# ON THE CONTINUITY OF PARTIAL ACTIONS OF HAUSDORFF GROUPS ON METRIC SPACES

J. GÓMEZ, H. PINEDO, AND C. UZCÁTEGUI

Abstract. We provide a sufficient condition for a topological partial action of a Hausdorff group on a metric space is continuous, provide that it is separately continuous.

## 1. Introduction

The notion of partial action of a group is a weakening of the classical concept of group action and was introduced in [6] and [11], and was then developed in [1] and [10], in which the authors provided examples in different guises. Since then partial actions have been an important tool in $C^*$-algebras and dynamical systems, and in the developing of new cohomological theories [4], [5] and [12]. Every partial action of a group $G$ on a set $\mathbb{X}$ can be obtained, roughly speaking, as a restriction of a global action (see [1] and [10]) on some bigger set $\mathbb{X}_G$, called the enveloping space of $\mathbb{X}$. Nevertheless, in the category of topological spaces, when $G$ acts partially on a space $\mathbb{X}$, the superspace $\mathbb{X}_G$ does not necessarily inherit its topological properties; for instance the globalization of a partial action of a group on a Hausdorff space is not in general Hausdorff (see e.g. [1, Example 1.4], [1, Proposition 1.2]).

On the other hand, actions of Polish groups have important connections with many areas of mathematics (see [2], [8] and the references therein). Recall that a Polish space is a topological space which is separable and completely metrizable, and a Polish group is a topological group whose topology is Polish. Partial actions of Polish groups have been recently considered in the works [9, 13, 14].

It is known that an action of a Polish group $G$ on a metric space $\mathbb{X}$ is continuous provided that it is separately continuous (see [8, Theorem 3.1.4]). In this note, under a mild restriction, we generalize [8, Theorem 3.1.4] in two directions, first we only assume that $G$ is Hausdorff and Baire, and second we only assume that $G$ acts partially on $\mathbb{X}$.

## 2. The notions

Let $G$ be a topological group with identity element 1 and $\mathbb{X}$ a topological space. A partially defined map $G \times \mathbb{X} \to \mathbb{X}$ is a map whose domain is a subset of $G \times \mathbb{X}$. Let $m\colon G \times \mathbb{X} \to \mathbb{X}$, $(g, x) \mapsto m(g, x) = g \cdot x \in \mathbb{X}$ be a partial map, that is $m$ is a map whose domain is contained in $G \times \mathbb{X}$. We write $\exists g \cdot x$ to mean that $(g, x)$ is in the domain of $m$. Then, following [1, 10] $m$ defines a (set theoretic) *partial action* of $G$ on $\mathbb{X}$, if for all $g, h \in G$ and $x \in \mathbb{X}$ the following assertions hold:

(PA1) $\exists g \cdot x$ implies $\exists g^{-1} \cdot (g \cdot x)$ and $g^{-1} \cdot (g \cdot x) = x$,

(PA2) $\exists g \cdot (h \cdot x)$ implies $\exists (gh) \cdot x$ and $g \cdot (h \cdot x) = (gh) \cdot x$,

(PA3) $\exists 1 \cdot x$, and $1 \cdot x = x$.

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Let $G * \mathbb{X} = \{(g,x) \in G \times \mathbb{X} \mid \exists g \cdot x\}$ be the domain of $m$, set $\mathbb{X}_{g^{-1}} = \{x \in \mathbb{X} \mid \exists g \cdot x\}$, and $m_g \colon \mathbb{X}_{g^{-1}} \ni x \mapsto g \cdot x \in \mathbb{X}_g$. Then a partial action $m \colon G * \mathbb{X} \to \mathbb{X}$ induces a family of bijections $\{m_g \colon \mathbb{X}_{g^{-1}} \to \mathbb{X}_g\}_{g \in G}$, and we denote $m = \{m_g \colon \mathbb{X}_{g^{-1}} \to \mathbb{X}_g\}_{g \in G}$.

**Proposition 2.1.** [15, Lemma 1.2] *A partial action $m$ of $G$ on $\mathbb{X}$ is a family $m = \{m_g \colon \mathbb{X}_{g^{-1}} \to \mathbb{X}_g\}_{g \in G}$, where $\mathbb{X}_g \subseteq \mathbb{X}$, $m_g \colon \mathbb{X}_{g^{-1}} \to \mathbb{X}_g$ is bijective, for all $g \in G$, and such that:*

(i) $\mathbb{X}_1 = \mathbb{X}$ *and* $m_1 = \mathrm{Id}_{\mathbb{X}}$;

(ii) $m_g(\mathbb{X}_{g^{-1}} \cap \mathbb{X}_h) = \mathbb{X}_g \cap \mathbb{X}_{gh}$;

