

THE GARDEN OF EDEN THEOREM: OLD AND NEW

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ABSTRACT. We review topics in the theory of cellular automata and dynamical systems that are related to the Moore-Myhill Garden of Eden theorem.

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1. INTRODUCTION

In the beginning, the Garden of Eden theorem, also known as the *Moore-Myhill theorem*, is a result in the theory of cellular automata which states that a cellular automaton is surjective if and only if it satisfies a weak form of injectivity, called pre-injectivity. The theorem was obtained by Moore and Myhill in the early 1960s for cellular automata with finite alphabet over the groups \mathbb{Z}^d . The fact that surjectivity implies pre-injectivity for such cellular automata was first proved by Moore in [39], and Myhill [43] established the converse implication shortly after. The proofs of Moore and Myhill appeared in two separate papers both published in 1963. The biblical terminology used to designate the Moore-Myhill theorem comes from the fact that configurations that are not in the image of a cellular automaton are called *Garden of Eden configurations* because, when considering the sequence of consecutive iterates of the cellular automaton applied to the set of configurations, they can only occur at time 0. Surjectivity of a cellular automaton is equivalent to absence of Garden of Eden configurations. In 1988 [49, Question 1], Schupp asked whether the class of groups for which the Garden of Eden theorem remains valid is precisely the class of virtually nilpotent groups. By a celebrated result of Gromov [29], a finitely generated group is virtually nilpotent if and only if it has polynomial growth. In 1993, Machì and Mignosi [36] proved that the Garden of Eden theorem is still valid over any finitely generated group with subexponential growth. As Grigorchuk [28], answering a longstanding open question raised by Milnor [38], gave examples of groups whose growth lies strictly between polynomial and exponential, it follows that the class of finitely generated groups satisfying the Garden of Eden theorem is larger than the class of finitely generated virtually nilpotent groups. Actually, it is even larger than the class of finitely generated groups with subexponential growth. Indeed, Machì, Scarabotti, and the first author [20] proved in 1999 that every amenable group satisfies the Garden of Eden theorem and it is a well known fact that there are finitely generated amenable groups, such as the solvable Baumslag-Solitar group $BS(1, 2) = \langle a, b : aba^{-1} = b^2 \rangle$, that are amenable

and have exponential growth. It was finally shown that the class of groups that satisfy the Garden of Eden theorem is precisely the class of amenable groups. This is a consequence of recent results of Bartholdi [2], Bartholdi and Kielak [3], who showed that none of the implications of the Garden of Eden theorem holds when the group is nonamenable.

In [30, Section 8], Gromov made an important contribution to the subject by providing a deep analysis of the role played by entropy in the proof of the Garden of Eden theorem and indicating new directions for extending it in many other interesting settings. He mentioned in particular [30, p 195] the possibility of proving an analogue of the Garden of Eden theorem for a suitable class of hyperbolic dynamical systems. Some results in that direction were subsequently obtained by the authors in [17], [16], and [18]. In particular, a version of the Garden of Eden theorem was established for Anosov diffeomorphisms on tori in [17] and for principal expansive algebraic actions of countable abelian groups in [18].

The present article is intended as a reasonably self-contained survey on the classical Garden of Eden theorem and some of its generalizations. Almost all results presented here have already appeared in the literature elsewhere but we sometimes give complete proofs when we feel they might be helpful to the reader. The general theory of cellular automata over groups is developed in our monograph [12]. The present survey is a kind of complement to our book since for instance cellular automata between subshifts are not considered in [12] while they are treated here.

The paper is organized as follows. Configuration spaces and shifts are presented in Section 2. Cellular automata are introduced in Section 3. Section 4 contains the proof of the Garden of Eden theorem in the case $G = \mathbb{Z}^d$ following Moore and Myhill. The proof of the Garden of Eden theorem in the case of an arbitrary countable amenable group is given in Section 5. Examples of cellular automata that do not satisfy the Garden of Eden theorem for groups containing nonabelian free subgroups are described in Section 6. We also discuss the results of Bartholdi and Kielak mentioned above, which, together with the Garden of Eden theorem, lead to characterizations of amenability in terms of cellular automata. Extensions of the Garden of Eden theorem to certain classes of subshifts are reviewed in Section 7. In Section 8, we present versions of the Garden of Eden theorem we obtained for certain classes of dynamical systems. The final section briefly discusses some additional topics and provides references for further readings.

2. CONFIGURATION SPACES AND SHIFTS

2.1. Notation. We use the symbol \mathbb{Z} to denote the set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$. The symbol \mathbb{N} denotes the set of nonnegative integers $\{0, 1, 2, \dots\}$. The cardinality of a finite set X is written $|X|$.

We use multiplicative notation for groups except for abelian groups such as

$$\mathbb{Z}^d = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}}_{d \text{ times}}$$

for which we generally prefer additive notation.

Let G be a group. We denote the identity element of G by 1_G . If A, B are subsets of G and $g \in G$, we write $AB := \{ab : a \in A, b \in B\}$, $A^{-1} := \{a^{-1} : a \in A\}$, $gA := \{g\}A$ and $Ag := A\{g\}$. A subset $A \subset G$ is said to be *symmetric* if it satisfies $A = A^{-1}$.

2.2. Configurations spaces. Let \mathcal{U} be a countable set, called the *universe*, and A a finite set, called the *alphabet*. Depending on the context, the elements of A are called *letters*, or *symbols*, or *states*, or *colors*. As usual, we denote by $A^{\mathcal{U}}$ the set consisting of all maps $x : \mathcal{U} \rightarrow A$. An element of $A^{\mathcal{U}}$ is called a *configuration* of the universe \mathcal{U} . Thus, a configuration is a way of attaching a letter of the alphabet to each element of the universe.

If $x \in A^{\mathcal{U}}$ is a configuration and $\mathcal{V} \subset \mathcal{U}$, we shall write $x|_{\mathcal{V}}$ for the *restriction* of x to \mathcal{V} , i.e., the element $x|_{\mathcal{V}} \in A^{\mathcal{V}}$ defined by $x|_{\mathcal{V}}(v) = x(v)$ for all $v \in \mathcal{V}$. If $X \subset A^{\mathcal{U}}$, we shall write

$$(2.1) \quad X_{\mathcal{V}} := \{x|_{\mathcal{V}} : x \in X\} \subset A^{\mathcal{V}}.$$

Two configurations $x, y \in A^{\mathcal{U}}$ are said to be *almost equal* if they coincide outside of a finite set, i.e., there is a finite subset $\Omega \subset \mathcal{U}$ such that $x|_{\mathcal{U} \setminus \Omega} = y|_{\mathcal{U} \setminus \Omega}$. Being almost equal clearly defines an equivalence relation on $A^{\mathcal{U}}$.

We equip the configuration set $A^{\mathcal{U}}$ with its *prodiscrete* topology, that is, the product topology obtained by taking the discrete topology on each factor A of $A^{\mathcal{U}} = \prod_{u \in \mathcal{U}} A$. A neighborhood base of a configuration $x \in A^{\mathcal{U}}$ is given by the sets

$$(2.2) \quad V(x, \Omega) = V(x, \Omega, \mathcal{U}, A) := \{y \in A^{\mathcal{U}} : x|_{\Omega} = y|_{\Omega}\},$$

where Ω runs over all finite subsets of \mathcal{U} . In this topology, two configurations are “close” if they coincide on a “large” finite subset of the universe.

Every finite discrete topological space is compact, totally disconnected, and metrizable. As a product of compact (resp. totally disconnected) topological spaces is itself compact (resp. totally disconnected) and a countable product of metrizable spaces is itself metrizable, it follows that $A^{\mathcal{U}}$ is a compact totally disconnected metrizable space. Note that $A^{\mathcal{U}}$ is homeomorphic to the Cantor set as soon as A contains more than one element and \mathcal{U} is infinite.

2.3. Group actions. An *action* of a group G on a set X is a map $\alpha : G \times X \rightarrow X$ satisfying $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$ and $\alpha(1_G, x) = x$ for all $g_1, g_2 \in G$ and $x \in X$. In the sequel, if α is an action of a group G on a set X , we shall simply write gx instead of $\alpha(g, x)$, if there is no risk of confusion.

Suppose that a group G acts on a set X . The orbit of a point $x \in X$ is the subset $Gx \subset X$ defined by $Gx := \{gx : g \in G\}$. A point $x \in X$ is called *periodic* if its orbit is finite. A subset $Y \subset X$ is called *invariant* if $Gy \subset Y$ for all $y \in Y$. One says that $Y \subset X$ is *fixed* by G if $gy = y$ for all $g \in G$ and $y \in Y$.

Suppose now that a group G acts on two sets X and Y . A map $f : X \rightarrow Y$ is called *equivariant* if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.

Let X be a topological space. An action of a group G on X is called *continuous* if the map $x \mapsto gx$ is continuous on X for each $g \in G$. Note that if G acts continuously on X then, for each $g \in G$, the map $x \mapsto gx$ is a homeomorphism of X with inverse $x \mapsto g^{-1}x$.

2.4. Shifts. From now on, our universe will be a countable group. So let G be a countable group and A a finite set. Given an element $g \in G$ and a configuration $x \in A^G$, we define the configuration $gx \in A^G$ by

$$gx := x \circ L_{g^{-1}},$$

where $L_g: G \rightarrow G$ is the left-multiplication by g . Thus

$$gx(h) = x(g^{-1}h) \quad \text{for all } h \in G.$$

Observe that, for all $g_1, g_2 \in G$ and $x \in A^G$,

$$g_1(g_2x) = x \circ L_{g_2^{-1}} \circ L_{g_1^{-1}} = x \circ L_{g_2^{-1}g_1^{-1}} = x \circ L_{(g_1g_2)^{-1}} = (g_1g_2)x,$$

and

$$1_Gx = x \circ L_{1_G} = x \circ \text{Id}_G = x.$$

Therefore the map

$$\begin{aligned} G \times A^G &\rightarrow A^G \\ (g, x) &\mapsto gx \end{aligned}$$

defines an action of G on A^G . This action is called the G -*shift*, or simply the *shift*, on A^G .

Observe that if two configurations $x, y \in A^G$ coincide on a subset $\Omega \subset G$, then, for every $g \in G$, the configurations gx and gy coincide on $g\Omega$. As the sets $V(x, \Omega)$ defined by (2.2) form a base of neighborhoods of $x \in A^G$ when Ω runs over all finite subsets of G , we deduce that the map $x \mapsto gx$ is continuous on A^G for each $g \in G$. Thus, the shift action of G on A^G is continuous.

2.5. Patterns. A *pattern* is a map $p: \Omega \rightarrow A$, where Ω is a finite subset of G . If $p: \Omega \rightarrow A$ is a pattern, we say that Ω is the *support* of p and write $\Omega = \text{supp}(p)$.

Let $\mathcal{P}(G, A)$ denote the set of all patterns. There is a natural action of the group G on $\mathcal{P}(G, A)$ defined as follows. Given $g \in G$ and a pattern $p \in \mathcal{P}(G, A)$ with support Ω , we define the pattern $gp \in \mathcal{P}(G, A)$ as being the pattern with support $g\Omega$ such that $gp(h) = p(g^{-1}h)$ for all $h \in g\Omega$. It is easy to check that this defines an action of G on $\mathcal{P}(G, A)$, i.e., $g_1(g_2p) = (g_1g_2)p$ and $1_Gp = p$ for all $g_1, g_2 \in G$ and $p \in \mathcal{P}(G, A)$. Observe that $\text{supp}(gp) = g\text{supp}(p)$ for all $g \in G$ and $p \in \mathcal{P}(G, A)$. Note also that if p is the restriction of a configuration $x \in A^G$ to a finite subset $\Omega \subset G$, then gp is the restriction of the configuration gx to $g\Omega$.

Example 2.1. Take $G = \mathbb{Z}$ and let A be a finite set. Denote by A^* the set of words on the alphabet A . We recall that A^* is the free monoid based on A and that any element $w \in A^*$ can be uniquely written in the form $w = a_1a_2 \cdots a_n$, where $a_i \in A$ for $1 \leq i \leq n$ and $n \in \mathbb{N}$ is the *length* of the word w . The monoid operation on A^* is the concatenation of words and the identity element is the empty word, that is, the unique word with length 0. Now let us fix some finite interval $\Omega \subset \mathbb{Z}$ of cardinality n , say $\Omega = \{m, m+1, \dots, m+n-1\}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then one can associate to each pattern $p: \Omega \rightarrow A$ the word

$$w = p(m)p(m+1) \cdots p(m+n-1) \in A^*.$$

This yields a one-to-one correspondence between the patterns supported by Ω and the words of length n on the alphabet A . This is frequently used to identify each pattern supported by Ω with the corresponding word.

2.6. Subshifts. A *subshift* is a subset $X \subset A^G$ that is invariant under the G -shift and closed for the prodiscrete topology on A^G .

Example 2.2. Take $G = \mathbb{Z}$ and $A = \{0, 1\}$. Then the subset $X \subset A^G$, consisting of all $x: \mathbb{Z} \rightarrow \{0, 1\}$ such that $(x(n), x(n+1)) \neq (1, 1)$ for all $n \in \mathbb{Z}$, is a subshift. This subshift is called the *golden mean subshift*.

Example 2.3. Take $G = \mathbb{Z}^d$ and $A = \{0, 1\}$. Then the subset $X \subset A^G$, consisting of all $x \in A^G$ such that $(x(g), x(g + e_i)) \neq (1, 1)$ for all $g \in G$, where $(e_i)_{1 \leq i \leq d}$ is the canonical basis of \mathbb{Z}^d , is a subshift. This subshift is called the *hard-ball model*. For $d = 1$, the hard-ball model is the golden mean subshift of the previous example.

Example 2.4. Take $G = \mathbb{Z}$ and $A = \{0, 1\}$. Then the subset $X \subset A^G$, consisting of all bi-infinite sequences $x: \mathbb{Z} \rightarrow \{0, 1\}$ such that there is always an even number of 0s between two 1s, is a subshift. This subshift is called the *even subshift*.

Example 2.5. Take $G = \mathbb{Z}^2$ and $A = \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$ (the integers modulo 2). Then the subset $X \subset A^G$, consisting of all $x: \mathbb{Z}^2 \rightarrow \{0, 1\}$ such that

$$x(m, n) + x(m+1, n) + x(m, n+1) = 0$$

for all $(m, n) \in \mathbb{Z}^2$, is a subshift. This subshift is called the *Ledrappier subshift*.

Remark 2.6. Every intersection of subshifts and every finite union of subshifts $X \subset A^G$ is itself a subshift. Therefore the subshifts $X \subset A^G$ are the closed subsets of a topology on A^G . This topology is coarser (it has less open sets) than the prodiscrete topology on A^G . It is not Hausdorff as soon as G is not trivial and A has more than one element.

Given a (possibly infinite) subset of patterns $P \subset \mathcal{P}(G, A)$, it is easy to see that the subset $X(P) \subset A^G$ defined by

$$X(P) := \{x \in A^G \text{ such that } (gx)|_{\text{supp}(p)} \neq p \text{ for all } g \in G \text{ and } p \in P\}$$

is a subshift.

Conversely, let $X \subset A^G$ be a subshift. One says that a pattern $p \in \mathcal{P}(G, A)$ *appears* in X if $p \in X_{\text{supp}(p)}$, i.e., if there is a configuration $x \in X$ such that $x|_{\text{supp}(p)} = p$. Then one easily checks that $X = X(P)$ for

$$P := \{p \in \mathcal{P}(G, A) \text{ such that } p \text{ does not appear in } X\}.$$

One says that a subshift $X \subset A^G$ is of *finite type* if there exists a finite subset $P \subset \mathcal{P}(G, A)$ such that $X = X(P)$. The hard-ball models (and hence in particular the golden mean subshift) and the Ledrappier subshifts are examples of subshifts of finite type. On the other hand, the even subshift is not of finite type.

3. CELLULAR AUTOMATA

3.1. Definition. Let G be a countable group and let A, B be finite sets. Suppose that $X \subset A^G$ and $Y \subset B^G$ are two subshifts.

