

CONNECTEDNESS OF THE SET OF CENTRAL LYAPUNOV EXPONENTS

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ABSTRACT. We show that there is a residual subset \mathcal{R} of $\text{Diff}^1(M)$ such that for any $f \in \mathcal{R}$ and any partially hyperbolic homoclinic class $H(p, f)$ with one dimensional center direction, the set of central Lyapunov exponents associated with the ergodic with either full support or positive entropy is an interval.

1. INTRODUCTION

The main question we try to answer is a reformulation of a fact initiated by Sigmund in a short article ([19]). There, he studied the connectedness of the set of hyperbolic ergodic measure. He established path connectedness of this space in the case of transitive topological Markov shifts and as a corollary, of transitive Axiom A diffeomorphisms. Since Sigmund's work the interest to this problem has been lost.

By definition, hyperbolic dynamical systems have nonzero Lyapunov exponents. However, they are not generic in the space of all dynamical systems. This necessitated weakening the notion of hyperbolicity. A wider class is that of partially hyperbolic systems studied by many authors (for a complete review, see [3, 13, 14]). A partially hyperbolic set may admit zero Lyapunov exponents along their central directions. As a recent effort, in Sigmund's direction, in [9] raised a question of whether the space of ergodic measures invariant under some partially hyperbolic systems is path connected. Our main question here is:

Question. Is the set of central Lyapunov vectors associated with the ergodic measures on a partially hyperbolic set convex?

In general, the answer is negative. In [16], the authors have studied destroying horseshoes via heterodimensional cycles. These maps are partially hyperbolic with a one dimensional center direction. They proved that every ergodic measure is hyperbolic, but the set of Lyapunov exponents in the central direction has a gap. As known, the existence of heterodimensional cycles isn't a generic phenomenon. There are examples of open sets of partially hyperbolic maps whose central Lyapunov exponents contains a convex set. Now, a question may be arisen:

what can be said in generic mode?

There are partial answers to the question in the case of generic "locally maximal" or "Lyapunov stable" homoclinic classes [2, 7]. Here, we give a positive answer to the question in the case of one dimensional central direction.

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Let M be a compact boundless Riemannian manifold and f be a diffeomorphism on it. For a periodic point p of f , we denote by $\pi(p)$ the period of p . For a hyperbolic periodic point p of f of period $\pi(p)$ the sets

$$W^s(p) = \{x \in M : f^{\pi(p)n}(x) \rightarrow p \text{ as } n \rightarrow \infty\}, \text{ and} \\ W^u(p) = \{x \in M : f^{-\pi(p)n}(x) \rightarrow p \text{ as } n \rightarrow \infty\};$$

are injectively immersed submanifolds of M . A point $x \in W^s(p) \cap W^u(p)$ is called a *homoclinic point* of f associated with p , and it is called a *transversal homoclinic point* of f if the above intersection is transversal at x . The closure of the transversal homoclinic points of f associated with the orbit of p is called the *homoclinic class* of p , and denoted by $H(p, f)$. Two saddles p and q are called *homoclinically related*, and write $p \sim q$, if

$$W^s(\mathcal{O}(p)) \bar{\cap} W^u(\mathcal{O}(q)) \neq \emptyset \text{ and } W^u(\mathcal{O}(p)) \bar{\cap} W^s(\mathcal{O}(q)) \neq \emptyset.$$

Note that, by *Smale's transverse homoclinic point theorem*, $H(p, f)$ coincides with the closure of the set of all hyperbolic periodic points q with $p \sim q$.

Definition 1.1. A compact invariant set Λ is partially hyperbolic if it admits a splitting $T_\Lambda M = E^{ss} \oplus E^c \oplus E^{uu}$ such that

$$\|Df|_{E^{ss}}\| < \|Df|_{E^c}\| < \|Df|_{E^{uu}}\|,$$

furthermore, E^{ss} is uniformly contraction and E^{uu} is uniformly expansion; it means

$$\|Df|_{E^{ss}}\| < 1 < \|Df|_{E^{uu}}\|.$$

The bundles E^{ss} and E^{uu} of a partially hyperbolic set are always uniquely integrable. Denote by $W_{loc}^{ss}(x)$ (resp. $W_{loc}^{uu}(x)$) the *local strong stable* (resp. *unstable*) manifold of x tangent to $E^{ss}(x)$ (resp. $E^{uu}(x)$) at x ([3, 13, 14]). Recall that if a diffeomorphism f has a partially hyperbolic set Λ then there is an open neighborhood U of Λ and a neighborhood \mathcal{U} of f such that the maximal invariant set $\tilde{\Lambda}_g$ of a diffeomorphism $g \in \mathcal{U}$ which is entirely contained in U is partially hyperbolic.

Denote by $\mathcal{M}_f(M)$ the set of f -invariant probability measures endowed with its usual topology; i.e., the unique metrizable topology such that $\mu_n \rightarrow \mu$ if and only if $\int f d\mu_n \rightarrow \int f d\mu$ for every continuous function $f : M \rightarrow \mathbb{R}$. The set of all ergodic elements of $\mathcal{M}_f(M)$ supported on an invariant set Λ is denoted by $\mathcal{M}_e(f|_\Lambda)$.

