ARITHMETIC OF BORCHERDS PRODUCTS

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ABSTRACT. We compute the divisors of Borcherds products on integral models of orthogonal Shimura varieties. As an application, we obtain an integral version of a theorem of Borcherds on the modularity of a generating series of special divisors.

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1. Introduction

In the series of papers [Bor95, Bor98, Bor99], Borcherds introduced a family of meromorphic modular forms on orthogonal Shimura varieties, whose zeroes and poles are prescribed linear combinations of *special divisors* arising from embeddings of smaller orthogonal Shimura varieties. These meromorphic modular forms are the Borcherds products of the title.

Following the work of Kisin [Kis10] on integral models of general Hodge and abelian type Shimura varieties, the theory of integral models of orthogonal Shimura varieties and their special divisors was developed further in [Mad16, AGHM17a, AGHMP17b].

The goal of this paper is to combine the above theories to compute the divisor of a Borcherds product on the integral model of an orthogonal Shimura variety. We show that such a divisor is given as a prescribed linear combination of special divisors, exactly as in the generic fiber.

The first such results were obtained by Bruinier, Burgos Gil, and Kühn, who worked on Hilbert modular surfaces (a special type of signature (2, 2)

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orthogonal Shimura variety). Those results were later extended to more general orthogonal Shimura varieties by Hörmann [Hör10, Hör14], but with some strong restrictions.

Our results extend Hörmann's, but with essentially no restrictions. For example, our results include cases where the integral model is not smooth, cases where the divisors in question may have irreducible components supported in nonzero characteristics, and even cases where the Shimura variety is compact (so that one cannot use q-expansions to study the divisor of the Borcherds product).

1.1. **The Shimura variety.** Suppose $n \ge 1$, and (V, Q) is a quadratic space over \mathbb{Q} of signature (n, 2).

From this quadratic space we construct in §4.1 a Shimura datum (G, \mathcal{D}) . The group G = GSpin(V) is a subgroup of the group of units in the Clifford algebra C(V), and sits in a short exact sequence

$$1 \to \mathbb{G}_m \to G \to \mathrm{SO}(V) \to 1.$$

The hermitian symmetric domain is

$$\mathcal{D} = \{ z \in V_{\mathbb{C}} : [z, z] = 0, [z, \overline{z}] < 0 \} / \mathbb{C}^{\times},$$

where [-,-] is the bilinear form on V determined by Q, extended \mathbb{C} -bilinearly to $V_{\mathbb{C}}$. A choice of compact open subgroup $K \subset G(\mathbb{A}_f)$ determines a complex Shimura variety

$$\operatorname{Sh}_K(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q}) \setminus (\mathcal{D} \times G(\mathbb{A}_f)/K),$$

whose canonical model is a smooth Deligne-Mumford stack over $\mathbb Q$ of dimension n.

Fix a \mathbb{Z} -lattice $V_{\mathbb{Z}} \subset V$ that is *maximal* in the sense that Q is \mathbb{Z} -valued on $V_{\mathbb{Z}}$, but there is no proper superlattice with this property. For the rest of the introduction we assume that

$$K \subset G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^{\times},$$

were $C(V_{\widehat{\mathbb{Z}}})$ is the Clifford algebra of the $\widehat{\mathbb{Z}}$ -quadratic space $V_{\widehat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$. We also fix a prime p, and assume that K factors as $K = K_p K^p$ with

$$K_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^{\times}.$$

For such choice of K, we recall in $\S 6.2$ the construction of a flat and normal integral model

$$S_K(G, \mathcal{D}) \to \operatorname{Spec}(\mathbb{Z}_{(p)})$$

of $\operatorname{Sh}_K(G,\mathcal{D})$. If the lattice $V_{\mathbb{Z}}$ is self-dual at p (or even almost self-dual in the sense of Definition 6.1.1) then the integral model is smooth, but in general it need not even be regular. However, at least if $n \geq 5$, Proposition 7.1.2 tells us that it has geometrically normal fibers.

The integral model carries over it a metrized line bundle

$$\widehat{\boldsymbol{\omega}} \in \widehat{\operatorname{Pic}}(\mathcal{S}_K(G, \mathcal{D}))$$

of weight one modular forms. This is defined first over the generic fiber in §4.4, and then extended to the integral model in §6.2. Under the complex uniformization of $Sh_K(G, \mathcal{D})$, this line bundle pulls back to the tautological bundle on \mathcal{D} , with the metric defined by (4.4.3).

The integral model also carries a family of effective Cartier divisors

$$\mathcal{Z}(m,\mu) \to \mathcal{S}_K(G,\mathcal{D})$$

indexed by positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. If $n \geq 3$ these divisors are $\mathbb{Z}_{(p)}$ -flat, at least under some extra mild hypotheses. See Proposition 7.1.4. If $n \in \{1,2,3\}$ one should not expect these divisors to be $\mathbb{Z}_{(p)}$ -flat. In these low-dimensional cases the orthogonal Shimura varieties and their special divisors can be interpreted as moduli space of abelian varieties with additional structure, as in the work of Kudla-Rapoport [KR00b, KR99, KR00a]. Already in the case of n=1, Kudla and Rapoport [KR00b] provide examples in which the special divisors are not flat.

1.2. Borcherds products. In §5.1, we recall the Weil representation

$$\rho_{V_{\mathbb{Z}}} : \widetilde{\mathrm{SL}}_{2}(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_{V_{\mathbb{Z}}})$$

of the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Z})$ on the finite-dimensional \mathbb{C} -vector space $S_{V_{\mathbb{Z}}} = \mathbb{C}[V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}]$. Any weakly holomorphic form

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c(m) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

valued in the complex-conjugate representation has Fourier coefficients $c(m) \in S_{V_{\mathbb{Z}}}$, and we denote by $c(m, \mu)$ the value of c(m) at the coset $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$.

Fix such an f, assume that all $c(m,\mu) \in \mathbb{Z}$, and define a Cartier divisor

$$\mathcal{Z}(f) = \sum_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m, \mu) \cdot \mathcal{Z}(m, \mu).$$

on $\mathcal{S}_K(G,\mathcal{D})$. After possibly replacing f by a positive integer multiple, Borcherds [Bor98] constructs a Green function $\Theta^{reg}(f)$ for $\mathcal{Z}(f)$ as a regularized theta lift, and a constructs a meromorphic section $\psi(f)$ of $\omega^{c(0,0)}$ satisfying

(1.2.1)
$$-2\log||\psi(f)|| = \Theta^{reg}(f).$$

In particular, this relation implies that the divisor of $\psi(f)$ is the analytic divisor $\mathcal{Z}(f)(\mathbb{C})$. The meromorphic section $\psi(f)$ is the *Borcherds product*, which we recall in §5.

Our main result, stated in the text as Theorem 7.3.2, concerns the arithmetic properties of $\psi(f)$. We prove that the Borcherds product is algebraic and defined over \mathbb{Q} , and has the expected divisor when viewed as a rational section of the line bundle $\omega^{c(0,0)}$ on the integral model $\mathcal{S}(G,\mathcal{D})$.

Theorem A. As above, assume that all $c(m, \mu) \in \mathbb{Z}$. After possibly replacing f by a positive integer multiple, there is a rational section $\psi(f)$ of the line bundle $\omega^{c(0,0)}$ on $S(G, \mathcal{D})$ satisfying (1.2.1), and satisfying

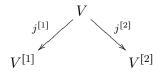
$$\operatorname{div}(\boldsymbol{\psi}(f)) = \mathcal{Z}(f).$$

As noted earlier, similar results can be found in the work of Hörmann [Hör10, Hör14]. Hörmann only considers self-dual lattices, so that the corresponding integral model $\mathcal{S}_K(G,\mathcal{D})$ is smooth, and always assumes that the quadratic space V admits an isotropic line. This allows him to prove the flatness of $\operatorname{div}(\psi(f))$ by examining the q-expansion of $\psi(f)$ at a cusp. As Hörmann's special divisors $\mathcal{Z}(m,\mu)$ (unlike ours) are defined as the Zariski closures of their generic fibers, the equality $\operatorname{div}(\psi(f)) = \mathcal{Z}(f)$ is then a formal consequence of the same equality in the generic fiber, which was proved by Borcherds.

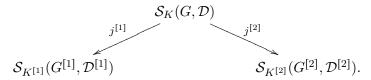
In contrast, we can prove the equality $\operatorname{div}(\psi(f)) = \mathcal{Z}(f)$ even in cases where the divisors may not be flat. More substantially, we can prove this equality even in cases where V contains no isotropic line, hence no theory of q-expansions is available. In the absence of q-expansions, even the fact that the Borcherds product is defined over \mathbb{Q} is a new result.

Indeed, we first prove the above theorem under the assumption that $n \geq 5$. This is Proposition 7.3.1 in the text. The assumption $n \geq 5$ has three crucial consequences. First, it guarantees the existence of an isotropic line $I \subset V$. Second, it guarantees that our integral model has geometrically normal special fiber, so that we may use the results of [Mad] to fix a toroidal compactification in such a way that every irreducible component of the special fiber of $S_K(G, \mathcal{D})$ meets a boundary stratum associated with I. Finally, it guarantees the flatness of all special divisors $\mathcal{Z}(m,\mu)$. By examining the q-expansion of $\psi(f)$ along the boundary, we see that its divisor is flat, and the equality of divisors $\operatorname{div}(\psi(f)) = \mathcal{Z}(f)$ once again follows from the corresponding equality in the generic fiber.

To explain how to remove the hypothesis $n \ge 5$, we must first recall how Borcherds constructs $\psi(f)$ in the complex fiber. If V contains an isotropic line, the construction of $\psi(f)$ ultimately amounts to explicitly writing down its q-expansion. Suppose now that V contains no such line. What Borcherds does is fix isometric embeddings



into (very particular) quadratic spaces of signature (n+24,2). These choices determine morphisms of orthogonal Shimura varieties

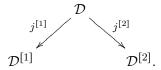


As both $V^{[1]}$ and $V^{[2]}$ contain isotropic lines, one can construct Borcherds products on their associated Shimura varieties. The idea is that one should define

(1.2.2)
$$\psi(f) = \frac{(j^{[2]})^* \psi(f^{[2]})}{(j^{[1]})^* \psi(f^{[1]})}$$

for (very particular) weakly holomorphic forms $f^{[1]}$ and $f^{[2]}$.

The problem is that the quotient on the right hadn side of (1.2.2) is nearly always either 0/0 or ∞/∞ , and so doesn't really make sense. Borcherds gets around this via an analytic construction on the level of hermitian domains



On the hermitian domain

$$\mathcal{D}^{[i]} = \{ z \in V_{\mathbb{C}}^{[i]} : [z, z] = 0, [z, \overline{z}] < 0 \} / \mathbb{C}^{\times} \subset \mathbb{P}(V_{\mathbb{C}}),$$

every irreducible component of every special divisor has the form

$$\mathcal{D}^{[i]}(x) = \{ z \in \mathcal{D}^{[i]} : z \perp x \}$$

for some $x \in V^{[i]}$. The dual of the tautological line bundle ω on $\mathcal{D}^{[i]}$ therefore admits a canonical section

$$\psi_x \in H^0(\mathcal{D}^{[i]}, \boldsymbol{\omega}^{-1})$$

with zero locus $\mathcal{D}^{[i]}(x)$, defined by $\psi_x(z) = [x, z]$.

Whenever there is an $x \in V^{[i]}$ such that $\mathcal{D} \subset \mathcal{D}^{[i]}(x)$, Borcherds multiplies $\psi(f^{[i]})$ by a suitable power of ψ_x in order to remove the component $\mathcal{D}^{[i]}(x)$ from $\operatorname{div}(\psi(f^{[i]})$. After modifying both $\psi(f^{[1]})$ and $\psi(f^{[2]})$ in this way, the quotient (1.2.2) is defined. This process is what Borcherds calls the *embedding trick* in [Bor98]. As understood by Borcherds, the embedding trick is a purely analytic construction. The sections ψ_x over $\mathcal{D}^{[i]}$ do not descend to the Shimura varieties, and have no obvious algebraic properties. In particular, even if one knows that the $\psi(f^{[i]})$ are defined over \mathbb{Q} , it is not obvious that the analytically defined (1.2.2) is defined over \mathbb{Q} .

The main new contribution of this paper is an algebraic analogue of the embedding trick, which works even on the level of integral models. This is based on the methods of improper intersection developed in [BHY15,

AGHM17a]. What we do, essentially, is construct an analogue of the section ψ_x , not over all of $\mathcal{S}_{K^{[i]}}(G^{[i]}, \mathcal{D}^{[i]})$, but only over the first order infinitesimal neighborhood of $\mathcal{S}_K(G, \mathcal{D})$ in $\mathcal{S}_{K^{[i]}}(G^{[i]}, \mathcal{D}^{[i]})$. This section, whose construction uses the Grothendieck-Messing deformation theory, is the *obstruction* to deforming x appearing in the proof of Lemma 7.2.4.

1.3. Modularity of the generating series of divisors. Now take

$$K = G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^{\times}$$

for the level structure. The integral models over $\mathbb{Z}_{(p)}$ can be glued together as p varies to define a normal and flat integral model $\mathcal{S}_K(G,\mathcal{D})$ over \mathbb{Z} , endowed with special divisors and a line bundle of weight one modular forms ω . We denote by

$$\mathcal{Z}(m,\mu) \in \operatorname{Pic}(\mathcal{S}_K(G,\mathcal{D}))$$

the associated line bundles for m>0 and $\mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. For notational consistency, we also set

$$\mathcal{Z}(0,0) = \boldsymbol{\omega}^{-1},$$

and
$$\mathcal{Z}(0,\mu) = 0$$
 if $\mu \neq 0$.

Exactly as in the work of Borcherds [Bor99], Theorem A produces enough relations in the group $\text{Pic}(\mathcal{S}_K(G,\mathcal{D}))$ to prove the modularity of the generating series of special divisors.

Theorem B. The formal generating series

$$\phi(\tau) = \sum_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \mathcal{Z}(m,\mu) \cdot q^{m}$$

is modular, in the sense that $\alpha(\phi) \in M_{1+\frac{n}{2}}(\rho_{V_{\mathbb{Z}}})$ for any \mathbb{Z} -linear map

$$\alpha : \operatorname{Pic}(\mathcal{S}_K(G, \mathcal{D})) \to \mathbb{C}.$$

Theorem B is stated in the text as Theorem 8.2.1. After endowing the special divisors with Green functions as in [Bru02], we also prove a modularity result in the group of metrized line bundles. See Theorem 8.3.1.

1.4. Organization of the paper. Ultimately, all arithmetic information about Borcherds products comes from their q-expansions, and so we must make heavy use of the arithmetic theory of toroidal compactifications of Shimura varieties of [Pin89, Mad]. This theory requires introducing a substantial amount of notation just to state the main results. Also, because Borcherds products are rational sections of powers the line bundle ω , we need to use the theory of automorphic vector bundles on toroidal compactifications. This theory is distributed across a series of papers of Harris [Har84, Har85, Har86, Har89] and Harris-Zucker [HZ94a, HZ94b, HZ01].

Accordingly, even before we begin to talk about orthogonal Shimura varieties, we first recall in §2 the main results on toroidal compactification from Pink's thesis [Pin89], and in §3 the results of Harris and Harris-Zucker

on automorphic vector bundles. All of this is in the generic fiber of fairly general Shimura varieties.

In $\S 4$ we specialize this theory to the case of orthogonal Shimura varieties, in order to formulate a q-expansion principle for detecting the field of definition of orthogonal modular forms.

In §5 we introduce the Borcherds products and prove that they descend to \mathbb{Q} , under the temporary assumption that V contains an isotropic line.

It is only in §6 that we at last introduce integral models of orthogonal Shimura varieties, their special divisors, and their toroidal compactifications. Much of this material is drawn from [Mad16, Mad, AGHM17a, AGHMP17b]. The culmination of the discussion is Corollary 6.4.4. This is an integral q-expansion principle that allows one to detect the flatness of the divisor of a rational section of ω by examining its q-expansion.

Finally, in §7 we prove Theorem A. As we have already indicated, the idea is to first prove the equality $\operatorname{div}(\psi(f)) = \mathcal{Z}(f)$ when $n \geq 5$, by proving that both divisors are flat. The general case then follows from our algebraic version of the embedding trick. The key result here is the pullback formula of Theorem 7.2.2, which describes how special divisors on an orthogonal Shimura variety decompose when they are intersected with a smaller embedded orthogonal Shimura variety. Crucially for the embedding trick, this description does not assume that the divisors have proper intersection with the smaller Shimura variety.

The modularity result of Theorem B, and its extension to the group of metrized line bundles, is contained in §8. The argument here is identical to that used by Borcherds [Bor99] to prove modularity in the complex fiber.

1.5. Notation and conventions. For every $a \in \mathbb{A}_f^{\times}$ there is a unique factorization

$$a = rat(a) \cdot unit(a)$$

in which $\operatorname{rat}(a)$ is a positive rational number and $\operatorname{unit}(a) \in \widehat{\mathbb{Z}}^{\times}$.

Class field theory provides us with a reciprocity map

$$\operatorname{rec}: \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times} \cong \operatorname{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}),$$

which we normalize as follows: Let μ_{∞} be the set of all roots of unity in \mathbb{C} , so that $\mathbb{Q}^{ab} = \mathbb{Q}(\mu_{\infty})$ is the maximal abelian extension of \mathbb{Q} . The group $(\mathbb{Z}/M\mathbb{Z})^{\times}$ acts on the set of M^{th} roots of unity in the usual way, by letting $u \in (\mathbb{Z}/M\mathbb{Z})^{\times}$ act by $\zeta \mapsto \zeta^{u}$. Passing to the limit yields an action of $\widehat{\mathbb{Z}}^{\times}$ on μ_{∞} , and the reciprocity map is characterized by

$$\zeta^{\mathrm{rec}(a)} = \zeta^{\mathrm{unit}(a)}$$

for all $a \in \mathbb{A}_f^{\times}$ and $\zeta \in \mu_{\infty}$.

We follow the conventions of [Del79] and [Pin89, Chapter 1] for Hodge structures and mixed Hodge structures. As usual, $\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m\mathbb{C}}$ is Deligne's torus, so that $\mathbb{S}(\mathbb{C}) = \mathbb{C}^{\times} \times \mathbb{C}^{\times}$, with complex conjugation acting by $(t_1, t_2) \mapsto (\bar{t}_2, \bar{t}_1)$. In particular, $\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^{\times}$ by $(t, \bar{t}) \mapsto t$. If V is a

rational vector space endowed with a Hodge structure $\mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$, then $V^{(p,q)} \subset V_{\mathbb{C}}$ is the subspace on which $(t_1,t_2) \in \mathbb{C}^{\times} \times \mathbb{C}^{\times} = \mathbb{S}(\mathbb{C})$ acts via $t_1^{-p}t_2^{-q}$. There is a distinguished cocharacter

$$\operatorname{wt}: \mathbb{G}_{m\mathbb{R}} \to \mathbb{S}$$

defined on complex points by $t \mapsto (t^{-1}, t^{-1})$. The composition

$$\mathbb{G}_{m\mathbb{R}} \xrightarrow{\mathrm{wt}} \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

encodes the weight grading on $V_{\mathbb{R}}$, in the sense that

weight grading on
$$V_{\mathbb{R}}$$
, in the sense that
$$\bigoplus_{p+q=k} V^{(p,q)} = \{ v \in V_{\mathbb{C}} : \operatorname{wt}(z) \cdot v = z^k \cdot v, \ \forall z \in \mathbb{C}^{\times} \}.$$

Now suppose that V is endowed with a mixed Hodge structure. This consists of an increasing weight filtration $\operatorname{wt}_{\bullet}V$ on V, and a decreasing Hodge filtration $F^{\bullet}V_{\mathbb{C}}$ on $V_{\mathbb{C}}$, whose induced filtration on every graded piece

$$\operatorname{gr}_k V = \operatorname{wt}_k V / \operatorname{wt}_{k-1} V$$

is a pure Hodge structure of weight k. By [PS08, Lemma-Definition 3.4] there is a canonical bigrading $V_{\mathbb{C}} = \bigoplus V^{(p,q)}$ with the property that

$$\operatorname{wt}_k V_{\mathbb C} = \bigoplus_{p+q \leqslant k} V^{(p,q)}, \quad F^i V_{\mathbb C} = \bigoplus_{p \geqslant i} V^{(p,q)}.$$

This bigrading is induced by a morphism $\mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$.

2. TOROIDAL COMPACTIFICATION

This section is a (relatively) short summary of Pink's thesis [Pin89] on toroidal compactifications of canonical models of Shimura varieties. See also [Hör10] and [HZ94a, HZ01]. We limit ourselves to what is needed in the sequel, and simplify the discussion somewhat by only dealing with those mixed Shimura varieties that appear at the boundary of pure Shimura varieties.

2.1. Shimura varieties. Throughout §2 and §3 we let (G, \mathcal{D}) be a (pure) Shimura datum in the sense of [Pin89, §2.1]. Thus G is a reductive group over \mathbb{Q} , and \mathcal{D} is a $G(\mathbb{R})$ -homogeneous space equipped with a finite-to-one $G(\mathbb{R})$ -equivariant map

$$h: \mathcal{D} \to \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$$

such that the pair $(G, h(\mathcal{D}))$ satisfies Deligne's axioms [Del79, (2.1.1.1)-(2.1.1.3)]. We often abuse notation and confuse $z \in \mathcal{D}$ with its image h(z). The weight cocharacter

(2.1.1)
$$w \stackrel{\text{def}}{=} \mathbf{h}(z) \circ \mathbf{wt} : \mathbb{G}_{m\mathbb{R}} \to G_{\mathbb{R}}$$

of (G, \mathcal{D}) is independent of $z \in \mathcal{D}$, and takes values in the center of $G_{\mathbb{R}}$.

Hypothesis 2.1.1. Because it will simplify much of what follows, and because it is assumed throughout [HZ01], we always assume that (G, \mathcal{D}) satisfies:

(1) The weight cocharacter (2.1.1) is defined over \mathbb{Q} .

(2) The connected center of G is isogenous to the product of a \mathbb{Q} -split torus with a torus whose group of real points is compact.

Suppose $K \subset G(\mathbb{A}_f)$ is any compact open subgroup. The associated Shimura variety

$$\operatorname{Sh}_K(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q}) \setminus (\mathcal{D} \times G(\mathbb{A}_f)/K)$$

is a complex orbifold. Its canonical model $\operatorname{Sh}_K(G,\mathcal{D})$ is a Deligne-Mumford stack over the reflex field $E(G,\mathcal{D}) \subset \mathbb{C}$. If K is neat in the sense of [Pin89, $\S 0.6$], then $\operatorname{Sh}_K(G,\mathcal{D})$ is a quasi-projective scheme. The image of a point $(z,g) \in \mathcal{D} \times G(\mathbb{A}_f)$ is denoted

$$[(z,g)] \in \operatorname{Sh}_K(G,\mathcal{D})(\mathbb{C}).$$

Remark 2.1.2. Let $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^{\times}$ act on the two point set

$$\mathcal{H}_0 \stackrel{\text{def}}{=} \{2\pi\epsilon \in \mathbb{C} : \epsilon^2 = -1\}$$

via the unique continuous transitive action: positive real numbers act trivially, and negative real numbers swap the two points. If we define $\mathcal{H}_0 \to \text{Hom}(\mathbb{S}, \mathbb{G}_{m\mathbb{R}})$ by sending both points to the norm map $\mathbb{C}^\times \to \mathbb{R}^\times$, then $(\mathbb{G}_m, \mathcal{H}_0)$ is a Shimura datum.

2.2. **Mixed Shimura varieties.** Toroidal compactifications of Shimura varieties are obtained by gluing together certain mixed Shimura varieties, which we now define.

Recall from [Pin89, Definition 4.5] the notion of an admissible parabolic subgroup $P \subset G$. If G^{ad} is simple, this just means that P is either a maximal proper parabolic subgroup, or is all of G. In general, it means that if we write $G^{ad} = G_1 \times \cdots \times G_s$ as a product of simple groups, then P is the preimage of a subgroup $P_1 \times \cdots \times P_s$, where each $P_i \subset G_i$ is an admissible parabolic.

Definition 2.2.1. A cusp label representative $\Phi = (P, \mathcal{D}^{\circ}, h)$ for (G, \mathcal{D}) is a triple consisting of an admissible parabolic subgroup P, a connected component $\mathcal{D}^{\circ} \subset \mathcal{D}$, and an $h \in G(\mathbb{A}_f)$.

As in [Pin89, §4.11 and §4.12], any cusp label representative $\Phi = (P, \mathcal{D}^{\circ}, h)$ determines a mixed Shimura datum $(Q_{\Phi}, \mathcal{D}_{\Phi})$, whose construction we now recall.

Let $W_{\Phi} \subset P$ be the unipotent radical, and let U_{Φ} be the center of W_{Φ} . According to [Pin89, §4.1] there is a distinguished central cocharacter λ : $\mathbb{G}_m \to P/W_{\Phi}$. The weight cocharacter $w: \mathbb{G}_m \to G$ is central, so takes values in P, and therefore determines a new central cocharacter

$$(2.2.1) w \cdot \lambda^{-1} : \mathbb{G}_m \to P/W_{\Phi}.$$

Suppose $G \to \mathrm{GL}(N)$ is a faithful representation on a finite dimensional \mathbb{Q} -vector space, then each point $z \in \mathcal{D}$ determines a Hodge filtration $F^{\bullet}N_{\mathbb{C}}$

on N. Any lift of (2.2.1) to a cocharacter $\mathbb{G}_m \to P$ determines a grading $N = \bigoplus N^k$, and the associated weight filtration

$$\operatorname{wt}_{\ell} N = \bigoplus_{k \leqslant \ell} N^k$$

is independent of the lift. The triple $(N, F^{\bullet}N_{\mathbb{C}}, \operatorname{wt}_{\bullet}N)$ is a mixed Hodge structure [Pin89, §4.12, Remark (i)], and its associated bigrading of $N_{\mathbb{C}}$ determines a morphism $h_{\Phi}(z) \in \operatorname{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$, which does not depend on the choice of N. As in [Pin89, §4.7], define $Q_{\Phi} \subset P$ to be the smallest closed normal subgroup through which every such $h_{\Phi}(z)$ factors. Thus

$$U_{\Phi} \lhd W_{\Phi} \lhd Q_{\Phi} \lhd P$$

and we have defined a map

$$h_{\Phi}: \mathcal{D} \to \operatorname{Hom}(\mathbb{S}_{\mathbb{C}}, Q_{\Phi\mathbb{C}}).$$

The cocharacter (2.2.1) takes values in Q_{Φ}/W_{Φ} , defining the weight cocharacter

$$(2.2.2) w_{\Phi}: \mathbb{G}_m \to Q_{\Phi}/W_{\Phi}.$$

Remark 2.2.2. Being an abelian unipotent group, $\text{Lie}(U_{\Phi}) \cong U_{\Phi}$ has the structure of a \mathbb{Q} -vector space. By [Pin89, Proposition 2.14], the conjugation action of Q_{Φ} on U_{Φ} is through a character

$$(2.2.3) \nu_{\Phi}: Q_{\Phi} \to \mathbb{G}_m.$$

By [Pin89, Proposition 4.15(a)], the map h_{Φ} restricts to an open immersion on every connected component of \mathcal{D} , and so the diagonal map

$$\mathcal{D} \hookrightarrow \pi_0(\mathcal{D}) \times \operatorname{Hom}(\mathbb{S}_{\mathbb{C}}, Q_{\Phi\mathbb{C}})$$

is a $P(\mathbb{R})$ -equivariant open immersion. The action of the subgroup $U_{\Phi}(\mathbb{R})$ on $\pi_0(\mathcal{D})$ is trivial, and we extend it to the trivial action of $U_{\Phi}(\mathbb{C})$ on $\pi_0(\mathcal{D})$. Now define

$$\mathcal{D}_{\Phi} = Q_{\Phi}(\mathbb{R})U_{\Phi}(\mathbb{C})\mathcal{D}^{\circ} \subset \pi_0(\mathcal{D}) \times \operatorname{Hom}(\mathbb{S}_{\mathbb{C}}, Q_{\Phi\mathbb{C}}).$$

Projection to the second factor defines a finite-to-one map

$$h_{\Phi}: \mathcal{D}_{\Phi} \to \operatorname{Hom}(\mathbb{S}_{\mathbb{C}}, Q_{\Phi\mathbb{C}}),$$

and we usually abuse notation and confuse $z \in \mathcal{D}_{\Phi}$ with its image $h_{\Phi}(z)$.

Having now defined the mixed Shimura datum $(Q_{\Phi}, \mathcal{D}_{\Phi})$, the compact open subgroup

$$K_{\Phi} \stackrel{\mathrm{def}}{=} hKh^{-1} \cap Q_{\Phi}(\mathbb{A}_f)$$

determines a mixed Shimura variety

$$(2.2.4) Sh_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C}) = Q_{\Phi}(\mathbb{Q}) \setminus (\mathcal{D}_{\Phi} \times Q_{\Phi}(\mathbb{A}_f)/K_{\Phi}),$$

which has a canonical model $\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})$ over its reflex field. Note that the reflex field is again $E(G, \mathcal{D})$, by [Pin89, Proposition 12.1]. The canonical model is a quasi-projective scheme if K (hence K_{Φ}) is neat.

Remark 2.2.3. If we choose our cusp label representative to have the form $\Phi = (G, \mathcal{D}^{\circ}, h)$, then $(Q_{\Phi}, \mathcal{D}_{\Phi}) = (G, \mathcal{D})$ and

$$\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) = \operatorname{Sh}_{hKh^{-1}}(G, \mathcal{D}) \cong \operatorname{Sh}_{K}(G, \mathcal{D}).$$

As a consequence, all of our statements about the mixed Shimura varieties $\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})$ include the Shimura variety $\operatorname{Sh}_{K}(G, \mathcal{D})$ as a special case.

2.3. The torsor structure. Define $\bar{Q}_{\Phi} = Q_{\Phi}/U_{\Phi}$ and $\bar{\mathcal{D}}_{\Phi} = U_{\Phi}(\mathbb{C})\backslash \mathcal{D}_{\Phi}$. The pair

$$(\bar{Q}_{\Phi}, \bar{\mathcal{D}}_{\Phi}) = (Q_{\Phi}, \mathcal{D}_{\Phi})/U_{\Phi}$$

is the quotient mixed Shimura datum in the sense of [Pin89, §2.9]; see especially the Remark of [loc. cit.]. Let \bar{K}_{Φ} be the image of K_{Φ} under the quotient map $Q_{\Phi}(\mathbb{A}_f) \to \bar{Q}_{\Phi}(\mathbb{A}_f)$, so that we have a canonical morphism

(2.3.1)
$$\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \to \operatorname{Sh}_{\bar{K}_{\Phi}}(\bar{Q}_{\Phi}, \bar{\mathcal{D}}_{\Phi}),$$

where the target mixed Shimura variety is defined in the same way as (2.2.4).

Proposition 2.3.1. Define a \mathbb{Z} -lattice in $U_{\Phi}(\mathbb{Q})$ by $\Gamma_{\Phi} = K_{\Phi} \cap U_{\Phi}(\mathbb{Q})$. The morphism (2.3.1) is canonically a torsor for the relative torus

$$T_{\Phi} \stackrel{\text{def}}{=} \Gamma_{\Phi}(-1) \otimes \mathbb{G}_m$$

with cocharacter group $\Gamma_{\Phi}(-1) = (2\pi i)^{-1}\Gamma_{\Phi}$.

Proof. This is proved in [Pin89, §6.6]. In what follows we only want to make the torsor structure explicit on the level of complex points.

The character (2.2.3) factors through a character $\bar{\nu}_{\Phi}: \bar{Q}_{\Phi} \to \mathbb{G}_m$. A pair $(z,g) \in \mathcal{D}_{\Phi} \times Q_{\Phi}(\mathbb{A}_f)$ determines points

$$[(z,g)] \in \operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C}), \quad [(\bar{z},\bar{g})] \in \operatorname{Sh}_{\bar{K}_{\Phi}}(\bar{Q}_{\Phi}, \bar{\mathcal{D}}_{\Phi})(\mathbb{C}),$$

and we define $\mathbf{T}_{\Phi}(\mathbb{C}) \to \mathrm{Sh}_{\bar{K}_{\Phi}}(\bar{Q}_{\Phi}, \bar{\mathcal{D}}_{\Phi})(\mathbb{C})$ as the relative torus with fiber

$$(2.3.2) U_{\Phi}(\mathbb{C})/(gK_{\Phi}g^{-1} \cap U_{\Phi}(\mathbb{Q})) = U_{\Phi}(\mathbb{C})/\mathrm{rat}(\bar{\nu}_{\Phi}(\bar{g})) \cdot \Gamma_{\Phi}$$

at $[(\bar{z}, \bar{g})]$. There is a natural action of $\mathbf{T}_{\Phi}(\mathbb{C})$ on (2.2.4) defined as follows: using the natural action of $U_{\Phi}(\mathbb{C})$ on \mathcal{D}_{Φ} , a point u in the fiber (2.3.2) acts as $[(z,g)] \mapsto [(uz,g)]$.

It now suffices to construct an isomorphism

$$\mathbf{T}_{\Phi}(\mathbb{C}) \cong T_{\Phi}(\mathbb{C}) \times \operatorname{Sh}_{\bar{K}_{\Phi}}(\bar{Q}_{\Phi}, \bar{\mathcal{D}}_{\Phi})(\mathbb{C}),$$

and this is essentially [Pin89, §3.16]. First choose a morphism

$$\bar{\mathcal{D}}_{\Phi} \xrightarrow{\bar{z} \mapsto 2\pi\epsilon(\bar{z})} \mathcal{H}_0$$

in such a way that it, along with the character $\bar{\nu}_{\Phi}$, induces a morphism of mixed Shimura data $(\bar{Q}_{\Phi}, \bar{\mathcal{D}}_{\Phi}) \to (\mathbb{G}_m, \mathcal{H}_0)$. Such a morphisms always exists, by the Remark of [Pin89, §6.8]. The fiber (2.3.2) is

$$U_{\Phi}(\mathbb{C})/\mathrm{rat}(\bar{\nu}_{\Phi}(\bar{g})) \cdot \Gamma_{\Phi} \xrightarrow{2\pi\epsilon(\bar{z})/\mathrm{rat}(\bar{\nu}_{\Phi}(\bar{g}))} U_{\Phi}(\mathbb{C})/\Gamma_{\Phi}(1),$$

and this identifies $\mathbf{T}_{\Phi}(\mathbb{C})$ fiber-by-fiber with the constant torus

$$U_{\Phi}(\mathbb{C})/\Gamma_{\Phi}(1) \cong \Gamma_{\Phi} \otimes \mathbb{C}/\mathbb{Z}(1) \cong \Gamma_{\Phi} \otimes \mathbb{C}^{\times} \xrightarrow{(-2\pi\epsilon^{\circ})^{-1}} \Gamma_{\Phi}(-1) \otimes \mathbb{C}^{\times}.$$

Here $2\pi\epsilon^{\circ}$ is the image of \mathcal{D}° under $\mathcal{D}_{\Phi} \to \bar{\mathcal{D}}_{\Phi} \to \mathcal{H}_{0}$, and the minus sign is included so that (2.6.5) holds below; compare with the definition of the function "ord" in [Pin89, §5.8].

Remark 2.3.2. Our \mathbb{Z} -lattice $\Gamma_{\Phi} \subset U_{\Phi}(\mathbb{Q})$ agrees with the seemingly more complicated lattice of [Pin89, §3.13], defined as the image of

$$\{(c,\gamma)\in Z_{\Phi}(\mathbb{Q})_0\times U_{\Phi}(\mathbb{Q}): c\gamma\in K_{\Phi}\}\xrightarrow{(c,\gamma)\mapsto\gamma} U_{\Phi}(\mathbb{Q}).$$

Here Z_{Φ} is the center of Q_{Φ} , and $Z_{\Phi}(\mathbb{Q})_0 \subset Z_{\Phi}(\mathbb{Q})$ is the largest subgroup acting trivially on \mathcal{D}_{Φ} (equivalently, acting trivially on $\pi_0(\mathcal{D}_{\Phi})$). This follows from the final comments of [loc. cit.] and the simplifying Hypothesis 2.1.1, which implies that the connected center of Q_{Φ}/U_{Φ} is isogenous to the product of a \mathbb{Q} -split torus and a torus whose group of real points is compact (see the proof of [Pin89, Corollary 4.10]).

