{2}}$$

$$= (\partial_{r}^{2}v^{\sharp})(r) + \frac{N-1}{r}(\partial_{r}v^{\sharp})(r)\Big|_{r=H_{0}(x)} = (\Delta v^{*})(y) \quad \text{with} \quad y = H_{0}(x)$$

$$(2.4)$$

for $x \in \mathbf{R}^N \setminus \{0\}$. Since $(\partial_r v^{\sharp})(0) = 0$, it follows from (2.3) that

$$H((\nabla v)(x))\nabla_{\xi_i}H((\nabla v)(x)) = (\partial_r^2 v^{\sharp})(0)x_i + o(|x|) \quad \text{near } x = 0,$$

where $i \in \{1, ..., N\}$. This implies that

$$(\Delta_H v)(0) = N(\partial_r^2 v^{\sharp})(0) = (\Delta v^*)(0). \tag{2.5}$$

By (2.4) and (2.5) we obtain assertion (a). Furthermore, we deduce from assertion (a) that

$$[\Delta_H(v+w)](x) = [\Delta(v^*+w^*)](|x|) = (\Delta v^*)(|x|) + (\Delta w^*)(|x|)$$

= $(\Delta_H v)(x) + (\Delta_H w)(x)$

for $x \in \mathbf{R}^N$. Thus assertion (b) follows, and the proof is complete. \square

Theorem 1.3 immediately follows from Theorem 1.1 (a) and (H3). Similarly, we can find the following H_0 -radially functions.

• (Finsler Gauss kernel) Let

$$G_{H_0}(x,t) := (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{H_0(x)^2}{4t}\right).$$

Then G_{H_0} is a solution of (1.4) in $\mathbf{R}^N \times (0, \infty)$. (See also [31, Example 4.3].)

• (Finsler Barenblatt solution) Let m > 1 and C > 0. Set

$$\mathcal{U}_{H_0}(x,t) := t^{-\alpha} (C - kH_0(x)^2 t^{-2\beta})_+^{\frac{1}{m-1}},$$

where $(s)_{+} := \max\{s, 0\}$ and

$$\alpha := \frac{N}{N(m-1)+2}, \qquad \beta := \frac{\alpha}{N}, \qquad k := \frac{\alpha(m-1)}{2mN}.$$

Then \mathcal{U}_{H_0} is a solution of the Finsler porous medium equation

$$\partial_t v = \Delta_H v^m$$
 in $\mathbf{R}^N \times (0, \infty)$.

• (Singular solutions to the m-th order Finsler-Laplace equation) Let $m \in \{1, 2, ...\}$. Set

$$v(x) := \begin{cases} H_0(x)^{-N+2m} & \text{if} \quad N - 2m \notin \{-2h : h = 0, 1, 2, \dots\}, \\ H_0(x)^{-N+2m} \log H_0(x) & \text{if} \quad N - 2m \in \{-2h : h = 0, 1, 2, \dots\}. \end{cases}$$

Then v satisfies

$$(\Delta_H)^m v = c\delta$$

for some constant $c \in \mathbf{R}$, where δ is the Dirac measure in \mathbf{R}^N .

• (Finsler Talenti function) Let p > 1. Set

$$w(x) := \left(A + BH_0(x)^{\frac{p}{p-1}}\right)^{1-\frac{N}{p}},$$

where A > 0 and B > 0. Then w satisfies

$$-\Delta_H w = w^p$$
 in \mathbf{R}^N .

Due to assertion (ii) of Theorem 1.1, we have a explicit representation of the H_0 -radially symmetric solution of (1.6). Set

$$I(z) := \int_{\mathbf{S}^{N-1}} e^{z\theta_1} \, d\theta.$$

Theorem 2.1 Let φ be an H_0 -radially symmetric, bounded and continuous function in \mathbf{R}^N . Then

$$u(x,t) := (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{H_0(x)^2}{4t}\right) \int_0^\infty I\left(\frac{H_0(x)r}{2t}\right) \exp\left(-\frac{r^2}{4t}\right) \varphi^{\sharp}(r) r^{N-1} dr \qquad (2.6)$$

is a solution of (1.4) such that $u(x,0) = \varphi(x)$ in \mathbf{R}^N .

Proof. Let $u^*(x,t) := e^{t\Delta} \varphi^*$, that is

$$u^{*}(x,t) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^{N}} \exp\left(-\frac{|x-y|^{2}}{4t}\right) \varphi^{*}(y) \, dy$$

$$= (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^{2}}{4t}\right) \int_{\mathbf{R}^{N}} \exp\left(\frac{x \cdot y}{2t}\right) \exp\left(-\frac{|y|^{2}}{4t}\right) \varphi^{\sharp}(|y|) \, dy$$

$$= (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^{2}}{4t}\right)$$

$$\times \int_{0}^{\infty} \int_{\mathbf{S}^{N-1}} \exp\left(\frac{r|x|}{2t} \frac{x}{|x|} \theta\right) \, d\theta \exp\left(-\frac{r^{2}}{4t}\right) \varphi^{\sharp}(r) r^{N-1} \, dr$$

$$= (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^{2}}{4t}\right) \int_{0}^{\infty} I\left(\frac{r|x|}{2t}\right) \exp\left(-\frac{r^{2}}{4t}\right) \varphi^{\sharp}(r) r^{N-1} \, dr.$$

This together with Theorem 1.1 implies that the function u defined by (2.6) is a solution of (1.4) such that $u(x,0) = \varphi(x)$ in \mathbb{R}^N . Thus Theorem 2.1 follows. \square

Remark 2.1 Let N = 2. Then

$$I(z) = \int_0^{2\pi} \exp(z\cos\theta) d\theta = \frac{1}{i} \oint_{\Gamma} \exp\left(\frac{z}{2}(\xi + \xi^{-1})\right) \frac{d\xi}{\xi}$$
$$= 2\pi \operatorname{Res}\left(\exp\left(\frac{z}{2}(\xi + \xi^{-1})\right) \xi^{-1}; 0\right), \quad z > 0,$$

where Γ denotes the unit circle centered at the origin in the complex plane. Noting that

$$\exp\left(\frac{z}{2}(\xi+\xi^{-1})\right)\xi^{-1} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!} \frac{1}{m!} \left(\frac{z}{2}\right)^{n+m} \xi^{n-m-1}$$

for $\xi \neq 0$, we see that

Res
$$\left(\exp\left(\frac{z}{2}(\xi+\xi^{-1})\right)\xi^{-1};0\right) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{z}{2}\right)^{2n} = I_0(z),$$

where I_0 denotes the modified Bessel function of the first kind defined by

$$I_0(z) := \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{(n!)^2} \quad \text{for } z \in \mathbb{C} \setminus (-\infty, 0].$$

Therefore we deduce that $I(z) = 2\pi I_0(z)$ for z > 0.

3 Cauchy-Dirichlet problem

Let $R \geq 1$, $\Omega = B_{H_0}(0,R)$ and $\phi \in C_0^{\infty}(\Omega)$. Consider the Cauchy-Dirichlet problem

$$\begin{cases}
\partial_t u = \Delta_H u & \text{in } \Omega \times (0, \infty), \\
u = 0 & \text{on } \partial\Omega \times (0, \infty), \\
u(\cdot, 0) = \phi & \text{in } \Omega.
\end{cases}$$
(3.1)

Problem (3.1) possesses a unique solution u in $\Omega \times [0, \infty)$ such that

$$u \in W^{1,2}(0,\infty : L^2(\Omega)) \cap C([0,\infty) : H_0^1(\Omega)).$$
 (3.2)

To prove this fact, we define a functional ψ on $\mathcal{H} := L^2(\Omega)$ by

$$\psi(u) := \begin{cases} \frac{1}{2} \int_{\Omega} H(\nabla u(x))^2 dx & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$
(3.3)

