High-frequency bounds for the Helmholtz equation under parabolic trapping and applications in numerical analysis

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Abstract

This paper is concerned with resolvent estimates on the real axis for the Helmholtz equation posed in the exterior of a bounded obstacle with Dirichlet boundary conditions when the obstacle is *trapping*. There are two resolvent estimates for this situation currently in the literature: (i) in the case of *elliptic trapping* the general "worst case" bound of exponential growth applies, and examples show that this growth can be realised through some sequence of wavenumbers; (ii) in the prototypical case of *hyperbolic trapping* where the Helmholtz equation is posed in the exterior of two strictly convex obstacles (or several obstacles with additional constraints) the nontrapping resolvent estimate holds with a logarithmic loss.

This paper proves the first resolvent estimate for *parabolic trapping* by obstacles, studying a class of obstacles the prototypical example of which is the exterior of two squares (in 2-d), or two cubes (in 3-d), whose sides are parallel. We show, via a novel application of the vector-field/positive-commutator argument of Morawetz, that a resolvent estimate holds with a polynomial loss over the nontrapping estimate. We use this bound, along with the other trapping resolvent estimates, to prove results about integral-equation formulations of the boundary value problem in the case of trapping, and to obtain convergence proofs for standard numerical methods (finite and boundary element methods) applied to these problems; these are the first wavenumber-explicit proofs of convergence for any numerical method for solving the Helmholtz equation in the exterior of a trapping obstacle.

Keywords: Helmholtz equation, high frequency, trapping, resolvent, scattering theory, semiclassical analysis, boundary integral equation.

AMS subject classifications: 35J05, 35J25, 35P25, 65N30, 65N38, 78A45

1 Introduction

1.1 Context, and informal discussion of the main results

Trapping and nontrapping are central concepts in scattering theory. In the case of the Helmholtz equation, $\Delta u + k^2 u = -f$, posed in the exterior of a bounded, Dirichlet obstacle Ω_- in 2- or 3-dimensions, Ω_- is *nontrapping* if all billiard trajectories starting in an exterior neighbourhood of Ω_- escape from that neighbourhood after some uniform time, and Ω_- is *trapping* otherwise (see Definitions 1.4 and 1.9 below for more precise statements, taking into account subtleties about diffraction from corners).

This paper is concerned with resolvent estimates (i.e. a priori bounds on the solution u in terms of the data f) for the exterior Dirichlet problem when k is real. We can write these in terms of the outgoing cut-off resolvent $\chi_1 R(k)\chi_2 : L^2(\Omega_+) \to L^2(\Omega_+)$ for $k \in \mathbb{R}\setminus\{0\}$, where $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$, $\chi_1, \chi_2 \in C^\infty_{\rm comp}(\overline{\Omega_+})$ and $R(k) := (\Delta + k^2)^{-1}$, with Dirichlet boundary conditions, is such that

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$R(k) : L^2(\Omega_+) \to L^2(\Omega_+)$ for $\Im k > 0$. When Ω_- is nontrapping, given $k_0 > 0$,

$$\|\chi_1 R(k)\chi_2\|_{L^2(\Omega_+)\to L^2(\Omega_+)} \lesssim \frac{1}{k} \quad \text{for all } k \geq k_0; \tag{1.1}$$

this classic result was first obtained by the combination of the results on propagation of singularities for the wave equation on manifolds with boundary by Melrose and Sjöstrand [71, 72] with either the parametrix method of Vainberg [90] (see [82]) or the methods of Lax and Phillips [58] (see [70]), following the proof by Morawetz, Ralston, and Strauss [75, 77] of the bound under a slightly-stronger condition than nontrapping.

In this situation of scattering by a (Dirichlet) obstacle, there are two resolvent estimates in the literature when Ω_- is trapping. The first is the general result of Burq [14, Theorem 2] that, given any smooth Ω_- and $k_0 > 0$, there exists $\alpha > 0$ such that

$$\|\chi_1 R(k)\chi_2\|_{L^2(\Omega_+)\to L^2(\Omega_+)} \lesssim \mathrm{e}^{\alpha k} \quad \text{for all } k \geq k_0. \tag{1.2}$$

If Ω_- has a ellipse-shaped cavity (see Figure 1.1(a)) then there exists a sequence of wavenumbers $0 < k_1 < k_2 < \ldots$, with $k_j \to \infty$, and $\alpha > 0$ such that

$$\|\chi_1 R(k_j)\chi_2\|_{L^2(\Omega_+)\to L^2(\Omega_+)} \gtrsim \mathrm{e}^{\alpha k_j} \quad j = 1, 2, \ldots, \tag{1.3}$$

see, e.g., [9, §2.5], and thus the bound (1.2) is sharp. More generally, if there exists an elliptic trapped ray (i.e. an elliptic closed broken geodesic), and $\partial\Omega_-$ is analytic in neighbourhoods of the vertices of the broken geodesic, then the resolvent can grow at least as fast as $\exp(\alpha k_j^q)$, through a sequence k_j as above and for some range of $q < 1$, by the quasimode construction of Cardoso and Popov [19] (note that Popov proved *superalgebraic* growth for certain elliptic trapped rays when $\partial\Omega_-$ is smooth in [81]).

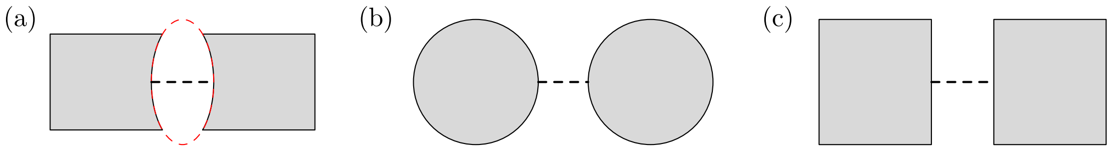

Figure 1.1: Examples of: (a) elliptic trapping; (b) hyperbolic trapping; (c) parabolic trapping.

The second trapping resolvent estimate in the literature concerns hyperbolic trapping, the standard example of which is when Ω_- equals two disjoint convex obstacles with strictly positive curvature; see Figure 1.1(b). The work of Ikawa on this problem (and its generalisation to a finite number of such obstacles satisfying additional conditions – see Definition 4.6 below) implies that there exists $N > 0$ such that

$$\|\chi_1 R(k)\chi_2\|_{L^2(\Omega_+)\to L^2(\Omega_+)} \lesssim k^N \quad \text{for all } k \geq k_0 \tag{1.4}$$

[50, Theorem 2.1], [16, Theorem 4.5], and this bound was later improved by Burq [16, Proposition 4.4] to

$$\|\chi_1 R(k)\chi_2\|_{L^2(\Omega_+)\to L^2(\Omega_+)} \lesssim \frac{\log(2+k)}{k} \quad \text{for all } k \geq k_0, \tag{1.5}$$

i.e. the trapping is so weak there is only a logarithmic loss over the nontrapping estimate (1.1).

Summary of the main results and their novelty. This paper considers the exterior Dirichlet problem for a certain class of parabolic trapping obstacles (see Definitions 1.1 and 1.6 below), the simplest example of which is two squares (in 2-d) or two cubes (in 3-d) with their sides parallel; see Figure 1.1(c). The heart of this paper is the proof that, for this class of obstacles,

$$\|\chi_1 R(k)\chi_2\|_{L^2(\Omega_+)\to L^2(\Omega_+)} \lesssim k \quad \text{for all } k \geq k_0 \tag{1.6}$$

(see Theorem 1.7 below). A simple construction involving the eigenfunctions of the Dirichlet Laplacian on an interval gives an example of a compactly-supported f such that $\|\chi_1 R(k) f\|_{L^2(\Omega_+)} \gtrsim$

$\|f\|_{L^2(\Omega_+)}$, so we expect that (1.4) is one power of k away from being sharp. Nevertheless, we believe (1.6) is the first resolvent estimate proved for parabolic trapping by obstacles. Furthermore, if *either* $\operatorname{supp}\chi_1$ *or* $\operatorname{supp}\chi_2$ is sufficiently far away from the "trapping region" (this is defined more precisely below, but in the example of two squares/cubes one can think of it as the region between the two obstacles), then $\|\chi_1 R(k)\chi_2\|_{L^2\to L^2} \lesssim 1$, and if both $\operatorname{supp}\chi_1$ and $\operatorname{supp}\chi_2$ are sufficiently far away from the trapping region, then the nontrapping estimate $\|\chi_1 R(k)\chi_2\|_{L^2\to L^2} \lesssim 1/k$ holds. These types of improvements in the k-dependence when one restricts attention to areas of phase space isolated from the trapped set have been established in the case of scattering by obstacles by Burq [15, Theorem 4] and Cardoso and Vodev [20, Theorem 1.1] (see, e.g., [34, Theorems 1.1, 1.2], [92, Theorem 1.1] for analogous results in the setting of scattering by a potential and/or by a metric).

We prove these resolvent estimates by adapting and extending the vector-field/positive commutator argument of Morawetz; this argument famously proves the estimate (1.1) for the Dirichlet resolvent for star-shaped domains [75, 76] (see also [27]) and (in $d = 2$) for a class of domains slightly more restrictive than nontrapping [75], [77, §4]. These type of arguments have also been used to prove a priori bounds on solutions of scattering by unbounded rough surfaces, and we bring in ideas from one of the authors works on these problems [26]. These methods are also similar to those used by Burq, Hassell, and Wunsch [17] to study spreading of quasimodes in the Bunimovich stadium. An advantage of these vector-field arguments is that, for $k \geq k_0$ for some explicitly given k_0, we have an expression for the omitted constant in (1.6) that is explicit in all parameters (in particular, the parameters describing the geometries of the domain, and the supports of the cut-off functions; see (3.30) and (3.37) below); thus our resolvent estimates are "quantitative" in the sense of, e.g., Rodnianski and Tao [85].

The resolvent estimate (1.6) has immediate implications for boundary-integral-equation formulations of the scattering problem, for the numerical analysis of these integral-equation formulations, and also for the numerical analysis of the finite element method (based on the standard domain-based variational formulation of the scattering problem); these implications are outlined in §1.4 and §1.5.3 below. In this sense, this paper follows the theme of [27], [86], and [6] of proving high-frequency estimates on the Helmholtz equation and then exploring their implications for integral equations and numerical analysis. In fact, the implications of the other two trapping resolvent estimates (1.2) and (1.4) in these contexts have not yet been explored (with the exception of the recent PhD of one of the authors [43, Theorem 5.5]) and §1.4 and §1.5.3 include the first wavenumber-explicit proofs of convergence for any numerical method for solving the Helmholtz equation in a trapping domain.

1.2 Statement of the main results

1.2.1 Geometric definitions

Let $\Omega_- \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz open set such that the open complement $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$ is connected, and let $\Gamma := \partial\Omega_+ = \partial\Omega_-$ and $R_\Gamma := \max_{x\in\Gamma}|x|$. Let $\gamma_\pm$ denote the trace operators from $\Omega_\pm$ to Γ, let $\partial_n^\pm$ denote the normal derivative trace operators (the normal pointing out of Ω_- and into Ω_+), and let ∇_S denote the surface gradient operator on Γ. Let $H^1_{\mathrm{loc}}(\Omega_+)$ denote the set of functions, v, such that v is locally integrable on Ω_+ and $\chi v \in H^1(\Omega_+)$ for every $\chi \in C^\infty_{\mathrm{comp}}(\overline{\Omega_+}) := \{\chi|_{\Omega_+} : \chi \in C^\infty(\mathbb{R}^d) \text{ is compactly supported}\}$. We abbreviate $r := |x|$, and x_j and $n_j(x)$ denote the jth components of x and $n(x)$, respectively, so that $n_j(x) = e_j \cdot n(x)$, where e_j is the unit vector in the x_j direction. Let $B_R(x) := \{y \in \mathbb{R}^d : |x - y| < R\}$ and $B_R := B_R(0)$. Finally, let $\Omega_R := \Omega_+ \cap B_R$.

Our results apply to particular classes of domains. Define $\psi \in C^{1,1}(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ by

$$\psi(t) = \begin{cases} (1-t)^3 - 1, & 0 \leq t \leq 1, \\ -1, & t > 1, \\ -\psi(-t), & t < 0. \end{cases} \tag{1.7}$$

Definition 1.1 ((R_0, R_1) obstacle) *For $0 < R_0 < R_1$ we say that Ω_- is an (R_0, R_1)* obstacle *if $Z(x) \cdot n(x) \geq 0$ for all $x \in \Gamma$ for which the normal $n(x)$ is defined, where*

$$Z(x) := e_d x_d \chi(r) + x\big(1 - \chi(r)\big), \quad x \in \mathbb{R}^d, \tag{1.8}$$

and

$$\chi(r) := \frac{1}{2} + \frac{1}{2}\psi\left(\frac{2r-(R_0+R_1)}{R_1-R_0}\right), \quad r \geq 0. \tag{1.9}$$

Remark 1.2 (The vector field Z (1.8)) *With χ given by (1.9), $\chi \in C^{1,1}[0,\infty)$, $\chi(r) = 1$ for $0 \leq r \leq R_0$, $\chi(r) = 0$ for $r \geq R_1$, and $\chi'(r) \leq 0$ and $0 \leq \chi(r) \leq 1$ for $r \geq 0$. Therefore, $Z(x) = x_d e_d$ for $|x| \leq R_0$ and $Z(x) = x$ for $|x| \geq R_1$. Thus, if Ω_- is an (R_0, R_1) obstacle, then, for all $x \in \Gamma$ for which $n(x)$ is defined, $x_d n_d(x) \geq 0$ if $|x| \leq R_0$, while $x \cdot n(x) \geq 0$ if $|x| \geq R_1$.*

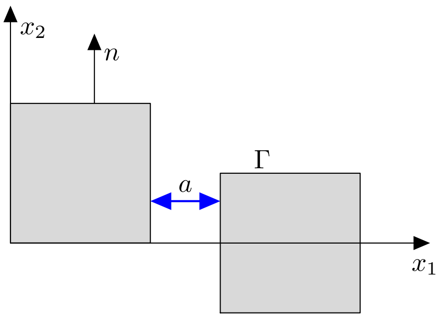

Figure 1.2: The obstacle Ω_- is the union of two parallel squares. For $R_1 > R_0 \geq R_\Gamma$, it is a 2-d example both of an (R_0, R_1) obstacle and of an (R_0, R_1, a) parallel trapping obstacle, since $x_2 n_2(x) \geq 0$ for all $x \in \Gamma$ for which $n(x)$ is defined.

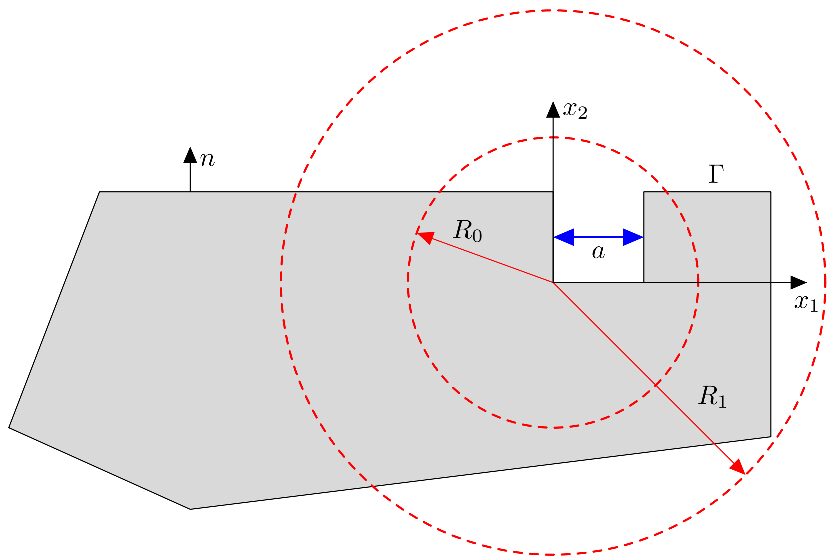

Figure 1.3: The grey-shaded obstacle Ω_- is a 2-d example both of an (R_0, R_1) obstacle and of an (R_0, R_1, a) parallel trapping obstacle, with the values of R_0, R_1, and a indicated, since, for $x \in \Gamma$, $x_2 n_2(x) \geq 0$ for $|x| \leq R_1$ and $x \cdot n(x) \geq 0$ for $|x| \geq R_0$.

To explore the different types of obstacles included in Definition 1.1, we recall the following geometric definitions.

Definition 1.3 (Star-shaped, and star-shaped with respect to a ball)
(i) Ω_- *is* star-shaped with respect to the point $x_0 \in \Omega_-$ *if, whenever $x \in \Omega_-$, the segment $[x_0, x] \subset \Omega_-$.*
(ii) Ω_- *is* star-shaped *if there exists a $x_0 \in \Omega_-$ such that Ω_- is star-shaped with respect to x_0.*
(iii) Ω_- *is* star-shaped with respect to the ball $B_a(x_0)$ *if it is star-shaped with respect to every point in $B_a(x_0)$.*

(iv) Ω_- *is* star-shaped with respect to a ball *if there exists* $a > 0$ *and* $x_0 \in \Omega_-$ *such that* Ω_- *is star-shaped with respect to the ball* $B_a(x_0)$.

Recall that if Ω_- is Lipschitz, then it is star-shaped with respect to x_0 if and only if $(x - x_0) \cdot n(x) \geq 0$ for all $x \in \Gamma$ for which $n(x)$ is defined, and Ω_- is star-shaped with respect to $B_a(x_0)$ if and only if $(x - x_0) \cdot n(x) \geq a$ for all $x \in \Gamma$ for which $n(x)$ is defined; see, e.g., [73, Lemma 5.4.1].

Definition 1.4 (Nontrapping) *We say that* $\Omega_- \subset \mathbb{R}^d$, $d = 2, 3$, *is* nontrapping *if* Γ *is smooth* (C^∞) *and, given* R *such that* $\overline{\Omega_-} \subset B_R$, *there exists a* $T(R) < \infty$ *such that all the billiard trajectories (in the sense of Melrose–Sjöstrand [72, Definition 7.20]) that start in* Ω_R *at time zero leave* Ω_R *by time* $T(R)$.

Example 1.5 (Examples of (R_0, R_1) obstacles) *As important example classes, we note that* Ω_- *is an* (R_0, R_1) *obstacle, for* $0 < R_0 < R_1$, *if any one of the following conditions holds.*

(i) $R_\Gamma \leq R_0$ *and* $x_d n_d(x) \geq 0$ *for all* $x \in \Gamma$ *for which* $n(x)$ *is defined (e.g.* Ω_- *is the union of two balls with centres in the plane* $x_d = 0$*, or the union of two parallel squares, see Figure 1.2).*

(ii) For all $x \in \Gamma$ *for which* $n(x)$ *is defined,* $x_d n_d(x) \geq 0$ *for* $|x| \leq R_1$*, and* $x \cdot n(x) \geq 0$ *for* $|x| \geq R_0$ *(see Figure 1.3).*

(iii) $\min_{x\in\Gamma} |x| \geq R_1$ *and* Ω_- *is star-shaped with respect to the origin.*

The third example shows that an (R_0, R_1) obstacle need not be trapping, and so it is convenient to define a class of (R_0, R_1) obstacles that *are* trapping.

Definition 1.6 ((R_0, R_1, a) parallel trapping obstacle) *For* $0 < R_0 < R_1$ *and* $a > 0$ *we say that* Ω_- *is an* (R_0, R_1, a) parallel trapping obstacle *if it is an* (R_0, R_1) *obstacle and there exists* $y, z \in \Gamma$ *with* $n_d(y) = 0$ *and* $n(z) = -n(y)$ *such that* $z = y + an(y)$ *and, for some* $\epsilon > 0$, $n(x) = n(y)$ *for* $x \in \Gamma \cap B_\epsilon(y)$, $n(x) = n(z)$ *for* $x \in \Gamma \cap B_\epsilon(z)$, *and*

$$\Omega_C := \big\{x + tn(x) : 0 < t < a \text{ and } x \in \Gamma \cap B_\epsilon(y)\big\} \subset \Omega_+.$$

The point of this definition is that $\Gamma \cap B_\epsilon(y)$ and $\Gamma \cap B_\epsilon(z)$ are parallel parts of Γ and that $\{x + tn(x) : 0 < t < a\}$ is a (trapped) billiard trajectory in Ω_+ for $x \in \Gamma \cap B_\epsilon(y)$. Figures 1.2 and 1.3 are examples of Definition 1.6. It is clear from a consideration of these and similar examples that (in any dimension d) there exists an (R_0, R_1, a) parallel trapping obstacle for every $a > 0$ and $R_1 > R_0 > a/2$.

1.2.2 Resolvent estimates and bounds on the Dirichlet-to-Neumann (DtN) map

In the following theorem χ is defined by (1.9) and, for $k > 0$ and $R > R_\Gamma$,

$$\|u\|^2_{H^1_k(\Omega_R)} := \int_{\Omega_R} \big(|\nabla u|^2 + k^2|u|^2\big)\,\mathrm{d}x \quad \text{and} \quad \|u\|^2_{H^1_k(\Omega_R;1-\chi)} := \int_{\Omega_R} \big(|\nabla u|^2 + k^2|u|^2\big)\,(1-\chi)\mathrm{d}x.$$

The notation $A \lesssim B$ (or $B \gtrsim A$) means that $A \leq CB$, where the constant $C > 0$ does not depend on k or f (but will depend on Ω_+, R, and k_0). We write $A \sim B$ if $A \lesssim B$ and $A \gtrsim B$.

Theorem 1.7 (Resolvent estimates) *Let* $f \in L^2(\Omega_+)$ *have compact support, and let* $u \in H^1_{\mathrm{loc}}(\Omega_+)$ *be a solution to the Helmholtz equation* $\Delta u + k^2 u = -f$ *in* Ω_+ *that satisfies the Sommerfeld radiation condition*

$$\frac{\partial u}{\partial r}(x) - \mathrm{i}ku(x) = o\left(\frac{1}{r^{(d-1)/2}}\right), \tag{1.10}$$

as $r \to \infty$*, uniformly in* $\widehat{x} := x/r$*, and the boundary condition* $\gamma_+ u = 0$*. If* Ω_- *is an* (R_0, R_1) *obstacle for some* $R_1 > R_0 > 0$*, and if* $R_1/R_0 \geq 121$*, then, for all* $R > \max_{x\in\Gamma\cup\mathrm{supp}(f)} |x|$*, given* $k_0 > 0$,

$$k^{-1}\|u\|_{H^1_k(\Omega_R)} + \|\partial_d u\|_{L^2(\Omega_R)} + \|u\|_{H^1_k(\Omega_R;1-\chi)} \lesssim k\|f\|_{L^2(\Omega_+)}, \tag{1.11}$$

for all $k \geq k_0$*. If the support of* f *does not intersect* B_{R_0} *and*

$$\|f\|_{L^2(\Omega_+;(1-\chi)^{-1})} := \left(\int_{\Omega_+} \frac{|f|^2}{1-\chi}\,\mathrm{d}x\right)^{1/2} < \infty,$$

then the bound (1.11) *holds with* $k\|f\|_{L^2(\Omega_+)}$ *replaced by* $\|f\|_{L^2(\Omega_+;(1-\chi)^{-1})}$.

This theorem contains the following important special cases.

1. Ω_- is star-shaped and $R_1 \le \inf_{x\in\Gamma}|x|$. In this case, since $\chi(r)=0$ for $r\ge R_1$, the bound recovers the standard bound when Ω_- is Lipschitz and star-shaped that is sharp in its dependence on k (see [27] and the discussion in §1.3), namely

$$\|u\|_{H^1_k(\Omega_R)} \lesssim \|f\|_{L^2(\Omega_+)}, \quad \text{for } k \ge k_0. \tag{1.12}$$

2. Ω_- is an (R_0,R_1,a) parallel trapping obstacle, such as those in Figures 1.2 and 1.3. Provided $R_1/R_0 \ge 121$ – and, at least in the Example 1.5(i) case that $x_d n_d(x) \ge 0$ almost everywhere on Γ, R_1 and R_0 can always be chosen to satisfy this constraint – it holds that

$$\|u\|_{H^1_k(\Omega_R)} + k\|u\|_{H^1_k(\Omega_R;1-\chi)} \lesssim k^2\|f\|_{L^2(\Omega_+)}. \tag{1.13}$$

Furthermore, if, for some $R' > R_0$, the support of f does not intersect $B_{R'}$ then

$$\|u\|_{H^1_k(\Omega_R)} + k\|u\|_{H^1_k(\Omega_R;1-\chi)} \lesssim k\|f\|_{L^2(\Omega_+)}. \tag{1.14}$$

The simple constructions at the end of [27, §3] show that, for every (R_0,R_1) obstacle, there exists an f supported outside B_{R_1} such that $\|u\|_{H^1_k(\Omega_R;1-\chi)} \gtrsim \|f\|_{L^2(\Omega_+)}$, so that the power of k in front of $\|u\|_{H^1_k(\Omega_R;1-\chi)}$ in (1.14) is sharp, and that, for every (R_0,R_1,a) parallel trapping obstacle, there exists a compactly supported f such that

$$\|u\|_{H^1_k(\Omega_R)} \gtrsim k\|f\|_{L^2(\Omega_+)}, \quad \text{for } k \in \{m\pi/a : m \in \mathbb{N}\} \tag{1.15}$$

(this quantisation condition is a requirement that the length a of the billiard orbits between the parallel sides of the trapping domain is a multiple of half the wavelength). These lower bounds show that the power of k on the right hand side of (1.13) can be reduced at most from k^2 to k, and that on the right hand side of (1.14) cannot be reduced.

