OUTERSTRING GRAPHS ARE χ-BOUNDED

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Abstract. An outerstring graph is an intersection graph of curves that lie in a common half-plane and have one endpoint on the boundary of that half-plane. We prove that the class of outerstring graphs is χ-bounded, which means that their chromatic number is bounded by a function of their clique number. This generalizes a series of previous results on χ-boundedness of outerstring graphs with various additional restrictions on the shape of curves or the number of times the pairs of curves can cross. The assumption that each curve has an endpoint on the boundary of the half-plane is justified by the known fact that triangle-free intersection graphs of straight-line segments can have arbitrarily large chromatic number.

1. Introduction

The *intersection graph* of a family of sets $\mathcal{F}$ is the graph with vertex set $\mathcal{F}$ and edge set comprised of the pairs of members of $\mathcal{F}$ that intersect. A *curve* is a homeomorphic image of the real interval $[0,1]$ in the plane. We consider finite families $\mathcal{F}$ of curves in a closed half-plane such that each curve $c \in \mathcal{F}$ has exactly one point on the boundary of the half-plane and that point is an endpoint of c. Such families of curves are called *grounded*, and their intersection graphs are known as *outerstring graphs*.[1]

For a graph G, we let $\chi(G)$ denote the chromatic number of G and $\omega(G)$ denote the clique number of G (the maximum size of a clique in G). A class of graphs $\mathcal{G}$ is *χ-bounded* if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leqslant f(\omega(G))$.

The main result of this paper is that the class of outerstring graphs is χ-bounded. Specifically, we establish the following bound.

Theorem. *Every outerstring graph G satisfies $\chi(G) \leqslant f(\omega(G))$ where $f(\omega) = 2^{O(2^{\omega(\omega-1)/2})}$.*

Outerstring graphs are special instances of *string graphs*—intersection graphs of arbitrary curves in the plane. It is known, however, that the class of string graphs is not χ-bounded [31].

Related work: coloring geometric intersection graphs. The study of chromatic number of intersection graphs of geometric objects in the plane was initiated in 1960 by Asplund and Grünbaum [3], who proved that intersection graphs of axis-parallel rectangles in the plane satisfy $\chi \leqslant 4\omega^2 - 3\omega$. This bound was later improved to $\chi \leqslant 3\omega^2 - 2\omega - 1$ by Hendler [15]. The best known lower bound construction of rectangle graphs, claimed by Kostochka [18], achieves $\chi = 3\omega$.

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[1] Alternatively, outerstring graphs are often defined as intersection graphs of curves in a closed disc with one endpoint (and no other points) on the boundary of that disc. The two definitions are equivalent up to a homeomorphism that transforms a closed half-plane to a closed disc without one boundary point.

Gyárfás [14] proved that circle graphs (intersection graphs of chords of a circle) satisfy $\chi \leqslant 2^\omega\omega^2(2^\omega - 2)$. This bound was later improved to $\chi \leqslant 2^\omega\omega(\omega + 2)$ by Kostochka [17], to $\chi \leqslant 50 \cdot 2^\omega - 32\omega - 64$ by Kostochka and Kratochvíl [19] (for the more general class of intersection graphs of polygons inscribed in a circle), and finally to $\chi \leqslant 21 \cdot 2^\omega - 24\omega - 24$ by Černý [6]. Using the probabilistic method, Kostochka [17, 18] proved that there exist circle graphs satisfying $\chi \geqslant \frac{1}{2}\omega \ln \omega - \omega$, and this is the best lower bound known to date.

Motivated by practical applications to channel assignment, Peeters [32] showed that intersection graphs of unit discs satisfy $\chi \leqslant 3\omega - 2$, whereas Malesińska, Piskorz, and Weißenfels [24] proved $\chi \leqslant 6\omega - 6$ for intersection graphs of discs of arbitrary sizes. More generally, Kim, Kostochka, and Nakprasit [16] showed that intersection graphs of homothets (uniformly scaled and translated copies) of a fixed convex compact set in the plane satisfy $\chi \leqslant 6\omega - 6$ while intersection graphs of translates of such a set satisfy $\chi \leqslant 3\omega - 2$. These results actually show a property stronger than χ-boundedness, namely, that average degree is bounded by a function of ω. The strongest such result on geometric intersection graphs, due to Fox and Pach [8], asserts that string graphs excluding a fixed bipartite graph as a subgraph have bounded average degree.

McGuinness [26] proved that intersection graphs of L-shapes[2] intersecting a common horizontal line satisfy $\chi = 2^{O(4^\omega)}$. He also proved in [27] that triangle-free intersection graphs of simple[3] grounded[4] families of compact arc-connected sets have bounded chromatic number. Suk [34] considered simple[5] families of x-monotone[6] curves grounded in a half-plane bounded by a vertical line; he proved that intersection graphs of such families satisfy $\chi = 2^{O(5^\omega)}$. Lasoń, Micek, Pawlik, and Walczak [22] generalized both these results, showing that intersection graphs of simple grounded families of compact arc-connected sets satisfy $\chi = 2^{O(2^\omega)}$. Our present result can be considered as a generalization of all the χ-boundedness results mentioned in this paragraph to grounded families of curves with no restriction on the number of pairwise intersections.

Krawczyk and Walczak [21] considered colorings of interval filament graphs (intersection graphs of continuous non-negative functions defined on closed intervals with zero-values at their endpoints), which form a subclass of the class of outerstring graphs. In particular, they constructed interval filament graphs with $\chi = \binom{\omega+1}{2}$. This seems to be the best lower bound construction known to date for outerstring graphs.

On the negative side, Burling [5] constructed triangle-free intersection graphs of axis-parallel boxes in $\mathbb{R}^3$ with arbitrarily large chromatic number. Using essentially the same construction, Pawlik et al. [30, 31] proved existence of triangle-free intersection graphs of line segments (and various other kinds of geometric shapes) in the plane with arbitrarily large chromatic number. These constructions show that some restriction on the layout of the geometric objects considered, like the one that the family is grounded, is indeed necessary to guarantee χ-boundedness.

The previous best upper bound on the chromatic number of outerstring graphs was of order $(\log n)^{O(\log \omega)}$, where n denotes the number of vertices. It was proved explicitly by Fox and Pach [11] for general string graphs, using a separator theorem due to Matoušek [25], but it also follows from an earlier separator theorem for outerstring graphs [9, Theorem 3.1] by the method already described in [8]. The above-mentioned result of Suk [34] implies that intersection graphs of

[2] An *L-shape* consists of a horizontal and a vertical segment joined to form the letter L.

[3] A family of compact arc-connected sets is *simple* if the intersection of any subset of them is arc-connected (possibly empty).

[4] A family of compact arc-connected sets is *grounded* if the sets are contained in a half-plane and the intersection of each of them with the boundary of the half-plane is a non-empty segment.

[5] A family of curves is *simple* if any two of them intersect in at most one point.

[6] A curve is *x-monotone* if it meets every vertical line in at most one point.

simple families of x-monotone curves have chromatic number $O_\omega(\log n)$.[7] On the other hand, the above-mentioned construction of Pawlik et al. [31] produces triangle-free segment intersection graphs with chromatic number $\Theta(\log\log n)$. Krawczyk and Walczak [21] further generalized it to a construction of string graphs with chromatic number $\Theta_\omega((\log\log n)^{\omega-1})$. It is possible that all string graphs have chromatic number of order $(\log\log n)^{f(\omega)}$ for some function f. So far, bounds of this kind have been established only for very special (still not χ-bounded) classes of string graphs [20, 21].

Related work: quasi-planarity. A *topological graph* is a graph drawn in the plane so that each vertex is a point and each edge is a curve connecting the two endpoints of that edge and avoiding all other vertices. Such a graph is *k-quasi-planar* if it has no k mutually crossing edges (where a common endpoint is not considered as a crossing). In particular, 2-quasi-planar graphs are just planar graphs. Any bound on the chromatic number of intersection graphs of curves with clique number less than k implies a bound on the number of edges in k-quasi-planar graphs, as follows. Given a k-quasi-planar topological graph G, if we shorten the edges a little at their endpoints so as to keep all crossings, then we obtain a family of curves with no clique of size k. A proper coloring of these curves with c colors yields an edge-decomposition of G into c planar graphs, whence the bound of $O(cn)$ on the number of edges of G follows.

