HAUSDORFF DIMENSION OF ASYMPTOTIC SELF-SIMILAR SETS

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Abstract. In this paper, we introduce the notion of asymptotic self-similar sets on general doubling metric spaces by extending the notion of self-similar sets, and determine their Hausdorff dimensions, which gives an extension of Balogh and Rohner 's result. This is carried out by introducing the notions of almost similarity maps and asymptotic similarity systems. These notions have an advantage of making geometric constructions possible. Actually, as an application, we determined the Hausdorff dimension of general Sierpinski gaskets on complete surfaces constructed by a geometric way in a natural manner.

1. Introduction

The notion of self-similar sets or general Cantor sets have played significant roles in fractal geometry. These sets are usually defined by means of iterated function systems $\{f_1, \cdots, f_k\}$ consisting of contracting similarity maps on a complete metric space as the unique nonempty compact set K, called an attractor or an invariant set, satisfying $K = \bigcup_{i=1}^{n} f_i(K)$. Hutchinson [10] (cf. Kigami [12], Schief [18]) introduced the notion of the open set condition and determined the Hausdorff dimension of self-similar sets in Euclidean space $\mathbb{R}^n$ satisfying the open set condition. Balogh and Rohner extended Hutchinson's result to doubling metric spaces ([2]). However, it is difficult to construct a similarity map in general metric spaces. Actually, similarity maps do not always exist on curved metric spaces. To overcome this difficulty, in the previous work [22], the first named author introduced the notion of (λ, c, ν)-almost similarity maps extending that of λ-similarity maps in order to construct generalized Cantor sets in general metric measure spaces, and determined the Hausdorff dimension of such a generalized Cantor set. However the basic subsets considered in [22]

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are assumed to be disjoint each other, and therefore generalized Cantor sets like Sierpinski gaskets are excluded in the results of [22].

In the present paper, we extend both Balogh and Rohner 's result and our previous result to the case when basic subsets may have intersections with their boundary by introducing a generalized open set condition. As an application, we determine the Hausdorff dimension of Sierpinski gaskets on complete surfaces defined via geometric way.

Let X be a proper complete metric space. We assume that X is doubling in the sense of [2] (see Section 2 for the precise definition). Complete Riemannian manifolds with Ricci curvature bounded from below are typical examples of doubling metric spaces (cf. [8]). Doubling metric spaces also appears in metric measure spaces satisfying a doubling condition. Nowadays, geomeric analysis on doubling metric measure spaces has been very active (see for instance Assouad [1], Gromov[8], Heinonen [9], Villani[20]), and therefore it is quite natural to study self-similarity sets in such doubling metric spaces.

Let $\bar{U} \supset \bar{V}$ be bounded domains in X homeomorphic to each other, where $\bar{U}$ and $\bar{V}$ denote the closures of the open subsets U and V. Fix constants $0 < \lambda < 1$, $0 < \nu < 1$ and a continuous increasing function $\varphi : (0, \infty) \to (0, \infty)$ with $\lim_{x \to +0} \varphi(x) = 0$. We call a homeomorphism $f : \bar{U} \to \bar{V}$ a $(\lambda, \varphi(|\bar{U}|), \nu)$-*almost similarity map* if for every $x, y \in \bar{U}$,

$$\left| \frac{d(f(x), f(y))}{d(x, y)} - \lambda \right| \le \lambda\varphi(|U|), \tag{1.1}$$

$$|V| \le \nu|U|. \tag{1.2}$$

where $|U|$ is the diameter of U. Then the set $\bar{V}$ is called a $(\lambda, \varphi(|\bar{U}|), \nu)$-almost similar set of $\bar{U}$.

In this paper, we assume the following conditions for φ:

$$\begin{aligned} &\varphi : (0, \infty) \to (0, \infty) \text{ is increasing with } \lim_{\mathrm{x} \to +0} \varphi(\mathrm{x}) = 0; \\ &\int_1^\infty \varphi(a\nu^x)\, dx < \infty \text{ for some constants a} > 0 \text{ and } 0 < \nu < 1. \end{aligned} \tag{1.3}$$

Note that the second condition (2) above does not depend on the choice of $a > 0$ and $0 < \nu < 1$, and that for any $\alpha > 0$ and any positive integer n, the following functions satisfy the above conditions:

$$\varphi(y) = y^\alpha, \;\; \varphi(y) = -(\log y)^{-1-\frac{2}{2n+1}}.$$

For a fixed positive integer k, we let $\mathcal{I} = \{1, 2, \dots, k\}$. We denote by $\mathcal{I}^*$ the set of all ordered multi-indices $I = i_1 \cdots i_n$ with $n \ge 1$, $i_j \in \mathcal{I}$ for every $1 \le j \le n$. We set $|I| = |i_1 \cdots i_n| = n$ and call it the length of I. Let $\mathcal{I}^n$ denote the set of all $I \in \mathcal{I}$ of length n.

In the present paper, we investigate an asymptotic self-similar set in X, which is defined under the following hypothesis: For $0 < \nu < 1$ and

$a > 0$, let $\varphi : (0, \infty) \to (0, \infty)$ be a continuous function satisfying the above conditions (1.3).

Definition 1.1. Suppose that *ratio coefficients* $0 < \lambda_i < 1$, $(i = 1, \dots, k)$ together with a non-empty open subset $V \subset X$ are given for which we have

(1) for each $i \in \mathcal{I}$, a $(\lambda_i, \varphi(|\bar{V}|), \nu)$-almost similarity map
$$f_i : \bar{V} \to \bar{V}_i \subset \bar{V},$$
is given in such a way that $V_i \cap V_j = \emptyset$ for every $i \neq j \in \mathcal{I}$, where $V_i := f_i(V)$;

(2) for each $ij \in \mathcal{I}^2$, a $(\lambda_j, \varphi(|\bar{V}_i|), \nu)$-almost similarity map
$$f_{ij} : \bar{V}_i \to \bar{V}_{ij} \subset \bar{V}_i,$$
is given in such a way that $V_{ij} \cap V_{ij'} = \emptyset$ for every $j \neq j' \in \mathcal{I}$, where $V_{ij} := f_{ij}(V_i)$;

(3) for each $I' \in \mathcal{I}^{n-1}$ and $i_n \in \mathcal{I}$ with $I := I'i_n$, a $(\lambda_{i_n}, \varphi(|\bar{V}_{I'}|), \nu)$-almost similarity map
$$f_I : \bar{V}_{I'} \to \bar{V}_I \subset \bar{V}_{I'},$$
is defined in such a way that $V_{I'i} \cap V_{I'j} = \emptyset$ for every $i \neq j \in \mathcal{I}$, where $V_I := f_I(V_{I'})$.

We call $\{(\bar{V}_I, f_I)\}_{I \in \mathcal{I}^*}$ an $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-*asymptotic similarity system.* Then the set K defined as
$$K = \bigcap_{n=1}^{\infty} \left(\bigcup_{I \in \mathcal{I}^n} \bar{V}_I \right),$$
is called an *asymptotic self-similar set* in X.

Let us consider the case of iterated function system of contracting similarity maps $\{f_1, \dots, f_k\}$ with open set condition

(1) $V \supset f_1(V) \cup \cdots \cup f_k(V)$;
(2) $f_i(V) \cap f_j(V) \neq \emptyset$ for every $i \neq j$;

for some non-empty open set $V \subset X$. In this case, for each $I = i_1 \cdots i_n \in \mathcal{I}^*$, let
$$V_I := f_{i_n} \circ \cdots \circ f_{i_1}(V), \ \ f_I := f_{i_n} : \bar{V}_{I'} \to \bar{V}_I.$$
Then this gives a $(\{\lambda_i\}_{i=1}^k, \varphi = 0, \lambda_{\max})$-asymptotic similarity system $\{(\bar{V}_I, f_I)\}_{I \in \mathcal{I}}$, where $\lambda_{\max} = \max \lambda_i$. Thus the notion of $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system is an extension of iterated function system of contracting similarity maps with open set condition.

Our main result in the present paper is stated as follows.

Theorem 1.2. *Let X be a complete doubling metric space and let K be the asymptotic self-similar set associated with a $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system $\{(\bar{V}_I, f_I)\}_{I\in\mathcal{I}}$. Then the Hausdorff and the box dimensions of K are given as*

$$\dim_H K = \dim_B K = s,$$

where s is a unique number satisfying $\displaystyle\sum_{i=1}^k \lambda_i^s = 1$.

