Low-dimensional representations of matrix groups and group actions on CAT(0) spaces and manifolds

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Abstract

We study low-dimensional representations of matrix groups over general rings, by considering group actions on CAT(0) spaces, spheres and acyclic manifolds.

1 Introduction

Low-dimensional representations are studied by many authors, such as Guralnick and Tiep [24] (for matrix groups over fields), Potapchik and Rapinchuk [30] (for automorphism group of free group), Doković and Platonov [18] (for $\operatorname{Aut}(F_2)$), Weinberger [35] (for $\operatorname{SL}_n(\mathbb{Z})$) and so on. In this article, we study low-dimensional representations of matrix groups over general rings. Let R be an associative ring with identity and $E_n(R)$ ($n \geq 3$) the group generated by elementary matrices (cf. Section 3.1). As motivation, we can consider the following problem.

Problem 1. For $n \geq 3$, is there any nontrivial group homomorphism $E_n(R) \rightarrow E_{n-1}(R)$?

Although this is a purely algebraic problem, in general it seems hard to give an answer in an algebraic way. In this article, we try to answer Problem 1 negatively from the point of view of geometric group theory. The idea is to find a good geometric object on which $E_{n-1}(R)$ acts naturally and nontrivially while $E_n(R)$ can only act in a special way. We study matrix group actions on CAT(0) spaces, spheres and acyclic manifolds. We prove that for low-dimensional CAT(0) spaces, a matrix group action always has a global fixed point (cf. Theorem 1) and that for low-dimensional spheres and acyclic manifolds, a matrix group action is always trivial (cf. Theorem 3). Based on these results, we show that the low-dimensional representation of matrix groups are quite constrained (cf. Corollary 2) and give a negative answer to Problem 1 for

some rings R (cf. Corollary 3). Moreover, these results give generalizations of a result of Farb [21] concerning Chevalley groups over commutative rings acting on CAT(0) spaces and that of Bridson and Vogtmann [15], Parwani [29] and Zimmermann [39] concerning the special linear groups $SL_n(\mathbb{Z})$ and symplectic groups $SL_n(\mathbb{Z})$ acting on spheres and acyclic manifolds.

We now consider group actions on CAT(0) spaces. A group G has Serre's property FA if any G-action on any simplicial tree T has a global fixed point. Recall from Farb [21] that for an integer $n \geq 1$, a group G is said to have property FA $_n$ if any isometric G-action on any n-dimensional CAT(0) cell complex X has a global fixed point. The property FA $_1$ is Serre's property FA. If a group G has property FA $_n$ then it has property FA $_n$ for all m < n. Farb [21] proves that when a reduced, irreducible root system Φ has rank $r \geq 2$ and R is a finitely generated commutative ring, the elementary subgroup $E(\Phi,R)$ of Chevalley group $G(\Phi,R)$ has property FA $_{r-1}$. This gives a generalization of a result obtained by Fukunaga [23] concerning groups acting on trees. The group actions on CAT(0) spaces and property FA $_n$ have also been studied by some other authors. For example, Bridson [10, 11] proves that the mapping class group of a closed orientable surface of genus g has property FA $_{g-1}$. The group action on CAT(0) spaces of automorphism groups of free groups is studied by Bridson [12]. Barnhill [7] considers the property FA $_n$ for Coxeter groups.

In this article, we prove the property FA_n for matrix groups over any ring (not necessary commutative). Without otherwise stated we assume that a ring is an associative ring with identity. Let R be such a ring and $n \geq 3$ an integer. Recall the definition of the elementary group $E_n(R)$ generated by elementary matrices and the unitary elementary group $EU_{2n}(R,\Lambda)$ generated by elementary unitary matrices from Section 3.1. When R is a ring of integers in a number field and $n \geq 3$, the group $E_n(R)$ is the special linear group $\mathrm{SL}_n(R)$. For different choices of parameters Λ , the group $EU_{2n}(R,\Lambda)$ contains as special cases the elementary symplectic groups, the elementary orthogonal groups and the elementary unitary groups.

Watatani [1] proves that a group with Kazhdan's property (T) has Serre's property FA. Ershov and Jaikin-Zapirain [19] proves that for a general ring R and an integer $n \geq 3$, the elementary group $E_n(R)$ has Kazhdan's property (T). It follows that $E_n(R)$ has Serre's property FA. Our first result is the following.

Theorem 1. Let R be any finitely generated ring and $n \geq 3$ an integer. Suppose that $E_n(R)$ (resp., $EU_{2n}(R,\Lambda)$) is the matrix group generated by all elementary matrices (resp., elementary unitary matrices). Then the group $E_n(R)$ (resp., $EU_{2n}(R,\Lambda)$) has property FA_{n-2} (resp., FA_{n-1}).

When R is commutative, Theorem 1 recovers partially the results obtained by Farb [21] for Chevalley groups. The dimension in Theorem 1 is sharp, since the group $\mathrm{SL}_n(\mathbb{Z}[1/p])$ acts without a global fixed point on the affine building associated to $\mathrm{SL}_n(\mathbb{Q}_p)$ and this building is an (n-1)-dimensional, nonpositively curved simplicial complex.

Remark 1. The property FA_{n-2} of $E_n(R)$ obtained in Theorem 1 can be viewed as a higher dimensional generalization of Serre's property FA for some

Kazhdan's groups. However, it is not clear that every unitary elementary group $EU_{2n}(R,\Lambda)$ also has Kazhdan's property (T) (for property (T) of groups defined by roots, see Ershov, Jaikin-Zapirain and Kassabov [20]).

We consider property FA_d for general linear groups $\operatorname{GL}_n(R)$ over a general ring R. For this, we have to introduce notions of K-groups $K_1(R)$, $KU_1(R,\Lambda)$, the stable range $\operatorname{sr}(R)$ and the unitary stable range $\operatorname{Asr}(R,\Lambda)$ (for details, see Section 3.2). The stable range is not bigger than most other famous dimensions of rings, e.g. absolute stable range, 1+ Krull dimension, 1+ maximal spectrum dimension, 1+ Bass-Serre dimension. When R is a Dedekind domain, the stable range $\operatorname{sr}(R) \leq 2$. When G is a finite group and $\mathbb{Z}[G]$ the integral group ring, the stable range $\operatorname{sr}(\mathbb{Z}[G]) \leq 2$. The next theorem gives a criterion when the general linear group $\operatorname{GL}_n(R)$ has property FA_d .

- **Theorem 2.** (i) Let R be a finitely generated ring with finite stable range $d = \operatorname{sr}(R)$. Suppose that $n \geq d+1$ and the K-group $K_1(R)$ has property FA_{n-2} (e.g. $K_1(R)$ is finite). Then the general linear group $\operatorname{GL}_n(R)$ has property FA_{n-2} .
- (ii) Let (R, Λ) be a form ring over a finitely generated associative ring R with a finite Λ -stable range $d = \Lambda sr(R)$. Suppose that $n \geq d+1$ and the K-group $KU_1(R, \Lambda)$ has property FA_{n-1} (e.g. $KU_1(R)$ is finite). Then the unitary group $U_{2n}(R, \Lambda)$ has property FA_{n-1} .

Note that the stable range of a ring A of integers in a number field is 2 and the group $K_1(A)$ is A^* , the group of invertible elements in A (cf. 11.37 in [26]). According to Theorem 2, for any ring A of integers in a number field with A^* finite, the general linear group $GL_n(A)$ has property FA_{n-2} for $n \geq 3$. Let G be a finite group and $\mathbb{Z}[G]$ the integral group ring over G. As a corollary to Theorem 2, we get a criterion when the general linear group $GL_n(\mathbb{Z}[G])$ has property FA_{n-2} .

Corollary 1. Suppose that G is a finite group with the same number of irreducible real representations and irreducible rational representations. Then $K_1(\mathbb{Z}[G])$ is finite and for any integer $n \geq 3$, the general linear group $\mathrm{GL}_n(\mathbb{Z}[G])$ has property FA_{n-2} .

For example, when G is any symmetric group (cf. page 14 in [28]), the general linear group $GL_n(\mathbb{Z}[G])$ has property FA_{n-2} for $n \geq 3$.

We consider the stable elementary groups E(R) and $EU(R,\Lambda)$ acting on a locally finite CAT(0) cell complex. Recall from Section 3.1 that the stable elementary group E(R) is a direct limit of $E_n(R)$ $(n \ge 2)$ and similarly the stable elementary unitary group $EU(R,\Lambda)$ is a direct limit of $EU_{2n}(R,\Lambda)$ $(n \ge 2)$. The following result is obtained:

Proposition 1. Let R be any finitely generated ring. Then any simplicial isometric action of E(R) or $EU(R,\Lambda)$ on a uniformly locally finite $CAT(\theta)$ cell complex is trivial.

When $R = \mathbb{Z}$ (so $E(R) = SL(\mathbb{Z})$), this is a result proved by Chatterji and Kassabov (cf. Corollary 4.5 in [16]).

As the representations of groups with property FA_n are quite constrained, we obtain that for integers $k \geq n$ the elementary group $E_{k+1}(R)$ and the unitary elementary group $EU_{2k}(R,\Lambda)$ are groups of integral n-representation type as follows. This theory was introduced and studied by Bass [6]. When the ring R in the second group of Problem 1 is a field, we have the following.