Definition 3.1. One says that a map $\tau: X \rightarrow Y$ is a *cellular automaton* if there exist a finite subset $S \subset G$ and a map $\mu: A^S \rightarrow B$ such that

$$(3.1) \quad \tau(x)(g) = \mu((g^{-1}x)|_S)$$

for all $x \in X$ and $g \in G$, where we recall that $(g^{-1}x)|_S$ denotes the restriction of the configuration $g^{-1}x \in X$ to S . Such a set S is called a *memory set* and μ is called a *local defining map* for τ .

It immediately follows from this definition that a map $\tau: X \rightarrow Y$ is a cellular automaton if and only if τ extends to a cellular automaton $\tilde{\tau}: A^G \rightarrow B^G$. Observe also that if S is a memory set for a cellular automaton $\tau: X \rightarrow Y$ and $g \in G$, then Formula (3.1) implies that the value taken by the configuration $\tau(x)$ at g only depends on the restriction of x to gS . Finally note that if S is a memory set for a cellular automaton τ , then any finite subset of G containing S is also a memory set for τ . Consequently, the memory set of a cellular automaton is not unique in general. However, every cellular automaton admits a unique memory set with minimal cardinality (this follows from the fact that if S_1 and S_2 are memory sets then so is $S_1 \cap S_2$).

Example 3.2. Take $G = \mathbb{Z}$ and $A = \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$. Then the map $\tau: A^G \rightarrow A^G$, defined by

$$\tau(x)(n) := x(n+1) + x(n)$$

for all $x \in A^G$ and $n \in \mathbb{Z}$, is a cellular automaton admitting $S := \{0, 1\} \subset \mathbb{Z}$ as a memory set and $\mu: A^S \rightarrow A$ given by

$$\mu(p) := p(0) + p(1)$$

for all $p \in A^S$, as a local defining map. Using the representation of patterns with support S by words of length 2 on the alphabet A (cf. Example 2.1), the map μ is given by

$$\mu(00) = \mu(11) = 0 \text{ and } \mu(01) = \mu(10) = 1.$$

Example 3.3 (Majority vote). Take $G = \mathbb{Z}$ and $A = \{0, 1\}$. Then the map $\tau: A^G \rightarrow A^G$, defined by

$$\tau(x)(n) := \begin{cases} 0 & \text{if } x(n-1) + x(n) + x(n+1) \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

for all $x \in A^G$ and $n \in \mathbb{Z}$, is a cellular automaton admitting $S := \{-1, 0, 1\} \subset \mathbb{Z}$ as a memory set and $\mu: A^S \rightarrow A$ given by

$$\mu(000) = \mu(001) = \mu(010) = \mu(100) = 0$$

and

$$\mu(011) = \mu(101) = \mu(110) = \mu(111) = 1$$

as local defining map. The cellular automaton τ is called the *majority vote* cellular automaton.

Remark 3.4. A cellular automaton $\tau: A^G \rightarrow A^G$, where $G = \mathbb{Z}$, $A = \{0, 1\}$, admitting $S = \{-1, 0, 1\}$ as a memory set is called an *elementary cellular automaton*. Each one of these cellular automata is uniquely determined by its local defining map $\mu: A^S \rightarrow A$, so that there are exactly $2^8 = 256$ elementary cellular automata. They are numbered from 0 to 255 according to a notation that was introduced by Wolfram (cf. [56]). To obtain the number n of an elementary cellular automaton τ , one proceeds as follows. One first lists all eight patterns $p \in A^S$ in increasing order from 000 to 111. The number n is the integer whose expansion in base 2 is $a_8 a_7 \dots a_1$, where a_k is the value taken by the local defining map of τ at the k -th pattern in the list. One also says that τ is *Rule n* . For instance, the elementary cellular automaton described in Example 3.2 is Rule 102 while the one described in Example 3.3 is Rule 232.

Example 3.5. Let G be a countable group, A a finite set, and $X \subset A^G$ a subshift. Then the identity map $\text{Id}_X: X \rightarrow X$ is a cellular automaton with memory set $S = \{1_G\}$ and local defining map $\mu = \text{Id}_A: A^S = A^{\{1_G\}} = A \rightarrow A$.

Example 3.6. Let G be a countable group, A a finite set, and $X \subset A^G$ a subshift. Let $s \in G$ and denote by R_s the right-multiplication by s , that is, the map $R_s: G \rightarrow G$ defined by $R_s(h) := hs$ for all $s \in G$. Then the subset $Y \subset A^G$ defined by

$$Y := \{x \circ R_s \text{ such that } x \in X\}$$

is a subshift. Moreover, the map $\tau: X \rightarrow Y$, defined by $\tau(x) := x \circ R_s$ for all $x \in X$, is a cellular automaton with memory set $S = \{s\}$ and local defining map $\mu = \text{Id}_A: A^S = A \rightarrow A$. Observe that if s is in the center of G , then $X = Y$ and $\tau: X \rightarrow X$ is the shift map $x \mapsto s^{-1}x$.

Example 3.7. Take $G = \mathbb{Z}$ and $A = \{0, 1\}$. Let $X \subset A^G$ and $Y \subset A^G$ denote respectively the golden mean subshift and the even subshift. For $x \in X$, define $\tau(x) \in A^G$ by

$$\tau(x)(n) := \begin{cases} 0 & \text{if } (x(n), x(n+1)) = (0, 1) \text{ or } (1, 0) \\ 1 & \text{if } (x(n), x(n+1)) = (0, 0) \end{cases}$$

for all $n \in \mathbb{Z}$. It is easy to see that $\tau(x) \in Y$ for all $x \in X$. The map $\tau: X \rightarrow Y$ is a cellular automaton admitting $S := \{0, 1\} \subset \mathbb{Z}$ as a memory set and the map $\mu: A^S \rightarrow A$, defined by

$$\mu(00) = \mu(11) = 1 \text{ and } \mu(01) = \mu(10) = 0.$$

Note that the map $\mu': A^S \rightarrow A$, defined by

$$\mu'(00) = 1 \text{ and } \mu'(01) = \mu'(10) = \mu'(11) = 0$$

is also a local defining map for τ . Thus, τ is the restriction to X of both Rule 153 and Rule 17.

3.2. The Curtis-Hedlund-Lyndon theorem. The definition of a cellular automaton given in the previous subsection is a local one. For τ to be a cellular automaton, it requires the existence of a rule, commuting with the shift, that allows one to evaluate the value taken by $\tau(x)$ at $g \in G$ by applying the rule to the restriction of x to a certain finite set, namely the left-translate by g of a memory set of the automaton. The following result, known as the *Curtis-Lyndon-Hedlund theorem* (see [31]), yields a global characterization of cellular automata involving only the shift actions and the prodiscrete topology on the configuration spaces.

Theorem 3.8. *Let G be a countable group and let A, B be finite sets. Let $\tau: X \rightarrow Y$ be a map from a subshift $X \subset A^G$ into a subshift $Y \subset B^G$. Then the following conditions are equivalent:*

- (a) τ is a cellular automaton;
- (b) τ is equivariant (with respect to the shift actions of G) and continuous (with respect to the prodiscrete topologies).

Proof. Suppose first that $\tau: X \rightarrow X$ is a cellular automaton. Let $S \subset G$ be a memory set and $\mu: A^S \rightarrow B$ a local defining map for τ . For all $g, h \in G$ and $x \in X$, we have that

$$\begin{aligned} \tau(gx)(h) &= \mu((h^{-1}gx)|_S) && \text{(by Formula (3.1))} \\ &= \mu(((g^{-1}h)^{-1}x)|_S) \\ &= \tau(x)(g^{-1}h) \\ &= g\tau(x)(h). \end{aligned}$$

Thus $\tau(gx) = g\tau(x)$ for all $g \in G$ and $x \in X$. This shows that τ is equivariant.

Now let Ω be a finite subset of G . Recall that Formula (3.1) implies that if two configurations $x, y \in X$ coincide on gS for some $g \in G$, then $\tau(x)(g) = \tau(y)(g)$. Therefore, if the configurations x and y coincide on the finite set ΩS , then $\tau(x)$ and $\tau(y)$ coincide on Ω . It follows that

$$\tau(X \cap V(x, \Omega S, G, A)) \subset V(\tau(x), \Omega, G, B).$$

This implies that τ is continuous. Thus (a) implies (b).

Conversely, suppose now that the map $\tau: X \rightarrow Y$ is equivariant and continuous. Let us show that τ is a cellular automaton. As the map $\varphi: X \rightarrow B$ defined by $\varphi(x) := \tau(x)(1_G)$ is continuous, we can find, for each $x \in X$, a finite subset $\Omega_x \subset G$ such that if $y \in X \cap V(x, \Omega_x, G, A)$, then $\tau(y)(1_G) = \tau(x)(1_G)$. The sets $X \cap V(x, \Omega_x, G, A)$ form an open cover of X . As X is compact, there is a finite subset $F \subset X$ such that the sets $V(x, \Omega_x, G, A)$, $x \in F$, cover X . Let us set $S = \cup_{x \in F} \Omega_x$ and suppose that two configurations $y, z \in X$ coincide on S . Let $x_0 \in F$ be such that $y \in V(x_0, \Omega_{x_0}, G, A)$, that is, $y|_{\Omega_{x_0}} = x_0|_{\Omega_{x_0}}$. As $\Omega_{x_0} \subset S$, we have that $y|_{\Omega_{x_0}} = z|_{\Omega_{x_0}}$ and therefore $\tau(y)(1_G) = \tau(x_0)(1_G) = \tau(z)(1_G)$. We deduce that there exists a map $\mu: A^S \rightarrow B$ such that $\tau(x)(1_G) = \mu(x|_S)$ for all $x \in X$.

Now, for all $x \in X$ and $g \in G$, we have that

$$\begin{aligned}\tau(x)(g) &= (g^{-1}\tau(x))(1_G) && \text{(by definition of the shift action on } B^G) \\ &= \tau(g^{-1}x)(1_G) && \text{(since } \tau \text{ is equivariant)} \\ &= \mu((g^{-1}x)|_S).\end{aligned}$$

This shows that τ is a cellular automaton with memory set S and local defining map μ . Thus (b) implies (a). \square

3.3. Operations on cellular automata.

Proposition 3.9. *Let G be a countable group and let A, B, C be finite sets. Suppose that $X \subset A^G$, $Y \subset B^G$, $Z \subset C^G$ are subshifts and that $\tau: X \rightarrow Y$, $\sigma: Y \rightarrow Z$ are cellular automata. Then the composite map $\sigma \circ \tau: X \rightarrow Z$ is a cellular automaton.*

Proof. This is an immediate consequence of the characterization of cellular automata given by the Curtis-Hedlund-Lyndon theorem (cf. Theorem 3.8) since the composite of two equivariant (resp. continuous) maps is itself equivariant (resp. continuous). \square

Remark 3.10. If we fix the countable group G , we deduce from Proposition 3.9 and Example 3.5 that the subshifts $X \subset A^G$, with A finite, are the objects of a concrete category \mathcal{C}_G in which the set of morphisms from $X \in \mathcal{C}_G$ to $Y \in \mathcal{C}_G$ consist of all cellular automata $\tau: X \rightarrow Y$ (cf. [15, Section 3.2]). In this category, the endomorphisms of $X \in \mathcal{C}_G$ consist of all cellular automata $\tau: X \rightarrow X$ and they form a monoid for the composition of maps.

Proposition 3.11. *Let G be a countable group and let A, B be finite sets. Suppose that $X \subset A^G$, $Y \subset B^G$ are subshifts and that $\tau: X \rightarrow Y$ is a bijective cellular automaton. Then the inverse map $\tau^{-1}: Y \rightarrow X$ is a cellular automaton.*

Proof. This is again an immediate consequence of the Curtis-Hedlund-Lyndon theorem since the inverse of a bijective equivariant map is an equivariant map and the inverse of a bijective continuous map between compact Hausdorff spaces is continuous. \square

3.4. Surjectivity of cellular automata, GOE configurations, and GOE patterns.

In what follows, we keep the notation introduced for defining cellular automata. Let $\tau: X \rightarrow Y$ be a cellular automaton. A configuration $y \in Y$ is called a *Garden of Eden* configuration for τ , briefly a *GOE configuration*, if it does not belong to the image of τ , i.e., there is no $x \in X$ such that $y = \tau(x)$. One says that a pattern $p \in \mathcal{P}(G, B)$ is a *Garden of Eden pattern* for τ , briefly a *GOE pattern*, if the pattern p appears in the subshift Y but not in the subshift $\tau(X)$, i.e., there exists $y \in Y$ such that $p = y|_{\text{supp}(p)}$ but there is no $x \in X$ such that $p = \tau(x)|_{\text{supp}(p)}$. Note that the set of GOE configurations (resp. of GOE patterns) is an invariant subset of Y (resp. of $\mathcal{P}(G, B)$). Observe also that if $p \in \mathcal{P}(G, A)$ is a GOE pattern for the cellular automaton $\tau: X \rightarrow Y$, then every configuration $y \in Y$ such that $y|_{\text{supp}(p)} = p$ is a GOE configuration for τ .

Example 3.12. It is easy to check that the pattern with support $\Omega := \{0, 1, 2, 3, 4\}$ associated with the word 01001 is a GOE pattern for the majority vote cellular automaton described in Example 3.3.

Proposition 3.13. *Let $\tau: X \rightarrow Y$ be a cellular automaton. Then the following conditions are equivalent:*

- (a) τ is surjective;
- (b) τ admits no GOE configurations;
- (c) τ admits no GOE patterns.

Proof. The equivalence of (a) and (b) as well as the implication (b) \implies (c) are trivial. The implication (c) \implies (b) easily follows from the compactness of $\tau(X)$. \square

3.5. Pre-injectivity of cellular automata and mutually erasable patterns. Let G be a countable group and let A and B be finite sets. Recall that two configurations $x, y \in A^G$ are called almost equal if they coincide outside of a finite subset of G .

Definition 3.14. Let $X \subset A^G$ and $Y \subset B^G$ be subshifts. One says that a cellular automaton $\tau: X \rightarrow Y$ is *pre-injective* if there are no distinct configurations $x_1, x_2 \in X$ that are almost equal and satisfy $\tau(x_1) = \tau(x_2)$.

A pair of configurations $(x_1, x_2) \in X \times X$ is called a *diamond* if x_1 and x_2 are distinct, almost equal, and have the same image under τ (cf. [32, Definition 8.1.15]). Thus, pre-injectivity is equivalent to the absence of diamonds. If (x_1, x_2) is a diamond, the nonempty finite subset

$$\{g \in G : x_1(g) \neq x_2(g)\} \subset G$$

is called the *support* of the diamond (x_1, x_2) .

Every injective cellular automaton is clearly pre-injective. The converse is false, as shown by the following examples.

Example 3.15. Take $G = \mathbb{Z}$, $A = \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$, and consider the cellular automaton $\tau: A^G \rightarrow A^G$ described in Example 3.2 (Rule 102 in Wolfram's notation). Then τ is pre-injective. Indeed, it is clear that if two configurations $x, x' \in A^G$ coincide on $\mathbb{Z} \cap (-\infty, n_0]$ for some $n_0 \in \mathbb{Z}$ and satisfy $\tau(x) = \tau(x')$ then $x = x'$. However, τ is not injective since the two constant configurations have the same image.

Example 3.16. Consider the cellular automaton $\tau: X \rightarrow Y$ from the golden mean subshift to the even subshift described in Example 3.7. It is easy to see that τ is pre-injective by an argument similar to the one used in the previous example. However, τ is not injective since the two sequences in X with exact period 2 have the same image under τ , namely the constant sequence with only 0s.