In [2], Balogh and Rohner suggested a problem. They considered an iterated function system of contracting *asymptotically similarity maps* in the sense that for all $I = i_i \cdots i_n \in \mathcal{I}$

$$c_1\lambda_I \le \frac{|f_I(x), f_I(y)|}{|x, y|} \le c_2\lambda_I,$$

where $f_I = f_{i_n} \circ \cdots \circ f_{i_1}$, $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_n}$ and c_1, c_2 are uniform positive constants. They posed a problem: What happens if an iterated function system of contracting similarity maps is replaced by one of contracting asymptotically similarity maps ? Rajala and Vilppolainen completely solved the above problem in Theorem 4.9 of [16] by introducing a more general notion of *a semiconformal iterated function system.* A $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system $\{(\bar{V}_I, f_I)\}_{I\in\mathcal{I}}$ is closely related with Balogh and Rohner's iterated function system of contracting asymptotically similarity maps and Rajala and Vilppolainen's semiconformal iterated function system under the open set condition. Actually our notion of asymptotic similarity system provides a controlled Moran construction in the sense of Rajala and Vilppolainen ([16]) (see Lemma 3.12). However an asymptotic self-similar set introduced in the present paper is constructed by means of a $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system, which consists of infinite series of almost similarity maps. Therefore in general, it is not simply defined by a finite iterated function system. For example, a generalized Sierpinski gasket on a general complete surfaces constructed in this paper is an asymptotic self-similar set. It would be an interesting question to determine whether a generalized Sierpinski gasket on a general complete surface can be defined by means of a finite iterated function system due to Balogh-Rohner or Rajala-Vilppolainen (see Section 4). Anyway the notion of asymptotic self-similar sets introduce in this paper has an advantage of making geometric constructions in general curved spaces much easier.

As indicated above, we consider a Sierpinski gasket on a complete surface M as an application of Theorem 1.2, which is naturally defined in a geometric way as follows.

Now let $\mathcal{I} = \{1, 2, 3\}$, and let Δ be a closed domain contained in a convex domain of M bounded by a geodesic triangle. By joining the midpoints of the edges of Δ by minimal geodesics, we divide Δ into four triangles, and remove the center triangle to get three geodesic triangles Δ_1, Δ_2 and Δ_3. Repeating this procedure for each Δ_i infinitely many times, we obtain a system of geodesic triangles $\{\Delta_I\}_{I\in\mathcal{I}^*}$. The *generalized Sierpinski gasket* K_Δ on M associated with Δ is defined as

$$K_\Delta = \bigcap_{n=1}^{\infty} \left(\bigcup_{I\in\mathcal{I}^n} \Delta_I \right),$$

We say that Δ is *asymptotically non-degenerate* if all the divided small triangles Δ_I are δ-non-degenerate for some constant $\delta > 0$. (See Section 4 for the precise definition). For example, every geodesic triangle region Δ of perimeter less than 2π on a unit sphere is asymptotically non-degenerate (see Example 4.3). We show that a small geodesic triangle region on a surface is asymptotically non-degenerate (see Lemma 4.9).

Theorem 1.3. *If a geodesic triangle domain Δ in a convex domain on a complete surface is asymptotically non-degenerate, then*

(1) *for some $0 < \nu < 1$ there exists a $(\{1/2, 1/2, 1/2\}, \varphi, \nu)$-asymptotic similarity system $\{(\Delta_I, f_I)\}_{I\in\mathcal{I}^*}$ associated with Δ, where $\varphi(x) = cx^2$ for some constant $c > 0$;*
(2) *the Hausdorff and box dimensions of the generalized Sierpinski gasket K_Δ associated with Δ are given by*

$$\dim_H K_\Delta = \dim_B K_\Delta = \frac{\log 3}{\log 2}. \tag{1.4}$$

The following result gives a condition for Δ to be asymptotically non-degenerate.

Corollary 1.4. *A geodesic triangle domain Δ in a convex domain on a complete surface is asymptotically non-degenerate if and only if for some $0 < \nu < 1$ there exists a $(\{1/2, 1/2, 1/2\}, \varphi, \nu)$-asymptotic similarity system $\{(\Delta_I, f_I)\}_{I\in\mathcal{I}^*}$ associated with Δ, where $\varphi(x) = cx^2$ for some constant $c > 0$.*

The organization of the present paper is as follows: In Section 2, we discuss some basic notions needed in the proof of the above results. In Section 3, we prove Theorem 1.2. In Section 4, we discuss generalized Sierpinski gaskets on complete surfaces, and prove Theorem 1.3 and Corollary 1.4.

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2. PRELIMINARIES

The distance between points x, y in a metric space will be denoted as $d(x, y)$. For $r > 0$, $B(x, r)$ denotes the open ball of radius r around x.

Definition 2.1. A metric space X is said to be *doubling* if there exists a positive integer C such that for any $x \in X$ and any $r > 0$, there exist $\{x_i\}_{i=1}^C \subset X$ such that

$$B(x, r) \subset \bigcup_{i=1}^{C} B(x_i, r/2)$$

Note that C, called the *doubling constant* of X, does not dependent on the choices of x or r.

For the proof of the following lemma, see Lemma 3.3 of [2].

Lemma 2.2. *Let X be a doubling metric space with doubling constant C. For any $0 < \delta < 1$, there exists a constant $C(\delta)$ such that the number of mutually disjoint balls $B(x_i, \delta r)$ in a ball $B(x, r)$ of X is bounded by $C(\delta)$.*

Definition 2.3. Let X be a metric space, $A \subset X$ and α be a non-negative real number. An *ϵ-cover* $\{U_i\}$ of A is a finite or countable collection of sets U_i covering A with $|U_i| \leq \epsilon$. Define $\mathcal{H}^\alpha_\epsilon(A)$ by

$$\mathcal{H}^\alpha_\epsilon(A) = \inf\Big\{ \sum_{i=1}^{\infty} |U_i|^\alpha \;\Big|\; \{U_i\} : \ \epsilon\text{-cover of } A \Big\}.$$

The *α-dimensional Hausdorff measure of* A is defined by

$$\mathcal{H}^\alpha(A) = \lim_{\epsilon \to 0} \mathcal{H}^\alpha_\epsilon(A),$$

and the *Hausdorff dimension* $\dim_H A$ *of* A is defined as

$$\dim_H A := \sup\{\alpha \geq 0 | \mathcal{H}^\alpha(A) = \infty\} = \inf\{\alpha \geq 0 | \mathcal{H}^\alpha(A) = 0\}.$$

Let A be a bounded subset of a metric space X. Let $N_\epsilon(A)$ denote the minimal number of subsets of diameter $\leq \epsilon$ needed to cover A. The *lower box dimension* and the *upper box dimension* of A are defined respectively as

$$\underline{\dim}_B A = \varliminf_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}, \quad \overline{\dim}_B A = \varlimsup_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}.$$

When both the lower and the upper box dimensions are equal, the common value

$$\dim_B A = \lim_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}$$

is called the *box dimension* of A.

The following is a standard fact (see [7] for instance):

$$\dim_H A \leq \underline{\dim}_B A \leq \overline{\dim}_B A. \tag{2.5}$$

Next we discuss self-similarity measures. In the rest of this section, we always assume that Y is a compact metric space unless otherwise stated.

Let $\mathcal{M}(Y)$ be the set of all Borel probability measures on Y. Consider the *Kantrovich-Rubinshtein metric* $d_{\mathcal{M}}$ and the *modified Kantrovich-Rubinshtein metric* $d^*_{\mathcal{M}}$ on $\mathcal{M}(Y)$ defined by

$$d_{\mathcal{M}}(\mu_1,\mu_2) = \sup\left\{\left|\int_Y \phi\, d\mu_1 - \int_Y \phi\, d\mu_2\right| : \phi \in \mathrm{Lip}_1(Y),\ \sup_{x\in Y}|\phi(x)| \le 1\right\},$$
$$d^*_{\mathcal{M}}(\mu_1,\mu_2) = \sup\left\{\left|\int_Y \phi\, d\mu_1 - \int_Y \phi\, d\mu_2\right| : \phi \in \mathrm{Lip}_1(Y)\right\},$$

where $\mathrm{Lip}_1(Y)$ denotes the set of all Lipschitz functions on Y with Lipschitz constant ≤ 1.

It is well known that $(\mathcal{M}(Y), d_{\mathcal{M}})$ is complete (see Theorem 8.10.43 of [3]). Further, we have from the definition

$$d_{\mathcal{M}}(\mu_1,\mu_2) \le d^*_{\mathcal{M}}(\mu_1,\mu_2) \le \max\{|Y|,1\} d_{\mathcal{M}}(\mu_1,\mu_2).$$

In particular, $(\mathcal{M}(Y), d^*_{\mathcal{M}})$ is also complete.

Let $\{f_i\}_{i=1}^m$ be a family of contracting maps in a compact metric space Y. Namely, there are some constants $0 < \lambda_i < 1$ such that

$$\frac{d(f_i(x), f_i(y))}{d(x,y)} \le \lambda_i < 1,$$

for every $x \ne y \in Y$ and $1 \le i \le m$.

Lemma 2.4. *(cf.* [11]*) Let Y and $\{f_i\}_{i=1}^m$ be as above. Then for any positive numbers a_i, $1 \le i \le m$, with $\sum_{i=1}^m a_i = 1$, there exists a unique Borel probability measure μ_0 such that*

$$\mu_0(A) = a_1\mu_0(f_1^{-1}(A)) + \cdots + a_m\mu_0(f_m^{-1}(A))$$

for every measurable subset $A \subset Y$. In other words,

$$\mu_0 = \sum_{i=1}^m a_i (f_i)_*(\mu_0),$$

where $(f_i)_*(\mu_0)$ is the push-forward measure of μ_0 by f_i.