(iii) $m_g m_h \colon \mathbb{X}_{h^{-1}} \cap \mathbb{X}_{h^{-1}g^{-1}} \to \mathbb{X}_g \cap \mathbb{X}_{gh}$, *and* $m_g m_h = m_{gh}$ *in* $\mathbb{X}_{h^{-1}} \cap \mathbb{X}_{h^{-1}g^{-1}}$;

*for all $g, h \in G$.*

Notice that conditions (ii) and (iii) in Lemma 2.1 say that $m_{gh}$ is an extension of $m_g m_h$ for all $g, h \in G$. We consider $G \times \mathbb{X}$ with the product topology and the subset $G * \mathbb{X}$ of $G \times \mathbb{X}$ inherits the subspace topology.

**Definition 2.2.** *A topological partial action of the group $G$ on the topological space $\mathbb{X}$ is a partial action $m = \{m_g \colon \mathbb{X}_{g^{-1}} \to \mathbb{X}_g\}_{g \in G}$ on the underlying set $\mathbb{X}$, such that each $\mathbb{X}_g$ is open in $\mathbb{X}$, and each $m_g$ is a homeomorphism, for any $g \in G$. If $m \colon G * \mathbb{X} \to \mathbb{X}$ is continuous, we say that the partial action is continuous.*

**Example 2.3. Induced partial action:** *Let $G$ be a topological group, and $\mathbb{Y}$ a topological space and let $u \colon G \times \mathbb{Y} \to \mathbb{Y}$ be a continuous action of $G$ on $\mathbb{Y}$ and $\mathbb{X} \subseteq \mathbb{Y}$ be an open set. For $g \in G$, set $\mathbb{X}_g = \mathbb{X} \cap u_g(\mathbb{X})$ and let $m_g = u_g \restriction \mathbb{X}_{g^{-1}}$ ( the restriction of $u_g$ to $\mathbb{X}_{g^{-1}}$). Then $m \colon G * \mathbb{X} \ni (g,x) \mapsto m_g(x) \in \mathbb{X}$ is a topological partial action of $G$ on $\mathbb{X}$.*

The interested reader may consult others examples of topological partial actions in [1, Example 1.2, Remark 1.1, Example 1.3, Example 1.4] and [10, p. 108].

## 3. Topological partial actions of Hausdorff groups on metric spaces

Let $m \colon G * \mathbb{X} \to \mathbb{X}$ be a topological partial action. For $x \in \mathbb{X}$, we set $G^x = \{g \in G \mid (g,x) \in G * \mathbb{X}\}$, then by (PA3) $1 \in G^x$, we also set $m^x \colon G^x \ni g \to m(g,x) \in \mathbb{X}$, $m$ is called *separately continuous* if the maps $m^x$ are continuous, for all $x \in \mathbb{X}$.[1] It is known that group actions of Polish groups on metric spaces are continuous, if and only if, they are separately continuous. We provide a mild condition on $G^x$, $x \in \mathbb{X}$, for which separately continuous partial actions of Hausdorff-Baire groups are continuous, the proof we present is inspired by the one given in [8, Theorem 3.1.4], for classical group actions.

**Theorem 3.1.** *Let $G$ be a Hausdorff group, $(\mathbb{X}, d)$ a metric space, and $m$ a topological partial action of $G$ on $\mathbb{X}$. Suppose that $G$ is Baire $G^x$ is open in $G$, for any $x \in \mathbb{X}$. Then $m$ is continuous, if and only if, it is separately continuous.*

*Proof.* It is clear that continuous partial actions are separately continuous. For the converse, suppose that $m$ is separately continuous and let $(g_0, x_0) \in G * \mathbb{X}$, we check that $m$ is continuous at $(g_0, x_0)$. Let $l, n \in \mathbb{N}$ and set

$$F_{n,l} = \{g \in G^{x_0} \mid \forall x \in \mathbb{X}_{g^{-1}}(d(x, x_0) < 2^{-n} \Rightarrow d(m(g,x), m(g,x_0)) \leq 2^{-l})\}.$$

We shall check that that $F_{n,l}$ is a closed subset of $G^{x_0}$. Indeed, let $g \in G^{x_0}$ and $\{g_i\}_{i \in I}$ a net with $g_i \to g$ such that $g_k \in F_{n,l}$, for all $k \in \mathbb{N}$. Then

$$(\forall k \in \mathbb{N}) \left(\forall x \in \mathbb{X}_{g_i^{-1}}\right) \left(d(x, x_0) < \frac{1}{2^n} \Rightarrow d\left(m(g_i, x), m(g_i, x_0)\right) \leq \frac{1}{2^l}\right).$$

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[1] Notice that for any $g \in G$ the map $m_g$ is continuous by definition of topological partial action.