A well-known conjecture (see e.g. [29] and [4, Problem 1 in Section 9.6]) asserts that k-quasi-planar topological graphs on n vertices have $O_k(n)$ edges. It was proved for 3-quasi-planar simple[8] topological graphs by Agarwal et al. [2], for all 3-quasi-planar topological graphs by Pach, Radoičić, and Tóth [28], and for 4-quasi-planar topological graphs by Ackerman [1]. Valtr [36] proved the bound of $O_k(n\log n)$ on the number of edges in k-quasi-planar simple topological graphs with edges drawn as x-monotone curves. An improvement of this result due to Fox, Pach, and Suk [12] provides the same conclusion without the simplicity assumption. They also proved the bound of $2^{\alpha(n)^c}n\log n$ on the number of edges in k-quasi-planar simple topological graphs, where α denotes the inverse Ackermann function and c depends only on k. Suk and Walczak [35] proved the same bound for k-quasi-planar topological graphs in which any two edges cross a bounded number of times, and they improved the bound for k-quasi-planar simple topological graphs to $O_k(n\log n)$. The best known general upper bound on the number of edges in k-quasi-planar topological graphs, due to Fox and Pach [9, 11], is $n(\log n)^{O(\log k)}$.

Corollaries and follow-up generalizations. An immediate consequence of our present result is that the class of intersection graphs of grounded families of compact arc-connected sets in the plane is χ-bounded, because each such set can be approximated by a grounded curve. This generalizes the result of Lasoń et al. [22] on simple grounded families of such sets. Another consequence is that the class of intersection graphs of families of curves each intersecting a fixed straight line in exactly one point is χ-bounded. Indeed, we can color the parts of curves lying on each side of the line independently, and then we can color the entire curves using the pairs of colors obtained on the two sides. This and a standard divide-and-conquer argument imply that intersection graphs of x-monotone curves have chromatic number $O_\omega(\log n)$, which yields an alternative proof of the result of Fox, Pach, and Suk [12] that k-quasi-planar topological graphs in which every edge is drawn as an x-monotone curve have $O_k(n\log n)$ edges.

By the same argument as is used in [35] for simple families of curves, our present result has the following corollary: the class of intersection graphs of curves all of which intersect a

[7] We write O_ω and Θ_ω to denote asymptotic growth rate with respect to n with ω fixed as a constant.
[8] A topological graph is *simple* if any two of its edges intersect in at most one point.

fixed curve c_0 in exactly one point is χ-bounded. In the follow-up paper [33], we prove the following generalization: for every integer $t \geqslant 1$, the class of intersection graphs of curves all of which intersect a fixed curve c_0 in at least one and at most t points is χ-bounded. Using that generalization, by the argument from [35], we obtain the bound of $O_{k,t}(n \log n)$ on the number of edges of k-quasi-planar topological graphs in which any two edges cross at most t times. The aforementioned generalization depends on the result of the current paper, which serves as the base case for induction. We refer the reader to [33] for more details.

2. Preliminaries

We let H^+ denote the upper closed half-plane determined by the horizontal axis. We call the horizontal axis the *baseline*. We can assume, without loss of generality, that H^+ is the underlying half-plane of any grounded family of curves that we consider. Accordingly, we call a family of curves *grounded* if every curve in the family has one endpoint on the baseline and all the remaining part above the baseline, and we call a curve with this property itself *grounded*. The *basepoint* of a grounded curve is the endpoint that lies on the baseline. We can assume, without loss of generality, that the basepoints of all curves in any grounded family that we consider are distinct. For if b is the common basepoint of several curves in a grounded family $\mathcal{F}$, then a sufficiently small neighborhood of b is disjoint from the other curves in $\mathcal{F}$ (because curves are compact sets and $\mathcal{F}$ is finite), and an appropriate perturbation of the curves within that neighborhood of b makes their basepoints distinct while keeping the curves pairwise intersecting.

A *proper coloring* of a grounded family of curves $\mathcal{F}$ is an assignment of colors to the curves in $\mathcal{F}$ such that no two intersecting curves receive the same color. A *clique* in $\mathcal{F}$ is a subfamily of $\mathcal{F}$ comprised of pairwise intersecting curves. We let $\chi(\mathcal{F})$ and $\omega(\mathcal{F})$ denote the minimum number of colors in a proper coloring of $\mathcal{F}$ and the maximum size of a clique in $\mathcal{F}$, respectively. Thus $\chi(\mathcal{F})$ is the chromatic number and $\omega(\mathcal{F})$ is the clique number of the intersection graph of $\mathcal{F}$.

For any two grounded curves c_1 and c_2, we let $c_1 \prec c_2$ denote that the basepoint of c_1 lies to the left of the basepoint of c_2. In view of the assumption above, $\prec$ is a total order on any grounded family of curves that we consider—the left-to-right order of the basepoints. The notation $\prec$ naturally extends to families of grounded curves: $\mathcal{F}_1 \prec \mathcal{F}_2$ denotes that $c_1 \prec c_2$ for all $c_1 \in \mathcal{F}_1$ and $c_2 \in \mathcal{F}_2$. For any two grounded curves c_1 and c_2 with $c_1 \prec c_2$ and any grounded family of curves $\mathcal{F}$, we let $\mathcal{F}(c_1, c_2) = \{c \in \mathcal{F} \colon c_1 \prec c \prec c_2\}$.

The following lemma is essentially due to McGuinness [26, Lemma 2.1]. Here, we adapt it to our setting and include its short proof for the reader's convenience.

Lemma 1. *If $\mathcal{F}$ is a grounded family of curves with $\chi(\mathcal{F}) > 2\alpha(\beta+1)$, where $\alpha, \beta \geqslant 0$, then there is a subfamily $\mathcal{H} \subseteq \mathcal{F}$ such that $\chi(\mathcal{H}) > \alpha$ and $\chi(\mathcal{F}(u,v)) > \beta$ for any two intersecting curves $u, v \in \mathcal{H}$ with $u \prec v$.*

Proof. Partition $\mathcal{F}$ into subfamilies $\mathcal{F}_0 \prec \cdots \prec \mathcal{F}_n$ so that $\chi(\mathcal{F}_i) = \beta + 1$ for $0 \leqslant i < n$ and $\chi(\mathcal{F}_n) \leqslant \beta + 1$. This is done greedily, by processing the curves in $\mathcal{F}$ in the order $\prec$, adding them to $\mathcal{F}_0$ until $\chi(\mathcal{F}_0) = \beta + 1$, then adding them to $\mathcal{F}_1$ until $\chi(\mathcal{F}_1) = \beta + 1$, and so on. For $0 \leqslant i \leqslant n$, a proper $(\beta+1)$-coloring of $\mathcal{F}_i$ yields a partition of $\mathcal{F}_i$ into color classes $\mathcal{F}_i^1, \ldots, \mathcal{F}_i^{\beta+1}$ each comprised of pairwise disjoint curves. Let $r \in \{1, \ldots, \beta+1\}$ be such that $\chi(\bigcup_{i=0}^n \mathcal{F}_i^r)$ is maximized. It follows that $\chi(\bigcup_{i=0}^n \mathcal{F}_i^r) \geqslant \chi(\mathcal{F})/(\beta+1) > 2\alpha$ and thus $\chi(\bigcup_{i \text{ even}} \mathcal{F}_i^r) > \alpha$ or $\chi(\bigcup_{i \text{ odd}} \mathcal{F}_i^r) > \alpha$. Let $\mathcal{H} = \bigcup_{i \text{ even}} \mathcal{F}_i^r$ or $\mathcal{H} = \bigcup_{i \text{ odd}} \mathcal{F}_i^r$ accordingly, so that $\chi(\mathcal{H}) > \alpha$. Now, if two curves $u, v \in \mathcal{H}$ with $u \prec v$ intersect, then $u \in \mathcal{F}_k^r$ and $v \in \mathcal{F}_\ell^r$ for two indices $k, \ell \in \{0, \ldots, n\}$ with $k < \ell$ (because no two curves in any $\mathcal{F}_i^r$ intersect) of the same parity, and therefore $\mathcal{F}_i$ is contained in $\mathcal{F}(u,v)$ for every (at least one) index $i \in \{k+1, \ldots, \ell-1\}$, witnessing $\chi(\mathcal{F}(u,v)) > \beta$. $\square$

A *cap-curve* is a curve in H^+ that has both endpoints on the baseline and does not intersect the baseline in any other point. It follows from the Jordan curve theorem that for every cap-curve γ, the set $H^+ \smallsetminus \gamma$ consists of two arc-connected components, one of which is bounded and denoted by $\operatorname{int}\gamma$ and the other is unbounded and denoted by $\operatorname{ext}\gamma$. A point of the baseline belongs to $\operatorname{int}\gamma$ if and only if it lies strictly between the two endpoints of γ.