Denoting by $\langle \cdot, \cdot \rangle : \Gamma_{\Phi}^{\vee}(1) \times \Gamma_{\Phi}(-1) \to \mathbb{Z}$ the tautological pairing, define an isomorphism

$$\Gamma_{\Phi}^{\vee}(1) \xrightarrow{\alpha \mapsto q_{\alpha}} \operatorname{Hom}(\Gamma_{\Phi}(-1) \otimes \mathbb{G}_m, \mathbb{G}_m) = \operatorname{Hom}(T_{\Phi}, \mathbb{G}_m)$$

by $q_{\alpha}(x \otimes z) = z^{\langle \alpha, x \rangle}$. This determines an isomorphism

$$T_{\Phi} \cong \operatorname{Spec}\left(\mathbb{Q}[q_{\alpha}]_{\alpha \in \Gamma_{\Phi}^{\vee}(1)}\right),$$

and hence, for any rational polyhedral cone¹ $\sigma \subset U_{\Phi}(\mathbb{R})(-1)$, a partial compactification

(2.3.3)
$$T_{\Phi}(\sigma) \stackrel{\text{def}}{=} \operatorname{Spec}\left(\mathbb{Q}[q_{\alpha}]_{\alpha \in \Gamma_{\Phi}^{\vee}(1)}\right).$$

More generally, the T_{Φ} -torsor structure on (2.3.1) determines, by the general theory of torus embeddings [Pin89, §5], a partial compactification

with a stratification by locally closed substacks

(2.3.5)
$$\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) = \bigsqcup_{\tau} Z_{K_{\Phi}}^{\tau}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)$$

indexed by the faces $\tau \subset \sigma$. The unique open stratum

$$Z_{K_{\Phi}}^{\{0\}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) = \operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})$$

¹By which we mean a *convex rational polyhedral cone* in the sense of [Pin89, §5.1]. In particular, each σ is a closed subset of the real vector space $U_{\Phi}(\mathbb{R})(-1)$.

corresponds to $\tau = \{0\}$. The unique closed stratum corresponds to $\tau = \sigma$.

2.4. Rational polyhedral cone decompositions. Let $\Phi = (P, \mathcal{D}^{\circ}, h)$ be a cusp label representative for (G, \mathcal{D}) , with associated mixed Shimura datum $(Q_{\Phi}, \mathcal{D}_{\Phi})$. We denote by $\mathcal{D}_{\Phi}^{\circ} = U_{\Phi}(\mathbb{C})\mathcal{D}^{\circ}$ the connected component of \mathcal{D}_{Φ} containing \mathcal{D}° .

Define the projection to the imaginary part $c_{\Phi}: \mathcal{D}_{\Phi} \to U_{\Phi}(\mathbb{R})(-1)$ by

$$c_{\Phi}(z)^{-1} \cdot z \in \pi_0(\mathcal{D}) \times \operatorname{Hom}(\mathbb{S}, Q_{\Phi\mathbb{R}})$$

for every $z \in \mathcal{D}_{\Phi}$. By [Pin89, Proposition 4.15] there is an open convex cone

$$(2.4.1) C_{\Phi} \subset U_{\Phi}(\mathbb{R})(-1)$$

characterized by $\mathcal{D}^{\circ} = \{z \in \mathcal{D}_{\Phi}^{\circ} : c_{\Phi}(z) \in C_{\Phi}\}.$

Definition 2.4.1. Suppose $\Phi = (P, \mathcal{D}^{\circ}, h)$ and $\Phi_1 = (P_1, \mathcal{D}_1^{\circ}, h_1)$ are cusp label representatives. A K-morphism

$$(2.4.2) \Phi \xrightarrow{(\gamma,q)} \Phi_1$$

is a pair $(\gamma, q) \in G(\mathbb{Q}) \times Q_{\Phi_1}(\mathbb{A}_f)$, such that

$$\gamma Q_{\Phi} \gamma^{-1} \subset Q_{\Phi_1}, \quad \gamma \mathcal{D}^{\circ} = \mathcal{D}_1^{\circ}, \quad \gamma h \in qh_1 K.$$

A K-morphism is a K-isomorphism if $\gamma Q_{\Phi} \gamma^{-1} = Q_{\Phi_1}$.

Remark 2.4.2. The Bailey-Borel compactification of $\operatorname{Sh}_K(G,\mathcal{D})$ admits a stratification by locally closed substacks, defined over the reflex field, whose strata are indexed by the K-isomorphism classes of cusp label representatives. Whenever there is a K-morphism $\Phi \to \Phi_1$, the stratum indexed by Φ is "deeper into the boundary" than the stratum indexed by Φ_1 , in the sense that the Φ -stratum is contained in the closure of the Φ_1 -stratum. The unique open stratum, which is just the Shimura variety $\operatorname{Sh}_K(G,\mathcal{D})$, is indexed by the K-isomorphism class consisting of all cusp label representatives of the form (G,\mathcal{D}°,h) as \mathcal{D}° and h vary.

Suppose we have a K-morphism (2.4.2) of cusp label representatives. It follows from [Pin89, Proposition 4.21] that $\gamma U_{\Phi_1} \gamma^{-1} \subset U_{\Phi}$, and the image of the open convex cone C_{Φ_1} under

$$(2.4.3) U_{\Phi_1}(\mathbb{R})(-1) \xrightarrow{u \mapsto \gamma u \gamma^{-1}} U_{\Phi}(\mathbb{R})(-1)$$

lies in the closure of the open convex cone C_{Φ} . Define, as in [Pin89, Definition-Proposition 4.22],

$$C_{\Phi}^* = \bigcup_{\Phi \to \Phi_1} \gamma C_{\Phi_1} \gamma^{-1} \subset U_{\Phi}(\mathbb{R})(-1),$$

where the union is over all K-morphisms with source Φ . This is a convex cone lying between C_{Φ} and its closure, but in general C_{Φ}^* is neither open nor closed. For every K-morphism $\Phi \to \Phi_1$ as above, the injection (2.4.3) identifies $C_{\Phi_1}^* \subset C_{\Phi}^*$.

Definition 2.4.3. A (rational polyhedral) partial cone decomposition of C_{Φ}^* is a collection $\Sigma_{\Phi} = \{\sigma\}$ of rational polyhedral cones $\sigma \subset U_{\Phi}(\mathbb{R})(-1)$ such that

- each $\sigma \in \Sigma_{\Phi}$ satisfies $\sigma \subset C_{\Phi}^*$,
- every face of every $\sigma \in \Sigma_{\Phi}$ is again an element of Σ_{Φ} ,
- the intersection of any $\sigma, \tau \in \Sigma_{\Phi}$ is a face of both σ and τ ,
- $\{0\} \in \Sigma_{\Phi}$.

We say that Σ_{Φ} is *smooth* if it is smooth, in the sense of [Pin89, §5.2], with respect to the lattice $\Gamma_{\Phi}(-1) \subset U_{\Phi}(\mathbb{R})(-1)$. It is *complete* if

$$C_{\Phi}^* = \bigcup_{\sigma \in \Sigma_{\Phi}} \sigma.$$

Definition 2.4.4. A K-admissible (rational polyhedral) partial cone decomposition $\Sigma = \{\Sigma_{\Phi}\}_{\Phi}$ for (G, \mathcal{D}) is a collection of partial cone decompositions Σ_{Φ} for C_{Φ}^* , one for every cusp label representative Φ , such that for any K-morphism $\Phi \to \Phi_1$, the induced inclusion $C_{\Phi_1}^* \subset C_{\Phi}^*$ identifies

$$\Sigma_{\Phi_1} = \{ \sigma \in \Sigma_{\Phi} : \sigma \subset C_{\Phi_1}^* \}.$$

We say that Σ is *smooth* if every Σ_{Φ} is smooth, and *complete* if every Σ_{Φ} is complete.

Fix a K-admissible, complete cone decomposition Σ of (G, \mathcal{D}) .

Definition 2.4.5. A toroidal stratum representative for (G, \mathcal{D}, Σ) is a pair (Φ, σ) in which Φ is a cusp label representative and $\sigma \in \Sigma_{\Phi}$ is a rational polyhedral cone whose interior is contained in C_{Φ} . In other words, σ is not contained in any proper subset $C_{\Phi_1}^* \subsetneq C_{\Phi}^*$ determined by a K-morphism $\Phi \to \Phi_1$.

We now extend Definition 2.4.1 from cusp label representatives to toroidal stratum representatives.

Definition 2.4.6. A K-morphism of toroidal stratum representatives

$$(\Phi, \sigma) \xrightarrow{(\gamma,q)} (\Phi_1, \sigma_1)$$

consists of a pair $(\gamma, q) \in G(\mathbb{Q}) \times Q_{\Phi_1}(\mathbb{A}_f)$ such that

$$\gamma Q_{\Phi} \gamma^{-1} \subset Q_{\Phi_1}, \quad \gamma \mathcal{D}^{\circ} = \mathcal{D}_1^{\circ}, \quad \gamma h \in qh_1 K,$$

and such that the injection (2.4.3) identifies σ_1 with a face of σ . Such a K-morphism is a K-isomorphism if $\gamma Q_{\Phi} \gamma^{-1} = Q_{\Phi_1}$ and $\gamma \sigma_1 \gamma^{-1} = \sigma$.

A K-isomorphism class of toroidal stratum representatives will be denoted $\Upsilon = [(\Phi, \sigma)]$. The set of all isomorphism classes is denoted $\operatorname{Strat}_K(G, \mathcal{D}, \Sigma)$.

Definition 2.4.7. We say that Σ is *finite* if $\#\text{Strat}_K(G, \mathcal{D}, \Sigma) < \infty$.

Definition 2.4.8. We say that Σ has the *no self-intersection property* if the following holds: whenever we are given toroidal stratum representatives (Φ, σ) and (Φ_1, σ_1) , and two K-morphisms

$$(\Phi, \sigma) \longrightarrow (\Phi_1, \sigma_1),$$

the two injections

$$U_{\Phi_1}(\mathbb{R})(-1) \longrightarrow U_{\Phi}(\mathbb{R})(-1)$$

of (2.4.3) send σ_1 to the same face of σ .

The no self-intersection property is just the condition of [Pin89, $\S7.12$], reworded. If Σ has the no self-intersection property then so does any refinement (in the sense of [Pin89, $\S5.1$]).

Remark 2.4.9. Any finite and K-admissible cone decomposition Σ for (G, \mathcal{D}) acquires the no self-intersection property after possibly replacing K by a smaller compact open subgroup [Pin89, §7.13]. Moreover, by examining the proof one can see that if K factors as $K = K_{\ell}K^{(\ell)}$ for some prime ℓ with $K_{\ell} \subset G(\mathbb{Q}_{\ell})$ and $K^{(\ell)} \subset G(\mathbb{A}_f^{(\ell)})$, then it suffices to shrink K_{ℓ} while holding $K^{(\ell)}$ fixed.

2.5. Functoriality of cone decompositions. Suppose that we have an embedding $(G, \mathcal{D}) \to (G', \mathcal{D}')$ of Shimura data.

As explained in [Mad, (2.1.28)], every cusp label representative

$$\Phi = (P, \mathcal{D}^{\circ}, q)$$

for (G, \mathcal{D}) determines a cusp label representative

$$\Phi' = (P', \mathcal{D}'^{, \circ}, g')$$

for (G', \mathcal{D}') . More precisely, we define g' = g, let $\mathcal{D}'^{,\circ} \subset \mathcal{D}'$ be the connected component containing \mathcal{D} , and let $P' \subset G'$ be the smallest admissible parabolic subgroup containing P. In particular,

$$Q_{\Phi} \subset Q_{\Phi'}, \quad U_{\Phi} \subset U_{\Phi'}, \quad C_{\Phi} \subset C_{\Phi'}.$$

If $K \subset G(\mathbb{A}_f)$ is a compact open subgroup contained in a compact open subgroup $K' \subset G'(\mathbb{A}_f)$, then every K-morphism

$$\Phi \xrightarrow{(\gamma,q)} \Phi_1$$

determines a K'-morphism

$$\Phi' \xrightarrow{(\gamma,q)} \Phi'_1.$$

Any K'-admissible rational cone decomposition Σ' for (G', \mathcal{D}') pulls back to a K-admissible rational cone decomposition Σ for (G, \mathcal{D}) , defined by

$$\Sigma_{\Phi} = \{ \sigma' \cap C_{\Phi}^* : \sigma' \in \Sigma_{\Phi'}' \}$$

for every cusp label representative Φ of (G, \mathcal{D}) . It is shown in [Har89, §3.3] that Σ is finite whenever Σ' is so. It is also not hard to check that Σ has

the no self-intersection property whenever Σ' does, and that it is complete when Σ' is so.

Given a cusp label representative Φ for (G, \mathcal{D}) and a $\sigma \in \Sigma_{\Phi}$, there is a unique rational polyhedral cone $\sigma' \in \Sigma'_{\Phi'}$ such that $\sigma \subset \sigma'$, but σ is not contained in any proper face of σ' . The assignment $(\Phi, \sigma) \mapsto (\Phi', \sigma')$ induces a map

$$\operatorname{Strat}_K(G,\mathcal{D},\Sigma) \xrightarrow{\Upsilon \mapsto \Upsilon'} \operatorname{Strat}_{K'}(G',\mathcal{D}',\Sigma')$$

on toroidal stratum representatives.

2.6. Compactification of canonical models. In this subsection we assume that $K \subset G(\mathbb{A}_f)$ is neat. Suppose Σ is a finite and K-admissible complete cone decomposition for (G, \mathcal{D}) .

The main result of [Pin89, §12] is the existence of a proper toroidal compactification

$$\operatorname{Sh}_K(G, \mathcal{D}) \hookrightarrow \operatorname{Sh}_K(G, \mathcal{D}, \Sigma),$$

in the category of algebraic spaces over $E(G, \mathcal{D})$, along with a stratification

(2.6.1)
$$\operatorname{Sh}_{K}(G, \mathcal{D}, \Sigma) = \bigsqcup_{\Upsilon \in \operatorname{Strat}_{K}(G, \mathcal{D}, \Sigma)} Z_{K}^{\Upsilon}(G, \mathcal{D}, \Sigma)$$

by locally closed subspaces, indexed by the finite set $\operatorname{Strat}_K(G, \mathcal{D}, \Sigma)$ defined above. The stratum indexed by $\Upsilon = [(\Phi, \sigma)]$ lies in the closure of the stratum indexed by $\Upsilon_1 = [(\Phi_1, \sigma_1)]$ if and only if there is a K-morphism of toroidal stratum representatives $(\Phi, \sigma) \to (\Phi_1, \sigma_1)$.

If Σ is smooth then so is the toroidal compacification.

Remark 2.6.1. Any finite K-admissible cone decomposition can be refined to a finite, K-admissible, smooth, complete cone decomposition Σ with the property that $\mathrm{Sh}_K(G,\mathcal{D},\Sigma)$ is a smooth projective variety. This is [Pin89, Theorem 9.21].

After possibly shrinking K, there exists a finite, K-admissible, complete cone decomposition Σ having the no-self intersection property. The no-self intersection property guarantees that the strata appearing in (2.6.1) have an especially simple shape. Fix one $\Upsilon = [(\Phi, \sigma)]$ and write $\Phi = (P, \mathcal{D}^{\circ}, h)$. Pink shows that there is a canonical isomorphism

$$(2.6.2) Z_{K_{\Phi}}^{\sigma}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) \xrightarrow{\cong} Z_{K}^{\Upsilon}(G, \mathcal{D}, \Sigma)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) \qquad \operatorname{Sh}_{K}(G, \mathcal{D}, \Sigma)$$

such that the two algebraic spaces in the bottom row become isomorphic after formal completion along their common locally closed subspace in the top row. See [Pin89, Corollary 7.17] and [Pin89, Theorem 12.4].

In other words, if we abbreviate

$$\widehat{\operatorname{Sh}}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) = \operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)^{\wedge}_{Z_{K_{\pi}}^{\sigma}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)}$$

for the formal completion of $\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)$ along its closed stratum, and abbreviate

$$\widehat{\operatorname{Sh}}_K(G, \mathcal{D}, \Sigma) = \operatorname{Sh}_K(G, \mathcal{D}, \Sigma)^{\wedge}_{Z_K^{\Upsilon}(G, \mathcal{D}, \Sigma)}$$

for the formal completion of $\operatorname{Sh}_K(G, \mathcal{D}, \Sigma)$ along the locally closed stratum $Z_K^{\Upsilon}(G, \mathcal{D}, \Sigma)$, there is an isomorphism of formal algebraic spaces

(2.6.3)
$$\widehat{\operatorname{Sh}}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) \cong \widehat{\operatorname{Sh}}_{K}(G, \mathcal{D}, \Sigma).$$

Remark 2.6.2. In [Pin89] the isomorphism (2.6.3) is constructed after the left hand side is replace by its quotient by a finite group action. Thanks to Hypothesis 2.1.1 and the assumption that K is neat, the finite group in question is trivial. See [Wil00, Lemma 1.7 and Remark 1.8].

We can make the above more explicit on the level of complex points. Suppose (Φ, σ) is a toroidal stratum representative with underlying cusp label representative $\Phi = (P, \mathcal{D}^{\circ}, h)$, and denote by $Q_{\Phi}(\mathbb{R})^{\circ} \subset Q_{\Phi}(\mathbb{R})$ the stabilizer of the connected component $\mathcal{D}^{\circ} \subset \mathcal{D}$. The complex manifold

$$\mathscr{U}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) = Q_{\Phi}(\mathbb{Q})^{\circ} \backslash (\mathcal{D}^{\circ} \times Q_{\Phi}(\mathbb{A}_f)/K_{\Phi})$$

sits in a diagram

$$(2.6.4) \mathcal{U}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \longrightarrow \operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C})$$

$$[(z,g)] \mapsto [(z,gh)] \downarrow \\ \operatorname{Sh}_{K}(G, \mathcal{D})(\mathbb{C})$$

in which the horizontal arrow is an open immersion, and the vertical arrow is a local isomorphism. This allows us to define a partial compactification

$$\mathscr{U}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \hookrightarrow \mathscr{U}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)$$

as the interior of the closure of $\mathscr{U}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})$ in $Sh_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)(\mathbb{C})$.

Any K-morphism as in Definition 2.4.6 induces a morphism of complex manifolds

$$\mathscr{U}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \xrightarrow{[(z,g)] \mapsto [(\gamma z, \gamma g \gamma^{-1} q)]} \mathscr{U}_{K_{\Phi_1}}(Q_{\Phi_1}, \mathcal{D}_{\Phi_1}),$$

which extends uniquely to

$$\mathscr{U}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) \to \mathscr{U}_{K_{\Phi_1}}(Q_{\Phi_1}, \mathcal{D}_{\Phi_1}, \sigma_1).$$

Complex analytically, the toroidal compactification is defined as the quotient

$$\operatorname{Sh}_K(G, \mathcal{D}, \Sigma)(\mathbb{C}) = \Big(\bigsqcup_{(\Phi, \sigma)} \mathscr{U}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)\Big) / \sim,$$

where \sim is the equivalence relation generated by the graphs of all such morphisms.

By [Pin89, §6.13] the closed stratum appearing in (2.3.5) satisfies

(2.6.5)
$$Z_{K_{\Phi}}^{\sigma}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)(\mathbb{C}) \subset \mathscr{U}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma).$$

The morphisms in (2.6.4) extend continuously to morphisms

$$(2.6.6) \mathcal{U}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) \longrightarrow \operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)(\mathbb{C})$$

$$\downarrow \\ \operatorname{Sh}_{K}(G, \mathcal{D}, \Sigma)(\mathbb{C})$$

in such a way that the vertical map identifies

$$Z^{\sigma}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)(\mathbb{C}) \cong Z^{\Upsilon}_{K}(G, \mathcal{D}, \Sigma)(\mathbb{C}).$$

This agrees with the analytification of the isomorphism (2.6.2).

Now pick any point $z \in Z_{\Phi}^{\sigma}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)(\mathbb{C})$. Let R be the completed local ring of $\operatorname{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$ at z, and let R_{Φ} be the completed local ring of $\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)_{/\mathbb{C}}$ at z. Each completed local ring can be computed with respect to the étale or analytic topologies, and the results are canonically identified. Working in the analytic topology, the morphisms in (2.6.6) induce an isomorphism $R \cong R_{\Phi}$, as they identify both rings with the completed local ring of $\mathscr{U}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)$ at z. This analytic isomorphism agrees with the one induced by the algebraic isomorphism (2.6.3).

3. Automorphic vector bundles

Throughout §3 we fix a Shimura datum (G, \mathcal{D}) satisfying Hypothesis 2.1.1, and a compact open subgroup $K \subset G(\mathbb{A}_f)$.

Our goal is to recall the theory of automorphic vector bundles on the Shimura variety $\operatorname{Sh}_K(G,\mathcal{D})$, on its toroidal compactification, and on the mixed Shimura varieties appearing along the boundary. The main reference is [HZ01].

3.1. Holomorphic vector bundles. Let $\Phi = (P, \mathcal{D}^{\circ}, h)$ be a cusp label representative for (G, \mathcal{D}) . As in §2, this determines a mixed Shimura datum $(Q_{\Phi}, \mathcal{D}_{\Phi})$ and a compact open subgroup $K_{\Phi} \subset Q_{\Phi}(\mathbb{A}_f)$.

Suppose we have a representation $Q_{\Phi} \to \operatorname{GL}(N)$ on a finite dimensional \mathbb{Q} -vector space. Given a point $z \in \mathcal{D}_{\Phi}$, its image under

$$\mathcal{D}_{\Phi} \to \operatorname{Hom}(\mathbb{S}_{\mathbb{C}}, Q_{\Phi\mathbb{C}})$$

determines a mixed Hodge structure $(N, F^{\bullet}N_{\mathbb{C}}, \operatorname{wt}_{\bullet}N)$. The weight filtration is independent of z, and is split by any lift $\mathbb{G}_m \to Q_{\Phi}$ of the weight cocharacter (2.2.2).

Denote by $(N_{dR}^{an}, F^{\bullet}N_{dR}^{an}, \operatorname{wt}_{\bullet}N_{dR}^{an})$ the doubly filtered holomorphic vector bundle on $\mathcal{D}_{\Phi} \times Q_{\Phi}(\mathbb{A}_f)/K_{\Phi}$ whose fiber at (z,g) is the vector space $N_{\mathbb{C}}$ endowed with the Hodge and weight filtrations determined by z. There is a natural action of $Q_{\Phi}(\mathbb{Q})$ on this doubly filtered vector bundle, covering the action on the base. By taking the quotient, we obtain a functor

$$(3.1.1) N \mapsto (\mathbf{N}_{dR}^{an}, F^{\bullet} \mathbf{N}_{dR}^{an}, \operatorname{wt}_{\bullet} \mathbf{N}_{dR}^{an})$$

from finite dimensional representations of Q_{Φ} to doubly filtered holomorphic vector bundles on $\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C})$. Ignoring the double filtration, this functor is simply

$$(3.1.2) N \mapsto \mathbf{N}_{dR}^{an} = Q_{\Phi}(\mathbb{Q}) \setminus (\mathcal{D}_{\Phi} \times N_{\mathbb{C}} \times Q_{\Phi}(\mathbb{A}_f) / K_{\Phi}).$$

Given a K_{Φ} -stable $\widehat{\mathbb{Z}}$ -lattice $N_{\widehat{\mathbb{Z}}} \subset N \otimes \mathbb{A}_f$, we may define a \mathbb{Z} -lattice

$$gN_{\mathbb{Z}} = gN_{\widehat{\mathbb{Z}}} \cap N$$

for every $g \in Q_{\Phi}(\mathbb{A}_f)$, along with a weight filtration

$$\operatorname{wt}_{\bullet}(gN_{\mathbb{Z}}) = gN_{\widehat{\mathbb{Z}}} \cap \operatorname{wt}_{\bullet}N.$$

Denote by $(N_{Be}, \operatorname{wt}_{\bullet} N_{Be})$ the filtered \mathbb{Z} -local system on $\mathcal{D}_{\Phi} \times Q_{\Phi}(\mathbb{A}_f)/K_{\Phi}$ whose fiber at (z, g) is $(gN_{\mathbb{Z}}, \operatorname{wt}_{\bullet}(gN_{\mathbb{Z}}))$. This local system has an obvious action of $Q_{\Phi}(\mathbb{Q})$, covering the action on the base. Passing to the quotient, we obtain a functor

$$N_{\widehat{\mathbb{Z}}} \mapsto (N_{Be}, \operatorname{wt}_{\bullet} N_{Be})$$

from K_{Φ} -stable $\widehat{\mathbb{Z}}$ -lattices in $N \otimes \mathbb{A}_f$ to filtered \mathbb{Z} -local systems on (2.2.4). By construction there is a canonical isomorphism

$$(3.1.3) (N_{dR}^{an}, \operatorname{wt}_{\bullet} N_{dR}^{an}) \cong (N_{Be} \otimes \mathcal{O}^{an}, \operatorname{wt}_{\bullet} N_{Be} \otimes \mathcal{O}^{an}),$$

where \mathcal{O}^{an} denotes the structure sheaf on $\mathrm{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C})$.

3.2. The Borel morphism. Suppose $G \to \operatorname{GL}(N)$ is any faithful representation of G on a finite dimensional \mathbb{Q} -vector space. A point $z \in \mathcal{D}$ determines a Hodge structure $\mathbb{S} \to \operatorname{GL}(N_{\mathbb{R}})$ on N, and we denote by $F^{\bullet}N_{\mathbb{C}}$ the induced Hodge filtration. As in [Mil90, §III.1] and [HZ01, §1], define the *compact dual*

$$(3.2.1) \check{M}(G, \mathcal{D})(\mathbb{C}) = \left\{ \begin{array}{c} \text{descending filtrations on } N_{\mathbb{C}} \\ \text{that are } G(\mathbb{C})\text{-conjugate to } F^{\bullet}N_{\mathbb{C}} \end{array} \right\}.$$

By construction, there is a canonical $G(\mathbb{R})$ -equivariant finite-to-one Borel morphism

$$\mathcal{D} \to \check{M}(G,\mathcal{D})(\mathbb{C})$$

sending a point of \mathcal{D} to the induced Hodge filtration on $N_{\mathbb{C}}$. The compact dual is the space of complex points of a smooth projective variety $\check{M}(G,\mathcal{D})$ defined over the reflex field $E(G,\mathcal{D})$, and admitting an action of $G_{E(G,\mathcal{D})}$ inducing the natural action of $G(\mathbb{C})$ on complex points. It is independent of the choice of z, and of the choice of faithful representation N.

More generally, there is an analogue of (3.2.1) for the mixed Shimura datum $(Q_{\Phi}, \mathcal{D}_{\Phi})$, as in [Hör10, Main Theorem 3.4.1] and [Hör14, Main Theorem 2.5.12]. Let $Q_{\Phi} \to \operatorname{GL}(N)$ be a faithful representation on a finite dimensional \mathbb{Q} -vector space. Any point $z \in \mathcal{D}_{\Phi}$ then determines a mixed Hodge structure $(N, F^{\bullet}N_{\mathbb{C}}, \operatorname{wt}_{\bullet}N)$, and we define the dual of $(Q_{\Phi}, \mathcal{D}_{\Phi})$ by

$$\check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C}) = \left\{ \begin{array}{c} \text{descending filtrations on } N_{\mathbb{C}} \\ \text{that are } Q_{\Phi}(\mathbb{C})\text{-conjugate to } F^{\bullet}N_{\mathbb{C}} \end{array} \right\}.$$

It is the space of complex points of an open $Q_{\Phi,E(G,\mathcal{D})}$ -orbit

$$\check{M}(Q_{\Phi}, \mathcal{D}_{\Phi}) \subset \check{M}(G, \mathcal{D}),$$

independent of the choice of $z \in \mathcal{D}_{\Phi}$ and N. By construction, there is a $Q_{\Phi}(\mathbb{C})$ -equivariant Borel morphism

$$\mathcal{D}_{\Phi} \to \check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C}).$$

3.3. The standard torsor. We want to give a more algebraic interpretation of the functor (3.1.1).

Harris and Zucker [HZ01, §1] prove that the mixed Shimura variety (2.2.4) carries a standard torsor². This consists of a diagram of $E(G, \mathcal{D})$ -stacks

in which a is a relative Q_{Φ} -torsor, and b is Q_{Φ} -equivariant. See also the papers of Harris [Har84, Har85, Har86], Harris-Zucker [HZ94a, HZ94b], and Milne [Mil88, Mil90]. Complex analytically, the standard torsor is the complex orbifold

$$J_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C}) = Q_{\Phi}(\mathbb{Q}) \setminus (\mathcal{D}_{\Phi} \times Q_{\Phi}(\mathbb{C}) \times Q_{\Phi}(\mathbb{A}_f) / K_{\Phi}),$$

with $Q_{\Phi}(\mathbb{C})$ acting by $s \cdot [(z, t, g)] = [(z, ts^{-1}, g)]$. The morphisms a and b are, respectively,

$$[(z,t,g)] \mapsto [(z,g)] \quad \text{and} \quad [(z,t,g)] \mapsto t^{-1}z.$$

Exactly as in [HZ01], we can use the standard torsor to define models of the vector bundles (3.1.1) over the reflex field. First, we require a lemma.

Lemma 3.3.1. Suppose $\check{N} \to \check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})$ is a Q_{Φ} -equivariant vector bundle; that is, a finite rank vector bundle endowed with an action of $Q_{\Phi,E(G,\mathcal{D})}$ covering the action on the base. There are canonical Q_{Φ} -equivariant filtrations $\operatorname{wt}_{\bullet}\check{N}$ and $F^{\bullet}\check{N}$ on \check{N} , and the construction

$$\check{N} \mapsto (\check{N}, F^{\bullet}\check{N}, \operatorname{wt}_{\bullet}\check{N})$$

is functorial in \check{N} .

Proof. Fix a faithful representation $Q_{\Phi} \to \operatorname{GL}(H)$. Suppose we are given an étale neighborhood $U \to \check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})$ of some geometric point x of $\check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})$. By the very definition of $\check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})$, U determines a $Q_{\Phi U}$ stable filtration $F^{\bullet}H_U$ on $H_U = H \otimes \mathcal{O}_U$. After possibly shrinking U we may choose a cocharacter $\mu_x : \mathbb{G}_m \to Q_{\Phi U}$ splitting this filtration.

As $Q_{\Phi U}$ acts on \check{N}_U , the cocharacter μ_x determines a filtration $F^{\bullet}\check{N}_U$, which does not depend on the choice of splitting. Glueing over an étale cover of $\check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})$ defines the desired filtration $F^{\bullet}\check{N}$. The definition of

²a.k.a. standard principal bundle

wt• \mathring{N} is similar, but easier: it is the filtration split by any lift $\mathbb{G}_m \to Q_{\Phi}$ of the weight cocharacter (2.2.2).

Now suppose we have a representation $Q_{\Phi} \to \mathrm{GL}(N)$ on a finite dimensional \mathbb{Q} -vector space. Applying Lemma 3.3.1 to the constant Q_{Φ} -equivariant vector bundle

$$\check{N} = \check{M}(Q_{\Phi}, \mathcal{D}_{\Phi}) \times_{\operatorname{Spec}(E(G,\mathcal{D}))} N_{E(G,\mathcal{D})}$$

yields a Q_{Φ} -equivariant doubly filtered vector bundle $(\check{N}, F^{\bullet}\check{N}, \operatorname{wt}_{\bullet}\check{N})$ on $\check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})$. The construction

$$(3.3.2) N \mapsto (\mathbf{N}_{dR}, F^{\bullet} \mathbf{N}_{dR}, \operatorname{wt}_{\bullet} \mathbf{N}_{dR}) = Q_{\Phi} \backslash b^{*}(\check{N}, F^{\bullet} \check{N}, \operatorname{wt}_{\bullet} \check{N})$$

defines a functor from representations of Q_{Φ} to doubly filtered vector bundles on $Sh_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})$. Passing to the complex fiber recovers the functor (3.1.1).

The following proposition extends the above functor to partial compactifications.

Proposition 3.3.2. For any rational polyhedral cone $\sigma \subset U_{\Phi}(\mathbb{R})(-1)$ there is a functor

$$N \mapsto (N_{dR}, F^{\bullet}N_{dR}, \operatorname{wt}_{\bullet}N_{dR}),$$

extending (3.3.2), from representations of Q_{Φ} on finite dimensional \mathbb{Q} -vector spaces to doubly filtered vector bundles on $\mathrm{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)$.

Proof. This is part of [HZ01, Definition-Proposition 1.3.5]. Here we sketch a different argument.

Recall the T_{Φ} -torsor structure on (2.3.1). On complex points, this action was deduced from the natural left action of $U_{\Phi}(\mathbb{C})$ on \mathcal{D}_{Φ} . Of course the group $U_{\Phi}(\mathbb{C})$ also acts on both factors of $\mathcal{D}_{\Phi} \times Q_{\Phi}(\mathbb{C})$ on the left, and imitating the proof of Proposition 2.3.1 yields action of the relative torus $T_{\Phi}(\mathbb{C})$ on the standard torsor $J_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C})$, covering the action on $\mathrm{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C})$.

To see that the action is algebraic and defined over the reflex field, one can reduce, exactly as in the proof of [HZ01, Proposition 1.2.4], to the case in which $(Q_{\Phi}, \mathcal{D}_{\Phi})$ is either a pure Shimura datum, or is a mixed Shimura datum associated with a Siegel Shimura datum. The pure case is vacuous (the relative torus is trivial). The Siegel mixed Shimura varieties are moduli spaces of polarized 1-motives, and it is not difficult to give a modulitheoretic interpretation of the torus action; see [Mad, (2.2.8)]. From this interpretation the descent to the reflex field is obvious.

In the diagram (3.3.1), the arrow a is T_{Φ} -equivariant, and the arrow b is constant on T_{Φ} -orbits. This is clear from the complex analytic description.

Taking the quotient of the standard torsor by this action, we obtain a diagram

$$T_{\Phi} \backslash J_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \xrightarrow{b} \check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})$$

$$\downarrow a \qquad \qquad \qquad \downarrow \\ \operatorname{Sh}_{\bar{K}_{\Phi}}(\bar{Q}_{\Phi}, \bar{\mathcal{D}}_{\Phi}),$$

in which a is a relative Q_{Φ} -torsor and b is Q_{Φ} -equivariant. Pulling back the quotient $T_{\Phi} \setminus J_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})$ along the diagonal arrow in (2.3.4) defines the upper left entry in the diagram

$$J_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) \xrightarrow{b} \check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})$$

$$\downarrow a \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

extending (3.3.1), in which a is a Q_{Φ} -torsor, and b is Q_{Φ} -equivariant. Now simply repeat the construction (3.3.2) to obtain the desired functor.

Remark 3.3.3. The proof actually shows more: because the standard torsor admits a canonical descent to $\operatorname{Sh}_{\bar{K}_{\Phi}}(\bar{Q}_{\Phi}, \bar{\mathcal{D}}_{\Phi})$, the same is true of all doubly filtered vector bundles (3.3.2). Compare with [HZ01, (1.2.11)].

3.4. Automorphic vector bundles on toroidal compactifications. Assume that K is neat, and that Σ is a finite K-admissible complete cone decomposition for (G, \mathcal{D}) having the no self-intersection property.

By results of Harris and Harris-Zucker, see especially [HZ01], one can glue together the diagrams in the proof of Proposition 3.3.2 as (Φ, σ) varies in order to obtain a diagram

$$(3.4.1) J_K(G, \mathcal{D}, \Sigma) \xrightarrow{b} \check{M}(G, \mathcal{D})$$

$$\downarrow a \\ Sh_K(G, \mathcal{D}, \Sigma)$$

in which a is a G-torsor and b is G-equivariant. This implies the following:

Theorem 3.4.1. There is a functor $N \mapsto (N_{dR}, F^{\bullet}N_{dR})$ from representations of G on finite dimensional \mathbb{Q} -vector spaces to filtered vector bundles on $Sh_K(G, \mathcal{D}, \Sigma)$, compatible, in the obvious sense, with the isomorphism

$$\widehat{\operatorname{Sh}}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) \cong \widehat{\operatorname{Sh}}_{K}(G, \mathcal{D}, \Sigma)$$

of (2.6.3) and the functor of Proposition 3.3.2, for every toroidal stratum representative

$$\Upsilon = [(\Phi, \sigma)] \in \operatorname{Strat}_K(G, \mathcal{D}, \Sigma).$$

In other words, there is an arithmetic theory of automorphic vector bundles on toroidal compactifications.

Remark 3.4.2. Over the open Shimura variety $\operatorname{Sh}_K(G,\mathcal{D})$ there is also a weight filtration $\operatorname{wt}_{\bullet} N_{dR}$ on N_{dR} , but it is not compatible with the weight filtrations along the boundary. It is also not very interesting. On an irreducible representation N the (central) weight cocharacter $w: \mathbb{G}_m \to G$ acts through $z \mapsto z^k$ for some k, and the weight filtration has a unique nonzero graded piece $\operatorname{gr}_k N_{dR}$.

3.5. A simple Shimura variety. Let $(\mathbb{G}_m, \mathcal{H}_0)$ be the Shimura datum of Remark 2.1.2. For any compact open subgroup $K \subset \mathbb{A}_f^{\times}$, we obtain a 0-dimensional Shimura variety

with a canonical model $\operatorname{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)$ over \mathbb{Q} .