In the following theorem we use the notation $\|\cdot\|_{H^s_k(\Gamma)}$ defined by equations (2.16)–(2.17) below.

Theorem 1.8 (Bounds on the DtN map) *Let $u \in H^1_{\rm loc}(\Omega_+)$ be a solution to the Helmholtz equation $\Delta u + k^2 u = 0$ in Ω_+ that satisfies the Sommerfeld radiation condition (1.10) and the boundary condition $\gamma_+ u = g$. If Ω_- is an (R_0,R_1) obstacle for some $R_1 > R_0 > 0$, and if $R_1/R_0 \ge 121$, then, for all $R > R_\Gamma$, given $k_0 > 0$,*

$$\|u\|_{H^1_k(\Omega_R)} + \|\partial_n^+ u\|_{L^2(\Gamma)} \lesssim k^2\,\|g\|_{H^1_k(\Gamma)}, \tag{1.16}$$

for all $k \ge k_0$ if $g \in H^1(\Gamma)$. Further, uniformly for $0 \le s \le 1$, provided $g \in H^s(\Gamma)$,

$$\|\partial_n^+ u\|_{H^{s-1}_k(\Gamma)} \lesssim k^2\|g\|_{H^s_k(\Gamma)}, \quad \textit{and} \quad \|\partial_n^+ u\|_{H^{s-1}(\Gamma)} \lesssim k^3\|g\|_{H^s(\Gamma)} \quad \textit{for } k \ge k_0. \tag{1.17}$$

If $g = -u^i|_\Gamma$, where u^i satisfies $\Delta u^i + k^2 u^i = 0$ in a neighbourhood G of $\overline{\Omega_- \cup B_{R_0}}$, then

$$\|u\|_{H^1_k(\Omega_R)} + \|\partial_n^+ u\|_{L^2(\Gamma)} \lesssim k^2 \sup_{x\in G}|u^i(x)|. \tag{1.18}$$

We derive the bounds (1.16) and (1.17) from the resolvent estimate in Theorem 1.7 using the method in Baskin et al. [6] (a sharpening of previous arguments in [56, 86]), which we capture below in Lemma 4.3. (This method was used in [6] to deduce the sharp DtN map bound $\|\partial_n^+ u\|_{L^2(\Gamma)} \lesssim \|g\|_{H^1_k(\Gamma)}$, when Ω_- is nontrapping, from the resolvent estimate (1.1)/(1.12).)

We also apply Lemma 4.3 to write down DtN bounds for the two other trapping configurations for which resolvent estimates are known, namely hyperbolic and elliptic trapping discussed in §1.1; see Corollaries 4.7 and 4.5 below.

The final bound (1.18) in Theorem 1.8 is derived from (1.14). To illustrate this result, suppose that u in Theorem 1.8 is the scattered field corresponding to an incident plane wave $u^i(x) = \exp(\mathrm{i}kx\cdot\hat{a})$, for some unit vector $\hat{a}$, with $\gamma_+ u = g = -u^i|_\Gamma$. Then $\|g\|_{H^1_k(\Gamma)} \sim k$ so that (1.16) implies $\|\partial_n^+ u\|_{L^2(\Gamma)} \lesssim k^3$ for $k \ge k_0$, while (1.18) implies the sharper $\|\partial_n^+ u\|_{L^2(\Gamma)} \lesssim k^2$.

1.3 Discussion of related results

In §1.1 we discussed the resolvent estimate in Theorem 1.7 in the context of the nontrapping resolvent estimate (1.1) and the resolvent estimates for elliptic trapping (1.2) and hyperbolic trapping (1.5), all in the obstacle case. In this section we discuss Theorem 1.7 in a slightly wider context.

Local energy decay and resonance-free regions. In the paper so far, we have only been concerned with resolvent estimates on the real axis (i.e. for k real), but establishing such an estimate is intimately related to (i) meromorphic continuation of the resolvent and resonance-free regions beneath the real axis, and (ii) local energy decay of the solution of the wave equation; results about the link between these three properties can be found in [91, Theorems 1.1 and 1.2], [16, Proposition 4.4 and Lemma 4.7], [11, Theorem 1.3], [33], [92, Theorem 1.5], and [52, Theorem 1], and overviews of results about resonances in obstacle scattering can be found in [95, Page 24], [36, Chapter 6]. In particular, the result of Datchev [33] (suitably translated from the setting of scattering from a potential to the setting of scattering by an obstacle) could be used to prove that the resolvent estimate of Theorem 1.7 holds for k in a prescribed neighbourhood below the real axis, but we do not pursue this here.

Trapping by diffraction from corners. When a ray hits a corner of, say, a polygon, it produces diffracted rays emanating from the corner, and in particular some that travel along the sides of the polygon. This means that there exist glancing rays that travel around the boundary of the polygon (hitting a corner and then either continuing on the next side or travelling back) and do not escape to infinity; thus the exterior of a polygon is, in this sense, a trapping domain. At each diffraction from a corner, however, these rays lose energy, and thus the trapping is in a weaker sense than having a closed path of rays. Baskin and Wunsch [7] proved that the nontrapping resolvent estimate (1.1) holds when Ω_- is a *nontrapping polygon*.

Definition 1.9 (Nontrapping polygon [7]) *$\Omega_- \subset \mathbb{R}^2$ is a* nontrapping polygon *if Ω_- is a finite union of disjoint polygons such that: (i) no three vertices are colinear; and (ii), given $R > R_\Gamma$, there exists a $T(R) < \infty$ such that all the billiard trajectories that start in Ω_R at time zero and miss the vertices leave Ω_R by time $T(R)$. (For a more precise statement of (ii) see [7, Section 5].)*

Parabolic and degenerate hyperbolic trapping by metrics. In the setting of scattering by metrics, Christianson and Wunsch [29] exhibited a sequence of metrics, indexed by $m = 1, 2, \ldots$, where the case $m = 1$ corresponds to a single trapped hyperbolic geodesic, but the hyperbolicity degenerates as m increases, and for $m \geq 2$ the sharp bound

$$\|\chi_1 R(k)\chi_2\|_{L^2\to L^2} \lesssim k^{-2/(m+1)} \tag{1.19}$$

holds (see also the review [93]). Observe that, as $m \to \infty$, the right-hand side of the bound tends to k^0, i.e. a constant. This case of infinite-degeneracy was studied by Christianson [28], who proved the bound

$$\|\chi_1 R(k)\chi_2\|_{L^2\to L^2} \lesssim k^{\varepsilon} \tag{1.20}$$

for any $\varepsilon > 0$ (where the omitted constant depends on ε) [28, Theorem 1 and Proposition 3.8]. The analogue of the situation in [29] in the obstacle setting is two strictly convex obstacles being flattened (in the neighbourhood of the trapped ray), and the bounds (1.19) and (1.20) are therefore consistent with our expectation that the sharp bound for an (R_0, R_1, a) parallel trapping obstacle should be $\|\chi_1 R(k)\chi_2\|_{L^2(\Omega_+)\to L^2(\Omega_+)} \lesssim 1$.

The situation of two convex obstacles being flattened was investigated by Ikawa in [49] and [51], with [51, Theorem 3.6.2] bounding a mapping related to the resolvent in a region below the real axis but excluding neighbourhoods of the resonances. Although this estimate depends on the order of the degeneracy, it is not a resolvent estimate per se and does not apply everywhere on the real axis, so it does not appear to lead to a bound similar to (1.19).

Obstacles rougher than Lipschitz. The vector-field/commutator method of Morawetz can be used to obtain resolvent estimates for rough domains under the assumption of star-shapedness. Indeed, essentially this method was used in Chandler-Wilde and Monk [27] to prove the nontrapping resolvent estimate (1.1), not only for Lipschitz star-shaped Ω_- in 2- and 3-d, but also for C^0 star-shaped Ω_-; indeed their proof of the resolvent estimate (1.1) assumes only that $\mathbb{R}^d\setminus\Omega_+$ is bounded, that $0 \notin \Omega_+$, and that if $x \in \Omega_+$ then $sx \in \Omega_+$ for every $s > 1$ [27, Lemma 3.8].

Parallel trapping domains in rough surface scattering. A resolvent estimate with the same k-dependence as (1.6) was proved for the Helmholtz equation posed above an unbounded rough surface in [26, Theorem 4.1]. Denoting the domain above the surface by Ω_+, the geometric assumption in [26, Theorem 4.1] is that $x \in \Omega_+$ implies that $x + se_d \in \Omega_+$ for all $s > 0$, and that, for some $h \leq H$, $U_H \subset \Omega_+ \subset U_h$, where $U_a := \{x : x_d > a\}$; these conditions allow square/cube-shaped cavities in the surface, and therefore allow the same type of parabolic trapping as present for (R_0, R_1, a) parallel trapping obstacles. At the beginning of §3 we discuss how the proof of Theorem 1.7 uses ideas from the proof of [26, Theorem 4.1].

1.4 Application of the above results to finite element discretisations

A standard reformulation of the problem studied in Theorem 1.7, and the starting point for discretisation by finite element methods (FEMs) (e.g., [69]), is the variational problem (5.1) below in which the unknown is $u_R := u|_{\Omega_R}$, for some $R > R_\Gamma$, which lies in the Hilbert space $V_R := \{w|_{\Omega_R} : w \in H^1_{\rm loc}(\Omega_+) \text{ and } \gamma_+ w = 0\}$. The following corollary bounds the inf-sup constant in this formulation, the upper bound (1.23) taken from [27].

Corollary 1.10 (Bound on the inf-sup constant) *If Ω_- is an (R_0, R_1) obstacle for some $R_1 > R_0 > 0$, and if $R_1/R_0 \geq 121$, then, for all $R > R_\Gamma$, given $k_0 > 0$,*

$$\beta_R := \inf_{0\neq u\in V_R} \sup_{0\neq v\in V_R} \frac{|a(u,v)|}{\|u\|_{H^1_k(\Omega_R)}\|v\|_{H^1_k(\Omega_R)}} \gtrsim k^{-3} \tag{1.21}$$

for all $k \geq k_0$, where

$$a(u,v) := \int_{\Omega_R} (\nabla u \cdot \overline{\nabla v} - k^2 u\bar{v})\,{\rm d}x - \int_{\Gamma_R} \overline{\gamma v}\, P^+_R \gamma u\,{\rm d}s, \quad \textit{for } u, v \in V_R, \tag{1.22}$$

is the sesquilinear form in (5.1). Here γ is the trace operator from Ω_R to $\Gamma_R := \partial\Omega_R$, and P^+_R is the DtN map in the case that $\Omega_- = B_R$ and $\Gamma = \Gamma_R$. Further, if Ω_+ is an (R_0, R_1, a) parallel trapping obstacle for some $a > 0$, then

$$\beta_R \lesssim k^{-2}, \quad \textit{for } k \in \{m\pi/a : m \in \mathbb{N}\}. \tag{1.23}$$

We point out in Remark 5.2 (and see Table 6.1) that the arguments (derived from [27]) that we use to derive the lower bound on β_R from the resolvent estimates in Theorem 1.7 apply whenever a resolvent estimate is available. Thus we can also write down lower bounds on β_R for the worst case of elliptic trapping, and for the case of the mild hyperbolic trapping between two smooth strictly convex obstacles: see Remark 5.2 and Table 6.1 for details.

Our results also prove the missing assumption needed to apply the wavenumber-explicit hp-finite element analysis of Melenk and Sauter [69] to problems of obstacle scattering when Ω_+ is trapping. Suppose that Γ is analytic, $R > R_\Gamma$, and let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω_R in the sense of [69, Assumption 5.1], with $h := \max_{K\in\mathcal{T}_h} \text{diam}(K)$ the maximum element diameter. Let $\mathcal{S}^{p,1}_0(\mathcal{T}_h) := \mathcal{S}^{p,1}(\mathcal{T}_h) \cap V_R$, where $\mathcal{S}^{p,1}(\mathcal{T}_h)$ is the space of continuous, piecewise polynomials of degree $\leq p$ on the triangulation $\mathcal{T}_h$ [69, Equation (5.1)]. Then a (Galerkin) finite element approximation, $u_{hp} \in \mathcal{S}^{p,1}_0(\mathcal{T}_h)$, to the solution u_R of (5.1) is defined by

$$a(u_{hp}, v_{hp}) = G(v_{hp}), \quad \text{for all } v_{hp} \in \mathcal{S}^{p,1}_0(\mathcal{T}_h),$$

where the anti-linear functional G is given by (5.2). Melenk and Sauter's results imply that if, given $k_0 > 0$, there exists $q \geq 1$ such that

$$\beta_R \gtrsim k^{-q}, \quad \text{for } k \geq k_0, \tag{1.24}$$

then the finite element method is quasi-optimal, i.e.

$$\|u_R - u_{hp}\|_{H^1_k(\Omega_R)} \leq C \inf_{v_{hp}\in\mathcal{S}_0^{p,1}(\mathcal{T}_h)} \|u_R - v_{hp}\|_{H^1_k(\Omega_R)}, \tag{1.25}$$

provided that p increases logarithmically with k, and N_{hp}, the degrees of freedom (the dimension of the subspace $\mathcal{S}_0^{p,1}(\mathcal{T}_h)$), increases with k so as to maintain a fixed number of degrees of freedom per wavelength (so that $N_{hp} \sim k^d$). This is a strong result, in particular the "pollution effect" [4] that arises with standard h-version finite element methods, which implies a requirement to increase N_{hp} at a faster rate than k^d, is avoided, and this analysis is fully wavenumber-explicit (the constant C in (1.25) is independent of k, h, and p). However, the result in [69] is established only for the case when Ω_- is star-shaped with respect to a ball. Corollary 1.10 implies that (1.24), and hence also (1.25), applies also for (R_0, R_1) obstacles with $R_1/R_0 \geq 121$ and Γ analytic. This class includes many domains Ω_+ that allow trapped periodic orbits, though not (R_0, R_1, a) parallel trapping obstacles for which Γ is not analytic. However, we expect that a version of (1.25) can be proved for (R_0, R_1, a) parallel trapping obstacles in 2-d that are polygonal, by combining Corollary 1.10 with the wavenumber-explicit hp-FEM analysis for non-convex polygonal domains in [38].

1.5 Application of the above results to integral equations

The results of Theorems 1.8 can be applied to integral equations. Our main result concerns the standard boundary integral equation formulations of the Helmholtz exterior Dirichlet problem.

If u is the solution to the Helmholtz exterior Dirichlet problem, the Neumann trace of u, $\partial_n^+ u$, satisfies the integral equation

$$A'_{k,\eta}\, \partial_n^+ u = f_{k,\eta} \tag{1.26}$$

on Γ, where the integral operator $A'_{k,\eta}$ is the so-called *combined-potential* or *combined-field* integral operator (defined by (6.7) below), the parameter η is a real constant different from zero, and $f_{k,\eta}$ is given in terms of the known Dirichlet data $\gamma_+ u$ (see (6.6)). The equation (1.26) also arises in so-called *sound soft* scattering problems in which u is interpreted as the scattered field corresponding to an incident field u^i, the total field $u^t := u + u^i$ satisfies $\gamma_+ u^t = 0$ on Γ, and (1.26) is satisfied by $\partial_n^+ u^t$ with $f_{k,\eta}$ given in terms of Dirichlet and Neumann traces of u^i on Γ; see (6.12). The other standard integral equation for the exterior Dirichlet problem ((6.8) below) takes the form $A_{k,\eta}\phi = h$, where $h = \gamma^+ u$ and $A_{k,\eta}$ is the adjoint of $A'_{k,\eta}$ with respect to the real inner product on $L^2(\Gamma)$ (as defined below (2.18)).

1.5.1 Bounds on $(A'_{k,\eta})^{-1}$ and $A^{-1}_{k,\eta}$.

The following corollary gives bounds on $(A'_{k,\eta})^{-1}$; bounds on $A^{-1}_{k,\eta}$ follow by duality (see (6.10) and (6.11) below).

Corollary 1.11 (Bounds on $(A'_{k,\eta})^{-1}$) *Suppose that the (finite number of) disjoint components of the Lipschitz open set Ω_- are each either star-shaped with respect to a ball or C^∞, that Ω_+ is an (R_0, R_1) obstacle for some $R_1 > R_0 > 0$, that $R_1/R_0 \geq 121$, and that $\eta = ck$, for some $c \in \mathbb{R}\setminus\{0\}$. Then, given $k_0 > 0$, for all $k \geq k_0$,*

$$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^2; \tag{1.27}$$

indeed

$$\|(A'_{k,\eta})^{-1}\|_{H^s_k(\Gamma)\to H^s_k(\Gamma)} \lesssim k^2 \quad \text{and} \quad \|(A'_{k,\eta})^{-1}\|_{H^s(\Gamma)\to H^s(\Gamma)} \lesssim k^{2-s}, \tag{1.28}$$

for $-1 \leq s \leq 0$. If Ω_+ is an (R_0, R_1, a) parallel trapping obstacle for some $a > 0$, then

$$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \gtrsim k, \quad \text{for } k \in \{m\pi/a : m \in \mathbb{N}\}. \tag{1.29}$$

Definition 1.12 (Piecewise smooth) *We say that the bounded Lipschitz open set Ω_- and its boundary Γ are* piecewise smooth *if Γ can be written as a finite union $\Gamma = \cup_{j=1}^M \overline{\Gamma_j}$ where each Γ_j is relatively open in Γ, the Γ_j are pairwise disjoint, $\Gamma_{\rm sing} := \Gamma \setminus \cup_{j=1}^M \Gamma_j$ has zero surface measure, and each $\Gamma_j \subset \widetilde{\Gamma}_j$, where $\widetilde{\Gamma}_j$ is the boundary of a bounded C^∞ open set.*

Remark 1.13 (Extensions of Corollary 1.11) *If the components of Ω_- are not all C^∞ or star-shaped with respect to a ball, using the bounds from Theorem 6.1 that apply in more general cases it follows that (1.27) and (1.28) still hold but with k^2 and k^{2-s} replaced by $k^{9/4}$ and $k^{9/4-s}$, respectively, if each component of Ω_- is piecewise smooth or star-shaped with respect to a ball, with k^2 and k^{2-s} replaced by $k^{5/2}$ and $k^{5/2-s}$, respectively, in the general case.*

Remark 1.14 (How sharp are the bounds in Corollary 1.11?) *The numerical computations in [9, §4.7] and [46, Example 5.2] give an example of an (R_0, R_1, a) parallel trapping domain for which, when $\eta = \pm k$, $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \sim k$ (at least for the range of k considered in the experiments), i.e., they indicate that the lower bound rate of k in (1.29) is sharp. The fact that the upper bound (1.27) appears to be a factor of k away from being sharp is consistent with the fact that we expect the resolvent estimate (1.6) to be a factor of k away from being sharp.*

Remark 1.15 (Previous upper bounds on $(A'_{k,\eta})^{-1}$) *There have been three previous upper bounds for $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ proved in the literature, all for nontrapping cases. The first is the bound*

$$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1 + \frac{k}{|\eta|}, \quad \textit{for } k > 0, \tag{1.30}$$

for the case when Ω_- is Lipschitz and star-shaped with respect to a ball [27, Theorem 4.3] ([27] assumes additionally that Γ is piecewise smooth, but this requirement can be avoided using density results from [74, Appendix A]; see [86, Remark 3.8]). The second is the bound $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1$ when Ω_+ is nontrapping (in the sense of Definition 1.4) and $|\eta| \sim k$ [6, Theorem 1.13]. The third is the bound, given $k_0 > 0$, that

$$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{5/4}\left(1 + \frac{k^{3/4}}{|\eta|}\right), \quad \textit{for } k \geq k_0, \tag{1.31}$$

when Ω_- is a nontrapping polygon [86, Theorem 1.11]. (In §6.4 below we improve this bound, when $|\eta| \sim k$, to $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{1/4}$ as a corollary of results in [6].)

The only other known bound on $(A'_{k,\eta})^{-1}$ appears in the thesis of the third author [43, Theorem 5.19] and we prove a sharpened and generalised form of it as (1.35) below.

The upper bounds on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ in Corollary 1.11 and Remark 1.15 use the representation (6.13) below that expresses $(A'_{k,\eta})^{-1}$ in terms of the exterior DtN map and an interior impedance to Dirichlet map (the use of this representation in [27] was implicit, and the representation was stated explicitly for the first time as [22, Theorem 2.33]). We prove the bound (1.27) in the same way, using the DtN map bound (1.16) along with existing bounds on the solution of the interior impedance problem; see §6.2 and §6.3 below. We extend this methodology to prove the bounds (1.28) on $(A'_{k,\eta})^{-1}$ also as an operator on $H^s(\Gamma)$, for $-1 \leq s \leq 0$. These arguments also show that, when $|\eta| \sim k$,

$$\|(A'_{k,\eta})^{-1}\|_{H^s_k(\Gamma)\to H^s_k(\Gamma)} \lesssim 1 \quad \text{and} \quad \|(A'_{k,\eta})^{-1}\|_{H^s(\Gamma)\to H^s(\Gamma)} \lesssim k^{-s}, \tag{1.32}$$

for $-1 \leq s \leq 0$ in the cases when Ω_- is *either* Lipschitz and star-shaped with respect to a ball *or* nontrapping.

The arguments from [27] and [6] are summarised in Lemma 6.3 as a general "recipe" where the input is a resolvent estimate for the exterior Dirichlet problem, and the output is a bound on $(A'_{k,\eta})^{-1}$ and $A^{-1}_{k,\eta}$. We apply this recipe to the two other existing resolvent estimates for trapping obstacles (1.2) and (1.5), showing that, when $|\eta| \sim k$,

$$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim \log(2 + k) \tag{1.33}$$

for the mild (hyperbolic) trapping case of a finite number of smooth convex obstacles with strictly positive curvature (additionally satisfying the conditions in Definition 4.6), and

$$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim \exp(\alpha k), \tag{1.34}$$

for some $\alpha > 0$, for the general C^∞ case, this "worst case" exponential growth achieved, as observed earlier in 2-d [9], when the geometry of Γ is such that there exists a stable (elliptic) periodic orbit. The same bounds hold on $\|(A'_{k,\eta})^{-1}\|_{H^s_k(\Gamma)\to H^s_k(\Gamma)}$ for $-1 \le s \le 0$, and they apply also to $\|(A'_{k,\eta})^{-1}\|_{H^s(\Gamma)\to H^s(\Gamma)}$ with the bounds increased by an additional factor k^{-s}. In particular, in the hyperbolic trapping case, we have that, when $|\eta| \sim k$, given $k_0 > 0$,

$$\|(A'_{k,\eta})^{-1}\|_{H^{-1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \lesssim k^{1/2}\log(2+k), \quad \text{for } k \ge k_0; \tag{1.35}$$

this is an improvement of the bound in [43, Theorem 5.19] by a factor $k^{1/2}$.

These new bounds on the norm of $(A'_{k,\eta})^{-1}$ in the cases of elliptic and hyperbolic trapping are of interest in their own right, but also contrast strongly with the new bounds for (R_0, R_1, a) parallel trapping obstacles in Corollary 1.11. The bound (1.33) is only worse than the nontrapping bound (1.30) by a log factor, while in the worst case of elliptic trapping the norm of $(A'_{k,\eta})^{-1}$ can grow exponentially through some sequence of wavenumbers [9, Theorem 2.8]. In between, for (R_0, R_1, a) parallel trapping obstacles, Corollary 1.11 proves polynomial growth (through a particular sequence of wavenumbers) at a rate between k and k^2.

1.5.2 Bounds on the condition numbers of $A'_{k,\eta}$ and $A_{k,\eta}$.

Many authors [55, 54, 2, 21, 9, 6] have studied, in addition to the norm of $A'_{k,\eta}$, its L^2 *condition number*, defined by

$$\mathrm{cond}(A'_{k,\eta}) := \|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}\,\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}. \tag{1.36}$$

This quantity is of interest because the condition number at a continuous level is closely related to the condition numbers of the matrices that arise in Galerkin-method discretisations. Indeed, if orthogonal basis functions are used and Γ is smooth enough, the condition number of the Galerkin matrix converges to (1.36) as the discretisation is refined [9, §3]. Thus, understanding the dependence of $\mathrm{cond}(A'_{k,\eta})$ on k and on the geometry provides quantitative information about condition numbers of matrices at a discrete level, which in turn is relevant to the stability of numerical methods and the convergence of iterative solvers (though see the discussion in [6, Section 7.2], [41] regarding related quantities that may be more informative still). In §6.5 we study $\mathrm{cond}(A'_{k,\eta})$ for trapping geometries, by combining bounds on $(A'_{k,\eta})^{-1}$ with known bounds on the norm of $A'_{k,\eta}$, notably those in Chandler-Wilde et al. [21] and those due to Galkowski and Smith [42, 47], proving the first upper bounds on the condition number for trapping obstacles, see Corollary 6.8.

1.5.3 k-explicit convergence of boundary element methods.

Along with bounding the condition number of $A'_{k,\eta}$, our results have another important application in the numerical solution of scattering problems by boundary integral equation methods. Recall that the *boundary element method (BEM)* is the standard term for the numerical solution of boundary integral equations by the Galerkin method when the finite-dimensional subspaces consist of piecewise polynomials. When convergence is achieved by *both* increasing the degree p of the polynomials *and* decreasing the mesh diameter h the method is called the *hp*-BEM; when only the mesh diameter h is decreased the method is called the h-BEM.