A point x in a partially hyperbolic set Λ is called *regular* if the following limit exists.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df^n(x)|_{E^c(x)}\|.$$

By Oseledec's theorem, given any invariant measure μ the limits exist μ -a.e.. The limits above are called *central Lyapunov exponent* of x . When μ is ergodic the limit doesn't depend on initial point μ -a.e.. We recall that an invariant set Λ of a diffeomorphism f is *Lyapunov stable* if for any neighborhood U of Λ there is a V of it such that $f^n(V) \subset U$, for any $n \geq 0$. Surely, for any hyperbolic periodic point p in a Lyapunov stable invariant set Λ we have $W^u(p) \subset \Lambda$. The set Λ is *bi-Lyapunov stable* if it is Lyapunov stable for f and f^{-1} .

Here, we try to build ergodic measure with middle Lyapunov exponents. There are two distinct strategies, approximation with periodic measures and flip-flop phenomenon. Although, ergodic measures constructed by the method of periodic approximations may have full support, they are highly repetitive and are likely to have zero entropy. For constructing measure of positive entropy we use the flip-flop family introduced in [5]. A disadvantage of

this method is that the nature of the obtained measure prevents the measures to have full support on the homoclinic class. In our main theorem, we benefit both method to build the desired measures.

Theorem A. *For a C^1 -generic diffeomorphism f and any partially hyperbolic homoclinic class $H(p, f)$ of f with the splitting $E^s \oplus E^c \oplus E^u$ such that*

- *either $\dim(E^c) = 1$ or*
- *$H(p, f)$ is bi-Lyapunov stable and E^c has a dominated splitting $E_1^c \oplus E_2^c$ into one dimensional bundles.*

The central Lyapunov exponents associated with the ergodic measures with support equal to $H(p, f)$ form a connected set. Furthermore, if $H(p, f)$ is bi-Lyapunov stable and $\dim(E^1) = 1$ then the set of central Lyapunov exponents associated with the ergodic measures supported on $H(p, f)$ and have positive topological entropy forms a connected set.

In fact, following [5], we prove that for any middle α there is an invariant set \mathbf{K}_α containing in the unstable lamination such that all points belong to it have Lyapunov exponents equal to α . In particular, any ergodic measure supported on \mathbf{K}_α has Lyapunov exponent α .

Problem 1.2. *In general, is the set of Lyapunov vectors associated with the set of ergodic measures on a generic homoclinic class convex?*

Section 2, describes two examples concerning our main question. The first example describe an open set of diffeomorphisms whose center Lyapunov exponents contains an interval. The second example provides a skew-product for which there is a gap in the center Lyapunov exponents. Both constructions benefit a realization of a skew-product over a finite generators. In section 3 we deal with the problem of approximating of ergodic measure by periodic ones. Proof of the Theorem A handled in sections 4 and 5.

2. ROBUST AND NON-GENERIC EXAMPLES

We need a bit of notations. Consider an iterated function system $\text{IFS}(f_1, \dots, f_k)$ generated by continuous maps f_1, \dots, f_k on X . A sequence of iterates $x_{n+1} = f_{\omega_n}(x_n)$ with $\omega_n \in \{1, \dots, k\}$ chosen randomly and independently is called *random orbit* starting at $x_0 = x$. The sequence of compositions $f_\omega^n = f_{\omega_n} \circ \dots \circ f_{\omega_1}$ is the *orbital branch* corresponding to $\omega = \omega_1 \omega_2 \dots \in \Sigma_k^+ = \{1, \dots, k\}^{\mathbb{N}}$. One can associate with the IFS a step skew-product $S : \Sigma_k^+ \times M \rightarrow \Sigma_k^+ \times M$ given by $S(\omega, x) = (\sigma(\omega), f_{\omega_0}(x))$. Considering a Markov partition with k element of a horseshoe in \mathbb{S}^2 , one can realize the skew-product as a diffeomorphism on $\mathbb{S}^2 \times M$ with a partially hyperbolic locally maximal invariant set homeomorphic to $\Sigma_k^+ \times M$.

2.1. A Robust Example. Suppose that f_0 is a Morse-Smale diffeomorphism on \mathbb{S}^1 with exactly one sink and one source and f_1 is an irrational rotation, also assume that both of them are sufficiently closed to the identity map in the C^1 -topology. It is known that

- the corresponding step skew-product is robustly transitive on the invariant set $\Lambda = \Sigma_2^+ \times \mathbb{S}^1$, in the sense that any perturbation of the initial skew-product is transitive on a locally maximal invariant set $\tilde{\Lambda}$ homomorphich to $\Sigma_2^+ \times \mathbb{S}^1$.
- $\tilde{\Lambda}$ is a local homoclinic class.

The conjugacy sends the central foliation of S to the central foliation of the perturbed diffeomorphism which are all circle. In fact, Gorodetski and Ilyashenko in [10] has proved the

Hölder continuity of the center manifolds of the perturbed diffeomorphisms (see also Theorem B in [20]). Using this key property, the authors in [15] have proved the existence of an ergodic measure with full support whose center Lyapunov exponent is zero. The same arguments can be adopted (with a duplicated proof) to fill an interval with the center Lyapunov exponents associated with the different ergodic measure. Furthermore, the ergodic measure may be chosen in such a way that their support equal to the whole of invariant set.