The action of $Aut(\mathbb{C})$ on its complex points satisfies

(3.5.2)
$$\tau \cdot [(2\pi\epsilon, a)] = [(2\pi\epsilon, aa_{\tau})]$$

whenever $\tau \in \operatorname{Aut}(\mathbb{C})$ and $a_{\tau} \in \mathbb{A}_f^{\times}$ are related by $\tau|_{\mathbb{Q}^{ab}} = \operatorname{rec}(a_{\tau})$. This implies that

$$\operatorname{Sh}_K(\mathbb{G}_m, \mathcal{H}_0) \cong \operatorname{Spec}(F),$$

where F/\mathbb{Q} is the abelian extension characterized by

$$\operatorname{rec}: \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times} / K \cong \operatorname{Gal}(F/\mathbb{Q}).$$

The following proposition shows that all automorphic vector bundles on (3.5.1) are canonically trivial. The particular trivializations will be essential in our later discussion of q-expansions. See especially Proposition 4.5.2.

Proposition 3.5.1. For any representation $\mathbb{G}_m \to \mathrm{GL}(N)$ there is a canonical isomorphism

$$N \otimes \mathcal{O}_{\operatorname{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)} \xrightarrow{n \otimes 1 \mapsto n} N_{dR}$$

of vector bundles. If \mathbb{G}_m acts on N through the character $z \mapsto z^k$, the global section $\mathbf{n} = n \otimes 1$ is given, in terms of the complex parametrization

$$\mathbf{N}_{dR}^{an} = \mathbb{Q}^{\times} \backslash (\mathcal{H}_0 \times N_{\mathbb{C}} \times \mathbb{A}_f^{\times} / K)$$

of (3.1.2), by

$$[(2\pi\epsilon, a)] \mapsto \left[\left(2\pi\epsilon, \frac{\operatorname{rat}(a)^k}{(2\pi\epsilon)^k} \cdot n, a \right) \right].$$

Proof. First set $N = \mathbb{Q}$ with \mathbb{G}_m acting via the identity character $z \mapsto z$, and set $N_{\widehat{\mathbb{Z}}} = \widehat{\mathbb{Z}}$. Recalling (3.1.3), the quotient $N_{Be} \setminus N_{dR}^{an}$ defines an analytic family of rank one tori over $\mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C})$, whose relative Lie algebra is the line bundle

$$\operatorname{Lie}(\mathbf{N}_{Be}\backslash \mathbf{N}_{dR}^{an}) = \mathbf{N}_{dR}^{an} = \mathbb{Q}^{\times}\backslash (\mathcal{H}_0 \times \mathbb{C} \times \mathbb{A}_f^{\times}/K).$$

Using this identification, we may identify the standard \mathbb{C}^{\times} -torsor

$$(3.5.3) J_K(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C}) = \mathbb{Q}^{\times} \setminus (\mathcal{H}_0 \times \mathbb{C}^{\times} \times \mathbb{A}_f^{\times} / K)$$

with the \mathbb{C}^{\times} -torsor of trivializations of Lie($N_{Be} \setminus N_{dR}^{an}$).

On the other hand, the isomorphisms

$$(N \cap aN_{\widehat{\mathbb{Z}}})\backslash N_{\mathbb{C}} = (\mathbb{Q} \cap a\widehat{\mathbb{Z}})\backslash \mathbb{C} \xrightarrow{2\pi\epsilon/\mathrm{rat}(a)} \mathbb{Z}(1)\backslash \mathbb{C} \xrightarrow{\exp} \mathbb{C}^{\times}$$

identify $N_{Be}\backslash N_{dR}^{an}$, fiber-by-fiber, with the constant torus \mathbb{C}^{\times} , and so identify (3.5.3) with the \mathbb{C}^{\times} -torsor of trivializations of $\text{Lie}(\mathbb{C}^{\times})$. The canonical model of (3.5.3) is now concretely realized as the \mathbb{G}_m -torsor

$$J_K(\mathbb{G}_m, \mathcal{H}_0) = \underline{\mathrm{Iso}}\big(\mathrm{Lie}(\mathbb{G}_m), \mathcal{O}_{\mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)}\big).$$

For any ring R, the Lie algebra of $\mathbb{G}_m = \operatorname{Spec}(R[q, q^{-1}])$ is canonically trivialized by the invariant derivation $q \cdot d/dq$. Thus the standard torsor admits a canonical section which, in terms of the uniformization (3.5.3), is

$$[(2\pi\epsilon, a)] \mapsto \left[\left(2\pi\epsilon, \frac{\operatorname{rat}(a)}{2\pi\epsilon}, a \right) \right].$$

This section trivializes the standard torsor, and induces the desired trivialization of any automorphic vector bundle. \Box

Remark 3.5.2. Let \mathbb{G}_m act on N via $z \mapsto z^k$. What the above proof actually shows is that there are canonical isomorphisms

$$N \otimes \mathcal{O}_{\operatorname{Sh}_K(\mathbb{G}_m,\mathcal{H}_0)} \cong N \otimes \operatorname{Lie}(\mathbb{G}_m)^{\otimes k} \cong \mathbf{N}_{dR}.$$

4. Orthogonal Shimura varieties

Now we specialize the preceding theory to the case of Shimura varieties of type GSpin.

4.1. The GSpin Shimura variety. Let (V, Q) be a quadratic space over \mathbb{Q} of signature (n, 2) with $n \ge 1$. The associated bilinear form is denoted

$$[x,y] = Q(x+y) - Q(x) - Q(y).$$

Let $G = \operatorname{GSpin}(V)$ as in [Mad16]. This is a reductive group over $\mathbb Q$ sitting in an exact sequence

$$1 \to \mathbb{G}_m \to G \to \mathrm{SO}(V) \to 1.$$

There is a character $\nu: G \to \mathbb{G}_m$, called the *spinor similitude*. Its kernel is the usual spin double cover of SO(V), and its restriction to \mathbb{G}_m is $z \mapsto z^2$.

The group $G(\mathbb{R})$ acts on the hermitian domain

(4.1.2)
$$\mathcal{D} = \{ z \in V_{\mathbb{C}} : [z, z] = 0 \text{ and } [z, \overline{z}] < 0 \} / \mathbb{C}^{\times}$$

in the obvious way. This hermitian domain has two connected components, interchanged by the action of any $\gamma \in G(\mathbb{R})$ with $\nu(\gamma) < 0$. The pair (G, \mathcal{D}) is the *GSpin Shimura datum*. Its reflex field is \mathbb{Q} .

By construction, G is a subgroup of the multiplicative group of the Clifford algebra C(V). As such, G has two distinguished representations. One is the standard representation $G \to SO(V)$, and the other is the faithful action on

H = C(V) defined by left multiplication in the Clifford algebra. These two representations are related by a G-equivariant injection

$$(4.1.3) V \to \operatorname{End}_{\mathbb{O}}(H)$$

defined by the left multiplication action of $V \subset C(V)$ on H.

A point $z \in \mathcal{D}$ determines a Hodge structure on V, given by

$$(4.1.4) F^2V_{\mathbb{C}} = 0, F^1V_{\mathbb{C}} = \mathbb{C}z, F^0V_{\mathbb{C}} = (\mathbb{C}z)^{\perp}, F^{-1}V_{\mathbb{C}} = V_{\mathbb{C}},$$

and a Hodge structure on H, given by

$$F^1H_{\mathbb{C}} = 0$$
, $F^0H_{\mathbb{C}} = zH_{\mathbb{C}}$, $F^{-1}H_{\mathbb{C}} = H_{\mathbb{C}}$.

Here we are using (4.1.3) to view $\mathbb{C}z \subset \operatorname{End}_{\mathbb{C}}(H_{\mathbb{C}})$.

In order to obtain a Shimura variety $\operatorname{Sh}_K(G,\mathcal{D})$ we must specify the level subgroup K. Fix a $\widehat{\mathbb{Z}}$ -lattice

$$(4.1.5) V_{\widehat{\mathcal{I}}} \subset V \otimes \mathbb{A}_f$$

on which Q is $\widehat{\mathbb{Z}}$ -valued, abbreviate $V_{\mathbb{Z}} = V_{\widehat{\mathbb{Z}}} \cap V$, and fix a compact open subgroup

$$(4.1.6) K \subset G(\mathbb{A}_f) \cap C(V_{\widehat{x}})^{\times},$$

the intersection taking place inside of $C(V \otimes \mathbb{A}_f)^{\times}$. The lattice (4.1.5) is K-stable, and hence so is its dual lattice relative to (4.1.1). By [Mad16, Lemma 2.6] the group K acts trivially on the quotient

$$(4.1.7) V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}} \cong V_{\widehat{\mathbb{Z}}}^{\vee}/V_{\widehat{\mathbb{Z}}}.$$

4.2. **The Kuga-Satake construction.** There is a natural choice of Hodge embedding

$$(G,\mathcal{D}) \to (G^{\operatorname{Sg}},\mathcal{D}^{\operatorname{Sg}})$$

into the Siegel Shimura datum determined by a symplectic space (H, ψ) over \mathbb{Q} . As above, let H = C(V) viewed as a representation of $G \subset C(V)^{\times}$ via left multiplication, and define a \mathbb{Z} -lattice

$$H_{\widehat{\mathbb{Z}}} = C(V_{\widehat{\mathbb{Z}}}) \subset H \otimes \mathbb{A}_f.$$

Lemma 4.2.1. There is symplectic form ψ on H such that ψ is $\widehat{\mathbb{Z}}$ -valued on $H_{\widehat{\mathbb{Z}}}$, and such that the action of G on H factors through

$$G^{\operatorname{Sg}} = \operatorname{GSp}(H, \psi)$$

and determines an embedding of Shimura data as above.

If V admits an isotropic vector, then we may choose ψ so that $H_{\mathbb{Z}_p}$ is self-dual for any prime p such that $V_{\mathbb{Z}_p}$ is almost self-dual.

Proof. We will use the decomposition

$$H = H^+ \oplus H^-$$

induced by the decomposition of C(V) into its even and odd parts.

By [Con14, §C.2], for any prime p such that $V_{\mathbb{Z}_p}$ is almost self-dual, $C^+(V_{\mathbb{Z}_p})$ is an Azumaya algebra over a finite étale cover of $\operatorname{Spec}(\mathbb{Z}_{(p)})$. In particular, the reduced trace map $\operatorname{Trd}:C^+(V)\to\mathbb{Q}$ induces a non-degenerate symmetric bilinear pairing

$$(x,y) \mapsto \operatorname{Trd}(xy)$$

on $C^+(V_{\mathbb{Z}_{(p)}})$.

If V admits an isotropic line, then, since $V_{\mathbb{Z}}$ is maximal, it contains a hyperbolic plane. Therefore, we can find $\delta \in C^+(V_{\mathbb{Z}})^{\times}$ such that $\delta^* = -\delta$, where * is the main involution on C(V): Indeed, choose elements $v, w \in V_{\mathbb{Z}}$ that are mutually orthogonal and satisfy Q(v) = 1, Q(w) = -1. Then $\delta = vw \in C^+(V_{\mathbb{Z}})$ answers to the requirement.

Now, as in [Mad16, §1.6], the pairing

$$\psi(x,y) = \operatorname{Trd}(x\delta y^*)$$

is a symplectic pairing on H^+ that takes integral values on $H_{\mathbb{Z}}$, and restricts to a non-degenerate pairing on $H_{\mathbb{Z}_{(p)}}^+$.

Moreover, right multiplication by the element v chosen above induces a G-equivariant isomorphism $H^+ \cong H^-$ that carries $H^+_{\mathbb{Z}}$ to $H^-_{\mathbb{Z}}$. Therefore, we can extend ψ to an alternating form ψ on $H_{\mathbb{Z}}$, non-degenerate on $H_{\mathbb{Z}_{(p)}}$, in such a way that the action of G on H factors through $G^{\operatorname{Sg}} = \operatorname{GSp}(H, \psi)$.

One can now check that this factoring determines an embedding of Shimura data. $\hfill\Box$

Fixing ψ as in Lemma 4.2.1, we obtain a morphism from $\operatorname{Sh}_K(G,\mathcal{D})$ to a moduli space of polarized abelian varieties of dimension 2^{n+1} . Pulling back the universal object defines the $Kuga\text{-}Satake\ abelian\ scheme$

$$\pi: A \to \operatorname{Sh}_K(G, \mathcal{D}).$$

The abelian scheme does not depend on the choice of ψ , but of course its polarization does.

As in (3.2.1), we may describe the compact dual $\check{M}(G,\mathcal{D})$ as a G-orbit of descending filtrations on the faithful representation H. It is more convenient to characterize the compact dual as the \mathbb{Q} -scheme with functor of points

$$\check{M}(G,\mathcal{D})(S) = \{\text{isotropic lines } z \subset V \otimes \mathcal{O}_S\},\$$

where line means local \mathcal{O}_S -module direct summand of rank one. In order to realize $\check{M}(G,\mathcal{D})$ as a space of filtrations on H, define

$$\check{M}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})(S) = \{ \operatorname{Lagrangian subsheaves} F^0 \subset H \otimes \mathcal{O}_S \}$$

and define a closed immersion

$$\check{M}(G,\mathcal{D}) \to \check{M}(G^{\operatorname{Sg}},\mathcal{D}^{\operatorname{Sg}})$$

by sending the isotropic line $z \subset V$ to the Lagrangian $zH \subset H$.

4.3. Cusp label representatives. Fix a cusp label representative

$$\Phi = (P, \mathcal{D}^{\circ}, h)$$

for the GSpin Shimura datum (G, \mathcal{D}) , and let $(Q_{\Phi}, \mathcal{D}_{\Phi})$ be the associated mixed Shimura datum. The parabolic subgroup $P \subset G$ is either all of G, or is the stabilizer of a totally isotropic subspace $I \subset V$ with $\dim(I) \in \{1, 2\}$.

Assume first that the parabolic subgroup $P \subset G$ is the stabilizer of an isotropic line $I \subset V$. We define P-stable weight filtrations on V and H by

$$\text{wt}_{-3}V = 0$$
, $\text{wt}_{-2}V = \text{wt}_{-1}V = I$, $\text{wt}_{0}V = \text{wt}_{1}V = I^{\perp}$, $\text{wt}_{2}V = V$,

$$wt_{-3}H = 0$$
, $wt_{-2}H = wt_{-1}H = IH$, $wt_0H = H$.

Here we are using the inclusion $I \subset \operatorname{End}_{\mathbb{Q}}(H)$ determined by (4.1.3), and setting

$$IH = \operatorname{Span}_{\mathbb{O}} \{ \ell x : \ell \in I, x \in H \}.$$

The subgroup $Q_{\Phi} \subset P$ in the mixed Shimura datum is

$$Q_{\Phi} = \ker(P \to \operatorname{GL}(\operatorname{gr}_0(H))).$$

The action $Q_{\Phi} \to SO(V)$ is faithful, and is given on the graded pieces of $\operatorname{wt}_{\bullet}V$ by the commutative diagram

$$Q_{\Phi} \xrightarrow{\nu_{\Phi}} \mathbb{G}_{m}$$

$$\downarrow \qquad \qquad \downarrow_{z \mapsto (z,1,z^{-1})}$$

$$P \longrightarrow \mathrm{GL}(I) \times \mathrm{SO}(I^{\perp}/I) \times \mathrm{GL}(V/I^{\perp}),$$

in which ν_{Φ} is the restriction to Q_{Φ} of the spinor similarity. This agrees with the character (2.2.3). The groups U_{Φ} and W_{Φ} are

$$U_{\Phi} = W_{\Phi} = \ker(\nu_{\Phi} : Q_{\Phi} \to \mathbb{G}_m),$$

and there is an isomorphism of \mathbb{Q} -vector spaces

$$(4.3.2) (I^{\perp}/I) \otimes I \cong U_{\Phi}(\mathbb{Q})$$

sending $v \otimes \ell \in (I^{\perp}/I) \otimes I$ to the unipotent transformation of V defined by

$$x \mapsto x + [x, \ell]v - [x, v]\ell - Q(v)[x, \ell]\ell.$$

The dual of $(Q_{\Phi}, \mathcal{D}_{\Phi})$ is the \mathbb{Q} -scheme with functor of points

$$\check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})(S) = \left\{ \begin{array}{l} \text{isotropic lines } z \subset V \otimes \mathcal{O}_S \text{ such that} \\ V \to V/I^{\perp} \\ \text{identifies } z \cong (V/I^{\perp}) \otimes \mathcal{O}_S \end{array} \right\}.$$

Each $z \in \mathcal{D}_{\Phi}$ determines a mixed Hodge structure on V of type (-1, -1), (0,0), (1,1), and $F^1V_{\mathbb{C}} \subset V_{\mathbb{C}}$ is an isotropic line. The Borel morphism

$$\mathcal{D}_{\Phi} \to \check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C})$$

sends $z \mapsto F^1 V_{\mathbb{C}}$, and defines an isomorphism

$$(4.3.3) \mathcal{D}_{\Phi} \cong \pi_0(\mathcal{D}) \times \check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C}).$$

Now assume that the parabolic subgroup $P \subset G$ is the stabilizer of a totally isotropic plane $I \subset V$. The plane I determines weight filtrations on V and H by

$$\text{wt}_{-2}V = 0$$
, $\text{wt}_{-1}V = I$, $\text{wt}_{0}V = I^{\perp}$, $\text{wt}_{1}V = V$

and

$$wt_{-3}H = 0$$
, $wt_{-2}H = I^2H$, $wt_{-1}H = IH$, $wt_0H = H$.

Here we view $I \subset \text{End}(H)$ using left multiplication in C(V), and set

$$\begin{split} IH &= \operatorname{Span}_{\mathbb{Q}} \{ \ell x : \ell \in I, \ x \in H \} \\ I^2H &= \operatorname{Span}_{\mathbb{Q}} \{ \ell \ell' x : \ell, \ell' \in I, \ x \in H \}. \end{split}$$

The subgroup $Q_{\Phi} \subset P$ defining the mixed Shimura datum is

$$Q_{\Phi} = \ker (P \to \operatorname{GL}(\operatorname{gr}_0(H))).$$

The natural action $Q_{\Phi} \to SO(V)$ is faithful, and is trivial on the quotient I^{\perp}/I . The groups $U_{\Phi} \lhd W_{\Phi} \lhd Q_{\Phi}$ are

$$W_{\Phi} = \ker(Q_{\Phi} \to \operatorname{GL}(I)),$$

and

$$U_{\Phi} \cong \bigwedge^2 I$$
,

where we identify $a \wedge b \in \bigwedge^2 I$ with the unipotent transformation of V defined by

$$x \mapsto x + [x, a]b - [x, b]a$$
.

The dual of $(Q_{\Phi}, \mathcal{D}_{\Phi})$ is the \mathbb{Q} -scheme with functor of points

$$\check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})(S) = \left\{ \begin{array}{l} \text{isotropic lines } z \subset V \otimes \mathcal{O}_S \text{ such that} \\ V \to V/I^{\perp} \text{ identifies } z \text{ with a rank one} \\ \text{local direct summand of } (V/I^{\perp}) \otimes \mathcal{O}_S \end{array} \right\}.$$

Each $z \in \mathcal{D}_{\Phi}$ determines a mixed Hodge structure on V of type (-1,0), (0,-1), (0,0), (0,1), (1,0), and again the Borel morphism

$$\mathcal{D}_{\Phi} \to \check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C})$$

sends $z \mapsto F^1 V_{\mathbb{C}}$. It identifies \mathcal{D}_{Φ} with the open subset

$$\mathcal{D}_{\Phi} = U_{\Phi}(\mathbb{C})\mathcal{D} \subset \pi_0(\mathcal{D}) \times \check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C}).$$

4.4. The line bundle of modular forms. Let $\Phi = (P, \mathcal{D}^{\circ}, h)$ be a cusp label representative for (G, \mathcal{D}) . As always, the associated mixed Shimura datum is denoted $(Q_{\Phi}, \mathcal{D}_{\Phi})$.

For any rational polyhedral cone $\sigma \subset U_{\Phi}(\mathbb{R})(-1)$, applying the functor of Proposition 3.3.2 to the standard representation $G \to SO(V)$ yields a filtered vector bundle $(V_{dR}, F^{\bullet}V_{dR})$ on the partially compactified mixed Shimura variety $Sh_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)$. The filtration has the form

$$0 = F^2 \mathbf{V}_{dR} \subset F^1 \mathbf{V}_{dR} \subset F^0 \mathbf{V}_{dR} \subset F^{-1} \mathbf{V}_{dR} = \mathbf{V}_{dR},$$

in which F^1V_{dR} is a line, isotropic with respect to the bilinear form

$$(4.4.1) V_{dR} \otimes V_{dR} \to \mathcal{O}_{\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)}$$

induced by (4.1.1). The filtration is completely determined by this isotropic line, as $F^0V_{dR} = (F^1V_{dR})^{\perp}$.

Definition 4.4.1. The line bundle of weight one modular forms on the partial compactification $\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)$ is $\boldsymbol{\omega} = F^1 V_{dR}$.

By Remark 2.2.3, we obtain as a special case a line bundle $\omega = F^1 V_{dR}$ of modular forms on the pure Shimura variety $\operatorname{Sh}_K(G, \mathcal{D})$. For any $g \in G(\mathbb{A}_f)$, the pullback of this line bundle via the complex uniformization

$$\mathcal{D} \xrightarrow{z \mapsto [(z,g)]} \operatorname{Sh}_K(G,\mathcal{D})(\mathbb{C})$$

is just the tautological bundle on the hermitian domain (4.1.2). This is clear from the complex analytic definition (3.1.1) of V_{dR}^{an} , and the explicit description of the Hodge filtration (4.1.4). In particular, the line bundle ω carries a metric, inherited from the metric

$$(4.4.2) ||z||_{\text{naive}}^2 = -[z, \overline{z}]$$

on the tautological line bundle over (4.1.2). We will often prefer to work with the rescaled metric

$$(4.4.3) ||z||^2 = -\frac{[z,\overline{z}]}{4\pi e^{\gamma}}$$

where $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant.

Now suppose that K is neat, and small enough that there exists a finite K-admissible complete cone decomposition Σ for (G, \mathcal{D}) having the no self-intersection property. The functor of Theorem 3.4.1 defines an extension of ω from $\mathrm{Sh}_K(G,\mathcal{D})$ to the toroidal compactification

(4.4.4)
$$\operatorname{Sh}_{K}(G, \mathcal{D}, \Sigma) = \bigsqcup_{\Upsilon \in \operatorname{Strat}_{K}(G, \mathcal{D}, \Sigma)} Z_{K}^{\Upsilon}(G, \mathcal{D}, \Sigma),$$

which we again denote by ω .

Remark 4.4.2. Of course ω now has multiple meanings, but no confusion should arise. Indeed, for every toroidal stratum representative $\Upsilon = [(\Phi, \sigma)]$ the canonical isomorphism

$$\widehat{\operatorname{Sh}}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) \cong \widehat{\operatorname{Sh}}_{K}(G, \mathcal{D}, \Sigma)$$

of (2.6.3) identifies the line bundle ω on $\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)$ with the eponymous line bundle on $\operatorname{Sh}_{K}(G, \mathcal{D}, \Sigma)$.

4.5. The q-expansion principle. Keep the notation of §4.4. In order to define q-expansions of sections of $\omega^{\otimes k}$ on the toroidal compactification (4.4.4), we need to make some additional choices.

The first choice, of course, is a toroidal stratum representative

$$\Upsilon = [(\Phi, \sigma)] \in \operatorname{Strat}_K(G, \mathcal{D}, \Sigma).$$

We choose this so that $\Phi = (P, \mathcal{D}^{\circ}, h)$ where P is the stabilizer of an isotropic line I, and so that the rational polyhedral cone $\sigma \subset U_{\Phi}(\mathbb{R})(-1)$ has top dimension. Thus the associated stratum of (4.4.4) is a closed subspace of dimension 0.

As \mathcal{D} has two connected components, there are exactly two continuous surjections $\epsilon: \mathcal{D} \to \mathcal{H}_0$. Fix one of them. It, along with the spinor similitude $\nu: G \to \mathbb{G}_m$, induces a morphism of Shimura data

$$(G, \mathcal{D}) \xrightarrow{\nu} (\mathbb{G}_m, \mathcal{H}_0).$$

Denote by $2\pi\epsilon^{\circ} \in \mathcal{H}_0$ the image of the component \mathcal{D}° . There is a unique continuous extension of $\epsilon: \mathcal{D} \to \mathcal{H}_0$ to $\epsilon: \mathcal{D}_{\Phi} \to \mathcal{H}_0$, and this determines a morphism of mixed Shimura data

$$(Q_{\Phi}, \mathcal{D}_{\Phi}) \xrightarrow{\nu_{\Phi}} (\mathbb{G}_m, \mathcal{H}_0),$$

where ν_{Φ} is the character of (4.3.1).

Now choose an auxiliary isotropic line $I_* \subset V$ with $[I, I_*] \neq 0$. This choice fixes a section

$$(Q_{\Phi}, \mathcal{D}_{\Phi}) \xrightarrow{s} (\mathbb{G}_m, \mathcal{H}_0).$$

The underlying morphism of groups $s: \mathbb{G}_m \to Q_{\Phi}$ sends, for any \mathbb{Q} -algebra $R, a \in R^{\times}$ to the orthogonal transformation

$$(4.5.1) s(a) \cdot x = \begin{cases} ax & \text{if } x \in I_R \\ a^{-1}x & \text{if } x \in I_{*,R} \\ x & \text{if } x \in (I \oplus I_*)_R^{\perp}. \end{cases}$$

Now use the isomorphism (4.3.3) to view

$$I_{*\mathbb{C}} \in \check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C})$$

as a point of $\mathcal{D}_{\Phi}^{\circ} = U_{\Phi}(\mathbb{C})\mathcal{D}^{\circ}$, and let $s: \mathcal{H}_0 \to \mathcal{D}_{\Phi}$ be the unique $Q_{\Phi}(\mathbb{R})$ -equivariant map sending $2\pi\epsilon^{\circ} \mapsto I_{*\mathbb{C}}$.

The section s determines a Levi decomposition $Q_{\Phi} = \mathbb{G}_m \ltimes U_{\Phi}$. Choose a compact open subgroup $K_0 \subset \mathbb{G}_m(\mathbb{A}_f)$ contained in K_{Φ} , and set

$$K_{\Phi 0} = K_0 \ltimes (U_{\Phi}(\mathbb{A}_f) \cap K_{\Phi}) \subset K_{\Phi}.$$

Proposition 4.5.1. The above choices determine a commutative diagram

$$\bigsqcup_{a \in \mathbb{Q}^{\times}_{>0} \backslash \mathbb{A}^{\times}_{f}/K_{0}} \widehat{T}_{\Phi}(\sigma)/\mathbb{C} \xrightarrow{\cong} \widehat{\operatorname{Sh}}_{K_{\Phi 0}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)/\mathbb{C}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\widehat{\operatorname{Sh}}_{K}(G, \mathcal{D}, \Sigma)/\mathbb{C} \xrightarrow{(2.6.3)} \widehat{\operatorname{Sh}}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)/\mathbb{C},$$

of formal algebraic spaces, in which the vertical arrows are formally étale surjections, and

(4.5.2)
$$\widehat{T}_{\Phi}(\sigma) \stackrel{\text{def}}{=} \operatorname{Spf}\left(\mathbb{Q}[[q_{\alpha}]]_{\substack{\alpha \in \Gamma_{\Phi}^{\vee}(1) \\ \langle \alpha, \sigma \rangle \geqslant 0}}\right)$$

is the formal completion of (2.3.3) along its closed stratum.

Proof. Consider the diagram

$$\operatorname{Sh}_{K_0}(\mathbb{G}_m, \mathcal{H}_0) \times_{\operatorname{Spec}(\mathbb{Q})} T_{\Phi} = = \operatorname{Sh}_{K_{\Phi_0}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \longrightarrow \operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})$$

$$\downarrow^{\nu_{\Phi}} \downarrow s \qquad \qquad \downarrow^{\nu_{\Phi}} \downarrow$$

$$\operatorname{Sh}_{K_0}(\mathbb{G}_m, \mathcal{H}_0) \longrightarrow \operatorname{Sh}_{\nu_{\Phi}(K_{\Phi})}(\mathbb{G}_m, \mathcal{H}_0)$$

in which the arrows labeled ν_{Φ} are the T_{Φ} -torsors of (2.3.1), and the isomorphism " = " is the trivialization induced by the section s.

There is a canonical bijection

$$\mathbb{Q}_{>0}^{\times}\backslash \mathbb{A}_{f}^{\times}/K_{0} \cong \mathrm{Sh}_{K_{0}}(\mathbb{G}_{m},\mathcal{H}_{0})(\mathbb{C})$$

defined by $a \mapsto [(2\pi\epsilon^{\circ}, a)]$. Using this, the top row of the above diagram exhibits $\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})_{/\mathbb{C}}$ as an étale quotient

exhibits
$$\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})/\mathbb{C}$$
 as an etale quotient
$$(4.5.3) \qquad \bigsqcup_{a \in \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times}/K_{0}} T_{\Phi/\mathbb{C}} \cong \operatorname{Sh}_{K_{\Phi 0}}(Q_{\Phi}, \mathcal{D}_{\Phi})/\mathbb{C} \to \operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})/\mathbb{C}.$$

This morphism extends to partial compactifications, and formally completing along the closed stratum yields a formally étale morphism

along the closed stratum yields a formally étale morphism
$$\bigsqcup_{a \in \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times} / K_{0}} \widehat{\mathrm{Th}}_{\Phi}(\sigma)_{/\mathbb{C}} \cong \widehat{\mathrm{Sh}}_{K_{\Phi 0}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)_{/\mathbb{C}} \to \widehat{\mathrm{Sh}}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)_{/\mathbb{C}}.$$

This defines the top horizontal arrow and the right vertical arrow in the diagram. The vertical arrow on the left is defined by the commutativity of the diagram. \Box

Applying the functor of Proposition 3.3.2 to the Q_{Φ} -representations $I \subset V$ determines vector bundles $I_{dR} \subset V_{dR}$ on $\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)$. On the other hand, there is an inclusion $\omega \subset V_{dR}$ by the very definition $\omega = F^1V_{dR}$. These inclusions, along with the bilinear form (4.4.1), determine a morphism

$$(4.5.4) [\cdot,\cdot]: \mathbf{I}_{dR} \otimes \boldsymbol{\omega} \to \mathcal{O}_{\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi},\mathcal{D}_{\Phi},\sigma)}.$$

This leads us to the final choice we must make: any nonzero vector $\ell \in I$ defines a section

$$\ell^{an} \in H^0(\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C}), I_{dR}^{an})$$

of the line bundle

$$\mathbf{I}_{dR}^{an} = Q_{\Phi}(\mathbb{Q}) \backslash \mathcal{D}_{\Phi} \times I_{\mathbb{C}} \times Q_{\Phi}(\mathbb{A}_f) / K_{\Phi}$$

by sending

$$[(z,g)] \mapsto \left[\left(z, \frac{\operatorname{rat}(\nu_{\Phi}(g))}{2\pi\epsilon(z)} \cdot \ell, g \right) \right].$$

Proposition 4.5.2. The holomorphic section ℓ^{an} extends uniquely to the partial compactification $\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)(\mathbb{C})$. This extension is algebraic and defined over \mathbb{Q} , and so arises from a unique global section

(4.5.5)
$$\ell \in H^0(\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma), \mathbf{I}_{dR}).$$

Moreover, (4.5.4) is an isomorphism, and induces an isomorphism

$$\omega \xrightarrow{\Psi \mapsto [\ell, \Psi]} \mathcal{O}_{\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)}.$$

Proof. As the action of Q_{Φ} on I is via $\nu_{\Phi}: Q_{\Phi}/U_{\Phi} \to \mathbb{G}_m$, the discussion of §3.5 (see especially Remark 3.5.2) identifies I_{dR} with the pullback of the line bundle $I \otimes \operatorname{Lie}(\mathbb{G}_m) \cong I \otimes \mathcal{O}_{\operatorname{Sh}_{\nu_{\Phi}}(K_{\Phi})}(\mathbb{G}_m, \mathcal{H}_0)$ via

$$\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) \xrightarrow{(2.3.4)} \operatorname{Sh}_{\bar{K}_{\Phi}}(\bar{Q}_{\Phi}, \bar{\mathcal{D}}_{\Phi}) = \operatorname{Sh}_{\nu_{\Phi}(K_{\Phi})}(\mathbb{G}_m, \mathcal{H}_0).$$

The section (4.5.5) is simply the pullback of the trivializing section

$$\ell \otimes 1 \in H^0(\operatorname{Sh}_{\nu_{\Phi}(K_{\Phi})}(\mathbb{G}_m, \mathcal{H}_0), I \otimes \mathcal{O}_{\operatorname{Sh}_{\nu_{\Phi}(K_{\Phi})}(\mathbb{G}_m, \mathcal{H}_0)}).$$

It now suffices to prove that (4.5.4) is an isomorphism. Recall from §4.3 that $\check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})$ has functor of points

$$\check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})(S) = \left\{ \begin{array}{l} \text{isotropic lines } z \subset V \otimes \mathcal{O}_S \text{ such that} \\ V \to V/I^{\perp} \text{ identifies } z \cong (V/I^{\perp}) \otimes \mathcal{O}_S \end{array} \right\}.$$

Let \check{I} and \check{V} be the (constant) Q_{Φ} -equivariant vector bundles on $\check{M}(Q_{\Phi}, \mathcal{D}_{\Phi})$ determined by the representations I and V. In the notation of Lemma 3.3.1, the line bundle $\check{\omega} = F^1\check{V}$ is the tautological bundle, and the bilinear form on V determines a Q_{Φ} -equivariant isomorphism

$$\check{I} \otimes \check{\omega} \to \check{V} \otimes \check{V} \xrightarrow{[\cdot,\cdot]} \mathcal{O}_{\check{M}(Q_{\Phi},\mathcal{D}_{\Phi})}.$$

By examining the construction of the functor in Proposition 3.3.2, the induced morphism (4.5.4) is also an isomorphism.

Propositions 4.5.1 and 4.5.2 are the basis for the theory of q-expansions. Taking tensor powers in Proposition 4.5.2 determines an isomorphism

$$[\ell^{\otimes k},\,\cdot\,]: \boldsymbol{\omega}^{\otimes k} \cong \mathcal{O}_{\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi},\mathcal{D}_{\Phi},\sigma)},$$

and hence any global section

$$\Psi \in H^0(\operatorname{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}, \boldsymbol{\omega}^{\otimes k})$$

determines a formal function $[\ell^{\otimes k}, \Psi]$ on

$$\widehat{\operatorname{Sh}}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}} \cong \widehat{\operatorname{Sh}}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)_{/\mathbb{C}}.$$

Now pull this formal function back via the formally étale surjection

$$\bigsqcup_{a \in \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times} / K_{0}} \widehat{T}_{\Phi}(\sigma)_{/\mathbb{C}} \to \widehat{\operatorname{Sh}}_{K}(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$$

of Proposition 4.5.1. By restricting the pullback to the copy of $\widehat{T}_{\Phi}(\sigma)_{/\mathbb{C}}$ indexed by a, we obtain a formal q-expansion (a.k.a. Fourier Jacobi expansion)

$$(4.5.6) FJ^{(a)}(\Psi) = \sum_{\substack{\alpha \in \Gamma_{\Phi}^{\vee}(1) \\ \langle \alpha, \sigma \rangle \geqslant 0}} FJ_{\alpha}^{(a)}(\Psi) \cdot q_{\alpha} \in \mathbb{C}[[q_{\alpha}]]_{\substack{\alpha \in \Gamma_{\Phi}^{\vee}(1) \\ \langle \alpha, \sigma \rangle \geqslant 0}}.$$

We emphasize that (4.5.6) depends on the choice of toroidal stratum representative Υ , as well as on the choices of $\mathcal{D}_{\Phi} \to \mathcal{H}$, I_* , and ℓ . These will always be clear from context.

For each $\tau \in \operatorname{Aut}(\mathbb{C})$, denote by $a_{\tau} \in \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times}$ the unique element with $\operatorname{rec}(a_{\tau}) = \tau|_{\mathbb{Q}^{ab}}$. The following is our version of the q-expansion principle of [Hör14, Theorem 2.8.7].

Proposition 4.5.3 (Rational q-expansion principle). For any $a \in \mathbb{A}_f^{\times}$ and $\tau \in \operatorname{Aut}(\mathbb{C})$, the q-expansion coefficients of Ψ and Ψ^{τ} are related by

$$\mathrm{FJ}_{\alpha}^{(aa_{\tau})}(\Psi^{\tau}) = \tau \big(\mathrm{FJ}_{\alpha}^{(a)}(\Psi)\big).$$

Moreover, Ψ is defined over a subfield $L \subset \mathbb{C}$ if and only if

$$FJ_{\alpha}^{(aa_{\tau})}(\Psi) = \tau (FJ_{\alpha}^{(a)}(\Psi))$$

for all $a \in \mathbb{A}_f^{\times}$, all $\tau \in \operatorname{Aut}(\mathbb{C}/L)$, and all $\alpha \in \Gamma_{\Phi}^{\vee}(1)$.

