SPANS, SIMPLICIAL FAMILIES AND THE FUNDAMENTAL PROGROUPOID

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ABSTRACT. In this paper we consider simplicial families, that is, simplicial objects indexed by a simplicial set. We develop a method to construct family hypercover refinements of a cover family based on the notion of *n-spans* that we introduce here. In [5] we had introduced the notion of *covering projection* in a topos. They are locally constant objects satisfying an additional condition which is valid in all locally constant objects when the topos is locally connected, and developed the theory of the fundamental groupoid of a general topos. Here we show that covering projections can be obtained as objects constructed from a descent datum of a simplicial set on a family of sets. We construct a groupoid $\mathbf{G}_{\mathcal{H}}$ such that the category $\mathcal{G}_{\mathcal{H}}$ of covering projections trivialized by $\mathcal{H}$ is its classifying topos. This determines a protopos $\{\mathcal{G}_{\mathcal{H}}\}_{\mathcal{H}}$ and a progroupoid $\{\mathbf{G}_{\mathcal{H}}\}_{\mathcal{H}}$, suitable indexed by a filtered poset of hypercovers. Then we show that this progroupoid classifies torsors. This construction is novel also in the case of locally connected topoi, showing that locally constant object in a locally connected topos are constructed by descent from a descent datum on a family of sets. The salient feature that distinguishes locally connected topoi is that the progroupoid is *strict*, that is, the transition morphisms are surjective on triangles, or, equivalently, the transition inverse image functors in the underlying indcategory are full and faithful.

CONTENTS

INTRODUCTION. Given a locally connected topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}et$, the category $\mathcal{P}_{\mathcal{U}}$ of locally constant objects trivialized by a fix cover $\mathcal{U}$ is the classifying topos of a groupoid $\mathbf{G}_{\mathcal{U}}$, $\mathcal{P}_{\mathcal{U}} = \beta\mathbf{G}_{\mathcal{U}}$. This determines a strict progroupoid $\pi_1(\mathcal{E})$ suitable indexed by a filtered poset of covers. This progroupoid is the fundamental progroupoid of $\mathcal{E}$ in the sense that for any (discrete) group K it classifies K-torsors ([1, expose IV], [10], [3], and for a resume of the theory [4, Appendix]).

For non-locally connected topos the category $\mathcal{P}_{\mathcal{U}}$ is not the classifying topos of a groupoid, it is not even an atomic topos. In [5] we introduced the notion of *covering projection* (locally constant objects with an additional property) and show how they can be used in place of locally constant objects to develop the theory for an arbitrary topos. However this is done using localic groupoids and the sophisticated results of [8].

In this paper we simplify the theory by considering *hypercovers*, in fact, family hypercover refinements of a cover family. When the topos is locally connected simplicial objects have a canonical indexing simplicial set $S_\bullet = \gamma!(H_\bullet)$, $H_\bullet \longrightarrow \gamma^* S_\bullet$. For non locally connected topoi the indexing has to be given explicitly as part of the datum, which leads to the notion of simplicial families. We show that the category $\mathcal{G}_{\mathcal{H}}$ of covering projections trivialized by a family hypercover is a descent topos on the indexing simplicial set, and as such the classifying topos of a groupoid whose existence does not depend of the results in [8]. An essential part of this paper is the construction of suitable family hypercover refinements of a cover family. We discover that the n-simplexes of a simplicial family furnish a notion of n-spans which is intimately related to the coskeleton functor, and which we use to construct these hypercover refinements.

We describe now in detail the contents of the paper. In Section 1 we define and examine the notion of *n-spans*, 1-spans are the usual spans. In Section 2 we review certain aspects of simplicial sets. In Section 3 we develop the concept of *simplicial families*, that is, simplicial objects in $\mathcal{E}$ indexed by a simplicial set, $H_\bullet \to \gamma^*(S_\bullet)$, and establish a correspondence between simplicial families and collections of n-spans associated to the n-simplices. In Section 4 we develop the concept of *family hypercover* refinements of a cover family. The principal result in this section is the construction of hypercover refinements of a cover family determined by a sets of objects and a sets of 1-spans. In Section 5 we construct and study certain groupoids associated to a simplicial family, and in Section 6 we establish results relating descent data on a simplicial family with left actions of the associated groupoid. In Section 7 we recall the notion of *covering projection* introduced in [5] and establish its basic properties. Then we prove the principal results of the paper, namely (1): For any covering projection X constant on a cover family $\mathcal{U} = U \longrightarrow \gamma^* S$ there is family hypercover refinement $\mathcal{H} = H_\bullet \to \gamma^*(S_\bullet)$ such that X is constant on $\mathcal{H}$, and (2): The category of covering projections constant on $\mathcal{H}$ is the topos of left action of a groupoid associated to the family hypercover. Finally, in Section 8 we apply all this to the construction of the fundamental progroupoid of a general topos.

I thank Matias de Hoyo for several fruitful conversations on the coskeleton construction.

CONTEXT. Throughout this paper $\mathcal{S} = Sets$ denotes the topos of sets, and *topos* means Grothendieck topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$. We argue in a way that should be valid if $\mathcal{S}$ is an arbitrary topos, but since we do not use *change of base*, we let the interested reader verify this.

1. Spans

A *1-span* is a diagram of the form

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \end{tikzcd}$$

A *2-span* is a commutative diagram of the form

$$\begin{tikzcd} & & \bullet \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & & \\ & & \bullet \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\ & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet & & \bullet \end{tikzcd}$$

The *dual* span is the span resulting from the symmetry respect the vertical axle. For example, the dual span of the span
$$\begin{tikzcd} & X \arrow[dl, "u"'] \arrow[dr, "v"] & \\ A & & B \end{tikzcd}$$
is
$$\begin{tikzcd} & X \arrow[dl, "v"'] \arrow[dr, "u"] & \\ B & & A \end{tikzcd}$$
In the same way we define the dual of a 2-span to be 2-span resulting from the symmetry respect the vertical axle. We stress the fact that spans are *ordered from left to right* structures.

A *n-span* is the commutative diagram resulting from the following procedure: At the top vertex stands a generic n-simplex (see section 2), then draw an arrow to each of its $n+1$ faces (considered ordered by the index). Then, repeat this procedure. At the bottom level stands the $n+1$ vertices of the generic n-simplex.

2. Simplicial sets

We fix some notation about simplicial sets, denoted by $S_\bullet$. A simplicial set has faces ∂_i and degeneracies σ_i as follows:

$$\begin{tikzcd} S_n \arrow[r, shift left, "\partial_i"] & S_{n-1} \arrow[l, shift left, "\sigma_i"] \end{tikzcd}, \quad n = 0,\, 1,\, \dots\, \infty, \quad \{\partial_i\}_{0 \leq i \leq n}, \quad \{\sigma_i\}_{0 \leq i \leq n-1}$$

subject to the usual equations.

We write $I = S_0$, the set of vertices or 0-simplexes. Given a 1-simplex $\ell \in S_1$ and $i, j \in I$, we write $i \xrightarrow{\ell} j$ to mean $i = \partial_1(\ell)$, $j = \partial_0(\ell)$.

Given a 2-simplex $w \in S_2$, we write

$$\begin{tikzcd} & j \arrow[dr, "r"] & \\ i \arrow[ur, "\ell"] \arrow[rr, "t"'] \arrow[r, phantom, "w" near end, shift left=3] & & k \end{tikzcd}$$

to mean that $\partial_2(w) = \ell$, $\partial_1(w) = t$, $\partial_0(w) = r$. We define $i = \varrho_2(w)$, $j = \varrho_1(w)$, $k = \varrho_0(w)$ to be the three pairs of equal composites of faces.

The simplicial equations show that this fits correctly. We say that the pair $i \xrightarrow{\ell} j \xrightarrow{r} k$ *compose.* Notice that we have:

$$\begin{tikzcd} & j \arrow[dr, "\sigma_0(j)"] & \\ i \arrow[ur, "\ell"] \arrow[rr, "\ell"'] & \sigma_1(\ell) & j \end{tikzcd} \qquad \begin{tikzcd} & i \arrow[dr, "\ell"] & \\ i \arrow[ur, "\sigma_0(i)"] \arrow[rr, "\ell"'] & \sigma_0(\ell) & j \end{tikzcd}$$

Recall that a category can be seen as a simplicial set such that given any pair $i \xrightarrow{\ell} j \xrightarrow{r} k$ there is a unique $w \in S_2$ such that $\partial_2(w) = \ell$, $\partial_0(w) = r$.