Corollary 2. Let R be a finitely generated ring and an integer $n \geq 2$. For an integer $k \geq n$, let Γ be the elementary group $E_{k+1}(R)$ or the unitary elementary group $EU_{2k}(R,\Lambda)$ (for $EU_{2k}(R,\Lambda)$, we assume that $k \geq \max\{n,3\}$). Let $\rho: \Gamma \to \operatorname{GL}_n(K)$ be any representation of degree n over a field K. Then

- (i) the eigenvalues of each of the matrices in $\rho(\Gamma)$ are integral. In particular they are algebraic integers if the characteristic char(K) = 0 and are roots of unity if the characteristic char(K) > 0; and
- (ii) for any algebraically closed field K, there are only finitely many conjugacy classes of irreducible representations of Γ into $GL_n(K)$.

We now consider group actions on manifolds. The following conjecture is from Farb and Shalen [22], which is related to Zimmer's program (see [37, 38]).

Conjecture 1. Any smooth action of a finite-index subgroup of $SL_n(\mathbb{Z})$, where n > 2, on a r-dimensional compact manifold M factors through a finite group action if r < n - 1.

Parwani [29] considers this conjecture for the group $SL_n(\mathbb{Z})$ itself and M is a sphere. The idea is to use the theory of compact transformation groups to show that some sufficiently large finite subgroups cannot act effectively on M, and then to use the Margulis finiteness theorem to show that any $SL_n(\mathbb{Z})$ -action on M must be finite. Such techniques are also used several times by many other authors, e.g. the proof of trivial actions of $SL_n(\mathbb{Z})$ on tori by Weinberger in [35], the proof of the trivial action of $SL(\mathbb{Z})$ on compact manifolds by Weinberger in [36] (Proposition 1), the proof of trivial actions of $SL_n(\mathbb{Z})$ on small finite sets by Chatterji and Kassabov in [16] (Lemma 4.2) and so on. Zimmermann [37] actually proves that any smooth action of $\mathrm{SL}_n(\mathbb{Z})$ on small spheres is trivial. It is natural to consider other kinds of group actions on compact manifolds. Zimmermann [40] proves a similar trivial action of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{Z})$. The group action of $Aut(F_n)$, the automorphism group of a free group, on spheres and acyclic manifolds is considered by Bridson and Vogtmann [15] and similar trivial-action results are obtained. More precisely, they show that for $n \geq 3$ and d < n - 1, any action of the special automorphism group $SAut(F_n)$ by homeomorphisms on a generalized d-sphere over \mathbb{Z}_2 or a (d+1)-dimensional \mathbb{Z}_2 -acyclic homology manifold over \mathbb{Z}_2 is trivial. Hence the group $\operatorname{Aut}(F_n)$ can act only via the determinant map det : $\operatorname{Aut}(F_n) \to \mathbb{Z}_2$. In this article, we notice that the Margulis finiteness theorem is not necessary for such problem. Actually, we get a much more general result for the actions of matrix groups over any general ring, as follows.

Theorem 3. Let R be any ring and $n \geq 3$ be an integer. Suppose that $E_n(R)$ (resp. $EU_{2n}(R,\Lambda)$) is the matrix group generated by all elementary matrices (resp. elementary unitary matrices). Then we have that

- (a)(i) for an integer $d \leq n-2$, any action of $E_n(R)$ by homeomorphisms on a generalized d-sphere over \mathbb{Z}_2 is trivial;
 - (ii) for an integer $d \leq n-1$, any action of $E_n(R)$ by homeomorphisms on a d-dimensional \mathbb{Z}_2 -acyclic homology manifold (i.e. has the \mathbb{Z}_2 -homology of a point) is trivial.
- (b)(i) for an integer $d \le n-2$ when n is even or $d \le n-3$ when n is odd, any action of $E_n(R)$ by homeomorphisms on a generalized d-sphere over \mathbb{Z}_3 is trivial;
 - (ii) for an integer $d \leq n-1$ when n is even or $d \leq n-2$ when n is odd, any action of $E_n(R)$ by homeomorphisms on a d-dimensional \mathbb{Z}_3 -acyclic homology manifold (i.e. has the \mathbb{Z}_3 -homology of a point) is trivial.
 - (c) The statements (a) and (b) also hold for $EU_{2n}(R,\Lambda)$ instead of $E_n(R)$.

When the ring $R = \mathbb{Z}$ and $E_n(R) = \operatorname{SL}_n(\mathbb{Z})$, the above theorem recovers the results obtained by Bridson and Vogtmann [15], Parwani [29] and Zimmermann [39]. The dimensions in (a) and those in (b) with even n of Theorem 3 are sharp, since the group $\operatorname{SL}_n(\mathbb{Z}) = E_n(\mathbb{Z})$ $(n \geq 3)$ can act nontrivially on the standard sphere S^{n-1} and the Euclidean space \mathbb{R}^n .

If the parameter Λ in the definition of form ring (R, Λ) contains the identity $1 \in R$, we can get an improvement of Theorem 3 as following.

Theorem 4. Let (R, Λ) be a form ring. Suppose that $1 \in \Lambda$. Then we have that

- (i) for an integer $d \leq 2n-2$, any action of $EU_{2n}(R,\Lambda)$ by homeomorphisms on a generalized d-sphere over \mathbb{Z}_3 is trivial;
- (ii) for an integer $d \leq 2n-1$, any action of $EU_{2n}(R,\Lambda)$ by homeomorphisms on a d-dimensional \mathbb{Z}_3 -acyclic homology manifold (i.e. has the \mathbb{Z}_3 -homology of a point) is trivial.

When the ring $R = \mathbb{Z}$ and $EU_{2n}(R, \Lambda) = \operatorname{Sp}_{2n}(\mathbb{Z})$, the above theorem recovers a result obtained by Zimmermann in [40]. Considering the nontrivial actions of the symplectic group $\operatorname{Sp}_{2n}(\mathbb{Z})$ on S^{2n-1} and \mathbb{R}^{2n} , we see that the dimensions in Theorem 4 are sharp.

As an easy corollary of Theorem 3 and Theorem 4, we have a negative answer to Problem 1 when R is a subring of the real numbers \mathbb{R} .

Corollary 3. Let R be a general ring and S a commutative ring. Assume that A is a subring of the real numbers \mathbb{R} and $n \geq 3$. Then

(i) any group homomorphism

$$E_n(R) \to E_{n-1}(A)$$

is trivial;

(ii) any group homomorphism

$$\operatorname{Sp}_{2n}(S) \to \operatorname{Sp}_{2(n-1)}(A)$$

is trivial.

As an easy corollary of Theorem 3 and Theorem 4, we see that the group E(R) or $EU(R,\Lambda)$ cannot act nontrivially by homeomorphisms on any generalized d-sphere or \mathbb{Z}_2 -acyclic homology manifold. Actually, on any compact manifold, the following theorem shows that there are no nontrivial actions of E(R) and $EU(R,\Lambda)$.

Theorem 5. Let R be any ring, E(R) and $EU(R,\Lambda)$ the stable elementary and unitary elementary groups. Then the group E(R) or $EU(R,\Lambda)$ does not act topologically, nontrivially, on any compact manifold, or indeed on any manifold whose homology with coefficients in a field of positive characteristic is finitely generated.

When $R = \mathbb{Z}$ and $E(R) = SL(\mathbb{Z})$, Theorem 5 is Proposition 1 in [36].

Remark 2. Let R be any ring and $n \geq 3$. The author believe that the same results in this article hold as well for the Steinberg group $St_n(R)$ (resp., the unitary Steinberg group $USt_n(R,\Lambda)$) instead of the elementary group $E_n(R)$ (resp., the unitary elementary group $EU_n(R,\Lambda)$). This is because in the proofs(cf. last section), only commutator formulas are used and these formulas hold as well for Steinberg groups (resp., unitary Steinberg groups).

In Section 2, we give some basic facts of CAT(0) spaces and homology manifolds. In Section 3, we introduce the notions of elementary groups $E_n(R)$, $EU_{2n}(R,\Lambda)$, Steinberg groups, the algebraic K-groups K_1 , KU_1 and the stable ranges. The results in the introduction will be proved in Section 4.

2 Notations and basic facts

Standard assumptions. In this article, we assume that all the rings are associative rings with identity. All the CAT(0) spaces are complete and all actions on them are isometric and semisimple. When we talk about groups with properties FA_n , we always assume that the groups are finitely generated.