Proof. Define the map $F^*(a_1,\dots,a_m) : (\mathcal{M}(Y), d^*_{\mathcal{M}}) \to (\mathcal{M}(Y), d^*_{\mathcal{M}})$ by

$$F^*(a_1,\dots,a_m)(\mu) = \sum_{i=1}^m a_i (f_i)_*(\mu).$$

If $\phi \in \mathrm{Lip}_1(Y)$, $\phi \circ f_i$ has Lipschitz constant $\le \lambda_{\max}$, where $\lambda_{\max} = \max\{\lambda_1,\dots,\lambda_m\}$. This implies that $F^*(a_1,\dots,a_m)$ is $\lambda_{\max}$-contracting, Since $(\mathcal{M}(Y), d^*_{\mathcal{M}})$ is complete, it has a fixed point μ_0 in $\mathcal{M}(K)$ by the contraction mapping theorem. This completes the proof. $\square$

3. Proof of Theorem 1.2

Let K be the asymptotic self-similar set in a complete doubling metric space X associated with a $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system $\{(\bar{V}_I, f_I)\}_{I\in\mathcal{I}^*}$. For each $I = i_1 \cdots i_n \in \mathcal{I}^*$, we set

$$g_I := f_I \circ \cdots \circ f_{i_1 i_2} \circ f_{i_1} : \bar{V} \to \bar{V}, \ \ \bar{V}_I := g_I(\bar{V}) \subset \bar{V}.$$

Note that

$$|V_I| \le \nu^{|I|}|V|. \tag{3.6}$$

Let s be a unique solution of $\displaystyle\sum_{i=1}^k \lambda_i^s = 1$

Lemma 3.1. *Let $\varphi : (0,\infty) \to [0,\infty)$ be a continuous function satisfying the conditions* (1.3). *Then*

$$\prod_{i=0}^{\infty}(1+\varphi(\nu^i|V|) < \infty, \quad \prod_{i=0}^{\infty}(1-\varphi(\nu^i|V|) > 0.$$

Proof. By the condition on φ, we have

$$\sum_{i=0}^{\infty}\log(1+\varphi(\nu^i|V|)) \le \sum_{i=0}^{\infty}\varphi(\nu^i|V|) < \infty.$$

Similarly we have

$$\sum_{i=0}^{\infty}\log(1-\varphi(\nu^i|V|)) \ge -2\sum_{i=0}^{\infty}\varphi(\nu^i|V|) > -\infty.$$

These complete the proof. □

Let $I = i_1 \cdots i_{m-1} i_m \in \mathcal{I}^*$. We use the notation

$$I_- = i_1 \cdots i_{m-1},$$

and write naturally like $I = I_- i_m$ as before.

Lemma 3.2. $\dim_H K \le s$

Proof. By the construction, we have $|V_{i_1\cdots i_n}| \le |V_{i_1\cdots i_{n-1}}|\nu$. For any $\epsilon > 0$ take a sufficiently large n such that $\mathcal{U}_n := \{\ V_I \mid I \in \mathcal{I}^n\}$ is an ϵ-cover of K. From the definition of $(\lambda_{i_n}, \varphi, \nu)$-almost similarity map $f_I : V_{I'} \to V_I$, $I = I' i_n$, we have

$$|V_I| \le \lambda_{i_n}(1+\varphi(|V_{I'}|)|V_{I'}|.$$

It follows from Lemma 3.1 that

$$
\begin{aligned}
\mathcal{H}^s_\epsilon(K) &\le \sum_{I\in\mathcal{I}^n} |V_I|^s \\
&= \sum_{I'\in\mathcal{I}^{n-1}} (\ |V_{I'1}|^s + \cdots + |V_{I'k}|^s\ ) \\
&\le \sum_{I'\in\mathcal{I}^{n-1}} (1+\varphi(|V_{I'}|))^s |V_{I'}|^s (\lambda_1^s + \cdots + \lambda_k^s) \\
&\le (1+\varphi(\nu^{n-1}|V|))^s \sum_{I'\in\mathcal{I}^{n-1}} |V_{I'}|^s \\
&\le \cdots < \prod_{i=0}^{\infty} (1+\varphi(\nu^i|V|))^s |V| < C|V|,
\end{aligned}
$$

where C is a constant, and therefore $\dim_H K \le s$. □

Lemma 3.3. *Let X be as in Theorem 1.2, and let $\mathcal{V} = \{V_i\}$ be a collection of disjoint open sets of X such that each V_i contains a closed ball of radius $c_1\rho$ and is included in a closed ball of radius $c_2\rho$ for some positive constants $c_1 < c_2$ and ρ. Then every closed ρ-ball $\bar{B}(x,\rho)$in X intersects at most $C(\delta)$ elements of $\bar{\mathcal{V}} = \{\bar{V}_i\}$, where $\delta = \frac{c_1}{c_1+4c_2+2}$ and $C(\delta)$ is a constant given in Lemma 2.2.*

Proof. Take $x_1^i, x_2^i \in X$ satisfying $\bar{B}(x_1^i, c_1\rho) \subset V_i \subset \bar{B}(x_2^i, c_2\rho)$. Let $\bar{V}_1, \cdots, \bar{V}_N$ intersect $\bar{B}(x,\rho)$.
Taking any point $z \in \bar{V}_i \cap \bar{B}(x,\rho)$, we have

$$d(x_1^i, x) \le d(x_1^i, z) + d(z,x) \le (2c_2+1)\rho.$$

Furthermore, for any $y \in B(x_1^i, c_1\rho)$, we have

$$d(y,x) \le d(y, x_1^i) + d(x_1^i, x) < (c_1+2c_2+1)\rho.$$

Thus we get

$$\bigcup_{i=1}^{N} B(x_1^i, c_1\rho) \subset B\big(x, (c_1+2c_2+1)\rho\big).$$

Since $B(x_1^i, c_1\rho)$ are mutually disjoint, from Lemma 2.2 we obtain the conclusion of the lemma. This completes the proof. □

The rest of this section is mainly devoted to prove the following.

Lemma 3.4. $\dim_H K \ge s$.

We set

$$\bar{V}^n := \bigcup_{I\in\mathcal{I}^n} \bar{V}_I.$$

Note that

$$K = \bigcap_{n=1}^{\infty} \bar{V}^n.$$

For a large n_0, fix an abitrary $I_0 = i_1 \cdots i_{n_0} \in \mathcal{I}^{n_0}$, and consider

$$\bar{V}_{I_0} := g_{I_0}(\bar{V}) = f_{I_0} \circ \cdots f_{i_1 i_2} \circ f_{i_1}(\bar{V}), \;\; K_{I_0} := K \cap V_{I_0}.$$

It suffices to prove that $\dim_H K_{I_0} \geq s$. Therefore we start with

$$W := V_{I_0},$$

instead of V.

For every $1 \leq i \leq k$, put

$$h_i := f_{I_0 i} : \bar{W} \to \bar{W}_i,$$

where

$$\bar{W}_i := h_i(\bar{W}) \subset \bar{W}.$$

Recall from the definition

$$\left| \frac{d(h_i(x), h_i(y))}{d(x,y)} - \lambda_i \right| < o(n_0),$$

for every $x \neq y \in \bar{W}$, where

$$o(n_0) = \lambda_{\max} \varphi(\nu^{n_0} |V|), \tag{3.7}$$

and therefore $\lim_{n_0 \to \infty} o(n_0) = 0$. For $J = j_1 \cdot \cdot j_m \in \mathcal{I}^*$ and every $1 \leq \ell \leq m$, we use the notation

$$h_{j_1 \cdot\cdot j_\ell} := f_{I j_1 \cdot\cdot j_\ell} : \bar{W}_{j_1 \cdot\cdot j_{\ell-1}} \to \bar{W}_{j_1 \cdot\cdot j_\ell},$$

as before, and define $g_J : \bar{W} \to \bar{W}_J$ by

$$g_J := h_J \circ \cdots \circ h_{j_1 j_2} \circ h_{j_1}.$$

Lemma 3.5. *For every $x \neq y \in \bar{W}$, we have*

$$\left| \frac{d(g_J(x), g_J(y))}{d(x,y)} - \lambda_J \right| < o(n_0) \lambda_J,$$

where $\lambda_J = \lambda_{j_1} \cdots \lambda_{j_m}$.

Proof. Put $J_\ell := j_1 \cdot \cdot j_\ell$ for each $1 \leq \ell \leq m$. From Lemma 3.1, we obtain

$$\begin{aligned}
\frac{d(g_J(x), g_J(y))}{d(x,y)} &= \frac{d(g_{J_m}(x), g_{J_m}(y))}{d(g_{J_{m-1}}(x), g_{J_{m-1}}(y))} \cdots \frac{d(g_{J_2}(x), g_{J_2}(y))}{d(g_{J_1}(x), g_{J_1}(y))} \frac{d(g_{J_1}(x), g_{J_1}(y))}{d(x,y)} \\
&\leq \lambda_J \prod_{\ell=0}^{\infty} (1 + \varphi(\nu^{n_0+\ell} |V|)) \\
&= \lambda_J (1 + o(n_0)).
\end{aligned}$$

An estimate from below is similar, and hence omitted. $\square$

For a small $\epsilon > 0$ compared with $|W|$, let $\{U_i\}$ be any ϵ-covering of

$$\tilde{K} := K_{I_0}.$$

Replacing U_i by balls B_i of radius $2|U_i|$, we have a covering $\{B_i\}$ of $\tilde{K}$. Thus

$$\sum |U_i|^s \geq 2^{-s} \sum |B_i|.$$

Fix B_i and take $c_1 > 0$ and $c_2 > 0$ such that W contains a ball of radius $c_1|W|$ and is contained in a ball of radius $c_2|W|$.