Let $\Omega \subset G$ be a finite set and $p_1, p_2 \in X_\Omega$ two patterns with support Ω appearing in X . One says that the patterns p_1 and p_2 are *mutually erasable* with respect to τ , briefly *ME*, provided the following hold:

(MEP-1) the set

$$X_{p_1, p_2} := \{(x_1, x_2) \in X \times X : x_1|_{\Omega} = p_1, x_2|_{\Omega} = p_2 \text{ and } x_1|_{G \setminus \Omega} = x_2|_{G \setminus \Omega}\}$$

is nonempty;

(MEP-2) for all $(x_1, x_2) \in X_{p_1, p_2}$ one has $\tau(x_1) = \tau(x_2)$.

Note that “being ME” is an equivalence relation on X_{Ω} . This equivalence relation is not trivial in general.

Example 3.17. The patterns with support $\Omega := \{0, 1, 2\}$ associated with the words 00000 and 00100 are ME patterns for the majority vote cellular automaton described in Example 3.3.

Observe that if the patterns p_1 and p_2 are ME, then so are gp_1 and gp_2 for all $g \in G$.

Proposition 3.18. *Let $\tau: X \rightarrow Y$ be a cellular automaton. Then the following conditions are equivalent:*

- (a) τ is pre-injective;
- (b) τ admits no distinct ME patterns.

Proof. Suppose first that τ admits two distinct ME patterns p_1 and p_2 . Let Ω denote their common support. Take $(x_1, x_2) \in X_{p_1, p_2}$. Then the configurations x_1 and x_2 are almost equal since they coincide outside of Ω . Moreover, they satisfy $x_1 \neq x_2$ and $\tau(x_1) = \tau(x_2)$. Therefore (x_1, x_2) is a diamond for τ . It follows that τ is not pre-injective.

Suppose now that τ is not pre-injective and let us show that τ admits two distinct ME patterns. By definition, τ admits a diamond $(x_1, x_2) \in X \times X$. Let Δ denote the support of this diamond and let $S \subset G$ be a memory set for τ with $1_G \in S$. Consider the set $\Omega := \Delta S^{-1}S$. We claim that the patterns $p_1 := x_1|_{\Omega}$ and $p_2 := x_2|_{\Omega}$ are distinct ME patterns. First observe that $p_1 \neq p_2$ since $\emptyset \neq \Delta \subset \Omega$. Moreover, X_{p_1, p_2} is nonempty since, by construction, $(x_1, x_2) \in X_{p_1, p_2}$. To complete the proof, we only need to show the following: if $(y_1, y_2) \in X_{p_1, p_2}$ then $\tau(y_1) = \tau(y_2)$. Let $g \in G$. Suppose first that $g \in G \setminus \Delta S^{-1}$. Then $gS \cap \Delta = \emptyset$. Since y_1 and y_2 coincide on $G \setminus \Delta$, we deduce that

$$(3.2) \quad \tau(y_1)(g) = \tau(y_2)(g) \text{ for all } g \in G \setminus \Delta S^{-1}.$$

Suppose now that $g \in \Delta S^{-1}$. Then $gS \subset \Delta S^{-1}S = \Omega$. As $y_1|_{\Omega} = p_1 = x_1|_{\Omega}$ (resp. $y_2|_{\Omega} = p_2 = x_2|_{\Omega}$), by construction, we deduce that

$$(3.3) \quad \tau(y_1)(g) = \tau(x_1)(g) = \tau(x_2)(g) = \tau(y_2)(g) \text{ for all } g \in \Delta S^{-1}.$$

From (3.2) and (3.3) we deduce that $\tau(y_1) = \tau(y_2)$. □

4. THE GARDEN OF EDEN THEOREM FOR \mathbb{Z}^d

In this section, we present a proof of the Garden of Eden theorem of Moore and Myhill for cellular automata over the group \mathbb{Z}^d . This is a particular case of the Garden of Eden theorem for amenable groups that will be established in the next section.

Theorem 4.1. *Let A be a finite set, $d \geq 1$ an integer, and $\tau: A^{\mathbb{Z}^d} \rightarrow A^{\mathbb{Z}^d}$ a cellular automaton. Then the following conditions are equivalent:*

- (a) τ is surjective;
- (b) τ is pre-injective.

The implication (a) \implies (b) is due to Moore [39] and the converse to Myhill [43]. We shall prove the contrapositive of each implication. Before undertaking the proof of Theorem 4.1, let us first introduce some notation and establish some preliminary results.

Let $S \subset \mathbb{Z}^d$ be a memory set for τ . Since any finite subset of \mathbb{Z}^d containing a memory set for τ is itself a memory set for τ , it is not restrictive to suppose that $S = \{0, \pm 1, \dots, \pm r\}^d$ for some integer $r \geq 1$.

Let us set, for each integer $m \geq 2r$,

$$\begin{aligned}\Omega_m &:= \{0, 1, \dots, m-1\}^d \\ \Omega_m^+ &:= \{-r, -r+1, \dots, m+r-1\}^d\end{aligned}$$

and

$$\Omega_m^- := \{r, r+1, \dots, m-r-1\}^d,$$

so that

$$(4.1) \quad |\Omega_m| = m^d, \quad |\Omega_m^+| = (m+2r)^d, \quad \text{and} \quad |\Omega_m^-| = (m-2r)^d.$$

Observe that if two configurations $x_1, x_2 \in A^{\mathbb{Z}^d}$ coincide on Ω_m (resp. Ω_m^+ , resp. $\mathbb{Z}^d \setminus \Omega_m^-$) then $\tau(x_1)$ and $\tau(x_2)$ coincide on Ω_m^- (resp. Ω_m , resp. $\mathbb{Z}^d \setminus \Omega_m$).

Also, let us set, for all integers $k, n \geq 1$

$$T_n^k := \{t = (t_1k, t_2k, \dots, t_dk) \in \mathbb{Z}^d : 0 \leq t_j \leq n-1\},$$

and observe that the n^d cubes $t + \Omega_k$, for t running over T_n^k , form a partition of the cube Ω_{nk} .

Finally, we introduce the following additional notation. Given a finite subset $\Omega \subset G$ and $p, q \in A^\Omega$ we write $p \sim q$ if and only if the patterns p and q are ME for τ . As usual, we denote by A^Ω / \sim the quotient set of A^Ω by \sim , i.e., the set of all ME-equivalence classes of patterns supported by Ω .

In the proof of Theorem 4.1 we shall make use of the following elementary result (here one should think of $a := |A|$ as to the cardinality of the alphabet set and $a^{(nk)^d}$ (resp. $a^{(nk-2r)^d}$, for $nk \geq 2r$) as to the number of all patterns supported by Ω_{nk} (resp. Ω_{nk}^-), cf. (4.1).

Lemma 4.2. *Let a, k, d, r be positive integers with $a \geq 2$. Then there exists $n_0 = n_0(a, k, d, r) \in \mathbb{N}$ such that*

$$(4.2) \quad \left(a^{k^d} - 1\right)^{n^d} < a^{(nk-2r)^d}$$

for all $n \geq n_0$.

Proof. Taking logarithms to base a , Inequality (4.2) is equivalent to

$$(4.3) \quad \log_a \left(a^{k^d} - 1\right) < \left(k - \frac{2r}{n}\right)^d.$$

Since

$$\log_a \left(a^{k^d} - 1 \right) < \log_a \left(a^{k^d} \right) = k^d = \lim_{n \rightarrow \infty} \left(k - \frac{2r}{n} \right)^d,$$

we deduce that there exists $n_0 \in \mathbb{N}$ such that (4.3) and therefore (4.2) are satisfied for all $n \geq n_0$. \square

Proof of Theorem 4.1. We can assume that $a := |A| \geq 2$.

Suppose first that τ is not pre-injective. Then, by Proposition 3.18, τ admits two distinct ME patterns, say p_1 and p_2 . Denote by $\Omega \subset \mathbb{Z}^d$ their common support. Since, for all $t \in \mathbb{Z}^d$, the patterns tp_1 and tp_2 (with support $t + \Omega$) are also distinct and ME, and any finite subset of \mathbb{Z}^d containing the support of two distinct ME patterns is itself the support of two distinct ME patterns, we may assume that $\Omega = \Omega_k$ for some integer $k \geq 2r$. As the patterns tp_1 and tp_2 are ME, we have that

$$(4.4) \quad |A^{t+\Omega_k} / \sim| \leq |A^{t+\Omega_k}| - 1 = a^{k^d} - 1$$

for all $t \in \mathbb{Z}^d$.

Now observe that two patterns with support Ω_{nk} are ME if their restrictions to $t + \Omega_k$ are ME for every $t \in T_n^k$. Using (4.4), we deduce that

$$|A^{\Omega_{nk}} / \sim| \leq \prod_{t \in T_n^k} |A^{t+\Omega_k} / \sim| \leq (a^{k^d} - 1)^{n^d}.$$

Taking $n \geq n_0(a, k, d, r)$, we then get

$$\left| \tau(A^{\mathbb{Z}^d})_{\Omega_{nk}^-} \right| \leq |A^{\Omega_{nk}} / \sim| \leq (a^{k^d} - 1)^{n^d} < a^{(nk-2r)^d} = |A^{\Omega_{nk}^-}|.$$

This implies that $\tau(A^{\mathbb{Z}^d})_{\Omega_{nk}^-} \subsetneq A^{\Omega_{nk}^-}$, so that there must exist a GOE pattern for τ with support Ω_{nk}^- . Consequently, τ is not surjective. This shows that (a) \implies (b).

Let us now turn to the proof of the converse implication. Suppose that τ is not surjective. Then, by Proposition 3.13, there exists a GOE pattern p for τ . Since tp is a GOE pattern for every $t \in \mathbb{Z}^d$, and any finite subset of \mathbb{Z}^d containing the support of a GOE pattern is itself the support of a GOE pattern, we can assume that p is supported by the cube Ω_k for some integer $k \geq 2r$.

Decompose again Ω_{nk} into the n^d translates $t + \Omega_k$, with $t \in T_n^k$, and observe that $tp \in A^{t+\Omega_k}$ is GOE for every $t \in T_n^k$. Any pattern $q \in A^{\Omega_{nk}}$ which is not GOE satisfies that $q|_{t+\Omega_k}$ is not GOE for every $t \in T_n^k$. As a consequence, we have that

$$(4.5) \quad \left| \tau(A^{\mathbb{Z}^d})_{\Omega_{nk}} \right| \leq \prod_{t \in T_n^k} \left| \tau(A^{\mathbb{Z}^d})_{t+\Omega_k} \right| \leq (a^{k^d} - 1)^{n^d}.$$

Let us fix now some element $a_0 \in A$ and consider the set X consisting of all configurations $x \in A^{\mathbb{Z}^d}$ that satisfy

$$x(g) = a_0 \quad \text{for all } g \in \mathbb{Z}^d \setminus \Omega_{nk}^-.$$

Observe that if x_1 and x_2 are in X , then $\tau(x_1)$ and $\tau(x_2)$ coincide on $\mathbb{Z}^d \setminus \Omega_{nk}$. It follows that

$$|\tau(X)| = |\tau(X)_{\Omega_{nk}}|.$$

On the other hand, taking $n \geq n_0(a, k, d, r)$ and using (4.5), we get

$$|\tau(X)_{\Omega_{nk}}| \leq \left| \tau(A^{\mathbb{Z}^d})_{\Omega_{nk}} \right| \leq (a^{k^d} - 1)^{n^d} < a^{(nk-2r)^d} = |A^{\Omega_{kn}^-}| = |X|$$

and hence

$$|\tau(X)| < |X|.$$

By the pigeon-hole principle, this implies that there exist two distinct configurations $x_1, x_2 \in X$ such that $\tau(x_1) = \tau(x_2)$. As all configurations in X are almost equal, we deduce that τ is not pre-injective. This shows that (b) \implies (a). \square

Remark 4.3. The proof of the implication (a) \implies (b) shows that if τ admits two distinct ME patterns supported by a cube of side $k \geq 2r$, then a cube of side $nk - 2r$, with $n \geq n_0(a, k, d, r)$, must support a GOE pattern. Conversely, a small addition to the proof of the implication (b) \implies (a) yields that if τ admits a GOE pattern supported by a cube of side $k \geq 2r$, then a cube of side $kn + 2r$, with $n \geq n_0(a, k, d, r)$, supports two distinct ME patterns. Indeed, the proof shows the existence of two configurations $x_1, x_2 \in A^{\mathbb{Z}^d}$ that coincide outside of Ω_{nk}^- and satisfy $\tau(x_1) = \tau(x_2)$. It then follows from the proof of the implication (b) \implies (a) in Proposition 3.18 that the set

$$(\Omega_{nk}^- + (-S)) + S = \Omega_{nk} + S = \Omega_{nk}^+$$

supports two distinct ME patterns.

5. THE GARDEN OF EDEN THEOREM FOR GENERAL AMENABLE GROUPS

5.1. **Amenability.** (cf. [27], [47], [12, Chapter 4], [21, Chapter 9])

Definition 5.1. A countable group G is called *amenable* if there exists a sequence $(F_n)_{n \in \mathbb{N}}$ of nonempty finite subsets of G such that

$$(5.1) \quad \lim_{n \rightarrow \infty} \frac{|F_n \setminus F_n g|}{|F_n|} = 0 \text{ for all } g \in G.$$

Such a sequence is called a *Følner sequence* for G .

Note that if A and B are finite sets with the same cardinality, then $|A \setminus B| = |B \setminus A|$ and $|A \triangle B| = |A \setminus B| + |B \setminus A| = 2|A \setminus B|$, where \triangle denotes symmetric difference of sets. As $|Fg| = |F|$ for every finite subset $F \subset G$ and any $g \in G$, it follows that Condition (5.1) is equivalent to each of the following conditions:

$$(5.2) \quad \lim_{n \rightarrow \infty} \frac{|F_n g \setminus F_n|}{|F_n|} = 0 \text{ for all } g \in G,$$

or

$$(5.3) \quad \lim_{n \rightarrow \infty} \frac{|F_n \triangle F_n g|}{|F_n|} = 0 \text{ for all } g \in G.$$

Example 5.2. All finite groups are amenable. Indeed, if G is a finite group, then the constant sequence, defined by $F_n := G$ for all $n \in \mathbb{N}$, is a Følner sequence for G since $F_n \setminus F_n g = \emptyset$ for every $g \in G$.

Example 5.3. The free abelian groups of finite rank \mathbb{Z}^d , $d \geq 1$, are also amenable. As a Følner sequence for \mathbb{Z}^d , one can take for instance the sequence of cubes

$$(5.4) \quad F_n := \{x \in \mathbb{Z}^d : \|x\|_\infty \leq n\} = \{0, \pm 1, \dots, \pm n\}^d,$$

where $\|x\|_\infty := \max_{1 \leq i \leq d} |x_i|$ for all $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ is the sup-norm. To see this, observe that

$$(5.5) \quad |F_n| = (2n + 1)^d$$

and, by the triangle inequality,

$$F_n + g \subset F_{n+\|g\|_\infty} \text{ for all } g \in \mathbb{Z}^d.$$

Since

$$F_n \subset F_{n+\|g\|_\infty},$$

this implies

$$(5.6) \quad |(F_n + g) \setminus F_n| \leq (2n + \|g\|_\infty + 1)^d - (2n + 1)^d.$$

As the right-hand side of (5.6) is a polynomial of degree $d-1$ in n while $|F_n|$ is a polynomial of degree d in n by (5.5), we conclude that

$$\lim_{n \rightarrow \infty} \frac{|(F_n + g) \setminus F_n|}{|F_n|} = 0,$$

which is (5.2) in additive notation.

Let G be a finitely generated group. If $S \subset G$ is a finite symmetric generating subset, the *Cayley graph* of G with respect to S is the graph $\mathcal{G}(G, S)$ whose set of vertices is G and two vertices $g, h \in G$ are joined by an edge if and only if $h = gs$ for some $s \in S$. Equip the set of vertices of $\mathcal{G}(G, S)$ with its graph metric and consider the ball $B_n \subset G$ of radius n centered at 1_G . It is easy to see that the sequence of positive integers $(|B_n|)_{n \in \mathbb{N}}$ is submultiplicative. Thus the limit

$$(5.7) \quad \gamma(G, S) := \lim_{n \rightarrow \infty} \sqrt[n]{|B_n|}$$

exists and satisfies $1 \leq \gamma(G, S) < \infty$. One says that the group G has *subexponential growth* if $\gamma(G, S) = 1$ and *exponential growth* if $\gamma(G, S) > 1$. The fact that G has subexponential (resp. exponential) growth does not depend on the choice of the finite generating subset $S \subset G$ although the value of $\gamma(G, S)$ does.

