THE GOLDMAN SYMPLECTIC FORM ON THE PSL(V)-HITCHIN COMPONENT

ZHE SUN AND TENGREN ZHANG

ABSTRACT. This article is the second of a pair of articles about the Goldman symplectic form on the PSL(V)-Hitchin component. We show that any ideal triangulation on a closed connected surface of genus at least 2, and any compatible bridge system determine a symplectic trivialization of the tangent bundle to the Hitchin component of the surface. Using this, we prove that a large class of flows defined in the companion paper [SWZ17] are Hamiltonian. We also construct an explicit collection of Hamiltonian vector fields on the Hitchin component that give a symplectic basis at every point.

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1. Introduction

Let S be a closed, oriented, connected surface of genus at least 2, and let Γ denote the fundamental group of S. The *Teichmüller space* of S, denoted $\mathcal{T}(S)$, is the deformation space of hyperbolic structures on S. By considering the holonomy representations of these hyperbolic structures, one can identify $\mathcal{T}(S)$ as the connected component

$$\big\{ [\rho] \in \mathcal{X}\big(\Gamma, \mathrm{PSL}(2, \mathbb{R})\big) : \rho \text{ is discrete and faithful} \big\}$$

of $\mathcal{X}\big(\Gamma, \mathrm{PSL}(2, \mathbb{R})\big) := \mathrm{Hom}\big(\Gamma, \mathrm{PSL}(2, \mathbb{R})\big) / \mathrm{PGL}(2, \mathbb{R})$.

This representation theoretic description of $\mathcal{T}(S)$ admits a natural generalization where we replace $\mathrm{PSL}(2, \mathbb{R})$ with a higher rank, semisimple Lie group G that is real split. These generalizations of $\mathcal{T}(S)$ are known as the *G-Hitchin components*, and are central objects of investigation in the study of higher Teichmüller theory. In this article, we consider the setting when $G = \mathrm{PSL}(V)$, where V is an n-dimensional real vector space (see Section 2.1). The $\mathrm{PSL}(V)$-Hitchin component, denoted $\mathrm{Hit}_V(S)$, was first studied by Hitchin [Hit92], who showed that $\mathrm{Hit}_V(S)$ is real-analytically diffeomorphic to $\mathbb{R}^{(n^2-1)(2g-2)}$. Later, Labourie [Lab06] proved that every representation in $\mathrm{Hit}_V(S)$ is Anosov, which implies in particular that these representations are all quasi-isometric embeddings from Γ (equipped with any word metric) to $\mathrm{PSL}(V)$ (equipped with any left-invariant Riemannian metric).

By specializing a general construction due to Goldman [Gol84], one can equip $\mathrm{Hit}_V(S)$ with a natural symplectic structure (see Section 2.4), known as the *Goldman symplectic form*. Goldman also proved [Gol86] that when $V = \mathbb{R}^2$, this symplectic form is a multiple of the Weil-Petersson symplectic form on $\mathcal{T}(S)$. The Goldman symplectic form on $\mathrm{Hit}_V(S)$ has been an object of interest in the last few years, and was previously studied by many other authors, including [Bri15], [LW16], [Lab12], [Nie13], [Sun17].

Perhaps the most famous result regarding this symplectic form on $\mathcal{T}(S)$ is a theorem of Wolpert [Wol82],[Wol83], which states that for any pants decomposition $\mathcal{P}$ on S, the Hamiltonian flows of the Fenchel-Nielsen coordinate functions corresponding to $\mathcal{P}$ are commuting flows whose tangent vector fields give a symplectic basis of $T_{[\rho]}\mathcal{T}(S)$ for every $[\rho] \in \mathcal{T}(S)$. This was partially generalized to $\mathrm{Hit}_V(S)$ when $\dim(V) = 3$ by Kim [Kim99], and when $\dim(V) = 4$ by H.T. Jung (in preparation). The key tool they used to prove these results is known as Fox calculus, which is a technique that allows one to compute the Goldman Poisson pairing between certain types of functions on $\mathrm{Hit}_V(S)$. However, the computations required to implement Fox calculus become more and more complicated as $\dim(V)$ gets large.

This article is the second of a pair of articles about the Goldman symplectic form on $\mathrm{Hit}_V(S)$, where the objective is to generalize Wolpert's theorem to $\mathrm{Hit}_V(S)$ for general V. In the companion article by Sun-Wienhard-Zhang [SWZ17], we show that given any ideal triangulation $\mathcal{T}$ on S and any compatible bridge system $\mathcal{J}$

(see Section 2.2 for definitions), one can construct a family of real-analytic flows on $\mathrm{Hit}_V(S)$ using the geometry of the representations in $\mathrm{Hit}_V(S)$. These are called the $(\mathcal{T}, \mathcal{J})$-parallel flows ([SWZ17, Section 5.1]), and satisfy the following properties:

- Any pair of $(\mathcal{T}, \mathcal{J})$-parallel flows commute.
- The set of $(\mathcal{T}, \mathcal{J})$-parallel flows form a $(n^2 - 1)(2g - 2)$-dimensional vector subspace of the space of flows on $\mathrm{Hit}_V(S)$.
- For any $[\rho] \in \mathrm{Hit}_V(S)$ and any $v \in T_{[\rho]} \mathrm{Hit}_V(S)$, there is a unique $(\mathcal{T}, \mathcal{J})$-parallel flow whose tangent vector at $[\rho]$ is v.

Special cases of these flows were previously constructed by Wienhard-Zhang [WZ17].

More informally, given a choice of some topological data on S (the choice of $\mathcal{T}$ and $\mathcal{J}$), the geometry of the representations in $\mathrm{Hit}_V(S)$ determine a vector space of commuting flows on $\mathrm{Hit}_V(S)$ that is naturally in bijection with $T_{[\rho]} \mathrm{Hit}_V(S)$ for every $[\rho] \in \mathrm{Hit}_V(S)$. In particular, this determines a trivialization of $T \mathrm{Hit}_V(S)$. The first goal of this article is to prove that this trivialization is symplectic.

Theorem 1.1 (Theorem 5.3)**.** *Let $\mathcal{T}$ be an ideal triangulation on S and let $\mathcal{J}$ be a compatible bridge system. If X_1 and X_2 are the tangent vector fields for a pair of $(\mathcal{T}, \mathcal{J})$-parallel flows on $\mathrm{Hit}_V(S)$, then the function $\mathrm{Hit}_V(S) \to \mathbb{R}$ given by*

$$[\rho] \mapsto \omega(X_1[\rho], X_2[\rho])$$

is constant.

Combining Theorem 1.1 with the properties of $(\mathcal{T}, \mathcal{J})$-parallel flows listed above, one can then prove the following corollary.

Corollary 1.2 (Corollary 5.4)**.** *Let $(\mathcal{T}, \mathcal{J})$ be as in Theorem 5.3. Any $(\mathcal{T}, \mathcal{J})$-parallel flow is a Hamiltonian flow on $\mathrm{Hit}_V(S)$.*

By Theorem 1.1, the vector space of $(\mathcal{T}, \mathcal{J})$-parallel flows admits a symplectic pairing induced by the Goldman symplectic form on $\mathrm{Hit}_V(S)$. Choose any symplectic basis for this vector space. By Corollary 5.4, the flows in the symplectic basis are Hamiltonian, so their Hamiltonian functions give global Darboux coordinates for $\mathrm{Hit}_V(S)$. As a consequence, we have the following corollary.

Corollary 1.3 (Corollary 5.5)**.** $\mathrm{Hit}_V(S)$ *is a complete integrable system.*

Behind the proof of Theorem 1.1 is a new method to compute the Goldman symplectic form, which is similar in flavor to the techniques used by Bonahon-Sozen [BS01] on $\mathcal{T}(S)$. This method uses the following theorem obtained by combining the work of Labourie [Lab06] and Guichard [Gui08].

Theorem 1.4 (Guichard, Labourie)**.** *There is canonical bijection*

$$\mathrm{Hit}_V(S) \simeq \left\{ \xi : \partial\Gamma \to \mathrm{PSL}(V) \,\middle|\, \begin{array}{l} \xi \text{ is Frenet and } \rho\text{-equivariant} \\ \text{for some } \rho \in \mathrm{Hom}\big(\Gamma, \mathrm{PSL}(V)\big) \end{array} \right\} \Big/ \mathrm{PGL}(V)$$

In other words, instead of thinking of $\mathrm{Hit}_V(S)$ as a space of conjugacy classes of representations, one can think of $\mathrm{Hit}_V(S)$ as a space of projective classes of Frenet curves (see Definition 2.3) satisfying an equivariance property.

We will now briefly describe this new computational method. Let $[\rho] \in \mathrm{Hit}_V(S)$ and let $[\nu] \in H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}) \simeq T_{[\rho]} \mathrm{Hit}_V(S)$. We first show that if one chooses $(\mathcal{T}, \mathcal{J})$, a hyperbolic metric on S to get a hyperbolic surface Σ, and a representative $\rho \in [\rho]$, then these choices naturally determine a cocycle representative $\mu_{\rho, [\nu]} \in [\nu]$

from the ρ-equivariant Frenet curve guaranteed by Theorem 1.4. We call this 1-cocycle the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-*tangent cocycle* in $[\nu]$ (see Section 2.5).

Next, we give a different way to construct the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle by making an additional choice of an orientation r on $\mathcal{T}$. From $\mathcal{T}$, we construct an associated *barrier system* $\widetilde{\mathcal{B}}$ (see Definition 2.8) which is a set that admits a Γ-action, and only depends on the topology of S. Using r and ρ, we then define the notion of an *admissible labelling* $L : \widetilde{\mathcal{B}} \to \mathfrak{sl}(V)$ (see Definition 3.7). The space of such admissible labellings, denoted by $\mathcal{A}(\rho, r, \mathcal{T})$, is a vector space of dimension $(n^2 - 1)(2g - 2)$. We also define an algebraic intersection number (see Section 2.3) between 1-simplices in $\widetilde{S}$ and barriers in $\widetilde{\mathcal{B}}$ by using r and Σ. Together with $\mathcal{J}$, this allows us to construct, for every $L \in \mathcal{A}(\rho, r, \mathcal{T})$, a 1-cocycle $\mu_L \in C^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ (see Section 4.2). The following theorem relates these 1-cocycles and the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycles.

Theorem 1.5 (Theorem 4.8). *Let $L \in \mathcal{A}(\rho, r, \mathcal{T})$ be an admissible labelling. Then μ_L is the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle in the cohomology class $[\mu_L]$.*

From this, we can deduce that the map $\Phi_\rho : \mathcal{A}(\rho, r, \mathcal{T}) \to H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ given by $L \mapsto [\mu_L]$ is a linear isomorphism that does not depend on Σ. Informally, this theorem converts the abstract definition of the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-cocycle into a concrete, explicit definition in terms of admissible labellings, which can be used to do computations. Hence, we can now compute the Goldman symplectic pairing between any pair of first cohomology classes in $H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ by using the cup product formula from simplicial cohomology.

The computational procedure above is the main tool used to prove Theorem 1.1. More precisely, for every $L \in \mathcal{A}(\rho, r, \mathcal{T})$, we define the *coeffecients* of L (see Definition 6.4). These are a collection of real numbers that determine L once we are given the ρ-equivariant Frenet curve. We then used Theorem 1.5 to compute that for any $L_1, L_2 \in \mathcal{A}(\rho, r, \mathcal{T})$, $\omega([\mu_{L_1}], [\mu_{L_2}])$ depends only on the coeffecients of L_1 and L_2. Theorem 1.1 then follows from the observation that the coeffecients of the admissible labellings corresponding to tangent vectors to any $(\mathcal{T}, \mathcal{J})$-parallel flow are constant on $\mathrm{Hit}_V(S)$. For more details, see the proof of Theorem 5.3.

The second goal of this article is to use the computational method described above to explicitly find a collection of $(n^2 - 1)(2g - 2)$ Hamiltonian flows on $\mathrm{Hit}_V(S)$ that commute, so that their tangent vector fields give a symplectic basis at every $[\rho] \in \mathrm{Hit}_V(S)$. To that end, we specialize to certain types of ideal triangulations $\mathcal{T}$ on S that contain a pants decomposition $\mathcal{P}$ of S (see Section 6.1). For these ideal triangulations, we found an explicit family of $(n^2 - 1)(2g - 2)$ admissible labellings in $\mathcal{A}(\rho, r, \mathcal{T})$, called *special admissible labellings*, that are defined by specifying their coeffecients. These consist of

- $(g - 1)(n - 1)(n - 2)$ *eruption labellings*, $\frac{(n-1)(n-2)}{2}$ for each $P \in \mathbb{P}$,
- $(g - 1)(n - 1)(n - 2)$ *hexagon labellings*, $\frac{(n-1)(n-2)}{2}$ for each $P \in \mathbb{P}$,
- $(3g - 3)(n - 1)$ *twist labellings*, $(n - 1)$ for each $c \in \mathcal{P}$,
- $(3g - 3)(n - 1)$ *length labellings*, $(n - 1)$ for each $c \in \mathcal{P}$.

Here, $\mathbb{P}$ denotes the set of pairs of pants given by $\mathcal{P}$. See Definition 6.2 and Definition 6.3 for the explicit description of these admissible labellings. We then prove the following theorem.

Theorem 1.6 (Theorem 6.5). *For any pants decomposition $\mathcal{P}$ of S, there is an ideal triangulation $\mathcal{T}$ of S containing $\mathcal{P}$ so that for any compatible bridge system*

$\mathcal{J}$ and any $[\rho] \in \mathrm{Hit}_V(S)$, the set

$$\{[\mu_L] \in H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}) : L \in \mathcal{A}(\rho, r, \mathcal{T}) \text{ is a special admissible labelling}\}$$

is a symplectic basis of $H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$.

Each tangent vector in this symplectic basis corresponds to a unique $(\mathcal{T}, \mathcal{J})$-parallel flow, which is Hamiltonian by Corollary 1.2. In the companion paper [SWZ17], we used this to give explicit global Darboux coordinates for $\mathrm{Hit}_V(S)$ by computing the Hamiltonian functions of the special $(\mathcal{T}, \mathcal{J})$-parallel flows. If we specialize to the case when $\dim(V) = 2$, then the Hamiltonian flows we found are essentially the Hamiltonian flows of the $6g-6$ Fenchel-Nielsen coordinate functions. This thus gives a generalization of Wolpert's theorem.

The rest of this paper is organized in the following way. In Section 2, we formally define the basic objects we need, including Frenet curves, Hitchin representations, ideal triangulations, barriers systems, and bridge systems. In this section, we also define a notion of algebraic intersection number between 1-simplices in S and oriented barriers, and construct the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle mentioned above. The notion of an admissible labelling is defined in Section 3. In Section 4, we construct the linear map $\Phi_\rho : \mathcal{A}(\rho, r, \mathcal{T}) \to H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ described above, prove that it does not depend on the choice of Σ on S, and that it is in fact an isomorphism. In Section 5, we use Φ_ρ to construct a symplectic trivialization of $T\,\mathrm{Hit}_V(S)$, and prove Theorem 1.1. Then in Section 6, we define the special admissible labellings and prove Theorem 1.6.

Some of the proofs in this article require long but elementary computations, which are completely written up in the appendices attached. Also, one should note that the results in this article depend only on the first five sections (but not Section 6) of the companion article [SWZ17].

2. Background

2.1. **Frenet curves, and** PSL(V)**-Hitchin representations.** In this section, we will recall some well-known facts about PSL(V)-Hitchin representations. We begin with the definition of these representations.

Definition 2.1. A representation $\rho : \Gamma \to \mathrm{PSL}(V)$ is a $\mathrm{PSL}(V)$-*Hitchin represen-tation* if it can be continuously deformed in $\mathrm{Hom}(\Gamma, \mathrm{PSL}(V))$ to a faithful representation whose image is a discrete subgroup that lies in the image of an irreducible representation from $\mathrm{PSL}_2(\mathbb{R})$ to $\mathrm{PSL}(V)$.

Let $\widetilde{\mathrm{Hit}}_V(S)$ denote the set of Hitchin representations in $\mathrm{Hom}(\Gamma, \mathrm{PSL}(V))$. Since $\mathrm{PSL}(V)$ is a real algebraic group, $\mathrm{Hom}(\Gamma, \mathrm{PSL}(V))$ is naturally a real algebraic variety. One can also topologize $\mathrm{Hom}(\Gamma, \mathrm{PSL}(V))$ by the compact-open topology. It is clear that $\widetilde{\mathrm{Hit}}_V(S)$ is a union of connected components of $\mathrm{Hom}(\Gamma, \mathrm{PSL}(V))$. Furthermore, since the condition for being a $\mathrm{PSL}(V)$-Hitchin representation is invariant under conjugation by elements in $\mathrm{PGL}(V)$, we can consider $\mathrm{Hit}_V(S) := \widetilde{\mathrm{Hit}}_V(S)/\mathrm{PGL}(V)$. This was first studied by Hitchin [Hit92], who used Higgs bundle techniques to prove that $\mathrm{Hit}_V(S)$ is homeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$. In particular, $\mathrm{Hit}_V(S)$ is connected, and is commonly known as the $\mathrm{PSL}(V)$-*Hitchin component*.

Note that in the case when $\dim(V) = 2$, $\mathrm{Hit}_V(S)$ is exactly the space of conjugacy classes of faithful representations from Γ to $\mathrm{PSL}(V)$ with discrete image. It is also classically known that every such conjugacy class is the holonomy of a unique hyperbolic structure on S, so $\mathrm{Hit}_V(S)$ can naturally be identified with the deformation space of hyperbolic structures on S. This phenomena was generalized by Choi-Goldman [CG93], who proved that when $\dim(V) = 3$, the $\mathrm{PSL}(V)$-Hitchin representations are exactly holonomies of convex real projective structures on S.

However, the geometric properties of $\mathrm{PSL}(V)$-Hitchin representations in general were poorly understood until a breakthrough by Labourie [Lab06], who used Frenet curves to study them. We will now give a brief account of Labourie's result.

Let V be an n-dimensional real vector space. A *(complete) flag* F in V is a nested sequence of subspaces

$$F^{(1)} \subset F^{(2)} \subset \cdots \subset F^{(n-1)} \subset V$$

so that $\dim F^{(i)} = i$. Let $\mathcal{F}(V)$ denote the space of flags in V. It is well-known that $\mathcal{F}(V)$ naturally has the structure of a real-analytic manifold, and $\mathrm{PGL}(V)$ acts transitively and real-analytically on $\mathcal{F}(V)$. We say that a k-tuple of flags $(F_1, \ldots, F_k) \in \mathcal{F}(V)^k$ is *transverse* if $F_1^{(i_1)} + \cdots + F_k^{(i_k)} = V$ for all $i_1, \ldots, i_k \geq 0$ so that $i_1 + \cdots + i_k = n$. Let $\mathcal{F}(V)^{(k)}$ denote the space of transverse k-tuples of flags. Note that this is an open set in $\mathcal{F}(V)^k$, and so naturally inherits a real-analytic structure which descends to a real-analytic structure on $\mathcal{F}(V)^{(k)}/\mathrm{PGL}(V)$.

Remark 2.2. In the rest of this paper, we will repeatedly use the following two well-known observations.

(1) Let $(F, G), (F', G') \in \mathcal{F}(V)^{(2)}$ and let $p, p' \in \mathbb{P}(V)$ so that

$$p + F^{(i)} + G^{(n-i-1)} = V = p' + F'^{(i)} + G'^{(n-i-1)}$$

for all $i = 0, \ldots, n-1$. Then there is a unique projective transformation $g \in \mathrm{PGL}(V)$ so that $g \cdot F = F'$, $g \cdot G = G'$ and $g \cdot p = p'$.

(2) Let $(F, G), (F, G') \in \mathcal{F}(V)^{(2)}$. Then there is a unique unipotent projective transformation $u \in \mathrm{PGL}(V)$ so that $u \cdot F = F$ and $u \cdot G = G'$.

Definition 2.3. A continuous map $\xi : S^1 \to \mathcal{F}(V)$ is *Frenet* if for all $k = 1, \ldots, n$, the following hold:

- If $x_1, \ldots, x_k \in S^1$ is a pairwise distinct k-tuple, then $\xi(x_1), \ldots, \xi(x_k)$ is a transverse k-tuple of flags in $\mathcal{F}(V)$.

- Let $\{(x_{1,i}, \ldots, x_{k,i})\}_{i=1}^{\infty}$ be a sequence of pairwise distinct k-tuples in S^1, so that $\lim_{i \to \infty} x_{j,i} = x \in S^1$ for all $j = 1, \ldots, k$. For any $i_1, \ldots, i_k \in \mathbb{Z}^+$ so that $i_1 + \cdots + i_k = m \leq n$, we have

$$\lim_{i \to \infty} \xi(x_{1,i})^{(i_1)} + \cdots + \xi(x_{k,i})^{(i_k)} = \xi^{(m)}(x).$$

It is well-known that Γ is a hyperbolic group, and that its Gromov boundary $\partial \Gamma$ is topologically a circle. Hence, it makes sense to say when a continuous map $\xi : \partial \Gamma \to \mathcal{F}(V)$ is Frenet. Labourie proved that every PSL(V)-Hitchin representation preserves a unique ρ-equivariant Frenet curve $\xi_\rho : \partial \Gamma \to \mathcal{F}(V)$. Using this, he deduces that every PSL(V)-Hitchin representation is an irreducible quasi-isometric embedding (with respect to any word metric on Γ and any left-invariant Riemannian metric on PSL(V)). He also conjectured that for any representation $\rho : \Gamma \to \text{PSL}(V)$, the existence of such a ρ-equivariant Frenet curve implies that ρ is a PSL(V)-Hitchin representation. This was later proven by Guichard, thus establishing the following theorem.

Theorem 2.4. [Gui08, Theorem 1], [Lab06, Theorem 1.4] *A representation $\rho : \Gamma \to \text{PSL}(V)$ is a PSL(V)-Hitchin representation if and only if there exists a ρ-equivariant Frenet curve $\xi_\rho : \partial \Gamma \to \mathcal{F}(V)$. If such a ρ-equivariant Frenet curve exists, then it is unique. Furthermore, for any pair ρ, ρ' of PSL(V)-Hitchin representations, $\xi_\rho = \xi_{\rho'}$ if and only if $\rho = \rho'$.*

In particular, this establishes a natural bijection between $\text{Hit}_V(S)$ and the set of projective classes of Frenet curves $\xi : \partial \Gamma \to \mathcal{F}(V)$ that are ρ-equivariant for some representation $\rho : \Gamma \to \text{PSL}(V)$. At about the same time, Fock-Goncharov [FG06] also gave a more algebraic version of the criterion in Theorem 2.4, where they replace the Frenet property with the notion of a "positive map".

Since every PSL(V)-Hitchin representation is irreducible, we see that $\widetilde{\text{Hit}}_V(S)$ lies in the smooth locus of the algebraic variety $\text{Hom}(\Gamma, \text{PSL}(V))$. As such, $\widetilde{\text{Hit}}_V(S)$ is naturally equipped with a real-analytic structure, which descends to a real-analytic structure on $\text{Hit}_V(S)$. With this real-analytic structure, Hitchin's proof in fact shows that $\text{Hit}_V(S)$ is real-analytically diffeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$.