***hp*-BEM.** Löhndorf and Melenk [61] provided the first wavenumber-explicit error analysis for *hp*-boundary element methods applied to the integral equations (1.26) and (6.8) under the assumption that Γ is analytic. Their convergence results however require that, for some $k_0 > 0$ and $\gamma \ge 0$,

$$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{\gamma}, \quad \text{for } k \ge k_0, \tag{1.37}$$

so that these convergence results have been proved to date only for nontrapping domains (see [61] [6, §1.4]). Corollary 1.11 above shows that (1.37) holds for all (R_0, R_1) obstacles with $R_1/R_0 \ge 121$, and the bound (1.33) shows that (1.37) holds also for an Ikawa-like union of convex obstacles (in the sense of Definition 4.6). Putting these results together with [61, Corollary 3.18] we have the following result. In this corollary we use the notation $\mathcal{S}^p(\mathcal{T})$ for the set of piecewise polynomials of degree p on the triangulation $\mathcal{T}$ in the sense of [61, Equation (3.17)].

Corollary 1.16 (Quasi-optimality of the hp-BEM) *Suppose that Γ is analytic, that $\mathcal{T}_h$ is a quasi-uniform triangulation with mesh size h of Γ in the sense of [61, Definition 3.15], that $\eta = ck$, for some non-zero real constant c, and that Ω_- is* either *nontrapping,* or *an Ikawa-like union of convex obstacles,* or *an (R_0, R_1) obstacle with $R_1/R_0 \geq 121$.*

Let $\partial_n^+ u$ be the solution of (1.26) and let $v_{hp} \in \mathcal{S}^p(\mathcal{T}_h)$ be its Galerkin-method approximation, defined by

$$(A'_{k,\eta} v_{hp}, v)_\Gamma = (f_{k,\eta}, v)_\Gamma, \quad \text{for all } v \in \mathcal{S}^p(\mathcal{T}_h), \tag{1.38}$$

where $(\cdot,\cdot)_\Gamma$ denotes the inner product on $L^2(\Gamma)$. Then, given $k_0 > 0$, there exist constants C_1, C_2, C_3 (independent of h, p, and k) such that, if $k \geq k_0$,

$$\frac{kh}{p} \leq C_1, \quad \text{and} \quad p \geq C_2 \log(2+k), \tag{1.39}$$

then the quasi-optimal error estimate

$$\|v_{hp} - \partial_n^+ u\|_{L^2(\Gamma)} \leq C_3 \inf_{v \in \mathcal{S}^p(\mathcal{T}_h)} \|v - \partial_n^+ u\|_{L^2(\Gamma)} \tag{1.40}$$

holds.

An attractive feature of this result is that it demonstrates, via the bounds (1.39), that it is enough to maintain a "fixed number of degrees of freedom per wavelength", meaning increasing the dimension N_{hp} of the approximating subspace $\mathcal{S}^p(\mathcal{T}_h)$ in proportion to k^{d-1}, in order to maintain accuracy as k increases, in agreement with much computational experience [62] (and the numerical results in [61] show that this requirement is sharp). This corollary applies to all (R_0, R_1) obstacles with $R_1/R_0 \geq 121$, including geometries that allow trapped periodic orbits, but does not apply to (R_0, R_1, a) parallel trapping obstacles for which Γ is not analytic.

h-BEM. It is commonly believed that, for nontrapping obstacles, the error estimate (1.40) holds (with C_3 independent of k) for the h-BEM when hk is sufficiently small, i.e., that a fixed number of degrees of freedom per wavelength is sufficient to maintain accuracy; this property can also be described by saying that the h-BEM does not suffer from the pollution effect [4]. However, the recent numerical experiments of Marburg [64], [65], [8] give examples of nontrapping situations where pollution appears to occur, and therefore determining the sharp threshold on h for the error estimate (1.40) to hold in general is an exciting open question.

The best results so far in this direction are by Galkowski et al. [41] (building on results in [46]). Indeed [41, Theorem 1.15] proves that (1.40) holds (with C_3 independent of k) if: (i) Ω_- is smooth with strictly positive curvature[1] and $hk^{4/3}$ is sufficiently small; and (ii) Ω_- is nontrapping and $hk^{3/2}$ is sufficiently small (2-d) and $hk^{3/2}\log(2+k)$ is sufficiently small (3-d).

The arguments and results in [41, 46], combined with the bounds on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ that we obtain in this paper, enable us to prove in the next corollary the first h-BEM convergence results for trapping obstacles. The bounds in this corollary are, unsurprisingly, weaker than the best results for nontrapping obstacles, but only by log factors for Ikawa-like unions of convex obstacles.

Corollary 1.17 (Quasi-optimality of the h-BEM) *Suppose that Ω_- is $C^{2,\alpha}$ for some $\alpha \in (0,1)$, that $\eta = ck$, for some non-zero real constant c, that $k_0 > 0$, that $p \geq 0$, and that $\mathcal{T}_h$ is a shape-regular triangulation of Γ in the sense of Definition 6.10, with $h > 0$ the maximum diameter of the elements $K \in \mathcal{T}_h$. Let $\partial_n^+ u$ be the solution of (1.26), and let $v_{hp} \in \mathcal{S}^p(\mathcal{T}_h)$ be the Galerkin-method approximation to $\partial_n^+ u$, defined by (1.38).*

(a) If Ω_- is an Ikawa-like union of convex obstacles then there exists $C > 0$ such that, provided $k \geq k_0$ and $hk^{4/3}\log(2+k) \leq C$, it holds that

$$\|v_{hp} - \partial_n^+ u\|_{L^2(\Gamma)} \lesssim \log(2+k) \inf_{v \in \mathcal{S}^p(\mathcal{T}_h)} \|v - \partial_n^+ u\|_{L^2(\Gamma)}. \tag{1.41}$$

[1] Here (and elsewhere in the paper), when $d = 3$ we say that a piecewise-smooth Γ has *strictly positive curvature* if there exists $c > 0$ such that, for almost every $x \in \Gamma$, the principal curvatures at x are $\geq c$. When $d = 2$ we say that Γ has strictly positive curvature if the above holds with the principal curvatures replaced by just the curvature.

(b) If Ω_- is a piecewise smooth (R_0, R_1) obstacle, with $R_1/R_0 \geq 121$, then there exists $C > 0$ such that, provided $k \geq k_0$ and $hk^{7/2}\log(2+k) \leq C$, it holds that

$$\|v_{hp} - \partial_n^+ u\|_{L^2(\Gamma)} \lesssim k^2 \inf_{v \in \mathcal{S}^p(\mathcal{T}_h)} \|v - \partial_n^+ u\|_{L^2(\Gamma)}. \tag{1.42}$$

The hidden constants in (1.41) and (1.42) are independent of h, p, and k.

1.6 Outline of paper

In §2 we establish notations and definitions and collect a few basic results that are used throughout the paper. In §3 we prove Theorem 1.7 (the resolvent estimates for (R_0, R_1) obstacles). In §4 we prove Theorem 1.8 (bounds on the DtN map for (R_0, R_1) obstacles), and deduce DtN bounds also for hyperbolic and elliptic trapping. In §5 we deduce bounds on the inf-sup constant for trapping confugurations, proving Corollary 1.10. We consider applications to boundary integral equations in §6, proving Corollaries 1.11 and 1.17, and discussing the other issues summarised in §1.5.3. We also, as an extension of the proof of the lower bound (1.29), provide in Remark 6.6 a counterexample to the conjecture of Betcke and Spence [10, Conjecture 6.2] that $A'_{k,\eta}$ is coercive uniformly in k for large k whenever Ω_- is nontrapping. Table 6.1 provides a useful summary of the results of this paper, and of the existing known sharpest bounds.

2 Preliminaries

2.1 Morawetz/Rellich-type identities and associated results

Lemma 2.1 (Morawetz-type identity) *Let $v \in C^2(D)$ for some open set $D \subset \mathbb{R}^d$, $d \geq 2$. Let $\mathcal{L}v := (\Delta + k^2)v$ with $k \in \mathbb{R}$. Let $Z \in (C^1(D))^d$ and $\beta, \alpha \in C^1(D)$ (i.e. Z is a vector and β and α are scalars) and let all three be* real-*valued. Let*

$$\mathcal{Z}v := Z \cdot \nabla v - \mathrm{i}k\beta v + \alpha v. \tag{2.1}$$

Then, with the usual summation convention,

$$\begin{aligned} 2\Re\big(\overline{\mathcal{Z}v}\,\mathcal{L}v\big) &= \nabla \cdot \Big[2\Re\big(\overline{\mathcal{Z}v}\,\nabla v\big) + \big(k^2|v|^2 - |\nabla v|^2\big)Z\Big] + \big(2\alpha - \nabla\cdot Z\big)\big(k^2|v|^2 - |\nabla v|^2\big) \\ &\quad - 2\Re\big(\partial_i Z_j \partial_i v \overline{\partial_j v}\big) - 2\Re\big(\overline{v}\,(\mathrm{i}k\nabla\beta + \nabla\alpha)\cdot\nabla v\big). \end{aligned} \tag{2.2}$$

Lemma 2.1 can be proved by expanding the divergence on the right-hand side; see [88, Proof of Lemma 2.1]. The identity (2.2) was essentially introduced by Morawetz in [75, §I.2]; see the bibliographic remarks in [88, Remark 2.7]. Identities arising from the multiplier $Z \cdot \nabla u$ are often called Rellich-type, due to Rellich's use of the multiplier $x \cdot \nabla v$ in [83] and the multiplier $e_d \cdot \nabla u$ in [84] (see, e.g., the discussion in [22, §5.3] and [74, §I.4]).

We now prove an integrated form of the identity (2.2); when we use this in the proof of Theorem 1.7, it turns out that we only need to consider constant β, and so we restrict attention to this case.

Lemma 2.2 (Integrated form of the identity (2.2)) *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain with outward-pointing unit normal ν, let γ denote the trace operator, and ∂_ν the normal derivative. If $Z \in (C^1(\overline{D}))^d$ and $\alpha \in W^{1,\infty}(D)$ are real-valued, $\beta \in \mathbb{R}$, and $v \in V(D)$, where*

$$V(D) := \Big\{ v \in H^1(D) : \Delta v \in L^2(D), \gamma v \in H^1(\partial D), \partial_\nu v \in L^2(\partial D) \Big\}, \tag{2.3}$$

then,

$$\begin{aligned} &\int_D 2\Re\big(\overline{\mathcal{Z}v}\,\mathcal{L}v\big) + 2\Re\big(\partial_i Z_j \partial_i v \overline{\partial_j v}\big) + 2\Re\big(\overline{v}\,\nabla\alpha\cdot\nabla v\big) - \big(2\alpha - \nabla\cdot Z\big)\big(k^2|v|^2 - |\nabla v|^2\big)\mathrm{d}x \\ &= \int_{\partial D} \Big[(Z\cdot\nu)\Big(|\partial_\nu v|^2 - |\nabla_S(\gamma v)|^2 + k^2|\gamma v|^2\Big) + 2\Re\Big(\big(Z\cdot\overline{\nabla_S(\gamma v)} + \mathrm{i}k\beta\overline{\gamma v} + \alpha\,\overline{\gamma v}\big)\partial_\nu v\Big)\Big]\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

Recall that, when D is Lipschitz, we can identify $W^{1,\infty}(D)$ with $C^{0,1}(\overline{D})$ (see, e.g., [39, §4.2.3, Theorem 5]), and understand α on ∂D in (2.4) by restriction without needing a trace operator.

Proof of Lemma 2.2. We first assume that Z, α, and β are as in the statement of the theorem, but $v \in \mathcal{D}(\overline{D}) := \{U|_D : U \in C^\infty(\mathbb{R}^d)\}$. Recall that the divergence theorem $\int_D \nabla \cdot F = \int_{\partial D} \gamma F \cdot \nu$ is valid when $F \in H^1(D)$ by [67, Theorems 3.29, 3.34, and 3.38]. Recall also that the product of an $H^1(D)$ function and a $W^{1,\infty}(D)$ function is in $H^1(D)$, and the usual product rule for differentiation holds. Thus $F = 2\Re\big(\overline{\mathcal{Z}v}\,\nabla v\big) + \big(k^2|v|^2 - |\nabla v|^2\big)Z$ is in $H^1(D)$ and $\nabla \cdot F$ is given by the integrand on the left-hand side of (2.4). Furthermore,

$$\gamma F \cdot \nu = (Z\cdot\nu)\left(\left|\frac{\partial v}{\partial \nu}\right|^2 + k^2|v|^2 - |\nabla_S v|^2\right) + 2\Re\left(\big(Z\cdot\overline{\nabla_S v} + \mathrm{i}k\beta\overline{v} + \alpha\overline{v}\big)\frac{\partial v}{\partial \nu}\right)$$

on ∂D, where we have used the fact that $\nabla v = \nu(\partial v/\partial\nu) + \nabla_S v$ on ∂D for $v \in \mathcal{D}(\overline{D})$; the identity (2.4) then follows from the divergence theorem.

The result for $v \in V(D)$ then follows from (i) the density of $\mathcal{D}(\overline{D})$ in $V(D)$ [74, Lemma A.1] and (ii) the fact that (2.4) is continuous in v with respect to the topology of $V(D)$. ■

Lemma 2.3 (Morawetz-Ludwig identity, [76, Equation 1.2]) *Let $v \in C^2(D)$ for some open $D \subset \mathbb{R}^d$, $d \geq 2$. Let $\mathcal{L}v := (\Delta + k^2)v$ and let*

$$\mathcal{M}_\alpha v := r\left(v_r - \mathrm{i}kv + \frac{\alpha}{r}v\right), \tag{2.5}$$

where $\alpha \in \mathbb{R}$ and $v_r = x \cdot \nabla v / r$. Then

$$2\Re(\overline{\mathcal{M}_\alpha v}\mathcal{L}v) = \nabla\cdot\left[2\Re\left(\overline{\mathcal{M}_\alpha v}\nabla v\right) + \left(k^2|v|^2 - |\nabla v|^2\right)x\right] \\ + \big(2\alpha - (d-1)\big)\big(k^2|v|^2 - |\nabla v|^2\big) - \big(|\nabla v|^2 - |v_r|^2\big) - \big|v_r - \mathrm{i}kv\big|^2. \tag{2.6}$$

The Morawetz-Ludwig identity is a particular example of the identity (2.2) with $Z = x$, $\beta = r$, and α a constant, and some further manipulation of the non-divergence terms (using the fact that $x = \beta\nabla\beta$). For a proof, see [76], [87, Proof of Lemma 2.2], or [88, Proof of Lemma 2.3].

The Morawetz-Ludwig identity (2.6) has two key properties. With this identity rearranged and written as $\nabla \cdot Q(v) = P(v)$, the key properties are:

1. If u is a solution of $\mathcal{L}u = 0$ in $\mathbb{R}^d \setminus \overline{B_{R_0}}$, for some $R_0 > 0$, satisfying the Sommerfeld radiation condition (1.10), then, where $\Gamma_R := \partial B_R$,
$$\int_{\Gamma_R} Q(u)\cdot\widehat{x}\,\mathrm{d}s \to 0 \quad \text{as } R \to \infty \tag{2.7}$$
(independent of the value of α in the multiplier $\mathcal{M}_\alpha u$); see [76, Proof of Lemma 5], [87, Lemma 2.4].

2. If $\mathcal{L}u = 0$ and $2\alpha = (d-1)$, then
$$P(u) \geq 0. \tag{2.8}$$

The two properties of the Morawetz-Ludwig identity above mean that if the multiplier that we use on the operator $\mathcal{L}$ is equal to $\mathcal{M}_{(d-1)/2}$ outside a large ball, then there is no contribution from infinity. A convenient way to encode this information is the following lemma due to Chandler-Wilde and Monk [27, Lemma 2.1].

Lemma 2.4 (Inequality on Γ_R used to deal with the contribution from infinity) *Let u be a solution of the homogeneous Helmholtz equation in $\mathbb{R}^d \setminus \overline{B_{R_0}}$, $d = 2, 3$, for some $R_0 > 0$, satisfying the Sommerfeld radiation condition (1.10). Let $\alpha \in \mathbb{R}$ with $2\alpha \geq d-1$. Then, for $R > R_0$,*

$$R\int_{\Gamma_R}\left(\left|\frac{\partial u}{\partial r}\right|^2 - |\nabla_S u|^2 + k^2|u|^2\right)\mathrm{d}s - 2kR\,\Im\int_{\Gamma_R}\overline{u}\frac{\partial u}{\partial r}\,\mathrm{d}s + 2\alpha\Re\int_{\Gamma_R}\overline{u}\frac{\partial u}{\partial r}\,\mathrm{d}s \leq 0, \tag{2.9}$$

where ∇_S is the surface gradient on $\Gamma_R := \partial B_R$.

We have purposely denoted the constant in (2.9) by α to emphasise the fact that the left-hand side of (2.9) is $\int_{\Gamma_R} Q(u)\cdot\widehat{x}\,\mathrm{d}s$ with $Q(u)$ arising from the multiplier $\mathcal{M}_\alpha u = x\cdot\nabla u - \mathrm{i}kru + \alpha u$. We'll see below that the Morawetz-Ludwig identity proves the inequality (2.9) when $2\alpha = d-1$, but it will be slightly more convenient to have this result for $2\alpha \geq d-1$. For the proof of this we need the following, slightly simpler, inequality on Γ_R.

Lemma 2.5 *Let u be a solution of the homogeneous Helmholtz equation in $\mathbb{R}^d\setminus\overline{B_{R_0}}$, $d=2,3$, for some $R_0>0$, satisfying the Sommerfeld radiation condition (1.10). Then, for $R>R_0$,*

$$\Re\int_{\Gamma_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s \leq 0. \tag{2.10}$$

Proof of Lemma 2.5. This result is proved in [79, Theorem 2.6.4, p.97] or [27, Lemma 2.1] using the explicit expression for the solution of the Helmholtz equation in the exterior of a ball (i.e. an expansion in either trigonometric polynomials, for $d=2$, or spherical harmonics, for $d=3$, with coefficients given in terms of Bessel and Hankel functions) and then proving bounds on particular combinations of Bessel and Hankel functions. ∎

Proof of Lemma 2.4. This result is proved in [27, Lemma 2.1] by using the explicit expression for the solution of the Helmholtz equation in the exterior of a ball, as in the proof of Lemma 2.5, and proving monotonicity properties of combinations of Bessel and Hankel functions. We provide here an alternative, shorter, proof via the Morawetz-Ludwig identity but note that in fact, [27, Lemma 2.1] is slightly stronger result than Lemma 2.4 when $d=3$, showing that (2.9) holds whenever $2\alpha\geq 1$.

By the inequality (2.10), it is sufficient to prove (2.9) with $2\alpha = d-1$. We now integrate (2.6) with $v=u$ and $2\alpha=d-1$ over $B_{R_1}\setminus B_R$, use the divergence theorem, and then let $R_1\to\infty$ (note that using the divergence theorem is allowed since u is C^∞ by elliptic regularity). The first key property of the Morawetz-Ludwig identity stated above (as (2.7)) implies that the surface integral on $|x|=R_1$ tends to zero as $R_1\to\infty$ [87, Lemma 2.4]. Then, using the decomposition $\nabla v = \nabla_S v + \widehat{x} v_r$ on the integral over Γ_R, we obtain that

$$\begin{aligned}\int_{\Gamma_R} Q(u)\cdot\widehat{x}\,\mathrm{d}s &= \int_{\Gamma_R} R\left(\left|\frac{\partial u}{\partial r}\right|^2 - |\nabla_S u|^2 + k^2|u|^2\right)\mathrm{d}s\\ &\qquad\qquad - 2kR\,\Im\int_{\Gamma_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s + (d-1)\Re\int_{\Gamma_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s\\ &= -\int_{\mathbb{R}^d\setminus B_R}\left(\left(|\nabla u|^2 - |u_r|^2\right) + |u_r - \mathrm{i}ku|^2\right)\mathrm{d}x \leq 0\end{aligned}$$

(where this last inequality is the second key property (2.8) above); i.e. we have established (2.9) with $2\alpha = d-1$ and we are done. ∎

The inequality (2.10) combined with Green's identity (i.e. pairing $\mathcal{L}v$ with v) has the following simple consequence, which we use later.

Lemma 2.6 *Let $f\in L^2(\Omega_+)$ have compact support, and let $u\in H^1_{\mathrm{loc}}(\Omega_+)$ be a solution to the Helmholtz equation $\Delta u + k^2 u = -f$ in Ω_+ that satisfies the Sommerfeld radiation condition (1.10) and the boundary condition $\gamma_+ u = 0$. For any $R>R_\Gamma$ such that $\mathrm{supp} f\subset B_R$, and for any $\varepsilon>0$,*

$$\|\nabla u\|^2_{L^2(\Omega_R)} \leq \left(1+\frac{1}{2\varepsilon}\right)k^2\,\|u\|^2_{L^2(\Omega_R)} + \frac{\varepsilon}{2k^2}\,\|f\|^2_{L^2(\Omega_R)}\,. \tag{2.11}$$

Proof. By multiplying $\mathcal{L}u = -f$ by $\bar{u}$ and integrating over Ω_R we have

$$\int_{\Omega_R}|\nabla u|^2\,\mathrm{d}x - k^2\int_{\Omega_R}|u|^2\,\mathrm{d}x - \int_{\Omega_R} f\bar{u}\,\mathrm{d}x = \int_{\Gamma_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s.$$

The result then follows by using (2.10) and the inequality

$$2ab \leq \epsilon a^2 + b^2/\epsilon \quad \text{for all } a,b,\epsilon>0. \tag{2.12}$$

∎

2.2 A Poincaré/Friedrichs-type inequality

Lemma 2.7 *For $R > 0$ and $v \in H^1(\mathbb{R}^d)$ it holds that*

$$\int_{B_{2R}} |v|^2 \, \mathrm{d}x \le 8 \int_{B_{\sqrt{13}R} \setminus B_{2R}} |v|^2 \, \mathrm{d}x + 4R^2 \int_{B_{\sqrt{13}R}} |\partial_d v|^2 \, \mathrm{d}x. \tag{2.13}$$

Proof. Suppose that $\phi \in C_0^\infty(\mathbb{R})$ and $h, H > 0$. Then, for $0 \le t \le h \le s \le h + H$,

$$\phi(t) = \phi(s) - \int_t^s \phi'(r) \, \mathrm{d}r$$

so that, by Cauchy-Schwarz and (2.12), for $\epsilon > 0$,

$$|\phi(t)|^2 \le (1+\epsilon)|\phi(s)|^2 + (1+\epsilon^{-1})(s-t) \int_t^s |\phi'(r)|^2 \, \mathrm{d}r.$$

Hence, for $0 \le h \le s \le h + H$,

$$\begin{aligned} \int_0^h |\phi(t)|^2 \, \mathrm{d}t &\le (1+\epsilon)h|\phi(s)|^2 + (1+\epsilon^{-1}) \int_0^h \left\{ (s-t) \int_0^{h+H} |\phi'(r)|^2 \, \mathrm{d}r \right\} \mathrm{d}t \\ &= (1+\epsilon)h|\phi(s)|^2 + \frac{1}{2}(1+\epsilon^{-1})h(2s-h) \int_0^{h+H} |\phi'(r)|^2 \, \mathrm{d}r, \end{aligned}$$

so that, integrating with respect to s from h to $h+H$ and dividing by H,

$$\int_0^h |\phi(t)|^2 \, \mathrm{d}t \le \frac{(1+\epsilon)h}{H} \int_h^{h+H} |\phi(s)|^2 \, \mathrm{d}s + \frac{(1+\epsilon^{-1})h(h+H)}{2} \int_0^{h+H} |\phi'(r)|^2 \, \mathrm{d}r. \tag{2.14}$$

For $h_1 < h_2$ and $A > 0$ let $U(h_1, h_2, A) := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : h_1 < x_d < h_2, |\widetilde{x}| < A\}$. Then, for $v \in C_0^\infty(\mathbb{R}^d)$ in the first instance, and then by density for all $v \in H^1(\mathbb{R}^d)$, it follows from (2.14) with $\epsilon = 3$ that, for $h, H, A > 0$,

$$\int_{U(0,h,A)} |v|^2 \, \mathrm{d}x \le \frac{4h}{H} \int_{U(h,h+H,A)} |v|^2 \, \mathrm{d}x + \frac{2h(h+H)}{3} \int_{U(0,h+H,A)} |\partial_d v|^2 \, \mathrm{d}x.$$

Similarly,

$$\int_{U(-h,0,A)} |v|^2 \, \mathrm{d}x \le \frac{4h}{H} \int_{U(-h-H,-h,A)} |v|^2 \, \mathrm{d}x + \frac{2h(h+H)}{3} \int_{U(-h-H,0,A)} |\partial_d v|^2 \, \mathrm{d}x,$$

for $v \in H^1(\mathbb{R}^d)$. Thus, for $v \in H^1(\mathbb{R}^d)$ it holds for $h > 0$ that

$$\begin{aligned} \int_{B_h} |v|^2 \, \mathrm{d}x &\le \int_{U(-h,h,h)} |v|^2 \, \mathrm{d}x \\ &\le \frac{4h}{H} \left\{ \int_{U(-h-H,-h,h)} |v|^2 \, \mathrm{d}x + \int_{U(h,h+H,h)} |v|^2 \, \mathrm{d}x \right\} \\ &\qquad + \frac{2h(h+H)}{3} \int_{U(-h-H,h+H,h)} |\partial_d v|^2 \, \mathrm{d}x \\ &\le \frac{4h}{H} \int_{B(h,\sqrt{h^2+(h+H)^2})} |v|^2 \, \mathrm{d}x + \frac{2h(h+H)}{3} \int_{B(0,\sqrt{h^2+(h+H)^2})} |\partial_d v|^2 \, \mathrm{d}x, \end{aligned}$$

where for $0 \le h_1 \le h_2$, $B(h_1, h_2) := B_{h_2} \setminus B_{h_1}$. Applying this bound with $h = 2R$ and $H = R$ we obtain the required result. ∎