Then ψ is convex and $D(\psi) := \{w \in \mathcal{H} : \psi(w) < +\infty\} = H_0^1(\Omega)$. Moreover, ψ is lower semicontinuous in \mathcal{H} . Indeed, let $\lambda \geq 0$ be arbitrarily fixed and let $u \in \mathcal{H}$ and $u_n \in [\psi \leq \lambda] := \{w \in \mathcal{H} : \psi(u_n) \leq \lambda\}$ be such that $u_n \to u$ strongly in \mathcal{H} . Then $\{u_n\}$ is bounded in $H_0^1(\Omega)$ by the equivalence of $H(\cdot)$ to the Euclidean norm of \mathbb{R}^N . Hence $u_n \to u$ weakly in $H_0^1(\Omega)$. Now, since the set $[\psi \leq \lambda]$ is convex and closed in the strong topology of $H_0^1(\Omega)$, so is it in the weak topology. Hence u belongs to $[\psi \leq \lambda]$. Thus $[\psi \leq \lambda]$ turns out to be closed in \mathcal{H} , and then, ψ is lower semicontinuous in \mathcal{H} . Next we define the subdifferential operator $\partial \psi : \mathcal{H} \to 2^{\mathcal{H}}$ of ψ by

$$\partial \psi(u) := \{ \xi \in \mathcal{H} : \psi(v) - \psi(u) \ge (\xi, v - u)_{\mathcal{H}} \text{ for all } v \in D(\psi) \},$$

where $(\cdot, \cdot)_{\mathcal{H}}$ is the inner product of \mathcal{H} . Here we remark that $\partial \psi$ is maximal monotone in \mathcal{H} since ψ is proper (i.e., $\psi \not\equiv +\infty$), lower semicontinuous and convex in \mathcal{H} . We note by (1.1) that $\psi|_{H_0^1(\Omega)}$ is Fréchet differentiable and its Fréchet derivative $d\psi|_{H_0^1(\Omega)}(u)$ coincides with $-\Delta_H u$ (in $H^{-1}(\Omega)$) for $u \in H_0^1(\Omega)$. Moreover, by the definition of subdifferentials we see that

$$\partial \psi(u) \subset \partial \psi|_{H_0^1(\Omega)}(u) = \{d\psi|_{H_0^1(\Omega)}(u)\}$$

for $u \in D(\partial \psi) := \{u \in H^1_0(\Omega) : \partial \psi(u) \neq \emptyset\}$. This implies that

$$\partial \psi(u) = \{-\Delta_H u\}$$
 in \mathcal{H}

for $u \in D(\partial \psi) = \{u \in H_0^1(\Omega) : \Delta_H u \in \mathcal{H}\}$. Hence (3.3) is reduced to an abstract Cauchy problem in the Hilbert space \mathcal{H} of the form

$$\partial_t u + \partial \psi(u) \ni 0 \text{ in } \mathcal{H}, \quad 0 < t < \infty, \quad u(0) = \phi,$$
 (3.4)

which has been well studied. In particular, the global-in-time well-posedness for (3.4) is guaranteed by Kōmura-Brézis theory (see e.g., [9, Theorem 3.6]). Thus we conclude that problem (3.1) has a unique strong solution u satisfying (3.2).

In the rest of this section we improve the arguments in [23, Section 4] to prove the following proposition.

Proposition 3.1 Assume (1.1) and (1.3). Let u be a solution of (3.1), $R \ge 1$ and $0 < \ell < 1/2$. Then there exist constants $C_* > 0$ and $T_* \in (0,1)$ such that

$$\sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} e^{-h(y,t)} |u(y,t)| \, dy \le C_* \sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} e^{-H_0(y)^2} |\phi(y)| \, dy \tag{3.5}$$

for all $t \in (0, T_*]$, where $h(x, t) := H_0(x)^2(1 + t^{\ell})$. Here C_* and T_* depend only on N and ℓ .

In what follows, the letter C denotes generic positive constants independent of Ω and it may have different values also within the same line. We start with proving an energy estimate of the solution of (3.1).

Lemma 3.1 Assume the same conditions as in Proposition 3.1. Set

$$z(x,t) := e^{-h(x,t)}u(x,t) = e^{-H_0(x)^2(1+t^\ell)}u(x,t).$$

Then there exist C > 0 and $T_1 \in (0,1)$ such that

$$\sup_{t_1 < s < t} \int_{\Omega \cap B_{H_0}(x, R_2)} z(y, s)^2 \zeta_x^2 \, dy + \iint_{Q_2} H(\nabla(z\zeta_x))^2 \, dy \, ds
\leq C[(R_2 - R_1)^{-2} + (t_1 - t_2)^{-1}] \iint_{Q_2} z(y, s)^2 \, dy \, ds$$
(3.6)

and

$$\iint_{Q_1} z^{2\kappa} \, dy \, ds \le C \left(\left[(R_2 - R_1)^{-2} + (t_1 - t_2)^{-1} \right] \iint_{Q_2} z^2 \, dy \, ds \right)^{\kappa}, \tag{3.7}$$

for all $x \in \Omega$, $0 < R_1 < R_2$ and $0 < t_2 < t_1 < t \le T_1$, where $\kappa := (N+2)/N$ and

$$\zeta_x(y,s) := \frac{1}{R_2 - R_1} \min\{\max\{R_2 - H_0(y - x), 0\}, R_2 - R_1\}
\times \frac{1}{t_1 - t_2} \min\{\max\{s - t_2, 0\}, t_1 - t_2\}$$
(3.8)

for $x, y \in \mathbf{R}^N$ and s > 0. Here

$$Q_1 := [\Omega \cap B_{H_0}(x, R_1)] \times (t_1, t), \qquad Q_2 := [\Omega \cap B_{H_0}(x, R_2)] \times (t_2, t).$$

Proof. It follows from (2.2) and (3.8) that

$$0 \le \zeta_x \le 1 \quad \text{in} \quad \mathbf{R}^N \times (0, \infty), \qquad \zeta_x = 1 \quad \text{on} \quad Q_1, \qquad \zeta_x = 0 \quad \text{on} \quad \partial_p Q_2,$$

$$H(\nabla \zeta_x) \le \frac{1}{R_2 - R_1} \quad \text{and} \quad 0 \le \partial_t \zeta_x \le \frac{1}{t_1 - t_2} \quad \text{in} \quad Q_2.$$
 (3.9)

Thanks to (1.1), (1.2) and (2.1), by (3.9) we have

$$\iint_{Q_{2}} H(\nabla u) \nabla_{\xi} H(\nabla u) \nabla [e^{-2h} u \zeta_{x}^{2}] \, dy \, ds \\
= \iint_{Q_{2}} H(\nabla u) \nabla_{\xi} H(\nabla u) [-2e^{-2h} \nabla h u \zeta_{x}^{2} + e^{-2h} \nabla u \zeta_{x}^{2} + 2e^{-2h} u \zeta_{x} \nabla \zeta_{x}] \, dy \, ds \\
\geq \iint_{Q_{2}} e^{-2h} H(\nabla u)^{2} \zeta_{x}^{2} \, dy \, ds - 2 \iint_{Q_{2}} e^{-2h} H(\nabla u) H_{0}(\nabla_{\xi} H(\nabla u)) H(\nabla h) u \zeta_{x}^{2} \, dy \, ds \\
- 2 \iint_{Q_{2}} e^{-2h} H(\nabla u) H_{0}(\nabla_{\xi} H(\nabla u)) H(\nabla \zeta_{x}) u \zeta_{x} \, dy \, ds \qquad (3.10)$$

$$= \iint_{Q_{2}} e^{-2h} H(\nabla u)^{2} \zeta_{x}^{2} \, dy \, ds - 2 \iint_{Q_{2}} e^{-2h} H(\nabla u) H(\nabla h) u \zeta_{x}^{2} \, dy \, ds \\
- 2 \iint_{Q_{2}} e^{-2h} H(\nabla u) H(\nabla \zeta_{x}) u \zeta_{x} \, dy \, ds$$

$$\geq \frac{1}{2} \iint_{Q_{2}} e^{-2h} H(\nabla u)^{2} \zeta_{x}^{2} \, dy \, ds - 4 \iint_{Q_{2}} e^{-2h} [H(\nabla h)^{2} \zeta_{x}^{2} + H(\nabla \zeta_{x})^{2}] u^{2} \, dy \, ds.$$

Since H is a norm of \mathbb{R}^N , it follows that

$$H(\nabla(z\zeta_x))^2 \le [e^{-h}u\zeta_x H(\nabla h) + e^{-h}H(\nabla u)\zeta_x + e^{-h}H(\nabla\zeta_x)u]^2$$

$$\le 3e^{-2h}[u^2\zeta_x^2 H(\nabla h)^2 + H(\nabla u)^2\zeta_x^2 + H(\nabla\zeta_x)^2u^2].$$

These imply that

$$\iint_{Q_2} H(\nabla u) \nabla_{\xi} H(\nabla u) \nabla [e^{-2h} u \zeta_x^2] \, dy \, ds$$

$$\geq \frac{1}{6} \iint_{Q_2} H(\nabla (z\zeta_x))^2 \, dy \, ds - C \iint_{Q_2} [H(\nabla h)^2 \zeta_x^2 + H(\nabla \zeta_x)^2] z^2 \, dy \, ds.$$