The description of Hitchin representations via Frenet curves also allows one to prove a striking positivity feature for Hitchin representations, which was first observed by Fock-Goncharov [FG06]. To describe this, we will define two well-known projective invariants associated to transverse pairs and triples of flags in $\mathcal{F}(V)$.

Definition 2.5.
- Let $E, F, G, H \in \mathcal{F}(V)$ be a quadruple of flags so that $(E, F, H), (E, G, H) \in \mathcal{F}(V)^{(3)}$. For any $i = 1, \ldots, n-1$, define the *cross ratio* by

$$C_i(E, F, G, H) := \frac{E^{(i)} \wedge H^{(n-i-1)} \wedge G^{(1)}}{E^{(i)} \wedge H^{(n-i-1)} \wedge F^{(1)}} \cdot \frac{E^{(i-1)} \wedge H^{(n-i)} \wedge F^{(1)}}{E^{(i-1)} \wedge H^{(n-i)} \wedge G^{(1)}}.$$

- Let $(F, G, H) \in \mathcal{F}(V)^{(3)}$ and let $i, j, k \in \mathbb{Z}^+$ so that $i + j + k = n$. Then define the *triple ratio*

$$T_{i,j,k}(F, G, H) := \frac{F^{(i+1)} \wedge G^{(j)} \wedge H^{(k-1)}}{F^{(i+1)} \wedge G^{(j-1)} \wedge H^{(k)}} \cdot \frac{F^{(i-1)} \wedge G^{(j+1)} \wedge H^{(k)}}{F^{(i)} \wedge G^{(j+1)} \wedge H^{(k-1)}} \cdot \frac{F^{(i)} \wedge G^{(j-1)} \wedge H^{(k+1)}}{F^{(i-1)} \wedge G^{(j)} \wedge H^{(k+1)}}.$$

We now explain the formulas used in Definition 2.5. Given any triple of flags F, G, H, choose three bases $\{f_1, \ldots, f_n\}$, $\{g_1, \ldots, g_n\}$, $\{h_1, \ldots, h_n\}$ of V so that

$F^{(l)} = \mathrm{Span}_{\mathbb{R}}(f_1, \ldots, f_l)$, $G^{(l)} = \mathrm{Span}_{\mathbb{R}}(g_1, \ldots, g_l)$ and $H^{(l)} = \mathrm{Span}_{\mathbb{R}}(h_1, \ldots, h_l)$ for all $l = 1, \ldots, n - 1$. For each $i, j, k \in \mathbb{Z}^+$ so that $i + j + k = n$, define

$$F^{(i)} \wedge G^{(j)} \wedge H^{(k)} := \det(f_1, \ldots, f_i, g_1, \ldots, g_j, h_1, \ldots, h_k),$$

and note that if $(F, G, H) \in \mathcal{F}(V)^{(3)}$, then $F^{(i)} \wedge G^{(j)} \wedge H^{(k)} \neq 0$. Of course, the definition of $F^{(i)} \wedge G^{(j)} \wedge H^{(k)}$ depends on the choice of the three bases, but one can easily verify that the cross ratio and triple ratio do not depend on these choices.

An important (but easily verified) property of the cross ratio and triple ratio is their projective invariance, i.e. $C_i(E, F, G, H) = C_i(g \cdot E, g \cdot F, g \cdot G, g \cdot H)$ and $T_{i,j,k}(F, G, H) = T_{i,j,k}(g \cdot F, g \cdot G, g \cdot H)$ for any $g \in \mathrm{PGL}(V)$. It is also easy to check the following symmetries from the definition:

- $C_i(E, F, G, H) = C_{n-i}(H, G, F, E) = \dfrac{1}{C_i(E, G, F, H)}$,
- $T_{i,j,k}(F, G, H) = T_{j,k,i}(G, H, F) = \dfrac{1}{T_{i,k,j}(F, H, G)}$.

By definition, $(\xi(x), \xi(y), \xi(z)) \in \mathcal{F}(V)^{(3)}$ for any triple $x, y, z \in S^1$ and any Frenet curve $\xi : S^1 \to \mathcal{F}(V)$. This allows us to evaluate the cross ratios and triple ratios along the image of ξ. The following theorem is essentially due to Fock-Goncharov [FG06] (also see [LM09, Appendix B] or [Zha15, Proposition 2.5.7]).

Theorem 2.6. *Let $\xi : S^1 \to \mathcal{F}(V)$ be Frenet.*

(1) *For any pairwise distinct triple of points $x, y, z \in S^1$, and for any $i, j, k \in \mathbb{Z}^+$ so that $i + j + k = n$, we have*

$$T_{i,j,k}\big(\xi(x), \xi(y), \xi(z)\big) > 0.$$

(2) *For any pairwise distinct quadruple of points $x, y, z, w \in S^1$ so that $x < y < z < w < x$ in the cyclic ordering on S^1, and for any $i = 1, \ldots, n - 1$, we have*

$$C_i\big(\xi(x), \xi(y), \xi(w), \xi(z)\big) < 0.$$

Another important feature of the triple ratio is that one can use them to parameterize the space of projective classes of triples of transverse flags. The following proposition was observed by Fock-Goncharov [FG06], and its proof is a straightforward computation which we omit.

Proposition 2.7. *The map $\mathcal{F}(V)^{(3)} / \mathrm{PGL}(V) \to (\mathbb{R} \setminus \{0\})^{\frac{(n-1)(n-2)}{2}}$ given by*

$$[F, G, H] \mapsto \big(T_{i,j,k}(F, G, H)\big)_{i,j,k}$$

is a real-analytic diffeomorphism.

2.2. **Ideal triangulations, barriers and bridges.** In this section, we introduce some terminology and notation regarding topological choices on S that we will make in this paper. While it is often convenient to choose a hyperbolic metric on S to realize these topological choices as geometric objects on S equipped with the chosen hyperbolic metric, it is important for our purposes that these are defined purely topologically.

For the rest of the article, if we choose a hyperbolic metric m on S, then we will use the notation $\Sigma := (S, m)$, while reserving the notation S to be used only to denote a topological surface. We will also abuse terminology by referring to Σ as a hyperbolic metric on S.

Let $\widetilde{S}$ denote the universal cover of S and $\pi : \widetilde{S} \to S$ the covering map. An *oriented geodesic in $\widetilde{S}$* is an ordered pair in $\mathcal{G}^o := \{(x,y) \in \partial\Gamma^2 : x \neq y\}$, and a *geodesic in $\widetilde{S}$* is an unordered pair in $\mathcal{G} := \mathcal{G}^o/\big((x,y) \sim (y,x)\big)$. We will use the set notation $\{x,y\}$ to denote the equivalence class in $\mathcal{G}$ containing (x,y). Two geodesics $\{x,y\}, \{x',y'\} \in \mathcal{G}$ are *transverse* if the two connected components of $\partial\Gamma \setminus \{x,y\}$ each contain a point in the set $\{x',y'\}$.

If we choose a hyperbolic metric Σ on S, then this induces a hyperbolic metric $\widetilde{\Sigma}$ on $\widetilde{S}$, and the visual boundary $\partial\widetilde{\Sigma}$ of $\widetilde{\Sigma}$ is canonically identified with $\partial\Gamma$. This in turn induces an identification of $\mathcal{G}^o$ and $\mathcal{G}$ respectively with the set of oriented hyperbolic geodesics in $\widetilde{\Sigma}$ and the set of hyperbolic geodesics in $\widetilde{\Sigma}$. More explicitly, $\{x,y\} \in \mathcal{G}$ is identified with the hyperbolic geodesic in $\widetilde{\Sigma}$ whose endpoints in $\partial\widetilde{\Sigma} = \partial\Gamma$ are x and y, and $(x,y) \in \mathcal{G}^o$ is identified with the oriented hyperbolic geodesic in $\widetilde{\Sigma}$ whose backward and forward endpoints are x and y respectively.

An *ideal triangulation of $\widetilde{S}$* is a maximal Γ-invariant subset $\widetilde{\mathcal{T}} \subset \mathcal{G}$ so that any two geodesics $\{x,y\}, \{x',y'\} \in \widetilde{\mathcal{T}}$ are not transverse, and for any geodesic $\{x,y\} \in \widetilde{\mathcal{T}}$, one of the following must hold:

(1) There is some $z \in \partial\Gamma$ so that $\{x,z\}$ and $\{y,z\}$ are geodesics in $\widetilde{\mathcal{T}}$,
(2) There is some group element $\gamma \in \Gamma$ so that $\gamma \cdot x = x$ and $\gamma \cdot y = y$.

The elements in $\widetilde{\mathcal{T}}$ are *edges*. If (1) holds, then $\{x,y\}$ is an *isolated edge* of $\widetilde{\mathcal{T}}$, and if (2) holds, then $\{x,y\}$ is a *closed edge* of $\widetilde{\mathcal{T}}$. Let $\widetilde{\mathcal{Q}} = \widetilde{\mathcal{Q}}_{\widetilde{\mathcal{T}}}$ denote the set of isolated edges in $\widetilde{\mathcal{T}}$ and let $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}_{\widetilde{\mathcal{T}}}$ denote the set of closed edges in $\widetilde{\mathcal{T}}$. For any $\{x,y\} \in \widetilde{\mathcal{T}}$, x and y are the *vertices* of the edge $\{x,y\}$. Also, any triple of $\{x,y,z\}$ of points in $\partial\Gamma$ so that $\{x,y\}, \{y,z\}, \{z,x\} \in \widetilde{\mathcal{T}}$ is an *ideal triangle* of $\widetilde{\mathcal{T}}$. Denote the set of ideal triangles of $\widetilde{\mathcal{T}}$ by $\widetilde{\Theta}_{\widetilde{\mathcal{T}}} = \widetilde{\Theta}$. If $e_1, e_2, e_3 \in \widetilde{\mathcal{T}}$ are the edges of an ideal triangle $T \in \widetilde{\Theta}$, we will also use the notation $T = \{e_1, e_2, e_3\}$.

If $\widetilde{\mathcal{T}}$ is an ideal triangulation of $\widetilde{S}$, then the quotient $\mathcal{T} := \widetilde{\mathcal{T}}/\Gamma$ is an *ideal triangulation of S*. Similarly, the elements in $\mathcal{T}$, $\mathcal{Q} = \mathcal{Q}_{\mathcal{T}} := \widetilde{\mathcal{Q}}/\Gamma$ and $\mathcal{P} = \mathcal{P}_{\mathcal{T}} := \widetilde{\mathcal{P}}/\Gamma$ are called *edges*, *isolated edges*, and *closed edges* respectively. The quotient $\Theta = \Theta_{\mathcal{T}} := \widetilde{\Theta}/\Gamma$ is the set of *ideal triangles* of $\mathcal{T}$.

By choosing a hyperbolic metric Σ on S, $\mathcal{T}$ is realized as an ideal triangulation of the hyperbolic surface Σ in the classical sense. Hence, $\widetilde{\mathcal{T}}$ is realized as a Γ-invariant lift of $\mathcal{T}$ to $\widetilde{\Sigma}$. This implies that $|\Theta| = 4g - 4$, $|\mathcal{Q}| = 9g - 9$, and $1 \leq |\mathcal{P}| \leq 3g - 3$, where g is the genus of S.

Let $\widetilde{\mathcal{T}}^o := \{(x,y) \in \mathcal{G}^o : \{x,y\} \in \widetilde{\mathcal{T}}\}$, and note that $\widetilde{\mathcal{T}}^o$ admits a natural Γ-action given by $\gamma \cdot (x,y) = (\gamma \cdot x, \gamma \cdot y)$ for any $\gamma \in \Gamma$ and any $(x,y) \in \widetilde{\mathcal{T}}^o$. Also, observe that there is an obvious two-to-one, Γ-equivariant map $\widetilde{\Pi} : \widetilde{\mathcal{T}}^o \to \widetilde{\mathcal{T}}$ given by $\widetilde{\Pi}(x,y) = \{x,y\}$. Let $\mathcal{T}^o := \widetilde{\mathcal{T}}^o/\Gamma$, and let $\Pi : \mathcal{T}^o \to \mathcal{T}$ denote the quotient map induced by $\widetilde{\Pi}$. We will also denote $\widetilde{\mathcal{P}}^o := \{(x,y) \in \mathcal{G}^o : \{x,y\} \in \widetilde{\mathcal{P}}\}$, $\widetilde{\mathcal{Q}}^o := \{(x,y) \in \mathcal{G}^o : \{x,y\} \in \widetilde{\mathcal{Q}}\}$, $\mathcal{P}^o := \widetilde{\mathcal{P}}^o/\Gamma$ and $\mathcal{Q}^o := \widetilde{\mathcal{Q}}^o/\Gamma$. An *orientation* on $\widetilde{\mathcal{T}}$ is a Γ-equivariant map $\widetilde{r} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}^o$ so that $\widetilde{\Pi} \circ \widetilde{r} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$ is the identity. This descends to a map $r : \mathcal{T} \to \mathcal{T}^o$, which is called an *orientation on $\mathcal{T}$*.

Given an ideal triangulation $\mathcal{T}$ of S, let

$$\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_{\widetilde{\mathcal{T}}} := \{(v, T) : T \in \widetilde{\Theta}, v \text{ a vertex of } T\}.$$

This set also admits an obvious Γ action, so we can consider $\mathcal{M} = \mathcal{M}_{\mathcal{T}} := \Gamma \backslash \widetilde{\mathcal{M}}$.
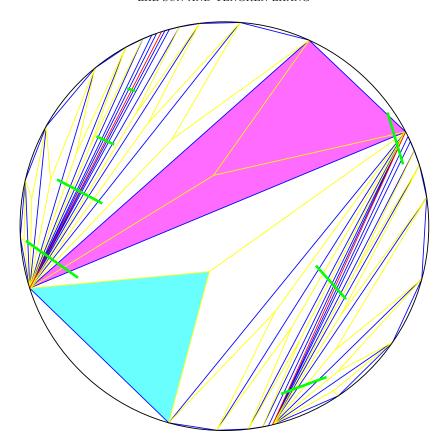
FIGURE 1. This is a drawing of $\widetilde{\mathcal{B}}$ and a compatible $\widetilde{\mathcal{J}}$ in the Klein model of $\widetilde{\Sigma}$. The closed edges in $\widetilde{\mathcal{P}}$ are drawn in red, the isolated edges in $\widetilde{\mathcal{Q}}$ are drawn in blue, and the non-edge barriers in $\widetilde{\mathcal{M}}$ are drawn in yellow. The bridges in $\widetilde{\mathcal{J}}$ are drawn in green. One ideal triangle in $\widetilde{\Theta}$ is drawn in violet, and one fragmented ideal triangle is drawn in turquoise.

Definition 2.8.

(1) Let $\widetilde{\mathcal{T}}$ be an ideal triangulation of $\widetilde{S}$. The *barrier system* associated to $\widetilde{\mathcal{T}}$ is $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}_{\widetilde{\mathcal{T}}} := \widetilde{\mathcal{T}} \cup \widetilde{\mathcal{M}}$. If $\widetilde{r}$ is an orientation on $\widetilde{\mathcal{T}}$, the pair $(\widetilde{r}, \widetilde{\mathcal{B}})$ is the *oriented barrier system* associated to $(\widetilde{r}, \widetilde{\mathcal{T}})$. Similarly, denote $\mathcal{B} = \mathcal{B}_{\mathcal{T}} := \widetilde{\mathcal{B}}/\Gamma$, and call $\mathcal{B}$ and $(r, \mathcal{B})$ the *barrier system* and *oriented barrier system* associated to $\mathcal{T}$ and $(r, \mathcal{T})$ respectively.

(2) The elements in $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are called *barriers*, and the elements in $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are called *non-edge barriers*.

(3) For any closed edge $\{x, y\} \in \widetilde{\mathcal{P}}$, let $\partial\Gamma_1$ and $\partial\Gamma_2$ be the connected components of $\partial\Gamma \setminus \{x, y\}$. A *bridge* across $\{x, y\}$ is an unordered pair $\{T_1, T_2\}$ so that for all $m = 1, 2$,

  - $T_m \in \widetilde{\Theta}$
  - one of the vertices of T_m is either x or y,

- the other two vertices of T_m lie in $\partial \Gamma_m$.
(4) A *bridge system compatible with* $\widetilde{\mathcal{T}}$ is a minimal Γ-invariant collection of bridges $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_{\widetilde{\mathcal{T}}}$ so that for every closed edge $e \in \widetilde{\mathcal{P}}$, there is a bridge $\{T_1, T_2\} \in \widetilde{\mathcal{J}}$ across e. If $\widetilde{\mathcal{J}}$ is a bridge system compatible with $\widetilde{\mathcal{T}}$, then $\mathcal{J} = \mathcal{J}_{\mathcal{T}} := \widetilde{\mathcal{J}}/\Gamma$ is a *bridge system compatible with* $\mathcal{T}$.

Observe that while the barrier system associated to $\widetilde{\mathcal{T}}$ (resp. $\mathcal{T}$) is determined by $\widetilde{\mathcal{T}}$ (resp. $\mathcal{T}$), this is not true for bridge systems compatible with $\widetilde{\mathcal{T}}$ (resp. $\mathcal{T}$).

If we choose a hyperbolic metric Σ on S, then each ideal triangle $\widehat{T} \in \Theta$ is realized as a hyperbolic ideal triangle in Σ and each $T \in \widetilde{\Theta}$ is realized as a hyperbolic ideal triangle in $\widetilde{\Sigma}$. For each hyperbolic ideal triangle $T \in \widetilde{\Theta}$, let p_T be the baricenter of T. Observe that this allows us to realize each non-edge barrier $(v, T) \in \widetilde{\mathcal{M}}$ as a geodesic ray in $\widetilde{\Sigma}$ from p_T to v (see Figure 1). Any non-edge barrier $[v, T] \in \mathcal{M}$ is then realized as a geodesic ray with source $p_{\widehat{T}}$, the baricenter of $\widehat{T}$, that "spirals" towards the closed geodesic on Σ that corresponds to some $\gamma \in \Gamma$ that has v as a fixed point in $\partial \Gamma$.

Note that $\widetilde{\Sigma} \setminus \widetilde{\mathcal{B}}$ is a union of triangles, each of which has two vertices in $\partial \widetilde{\Sigma}$ and a vertex in $\widetilde{\Sigma}$. We call any such triangle a *fragmented ideal triangle* (see Figure 1). Any fragmented ideal triangle T' lies in a unique ideal triangle T in $\widetilde{\Theta}$, and T', T have two vertices in common. Similarly, a connected component of $\Sigma \setminus \mathcal{B}$ is also called a *fragmented ideal triangle*.
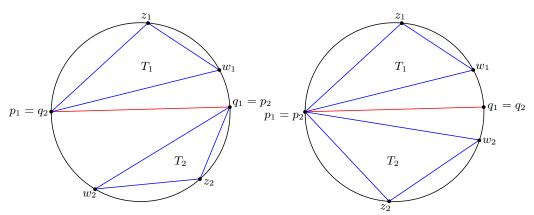
FIGURE 2. Notation 2.9

Notation 2.9. For each bridge $J = \{T_1, T_2\}$ across $\{x, y\} \in \widetilde{\mathcal{P}}$, let p_m denote the point in $\{x, y\}$ that is a vertex of T_m for $m = 1, 2$, and let q_m denote the point in $\{x, y\}$ that is not a vertex of T_m. Also, let z_m and w_m be the vertices of T_m that are not p_m, so that $\{w_m, p_m\}$ intersects $\{z_m, q_m\}$ transversely (see Figure 2).

With our choice of a hyperbolic metric Σ on S, the bridge $\{T_1, T_2\}$ can be realized as an unoriented geodesic segment in $\widetilde{\Sigma}$ with one endpoint in the fragmented ideal triangle $T_1' \subset \widetilde{\Sigma}$ with vertices p_{T_1}, p_1, w_1 and the other endpoint in the fragmented ideal triangle $T_2' \subset \widetilde{\Sigma}$ with vertices p_{T_2}, p_2, w_2 (see Figure 1).

We can define orientations on a bridge system in the same way we defined orientations on ideal triangulations. Namely, let $\widetilde{\mathcal{J}}^o := \{(T_1, T_2) : \{T_1, T_2\} \in \widetilde{\mathcal{J}}\}$, then

Γ acts diagonally on $\widetilde{\mathcal{J}}^o$, and the map $\widetilde{\Pi} : \widetilde{\mathcal{J}}^o \to \widetilde{\mathcal{J}}$ given by $\widetilde{\Pi}(T_1, T_2) = \{T_1, T_2\}$ is a Γ-equivariant two-to-one projection. Hence, if we define $\mathcal{J}^o := \Gamma \backslash \widetilde{\mathcal{J}}^o$, then $\widetilde{\Pi}$ descends to $\Pi : \mathcal{J}^o \to \mathcal{J}$.

Definition 2.10. An *orientation* on a bridge system $\widetilde{\mathcal{J}}$ is a Γ-equivariant map $\widetilde{s} : \widetilde{\mathcal{J}} \to \widetilde{\mathcal{J}}^o$ so that $\widetilde{\Pi} \circ \widetilde{s} : \widetilde{\mathcal{J}} \to \widetilde{\mathcal{J}}$ is the identity. The orientation $\widetilde{s} : \widetilde{\mathcal{J}} \to \widetilde{\mathcal{J}}^o$ descends to a map $s : \mathcal{J} \to \mathcal{J}^o$, which is called an *orientation on $\mathcal{J}$*.

2.3. **Algebraic intersection number.** Choose an oriented barrier system $(r, \mathcal{B})$ on S, a compatible bridge system $\mathcal{J}$, and a hyperbolic metric Σ on S. The choice of Σ realizes the oriented edges in $\widetilde{\mathcal{T}}^o$, non-edge barriers in $\widetilde{\mathcal{M}}$ and oriented bridges in $\widetilde{\mathcal{J}}^o$ as oriented geodesics, geodesic rays, and oriented geodesic segments in $\widetilde{\Sigma}$. Using this, we define the notion of an algebraic intersection between 1-simplices $g : [0, 1] \to \widetilde{\Sigma}$ and elements $X \in \widetilde{\mathcal{J}}^o \cup \widetilde{\mathcal{T}}^o \cup \widetilde{\mathcal{M}}$ that satisfy $\left| g[0, 1] \cap X \right| < \infty$.