Most coloring arguments for geometric intersection graphs make essential use of the idea, originally due to Gyárfás [14], to use distance levels to guarantee that each object to be colored lies within some restricted region and has a neighbor that crosses the boundary of that region. The following lemma adapts this idea to our setting.

Lemma 2. *For every grounded family of curves $\mathcal{F}$ with $\omega(\mathcal{F}) \geqslant 2$, there are a cap-curve γ and a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $\chi(\mathcal{G}) \geqslant \chi(\mathcal{F})/2$ such that every curve in $\mathcal{G}$ is entirely contained in* $\operatorname{int}\gamma$ *and intersects some curve in $\mathcal{F}$ that intersects γ.*

Proof. Since the chromatic number of a graph is the maximum of the chromatic numbers of its connected components, we can assume without loss of generality that the intersection graph of $\mathcal{F}$ is connected (otherwise we can restrict $\mathcal{F}$ to the component with maximum chromatic number).

Let c_0 be the curve in $\mathcal{F}$ with leftmost basepoint. For $i \geqslant 0$, let $\mathcal{F}_i$ denote the family of curves in $\mathcal{F}$ that are at distance i from c_0 in the intersection graph of $\mathcal{F}$. It follows that $\mathcal{F}_0 = \{c_0\}$ and every curve in $\mathcal{F}_i$ is disjoint from every curve in $\mathcal{F}_j$ whenever $|i-j| \geqslant 2$. Since $\bigcup_{i=0}^{\infty} \mathcal{F}_i = \mathcal{F}$, we have $\chi(\bigcup_{i \text{ even}} \mathcal{F}_i) \geqslant \chi(\mathcal{F})/2$ or $\chi(\bigcup_{i \text{ odd}} \mathcal{F}_i) \geqslant \chi(\mathcal{F})/2$, and therefore there is an index $d \geqslant 1$ such that $\chi(\mathcal{F}_d) \geqslant \chi(\mathcal{F})/2$. There is a subfamily $\mathcal{G} \subseteq \mathcal{F}_d$ such that $\chi(\mathcal{G}) = \chi(\mathcal{F}_d) \geqslant \chi(\mathcal{F})/2$ and the intersection graph of $\mathcal{G}$ is connected. The latter implies that the union of the curves in $\mathcal{G}$ contains a cap-curve ν connecting the leftmost and the rightmost basepoints of curves in $\mathcal{G}$.

Let b_0 be the basepoint of c_0. Let $E = \{b_0\}$ when $d = 1$ and E be the union of the curves in $\bigcup_{i=0}^{d-2} \mathcal{F}_i$ when $d \geqslant 2$ (in particular, $b_0 \in E$). Let G be the union of the curves in $\mathcal{G}$. Since every curve in $\bigcup_{i=0}^{d-2} \mathcal{F}_i$ is disjoint from every curve in $\mathcal{G}$ and the intersection graphs of both families are connected, the sets E and G are disjoint arc-connected subsets of H^+. Furthermore, the set E lies in the unbounded arc-connected component of $H^+ \smallsetminus G$, because so does the point b_0. Therefore, there is a cap-curve γ separating E and G in H^+ so that $G \subset \operatorname{int}\gamma$ and $E \subset \operatorname{ext}\gamma$.

For every curve $c \in \mathcal{G}$, since $c \in \mathcal{F}_d$, there is a curve $s \in \mathcal{F}_{d-1}$ that intersects c. Since $s \in \mathcal{F}_{d-1}$, either $s = c_0$ (when $d = 1$) or s intersects some curve in $\mathcal{F}_{d-2}$ (when $d \geqslant 2$). In either case, s intersects both E and G. This, $E \subset \operatorname{ext}\gamma$, and $G \subset \operatorname{int}\gamma$ imply that s intersects γ. We conclude that every curve in $\mathcal{G}$ intersects some curve in $\mathcal{F}$ that intersects γ. $\square$

Complements of comparability graphs have equivalent representations as intersection graphs of appropriately restricted families of curves [13, 23]. The next lemma combines a variant of this result (essentially [10, Lemma 4.2]) with the fact (a consequence of Dilworth's theorem [7]) that complements of comparability graphs are perfect.

Lemma 3. *If γ is a cap-curve and $\mathcal{U}$ is a grounded family of curves each having one endpoint (other than the basepoint) on γ and all the remaining part in* $\operatorname{int}\gamma$*, then $\chi(\mathcal{U}) = \omega(\mathcal{U})$.*

Proof. Consider a binary relation $<$ on $\mathcal{U}$ defined as follows, for any $u_1, u_2 \in \mathcal{U}$:

$$u_1 < u_2 \quad \text{if and only if} \quad u_1 \prec u_2 \quad \text{and} \quad u_1 \cap u_2 = \emptyset.$$

We claim that $<$ is a strict partial order on $\mathcal{U}$. Clearly, it is an irreflexive and antisymmetric relation. For transitivity, suppose $u_1 < u_2$ and $u_2 < u_3$ but not $u_1 < u_3$, for some $u_1, u_2, u_3 \in \mathcal{U}$. Since $u_1 \prec u_2 \prec u_3$, it follows that $u_1 \cap u_3 \neq \emptyset$. Therefore, there is a cap-curve $\nu \subseteq u_1 \cup u_3$ that connects the basepoints of u_1 and u_3. The basepoint of u_2 lies in $\operatorname{int}\nu$, while the other endpoint of u_2 lies in $\operatorname{ext}\nu$, as it lies on γ and $\nu \subset \operatorname{int}\gamma$. It follows that u_2 intersects ν. This yields $u_1 \cap u_2 \neq \emptyset$ or $u_2 \cap u_3 \neq \emptyset$, either of which is a contradiction.

Since the antichains of the order $<$ on $\mathcal{U}$ are the cliques in the intersection graph of $\mathcal{U}$, the width of $<$ is $\omega(\mathcal{U})$. Therefore, by Dilworth's theorem, the family $\mathcal{U}$ can be partitioned into $\omega(\mathcal{U})$ chains with respect to $<$. Such a partition is equivalent to a proper $\omega(\mathcal{U})$-coloring of $\mathcal{U}$. $\square$

3. Proof

Setup. Here is our main theorem in the form that we are going to prove.

Theorem (rephrased)**.** *There is a function $f\colon \mathbb{N} \to \mathbb{N}$ with $f(k) = 2^{O(2^{k(k-1)/2})}$ such that for every $k \in \mathbb{N}$, every grounded family of curves $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ satisfies $\chi(\mathcal{F}) \leqslant f(k)$.*

The proof proceeds by induction on k. We let $f(1) = 1$, which clearly satisfies the conclusion of the theorem for $k = 1$. For $k \geqslant 2$, we assume that $f(k-1)$ is an upper bound on the chromatic number of grounded families with clique number at most $k-1$, and we use it to derive an upper bound $f(k)$ on the chromatic number of grounded families of curves with clique number at most k. The induction hypothesis is only applied as follows: if $\mathcal{F}$ is a grounded family of curves with $\omega(\mathcal{F}) \leqslant k$ and $c \in \mathcal{F}$, then the family of curves in $\mathcal{F} \smallsetminus \{c\}$ that intersect c (that is, the neighborhood of c in the intersection graph of $\mathcal{F}$) has clique number at most $k-1$ and therefore has chromatic number at most $f(k-1)$.

This motivates the following definition: for $\xi \in \mathbb{N}$, a *ξ-family* is a grounded family of curves $\mathcal{F}$ such that for every curve $c \in \mathcal{F}$, the family of curves in $\mathcal{F} \smallsetminus \{c\}$ that intersect c has chromatic number at most ξ. In the rest of the paper, we prove the following lemma.

Lemma 4. *There is a constant $\zeta = 2^{O(k2^{2k})}\xi^{2^{k-1}-1}$ such that every ξ-family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ satisfies $\chi(\mathcal{F}) \leqslant \zeta$.*

Here and through the rest of the paper, for clarity of presentation, we treat k and ξ as fixed integer constants (except in statements on asymptotic growth rate) such that $k \geqslant 2$ and $\xi \geqslant 1$.

To complete the induction step in the proof of our main theorem, we fix $\xi = f(k-1)$, and we let $f(k)$ be the constant ζ claimed by Lemma 4 for k and ξ. By the induction hypothesis, every grounded family of curves with clique number at most k is an $f(k-1)$-family and therefore, by Lemma 4, has chromatic number at most $f(k)$.