We recall now the construction of a category and a groupoid associated to a simplicial set, which involve only the first three terms.

$$\begin{tikzcd} S_2 \arrow[r, shift left=4, "\partial_0"] \arrow[r, "\partial_1"] \arrow[r, shift right=4, "\partial_2"'] & S_1 \arrow[l, shift left=2, "\sigma_0"] \arrow[l, shift right=2, "\sigma_1"'] \arrow[r, shift left=2, "\partial_0"] \arrow[r, shift right=2, "\partial_1"'] & S_0 \arrow[l, "\sigma_0"] \end{tikzcd}$$

2.1. **Proposition** (Fundamental category and groupoid of a simplicial set, [6])**.**

Objects: The set of objects is the set $I = S_0$.

Premorphisms: Basic premorphisms $i \xrightarrow{\ell} j$, $i, j \in S_0$, are 1-simplexes $\ell \in S_1$, $\partial_0(\ell) = j$, $\partial_1(\ell) = i$. A general premorphism $i \xrightarrow{\phi} j$ is a sequence $\phi = (\ell_n \, \ldots \, \ell_2 \, \ell_1)$, $\ell_k \in S_1$, $\partial_1(\ell_1) = i$, $\partial_1(l_{k+1}) = \partial_0(\ell_k)$, $n \geq 1$, $1 \leq k \leq n-1$, $\partial_0(\ell_n) = j$. When $n = 1$ we write $(\ell_1) = \ell$. Premorphisms compose by concatenation.

Morphisms: The set of morphisms is the quotient of the set of premorphisms by the equivalent relation generated by the following basic *pairs:*

(1a) Given $i \xrightarrow{\ell} j \xrightarrow{r} k$, then $i \xrightarrow{(r\,\ell)} k \;\sim\; i \xrightarrow{t} k$ if there is $w \in S_2$ such that $\partial_2(w) = \ell$, $\partial_1(w) = t$, $\partial_0(w) = r$. That is, for each $w \in S_2$ we establish $\partial_1(w) \sim (\partial_0(w)\, \partial_2(w))$.

The arrow $i \xrightarrow{\sigma_0(i)} i$ becomes the identity morphism of i, $id_i = \sigma_0(i)$.

(1b) The groupoid is obtained by formally inverting all the arrows of the category. $\square$

2.2. **Definition.** *A contravariant simplicial morphism $S_\bullet \xrightarrow{h_\bullet} T_\bullet$ between two simplicial sets is a family of maps $S_n \xrightarrow{h_n} T_n$ such that*

$$\partial_i(h_n(w)) = h_{n-1}\partial_{n-i}(w), \quad \sigma_i(h_{n-1}(w)) = h_n(\sigma_{n-1-i}(w)).$$

2.3. **Definition.** *A* strict duality *in a simplicial set $S_\bullet$ is a contravariant simplicial isomorphism $S_\bullet \xrightarrow{\tau_\bullet} S_\bullet$ with $\tau_0 = id$. We denote τ in both directions $\tau \circ \tau^{-1} = id$. A simplicial set with a strict duality is said to be* self-dual.

For any vertex i, $\tau_0(i) = i$. For any n-simplex w, $n > 0$, we will denote $\tau_n(w) = w^{op}$, omitting the n.

$$w \in S_n : \;\; \partial_i(w^{op}) = \partial_{n-i}(w)^{op}, \quad \sigma_i(w^{op}) = (\sigma_{n-1-i}(w)^{op}.$$

2.4. **Remark.** Clearly, the notion of strict duality applies to a simplicial object in any category.

The following is clear:

2.5. **Proposition.** *Let $S_\bullet$ be a self-dual simplicial set such that:*

$$\forall\, i \xrightarrow{\ell} j \in S_1 \;\exists\, w \in S_2, \qquad \begin{tikzcd} & j \arrow[dr, "\ell^{op}"] & \\ i \arrow[ur, "\ell"] \arrow[rr, "\partial_1(w)"'] \arrow[rr, phantom, shift left=4, "w"] & & i \end{tikzcd}, \qquad \partial_1(w) = \sigma_0(i).$$

Then the fundamental category is already groupoid. □

2.6. **Example** (Cech nerve)**.** Given a family $\mathcal{U} = (U, I, \zeta)$, $\{U_i\}_{i \in I}$, $U \xrightarrow{\zeta} \gamma^* I$, in a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$, the *Cech simplicial set* [1] is the simplicial set $N_\bullet$ whose n-simplexes are given by $N_n = \{(i_0, i_1, \dots i_n) \mid U_{i_0} \times U_{i_1} \dots \times U_{i_n} \neq \emptyset\} \subset I^{n+1}$, in particular $N_0 = I$, $N_1 = \{(i,\, j) \,|\, U_i \times U_j \neq \emptyset\}$, $N_2 = \{(i,\, j,\, k) \,|\, U_i \times U_j \times U_k \neq \emptyset\}$.

The reader can check that it is a self-dual simplicial set. For $i \in N_0$, $w = (i,\, j,\, k) \in N_2$, $\sigma_0(i) = (i,\, i)$, and $\partial_2(w) = (i,\, j)$, $\partial_0(w) = (j,\, k)$, $\partial_1(w) = (i,\, k)$. Given $\ell = (i,\, j) \in N_1$, $\ell^{op} = (j,\, i)$. Then $w = (i,\, j,\, i)$ establish the condition in proposition 2.5. Thus the fundamental category is a groupoid. □

3. Simplicial families

Recall that a family in a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$ is an arrow $\zeta \colon H \to \gamma^* S$. In alternative notation we write $\mathcal{H} = \{H_i\}_{i \in S}$. We say that the objects H_i are the *components* of H. Families are 3-tuples $\mathcal{H} = (H, S, \zeta)$, and H is the coproduct $H = \sum_{i \in S} H_i$ in $\mathcal{E}$.

Remark that the same object H can be indexed by a different set, having then a different set of components.

A *morphism of families* $(Y,\, J,\, \xi) \xrightarrow{(h, \alpha)} (H,\, S,\, \zeta)$, is a pair $Y \xrightarrow{h} H$, $J \xrightarrow{\alpha} S$, making the following square commutative:

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "\xi"] & H \arrow[d, "\zeta"] \\ \gamma^* J \arrow[r, "\gamma^* \alpha"] & \gamma^* S \end{tikzcd}$$

In alternative notation, this corresponds to $h = \{Y_i \xrightarrow{h_i} H_{\alpha(i)}\}_{i \in J}$.

We also say that $\mathcal{Y}$ is a *refinement* of $\mathcal{H}$.

Assumption. *We shall assume always that the components of the families are non empty, $H_i \neq \emptyset$ for all $i \in I$.*

3.1. **Definition.** *A* simplicial family *is a 3-tuple $\mathcal{H}_\bullet = (H_\bullet, S_\bullet, \zeta_\bullet)$, where $H_\bullet$, $S_\bullet$ are simplicial objects in $\mathcal{E}$, $\mathcal{S}$ respectively, and $H_\bullet \xrightarrow{\zeta_\bullet} \gamma^*(S_\bullet)$ is a morphism of simplicial objects in $\mathcal{E}$. Remark that we assume that $(H_n)_w \neq \emptyset$.*

[1] Often called the *nerve* of $\mathcal{U}$

A morphism of simplicial families $(Y_\bullet, J_\bullet, \zeta_\bullet) \xrightarrow{(h_\bullet, \alpha_\bullet)} (H_\bullet, S_\bullet, \zeta_\bullet)$, *is a pair* $Y_\bullet \xrightarrow{h_\bullet} H_\bullet$, $J_\bullet \xrightarrow{\alpha_\bullet} S_\bullet$, *of simplicial morphisms making the following square commutative:*

$$\begin{tikzcd} Y_\bullet \arrow[r, "h_\bullet"] \arrow[d, "\xi_\bullet"] & H_\bullet \arrow[d, "\zeta_\bullet"] \\ \gamma^* J_\bullet \arrow[r, "\gamma^*\alpha_\bullet"] & \gamma^* S_\bullet \end{tikzcd}$$

We also say that $\mathcal{Y}_\bullet$ *is a* refinement *of* $\mathcal{H}_\bullet$.

In alternative notation, $h_n = \{(Y_n)_w \xrightarrow{(h_n)_w} (H_n)_{\alpha_n(w)}\}_{w \in J_n}$.