Let $I \subset g[0, 1]$ be a closed subinterval whose interior is a component of

$$g[0, 1] \setminus \bigcup_{X \in \widetilde{\mathcal{J}}^o \cup \widetilde{\mathcal{T}}^o \cup \widetilde{\mathcal{M}}} X.$$

Denote the backward and forward endpoints of I by I^- and I^+ respectively. For any $p \in \{I^-, I^+\}$, we say that I lies *to the right (resp. left)* of X *at* p if $p \in X$, and there is some closed subinterval $I' \subset I$ containing p so that I' lies to the right (resp. left) of the oriented (bi-infinite) geodesic containing X. If I^- is not the source of a non-edge barrier, define

$$\epsilon(I^-, X) := \begin{cases} \dfrac{1}{2} & \text{if } I \text{ lies to the right of } X \text{ at } I^-; \\ -\dfrac{1}{2} & \text{if } I \text{ lies to the left of } X \text{ at } I^-; \\ 0 & \text{if } I^- \notin X. \end{cases}$$

Similarly, if I^+ is not the source of a non-edge barrier, define

$$\epsilon(I^+, X) := \begin{cases} \dfrac{1}{2} & \text{if } I \text{ lies to the left of } X \text{ at } I^+; \\ -\dfrac{1}{2} & \text{if } I \text{ lies to the right of } X \text{ at } I^+; \\ 0 & \text{if } I^+ \notin X. \end{cases}$$

On the other hand, suppose that $p \in \{I^-, I^+\}$ is the source of a non-edge barrier. Let $X_1, X_2, X_3 \in \widetilde{\mathcal{M}}$ be the three non-edge barriers whose source is p, so that the endpoint v_i of X_i in $\partial\Gamma$ satisfy $v_1 < v_2 < v_3 < v_1$. Let $U_{1,p}, U_{2,p}, U_{3,p}$ be the three connected components of $\widetilde{\Sigma} \setminus (X_1 \cup X_2 \cup X_3)$ so that $X_{i-1}, X_{i+1} \subset \overline{U_{i,p}}$. Here, the arithmetic in the subscripts is done modulo 3. Then define

$$\epsilon(I^-, X) := \begin{cases} \dfrac{1}{3} & \text{if } X = X_i \text{ and } I \subset U_{i-1, I^-}; \\ -\dfrac{1}{3} & \text{if } X = X_i \text{ and } I \subset U_{i+1, I^-}; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\epsilon(I^+, X) := \begin{cases} \dfrac{1}{3} & \text{if } X = X_i \text{ and } I \subset U_{i+1, I^+}; \\ -\dfrac{1}{3} & \text{if } X = X_i \text{ and } I \subset U_{i-1, I^+}; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.11. Let $g : [0,1] \to \widetilde{\Sigma}$ be a 1-simplex and $X \in \widetilde{\mathcal{J}}^o \cup \widetilde{\mathcal{T}}^o \cup \widetilde{\mathcal{M}}$ satisfying $\left| g[0,1] \cap X \right| < \infty$. Let $I_1, \ldots, I_k$ be the closed subintervals of $g[0,1]$ whose interiors are the connected components of $g[0,1] \setminus X$. The *algebraic intersection number* (see Figure 3) between any g and X is defined to be

$$\hat{i}(g,X) = \hat{i}_\Sigma(g,X) := \sum_{l=1}^{k} \epsilon(I_k^-, X) + \epsilon(I_k^+, X).$$
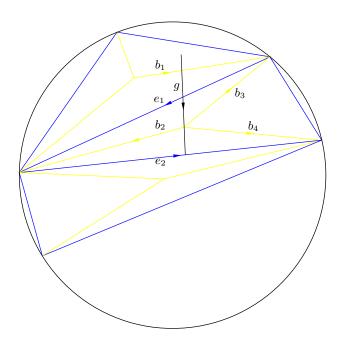
FIGURE 3. $\hat{i}(g,b_1) = 1$, $\hat{i}(g,b_2) = -\frac{2}{3}$, $\hat{i}(g,b_3) = \frac{1}{3}$, $\hat{i}(g,b_4) = \frac{1}{3}$, $\hat{i}(g,e_1) = -1$, $\hat{i}(g,e_2) = \frac{1}{2}$.

Let $\pi : \widetilde{S} \to S$ be the covering map. Observe that when $g : [0,1] \to \widetilde{S}$ is a 1-simplex so that $\pi \circ g : [0,1] \to S$ is a closed curve and $e \in \widetilde{\mathcal{P}}^o$, $\hat{i}(g,e)$ is the usual algebraic intersection number on $H_1(S, \mathbb{Z})$ between the homology classes $[\pi \circ g]$ and $[\pi \circ e]$, which is in particular independent of Σ. However, $\hat{i}$ in general depends on the choice of Σ.

2.4. **The tangent space to** $\mathrm{Hit}_V(S)$ **and the Goldman symplectic form.** Recall that the Lie algebra $\mathfrak{sl}(V)$ of $\mathrm{PSL}(V)$ is naturally identified with the traceless endomorphisms of V. Hence, one can define the *trace form*

$$B : \mathfrak{sl}(V) \times \mathfrak{sl}(V) \to \mathbb{R}$$

by $B(X,Y) := \mathrm{tr}(XY)$. This pairing is well-known to be bilinear, symmetric and non-degenerate (it is a multiple of the Killing form on $\mathfrak{sl}(V)$).

Using the trace form, we can apply a general construction by Goldman [Gol84] to define a symplectic form on $\mathrm{Hit}_V(S)$, which we will refer to as the *Goldman symplectic form*. From the work of Atiyah-Bott [AB83], one also has a description of this symplectic form using the language of flat bundles over S. In the case when

$\dim(V) = 2$, Goldman [Gol84, Proposition 2.5] showed that this symplectic form agrees with the Weil-Petersson symplectic form on the Teichmüller space of S.

We will now recall the construction of the Goldman symplectic form. First, Goldman observed that for any $[\rho] \in \mathrm{Hit}_V(S)$ and any representative ρ of $[\rho]$, one has the following canonical isomorphisms:

$$T_{[\rho]} \mathrm{Hit}_V(S) \simeq H^1(\Gamma, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}) \simeq H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$$

The first linear isomorphism $T_{[\rho]} \mathrm{Hit}_V(S) \simeq H^1(\Gamma, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ can be described explicitly in the following way. Let $t \mapsto [\rho_t]$ be a smooth curve in $\mathrm{Hit}_V(S)$ so that $[\rho_0] = [\rho]$, and denote its tangent vector at $[\rho_0]$ by $X \in T_{[\rho_0]} \mathrm{Hit}_V(S)$. Choose any smooth curve $t \mapsto \rho_t$ in $\widetilde{\mathrm{Hit}}_V(S)$ so that $\Pi(\rho_t) = [\rho_t]$, where $\Pi : \widetilde{\mathrm{Hit}}_V(S) \to \mathrm{Hit}_V(S)$ is the quotient map. For any $\gamma \in \Gamma$, let $t \mapsto g_{t,\gamma}$ be the smooth curve in $\mathrm{PSL}(V)$ defined by $\rho_t(\gamma) = g_{t,\gamma} \cdot \rho_0(\gamma)$. The fact that ρ_t is a homomorphism for all t then implies that the map $\mu_X : \Gamma \to \mathfrak{sl}(V)$ defined by $\mu_X : \gamma \mapsto \frac{d}{dt}|_{t=0} g_{t,\gamma}$ is a group cocycle, and thus defines a cohomology class $[\mu_X] \in H^1(\Gamma, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$. Goldman [Gol84, Section 1.4] then showed that the map $T_{[\rho]} \mathrm{Hit}_V(S) \to H^1(\Gamma, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ given by $X \mapsto [\mu_X]$ is a well-defined linear isomorphism.

The second linear isomorphism $H^1(\Gamma, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}) \simeq H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ can also be described explicitly. Let $\mu \in C^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ be a cocycle in the cohomology class $[\mu] \in H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$, and let $\widetilde{\mu} : C_1(\widetilde{S}, \mathbb{Z}) \to \mathfrak{sl}(V)$ be its $\mathrm{Ad} \circ \rho$-equivariant lift. Pick any point $p \in \widetilde{S}$. For any $\gamma \in \Gamma$, let $h_\gamma : [0,1] \to \widetilde{S}$ be a 1-simplex so that $h_\gamma(0) = p$ and $h_\gamma(1) = \gamma \cdot p$. Then define $\bar{\mu} : \Gamma \to \mathfrak{sl}(V)$ by $\bar{\mu} : \gamma \mapsto \mu(h_\gamma)$. One can verify that $\bar{\mu}$ is a group cocycle, and that the map $H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}) \to H^1(\Gamma, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ given by $[\mu] \mapsto [\bar{\mu}]$ is a well-defined linear isomorphism.

With these identifications, we can define the Goldman symplectic form.

Definition 2.12. The *Goldman symplectic form* is the pairing

$$\omega : H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}) \times H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}) \to H^2(S, \mathbb{R}) \to \mathbb{R}$$

where the first map is the cup product on cohomology while pairing the $\mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}$ coeffecients using the trace form, and the second map is taking the cap product with the fundamental class $[S] \in H_2(S, \mathbb{R})$.

Goldman [Gol84, Theorem 1.7] proved that this bilinear pairing defines a symplectic form on $\mathrm{Hit}_V(S)$.

The Goldman symplectic form can be described more explicitly if one makes some topological choices. Choose a finite triangulation $\mathbb{T}$ (NOT an ideal triangulation described in Section 2.2) of S so that the three vertices of every triangle given by $\mathbb{T}$ are pairwise distinct. Enumerate the vertices of $\mathbb{T}$ by $v_1, \ldots, v_k$. Then for any triangle $\delta \in \mathbb{T}$ with vertices v_a, v_b, v_c so that $a < b < c$, let $e_{\delta,1}$ denote the edge of δ with endpoints v_a, v_b, oriented from v_a to v_b. Similarly, let $e_{\delta,2}$ denote the edge of δ with endpoints v_b, v_c, oriented from v_b to v_c. Let $\widetilde{\delta}$ denote a triangle in $\widetilde{S}$ so that $\pi(\widetilde{\delta}) = \delta$ (recall that $\pi : \widetilde{S} \to S$ is the covering map), and for $m = 1, 2$, let $e_{\widetilde{\delta},m}$ denote the oriented boundary edge of $\widetilde{\delta}$ so that $\pi(e_{\widetilde{\delta},m}) = e_{\delta,m}$.

If $\mu_1, \mu_2 \in C^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ are cocycles with $\mathrm{Ad} \circ \rho$-equivariant lifts $\widetilde{\mu}_1, \widetilde{\mu}_2 : C_1(\widetilde{S}, \mathbb{Z}) \to \mathfrak{sl}(V)$, then the cup product formula in simplicial cohomology allows us to write the Goldman symplectic form as

$$(2.1) \qquad \omega\big([\mu_1], [\mu_2]\big) = \sum_{\delta \in \mathbb{T}} \mathrm{sgn}(\delta) \cdot \mathrm{tr}\big(\widetilde{\mu}_1(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_2(e_{\widetilde{\delta},2})\big).$$

Here, $\mathrm{sgn}(\delta)$ is 1 if the enumeration of the vertices of δ is increasing in the clockwise order, and is -1 otherwise. It is a standard exercise in algebraic topology that this formula does not depend on any of the choices made.

2.5. **The tangent cocycle.** The formula for the Goldman symplectic form given by (2.1) shows that if we can find explicit $\mathrm{Ad}\circ\rho$-equivariant cocycles $\widetilde{\mu}_1, \widetilde{\mu}_2 :$ $C_1(\widetilde{S}, \mathbb{Z}) \to \mathfrak{sl}(V)$ representing the cohomology classes $[\mu_1], [\mu_2] \in H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad}\circ\rho})$, then we can explicitly compute the Goldman symplectic form between the tangent vectors in $T_{[\rho]} \mathrm{Hit}_V(S)$ corresponding to $[\mu_1]$ and $[\mu_2]$. It turns out that one way to systematically find these cocycle representatives is to use Theorem 2.4 to view the cohomology classes in $H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad}\circ\rho})$ as describing infinitesimal deformations of Frenet curves instead of representations. This gives rise to the notion of the tangent cocycle which we will now define.

Choose an oriented ideal triangulation $\widetilde{\mathcal{T}}$ on $\widetilde{S}$, let $\widetilde{\mathcal{B}}$ be the associated barrier system, and choose a compatible bridge system $\widetilde{\mathcal{J}}$. Also, choose a hyperbolic metric Σ on S. This realizes the elements in $\widetilde{\mathcal{B}}$ as geodesics (edges in $\widetilde{\mathcal{T}}$) or geodesic rays (non-edge barriers in $\widetilde{\mathcal{M}}$) in $\widetilde{\Sigma}$, and also realizes the elements in $\widetilde{\mathcal{J}}$ as geodesic segments that are transverse to the closed edges in $\widetilde{\mathcal{P}}$ (see Section 2.2). Given $\rho \in \widetilde{\mathrm{Hit}}_V(S)$ and a cohomology class $[\nu] \in H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad}\circ\rho})$, we will use these choices to define a particular cocycle representative $\mu_{\rho,[\nu]} \in [\nu]$.

It is sufficient to define $\widetilde{\mu}_{\rho,[\nu]}$ on 1-simplices in $\widetilde{S}$. We will do so in four steps. In the first step, we define $\widetilde{\mu}_{\rho,[\nu]}(h)$ when h is a 1-simplex with endpoints that do not lie in any of the barriers in $\widetilde{\mathcal{B}}$. In the second step, we extend the definition of $\widetilde{\mu}_{\rho,[\nu]}(h)$ to the case when h has some endpoint in an isolated edge in $\widetilde{\mathcal{Q}}$ or in the interior of a non-edge barrier in $\widetilde{\mathcal{M}}$. Then in the third step, we further extend $\widetilde{\mu}_{\rho,[\nu]}(h)$ to the case when h has some endpoint as the source of a non-edge barrier. Finally, in the fourth step we extend the definition of $\widetilde{\mu}_{\rho,[\nu]}(h)$ to the case when h has an endpoint in a closed edge.

Choose a smooth path $[\rho_t] \in \mathrm{Hit}_V(S)$ so that $[\rho_0] = [\rho]$ and

$$\left. \frac{d}{dt} \right|_{t=0} [\rho_t] = [\nu] \in T_{[\rho]} \mathrm{Hit}_V(S) = H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad}\circ\rho}).$$

It will be clear from our definition that $\widetilde{\mu}_{\rho,[\nu]}$ will not depend on the choice of $[\rho_t]$, but only on the tangent vector $[\nu]$ to $[\rho_t]$ at $t = 0$.

Step 1: Suppose that $h : [0, 1] \to \widetilde{\Sigma}$ is a 1-simplex with the property that the endpoints of h do not lie in any of the geodesics or geodesic rays in $\widetilde{\mathcal{B}}$. Observe that for such h, there are unique fragmented ideal triangles $T'_{h,0}$ and $T'_{h,1}$ that contain $h(0)$ and $h(1)$ respectively. For $m = 0, 1$, let $T_{h,m} \in \widetilde{\Theta}$ be the ideal triangle that contains $T'_{h,m}$, and let $x_{h,m}, y_{h,m}, z_{h,m}$ be the vertices of $T_{h,m}$ so that $x_{h,m} < y_{h,m} < z_{h,m} < x_{h,m}$ along $\partial\Gamma$, and $z_{h,m}$ is not a vertex of $T'_{h,m}$ (see Figure 4).

Let $\rho_t \in \widetilde{\mathrm{Hit}}_V(S)$ be the smooth lift of $[\rho_t]$ so that $\rho_0 = \rho$, and the ρ_t-equivariant Frenet curve $\xi_t : \partial\Gamma \to \mathcal{F}(V)$ guaranteed by Theorem 2.4 satisfy $\xi_t(x_{h,0}) = \xi_0(x_{h,0})$, $\xi_t(y_{h,0}) = \xi_0(y_{h,0})$, and $\xi_t^{(1)}(z_{h,0}) = \xi_0^{(1)}(z_{h,0})$. Such a lift exists uniquely by Remark 2.2(1). Again by Remark 2.2(1), there is a unique projective transformation $g_{t,h} \in \mathrm{PSL}(V)$ so that

$$g_{t,h} \cdot \xi_0(x_{h,1}) = \xi_t(x_{h,1}), \quad g_{t,h} \cdot \xi_0(y_{h,1}) = \xi_t(y_{h,1}), \quad g_{t,h} \cdot \xi_0^{(1)}(z_{h,1}) = \xi_t^{(1)}(z_{h,1}).$$
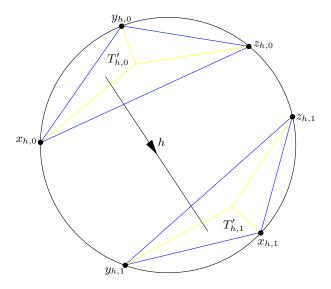
FIGURE 4. 1-simplices considered in Step 1.

Observe that $t \mapsto g_{t,h}$ is a smooth path in $\mathrm{PSL}(V)$ with $g_{0,h} = \mathrm{id}$. We define

$$\widetilde{\mu}_{\rho,[\nu]}(h) := \left.\frac{d}{dt}\right|_{t=0} g_{t,h}.$$
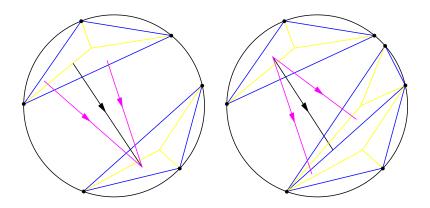
FIGURE 5. In Step 2, $\widetilde{\mu}_{\rho,[\nu]}$ evaluated on the black 1-simplex is the average of $\widetilde{\mu}_{\rho,[\nu]}$ evaluated on the violet 1-simplices, which was defined in Step 1.

Step 2: Suppose now that $h : [0,1] \to \widetilde{\Sigma}$ is a 1-simplex so that $h(m)$ lies in the interior of an oriented geodesic (isolated edge) or geodesic ray (bridge) $b \in \widetilde{\mathcal{M}} \cup \widetilde{\mathcal{Q}}$ for some $m = 0, 1$. Let $h_1, h_2 : [0,1] \to \widetilde{\Sigma}$ be 1-simplices so that (see Figure 5)

- $h_1(1-m) = h_2(1-m) = h(1-m)$,
- $h_1(m)$ and $h_2(m)$ lie in the two fragmented ideal triangles that contain $h(m)$ in their boundaries

and define $\widetilde{\mu}_{\rho,[\nu]}(h) := \frac{1}{2}\left(\widetilde{\mu}_{\rho,[\nu]}(h_1) + \widetilde{\mu}_{\rho,[\nu]}(h_2)\right)$. Note that $\widetilde{\mu}_{\rho,[\nu]}(h)$ does not depend on the choice of h_1 or h_2. This, together with Step 1, iteratively defines $\widetilde{\mu}_{\rho,[\nu]}(h)$ for all 1-simplices $h : [0,1] \to \widetilde{\Sigma}$ where neither $h(0)$ nor $h(1)$ is the source of a non-edge barrier in $\widetilde{\mathcal{M}}$, or lies in a closed edge in $\widetilde{\mathcal{P}}$.

Step 3: Next, consider the case where $h : [0,1] \to \widetilde{\Sigma}$ is a 1-simplex so that $h(m)$ is the source of a non-edge barrier in $\widetilde{\mathcal{M}}$ for some $m = 0, 1$. Let $b_1 = (v_1, T)$, $b_2 = (v_2, T)$, $b_3 = (v_3, T)$ be the three non-edge barriers whose source is $h(m)$ so that $v_1 < v_2 < v_3 < v_1$ in $\partial\Gamma$. Also, let $U_{1,h(m)}, U_{2,h(m)}, U_{3,h(m)}$ be the three connected components of $\widetilde{\Sigma} \setminus (b_1 \cup b_2 \cup b_3)$ so that the closure $\overline{U_{i,h(m)}}$ of $U_{i,h(m)}$ in $\widetilde{\Sigma}$ contains both b_{i-1} and b_{i+1}.

If $h(0) = h(1)$, then define $\widetilde{\mu}_{\rho,[\nu]}(h) := 0$. If $h(1-m)$ lies in $U_{i,h(m)}$, let $\bar{h}_m$ be a 1-simplex so that $\bar{h}_m(1-m) = h(1-m)$ and $\bar{h}_m(m)$ lies in the fragmented ideal triangle $U_{i,h(m)} \cap T$. Also, for $i,j = 1,2,3$, let $l_{i,j,h(m)} : [0,1] \to \widetilde{S}$ be the 1-simplex with $l_{i,j,h(m)}(0) \in U_{i,h(m)} \cap T$ and $l_{i,j,h(m)}(1) \in U_{j,h(m)} \cap T$ (see Figure 6). If $m = 0$ and $h(1)$ lies in $U_{i,h(0)}$, define

$$\widetilde{\mu}_{\rho,[\nu]}(h) := \widetilde{\mu}_{\rho,[\nu]}(\bar{h}_0) + \frac{1}{3}\left(\widetilde{\mu}_{\rho,[\nu]}(l_{i-1,i,h(0)}) + \widetilde{\mu}_{\rho,[\nu]}(l_{i+1,i,h(0)})\right),$$

and if $m = 1$ and $h(0)$ lies in $U_{i,h(1)}$, define

$$\widetilde{\mu}_{\rho,[\nu]}(h) := \widetilde{\mu}_{\rho,[\nu]}(\bar{h}_1) + \frac{1}{3}\left(\widetilde{\mu}_{\rho,[\nu]}(l_{i,i-1,h(1)}) + \widetilde{\mu}_{\rho,[\nu]}(l_{i,i+1,h(1)})\right),$$

Here, arithmetic in the subscripts is done modulo 3. This, together with Step 1 and Step 2, iteratively defines $\widetilde{\mu}_{\rho,[\nu]}(h)$ for all 1-simplices $h : [0,1] \to \widetilde{\Sigma}$ whose endpoints do not lie in a closed edge in $\widetilde{\mathcal{P}}$.
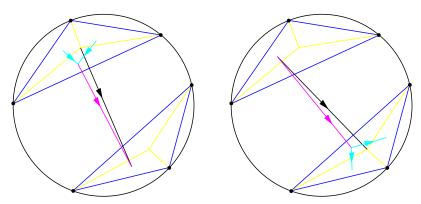

FIGURE 6. In Step 3, $\widetilde{\mu}_{\rho,[\nu]}$ evaluated on the black 1-simplex is the sum of $\widetilde{\mu}_{\rho,[\nu]}$ evaluated on the violet 1-simplex, and a third of the sum of $\widetilde{\mu}_{\rho,[\nu]}$ evaluated on the turquoise 1-simplices.

Step 4: We will now consider the case where $h : [0,1] \to \widetilde{\Sigma}$ is a 1-simplex so that $h(m)$ lies in a closed edge $e = \{x, y\} \in \widetilde{\mathcal{P}}$ for some $m = 0, 1$. The strategy to define $\widetilde{\mu}_{\rho,[\nu]}(h)$ in this case is to first define $\widetilde{\mu}_{\rho,[\nu]}(h)$ when $h[0,1]$ lies entirely in e or entirely in a bridge across e. These are defined in Step 4.1 and Step 4.2 below respectively. Then using this and Step 3, we can extend the definition of $\widetilde{\mu}_{\rho,[\nu]}(h)$ to all 1-simplices with an endpoint in e.

Step 4.1: Suppose that $h(0)$ and $h(1)$ both lie in e. Let $\gamma \in \Gamma$ be the primitive group element whose axis is e, so that γ translates along e in the the direction of h. Also, choose a smooth lift $\rho_t \in \widetilde{\mathrm{Hit}}_V(S)$ of $[\rho_t]$ so that $\rho_0 = \rho$ and the ρ_t-equivariant Frenet curve ξ_t satisfies $\xi_t(x) = \xi_0(x)$ and $\xi_t(y) = \xi_0(y)$. Note that the (possibly empty) set of bridges that intersect $h(0)$ and $h(1)$ are naturally ordered according to the orientation of h. Let k_1 be the number of bridges that intersect the interior of h, let k_2 be the number of bridges that contain the endpoints of h, and let $k := k_1 + \frac{k_2}{2}$. Then define

$$\widetilde{\mu}_{\rho,[\nu]}(h) := k \cdot \frac{d}{dt}\bigg|_{t=0} \rho_t(\gamma)\rho(\gamma)^{-1} \in \mathfrak{sl}(V).$$

It is easy to see that this does not depend on the choice ρ_t.

