The coupling method for inhomogeneous random intersection graphs.

Katarzyna Rybarczyk

Faculty of Mathematics and Computer Science,

Adam Mickiewicz University, 60–769 Poznań, Poland

`kryba@amu.edu.pl`

Abstract

We present new results concerning threshold functions for a wide family of random intersection graphs. To this end we improve and generalise the coupling method introduced for random intersection graphs so that it may be used for a wider range of parameters. Using the new approach we are able to sharpen the best known results concerning homogeneous random intersection graphs and establish threshold functions for some monotone properties of inhomogeneous random intersection graphs. Considered properties are: k-connectivity, matching containment and hamiltonicity.

1 Introduction

Since their introduction by Karoński, Scheinerman, and Singer–Cohen [13] random intersection graphs have been attracting attention due to their interesting structure and wide applications. The random intersection graph model appears in problems concerning for example "gate matrix layout" for VLSI design (see e.g. [13]), cluster analysis and classification (see e.g. [11]), analysis of complex networks (see e.g. [2, 9]), secure wireless networks (see e.g. [5]) or epidemics ([8]). Several generalisations of the model has been proposed, mainly in order to adapt it to use for a specific purpose. In this paper we consider the $\mathcal{G}(n, m, \overline{p})$ model studied for example in [1, 4, 7, 14]. Alternative ways of generalizing the model defined in [13] are given for example in [9] and [11].

1

In a random intersection graph $\mathcal{G}(n, m, \overline{p})$ there is a set of n vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$, an auxiliary set of $m = m(n)$ features $\mathcal{W} = \{w_1, \ldots, w_{m(n)}\}$, and a vector $\overline{p} = \overline{p}(n) = (p_1, \ldots, p_{m(n)})$ such that $p_i \in (0; 1)$, for each $1 \leq i \leq m$. To each vertex $v \in \mathcal{V}$ we attribute a set of its features $W(v) \subseteq \mathcal{W}$ such that for each i, $1 \leq i \leq m$, $w_i \in W(v)$ with probability p_i independently of all other features and vertices. If $w \in W(v)$ then we say that v *has chosen* w. Any two vertices $v, v' \in \mathcal{V}$ are connected by an edge in $\mathcal{G}(n, m, \overline{p})$ if $W(v)$ and $W(v')$ intersect. If $\overline{p}(n) = (p, \ldots, p)$ for some $p \in (0; 1)$ then $\mathcal{G}(n, m, \overline{p})$ is a random intersection graph defined in [13]. We denote it by $\mathcal{G}(n, m, p)$.

The random intersection graph model is very flexible and its properties change a lot if we alter the parameters. For example $\mathcal{G}(n, m, p)$ for some ranges of parameters behaves similarly to a random graph with independent edges (see [10, 16]) but in some cases it exhibit large dependencies between edge appearance (see for example [13, 18]). It was proved in [15] that in both cases $\mathcal{G}(n, m, p)$ may be coupled with a random graph with independent edges so that with probability tending to 1 as $n \to \infty$, $\mathcal{G}(n, m, p)$ is an overgraph of a graph with independent edges. It is also explained how this coupling may be used to obtain sharp results on threshold functions for $\mathcal{G}(n, m, p)$. Such properties as connectivity, a Hamilton cycle containment or a matching containment are given as examples. In general, the coupling technique provides a very elegant method to get bounds on threshold functions for random intersection graphs for a large class of properties.

The result presented in [15] is not sharp for some values of n, m and p. Therefore it cannot be straightforward generalised to $\mathcal{G}(n, m, \overline{p})$ with arbitrary $\overline{p}(n)$. In particular the method does not give sharp results for np tending to a constant. In this article we modify and extend the techniques used in [15] in order to overcome these constraints. First of all, to get the general result, we couple $\mathcal{G}(n, m, \overline{p})$ with an auxiliary random graph which does not have fully independent edges. Therefore we need to prove some additional facts about the auxiliary random graph model. Moreover we need sharp bounds on the minimum degree threshold function for $\mathcal{G}(n, m, \overline{p})$. Due to edge dependencies, estimation of moments of the random variable counting vertices with a given degree in $\mathcal{G}(n, m, \overline{p})$ is complicated. Therefore we suggest a different approach to resolve the problem. We divide $\mathcal{G}(n, m, \overline{p})$ into subgraphs so that the solution of a coupon collector problem combined with the method of moments provide the answer. The new approach to the coupling method allows to obtain better results on threshold functions for $\mathcal{G}(n, m, p)$ and by this means resolve open problems left over in [15].

Concluding, we provide a general method to establish bounds on threshold functions for many properties for $\mathcal{G}(n, m, \overline{p})$. By means of the method we are able to obtain sharp thresholds for k–connectivity, perfect matching containment and hamiltonicity for the general model. Last but not least we considerably improve known results concerning $\mathcal{G}(n, m, p)$.

All limits in the paper are taken as $n \to \infty$. Throughout the paper we use standard asymptotic notation $o(\cdot)$, $O(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$, $\sim$, $\ll$, and $\gg$ defined as in [12]. By $\text{Bin}(n, p)$ and $\text{Po}(\lambda)$ we denote the binomial distribution with parameters n, p and the Poisson distribution with expected value λ, respectively. We also use the phrase "with high probability" to say with probability tending to one as n tends to infinity. All inequalities hold for n large enough. If it does not influence the reasoning, for clarity we omit $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$.

2 Main Results

In the article we compare random intersection graph $\mathcal{G}(n, m, \overline{p})$ with a sum of a random graph with independent edges $G_2(n, \hat{p}_2)$ and a random graph $G_3(n, \hat{p}_3)$ constructed on the basis of a random 3–uniform hypergraph with independent hyperedges. Generally, for any $\hat{p} = \hat{p}(n) \in [0; 1]$ and $i = 2, \ldots, n$, let $H_i(n, \hat{p})$ be an i–uniform hypergraph with the vertex set $\mathcal{V}$ in which each i–element subset of $\mathcal{V}$ is added to the hyperedge set independently with probability $\hat{p}$. $G_i(n, \hat{p})$ is a graph with the vertex set $\mathcal{V}$ and $\{v, v'\}$, $v, v' \in \mathcal{V}$, is an edge in $G_i(n, \hat{p})$ if there exists a hyperedge in $H_i(n, \hat{p})$ containing v and v'.

We consider monotone graph properties of random graphs. For the family $\mathcal{G}$ of all graphs with the vertex set $\mathcal{V}$, we call $\mathcal{A} \subseteq \mathcal{G}$ a property if it is closed under isomorphism. Moreover $\mathcal{A}$ is increasing if $G \in \mathcal{A}$ implies $G' \in \mathcal{A}$ for all $G' \in \mathcal{G}$ such that $E(G) \subseteq E(G')$. Examples of increasing properties are: k–connectivity, containing a perfect matching and containing a Hamilton cycle.

Let $\overline{p} = (p_1, \ldots, p_m)$ be such that $p_i \in (0,1)$, for all $1 \leq i \leq m$. Define

(1)
$$S_1 = \sum_{i=1}^{m} np_i \left(1 - (1 - p_i)^{n-1}\right);$$

$$S_2 = \sum_{i=1}^{m} np_i \left(1 - \frac{1 - (1 - 2p_i)^n}{2np_i}\right);$$

$$S_3 = \sum_{i=1}^{m} np_i \left(\frac{1 - (1 - 2p_i)^n}{2np_i} - (1 - p_i)^{n-1}\right);$$

$$S_{1,t} = \sum_{i=1}^{m} t \binom{n}{t} p_i^t (1 - p_i)^{n-t}, \text{ for } t = 2, 3, \ldots, n.$$

The following theorem is an extension of the result obtained in [15].

Theorem 1. *Let $\overline{p} = (p_1, \ldots, p_m)$ be such that $p_i \in (0,1)$, for all $1 \leq i \leq m$ and S_1, S_2, and S_3 be given by (1). For a function ω tending to infinity let*

(2)
$$\hat{p} = \frac{S_2 - \omega\sqrt{S_2} - 2S_2^2 n^{-2}}{2\binom{n}{2}};$$

$$\hat{p}_2 = \begin{cases} \frac{S_1 - 3S_3 - \omega\sqrt{S_1} - 2S_1^2 n^{-2}}{2\binom{n}{2}}, & \text{for } S_3 \gg \sqrt{S_1} \text{ and } \omega \ll S_3/\sqrt{S_1}; \\ \frac{S_1 - \omega\sqrt{S_1} - 2S_1^2 n^{-2}}{2\binom{n}{2}}, & \text{for } S_3 = O(\sqrt{S_1}); \end{cases}$$

$$\hat{p}_3 = \begin{cases} \frac{S_3 - \omega\sqrt{S_1} - 6S_3^2 n^{-3}}{\binom{n}{3}}, & \text{for } S_3 \gg \sqrt{S_1} \text{ and } \omega \ll S_3/\sqrt{S_1}; \\ 0, & \text{for } S_3 = O(\sqrt{S_1}). \end{cases}$$

If $S_1 \to \infty$ and $S_1 = o(n^2)$ then for any increasing property $\mathcal{A}$.

(3)
$$\liminf_{n \to \infty} \Pr\{G_2(n, \hat{p}) \in \mathcal{A}\} \leq \limsup_{n \to \infty} \Pr\{\mathcal{G}(n, m, \overline{p}) \in \mathcal{A}\},$$

(4)
$$\liminf_{n \to \infty} \Pr\{G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \in \mathcal{A}\} \leq \limsup_{n \to \infty} \Pr\{\mathcal{G}(n, m, \overline{p}) \in \mathcal{A}\}.$$

Remark 1. Assumption $S_1 \to \infty$ is natural since for $S_1 = o(1)$ with high probability $\mathcal{G}(n, m, \overline{p})$ is an empty graph.

Remark 2. S_3 is the expected number of edges in $G_3(n, \hat{p}_3)$. If $S_3 = O(\sqrt{S_1})$ then by Markov's inequality with high probability the number of edges in $G_3(n, \hat{p}_3)$ is at most $\omega\sqrt{S_1}$. Thus $S_2 = S_1 - S_3 = S_1 + O(\omega\sqrt{S_1})$ and the bound provided by (3) is as good as the one taking into consideration the edges from $G_3(n, \hat{p}_3)$.

Remark 3. Theorem is also valid for $S_1 = \Omega(n^2)$ but with

$$\hat{p} = 1 - \exp\left(-\frac{S_2 - \omega\sqrt{S_2}}{2\binom{n}{2}}\right);$$

$$\hat{p}_2 = \begin{cases} 1 - \exp\left(-\frac{S_1 - 3S_3 - \omega\sqrt{S_1}}{2\binom{n}{2}}\right), & \text{for } S_3 \gg \sqrt{S_1} \text{ and } \omega \ll S_3/\sqrt{S_1}; \\ 1 - \exp\left(-\frac{S_1 - \omega\sqrt{S_1}}{2\binom{n}{2}}\right), & \text{for } S_3 = O(\sqrt{S_1}); \end{cases}$$

$$\hat{p}_3 = \begin{cases} 1 - \exp\left(-\frac{S_3 - \omega\sqrt{S_1}}{\binom{n}{3}}\right), & \text{for } S_3 \gg \sqrt{S_1} \text{ and } \omega \ll S_3/\sqrt{S_1}; \\ 0, & \text{for } S_3 = O(\sqrt{S_1}). \end{cases}$$

Denote by $\mathcal{C}_k$, $\mathcal{PM}$ and $\mathcal{HC}$ the following graph properties: a graph is k–connected, has a perfect matching and has a Hamilton cycle, respectively. We will use Theorem 1 to establish threshold functions for $\mathcal{C}_k$, $\mathcal{PM}$ and $\mathcal{HC}$ in $\mathcal{G}(n, m, \overline{p})$. By $\mathcal{C}_k$ we denote the vertex connectivity. From the proof it follows that the threshold function for the edge connectivity is the same as this for $\mathcal{C}_k$.

For any sequence c_n with a limit we write

$$(5) \qquad f(c_n) = \begin{cases} 0 & \text{for } c_n \to -\infty; \\ e^{-e^{-c}} & \text{for } c_n \to c \in (-\infty; \infty); \\ 1 & \text{for } c_n \to \infty. \end{cases}$$

Theorem 2. *Let $\max_{1 \le i \le m} p_i = o((\ln n)^{-1})$ and S_1 and $S_{1,2}$ be given by (1).*

(i) If $S_1 = n(\ln n + c_n)$, then

$$\lim_{n \to \infty} \Pr\{\mathcal{G}(n, m, \overline{p}) \in \mathcal{C}_1\} = f(c_n),$$

where $f(c_n)$ is given by (5).

(ii) Let k be a positive integer and $a_n = \frac{S_{1,2}}{S_1}$. If

$$S_1 = n(\ln n + (k-1)\ln\ln n + c_n),$$

then

$$\lim_{n \to \infty} \Pr\{\mathcal{G}(n, m, \overline{p}) \in \mathcal{C}_k\} = \begin{cases} 0 & \text{for } c_n \to -\infty \text{ and } a_n \to a \in (0; 1]; \\ 1 & \text{for } c_n \to \infty. \end{cases}$$

5

Assumption $\max_{1\le i\le m} p_i = o((\ln n)^{-1})$ is necessary to avoid awkward cases. The problem is explained in more detail in Section 4. A straightforward corollary of the above theorem is that for $S_1 = n(\ln n + c_n)$, $c_n \to -\infty$ and any $k = 1, 2, \ldots, n$.

$$\lim_{n\to\infty} \Pr\{\mathcal{G}(n, m, \overline{p}) \in \mathcal{C}_k\} = 0 \quad \text{and} \quad \lim_{n\to\infty} \Pr\{\mathcal{G}(n, m, \overline{p}) \in \mathcal{HC}\} = 0.$$

Theorem 3. *Let* $\max_{1\le i\le m} p_i = o((\ln n)^{-1})$ *and* S_1 *be given by* (1). *If* $S_1 = n(\ln n + c_n)$ *then*

$$\lim_{n\to\infty} \Pr\{\mathcal{G}(2n, m, \overline{p}(2n)) \in \mathcal{PM}\} = f(c_{2n}),$$

where $f(\cdot)$ *is given by* (5).

Theorem 4. *Let* $\max_{1\le i\le m} p_i = o((\ln n)^{-1})$, S_1 *and* $S_{1,2}$ *be given by* (1) *and* $a_n = \frac{S_{1,2}}{S_1}$. *If* $S_1 = n(\ln n + \ln\ln n + c_n)$, *then*

$$\lim_{n\to\infty} \Pr\{\mathcal{G}(n, m, \overline{p}) \in \mathcal{HC}\} = \begin{cases} 0 & \text{for } c_n \to -\infty \text{ and } a_n \to a \in (0; 1]; \\ 1 & \text{for } c_n \to \infty. \end{cases}$$

Already simple corollaries of Theorems 2–4 give sharp threshold functions for $\mathcal{G}(n, m, p)$. For example.