Notice that faces and degeneracy operators are morphisms of families:

$$\begin{tikzcd} H_n \arrow[r, shift left, "d_i"] \arrow[d, "\zeta_n"] & H_{n-1} \arrow[l, shift left, "s_i"] \arrow[d, "\zeta_{n-1}"] \\ \gamma^* S_n \arrow[r, shift left, "\gamma^*\partial_i"] & \gamma^* S_{n-1} \arrow[l, shift left, "\gamma^*\sigma_i"] \end{tikzcd}$$

and that in alternative notation correspond to families of maps

$$\{\, (H_n)_w \xrightarrow{(d_i)_w} (H_{n-1})_{\partial_i(w)} \,\}_{w \in S_n} \qquad \{\, (H_n)_{\sigma_i(w)} \xleftarrow{(s_i)_w} (H_{n-1})_w \,\}_{w \in S_{n-1}}$$

We make now some considerations involving the first three terms.

$$\begin{tikzcd}[column sep=huge] H_2 \arrow[r, shift left=4, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=4, "d_2"'] \arrow[d, "\zeta_2"] & H_1 \arrow[l, shift left=2, "s_0"'] \arrow[l, shift right=2, "s_1"] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "\zeta_1"] & H_0 \arrow[l, "s_0"'] \arrow[d, "\zeta_0"] \\ \gamma^* S_2 \arrow[r, shift left=4, "\partial_0"] \arrow[r, "\partial_1"] \arrow[r, shift right=4, "\partial_2"'] & \gamma^* S_1 \arrow[l, shift left=2, "\sigma_0"'] \arrow[l, shift right=2, "\sigma_1"] \arrow[r, shift left=2, "\partial_0"] \arrow[r, shift right=2, "\partial_1"'] & \gamma^* S_0 \arrow[l, "\sigma_0"'] \end{tikzcd}$$

3.2. **Remark.**

(1) Each 1-simplex $\ell \in S_1$, $i \xrightarrow{\ell} j$ determines a 1-span

$$\begin{tikzcd} & (H_1)_\ell \arrow[dl, "(d_1)_\ell"'] \arrow[dr, "(d_0)_\ell"] & \\ (H_0)_i & & (H_0)_j \end{tikzcd}$$

(2) For each vertex $i \in S_0$ we have a morphism of spans:

$$\begin{tikzcd}
& (H_0)_i \arrow[dl, "id"'] \arrow[dr, "id"] \arrow[dd, "(s_0)_i"] & \\
(H_0)_i & & (H_0)_i \\
& (H_1)_{\sigma_0(i)} \arrow[ul, "(d_1)_{\sigma_0(i)}"] \arrow[ur, "(d_0)_{\sigma_0(i)}"'] &
\end{tikzcd}$$

(3)

Each 2-simplex $w \in S_2$ $\begin{tikzcd} & j \arrow[dr, "r"] & \\ i \arrow[ur, "\ell"] \arrow[rr, "t"'] \arrow[rr, phantom, shift left=3ex, "w"] & & k \end{tikzcd}$ determines a 2-span

$$\begin{tikzcd}
& & (H_2)_w \arrow[dl, bend right, "(d_2)_w"'] \arrow[d, "(d_1)_w"] \arrow[dr, bend left, "(d_0)_w"] & & \\
& & (H_1)_t \arrow[dll, bend right, "(d_1)_t"'] \arrow[drr, bend left, "(d_0)_t"] & & \\
& (H_1)_\ell \arrow[dl, "(d_1)_\ell"] \arrow[dr, "(d_0)_\ell"'] & & (H_1)_r \arrow[dl, "(d_1)_r"] \arrow[dr, "(d_0)_r"'] & \\
(H_0)_i & & (H_0)_j & & (H_0)_k
\end{tikzcd}$$

The composites determine three maps $\begin{tikzcd} & (H_2)_w \arrow[dl, "(p_2)_w"'] \arrow[d, "(p_1)_w"] \arrow[dr, "(p_0)_w"] & \\ (H_0)_i & (H_0)_j & (H_0)_k \end{tikzcd}$

□

3.3. **Definition.** *A* self-dual *simplicial family is a simplicial family $\mathcal{H}_\bullet = (H_\bullet, S_\bullet, \zeta_\bullet)$ together with a pair of strict self-dualities τ such that $\zeta_\bullet \circ \tau_\bullet = \gamma^*\tau_\bullet \circ \zeta_\bullet$. That is, for $w \in S_n$, $(H_n)_w \xrightarrow{\tau_n} (H_n)_{w^{op}}$.*

$$\begin{tikzcd}
H_n \arrow[r, "d_{n-i}"] \arrow[d, "\tau_n"] & H_{n-1} \arrow[d, "\tau_{n-1}"] \\
H_n \arrow[r, "d_i"] & H_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
(H_n)_w \arrow[r, "(d_{n-i})_w"] \arrow[d, "(\tau_n)_w"] & (H_{n-1})_{\partial_{n-i}(w)} \arrow[d, "(\tau_{n-1})_{\partial_{n-i}(w)}"] \\
(H_n)_{w^{op}} \arrow[r, "(d_i)_{w^{op}}"] & (H_{n-1})_{\partial_{n-i}(w)^{op}}
\end{tikzcd}$$

$\partial_{n-i}(w)^{op} = \partial_i(w^{op})$. This means that τ establishes an isomorphism between the span of w^{op} and the dual span of the span of w.

3.4. **Remark.**

(1) For $\ell \in S_1$, if $i \xrightarrow{\ell} j$, then $j \xrightarrow{\ell^{op}} i$, and $(H_1)_{\ell^{op}} \cong (H_1)_\ell^{op}$ in the sense that $(\tau_1)_\ell$ establishes an isomorphism between the span ℓ^{op} and the dual span of ℓ :

$$\begin{tikzcd}
& (H_1)_{\ell^{op}} \arrow[dl, "(d_1)_{\ell^{op}}"'] \arrow[dr, "(d_0)_{\ell^{op}}"] \arrow[dd, "(\tau_1)_\ell", "\cong"'] & \\
(H_0)_j & & (H_0)_i \\
& (H_1)_\ell \arrow[ul, "(d_0)_\ell"] \arrow[ur, "(d_1)_\ell"'] &
\end{tikzcd}$$

(2)

For w $\in$ S_2, if
$$\begin{tikzcd} & j \arrow[dr, "r"] & \\ i \arrow[ur, "\ell"] \arrow[rr, "t"'] & w & k \end{tikzcd}$$
then
$$\begin{tikzcd} & j \arrow[dr, "\ell^{op}"] & \\ k \arrow[ur, "r^{op}"] \arrow[rr, "t^{op}"'] & w^{op} & i \end{tikzcd}$$
and

$(H_2)_{w^{op}} \cong (H_2)_w^{op}$ in the sense that that $(\tau_2)_w$ and $(\tau_1)_l$, $(\tau_1)_t$, $(\tau_1)_r$ establish an isomorphism between the 2-span of w^{op} and the dual 2-span of w. We leave to the interested reader to draw the corresponding diagram . □

3.5. **Example** (Cech nerve simplicial family)**.** Consider a family $U \xrightarrow{\zeta} \gamma^* I$ in a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$, and the Cech simplicial set $N_\bullet$, see example 2.6.

The *canonical simplicial object* $U_\bullet$ is the simplicial object in $\mathcal{E}$ whose n-simplexes are given by $U_n = U \times U \times \cdots U$ ($n+1$ times), $U_n = \sum_{(i_0, i_1, \dots i_n) \in N_n} U_{i_0} \times U_{i_1} \dots \times U_{i_n}$, in particular

$$U_0 = \sum_{i \in N_0} U_i \,, \quad U_1 = \sum_{(i,j) \in N_1} U_i \times U_j \,, \quad U_2 = \sum_{(i,j,k) \in N_2} U_i \times U_j \times U_k$$

It is easy to see that $U_\bullet$ is a self-dual simplicial object with faces given by the appropriate projections ans degeneracies by the appropriate diagonals. As for the self-duality, τ_1 is the usual symmetry of the cartesian product, τ_2 permutes the first and third factors, and leave unchanged the middle one, etc. The map $U \xrightarrow{\zeta} \gamma^* I$ determines a self-dual simplicial family $U_\bullet \xrightarrow{\zeta_\bullet} \gamma^*(N_\bullet)$ that we call *Cech simplicial family.* □

3.6. **Proposition.** *Given any simplicial family* $\mathcal{H}_\bullet = (H_\bullet, S_\bullet, \zeta_\bullet)$*, there is a canonical morphism of simplicial families*

$$\begin{tikzcd} H_\bullet \arrow[r, "h_\bullet"] \arrow[d, "\zeta_\bullet"] & U_\bullet \arrow[d, "\zeta_\bullet"] \\ \gamma^* S_\bullet \arrow[r, "\gamma^* \alpha_\bullet"] & \gamma^* N_\bullet \end{tikzcd}$$

where $(U \xrightarrow{\zeta} \gamma^* I) = (H_0 \xrightarrow{\zeta_0} \gamma^* S_0)$*. If the family is self-dual,* h *and* α *commute with the dualities.*

Proof. For the first three terms the proof should be clear by the remark 3.2: Given $\ell \in S_1$ and $w \in S_2$, $\alpha_1(\ell) = (\partial_1(\ell),\, \partial_0(\ell))$, $(h_1)_\ell = ((d_1)_\ell,\, (d_0)_\ell)$, $\alpha_2(w) = (\varrho_2(w),\, \varrho_1(w),\, \varrho_0(w))$, $(h_2)_w = ((p_2)_w,\, (p_1)_w,\, (p_0)_w)$. The second assertion follows by remark 3.4. For the higher simplexes the proof is the same and the interested reader can deduce the necessary calculations. □

4. Family hypercovers

We consider now $U \xrightarrow{\zeta} \gamma^* I$ to be a cover, that is the map $U \to 1$ an epimorphism, and we will establish the condition that says that $H_\bullet \xrightarrow{h_\bullet} U_\bullet$ is a hypercover in the sense of [2].