2.1 CAT(0) spaces and property FA_n

Let (X,d_X) be a geodesic metric space. For three points $x,y,z\in X$, the geodesic triangle $\Delta(x,y,z)$ consists of the three vertices x,y,z and the three geodesics [x,y],[y,z] and [z,x]. Let \mathbb{R}^2 be the Euclidean plane with the standard distance $d_{\mathbb{R}^2}$ and $\bar{\Delta}$ a triangle in \mathbb{R}^2 with the same edge lengths as Δ . Denote by $\varphi:\Delta\to \bar{\Delta}$ the map sending each edge of Δ to the corresponding edge of $\bar{\Delta}$. The space X is called a CAT(0) space if for any triangle Δ and two elements $a,b\in\Delta$, we have the inequality

$$d_X(a,b) \le d_{\mathbb{R}^2}(\varphi(a),\varphi(b)).$$

The typical examples of CAT(0) spaces include simplicial trees, hyperbolic spaces, products of CAT(0) spaces and so on. From now on, we assume that X is a complete CAT(0) space. Denote by Isom(X) the isometry group of X. For any $g \in \text{Isom}(X)$, let

$$Minset(g) = \{x \in X : d(x, gx) \le d(y, gy) \text{ for any } y \in X\}$$

and let $\tau(g) = \inf_{x \in X} d(x, gx)$ be the translation length of g. When the fixed-point set $\operatorname{Fix}(g) \neq \emptyset$, we call g elliptic. When $\operatorname{Minset}(g) \neq \emptyset$ and $d_X(x, gx) = \tau(g) > 0$ for any $x \in \operatorname{Minset}(g)$, we call g hyperbolic. The group element g is called semisimple if the minimal set $\operatorname{Minset}(g)$ is not empty, i.e. it is either elliptic or hyperbolic. By a $\operatorname{CAT}(0)$ complex, we mean a $\operatorname{CAT}(0)$ cell complex of piecewise constant curvature with only finitely many isometry types of cells. For more details on $\operatorname{CAT}(0)$ spaces, see the book of Bridson and Haefliger [14].

The following definition of property FA_n and strong FA_n property were given by Farb [21] as a generalization of Serre's property FA.

Definition 1. Let $n \geq 1$ be an integer. A group Γ is said to have property FA_n if any isometric Γ -action on any n-dimensional, $\mathrm{CAT}(0)$ cell complex X has a global fixed point. A group Γ is said to have strong FA_n property if any Γ -action on a complete $\mathrm{CAT}(0)$ space X satisfying the following two properties has a global fixed point.

- (i) n-dimensionality: The reduced homology group $\tilde{H}_n(Y;\mathbb{Z}) = 0$ for all open subsets $Y \subseteq X$.
- (ii) Semisimplicity: The action of Γ on X is semisimple, i.e., the translation length of each $g \in \Gamma$ is realized by some $x \in X$.

When n = 1, the property FA₁ corresponds with Serre's property FA. Since any isometric action on a CAT(0) cell complex must be semisimple (cf. page 231 in [14]), we see that strong FA_n implies FA_n. The following lemma contains some general facts on FA_n (see pages 1578-1579 in [21] for more details).

Lemma 1. The following properties hold:

- (1) If G has property FA_n then G has FA_m for all $m \leq n$.
- (2) If G has FA_n then so does every quotient group of G.

- (3) Let H be a normal subgroup of G. If H and G/H have FA_n then so does G.
- (4) If some finite index subgroup H of G has FA_n , then so does G.

2.2 Homology manifolds and Smith theory

Since the fixed-point set of a finite-period homeomorphism of a manifold is not necessary a manifold any more, we are working with generalized manifolds. All homology groups in this subsection are Borel-Moore homology with compact supports and coefficients in a sheaf \mathcal{A} of modules over a principal ideal domain L. All the concepts below are from Bredon's book [9]. Let X be a locally finite CW-complex and \mathcal{A} be the constant sheaf $X \times L$ (simply denoted by L). The homology groups $H_c^*(X)$ of X are isomorphic to the singular homology groups with coefficients in L (cf. page 279 in [9]).

Let L be the integers \mathbb{Z} or the finite field \mathbb{Z}_p for a prime p. The following definition is from page 329 in [9] (see also Definition 4.1 of [15]).

Definition 2. An m-dimensional homology manifold over L (denoted m-hm $_L$) is a locally compact Hausdorff space X with finite homological dimension over L that has the local homology properties of a manifold of dimension m.

The homology spheres and homology acyclic manifolds are defined as follows (cf. Definition 4.2 and 4.3 of [15]).

Definition 3. Let S^n be the standard n-dimensional sphere. If X is an m-hm_L and $H^c_*(X;L) \cong H^c_*(S^m;L)$ then X is called a generalized m-sphere over L. If X is an m-hm_L with $H^c_0(X;L) = L$ and $H^c_k(X;L) = 0$ for k > 0, then X is said to be L-acyclic.

The following "global" Smith theorem was originally proved by P.A. Smith ([31], [32]). Here we follow the exposition in Bredon's book [9]. The following lemma is a combination of Corollary 19.8 and Corollary 19.9 (page 144) in [9] (see also Theorem 4.5 in [15]).

Lemma 2. Let p be a prime and X be a locally compact Hausdorff space of finite dimension over \mathbb{Z}_p . Suppose that \mathbb{Z}_p acts on X with fixed-point set F.

- (i) If $H^c_*(X; \mathbb{Z}_p) \cong H^c_*(S^m; \mathbb{Z}_p)$, then $H^c_*(F; \mathbb{Z}_p) \cong H^c_*(S^r; \mathbb{Z}_p)$ for some r with $-1 \leq r \leq m$. If p is odd, then r m is even.
- (ii) If X is \mathbb{Z}_p -acyclic, then F is \mathbb{Z}_p -acyclic (in particular nonempty and connected).

3 Elementary groups and K-theory

3.1 Elementary groups and Steinberg groups

In this subsection, we briefly recall the definitions of the elementary subgroups $E_n(R)$ of the general linear group $\mathrm{GL}_n(R)$, the unitary elementary subgroup

 $EU_{2n}(R,\Lambda)$ of the unitary group $U_{2n}(R,\Lambda)$ and the Steinberg groups $\operatorname{St}_n(R)$. For more details, see the book of Magurn [26], the book of Hahn and O'Meara [25] and the book of Bak [2]. We define the groups $\operatorname{GL}_n(R)$ and $E_n(R)$ first. Let R be an associative ring with identity and $n \geq 2$ be an integer. The general linear group $\operatorname{GL}_n(R)$ is the group of all $n \times n$ invertible matrices with entries in R. For an element $r \in R$ and any integers i, j such that $1 \leq i \neq j \leq n$, denote by $e_{ij}(r)$ the elementary $n \times n$ matrix with 1 in the diagonal positions and r in the (i,j)-th position and zeros elsewhere. The group $E_n(R)$ is generated by all such $e_{ij}(r)$, i.e.

$$E_n(R) = \langle e_{ij}(r) | 1 \le i \ne j \le n, r \in R \rangle.$$

Denote by I_n the identity matrix and by [a,b] the commutator $aba^{-1}b^{-1}$.

The following lemma displays the commutator formulas for $E_n(R)$ (cf. Lemma 9.4 in [26]).

Lemma 3. Let R be a ring and $r, s \in R$. Then for distinct integers i, j, k, l with $1 \le i, j, k, l \le n$, the following hold:

- (1) $e_{ij}(r+s) = e_{ij}(r)e_{ij}(s);$
- (2) $[e_{ij}(r), e_{jk}(s)] = e_{ik}(rs);$
- (3) $[e_{ij}(r), e_{kl}(s)] = I_n$.

By Lemma 3, the group $E_n(R)$ is finitely generated when the ring R is finitely generated. Moreover, when $n \geq 3$, the group $E_n(R)$ is normally generated by any elementary matrix $e_{ij}(1)$. We will use such fact several times in Section 4.

The commutator formulas can be used to define Steinberg group as follows. For $n \geq 3$, the Steinberg group $\operatorname{St}_n(R)$ is the group generated by the symbols $\{x_{ij}(r): 1 \leq i \neq j \leq n, r \in R\}$ subject to the following relations:

- (St1) $x_{ij}(r+s) = x_{ij}(r)x_{ij}(s);$
- (St2) $[e_{ij}(r), e_{jk}(s)] = e_{ik}(rs)$ for $i \neq k$;
- (St3) $[e_{ij}(r), e_{kl}(s)] = 1$ for $i \neq l, j \neq k$.

There is an obvious surjection $\operatorname{St}_n(R) \to E_n(R)$ defined by $x_{ij}(r) \longmapsto e_{ij}(r)$.

For any ideal $I \triangleleft R$, let $p: R \to R/I$ be the quotient map. Then the map p induces a group homomorphism $p_*: \operatorname{St}_n(R) \to \operatorname{St}_n(R/I)$. Denote by $\operatorname{St}_n(R,I)$ (resp., $E_n(R,I)$) the subgroup of $\operatorname{St}_n(R)$ (resp., $E_n(R)$) normally generated by elements of the form $x_{ij}(r)$ (resp., $e_{ij}(r)$) for $r \in I$ and $1 \le i \ne j \le n$. In fact, $\operatorname{St}_n(R,I)$ is the kernel of p_* (cf. Lemma 13.18 in Magurn [26] and its proof). However, $E_n(R,I)$ may not be the kenel of $E_n(R) \to E_n(R/I)$ induced by p.

