CHERN CLASSES IN DELIGNE COHOMOLOGY FOR COHERENT ANALYTIC SHEAVES

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Abstract. In this article, we construct Chern classes in rational Deligne cohomology for coherent sheaves on a smooth complex compact manifold. We prove that these classes satisfy the functoriality property under pullbacks, the Whitney formula and the Grothendieck-Riemann-Roch theorem for projective morphisms between smooth complex compact manifolds.

Contents

1. Introduction

Let X be a smooth differentiable manifold and E be a complex vector bundle of rank r on X. The Chern-Weil theory (see [Gri-Ha, Ch. 3 § 3]) constructs classes $c_i(E)^{\mathrm{top}}$, $1 \leq i \leq r$, with values in the de Rham cohomology $H^{2i}(X, \mathbb{R})$, which generalize the first Chern class of a line bundle in $H^2(X, \mathbb{Z})$ obtained by the exponential exact sequence. These classes are compatible with pullbacks under smooth morphisms and verify the Whitney sum formula

$$c_k(E\oplus F)^{\mathrm{top}}=\sum_{i+j=k} c_i(E)^{\mathrm{top}}\, c_j(F)^{\mathrm{top}}.$$

There exist more refined ways of defining $c_i(E)^{\mathrm{top}}$ in $H^{2i}(X,\mathbb{Z})$. The first method is due to Chow (see the introduction of [Gro1]). The idea is to define explicitly the Chern classes of the universal bundles of the grassmannians and to write any complex vector bundle as a quotient of a trivial vector bundle. Of course, computations have to be done on the grassmannians to check the compatibilities. Note that in the holomorphic or in the algebraic context, a vector bundle is not in general a quotient of a trivial vector bundle. Nevertheless, if X is projective, this is true after tensorising by a sufficiently high power of an ample line bundle and the construction can be adapted (see [Br]).

A more intrinsic construction is the splitting method, introduced by Grothendieck in [Gro1]. Let us briefly recall how it works. By the Leray-Hirsh theorem, we know that $H^*(\mathbb{P}(E),\mathbb{Z})$ is a free module over $H^*(X,\mathbb{Z})$ with basis $1,\alpha,\dots,\alpha^{r-1}$, where α is the opposite of the first Chern class of the relative Hopf bundle on $\mathbb{P}(E)$. Now the Chern classes of E are uniquely defined by the relation

$$\alpha^r+p^*c_1(E)^{\mathrm{top}}\,\alpha^{r-1}+\cdots+p^*c_{r-1}^{\mathrm{top}}(E)\,\alpha+p^*c_r(E)^{\mathrm{top}}=0.$$

(see Grothendieck [Gro1], Voisin [Vo1, Ch. 11 § 2], and Zucker [Zu, § 1]).

The splitting method works amazingly well in various contexts, provided that we have

- the definition of the first Chern class of a line bundle,
- a structure theorem for the cohomology ring of a projective bundle considered as a module over the cohomology ring of the base.

Let us now examine the algebraic case. Let X be a smooth algebraic variety over a field k of characteristic zero, and E be an algebraic bundle on X. Then the splitting principle allows to define $c_i(E)^{\mathrm{alg}}$

- in the Chow ring $CH^i(X)$ if X is quasi-projective,
- in the algebraic de Rham cohomology group $H^{2i}_{\mathrm{DR}}(X/k)$.

Suppose now that $k=\mathbb{C}$. Then Grothendieck's comparison theorem (see [Gro3]) says that we have a canonical isomorphism between $H^{2i}_{DR}(X/\mathbb{C})$ and $H^{2i}(X^{\mathrm{an}},\mathbb{C})$. It is important to notice that the class $c_i(E)^{\mathrm{alg}}$ is mapped to $(2\pi\sqrt{-1})^i c_i(E)^{\mathrm{top}}$ by this morphism.

Next, we consider the problem in the abstract analytic setting. Let X be a smooth complex analytic manifold and E be a holomorphic vector bundle on X. We denote by $\mathcal{A}^{p,q}_{\mathbb{C}}(X)$ the space of complex differential forms of type (p,q) on X and we put $\mathcal{A}_{\mathbb{C}}(X)=\bigoplus_{p,q}\mathcal{A}^{p,q}_{\mathbb{C}}(X)$. The Hodge filtration on $\mathcal{A}_{\mathbb{C}}(X)$ is defined by $F^i\mathcal{A}_{\mathbb{C}}(X)=\bigoplus_{p\geq i,q}\mathcal{A}^{p,q}_{\mathbb{C}}(X)$. It induces a filtration $F^iH^k(X,\mathbb{C})$ on $H^k(X,\mathbb{C})$. For a detailed exposition see [Vo1, Ch. 7 and 8]. Let $\Omega^{\bullet}_X$ be the holomorphic de Rham complex on X. This is a complex of locally free sheaves. We can consider the analytic de Rham cohomology $\mathbb{H}^{k+i}(X,\Omega_X^{\bullet\geqslant i})$ which is the hypercohomology of the truncated de Rham complex. The maps of complexes $\Omega_X^{\bullet\geqslant i}\longrightarrow\Omega_X^{\bullet}$ and $\Omega_X^{\bullet\geqslant i}\longrightarrow\Omega_X^i[-i]$ give two maps $\mathbb{H}^{k+i}(X,\Omega_X^{\bullet\geqslant i})\longrightarrow F^iH^k(X,\mathbb{C})$ and $\mathbb{H}^{k+i}(X,\Omega_X^{\bullet\geqslant i})\longrightarrow H^k(X,\Omega_X^i)$. In the compact Kähler case, the first map is an isomorphism, but it is no longer true in the general case. We will denote by $H^{p,q}(X)$ the cohomology classes in $H^{p+q}(X,\mathbb{C})$ which admit a representative in $\mathcal{A}^{p,q}(X)$.

If E is endowed with the Chern connection associated to a hermitian metric, the de Rham representative of $c_i(E)^{\mathrm{top}}$ obtained by Chern-Weil theory is of type (i,i) and is unique modulo $d(F^i\mathcal{A}_X^{2i-1})$. This allows to define $c_i(E)$ in $\mathbb{H}^{2i}(X,\Omega_X^{\bullet\geqslant i})$, and then in $H^{i,i}(X)$ and $H^i(X,\Omega^i_X)$. The notations for these three classes will be $c_i(E)^{\mathrm{an}}$, $c_i(E)^{\mathrm{hodge}}$ and $c_i(E)^{\mathrm{dolb}}$.

If we forget the holomorphic structure of E, we can consider its topological Chern classes $c_i(E)^{\mathrm{top}}$ in the Betti cohomology groups $H^{2i}(X,\mathbb{Z})$. The image of $c_i(E)^{\mathrm{top}}$ in $H^{2i}(X,\mathbb{C})$ is $c_i(E)^{\mathrm{hodge}}$. Thus $c_i(E)^{\mathrm{top}}$ is an integral cohomology class whose image in $H^{2i}(X,\mathbb{C})$ lies in $F^iH^{2i}(X,\mathbb{C})$. Such classes are called *Hodge classes* of weight $2i$.
The Chern classes of E in $H^{2i}(X,\mathbb{Z})$, $F^iH^{2i}(X,\mathbb{C})$, $\mathbb{H}^{2i}\big(X,\Omega_X^{\bullet\geqslant i}\big)$ and $H^i\big(X,\Omega_X^i\big)$ appear in the following diagram:

$$\begin{tikzcd}
 & H^{2i}(X,\mathbb{Z}) \ni c_i(E)^{\mathrm{top}} \arrow[d] \\
c_i(E)^{\mathrm{an}} \in \mathbb{H}^{2i}(X,\Omega_X^{\bullet\geqslant i}) \arrow[r] \arrow[d] & F^iH^{2i}(X,\mathbb{C}) \ni c_i(E)^{\mathrm{hodge}} \\
c_i(E)^{\mathrm{dolb}} \in H^i(X,\Omega_X^i) &
\end{tikzcd}$$

This means in particular that these different classes are compatible with the Hodge decomposition in the compact Kähler case, and in general via the Hodge $\longrightarrow$ de Rham spectral sequence. Furthermore, the knowledge of $c_i(E)^{\mathrm{an}}$ allows to obtain the two other classes $c_i(E)^{\mathrm{dolb}}$ and $c_i(E)^{\mathrm{hodge}}$, but the converse is not true. Thus $c_i(E)^{\mathrm{an}}$ contains more information (except torsion) than the other classes in the diagram.

Recall now the Deligne cohomology groups $H_D^p(X,\mathbb{Z}(q))$ (see [Es-Vi, § 1] and Section 3.1). We will be mainly interested in the cohomology groups $H_D^{2i}(X,\mathbb{Z}(i))$. They admit natural maps to $H^{2i}(X,\mathbb{Z})$ and to $\mathbb{H}^{2i}(X,\Omega_X^{\bullet\geqslant i})$, which are compatible with the diagram above. Furthermore, there is an exact sequence $0\longrightarrow \mathbb{H}^{2i-1}(X,\Omega_X^{\bullet\leqslant i-1})\big/H^{2i-1}(X,\mathbb{Z}) \longrightarrow H_D^{2i}(X,\mathbb{Z}(i))\longrightarrow H^{2i}(X,\mathbb{Z})$, (see [Vo1] and Proposition 3.3 (i)). Thus a Deligne class is a strong refinement of any of the above mentioned classes. The splitting method works for the construction of $c_i(E)$ in $H_D^{2i}(X,\mathbb{Z}(i))$, as explained in [Zu, § 4], and [Es-Vi, § 8]. These Chern classes, as all the others constructed above, satisfy the following properties:

- they are functorial with respect to pullbacks.
- if $0\longrightarrow E\longrightarrow F\longrightarrow G\longrightarrow 0$ is an exact sequence of vector bundles, then for all i
$$c_i(F)=\sum_{p+q=i} c_p(E)\,c_q(G).$$

The last property means that the total Chern class $c=1+c_1+\cdots+c_n$ is defined on the Grothendieck group $K(X)$ of holomorphic vector bundles on X and satisfies the additivity property $c(x+x')=c(x)c(x')$.

Now, what happens if we work with coherent sheaves instead of locally free ones? If X is quasi-projective and $\mathcal{F}$ is an algebraic coherent sheaf on X, there exists a locally free resolution

$$0\longrightarrow E_1\longrightarrow\cdots\longrightarrow E_N\longrightarrow\mathcal{F}\longrightarrow 0.$$

(This is still true under the weaker assumption that X is a regular separated scheme over $\mathbb{C}$ by Kleiman's lemma; see [SGA 6, II, 2.2.7.1]). The total Chern class of $\mathcal{F}$ is defined by

$$c(\mathcal{F})=c(E_N)\,c(E_{N-1})^{-1}c(E_{N-2})\dots$$

The class $c(\mathcal{F})$ does not depend on the locally free resolution (see [Bo-Se, § 4 and 6]). More formally, if $G(X)$ is the Grothendieck group of coherent sheaves on X, the canonical map $\iota\colon K(X)\longrightarrow G(X)$ is an isomorphism. The inverse is given by $[\mathcal{F}]\longrightarrow[E_N]-[E_{N-1}]+[E_{N-2}]-\dots$

In what follows, we consider the complex analytic case. The problem of the existence of global locally free resolutions in the analytic case has been opened for a long time. For smooth complex surfaces, such resolutions always exist by [Sch]. More recently, Schröer and Vezzosi proved in [ScVe] the same result for singular separated surfaces. Nevertheless, for varieties of dimension at least 3, a negative answer to the question is provided by the following counterexample of Voisin:

Theorem [Vo2] *On any generic complex torus of dimension greater than* 3*, the ideal sheaf of a point does not admit a global locally free resolution.*

Worse than that, even if $\mathcal{F}$ admits a globally free resolution $E^{\bullet}$, the method of Borel and Serre [Bo-Se] does not prove that $c(E_N)\, c(E_{N-1})^{-1} c(E_{N-2}) \dots$ is independent of $E^{\bullet}$. In fact, the crucial point in their argument is that *every* coherent sheaf should have a resolution.

Nevertheless Borel-Serre's method applies in a weaker context if we consider Chern classes in $H^*(X,\mathbb{Z})$. Indeed, if $\mathcal{F}$ is a coherent sheaf on X and $\mathcal{C}^{\omega}_X$ is the sheaf of real-analytic functions on X, then $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{C}^{\omega}_X$ admits a locally free real-analytic resolution by the Grauert vanishing theorem [Gra]. We obtain by this method topological Chern classes $c_i(\mathcal{F})^{\mathrm{top}}$ in $H^{2i}(X,\mathbb{Z})$.

It is natural to require that c_i should take its values in more refined rings depending on the holomorphic structure of $\mathcal{F}$ and X. Such a construction has been carried out by Atiyah for the Dolbeault cohomology ring in [At]. Let us briefly describe his method: the exact sequence

$$0 \longrightarrow \mathcal{F}\otimes\Omega^1_X \longrightarrow \wp^1_X(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow 0$$

of principal parts of $\mathcal{F}$ of order one (see [EGA IV, § 16.7]) gives an extension class (the Atiyah class) $a(\mathcal{F})$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}\otimes\Omega^1_X)$. Then $c_p(\mathcal{F})$ is the trace of the p-th Yoneda product of $a(\mathcal{F})$. These classes are used by O'Brian, Toledo and Tong in [OB-To-To1] and [OB-To-To2] to prove the Grothendieck-Riemann-Roch theorem on abstract manifolds in the Hodge ring. The Atiyah class has been constructed by Grothendieck and Illusie for perfect complexes (see [Ill, Ch. 5]). Nevertheless, if X is not a Kähler manifold, there is no good relation between $H^p(X,\Omega^p_X)$ and $\mathbb{H}^{2p}(X,\Omega^{\bullet\geqslant p}_X)$, as the Frölicher spectral sequence may not degenerate at E_1 for example.

In this context, the most satisfactory construction was obtained by Green in his unpublished thesis (see [Gre] and [To-To]). He proved the following theorem

Theorem 1 [Gre], [To-To] *Let $\mathcal{F}$ be a coherent sheaf on X. Then there exist Chern classes $c_i(\mathcal{F})^{\mathrm{Gr}}$ in the analytic de Rham cohomology groups $\mathbb{H}^{2i}(X,\Omega^{\bullet\geqslant i}_X)$ which are compatible with Atiyah Chern classes and topological Chern classes.*

In order to avoid the problem of nonexistence of locally free resolutions, he introduced the notion of a *simplicial resolution* by simplicial vector bundles with respect to a given covering. Green's basic result is the following:

Theorem 2 [Gre], [To-To] *Any coherent sheaf on a smooth complex compact manifold admits a finite simplicial resolution by simplicial holomorphic vector bundles.*

The next step in order to obtain Theorem 1 above, is to define the Chern classes of a simplicial vector bundle. For this, Green uses Bott's construction (see [Bott]) which can be adapted to the simplicial context. Though, it is not clear how to extend Green's method to Deligne cohomology.

Let us now state the main result of this article:

Theorem 1.1. *Let X be a complex compact manifold. For every coherent sheaf $\mathcal{F}$ on X, we can define classes $c_p(\mathcal{F})$ and $\mathrm{ch}_p(\mathcal{F})$ in $H^{2p}_D(X,\mathbb{Q}(p))$ such that:*

- (i) *For every exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ of coherent sheaves on X, we have $c(\mathcal{G}) = c(\mathcal{F})c(\mathcal{H})$ and $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$. The total Chern class $c\colon G(X) \longrightarrow H^*_D(X,\mathbb{Q})^{\times}$ is a group morphism and the Chern character $\mathrm{ch}\colon G(X) \longrightarrow H^*_D(X,\mathbb{Q})$ is a ring morphism.*

(ii) *If* $f : X \longrightarrow Y$ *is holomorphic and* y *is an element of* $G(Y)$*, then* $c(f^! y) = f^* c(y)$*, where* $f^! : G(Y) \longrightarrow G(X)$ *is the pullback in analytic* K*-theory.*

(iii) *If* $\mathcal{E}$ *is a locally free sheaf, then* $c(\mathcal{E})$ *is the usual Chern class in rational Deligne cohomology.*

(iv) *If* Z *is a smooth closed submanifold of* X *and* $\mathcal{F}$ *a coherent sheaf on* Z*, the Grothendieck-Riemann-Roch (GRR) theorem is valid for* $\big(i_Z, \mathcal{F}\big)$*, namely*

$$\mathrm{ch}\big(i_{Z*}\mathcal{F}\big) = i_{Z*}\Big(\mathrm{ch}(\mathcal{F})\ \mathrm{td}\big(N_{Z/X}\big)^{-1}\Big).$$

(v) *If* $f : X \longrightarrow Y$ *is a projective morphism between smooth complex compact manifolds, for every coherent sheaf* $\mathcal{F}$ *on* X*, we have the Grothendieck-Riemann-Roch theorem*

$$\mathrm{ch}\big(f_![\mathcal{F}]\big)\,\mathrm{td}(Y) = f_*[\mathrm{ch}(\mathcal{F})]\,\mathrm{td}(X).$$

Our approach is completely different from [Bo-Se]. Indeed, Voisin's result prevents from using locally free resolutions. Our geometric starting point, which will be used instead of locally free resolutions, is the following (Theorem 4.11):

Theorem *Let* $\mathcal{F}$ *be a coherent sheaf on* X *of generic rank* r*. Then there exists a bimeromorphic morphism* $\pi : \widetilde{X} \longrightarrow X$ *and a locally free sheaf* $\mathcal{Q}$ *on* $\widetilde{X}$ *of rank* r*, together with a surjective map* $\pi^*\mathcal{F} \longrightarrow \mathcal{Q}$ *.*

It follows that up to torsion sheaves, $\pi^![\mathcal{F}]$ is locally free. This will allow us to define our Chern classes by induction on the dimension of the base. Of course, we will need to show that our Chern classes satisfy the Whitney formula and are independent of the bimeromorphic model $\widetilde{X}$.

The theorem above is a particular case of Hironaka's flattening theorem (see [Hiro2] and in the algebraic case [Gr-Ra]). Indeed, if we apply Hironaka's result to the couple $(\mathcal{F}, \mathrm{id})$, there exists a bimeromorphic map $\sigma : \widetilde{X} \longrightarrow X$ such that $\sigma^*\mathcal{F} \;/\; \big(\sigma^*\mathcal{F}\big)_{\mathrm{tor}}$ is flat with respect to the identity morphism, and thus locally free. For the sake of completeness, we include an elementary proof of Theorem 4.11.

Property (iv) of Theorem 1.1 is noteworthy. The lack of global resolutions (see [Vo2]) prevents from using the proofs of Borel, Serre and of Baum, Fulton and McPherson (see [Fu, Ch. 15 § 2]). The equivalent formula in the topological setting is proved in [At-Hi]. In the holomorphic context, O'Brian, Toledo and Tong ([OB-To-To3]) prove this formula for the Atiyah Chern classes when there exists a retraction from X to Z, then they establish (GRR) for a projection and they deduce that (GRR) is valid for any holomorphic map, so *a posteriori* for an immersion (see [OB-To-To2]). Nevertheless, our result does not give a new proof of (GRR) formula for an immersion in the case of the Atiyah Chern classes. Indeed, the compatibility between our construction and the Atiyah Chern classes is a consequence of the (GRR) theorem for an immersion in both theories, as explained further.

Property (v) is an immediate consequence of (iv), as originally noticed in [Bo-Se], since the natural map from $G(X) \otimes_{\mathbb{Z}} G(\mathbb{P}^N)$ to $G(X \times \mathbb{P}^N)$ is surjective (see [SGA 6, Exposé VI] and [Bei]). Yet, we do not obtain the (GRR) theorem for a general holomorphic map between smooth complex compact manifolds.

Remark that $c_p(\mathcal{F})$ is only constructed in the rational Deligne cohomology group $H^{2p}_{\mathrm{D}}(X, \mathbb{Q}(p))$. The reason is that we make full use of the Chern character, which has denominators, and thus determines the total Chern class only up to torsion classes. We think that it could be possible to define $c_p(\mathcal{F})$ in $H^{2p}_{\mathrm{D}}(X, \mathbb{Z}(p))$ following our approach, but with huge computations. For $p = 1$, $c_1(\mathcal{F})$ can be easily constructed in $H^2_D(X, \mathbb{Z}(1))$. Indeed, it suffices to define $c_1(\mathcal{F}) = c_1(\det \mathcal{F})$, where $\det \mathcal{F}$ is the determinant line bundle of $\mathcal{F}$ (see [Kn-Mu]).

It is interesting to compare the classes of Theorem 1.1 with other existing theories. We adopt a more general setting by using Theorem 4.11. We prove that, for any cohomology ring satisfying reasonable properties, a theory of Chern classes can be completely determined if we suppose that the GRR formula is valid for immersions. More precisely, our statement is the following (Theorem 6.3):

Theorem *Under the hypotheses* (α)-(δ) *of page 42 on the cohomology ring, a theory of Chern classes for coherent sheaves on smooth complex compact manifolds which satisfies the GRR theorem for immersions, the Whitney additivity formula and the functoriality formula is completely determined by the first Chern class of holomorphic line bundles.*

This theorem yields compatibility results (Corollary 6.5):

Corollary *The classes of Theorem 1.1 are compatible with the rational topological Chern classes and the Atiyah Chern classes.*

Nevertheless, since GRR for immersions does not seem to be known for the Green Chern classes if X is not Kähler, the theorem above does not give the compatibility in this setting. In fact, the compatibility is *equivalent* to the GRR theorem for immersions for the Green Chern classes.

Let us mention the link of our construction with secondary characteristic classes.

We can look at a subring of the ring of Cheeger-Simons characters on X which are the "holomorphic" characters (that is the G-cohomology defined in [Es, § 4], or equivalently the restricted differential characters defined in [Br, § 2]). This subring can be mapped onto the Deligne cohomology ring, but not in an injective way in general. When E is a holomorphic vector bundle with a compatible connection, the Cheeger-Simons theory (see [Ch-Si]) produces Chern classes with values in this subring. It is known that these classes are the same as the Deligne classes (see [Br] in the algebraic case and [Zu, § 5] for the general case). When E is topologically trivial, this construction gives the so-called secondary classes with values in the intermediate jacobians of X (see [Na] for a different construction, and [Ber] who proves the link with the generalized Abel-Jacobi map). The intermediate jacobians of X have been constructed in the Kähler case by Griffiths. They are complex tori (see [Vo1, Ch. 12 § 1], and [Es-Vi, § 7 and 8]). If X is not Kähler, intermediate jacobians can still be defined but they are no longer complex tori.

Our result provides similarly refined Chern classes for coherent sheaves, and in particular, secondary invariants for coherent sheaves with trivial topological Chern classes.

The organization of the paper is the following. We recall in Part 3 the basic properties of Deligne cohomology and Chern classes for locally free sheaves in Deligne cohomology. The necessary results of analytic K-theory with support are grouped in Appendix 7; they will be used extensively throughout the paper. The rest of the article is devoted to the proof of Theorem 1.1. The construction of the Chern classes is achieved by induction on $\dim X$. In Part 4 we perform the induction step for torsion sheaves using the (GRR) formula for the immersion of smooth divisors; then we prove a dévissage theorem which enables us to break any coherent sheaf into a locally free sheaf and a torsion sheaf on a suitable modification of X; this is the key of the construction of $c_i(\mathcal{F})$ when $\mathcal{F}$ has strictly positive rank. The Whitney formula, which is a part of the induction process, is proved in Part 5. After several reductions, we use a deformation argument which leads to the deformation space of the normal cone of a smooth hypersurface. We establish the (GRR) theorem for the immersion of a smooth hypersurface in Part 4, in Part 6 we recall how to deduce the general (GRR) theorem for an immersion from this particular case, using excess formulae, then we deduce uniqueness results using Theorem 4.11.

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2. Notations and conventions

All manifolds are complex smooth analytic connected manifolds. All the results are clearly valid for non connected ones, by reasoning on the connected components.

Except in Section 3, all manifolds are compact. By *submanifold*, we always mean a *closed* submanifold.

Holomorphic vector bundles

The rank of a holomorphic vector bundle is well defined since the manifolds are connected. If E is a holomorphic vector bundle, we denote by $\mathcal{E}$ the associated locally free sheaf. The letter $\mathcal{E}$ always denotes a locally free sheaf.

Coherent sheaves

If $\mathcal{F}$ is a coherent analytic sheaf on a smooth connected manifold X, then $\mathcal{F}$ is locally free outside a proper analytic subset S of X (see [Gra-Re]). Then $U = X \backslash S$ is connected. By definition, the generic rank of $\mathcal{F}$ is the rank of the locally free sheaf $\mathcal{F}_{|U}$. If the generic rank of $\mathcal{F}$ vanishes, $\mathcal{F}$ is supported in a proper analytic subset Z of X, it is therefore annihilated by the action of a sufficiently high power of the ideal sheaf $\mathcal{I}_Z$. Conversely, if $\mathcal{F}$ is a torsion sheaf, $\mathcal{F}$ is identically zero outside a proper analytic subset of X, so it has generic rank zero. The letter $\mathcal{T}$ always denotes a torsion sheaf.

Divisors

A strict normal crossing divisor D in X is a formal sum $m_1 D_1 + \cdots + m_N D_N$, where D_i, $1 \leq i \leq N$, are smooth transverse hypersurfaces and m_i, $1 \leq i \leq N$, are nonzero integers. If all the coefficients m_i are positive, D is *effective*. In that case, the associated reduced divisor D^{red} is the effective divisor $D_1 + \cdots + D_N$. A strict simple normal crossing divisor is *reduced* if for all i, $m_i = 1$. We make no difference between a reduced divisor and its support.

By a *normal crossing divisor* we always mean a *strict* normal crossing divisor. If D is an effective simple normal crossing divisor, it defines an ideal sheaf $\mathcal{I}_D = \mathcal{O}_X(-D)$. The associated quotient sheaf is denoted by $\mathcal{O}_D$.

We use frequently Hironaka's desingularization theorem [Hiro1] for complex spaces as stated in [An-Ga, Th. 7.9 and 7.10].

Tor sheaves

Let $f\colon X \longrightarrow Y$ be a holomorphic map and $\mathcal{F}$ be a coherent sheaf on Y. We denote by $\mathrm{Tor}_i(\mathcal{F}, f)$ the sheaf $\mathrm{Tor}_i^{f^{-1}\mathcal{O}_Y}\big(f^{-1}\mathcal{F}, \mathcal{O}_X\big)$.

Grothendieck groups

The Grothendieck group of coherent analytic sheaves (resp. of torsion coherent analytic sheaves) on a complex space X is denoted by $G(X)$ (resp. $G_{\mathrm{tors}}(X)$). If $\mathcal{F}$ is a coherent analytic sheaf on X, $[\mathcal{F}]$ denotes its class in $G(X)$. The notation $G_Z(X)$ is defined in Appendix 7. In order to avoid subtle confusions, we never use here the Grothendieck group of locally free sheaves.