Proof. The formal scheme (4.5.2) has a distinguished \mathbb{Q} -valued point defined by $q_{\alpha} = 0$ (*i.e.* the unique point of the underlying reduced \mathbb{Q} -scheme), and hence has a distinguished \mathbb{C} -valued point. Hence, using the morphisms of Proposition 4.5.1, each $a \in \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times}$ determines a distinguished point

$$\operatorname{cusp}_{K_{\Phi 0}}^{(a)} \in \widehat{\operatorname{Sh}}_{K_{\Phi 0}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)(\mathbb{C}).$$

By examining the proof of Proposition 4.5.1, the reciprocity law (3.5.2) implies that

$$\operatorname{cusp}_{K_{\Phi_0}}^{(aa_{\tau})} = \tau(\operatorname{cusp}_{K_{\Phi_0}}^{(a)})$$

for any $\tau \in \operatorname{Aut}(\mathbb{C})$, and the q-expansion (4.5.6) is, tautologically, the image of the formal function $[\ell^{\otimes k}, \Psi]$ in the completed local ring at $\operatorname{cusp}_{K_{\Phi 0}}^{(a)}$. The first claim is now a consequence of the equality

$$[\boldsymbol{\ell}^{\otimes k}, \Psi]^{\tau} = [\boldsymbol{\ell}^{\otimes k}, \Psi^{\tau}]$$

of formal functions on

$$\widehat{\operatorname{Sh}}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) \cong \widehat{\operatorname{Sh}}_{K}(G, \mathcal{D}, \Sigma).$$

The second claim follows from the first, and the observation that two rational sections Ψ_1 and Ψ_2 are equal if and only if $\mathrm{FJ}^{(a)}(\Psi_1) = \mathrm{FJ}^{(a)}(\Psi_2)$ for all a. Indeed, to check that $\Psi_1 = \Psi_2$, it suffices to check this in a formal neighborhood of one point on each connected component of $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$. Using strong approximation for the simply connected group

$$Spin(V) = \ker(\nu : G \to \mathbb{G}_m),$$

one can show that the fibers of

$$\operatorname{Sh}_K(G,\mathcal{D})(\mathbb{C}) \to \operatorname{Sh}_{\nu(K)}(\mathbb{G}_m,\mathcal{H}_0)(\mathbb{C})$$

are connected. This implies that the images of the points $\operatorname{cusp}_{K_{\Phi 0}}^{(a)}$ under

$$\widehat{\operatorname{Sh}}_{K_{\Phi 0}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)(\mathbb{C}) \to \widehat{\operatorname{Sh}}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)(\mathbb{C}) \cong \widehat{\operatorname{Sh}}_{K}(G, \mathcal{D}, \Sigma)(\mathbb{C})$$
 hit every connected component of $\operatorname{Sh}_{K}(G, \mathcal{D}, \Sigma)(\mathbb{C})$.

5. Rational theory of Borcherds products

We turn to the theory of Borcherds products [Bor98, Bru02], using the adelic formulation as in [Kud03].

For the rest of this article we fix \mathbb{Q} -quadratic space (V,Q) of signature (n,2) with $n \geq 1$, and denote by (G,\mathcal{D}) the associated GSpin Shimura datum as in §4.1. We also fix $V_{\widehat{\mathbb{Z}}}$ and K as in (4.1.5) and (4.1.6).

5.1. Weakly holomorphic forms. Let $S(V_{\mathbb{A}_f})$ be the Schwartz space of locally constant \mathbb{C} -valued compactly supported functions on $V_{\mathbb{A}_f} = V \otimes \mathbb{A}_f$. For any $g \in G(\mathbb{A}_f)$ abbreviate

$$gV_{\mathbb{Z}} = gV_{\widehat{\mathbb{Z}}} \cap V.$$

Denote by $S_{V_{\mathbb{Z}}} \subset S(V_{\mathbb{A}_f})$ the finite dimensional subspace of functions invariant under $V_{\widehat{\mathbb{Z}}}$, and supported on its dual lattice; we often identify it with the space

$$S_{V_{\mathbb{Z}}} = \mathbb{C}[V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}]$$

of functions on $V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. The metaplectic double cover $\widetilde{\mathrm{SL}}_{2}(\mathbb{Z})$ of $\mathrm{SL}_{2}(\mathbb{Z})$ acts via the Weil representation

$$\rho_{V_{\mathbb{Z}}} : \widetilde{\mathrm{SL}}_{2}(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_{V_{\mathbb{Z}}})$$

as in [Bor98, Bru02, BF04]. Define the complex conjugate representation by

$$\overline{\rho}_{V_{\mathbb{Z}}}(\gamma) \cdot \varphi = \overline{(\rho_{V_{\mathbb{Z}}}(\gamma) \cdot \overline{\varphi})},$$

for $\gamma \in \widetilde{\mathrm{SL}}_2(\mathbb{Z})$ and $\varphi \in S_{V_{\mathbb{Z}}}$.

Remark 5.1.1. The complex conjugate $\overline{\rho}_{V_{\mathbb{Z}}}$ is isomorphic to the contragredient of $\rho_{V_{\mathbb{Z}}}$. It agrees with the representation denoted $\omega_{V_{\mathbb{Z}}}$ in [AGHM17a, AGHMP17b].

Denote by $M_{1-n/2}^!(\overline{\rho}_{V_{\mathbb{Z}}})$ the space of weakly holomorphic forms for $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ of weight 1-n/2 and representation $\overline{\rho}_{V_{\mathbb{Z}}}$, as in [Bor98, Bru02, BF04]. In particular, any

(5.1.1)
$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c(m) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

is an $S_{V_{\mathbb{Z}}}$ -valued holomorphic function on the complex upper half-plane \mathbb{H} . Each Fourier coefficient $c(m) \in S_{V_{\mathbb{Z}}}$ is determined by its values $c(m,\mu)$ at the various cosets $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. Moreover, $c(m,\mu) \neq 0$ implies $m \equiv Q(\mu)$ modulo \mathbb{Z} .

Definition 5.1.2. The weakly holomorphic form (5.1.1) is *integral* if

$$c(m,\mu) \in \mathbb{Z}$$

for all $m \in \mathbb{Q}$ and all $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$.

It is a theorem of McGraw [McG03] that the space of all forms (5.1.1) has a C-basis of integral forms.

5.2. Borcherds products and regularized theta lifts. We now briefly recall the construction of Borcherds products as in [Bor98, Bru02, Kud03].

Fix a morphism of Shimura data $(G, \mathcal{D}) \to (\mathbb{G}_m, \mathcal{H}_0)$, and a connected component $\mathcal{D}^{\circ} \subset \mathcal{D}$. Let $G(\mathbb{R})^{\circ} \subset G(\mathbb{R})$ be the stabilizer of \mathcal{D}° . Equivalently, $G(\mathbb{R})^{\circ}$ is the subgroup of elements with positive spinor similitude. Define $G(\mathbb{Q})^{\circ} = G(\mathbb{Q}) \cap G(\mathbb{R})^{\circ}$.

The image $2\pi i \in \mathcal{H}_0$ of \mathcal{D}° determines an isomorphism from the hermitian domain \mathcal{D} defined in (4.1.2) to the space of oriented negative definite planes in $V_{\mathbb{R}}$, by sending the isotropic vector $z = x + iy \in \mathcal{D}$ to the plane $\mathbb{R}x + \mathbb{R}y$ with its orientation determined by the ordered basis x, y.

Write $\tau = u + iv \in \mathbb{H}$ for the variable on the upper half-plane. For each $\varphi \in S(V_{\mathbb{A}_f})$ there is a Siegel theta function

$$\vartheta(\tau, z, q; \varphi) : \mathbb{H} \times \mathcal{D} \times G(\mathbb{A}_f) \to \mathbb{C},$$

as in [Kud03, (1.37)], satisfying the transformation law

$$\vartheta(\tau, \gamma z, \gamma g h; \varphi) = \vartheta(\tau, z, g; \varphi \circ h^{-1})$$

for any $\gamma \in G(\mathbb{Q})$ and any $h \in G(\mathbb{A}_f)$.

Given a weakly holomorphic form (5.1.1) one can regularize the divergent integral

(5.2.1)
$$\Theta^{reg}(f)(z,g) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{H}} f(\tau)\vartheta(\tau,z,g) \, \frac{du \, dv}{v^2}$$

as in [Bor98, Bru02, Kud03]. Here we are viewing ϑ as a function

$$\vartheta(\tau, z, g) : \mathbb{H} \times \mathcal{D} \times G(\mathbb{A}_f) \to S(V_{\mathbb{A}_f})^{\vee},$$

and implicitly making use of the tautological pairing

$$S(V_{\mathbb{A}_f}) \otimes S(V_{\mathbb{A}_f})^{\vee} \to \mathbb{C}$$

to obtain an $SL_2(\mathbb{Z})$ -invariant scalar-valued integrand $f(\tau)\vartheta(\tau,z,g)$.

The regularized theta lift $\Theta^{reg}(f)$ is a real analytic function defined on an open subset of $\mathcal{D} \times G(\mathbb{A}_f)$. More precisely, it is defined on the complement of a prescribed analytic divisor, and has logarithmic singularities along that divisor. Our $\Theta^{reg}(f)$ is usually denoted $\Phi(f)$ in the literature. We have strayed from the standard notation to avoid confusion with cusp label representatives.

As the subgroup K acts trivially on the quotient (4.1.7), the subspace $S_{V_{\mathbb{Z}}} \subset S(V_{\mathbb{A}_f})$ is K-invariant. It follows that the regularized theta lift satisfies

(5.2.2)
$$\Theta^{reg}(f)(\gamma z, \gamma gk) = \Theta^{reg}(f)(z, g)$$

for any $\gamma \in G(\mathbb{Q})$ and any $k \in K$. This allows us to view $\Theta^{reg}(f)$ as a function on $\operatorname{Sh}_K(G,\mathcal{D})(\mathbb{C})$. For any $g \in G(\mathbb{A}_f)$, its restriction to the connected component

$$(5.2.3) (G(\mathbb{Q})^{\circ} \cap gKg^{-1}) \setminus \mathcal{D}^{\circ} \xrightarrow{z \mapsto [(z,g)]} \operatorname{Sh}_{K}(G,\mathcal{D})(\mathbb{C})$$

is denoted

$$\Theta_g^{reg}(f)(z) \stackrel{\text{def}}{=} \Theta^{reg}(f)(z,g).$$

Denote by $\omega_{\mathcal{D}^{\circ}}$ the restriction to \mathcal{D}° of the tautological line bundle on (4.1.2). It carries an action of $G(\mathbb{R})^{\circ}$ covering the action on the base, and a $G(\mathbb{R})^{\circ}$ invariant metric (4.4.2).

Theorem 5.2.1 (Borcherds [Bor98, Bru02]). Assume that f is integral, and that $c(0,0) \in 2\mathbb{Z}$. There is a meromorphic section $\Psi_g(f)$ of $\boldsymbol{\omega}_{\mathcal{D}^{\circ}}^{\otimes c(0,0)/2}$ such that

(5.2.4)
$$-4\log||\Psi_g(f)||_{\text{naive}} = \Theta_g^{reg}(f) + c(0,0)\log(\pi) + c(0,0)\Gamma'(1).$$

Here $\Gamma'(s)$ is the derivative of the usual Gamma function.

We call $\Psi_g(f)$ the Borcherds product (or Borcherds lift) of f to \mathcal{D}° , but note that the relation (5.2.4) only determines $\Psi_g(f)$ up to scaling by a complex number of absolute value 1. The linearity of $f \mapsto \Theta_g^{reg}(f)$ implies the multiplicativity

$$\Psi_g(f_1 + f_2) = \Psi_g(f_1) \otimes \Psi_g(f_2)$$

of the Borcherds product, up to the ambiguity just noted.

The transformation law (5.2.2) implies the invariance of $\Theta_g^{reg}(f)$ under any $\gamma \in G(\mathbb{Q})^{\circ} \cap gKg^{-1}$. This in turn implies the relation

$$\xi_g(\gamma) = \frac{\Psi_g(f)(\gamma z)}{\gamma \cdot \Psi_g(f)(z)}$$

for some unitary character

$$\xi_g: G(\mathbb{Q})^\circ \cap gKg^{-1} \to \mathbb{C}^\times,$$

which depends only on the double coset $G(\mathbb{Q})^{\circ}gK$.

Definition 5.2.2. If ξ_g is trivial for all $g \in G(\mathbb{A}_f)$, we will say that f has trivial multiplier system with respect to K.

Lemma 5.2.3. Suppose (5.1.1) is integral. We may replace f with a positive integer multiple in such a way that

- (1) $c(0,0) \in 2^{n+2}\mathbb{Z}$,
- (2) f has trivial multiplier system with respect to K.

Proof. This is clear from the main result of [Bor00], which asserts that the characters ξ_q are of finite order.

We assume until the end of $\S 5.4$ that the following hypotheses are satisfied. Recall that (n,2) is the signature of (V,Q).

Hypothesis 5.2.4. Either

- (1) $n \geqslant 3$, or
- (2) n=2 and V has Witt index 1.

Hypothesis 5.2.5. The weakly holomorphic form f of (5.1.1) is integral, has trivial multiplier system with respect to K, and satisfies $c(0,0) \in 2^{n+2}\mathbb{Z}$.

Hypothesis 5.2.4 guarantees that V contains an isotropic line, so that we may form q-expansions. It also implies that the boundary of the Bailey-Borel compactification of $\operatorname{Sh}_K(G,\mathcal{D})$ lies in codimension ≥ 2 , and so Koecher's principle applies. The condition $c(0,0) \in 2^{n+2}\mathbb{Z}$ in Hypothesis 5.2.5 implies that the line bundle $\omega^{\otimes c(0,0)/2}$ on $\operatorname{Sh}_K(G,\mathcal{D})$ extends to the Baily-Borel compactification³, as we will see in the proof of Proposition 5.2.7 below.

The assumption of trivial multiplier system in Hypothesis 5.2.5 guarantees that each $\Psi_g(f)$ descends to a meromorphic section on the connected component (5.2.3). This allows us to make the following definition.

Definition 5.2.6. The Borcherds product $\Psi(f)$ is the meromorphic section of the line bundle $(\boldsymbol{\omega}^{an})^{\otimes c(0,0)/2}$ on $\operatorname{Sh}_K(G,\mathcal{D})(\mathbb{C})$ satisfying

$$-4\log||\Psi(f)||_{\text{naive}} = \Theta^{reg}(f) + c(0,0)\log(\pi) + c(0,0)\Gamma'(1).$$

It is well-defined up to rescaling by a complex number of absolute value 1 on every connected component. Its restriction to each component (5.2.3) is equal to $\Psi_q(f)$, again up to scaling by a constant of absolute value 1.

Proposition 5.2.7. The Borcherds product $\Psi(f)$, a priori a meromorphic section on $\operatorname{Sh}_K(G,\mathcal{D})(\mathbb{C})$, is the analytification of a rational section on $\operatorname{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}$.

 $^{^3\}mathrm{It}$ is presumably true that $\boldsymbol{\omega}$ already extends to the Baily-Borel compactification, but we will never need this.

Proof. It suffices to prove this after shrinking K, so we may assume that K is neat and $\operatorname{Sh}_K(G,\mathcal{D})$ is a quasi-projective variety. Hypothesis 5.2.4 guarantees that the boundary of the (normal and projective) Bailey-Borel compacification

$$\operatorname{Sh}_K(G, \mathcal{D}) \hookrightarrow \operatorname{Sh}_K(G, \mathcal{D})^{\operatorname{BB}}$$

lies in codimension ≥ 2 .

Let $\pi: A \to \operatorname{Sh}_K(G, \mathcal{D})$ be the Kuga-Satake abelian scheme of §4.2, and define the *Hodge bundle*

$$\omega_{\mathrm{Hodge}} = \pi_* \Omega^{\dim(A)}_{A/\mathrm{Sh}_K(G,\mathcal{D})}$$

on $\operatorname{Sh}_K(G,\mathcal{D})$. According to [Mad16, Proposition 4.18] there is an isomorphism

$$oldsymbol{\omega}_{ ext{Hodge}}^{igotimes 2} \cong oldsymbol{\omega}^{igotimes 2^{n+1}}.$$

The Hodge bundle extends to the Baily-Borel compactification. Indeed, this follows from functoriality of the Baily-Borel compactification and the analogous extension result [FC90, Chapter V] on the Siegel modular variety from which A was pulled back. By Hypothesis 5.2.5 the line bundle $\omega^{\otimes c(0,0)/2}$ also extends to the Baily-Borel compactification.

By a suitable generalization of the Koecher principle (more precisely, Levi's generalization [GR84, §9.5] of Hartogs' theorem to meromorphic functions on normal complex analytic spaces), the Borcherds product $\Psi(f)$ extends to a meromorphic section of $(\omega^{an})^{\otimes c(0,0)/2}$ over the Baily-Borel compactification.

By Chow's theorem on the algebraicity of analytic divisors on projective varieties, the analytically defined divisor $D = \operatorname{div}(\Psi(f))$ is actually an algebraic divisor on $\operatorname{Sh}_K(G,\mathcal{D})^{\operatorname{BB}}_{/\mathbb{C}}$. Now view $\Psi(f)$ as a holomorphic section of the analytification of the line bundle

$$\boldsymbol{\omega}^{\otimes c(0,0)/2} \otimes \mathcal{O}(-D)$$

on $\operatorname{Sh}_K(G,\mathcal{D})^{\operatorname{BB}}_{/\mathbb{C}}$. By GAGA this section is algebraic, as desired. \square

Remark 5.2.8. Borcherds does not work adelically. Instead, for every input form (5.1.1) he constructs a single meromorphic section $\Psi_{\text{classical}}(f)$ over \mathcal{D}° . However, each $g \in G(\mathbb{A}_f)$ determines an isomorphism $V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}} \to gV_{\mathbb{Z}}^{\vee}/gV_{\mathbb{Z}}$, which induces an isomorphism

$$M_{1-\frac{n}{2}}^!(\overline{\rho}_{V_{\mathbb{Z}}}) \xrightarrow{f \mapsto g \cdot f} M_{1-\frac{n}{2}}^!(\overline{\rho}_{gV_{\mathbb{Z}}}).$$

Replacing the pair $(V_{\mathbb{Z}}, f)$ by $(gV_{\mathbb{Z}}, gf)$ yields another meromorphic section $\Psi_{\text{classical}}(gf)$ over \mathcal{D}° , and

$$\Psi_g(f) = \Psi_{\text{classical}}(gf).$$

5.3. The product expansion I. For a fixed $h \in G(\mathbb{A}_f)$, we will recall the product formula for $\Psi_h(f)$ due to Borcherds.

Fix an isotropic line $I \subset V$. Fix also an isotropic line $I_* \subset V$ with $[I, I_*] \neq 0$, but do this in a particular way: first choose a \mathbb{Z} -module generator $\ell \in I \cap hV_{\mathbb{Z}}$, and then choose a $k \in hV_{\mathbb{Z}}^{\vee}$ such that $[\ell, k] = 1$. Now take I_* be the span of the isotropic vector

$$(5.3.1) \qquad \ell_* = k - Q(k)\ell.$$

Obviously $[\ell, \ell_*] = 1$, but we need not have $\ell_* \in hV_{\mathbb{Z}}^{\vee}$.

Abbreviate $V_0 = I^{\perp}/I$. This is a \mathbb{Q} -vector space endowed with a quadratic form of signature (n-1,1), and a \mathbb{Z} -lattice

$$(5.3.2) V_{0\mathbb{Z}} = (I^{\perp} \cap hV_{\mathbb{Z}})/(I \cap hV_{\mathbb{Z}}) \subset V_0.$$

Denote by

$$LightCone(V_{0\mathbb{R}}) = \{ w \in V_{0\mathbb{R}} : Q(w) < 0 \}$$

the light cone in $V_{0\mathbb{R}}$. It is a disjoint union of two open convex cones. Every $v \in I_{\mathbb{C}}^{\perp}$ determines an isotropic vector

$$\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in V_{\mathbb{C}},$$

depending only on the image $v \in V_{0\mathbb{C}}$. The resulting injection $V_{0\mathbb{C}} \to \mathbb{P}^1(V_{\mathbb{C}})$ restricts to an isomorphism

$$V_{0\mathbb{R}} + (2\pi i)^{-1} \text{LightCone}(V_{0\mathbb{R}}) \cong \mathcal{D},$$

and we let $LightCone^{\circ}(V_{0\mathbb{R}}) \subset LightCone(V_{0\mathbb{R}})$ be the connected component with

$$V_{0\mathbb{R}} + (2\pi i)^{-1} \text{LightCone}^{\circ}(V_{0\mathbb{R}}) \cong \mathcal{D}^{\circ}.$$

There is an action $\rho_{V_{0\mathbb{Z}}}$ of $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ on the finite dimensional \mathbb{C} -vector space $S_{V_{0\mathbb{Z}}}$, exactly as in §5.1, and a weakly holomorphic modular form

$$f_0(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} \sum_{\lambda \in V_{0\mathbb{Z}}^{\vee}/V_{0\mathbb{Z}}} c_0(m,\lambda) \cdot q^m \in M_{1-\frac{n}{2}}^!(\overline{\rho}_{V_{0\mathbb{Z}}})$$

whose coefficients are defined by

$$c_0(m,\lambda) = \sum_{\substack{\mu \in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \\ \mu \sim \lambda}} c(m,h^{-1}\mu).$$

Here we understand $h^{-1}\mu$ to mean the image of μ under the isomorphism $hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \to V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$ defined by multiplication by h^{-1} . The notation $\mu \sim \lambda$ requires explanation: denoting by

$$p: (I^{\perp} \cap hV_{\mathbb{Z}}^{\vee})/(I^{\perp} \cap hV_{\mathbb{Z}}) \to V_{0\mathbb{Z}}^{\vee}/V_{0\mathbb{Z}}$$

the natural map, $\mu \sim \lambda$ means that there is a

(5.3.3)
$$\tilde{\mu} \in I^{\perp} \cap (\mu + hV_{\mathbb{Z}})$$

such that $p(\tilde{\mu}) = \lambda$.

Every vector $x \in V_0$ of positive length determines a hyperplane $x^{\perp} \subset V_{0\mathbb{R}}$. For each $m \in \mathbb{Q}_{>0}$ and $\lambda \in V_{0\mathbb{Z}}^{\vee}/V_{0\mathbb{Z}}$ define a formal sum of hyperplanes

$$H(m,\lambda) = \sum_{\substack{x \in \lambda + V_{0\mathbb{Z}} \\ O(x) = m}} x^{\perp},$$

in $V_{0\mathbb{R}}$, and set

$$H(f_0) = \sum_{\substack{m \in \mathbb{Q}_{>0} \\ \lambda \in V_0^{\vee}/V_{0\mathbb{Z}}}} c_0(-m, \lambda) \cdot H(m, \lambda).$$

Definition 5.3.1. A Weyl chamber for f_0 is a connected component

(5.3.4)
$$\mathscr{W} \subset \text{LightCone}^{\circ}(V_{0\mathbb{R}}) \setminus \text{Support}(H(f_0)).$$

Let N be the positive integer determined by $N\mathbb{Z} = [hV_{\mathbb{Z}}, I \cap hV_{\mathbb{Z}}]$, and note that $\ell/N \in hV_{\mathbb{Z}}^{\vee}$. Set

(5.3.5)
$$A = \prod_{\substack{x \in \mathbb{Z}/N\mathbb{Z} \\ r \neq 0}} \left(1 - e^{2\pi i x/N}\right)^{c(0,xh^{-1}\ell/N)}.$$

Tautologically, every fiber of $\omega_{\mathcal{D}^{\circ}}$ is a line in $V_{\mathbb{C}}$, and each such fiber pairs nontrivially with the isotropic line $I_{\mathbb{C}}$. Using the nondegenerate pairing

$$[\cdot,\cdot]:I_{\mathbb{C}}\otimes\omega_{\mathcal{D}^{\circ}}\to\mathcal{O}_{\mathcal{D}^{\circ}},$$

the Borcherds product $\Psi_h(f)$ and the isotropic vector $\ell \in I$ determine a meromorphic function $[\ell^{\otimes c(0,0)/2}, \Psi_h(f)]$ on \mathcal{D}° . It is this function that Borcherds expresses as an infinite product.

Theorem 5.3.2 (Borcherds [Bor98, Bru02]). For each Weyl chamber \mathcal{W} there is a vector $\varrho \in V_0$ with the following property: For all

$$v \in V_{0\mathbb{R}} + (2\pi i)^{-1} \mathscr{W} \subset V_{0\mathbb{C}}$$

with $|Q(\operatorname{Im}(v))| \gg 0$, the value of $[\ell^{\otimes c(0,0)/2}, \Psi_h(f)]$ at the isotropic line

$$\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}^\circ$$

is given by the (convergent) infinite product

$$uA \cdot e^{2\pi i [\varrho,v]} \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^{\vee} \\ [\lambda,\mathcal{W}] > 0}} \prod_{\substack{\mu \in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \\ \mu \sim \lambda}} \left(1 - \zeta_{\mu} \cdot e^{2\pi i [\lambda,v]}\right)^{c(-Q(\lambda),h^{-1}\mu)}$$

for some complex number u of absolute value 1. Here, recalling the vector $k \in hV_{\mathbb{Z}}^{\vee}$ appearing in (5.3.1), we have set

$$\zeta_{\mu} = e^{2\pi i [\mu, k]}.$$

Remark 5.3.3. The vector $\varrho \in V_0$ of the theorem is the Weyl vector. It is completely determined by the weakly holomorphic form f_0 and the choice of Weyl chamber \mathcal{W} .

5.4. The product expansion II. Our goal is to connect the product expansion of Theorem 5.3.2 with the algebraic theory of q-expansions from $\S 4.5$. First we make a good choice of toroidal compactification of $\operatorname{Sh}_K(G, \mathcal{D})$. Throughout $\S 5.4$ we assume that K is neat.

Begin by choosing a cusp label representative $\Phi = (P, \mathcal{D}^{\circ}, h)$ for which P is the stabilizer of an isotropic line I. Let $\ell \in I \cap hV_{\mathbb{Z}}$ be a generator, let ℓ_* be as in (5.3.1), and let $I_* = \mathbb{Q}\ell_*$. Set $V_0 = I^{\perp}/I$ as before, and recall the isomorphism $V_0 \otimes I \cong U_{\Phi}(\mathbb{Q})$ of (4.3.2). This induces an isomorphism

$$V_{0\mathbb{R}} \xrightarrow{\otimes (2\pi i)^{-1}\ell} V_{0\mathbb{R}} \otimes I(-1) \cong U_{\Phi}(\mathbb{R})(-1)$$

identifying LightCone° $(V_{0\mathbb{R}})$ with the open convex cone C_{Φ} of (2.4.1).

Lemma 5.4.1. Fix a Weyl chamber \mathcal{W} as (5.3.4). After possibly shrinking K, there exists a K-admissible, complete cone decomposition Σ of (G, \mathcal{D}) having the no self-intersection property, and such that the following holds: there is some top-dimensional rational polyhedral cone $\sigma \in \Sigma_{\Phi}$ whose interior is identified with an open subset of \mathcal{W} under the above isomorphism

$$C_{\Phi} \cong \text{LightCone}^{\circ}(V_{0\mathbb{R}}).$$

Proof. This is an elementary exercise. Using Remark 2.6.1, we first shrink K in order to find some K-admissible, complete cone decomposition Σ of (G, \mathcal{D}) having the no self-intersection property. We may furthermore choose Σ to be smooth, and applying barycentric subdivision [Pin89, §5.24] finitely many times yields a refinement of Σ with the desired properties.

For the remainder of §5.4 we assume that K, Σ , \mathscr{W} , and $\sigma \subset U_{\Phi}(\mathbb{R})(-1)$ are as in Lemma 5.4.1. As in §4.4, the line bundle ω on $\operatorname{Sh}_K(G, \mathcal{D})$ has a canonical extension to $\operatorname{Sh}_K(G, \mathcal{D}, \Sigma)$, and we view $\Psi(f)$ as a rational section over $\operatorname{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$.

By setting $\Upsilon = [(\Phi, \sigma)] \in \text{Strat}_K(G, \mathcal{D}, \Sigma)$, the top-dimensional cone σ singles out a 0-dimensional stratum

$$Z_K^{\Upsilon}(G, \mathcal{D}, \Sigma) \subset \operatorname{Sh}_K(G, \mathcal{D}, \Sigma)$$

as in §2.6. Completing along this stratum, Proposition 4.5.1 provides us with a formally étale surjection

$$\bigsqcup_{\substack{a \in \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times} / K_{0}}} \operatorname{Spf}\left(\mathbb{C}[[q_{\alpha}]]_{\substack{\alpha \in \Gamma_{\Phi}^{\times}(1) \\ \langle \alpha, \sigma \rangle \geqslant 0}}\right) \to \widehat{\operatorname{Sh}}_{K}(G, \mathcal{D}, \Sigma)_{/\mathbb{C}},$$

where $K_0 \subset \mathbb{A}_f^{\times}$ is chosen small enough that the section (4.5.1) satisfies $s(K_0) \subset K_{\Phi}$. As in (4.5.6), the Borcherds product $\Psi(f)$ and the isotropic vector ℓ determine a rational formal function $[\ell^{\otimes c(0,0)/2}, \Psi(f)]$ on the target, which pulls back to a rational formal function

(5.4.1)
$$\operatorname{FJ}^{(a)}(\Psi(f)) \in \operatorname{Frac}\left(\mathbb{C}[[q_{\alpha}]]_{\alpha \in \Gamma_{\Phi}^{\vee}(1)}\right)$$
$$\underset{\langle \alpha, \sigma \rangle \geqslant 0}{\langle \alpha, \sigma \rangle}$$

for every index a. The following proposition explains how this formal q-expansion varies with a.

Proposition 5.4.2. Let $F \subset \mathbb{C}$ be the abelian extension of \mathbb{Q} determined by

$$\operatorname{rec}: \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times} / K_{0} \cong \operatorname{Gal}(F/\mathbb{Q}).$$

The rational formal function (5.4.1) has the form

$$(5.4.2) (2\pi i)^{c(0,0)/2} \cdot \mathrm{FJ}^{(a)}(\Psi(f)) = u^{(a)} A^{\mathrm{rec}(a)} q_{\varrho \otimes 2\pi i \ell_*} \cdot \mathrm{BP}^{\mathrm{rec}(a)}.$$

Here $u^{(a)} \in \mathbb{C}$ is some constant of absolute value 1. The power series BP (<u>B</u>orcherds <u>P</u>roduct) is the infinite product

$$\mathrm{BP} = \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^{\vee} \\ [\lambda, \mathscr{W}] > 0}} \prod_{\substack{\mu \in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}}}} \left(1 - \zeta_{\mu} \cdot q_{\lambda \otimes 2\pi i\ell_{*}}\right)^{c(-Q(\lambda), h^{-1}\mu)} \in \mathcal{O}_{F}[[q_{\alpha}]]_{\substack{\alpha \in \Gamma_{\Phi}^{\vee}(1) \\ \langle \alpha, \sigma \rangle \geqslant 0}}.$$

The constant A and the roots of unity ζ_{μ} have the same meaning as in Theorem 5.3.2, and these constants lie in \mathcal{O}_F . The meaning of $q_{\lambda\otimes 2\pi i\ell_*}$ is as follows: dualizing the isomorphism $V_0\otimes I\cong U_{\Phi}(\mathbb{Q})$ of (4.3.2) yields an isomorphism $V_0\otimes V/I^{\perp}\cong U_{\Phi}(\mathbb{Q})^{\vee}$, and each

$$(5.4.3) \alpha = \lambda \otimes 2\pi i \ell_* \in V_0 \otimes (V/I^{\perp})(1) \cong U_{\Phi}(\mathbb{Q})^{\vee}(1)$$

appearing in the product actually lies in the lattice $\Gamma_{\Phi}^{\vee}(1) \subset U_{\Phi}(\mathbb{Q})^{\vee}(1)$. The condition $[\lambda, \mathcal{W}] > 0$ implies that $\langle \alpha, \sigma \rangle > 0$. Of course $q_{\varrho \otimes 2\pi i\ell_*}$ has the same meaning, with ϱ the Weyl vector of Theorem 5.3.2, although

$$\varrho \otimes 2\pi i \ell_* \in \Gamma_\Phi^\vee(1)$$

need not satisfy the positivity condition $\langle \varrho \otimes 2\pi i \ell_*, \sigma \rangle > 0$.

Proof. First we address the field of definition of the constants A and ζ_{μ} .

Lemma 5.4.3. The constant A of (5.3.5) lies in \mathcal{O}_F , and $\zeta_{\mu} \in \mathcal{O}_F$ for every μ appearing in the above product.

Proof. Suppose $a \in K_0$. It follows from the discussion preceding (4.1.7) that $s(a) \in hKh^{-1}$ stabilizes the lattice $hV_{\mathbb{Z}}$, and acts trivially on the quotient $hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}}$. In particular, s(a) acts trivially on the vector $\ell/N \in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}}$. On the other hand, by its very definition (4.5.1) we know that s(a) acts by a on this vector. It follows that $(a-1)\ell/N \in hV_{\mathbb{Z}}$, from which we deduce first $a-1 \in N\widehat{\mathbb{Z}}$, and then $A^{\operatorname{rec}(a)} = A$.

Suppose $\mu \in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}}$ satisfies $\mu \sim \lambda$ for some $\lambda \in V_{0\mathbb{Z}}^{\vee}$. By (5.3.3) we may fix some $\tilde{\mu} \in I^{\perp} \cap (\mu + hV_{\mathbb{Z}})$. This allows us to compute, using (5.3.1),

$$\begin{split} \zeta_{\mu}^{\text{rec}(a)} &= e^{2\pi i [\tilde{\mu}, ak]} = e^{2\pi i [\tilde{\mu}, a\ell_*]} e^{2\pi i Q(k) \cdot [\tilde{\mu}, a\ell]} \\ &= e^{2\pi i [\tilde{\mu}, s(a)^{-1} \ell_*]} e^{2\pi i Q(k) \cdot [\tilde{\mu}, s(a)\ell]}. \end{split}$$

As
$$[\tilde{\mu}, \ell] = 0$$
, we have $[\tilde{\mu}, s(a)\ell] = 0 = [\tilde{\mu}, s(a)^{-1}\ell]$. Thus
$$\zeta_{\mu}^{\text{rec}(a)} = e^{2\pi i [\tilde{\mu}, s(a)^{-1}\ell_*]} e^{2\pi i Q(k) \cdot [\tilde{\mu}, s(a)^{-1}\ell]} = e^{2\pi i [\tilde{\mu}, s(a)^{-1}k]} = e^{2\pi i [s(a)\tilde{\mu}, k]}.$$

As above, s(a) acts trivially on $hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}}$, and we conclude that

$$\zeta_{\mu}^{\text{rec}(a)} = e^{2\pi i [\tilde{\mu}, k]} = \zeta_{\mu}.$$

Suppose $a \in \mathbb{A}_f^{\times}$. The image of the discrete group

$$\Gamma_{\Phi}^{(a)} = s(a)K_{\Phi}s(a)^{-1} \cap Q_{\Phi}(\mathbb{Q})^{\circ}$$

under $\nu_{\Phi}: Q_{\Phi} \to \mathbb{G}_m$ is contained in $\widehat{\mathbb{Z}}^{\times} \cap \mathbb{Q}_{>0}^{\times} = \{1\}$, and hence $\Gamma_{\Phi}^{(a)}$ is contained in $\ker(\nu_{\Phi}) = U_{\Phi}$. Recalling that the conjugation action of Q_{Φ} on U_{Φ} is by ν_{Φ} , we find that

$$\Gamma_{\Phi}^{(a)} = \operatorname{rat}(\nu_{\Phi}(s(a))) \cdot \left(K_{\Phi} \cap U_{\Phi}(\mathbb{Q})\right) = \operatorname{rat}(a) \cdot \Gamma_{\Phi}$$

as lattices in $U_{\Phi}(\mathbb{Q})$.