We define the groups $U_{2n}(R, \Lambda)$ and $EU_{2n}(R, \Lambda)$ as follows. Let R be a general ring and assume that an anti-automorphism $*: x \mapsto x^*$ is defined on R such that $x^{**} = \varepsilon x \varepsilon^*$ for some unit $\varepsilon = \varepsilon^{-1}$ of R and every x in R. It determines an anti-automorphism of the ring $M_n R$ of all $n \times n$ matrices (x_{ij}) by $(x_{ij})^* = (x_{ii}^*)$.

Set $R_{\varepsilon} = \{x - x^* \varepsilon | x \in R\}$ and $R^{\varepsilon} = \{x \in R | x = -x^* \varepsilon\}$. If some additive subgroup Λ of (R, +) satisfies:

- (i) $r^*\Lambda r \subset \Lambda$ for all $r \in R$;
- (ii) $R_{\varepsilon} \subset \Lambda \subset R^{\varepsilon}$,

we will call Λ a form and $(\Lambda, *, \varepsilon)$ a form parameter on R. Usually (R, Λ) is called a form ring. Let $\Lambda_n = \{(a_{ij}) \in M_n R | a_{ij} = -a_{ji}^* \varepsilon \text{ for } i \neq j \text{ and } a_{ii} \in \Lambda\}$. As in [2], for an integer $n \geq 1$ we define the unitary group

$$U_{2n}(R,\Lambda) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_{2n}R \mid \alpha^*\delta + \gamma^*\varepsilon\beta = I_n, \ \alpha^*\gamma, \ \beta^*\delta \in \Lambda_n \right\}.$$

Sometimes, the unitary group $U_{2n}(R,\Lambda)$ is also called the quadratic group [2] or the pseudo-orthogonal group [34].

It can be easily seen that the inverse of a unitary matrix has the form

$$\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right)^{-1} = \left(\begin{array}{cc} \varepsilon^* \delta^* \varepsilon & \varepsilon^* \beta^* \\ \gamma^* \varepsilon & \alpha^* \end{array} \right).$$

The unitary group $U_{2n}(R,\Lambda)$ has many important special cases, as follows.

- When $\Lambda = R$, $U_{2n}(R, \Lambda)$ is the symplectic group. This can only happen when $\varepsilon = -1$ and $* = \mathrm{id}_R$ (R is commutative) the trivial anti-automorphism.
- When $\Lambda = 0$, $U_{2n}(R, \Lambda)$ is the ordinary orthogonal group. This can only happen when $\varepsilon = 1$ and $* = \mathrm{id}_R$ (R is commutative) as well.
- When $\Lambda = R^{\varepsilon}$ and $* \neq \mathrm{id}_R$, $U_{2n}(R,\Lambda)$ is the classical unitary group

$$U_{2n} = \{ A \in GL_{2n}R | A^*\varphi_n A = \varphi_n \},$$

where

$$\varphi_n = \left(\begin{array}{cc} 0 & I_n \\ \varepsilon I_n & 0 \end{array} \right).$$

Let E_{ij} denote the $n \times n$ matrix with 1 in the (i, j)-th position and zeros elsewhere. Then $e_{ij}(a) = I_n + aE_{ij}$ is an elementary matrix, where I_n is the identity matrix of size n. With n fixed, for any integer $1 \le k \le 2n$, set $\sigma k = k+n$ if $k \le n$ and $\sigma k = k - n$ if k > n. For $a \in R$ and $1 \le i \ne j \le 2n$, we define the elementary unitary matrices $\rho_{i,\sigma i}(a)$ and $\rho_{ij}(a)$ with $j \ne \sigma i$ as follows:

- $\rho_{i,\sigma i}(a) = I_{2n} + aE_{i,\sigma i}$ with $a \in \Lambda$ when $n+1 \le i$ and $a^* \in \Lambda$ when $i \le n$;
- $\rho_{ij}(a) = \rho_{\sigma j,\sigma i}(-a') = I_{2n} + aE_{ij} a'E_{\sigma j,\sigma i}$ with $a' = a^*$ when $i, j \leq n$; $a' = \varepsilon^* a^*$ when $i \leq n < j$; $a' = a^* \varepsilon$ when $j \leq n < i$; and $a' = \varepsilon^* a^* \varepsilon$ when $n+1 \leq i,j$.

The following lemma displays the commutator formulas for $EU_n(R, \Lambda)$ (cf. Lemma 2.1 in [34])

Lemma 4. The following identities hold for elementary unitary matrices $(1 \le i \ne j \le 2n)$:

- 1). $\rho_{ij}(a+b) = \rho_{ij}(a)\rho_{ij}(b);$
- 2). $[\rho_{ij}(a), \rho_{jk}(b)] = \rho_{ik}(ab)$ when $i, j, k, \sigma i, \sigma j, \sigma k$ are distinct;
- 3). $[\rho_{ij}(a), \rho_{j,\sigma i}(b)] = \rho_{i,\sigma i}(ab-c)$ when $j \neq \sigma i$, where $c = b^*a^*\epsilon$ when $n+1 \leq i$ and $c = \epsilon^*b^*a^*$ when $i \leq n$;
- 4). $[\rho_{ij}(a), \rho_{j,\sigma j}(b)] = \rho_{i,\sigma j}(ab)\rho_{i,\sigma i}(c)$ when $j \neq \sigma i$, where $b^* \in \Lambda$ and $c = aba^*$ when $i, j \leq n$, $b^* \in \Lambda$ and $c = aba^*\epsilon$ when $j \leq n < i$, $b \in \Lambda$ and $c = -ab^*a^*$ when $i \leq n < j$, $b \in \Lambda$ and $c = -ab^*a^*\epsilon$ when $n + 1 \leq i, j$.

When the ring R is finitely generated and Λ/R_{ε} is a finitely generated Rmodule by right multiplications, the above commutator formulas show that $E_n(R,\Lambda)$ is finitely generated (cf. [25], Section 9.2B). Our later discussions will
base on the following lemma.

Lemma 5. Let R be a ring and assume that the characteristic of R is not 2. For two integers i, j such that $1 \le i \ne j \le n$, let A_{ij} be the diagonal matrix whose (i, i)-th and (j, j)-th entries are -1 and other diagonal entries are 1. Then the subgroup generated by the elements

$$A_{12}, A_{23}, \ldots, A_{n-1,n}$$

in $E_n(R)$ is isomorphic to the abelian group $\mathbb{Z}_2^{n-1} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, i.e. n-1 copies of groups of two elements.

Proof. First note that there is an equality

$$A_{12} = e_{12}(1)e_{21}(-1)e_{12}(1)e_{12}(1)e_{21}(-1)e_{12}(1),$$

which shows A_{12} in an element of $E_n(R)$. Similar arguments show that all other elements $A_{i,i+1}$ are also in $E_n(R)$. It is not hard to see that the elements are pairwise commutative and the subgroup generated is isomorphic to the n-1 copies of \mathbb{Z}_2 .

Theorem 6. Let R be a ring with identity and $n \geq 3$ an integer. Suppose that the characteristic of R is not 2 and G is a normal subgroup in $E_n(R)$ containing a noncentral element in the subgroup generated \mathbb{Z}_2^{n-1} by $A_{12}, A_{23}, \ldots, A_{n-1,n}$ in Lemma 5. Then G contains $E_n(R, 2R)$ as a normal subgroup, i.e.,

$$E_n(R, 2R) \leq G$$
.

Proof. Let $A \in G$ be a noncentral element of $E_n(R)$ in \mathbb{Z}_2^{n-1} , the subgroup generated by $A_{12}, A_{23}, \ldots, A_{n-1,n}$. In other words, $A \neq \operatorname{diag}(1, \ldots, 1)$, $\operatorname{diag}(-1, \ldots, -1)$. Without loss of generality, we assume that the first three diagonal entries of A are 1, -1, -1 in order. Then for any element $r \in R$, we have that the matrix $e_{12}(2r)$ is the product

$$e_{12}(r) \cdot A \cdot e_{12}(-r) \cdot A^{-1},$$

which is an element in G. By the commutator formulas in Lemma 3, we see that for any two integers i, j with $1 \le i \ne j \le n$, the matrix $e_{ij}(2r) \in G$. This shows that $E_n(R, 2R)$ is a normal subgroup of G.

3.2 *K*-theory and stable ranges

In this subsection, we briefly recall the definitions of algebraic K_1 and unitary KU_1 groups. The standard references are also the textbook of Magurn [26] (for K_1), the book of Hahn and O'Meara [25] and the book of Bak [2] (for KU_1).

We define K_1 first. For a ring R, let $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$ be the inclusion defined by

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

The group GL(R) is defined to be the direct limit of

$$\operatorname{GL}_1(R) \subset \operatorname{GL}_2(R) \subset \cdots \subset \operatorname{GL}_n(R) \subset \cdots$$
.

Similarly, the group E(R) is the direct limit of

$$E_2(R) \subset E_3(R) \subset \cdots \subset E_n(R) \subset \cdots$$
.

According to the Whitehead lemma, the group E(R) is normal in GL(R). The K-theory group $K_1(R)$ is defined as GL(R)/E(R).