To see that the bound on ζ from Lemma 4 implies the bound on f from our main theorem, suppose $k \geqslant 2$ and $\log_2 f(k-1) \leqslant C \cdot 2^{(k-1)(k-2)/2}$ for some large constant C. Lemma 4 applied with $\xi = f(k-1)$ then provides the following bound:

$$\begin{aligned}\log_2 \chi(\mathcal{F}) &\leqslant C \cdot 2^{(k-1)(k-2)/2} \cdot (2^{k-1} - 1) + O(k2^{2k}) \\ &= C \cdot 2^{k(k-1)/2} - C \cdot 2^{(k-1)(k-2)/2} + O(k2^{2k}) \leqslant C \cdot 2^{k(k-1)/2},\end{aligned}$$

where the last inequality holds when C is large enough. This completes the proof of our main theorem provided that we have Lemma 4. It remains to prove the lemma.

Here is an overview of the proof of Lemma 4. Assuming that $\mathcal{F}$ is a ξ-family with clique number at most k and sufficiently large chromatic number, we show that some specific configurations can be found in $\mathcal{F}$. Our goal is to reach a contradiction by finding a clique of size $k+1$ in $\mathcal{F}$.

First, we show that every ξ-family with clique number at most k and sufficiently large chromatic number contains a subfamily with large chromatic number *supported by a skeleton* or *supported from outside*. Iterating this step on the successive subfamilies, we construct a long chain of subfamilies supported by skeletons or supported from outside. Then, the proof splits into two parts. In the first part, we show that a long chain of subfamilies supported from outside contains a long *bracket system*, which then contains a large clique. In the second part, we show that a long chain of subfamilies supported by skeletons contains a long *tree-configuration*, which then contains a large clique.

Skeletons. A *skeleton* is a pair $(\gamma,\mathcal{U})$ such that γ is a cap-curve and $\mathcal{U}$ is a family of pairwise disjoint grounded curves each having one endpoint (other than the basepoint) on γ and all the remaining part in $\operatorname{int}\gamma$. For a grounded family of curves $\mathcal{F}$, a skeleton $(\gamma,\mathcal{U})$ is an *$\mathcal{F}$-skeleton* if every curve in $\mathcal{U}$ is a subcurve of some curve in $\mathcal{F}$. A grounded family of curves $\mathcal{G}$ is *supported* by a skeleton $(\gamma,\mathcal{U})$ if every curve in $\mathcal{G}$ lies entirely in $\operatorname{int}\gamma$ and intersects some curve in $\mathcal{U}$. A grounded family of curves $\mathcal{H}$ is *supported from outside* by a grounded family of curves $\mathcal{S}$ if every curve in $\mathcal{H}$ intersects some curve in $\mathcal{S}$ and every curve $s \in \mathcal{S}$ satisfies $s \prec \mathcal{H}$ or $\mathcal{H} \prec s$.

Lemma 5. *For any $\alpha,\beta \in \mathbb{N}$, every ξ-family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ and $\chi(\mathcal{F}) > 2(k\alpha+\beta)$ contains at least one of the following configurations:*

- *a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $\chi(\mathcal{G}) > \alpha$ supported by an $\mathcal{F}$-skeleton,*
- *a subfamily $\mathcal{H} \subseteq \mathcal{F}$ with $\chi(\mathcal{H}) > \beta$ supported from outside by another subfamily of $\mathcal{F}$.*

Proof. Apply Lemma 2 to obtain a cap-curve γ and a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $\chi(\mathcal{G}) > k\alpha+\beta$ such that every curve in $\mathcal{G}$ lies entirely in $\operatorname{int}\gamma$ and intersects some curve in $\mathcal{S}$, where $\mathcal{S}$ are the curves in $\mathcal{F}$ that intersect γ. Let $\mathcal{U}$ be the grounded subcurves of the curves in $\mathcal{S}$ with one endpoint (other than the basepoint) on γ and all the remaining part in $\operatorname{int}\gamma$. Lemma 3 yields $\chi(\mathcal{U}) = \omega(\mathcal{U}) \leqslant k$. A proper k-coloring partitions $\mathcal{U}$ into families $\mathcal{U}_1,\ldots,\mathcal{U}_k$ each consisting of pairwise disjoint curves. It follows that $(\gamma,\mathcal{U}_1),\ldots,(\gamma,\mathcal{U}_k)$ are $\mathcal{F}$-skeletons.

For $1 \leqslant i \leqslant k$, let $\mathcal{G}_i$ be the curves in $\mathcal{G}$ that intersect some curve in $\mathcal{U}_i$, so that the family $\mathcal{G}_i$ is supported by the skeleton $(\gamma,\mathcal{U}_i)$. If $\chi(\mathcal{G}_i) > \alpha$, then $\mathcal{G}_i$ and $(\gamma,\mathcal{U}_i)$ satisfy the first condition in the conclusion of the lemma, so suppose $\chi(\mathcal{G}_i) \leqslant \alpha$, for $1 \leqslant i \leqslant k$. Let $\mathcal{G}' = \mathcal{G} \smallsetminus (\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_k)$. It follows that $\chi(\mathcal{G}') \geqslant \chi(\mathcal{G}) - k\alpha > \beta$.

Since the chromatic number of a graph is the maximum of the chromatic numbers of its connected components, there is a subfamily $\mathcal{H} \subseteq \mathcal{G}'$ such that $\chi(\mathcal{H}) = \chi(\mathcal{G}') > \beta$ and the intersection graph of $\mathcal{H}$ is connected. The latter implies that the union of the curves in $\mathcal{H}$ contains a cap-curve ν connecting the leftmost and the rightmost basepoints of curves in $\mathcal{H}$. Suppose there is a curve $s \in \mathcal{S}$ with basepoint between the leftmost and the rightmost basepoints of curves in $\mathcal{H}$. The part of s from the basepoint to the first intersection point with γ is a curve in some $\mathcal{U}_i$ $(1 \leqslant i \leqslant k)$ that must intersect ν (as $\nu \subset \operatorname{int}\gamma$) and thus some curve in $\mathcal{G}'$ (as ν lies in the union of curves in $\mathcal{G}'$). This yields $\mathcal{G}' \cap \mathcal{G}_i \neq \emptyset$, which is a contradiction. Thus $s \prec \mathcal{H}$ or $\mathcal{H} \prec s$. We conclude that $\mathcal{H}$ and $\mathcal{S}$ satisfy the second condition in the conclusion of the lemma. □

Brackets and bracket systems. A *bracket* is a pair $(\mathcal{H},\mathcal{S})$ of families of grounded curves with the following properties:

- $\mathcal{H} \prec \mathcal{S}$ or $\mathcal{S} \prec \mathcal{H}$,
- every curve in $\mathcal{H}$ intersects some curve in $\mathcal{S}$.

See Figure 1 for an illustration. For a bracket $(\mathcal{H},\mathcal{S})$ and a curve $c \in \mathcal{H}$, we let

- $p(c)$ denote the first intersection point of c with a curve in $\mathcal{S}$ as going from the basepoint of c,
- $s(c)$ denote an arbitrarily chosen curve in $\mathcal{S}$ that contains the point $p(c)$,
- c' denote the part of c from the basepoint to $p(c)$ excluding the point $p(c)$,
- $\nu(c)$ be the cap-curve formed by the union of c' and the part of $s(c)$ from the basepoint to $p(c)$,
- $I(c)$ be the closed bounded region determined by $\nu(c)$ and the part of the baseline between the two endpoints of $\nu(c)$.

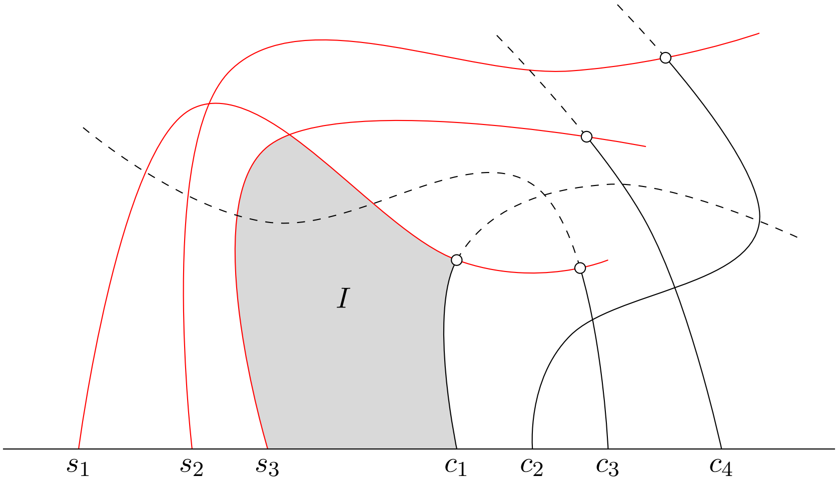

FIGURE 1. A bracket $(\{c_1, c_2, c_3, c_4\}, \{s_1, s_2, s_3\})$ with internal region I and with $s_1 = s(c_1) = s(c_3)$, $s_2 = s(c_2)$, $s_3 = s(c_4)$. The part c_i' of each c_i is drawn solid.