Definition 3.6. We denote by $\mathcal{I}^\infty$ the set of all infinite sequences $J = j_1 j_2 \cdots$ with $j_\ell \in \mathcal{I}$ for all $\ell \geq 1$. We call a finite subset $\mathcal{S}$ of $\mathcal{I}^*$ a *simple family* if for each $J = j_1 j_2 \cdots \in \mathcal{I}^\infty$, there is a unique m such that $J_m = j_1 j_2 \cdots j_m \in \mathcal{S}$.

For instance, $\mathcal{I}^m$ is a simple family for every $m \geq 1$.

Lemma 3.7. *For every simple family $\mathcal{S}$, we have*

$$\sum_{I \in \mathcal{S}} \lambda_I^s = 1.$$

Proof. Let $m := \max_{I \in \mathcal{S}} |I|$. We prove the lemma by the reverse induction on m. Take $I \in \mathcal{S}$ with $|I| = m$, and let $I = i_1 \cdots i_m$. Recall $I_- = i_1 \cdots i_{m-1}$ and note that $I_- j \in \mathcal{S}$ for all $j \in \mathcal{I}$. It follows that

$$\sum_{j=1}^{k} \lambda_{I_- j}^s = \lambda_{I_-}^s.$$

Set

$$\mathcal{S}_m := \mathcal{S} \cap \mathcal{I}^m, \ \ \mathcal{S}' := (\mathcal{S} \setminus \mathcal{S}_m) \cup \{I_- \mid I \in \mathcal{S}_m\}.$$

Since $\mathcal{S}'$is a simple family, it follows from the inductive hypothesis that

$$\sum_{I \in \mathcal{S}} \lambda_I^s = \sum_{I \in \mathcal{S}'} \lambda_I^s = 1$$

□

Assertion 3.8. *For each i, there is a simple family $\mathcal{S} = \mathcal{S}_i$ consisting of J satisfying that $\bar{W}_J$ is contained in a ball of radius $c_2|B_i|$ and contains a ball of radius $\tilde{\lambda}_{min} c_1 c_2 |B_i|$ for some uniform constant $0 < \tilde{\lambda}_{min} \leq \lambda_{\min}$.*

Proof. For each $J = j_1 j_2 \cdots \in \mathcal{I}^\infty$, there is a unique m such that

$$|W_{j_1 \cdots j_{m-1}}| > c_2 |B_i|, \ \ |W_{j_1 \cdots j_m}| \leq c_2 |B_i|. \tag{3.8}$$

Set $J_m := j_1 \cdots j_m$. Obviously, W_{J_m} is contained in a ball of rdius $c_2|B_i|$. Since W contains a ball of radius $c_1|W|$ and since W_{J_m} is open, W_{J_m} contains a ball of radius $(1 - o(n_0))\lambda_J c_1 |W|$. From the choice of J_m,

$$(1 - o(n_0))\lambda_J c_1 |W| \geq (1 - o(n_0))^2 \lambda_{j_m} c_1 c_2 |B_i|.$$

Let $\mathcal{S}$ be the set of all $J_m \in \mathcal{I}^*$ when J runs over $\mathcal{I}^\infty$. (3.16) implies that $\nu^{m-1} \geq c_2 |B_i| / |W|$, and therefore $\mathcal{S}$ is finite. This completes the proof. □

Applying Lemma 2.4 to the contracting maps $g_I : \bar{W} \to \bar{W}$, $I \in \mathcal{S}$, we have

Assertion 3.9. *Let $\mathcal{S} = \mathcal{S}_i$ be as in Assertion 3.8. Then there is a unique Borel probability measure $\mu = \mu_{\mathcal{S}}$ in $\mathcal{M}(\bar{W})$ such that*

$$\mu = \sum_{I \in \mathcal{S}} \lambda_I^s (g_I)_*(\mu),$$

where $\lambda_I^s = (\lambda_I)^s$.

Since $\bar{W} \supset \tilde{K}$, it follows from Lemma 3.5 and the property of $\mathcal{S}$ that for any $J \in \mathcal{S}$,

$$2^s c_2^s |B_i|^s \geq |W_J|^s \geq |\tilde{K}_J|^s \geq (1 - o(n_0))\lambda_J^s |\tilde{K}|^s. \tag{3.9}$$

By Lemma 3.3, the number of $\bar{W}_J$ with $J \in \mathcal{S}$ meeting B_i is uniformly bounded by some constant $C = C(\delta)$, where $\delta = \delta(c_1, c_2, \tilde{\lambda}_{\min})$. Let μ be the measure constructed in Assertion 3.9. Then we have

$$\begin{aligned} \mu(B_i) &= \sum_{I \in \mathcal{S}} \lambda_I^s (g_I)_*(\mu)(B_i) = \sum_{I \in \mathcal{S}} \lambda_I^s (g_I)_*(\mu)(B_i \cap \bar{W}_I) \\ &\leq C(\delta) \max_{I \in \mathcal{S}, \bar{W}_I \cap B_i \neq \phi} \lambda_I^s. \end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$c_2^s |B_i|^s \geq (1 - o(n_0)) C(\delta)^{-1} |K|^s \mu(B_i). \tag{3.11}$$

Since

$$\sum_{|J|=m} \lambda_J^s = 1, \tag{3.12}$$

for each $m \geq 1$, applying Lemma 2.4 to the contracting maps $g_J : \bar{W} \to \bar{W}$, $J \in \mathcal{I}^m$, we have a unique measure $\mu_m \in \mathcal{M}(\bar{W})$ such that

$$\mu_m = \sum_{|J|=m} \lambda_J^s (g_J)_*(\mu_m).$$

Assertion 3.10. *For $m > \max_{I \in \mathcal{S}} |I|$, we have $\mu = \mu_m$.*

Proof. For each $J \in \mathcal{I}^m$, there are unique $I \in \mathcal{S}$ and $J_\alpha \in \mathcal{I}^*$ such that $J = IJ_\alpha$. Let A_I be the set of all the indices α with $J = IJ_\alpha$ for some $J \in \mathcal{I}^m$ We can write as

$$\mu_m = \sum_{I \in \mathcal{S}, \alpha \in A_I} \lambda_{IJ_\alpha}^s (g_{IJ_\alpha})_*(\mu_m).$$

By iterating ℓ-times, we have

$$\begin{aligned} \mu_m &= \sum_{J_1, \dots, J_\ell \in \mathcal{I}^m} \lambda_{J_1}^s \cdots \lambda_{J_\ell}^s (g_{J_1} \circ \cdots \circ g_\ell)_*(\mu_m) \\ &= \sum_{I_i \in \mathcal{S}, \alpha_i \in A_{I_i}} \lambda_{I_1 J_{\alpha_1}}^s \cdots \lambda_{I_\ell J_{\alpha_\ell}}^s (g_{J_1} \circ \cdots \circ g_{J_\ell})_*(\mu_m). \end{aligned}$$

Since $A_I = \mathcal{I}^{m-|I|}$, similarly to (3.12) we see

$$\sum_{\alpha\in A_I} \lambda_{J_\alpha}^s = 1. \tag{3.13}$$

It follows that

$$\mu = \sum_{I\in\mathcal{S}} \lambda_I^s (g_I)_*(\mu) = \sum_{I\in\mathcal{S},\alpha\in A_I} \lambda_{IJ_\alpha}^s (g_I)_*(\mu).$$

By iterating ℓ-times, we obtain

$$\mu = \sum_{I_i\in\mathcal{S},\alpha_i\in A_{I_i}} \lambda_{I_1J_{\alpha_1}}^s \cdots \lambda_{I_\ell J_{\alpha_\ell}}^s (g_{I_1}\circ\cdots\circ g_{I_\ell})_*(\mu).$$

It follows that

$$d_{\mathcal{M}}^*(\mu,\mu_m) \le \sum_{I_i\in\mathcal{S},\alpha_i\in A_{I_i}} \lambda_{I_1J_{\alpha_1}}^s \cdots \lambda_{I_\ell J_{\alpha_\ell}}^s$$
$$\sup_{L(\phi)\le 1} \left| \int \phi\circ g_{I_\ell}\circ\cdots\circ g_{I_1}\, d\mu - \int \phi\circ g_{J_\ell}\circ\cdots\circ g_{J_1}\, d\mu_m \right|$$