Example 5.4. The groups \mathbb{Z}^d have subexponential growth. Indeed, if e_1, \dots, e_d is the canonical basis of \mathbb{Z}^d , and we take $S := \{\pm e_1, \dots, \pm e_d\}$, then the graph distance between two vertices g and h of $\mathcal{G}(\mathbb{Z}^d, S)$ is $\|g - h\|_1$, where we write $\|x\|_1 := \sum_{1 \leq i \leq d} |x_i|$ for the 1-norm of $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$. We then have

$$B_n = \{g \in \mathbb{Z}^d : \|g\|_1 \leq n\} \subset \{0, \pm 1, \dots, \pm n\}^d$$

and hence $|B_n| \leq (2n+1)^d$. This implies that \mathbb{Z}^d has subexponential growth. Here it can be checked that the sequence $(B_n)_{n \in \mathbb{N}}$ is also a Følner sequence for \mathbb{Z}^d .

When G is an arbitrary finitely generated group of subexponential growth and S is a finite symmetric generating set for G , it can be shown that one can always extract a Følner sequence from the sequence $(B_n)_{n \in \mathbb{N}}$. Consequently, every finitely generated group with subexponential growth is amenable.

Example 5.5. A (nonabelian) free group on two generators has exponential growth and is not amenable. Indeed, let G be a free group based on two generators a and b . Consider the finite symmetric generating subset $S \subset G$ defined by

$$S := \{a, b, a^{-1}, b^{-1}\}.$$

Then every element $g \in G$ can be uniquely written in *reduced form*, i.e., in the form

$$(5.8) \quad g = s_1 s_2 \dots s_n,$$

where $n \geq 0$, $s_i \in S$ for all $1 \leq i \leq n$, and $s_{i+1} \neq s_i^{-1}$ for all $1 \leq i \leq n-1$. The integer $\ell_S(g) := n$ is called the *length* of g with respect to the generators a and b . It is equal to the distance from g to 1_G in the Cayley graph $\mathcal{G}(G, S)$. We deduce that $|B_n| = 4 \cdot 3^{n-1}$ for all $n \geq 1$ so that

$$\gamma(G, S) = \lim_{n \rightarrow \infty} \sqrt[n]{4 \cdot 3^{n-1}} = 3 > 1.$$

This shows that G has exponential growth.

Now suppose by contradiction that $(F_n)_{n \in \mathbb{N}}$ is a Følner sequence for G and choose some positive real number $\varepsilon < 1/2$. Since the sequence $(F_n)_{n \in \mathbb{N}}$ is Følner, it follows from (5.1) that there exists an integer $N \geq 0$ such that the set $F := F_N$ satisfies

$$(5.9) \quad |F \setminus Fs| \leq \varepsilon |F| \quad \text{for all } s \in S.$$

Denote, for each $s \in S$, by G_s the subset of G consisting of all elements $g \neq 1_G$ whose reduced form ends with the letter s^{-1} . The four sets G_s , $s \in S$, are pairwise disjoint so that

$$(5.10) \quad \sum_{s \in S} |F \cap G_s| \leq |F|.$$

On the other hand, for each $s \in S$, we have that

$$(5.11) \quad |F| = |F \setminus G_s| + |F \cap G_s| = |(F \setminus G_s)s| + |F \cap G_s|.$$

We now observe that

$$(G \setminus G_s)s \subset G_{s^{-1}}$$

so that

$$(F \setminus G_s)s \subset (Fs \setminus F) \cup (F \cap G_{s^{-1}})$$

and hence

$$\begin{aligned}
|(F \setminus G_s)s| &\leq |Fs \setminus F| + |F \cap G_{s^{-1}}| \\
&= |F \setminus Fs| + |F \cap G_{s^{-1}}| \\
&\leq \varepsilon|F| + |F \cap G_{s^{-1}}| \quad (\text{by (5.9)}).
\end{aligned}$$

By using (5.11), we deduce that

$$|F| \leq \varepsilon|F| + |F \cap G_{s^{-1}}| + |F \cap G_s|$$

for all $s \in S$. After summing up over all $s \in S$, this yields

$$\begin{aligned}
4|F| &\leq 4\varepsilon|F| + \sum_{s \in S} (|F \cap G_{s^{-1}}| + |F \cap G_s|) \\
&= 4\varepsilon|F| + 2 \sum_{s \in S} |F \cap G_s|.
\end{aligned}$$

Finally, combining with (5.10), we obtain

$$4|F| \leq 4\varepsilon|F| + 2|F|$$

and hence $|F| \leq 2\varepsilon|F|$, which is a contradiction since $F \neq \emptyset$ and $\varepsilon < 1/2$. This proves that G is not amenable.

The class of amenable groups is closed under the operations of taking subgroups, quotients, extensions (this means that if $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ is an exact sequence with both H and K amenable, so is G), and inductive limits. Consequently, all locally finite groups, all abelian groups and, more generally, all solvable groups are amenable. On the other hand every group containing a free subgroup on two generators is nonamenable. This implies for instance that all nonabelian free groups and the groups $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 2$, are nonamenable. However, there are groups containing no free subgroups on two generators that are nonamenable. The first examples of such a group was given in [45] where Ol'sanskii constructed a nonamenable *monster group* in which every proper subgroup is cyclic.

Let us note that there are finitely generated groups of exponential growth that are amenable. For example, the Baumslag-Solitar group $BS(1, 2)$, i.e., the group with presentation $\langle a, b : aba^{-1} = b^2 \rangle$, and the lamplighter group, i.e., the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$, have exponential growth but are both solvable and hence amenable (cf. [23]).

The original definition of amenability that was given by von Neumann [44] in 1929 is that a group G is amenable if there exists a finitely additive invariant probability measure defined on the set of all subsets of G . A key observation due to Day [22] is that this is equivalent to the existence of an invariant mean on the Banach space $\ell^\infty(G)$ of bounded real-valued functions on G . It is also in [22] that the term *amenable* occurred for the first time (see [46, p. 137]). The fact that nonabelian free groups are not amenable is related to the Hausdorff-Banach-Tarski paradox which actually was the motivation of von Neumann for introducing the notion of amenability.

5.2. Entropy. Let G be countable group, A a finite set, and X a subset of A^G (not necessarily a subshift). Given a finite subset $\Omega \subset G$, recall (cf. (2.1)) that

$$X_\Omega := \{x|_\Omega : x \in X\} \subset A^\Omega.$$

Suppose now that the group G is amenable and fix a Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ for G . The *entropy* of X (with respect to \mathcal{F}) is defined by

$$(5.12) \quad h_{\mathcal{F}}(X) := \limsup_{n \rightarrow \infty} \frac{\log |X_{F_n}|}{|F_n|}.$$

Since $X_F \subset A^F$ and hence $\log |X_F| \leq |F| \cdot \log |A|$ for every finite subset $F \subset G$, we always have

$$h_{\mathcal{F}}(X) \leq h_{\mathcal{F}}(A^G) = \log |A|.$$

Example 5.6. Take $G = \mathbb{Z}$ and $A = \{0, 1\}$. Let us compute the entropy of the golden mean subshift $X \subset A^G$ (cf. Example 2.2) with respect to the Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$, where $F_n := \{0, 1, \dots, n\}$. We observe that $u_n := |X_{F_n}|$ satisfies $u_0 = 2$, $u_1 = 3$, and $u_{n+2} = u_{n+1} + u_n$ for all $n \geq 2$. Thus, the sequence $(u_n)_{n \in \mathbb{N}}$ is a Fibonacci sequence and, by Binet's formula,

$$u_n = \frac{1}{\sqrt{5}} (\varphi^{n+3} - (1 - \varphi)^{n+3}),$$

where $\varphi := (1 + \sqrt{5})/2$ is the golden mean (this is the origin of the name of this subshift). It follows that

$$h_{\mathcal{F}}(X) = \limsup_{n \rightarrow \infty} \frac{\log u_n}{n+1} = \log \varphi.$$

Example 5.7. Take again $G = \mathbb{Z}$ and $A = \{0, 1\}$, and let us compute now the entropy of the even subshift $X \subset A^G$ (cf. Example 2.4) with respect to the Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$, where $F_n := \{0, 1, \dots, n\}$. We observe that $u_n := |X_{F_n}|$ satisfies $u_0 = 2$, $u_1 = 4$, and $u_{n+2} = 1 + u_{n+1} + u_n$ for all $n \geq 2$. Thus, the sequence $(v_n)_{n \in \mathbb{N}}$, defined by $v_n := 1 + u_n$ for all $n \in \mathbb{N}$, is a Fibonacci sequence. As $v_0 = 3$ and $v_1 = 5$, by applying again Binet's formula, we get

$$u_n = -1 + v_n = -1 + \frac{1}{\sqrt{5}} (\varphi^{n+5} - (1 - \varphi)^{n+5}),$$

where φ is the golden mean. It follows that

$$h_{\mathcal{F}}(X) = \limsup_{n \rightarrow \infty} \frac{\log u_n}{n+1} = \log \varphi.$$

Thus, the even subshift has the same entropy as the golden mean subshift with respect to \mathcal{F} .

Example 5.8. Take $G = \mathbb{Z}^2$ and $A = \mathbb{Z}/2\mathbb{Z}$. Let us compute the entropy of the Ledrappier subshift $X \subset A^G$ (cf. Example 2.5) with respect to the Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$, where $F_n := \{0, 1, \dots, n\}^2$. We observe that, for each $x \in X$, the pattern $x|_{F_n}$ is entirely

determined by $x|_{H_n}$, where $H_n \subset \mathbb{Z}^2$ is the horizontal interval $H_n := \{0, 1, \dots, 2n\} \times \{0\}$. Therefore, $u_n := |X_{F_n}|$ satisfies

$$\log u_n \leq |H_n| \cdot \log |A| = (2n + 1) \log 2.$$

This gives us

$$h_{\mathcal{F}}(X) = \limsup_{n \rightarrow \infty} \frac{\log u_n}{(n + 1)^2} = 0.$$

Remark 5.9. It is a deep result due to Ornstein and Weiss [46] that, when G is a countable amenable group, A a finite set, and $X \subset A^G$ a subshift, then the lim sup in (5.12) is actually a true limit and does not depend on the particular choice of the Følner sequence \mathcal{F} for G . However, we shall not need it in the sequel.

An important property of cellular automata that we shall use in the proof of the Garden of Eden theorem below is that they cannot increase entropy. More precisely, we have the following result.

Proposition 5.10. *Let G be a countable amenable group with Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ and A, B finite sets. Suppose that $\tau: A^G \rightarrow B^G$ is a cellular automaton and X is a subset of A^G . Then one has $h_{\mathcal{F}}(\tau(X)) \leq h_{\mathcal{F}}(X)$.*

For the proof, we shall use the following general property of Følner sequences.

Lemma 5.11. *Let G be a countable amenable group with Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ and let S be a finite subset of G . Then one has*

$$(5.13) \quad \lim_{n \rightarrow \infty} \frac{|F_n S \setminus F_n|}{|F_n|} = 0.$$

Proof. Observe that

$$F_n S \setminus F_n = \bigcup_{s \in S} (F_n s \setminus F_n)$$

so that

$$|F_n S \setminus F_n| = \left| \bigcup_{s \in S} (F_n s \setminus F_n) \right| \leq \sum_{s \in S} |F_n s \setminus F_n|.$$

Thus we get

$$\frac{|F_n S \setminus F_n|}{|F_n|} \leq \sum_{s \in S} \frac{|F_n s \setminus F_n|}{|F_n|}$$

for all $n \in \mathbb{N}$. As

$$\lim_{n \rightarrow \infty} \frac{|F_n s \setminus F_n|}{|F_n|} = 0$$

for each $s \in S$ by (5.2), this gives us (5.13). □

Proof of Proposition 5.10. Let us set $Y := \tau(X)$ and let $S \subset G$ be a memory set for τ with $1_G \in S$. Recall that it immediately follows from (3.1) that if two configurations coincide on gS for some $g \in G$ then their images by τ take the same value at g . We deduce that

$$(5.14) \quad |Y_\Omega| \leq |X_{\Omega S}|$$

for every finite subset $\Omega \subset G$. Now observe that

$$X_{\Omega S} \subset X_\Omega \times X_{\Omega S \setminus \Omega} \subset X_\Omega \times A^{\Omega S \setminus \Omega},$$

so that we get

$$\log |Y_\Omega| \leq \log |X_\Omega| + |\Omega S \setminus \Omega| \cdot \log |A|.$$

After replacing Ω by F_n and dividing both sides by $|F_n|$, this inequality becomes

$$(5.15) \quad \frac{\log |Y_{F_n}|}{|F_n|} \leq \frac{\log |X_{F_n}|}{|F_n|} + \frac{|F_n S \setminus F_n|}{|F_n|} \cdot \log |A|.$$

As

$$\lim_{n \rightarrow \infty} \frac{|F_n S \setminus F_n|}{|F_n|} = 0$$

by Lemma 5.11, taking the limsup in (5.15) finally gives the required inequality $h_{\mathcal{F}}(Y) \leq h_{\mathcal{F}}(X)$. \square

Corollary 5.12. *Let G be a countable amenable group and let A, B be finite sets with $|A| < |B|$. Then there exists no surjective cellular automaton $\tau: A^G \rightarrow B^G$.*

Proof. This is an immediate consequence of Proposition 5.10 since

$$h_{\mathcal{F}}(A^G) = \log |A| < \log |B| = h_{\mathcal{F}}(B^G).$$

\square

The following example (cf. [46, p. 138]) shows that Corollary 5.12 becomes false if the amenability hypothesis is removed.

Example 5.13. Let G be the free group on two generators a and b . Take $A := \mathbb{Z}/2\mathbb{Z}$ and $B := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so that $|A| = 2$ and $|B| = 4$. Consider the map $\tau: A^G \rightarrow B^G$ defined by

$$\tau(x)(g) = (x(g) + x(ga), x(g) + x(gb))$$

for all $x \in A^G$ and $g \in G$. Observe that τ is a cellular automaton with memory set $S = \{1_G, a, b\}$ and local defining map $\mu: A^S \rightarrow B$ given by

$$\mu(p) = (p(1_G) + p(a), p(1_G) + p(b))$$

for all $p \in A^S$. It is easy to check that τ is surjective. Note that A^G and B^G are totally disconnected compact abelian topological groups and τ is a continuous group morphism whose kernel consists of the two constant configurations in A^G .

5.3. Tilings. Let G be a group. Given a finite subset $E \subset G$, let us say that a subset $T \subset G$ is an E -tiling of G provided the sets tE , $t \in T$, are pairwise disjoint and there exists a finite subset $E' \subset G$ such that the sets tE' , $t \in T$, cover G .

Example 5.14. Take $G = \mathbb{Z}^d$ and $E = \{0, \pm 1, \pm 2, \dots, \pm m\}^d$ for some $m \in \mathbb{N}$, then $T := ((2m+1)\mathbb{Z})^d \subset \mathbb{Z}^d$ is an E -tiling (here one can take $E' = E$).