It follows that c' is disjoint from every curve in $\mathcal{S}$, for every $c \in \mathcal{H}$. Finally, we let $I = \bigcap_{c\in\mathcal{H}} I(c)$, and we call I the *internal region* of the bracket $(\mathcal{H}, \mathcal{S})$.

Lemma 6. *In the setting above, there are two cliques $\mathcal{H}_I \subseteq \mathcal{H}$ and $\mathcal{S}_I \subseteq \mathcal{S}$ such that every curve with basepoint in I lies entirely in I or intersects at least one of the curves in $\mathcal{H}_I \cup \mathcal{S}_I$.*

Proof. Assume $\mathcal{S} \prec \mathcal{H}$; the case that $\mathcal{H} \prec \mathcal{S}$ is analogous. Let $\mathcal{H}_I = \{c \in \mathcal{H} \colon c' \cap I \neq \emptyset\}$ and $\mathcal{S}_I = \{s(c) \in \mathcal{S} \colon c \in \mathcal{H} \text{ and } s(c) \cap I \neq \emptyset\}$. To see that $\mathcal{H}_I$ is a clique, suppose there are two disjoint curves $c_1, c_2 \in \mathcal{H}_I$ with $c_1 \prec c_2$. It follows that c_2' is disjoint from c_1' and $s(c_1)$. This and $s(c_1) \prec c_1 \prec c_2$ imply that c_2' is disjoint from $I(c_1)$. This and $I \subseteq I(c_1)$ contradict the assumption that $c_2' \cap I \neq \emptyset$. To see that $\mathcal{S}_I$ is a clique, suppose there are $c_1, c_2 \in \mathcal{H}$ such that $s(c_1)$ and $s(c_2)$ are two disjoint curves in $\mathcal{S}_I$ with $s(c_1) \prec s(c_2)$. This, the fact that $s(c_1)$ is also disjoint from c_2', and $s(c_1) \prec s(c_2) \prec c_2$ imply that $s(c_1)$ is disjoint from $I(c_2)$. This and $I \subseteq I(c_2)$ contradict the assumption that $s(c_1) \cap I \neq \emptyset$.

Let z be a curve with basepoint in I that does not lie entirely in I. It follows that $z \not\subseteq I(c)$ and therefore z intersects $\nu(c)$ for at least one curve $c \in \mathcal{H}$. Let $c \in \mathcal{H}$ be such that $\nu(c)$ is the first curve of this form intersected by z, and let p be the first intersection point of z with $\nu(c)$. This yields $p \in I$, as I is a closed set. The fact that $p \in \nu(c) \subseteq c' \cup s(c)$ implies $p \in c'$, in which case $c \in \mathcal{H}_I$, or $p \in s(c)$, in which case $s(c) \in \mathcal{S}_I$. In either case, z intersects a curve in $\mathcal{H}_I \cup \mathcal{S}_I$. $\square$

A *bracket system* is a sequence of brackets $((\mathcal{H}_0, \mathcal{S}_0), \ldots, (\mathcal{H}_n, \mathcal{S}_n))$ with internal regions $I_0, \ldots, I_n$, respectively, and with the following properties, for $0 \leqslant i \leqslant n-1$:

- every curve in $\mathcal{H}_i$ lies is entirely in $I_{i+1} \cap \cdots \cap I_n$,
- $\mathcal{S}_i \prec (\mathcal{H}_{i+1} \cup \mathcal{S}_{i+1}) \cup \cdots \cup (\mathcal{H}_n \cup \mathcal{S}_n)$ or $(\mathcal{H}_{i+1} \cup \mathcal{S}_{i+1}) \cup \cdots \cup (\mathcal{H}_n \cup \mathcal{S}_n) \prec \mathcal{S}_i$.

Lemma 7. *Let $((\mathcal{H}_0, \mathcal{S}_0), \ldots, (\mathcal{H}_n, \mathcal{S}_n))$ be a bracket system in a ξ-family $\mathcal{F}$. If $\chi(\mathcal{H}_i) > i\xi$ for $0 \leqslant i \leqslant n$, then there is a clique $\{s_0, \ldots, s_n\}$ with $s_i \in \mathcal{S}_i$ for $0 \leqslant i \leqslant n$.*

Proof. We proceed by induction on n. The assumption that $\chi(\mathcal{H}_0) > 0$ yields $\mathcal{H}_0 \neq \emptyset$ and thus $\mathcal{S}_0 \neq \emptyset$. Choose any $s_0 \in \mathcal{S}_0$. This already completes the proof for the base case of $n = 0$, as $\{s_0\}$ is a clique. Therefore, for the rest of the proof, suppose that $n \geqslant 1$ and the lemma holds for $n-1$. By the first property of a bracket system, since s_0 intersects a curve in $\mathcal{H}_0$, s_0 intersects

$I_1 \cap \dots \cap I_n$. This and the definition of the internal region of a bracket imply that s_0 intersects $I(c)$ for all $c \in \mathcal{H}_1 \cup \dots \cup \mathcal{H}_n$. This and the second property of a bracket system imply that s_0 intersects $\nu(c)$ for all $c \in \mathcal{H}_1 \cup \dots \cup \mathcal{H}_n$. For $1 \leqslant i \leqslant n$, let $\mathcal{H}'_i$ be the curves in $\mathcal{H}_i$ that do not intersect s_0, and let $\mathcal{S}'_i = \{s(c) \colon c \in \mathcal{H}'_i\}$. It follows that $\chi(\mathcal{H}_i \setminus \mathcal{H}'_i) \leqslant \xi$ (as $\mathcal{F}$ is a ξ-family) and thus $\chi(\mathcal{H}'_i) \geqslant \chi(\mathcal{H}_i) - \xi > (i-1)\xi$ for $1 \leqslant i \leqslant n$. Therefore, we can apply the induction hypothesis to the bracket system $((\mathcal{H}'_1, \mathcal{S}'_1), \dots, (\mathcal{H}'_n, \mathcal{S}'_n))$ to find a clique $\{s_1, \dots, s_n\}$ with $s_i \in \mathcal{S}'_i$ for $1 \leqslant i \leqslant n$. For every $c \in \mathcal{H}'_1 \cup \dots \cup \mathcal{H}'_n$, since s_0 intersects $\nu(c)$ but not c, it intersects $s(c)$. In particular, s_0 intersects each of $s_1, \dots, s_n$, and therefore $\{s_0, \dots, s_n\}$ is a clique. $\square$

Lemma 8. *There is a function $g \colon \mathbb{N} \to \mathbb{N}$ with $g(\alpha) = 2^{O(k)}(\xi + \alpha)$ such that for every $\alpha \in \mathbb{N}$, every ξ-family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ and $\chi(\mathcal{F}) > g(\alpha)$ contains a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $\chi(\mathcal{G}) > \alpha$ supported by an $\mathcal{F}$-skeleton.*

Proof. Fix $\alpha \in \mathbb{N}$, and let

$$\begin{aligned} \gamma_1 &= 0, & \gamma_{i+1} &= 2(\gamma_i + 2k\xi + i\xi + 1) && \text{for } 1 \leqslant i \leqslant k, \\ \beta_{k+1} &= \gamma_{k+1}, & \beta_i &= 2(k\alpha + \beta_{i+1}) && \text{for } 0 \leqslant i \leqslant k. \end{aligned}$$

Let $g(\alpha) = \beta_0$. It easily follows from the definition above that g has the required order of magnitude. It remains to prove that g has the property claimed by the lemma.

Let $\mathcal{F}$ be a ξ-family with $\omega(\mathcal{F}) \leqslant k$ and $\chi(\mathcal{F}) > \beta_0$. Suppose for the sake of contradiction that every subfamily of $\mathcal{F}$ supported by an $\mathcal{F}$-skeleton has chromatic number at most α. Let $\mathcal{F}_0 = \mathcal{F}$. Apply Lemma 5 (and the second conclusion thereof) $k+1$ times to find families $\mathcal{F}_1, \dots, \mathcal{F}_{k+1}$ with the following properties:

- $\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \dots \supseteq \mathcal{F}_{k+1}$,
- for $0 \leqslant i \leqslant k$, the family $\mathcal{F}_{i+1}$ is supported from outside by $\mathcal{F}_i$,
- for $0 \leqslant i \leqslant k+1$, we have $\chi(\mathcal{F}_i) > \beta_i$.