Let $x \in \mathbb{X}_{g^{-1}}$, that is $g \in G^x$, since $G^x$ is open one may assume that $\{g_i\}_{i\in I} \subseteq G^x$, and $x \in \mathbb{X}_{g^{-1}} \cap \mathbb{X}_{g_i^{-1}}$, for all $i \in I$. Thus, by the continuity of $m^x$ and $m^{x_0}$, we have that $m(g_i, x) \to m(g, x)$ and $m(g_i, x_0) \to m(g, x_0)$, which gives

$$(\forall x \in \mathbb{X}_{g^{-1}}) \left( d(x, x_0) < \frac{1}{2^n} \Rightarrow d\left(m(g, x), m(g, x_0)\right) \leq \frac{1}{2^l} \right),$$

and we conclude that $g \in F_{n,l}$. Now, we check that

$$G^{x_0} = \bigcap_l \bigcup_n F_{n,l}. \tag{3.1}$$

It is clear that $G^{x_0} \supseteq \bigcap_l \bigcup_n F_{n,l}$. Conversely, take $g \in G^{x_0}$ and $l \in \mathbb{N}$. Since $m_g$ is continuous at $x_0$, for $\varepsilon = \dfrac{1}{2^l}$, there exists $\delta > 0$ such that

$$(\forall x \in \mathbb{X}_{g^{-1}}) \left( d(x, x_0) < \delta \Rightarrow d\left(m(g, x), m(g, x_0)\right) \leq \frac{1}{2^l} \right),$$

Let $n \in \mathbb{N}$ with $\dfrac{1}{2^n} < \delta$, then $g \in F_{n,l}$. Thus $G^{x_0} \subseteq \bigcap_l \bigcup_n F_{n,l}$, and we have (3.1).

Since $F_{l,n}$ is closed in $G^{x_0}$, for all $n, l$; the set $D = \bigcup_l \bigcup_n (F_{n,l} \setminus \operatorname{int}(F_{n,l}))$ is meager. But $G^{x_0}$ is a non empty open set, since $G$ is Baire, there is $g_1 \in G^{x_0} \setminus D$. We shall check that $m$ is continuous at $(g_1, x_0)$. Indeed, let $\{(h_\alpha, y_\alpha)\}_{\alpha\in\Lambda} \subseteq G*\mathbb{X}$ be a net converging to $(g_1, x_0)$. Take $\varepsilon > 0$ and $l \in \mathbb{N}$ such that $\dfrac{1}{2^{l-1}} < \varepsilon$. By (3.1), there exists $n \in \mathbb{N}$ such that $g_1 \in F_{n,l}$. Since $g_1 \notin D$ then $g_1 \in \operatorname{int}(F_{n,l})$. The fact that $h_\alpha \to g_1$, implies that there is $\alpha_1 \in \Lambda$ such that $h_\alpha \in \operatorname{int}(F_{n,l})$, for all $\alpha \geq \alpha_1$. Also, since $y_\alpha \to x_0$, there exists $\alpha_2 \in \mathbb{N}$ such that $d(y_\alpha, x_0) < \dfrac{1}{2^n}$, for all $\alpha \geq \alpha_2$. Additionally, by the continuity of $m^{x_0}$, we have that $m(h_\alpha, x_0) \to m(g_1, x_0)$. Thus, there exists $\alpha_3 \in \Lambda$ such that $d(m(h_\alpha, x_0), m(g_1, x_0)) < \dfrac{1}{2^l}$, for all $\alpha \geq \alpha_3$.

Let $\alpha \geq \max\{\alpha_1, \alpha_2, \alpha_3\}$. Then $h_\alpha \in F_{n,l}$ and $d(y_\alpha, x_0) < \dfrac{1}{2^n}$, hence

$$d(m(h_\alpha, y_\alpha), m(g_1, x_0)) \leq d(m(h_\alpha, y_\alpha), m(h_\alpha, x_0)) + d(m(h_\alpha, x_0), m(g_1, x_0)) < \frac{1}{2^{l-1}} < \varepsilon.$$

Thus, $m$ is continuous at $(g_1, x_0)$.