Corollary 2.8 *If $v \in H^1(\Omega_R)$ with $\gamma_+ v = 0$ on Γ, and $R \ge \sqrt{13}R_0$, then*

$$\int_{\Omega_{2R_0}} |v|^2 \, \mathrm{d}x \le 8 \int_{\Omega_R \setminus \Omega_{2R_0}} |v|^2 \, \mathrm{d}x + 4R^2 \int_{\Omega_R} |\partial_d v|^2 \, \mathrm{d}x. \tag{2.15}$$

Proof. This follows from Lemma 2.7 since, given $v \in H^1(\Omega_R)$ with $\gamma_+ v = 0$ on Γ, we can extend the definition of v to $\mathbb{R}^d$ so that $v \in H^1(\mathbb{R}^d)$ and $v = 0$ in Ω_-. ∎

2.3 Boundary Sobolev spaces and interpolation

We use boundary Sobolev spaces $H^s(\Gamma)$ defined in the usual way (see, e.g., [67, Pages 98 and 99]), and denote by $H^s_k(\Gamma)$ the space $H^s(\Gamma)$ equipped with a wavenumber dependent norm $\|\cdot\|_{H^s_k(\Gamma)}$. Precisely, we equip $H^0(\Gamma) = H^0_k(\Gamma) = L^2(\Gamma)$ with the $L^2(\Gamma)$ norm. We define $\|\cdot\|_{H^s(\Gamma)}$ and $\|\cdot\|_{H^s_k(\Gamma)}$ for $s = 1$ by

$$\|\phi\|^2_{H^1(\Gamma)} = \|\nabla_S\phi\|^2_{L^2(\Gamma)} + \|\phi\|^2_{L^2(\Gamma)} \quad \text{and} \quad \|\phi\|^2_{H^1_k(\Gamma)} = \|\nabla_S\phi\|^2_{L^2(\Gamma)} + k^2\|\phi\|^2_{L^2(\Gamma)}, \tag{2.16}$$

and for $0 < s < 1$ by interpolation, choosing the specific norm given by the complex interpolation method (equivalently, by real methods of interpolation appropriately defined and normalised; see [66], [25, Remark 3.6]). We then define the norms on $H^s(\Gamma)$ and $H^s_k(\Gamma)$ for $-1 \leq s < 0$ by duality,

$$\|\phi\|_{H^s(\Gamma)} := \sup_{0\neq\psi\in H^{-s}(\Gamma)} \frac{|\langle\phi,\psi\rangle_\Gamma|}{\|\psi\|_{H^{-s}(\Gamma)}} \quad \text{and} \quad \|\phi\|_{H^s_k(\Gamma)} := \sup_{0\neq\psi\in H^{-s}(\Gamma)} \frac{|\langle\phi,\psi\rangle_\Gamma|}{\|\psi\|_{H^{-s}_k(\Gamma)}}, \tag{2.17}$$

for $\phi \in H^s(\Gamma)$, where $\langle\phi,\psi\rangle_\Gamma$ denotes the standard duality pairing that reduces to $(\phi,\psi)_\Gamma$, the inner product on $L^2(\Gamma)$, when $\psi \in L^2(\Gamma)$. In the terminology of [25, Remark 3.8], with the norms we have selected, $\{H^s(\Gamma) : -1 \leq s \leq 1\}$ and $\{H^s_k(\Gamma) : -1 \leq s \leq 1\}$ are *exact interpolation scales*, so that, if $B : H^{s_j}_k(\Gamma) \to H^{t_j}_k(\Gamma)$ is a bounded linear operator and

$$\|B\|_{H^{s_j}_k(\Gamma)\to H^{t_j}_k(\Gamma)} \leq C_j, \quad \text{for } j = 1, 2,$$

with $s_j, t_j \in [-1, 1]$, then $B : H^s_k(\Gamma) \to H^t_k(\Gamma)$ and

$$\|B\|_{H^s_k(\Gamma)\to H^t_k(\Gamma)} \leq C_1^{1-\theta} C_2^\theta, \quad \text{for } s = \theta s_1 + (1-\theta)s_2 \text{ and } t = \theta t_1 + (1-\theta)t_2 \text{ with } 0 < \theta < 1. \tag{2.18}$$

Moreover (by definition) $H^{-s}(\Gamma)$ is an isometric realisation of $(H^s(\Gamma))'$, the dual space of $H^s(\Gamma)$, for $-1 \leq s \leq 1$, so that, if $A : H^s_k(\Gamma) \to H^t_k(\Gamma)$ is bounded and B is the adjoint of A with respect to the $L^2(\Gamma)$ inner product, or with respect to the *real* inner product $(\cdot,\cdot)^r_\Gamma$, defined by $(\phi,\psi)^r_\Gamma = \int_\Gamma \phi\psi \mathrm{d}s$, then $B : H^{-t}_k(\Gamma) \to H^{-s}_k(\Gamma)$ is bounded and

$$\|B\|_{H^{-t}_k(\Gamma)\to H^{-s}_k(\Gamma)} = \|A\|_{H^s_k(\Gamma)\to H^t_k(\Gamma)}. \tag{2.19}$$

Combining these observations, if $A : H^s_k(\Gamma) \to H^t_k(\Gamma)$ is bounded and self-adjoint, or is self-adjoint with respect to the real inner product, meaning that $(A\phi,\psi)^r_\Gamma = (\phi, A\psi)^r_\Gamma$, for $\phi,\psi \in H^1(\Gamma)$, then

$$\|A\|_{H^\sigma_k(\Gamma)\to H^\tau_k(\Gamma)} \leq \|A\|_{H^s_k(\Gamma)\to H^t_k(\Gamma)}, \tag{2.20}$$

for $\sigma = \theta s - (1-\theta)t$, $\tau = \theta t - (1-\theta)s$, and $0 \leq \theta \leq 1$.

3 Proof of Theorem 1.7 on resolvent estimates

3.1 The ideas behind the proof

The proof is based on the Morawetz-type identity (2.2). Recall that in [75], [77, §4], Morawetz and co-workers showed that if there exists a vector field $Z(x)$, $R > 0$, and $c_1 > 0$ such that

$$Z(x) = x \quad \text{in a neighbourhood of } \Gamma_R, \tag{3.1}$$

$$\Re\big(\partial_i Z_j(x)\xi_j\overline{\xi_j}\big) \geq 0 \text{ for all } \xi \in \mathbb{C}^d \text{ and } x \in \Omega_R, \quad \text{and } Z(x)\cdot n(x) \geq c_1 \quad \text{for all } x \in \Gamma, \tag{3.2}$$

then (2.2) can be used to prove the resolvent estimate (1.1) ([77, §4] then constructed such a Z for a class of obstacles slightly more restrictive than nontrapping). Implicit in [75] is the fact that one can replace the two conditions (3.2) with

$$\Re\big(\partial_i Z_j(x)\xi_j\overline{\xi_j}\big) \geq c_2|\xi|^2 \text{ for all } \xi \in \mathbb{C}^d \text{ and } x \in \Omega_R, \quad \text{and } Z(x)\cdot n(x) \geq 0 \quad \text{for all } x \in \Gamma, \tag{3.3}$$

for some $c_2 > 0$; i.e., one needs strict positivity *either* in Ω_R *or* on Γ. Note that $Z(x) = x$ satisfies this second set of conditions, implying that the resolvent estimate holds for Ω_- that are star-shaped (see also [76, 27]).

We cannot expect to satisfy one of these sets of conditions on Z (either (3.1) and (3.2) or (3.1) and (3.3)) for an (R_0, R_1) obstacle, since we know the nontrapping resolvent estimate (1.1) does not hold in this case. The Z that we use in our arguments is the one in the definition of (R_0, R_1) obstacles, namely (1.8). By the definition of (R_0, R_1) obstacles, we have $Z(x)\cdot n(x) \geq 0$ for all $x \in \Gamma$, but now, for $r < R_0$ at least, $\Re\big(\partial_i Z_j(x)\xi_j\overline{\xi_j}\big) = |\xi_d|^2$, which is only positive semi-definite. (Note that the vector field $e_d x_d$ is often used in arguments involving Rellich/Morawetz-type identities in rough surface scattering; see [84, 94, 26, 59], [22, §5.3], [74, §I.4], and the references therein.)

Applying the Morawetz-type identity (2.2) in Ω_R with Z given by (1.8), $\beta = R$, and α defined by (3.5) below, and then using Lemma 2.4 to deal with the contribution at infinity, we find in Lemma 3.1 below that

$$\begin{aligned}
&\int_{\Omega_R} \Big(2|\partial_d u|^2\chi(r) + |\nabla u|^2(2-q)\big(1-\chi(r)\big) + qk^2|u|^2\big(1-\chi(r)\big) - 2r|\partial_r u|^2\chi'(r)\Big)\,\mathrm{d}x \\
&+ 2\Re\int_{\Omega_R} \bar{u}\nabla\alpha\cdot\nabla u\,\mathrm{d}x + 2\Re\int_{\Omega_R} x_d\partial_d\bar{u}\partial_r u\chi'(r)\,\mathrm{d}x \\
&\qquad\leq -2kR\Im\int_{\Omega_R} f\bar{u}\,\mathrm{d}x + \Re\int_{\Omega_R} f\big(2x_d\partial_d\bar{u}\chi(r) + 2r\partial_r\bar{u}\big(1-\chi(r)\big) + 2\alpha\bar{u}\big)\,\mathrm{d}x
\end{aligned} \tag{3.4}$$

for any $q \in [0,1]$. We see that in the "trapping region", namely when $\chi = 1$, we only have control of $|\partial_d u|^2$, but in the "nontrapping region" (in $\operatorname{supp}(1-\chi)$) we have control of $|\nabla u|^2 + k^2|u|^2$ (as expected, since here $Z = x$).

We then proceed via a series of lemmas. In Lemma 3.2 we get rid of the sign-indefinite "cross" terms on the second line of (3.4). In Lemma 3.3 we use the Poincaré-Friedrichs inequality of Corollary 2.8 to put $|u|^2$ back in the trapping region (here is the place where we lose powers of k compared to the nontrapping estimate). Finally, in Lemma 3.4 we use Lemma 2.6 to gain control of $|\nabla u|^2$ in the trapping region, resulting in the resolvent estimate (1.11). Lemmas 3.2 and 3.3 impose restrictions on the ratio of R_1/R_0, in the first case from carefully dealing with the contribution from the cut-off function in the transition between the trapping and nontrapping regions, and in the second case from the requirements needed to use the Poincaré-Friedrichs inequality.

As discussed in §1.3, the analogue of the resolvent estimate (1.11) in the case of rough surface scattering was proved in [26, Theorem 4.1]; the proof of this estimate uses $Z(x) = e_d x_d$, along with the analogue of Lemma 2.4 in this case [26, Lemma 2.2], avoiding the subtleties of transitioning between the vector fields $e_d x_d$ and x that we encounter here.

3.2 Lemmas 3.1–3.4, their proofs, and the proof of Theorem 1.7

Lemma 3.1 *Let Ω_- be an (R_0, R_1) obstacle, let Z be defined by (1.8), let χ be defined by (1.9), and let*

$$2\alpha := \nabla\cdot Z - q\big(1-\chi(r)\big), \tag{3.5}$$

for some $q \in [0,1]$. If u is the solution of the exterior Dirichlet problem described in Theorem 1.7 and $R > R_1$, then (3.4) holds.

Proof. The regularity result of Nečas [78, §5.1.2], [67, Theorem 4.24(ii)] (stated for u the solution of the exterior Dirichlet problem in [86, Lemma 3.5]) implies that $u \in V(\Omega_R)$ defined by (2.3). With $R > R_1$, we use the integrated Morawetz identity (2.4) with $D = \Omega_R$, Z the vector field in (1.8) (observe that this is in $C^1(\overline{\Omega_R})$ by Remark 1.2), $\beta = R$, and $\alpha \in W^{1,\infty}(\Omega_R)$. We fix α specifically later, but assume at this stage that $2\alpha \geq d-1$ on $\Gamma_R := \partial B_R$.

Using the fact that $\gamma_+ u = 0$ on Γ, we obtain

$$\begin{aligned}
&\int_{\Omega_R} \Big[-2\Re\big(\overline{\mathcal{Z}u}\,f\big) + 2\Re\big(\partial_i Z_j\partial_i u\overline{\partial_j u}\big) + 2\Re\big(\overline{u}\,\nabla\alpha\cdot\nabla u\big) - \big(2\alpha - \nabla\cdot Z\big)\big(k^2|u|^2 - |\nabla u|^2\big)\Big]\mathrm{d}x \\
&+ \int_\Gamma (Z\cdot n)\,|\partial_n u|^2\,\mathrm{d}s = \int_{\Gamma_R} R\left(\left|\frac{\partial u}{\partial r}\right|^2 - |\nabla_S u|^2 + k^2|u|^2\right)\mathrm{d}s + 2\Re\left(\big(\mathrm{i}kR\overline{u} + \alpha\overline{u}\big)\frac{\partial u}{\partial r}\right)\mathrm{d}s.
\end{aligned} \tag{3.6}$$

Since $\chi(r) = 0$ for $r \geq R_1$, we have that $Z = x$ on Γ_R; this fact, along with the assumption that $2\alpha \geq d-1$ on Γ_R, implies that the right-hand side of (3.6) is ≤ 0 by Lemma 2.4. Then, since $Z \cdot n \geq 0$ on Γ from the definition of an (R_0, R_1) obstacle, we have

$$\int_{\Omega_R} \Big[2\Re\big(\partial_i Z_j \partial_i u \overline{\partial_j u}\big) + 2\Re\big(\overline{u}\,\nabla\alpha \cdot \nabla u\big) - \big(2\alpha - \nabla\cdot Z\big)\big(k^2|u|^2 - |\nabla u|^2\big)\Big] \mathrm{d}x$$
$$\leq 2\Re \int_{\Omega_R} \big(Z \cdot \overline{\nabla u} + \mathrm{i}kR\overline{u} + \alpha\overline{u}\big)\, f \,\mathrm{d}x. \tag{3.7}$$

Simple calculations imply that, with the summation convention for the indices i and j but not d,

$$Z_j = \delta_{jd} x_d \chi(r) + x_j\big(1-\chi(r)\big), \tag{3.8}$$
$$\partial_i Z_j = \delta_{id}\delta_{jd}\chi(r) + \delta_{ij}\big(1-\chi(r)\big) + (\delta_{jd}x_d - x_j)\, x_i \chi'(r)/r, \tag{3.9}$$
$$\nabla \cdot Z = d - (d-1)\chi(r) + (x_d^2 - r^2)\chi'(r)/r, \tag{3.10}$$
$$Z \cdot \nabla v = x_d \partial_d v \chi(r) + r\partial_r v\big(1-\chi(r)\big), \tag{3.11}$$
$$\partial_i Z_j \partial_i v \overline{\partial_j v} = |\partial_d v|^2 \chi(r) + |\nabla v|^2\big(1-\chi(r)\big) + \big(x_d \partial_d \bar{v} \partial_r v - r|\partial_r v|^2\big)\,\chi'(r), \quad \text{and} \tag{3.12}$$
$$\begin{aligned}\nabla(\nabla\cdot Z) &= \big(2x_d e_d - x(d + x_d^2/r^2)\big)\,\chi'(r)/r - x(r^2 - x_d^2)\chi''(r)/r^2 \\ &= 2x_d e_d \chi'(r)/r - (x/r)\left[(d + x_d^2/r^2)\chi'(r) + (r^2 - x_d^2)\chi''(r)/r\right].\end{aligned} \tag{3.13}$$

We now choose α as in (3.5). By Remark 1.2, both χ and $\chi' \in W^{1,\infty}(0,R)$, and $\chi = 1$ in a neighbourhood of zero. Thus $\nabla \cdot Z \in W^{1,\infty}(\Omega_R)$ and so is α, as required to use the integrated identity (2.4).

The rationale behind this choice of α is that: (i) we want 2α to equal $\nabla\cdot Z - q$, for some $q \in [0,1]$, on Γ_R so that $2\alpha \geq d-1$, allowing the application above of Lemma 2.4; and (ii) we want $2\alpha = \nabla \cdot Z$ in the trapping region to kill the sign-indefinite combination $k^2|u|^2 - |\nabla u|^2$ in (3.7), and leave $2\Re(\partial_i Z_j \partial_i u \overline{\partial_j u})$ as the only volume term in this region.

With α defined by (3.5), (3.13) implies that

$$2\nabla\alpha = 2x_d e_d \chi'(r)/r - (x/r)\Big[(d - q + x_d^2/r^2)\chi'(r) + (r^2 - x_d^2)\chi''(r)/r\Big]. \tag{3.14}$$

Using (3.8)–(3.13) in (3.7), we find

$$2\Re\int_{\Omega_R} \Big(|\partial_d u|^2\chi(r) + |\nabla u|^2\big(1-\chi(r)\big) - \big(r|\partial_r u|^2 - x_d \partial_d \bar{u}\partial_r u\big)\,\chi'(r)\Big)\,\mathrm{d}x$$
$$+ 2\Re\int_{\Omega_R} \bar{u}\nabla\alpha\cdot\nabla u \,\mathrm{d}x - q\int_{\Omega_R}\big(1-\chi(r)\big)(|\nabla u|^2 - k^2|u|^2)\,\mathrm{d}x$$
$$\leq -2kR\Im\int_{\Omega_R} f\bar{u}\,\mathrm{d}x + \Re\int_{\Omega_R} f\Big(2x_d\partial_d\bar{u}\chi(r) + 2r\partial_r\bar{u}\big(1-\chi(r)\big) + 2\alpha\bar{u}\Big)\mathrm{d}x$$

(to keep the expression compact, we do not use the expression for $\nabla\alpha$, (3.14), yet), which rearranges to the result (3.4). ∎

Lemma 3.2 *Let Ω_- be an (R_0, R_1) obstacle, Z be defined by (1.8), χ be defined by (1.9), and α be defined by (3.5). Let $\rho := R_1/R_0$ and assume that $R > R_1$. If $v \in H^1(\Omega_R)$ and $\rho > 10$, then*

$$\int_{\Omega_R}\Big[2|\partial_d v|^2\chi(r) + |\nabla v|^2(2-q)\big(1-\chi(r)\big) + qk^2|v|^2\big(1-\chi(r)\big) - 2r|\partial_r v|^2\chi'(r)\Big]\,\mathrm{d}x$$
$$+ 2\Re\int_{\Omega_R}\bar{v}\nabla\alpha\cdot\nabla v\,\mathrm{d}x + 2\Re\int_{\Omega_R} x_d\partial_d\bar{v}\partial_r v\chi'(r)\,\mathrm{d}x$$
$$\geq \left(\frac{7}{24} - q - \nu\right)\int_{\Omega_R}|\partial_d v|^2\,\mathrm{d}x + \nu\int_{\Omega_R}|\nabla v|^2\big(1-\chi(r)\big)\,\mathrm{d}x$$
$$+ qk^2\int_{\Omega_R}|v|^2\big(1-\chi(r)\big)\,\mathrm{d}x - \frac{F(\rho)}{R_0^2}\int_{\Omega_R}|v|^2\,\mathrm{d}x, \tag{3.15}$$

for all $\nu > 0$, where

$$F(\rho) := 3\left(\frac{5(d-q+1)^2}{(\rho+1)(\rho-10)} + \frac{72}{(\rho-1)^2} + \frac{40\rho}{(\rho-1)^2(\rho-10)}\right). \tag{3.16}$$

Proof. The plan is to bound the two sign-indefinite terms on the left-hand side of (3.4) (on the second line) by the four sign-definite terms (on the first line).

By the Cauchy–Schwarz inequality and the inequality (2.12), for all $\varepsilon > 0$,

$$\left|2\Re\int_{\Omega_R} x_d \partial_d \bar{v} \partial_r v \chi'(r) \mathrm{d}x\right| \leq \epsilon \int_{\Omega_R} r|\partial_d v|^2 |\chi'(r)| \,\mathrm{d}x + \epsilon^{-1}\int_{\Omega_R} r|\partial_r v|^2|\chi'(r)| \,\mathrm{d}x.$$

From (3.14) we see that

$$2\nabla\alpha\cdot\nabla v = 2x_d\partial_d v\chi'(r)/r - \partial_r v\Big[(d-q+x_d^2/r^2)\chi'(r) + (r^2-x_d^2)\chi''(r)/r\Big]$$

so that, for any $\eta, \theta, \phi > 0$,

$$\begin{aligned}
&\left|2\Re\int_{\Omega_R} \bar{v}\nabla\alpha\cdot\nabla v \,\mathrm{d}x\right| \\
&\quad\leq 2\int_{\Omega_R} |v||\chi'(r)|\,|\partial_d v|\,\mathrm{d}x + \int_{\Omega_R}|v||\partial_r v|\Big[|d-q+1|\,|\chi'(r)| + r|\chi''(r)|\Big]\,\mathrm{d}x \\
&\quad\leq \eta\int_{\Omega_R}|v|^2|\chi'(r)|^2\,\mathrm{d}x + \eta^{-1}\int_{\Omega_R}|\partial_d v|^2\,\mathrm{d}x \\
&\qquad + \frac{\theta|d-1+q|}{2}\int_{\Omega_R}|v|^2 r^{-1}|\chi'(r)|\,\mathrm{d}x + \frac{|d-q+1|}{2\theta}\int_{\Omega_R} r|\partial_r v|^2|\chi'(r)|\,\mathrm{d}x \\
&\qquad + \frac{\phi}{2}\int_{\Omega_R} r|v|^2\frac{|\chi''(r)|^2}{|\chi'(r)|}\,\mathrm{d}x + \frac{1}{2\phi}\int_{\Omega_R} r|\partial_r v|^2|\chi'(r)|\,\mathrm{d}x
\end{aligned}$$

so

$$\begin{aligned}
&\left|2\Re\int_{\Omega_R} x_d\partial_d\bar{v}\partial_r v\chi'(r)\mathrm{d}x\right| + \left|2\Re\int_{\Omega_R}\bar{v}\nabla\alpha\cdot\nabla v\,\mathrm{d}x\right| \\
&\quad\leq \int_{\Omega_R}|\partial_d v|^2|\left(\epsilon r|\chi'(r)| + \eta^{-1}\right)\,\mathrm{d}x + \left(\frac{1}{\epsilon} + \frac{|d-q+1|}{2\theta} + \frac{1}{2\phi}\right)\int_{\Omega_R} r|\partial_r v|^2|\chi'(r)|\,\mathrm{d}x \\
&\quad + \int_{\Omega_R}|v|^2\left(|\chi'(r)|\frac{\theta|d-q+1|}{2r} + \eta|\chi'(r)|^2 + \frac{\phi}{2}r\frac{|\chi''(r)|^2}{|\chi'(r)|}\right)\,\mathrm{d}x. \qquad (3.17)
\end{aligned}$$

The plan is to choose θ, ϕ as functions of $d-q+1$ and ϵ so that

$$\left(\frac{1}{\epsilon} + \frac{|d-q+1|}{2\theta} + \frac{1}{2\phi}\right) = 2,$$

i.e. θ and ϕ are chosen to "use up" all of the $-2r|\partial_r v|^2\chi'(r)$ term in (3.4) (recalling that $\chi'(r) \leq 0$). We therefore choose

$$\frac{|d-q+1|}{2\theta} = \frac{1}{2\phi} = 1 - \frac{1}{2\epsilon}. \qquad (3.18)$$

We then need to choose ϵ and η so that: (i) $\epsilon^{-1} < 2$ (so (3.18) makes sense); and (ii) the coefficient of $|\partial_d v|^2$ in (3.17) is less than 2, i.e.

$$\left(\epsilon r|\chi'(r)| + \eta^{-1}\right) < 2. \qquad (3.19)$$

We now bound χ and its derivatives, both to proceed with (3.19) and to understand better the final term on the right-hand side of (3.17). The definition of ψ (1.7) implies that

$$\psi'(t) = -3(1-|t|)^2, \quad -1 \leq t \leq 1, \quad \psi''(t) = 6\,\mathrm{sign}(t)(1-|t|), \quad t \in [-1,0)\cup(0,1].$$

and then the definition of χ (1.9) implies that

$$|\chi'(r)| = \frac{-1}{R_1-R_0}\,\psi'\left(\frac{2r-(R_0+R_1)}{R_1-R_0}\right) \leq \frac{3}{R_1-R_0}, \qquad (3.20)$$

$$|\chi''(r)| = \frac{2}{(R_1-R_0)^2}\left|\psi''\left(\frac{2r-(R_0+R_1)}{R_1-R_0}\right)\right| \le \frac{12}{(R_1-R_0)^2}$$

and, for $R_0 < r < R_1$,

$$\frac{|\chi''(r)|^2}{|\chi'(r)|} = \frac{4}{(R_1-R_0)^3}\left[\psi''\left(\frac{2r-(R_0+R_1)}{R_1-R_0}\right)\right]^2\left|\psi'\left(\frac{2r-(R_0+R_1)}{R_1-R_0}\right)\right|^{-1} = \frac{48}{(R_1-R_0)^3}.$$

Furthermore, one can show that the maximum of $|\chi'(r)|/r$ occurs at $r = (R_0+R_1)/2$ (this follows from the fact that the maximum of $(1-|t|)^2/((R_1-R_0)t + R_0 + R_1)$ for $|t| \le 1$ occurs at $t = 0$).