Theorem 2.1. *For any perturbation of S , there is a non-degenerate interval I containing in the set of central Lyapunov exponents associated with the ergodic measure with full support.*

2.2. A Non-generic Example with a Center Gap. The crucial step of the example below (Lemma 2.2) borrowed from [16]. Let $f_0 : [0, 1] \rightarrow [0, 1]$ be a map satisfying the following properties

- $f_0(0) = 0, f_0(1) = 1$ and $f_0(x) > x$, for any $x \in (0, 1)$;
- f'_0 is decreasing, $f'_0(1) = 1/\alpha$ and $f'_0(0) = \alpha$, for some $\alpha > 1$;

and for some $\beta > \alpha$, $f_1(x) = \beta(1 - x)$ (see Figure 1).

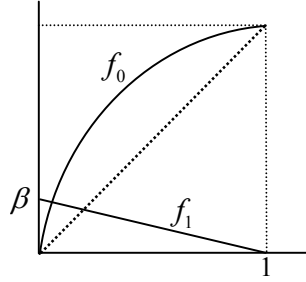


FIGURE 1. The graph of f_0 and f_1

Lemma 2.2. *Let $\omega \in \Sigma_2^+$ and n_i be the position of i -th 1 in ω . There are $0 < \lambda < 1$ and $C(\omega)$ such that for any $x \in [0, 1]$,*

$$|Df_\omega^{n_i}(x)| \leq C(\omega)\lambda^i.$$

Putting $\bar{0} = (0, 0, \dots) \in \Sigma_2^+$, the skew-product has two fixed points $p = (\bar{0}, 0)$ and $q = (\bar{0}, 1)$. Suppose that μ is an ergodic measure for the skew-product. Then $\pi^*\mu$ is an ergodic measure for the shift map. If $\pi^*\mu(\bar{0}) = 1$ then μ is δ_p or δ_q and hence has negative or positive center Lyapunov exponents. If $\pi^*\mu(\bar{0}) < 1$ then for μ -almost every $(\omega, x) \in \Sigma_2^+ \times M$, 1 appears with positive frequency γ in ω . By the lemma above,

$$\liminf_{n_i} \frac{1}{n_i} \log |Df_\omega^{n_i}(x)| \leq \liminf_{n_i} \frac{1}{n_i} \log C\lambda^i = \gamma \log \lambda < 0.$$

that is, any ergodic measure is hyperbolic and there are ergodic measures with positive and negative center Lyapunov exponents.

Theorem 2.3. *For the skew-product generated by $IFS(f_0, f_1)$, any ergodic measure is hyperbolic and there are ergodic measures whose center Lyapunov exponents have different sign. In particular, the set of center Lyapunov exponents has a gap.*

3. ERGODIC MEASURE APPROXIMATED BY PERIODIC ONES

First, we state two following lemmas to guarantee the existence of hyperbolic periodic points with desired central Lyapunov exponents inside the homoclinic class.

Lemma 3.1. (*Pliss's Lemma* [17]) *Given $\lambda_0 < \lambda_1$ and $\mathcal{O}(p) \subset \tilde{\Lambda}$ there is some natural number m such that if $\prod_{j=0}^t \|Df|_{E^c(f^j(p_n))}\| \leq \lambda_0^t$ for some $t \geq m$ then there is a sequence $n_1 < n_2 < \dots < n_k \leq t$ such that $\prod_{j=n_r}^t \|Df|_{E^c(f^j(p_n))}\| \leq \lambda_1^{t-n_r}$, for any $1 \leq r \leq k$.*

Lemma 3.2. ([18]) *For any $0 < \lambda < 1$, there is $\epsilon > 0$ such that for $x \in \tilde{\Lambda}$ which satisfies $\prod_{j=0}^{n-1} \|Df|_{E^c(f^j(p_n))}\| \leq \lambda^n$, for all $n > 0$, we have $\text{diam}(f^n(W_\epsilon^{cs}(x))) \rightarrow 0$. In other word, the center stable manifold of x with size ϵ is in fact stable manifold.*

To prove Theorem A, we need to recall some C^1 generic statements. For a partially hyperbolic homoclinic class $H(p, f)$ and $\mu \in \mathcal{M}_e(f|_{H(p, f)})$, Let λ_μ^c be the Lyapunov exponent of μ along the central direction. For a hyperbolic periodic point $q \in H(p, f)$ of period $\pi(q)$, $\lambda_q^c = \log |\nu| / \pi(q)$ where $\nu = Df^{\pi(q)}(q)|_{E^c(q)}$. Put

$$\mathcal{LE}^c(H(p, f)) = \overline{\{\lambda_q^c; q \in H(p, f) \cap \text{Per}(f)\}}$$

Proposition 3.3. *There is a residual subset \mathcal{R}_0 of $\text{Diff}^1(M)$ such that for any $f \in \mathcal{R}$ and any hyperbolic periodic point p of f ,*