The stable range $\operatorname{sr}(R)$ is defined as follows. Let n be a positive integer and R^n the free R-module of rank n with standard basis. A vector (a_1,\ldots,a_n) in R^n is called $\operatorname{right}\ unimodular$ if there are elements $b_1,\ldots,b_n\in R$ such that $a_1b_1+\cdots+a_nb_n=1$. The $\operatorname{stable}\ range\ condition\ \operatorname{sr}_m$ says that if (a_1,\ldots,a_{m+1}) is a right unimodular vector then there exist elements $b_1,\ldots,b_m\in R$ such that $(a_1+a_{m+1}b_1,\ldots,a_m+a_{m+1}b_m)$ is right unimodular. It follows easily that $\operatorname{sr}_m\Rightarrow\operatorname{sr}_n$ for any $n\geq m$. The $\operatorname{stable}\ range\ \operatorname{sr}(R)$ of R is the smallest number m such that sr_m holds. If R is commutative, the Krull dimension $\operatorname{Kdim}(R)$ of R is the number of steps r in a longest chain of prime ideals

$$A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_r$$

in R. It is well-known that $\operatorname{sr}(R) \leq \operatorname{Kdim}(R) + 1$ (cf. Section 4E of [26]). When R is a Dedekind domain, then $\operatorname{sr}(R) \leq 2$. When G is finite and R is a Dedekind domain, the stable range $\operatorname{sr}(R[G]) \leq 2$ (cf. 41.23 of page 98 in [17]). The stable range is not bigger than most other famous ranges, e.g. absolute stable range, 1+ maximal spectrum dimension, 1+ Bass-Serre dimension (cf. [4]) and so on.

The following result on stabilization of K_1 is Theorem 10.15 in [26].

Lemma 6. Let R be a ring of finite stable range sr(R). Then for an integer $n \ge sr(R) + 1$, the natural map

$$\operatorname{GL}_n(R)/E_n(R) \to K_1(R)$$

is an isomorphism.

We define the unitary K-group KU_1 as follows. There is an obvious embedding

$$U_{2n}(R,\Lambda) \to U_{2(n+1)}(R,\Lambda),$$

$$\left(\begin{array}{ccc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \longmapsto \left(\begin{array}{cccc} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Using this map, we shall consider $U_{2n}(R,\Lambda)$ as a subgroup of $U_{2(n+1)}(R,\Lambda)$. Similarly, define $U(R,\Lambda)$ as the direct limit of

$$U_2(R,\Lambda) \subset U_4(R,\Lambda) \subset \cdots \subset U_{2n}(R,\Lambda) \subset \cdots$$

and $E(R,\Lambda)$ as the direct limit of

$$EU_2(R) \subset EU_4(R) \subset \cdots \subset EU_{2n}(R) \subset \cdots$$
.

The unitary K-theory group $KU_1(R,\Lambda)$ is defined as $U(R,\Lambda)/E(R,\Lambda)$.

The Λ -stable range condition Λsr_m of Bak and Tang [3] says that R satisfies sr_m and given any unimodular vector $(a_1, \ldots, a_{m+1}, b_1, \ldots, b_{m+1}) \in R^{2(m+1)}$ there exists a matrix $\gamma \in \Lambda_{m+1}$ such that $(a_1, \ldots, a_{m+1}) + (b_1, \ldots, b_{m+1})\gamma$ is unimodular. By [3], $\Lambda sr_m \Rightarrow \Lambda sr_n$ for all $n \geq m$. The Λ -stable range $\Lambda sr(R)$ of (R, Λ) is the smallest number m such that Λsr_m holds. In general, the Λ -stable range is also not bigger than 1+ Bass-Serre dimension (cf. [4]).

The following result on stabilization of KU_1 was proved by Bak and Tang in [3].

Lemma 7. Let (R, Λ) be a form ring with finite Λ -stable range $\Lambda sr(R)$. Then for an integer $n \geq \Lambda sr(R) + 1$ the natural map

$$U_{2n}(R,\Lambda)/EU_{2n}(R,\Lambda) \to KU_1(R,\Lambda)$$

is an isomorphism.

4 Proof of Theorems

In this section, we prove the results presented in Section 1.

4.1 Group actions on CAT(0) spaces

In order to prove Theorem 1, we need the following lemma. This is a generalization of Proposition 2 in [23], which is stated for Chevalley groups over commutative rings. Recall that the permutation σ is defined in Section 3.1.

- **Lemma 8.** (i) Let R be a general ring. Then for any integer $n \geq 3$, $1 \leq i \neq j \leq n$, an element $r \in R$ and any elementary matrix $e_{ij}(r) \in E_n(R)$, there exists a nilpotent subgroup $U \subset E_n(R)$ such that $e_{ij}(r) \in [U, U]$.
- (ii) Let (R, Λ) be a form ring over a general ring R. Then for any integer $n \geq 3$, $1 \leq i \neq j \leq 2n$, an element $r \in R$ and any elementary matrix $\rho_{ij}(r) \in EU_{2n}(R, \Lambda)$ (when $i = \sigma j$, we assume that $r \in \Lambda$ or Λ^*) there exists a nilpotent subgroup $U \subset EU_n(R)$ such that $\rho_{ij}(r) \in [U, U]$.

Proof. These are easy consequences of commutator formulas. For example, we have $e_{12}(r) = [e_{13}(1), e_{32}(r)]$. We choose U to be the subgroup generated by all elementary matrices $e_{13}(x), e_{32}(y)$ with $x, y \in R$. Since the commutator $[e_{13}(x), e_{32}(y)] = e_{12}(xy)$ is central in U, it is clear that this is a nilpotent subgroup. Other cases are similar. For the group $EU_{2n}(R, \Lambda)$ and $i \neq \sigma j$, the statement for $\rho_{ij}(r)$ is similar to that of $e_{12}(r)$ in $E_n(R)$. When $i = \sigma j$, for example $\rho_{1,n+1}(r)$ with $r \in \Lambda^*$, we have identities

$$\begin{array}{lcl} \rho_{1,n+1}(r) & = & \rho_{1,n+2}(-r)[\rho_{12}(1),\rho_{2,n+2}(r)] \\ & = & [\rho_{13}(1),\rho_{3,n+2}(-r)][\rho_{12}(1),\rho_{2,n+2}(r)] \end{array}$$

by (4) and (2) of Lemma 4. Take U to be the subgroup generated by all unitary elementary matrices $\rho_{12}(x)$, $\rho_{13}(y)$, $\rho_{3,n+2}(z)$, $\rho_{2,n+2}(a)$ with $x, y, z \in R$ and $a \in \Lambda^*$. This is also a nilpotent group by the commutator formulas for unitary groups in Lemma 4, since the commutators of these matrices are all upper triangular matrices.

Our proof of Theorem 1 will be based on the following general fixed-point theorem, which is Theorem 5.1 in Farb [21].

Lemma 9. Let Γ be a finitely generated group, and let $C = \{\Gamma_1, \Gamma_2, \dots, \Gamma_{r+1}\}$ be a collection of finitely generated nilpotent subgroups of Γ . Suppose that:

- (1) C generates a finite index subgroup of Γ .
- (2) Any proper subset of C generates a nilpotent group.
- (3) There exists m > 0 so that for any element g of any Γ_i there is a nilpotent subgroup $N < \Gamma$ with $g^m \in [N, N]$.

Then Γ has the strong property FA_{r-1} .

Proof of Theorem 1. We first prove that the elementary group $E_n(R)$ has the property FA_{n-2} . If a group has strong property FA_r , then so do all its quotient groups (cf. (2) of Lemma 1). Therefore, we may assume that the ring R is the free noncommutative ring $\mathbb{Z}\langle x_1, x_2, \ldots, x_k \rangle$ generated by elements x_1, x_2, \ldots, x_k . For $1 \leq i \leq n-1$, let Γ_i be the subgroup generated by all matrices $e_{i,i+1}(x)$ with $x \in R$. Denote by Γ_n the subgroup generated by all matrices $e_{n1}(x)$ with $x \in R$. Then the set $C := \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ generates the whole group $E_n(R)$, as follows. Denote by $\langle C \rangle$ the subgroup generated by C in $E_n(R)$. By the commutator formulas in Lemma 3, when $x \in R$ and $x \in R$ and $x \in R$ we have that

$$e_{ij}(r) = [e_{i,i+1}(r), e_{i+1,j}(1)] = [e_{i,i+1}(r), [e_{i+1,i+2}(1), e_{i+2,j}(1)]]$$
$$= \cdots = [e_{i,i+1}(r), [\cdots, e_{j-1,j}(1)] \cdots] \in \langle C \rangle$$

and

$$e_{ji}(r) = [e_{jn}(r), e_{ni}(1)] = [e_{jn}(r), [e_{n1}(1), e_{1i}(1)]] \in \langle C \rangle.$$

This checks (1) of Lemma 9. It is obvious that (2) also holds. By Lemma 8, the condition (3) holds as well for m = 1. Therefore, Lemma 9 implies that $E_n(R)$ has the strong property FA_{n-2} .