Step 4.2: Suppose that $h(0)$ is the endpoint of a bridge $J \in \widetilde{\mathcal{J}}$ and $h(1)$ is the intersection of J with a closed edge $\{x, y\} \in \widetilde{\mathcal{P}}$. Let $h' : [0, 1] \to \widetilde{\Sigma}$ be the 1-simplex so that $h'(0)$ is the endpoint of J that is not $h(0)$, and $h'(1) = h(1)$. Observe that the concatenation of $h \cdot h'^{-1}$ is J equipped with the orientation from $h(0)$ to $h'(0)$ (see Figure 7). Let $\rho_t, \rho'_t \in \widetilde{\mathrm{Hit}}_V(S)$ be the smooth lifts of $[\rho_t]$ so that $\rho_0 = \rho'_0 = \rho$, and the corresponding equivariant Frenet curves ξ_t, ξ'_t respectively satisfy

$$\xi_t(x_{h,0}) = \xi_0(x_{h,0}), \quad \xi_t(y_{h,0}) = \xi_0(y_{h,0}), \quad \xi_t^{(1)}(z_{h,0}) = \xi_0^{(1)}(z_{h,0}),$$
$$\xi'_t(x_{h',0}) = \xi'_0(x_{h',0}), \quad \xi'_t(y_{h',0}) = \xi'_0(y_{h',0}), \quad \xi_t'^{(1)}(z_{h',0}) = \xi_0'^{(1)}(z_{h',0}).$$
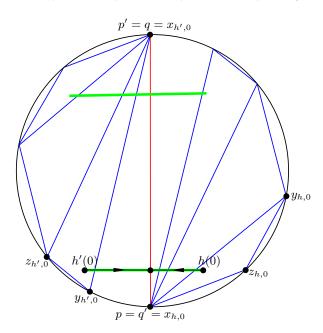
FIGURE 7. The 1-simplex h in Step 4.2.

Let p be the common vertex shared by e and $T_{h,0}$, and let q be the other endpoint of e. Similarly, let p' be the common vertex shared by e and $T_{h',0}$ and let q' be the other endpoint of e. Then let $u_{t,h}, u_{t,h'} \in \mathrm{PSL}(V)$ be the unique unipotent projective transformations so that

$$u_{t,h} \cdot \xi_0(p) = \xi_t(p) = \xi_0(p) \quad \text{and} \quad u_{t,h} \cdot \xi_0(q) = \xi_t(q),$$

$$u_{t,h'} \cdot \xi_0'(p') = \xi_t'(p') = \xi_0'(p') \quad \text{and} \quad u_{t,h'} \cdot \xi_0'(q') = \xi_t'(q').$$

Let $d_{t,h} \in \mathrm{PGL}(V)$ be the unique projective transformation that sends $u_{t,h'}^{-1} \cdot \xi_t'$ to $u_{t,h}^{-1} \cdot \xi_t$, and observe that $d_{t,h}$ fixes both $\xi(x)$ and $\xi(y)$. Define

$$\widetilde{\mu}_{\rho,[\nu]}(h) := \frac{d}{dt}\bigg|_{t=0} u_{t,h} + \frac{1}{2} \cdot \frac{d}{dt}\bigg|_{t=0} d_{t,h} \in \mathfrak{sl}(V).$$

Finally, for any arbitrary $h : [0,1] \to \widetilde{\Sigma}$ so that $h(1)$ lies in a closed edge $e \in \widetilde{\mathcal{P}}$, we can homotope h to the concatenation $h_1 \cdot h_2 \cdot h_3$, where

- $h_1(0) = h(0)$,
- $h_1(1) = h_2(0)$ is the endpoint of a bridge $J \in \widetilde{\mathcal{J}}$ that intersects e,
- $h_2(1) = h_3(0)$ is the intersection of J with e
- $h_3(1) = h(1)$.

Then define

$$\widetilde{\mu}_{\rho,[\nu]}(h) := \widetilde{\mu}_{\rho,[\nu]}(h_1) + \widetilde{\mu}_{\rho,[\nu]}(h_2) + \widetilde{\mu}_{\rho,[\nu]}(h_3)$$

and define $\widetilde{\mu}_{\rho,[\nu]}(h^{-1}) := -\widetilde{\mu}_{\rho,[\nu]}(h)$. Of course, one needs to verify that $\widetilde{\mu}_{\rho,[\nu]}(h)$ does not depend on the choice of h_1, h_2 and h_3. In fact, we prove the following.

Proposition 2.13. *The map* $\widetilde{\mu}_{\rho,[\nu]} : C_1(\widetilde{S}, \mathbb{Z}) \to \mathfrak{sl}(V)$ *as described above is a well-defined,* $\mathrm{Ad} \circ \rho$-*equivariant* 1-*cocycle.*

The proof of Proposition 2.13 is a straightforward argument which we include as Appendix A. By the description of the linear isomorphisms

$$T_{[\rho]} \mathrm{Hit}_V(S) \simeq H^1(\Gamma, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}) \simeq H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$$

given above, it is clear that $[\mu_{\rho,[\nu]}] = [\nu]$. Hence, we can make the following definition.

Definition 2.14. Let Σ be a hyperbolic metric on S, let $\mathcal{T}$ be an ideal triangulation of S and let $\mathcal{J}$ be a compatible bridge system. Also, let $\rho \in \widetilde{\mathrm{Hit}}_V(S)$ and let $[\nu] \in H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$. The cocycle $\mu_{\rho,[\nu]}$ defined above is the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-*tangent cocycle* in $[\nu]$.

For any choice of a hyperbolic metric Σ on S, an oriented ideal triangulation $\mathcal{T}$ for S and a compatible bridge system $\mathcal{J}$, we see that every cohomology class in $H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ contains a unique $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle. Later in Section 4, we will prove that these tangent cocycles arise from admissible labellings, and thus can be used to explicitly compute the Goldman symplectic form on $\mathrm{Hit}_V(S)$.

3. The vector space of admissible labellings

The goal of this section is to define the vector space of admissible labellings. An admissible labelling is a particular way to label any barrier system on $\widetilde{S}$ with elements in $\mathfrak{sl}(V) = \{X \in \mathrm{End}(V) : \mathrm{tr}(X) = 0\}$. These admissible labellings will eventually give us an explicit way to describe all tangent cocycles. As such, it gives us an explicit description of the tangent space to $\mathrm{Hit}_V(S)$, which allows one to explicitly compute the Goldman symplectic pairing between any two vectors that are given to us as admissible labellings. We will later (Section 5) also show that the admissible labellings give us a symplectic identification between the tangent spaces to any pair of points in $\mathrm{Hit}_V(S)$, which allows us to define large families of Hamiltonian vector fields for the Goldman symplectic form.

3.1. **Legal labellings.** To define admissible labellings, we first give a preliminary definition of a legal labelling. An important piece of this definition is a particular family of elements in $\mathfrak{sl}(V)$, called eruption and shearing endomorphisms.

First, we define the eruption endomorphisms.

Definition 3.1. The (i, j, k)-*eruption endomorphism* with respect to the triple $(F, G, H) \in \mathcal{F}(V)^{(3)}$ is the Lie algebra element $A_{F,G,H}^{i,j,k} \in \mathfrak{sl}(V)$ with eigenspaces $F^{(i)}$ and $G^{(j)} + H^{(k)}$ corresponding to eigenvalues $\frac{n-i}{n}$ and $-\frac{i}{n}$ respectively.

For the triple $(F, G, H) \in \mathcal{F}(V)^{(3)}$, and let $\{f_1, \ldots, f_n\}, \{g_1, \ldots, g_n\}, \{h_1, \ldots, h_n\}$ be three bases for V so that for all $l = 1, \ldots, n$,

$$F^{(l)} = \mathrm{Span}_{\mathbb{R}}\{f_1, \ldots, f_l\}, \quad G^{(l)} = \mathrm{Span}_{\mathbb{R}}\{g_1, \ldots, g_l\}, \quad H^{(l)} = \mathrm{Span}_{\mathbb{R}}\{h_1, \ldots, h_l\}.$$

Since $(F, G, H) \in \mathcal{F}(V)^{(3)}$, $B_{F,G,H}^{i,j,k} := \{f_1, \ldots, f_i, g_1, \ldots, g_j, h_1, \ldots, h_k\}$ is an ordered basis of V for any $i, j, k \in \mathbb{Z}^+$ so that $i + j + k = n$. In this ordered basis, we can write $A_{F,G,H}^{i,j,k}$ as the matrix

$$A_{F,G,H}^{i,j,k} = \left[\begin{array}{cc} \frac{n-i}{n} \cdot \mathrm{id}_i & 0 \\ 0 & -\frac{i}{n} \cdot \mathrm{id}_{n-i} \end{array} \right],$$

where id_l is the $l \times l$ identity matrix.

The next proposition states some important properties of $A_{F,G,H}^{i,j,k}$.

Proposition 3.2. *Let* $(F, G, H) \in \mathcal{F}(V)^{(3)}$ *and let* $i, j, k \in \mathbb{Z}^+$ *so that* $i+j+k = n$. *The following statements hold.*

(1) $A_{F,G,H}^{i,j,k}$ *fixes* F.

(2) *There is a projective transformation* $g \in \mathrm{PGL}(V)$ *so that*

$$\left(F, \exp\left(t A_{F,G,H}^{i,j,k} \right) \cdot G, H \right) = g \cdot \left(F, G, \exp\left(t A_{G,H,F}^{j,k,i} \right) \cdot H \right).$$

(3) *For all* $i', j', k' \in \mathbb{Z}^+$ *so that* $i' + j' + k' = n$,

$$T_{i',j',k'}\left(F, \exp\left(t A_{F,G,H}^{i,j,k} \right) G, H \right) = \left\{ \begin{array}{ll} T_{i',j',k'}(F, G, H) & \text{if } (i, j, k) \neq (i', j', k') \\ e^t \cdot T_{i,j,k}(F, G, H) & \text{if } (i, j, k) = (i', j', k') \end{array} \right. .$$

(4) $\left\{ A_{F,G,H}^{i,j,k} : i, j, k \in \mathbb{Z}^+, i + j + k = n \right\}$ *is linearly independent in* $\mathfrak{sl}(V)$.

Proof. (1) It is clear that $A_{F,G,H}^{i,j,k}$ fixes $F^{(m)}$ for all $m = 1, \ldots, i$. On the other hand, for any $m > i$, write the vector f_m in the basis $B_{F,G,H}^{i,j,k}$, i.e.

$$f_m = \sum_{l=1}^{i} a_l f_l + \sum_{l=1}^{j} b_l g_l + \sum_{l=1}^{k} c_l h_l$$

for some constants $a_l, b_l, c_l \in \mathbb{R}$. Then

$$(3.1) \qquad \begin{aligned} A_{F,G,H}^{i,j,k} \cdot f_m &= \sum_{l=1}^{i} \frac{n-i}{n} a_l f_l - \sum_{l=1}^{j} \frac{i}{n} b_l g_l - \sum_{l=1}^{k} \frac{i}{n} c_l h_l \\ &= \sum_{l=1}^{i} a_l f_l - \frac{i}{n} f_m. \end{aligned}$$

This implies that for all $m = 1, \ldots, n - 1$, $A_{F,G,H}^{i,j,k} \cdot F^{(m)} = F^{(m)}$.

(2) For any $i, j, k \in \mathbb{Z}^+$ so that $i + j + k = n$ and any $(F, G, H) \in \mathcal{F}(V)^{(3)}$, let $X_{F,G,H}^{i,j,k} \in \mathfrak{sl}(V)$ be the endomorphism that is the zero endomorphism on $F^{(i)}$, scales $G^{(j)}$ by $\frac{k}{n}$ and scales $H^{(k)}$ by $-\frac{j}{n}$. In other words, in the basis $B_{F,G,H}^{i,j,k}$, $X_{F,G,H}^{i,j,k}$ is represented by the matrix

$$X_{F,G,H}^{i,j,k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k}{n} \cdot \mathrm{id}_j & 0 \\ 0 & 0 & -\frac{j}{n} \cdot \mathrm{id}_k \end{bmatrix}.$$

One can then compute in the basis $B_{F,G,H}^{i,j,k}$ that

$$X_{H,F,G}^{k,i,j} - X_{G,H,F}^{j,k,i} = \begin{bmatrix} \frac{j}{n}\,\mathrm{id}_i & 0 & 0 \\ 0 & -\frac{i}{n} \cdot \mathrm{id}_j & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -\frac{k}{n}\,\mathrm{id}_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{i}{n} \cdot \mathrm{id}_k \end{bmatrix} = A_{F,G,H}^{i,j,k}.$$

It is clear that $X_{H,F,G}^{k,i,j}$ and $X_{G,H,F}^{j,k,i}$ commute, so (1) implies that the product $\exp\left(tX_{H,F,G}^{k,i,j}\right)\exp\left(-tX_{G,H,F}^{j,k,i}\right) = \exp\left(tA_{F,G,H}^{i,j,k}\right) \in \mathrm{PSL}(V)$ fixes the flag F, i.e. $\exp\left(tX_{H,F,G}^{k,i,j}\right) \cdot F = \exp\left(tX_{G,H,F}^{j,k,i}\right) \cdot F$. Similarly, $\exp\left(tX_{F,G,H}^{i,j,k}\right) \cdot G = \exp\left(tX_{H,F,G}^{k,i,j}\right) \cdot G$ and $\exp\left(tX_{G,H,F}^{j,k,i}\right) \cdot H = \exp\left(tX_{F,G,H}^{i,j,k}\right) \cdot H$. Hence,

$$\left(F, \exp\left(tA_{F,G,H}^{i,j,k}\right) \cdot G, H\right)$$
$$= \left(\exp\left(tA_{F,G,H}^{i,j,k}\right) \cdot F, \exp\left(tA_{F,G,H}^{i,j,k}\right) \cdot G, H\right)$$
$$= \exp\left(tA_{F,G,H}^{i,j,k}\right) \cdot \left(F, G, \exp\left(-tA_{F,G,H}^{i,j,k}\right)\exp\left(tA_{H,F,G}^{k,i,j}\right) \cdot H\right)$$
$$= \exp\left(tA_{F,G,H}^{i,j,k}\right) \cdot \left(F, G, \exp\left(tA_{G,H,F}^{j,k,i}\right) \cdot H\right).$$

(3) First, consider the case where $(i, j, k) \neq (i', j', k')$, then either $i > i'$, $j > j'$ or $k > k'$. Suppose that $j > j'$, and observe that

$$\exp(tA_{F,G,H}^{i,j,k}) = \begin{bmatrix} e^{\frac{(n-i)t}{n}} \cdot \mathrm{id}_i & 0 \\ 0 & e^{-\frac{it}{n}} \cdot \mathrm{id}_{n-i} \end{bmatrix}$$

when written as a matrix in the basis $B_{F,G,H}^{i,j,k}$. In particular, $\exp\left(tA_{F,G,H}^{i,j,k}\right)$ fixes the subspaces $G^{(j'+1)}$, $G^{(j')}$ and $G^{(j'-1)}$. This means that

$$T_{i',j',k'}\left(F, \exp\left(tA_{F,G,H}^{i,j,k}\right) \cdot G, H\right)$$
$$= \frac{F^{(i'+1)} \wedge G^{(j')} \wedge H^{(k'-1)}}{F^{(i'+1)} \wedge G^{(j'-1)} \wedge H^{(k')}} \cdot \frac{F^{(i'-1)} \wedge G^{(j'+1)} \wedge H^{(k')}}{F^{(i')} \wedge G^{(j'+1)} \wedge H^{(k'-1)}} \cdot \frac{F^{(i')} \wedge G^{(j'-1)} \wedge H^{(k'+1)}}{F^{(i'-1)} \wedge G^{(j')} \wedge H^{(k'+1)}}$$
$$= T_{i',j',k'}(F, G, H).$$

Since the triple ratio is a projective invariant, (2) implies that

$$T_{i',j',k'}\left(F, \exp\left(tA_{F,G,H}^{i,j,k}\right) \cdot G, H\right) = T_{i',j',k'}\left(F, G, \exp\left(tA_{G,H,F}^{j,k,i}\right) \cdot H\right)$$
$$= T_{i',j',k'}\left(\exp\left(tA_{H,F,G}^{k,i,j}\right) \cdot F, G, H\right).$$

Thus, if $k > k'$ or $i > i'$, we can run the same argument as we did above, using $T_{i',j',k'}\left(F, G, \exp\left(tA_{G,H,F}^{j,k,i}\right) \cdot H\right)$ or $T_{i',j',k'}\left(\exp\left(tA_{H,F,G}^{k,i,j}\right) \cdot F, G, H\right)$ respectively in place of $T_{i',j',k'}\left(F, \exp\left(tA_{F,G,H}^{i,j,k}\right) \cdot G, H\right)$. This will prove that if $(i,j,k) \neq (i',j',k')$, then $T_{i',j',k'}\left(F, \exp\left(tA_{F,G,H}^{i,j,k}\right) \cdot G, H\right) = T_{i',j',k'}(F,G,H)$.

Next, consider the case where $(i',j',k') = (i,j,k)$. In this case, observe that

$$
\begin{aligned}
& T_{i,j,k}\left(F, \exp\left(tA_{F,G,H}^{i,j,k}\right) \cdot G, H\right) \\
=\ & \frac{F^{(i+1)} \wedge G^{(j)} \wedge H^{(k-1)}}{F^{(i+1)} \wedge G^{(j-1)} \wedge H^{(k)}} \cdot \frac{e^{\frac{(n-i)t}{n}} \cdot F^{(i-1)} \wedge G^{(j+1)} \wedge H^{(k)}}{e^{-\frac{it}{n}} \cdot F^{(i)} \wedge G^{(j+1)} \wedge H^{(k-1)}} \cdot \frac{F^{(i)} \wedge G^{(j-1)} \wedge H^{(k+1)}}{F^{(i-1)} \wedge G^{(j)} \wedge H^{(k+1)}} \\
=\ & e^{t} \cdot T_{i,j,k}(F,G,H).
\end{aligned}
$$

(4) Suppose for contradiction that there is some $i_0, j_0, k_0 \in \mathbb{Z}^+$ so that $i_0 + j_0 + k_0 = n$ and

(3.2) $$A_{F,G,H}^{i_0,j_0,k_0} = \sum_{(i,j,k) \neq (i_0,j_0,k_0)} \alpha^{i,j,k} A_{F,G,H}^{i,j,k}$$

for some $\alpha^{i,j,k} \in \mathbb{R}$. By (3), we have that

$$T_{i,j,k}\left(F, \exp\left(tA_{F,G,H}^{i_0,j_0,k_0}\right) \cdot G, H\right) = \begin{cases} T_{i,j,k}(F,G,H) & \text{if } (i,j,k) \neq (i_0,j_0,k_0); \\ e^{t} \cdot T_{i_0,j_0,k_0}(F,G,H) & \text{if } (i,j,k) = (i_0,j_0,k_0). \end{cases}$$

On the other hand, (3.2) and (3) together imply that

$$T_{i,j,k}\left(F, \exp\left(tA_{F,G,H}^{i_0,j_0,k_0}\right) \cdot G, H\right) = \begin{cases} e^{\alpha^{i,j,k}t} \cdot T_{i,j,k}(F,G,H) & \text{if } (i,j,k) \neq (i_0,j_0,k_0); \\ T_{i_0,j_0,k_0}(F,G,H) & \text{if } (i,j,k) = (i_0,j_0,k_0). \end{cases}$$

This is clearly a contradiction. $\qquad\square$

Similarly, since $(F,G) \in \mathcal{F}(V)^{(2)}$, we see that $B_{F,G}^{i,n-i} := \{f_1, \ldots, f_i, g_1, \ldots, g_{n-i}\}$ is a basis of V as well. This allows us to define the shearing endomorphisms.

Definition 3.3. The $(i, n-i)$-*shearing endomorphism* with respect to the pair $(F,G) \in \mathcal{F}(V)^{(2)}$ is the endomorphism $A_{F,G}^{i,n-i} \in \mathfrak{sl}(V)$ with eigenspaces $F^{(i)}$ and $G^{(n-i)}$ corresponding to eigenvalues $\frac{n-i}{n}$ and $-\frac{i}{n}$ respectively. Also, if $F, G, H \in \mathcal{F}(V)^{(3)}$, we will use the notation $A_{F,G,H}^{i,n-i,0} := A_{F,G}^{i,n-i}$.

If we write $A_{F,G}^{i,n-i}$ as a matrix in the basis $B_{F,G}^{i,n-i}$, then

$$A_{F,G}^{i,n-i} = \begin{bmatrix} \frac{n-i}{n} \cdot \mathrm{id}_i & 0 \\ 0 & -\frac{i}{n} \cdot \mathrm{id}_{n-i} \end{bmatrix}.$$

Furthermore, it is clear that the following proposition holds.

Proposition 3.4. *Let F and G be a transverse pair of flags in $\mathcal{F}(V)$, then the following hold.*

(1) *For all $i = 1, \ldots, n-1$, $A_{F,G}^{i,n-i}$ preserve the flags F and G.*

(2) *$\left\{A_{F,G}^{i,n-i} : i = 1, \ldots, n-1\right\}$ is a linearly independent collection in $\mathfrak{sl}(V)$.*

We are now ready to use the eruption and shearing endomophisms to define legal labellings. To do so, choose an ideal triangulation $\mathcal{T}$ on S. This determines a barrier system $\mathcal{B}$ on S, which lifts to a barrier system $\widetilde{\mathcal{B}}$. Note that the orientation on S induces a cyclic ordering on $\partial\Gamma$.

Definition 3.5. Let $\rho : \Gamma \to \mathrm{PSL}(V)$ be a Hitchin representation with Frenet curve $\xi : \partial\Gamma \to \mathcal{F}(V)$. A *legal labeling at ρ* is a $\mathrm{Ad}\circ\rho$-equivariant map $L : \widetilde{\mathcal{B}} \to \mathfrak{sl}(V)$ so that the following conditions hold:

(1) For every non-edge barrier $(x,T) \in \widetilde{\mathcal{M}}$ where $T = \{x,y,z\}$ so that $x < y < z < x$ in the cyclic ordering on $\partial\Gamma$, we have

$$L(x,T) = \sum_{i+j+k=n; i,j,k \in \mathbb{Z}^+} a_{i,j,k}(x,T) \cdot A^{i,j,k}_{\xi(x),\xi(y),\xi(z)},$$

for some $a_{i,j,k}(x,T) = a_{i,j,k}(x,T,L) \in \mathbb{R}$.

(2) For all $T = \{x,y,z\} \in \widetilde{\Theta}$ so that $x < y < z < x$ in the cyclic ordering on $\partial\Gamma$, $a_{i,j,k}(x,T) = a_{j,k,i}(y,T) = a_{k,i,j}(z,T)$.

(3) For every edge $e = \{x,y\} \in \widetilde{\mathcal{T}}$, we have

$$L(e) = \sum_{i=1}^{n-1} a_i(y,x) \cdot A^{i,n-i}_{\xi(x),\xi(y)} = \sum_{i=1}^{n-1} a_i(x,y) \cdot A^{i,n-i}_{\xi(y),\xi(z)},$$

for some $a_i(y,x) = a_i(y,x,L) = -a_{n-i}(y,x) = -a_{n-i}(y,x,L) \in \mathbb{R}$.