In particular, by the last condition, we have $\chi(\mathcal{F}_{k+1}) > \beta_{k+1} = \gamma_{k+1}$.

We claim that there are families $\mathcal{G}_1, \dots, \mathcal{G}_k$ and brackets $(\mathcal{H}_0, \mathcal{S}_0), \dots, (\mathcal{H}_k, \mathcal{S}_k)$ with internal regions $I_0, \dots, I_k$, respectively, and with the following properties:

- $\mathcal{G}_1 \subseteq \dots \subseteq \mathcal{G}_k \subseteq \mathcal{G}_{k+1} = \mathcal{F}_{k+1}$,
- for $0 \leqslant i \leqslant k$, we have $\mathcal{H}_i \subseteq \mathcal{G}_{i+1}$ and $\chi(\mathcal{H}_i) = i\xi + 1$,
- for $0 \leqslant i \leqslant k$, we have $\mathcal{S}_i \subseteq \mathcal{F}_i$ and either $\mathcal{S}_i \prec \mathcal{F}_{i+1}$ or $\mathcal{F}_{i+1} \prec \mathcal{S}_i$,
- for $1 \leqslant i \leqslant k$, every curve in $\mathcal{G}_i$ is entirely contained in $I_i \cap \dots \cap I_k$,
- for $1 \leqslant i \leqslant k+1$, we have $\chi(\mathcal{G}_i) > \gamma_i$.

This suffices for the proof of the lemma—the properties above imply that $((\mathcal{H}_0, \mathcal{S}_0), \dots, (\mathcal{H}_k, \mathcal{S}_k))$ is a bracket system of length $k+1$ that satisfies the assumption of Lemma 7 and therefore (by Lemma 7) contains a clique of size $k+1$, contradicting the assumption that $\omega(\mathcal{F}) \leqslant k$.

Let $\mathcal{G}_{k+1} = \mathcal{F}_{k+1}$. For $1 \leqslant i \leqslant k$ (considered in the order from k to 1), we assume that we have already found $\mathcal{G}_{i+1}$, and we show how to find $\mathcal{H}_i$, $\mathcal{S}_i$, and $\mathcal{G}_i$. Let

$$\begin{aligned} \mathcal{F}_i^L &= \{s \in \mathcal{F}_i \colon s \prec \mathcal{F}_{i+1}\}, & \mathcal{G}_{i+1}^L &= \{c \in \mathcal{G}_{i+1} \colon c \text{ intersects a curve in } \mathcal{F}_i^L\}, \\ \mathcal{F}_i^R &= \{s \in \mathcal{F}_i \colon \mathcal{F}_{i+1} \prec s\}, & \mathcal{G}_{i+1}^R &= \{c \in \mathcal{G}_{i+1} \colon c \text{ intersects a curve in } \mathcal{F}_i^R\}. \end{aligned}$$

The fact that $\mathcal{F}_{i+1}$ is supported from outside by $\mathcal{F}_i$ implies $\mathcal{G}_{i+1} = \mathcal{G}_{i+1}^L \cup \mathcal{G}_{i+1}^R$ and thus $\chi(\mathcal{G}_{i+1}^L) \geqslant \chi(\mathcal{G}_{i+1})/2$ or $\chi(\mathcal{G}_{i+1}^R) \geqslant \chi(\mathcal{G}_{i+1})/2$. Assume the former (the other case is analogous). This and $\chi(\mathcal{G}_{i+1}) > \gamma_{i+1} = 2(\gamma_i + 2k\xi + i\xi + 1)$ imply $\chi(\mathcal{G}_{i+1}^L) > \gamma_i + 2k\xi + i\xi + 1$. Let $\mathcal{H}_i$ be a subfamily of $\mathcal{G}_{i+1}^L$ such that $\mathcal{G}_{i+1}^L \setminus \mathcal{H}_i \prec \mathcal{H}_i$ and $\chi(\mathcal{H}_i) = i\xi + 1$. It can be found greedily, by

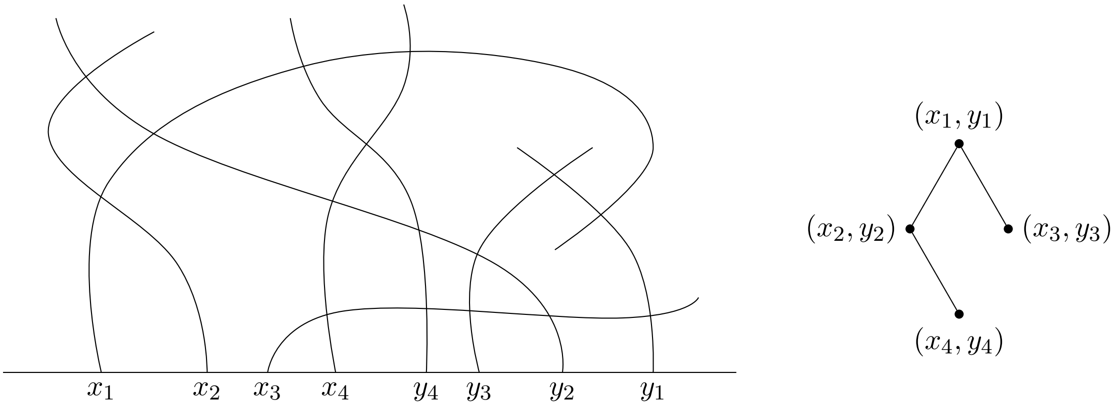

FIGURE 2. A tree-configuration $\bigl((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\bigr)$ and its unique witness

processing the curves in $\mathcal{G}^L_{i+1}$ in the order opposite to $\prec$ (from right to left), adding them to $\mathcal{H}_i$ until $\chi(\mathcal{H}_i) = i\xi + 1$. Let $\mathcal{S}_i = \mathcal{F}^L_i$. By the definition of $\mathcal{G}^L_{i+1}$, every curve in $\mathcal{H}_i$ intersects some curve in $\mathcal{S}_i$. It follows that $(\mathcal{H}_i, \mathcal{S}_i)$ is a bracket with $\mathcal{S}_i \prec \mathcal{G}^L_{i+1} \smallsetminus \mathcal{H}_i \prec \mathcal{H}_i$. Let I_i be the internal region of the bracket $(\mathcal{H}_i, \mathcal{S}_i)$. Let $\mathcal{G}_i$ be the curves in $\mathcal{G}^L_{i+1} \smallsetminus \mathcal{H}_i$ that are entirely contained in I_i. Since $\omega(\mathcal{G}^L_{i+1} \smallsetminus \mathcal{H}_i) \leqslant \omega(\mathcal{F}) \leqslant k$, Lemma 6 provides a set of at most $2k$ curves in $\mathcal{H}_i \cup \mathcal{S}_i$ such that every curve in $(\mathcal{G}^L_{i+1} \smallsetminus \mathcal{H}_i) \smallsetminus \mathcal{G}_i$ intersects at least one of them. Since $\mathcal{F}$ is a ξ-family, it follows that $\chi((\mathcal{G}^L_{i+1} \smallsetminus \mathcal{H}_i) \smallsetminus \mathcal{G}_i) \leqslant 2k\xi$ and thus $\chi(\mathcal{G}_i) \geqslant \chi(\mathcal{G}^L_{i+1} \smallsetminus \mathcal{H}_i) - 2k\xi \geqslant \chi(\mathcal{G}^L_{i+1}) - (i\xi + 1) - 2k\xi > \gamma_i$. We have thus found families $\mathcal{H}_i$, $\mathcal{S}_i$, and $\mathcal{G}_i$ with the requested properties.