- (1) *f is Kupka-Smale,*
- (2) *the set $\mathcal{LE}^c(H(p, f))$ is connected ([2]),*
- (3) *for any periodic point $q \in H(p, f)$ with $\text{index}(q) = \text{index}(p)$, we have $q \sim p$, ([7, 8]),*
- (4) *for any ergodic measure μ of f , there is a sequence of periodic point p_n such that $\mu_{p_n} \rightarrow \mu$ in weak* topology and $\mathcal{O}(p_n) \rightarrow \text{Supp}(\mu)$ in Housdorff metric ([1]).*
- (5) *If a homoclinic class $H(p, f)$ is bi-Lyapunov stable then for any sufficiently close g , $H(p_g, g)$ is bi-Lyapunov stable, where p_g is the continuation of p .*

Using arguments above we prove that any hyperbolic ergodic measure supported on a C^1 -generic partially hyperbolic homoclinic class $H(p, f)$ can be approximated by periodic measures also supported on the homoclinic class. In fact, this is a special case of Bonatti's conjecture.

Lemma 3.4, below, gives a positive answer to the conjecture in the case of hyperbolic measures on a partially hyperbolic homoclinic class.

Lemma 3.4. *For any $f \in \mathcal{R}_0$, any partially hyperbolic homoclinic class $H(p, f)$, any hyperbolic measure $\mu \in \mathcal{M}_e(f|_{H(p, f)})$ and any $\epsilon > 0$, there is a periodic point $q \in H(p, f)$ such that for any one dimensional center bundle E^c ,*

$$|\lambda_p^c - \lambda_\mu^c| < \epsilon.$$

Proof. Choose $\lambda_0 = \lambda_\mu^c + \epsilon < \lambda_1 < 0$. By Proposition 3.3, for any $\mu \in \mathcal{M}_e(f|_{H(p, f)})$, there is a sequence $\{p_n\} \subset \tilde{\Lambda}$ such that $\lambda_{p_n}^c \rightarrow \lambda_\mu^c$. Hence, for sufficiently large n , $\lambda_{p_n}^c < \lambda_0 = \lambda_\mu^c + \epsilon < 0$ and so $\prod_{j=0}^{\pi(p_n)} \|Df|_{E^c(f^j(p_n))}\| \leq (\exp^{\lambda_0})^{\pi(p_n)}$. Now, by Pliss Lemma, for large n , there is a point $q_n \in \mathcal{O}(p_n)$ such that $\prod_{j=0}^t \|Df|_{E^c(f^j(q_n))}\| \leq (\exp^{\lambda_1})^t$ for any $t \leq \pi(p_n)$. By Lemma 3.2, the center-stable manifolds of q_n 's are in fact stable manifolds and have uniform size. This implies that for large $n, m \in \mathbb{N}$, $p_n \sim p_m$ and $p_n \in H(p, f)$. \square

In what follows, we first state a lemma which is a modification of some recent results on C^1 -generic dynamics in our context ([7]). The second part of the Lemma helps us to approximate ergodic measures, not necessarily hyperbolic, supported on bi-Lyapunov stable homoclinic class by periodic ones, Theorem 3.6 below. We recall that a periodic point p of a diffeomorphism f has *weak Lyapunov exponent* along a center bundle E^c if for some small $\delta > 0$, $|\lambda^c| < \delta$.

Lemma 3.5. *There is a residual subset \mathcal{R}_1 of $\text{Diff}^1(M)$ such that for any $f \in \mathcal{R}_1$ and any homoclinic class $H(p, f)$ of f having a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ with non-hyperbolic center bundle E^c , we have*

- *if E^c is one-dimensional then there is a periodic point in the homoclinic class with weak Lyapunov exponent along E^c ,*
- *if E^c has a dominated splitting $E^c = E_1^c \oplus E_2^c$ into one dimensional subbundles then either $H(p, f)$ contains a periodic point whose index is equal to $\dim(E^s)$ or it contains a periodic point with weak Lyapunov exponent along E_1^c .*

Theorem 3.6. *For C^1 -generic bi-Lyapunov stable homoclinic class $H(p, f)$, satisfying Proposition 2.3 and Lemma 2.5, any ergodic measure can be approximate by periodic measure inside the homoclinic class.*

Proof. As before, if an ergodic measure is hyperbolic then nothing remains. Hence, suppose that μ is a non-hyperbolic ergodic measure supported on the homoclinic class $H(p, f)$. Using the forth part of Proposition 3.3, one can find a sequence $\{p_n\}$ of periodic point such that $\mathcal{O}(p_n) \rightarrow \text{Supp}(\mu)$ in Housdorff metric and also the periodic measures associated with p_n tends toward μ in weak topology. Now, if the homoclinic class contains a periodic point q of index $s = \dim(E^s)$ then by the partial hyperbolicity, for large n , and positive iterates m_n , $W^u(q) \cap W^s(f^{m_n}(p_n)) \neq \emptyset$. Hence, by the Lyapunov stability, $p_n \in H(p, f)$. Thus, suppose that for any periodic point in the homoclinic class its index equals to $s + 1$.