Denote the set of legal labellings at ρ by $\mathcal{L}(\rho) = \mathcal{L}(\rho, \mathcal{T})$. We refer to the numbers

$$\{a_{i,j,k}(x,T,L) : (x,T) \in \widetilde{\mathcal{M}}\}_{i+j+k=n; i,j,k>0} \cup \{a_i(y,x,L) : (y,x) \in \widetilde{\mathcal{T}}^o\}_{i=1,\ldots,n-1}$$

as the *coeffecients* of L.

Let $L \in \mathcal{L}(\rho)$ be any legal labelling. Definition 3.5(2) and Proposition 3.2(4) ensure that for any ideal triangle $T = \{x,y,z\} \in \widetilde{\Theta}$, any one of $L(x,T)$, $L(y,T)$, $L(z,T)$ determines the other two.

The set $\mathcal{L}(\rho)$ is naturally a real vector space, whose dimension we can compute. Suppose that $\mathcal{T}$ has k closed edges, then it is easy to see that $|\mathcal{T}| = 6g - 6 + k$ and $|\mathcal{M}| = 12g - 12$. By Proposition 3.2(4) and Proposition 3.4(2),

$$
\begin{aligned}
\dim \mathcal{L}(\rho) &= (6g - 6 + k)(n-1) + \frac{1}{3}(12g - 12)\frac{(n-1)(n-2)}{2} \\
&= (n^2 - 1)(2g - 2) + k(n-1).
\end{aligned}
$$

3.2. Admissible labellings. Next, we will define the notion of an admissible labelling. For that, we further choose an orientation r on $\mathcal{T}$. This lifts to an orientation $\widetilde{r}$ on $\widetilde{\mathcal{T}}$. We will also need the indicator function α defined as follows. For any edge $e = \{x,y\} \in \widetilde{\mathcal{T}}$, define

$$\alpha(y,x) := \begin{cases} 1 & \text{if } \widetilde{r}(e) = (y,x); \\ -1 & \text{if } \widetilde{r}(e) = (x,y). \end{cases}$$

Notation 3.6. For every closed edge $e = \{x,y\} \in \widetilde{\mathcal{P}}$, let $J = \{T_1, T_2\}$ be any bridge across e so that T_1 and T_2 lie to the right and left of $\widetilde{r}(e) = (x,y)$ respectively. For $m = 1, 2$, let $\gamma_m \in \Gamma$ be the primitive group element that has q_m as its attracting fixed point and p_m as its repelling fixed point (see Notation 2.9). Let $T_m =: T_{m,1}, \ldots, T_{m,C_m+1} := \gamma_m \cdot T_m$ be the finite sequence of triangles in $\widetilde{\Theta}$ so that $T_{m,c}$

shares a common edge with $T_{m,c+1}$ for all $c = 1, \ldots, C_m$ (see Figure 8). Also, let $e_{m,c}$ be the common edge of $T_{m,c}$ and $T_{m,c+1}$. Then define

$$\widetilde{\Theta}(J, T_m) := \{T_{m,1}, \ldots, T_{m,C_m}\} \quad \text{and} \quad \widetilde{\mathcal{T}}(J, T_m) := \{e_{m,1}, \ldots, e_{m,C_m}\}.$$
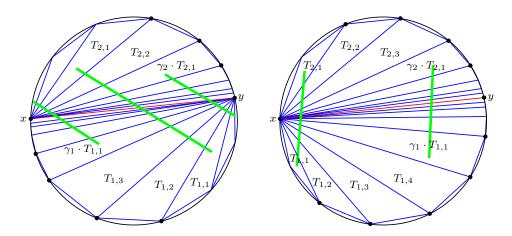
FIGURE 8. In both diagrams above, the closed edge $e = \{x, y\}$ is drawn in red, and the $\langle \gamma_1 \rangle = \langle \gamma_2 \rangle$ orbit of the bridge $J = \{T_1, T_2\}$ is drawn in green. On the left, $C_1 = 3$ and $C_2 = 2$. On the right, $C_1 = 4$ and $C_2 = 3$.

Observe that in Notation 3.6, the triangles in $\widetilde{\Theta}(J, T_m)$ and the edges in $\widetilde{\mathcal{T}}(J, T_m)$ all share p_m as a common vertex.

Definition 3.7. A legal labelling $L : \widetilde{\mathcal{B}} \to \mathfrak{sl}(V)$ at ρ is *admissible* if for any closed edge $e = \{x, y\} \in \widetilde{\mathcal{P}}$ and any bridge $J = \{T_1, T_2\}$ across e so that T_1 and T_2 lie to the right and left of $\widetilde{r}(e) = (x, y)$ respectively, the following hold for all $i = 1, \ldots, n-1$:

- If $p_1 \neq p_2$, then

$$\sum_{\{p_1, z\} \in \widetilde{\mathcal{T}}(J, T_1)} \alpha(z, p_1) a_i(z, p_1, L) + \sum_{T \in \widetilde{\Theta}(J, T_1)} \sum_{j+k=n-i} a_{i,j,k}(p_1, T, L)$$

$$= \sum_{\{p_2, z\} \in \widetilde{\mathcal{T}}(J, T_2)} \alpha(z, p_2) a_{n-i}(z, p_2, L) + \sum_{T \in \widetilde{\Theta}(J, T_2)} \sum_{j+k=i} a_{n-i,j,k}(p_2, T, L).$$

- If $p_1 = p_2$, then

$$\sum_{\{p_1, z\} \in \widetilde{\mathcal{T}}(J, T_1)} \alpha(z, p_1) a_i(z, p_1, L) + \sum_{T \in \widetilde{\Theta}(J, T_1)} \sum_{j+k=n-i} a_{i,j,k}(p_1, T, L)$$

$$= - \left(\sum_{\{p_2, z\} \in \widetilde{\mathcal{T}}(J, T_2)} \alpha(z, p_2) a_i(z, p_2, L) + \sum_{T \in \widetilde{\Theta}(J, T_2)} \sum_{j+k=n-i} a_{i,j,k}(p_2, T, L) \right).$$

Let $\mathcal{A}(\rho) = \mathcal{A}(\rho, r, \mathcal{T})$ denote the set of admissible labellings at ρ.

Observe that the $\mathrm{Ad} \circ \rho$-equivariance of L and the Γ-invariance of the orientation $\widetilde{r}$ ensure that if the admissibility condition holds for some bridge across the closed

edge e, then it automatically holds for all bridges across e. We will now explain the admissibility condition geometrically.
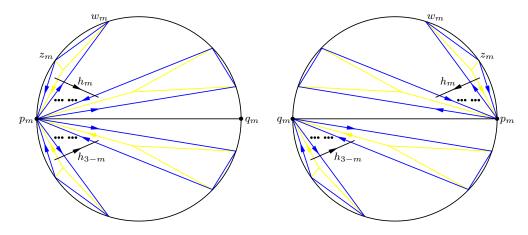
FIGURE 9. The case where $p_1 = p_2$ is drawn on the left, while the case where $p_1 = q_2$ is drawn on the right.

Let $e = \{x, y\} \in \widetilde{\mathcal{P}}$ be any closed edge and let $J = \{T_1, T_2\}$ be any bridge across e so that T_1 and T_2 lie to the right and left of $\widetilde{r}(e)$ respectively. If $q_m < w_m < z_m < p_m < q_m$ (see Notation 2.9), define

$$B(J, T_m) := \sum_{\{p_m, z\} \in \widetilde{\mathcal{T}}(J, T_m)} \alpha(z, p_m) L\{p_m, z\} + \sum_{T \in \widetilde{\Theta}(J, T_m)} L(p_m, T),$$

and if $p_m < z_m < w_m < q_m < p_m$, define

$$B(J, T_m) := -\left(\sum_{\{p_m, z\} \in \widetilde{\mathcal{T}}(J, T_m)} \alpha(z, p_m) L\{p_m, z\} + \sum_{T \in \widetilde{\Theta}(J, T_m)} L(p_m, T) \right).$$

More geometrically, $B(J, T_m)$ can be described as follows. Consider a geodesic 1-simplex $h_m : [0, 1] \to \widetilde{\Sigma}$ so that $h_m(0)$ lies in the fragmented ideal triangle in $T_{m,1}$ that has $e_{m,1}$ as an edge, and $h_m(1) = \gamma_m \cdot h_m(0)$ (see Figure 9). Then

$$B(J, T_m) = \sum_{e \in \widetilde{\mathcal{T}}(J, T_m)} \hat{i}\big(h_m, \widetilde{r}(e)\big) L(e) + \sum_{T \in \widetilde{\Theta}(J, T_m)} \hat{i}\big(h_m, (p_m, T)\big) L(p_m, T).$$

By Proposition 3.2(1) and Proposition 3.4(1), we see that $B(J, T_m)$ is an endomorphism that preserves the flag $\xi(p_m)$, i.e. $B(J, T_m) \cdot \xi^{(l)}(p_m) \subset \xi^{(l)}(p_m)$ for all $l = 1, \ldots, n-1$. In particular, we can write

$$B(J, T_m) = N(J, T_m) + D(J, T_m),$$

where $N(J, T_m)$ is a nilpotent endomorphism that preserves $\xi(p_m)$, and $D(J, T_m)$ is an endomorphism that preserves both $\xi(x)$ and $\xi(y)$. The next proposition explains the reason we impose the admissibility condition on the space of legal labellings.

Proposition 3.8. *Let $L \in \mathcal{L}(\rho)$ be any legal labelling, let $e = \{x, y\} \in \widetilde{\mathcal{P}}$, and let $J = \{T_1, T_2\}$ be any bridge across e so that T_1 and T_2 lie to the right and left of*

$\widetilde{r}(e)$ *respectively. For any $m = 1, 2$,*

$$
D(J, T_m) = \sum_{i=1}^{n-1} \left(\sum_{\{p_m, z\} \in \widetilde{\mathcal{T}}(J, T_m)} \alpha(z, p_m) a_i(z, p_m) \right.
$$

$$
\left. + \sum_{T \in \widetilde{\Theta}(J, T_m)} \sum_{j+k=n-i} a_{i,j,k}(p_m, T) \right) \cdot A^{i,n-i}_{\xi(p_m), \xi(q_m)}.
$$

if $q_m < w_m < z_m < p_m < q_m$, and

$$
D(J, T_m) = -\sum_{i=1}^{n-1} \left(\sum_{\{p_m, z\} \in \widetilde{\mathcal{T}}(J, T_m)} \alpha(z, p_m) a_i(z, p_m) \right.
$$

$$
\left. + \sum_{T \in \widetilde{\Theta}(J, T_m)} \sum_{j+k=n-i} a_{i,j,k}(p_m, T) \right) \cdot A^{i,n-i}_{\xi(p_m), \xi(q_m)}.
$$

if $p_m < z_m < w_m < q_m < p_m$.

Remark 3.9. In particular, unlike $B(J, T_m)$ and $N(J, T_m)$, $D(J, T_m)$ does not depend on the choice of J nor T_m, but only on the closed edge e. To emphasize this, we will henceforth denote $D(J, T_m)$ by $D_m(e) = D_m(e, L)$. Furthermore, Proposition 3.8 implies that the admissibility condition on a legal labelling $L \in \mathcal{A}(\rho)$ is simply requiring

$$
D_1(e) = \begin{cases} -D_2(e) & \text{if } p_1 \neq p_2; \\ D_2(e) & \text{if } p_1 = p_2. \end{cases}
$$

Proof. For all $l = 1, \ldots, n$ and $m = 1, 2$, let $f_{l,m} \in V$ so that $\mathrm{Span}_{\mathbb{R}}(f_{l,m}) = \xi^{(l)}(p_m) \cap \xi^{(n-l+1)}(q_m)$. For each $T \in \widetilde{\Theta}(J, T_m)$, let $a = a(T)$ and $b = b(T)$ be the two vertices of T that are not p_m, so that $p_m < a < b < p_m$ along $\partial\Gamma$. Let $A_i(p_m) \in \mathfrak{sl}(V)$ be any endomorphism of the form $A^{i,j,k}_{\xi(p_m), \xi(a), \xi(b)}$ or $A^{i,n-i}_{\xi(p_m), \xi(a)}$.

By Proposition 3.2(1) and Proposition 3.4(1), we see that the endomorphism $A_i(p_m)$ preserves the flag $\xi(p_m)$. Thus, as a matrix in the basis $\{f_{1,m}, \ldots, f_{m,n}\}$, $A_i(p_m)$ is upper triangular. Furthermore, $A_i(p_m)$ has an eigenbasis in V whose corresponding eigenvalues are either $\frac{n-i}{n}$ or $-\frac{i}{n}$, and it acts as scaling by $\frac{n-i}{n}$ on $\xi^{(i)}(p_m)$. In other words, the diagonal entries of $A_i(p_m)$ give the matrix representing the endomorphism $A^{i,n-i}_{\xi(p_m), \xi(q_m)}$ in the basis $\{f_{1,m}, \ldots, f_{m,n}\}$.

This implies that for all $i = 1, \ldots, n-1$, the endomorphism

$$
S_i := \left(\sum_{\{p_m, z\} \in \widetilde{\mathcal{T}}(J, T_m)} \alpha(z, p_m) a_i(z, p_m) A^{i,n-i}_{\xi(p_m), \xi(z)} \right.
$$

$$
\left. + \sum_{T \in \widetilde{\Theta}(J, T_m)} \sum_{j+k=n-i} a_{i,j,k}(p_m, T) A^{i,j,k}_{\xi(p_m), \xi(a(T)), \xi(b(T))} \right)
$$

when written as a matrix in the basis $\{f_{1,m}, \ldots, f_{n,m}\}$, is upper triangular, and its diagonal is of the form

$$\left(\sum_{\{p_m, z\} \in \widetilde{\mathcal{T}}(J, T_m)} \alpha(z, p_m) a_i(z, p_m) + \sum_{T \in \widetilde{\Theta}(J, T_m)} \sum_{j+k=n-i} a_{i,j,k}(p_m, T) \right) \cdot A^{i,n-i}_{\xi(p_m), \xi(q_m)}.$$

Since

$$B(J, T_m) = \begin{cases} \displaystyle\sum_{i=1}^{n-1} S_i & \text{if } q_m < w_m < z_m < p_m < q_m; \\ \displaystyle-\sum_{i=1}^{n-1} S_i & \text{if } p_m < z_m < w_m < q_m < p_m. \end{cases}$$

and $D(J, T_m)$ is the diagonal part of $B(J, T_m)$, the proposition follows. $\qquad \square$

The ρ-equivariance of L ensures that if $e = \rho(\gamma) \cdot e' \in \widetilde{\mathcal{P}}$ for some $\gamma \in \Gamma$, then the $n-1$ equations associated to e and e' in Definition 3.7 agree. Hence, if $\mathcal{T}$ has k closed edges, then $\mathcal{A}(\rho) \subset \mathcal{L}(\rho)$ is a vector subspace determined by $k(n-1)$ linear equations. In particular, $\dim \mathcal{A}(\rho) \geq \dim \mathcal{L}(\rho) - k(n-1) \geq (n^2-1)(2g-2)$. In fact, we will later show that $\dim \mathcal{A}(\rho) = (n^2-1)(2g-2)$, so these $k(n-1)$ linear equations are linearly independent. Also, we will verify in Section 4.3 that the linear equations determined by the admissibility condition, when interpreted appropriately, are exactly the closed leaf equalities found by Bonahon-Dreyer [BD14].

4. The tangent space to $\mathrm{Hit}_V(S)$ using admissible labellings

Let $\rho : \Gamma \to \mathrm{PSL}(V)$ be a Hitchin representation. We saw previously that by choosing an oriented ideal triangulation $(r, \mathcal{T})$ of S, we can define the vector space of admissible labellings $\mathcal{A}(\rho)$. The goal of this section is to establish a linear isomorphism between $\mathcal{A}(\rho)$ and $T_{[\rho]} \mathrm{Hit}_V(S)$ that depends on an additional choice of a bridge system $\mathcal{J}$ that is compatible with $\mathcal{T}$. To do so, choose a hyperbolic metric Σ on S. We then use the data of Σ, $(r, \mathcal{T})$ and $\mathcal{J}$ to construct a cocycle $\mu_L \in C^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ from any admissible labelling $L \in \mathcal{A}(\rho)$. We will later see (Section 4.3) that even though the cocycle μ_L depends on Σ, it's cohomology class $[\mu_L] \in H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ does not. This thus defines a linear map $\Phi_\rho : \mathcal{A}(\rho) \to H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}) = T_{[\rho]} \mathrm{Hit}_V(S)$ by $\Phi_\rho : L \mapsto [\mu_L]$ that depends only on $(r, \mathcal{T})$ and $\mathcal{J}$.

The rough idea behind constructing the cocycle $\mu_L \in C^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$, or equivalently an $\mathrm{Ad} \circ \rho$-equivariant linear map $\widetilde{\mu}_L : C_1(\widetilde{S}, \mathbb{Z}) \to \mathfrak{sl}(V)$, is the following. Let $\widetilde{\mathcal{B}}$ be the barrier system associated to $(r, \widetilde{\mathcal{T}})$, and let $h : [0, 1] \to \widetilde{\Sigma}$ be a 1-simplex. We want to define $\widetilde{\mu}_L(h)$ to be the sum

$$\widetilde{\mu}_L(h) = \sum_{b \in \widetilde{\mathcal{B}}} \hat{i}(h, \widetilde{r}(b)) L(b).$$

In general however, this is an infinite sum that does not converge. Fortunately, we can use our choice of the bridge system $\widetilde{\mathcal{J}}$ to tweak the definition of $\widetilde{\mu}_L$ in order to ensure it is well-defined (see Section 4.2).

To prove that Φ_ρ is indeed an isomorphism of vector spaces, we will show that each μ_L is the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle in the cohomology class $[\mu_L]$. This will imply that the map Φ_ρ is an injective linear map, which in turn implies that Φ_ρ is an isomorphism of vector spaces because we have already established that

$\dim \mathcal{A}(\rho) \leq (n^2 - 1)(2g - 2) = \dim H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$. The proof that each μ_L is a $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle heavily depends on the notion of a $(\mathcal{T}, \mathcal{J})$-parallel flow on $\mathrm{Hit}_V(S)$, which was defined in Sun-Wienhard-Zhang [SWZ17].

For the rest of this section, we will fix an oriented ideal triangulation $(r, \mathcal{T})$ on S, a compatible bridge system $\widetilde{\mathcal{J}}$, and a hyperbolic metric Σ.

4.1. **The maps K_0 and K_1.** In order to formally define the cocycle $\widetilde{\mu}_L$, it is convenient to consider the pair of maps

$$K_0 : \widetilde{\mathcal{P}}^o \to \mathfrak{sl}(V) \quad \text{and} \quad K_1 : \widetilde{\mathcal{J}}^o \to \mathfrak{sl}(V)$$

which we will now define. For any $e = \{x, y\} \in \widetilde{\mathcal{P}}$, let $J = \{T_1, T_2\} \in \widetilde{\mathcal{J}}$ be a bridge across e so that T_1 and T_2 are on the right and left of $\widetilde{r}(e)$ respectively. Also, for $m = 1, 2$, let $p_m, q_m \in \partial\Gamma$ be as defined in Notation 2.9 and let $\gamma_m \in \Gamma$ be as defined in Notation 3.6. Define $B(J, T_m)$, $N(J, T_m)$ and $D_m(e)$ as we did in Section 3.2.

Recall that Proposition 3.8 ensures

$$D_1(e) = \begin{cases} -D_2(e) & \text{if } p_1 \neq p_2; \\ D_2(e) & \text{if } p_1 = p_2. \end{cases}$$

so we can define K_0 by

$$K_0(x, y) := \begin{cases} D_m(e) & \text{if } (x, y) = (p_m, q_m); \\ -D_m(e) & \text{if } (x, y) = (q_m, p_m). \end{cases}$$

Also, for $m = 1, 2$, define K_1 by

$$K_1(T_m, T_{3-m}) := \sum_{l=0}^{\infty} \rho(\gamma_m^l) N(J, T_m) \rho(\gamma_m^{-l}).$$

Proposition 4.1. *The map K_1 is well-defined, i.e. for any bridge $\{T_1, T_2\} \in \widetilde{\mathcal{J}}$, the sum $K_1(T_m, T_{3-m})$ converge for $m = 1, 2$.*

Proof. Recall that $N(J, T_m)$ is a nilpotent endomorphism that preserves $\xi(p_m)$, and $\rho(\gamma_m)$ has $\xi(p_m)$ and $\xi(q_m)$ as its repelling and attracting flags respectively. Let $\{f_{1,m}, \ldots, f_{n,m}\}$ be a basis of V so that $\mathrm{Span}_{\mathbb{R}}(f_{l,m}) = \xi^{(l)}(p_m) \cap \xi^{(n-l+1)}(q_m)$ for all $l = 1, \ldots, n$. In this basis, $N(J, T_m)$ and $\rho(\gamma_m)$ can be represented by the matrices

$$N(J, T_m) = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \ldots & a_{1,n} \\ 0 & 0 & a_{2,3} & \ldots & a_{2,n} \\ 0 & 0 & 0 & \ldots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad \rho(\gamma_m) = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{pmatrix},$$

where $a_{i,j} \in \mathbb{R}$ for all $i < j$ and $\lambda_1 < \cdots < \lambda_n$. One can then compute that

$$\rho(\gamma^l) \cdot N(J, T_m) \cdot \rho(\gamma^{-l}) = \begin{pmatrix} 0 & \left(\frac{\lambda_1}{\lambda_2}\right)^l \cdot a_{1,2} & \left(\frac{\lambda_1}{\lambda_3}\right)^l \cdot a_{1,3} & \ldots & \left(\frac{\lambda_1}{\lambda_n}\right)^l \cdot a_{1,n} \\ 0 & 0 & \left(\frac{\lambda_2}{\lambda_3}\right)^l \cdot a_{2,3} & \ldots & \left(\frac{\lambda_2}{\lambda_n}\right)^l \cdot a_{2,n} \\ 0 & 0 & 0 & \ldots & \left(\frac{\lambda_3}{\lambda_n}\right)^l \cdot a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

for all $l = 0, \ldots, \infty$, which implies that

$$K_1(T_m, T_{3-m}) = \begin{pmatrix} 0 & \frac{\lambda_2}{\lambda_2 - \lambda_1} \cdot a_{1,2} & \frac{\lambda_3}{\lambda_3 - \lambda_1} \cdot a_{1,3} & \cdots & \frac{\lambda_n}{\lambda_n - \lambda_1} \cdot a_{1,n} \\ 0 & 0 & \frac{\lambda_3}{\lambda_3 - \lambda_2} \cdot a_{2,3} & \cdots & \frac{\lambda_n}{\lambda_n - \lambda_2} \cdot a_{2,n} \\ 0 & 0 & 0 & \cdots & \frac{\lambda_n}{\lambda_n - \lambda_3} \cdot a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

$\square$

The ρ-equivariance of L ensures that K_0 and K_1 are also ρ-equivariant.