Given any nonempty finite subset E of a group G , we can use Zorn's lemma to prove that there always exists an E -tiling $T \subset G$. Indeed, consider the set $\mathcal{S}(E)$ consisting of all subsets $S \subset G$ such that the sets sE , $s \in S$, are pairwise disjoint. We first observe that $\mathcal{S}(E)$ is nonempty since $\{1_G\} \in \mathcal{S}(E)$. On the other hand, $\mathcal{S}(E)$ is inductive with respect to set inclusion since if $\mathcal{S}' \subset \mathcal{S}(E)$ is a chain, then $M := \cup_{S \in \mathcal{S}'} S$ belongs to $\mathcal{S}(E)$ and is an upper bound for \mathcal{S}' . By Zorn's lemma, there exists a maximal element $T \in \mathcal{S}(E)$. As $T \in \mathcal{S}(E)$, the sets tE , $t \in T$, are pairwise disjoint. Now, given any $g \in G$, we can find, by maximality of T , an element $t = t(g) \in T$ such that $gE \cap tE \neq \emptyset$ and hence $g \in tEE^{-1}$. It follows that the sets tEE^{-1} , $t \in T$, cover G . Since the set $E' := EE^{-1}$ is finite, this shows that T is an E -tiling of G .

For the proof of the Garden of Eden theorem in the next subsection, we shall use some technical results about tilings in amenable groups.

Lemma 5.15. *Let G be a countable amenable group with Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$. Let $E \subset G$ be a nonempty finite subset and $T \subset G$ an E -tiling. Define, for $n \in \mathbb{N}$, the subset $T_n \subset T$ by*

$$T_n := \{t \in T : tE \subset F_n\}.$$

Then there exist a constant $\alpha = \alpha(\mathcal{F}, T) > 0$ and $n_0 \in \mathbb{N}$ such that

$$(5.16) \quad |T_n| \geq \alpha |F_n| \quad \text{for all } n \geq n_0.$$

Proof. Since T is a E -tiling, there exists a finite subset $E' \subset G$ such that the sets tE' , $t \in T$, cover G . After replacing E' by $E' \cup E$, if necessary, we may assume that $E \subset E'$. Define, for $n \in \mathbb{N}$,

$$T_n^+ := \{t \in T : tE' \cap F_n \neq \emptyset\}.$$

Clearly $T_n \subset T_n^+$. As the sets tE' , $t \in T_n^+$, cover F_n , we have $|F_n| \leq |T_n^+| \cdot |E'|$ so that

$$(5.17) \quad \frac{|T_n^+|}{|F_n|} \geq \frac{1}{|E'|}$$

for all $n \in \mathbb{N}$. Now observe that

$$T_n^+ = T \cap \left(\bigcup_{g \in E'} F_n g^{-1} \right) \quad \text{and} \quad T_n = T \cap \left(\bigcap_{h \in E} F_n h^{-1} \right),$$

so that

$$\begin{aligned} T_n^+ \setminus T_n &= T \cap \left(\bigcup_{g \in E'} F_n g^{-1} \setminus \bigcap_{h \in E} F_n h^{-1} \right) \\ &\subset \bigcup_{g \in E'} F_n g^{-1} \setminus \bigcap_{h \in E} F_n h^{-1} \\ &= \bigcup_{g \in E', h \in E} (F_n g^{-1} \setminus F_n h^{-1}). \end{aligned}$$

We deduce that

$$|T_n^+ \setminus T_n| \leq \sum_{g \in E', h \in E} |F_n g^{-1} \setminus F_n h^{-1}| = \sum_{g \in E', h \in E} |F_n \setminus F_n h^{-1} g|.$$

As

$$\lim_{n \rightarrow \infty} \frac{|F_n \setminus F_n h^{-1} g|}{|F_n|} = 0$$

for all $g \in E'$ and $h \in E$ by (5.1), it follows that

$$\frac{|T_n^+| - |T_n|}{|F_n|} = \frac{|T_n^+ \setminus T_n|}{|F_n|} \rightarrow 0$$

as $n \rightarrow \infty$. Using (5.17) and taking $\varepsilon := \frac{1}{2|E'|}$, we deduce that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{|T_n|}{|F_n|} = \frac{|T_n^+|}{|F_n|} - \frac{|T_n^+| - |T_n|}{|F_n|} \geq \frac{1}{|E'|} - \varepsilon = \alpha,$$

where $\alpha := \frac{1}{2|E'|}$, for all $n \geq n_0$. □

Proposition 5.16. *Let G be a countable amenable group with Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ and let A be a finite set. Let $X \subset A^G$ be a subset and suppose there exist a nonempty finite subset $E \subset G$ and an E -tiling $T \subset G$ such that $X_{tE} \not\subseteq A^{tE}$ for all $t \in T$. Then $h_{\mathcal{F}}(X) < \log |A|$.*

Proof. Let us set, as above, $T_n := \{t \in T : tE \subset F_n\}$ and write

$$F_n^* := F_n \setminus \bigcup_{t \in T_n} tE,$$

for all $n \in \mathbb{N}$. Observe that $\bigcup_{t \in T_n} tE \subset F_n$ so that

$$X_{F_n} \subset A^{F_n^*} \times \prod_{t \in T_n} X_{tE}$$

and

$$(5.18) \quad |F_n| = |F_n^*| + |T_n| \cdot |E|.$$

It follows that

$$\begin{aligned}
\log |X_{F_n}| &\leq |F_n^*| \cdot \log |A| + \sum_{t \in T_n} \log |X_{tE}| \\
&\leq |F_n^*| \cdot \log |A| + \sum_{t \in T_n} \log (|A^{tE}| - 1) \\
&= |F_n^*| \cdot \log |A| + |T_n| \cdot \log (|A|^{|E|} - 1) \\
&= |F_n^*| \cdot \log |A| + |T_n| \cdot |E| \cdot \log |A| + |T_n| \cdot \log (1 - |A|^{-|E|}) \\
&= |F_n| \cdot \log |A| + |T_n| \cdot \log (1 - |A|^{-|E|}),
\end{aligned}$$

where the last equality follows from (5.18). Setting $c := -\log(1 - |A|^{-|E|}) > 0$, we deduce that

$$h_{\mathcal{F}}(X) = \limsup_{n \rightarrow \infty} \frac{\log |X_{F_n}|}{|F_n|} \leq \log |A| - c\alpha < \log |A|,$$

where $\alpha = \alpha(\mathcal{F}, T)$ is as in (5.16). \square

Corollary 5.17. *Let G be a countable amenable group with Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ and let A be a finite set. Let $X \subset A^G$ be a subshift and suppose that there exists a nonempty finite subset $E \subset G$ such that $X_E \subsetneq A^E$. Then one has $h_{\mathcal{F}}(X) < \log |A|$.*

Proof. If T is an E -tiling of G , we deduce from the shift-invariance of X that $X_{tE} \subsetneq A^{tE}$ for all $t \in T$, so that Proposition 5.16 applies. \square

5.4. The Garden of Eden theorem for amenable groups. The following result is due to Machì, Scarabotti, and the first author [20]. Since the groups \mathbb{Z}^d are all amenable, it extends Theorem 4.1.

Theorem 5.18. *Let G be a countable amenable group with Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ and A a finite set. Suppose that $\tau: A^G \rightarrow A^G$ is a cellular automaton. Then the following conditions are equivalent:*

- (a) τ is surjective;
- (b) $h_{\mathcal{F}}(\tau(A^G)) = \log |A|$;
- (c) τ is pre-injective.

Proof. The implication (a) \implies (b) is obvious since $h_{\mathcal{F}}(A^G) = \log |A|$.

In order to show the converse implication, let us suppose that τ is not surjective, that is, the image subshift $X := \tau(A^G)$ is such that $X \subsetneq A^G$. Since X is closed in A^G , there exists a finite subset $E \subset G$ such that $X_E \subsetneq A^E$. By applying Corollary 5.17, we deduce that $h_{\mathcal{F}}(X) < \log |A|$. This shows (b) \implies (a).

Let $S \subset G$ be a memory set for τ such that $1_G \in S$.

Let us show (b) \implies (c). Suppose that τ is not pre-injective. By virtue of Proposition 3.18, we can find a nonempty finite subset $\Omega \subset G$ and two patterns $p_1, p_2 \in A^\Omega$ that are mutually erasable for τ . Let $E := \Omega S^{-1}S$. Observe that $\Omega \subset E$ since $1_G \in S$. Let $T \subset G$ be an E -tiling of G . Consider the subset $Z \subset A^G$ defined by

$$Z := \{z \in A^G : z|_{t\Omega} \neq tp_1 \text{ for all } t \in T\}.$$

Observe that $Z_{tE} \subsetneq A^{tE}$ for all $t \in T$. By using Proposition 5.10 and Proposition 5.16, we deduce that $h_{\mathcal{F}}(\tau(Z)) \leq h_{\mathcal{F}}(Z) < \log |A|$. We claim that $\tau(Z) = \tau(A^G)$. Let $x \in A^G$. Let $T_x := \{t \in T : x|_{t\Omega} = tp_1\}$ and define $z \in Z$ by setting, for all $g \in G$,

$$z(g) := \begin{cases} tp_2(g) & \text{if } g \in t\Omega \text{ for some } t \in T_x \\ x(g) & \text{otherwise.} \end{cases}$$

Let us check that $\tau(z) = \tau(x)$. Let $g \in G$. If $g \notin \cup_{t \in T_x} t\Omega S^{-1}$, then $gS \cap t\Omega = \emptyset$ for all $t \in T_x$ and therefore $z|_{gS} = x|_{gS}$, so that $\tau(z)(g) = \tau(x)(g)$. Suppose now that $g \in t\Omega S^{-1}$ for some (unique) $t = t(g) \in T_x$ and consider the configuration $y \in A^G$ defined by setting, for all $h \in G$,

$$y(h) := \begin{cases} tp_2(h) & \text{if } h \in t\Omega \\ x(h) & \text{otherwise.} \end{cases}$$

Observe that $x|_{G \setminus t\Omega} = y|_{G \setminus t\Omega}$. Since the patterns $x|_{t\Omega} = tp_1$ and $y|_{t\Omega} = tp_2$ are mutually erasable, we deduce that $\tau(y) = \tau(x)$. Moreover, as $gS \subset t\Omega S^{-1}S = tE$, we have $z|_{gS} = y|_{gS}$, and therefore $\tau(z)(g) = \tau(y)(g) = \tau(x)(g)$. This shows that $\tau(z) = \tau(x)$, and the claim follows. We conclude that $h_{\mathcal{F}}(\tau(A^G)) = h_{\mathcal{F}}(\tau(Z)) < \log |A|$. This shows the implication (b) \implies (c).

Finally, let us show (c) \implies (b). Let us set as above $X := \tau(A^G)$ and suppose that $h_{\mathcal{F}}(X) < \log |A|$. As $1_G \in S$, we have $F_n \subset F_n S^{-1}$ so that

$$X_{F_n S^{-1}} \subset X_{F_n} \times A^{F_n S^{-1} \setminus F_n},$$

for all $n \in \mathbb{N}$. We deduce that

$$(5.19) \quad \frac{\log |X_{F_n S^{-1}}|}{|F_n|} \leq \frac{\log |X_{F_n}|}{|F_n|} + \frac{|F_n S^{-1} \setminus F_n|}{|F_n|} \log |A|.$$

As

$$\lim_{n \rightarrow \infty} \frac{|F_n S^{-1} \setminus F_n|}{|F_n|} = 0$$

by Lemma 5.11, we deduce from (5.19) that

$$\limsup_{n \rightarrow \infty} \frac{\log |X_{F_n S^{-1}}|}{|F_n|} \leq \limsup_{n \rightarrow \infty} \frac{\log |X_{F_n}|}{|F_n|} = h_{\mathcal{F}}(X) < \log |A|.$$

Consequently, we can find $n_0 \in \mathbb{N}$ such that,

$$(5.20) \quad |X_{F_{n_0} S^{-1}}| < |A|^{|F_{n_0}|}.$$

Fix $a_0 \in A$ and consider the subset $Z \subset A^G$ defined by

$$Z := \{z \in A^G : z(g) = a_0 \text{ for all } g \in G \setminus F_{n_0}\}.$$

Note that $|Z| = |A|^{|F_{n_0}|}$. Let $z_1, z_2 \in Z$. If $g \in G \setminus F_{n_0} S^{-1}$, then z_1 and z_2 coincide on $gS \subset G \setminus F_{n_0}$ so that $\tau(z_1)(g) = \tau(z_2)(g)$. Therefore $\tau(z_1)$ and $\tau(z_2)$ coincide on $G \setminus F_{n_0} S^{-1}$. This implies that $|\tau(Z)| \leq |X_{F_{n_0} S^{-1}}|$. Using (5.20), we deduce that $|\tau(Z)| < |Z|$. By the pigeon-hole principle, there exist two distinct elements $z_1, z_2 \in Z$ such that $\tau(z_1) = \tau(z_2)$.

As all elements in Z are almost equal (they coincide outside of the finite set F_{n_0}), we conclude that τ is not pre-injective. \square

6. FAILURE OF THE GARDEN OF EDEN THEOREM FOR NONAMENABLE GROUPS

Let us say that a countable group G has the *Moore property* if every surjective cellular automaton $\tau: A^G \rightarrow A^G$ with finite alphabet A over G is pre-injective and that it has the *Myhill property* if every pre-injective cellular automaton $\tau: A^G \rightarrow A^G$ with finite alphabet A over G is surjective. Also let us say that a countable group G satisfies the *Garden of Eden theorem* if G has both the Moore and the Myhill properties. Theorem 5.18 tells us that every countable amenable group satisfies the Garden of Eden theorem. The examples below, essentially due to Muller [42] (see also [36, Section 6], [20, Section 6], [12, Chapter 5]), show that neither the Moore nor the Myhill property holds for countable groups containing nonabelian free subgroups.

Example 6.1. Let G be a countable group and suppose that G contains two elements a and b generating a nonabelian free subgroup $H \subset G$. Take $A = \{0, 1\}$ and let $S := \{a, a^{-1}, b, b^{-1}\}$. Consider the cellular automaton $\tau: A^G \rightarrow A^G$ with memory set $\{1_G\} \cup S$ defined by

$$\tau(x)(g) := \begin{cases} 0 & \text{if } x(g) + x(ga) + x(ga^{-1}) + x(gb) + x(gb^{-1}) \leq 2 \\ 1 & \text{otherwise} \end{cases}$$

for all $x \in A^G$ and $g \in G$.

The pair of configurations $(x_1, x_2) \in A^G \times A^G$, defined by $x_1(g) = 0$ for all $g \in G$, and $x_2(g) = 0$ for all $g \in G \setminus \{1_G\}$ and $x_2(1_G) = 1$, is a diamond for τ . Therefore τ is not pre-injective. However, τ is surjective. To see this, let $y \in A^G$. Let us show that there exists $x \in A^G$ such that $\tau(x) = y$. Let $R \subset G$ be a complete set of representatives of the left cosets of H in G . We define x as follows. Every element $g \in G$ can be uniquely written in the form $g = rh$ with $r \in R$ and $h \in H$. If $g \in R$, i.e., $h = 1_G$, we set $x(g) := 0$. Otherwise, we set $x(g) := y(rh^-)$, where h^- is the *predecessor* of h in H , i.e., the unique element $h^- \in H$ such that $\ell_S(h^-) = \ell_S(h) - 1$ and $h = h^-s$ for some $s \in S$ (here $\ell_S(\cdot)$ denotes the length of the reduced form for elements of H , see Example 5.5). One easily checks that $\tau(x) = y$. This shows that τ is surjective. Thus the Moore implication fails to hold for groups containing nonabelian free subgroups.