Using these facts, we have that, for $R_0 < r < R_1$,

$$|\chi'(r)|\frac{\theta|d-q+1|}{2r} + \eta|\chi'(r)|^2 + \frac{\phi}{2}r\frac{|\chi''(r)|^2}{|\chi'(r)|} \le \frac{3\theta|d-q+1|}{(R_1^2-R_0^2)} + \frac{9\eta}{(R_1-R_0)^2} + \frac{24\phi R_1}{(R_1-R_0)^3} \tag{3.21}$$

and

$$\epsilon r|\chi'(r)| \le \frac{3\epsilon R_1}{R_1-R_0}.$$

This last bound shows that the condition (3.19) holds if

$$\frac{3\epsilon R_1}{R_1-R_0} = 5/3 \quad \text{and} \quad \eta = 24. \tag{3.22}$$

With this choice of ϵ and the definition $\rho = R_1/R_0$, we have that $\epsilon^{-1} = 9\rho/(5(\rho-1))$, which is less than 2 if $\rho/(\rho-1) < 10/9$, i.e., $\rho > 10$. Then the first two terms on the right-hand side of (3.17) satisfy

$$\int_{\Omega_R}|\partial_d v|^2\left(\epsilon r|\chi'(r)| + \eta^{-1}\right)\mathrm{d}x + \left(\frac{1}{\epsilon} + \frac{(d-q+1)}{2\theta} + \frac{1}{2\phi}\right)\int_{\Omega_R} r|\partial_r v|^2|\chi'(r)|\,\mathrm{d}x$$
$$\le \frac{41}{24}\int_{\Omega_R}|\partial_d v|^2\,\mathrm{d}x + 2\int_{\Omega_R} r|\partial_r v|^2|\chi'(r)|\,\mathrm{d}x. \tag{3.23}$$

The choices of θ and ϕ in (3.18) and the choice of ϵ in (3.22) imply that

$$\theta = \frac{5(\rho-1)(d-q+1)}{\rho-10}, \quad \phi = \frac{5(\rho-1)}{\rho-10}$$

and then using these in (3.21) we have that, for $R_0 < r < R_1$,

$$|\chi'(r)|\frac{\theta|d-q+1|}{2r} + \eta|\chi'(r)|^2 + \frac{\phi}{2}r\frac{|\chi''(r)|^2}{|\chi'(r)} \le \frac{1}{R_0^2}\left[\frac{3\theta(d-q+1)}{\rho^2-1} + \frac{9\eta}{(\rho-1)^2} + \frac{24\phi\rho}{(\rho-1)^3}\right] \le \frac{F(\rho)}{R_0^2}, \tag{3.24}$$

where $F(\rho)$ is defined by (3.16). Using this in (3.17) along with (3.23), we have

$$\left|2\Re\int_{\Omega_R} x_d\partial_d\bar{v}\partial_r v\chi'(r)\mathrm{d}x\right| + \left|2\Re\int_{\Omega_R}\bar{v}\nabla\alpha\cdot\nabla v\,\mathrm{d}x\right| \le \frac{41}{24}\int_{\Omega_R}|\partial_d v|^2\,\mathrm{d}x + 2\int_{\Omega_R} r|\partial_r v|^2|\chi'(r)|\,\mathrm{d}x$$
$$+ \frac{F(\rho)}{R_0^2}\int_{\Omega_R}|v|^2\,\mathrm{d}x. \tag{3.25}$$

Finally, to use (3.25) in (3.4) we need to create a $|\partial_d v|^2$ on the left-hand side of (3.4) via both the $|\partial_d v|^2\chi$ and $|\nabla v|^2$ terms. Indeed, for any $\nu > 0$,

$$2|\partial_d v|^2\chi + (2-q)|\nabla v|^2(1-\chi) = 2|\partial_d v|^2\chi + (2-q-\nu)|\nabla v|^2(1-\chi) + \nu|\nabla v|^2(1-\chi),$$
$$\ge (2-q-\nu)|\partial_d v|^2 + \nu|\nabla v|^2(1-\chi).$$

Using this along with (3.25) in (3.4), the result (3.15) follows. ■

Lemma 3.3 *Let Ω_- be an (R_0, R_1) obstacle, let $R > R_1$, and let $v \in H^1(\Omega_R)$ with $\gamma_+ v = 0$. Given $q > 0$, $p > 0$, $R_0 > 0$, and $\rho > \sqrt{13}$, if k is large enough so that*

$$2qk^2 R_0^2 \geq 9(F(\rho) + p)(\rho - 1)^3, \tag{3.26}$$

where $F(\rho)$ is defined by (3.16), then,

$$\begin{aligned}
&\left(\frac{7}{24} - q\right) \int_{\Omega_R} |\partial_d v|^2 \, dx + qk^2 \int_{\Omega_R} |v|^2 \big(1 - \chi(r)\big) \, dx - \frac{F(\rho)}{R_0^2} \int_{\Omega_R} |v|^2 \, dx \\
&\geq \left(\frac{7}{24} - q - 4p - 4F(\rho)\right) \int_{\Omega_R} |\partial_d v|^2 \, dx + \frac{qk^2}{2} \int_{\Omega_R} |v|^2 \big(1 - \chi(r)\big) \, dx + \frac{p}{R_0^2} \int_{\Omega_R} |v|^2 \, dx.
\end{aligned} \tag{3.27}$$

Proof. Since $\chi(r) \leq \chi(2R_0)$ for all $r > 2R_0$, the left-hand side of (3.27) is

$$\begin{aligned}
&\geq \left(\frac{7}{24} - q\right) \int_{\Omega_R} |\partial_d v|^2 \, dx + \frac{qk^2}{2}(1 - \chi(2R_0)) \int_{\Omega_R \setminus \Omega_{2R_0}} |v|^2 \, dx \\
&\quad + \frac{qk^2}{2} \int_{\Omega_R} |v|^2 \big(1 - \chi(r)\big) \, dx - \frac{F(\rho)}{R_0^2} \int_{\Omega_R} |v|^2 \, dx.
\end{aligned} \tag{3.28}$$

Given $p > 0$, the Friedrichs-type inequality (2.15) implies that, if $\rho > \sqrt{13}$,

$$\frac{(F(\rho) + p)}{R_0^2} \int_{\Omega_{2R_0}} |v|^2 \, dx \leq 8 \frac{(F(\rho) + p)}{R_0^2} \int_{\Omega_R \setminus \Omega_{2R_0}} |v|^2 \, dx + 4(F(\rho) + p) \int_{\Omega_R} |\partial_d v|^2 \, dx.$$

Using this in (3.28), we find that the left-hand side of (3.27) is

$$\begin{aligned}
&\geq \left(\frac{7}{24} - q - 4p - 4F(\rho)\right) \int_{\Omega_R} |\partial_d v|^2 \, dx + \left(\frac{qk^2}{2}(1 - \chi(2R_0)) - \frac{8(F(\rho) + p)}{R_0^2}\right) \int_{\Omega_R \setminus \Omega_{2R_0}} |v|^2 \, dx \\
&\quad + \frac{qk^2}{2} \int_{\Omega_R} |v|^2 \big(1 - \chi(r)\big) \, dx + \frac{F(\rho) + p}{R_0^2} \int_{\Omega_{2R_0}} |v|^2 \, dx - \frac{F(\rho)}{R_0^2} \int_{\Omega_R} |v|^2 \, dx \\
&\geq \left(\frac{7}{24} - q - 4p - 4F(\rho)\right) \int_{\Omega_R} |\partial_d v|^2 \, dx + \left(\frac{2qk^2}{(\rho - 1)^3} - \frac{8(F(\rho) + p)}{R_0^2}\right) \int_{\Omega_R \setminus \Omega_{2R_0}} |v|^2 \, dx \\
&\quad + \frac{qk^2}{2} \int_{\Omega_R} |v|^2 \big(1 - \chi(r)\big) \, dx + \frac{F(\rho) + p}{R_0^2} \int_{\Omega_{2R_0}} |v|^2 \, dx - \frac{F(\rho)}{R_0^2} \int_{\Omega_R} |v|^2 \, dx,
\end{aligned}$$

where we have used the fact that

$$1 - \chi(2R_0) = \frac{1}{2} - \frac{1}{2}\psi\left(\frac{3 - \rho}{\rho - 1}\right) = 4\left(\frac{1}{\rho - 1}\right)^3,$$

since $2 < (1 + \rho)/2$.

If (3.26) holds, then

$$\begin{aligned}
&\left(\frac{2qk^2}{(\rho - 1)^3} - \frac{8(F(\rho) + p)}{R_0^2}\right) \int_{\Omega_R \setminus \Omega_{2R_0}} |v|^2 \, dx + \frac{F(\rho) + p}{R_0^2} \int_{\Omega_{2R_0}} |v|^2 \, dx - \frac{F(\rho)}{R_0^2} \int_{\Omega_R} |v|^2 \, dx \\
&= \left(\frac{2qk^2}{(\rho - 1)^3} - \frac{(9F(\rho) + 8p)}{R_0^2}\right) \int_{\Omega_R \setminus \Omega_{2R_0}} |v|^2 \, dx + \frac{p}{R_0^2} \int_{\Omega_{2R_0}} |v|^2 \, dx \geq \frac{p}{R_0^2} \int_{\Omega_R} |v|^2 \, dx,
\end{aligned}$$

and the result (3.27) follows. ∎

Lemma 3.4 *Let Ω_- be an (R_0, R_1) obstacle with $\rho := R_1/R_0 \geq \rho_0$, where ρ_0 is the unique solution of $F(\rho) = 1/24$ in $(10, \infty)$, and F is defined by (3.16) with $q = 1/48$. Let u be the solution of the exterior Dirichlet problem described in Theorem 1.7. If $k > 0$ is large enough so that*

$$k^2 R_0^2 \geq \frac{45}{4}(\rho - 1)^3, \tag{3.29}$$

then, for all $R > R_1$,

$$\frac{1}{1+384k^2R_0^2}\int_{\Omega_R}\big(|\nabla u|^2+k^2|u|^2\big)\mathrm{d}x+\frac{1}{48}\int_{\Omega_R}|\partial_d u|^2\,\mathrm{d}x+\frac{1}{96}\int_{\Omega_R}\big(|\nabla u|^2+k^2|u|^2\big)\big(1-\chi(r)\big)\,\mathrm{d}x$$
$$\leq 48\Big(R_0^2\Big(2kR+d+\frac{31}{10}\Big)^2+R_0^2+R_1^2+2R^2\Big)\int_{\Omega_R}|f|^2\,\mathrm{d}x. \tag{3.30}$$

Remark 3.5 (Size of ρ_0) *It is clear from its definition (3.16) that $F(\rho)$ is decreasing on $(10,\infty)$ and that $F(\rho)\to 0$ as $\rho\to\infty$ and $F(\rho)\to\infty$ as $\rho\to 10^+$. Thus ρ_0 in the above lemma is well-defined, and straightforward calculations yield $109<\rho_0<110$ if $d=2$ and $120<\rho_0<121$ if $d=3$. Thus the above lemma holds (for both $d=2$ and 3) for $\rho\geq 121$. (Note that with $\rho=121$ the condition (3.29) becomes $kR_0\geq\sigma$, with $\sigma\approx 4409$.)*

Proof of Lemma 3.4. Combining (3.27) with (3.15) and (3.4), we have

$$\Big(\frac{7}{24}-q-\nu-4p-4F(\rho)\Big)\int_{\Omega_R}|\partial_d u|^2\,\mathrm{d}x+\nu\int_{\Omega_R}|\nabla u|^2\big(1-\chi(r)\big)\,\mathrm{d}x$$
$$+\frac{qk^2}{2}\int_{\Omega_R}|u|^2\big(1-\chi(r)\big)\,\mathrm{d}x+\frac{p}{R_0^2}\int_{\Omega_R}|u|^2\,\mathrm{d}x$$
$$\leq -2kR\Im\int_{\Omega_R}f\bar{u}\,\mathrm{d}x+\Re\int_{\Omega_R}f\big(2x_d\partial_d\bar{u}\chi(r)+2r\partial_r\bar{u}\big(1-\chi(r)\big)+2\alpha\bar{u}\big)\,\mathrm{d}x. \tag{3.31}$$

From the definition of α (3.5) and the expression (3.10), we have

$$2\alpha=d-q-(d-1-q)\chi(r)+(x_d^2-r^2)\chi'(r)/r\leq d-q+r|\chi'(r)|$$

and then, using the bound on χ' (3.20), the right-hand side of (3.31) is

$$\leq 2kR\int_{\Omega_R}|f||u|\,\mathrm{d}x+\int_{\Omega_R}|f|\Big(2R_1|\partial_d u|\chi(r)+2r|\partial_r u|\big(1-\chi(r)\big)+\big(d-q+r|\chi'(r)|\big)|u|\Big)\,\mathrm{d}x$$
$$\leq\Big(2kR+d-q+\frac{3\rho}{(\rho-1)}\Big)\int_{\Omega_R}|f||u|\,\mathrm{d}x+2R_1\int_{\Omega_R}|f|\,|\partial_d u|\,\mathrm{d}x+2R\int_{\Omega_R}|f|\,|\partial_r u|\big(1-\chi(r)\big)\,\mathrm{d}x. \tag{3.32}$$

We now choose $p=1/96$, $q=\nu=1/48$. We then take $\rho\geq\rho_0$ so that $F(\rho)\leq 1/24$. Observe that the condition (3.26) then becomes

$$k^2R_0^2\geq\frac{9}{4}\big(96F(\rho)+1\big)(\rho-1)^3,$$

which is ensured by (3.29).

The inequalities (3.31) and (3.32) then imply, since $\rho\geq\rho_0>109$ by Remark 3.5, that

$$\frac{1}{24}\int_{\Omega_R}|\partial_d u|^2\,\mathrm{d}x+\frac{1}{48}\int_{\Omega_R}|\nabla u|^2\big(1-\chi(r)\big)\,\mathrm{d}x+\frac{k^2}{96}\int_{\Omega_R}|u|^2\big(1-\chi(r)\big)\,\mathrm{d}x+\frac{1}{96R_0^2}\int_{\Omega_R}|u|^2\,\mathrm{d}x$$
$$\leq\Big(2kR+d+\frac{31}{10}\Big)\int_{\Omega_R}|f||u|\,\mathrm{d}x+2R_1\int_{\Omega_R}|f|\,|\partial_d u|\,\mathrm{d}x+2R\int_{\Omega_R}|f|\,|\partial_r u|\big(1-\chi(r)\big)\,\mathrm{d}x \tag{3.33}$$
$$\leq\frac{1}{192R_0^2}\int_{\Omega_R}|u|^2\,\mathrm{d}x+48R_0^2\Big(2kR+d+\frac{31}{10}\Big)^2\int_{\Omega_R}|f|^2\,\mathrm{d}x$$
$$+\frac{1}{48}\int_{\Omega_R}|\partial_d u|^2\,\mathrm{d}x+48R_1^2\int_{\Omega_R}|f|^2\,\mathrm{d}x+\frac{1}{96}\int_{\Omega_R}|\partial_r u|^2\big(1-\chi(r)\big)\,\mathrm{d}x+96R^2\int_{\Omega_R}|f|^2\,\mathrm{d}x,$$

so that

$$\frac{1}{192R_0^2}\int_{\Omega_R}|u|^2\,\mathrm{d}x+\frac{1}{48}\int_{\Omega_R}|\partial_d u|^2\,\mathrm{d}x+\frac{1}{96}\int_{\Omega_R}\big(|\nabla u|^2+k^2|u|^2\big)\big(1-\chi(r)\big)\,\mathrm{d}x$$

$$\leq 24R_0^2\left(2\big(2kR+d+\frac{31}{10}\big)^2+2\rho^2+4R^2/R_0^2\right)\int_{\Omega_R}|f|^2\,\mathrm{d}x,$$
$$= 48\left(R_0^2\left(2kR+d+\frac{31}{10}\right)^2+R_1^2+2R^2\right)\int_{\Omega_R}|f|^2\,\mathrm{d}x. \tag{3.34}$$

Now the inequality (2.11) implies that

$$\left(2+\frac{1}{2\varepsilon}\right)k^2\int_{\Omega_R}|u|^2\,\mathrm{d}x \geq \int_{\Omega_R}\left(|\nabla u|^2+k^2|u|^2\right)\mathrm{d}x-\frac{\varepsilon}{2k^2}\int_{\Omega_R}|f|^2\mathrm{d}x;$$

choosing $\varepsilon = 96k^2R_0^2$ it follows that

$$\frac{1}{192R_0^2}\int_{\Omega_R}|u|^2\,\mathrm{d}x \geq \frac{1}{1+384k^2R_0^2}\int_{\Omega_R}\left(|\nabla u|^2+k^2|u|^2\right)\mathrm{d}x-48R_0^2\int_{\Omega_R}|f|^2\mathrm{d}x. \tag{3.35}$$

Using this in (3.34) we obtain the result (3.30). ■

Proof of Theorem 1.7 from Lemma 3.4. For $\rho = R_1/R_0 \geq \rho_0$, the bound (3.30) implies that there exists an absolute constant $C > 0$ such that

$$\frac{1}{kR_0}\|u\|_{H_k^1(\Omega_R)}+\|\partial_d u\|_{L^2(\Omega_R)}+\|u\|_{H_k^1(\Omega_R;1-\chi)} \leq C\left(R_0\big(kR+d\big)+R_1+R\right)\|f\|_{L^2(\Omega_+)}, \tag{3.36}$$

for $k \geq k_1$ and $R > R_1$, where $k_1 > 0$ is given by (3.29), i.e. by $4k_1^2R_0^2 = 45(\rho-1)^3$. Clearly (3.36) implies that the same bound holds also for $R_\Gamma < R \leq R_1$, with C replaced by $2C$. Thus (1.11) holds for $k \geq k_1$. Given $k_0 > 0$, that the bound (1.11) holds for $k \in (k_0, k_1)$ follows by standard arguments; see Remark 4.2.

For the proof of the result when f is supported in $\Omega_R \setminus B_{R_0}$ and $\|f\|_{L^2(\Omega_+;(1-\chi)^{-1})} < \infty$, we use the Cauchy–Schwarz inequality and the inequality (2.12) in (3.33) to obtain that

$$\frac{1}{24}\int_{\Omega_R}|\partial_d u|^2\,\mathrm{d}x+\frac{1}{48}\int_{\Omega_R}|\nabla u|^2\big(1-\chi(r)\big)\,\mathrm{d}x+\frac{k^2}{96}\int_{\Omega_R}|u|^2\big(1-\chi(r)\big)\,\mathrm{d}x+\frac{1}{96R_0^2}\int_{\Omega_R}|u|^2\,\mathrm{d}x$$
$$\leq \frac{k^2}{288}\int_{\Omega_R}|u|^2\big(1-\chi(r)\big)\,\mathrm{d}x+\frac{72}{k^2}\left(2kR+d-\frac{1}{48}+\frac{3\rho}{(\rho-1)}\right)^2\int_{\Omega_R}|f|^2\big(1-\chi(r)\big)^{-1}\,\mathrm{d}x$$
$$+\frac{1}{48}\int_{\Omega_R}|\partial_d u|^2\,\mathrm{d}x+48R_1^2\int_{\Omega_R}|f|^2\,\mathrm{d}x+\frac{1}{72}\int_{\Omega_R}|\partial_r u|^2\big(1-\chi(r)\big)\,\mathrm{d}x+72R^2\int_{\Omega_R}|f|^2\,\mathrm{d}x,$$

so that, since $\rho \geq \rho_0 > 109$,

$$\frac{1}{48}\int_{\Omega_R}|\partial_d v|^2\,\mathrm{d}x+\frac{1}{144}\int_{\Omega_R}(|\nabla v|^2+k^2|v|^2)\big(1-\chi(r)\big)\,\mathrm{d}x+\frac{1}{96R_0^2}\int_{\Omega_R}|v|^2\,\mathrm{d}x$$
$$\leq \frac{72}{k^2}\left(2kR+d+\frac{31}{10}\right)^2\int_{\Omega_R}|f|^2\big(1-\chi(r)\big)^{-1}\,\mathrm{d}x+18(3R_1^2+4R^2)\int_{\Omega_R}|f|^2\,\mathrm{d}x.$$

Using (3.35), we then have

$$\frac{1}{48}\int_{\Omega_R}|\partial_d v|^2\,\mathrm{d}x+\frac{1}{144}\int_{\Omega_R}(|\nabla v|^2+k^2|v|^2)\big(1-\chi(r)\big)\,\mathrm{d}x+\frac{2}{1+384k^2R_0^2}\int_{\Omega_R}(|\nabla v|^2+k^2|v|^2)\,\mathrm{d}x$$
$$\leq \frac{72}{k^2}\left(2kR+d+\frac{31}{10}\right)^2\int_{\Omega_R}|f|^2\big(1-\chi(r)\big)^{-1}\,\mathrm{d}x+18\left(6R_0^2+3R_1^2+4R^2\right)\int_{\Omega_R}|f|^2\,\mathrm{d}x. \tag{3.37}$$

Again this holds provided $\rho \geq \rho_0$ and (3.29) holds, i.e. provided $k \geq k_1$. The bound (3.37) implies that there exists an absolute constant $C > 0$ such that

$$\frac{1}{kR_0}\|v\|_{H_k^1(\Omega_R)}+\|\partial_d v\|_{L^2(\Omega_R)}+\|v\|_{H_k^1(\Omega_R;1-\chi)} \leq C\left(R+\frac{d}{k}+R_1\right)\|f\|_{L^2(\Omega_+;(1-\chi)^{-1})},$$

for $k \geq k_1$ and $R > R_\Gamma$, and the result follows. ■

4 Proof of Theorem 1.8 on the exterior DtN map

Definition 4.1 (K resolvent estimate) *For $K \in C[0,\infty)$, with $K(k) \geq 1$ for $k > 0$, we say that Ω_+* satisfies a K resolvent estimate *if, whenever $u \in H^1_{\mathrm{loc}}(\Omega_+)$ satisfies the radiation condition (1.10), the boundary condition $\gamma_+ u = 0$, and the Helmholtz equation $\Delta u + k^2 u = -f$ in Ω_+, with $f \in L^2(\Omega_+)$ compactly supported, it holds for all $R > \max_{x\in\Gamma\cup\mathrm{supp}(f)} |x|$ that*

$$\|u\|_{H^1_k(\Omega_R)} \lesssim K(k)\|f\|_{L^2(\Omega_+)}, \quad \textit{for } k > 0. \tag{4.1}$$

Remark 4.2 *To show that Ω_+ satisfies a K resolvent estimate it is enough to show that (4.1) holds for all sufficiently large k. For, as observed at the end of §5 below in (5.7) and Lemma 5.1, the bound (4.1) holds for every Ω_+ for all sufficiently small $k > 0$. Further, for every $k_0 > 0$, it then follows, by continuity arguments and well-posedness at every fixed $k > 0$, that (4.1) holds for $0 < k \leq k_0$. (Concretely, one route to carrying out these latter arguments is to note that the inf-sup constant β_R, given by (5.4) below, is positive for each fixed k and depends continuously on k, and then apply Lemma 5.1.)*

The bounds (1.16) and (1.17) in Theorem 1.8 will follow from Theorem 1.7 combined with the following lemma; this lemma encapsulates the method laid out in [6, §3] for deriving wavenumber-explicit bounds on the exterior DtN map from resolvent estimates in the exterior domain.

Lemma 4.3 (From resolvent estimates to DtN map bounds) *Suppose that Ω_+ satisfies a K resolvent estimate, for some $K \in C[0,\infty)$ with $K(k) \geq 1$ for $k > 0$. Then, whenever $u \in H^1_{\mathrm{loc}}(\Omega_+)$ satisfies the radiation condition (1.10) and $\Delta u + k^2 u = 0$ in Ω_+, it holds for all $R > R_\Gamma$ that, given $k_0 > 0$,*

$$\|u\|_{H^1_k(\Omega_R)} + \|\partial_n^+ u\|_{L^2(\Gamma)} \lesssim K(k)\,\|g\|_{H^1_k(\Gamma)}, \quad \textit{for } k \geq k_0, \tag{4.2}$$

provided $g := \gamma_+ u \in H^1(\Gamma)$. Moreover, for $k \geq k_0$,

$$\|\partial_n^+ u\|_{H^{s-1}_k(\Gamma)} \lesssim K(k)\|g\|_{H^s_k(\Gamma)} \quad \textit{and} \quad \|\partial_n^+ u\|_{H^{s-1}(\Gamma)} \lesssim kK(k)\|g\|_{H^s(\Gamma)}, \tag{4.3}$$

uniformly for $0 \leq s \leq 1$, assuming, in the case $s > 1/2$, that $g \in H^s(\Gamma)$.

