AN OBSERVATION ON (-1)-CURVES ON RATIONAL SURFACES

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Abstract. We give an effective iterative characterization of the classes of (smooth, rational) (-1)-curves on the blowup of the projective plane at general points. Such classes are characterized as having self-intersection -1, arithmetic genus 0, and intersecting every (-1)-curve of smaller degree nonnegatively.

1. Introduction

Let S be a smooth projective surface. We will call an irreducible curve C on S a (-1)**-curve** if it is smooth and rational, and $C \cdot C = -1$. The (-1)-curves are of fundamental importance in various aspects of surface theory, but we are especially interested in their role in the interpolation problem. Specifically, suppose that we have n points $P_1, \dots, P_n$ on $\mathbb{P}^2$. Given also positive integers $d, m_1, \dots, m_n$, we can ask the following basic question:

Question 1.1. What is the dimension of the space of homogeneous polynomials of degree d in three variables which vanish to order at least m_i at P_i for $i = 1, \dots, n$?

The naive expected dimension is given by

$$\text{expdim}(d, m_1, \dots, m_n) := \max \left\{ 0, \binom{d+2}{2} - \sum_{i=1}^{n} \binom{m_i+1}{2} \right\}. \tag{1.1}$$

The actual dimension is always at least the expected one, but one quickly sees that the actual dimension may be larger, even when the P_i are general: the first example is the case $d = n = m_1 = m_2 = 2$, where the expected dimension is 0, but in fact the space of polynomials is 1-dimensional, containing (the polynomial whose zero set is) the doubled line through P_1 and P_2.

Now, let S be the blowup of $\mathbb{P}^2$ at $P_1, \dots, P_n$, and let $H, E_1, \dots, E_n$ denote the classes of the hyperplane and the exceptional divisors, respectively. Then our space of homogeneous polynomials can be reinterpreted as $\Gamma(S, \mathscr{O}(dH - m_1E_1 - \cdots - m_nE_n))$, and we observe that the strict transform of a line through P_1 and P_2 is a (-1)-curve on S, which occurs twice in the base locus of $\mathscr{O}(2H - 2E_1 - 2E_2)$. The fundamental conjecture in the field is that this phenomenon is the only one causing the expected dimension to differ from the actual dimension:

Conjecture 1.2. *(Gimigliano-Harbourne-Hirschowitz* [Gim89] [Har86] [Hir89]*) Assume the situation is as above, and the P_i are general. Let $\mathcal{L} = \mathscr{O}(dH - m_1E_1 - \cdots - m_nE_n)$. Then we have that*

$$\dim \Gamma(S, \mathcal{L}) = \text{expdim}(d, m_1, \dots, m_n)$$

unless there is some (-1)-curve C in S such that $C \cdot \mathcal{L} \leq -2$.

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Thus, it becomes important to understand the (-1)-curves in surfaces S obtained from $\mathbb{P}^2$ by blowing up general points. An effective but iterative description of such curves as iterated images under Cremona transformations of exceptional divisors is exposited in Dolgachev (Corollary 1 of [Dol83]). The purpose of this note is to give an alternative description, which is also effective but iterative, but which may be more satisfying conceptually. In essence, it says that a class which numerically looks like the class of a (-1)-curve is in fact the class of a (-1)-curve unless it contains a smaller-degree (-1)-curve in its base locus. More specifically, our result is the following:

Theorem 1.3. *Let S be the blowup of $\mathbb{P}^2$ at very general points $P_1, \dots, P_n$. A divisor class on S is the class of a (-1)-curve if and only if either it is one of the E_i, or it is of the form $dH - m_1E_1 - \cdots - m_nE_n$, with $d > 0$, and $m_i \geq 0$ for all i, and the following conditions are satisfied:*

(i) *(self-intersection -1) $d^2 - \sum_i m_i^2 = -1$;*
(ii) *(arithmetic genus 0) $\frac{(d-1)(d-2)}{2} - \sum_i \frac{m_i(m_i-1)}{2} = 0$;*
(iii) *for all $0 < d' < d$, and all (-1)-curves C' of degree d' on S, we have $C \cdot C' \geq 0$.*

Thus, we get a purely numerical (albeit inductive) criterion for describing all the classes of (-1)-curves on S. If we consider a single class at a time, it suffices to assume that the P_i are general; we impose that they are very general in order to treat all classes simultaneously.