Corollary 1. *Let* $m \gg \ln^2 n$ *and* $p(1 - (1 - p)^{n-1}) = (\ln n + c_n)/m$. *Then*

$$\lim_{n\to\infty} \Pr\{\mathcal{G}(n, m, p) \in \mathcal{C}_1\} = f(c_n)$$

and

$$\lim_{n\to\infty} \Pr\{\mathcal{G}(2n, m, p) \in \mathcal{PM}\} = f(c_{2n}),$$

where $f(\cdot)$ *is given by* (5).

In particular we may state the following extension of the result from [19].

Corollary 2. *Let* b_n *be a sequence,* β *and* γ *be constants such that* $\beta\gamma(1 - e^{-\gamma}) = 1$. *If*

$$m = \beta n \ln n \quad \text{and} \quad p = \frac{\gamma}{n}\left(1 + \frac{b_n}{\ln n}\right) \quad \text{then}$$

6

$$\lim_{n\to\infty} \Pr\{\mathcal{G}(n,m,p) \in \mathcal{C}_1\} = f\left(\left(1 + \frac{e^{-\gamma}\gamma}{1-e^{-\gamma}}\right)b_n\right)$$

and

$$\lim_{n\to\infty} \Pr\{\mathcal{G}(2n,m,p(2n)) \in \mathcal{PM}\} = f\left(\left(1 + \frac{e^{-\gamma}\gamma}{1-e^{-\gamma}}\right)b_{2n}\right),$$

where $f(\cdot)$ is given by (5).

Sometimes the method of the proof enables to improve results concerning $\mathcal{G}(n,m,p)$ even more.

Theorem 5. *Let $m \gg \ln^2 n$ and*

$$(6) \qquad p(1-(1-p)^{n-1}) = \frac{\ln n + \ln\left(\max\left\{1, \left(\frac{npe^{-np}\ln n}{1-e^{-np}}\right)\right\}\right) + c_n}{m}.$$

Then

$$\lim_{n\to\infty} \Pr\{\mathcal{G}(n,m,p) \in \mathcal{HC}\} = \begin{cases} 0 & \text{for } c_n \to -\infty; \\ 1 & \text{for } c_n \to \infty. \end{cases}$$

Theorem 6. *Let $m \gg \ln^2 n$, k be a positive integer, and*

$$a_n = (np)^{k-1}\left(\left(\frac{e^{-np}\ln n}{1-e^{-np}}\right)^{k-1} + \frac{e^{-np}\ln n}{1-e^{-np}}\right)$$

If

$$p(1-(1-p)^{n-1}) = \frac{\ln n + \ln\left(\max\left\{1, a_n\right\}\right) + c_n}{m},$$

then

$$\lim_{n\to\infty} \Pr\{\mathcal{G}(n,m,p) \in \mathcal{C}_k\} = \begin{cases} 0 & \text{for } c_n \to -\infty; \\ 1 & \text{for } c_n \to \infty. \end{cases}$$

One of the question posed in [15] concerned the range of $m = m(n)$ for which the threshold function for $\mathcal{C}_k$ for $\mathcal{G}(n,m,p)$ coincides with this for $\delta(\mathcal{G}(n,m,p)) \geq 1$. Moreover we may ask when threshold function for $\mathcal{C}_k$ for $\mathcal{G}(n,m,p)$ is the same as this for $\mathcal{C}_k$ for $G_2(n,\hat{p})$ with $\hat{p} = mp^2$. Theorem 6 gives a final answer to these questions.

Corollary 3. *Let k be a positive integer. If*

$$p(1 - (1-p)^{n-1}) = \begin{cases} \frac{\ln n + c_n}{m}, & \text{for } \ln^2 n \ll m \ll \frac{n \ln n}{\ln \ln n} \\ \frac{\ln n + (k-1) \ln \ln n + c_n}{m}, & \text{for } m = \Omega(n \ln n); \end{cases}$$

then

$$\lim_{n \to \infty} \Pr\{\mathcal{G}(n, m, p) \in \mathcal{C}_k\} = \begin{cases} 0 & \text{for } c_n \to -\infty; \\ 1 & \text{for } c_n \to \infty. \end{cases}$$

The remaining part of the article is organised as follows. Section 3 is devoted to a construction of the coupling of $\mathcal{G}(n, m, \overline{p})$ and $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$. The coupling is used in Section 3 to prove Theorem 1. Moreover in the subsequent sections the coupling will be applied in the proofs of the remaining theorems, especially Theorems 5 and 6. In Section 4 we give the threshold function for the minimum degree at least k in $\mathcal{G}(n, m, \overline{p})$. To this end we construct a coupling of a construction of $\mathcal{G}(n, m, \overline{p})$ and a coupon collector process. Results included in Section 4 will be mainly used to prove 0–statements of Theorems 2–6. Section 5 is dedicated to the properties of $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$. Results gathered in this section will be crucial in showing 1– statements of Theorems 2–6. In Section 6 we complete the proofs of Theorems 2–6 using the results obtained in the previous sections.

3 Coupling

In this section we present a proof of Theorem 1. In the proof we use auxiliary random graph models $\mathbb{G}_{*i}(n, M)$, $i = 2, 3, \ldots, n$, in which M is a random variable with non–negative integer values. For $i = 2, \ldots, n$, $\mathbb{G}_{*i}(n, M)$ is constructed on the basis of a random hypergraph $\mathbb{H}_{*i}(n, M)$. $\mathbb{H}_{*i}(n, M)$ is a random hypergraph with the vertex set $\mathcal{V}$ in which a hyperedge set is constructed by sampling M times with repetition elements from the set of all i–element subsets of $\mathcal{V}$ (all sets which are chosen several times are added only once to the hyperedge set). $\mathbb{G}_{*i}(n, M)$ is a graph with the vertex set $\mathcal{V}$ in which $v, v' \in \mathcal{V}$ are connected by an edge if $\{v, v'\}$ is contained in at least one of the hyperedges of $\mathbb{H}_{*i}(n, M)$. For simplicity of notation, if M equals a constant t with probability one or has the Poisson distribution, we write $\mathbb{G}_{*i}(n, t)$ or $\mathbb{G}_{*i}(n, \text{Po}(\cdot))$, respectively. Recall that similarly $G_i(n, \hat{p})$ is constructed on the basis of $H_i(n, \hat{p})$ – a hypergraph with independent hyperedges (see definitions at the beginning of Section 2).

In this paper we treat random graphs as random variables. By a coupling $(\mathbb{G}_1, \mathbb{G}_2)$ of two random variables $\mathbb{G}_1$ and $\mathbb{G}_2$ we mean a choice of a probability space on which a random vector $(\mathbb{G}_1', \mathbb{G}_2')$ is defined and $\mathbb{G}_1'$ and $\mathbb{G}_2'$ have the same distributions as $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. For simplicity of notation we will not differentiate between $(\mathbb{G}_1', \mathbb{G}_2')$ and $(\mathbb{G}_1, \mathbb{G}_2)$. For two random graphs $\mathbb{G}_1$ and $\mathbb{G}_2$ we write

$$\mathbb{G}_1 \preceq \mathbb{G}_2$$

if there exists a coupling $(\mathbb{G}_1, \mathbb{G}_2)$, such that in the probability space of the coupling $\mathbb{G}_1$ is a subgraph of $\mathbb{G}_2$ with probability 1. Moreover, we write

$$\mathbb{G}_1 = \mathbb{G}_2,$$

if $\mathbb{G}_1$ and $\mathbb{G}_2$ have the same probability distribution (equivalently there exists a coupling $(\mathbb{G}_1, \mathbb{G}_2)$ such that $\mathbb{G}_1 = \mathbb{G}_2$ with probability one). For two sequences of random graphs $\mathbb{G}_1 = \mathbb{G}_1(n)$ and $\mathbb{G}_2 = \mathbb{G}_2(n)$ we write

$$\mathbb{G}_1 \preceq_{1-o(1)} \mathbb{G}_2,$$

if there exists a sequence of couplings $(\mathbb{G}_1(n), \mathbb{G}_2(n))$, such that in the probability space of the coupling $\mathbb{G}_1(n)$ is a subgraph of $\mathbb{G}_2(n)$ with probability $1 - o(1)$, respectively.

Note that, for any λ, in $\mathbb{H}_{*i}(n, \text{Po}(\lambda))$ each edge appears independently with probability $1 - \exp(-\lambda/\binom{n}{i})$ (see [10]). Thus

(7) $$\mathbb{G}_{*i}(n, \text{Po}(\lambda)) = G_i\left(n, 1 - \exp\left(-\frac{\lambda}{\binom{n}{i}}\right)\right).$$

We gather here a few useful facts concerning couplings of random graphs. For proofs see [15, 16].

Fact 1. *Let M_n be a sequence of random variables and let t_n be a sequence of positive integers.*

(i) If $\Pr\{M_n \geq t_n\} = o(1)$ then $\mathbb{G}_{*i}(n, M_n) \preceq_{1-o(1)} \mathbb{G}_{*i}(n, t_n)$.
(ii) If $\Pr\{M_n \leq t_n\} = o(1)$ then $\mathbb{G}_{*i}(n, t_n) \preceq_{1-o(1)} \mathbb{G}_{*i}(n, M_n)$.

Fact 2. *Let $(\mathbb{G}_i)_{i=1,\ldots,m}$ and $(\mathbb{G}_i')_{i=1,\ldots,m}$ be sequences of independent random graphs. If*

$$\mathbb{G}_i \preceq \mathbb{G}_i', \text{ for all } i = 1, \ldots, m,$$

then

$$\bigcup_{i=1}^{m} \mathbb{G}_i \preceq \bigcup_{i=1}^{m} \mathbb{G}_i'.$$

Fact 3. *Let* $\mathbb{G}_1 = \mathbb{G}_1(n)$, $\mathbb{G}_2 = \mathbb{G}_2(n)$, *and* $\mathbb{G}_3 = \mathbb{G}_3(n)$ *be random graphs. If*

$$\mathbb{G}_1 \preceq_{1-o(1)} \mathbb{G}_2 \quad and \quad \mathbb{G}_2 \preceq_{1-o(1)} \mathbb{G}_3$$

then

$$\mathbb{G}_1 \preceq_{1-o(1)} \mathbb{G}_3.$$

Fact 4. *Let* $\mathbb{G}_1 = \mathbb{G}_1(n)$ *and* $\mathbb{G}_2 = \mathbb{G}_2(n)$ *be two random graphs, such that*

$$(8) \qquad\qquad \mathbb{G}_1 \preceq_{1-o(1)} \mathbb{G}_2.$$

Then for any increasing property $\mathcal{A}$

$$\liminf_{n\to\infty} \Pr\left\{\mathbb{G}_1(n) \in \mathcal{A}\right\} \le \limsup_{n\to\infty} \Pr\left\{\mathbb{G}_2(n) \in \mathcal{A}\right\}.$$

Proof. Define event $\mathcal{E} := \{\mathbb{G}_1 \subseteq \mathbb{G}_2\}$ on a probability space of a coupling $(\mathbb{G}_1, \mathbb{G}_2)$ existing by (8). Then for any increasing property $\mathcal{A}$

$$
\begin{aligned}
\Pr\left\{\mathbb{G}_2 \in \mathcal{A}\right\} &\ge \Pr\left\{\mathbb{G}_2 \in \mathcal{A}|\mathcal{E}\right\}\Pr\{\mathcal{E}\} \\
&\ge \Pr\left\{\mathbb{G}_1 \in \mathcal{A}|\mathcal{E}\right\}\Pr\{\mathcal{E}\} \\
&= \Pr\left\{\{\mathbb{G}_1 \in \mathcal{A}\} \cap \mathcal{E}\right\} \\
&= \Pr\left\{\mathbb{G}_1 \in \mathcal{A}\right\} + \Pr\left\{\mathcal{E}\right\} - \Pr\left\{\{\mathbb{G}_1 \in \mathcal{A}\} \cup \mathcal{E}\right\} \\
&\ge \Pr\left\{\mathbb{G}_1 \in \mathcal{A}\right\} + \Pr\left\{\mathcal{E}\right\} - 1 \\
&= \Pr\left\{\mathbb{G}_1 \in \mathcal{A}\right\} + o(1).
\end{aligned}
$$

The result follows by taking $n \to \infty$ $\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of Theorem 1. We will show only (4) in the case $S_3 \gg \sqrt{S_1}$. The remaining cases follow by similar arguments. Recall that $S_2 = S_1 - S_3$ and $S_2 = \Theta(S_1)$.

Let $w_i \in \mathcal{W}$. Denote by V_i the set of vertices which have chosen feature w_i (i.e. $V_i = \{v \in \mathcal{V} : w_i \in W(v)\}$). Let

$$(9) \qquad\qquad V_i' = \begin{cases} V_i & \text{for } |V_i| \ge 2; \\ \emptyset & \text{otherwise.} \end{cases}$$

For each i, let $\mathcal{G}[V_i']$ be a graph with the vertex set $\mathcal{V}$ and an edge set containing those edges from $\mathcal{G}(n, m, \overline{p})$ which have both ends in V_i' (i.e. its

edges form a clique with the vertex set V_i'). In the proof we will use the fact that $\bigcup_{i=1}^{m} \mathcal{G}[V_i'] = \mathcal{G}(n, m, \overline{p})$. First, for each i we will construct a coupling of $\mathcal{G}[V_i']$ with an auxiliary random graph and then we will couple a sum of the auxiliary random graphs with $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$.