We prove the property FA_{n-1} of the elementary unitary group $EU_{2n}(R,\Lambda)$ as follows. The idea is the same as the proof for $E_n(R)$. For $1 \le i \le n-1$, let Γ_i be the subgroup generated by all $\rho_{i,i+1}(x)$ with $x \in R$. Denote by Γ_n the subgroup generated by all $\rho_{n,2n-1}(r)\rho_{n,2n}(x)$ with $r \in R$, $x \in \Lambda^*$ and by Γ_{n+1} the subgroup generated by all $\rho_{n+1,2}(r)\rho_{n+1,1}(x)$ with $r \in R$, $x \in \Lambda$. Let C' be the set of subgroups $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_{n+1}\}$. It is sufficient to check that all the conditions in Lemma 9 are satisfied. By Lemma 8, for any integer $1 \le i \le n-1$, the group Γ_i satisfies the condition (3). Note that for any $r \in R$, $x \in \Lambda^*$, by Lemma 4 we have that

$$\begin{array}{lcl} \rho_{n,2n-1}(r)\rho_{n,2n}(x) & = & \rho_{n,2n-1}(r-x)[\rho_{n,n-1}(1),\rho_{n-1,2n-1}(x)] \\ & = & [\rho_{n1}(r-x),\rho_{1,2n-1}(1)][\rho_{n,n-1}(1),\rho_{n-1,2n-1}(x)]. \end{array}$$

Therefore, any element of the group Γ_n lies in the commutator subgroup of the nilpotent subgroup generated by all matrices $\rho_{n1}(r_1)$, $\rho_{n,n-1}(r_2)$, $\rho_{1,2n-1}(r_3)$ and $\rho_{n-1,2n-1}(x)$ with $r_1, r_2, r_3 \in R$ and $x \in \Lambda^*$. A similar argument shows that Γ_{n+1} satisfies the condition (3) as well. We now check the condition (1). Denote by $\langle C' \rangle$ the subgroup generated by C' in $EU_{2n}(R, \Lambda)$. According to the commutator formulas in Lemma 4, for any $r \in R$ and $1 \le i < j \le n$ we have that $\rho_{ij}(r) \in \langle C' \rangle$ and

$$\rho_{i,2n}(r) = [\rho_{i,n-1}(1), \rho_{n-1,2n}(r)] \in \langle C' \rangle.$$

Note that $\rho_{i,2n}(r) = \rho_{n,n+i}(-\varepsilon^*r^*)$. When $1 \le i < n < j \le 2n$ with $i \ne n-j$ and $r \in R$, we have that

$$\rho_{ij}(r) = [\rho_{in}(1), \rho_{nj}(r)] \in \langle C' \rangle.$$

Since all the matrices $\rho_{i,\sigma i}(x)$ can be generated by $\rho_{ij}(1)$ with $i \neq \sigma j$ and $\rho_{n,2n}(x)$ (cf. (4) in Lemma 4), we get that all the upper triangular elementary unitary matrices belong to $\langle C' \rangle$. For any $r \in R, 1 < i \leq 2n$ with $i \neq n+1, n+2$, we have that

$$\rho_{n+1,i}(r) = [\rho_{n+1,2}(r), \rho_{2,i}(1)] \in \langle C' \rangle.$$

Note that for any $r \in R$ and $i \neq 1, n+1$, the matrix $\rho_{i,1}(r) = \rho_{n+1,\sigma i}(x)$ for some $x \in R$. Therefore for any $r \in R$ and all $1 < i, j \leq 2n$ with $i, j, \sigma i, \sigma j$ distinct and $i, j \neq n+1$, we have

$$\rho_{ij}(r) = [\rho_{i1}(1), \rho_{1j}(r)].$$

This proves that the subgroup generated by C' is $EU_{2n}(R,\Lambda)$ and the condition (1) in Lemma 9 is satisfied. It can be directly checked that condition (2) holds. Therefore, the group $EU_{2n}(R,\Lambda)$ has property FA_{n-1} by Lemma 9.

Proof of Theorem 2. Recall the stabilization of K_1 from Lemma 6. When $n \ge \operatorname{sr}(R) + 1$, the group $E_n(R)$ is normal in $\operatorname{GL}_n(R)$ and there is an isomorphism $\operatorname{GL}_n(R)/E_n(R) \to K_1(R)$. When $n \ge \max\{3, \operatorname{sr}(R) + 1\}$, the group $E_n(R)$ has property FA_{n-2} by Theorem 1. By assumption, the quotient group $\operatorname{GL}_n(R)/E_n(R) \cong K_1(R)$ has property FA_{n-2} . Therefore, the group $\operatorname{GL}_n(R)$ has property FA_{n-2} according to (3) of Lemma 1. The second part for $U_{2n}(R,\Lambda)$ can be proved similarly using Lemma 7 and Theorem 1.

Proof of Corollary 1. When G is finite and R is a Dedekind domain, the stable range $\operatorname{sr}(R[G]) \leq 2$ (41.23 of [17], page 98). When $R = \mathbb{Z}$, the abelian group $K_1(\mathbb{Z}[G])$ is finitely generated of rank equal to the number of irreducible real representations of G minus the number of irreducible rational representations (Theorem 7.5 of [5], page 625). By assumption, we have that the group $K_1(\mathbb{Z}[G])$ is finite. By Lemma 2.1 in [21], any finite group action on a CAT(0) space has a global fixed point and thus has property FA_{n-2} . This finishes the proof by Theorem 2.

In order to prove Proposition 1, we need the following lemma, which was pointed out to the author by A.J. Berrick. This is a generalization of Lemma 4.2 in [16] which is stated for $R = \mathbb{Z}$.

Lemma 10. Let $n \geq 3$ and R a general ring. Then any action of $E_n(R)$ on a finite set with less than n points is trivial.

Proof. Let Sym(k) be the permutation group of k elements. Any group action of $E_n(R)$ on a finite set of k elements corresponds a group homomorphism

$$\varphi: E_n(R) \to \operatorname{Sym}(k)$$
.

When $k \leq n-1$ and $n \geq 5$, the alternating group A_n is simple and there is no nontrivial map from A_n to $\operatorname{Sym}(k)$ by considering the cardinalities. Since A_n normally generates $E_n(R)$ (cf. Berrick [8], 9.4), any map φ is trivial. For n=3 and n=4, the triviality of φ follows from the fact that $E_n(R)$ is perfect and $\operatorname{Sym}(k)$ is soluble.

Proof of Proposition 1. Let X be a uniformly finite CAT(0) cell complex. Assume that the degree of each vertex is less N for some positive integer N. For an integer $n \geq \max\{\dim(X) - 2, 3\}$, let G be a copy of $E_n(R)$ sitting inside of E(R) (or inside of $EU(R,\Lambda)$ by the hyperbolic embedding defined by $A \longmapsto \operatorname{diag}(A,A^{*-1})$). We may assume that n > N. By Theorem 1, there is a fixed point $x_0 \in X$ under the G-action. Denote by $\operatorname{Fix}(G)$ the set of fixed points of G-action in X. Then G acts on the link of x_0 , which is a finite set with less than N elements. By Lemma 10, the group G action is trivial, which shows that any neighbor of x_0 is also in $\operatorname{Fix}(G)$. Therefore, the group G acts

trivially on all vertices of X. According to the commutator formulas in Lemma 3 and Lemma 7, the group E(R) and $EU(R,\Lambda)$ are normally generated by G and hence act trivially on the whole space X.

Proof of Corollary 2. It is proved by Farb in Theorem 1.7 and Theorem 1.8 of [21] that any group Γ with property FA_{n-1} is of integral n-representation type. Then the corollary is a direct consequence of Theorem 1.

4.2 Group actions on spheres and acyclic manifolds

Recall that a group G action on a space X is effective if the subgroup that fixes all elements of X is trivial. In order to prove Theorem 3, we need two lemmas from Bridson and Vogtmann [15].

Lemma 11 ([15], Theorem 4.7). Let m and n be two integers with m < n - 1. Then the group \mathbb{Z}_2^n , n copies of groups of two elements, cannot act effectively by homeomorphisms on a generalized m-sphere over \mathbb{Z}_2 or a \mathbb{Z}_2 -acyclic (m+1)- $hm_{\mathbb{Z}_2}$.

If m < 2n-1 and p is an odd prime, then \mathbb{Z}_p^n cannot act effectively by homeomorphisms on a generalized m-sphere over \mathbb{Z}_p or a \mathbb{Z}_p -acyclic (m+1)- $hm_{\mathbb{Z}_p}$.

Lemma 12 ([15], Lemma 4.12). Let X be a generalized m-sphere over \mathbb{Z}_2 or a \mathbb{Z}_2 -acyclic (m+1)-hm \mathbb{Z}_2 and G be a group acting by homeomorphisms on X. Suppose G contains a subgroup $P = \mathbb{Z}_2 \times \mathbb{Z}_2$ all of whose nontrivial elements are conjugate in G. If P acts nontrivially, then the fixed-point sets of its nontrivial elements have codimension $m \geq 2$.

Proof of Theorem 3. We only give the proof of group actions on generalized d-spheres, while that of group actions on acyclic homology manifolds is similar. Suppose that $E_n(R)$ acts by homeomorphisms on some generalized d-sphere X. This means that there is a group homomorphism $f: E_n(R) \to \text{Homeo}(X)$. We prove (a)(i) in two cases.