4.2. Obtaining cocycles from admissible labellings.

Let $L \in \mathcal{A}(\rho)$. We will now construct the cocycle $\mu_L \in C^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ that depends on Σ, $(r, \mathcal{T})$ and $\mathcal{J}$. Observe that constructing a cochain $\mu_L \in C^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ is equivalent to constructing an $\mathrm{Ad} \circ \rho$-equivariant linear map $\widetilde{\mu}_L : C_1(\widetilde{S}, \mathbb{Z}) \to \mathfrak{sl}(V)$. It is sufficient to define $\widetilde{\mu}_L$ on the set of 1-simplices, and extend this to $C_1(\widetilde{S}, \mathbb{Z})$ by linearity.

Let $h : [0, 1] \to \widetilde{\Sigma}$ be a 1-simplex. Before defining $\widetilde{\mu}_L(h)$ formally, we will give more colloquial description of $\widetilde{\mu}_L(h)$ to motivate the formalism we use. Homotope h (relative end points) to a smooth curve $f : [0, 1] \to \widetilde{\Sigma}$ that intersects every barrier in $\widetilde{\mathcal{B}}$ only finitely many times. In the case when $h[0, 1]$ does not intersect any $e \in \widetilde{\mathcal{P}}$, then $f[0, 1]$ only intersects finitely barriers in $\widetilde{\mathcal{B}}$, so we can define

(4.1)
$$\widetilde{\mu}_L(h) := \sum_{b \in \widetilde{\mathcal{B}}} \hat{i}(f, \widetilde{r}(b)) L(b).$$

Of course, one needs to check that $\widetilde{\mu}_L(h)$ does not depend on the choice of f. This is a consequence of Proposition 4.6.

On the other hand, if $h[0, 1]$ intersects $e \in \widetilde{\mathcal{P}}$, then any 1-simplex that is homotopic to h (relative endpoints) will intersect infinitely many barriers in $\widetilde{\mathcal{B}}$, and the sum given in (4.1) does not converge for a general admissible labelling L. In this case, we further homotope h to a piecewise geodesic path $f : [0, 1] \to \widetilde{\Sigma}$ with the property that if $t \in [0, 1]$ and $f(t) \in e$, then there is some subinterval of $I \subset [0, 1]$ so that $t \in I$ and $f(I) = J$ for some bridge $J \in \widetilde{\mathcal{J}}$. More informally, we require that f crosses closed edges only along bridges.

This allows us to decompose $f[0, 1]$ into subsegments $f_1, \ldots, f_k$ that are bridges, and subsegments $f_1', \ldots, f_l'$ that do not intersect any closed edges. We then use (4.1) to assign to each f_i' the vector $\widetilde{\mu}_L(f_i') \in \mathfrak{sl}(V)$, and we use the functions K_0 and K_1 to assign to each f_i a vector $\widetilde{\mu}_L(f_i) \in \mathfrak{sl}(V)$. We can then define

$$\widetilde{\mu}_L(h) := \sum_{i=1}^{k} \widetilde{\mu}_L(f_i) + \sum_{i=1}^{l} \widetilde{\mu}_L(f_i').$$

Again, one needs to prove that $\widetilde{\mu}_L(h)$ does not depend on the choice of f, and that $\widetilde{\mu}_L$ defined this way is indeed $\mathrm{Ad} \circ \rho$-equivariant, and satisfies the cocycle condition.

We will now formally define $\widetilde{\mu}_L(h)$. Note that the orientation $r : \mathcal{T} \to \mathcal{T}^o$ induces an orientation $s : \mathcal{J} \to \mathcal{J}^o$ with the property that if $\widetilde{s} : \widetilde{\mathcal{J}} \to \widetilde{\mathcal{J}}^o$ is the lift of s, then for all $J \in \widetilde{\mathcal{J}}$ that intersects the closed edge $e \in \widetilde{\mathcal{P}}$, $\widetilde{s}(J)$ crosses from the right to the left of $\widetilde{r}(e)$. Recall that we have chosen a hyperbolic metric Σ on S to define

the algebraic intersection number (see Section 2.3). We first give the definition of $\widetilde{\mu}_L(h)$ in the three special cases described below.
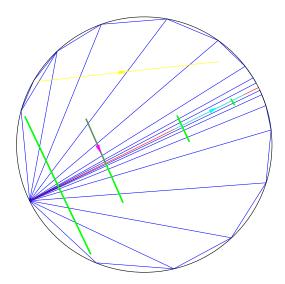

FIGURE 10. In the above picture, the red edge is a closed edge, the blue edges are isolated edges, and the bridges are drawn in green. A Type A, Type B, Type C 1-simplex is drawn in turqoise, violet and yellow respectively.

Definition 4.2. A 1-simplex $h : [0,1] \to \widetilde{\Sigma}$ is a *Type A 1-simplex* if it is a geodesic segment that lies in some closed edge $e = \{x, y\} \in \widetilde{P}$ (see Figure 10).

If h is a Type A 1-simplex that lies in the closed edge $e = \{x, y\} \in \widetilde{P}$, define

$$\widetilde{\mu}_L(h) := \sum_{J \in \widetilde{\mathcal{J}}} \hat{i}\big(h, \widetilde{s}(J)\big) K_0(\widetilde{r}(e)).$$

Definition 4.3. A 1-simplex $h : [0,1] \to \widetilde{\Sigma}$ is a *Type B 1-simplex* if it is a geodesic segment that lies in a bridge $J \in \widetilde{\mathcal{J}}$ that intersects the closed edge $e \in \widetilde{P}$, so that one of the endpoints of h is an endpoint of J, and the other endpoint of h is the point of intersection of e and J (see Figure 10).

If h is a Type B 1-simplex that lies in the bridge J which intersects the closed edge $e \in \widetilde{\mathcal{P}}$, let v be the common endpoint of J and $g[0,1]$, and equip J with the orientation $\overrightarrow{J}$ so that v is the backward vertex of $\overrightarrow{J}$. Then define

$$\widetilde{\mu}_L(h) := \begin{cases} K_1(\overrightarrow{J}) + \hat{i}\big(h, \widetilde{r}(e)\big) \cdot L(e) & \text{if } h(0) = v \text{ and } h(1) = e \cap J; \\ -K_1(\overrightarrow{J}) + \hat{i}\big(h, \widetilde{r}(e)\big) \cdot L(e) & \text{if } h(0) = e \cap J \text{ and } h(1) = v. \end{cases}$$

Definition 4.4. A 1-simplex $h : [0,1] \to \widetilde{\Sigma}$ is a *Type C 1-simplex* if it satisfies the following properties (see Figure 10):

- $h[0,1]$ intersects only finitely many barriers $b_1, \dots, b_k \in \widetilde{\mathcal{B}}$,
- $\big| h[0,1] \cap b_l \big| < \infty$ for all $l = 1, \dots, k$.

In particular, Type C 1-simplices do not intersect closed edges in $\widetilde{\mathcal{P}}$. If h is a Type C 1-simplex, define

$$\widetilde{\mu}_L(h) := \sum_{b \in \widetilde{\mathcal{B}}} \hat{i}\big(h, \widetilde{r}(b)\big) \cdot L(b).$$

Definition 4.5. The Type A, Type B and Type C 1-simplices are collectively called *elementary* 1-*simplices*.

Using the elementary 1-simplices, we will now define $\widetilde{\mu}_L(h)$ for a general 1-simplex $h : [0,1] \to \widetilde{\Sigma}$. Note that h can be homotoped relative endpoints to a concatenation $h_1 \cdot \ldots \cdot h_k$, where each h_i is an elementary 1-simplex. Define

$$\widetilde{\mu}_L(h) := \sum_{i=1}^{k} \widetilde{\mu}_L(h_i).$$

Proposition 4.6. *For any simplex $h : [0,1] \to \widetilde{\Sigma}$, $\widetilde{\mu}_L(h)$ is well-defined. Furthermore, if h_1, h_2, h_3 are 1-simplices so that $h_{i+1}(0) = h_i(1)$ for $i = 1, 2, 3$, then $\widetilde{\mu}_L(h_1 \cdot h_2 \cdot h_3) = 0$. In particular, $\widetilde{\mu}_L \in C^1(\widetilde{S}, \mathfrak{sl}(V))$ is a cocycle.*

In the Proposition 4.6, arithmetic in the subscripts are done modulo 3. The proof of Proposition 4.6 is a long but elementary argument, which we postpone to Appendix B. The $\mathrm{Ad} \circ \rho$-equivariance of L implies that $\widetilde{\mu}_L$ is an $\mathrm{Ad} \circ \rho$-equivariant cocycle, so $\widetilde{\mu}_L$ descends to a cocycle $\mu_L \in C^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$. Hence, we can define

$$\Phi_\rho = \Phi_{\rho, r, \mathcal{T}, \mathcal{J}} : \mathcal{A}(\rho) \to H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}) = T_{[\rho]} \mathrm{Hit}_V(S)$$

by $L \mapsto [\mu_L]$. It is easy to see that Φ_ρ is a linear map.

Recall that in the construction of the cocycle μ_L, we chose a hyperbolic metric Σ on S. We will now prove that the cohomology class $[\mu_L]$ does not depend on this choice. In particular, the linear map Φ_ρ does not depend on the choice of Σ.

Proposition 4.7. *Let $L \in \mathcal{A}(\rho)$ and let Σ, Σ' be two hyperbolic metrics on S. Also, let $\mu_L, \mu'_L \in C^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho})$ be the two cocycles constructed as above using Σ, Σ' respectively. Then $[\mu_L] = [\mu'_L]$.*

Proof. Let $\gamma \in \Gamma \backslash \{\mathrm{id}\}$ and let $\gamma^-, \gamma^+ \in \partial \Gamma$ denote the repelling and attracting fixed points of γ respectively. Let $e \in \widetilde{\mathcal{T}}$ be an edge that intersects $\{\gamma^-, \gamma^+\}$ transversely, and let $e_\Sigma, e_{\Sigma'}$ be the hyperbolic geodesics in Σ, Σ' respectively that realize the geodesic e. Then let p_Σ be the point of intersection between e_Σ and the axis of γ in Σ. Similarly, let $p_{\Sigma'}$ be the point of intersection between $e_{\Sigma'}$ and the axis of γ in Σ'. Define $g_\Sigma : [0,1] \to \widetilde{\Sigma}$ to be the geodesic segment with endpoints p_Σ and $\gamma \cdot p_\Sigma$ and $g_{\Sigma'} : [0,1] \to \widetilde{\Sigma'}$ to be the geodesic segment with endpoints $p_{\Sigma'}$ and $\gamma \cdot p_{\Sigma'}$.

From the coboundary condition, it is sufficient to show that $\widetilde{\mu}_L(g_\Sigma) = \widetilde{\mu}'_L(g_{\Sigma'})$. Let $z, w \in \partial \Gamma$ so that z and w lie to the left and right of (γ^-, γ^+) respectively, and $\{z, w\} = e$. By definition, $\gamma \cdot e_\Sigma$ contains $g_\Sigma(1)$ and $\gamma \cdot e_{\Sigma'}$ contains $g_{\Sigma'}(1)$. Hence, if $e' \in \widetilde{\mathcal{T}}$, and $\widetilde{r}(e') = (q^-, q^+)$, then

$$\hat{i}_\Sigma(g_\Sigma, \widetilde{r}(e')) = \hat{i}_{\Sigma'}(g_{\Sigma'}, \widetilde{r}(e')) = \begin{cases} 1 & \text{if } z < q^+ < \gamma \cdot z < \gamma \cdot w < q^- < w; \\ -1 & \text{if } z < q^- < \gamma \cdot z < \gamma \cdot w < q^+ < w; \\ \dfrac{1}{2} & \text{if } \widetilde{r}(e) = (w, z) \text{ or } \gamma \cdot (w, z); \\ -\dfrac{1}{2} & \text{if } \widetilde{r}(e) = (z, w) \text{ or } \gamma \cdot (z, w); \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, if $T \in \widetilde{\Theta}$ is an ideal triangle with vertices $v_1 < v_2 < v_3 < v_1$, let $b_i := (T, v_i) \in \widetilde{\mathcal{M}}$. Then

$$\sum_{i=1}^{3} \hat{i}_{\Sigma}(g_{\Sigma}, b_i) L(b_i)$$

$$= \begin{cases} L(b_i) & \text{if } z < v_i < \gamma \cdot z < \gamma \cdot w < v_{i+1} < v_{i-1} < w < z \text{ for some } i = 1, 2, 3; \\ -L(b_i) & \text{if } z < v_{i+1} < v_{i-1} < \gamma \cdot z < \gamma \cdot w < v_i < w < z \text{ for some } i = 1, 2, 3; \\ 0 & \text{otherwise.} \end{cases}$$

$$= \sum_{i=1}^{3} \hat{i}_{\Sigma'}(g_{\Sigma'}, b_i) L(b_i).$$

It immediately follows from the definition of $\widetilde{\mu}_L$ and $\widetilde{\mu}'_L$ that $\widetilde{\mu}_L(g_{\Sigma}) = \widetilde{\mu}'_L(g_{\Sigma'})$.
□

4.3. **Admissible labellings and tangent cocycles.** Our next goal is to prove the following theorem.

Theorem 4.8. *For any admissible labelling $L \in \mathcal{A}(\rho, r, \mathcal{T})$, μ_L is the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle in the cohomology class $[\mu_L]$.*

Theorem 4.8 has the following corollary, which is the key that allows us to compute the Goldman symplectic pairing between tangent vectors to $\mathrm{Hit}_V(S)$.

Corollary 4.9. *The linear map $\Phi_{\rho} : \mathcal{A}(\rho) \to T_{[\rho]} \mathrm{Hit}_V(S)$ is a linear isomorphism.*

Proof. Recall we previously computed that $\dim \mathcal{A}(\rho) \geq \dim T_{[\rho]} \mathrm{Hit}_V(S)$, so it is sufficient to show that $\ker \Phi_{\rho} = \{0\}$. Suppose that $L \in \ker \Phi_{\rho}$. By Theorem 4.8, the cocycle μ_L is a $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle whose cohomology class is the tangent vector to the constant path $t \mapsto [\rho]$ in $\mathrm{Hit}_V(S)$ at $t = 0$. It then follows from the definition of the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle that $\mu_L = 0$, which in turn implies that $L = 0$.
□

To prove that μ_L is the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle in $[\mu_L]$, it is convenient to specify a path $t \mapsto [\rho_t]$ in the Hitchin component that is tangential to $[\mu_L] \in H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad} \circ \rho}) \simeq T_{[\rho]} \mathrm{Hit}_V(S)$ so that ρ is a representative of $[\rho_0]$. It turns out that good candidates for such paths are the flow lines of the $(\mathcal{T}, \mathcal{J})$-parallel flows that are the main objects studied in the companion paper [SWZ17]. We will now give a brief description of these flows. For more details, see Section 5 of [SWZ17].

Following the work of Fock-Goncharov [FG06] and Bonahon-Dreyer [BD14], one can define some real-analytic functions on $\mathrm{Hit}_V(S)$ associated to the ideal triangles in Θ and the edges in $\mathcal{T}$. First, for any ideal triangle $\widehat{T} = [x, y, z] \in \Theta$, choose a representative $\{x, y, z\} \in \widetilde{\mathcal{T}}$ of $[x, y, z]$, and assume without loss of generality that $x < y < z < x$ in $\partial\Gamma$. Then for any $i, j, k \in \mathbb{Z}^+$ so that $i + j + k = n$, define the function $\tau_{x,y,z}^{i,j,k} : \mathrm{Hit}_V(S) \to \mathbb{R}$ by

$$\tau_{x,y,z}^{i,j,k}[\rho] := \log \left(T_{i,j,k}\big(\xi_{\rho}(x), \xi_{\rho}(y), \xi_{\rho}(z)\big) \right),$$

where ξ_{ρ} is the ρ-equivariant Frenet curve for some representative ρ in $[\rho]$ and $T_{i,j,k}$ is the triple ratio defined in Definition 2.5. This is well-defined because $T_{i,j,k}\big(\xi_{\rho}(x), \xi_{\rho}(y), \xi_{\rho}(z)\big) > 0$ by Theorem 2.6. Furthermore, the projective invariance of the triple ratio implies that $\tau_{x,y,z}^{i,j,k}[\rho]$ does not depend on the choice of representative $\{x, y, z\}$ of $[x, y, z]$, nor on the choice of representative ρ of $[\rho]$, and

the symmetries of the triple ratio imply that the $\tau_{x,y,z}^{i,j,k}[\rho] = \tau_{y,z,x}^{j,k,i}[\rho] = \tau_{x,z,y}^{i,k,j}[\rho]$. We refer to any such function as a *triangle invariant*.

Similarly, for any isolated edge $e = [x,y] \in \mathcal{Q}$, choose a representative $\{x,y\} \in \widetilde{\mathcal{Q}}$ of $[x,y]$, and let $z,w \in \partial\Gamma$ so that $\{x,y,z\}, \{x,y,w\} \in \widetilde{\Theta}$ and $x < z < y < w < x$ in $\partial\Gamma$. Then for any $i = 1, \ldots, n-1$, define the function $\sigma_{x,y}^{i,n-i} : \mathrm{Hit}_V(S) \to \mathbb{R}$ by

$$\sigma_{x,y}^{i,n-i}[\rho] := \log\left(-C_i\big(\xi_\rho(x), \xi_\rho(z), \xi_\rho(w), \xi_\rho(y)\big)\right),$$

where ξ_ρ is as above. Again, Theorem 2.6 implies that $\sigma_{x,y}^{i,n-i}[\rho]$ is well-defined. Also, the projective invariance of the cross ratio implies that $\sigma_{x,y}^{i,n-i}[\rho]$ does not depend on the choice of representatives $\{x,y\}$ of $[x,y]$ and ρ of $[\rho]$, and the symmetries of the cross ratio imply that $\sigma_{x,y}^{i,n-i}[\rho] = \sigma_{y,x}^{n-i,i}[\rho]$. We refer to any such function as an *edge invariant* along the isolated edge e.

Finally, for any closed edge $e = [x,y] \in \mathcal{P}$ and any $i = 1, \ldots, n-1$, choose a representative $\{x,y\} \in \widetilde{\mathcal{Q}}$ of $[x,y]$ and a bridge $J = \{T_1, T_2\} \in \widetilde{\mathcal{J}}$ so that T_1 and T_2 lie to the right and left of (x,y) respectively. For $m = 1,2$, let p_m, q_m, z_m, w_m be as defined in Notation 2.9, and let u_m be the unique unipotent projective transformation in $\mathrm{PSL}(V)$ that fixes p_m and sends z_m to q_m. Then define the function $\alpha_{x,y}^{i,n-i} : \mathrm{Hit}_V(S) \to \mathbb{R}$ by

$$\alpha_{x,y}^{i,n-i}[\rho] := \log\left(-C_i\big(\xi_\rho(x), u_1 \cdot \xi_\rho(w_1), u_2 \cdot \xi_\rho(w_2), \xi_\rho(y)\big)\right).$$

By [SWZ17, Proposition 4.11] that $\alpha_{x,y}^{i,n-i}[\rho]$ is well-defined, and the projective invariance of the cross ratio implies that $\alpha_{x,y}^{i,n-i}[\rho]$ does not depend on the choices of $\{x,y\}$, ξ_ρ and J. Furthermore, the symmetries of the cross ratio also implies that $\alpha_{x,y}^{i,n-i}[\rho] = \alpha_{y,x}^{n-i,i}[\rho]$. This function is called the *symplectic closed edge invariant* along the closed edge e.

By slightly modifying the work of Bonahon-Dreyer [BD14] it was shown in [SWZ17] that the triangle invariant, edge invariant along isolated edges, and symplectic closed edge invariants along closed edges give a real-analytic parameterization of $\mathrm{Hit}_V(S)$. More precisely, the map

$$\begin{array}{rlcl} \Omega = \Omega_{\mathcal{T},\mathcal{J}} & : \mathrm{Hit}_V(S) & \to & \mathbb{R}^{(n^2-1)(2g-2)+|\mathcal{P}|(n-1)} \\ & [\rho] & \mapsto & (\Sigma_1, \Sigma_2, \Sigma_3) \end{array}$$

where

$$\begin{array}{rcl} \Sigma_1 & := & \big(\sigma_{x,y}^{i,n-i}[\xi]\big)_{i=1,\ldots,n-1; [x,y]\in\mathcal{Q}} \\[4pt] \Sigma_2 & := & \big(\alpha_{x,y}^{i,n-i}[\xi]\big)_{i=1,\ldots,n-1; [x,y]\in\mathcal{P}} \\[4pt] \Sigma_3 & := & \big(\tau_{x,y,z}^{i,j,k}[\xi]\big)_{i,j,k\in\mathbb{Z}^+; i+j+k=n; [x,y,z]\in\Theta} . \end{array}$$

is a real-analytic diffeomorphism onto a convex polytope $H_{\mathcal{T},\mathcal{J}} = H$ of dimension $(n^2-1)(2g-2)$.

Let $W_{\mathcal{T},\mathcal{J}} = W$ be the $(n^2-1)(2g-2)$-dimensional subspace of $\mathbb{R}^{(n^2-1)(2g-2)+|\mathcal{P}|(n-1)}$ containing H, and note that W can be thought of as the tangent space to any point in $\Omega\big(\mathrm{Hit}_V(S)\big)$. The $|\mathcal{P}|(n-1)$ linear equations on $\mathbb{R}^{(n^2-1)(2g-2)+|\mathcal{P}|(n-1)}$ that cut out W are known as the *closed leaf equalities*, and were computed explicitly by Bonahon-Dreyer [BD14]. We will now describe these equalities.

For any closed edge $\{x,y\} \in \widetilde{\mathcal{P}}$, let $\{T_1, T_2\}$ be any bridge across $\{x,y\}$ so that T_1 and T_2 lie to the right and left of (x,y) respectively. For $m = 1,2$, let p_m, q_m,

z_m, w_m be as defined in Notation 2.9, and let $\widetilde{\mathcal{T}}(J, T_m)$, $\widetilde{\Theta}(J, T_m)$ and C_m be as defined in Notation 3.6.

Notation 4.10. For $m = 1, 2$, let $z_{m,0}, z_{m,1}, \ldots, z_{m,C_m} \in \partial\Gamma$ be the points so that $T_{m,c} = \{p_m, z_{m,c}, z_{m,c-1}\}$ (see Notation 3.6) for all $c = 1, \ldots, C_m$.

Denote an arbitrary vector $\mu \in \mathbb{R}^{(n^2-1)(2g-2)+|\mathcal{P}|(n-1)}$ in coordinates by

$$\mu = \left(\left(\mu_{x,y}^{i,n-i} = \mu_{y,x}^{n-i,i} \right)_{i=1,\ldots,n-1;[x,y]\in\mathcal{Q}}, \left(\mu_{x,y}^{i,n-i} = \mu_{y,x}^{n-i,i} \right)_{i=1,\ldots,n-1;[x,y]\in\mathcal{P}}, \right.$$
$$\left. \left(\mu_{x,y,z}^{i,j,k} = \mu_{y,z,x}^{j,k,i} = \mu_{z,x,y}^{k,i,j} \right)_{i,j,k\in\mathbb{Z}^+;i+j+k=n;[x,y,z]\in\Theta} \right).$$

For all $i = 1, \ldots, n-1$, the following linear equation on $\mathbb{R}^{(n^2-1)(2g-2)+|\mathcal{P}|(n-1)}$ is called a *closed leaf equality*.