4.1 (**The coskeleton).**

By construction of the coskeleton we have, for $(\ell,\, t,\, r) \in S_1 \times S_1 \times S_1$,

$$(cosk_1 S_\bullet)_2 = \{(\ell,\, t,\, r) \mid \begin{tikzcd}[column sep=small] & j \arrow[dr, "r"] & \\ i \arrow[ur, "\ell"] \arrow[rr, "t"] & & k \end{tikzcd} \text{ (no w filling the triangle) } \}$$
$$= \{(\ell,\, t,\, r) \mid \partial_0(\ell) = \partial_1(r) = j,\, \partial_1(t) = \partial_1(\ell) = i,\, \partial_0(t) = \partial_0(r) = k\,\}$$

Let P_{ij} and $P_{\ell tr}$ be limit cones as follows:

$$\begin{tikzcd} & P_{ij} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_0"] & \\ (H_0)_i & & (H_0)_j \,, \end{tikzcd} \qquad \begin{tikzcd} & & & P_{\ell tr} \arrow[d, "\pi_1"] \arrow[ddll, bend right, "\pi_2"'] \arrow[ddrr, bend left, "\pi_0"] & & & \\ & & & (H_1)_t \arrow[dddlll, bend right, "(d_1)_t"'] \arrow[dddrrr, bend left, "(d_0)_t"] & & & \\ & (H_1)_\ell \arrow[dl, "(d_1)_\ell"] \arrow[drr, "(d_0)_\ell"'] & & & & (H_1)_r \arrow[dll, "(d_1)_r"] \arrow[dr, "(d_0)_r"'] & \\ (H_0)_i & & & (H_0)_j & & & (H_0)_k \end{tikzcd}$$

$P_{ij} = (H_0)_i \times (H_0)_j$, $N_1 = \{(i,\, j) \,|\, P_{ij} \neq \emptyset\,\}$. Let T_2 be $T_2 \subset (cosk_1 S_\bullet)_2$, $T_2 = \{(\ell,\, t,\, r) \,|\, P_{\ell tr} \neq \emptyset\,\}$. It follows $(i,\, j,\, k) \in N_2$, thus there is a function $T_2 \longrightarrow N_2$. Clearly for $w \in S_2$, $(\partial_2(w),\, \partial_1(w),\, \partial_0(w)) \in T_2$, defining a function $S_2 \longrightarrow T_2$. By construction of the coskeleton, we have:

$(H_0)_i = U_i$, indexed by S_0,

$((cosk_0 H_\bullet)_1)_{ij} = P_{ij}$, indexed by N_1,

$((cosk_1 H_\bullet)_2)_{\ell tr} = P_{\ell tr}$, indexed by T_2.

$$H_0 = U_0, \qquad (cosk_0 H_\bullet)_1 = U_1 = \sum_{(i,\, j) \in N_1} P_{ij},$$

$$(cosk_1 H_\bullet)_2 \;=\; \sum_{(\ell, t, r) \in T_2} P_{\ell tr} \;=\; \sum_{(i, j, k) \in N_2} \; \sum_{(\ell, t, r) \in (T_2)_{ijk}} P_{\ell tr}\,.$$

Clearly, for each $\ell \in S_1$, $i = \partial_1(\ell)$, $j = \partial_0(\ell)$, there is a map $(H_1)_\ell \longrightarrow P_{ij}$, and for each $w \in S_2$, $\ell = \partial_2(w)$, $t = \partial_1(w)$, $r = \partial_0(w)$, there is a map $(H_2)_w \longrightarrow P_{\ell tr}$, which are the components of the maps $H_1 \longrightarrow (cosk_0 H_\bullet)_1$ and $H_2 \longrightarrow (cosk_1 H_\bullet)_2$. From these considerations it follows:

4.2. **Definition.** *A indexed hypercover refinement of a cover* $(U \xrightarrow{\zeta} \gamma^* I)$ *is a simplicial family* $H_\bullet \xrightarrow{\zeta} S_\bullet$, $S_0 = I$, $H_0 = U$ *such that the canonical morphism* $H_\bullet \longrightarrow U_\bullet$ *is a hypercover, that is, the maps* $H_k \longrightarrow (cosk_{k-1} H_\bullet)_k$ *are epimorphic* ([2]). *This is the case when for each* $(i,\, j) \in N_1$ *and* $(\ell, t, r) \in T_2$, *the families* $\{(H_1)_\ell \longrightarrow P_{ij}\}_{\ell \in (S_1)_{ij}}$ *and* $\{(H_2)_w \longrightarrow P_{\ell tr}\}_{w \in (S_2)_{\ell tr}}$ *are epimorphic.* □

As we have seen in remark 3.2, a simplicial family $H_\bullet \xrightarrow{\zeta} S_\bullet$ determines a collection of spans in each dimension. The n-spans in this collection are in one to one correspondence with the set S_n of n-simplexes of the index simplicial set, while the object H_n of the simplicial object is the coproduct of the vertices of all the n-spans indexed by S_n. This collection of spans conform a set of data which is equivalent to the simplicial family. Thus, we can determine a simplicial family by specifying a suitable collections of spans.

Consider any family $U \xrightarrow{\zeta} \gamma^* I$. We will construct self-dual simplicial refinements of the Cech simplicial family

$$\begin{tikzcd} H_\bullet \arrow[r, "h_\bullet"] \arrow[d, "\xi_\bullet"] & U_\bullet \arrow[d, "\zeta_\bullet"] \\ \gamma^* S_\bullet \arrow[r, "\gamma^*\alpha_\bullet"] & \gamma^* N_\bullet \end{tikzcd}$$

determined by a given set of spans. This is similar to the construction of the coskeleton functor.

4.3. **Construction** (0-Span refinements of the Cech simplicial family)**.**

Let $\mathcal{C}$ be a set of *non empty* objects closed under isomorphism and such that $U_i \in \mathcal{C}$, all i. We will construct a self-dual simplicial refinement such that for any $w \in S_n$, the components $(H_n)_w$ are in $\mathcal{C}$. We describe in detail the first three terms, where the procedure is best understood.

The simplicial set $S_\bullet$:

(1) $S_0 = N_0$, $h_0 = id$.

(2) S_1 is the set of all 1-spans with vertex in $\mathcal{C}$ over the objects U_i. We write $S_1 \xrightarrow{\alpha_1} N_1$, and for $(i,\, j) \in N_1$, define the fibers of α_1 as:

$$(S_1)_{ij} \;=\; \left\{\, \ell \;=\; \begin{tikzcd}[column sep=small] & V \arrow[dl, "u"'] \arrow[dr, "v"] & \\ U_i & & U_j \end{tikzcd}, \quad V \in \mathcal{C} \right\}$$

and $\partial_0(\ell) = j$, $\partial_1(\ell) = i$, $\sigma_0(i) = (U_i \xleftarrow{id} U_i \xrightarrow{id} U_i)$.

(3) S_2 is the set of all 2-spans over the objects U_i determined by the objects of $\mathcal{C}$. We write $S_2 \xrightarrow{\alpha_2} N_2$, and for $(i,\, j,\, k) \in N_2$, define the fibers of α_2 as:

$$(S_2)_{ijk} \;=\; \left\{ w = \begin{tikzcd} & & W \arrow[ddl, bend right, "x"'] \arrow[d, "y"] \arrow[ddr, bend left, "z"] & & \\ & & Y \arrow[ddll, bend right, "u_b"'] \arrow[ddrr, bend left, "v_b"] & & \\ & X \arrow[dl, "u_a"] \arrow[dr, "v_a"'] & & Z \arrow[dl, "u_c"] \arrow[dr, "v_c"'] & \\ U_i & & U_j & & U_k \end{tikzcd} ,\; X,\, Y,\, Z,\, W \in \mathcal{C} \right\}$$

Taking the respective composites we have three maps $\begin{tikzcd}[column sep=small] & W \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ U_i & U_j & U_k \end{tikzcd}$.