Example 6.2. Let G be a countable group and suppose that G contains two elements a and b generating a nonabelian free subgroup $H \subset G$. Let $A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ be the Klein four-group and consider the group endomorphisms p and q of A respectively defined by $p(\alpha, \beta) = (\alpha, 0)$ and $q(\alpha, \beta) = (\beta, 0)$ for all $(\alpha, \beta) \in A$. Let $\tau: A^G \rightarrow A^G$ be the cellular automaton with memory set $S := \{a, a^{-1}, b, b^{-1}\}$ defined by

$$\tau(x)(g) = p(x(ga)) + p(x(ga^{-1})) + q(x(gb)) + q(x(gb^{-1}))$$

for all $x \in A^G$ and $g \in G$. The image of τ is contained in $(\mathbb{Z}/2\mathbb{Z} \times \{0\})^G$. Therefore τ is not surjective. We claim that τ is pre-injective. As τ is a group endomorphism of

A^G , it suffices to show that there is no configuration with finite support in the kernel of τ . Assume on the contrary that there is an element $x \in A^G$ with nonempty finite support $\Omega := \{g \in G : x(g) \neq 0_A\} \subset G$ such that $\tau(x) = 0$. Let $R \subset G$ be a complete set of representatives of the left cosets of H in G . Let us set $\Omega_r := \Omega \cap rH$ for all $r \in R$. Then Ω is the disjoint union of the sets Ω_r , $r \in R$. Let $r \in R$ such that $\Omega_r \neq \emptyset$ and consider an element $g = rh \in \Omega_r$ with $h \in H$ at maximal distance from the identity in the Cayley graph of (H, S) (i.e., with $\ell_S(h)$ maximal). We have that $x(g) = (\alpha, \beta) \neq (0, 0) = 0_A$. Suppose first that $\alpha \neq 0$. We can find $s \in \{a, a^{-1}\}$ such that $\ell_S(hs) = \ell_S(h) + 1$. For all $t \in S \setminus \{s^{-1}\}$, we have that $\ell_S(hst) = \ell_S(h) + 2$ and hence $x(gst) = 0_A$ by maximality. It follows that

$$\tau(x)(gs) = p(x(g)) = (\alpha, 0) \neq 0_A,$$

which contradicts the fact that x is in the kernel of τ . Suppose now that $\alpha = 0$. Then $\beta \neq 0$. We take now $s \in \{b, b^{-1}\}$ such that $\ell_S(hs) = \ell_S(h) + 1$. By an argument similar to the one that we used in the first case, we get

$$\tau(x)(gs) = q(x(g)) = (\beta, 0) \neq 0_A,$$

so that we arrive at a contradiction also in this case. Thus τ is pre-injective. This shows that the Myhill implication fails to hold for groups containing nonabelian free subgroups.

As mentioned in Subsection 5.1, there are nonamenable countable groups containing no nonabelian free subgroups. However, Bartholdi [2] (see [12, Chapter 5]) proved that the Moore property fails to hold for all nonamenable countable groups. Recently, Bartholdi and Kielak [3] also proved that the Myhill property fails to hold for all nonamenable countable groups. Combining these results with the Garden of Eden theorem for amenable groups (Theorem 5.18), this yields the following characterization of amenability in terms of cellular automata.

Theorem 6.3. *Let G be a countable group. Then the following conditions are equivalent:*

- (a) G is amenable;
- (b) G has the Moore property;
- (c) G has the Myhill property;
- (d) G satisfies the Garden of Eden theorem.

7. THE GARDEN OF EDEN THEOREM FOR SUBSHIFTS

7.1. Strongly irreducible subshifts. Let G be a countable group and A a finite set.

A subshift $X \subset A^G$ is called *strongly irreducible* if there is a finite subset $\Delta \subset G$ satisfying the following property: if Ω_1 and Ω_2 are finite subsets of G such that $\Omega_1 \Delta$ does not meet Ω_2 , then, given any two configurations $x_1, x_2 \in X$, there exists a configuration $x \in X$ which coincides with x_1 on Ω_1 and with x_2 on Ω_2 .

Example 7.1. The full shift A^G is strongly irreducible (one can take $\Delta = \{1_G\}$).

Example 7.2. The even subshift $X \subset \{0, 1\}^{\mathbb{Z}}$, described in Example 2.4, is strongly irreducible (one can take $\Delta = \{-2, -1, 0, 1, 2\}$).

Example 7.3. The hard-ball model, described in Example 2.3, is strongly irreducible (one can take $\Delta = \{0, \pm e_1, \dots, \pm e_d\}$). In particular ($d = 1$), the golden mean subshift is strongly irreducible.

Example 7.4. The Ledrappier subshift, described in Example 2.5, is not strongly irreducible.

Fiorenzi [25, Theorem 4.7] obtained the following extension of Theorem 5.18.

Theorem 7.5. *Let G be a countable amenable group with Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ and A, B finite sets. Suppose that $X \subset A^G$ is a strongly irreducible subshift of finite type and $Y \subset B^G$ is a strongly irreducible subshift with $h_{\mathcal{F}}(X) = h_{\mathcal{F}}(Y)$ and that $\tau: X \rightarrow Y$ is a cellular automaton. Then the following conditions are equivalent:*

- (a) τ is surjective;
- (b) $h_{\mathcal{F}}(\tau(X)) = h_{\mathcal{F}}(Y)$;
- (c) τ is pre-injective.

Example 7.6. The cellular automaton $\tau: X \rightarrow Y$ from the golden mean subshift to the even subshift described in Example 3.7 satisfies all the hypotheses in the previous theorem. As τ is pre-injective (cf. Example 3.16), we deduce that τ is surjective. Note that here one might also easily obtain surjectivity of τ by a direct argument.

7.2. The Moore and the Myhill properties for subshifts. Let G be a countable group, A a finite set, and $X \subset A^G$ a subshift. One says that the subshift X has the *Moore property* if every surjective cellular automaton $\tau: X \rightarrow X$ is pre-injective and that it has the *Myhill property* if every pre-injective cellular automaton $\tau: X \rightarrow X$ is surjective. One says that X has the *Moore-Myhill property* or that it satisfies the *Garden of Eden theorem* if it has both the Moore and the Myhill properties.

From Theorem 7.5, we immediately deduce the following.

Corollary 7.7. *Let G be a countable amenable group and A a finite set. Then every strongly irreducible subshift of finite type $X \subset A^G$ has the Moore-Myhill property.*

Example 7.8. Let $G = \mathbb{Z}^d$ and $A = \{0, 1\}$. Consider the hard-ball model $X \subset A^G$ described in Example 2.3. As \mathbb{Z}^d is amenable and X is both strongly irreducible and of finite type, we deduce from Corollary 7.7 that X has the Moore-Myhill property. In particular ($d = 1$), the golden mean subshift has the Moore-Myhill property.

Example 7.9 (Fiorenzi). Let $A = \{0, 1\}$ and let $X \subset A^{\mathbb{Z}}$ be the even subshift (cf. Example 2.4). Consider the cellular automaton $\sigma: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ with memory set $S = \{0, 1, 2, 3, 4\}$ and local defining map $\mu: A^S \rightarrow A$ given by

$$\mu(y) = \begin{cases} 1 & \text{if } y(0)y(1)y(2) \in \{000, 111\} \text{ or } y(0)y(1)y(2)y(3)y(4) = 00100 \\ 0 & \text{otherwise.} \end{cases}$$

Then one has $\sigma(X) \subset X$, and the cellular automaton $\tau := \sigma|_X: X \rightarrow X$ is not pre-injective. Indeed, the configurations $x_1, x_2 \in X$ defined by

$$x_1 = \dots 0 \dots 00(100)100 \dots 0 \dots$$

and

$$x_2 = \cdots 0 \cdots 00(011)100 \cdots 0 \cdots$$

satisfy

$$\tau(x_1) = \cdots 1 \cdots 11(100)10011 \cdots 1 \cdots = \tau(x_2).$$

Observe, alternatively, that the patterns p, q with support $\Omega := \{0, 1, \dots, 12\}$ defined by

$$p(n) = \begin{cases} 1 & \text{if } n = 6, 9 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad q(n) = \begin{cases} 1 & \text{if } n = 7, 8, 9 \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \in \Omega$, are ME.

From a case-by-case analysis, one can show that τ is surjective. It follows that X does not have the Moore property.

We refer to [24, Section 3] and [19, Counterexample 2.18] for more details.

As the even subshift is strongly irreducible and \mathbb{Z} is amenable, the previous example shows that the hypothesis that X is of finite type cannot be removed from Corollary 7.7. However, we have the following (cf. [14]).

Theorem 7.10. *Let G be a countable amenable group and A a finite set. Then every strongly irreducible subshift $X \subset A^G$ has the Myhill property.*

Example 7.11. The even subshift $X \subset \{0, 1\}^{\mathbb{Z}}$ has the Myhill property since it is strongly irreducible and \mathbb{Z} is amenable.

Example 7.12. Let $A = \{0, 1\}$. Let $x_0, x_1 \in A^{\mathbb{Z}}$ denote the two constant configurations respectively defined by $x_0(n) = 0$ and $x_1(n) = 1$ for all $n \in \mathbb{Z}$. Note that $X = \{x_0, x_1\}$ is a subshift of finite type. The map $\tau: X \rightarrow X$ given by $\tau(x_0) = \tau(x_1) = x_0$ is a cellular automaton which is pre-injective but not surjective. It follows that X does not have the Myhill property. This very simple example shows that the hypothesis that X is strongly irreducible cannot be removed neither from Corollary 7.7 nor from Theorem 7.10. Note that X has the Moore property since X is finite, so that every surjective self-mapping of X is injective and therefore pre-injective.

Example 7.13 (Fiorenzi). Let $A = \{0, 1, 2\}$ and let $X \subset A^{\mathbb{Z}}$ be the subshift of finite type consisting of all $x \in A^{\mathbb{Z}}$ such that

$$x(n)x(n+1) \notin \{01, 02\} \quad \text{for all } n \in \mathbb{Z}.$$

Thus a configuration $x: \mathbb{Z} \rightarrow A$ is in X if and only if one of the following conditions is satisfied:

- (i) $x(n) = 0$ for all $n \in \mathbb{Z}$;
- (ii) $x(n) \neq 0$ for all $n \in \mathbb{Z}$;
- (iii) there exists $n_0 \in \mathbb{N}$ such that $x(n) \in \{1, 2\}$ for all $n \leq n_0$ and $x(n) = 0$ for all $n > n_0$.

Consider the cellular automaton $\sigma: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ with memory set $S = \{0, 1\}$ and local defining map

$$\mu(y) = \begin{cases} y(0) & \text{if } y(1) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\sigma(x) = x$ if $x \in X$ is of type (i) or (ii) while, if $x \in X$ is of type (iii), then $\sigma(x)$ is obtained from x by replacing its rightmost nonzero term by 0. We deduce that $\sigma(X) \subset X$ and that the cellular automaton $\tau := \sigma|_X: X \rightarrow X$ is surjective but not pre-injective (see [24, Counterexample 4.27]). It follows that X does not have the Moore property.

It turns out that X does not have the Myhill property either. Indeed, consider now the cellular automaton $\sigma': A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ with memory set $S' = \{-1, 0\}$ and local defining map

$$\mu'(y) = \begin{cases} y(0) & \text{if } y(-1)y(0) \notin \{10, 20\} \\ y(-1) & \text{otherwise.} \end{cases}$$

Observe that $\sigma'(x) = x$ if $x \in X$ is of type (i) or (ii) while, if $x \in X$ is of type (iii), then $\sigma'(x)$ is obtained from x by repeating on its right its rightmost nonzero term. We deduce that $\sigma'(X) \subset X$ and that the cellular automaton $\tau' := \sigma'|_X: X \rightarrow X$ is injective and hence pre-injective. However, τ' is not surjective (observe for instance that the pattern $p \in A^{\{-1,0,1\}}$ defined by $p(-1)p(0)p(1) = 120$ is a Garden of Eden pattern for τ').

Let A be a finite set and $X \subset A^{\mathbb{Z}}$ a subshift. One says that a word $u \in A^*$ of length n *appears* in X if there is a configuration $x \in X$ and $m \in \mathbb{Z}$ such that $u = x(m)x(m+1)\cdots x(m+n-1)$. The subset $L(X) \subset A^*$ consisting of all words that appear in X is called the *language* of X . One says that the subshift X is *irreducible* if given any two words $u, v \in L(X)$ there exists a word $w \in L(X)$ such that $uwv \in L(X)$. Clearly every strongly irreducible subshift $X \subset A^{\mathbb{Z}}$ is irreducible. The converse is false as shown by the following example.

Example 7.14. Let $A = \{0, 1\}$ and consider the subshift $X \subset A^{\mathbb{Z}}$ consisting of the two configurations $x \in A^{\mathbb{Z}}$ that satisfy $x(n) \neq x(n+1)$ for all $n \in \mathbb{Z}$. It is clear that X is irreducible but not strongly irreducible. Observe that X is of finite type.

The following result is an immediate consequence of [32, Theorem 8.1.16] (cf. [24, Corollary 2.19]).

Theorem 7.15. *Let A be a finite set. Then every irreducible subshift of finite type $X \subset A^{\mathbb{Z}}$ has the Moore-Myhill property.*

8. GARDEN OF EDEN THEOREMS FOR OTHER DYNAMICAL SYSTEMS

8.1. Dynamical systems. By a *dynamical system*, we mean a triple (X, G, α) , where X is a compact metrizable space, G is a countable group, and α is a continuous action of G on X . The space X is called the *phase space* of the dynamical system. If there is no risk of confusion, we shall write (X, G) , or even sometimes simply X , instead of (X, G, α) .

Example 8.1. Let G be a countable group and A a compact metrizable topological space (e.g. a finite set with its discrete topology). Equip $A^G = \{x: G \rightarrow A\}$ with the product topology. The *shift action* σ of G on A^G is the action defined by $\sigma(g, x) = gx$, where

$$(gx)(h) = x(g^{-1}h)$$

for all $x \in A^G$ and $g, h \in G$. Then (A^G, G, σ) is a dynamical system.

Example 8.2. If (X, G, α) is a dynamical system and $Y \subset X$ is a closed α -invariant subset, then $(Y, G, \alpha|_Y)$, where $\alpha|_Y$ denotes the action of G on Y induced by restriction of α , is a dynamical system. In particular, if G is a countable group, A a finite set, and $X \subset A^G$ a subshift, then $(X, G, \sigma|_X)$ is a dynamical system.

Example 8.3. Let $f: X \rightarrow X$ be a homeomorphism of a compact metrizable space X . The dynamical system *generated* by f is the dynamical system $(X, \mathbb{Z}, \alpha_f)$, where α_f is the action of \mathbb{Z} on X given by $\alpha_f(n, f) := f^n(x)$ for all $n \in \mathbb{Z}$ and $x \in X$. We shall also write (X, f) to denote the dynamical system generated by f .

Remark 8.4. If we fix the countable group G , the dynamical systems (X, G) are the objects of a concrete category \mathcal{D}_G in which the morphisms from an object $X \in \mathcal{D}_G$ to another object $Y \in \mathcal{D}_G$ consist of all equivariant continuous maps $\tau: X \rightarrow Y$. It follows from the Curtis-Hedlund-Lyndon theorem (cf. Theorem 3.8) that the category \mathcal{C}_G described in Remark 3.10 is a full subcategory of the category \mathcal{D}_G .

Let (X, G) and (\tilde{X}, G) be two dynamical systems.

One says that the dynamical systems (X, G) and (\tilde{X}, G) are *topologically conjugate* if they are isomorphic objects in the category \mathcal{D}_G , i.e., if there exists an equivariant homeomorphism $h: \tilde{X} \rightarrow X$.

One says that (X, G) is a *factor* of (\tilde{X}, G) if there exists an equivariant continuous surjective map $\theta: \tilde{X} \rightarrow X$. Such a map θ is then called a *factor map*. A factor map $\theta: \tilde{X} \rightarrow X$ is said to be *finite-to-one* if the pre-image set $\theta^{-1}(x)$ is finite for each $x \in X$. A finite-to-one factor map is said to be *uniformly bounded-to-one* if there is an integer $K \geq 1$ such that $|\theta^{-1}(x)| \leq K$ for all $x \in X$.

8.2. Expansiveness. One says that a dynamical system (X, G) is *expansive* if there exists a neighborhood $W \subset X \times X$ of the diagonal

$$\Delta_X := \{(x, x) : x \in X\} \subset X \times X$$

such that, for every pair of distinct points $x, y \in X$, there exists an element $g = g(x, y) \in G$ such that $(gx, gy) \notin W$. Such a set W is then called an *expansiveness neighborhood* of the diagonal.

If d is a metric on X compatible with the topology, the fact that (X, G) is expansive is equivalent to the existence of a constant $\delta > 0$ such that, for every pair of distinct points $x, y \in X$, there exists an element $g = g(x, y) \in G$ such that $d(gx, gy) \geq \delta$.