For each i, $1 \leq i \leq m$, let

(10)
$$
\begin{aligned}
X_i &= |V_i|; \\
Y_i &= |V_i'|; \\
Z_i &= \mathbb{I}_{\{Y_i \text{ is odd}\}},
\end{aligned}
$$

where $\mathbb{I}_A$ is an indicator random variable of the event A. Note that X_i, $1 \leq i \leq m$, are independent random variables with the binomial distributions $\mathrm{Bin}(n, p_i)$, $1 \leq i \leq m$. For each i, $1 \leq i \leq m$, we construct independently a coupling of $\mathbb{G}_{*2}\left(n, \frac{Y_i - 3Z_i}{2}\right) \cup \mathbb{G}_{*3}(n, Z_i)$ and $\mathcal{G}[V_i']$. Given $Y_i = y_i$ and $Z_i = z_i$, for each i independently, we generate instances of $\mathbb{G}_{*2}\left(n, \frac{y_i - 3z_i}{2}\right)$ and $\mathbb{G}_{*3}(n, z_i)$. Let $Y_i' = y_i'$ be the number of non-isolated vertices in the constructed instance of $\mathbb{G}_{*2}\left(n, \frac{y_i - 3z_i}{2}\right) \cup \mathbb{G}_{*3}(n, z_i)$. By definition $y_i' \leq y_i$. Set now V_i' to be the union of the set of non–isolated vertices of the constructed instance of $\mathbb{G}_{*2}\left(n, \frac{y_i - 3z_i}{2}\right) \cup \mathbb{G}_{*3}(n, z_i)$ and $y_i - y_i'$ vertices chosen uniformly at random from the remaining vertices. This coupling implies

$$
\mathbb{G}_{*2}\left(n, \frac{Y_i - 3Z_i}{2}\right) \cup \mathbb{G}_{*3}(n, Z_i) \preceq \mathcal{G}[V_i].
$$

Now let
$$
M_2 = \sum_{1 \leq i \leq m} \frac{Y_i - 3Z_i}{2} \quad \text{and} \quad M_3 = \sum_{1 \leq i \leq m} Z_i.
$$

Graphs $\mathbb{G}_{*2}\left(n, \frac{Y_i - 3Z_i}{2}\right) \cup \mathbb{G}_{*3}(n, Z_i)$, $1 \leq i \leq m$, are independent and $\mathcal{G}[V_i]$, $1 \leq i \leq m$, are independent. Therefore by Fact 2, the definition of $\mathcal{G}(n, m, \overline{p})$, and the definitions of $\mathbb{G}_{*2}(n, \cdot)$ and $\mathbb{G}_{*3}(n, \cdot)$ we have

(11)
$$
\begin{aligned}
\mathbb{G}_{*2}(n, M_2) \cup \mathbb{G}_{*3}(n, M_3) &= \bigcup_{1 \leq i \leq m} \left(\mathbb{G}_{*2}\left(n, \frac{Y_i - 3Z_i}{2}\right) \cup \mathbb{G}_{*3}(n, Z_i) \right) \\
&\preceq \bigcup_{1 \leq i \leq m} \mathcal{G}[V_i] \\
&= \mathcal{G}(n, m, \overline{p}).
\end{aligned}
$$

Note that by definition

$$\mathbb{E} \sum_{i=1}^{m} Y_i = \mathbb{E} \sum_{i=1}^{m} (X_i - \mathbb{I}_{X_i=1}) = S_1 \quad \text{and}$$

$$\mathbb{E} \sum_{i=1}^{m} Z_i = \mathbb{E} \sum_{i=1}^{m} (\mathbb{I}_{X_i \, is \, odd} - \mathbb{I}_{X_i=1}) = S_3,$$

where S_1 and S_3 are defined by (1). Therefore

$$\mathbb{E} M_2 = \frac{S_1 - 3S_3}{2} \quad \text{and} \quad \mathbb{E} M_3 = S_3.$$

We will prove that M_2 and M_3 with high probability are concentrated around their expected values. Since for each i, $1 \le i \le m$, X_i is a random variable with the binomial distribution $\text{Bin}\,(n, p_i)$ and $\mathbb{E}(X_i \mathbb{I}_{X_i=1}) = \mathbb{E} \mathbb{I}_{X_i=1}$ we have

$$
\begin{aligned}
\text{Var} \sum_{i=1}^{m} Y_i &= \sum_{i=1}^{m} \text{Var}\,(X_i - \mathbb{I}_{X_i=1}) \\
&= \sum_{i=1}^{m} (\text{Var} X_i + \text{Var} \mathbb{I}_{X_i=1} - 2(\mathbb{E}(X_i \mathbb{I}_{X_i=1}) - \mathbb{E} X_i \mathbb{E} \mathbb{I}_{X_i=1})) \\
&\le \sum_{i=1}^{m} (\mathbb{E} X_i - \mathbb{E} \mathbb{I}_{X_i=1} + 2\mathbb{E} X_i \mathbb{E} \mathbb{I}_{X_i=1}) \\
&= \sum_{i=1}^{m} \left(\mathbb{E} Y_i + 2(np_i)^2 (1 - p_i)^{n-1} \right) \\
&\le \sum_{i=1}^{m} \left(\mathbb{E} Y_i + 3(np_i - np_i(1 - p_i)^{n-1}) \right) \\
&= 4S_1.
\end{aligned}
$$

In the second last line we use the fact that function $(2nx + 3)(1 - x)^{n-1}$ is

decreasing for $x \in [0,1)$. Similarly

$$
\begin{aligned}
\operatorname{Var} \sum_{i=1}^{m} Z_i &= \sum_{i=1}^{m} \operatorname{Var}\left(\mathbb{I}_{X_i odd} - \mathbb{I}_{X_i=1}\right) \\
&= \sum_{i=1}^{m} \left(\operatorname{Var}\mathbb{I}_{X_i\, odd} + \operatorname{Var}\mathbb{I}_{X_i=1} - 2(\mathbb{E}(\mathbb{I}_{X_i\, odd}\mathbb{I}_{X_i=1}) - \mathbb{E}\mathbb{I}_{X_i\, odd}\mathbb{E}\mathbb{I}_{X_i=1}))\right) \\
&\leq \sum_{i=1}^{m} \left(\mathbb{E}\mathbb{I}_{X_i\, odd} - \mathbb{E}\mathbb{I}_{X_i=1} + 2\mathbb{E}\mathbb{I}_{X_i\, odd}\mathbb{E}\mathbb{I}_{X_i=1}\right) \\
&\leq \sum_{i=1}^{m} \left(\mathbb{E}X_i - \mathbb{E}\mathbb{I}_{X_i=1} + 2\mathbb{E}X_i\mathbb{E}\mathbb{I}_{X_i=1}\right) \\
&\leq 4S_1.
\end{aligned}
$$

Therefore by Chebyshev's inequality, for any function ω' tending to infinity with high probability

$$
\text{(12)} \qquad \Pr\left\{\left|\sum_{i=1}^{m} Y_i - S_1\right| \geq \omega'\sqrt{S_1}\right\} \leq \frac{4}{(\omega')^2} = o(1),
$$

$$
\Pr\left\{\left|\sum_{i=1}^{m} Z_i - S_3\right| \geq \omega'\sqrt{S_1}\right\} \leq \frac{4}{(\omega')^2} = o(1).
$$

Thus with with high probability for $\omega' \ll S_3/\sqrt{S_1}$ (the assumption is to ensure that for large n r.h.s. is positive)

$$
M_2 \geq \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2},
$$

$$
M_3 \geq S_3 - \omega'\sqrt{S_1}.
$$

Therefore by Fact 1 and (11)

$$
\mathbb{G}_{*2}\left(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2}\right) \cup \mathbb{G}_{*3}\left(n, S_3 - \omega'\sqrt{S_1}\right)
$$

$$
\preceq_{1-o(1)} \mathbb{G}_{*2}\left(n, M_2\right) \cup \mathbb{G}_{*3}\left(n, M_3\right) \preceq \mathcal{G}\left(n, m, \overline{p}\right).
$$

We may assume that in the above coupling $\mathbb{G}_{*2}\left(n, \frac{S_1-3S_3-4\omega'\sqrt{S_1}}{2}\right)$ and $\mathbb{G}_{*3}\left(n, S_3 - \omega'\sqrt{S_1}\right)$ are independent. The main reason for this is the fact that even though

M_2 and M_3 are dependent (i.e. also $\mathbb{G}_{*2}(n, M_2)$ and $\mathbb{G}_{*3}(n, M_3)$ are dependent), the choices of hyperedges of $\mathbb{H}_{*i}(n, \cdot)$ in distinct draws are independent. Moreover note that in the coupling, in order to get $\mathbb{G}_{*2}(n, M_2) \cup \mathbb{G}_{*3}(n, M_3)$ from a sum of independent graphs $\mathbb{G}_{*2}\left(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2}\right) \cup \mathbb{G}_{*3}\left(n, S_3 - \omega'\sqrt{S_1}\right)$ we may proceed in the following way. Given independent instances of $\mathbb{H}_{*2}(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2})$ and $\mathbb{H}_{*3}(n, S_3 - \omega'\sqrt{S_1})$, $M_2 = m_2$ and $M_3 = m_3$:

– if m_2 (or m_3 resp.) is larger than $(S_1 - 3S_3 - 4\omega'\sqrt{S_1})/2$ (or $S_3 - \omega'\sqrt{S_1}$ resp.) then we make $m_2 - (S_1 - 3S_3 - 4\omega'\sqrt{S_1})/2$ (or $m_3 - (S_3 - \omega'\sqrt{S_1})$) additional draws and add hyperedges to $\mathbb{H}_{*2}(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2})$ ($\mathbb{H}_{*3}(n, S_3 - \omega'\sqrt{S_1})$ resp.)

– if m_2 (m_3 resp.) is smaller than $(S_1 - 3S_3 - 4\omega'\sqrt{S_1})/2$ (or $S_3 - \omega'\sqrt{S_1}$ resp.) then we delete from $\mathbb{H}_{*2}(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2})$ ($\mathbb{H}_{*3}(n, S_3 - \omega'\sqrt{S_1})$ resp.) hypredges attributed to the last draws to get exactly m_i, $i = 2, 3$, draws.

Let M_2' and M_3' be independent random variables with the Poisson distributions

$$\text{Po}\left(\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2}\right) \quad \text{and} \quad \text{Po}\left(S_3 - 2\omega'\sqrt{S_1}\right), \quad \text{respectively.}$$

Then by a sharp concentration of the Poisson distribution

$$\Pr\left\{M_2' \leq \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2}\right\} = 1 - o(1) \quad \text{and}$$
$$\Pr\left\{M_3' \leq S_3 - \omega'\sqrt{S_1}\right\} = 1 - o(1).$$

Therefore by Fact 1 and (7)

$$G_2\left(n, 1 - e^{\left(-\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2\binom{n}{2}}\right)}\right) \cup G_3\left(n, 1 - e^{\left(-\frac{S_3 - 2\omega'\sqrt{S_1}}{\binom{n}{3}}\right)}\right)$$
$$= \mathbb{G}_{*2}(n, M_2') \cup \mathbb{G}_{*3}(n, M_3')$$
$$\preceq_{1-o(1)} \mathbb{G}_{*2}\left(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2}\right) \cup \mathbb{G}_{*3}\left(n, S_3 - \omega'\sqrt{S_1}\right)$$
$$\preceq_{1-o(1)} \mathcal{G}(n, m, \overline{p}).$$

14

For $S_1 = o(n^2)$ (then also $S_3 \leq S_1 = o(n^3)$), $\omega = 5\omega'$, and $\hat{p}_2$ and $\hat{p}_3$ defined by (2) we have

$$\hat{p}_2 = \frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1} - \frac{2S_1^2}{n^2}}{2\binom{n}{2}} \leq 1 - e^{\left(-\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2\binom{n}{2}}\right)};$$

$$\hat{p}_3 \leq \frac{S_3 - 2\omega'\sqrt{S_1} - \frac{6S_3^2}{n^3}}{\binom{n}{3}} \leq 1 - e^{\left(-\frac{S_3 - 2\omega'\sqrt{S_1}}{\binom{n}{3}}\right)}.$$

Therefore using standard couplings of $G_2(n, \cdot)$ and $H_3(n, \cdot)$ finally we get

$$G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$$

$$\preceq G_2\left(n, 1 - e^{\left(-\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2\binom{n}{2}}\right)}\right) \cup G_3\left(n, 1 - e^{\left(-\frac{S_3 - 2\omega'\sqrt{S_1}}{\binom{n}{3}}\right)}\right)$$

$$= \mathbb{G}_{*2}\left(n, \mathrm{Po}\left(\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2}\right)\right) \cup \mathbb{G}_{*3}\left(n, \mathrm{Po}\left(S_3 - 2\omega'\sqrt{S_1}\right)\right)$$

$$\preceq_{1-o(1)} \mathcal{G}(n, m, \overline{p}).$$

The result follows by Facts 3 and 4. $\qquad\square$

4 Vertex degrees in $\mathcal{G}(n, m, \overline{p})$

For any graph G denote by $\delta(G)$ the minimum vertex degree in G.

Lemma 1. *Let c_n be a sequence of real numbers, $\overline{p} = (p_1, \ldots, p_m)$ be such that $\max_{1 \leq i \leq m} p_i = o((\ln n)^{-1})$, and S_1 be given by (1).*
If

$$S_1 = n(\ln n + c_n)$$

then

$$\lim_{n \to \infty} \mathrm{Pr}\left\{\delta(\mathcal{G}(n, m, \overline{p})) \geq 1\right\} = f(c_n),$$

where $f(\cdot)$ is given by (5).

Note that the condition $\max_{1 \leq i \leq m} p_i = o((\ln n)^{-1})$ is necessary. Otherwise the number of vertices of a given degree depends more on the fluctuations of

the values of the vector $\bar{p}$. For example let $m = n^2$ and $\bar{p}$ equals

$$\Big(\underbrace{\frac{b_n}{\ln n}, \ldots, \frac{b_n}{\ln n}}_{\ln n(\ln n + c_n)/b_n}, \frac{1}{nm}, \ldots, \frac{1}{nm}\Big) \quad \text{or}$$

$$\Big(\frac{1}{\sqrt{\ln n}}, \sqrt{\frac{\ln n + c_n}{nm}}, \ldots, \sqrt{\frac{\ln n + c_n}{nm}}\Big),$$

for some $b_n, c_n = o(\ln n)$. In both cases

$$\sum_{i=1}^{m} np_i \left(1 - (1 - p_i)^{n-1}\right) = n(\ln n + c_n + o(1))$$

but the expected number of vertices of degree 0 in $\mathcal{G}(n, m, \bar{p})$ is

$$(1 + o(1))\exp\left(-c_n - \frac{b_n}{2}\right) \quad \text{and} \quad (1 + o(1))\exp(-c_n),$$

respectively.

Lemma 2. *Let c_n be a sequence of real numbers, k be a positive integer, $\bar{p} = (p_1, \ldots, p_m)$ be such that $\max_{1 \le i \le m} p_i = o(\ln n^{-1})$, and S_1 and $S_{1,t}$, $t = 2 \ldots, k$, be given by (1).*

(i) If

$$S_1 = n\left(\ln n + (k-1)\ln\left(\max\left\{1, \left(\frac{S_{1,2}}{S_1}\ln n\right)\right\}\right) + c_n\right)$$

then

$$\lim_{n \to \infty} \Pr\left\{\delta(\mathcal{G}(n, m, \bar{p})) \ge k\right\} = 0 \quad \text{for } c_n \to -\infty.$$

(ii) If

$$S_1 - \sum_{t=3}^{k} S_{1,t} = n\left(\ln n + (k-1)\ln\left(\max\left\{1, \left(\frac{S_{1,2}}{S_1}\ln n\right)\right\}\right) + c_n\right)$$

then

$$\lim_{n \to \infty} \Pr\left\{\delta(\mathcal{G}(n, m, \bar{p})) \ge k\right\} = 1 \quad \text{for } c_n \to \infty.$$

16

Here and in the proof we assume that $\sum_{t=3}^{2} S_{1,t} = 0$.

If we assume that $p_i = p$, for all $1 \leq i \leq m$, then with a little more work the result of Lemma 2 may be improved.