The three conditions of the theorem are clearly necessary. Conversely, if conditions (i) and (ii) are satisfied, it is easy to see (Proposition 2.1 below) that the class is effective, so the heart of the matter is to show that if a curve in the class is not irreducible, it must have negative product with a (-1)-curve having smaller degree.

Remark 1.4. The last condition of the theorem is necessary: the lowest-degree example is in the case $d = 5$, $n = 10$, with the class $5H - 3E_1 - 3E_2 - E_3 - \cdots - E_{10}$. This satisfies conditions (i) and (ii), but it is not the class of a (-1)-curve; indeed, it can be written as the sum of the classes $H - E_1 - E_2$ and $4H - 2E_1 - 2E_2 - E_3 - \cdots - E_{10}$, which represent disjoint genus-0 and genus-1 curves, respectively.

In addition, the generality of points is necessary: for instance, if P_1, P_2, P_3 are collinear, then the class $H - E_1 - E_2$ satisfies the hypotheses of the theorem, but is no longer the class of a (-1)-curve: rather, it is the class of the line through P_1, P_2, P_3 together with E_3.

The proof of Theorem 1.3 is brief, and relies on the same techniques (namely, a lemma of Noether allowing inductive application of Cremona transformations) as the aforementioned criterion in [Dol83].

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2. Proof

Our proofs rely on two basic background elements: a numerical lemma of Noether, and the theory of standard Cremona transformations of the plane. Both of these are described in Dolgachev [Dol83]; however, because their proofs are short and our hypotheses for Noether's lemma are slightly more general, we have elected to give a self-contained presentation, nonetheless following the arguments in [Dol83].

Our first two results are purely numerical, and do not involve any generality conditions on the points P_i.

Proposition 2.1. *Let $C = dH - m_1E_1 - \cdots - m_nE_n$ be an arbitrary curve class on S. Then any two of the following conditions imply the remaining two conditions.*

(a) *($C \cdot C = -1$) $d^2 - \sum_i m_i^2 = -1$;*
(b) *(C has arithmetic genus 0) $\frac{(d-1)(d-2)}{2} - \sum_i \frac{m_i(m_i-1)}{2} = 0$;*
(c) *($C \cdot (-K) = 1$) $3d - \sum_i m_i = 1$;*
(d) *(expdim $= 1$) $\frac{(d+2)(d+1)}{2} - \sum_i \frac{(m_i+1)m_i}{2} = 1$.*

Proof. If K is the canonical class of S, we have $K = -3H + E_1 + \cdots + E_n$, and by adjunction the arithmetic genus of C is $\frac{2+C\cdot(C+K)}{2}$. We therefore see that we can re-express all our conditions in terms of intersection theory: (a) is $C \cdot C = -1$, (b) is $C \cdot C + C \cdot K = -2$, (c) is $C \cdot K = -1$, and (d) is $C \cdot C = C \cdot K$. The proposition follows trivially. $\square$

We now prove the aforementioned lemma of Noether; our proof follows [Dol83]. We will only use the case $C \cdot C = -1$, but we have included the more general statement because the proof is the same.