3. Deligne cohomology and Chern classes for locally free sheaves

In this section, we will expose the basics of Deligne cohomology for the reader's convenience. For a more detailed exposition, see [Es-Vi, § 1, 6, 7, 8], [Vo1, Ch. 12], and [EZZ].

3.1. **Deligne cohomology.**

Definition 3.1. Let X be a smooth complex manifold and let p be a nonnegative integer. Then

- The Deligne complex $\mathbb{Z}_{D,X}(p)$ of X is the following complex of sheaves

$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{(2i\pi)^p} \mathcal{O}_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{p-1},$$

  where $\mathbb{Z}_X$ is in degree zero. Similarly, the rational Deligne complex $\mathbb{Q}_{D,X}(p)$ is the same complex as above with $\mathbb{Z}_X$ replaced by $\mathbb{Q}_X$.

– The Deligne cohomology groups $H^i_D(X, \mathbb{Z}(p))$ are the hypercohomology groups defined by
$$H^i_D(X, \mathbb{Z}(p)) = \mathbb{H}^i(X, \mathbb{Z}_{D,X}(p)).$$
The rational Deligne cohomology groups are defined by the same formula as the hypercohomology groups of the rational Deligne complex.

– The same definition holds for the Deligne cohomology with support in a closed subset Z:
$$H^i_{D,Z}(X, \mathbb{Z}(p)) = \mathbb{H}^i_Z(X, \mathbb{Z}_{D,X}(p)).$$

– The total Deligne cohomology group of X is $H^*_D(X) = \bigoplus\limits_{k,p} H^k_D(X, \mathbb{Z}(p))$. We will denote by $H^*_D(X, \mathbb{Q})$ the total rational Deligne cohomology group.

Example 3.2.

– $H^i_D(X, \mathbb{Z}(0))$ is the usual Betti cohomology group $H^i(X, \mathbb{Z})$.

– $\mathbb{Z}_{D,X}(1)$ is quasi-isomorphic to $\mathcal{O}^*_X[-1]$ by the exponential exact sequence. Thus we have a group isomorphism $H^2_D(X, \mathbb{Z}(1)) \simeq H^1\big(X, \mathcal{O}^*_X\big) \simeq \mathrm{Pic}(X)$. The first Chern class of a line bundle L in $H^2_D(X, \mathbb{Z}(1))$ is the element of $\mathrm{Pic}(X)$ defined by $c_1(L) = \{L\}$.

– $H^2_D(X, \mathbb{Z}(2))$ is the group of flat holomorphic line bundles, i.e. holomorphic line bundles with a holomorphic connection (see [Es-Vi, § 1] and [Es]).

For geometric interpretations of higher Deligne cohomology groups, we refer the reader to [Ga].

Some fundamental properties of Deligne cohomology are listed below:

Proposition 3.3.

(i) *We have an exact sequence* $0 \longrightarrow \Omega^{\bullet \leqslant p-1}_X[-1] \longrightarrow \mathbb{Z}_{D,X}(p) \longrightarrow \mathbb{Z}_X \longrightarrow 0\,.$

In particular, $H^{2p}_D(X, \mathbb{Z}(p))$ *fits into the exact sequence*
$$H^{2p-1}(X, \mathbb{Z}) \longrightarrow \mathbb{H}^{2p-1}\big(X, \Omega^{\bullet \leqslant p-1}_X\big) \longrightarrow H^{2p}_D(X, \mathbb{Z}(p)) \longrightarrow H^{2p}(X, \mathbb{Z})\,.$$

(ii) *The complex* $\mathbb{Z}_{D,X}(p)[1]$ *is quasi-isomorphic to the cone of the morphism*
$$\mathbb{Z}_X \oplus \Omega^{\bullet \geqslant p}_X \xrightarrow{(2i\pi)^p,\, i} \Omega^{\bullet}_X\;.$$
Thus we have a long exact sequence:
$$\cdots \longrightarrow H^{k-1}(X, \mathbb{C}\,) \longrightarrow H^k_D(X, \mathbb{Z}(p)) \longrightarrow \mathbb{H}^k\big(X, \Omega^{\bullet \geqslant p}_X\big) \oplus H^k(X, \mathbb{Z}) \longrightarrow H^k(X, \mathbb{C}\,) \longrightarrow \cdots$$
and a similar exact sequence can be written with support in a closed subset Z.

(iii) *A cup-product*
$$H^i_D(X, \mathbb{Z}(p)) \otimes_{\mathbb{Z}} H^j_D(X, \mathbb{Z}(q)) \longrightarrow H^{i+j}_D(X, \mathbb{Z}(p+q))$$
is defined and endows $H^*_D(X)$ *with a ring structure.*

(iv) *If* $f\colon X \longrightarrow Y$ *is a holomorphic map between two smooth complex manifolds, we have a pullback morphism* $f^*\colon H^i_D(Y, \mathbb{Z}(p)) \longrightarrow H^i_D(X, \mathbb{Z}(p))$ *which is a ring morphism.*

(v) *If* X *is smooth, compact, and if* E *is a holomorphic vector bundle on* X *of rank* r*, then* $H^*_D(\mathbb{P}(E))$ *is a free* $H^*_D(X)$*-module with basis* 1, $c_1(\mathcal{O}_E(1)), \dots, c_1(\mathcal{O}_E(1))^{r-1}$.

(vi) *For every* t *in* $\mathbb{P}^1$*, let* j_t *be the inclusion* $X \simeq X \times \{t\} \hookrightarrow X \times \mathbb{P}^1$ *. Then the pullback morphism* $j_t^*\colon H^*_D(X \times \mathbb{P}^1) \longrightarrow H^*_D(X)$ *is independent of* t *(homotopy principle).*

The assertions (i) and (ii) are obvious. The cup product in (iii) comes from a morphism of complexes $\mathbb{Z}(p)\otimes_{\mathbb{Z}}\mathbb{Z}(q)\longrightarrow\mathbb{Z}(p+q)$, see [Es-Vi, § 1]. This morphism if functorial with respect to pullbacks, which gives (iv). Property (v) is proved by dévissage using the exact sequence

$$0\longrightarrow\Omega_X^{p-1}[-p]\longrightarrow\mathbb{Z}_{D,X}(p)\longrightarrow\mathbb{Z}_{D,X}(p-1)\longrightarrow 0$$

and the five lemma (see [Es-Vi, § 8]). Property (vi) is a consequence of (v): if α is a Deligne class in $H_D^*(X\times\mathbb{P}^1)$, we can write $\alpha=\mathrm{pr}_1^*\lambda+\mathrm{pr}_1^*\mu\,.\,c_1\big(\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(1)\big)$. Thus $j_t^*\alpha=\lambda$.

Remark that (vi) is false if we replace $\mathbb{P}^1(\mathbb{C})$ by $\mathbb{C}$, in contrast with the algebraic case. Indeed, take an elliptic curve S and choose an isomorphism $\phi\colon S\longrightarrow\mathrm{Pic}^0(S)$. There is a universal line bundle $\mathcal{L}$ on $S\times S$ such that for all x in S, $\mathcal{L}_{|S\times x}\simeq\phi(x)$. Let $\pi:\mathbb{C}\longrightarrow S$ be the universal covering map of S. Consider the class $\alpha=c_1\big[(\mathrm{id},\pi)^*\mathcal{L}\big]$, then for all t in $\mathbb{C}$, $j_t^*\alpha=c_1\big[\phi(\pi(t))\big]$.

We will now consider more refined properties of Deligne cohomology.

Proposition 3.4 (see [EZZ, § 2])**.**

(i) *If X is a smooth complex manifold and Z is a smooth submanifold of X of codimension d, there exists a cycle class $\{Z\}_D$ in $H^{2d}_{D,Z}(X,\mathbb{Z}(d))$ compatible with the Bloch cycle class (see* [Bl, § 5], [Es-Vi, § 6]*) and the topological cycle class. If Z and Z' intersect transversally, $\{Z\cap Z'\}_D=\{Z\}_D\,.\,\{Z'\}_D$. If Z is a smooth hypersurface of X, the image of $\{Z\}_D$ in $H^2_D(X,\mathbb{Z}(1))\simeq\mathrm{Pic}(X)$ is the class of $\mathcal{O}_X(Z)$.*

(ii) *More generally, let $f\colon X\longrightarrow Y$ be a proper holomorphic map between smooth complex manifolds and $d=\dim Y-\dim X$. Then there exists a Gysin morphism*

$$f_*\colon H^{2p}_D(X,\mathbb{Z}(q))\longrightarrow H_D^{2(p+d)}(Y,\mathbb{Z}(q+d))$$

compatible with the usual Gysin morphisms in integer and analytic de Rham cohomology. If Z is a smooth submanifold of codimension d of X and $i_Z\colon Z\longrightarrow X$ is the canonical inclusion, then $i_{Z*}(1)$ is the image of $\{Z\}_D$ in $H^{2d}_D(X,\mathbb{Z}(d))$.

The point (i) is easy to understand. By Proposition 3.3 (ii), since $H^{2d-1}_Z(X,\mathbb{Z})=0$, we have an exact sequence

$$0\longrightarrow H^{2d}_{D,Z}(X,\mathbb{Z}(d))\longrightarrow H^{2d}_Z(X,\Omega_X^{\bullet\geqslant d})\oplus H^{2d}_Z(X,\mathbb{Z})\longrightarrow H^{2d}_Z(X,\mathbb{C}).$$

The couple $\big((2i\pi)^d\{Z\}_{\mathrm{Bloch}},\{Z\}_{\mathrm{top}}\big)$ is mapped to 0 in $H^{2d}_Z(X,\mathbb{C})$ (see [Es-Vi, § 7]). Therefore, it defines a unique element $\{Z\}_D$ in $H^{2d}_{D,Z}(X,\mathbb{Z}(d))$.

For (ii), we introduce the sheaves $\mathcal{D}^k_{X,\mathbb{Z}}$ of locally integral currents of degree k as done in [EZZ, § 2], and [Gi-So, § 2.2]. These sheaves are, in a way to be properly defined, a completion of the currents induced by smooth integral chains on X (see [Ki, § 2.1] and [Fe, § 4.1.24]). Then

– $\mathcal{D}^\bullet_{X,\mathbb{Z}}$ is a soft resolution of $\mathbb{Z}_X$.
– $\mathcal{D}^k_{X,\mathbb{Z}}$ is a subsheaf of $\mathcal{D}^k_X$ stable by push-forward under proper C^∞ maps, where $\mathcal{D}^k_X$ is the sheaf of usual currents of degree k on X.

Thus $\mathbb{Z}_{D,X}(p)[1]$ is quasi-isomorphic to the cone of the morphism $\big([2i\pi]^p,i\big)\colon\mathcal{D}^\bullet_{X,\mathbb{Z}}\oplus F^p\mathcal{D}^\bullet_X\longrightarrow\mathcal{D}^\bullet_X$. We will denote by $\widetilde{\mathbb{Z}}_{D,X}(p)$ this cone shifted by minus one. Since the sheaves $\mathcal{D}^k_{X,\mathbb{Z}}$, $F^p\mathcal{D}^k_X$ and $\mathcal{D}^k_X$ are acyclic, $Rf_*\mathbb{Z}_{D,X}(p)[1]$ is quasi-isomorphic to the cone of the morphism $\big([2i\pi]^p,i\big)\colon f_*\mathcal{D}^\bullet_{X,\mathbb{Z}}\oplus f_*F^p\mathcal{D}^\bullet_X\longrightarrow f_*\mathcal{D}^\bullet_X$.

The push-forward of currents by f gives an explicit morphism $f_*\colon f_*\widetilde{\mathbb{Z}}_{D,X}(p)\longrightarrow\widetilde{\mathbb{Z}}_{D,Y}(p+d)[2d]$ and then a morphism $f_*\colon Rf_*\mathbb{Z}_{D,X}(p)\longrightarrow\mathbb{Z}_{D,Y}(p+d)[2d]$ in the derived category $\mathcal{D}^b\big(\mathrm{Mod}(\mathbb{Z}_Y)\big)$. We get the Gysin morphism by taking the hypercohomology on Y.

The compatibility between $\{Z\}_D$ and $i_{Z*}(1)$ is shown in [EZZ].

We now state all the properties of the Gysin morphism needed here. The points (vi) and (vii) use Chern classes of vector bundles. They will be defined in the next section.

Proposition 3.5.

(i) *f_* is compatible with the composition of maps and satisfies the projection formula.*

$$f_*\big(x\,.\,f^*y\big)=f_*x\,.\,y.$$

In particular, if $\Gamma_f\subseteq X\times Y$ is the graph of f and if X is compact, then for every Deligne class α in X,

$$f_*\alpha=p_{2*}\Big(p_1^*\alpha\,.\,\{\Gamma_f\}_D\Big).$$

(ii) *Consider the cartesian diagram*

$$\begin{tikzcd} Y\times Z \arrow[r, hook, "i_{Y\times Z}"] \arrow[d, "p"'] & X\times Z \arrow[d, "q"] \\ Y \arrow[r, hook, "i_Y"'] & X \end{tikzcd}$$

Then $q^*i_{Y*}=i_{Y\times Z*}\,p^*$.

(iii) *If $f\colon X\longrightarrow Y$ is proper and generically finite of degree d, then $f_*f^*=d\times\mathrm{id}$.*

(iv) *Consider the cartesian diagram, where Y and Z are compact and intersect transversally:*

$$\begin{tikzcd} W \arrow[r, hook, "i_{W\to Y}"] \arrow[d, hook, "i_{W\to Z}"'] & Y \arrow[d, hook, "i_Y"] \\ Z \arrow[r, hook, "i_Z"'] & X \end{tikzcd}$$

Then $i_Y^*\,i_{Z*}=i_{W\longrightarrow Y*}\,i^*_{W\longrightarrow Z}$.

(v) *Let $f\colon X\longrightarrow Y$ be a surjective map between smooth complex compact manifolds, and let D be a smooth hypersurface of Y such that $f^{-1}(D)$ is a simple normal crossing divisor. Let us write $f^*D=m_1\widetilde{D}_1+\cdots+m_N\widetilde{D}_N$. Let $\overline{f}_i\colon\widetilde{D}_i\longrightarrow D$ be the restriction of f to $\widetilde{D}_i$. Then*

$$f^*\,i_{D*}=\sum_{i=1}^{N}m_i\,i_{\widetilde{D}_i*}\overline{f}_i{}^*.$$

(vi) *Let X be compact, smooth, and let Y be a smooth submanifold of codimension d of X. Let $\widetilde{X}$ be the blowup of X along Y, as shown in the following diagram:*

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "q"'] & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

Then we have an isomorphism

$$\begin{aligned} H^*_D(X) \oplus \bigoplus_{i=1}^{d-1} H^*_D(Y) &\longrightarrow H^*_D(\widetilde{X}) \\ \big(x, (y_i)_{1\leq i\leq d-1}\big) &\longmapsto p^*x + \sum_{i=1}^{d-1} j_*\Big[y_i\, c_1\big(\mathcal{O}_{N_{Y/X}}(-1)\big)^{i-1}\Big]. \end{aligned}$$

In particular, if α is a Deligne class on X such that $j^*\alpha$ is the pullback of a Deligne class on Y, then α is the pullback of a unique Deligne class on X.

Moreover, if F is the excess conormal bundle defined by the exact sequence

$$0 \longrightarrow F \longrightarrow q^*N^*_{Y/X} \longrightarrow N^*_{E/\widetilde{X}} \longrightarrow 0,$$

we have the excess formula $p^*\, i_*\, \alpha = j_*\big(q^*\alpha\; c_{d-1}(F^*)\big)$.

(vii) *If Y is a smooth compact submanifold of X of codimension d, we have the auto-intersection formula*

$$i_Y^*\, i_{Y*}\alpha = \alpha\, c_d\big(N_{Z/X}\big).$$

Proof. (i) and (ii) hold at the level of complexes. More precisely, the Gysin morphism f_* is functorial at the complexes level $\widetilde{\mathbb{Z}}_D(\,.\,)$. For the projection formula, we use the complex $\widetilde{\mathbb{Z}}_{D,X}(\,.\,)$ for the variable x and the complex $\mathbb{Z}_{D,Y}(\,.\,)$ for the variable y. To prove the last formula of (i), we remark that p_2 is proper since X is compact. Then we write

$$\begin{aligned} f_*\alpha = p_{2*}(\mathrm{id}, f)_*\alpha &= p_{2*}(\mathrm{id}, f)_*\big[(\mathrm{id}, f)^*p_1^*\alpha\big] \\ &= p_{2*}\big(p_1^*\alpha\,.\,(\mathrm{id}, f)_*(1)\big) = p_{2*}\big(p_1^*\alpha\,.\,\{\Gamma_f\}_D\big). \end{aligned}$$

For (ii), we can pull-back currents under p and q since these morphisms are submersions. Then p^* and q^* are defined for the complexes $\widetilde{\mathbb{Z}}_D(\,.\,)$ and (ii) is an equality of complexes morphisms.

For (iii), it is enough by the projection formula to prove that $f_*(1) = d$, which is well known.

The formulae (iv) and (v) are of the same type. Let us prove (v) for instance. We will define first some notations: Let Γ be the graph of $i_D : D \hookrightarrow Y$ and Γ_i be the graph of $i_{\widetilde{D}_i} : \widetilde{D}_i \hookrightarrow X$. We define $\Gamma'_i = \big(\overline{f}_i, \mathrm{id}\big)_*\big(\Gamma_i\big) \subseteq D \times X$. We call $p_1 : D \times Y \longrightarrow D$ and $p_2 : D \times Y \longrightarrow Y$ the first and second projections. In the same manner, we define the projections $p'_1 : D \times X \longrightarrow D$, $p'_2 : D \times X \longrightarrow X$, $p'_{1,i} : \widetilde{D}_i \times X \longrightarrow \widetilde{D}_i$, and $p'_{2,i} : \widetilde{D}_i \times X \longrightarrow X$.

We have $(\mathrm{id}, f)^*\{\Gamma\}_D = \sum_{i=1}^{N} m_i \, \{\Gamma'_i\}_D$ (this can be seen using explicit description of the Bloch cycle class, see [Bl]). Then

$$\begin{aligned}
f^*\, i_{D*}\, \alpha &= f^*\, p_{2*}\left(p_1^*\alpha \,.\, \{\Gamma\}_D\right) && \text{by (i)}\\
&= p'_{2*}(\mathrm{id}, f)^*\left(p_1^*\,\alpha \,.\, \{\Gamma\}_D\right) && \text{by (ii)}\\
&= \sum_{i=1}^{N} m_i\, p'_{2*}\left(p_1'^{\,*}\alpha \,.\, \{\Gamma'_i\}_D\right) && \text{by the projection formula}\\
&= \sum_{i=1}^{N} m_i\, p'_{2*}\left(p_1'^{\,*}\alpha \,.\, (\overline{f}_i, \mathrm{id})_*\{\Gamma_i\}_D\right)\\
&= \sum_{i=1}^{N} m_i\, p'_{2,i*}\left(p_{1,i}'^{\,*}\overline{f}_i^{\,*}\alpha \,.\, \{\Gamma_i\}_D\right) && \text{by the projection formula}\\
&= \sum_{i=1}^{N} m_i\, i_{\widetilde{D}_i *}\, \overline{f}_i^{\,*}\, \alpha.
\end{aligned}$$

Before dealing with (vi), we prove (vii) when Y is a hypersurface. In the case of the étale cohomology, it is possible to assume that $\alpha = 1$ (see [SGA 5, Exposé VII, § 4] and [SGA $4\frac{1}{2}$, Cycle § 1.2]). Remark that this is no longer possible here, for there is no purity theorem in Deligne cohomology.

We use the deformation to the normal cone, an idea which goes back to Mumford (see [LMS] and [SGA 5, Exposé VII § 9]). The aim was originally to prove the same formula in the Chow groups. Let $M_{Y/X}$ be the blowup of $X \times \mathbb{P}^1$ along $Y \times \{0\}$, $\widetilde{X}$ be the blowup of X along Y, and $M^\circ_{Y/X} = M_{Y/X} \backslash \widetilde{X}$. Then we have an injection $F : Y \times \mathbb{P}^1 \hookrightarrow M^0_{Y/X}$ over $\mathbb{P}^1$ (see [Fu, Ch. 5] § 5.1). We denote the inclusions $N_{Y/X} \hookrightarrow M^\circ_{Y/X}$ and $Y \hookrightarrow N_{Y/X}$ by j_0 and i, the projections of $\left(Y \times \mathbb{P}^1\right) \times M^\circ_{Y/X}$ (resp. $Y \times \mathbb{P}^1$, resp. $\left(Y \times \mathbb{P}^1\right) \times N_{Y/X}$, resp. $Y \times N_{Y/X}$) on its first and second factor by pr_1 and pr_2 (resp $\widetilde{\mathrm{pr}}_1$ and $\widetilde{\mathrm{pr}}_2$, resp. pr'_1 and pr'_2, resp. pr''_1 and pr''_2). Besides, $\Gamma \subseteq Y \times \mathbb{P}^1 \times M^\circ_{Y/X}$ is the graph of F, and $\Gamma' \subseteq Y \times N_{Y/X}$ is the graph of i. Finally, $\Gamma'' = \left(i_0, \mathrm{id}_{N_{Y/X}}\right)_* \Gamma' \subseteq Y \times \mathbb{P}^1 \times N_{Y/X}$, where $i_0 : Y \times \{0\} \hookrightarrow Y \times \mathbb{P}^1$ is the injection of the central fiber. Remark that pr'_2 and pr''_2 are proper maps since Y is compact.

We have $\left(i_0, \mathrm{id}_{N_{Y/X}}\right)_*\{\Gamma'\}_D = \{\Gamma''\}_D$ and $\left(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0\right)^*\{\Gamma\}_D = \{\Gamma''\}_D$. Let $\gamma = F_*\left(\widetilde{\mathrm{pr}}_1^*\alpha\right)$. Then

$$\begin{aligned}
j_0^*\gamma &= j_0^*\,\mathrm{pr}_{2*}\left(\mathrm{pr}_1^*\widetilde{\mathrm{pr}}_1^*\alpha \,.\, \{\Gamma\}_D\right) = \mathrm{pr}'_{2*}\Big[\left(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0\right)^*\left(\mathrm{pr}_1^*\widetilde{\mathrm{pr}}_1^*\alpha \,.\, \{\Gamma\}_D\right)\Big] \quad \text{by (ii)}\\
&= \mathrm{pr}'_{2*}\Big[\left(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0\right)^*\ \mathrm{pr}_1^*\widetilde{\mathrm{pr}}_1^*\alpha \,.\, \{\Gamma''\}_D\Big]\\
&= \mathrm{pr}'_{2*}\left(i_0, \mathrm{id}_{N_{Y/X}}\right)_*\Big(\left(i_0, \mathrm{id}_{N_{Y/X}}\right)^*\ \left(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0\right)^*\ \mathrm{pr}_1^*\widetilde{\mathrm{pr}}_1^*\alpha \,.\, \{\Gamma'\}_D\Big) \quad \text{by the projection formula}\\
&= \mathrm{pr}''_{2*}\left(\mathrm{pr}_1''^{*}\alpha \,.\, \{\Gamma'\}_D\right) = i_*\alpha.
\end{aligned}$$

By the homotopy principle (Proposition 3.3 (vi)), the class $F^*\gamma_{|Y\times\{t\}}$ is independent of t. If $t \neq 0$, we have clearly $F^*\gamma_{|Y\times\{t\}} = i_Y^* i_{Y*}\alpha$. For $t = 0$, $F^*\gamma_{|Y\times\{0\}} = i^* j_0^*\gamma = i^* i_*\alpha$. Let $\pi : N_{Y/X} \longrightarrow Y$ be the projection of $N_{Y/X}$ on Y. Then $\alpha = i^*\pi^*\alpha$. Thus

$$i^* i_*\alpha = i^* i_*(i^*\pi^*\alpha) = i^*\left(\pi^*\alpha \,.\, \overline{\{Y\}}_D\right) = \alpha \,.\, i^*\overline{\{Y\}}_D,$$

where $\overline{\{Y\}}_D$ is the cycle class of Y in $N_{Y/X}$.
Now $\overline{\{Y\}}_D = c_1\big(\mathcal{O}_{N_{Y/X}}(Y)\big)$, so that $i^*\overline{\{Y\}}_D = c_1\big(N_{Y/N_{Y/X}}\big) = c_1\big(N_{Y/X}\big)$.

We can now prove (vi). Its first part is straightforward using dévissage as in Proposition 3.3 (v) and the analogous result in Dolbeault cohomology and in integer cohomology.

If α is a Deligne class on $\widetilde{X}$, we can write $\alpha = p^*x + \sum_{i=1}^{d-1} j_*\Big[y_i\, c_1\big(\mathcal{O}_{N_{Y/X}}(-1)\big)^{i-1}\Big]$. Since E is a hypersurface of $\widetilde{X}$, by the formula proved above $j^*j_*\,\lambda = \lambda\, c_1\big(N_{E/\widetilde{X}}\big) = \lambda\; c_1\big(\mathcal{O}_{N_{Y/X}}(-1)\big)$ for any Deligne class λ on E. We obtain $j^*\alpha = q^*i^*x + \sum_{i=1}^{d-1}(-1)^i y_i\, c_1\big(\mathcal{O}_{N_{Y/X}}(1)\big)^i$. Since $j^*\alpha = q^*\delta$, all the classes y_i vanish by Proposition 3.3 (v). Thus $\alpha = p^*x$. By Proposition 3.5 (iii), $x = p_*\alpha$.

For the excess formula, let α be a Deligne class on Y. We define $\beta = j_*\big(q^*\alpha\; c_{d-1}(F^*)\big)$. Then, by Proposition 3.7 (i) and (ii),

$$j^*\beta = \big[q^*\alpha\; c_{d-1}(F^*)\big]\, c_1\big(N_{E/\widetilde{X}}\big) = q^*\big[\alpha\, c_d\big(N_{Y/X}\big)\big].$$

By the discussion above, β comes from the base, so that

$$\beta = p^*p_*\beta = p^*i_*q_*\big(q^*\alpha\, c_{d-1}(F^*)\big) = p^*i_*\big[\alpha\; q_*\big(c_{d-1}(F^*)\big)\big] = p^*i_*\alpha$$

for $q_*\big(c_{d-1}(F^*)\big) = 1$ (see [Bo-Se, Lemme 19.b]).