Proof. We sketch the proof, which is essentially contained in [6, §3]. Suppose that Ω_+ satisfies a K resolvent estimate, with the given conditions on K, and that $u \in H^1_{\mathrm{loc}}(\Omega_+)$ satisfies (1.10), $\Delta u + k^2 u = 0$ in Ω_+, and $g := \gamma_+ u \in H^1(\Gamma)$. Let $w \in H^1(\Omega_+)$ satisfy $\Delta w + (k^2 + \mathrm{i}k)w = 0$ in Ω_+ and the boundary condition $\gamma_+ w = g$. Green's identity can then be used to show that, given $k_0 > 0$,

$$\|w\|_{H^1_k(\Omega_+)} \lesssim \|g\|_{H^1_k(\Gamma)}, \quad \text{for } k \geq k_0 \tag{4.4}$$

[6, Lemma 3.3]. Choose $\psi \in C_0^\infty(\mathbb{R}^d)$ that is equal to one on Ω_-, and define $v := u - \psi w$. Then $v \in H^1_{\mathrm{loc}}(\Omega_+)$ satisfies (1.10), $\gamma_+ v = 0$, and $\Delta v + k^2 v = h := \mathrm{i}k\psi w - w\Delta\psi - 2\nabla w \cdot \nabla\psi \in L^2(\Gamma)$, and h is compactly supported. Thus, for all $R > R_\Gamma$,

$$\|v\|_{H^1_k(\Omega_R)} \lesssim K(k)\|h\|_{L^2(\Omega_+)} \lesssim K(k)\|w\|_{H^1_k(\Omega_+)}, \tag{4.5}$$

for $k \geq k_0$. Combining (4.4) and (4.5) we see that, for all $R > R_\Gamma$,

$$\|u\|_{H^1_k(\Omega_R)} \lesssim K(k)\|g\|_{H^1_k(\Gamma)}, \quad \text{for } k \geq k_0.$$

That $\|\partial_n^+ u\|_{L^2(\Gamma)}$ is also bounded by the right hand side of this last equation follows from [6, Lemma 2.3] (essentially Nečas' regularity result [78, §5.1.2], [67, Theorem 4.24(ii)], proved using a Rellich identity). Using the notation $P^+_{DtN} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ to denote the DtN map for the exterior domain Ω_+, we see equation (4.2) implies that

$$\|P^+_{DtN}\|_{H^1_k(\Gamma)\to L^2(\Gamma)} \lesssim K(k) \quad \text{so} \quad \|P^+_{DtN}\|_{H^1(\Gamma)\to L^2(\Gamma)} \lesssim kK(k),$$

for $k \geq k_0$. It is well known (e.g., [22, Theorem 2.31]) that P^+_{DtN} can be extended uniquely to a bounded mapping from $H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$ for $|s| \leq 1/2$. Since P^+_{DtN} is self-adjoint with respect to the real inner product $(\cdot,\cdot)^r_\Gamma$ (see [22, Section 2.7]), (4.3) follows from (2.20) (cf. [86, Lemma 2.3]). ■

Remark 4.4 (Previous uses of the arguments in Lemma 4.3) *The method in Lemma 4.3 is a sharpening of arguments used to obtain bounds on the DtN map from resolvent estimates in [56, 86], with this type of argument going back at least to [57, §5]. Indeed, in [56, 86] the equation* $\Delta w + (k^2 + \mathrm{i}k)w = 0$ *in the proof of Lemma 4.3 below is replaced by* $\Delta w - k^2 w = 0$*, losing a factor* k *in the final estimates.*

To prove the last part of Theorem 1.8, namely the bound (1.18), we use the interior elliptic regularity estimate that if, for some $x \in \mathbb{R}^d$ and $\varepsilon > 0$, $v \in C^2(B_\epsilon(x))$, $\Delta v = f$ in $B_\varepsilon(x)$ and $v, f \in L^\infty(B_\varepsilon(x))$, then [45, Theorem 3.9], for some constant $C_d > 0$ that depends only on d,

$$|\nabla v(x)| \leq \frac{C_d}{\varepsilon}\left(\|v\|_{L^\infty(B_\varepsilon)} + \varepsilon^2 \|f\|_{L^\infty(B_d(x))}\right).$$

In the particular case that $f = -k^2 v$, so that $\Delta v + k^2 v = 0$, this estimate is

$$|\nabla v(x)| \leq C_d \,\frac{(1 + k^2\varepsilon^2)}{\varepsilon}\, \|u\|_{L^\infty(B_\varepsilon(x))}. \tag{4.6}$$

Proof of Theorem 1.8. The bounds (1.16) and (1.17) follow immediately from Lemma 4.3 and Theorem 1.7, which shows (under the conditions on Ω_+ and R_1/R_0, and taking into account Remark 4.2) that Ω_+ satisfies a K resolvent estimate with $K(k) = 1 + k^2$. The bound (1.16) and (4.6) imply a version of (1.18), but with k^2 replaced by k^3. To show the sharper bound (1.18), choose $\psi \in C_0^\infty(\mathbb{R}^d)$ supported in G that is equal to one on $\Omega_- \cup B_{R'}$, for some $R' > R_0$, and let $w := u + \psi u^i$. Then w satisfies (1.10), $\gamma_+ w = 0$, and $\Delta w + k^2 w = h := 2\nabla u^i \cdot \nabla\psi + u^i \Delta\psi$ in Ω_+. Further, $h \in L^2(\Omega_+)$ is compactly supported, $h = 0$ in $B_{R'} \cap \Omega_+$, and, applying (4.6) with $\varepsilon = \min(\epsilon, k^{-1})$ for some sufficiently small ϵ, we see that

$$\|h\|_{L^2(\Omega_+)} \lesssim (1 + k) \max_{x\in G} |u^i(x)|, \quad \text{for } k > 0.$$

It follows from Theorem 1.7 (see (1.14)) that (1.18) holds. ∎

Using Lemma 4.3 we can derive other bounds on the exterior DtN map that apply to classes of trapping domains, using the two other trapping resolvent estimates in the literature, which we discussed in §1.1.

Corollary 4.5 (Worst case bounds on the DtN map) *Let* $u \in H^1_{\mathrm{loc}}(\Omega_+)$ *be a solution to the Helmholtz equation* $\Delta u + k^2 u = 0$ *in* Ω_+ *that satisfies* (1.10) *and* $\gamma_+ u = g$*. If* Ω_+ *is* C^∞ *there exists* $\alpha > 0$ *such that, given* $k_0 > 0$,

$$\|\partial_n^+ u\|_{L^2(\Gamma)} \lesssim \exp(\alpha k)\, \|g\|_{H^1_k(\Gamma)}, \tag{4.7}$$

for all $k \geq k_0$ *if* $g \in H^1(\Gamma)$*. In fact, for* $k \geq k_0$,

$$\|\partial_n^+ u\|_{H_k^{s-1}(\Gamma)} \lesssim \exp(\alpha k)\|g\|_{H_k^s(\Gamma)} \quad \textit{and} \quad \|\partial_n^+ u\|_{H^{s-1}(\Gamma)} \lesssim k\exp(\alpha k)\|g\|_{H^s(\Gamma)}, \tag{4.8}$$

uniformly for $0 \leq s \leq 1$*, assuming, in the case* $s > 1/2$*, that* $g \in H^s(\Gamma)$*.*

The second resolvent estimate, developed by Ikawa [48, 50] and Burq [16], is for mild, hyperbolic trapping, where Ω_- is an *Ikawa-like union of convex obstacles* in the following sense.

Definition 4.6 (Ikawa-like union of convex obstacles [50, 16]) *We say that* Ω_- *is an* Ikawa-like union of convex obstacles *if:*

(i) for some $M \in \mathbb{N}$, $\overline{\Omega_-} = \bigcup_{i=1}^N \Theta_i$*, where* $\Theta_1, ..., \Theta_N \subset \mathbb{R}^d$ *are disjoint compact* C^∞ *strictly convex sets with* $\kappa > 0$*, where* κ *is the infimum of the principal curvatures of the boundaries of the obstacles* Θ_i*;*

(ii) for $1 \le i, j, \ell \le N$, $i \neq j$, $j \neq \ell$, $\ell \neq i$,

$$\text{Convex hull}(\Theta_i \cup \Theta_j) \cap \Theta_\ell = \emptyset;$$

(iii) if $N > 2$, $\kappa L > N$, where L denotes the minimum of the distances between pairs of obstacles.

Corollary 4.7 (DtN map for Ikawa-like union of convex obstacles) *Let $u \in H^1_{\rm loc}(\Omega_+)$ be a solution to the Helmholtz equation $\Delta u + k^2 u = 0$ in Ω_+ that satisfies (1.10) and $\gamma_+ u = g$. If Ω_- is an Ikawa-like union of convex obstacles then, given $k_0 > 0$,*

$$\|\partial_n^+ u\|_{L^2(\Gamma)} \lesssim \log(2+k)\, \|g\|_{H^1_k(\Gamma)}, \tag{4.9}$$

for all $k \ge k_0$ if $g \in H^1(\Gamma)$. In fact, for $k \ge k_0$,

$$\|\partial_n^+ u\|_{H^{s-1}_k(\Gamma)} \lesssim \log(2+k)\|g\|_{H^s_k(\Gamma)} \quad \text{and} \quad \|\partial_n^+ u\|_{H^{s-1}(\Gamma)} \lesssim k\log(2+k)\|g\|_{H^s(\Gamma)}, \tag{4.10}$$

uniformly for $0 \le s \le 1$, assuming, in the case $s > 1/2$, that $g \in H^s(\Gamma)$.

5 Proof of Corollary 1.10 on the inf-sup constant

Since Ω_+ is unbounded, standard FEMs cannot be applied directly to the exterior Dirichlet problem. A standard fix is to reformulate the exterior Dirichlet problem as a variational problem in the truncated domain Ω_R, for some $R > R_\Gamma$. The effect of the rest of Ω_+, i.e. of $\Omega_R^+ := \Omega_+ \setminus \overline{\Omega_R}$, is replaced by the exact DtN map on Γ_R for Ω_R^+, abbreviated as P_R^+ (our notation as in Corollary 1.10). As Ω_R^+ is a geometry in which the Helmholtz equation separates, the action of P_R^+ can be computed analytically (e.g. [27, Equations (3.5)–(3.6)]).

Given $f \in L^2(\Omega_+)$ with compact support, consider the problem of finding $u \in H^1_{\rm loc}(\Omega_+)$ such that u satisfies the radiation condition (1.10), the Helmholtz equation $\Delta u + k^2 u = -f$ in Ω_+, and $\gamma_+ u = 0$ on Γ. It is well-known that a variational formulation in Ω_R can be obtained by multiplying the Helmholtz equation by a test function $v_R \in V_R$, integrating by parts, and applying the boundary condition $\gamma_+ u = 0$. In particular (e.g., [79]), if the support of f lies in Ω_R, u satisfies this BVP in Ω_+ if and only if $u_R := u|_{\Omega_R} \in V_R$ and

$$a(u_R, v_R) = G(v_R), \quad \text{for all } v_R \in V_R, \tag{5.1}$$

where $a(\cdot,\cdot)$ is defined in (1.22) and

$$G(v) := \int_{\Omega_R} \bar{v} f \,{\rm d}x, \quad \text{for } v \in V_R. \tag{5.2}$$

The following lemma is proved as [27, Lemmas 3.3, 3.4].

Lemma 5.1 (Link between resolvent estimates and bounds on the inf-sup constant) *Suppose that $R > R_\Gamma$, $L > 0$, $k > 0$, and that*

$$\|u\|_{H^1_k(\Omega_R)} \le L \|f\|_{L^2(\Omega_+)}, \tag{5.3}$$

for all $f \in L^2(\Omega_+)$ supported in Ω_R, where $u \in H^1_{\rm loc}(\Omega_+)$ is the solution of $\Delta u + k^2 u = -f$ in Ω_+ that satisfies (1.10) and $\gamma_+ u = 0$. Then

$$\beta_R := \inf_{0 \neq u \in V_R} \sup_{0 \neq v \in V_R} \frac{|a(u,v)|}{\|u\|_{H^1_k(\Omega_R)} \|v\|_{H^1_k(\Omega_R)}} \ge \alpha, \tag{5.4}$$

where $\alpha = (1 + 2kL)^{-1}$. Conversely, if (5.4) holds for some $\alpha > 0$, then (5.3) holds for all $f \in L^2(\Omega_+)$ supported in Ω_R, with $L = \alpha^{-1} \min(k^{-1}, c_R)$, where

$$c_R := \sup_{0 \neq v \in V_R} \frac{\|v\|_{L^2(\Omega_R)}}{\|\nabla v\|_{L^2(\Omega_R)}}. \tag{5.5}$$

Corollary 1.10 follows immediately from Theorem 1.7 and Lemma 5.1.

We remark also that (see [79] or [27, Lemma 2.1])

$$\beta_R \geq \inf_{0\neq v\in V_R} \frac{\Re(a(u,v))}{\|v\|^2_{H^1_k(\Omega_R)}} \geq \inf_{0\neq v\in V_R} \frac{\int_{\Omega_R}(|\nabla v|^2 - k^2|v|^2)\mathrm{d}x}{\int_{\Omega_R}(|\nabla v|^2 + k^2|v|^2)\mathrm{d}x} \geq \frac{1-k^2c_R^2}{1+k^2c_R^2}. \tag{5.6}$$

This, combined with Lemma 5.1, shows that, if $kc_R < 1$, (5.3) holds for all $f \in L^2(\Omega_+)$ supported in Ω_R, with

$$L = c_R \frac{1+k^2c_R^2}{1-k^2c_R^2}. \tag{5.7}$$

Remark 5.2 (Bound on β_R^{-1} from a K resolvent estimate) *In the language of Definition 4.1, Lemma 5.1 tells us that Ω_+ satisfies a K resolvent estimate (with K satisfying the conditions of Definition 4.1) if and only if the inf-sup constant satisfies*

$$\beta_R^{-1} \lesssim (1+k)K(k), \quad \text{for } k>0, \tag{5.8}$$

for all $R > R_\Gamma$. Table 6.1 lists the known resolvent estimates for scattering by an obstacle, as well as the bounds β_R^{-1} that follows from these.

Remark 5.3 (Upper bound on β_R) *The simple constructions in [27, Lemma 3.10] (see also [86, Lemma 4.12]) show that for every Ω_+ and every $R > R_\Gamma$,*

$$\beta_R \lesssim (1+k)^{-1}, \quad \text{for } k>0; \tag{5.9}$$

and the nontrapping resolvent estimate combined with (5.8) shows that this is sharp.

6 Combined-potential integral equation formulations and the proof of Corollary 1.11.

Integral equation methods are widely used for both the theoretical analysis and the numerical solution of direct and inverse acoustic scattering problems (e.g., [30, 31, 22]). In this section we recall the standard integral equation formulations for the exterior Dirichlet problem, and derive new wavenumber-explicit bounds in the case when Ω_- is trapping, combining the resolvent and DtN estimates in Theorems 1.7 and 1.8 (proved in Sections 3 and 4) with the sharp bounds for the interior impedance problem recently obtained in [6].

6.1 Integral equations for the exterior Dirichlet problem

If u is a solution of $\Delta u + k^2 u = 0$ in Ω_+ that satisfies the radiation condition (1.10) then Green's representation theorem (see, e.g., [22, Theorem 2.21]) gives

$$u(x) = -\int_\Gamma \Phi_k(x,y)\partial_n^+ u(y)\,\mathrm{d}s(y) + \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n(y)}\gamma_+ u(y)\,\mathrm{d}s(y), \quad x\in\Omega_+, \tag{6.1}$$

where $\Phi_k(x,y)$ is the fundamental solution of the Helmholtz equation given by

$$\Phi_k(x,y) := \frac{\mathrm{i}}{4}\left(\frac{k}{2\pi|x-y|}\right)^{(d-2)/2} H^{(1)}_{(d-2)/2}\big(k|x-y|\big) = \begin{cases} \dfrac{\mathrm{i}}{4}H_0^{(1)}\big(k|x-y|\big), & d=2,\\ \dfrac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, & d=3,\end{cases} \tag{6.2}$$

where $H_\nu^{(1)}$ denotes the Hankel function of the first kind of order ν. Taking the exterior Dirichlet and Neumann traces of (6.1) on Γ and using the jump relations for the single- and double-layer potentials (e.g. [22, Equation 2.41]) we obtain the integral equations

$$S_k\partial_n^+ u = \left(-\frac{1}{2}I + D_k\right)\gamma_+ u \quad \text{and} \quad \left(\frac{1}{2}I + D'_k\right)\partial_n^+ u = H_k\gamma_+ u, \tag{6.3}$$

where S_k, D_k are the single- and double-layer operators, D'_k is the adjoint double-layer operator, and H_k is the hypersingular operator. These four integral operators are defined for $\phi \in L^2(\Gamma)$, $\psi \in H^1(\Gamma)$, and almost all $x \in \Gamma$ by

$$S_k\phi(x) := \int_\Gamma \Phi_k(x,y)\phi(y)\,\mathrm{d}s(y), \qquad D_k\phi(x) := \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n(y)}\phi(y)\,\mathrm{d}s(y), \tag{6.4}$$

$$D'_k\phi(x) := \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n(x)}\phi(y)\,\mathrm{d}s(y), \quad H_k\psi(x) := \frac{\partial}{\partial n(x)}\int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n(y)}\psi(y)\,\mathrm{d}s(y). \tag{6.5}$$

When Γ is Lipschitz, the integrals defining D_k and D'_k must be understood as Cauchy principal value integrals and even when Γ is smooth there are subtleties in defining $H_k\psi$ for $\psi \in H^1(\Gamma)$ which we ignore here (see, e.g., [22, §2.3]).

For the exterior Dirichlet problem, the integral equations (6.3) are both equations for the unknown Neumann trace $\partial_n^+ u$. However the first of these equations is not uniquely solvable when $-k^2$ is a Dirichlet eigenvalue of the Laplacian in Ω_-, and the second is not uniquely solvable when $-k^2$ is a Neumann eigenvalue of the Laplacian in Ω_-; see, e.g., [22, Theorem 2.25].

One standard way to resolve this difficulty (going back to the work of [18]) is to take a linear combination of the two equations, which yields the integral equation

$$A'_{k,\eta}\partial_n^+ u = B_{k,\eta}\gamma_+ u \tag{6.6}$$

where

$$A'_{k,\eta} := \frac{1}{2}I + D'_k - \mathrm{i}\eta S_k \quad \text{and} \quad B_{k,\eta} := H_k + \mathrm{i}\eta\left(\frac{1}{2}I - D_k\right). \tag{6.7}$$

If $\eta \in \mathbb{R}\setminus\{0\}$ then the integral operator $A'_{k,\eta}$ is invertible (on appropriate Sobolev spaces) and so (6.6) can be used to solve the exterior Dirichlet problem for all $k > 0$. Indeed, if $\eta \in \mathbb{R}\setminus\{0\}$ then $A'_{k,\eta}$ is a bounded invertible operator from $H^s(\Gamma)$ to itself for $-1 \leq s \leq 0$; [22, Theorem 2.27].

An alternative resolution (proposed essentially simultaneously by [12, 60, 80]) is to work with a so-called *indirect* formulation, looking for a solution to the exterior Dirichlet problem as the *combined double- and single-layer potential*

$$u(x) = \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n(y)}\phi(y)\,\mathrm{d}s(y) - \mathrm{i}\eta\int_\Gamma \Phi_k(x,y)\phi(y)\,\mathrm{d}s(y), \quad x \in \Omega_+,$$

for some $\phi \in H^{1/2}(\Gamma)$ and $\eta \in \mathbb{R}\setminus\{0\}$. It follows from the jump relations [22, Equation 2.41] that this ansatz satisfies the exterior Dirichlet problem with Dirchlet data $h = \gamma_+ u \in H^{1/2}(\Gamma)$ if and only if

$$A_{k,\eta}\phi = h, \tag{6.8}$$

where

$$A_{k,\eta} := \frac{1}{2}I + D_k - \mathrm{i}\eta S_k. \tag{6.9}$$

If $\eta \in \mathbb{R}\setminus\{0\}$ then $A_{k,\eta}$ is a bounded invertible operator from $H^s(\Gamma)$ to itself for $0 \leq s \leq 1$; [22, Theorem 2.27]. The operators $A'_{k,\eta}$ and $A_{k,\eta}$ are closely related in that $A'_{k,\eta}$ is the adjoint of $A_{k,\eta}$ with respect to the real L^2 inner product on Γ, i.e. $(A_{k,\eta}\phi,\psi)^r_\Gamma = (\phi, A'_{k,\eta}\psi)^r_\Gamma$, for all $\phi,\psi \in L^2(\Gamma)$. Thus, by (2.19),

$$\|(A'_{k,\eta})^{-1}\|_{H_k^{-s}(\Gamma)\to H_k^{-s}(\Gamma)} = \|A^{-1}_{k,\eta}\|_{H_k^{s}(\Gamma)\to H_k^{s}(\Gamma)} \quad \text{and} \tag{6.10}$$

$$\|(A'_{k,\eta})^{-1}\|_{H^{-s}(\Gamma)\to H^{-s}(\Gamma)} = \|A^{-1}_{k,\eta}\|_{H^{s}(\Gamma)\to H^{s}(\Gamma)} \quad \text{for } 0 \leq s \leq 1. \tag{6.11}$$

For the general exterior Dirichlet problem it is natural to pose Dirichlet data in $H^{1/2}(\Gamma)$ (since $\gamma_+ u \in H^{1/2}(\Gamma)$). The mapping properties of H_k and D_k (see [22, Theorems 2.17, 2.18]) imply that $B_{k,\eta} : H^{s+1}(\Gamma) \to H^s(\Gamma)$ for $-1 \leq s \leq 0$, and thus $B_{k,\eta}\gamma_+ u \in H^{-1/2}(\Gamma)$. Thus, for Dirichlet data in $H^{1/2}(\Gamma)$, the invertibility of $A'_{k,\eta}$ on $H^{-1/2}(\Gamma)$ is particularly relevant and, for the solution of (6.8), the invertibility of $A_{k,\eta}$ on $H^{1/2}(\Gamma)$. The major application of (6.6), however, is the solution of problems of sound soft acoustic scattering (see [22, Definition 2.11, Theorem 2.46]), in which u is interpreted as the *scattered field* corresponding to an *incident field* u^i that satisfies $\Delta u^i + k^2 u^i = 0$

in some neighbourhood G of $\overline{\Omega_-}$, and here the Dirichlet data $\gamma_+ u = -u^i|_\Gamma \in H^1(\Gamma)$ is smoother, so that $B_{k,\eta}\gamma_+ u \in L^2(\Gamma)$. Indeed, in this case [22, Theorem 2.46], the unknown $\partial_n^+ u^t$ satisfies the integral equation,

$$A'_{k,\eta}\partial_n^+ u^t = f_{k,\eta} := \partial_n^+ u^i - \mathrm{i}\eta\gamma_+ u^i \in L^2(\Gamma). \tag{6.12}$$

where $u^t := u + u^i$ is the so-called *total field* satisfying $\gamma_+ u^t = 0$. Therefore, in applications to acoustic scattering, the invertibility of $A'_{k,\eta}$ on $L^2(\Gamma)$ is key. $L^2(\Gamma)$ is also a natural function space setting for implementation and analysis of Galerkin numerical methods for the solution of the direct equations (6.6) and (6.12), and the indirect equation (6.8) (e.g., [61, 24, 46, 37, 41] and recall the discussion in §1.5.3).

6.2 Inverses of the combined-field operators in terms of the exterior DtN and the interior impedance to Dirichlet maps

We introduced in the proof of Lemma 4.3 the notation P^+_{DtN} for the exterior DtN map. Similarly, for the Lipschitz open set Ω_-, let $P^{-,\eta}_{ItD} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ denote the interior impedance-to-Dirichlet map, that takes impedance data $g \in H^{-1/2}(\Gamma)$ to $\gamma_- u \in H^{1/2}(\Gamma)$, where u is the solution of $\Delta u + k^2 u = 0$ in Ω_- that satisfies the impedance boundary condition (6.14) below. $P^{-,\eta}_{ItD}$ extends uniquely to a bounded mapping from $H^s(\Gamma) \to H^{s+1}(\Gamma)$ for $-1 \le s \le 0$ (see [22, Theorem 2.32]).

The inverse of $A'_{k,\eta}$ can be written in terms of P^+_{DtN} and $P^{-,\eta}_{ItD}$ as

$$(A'_{k,\eta})^{-1} = I - P^+_{DtN}P^{-,\eta}_{ItD} + \mathrm{i}\eta P^{-,\eta}_{ItD} \tag{6.13}$$

[22, Theorem 2.33]. The fact that $P^{-,\eta}_{ItD}$, as well as P^+_{DtN}, appears in this formula is because a boundary integral equation formulation of the interior impedance problem leads to the same operator $A'_{k,\eta}$; see [22, Theorem 2.38]. To use (6.13) to bound $(A'_{k,\eta})^{-1}$ one therefore needs bounds on the exterior DtN map, provided for (R_0, R_1) obstacles in Theorem 1.8, but also bounds on the interior impedance to Dirichlet map.

The known bounds on the interior impedance to Dirichlet map are summarised in the following theorem.

Theorem 6.1 *Let Ω_- be a bounded Lipschitz domain. Given $g \in L^2(\Gamma)$ let $u \in H^1(\Omega_-)$ denote the solution to $\Delta u + k^2 u = 0$ in Ω_- that satisfies*

$$\partial_n^- u - \mathrm{i}\eta\gamma_- u = g \quad \text{on } \Gamma, \tag{6.14}$$

with $\eta = ck$, for some $c \in \mathbb{R} \setminus \{0\}$. Then, for $k \ge 0$,

$$\|\partial_n^- u\|_{L^2(\Gamma)} + k\|\gamma_- u\|_{L^2(\Gamma)} \lesssim \|g\|_{L^2(\Gamma)}. \tag{6.15}$$

If Ω_- is either star-shaped with respect to a ball or C^∞, then

$$\|\nabla_S(\gamma_- u)\|_{L^2(\Gamma)} \lesssim \|g\|_{L^2(\Gamma)}. \tag{6.16}$$

If Ω_- is piecewise smooth, then, given $k_0 > 0$, the bound (6.16) holds for $k \ge k_0$, but with $\|g\|_{L^2(\Gamma)}$ replaced by $k^{1/4}\|g\|_{L^2(\Gamma)}$. In the general Lipschitz case (6.16) holds for $k \ge k_0$ with $\|g\|_{L^2(\Gamma)}$ replaced by $k^{1/2}\|g\|_{L^2(\Gamma)}$.