Here,

$$\begin{aligned} &|\int \phi\circ g_{I_\ell}\circ\cdots\circ g_{I_1}\, d\mu - \int \phi\circ g_{J_\ell}\circ\cdots\circ g_{J_1}\, d\mu_m| \\ &\qquad \le \left| \int \phi\circ g_{I_\ell}\circ\cdots\circ g_{I_1}\, d\mu - \int \phi\circ g_{I_\ell}\circ\cdots\circ g_{I_1}\, d\mu_m \right| \\ &\qquad + \left| \int \phi\circ g_{I_\ell}\circ\cdots\circ g_{I_1}\, d\mu_m - \int \phi\circ g_{J_\ell}\circ\cdots\circ g_{J_1}\, d\mu_m \right|. \end{aligned}$$

For a constant $\tilde{\lambda}$ with $\lambda_{\max} < \tilde{\lambda} < 1$, choose a large n_0 such that $(1+o(n_0))\lambda_{\max} < \tilde{\lambda} < 1$. Then the Lipschitz constant of $g_{I_\ell}\circ\cdots\circ g_{I_1}$ satisfies

$$L(g_{I_\ell}\circ\cdots\circ g_{I_1}) \le (1+o(n_0))^\ell \lambda_{I_\ell}\cdots\lambda_{I_1} < \tilde{\lambda}_{I_1\cdots I_\ell},$$

where we put $\tilde{\lambda}_{I_1\cdots I_\ell} := (\tilde{\lambda})^{|I_1|+\cdots+|I_\ell|}$. Therefore we obtain

$$\begin{aligned} &|\int \phi\circ g_{I_\ell}\circ\cdots\circ g_{I_1}\, d\mu - \int \phi\circ g_{I_\ell}\circ\cdots\circ g_{I_1}\, d\mu_m| \\ &\qquad \le \tilde{\lambda}_{I_1\cdots I_\ell} d_{\mathcal{M}}^*(\mu,\mu_m). \end{aligned}$$

On the other hand, from the inclusion

$$g_{I_\ell}\circ\cdots\circ g_{I_1}(\bar{W}) \supset g_{J_\ell}\circ\cdots\circ g_{J_1}(\bar{W}),$$

we have

$$\begin{aligned} &\sup_{x\in\bar{W}} |\phi\circ g_{I_\ell}\circ\cdots\circ g_{I_1}(x) - \phi\circ g_{J_\ell}\circ\cdots\circ g_{J_1}(x)| \\ &\qquad \le |g_{I_\ell}\circ\cdots\circ g_{I_1}(\bar{W})| \\ &\qquad \le (1+o(n_0))^\ell \lambda_{I_\ell}\cdots\lambda_{I_1} < \tilde{\lambda}_{I_1\cdots I_\ell} \end{aligned}$$

Thus letting $n = \min_{I\in\mathcal{S}} |I|$ together with (3.13), we have

$$\begin{aligned}
d^*_{\mathcal{M}}(\mu,\mu_m) &\le \sum_{I_1,\dots,I_\ell,\alpha_1,\dots,\alpha_\ell} \lambda^s_{I_1J_{\alpha_1}}\cdots\lambda^s_{I_\ell J_{\alpha_\ell}}\tilde\lambda_{I_1\cdots I_\ell}(d^*_{\mathcal{M}}(\mu,\mu_m)+1)\\
&\le \tilde\lambda^{n\ell}\sum_{I_1,\dots,I_\ell,\alpha_1,\dots,\alpha_\ell} \lambda^s_{I_1J_{\alpha_1}}\cdots\lambda^s_{I_\ell J_{\alpha_\ell}}(d^*_{\mathcal{M}}(\mu,\mu_m)+1)\\
&= \tilde\lambda^{n\ell}\sum_{I_1,\dots,I_\ell\in\mathcal{S}} \lambda^s_{I_1}\cdots\lambda^s_{I_\ell}(d^*_{\mathcal{M}}(\mu,\mu_m)+1)\\
&= \tilde\lambda^{n\ell}(d^*_{\mathcal{M}}(\mu,\mu_m)+1),
\end{aligned}$$

which yields

$$d^*_{\mathcal{M}}(\mu,\mu_m) < \frac{1}{1-\tilde\lambda^{n\ell}}\tilde\lambda^{n\ell}.$$

Letting $\ell\to\infty$, we conclude that $\mu=\mu_m$. $\square$

Proof of Lemma 3.4. From the last assertion, we have

$$\mathrm{supp}(\mu)\subset\bigcap_{m=1}^{\infty}\left(\bigcup_{|J|=m} g_J(\bar W)\right)=\tilde K.$$

It follows from (3.11) that

$$\begin{aligned}
\sum 2^{-s}|B_i|^s &\ge (1-o(n_0))4^{-s}c_2^{-s}C(\delta)^{-1}|\tilde K|\sum\mu(B_i)\\
&\ge (1-o(n_0))4^{-s}c_2^{-s}C(\delta)^{-1}|\tilde K|.
\end{aligned}$$

This shows that $\dim_H\tilde K\ge s$. We have completed the proof of lemma 3.4. $\square$

Finally we show that

Lemma 3.11. $\overline{\dim}_B K\le s$.

Proof. For every $\epsilon>0$ and $J_\infty=j_1j_2\cdots\in\mathcal{I}^\infty$, take a minimal m satisfying $|W_J|\le\epsilon$ for $J:=J_m=j_1\cdots j_m$. Note that

$$|W_J|\ge\lambda_{\min}/2|W_{J_-}|\ge\epsilon\lambda_{\min}/2. \tag{3.14}$$

Thus we have a simple family $\mathcal{S}=\{J\mid J_\infty\in\mathcal{I}^\infty\}$. By Lemma 3.7, we have

$$\sum_{J\in\mathcal{S}}\lambda_J^s=1. \tag{3.15}$$

By Lemma 3.5, we have

$$\left|\frac{|W_J|}{|W|}-\lambda_J\right|<\lambda_J o(n_0). \tag{3.16}$$

It follows from (3.14) and (3.16) that

$$(\epsilon\lambda_{\min}/2)^s\le 2^s\lambda_J^s|W|^s.$$

Using (3.15), we obtain

$$\sum_{J\in\mathcal{S}}(\epsilon\lambda_{\min}/2)^s \le 2^s|W|^s.$$

Since $\{W_J \mid J \in \mathcal{S}\}$ is disjoint, we conclude that

$$N_\epsilon(\tilde{K}) \le 2^s|W|^s(\epsilon\lambda_{\min}/2)^{-s}.$$

This shows that $\overline{\dim}_B \tilde{K} \le s$, and the conclusion of the lemma follows. $\square$

It follows from Lemmas 3.4, 3.11 and (2.5) that $\dim_H K = \dim_B K = s$. This completes the proof of Theorem 1.2.

Finally we point out that our notion of asymptotic similarity system provides a controlled Moran construction defined in Rajala and Vilppolainen [16]:

Lemma 3.12. *Let $\{(\bar{V}_I, f_I)\}_{I\in\mathcal{I}^*}$ be a $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system. Then $\{\bar{V}_I\}_{I\in\mathcal{I}^*}$ is a controlled Moran construction defined in Rajala and Vilppolainen ([16]). Namely, there exists a constant $D \ge 1$ such that for every $I, J \in \mathcal{I}^*$*

1. $\bar{V}_I \subset \bar{V}_{I^-}$;
2. *there exists a positive integer n such that*
$$\max_{I\in\mathcal{I}^n} |\bar{V}_I| < D^{-1};$$
3. $D^{-1} \le \dfrac{|\bar{V}_{IJ}|}{|\bar{V}_I||\bar{V}_J|} \le D.$

Proof. (1) is clear. In view of (3.6), (2) is obvious. To show (3), we go back to the situation of Lemma 3.5. Let $o(n_0)$ be as in (3.7). For a large n_0, fix an abitrary $I_0 = i_1 \cdots i_{n_0} \in \mathcal{I}^{n_0}$, and consider $W = V_{I_0}$. If we take n_0 with $o(n_0) < 1/2$, we have from Lemma 3.5,

$$\frac{1}{2}\lambda_I|\bar{W}| < |\bar{W}_I| < 2\lambda_I\bar{W}|, \quad \frac{1}{2}\lambda_J|\bar{W}| < |\bar{W}_J| < 2\lambda_J|\bar{W}|,$$

which imply

$$\frac{1}{4|\bar{W}|}|\bar{W}_I||\bar{W}_J| < |\bar{W}_{IJ}| < \frac{4}{|\bar{W}|}|\bar{W}_I||\bar{W}_J|.$$

Now (3) is immediate, since we have only finitely many choices for I_0. $\square$

4. Sierpinski gaskets on surfaces

Let D be a domain in a complete surface M. We assume that D is convex in the sense that for every two points of D there exits a unique minimal geodesic joining them and it is contained in D. For simplicity, we assume that the absolute value of the Gaussian curvature of M is at most 1 on D. Let Δ be a domain in D bounded by a geodesic triangle $(\gamma_1, \gamma_2, \gamma_3)$. We call Δ a *geodesic triangle region.* The set of lengths $\{L(\gamma_i)\}_{i=1}^3$ is called the side-length of Δ.