— Proof of (vii). The formula is already true for $d = 1$. We blowup X along Y, use the excess formula (vi) and we obtain:

$$\begin{aligned} q^*i_Y^*i_{Y*}\,\alpha &= j^*p^*i_{Y*}\,\alpha = j^*\big[j_*\big(q^*\alpha\, c_{d-1}(F^*)\big)\big] \\ &= q^*\alpha\; c_{d-1}\big(F^*\big)\, c_1\big(N_{E/\widetilde{X}}\big) = q^*\big(\alpha\, c_d\big(N_{Y/X}\big)\big). \end{aligned}$$

Since q^* is injective, we get the result. □

3.2. **Chern classes for holomorphic vector bundles.** We refer to [Zu, § 4] and [Es-Vi, § 8] for all this section. From now on, we will suppose that X is smooth of dimension n. Let E be a holomorphic bundle on X of rank r, and $\mathbb{P}(E)$ be the projective bundle of E endowed with the line bundle $\mathcal{O}_E(1)$. Let $\alpha = c_1\big(\mathcal{O}_E(1)\big)$. By property (v) of Proposition 3.3, we can define $\big(c_i(E)\big)_{1\le i\le r}$ by the relation

$$\alpha^r + p^*c_1(E)\,\alpha^{r-1} + \cdots + p^*c_r(E) = 0 \quad \text{in } H_D^{2r}\big(\mathbb{P}(E), \mathbb{Z}(r)\big).$$

Thus $c_i(E)$ is an element of $H_D^{2i}(X, \mathbb{Z}(i))$. The knowledge of the Chern classes $c_i(E)$ allows to construct exponential Chern classes $\mathrm{ch}_i(E)$, $0 \le i \le n$, in $H_D^{2i}(X, \mathbb{Q}(i))$. These classes are obtained as the values of certain universal polynomials with rational coefficients on $c_1(E), \ldots, c_r(E)$ (see [Hirz]). They can also be constructed with the splitting principle using projective towers (see [Gro1]). They are completely characterized by the following facts:

- they satisfy the Whitney additivity formula (Proposition 3.7 (i));
- they satisfy the functoriality formula under pullbacks (Proposition 3.7 (ii));
- if L is a line bundle, $\mathrm{ch}(L) = e^{c_1(L)}$.

Definition 3.6. The total Chern class of E is the element $c(E)$ of $H_D^*(X)$ defined by

$$c(E) = 1 + c_1(E) + \cdots + c_r(E).$$

The Chern character of E is the element $\mathrm{ch}(E)$ of $H_D^*(X, \mathbb{Q})$ defined by

$$\mathrm{ch}(E) = \mathrm{ch}_0(E) + \cdots + \mathrm{ch}_n(E).$$

The splitting machinery gives the following proposition:

Proposition 3.7.

(i) *If* $0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$ *is an exact sequence of vector bundles on* X*, the Whitney formula holds:* $c(F) = c(E)c(G)$ *and* $\mathrm{ch}(F) = \mathrm{ch}(E) + \mathrm{ch}(G)$*.*

(ii) *If* f *is a holomorphic map between* X *and* Y *and if* E *is a holomorphic bundle on* Y*, we have* $c\big(f^*E\big) = f^*c(E)$ *and* $\mathrm{ch}\big(f^*E\big) = f^*\,\mathrm{ch}(E)$*.*

(iii) *If* E *and* F *are two holomorphic vector bundles on* X*, then* $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$*.*

Notation 3.8. From now on, if $\mathcal{E}$ is a locally free sheaf and E is the associated holomorphic vector bundle, we will denote by $\overline{\mathrm{ch}}(\mathcal{E})$ the Chern character $\mathrm{ch}(E)$. Thus $\overline{\mathrm{ch}}$ is well defined on a basis of any dimension but only for locally free sheaves, and we will make use of it in our construction.

4. Construction of Chern classes

The construction of exponential Chern classes $\mathrm{ch}_p(\mathcal{F})$ in $H_D^{2p}(X, \mathbb{Q}(p))$ for an arbitrary coherent sheaf $\mathcal{F}$ on X will be done by induction on $\dim X$. If $\dim X = 0$, X is a point and everything is obvious.
Let us now precisely state the induction hypotheses (H_n):

(E_n) If $\dim X \le n$ and $\mathcal{F}$ is a coherent analytic sheaf on X, then the Chern classes $\mathrm{ch}_p(\mathcal{F})$ are defined in $H_D^{2p}(X, \mathbb{Q}(p))$.

(W_n) If $\dim X \le n$ and $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is an exact sequence of analytic sheaves on X, then $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$. This means that ch is defined on $G(X)$ and is a group morphism.

(F_n) If $\dim X \le n$, $\dim Y \le n$, and if $f : X \longrightarrow Y$ is a holomorphic map, then for all y in $G(Y)$, $\mathrm{ch}(f^!y) = f^*\,\mathrm{ch}(y)$.

(C_n) If $\dim X \le n$, the Chern classes are compatible with those constructed in Part 3 on locally free sheaves, i.e. $\mathrm{ch}(\mathcal{F}) = \overline{\mathrm{ch}}(\mathcal{F})$ for all locally free sheaf $\mathcal{F}$.

(P_n) If $\dim X \le n$, ch is a ring morphism: if x, y are two elements of $G(X)$, $\mathrm{ch}(x\,.\,y) = \mathrm{ch}(x)\,\mathrm{ch}(y)$ and $\mathrm{ch}(1) = 1$.

(RR_n) If Z is a smooth hypersurface of X, where $\dim X \le n$, then the (GRR) theorem holds for i_Z: for every coherent sheaf $\mathcal{F}$ on Z, $\mathrm{ch}(i_{Z*}\mathcal{F}) = i_{Z*}\big(\mathrm{ch}^Z(\mathcal{F})\,\mathrm{td}(N_{Z/X})^{-1}\big)$.

For the definition of analytic K-theory and related operations we refer to [Bo-Se].

Remark 4.1. To avoid any confusion, for a coherent sheaf on $Z \subsetneq X$, we use the notation $\mathrm{ch}^Z(\mathcal{F})$, emphasizing the fact that the class is taken on Z.

From now on, we will suppose that all the properties of the induction hypotheses (H_{n-1}) above are true.

Theorem 4.2. *Assuming hypotheses* (H_{n-1})*, we can define a Chern character for analytic coherent sheaves on compact complex manifolds of dimension* n*. It further satisfies* (P_n)*,* (F_n)*,* (RR_n)*,* (W_n) *and* (C_n)*.*

Let us briefly explain the organization of the proof of this theorem. In § 4.1, we construct the Chern character for torsion sheaves. In § 4.3, we construct the Chern character for arbitrary coherent sheaves, using the results of § 4.2. Properties (RR_n) for a smooth hypersurface and (C_n) will be obvious consequences of the construction. In § 5.3, we prove (W_n) and then (F_n) and (P_n) using the preliminary results of § 5.1 and § 5.2. Finally, we prove (RR_n) in § 6.

4.1. **Construction for torsion sheaves.** In this section, we define Chern classes for torsion sheaves by forcing the Grothendieck-Riemann-Roch formula for immersions of smooth hypersurfaces. Let $G_{\mathrm{tors}}(X)$ denote the Grothendieck group of the abelian category of torsion sheaves on X. We will prove the following version of Theorem 4.2 for torsion sheaves:

Proposition 4.3. *We can define exponential Chern classes for torsion sheaves on any* n*-dimensional complex manifold such that:*

(i) (W_n) *If* $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ *is an exact sequence of torsion sheaves on X with* $\dim X \leq n$*, then* $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$*. This means that* ch *is a group morphism defined on* $G_{\mathrm{tors}}(X)$*.*

(ii) (P_n) *Let $\mathcal{E}$ be a locally free sheaf and x be an element of $G_{\mathrm{tors}}(X)$. Then* $\mathrm{ch}([\mathcal{E}].\, x) = \overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}(x)$*.*

(iii) (F_n) *Let* $f\colon X \longrightarrow Y$ *be a holomorphic map where* $\dim X \leq n$ *and* $\dim Y \leq n$*, and $\mathcal{F}$ be a coherent sheaf on Y such that $\mathcal{F}$ and $f^*\mathcal{F}$ are torsion sheaves. Then* $\mathrm{ch}(f^{\,!}[\mathcal{F}]) = f^*\,\mathrm{ch}(\mathcal{F})$*.*

(iv) (RR_n) *If Z is a smooth hypersurface of X and $\mathcal{F}$ is coherent on Z, then*

$$\mathrm{ch}\big(i_{Z*}\mathcal{F}\big) = i_{Z*}\big(\mathrm{ch}^Z(\mathcal{F})\,\mathrm{td}(N_{Z/X})^{-1}\big).$$

We will proceed in three steps. In § 4.1.1, we perform the construction for coherent sheaves supported in a smooth hypersurface. In § 4.1.2, we deal with sheaves supported in a simple normal crossing divisor. In § 4.1.3, we study the general case.

4.1.1. Let Z be a smooth hypersurface of X where $\dim X \leq n$. For $\mathcal{G}$ coherent on Z, we define $\mathrm{ch}\big(i_{Z*}\mathcal{G}\big)$ by the GRR formula $\mathrm{ch}\big(i_{Z*}\mathcal{G}\big) = i_{Z*}\big(\mathrm{ch}^Z(\mathcal{G})\,\mathrm{td}(N_{Z/X})^{-1}\big)$, where $\mathrm{ch}^Z(\mathcal{G})$ is defined by (E_{n-1}).

If $0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$ is an exact sequence of coherent sheaves on Z, by (W_{n-1}), we have $\mathrm{ch}^Z(\mathcal{G}) = \mathrm{ch}^Z(\mathcal{G}') + \mathrm{ch}^Z(\mathcal{G}'')$. Thus $\mathrm{ch}\big(i_{Z*}\mathcal{G}\big) = \mathrm{ch}\big(i_{Z*}\mathcal{G}'\big) + \mathrm{ch}\big(i_{Z*}\mathcal{G}''\big)$. We obtain now a well-defined morphism

$$\begin{array}{ccc} G(Z) & \xrightarrow[i_{Z*}]{\sim} & G_Z(X) \\ & & \big\downarrow{\scriptstyle \mathrm{ch}_Z} \\ & & H^*_D(X,\mathbb{Q}) \end{array}$$

Remark that if $\mathcal{G}$ is a coherent sheaf on X which can be written $i_{Z*}\mathcal{F}$, then the hypersurface Z is not necessarily unique. If Z is chosen, $\mathcal{F}$ is of course unique. This is the reason why we use the notation $\mathrm{ch}_Z(\mathcal{G})$. We will see in Proposition 4.8 that $\mathrm{ch}_Z(\mathcal{G})$ is in fact independent of Z.

The assertions of the following proposition are particular cases of (C_n), (F_n), and (P_n).

Proposition 4.4. *Let Z be a smooth hypersurface of X.*

(i) *For all x in $G_Z(X)$,* $\mathrm{ch}^Z\big(i_Z^!x\big) = i_Z^!\,\mathrm{ch}_Z(x)$*.*

(ii) *If $\mathcal{E}$ is a locally free sheaf on X and x is an element of $G_Z(X)$, then*

$$\mathrm{ch}_Z([\mathcal{E}].\, x) = \overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}_Z(x).$$

Proof. (i) We have $x = \overline{x}\,\bullet_Z\,[i_{Z*}\mathcal{O}_Z]$ in $G_Z(X)$, where $\overline{x}$ is defined in Appendix 7.2. Thus,

$$\begin{aligned} i_Z^*\,\mathrm{ch}_Z(x) &= i_Z^* i_{Z*}\big(\mathrm{ch}^Z(\overline{x})\,\mathrm{td}\big(N_{Z/X}\big)^{-1}\big) \\ &= \mathrm{ch}^Z(\overline{x})\,\mathrm{td}\big(N_{Z/X}\big)^{-1} c_1\big(N_{Z/X}\big) && \text{by Proposition 3.5 (vii)} \\ &= \mathrm{ch}^Z(\overline{x})\Big[1 - e^{-c_1\left(N_{Z/X}\right)}\Big] \\ &= \mathrm{ch}^Z(\overline{x})\,\mathrm{ch}^Z\big(i_Z^! i_{Z*}\mathcal{O}_Z\big) && \text{by } (\mathrm{C}_{n-1}) \\ &= \mathrm{ch}^Z\big(\overline{x}\,.\,i_Z^! i_{Z*}\mathcal{O}_Z\big) && \text{by } (\mathrm{P}_{n-1}) \\ &= \mathrm{ch}^Z\big(i_Z^! x\big). \end{aligned}$$

(ii) We have

$$
\begin{aligned}
\mathrm{ch}_Z\big([\mathcal{E}]\,.\,x\big) &= \mathrm{ch}_Z\big(i_{Z*}\big(i_Z^{!}[\mathcal{E}]\,.\,\overline{x}\big)\big)\\
&= i_{Z*}\Big(\mathrm{ch}^Z\big(i_Z^{!}\,[\mathcal{E}]\,.\,\overline{x}\big)\,\mathrm{td}\big(N_{Z/X}\big)^{-1}\Big)\\
&= i_{Z*}\Big(i_Z^{*}\,\overline{\mathrm{ch}}(\mathcal{E})\,\mathrm{ch}^Z(\overline{x})\,\mathrm{td}\big(N_{Z/X}\big)^{-1}\Big) && \text{by Proposition 3.7 (ii), } (\mathrm{P}_{n-1}) \text{ and } (\mathrm{C}_{n-1})\\
&= \overline{\mathrm{ch}}(\mathcal{E})\; i_{Z*}\Big(\mathrm{ch}^Z(\overline{x})\,\mathrm{td}\big(N_{Z/X}\big)^{-1}\Big) && \text{by the projection formula}\\
&= \overline{\mathrm{ch}}(\mathcal{E})\;\mathrm{ch}_Z(x).
\end{aligned}
$$

□

4.1.2. Let D be a divisor in X with simple normal crossing. By Proposition 7.8, we have an exact sequence:

$$\bigoplus_{i<j} G_{D_{ij}}(X) \longrightarrow \bigoplus_{i} G_{D_i}(X) \longrightarrow G_D(X) \longrightarrow 0.$$

Let us consider the morphism $\bigoplus_i \mathrm{ch}_{D_i}$. If $\mathcal{F}$ belongs to $G(D_{ij})$, then

$$\mathrm{ch}_{D_i}\big(i_{D_{ij}*}\mathcal{F}\big) = i_{D_i*}\Big(\mathrm{ch}^{D_i}\big(i_{D_{ij} \longrightarrow D_i*}\mathcal{F}\big)\,\mathrm{td}\big(N_{D_i/X}\big)^{-1}\Big) = i_{D_{ij}*}\Big(\mathrm{ch}^{D_{ij}}(\mathcal{F})\,\mathrm{td}\big(N_{D_{ij}/X}\big)^{-1}\Big)$$

because of (RR_{n-1}) and the multiplicativity of the Todd class.
Thus $\mathrm{ch}_{D_i}\big(i_{D_{ij}*}\mathcal{F}\big) = \mathrm{ch}_{D_j}\big(i_{D_{ij}*}\mathcal{F}\big)$, and we get a map $\mathrm{ch}_D : G_D(X) \longrightarrow H^*_D(X,\mathbb{Q})$ such that the diagram

$$
\begin{tikzcd}
\bigoplus_i G_{D_i}(X) \arrow[r] \arrow[dr, "\bigoplus_i \mathrm{ch}_{D_i}"'] & G_D(X) \arrow[r] \arrow[d, "\mathrm{ch}_D"] & 0\\
& H^*_D(X,\mathbb{Q}) &
\end{tikzcd}
$$

is commutative.

Proposition 4.5. *The classes* ch_D *have the following properties:*

(i) *If $\mathcal{E}$ is a locally free sheaf on X and x is an element of $G_D(X)$, then*

$$\mathrm{ch}_D([\mathcal{E}]\,.\,x) = \overline{\mathrm{ch}}(\mathcal{E}).\ \mathrm{ch}_D(x).$$

(ii) *Let $\widetilde{D}$ be an effective simple normal crossing divisor in X such that $\widetilde{D}^{red} = D$. Then*

$$\mathrm{ch}_D\big(\mathcal{O}_{\widetilde{D}}\big) = 1 - \overline{\mathrm{ch}}\big(\mathcal{O}_X(-\widetilde{D})\big).$$

(iii) (First lemma of functoriality) *Let $f : X \longrightarrow Y$ be a surjective map. Let D be a reduced divisor in Y with simple normal crossing such that $f^{-1}(D)$ is also a divisor with simple normal crossing in X. Then for all y in $G_D(Y)$, $\mathrm{ch}_{f^{-1}(D)}\big(f^{\,!}y\big) = f^*\,\mathrm{ch}_D(y)$.*

(iv) (Second lemma of functoriality) *Let Y be a smooth submanifold of X and D be a reduced divisor in X with simple normal crossing. Then, for every x in $G_D(X)$, $\mathrm{ch}^Y\big(i_Y^{!}x\big) = i_Y^*\,\mathrm{ch}_D(x)$.*

Proof. We start with two technical lemmas which will be crucial for the proof of (ii) and (iii).

Lemma 4.6. *Let $D = m_1 D_1 + \cdots + m_N D_N$ be an effective simple normal crossing divisor in X, and μ be the element of $H^*_D(X,\mathbb{Q})$ defined by*

$$\mu = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k!}\Big(m_1\{D_1\}_D + \cdots + m_N\{D_N\}_D\Big)^{k-1}.$$

Then there exist u_i in $G_{D_i}(X)$, $1 \leq i \leq N$, and ζ_{ij} in $H^*_D(D_{ij})$, $1 \leq i, j \leq N$, $i \neq j$, such that

(a) $u_1+\cdots+u_N=\mathcal{O}_D$ *in* $G_{D^{\mathrm{red}}}(X)$.
(b) $\zeta_{ij}=-\zeta_{ji}$, $1\le i,j\le N$, $i\ne j$.
(c) $\mathrm{ch}(\overline{u}_i)\,\mathrm{td}\big(N_{D_i/X}\big)^{-1}-m_i\,i^*_{D_i}\,\mu=\sum_{\substack{j=1\\ j\ne i}}^{N} i_{D_{ij}\longrightarrow D_i*}\zeta_{ij}$, $\quad 1\le i\le N$.

Proof. We proceed by induction on the number N of branches of D^{red}.
If $N=1$, we must prove that $\mathrm{ch}(\overline{u}_1)\,\mathrm{td}\big(N_{D_1/X}\big)^{-1}=m_1\,i^*_{D_1}\,\mu$, where $u_1=\mathcal{O}_{m_1D_1}$. In $G_{D_1}(X)$ we have $\mathcal{O}_{m_1D_1}=\sum_{q=0}^{m_1-1} i_{D_1*}\big(N^{*\otimes q}_{D_1/X}\big)$, thus $\quad\overline{u}_1=\sum_{q=0}^{m_1-1}N^{*\otimes q}_{D_1/X}$. Therefore

$$\begin{aligned}\mathrm{ch}(\overline{u}_1)\,\mathrm{td}\big(N_{D_1/X}\big)^{-1}&=\left(\sum_{q=0}^{m_1-1}e^{-q\,c_1(N_{D_1/X})}\right)\frac{1-e^{-c_1(N_{D_1/X})}}{c_1\big(N_{D_1/X}\big)}\\&=\frac{1-e^{-m_1c_1(N_{D_1/X})}}{c_1\big(N_{D_1/X}\big)}=m_1\,i^*_{D_1}\,\mu.\end{aligned}$$

Suppose that the lemma holds for divisors D' such that D'_{red} has $N-1$ branches. Let $D=m_1D_1+\cdots+m_ND_N$ and $D'=m_1D_1+\cdots+m_{N-1}D_{N-1}$. By induction, there exist u'_i in $G_{D_i}(X)$, $1\le i\le N-1$, and ζ'_{ij} in $H^*_D(D_{ij})$, $1\le i,j\le N-1$, $i\ne j$, satisfying properties (a), (b), and (c) of Lemma 4.6. For $0\le k\le m_N$, we introduce the divisors $Z_k=m_1D_1+\cdots+m_{N-1}D_{N-1}+kD_N$. We have exact sequences

$$0\longrightarrow i^*_{D_N}\mathcal{O}_X(-Z_k)\longrightarrow\mathcal{O}_{Z_{k+1}}\longrightarrow\mathcal{O}_{Z_k}\longrightarrow 0.$$

Thus, in $G_{D^{\mathrm{red}}}(X)$, we have

$$\mathcal{O}_D=\mathcal{O}_{D'}+i_{D_N*}\left[\sum_{q=0}^{m_{N-1}}i^*_{D_N}\mathcal{O}_X(-Z_q)\right]=\mathcal{O}_{D'}+i_{D_N*}i^*_{D_N}\left[\mathcal{O}_X(-D')\sum_{q=0}^{m_{N-1}}\mathcal{O}_X\big(-qD_N\big)\right].$$

We choose $\begin{cases}u_N=i_{D_N*}i^*_{D_N}\Big[\mathcal{O}_X(-D')\sum_{q=0}^{m_{N-1}}\mathcal{O}_X\big(-qD_N\big)\Big]\\ u_i=u'_i\qquad\text{for } 1\le i\le N-1.\end{cases}$

Let i be such that $1\le i\le N-1$. Then

$$\begin{aligned}&\mathrm{ch}(\overline{u}_i)\,\mathrm{td}\big(N_{D_i/X}\big)-m_i\,i^*_{D_i}\,\mu=\mathrm{ch}(\overline{u}'_i)\,\mathrm{td}\big(N_{D_i/X}\big)-m_i\,i^*_{D_i}\,\mu'+m_i\,i^*_{D_i}\,(\mu'-\mu)\\
&\quad=\sum_{\substack{j=1\\ j\ne i}}^{N-1}i_{D_{ij}\longrightarrow D_i*}\zeta'_{ij}+m_i\;i^*_{D_i}\left[\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{j=1}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}_D+\cdots\right.\\
&\qquad\qquad\left.\cdots+m_{N-1}\{D_{N-1}\}_D\big)^{k-1-j}\big(m_N\{D_N\}_D\big)^j\right]\qquad\text{by induction}\\
&\quad=\sum_{\substack{j=1\\ j\ne i}}^{N-1}i_{D_{ij}\longrightarrow D_i*}\zeta'_{ij}+m_i\;i_{D_{iN}\longrightarrow D_i*}i^*_{D_{iN}}\left[\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{j=1}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}_D+\cdots\right.\\
&\qquad\qquad\left.\cdots+m_{N-1}\{D_{N-1}\}_D\big)^{k-1-j}m_N^j\{D_N\}_D^{j-1}\right].\end{aligned}$$

For the last equality, we have used that

$$i^*_{D_i}\big(\alpha\,\{D_N\}_D\big)=i^*_{D_i}\alpha\,\{D_{iN}\}_D=i_{D_{iN}\longrightarrow D_i *}\big(i^*_{D_{iN}}\alpha\big)$$

where $\{D_{iN}\}_D$ is the cycle class of D_{iN} in D_i.

Let us define

$$\begin{cases}\zeta_{ij}=\zeta'_{ij} & \text{if } 1\leq i,j\leq N-1,\ i\neq j\\ \zeta_{iN}=m_i\ i^*_{D_{iN}}\left[\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{j=1}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}_D+\cdots\right. & \\ \qquad\qquad \left.\cdots+m_{N-1}\{D_{N-1}\}_D\big)^{k-1-j}m_N^j\{D_N\}_D^{j-1}\right] & \text{if } 1\leq i\leq N-1\\ \zeta_{Nj}=-\zeta_{jN} & \text{if } 1\leq j\leq N-1.\end{cases}$$

Properties (a) and (b) of Lemma 4.6 hold, and property (c) of the same lemma hold for $1\leq i\leq N-1$. For $i=N$, let us now compute both members of (c). We have

$$\sum_{l=1}^{N-1}i_{D_{Nl}\longrightarrow D_N *}\zeta_{Nl}=\sum_{l=1}^{N-1}m_l\ i^*_{D_N}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\sum_{j=1}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}_D+\cdots\right.$$
$$\left.\cdots+m_{N-1}\{D_{N-1}\}\big)^{k-1-j}m_N^j\{D_N\}_D^{j-1}\{D_l\}\right]$$
$$(*)\qquad =i^*_{D_N}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\sum_{j=1}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}_D+\cdots+m_{N-1}\{D_{N-1}\}_D\big)^{k-j}m_N^j\{D_N\}_D^{j-1}\right].$$

In the first equality, we have used

$$i_{D_{Nl}\longrightarrow D_N *}\,i^*_{D_{lN}}\alpha=i^*_{D_N}\alpha\,\{D_{lN}\}_D=i^*_{D_N}\left(\alpha\,\{D_l\}_D\right)$$

where $\{D_{lN}\}_D$ is the cycle class of D_{lN} in D_N.

Now,

$$\mathrm{ch}(\overline{u}_N)\,\mathrm{td}\big(N_{D_N/X}\big)^{-1}-m_N\,i^*_{D_N}\,\mu$$
$$=i^*_{D_N}\left[e^{-m_1\{D_1\}-\cdots-m_{N-1}\{D_{N-1}\}}\left(\sum_{q=0}^{m_{N-1}}e^{-q\{D_N\}}\right)\frac{1-e^{-\{D_N\}}}{\{D_N\}}\right]-m_N\,i^*_{D_N}\,\mu$$

by (C_{n-1}) and Proposition 3.7 (ii) and (iii)

$$=i^*_{D_N}\left[e^{-m_1\{D_1\}-\cdots-m_{N-1}\{D_{N-1}\}}\ \frac{1-e^{-m_N\{D_N\}}}{\{D_N\}}-m_N\,\mu\right]$$
$$=i^*_{D_N}\left[m_N\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\frac{(-1)^{r+q-1}}{r!\,q!}\big(m_1\{D_1\}+\cdots+m_{N-1}\{D_{N-1}\}\big)^r\big(m_N\{D_N\}\big)^{q-1}\right.$$
$$\left.-\,m_N\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\sum_{j=0}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}+\cdots+m_{N-1}\{D_{N-1}\}\big)^{k-1-j}\big(m_N\{D_N\}\big)^j\right].$$

In the first term, we put $k = q + r$, $p = q - 1$ and we obtain

$$m_N\, i^*_{D_N} \left[\sum_{k=1}^{\infty} \sum_{p=0}^{k-1} \frac{(-1)^{k-1}}{k!} \left(\binom{k}{p+1} - \binom{k-1}{p} \right) \big(m_1\{D_1\} + \cdots \right.$$

$$(**) \qquad \left. \cdots + m_{N-1}\{D_{N-1}\}\big)^{k-1-p} \big(m_N\{D_N\}\big)^p \right].$$

Now $\binom{k}{p+1} - \binom{k-1}{p}$ is equal to $\binom{k-1}{p+1}$ for $p \leq k-2$ and to zero for $p = k-1$. It suffices to put $j = p+1$ in the sum to obtain the equality of $(*)$ and $(**)$. □

Lemma 4.7. *Using the same notations as in Lemma 4.6, let α_i in $H^*_D(D_i)$, $1 \leq i \leq N$, be such that $i^*_{D_{ij} \longrightarrow D_i} \alpha_i = i^*_{D_{ij} \longrightarrow D_j} \alpha_j$. Then there exist u_i in $G_{D_i}(X)$, satisfying $u_1 + \cdots + u_N = \mathcal{O}_D$ in $G_{D^{\mathrm{red}}}(X)$, such that*

$$\sum_{i=1}^{N} i_{D_i *}\Big(\alpha_i \ \mathrm{ch}(\overline{u}_i)\, \mathrm{td}\big(N_{D_i/X}\big)^{-1}\Big) = \left(\sum_{i=1}^{N} m_i\, i_{D_i *}(\alpha_i) \right) \mu.$$

Proof. We pick $u_1, \ldots, u_N$ given by Lemma 4.6. Then

$$\sum_{i=1}^{N} i_{D_i *}\Big(\alpha_i \ \mathrm{ch}(\overline{u}_i)\ \mathrm{td}\big(N_{D_i/X}\big)^{-1}\Big) - \left(\sum_{i=1}^{N} m_i\, i_{D_i *}(\alpha_i) \right) \mu$$

$$= \sum_{i=1}^{N} i_{D_i *}\Big[\alpha_i \Big(\mathrm{ch}(\overline{u}_i)\, \mathrm{td}\big(N_{D_i/X}\big)^{-1} - m_i\, i^*_{D_i}\, \mu\Big)\Big] \qquad \text{by the projection formula}$$

$$= \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} i_{D_i *}\Big[\alpha_i \ i_{D_{ij} \longrightarrow D_i *}\, \zeta_{ij}\Big]$$

$$= \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} i_{D_{ij} *}\Big(i^*_{D_{ij} \longrightarrow D_i}\, \alpha_i \ \zeta_{ij}\Big) \qquad \text{by the projection formula.}$$

Grouping the terms (i, j) and (j, i), we get 0, since $\zeta_{ij} = -\zeta_{ji}$. □

We now prove Proposition 4.5.