Example 8.5. Let G be a countable group and A a finite set. Then the G -shift on A^G is expansive. Indeed, it is clear that

$$W := \{(x, y) \in A^G \times A^G : x(1_G) = y(1_G)\}$$

is an expansiveness neighborhood of Δ_{A^G} .

Example 8.6. If (X, G) is an expansive dynamical system and $Y \subset X$ is a closed invariant subset, then (Y, G) is expansive. Indeed, if W is an expansiveness neighborhood of Δ_X , then $W \cap (Y \times Y)$ is an expansiveness neighborhood of Δ_Y . In particular, if G is a countable

group, A a finite set, and $X \subset A^G$ a subshift, then the dynamical system $(X, G, \sigma|_X)$ is expansive.

8.3. Homoclinicity. Let (X, G) be a dynamical system. Two points $x, y \in X$ are called *homoclinic* with respect to the action of G on X , ore more briefly, *G -homoclinic*, if for any neighborhood $W \subset X \times X$ of the diagonal Δ_X , there is a finite set $F = F(W, x, y) \subset G$ such that $(gx, gy) \in W$ for all $g \in G \setminus F$.

If d is a metric on X that is compatible with the topology, two points $x, y \in X$ are homoclinic if and only if

$$\lim_{g \rightarrow \infty} d(gx, gy) = 0,$$

where ∞ is the point at infinity in the one-point compactification of the discrete group G . This means that, for every $\varepsilon > 0$, there is a finite subset $F = F(\varepsilon, d, x, y) \subset G$ such that $d(gx, gy) < \varepsilon$ for all $g \in G \setminus F$.

Homoclinicity clearly defines an equivalence relation on X (transitivity follows from the triangle inequality). The equivalence classes of this relation are called the *G -homoclinicity classes* of X .

Definition 8.7. Let (X, G) be a dynamical system and Y a set. One says that a map $\tau: X \rightarrow Y$ is *pre-injective* if its restriction to each G -homoclinicity class is injective.

Example 8.8. Let G be a countable group and A a finite set. Two configurations $x, y \in A^G$ are homoclinic with respect to the shift action of G on A^G if and only if they are almost equal (see e.g. [16, Proposition 2.5]). Indeed, first observe that the sets

$$W_\Omega := \{(x, y) \in A^G \times A^G : x|_\Omega = y|_\Omega\},$$

where Ω runs over all finite subsets of G , form a neighborhood base of the diagonal $\Delta_{A^G} \subset A^G \times A^G$ (this immediately follows from the definition of the product topology). Now, if $x, y \in A^G$ are almost equal, then the set $D \subset G$ consisting of all $g \in G$ such that $x(g) \neq y(g)$ is finite, so that ΩD^{-1} is also finite for every finite subset $\Omega \subset G$. As $(gx, gy) \in W_\Omega$ for every $g \in G \setminus \Omega D^{-1}$, this implies that x and y are homoclinic. Conversely, suppose that $x, y \in A^G$ are homoclinic. Then there exists a finite subset $F \subset G$ such that $(gx, gy) \in W_{\{1_G\}}$ for all $g \in G \setminus F$. This implies that $x(g) = y(g)$ for all $g \in G \setminus F^{-1}$, so that x and y are almost equal.

Example 8.9. Let (X, G, α) be a dynamical system and $Y \subset X$ a closed invariant subset. Denote as above by $\alpha|_Y$ the restriction of α to Y . Then two points $x, y \in Y$ are homoclinic with respect to $\alpha|_Y$ if and only if they are homoclinic with respect to α . In particular, if G is a countable group, A a finite set, and $X \subset A^G$ a subshift, then two configurations $x, y \in X$ are homoclinic with respect to $\sigma|_X$ if and only if they are almost equal. It follows that the definition of pre-injectivity for cellular automata between subshifts given in Definition 3.14 agrees with the one given in Definition 8.7 above.

8.4. The Moore and the Myhill properties for dynamical systems. Let (X, G) be a dynamical system.

An *endomorphism* of (X, G) is a continuous equivariant map $\tau: X \rightarrow X$.

One says that the dynamical system (X, G) has the *Moore property* if every surjective endomorphism of (X, G) is pre-injective and that it has the *Myhill property* if every pre-injective endomorphism of (X, G) is surjective. One says that (X, G) has the *Moore-Myhill property* or that it satisfies the *Garden of Eden theorem* if it has both the Moore and the Myhill properties.

Observe that all these properties are invariants of topological conjugacy in the sense that if the dynamical systems (X, G) and (Y, G) are topologically conjugate then (X, G) has the Moore (resp. the Myhill, resp. the Moore-Myhill) property if and only if (Y, G) has the Moore (resp. the Myhill, resp. the Moore-Myhill) property.

Remark 8.10. In the particular case when (X, G) is a subshift, it immediately follows from Theorem 3.8 and Example 8.9 that these definitions are equivalent to the ones given in Subsection 7.2.

8.5. Anosov diffeomorphisms. Let $f: M \rightarrow M$ be a diffeomorphism of a compact smooth manifold M . One says that f is an *Anosov diffeomorphism* (see e.g. [51], [7], [50]) if the tangent bundle TM of M continuously splits as a direct sum $TM = E_s \oplus E_u$ of two df -invariant subbundles E_s and E_u such that, with respect to some (or equivalently any) Riemannian metric on M , the differential df is exponentially contracting on E_s and exponentially expanding on E_u , i.e., there are constants $C > 0$ and $0 < \lambda < 1$ such that

- (i) $\|df^n(v)\| \leq C\lambda^n\|v\|$,
- (ii) $\|df^{-n}(w)\| \leq C\lambda^n\|w\|$

for all $x \in M$, $v \in E_s(x)$, $w \in E_u(x)$, and $n \geq 0$.

Example 8.11 (Arnold's cat). Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and the diffeomorphism f of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ given by

$$f(x) := Ax = \begin{pmatrix} x_2 \\ x_1 + x_2 \end{pmatrix} \quad \text{for all } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{T}^2.$$

The dynamical system (\mathbb{T}^2, f) is known as *Arnold's cat*. The diffeomorphism f is Anosov.

Indeed, the eigenvalues of A are $\lambda_1 = -\frac{1}{\varphi}$, and $\lambda_2 = \varphi$, where $\varphi := \frac{1+\sqrt{5}}{2}$ is the golden mean. As $-1 < \lambda_1 < 0$ and $1 < \lambda_2$, it follows that $df = A$ is exponentially contracting in the direction of the eigenline associated with λ_1 and uniformly expanding in the direction of the eigenline associated with λ_2 .

Example 8.12 (Hyperbolic toral automorphism). More generally, if $A \in \text{GL}_n(\mathbb{Z})$ has no eigenvalue on the unit circle, then the diffeomorphism f of the n -torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, defined by $f(x) = Ax$ for all $x \in \mathbb{T}^n$, is Anosov. Such a diffeomorphism is called a *hyperbolic toral automorphism*.

In [17, Theorem 1.1], we obtained the following result.

Theorem 8.13. *Let f be an Anosov diffeomorphism of the n -dimensional torus \mathbb{T}^n . Then the dynamical system (\mathbb{T}^n, f) has the Moore-Myhill property.*

The proof given in [17] uses two classical results. The first one is the Franks-Manning theorem [26], [37] which states that (\mathbb{T}^n, f) is topologically conjugate to a hyperbolic toral automorphism. The second one is a result of Walters [55] which says that every endomorphism of a hyperbolic toral automorphism is affine.

We do not know if the dynamical system (M, f) has the Moore-Myhill property whenever f is an Anosov diffeomorphism of a compact smooth manifold M . However, we have obtained in [16, Theorem 1.1] the following result.

Theorem 8.14. *Let X be a compact metrizable space equipped with a continuous action of a countable amenable group G . Suppose that the dynamical system (X, G) is expansive and that there exist a finite set A , a strongly irreducible subshift $\tilde{X} \subset A^G$, and a uniformly bounded-to-one factor map $\theta: \tilde{X} \rightarrow X$. Then the dynamical system (X, G) has the Myhill property.*

A homeomorphism f of a topological space X is called *topologically mixing* if, given any two nonempty open subsets $U, V \subset X$, there exists an integer $N \geq 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \in \mathbb{Z}$ that satisfy $|n| \geq N$. By the classical work of Bowen (cf. [6, Theorem 28 and Proposition 30] and [5, Proposition 10]), dynamical systems generated by topologically mixing Anosov diffeomorphisms satisfy the hypotheses of Theorem 8.14. As a consequence (cf. [17, Corollary 4.4]), we get the following partial extension of Theorem 8.13.

Corollary 8.15. *Let f be a topologically mixing Anosov diffeomorphism of a compact smooth manifold M . Then the dynamical system (M, f) has the Myhill property.*

Remark 8.16. All known examples of Anosov diffeomorphisms are topologically mixing. Also, all compact smooth manifolds that are known to admit Anosov diffeomorphisms are *infra-nilmanifolds*. We recall that a *nilmanifold* is a manifold of the form N/Γ , where N is a simply-connected nilpotent Lie group and Γ is a discrete cocompact subgroup of N and that a *infra-nilmanifold* is a manifold that is finitely covered by some nilmanifold.

8.6. Principal algebraic dynamical systems. An *algebraic dynamical system* is a dynamical system of the form (X, G) , where X is a compact metrizable abelian topological group and G is a countable group acting on X by continuous group morphisms. Note that if (X, G) is an algebraic dynamical system, then, for each $g \in G$, the map $x \mapsto gx$ is a continuous group automorphism of G with inverse $x \mapsto g^{-1}x$.

Example 8.17. Let G be a countable group and A a compact metrizable topological group (for example a finite discrete abelian group, or the n -dimensional torus \mathbb{T}^n , or the infinite-dimensional torus $\mathbb{T}^{\mathbb{N}}$, or the group \mathbb{Z}_p of p -adic integers for some prime p). Then the G -shift (A^G, G) is an algebraic dynamical system.

Example 8.18. Let X be a compact metrizable abelian group and $f: X \rightarrow X$ a continuous group automorphism (for example $X = \mathbb{T}^n$ and $f \in \mathrm{GL}_n(\mathbb{Z})$). Then the dynamical system (X, f) generated by f is an algebraic dynamical system.

Let (X, G) be an algebraic dynamical system.

If d is a metric on X that is compatible with the topology then a point $x \in X$ is homoclinic to 0_X if and only if one has

$$\lim_{g \rightarrow \infty} d(gx, 0_X) = 0.$$

The set $\Delta(X, G)$ consisting of all points of X that are homoclinic to 0_X is an G -invariant additive subgroup of X , called the *homoclinic group* of (X, G) (cf. [33]). Two points $x, y \in X$ are homoclinic if and only if $x - y \in \Delta(X, G)$. It follows that the set of homoclinicity classes of (X, G) can be identified with the quotient group $X/\Delta(X, G)$.

Consider now the Pontryagin dual \widehat{X} of X . We recall that if L is a locally compact abelian group, the elements of its Pontryagin dual \widehat{L} are the *characters* of L , i.e., the continuous group morphisms $\chi: L \rightarrow \mathbb{T}$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, and that the topology on \widehat{L} is the topology of uniform convergence on compact subsets. (see e.g. [41]). As the abelian group X is compact and metrizable, \widehat{X} is a discrete countable abelian group. There is also a natural dual action of G on \widehat{X} defined by

$$g\chi(x) := \chi(g^{-1}x)$$

for all $g \in G$, $\chi \in \widehat{X}$, and $x \in X$. Note that $\chi \mapsto g\chi$ is a group automorphism of \widehat{X} for each $g \in G$.

We recall that the *integral group ring* $\mathbb{Z}[G]$ of G consists of all formal series

$$r = \sum_{g \in G} r_g g,$$

where $r_g \in \mathbb{Z}$ for all $g \in G$ and $r_g = 0$ for all but finitely many $g \in G$, and the operations on $\mathbb{Z}[G]$ are defined by the formulas

$$\begin{aligned} r + s &= \sum_{g \in G} (r_g + s_g)g, \\ rs &= \sum_{g_1, g_2 \in G} r_{g_1} s_{g_2} g_1 g_2 \end{aligned}$$

for all

$$r = \sum_{g \in G} r_g g, \quad s = \sum_{g \in G} s_g g \in \mathbb{Z}[G].$$

By linearity, the action of G on \widehat{X} extends to a left $\mathbb{Z}[G]$ -module structure on \widehat{X} .

Conversely, if M is a countable left $\mathbb{Z}[G]$ -module and we equip M with its discrete topology, then its Pontryagin dual \widehat{M} is a compact metrizable abelian group. The left $\mathbb{Z}[G]$ -module structure on M induces by restriction an action of G on M , and, by dualizing, we get an action of G on \widehat{M} by continuous group morphisms, so that (\widehat{M}, G) is an algebraic dynamical system.

Using the fact that every locally compact abelian group is isomorphic to its bidual, one shows that Pontryagin duality yields a one-to-one correspondence between algebraic

dynamical systems with acting group G and countable left $\mathbb{Z}[G]$ -modules (see [48], [34], [35]).

Let $f \in \mathbb{Z}[G]$ and consider the cyclic left $\mathbb{Z}[G]$ -module $M_f := \mathbb{Z}[G]/\mathbb{Z}[G]f$ obtained by quotienting the ring $\mathbb{Z}[G]$ by the principal left ideal generated by f . The algebraic dynamical system associated by Pontryagin duality with M_f is denoted by (X_f, G) and is called the *principal algebraic dynamical system* associated with f .

In [18], we obtained the following result.

Theorem 8.19. *Let G be a countable abelian group and $f \in \mathbb{Z}[G]$. Suppose that the principal algebraic dynamical system (X_f, G) associated with f is expansive and that X_f is connected. Then the dynamical system (X_f, G) satisfies the Moore-Myhill property.*

The proof uses two main ingredients. The first one is a result of Lind and Schmidt [33] which asserts that the homoclinic group of (X_f, G) , equipped with the discrete topology and the induced action of G , is conjugate to the Pontryagin dual $\mathbb{Z}[G]/\mathbb{Z}[G]f$ of X_f . The second is a rigidity result of Bhattacharya [4] according to which every endomorphism of (X_f, G) is affine (see [18] for more details).

In the case $G = \mathbb{Z}^d$, the group ring $\mathbb{Z}[G]$ can be identified with the ring $\mathbb{Z}[u_1, u_1^{-1}, \dots, u_d, u_d^{-1}]$ of Laurent polynomials with integral coefficients on d commuting indeterminates.

Example 8.20. For $G = \mathbb{Z}$ and $f = u^2 - u - 1 \in \mathbb{Z}[u, u^{-1}] = \mathbb{Z}[G]$, one can check that the associated principal algebraic dynamical system (X_f, \mathbb{Z}) is topologically conjugate to Arnold's cat on \mathbb{T}^2 (see e.g. [48, Example 2.18.(2)]). Thus, we recover from Theorem 8.19 that Arnold's cat satisfies the Moore-Myhill property.

9. SOME ADDITIONAL TOPICS

9.1. Infinite alphabets and uncountable groups. The notion of a subshift and that of a cellular automaton between subshifts can be extended to the case where the alphabet sets are infinite and the group is not countable.

More specifically, let G be a (possibly uncountable) group and A a (possibly infinite) set. The *prodiscrete topology* on A^G is the product topology obtained by taking the discrete topology on each factor A of $A^G = \prod_{g \in G} A$. The prodiscrete topology on A^G is not metrizable as soon as G is uncountable and A contains more than one element. However, this topology is induced by the *prodiscrete uniform structure* on A^G , that is, the product uniform structure on A^G obtained by taking the discrete uniform structure on each factor A of A^G (see [12, Appendix B] for more details). A subset $X \subset A^G$ is called a *subshift* if X is invariant under the shift action and closed for the prodiscrete topology.