By Lemma 3.5, there is a periodic point q in the homoclinic class with weak Lyapunov exponents along the central direction E_1^c . We note that by the assumption $\text{index}(q) = \text{index}(p) = s + 1$ and so $q \sim p$. Now, using Gourmelon's perturbation lemma [12], which allows us to perturb the derivative controlling the invariant manifolds, we can produce a periodic point q' of index s such that $W^s(p) \cap W^u(q') \neq \emptyset$. Again, by the Lyapunov stability, $q' \in H(p, f)$. This is contradiction. \square

3.1. Remarks in the case $E^c = E_1^c \oplus E_2^c$. In general, we don't know much about the topological properties of the set of ergodic measures supported on a homoclinic class. Since the central directions are one dimensional the Lyapunov exponents along them define a continues map with in weak-topology. As before, one can deduce that the closure of the set of central Lyapunov exponents associated with the hyperbolic ergodic measures forms a convex set. One should notice that by the domination, the Lyapunov exponent along E_1^c is strictly less than the Lyapunov exponents along E_2^c . Hence, in the case of non-hyperbolic ergodic measures two disjoint cases may be occurred, either $\lambda_1^\mu < 0 = \lambda_2^\mu$ or $\lambda_1^\mu = 0 < \lambda_2^\mu$. The overall picture of the set of central Lyapunov exponents is illustrated as the general position in Figure 2, left. The shadowed part is the closure of the set of central Lyapunov exponents associated with the hyperbolic ergodic measures supported on $H(p, f)$ which is in fact the set $\mathcal{LE}^c(H(p, f))$. Using the central model for chain transitive sets, S. Crovisier

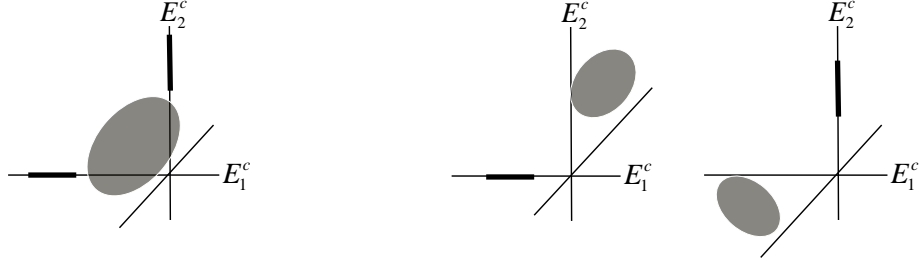


FIGURE 2. Left: General Position, Right: Special Position

et al. have obtained the following version of the Mane's ergodic closing lemma inside the homoclinic class ([7]).

Proposition 3.7. *For any C^1 -generic diffeomorphism f , let $H(p, f)$ be a partially hyperbolic homoclinic class with one dimensional central bundles. Let i be the minimal stable dimension of its periodic orbits. If $H(p, f)$ contains periodic point with i^{th} weak Lyapunov exponent, then for any ergodic measure supported on $H(p, f)$ whose $(i-1)^{\text{th}}$ Lyapunov exponent is zero, there exists periodic orbits contained in $H(p, f)$ whose associated measures converge for the weak topology towards the measure.*

The lemma suggests to special situations showed in the Figure 2, right.

4. GENERATING A MEASURE WITH FULL SUPPORT

Now, we follow the approach suggested in [11] to provide sufficient conditions for the existence of desired ergodic invariant measure as a limit of periodic ones.

Definition 4.1. *A periodic orbit X is a (γ, χ) -good approximation of a periodic orbit Y if the following holds*

- for a subset Γ of X and a projection $\rho : \Gamma \rightarrow Y$

$$d(f^j(x), f^j(\rho(x))) < \gamma,$$

for any $x \in \Gamma$ and any $j = 0, 1, \dots, \pi(Y) - 1$;

- $\text{card}(\Gamma)/\text{card}(X) \geq \chi$;
- $\text{card}(\rho^{-1}(y))$ is the same for all $y \in Y$.

The next lemma is a key point in the proof of the ergodicity of a limit measure. This lemma was suggested by Yu. Ilyashenko [11]. The following modified version, borrowed from [4], determines the support of the obtained ergodic measure.

Lemma 4.2. *Let $\{X_n\}$ be a sequence of periodic orbits with increasing periods $\pi(X_n)$ of f . Assume that there are two sequences of numbers $\{\gamma_n\}$, $\gamma_n > 0$ and $\{\chi_n\}$, $\chi_n \in (0, 1]$, such that*

1. for any $n \in \mathbb{N}$ the orbit X_{n+1} is a (γ_n, χ_n) -good approximation of X_n ;
2. $\sum_{n=1}^{\infty} \gamma_n < \infty$;
3. $\prod_{n=1}^{\infty} \chi_n \in (0, 1]$.