References for the proof of Theorem 6.1. The sharp bound (6.15) is an easy energy estimate from Green's theorem (e.g., [86, Lemma 4.2]). The sharp wavenumber-explicit bound (6.16) for the impedance to Dirichlet map for the case where Ω_- is star-shaped with respect to a ball is given by [74, Equation 3.12]; note that this bound also follows from the sharp bound on the $H^1_k(\Omega_-)$ norm of the solution proved in [68, Proposition 8.1.4] and [32, Theorem 1] combined with the Rellich-identity argument of Nečas [78, §5.1.2]. The bound (6.16) when Ω_- is C^∞ is [6, Corollary 1.9], obtained using results on exponential decay of the energy of solutions of the wave equation with damped boundary conditions [5]. The weaker bounds when Ω_- is just Lipschitz, or is Lipschitz and piecewise smooth, are [86, Lemma 4.6], obtained by combining (6.15) with wavenumber-explicit estimates for single and double-layer potentials from [86] in the general Lipschitz case, from [47] in the piecewise smooth case. ■

Corollary 6.2 *If Ω_- is either star-shaped with respect to a ball or C^∞ then*

$$\|P^{-,\eta}_{ItD}\|_{H^s_k(\Gamma)\to H^{s+1}_k(\Gamma)} \lesssim 1, \quad \textit{for } k \geq 0, \tag{6.17}$$

uniformly for $-1 \leq s \leq 0$, and

$$\|P^{-,\eta}_{ItD}\|_{H^s_k(\Gamma)\to H^s_k(\Gamma)} \lesssim k^{-1}, \quad \textit{for } k > 0, \tag{6.18}$$

uniformly for $0 \leq s \leq 1$. Equations (6.17) and (6.18) hold, but with 1 and k^{-1} replaced by $k^{1/4}$ and $k^{-3/4}$, respectively, if Ω_- is merely piecewise smooth, and replaced by $k^{1/2}$ and $k^{-1/2}$, respectively, if Ω_- is merely Lipschitz.

Proof. Equations (6.15) and (6.16) imply that (6.17) holds for $s = 0$. Since [22, p. 130] $P^{-,\eta}_{ItD}$ is self-adjoint with respect to the real inner product $(\cdot,\cdot)^r_\Gamma$, it follows from (2.20) that (6.17) holds for $-1 \leq s \leq 0$, uniformly in s. Furthermore, from (6.15) and (6.16) it follows that (6.18) holds for $s = 0, 1$. By interpolation (see (2.18)) (6.18) holds in fact for $0 \leq s \leq 1$, uniformly in s.

The adjustments to (6.17) and (6.18) for Ω_- piecewise smooth or merely Lipschitz follow from the adjustments to (6.15) and (6.16) in these cases, as described in Theorem 6.1. ■

6.3 From resolvent estimates to Corollary 1.11

The following lemma captures arguments made in [6] for the nontrapping case (where $K(k) = 1$ for $k \geq 0$), and provides a general recipe for bounding $(A'_{k,\eta})^{-1}$ as a corollary of resolvent estimates in Ω_+. We note that bounds on $A^{-1}_{k,\eta}$ (as opposed to $(A'_{k,\eta})^{-1}$) then follow immediately from (6.10) and (6.11).

Lemma 6.3 *Suppose that Ω_+ satisfies a K resolvent estimate, for some $K \in C[0,\infty)$ with $K(k) \geq 1$ for $k > 0$, and that $\eta = ck$, for some $c \in \mathbb{R}\setminus\{0\}$. Then, given $k_0 > 0$, provided each component of Ω_- is either star-shaped with respect to a ball or C^∞,*

$$\|(A'_{k,\eta})^{-1}\|_{H^s_k(\Gamma)\to H^s_k(\Gamma)} \lesssim K(k) \quad \textit{and} \quad \|(A'_{k,\eta})^{-1}\|_{H^s(\Gamma)\to H^s(\Gamma)} \lesssim k^{-s}K(k), \quad \textit{for } k \geq k_0, \tag{6.19}$$

uniformly for $-1 \leq s \leq 0$. The bounds (6.19) hold with $K(k)$ and $k^{-s}K(k)$ replaced by $k^{1/4}K(k)$ and $k^{1/4-s}K(k)$, respectively, if each component of Ω_- is either star-shaped with respect to a ball (and Lipschitz) or piecewise smooth. They hold with $K(k)$ and $k^{-s}K(k)$ replaced by $k^{1/2}K(k)$ and $k^{1/2-s}K(k)$, respectively, in the general Lipschitz case.

Proof. The first bound in (6.19) follows immediately from Lemma 4.3, (6.13), and Corollary 6.2. The second bound in (6.19) when $s = 0$ follows from the first, and the second bound when $s = -1$ follows from: (i) the second bound in (4.3) when $s = 0$, and (ii) the bounds $\|P^{-,\eta}_{ItD}\|_{H^{-1}(\Gamma)\to L^2(\Gamma)} \lesssim 1$ and $\|P^{-,\eta}_{ItD}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1/k$, which follow from Corollary 6.2. The second bound in (6.19) for $-1 < s < 0$ then follows from the cases $s = 0, 1$ by the interpolation bound (2.18). ■

The upper bounds (1.27) and (1.28) in Corollary 1.11 and the comments in Remark 1.13 follow immediately from combining Lemma 6.3 and Theorem 1.7. The following lemma proves the bound (1.29) and so completes the proof of Corollary 1.11. In this lemma, for $x = (x_1, ..., x_d) \in \mathbb{R}^d$ we write $\widetilde{x} := (x_2, ..., x_d) \in \mathbb{R}^{d-1}$, so that $x = (x_1, \widetilde{x})$.

Lemma 6.4 *Suppose that $\epsilon > 0$, $a_2 > a_1$, and that $\Gamma_1 \cup \Gamma_2 \subset \Gamma$ and $\Omega_C \subset \Omega_+$, where $\Gamma_j := \{x = (a_j, \widetilde{x}) : |\widetilde{x}| < \epsilon\}$, for $j = 1, 2$, and $\Omega_C := \{x = (x_1, \widetilde{x}) : |\widetilde{x}| < \epsilon$ and $a_1 < x_1 < a_2\}$. Then, where $a := a_2 - a_1$, provided $|\eta| \lesssim k$,*

$$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \gtrsim k, \quad \textit{for } k \in Q := \{m\pi/a : m \in \mathbb{N}\}. \tag{6.20}$$

Observe that the geometric assumptions in Lemma 6.4 include every (R_0, R_1, a) parallel trapping obstacle, if necessary after an appropriate change of coordinate system.

Proof of Lemma 6.4. Let $S := \{\widetilde{x} \in \mathbb{R}^{d-1} : |\widetilde{x}| < \epsilon/2\}$. Choose a non-zero $\chi \in C_0^\infty(\mathbb{R}^{d-1})$ supported in S. For some $c_j \in \mathbb{C}$ with $|c_1| = |c_2| = 1$, let $\phi_j((a_j, \widetilde{x})) := c_j \chi(\widetilde{x})$, $\widetilde{x} \in \mathbb{R}^{d-1}$, for $j = 1, 2$. Let $\phi \in C^1(\Gamma)$ be defined by $\phi(x) = \phi_j(x)$, $x \in \Gamma_j$, for $j = 1, 2$, $\phi(x) = 0$ otherwise, and define $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \cap C(\mathbb{R}^d) \cap C^2(\mathbb{R}^d \setminus \mathrm{supp}(\phi))$ by

$$u(x) := \int_\Gamma \Phi_k(x, y)\phi(y)\, \mathrm{d}s(y), \quad \text{for } x \in \mathbb{R}^d. \tag{6.21}$$

Using the standard jump relations [22, p. 115], we see that

$$A'_{k,\eta}\phi = f_{k,\eta} := \partial_n^- u - \mathrm{i}\eta\gamma_- u. \tag{6.22}$$

Clearly, $\|\phi\|_{L^2(\Gamma)} \gtrsim 1$. We will prove the lemma by showing that $\|f_{k,\eta}\|_{L^2(\Gamma)} \lesssim k^{-1}$ if $k \in Q$ and $|\eta| \lesssim k$, provided we choose the phase of c_2/c_1 correctly.

Let $\widehat{\chi}$ denote the Fourier transform of χ, given by

$$\widehat{\chi}(\xi) := \int_{\mathbb{R}^{d-1}} \chi(\widetilde{x})\mathrm{e}^{-\mathrm{i}\widetilde{x}\cdot\xi}\, \mathrm{d}\widetilde{x}, \quad \xi \in \mathbb{R}^{d-1}.$$

Clearly $u = u^{(1)} + u^{(2)}$, where

$$u^{(j)}(x) := \int_{\Gamma_j} \Phi_k(x, y)\phi_j(y)\, \mathrm{d}s(y) \tag{6.23}$$

$$= \frac{\mathrm{i}c_j}{2(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{\widehat{\chi}(\xi)}{\sqrt{k^2 - \xi^2}} \exp\left(\mathrm{i}\big(\widetilde{x}\cdot\xi + |x_1 - a_j|\sqrt{k^2 - \xi^2}\,\big)\right) \mathrm{d}\xi, \tag{6.24}$$

for $j = 1, 2$ and $x \in \mathbb{R}^d$, with $\sqrt{k^2 - \xi^2} = \mathrm{i}\sqrt{\xi^2 - k^2}$ for $|\xi| > k$. The fact that (6.23) and (6.24) are equivalent follows from Fourier representations for layer potentials and BIOs; see, e.g., [23, Theorem 3.1].

For $x \in \mathbb{R}^d$,

$$u^{(j)}(x) = \frac{\mathrm{i}c_j}{2k}\, \chi(\widetilde{x})\, \mathrm{e}^{\mathrm{i}k|x_1 - a_j|} + v^{(j)}(x)$$

where

$$v^{(j)}(x) = \frac{\mathrm{i}c_j}{2(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} \widehat{\chi}(\xi)\, \mathrm{e}^{\mathrm{i}\widetilde{x}\cdot\xi} \left(\frac{\exp(\mathrm{i}|x_1 - a_j|\sqrt{k^2 - \xi^2}\,)}{\sqrt{k^2 - \xi^2}} - \frac{\mathrm{e}^{\mathrm{i}k|x_1 - a_j|}}{k} \right) \mathrm{d}\xi.$$

The point of this decomposition is that $v^{(j)}(x) = \mathcal{O}(k^{-2})$ as $k \to \infty$, uniformly on every bounded subset of $\mathbb{R}^d$, and, provided $k \in Q$, one can choose c_1 and c_2 such that

$$\frac{\mathrm{i}c_1}{2k}\, \chi(\widetilde{x})\, \mathrm{e}^{\mathrm{i}k|x_1 - a_1|} + \frac{\mathrm{i}c_2}{2k}\, \chi(\widetilde{x})\, \mathrm{e}^{\mathrm{i}k|x_1 - a_2|} \tag{6.25}$$

is zero for $x \in \Gamma$ (indeed for all $x \notin \Omega_C$); these observations will lead to the required estimate $\|f_{k,\eta}\|_{L^2(\Gamma)} = \mathcal{O}(k^{-1})$.

To obtain the bound on $v^{(j)}(x)$ we observe that, for $x \in \mathbb{R}^d$,

$$\begin{aligned} |v^{(j)}(x)| &\le \frac{1}{2(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} |\widehat{\chi}(\xi)| \left(\left| \frac{\exp(\mathrm{i}|x_1 - a_j|(\sqrt{k^2 - \xi^2} - k)) - 1}{\sqrt{k^2 - \xi^2}} \right| + \frac{|k - \sqrt{k^2 - \xi^2}|}{k|\sqrt{k^2 - \xi^2}|} \right) \mathrm{d}\xi \\ &\le \frac{k|x_1 - a_j| + 3}{2(2\pi)^{d-1}k^2} \int_{\mathbb{R}^{d-1}} \frac{|\widehat{\chi}(\xi)|\xi^2}{|\sqrt{k^2 - \xi^2}|}\, \mathrm{d}\xi, \end{aligned}$$

since $|\mathrm{e}^{\mathrm{i}t} - 1| \le |t|$ for $t \in \mathbb{R}$,

$$|\sqrt{k^2 - \xi^2} - k| = \xi^2/|\sqrt{k^2 - \xi^2} + k| \le \xi^2/k \quad \text{for } \xi \in \mathbb{R}^{d-1}, \tag{6.26}$$

and, for $|\xi| > k$ and $b \ge 0$,

$$\left|\exp(\mathrm{i}b(\sqrt{k^2 - \xi^2} - k)) - 1\right| \le 2 \le 2\xi^2/k^2. \tag{6.27}$$

Moreover, since $\hat{\chi}$ is in the Schwartz space $\mathcal{S}(\mathbb{R}^{d-1})$, it vanishes rapidly at infinity, and thus for some $C > 0$ we have $|\hat{\chi}(\xi)| \leq C(1 + |\xi|)^{-2-d}$ for $\xi \in \mathbb{R}^{d-1}$, so that, for some $C', C'' > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}} \frac{|\hat{\chi}(\xi)|\xi^2}{|\sqrt{k^2 - \xi^2}|} \,\mathrm{d}\xi \quad &\leq \quad C' \int_0^\infty \frac{\mathrm{d}r}{|\sqrt{k^2 - r^2}|(1+r)^2} \\
&\leq \quad C'' \left(\frac{1}{k^2} \int_{k/2}^{3k/2} \frac{\mathrm{d}r}{|\sqrt{k^2 - r^2}|} + \frac{1}{k} \int_0^\infty (1+r)^{-2} \,\mathrm{d}r \right) = \mathcal{O}(k^{-1}).
\end{aligned}$$

Thus $v^{(j)}(x) = \mathcal{O}(k^{-2})$ as $k \to \infty$ for $j = 1, 2$, uniformly in x in every bounded subset of $\mathbb{R}^d$.

Since $\Omega_C \subset \Omega_+$, we have $\Gamma \subset \overline{\Omega_*}$ and $\Gamma \setminus (\Gamma_1 \cup \Gamma_2) \subset \Omega_*$, where $\Omega_* := \{x \in \mathbb{R}^d : x_1 < a_1 \text{ or } x_1 > a_2 \text{ or } |\widetilde{x}| > \epsilon/2\}$. Choosing $c_1 = 1$ and $c_2 = -\mathrm{e}^{\mathrm{i}ka}$, we see that (6.25) equals zero for $x \in \overline{\Omega_*}$ and $k \in Q$, and thus

$$u(x) = u^{(1)}(x) + u^{(2)}(x) = v^{(1)}(x) + v^{(2)}(x), \quad \text{so that} \quad u(x) = \mathcal{O}(k^{-2}) \tag{6.28}$$

for $k \in Q$, uniformly on bounded subsets of $\overline{\Omega_*}$, in particular uniformly on Γ. Since $\Delta u + k^2 u = 0$ in Ω_*, it follows using (4.6) that $\nabla u(x) = \mathcal{O}(k^{-1})$ for $k \in Q$, uniformly for $x \in \Gamma \setminus (\Gamma_1 \cup \Gamma_2)$. Finally, from (6.24) we have that, with this choice of c_1 and c_2 and $k \in Q$, for $x \in \Gamma_1 \cup \Gamma_2$,

$$\begin{aligned}
|\partial_n^- u(x)| &= \frac{1}{2(2\pi)^{d-1}} \left| \int_{\mathbb{R}^{d-1}} \widehat{\chi}(\xi) \mathrm{e}^{\mathrm{i}\widetilde{x}\cdot\xi} \left(\exp(\mathrm{i}a(\sqrt{k^2 - \xi^2} - k)) - 1 \right) \mathrm{d}\xi \right| \\
&\leq \frac{a}{2(2\pi)^{d-1}k} \int_{\mathbb{R}^{d-1}} |\widehat{\chi}(\xi)|\xi^2 \,\mathrm{d}\xi,
\end{aligned} \tag{6.29}$$

using (6.26), (6.27), and that $\pi \leq ka$ for $k \in Q$. Putting these bounds together in (6.22), we have shown that $f_{k,\eta}(x) = \mathcal{O}(k^{-1})$ as $k \to \infty$ with $k \in Q$ and $|\eta| \lesssim k$, uniformly on Γ. ■

The proof of Lemma 6.4 was inspired by the billiard-type arguments used to construct high-frequency quasimodes, going back to Keller and Rubinow [53]; see, e.g., [3] and the references therein. We also expect that lower bounds on $\|(A'_{k,\eta})^{-1}\|$ similar to that in Lemma 6.4 can be obtained when Ω_+ supports arbitrary closed finite billiards and Γ is flat in the neighbourhood of each reflection.

Remark 6.5 *In the proof of Lemma 6.4, $f_{k,\eta}$ is bounded via its representation (6.22) as boundary data for an interior impedance problem satisfied by u. In [21] a less-sharp bound is obtained in 2-d, that $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \gtrsim k^{9/10}$ for $k \in Q$, via an alternative formula for $f_{k,\eta}$. Precisely, with ϕ and $f_{k,\eta}$ as in the above proof, it is shown that $\phi = \partial_n^+ u^t$ is the normal derivative of the total field for sound soft scattering when the incident field is*

$$u^i(x) = \int_{\Omega_+} \Phi_k(x, y) f(y) \,\mathrm{d}s(y), \quad x \in \mathbb{R}^d, \tag{6.30}$$

with f supported in $\Omega_C \subset \Omega_+$ given by $f(x) := k^{-1} \sin(kx_1) \widetilde{\Delta} \chi(\widetilde{x})$, for $x \in \Omega_C$, where $\widetilde{\Delta}$ is the Laplacian in $\mathbb{R}^{d-1}$. It follows from (6.12) that $f_{k,\eta} = \partial_n^+ u^i - \mathrm{i}\eta\gamma_+ u^i$. This, together with (6.30), is a formula for $f_{k,\eta}$ as an oscillatory integral over $\mathrm{supp}(f) \subset \Omega_C$. *Estimating this integral (suboptimally) in [21, Theorem 5.1] led to the bound $\|f_{k,\eta}\|_{L^2(\Gamma)} \lesssim k^{-9/10}$.*

Remark 6.6 (Counterexample to a conjecture on coercivity) *Under the assumptions that Ω_- is C^3, piecewise analytic, and has strictly positive curvature, [88] shows that there exists an $\eta_0 > 0$ (equal to one when Ω_- is a ball) and $k_0 > 0$ such that if $\eta \geq \eta_0 k$ then $A'_{k,\eta}$* is coercive *uniformly in k for $k \geq k_0$, meaning that*

$$\left| \left(A'_{k,\eta}\phi, \phi \right)_\Gamma \right| \geq c_k \|\phi\|^2_{L^2(\Gamma)}, \quad \textit{for all } k \geq k_0 \textit{ and } \phi \in L^2(\Gamma), \quad \textit{with } c_k \gtrsim 1; \tag{6.31}$$

this is shown via a novel use of Morawetz identities in [88], generalising an earlier result for the case of a circle/sphere obtained via Fourier analysis [35]. This result implies that $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1$, but the converse is not true.

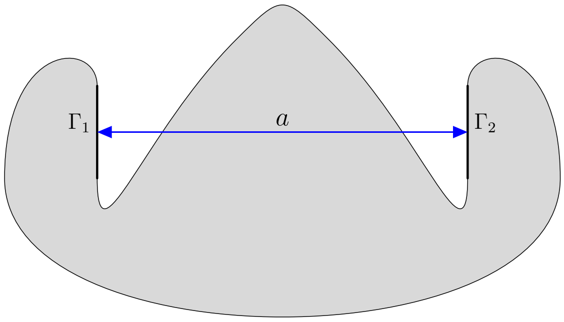

Figure 6.1: The obstacle Ω_- shaded grey is nontrapping, so that $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1$ if $\eta = ck$, for some constant $c \in \mathbb{R}\setminus\{0\}$. However, Remark 6.6 shows that $A'_{k,\eta}$ is not coercive uniformly in k.

The advantage of coercivity, as opposed to just boundedness of the inverse, for the numerical analysis of Galerkin methods is discussed in [88]; for example, the coercivity result in [88] completes the numerical analysis of high frequency numerical-asymptotic boundary element methods for scattering by convex obstacles [35, 37].

Based on computations of the numerical range (an operator is coercive if and only if zero is not in the closure of its numerical range), [10] conjectured that, if Ω_- is nontrapping, then (6.31) *holds with $\eta = k$ (i.e. $A'_{k,k}$ is coercive uniformly in k) [10, Conjecture 6.2]. This conjecture implies that $\|(A'_{k,k})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1$ for nontrapping domains, and this result was recently proved in [6, Theorem 1.13]. The calculations in Lemma 6.4, however, show that the conjecture is false.*

Suppose that Ω_+ satisfies the conditions of Lemma 6.4, except that we no longer require that $\Omega_C \subset \Omega_+$, instead we require that Ω_- is nontrapping (which implies that Γ passes through Ω_C), and we require that $n(x) = e_1$ on Γ_1, $n(x) = -e_1$ on Γ_2. An example is Figure 6.1. Define $\phi \in L^2(\Gamma)$ as in the proof of Lemma 6.4, so that the value of $\|\phi\|_{L^2(\Gamma)} \neq 0$ is independent of k. Equations (6.21), (6.22), (6.28) *and* (6.29) *still hold, and still imply that $f_{k,\eta}(x) = \mathcal{O}(k^{-1})$ for $k \in Q = \{m\pi/a : m \in \mathbb{N}\}$ with $|\eta| \lesssim k$, uniformly on $\Gamma_1 \cup \Gamma_2$ (but now not on all of Γ since $\Gamma \not\subset \overline{\Omega_*}$). Thus, provided $|\eta| \lesssim k$, since* $\mathrm{supp}(\phi) \subset \Gamma_1 \cup \Gamma_2$,

$$\left(A'_{k,\eta}\phi, \phi\right)_\Gamma = \int_{\Gamma_1\cup\Gamma_2} f_{k,\eta}\overline{\phi}\,\mathrm{d}s = \mathcal{O}(k^{-1}),$$

as $k \to \infty$ through the sequence Q, so that (6.31) *is false in this case. It may still hold that $A'_{k,\eta}$ is coercive, but if this is the case then the coercivity constant $c_k = \mathcal{O}(k^{-1})$ as $k \to \infty$ through the sequence Q.*

6.4 Summary of wavenumber-explicit bounds on $(A'_{k,\eta})^{-1}$

Table 6.1 below summarises: (i) the (sharpest) known resolvent estimates for scattering by obstacles, discussed in §1; (ii) the sharpest known bounds on the DtN map, taken from [6] for the nontrapping cases, proved as corollaries in §4 for the trapping cases; (iii) the bounds on the inf-sup constant obtained from the resolvent estimates (as discussed in Remark 5.2); and (iv) the upper bounds on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ that follow from the resolvent estimates by the general Lemma 6.3. (The bounds that were already known have been discussed earlier in §1.5.1, the other bounds are stated here for the first time as corollaries of the resolvent estimates and Lemma 6.3.) The upper bounds in the last column, by Lemma 6.3, are also upper bounds for $\|(A'_{k,\eta})^{-1}\|_{H^s_k(\Gamma)\to H^s_k(\Gamma)}$, uniformly for $-1 \leq s \leq 0$, and the same bounds, multiplied by a factor k^{-s}, are upper bounds for $\|(A'_{k,\eta})^{-1}\|_{H^s(\Gamma)\to H^s(\Gamma)}$. Further, bounds on $A^{-1}_{k,\eta}$ follow immediately from (6.10) and (6.11).

In the last column of Table 6.1 and in row 6 we include lower as well as upper bounds. Each lower bound holds *for at least one example* in the class indicated and *for at least some unbounded sequence of wavenumbers*. (The particular bound $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \gtrsim 1$ [21, Lemma 4.1] holds for $k \geq k_0$ whenever part of Γ is C^1.) The lower bounds in the last row and column of the table, and their relationship to the upper bound, should be interpreted as follows. Firstly, that in 2-d there exists an Ω_+ that is C^∞ ([9] gives specific elliptic-cavity trapping examples of which Figure 1.1(a) is typical) and positive constants $\alpha_2 \geq \alpha_1$ such that, with $\eta = ck$ for some $c \in \mathbb{R} \setminus \{0\}$.

$$\exp(\alpha_1 k) \lesssim \|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim \exp(\alpha_2 k)$$

as $k \to \infty$ through some positive, unbounded sequence of wavenumbers. Secondly, that in both 2-d and 3-d, whenever Ω_- permits elliptic trapping, allowing an elliptic closed broken geodesic γ, provided Γ is analytic in neighbourhoods of the vertices of γ and the local Poincaré map near γ satisfies the additional conditions of [19, (H1)], it holds for every $q < 2/11$ ($d = 2$), $q < 1/7$ ($d = 3$), that there exists $\alpha_3 > 0$ such that

$$\exp(\alpha_3 k^q) \lesssim \|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \tag{6.32}$$

as $k \to \infty$ through some positive, unbounded sequence of wavenumbers. The lower bound (6.32) follows immediately from Theorem 1 in Cardoso and Popov [19], which shows the existence, under these assumptions, of exponentially small quasi-modes, which moreoever can be constructed to be localised arbitrarily close to γ, and [22, Equation (5.39)], which converts exponentially small quasi-modes into lower bounds on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$.