Lemma 3. *Let $p \in (0; 1)$, k be a positive integer, $m = m(n)$ be such that $m \gg \ln^2 n$, and*

$$a_n = (np)^{k-1} \left(\left(\frac{e^{-np} \ln n}{1 - e^{-np}} \right)^{k-1} + \frac{e^{-np} \ln n}{1 - e^{-np}} \right).$$

If

$$p(1 - (1-p)^{n-1}) = \frac{\ln n + \ln\left(\max\{1, a_n\}\right) + c_n}{m},$$

then

$$\lim_{n \to \infty} \Pr\left\{\delta(\mathcal{G}(n, m, p)) \geq k\right\} = \begin{cases} 0 & \text{for } c_n \to -\infty; \\ 1 & \text{for } c_n \to \infty. \end{cases}$$

Proof of Lemma 1. In the proof we assume that $c_n = o(\ln n)$. Note that if $\overline{p}' = (p_1', \ldots, p_m')$ and $\overline{p} = (p_1, \ldots, p_m)$ are such that $p_i' \leq p_i$, for all $1 \leq i \leq m$, then $\mathcal{G}(n, m, \overline{p}') \preceq \mathcal{G}(n, m, \overline{p})$ and $\sum_{i=1}^{m} np_i'(1 - (1 - p_i')^{n-1}) \leq \sum_{i=1}^{m} np_i(1 - (1 - p_i)^{n-1})$. Therefore by monotonicity of the considered property, analysis of the case $c_n = o(\ln n)$ is enough to prove the general result stated above.

Consider a coupon collector process in which in each draw one choose one coupon uniformly at random from $\mathcal{V}$. In order to determine the minimum degree we establish a coupling of the coupon collector process on $\mathcal{V}$ and the construction of $\mathcal{G}(n, m, \overline{p})$. Define V_i' and Y_i as in (9) and (10). We consider a process in which we collect coupons from $\mathcal{V}$ and at the same time we construct a family of sets $\{V_i' : i = 1, \ldots, m\}$ (i.e. equivalently we construct an instance of $\mathcal{G}(n, m, \overline{p})$). Assume that we have a given vector

$$(Y_1, Y_2, \ldots, Y_m) = (y_1, y_2, \ldots, y_m)$$

chosen so that Y_i are independent random variables with distribution of the random variables defined in (10). We divide the process of collecting coupons into m phases. In the i–th phase, $1 \leq i \leq m$, we draw independently one by one vertices uniformly at random from $\mathcal{V}$ until within the phase we get y_i distinct vertices. Let V_i' be the set of vertices chosen in the i–th phase. We construct an instance of $\mathcal{G}(n, m, \overline{p})$ by connecting by edges all pairs of vertices within V_i', for all $1 \leq i \leq m$. After the m–th phase $\bigcup_{1 \leq i \leq m} V_i'$ is the

set of non-isolated vertices in $\mathcal{G}\left(n, m, \overline{p}\right)$. Denote by T_i the number of draws in the i–th phase. If $Y_i = y_i = 0$ then $T_i = 0$. If $Y_i = y_i \geq 2$ and in the i–th phase we have already collected $j < y_i$ vertices then the number of draws to collect the $j+1$–st vertex has the geometric distribution with parameter $\frac{n-j}{n}$. Therefore

$$
\mathbb{E}(T_i - Y_i | Y_i = y_i) = \left(\sum_{j=0}^{y_i-1} \frac{n}{n-j} \right) - y_i
$$

$$
= \sum_{j=0}^{y_i-1} \frac{j}{n-j}
$$

$$
\leq \begin{cases} \sum_{j=0}^{y_i-1} \frac{2j}{n} = \frac{y_i(y_i-1)}{n} & \text{for } 2 \leq y_i < n/2; \\ n \ln n & \text{for } y_i \geq n/2. \end{cases}
$$

Note that

$$
\Pr\left\{ Y_i \geq \frac{n}{2} \right\} \leq \binom{n}{\frac{n}{2}} p_i^{\frac{n}{2}} \leq \left(\frac{enp_i}{\frac{n}{2}} \right)^{\frac{n}{2}} \leq \frac{p_i^2}{\ln n}.
$$

Thus

$$
\mathbb{E}(T_i - Y_i) \leq \sum_{y_i=2}^{n/2} \frac{y_i(y_i-1)}{n} \binom{n}{y_i} p_i^{y_i} (1-p_i)^{n-y_i} + n \ln n \Pr\left\{ Y_i \geq \frac{n}{2} \right\}
$$

$$
\leq 2np_i^2.
$$

Therefore

$$
\mathbb{E}\left(\sum_{i=1}^{m} (T_i - Y_i) \right) \leq \sum_{i=1}^{m} 2np_i^2
$$

$$
= \sum_{i=1}^{m} 2 \max\left\{ p_i, \frac{1}{n} \right\} \min\{np_i, n^2 p_i^2\}
$$

$$
\leq 4 \max\left\{ p_1, \ldots, p_m, \frac{1}{n} \right\} \sum_{i=1}^{m} \frac{1}{2} \min\{np_i, n^2 p_i^2\}
$$

$$
= o\left(\frac{1}{\ln n} \right) \sum_{i=1}^{m} np_i \left(1 - (1-p_i)^{n-1} \right)
$$

$$
= o\left(\frac{S_1}{\ln n} \right),
$$

18

where S_1 is defined by (1). In order to get the last but one line consider two cases $p(n-1) \leq 1/2$ and $p(n-1) > 1/2$.

Moreover by Markov's inequality for any ω

$$\Pr\left\{\sum_{i=1}^{m}(T_i - Y_i) \geq \frac{S_1}{\omega \ln n}\right\} \leq \mathbb{E}\left(\sum_{i=1}^{m}(T_i - Y_i)\right)\frac{\omega \ln n}{S_1}.$$

Therefore with high probability

$$(13) \quad \sum_{i=1}^{m}(T_i - Y_i) \leq \frac{S_1}{\omega \ln n} \quad, \text{ for any } 1 \ll \omega \ll \frac{S_1}{(\ln n \ \mathbb{E}(\sum_{i=1}^{m}(T_i - Y_i)))}.$$

Given $1 \ll \omega \ll \min\{S_1/(\ln n \ \mathbb{E}(\sum_{i=1}^{m}(T_i - Y_i))); \sqrt{S_1}/\ln n\}$ let

$$T_- = S_1 - \omega\sqrt{S_1} \quad \text{and} \quad T_+ = S_1 + \omega\sqrt{S_1} + \frac{S_1}{\omega \ln n}.$$

In the probability space of the coupling described above define events:

$\mathcal{A}_-$ –   all coupons are collected in at most T_- draws;

$\mathcal{A}_+$ –   all coupons are collected in at most T_+ draws;

$\mathcal{A}$ – $\delta(\mathcal{G}(n,m,\overline{p})) \geq 1$;

$\mathcal{B}$ –   the construction of $\mathcal{G}(n,m,\overline{p})$ is finished between T_--th and T_+-th draw;

$\mathcal{B}_1$ –   the construction of $\mathcal{G}(n,m,\overline{p})$ is finished in at most $\sum_{i=1}^{m}Y_i + S_1/(\omega \ln n)$ draws;

$\mathcal{B}_2$ –   in $\mathcal{G}(n,m,\overline{p})$ we have $S_1 - \omega\sqrt{S_1} \leq \sum_{i=1}^{m}Y_i \leq S_1 + \omega\sqrt{S_1}$.

By definition

$$\mathcal{A}_+ \cap \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}_- \cap \mathcal{B}.$$

Therefore

$$(14) \qquad \Pr\{\mathcal{A}_+\} - \Pr\{\mathcal{B}^c\} \leq \Pr\{\mathcal{A}\} \leq \Pr\{\mathcal{A}_-\} + \Pr\{\mathcal{B}^c\},$$

where for any event $\mathcal{C}$ we denote by $\mathcal{C}^c$ its complement.

Let $S_1 = n(\ln n + c_n)$, then

$$T_\pm = n(\ln n + c_n + o(1)).$$

Therefore by the classical results on the coupon collector problem

$$\Pr\{\mathcal{A}_-\} \sim \Pr\{\mathcal{A}_+\} \sim f(c_n)$$

19

and by (12) and (13)

$$\Pr\{\mathcal{B}^c\} \leq \Pr\{\mathcal{B}_1^c \cup \mathcal{B}_2^c\} = o(1).$$

Thus the lemma follows by substituting the above values to (14). □

The proofs of Lemmas 2 and 3 rely on a coupling with the coupon collector process similar to this used in the proof of Lemma 1 and a technique of dividing $\mathcal{G}(n, m, \overline{p})$ into auxiliary subgraphs. We will give here an idea of the proofs of Lemmas 2 and 3 and leave purely technical details to Appendix.

First we gather some simple facts concerning the division of $\mathcal{G}(n, m, \overline{p})$. Let $k \geq 2$ be an integer. For each $2 \leq t \leq k$, let $\mathcal{G}_t(n, m, \overline{p})$ be a random graph with a vertex set $\mathcal{V}$ and an edge set consisting of those edges from $\mathcal{G}(n, m, \overline{p})$, which are contained in at least one of the sets $\{V_i : |V_i| = t\}$, where V_i is defined as in the proof of Theorem 1. Moreover let $\mathcal{G}_{k+1}(n, m, \overline{p})$ be a subgraph of $\mathcal{G}(n, m, \overline{p})$ containing only those edges which are subsets of at least one of the sets $\{V_i : |V_i| \geq k + 1\}$. Let

$$Q_t = \sum_{i=1}^{m} \mathbb{I}_{Y_i = t}, \text{ for all } t = 2, \ldots, k, \quad \text{and} \quad Q_{k+1} = \sum_{i=1}^{m} Y_i - \sum_{t=2}^{k} t Q_t.$$

Note that, for all $t = 2, \ldots, k$, we have $\mathcal{G}_t(n, m, \overline{p}) = \mathbb{G}_{*t}(n, Q_t)$, where $\mathbb{G}_{*t}(n, \cdot)$ is defined as in Section 3. Moreover

$$\mathbb{E}Q_t = \frac{S_{1,t}}{t} \text{ for all } t = 2, \ldots, k, \quad \text{and} \quad \mathbb{E}Q_{k+1} = S_1 - \sum_{t=2}^{k} S_{1,t},$$

where S_1 and $S_{1,t}$ are defined by (1). For all $1 \leq t \leq k$, Q_t is a sum of independent Bernoulli random variables and $S_{1,t} \leq S_1$. Therefore by Chebyshev's inequality, Markov's inequality, and by (12) for any $\omega \to \infty$ with high probability

$$(15) \quad \max\left\{0, \frac{S_{1,t} - \omega\sqrt{S_1}}{t}\right\} \leq Q_t \leq \frac{S_{1,t} + \omega\sqrt{S_1}}{t}, \text{ for all } t = 2, 3, \ldots, k$$

$$(16) \quad Q_{k+1} \geq S_1 - \sum_{t=2}^{k} S_{1,t} - k\omega\sqrt{S_1},$$

$$(17) \quad Q_{k+1} + \sum_{j=2}^{k} j Q_j - t Q_t \leq S_1 - S_{1,t} + 2\omega\sqrt{S_1}, \text{ for all } t = 2, 3, \ldots, k.$$

20

Moreover, for any $t = 2, 3, \ldots, k$, if $S_{1,t} \to \infty$ then with high probability

(18)
$$\frac{S_{1,t} - \omega\sqrt{S_{1,t}}}{t} \leq Q_t \leq \frac{S_{1,t} + \omega\sqrt{S_{1,t}}}{t}.$$

Let ω be a function tending slowly to infinity. In the proofs we will assume that ω is small enough to get the needed bounds.

For $t = 2, \ldots, k$ let

$$T_{t+} = S_1 - S_{1,t} + 2\omega\sqrt{S_1} + \frac{S_1}{\omega \ln n};$$

$$T_- = \max\left\{0, S_1 - \sum_{t=2}^{k} S_{1,t} - k\omega\sqrt{S_1}, \right\};$$

(19)
$$\hat{p}_{t-} = \frac{S_{1,t} - 2\omega\sqrt{S_{1,t}} - t!S_{1,t}^2 n^{-t}}{t\binom{n}{t}};$$

$$\hat{p}_{t+} = \frac{S_{1,t} + 2\omega\sqrt{S_{1,t}}}{t\binom{n}{t}};$$

$$\hat{q}_t = \begin{cases} 0, & \text{for } S_{1,t} = O(\omega\sqrt{S_1}); \\ \frac{S_{1,t} - 2\omega\sqrt{S_1} - t!S_{1,t}^2 n^{-2}}{t\binom{n}{t}}, & \text{otherwise.} \end{cases}$$

For all $t = 2, \ldots, k$, $\mathcal{G}_t(n, m, \overline{p}) = \mathbb{G}_{*t}(n, M_2)$, thus by (15) and (18), using the same methods as in the proof of Theorem 1 we can show the following fact.

Fact 5. *Let $\hat{q}_t$ and $\hat{p}_{t\pm}$ be defined as in* (19). *Then*

$$G_t(n, \hat{q}_t) \preceq_{1-o(1)} \mathcal{G}_t(n, m, \overline{p}),$$

where $G_t(n, \hat{q}_t)$ is independent from $(\mathcal{G}_j(n, m, \overline{p}))_{j=2,\ldots k+1, j\neq t}$.
Moreover if $S_{1,t} \to \infty$ then

$$G_t(n, \hat{p}_{t-}) \preceq_{1-o(1)} \mathcal{G}_t(n, m, \overline{p}) \preceq_{1-o(1)} G_t(n, \hat{p}_{t+}).$$

where $G_t(n, \hat{p}_{t\pm})$ is independent from $(\mathcal{G}_j(n, m, \overline{p}))_{j=2,\ldots k+1, j\neq t}$.

As in the proof of Lemma 1 we use the coupling of the coupon collector process on $\mathcal{V}$ and the construction of $\mathcal{G}(n, m, \overline{p})$. However here in the process we omit rounds in which $Y_i = t$, $2 \leq t \leq k$. Therefore we construct

21

$\bigcup_{j=2,\ldots,k+1;j\neq t}\mathcal{G}_t(n,m,\overline{p})$ instead of $\mathcal{G}(n,m,\overline{p})$. Reasoning in the same way as in the proof of Lemma 1, by the definition of Q_t, (13), and (17) we get that with high probability the construction of $\bigcup_{j=2,\ldots,k+1;j\neq t}\mathcal{G}_t(n,m,\overline{p})$ is finished before the T_{t+}–th draw. Similarly, if in the process we omit rounds with $Y_i \leq k$, then with high probability the construction of $\mathcal{G}_{k+1}(n,m,\overline{p})$ is not finished before the T_- draw. Therefore an analogous reasoning to this used in the proof of Lemma 1 gives the following facts.

Fact 6. *There exists a coupling of the coupon collector process with the set of coupons $\mathcal{V}$ and the construction of $\bigcup_{j=2,\ldots,k+1;j\neq t}\mathcal{G}_j(n,m,\overline{p})$ such that with high probability the number of collected coupons in T_{t+} draws is at least the number of non–isolated vertices in $\bigcup_{j=2,\ldots,k+1;j\neq t}\mathcal{G}_j(n,m,\overline{p})$.*

Fact 7. *There exists a coupling of the coupon collector process with the set of coupons $\mathcal{V}$ and the construction of $\mathcal{G}_{k+1}(n,m,\overline{p})$ such that with high probability the number of collected coupons in T_- draws is at most the number of non–isolated vertices in $\mathcal{G}_{k+1}(n,m,\overline{p})$.*

If there exists a vertex which is isolated in $\bigcup_{t=3}^{k+1}\mathcal{G}_t(n,m,\overline{p})$ and has degree at most $k-1$ in $\mathcal{G}_2(n,m,\overline{p})$, then $\delta(\mathcal{G}(n,m,\overline{p})) \leq k-1$ and if each vertex in $\mathcal{G}_{k+1}(n,m,\overline{p})$ is non-isolated or has degree at least k in $\mathcal{G}_2(n,m,\overline{p})$, then $\delta(\mathcal{G}(n,m,\overline{p})) \geq k$. The lemmas follow by the above facts and by the first and the second moment method, which is used to count the vertices, which are isolated in a considered auxiliary graph or have degree at least or less than k in another auxiliary graphs. Technical details of the proofs are postponed to Appendix.