Therefore, multiplying (1.4) by $e^{-2h}u\zeta_x^2$ and integrating it on Q_2 , we obtain

$$\frac{1}{2} \int_{\Omega \cap B_{H_0}(x,R_2)} z(y,t)^2 \zeta_x(y,t)^2 dy + \iint_{Q_2} z^2 [(\partial_t h) \zeta_x^2 - \zeta_x \partial_t \zeta_x] dy ds
+ \frac{1}{6} \iint_{Q_2} H(\nabla(z\zeta_x))^2 dy ds \le C \iint_{Q_2} [H(\nabla h)^2 \zeta_x^2 + H(\nabla \zeta_x)^2] z^2 dy ds.$$
(3.11)

Let $0 < T_1 \le 1$. Since $h(x,t) := H_0(x)^2(1+t^{\ell})$, it follows from (2.2) that

$$\partial_t h \ge \ell T_1^{\ell-1} H_0(x)^2, \qquad H(\nabla h)^2 = 4H_0(x)^2 (1+t^{\ell})^2 \le 16H_0(x)^2,$$
 (3.12)

for all $x \in \mathbf{R}^N$ and $0 < t < T_1$. Since $0 < \ell < 1/2$, by (3.11) and (3.12), taking a sufficiently small $T_1 > 0$ if necessary, we obtain

$$\frac{1}{2} \int_{\Omega \cap B_{H_0}(x,R_2)} z(y,t)^2 \zeta_x^2 \, dy + \frac{1}{6} \iint_{Q_2} H(\nabla(z\zeta_x))^2 \, dy \, ds$$

$$\leq \iint_{Q_2} [CH(\nabla \zeta_x)^2 + \zeta_x \partial_t \zeta_x] z^2 \, dy \, ds,$$

which together with (3.9) implies (3.6).

Since H is an equivalent norm to the Euclidean norm of \mathbb{R}^N , by the Gagliardo-Nirenberg inequality we have

$$\int_{t_{1}}^{t} \int_{\Omega \cap B_{H_{0}}(x,R_{2})} (z\zeta_{x})^{2\kappa} dy ds
\leq C \int_{t_{1}}^{t} \int_{\Omega \cap B_{H_{0}}(x,R_{2})} |\nabla(z\zeta_{x})|^{2} dy \left(\int_{\Omega \cap B_{H_{0}}(x,R_{2})} (z\zeta_{x})^{2} dy \right)^{\frac{2}{N}} ds
\leq C \int_{t_{1}}^{t} \int_{\Omega \cap B_{H_{0}}(x,R_{2})} H(\nabla(z\zeta_{x}))^{2} dy ds \sup_{t_{1} < s < t} \left(\int_{\Omega \cap B_{H_{0}}(x,R_{2})} (z\zeta_{x})^{2} (y,s) dy \right)^{\frac{2}{N}} ds$$

for all $x \in \Omega$ and $t > t_1$. This together with (3.6) and (3.9) implies the desired inequality and Lemma 3.1 follows. \square

By Lemma 3.1 and the Besicovitch covering theorem we prove the following lemma.

Lemma 3.2 Assume the same conditions as in Lemma 3.1. Then, for any $\delta \in (0,1]$, there exists a positive constant C such that

$$\sup_{x \in \Omega} \|z(t)\|_{L^{1+\delta}(\Omega \cap B_{H_0}(x,1))} \le Ct^{-\frac{N}{2}(1-\frac{1}{1+\delta})} \sup_{0 < s < t} \sup_{x \in \Omega} \|z(s)\|_{L^1(\Omega \cap B_{H_0}(x,1))}$$
(3.13)

for all $0 < t < T_1$, where T_1 is as in Lemma 3.1.

Proof. Let $x \in \Omega$, $\rho > 0$ and $0 < t < T_1$. Set

$$\sigma_n := \sum_{i=1}^n 2^{-i}, \qquad Q_n := \left[\Omega \cap B_{H_0}(x, (1+\sigma_n)\rho)\right] \times \left(\frac{t}{4} \left(1 - \frac{1}{2}\sigma_n\right), t\right).$$

It follows that

$$Q_0 := \left[\Omega \cap B_{H_0}(x,\rho)\right] \times \left(\frac{t}{4},t\right) \subset Q_n \subset Q_{n+1} \subset Q_\infty = \left[\Omega \cap B_{H_0}(x,2\rho)\right] \times \left(\frac{t}{8},t\right)$$

for $n = 1, 2, \ldots$ Then we apply Lemma 3.1 with Q_1 and Q_2 replaced by Q_n and Q_{n+1} , respectively, and obtain

$$||z||_{L^{2\kappa}(Q_n)} \le CD_n^{1/2} ||z||_{L^2(Q_{n+1})},$$
 (3.14)

where $D_n := 2^{2n}(\rho^{-2} + t^{-1})$. Furthermore, for any $\epsilon > 0$, the Hölder and Young inequalities imply that

$$CD_{n}^{1/2} \|z\|_{L^{2}(Q_{n+1})} \leq CD_{n}^{1/2} \|z\|_{L^{1}(Q_{n+1})}^{\theta} \|z\|_{L^{2\kappa}(Q_{n+1})}^{1-\theta}$$

$$\leq \epsilon \|z\|_{L^{2\kappa}(Q_{n+1})} + \epsilon^{-(1-\theta)/\theta} \left[CD_{n}^{1/2} \|z\|_{L^{1}(Q_{n+1})}^{\theta} \right]^{\frac{1}{\theta}}$$

$$\leq \epsilon \|z\|_{L^{2\kappa}(Q_{n+1})} + \epsilon^{-(1-\theta)/\theta} C^{1/\theta} D_{n}^{1/2\theta} \|z\|_{L^{1}(Q_{n+1})},$$
(3.15)

where

$$\frac{1}{2} = \theta + \frac{1 - \theta}{2\kappa}.$$

By (3.14) and (3.15) we see that

$$||z||_{L^{2\kappa}(Q_n)} \le \epsilon ||z||_{L^{2\kappa}(Q_{n+1})} + C\epsilon^{-(1-\theta)/\theta} D_n^{1/2\theta} ||z||_{L^1(Q_{n+1})}$$

for $n = 0, 1, 2, \ldots$ Then it follows that

$$||z||_{L^{2\kappa}(Q_0)} \leq \epsilon^k ||z||_{L^{2\kappa}(Q_k)} + C\epsilon^{-(1-\theta)/\theta} \sum_{n=0}^{k-1} \epsilon^n D_n^{1/2\theta} ||z||_{L^1(Q_{n+1})}$$

$$\leq \epsilon^k ||z||_{L^{2\kappa}(Q_k)} + C\epsilon^{-(1-\theta)/\theta} \sum_{n=0}^{k-1} \epsilon^n \left[2^{2n} (\rho^{-2} + t^{-1}) \right]^{1/2\theta} ||z||_{L^1(Q_{n+1})}$$
(3.16)

for $k = 1, 2, \ldots$ Taking a sufficiently small $\epsilon > 0$ so that $\epsilon 2^{1/\theta} \le 1/2$ and passing to the limit, by (3.16) we obtain

$$||z||_{L^{2\kappa}([\Omega \cap B_{H_0}(x,\rho)] \times (\frac{t}{4},t))} = ||z||_{L^{2\kappa}(Q_0)}$$

$$\leq C[\rho^{-2} + t^{-1}]^{1/2\theta} ||z||_{L^1(Q_\infty)} = C[\rho^{-2} + t^{-1}]^{1/2\theta} ||z||_{L^1(B_{H_0}(x,2\rho) \times (\frac{t}{2},t))}$$
(3.17)

for $x \in \Omega$, $\rho > 0$ and $0 < t < T_1$.