Let G be a group and let A, B be sets. Suppose that $X \subset A^G$ and $Y \subset B^G$ are two subshifts. One defines cellular automata between X and Y exactly as in Definition 3.1. Every cellular automaton $\tau: X \rightarrow Y$ is continuous with respect to the topologies on X and Y induced by the prodiscrete topologies on A^G and B^G . The converse is false in general [10, Section 4], [12, Example 1.8.2]. However, the Curtis-Hedlund-Lyndon theorem (cf. Theorem 3.8) admits the following generalization [10, Theorem 1.1], [12, Theorem 1.8.1]: a map $\tau: X \rightarrow Y$ is a cellular automaton if and only if it is equivariant (with respect to the

G -shift actions) and uniformly continuous (for the uniform structures on X and Y induced by the prodiscrete uniform structures on A^G and B^G).

One can extend the notion of amenability defined only for countable groups in Section 5.1 by declaring that a general group G is *amenable* if all of its finitely generated subgroups are amenable. This extension makes sense since every finitely generated group is countable and every subgroup of a countable amenable group is itself amenable. The Garden of Eden theorem (cf. Theorem 5.18) remains valid in this more general setting: if G is a (possibly uncountable) amenable group, A a finite set, and $\tau: A^G \rightarrow A^G$ a cellular automaton, then $\tau: A^G \rightarrow A^G$ is surjective if and only if it is pre-injective. The proof can be reduced to the case when the group G is finitely generated (and hence countable) by using the operations of *restriction* and *induction* for cellular automata (see [11] and [12, Section 1.7]). One can also directly follow the proof given above for Theorem 5.18 by replacing the Følner sequence by a Følner net (see [12, Theorem 5.8.1]).

9.2. Linear cellular automata. Let G be a group and let K be a field. Let A be a finite-dimensional vector space over K and set $d = \dim_K(A)$. Observe that A is infinite as soon as the field K is infinite (e.g. $K = \mathbb{R}$) and $d \neq 0$. Taking A as an alphabet, the configuration set A^G inherits a natural product vector space structure. The support of a configuration $x \in A^G$ is the subset $\text{supp}(x) := \{g \in G : x(g) \neq 0\} \subset G$. Thus, $x \in A^G$ has finite support if and only if it is almost equal to the constant zero-configuration. We denote by $A[G] \subset A^G$ the vector subspace consisting of all configurations with finite support.

Recall that an *involutive K -algebra* is a K -algebra equipped with an involution that is a K -algebra anti-automorphism.

The vector space $K[G]$ has a natural additional structure of an involutive K -algebra. The multiplication on $K[G]$ is the *convolution product* defined by

$$(\alpha\beta)(g) := \sum_{\substack{g_1, g_2 \in G: \\ g_1 g_2 = g}} \alpha(g_1)\beta(g_2) = \sum_{h \in G} \alpha(h)\beta(h^{-1}g)$$

for all $\alpha, \beta \in K[G]$ and $g \in G$, and the involution is the map $\alpha \mapsto \alpha^*$ given by

$$\alpha^*(g) := \alpha(g^{-1})$$

for all $\alpha \in K[G]$ and $g \in G$. This involutive K -algebra is called the *group algebra* of the group G with coefficients in K . Note that the group G embeds as a subgroup of the group of invertible elements of $K[G]$ via the map $g \mapsto \delta_g$, where $\delta_g \in K[G]$ is defined by $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for all $h \in G$ with $h \neq g$, and that $G \subset K[G]$ is a base for the vector space $K[G]$.

The G -shift action on A^G is clearly K -linear, so that it yields a left $K[G]$ -module structure on A^G . Observe that $A[G]$ is a submodule of A^G .

A *linear cellular automaton* over the group G and the alphabet A is a cellular automaton $\tau: A^G \rightarrow A^G$ that is K -linear with respect to the vector space structure on A^G (if S is any memory set for τ and $\mu: A^S \rightarrow A$ is the associated local defining map, this is equivalent to requiring that μ is K -linear).

Let us denote by $\text{LCA}(G; A)$ the vector space consisting of all linear cellular automata $\tau: A^G \rightarrow A^G$.

Let $\tau \in \text{LCA}(G; A)$. Note that $A[G]$ is stable under τ . Indeed, if $S \subset G$ is a memory set for τ , then $\text{supp}(\tau(x)) \subset \text{supp}(x)S^{-1}$ for all $x \in A^G$ (see [12, Proposition 8.2.3]). Moreover, τ is pre-injective if and only if $\tau|_{A[G]}: A[G] \rightarrow A[G]$ is injective (cf. [12, Proposition 8.2.5]). Observe also that τ is a $K[G]$ -module endomorphism of A^G and hence of $A[G]$.

The vector space $\text{LCA}(G; A)$ has a natural structure of a K -algebra with the composition of maps as the multiplicative operation. Furthermore, the restriction map $\tau \mapsto \tau|_{A[G]}$ yields a K -algebra isomorphism from $\text{LCA}(G; A)$ onto $\text{End}_{K[G]}(A[G])$, the endomorphism K -algebra of the $K[G]$ -module $A[G]$ (cf. [12, Theorem 8.7.6]). It turns out that $A[G]$ is a free $K[G]$ -module with rank d . Actually, if $(e_i)_{1 \leq i \leq d}$ is a base for the vector space A , then the family $(x_i)_{1 \leq i \leq d}$, where $x_i \in A[G]$ is the configuration defined by $x_i(1_G) = e_i$ and $x_i(g) = 0$ for $g \neq 1_G$, is a free base for the $K[G]$ -module $A[G]$ (cf. [12, Proposition 8.7.3]). One deduces that $\text{End}_{K[G]}(A[G])$ is isomorphic, as a K -algebra, to the K -algebra $\text{Mat}_d(K[G])$ of $d \times d$ matrices with coefficients in the group algebra $K[G]$. It follows that $\text{LCA}(G; A)$ and $\text{Mat}_d(K[G])$ are isomorphic as K -algebras (cf. [12, Corollary 8.7.8]). For instance, the map $\Phi: \text{Mat}_d(K[G]) \rightarrow \text{LCA}(G; A)$, sending each matrix $\alpha = (\alpha_{ij})_{1 \leq i, j \leq d} \in \text{Mat}_d(K[G])$ to the unique linear cellular automaton $\tau \in \text{LCA}(G; A)$ such that

$$\tau(x_i) = \sum_{1 \leq j \leq d} \alpha_{ji}^* x_j$$

for all $1 \leq i \leq d$, is a K -algebra isomorphism.

The *adjoint* of a matrix $\alpha \in \text{Mat}_d(K[G])$ is the matrix $\alpha^* \in \text{Mat}_d(K[G])$ given by $(\alpha^*)_{ij} := \alpha_{ji}^* \in K[G]$ for all $1 \leq i, j \leq d$. The involution $\alpha \mapsto \alpha^*$ makes $\text{Mat}_d(K[G])$ into an involutive K -algebra. Let us transport this involution to $\text{LCA}(G; A)$ via Φ . Thus, $\text{LCA}(G; A)$ becomes an involutive K -algebra with involution $\tau \mapsto \tau^*$ satisfying

$$\tau^*(x_i) = \sum_{1 \leq j \leq d} \alpha_{ij} x_j$$

for all $1 \leq i \leq d$ and $\tau = \Phi(\alpha) \in \text{LCA}(G; A)$. Note that this involution on $\text{LCA}(G; A)$ depends on the choice of a base for A .

Consider now the non-degenerate K -bilinear symmetric map $A \times A \rightarrow K$ defined by

$$a \cdot b = \sum_{1 \leq i \leq d} a_i b_i \quad \text{for all } a = \sum_{1 \leq i \leq d} a_i e_i \text{ and } b = \sum_{1 \leq i \leq d} b_i e_i, \text{ with } a_i, b_i \in K \text{ for } 1 \leq i \leq d.$$

Then the K -bilinear map $A[G] \times A^G \rightarrow K$, defined by

$$\langle x, y \rangle := \sum_{g \in G} x(g) \cdot y(g)$$

for all $x \in A[G]$ and $y \in A^G$, is non-degenerate in both arguments. Given a linear cellular automaton $\tau: A^G \rightarrow A^G$, Bartholdi [1] (see also [52]) observed that

$$\langle \tau(x), y \rangle = \langle x, \tau^*(y) \rangle$$

for all $x \in A[G]$ and $y \in A^G$, and used this to show that τ is pre-injective (resp. surjective) if and only if τ^* is surjective (resp. pre-injective).

In [8] (see also [12, Theorem 8.9.6]) a linear version of the Garden of Eden theorem is proved, namely that if G is amenable and $\tau \in \text{LCA}(G; A)$, then τ is surjective if and only if it is pre-injective.

Let G be a nonamenable group. In [3, Theorem 1.1] Bartholdi showed that there exists a finite field K (in [1] he actually observed that the field K can be arbitrary), a finite dimensional vector space A over K , and a pre-injective linear cellular automaton $\tau \in \text{LCA}(G; A)$ which is not surjective. As a consequence ([1, Corollary]), the cellular automaton $\tau^* \in \text{LCA}(G; A)$ is surjective but not pre-injective. These two facts, in combination with the linear version of the Garden of Eden theorem in [8], yield a characterization of group amenability in terms of linear cellular automata.

The linear version of the Garden of Eden theorem has been extended in [13] to linear cellular automata $\tau: X \rightarrow X$ with $X \subset A^G$ a strongly irreducible linear subshift of finite type, and in [9] to the case when the alphabet A is a semi-simple left-module of finite length over a (possibly noncommutative) ring.

9.3. Gromov's Garden of Eden theorem. In [30, Subsection 8.F'], Gromov proved a Garden of Eden type theorem generalizing Theorem 5.18 under several aspects. First of all, the alphabet set A is only assumed to be countable, not necessarily finite. In addition, the universe is (the vertex set V of) a connected simplicial graph $\mathcal{G} = (V, E)$ of bounded degree with a natural homogeneity condition (to admit a *dense pseudogroup of partial isometries*). The classical case corresponds to $\mathcal{G} = \mathcal{C}(G, S)$ being the Cayley graph of a finitely generated group G with respect to a finite and symmetric generating subset $S \subset G \setminus \{1_G\}$. The dense pseudogroup of partial isometries is, in this particular case, given by partial left-multiplication by group elements. In this more general setting, the category corresponding to that of cellular automata consists now of the following:

- *stable spaces*, i.e. (stable) projective limits of locally-finite projective systems (X_Ω) of A -valued maps on (subsets of) V with a suitable finiteness and irreducibility condition (*bounded propagation*) and admitting a *dense holonomy* (corresponding to shift-invariance in the classical case) as objects, and
- *maps of bounded propagation* (this condition corresponds to continuity) admitting a *dense holonomy* (this corresponds to G -equivariance), as morphisms.

The notion of a Følner sequence and of amenability for simplicial graphs, together with the corresponding notion of entropy (for the above-mentioned spaces of A -valued maps), carry verbatim from the group theoretical framework. All this said, Gromov's theorem states as follows.

Let $\mathcal{G} = (V, E)$ be an amenable simplicial connected graph of bounded degree admitting a dense pseudogroup of partial isometries and let A be a finite or countably infinite alphabet set. Suppose that $X, Y \subset A^V$ are stable spaces of bounded propagation with the same entropy. Let $\tau: X \rightarrow Y$ be a map of bounded propagation admitting a dense holonomy. Then τ is surjective if and only if it is pre-injective.

In [19, Lemma 3.11] it is shown that a stable space of bounded propagation is strongly irreducible (cf. Section 7) and of finite type (cf. Section 2). However, as shown in [19, Counterexample 3.13], the converse fails to hold: strong irreducibility and finite type conditions do not imply, in general, bounded propagation. As a consequence, the following theorem (cf. [19, Theorem B]) improves on Gromov's theorem.

Let $\mathcal{G} = (V, E)$ be an amenable simplicial connected graph of bounded degree admitting a dense pseudogroup of partial isometries and let A be a finite or countably infinite alphabet set. Suppose that $X, Y \subset A^V$ are strongly irreducible stable spaces of finite type with the same entropy. Let $\tau: X \rightarrow Y$ be a map of bounded propagation admitting a dense holonomy. Then τ is surjective if and only if it is pre-injective.

Note that this last result also covers Theorem 7.5.

9.4. Cellular automata over homogeneous sets. Cellular automata where the universe is a set endowed with a transitive group action have been investigated by Moriceau [40]. Versions of the Curtis-Hedlund-Lyndon theorem and of the Garden of Eden theorem in this more general setting have been obtained by Wacker [53], [54].

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ADDED IN PROOF

Recently, after our preprint [18] had circulated, Hanfeng Li posted his paper [60] containing an impressive Garden of Eden type theorem generalizing several results mentioned above in our survey. The key notion in Li's paper is that of *specification*, a strong orbit tracing property which was introduced by Rufus Bowen for \mathbb{Z} -actions in relation to his studies on Axiom A diffeomorphisms in [57] (see also [59, Definition 21.1]) and was extended to \mathbb{Z}^d -actions by Ruelle in [61]. Several versions and generalizations of specification have appeared in the literature (see, in particular, [33, Definition 5.1] and [58, Definition 6.1]). Here is the one we need (cf. [58, Definition 6.1]).

Definition 9.1. A dynamical system (X, G) has the *weak specification property* if for any $\varepsilon > 0$ there exists a nonempty symmetric finite subset $\Delta \subset G$ satisfying the following property: if $(\Omega_i)_{i \in I}$ is any finite family of finite subsets of G such that $\Delta\Omega_i \cap \Omega_j = \emptyset$ for all distinct $i, j \in I$, and $(x_i)_{i \in I}$ is any family of points in X , then there exists $x \in X$ such that

$$d(sx, sx_i) \leq \varepsilon \quad \text{for all } i \in I \text{ and } s \in \Omega_i,$$

where d is any metric compatible with the topology on X .

It is straightforward (cf. [60, Proposition A.1]) to check that if G is a countable group, A is a finite alphabet set, and $X \subset A^G$ is a subshift, then the shift dynamical system (X, G) has the weak

specification property if and only if it is strongly irreducible (cf. Section 7). Also, it is easy to see that the weak specification property passes to factors.

Li [60, Theorem 1.1] proved the following:

Theorem 9.2. *Let (X, G) be a dynamical system. Suppose that the group G is amenable and that (X, G) is expansive and has the weak specification property. Then (X, G) has the Myhill property.*

Note that Theorem 9.2 covers Theorem 8.14, by virtue of the remarks following Definition 9.1.

Recall (cf. Example 7.9) that if $X \subset \{0, 1\}^{\mathbb{Z}}$ denotes the even subshift (cf. Example 2.4), then (X, \mathbb{Z}) is expansive, has the weak specification property (since X is strongly irreducible), but does not have the Moore property. This shows that from the hypotheses of Theorem 9.2 one cannot deduce the Moore property, in general. However, Li [60, Theorem 1.2] proved that when restricting to the class of algebraic dynamical systems (cf. Section 8.6) with amenable acting group, the Moore property follows from expansiveness and weak specification:

Theorem 9.3. *Let (X, G) be an algebraic dynamical system. Suppose that the group G is amenable and that (X, G) is expansive and has the weak specification property. Then (X, G) has the Moore property.*

As an immediate consequence of Theorems 9.2 and 9.3, one deduces the following (cf. [60, Theorem 1.3]):

Corollary 9.4. *Let (X, G) be an algebraic dynamical system. Suppose that the group G is amenable and that (X, G) is expansive and has the weak specification property. Then (X, G) has the Moore-Myhill property.*

Moreover, since (cf. [60, Lemma 2.1], see also [58, Theorem 1.2]) every expansive principal algebraic action has the weak specification property, one immediately deduces (cf. [60, Theorem 1.3]):

Corollary 9.5. *Let (X, G) be a principal algebraic dynamical system. Suppose that the group G is amenable and that (X, G) is expansive. Then (X, G) has the Moore-Myhill property.*

Note that Corollary 9.5 covers Theorem 8.19 (since every abelian group is amenable and no assumption on connectedness of the phase space is required).

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