Then the sequence $\{\mu_{X_n}\}$ of atomic measures has a limit μ which is ergodic and

$$(1) \quad \text{Supp}(\mu) = \bigcap_{k=1}^{\infty} \overline{\bigcup_{l=k}^{\infty} X_l}$$

Lemma 4.3. (lemma 3.5 in [4]) *There is a residual subset \mathcal{R}_2 of $\text{Diff}^1(M)$ such that for any $f \in \mathcal{R}_2$, any partially hyperbolic homoclinic class $H(p, f)$, any one dimensional center bundle E^c , any saddle q in $H(p, f)$ and any $\epsilon > 0$, $H(p, f)$ contains a saddle q_1 which is homoclinically related to q , its orbit is ϵ -dense in the homoclinic class $H(p, f)$ and*

$$|\lambda_q^c - \lambda_{q_1}^c| < \epsilon.$$

4.1. Full Support, First Part of Theorem A. We first proceed with the first case of the theorem. Let $f \in \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}_2$ and $H(p, f)$ be a partially hyperbolic homoclinic class of f with one dimensional center direction. Put

$$\lambda_{\min}^c = \min \lambda^c(\mu), \quad \lambda_{\max}^c = \max \lambda^c(\mu),$$

where “min” and “max” are given over all f -invariant ergodic measures supported on $H(P, f)$. We consider three possible cases.

At the first consider the case in which $\lambda_{\min}^c < 0 < \lambda_{\max}^c$. Let $\alpha \in (\lambda_{\min}^c, \lambda_{\max}^c)$. We use Lemma 4.2 to construct an ergodic measure μ with $\text{Supp}(\mu) = H(p, f)$ such that $\lambda_{\mu}^c = s$. By Lemma 4.3, there are two saddles p and q_0 such that $\lambda_p^c < 0 < \alpha < \lambda_{q_0}^c$. Two cases may be occurred.

Case 1. $\alpha = 0$. This case is deduced from [4]. In fact, the authors in [4] have proved that if a C^1 -generic homoclinic class which has partial splitting with one dimensional center direction has periodic points with different index then it can admit a non-hyperbolic ergodic measure whose support equals to the whole of the homoclinic class.

Case 2. $\alpha \neq 0$. Suppose that $\alpha > 0$. By the second item of Proposition 3.3, there is a periodic point $p_0 \in H(p, f)$ such that $0 < \lambda_{p_0}^c < \alpha < \lambda_{q_0}^c$. Inductively, we find two sequences of $\{p_n\}$ and $\{q_n\}$ of the same index such that

- (a) $0 < \lambda_{p_{n-1}}^c < \lambda_{p_n}^c < (s + \lambda_{p_{n-1}}^c)/2 < \alpha < \lambda_{q_n}^c < (s + \lambda_{q_{n-1}}^c)/2 < \lambda_{q_{n-1}}^c$,
- (b) q_n is $\epsilon/2^n$ -dense in $H(p, f)$,
- (c) q_n is $(\epsilon/2^n, (1 - c/2^n))$ -good approximation of q_{n-1} , for some positive constant c .

Let $0 < \lambda_{p_{n-1}}^c < \alpha < \lambda_{q_{n-1}}^c$. By Lemma 4.3, there is a saddle $p_n \in H(p, f)$ with $\epsilon/2^n$ -dense orbit such that $0 < \lambda_{p_{n-1}}^c < \lambda_{p_n}^c < (\alpha + \lambda_{p_{n-1}}^c)/2 < \alpha$. Since p_n and q_{n-1} have the same index they are homoclinically related. Let $x \in W^s(\mathcal{O}(q_{n-1})) \cap W^u(\mathcal{O}(p_n))$ and $y \in W^u(\mathcal{O}(q_{n-1})) \cap W^s(\mathcal{O}(p_n))$. Consider a locally maximal hyperbolic set Λ_n containing $\mathcal{O}(p_n) \cup \mathcal{O}(x) \cup \mathcal{O}(y) \cup \mathcal{O}(q_{n-1})$. For some suitable $m_n \in \mathbb{N}$, put

$$\mathcal{PO}_n = \{ \underbrace{\mathcal{O}(p_n)}_{t_n\text{-times}}, f^{-m_n}(x), \dots, f^{m_n}(x), \underbrace{\mathcal{O}(q_{n-1})}_{s_n\text{-times}}, f^{-m_n}(y), \dots, f^{m_n}(y) \},$$

where $2m_n M / (t_n \pi(p_n) \lambda_{p_n}^c + s_n \pi(q_{n-1}) \lambda_{q_{n-1}}^c + 2m_n M) < \epsilon/2^n$. If \mathcal{PO}_n is $\epsilon/2^n$ -shadowed by a saddle q_n then $\pi(q_n) = t_n \pi(p_n) + s_n \pi(q_{n-1}) + 2m_n$ and for suitable t_n and s_n one has

$$\alpha < \lambda_{q_n}^c \approx \frac{t_n \pi(p_n) \lambda_{p_n}^c + s_n \pi(q_{n-1}) \lambda_{q_{n-1}}^c + 2m_n M}{t_n \pi(p_n) + s_n \pi(q_{n-1}) + 2m_n} < \frac{\alpha + \lambda_{q_{n-1}}^c}{2},$$

where $M = \max_{x \in M} \log \|Df(x)\|$. It is't difficult to see that

$$\chi_n = \frac{s_n \pi(p_n)}{t_n \pi(q_{n-1}) + s_n \pi(p_n) + 2m} = 1 - \frac{t_n \pi(q_{n-1}) + 2m}{t_n \pi(q_{n-1}) + s_n \pi(p_n) + 2m} > 1 - c/2^n,$$

for some positive constant c . Now, the sequence $\{q_n\}$ satisfies in the assumption of Lemma 4.2. Hence, the weak* limit μ of the atomic measures μ_{q_n} is ergodic and

$$\lambda_\mu^c = \int |Df|_{E^c} d\mu = \lim_{n \rightarrow \infty} \int |Df|_{E^c} d\mu_{q_n} = \lim_{n \rightarrow \infty} \lambda_{q_n}^c = \alpha.$$

Furthermore, by (1), $\text{Supp}(\mu) = H(p, f)$.