5   Structural properties of $G_2(n,\hat{p}_2) \cup G_3(n,\hat{p}_3)$

In this section we give several structural results concerning $G_2(n,\hat{p}_2) \cup G_3(n,\hat{p}_3)$, which are analogous to those known for $G_2(n,\hat{p}_2)$. Then we will show how combine those properties with the coupling constructed in the proof of Theorem 1 in order to get the threshold functions of $\mathcal{G}(n,m,\overline{p})$.

For any graph G and any set $S \subseteq V(G)$ denote by $N_G(S)$ the set of neighbours of vertices form S contained in $V(G) \setminus S$. For simplicity we write $N_2(\cdot) = N_{G_2(n,\hat{p}_2)}(\cdot)$ and $N_3(\cdot) = N_{G_2(n,\hat{p}_2)\cup G_3(n,\hat{p}_3)}(\cdot)$. Note that given $G_2(n,\hat{p}_2) \cup G_3(n,\hat{p}_3)$ we have $|N_2(S)| \leq |N_3(S)|$.

Lemma 4. *Let k and C be positive integers and*

$$\hat{p}_2 + \frac{n}{2}\hat{p}_3 = \frac{\theta_n \ln n}{n}, \quad \textit{where } \liminf_{n \to \infty} \theta_n = \theta > \frac{1}{2} \textit{ and } \theta_n = O(1),$$

$$\hat{p}_2 = \frac{\theta'_n \ln n}{n}, \quad \textit{where } \liminf_{n \to \infty} \theta'_n = \theta' > \frac{1}{2} \textit{ and } \theta'_n = O(1).$$

Let moreover γ be a real number such that $1 - \theta' < \gamma < 1$. Then with high probability $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ has the following properties:

(i) $\mathcal{B}_1$ – *for all $S \subseteq \mathcal{V}$ such that $n^\gamma \leq |S| \leq \frac{1}{4}n$*

$$|N_3(S)| > 2|S|.$$

(ii) $\mathcal{B}_2$ – *for all $S \subseteq \mathcal{V}$ such that $n^\gamma \leq |S| \leq \frac{2}{3}n$*

$$|N_3(S)| > \min\{|S|, 4n \ln\ln n / \ln n\}$$
$$> \min\{|S|, \alpha(G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3))\},$$

where $\alpha(G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3))$ is the stability number of $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$.

(iii) $\mathcal{B}_{3,k}$ – *for all $S \subseteq \mathcal{V}$ if $1 \leq |S| \leq n^\gamma$ and all vertices in S have degree at least $4k + 14$ in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ we have*

$$|N_3(S)| \geq 2k|S|.$$

(iv) $\mathcal{B}_{4,C}$ – *any two vertices of degree at most C are at distance at least 6.*

(v) $\mathcal{B}_5$ – *contains a path of length at least*

$$\left(1 - \frac{8 \ln 2}{\ln n}\right) n.$$

Proof. (i) and (ii) By Chernoff's inequality (for the proof see for example Theorem 2.1 in [12]) for any $\delta = \delta(n) = o(1)$ and any binomial random variable $X = X_n$ we have

$$\Pr\{X \leq \delta \mathbb{E}X\} \leq \exp(-\mathbb{E}X(\delta \ln \delta + 1 - \delta)) = \exp(-(1 + o(1))\mathbb{E}X).$$

First consider the case $n^\gamma \leq |S| \leq 4n \ln\ln n / \ln n$. Let $s = |S|$. Then $|N_2(S)|$ has the binomial distribution $\mathrm{Bin}(n - s, 1 - (1 - \hat{p}_2)^s)$ with the expected

23

value $\mathbb{E}|N_2(S)| \gg 2|S|$. Denote by X the number of sets S of cardinality $n^\gamma \le |S| \le 4n\ln\ln n/\ln n$ such that $|N_2(S)| \le 2|S|$. Using Chernoff's inequality we get

$$\mathbb{E}X \le \sum_{s=n^\gamma}^{4n\ln\ln n/\ln n} \binom{n}{s} \exp\left(-(1+o(1))(n-s)(1-(1-\hat{p}_2)^s)\right)$$

$$\le \sum_{s=n^\gamma}^{n/(\ln n \ln\ln\ln n)} \exp\left(s\left(1+\ln\frac{n}{s}\right) - (1+o(1))sn\hat{p}_2\right)$$

$$+ \sum_{s=n/(\ln n \ln\ln\ln n)}^{4n\ln\ln n/\ln n} \exp\left(s\left(1+\ln\frac{n}{s}\right) - n\frac{\theta'}{2\ln\ln n}\right)$$

$$\le \sum_{s=n^\gamma}^{n/(\ln n \ln\ln\ln n)} \exp\left(s\left(1+(1-\gamma)\ln n - (1+o(1))\theta'\ln n\right)\right)$$

$$+ \sum_{s=n/(\ln n \ln\ln\ln n)}^{4n\ln\ln n/\ln n} \exp\left(s\left((1+o(1))\ln\ln n - \frac{\theta'\ln n}{8(\ln\ln n)^2}\right)\right)$$

$$= o(1).$$

Therefore with high probability for all sets $S \subseteq \mathcal{V}$, $n^\gamma \le |S| \le 4n\ln\ln n/\ln n$, we have

$$|N_2(S)| \ge 2|S|.$$

Now consider the case $|S| \ge 4n\ln\ln n/\ln n$. Let $r = 4n\ln\ln n/\ln n$. Let moreover $\overline{K}_r$ and $\overline{K}_{r,r}$ be the complement of the complete graph on r vertices and the complement of the complete bipartite graph with each set of bipartition of cardinality r. Denote by X_r and $X_{r,r}$ the number of $\overline{K}_r$ and $\overline{K}_{r,r}$ in $G_2(n,\hat{p}_2) \cup G_3(n,\hat{p}_3)$, respectively. Then

$$\mathbb{E}X_r = \binom{n}{r}(1-\hat{p}_2)^{\binom{r}{2}}(1-\hat{p}_3)^{\binom{r}{3}+\binom{r}{2}(n-r)}$$

$$\le \left(\frac{en}{r}\exp\left(-\frac{r}{2}(\hat{p}_2 + (n-r)\hat{p}_3) + o(1)\right)\right)^r$$

$$\le \left(\frac{e\ln n}{4\ln\ln n}\exp\left(-2\theta_n\ln\ln n\right)\right)^r = o(1)$$

and

$$\mathbb{E}X_{r,r} \leq \binom{n}{r}^2 (1-\hat{p}_2)^{r^2} (1-\hat{p}_3)^{2\binom{r}{2}r + r^2(n-2r)}$$

$$\leq \left(\frac{en}{r} \exp\left(-\frac{r}{2}(\hat{p}_2 + (n-2r)\hat{p}_3) + o(1)\right)\right)^{2r} = o(1).$$

Therefore with high probability $X_r = 0$ which implies that with high probability

$$\alpha(G_2(n,\hat{p}_2) \cup G_3(n,\hat{p}_3)) \leq r.$$

Moreover with high probability $X_{r,r} = 0$ thus with high probability for any $S \subseteq \mathcal{V}$ such that $r \leq |S|$ we have

$$N_3(S) \geq n - |S| - r.$$

(Otherwise $G_2(n,\hat{p}_2) \cup G_3(n,\hat{p}_3)$ would contain $\overline{K}_{r,r}$.)
Therefore with high probability for any $S \subseteq \mathcal{V}$ such that $r \leq |S| \leq 2n/3$

$$N_3(S) \geq \min\{|S|, r\} \geq \min\{|S|, \alpha(G_2(n,\hat{p}_2) \cup G_3(n,\hat{p}_3))\}.$$

Moreover with high probability for any $S \subseteq \mathcal{V}$ such that $r \leq |S| \leq n/4$ we have

$$N_3(S) \geq n - |S| - r \geq 2|S|.$$

This finishes the proof of (i) and (ii).
(iii) For any two disjoint sets $S \subseteq \mathcal{V}$ and $S' \subseteq \mathcal{V}$ we denote by $e(S; S')$ the number of edges in $G_2(n,\hat{p}_2) \cup G_3(n,\hat{p}_3)$ with one end in S and one end in S' and by $e(S)$ the number of edges in $G_2(n,\hat{p}_2) \cup G_3(n,\hat{p}_3)$ with both ends in S. We will bound the numbers $e(S)$ and $e(S, S')$ for $S \subseteq \mathcal{V}$ and $S' \subseteq \mathcal{V} \setminus S$ such that $1 \leq |S| \leq n^\gamma$ and $|S'| = O(|S|)$. Let k be a positive integer, $S \subseteq \mathcal{V}$, $S' \subseteq \mathcal{V} \setminus S$, $|S| = s$, $1 \leq s \leq n^\gamma$ and $|S'| = 2ks$. A pair $\{v, v'\}$ ($v \in S$ and $v' \in S'$) is an edge in $G_2(n,\hat{p}_2) \cup G_3(n,\hat{p}_3)$ if at least one of the three following events occurs:

- $\{v, v'\}$ is an edge in $G_2(n,\hat{p}_2)$,

- there is a hyperedge $\{v, v', v''\}$ with $v'' \in \mathcal{V} \setminus (S \cup S')$ in $H_3(n,\hat{p}_3)$ associated with $G_3(n,\hat{p}_3)$,

- there is a hyperedge $\{v, v', v''\}$ with $v'' \in S \cup S'$ in $H_3(n,\hat{p}_3)$ associated with $G_3(n,\hat{p}_3)$.

25

The union of the two first events occurs with probability at most $\hat{p}_2 + (n - (2k+1)s)\hat{p}_3 = O(\ln n/n)$ independently for all $v \in S$ and $v' \in S'$. Moreover each hyperedge of $H_3(n, \hat{p}_3)$ with all vertices in $S \cup S'$ appears independently with probability $\hat{p}_3 = O(\ln n/n^2)$ and generates two edges between S and S' in $G_3(n, \hat{p}_3)$. Therefore

$$
\Pr\left\{ \exists_{\substack{S,S' \subseteq \mathcal{V}, \\ |S'|=2k|S| \\ 1 \le |S| \le n^\gamma}} e(S, S') \ge 4(k+1)|S| \right\}
$$

$$
\le \sum_{s=1}^{n^\gamma} \binom{n}{s}\binom{n-s}{2ks}\left[\binom{2ks^2}{(2k+2)s}(\hat{p}_2 + (n-(2k+1)s)\hat{p}_3)^{(2k+2)s} \right.
$$

$$
\left. + \binom{\binom{s}{2}2ks + \binom{2ks}{2}s}{(k+1)s}\hat{p}_3^{(k+1)s} \right]
$$

$$
\le \sum_{s=1}^{n^\gamma}\left[\left(O(1)\left(\frac{n}{s}\right)^{2k+1}\left(\frac{s\ln n}{n}\right)^{2k+2} \right)^s \right.
$$

$$
\left. + \left(O(1)\left(\frac{n}{s}\right)^{2k+1} s^{k+1}\left(\frac{\ln n}{n^2}\right)^{k+1} \right)^s \right]
$$

$$
\le \sum_{s=1}^{n^\gamma}\left[\left(O(1)\frac{\ln^{2k+2} n}{n^{1-\gamma}} \right)^s + \left(O(1)\frac{\ln^{k+1} n}{n} \right)^s \right]
$$

$$
= o(1).
$$

Similarly

$$
\Pr\left\{ \exists_{\substack{S \subseteq \mathcal{V} \\ 1 \le |S| \le n^\gamma}} e(S) \ge 5|S| \right\}
$$

$$
\le \sum_{s=1}^{n^\gamma} \binom{n}{s}\left[\binom{\binom{s}{2}}{2s}(\hat{p}_2 + (n-s)\hat{p}_3)^{2s} + \binom{\binom{s}{3}}{s}\hat{p}_3^s \right]
$$

$$
\le \sum_{s=1}^{n^\gamma}\left(O(1)\frac{\ln^2 n}{n^{1-\gamma}} \right)^s + \left(O(1)\frac{\ln n}{n^{1-\gamma}} \right)^s
$$

$$
= o(1).
$$

Therefore with high probability for any set S $(1 \le |S| \le n^\gamma)$ of vertices of

degree at least $4k + 14$ in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ we have

$$|N_3(S)| \geq 2k|S|.$$

Otherwise for $S' = N_3(S)$ there would be $|S'| = |N_3(S)| \leq 2ks$ and $e(S, N_3(S)) + 2e(S) \geq (4k + 14)|S|$, i.e. $e(S, N_3(S)) \geq 4(k + 1)|S|$ or $e(S) \geq 5|S|$.

(iv) The probability that in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ there are two vertices of degree at most C at distance at most 5 is at most

$$n^2 \sum_{l=0}^{4} n^l (\hat{p}_2 + n\hat{p}_3)^{l+1}$$

$$\cdot \left(\sum_{i=0}^{C} \sum_{j=0}^{\lfloor i/2 \rfloor} \binom{\binom{n-1-l}{2}}{j} \hat{p}_3^j \binom{n-1-l}{i-2j} \hat{p}_2^{i-2j} \right)$$

$$\cdot (1 - \hat{p}_2)^{n-1-i-l} (1 - \hat{p}_3)^{\binom{n-1-l}{2} - \binom{i+l}{2}} \right)^2$$

$$= O(1) n (\ln n)^{2C+5} e^{-2n(\hat{p}_2 + \frac{n}{2}\hat{p}_3)} = o(1).$$

(v) Follows by Theorem 8.1 from [6]. $\qquad\qquad\qquad\qquad\square$

Lemma 5. *Let k be a positive integer and $\hat{p}_2 = \hat{p}_2(n) \in (0, 1)$, $\hat{p}_3 = \hat{p}_3(n) \in (0, 1)$ be such that*

$$\hat{p}_2 + \frac{n}{2}\hat{p}_3 = \frac{\ln n + (k - 1) \ln \ln n + c_n}{n}.$$

(i) If $k = 1$ then

$$\Pr\left\{ \delta(G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)) \geq 1 \right\} \to f(c_n),$$

where $f(\cdot)$ is defined by (5).

(ii) If $c_n \to \infty$ then

$$\Pr\left\{ \delta(G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)) \geq k \right\} \to 1.$$

Proof. (i) Follows by Theorem 3.10 from [17].