(1) The characteristic of R is 2.

When n=3, the elements $e_{12}(1), e_{13}(1)$ generate a subgroup which is isomorphic to $G:=\mathbb{Z}_2^2$ in $E_n(R)$. Note that $e_{12}(1)$ and $e_{13}(1)$ are conjugate by a permutation matrix and that

$$e_{12}(1)e_{13}(1) = e_{23}(1)e_{12}(1)e_{23}(1).$$

We conclude from Lemma 12 that if the action of G is not trivial then the fixedpoint set of any nontrivial element is at least of codimension 2. Since $d \leq 1$, this shows that the action of G is free. However, a classical result of Smith says that $\mathbb{Z}_p \times \mathbb{Z}_p$ cannot act freely on any generalized sphere over \mathbb{Z}_p for any prime number p (cf. [33]). This implies that the action of G is trivial. By the commutator formulas in Lemma 3, the group $E_n(R)$ is normally generated by G. This shows that the action of $E_n(R)$ is trivial. When $n \geq 4$, the matrices $e_{ij}(1)$ $(1 \leq i \leq n/2, n/2 < j \leq n)$ generate an abelian group \mathbb{Z}_2^k , where in general $k \geq n$. By Lemma 11, the action of \mathbb{Z}_2^k is not effective on the generalized d-sphere X over \mathbb{Z}_2 . Choose a nontrivial element $M \in \mathbb{Z}_2^k$ acting trivially on X. Without loss of generality, we may assume that $M = e_{1n}(1)$. By the commutator formulas in Lemma 3 again, the group $E_n(R)$ is normally generated by such M. This shows that the action of $E_n(R)$ is trivial. The same argument with $x_{ij}(1)$ instead of $e_{ij}(1)$ show that any action of $\operatorname{St}_n(R)$ on X is also trivial.

(2) The characteristic of R is not 2.

Let $A_{12}, A_{23}, \ldots, A_{n-1,n}$ be the elements in $E_n(R)$ defined in Lemma 5. By Lemma 5, they generate a subgroup which is isomorphic to \mathbb{Z}_2^{n-1} . Suppose that we can find a noncentral element A of $E_n(R)$ in \mathbb{Z}_2^{n-1} such that the action of A is trivial. According to Theorem 6, the normal subgroup generated by A contains the subgroup $E_n(R, 2R)$. Note that the action of any element in $E_n(R, 2R)$ is trivial. When 2 is invertible in R, we have that $E_n(R, 2R) = E_n(R)$. This implies that any element in $E_n(R)$ acts trivially on X. When 2 is not invertible, the action of $E_n(R)$ factors through that of $E_n(R)/E_n(R, 2R)$. Note that there is a commutative diagram

$$1 \to \operatorname{St}_n(R, 2R) \to \operatorname{St}_n(R) \to \operatorname{St}_n(R/2R) \to 1$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$1 \to E_n(R, 2R) \to E_n(R) \to E_n(R)/E_n(R, 2R) \to 1,$$

where the two horizontal sequences are exact (for the exactness of the first one, see Lemma 13.18 and its proof in Magurn [26]). Then the action of $E_n(R)/E_n(R,2R)$ on X can be lifted as an action of $\operatorname{St}_n(R/2R)$. Since the quotient ring R/2R is of characteristic 2, this case is already proved in case (1). Therefore, it is enough to find such element A in \mathbb{Z}_2^{n-1} such that the action of A is trivial. It is not hard to see that for each integer $1 \leq i \leq n-2$, the elements $A_{i,i+1}$, $A_{i+1,i+2}$ and $A_{i,i+1}A_{i+1,i+2}$ are conjugate by some permutation matrices. We will finish the proof by induction on n (cf. the proof of Theorem 1.1 in Bridson and Vogtmann [15]).

When n=3, using a similar argument as that of case (1), we see that the group generated by A_{12} and A_{23} cannot act effectively on the generalized d-sphere X. Therefore such element A exists.

When n=4, if the action of A_{12} is trivial, we are done. Otherwise, Lemma 12 and Lemma 2 show that the fixed-point set $\operatorname{Fix}(A_{12})$ of A_{12} is a generalized sphere over \mathbb{Z}_2 of dimension 0 (note: the fixed-point set is not empty). Then the abelian group \mathbb{Z}_2^2 generated by A_{23} and A_{34} acts on $\operatorname{Fix}(A_{12})$. By Lemma 11, there exists a nontrivial element γ with trivial action on $\operatorname{Fix}(A_{12})$. Since γ and A_{12} are conjugate, we have that $\operatorname{Fix}(A_{12}) = \operatorname{Fix}(\gamma)$. By Theorem 4.8 in [15], A_{12} and γ have the same image in $\operatorname{Homeo}(X)$. If $\gamma \neq A_{34}$, $A_{12}\gamma^{-1}$ is noncentral in $E_n(R)$ and we can take $A = A_{12}\gamma^{-1}$. If $\gamma = A_{34}$, the group homomorphism f factors through

$$\bar{f}: E_n(R)/\langle \pm I_n \rangle \to \operatorname{Homeo}(X).$$

In $E_n(R)/\langle \pm I_n \rangle$, the images of A_{12} , A_{23} , $e_{12}(1)e_{21}(-1)e_{12}(1)e_{34}(1)e_{43}(-1)e_{34}(1)$ and $e_{13}(1)e_{31}(-1)e_{13}(1)e_{24}(1)e_{42}(-1)e_{24}(1)$ generate an abelian group \mathbb{Z}_2^4 . By Lemma 11, there exists a nontrivial element having trivial action on X. The preimage of such an element normally generates $E_n(R, 2R)$. By case (1), we are done

We now consider the general case when $n \geq 5$. If the action of $A_{n-1,n}$ is trivial, we are done. Otherwise, Lemma 12 and Lemma 2 show that the fixed-point set $\operatorname{Fix}(A_{n-1,n})$ is a generalized sphere over \mathbb{Z}_2 of codimension at least 2. The elements in the subgroup $E_{n-2}(R)$ in the upper left corner of $E_n(R)$ are centralizers of $A_{n-1,n}$. By induction assumption, the action of $E_{n-2}(R)$ on $\operatorname{Fix}(A_{n-1,n})$ is trivial. This shows that $\operatorname{Fix}(A_{n-1,n}) \subset \operatorname{Fix}(A_{12})$. Similarly, the converse holds. This implies that $f(A_{12}) = f(A_{n-1,n})$ (cf. Theorem 4.8 in [15]). Take $A = A_{12}^{-1}A_{n-1,n}$. This finishes the proof of (a).

We prove (b)(i) as follows. Since $E_{n-1}(R)$ normally generates $E_n(R)$ when n > 2, it is enough to prove (ii) when n = 2k for some $k \ge 2$. Construct an abelian subgroup \mathbb{Z}_3^k in $E_n(R)$, as follows. For each integer i (i = 1, 2, ..., k), denote by B_i the matrix

$$e_{2i-1,2i}(1)e_{2i,2i-1}(-1)e_{2i-1,2i}(1)e_{2i,2i-1}(-1) \in E_n(R).$$

For example, B_1 looks like the matrix

$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \\ & I_{n-2} \end{pmatrix}$$
.

It is obvious that each matrix B_i has order 3 and together they generate an abelian subgroup \mathbb{Z}_3^k in $E_n(R)$. By Lemma 11, for an integer $d \leq 2k-2$ the group \mathbb{Z}_3^k cannot act effectively by homeomorphisms on a generalized d-sphere over \mathbb{Z}_3 . Without loss of generality, we may assume that the action of B_1 is trivial. Note that

$$[e_{32}(1), B_1] = e_{31}(-1)e_{32}(2)$$

and

$$[e_{31}(-1)e_{32}(2), e_{12}(-1)] = e_{32}(1).$$

The matrix $e_{32}(1)$ normally generates the whole group $E_n(R)$. This shows that the group action of $E_n(R)$ is trivial.

Now we prove (c). Suppose that the group $EU_{2n}(R,\Lambda)$ acts by homeomorphisms on a generalized d-homology sphere over \mathbb{Z}_2 or \mathbb{Z}_3 . There is a group homomorphism $E_n(R) \to EU_{2n}(R,\Lambda)$ defined by the hyperbolic embedding

$$A \longmapsto \operatorname{diag}(A, A^{*-1})$$

for any element $A \in E_n(R)$. By the commutator formulas in Lemma 4, we see that $EU_{2n}(R,\Lambda)$ is normally generated by the image of $E_n(R)$. Since the action of $E_n(R)$ is trivial, the action of $EU_{2n}(R,\Lambda)$ is trivial as well.

Remark 3. If the generalized spheres in Theorem 3 are smooth manifolds and the actions are smooth, the proof is much easier by noting the fact that \mathbb{Z}^k cannot act effectively by orientation-preserving diffeomorphisms on a d-sphere for $d \leq k-1$ (cf. the proof of Theorem 2.1 in [15]). When we know that Theorem 3 is true for $R = \mathbb{Z}$, the general-ring case can also be proved by using the normal generation of $E_n(R)$ by the image of $E_n(\mathbb{Z})$. Our intent here is to avoid the Margulis finiteness theorem. Moreover, the proof given here works for Steinberg groups as well.