Proof. (i) We write $x = x_1 + \cdots + x_N$ in $G_{D^{\mathrm{red}}}(X)$, where x_i is an element of $G_{D_i}(X)$. Then

$$\mathrm{ch}_D([\mathcal{E}]\,.\,x) = \sum_{i=1}^{N} \mathrm{ch}_{D_i}([\mathcal{E}]\,.\,x_i) = \sum_{i=1}^{N} \overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}_{D_i}(x_i) \qquad \text{by Proposition 4.4 (ii)}$$

$$= \overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}_D(x) \qquad \text{by the very definition of } \mathrm{ch}_D(x).$$

(ii) We choose $u_1, \ldots, u_N$ such that Lemma 4.6 holds. Then

$$\mathrm{ch}\big(\mathcal{O}_{\widetilde{D}}\big) = \sum_{i=1}^{N} \mathrm{ch}(u_i) = \sum_{i=1}^{N} i_{D_i *}\Big(\mathrm{ch}(\overline{u}_i)\, \mathrm{td}\big(N_{\widetilde{D}_i/X}\big)^{-1}\Big)$$

$$= \left(\sum_{i=1}^{N} m_i \{\widetilde{D}_i\}_D \right) \mu = 1 - e^{-\big(m_1\{\widetilde{D}_1\}_D + \cdots + m_N\{\widetilde{D}_N\}_D\big)} = 1 - \overline{\mathrm{ch}}\big(\mathcal{O}_X(-\widetilde{D})\big).$$

(iii) By dévissage we can suppose that D is a smooth hypersurface of Y. Let $\overline{f}_i$ be defined by the diagram

$$\begin{tikzcd} \widetilde{D}_i \arrow[r] \arrow[d, "\overline{f}_i"'] & X \arrow[d, "f"] \\ D \arrow[r] & Y \end{tikzcd}$$

and let y be an element of $G(D)$. We put $\alpha_i = \overline{f}_i^{\,*} \operatorname{ch}^D(y)$. By the functoriality property (F_{n-1}) we have $i^*_{\widetilde{D}_{ij} \longrightarrow \widetilde{D}_i} \alpha_i = i^*_{\widetilde{D}_{ij} \longrightarrow \widetilde{D}_i} \alpha_j$. We choose again $u_1, \dots, u_N$ such that Lemma 4.7 holds. By Proposition 7.7 of Appendix 7 we can write $f^{\,!} i_{D*} y = \sum_{i=1}^N (\overline{f}_i^{\,!} y) \,\raisebox{0.1em}{.}_{\widetilde{D}_i}\, u_i$. Thus

$$\begin{aligned}
\operatorname{ch}_{\widetilde{D}}\big(f^{\,!} i_{D*} y\big) &= \sum_{i=1}^N i_{\widetilde{D}_i *}\Big(\operatorname{ch}^{\widetilde{D}_i}\big(\overline{f}_i^{\,!} y\big) \operatorname{ch}^{\widetilde{D}_i}(\overline{u}_i) \operatorname{td}\big(N_{\widetilde{D}_i/X}\big)^{-1}\Big) && \text{by } (\mathrm{P}_{n-1}) \\
&= \sum_{i=1}^N i_{\widetilde{D}_i *}\Big(\alpha_i \operatorname{ch}^{\widetilde{D}_i}(\overline{u}_i) \operatorname{td}\big(N_{\widetilde{D}_i/X}\big)^{-1}\Big) && \text{by } (\mathrm{F}_{n-1}) \\
&= \left(\sum_{i=1}^N m_i \; i_{\widetilde{D}_i *}(\alpha_i)\right) \mu && \text{by Lemma 4.7} \\
&= \left[\sum_{i=1}^N m_i \; i_{\widetilde{D}_i *}\big(\overline{f}_i^{\,*} \operatorname{ch}^D(y)\big)\right] f^{\,*}\left(\frac{1 - e^{-\{D\}_D}}{\{D\}_D}\right) \\
&= f^{\,*}\left[i_{D*}\big(\operatorname{ch}^D(y)\big) \cdot \frac{1 - e^{-\{D\}_D}}{\{D\}_D}\right] && \text{by Proposition 3.5 (v)} \\
&= f^{\,*} i_{D*}\Big(\operatorname{ch}^D(y) \operatorname{td}\big(N_{D/Y}\big)^{-1}\Big) && \text{by the projection formula} \\
&= f^{\,*} \operatorname{ch}_D\big(i_{D*} y\big).
\end{aligned}$$

□

(iv) We will first prove it under the assumption that, for all i, either Y and D_i intersect transversally, or $Y = D_i$. By dévissage, we can suppose that D has only one branch and that Y and D intersect transversally, or $Y = D$. We deal with both cases separately.

– If Y and D intersect transversally, then $i_Y^! \big[i_{D*} \mathcal{O}_D\big] = \big[i_{Y \cap D \longrightarrow Y *} \mathcal{O}_{Y \cap D}\big]$. Thus, by Proposition 7.5 of Appendix 7,

$$i_Y^! x = i_Y^! \big(\overline{x} \,\raisebox{0.1em}{.}_D\, [i_{D*} \mathcal{O}_D]\big) = i^!_{Y \cap D \longrightarrow D} \overline{x} \,\raisebox{0.1em}{.}_{Y \cap D}\, \big[i_{Y \cap D \longrightarrow Y *} \mathcal{O}_{Y \cap D}\big] = i_{Y \cap D \longrightarrow Y *}\big(i^!_{Y \cap D \longrightarrow D} \overline{x}\big)$$

and we obtain

$$\begin{aligned}
\operatorname{ch}^Y\big(i_Y^! x\big) &= i_{Y \cap D \longrightarrow Y *}\Big(\operatorname{ch}^{Y \cap D}\big(i^!_{Y \cap D \longrightarrow D} \overline{x}\big) \operatorname{td}\big(N_{Y \cap D/Y}\big)^{-1}\Big) && \text{by } (\mathrm{RR}_{n-1}) \\
&= i_{Y \cap D \longrightarrow Y *}\Big(i^*_{Y \cap D \longrightarrow D} \operatorname{ch}^D(\overline{x}) \; i^*_{Y \cap D \longrightarrow D} \operatorname{td}\big(N_{D/X}\big)^{-1}\Big) && \text{by } (\mathrm{F}_{n-1}) \\
&= i_Y^* \, i_{D*}\Big(\operatorname{ch}^D(\overline{x}) \operatorname{td}\big(N_{D/X}\big)^{-1}\Big) && \text{by Proposition 3.5 (iv)} \\
&= i_Y^* \operatorname{ch}_D(x).
\end{aligned}$$

– If $Y = D$, $i_Y^![i_{D*}\mathcal{O}_D] = [\mathcal{O}_Y] - [N^*_{Y/X}]$. Thus $i_Y^!\,x = \overline{x} - \overline{x}\,.\,[N^*_{Y/X}]$ and

$$\begin{aligned}
\mathrm{ch}^Y\big(i_Y^!x\big) &= \mathrm{ch}^Y(\overline{x}) - \mathrm{ch}^Y(\overline{x})\overline{\mathrm{ch}}\big(N^*_{Y/X}\big) && \text{by } (\mathrm{P}_{n-1}) \text{ and } (\mathrm{C}_{n-1})\\
&= \mathrm{ch}^Y(\overline{x})\Big(1 - e^{-c_1(N_{Y/X})}\Big)\\
&= \mathrm{ch}^Y(\overline{x})\ \mathrm{td}\big(N_{Y/X}\big)^{-1} c_1\big(N_{Y/X}\big)\\
&= i_Y^* i_{Y*}\Big(\mathrm{ch}^Y(\overline{x})\,\mathrm{td}\big(N_{Y/X}\big)^{-1}\Big) && \text{by Proposition 3.5 (vii)}\\
&= i_Y^*\,\mathrm{ch}_Y(x).
\end{aligned}$$

We examine now the general case. By Hironaka's desingularization theorem, we can desingularize $Y \cup D$ by a succession τ of k blowups with smooth centers such that $\tau^{-1}(Y \cup D)$ is a divisor with simple normal crossing. By first blowing up X along Y, we can suppose that $\tau^{-1}(Y) = \breve{D}$ is a subdivisor of $\widetilde{D} = \tau^{-1}(Y \cup D)$. We have the following diagram:

$$\begin{tikzcd}
\breve{D}_j \arrow[r, "i_{\breve{D}_j}"] \arrow[d, "q_j"'] & \widetilde{X} \arrow[d, "\tau"] \\
Y \arrow[r, "i_Y"'] & X
\end{tikzcd}$$

Then

$$\begin{aligned}
q_j^*\,\mathrm{ch}^Y\big(i_Y^!x\big) &= \mathrm{ch}^{\breve{D}_j}\big(q_j^! i_Y^! x\big) && \text{by } (\mathrm{F}_{n-1})\\
&= \mathrm{ch}^{\breve{D}_j}\big(i^!_{\breve{D}_j}\tau^{\,!}x\big)\\
&= i^*_{\breve{D}_j}\,\mathrm{ch}_{\widetilde{D}}\big(\tau^{\,!}x\big) && \text{since } \breve{D}_j \text{ and } \widetilde{D}_i \text{ intersect transversally, or } \breve{D}_j = \widetilde{D}_i\\
&= i^*_{\breve{D}_j}\tau^*\,\mathrm{ch}_D(x) && \text{by the first lemma of functoriality 4.5 (iii)}\\
&= q_j^* i_Y^*\,\mathrm{ch}_D(x).
\end{aligned}$$

We can now write q_j as $\delta \circ \mu_j$, where E is the exceptional divisor of the blowup of X along Y, $\delta\colon E \longrightarrow Y$ is the canonical projection and $\mu_j\colon \breve{D}_j \longrightarrow E$ is the restriction of the last $k-1$ blowups to $\breve{D}_j$. Write $\tau = \tau_k \circ \tau_{k-1} \circ \cdots \circ \tau_1$ where τ_i are the blowups. Let us define a sequence of divisors $\big(E_i\big)_{0 \le i \le k}$ by induction: $E_0 = E$, and E_{i+1} is the strict transform of E_i under τ_{i+1}. Since the E_i are smooth divisors, all the maps $\tau_{i+1} : E_{i+1} \longrightarrow E_i$ are isomorphisms. There exists j such that $E_k = \breve{D}_j$. We deduce that $\mu_j = \tau_{|\breve{D}_j} : \breve{D}_j \longrightarrow D$ is an isomorphism. Since δ is the projection of the projective bundle $\mathbb{P}\big(N_{Y/X}\big) \longrightarrow Y$, δ^* is injective. Thus $q_j^* = \mu_j^*\delta^*$ is injective and we get $\mathrm{ch}^Y\big(i_Y^!x\big) = i_Y^*\,\mathrm{ch}_D(x)$. □

Now, we can clear up the problem of the dependence with respect to D of $\mathrm{ch}_D(\mathcal{F})$.

Proposition 4.8. *If D_1 and D_2 are two divisors of X with simple normal crossing such that* $\operatorname{supp}\mathcal{F} \subseteq D_1$ *and* $\operatorname{supp}\mathcal{F} \subseteq D_2$*, then* $\mathrm{ch}_{D_1}(\mathcal{F}) = \mathrm{ch}_{D_2}(\mathcal{F})$.

Proof. This property is clear if $D_1 \subseteq D_2$. We will reduce the general situation to this case. By Hironaka's theorem, there exists $\tau\colon \widetilde{X} \longrightarrow X$ such that $\tau^{-1}(D_1 \cup D_2)$ is a divisor with simple normal crossing. Let $\widetilde{D}_1 = \tau^{-1}D_1$ and $\widetilde{D}_2 = \tau^{-1}D_2$. By the first functoriality lemma 4.5 (iii), since $\widetilde{D}_1 \subseteq \widetilde{D}$, we have $\tau^*\,\mathrm{ch}_{D_1}(\mathcal{F}) = \mathrm{ch}_{\widetilde{D}_1}\big(\tau^{\,!}[\mathcal{F}]\big) = \mathrm{ch}_{\widetilde{D}}\big(\tau^{\,!}[\mathcal{F}]\big)$. The same property holds for D_2. The map τ is a succession of blowups, thus τ^* is injective and we get $\mathrm{ch}_{D_1}(\mathcal{F}) = \mathrm{ch}_{D_2}(\mathcal{F})$. □

Definition 4.9. If $\mathrm{supp}(\mathcal{F}) \subseteq D$ where D is a normal simple crossing divisor, $\mathrm{ch}(\mathcal{F})$ is defined as $\mathrm{ch}_D(\mathcal{F})$.

By Proposition 4.8, this definition makes sense.

4.1.3. We can now define $\mathrm{ch}(\mathcal{F})$ for an arbitrary torsion sheaf.

Let $\mathcal{F}$ be a torsion sheaf. We say that a succession of blowups with smooth centers $\tau : \widetilde{X} \longrightarrow X$ is a desingularization of $\mathcal{F}$ if there exists a divisor with simple normal crossing D in $\widetilde{X}$ such that $\tau^{-1}\big(\mathrm{supp}(\mathcal{F})\big) \subseteq D$. By Hironaka's theorem applied to $\mathrm{supp}(\mathcal{F})$, there always exists such a τ. We say that $\mathcal{F}$ can be desingularized in d steps if there exists a desingularization τ of $\mathcal{F}$ consisting of at most d blowups. In that case, $\mathrm{ch}(\tau^{!}[\mathcal{F}])$ is defined by Definition 4.9.

Proposition 4.10. *There exists a class* $\mathrm{ch}(\mathcal{F})$ *uniquely determined by* $\mathcal{F}$ *such that*

(i) *If* τ *is a desingularization of* $\mathcal{F}$*, then* $\tau^* \mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\tau^{!}[\mathcal{F}])$.
(ii) *If* Y *is a smooth submanifold of* X*, then* $\mathrm{ch}^Y\big(i_Y^{!}[\mathcal{F}]\big) = i_Y^* \mathrm{ch}(\mathcal{F})$.

Proof. Let d be the number of blowups necessary to desingularize $\mathcal{F}$. Both assertions will be proved at the same time by induction on d.

If $d = 0$, $\mathrm{supp}(\mathcal{F})$ is a subset of a divisor with simple normal crossing D. The properties (i) and (ii) are immediate consequences of the two lemmas of functoriality 4.5 (iii) and (iv).

Suppose now that Proposition 4.10 is proved for torsion sheaves which can be desingularized in $d-1$ steps. Let $\mathcal{F}$ be a torsion sheaf which can be desingularized with at most d blowups. Let $(\widetilde{X}, \tau)$ be such a desingularization. We write τ as $\widetilde{\tau} \circ \tau_1$, where $\widetilde{\tau}$ is the first blowup in τ with E as exceptional divisor, as shown in the following diagram:

$$\begin{tikzcd}
 & \widetilde{X} \arrow[d, "\tau_1"] \\
E \arrow[r, "i_E"] \arrow[d, "q"'] & \widetilde{X}_1 \arrow[d, "\widetilde{\tau}"] \\
Y \arrow[r, "i_Y"'] & X
\end{tikzcd}$$

Then τ_1 consists of at most $d-1$ blowups and is a desingularization of the sheaves $\mathrm{Tor}_j(\mathcal{F}, \widetilde{\tau})$, $0 \leq j \leq n$. By induction, we can consider the following expression in $H^*_D(\widetilde{X}_1, \mathbb{Q})$:

$$\gamma\big(\widetilde{X}_1, \mathcal{F}\big) = \sum_{j=0}^{n} (-1)^j \, \mathrm{ch}\Big(\mathrm{Tor}_j(\mathcal{F}, \widetilde{\tau})\Big).$$

We have

$$\begin{aligned}
i_E^* \gamma\big(\widetilde{X}_1, \mathcal{F}\big) &= \sum_{j=0}^{n} (-1)^j \, \mathrm{ch}^E\Big(i_E^{!}\big[\mathrm{Tor}_j(\mathcal{F}, \widetilde{\tau})\big]\Big) && \text{by induction, property (ii)} \\
&= \mathrm{ch}^E\big(i_E^{!}\widetilde{\tau}^{!}[\mathcal{F}]\big) && \text{by } (\mathrm{W}_{n-1}) \\
&= \mathrm{ch}^E\big(q^{!} i_Y^{!}[\mathcal{F}]\big) = q^* \mathrm{ch}^Y\big(i_Y^{!}[\mathcal{F}]\big) && \text{by } (\mathrm{F}_{n-1}).
\end{aligned}$$

By Proposition 3.5 (vi), there exists a unique class $\mathrm{ch}(\mathcal{F},\tau)$ on X such that $\gamma\big(\widetilde{X}_1,\mathcal{F}\big)=\widetilde{\tau}^*\,\mathrm{ch}\big(\mathcal{F},\tau\big)$. Now

$$\begin{aligned}
\tau^*\,\mathrm{ch}(\mathcal{F},\tau) &= {\tau_1}^*\gamma\big(\widetilde{X}_1,\mathcal{F}\big)\\
&= \sum_{j=0}^{n}(-1)^j\,\mathrm{ch}\Big(\tau_1^{\,!}\big[\mathrm{Tor}_j(\mathcal{F},\widetilde{\tau})\big]\Big) && \text{by induction, property (i)}\\
&= \mathrm{ch}\big(\tau_1^{\,!}\widetilde{\tau}^{\,!}[\mathcal{F}]\big)=\mathrm{ch}\big(\tau^{\,!}[\mathcal{F}]\big).
\end{aligned}$$

Suppose that we have two resolutions. We dominate them by a third one, according to the diagram:

$$\begin{tikzcd}
 & W \arrow[dl, "\mu"'] \arrow[dd, "\delta"] \arrow[dr, "\breve{\mu}"] & \\
\widetilde{X}_1 \arrow[dr, "\tau"'] & & \widetilde{X}_2 \arrow[dl, "\breve{\tau}"] \\
 & X &
\end{tikzcd}$$

Then

$$\begin{aligned}
\delta^*\,\mathrm{ch}(\mathcal{F},\tau) &= \mu^*\,\mathrm{ch}\big(\tau^{\,!}[\mathcal{F}]\big)\\
&= \mathrm{ch}\big(\mu^{\,!}\tau^{\,!}[\mathcal{F}]\big) && \text{by the first lemma of functoriality 4.5 (iii)}\\
&= \mathrm{ch}\big(\delta^{\,!}[\mathcal{F}]\big)=\delta^*\,\mathrm{ch}(\mathcal{F},\breve{\tau}) && \text{by symmetry.}
\end{aligned}$$

The map δ^* being injective, $\mathrm{ch}(\mathcal{F},\tau)=\mathrm{ch}(\mathcal{F},\breve{\tau})$, and we can therefore define $\mathrm{ch}(\mathcal{F})$ by $\mathrm{ch}(\mathcal{F})=\mathrm{ch}(\mathcal{F},\tau)$ for any desingularization τ of $\mathcal{F}$ with at most d blowups.

We have shown that (i) is true when τ consists of at most d blowups. In the general case, let $\tau\colon\widetilde{X}_1\longrightarrow X$ be an arbitrary desingularization of $\mathcal{F}$ and $\breve{\tau}$ be a desingularization of $\mathcal{F}$ with at most d blowups. We can find W, μ and $\breve{\mu}$ as before. Then

$$\begin{aligned}
\mu^*\,\mathrm{ch}\big(\tau^{\,!}[\mathcal{F}]\big) &= \mathrm{ch}\big(\delta^{\,!}[\mathcal{F}]\big) && \text{by the first functoriality lemma 4.5 (iii)}\\
&= \breve{\mu}^*\,\mathrm{ch}\big(\breve{\tau}^{\,!}[\mathcal{F}]\big) && \text{by the first functoriality lemma 4.5 (iii)}\\
&= \breve{\mu}^*\breve{\tau}^*\,\mathrm{ch}(\mathcal{F}) && \text{since } \breve{\tau} \text{ consists of at most } d \text{ blowups}\\
&= \mu^*\tau^*\,\mathrm{ch}(\mathcal{F}).
\end{aligned}$$

It remains to show (ii). For this, we desingularize $\mathrm{supp}(\mathcal{F})\cup Y$ exactly as in the proof of the second lemma of functoriality 4.5 (iv). We have a diagram

$$\begin{tikzcd}
\breve{D}_i \arrow[r, "i_{\breve{D}_i}"'] \arrow[d, "q_i"'] & \widetilde{X} \arrow[d, "\tau"] \\
Y \arrow[r, "i_Y"'] & X
\end{tikzcd}$$

where q_i^* is injective for at least one i. Then

$$\begin{aligned}
q_i^*\big(i_Y^*\,\mathrm{ch}(\mathcal{F})\big) &= i_{\breve{D}_i}^*\tau^*\,\mathrm{ch}(\mathcal{F})=i_{\breve{D}_i}^*\,\mathrm{ch}(\tau^{\,!}[\mathcal{F}]) && \text{by (i)}\\
&= \mathrm{ch}^{\breve{D}_i}\big(i_{\breve{D}_i}^{\,!}\tau^{\,!}[\mathcal{F}]\big) && \text{by the second lemma of functoriality 4.5 (iv)}\\
&= \mathrm{ch}^{\breve{D}_i}\big(q_i^{\,!}\,i_Y^{\,!}[\mathcal{F}]\big)=q_i^*\,\mathrm{ch}^Y\big(i_Y^{\,!}[\mathcal{F}]\big) && \text{by } (\mathrm{F}_{n-1}).
\end{aligned}$$

Thus $i_Y^*\,\mathrm{ch}(\mathcal{F})=\mathrm{ch}^Y\big(i_Y^{\,!}\,[\mathcal{F}]\big)$, which proves (ii). $\square$

We have now completed the existence part of Theorem 4.2 for torsion sheaves.

We turn to the proof of Proposition 4.3. So doing, we establish almost all the properties listed in the induction hypotheses for torsion sheaves.

Proof. (i) Let $(\widetilde{X},\tau)$ be a desingularization of $\mathrm{supp}(\mathcal{F})\cup\mathrm{supp}(\mathcal{H})$ and D be the associated simple normal crossing divisor. Then $\tau^{\,!}\mathcal{F}$, $\tau^{\,!}\mathcal{G}$ and $\tau^{\,!}\mathcal{H}$ belong to $G_D(\widetilde{X})$ and $\tau^{\,!}\mathcal{F}+\tau^{\,!}\mathcal{H}=\tau^{\,!}\mathcal{G}$ in $G_D(X)$. Thus, by Proposition 4.10 (i),

$$\tau^*\big[\mathrm{ch}(\mathcal{F})+\mathrm{ch}(\mathcal{H})\big]=\mathrm{ch}\big(\tau^{\,!}[\mathcal{F}]\big)+\mathrm{ch}\big(\tau^{\,!}[\mathcal{H}]\big)=\mathrm{ch}\big(\tau^{\,!}[\mathcal{G}]\big)=\tau^*\,\mathrm{ch}(\mathcal{G}).$$

The map τ^* being injective, we get the Whitney formula for torsion sheaves.

(ii) The method is the same: let $x=[\mathcal{G}]$ and let τ be a desingularization of $\mathcal{G}$. Then, by Proposition 4.10 (i) and Proposition 4.5 (i),

$$\tau^*\,\mathrm{ch}([\mathcal{E}]\,.\,[\mathcal{G}])=\mathrm{ch}\big(\tau^{\,!}[\mathcal{E}]\,.\,\tau^{\,!}[\mathcal{G}]\big)=\overline{\mathrm{ch}}\big(\tau^{\,!}[\mathcal{E}]\big)\,.\,\mathrm{ch}\big(\tau^{\,!}[\mathcal{G}]\big)=\tau^*\big(\overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}(\mathcal{G})\big).$$

(iii) This property is known when f if the immersion of a smooth submanifold and when f is a bimeromorphic morphism by Proposition 4.10. Let us consider now the general case. By Grauert's direct image theorem, $f(X)$ is an irreducible analytic subset of Y. We desingularize $f(X)$ as an abstract complex space. We get a connected smooth manifold W and a bimeromorphic morphism $\tau\colon W\longrightarrow f(X)$ obtained as a succession of blowups with smooth centers in $f(X)$. We perform a similar sequence of blowups, starting from $Y_1=Y$ and blowing up at each step in Y_i the smooth center blown up at the i-th step of the desingularization of $f(X)$. Let $\pi_Y\colon\widetilde{Y}\longrightarrow Y$ be this morphism. The strict transform of $f(X)$ is W. The map $\tau:\tau^{-1}\big(f(X)_{\mathrm{reg}}\big)\xrightarrow{\sim} f(X)_{\mathrm{reg}}$ is an isomorphism. So we get a morphism $f(X)_{\mathrm{reg}}\longrightarrow W$ which is in fact a meromorphic map from $f(X)$ to W, and finally, after composition on the left by f, from X to W. We desingularize this morphism:

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "\pi_X"'] \arrow[dr, "\widetilde{f}"] & \\
X \arrow[r, dotted] & W
\end{tikzcd}$$

and we get the following global diagram, where π_X is a bimeromorphic map:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_X"'] & W \arrow[r, "i_W"] \arrow[d, "\tau"] & \widetilde{Y} \arrow[d, "\pi_Y"] \\
X \arrow[r, "f"'] & f(X) \arrow[r] & Y
\end{tikzcd}$$

Now $f\circ\pi_X=\pi_Y\circ\big(i_W\circ\widetilde{f}\,\big)$, and we know the functoriality formula for π_X, π_Y and i_W by Proposition 4.10. Since π_X^* is injective, it is enough to show the functoriality formula for $\widetilde{f}$. So we will assume that f is onto.