27

(ii) Let X_t be the number of vertices of degree $0 \leq t \leq k - 1$ in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$. Then

$$\mathbb{E}X_t \leq n\binom{n-1}{t}\sum_{i=0}^{t}\binom{\binom{t}{2}}{i}\hat{p}_3^i\hat{p}_2^{\max\{t-2i,0\}}(1-\hat{p}_2)^{n-t-1}(1-\hat{p}_3)^{\binom{n-1}{2}-\binom{t}{2}}$$

$$= O(1)n(\ln n)^t \exp(-\ln n + (k-1)\ln\ln n + c_n) = o(1).$$

Thus with high probability $X_t = 0$ for all $t \leq k - 1$. $\qquad\square$

Lemma 6. *Let k be a positive integer, $\liminf_{n\to\infty}\theta_n = \theta > \frac{1}{2}$, $\liminf_{n\to\infty}\theta'_n = \theta' > \frac{1}{2}$ and $\theta, \theta' = O(1)$. Let $\mathbb{G}(n)$ be a random graph such that for*

$$(20) \qquad\qquad \hat{p}_2 + \frac{n}{2}\hat{p}_3 = \frac{\theta_n \ln n}{n} \quad\text{and}\quad \hat{p}_2 = \frac{\theta'_n \ln n}{n}$$

we have

$$(21) \qquad\qquad G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \preceq_{1-o(1)} \mathbb{G}(n)$$

and in the probability space of the coupling with high probability all vertices of degree at most $4k + 13$ in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ are at distance at least 6 in $\mathbb{G}(n)$. If $\Pr\{\delta(\mathbb{G}(n) \geq k)\}$ is bounded away from zero by a constant then

$$(22) \qquad\qquad \Pr\{\mathbb{G}(n) \in \mathcal{C}_k | \delta(\mathbb{G}(n)) \geq k\} \to 1,$$
$$(23) \qquad\qquad \Pr\{\mathbb{G}(2n) \in \mathcal{PM} | \delta(\mathbb{G}(2n)) \geq 1\} \to 1.$$

In particular, if we substitute $\mathbb{G}(n) = G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ then by Lemmas 5 and 6 we obtain the following result.

Lemma 7. *Let $\hat{p}_2$ and $\hat{p}_3$ fulfil (20).*

(i) *If $\hat{p}_2 + \frac{n}{2}\hat{p}_3 = (\ln n + c_n)/n$ then*

$$\Pr\{G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \in \mathcal{C}_1\} \to f(c_n);$$
$$\Pr\{G_2(2n, \hat{p}_2(2n)) \cup G_3(2n, \hat{p}_3(2n)) \in \mathcal{PM}\} \to f(c_{2n}),$$

where $f(\cdot)$ is defined by (5).

(ii) *If $\hat{p}_2 + \frac{n}{2}\hat{p}_3 = (\ln n + (k-1)\ln\ln n + c_n)/n$ and $c_n \to \infty$ then*

$$\Pr\{G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \in \mathcal{C}_k\} \to 1.$$

28

Proof of Lemma 6. Denote by $\mathbb{G}(n)_{\delta \geq k}$ a graph $\mathbb{G}(n)$ under condition $\delta(\mathbb{G}(n)) \geq k$. Let $\mathcal{A}$ ba a graph property. In what follows we will use the fact that if $\Pr\{\delta(\mathbb{G}(n) \geq k)\}$ is bounded away from zero and $\mathbb{G}(n)$ has $\mathcal{A}$ with high probability, then also $\mathbb{G}(n)_{\delta \geq k}$ has property $\mathcal{A}$ with high probability.

From now on we assume that $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ and $\mathbb{G}(n)$ are defined on the same probability space existing by (21). We call a vertex $v \in \mathcal{V}$ small if its degree in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ is at most $4k + 13$. Otherwise we call v a large vertex. Let $S \subseteq \mathcal{V}$ and $|S| \leq n^\gamma$. Denote by S^+ and S^- the subset of large and small vertices of S, respectively. Then by Lemma 4(iii) with high probability

$$|N_{\mathbb{G}(n)_{\delta \geq k}}(S^+)| \geq |N_3(S^+)| \geq 2k|S^+|.$$

Moreover in $\mathbb{G}(n)_{\delta \geq k}$ all vertices in S^- have degree at least k and with high probability are at distance at least 6. Therefore with high probability in $\mathbb{G}(n)_{\delta \geq k}$ no two vertices in S^- are connected by an edge or have a common neighbour and at most $|S^+|$ of them have neighbours in $N_3(S^+) \cup S^+$ (otherwise they would be connected by a path of length at most 5). Thus with high probability

$$\forall_{S \subseteq \mathcal{V}, 1 \leq |S| \leq n^\gamma} |N_{\mathbb{G}(n)_{\delta \geq k}}(S)| \geq |N_{\mathbb{G}(n)_{\delta \geq k}}(S^+)| + k \max\{|S^-| - |S^+|, 0\} \geq k|S|.$$

If we combine this with Lemma 4(i) and (ii) we get that with high probability

(24) $\quad |N_{\mathbb{G}(n)_{\delta \geq k}}(S)| \geq k, \qquad\qquad$ for all $S \subseteq \mathcal{V}$, $1 \leq |S| \leq \dfrac{n}{2}$;

(25) $\quad |N_{\mathbb{G}(n)_{\delta \geq 1}}(S)| \geq \min\{|S|, 4n \ln \ln n / \ln n\} \geq \min\{|S|, \alpha(\mathbb{G}(n)_{\delta \geq k})\},$

$$\text{for all } S \subseteq \mathcal{V}, \ 1 \leq |S| \leq \frac{2n}{3};$$

(26) $\quad |N_{\mathbb{G}(n)_{\delta \geq 2}}(S)| \geq 2|S|, \qquad\qquad$ for all $S \subseteq \mathcal{V}$, $1 \leq |S| \leq \dfrac{n}{4}$.

Finally (22) follows immediately by (24). Moreover if (25) is fulfilled then $\mathbb{G}(n)_{\delta \geq 1}$ has a perfect matching (for the proof see for example Lemma 1 in [3]). Therefore (23) follows. (26) will be used later to establish threshold function for a Hamilton cycle. $\qquad\square$

Lemma 8. *Let k be a positive integer, θ_n and θ'_n be sequences such that* $\liminf_{n \to \infty} \theta_n = \theta > \frac{1}{2}$, $\liminf_{n \to \infty} \theta'_n = \theta' > \frac{1}{2}$, $\theta, \theta' = O(1)$, *and*

$$\hat{p}_2 + \frac{n}{2}\hat{p}_3 = \frac{\theta_n \ln n}{n}, \ \ \hat{p}_2 = \frac{\theta'_n \ln n}{n} \ \ and \ \ \hat{p}_4 = \frac{512}{n}.$$

Let moreover $\mathbb{G}(n)$ be a random graph such that:

(i) $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \preceq_{1-o(1)} \mathbb{G}(n)$;

(ii) with high probability $\delta(\mathbb{G}(n)) \geq 2$;

(iii) in a probability space existing by (i) all vertices of degree at most 21 in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ *are at distance at least 6 in* $\mathbb{G}(n)$.

Then

$$\Pr\{\mathbb{G}(n) \cup G_2(n, \hat{p}_4) \in \mathcal{HC}\} \to 1.$$

In particular by Lemma 4 we get the following simple corollary of Lemma 8.

Corollary 4. *Let $\hat{p}_2$ and $\hat{p}_3$ be as in Lemma 8. If moreover*

$$\hat{p}_2 + \frac{n}{2}\hat{p}_3 = (\ln n + \ln \ln n + c_n)/n \text{ where } c_n \to \infty$$

then

$$\Pr\{G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3) \in \mathcal{HC}\} \to 1.$$

Proof of Lemma 8. We will use the heuristic of the proof of Theorem 8.9 [6]. Let $t = 8n/\ln n$ and $\hat{p}_{4,0} = 64 \ln n/n^2$. Then $t\hat{p}_{4,0} = \hat{p}_4$. For any graph G let $l(G)$ be the length of the longest path in G and $l(G) = n$ if G has a Hamilton cycle. We say that G has property $\mathcal{Q}$ if

$$G \text{ is connected} \quad \text{and} \quad |N_G(S)| \geq 2|S|, \text{ for all } S \subseteq \mathcal{V}, |S| \leq n/4.$$

In the proof we assume that $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ and $\mathbb{G}(n)$ are defined on the probability space of the coupling existing by (i). Let $\mathbb{G}_0 = \mathbb{G}(n)$ and

$$\mathbb{G}_i = \mathbb{G}_{i-1} \cup G_2(n, \hat{p}_{4,0}), \quad \text{for } 1 \leq i \leq t.$$

By Lemma 6, assumptions (i)–(iii), and (26) with high probability $\mathbb{G}(n)$ has property $\mathcal{Q}$. Moreover by Lemma 4(v) with high probability

$$l(\mathbb{G}_0) \geq \left(1 - \frac{8\ln 2}{\ln n}\right) n.$$

It may be shown (see (8.7) in [6]) that

$$\Pr\{l(\mathbb{G}_i) = n - t + i - 1 | l(\mathbb{G}_{i-1}) = n - t + i - 1 \text{ and } \mathbb{G}_{i-1} \text{ has } \mathcal{Q}\}$$
$$\leq (1 - \hat{p}_{4,0})^{n^2/32} \leq n^{-2}.$$

Thus
$$\Pr\left\{l(\mathbb{G}_t) = n\right\} \geq 1 - \frac{t}{n^2} - o(1) = 1 - o(1).$$

Since
$$\mathbb{G}_t \preceq \mathbb{G}(n) \cup G_2\left(n, t\hat{p}_{4,0}\right),$$

this finishes the proof. $\qquad\square$

6 Sharp thresholds

Proof of Theorems 2–4. First let $S_1 = n(\ln n + c_n)$ and $\hat{p}_2$ and $\hat{p}_3$ be given by (2). Note that then $\hat{p}_2 > \frac{1+\varepsilon}{2}\ln n$ for some constant $\varepsilon > 0$. Therefore by Theorem 1 and Lemma 7(i)

$$\lim_{n\to\infty} \Pr\left\{\mathcal{G}\left(n,m,\overline{p}\right) \in \mathcal{C}_1\right\} \geq \lim_{n\to\infty} \Pr\left\{G_2\left(n,\hat{p}_2\right) \cup G_3\left(n,\hat{p}_3\right) \in \mathcal{C}_1\right\} = f(c_n)$$

and by Lemma 1

$$\lim_{n\to\infty} \Pr\left\{\mathcal{G}\left(n,m,\overline{p}\right) \in \mathcal{C}_1\right\} \leq \lim_{n\to\infty} \Pr\left\{\delta(\mathcal{G}\left(n,m,\overline{p}\right)) \geq 1\right\} = f(c_n).$$

This implies Theorem 2(i). Similarly Theorem 2(ii) follows by Theorem 1, Lemma 7(ii), and Lemma 2; Theorem 3 by Theorem 1, Lemma 7(i), and Lemma 1; Theorem 4 by Theorem 1, Corollary 4 and Lemma 2. $\qquad\square$

In the following proofs we will assume that $c_n = O(\ln n)$. In the other cases the theorems follow by Lemma 1 and Theorem 1 (3) combined with known results concerning $G_2\left(n,\hat{p}\right)$.

Proof of Theorem 5. Let $c_n \to -\infty$, then by Lemma 3

$$\Pr\left\{\mathcal{G}\left(n,m,p\right) \in \mathcal{HC}\right\} \leq \Pr\left\{\delta(\mathcal{G}\left(n,m,p\right)) \geq 2\right\} \to 0.$$

Let now $c_n \to \infty$. We will consider two cases:
CASE 1: $m = \Omega(n/\ln n)$.
Let
$$m' = m\left(1 + \frac{700}{\ln n}\right)^{-1} \quad \text{and} \quad m'' = \frac{700}{\ln n}m'.$$

Then for p given by (6)

$$m'p(1 - (1-p)^{n-1}) = \ln n + \ln\max\left\{1, \frac{npe^{-np}\ln n}{1 - e^{-np}}\right\} - 700 + o(1) + c_n$$

31

and

$$
(27) \qquad nm'p^2 = \begin{cases} O(\ln n) & \text{for } np = O(1), \\ \frac{n}{m'}(m'p)^2 = O(\ln^3 n) & \text{for } np \to \infty. \end{cases}
$$

Let $\omega \to \infty$. Define

$$
\hat{p}_2 = \begin{cases} \dfrac{m'p(1-(1-p)^{n-1})-3m'p\left(\frac{1-(1-2p)^n}{2np}-(1-p)^{n-1}\right)-\omega\sqrt{\frac{\ln n}{n}}}{n}, \\ \qquad\qquad \text{for } nm'p\left(\frac{1-(1-2p)^n}{2np}-(1-p)^{n-1}\right) \gg \sqrt{n\ln n}; \\ \dfrac{m'p(1-(1-p)^{n-1})-\omega\sqrt{\frac{\ln n}{n}}}{n}, \qquad \text{otherwise ;} \end{cases}
$$

$$
\hat{p}_3 = \begin{cases} \dfrac{m'p\left(\frac{1-(1-2p)^n}{2np}-(1-p)^{n-1}\right)-\omega\sqrt{\frac{\ln n}{n}}}{\frac{n^2}{6}}, \\ \qquad\qquad \text{for } nm'p\left(\frac{1-(1-2p)^n}{2np}-(1-p)^{n-1}\right) \gg \sqrt{n\ln n}; \\ 0, \qquad\qquad \text{otherwise ;} \end{cases}
$$

$$
\hat{p}_4 = \dfrac{m''p\left(1-\frac{1-(1-2p)^n}{2np}\right)-\omega\sqrt{\frac{1}{n}}}{n} \geq \frac{512}{n}.
$$

Let now $\mathbb{G}(n) = \mathcal{G}(n, m', p)$. Then by Theorem 1

$$
(28) \qquad G_2\left(n, \hat{p}_2\right) \cup G_3\left(n, \hat{p}_3\right) \preceq_{1-o(1)} \mathcal{G}\left(n, m', p\right) = \mathbb{G}(n);
$$
$$
G_2\left(n, \hat{p}_4\right) \preceq_{1-o(1)} \mathcal{G}\left(n, m'', p\right).
$$

Moreover by Lemma 3 with high probability $\delta(\mathbb{G}(n)) \geq 2$. Therefore assumptions (i) and (ii) in Lemma 8 are fulfilled.