Lemma 2.2. *Let $C = dH - m_1E_1 - \cdots - m_nE_n$ be a curve class with $d > 0$ and all $m_i \geq 0$, and satisfying condition (b) of Proposition 2.1, and also having $-2 \leq C \cdot C \leq 1$. If $d = 1$, suppose further that $C \cdot C < 0$. Then there exist $i_1 < i_2 < i_3$ such that*

$$m_{i_1} + m_{i_2} + m_{i_3} > d.$$

Proof. Without loss of generality, we may suppose that we have ordered our points so that $m_1 \geq m_2 \geq \cdots \geq m_n$, and we thus want simply to show that $m_1+m_2+m_3 > d$. If $d = 1$, then the hypothesis $C \cdot C < 0$ means that we cannot have $m_1 = m_2 = 0$ or $m_1 = 1$ and $m_2 = 0$, so the lemma holds in this case. We therefore assume that $d > 1$.

Now, for $1 \leq j \neq n$, define

$$q_j = \frac{\sum_{i=j}^n m_i^2}{\sum_{i=j}^n m_i}.$$

Then because $m_j \geq m_i$ for $i \geq j$, we have $m_j \geq q_j$ for any j. Set $r_j = m_j - q_j$ for each j. For brevity, set $c = C \cdot C$, and observe (using condition (b) of Proposition 2.1) that we have the following expressions:

$$q_1 = \frac{d^2 - c}{3d - 2 - c}, \text{ and } q_i = q_{i-1} - r_{i-1}\frac{m_{i-1}}{m_i + \cdots + m_n} \text{ for } i > 1.$$

Condition (b) of Proposition 2.1 also gives us first that $m_i < d$ for all i (because $d > 1$), and second that we have $c = C\cdot C = -2 - C\cdot K$, so that $3d - m_1 - \cdots - m_n = c + 2$. Because we have $m_i < d$ for $i = 1, 2$, it follows that we have

$$m_2+\cdots+m_n = 3d-m_1-c-2 \geq 2d-c-1, \text{ and } m_3+\cdots+m_n = 3d-m_1-m_2-c-2 \geq d-c.$$

Again using $m_i < d$ for $i = 1, 2$, we thus find that

$$q_2 = q_1 - r_1\frac{m_1}{m_2 + \cdots + m_n} \geq q_1 - r_1\frac{d-1}{2d-c-1},$$

and

$$q_3 = q_2 - r_2 \frac{m_2}{m_3 + \cdots + m_n} \geq q_2 - r_2 \frac{d-1}{d-c}.$$

We finally conclude that

$$\begin{aligned}
m_1 + m_2 + m_3 &\geq q_1 + r_1 + q_2 + r_2 + q_3 \geq q_1 + r_1 + 2q_2 + r_2(1 - \frac{d-1}{d-c}) \\
&\geq 3q_1 + r_1(1 - 2\frac{d-1}{2d-c-1}) + r_2(1 - \frac{d-1}{d-c}) \\
&\geq 3q_1 = 3\frac{d^2 - c}{3d-2-c} > d,
\end{aligned}$$

where we have used our hypothesis that $c \leq 1$ in the last two inequalities, and also that $d > 1$ and $c \geq -2$ for the final inequality. □

For our purposes, we will be interested only in the Cremona transformations which are standard Cremona transformations based in some three of our points P_i. One can use the geometric Cremona transformation to realize S also as a blowup of $\mathbb{P}^2$ in a different set of points; from this point of view, the transformation induces a change of basis of $\mathrm{Pic}(S)$. However, for our applications we will be interested instead in a more artificial construction, in which we keep our fixed basis of $\mathrm{Pic}(S)$, and use a numerical form of the Cremona transformation to construct an automorphism on $\mathrm{Pic}(S)$. This leads to the following definition.

Definition 2.3. Given $1 \leq i_1 < i_2 < i_3 \leq n$, the (i_1, i_2, i_3)-Cremona transformation on S is the automorphism of $\mathrm{Pic}(S)$ which sends a class $dH - m_1E_1 - \cdots - m_nE_n$ to $d'H - m'_1E_1 - \cdots - m'_nE_n$, where $d' = 2d - m_{i_1} - m_{i_2} - m_{i_3}$, $m'_i = m_i$ for $i \neq i_1, i_2, i_3$, and $m'_{i_1} = d - m_{i_2} - m_{i_3}$, $m'_{i_2} = d - m_{i_1} - m_{i_3}$, and $m'_{i_3} = d - m_{i_1} - m_{i_2}$.