Let $(\tau,\widetilde{Y})$ be a desingularization of $\mathcal{F}$. We have the diagram

$$\begin{tikzcd}
X\times_Y\widetilde{Y} \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d, "\tau"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where $\widetilde{\tau}^{-1}(\operatorname{supp}\mathcal{F}) = D \subseteq \widetilde{Y}$ is a divisor with simple normal crossing and the map $X \times_Y \widetilde{Y} \longrightarrow X$ is a bimeromorphic morphism. We have a meromorphic map $X \dashrightarrow X \times_Y \widetilde{Y}$, and we desingularize it by a morphism $T \longrightarrow X \times_Y \widetilde{Y}$. Therefore, we obtain the following commutative diagram, where $\pi\colon T \longrightarrow X$ is a bimeromorphic map:

$$\begin{tikzcd} T \arrow[r, "\widetilde{f}"] \arrow[d, "\pi"'] & \widetilde{Y} \arrow[d, "\tau"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Therefore we can assume that $\operatorname{supp}(\mathcal{F})$ is included in a divisor with simple normal crossing D. We desingularize $f^{-1}(D)$ so that we are led to the case $\operatorname{supp}(\mathcal{F}) \subseteq D$, where D and $f^{-1}(D)$ are divisors with simple normal crossing in Y and X respectively. In this case, we can use the first lemma of functoriality 4.5 (iii). □

4.2. **The case of sheaves of positive rank.** In this section, we consider the case of sheaves of arbitrary rank. We are going to introduce the main tool of the construction, namely a dévissage theorem for coherent analytic sheaves. Let X be a complex compact manifold and $\mathcal{F}$ an analytic coherent sheaf on X. We have seen in section 4.1 how to define $\operatorname{ch}(\mathcal{F})$ when $\mathcal{F}$ is a torsion sheaf.

Suppose that $\mathcal{F}$ has strictly positive generic rank. When $\mathcal{F}$ admits a global locally free resolution, we could try to define $\operatorname{ch}(\mathcal{F})$ the usual way. As explained in the introduction, this condition on $\mathcal{F}$ is not necessarily fulfilled. Even if such a resolution exists, the definition of $\operatorname{ch}(\mathcal{F})$ depends a priori on this resolution. A good substitute for a locally free resolution is a locally free quotient with maximal rank, since the kernel is then a torsion sheaf. Let $\mathcal{F}_{\mathrm{tor}} \subseteq \mathcal{F}$ be the maximal torsion subsheaf of $\mathcal{F}$. Then $\mathcal{F}$ admits a locally free quotient $\mathcal{E}$ of maximal rank if and only if $\mathcal{F} \big/ \mathcal{F}_{\mathrm{tor}}$ is locally free. In this case, $\mathcal{E} = \mathcal{F} \big/ \mathcal{F}_{\mathrm{tor}}$.

Unfortunately, the existence of such a quotient is not assured (for instance, take a torsion-free sheaf which is not locally free), but we will show that it exists up to a bimeromorphic morphism.

Theorem 4.11. *Let X be a complex compact manifold and $\mathcal{F}$ a coherent analytic sheaf on X. There exists a bimeromorphic morphism $\sigma : \widetilde{X} \longrightarrow X$, which is a finite composition of blowups with smooth centers, such that $\sigma^*\mathcal{F}$ admits a locally free quotient of maximal rank on $\widetilde{X}$. Such quotients are unique, up to a unique isomorphism.*

Proof. Let r be the rank of $\mathcal{F}$. We define a universal set $\widetilde{X}$ by $\widetilde{X} = \coprod_{x \in X} \operatorname{Gr}^*\big(r, \mathcal{F}_{|x}\big)$ where $\operatorname{Gr}^*\big(r, \mathcal{F}_{|x}\big)$ is the dual grassmannian of quotients of $\mathcal{F}_{|x}$ with rank r. The set $\widetilde{X}$ is the disjoint union of all the quotients of rank r of all fibers of $\mathcal{F}$. The canonical map $\sigma\colon \widetilde{X} \longrightarrow X$ is an isomorphism on $\sigma^{-1}\big(\mathcal{F}_{\mathrm{reg}}\big)$. We will now endow $\widetilde{X}$ with the structure of a reduced complex space.

Let us first argue locally. Let $\mathcal{O}^p_{|U} \xrightarrow{M} \mathcal{O}^q_{|U} \longrightarrow \mathcal{F}_{|U} \longrightarrow 0$ be a presentation of $\mathcal{F}$ on an open set U. Here, M is an element of $\mathfrak{M}_{q,p}\big(\mathcal{O}_U\big)$. Then, for every x in U, we get the exact sequence

$$\mathbb{C}^p \xrightarrow{M(x)} \mathbb{C}^q \xrightarrow{\pi_x} \mathcal{F}_{|x} \longrightarrow 0.$$

We have an inclusion

$$\operatorname{Gr}^*\big(r, \mathcal{F}_{|x}\big) \hookrightarrow \operatorname{Gr}^*\big(r, \mathbb{C}^q\big) \xrightarrow{\sim} \operatorname{Gr}(q-r, q)$$

given by

$$(q,Q)\longmapsto (q\circ\pi_x,Q)\longmapsto \ker(q\circ\pi_x),$$

where q and Q appear in the following diagram:

$$\begin{tikzcd}
\mathbb{C}^p \arrow[r, "M(x)"] & \mathbb{C}^q \arrow[r, "\pi_x"] \arrow[dr, dotted, "\widetilde{q}"'] & \mathcal{F}_{|x} \arrow[r] \arrow[d, "q"] & 0 \\
& & Q \arrow[d] & \\
& & 0 &
\end{tikzcd}$$

Therefore, we have a fibered inclusion $\sigma^{-1}(U)\hookrightarrow U\times \mathrm{Gr}(q-r,q)$. We know that

$$\sigma^{-1}(U)=\big\{(x,E)\in U\times \mathrm{Gr}(q-r,q) \text{ such that } \operatorname{Im} M(x)\subseteq E\big\}.$$

Let $(e_1,\dots,e_q)$ be the canonical basis of $\mathbb{C}^q$, and $e_1^*,\dots,e_q^*$ its dual basis. We can suppose that $e_1^*,\dots,e_{q-r}^*$ are linearly independent on E. We parametrize $\mathrm{Gr}(q-r,q)$ in the neighborhood of x by a matrix $A=\big(a_{i,j}\big)\in\mathfrak{M}_{r,q-r}(\mathbb{C})$. The associated vector space will be spanned by the columns of the matrix $\begin{pmatrix}\mathrm{id}_{q-r}\\ A\end{pmatrix}$. Writing $M=\Big(M_j^i\Big)_{\substack{1\le j\le q\\ 1\le i\le p}}$, $\operatorname{Im}(x)$ is a subspace of E if and only if for all i, $1\le i\le p$,

$$\Big(M_1^i(x)e_1+\dots+M_q^i(x)e_q\Big)\wedge\bigwedge_{l=1}^{q-r}\big(e_l+a_{1,l}e_{q-r+1}+\dots+a_{r,l}e_q\big)=0$$

in $\bigwedge^{q-r+1}\mathbb{C}^q$. This is clearly an analytic condition in the variables x and a_{ij}, thus $\sigma^{-1}(U)$ is an analytic subset of $U\times\mathrm{Gr}(q-r,q)$. We endow $\sigma^{-1}(U)$ with the associated *reduced* structure.

We must check carefully that the structure defined above does not depend on the chosen presentation.

Let us consider two resolutions of $\mathcal{F}$ on U

$$\mathcal{O}^p_{|U}\xrightarrow{M}\mathcal{O}^q_{|U}\xrightarrow{\pi}\mathcal{F}_{|U}\longrightarrow 0$$

$$\mathcal{O}^{p'}_{|U}\xrightarrow{M'}\mathcal{O}^{q'}_{|U}\xrightarrow{\pi'}\mathcal{F}_{|U}\longrightarrow 0$$

and a (q',q) matrix $\alpha\colon\mathcal{O}^q_{|U}\longrightarrow\mathcal{O}^{q'}_{|U}$ such that the diagram

$$\begin{tikzcd}
\mathcal{O}^p_{|U} \arrow[r, "M"] & \mathcal{O}^q_{|U} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \mathcal{F}_{|U} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\mathcal{O}^{p'}_{|U} \arrow[r, "M'"] & \mathcal{O}^{q'}_{|U} \arrow[r, "\pi'"] & \mathcal{F}_{|U} \arrow[r] & 0
\end{tikzcd}$$

commutes. The morphism

$$\begin{tikzcd}
\sigma^{-1}(U) \arrow[r, "\mathrm{id}"] \arrow[d, hook] & \sigma^{-1}(U) \arrow[d, hook] \\
U\times\mathrm{Gr}(q-r,q) \arrow[r] & U\times\mathrm{Gr}(q'-r,q')
\end{tikzcd}$$

is given by $(x,E)\longmapsto \big(x,{\pi'_x}^{-1}\pi_x(E)\big)$ according to the following diagram:

$$\begin{tikzcd}
\mathbb{C}^p \arrow[r, "M(x)"] & \mathbb{C}^q \arrow[r, "\pi_x"] \arrow[d, "\alpha(x)"] & \mathcal{F}_{|x} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\mathbb{C}^{p'} \arrow[r, "M'(x)"'] & \mathbb{C}^{q'} \arrow[r, "\pi'_x"'] & \mathcal{F}_{|x} \arrow[r] & 0
\end{tikzcd}$$

Since $\pi'_x\big(\alpha(x)(E)\big)=\pi_x(E)$, we have ${\pi'_x}^{-1}\pi_x(E)={\pi'_x}^{-1}\pi'_x\big(\alpha(x)(E)\big)$. We can write this

$${\pi'_x}^{-1}\pi_x(E)=\alpha(x)(E)+\ker\pi'_x=\alpha(x)(E)+\operatorname{Im}M'(x).$$

If (x_0,E_0) is an element of $\sigma^{-1}(U)\times\mathrm{Gr}(q-r,q)$, then $\alpha(x_0)(E_0)+\operatorname{Im}M'(x_0)$ belongs to $\mathrm{Gr}(q'-r,q')$. We can suppose as above that $e_1^*,\dots,e_{q-r}^*$ are linearly independent on E_0. Therefore, E_0 is spanned by $q-r$ vectors $A_1,\dots,A_{q-r}$ where $A_i=\big(0,\dots,1,\dots,a_{1,i},\dots,a_{q-r,i}\big)^{\mathrm{T}}$, the first "1" having index i. Then $\alpha(x_0)(E_0)+\operatorname{Im}M'(x_0)$ is spanned by the vectors $\big(\alpha(x_0)(A_i)+M'^j(x_0)\big)_{\substack{1\le i\le q-r\\1\le j\le p'}}$, where the M'^j are the columns of M'. We can find $q'-r$ independent vectors in this family, and this property holds also in a neighborhood of (x_0,E_0). We denote these vectors by $\big(\alpha(x)(A_{i_k})+M'^{j_k}(x)\big)_{1\le k\le q'-r}$. Let us define

$$f\big(A_1,\dots,A_{q-r},x\big)=\operatorname{span}\big(\alpha(x)(A_{i_k})+M'^{j_k}(x)\big)_{1\le k\le q'-r}.$$

This is a holomorphic map defined in a neighborhood of (x_0,E_0) with values in $U\times\mathrm{Gr}(q'-r,q')$. On the same pattern, we can define another map g on a neighborhood of $f\big(x_0,E_0\big)$ with values in $U\times\mathrm{Gr}(q-r,q)$. The couple (f,g) defines an isomorphism of complex spaces. This proves that $\widetilde{X}$ is endowed with the structure of an intrinsic reduced complex analytic space (not generally smooth).

We define a subsheaf $\mathcal{N}$ of $\sigma^*\mathcal{F}$ by

$$\mathcal{N}(V)=\big\{s\in\sigma^*\mathcal{F}(V)\text{ such that }\forall\big(x,\{q,Q\}\big)\in V,\ s_x\in\ker q\big\}.$$

Remark that $\mathcal{N}$ is supported in the singular locus of $\sigma^*\mathcal{F}$.

Lemma 4.12. $\mathcal{N}$ *satisfies the following properties:*

(i) $\mathcal{N}$ *is a coherent subsheaf of* $\sigma^*\mathcal{F}$.

(ii) $\sigma^*\mathcal{F}/_{\mathcal{N}}$ *is locally free and* $\operatorname{rank}\big(\sigma^*\mathcal{F}/_{\mathcal{N}}\big)=\operatorname{rank}(\mathcal{F})$.

Proof. We take a local presentation $\mathcal{O}^p_{|U}\xrightarrow{M}\mathcal{O}^q_{|U}\xrightarrow{\pi}\mathcal{F}_{|U}\longrightarrow 0$ of $\mathcal{F}$. Then $\sigma^*\mathcal{F}$ has the presentation

$$\mathcal{O}^p_{|\sigma^{-1}(U)}\xrightarrow{M\circ\sigma}\mathcal{O}^q_{|\sigma^{-1}(U)}\longrightarrow\sigma^*\mathcal{F}_{|\sigma^{-1}(U)}\longrightarrow 0.$$

Let $(x,E)\longmapsto\big(f_1(x,E),\dots,f_q(x,E)\big)$ be a section of $\sigma^*\mathcal{F}$ on $V\subseteq\sigma^{-1}(U)$. Then s is a section of $\mathcal{N}$ if and only if for every (x,E) in V, $\big(f_1(x,E),\dots,f_q(x,E)\big)$ is an element of E. Let $U_{q-r,q}$ be the universal bundle on $\mathrm{Gr}(q-r,q)$. Then $U_{q-r,q\,|V}$ is a subbundle of $\mathcal{O}^q_{|V}$ and $\mathcal{N}_{|V}$ is the image of $U_{q-r,q\,|V}$ by the morphism $U_{q-r,q\,|V}\hookrightarrow\mathcal{O}^q_{|\sigma^{-1}(U)}\xrightarrow{\pi}\sigma^*\mathcal{F}_{|\sigma^{-1}(U)}$. So $\mathcal{N}$ is coherent.

(ii) Let us define $\mathcal{E}$ by $\mathcal{E}=\sigma^*\mathcal{F}/_{\mathcal{N}}$. For all (x,E) in V, we have an exact sequence

$$\mathcal{N}_{|(x,E)}\longrightarrow\mathcal{F}_{|x}\longrightarrow\mathcal{E}_{|(x,E)}\longrightarrow 0$$

and a commutative diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & \mathbb{C}^q \arrow[d, "\pi_x"] \\ \mathcal{N}_{|(x,E)} \arrow[r] & \mathcal{F}_{|x} \end{tikzcd}$$

The first vertical arrow is the morphism $U_{q-r,q\,|U} \longrightarrow \mathcal{N}$ restricted at (x,E), so it is onto. Thus $\pi_x(E)$ is the image of the morphism $\mathcal{N}_{|(x,E)} \longrightarrow \mathcal{F}_{|x}$. Since we have an exact sequence

$$0 \longrightarrow \pi_x(E) \longrightarrow \mathcal{F}_{|x} \xrightarrow{q} Q \longrightarrow 0$$

where $(x,E) = (x,Q)$, then $\dim \pi_x(E) = \dim \mathcal{F}_{|x} - r$. Since $\dim \mathcal{F}_{|x} = \dim \pi_x(E) + \dim \mathcal{E}_{|(x,E)}$, we get $\dim \mathcal{E}_{|(x,E)} = r$. We can see that $\mathcal{N}$ is a torsion sheaf, for $\mathcal{E}$ is locally free of rank r. $\square$

We can now finish the proof of Theorem 4.11. Using Hironaka's theorem, we desingularize the complex space $\widetilde{X}$. We get a succession of blowups with smooth centers $\widetilde{\sigma}\colon \widetilde{X}' \longrightarrow \widetilde{X}$ where $\widetilde{X}'$ is smooth. By the Hironaka-Chow lemma (see [An-Ga, Th. 7.8]), we can suppose that $\tau = \sigma \circ \widetilde{\sigma}$ is a succession of blowups with smooth centers. Since $\mathcal{E}$ is locally free, the following sequence is exact:

$$0 \longrightarrow \widetilde{\sigma}^*\mathcal{N} \longrightarrow \tau^*\mathcal{F} \longrightarrow \widetilde{\sigma}^*\mathcal{E} \longrightarrow 0\,.$$

Therefore $\tau^*\mathcal{F}$ admits a locally free quotient of maximal rank. The unicity is clear. This finishes the proof. $\square$

4.3. **Construction of the classes in the general case.** Let X be a complex compact manifold of dimension n.

4.3.1. Let $\mathcal{F}$ be a coherent sheaf on X which has a locally free quotient of maximal rank. We have an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$

where $\mathcal{T}$ is a torsion sheaf and $\mathcal{E}$ is locally free. Then we define $\mathrm{ch}(\mathcal{F})$ by $\mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\mathcal{T}) + \overline{\mathrm{ch}}(\mathcal{E})$, where $\mathrm{ch}(\mathcal{T})$ has been constructed in part 4.1. Remark that $\mathrm{ch}(\mathcal{F})$ depends only on $\mathcal{F}$, the exact sequence $0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$ being unique up to (a unique) isomorphism.

We state now the Whitney formulae which apply to the Chern characters we have defined above.

Proposition 4.13. *Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ be an exact sequence of coherent analytic sheaves on X. Then $\mathrm{ch}(\mathcal{F})$, $\mathrm{ch}(\mathcal{G})$ and $\mathrm{ch}(\mathcal{H})$ are well defined and verify $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$ under any of the following hypotheses:*

(i) *$\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are locally free sheaves on X.*
(ii) *$\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are torsion sheaves.*
(iii) *$\mathcal{G}$ admits a locally free quotient of maximal rank and $\mathcal{F}$ is a torsion sheaf.*

Proof. (i) If $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are locally free sheaves on X, then $\mathrm{ch}(\mathcal{F}) = \overline{\mathrm{ch}}(\mathcal{F})$, $\mathrm{ch}(\mathcal{G}) = \overline{\mathrm{ch}}(\mathcal{G})$, $\mathrm{ch}(\mathcal{H}) = \overline{\mathrm{ch}}(\mathcal{H})$ and we use Proposition 3.7 (i).

(ii) This is Proposition 4.3 (i).

(iii) Let $\mathcal{E}$ be the locally free quotient of maximal rank of $\mathcal{G}$. We have an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow 0$$

where $\mathcal{T}$ is a torsion sheaf. Since $\mathcal{F}$ is a torsion sheaf, the morphism $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}$ is identically zero. Let us define $\mathcal{T}'$ by the exact sequence

$$0 \longrightarrow \mathcal{T}' \longrightarrow \mathcal{H} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Then $\mathcal{T}'$ is a torsion sheaf and we have the exact sequence of torsion sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}' \longrightarrow 0.$$

Thus $\mathcal{H}$ admits a locally free quotient of maximal rank, so that $\mathrm{ch}(\mathcal{H})$ is defined and

$$\begin{aligned}\mathrm{ch}(\mathcal{H}) &= \overline{\mathrm{ch}}(\mathcal{E}) + \mathrm{ch}(\mathcal{T}') = \overline{\mathrm{ch}}(\mathcal{E}) + \mathrm{ch}(\mathcal{T}) - \mathrm{ch}(\mathcal{F}) && \text{by (ii)} \\ &= \mathrm{ch}(\mathcal{G}) - \mathrm{ch}(\mathcal{F}).\end{aligned}$$

□

Let us now look at the functoriality properties with respect to pullbacks.

Proposition 4.14. *Let $f\colon X \longrightarrow Y$ be a holomorphic map. We assume that*

- $\dim Y = n$ *and* $\dim X \leq n$,
- *if* $\dim X = n$, f *is surjective.*

Then for every coherent sheaf on Y which admits a locally free quotient of maximal rank, the following properties hold:

(i) *The Chern characters* $\mathrm{ch}\big(\mathrm{Tor}_i(\mathcal{F}, f)\big)$ *are well defined.*

(ii) $f^*\,\mathrm{ch}(\mathcal{F}) = \sum_{i\geq 0}(-1)^i\,\mathrm{ch}\big(\mathrm{Tor}_i(\mathcal{F}, f)\big)$.

Proof. (i) If $\dim X < n$, the classes $\mathrm{ch}\big(\mathrm{Tor}_i(\mathcal{F}, f)\big)$ are defined by the induction property (E_{n-1}). If $\dim X = n$ and f is surjective, then f is generically finite. Thus all the sheaves $\mathrm{Tor}_i(\mathcal{F}, f)$, $i \geq 1$, are torsion sheaves on X, so their Chern classes are defined by Proposition 4.3. The sheaf $f^*\mathcal{F}$ admits on X a locally free quotient of maximal rank so that $\mathrm{ch}(f^*\mathcal{F})$ is well defined.

(ii) We have an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$

where $\mathcal{T}$ is a torsion sheaf and $\mathcal{E}$ is a locally free sheaf. Remark that, for $i \geq 1$, $\mathrm{Tor}_i(\mathcal{F}, f) \simeq \mathrm{Tor}_i(\mathcal{T}, f)$. Thus, by Proposition 3.7 (ii) and Proposition 4.3 (iii),

$$\begin{aligned}\sum_{i\geq 0}(-1)^i\,\mathrm{ch}\big(\mathrm{Tor}_i(\mathcal{F}, f)\big) &= \overline{\mathrm{ch}}\big(f^*\mathcal{E}\big) + \mathrm{ch}\big(f^*\mathcal{T}\big) + \sum_{i\geq 1}(-1)^i\,\mathrm{ch}\Big(\mathrm{Tor}_i(\mathcal{T}, f)\Big) \\ &= f^*\overline{\mathrm{ch}}(\mathcal{E}) + \mathrm{ch}\big(f^![\mathcal{T}]\big) = f^*\big(\overline{\mathrm{ch}}(\mathcal{E}) + \mathrm{ch}(\mathcal{T})\big) = f^*\,\mathrm{ch}(\mathcal{F}).\end{aligned}$$

□

4.3.2. We consider now an arbitrary coherent sheaf $\mathcal{F}$ on X. By Theorem 4.11, there exists $\sigma\colon \widetilde{X} \longrightarrow X$ obtained as a finite composition of blowups with smooth centers such that $\sigma^*\mathcal{F}$ admits a locally free quotient of maximal rank. This is the key property for the definition of $\mathrm{ch}(\mathcal{F})$ in full generality.

Theorem 4.15. *There exists a class $\mathrm{ch}(\mathcal{F})$ on X uniquely determined by $\mathcal{F}$ such that:*

(i) *If $\sigma\colon \widetilde{X} \longrightarrow X$ is a succession of blowups with smooth centers such that $\sigma^*\mathcal{F}$ admits a locally free quotient of maximal rank, then* $\sigma^*\,\mathrm{ch}(\mathcal{F}) = \sum_{i\geq 0}(-1)^i\,\mathrm{ch}\big(\mathrm{Tor}_i(\mathcal{F}, \sigma)\big)$.

(ii) *If Y is a smooth submanifold of X,* $\mathrm{ch}^Y\big(i_Y^![\mathcal{F}]\big) = i_Y^*\,\mathrm{ch}(\mathcal{F})$.

Remark 4.16. By Proposition 4.14 (i), all the terms in (i) are defined.

Proof. The proof of (i) will use Lemmas 4.17, 4.18, and 4.19. We will prove the result by induction on the number d of blowups in σ. If $d=0$, $\mathcal{F}$ admits a locally free quotient of maximal rank and we can use Proposition 4.14.

Suppose now that (i) and (ii) hold at step $d-1$. As usual, we look at the first blowup in σ

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "\sigma_1"] \arrow[dd, bend left, "\sigma"] \\ E \arrow[r, "i_E"] \arrow[d, "q"'] & \widetilde{X}_1 \arrow[d, "\widetilde{\sigma}"] \\ Y \arrow[r, "i_Y"'] & X. \end{tikzcd}$$

The sheaves $\mathrm{Tor}_j(\mathcal{F},\sigma)$ are torsion sheaves for $j\geq 1$ and $\sigma_1^*\,\mathrm{Tor}_0(\mathcal{F},\widetilde{\sigma})=\sigma^*\mathcal{F}$ admits a locally free quotient of maximal rank. Since σ_1 consists of $d-1$ blowups, we can define by induction a class $\gamma(\mathcal{F})$ in $H^*_D(\widetilde{X}_1,\mathbb{Q})$ as follows:

$$\gamma(\mathcal{F})=\sum_{j\geq 0}(-1)^j\,\mathrm{ch}\Big(\mathrm{Tor}_j(\mathcal{F},\widetilde{\sigma})\Big).$$

Lemma 4.17. $\sigma_1^*\gamma(\mathcal{F})=\displaystyle\sum_{i\geq 0}(-1)^i\,\mathrm{ch}\big(\mathrm{Tor}_i(\mathcal{F},\sigma)\big)$.

Proof. We have by induction

$$\sigma_1^*\gamma(\mathcal{F})=\sum_{p,q\geq 0}(-1)^{p+q}\,\mathrm{ch}\Big[\mathrm{Tor}_p(\mathrm{Tor}_q(\mathcal{F},\widetilde{\sigma}),\sigma_1)\Big]=\sum_{p,q\geq 0}(-1)^{p+q}\,\mathrm{ch}(E_2^{p,q})$$

where the Tor spectral sequence satisfies

$$\begin{aligned} E_2^{p,q}&=\mathrm{Tor}_p(\mathrm{Tor}_q(\mathcal{F},\widetilde{\sigma}),\sigma_1)\\ E_\infty^{p,q}&=\mathrm{Gr}^p\ \mathrm{Tor}_{p+q}(\mathcal{F},\sigma). \end{aligned}$$

All the $E_r^{p,q}$, $2\leq r\leq\infty$, are torsion sheaves except perhaps $E_r^{0,0}$. Remark that no arrow $d_r^{p,q}$ starts or arrives at $E_r^{0,0}$. Thus we have

$$\sum_{\substack{p,q\\ p+q\geq 1}}(-1)^{p+q}[E_2^{p,q}]=\sum_{\substack{p,q\\ p+q\geq 1}}(-1)^{p+q}[E_\infty^{p,q}]=\sum_{i\geq 1}(-1)^i\big[\mathrm{Tor}_i(\mathcal{F},\sigma)\big]$$

in $G_{\mathrm{tors}}(X)$. Using Proposition 4.13 (ii), we get

$$\begin{aligned}\sigma_1^*\gamma(\mathcal{F})&=\mathrm{ch}\big(E_2^{0,0}\big)+\mathrm{ch}\Big(\sum_{i\geq 1}(-1)^i\,\mathrm{Tor}_i(\mathcal{F},\sigma)\Big)\\ &=\sum_{i\geq 0}(-1)^i\,\mathrm{ch}\Big(\mathrm{Tor}_i(\mathcal{F},\sigma)\Big).\end{aligned}$$

□

Lemma 4.18. *There exists a unique class* $\mathrm{ch}(\mathcal{F},\sigma)$ *on* X *such that* $\gamma(\mathcal{F})=\widetilde{\sigma}^*\,\mathrm{ch}(\mathcal{F},\sigma)$.