The face operators are clear:

$$\partial_2(w) \;= \begin{tikzcd}[column sep=small] & X \arrow[dl, "u_a"'] \arrow[dr, "v_a"] & \\ U_i & & U_j \end{tikzcd}, \quad \partial_1(w) \;= \begin{tikzcd}[column sep=small] & Y \arrow[dl, "u_b"'] \arrow[dr, "v_b"] & \\ U_i & & U_k \end{tikzcd}, \quad \partial_0(w) \;= \begin{tikzcd}[column sep=small] & Z \arrow[dl, "u_c"'] \arrow[dr, "v_c"] & \\ U_j & & U_k \end{tikzcd}$$

Since $W \neq \emptyset$, $(\partial_2(w),\, \partial_1(w),\, \partial_0(w)) \in T_2$ (see 4.1), $S_2 \xrightarrow{\alpha_2} N_2$ factors $S_2 \longrightarrow T_2 \longrightarrow N_2$.

Given $\ell \in S_1$, the degeneracy operators $\sigma_0(\ell)$ and $\sigma_1(\ell)$ are given by:

$$
\begin{tikzcd}
 & & V \arrow[d, "id"] \arrow[ddl, bend right, "id"'] \arrow[ddr, bend left, "id"] & & \\
 & & V \arrow[ddll, bend right, "u"'] \arrow[ddrr, bend left, "v"] & & \\
 & V \arrow[dl, "u"] \arrow[dr, "v"] & & V \arrow[dl, "v"] \arrow[dr, "v"] & \\
 U_i & & U_j & & U_j
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & V \arrow[d, "id"] \arrow[ddl, bend right, "id"'] \arrow[ddr, bend left, "id"] & & \\
 & & V \arrow[ddll, bend right, "u"'] \arrow[ddrr, bend left, "v"] & & \\
 & V \arrow[dl, "u"] \arrow[dr, "u"] & & V \arrow[dl, "u"] \arrow[dr, "v"] & \\
 U_i & & U_i & & U_j
\end{tikzcd}
$$

The simplicial object $H_\bullet$:

It is determined by talking the coproducts of the vertices of the spans:

(1) $H_0 \xrightarrow{\zeta_0} \gamma^* S_0$ is defined by:

$$(H_0)_i = U_i\,,\ H_0 = \sum_{i \in S_0} U_i$$

(2) $H_1 \xrightarrow{\xi_1} S_1$ is defined by:

$$(H_1)_\ell = V, \qquad H_1 \ = \sum_{\ell \in S_1} V \ = \sum_{(i,j) \in N_1} \ \sum_{\ell \in (S_1)_{ij}} V.$$

Then $(s_0)_i = id_{U_i}$, $(d_0)_\ell = v$, and $(d_1)_\ell = u$. The map $V \xrightarrow{(u,v)} U_i \times U_j$ induce a map $H_1 \xrightarrow{h_1} U_1$ commuting with the simplicial structure.

(3) $H_2 \xrightarrow{\xi_2} S_2$ is defined by:

$$(H_2)_w = W\,, \qquad H_2 \ = \sum_{w \in S_2} W \ = \sum_{(\ell,t,r) \in T_2} \ \sum_{w \in (S_2)_{\ell t r}} W$$

$$= \sum_{(i,j,k) \in N_2} \ \sum_{(\ell,t,r) \in (T_2)_{ijk}} \ \sum_{w \in (S_2)_{\ell t r}} W\,.$$

Define $(s_0)_\ell = id_V$, $(s_1)_\ell = id_V$, $(d_0)_w = z$, $(d_1)_w = y$, $(d_2)_w = x$. The map $W \xrightarrow{(f,g,h)} U_i \times U_j \times U_k$ induce a map $H_2 \xrightarrow{h_2} U_2$ commuting with the simplicial structure.

The self-duality τ :

Recall that $\tau_0 = id$. Given $\ell \in S_1$, we define $\tau_1(\ell) = \ell^{op}$ as the pullback on the right below:

$$
\begin{tikzcd}
 & & V^{op} \arrow[dl, "u^{op}"'] \arrow[dr, "v^{op}"] & \\
\ell^{op} \ = & U_j & & U_i\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
V^{op} \arrow[r, "(u^{op},v^{op})"] \arrow[d, "(\tau_1)_\ell"] & U_j \times U_i \arrow[d, "\tau_1"] \\
V \arrow[r, "(u,v)"] & U_i \times U_j
\end{tikzcd}
$$

It follows $u^{op} = v \circ (\tau_1)_\ell$ and $v^{op} = u \circ (\tau_1)_\ell$, so that $(\tau_1)_\ell$ establishes an isomorphism between the span ℓ^{op} and the dual span of ℓ. This shows that τ_1 commutes with the family structure, $\xi_1 \circ \tau_1 = \gamma^*(\tau_1) \circ \xi_1$ (see 3.3).

Given $w \in S_2$, we define $\tau_2(w) = w^{op}$ as the following 2-span:

$$\begin{tikzcd}
 & & W^{op} \arrow[dl, bend right, "x^{op}"'] \arrow[d, "y^{op}"] \arrow[dr, bend left, "z^{op}"] & & \\
 & & Y^{op} \arrow[dll, bend right, "u_b^{op}"'] \arrow[drr, bend left, "v_b^{op}"] & & \\
 & Z^{op} \arrow[dl, "u_c^{op}"] \arrow[dr, "v_c^{op}"'] & & X^{op} \arrow[dl, "u_a^{op}"] \arrow[dr, "v_a^{op}"'] & \\
U_k & & U_j & & U_i
\end{tikzcd}$$

Where W^{op} is defined as the pullback on the right below:

$$\begin{tikzcd}
 & W^{op} \arrow[dl, "f^{op}"'] \arrow[d, "h^{op}"] \arrow[dr, "g^{op}"] & \\
U_k & U_j & U_i\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
W^{op} \arrow[r, "(f^{op},\, h^{op},\, g^{op})"] \arrow[d, "(\tau_2)_w"] & U_k \times U_j \times U_i \arrow[d, "\tau_2"] \\
W \arrow[r, "(f,\, h,\, g)"] & U_i \times U_j \times U_k
\end{tikzcd}$$

We have $f^{op} = g \circ (\tau_2)_w$, $h^{op} = h \circ (\tau_2)_w$ and $g^{op} = f \circ (\tau_2)_w$. It follows there are maps x^{op}, y^{op}, z^{op} (as indicated in the span diagram above) which satisfy the equations:

$(\tau_1)_r \circ x^{op} = z \circ (\tau_2)_w,\ \ (\tau_1)_t \circ y^{op} = y \circ (\tau_2)_w$ *and* $(\tau_1)_\ell \circ z^{op} = x \circ (\tau_2)_w\,.$

This shows that $(\tau_2)_w$ and $(\tau_1)_\ell$, $(\tau_1)_t$, $(\tau_1)_r$ establish an isomorphism between the 2-span w^{op} and the dual 2-span of w. That is, τ_2 commutes with the family structure, $\xi_2 \circ \tau_2 = \gamma^*(\tau_2) \circ \xi_2$ (see 3.3). □

4.4. **Construction** (1-Span refinements of the Cech simplicial family)**.**

Let $\mathcal{C}_{sp}$ be a set of *non empty* 1-spans closed under isomorphisms, the dual span, such that $(U_i \xleftarrow{id} U_i \xrightarrow{id} U_i) \in \mathcal{C}_{sp}$, all i, and such that $U_i \xleftarrow{u} V \xrightarrow{u} U_i\,,\ U_j \xleftarrow{v} V \xrightarrow{v} U_j \in \mathcal{C}_{sp}$ for all $U_i \xleftarrow{u} V \xrightarrow{v} U_j \in \mathcal{C}_{sp}$. Let $\mathcal{C}$ be the set of vertices of the spans in $\mathcal{C}_{sp}$. We will construct a selfdual simplicial refinement such that the set of 1-spans determined by the 1-simplexes is the set $\mathcal{C}_{sp}$ (and for $w \in S_n$, the component $(H_n)_w$ is in $\mathcal{C}$).