On the other hand, by the Hölder inequality and (3.6) we have

$$||z(t)||_{L^{1+\delta}(\Omega \cap B_{H_0}(x,\sqrt{t}))} \leq |B_{H_0}(x,\sqrt{t})|^{\frac{1}{1+\delta}-\frac{1}{2}}||z(t)||_{L^2(\Omega \cap B_{H_0}(x,\sqrt{t}))}$$

$$\leq Ct^{\frac{N}{2}\left(\frac{1}{1+\delta}-\frac{1}{2}\right)}t^{-\frac{1}{2}}\left(\int_{t/4}^{t}\int_{\Omega \cap B_{H_0}(x,2\sqrt{t})}z^2\,dy\,ds\right)^{\frac{1}{2}}$$

$$\leq Ct^{\frac{N}{2}\left(\frac{1}{1+\delta}-\frac{1}{2}\right)}t^{-\frac{1}{2}}\left(t|B_{H_0}(x,\sqrt{t})|\right)^{\frac{1}{2}(1-\frac{1}{\kappa})}\left(\int_{t/4}^{t}\int_{\Omega \cap B_{H_0}(x,2\sqrt{t})}z^{2\kappa}\,dy\,ds\right)^{\frac{1}{2\kappa}}.$$

$$(3.18)$$

Applying (3.17) with $\rho = 2\sqrt{t}$ to (3.18), we obtain

$$||z(t)||_{L^{1+\delta}(\Omega \cap B_{H_0}(x,\sqrt{t}))} \leq Ct^{\frac{N}{2}\left(\frac{1}{1+\delta}-\frac{1}{2}\right)}t^{-\frac{1}{2}}(t^{1+\frac{N}{2}})^{\frac{1}{2}(1-\frac{1}{\kappa})}t^{-\frac{1}{2\theta}} \int_{t/8}^{t} \int_{\Omega \cap B_{H_0}(x,4\sqrt{t})} |z| \, dy \, ds$$

$$\leq Ct^{-\frac{N}{2}\left(1-\frac{1}{1+\delta}\right)} \sup_{t/8 < s < t} \int_{\Omega \cap B_{H_0}(x,4\sqrt{t})} |z(y,s)| \, dy$$

$$(3.19)$$

for all $x \in \Omega$ and $0 < t < T_1$.

Let $x_0 \in \Omega$. By the Besicovitch covering theorem (see e.g., [17]) we can find

$$\mathcal{G}_{1} := \{ \overline{B_{H_{0}}(y_{1,k}, \sqrt{t})} \}, \quad \dots, \quad \mathcal{G}_{n} := \{ \overline{B_{H_{0}}(y_{n,k}, \sqrt{t})} \}$$
$$\subset \mathcal{F} := \{ \overline{B_{H_{0}}(y, \sqrt{t})} : y \in \overline{B_{H_{0}}(x_{0}, 1)} \}$$

such that each \mathcal{G}_i $(i=1,\ldots,n)$ is a countable correction of disjoint balls in \mathcal{F} and

$$\overline{B_{H_0}(x_0, 1)} \subset \bigcup_{i=1}^n \bigcup_k \overline{B_{H_0}(y_{i,k}, \sqrt{t})},$$
 (3.20)

where n is an integer depending only on N. Then there exists an integer m depending only on N such that

$$\# \left\{ y_{j,\ell} : B_{H_0}(y_{j,\ell}, 4\sqrt{t}) \cap B_{H_0}(y_{i,k}, 4\sqrt{t}) \neq \emptyset \right\} \le m$$
 (3.21)

for any $y_{i,k}$. Therefore, by (3.19), (3.20) and (3.21) we obtain

$$\begin{split} \|z(t)\|_{L^{1+\delta}(\Omega\cap B_{H_0}(x_0,1))} &\leq \left\|\sum_{i=1}^n \sum_k |z(t)| \chi_{B_{H_0}(y_{i,k},\sqrt{t})} \right\|_{L^{1+\delta}(\Omega\cap B_{H_0}(x_0,1))} \\ &\leq \sum_{i=1}^n \sum_k \|z(t)\|_{L^{1+\delta}(\Omega\cap B_{H_0}(y_{i,k},\sqrt{t}))} \\ &\leq Ct^{-\frac{N}{2}\left(1-\frac{1}{1+\delta}\right)} \sum_{i=1}^n \sum_k \sup_{t/8 < s < t} \int_{\Omega\cap B_{H_0}(y_{i,k},4\sqrt{t})} |z(y,s)| \, dy \\ &\leq Ct^{-\frac{N}{2}\left(1-\frac{1}{1+\delta}\right)} \sup_{t/8 < s < t} \int_{\Omega\cap B_{H_0}(x_0,1+4\sqrt{t})} |z(y,s)| \, dy. \end{split}$$

Since x_0 is arbitrary, we deduce that

$$\sup_{x \in \Omega} ||z(t)||_{L^{1+\delta}(\Omega \cap B_{H_0}(x,1))} \le Ct^{-\frac{N}{2}\left(1-\frac{1}{1+\delta}\right)} \sup_{0 < s < t} \sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1+4\sqrt{t})} |z(y,s)| \, dy$$

$$\le Ct^{-\frac{N}{2}\left(1-\frac{1}{1+\delta}\right)} \sup_{0 < s < t} \sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} |z(y,s)| \, dy$$

for all $0 < t < T_1$. Thus (3.13) holds. The proof is complete. \square

Next, applying a similar argument as in [23, Lemma 4.2], we have:

Lemma 3.3 Assume the same conditions as in Lemma 3.1. Let $x \in \Omega$ and $\eta(y) := \min\{\max\{2 - H_0(y - x), 0\}, 1\}$. For $\epsilon > 0$, $\sigma > 0$ and $\delta > 0$, set

$$J_{\pm}^{\epsilon}(x,t) := \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} e^{-h} \frac{H(\nabla u_{\pm})^{2}}{u_{\pm} + \epsilon} [e^{-h}(u_{\pm} + \epsilon)]^{\delta} \eta^{2} \, dy \, ds,$$

$$u_{\pm} := \max\{\pm u, 0\}, \qquad z_{\pm} = e^{-h} u_{\pm}.$$

Then, for any $\sigma \in (0, 1/2)$ and $\delta \in (0, 2\sigma/N)$, there exist positive constants C and T_2 such that

$$\limsup_{\epsilon \to 0} J_{\pm}^{\epsilon}(x,t) \le C \int_0^t \int_{\Omega \cap B_{H_0}(x,2)} s^{\sigma-1} z_{\pm}^{1+\delta} \, dy \, ds \tag{3.22}$$

for all $0 < t < T_2$ and $x \in \Omega$.

Proof. For any $\epsilon > 0$, set $z_{\pm}^{\epsilon} := e^{-h}(u_{\pm} + \epsilon)$ and $\psi_{\pm}^{\epsilon} := s^{\sigma}e^{-h}(z_{\pm}^{\epsilon})^{\delta}\eta^{2}$. Then

$$\lim_{\epsilon \to 0} \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} (\partial_{t}u) \psi_{\pm}^{\epsilon} dy ds
= \lim_{\epsilon \to 0} \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} [e^{h} \partial_{t} z_{\pm} + (u_{\pm} + \epsilon) \partial_{t} h] s^{\sigma} e^{-h} (z_{\pm}^{\epsilon})^{\delta} \eta^{2} dy ds
= \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} [s^{\sigma} (\partial_{t} z_{\pm}) z_{\pm}^{\delta} \eta^{2} + s^{\sigma} z_{\pm}^{1+\delta} \eta^{2} \partial_{t} h] dy ds
= \frac{1}{1+\delta} \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} \eta^{2} \partial_{t} (z_{\pm}^{1+\delta}) dy ds + \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} z_{\pm}^{1+\delta} \eta^{2} \partial_{t} h dy ds
\geq -\frac{\sigma}{1+\delta} \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma-1} z_{\pm}^{1+\delta} \eta^{2} dy ds + \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} z_{\pm}^{1+\delta} \eta^{2} \partial_{t} h dy ds.$$
(3.23)

Furthermore, by (1.2), (2.1) and (2.2) we obtain

$$\int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} H(\nabla u_{\pm}) \nabla_{\xi} H(\nabla u_{\pm}) \nabla \psi_{\pm}^{\epsilon} \, dy \, ds
= \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} H(\nabla u_{\pm}) \nabla_{\xi} H(\nabla u_{\pm}) \nabla (s^{\sigma} e^{-h} (z_{\pm}^{\epsilon})^{\delta} \eta^{2}) \, dy \, ds
\geq \frac{\delta}{2} \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} e^{-2h} H(\nabla u_{\pm})^{2} (z_{\pm}^{\epsilon})^{-1+\delta} \eta^{2} \, dy \, ds
- C \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} (z_{\pm}^{\epsilon})^{1+\delta} [H(\nabla h)^{2} \eta^{2} + H(\nabla \eta)^{2}] \, dy \, ds
\geq \frac{\delta}{2} \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} e^{-h} \frac{H(\nabla u_{\pm})^{2}}{u_{\pm} + \epsilon} [e^{-h} (u_{\pm} + \epsilon)]^{\delta} \eta^{2} \, dy \, ds
- C \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} (z_{\pm}^{\epsilon})^{1+\delta} [H(\nabla h)^{2} \eta^{2} + 1] \, dy \, ds.$$
(3.24)