(1) If $p_1 = x_1$ and $p_2 = x_2$, then for all $i = 1, \ldots, n-1$,

$$\sum_{c=1}^{C_1} \left(\mu_{p_1,z_{1,c}}^{i,n-i} + \sum_{j+k=n-i} \mu_{p_1,z_{1,c},z_{1,c-1}}^{i,j,k} \right) = \sum_{c=1}^{C_2} \left(\mu_{p_2,z_{2,c}}^{n-i,i} + \sum_{j+k=i} \mu_{p_2,z_{2,c},z_{2,c-1}}^{n-i,j,k} \right).$$

(2) If $p_1 = x_2$ and $p_2 = x_1$, then for all $i = 1, \ldots, n-1$,

$$\sum_{c=1}^{C_1} \left(\mu_{p_1,z_{1,c}}^{i,n-i} + \sum_{j+k=n-i} \mu_{p_1,z_{1,c-1},z_{1,c}}^{i,j,k} \right) = \sum_{c=1}^{C_2} \left(\mu_{p_2,z_{2,c}}^{n-i,i} + \sum_{j+k=i} \mu_{p_2,z_{2,c-1},z_{2,c}}^{n-i,j,k} \right).$$

(3) If $p_1 = p_2 = x_1$, then for all $i = 1, \ldots, n-1$,

$$\sum_{c=1}^{C_1} \left(\mu_{p_1,z_{1,c}}^{i,n-i} + \sum_{j+k=n-i} \mu_{p_1,z_{1,c},z_{1,c-1}}^{i,j,k} \right) = -\sum_{c=1}^{C_2} \left(\mu_{p_2,z_{2,c}}^{i,n-i} + \sum_{j+k=n-i} \mu_{p_2,z_{2,c-1},z_{2,c}}^{i,j,k} \right).$$

(4) If $p_1 = p_2 = x_2$, then for all $i = 1, \ldots, n-1$,

$$\sum_{c=1}^{C_1} \left(\mu_{p_1,z_{1,c}}^{i,n-i} + \sum_{j+k=n-i} \mu_{p_1,z_{1,c-1},z_{1,c}}^{i,j,k} \right) = -\sum_{c=1}^{C_2} \left(\mu_{p_2,z_{2,c}}^{i,n-i} + \sum_{j+k=n-i} \mu_{p_2,z_{2,c},z_{2,c-1}}^{i,j,k} \right).$$

There are $|\mathcal{P}|(n-1)$ such equations.

Theorem 4.11. [BD14] *The $|\mathcal{P}|(n-1)$ closed leaf equalities cut out the vector space $W \subset \mathbb{R}^{(n^2-1)(2g-2)+|\mathcal{P}|(n-1)}$. In particular, they are linearly independent.*

Given any ideal triangulation $\mathcal{T}$ on S and any compatible bridge system $\mathcal{J}$, Sun-Wienhard-Zhang [SWZ17, Theorem 5.8] constructed, for every $\mu \in W_{\mathcal{T},\mathcal{J}}$, a particular real-analytic flow

$$\left(\phi_{\mathcal{T},\mathcal{J}}^\mu \right)_t = \phi_t^\mu : \mathrm{Hit}_V(S) \to \mathrm{Hit}_V(S)$$

satisfying $\phi_t^\mu[\rho] = \Omega^{-1}(\Omega[\rho] + t\mu)$ using the Frenet curve ξ corresponding to ρ. This flow was called the $(\mathcal{T}, \mathcal{J})$-*parallel flow* corresponding to $\mu \in W$. In particular, $\frac{d}{dt}|_{t=0}\Omega \circ \phi_t^\mu[\rho] = \mu$ for every $[\rho] \in \mathrm{Hit}_V(S)$. This flow is known as the $(\mathcal{T}, \mathcal{J})$-*parallel flow* corresponding to μ. For the purposes of this paper, the actual construction of this flow is not so important, so we will not describe it here. The main properties we need are summarized in the following pair of propositions. The first is an immediate consequence of [SWZ17, Proposition 5.11] while the second follows easily from [SWZ17, Proposition 5.12].

Proposition 4.12. *Let $\{x, y, z\} = T, \{x', y', z'\} = T' \in \widetilde{\Theta}$, let $\rho \in \widetilde{\mathrm{Hit}}_V(S)$, and let $\mu \in W$. Also, let $t \mapsto \rho_t$ be a lift in $\widetilde{\mathrm{Hit}}_V(S)$ of $t \mapsto \phi_t^\mu[\rho]$ in $\mathrm{Hit}_V(S)$ so that $\rho_0 = \rho$, and the ρ_t-equivariant Frenet curve ξ_t satisfies $\xi_t(x) = \xi(x)$, $\xi_t(y) = \xi(y)$, and $\xi_t^{(1)}(z) = \xi^{(1)}(z)$. Let $g_t \in \mathrm{PSL}(V)$ be the group element so that $\xi_t(x') = g_t \cdot \xi(x')$, $\xi_t(y') = g_t \cdot \xi(y')$, $\xi_t^{(1)}(z') = g_t \cdot \xi^{(1)}(z')$.*

(1) If $e = \{x, y\} = \{x', y'\}$, so that z and z' lies to the right and left of (x, y) then

$$\frac{d}{dt}\bigg|_{t=0} g_t = \sum_{i=1}^{n-1} \mu_{x,y}^{i,n-i} A_{\xi(x),\xi(y)}^{i,n-i}.$$

(2) Suppose that $x < y < z < x$ and $x' = y$, $y' = z$ and $z' = x$. Then

$$\frac{d}{dt}\bigg|_{t=0} g_t = \sum_{i+j+k=n} \mu_{y,z,x}^{j,k,i} A_{\xi(y),\xi(z),\xi(x)}^{j,k,i}.$$

Proposition 4.13. *Let $\{T_1, T_2\} \in \widetilde{\mathcal{J}}$ be a bridge across the closed edge $e = \{x, y\}$ so that T_1 and T_2 lie to the right and left of (x, y) respectively. Also, for $m = 1, 2$, let p_m, q_m, z_m, w_m be as defined in Notation 2.9, and let $\widetilde{\Theta}(J, T_m)$ and $\widetilde{\mathcal{T}}(J, T_m)$ be as defined in Notation 3.6. Let $t \mapsto \rho_t$ be a lift in $\widetilde{\mathrm{Hit}}_V(S)$ of $t \mapsto \phi_t^\mu[\rho]$ in $\mathrm{Hit}_V(S)$ so that $\rho_0 = \rho$, and the ρ_t-equivariant Frenet curve ξ_t satisfies $\xi_t(p_1) = \xi(p_1)$, $\xi_t(w_1) = \xi(w_1)$, and $\xi_t^{(1)}(z_1) = \xi^{(1)}(z_1)$. Let $g_t, u_t \in \mathrm{PSL}(V)$ be the group elements so that $\xi_t(p_2) = g_t \cdot \xi(p_2)$, $\xi_t(w_2) = g_t \cdot \xi(w_2)$, $\xi_t^{(1)}(z_2) = g_t \cdot \xi^{(1)}(z_2)$, $\xi_t(p_1) = u_t \cdot \xi(p_1) = \xi(p_1)$ and $\xi_t(q_1) = u_t \cdot \xi(q_1)$. Then*

$$\frac{d}{dt}\bigg|_{t=0} u_t = K_1(T_1, T_2)$$

and

$$\frac{d}{dt}\bigg|_{t=0} g_t = K_1(T_1, T_2) - K_1(T_2, T_1) + \sum_{i=1}^{n-1} \mu_{x,y}^{i,n-i} A_{\xi(x),\xi(y)}^{i,n-i}.$$

Notation 4.14. For any admissible labelling $L \in \mathcal{A}(\rho)$, let $\mu = \mu(L) \in W$ be the vector defined in the following way:

- For every $T = \{x, y, z\} \in \widetilde{\Theta}$ so that $x < y < z < x$, define

$$\mu_{x,y,z}^{i,j,k} := a_{i,j,k}(x, T, L),$$

- For every $\{x, y\} \in \widetilde{\mathcal{T}}$, define

$$\mu_{x,y}^{i,n-i} = \alpha(x, y)a_{i,n-i}(x, y, L).$$

Recall from Definition 3.5 that $a_{i,j,k}(x, T, L) = a_{j,k,i}(y, T, L) = a_{k,i,j}(z, T, L)$ and $\alpha(x, y)a_{i,n-i}(x, y, L) = \alpha(y, x)a_{n-i,i}(y, x, L)$. Also, it is easy to verify that the condition of $L \in \mathcal{A}(\rho)$ is equivalent to the condition that $\mu(L) \in W$, so Notation 4.14 defines a linear bijection between $\mathcal{A}(\rho)$ and W. As a corollary of Proposition 4.12, Proposition 4.13 and the definition of the tangent cocycle, we have the following.

Corollary 4.15. *Let $\rho \in \widetilde{\mathrm{Hit}}_V(S)$, Σ be a hyperbolic metric on S, $(r, \mathcal{T})$ be an oriented ideal triangulation on S, and $\mathcal{J}$ a compatible bridge system. For any admissible labelling $L \in \mathcal{A}(\rho, r, \mathcal{T})$, let $\mu(L) \in W$ be the vector defined as above.*

Then the cocycle $\mu_L \in C^1(S, \mathfrak{sl}(V)_{\mathrm{Ad}\circ\rho})$ is the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle in the cohomology class

$$[\nu] := \left.\frac{d}{dt}\right|_{t=0} \phi_t^{\mu(L)}[\rho] \in H^1(S, \mathfrak{sl}(V)_{\mathrm{Ad}\circ\rho}).$$

Proof. One needs only to verify that μ_L and the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle in $[\nu]$ agree on Type B and Type C 1-simplices (see Section 4.2). The fact that they agree on Type B 1-simplices follows immediately from Proposition 4.13 and the definition (Step 4) of the $(\rho, \Sigma, \mathcal{T}, \mathcal{J})$-tangent cocycle, while the fact that they agree on Type C 1-simplices is an immediate consequence of Proposition 4.12. □

Theorem 4.8 follows immediately from this corollary.

5. Computing the Goldman symplectic form on $\mathrm{Hit}_V(S)$

Let $[\rho], [\rho']$ be a pair of representations in $\mathrm{Hit}_V(S)$. In this section, we will describe a trivialization of $T\,\mathrm{Hit}_V(S)$ that depends only on the choice of an ideal triangulation $\mathcal{T}$ on S and a compatible the bridge system $\mathcal{J}$. We will also prove that this trivialization is symplectic with respect to the Goldman symplectic form.

5.1. **The $(\mathcal{T}, \mathcal{J})$-trivialization of $T\,\mathrm{Hit}_V(S)$.** Let $\rho, \rho' \in \widetilde{\mathrm{Hit}}_V(S)$ and let $\xi, \xi' : \partial\Gamma \to \mathcal{F}(V)$ be the respective equivariant Frenet curves. Fix an ideal triangulation $\mathcal{T}$ on S and a compatible bridge system $\mathcal{J}$. Choose an orientation r on $\mathcal{T}$ and define $\Psi_{\rho,\rho'} : \mathcal{A}(\rho) \to \mathcal{A}(\rho')$ to be the map so that for every $L \in \mathcal{A}(\rho)$, $L' := \Psi_{\rho,\rho'}(L) \in \mathcal{A}(\rho')$ has the same coeffecients as L (see Definition 3.5), i.e.

- For all $(x, T) \in \widetilde{\mathcal{M}}$ and $i, j, k \in \mathbb{Z}^+$ so that $i + j + k = n$,

$$a_{i,j,k}(x, T, L) = a_{i,j,k}(x, T, L').$$

- For all $\{x, y\} \in \widetilde{\mathcal{T}}$ and for all $i = 1, \ldots, n-1$,

$$a_i(x, y, L) = a_i(x, y, L').$$

It is clear that $\Psi_{\rho,\rho'}$ is a linear isomorphism. Moreover, if $[\rho] = [\rho']$, then $\xi = g \cdot \xi'$ for some $g \in \mathrm{PGL}(V)$, so $\Phi_{\rho'} \circ \Psi_{\rho,\rho'} \circ \Phi_\rho^{-1} : T_{[\rho]}\,\mathrm{Hit}_V(S) \to T_{[\rho']}\,\mathrm{Hit}_V(S)$ is the identity map. As such, for any pair $[\rho], [\rho'] \in \mathrm{Hit}_V(S)$, the linear isomorphism

$$\Psi_{[\rho],[\rho']} := \Phi_{\rho'} \circ \Psi_{\rho,\rho'} \circ \Phi_\rho^{-1} : T_{[\rho]}\,\mathrm{Hit}_V(S) \to T_{[\rho']}\,\mathrm{Hit}_V(S).$$

does not depend on the choice of representatives ρ of $[\rho]$ and ρ' of $[\rho']$. Furthermore, it is straightforward to verify that $\Psi_{[\rho'],[\rho'']} \circ \Psi_{[\rho],[\rho']} = \Psi_{[\rho],[\rho'']}$. This thus defines a trivialization of $T\,\mathrm{Hit}_V(S)$ that depends only on $(r, \mathcal{T})$ and $\mathcal{J}$. We will soon see that this trivialization does not depend on r, thus justifying the following definition.

Definition 5.1. Let $(r, \mathcal{T})$ be an oriented ideal triangulation on S and $\mathcal{J}$ be a compatible bridge system. The trivialization of $T\,\mathrm{Hit}_V(S)$ described above is called the $(\mathcal{T}, \mathcal{J})$-*trivialization*. A vector field X on $\mathrm{Hit}_V(S)$ is a $(\mathcal{T}, \mathcal{J})$-*parallel vector field* if $X[\rho'] = \Psi_{[\rho],[\rho']}(X[\rho])$ for all $[\rho], [\rho'] \in \mathrm{Hit}_V(S)$.

In other words, a $(\mathcal{T}, \mathcal{J})$-parallel vector field is a parallel vector field with respect to the flat connection on $T\,\mathrm{Hit}_V(S)$ induced by the $(\mathcal{T}, \mathcal{J})$-trivialization. The next proposition relates the notion of a $(\mathcal{T}, \mathcal{J})$-parallel flow described studied in the companion article [SWZ17] (described after Theorem 4.11), and the notion of a $(\mathcal{T}, \mathcal{J})$-parallel vector field.

Proposition 5.2. *Choose an orientation r on $\mathcal{T}$, let $\rho \in \widetilde{\mathrm{Hit}}_V(S)$ and let $L \in \mathcal{A}(\rho) = \mathcal{A}(\rho, r, \mathcal{T})$. Then the $(\mathcal{T}, \mathcal{J})$-parallel flow $\phi_t^{\mu(L)}$ is the integral flow of the $(\mathcal{T}, \mathcal{J})$-parallel vector field X so that $X[\rho] = [\mu_L]$. (Recall that $\mu(L) \in W_{\mathcal{T}, \mathcal{J}}$ is the vector defined after Proposition 4.13).*

Proof. By definition, $\mu\left(\Psi_{\rho,\rho'}(L)\right) = \mu(L)$ for any $L \in \mathcal{A}(\rho)$. The proposition follows immediately from Corollary 4.15. $\qquad\square$

Since the $(\mathcal{T}, \mathcal{J})$-parallel flows do not depend on the choice of an orientation on $\mathcal{T}$, the $(\mathcal{T}, \mathcal{J})$-parallel vector fields also do not.

The goal of this section is to prove that any $(\mathcal{T}, \mathcal{J})$-trivialization of $T \mathrm{Hit}_V(S)$ is symplectic, i.e. the parallel transport of the flat connection on $T \mathrm{Hit}_V(S)$ induced by this trivialization preserves the Goldman symplectic form. This is equivalent to proving the following theorem.

Theorem 5.3. *Let $\mathcal{T}$ be an ideal triangulation on S and $\mathcal{J}$ a compatible bridge system. Let X_1 and X_2 be a pair of $(\mathcal{T}, \mathcal{J})$-parallel vector fields on $\mathrm{Hit}_V(S)$, then the map $\mathrm{Hit}_V(S) \to \mathbb{R}$ given by*

$$[\rho] \mapsto \omega(X_1[\rho], X_2[\rho])$$

is constant.

Theorem 5.3 will be proven in Section 5.3. By the companion article [SWZ17, Theorem 5.8], any pair of $(\mathcal{T}, \mathcal{J})$-parallel flows commute. Combining this with Theorem 5.3 yields the following corollary.

Corollary 5.4. *Let $\mathcal{T}$ be an ideal triangulation on S and $\mathcal{J}$ a compatible bridge system. Every $(\mathcal{T}, \mathcal{J})$-parallel vector field on $\mathrm{Hit}_V(S)$ is a Hamiltonian vector field.*

Proof. Let $X = X_1$ be any $(\mathcal{T}, \mathcal{J})$-parallel vector field on $\mathrm{Hit}_V(S)$. By Theorem 5.3, there are $(\mathcal{T}, \mathcal{J})$-parallel vector fields $X_2, \ldots, X_{(n^2-1)(2g-2)}$ on $\mathrm{Hit}_V(S)$ so that for $m, l \in \{1, \ldots, (n^2 - 1)(2g - 2)\}$

$$\omega(X_m, X_l) = \begin{cases} 1 & \text{if } m \in \{1, \ldots, (n^2 - 1)(g - 1)\} \text{ and } l = (n^2 - 1)(g - 1) + m; \\ -1 & \text{if } l \in \{1, \ldots, (n^2 - 1)(g - 1)\} \text{ and } m = (n^2 - 1)(g - 1) + l; \\ 0 & \text{otherwise.} \end{cases}$$

For all $m = 1, \ldots, (n^2-1)(2g-2)$, let $\mu_m \in W$ so that $\phi_t^{\mu_m}$ is the $(\mathcal{T}, \mathcal{J})$-parallel flow whose tangent field is X_m. Since $\{\phi_t^{\mu_m}\}_{m=1}^{(n^2-1)(2g-2)}$ is a collection of commuting flows, the Frobenius theorem thus allows us to integrate the distribution on $\mathrm{Hit}_V(S)$ spanned by

$$\left\{X_1, \ldots, X_{(n^2-1)(g-1)}, X_{(n^2-1)(g-1)+2}, \ldots, X_{(n^2-1)(2g-2)}\right\}$$

at every $[\rho]$ into a foliation $\mathfrak{F}$ of $\mathrm{Hit}_V(S)$ whose leaves are of codimension 1.

Since $\phi_t^{\mu}[\rho] = \Omega^{-1}(\Omega[\rho] + t\mu)$ for all $\mu \in W$ and Ω is real-analytic, we see that there is real-analytic function $f : \mathrm{Hit}_V(S) \to \mathbb{R}$ that is constant on the leaves of $\mathfrak{F}$, and satisfies $f(\phi_t^{\mu_b}[\rho]) = f[\rho] + t$, where $b = (n^2 - 1)(g - 1) + 1$. In particular,

$$\omega(X_1, X_m) = X_m(f) = \begin{cases} 1 & \text{if } m = b; \\ 0 & \text{otherwise.} \end{cases}$$

So f is the Hamiltonian function of X_1. $\qquad\square$

Corollary 5.4 also gives the following consequence.

Corollary 5.5. $\mathrm{Hit}_V(S)$ *equipped with ω is a complete integrable system.*

Proof. Let $\{\phi_t^{\mu_m}\}_{m=1}^{(n^2-1)(2g-2)}$ be the collection of commuting flows defined in the proof of Corollary 5.4. For all $m = 1, \ldots, (n^2-1)(2g-2)$, let f_m be the Hamiltonian function of $\phi_t^{\mu_m}$ (these exist by Corollary 5.4). Then by definition, the functions $\{f_m\}_{m=1}^{(n^2-1)(g-1)}$ are a half-dimensional family of Poisson commuting functions. $\square$

5.2. **A triangulation of S.** In order to prove Theorem 5.3, we need to compute the Goldman symplectic form using (2.1), and to do so, we need to choose a triangulation $\mathbb{T}$ of the surface that is well adapted to $\mathcal{T}$ and $\mathcal{J}$. We will describe such a triangulation in this section.

Choose a hyperbolic metric Σ on S, then each edge in $\widetilde{\mathcal{T}}$ is realized as a simple bi-infinite hyperbolic geodesic in $\widetilde{\Sigma}$. For each isolated edge $e \in \widetilde{\mathcal{Q}}$, choose a point q_e on e so that for all $\gamma \in \Gamma$, $\gamma \cdot q_e = q_{\gamma \cdot e}$. Then for each ideal triangle $T = \{e_1, e_2, e_3\} \in \widetilde{\Theta}$, let $\delta(T)$ be the geodesic triangle in T whose vertices are $q_{e_1}, q_{e_2}, q_{e_3}$ (see Figure 11). Observe that for all $\gamma \in \Gamma$, $\gamma \cdot \delta(T) = \delta(\gamma \cdot T)$. Thus, if $\pi : \widetilde{\Sigma} \to \Sigma$ is the covering map, then denote $\delta(\pi(T)) := \pi(\delta(T))$.
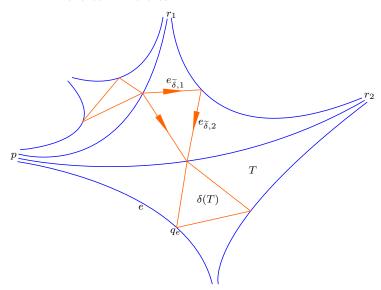
FIGURE 11. The isolated edges of $\mathcal{T}$ are drawn in blue, while the edges of $\mathbb{T}$ are drawn in orange.

Note that $\Sigma \setminus \left(\bigcup_{T \in \mathcal{T}} \overline{\delta(T)} \right)$ is topologically a union of cylinders, each of which deformation retracts onto a closed edge in $\mathcal{T}$. Denote the cylinder corresponding to the closed edge $[e] \in \mathcal{T}$ by $C = C_{[e]}$, and let $\partial_1 C$ and $\partial_2 C$ denote the boundary components of C that lie to the right and left of $r[e]$ respectively. Let $e \in \widetilde{\mathcal{T}}$ be a closed edge representative of $[e]$, and let $\gamma \in \Gamma$ be the primitive group element whose attracting and repelling fixed points are the forward and backward endpoints of $\widetilde{r}(e)$ respectively. For $m = 1, 2$, let $\mathbb{G}_m := \{\hat{g}_{m,1}, \ldots, \hat{g}_{m,k_m}\}$ denote the collection of geodesic segments whose union is $\partial_m C$.