Recalling (4.5.3) and (2.6.4), consider the following commutative diagram of complex analytic spaces

$$(5.4.4) \qquad \bigsqcup_{a} \Gamma_{\Phi}^{(a)} \backslash \mathcal{D}^{\circ} - - - - - \succ \bigsqcup_{a} T_{\Phi}(\mathbb{C}) = = \bigsqcup_{a} \Gamma_{\Phi}(-1) \otimes \mathbb{C}^{\times}$$

$$\cong \bigvee_{z \mapsto [(z,s(a))]} \qquad \cong$$

$$\mathscr{U}_{K_{\Phi 0}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \longrightarrow \operatorname{Sh}_{K_{\Phi 0}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C})$$

$$\bigvee_{K_{\Phi}} (Q_{\Phi}, \mathcal{D}_{\Phi}) \longrightarrow \operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C})$$

$$\downarrow_{[(z,g)] \mapsto [(z,gh)]} = \operatorname{Sh}_{K}(G, \mathcal{D})(\mathbb{C}),$$

in which all horizontal arrows are open immersions, all vertical arrows are local isomorphisms on the source, and the disjoint unions are over a set of coset representatives $a \in \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times} / K_{0}$. The dotted arrow is, by definition, the unique open immersion making the upper left square commute.

Lemma 5.4.4. Fix an $\alpha \in \Gamma_{\Phi}^{\vee}(1)$, and write $\alpha = \lambda \otimes 2\pi i \ell_*$ as in (5.4.3). Suppose $v \in V_{0\mathbb{R}} + (2\pi i)^{-1} \text{LightCone}^{\circ}(V_{0\mathbb{R}})$. If we restrict the character

$$q_{\alpha}: \Gamma_{\Phi}(-1) \otimes \mathbb{C}^{\times} \to \mathbb{C}^{\times}$$

to a function $\Gamma_{\Phi}^{(a)} \backslash \mathcal{D}^{\circ} \to \mathbb{C}^{\times}$ via the open immersion in the top row of (5.4.4), its value at the isotropic vector $\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}^{\circ}$ is

$$q_{\lambda \otimes 2\pi i \ell_*} = e^{2\pi i [\lambda, v]/\mathrm{rat}(a)}$$

Proof. The proof is a (slightly tedious) exercise in tracing through the definitions. The dotted arrow in the diagram above is induced by the open immersion $\mathcal{D}^{\circ} \subset U_{\Phi}(\mathbb{C})\mathcal{D}^{\circ} = \mathcal{D}_{\Phi}^{\circ}$ and the isomorphisms

$$\bigsqcup_{a} \Gamma_{\Phi}^{(a)} \backslash \mathcal{D}_{\Phi}^{\circ} \cong \operatorname{Sh}_{K_{\Phi_0}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C}) \cong \bigsqcup_{a} T_{\Phi}(\mathbb{C}).$$

The second isomorphism is the trivialization of the $T_{K_{\Phi 0}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C})$ -torsor

$$\operatorname{Sh}_{K_{\Phi_0}}(Q_{\Phi}, \mathcal{D}_{\Phi})(\mathbb{C}) \to \operatorname{Sh}_{K_0}(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C})$$

induced by the section $s: (\mathbb{G}_m, \mathcal{H}_0) \to (Q_{\Phi}, \mathcal{D}_{\Phi})$, as in the proof of Proposition 4.5.1.

On the component indexed by a, this isomorphism is obtained by combining the isomorphism

$$(5.4.5) U_{\Phi}(\mathbb{C})/\Gamma_{\Phi} \cong \Gamma_{\Phi}(-1) \otimes \mathbb{C}/\mathbb{Z}(1) \xrightarrow{\mathrm{id} \otimes \exp} \Gamma_{\Phi}(-1) \otimes \mathbb{C}^{\times} = T_{\Phi}(\mathbb{C})$$

with the isomorphism

$$U_{\Phi}(\mathbb{C})/\Gamma_{\Phi} \xrightarrow{\operatorname{rat}(a)} U_{\Phi}(\mathbb{C})/\Gamma_{\Phi}^{(a)} \cong \Gamma_{\Phi}^{(a)} \backslash \mathcal{D}_{\Phi}^{\circ},$$

where we use the isotropic line $\ell_* = s(2\pi i) \in \mathcal{D}_{\Phi}^{\circ}$ to trivialize $\Gamma_{\Phi}^{(a)} \backslash \mathcal{D}_{\Phi}^{\circ}$ as a $U_{\Phi}(\mathbb{C})/\Gamma_{\Phi}^{(a)}$ -torsor.

Now identify

$$V_{0\mathbb{C}} \otimes I \cong U_{\Phi}(\mathbb{C}) \xrightarrow{u \mapsto (\operatorname{rat}(a)u) \cdot \ell_*} \mathcal{D}_{\Phi}^{\circ}$$

using (4.3.2). On the one hand, a point $(v \otimes \ell)/\mathrm{rat}(a) \in V_{0\mathbb{C}} \otimes I$ is sent to the isotropic vector

$$\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}_{\Phi}^{\circ}.$$

On the other hand, the image of $(v \otimes \ell)/\text{rat}(a)$ under the composition of (5.4.5) with $q_{\alpha} = q_{\lambda \otimes 2\pi i \ell_*}$ is

$$e^{[\lambda,v][2\pi i\ell_*,\ell]/\mathrm{rat}(a)} = e^{2\pi i[\lambda,v]/\mathrm{rat}(a)}$$

as desired. \Box

Lemma 5.4.5. Suppose $v \in V_{0\mathbb{R}} + (2\pi i)^{-1} \mathcal{W}$ with $|Q(\operatorname{Im}(v))| \gg 0$. The value of the meromorphic function

$$\mathrm{rat}(a)^{c(0,0)/2} \cdot [\ell^{c(0,0)/2}, \Psi_{s(a)h}(f)]$$

at the isotropic line $\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}^{\circ}$ is

$$A_{\Phi}^{\mathrm{rec}(a)} \cdot e^{2\pi i[\varrho,v]/\mathrm{rat}(a)}$$

$$\times \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^{\vee} \\ [\lambda, \mathscr{W}] > 0}} \prod_{\substack{\mu \in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \\ \mu \sim \lambda}} \left(1 - \zeta_{\mu}^{\mathrm{rec}(a)} \cdot e^{2\pi i [\lambda, v]/\mathrm{rat}(a)}\right)^{c(-Q(\lambda), h^{-1}\mu)},$$

up to scaling by a complex number of absolute value 1.

Proof. The proof amounts to carefully keeping track of how Theorem 5.3.2 changes when $\Psi_h(f)$ is replaced by $\Psi_{s(a)h}(f)$. The main source of confusion is that the vectors ℓ and ℓ_* appearing in Theorem 5.3.2 were chosen to have nice properties with respect to the lattice $hV_{\mathbb{Z}}$, and so we must first pick new isotropic vectors $\ell^{(a)}$ and $\ell^{(a)}_*$ having similarly nice properties with respect to $s(a)hV_{\mathbb{Z}}$.

Set $\ell^{(a)} = \operatorname{rat}(a)\ell$. This is a generator of

$$I \cap s(a)hV_{\mathbb{Z}} = \operatorname{rat}(a) \cdot (I \cap hV_{\mathbb{Z}}).$$

Now choose a $k^{(a)} \in s(a)hV_{\mathbb{Z}}^{\vee}$ such that $[\ell^{(a)}, k^{(a)}] = 1$, and let $I_*^{(a)} \subset V$ be the span of the isotropic vector

$$\ell_*^{(a)} = k^{(a)} - Q(k^{(a)})\ell^{(a)}$$

Using the fact that Q_{Φ} acts trivially on the quotient I^{\perp}/I , it is easy to see that the lattice

$$V_{0\mathbb{Z}}^{(a)} = (I^{\perp} \cap s(a)hV_{\mathbb{Z}})/(I \cap s(a)hV_{\mathbb{Z}}) \subset I^{\perp}/I$$

is equal, as a subset of I^{\perp}/I , to the lattice $V_{0\mathbb{Z}}$ of (5.3.2). Thus replacing $hV_{\mathbb{Z}}$ by $s(a)hV_{\mathbb{Z}}$ has no effect on the construction of the modular form f_0 , or on the formation of Weyl chambers or their corresponding Weyl vectors.

Similarly, as Q_{Φ} stabilizes I, the ideal $N\mathbb{Z} = [hV_{\mathbb{Z}}, I \cap hV_{\mathbb{Z}}]$ is unchanged if h is replaced by s(a)h. Replacing h by s(a)h in the definition of A now determines a new constant

$$\begin{split} A^{(a)} &= \prod_{\substack{x \in \mathbb{Z}/N\mathbb{Z} \\ x \neq 0}} \left(1 - e^{2\pi i x/N}\right)^{c(0,x \cdot s(a)^{-1}h^{-1}\ell^{(a)}/N)} \\ &= \prod_{\substack{x \in \mathbb{Z}/N\mathbb{Z} \\ x \neq 0}} \left(1 - e^{2\pi i x/N}\right)^{c(0,x \cdot \text{unit}(a)^{-1}h^{-1}\ell/N)} \\ &= \prod_{\substack{x \in \mathbb{Z}/N\mathbb{Z} \\ x \neq 0}} \left(1 - e^{2\pi i x \cdot \text{unit}(a)/N}\right)^{c(0,ah^{-1}\ell/N)} \\ &= A^{\text{rec}(a)}. \end{split}$$

Citing Theorem 5.3.2 with h replaced by s(a)h everywhere, and using the isomorphism

$$s(a)hV_{\mathbb{Z}}^{\vee}/s(a)hV_{\mathbb{Z}} \cong hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}}$$

induced by the action of $s(a)^{-1}$, we find that the value of

(5.4.6)
$$[(\ell^{(a)})^{c(0,0)/2}, \Psi_{s(a)h}(f)] = \operatorname{rat}(a)^{c(0,0)/2} [\ell^{c(0,0)/2}, \Psi_{s(a)h}(f)]$$

at the isotropic line

$$\ell_*^{(a)} + v - [\ell_*^{(a)}, v]\ell^{(a)} - Q(v)\ell^{(a)} \in \mathcal{D}^\circ$$

is given by the infinite product

$$A_{\Phi}^{(a)} \cdot e^{2\pi i [\varrho,v]} \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^{\vee} \\ [\lambda,\mathscr{W}] > 0}} \prod_{\substack{\mu \in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \\ \mu \sim \lambda}} \left(1 - e^{2\pi i [s(a)\mu,k^{(a)}]} \cdot e^{2\pi i [\lambda,v]}\right)^{c(-Q(\lambda),h^{-1}\mu)}.$$

Now make a change of variables. If we set $v^{(a)} = \ell_* - \operatorname{rat}(a)\ell_*^{(a)} \in V_0$, we find that the value of (5.4.6) at the isotropic line

$$\ell_* + v - [\ell_*, v]\ell - Q(v)\ell$$

$$= \ell_*^{(a)} + \left(\frac{v + v^{(a)}}{\operatorname{rat}(a)}\right) - \left[\ell_*^{(a)}, \left(\frac{v + v^{(a)}}{\operatorname{rat}(a)}\right)\right]\ell^{(a)} - Q\left(\frac{v + v^{(a)}}{\operatorname{rat}(a)}\right)\ell^{(a)}$$

is

$$A_{\Phi}^{(a)} \cdot e^{2\pi i [\varrho, v + v^{(a)}]/\mathrm{rat}(a)} \times \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^{\vee} \mid h \in V_{\mathbb{Z}}^{\vee} / h V_{\mathbb{Z}} \\ [\lambda, \mathscr{W}] > 0}} \prod_{\substack{\mu \in h V_{\mathbb{Z}}^{\vee} / h V_{\mathbb{Z}} \\ \mu \sim \lambda}} \left(1 - e^{2\pi i [s(a)\mu, k^{(a)}]} e^{2\pi i [\lambda, v + v^{(a)}]/\mathrm{rat}(a)}\right)^{c(-Q(\lambda), h^{-1}\mu)}.$$

Assuming that $\mu \sim \lambda$, we may lift $\lambda \in I^{\perp}/I$ to $\tilde{\mu} \in I^{\perp} \cap (\mu + hV_{\mathbb{Z}})$. Computing in $\mathbb{Q}/\mathbb{Z} \cong \widehat{\mathbb{Q}}/\widehat{\mathbb{Z}}$, and using $[\tilde{\mu}, \ell] = 0 = [\tilde{\mu}, \ell_*]$ and the fact that $s(a) \in Q_{\Phi}(\mathbb{A}_f)$ acts trivially on $(I^{\perp}/I) \otimes \mathbb{A}_f$, we have

$$[\lambda, v^{(a)}]/\operatorname{rat}(a) = [s(a)\tilde{\mu}, v^{(a)}]/\operatorname{rat}(a)$$

$$= [s(a)\tilde{\mu}, \operatorname{rat}(a)^{-1}\ell_* - \ell_*^{(a)}]$$

$$= [\tilde{\mu}, s(a)^{-1}\operatorname{rat}(a)^{-1}\ell_* - s(a)^{-1}\ell_*^{(a)}]$$

$$= [\tilde{\mu}, a \cdot \operatorname{rat}(a)^{-1}\ell_* - s(a)^{-1}\ell_*^{(a)}]$$

$$= [\tilde{\mu}, \operatorname{unit}(a)k - s(a)^{-1}k^{(a)}]$$

$$= [\mu, \operatorname{unit}(a)k - s(a)^{-1}k^{(a)}],$$

where the final equality follows from

$$\operatorname{unit}(a)k - s(a^{-1})k^{(a)} \in \widehat{I}^{\perp} \cap hV_{\widehat{\mathbb{Z}}}^{\vee}.$$

Thus

$$\begin{split} e^{2\pi i [s(h)\mu,k^{(a)}]} e^{2\pi i [\lambda,v+v^{(a)}]/\mathrm{rat}(a)} &= e^{2\pi i [\mu,s(a)^{-1}k^{(a)}]} e^{2\pi i [\lambda,v+v^{(a)}]/\mathrm{rat}(a)} \\ &= \zeta_{\mu}^{\mathrm{unit}(a)} \cdot e^{2\pi i [\lambda,v]/\mathrm{rat}(a)}. \end{split}$$

Finally, the equality

$$e^{2\pi i[\varrho,v+v^{(a)}]/\mathrm{rat}(a)} = e^{2\pi i[\varrho,v]/\mathrm{rat}(a)}$$

holds up to a root of unity, simply because $[\varrho, v^{(a)}] \in \mathbb{Q}$.

Working on one connected component

$$\Gamma_{\Phi}^{(a)} \backslash \mathcal{D}^{\circ} \hookrightarrow \mathscr{U}_{K_{\Phi 0}}(Q_{\Phi}, \mathcal{D}_{\Phi}),$$

the pullback of $\Psi(f)$ is $\Psi_{s(a)h}(f)$. The pullback of the section ℓ^{an} of the constant vector bundle I_{dR}^{an} determined by $I_{\mathbb{C}}$ is, by the definition preceding Proposition 4.5.2, the constant section determined by

$$\frac{\mathrm{rat}(a)}{2\pi i} \cdot \ell \in I_{\mathbb{C}}.$$

Thus on $\Gamma_{\Phi}^{(a)} \backslash \mathcal{D}^{\circ}$ we have the equality of meromorphic functions

$$(2\pi i)^{c(0,0)/2} \cdot [\boldsymbol{\ell}^{\otimes c(0,0)/2}, \Psi(f)] = \mathrm{rat}(a)^{c(0,0)/2} \cdot [\boldsymbol{\ell}^{\otimes c(0,0)/2}, \Psi_{s(a)h}(f)],$$

and the stated q-expansion (5.4.2) follows from the above lemmas. The integrality conditions $\lambda \otimes 2\pi i \ell_* \in \Gamma_{\Phi}^{\vee}(1)$ of the proposition (and similarly for the Weyl vector) are deduced after the fact, using the invariance of this meromorphic function under $\Gamma_{\Phi}^{(a)}$. This completes the proof of Proposition 5.4.2.

5.5. Descent to the reflex field. In §5.5 we assume that the $\hat{\mathbb{Z}}$ -lattice (4.1.5) is chosen so that $V_{\mathbb{Z}} \subset V$ is maximal with respect to the quadratic form Q. That is, $V_{\mathbb{Z}}$ is not properly contained in any \mathbb{Z} -lattice on which Q is \mathbb{Z} -valued. Again, K is any compact open subgroup satisfying (4.1.6).

We now use Proposition 4.5.3 to show that the Borcherds product of Proposition 5.2.7, after slight modification, descends to \mathbb{Q} .

Theorem 5.5.1. Assume that (V,Q) satisfies Hypothesis 5.2.4, and let

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c(m) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

be any weakly holomorphic form that is integral, in the sense of Definition 5.1.2. After multiplying f by any sufficiently divisible positive integer, there is a rational section $\psi(f)$ of $\omega^{\otimes c(0,0)}$ over $\operatorname{Sh}_K(G,\mathcal{D})$ satisfying

(5.5.1)
$$-2\log||\psi(f)|| = \Theta^{reg}(f),$$

where the metric on $\boldsymbol{\omega}^{\otimes c(0,0)}$ is (4.4.3).

Proof. It suffices to prove this when $K = G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^{\times}$, for then we can pull back $\psi(f)$ to any smaller level structure. For this choice of level structure the Shimura variety $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$ is connected [AGHMP17b, Proposition 4.1.1].

By Lemma 5.2.3 we may replace $f(\tau)$ by a positive integer multiple so that Hypothesis 5.2.5 is satisfied. Applying Proposition 5.2.7 to the form 2f gives us a rational section

$$\psi(f) = (2\pi i)^{c(0,0)} \Psi(2f)$$

of $\omega^{\otimes c(0,0)}$ over $Sh_K(G,\mathcal{D})_{/\mathbb{C}}$. The relation (5.5.1) is just a restatement of (5.2.4), and the only thing to prove is that $\psi(f)$ is, after rescaling by a constant of absolute value 1, defined over \mathbb{Q} .

Fix any cusp label representative $\Phi = (P, \mathcal{D}^{\circ}, h)$ with P the stabilizer of an isotropic line I. As in §5.3, let $\ell \in I \cap hV_{\mathbb{Z}}$ be a generator.

Let N be as in (5.3.5). Our assumption that $V_{\mathbb{Z}}$ is maximal implies that N=1. Indeed, the finite quadratic space $V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}\cong hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}}$, endowed with the \mathbb{Q}/\mathbb{Z} -valued quadratic form induced by Q, has no nonzero isotropic vectors. As $\ell/N \in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}}$ is isotropic, we deduce first that $\ell/N \in hV_{\mathbb{Z}}$, and then that N=1.

By the very definition of N, we may now choose a $k \in hV_{\mathbb{Z}}$ with $[\ell, k] = 1$. With this choice of k, let $\ell_* \in hV_{\mathbb{Z}}$ be as in (5.3.1), and set $I_* = \mathbb{Q}\ell_*$.

Lemma 5.5.2. The section (4.5.1) determined by I_* satisfies

$$(5.5.2) s(\widehat{\mathbb{Z}}^{\times}) \subset K_{\Phi}.$$

Proof. Note that $[\ell, \ell_*] = 1$, and so $H = \mathbb{Q}\ell + \mathbb{Q}\ell_* \subset V$ is a hyperbolic plane. Its corresponding spinor similar group GSpin(H) is just the unit group of the even Clifford algebra $C^+(H)$.

The natural inclusion $GSpin(H) \to G$ takes values in the subgroup Q_{Φ} , and the cocharacter (4.5.1) factors as

$$\mathbb{G}_m \xrightarrow{s} \mathrm{GSpin}(H) \to Q_{\Phi}$$

where the first arrow sends $a \in \mathbb{Q}^{\times}$ to $s(a) = a^{-1}\ell_{*}\ell + \ell\ell_{*}$, viewed as an element in the even Clifford algebra of H. From this explicit formula and the inclusion

$$H_{\widehat{\mathbb{Z}}} = \widehat{\mathbb{Z}}\ell \oplus \widehat{\mathbb{Z}}\ell_* \subset hV_{\widehat{\mathbb{Z}}},$$

it is clear that (4.5.1) satisfies

$$s(\widehat{\mathbb{Z}}) \subset C^+(H_{\widehat{\mathbb{Z}}})^{\times} \subset Q_{\Phi}(\mathbb{A}_f) \cap C(hV_{\widehat{\mathbb{Z}}})^{\times} = K_{\Phi}.$$

Fix a neat compact open subgroup $\tilde{K} \subset K$ small enough that there is a \tilde{K} -admissible complete cone decomposition Σ for (G, \mathcal{D}) satisfying the conclusion of Lemma 5.4.1, for some fixed Weyl chamber \mathscr{W} . In particular, Lemma 5.4.1 singles out a top-dimensional rational polyhedral cone $\sigma \in \Sigma_{\Phi}$.

Letting $\Upsilon = [(\Phi, \sigma)] \in \operatorname{Strat}_{\tilde{K}}(G, \mathcal{D}, \Sigma)$, we obtain, as in Proposition 4.5.1, a formally étale surjection

$$\bigsqcup_{\substack{a \in \mathbb{Q}_{>0}^{\times} \setminus \mathbb{A}_{\ell}^{\times}/\tilde{K}_{0}}} \operatorname{Spf}\left(\mathbb{C}[[q_{\alpha}]]_{\alpha \in \Gamma_{\Phi}^{\times}(1)}\right) \to \widehat{\operatorname{Sh}}_{\tilde{K}}(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$$

for some $\tilde{K}_0 \subset \mathbb{A}_f^{\times}$. Letting $\tilde{\psi}(f)$ be the pullback of $\psi(f)$ to a rational section on $\mathrm{Sh}_{\tilde{K}}(G,\mathcal{D},\Sigma)_{/\mathbb{C}}$, the q-expansion

(5.5.3)
$$\operatorname{FJ}^{(a)}(\tilde{\psi}(f)) \in \mathbb{C}[[q_{\alpha}]]_{\substack{\alpha \in \Gamma_{\Phi}^{\vee}(1) \\ \langle \alpha, \sigma \rangle \geqslant 0}}$$

is actually independent of a. Indeed, using the notation of (5.4.4), with K replaced by \tilde{K} throughout, these q-expansions can be computed in terms of the pullback of $\psi(f)$ to the upper left corner in

$$\bigsqcup_{\substack{a \in \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times} / \tilde{K}_{0} \\ \downarrow \\ (K_{\Phi} \cap U_{\Phi}(\mathbb{Q})) \backslash \mathcal{D}^{\circ}}} \tilde{\Gamma}_{\Phi}^{(a)} \backslash \mathcal{D}^{\circ} \xrightarrow{z \mapsto [(z,s(a)h)]} \longrightarrow \operatorname{Sh}_{\tilde{K}}(G,\mathcal{D})(\mathbb{C})$$

Here we have chosen our coset representatives $a \in \widehat{\mathbb{Z}}^{\times}$. This implies, by (5.5.2), that $s(a) \in K_{\Phi} \subset hKh^{-1}$, and so

$$\tilde{\Gamma}_{\Phi}^{(a)} = s(a)\tilde{K}_{\Phi}s(a)^{-1} \cap U_{\Phi}(\mathbb{Q}) \subset K_{\Phi} \cap U_{\Phi}(\mathbb{Q})$$

and s(a)hK = hK. It follows that the pullback of $\psi(f)$ to the upper left corner is the same on every copy of \mathcal{D}° .

The constants A and ζ_{μ} appearing in Proposition 5.4.2 are equal to 1. This is clear from our calculation N=1 above, and the fact that we chose $k \in hV_{\mathbb{Z}}$. By Proposition 5.4.2 we may rescale $\psi(f)$ by a constant of absolute value 1 so that (5.5.3) has integer coefficients for one, hence all choices of a. Moreover, it is clear from Proposition 5.4.2 that at least one coefficient of (5.5.3) is equal to 1. (This last observation is not needed here, but will be used in the proof of Proposition 7.3.1.)

Proposition 4.5.3 now implies that $\psi(f)$ is defined over \mathbb{Q} , and the same is therefore true of $\psi(f)$.

Remark 5.5.3. We will see later that Theorem 5.5.1 holds word-for-word if Hypothesis 5.2.4 is omitted. Indeed, if $K = G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^{\times}$ this follows by specializing Theorem 7.3.2 below to the generic fiber. The desired $\psi(f)$ for smaller K is then obtained by pullback.

6. Integral models of orthogonal Shimura varieties

Fix a prime p. We assume throughout §6 that $V_{\mathbb{Z}}$ is a maximal lattice in our fixed quadratic space (V,Q), and that the compact open subgroup (4.1.6) factors as $K = K_p K^p$ with

$$K_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^{\times}.$$

In this section we will define an integral model

$$\mathcal{S}_K(G,\mathcal{D}) \to \operatorname{Spec}(\mathbb{Z}_{(p)})$$

of $Sh_K(G, \mathcal{D})$, and construct toroidal compactifications of $\mathcal{S}_K(G, \mathcal{D})$. We will also show that the line bundle of weight one modular forms ω extends in a canonical way to these compactified integral models.

6.1. **Isometric embeddings.** First, we need a few preliminaries about quadratic lattices.

Definition 6.1.1. We will say that $V_{\mathbb{Z}_p}$ is almost self-dual if one of the following holds:

- $V_{\mathbb{Z}_p}$ is self-dual; p=2, $\dim_{\mathbb{Q}}(V)$ is odd, and $[V_{\mathbb{Z}_2}^{\vee}:V_{\mathbb{Z}_2}]$ is not divisible by 4.

Remark 6.1.2. This definition is equivalent to requiring that the quadric over \mathbb{Z}_p that parameterizes isotropic lines in $V_{\mathbb{Z}_p}$ is smooth. Here, an isotropic line in V_R for an \mathbb{Z}_p -algebra R is a local direct summand $I \subset V_R$ of rank 1 that is locally generated by an element $v \in I$ satisfying Q(v) = 0.

Remark 6.1.3. The almost self-duality of $V_{\mathbb{Z}_2}$ is equivalent to requiring that the radical of the bilinear form on $V_{\mathbb{F}_2}$ has dimension at most 1.

We will repeatedly find ourselves in the following situation. Suppose we have another quadratic space $(V^{\diamond}, Q^{\diamond})$ of signature $(n^{\diamond}, 2)$, and an isometric embedding $V \hookrightarrow V^{\diamond}$. This induces a morphism of Clifford algebras $C(V) \rightarrow$ $C(V^{\diamond})$, which induces a morphism of GSpin Shimura data

$$(G, \mathcal{D}) \to (G^{\diamond}, \mathcal{D}^{\diamond}).$$

Exactly as we assume for (V, Q), suppose we are given a $\hat{\mathbb{Z}}$ -lattice

$$V_{\widehat{\mathbb{Z}}}^{\diamond} \subset V^{\diamond} \otimes \mathbb{A}_f$$

on which Q^{\diamond} is $\widehat{\mathbb{Z}}$ -valued, and such that $V_{\mathbb{Z}}^{\diamond} = V^{\diamond} \cap V_{\widehat{\mathbb{Z}}}^{\diamond}$ is maximal. Let

$$K^{\diamond} = K_p^{\diamond} K^{\diamond,p} \subset G^{\diamond}(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}}^{\diamond})^{\times}$$

be a compact open subgroup with $K_p^{\diamond} = G^{\diamond}(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p}^{\diamond})^{\times}$.

Assume that $V_{\mathbb{Z}} \subset V_{\mathbb{Z}}^{\diamond}$, and that $K \subset K^{\diamond}$ so that we have a finite and unramified morphism

$$(6.1.1) j: \operatorname{Sh}_{K}(G, \mathcal{D}) \to \operatorname{Sh}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$$

of canonical models. Note that our choices imply that $K_p = K_p^{\diamond} \cap G(\mathbb{Q}_p)$, and (using the assumption that $V_{\mathbb{Z}}$ is maximal) that $V_{\mathbb{Z}} = V \cap V_{\mathbb{Z}}^{\diamond}$.

Lemma 6.1.4.

- (1) It is possible to choose $(V^{\diamond}, Q^{\diamond})$ and $V_{\widehat{\mathcal{Z}}}^{\diamond}$ as above in such a way that
- $V_{\mathbb{Z}}^{\diamond}$ is self-dual. (2) It is possible to choose $(V^{\diamond}, Q^{\diamond})$ and $V_{\widehat{\mathbb{Z}}}^{\diamond}$ as above in such a way that $V_{\mathbb{Z}_p}^{\diamond}$ is almost self-dual, and $V \subset V^{\diamond}$ with codimension at most 2.

Proof. An exercise in the classification of quadratic spaces over \mathbb{Q} shows that we may choose a positive definite quadratic space W in such a way that the orthogonal direct sum $V^{\diamond} = V \oplus W$ admits a self-dual lattice locally at every finite prime (for example, we may arrange for V^{\diamond} to be a sum of hyperbolic planes locally at every finite prime). From Eichler's theorem that any two maximal lattices in a \mathbb{Q}_p -quadratic space are isometric, it follows that any maximal lattice in V^{\diamond} is self-dual. Enlarging $V_{\mathbb{Z}}$ to a maximal lattice $V_{\mathbb{Z}}^{\diamond} \subset V^{\diamond}$ proves the first claim.

A similar argument also gives the second claim.

6.2. Definition of the integral model. Assume that $V_{\mathbb{Z}_p}$ is almost selfdual in the sense of Definition 6.1.1.

This implies, by [Con14, §C.4], that $\mathcal{G} = \operatorname{GSpin}(V_{\mathbb{Z}_p})$ is a reductive group scheme over \mathbb{Z}_p , and hence that K_p is a hyperspecial compact open subgroup of $G(\mathbb{Q}_p)$. Thus $Sh_K(G,\mathcal{D})$ admits a canonical smooth integral model $\mathcal{S}_K(G,\mathcal{D})$ over $\mathbb{Z}_{(p)}$ by the results of Kisin [Kis10] (and [KM16] if p=2).

Remark 6.2.1. The notion of almost self-duality does not appear anywhere in our main references [Mad16, AGHM17a, AGHMP17b] on integral models of $\operatorname{Sh}_K(G,\mathcal{D})$. This is due to an oversight on the authors' part: we did not realize that one could obtain smooth integral models even if $V_{\mathbb{Z}_p}$ fails to be self-dual.

According to [KM16, Proposition 3.7] there is a functor

$$(6.2.1) N \mapsto (\mathbf{N}_{dR}, F^{\bullet} \mathbf{N}_{dR})$$

from representations $\mathcal{G} \to \mathrm{GL}(N)$ on free $\mathbb{Z}_{(p)}$ -modules of finite rank to filtered vectors bundles on $\mathcal{S}_K(G,\mathcal{D})$, restricting to the functor (3.3.2) in the generic fiber.⁴

Applying this functor to the representation $H_{\mathbb{Z}_{(p)}} = C(V_{\mathbb{Z}_{(p)}})$ as in §4.2 yields a filtered vector bundle $(\mathbf{H}_{dR}, F^{\bullet}\mathbf{H}_{dR})$. Applying the functor to the representation $V_{\mathbb{Z}_{(p)}}$ yields a filtered vector bundle $(\mathbf{V}_{dR}, F^{\bullet}\mathbf{V}_{dR})$. The inclusion (4.1.3) determines an injection $\mathbf{V}_{dR} \to \underline{\operatorname{End}}(\mathbf{H}_{dR})$ onto a local direct summand.

Composition in $\underline{\operatorname{End}}(H_{dR})$ induces a non-degenerate quadratic form

$$Q: V_{dR} \to \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})}$$

such that, for any section x of V_{dR} , the endomorphism $x \circ x$ of H_{dR} is multiplication by the scalar Q(x). We also have the associated bilinear form [x, y] = Q(x + y) - Q(x) - Q(y).

The filtration on V_{dR} has the form

$$0 = F^2 \mathbf{V}_{dR} \subset F^1 \mathbf{V}_{dR} \subset F^0 \mathbf{V}_{dR} \subset F^{-1} \mathbf{V}_{dR} = \mathbf{V}_{dR},$$

in which F^1V_{dR} is an isotropic line, and $F^0V_{dR} = (F^1V_{dR})^{\perp}$. The line bundle of weight one modular forms on $\mathcal{S}_K(G,\mathcal{D})$ is

$$\boldsymbol{\omega} = F^1 \boldsymbol{V}_{dR}.$$

If $V_{\mathbb{Z}_p}$ is not almost self-dual then choose auxiliary data $(V^{\diamond}, Q^{\diamond})$ as in §6.1 in such a way that \mathbb{Z}_p -quadratic space $V_{\mathbb{Z}_p}^{\diamond}$ is almost self-dual. This determines a commutative diagram

(6.2.2)
$$\mathcal{S}_{K}(G,\mathcal{D}) \longleftarrow \operatorname{Sh}_{K}(G,\mathcal{D})$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond}) \longleftarrow \operatorname{Sh}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond})$$

in which the lower left corner is the canonical integral model of $\operatorname{Sh}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$, and $\mathcal{S}_{K}(G, \mathcal{D})$ is defined as its normalization [AGHMP17b, Definition 4.2.1] in $\operatorname{Sh}_{K}(G, \mathcal{D})$. By construction, $\mathcal{S}_{K}(G, \mathcal{D})$ is normal and flat over $\mathbb{Z}_{(p)}$.

We now define

(6.2.3)
$$\boldsymbol{\omega} = \boldsymbol{\omega}^{\diamond}|_{\mathcal{S}_K(G,\mathcal{D})},$$

⁴There is also a weight filtration on N_{dR} , but, as noted in Remark 3.4.2, it is not very interesting over the pure Shimura variety.

where ω^{\diamond} is the line bundle of weight one modular forms on $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ constructed above. This extends the line bundle in the generic fiber defined in §4.4.

Proposition 6.2.2. The $\mathbb{Z}_{(p)}$ -stack $\mathcal{S}_K(G,\mathcal{D})$ and the line bundle ω are independent of the auxiliary choices of $(V^{\diamond}, Q^{\diamond})$, $V_{\widehat{\mathbb{Z}}}^{\diamond}$, and K^{\diamond} used in their construction, and the Kuga-Satake abelian scheme of §4.2 extends uniquely to an abelian scheme $\mathcal{A} \to \mathcal{S}_K(G,\mathcal{D})$.

Proof. This is [AGHMP17b, Proposition 4.4.1]. \Box

Remark 6.2.3. If $K = G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^{\times}$ then our integral model $\mathcal{S}_K(G, \mathcal{D})$ agrees with the $\mathbb{Z}_{(p)}$ -stack $\mathcal{M}_{(p)} = \mathcal{M} \otimes \mathbb{Z}_{(p)}$ of [AGHMP17b, §4.4]. For smaller K^p , our $\mathcal{S}_K(G, \mathcal{D})$ is a finite étale cover of $\mathcal{M}_{(p)}$.

Remark 6.2.4. Our $\mathcal{S}_K(G,\mathcal{D})$ is not the same as the integral model of [AGHM17a]. That integral model is obtained from $\mathcal{S}_K(G,\mathcal{D})$ by deleting certain closed substacks supported in characteristics p for which p^2 divides $[V_{\mathbb{Z}}^{\vee}:V_{\mathbb{Z}}]$. The point of deleting such substacks is that the vector bundle V_{dR} on $\mathrm{Sh}_K(G,\mathcal{D})$ of §4.4 then extends canonically to the remaining open substack. In the present work, as in [AGHMP17b], the only automorphic vector bundle required on $\mathcal{S}_K(G,\mathcal{D})$ is the line bundle of modular forms ω just constructed; we have no need of an extension of V_{dR} to $\mathcal{S}_K(G,\mathcal{D})$.

6.3. Toroidal compactification. We will now consider integral models of the toroidal compactifications described in §2. In the case where V is anisotropic, then [Mad, Corollary 4.1.7] shows that $\mathcal{S}_K(G,\mathcal{D})$ is already compact. Therefore, in this subsection, we will assume that V admits an isotropic vector.

Fix auxiliary data $(V^{\diamond}, Q^{\diamond})$, $V_{\widehat{\mathbb{Z}}}^{\diamond}$, and K^{\diamond} as in §6.1, in such a way that $V_{\widehat{\mathbb{Z}}}^{\diamond}$ is almost self-dual at p. By construction, there is a finite morphism $S_K(G, \mathcal{D}) \to S_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$, under which ω^{\diamond} pulls back to ω .

Fix a Hodge embedding

$$(G^\diamond, \mathcal{D}^\diamond) \to (G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})$$

into the Siegel Shimura datum determined by a symplectic space $(H^{\diamond}, \psi^{\diamond})$ over \mathbb{Q} . This also fixes a morphism $(G^{\diamond}, \mathcal{D}^{\diamond}) \to (\mathbb{G}_m, \mathcal{H}_0)$. Although it is not strictly necessary, it will be convenient to choose the Hodge embedding as in Lemma 4.2.1. That is, we take $H^{\diamond} = C(V^{\diamond})$ and fix a symplectic form ψ^{\diamond} so that the p-component of

$$H_{\widehat{\mathcal{D}}}^{\diamond} = C(V_{\widehat{\mathcal{D}}}^{\diamond}) \subset H^{\diamond} \otimes \mathbb{A}_f$$

is self-dual.