Proof. We have

$$
\begin{aligned}
i_E^*\gamma(\mathcal{F}) &= i_E^*\Big(\sum_{j\geq 0}(-1)^j \operatorname{ch}\big(\operatorname{Tor}_j(\mathcal{F},\sigma)\big)\Big) \\
&= \sum_{j\geq 0}(-1)^j \operatorname{ch}^E\big(i_E^{\,!}\,[\operatorname{Tor}_j(\mathcal{F},\sigma)]\big) && \text{by induction property (ii)} \\
&= \operatorname{ch}^E\big(i_E^{\,!}\widetilde{\sigma}^{\,!}[\mathcal{F}]\big) = \operatorname{ch}^E\big(q^{\,!}i_Y^{\,!}[\mathcal{F}]\big) = q^{\,*}\operatorname{ch}^Y\big(i_Y^{\,!}[\mathcal{F}]\big) && \text{by } (\mathrm{F}_{n-1}).
\end{aligned}
$$

By Proposition 3.5 (vi), there exists a unique class $\operatorname{ch}(\mathcal{F},\sigma)$ on X such that $\gamma(\mathcal{F}) = \widetilde{\sigma}^{\,*}\operatorname{ch}(\mathcal{F},\sigma)$. □

Putting Lemma 4.17 and Lemma 4.18 together

$$\sigma^{\,*}\operatorname{ch}(\mathcal{F},\sigma) = \sigma_1^{\,*}\gamma(\mathcal{F}) = \sum_{i\geq 0}(-1)^i \operatorname{ch}\big(\operatorname{Tor}_i(\mathcal{F},\sigma)\big).$$

Lemma 4.19. *The class* $\operatorname{ch}(\mathcal{F},\sigma)$ *does not depend on* σ*.*

Proof. As usual, if we have two resolutions, we dominate them by a third one, as shown in the diagram

$$
\begin{tikzcd}
 & \widetilde{X} \arrow[dl, "\sigma_1"'] \arrow[dd, "\sigma"] \arrow[dr, "\sigma_1'"] & \\
\widetilde{X}_1 \arrow[dr, "\widetilde{\sigma}"'] & & \widetilde{X}_1' \arrow[dl, "\widetilde{\sigma}'"] \\
 & X &
\end{tikzcd}
$$

Now

$$
\begin{aligned}
\sigma^{\,*}\operatorname{ch}(\mathcal{F},\widetilde{\sigma}) &= \sigma_1^{\,*}\gamma(\mathcal{F}) && \text{by Lemma 4.18} \\
&= \sum_{i\geq 0}(-1)^i \operatorname{ch}\big(\operatorname{Tor}_i(\mathcal{F},\sigma)\big) && \text{by Lemma 4.17.}
\end{aligned}
$$

By symmetry $\sigma^{\,*}\operatorname{ch}(\mathcal{F},\widetilde{\sigma}) = \sigma^{\,*}\operatorname{ch}(\mathcal{F},\widetilde{\sigma}')$ and we get the result. □

We proved the existence statement and part (i) of Theorem 4.15 if σ consists of at most d blowups. The general case follows using the diagram above.

We must now prove Theorem 4.15 (ii). Let Y be a smooth submanifold of X. We choose $\sigma\colon \widetilde{X} \longrightarrow X$ such that $\sigma^{\,*}\mathcal{F}$ admits a locally free quotient of maximal rank and $\sigma^{-1}(Y)$ is a simple normal crossing divisor with branches D_j. We choose as usual j such that q_j^* is injective, q_j being defined by the diagram

$$
\begin{tikzcd}
D_j \arrow[r, "i_{D_j}"] \arrow[d, "q_j"'] & \widetilde{X} \arrow[d, "\sigma"] \\
Y \arrow[r, "i_Y"] & X
\end{tikzcd}
$$

We have

$$
\begin{aligned}
q_j^*\operatorname{ch}^Y\big(i_Y^{\,!}[\mathcal{F}]\big) = \operatorname{ch}^{D_j}\big(q_j^{\,!}i_Y^{\,!}[\mathcal{F}]\big) &= \operatorname{ch}^{D_j}\big(i_{D_j}^{\,!}\sigma^{\,!}[\mathcal{F}]\big) \\
&= \sum_{i\geq 0}(-1)^i\, i_{D_j}^*\operatorname{ch}\big(\operatorname{Tor}_i(\mathcal{F},\sigma)\big) && \text{by Proposition 4.14 (ii).}
\end{aligned}
$$

Now, by the point (i), we have $\sum_{i\geq 0}(-1)^i \operatorname{ch}\big(\operatorname{Tor}_i(\mathcal{F},\sigma)\big) = \sigma^{\,*}\operatorname{ch}(\mathcal{F})$. Thus we get

$$q_j^*\operatorname{ch}^Y(i_Y^{\,!}[\mathcal{F}]) = i_{D_j}^*\sigma^{\,*}\operatorname{ch}(\mathcal{F}) = q_j^*\big(i_Y^*\operatorname{ch}(\mathcal{F})\big).$$

Therefore $\mathrm{ch}\big(i_Y^!\,[\mathcal{F}]\big) = i_Y^*\,\mathrm{ch}(\mathcal{F})$ and the proof is complete. $\square$

5. The Whitney formula

In the previous section, we achieved an important step in the induction process by defining the classes $\mathrm{ch}(\mathcal{F})$ when $\mathcal{F}$ is any coherent sheaf on a n-dimensional manifold. To conclude the proof of Theorem 4.2, it remains to check properties (W_n), (F_n) and (P_n). The crux of the proof is in fact property (W_n). The main result of this section is Theorem 5.1. The other induction hypotheses will be proved in Theorem 5.14.

Theorem 5.1. (W_n) *holds.*

To prove Theorem 5.1, we need several reduction steps.

5.1. Reduction to the case where $\mathcal{F}$ and $\mathcal{G}$ are locally free and $\mathcal{H}$ is a torsion sheaf.

Proposition 5.2. *Suppose that* (W_n) *holds when* $\mathcal{F}$ *and* $\mathcal{G}$ *are locally free sheaves and* $\mathcal{H}$ *is a torsion sheaf. Then* (W_n) *holds for arbitrary sheaves.*

Proof. We proceed by successive reductions.

Lemma 5.3. *It is sufficient to prove* (W_n) *when* $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ *admit a locally free quotient of maximal rank.*

Proof. We take a general exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$. Let $\sigma\colon \widetilde{X} \longrightarrow X$ be a bimeromorphic morphism such that $\sigma^*\mathcal{F}$, $\sigma^*\mathcal{G}$ and $\sigma^*\mathcal{H}$ admit locally free quotients of maximal rank (we know that such a σ exists by Theorem 4.11). We have an exact sequence defining $\mathcal{Q}$ and $\mathcal{T}_1$:

$$\begin{tikzcd}
&&&&&& 0 \arrow[dr] && 0 &&&& \\
&&&&&&& \mathcal{Q} \arrow[ur] \arrow[dr] &&&&& \\
\cdots \arrow[rr] && \mathrm{Tor}_1(\mathcal{G},\sigma) \arrow[rr] && \mathrm{Tor}_1(\mathcal{H},\sigma) \arrow[rr] \arrow[dr] && \sigma^*\mathcal{F} \arrow[rr] \arrow[ur] && \sigma^*\mathcal{G} \arrow[rr] && \sigma^*\mathcal{H} \arrow[rr] && 0 \\
&&&&& \mathcal{T}_1 \arrow[ur] \arrow[dr] &&&&&&& \\
&&&& 0 \arrow[ur] && 0 &&&&&&
\end{tikzcd}$$

Remark that $\mathcal{T}_1$ is a torsion sheaf. By Proposition 4.13 (iii), $\mathcal{Q}$ admits a locally free quotient of maximal rank and $\mathrm{ch}(\sigma^*\mathcal{F}) = \mathrm{ch}(\mathcal{T}_1) + \mathrm{ch}(\mathcal{Q})$. Furthermore,

$$[\mathcal{T}_1] - \big[\mathrm{Tor}_1(\mathcal{H},\sigma)\big] + \big[\mathrm{Tor}_1(\mathcal{G},\sigma)\big] - \cdots = 0 \text{ in } G_{\mathrm{tors}}(\widetilde{X}).$$

Then by Proposition 4.10 (i) and Proposition 4.13 (ii),

$$\begin{aligned}
\sigma^*\big(\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \mathrm{ch}(\mathcal{G})\big) &= \sum_{i\geq 0}(-1)^i\Big[\mathrm{ch}\big(\mathrm{Tor}_i(\mathcal{F},\sigma)\big) + \mathrm{ch}\big(\mathrm{Tor}_i(\mathcal{H},\sigma)\big) - \mathrm{ch}\big(\mathrm{Tor}_i(\mathcal{G},\sigma)\big)\Big] \\
&= \mathrm{ch}\big(\sigma^*\mathcal{F}\big) + \mathrm{ch}\big(\sigma^*\mathcal{H}\big) - \mathrm{ch}\big(\sigma^*\mathcal{G}\big) - \mathrm{ch}(\mathcal{T}_1) \\
&= \mathrm{ch}(\mathcal{Q}) + \mathrm{ch}\big(\sigma^*\mathcal{H}\big) - \mathrm{ch}\big(\sigma^*\mathcal{G}\big).
\end{aligned}$$

Since σ^* is injective, Lemma 5.3 is proved. $\square$

Lemma 5.4. *It is sufficient to prove* (W_n) *when* $\mathcal{F}$, $\mathcal{G}$ *admit a locally free quotient of maximal rank and* $\mathcal{H}$ *is a torsion sheaf.*

Proof. By Lemma 5.3, we can assume that $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ admit a locally free quotient of maximal rank. In the sequel, the letter "$\mathcal{T}$" will denote a torsion sheaf and the letter "$\mathcal{E}$" a locally free sheaf. Let $\mathcal{E}_1$ be the locally free quotient of maximal rank of $\mathcal{H}$, so we have an exact sequence

$$0 \longrightarrow \mathcal{T}_1 \longrightarrow \mathcal{H} \longrightarrow \mathcal{E}_1 \longrightarrow 0.$$

We define $\mathcal{F}_1$ by the exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}_1 \longrightarrow 0.$$

Then we get a third exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{T}_1 \longrightarrow 0.$$

We have by definition $\mathrm{ch}(\mathcal{H}) = \overline{\mathrm{ch}}(\mathcal{E}_1) + \mathrm{ch}(\mathcal{T}_1)$. Thus,

$$\begin{aligned}
\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \mathrm{ch}(\mathcal{G}) &= \big(\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{T}_1) - \mathrm{ch}(\mathcal{F}_1)\big) + \big(\mathrm{ch}(\mathcal{F}_1) + \overline{\mathrm{ch}}(\mathcal{E}_1) - \mathrm{ch}(\mathcal{G})\big) \\
&\quad - \big(\mathrm{ch}(\mathcal{T}_1) + \overline{\mathrm{ch}}(\mathcal{E}_1) - \mathrm{ch}(\mathcal{H})\big) \\
&= \big(\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{T}_1) - \mathrm{ch}(\mathcal{F}_1)\big) + \big(\mathrm{ch}(\mathcal{F}_1) + \overline{\mathrm{ch}}(\mathcal{E}_1) - \mathrm{ch}(\mathcal{G})\big).
\end{aligned}$$

Let $\mathcal{E}_2$ be the locally free quotient of maximal rank of $\mathcal{G}$. We define $\mathcal{T}_2$ by the exact sequence

$$0 \longrightarrow \mathcal{T}_2 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}_2 \longrightarrow 0.$$

The morphism from $\mathcal{G}$ to $\mathcal{E}_1$ (via $\mathcal{H}$) induces a morphism $\mathcal{E}_2 \longrightarrow \mathcal{E}_1$ which remains of course surjective. Let $\mathcal{E}_3$ be the kernel of this morphism, then $\mathcal{E}_3$ is a locally free sheaf. We get an exact sequence

$$0 \longrightarrow \mathcal{T}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E}_3 \longrightarrow 0.$$

Therefore $\mathcal{F}_1$ admits a locally free quotient of maximal rank and $\mathrm{ch}(\mathcal{F}_1) = \mathrm{ch}(\mathcal{T}_2) + \overline{\mathrm{ch}}(\mathcal{E}_3)$. On the other hand, by Proposition 4.13 (i), $\overline{\mathrm{ch}}\big(\mathcal{E}_1\big) + \overline{\mathrm{ch}}\big(\mathcal{E}_3\big) = \overline{\mathrm{ch}}\big(\mathcal{E}_2\big)$, and we obtain $\mathrm{ch}\big(\mathcal{F}_1\big) + \overline{\mathrm{ch}}\big(\mathcal{E}_1\big) - \mathrm{ch}(\mathcal{G}) = \big(\mathrm{ch}(\mathcal{T}_2) + \overline{\mathrm{ch}}(\mathcal{E}_3)\big) + \big(\overline{\mathrm{ch}}(\mathcal{E}_2) - \overline{\mathrm{ch}}(\mathcal{E}_3)\big) - \big(\mathrm{ch}(\mathcal{T}_2) + \overline{\mathrm{ch}}(\mathcal{E}_2)\big) = 0$. Therefore,

$$\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{T}_1) - \mathrm{ch}(\mathcal{F}_1).$$

Since $\mathcal{T}_1$ is a torsion sheaf, we are done. □

We can now conclude the proof of Proposition 5.2.

By Lemma 5.4, we can suppose that $\mathcal{F}$, $\mathcal{G}$ admit locally free quotients of maximal rank and $\mathcal{H}$ is a torsion sheaf. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the locally free quotients of maximal rank of $\mathcal{F}$ and $\mathcal{G}$. We define $\mathcal{T}_1$ and $\mathcal{T}_2$ by the two exact sequences

$$0 \longrightarrow \mathcal{T}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}_1 \longrightarrow 0$$
$$0 \longrightarrow \mathcal{T}_2 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}_2 \longrightarrow 0.$$

The morphism $\mathcal{F} \longrightarrow \mathcal{G}$ induces a morphism $\mathcal{T}_1 \longrightarrow \mathcal{T}_2$. We get a morphism $\mathcal{E}_1 \longrightarrow \mathcal{E}_2$ with torsion kernel and cokernel. Since $\mathcal{E}_1$ is a locally free sheaf, this morphism is injective. In the following diagram, we introduce the cokernels $\mathcal{T}_3$ and $\mathcal{T}_4$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{T}_1 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{T}_2 \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{E}_2 \arrow[r] \arrow[d] & 0 \\
 & \mathcal{T}_3 \arrow[d] & \mathcal{H} \arrow[d] & \mathcal{T}_4 \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By the nine lemma, $0 \longrightarrow \mathcal{T}_3 \longrightarrow \mathcal{H} \longrightarrow \mathcal{T}_4 \longrightarrow 0$ is an exact sequence of torsion sheaves. Then by Proposition 4.3 (i),

$$\begin{aligned}
\operatorname{ch}(\mathcal{F}) + \operatorname{ch}(\mathcal{H}) - \operatorname{ch}(\mathcal{G}) &= \operatorname{ch}\big(\mathcal{T}_1\big) + \overline{\operatorname{ch}}\big(\mathcal{E}_1\big) + \operatorname{ch}\big(\mathcal{T}_3\big) + \operatorname{ch}\big(\mathcal{T}_4\big) - \operatorname{ch}\big(\mathcal{T}_2\big) - \overline{\operatorname{ch}}\big(\mathcal{E}_2\big) \\
&= \overline{\operatorname{ch}}\big(\mathcal{E}_1\big) + \operatorname{ch}\big(\mathcal{T}_4\big) - \overline{\operatorname{ch}}\big(\mathcal{E}_2\big).
\end{aligned}$$

This finishes the proof. □

5.2. **A structure theorem for coherent torsion sheaves of projective dimension one.** In section 5.1 we have reduced the Whitney formula to the particular case where $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is a torsion sheaf. We are now going to prove that it is sufficient to suppose that $\mathcal{H}$ is the push-forward of a locally free sheaf on a smooth hypersurface of X. The main tool of this section is the following proposition:

Proposition 5.5. *Let $\mathcal{H}$ be a torsion sheaf which admits a global locally free resolution of length two. Then there exist a bimeromorphic morphism $\sigma\colon \widetilde{X} \longrightarrow X$ obtained by a finite number of blowups with smooth centers, a simple normal crossing divisor D, $D \subseteq X$, and an increasing sequence $\big(D_i\big)_{1\le i\le r}$ of subdivisors of D such that $\sigma^*\mathcal{H}$ is everywhere locally isomorphic to $\bigoplus_{i=1}^{r} \mathcal{O}_{\widetilde{X}}/\mathcal{I}_{D_i}$.*

Proof. Let $0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{H} \longrightarrow 0$ be a locally free resolution of $\mathcal{H}$, such that $\operatorname{rank}\big(\mathcal{E}_1\big) = \operatorname{rank}\big(\mathcal{E}_2\big) = r$. Recall that the kth Fitting ideal of $\mathcal{H}$ is the coherent ideal sheaf generated by the determinants of all the $k \times k$ minors of M when M is any local matrix realization in coordinates of the morphism $\mathcal{E}_1 \longrightarrow \mathcal{E}_2$ (for a general presentation of the Fitting ideals, see [Ei]). We have

$$\operatorname{Fitt}_1(\mathcal{H}) \supseteq \operatorname{Fitt}_2(\mathcal{H}) \supseteq \cdots \supseteq \operatorname{Fitt}_r(\mathcal{H}) \supsetneq \{0\}.$$

These ideals have good functoriality properties. Indeed, if $\sigma\colon \widetilde{X} \longrightarrow X$ is a bimeromorphic morphism, the sequence $0 \longrightarrow \sigma^*\mathcal{E}_1 \longrightarrow \sigma^*\mathcal{E}_2 \longrightarrow \sigma^*\mathcal{H} \longrightarrow 0$ is exact and $\operatorname{Fitt}_j\big(\sigma^*\mathcal{H}\big) = \sigma^* \operatorname{Fitt}_j(\mathcal{H})$ (by $\sigma^* \operatorname{Fitt}_j(\mathcal{H})$, we mean of course its image in $\mathcal{O}_{\widetilde{X}}$). By the Hironaka theorem, we can suppose, after taking a finite number of pullbacks under blowups with smooth centers, that all the Fitting ideals $\operatorname{Fitt}_k(\mathcal{F})$ are ideal sheaves associated with effective normal crossing divisors D'_k. Now, take an element x of X. Consider an exact sequence

$$\mathcal{O}_U^r \xrightarrow[M]{} \mathcal{O}_U^r \longrightarrow \mathcal{H}_{|U} \longrightarrow 0$$

in a neighbourhood of x. The matrix M is a $r \times r$ matrix of holomorphic functions on U. Let $\{\phi_1 = 0\}$ be an equation of D_1 around x. Then we can write $M = \phi_1 M_1$ and the coefficients of M_1 generate $\mathcal{O}_x$. Thus, at least one of these coefficients does not vanish at x. We can suppose that it is the upper-left one. By Gauss elimination process, we can write

$$M = \phi_1 \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boxed{M_2} & \\ 0 & & & \end{pmatrix}.$$

Then, since $\mathrm{Fitt}_k(\mathcal{F})_{|U} = \mathrm{Fitt}_k(M)$, we get $\mathrm{Fitt}_2(M) = \phi_1^2 \,\mathrm{Fitt}_1(M_2)$. Since $\mathrm{Fitt}_2(M)$ is principal, so is $\mathrm{Fitt}_1(M_2)$ and we write $\mathrm{Fitt}_1(M_2) = (\phi_2)$. Then, by the same argument as above, we can write

$$M = \begin{pmatrix} \phi_1 & 0 & \cdots & \cdots & 0 \\ 0 & \phi_1\phi_2 & 0 & \cdots & 0 \\ \vdots & 0 & & & \\ \vdots & \vdots & & \boxed{M_3} & \\ 0 & 0 & & & \end{pmatrix}.$$

By this algorithm, we get

$$M = \begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_1\phi_2 & & \\ \vdots & & \ddots & \\ 0 & & & \phi_1 \cdots \phi_r \end{pmatrix}$$

and then $\mathcal{F}_{|U} \simeq \Big(\mathcal{O}_X \big/ \phi_1 \mathcal{O}_X\Big)_{|U} \oplus \cdots \oplus \Big(\mathcal{O}_X \big/ \phi_1 \ldots \phi_r \mathcal{O}_X\Big)_{|U}$. Thus, if $D_1, \ldots, D_r$ are the divisors of $\phi_1, \phi_1\,\phi_2, \ldots, \phi_1\,\phi_2 \ldots \phi_r$, we have $D_k = D'_k - D'_{k-1}$, which shows that the divisors D_k are intrinsically defined by $\mathcal{F}$. □

From now on, we will say that a torsion sheaf $\mathcal{H}$ is *principal* if it is everywhere locally isomorphic to a fixed sheaf $\bigoplus_{i=1}^{r} \mathcal{O}_X / \mathcal{I}_{D_i}$ where the D_i are (non necessarily reduced) effective normal crossing divisors and $D_1 \leq D_2 \leq \cdots \leq D_r$. We will denote by $\nu(\mathcal{H})$ the number of branches of D, counted with their multiplicities.

Proposition 5.6. *It suffices to prove the Whitney formula when $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is the push-forward of a locally free sheaf on a smooth hypersuface.*

Proof. We proceed in several steps.

Lemma 5.7. *Consider an exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow i_{Y*}\mathcal{E} \longrightarrow 0$ where Y is a smooth hypersurface of X, $\mathcal{G}$ is a locally free sheaf on X and $\mathcal{E}$ is a locally free sheaf on Y. Then $\mathcal{F}$ is locally free on X.*

Proof. Let m_x be the maximal ideal of the local ring $\mathcal{O}_x$. By Nakayama's lemma, it suffices to show that for every x in X, $\mathrm{Tor}_1^{\mathcal{O}_x}\big(\mathcal{F}_x, \mathcal{O}_x/m_x\big) = 0$. Since Y is a hypersurface, $i_{Y*}\mathcal{E}$ admits a locally free resolution of length two. Thus $\mathrm{Tor}_1^{\mathcal{O}_x}\big(\mathcal{F}_x, \mathcal{O}_x/m_x\big) \simeq \mathrm{Tor}_2^{\mathcal{O}_x}\big((i_{Y*}\mathcal{E})_x, \mathcal{O}_x/m_x\big) = 0$. □

Lemma 5.8. *It suffices to prove* (W_n) *when $\mathcal{F}$, $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is principal.*

Proof. By Proposition 5.2, it is enough to prove the Whitney formula when $\mathcal{F}$, $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is a torsion sheaf. So we suppose that $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ verify these hypotheses. By Proposition 5.5, there exists a bimeromorphic morphism $\sigma\colon \widetilde{X} \longrightarrow X$ such that $\sigma^*\mathcal{H}$ is principal. We have an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1(\mathcal{H},\sigma) \longrightarrow \sigma^*\mathcal{F} \longrightarrow \sigma^*\mathcal{G} \longrightarrow \sigma^*\mathcal{H} \longrightarrow 0.$$

But $\mathrm{Tor}_1(\mathcal{H},\sigma)$ is a torsion sheaf and $\sigma^*\mathcal{F}$ is locally free, so we get an exact sequence

$$0 \longrightarrow \sigma^*\mathcal{F} \longrightarrow \sigma^*\mathcal{G} \longrightarrow \sigma^*\mathcal{H} \longrightarrow 0.$$

and we have $\mathrm{Tor}_i(\mathcal{H},\sigma)=0$ for $i\geq 1$. By Proposition 3.7 (ii), we obtain the equalities $\overline{\mathrm{ch}}(\sigma^*\mathcal{F})=\sigma^*\overline{\mathrm{ch}}(\mathcal{F})$ and $\overline{\mathrm{ch}}(\sigma^*\mathcal{G})=\sigma^*\overline{\mathrm{ch}}(\mathcal{G})$, and by Proposition 4.3 (iii) we get $\mathrm{ch}(\sigma^*\mathcal{H})=\sigma^*\,\mathrm{ch}(\mathcal{H})$. Thus

$$\sigma^*\big(\overline{\mathrm{ch}}(\mathcal{F})+\mathrm{ch}(\mathcal{H})-\overline{\mathrm{ch}}(\mathcal{G})\big)=\overline{\mathrm{ch}}(\sigma^*\mathcal{F})+\mathrm{ch}(\sigma^*\mathcal{H})-\overline{\mathrm{ch}}(\sigma^*\mathcal{G}).$$

□

Lemma 5.9. *Suppose that* (W_n) *holds if* $\mathcal{F}$, $\mathcal{G}$ *are locally free sheaves and* $\mathcal{H}$ *is the push-forward of a locally free sheaf on a smooth hypersurface. Then* (W_n) *holds when* $\mathcal{F}$, $\mathcal{G}$ *are locally free sheaves and* $\mathcal{H}$ *is principal.*

Proof. We argue by induction on $\nu(\mathcal{H})$. If $\nu(\mathcal{H})=0$, $\mathcal{H}=0$ and $\mathcal{F}\simeq\mathcal{G}$. If $\nu(\mathcal{H})=1$, $\mathcal{H}$ is the push-forward of a locally free sheaf on a smooth hypersurface and there is nothing to prove.