Definition 4.1. We say that Δ is δ*-non-degenerate* if each angle $\tilde{\alpha}$ of a comparison triangle $\tilde{\Delta}$ of Δ in $\mathbb{R}^2$ satisfies $\delta < \tilde{\alpha} < \pi - \delta$, where a comparison triangle means that $\tilde{\Delta}$ has the same side-length as Δ.

In this section, we let $\mathcal{I} = \{1, 2, 3\}$. Let $\{\Delta_I\}_{I\in\mathcal{I}^*}$ be the system of geodesic triangles obtained by dividing Δ into smaller triangles Δ_I consecutively, as stated in Introduction.

Definition 4.2. We say that the system $\{\Delta_I\}_{I\in\mathcal{I}^*}$ is *non-degenerate* if there is a $\delta > 0$ such that Δ_I is δ-non-degenerate for every $I \in \mathcal{I}^*$. In this case, we also say that Δ is *asymptotically non-degenerate.*

Example 4.3. Let $\mathbb{S}^2$ denote the unit sphere around the origin in $\mathbb{R}^3$, and let Δ be a geodesic triangle domain on $\mathbb{S}^2$ of perimeter less than 2π. Joining the vertexes p_1, p_2, p_3 of Δ by shortest segments in $\mathbb{R}^3$, we have a geodesic triangle region $\hat{\Delta}$ on the plane through p_1, p_2, p_3. By the projection along the rays from the origin of $\mathbb{R}^3$, we have a canonical map

$$\pi : \Delta \to \hat{\Delta},$$

which is a bi-Lipschitz homeomorphism. From a system of geodesic triangles $\{\Delta_I\}_{I\in\mathcal{I}^*}$ of Δ, setting $\hat{\Delta}_I := \pi(\Delta_I)$, we have the system of geodesic triangles $\{\hat{\Delta}_I\}_{I\in\mathcal{I}^*}$ of $\tilde{\Delta}$. Note that each $\hat{\Delta}_I$ is $2^{-|I|}$-similar to $\hat{\Delta}$ in the usual sense. Since Δ_I is bi-Lipschitz homeomorphic to $\hat{\Delta}_I$,

$$\mathrm{Area}(\Delta_I) \geq L^{-2}\mathrm{Area}(\hat{\Delta}_I),$$

where L is the bi-Lipschitz constant of π. It follows that Δ is asymptotically non-degenerate. Now we have the formula (1.4) for the Sierpinsli gasket K_Δ associated with Δ by two reasons. One is by Theorem 1.3 and the other one is due to the well-known formula for $K_{\hat{\Delta}}$.

Example 4.3 is the special case. For a geodesic triangle region on a general complete surface, it seems impossible to reduce the problem to a triangle region in $\mathbb{R}^2$.

The main purpose of this section is to prove the following result.

Theorem 4.4. *For every $\delta > 0$ there exists an $r > 0$ such that*

(1) *every geodesic triangle region Δ on D with $|\Delta| \leq r$ is asymptotically non-degenerate;*

(2) *the Hausdorff and box dimensions of the Sierpinski gasket K_Δ associated with Δ are given by* (1.4).

If Δ be asymptotically non-degenerate as in Theorem 1.3, we can apply Theorem 4.4 to Δ_I for each $I \in \mathcal{I}^*$ with large enough $|I|$. Therefore Theorem 4.4 yields Theorem 1.3.

The following lemma is a consequence of law of cosine, and hence is omitted.

Lemma 4.5. *For any $\delta > 0$ there exists an $\epsilon > 0$ such that if a geodesic triangle Δ of side length (a_1, a_2, a_3) is δ-non-degenerate, and if the side length (a_1', a_2', a_3') of a geodesic triangle Δ' satisfies*

$$(1-\epsilon)\frac{a_j}{a_i} < \frac{a_j'}{a_i'} < (1+\epsilon)\frac{a_j}{a_i}, \tag{4.17}$$

for any $i \neq j$, then Δ' is $\delta/2$-non-degenerate.

Proof. We may assume that Δ and Δ' are triangles in $\mathbb{R}^2$. Set $(a, b, c) := (a_1, a_2, a_3)$ and $(a', b', c') := (a_1', a_2', a_3')$ for simplicity. Rescaling Δ', we may assume that $c = c'$. It suffices to show that if Δ' has side-length $(a', b', c') = (a', b, c)$ satisfying (4.17), then the angles α, β (resp. α', β') opposite to the edges of length a and b in Δ (resp a' and b in Δ') satisfy that $|\alpha' - \alpha| < \delta/4$ and $|\beta' - \beta| < \delta/4$ for a suitable $\epsilon = \epsilon(\delta) > 0$.

Sublemma 4.6. *If a geodesic triangle Δ of side lengths (a_1, a_2, a_3) is δ-non-degenerate, then there exists a constant $C(\delta)$ such that*

$$C(\delta)^{-1} < \frac{a_j}{a_i} < C(\delta),$$

for every $1 \leq i, j \leq 3$.

Proof. This is an immediate consequence of the law of sines. One can take $C(\delta) = 1/\sin\delta$. □

By trigonometry, we have

$$\sin^2\alpha/2 = (a+c)(a+b)/bc, \ \sin^2\alpha'/2 = (a'+c)(a'+b)/bc.$$

It follows from the assumption and Sublemma 4.6 with $|a' - a| < \epsilon a$ that

$$|\sin^2\alpha'/2 - \sin^2\alpha/2| \leq a(a + a'b + c)\epsilon/bc \leq 5C(\delta)^2\epsilon. \tag{4.18}$$

Since $\sin\alpha'/2 + \sin\alpha/2 > \sin(\delta/2)$, we obtain

$$|\sin\alpha'/2 - \sin\alpha/2| \leq 5C(\delta)^2\epsilon/\sin(\delta/2).$$

From $\alpha < \pi - 2\delta$, we have $\cos\frac{\alpha'+\alpha}{4} > \sin(\delta/4)$. It follows that

$$|\alpha' - \alpha| \leq 8\left|\sin\frac{\alpha'-\alpha}{4}\right| < 5C(\delta)^2\epsilon/\sin^2(\delta/4). \tag{4.19}$$

Similarly we have

$$\begin{aligned}|\sin^2\beta'/2-\sin^2\beta/2| &= |a-a'|b(b+c)/aa'c \le b(b+c)\epsilon/ca' \\ &\le \frac{\epsilon}{1-\epsilon}\frac{b(b+c)}{a} \le \frac{\epsilon}{1-\epsilon}2C(\delta)^2,\end{aligned}$$

which implies

$$|\beta'-\beta| < \frac{8\epsilon}{1-\epsilon}\left(\frac{C(\delta)}{\sin(\delta/2)}\right)^2. \tag{4.20}$$

Thus from (4.19), (4.20), we obtain $|\alpha'-\alpha|<\delta/4$ and $|\beta'-\beta|<\delta/4$ for a suitable $\epsilon \le \epsilon(\delta)$. This completes the proof. □

Let Δ be a geodesic triangle region on D bounded by a geodesic triangle $(\gamma_1,\gamma_2,\gamma_3)$ with vertices p_1,p_2,p_3. By the convexity of D, we have

$$|\Delta| = \max_{1\le i\le 3} a_i,$$

where we put $a_i := L(\gamma_i)$. Fix a vertex p_1 and let γ_i be parametrized on $[0,1]$ in such a way that $\gamma_2(0)=\gamma_3(0)=p_1$. Let $\varphi:[0,1]\times[0,1]\to\Delta$ be a parametrization of Δ such that $t\to\varphi(t,s)$, $0\le t\le 1$, is the geodesic, denoted by σ_s, from $\gamma_2(s)$ to $\gamma_3(s)$ for each $s\in[0,1]$. Namely $\varphi(t,s)=\sigma_s(t)$. We set

$$a_1(s) := L(\sigma_s).$$

Now define the map $f_1:\Delta\to\Delta$ by

$$f_1(\varphi(t,s)) = \varphi(t,s/2).$$

Note that the image Δ_1 of f_1 is the geodesic triangle region bounded by $(\gamma_2|_{[0,1/2]},\gamma_3|_{[0,1/2]},\sigma_{1/2})$ and that Δ_1 has side-length $(a_1(1/2),a_2/2,a_3/2)$. We put

$$r := |\Delta|.$$

Lemma 4.7. *For any $s\in(0,1)$, we have*

$$1-r^2 < \frac{a_1(s)}{sa_1} < 1+r^2.$$

In particular, $|\Delta_1| \le \frac{1}{2}(1+r^2)|\Delta|$.