Define reductive groups over $\mathbb{Z}_{(p)}$ by

$$\mathcal{G}^{\diamond} = \operatorname{GSpin}(V_{\mathbb{Z}_{(p)}}^{\diamond}), \quad \mathcal{G}^{\operatorname{Sg}} = \operatorname{GSp}(H_{\mathbb{Z}_{(p)}}^{\diamond}),$$

so that $G^{\diamond} \to G^{\operatorname{Sg}}$ extends to a closed immersion $\mathcal{G}^{\diamond} \to \mathcal{G}^{\operatorname{Sg}}$. Fix a compact open subgroup

$$K^{\operatorname{Sg}} = K_p^{\operatorname{Sg}} K^{\operatorname{Sg},p} \subset G^{\operatorname{Sg}}(\mathbb{A}_f)$$

containing K^{\diamond} and satisfying $K_p^{\operatorname{Sg}} = \mathcal{G}^{\operatorname{Sg}}(\mathbb{Z}_p)$. After shrinking the prime-to-p parts of

$$K \subset K^{\diamond} \subset K^{\operatorname{Sg}}$$
,

we assume that all three are neat.

We can construct a toroidal compactification of $\mathcal{S}_K(G,\mathcal{D})$ as follows. Fix a finite, complete K^{Sg} -admissible cone decomposition $\Sigma^{\operatorname{Sg}}$ for $(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})$. As explained in §2.5, it pulls back to a finite, complete, K^{\diamond} -admissible polyhedral cone decomposition Σ^{\diamond} for $(G^{\diamond}, \mathcal{D}^{\diamond})$, and a finite, complete, Kadmissible polyhedral cone decomposition Σ for (G, \mathcal{D}) . If $\Sigma^{\operatorname{Sg}}$ has the no self-intersection property, then so do the decompositions induced from it.

Assume that K^{Sg} and $\Sigma^{\operatorname{Sg}}$ are chosen so that $\Sigma^{\operatorname{Sg}}$ is smooth and satisfies the no self-intersection property. We obtain a commutative diagram

$$(6.3.1) \qquad \mathcal{S}_{K}(G, \mathcal{D}, \Sigma) \longleftarrow \operatorname{Sh}_{K}(G, \mathcal{D}, \Sigma)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad$$

where $\mathcal{S}_{K^{\operatorname{Sg}}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}}, \Sigma^{\operatorname{Sg}})$ is the toroidal compactification of $\mathcal{S}_{K^{\operatorname{Sg}}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})$ constructed by Faltings-Chai and the two schemes above it are defined by normalization, exactly as in (6.2.2).

According to [Mad, Theorem 4.1.5], the $\mathbb{Z}_{(p)}$ -scheme $\mathcal{S}_K(G, \mathcal{D}, \Sigma)$ is proper, and admits a stratification

(6.3.2)
$$\mathcal{S}_K(G, \mathcal{D}, \Sigma) = \bigsqcup_{\Upsilon \in \text{Strat}_K(G, \mathcal{D}, \Sigma)} \mathcal{Z}_K^{\Upsilon}(G, \mathcal{D}, \Sigma)$$

by locally closed subschemes, extending (2.6.1), in which every stratum is flat over $\mathbb{Z}_{(p)}$. The unique open stratum is $\mathcal{S}_K(G, \mathcal{D})$, and its complement is a Cartier divisor.

Fix $\Upsilon = [(\Phi, \sigma)] \in \operatorname{Strat}_K(G, \mathcal{D}, \Sigma)$ in such a way that the parabolic subgroup underlying Φ is the stabilizer of an isotropic line. As in §2.3, the cusp label representative Φ determines a T_{Φ} -torsor

$$\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \to \operatorname{Sh}_{\nu_{\Phi}(K_{\Phi})}(\mathbb{G}_m, \mathcal{H}_0),$$

and the rational polyhedral cone σ determines a partial compactification

$$\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \hookrightarrow \operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma).$$

The base $\operatorname{Sh}_{\nu_{\Phi}(K_{\Phi})}(\mathbb{G}_m, \mathcal{H}_0)$ of the T_{Φ} -torsor, being a zero dimensional normal scheme over \mathbb{Q} , has a canonical finite normal integral model defined

as the normalization of $\operatorname{Spec}(\mathbb{Z}_{(p)})$. The picture is

$$S_{\nu_{\Phi}(K_{\Phi})}(\mathbb{G}_m, \mathcal{H}_0) \longleftarrow \operatorname{Sh}_{\nu_{\Phi}(K_{\Phi})}(\mathbb{G}_m, \mathcal{H}_0)$$

$$\downarrow \qquad \qquad \downarrow$$

$$\operatorname{Spec}(\mathbb{Z}_{(p)}) \longleftarrow \operatorname{Spec}(\mathbb{Q}).$$

Proposition 6.3.1. Define an integral model

$$\mathcal{T}_{\Phi} = \operatorname{Spec}\left(\mathbb{Z}_{(p)}[q_{\alpha}]_{\alpha \in \Gamma_{\Phi}^{\vee}(1)}\right)$$

of the torus T_{Φ} of §2.3.

(1) The \mathbb{Q} -scheme $\operatorname{Sh}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi})$ admits a canonical integral model

$$\mathcal{S}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \to \operatorname{Spec}(\mathbb{Z}_{(p)}),$$

endowed with the structure of a relative \mathcal{T}_{Φ} -torsor

$$S_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \to S_{\nu_{\Phi}(K_{\Phi})}(\mathbb{G}_m, \mathcal{H}_0)$$

compatible with the torsor structure (2.3.1) in the generic fiber.

(2) There is a canonical isomorphism

$$\widehat{\mathcal{S}}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma) \cong \widehat{\mathcal{S}}_{K}(G, \mathcal{D}, \Sigma)$$

of formal schemes extending (2.6.3).

Here $\mathcal{S}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}) \hookrightarrow \mathcal{S}_{K_{\Phi}}(Q_{\Phi}, \mathcal{D}_{\Phi}, \sigma)$ is the partial compactification determined by the rational polyhedral cone

$$\sigma \subset U_{\Phi}(\mathbb{R})(-1) = \operatorname{Hom}(\mathbb{G}_m, \mathcal{T}_{\Phi})_{\mathbb{R}}$$

and the formal scheme on the left hand side is its completion along its unique closed stratum. On the right,

$$\hat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma) = \mathcal{S}_K(G, \mathcal{D}, \Sigma)^{\wedge}_{\mathcal{Z}_K^{\Upsilon}(G, \mathcal{D}, \Sigma)}$$

is the formal completion along the stratum indexed by Υ .

Proof. This is a consequence of [Mad, Theorem 4.1.5].

By [Mad, Theorem 2] and [FC90], both $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$ and the Faltings-Chai compactification are proper. They admit stratifications

$$\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}) = \bigsqcup_{\Upsilon^{\diamond} \in \operatorname{Strat}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})} \mathcal{Z}_{K^{\diamond}}^{\Upsilon^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}),$$

and

$$\mathcal{S}_{K^{\operatorname{Sg}}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}}, \Sigma^{\operatorname{Sg}}) = \bigsqcup_{\Upsilon^{\operatorname{Sg}} \in \operatorname{Strat}_{K^{\operatorname{Sg}}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})} \mathcal{Z}_{K^{\operatorname{Sg}}}^{\Upsilon^{\operatorname{Sg}}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}}, \Sigma^{\operatorname{Sg}}).$$

analogous to (6.3.2). By [Mad, (4.1.13)], these stratifications satisfy a natural compatibility: if $\Upsilon \in \text{Strat}_K(G, \mathcal{D}, \Sigma)$ has image strata Υ^{\diamond} and Υ^{Sg} , in the sense of §2.5, then the maps in (6.3.1) induce maps on strata

$$\mathcal{Z}_{K}^{\Upsilon}(G,\mathcal{D},\Sigma) \to \mathcal{Z}_{K^{\Diamond}}^{\Upsilon^{\Diamond}}(G^{\Diamond},\mathcal{D}^{\Diamond},\Sigma^{\Diamond}) \to \mathcal{Z}_{K^{\operatorname{Sg}}}^{\Upsilon^{\operatorname{Sg}}}(G^{\operatorname{Sg}},\mathcal{D}^{\operatorname{Sg}},\Sigma^{\operatorname{Sg}}).$$

Applying the functor of Proposition 6.3.2 below to the \mathcal{G}^{\diamond} -representation $V_{\mathbb{Z}_{(p)}}^{\diamond}$ yields a line bundle $\boldsymbol{\omega}^{\diamond} = F^{1}\boldsymbol{V}_{dR}^{\diamond}$ on $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$, which we pull back to a line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{K}(G, \mathcal{D}, \Sigma)$. Thus the line bundle (6.2.3) admits a canonical extension to the toroidal compactification.

Proposition 6.3.2. There is a functor

$$N \mapsto (\mathbf{N}_{dR}, F^{\bullet} \mathbf{N}_{dR})$$

from representations $\mathcal{G}^{\diamond} \to \operatorname{GL}(N)$ on free $\mathbb{Z}_{(p)}$ -modules of finite rank to filtered vectors bundles on $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$, extending the functor (6.2.1) on the open stratum, and the functor of Theorem 3.4.1 in the generic fiber.

Proof. Consider the filtered vector bundle $(\mathbf{H}_{dR}^{\diamond}, F^{\bullet}\mathbf{H}_{dR}^{\diamond})$ over $\operatorname{Sh}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ obtained by applying the functor (3.3.2) to the representation

$$G^{\diamond} \to G^{\operatorname{Sg}} = \operatorname{GSp}(H^{\diamond}, \psi^{\diamond}).$$

It is endowed with an alternating form

$$\psi^{\diamond}: \mathbf{H}_{dR}^{\diamond} \otimes \mathbf{H}_{dR}^{\diamond} \to \mathrm{Lie}(\mathbb{G}_m)$$

induced by the G^{\diamond} -equivariant morphism $\psi^{\diamond}: H \otimes H \to \mathbb{Q}(\nu^{\diamond})$, where ν^{\diamond} is the spinor similitude on G^{\diamond} . See Remark 3.5.2. The nontrivial step $F^0H_{dR}^{\diamond}$ in the filtration is a Lagrangian subsheaf with respect to this pairing.

The vector bundle $\mathbf{H}_{dR}^{\diamond}$ is canonically identified with the pullback to $\operatorname{Sh}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ of the first relative homology

$$\boldsymbol{H}_{dR}^{\operatorname{Sg}} = \underline{\operatorname{Hom}} \big(R^1 \pi_* \Omega_{A^{\operatorname{Sg}}/\operatorname{Sh}_K^{\operatorname{Sg}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})}^{\bullet}, \mathcal{O}_{\operatorname{Sh}_K^{\operatorname{Sg}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})} \big)$$

of the universal polarized abelian scheme $\pi: A^{\operatorname{Sg}} \to \operatorname{Sh}_{K^{\operatorname{Sg}}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})$. As the universal abelian scheme extends canonically to the integral model, so does $\mathbf{H}^{\operatorname{Sg}}$. Its pullback defines an extension of $\mathbf{H}_{dR}^{\diamond}$, along with its filtration and alternating form, to the integral model $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$.

Now fix a family of tensors

$$\{s_{\alpha}\}\subset H_{\mathbb{Z}_{(p)}}^{\diamond,\otimes}$$

that cut the reductive subgroup $\mathcal{G}^{\diamond} \subset \mathcal{G}^{\operatorname{Sg}}$. The functoriality of (3.3.2) implies that these tensors define global sections $\{s_{\alpha,dR}\}$ of $\mathcal{H}_{dR}^{\diamond,\otimes}$ over the generic fiber. By [Kis10, Corollary 2.3.9], they extend (necessarily uniquely) to sections over the integral model $\mathcal{S}_{K^{\diamond}}(\mathcal{G}^{\diamond}, \mathcal{D}^{\diamond})$.

By [Mad, Prop. 4.3.7], the filtered vector bundle $(\mathbf{H}_{dR}^{\diamond}, F^{\bullet} \mathbf{H}_{dR}^{\diamond})$ admits a canonical extension to $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$. The alternating form ψ^{\diamond} and the sections $\mathbf{s}_{\alpha,dR}$ extend (necessarily uniquely) to the toroidal compactification.

This allows us to define a \mathcal{G}^{\diamond} -torsor

$$\mathcal{J}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}) \xrightarrow{a} \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$$

whose functor of points assigns to a scheme $S \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$ the set of all pairs (f, f_0) of isomorphisms

$$(6.3.3) f: \mathbf{H}_{dR/S}^{\diamond} \cong H_{\mathbb{Z}_{(p)}}^{\diamond} \otimes \mathcal{O}_{S}, \quad f_{0}: \operatorname{Lie}(\mathbb{G}_{m})_{/S} \cong \mathcal{O}_{S}$$

satisfying $f(\mathbf{s}_{\alpha,dR}) = \mathbf{s}_{\alpha} \otimes 1$ for all α , and making the diagram

$$\begin{array}{ccc} \boldsymbol{H}_{dR/S}^{\diamond} \otimes \boldsymbol{H}_{dR/S}^{\diamond} & & \psi^{\diamond} & & \operatorname{Lie}(\mathbb{G}_{m}) \\ & & & f \otimes f \downarrow & & \downarrow f_{0} \\ (H_{\mathbb{Z}_{(p)}}^{\diamond} \otimes \mathcal{O}_{S}) \otimes (H_{\mathbb{Z}_{(p)}}^{\diamond} \otimes \mathcal{O}_{S}) & & & \mathcal{O}_{S} \end{array}$$

commute.

Define smooth $\mathbb{Z}_{(p)}$ -schemes $\check{\mathcal{M}}^{\diamond}$ and $\check{\mathcal{M}}^{\operatorname{Sg}}$ with functors of points

$$\check{\mathcal{M}}(G^{\diamond}, \mathcal{D}^{\diamond})(S) = \{\text{isotropic lines } z \subset V_{\mathbb{Z}_{(p)}}^{\diamond} \otimes \mathcal{O}_S\}$$
$$\check{\mathcal{M}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})(S) = \{\text{Lagrangian subsheaves } F^0 \subset H_{\mathbb{Z}_{(p)}}^{\diamond} \otimes \mathcal{O}_S\}.$$

These are integral models of the compact duals $\check{M}(G^{\diamond}, \mathcal{D}^{\diamond})$ and $\check{M}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})$ of §4.2, and are related, using (4.1.3), by a closed immersion

$$(6.3.4) \qquad \qquad \check{\mathcal{M}}(G^{\diamond}, \mathcal{D}^{\diamond}) \to \check{\mathcal{M}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})$$

sending the isotropic line $z \subset V_{\mathbb{Z}_{(p)}}^{\diamond}$ to the Lagrangian $zH_{\mathbb{Z}_{(p)}}^{\diamond} \subset H_{\mathbb{Z}_{(p)}}^{\diamond}$. We now have a diagram

in which a is a \mathcal{G}^{\diamond} -torsor and b is \mathcal{G}^{\diamond} -equivariant, extending the diagram (3.4.1) already constructed in the generic fiber. To define the morphism b we first define a morphism

$$\mathcal{J}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}) \to \check{\mathcal{M}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})$$

by sending an S-point (f, f_0) to the Lagrangian subsheaf

$$f(F^0 \mathbf{H}_{dR/S}) \subset H_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S.$$

This morphism factors through (6.3.4). Indeed, as (6.3.4) is a closed immersion, this is a formal consequence of the fact that we have such a factorization in the generic fiber, as can be checked using the analogous complex analytic construction.

With the diagram (6.3.5) in hand, the construction of the desired functor proceeds by simply imitating the construction (3.3.2) used in the generic fiber.

6.4. **Integral** q-expansions. Continue with the assumptions of $\S 6.3$, and now fix a toroidal stratum representative

$$\Upsilon = [(\Phi, \sigma)] \in \operatorname{Strat}_K(G, \mathcal{D}, \Sigma)$$

as in §4.5. Thus $\Phi = (P, \mathcal{D}^{\circ}, h)$ with P the stabilizer of an isotropic line $I \subset V$, and $\sigma \in \Sigma_{\Phi}$ is a top dimensional rational polyhedral cone. Let

$$\Upsilon^{\diamond} \in \operatorname{Strat}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$$

be the image of Υ , in the sense of §2.5.

The corresponding strata

(6.4.1)
$$\mathcal{Z}_{K}^{\Upsilon}(G, \mathcal{D}, \Sigma) \subset \mathcal{S}_{K}(G, \mathcal{D}, \Sigma)$$
$$\mathcal{Z}_{K^{\diamond}}^{\Upsilon^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}) \subset \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}),$$

are flat over $\mathbb{Z}_{(p)}$ of relative dimension 0. The formal completions along these strata are denoted

$$\begin{split} \widehat{\mathcal{S}}_K(G,\mathcal{D},\Sigma) &= \mathcal{S}_K(G,\mathcal{D},\Sigma)^{\wedge}_{\mathcal{Z}_K^{\Upsilon}(G,\mathcal{D},\Sigma)} \\ \widehat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond},\Sigma^{\diamond}) &= \mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond},\Sigma^{\diamond})^{\wedge}_{\mathcal{Z}_{K^{\diamond}}^{\Upsilon^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond},\Sigma^{\diamond})}. \end{split}$$

These are formal $\mathbb{Z}_{(p)}$ -schemes related by a finite morphism

$$(6.4.2) \hat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma) \to \hat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}).$$

Fix a $\mathbb{Z}_{(p)}$ -module generator $\ell \in I \cap V_{\mathbb{Z}_{(p)}}$. Recall from the discussion leading to (4.5.6) that such an ℓ determines an isomorphism

$$[\ell^{\otimes k},\,\cdot\,]: \boldsymbol{\omega}^{\otimes k} \to \mathcal{O}_{\widehat{\operatorname{Sh}}_K(G,\mathcal{D},\Sigma)}$$

of line bundles on $\widehat{\operatorname{Sh}}_K(G, \mathcal{D}, \Sigma)$.

Proposition 6.4.1. The above isomorphism extends uniquely to an isomorphism

$$[\ell^{\otimes k},\,\cdot\,]:\omega^{\otimes k}\to\mathcal{O}_{\widehat{\mathcal{S}}_K(G,\mathcal{D},\Sigma)}$$

of line bundles on the integral model $\hat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma)$.

Proof. The maximality of $V_{\mathbb{Z}_p}$ implies that $V_{\mathbb{Z}_p} \subset V_{\mathbb{Z}_p}^{\diamond}$ is a \mathbb{Z}_p -module direct summand. In particular,

$$I_{\mathbb{Z}_{(p)}} = I \cap V_{\mathbb{Z}_{(p)}} = I \cap V_{\mathbb{Z}_{(p)}}^{\diamond} \subset V_{\mathbb{Z}_{(p)}}^{\diamond}$$

is a $\mathbb{Z}_{(p)}$ -module direct summand generated by ℓ . Because ω is defined as the pullback of ω^{\diamond} , and because the uniqueness part of the claim is obvious, it suffices to construct an isomorphism

$$[\ell,\cdot]:\omega^{\diamond}\to\mathcal{O}_{\widehat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond},\Sigma^{\diamond})}$$

extending the one in the generic fiber, and then pull back along (6.4.2).

We return to the notation of the proof of Proposition 6.3.2. Let $\mathcal{P}^{\diamond} \subset \mathcal{G}^{\diamond}$ be the stabilizer of the isotropic line $I_{\mathbb{Z}_{(p)}} \subset V_{\mathbb{Z}_{(p)}}^{\diamond}$, define a \mathcal{P}^{\diamond} -stable weight filtration

$$\text{wt}_{-3}H_{\mathbb{Z}_{(p)}}^{\diamondsuit} = 0, \quad \text{wt}_{-2}H_{\mathbb{Z}_{(p)}}^{\diamondsuit} = \text{wt}_{-1}H_{\mathbb{Z}_{(p)}}^{\diamondsuit} = I_{\mathbb{Z}_{(p)}}H_{\mathbb{Z}_{(p)}}^{\diamondsuit}, \quad \text{wt}_{0}H_{\mathbb{Z}_{(p)}}^{\diamondsuit} = H_{\mathbb{Z}_{(p)}}^{\diamondsuit},$$
 and set

$$\mathcal{Q}_{\Phi}^{\diamond} = \ker \left(\mathcal{P}^{\diamond} \to \mathrm{GL}(\mathrm{gr}_0(H_{\mathbb{Z}_{(p)}}^{\diamond})) \right).$$

Compare with the discussion of $\S4.3$.

The $\mathbb{Z}_{(p)}$ -schemes of (6.3.4) sit in a commutative diagram

$$\check{\mathcal{M}}_{\Phi}^{\diamond} \longrightarrow \check{\mathcal{M}}_{\Phi}^{\operatorname{Sg}} \\
\downarrow \qquad \qquad \downarrow \\
\check{\mathcal{M}}(G^{\diamond}, \mathcal{D}^{\diamond}) \longrightarrow \check{\mathcal{M}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})$$

in which the horizontal arrows are closed immersions, and the horizontal arrows are open immersions. The $\mathbb{Z}_{(p)}$ -schemes schemes in the top row are defined by their functors of points, which are

$$\check{\mathcal{M}}_{\Phi}^{\diamond}(S) = \left\{ \begin{array}{l} \text{isotropic lines } z \subset V_{\mathbb{Z}_{(p)}}^{\diamond} \otimes \mathcal{O}_{S} \text{ such that} \\ V_{\mathbb{Z}_{(p)}}^{\diamond} \to V_{\mathbb{Z}_{(p)}}^{\diamond} / I_{\mathbb{Z}_{(p)}}^{\perp} \\ \text{identifies } z \cong (V_{\mathbb{Z}_{(p)}}^{\diamond} / I_{\mathbb{Z}_{(p)}}^{\perp}) \otimes \mathcal{O}_{S} \end{array} \right\}$$

and

$$\check{\mathcal{M}}_{\Phi}^{\operatorname{Sg}}(S) = \left\{ \begin{array}{c} \operatorname{Lagrangian \ subsheaves} \ F^0 \subset H_{\mathbb{Z}(p)}^{\diamond} \otimes \mathcal{O}_S \ \text{such that} \\ H_{\mathbb{Z}(p)}^{\diamond} \to \operatorname{gr}_0(H_{\mathbb{Z}(p)}^{\diamond}) \\ \operatorname{identifies} \ F^0 \cong \operatorname{gr}_0(H_{\mathbb{Z}(p)}^{\diamond}) \otimes \mathcal{O}_S \end{array} \right\}.$$

By passing to formal completions, the diagram (6.3.5) determines a diagram of formal $\mathbb{Z}_{(p)}$ -schemes

$$(6.4.4) \qquad \qquad \widehat{\mathcal{J}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}) \xrightarrow{b} \check{\mathcal{M}}(G^{\diamond}, \mathcal{D}^{\diamond})$$

$$\downarrow a \qquad \qquad \qquad \hat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$$

in which a is a \mathcal{G}^{\diamond} -torsor and b is \mathcal{G}^{\diamond} -equivariant, and $\widehat{\mathcal{J}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$ is the formal completion of $\mathcal{J}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$ along the fiber over the stratum (6.4.1).

Lemma 6.4.2. The \mathcal{G}^{\diamond} -torsor in (6.4.4) admits a canonical reduction of structure to a $\mathcal{Q}_{\Phi}^{\diamond}$ -torsor $\mathcal{J}_{\Phi}^{\diamond}$, sitting in a diagram

$$\begin{array}{ccc}
\mathcal{J}_{\Phi}^{\diamond} & \xrightarrow{b} & \check{\mathcal{M}}_{\Phi}^{\diamond} \\
\downarrow & & \downarrow \\
\hat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}).
\end{array}$$

Proof. The essential point is that the filtered vector bundle $(\mathbf{H}_{dR}^{\diamond}, F^{\bullet}\mathbf{H}_{dR}^{\diamond})$ on $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$ used in the construction of the \mathcal{G}^{\diamond} -torsor

$$\mathcal{J}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$$

acquires extra structure after restriction to $\hat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$. Namely, it acquires a weight filtration

$$\operatorname{wt}_{-3}\boldsymbol{H}_{dR}^{\diamond} = 0$$
, $\operatorname{wt}_{-2}\boldsymbol{H}_{dR}^{\diamond} = \operatorname{wt}_{-1}\boldsymbol{H}_{dR}^{\diamond}$, $\operatorname{wt}_{0}\boldsymbol{H}_{dR}^{\diamond} = \boldsymbol{H}_{dR}^{\diamond}$,

along with distinguished isomorphisms

$$\operatorname{gr}_{-2}(\boldsymbol{H}_{dR}^{\diamond}) \cong \operatorname{gr}_{-2}(H_{\mathbb{Z}_{(p)}}^{\diamond}) \otimes \operatorname{Lie}(\mathbb{G}_m)$$
$$\operatorname{gr}_{0}(\boldsymbol{H}_{dR}^{\diamond}) \cong \operatorname{gr}_{0}(H_{\mathbb{Z}_{(p)}}^{\diamond}) \otimes \mathcal{O}_{\widehat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})}.$$

This follows from the discussion of [Mad, (4.3.1)]. The essential point is that, over the formal completion $\hat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$, there is a canonical degenerating abelian scheme, and the desired extension of $\mathbf{H}_{dR}^{\diamond}$ is its de Rham realization. The extension of the weight and Hodge filtrations is also a consequence of this observation; see [Mad, §1], and in particular [Mad, Proposition 1.3.5].

The desired reduction of structure $\mathcal{J}_{\Phi}^{\diamond} \subset \widehat{\mathcal{J}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$ is now defined as the closed formal subscheme parametrizing pairs of isomorphisms (f, f_0) as in (6.3.3) that respect this additional structure.

Moreover, after restricting $\mathbf{H}_{dR}^{\diamond}$ to $\widehat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$, the surjection $\mathbf{H}_{dR}^{\diamond} \to \operatorname{gr}_{0}\mathbf{H}_{dR}^{\diamond}$ identifies $F^{0}\mathbf{H}_{dR}^{\diamond} \cong \operatorname{gr}_{0}\mathbf{H}_{dR}^{\diamond}$. Indeed, in the language of [Mad, §1], this just amounts to the observation that the de Rham realization of a 1-motive with trivial abelian part has trivial weight and Hodge filtrations.

As the composition

$$\mathcal{J}_{\Phi}^{\diamond} \subset \widehat{\mathcal{J}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}) \xrightarrow{b} \check{\mathcal{M}}(G^{\diamond}, \mathcal{D}^{\diamond}) \subset \check{\mathcal{M}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}})$$

sends $(f, f_0) \mapsto f(F^0 \mathbf{H}_{dR}^{\diamond})$, it takes values in the open subscheme

$$\check{\mathcal{M}}_{\Phi}^{\operatorname{Sg}} \subset \check{\mathcal{M}}(G^{\operatorname{Sg}}, \mathcal{D}^{\operatorname{Sg}}).$$

It therefore take values in the closed subscheme $\check{\mathcal{M}}_{\Phi}^{\diamond} \subset \check{\mathcal{M}}_{\Phi}^{\operatorname{Sg}}$, as this can be checked in the generic fiber, where it follows from the analogous complex analytic constructions.

Returning to the main proof, let $\check{I} \subset \check{V}^{\diamond}$ be the constant $\mathcal{Q}_{\Phi}^{\diamond}$ -equivariant line bundles on $\check{\mathcal{M}}_{\Phi}^{\diamond}$ determined by the representations $I_{\mathbb{Z}_{(p)}} \subset V_{\mathbb{Z}_{(p)}}^{\diamond}$, and let $\check{\omega}^{\diamond} \subset \check{V}^{\diamond}$ be the tautological line bundle. The self-duality of $V_{\mathbb{Z}_{(p)}}^{\diamond}$ guarantees that the bilinear pairing on \check{V}^{\diamond} restricts to an isomorphism

$$[\cdot,\cdot]:\check{I}\otimes\check{\omega}^{\diamond}\to\mathcal{O}_{\check{\mathcal{M}}_{\Phi}^{\diamond}}.$$

Pulling back these line bundles to $\mathcal{J}_{\Phi}^{\diamond}$ and taking the quotient by $\mathcal{Q}_{\Phi}^{\diamond}$, we obtain an isomorphism

$$[\cdot,\cdot]: \mathbf{I}_{dR} \otimes \boldsymbol{\omega}^{\diamond} \to \mathcal{O}_{\widehat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond},\Sigma^{\diamond})}$$

of line bundles on $\widehat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$.

On the other hand, the action of $\mathcal{Q}_{\Phi}^{\diamond}$ on $I_{\mathbb{Z}_{(p)}}$ is through the character ν_{Φ}^{\diamond} , which agrees with the restriction of $\nu^{\diamond}: \mathcal{G}^{\diamond} \to \mathbb{G}_m$ to $\mathcal{Q}_{\Phi}^{\diamond}$. The canonical morphism

$$\mathcal{J}_{\Phi}^{\diamond} \to \widehat{\mathcal{J}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond}) \xrightarrow{(f, f_0) \mapsto f_0} \underline{\operatorname{Iso}} \big(\operatorname{Lie}(\mathbb{G}_m), \mathcal{O}_{\widehat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})} \big)$$

of formal $\hat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$ -schemes identifies $\ker(\nu_{\Phi}^{\diamond}) \setminus \mathcal{J}_{\Phi}^{\diamond}$ with the trivial \mathbb{G}_{m} -torsor

$$\underline{\mathrm{Iso}}\big(\mathrm{Lie}(\mathbb{G}_m),\mathcal{O}_{\widehat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond},\Sigma^{\diamond})}\big) \cong \underline{\mathrm{Aut}}\big(\mathcal{O}_{\widehat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond},\Sigma^{\diamond})}\big)$$

over $\widehat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})$. As the action of \mathcal{G}^{\diamond} on $I_{\mathbb{Z}_{(p)}}$ is via ν_{Φ}^{\diamond} , this trivialization fixes an isomorphism

$$I_{dR} = \mathcal{Q}_{\Phi}^{\diamond} \backslash \left(I_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_{\mathcal{J}_{\Phi}^{\diamond}} \right)$$

$$= \mathbb{G}_{m} \backslash \left(I_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_{\ker(\nu_{\Phi}^{\diamond}) \backslash \mathcal{J}_{\Phi}^{\diamond}} \right)$$

$$\cong I_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_{\widehat{\mathcal{S}}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}, \Sigma^{\diamond})}.$$

The generator $\ell \in I_{\mathbb{Z}_{(p)}}$ now determines a trivializing section $\ell = \ell \otimes 1$ of I_{dR} , defining the desired isomorphism (6.4.3). This completes the proof of Proposition 6.4.1.

Let I_* and $s: \mathbb{G}_m \to Q_{\Phi}$ be as in (4.5.1). As in the discussion preceding Proposition 4.5.1, choose a compact open subgroup $K_0 \subset \mathbb{A}_f^{\times}$ small enough that $s(K_0) \subset K_{\Phi}$. Let F/\mathbb{Q} be the abelian extension of \mathbb{Q} determined by

$$\operatorname{rec}: \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times} / K_{0} \cong \operatorname{Gal}(F/\mathbb{Q}).$$

Fix a prime $\mathfrak{p} \subset \mathcal{O}_F$ above p, and let $R \subset F$ be the localization of \mathcal{O}_F at \mathfrak{p} .

Proposition 6.4.3. If we set

$$\widehat{\mathcal{T}}_{\Phi}(\sigma) = \operatorname{Spf}\left(\mathbb{Z}_{(p)}[[q_{\alpha}]]_{\alpha \in \Gamma_{\Phi}^{\vee}(1)}\right),$$

there is a unique morphism

$$\bigsqcup_{a \in \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_{f}^{\times} / K_{0}} \widehat{\mathcal{T}}_{\Phi}(\sigma)_{/R} \to \widehat{\mathcal{S}}_{K}(G, \mathcal{D}, \Sigma)_{/R}$$

of formal R-schemes whose base change to \mathbb{C} agrees with the morphism of Proposition 4.5.1. Moreover, if t is any point of the source and s is its image in $\hat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma)_{/R}$, the induced map of local rings $\mathcal{O}_s \to \mathcal{O}_t$ is faithfully flat.

Proof. The uniqueness of such a morphism is clear. We have to show existence. The proof of this proceeds just as that of Proposition 4.5.1, except that it uses Proposition 6.3.1 as input. The only additional observation required is that p is unramified in F, and so we have an isomorphism

$$\mathcal{S}_{\nu(K_{\Phi})}(Q_{\Phi}, \mathcal{D}_{\Phi})_{/R} \cong \bigsqcup_{a \in \mathbb{Q}_{>0}^{\times} \setminus \mathbb{A}_{f}^{\times}/K_{0}} \operatorname{Spec}(R)$$

of R-schemes.

Suppose Ψ is a section of the line bundle $\omega^{\otimes k}$ on $\operatorname{Sh}_K(G,\mathcal{D})_{/F}$. It follows from Proposition 4.5.3 that the q-expansion (4.5.6) of Ψ has coefficients in F for every $a \in \mathbb{A}_f^{\times}$. If we view Ψ as a rational section on $\mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$, the following result gives a criterion for testing flatness of its divisor.

Corollary 6.4.4. Assume that the special fiber of $S_K(G, \mathcal{D})_{/R}$ is geometrically normal, and for every $a \in \mathbb{A}_f^{\times}$ the q-expansion (4.5.6) satisfies

$$\mathrm{FJ}^{(a)}(\Psi) \in R[[q_{\alpha}]]_{\substack{\alpha \in \Gamma_{\Phi}^{\times}(1) \\ \langle \alpha, \sigma \rangle \geqslant 0}}$$

If this q-expansion is nonzero modulo \mathfrak{p} for all a, then $\operatorname{div}(\Psi)$ is R-flat.

Proof. Since R is a discrete valuation ring, and since $\mathcal{S}_K(G, \mathcal{D}, \Sigma)_{/R}$ is flat over R, to show that $\operatorname{div}(\Psi)$ is an R-flat divisor on $\mathcal{S}_K(G, \mathcal{D}, \Sigma)_{/R}$, it is enough to show that this divisor does not contain any irreducible components of the special fiber in its support.

Every connected component

$$C \subset \mathcal{S}_K(G, \mathcal{D}, \Sigma)_{/R}$$
.

has irreducible special fiber. Indeed, we have assumed that the special fiber of $\mathcal{S}_K(G,\mathcal{D})_{/R}$ is geometrically normal, and it follows from [Mad, Theorem 1] and its proof, the special fiber of $\mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$ is geometrically normal. On the other hand, [Mad, Corollary 4.1.11] shows that C has geometrically connected special fiber. Therefore the special fiber of C is both connected and normal, and hence is irreducible.

As in the proof of Proposition 4.5.3, the closed stratum

$$\mathcal{Z}_K^{\Upsilon}(G, \mathcal{D}, \Sigma)_{/R} \subset \mathcal{S}_K(G, \mathcal{D}, \Sigma)_{/R}$$

meets every connected component. Pick a closed point s of this stratum lying on the connected component C. By the definition of $\mathrm{FJ}^{(a)}(\Psi)$, and from Proposition 6.4.3, our hypothesis on the q-expansion implies that the restriction of Ψ to the completed local ring \mathcal{O}_s of s defines a rational section of $\omega^{\otimes k}$ whose divisor is an R-flat Cartier divisor on $\mathrm{Spf}(\mathcal{O}_s)$.

It follows that $\operatorname{div}(\Psi)$ does not contain the special fiber of C, and varying C shows that $\operatorname{div}(\Psi)$ contains no irreducible components of the special fiber of $\mathcal{S}_K(G, \mathcal{D}, \Sigma)_{/R}$.

Remark 6.4.5. If $V_{\mathbb{Z}_p}$ is almost self-dual, then $S_K(G, \mathcal{D})$ is smooth over $\mathbb{Z}_{(p)}$, and hence has geometrically normal special fiber. Without the assumption of almost self-duality, the special fiber is geometrically normal whenever $n \geq 5$. This is Proposition 7.1.2 below.

7. Integral theory of Borcherds products

As in §6, fix a maximal lattice $V_{\mathbb{Z}}$ in the quadratic space (V, Q), a prime p, and a compact open subgroup (4.1.6) that factors as $K = K_p K^p$ with

$$K_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^{\times}.$$

7.1. **Special divisors.** For $m \in \mathbb{Q}_{>0}$ and $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$ there is a Cartier divisor $\mathcal{Z}(m,\mu)$ on $\mathcal{S}_K(G,\mathcal{D})$, defined in [AGHM17a, AGHMP17b]. We quickly review the essentials, referring the reader to [loc. cit.] for details.