After completing their construction for $1 \leqslant i \leqslant k$, we choose any $c_0 \in \mathcal{G}_1$, which exists because $\chi(\mathcal{G}_1) > 0$, we choose any $s_0 \in \mathcal{F}_0$ with $s_0 \prec \mathcal{F}_1$ or $\mathcal{F}_1 \prec s_0$ that intersects c_0, which exists because $\mathcal{G}_1 \subseteq \mathcal{F}_1$ and $\mathcal{F}_1$ is supported from outside by $\mathcal{F}_0$, and we let $\mathcal{H}_0 = \{c_0\}$ and $\mathcal{S}_0 = \{s_0\}$. This yields a bracket $(\mathcal{H}_0, \mathcal{S}_0)$ with the requested properties and completes the proof of the claim. $\square$

Tree-configurations. A *binary tree* is a rooted tree in which every node has at most one *left child* and at most one *right child*. A *descendant* of a node v in such a tree is any node u such that $u \neq v$ and v lies on the path from u to the root of the tree. A *left descendant* of a node v is the left child of v or any descendant of the left child of v, and a *right descendant* of v is the right child of v or any descendant of the right child of v. A *tree-configuration* is a sequence $\bigl((x_1, y_1), \ldots, (x_n, y_n)\bigr)$ of pairs of grounded curves such that the following conditions are satisfied:

- $x_1 \prec \cdots \prec x_n \prec y_n \prec \cdots \prec y_1$,
- for $1 \leqslant i \leqslant n$, the curves x_i and y_i intersect,

and the pairs $(x_1, y_1), \ldots, (x_n, y_n)$ can be arranged into a binary tree T so that

- (x_1, y_1) is the root of T,
- for $2 \leqslant i \leqslant n$, the parent of (x_i, y_i) in T is one of $(x_1, y_1), \ldots, (x_{i-1}, y_{i-1})$,
- for $1 \leqslant i < j \leqslant n$, if (x_j, y_j) is a left descendant of (x_i, y_i), then both x_j and y_j intersect x_i, and if (x_j, y_j) is a right descendant of (x_i, y_i), then both x_j and y_j intersect y_i.

The number n is the *length* of the tree-configuration $\bigl((x_1, y_1), \ldots, (x_n, y_n)\bigr)$, and any tree T satisfying the conditions above is a *witness* of the tree-configuration. See Figure 2 for an illustration. The following lemma explains the role of tree-configurations.

Lemma 9. *Every tree-configuration of length 2^{k-1} contains a clique of size $k + 1$.*

Proof. Let $((x_1, y_1), \ldots, (x_n, y_n))$ be a tree-configuration with $n = 2^{k-1}$, and let T be its witness. Since every binary tree with height less than k has fewer than 2^{k-1} nodes, the height of T is at least k. In particular, there are indices $1 = i_1 \leqslant \cdots \leqslant i_k \leqslant n$ such that $(x_{i_{r+1}}, y_{i_{r+1}})$ is a child of (x_{i_r}, y_{i_r}) for $1 \leqslant r < k$. A clique of size $k+1$ arises by taking x_{i_k}, y_{i_k}, and either x_{i_r} or y_{i_r} for $1 \leqslant r < k$ according to the following rule:

- if $(x_{i_{r+1}}, y_{i_{r+1}})$ is the left child of (x_{i_r}, y_{i_r}), then take x_{i_r},
- if $(x_{i_{r+1}}, y_{i_{r+1}})$ is the right child of (x_{i_r}, y_{i_r}), then take y_{i_r}.

The definition of a tree-configuration guarantees that the respective x_{i_r} or y_{i_r} (whichever is taken) intersects all $x_{i_{r+1}}, y_{i_{r+1}}, \ldots, x_{i_k}, y_{i_k}$. □

The proof of Lemma 4 proceeds by inductive construction of a tree-configuration of length 2^{k-1} in a ξ-family of sufficiently large chromatic number, which then contains a clique of size $k+1$, by Lemma 9. First, we present a way of extending a tree-configuration by one pair of curves.

An *attachment point* of a binary tree T is a pair (u, d) such that u is a node of T and $d \in \{\text{left}, \text{right}\}$ is a direction in which u has no child. If (u, d) is an attachment point of T, then a new node can be added to T becoming a child of u in direction d. A *left attachment point* for a node v of T is an attachment point (u, d) such that $(u, d) = (v, \text{left})$ or u is a left descendant of v, and a *right attachment point* for v is an attachment point (u, d) such that $(u, d) = (v, \text{right})$ or u is a right descendant of v. Thus, a new node added to T at a left or right attachment point for v becomes a left or right descendant of v, respectively. A binary tree with n nodes has exactly $n+1$ attachment points.

Let T be a witness of a tree-configuration $((x_1, y_1), \ldots, (x_n, y_n))$, and let a be an attachment point of T. A grounded curve c is *valid* for a if $x_n \prec c \prec y_n$, c intersects x_i for every node (x_i, y_i) of T whose left attachment point is a, and c intersects y_i for every node (x_i, y_i) of T whose right attachment point is a. Any pair of intersecting grounded curves that are valid for the same attachment point of T can be added to the end of $((x_1, y_1), \ldots, (x_n, y_n))$ to form a tree-configuration of length $n+1$ whose witness is obtained from T by adding that pair as a new node at that attachment point.

Lemma 10. *Let T be a witness of a tree-configuration $((x_1, y_1), \ldots, (x_n, y_n))$ and γ be a cap-curve such that $x_1, y_1, \ldots, x_n, y_n \subset \operatorname{int} \gamma$.*

(1) *Every grounded curve c with $x_n \prec c \prec y_n$ that intersects γ is valid for at least one attachment point of T.*
(2) *For any mutually disjoint grounded curves c_1, c_2, and c with $x_n \prec c_1 \prec c \prec c_2 \prec y_n$ that intersect γ, if c_1 and c_2 are valid for an attachment point a of T, then c is also valid for a.*

Proof. For $1 \leqslant i \leqslant n$, since x_i and y_i intersect, there is a cap-curve $\nu_i \subseteq x_i \cup y_i$ connecting the basepoints of x_i and y_i; it follows that $\gamma \subset \operatorname{ext} \nu_i$ while the basepoint of c lies in $\operatorname{int} \nu_i$, so c intersects x_i or y_i. We find an attachment point (u, d) of T such that c is valid for (u, d) as follows. We start from the root (x_1, y_1) and repeatedly move to the left child of the current node (x_i, y_i) if c intersects x_i or to the right child if c intersects y_i (choosing any child if c intersects both x_i and y_i) until the current node has no child in the requested direction. In the latter case, we let u be the current node and d be the requested direction.

Now, suppose c_1 and c_2 intersect a curve z contained in $\operatorname{int} \gamma$. It follows that there is a cap-curve $\nu \subseteq c_1 \cup z \cup c_2$ connecting the basepoints of c_1 and c_2. If $c_1 \prec c \prec c_2$, then the basepoint of c lies in $\operatorname{int} \nu$, while $\gamma \subset \operatorname{ext} \nu$ (as $c_1, z, c_2 \subset \operatorname{int} \gamma$), so c intersects z. This observation applied to every $z \in \{x_1, y_1, \ldots, x_n, y_n\}$ yields the second statement. □

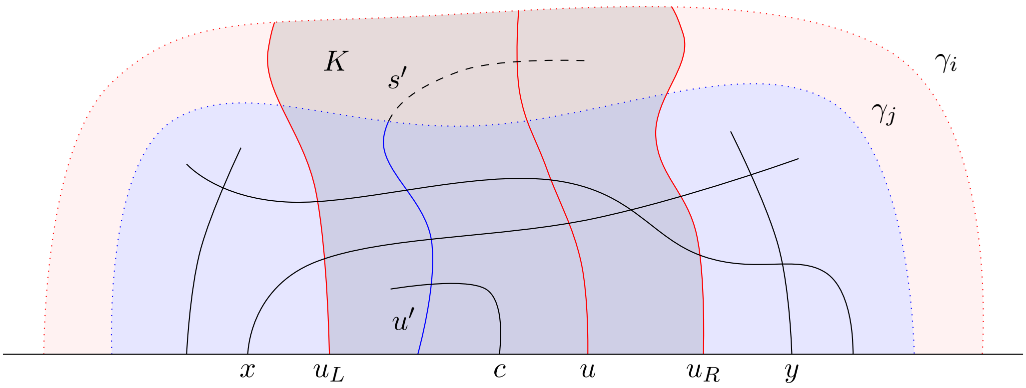

FIGURE 3. Illustration for the final part of the proof of Lemma 11

Lemma 11. *There is a function $h\colon \mathbb{N} \to \mathbb{N}$ with $h(n) = 2^{O(kn^2)}\xi^{n-1}$ such that for every $n \in \mathbb{N}$, every ξ-family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ and $\chi(\mathcal{F}) > h(n)$ contains a tree-configuration of length n.*

Proof. We define the function h by induction, as follows. We set $h(1) = 1$; if $\chi(\mathcal{F}) > 1$, then $\mathcal{F}$ contains two intersecting curves, which form a tree-configuration of length 1. For the induction step, fix $n \geqslant 1$, and assume that $h(n)$ is defined so that every ξ-family $\mathcal{H}$ with $\chi(\mathcal{H}) > h(n)$ contains a tree-configuration of length n. Let g be the function claimed by Lemma 8. Let

$$\alpha = h(n), \qquad \beta = 2(n+1)\tbinom{n+2}{2}\xi + 2(n+2)\xi, \qquad h(n+1) = g^{(n+2)}(2\alpha(\beta+1)),$$

where $g^{(m)}$ denotes the m-fold composition of g. Let $\mathcal{F}$ be a ξ-family with $\omega(\mathcal{F}) \leqslant k$ and $\chi(\mathcal{F}) > h(n+1)$. We claim that $\mathcal{F}$ contains a tree-configuration of length $n+1$.