With the notation in construction 4.3, the 0-term is the same than in 4.3. The set S_1 is just defined to be the set $\mathcal{C}_{sp}$ with the same simplicial structure, and S_2 is also defined in the same way, but with the assumption that the three 1-spans with vertices X, Y, Z should be in $\mathcal{C}_{sp}$. The simplicial object $H_\bullet$ is defined exactly as in 4.3, and from the fact that $\mathcal{C}_{sp}$ is closed under the dual span and isomorphisms it follows that the definition of the selfduality τ in 4.3 also applies here. □

From definition 4.2 we have:

4.5. **Proposition.** *Given a cover* $U \xrightarrow{\zeta} \gamma^* I$*, if for each* $(i,\, j) \in N_1$ *and* $(\ell,\, t,\, r) \in T_2$ *we have:*

(1) If $\mathcal{C}$ as in construction 4.3 is such that the families of all maps $\{W \longrightarrow P_{ij}\}_{W \in \mathcal{C}}$ and $\{W \longrightarrow P_{\ell tr}\}_{W \in \mathcal{C}}$ are epimorphic. Or

(2) If $\mathcal{C}_{sp}$ and $\mathcal{C}$ as in construction 4.4 are such that the families of all maps $\{W \xrightarrow{(u,\,v)} P_{ij}\}_{(u,\,v) \in \mathcal{C}_{sp}}$ and $\{W \longrightarrow P_{\ell tr}\}_{W \in \mathcal{C}}$ are epimorphic.

Then the span refinement $H_\bullet \xrightarrow{h_\bullet} U_\bullet$ *is a hypercover.* □

4.6. **Example** (Canonical hypercover refinement of the Cech simplicial family by connected objects)**.**

We assume the topos (or the site) to be *locally connected* ([2], [9]). Consider a cover $\mathcal{U} = (U, S, \zeta)$, $U \xrightarrow{\zeta} \gamma^* I$ such that all the U_i are connected objects. Then, taking as $\mathcal{C}$ any set of connected generators the construction 4.3 yields an hypercover refinement of the Cech simplicial family in which all the components are connected. □

5. Fundamental groupoid of a simplicial family

We will now associate a groupoid in $\mathcal{S}$ to any self-dual simplicial family satisfying the following filling condition. We remark that this condition does not hold for the Cech simplicial family but it will hold for the span refinements.

5.1. **Definition.** *Let* $H_\bullet \xrightarrow{\xi} \gamma^*(S_\bullet)$, τ, *be any self-dual simplicial family, we say that* condition G *is satisfied if the following holds:*

For all $i \xrightarrow{\ell} j \in S_1$ *there exists a 2-simplex* $w \in S_2$,

$$\begin{tikzcd} & j \arrow[dr, "\ell^{op}"] \arrow[d, phantom, "w"] & \\ i \arrow[ur, "\ell"] \arrow[rr, "\partial_1(w)"] & {} & i \end{tikzcd},$$

such that $(d_1)_{\partial_1(w)} = (d_0)_{\partial_1(w)}$.

5.2. **Proposition.** *Any* 0*-span or* 1*-span simplicial refinement* $H_\bullet \xrightarrow{\xi} \gamma^*(S_\bullet)$ *of a family* $U \xrightarrow{\zeta} \gamma^* I$ *as in constructions 4.3 or 4.4 satisfies condition G.*

Proof. Let $\ell \in S_1$ be any 1-simplex, and let $w \in S_2$ be as follows:

$$\ell \;=\; \begin{tikzcd} & V \arrow[dl, "u"'] \arrow[dr, "v"] & \\ U_i & & U_j \end{tikzcd},$$

$$w \;=\; \begin{tikzcd} & & V \arrow[ddl, bend right, "id"'] \arrow[d, "id"] \arrow[ddr, bend left, "(\tau_1)_\ell"] & & \\ & & V \arrow[ddll, bend right, "u"'] \arrow[ddrr, bend left, "u"] & & \\ & V \arrow[dl, "u"] \arrow[dr, "v"'] & & V^{op} \arrow[dl, "u^{op}"] \arrow[dr, "v^{op}"'] & \\ U_i & & U_j & & U_i \end{tikzcd}$$

It is clear that w meets the requirements of condition G. □

5.3. **Proposition** (G-Fundamental Groupoid of a simplicial family)**.**

Let $H_\bullet \xrightarrow{\xi} \gamma^*(S_\bullet)$, τ, *be a self dual simplicial family satisfying condition G. Consider the fundamental category of the simplicial set* $S_\bullet$ *(proposition 2.1). Add to the equivalence relation that defines the morphism the following pairs:*

(2) $(i \xrightarrow{\ell} j) \sim (i \xrightarrow{t} j)$ *if there is a morphism of spans:*

$$\begin{tikzcd} & (H_1)_\ell \arrow[dl, "(d_0)_\ell"'] \arrow[dr, "(d_1)_\ell"] \arrow[dd] & \\ (H_0)_i & & (H_0)_j \\ & (H_1)_t \arrow[ul, "(d_0)_t"] \arrow[ur, "(d_1)_t"'] & \end{tikzcd}$$

Then, the resulting category is groupoid.

Proof. We have:

$$\begin{tikzcd} & (H_1)_{\partial_1(w)} \arrow[dl, "(d_1)_{\partial_1(w)}"'] \arrow[dr, "(d_0)_{\partial_1(w)}"] \arrow[d] & \\ (H_0)_i & (H_0)_i \arrow[l, "id"'] \arrow[r, "id"] \arrow[d, "(s_0)_i"] & (H_0)_i \\ & (H_1)_{\sigma_0(i)} \arrow[ul, "(d_1)_{\sigma_0(i)}"] \arrow[ur, "(d_0)_{\sigma_0(i)}"'] & \end{tikzcd}$$

This shows that $(\ell^{op}\,\ell) \sim \partial_1(w) \sim \sigma_0(i)$. We use the assumption in the 1-simplex ℓ^{op} to show $(\ell\,\ell^{op}) \sim \sigma_0(j)$. $\square$

The G-fundamental groupoid of a family does not coincide with the fundamental groupoid of the index simplicial set. Because of this we have added the letter G to the word $fundamental$ for the groupoid constructed in proposition 5.3.

6. Descent

Let $S_\bullet$ be a simplicial set, $S_0 = I$,

6.1. **Definition.** *A $S_\bullet$-descent datum on a family indexed by I, $R \longrightarrow I$, is an isomorphism in $\mathcal{S}_{/I}$, and it consists of the following data:*

For each 1-simplex $i \xrightarrow{\ell} j$ a bijection $R_i \xrightarrow{s_\ell} R_j$ such that:

1) For each vertex $i \in I = S_0$, $\quad s_{\sigma_0(i)} = id_{R_i}$.

2) For each 2-simplex $w \in S_2$, $\quad s_{\partial_1(w)} = s_{\partial_0(w)} \circ s_{\partial_2(w)}$.

Recall that a *(left) action* of a small category with set of objects I in a I-indexed family $R \longrightarrow I$ is a (covariant) set-valued functor R, $R(i) = R_i$, and for $x \in R_i$, $\ell_\bullet x = R(\ell)(x)$. We say that the action is *by isomorphisms* if $R(\ell)$ is a bijection for all ℓ. Actions by isomorphisms are the same thing that actions of the groupoid resulting by formally inverting all the arrows of the category. The following is straightforward. For the record:

6.2. **Proposition.** *For any simplicial set $S_\bullet$, there is a one to one correspondence between $S_\bullet$-descent data and left actions by isomorphisms of the fundamental category, that is, actions of the fundamental groupoid. With the obvious definition of morphisms this bijection extends to an isomorphism of categories (and of topoi).* $\square$

Let $H_\bullet \longrightarrow S_\bullet$ be any simplicial family, $S_0 = I$, $H_0 = U$, 6.3. **Definition.** *A $H_\bullet$-descent datum σ on an object $\gamma^*(R) \times_{\gamma^*(I)} U \longrightarrow U$ (where $R \longrightarrow I$ is a family indexed by I), is an isomorphism in $\mathcal{E}_{/U}$, and it consists of the following data:*

For each 1-simplex $i \xrightarrow{\ell} j$ an isomorphism σ_ℓ:

$$\begin{tikzcd} \gamma^* R_i \times (H_1)_\ell \arrow[r, "\sigma_\ell"] \arrow[d] & \gamma^* R_j \times (H_1)_{\ell^{op}} \arrow[d] \\ (H_1)\ell \arrow[r, "(\tau_1)_\ell"] & (H_1)_{\ell^{op}} \end{tikzcd}$$

Notice that σ_ℓ is completely determined by its first projection that we denote with the same letter $\gamma^* R_i \times (H_1)_\ell \xrightarrow{\sigma_\ell} \gamma^* R_j$, or $(H_1)_\ell \xrightarrow{\widehat{\sigma}_\ell} {\gamma^* R_j}^{\gamma^* R_i}$. The identity and cocycle conditions are:

(1) For each $i \in S_0$, $y \in \gamma^* R_i$, $x \in (H_0)_i$:
$$\sigma_{\sigma_0(i)}(y,\, (s_0)_i(x)) \;=\; (y,\, (s_0)_i(x)) : \quad \widehat{\sigma}_{\sigma_0(i)}((s_0)_i(x)) \;=\; id_{\gamma^* R_i},$$

(2) For each $w \in S_2$, $y \in \gamma^* R_i$, $x \in (H_2)_w$:
$$\sigma_{\partial_0(w)}(\sigma_{\partial_2(w)}(y,\, (d_2)_w(x)),\, (d_0)_w(x)) \;=\; \sigma_{\partial_1(w)}(y,\, (d_1)_w(x)) :$$
$$\widehat{\sigma}_{\partial_0(w)}(d_0(x)) \circ \widehat{\sigma}_{\partial_2(w)}(d_2(x)) \;=\; \widehat{\sigma}_{\partial_1(w)}(d_1(x)),$$

The equations above correspond to commutative diagrams, the letters x, y can be thought as internal variables, or simply as a way to indicate how to construct the diagram.