First assume that $V_{\mathbb{Z}}$ is self-dual, so that $V_{\mathbb{Z}} = V_{\mathbb{Z}}^{\vee}$, and $\mathcal{S}_K(G, \mathcal{D})$ is the smooth canonical integral model. Recall the Kuga-Satake abelian scheme

$$\mathcal{A} \to \mathcal{S}_K(G, \mathcal{D})$$

of Proposition 7.1.2. It has a right action of the Clifford algebra $C(V_{\mathbb{Z}})$. For any scheme $S \to \mathcal{S}_K(G, \mathcal{D})$ there is a distinguished \mathbb{Z} -submodule

$$V(\mathcal{A}_S) \subset \operatorname{End}_{C(V_{\mathbb{Z}})}(\mathcal{A}_S)$$

of special endomorphisms. It is endowed with a positive definite \mathbb{Z} -valued quadratic form Q, satisfying $Q(x) = x \circ x$, and the functor

$$\mathcal{Z}(m)(S) = \{ x \in V(\mathcal{A}_S) : Q(x) = m \}.$$

on $\mathcal{S}_K(G,\mathcal{D})$ -schemes is representable by a finite and unramified $\mathcal{S}_K(G,\mathcal{D})$ -stack $\mathcal{Z}(m)$. In fact, étale locally on the target $\mathcal{Z}(m) \to \mathcal{S}_K(G,\mathcal{D})$ is an effective Cartier divisor. It will be helpful to recall from [AGHM17a, Proposition 2.7.4] how this is proved.

As in the discussion of §6.2, there is a filtered vector sub-bundle

$$0 \subset F^1 V_{dR} \subset F^0 V_{dR} \subset V_{dR}$$

of $\underline{\operatorname{End}}(\boldsymbol{H}_{dR})$ over $\mathcal{S}_K(G,\mathcal{D})$. Composition in $\underline{\operatorname{End}}(\boldsymbol{H}_{dR})$ endows \boldsymbol{V}_{dR} with a non-degenerate quadratic form

$$Q: V_{dR} \to \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})},$$

for which F^1V_{dR} is isotropic, with F^0V_{dR} its orthogonal complement.

Suppose that we are given an algebraically closed field k, and a point of $\mathcal{Z}(m)(k)$ corresponding to a point $t \in \mathcal{S}_K(G,\mathcal{D})(k)$ and a special endomorphism $x \in V(\mathcal{A}_t)$ with Q(x) = m. Suppose that R_t is the complete local ring for $\mathcal{S}_K(G,\mathcal{D})$ at t, and let $\mathfrak{m}_t \subset R_t$ be its maximal ideal. Write $\tilde{t} \in \mathcal{S}_K(G,\mathcal{D})(R_t/\mathfrak{m}_t^2)$ for the tautological lift of t.

The de Rham realization x_{dR} lies in the subspace

$$V_{dR,t} \subset \operatorname{End}(\boldsymbol{H}_{dR,t}),$$

and it propagates by parallel transport to a section

$$\tilde{x}_{dR,\tilde{t}} \in V_{dR,\tilde{t}}$$
.

Let $\bar{I}_x \subset R_t/\mathfrak{m}_t^2$ be the ideal generated by the pairing with $\tilde{x}_{dR,\tilde{t}}$ of any generator of the line $F^1V_{dR,\tilde{t}}$, and let $I_x \subset R_t$ be the pre-image of \bar{I}_x . Then I_x is a principal ideal and R_t/I_x is identified with the complete local ring of $\mathcal{Z}(m)$ at the point associated with the pair (t,x).

For a general maximal lattice $V_{\mathbb{Z}}$ choose, as in §6.1, an isometric embedding of V into a larger quadratic space V^{\diamond} admitting a self-dual lattice

 $V_{\mathbb{Z}}^{\diamond} \supset V_{\mathbb{Z}}$. For a suitable choice of level subgroup $K^{\diamond} \supset K$ we obtain a morphism of integral models

$$j: \mathcal{S}_K(G, \mathcal{D}) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$$

as in (6.2.2), and the source and target carry Kuga-Satake abelian schemes \mathcal{A} and \mathcal{A}^{\diamond} , respectively.

For any scheme $S \to \mathcal{S}_K(G, \mathcal{D})$ the pullbacks \mathcal{A}_S and \mathcal{A}_S^{\diamond} are related by a canonical $C(V_{\mathbb{Z}}^{\diamond})$ -linear isomorphism

$$\mathcal{A}_S \otimes_{C(V_{\mathbb{Z}})} C(V_{\mathbb{Z}}^{\diamond}) \cong \mathcal{A}_S^{\diamond}.$$

This isomorphism induces an inclusion

$$\operatorname{End}_{C(V_{\mathbb{Z}})}(\mathcal{A}_S) \subset \operatorname{End}_{C(V_{\mathbb{Z}}^{\diamond})}(\mathcal{A}_S^{\diamond}),$$

and, having already defined $V(\mathcal{A}_S^{\diamond}) \subset \operatorname{End}_{C(V_{\mathbb{Z}}^{\diamond})}(\mathcal{A}_S^{\diamond})$, we may now define the \mathbb{Z} -module

$$V(\mathcal{A}_S) = \operatorname{End}_{C(V_{\mathbb{Z}})}(\mathcal{A}_S) \cap V(\mathcal{A}_S^{\diamond})$$

of special endomorphisms of A_S . It is a finite free \mathbb{Z} -module endowed with the quadratic form $Q(x) = x \circ x$ inherited from $V(A_S^{\diamond})$.

We next define a subset $V_{\mu}(\mathcal{A}_S) \subset V(\mathcal{A}_S)_{\mathbb{Q}}$ for each coset $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. First, define a positive definite \mathbb{Z} -quadratic space

(7.1.1)
$$\Lambda = \{ \lambda \in V_{\mathbb{Z}}^{\diamond} : \lambda \perp V_{\mathbb{Z}} \},$$

so that $V_{\mathbb{Q}}^{\diamond} = V_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}}$. The orthogonal projections to the two factors restrict to maps $V_{\mathbb{Z}}^{\diamond} \to V_{\mathbb{Z}}^{\diamond}$ and $V_{\mathbb{Z}}^{\diamond} \to \Lambda^{\vee}$, which induce isomorphisms

$$V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}} \cong V_{\mathbb{Z}}^{\diamond}/(V_{\mathbb{Z}} \oplus \Lambda) \cong \Lambda^{\vee}/\Lambda.$$

Denote by $\tilde{\mu} \in \Lambda^{\vee}/\Lambda$ the image of μ . According to [AGHM17a, Proposition 2.6.4], there is a canonical isometric embedding $\Lambda \to V(\mathcal{A}_S^{\diamond})$ satisfying

$$V(\mathcal{A}_S) = \{ x \in V(\mathcal{A}_S^{\diamond}) : x \perp \Lambda \}.$$

In particular, there is an isometric embedding

$$(7.1.2) V(\mathcal{A}_S) \oplus \Lambda \subset V(\mathcal{A}_S^{\diamond})$$

with finite cokernel, which allows us to define

$$V_{\mu}(\mathcal{A}_S) = \{ x \in V(\mathcal{A}_S)_{\mathbb{Q}} : x + \tilde{\mu} \in V(\mathcal{A}_S^{\diamond}) \}.$$

Remark 7.1.1. The set $V_{\mu}(\mathcal{A}_S)$ is independent of the choice of auxiliary data of $(V^{\diamond}, Q^{\diamond})$, $V_{\mathbb{Z}}^{\diamond}$, and K^{\diamond} used in its definition. Of course if $\mu = 0$ we recover the \mathbb{Z} -quadratic space $V(\mathcal{A}_S)$ defined above.

Proposition 7.1.2. If $n \geq 5$, then $S_K(G, \mathcal{D})$ has geometrically normal special fiber.

Proof. When p > 2, this is part of [AGHMP17b, Theorem 4.4.5]. The same proof, with minor changes, applies when p = 2 as well. For the convenience of the reader, we provide a sketch here, indicating the places where modifications need to be made.

For any positive definite quadratic space (M, q) over \mathbb{Z} , consider the stack $\mathcal{Z}(M) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ with functor of points

$$\mathcal{Z}(M)(T) = \{\text{isometric embeddings } \iota : M \to V(\mathcal{A}_T^{\diamond})\}$$

for any morphism $T \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$. As observed in [AGHMP17b, §4.4], this is an algebraic stack over $\mathbb{Z}_{(p)}$, finite and unramified over $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$, and with smooth generic fiber.

As in Lemma 6.16 of [Mad16], there is a canonical open substack

$$\mathcal{Z}^{\mathrm{pr}}(M) \subset \mathcal{Z}(M)$$

characterized by the property that, for any scheme $T \to \mathcal{Z}^{pr}(M)$, the de Rham realization of the associated embedding $M \hookrightarrow V(\mathcal{A}_T^{\diamond})$ generates a local direct summand of $V_{dR,T}$. In particular, it contains the generic fiber, and we have a canonical local direct summand

$$M_{dR} \subset V_{dR,\mathcal{Z}^{\mathrm{pr}}(M)}$$

generated by the tautological embedding of M into the space of special endomorphisms over $\mathcal{Z}^{pr}(M)$. In particular, when M is maximal and rank(M) = 1, then we in fact have

$$\mathcal{Z}^{\operatorname{pr}}(M) = \mathcal{Z}(M).$$

Lemma 7.1.3. Suppose that $M_{\mathbb{Z}_p}$ is maximal of rank at most 2. Then:

- (1) The special fiber $\mathcal{Z}^{pr}(M)$ is generically smooth.
- (2) Let η be a generic point of $\mathcal{Z}^{pr}(M)_{\mathbb{F}_p}$ with algebraically closed residue field. Then the tautological map

$$M \to V(\mathcal{A}_n^{\diamond})$$

is an isomorphism.

(3) $\mathcal{Z}(M)_{\mathbb{F}_p}$ is a local complete intersection of dimension n.

Proof. When p > 2, the first assertion follows from Corollary 6.22 of [Mad16], where it is shown using the method of local models.

We can in fact give a direct proof that applies also when p=2. For this, consider the orthogonal complement of M_{dR}

$$oldsymbol{M}_{dR}^{\perp} \subset oldsymbol{V}_{dR,\mathcal{Z}^{\mathrm{pr}}(M)}.$$

Since M_{dR} is generated by realizations of special endomorphisms, and since such realizations must preserve the Hodge filtration on H_{dR}^{\diamond} , one finds that

$$F^1V_{dR,\mathcal{Z}^{\operatorname{pr}}(M)} \subset M_{dR}^{\perp}$$
.

Let k be the field of definition of s. The lifts of t over $k[\epsilon]$ are in canonical bijection with the isotropic lines in $V_{dR,t} \otimes_k k[\epsilon]$ that lift $F^1V_{dR,t}$. Moreover, the ones that in addition admit a lift of the embedding $M \hookrightarrow V(\mathcal{A}_t)$ are in bijection with the subspace of isotropic lines that are orthogonal to M_{dR} . This is shown just as in [Mad16, Proposition 5.16].

It can be easily deduced from this that the smooth locus of $\mathcal{Z}^{pr}(M)$ contains (and in fact coincides with) the open locus where F^1V_{dR} maps isomorphically onto a local direct summand of $V_{dR,\mathcal{Z}^{pr}(M)}/M_{dR}$.

The complement of this locus therefore is supported over the locus in $S_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ where $F^{1}V_{dR}$ is contained in some rank 2 local direct summand of V_{dR} that is *horizontal* for the connection on V_{dR} induced from the Gauss-Manin connection on H_{dR} .

However, the Kodaira-Spencer map for the connection induces an injective map

$$F^1V_{dR} \to (F^0V_{dR}/F^1V_{dR}) \otimes \Omega^1_{\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})/\mathbb{Z}_{(p)}},$$

whose image is a local direct summand. For this, see [Mad16, Proposition 5.16], whose proof applies also when p=2 and $V_{\mathbb{Z}_p}$ is almost selfdual. One has to replace appeals to results from [Kis10] with ones to results from [KM16].

From this, one can check that every component of the non-smooth locus has dimension at most 2, and therefore that $\mathcal{Z}^{pr}(M)$ is generically smooth.⁵

Now, let us look at the second assertion. When p > 2, this is [Mad16, Proposition 6.17], and the same proof applies verbatim without any assumption on p.

Finally, we move on to the last assertion. In general, it is clear from the deformation theory of special endomorphisms described above that the complete local rings of $\mathcal{Z}(M)_{\mathbb{F}_p}$ are quotients of those of $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ by at most $r = \operatorname{rank}(M)$ equations.

Therefore, to show that $\mathcal{Z}(M)_{\mathbb{F}_p}$ is a local complete intersection, it is enough to show that it has dimension at most n-r. If r=0, then this is clear. If r=1, it amounts to the assertion that no generic point of $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ admits a non-zero special endomorphism, which is a special case of (2) with M=0.

If r = 2, choose a basis v, w for M, and set $M_1 = \langle v \rangle$. It is now enough to show that $\mathcal{Z}(M)$ does not contain any components of $\mathcal{Z}(M_1)_{\mathbb{F}_p}$. But this is again a special case of (2).

Returning to the proof of Proposition 7.1.2, choose an embedding $V_{\mathbb{Z}} \hookrightarrow V_{\mathbb{Z}}^{\diamond}$ as in Lemma 6.1.4, so that $V_{\mathbb{Z}_{p}}^{\diamond}$ is almost self-dual, and

$$\Lambda = V_{\mathbb{Z}}^{\perp} \subset V_{\mathbb{Z}}^{\diamond}$$

is a maximal lattice of rank at most 2. The map

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond})$$

factors through a canonical map

$$\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}) \to \mathcal{Z}(\Lambda)$$

whose restriction to the generic fibers is an open and closed immersion. It is now enough to show that $\mathcal{Z}(\Lambda)$ has geometrically normal special fiber.

⁵This argument is derived from [Ogu79] and is also used in the proof of (2) in [Mad16].

By Serre's criterion for normality and assertions (1) and (3) of Lemma 7.1.3, we can now finish by showing that the complement of $\mathcal{Z}^{pr}(\Lambda)_{\mathbb{F}_p}$ in $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ has dimension at most n.

The argument from [Mad16, §6.27] shows that this complement is supported over the supersingular locus of $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$. Therefore, we only have to show that this latter locus has dimension at most n-1. In fact, when p>2, it is shown in [HP17] that this locus has dimension at most

$$\left| \frac{n+4}{2} \right| \leqslant n-1.$$

The same bound is also valid when p=2: One can appeal instead to the results of Ogus from [Ogu01]. More precisely, one applies Corollary 8 and the argument from Proposition 11 of [loc. cit.] to the K3 crystal obtained from the crystalline realization V_{cris} of V. This completes the proof of Proposition 7.1.2.

Define $\mathcal{Z}(m,\mu) \to \mathcal{S}_K(G,\mathcal{D})$ to be the stack whose functor of points is

$$\mathcal{Z}(m,\mu)(S) = \{x \in V_{\mu}(\mathcal{A}_S) : Q(x) = m\}$$

for any scheme $S \to \mathcal{S}_K(G, \mathcal{D})$. The morphism

$$\mathcal{Z}(m,\mu) \to \mathcal{S}_K(G,\mathcal{D})$$

is finite and unramified, and so around every geometric point of $\mathcal{S}_K(G,\mathcal{D})$ one can find an étale neighborhood $U \to \mathcal{S}_K(G,\mathcal{D})$ such that $\mathcal{Z}(m,\mu)_{/U}$ is isomorphic to a disjoint union of closed subschemes of U, each defined locally by a single nonzero equation. In this way $\mathcal{Z}(m,\mu)_{/U}$ defines a Cartier divisor on U, and by glueing over an étale cover we obtain an effective Cartier divisor on $\mathcal{S}_K(G,\mathcal{D})$, again denoted $\mathcal{Z}(m,\mu)$.

Proposition 7.1.4. Assume that $n \ge 3$. Then the special divisor $\mathcal{Z}(m,\mu)$ is flat over $\mathbb{Z}_{(p)}$.

Proof. When p > 2 this is [AGHMP17b, Proposition 4.5.8]. We explain how to extend the proof to the generl case.

As in the proof of Proposition 7.1.2, fix an embedding $V_{\mathbb{Z}} \hookrightarrow V_{\mathbb{Z}}^{\diamond}$ with $V_{\mathbb{Z}_p}^{\diamond}$ self-dual, and such that

$$\Lambda = V_{\mathbb{Z}}^{\perp} \subset V_{\mathbb{Z}}^{\diamond}$$

is maximal of rank at most 2. Fix $m^{\diamond} \in \mathbb{Q}_{>0}$ and $\mu^{\diamond} \in V_{\mathbb{Z}}^{\diamond,\vee}/V_{\mathbb{Z}}^{\diamond}$, and consider the associated finite unramified morphism

$$\mathcal{Z}^{\diamond}(m^{\diamond}, \mu^{\diamond}) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}).$$

Consider the finite unramified stack

$$\mathcal{Z}(\Lambda) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$$

parameterizing morphisms $T \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ along with isometric embeddings $\Lambda \to V(\mathcal{A}_T^{\diamond})$, and the closed substack

$$\mathcal{W} \subset \mathcal{Z}^{\diamond}(m^{\diamond}, \mu^{\diamond}) \times_{\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})} \mathcal{Z}(\Lambda)$$

such that $\mathcal{W}(T)$ consists of pairs (x, ι) with

$$x \in \iota(\Lambda)^{\perp} \subset V(\mathcal{A}_T^{\diamond})_{\mathbb{O}}.$$

Then, arguing as in [AGHMP17b, Proposition 4.5.8], it is now enough to show that the image of the morphism

$$\mathcal{W} \to \mathcal{Z}(\Lambda)$$

does not contain any irreducible components of $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$. To see this, note that, by the proof of Proposition 7.1.2, for every generic point η of $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$, we have

$$\iota_{\eta}: \Lambda \xrightarrow{\simeq} V(\mathcal{A}_{\eta}).$$

In particular, $V(\mathcal{A}_{\eta})^{\perp} = 0$, and hence η is not in the image of \mathcal{W} .

Recalling the line bundle of weight one modular forms ω from §6.2, we also define a line bundle

(7.1.3)
$$\mathcal{Z}(0,\mu) = \begin{cases} \boldsymbol{\omega}^{-1} & \text{if } \mu = 0\\ 0 & \text{otherwise,} \end{cases}$$

on $\mathcal{S}_K(G,\mathcal{D})$, and, for convenience, set $\mathcal{Z}(m,\mu)=0$ whenever m<0.

7.2. The pullback formula for special divisors. Suppose we are in the general situation of §6.1 (in particular, we impose no assumption of self-duality on $V_{\mathbb{Z}}^{\diamond}$), so that we have a morphism of Shimura varieties

$$\operatorname{Sh}_K(G,\mathcal{D}) \to \operatorname{Sh}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond})$$

as in (6.1.1).

The larger Shimura variety $\operatorname{Sh}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ has its own integral model $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ over $\mathbb{Z}_{(p)}$, obtained by repeating the construction of §6.2 with (G, \mathcal{D}) replaced by $(G^{\diamond}, \mathcal{D}^{\diamond})$. That is, choose an isometric embedding $V^{\diamond} \subset V^{\diamond \diamond}$ into a larger quadratic space that admits a self-dual lattice at p, and define $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ as a normalization. Moreover, $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ has its own line bundle ω^{\diamond} , its own Kuga-Satake abelian scheme, and its own collection of special divisors $\mathcal{Z}^{\diamond}(m, \mu)$. These are closely related to the analogous structures on $\mathcal{S}_{K}(G, \mathcal{D})$, as we now explain.

Proposition 7.2.1. The morphism (6.1.1) extends uniquely to a finite morphism

(7.2.1)
$$\mathcal{S}_K(G, \mathcal{D}) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}).$$

Moreover, for any scheme $S \to \mathcal{S}_K(G, \mathcal{D})$ there is a canonical $C(V_{\mathbb{Z}}^{\diamond})$ -linear isomorphism

$$\mathcal{A}_S \otimes_{C(V_{\mathbb{Z}})} C(V_{\mathbb{Z}}^{\diamond}) \cong \mathcal{A}_S^{\diamond},$$

where A_S and A_S^{\diamond} are the pullbacks to S of the Kuga-Satake abelian schemes

$$\mathcal{A} \to \mathcal{S}_K(G, \mathcal{D}), \quad \mathcal{A}^{\diamond} \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}).$$

Proof. All claims are part of [AGHM17a, Proposition 2.5.1].

The positive definite \mathbb{Z} -quadratic space Λ of (7.1.1) satisfies

$$V_{\mathbb{Z}} \oplus \Lambda \subset V_{\mathbb{Z}}^{\diamond} \subset (V_{\mathbb{Z}}^{\diamond})^{\vee} \subset V_{\mathbb{Z}}^{\vee} \oplus \Lambda^{\vee},$$

with all inclusions of finite index. For any $m \in \mathbb{Q}$ and $\mu \in \Lambda^{\vee}/\Lambda$, set

$$R_{\Lambda}(m,\mu) = \{\lambda \in \mu + \Lambda : Q(\lambda) = m\}$$

and $r_{\Lambda}(m,\mu) = \#R_{\Lambda}(m,\mu)$.

Theorem 7.2.2. For any rational number $m \ge 0$ and any $\mu \in (V_{\mathbb{Z}}^{\diamond})^{\vee}/V_{\mathbb{Z}}^{\diamond}$, there is a canonical isomorphism of line bundles

$$\mathcal{Z}^{\diamond}(m,\mu)|_{\mathcal{S}_K(G,\mathcal{D})} \cong \bigotimes_{\substack{m_1+m_2=m\\\mu_1+\mu_2\in\mu}} \mathcal{Z}(m_1,\mu_1)^{\otimes r_{\Lambda}(m_2,\mu_2)},$$

where the product over $\mu_1 + \mu_2 \in \mu$ is understood to mean the product over all pairs

$$(\mu_1, \mu_2) \in (V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}) \oplus (\Lambda^{\vee}/\Lambda)$$

satisfying $\mu_1 + \mu_2 \in (\mu + V_{\mathbb{Z}}^{\diamond})/(V_{\mathbb{Z}} \oplus \Lambda)$.

Proof. If m = 0 this amounts to checking that

$$oldsymbol{\omega}^{\diamond}|_{\mathcal{S}_K(G,\mathcal{D})}\congoldsymbol{\omega}.$$

This is clear directly from the definitions and Proposition 7.1.2, which allow us to identify both of these line bundles with the pullback of ω^{\otimes} for some morphisms

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond}) \to \mathcal{S}_{K^{\diamond\diamond}}(G^{\diamond\diamond},\mathcal{D}^{\diamond\diamond})$$

to a third Shimura variety defined by embedding V^{\diamond} into a quadratic space $V^{\diamond\diamond}$ admitting a lattice that is self-dual at p. Thus we may assume that m>0.

The isometric embedding (7.1.2) induces, for every $\mu \in (V_{\mathbb{Z}}^{\diamond})^{\vee}/V_{Z}^{\diamond}$, a decomposition

(7.2.2)
$$V_{\mu}(\mathcal{A}_{S}^{\diamond}) = \bigsqcup_{\mu_{1} + \mu_{2} \in \mu} V_{\mu_{1}}(\mathcal{A}_{S}) \times (\mu_{2} + \Lambda),$$

which in turn induces an isomorphism of $\mathcal{S}_K(G,\mathcal{D})$ -stacks⁶

$$(7.2.3) \quad \mathcal{Z}^{\diamond}(m,\mu)_{/\mathcal{S}_{K}(G,\mathcal{D})} \cong \bigsqcup_{\substack{m_{1}+m_{2}=m\\\mu_{1}+\mu_{2}\in\mu\\m_{1}>0\\\lambda\in R_{\Lambda}(m_{2},\mu_{2})}} \mathcal{Z}(m_{1},\mu_{1}) \sqcup \bigsqcup_{\substack{\mu_{2}\in\mu\\\lambda\in R_{\Lambda}(m,\mu_{2})\\\lambda\in R_{\Lambda}(m,\mu_{2})}} \mathcal{S}_{K}(G,\mathcal{D}).$$

The condition $\mu_2 \in \mu$ means that

$$0 + \mu_2 \in (V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}) \oplus (\Lambda^{\vee}/\Lambda_{\mathbb{Z}})$$

⁶Up to a change of notation, this is the decomposition of [AGHM17a, Proposition 2.7.1]. In [loc. cit.] the notation $\mathcal{Z}(0,\mu)$ has a different meaning: it is the stack $\mathcal{S}_K(G,\mathcal{D})$ if $\mu=0$, and is \varnothing otherwise.

lies in the subset $(\mu + V_{\mathbb{Z}}^{\diamond})/(V_{\mathbb{Z}} \oplus \Lambda)$.

Explicitly, an S-point of the first disjoint union in (7.2.3) consists of a morphism $S \to \mathcal{S}_K(G, \mathcal{D})$ along with a pair

$$(x,\lambda) \in V_{\mu_1}(\mathcal{A}_S) \times (\mu_2 + \Lambda)$$

satisfying $m_1 = Q(x)$ and $m_2 = Q(\lambda)$. Using (7.2.2) we obtain a special quasi-endomorphism

$$x^{\diamond} = x + \lambda \in V_{\mu}(\mathcal{A}_{S}^{\diamond}).$$

Similarly, an S-point of the second disjoint union consists of a morphism $S \to \mathcal{S}_K(G, \mathcal{D})$ along with a $\lambda \in \mu_2 + \Lambda$ satisfying $m = Q(\lambda)$. This determines a special quasi-endomorphism

$$x^{\diamond} = 0 + \lambda \in V_{\mu}(\mathcal{A}_{S}^{\diamond}).$$

In either case $Q(x^{\diamond}) = m$, and the data of $S \to \mathcal{S}_K(G, \mathcal{D})$ and x^{\diamond} defines an S-point of $\mathcal{Z}^{\diamond}(m, \mu)_{/\mathcal{S}_K(G, \mathcal{D})}$.

If Λ^{\vee} does not represent m, then $R_{\Lambda}(m, \mu_2) = \emptyset$ for all choices of μ_2 , and the desired isomorphism of line bundles

$$\begin{split} \mathcal{Z}^{\diamond}(m,\mu)|_{\mathcal{S}_{K}} &\cong \bigotimes_{\substack{m_{1}+m_{2}=m\\\mu_{1}+\mu_{2}\in\mu\\m_{1}>0}} \mathcal{Z}(m_{1},\mu_{1})^{\otimes r_{\Lambda}(m_{2},\mu_{2})} \\ &\cong \bigotimes_{\substack{m_{1}+m_{2}=m\\\mu_{1}+\mu_{2}\in\mu}} \mathcal{Z}(m_{1},\mu_{1})^{\otimes r_{\Lambda}(m_{2},\mu_{2})}, \end{split}$$

follows immediately from (7.2.3).

If Λ^{\vee} does represent m then $\mathcal{Z}^{\wedge}(m,\mu)_{/\mathcal{S}_K(G,\mathcal{D})}$ may contain connected components isomorphic to $\mathcal{S}_K(G,\mathcal{D})$. In other words, the support of the divisor $\mathcal{Z}^{\wedge}(m,\mu)$ may contain the image of $\mathcal{S}_K(G,\mathcal{D})$, and we must compute the improper intersection.

Fix a geometric point $z \to \mathcal{S}_K(G, \mathcal{D})$. Because $\mathcal{Z}^{\diamond}(m, \mu) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ is finite and unramified, we may choose an étale neighborhood

$$U^{\diamond} \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$$

of z small enough that the morphism

$$\mathcal{Z}^{\diamond}(m,\mu)_{/U^{\diamond}} \to U^{\diamond}$$

restricts to a closed immersion on every connected component of the domain, and so that these connected components are in bijection with the set of lifts

$$z \xrightarrow{\mathcal{Z}^{\diamond}(m,\mu)_{/U^{\diamond}}} U^{\diamond}.$$

Having so chosen U^{\diamond} , we then choose an étale neighborhood

$$U \to \mathcal{S}_K(G, \mathcal{D})$$

of z small enough that there exists a lift

$$U \xrightarrow{U} V^{\diamond} \downarrow \qquad \qquad \downarrow \\ S_K(G, \mathcal{D}) \longrightarrow S_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}).$$

and so that in the cartesian diagram

each of the vertical arrows restricts to a closed immersion on every connected component of its source, and the top horizontal arrow induces a bijection on connected components.

The decomposition (7.2.3) induces a decomposition

$$\mathcal{Z}^{\diamond}(m,\mu)_{/U} \cong \bigsqcup_{\substack{m_1+m_2=m\\\mu_1+\mu_2\in\mu\\m_1>0\\\lambda\in R_{\Lambda}(m_2,\mu_2)}} \mathcal{Z}(m_1,\mu_1)_{/U} \sqcup \bigsqcup_{\substack{\mu_2\in\mu\\\lambda\in R_{\Lambda}(m,\mu_2)}} U_{\lambda}.$$

The first disjoint union defines a Cartier divisor on U, while the second is a disjoint union of copies of U. Each U_{λ} is defined as the image of the closed immersion $U \to \mathcal{Z}^{\diamond}(m,\mu)_{/U}$ determined by the special quasi-endomorphism

$$x^{\diamond} = 0 + \lambda \in V_0(\mathcal{A}_U) \times (\mu_2 + \Lambda) \subset V_{\mu}(\mathcal{A}_U^{\diamond}).$$

There is now a canonical decomposition of schemes

(7.2.4)
$$\mathcal{Z}^{\diamond}(m,\mu)_{/U^{\diamond}} = \mathcal{Z}^{\diamond}_{\text{prop}} \sqcup \bigsqcup_{\substack{\mu_2 \in \mu \\ \lambda \in R_{\Lambda}(m,\mu_2)}} \mathcal{Z}^{\diamond}_{\lambda},$$

where $\mathcal{Z}_{\lambda}^{\diamond} \subset \mathcal{Z}^{\diamond}(m,\mu)_{/U^{\diamond}}$ is the connected component containing U_{λ} , and $\mathcal{Z}_{\text{prop}}^{\diamond}$ is the union of all connected components not of this form. Thus $\mathcal{Z}_{\text{prop}}^{\diamond}$ is the part of the divisor $\mathcal{Z}^{\diamond}(m,\mu)_{/U^{\diamond}}$ that meets the image of $U \to U^{\diamond}$ properly, while each of the remaining factors $\mathcal{Z}_{\lambda}^{\diamond}$ contains the image of U.

Lemma 7.2.3. Let $\mathcal{O}(\mathcal{Z}_{\text{prop}}^{\diamond})$ be the line bundle defined by the Cartier divisor $\mathcal{Z}_{\text{prop}}^{\diamond} \hookrightarrow U^{\diamond}$. There is a canonical isomorphism of line bundles

$$\mathcal{O}(\mathcal{Z}_{\text{prop}}^{\diamond})|_{U} \cong \bigotimes_{\substack{m_1 + m_2 = m \\ \mu_1 + \mu_2 \in \mu \\ m_1 > 0}} \mathcal{Z}(m_1, \mu_1)|_{U}^{\otimes r_{\Lambda}(m_2, \mu_2)}.$$

Proof. The definition of $\mathcal{Z}_{\text{prop}}^{\diamond}$ is made in such way that

$$\mathcal{Z}_{\text{prop}/U}^{\diamond} \cong \bigsqcup_{\substack{m_1 + m_2 = m \\ \mu_1 + \mu_2 \in \mu \\ m_1 > 0 \\ \lambda \in R_{\Lambda}(m_2, \mu_2)}} \mathcal{Z}(m_1, \mu_1)_{/U},$$

as *U*-schemes, from which the claim follows immediately.

Let $\mathcal{O}(\mathcal{Z}_{\lambda}^{\diamond})$ be the line bundle defined by the Cartier divisor $\mathcal{Z}_{\lambda}^{\diamond} \hookrightarrow U^{\diamond}$. We will now compute the pullback of this line bundle via

$$U \cong U_{\lambda} \to \mathcal{Z}_{\lambda}^{\diamond} \hookrightarrow U^{\diamond}.$$

The analogous calculation in the context of unitary Shimura varieties is [BHY15, Theorem 7.10], and the general strategy of the following proof is the same.

Lemma 7.2.4. There is a canonical isomorphism $\mathcal{O}(\mathcal{Z}_{\lambda}^{\diamond})|_{U} \cong \boldsymbol{\omega}|_{U}^{-1}$.

Proof. Let

$$T(\mathcal{Z}_{\lambda}^{\diamond}) \subset U^{\diamond}$$

be the first order infinitesimal tube around the closed subscheme $\mathcal{Z}_{\lambda}^{\diamond} \subset U_{\diamond}$. That is to say, if $J_{\lambda} \subset \mathcal{O}_{U^{\diamond}}$ is the ideal sheaf defining $\mathcal{Z}_{\lambda}^{\diamond}$, then $T(\mathcal{Z}_{\lambda}^{\diamond})$ is defined by J_{λ}^{2} . We now have morphisms

$$U \cong U_{\lambda} \to \mathcal{Z}_{\lambda}^{\diamond} \hookrightarrow T(\mathcal{Z}_{\lambda}^{\diamond}) \hookrightarrow U^{\diamond} \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}).$$

The crux of the proof is the observation that, after restriction to the tube $T(\mathcal{Z}_{\lambda}^{\diamond})$, the line bundles $\mathcal{O}(\mathcal{Z}_{\lambda}^{\diamond})$ and $\boldsymbol{\omega}^{\diamond}|_{U^{\diamond}}^{-1}$ on U^{\diamond} each admit a canonical section with (scheme-theoretic) zero locus $\mathcal{Z}_{\lambda}^{\diamond}$.

For the line bundle $\mathcal{O}(\mathcal{Z}_{\lambda}^{\diamond}) \supset \mathcal{O}_{U^{\diamond}}$, this section is just the restriction of the constant function $1 \in H^0(U^{\diamond}, \mathcal{O}_{U^{\diamond}}) \subset H^0(U^{\diamond}, \mathcal{O}(\mathcal{Z}_{\lambda}^{\diamond}))$. Call this section

$$(7.2.5) s_{\lambda} \in H^{0}(T(\mathcal{Z}_{\lambda}^{\diamond}), \mathcal{O}(\mathcal{Z}_{\lambda}^{\diamond})|_{T(\mathcal{Z}_{\lambda}^{\diamond})}).$$

For the line bundle $\boldsymbol{\omega}^{\diamond}|_{U^{\diamond}}^{-1}$ the construction of the section uses deformation theory. By the very definition of $\mathcal{Z}^{\diamond}(m,\mu)$, the Kuga-Satake abelian scheme $\mathcal{A}^{\diamond} \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ acquires a tautological special endomorphism

$$x^{\diamond} \in V_{\mu}(\mathcal{A}_{\mathcal{Z}_{\lambda}^{\diamond}}^{\diamond})$$

when restricted to $\mathcal{Z}_{\lambda}^{\diamond} \subset \mathcal{Z}^{\diamond}(m,\mu)_{U^{\diamond}}$, and it is the deformation locus of x^{\diamond} inside the tube $T(\mathcal{Z}_{\lambda}^{\diamond})$ that we will examine. We will be following the methods of the proof of [AGHM17a, Proposition 2.7.4], to which we direct the reader for more details.

First, let us assume that $V_{\mathbb{Z}}^{\diamond}$ is self-dual, so that $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ is the canonical smooth integral model (and $\mu = 0$). As in the discussion of §7.1, there is a filtered vector bundle

$$0 \subset F^1 V_{dR}^{\diamond} \subset F^0 V_{dR}^{\diamond} \subset V_{dR}^{\diamond}$$

over $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$, endowed with a bilinear form

$$V_{dR}^{\diamond} \otimes V_{dR}^{\diamond} \to \mathcal{O}_{\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})}$$

inducing an isomorphism

$$(\boldsymbol{\omega}^{\diamond})^{-1} = (F^1 \boldsymbol{V}_{dR}^{\diamond})^{-1} \cong \boldsymbol{V}_{dR}^{\diamond} / F^0 \boldsymbol{V}_{dR}^{\diamond}.$$

The de Rham realization of x^{\diamond} determines a global section

$$x_{dR}^{\diamond} \in H^0(\mathcal{Z}_{\lambda}^{\diamond}, F^0 V^{\diamond}|_{\mathcal{Z}_{\lambda}^{\diamond}}) \subset H^0(\mathcal{Z}_{\lambda}^{\diamond}, V^{\diamond}|_{\mathcal{Z}_{\lambda}^{\diamond}}).$$

Moreover, the vector bundle V^{\diamond} is endowed with a canonical flat connection, with respect to which x_{dR}^{\diamond} is parallel.

Parallel transport through the square-zero thickening $\mathcal{Z}^{\diamond}_{\lambda} \subset T(\mathcal{Z}^{\diamond}_{\lambda})$ defines an extension

$$T(x_{dR}^{\diamond}) \in H^0(T(\mathcal{Z}_{\lambda}^{\diamond}), \mathbf{V}^{\diamond}|_{T(\mathcal{Z}_{\lambda}^{\diamond})}),$$

and we define the obstruction to deforming x^{\diamond} to be its image

$$(7.2.6) \quad \text{obst}_{\lambda} \in H^{0}\left(T(\mathcal{Z}_{\lambda}^{\diamond}), (\mathbf{V}^{\diamond}/F^{0}\mathbf{V}^{\diamond})|_{T(\mathcal{Z}_{\lambda}^{\diamond})}\right) \cong H^{0}\left(T(\mathcal{Z}_{\lambda}^{\diamond}), \boldsymbol{\omega}^{\diamond}|_{T(\mathcal{Z}_{\lambda}^{\diamond})}^{-1}\right).$$

The zero locus of obst_{λ} is the largest closed subscheme of $T(\mathcal{Z}_{\lambda}^{\diamond})$ to which x^{\diamond} extends, which is just $\mathcal{Z}_{\lambda}^{\diamond}$ again.