In the general case, let Y be a branch of D_1. Since $Y\leq D_i$ for every i with $1\leq i\leq r$, we can see that $\mathcal{E}=\mathcal{H}_{|Y}$ is locally free on Y. Besides, if we define $\widetilde{\mathcal{H}}$ by the exact sequence

$$0 \longrightarrow \widetilde{\mathcal{H}} \longrightarrow \mathcal{H} \longrightarrow i_{Y*}\mathcal{E} \longrightarrow 0,$$

$\widetilde{\mathcal{H}}$ is everywhere locally isomorphic to $\bigoplus_{i=1}^{r}\mathcal{O}_X/_{\mathcal{I}_{D_i-Y}}$. Thus $\widetilde{\mathcal{H}}$ is principal and $\nu(\widetilde{\mathcal{H}})=\nu(\mathcal{H})-1$. We define $\widetilde{\mathcal{E}}$ by the exact sequence:

$$0 \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \mathcal{G} \longrightarrow i_{Y*}\mathcal{E} \longrightarrow 0.$$

By Lemma 5.7, $\widetilde{\mathcal{E}}$ is locally free. Furthermore, we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathcal{H}} \longrightarrow 0.$$

By induction, $\overline{\mathrm{ch}}\big(\widetilde{\mathcal{E}}\big)=\overline{\mathrm{ch}}(\mathcal{F})+\mathrm{ch}\big(\widetilde{\mathcal{H}}\big)$ and by our hypothesis $\mathrm{ch}(\mathcal{G})=\overline{\mathrm{ch}}\big(\widetilde{\mathcal{E}}\big)+\mathrm{ch}\big(i_{Y*}\mathcal{E}\big)$. Since $\widetilde{\mathcal{H}}$, $\mathcal{H}$ and $i_{Y*}\mathcal{E}$ are torsion sheaves, $\mathrm{ch}(\mathcal{H})=\mathrm{ch}\big(\widetilde{\mathcal{H}}\big)+\mathrm{ch}\big(i_{Y*}\mathcal{E}\big)$ and we get $\overline{\mathrm{ch}}(\mathcal{G})=\overline{\mathrm{ch}}(\mathcal{F})+\mathrm{ch}(\mathcal{H})$. □

Putting the two lemmas together, we obtain Proposition 5.6. □

5.3. **Proof of the Whitney formula.** We are now ready to prove Theorem 5.1.

In the sections 5.1 and 5.2, we have made successive reductions in order to prove the Whitney formula in a tractable context, so that we are reduced to the case where $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves and $\mathcal{H}=i_{Y*}\mathcal{E}$, where Y is a smooth hypersurface of X and $\mathcal{E}$ is a locally free sheaf on Y. Our working hypotheses will be these.

Let us briefly explain the sketch of the argument. We consider the sheaf $\widetilde{\mathcal{G}}$ on $X\times\mathbb{P}^1$ obtained by deformation of the second extension class of the exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$ Then $\widetilde{\mathcal{G}}_{|X\times\{0\}}\simeq\mathcal{F}\oplus\mathcal{H}$ and $\widetilde{\mathcal{G}}_{|X\times\{t\}}\simeq\mathcal{G}$ for $t\neq 0$. It will turn out that $\widetilde{\mathcal{G}}$ admits a locally free quotient of maximal rank $\mathcal{Q}$ on the blowup of $X\times\mathbb{P}^1$ along $Y\times\{0\}$, and the associated kernel $\mathcal{N}$ will be the push-forward of a locally free sheaf on the exceptional divisor E, say $\mathcal{N}=i_{E*}\mathcal{L}$. Then we consider the class $\alpha=\overline{\mathrm{ch}}(\mathcal{Q})+i_{E*}\big(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}(N_{E/X})^{-1}\big)$ on the blowup. After explicit computations, it will appear that

α is the pullback of a form β on the base $X \times \mathbb{P}^1$. By the $\mathbb{P}^1$-homotopy invariance of Deligne cohomology (Proposition 3.3 (vi)), $\beta_{|X\times\{t\}}$ does not depend on t. This will give the desired result.

Let us first introduce some notations. The morphism $\mathcal{F} \longrightarrow \mathcal{G}$ will be denoted by γ. Let s be a global section of $\mathcal{O}_{\mathbb{P}^1}(1)$ which vanishes exactly at $\{0\}$. Let $\mathrm{pr}_1 \colon X \times \mathbb{P}^1 \longrightarrow X$ be the projection on the first factor. The relative $\mathcal{O}(1)$, namely $\mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$, will still be denoted by $\mathcal{O}(1)$. We define a sheaf $\widetilde{\mathcal{G}}$ on $X \times \mathbb{P}^1$ by the exact sequence

$$0 \longrightarrow \mathrm{pr}_1^*\,\mathcal{F} \xrightarrow[(\mathrm{id}\otimes s,\,\gamma)]{} \mathrm{pr}_1^*\,\mathcal{F}(1) \oplus \mathrm{pr}_1^*\,\mathcal{G} \longrightarrow \widetilde{\mathcal{G}} \longrightarrow 0.$$

Remark that $\widetilde{\mathcal{G}}_0 \simeq \mathcal{F} \oplus \mathcal{H}$ and $\widetilde{\mathcal{G}}_t \simeq \mathcal{G}$ if $t \neq 0$.

Lemma 5.10. *There exist two exact sequences*

(i) $$0 \longrightarrow \mathrm{pr}_1^*\,\mathcal{F}(1) \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathrm{pr}_1^*\,\mathcal{H} \longrightarrow 0$$

(ii) $$0 \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathrm{pr}_1^*\,\mathcal{G}(1) \longrightarrow i_{X_0 *}\mathcal{H} \longrightarrow 0.$$

Remark 5.11. (i) implies that $\widetilde{\mathcal{G}}$ is flat over $\mathbb{P}^1$.

Proof. (i) The morphism $\mathrm{pr}_1^*\,\mathcal{F}(1) \oplus \mathrm{pr}_1^*\,\mathcal{G} \twoheadrightarrow \mathrm{pr}_1^*\,\mathcal{G} \twoheadrightarrow \mathrm{pr}_1^*\,\mathcal{H}$ induces a morphism $\widetilde{\mathcal{G}} \twoheadrightarrow \mathrm{pr}_1^*\,\mathcal{H}$. If $\mathcal{K}$ is the kernel of this morphism, we obtain an exact sequence

$$0 \longrightarrow \mathrm{pr}_1^*\,\mathcal{F} \xrightarrow[(\mathrm{id}\otimes s,\,\mathrm{id})]{} \mathrm{pr}_1^*\,\mathcal{F}(1) \oplus \mathrm{pr}_1^*\,\mathcal{F} \longrightarrow \mathcal{K} \longrightarrow 0.$$

Thus $\mathcal{K} = \mathrm{pr}_1^*\,\mathcal{F}(1)$.

(ii) We consider the morphism $\mathrm{pr}_1^*\,\mathcal{F}(1) \oplus \mathrm{pr}_1^*\,\mathcal{G} \twoheadrightarrow \mathrm{pr}_1^*\,\mathcal{G}(1)$

$$f + g \longrightarrow \gamma(f) - g \otimes s.$$

It induces a morphism $\phi \colon \widetilde{\mathcal{G}} \longrightarrow \mathrm{pr}_1^*\,\mathcal{G}(1)$. The last morphism of (ii) is just the composition

$$\mathrm{pr}_1^*\,\mathcal{G}(1) \twoheadrightarrow i_{X_0 *}\mathcal{G} \twoheadrightarrow i_{X_0 *}\mathcal{H}.$$

The cokernel of this morphism has support in $X \times \{0\}$. Besides, the action of t on this cokernel is zero. The restriction of ϕ to the fiber $X_0 = X \times \{0\}$ is the morphism $\mathcal{F} \oplus \mathcal{H} \longrightarrow \mathcal{G}$, thus the sequence $\widetilde{\mathcal{G}} \longrightarrow \mathrm{pr}_1^*\,\mathcal{G}(1) \longrightarrow i_{X_0 *}\mathcal{H} \longrightarrow 0$ is exact. The kernel of ϕ, as its cokernel, is an $\mathcal{O}_{X_0}$-module. Thus we can find $\mathcal{Z}$ such that $\ker\phi = i_{X_0 *}\mathcal{Z}$. Since X_0 is a hypersurface of $X \times \mathbb{P}^1$, for every coherent sheaf $\mathcal{L}$ on $X \times \mathbb{P}^1$, we have $\mathrm{Tor}_2(\mathcal{L}, i_{X_0}) = 0$. Applying this to $\mathcal{L} = \widetilde{\mathcal{G}}/_{i_{X_0 *}\mathcal{Z}}$ and using Remark 5.11, we get

$$\mathrm{Tor}_1(i_{X_0 *}\mathcal{Z}, i_{X_0}) \subseteq \mathrm{Tor}_1(\widetilde{\mathcal{G}}, i_{X_0}) = \{0\}.$$

But $\mathrm{Tor}_1(i_{X_0 *}\mathcal{Z}, i_{X_0}) \simeq \mathcal{Z} \otimes N^*_{X_0/X\times\mathbb{P}^1} \simeq \mathcal{Z}$, so $\mathcal{Z} = \{0\}$. □

Recall now that $\mathcal{H} = i_{Y*}\mathcal{E}$ where Y is a smooth hypersurface of X and $\mathcal{E}$ is a locally free sheaf on Y. We consider the space $M_{Y/X}$ of the deformation of the normal cone of Y in X (see [Fu]). Basically, $M_{Y/X}$ is the blowup of $X \times \mathbb{P}^1$ along $Y \times \{0\}$. Let $\sigma \colon M_{Y/X} \longrightarrow X \times \mathbb{P}^1$ be the canonical morphism. Then $\sigma^* X_0$ is a Cartier divisor in $M_{Y/X}$ with two simple branches: $E = \mathbb{P}\big(N_{Y/X} \oplus \mathcal{O}_Y\big)$ and $D = \mathrm{Bl}_Y\, X \simeq X$, which intersect at $\mathbb{P}\big(N_{Y/X}\big) \simeq Y$. The projection of the blowup from E to $Y \times \{0\}$ will be denoted by q, and the canonical isomorphism from D to $X \times \{0\}$ will be denoted by μ.

We now show:

Lemma 5.12. *The sheaf $\sigma^*\widetilde{\mathcal{G}}$ admits a locally free quotient with maximal rank on $M_{Y/X}$, and the associated kernel $\mathcal{N}$ is the push-forward of a locally free sheaf on E. More explicitly, if F is the excess conormal bundle of q, $\mathcal{N} = i_{E*}\big(q^*\mathcal{E}\otimes F\big)$.*

Proof. We start from the exact sequence $0 \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathrm{pr}_1^*\,\mathcal{G}(1) \longrightarrow i_{X_0*}\mathcal{H} \longrightarrow 0$. We define $\mathcal{Q}$ by the exact sequence $0 \longrightarrow \mathcal{Q} \longrightarrow \sigma^*\,\mathrm{pr}_1^*\,\mathcal{G}(1) \longrightarrow \sigma^* i_{X_0*}\mathcal{H} \longrightarrow 0$. Since $\sigma^* i_{X_0*}\mathcal{H}$ is the push-forward of a locally free sheaf on E, by Lemma 5.7, the sheaf $\mathcal{Q}$ is locally free on $M_{Y/X}$. The sequence

$$0 \longrightarrow \mathrm{Tor}_1(i_{X_0*}\mathcal{H},\sigma) \longrightarrow \sigma^*\widetilde{\mathcal{G}} \longrightarrow \mathcal{Q} \longrightarrow 0$$

is exact. The first sheaf being a torsion sheaf, $\mathcal{Q}$ is a locally free quotient of $\widetilde{\mathcal{G}}$ with maximal rank. Besides, using the notations given in the following diagram

$$\begin{tikzcd} E \arrow[r, hook, "i_E"] \arrow[d, "q"'] & M_{Y/X} \arrow[d, "\sigma"] \\ Y\times\{0\} \arrow[r, hook, "i_{Y\times\{0\}}"'] & X\times\mathbb{P}^1 \end{tikzcd}$$

we have $\mathrm{Tor}_1(i_{X_0*}\mathcal{H},\sigma) = i_{E*}\big(q^*\mathcal{E}\otimes F\big)$ where F is the excess conormal bundle of q (see [Bo-Se, § 15]. Be aware of the fact that what we note F is F^* in [Bo-Se]). This finishes the proof. □

We consider now the exact sequence $0 \longrightarrow \mathcal{N} \longrightarrow \sigma^*\widetilde{\mathcal{G}} \longrightarrow \mathcal{Q} \longrightarrow 0$ where $\mathcal{Q}$ is locally free on $M_{Y/X}$ and $\mathcal{N} = i_{E*}\big(q^*\mathcal{E}\otimes F\big) = i_{E*}\mathcal{L}$. We would like to introduce the form $\mathrm{ch}(\sigma^*\widetilde{\mathcal{G}})$, but it is not defined since $M_{Y/X}$ is of dimension $n+1$. However, $\sigma^*\widetilde{\mathcal{G}}$ fits in a short exact sequence where the Chern classes of the two other sheaves can be defined. Remark that we need Lemma 5.12 to perform this trick, since it cannot be done on $X\times\mathbb{P}^1$.

Lemma 5.13. *Let α be the Deligne class on $M_{Y/X}$ defined by $\alpha = \overline{\mathrm{ch}}(\mathcal{Q}) + i_{E*}\Big(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\big(N_{E/M_{Y/X}}\big)^{-1}\Big)$.*

(i) *The class α is the pullback of a Deligne class on $X\times\mathbb{P}^1$.*
(ii) *We have $i_D^*\alpha = \mu^*\,\mathrm{ch}^{X_0}\big(\widetilde{\mathcal{G}}_0\big)$.*

Proof. We compute explicitly $i_E^*\alpha$.

$$\begin{aligned} i_E^*\, i_{E*}\Big(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\big(N_{E/M_{Y/X}}\big)^{-1}\Big) &= \overline{\mathrm{ch}}(\mathcal{L})\;\,\mathrm{td}\big(N_{E/M_{Y/X}}\big)^{-1}\; c_1\big(N_{E/M_{Y/X}}\big) && \text{by Proposition 3.5 (vii)} \\ &= \overline{\mathrm{ch}}(\mathcal{L})\Big(1 - e^{-c_1\left(N_{E/M_{Y/X}}\right)}\Big) \\ &= \overline{\mathrm{ch}}(\mathcal{L}) - \overline{\mathrm{ch}}\Big(\mathcal{L}\otimes N^*_{E/M_{Y/X}}\Big) && \text{by Proposition 3.7 (iii)} \\ &= \overline{\mathrm{ch}}\big(i_E^*\mathcal{N}\big) - \overline{\mathrm{ch}}\Big(\mathcal{L}\otimes N^*_{E/M_{Y/X}}\Big). \end{aligned}$$

From the exact sequence $0 \longrightarrow \mathcal{N} \longrightarrow \sigma^*\widetilde{\mathcal{G}} \longrightarrow \mathcal{Q} \longrightarrow 0$, we get the exact sequence of locally free sheaves on E: $0 \longrightarrow i_E^*\mathcal{N} \longrightarrow i_E^*\,\sigma^*\widetilde{\mathcal{G}} \longrightarrow i_E^*\mathcal{Q} \longrightarrow 0$. Since $i_E^*\,\overline{\mathrm{ch}}(\mathcal{Q}) = \overline{\mathrm{ch}}\big(i_E^*\mathcal{Q}\big)$, we obtain

$$\begin{aligned}
i_E^*\alpha &= \overline{\mathrm{ch}}\big(i_E^*\mathcal{Q}\big) + \overline{\mathrm{ch}}\big(i_E^*\mathcal{N}\big) - \overline{\mathrm{ch}}\big(\mathcal{L}\otimes N^*_{E/M_{Y/X}}\big) \\
&= \overline{\mathrm{ch}}\big(i_E^*\sigma^*\widetilde{\mathcal{G}}\big) - \overline{\mathrm{ch}}\big(\mathcal{L}\otimes N^*_{E/M_{Y/X}}\big) && \text{by Proposition 3.7 (i)} \\
&= \overline{\mathrm{ch}}\big(q^*i_Y^*\mathcal{F}\big) + \overline{\mathrm{ch}}\big(q^*i_Y^*\mathcal{H}\big) - \overline{\mathrm{ch}}\big(\mathcal{L}\otimes N^*_{E/M_{Y/X}}\big) \\
&= q^*\overline{\mathrm{ch}}\big(i_Y^*\mathcal{F}\big) + q^*\overline{\mathrm{ch}}(\mathcal{E}) - \overline{\mathrm{ch}}\big(q^*\mathcal{E}\otimes F\otimes N^*_{E/M_{Y/X}}\big) && \text{by Proposition 3.7 (ii).}
\end{aligned}$$

Recall that the conormal excess bundle F is the line bundle defined by the exact sequence

$$0 \longrightarrow F \longrightarrow q^*N^*_{Y/X\times\mathbb{P}^1} \longrightarrow N^*_{E/M_{Y/X}} \longrightarrow 0.$$

Thus, we have $\det\big(q^*N^*_{Y/X\times\mathbb{P}^1}\big) = F\otimes N^*_{E/M_{Y/X}}$. Since $\det\big(q^*N^*_{Y/X\times\mathbb{P}^1}\big) = q^*\det\big(N^*_{Y/X\times\mathbb{P}^1}\big)$, we get by Proposition 3.7 (ii) again

$$i_E^*\alpha = q^*\big[\,\overline{\mathrm{ch}}\big(i_Y^*\mathcal{F}\big) + \overline{\mathrm{ch}}(\mathcal{E}) - \overline{\mathrm{ch}}\big(\mathcal{E}\otimes\det\big(N^*_{Y/X\times\mathbb{P}^1}\big)\big)\big].$$

This proves (i).

(ii) The divisors E and D meet transversally. Then, by Proposition 3.5 (iv),

$$\begin{aligned}
i_D^*\alpha &= i_D^*\overline{\mathrm{ch}}(\mathcal{Q}) + i_D^*i_{E*}\Big(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\big(N_{E/M_{Y/X}}\big)^{-1}\Big) \\
&= \overline{\mathrm{ch}}\big(i_D^*\mathcal{Q}\big) + i_{E\cap D\longrightarrow D*}\; i^*_{E\cap D\longrightarrow E}\Big(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\big(N_{E/M_{Y/X}}\big)^{-1}\Big) \\
&= \overline{\mathrm{ch}}\big(i_D^*\mathcal{Q}\big) + i_{E\cap D\longrightarrow D*}\Big(\overline{\mathrm{ch}}\big(i^*_{E\cap D\longrightarrow E}\mathcal{L}\big)\,\mathrm{td}\big(N_{E\cap D/D}\big)^{-1}\Big).
\end{aligned}$$

We remark now that $i^*_{E\cap D\longrightarrow E}\mathcal{L} = i^*_{D\cap E}\,\mathcal{N}$. Since $\dim(E\cap D) = n-1$, we obtain

$$i_D^*\alpha = \overline{\mathrm{ch}}\big(i_D^*\mathcal{Q}\big) + \mathrm{ch}^D\big(i_{E\cap D\longrightarrow D*}\,i^*_{E\cap D}\mathcal{N}\big) = \overline{\mathrm{ch}}\big(i_D^*\mathcal{Q}\big) + \mathrm{ch}^D\big(i_D^*\mathcal{N}\big).$$

We have the exact sequence on D: $0 \longrightarrow i_D^*\mathcal{N} \longrightarrow i_D^*\,\sigma^*\widetilde{\mathcal{G}} \longrightarrow i_D^*\mathcal{Q} \longrightarrow 0$. Therefore $i_D^*\,\sigma^*\widetilde{\mathcal{G}}$ admits a locally free quotient of maximal rank and, μ being an isomorphism,

$$\overline{\mathrm{ch}}\big(i_D^*\mathcal{Q}\big) + \mathrm{ch}^D(i_D^*\mathcal{N}) = \mathrm{ch}^D\big(i_D^*\,\sigma^*\widetilde{\mathcal{G}}\big) = \mathrm{ch}^D\big(\mu^*\widetilde{\mathcal{G}}_0\big) = \mu^*\,\mathrm{ch}^{X_0}\big(\widetilde{\mathcal{G}}_0\big).$$

□

We are now ready to use the homotopy property for Deligne cohomology (Proposition 3.3 (vi)).

Let α be the form defined in Lemma 5.13. Using (i) of this lemma and Proposition 3.5 (vi), we can write $\alpha = \sigma^*\beta$. Thus $i_D^*\,\alpha = i_D^*\,\sigma^*\beta = \mu^*i^*_{X_0}\beta$. By (ii) of the same lemma, $i_D^*\alpha = \mu^*\,\mathrm{ch}^{X_0}\big(\widetilde{\mathcal{G}}_0\big)$ and we get $i^*_{X_0}\beta = \mathrm{ch}^{X_0}\big(\widetilde{\mathcal{G}}_0\big)$. If $t\in\mathbb{P}^1\backslash\{0\}$, we have clearly $\beta_{|X_t} = \overline{\mathrm{ch}}(\mathcal{G})$. Since $\beta_{|X_t} = \beta_{|X_0}$ we obtain

$$\overline{\mathrm{ch}}(\mathcal{G}) = \mathrm{ch}^{X_0}\big(\widetilde{\mathcal{G}}_0\big) = \overline{\mathrm{ch}}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}).$$

□

We can now establish the remaining induction properties.

Theorem 5.14. *The following assertions are valid:*

(i) *Property* (F_n) *holds.*
(ii) *Property* (P_n) *holds.*

Proof. (i) We take $y=[\mathcal{F}]$. Let us first suppose that f is a bimeromorphic map. Then there exists a bimeromorphic map $\sigma\colon \widetilde{X}\longrightarrow X$ such that $(f\circ\sigma)^*\mathcal{F}$ admits a locally free quotient of maximal rank. Then by Theorem 4.15 (i),

$$\sigma^*\,\mathrm{ch}\big(f^{\,!}[\mathcal{F}]\big)=\mathrm{ch}\big(\sigma^{\,!}\,f^{\,!}[\mathcal{F}]\big)=(f\circ\sigma)^*\,\mathrm{ch}\,\mathcal{F}=\sigma^*\big[f^*\,\mathrm{ch}(\mathcal{F})\big].$$

Suppose now that f is a surjective map. Then there exist two bimeromorphic maps $\pi_X:\widetilde{X}\longrightarrow X$, $\pi_Y\colon\widetilde{Y}\longrightarrow Y$ and a surjective map $\widetilde{f}\colon\widetilde{X}\longrightarrow\widetilde{Y}$ such that:

– the diagram $\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_X"'] & \widetilde{Y} \arrow[d, "\pi_Y"'] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$ is commutative.

– the sheaf $\pi_Y^*\mathcal{F}$ admits a locally free quotient $\mathcal{E}$ of maximal rank.

We can write $\pi_Y^![\mathcal{F}]=[\mathcal{E}]+\widetilde{y}$ in $G(\widetilde{Y})$, where $\widetilde{y}$ is in the image of the natural map $\iota\colon G_{\mathrm{tors}}(\widetilde{Y})\longrightarrow G(\widetilde{Y})$. The functoriality property being known for bimeromorphic maps, it holds for π_X and π_Y. The result is now a consequence of Proposition 3.7 (ii) and Proposition 4.3 (iii).

In the general case, we consider the diagram used in the proof of Proposition 4.3 (iii)

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_X"'] & W \arrow[r, "i_W"] \arrow[d, "\tau"] & \widetilde{Y} \arrow[d, "\pi_Y"] \\ X \arrow[r, "f"'] & f(X) \arrow[r] & Y \end{tikzcd}$$

where $\widetilde{f}$ is surjective. Then the functoriality property holds for $\widetilde{f}$ by the argument above and for i_W by Theorem 4.15 (ii). This finishes the proof.

(ii) We can suppose that $x=[\mathcal{F}]$, $y=[\mathcal{G}]$ and that $\mathcal{F}$ and $\mathcal{G}$ admit locally free quotients $\mathcal{E}_1$, $\mathcal{E}_2$ of maximal rank. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the associated kernels. We can also suppose that $\mathrm{supp}(\mathcal{T}_1)$ lies in a simple normal crossing divisor. Then

$$\begin{aligned}\mathrm{ch}([\mathcal{F}].[\mathcal{G}])&=\overline{\mathrm{ch}}([\mathcal{E}_1].[\mathcal{E}_2])+\mathrm{ch}([\mathcal{E}_1].[\mathcal{T}_2])+\mathrm{ch}([\mathcal{E}_2].[\mathcal{T}_1])+\mathrm{ch}([\mathcal{T}_1].[\mathcal{T}_2])\\&=\overline{\mathrm{ch}}(\mathcal{E}_1)\overline{\mathrm{ch}}(\mathcal{E}_2)+\overline{\mathrm{ch}}(\mathcal{E}_1)\,\mathrm{ch}(\mathcal{T}_2)+\overline{\mathrm{ch}}(\mathcal{E}_2)\,\mathrm{ch}(\mathcal{T}_1)+\mathrm{ch}([\mathcal{T}_1].[\mathcal{T}_2])\end{aligned}$$

by (W_n), Proposition 4.3 (ii) and Proposition 3.7 (iii). By dévissage, we can suppose that $\mathcal{T}_1$ is a $\mathcal{O}_Z$-module, where Z is a smooth hypersurface of X. We write $[\mathcal{T}_1]=i_{Z!}u$ and $[\mathcal{T}_2]=v$. Then $[\mathcal{T}_1]\,.\,[\mathcal{T}_2]=i_{Z!}\big(u\,.\,i_Z^!v\big)$. So

$$\begin{aligned}\mathrm{ch}\big([\mathcal{T}_1].[\mathcal{T}_2]\big)&=i_{Z*}\Big(\mathrm{ch}^Z(u\,.\,i_Z^!v)\ \ \mathrm{td}\big(N_{Z/X}\big)^{-1}\Big)\\&=i_{Z*}\Big(\mathrm{ch}^Z(u)\ i_Z^*\,\mathrm{ch}(v)\ \ \mathrm{td}\big(N_{Z/X}\big)^{-1}\Big)&&\text{by }(\mathrm{P}_{n-1})\text{ and Proposition 4.10 (ii)}\\&=i_{Z*}\Big(\mathrm{ch}^Z(u)\ \ \mathrm{td}\big(N_{Z/X}\big)^{-1}\Big)\,\mathrm{ch}(v)&&\text{by the projection formula}\\&=\mathrm{ch}\big(i_{Z!}u\big)\,\mathrm{ch}(v)=\mathrm{ch}\big([\mathcal{T}_1]\big)\,\mathrm{ch}\big([\mathcal{T}_2]\big)\end{aligned}$$

□

The proof of Theorem 4.2 is now concluded with the exception of property (RR_n).

6. The Grothendieck-Riemann-Roch theorem for projective morphisms

6.1. **Proof of the GRR formula.** We have already obtained the (GRR) formula for the immersion of a smooth divisor. We reduce now the general case to the divisor case by a blowup. This construction is classical (see [Bo-Se]).

Theorem 6.1. *Let Y be a smooth submanifold of X. Then, for all y in $G(Y)$, we have*

$$\mathrm{ch}\big(i_{Y!}y\big) = i_{Y*}\big(\mathrm{ch}(y)\,\mathrm{td}(N_{Z/X})^{-1}\big).$$

Proof. We blow up Y along X as shown below, where E is the exceptional divisor.

$$\begin{tikzcd} E \arrow[r, "i_E"] \arrow[d, "q"'] & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[r, "i_Y"'] & X \end{tikzcd}$$

Let F be the excess conormal bundle of q defined by the exact sequence

$$0 \longrightarrow F \longrightarrow q^*N^*_{Y/X} \longrightarrow N^*_{E/\widetilde{X}} \longrightarrow 0.$$

If d is the codimension of Y in X, then $\mathrm{rank}(F) = d-1$. Recall the following formulae:

(a) Excess formula in K-theory (Proposition 7.6 (ii)): for all y in $G(Y)$, $p^!\, i_! y = i_{E!}\big(q^! y\,.\,\lambda_{-1}F\big)$.