We are left with proving that (iii) is fulfilled. Let C be a positive integer. We will show that in the probability space of the coupling (28) with high probability any two vertices of degree at most C in $G_2\left(n, \hat{p}_2\right) \cup G_3\left(n, \hat{p}_3\right)$ are at distance at least 6 in $\mathcal{G}\left(n, m', p\right) = \mathbb{G}(n)$. Therefore we need to study the couplings described in the proof of Theorem 1. Recall that in the proof of Theorem 1 it was shown that, if in all definitions of S_1, S_3, M_2 and M_3 we

32

replace m by m', then

$$G_2\left(n, \hat{p}_2\right) \cup G_3\left(n, \hat{p}_3\right)$$

$$= \mathbb{G}_{*2}\left(n, \mathrm{Po}\left(\binom{n}{2}\ln(1-\hat{p}_2)^{-1}\right)\right) \cup \mathbb{G}_{*3}\left(n, \mathrm{Po}\left(\binom{n}{3}\ln(1-\hat{p}_3)^{-1}\right)\right)$$

$$\preceq^{(i)} \mathbb{G}_{*2}\left(n, \mathrm{Po}\left(\frac{S_1 - 3S_3 - 5\omega'\sqrt{S_1}}{2}\right)\right) \cup \mathbb{G}_{*3}\left(n, \mathrm{Po}\left(S_3 - 2\omega'\sqrt{S_1}\right)\right)$$

$$\preceq^{(ii)}_{1-o(1)} \mathbb{G}_{*2}\left(n, \frac{S_1 - 3S_3 - 4\omega'\sqrt{S_1}}{2}\right) \cup \mathbb{G}_{*3}\left(n, S_3 - \omega'\sqrt{S_1}\right)$$

$$\preceq^{(iii)}_{1-o(1)} \mathbb{G}_{*2}\left(n, M_2\right) \cup \mathbb{G}_{*3}\left(n, M_3\right) = \bigcup_{1 \le i \le m'}\left(\mathbb{G}_{*2}\left(n, \tfrac{Y_i - 3Z_i}{2}\right) \cup \mathbb{G}_{*3}\left(n, Z_i\right)\right)$$

$$\preceq^{(iv)} \mathcal{G}\left(n, m', p\right)$$

for some $\omega' = \Theta(\omega)$.

In the probability space of the above couplings, we will find a relation between the degree of a vertex $v \in \mathcal{V}$ in $G_2\left(n, \hat{p}_2\right) \cup G_3\left(n, \hat{p}_3\right)$ and the number of those features $w_i \in W(v)$ that are chosen by at least two vertices (i.e. that contribute to at least one edge in $\mathcal{G}\left(n, m', p\right)$). For any $v \in \mathcal{V}$ in $\mathcal{G}\left(n, m', p\right)$ let

$$W'(v) = \{w_i \in W(v) : |V_i'| \ge 2\},$$

where V_i' is defined in (9). Therefore we will find the relation between the degree of a vertex v in $G_2\left(n, \hat{p}_2\right) \cup G_3\left(n, \hat{p}_3\right)$ and $|W'(v)|$ in $\mathcal{G}\left(n, m', p\right)$.

To this end we will need an insight into the construction of the intermediary couplings (i)–(iii). Recall that in the couplings, the construction of $\mathbb{G}_{*2}\left(n, M_2\right) \cup \mathbb{G}_{*3}\left(n, M_3\right)$ proceeds in m' rounds. In the i-th round $\mathbb{G}_{*2}\left(n, \frac{Y_i - 3Z_i}{2}\right) \cup \mathbb{G}_{*3}\left(n, Z_i\right)$ is constructed in $\frac{Y_i - 3Z_i}{2} + Z_i$ draws. In each draw a hyperedge of size 2 or 3 in $\mathbb{H}_{*2}(\frac{Y_i - 3Z_i}{2}) \cup \mathbb{H}_{*3}(Z_i)$ is chosen.

Denote by $\mathcal{A}_v$ the event that while constructing $\mathbb{G}_{*2}\left(n, M_2\right) \cup \mathbb{G}_{*3}\left(n, M_3\right)$ there are less than $|W'(v)| - 1$ draws in which a hyperedge containing v is chosen. Given $v \in \mathcal{V}$ and $1 \le i \le m'$, denote by $\mathcal{A}_{v,i,1}$ the event that v is an isolated vertex in $\mathbb{G}_{*2}\left(n, (Y_i - 3Z_i)/2\right) \cup \mathbb{G}_{*3}\left(n, Z_i\right)$ and $\mathcal{A}_{v,i,2}$ the event that $v \in V_i'$. Then

$$\mathcal{A}_v \subseteq \bigcup_{i \ne j}(\mathcal{A}_{v,i,1} \cap \mathcal{A}_{v,i,2}) \cap (\mathcal{A}_{v,j,1} \cap \mathcal{A}_{v,j,2}).$$

For $i \ne j$, $(\mathcal{A}_{v,i,1} \cap \mathcal{A}_{v,i,2})$ and $(\mathcal{A}_{v,j,1} \cap \mathcal{A}_{v,j,2})$ are independent and

$$\Pr\{\mathcal{A}_{v,i,1} \cup \mathcal{A}_{v,i,2}\} = 1,$$

thus

$$\Pr\{\mathcal{A}_v\} \le \binom{m'}{2}\left(\Pr\{\mathcal{A}_{v,1,1} \cap \mathcal{A}_{v,1,2}\}\right)^2$$

$$= \binom{m'}{2}\left(\Pr\{\mathcal{A}_{v,1,1}\} + \Pr\{\mathcal{A}_{v,1,2}\} - 1\right)^2$$

$$\le \binom{m'}{2}\Bigg((1-p)^n + np(1-p)^{n-1}$$

$$+ \sum_{y=2}^{n-1}\binom{n}{y}p^y(1-p)^{n-y}\left(1-\frac{1}{n}\right)^y + p^n \cdot 0$$

$$+ \sum_{y=2}^{n}\binom{n}{y}p^y(1-p)^{n-y}\frac{y}{n} - 1\Bigg)^2$$

$$= \binom{m'}{2}\Bigg(\sum_{y=0}^{n}\binom{n}{y}p^y(1-p)^{n-y}\left(1-\frac{1}{n}\right)^y - p^n$$

$$+ \sum_{y=1}^{n}\binom{n}{y}p^y(1-p)^{n-y}\frac{y}{n} - 1\Bigg)^2$$

$$\le \binom{m'}{2}\left(\left(1-\frac{p}{n}\right)^n + p - 1\right)^2 \le (m')^2 p^4$$

Therefore by (27)

$$(29) \qquad \Pr\left\{\bigcup_{v\in\mathcal{V}}\mathcal{A}_v\right\} \le n(m')^2 p^4 = O\left(\frac{\ln^6 n}{n}\right) = o(1).$$

Thus in the probability space of the couplings, with high probability for all $v \in \mathcal{V}$ in the construction of $\mathbb{G}_{*2}(n, M_2) \cup \mathbb{G}_{*3}(n, M_3)$ the number of draws in which a hyperedge containing v is chosen is at least $|W'(v)| - 1$.

Now we will find the relation between $|W'(v)| - 1$ and the number of draws in the construction of

$$G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$$
$$= \mathbb{G}_{*2}\left(n, \mathrm{Po}\left(\binom{n}{2}\ln(1-\hat{p}_2)^{-1}\right)\right) \cup \mathbb{G}_{*3}\left(n, \mathrm{Po}\left(\binom{n}{3}\ln(1-\hat{p}_3)^{-1}\right)\right).$$

Note that in the couplings (i)–(iii) in order to construct a graph $\mathbb{G}_{*2}(n, M_2) \cup \mathbb{G}_{*3}(n, M_3)$ from $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ one make some additional draws of hyperedges in auxiliary hypergraphs $\mathbb{H}_{*2}(n, \cdot) \cup \mathbb{H}_{*3}(n, \cdot)$. By the sharp concentration of the Poisson distribution and (12) with high probability the number of additional draws is at most $K\omega\sqrt{n \ln n}$ for some constant K. Under condition that the number of additional draws is at most $K\omega\sqrt{n \ln n}$, probability that a hyperedge containing v is chosen in at least 3 of the additional draws is at most.

$$\binom{K\omega\sqrt{n \ln n}}{3}\left(\frac{2}{n}\right)^3 = o(n^{-1}).$$

Therefore by the union bound and (29) with high probability for all $v \in \mathcal{V}$ in the construction of $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ the number of draws in which a hyperedge containing v is chosen is at least $|W'(v)| - 3$.

Now we will show the relation between the number of draws and the number of edges incident to v in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$. Note that the number of draws made in the construction of $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ is a random variable with the expected value at most $S_1 = O(n \ln n)$. Therefore by Markov's inequality with high probability the number of draws made in the construction of $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ is at most $n \ln^2 n$. In addition the probability that a pair $\{v, v'\}$ is contained in a hyperedge chosen in a given draw is at most $6/n(n-1)$. Therefore the probability that there exists an edge in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ drawn at least 3 times during the construction is at most

$$\binom{n}{2}\binom{n \ln^2 n}{3}\left(\frac{6}{n(n-1)}\right)^3 + o(1) = o(1)$$

Concluding, we have that with high probability for all $v \in \mathcal{V}$ the number of edges incident to v in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ is at least half of the number of drawn hyperedges containing v, which is with high probability at least $\frac{|W'(v)|-3}{2}$. Thus in the probability space of the couplings with high probability for all $v \in \mathcal{V}$ of degree at most C in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ we have $|W'(v)| \leq 2C + 6$.

Finally, probability that there are two vertices v, v' such that $|W'(v)| \leq 2C + 6$ and $|W'(v')| \leq 2C + 6$ connected by a path of length at most 5 in

$\mathcal{G}(n, m', p)$ is at most

$$n^2 \sum_{t=1}^{5} (m')^t n^{t-1} p^{2t}$$

$$\cdot \left(\sum_{l=0}^{2C+6} \binom{m' - t}{l} (p - p(1-p)^{n-1})^l \left[(1-p) + p(1-p)^{n-1} \right]^{m' - O(1)} \right)^2$$

$$= O(1) n (\ln n)^{O(1)} \exp \left(-2m'p(1 - (1-p)^{n-1}) \right)$$

$$= o(1).$$

This shows that assumption (iii) of Lemma 8 is fulfilled.

Therefore by Lemma 8 with high probability

$$\mathbb{G}(n) \cup G_2 \left(n, \frac{512}{n} \right) \in \mathcal{HC}.$$

Thus in the case $m = \Omega(n/\ln n)$ the theorem follows by (28) and a straight forward coupling of random intersection graphs.

$$\mathbb{G}(n) \cup G_2 \left(n, \frac{512}{n} \right) \preceq_{1-o(1)} \mathcal{G}(n, m', p) \cup \mathcal{G}(n, m'', p) \preceq \mathcal{G}(n, m, p)$$

CASE 2: $m = o(n/\ln n)$.

Note that in this case (6) and $c_n = O(\ln n)$ imply that

$$p = \frac{\ln n + c_n'}{m}$$

for some $c_n' \to \infty$. Let m' be such that m divides m' and $m' \sim n/\ln n$. Take an instance of $\mathcal{G}(n, m', p')$ with the set of features $\mathcal{W}'$ of size m' and

$$p' = \frac{\ln n + c_n'}{m'}.$$

Divide $\mathcal{W}'$ into m groups of m'/m features. Denote by A_i the set of vertices which have chosen features from the i–th, $1 \le i \le m$, group in $\mathcal{G}(n, m', p')$. $|A_i|$ has the binomial distribution $\text{Bin}(n, p'')$ for some $p'' \le \frac{m'}{m} p' = p$. Now take an instance of $\mathcal{G}(n, m', p')$ and construct the sets V_i, $1 \le i \le m$, in $\mathcal{G}(n, m, p)$ by taking A_i and additionally adding to V_i each vertex from $\mathcal{V} \setminus A_i$ independently with probability $(p - p'')/(1 - p'')$. This coupling implies

$$\mathcal{G}(n, m', p') \preceq \mathcal{G}(n, m, p).$$

From the considerations concerning the first case we have that with high probability $\mathcal{G}(n, m', p') \in \mathcal{HC}$. Therefore also $\mathcal{G}(n, m, p) \in \mathcal{HC}$ with high probability . $\qquad\square$

Proof of Theorem 6. The technique of the proof is analogous to this of the proof of Theorem 5. We consider two cases $m = \Omega(n/\ln n)$ and $m = o(n/\ln n)$. In the first case the proof relies on Lemma 3 and Lemma 6. Moreover we have to use the fact shown in the proof of Theorem 5, that with high probability in the coupling constructed in the proof of Theorem 1 with high probability the vertices of degree bounded by a constant in $G_2(n, \hat{p}_2) \cup G_3(n, \hat{p}_3)$ are at distance at least 6 in $\mathcal{G}(n, m, p)$. The proof of the case $m = o(n/\ln n)$ is the same as in the proof of Theorem 5. $\qquad\square$

Acknowledgements

References

[1] BARBOUR, A.D. and REINERT, G. (2011) The shortest distance in random multi-type intersection graphs. *Random Structures & Algorithms* **39** 179–209.

[2] BLOZNELIS, M. (2013+). Degree and clustering coefficient in sparse random intersection graphs. *Ann. Appl. Probab.* **23** 1254-1289.

[3] BLOZNELIS, M. and ŁUCZAK, T. (2013). Perfect matchings in random intersection graphs. *Acta Math. Hungar* **138** 15–33.

[4] BLOZNELIS, M. and DAMARACKAS, J. (2012+). Degree distribution of an inhomogeneous random intersection graph. *submitted* .

[5] BLOZNELIS, M., JAWORSKI, J., and RYBARCZYK, K. (2009). Component evolution in a secure wireless sensor network. *Networks* **53** 19–26.

[6] BOLLOBÁS, B. (1985). *Random Graphs*, Academic Press.

[7] BRADONJIĆ, M. , HAGBERGY, A., HENGARTNERZ, N. W., LEMONS, N. and PERCUS, A. G. (2010). Component Evolution in General Random Intersection Graphs. *Algorithms and Models for the Web-Graph*, LNCS 6516, Springer, 36–49.

[8] BRITTOM, T., DEIJFEN, M., LAGERÅS, A. N., and LINDHOLM, M. (2008). Epidemics on random graphs with tunable clustering. *J. Appl. Probab* **45** 743–756.

[9] DEIJFEN, M. and KETS, W. (2009). Random Intersection Graphs with Tunable Degree Distribution and Clustering. *Probab. Engrg. Inform. Sci.* **23** 661–674.

[10] FILL, J. A., SCHEINERMAN, E. R., and SINGER–COHEN K. B. (2000). Random Intersection Graphs when $m = \omega(n)$: An Equivalence Theorem Relating the Evolution of the G(n, m, p) and G(n, p) Models. *Random Structures Algorithms* **16** 156–176.

[11] GODEHARDT, E. and JAWORSKI, J. (2003). Two Models of Random Intersection Graphs for Classification. *Stud. Classication Data Anal. Knowledge Organ.* **22**, Springer, 67–81.

[12] JANSON, S., ŁUCZAK, T., and RUCIŃSKI, A. (2001). *Random Graphs*, Wiley.

[13] KAROŃSKI, M., SCHEINERMAN, E. R., and SINGER-COHEN, K.B. (1999). On random intersection graphs: The subgraph problem. *Combin. Probab. Comput.* **8** 131–159.

[14] NIKOLETSEAS, S., RAPTOPOULOS, C., and SPIRAKIS, P. (2008). Large independent sets in general random intersection graphs. *Theoret. Comput. Sci.* **406** 215-224.

[15] RYBARCZYK, K. (2011). Sharp Threshold Functions for Random Intersection Graphs via a Coupling Method. *Electron. J. Combin* **18** P36.