Now, we are dealing with the remaining cases. First Consider the case $\lambda_{\min}^c > 0$ or $\lambda_{\max}^c < 0$. In this case, the central direction is hyperbolic ([6]). Another cases that may be happened are $\lambda_{\min}^c = 0$ or $\lambda_{\max}^c = 0$. In these cases, by the first part of Lemma 3.5, the problem is reduced to the previous ones.

5. GENERATING A MEASURE WITH POSITIVE ENTROPY

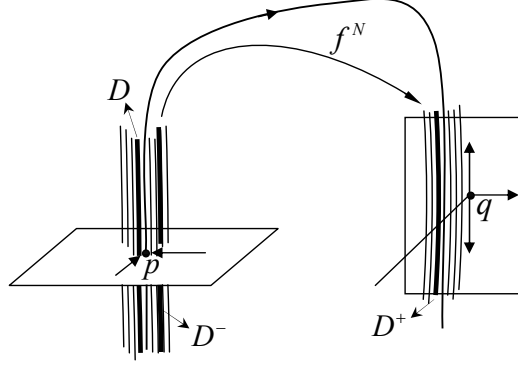
As it is seen in the previous section the crucial is the existence of periodic points of different indices. Here, we only deal with this case. This section is essentially owed to [5]. We start by stating some preliminarily notions and definitions.

5.1. Flip-flop Family. Let (X, d) be a metric space, $f : X \rightarrow X$ be a continuous map, K a compact subset of X , $\phi : K \rightarrow \mathbb{R}$ a continuous function and α a real number. A flip-flop family with respect to couple (ϕ, α) is a family $\mathfrak{F}_{(\phi, \alpha)}$ of compact subsets of K with bounded diameter such that $\mathfrak{F}_{(\phi, \alpha)} = \mathfrak{F}_{(\phi, \alpha)}^- \cup \mathfrak{F}_{(\phi, \alpha)}^+$ into disjoint families such that

- (1) There exists a constant $\chi_0 > 0$ such that for any $D^+ \in \mathfrak{F}_{(\phi, \alpha)}^+$ and $D^- \in \mathfrak{F}_{(\phi, \alpha)}^-$,
$$\phi|_{D^-} < \alpha - \chi_0 < \alpha + \chi_0 < \phi|_{D^+};$$
- (2) For any $D \in \mathfrak{F}_{(\phi, \alpha)}$, there exist two subsets D^+ and D^- of D such that $f(D^-) \in \mathfrak{F}_{(\phi, \alpha)}^-$ and $f(D^+) \in \mathfrak{F}_{(\phi, \alpha)}^+$;
- (3) There exists a constant $\lambda > 1$ such that if x, y belong to the same element D_0 of $\mathfrak{F}_{(\phi, \alpha)}$ and if $f(x)$ and $f(y)$ belong also to the same element D_1 of $\mathfrak{F}_{(\phi, \alpha)}$ then $d(f(x), f(y)) \geq \lambda d(x, y)$.

Now, suppose that $H(p, f)$ is a partially hyperbolic homoclinic class and $\text{index}(q) < \text{index}(p)$ for some $q \in H(p, f)$. We say that $H(p, f)$ admits a flip-flop family with respect to a continuous function $\phi(x)$ and α if there is a flip-flop family $\mathfrak{F}_{(\phi, \alpha)} = \mathfrak{F}_{(\phi, \alpha)}^- \cup \mathfrak{F}_{(\phi, \alpha)}^+$ both contained in the local unstable lamination. Using the notion of blender, the authors in [5] proved the following fact. Let \mathcal{U} be an open set of diffeomorphisms such that for any $f \in \mathcal{U}$, there exist two hyperbolic periodic points p_f, q_f with $\text{index}(q) < \text{index}(p)$, depending continuously on f and in the same partially hyperbolic homoclinic class. Then there exists an open and dense subset \mathcal{V} of \mathcal{U} such that for any $f \in \mathcal{V}$ the homoclinic class $H(p_f, f)$ admits a Flop-flop family associated with $(\phi := \log |Df^N|_{E^c}, \alpha)$ with $\lambda^c(q) < \alpha < \lambda^c(p)$ (see Figure 3). By a classical generic argument, one can deduce the following generic version.

Theorem 5.1. *There is a residual subset \mathcal{R} of $\text{Diff}^r(M)$, $r \geq 1$, such that for any $f \in \mathcal{R}$, any partially hyperbolic homoclinic class $H(p, f)$, any periodic point q of different index of that of p and any $\lambda^c(p) < \alpha < \lambda^c(q)$ there is a constant N and flip-flop family $\mathfrak{F}_{(\phi, \alpha)} = \mathfrak{F}_{(\phi, \alpha)}^+ \cup \mathfrak{F}_{(\phi, \alpha)}^-$*

FIGURE 3. Flip-flop family associated with p and q

for f^N and function $\phi := \log |Df^N|_{E^c}|$. Furthermore, \mathfrak{F}^- and \mathfrak{F}^+ may be chosen in such a way that \mathfrak{F}^- and \mathfrak{F}^+ happen respectively in a neighborhood of p and q .