On the other hand, by [13, Chapter II, Section 1] we see that u_+ and u_- are subsolutions of (3.1). Then, similarly to (3.11), we deduce from (3.23) and (3.24) that

$$\frac{\delta}{2} \limsup_{\epsilon \to 0} \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} e^{-h} \frac{H(\nabla u_{\pm})^{2}}{u_{\pm} + \epsilon} [e^{-h}(u_{\pm} + \epsilon)]^{\delta} \eta^{2} \, dy \, ds$$

$$\leq \frac{\sigma}{1 + \delta} \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma - 1} z_{\pm}^{1 + \delta} \eta^{2} \, dy \, ds$$

$$+ \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} z_{\pm}^{1 + \delta} [CH(\nabla h)^{2} \eta^{2} - (\partial_{t} h) \eta^{2} + C] \, dy \, ds$$

$$\leq C \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma - 1} z_{\pm}^{1 + \delta} \, dy \, ds$$

$$+ \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} z_{\pm}^{1 + \delta} [CH(\nabla h)^{2} \eta^{2} - (\partial_{t} h) \eta^{2}] \, dy \, ds.$$
(3.25)

Furthermore, by (3.12) we can find $T_2 \in (0,1)$ such that

$$CH(\nabla h)^2 - \partial_t h \le 0$$
 in $\mathbf{R}^N \times (0, T_2)$.

This together with (3.25) implies (3.22). The proof is complete. \square

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let $x \in \mathbb{R}^N$ and let η be as in Lemma 3.3. Let δ be a sufficiently small positive constant. Following [23] and Lemma 3.3, we set

$$F(v) := \int_0^v \min\{\tau^{\delta}, 1\} d\tau = \begin{cases} v^{1+\delta}/(1+\delta) & \text{for } 0 \le v \le 1, \\ v - \delta/(1+\delta) & \text{for } v > 1, \end{cases}$$
$$\varphi_{\pm}^{\epsilon}(y, s) := e^{-h} F'(e^{-h}(u_{\pm} + \epsilon))\eta^2, \quad z_{\pm} = e^{-h}u_{\pm}, \quad z_{\epsilon} = e^{-h}(u_{\pm} + \epsilon).$$

Then $F'(v) = v^{\delta}$ for $0 \le v \le 1$ and F'(v) = 1 for v > 1. Taking a sufficiently small $\delta > 0$ if necessary, we can assume that

$$F(v) \le v \quad \text{for} \quad v \ge 0 \quad \text{and} \quad v \le 2F(v) \quad \text{for} \quad v \ge 1.$$
 (3.26)

Let $0 < \sigma < 1 - \ell$. Then we have

$$\lim_{\epsilon \to 0} \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} \partial_{t} u_{\pm} \varphi_{\pm}^{\epsilon} \, dy \, ds = \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} [\partial_{t} z_{\pm} + z_{\pm} \partial_{t} h] F'(z_{\pm}) \eta^{2} \, dy \, ds$$

$$= \int_{\Omega \cap B_{H_{0}}(x,2)} F(z_{\pm}(s)) \eta^{2} \, dy \Big|_{s=0}^{s=t}$$

$$+ \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} \left[z_{\pm}^{1+\delta} \chi_{\{z_{\pm} \le 1\}} + z_{\pm} \chi_{\{z_{\pm} > 1\}} \right] (\partial_{t} h) \eta^{2} \, dy \, ds.$$

Furthermore, similarly to (3.24), we obtain

$$\begin{split} &\int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} H(\nabla u_{\pm}) \nabla_{\xi} H(\nabla u_{\pm}) \nabla \varphi_{\pm}^{\epsilon} \, dy \, ds \\ &= \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} H(\nabla u_{\pm}) \nabla_{\xi} H(\nabla u_{\pm}) e^{-h} [-z_{\pm}^{\epsilon} \nabla h + e^{-h} \nabla u_{\pm}] \delta(z_{\pm}^{\epsilon})^{-1+\delta} \eta^{2} \chi_{\{z_{\pm}^{\epsilon} \leq 1\}} \, dy \, ds \\ &\quad + \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} H(\nabla u_{\pm}) \nabla_{\xi} H(\nabla u_{\pm}) [-e^{-h} \nabla h \eta^{2} + e^{-h} 2 \eta \nabla \eta] F'(z_{\pm}^{\epsilon}) \, dy \, ds \\ &\geq \delta \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} e^{-2h} H(\nabla u_{\pm})^{2} (z_{\pm}^{\epsilon})^{-1+\delta} \eta^{2} \chi_{\{z_{\pm}^{\epsilon} \leq 1\}} \, dy \, ds \\ &\quad - \delta \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} e^{-h} H(\nabla u_{\pm}) H(\nabla h) (z_{\pm}^{\epsilon})^{\delta} \eta^{2} \chi_{\{z_{\pm}^{\epsilon} \leq 1\}} \, dy \, ds \\ &\quad - \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} e^{-h} H(\nabla u_{\pm}) (H(\nabla h) \eta^{2} + 2 \eta H(\nabla \eta)) [(z_{\pm}^{\epsilon})^{\delta} \chi_{\{z_{\pm}^{\epsilon} \leq 1\}} + \chi_{\{z_{\pm}^{\epsilon} > 1\}}] \, dy \, ds \\ &\geq - \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} e^{-2h} H(\nabla u_{\pm})^{2} (z_{\pm}^{\epsilon})^{-1+\delta} \eta^{2} \, dy \, ds \\ &\quad - C \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{-\sigma} (z_{\pm}^{\epsilon})^{1+\delta} [H(\nabla h)^{2} \eta^{2} + H(\nabla \eta)^{2}] \chi_{\{z_{\pm}^{\epsilon} \leq 1\}} \, dy \, ds \\ &\quad - C \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{-\sigma} (z_{\pm}^{\epsilon})^{1-\delta} [H(\nabla h)^{2} \eta^{2} + H(\nabla \eta)^{2}] \chi_{\{z_{\pm}^{\epsilon} > 1\}} \, dy \, ds. \end{split}$$

Similarly to (3.25), these imply that

$$\int_{\Omega \cap B_{H_{0}}(x,2)} F(z_{\pm}(s)) \eta^{2} dy \Big|_{s=0}^{s=t}$$

$$\leq \lim \sup_{\epsilon \to 0} \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{\sigma} e^{-2h} H(\nabla u_{\pm})^{2} (z_{\pm}^{\epsilon})^{-1+\delta} \eta^{2} dy ds$$

$$+ C \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} s^{-\sigma} [z_{\pm}^{1+\delta} \chi_{\{z_{\pm} \le 1\}} + z_{\pm}^{1-\delta} \chi_{\{z_{\pm} > 1\}}] [H(\nabla h)^{2} \eta^{2} + H(\nabla \eta)^{2}] dy ds$$

$$- \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,2)} [z_{\pm}^{1+\delta} \chi_{\{z_{\pm} \le 1\}} + z_{\pm} \chi_{\{z_{\pm} > 1\}}] (\partial_{t} h) \eta^{2} dy ds.$$
(3.27)