Recall now from [2, 10.3].

6.4. **Proposition.** *Given a cover $U \xrightarrow{\zeta} \gamma^* I$, a simplicial hypercover refinement of the Cech simplicial family (see proposition 3.6).*

$$\begin{tikzcd} H_\bullet \arrow[r, "h_\bullet"] \arrow[d, "\zeta_\bullet"] & U_\bullet \arrow[d, "\zeta_\bullet"] \\ \gamma^* S_\bullet \arrow[r, "\gamma^* \alpha_\bullet"] & \gamma^* N_\bullet \end{tikzcd}$$

and a family of sets $R \longrightarrow I$ indexed by I, consider for all $i \xrightarrow{\ell} j \in S_1$, $(i,\, j) = \alpha_1(\ell)$, the following diagrams:

$$\begin{tikzcd} \gamma^* R_i \times U_i \times U_j \arrow[r, "\sigma_{j,\,i}"] & \gamma^* R_j \times U_j \times U_i \\ \gamma^* R_i \times (H_1)_\ell \arrow[r, "\sigma_\ell"] \arrow[u, "\gamma^* R_i \times (h_1)_\ell"] & \gamma^* R_j \times (H_1)_{\ell^{op}} \arrow[u, "\gamma^* R_j \times (h_1)_{\ell^{op}}"] \end{tikzcd} \qquad \begin{tikzcd} U_i \times U_j \arrow[dr, "\widehat{\sigma}_{j,\,i}"] & \\ (H_1)_\ell \arrow[u, "(h_1)_\ell"] \arrow[r, "\widehat{\sigma}_\ell"] & {\gamma^* R_j}^{\gamma^* R_i} \end{tikzcd}$$

Then, composing with $(h_1)_\ell$, that is the correspondence $\sigma_{j,\,i} \mapsto \sigma_\ell$, where σ_ℓ is such that $\sigma_{j,\,i} \circ \gamma^* R_i \times (h_1)_\ell = \gamma^* R_j \times (h_1)_{\ell^{op}} \circ \sigma_\ell$ or $\widehat{\sigma}_{j,\,i} \mapsto \widehat{\sigma}_\ell = \widehat{\sigma}_{j,\,i} \circ (h_1)_\ell$, induces a bijection between $H_\bullet$-descent data σ_ℓ and $U_\bullet$-descent data $\sigma_{j,\,i}$ on objects of the form $\gamma^*(R) \times_{\gamma^*(I)} U \longrightarrow U$. This actually establishes an isomorphism of the respective categories.

Proof. We give only an sketch of the proof. The correspondence is injective since the family $\{(h_1)_\ell\}_{\ell \in (S_1)_{ij}}$ is epimorphic. Given a $H_\bullet$-descent datum σ_ℓ, it can be seen that the family $\{\widehat{\sigma}_\ell\}_{\ell \in (S_1)_{ij}}$ is compatible, thus there exists a unique $\widehat{\sigma}_{j,\,i}$ such that $\widehat{\sigma}_\ell = \widehat{\sigma}_{j,\,i} \circ (h_1)_\ell$ all $\ell \in (S_1)_{ij}$. The cocycle and identity equations follow the fact that the family (remark 3.2, 3.)

$$\{(H_2)_w \xrightarrow{((p_2)_w,\,(p_1)_w,\,(p_0)_w)} U_i \times U_j \times U_k\}_{w \in (S_2)_{ijk}} \quad \text{is epimorphic.} \qquad \square$$

6.5. **Remark.** A $S_\bullet$ descent datum s_ℓ on a family $R \longrightarrow I$ induces a $H_\bullet$-descent datum $\sigma_\ell = \gamma^*(s_\ell) \times (\tau_1)_\ell$ on the object $\gamma^*(R) \times_{\gamma^*(I)} U$. $\square$

6.6. We say that a $S_\bullet$-descent datum (as in definition 6.1) is *consistent* if for each pair of 1-simplexes $\ell,\, t \in S_1$ as in (2) proposition 5.3, $s_\ell = s_t$.

We have (compare with proposition 6.2):

6.7. **Proposition.** *Given any self dual simplicial family $H_\bullet \xrightarrow{\xi} \gamma^*(S_\bullet)$ satisfying condition G, there is a one to one correspondence between consistent $S_\bullet$-descent data and left actions of the G-fundamental groupoid of the family (Proposition 5.3). With the obvious definition of morphisms this bijection extends to an isomorphism of categories (and of topoi).* $\square$

7. Covering projections

We recall now the concept of *covering projection* introduced in [5], for details we refer the reader to this source. Consider a cover $\mathcal{U} = U \xrightarrow{\zeta} \gamma^* I$ in a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$, and the Cech simplicial family $U_\bullet \xrightarrow{\zeta_\bullet} \gamma^*(N_\bullet)$ (example 3.5). A locally constant object is an object X together with a trivialization structure θ. This structure consists in a family of isomorphisms $\{\theta_i\}_{i \in I}$:

$$\begin{tikzcd} \gamma^* R_i \times U_i \arrow[rr, "\theta_i"] \arrow[dr] & & X \times U_i \arrow[dl] \\ & U_i & \end{tikzcd}$$

where $R \longrightarrow I$, $\{R_i\}_{i \in I}$ is a family of sets. These objects are constructed by descent (see [7]) on a $U_\bullet$-descent datum σ on an object in $\mathcal{E}_{/U}$ of the form $\gamma^* R \times_{\gamma^* I} U \longrightarrow U$, $\{\gamma^* R_i \times U_i \longrightarrow U_i\}_{i \in I}$ (definition 6.3). Such a descent datum consists of a family of isomorphisms, $\{\sigma_{j,\,i}\}_{(i,\,j) \in N_1}$:

$$\begin{tikzcd} \gamma^* R_i \times U_i \times U_j \arrow[r, "\sigma_{j,\,i}"] \arrow[d] & \gamma^* R_j \times U_j \times U_i \arrow[d] \\ U_i \times U_j \arrow[r, "\tau"] & U_j \times U_i \end{tikzcd}$$

satisfying the corresponding identity and cocycle conditions in definition 6.3. The relationship between the trivialization structure θ and the descent datum σ is given in the following commutative diagram:

$$\begin{tikzcd} \gamma^* R_i \times U_i \times U_j \arrow[r, "\sigma_{j,\,i}"] \arrow[d, "\theta_i \times U_j"] & \gamma^* R_j \times U_j \times U_i \arrow[d, "\theta_j \times U_i"] \\ X \times U_i \times U_j \arrow[r, "X \times \tau"] & X \times U_j \times U_i \end{tikzcd}$$

Let $X \sim (R \to I,\, \sigma)$ be a locally constant object determined by a $U_\bullet$-descent datum σ on a cover $U \to \gamma^* I$. Recall from [5] the following

7.1. **Definition.** *An* action span *for* X *is a span* $\ell = \begin{tikzcd}[column sep=small] & V \arrow[dl, "u"'] \arrow[dr, "v"] & \\ U_i & & U_j \end{tikzcd}$,
with $V \neq \emptyset$*, and such that there is a bijection* $S_i \xrightarrow{s_\ell} S_j$ *(necessarily unique) such that the following diagram commutes*

$$\begin{tikzcd} \gamma^* R_i \times U_i \times U_j \arrow[r, "\sigma_{j,\,i}"] & \gamma^* R_j \times U_j \times U_i \\ \gamma^* R_i \times V \arrow[u, "\gamma^* S_i \times (u,v)"'] \arrow[r, "\gamma^* s_\ell \times V"] & \gamma^* R_j \times V \arrow[u, "\gamma^* S_j \times (v,u)"'] \end{tikzcd}$$