We want to prove the main result of this section using the characteristic property of Flip-flop family. Let us return to abstract definition above.

- ($\mathfrak{F}_{(\phi, \alpha)}$ -segment). Let $T \in \mathbb{N}^\infty$. A $\mathfrak{F}_{(\phi, \alpha)}$ -segment of length T is a sequence $\mathbf{D} = \{D_i\}_{i=0}^T$ such that $f(D_i) = D_{i+1}$, each D_i contained in a member of $\mathfrak{F}_{(\phi, \alpha)}$ and D_T is also a member of $\mathfrak{F}_{(\phi, \alpha)}$. D_0 and D_T are respectively called the entrance and the exit of \mathbf{D} .
- (β -controlling). For $\beta > 0$ and $t \leq T$, \mathbf{D} is said to be (β, t, T, α) controlled if there is a set of control times $\mathcal{P} \subset \{0, 1, \dots, T\}$ such that
 - $0, T \in \mathcal{P}$;
 - if $k < l$ and two consecutive elements of \mathcal{P} then $l - k < t$ and $\frac{1}{l-k} |\phi_{l-k}(f^k(x)) - \alpha| < \beta$, for all $x \in D_0$.
- (Following τ -pattern). For $\tau > 1$, \mathbf{D} follows τ -pattern $s \in \{-, +\}^\mathbb{N}$ up to time T if for any $n \in [0, T]$ with $n \equiv 0 \pmod{\tau}$, one has $f^{n+1}(D_0)$ is contained in $\mathfrak{F}_{(\phi, \alpha)}^{s_n}$.

We can now state the distinctive property of flip-flop family. The following theorem borrows essentially from [5].

Theorem 5.2. Fix any member D of the Flip-flop family \mathfrak{F} and a pattern $s \in \{-, +\}^\mathbb{N}$. There are a monotone sequence $\{\beta_i\}$ of positive numbers converging to zero, a monotone sequence $\{t_i\}$ of positive integers converging to infinity, and a sequence of \mathfrak{F} -segments (\mathbf{D}_k) such that:

- C1. the entrance of \mathbf{D}_k is contained in D ;
- C2. the length T_k of the segment \mathbf{D}_k goes to infinity with k ;
- C3. every point in the entrance of \mathbf{D}_k is $(\beta_i, t_i, T_k, \alpha)$ -controlled, for every $1 \leq i \leq k$.
- C4. \mathbf{D}_k follows τ -ppattern s up to time T_k .

5.2. Positive Entropy, Second Part of Theorem A. Now, suppose that $s \in \{-, +\}^\mathbb{N}$ has dense orbit under the shift map. and let $x_k \in \mathbf{D}_k$, $\mathbf{D}_k \subset D \in \mathfrak{F}$, is $(\beta_k, t_k, T_k, \alpha)$ -controlledd for $t_k \rightarrow \infty$ and $\beta_k \rightarrow 0$ provided by C1-C4. Let $x_k \rightarrow x \in D$. It is not difficult to see that x is controlled in any scale and follows τ -pattern s up to infinity. In particular the central Lyapunov exponent of x is equal to α .

Now, for any $y \in \omega(x)$, define $\pi(y) \in \{-, +\}^{\mathbb{N}}$ by

$$(\pi(y))_i := \begin{cases} - & f^i(y) \in \phi^{-1}([\alpha + \chi_0, +\infty]), \\ + & f^i(y) \in \phi^{-1}([-\infty, \alpha - \chi_0]). \end{cases}$$

Then π is continuous. The mapping π is also onto. For the proof, suppose that $\eta = (\eta_{-L}, \dots, \eta_0, \dots, \eta_L)$ is a finite word of $-$ and $+$. Since the orbit of s is dense under the shift map, there is a sequence ℓ_i with $\ell_i \rightarrow \infty$ such that $(s_{\ell_i}, \dots, s_{2L+\ell_i}) = \eta$. Let $f^{\tau \ell_i}(x) \rightarrow y \in \omega(x)$, then $\pi(y) = \eta$. Since $y \in \omega(x)$, the central Lyapunov exponent of y is also equals to α .

Suppose that \mathcal{K}_α be the set all $x \in K$ with the following property: $x \in D \in \mathcal{F}$ is controlled at any scale and follows a s -pattern. Put

$$\mathbf{K}_\alpha = \bigcup_{x \in \mathcal{K}_\alpha} \omega(x).$$

Then all point in \mathbf{K}_α has central Lyapunov exponent equal to α and also $h_{\text{top}}(f|_{\mathbf{K}_\alpha}) \geq \log 2$. The variational principle for entropy implies the existence of measure of positive entropy supported on the homoclinic class.

Corollary 5.3. *The set \mathbf{K}_α is contained in local unstable manifold of $H(p, f)$. In particular, \mathbf{K}_α is contained in $H(p, f)$ provided that $H(p, f)$ is an bi-Lyapunov stable.*

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