On the other hand, since $\ell + \sigma < 1$, we can find a constant $T_2 \in (0, T_1)$ such that

$$\partial_t h = \ell s^{\ell-1} H_0(y)^2 \ge \ell T_2^{\ell+\sigma-1} s^{-\sigma} H_0(y)^2,
C s^{-\sigma} H(\nabla h)^2 = 4C(1+s^{\ell})^2 s^{-\sigma} H_0(y)^2 \le 4C(1+T_2^{\ell})^2 s^{-\sigma} H_0(y)^2 \le \partial_t h,$$
(3.28)

for $y \in \mathbf{R}^N$ and $0 < t < T_2$. Since $z_{\pm}^{1-\delta} \chi_{\{z_{\pm} > 1\}} \le z_{\pm} \chi_{\{z_{\pm} > 1\}}$, by (3.27) and (3.28) we obtain

$$\int_{\Omega \cap B_{H_0}(x,2)} F(z_{\pm}(s)) \eta^2 \, dy \Big|_{s=0}^{s=t}$$

$$\leq \limsup_{\epsilon \to 0} J_{\pm}^{\epsilon}(x,t) + C \int_0^t \int_{\Omega \cap B_{H_0}(x,2)} s^{-\sigma} [z_{\pm}^{1+\delta} \chi_{\{z_{\pm} \le 1\}} + z_{\pm} \chi_{\{z_{\pm} > 1\}}] H(\nabla \eta)^2 \, dy \, ds$$

$$\leq \limsup_{\epsilon \to 0} J_{\pm}^{\epsilon}(x,t) + C \int_0^t \int_{\Omega \cap B_{H_0}(x,2)} s^{-\sigma} F(z_{\pm}) \, dy \, ds$$

for $x \in \Omega$ and $0 < t < T_2$. This together with Lemma 3.3 implies that

$$\int_{\Omega \cap B_{H_0}(x,1)} F(|z(t)|) \, dy \le \int_{\Omega \cap B_{H_0}(x,2)} F(|z(0)|) \, dy
+ C \int_0^t \int_{\Omega \cap B_{H_0}(x,2)} s^{\sigma-1} |z|^{1+\delta} \, dy \, ds + C \int_0^t \int_{\Omega \cap B_{H_0}(x,2)} s^{-\sigma} F(z) \, dy \, ds$$
(3.29)

for $x \in \Omega$ and $0 < t < T_2$.

Set

$$I(t) := \sup_{0 \le s \le t} \sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} |z(y,s)| \, dy, \qquad t > 0.$$

If I(0) = 0, then $u \equiv 0$ in $\Omega \times (0, \infty)$ and Proposition 3.1 holds. So it suffices to consider the case $I(0) \neq 0$. Furthermore, thanks to (1.5), we can assume, without loss of generality, that

$$I(0) = \sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} |z(y,0)| \, dy = \sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} e^{-H_0(y)^2} |\phi(y)| \, dy = 1.$$
 (3.30)

On the other hand, for any $x \in \mathbf{R}^N$, there exist an integer M depending only on N and a set $\{x_j\}_{j=1}^M$ such that

$$B_{H_0}(x,2) \subset \bigcup_{j=1}^{M} B_{H_0}(x_j,1)$$
 (3.31)

(see e.g., [26, Lemma 2.1]). Then, by (3.26), (3.29), (3.30) and (3.31) we see that

$$\sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} F(|z(y,t)|) \, dy$$

$$\leq \sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,2)} |z(y,0)| \, dy + C \sup_{x \in \Omega} \int_0^t \int_{\Omega \cap B_{H_0}(x,2)} s^{\sigma-1} |z(y,s)|^{1+\delta} \, dy \, ds$$

$$+ C \sup_{x \in \Omega} \int_0^t \int_{\Omega \cap B_{H_0}(x,2)} s^{-\sigma} |z(y,s)| \, dy \, ds$$

$$\leq M \sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} |z(y,0)| \, dy + CM \sup_{x \in \Omega} \int_0^t \int_{\Omega \cap B_{H_0}(x,1)} s^{\sigma-1} |z(y,s)|^{1+\delta} \, dy \, ds$$

$$+ CM \sup_{x \in \Omega} \int_0^t \int_{\Omega \cap B_{H_0}(x,1)} s^{-\sigma} |z(y,s)| \, dy \, ds$$

for $0 < t < T_2$. This together with Lemma 3.2 implies that

$$\sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} F(|z(y,t)|) \, dy$$

$$\leq MI(0) + C \int_0^t s^{\sigma - 1} \sup_{x \in \Omega} ||z(s)||_{L^{1+\delta}(\Omega \cap B_{H_0}(x,1))}^{1+\delta} \, ds + C \int_0^t s^{-\sigma} I(s) \, ds$$

$$\leq M + C \int_0^t s^{\sigma - 1 - \frac{\delta N}{2}} I(s)^{1+\delta} \, ds + C \int_0^t s^{-\sigma} I(s) \, ds$$
(3.32)

for all $0 < t < T_2$. Since I(0) = 1 < 5M, we can define

$$T_* := \sup \{ t \in (0, T_2] : I(t) \le L := 5M + |B_{H_0}(0, 1)| \}.$$

Since $\delta N/2 < \sigma < 1$ (see Lemma 3.3), taking a sufficiently small $T_2 > 0$ if necessary, by (3.32) we obtain

$$\sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} F(|z(y,t)|) \, dy \le M + CL^{1+\delta} \int_0^t s^{\sigma - 1 - \frac{\delta N}{2}} \, ds + CL \int_0^t s^{-\sigma} \, ds \le 2M$$
(3.33)

for all $0 < t \le T_2$. On the other hand, it follows from (3.26) that

$$\sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} |z(y,t)| \, dy$$

$$\leq \sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} \chi_{\{|z(y,t)|<1\}}(y) \, dy + \sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} |z(y,t)| \chi_{\{|z(y,t)|\geq1\}}(y) \, dy \quad (3.34)$$

$$\leq |B_{H_0}(0,1)| + 2 \sup_{x \in \mathbb{R}^N} \int_{\Omega \cap B_{H_0}(x,1)} F(|z(y,t)|) \, dy$$

for all t > 0. Therefore we deduce from (3.33) and (3.34) that

$$I(t) \le |B_{H_0}(0,1)| + 4M$$

for all $0 < t \le T_*$. This means that $T_* = T_2$ and $I(t) \le L = LI(0)$ for all $0 < t \le T_2$. Thus (3.5) holds for $t \in (0, T_2]$ and Proposition 3.1 follows. \square

By Proposition 3.1 we have:

Proposition 3.2 Assume the same conditions as in Proposition 3.1. Then there exist positive constants C_1 and σ' such that

$$\sup_{x \in \Omega} \int_{0}^{t} \int_{\Omega \cap B_{H_{0}}(x,1)} e^{-h(y,s)} H(\nabla u(y,s)) \, dy \, ds$$

$$\leq C_{1} t^{\sigma'} \sup_{x \in \Omega} \int_{\Omega \cap B_{H_{0}}(x,1)} e^{-H_{0}(y)^{2}} |\phi(y)| \, dy$$
(3.35)

for all $t \in (0, T_*)$, where T_* is as in Proposition 3.1. Furthermore, there exists a positive constant C_2 such that

$$\sup_{x \in \Omega} \left(\int_{\Omega \cap B_{H_0}(x,1)} |e^{-h(y,t)} u(y,t)|^2 dy \right)^{\frac{1}{2}}$$

$$\leq C_2 t^{-\frac{N}{4}} \sup_{x \in \Omega} \int_{\Omega \cap B_{H_0}(x,1)} e^{-H_0(y)^2} |\phi(y)| dy$$
(3.36)

for all $t \in (0, T_*)$.

Proof. We use the same notation and assume I(0) = 1 as in the proof of Proposition 3.1. By Lemma 3.3 we have

$$\begin{split} & \int_0^t \int_{\Omega \cap B_{H_0}(x,1)} e^{-h(y,s)} H(\nabla u(y,s)) \, dy \, ds \\ & \leq \frac{1}{2} \limsup_{\epsilon \to 0} \left[J_+^{\epsilon}(x,t) + J_-^{\epsilon}(x,t) \right] + \frac{1}{2} \int_0^t \int_{\Omega \cap B_{H_0}(x,2)} s^{-\sigma} |z|^{1-\delta} \eta^2 \, dy \, ds \\ & \leq C \int_0^t \int_{\Omega \cap B_{H_0}(x,2)} s^{\sigma-1} |z|^{1+\delta} \, dy \, ds + \frac{M}{2} \int_0^t \int_{\Omega \cap B_{H_0}(x,1)} s^{-\sigma} [1+|z|] \, dy \, ds \end{split}$$

for all $x \in \Omega$ and $0 < t < T_* = T_2$. Then, similarly to (3.32), by Lemma 3.2 and (3.31) we obtain

$$\sup_{x \in \Omega} \int_0^t \int_{\Omega \cap B_{H_0}(x,1)} e^{-h(y,s)} H(\nabla u(y,s)) \, dy \, ds
\leq C \int_0^t s^{\sigma - 1 - \frac{\delta N}{2}} I(s)^{1+\delta} \, ds + \frac{M}{2} \int_0^t s^{-\sigma} [I(s) + |B_{H_0}(x,1)|] \, ds$$

for all $0 < t < T_* = T_2$. This together with Proposition 3.1 implies that

$$\sup_{x \in \Omega} \int_0^t \int_{\Omega \cap B_{H_0}(x,1)} e^{-h(y,s)} H(\nabla u(y,s)) \, dy \, ds \le C \left[t^{\sigma - \frac{\delta N}{2}} + t^{1-\sigma} + t \right] I(0) \le C t^{\sigma'} I(0)$$

for all $0 < t < T_* = T_2$, where $\sigma' = \min\{\sigma - \delta N/2, 1 - \sigma, 1\}$. Thus (3.35) holds. On the other hand, by Lemma 3.2 and Proposition 3.1 we have

$$\sup_{x \in \Omega} \|z(t)\|_{L^2(\Omega \cap B_{H_0}(x,1))} \le Ct^{-\frac{N}{4}} \sup_{0 < s < t} \sup_{x \in \Omega} \|z(s)\|_{L^1(\Omega \cap B_{H_0}(x,1))} \le Ct^{-\frac{N}{4}} I(0)$$

for all $0 < t < T_*$. This implies (3.36). Thus Proposition 3.2 follows. \square

4 Proof of Theorem 1.2

In this section we prove Theorem 1.2. We prepare the following proposition.