7.2. **Remark.** Given a morphism of non empty spans $\ell' \to \ell$, $V' \to V$, if ℓ is an action span, then so it is ℓ', and $s_{\ell'} = s_\ell$. $\square$

7.3. **Definition.** [5, 2.12] *We say that a locally constant object* $X \sim (S \to I,\, \sigma)$ *trivialized by a cover* $U \to \gamma^* I$ *is a* covering projection *if, for each* $(i,\, j) \in N_1$*, the family* $V \xrightarrow{(u,v)} U_i \times U_j$ *is an epimorphic family, where* $(V,\, u,\, v)$ *ranges over all action spans. By remark 7.2 it is equivalent to restrict* V *to a site of definition.*

Every span with connected vertex is an action span, thus we have:

7.4. **Proposition.** *In a locally connected topos every locally constant object is a covering projection.* $\square$

7.5. **Proposition.** *Let* $X \sim (S \to I,\, \sigma)$ *be a locally constant object trivialized by a cover* $U \to \gamma^* I$*, let* $\mathcal{C}_{sp}$ *be the set of all action spans with* V *in a site of definition, and* $\mathcal{C}$ *be the set of vertices of the spans in* $\mathcal{C}_{sp}$*. Then:*

(1) The conditions in construction 4.4 are satisfied.

(2) The map $H_2 \longrightarrow (cosk_1 H_\bullet)_2$ *is an epimorphism.*

(3) If X *is a covering projection, then the map* $H_1 \longrightarrow (cosk_0 H_\bullet)_1$ *is an epimorphism. Thus* $H_\bullet \xrightarrow{h_\bullet} U_\bullet$ *is an hypercovering.*

Proof. (1) Observe that the dual span of an action span is an action span with inverse bijection s_ℓ^{-1}. The other requirements follow from the descent identity condition (1) in definition 6.3 and remark 7.2. Recall that in this case $(s_0)_i$ is the diagonal of U_i.

We refer now to proposition 4.5, 2:

(2) From remark 7.2 it follows that for any map $\emptyset \neq V \longrightarrow P_{\ell tr}$, V is the vertex of a (in fact three) action spans, thus it is in $\mathcal{C}$.

(3) It holds by definition of covering projection. $\square$

7.6. **Remark.** From remark 7.2 it follows that the simplicial family $H_\bullet \longrightarrow \gamma^*(S_\bullet)$ is a *sieve* in the sense that given any $V \in \mathcal{C}$, $V \subset (H_1)_\ell$, there exists $t \in S_1$ such that $(H_1)_t = V$.

Finally, from remark 7.2 and propositions 5.2, 6.4 and 7.4 we have:

7.7. **Theorem.** *Let* $X \sim (R \to I,\, \sigma)$ *be a covering projection trivialized by a cover* $U \to \gamma^* I$*. Then there exist a self-dual simplicial hypercover refinement* *of the Cech simplicial family (see proposition 3.6)*

$$\begin{tikzcd} H_\bullet \arrow[r, "h_\bullet"] \arrow[d, "\zeta_\bullet"] & U_\bullet \arrow[d, "\zeta_\bullet"] \\ \gamma^* S_\bullet \arrow[r, "\gamma^*\alpha_\bullet"] & \gamma^* N_\bullet \end{tikzcd}$$

satisfying condition G (definition 5.1), and a consistent (cf 6.6) $S_\bullet$-descent datum $\{s_\ell\}_{\ell \in S_1}$ on the family $R \to I$ such that the corresponding $H_\bullet$-descent datum σ_ℓ (proposition 6.4) is of the form $\sigma_\ell = \gamma^*(s_\ell) \times (\tau_1)_\ell$ (remark 6.5). Vice-versa, any such descent datum on a self-dual simplicial hypercover refinement of the Cech simplicial family determines a covering projection trivialized by the cover $U \to \gamma^* I$. □

We will also say that the covering projection is trivialized by the hypercover $H_\bullet \longrightarrow \gamma^*(S_\bullet)$ With the notation in the theorem above, from proposition 6.7 we have:

7.8. **Theorem.** *Given a cover $U \to \gamma^* I$ and a self-dual simplicial hypercover refinement of the Cech simplicial family satisfying condition G, then the category of covering projections trivialized by a consistent $S_\bullet$-descent datum $\{s_\ell\}_{\ell \in S_1}$ on a family $R \to I$, is isomorphic to the category (topos) of left actions of the G-fundamental groupoid of the family (Proposition 5.3).* □

In the case of a locally connected topos, taking into account construction 4.3, example 4.6 and proposition 7.4 it follows:

7.9. **Theorem.** *Given any locally connected topos $\mathcal{E}$, the statement in theorem 7.7 holds for any locally constant object $X \sim (R \to I,\, \sigma)$ trivialized by a cover $U \to \gamma^* I$. Thus X can be constructed by a $S_\bullet$-descent datum $\{s_\ell\}_{\ell \in S_1}$ on the family $R \to I$, where $S_\bullet$ is the simplicial set constructed in 4.3 with any set of connected generators.* □

8. Fundamental progroupoid of a topos

This section is rather sketchy and we refer the reader to [5] for details and complete proofs. Given a cover family $\mathcal{U} = U \longrightarrow \gamma^* S$ and a family hypercover refinement $\mathcal{H} = H_\bullet \longrightarrow \gamma^*(S_\bullet)$, there is clear definition of morphisms of covering projections constant on $\mathcal{U}$ (constant on $\mathcal{H}$), [5, 1.4]. This determines categories $\mathcal{G}_\mathcal{U}$ ($\mathcal{G}_\mathcal{H}$), furnished with a faithful (but not full) functor $\mathcal{G}_\mathcal{U} \longrightarrow \mathcal{E}$ ($\mathcal{G}_\mathcal{H} \longrightarrow \mathcal{E}$). With this it is easy to construct the colimit (inside $\mathcal{E}$) of the categories $\mathcal{G}_\mathcal{U}$ ($\mathcal{G}_\mathcal{H}$) indexed by $\mathcal{U}$ (indexed by $\mathcal{H}$), [5, 1.4]. It follows from theorem 7.7 that every covering projection is in some $\mathcal{G}_\mathcal{H}$, so these two colimits are equal. We denote this category $c\mathcal{G}(\mathcal{E})$, it is the category of all covering projections. We have $\mathcal{G}_\mathcal{H} \to c\mathcal{G}(\mathcal{E}) \to \mathcal{E}$. It follows from theorem 7.8 that the category $\mathcal{G}_\mathcal{H}$ is the classifying topos of a groupoid $\mathbf{G}_\mathcal{H}$ (the G-fundamental groupoid of the simplicial family) $\mathcal{G}_\mathcal{H} = \beta \mathbf{G}_\mathcal{H}$. This determines a protopos $\mathcal{G}(\mathcal{E}) = \{\mathcal{G}_\mathcal{H}\}_\mathcal{H}$ and a progroupoid $\pi_1(\mathcal{E}) = \{\mathbf{G}_\mathcal{H}\}_\mathcal{H}$, and we have $\beta\pi_1(\mathcal{E}) = \mathcal{G}(\mathcal{E})$. The inverse limit topos of this protopos is the topos of sheaves for a subcanonical Grothendieck topology on the category $c\mathcal{G}(\mathcal{E})$ [5, 4.1, 4.4].

Given a group K, recall that a *K-torsor* in a topos $\mathcal{E}$ is an object $T \in \mathcal{E}$, $T \to 1$ epi, together with an action $\gamma^*K \times T \longrightarrow T$ such that the arrow $\gamma^*K \times T \xrightarrow{\varepsilon} T \times T$ defined by $\varepsilon(x, u) = (x \cdot u, u)$ is an isomorphism. Clearly any torsor T determines in a canonical way a locally constant object $T = (T, K, \varepsilon)$ split by the (singleton family) cover $T \to 1$, which in fact is a covering projection. Following exactly the same lines that in [5, Section 6], it can be proved that $\pi_1(\mathcal{E})$ classifies torsors. If we denote $pro\mathcal{G}rpd$, the 2-category of progroupoids, we have:

There is an equivalence of categories $pro\mathcal{G}rpd[\pi_1(\mathcal{E}), K] \cong K\text{-}\mathcal{T}ors(\mathcal{E})$.

Note that this furnish an explicit construction of the fundamental progroupoid $\pi(\mathcal{E})$, to be compared in the locally connected case with the construction in [2, Section 10].

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