[16] RYBARCZYK, K. (2011). Equivalence of the random intersection graph and G(n,p). *Random Structures Algorithms* **38** 205–234.

[17] RYBARCZYK, K. (2005). *O pewnych zastosowaniach hipergrafów losowych (On some applications of random intersection graphs)* Master

Thesis, Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań.

[18] RYBARCZYK, K. and STARK, D. (2010). Poisson approximation of the number of cliques in the random intersection graph. *J. Appl. Probab* **47** 826 – 840.

[19] SINGER, K. B. (1995). *Random intersection graphs.* PhD Thesis, Department of Mathematical Sciences, The Johns Hopkins University.

A   Proofs of Lemma 2 and 3

Proof of Lemma 2. In the proof we use notation introduced in (19). Moreover let $a_n = S_{1,2}/S_1$.

(i) We restrict attention to the case $c_n = o(\ln n)$. In the latter case the result follows by Lemma 1.

First consider the case $a_n = S_{1,2}/S_1 \gg 1/\ln n$. Then $S_{1,2} \to \infty$. Take any probability space on which we define the coupon collector process on $\mathcal{V}$ and $G_2(n, \hat{p}_{2+})$, in such a manner that they are independent. Let X_+ be a random variable counting vertices which have not been collected during the coupon collector process in T_{2+} draws and have degree at most $k-1$ in $G_2(n, \hat{p}_{2+})$. If

$$S_1 = n(\ln n + (k-1)\ln(a_n \ln n) + c_n)$$

then for ω tending to infinity slowly enough

$$\frac{1}{n}T_{2+} + (n-1)\hat{p}_{2+} = \frac{1}{n}\left(S_1 + O\left(\omega\sqrt{S_1} + \frac{S_1}{\omega \ln n}\right)\right)$$
$$= \ln n + (k-1)\ln(a_n \ln n) + c_n + o(1)$$

and

$$\hat{p}_{2+} \sim \frac{a_n \ln n}{n}.$$

Therefore

$$\mathbb{E}X_+ = n\left(1 - \frac{1}{n}\right)^{T_{2+}} \sum_{i=0}^{k-1} \binom{n-1}{i} \hat{p}_{2+}^i (1 - \hat{p}_{2+})^{n-i-1}$$

$$\sim n\frac{1}{(k-1)!}(a_n \ln n)^{k-1} \exp(-\ln n - (k-1)\ln(a_n \ln n) - c_n)$$

$$\sim \frac{e^{-c_n}}{(k-1)!}$$

and

$$\mathbb{E}X_+(X_+ - 1) \sim n^2\left(1 - \frac{2}{n}\right)^{T_{2+}}(1 - \hat{p}_{2+})\left(\sum_{i=0}^{k-1}\binom{n-2}{i}\hat{p}_{2+}^i(1-\hat{p}_{2+})^{n-i-2}\right)^2$$

$$+ n^2\left(1 - \frac{2}{n}\right)^{T_{2+}}\hat{p}_{2+}\left(\sum_{i=0}^{k-2}\binom{n-2}{i}\hat{p}_{2+}^i(1-\hat{p}_{2+})^{n-i-2}\right)^2$$

$$\sim \left(\frac{e^{-c_n}}{(k-1)!}\right)^2.$$

Thus by the second moment method with high probability $X_+ > 0$ as $c_n \to -\infty$. Note that if there is a vertex which is isolated in $\bigcup_{t=3}^{k+1} \mathcal{G}_t(n, m, \overline{p})$ and has degree at most $k-1$ in $\mathcal{G}_2(n, m, \overline{p})$, then $\delta(\mathcal{G}(n, m, \overline{p})) \leq k - 1$. By Facts 5 and 6 there is a probability space such that with high probability $\mathcal{G}_2(n, m, \overline{p}) \subseteq G_2(n, \hat{p}_{2+})$, the number of isolated vertices in $\bigcup_{t=3}^{k+1} \mathcal{G}_t(n, m, \overline{p})$ is at least the number of non–collected coupons after T_{2+} draws, and the coupon collector process and $G_2(n, \hat{p}_{2+})$ are independent. Thus $X_+ > 0$ implies that with high probability $\delta(\mathcal{G}(n, m, \overline{p})) \leq k - 1$.

Now let $S_{1,2}/S_1 = O((\ln n)^{-1})$. Then $S_1 = n(\ln n + c_n + O(1))$. Therefore for the result follows by Lemma 1.

(ii) In the proof we restrict our attention to the case $c_n = O(\ln n)$. In the latter case the result follows after combining (3) with known results on $G_2(n, \hat{p})$.

Let $a_n = S_{1,2}/S_1 \gg 1/\ln n$ (thus $S_{1,2} \to \infty$ and $a_n \ln n \to \infty$). Consider any probability space on which we define the coupon collector process on $\mathcal{V}$ and $G_2(n, \hat{p}_{2-})$, in such a manner that they are independent. Let X_- be a random variable defined on this probability space and counting vertices which have not been collected during the coupon collector process with T_-

draws and have degree at most $k-1$ in $G_2(n, \hat{p}_{2-})$. If $S_1 - \sum_{t=3}^{k} S_{1,t} = n(\ln n + (k-1)\ln(a_n \ln n) + c_n)$ then for ω tending to infinity slowly enough

$$\frac{1}{n}T_- + (n-1)\hat{p}_{2-} \geq \frac{1}{n}\left(S_1 - \sum_{t=3}^{k-1} S_{1,t} + O\left(\omega\sqrt{S_1} + S_1^2 n^{-2}\right)\right)$$
$$= \ln n + (k-1)\ln(a_n \ln n) + c_n + o(1).$$

Thus

$$\mathbb{E}X_- = n\left(1 - \frac{1}{n}\right)^{T_-}\left(\sum_{i=0}^{k-1}\binom{n-1}{i}\hat{p}_{2-}^i(1-\hat{p}_{2-})^{k-1-i}\right)(1-\hat{p}_{2-})^{n-k}$$
$$= O(1)n\frac{1}{(k-1)!}(a_n \ln n)^{k-1}\exp(-\ln n - (k-1)\ln(a_n \ln n) - c_n)$$
$$= O(1)\frac{e^{-c_n}}{(k-1)!}$$

Therefore with high probability $X_- = 0$, i.e. with high probability each vertex is collected in T_- draws or has degree at least k in $G_2(n, \hat{p}_{2-})$. Note that if each vertex is non-isolated in $\mathcal{G}_{k+1}(n, m, \overline{p})$ or has degree at least k in $\mathcal{G}_2(n, m, \overline{p})$, then $\delta(\mathcal{G}(n, m, \overline{p})) \geq k$. Therefore Facts 5 and 7 imply that with high probability $\delta(\mathcal{G}(n, m, \overline{p})) \geq k$.

Now let $S_{1,2}/S_1 = O((\ln n)^{-1})$, i.e. $S_{1,2} = O(n)$. Then

$$S_1 - \sum_{t=2}^{k-1} S_{1,t} = n(\ln n + c_n + O(1))$$

and

$$T_- = n(\ln n + c_n + O(1)).$$

Thus by Fact 7 with high probability there are no isolated vertices in $\mathcal{G}_{k+1}(n, m, \overline{p})$, i.e. with high probability $\delta(\mathcal{G}(n, m, \overline{p})) \geq k$. $\qquad\square$

Proof of Lemma 3. We use notation from (19). Moreover let

$$a_n = (np)^{k-1}\left(\left(\frac{e^{-np}\ln n}{1 - e^{-np}}\right)^{k-1} + \frac{e^{-np}\ln n}{1 - e^{-np}}\right).$$

Note that

$$(30) \quad S_{1,t} \sim m\frac{(np)^t}{(t-1)!}(1-p)^n \sim ((t-1)!)^{-1}\frac{(np)^{t-1}(1-p)^n}{1-(1-p)^{n-1}}S_1,$$

$$\text{for } t = 2, \ldots, k,$$

where S_1 and $S_{1,t}$ are defined by (1). Therefore if $np^2 = o(1)$ then

$$\left(\frac{S_{1,2}}{S_1}\ln n\right)^{k-1} + (k-1)!\frac{S_{1,k}}{S_1}\ln n \sim a_n.$$

We will divide the proof into cases depending on the value of $S_{1,t}/S_1$, for $t = 2, k$.

CASE 1. $S_{1,2}/S_1 = O(1/\ln n)$ and $S_{1,k}/S_1 = O(1/\ln n)$.

Note that by (30) this case includes $np^2 = \Omega(1)$. Moreover in this case $a_n = O(1)$ and $\frac{S_{1,t}}{S_1}\ln n = O(1)$ for all $2 \leq t \leq k$. Therefore the lemma follows by Lemma 2.

CASE 2. $S_{1,2}/S_1 \gg 1/\ln n$ or $S_{1,k}/S_1 \gg 1/\ln n$.

Let $c_n \to -\infty$. If $S_{1,k} = O(S_{1,2})$ then the result follows by Lemma 2. In the case $S_{1,k} \gg S_{1,2}$ we may restrict attention to $c_n = o(\ln n)$ (for explanation see the proof of Lemma 2(i)). Since $S_1 = \Theta(n\ln n)$ and $S_{1,k} \gg S_1/\ln n$ we have $S_{1,k} \to \infty$. Thus we may apply the second part of Fact 5.

Take any probability space on which we define independent coupon collector process on $\mathcal{V}$ and $G_k(n, \hat{p}_{k+})$. Let X_+ be a random variable equal to the number of vertices which have not been collected during the coupon collector process in T_{k+} draws and have degree at most $k-1$ in $G_k(n, \hat{p}_{k+})$. For ω tending to infinity slowly enough

$$\frac{1}{n}T_{k+} + \binom{n-1}{k-1}\hat{p}_{k+} = \frac{1}{n}\left(S_1 + O\left(\omega\sqrt{S_1} + S_1^2 n^{-2} + \frac{S_1}{\omega\ln n}\right)\right)$$

$$= \ln n + \ln a_n + c_n + o(1),$$

thus

$$
\mathbb{E}X_+ = n\left(1 - \frac{1}{n}\right)^{T_{k+}}
$$
$$
\cdot \left((1-\hat{p}_{k+})^{\binom{n-1}{k-1}} + \binom{n-1}{k-1}\hat{p}_{k+}(1-\hat{p}_{k+})^{\binom{n-1}{k-1}-1}\right)
$$
$$
\sim n\frac{1}{(k-1)!}a_n\exp(-\ln n - \ln a_n - c_n) \sim \frac{e^{-c_n}}{(k-1)!}
$$

and

$$
\mathbb{E}X_+(X_+-1) \sim n^2\left(1 - \frac{2}{n}\right)^{T_{k+}}
$$
$$
\cdot \left[\left(1 + \binom{n-2}{k-1}\hat{p}_{k+}(1-\hat{p}_{k+})^{-1}\right)^2 (1-\hat{p}_{k+})^{2\binom{n-1}{k-1}-\binom{n-2}{k-2}}\right.
$$
$$
\left. + \binom{n-2}{k-2}\hat{p}_{k+}(1-\hat{p}_{k+})^{2\binom{n-1}{k-1}-\binom{n-2}{k-2}-1}\right]
$$
$$
\sim \left(\frac{e^{-c_n}}{(k-1)!}\right)^2.
$$

Thus with high probability $X_+ > 0$ as $c_n \to -\infty$ (i.e. with high probability there exists a vertex, which has degree at most $k-1$ in $G_k(n,\hat{p}_{k+})$ or has not been collected in T_{k+} draws). Thus by Facts 5 and 6 with high probability $\delta(\mathcal{G}(n,m,p)) \le k-1$.

Now assume that $c_n \to \infty$. As it is explained in the proof of Lemma 2 we may restrict considerations to the case $c_n = O(\ln n)$. Recall that we assume that $S_{1,2} \gg S_1/\ln n$ or $S_{1,t} \gg S_1/\ln n$, i.e. $a_n \to \infty$. Note that by definition

$$
\hat{q}_t \le \frac{S_{1,t}}{t\binom{n}{t}} = O\left(mp^t e^{-np}\right).
$$

Take any probability space on which we may define independent coupon collector process with the set of coupons $\mathcal{V}$ and $\bigcup_{t=2}^k G_t(n,\hat{q}_t)$. Let X_- be a random variable equal to the number of vertices which have not been collected during the coupon collector process in T_- draws and have degree at most $k-1$ in $\bigcup_{t=2}^k G_t(n,\hat{q}_t)$. Note that if v has degree at most $k-1$ in $\bigcup_{t=2}^k G_t(n,\hat{q}_t)$, then for some $0 \le k_0 \le k-1$ and some sequence of integers $r_2,\ldots,r_t$ such that $\sum_{t=2}^k (t-1)r_t = k_0$

43

(i) there is a set $\mathcal{V}' \subseteq \mathcal{V} \setminus \{v\}$ of k_0 vertices such that for each t there are r_t hyperedges in $H_t(n, \hat{q}_t)$ (generating $G_t(n, \hat{q}_t)$) containing v and contained in $\mathcal{V}' \cup \{v\}$.

(ii) and all hyperedges in $\bigcup_{t=2}^k H_t(n, \hat{q}_t)$ containing v are subsets of $\mathcal{V}' \cup \{v\}$.

Let $r = \sum_{t=2}^k r_t$, then for $c_n = O(\ln n)$ event (i) occurs with probability at most

$$\binom{n-1}{k_0} \prod_{t=2}^k \binom{k_0-1}{t-1}^{r_t} \left(O(1)(mp^t)e^{-np} \right)^{r_t}$$
$$= O(1)n^{k_0}m^r p^{k_0+r} \left(e^{-np} \right)^r$$
$$= O(1)(np)^{k_0}(mpe^{-np})^r$$
$$= O(1)(np)^{k_0} \left(\frac{\ln n}{(1-e^{-np})}e^{-np} \right)^r$$
$$= O(a_n).$$

There are $O(1)$ sequences of integers $r_2, \ldots, r_t$ such that $\sum_{t=2}^k (t-1)r_t \leq k-1$, thus the probability that there exists such sequence $r_2, \ldots, r_t$ that (i) and (ii) occur is at most

$$O(a_n) \prod_{t=2}^k (1 - \hat{q}_t)^{\binom{n-1}{t-1} - \binom{k-1}{t-1}}.$$

Moreover

$$\frac{1}{n}T_- + \sum_{t=2}^k \binom{n-1}{t-1}\hat{q}_t \geq \frac{1}{n}\left(S_1 - O\left(\omega\sqrt{S_1} + S_1^2 n^{-2} \right) \right)$$
$$= \ln n + \ln a_n + c_n + o(1).$$

Therefore finally

$$\mathbb{E}X_- = n \left(1 - \frac{1}{n} \right)^{T_-} O(a_n) \prod_{t=2}^k (1 - \hat{q}_t)^{\binom{n-1}{t-1} - \binom{k-1}{t-1}} = o(1).$$

Concluding, with high probability $X_- = 0$. By Facts 5 and 7 with high probability $\delta(\mathcal{G}(n, m, p)) \geq k$. $\qquad\square$