Proposition 4.1 Assume (1.1) and (1.3). Let u be a solution of (3.1). Then

$$\sup_{0 < t < S_{\lambda}} \int_{\Omega} e^{-2g(y,t)} u(y,t)^{2} dy \le \int_{\Omega} e^{-2\lambda H_{0}(y)^{2}} \phi(y)^{2} dy, \tag{4.1}$$

where $\lambda > 0$, $g(x,t) := \lambda H_0(x)^2/(1-4\lambda t)$ and $S_{\lambda} = 1/4\lambda$.

Proof. Similarly to (3.10), it follows that

$$\int_0^t \int_{\Omega} H(\nabla u) \nabla_{\xi} H(\nabla u) \nabla [e^{-2g} u] \, dy \, ds$$

$$\geq \int_0^t \int_{\Omega} e^{-2g} H(\nabla u)^2 \, dy \, ds - 2 \int_0^t \int_{\Omega} e^{-2g} H(\nabla u) H(\nabla g) u \, dy \, ds$$

$$\geq (1 - \mu) \int_0^t \int_{\Omega} e^{-2g} H(\nabla u)^2 \, dy \, ds - \mu^{-1} \int_0^t \int_{\Omega} e^{-2g} u^2 H(\nabla g)^2 \, dy \, ds$$

for all $t \in (0, S_{\lambda})$, where $\mu > 0$. Then, similarly to (3.11), by (3.1) we have

$$\frac{1}{2} \int_{\Omega} e^{-2g} u(y,s)^2 dy \Big|_{s=0}^{s=t} + (1-\mu) \int_{0}^{t} \int_{\Omega} e^{-2g} H(\nabla u)^2 dy ds
\leq \int_{0}^{t} \int_{\Omega} e^{-2g} u^2 \left[\mu^{-1} H(\nabla g)^2 - \partial_t g \right] dy ds$$
(4.2)

for $t \in (0, S_{\lambda})$. Setting $\mu = 1$, by (2.2) we see that

$$\mu^{-1}H(\nabla g)^2 - \partial_t g = \frac{4\lambda^2}{(1 - 4\lambda t)^2}H_0(y)^2 - \frac{4\lambda^2 H_0(y)^2}{(1 - 4\lambda t)^2} = 0.$$

This together with (4.2) implies that

$$\frac{1}{2} \int_{\Omega} e^{-2g(y,t)} u(y,t)^2 dy \le \frac{1}{2} \int_{\Omega} e^{-2g(y,0)} u(y,0)^2 dy$$

for all $t \in (0, S_{\lambda})$. Thus (4.1) holds. The proof is complete. \square

Now we are ready to complete the proof of assertion (ii) of Theorem 1.2.

Proof of assertion (ii) of Theorem 1.2. It suffices to consider the case $|\mu|(\mathbf{R}^N) \neq 0$.

Let $\lambda > \Lambda > 0$ and assume (1.12). Due to (1.5), we can assume, without loss of generality, that $\lambda > 1 > \Lambda > 0$.

By the Jordan decomposition theorem there exist two nonnegative Radon measure μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$. Furthermore, we can find sequences $\{\mu_n^{\pm}\} \subset C^{\infty}(\mathbf{R}^N)$ such that

$$\lim_{n \to \infty} \int_{\mathbf{R}^N} \mu_n^{\pm}(x) \psi(x) \, dx = \int_{\mathbf{R}^N} \psi(x) \, d\mu^{\pm}(y) \quad \text{for} \quad \psi \in C_0(\mathbf{R}^N). \tag{4.3}$$

For any $m = 1, 2, ..., \text{ let } \zeta_m \in C_0^{\infty}(\mathbf{R}^N)$ be such that

$$\zeta_m = 1$$
 in $B_{H_0}(0, m/2)$, supp $\zeta_m \subset B_{H_0}(0, m)$, $0 \le \zeta_m \le 1$ in \mathbf{R}^N .

Let $u_{m,n}$ be a solution of (3.1) with Ω and ϕ replaced by $B_m := B_{H_0}(0,m)$ and $\mu_{m,n} := \zeta_m(\mu_n^+ - \mu_n^-) \in C_0^{\infty}(B_m)$, respectively. Setting $u_{m,n}(\cdot,t) = 0$ outside B_m for any t > 0 and applying Propositions 3.1 and 3.2, we obtain

$$\sup_{0 < t < T_*} \sup_{x \in \mathbf{R}^N} \int_{B_m \cap B_{H_0}(x,1)} e^{-h(y,t)} |u_{m,n}(y,t)| \, dy
+ \sup_{0 < t < T_*} t^{-\sigma'} \sup_{x \in \mathbf{R}^N} \int_0^t \int_{B_m \cap B_{H_0}(x,1)} e^{-h(y,s)} H(\nabla u_{m,n}(y,s)) \, dy \, ds
+ \sup_{0 < t < T_*} t^{\frac{N}{4}} \sup_{x \in \mathbf{R}^N} \left(\int_{B_m \cap B_{H_0}(x,1)} |e^{-h(y,t)} u_{m,n}(y,t)|^2 \, dy \right)^{\frac{1}{2}}
\leq C I_{m,n} \quad \text{with} \quad I_{m,n} := \sup_{x \in \mathbf{R}^N} \int_{B_m \cap B_{H_0}(x,1)} e^{-H_0(y)^2} |\mu_{m,n}(y)| \, dy$$
(4.4)

for all m, n = 1, 2, ..., where T_* and σ' are as in Propositions 3.1 and 3.2, respectively, and $h(x,t) := H_0(x)^2(1+t^\ell)$. Here and in what follows, C denotes generic positive constant independent of m and n.

Let $1 < \lambda' < \lambda$. Then we can find $t_* \in (0, T_*)$ such that $1 + t_*^{\ell} < \lambda'$. Then, by (4.4) we see that

$$\int_{\mathbf{R}^N} e^{-2\lambda' H_0(y)^2} u_{m,n}(y,\tau)^2 dy = \int_{B_m} e^{-2\lambda' H_0(y)^2} u_{m,n}(y,\tau)^2 dy$$

$$\leq C \sup_{x \in \mathbf{R}^N} \int_{B_m \cap B_{H_0}(x,1)} e^{-2h(y,\tau)} u_{m,n}(y,\tau)^2 dy \leq C\tau^{-\frac{N}{2}} I_{m,n}^2$$

for $\tau \in (0, t_*]$. Applying Proposition 4.1 with $u(y, t) = u_{m,n}(y, t + \tau)$ and $\lambda = \lambda'$, we obtain

$$\sup_{\tau < t < S_{\lambda'} + \tau} \int_{\mathbf{R}^N} e^{-\frac{2\lambda' H_0(y)^2}{1 - 4\lambda'(t - \tau)}} u_{m,n}(y, t)^2 dy \le C \int_{\mathbf{R}^N} e^{-2\lambda' H_0(y)^2} u_{m,n}(y, \tau)^2 dy$$

$$\le C \tau^{-\frac{N}{2}} I_{m,n}^2$$

$$(4.5)$$

for all $\tau \in (0, t_*]$. Taking a sufficiently small $t_* \in (0, T_*)$ if necessary, we see that

$$1 + t^{\ell} \le 1 + t_{*}^{\ell} < \lambda' < \lambda < \frac{\lambda}{1 - 4\lambda t} \quad \text{for} \quad 0 < t < t_{*},$$

$$\frac{1}{1 - 4\lambda'(t - t_{*})} < \frac{1}{1 - 4\lambda t} - 2\lambda t_{*} \quad \text{for} \quad 0 < t < S_{\lambda}.$$