(b) Excess formula in Deligne cohomology (Proposition 3.5 (vi)): for each Deligne class β on Y,

$$p^*\, i_{Y*}\,\beta = i_{E*}\big(q^*\beta\; c_{d-1}(F^*)\big).$$

(c) If G is a vector bundle of rank r, then $\mathrm{ch}\big(\lambda_{-1}[G]\big) = c_r(G^*)\,\mathrm{td}(G^*)^{-1}$ ([Bo-Se, Lemme 18]).

We compute now

$$\begin{aligned}
p^*\,\mathrm{ch}\big(i_{Y!}y\big) &= \mathrm{ch}\big(p^!\, i_{Y!}y\big) = \mathrm{ch}\big(i_{E!}(q^! y\,.\,\lambda_{-1}[F])\big) && \text{by } (\mathrm{F}_n) \text{ and (a)}\\
&= i_{E*}\Big(\mathrm{ch}^E\big(q^! y\,.\,\lambda_{-1}[F]\big)\,\mathrm{td}\big(N_{E/\widetilde{X}}\big)^{-1}\Big) && \text{by (GRR) for } i_E\\
&= i_{E*}\Big(q^*\,\mathrm{ch}^Y(y)\,\mathrm{ch}\big(\lambda_{-1}[F]\big)\; q^*\,\mathrm{td}\big(N_{Y/X}\big)^{-1}\,\mathrm{td}\big(F^*\big)\Big) && \text{by } (\mathrm{F}_n),\ (\mathrm{P}_n) \text{ and Proposition 3.7 (i)}\\
&= i_{E*}\Big(q^*\big(\mathrm{ch}^Y(y)\,\mathrm{td}\big(N_{Y/X}\big)^{-1}\big)\; c_{d-1}(F^*)\Big) && \text{by (c)}\\
&= p^*\, i_{Y*}\Big(\mathrm{ch}^Y(y)\,\mathrm{td}\big(N_{Y/X}\big)^{-1}\Big) && \text{by (b).}
\end{aligned}$$

Thus $\mathrm{ch}\big(i_{Y!}y\big) = i_{Y*}\big(\mathrm{ch}^Y(y)\,\mathrm{td}\big(N_{Y/X}\big)^{-1}\big)$. ☐

Now we can prove a more general Grothendieck-Riemann-Roch theorem:

Theorem 6.2. *The GRR theorem holds in rational Deligne cohomology for projective morphisms between smooth complex compact manifolds.*

Proof. Let $f : X \longrightarrow Y$ be a projective morphism. Then we can write f as the composition of an immersion $i\colon X \longrightarrow Y \times \mathbb{P}^N$ and the second projection $p\colon Y \times \mathbb{P}^N \longrightarrow Y$. By Theorem 6.1, GRR is true for i. Now the arguments in [Bei] show that the canonical map from $G(Y) \otimes_{\mathbb{Z}} G(\mathbb{P}^N)$ to $G(Y \times \mathbb{P}^N)$ is surjective. Therefore, it is enough to prove GRR for p with elements of the form $y\,.\,w$, where y belongs to $G(Y)$ and w belongs to $G(\mathbb{P}^N)$. By the product formula for the Chern character, we are led to the Hirzebruch-Riemann-Roch formula for $\mathbb{P}^N$, which is well known. ☐

6.2. **Compatibility of Chern classes and the GRR formula.** We will show that the GRR formula for immersions combined with some basic properties can be sufficient to characterize completely a theory of Chern classes. This will give various compatibility theorems.

We assume to be given for each smooth complex compact manifold X a graded commutative cohomology ring $A(X) = \oplus_{i=0}^{\dim X} A^i(X)$ which is an algebra over $\mathbb{Q}$, $\mathbb{Q} \subset A^0(X)$, with the following properties:

(α) For each holomorphic map $f : X \longrightarrow Y$, there is a pull-back morphism $f^* : A(Y) \longrightarrow A(X)$ which is functorial and compatible with the products and the gradings.

(β) If σ is the blowup of a smooth complex compact manifold along a smooth submanifold, then σ^* is injective.

(γ) If E is a holomorphic vector bundle on X and $\pi : \mathbb{P}(E) \longrightarrow X$ is the projection of the associated projective bundle, then π^* is injective.

(δ) If X is a smooth complex compact manifold and Y is a smooth submanifold of codimension d, then there is a Gysin morphism $i_* : A^*(Y) \longrightarrow A^{*+d}(X)$.

The main examples are:

$$A^i(X) = H^{2i}_D(X, \mathbb{Q}(i)),\ \mathbb{H}^{2i}(X, \Omega_X^{\bullet \geqslant i}),\ H^i(X, \Omega^i_X),\ F^i H^{2i}(X, \mathbb{C}),\ H^{2i}(X, \mathbb{Q}) \text{ and } H^{2i}(X, \mathbb{C}).$$

Then we have the following theorem:

Theorem 6.3. *Suppose that we have two theories of Chern classes c_i and c'_i for coherent sheaves on any smooth complex compact manifold X with values in $A^i(X)$ such that $c_0 = c'_0 = 1$ and*

(i) *The Whitney formula holds for the total classes c and c'.*

(ii) *The functoriality formula holds for c and c'.*

(iii) *If L is a holomorphic line bundle, then*

$$c(L) = 1 + c_1(L) = c'(L) = 1 + c'_1(L).$$

(iv) *In both theories, the GRR theorem holds for immersions.*

Then for every coherent sheaf $\mathcal{F}$, $c(\mathcal{F})$ and $c'(\mathcal{F})$ are equal.

Remark 6.4. 1. The same conclusion holds for cohomology algebras over $\mathbb{Z}$ if we assume GRR *without denominators*.

2. If X is projective, (i) and (iii) are sufficient to imply the equality of c and c' because of the existence of global locally free resolutions.

3. In (iv), the Todd classes of the normal bundle are defined in $A(X)$ since $A(X)$ is a $\mathbb{Q}$-algebra.

Proof. We start by proving that for any holomorphic vector bundle E, $c(E)$ and $c'(E)$ are equal. Actually, this proof is a variant of the splitting principle. We argue by induction on the rank of E. Let $\pi : \mathbb{P}(E) \longrightarrow X$ be the projective bundle of E. Then we have the exact sequence

$$0 \longrightarrow \mathcal{O}_E(-1) \longrightarrow \pi^* E \longrightarrow F \longrightarrow 0$$

on $\mathbb{P}(E)$, where F is a holomorphic vector bundle on $\mathbb{P}(E)$ whose rank is the rank of E minus one. By induction, $c(F) = c'(F)$ and by (iii), $c(\mathcal{O}_E(-1)) = c'(\mathcal{O}_E(-1))$. By (i), $c(\pi^* E) = c'(\pi^* E)$ and by (ii), $\pi^*[c(E) - c'(E)] = 0$. By (γ), $c(E) = c'(E)$.

We can now prove Theorem 6.3. As usual, we deal with exponential Chern classes. The proof proceeds by induction on the dimension of the base manifold X.

Let $\mathcal{F}$ be a coherent sheaf on X. By Theorem 4.11 there exists a bimeromorphic morphism $\sigma : \widetilde{X} \longrightarrow X$ which is a finite composition of blowups with smooth centers and a locally free sheaf $\mathcal{E}$ on $\widetilde{X}$ which is a quotient of maximal rank of $\sigma^* \mathcal{F}$. Furthermore, by Hironaka's theorem, we can suppose that the exceptional locus of σ and the kernel of the morphism $\sigma^*\mathcal{F} \longrightarrow \mathcal{E}$ are both contained in a simple normal crossing divisor D of $\widetilde{X}$. Thus

$$\sigma^![\mathcal{F}] = \sum_{i=0}^{n}(-1)^i\big[\mathrm{Tor}_i(\mathcal{F},\sigma)\big] = [\mathcal{E}] + \sum_{i=1}^{n}(-1)^i\big[\mathrm{Tor}_i(\mathcal{F},\sigma)\big]$$

and then $\sigma^![\mathcal{F}]$ belongs to $[\mathcal{E}] + G_D(\widetilde{X})$. Now there is a surjective morphism

$$\bigoplus_{i=1}^{N} G_{D_i}(\widetilde{X}) \longrightarrow G_D(\widetilde{X}) \ .$$

Moreover, by Proposition 7.2, $G(D_i)$ is isomorphic to $G_{D_i}(\widetilde{X})$. Remark that $\mathrm{td}\big(N_{D_i/\widetilde{X}}\big) = \mathrm{td}'\big(N_{D_i/\widetilde{X}}\big)$. By the GRR formulae (iv) and the induction hypothesis, ch and ch$'$ are equal on each $G_{D_i}(\widetilde{X})$. By the first part of the proof, $\mathrm{ch}(\mathcal{E}) = \mathrm{ch}'(\mathcal{E})$. Thus $\mathrm{ch}\big(\sigma^![\mathcal{F}]\big) = \mathrm{ch}'\big(\sigma^![\mathcal{F}]\big)$. By (ii), $\sigma^*\big[\mathrm{ch}(\mathcal{F}) - \mathrm{ch}'(\mathcal{F})\big] = 0$. Since σ^* is injective by (β), $\mathrm{ch}(\mathcal{F}) = \mathrm{ch}'(\mathcal{F})$. □

Corollary 6.5. *Let $\mathcal{F}$ be a coherent analytic sheaf on X. Then:*

(i) *The classes $c_i(\mathcal{F})$ in $H^{2i}_D(X,\mathbb{Q}(i))$ and $c_i(\mathcal{F})^{top}$ in $H^{2i}(X,\mathbb{Z})$ have the same image in $H^{2i}(X,\mathbb{Q})$.*

(ii) *The image of $c_i(\mathcal{F})$ via the natural morphism from $H^{2i}_D(X,\mathbb{Q}(i))$ to $H^i(X,\Omega^i_X)$ is the ith Atiyah Chern class of $\mathcal{F}$*

Proof. If L is a holomorphic line bundle, then (i) and (ii) hold for L. Indeed, using the isomorphism between $\mathrm{Pic}(X)$ and $H^2_D(X,\mathbb{Z}(1))$, the topological and Atiyah first Chern classes are obtained by the two morphisms of complexes

$$\begin{array}{ccc} \mathbb{Z} \xrightarrow{2i\pi} \mathcal{O} \\ \downarrow \\ \mathbb{Z} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{Z} \xrightarrow{2i\pi} \mathcal{O} \\ \qquad\ \ \downarrow{\scriptstyle d/2i\pi} \\ \qquad\ \ \Omega^1_X \end{array}$$

On the other hand, GRR for immersions holds for topological Chern classes by [At-Hi] and for Atiyah Chern classes by [OB-To-To3]. □

Remark 6.6. If X is a Kähler complex manifold, the Green Chern classes are the same as the Atiyah Chern classes and the complex topological Chern classes. If X is non Kähler, GRR does not seem to be known for the Green Chern classes, except for a constant morphism (see [To-To]). If this were true for immersions, it would imply the compatibility of $c_i(\mathcal{F})$ and $c_i(\mathcal{F})^{\mathrm{Gr}}$, via the map from $H^{2i}_D(X,\mathbb{Q}(i))$ to $\mathbb{H}^{2i}(X,\Omega_X^{\bullet\geqslant i})$. On the other hand, if this compatibility holds, it implies GRR for immersions for the Green Chern classes.

7. Appendix. Analytic K-theory with support

Our aim in this appendix is to answer the following question: if $f\colon X \longrightarrow Y$ is a morphism, $\mathcal{F}$ a torsion sheaf on Y with support W, and $Z = f^{-1}(W)$, can we express the sheaves $\mathrm{Tor}_i(\mathcal{F},f)$ in terms of the sheaves $\mathrm{Tor}_i(\mathcal{F}_{|W}, f_{|Z})$?

7.1. **Definition of the analytic K-theory with support.** Let X be a smooth compact complex manifold and Z be an analytic subset of X. We will denote by $\mathrm{coh}_Z(X)$ the abelian category of coherent sheaves on X with support in Z. Then, X being compact,

$$\mathrm{coh}_Z(X) = \big\{\mathcal{F},\ \mathcal{F} \in \mathrm{coh}(X) \text{ such that } \mathcal{I}_Z^n\mathcal{F} = 0 \text{ for } n \gg 0\big\}.$$

Definition 7.1. We define $G_Z(X)$ as the quotient of $\mathbb{Z}\big[\mathrm{coh}_Z(X)\big]$ by the relations: $\mathcal{F}+\mathcal{H}=\mathcal{G}$ if there exists an exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ with $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ elements of $\mathrm{coh}_Z(X)$. In other words, $G_Z(X)$ is the Grothendieck group of $\mathrm{coh}_Z(X)$. The image of an element $\mathcal{F}$ of $\mathrm{coh}_Z(X)$ in $G_Z(X)$ will be denoted by $[\mathcal{F}]$.

We first prove

Proposition 7.2. *The map* $i_{Z*}\colon G(Z) \longrightarrow G_Z(X)$ *is a group isomorphism.*

Proof. We consider the inclusion $\mathrm{coh}(Z) \subseteq \mathrm{coh}_Z(X)$. If $\mathcal{F}$ is in $\mathrm{coh}_Z(X)$, we can filter $\mathcal{F}$ by the formula $F^i\mathcal{F} = \mathcal{I}_Z^i\mathcal{F}$. This is a finite filtration and the associated quotients $F^i\mathcal{F}/_{F^{i+1}\mathcal{F}}$ are in $\mathrm{coh}(Z)$. Therefore, we can use the dévissage theorem for the Grothendieck group, which is in fact valid for all K-theory groups (see [Qui, § 5, Theorem 4]). $\square$

7.2. **Product on the K-theory with support.** Let Z be a *smooth* submanifold of X. For any x in $G_Z(X)$, by Proposition 7.2, there exists a unique $\overline{x}$ in $G(Z)$ such that $i_{Z*}\overline{x} = x$.

Definition 7.3. We define a product $G(Z) \otimes_{\mathbb{Z}} G_Z(X) \xrightarrow[\cdot_Z]{} G_Z(X)$ by $x \cdot_Z y = i_{Z*}(x.\overline{y})$.

In other words, the following diagram is commutative:

$$\begin{tikzcd} G(Z) \otimes_{\mathbb{Z}} G_Z(X) \arrow[r, "\cdot_Z"'] \arrow[d, "\simeq"] & G_Z(X) \arrow[d, "\simeq"] \\ G(Z) \otimes_{\mathbb{Z}} G(Z) \arrow[r, "\cdot"'] & G(Z) \end{tikzcd}$$

Remark 7.4. Definition 7.3 has some obvious consequences:

1. For any x in $G(Z)$, $i_{Z*}x = x \cdot_Z [i_{Z*}\mathcal{O}_Z]$.
2. More generally, $i_{Z*}(x.\,y) = x \cdot_Z i_{Z*}y$.
3. The product $\cdot_Z$ endows $G_Z(X)$ with the structure of a $G(Z)$-module.

7.3. **Functoriality.** Let $f\colon X \longrightarrow Y$ be a holomorphic map and W be a smooth submanifold of Y such that $Z = f^{-1}(W)$ is smooth. We can define a morphism $f^{\,!}\colon G_W(Y) \longrightarrow G_Z(X)$ by the usual formula $f^{\,*}[\mathcal{K}] = \sum_{i\geq 0}(-1)^i\Big[\mathrm{Tor}_i(\mathcal{K}, f)\Big]$. Let $\overline{f}$ be the restriction of f to Z, as shown in the following diagram:

$$\begin{tikzcd} Z \arrow[r, "i_Z"] \arrow[d, "\overline{f}"'] & X \arrow[d, "f"] \\ W \arrow[r, "i_W"'] & Y \end{tikzcd}$$

Then we have the following proposition:

Proposition 7.5. *For all x in $G(W)$ and for all y in $G_W(Y)$, we have* $f^{\,!}\big(x \cdot_W y\big) = \overline{f}^{\,!}x \cdot_Z f^{\,!}y$.

Recall that if E is a holomorphic vector bundle on X, $\lambda_{-1}[E]$ is the element of $G(X)$ defined by

$$\lambda_{-1}[E] = 1 - [E] + \Big[\bigwedge^2 E\Big] - \Big[\bigwedge^3 E\Big] + \cdots$$

Proposition 7.6. (In the algebraic case, see [Bo-Se, Proposition 12 and Lemme 19 c.])

(i) *If Z is a smooth submanifold of X, then for all x in $G(Z)$,* $i_Z^!\, i_{Z*}x = x\,.\,\lambda_{-1}[N^*_{Z/X}]$.

(ii) *Consider the blowup $\widetilde{X}$ of X along a smooth submanifold Y, and let E be the exceptional divisor as shown in the following diagram:*

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "q"'] & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

Let F be the excess conormal bundle on E defined by the exact sequence

$$0 \longrightarrow F \longrightarrow q^* N^*_{Y/X} \longrightarrow N^*_{E/\widetilde{X}} \longrightarrow 0.$$

Then for all x in $G(Y)$, $p^! i_! x = j_!\big(q^! x \,.\, \lambda_{-1} F\big)$.

The assertion (ii) of Proposition 7.6 is the *excess formula* in K-theory.

Proof. (i) We write $i_Z^! i_{Z*} x = i_Z^!\big(x \,\textbf{.}_Z\, [i_{Z*}\mathcal{O}_Z]\big) = x \,.\, i_Z^! [i_{Z*}\mathcal{O}_Z]$. Now $\mathrm{Tor}_i(i_{Z*}\mathcal{O}_Z, i_Z) = \bigwedge^i N^*_{Z/X}$ ([Bo-Se, Proposition 12]) and we are done.

(ii) The equality $p^!\big[i_*\mathcal{O}_Y\big] = j_!(\lambda_{-1}[F])$ is proved in [Bo-Se, Lemme 19.c]. Thus

$$p^! i_! x = p^!\big(x \,\textbf{.}_Y\, [i^*\mathcal{O}_Y]\big) = q^! x \,\textbf{.}_{\widetilde{Y}}\, p^! [i_* \mathcal{O}_Y] = q^! x \,\textbf{.}_{\widetilde{Y}}\, j_!\big(\lambda_{-1}[F]\big) = j_*\big(q^! x \,.\, \lambda_{-1}[F]\big).$$

□

We will now need a statement similar to Proposition 7.5, with the hypothesis that Z is a divisor with simple normal crossing.

Let $f\colon X \longrightarrow Y$ be a surjective holomorphic map, D be a smooth hypersurface of Y, and $\widetilde{D} = f^*(D)$. We suppose that $\widetilde{D}$ is a divisor of X with simple normal crossing. Let $\widetilde{D}_k$, $1 \leq k \leq N$, be the branches of $\widetilde{D}^{\mathrm{red}}$. We denote by $\overline{f}_k$ the restricted map $f\colon \widetilde{D}_k \longrightarrow D$.

Proposition 7.7. *For all elements u_k in $G_{\widetilde{D}_k}(X)$ such that $\mathcal{O}_{\widetilde{D}} = u_1 + \cdots + u_N$ and for all y in $G(D)$,*

$$f^!\big(i_{D!}\, y\big) = \sum_{k=1}^{N} \overline{f}_k^{\,!}(y) \,\textbf{.}_{\widetilde{D}_k}\, u_k$$

7.4. **Analytic K-theory with support in a divisor with simple normal crossing.** Let X be a smooth complex compact manifold and D be a reduced divisor with simple normal crossing. The branches of D will be denoted by $D_1, \ldots, D_N$ and $D_{ij} = D_i \cap D_j$. By a dévissage argument, the canonical map from $\bigoplus_{i=1}^n G_{D_i}(X)$ to $G_D(X)$ is surjective. If x belongs to $G_{D_{ij}}(X)$, then the element $(x, -x)$ of $G_{D_i}(X) \oplus G_{D_j}(X)$ is in the kernel of this map. We will show that this kernel is generated by these elements:

Proposition 7.8. *There is an exact sequence:*

$$\textstyle\bigoplus_{i<j} G_{D_{ij}}(X) \longrightarrow \bigoplus_i G_{D_i}(X) \longrightarrow G_D(X) \longrightarrow 0.$$

Proof. We will deal with the exact sequence

$$\textstyle\bigoplus_{i<j} G(D_{ij}) \longrightarrow \bigoplus_i G(D_i) \longrightarrow G(D) \longrightarrow 0.$$

By the dévissage theorem 7.2, this sequence is isomorphic to the initial one. We proceed by induction on the number N of the branches of D. Let D' be the divisor whose branches are $D_1, \ldots, D_{N-1}$. We have a complex

$$(*) \qquad G(D' \cap D_N) \longrightarrow G(D') \oplus G(D_N) \xrightarrow{\ \pi\ } G(D) \longrightarrow 0$$

where the first map is given by $\alpha \longmapsto (\alpha, -\alpha)$. Let us verify that this complex is exact. Consider the map $\psi : \mathbb{Z}[\,\mathrm{coh}(D)\,] \longrightarrow G(D') \oplus G(D_N) \,/\, G(D' \cap D_N)$ defined by $\psi(\mathcal{F}) = [i^*_{D'}\mathcal{F}] + [\mathcal{I}_{D'}\mathcal{F}]$. Remark that $\mathcal{I}_{D'}$ is a sheaf of $\mathcal{O}_{D_N}$-modules, namely the sheaf of ideals of $D' \cap D_N$ in D_N extended to D by zero. Let us show that ψ can be defined in K-theory.

We consider an exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ of coherent sheaves on D. Let us define the sheaf $\mathcal{N}$ by the exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow i^*_{D'}\mathcal{F} \longrightarrow i^*_{D'}\mathcal{G} \longrightarrow i^*_{D'}\mathcal{H} \longrightarrow 0.$$

It is clear that $\mathcal{N}$ is a sheaf of $\mathcal{O}_{D'}$-modules with support in $D' \cap D_N$, and $[i^*_{D'}\mathcal{G}] - [i^*_{D'}\mathcal{F}] - [i^*_{D'}\mathcal{H}] = -[\mathcal{N}]$ in $G(D')$. Let us consider the following exact sequence of complexes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_{D'}\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & i^*_{D'}\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_{D'}\mathcal{G} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & i^*_{D'}\mathcal{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_{D'}\mathcal{H} \arrow[r] & \mathcal{H} \arrow[r] & i^*_{D'}\mathcal{H} \arrow[r] & 0
\end{tikzcd}$$

Let $\mathcal{C}$ be the first complex, that is the first column of the diagram above. It is a complex of $\mathcal{O}_{D_N}$-modules. If we denote by $\mathcal{H}^k(\mathcal{C})$, $0 \leq k \leq 2$, the cohomology sheaves of $\mathcal{C}$, we have the long exact sequence

$$0 \longrightarrow \mathcal{H}^0(\mathcal{C}) \longrightarrow 0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{H}^1(\mathcal{C}) \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{H}^2(\mathcal{C}) \longrightarrow 0.$$

Since $\mathcal{H}^1(\mathcal{C})$ is a sheaf of $\mathcal{O}_{D_N}$-modules, $\mathcal{N}$ is also a sheaf of $\mathcal{O}_{D_N}$-modules. Therefore, $\mathcal{N}$ is a sheaf of $\mathcal{O}_{D' \cap D_N}$-modules.

In $G(D_N)$ we have $[\mathcal{I}_{D'}\mathcal{F}] - [\mathcal{I}_{D'}\mathcal{G}] + [\mathcal{I}_{D'}\mathcal{H}] = [\mathcal{H}^0(\mathcal{C})] - [\mathcal{H}^1(\mathcal{C})] + [\mathcal{H}^2(\mathcal{C})] = -[\mathcal{N}]$. Thus $\psi(\mathcal{F}) - \psi(\mathcal{G}) + \psi(\mathcal{H}) = ([\mathcal{N}], -[\mathcal{N}]) = 0$ in the quotient.

If $\mathcal{F}$ belongs to $G(D)$, then $[\mathcal{F}] = [i^*_{D'}\mathcal{F}] + [\mathcal{I}_{D'}\mathcal{F}]$ in $G(D)$. This means that $\pi \circ \psi = \mathrm{id}$. We consider now $\mathcal{H}$ in $G(D')$ and $\mathcal{K}$ in $G(D_N)$. Then

$$\psi(\pi(\mathcal{H}, \mathcal{K})) = ([i^*_{D'}\mathcal{H}] + [i^*_{D'}\mathcal{K}]) \oplus ([\mathcal{I}_{D'}\mathcal{H}] + [\mathcal{I}_{D'}\mathcal{K}]) = ([\mathcal{H}] + [\mathcal{K}_{|D' \cap D_N}]) \oplus [\mathcal{I}_{D' \cap D_N}\mathcal{K}].$$

Remark that $[\mathcal{I}_{D' \cap D_N}\mathcal{K}] = [\mathcal{K}] - [\mathcal{K}_{|D' \cap D_N}]$ in $G\big(D_N\big)$. Thus $([\mathcal{H}] + [\mathcal{K}_{|D' \cap D_N}]) \oplus ([\mathcal{K}] - [\mathcal{K}_{|D' \cap D_N}]) = [\mathcal{H}] \oplus [\mathcal{K}]$ modulo $G(D' \cap D_N)$, so that $\psi \circ \pi = \mathrm{id}$. This proves that $(*)$ is exact.

We can now use the induction hypothesis with D'. We obtain the following diagram, where the columns as well as the first line are exact:

$$\begin{tikzcd}
0 & 0 & & \\
G(D' \cap D_N) \arrow[r, "r"] \arrow[u] & G(D') \oplus G(D_N) \arrow[r, "u"] \arrow[u] & G(D) \arrow[r] & 0 \\
\bigoplus_{i<N} G(D_{iN}) \arrow[r, "s"] \arrow[u, "q"] & \bigoplus_{i<N} G(D_i) \oplus G(D_N) \arrow[u, "p"] & & \\
 & \bigoplus_{i<j<N} G(D_{ij}) \arrow[u, "t"] & &
\end{tikzcd}$$

The map $\bigoplus_i G(D_i) \xrightarrow{\pi} G(D)$ is clearly onto. Let α be an element of $\bigoplus_i G(D_i)$ such that $\pi(\alpha) = 0$. Then $u(p(\alpha)) = 0$, so that there exists β such that $r(\beta) = p(\alpha)$. There exists γ such that $q(\gamma) = \beta$. Then $p(\alpha - s(\gamma)) = p(\alpha) - r(q(\gamma)) = 0$. So there exists δ such that $\alpha = s(\gamma) + t(\delta)$. It follows that α is in the image of $\bigoplus_{i<j} G(D_{ij})$. Hence we have the exact sequence

$$\bigoplus_{i<j} G(D_{ij}) \xrightarrow{s+t} \bigoplus_i G(D_i) \xrightarrow{\pi} G(D) \longrightarrow 0,$$

which finishes the proof. □

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