Note that this map is an automorphism because it is visibly a homomorphism, and it is an involution. Geometrically speaking, when no three of the P_i are collinear, this map is motivated by replacing the exceptional divisors $E_{i_1}, E_{i_2}, E_{i_3}$ with the (-1)-curves $H - E_{i_2} - E_{i_3}$, $H - E_{i_1} - E_{i_3}$, and $H - E_{i_1} - E_{i_2}$, and changing the P_j for $j \neq i_1, i_2, i_3$ by the induced geometric Cremona transformation. Although our purely numerical definition is a distinct construction, in the below proposition we leverage the connection between our definition and its geometric motivation to show that when the P_i are general, our definition has good behavior.

The following proposition is essentially one half of Theorem 1 of [Dol83].

Proposition 2.4. *Assume that $P_1, \dots, P_r$ are very general in $\mathbb{P}^2$. Then Cremona transformations on S preserve the following:*

(i) *Intersection products;*
(ii) *The canonical class;*
(iii) *Effective classes.*

If we consider any given class, it suffices to assume that the P_i are general; we assume they are very general in order to treat all classes simultaneously.

Proof. That Cremona transformations preserve intersection products and the canonical class is an elementary calculation which holds regardless of generality of the P_i. For the preservation of effective classes, we will more precisely show the following: if we fix a class $C = dH - m_1E_1 - \cdots - m_nE_n$, and $i_1 < i_2 < i_3$, for general choice of the P_i, we will have that C is effective if and only if the (i_1, i_2, i_3)-Cremona image of C is effective. In fact, since the (i_1, i_2, i_3)-Cremona map is an involution on curve classes, it is enough to show that the image of an effective class is effective. Fix any choice of $P_{i_1}, P_{i_2}, P_{i_3}$ which are not collinear, and let $U \subseteq \mathbb{P}^2$ be the complement of the three lines through pairs of the $P_{i_1}, P_{i_2}, P_{i_3}$, so that U^{n-3} parametrizes the choices of the remaining P_j such that none of the P_j are collinear with any two of $P_{i_1}, P_{i_2}, P_{i_3}$. Thus, the geometric Cremona transformation based at $P_{i_1}, P_{i_2}, P_{i_3}$ can be thought of as an automorphism $\varphi : U \overset{\sim}{\to} U$, inducing an automorphism $\varphi^j : U^j \overset{\sim}{\to} U^j$. If $U' \subseteq U^j$ is the complement of the pairwise diagonals, then φ^j maps U' into itself, inducing an automorphism $\varphi' : U' \to U'$. Let $\mathcal{S}$ be the surface over U' obtained from $\mathbb{P}^2 \times U'$ by blowing up $P_{i_1}, P_{i_2}, P_{i_3}$ (considered as constant sections) together with the j disjoint universal sections over U'. Then via the geometry of the Cremona transformation, the fiber S of $\mathcal{S}$ over a given choice of $(P_j)_j \in U'$ can simultaneously be realized as the fiber over $(\varphi(P_j))_j$, and the resulting change of basis of $\operatorname{Pic}(S)$ is precisely the (numerical) (i_1, i_2, i_3)-Cremona transformation.