Proof of Theorem 4. The strategy of the proof is similar to that of Theorem 3. We construct an abelian subgroup \mathbb{Z}_3^n of $EU_{2n}(R,\Lambda)$ as follows. For $i=1,2,\ldots,n$, let

$$C_i = \rho_{i,n+i}(1)\rho_{n+i,i}(-1)\rho_{i,n+i}(1)\rho_{n+i,i}(-1) \in EU_{2n}(R).$$

It is obvious that the order of C_i is 3 and the subgroup generated by C_i (i = 1, 2, ..., n) is \mathbb{Z}_3^n . The remainder of the proof of (i) is the same as that of (b)(i) in Theorem 3.

Proof of Corollary 3. Let $E_{n-1}(A)$ act on the space \mathbb{R}^{n-1} by matrix multiplications. According to Theorem 3 a(ii), the image of $E_n(R)$ in $E_{n-1}(A)$ acts trivially on \mathbb{R}^{n-1} . This implies that the image in (i) is the identity matrix. The second part can be proved similarly by using Theorem 4 and considering the group $\operatorname{Sp}_{2(n-1)}(A)$ action on the space $\mathbb{R}^{2(n-1)}$.

Proof of Theorem 5. For the group E(R), the proof is similar to that of Lemma 1 in [36]. The idea is as follows. For sufficiently large k, the abelian group \mathbb{Z}_2^k cannot act effectively on the manifolds in Theorem 5. When the characteristic of R is 2, we take such \mathbb{Z}_2^k as the subgroup in E(R) generated by $e_{1j}(1)$ for $2 \leq j \leq k+1$. By commutator formulas (cf. Lemma 3), any nontrivial element in \mathbb{Z}_2^k normally generates E(R). This shows that the action of E(R) is trivial. When the characteristic of R is not 2, we take such \mathbb{Z}_2^k as the subgroup generated by $A_{i,i+1}$ defined in Lemma 5 for $1 \leq i \leq k$. Any nontrivial element in such \mathbb{Z}_2^k is noncentral in E(R). By Lemma 5, any noncentral element in such \mathbb{Z}_2^k generates a normal subgroup containing E(R, 2R). Therefore the action of E(R) factors through that of E(R/2R), which is already proved since the characteristic of R/2R is 2.

For the group $EU(R, \Lambda)$, note that there is a hyperbolic embedding $E(R) \to EU(R, \Lambda)$ defined by $A \longmapsto \operatorname{diag}(A, A^{*-1})$. The action of $EU(R, \Lambda)$ is trivial since E(R) normally generates $EU(R, \Lambda)$.

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References

- R. Alperin, Locally compact groups acting on trees and property T, *Monatsh. Math.* 93 (1982), 261-265.
- [2] A. Bak, *K-theory of forms*. Ann. of Math. Stud., vol. 98. Princeton Univ. Press, Princeton, 1981.
- [3] A. Bak, G.P. Tang, Stability for Hermitian K_1 , J. Pure Appl. Algebra 150 (2000), 107-121.
- [4] A. Bak, V. Petrov and G.P. Tang, Stability for quadratic K_1 , K-theory **30** (2003), 1-11.
- [5] H. Bass, Algebraic K-theory, Benjamin, New York, 1968.
- [6] H. Bass, Groups of integral representation type, Pacific Journal of Math., 86 (1980), 15-51.
- [7] A. Barnhill, The FA_n conjecture for Coxeter groups, Algebraic & Geometric topology 6 (2006), 2117-2150.
- [8] A.J. Berrick, An approach to algebraic K-Theory, Pitman Research Notes in Math 56, London, 1982.
- [9] G.E. Bredon, *Sheaf Theory*, second edition. Graduate Texts in Mathematics, 170. Springer-Verlag, New York, 1997. xii+502 pp.
- [10] M.R. Bridson, Semisimple actions of mapping class groups on CAT(0) spaces, to appear in "The Geometry of Riemann Surfaces" (F. P. Gardiner, G. Gonzalez-Diez and C. Kourouniotis, eds.), LMS Lecture Notes 368, Cambridge Univ. Press, Cambridge, 2010, pp. 1–14.
- [11] M.R. Bridson, On the dimension of CAT(0) spaces where mapping class groups act, *Journal für reine und angewandte Mathematik*, to appear. arXiv:0908.0690.
- [12] M.R. Bridson, Helly's theorem, CAT(0) spaces, and actions of automorphism groups of free groups, preprint 2007.
- [13] M.R. Bridson, The rhombic dodecahedron and semisimple actions of $\operatorname{Aut}(F_n)$ on $\operatorname{CAT}(0)$ spaces, $\operatorname{arXiv:}1102.5664v1$.
- [14] M.R. Bridson and A. Haefliger, *Metric spaces of nonpositive curvature*, Grundlehren der Math. Wiss. 319, Springer-Verlag, Berlin, 1999.

- [15] M.R. Bridson, K. Vogtmann, Actions of automorphism groups of free groups on homology spheres and acyclic manifolds. *Commentarii Mathematici Helvetici* 86 (2011), 73-90.
- [16] I. Chatterji and M. Kassabov, New examples of finitely presented groups with strong fixed point properties. *Journal of Topology and Analysis* 1 (2009), 1-12.
- [17] C.W. Curtis and I. Reiner, Methods of Representation theory with applications to finite groups and orders (Volume 2), Wiley-Interscience Publication, J. Wiley & Sons, Inc., New York, 1987.
- [18] D. Doković, V.P. Platonov, Low-dimensional representations of $Aut(F_2)$, Manuscripta Math., **89** (1996) 475-509.
- [19] M. Ershov and A. Jaikin-Zapirain, Property (T) for noncommutative universal lattices, *Inventiones Mathematicae* **179** (2010), 303-347.
- [20] M. Ershov, Andrei Jaikin-Zapirain and M. Kassabov, Property (T) for groups graded by root systems, arXiv:1102.0031.
- [21] B. Farb, Group actions and Helly's theorem, Advances in Mathematics 222 (2009), 1574-1588.
- [22] B. Farb and P. Shalen, Real-analytic actions of lattices, *Inventiones Mathematicae* **135** (1999), 273-296.
- [23] M. Fukunaga, Fixed points of elementary subgroups of Chevalley groups acting on trees, *Tsukuba J. Math.*, **3** (1979), 7-16.
- [24] R.M. Guralnick, P.H. Tiep, Low-dimensional representations of special linear groups in cross characteristics, Proc. London Math. Soc. **78** (1999), 116-138.
- [25] A.J. Hahn, O.T. O'Meara, The classical groups and K-theory, Springer-Verlag, Berlin, 1989.
- [26] B.A. Magurn. An algebraic introduction to K-theory, Cambridge University Press, 2002.
- [27] G.A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergeb. Math. Grenzgeb. (3) 17, Springer-Verlag, Berlin, 1991.
- [28] R. Oliver, Whitehead groups of finite groups, London Mathematical Society Lecture Note Series .132, Cambridge University Press, 1988.
- [29] K. Parwani, Actions of $SL(n, \mathbb{Z})$ on homology spheres, Geom. Dedicata 112 (2005), 215-223.
- [30] A. Potapchik, A. Rapinchuk, Low-dimensional linear representations of $\operatorname{Aut}(F_n)$, $n \geq 3$, Trans. AMS. **352** (2000), 1437-1451.

- [31] P.A. Smith, Transformations of finite period, Ann. Math. 39 (1938), 137-164
- [32] P.A. Smith, Transformations of finite period II, Ann. Math. 40 (1939), 690-711.
- [33] P.A. Smith, Permutable periodic transformations, *Proc. Nat. Acad. Sci. U.S.A.* **30** (1944), 105-108.
- [34] L.N. Vaserstein and H. You, Normal subgroups of classical groups over rings, *J. Pure Appl. Algebra* **105** (1995), 93-105.
- [35] S. Weinberger, $SL(n,\mathbb{Z})$ cannot act on small tori. Geometric topology (Athens, GA, 1993), 406–408, AMS/IP Stud. Adv. Math., Amer. Math. Soc., Providence, RI, 1997.
- [36] S. Weinberger. Some remarks inspired by the C^0 Zimmer program. in *Rigidity and Group actions*, Chicago lecture notes, 2011.
- [37] R. Zimmer, Actions of semisimple groups and discrete subgroups, *Proc. I.C.M.*, Berkeley 1986, 1247-1258
- [38] R. Zimmer, Lattices in semisimple groups and invariant geometric structures on compact manifolds, *Discrete groups in geometry and analysis* (New Haven, Conn.,1984), 152-210, Progr. Math., Vol. 67, Birkhäuser Boston, Boston, MA, 1987.
- [39] B.P. Zimmermann, $SL(n,\mathbb{Z})$ cannot act on small spheres, Topology and its Applications 156 (2009), 1167-1169.
- [40] B.P. Zimmermann, A note on actions of the symplectic group $\operatorname{Sp}(2g,\mathbb{Z})$ on homology spheres, arxiv.org/abs/0903.2946v1.

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