We now explain how to construct the section (7.2.6) without assuming that $V_{\mathbb{Z}}^{\diamond}$ is self-dual. First fix an isometric embedding $V_{\mathbb{Z}}^{\diamond} \subset V_{\mathbb{Z}}^{\diamond \diamond}$ into a self-dual quadratic lattice of signature $(n^{\diamond \diamond}, 2)$, let $(G^{\diamond \diamond}, \mathcal{D}^{\diamond \diamond})$ be the associated GSpin Shimura datum, and choose a level structure $K^{\diamond \diamond} \subset G^{\diamond \diamond}(\mathbb{A}_f)$ containing K^{\diamond} . Thus we have morphisms of integral models

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond}) \to \mathcal{S}_{K^{\diamond\diamond}}(G^{\diamond\diamond},\mathcal{D}^{\diamond\diamond}).$$

Over $\mathcal{S}_{K^{\diamond\diamond}}(G^{\diamond\diamond}, \mathcal{D}^{\diamond\diamond})$ there is a filtered vector bundle $V_{dR}^{\diamond\diamond}$ endowed with a flat connection and an isomorphism

$$(\boldsymbol{\omega}^{\diamond\diamond})^{-1} \cong \boldsymbol{V}_{dR}^{\diamond\diamond}/F^0\boldsymbol{V}_{dR}^{\diamond\diamond}.$$

If we set $\Lambda^{\diamond} = \{\lambda \in V_{\mathbb{Z}}^{\diamond \diamond} : \lambda \perp V_{\mathbb{Z}}^{\diamond}\}$, there is a canonical isometry

$$V(\mathcal{A}_S^{\diamond})_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}}^{\diamond} \cong V(\mathcal{A}_S^{\diamond \diamond})_{\mathbb{Q}}$$

for any scheme $S \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$. Applying this with $S = \mathcal{Z}_{\lambda}^{\diamond}$ and recalling the definition of $V_{\mu}(\mathcal{A}_{\mathcal{Z}_{\lambda}^{\diamond}}^{\diamond})$, we see that there exists a $\tilde{\mu} \in (\Lambda^{\diamond})^{\vee}$ such that

$$x^{\diamond\diamond} \stackrel{\text{def}}{=} x^{\diamond} + \tilde{\mu} \in V(\mathcal{A}_{\mathcal{Z}_{\lambda}^{\diamond}}^{\diamond\diamond}).$$

Exactly as above, the de Rham realization

$$x_{dR}^{\diamond\diamond} \in H^0(\mathcal{Z}_{\lambda}^{\diamond}, \mathbf{V}^{\diamond\diamond}|_{\mathcal{Z}_{\lambda}^{\diamond}})$$

admits a parallel transport through the square-zero thickening $\mathcal{Z}_{\lambda}^{\diamond} \subset T(\mathcal{Z}_{\lambda}^{\diamond})$, whose image

$$\mathrm{obst}_{\lambda} \in H^0\big(T(\mathcal{Z}_{\lambda}^{\diamond}), (\boldsymbol{V}^{\diamond \diamond}/F^0\boldsymbol{V}^{\diamond \diamond})|_{T(\mathcal{Z}_{\lambda}^{\diamond})}\big) \cong H^0\big(T(\mathcal{Z}_{\lambda}^{\diamond}), \boldsymbol{\omega}^{\diamond \diamond}|_{T(\mathcal{Z}_{\lambda}^{\diamond})}^{-1}\big),$$

the obstruction to deforming $x^{\diamond\diamond}$, has zero locus $\mathcal{Z}^{\diamond}_{\lambda}$ (the largest closed subscheme of $T(\mathcal{Z}^{\diamond}_{\lambda})$ to which $x^{\diamond\diamond}$, equivalently x^{\diamond} , deforms).

Having constructed the sections s_{λ} and obst_{λ} over $T(\mathcal{Z}_{\lambda}^{\diamond})$, the idea is roughly that the equality of the scheme-theoretic zero loci of these sections should imply the existence of an isomorphism

$$\omega|_{T(\mathcal{Z}_{\lambda}^{\diamond})}^{-1} \xrightarrow{\mathrm{obst}_{\lambda} \mapsto s_{\lambda}} \mathcal{O}(\mathcal{Z}_{\lambda}^{\diamond})|_{T(\mathcal{Z}_{\lambda}^{\diamond})},$$

which we would then restrict to U.

However, because $T(\mathcal{Z}_{\lambda}^{\diamond})$ is not integral, there is no reason to believe that such an isomorphism exists, and even if it does exist it need not be unique. The way around this detail is this: arguing exactly as in the proof of [BHY15, Theorem 7.10], a morphism as above does exist Zariski locally on $T(\mathcal{Z}_{\lambda}^{\diamond})$, and any two such local morphisms restrict to the same isomorphism over U.

Indeed, working Zariski locally, we can assume that

$$U = \operatorname{Spec}(R), \quad U^{\diamond} = \operatorname{Spec}(R^{\diamond})$$

for $\mathbb{Z}_{(p)}$ -flat integral domains R and R^{\diamond} , and

$$\mathcal{Z}_{\lambda}^{\diamond} = \operatorname{Spec}(R^{\diamond}/J), \quad T(\mathcal{Z}_{\lambda}^{\diamond}) = \operatorname{Spec}(R^{\diamond}/J^2).$$

The morphisms $U \to \mathcal{Z}^{\diamond}_{\lambda} \to U^{\diamond}$ then correspond to homomorphisms

$$R^{\diamond} \to R^{\diamond}/J \to R$$
.

Let $\mathfrak{p} \subset R^{\diamond}$ be the kernel of this composition, so that $J \subset \mathfrak{p}$.

We assume that we have chosen trivializations of the line bundles $\omega|_{U^{\diamond}}$ and $\mathcal{O}(\mathcal{Z}_{\lambda}^{\diamond})$, so that our sections obst_{λ} and s_{λ} are identified with elements

$$a, b \in R^{\diamond}/J^2$$
,

respectively. Each of these elements generates the same ideal J/J^2 .

Lemma 7.2.5. If $x \in R^{\diamond}/J^2$ satisfies bx = 0, then $x \in \mathfrak{p}/J^2$.

Proof. By way of contradiction, suppose $x \notin \mathfrak{p}/J^2$.

The section s_{λ} of (7.2.5), by its very construction, admits a canonical extension to

$$s_{\lambda} \in H^0(U^{\diamond}, \mathcal{O}(\mathcal{Z}_{\lambda}^{\diamond}))$$

with zero locus $\mathcal{Z}_{\lambda}^{\diamond}$. This determines a lift of b to an element $b \in R^{\diamond}$, and we fix any lift $x \in R^{\diamond}$. By assumption $x \notin \mathfrak{p}$, and so x becomes a unit in the localization $R_{\mathfrak{p}}^{\diamond}$. As $bx \in J^2$, we obtain

$$(7.2.7) b \in \mathfrak{p}^2 R_{\mathfrak{p}}^{\diamond}.$$

As R is $\mathbb{Z}_{(p)}$ -flat, the kernel \mathfrak{p} of $R^{\diamond} \to R$ does not contain p, and so the localization map $R^{\diamond} \to R_{\mathfrak{p}}^{\diamond}$ factors through $R^{\diamond}[1/p]$. Both $\mathbb{Z}_{(p)}$ -stacks in

$$\mathcal{Z}^{\diamond}(m,\mu) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond})$$

have smooth generic fibers, and so

$$R^{\diamond}[1/p] \to (R^{\diamond}/bR^{\diamond})[1/p]$$

is a morphism of smooth Q-algebras. This implies that

$$R_{\mathfrak{p}}^{\diamond} \to R_{\mathfrak{p}}^{\diamond}/bR_{\mathfrak{p}}^{\diamond}$$

is a morphism of regular local rings. By (7.2.7), this morphism induces an isomorphism on tangent spaces, and so is itself an isomorphism. It follows that b = 0, contradicting the fact that b generates J/J^2 .

Lemma 7.2.6. There exists $u \in R^{\diamond}/J^2$ such that ua = b. The image of any such u in $R^{\diamond}/\mathfrak{p} \subset R$ is a unit. If also u'a = b, then u = u' in $R^{\diamond}/\mathfrak{p} \subset R$.

Proof. As a and b generate both generate J/J^2 , there exist $u, v \in R^{\diamond}/J^2$ such that ua = b and vb = a. Obviously $b \cdot (1 - uv) = 0$, and so the previous lemma implies $1 - uv \in \mathfrak{p}/J^2$. Thus the image of u in $R^{\diamond}/\mathfrak{p}$ is a unit with inverse v. If also u'a = b, the same argument shows that the image of u' in $R^{\diamond}/\mathfrak{p}$ is a unit with inverse v, and hence u = u' in $R^{\diamond}/\mathfrak{p}$.

The discussion above provides us with a canonical isomorphism

$$\mathcal{O}(\mathcal{Z}_{\lambda}^{\diamond})|_{U} \cong \boldsymbol{\omega}|_{U}^{-1}$$

Zariski locally on U, and glueing over an open cover completes the proof of Lemma 7.2.4.

We now complete the proof of Theorem 7.2.2. Recalling the definition of $\mathcal{Z}(0, \mu_1)$ from (7.1.3), Lemmas 7.2.3 and 7.2.4, along with the decomposition (7.2.4) provide us with canonical isomorphisms

$$\mathcal{Z}^{\diamond}(m,\mu)|_{U} \cong \left(\bigotimes_{\substack{m_{1}+m_{2}=m\\\mu_{1}+\mu_{2}\in\mu\\m_{1}>0}} \mathcal{Z}(m_{1},\mu_{1})|_{U}^{\otimes r_{\Lambda}(m_{2},\mu_{2})}\right) \otimes \left(\bigotimes_{\mu_{2}\in\mu} \boldsymbol{\omega}|_{U}^{-r_{\Lambda}(m,\mu_{2})}\right)$$

$$\cong \bigotimes_{\substack{m_{1}+m_{2}=m\\\mu_{1}+\mu_{2}=\mu}} \mathcal{Z}(m_{1},\mu_{1})|_{U}^{\otimes r_{\Lambda}(m_{2},\mu_{2})}$$

of line bundles over some étale neighborhood U of $z \to \mathcal{S}_K(G, \mathcal{D})$. Now let U vary over an étale cover and apply descent.

7.3. The main theorem. Suppose

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c(m) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

is a weakly holomorphic form as in (5.1.1), and assume f is integral in the sense of Definition 5.1.2.

Proposition 7.3.1. If $n \ge 5$ (in particular Hypothesis 5.2.4 holds) the Borcherds product $\psi(f)$ of Theorem 5.5.1, viewed as a rational section of the line bundle $\omega^{\otimes c(0,0)}$ on the integral model $\mathcal{S}_K(G,\mathcal{D})$, satisfies

$$\operatorname{div}(\boldsymbol{\psi}(f)) = \sum_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m, \mu) \cdot \mathcal{Z}(m, \mu).$$

Proof. Certainly an equality of the form

$$\operatorname{div}(\boldsymbol{\psi}(f)) = \sum_{\substack{m>0\\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot \mathcal{Z}(m,\mu) + \mathcal{E}_p(f)$$

holds for some Cartier divisor $\mathcal{E}_p(f)$ supported in the special fiber; this follows by comparing the calculation of the divisor of $\operatorname{div}(\psi(f))$ in the complex

fiber found in [Bor98] with the complex analytic description of the special divisors $\mathcal{Z}(m,\mu)(\mathbb{C})$ found in [AGHM17a, Lemma 2.7.3]. We need only to check that $\mathcal{E}_p(f) = 0$ for every prime p.

As in the proof of Theorem 5.5.1, we shrink the prime-to-p part of K, and work with a level subgroup $\tilde{K} \subset K$ that is neat, and for which there is a toroidal compactification $\mathcal{S}_{\tilde{K}}(G,\mathcal{D},\Sigma)$ chosen so that Σ satisfies the conclusion of Lemma 5.4.1, for some fixed choice of Weyl chamber.

We now apply the results of §6.4 with K replaced by \tilde{K} throughout. Proposition 7.1.2 tells us that $\mathcal{S}_{\tilde{K}}(G,\mathcal{D})$ has geometrically normal special fiber. Combining Corollary 6.4.4 with the analysis of the q-expansion of $\psi(f)$ found in the proof of Theorem 5.5.1, we find a faithfully flat $\mathbb{Z}_{(p)}$ -algebra R such that the divisor of $\psi(f)$, computed on $\mathcal{S}_{\tilde{K}}(G,\mathcal{D},\Sigma)_{/R}$, is flat. Hence this divisor was already flat over $\mathbb{Z}_{(p)}$. Similarly Proposition 7.1.4 implies that the divisors $\mathcal{Z}(m,\mu)$ are flat, and so $\mathcal{E}_p(f)=0$.

We now strengthen Proposition 7.3.1 by dropping the hypothesis $n \ge 5$. In the following, we assume only that $n \ge 1$.

Theorem 7.3.2. After multiplying f by any sufficiently divisible positive integer, there is a rational section $\psi(f)$ of $\omega^{\otimes c(0,0)}$ over $\mathcal{S}_K(G,\mathcal{D})$ satisfying (5.5.1), and with divisor of the form

$$\operatorname{div}(\boldsymbol{\psi}(f)) = \sum_{\substack{m>0\\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m, \mu) \cdot \mathcal{Z}(m, \mu).$$

Proof. We will use an integral version of the embedding trick of [Bor98, Lemma 8.1]. According to [loc. cit.] there exist self-dual \mathbb{Z} -quadratic spaces $\Lambda^{[1]}$ and $\Lambda^{[2]}$ of signature (24,0), whose corresponding theta series

$$\vartheta^{[i]}(\tau) = \sum_{x \in \Lambda^{[i]}} q^{Q(x)} \in M_{12}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C})$$

are related by

(7.3.1)
$$\vartheta^{[2]}(\tau) - \vartheta^{[1]}(\tau) = 24\Delta(\tau).$$

Here Δ is Ramanujan's modular discriminant. Denote by

$$r^{[i]}(m) = \#\{x \in \Lambda^{[i]} : Q(x) = m\}$$

the m^{th} Fourier coefficient of $\vartheta^{[i]}$.

For $i \in \{1, 2\}$, define a quadratic space $V^{[i]} = V \oplus \Lambda^{[i]}_{\mathbb{Q}}$ and a $\widehat{\mathbb{Z}}$ -lattice

$$V_{\widehat{\mathbb{Z}}}^{[i]} = V_{\widehat{\mathbb{Z}}} \oplus \Lambda_{\widehat{\mathbb{Z}}}^{[i]}$$

in $V^{[i]} \otimes \mathbb{A}_f$. In the notation of §5.1, the inclusion $V_{\mathbb{Z}} \hookrightarrow V_{\mathbb{Z}}^{[i]}$ induces an isomorphism of $\widetilde{\operatorname{SL}}_2(\mathbb{Z})$ -representations

$$S_{V_{\mathbb{Z}}} = \mathbb{C}[V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}] \cong \mathbb{C}[(V_{\mathbb{Z}}^{[i]})^{\vee}/V_{\mathbb{Z}}^{[i]}] = S_{V_{\mathbb{Z}}^{[i]}},$$

and so $f^{[i]} = f/(24\Delta)$ defines a weakly holomorphic form

$$f^{[i]}(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c^{[i]}(m) \cdot q^m \in M^!_{-11 - \frac{n}{2}}(\overline{\rho}_{V^{[i]}_{\mathbb{Z}}}).$$

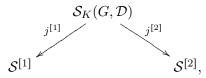
The relation $f = \vartheta^{[2]} f^{[2]} - \vartheta^{[1]} f^{[1]}$ implies the equality of Fourier coefficients

(7.3.2)
$$c(m,\mu) = \sum_{k\geq 0} r^{[2]}(k) \cdot c^{[2]}(m-k,\mu) - \sum_{k\geq 0} r^{[1]}(k) \cdot c^{[1]}(m-k,\mu).$$

Each $V^{[i]}$ determines a GSpin Shimura datum $(G^{[i]}, \mathcal{D}^{[i]})$. By choosing

$$K^{[i]} = G^{[i]}(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}}^{[i]})^{\times}$$

for our compact open subgroups, we put ourselves in the situation of $\S6.1$ and $\S7.1$. In particular, Proposition 7.2.1 provides us with finite unramified morphisms



where we abbreviate $\mathcal{S}^{[i]} = \mathcal{S}_{K^{[i]}}(G^{[i]}, \mathcal{D}^{[i]}).$

As $n^{[i]} = n + 24 \ge 5$ we may apply Proposition 7.3.1 to the form $f^{[i]}$. After replacing f by a positive multiple, we obtain a Borcherds product $\psi(f^{[i]})$ on $\mathcal{S}^{[i]}$ with divisor

$$\operatorname{div}(\boldsymbol{\psi}(f^{[i]}) = \sum_{\substack{m>0\\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c^{[i]}(-m,\mu) \cdot \mathcal{Z}^{[i]}(m,\mu).$$

At least formally, we wish to define

$$\psi(f) = \frac{(j^{[2]})^* \psi(f^{[2]})}{(j^{[1]})^* \psi(f^{[1]})}.$$

The problem is that the image of $j^{[i]}$ will typically be contained in the support of the divisor of $\psi(f^{[i]})$, and so the quotient on the right will typically be either ∞/∞ or 0/0.

The key to making sense of this quotient is to combine the pullback formula of Theorem 7.2.2 with the following lemma.

Lemma 7.3.3. The Borcherds product $\psi(f^{[i]})$ determines an isomorphism of line bundles

$$\mathcal{O}_{\mathcal{S}^{[i]}} \cong \bigotimes_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^{\times}/V_{\mathbb{Z}}}} \mathcal{Z}^{[i]}(m,\mu)^{\otimes c^{[i]}(-m,\mu)}.$$

Proof. If m > 0, the line bundle $\mathcal{Z}^{[i]}(m,\mu)$ admits a canonical global section $s^{[i]}(m,\mu)$ whose divisor is the Cartier divisor $\mathcal{Z}^{[i]}(m,\mu)$. This is just the constant function 1, viewed as a section of $\mathcal{O}_{\mathcal{S}^{[i]}} \subset \mathcal{Z}^{[i]}(m,\mu)$. By comparing divisors, there is a unique isomorphism

$$\boldsymbol{\omega}^{c^{[i]}(0,0)} \cong \bigotimes_{\substack{m>0\\ \mu \in V_{\mathbb{Z}}^{^{\vee}}/V_{\mathbb{Z}}}} \mathcal{Z}^{[i]}(m,\mu)^{\otimes c^{[i]}(-m,\mu)}$$

sending

$$\psi(f^{[i]}) \mapsto \bigotimes_{\substack{m>0\\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} s^{[i]}(m,\mu)^{\otimes c^{[i]}(-m,\mu)},$$

and the claim follows from (7.1.3).

If we pull back the isomorphism of Lemma 7.3.3 via $j^{[i]}$ and use Theorem 7.2.2, we obtain isomorphisms

$$\mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} \cong \bigotimes_{\substack{m_1,m_2 \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \mathcal{Z}(m_1,\mu)^{\otimes r^{[i]}(m_2) \cdot c^{[i]}(-m_1-m_2,\mu)}$$

for $i \in \{1, 2\}$. These two isomorphisms, along with (7.3.2), determine an isomorphism

$$\mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} \cong \bigotimes_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \mathcal{Z}(m,\mu)^{\otimes c(-m,\mu)}.$$

Now simply reverse the reasoning in the proof of Lemma 7.3.3. Using the notation of that lemma, rewrite this last isomorphism as

$$\boldsymbol{\omega}^{c(0,0)} \cong \bigotimes_{\substack{m>0\\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \mathcal{Z}(m,\mu)^{\otimes c(-m,\mu)},$$

and define the desired rational section of $\omega^{c(0,0)}$ by

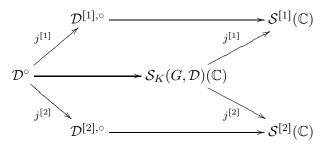
$$\psi(f) = \bigotimes_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^{\vee} / V_{\mathbb{Z}}}} s(m, \mu)^{\otimes c(-m, \mu)}.$$

The next lemma completes the proof of Theorem 7.3.2.

Lemma 7.3.4. The section $\psi(f)$ just constructed satisfies (5.5.1).

Proof. We return to the notation of §5.2. In particular, we fix a connected component $\mathcal{D}^{\circ} \subset \mathcal{D}$, and let $\mathcal{D}^{[i],\circ} \subset \mathcal{D}^{[i]}$ be the connected component that

contains it. Fix a $g \in G(\mathbb{A}_f)$, and consider the complex uniformizations



in which all horizontal arrows send $z \mapsto [(z, g)]$.

Denote by $\psi_g(f)$ the pullback of $\psi(f)$ to \mathcal{D}° , and define $\psi_g(f^{[i]})$ similarly. According to Theorem 5.5.1, these sections satisfy

$$-2\log||\psi_q(f^{[i]})|| = \Theta_q^{reg}(f^{[i]}),$$

where $\Theta_g^{reg}(f^{[i]}) = \Theta^{reg}(f^{[i]}, g)$ is the regularized theta lift of §5.2.

Every $x \in V^{[i]}$ with Q(x) > 0 determines a global section of the constant holomorphic vector bundle

$$V_{dR}^{[i],an} = V_{\mathbb{C}}^{[i]} \otimes \mathcal{O}_{\mathcal{D}^{[i],\circ}}$$

Recall that this vector bundle admits a filtration

$$0 \subset F^{1} \boldsymbol{V}_{dR}^{[i],an} \subset F^{0} \boldsymbol{V}_{dR}^{[i],an} \subset \boldsymbol{V}_{dR}^{[i],an}$$

whose fiber at the point $z \in \mathcal{D}^{[i]}$ is $0 \subset \mathbb{C}z \subset (\mathbb{C}z)^{\perp} \subset V_{\mathbb{C}}^{[i]}$. Denote by

$$\mathrm{obst}_x \in H^0\left(\mathcal{D}^{[i],\circ}, (\boldsymbol{\omega}_{an}^{[i]})^{-1}\right) \cong H^0\left(\mathcal{D}^{[i],\circ}, \boldsymbol{V}_{dR}^{[i],an}/F^0\boldsymbol{V}_{dR}^{[i],an}\right)$$

the image of the constant section $x \otimes 1$ of $V_{dR}^{[i],an}$. Its zero locus is the analytic divisor

$$Z_x^{[i]} = \{z \in \mathcal{D}^{[i],\circ} : x \perp z\}.$$

The pullback of $\mathcal{Z}^{[i]}(m,\mu)(\mathbb{C})$ to $\mathcal{D}^{[i],\circ}$, denoted the same way, is now given explicitly by the locally finite sum of analytic divisors

$$\mathcal{Z}^{[i]}(m,\mu)(\mathbb{C}) = \sum_{\substack{x \in g\mu + gV_{\mathbb{Z}}^{[i]} \\ Q(x) = m}} Z_x^{[i]}.$$

Define the *renormalized* Borcherds product

$$\tilde{\psi}_g(f^{[i]}) = \psi_g(f^{[i]}) \otimes \bigotimes_{m>0} \bigotimes_{\substack{\lambda \in \Lambda^{[i]} \\ Q(x)=m}} \operatorname{obst}_{\lambda}^{\otimes -c^{[i]}(-m,0)}.$$

This is meromorphic section of $\bigotimes_{m\geqslant 0} \left(\boldsymbol{\omega}^{[i],an}\right)^{\otimes r^{[i]}(m)c^{[i]}(-m,0)}$ with divisor

$$\operatorname{div}(\tilde{\psi}_{g}(f^{[i]})) = \sum_{\substack{m>0\\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c^{[i]}(-m,\mu) \sum_{\substack{x \in g\mu + gV_{\mathbb{Z}}^{[i]}\\ Q(x) = m\\ x \notin \Lambda^{[i]}}} Z_{x}^{[i]}.$$

Note that each divisor $Z_x^{[i]}$ appearing on the right hand side intersects the subspace $\mathcal{D}^{\circ} \subset \mathcal{D}^{[i],\circ}$ properly. Indeed, If we decompose $x = y + \lambda$ with $y \in g\mu + gV_{\mathbb{Z}}$ and $\lambda \in \Lambda$, then

$$Z_x^{[i]} \cap \mathcal{D}^\circ = \begin{cases} Z_y & \text{if } Q(y) > 0 \\ \emptyset & \text{else,} \end{cases}$$

where $Z_y = \{z \in \mathcal{D}^\circ : y \perp z\}.$

This is the point: by renormalizing the Borcherds product we have removed precisely the part of its divisor that intersects \mathcal{D}° improperly, and so the renormalized Borcherds product has a well-defined pullback to \mathcal{D}° . Indeed, using the relation (7.3.2), we see that

(7.3.3)
$$\psi_g(f) = \frac{(j^{[2]})^* \tilde{\psi}_g(f^{[2]})}{(j^{[1]})^* \tilde{\psi}_g(f^{[1]})}$$

is a section of the line bundle $(\omega^{an})^{\otimes c(0,0)}$ over \mathcal{D}° . By directly comparing the algebraic and analytic constructions, one can check that it agrees with the $\psi_q(f)$ defined at the beginning of the proof.

Define the renormalized theta lift

$$\tilde{\Theta}_g^{reg}(f^{[i]}) = \Theta_g^{reg}(f^{[i]}) + 2\sum_{m>0} c^{[i]}(-m,0) \sum_{\substack{\lambda \in \Lambda^{[i]} \\ Q(\lambda) = m}} \log||\mathrm{obst}_{\lambda}||$$

so that

$$(7.3.4) -2\log||\tilde{\psi}_g(f^{[i]})|| = \tilde{\Theta}_g^{reg}(f^{[i]}).$$

Combining this with (7.3.3) yields

$$-2\log||\psi_g(f)|| = (j^{[2]})^* \tilde{\Theta}_g^{reg}(f^{[2]}) - (j^{[1]})^* \tilde{\Theta}_g^{reg}(f^{[1]}).$$

The regularized theta lift $\Theta_g^{reg}(f^{[i]})$ is over-regularized, in the sense that its definition makes sense at every point of $\mathcal{D}^{[i],\circ}$, even at points of the divisor along which $\Theta_g^{reg}(f^{[i]})$ has its logarithmic singularities. In other words, the regularized theta lift is defined (but discontinuous) on all of $\mathcal{D}^{[i],\circ}$. By [AGHM17a, Proposition 5.5.1], its values along the discontinuity agree with the values of $\tilde{\Theta}_g^{reg}(f^{[i]})$, and in fact we have

$$(j^{[i]})^*\Theta_g^{reg}(f^{[i]}) = (j^{[i]})^*\tilde{\Theta}_g^{reg}(f^{[i]})$$

as functions on \mathcal{D}° .

On the other hand, for each $i \in \{1, 2\}$, the regularized theta lift has the form

$$\Theta_g^{reg}(f^{[i]})(z) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{H}} f^{[i]}(\tau) \vartheta^{[i]}(\tau, z, g) \, \frac{du \, dv}{v^2}$$

as in (5.2.1). As in [BY09, (4.16)], when we restrict the variable z to $\mathcal{D} \subset \mathcal{D}^{[i]}$ the theta kernel factors as

$$\vartheta^{[i]}(\tau, z, g) = \vartheta(\tau, z, g) \cdot \vartheta^{[i]}(\tau),$$

where $\vartheta(\tau, z, g)$ is the theta kernel defining $\Theta_q^{reg}(f)$. Thus

$$(j^{[i]})^* \Theta_g^{reg}(f^{[i]}) = \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} f(\tau) \vartheta(\tau, z, g) \cdot \frac{\vartheta^{[i]}(\tau)}{24\Delta} \frac{du \, dv}{v^2}$$

Combining this last equality with (7.3.1) proves the first equality in

$$\begin{split} \Theta_g^{reg}(f) &= (j^{[2]})^* \Theta_g^{reg}(f^{[2]}) - (j^{[1]})^* \Theta_g^{reg}(f^{[1]}) \\ &= (j^{[2]})^* \tilde{\Theta}_g^{reg}(f^{[2]}) - (j^{[1]})^* \tilde{\Theta}_g^{reg}(f^{[1]}), \end{split}$$

which is just a more explicit statement of [Bor98, Lemma 8.1]. Combining this with (7.3.4) complete the proof of Lemma 7.3.4.

8. Modularity of the generating series of special divisors

Assume that $V_{\mathbb{Z}}$ is a maximal lattice in our fixed quadratic space (V, Q) of signature (n, 2), and that the compact open subgroup (4.1.6) is

$$K = G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^{\times}.$$

8.1. The integral model over \mathbb{Z} . In §6.2 and §7.1 we constructed, for every prime p, an integral model over $\mathbb{Z}_{(p)}$ of the orthogonal Shimura $\operatorname{Sh}_K(G, \mathcal{D})$, along with a family of special divisors and a line bundle of weight one modular forms. As in [AGHM17a, §2.4] and [AGHMP17b, §4.5], these models may be glued together as p varies to define a flat and normal integral model

$$S_K(G, \mathcal{D}) \to \operatorname{Spec}(\mathbb{Z})$$

over \mathbb{Z} , along with a family of special divisors $\mathcal{Z}(m,\mu)$, and a line bundle of weight one modular forms $\boldsymbol{\omega}$.

Theorem 8.1.1. Suppose

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c(m) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

is a weakly holomorphic form as in (5.1.1), and assume f is integral in the sense of Definition 5.1.2. After multiplying f by any sufficiently divisible positive integer, there is a rational section $\psi(f)$ of $\omega^{\otimes c(0,0)}$ over $\mathcal{S}_K(G,\mathcal{D})$ satisfying (5.5.1), and with divisor

$$\operatorname{div}(\boldsymbol{\psi}(f)) = \sum_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m, \mu) \cdot \mathcal{Z}(m, \mu).$$

Proof. In Theorem 7.3.2 we constructed a Borcherds product $\psi(f)$ on the integral model $\mathcal{S}_K(G,\mathcal{D})_{/\mathbb{Z}_{(p)}}$, for every prime p. It is clear from the construction that these Borcherds products, one for each prime p, agree in the

generic fiber, so define a Borcherds product on the integral model over $\mathbb Z$ satisfying

$$\operatorname{div}(\boldsymbol{\psi}(f)) = \sum_{\substack{m>0\\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m, \mu) \cdot \mathcal{Z}(m, \mu).$$

8.2. Modularity of the generating series. For any positive $m \in \mathbb{Q}$ and any $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$, we denote again by $\mathcal{Z}(m,\mu)$ the line bundle defined by the special divisor of the same name. Define $\mathcal{Z}(0,\mu)$ as in (7.1.3).

Theorem 8.2.1. The formal generating series

$$\phi(\tau) = \sum_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \mathcal{Z}(m,\mu) \cdot q^{m}$$

is modular, in the sense that $\alpha(\phi) \in M_{1+\frac{n}{2}}(\rho_{V_{\mathbb{Z}}})$ for any \mathbb{Z} -linear map

$$\alpha: \operatorname{Pic}(\mathcal{S}_K(G,\mathcal{D})) \to \mathbb{C}.$$

Proof. According to the modularity criterion of [Bor99, Theorem 3.1], a formal q-expansion

$$\sum_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{T}}^{\vee} / V_{\mathbb{Z}}}} a(m, \mu) \cdot q^m$$

defines an element of $M_{1+\frac{n}{2}}(\rho_{V_{\mathbb{Z}}})$ if and only if

(8.2.1)
$$0 = \sum_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m, \mu) \cdot a(m, \mu)$$

for every

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(m, \mu) \cdot q^m \in M_{1 - \frac{n}{2}}^!(\overline{\rho}_{V_{\mathbb{Z}}}).$$

By the main result of [McG03], it suffices to verify (8.2.1) only for $f(\tau)$ that are integral, in the sense of Definition 5.1.2.

For any integral $f(\tau)$, Theorem 8.1.1 implies that

$$\boldsymbol{\omega}^{c(0,0)} = \sum_{\substack{m>0\\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot \mathcal{Z}(m,\mu)$$

up to a torsion element in $Pic(S_K(G, \mathcal{D}))$, and hence

$$\sum_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m, \mu) \cdot \mathcal{Z}(m, \mu) \in \operatorname{Pic}(\mathcal{S}_K(G, \mathcal{D}))$$

is killed by any \mathbb{Z} -linear map $\alpha : \operatorname{Pic}(\mathcal{S}_K(G, \mathcal{D})) \to \mathbb{C}$. Thus the claimed modularity follows from the result of Borcherds cited above. 5.1.2.

8.3. Modularity of the arithmetic generating series. Bruinier [Bru02] has defined a Green function $\Theta^{reg}(F_{m,\mu})$ for the divisor $\mathcal{Z}(m,\mu)$. This Green function is constructed, as in (5.2.1), as the regularized theta lift of a harmonic Hejhal-Poincare series

$$F_{m,\mu} \in H_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

whose holomorphic part, in the sense of [BF04, §3], has the form

$$F_{m,\mu}^+(\tau) = \left(\frac{\phi_{\mu} + \phi_{-\mu}}{2}\right) \cdot q^{-m} + O(1),$$

where $\phi_{\mu} \in S_{V_{\mathbb{Z}}}$ is the characteristic function of the coset $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. See [AGHM17a, §3.2] and the references therein.

This Green function determines a metric on the corresponding line bundle, and so determines a class

$$\widehat{\mathcal{Z}}(m,\mu) = (\mathcal{Z}(m,\mu), \Theta^{reg}(F_{m,\mu})) \in \widehat{\operatorname{Pic}}(\mathcal{S}_K(G,\mathcal{D}))$$

for every positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. Recall that that we have defined a metric (4.4.3) on the line bundle ω , and so obtain a class

$$\widehat{\boldsymbol{\omega}} \in \widehat{\operatorname{Pic}}(\mathcal{S}_K(G, \mathcal{D}))$$

in the group of metrized line bundles. We define

$$\widehat{\mathcal{Z}}(0,\mu) = \begin{cases} \widehat{\boldsymbol{\omega}}^{-1} & \text{if } \mu = 0\\ 0 & \text{otherwise.} \end{cases}$$

Theorem 8.3.1. If $n \ge 3$, the formal generating series

$$\widehat{\phi}(\tau) = \sum_{\substack{m \geqslant 0\\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \widehat{\mathcal{Z}}(m,\mu) \cdot q^{m}$$

is modular, in the sense that $\alpha(\widehat{\phi}) \in M_{1+\frac{n}{2}}(\rho_{V_{\mathbb{Z}}})$ for any \mathbb{Z} -linear functional

$$\alpha : \widehat{\operatorname{Pic}}(\mathcal{S}_K(G, \mathcal{D})) \to \mathbb{C}.$$

Proof. The assumption that $n \ge 3$ implies that any form

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(m, \mu) \cdot q^m \in M_{1 - \frac{n}{2}}^!(\overline{\rho}_{V_{\mathbb{Z}}}).$$

has negative weight. As in [BF04, Remark 3.10], this implies that any such f is a linear combination of the Hejhal-Poincare series $F_{m,\mu}$, and in fact

$$f = \sum_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m, \mu) \cdot F_{m, \mu}.$$

This last equality follows, as in the proof of [BHY15, Lemma 3.10], from the fact that the difference between the two sides is a harmonic weak Maass form whose holomorphic part is O(1). In particular, we have the equality of regularized theta lifts

$$\Theta^{reg}(f) = \sum_{\substack{m>0\\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m, \mu) \cdot \Theta^{reg}(F_{m, \mu}).$$

Now assume that f is integral. After replacing f by a positive integer multiple, Theorem 8.1.1 provides us with Borcherds product $\psi(f)$ with arithmetic divisor

$$\widehat{\operatorname{div}}(\boldsymbol{\psi}(f)) = (\operatorname{div}(\boldsymbol{\psi}(f)), -\log||\boldsymbol{\psi}(f)||^2) = \sum_{\substack{m>0\\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m, \mu) \cdot \widehat{\mathcal{Z}}(m, \mu).$$

On the other hand, in the group of metrized line bundles we have the relation

$$\widehat{\operatorname{div}}(\boldsymbol{\psi}(f)) = \widehat{\boldsymbol{\omega}}^{\otimes c(0,0)} = -c(0,0) \cdot \widehat{\mathcal{Z}}(0,0).$$

The above relations show that

$$\sum_{\substack{m\geqslant 0\\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot \widehat{\mathcal{Z}}(m,\mu) \in \widehat{\mathrm{Pic}}(\mathcal{S}_{K}(G,\mathcal{D}))$$

is a torsion element for any integral f. Exactly as in the proof of Theorem 8.2.1, modularity of the generating series $\widehat{\phi}(\tau)$ now follows from the modularity criterion of Borcherds.

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