Proof. Let $\tilde\gamma_i(s) := \exp_{p_1}^{-1}(\gamma_i(s))$, $i=2,3$. The Rauch comparison theorem (see [5]) implies

$$\frac{\sin r}{r} < \frac{a_1}{d(\tilde\gamma_2(1),\tilde\gamma_3(1))} < \frac{\sinh r}{r} \tag{4.21}$$

$$\frac{\sin r}{r} < \frac{a_1(s))}{d(\tilde\gamma_2(s),\tilde\gamma_3(s))} < \frac{\sinh r}{r}. \tag{4.22}$$

Since $d(\tilde\gamma_2(s),\tilde\gamma_3(s)) = sd(\tilde\gamma_2(1),\tilde\gamma_3(1)$, the conclusion follows. □

Let us denote by $(a_{1,1}, a_{1,2}, a_{1,3})$ the side length $(a_1(1/2), a_2/2, a_3/2)$ of Δ_1. Lemma 4.7 implies that

$$(1 - r^2)\frac{a_i}{a_j} < \frac{a_{1,i}}{a_{1,j}} < (1 + r^2)\frac{a_i}{a_j}, \tag{4.23}$$

for every $1 \leq i, j \leq 3$.

In a similar way, we construct a map $f_{i_1} : \Delta \to \Delta_{i_1} \subset \Delta$ for each $1 \leq i_1 \leq 3$. Repeating this procedure for each Δ_i inductively, for each multi-index $I = i_1 \cdots i_{n-1} i_n$, we have a geodesic triangle region Δ_I and a map $f_I : \Delta_{I'} \to \Delta_I$, where $I' = i_1 \cdots i_{n-1}$. The side-length $(a_{I,1}, a_{I,2}, a_{I,3})$ of Δ_I is also suitably defined inductively. Take $r < 1$ and set

$$\nu := \frac{1}{2}(1 + r^2) < 1.$$

Lemma 4.8. *There exists an $L(r) > 1$ such that for every I and $1 \leq i, j \leq 3$*

$$L(r)^{-1}\frac{a_i}{a_j} < \frac{a_{I,i}}{a_{I,j}} < L(r)\frac{a_i}{a_j}.$$

Proof. Repeating use of (4.23) and Lemma 4.7 applied to $s = 1/2$ implies that for each $I = i_1 \cdots i_m$,

$$\begin{aligned}(1 - r_m^2) \cdots (1 - r_1^2)(1 - r^2)&\frac{a_i}{a_j} \\ &< \frac{a_{I,i}}{a_{I,j}} < (1 + r_m^2) \cdots (1 + r_1^2)(1 + r^2)\frac{a_i}{a_j}\end{aligned}$$

for every $1 \leq i, j \leq 3$, where $r_k := |\Delta_{i_1 \cdots i_k}|$, $1 \leq k \leq m$. Since

$$r_k \leq \frac{1}{2}(1 + r_{k-1}^2) r_{k-1} < \nu r_{k-1} < \cdots < \nu^k r.$$

it follows that

$$\Pi_{m=0}^{\infty}\left(1 - \nu^{2m} r^2\right)\frac{a_i}{a_j} < \frac{a_{I,i}}{a_{I,j}} < \Pi_{m=1}^{\infty}\left(1 + \nu^{2m} r^2\right)\frac{a_i}{a_j}. \tag{4.24}$$

This completes the proof. $\square$

From (4.24), one can take $L(r)$ as

$$L(r) := e^{\frac{2r^2}{1-\nu^2}}.$$

For every $s \in (0, 1]$ we denote by $\Delta(1 : s)$ the geodesic triangle $(\gamma_2|_{[0,s]}, \gamma_3|_{[0,s]}, \sigma_s)$. Similarly, $\Delta(i : s)$ and $\Delta_I(i : s)$ are defined for every $1 \leq i \leq 3$ and every multi-index $I \in \mathcal{I}^*$.

Lemmas 4.5, 4.7 and 4.8 imply

Lemma 4.9. *For every $\delta > 0$, there exists a positive number r such that if Δ is δ-non-degenerate and the diameter $|\Delta|$ of Δ is less than r, then Δ_I as well as $\Delta_I(i : s)$ is $\delta/2$-non-degenerate for every multi-index I, $1 \leq i \leq 3$ and $s \in (0, 1)$.*

By Lemma 4.9, we get the conclusion (1) of Theorem 4.4. In view of Theorem 1.2, to prove the conclusion (2) of Theorem 4.4, it suffices to prove the following.

Theorem 4.10. *There is a positive numbers $c = c(\delta)$ such that $\{(\Delta_I, f_I)\}_{I\in\mathcal{I}^*}$ gives a $(1/2, \varphi_c, \nu)$-asynptotic similarity system, where $\varphi_c(x) = cx^2$.*

Proof. In view of Lemma 4.9, it suffices to prove that the map $f := f_1 : \Delta \to \Delta_1 \subset \Delta$ is a $(1/2, \varphi_c, \nu)$-almost similarity map for a uniform positive constant $c = c(\delta)$. Note that $J_s(t) := \frac{\partial\varphi}{\partial s}(t, s)$ is a Jacobi field along σ_s. Set $T_s(t) := \frac{\partial\varphi}{\partial t}(t, s) = \dot\sigma_s(t)$. Observe that

$$df(T_s(t)) = T_{s/2}(t),\ \ df(J_s(t)) = \frac{1}{2}J_{s/2}(t). \tag{4.25}$$

Lemma 4.7 shows that

$$\left|\frac{L(\sigma_{s/2})}{L(\sigma_s)} - \frac{1}{2}\right| < 3r^2,$$

which implies that

$$\left|\frac{|df(T_s)|}{|T_s|} - \frac{1}{2}\right| < 3r^2. \tag{4.26}$$

Next we show

Lemma 4.11. *For every $s, u \in (0, 1]$ and $t \in [0, 1]$, we have*

$$\left|\frac{|J_u(t)|}{|J_s(t)|} - 1\right| < C(\delta)r^2. \tag{4.27}$$

From now on, we shall use the general symbols $C(\delta)$ or $c(\delta)$ to denote constants depending only on δ unless otherwise stated.

Proof. For any fixed s, take unique Jacobi fields Y_1 and Y_2 along σ_s and the reverse geodesic $\sigma_s^-(t) := \sigma(1 - t)$ respectively such that

$$Y_1(0) = 0,\ Y_1(1) = J_s(1),\ \ Y_2(1) = J_s(0),\ Y_2(0) = 0,$$

to have

$$J_s(t) = Y_1(t)) + Y_2(1 - t).$$

We dente by $\mathbb{S}^2$ and $\mathbb{H}^2$ the sphere and the hyperbolic plane of constant curvature 1 and -1 respectively.

Recall that Δ is a δ-non-degenerate geodesic triangle region of side lengths (a_1, a_2, a_3) in D whose diameter is denoted by r.

Lemma 4.12. *Let α_{i+} and α_{i-}be the angles of comparison triangles Δ_+ and Δ_- of Δ in $\mathbb{S}^2$ and $\mathbb{H}^2$ respectively at the vertices opposite to the edge of length a_i. Then we have*

$$|\alpha_{i+} - \alpha_{i-}| < C(\delta)r^2.$$

Proof. Put $(a,b,c) := (a_1,a_2,a_3)$, and let α_+, α_- and α be the angles of comparison triangles of Δ in $\mathbb{S}^2$, $\mathbb{H}^2$ and $\mathbb{R}^2$ respectively at the vertices opposite to the edge of length a. By the laws of cosines, we have

$$\begin{aligned}\sin b \sin c \cos \alpha_+ &= \cos a - \cos b \cos c \\ \sinh b \sinh c \cos \alpha_- &= \cosh b \cosh c - \cosh a \\ 2bc \cos \alpha &= b^2 + c^2 - a^2,\end{aligned}$$

which imply

$$\begin{aligned}2bc\cos\alpha_+ &= 2bc\cos\alpha + O(b^3c) + O(bc^3) + O(b^2c^2) + O(a^4) \\ 2bc\cos\alpha_- &= 2bc\cos\alpha + O(b^3c) + O(bc^3) + O(b^2c^2) + O(a^4).\end{aligned}$$

It follows from Sublemma4.6 that

$$\begin{aligned}|\cos\alpha_+ - \cos\alpha| &\le O(b^2) + O(c^2) + O(bc) + O(a^4/bc) \\ &\le C(\delta)r^2.\end{aligned}$$

Since $\delta < \alpha < \pi - \delta$, we obtain $|\alpha_+ - \alpha| \le C(\delta)r^2$. Similarly we get $|\alpha_- - \alpha| \le C(\delta)r^2$, and hence $|\alpha_+ - \alpha_-| \le C(\delta)r^2$. □

Let α_s and β_s be the angle of the geodesic triangle $\Delta(1:s) = (\gamma_2|_{[0,s]}, \gamma_3|_{[0,s]}, \sigma_s)$ at $\gamma_2(s)$ and $\gamma_3(s)$ respectively.

Lemma 4.13.

$$|\alpha_s - \alpha_t| < c(\delta)r^2, \ |\beta_s - \beta_t| < c(\delta)r^2,$$

for every $s,t \in (0,1]$.