Now, since $x_0 \in \mathbb{X}_{g_1^{-1}} \cap \mathbb{X}_{g_0^{-1}}$, we have, by Proposition 2.1(ii), that $g_1 \cdot x_0 \in \mathbb{X}_{g_1} \cap \mathbb{X}_{g_1 g_0^{-1}}$, then $(g_0 g_1^{-1}) \cdot (g_1 \cdot x_0)$ is defined and $(g_0 g_1^{-1}) \cdot (g_1 \cdot x_0) = g_0 \cdot x_0$, by (PA2). That is

$$m(g_0, x_0) = m(g_0 g_1^{-1}, m(g_1, x_0)). \tag{3.2}$$

Finally take a net $\{(h_j, y_j)\}_{j\in J}$ in $G * \mathbb{X}$ converging to $(g_0, x_0)$. Then $g_1 g_0^{-1} h_j \to g_1 \in G^{x_0}$, and since $G^{x_0}$ is open we assume that the net $\{g_1 g_0^{-1} h_j\}_{j\in J}$ is contained in $G^{x_0}$, thus $x_0 \in \mathbb{X}_{(g_1 g_0^{-1} h_j)^{-1}}$, for all $j \in J$. But $y_j \to x_0$, then there is $j_0 \in J$ such that $y_j \in \mathbb{X}_{(g_1 g_0^{-1} h_j)^{-1}}$, for all $j \geq j_0$. Now $y_j \in \mathbb{X}_{(g_1 g_0^{-1} h_j)^{-1}} \cap \mathbb{X}_{h_j^{-1}}$, then $h_j \cdot y_j \in \mathbb{X}_{(g_1 g_0^{-1})^{-1}}$ (by Proposition 2.1(ii)). By (PA2) we get

$$(g_1 g_0^{-1} h_j) \cdot y_j = (g_1 g_0^{-1}) \cdot (h_j \cdot y_j) \in \mathbb{X}_{g_1 g_0^{-1}}.$$

Thus, by (PA1),

$$h_j \cdot y_j = (g_0 g_1^{-1}) \cdot [(g_1 g_0^{-1} h_j) \cdot y_j].$$

Notice that $(g_1{g_0}^{-1}h_j, y_j) \to (g_1, x_0)$ . Then by continuity of $m_{g_0g_1^{-1}}$ and the continuity of $m$ at $(g_1, x_0)$ one gets

$$m(h_j, y_j) = (g_0g_1^{-1}) \cdot [(g_1g_0^{-1}h_j) \cdot y_j] \to (g_0g_1^{-1}) \cdot (g_1 \cdot x_0) = m(g_0g_1^{-1}, m(g_1, x_0)),$$

and by (3.2) we obtain $m(h_j, y_j) \to m(g_0, x_0)$. $\square$

**Corollary 3.2.** *Let $G$ be a topological Hausdorff group, and $a\colon G \times \mathbb{X} \to \mathbb{X}$ an action of $G$ on a metric space $\mathbb{X}$. If $G$ is Baire, then $a$ is continuous, if and only if, $a$ is separately continuous.*

**Corollary 3.3.** *Let $G$ be a countable discrete group and $m$ a topological partial action of $G$ on a metric space $\mathbb{X}$. Then $m$ is continuous, if and only if, $m$ is separately continuous.*

**Example 3.4. Möbius transfromations** [3, p. 175] *The group $G = \mathrm{GL}(2, \mathbb{R})$ is Polish and acts partially on $\mathbb{R}$ by setting*

$$g \cdot x = \frac{ax + b}{cx + d}, \quad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Notice that for all $g \in G$ the set $\mathbb{R}_g = \{x \in \mathbb{R} \mid cx + d \neq 0\}$ is open, and $m = \{m_g \colon \mathbb{R}_{g^{-1}} \ni x \to g \cdot x \in \mathbb{R}_g\}_{g \in G}$ is a topological partial action. For $x \in \mathbb{R}$ let $t_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, then for $y \in \mathbb{R}$ one has that $t_{y-x} \cdot x = y$. Since

$$G^0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : d \neq 0 \right\}$$

is open, and

$$G^x = G^{t_x \cdot 0} = G^0 t_x^{-1} = G^0 \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix},$$

then $G^x$ is open. Finally, since

$$m^x \colon G^x \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{ax + b}{cx + d} \in \mathbb{R},$$

is continuous, then $m$ is continuous.

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Escuela de Matemáticas, Universidad Industrial de Santander, Cra. 27 Calle 9 UIS Edificio 45, Bucaramanga, Colombia
*E-mail address*: jorge.gomez14@correo.uis.edu.co

Escuela de Matemáticas, Universidad Industrial de Santander, Cra. 27 Calle 9 UIS Edificio 45, Bucaramanga, Colombia
*E-mail address*: hpinedot@uis.edu.co

Escuela de Matemáticas, Universidad Industrial de Santander, Cra. 27 Calle 9 UIS Edificio 45, Bucaramanga, Colombia
*E-mail address*: cuzcatea@uis.edu.co