Geometry of Ω_-	$K(k)$	$\|P^+_{DtN}\|_{H^1\to L^2}$	β_R^{-1}	$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$
1. C^∞ and nontrapping	$\lesssim 1$ [90, 72]	$\lesssim 1$ [6]	$\lesssim k$ [86]	$\lesssim 1$ [6] ($\gtrsim 1$ [21])
2. nontrapping polygon	$\lesssim 1$ [7]	$\lesssim 1$ [6]	$\lesssim k$ [86]	$\lesssim k^{1/4}$ ($\gtrsim 1$ [21])
3. Star-shaped and Lipschitz	$\lesssim 1$ [75, 27]	$\lesssim 1$ [6]	$\lesssim k$ [27]	$\lesssim k^{\beta}$ ($\gtrsim 1$ [21])
4. Star-shaped with respect to a ball and Lipschitz	$\lesssim 1$ [75, 27]	$\lesssim 1$ [6]	$\lesssim k$ [27]	$\lesssim 1$ [27, 86] ($\gtrsim 1$ [21])
5. Ikawa-like union of convex obstacles	$\lesssim \log(2+k)$ [16]	$\lesssim \log(2+k)$	$\lesssim k\log(2+k)$	$\lesssim \log(2+k)$ ($\gtrsim 1$ [21])
6. (R_0, R_1) obstacle	$\lesssim k^2$ ($\gtrsim k$ [27])	$\lesssim k^2$	$\lesssim k^3$ ($\gtrsim k^2$[27])	$\lesssim k^{2+\beta}$ ($\gtrsim k$)
7. Arbitrary C^∞	$\lesssim e^{\alpha k}$ [14]	$\lesssim e^{\alpha k}$	$\lesssim k e^{\alpha k}$	$\lesssim e^{\alpha k}$ ($\gtrsim e^{\alpha k}$ (2-d) [9], $\gtrsim e^{\alpha k^q}$)

Table 6.1: Summarising the known wavenumber-explicit upper bounds that hold for $k \geq k_0 > 0$; in the last column and in row 6 we also show the known lower bounds. Rows 1-4 apply in nontrapping cases. Rows 5-7 apply to trapping geometries, row 6 in particular to (R_0, R_1, a) parallel trapping obstacles. In the last column we assume that $\eta = ck$, for some non-zero real constant c, and $\beta = 0$ if each component of Ω_- is C^∞ or star-shaped with respect to a ball, $\beta = 1/4$ if each component is merely piecewise smooth or star-shaped with respect to a ball, $\beta = 1/2$ for general Lipschitz Ω_-. The bounds without citations are stated explicitly for the first time in this paper.

6.5 Bounds on $\mathrm{cond}(A'_{k,\eta})$

Putting the bounds in the last column of Table 6.1 together with known bounds on the norm of $A'_{k,\eta}$ one can deduce upper and lower bounds on the growth of the condition number of $A'_{k,\eta}$ as an operator on $L^2(\Gamma)$, defined by (1.36). The study of the conditioning of $A'_{k,\eta}$, and its dependence on the choice of the coupling parameter η, and latterly also on the geometry of Γ, has a long history, dating back to the original studies by Kress and Spassov [55, 54] for the case where Ω_- is a circle or sphere, these studies focussed on the low-wavenumber limit. The first rigorous (and sharp) high frequency bounds on $\mathrm{cond}(A'_{k,\eta})$, specifically for a circle/sphere and carried out using the Fourier analysis framework of [55], were obtained in [35], and the first rigorous results for high frequency for more general geometries were obtained in [21] and [9]. Bounds on $\mathrm{cond}(A'_{k,\eta})$ for C^∞ nontrapping domains were obtained recently in [6, §7.1], by combining upper bounds on $(A'_{k,\eta})^{-1}$ with bounds on $A'_{k,\eta}$ from [21] and [47].

To bound $A'_{k,\eta}$ it is clearly sufficient to obtain bounds on the operators S_k and D'_k (and note that D'_k has the same norm as D_k as an operator on $L^2(\Gamma)$ as D'_k is the adjoint of D_k with respect to the real inner product on $L^2(\Gamma)$; see, e.g., [22, Equation 2.37]). The following theorem summarises the known bounds on these operators.

Theorem 6.7 *Given $k_0 > 0$, for $k \geq k_0$,*

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{(d-3)/2} \quad \text{and} \quad \|D'_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{(d-1)/2} \tag{6.33}$$

for every Lipschitz Γ;

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{-1/2}\log(2+k) \quad \text{and} \quad \|D'_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{1/4}\log(2+k) \tag{6.34}$$

if Γ is piecewise smooth;

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{-2/3}\log(2+k) \quad \text{and} \quad \|D'_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{1/6}\log(2+k) \tag{6.35}$$

if Γ is piecewise smooth and has strictly positive curvature.

The bounds (6.34) and (6.35) are sharp modulo the log factors, in the sense that there exists, for each bound, a Γ in the class indicated for which the bound holds for $k \geq k_0$ with the $\lesssim$ replaced by $\gtrsim$ and with the log factor deleted. In fact, $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \gtrsim k^{-2/3}$ whenever some relatively open subset of Γ is C^2, while $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \gtrsim k^{-1/2}$ whenever Γ contains a line segment and is C^2 in a neighbourhood thereof. Thus, in the 2-d case ($d = 2$), the first bound in (6.33) is sharp. The second bound is also sharp in 2-d: if $c(k) = o(k^{1/2})$ as $k \to \infty$, then (for every dimension d) there exists a Lipschitz Γ for which $\|D'_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \gtrsim c(k)$ for $k \geq k_0$.

The bounds (6.33) are from [21, Theorems 3.3, 3.5] and ignore the oscillation in the kernels of S_k and D'_k; nevertheless these bounds are sharp in the 2-d case ($d = 2$) [21, Theorems 4.2, 4.3] in the senses indicated in the theorem. The bounds on S_k in (6.34) and (6.35) are taken from [42, Theorem 1.2], and the bounds on D'_k are taken from the appendix by Galkowski in [47]. In the same appendix the bounds (6.34) and (6.35) are shown to be sharp, modulo log factors. The fact that the bounds on S_k are sharp, in precisely the sense stated in the theorem, is shown in [21, Theorems 4.2, 4.4] in the 2-d case and [41, §3] in the 3-d case.

Proof of Theorem 6.7. From the remarks above all that remains to be shown is that the last sentence holds in dimensions higher than two. We prove this below; our proof, which extends the argument for the simpler 2-d case in [21, Theorem 4.3], applies in dimensions $d \geq 3$.

Suppose $\widetilde{F} \in C^1(\mathbb{R})$ and $\widetilde{F}(s) = 0$ for $s < 0$, $0 < \widetilde{F}'(s) \leq 1$ for $s > 0$. Choose $a > 0$ and $L > 1$, and let $F \in C^{0,1}[-a, a]$ be a Lipschitz continuous function such that $0 \leq F(s) \leq \widetilde{F}(s)$, for $-a \leq s \leq a$, and such that $|F'(s)| = L$, for almost all $s \in (0, a)$ (see [21, Theorem 4.3] for an example). Let $\widetilde{\Gamma} := \{(\widetilde{x}, F(x_1)) : |x_j| \leq a, \text{ for } j = 1, ..., d-1\}$, and choose the Lipschitz domain Ω_- so that $\widetilde{\Gamma} \subset \Gamma = \partial\Omega_-$.

Writing $x = (x_1, \widehat{x}, x_d)$, where $\widehat{x} = (x_2, ..., x_{d-1})$, given $c > 0$ and $\epsilon_k \in (0, a)$, define $\phi_k \in L^2(\Gamma)$ by $\phi_k(x) = \exp(-\mathrm{i}kx_1)F'(x_1)$, if $x \in \widetilde{\Gamma}$, $0 < x_1 < \epsilon_k$, and $|\widehat{x}| < \max\left(a, ck^{-1/2}\right)$, and by $\phi_k(x) = 0$, otherwise, and set $\psi_k := D_k\phi_k$. We will show that, provided we choose $\widetilde{F}$ so that $\widetilde{F}(s) \to 0$ sufficiently rapidly as $s \to 0^+$, choose c small enough, and choose ϵ_k so that $\epsilon_k \to 0$ as $k \to \infty$, but at a sufficiently slow rate, it will hold that $\|\psi_k\|_{L^2(\Gamma)}/\|\phi_k\|_{L^2(\Gamma)} \geq c(k)$ for all sufficiently large k, which will prove the result.

For $x \in \widetilde{\Gamma}$ with $x_1 < 0$, by (6.2) and (6.4), where $O_k := \{(x_1, \widehat{x}) : 0 < x_1 < \epsilon_k, |\widehat{x}| < ck^{-1/2}\}$ and $\rho(\widetilde{x}, \widetilde{y}) := \left((\widetilde{x} - \widetilde{y})^2 + (F(y_1))^2\right)^{1/2}$,

$$\begin{aligned} |\psi_k(x)| \;=\; \frac{1}{4}\left(\frac{k}{2\pi}\right)^{(d-2)/2} \Bigg| \int_{O_k} \Bigg[& k\rho(\widetilde{x},\widetilde{y}) H^{(1)'}_{(d-2)/2}(k\rho(\widetilde{x},\widetilde{y})) - \frac{d-2}{2} H^{(1)}_{(d-2)/2}(k\rho(\widetilde{x},\widetilde{y})) \Bigg] \\ & \frac{(x_1 - y_1)F'(y_1) + F(y_1)}{(\rho(\widetilde{x},\widetilde{y}))^{(d+2)/2}}\, \mathrm{e}^{-\mathrm{i}ky_1} F'(y_1)\, \mathrm{d}\widetilde{y} \Bigg|. \end{aligned}$$

Since [1, Equations (9.2.7), (9.2.13)]

$$H_\nu^{(1)}(t) = \sqrt{\frac{2}{\pi t}} \mathrm{e}^{\mathrm{i}(t-\nu\pi/2-\pi/4)} + \mathcal{O}(t^{-3/2}) \quad \text{and} \quad H_\nu^{(1)\prime}(t) = \sqrt{\frac{2}{\pi t}} \mathrm{e}^{\mathrm{i}(t-\nu\pi/2+\pi/4)} + \mathcal{O}(t^{-3/2})$$

as $t \to \infty$, $(F'(y_1))^2 = L^2$ and $0 \le F(y_1) \le \widetilde{F}(y_1) \le y_1 \le \epsilon_k$, for $\widetilde{y} \in O_k$, and $\epsilon_k \to 0$ as $k \to \infty$, it follows that

$$|\psi_k(x)| \;=\; \frac{1}{2}\left(\frac{k}{2\pi}\right)^{(d-1)/2} (-x_1 L^2) \left| \int_{O_k} \frac{\exp(\mathrm{i}k(\rho(\widetilde{x},\widetilde{y}) + x_1 - y_1))}{(\rho(\widetilde{x},\widetilde{y}))^{(d+1)/2}} \,\mathrm{d}\widetilde{y} \right| + o(\epsilon_k k^{1/2})$$

as $k \to \infty$, uniformly for $x \in \widetilde{\Gamma}$ with $x_1 \le -a/2$. Now, provided $\widetilde{x} \in O_k' := \{(x_1, \widehat{x}) : -a \le x_1 \le -a/2 \text{ and } |\widehat{x}| < ck^{-1/2}\}$ and $\widetilde{y} \in O_k$,

$$0 \le \rho(\widetilde{x},\widetilde{y}) + x_1 - y_1 = \frac{(\widehat{x}-\widehat{y})^2 + (F(y_1))^2}{\rho(\widetilde{x},\widetilde{y}) + y_1 - x_1} \le \frac{4c^2/k + (\widetilde{F}(\epsilon_k))^2}{a}.$$

Further, $\rho(\widetilde{x},\widetilde{y}) = (-x_1) + \mathcal{O}(\epsilon_k) + \mathcal{O}(k^{-1})$, as $k \to \infty$, uniformly for $\widetilde{x} \in O_k'$ and $\widetilde{y} \in O_k$. Thus, provided $c < \sqrt{\pi a}/4$ and $k^{1/2}\widetilde{F}(\epsilon_k) \to 0$ as $k \to \infty$, it holds that $\cos(k(\rho(\widetilde{x},\widetilde{y}) + x_1 - y_1)) \ge 1/\sqrt{2}$ for all sufficiently large k, for all $\widetilde{x} \in O_k'$ and $\widetilde{y} \in O_k$, so that

$$\begin{aligned} |\psi_k(x)| \;&\ge\; \frac{L^2}{2\sqrt{2}}\left(\frac{k}{2\pi(-x_1)}\right)^{(d-1)/2} \int_{O_k} \mathrm{d}\widetilde{y} + o(\epsilon_k k^{1/2}) \\ &=\; \epsilon_k k^{1/2} \frac{L^2}{2\sqrt{2\pi}} \frac{c^{d-2}}{\Gamma(d/2)(-2x_1)^{(d-1)/2}} + o(\epsilon_k k^{1/2}) \end{aligned}$$

as $k \to \infty$, uniformly for $x \in \widetilde{\Gamma}$ with $\widetilde{x} \in O_k'$. Thus

$$\frac{\|\psi_k\|_{L^2(\Gamma)}}{\|\phi_k\|_{L^2(\Gamma)}} \ge \frac{\left(\int_{O_k'} |\psi((\widetilde{x},0))|^2 \,\mathrm{d}\widetilde{x}\right)^{1/2}}{\|\phi_k\|_{L^2(\Gamma)}} \ge \frac{(\epsilon_k k)^{1/2}\sqrt{a}L}{4\sqrt{\pi}(1+L^2)^{1/4}} \frac{c^{d-2}}{\Gamma(d/2)(-2x_1)^{(d-1)/2}} + o((\epsilon_k k)^{1/2}),$$

as $k \to \infty$. Since we can choose ϵ_k to reduce so slowly that $c(k) = o((\epsilon_k k)^{1/2})$ as $k \to \infty$, and then choose $\widetilde{F}(s)$ to reduce so rapidly as $s \to 0$ that $k^{1/2}\widetilde{F}(\epsilon_k) \to 0$ as $k \to \infty$, we see that we have proved the claimed result. ■

Combining the bounds in Theorem 6.7 with those in Table 6.1, we obtain the following corollary describing the growth of the condition number of $A_{k,\eta}'$ as k increases, and how this depends on the geometry of Ω_-.

Corollary 6.8 (Bounds on the condition number) *Suppose that $\eta = ck$, for some non-zero real constant c, and that $k_0 > 0$.*

(i) Let Ω_- be C^∞ and nontrapping, or star-shaped with respect to a ball and piecewise smooth, and suppose that Γ has strictly positive curvature. Then, for $k \ge k_0$,

$$k^{1/3} \lesssim \mathrm{cond}(A_{k,\eta}') \lesssim k^{1/3}\log(2+k); \quad \textit{indeed} \quad \mathrm{cond}(A_{k,\eta}') \sim k^{1/3} \tag{6.36}$$

if Ω_- is a ball in 2-d or 3-d (i.e., a circle or sphere).

(ii) Let Ω_- be C^∞ and nontrapping, or star-shaped with respect to a ball and piecewise smooth. Then, for $k \ge k_0$,

$$k^{1/3} \lesssim \mathrm{cond}(A_{k,\eta}') \lesssim k^{1/2}\log(2+k); \quad \textit{indeed} \quad k^{1/2} \lesssim \mathrm{cond}(A_{k,\eta}') \lesssim k^{1/2}\log(2+k) \tag{6.37}$$

if Γ contains a line segment. Moreover these bounds hold without the log factors in 2-d; in particular $\mathrm{cond}(A_{k,\eta}') \sim k^{1/2}$ in 2-d if Ω_- is C^∞ and nontrapping and Γ contains a line segment.

(iii) Let Ω_- be a nontrapping polygon. Then, for $k \geq k_0$,

$$k^{1/2} \lesssim \mathrm{cond}(A'_{k,\eta}) \lesssim k^{3/4}; \quad \textit{indeed} \quad \mathrm{cond}(A'_{k,\eta}) \sim k^{1/2} \tag{6.38}$$

if Ω_- is star-shaped with respect to a ball.

(iv) Let Ω_- be an Ikawa-like union of convex obstacles. Then, for $k \geq k_0$,

$$k^{1/3} \lesssim \mathrm{cond}(A'_{k,\eta}) \lesssim k^{1/3}[\log(2+k)]^2. \tag{6.39}$$

(v) Let Ω_- be an (R_0, R_1, a) parallel trapping obstacle with $R_1/R_0 \geq 121$. Then, where the upper bounds hold for all $k \geq k_0$ while the lower bounds apply specifically for $k \in Q := \{m\pi/a : m \in \mathbb{N}\}$, it holds that

$$k^{5/2} \lesssim \mathrm{cond}(A'_{k,\eta}) \lesssim k^{3+d/2}; \quad \textit{indeed } k^{5/2} \lesssim \mathrm{cond}(A'_{k,\eta}) \lesssim k^{7/2+\beta} \log(2+k) \tag{6.40}$$

if Γ is piecewise smooth, with $\beta = 0$ if each component of Ω_- is either C^∞ or star-shaped with respect to a ball, $\beta = 1/4$ otherwise. For all $k \geq k_0$ the weaker lower bound holds that $\mathrm{cond}(A'_{k,\eta}) \gtrsim k^{1/2}$.

(vi) Let Ω_- be C^∞. Then there exists $\alpha > 0$ such that, for $k \geq k_0$,

$$k^{1/3} \lesssim \mathrm{cond}(A'_{k,\eta}) \lesssim \exp(\alpha k). \tag{6.41}$$

Further, whenever Ω_- permits elliptic trapping, allowing an elliptic closed broken geodesic γ, provided Γ is analytic in neighbourhoods of the vertices of γ and the local Poincaré map near γ satisfies the additional conditions of [19, (H1)], it holds for every $q < 2/11$ ($d = 2$), $q < 1/7$ ($d = 3$), that there exists $\alpha' > 0$ such that

$$\mathrm{cond}(A'_{k,\eta}) \gtrsim \exp(\alpha' k^q), \tag{6.42}$$

for some unbounded sequence of positive wavenumbers k. Moreover, (6.42) holds with $q = 1$ in the 2-d case of an elliptic cavity in the sense of [9, Theorem 2.8] (an example is Figure 1.1(a)).

Proof of Corollary 6.8. The definition of $A'_{k,\eta}$ (6.7) and Theorem 6.7 imply that, with $\eta = ck$, $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{1/3}\log(2+k)$ if Γ is piecewise smooth with each piece having strictly positive curvature; is $\lesssim k^{1/2}\log(2+k)$ if Γ is piecewise smooth; and is $\lesssim k^{(d-1)/2}$ in general. These same results imply that $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)} \gtrsim k^{1/3}$ if Γ is piecewise smooth; is $\gtrsim k^{1/2}$ if Γ contains a line segment and is C^2 in a neighbourhood thereof. Further, it is known [44, 35] that $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)} \sim k^{1/3}$ for a ball (in 2-d and 3-d), and that, because of the compactness of D'_k (and S_k) on $L^2(\Gamma)$ when Γ is C^1 [40], if a part of Γ is C^1 then $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)} \geq 1/2$ and $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \geq 2$ for $k > 0$ [21, Lemma 4.1]. The corollary follows by combining these estimates with the bounds on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ summarised in Table 6.1 (and recall the discussion of case (vi) in §6.4). ∎

Remark 6.9 (Other choices of coupling parameter η) *Corollary 6.8 focused on the case when the coupling parameter η is chosen proportional to k as this is the recommendation from various computational and theoretical studies [54, 2, 13, 21, 41]. For discussions of conditioning for other choices of η, and of the effect of choices of η on the condition number and other aspects of the effectiveness of numerical solution methods, see [54, 2, 13, 21, 9, 63, 6, 41].*

Corollary 6.8 contains bounds for trapping cases, which appear for the first time in this paper, and bounds for nontrapping cases, many of which have appeared already in the literature. Specifically: the bounds in (6.37) for the 2-d star-shaped case previously appeared in [21, §6] (indeed in sharper versions without the log factors); the second of the estimates in (6.38) also appeared in [21, §6]; the second (lower) bound in (6.41) appeared in [9, Theorem 2.8] for a particular 2-d elliptic trapping case; and the upper bounds in (6.36) and (6.37) appeared in [6, Theorem 7.2].

The corollary makes clear that the conditioning of $A'_{k,\eta}$ (with η proportional to k) depends strongly on the type of trapping. When Ω_- is a ball the conditioning grows precisely as $k^{1/3}$. The conditioning is worse than this for the mild hyperbolic trapping of an Ikawa-like union of convex obstacles, but at most by logarithmic factors. A C^∞ nontrapping obstacle has slightly higher growth in condition number (proportional to $k^{1/2}$) if Γ contains a line segment.

By contrast, (R_0, R_1, a) parallel trapping obstacles (the main focus of this paper), have only polynomial growth in condition number, but at a faster rate than all the nontrapping cases considered in the above corollary, at least as fast as $k^{5/2}$ as k increases through a particular unbounded sequence. Finally, if the obstacle allows a stable (elliptic) periodic orbit, then the condition number grows exponentially as k increases through some unbounded sequence.

6.6 Proof of Corollary 1.17 (convergence of the h-BEM)

Definition 6.10 (Shape-regular triangulation) *Suppose $\mathcal{T}$ is a triangulation of Γ in the sense, e.g., of [61], so that each element $K \in \mathcal{T}$ (with $K \subset \Gamma$) is the image of a reference element $\widehat{K} = \{\xi \in \mathbb{R}^{d-1} : 0 < \xi_i < 1, \sum_{i=1}^{d-1} \xi_i < 1\}$ under a C^1-diffeomorphism $F_K : \overline{\widehat{K}} \to \overline{K}$, with Jacobian $J_K := DF_K$. Then $\mathcal{T}$ is* shape-regular *if there exists a constant $c_S > 0$ such that, for every $K \in \mathcal{T}$,*

$$\frac{\sup_{\xi \in K} \lambda_K^{\max}(\xi)}{\inf_{\xi \in K} \lambda_K^{\min}(\xi)} \leq c_S, \tag{6.43}$$

where $\lambda_K^{\max}$ and $\lambda_K^{\min}$ denote the maximum and minimum eigenvalues of $J_K^T J_K$.

The heart of the k-explicit analysis of the h-BEM in [41] and [46] is the following lemma (see [46, Lemma 4.1], [41, Lemma 4.3]).

Lemma 6.11 *Let $L_{k,\eta} := D'_k - \mathrm{i}\eta S_k$ and $\lambda := 1/2$, so that $A'_{k,\eta}$ defined by (6.7) can be written as $A'_{k,\eta} = \lambda I + L_{k,\eta}$. Let $\mathcal{T}_h$ be a shape-regular triangulation of Γ, with maximum element diameter h, let $p \geq 0$, define the boundary element space $\mathcal{S}^p(\mathcal{T}_h)$ as in §1.5.3, and let $P_{hp} : L^2(\Gamma) \to \mathcal{S}^p(\mathcal{T}_h)$ be orthogonal projection. If, for some $\delta > 0$,*

$$\|I - P_{hp}\|_{H^1(\Gamma)\to L^2(\Gamma)} \|L_{k,\eta}\|_{L^2(\Gamma)\to H^1(\Gamma)} \|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \frac{\delta}{1+\delta}, \tag{6.44}$$

then the Galerkin solution v_{hp} of the variational problem (1.38) is well-defined and (1.40) holds with

$$C_3 = \lambda(1+\delta)\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}. \tag{6.45}$$

Since $\mathcal{S}^p(\mathcal{T}_h) \subset \mathcal{S}^0(\mathcal{T}_h)$ it is clear that $\|I - P_{hp}\|_{H^1(\Gamma)\to L^2(\Gamma)} \leq \|I - P_{h0}\|_{H^1(\Gamma)\to L^2(\Gamma)}$. The approximation result $\|I - P_{h0}\|_{H^1(\Gamma)\to L^2(\Gamma)} \leq Ch$, with $C > 0$ dependent only on the constant c_S in (6.43), is proved in [89, Theorem 1.4] for the case when Γ is piecewise smooth and each element $K \in \mathcal{T}_h$ is flat, and the argument extends to the case when Γ is piecewise C^1.

Corollary 1.17 follows from combining: (i) this approximation result; (ii) Lemma 6.11; (iii) the bounds on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ in (1.27) and (1.33); and (iv) the following theorem.

Theorem 6.12 *Suppose that $|\eta| \sim k$ and that Ω_- is $C^{2,\alpha}$ for some $\alpha \in (0,1)$. Given $k_0 > 0$, for all $k \geq k_0$:*

$$\|L_{k,\eta}\|_{L^2(\Gamma)\to H^1(\Gamma)} \lesssim k^{(d+1)/2}; \tag{6.46}$$

if Γ is additionally piecewise smooth, then

$$\|L_{k,\eta}\|_{L^2(\Gamma)\to H^1(\Gamma)} \lesssim k^{3/2}\log(2+k); \tag{6.47}$$

if Γ additionally has strictly positive curvature, then

$$\|L_{k,\eta}\|_{L^2(\Gamma)\to H^1(\Gamma)} \lesssim k^{4/3}\log(2+k). \tag{6.48}$$

References for the proof of Theorem 6.12. The bound (6.46) is [46, Theorem 1.6], and the bounds (6.47) and (6.48) follow from [41, Theorem 1.10]. ■

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