Let $\mathcal{F}_0 = \mathcal{F}$. Lemma 8 applied $n+2$ times provides families $\mathcal{F}_1, \ldots, \mathcal{F}_{n+2}$ and skeletons $(\gamma_1, \mathcal{U}_1), \ldots, (\gamma_{n+2}, \mathcal{U}_{n+2})$ with the following properties:

- $\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots \supseteq \mathcal{F}_{n+2}$,
- for $1 \leqslant i \leqslant n+2$, $(\gamma_i, \mathcal{U}_i)$ is an $\mathcal{F}_{i-1}$-skeleton supporting $\mathcal{F}_i$,
- for $1 \leqslant i \leqslant n+2$, we have $\chi(\mathcal{F}_i) > g^{(n+2-i)}(2\alpha(\beta+1))$.

In particular, by the last condition, we have $\chi(\mathcal{F}_{n+2}) > 2\alpha(\beta+1)$. Therefore, by Lemma 1, there is a subfamily $\mathcal{H} \subseteq \mathcal{F}_{n+2}$ such that $\chi(\mathcal{H}) > \alpha$ and $\chi(\mathcal{F}_{n+2}(x,y)) > \beta$ for any two intersecting curves $x, y \in \mathcal{H}$. Since $\chi(\mathcal{H}) > \alpha$, there is a tree-configuration $((x_1,y_1), \ldots, (x_n,y_n))$ of length n in $\mathcal{H}$. Let $x = x_n$ and $y = y_n$. Thus $\chi(\mathcal{F}_{n+2}(x,y)) > \beta = 2(n+1)\tbinom{n+2}{2}\xi + 2(n+2)\xi$.

Let $\mathcal{G}$ be the family of curves in $\mathcal{F}_{n+2}(x,y)$ that intersect some curve in $\mathcal{U}_i(x,y)$ for every i with $1 \leqslant i \leqslant n+2$. If a curve $c \in \mathcal{F}_{n+2}(x,y)$ intersects no curve in $\mathcal{U}_i(x,y)$, where $1 \leqslant i \leqslant n+2$, then c intersects the curve in $\mathcal{U}_i$ with rightmost basepoint to the left of the basepoint of x (if such a curve exists) or the curve in $\mathcal{U}_i$ with leftmost basepoint to the right of the basepoint of y (if such a curve exists). This gives at most $2(n+2)$ curves such that every curve in $\mathcal{F}_{n+2}(x,y) \smallsetminus \mathcal{G}$ intersects at least one of them. The fact that $\mathcal{F}$ is a ξ-family implies $\chi(\mathcal{F}_{n+2}(x,y) \smallsetminus \mathcal{G}) \leqslant 2(n+2)\xi$ and thus $\chi(\mathcal{G}) \geqslant \chi(\mathcal{F}_{n+2}(x,y)) - 2(n+2)\xi > 2(n+1)\tbinom{n+2}{2}\xi$.

Let T be a witness of the tree-configuration $((x_1,y_1), \ldots, (x_n,y_n))$. For $1 \leqslant i \leqslant n+2$, let $\mathcal{U}_i^a$ be the curves in $\mathcal{U}_i(x,y)$ that are valid for an attachment point a of T. Thus $\mathcal{U}_i(x,y) = \bigcup_a \mathcal{U}_i^a$, by Lemma 10 (1). For a triple $\sigma = (i,j,a)$ such that $1 \leqslant i < j \leqslant n+2$ and a is an attachment point of T, let $\mathcal{G}_\sigma$ be the curves in $\mathcal{G}$ that intersect a curve in $\mathcal{U}_i^a$ and a curve in $\mathcal{U}_j^a$. By the pigeonhole principle, since T has $n+1$ attachment points, we have $\mathcal{G} = \bigcup_\sigma \mathcal{G}_\sigma$, and since there are $(n+1)\tbinom{n+2}{2}$ distinct triples σ and $\chi(\mathcal{G}) > 2(n+1)\tbinom{n+2}{2}\xi$, there is a triple $\sigma = (i,j,a)$ such that $\chi(\mathcal{G}_\sigma) > 2\xi$.

The rest of the argument is illustrated in Figure 3. Let u_L and u_R be the curves in $\mathcal{U}_i^a$ with leftmost and rightmost basepoints, respectively. Every curve in $\mathcal{U}_i^a$ lies in the closed region K bounded by u_L, u_R, the segment of the baseline between the basepoints of u_L and u_R, and the part of γ_i between its intersection points with u_L and u_R. Since $\mathcal{F}$ is a ξ-family, the curves in $\mathcal{G}_\sigma$ that intersect u_L have chromatic number at most ξ, and so do the curves in $\mathcal{G}_\sigma$ that intersect u_R. Since $\chi(\mathcal{G}_\sigma) > 2\xi$, there is a curve $c \in \mathcal{G}_\sigma$ that intersects neither u_L nor u_R. This and the fact that c intersects some curve in $\mathcal{U}_i^a$ implies $c \subset K$. The curve c also intersects some curve $u' \in \mathcal{U}_j^a$, which is a subcurve of some curve $s' \in \mathcal{F}_{j-1}$ valid for a. The facts that $c \subset K$ and $s' \subset \operatorname{int}\gamma_i$ imply $s' \subset K$ unless s' intersects u_L or u_R. Since $s' \in \mathcal{F}_{j-1} \subseteq \mathcal{F}_i$ and $\mathcal{F}_i$ is supported by $(\gamma_i, \mathcal{U}_i)$, the curve s' intersects some curve $u \in \mathcal{U}_i$, and by the above, u can be chosen so that $u_L \preceq u \preceq u_R$. Lemma 10 (2) implies that u is also valid for a. The curve u is a subcurve of some curve $s \in \mathcal{F}_{i-1}$ valid for a. Since s and s' intersect and are both valid for a, they can be used to extend the tree-configuration $\bigl((x_1, y_1), \ldots, (x_n, y_n)\bigr)$. Specifically, if $(x_{n+1}, y_{n+1}) = (s, s')$ or $(x_{n+1}, y_{n+1}) = (s', s)$ so that $x_{n+1} \prec y_{n+1}$, then $\bigl((x_1, y_1), \ldots, (x_{n+1}, y_{n+1})\bigr)$ is a tree-configuration of length $n+1$ in $\mathcal{F}$.

It remains to derive the claimed bound on h. By Lemma 8, we have $g(\alpha) = 2^{O(k)}(\xi + \alpha)$, which yields $g^{(n+2)}(\alpha) = 2^{O(kn)}(\xi + \alpha)$. This yields the following, for $n \geqslant 1$:

$$h(n+1) = g^{(n+2)}(\operatorname{poly}(n)\xi h(n)) = 2^{O(kn)}\xi + 2^{O(kn)}\xi h(n) = 2^{O(kn)}\xi h(n),$$

where the last bound follows from $h(n) \geqslant 1$. This and $h(1) = 1$ yield $h(n) = 2^{O(kn^2)}\xi^{n-1}$. $\square$

Proof of Lemma 4. Let $\zeta = h(2^{k-1})$, where h is the function claimed by Lemma 11. It follows that every ξ-family $\mathcal{F}$ with $\chi(\mathcal{F}) > \zeta$ contains a tree-configuration of length 2^{k-1} and thus, by Lemma 9, a clique of size $k+1$, which is not possible when $\omega(\mathcal{F}) \leqslant k$. Therefore, every ξ-family $\mathcal{F}$ with $\omega(\mathcal{F}) \leqslant k$ satisfies $\chi(\mathcal{F}) \leqslant \zeta$. The bound on h from Lemma 11 immediately yields the claimed bound on ζ. $\square$

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