Then, by (4.4) and by applying the Hölder inequality to (4.5) with $\tau = t_*$, we obtain

$$\sup_{0 < t < S_{\lambda}} \int_{\mathbf{R}^{N}} e^{-\lambda H_{0}(y)^{2}/(1 - 4\lambda t)} |u_{m,n}(y,t)| \, dy \le C I_{m,n} \tag{4.6}$$

for m, n = 1, 2,

On the other hand, it follows from (1.12) with $\Lambda < 1$ and (4.3) that

$$\lim_{n \to \infty} \sup I_{m,n} \le \lim_{n \to \infty} \int_{B_m} e^{-H_0(y)^2} |\mu_{m,n}(y)| \, dy$$

$$= \lim_{n \to \infty} \int_{\mathbf{R}^N} e^{-H_0(y)^2} \zeta_m(y) [\mu_n^+(y) + \mu_n^-(y)] \, dy = \int_{\mathbf{R}^N} e^{-H_0(y)^2} \zeta_m(y) \, d|\mu|(y)$$

$$\le \int_{\mathbf{R}^N} e^{-H_0(y)^2} \, d|\mu|(y) \le C \sup_{x \in \mathbf{R}^N} \int_{B_{H_0}(x,1/\sqrt{\Lambda})} e^{-\Lambda H_0(y)^2} \, d|\mu|(y) < \infty$$

for $m = 1, 2, \ldots$ Since $|\mu|(\mathbf{R}^N) \neq 0$, by the diagonal argument we can find a sequence $\{n_m\}_{m=1}^{\infty}$ such that

$$\sup_{m} I_{m,n_m} \le C \sup_{x \in \mathbf{R}^N} \int_{B_{H_0}(x,1/\sqrt{\Lambda})} e^{-\Lambda H_0(y)^2} d|\mu|(y) < \infty.$$
 (4.7)

This together with (4.5) implies that

$$\sup_{m} \|u_{m,n_m}\|_{L^2(K)} < \infty$$

for any compact set $K \subset \mathbf{R}^N \times (0, S_\lambda)$ and $m = 1, 2, \ldots$. Then, by (1.15) we apply the standard parabolic regularity theorems to the solution u_{m,n_m} and we can find $\alpha \in (0,1)$ such that

$$\sup_{m} \|u_{m,n_m}\|_{C^{1,\alpha;0,\alpha/2}(K)} < \infty \tag{4.8}$$

for any compact set $K \subset \mathbf{R}^N \times (0, S_\lambda)$. See [30, Chapter III]. (See also [14], [28], [29] and the last comment in [10].) Therefore, by the Ascoli-Arzelà theorem and the diagonal argument, taking a subsequence if necessary, we can find a function $u \in C^{1,\alpha;0,\alpha/2}(\mathbf{R}^N \times (0,S_\lambda))$ such that

$$\lim_{m \to \infty} u_{m,n_m}(x,t) = u(x,t), \quad \lim_{m \to \infty} \nabla u_{m,n_m}(x,t) = \nabla u(x,t), \tag{4.9}$$

uniformly on any compact set $K \subset \mathbf{R}^N \times (0, S_\lambda)$. Furthermore, due to (4.4), (4.7) and (4.9), the Fatou lemma implies that $u \in L^1_{loc}([0, S_\lambda) : W^{1,1}(B_{H_0}(0, R)))$ for any R > 0. Moreover, by (4.4) and (4.7) we see that

$$\int_0^{\epsilon} \int_{B_{H_0}(0,R)} \left[|u_{m,n_m}| + H(\nabla u_{m,n_m})| \right] dy ds \le C[\epsilon + \epsilon^{\sigma'}]$$
(4.10)

for all sufficiently small $\epsilon > 0$. Then, by (4.8), (4.9) and (4.10) we apply the Lebesgue

dominated convergence theorem to obtain

$$\lim_{m \to \infty} \int_0^t \int_{\mathbf{R}^N} \left[-u_{m,n_m} \partial_t \varphi + H(\nabla u_{m,n_m}) \nabla_{\xi} H(\nabla u_{m,n_m}) \nabla \varphi \right] \, dy \, ds$$

$$= \lim_{m \to \infty} \int_0^{\epsilon} \int_{\mathbf{R}^N} \left[-u_{m,n_m} \partial_t \varphi + H(\nabla u_{m,n_m}) \nabla_{\xi} H(\nabla u_{m,n_m}) \nabla \varphi \right] \, dy \, ds$$

$$+ \lim_{m \to \infty} \int_{\epsilon}^t \int_{\mathbf{R}^N} \left[-u_{m,n_m} \partial_t \varphi + H(\nabla u_{m,n_m}) \nabla_{\xi} H(\nabla u_{m,n_m}) \nabla \varphi \right] \, dy \, ds$$

$$= O(\epsilon + \epsilon^{\sigma'}) + \int_{\epsilon}^t \int_{\mathbf{R}^N} \left[-u \partial_t \varphi + H(\nabla u) \nabla_{\xi} H(\nabla u) \nabla \varphi \right] \, dy \, ds$$

for any $\varphi \in C_0^{\infty}(\mathbf{R}^N \times [0, S_{\lambda}))$ and $0 < t < S_{\lambda}$. Since $u \in L^1_{loc}([0, S_{\lambda}) : W^{1,1}(B_{H_0}(0, R)))$ for any R > 0 and ϵ is arbitrary, we deduce that

$$\lim_{m \to \infty} \int_0^t \int_{\mathbf{R}^N} \left[-u_{m,n_m} \partial_t \varphi + H(\nabla u_{m,n_m}) \nabla_{\xi} H(\nabla u_{m,n_m}) \nabla \varphi \right] \, dy \, ds$$

$$= \int_0^t \int_{\mathbf{R}^N} \left[-u \partial_t \varphi + H(\nabla u) \nabla_{\xi} H(\nabla u) \nabla \varphi \right] \, dy \, ds.$$

Furthermore, recalling the weak formulation of (3.1) with $u = u_{m,n_m}$, we see that

$$\lim_{m \to \infty} \int_{\mathbf{R}^N} u_{m,n_m}(y,t) \varphi(y,t) \, dy + \int_0^t \int_{\mathbf{R}^N} \left[-u \partial_t \varphi + H(\nabla u) \nabla_\xi H(\nabla u) \nabla \varphi \right] \, dy \, ds$$

$$= \lim_{m \to \infty} \int_{\mathbf{R}^N} \varphi(y,0) \mu_{m,n_m}(y) \, dy$$

for all $\varphi \in C_0^{\infty}(B_{H_0}(0,R) \times [0,S_{\lambda}))$ and $0 < t < S_{\lambda}$. This together with (4.3) and (4.9) implies that

$$\int_{\mathbf{R}^{N}} u(y,t)\varphi(y,t) \, dy + \int_{0}^{t} \int_{\mathbf{R}^{N}} \left[-u\partial_{t}\varphi + H(\nabla u)\nabla_{\xi}H(\nabla u)\nabla\varphi \right] \, dy \, ds$$

$$= \int_{\mathbf{R}^{N}} \varphi(y,0) \, d\mu(y)$$

for all $0 < t < S_{\lambda}$ and $\varphi \in C_0^{\infty}(B_{H_0}(0, R) \times [0, S_{\lambda}))$. In addition, by (4.6), (4.7) and (4.9) we apply the Fatou lemma again and see that u satisfies (1.13). Therefore u is the desired solution of (1.6). Thus assertion (ii) of Theorem 1.2 follows. \square

Proof of assertion (i) of Theorem 1.2. Let u be a nonnegative solution of (1.4) in $\mathbf{R}^N \times (0,T)$ for some T>0. By (1.15), applying the same argument as in [25], we can find a unique Radon measure μ in \mathbf{R}^N such that μ satisfies (1.11) and

$$\int_{\mathbf{R}^N} e^{-C|y|^2} d\mu(y) < \infty$$

for some constant C > 0. Since H_0 is an equivalent norm to the Euclidean norm $|\cdot|$ of \mathbf{R}^N , we obtain

$$\int_{\mathbf{R}^N} e^{-C'H_0(y)^2} d\mu(y) < \infty.$$

Thus assertion (i) of Theorem 1.2 follows. \Box

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