Now, a class $C = dH - m_1E_1 - \cdots - m_nE_n$ induces a line bundle on $\mathcal{S}$, and since $\mathcal{S}$ is flat and projective over U', by the semicontinuity theorem either C is effective over all points of U', or there is a nonempty open subset $U'' \subseteq U'$ over which C is not effective. But we see by construction that C is effective over a given point $(P_j)_j \in U'$ if and only if its (i_1, i_2, i_3)-Cremona transformation is effective over its Cremona image $\varphi'((P_j)_j)$. Thus, if C is effective over a general point of U', it is effective over all of U', and its (i_1, i_2, i_3)-Cremona transformation is likewise effective over all of U'. On the other hand, if C fails to be effective over $U'' \subseteq U'$, then provided we choose $(P_j)_j \in U''$, the statement of the proposition will still be satisfied. □

Example 2.5. Note that we need some generality hypothesis on the P_i in order for Proposition 2.4 to hold, as otherwise we could have for instance $P_1, \ldots, P_6$ all lying on a conic, so that $2H - E_1 - \cdots - E_6$ is effective, but the Cremona transformation at $(1, 2, 3)$ gives $H - E_4 - E_5 - E_6$, which is not effective unless P_4, P_5, P_6 are collinear.

We are now ready to prove our main theorem.

Proof of Theorem 1.3. We first observe that a curve class is the class of a (-1)-curve if and only if it satisfies conditions (i) and (ii) of the theorem, and cannot be written as a sum of effective classes. Certainly, these conditions are necessary, and conversely, if conditions (i) and (ii) are satisfied, Proposition 2.1 gives us that the expected dimension is positive, so the class is effective. If it is not a sum of effective classes, any representative is an integral curve, hence a (-1)-curve. Consequently, Proposition 2.4 also implies that Cremona transformations preserve classes of (-1)-curves.

Now, we see that the class of a (-1)-curve must satisfy conditions (i)-(iii) of the theorem, and we will prove the converse by induction on d. For the base case is $d = 0$, we prove a stronger statement: even without the hypothesis $m_i \geq 0$, we note that conditions (i) and (ii) already imply that C is the class of one of the E_i, which is indeed a (-1)-curve. Now, suppose that $d > 0$, and we know the theorem for all $d' < d$. Given a class $C = dH - m_1E_1 - \cdots - m_nE_n$ satisfying conditions (i) and (ii) of the statement, by our above observations, it suffices to show that if C is a sum of effective classes, then there is a (-1)-curve of smaller degree intersecting

C negatively. In fact, we observe that the smaller-degree condition is automatic: since the (-1)-curve would necessarily occur in the base locus of C, its degree is at most d. If its degree is exactly d, the difference is a positive combination of the E_i, but we see immediately from Proposition 2.1 that we cannot have two curve classes satisfying conditions (i) and (ii) which differ by a positive combination of the E_i.

Now, according to Lemma 2.2, there exist $i_1 < i_2 < i_3$ such that $m_{i_1} + m_{i_2} + m_{i_3} > d$. We therefore apply an (i_1, i_2, i_3)-Cremona transformation, yielding a curve class $C' = d'H - m'_1E_1 - \cdots - m'_nE_n$, where $d' = 2d - m_{i_1} - m_{i_2} - m_{i_3} < d$. Then by Proposition 2.4, we have that C' still satisfies conditions (i) and (ii) of the theorem, and is still effective. In particular, $d' \geq 0$. If $C = C_1 + C_2$ is a sum of effective classes, applying the same Cremona transformation to C_1 and C_2, we write $C' = C'_1 + C'_2$, with C'_1 and C'_2 still effective by Proposition 2.4, so in particular C' is not the class of a (-1)-curve. First note that if $d' = 0$, then by the base case we have that $C' = \pm E_i$, contradicting that C' is a not a (-1)-curve. On the other hand, if $d' > 0$, we conclude that there is a (-1)-curve C'' of degree $d'' < d'$ such that $C'' \cdot C' < 0$. If $m'_i \geq 0$ for all i, this follows from the induction hypothesis, while if some $m'_i < 0$, we set $C'' = E_i$. Applying the same Cremona transformation to C'' produces a (-1)-curve (of distinct class from C) intersecting C negatively, as desired. □

Thus, if we argue iteratively instead of inductively, we see that if we start with a non-(-1)-curve class, the process always ends with one of the m_i going negative.

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