Proof. Let α_s^+ , α_s^- , α_s^0 denote the angles of comparison triangles in $\mathbb{S}^2$, $\mathbb{H}^2$, and $\mathbb{R}^2$ respectively at the vertices coresponding $\gamma_2(s)$. By Toponogov's theorem (cf. [5]), we have

$$\alpha_s^- \le \alpha_s, \, \alpha_s^0 \le \alpha_s^+. \tag{4.28}$$

By the law of cosines, we have

$$\begin{aligned}\cos\alpha_s^0 &= \frac{a_2^2 + (a_1(s)/s)^2 - a_3^2}{2a_2(a_1(s)/s)} \\ \cos\alpha_t^0 &= \frac{a_2^2 + (a_1(t)/t)^2 - a_3^2}{2a_2(a_1(t)/t)},\end{aligned}$$

which imply with Lemma4.7

$$\begin{aligned}
&\cos\alpha_s^0 - \cos\alpha_t^0 \\
&\qquad \le \frac{a_2^2 + a_1^2(1+r^2) - a_3^2}{2a_2a_1(1-r^2)} - \frac{a_2^2 + a_1^2(1-r^2) - a_3^2}{2a_2a_1(1+r^2)} \\
&\qquad = \frac{r^2(2a_1^2 + a_2^2 - a_3^2)}{a_1a_2(1-r^2)(1+r^2)} \\
&\qquad = \frac{r^2}{1-r^4}\left(\frac{2a_1}{a_2} + \frac{a_2}{a_1} - \frac{a_3^2}{a_1a_2}\right) \\
&\qquad \le C(\delta)r^2.
\end{aligned}$$

Revercing the role of s and t, we have

$$|\cos\alpha_s^0 - \cos\alpha_t^0| \le C(\delta)r^2.$$

By Lemma 4.9, we have $\delta/2 < (\alpha_s^0 + \alpha_t^0)/2 < \pi - \delta/2$, which implies $\sin\frac{\alpha_s^0+\alpha_t^0}{2} > \sin(\delta/2)$. Therefore we conclude that

$$|\alpha_s^0 - \alpha_t^0| \le 4\left|\sin\left(\frac{\alpha_s^0 - \alpha_t^0}{2}\right)\right| \le C_1(\delta)r^2.$$

where $C_1(\delta) := \frac{2C(\delta)}{\sin(\delta/2)}$ Using (4.28) and Lemma 4.12, we see

$$\begin{aligned}
\alpha_s &\le \alpha_s^0 + C(\delta)r^2 \\
&\le \alpha_t^0 + C(\delta)r^2 + C_1(\delta)r^2 \\
&\le \alpha_t + 2C(\delta)r^2 + C_1(\delta)r^2.
\end{aligned}$$

Reversing the role of s and t completes the proof. $\square$

Next we analyze the behavior of the norm of Jacobi field J_s. For a fixed $s \in (0,1]$, let $Y_i(t) = Y_i^N(t) + Y_i^T(t)$, $i = 1,2$, be the orthogonal decompositions of Y_i to the normal and tangential components to $\dot\sigma_s$. We can write $Y_i(t)$ and $Y_i(t)^N$ as

(4.29)
$$Y_1(t) = d\exp_{\gamma_2(s)}(t(V_1)_{t\dot\sigma_s(0)}),\ Y_2(t) = d\exp_{\gamma_3(s)}(t(V_2)_{t\dot\sigma_s^-(0)}),$$

(4.30)
$$Y_1^N(t) = d\exp_{\gamma_2(s)}(t(V_1^N)_{t\dot\sigma_s(0)}),\ Y_2^N(t) = d\exp_{\gamma_3(s)}(t(V_2^N)_{t\dot\sigma_s^-(0)}),$$

where V_1 and V_2 are some parallel vector fields on the tangent spaces satisfying

$$d\exp_{\gamma_2(s)}((V_1)_{\dot\sigma_s(0)}) = \dot\gamma_3(s),\ d\exp_{\gamma_3(s)}((V_2)_{\dot\sigma_s^-(0)}) = \dot\gamma_2(s).$$

The Rauch comparison theorem shows that

$$|Y_1^N(t)| \fallingdotseq t|V_1^N| \fallingdotseq t|\dot\gamma_3(t)^N|,\ |Y_2^N(1-t)| \fallingdotseq (1-t)|V_2^N| \fallingdotseq (1-t)|\dot\gamma_2(t)^N|.$$

Here and hereafter we use the symbol $a \risingdotseq b$ whenever $\left|\frac{a}{b} - 1\right| < C(\delta)r^2$. It follows from $\dim M = 2$ that

$$|J_s^N(t)| = |Y_1^N(t)| + |Y_2^N(1-t)| \tag{4.31}$$
$$\risingdotseq t|\dot\gamma_3(t)^N| + (1-t)|\dot\gamma_2(t)^N| \tag{4.32}$$
$$= t\sin\beta_s a_3 + (1-t)\sin\alpha_s a_2, \tag{4.33}$$

where we recall $a_i = L(\gamma_i) = |\dot\gamma_i(t))|$. Similarly we have

$$|J_u^N(t)| \risingdotseq t\sin\beta_u a_3 + (1-t)\sin\alpha_u a_2.$$

It follows from that

$$|J_s^N(t)| \risingdotseq |J_u^N(t)|. \tag{4.34}$$

Next we show that

$$|J_s^T(t)| \risingdotseq |J_u^T(t)|. \tag{4.35}$$

We use the expression (4.29) with Gauss's lemma to obtain

$$\langle Y_1(t), T_s(t)\rangle = ta_3|T_s|\cos\beta_s,$$
$$\langle Y_2(t), T_s(t)\rangle = -(1-t)a_2|T_s|\cos\alpha_s.$$

Thus we get

$$|J_s^T(t)| = |ta_3\cos\beta_s - (1-t)a_2\cos\alpha_s|.$$

From an inequality for $|J_u^T(t)|$ similar to the above and Lemma 4.13, we have (4.35). Now (4.27) follows from (4.34), (4.35). Thus we have completed the proof of Lemma 4.11. □

The expression (4.29) also yields

$$|Y_1(t)| \risingdotseq t|V_1| \risingdotseq ta_3,\ |Y_2(1-t)| \risingdotseq (1-t)|V_2| \risingdotseq (1-t)a_2.$$

In particular we have

$$|J_s(t)| \le 2r. \tag{4.36}$$

Since $|J_s^N(t)| \ge c(\delta)r$ from (4.33), (4.36) implies that the angle $\theta_s(t) := \angle(J_s(t), T_s(t))$ has definite lower and upper bounds:

$$0 < c(\delta) \le \theta_s(t) \le \pi - c(\delta). \tag{4.37}$$

(4.25), (4.26), (4.27) and (4.37) yield that

$$\left|\frac{|df(v)|}{|v|} - \frac{1}{2}\right| < C(\delta)r^2,$$

for every tangent vector v. Thus we conclude that $f : \Delta \to \Delta_1$ is a $(1/2, \varphi_{C(\delta)}, \nu)$-almost similarity map, with $\varphi_{C(\delta)}(x) = C(\delta)x^2$. This completes the proof of Theorem (2) 4.10. □

Proof of Corollary 1.4. In view of Theorem 1.2, it suffices to show that for a geodesic triangle region Δ on a convex domain of a complete surface, if the collection $\{(\Delta_I, f_I)\}_{I\in\mathcal{I}^*}$ gives a $(\{1/2, 1/2, 1/2\}, \varphi_C, \nu)$-asymptotic similarity system with $\varphi_C(x) = Cx^2$ and $0 < \nu < 1$, then Δ is asymptotically non-degenerate.

For a large n_0, fix an abitrary $I_0 = i_1 \cdots i_{n_0} \in \mathcal{I}_{n_0}$, and set

$$W := \Delta_{I_0} = g_{I_0}(\Delta) = f_{I_0} \circ \cdots f_{i_1 i_2} \circ f_{i_1}(\Delta).$$

For every $1 \le i \le k$, put

$$h_i := f_{I_0 i} : W \to W_i = h_i(W) \subset W,$$

and recall from the definition

$$\left| \frac{d(h_i(x), h_i(y))}{d(x,y)} - \lambda_i \right| < o(n_0),$$

where $o(n_0) = \lambda_i \varphi(\nu^{n_0}|\Delta|)$ and therefore $\lim_{n_0\to\infty} o(n_0) = 0$. For $J = j_1 \cdot \cdot j_m$, define $g_J : W \to W_J$ by

$$g_J := h_J \circ \cdots \circ h_{j_1 j_2} \circ h_{j_1},$$

where we use the notation

$$h_{j_1 \cdot\cdot j_\ell} := f_{I j_1 \cdot\cdot j_\ell} : W_{j_1 \cdot\cdot j_{\ell-1}} \to W_{j_1 \cdot\cdot j_\ell},$$

as before. By Lemma 3.5, we have

$$\left| \frac{d(g_J(x), g_J(y))}{d(x,y)} - \lambda_J \right| < o(n_0)\lambda_J,$$

for every $x, y \in W$. We denote by $\mathrm{inrad}(W)$, the inradius of W, the largest $r > 0$ such that an r-ball is contained in W. It follows that

$$\frac{|W_J|}{\mathrm{inrad}(W_J)} \le \frac{1 + o(n_0)}{1 - o(n_0)} \frac{|W|}{\mathrm{inrad}(W)},$$

for every $J \in \mathcal{I}^*$. This implies that there exists a $\delta > 0$ such that Δ_I is δ-nondegenerate for every $I \in \mathcal{I}^*$. $\square$

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