Let $J = \{T_1, T_2\} \in \widetilde{\mathcal{J}}$ be a bridge across the closed edge $e \in \widetilde{\mathcal{P}}$ so that T_1 and T_2 lie to the right and left of $\widetilde{r}(e)$ respectively. Since we have chosen the hyperbolic

FIGURE 12. The two isolated edges of $\widetilde{\mathcal{T}}$ are drawn in blue, while the closed edge $e \in \widetilde{\mathcal{P}}$ is drawn in red. The bridge $J = \{T_1, T_2\}$ across e is drawn in green. The orange lines are the edges in
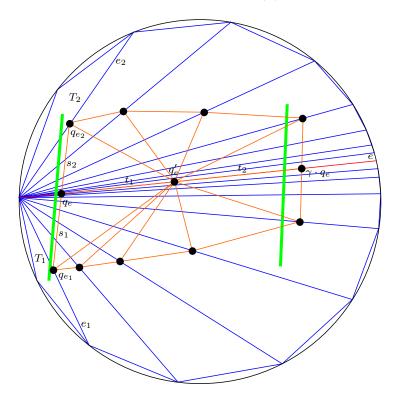
$$\bigcup_{m=1}^{2} \left(\widetilde{\mathbb{G}}_m(J,e) \cup \widetilde{\mathbb{F}}_m(J,e) \cup \{s_m, \gamma \cdot s_m\} \right) \cup \{t_1, t_2\}.$$

metric Σ on S, J is realized as a hyperbolic geodesic segment in $\widetilde{\Sigma}$ with endpoints in T'_1 and T'_2, where each T'_m is a fragmented ideal triangle in T_m that has p_{T_m} as a vertex (see Section 2.2). For $m = 1, 2$, let $e_m \in \widetilde{\mathcal{Q}}$ be the common edge of T_m and T'_m, and let s denote the geodesic segment in $\widetilde{\Sigma}$ with q_{e_1} and q_{e_2} as endpoints. Also, let $q_e := s \cap e$, and let $s_m(J,e) = s_m$ be the subsegment of s with endpoints q_{e_m} and q_e. Choose a point q'_e on the geodesic e between q_e and $\gamma \cdot q_e$. Let $t_1(J, q'_e) = t_1$ be the geodesic segment in $\widetilde{\Sigma}$ with endpoints q_e and q'_e, and let $t_2(J, q'_e) = t_2$ be the geodesic segment in $\widetilde{\Sigma}$ with endpoints q'_e and $\gamma \cdot q_e$ (see Figure 12).

For each $l = 1, \ldots, k_m$, let $g_{m,l}$ be the lift of $\hat{g}_{m,l}$ to $\widetilde{\Sigma}$, so that $\bigcup_{l=1}^{k_m} g_{m,l}$ is a connected piecewise geodesic curve whose endpoints are q_{e_m} and $\gamma \cdot q_{e_m}$. Let

$$\widetilde{\mathbb{G}}_m(J,e) := \{g_{m,l} : l = 1, \ldots, k_m\},$$

let $\widetilde{\mathbb{V}}_m(J,e)$ denote the set of endpoints of the geodesic segments in $\widetilde{\mathbb{G}}_m(J,e)$, and let $\widetilde{\mathbb{F}}_m(J,e)$ be the set of geodesic segments in $\widetilde{\Sigma}$ that have q'_e and a point in $\widehat{\mathbb{V}}_m(J,e)$
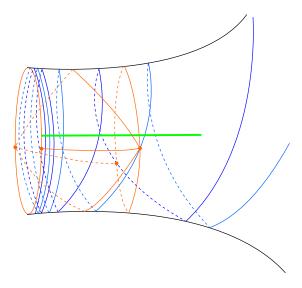
FIGURE 13. The two isolated edges of $\mathcal{T}$ "spiralling" towards the closed edge $[e] \in \mathcal{P}$ are drawn in blue, while the edges of $\mathbb{T}$ in half of $C_{[e]}$ are drawn in orange. The bridge across $[e]$ in this half of $C_{[e]}$ is drawn in green, and the edges of $\mathbb{T}_m[e]$ are drawn in orange.

as endpoints. Observe that

$$\bigcup_{l=1}^{k_m} g_{m,l} \cup s_m \cup (\gamma \cdot s_m) \cup t_1 \cup t_2$$

is a closed, piecewise geodesic curve that bounds a simply connected polygon $\widetilde{C}_m \subset \widetilde{\Sigma}$, and the geodesic segments in $\widetilde{\mathbb{F}}_m(J, e)$ lie in $\widetilde{C}_m$. In fact, we have constructed a triangulation $\widetilde{\mathbb{T}}_m(J, e)$ of $\widetilde{C}_m$ where the set of vertices is $\widetilde{\mathbb{V}}_m(J, e) \cup \{q_e, q'_e, \gamma \cdot q_e\}$ and the set of edges is $\widetilde{\mathbb{G}}_m(J, e) \cup \widetilde{\mathbb{F}}_m(J, e) \cup \{s_m, \gamma \cdot s_m, t_1, t_2\}$. Furthermore, $\widetilde{C}_1$ and $\widetilde{C}_2$ intersect along $t_1 \cup t_2$, so $\widetilde{\mathbb{T}}_1(J, e) \cup \widetilde{\mathbb{T}}_2(J, e)$ is a triangulation of $\widetilde{C}_1 \cup \widetilde{C}_2$.

It is clear from our definitions that $\widetilde{C}_1 \cup \widetilde{C}_2$ is a lift of C to $\widetilde{S}$. Also, it is easy to see that the triangulation $\widetilde{\mathbb{T}}_1(J, e) \cup \widetilde{\mathbb{T}}_2(J, e)$ of $\widetilde{C}_1 \cup \widetilde{C}_2$ descends to a triangulation $\mathbb{T}[e] = \mathbb{T}_1[e] \cup \mathbb{T}_2[e]$ of C, where $\mathbb{T}_m[e] := \pi\big(\widetilde{\mathbb{T}}_m(J, e)\big)$ (see Figure 13). Hence,

$$\mathbb{T} = \mathbb{T}(\mathcal{T}, \mathcal{J}) := \{\delta(T) : T \in \Theta\} \cup \bigcup_{[e] \in \mathcal{P}} \mathbb{T}[e]$$

is a finite triangulation of S.

Choose an enumeration on the set of vertices $\{v_1, \ldots, v_{6g-6+2|\mathcal{P}|}\}$ of the triangulation $\mathbb{T}$, where g is the genus of S. This induces an orientation on every edge of $\mathbb{T}$, so that if v_i and v_j are the endpoints of an edge of $\mathbb{T}$, then v_i is the backward endpoint and v_j is the forward endpoint if and only if $i < j$. Also, for any triangle $\delta \in \mathbb{T}$ with vertices v_i, v_j, v_k so that $i < j < k$, let $e_{\delta,1}$ denote the oriented edge of δ with endpoints v_i, v_j. Similarly, let $e_{\delta,2}$ denote the oriented edge of δ with endpoints v_j, v_k. Let $\widetilde{\delta}$ denote a triangle in $\widetilde{\mathbb{T}}$ so that $\pi(\widetilde{\delta}) = \delta$, and for $m = 1, 2$, let $e_{\widetilde{\delta},m}$ denote the oriented boundary edge of $\widetilde{\delta}$ so that $\pi(e_{\widetilde{\delta},m}) = e_{\delta,m}$. If L_1 and

L_2 are admissible labellings in $\mathcal{A}(\rho, \mathcal{T}, r)$, then (2.1) implies that

$$(5.1) \qquad \omega\big([\mu_{L_1}], [\mu_{L_2}]\big) = \sum_{\delta \in \mathbb{T}} \mathrm{sgn}(\delta)\, \mathrm{tr}\,\big(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\big).$$

5.3. The cup product of parallel vector fields.
In this section, we will prove Theorem 5.3. For simplicity, choose the enumeration of the vertices of $\mathbb{T}$ so that if v_i is a vertex that lies on a closed edge of $\mathcal{T}$ and v_j is a vertex that does not, then $i > j$. In other words, if an edge e of $\mathbb{T}$ has exactly one vertex that lies in a closed edge of $\mathcal{T}$, then the orientation on e induced by the enumeration of the vertices of $\mathbb{T}$ points towards the closed edge.

The following lemma is the key computation that motivates our choice of triangulation to compute the Goldman symplectic form.

Notation 5.6. For any $r \in \mathbb{R}$, we will denote $[r]_+ := \max\{0, r\}$.

Lemma 5.7. *Let $(F, G, H) \in \mathcal{F}(V)^3$ be a transverse triple of flags. Also, let $i, j, k, i', j', k' \in \{0, \ldots, n-1\}$ so that $i + j + k = n = i' + j' + k'$ (see Definition 3.3). Then the following hold.*

(1) $\mathrm{tr}\left(A_{F,G,H}^{i,j,k} \cdot A_{G,H,F}^{j',k',i'}\right) = [\min\{i - i', j' - j\}]_+ - \dfrac{ij'}{n}.$

(2) $\mathrm{tr}\left(A_{F,G,H}^{i,j,k} \cdot A_{F,G,H}^{i',j',k'}\right) = \min\{i, i'\} - \dfrac{ii'}{n}.$

In particular, these numbers do not depend on F, G, H.

Proof. Choose bases $\{f_1, \ldots, f_n\}$, $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_n\}$ of V so that

$$F^{(l)} = \mathrm{Span}\{f_1, \ldots, f_l\}, \quad G^{(l)} = \mathrm{Span}\{g_1, \ldots, g_l\} \quad \text{and} \quad H^{(l)} = \mathrm{Span}\{h_1, \ldots, h_l\}$$

for all $l = 1, \ldots, n-1$.

(1) Observe that $A_{F,G,H}^{i,j,k} = Y_{F,G,H}^{i,j,k} - \frac{i}{n} \cdot \mathrm{id}$, where $Y_{F,G,H}^{i,j,k} \in \mathrm{End}(V)$ is the endomorphism that has $F^{(i)}$ and $G^{(j)} + H^{(k)}$ as eigenspaces corresponding to eigenvalues 1 and 0 respectively. Similarly, $A_{G,H,F}^{j',k',i'} = Y_{G,H,F}^{j',k',i'} - \frac{j'}{n} \cdot \mathrm{id}$. This implies that

$$(5.2) \qquad \mathrm{tr}\left(A_{F,G,H}^{i,j,k} \cdot A_{G,H,F}^{j',k',i'}\right) = \mathrm{tr}\left(Y_{F,G,H}^{i,j,k} \cdot Y_{G,H,F}^{j',k',i'} \cdot\right) - \frac{ij'}{n}$$

The proof of (1) will be divided into the following four cases:

(i) $i \leq i'$,
(ii) $j' \leq j$,
(iii) $i \geq i'$, $j' \geq j$ and $k \geq k'$,
(iv) $i \geq i'$, $j' \geq j$ and $k' \geq k$.

Note that if (i) holds, then every vector in the basis

$$B_{F,G,H}^{i,j,k} := \{f_1, \ldots, f_i, g_1, \ldots, g_j, h_1, \ldots, h_k\}$$

lies in the kernel of the product $Y_{G,H,F}^{j',k',i'} \cdot Y_{F,G,H}^{i,j,k}$, so $Y_{G,H,F}^{j',k',i'} \cdot Y_{F,G,H}^{i,j,k} = 0$. Similarly, if (ii) holds, then every vector in the basis

$$B_{G,H,F}^{j',k',i'} := \{g_1, \ldots, g_{j'}, h_1, \ldots, h_{k'}, f_1, \ldots, f_{i'}\}$$

of V lies in the kernel of the product $Y_{F,G,H}^{i,j,k} \cdot Y_{G,H,F}^{j',k',i'}$, so $Y_{F,G,H}^{i,j,k} \cdot Y_{G,H,F}^{j',k',i'} = 0$. Thus, in either of these cases,

$$\mathrm{tr}\left(Y_{F,G,H}^{i,j,k} \cdot Y_{G,H,F}^{j',k',i'}\right) = \mathrm{tr}\left(Y_{G,H,F}^{j',k',i'} \cdot Y_{F,G,H}^{i,j,k}\right) = 0.$$

Now suppose that (iii) holds. In the basis $B_{F,G,H}^{i,j,k}$, the endomorphism $Y_{F,G,H}^{i,j,k}$ can be written as the diagonal matrix

$$Y_{F,G,H}^{i,j,k} = \begin{pmatrix} \mathrm{id}_i & 0 \\ 0 & 0 \cdot \mathrm{id}_{j+k} \end{pmatrix}.$$

For any $i_0 \in \{i' + 1, i' + 2, \ldots, i\}$ (this is empty if $i' = i$), we may write f_{i_0} as a linear combination of the vectors in the basis $B_{F,G,H}^{i',j',k'}$, i.e.

$$f_{i_0} = \sum_{s=1}^{i'} \alpha_{i_0,s} f_s + \sum_{s=1}^{j'} \beta_{i_0,s} g_s + \sum_{s=1}^{k'} \gamma_{i_0,s} h_s$$

for some $\alpha_{i_0,1}, \ldots, \alpha_{i_0,i'}, \beta_{i_0,1}, \ldots, \beta_{i_0,j'}, \gamma_{i_0,1}, \ldots, \gamma_{i_0,k'} \in \mathbb{R}$. One can compute that

$$Y_{G,H,F}^{j',k',i'} \cdot f_{i_0} = f_{i_0} - \left(\sum_{s=1}^{i'} \alpha_{i_0,s} f_s + \sum_{s=1}^{k'} \gamma_{i_0,s} h_s \right).$$

Similarly, for all $k_0 \in \{k' + 1, k' + 2, \ldots, k\}$ (this is empty of $k' = k$), we also have

$$Y_{G,H,F}^{j',k',i'} \cdot h_{k_0} = h_{k_0} - \left(\sum_{s=1}^{i'} \alpha'_{k_0,s} f_s + \sum_{s=1}^{k'} \gamma'_{k_0,s} h_s \right)$$

for some $\alpha'_{k_0,1}, \ldots, \alpha'_{k_0,i'}, \gamma'_{k_0,1}, \ldots, \gamma'_{k_0,k'} \in \mathbb{R}$. This implies that in the basis $B_{F,G,H}^{i,j,k}$, the endomorphism $Y_{G,H,F}^{j',k',i'}$ can be written as the matrix

$$Y_{G,H,F}^{j',k',i'} = \begin{pmatrix} 0 \cdot \mathrm{id}_{i'} & * & 0 & 0 & * \\ 0 & \mathrm{id}_{i-i'} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_j & 0 & 0 \\ 0 & * & 0 & 0 \cdot \mathrm{id}_{k'} & * \\ 0 & 0 & 0 & 0 & \mathrm{id}_{k-k'} \end{pmatrix}.$$

It is then a straightforward computation to show that

$$\mathrm{tr}\left(Y_{F,G,H}^{i,j,k} \cdot Y_{G,H,F}^{j',k',i'} \right) = i - i'.$$

By switching the roles of the bases $B_{F,G,H}^{i,j,k}$ and $B_{F,G,H}^{i',j',k'}$, we can use the same argument to prove that if (iv) holds then $\mathrm{tr}\left(Y_{F,G,H}^{i,j,k} \cdot Y_{G,H,F}^{j',k',i'} \right) = j' - j$. This then proves that in all four cases, $\mathrm{tr}\left(X_{F,G,H}^{i,j,k} \cdot X_{G,H,F}^{j',k',i'} \right) = [\min\{i - i', j' - j\}]_+$. Combining this with (5.2) proves (1).

(2) By (1) of Proposition 3.2, both $A_{F,G,H}^{i,j,k}$ and $A_{F,G,H}^{i',j',k'}$ fix the flag F. Thus, in the basis $B_{F,G,H}^{n,0,0} := \{f_1, \ldots, f_n\}$, they can be represented by the matrices

$$A_{F,G,H}^{i,j,k} = \begin{pmatrix} \frac{n-i}{n} \mathrm{id}_i & * \\ 0 & U_{n-i} \end{pmatrix} \text{ and } A_{F,G,H}^{i',j',k'} = \begin{pmatrix} \frac{n-i'}{n} \mathrm{id}_{i'} & * \\ 0 & U_{n-i'} \end{pmatrix}$$

where U_{n-i} (resp. $U_{n-i'}$) is a $(n-i) \times (n-i)$ (resp. $(n-i') \times (n-i')$) upper triangular matrix whose diagonal entries are all $-\frac{i}{n}$ (resp. $-\frac{i'}{n}$). Then

$$
\begin{aligned}
& \operatorname{tr}\left(A_{F,G,H}^{i,j,k} \cdot A_{F,G,H}^{i',j',k'} \right) \\
= \ & \operatorname{tr}\left(\begin{bmatrix} \frac{n-i}{n}\operatorname{id}_i & 0 \\ 0 & -\frac{i}{n}\operatorname{id}_{n-i} \end{bmatrix} \cdot \begin{bmatrix} \frac{n-i'}{n}\operatorname{id}_{i'} & 0 \\ 0 & -\frac{i'}{n}\operatorname{id}_{n-i'} \end{bmatrix} \right) \\
= \ & \operatorname{tr}\left(\left(\begin{bmatrix} \operatorname{id}_i & 0 \\ 0 & 0 \end{bmatrix} - \frac{i}{n}\operatorname{id} \right) \cdot \left(\begin{bmatrix} \operatorname{id}_{i'} & 0 \\ 0 & 0 \end{bmatrix} - \frac{i'}{n}\operatorname{id} \right) \right) \\
= \ & \min\{i', i\} - \frac{i'i}{n}.
\end{aligned}
$$

$\square$

Let V_1, V_2 be a pair of $(\mathcal{T}, \mathcal{J})$-parallel vector fields on $\operatorname{Hit}_V(S)$. For $\rho \in \widetilde{\operatorname{Hit}}_V(S)$, let $L_1, L_2 \in \mathcal{A}(\rho)$ be the admissible labellings corresponding to $V_1[\rho]$, $V_2[\rho]$ respectively. By (5.1) and the notation developed in Section 5.2, we can write

$$
\omega\left([\mu_{L_1}],[\mu_{L_2}]\right) = \sum_{e \in \mathcal{P}} \sum_{\delta \in \mathbb{T}[e]} \operatorname{sgn}(\delta) \operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2}) \right)
$$
$$
+ \sum_{T \in \Theta} \operatorname{sgn}(\delta(T)) \operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\delta(T),1}) \cdot \widetilde{\mu}_{L_2}(e_{\delta(T),2}) \right).
$$

To prove Theorem 5.3, it is thus sufficient to show that each of the maps

$$
[\rho] \mapsto \sum_{e \in \mathcal{P}} \sum_{\delta \in \mathbb{T}[e]} \operatorname{sgn}(\delta) \operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2}) \right)
$$

and

$$
[\rho] \mapsto \sum_{T \in \Theta} \operatorname{sgn}(\delta(T)) \operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\delta(T),1}) \cdot \widetilde{\mu}_{L_2}(e_{\delta(T),2}) \right)
$$

are constant on $\operatorname{Hit}_V(S)$. We will prove these separately in the next two lemmas.

Lemma 5.8. *Let $[e] \in \mathcal{P}$ and let V_1, V_2 be a pair of $(\mathcal{T}, \mathcal{J})$-parallel vector fields on $\operatorname{Hit}_V(S)$. For $\rho \in \widetilde{\operatorname{Hit}}_V(S)$, let $L_1, L_2 \in \mathcal{A}(\rho)$ be the admissible labellings corresponding to $V_1[\rho]$, $V_2[\rho]$ respectively. Then the function $\operatorname{Hit}_V(S) \to \mathbb{R}$ given by*

$$
[\rho] \mapsto \sum_{\delta \in \mathbb{T}[e]} \operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2}) \right)
$$

is constant.

Remark 5.9. It is important to note that for $m = 1, 2$, the coeffecients $a_i(\widetilde{r}(e'), L_m)$ for every $e' \in \widetilde{\mathcal{T}}$, and $a_{i,j,k}(x, T, L_m)$ for every $(x, T) \in \widetilde{\mathcal{M}}$ do not depend on ρ, even though L_m does.

Proof. Let $e \in \widetilde{\mathcal{P}}$ be a lift of $[e] \in \mathcal{P}$ and let $J \in \widetilde{\mathcal{J}}$ be a bridge across e. Since $\widetilde{\mu}_{L_l}$ satisfies the cocycle condition for $l = 1, 2$, we see that for any $m = 1, 2$ and any $\delta \in \mathbb{T}_m[e]$, $\widetilde{\mu}_{L_l}(e_{\widetilde{\delta},l})$ is a linear combination of endomorphisms in the set

$$
\left\{ \widetilde{\mu}_{L_l}(h) : h \in \widetilde{\mathbb{G}}_m(J, e) \cup \{t_1(J, e), t_2(J, e), s_m(J, e)\} \right\},
$$

where the coeffecients do not depend on ρ. (We use the notation from Section 5.2.)

Thus, it is sufficient to show that for all h_1, h_2 in

$$
\widetilde{\mathbb{G}}_m(J, e) \cup \left\{ t_1(J, e) \cup t_2(J, e) \cup s_m(J, e) \right\}
$$

and for any $\rho \in \widetilde{\mathrm{Hit}}_V(S)$, the number $\mathrm{tr}\left(\widetilde{\mu}_{L_1}(h_1) \cdot \widetilde{\mu}_{L_2}(h_2)\right)$ does not depend on ρ. We may assume that h_1, h_2 lies in $\widetilde{\mathbb{G}}_1(J,e) \cup \{t_1(J,e), t_2(J,e), s_1(J,e)\}$; the case when h_1, h_2 lies in $\widetilde{\mathbb{G}}_2(J,e) \cup \{t_1(J,e), t_2(J,e), s_2(J,e)\}$ is similar.

Let ξ be the ρ-equivariant Frenet curve, and choose a basis $\{f_1, \ldots, f_n\}$ of V so that $[f_l] = \xi^{(l)}(p_1) \cap \xi^{(n-l+1)}(q_1)$ for all $l = 1, \ldots, n$ (recall that p_1 and q_1 are the vertices of e as defined in Notation 2.9). Note that the barriers that intersect any $h \in \widetilde{\mathbb{G}}_1(J,e) \cup \{t_1(J,e), t_2(J,e), s_1(J,e)\}$ all have p_1 as a common vertex. By (1) of Proposition 3.2 and (1) of Proposition 3.4, we see that for all $L \in \mathcal{A}(\rho)$, $\widetilde{\mu}_L(h)$ is an endomorphism that fixes $\xi(p_1)$, so it is represented by an upper triangular matrix in the basis $\{f_1, \ldots, f_n\}$.

In particular, if we write

$$\widetilde{\mu}_{L_m}(h) = D(\widetilde{\mu}_{L_m}(h)) + N(\widetilde{\mu}_{L_m}(h)),$$

where $D(\widetilde{\mu}_{L_m}(h))$ is a diagonal matrix and $N(\widetilde{\mu}_{L_m}(h))$ is a nilpotent, upper triangular matrix in the basis $\{f_1, \ldots, f_n\}$, then

$$\mathrm{tr}\left(\widetilde{\mu}_{L_1}(h_1) \cdot \widetilde{\mu}_{L_2}(h_2)\right) = \mathrm{tr}\left(D(\widetilde{\mu}_{L_1}(h_1)) \cdot D(\widetilde{\mu}_{L_2}(h_2))\right).$$

By Lemma 5.7, we know that

$$\mathrm{tr}\left(A^{i_1, n-i_1}_{\xi(p_1), \xi(q_1)} \cdot A^{i_2, n-i_2}_{\xi(p_1), \xi(q_1)}\right) = \min\{i_1, i_2\} - \frac{i_1 i_2}{n},$$

which does not depend on $[\rho]$. Hence, to prove the lemma, we need only to show that for all $h \in \widetilde{\mathbb{G}}_1(J,e) \cup \{t_1(J,e), t_2(J,e), s_1(J,e)\}$ and for all $m = 1, 2$, when we write $D(\widetilde{\mu}_{L_m}(h))$ as a linear combination of

$$A^{1,n-1}_{\xi(p_1), \xi(q_1)}, \quad A^{2,n-2}_{\xi(p_1), \xi(q_1)}, \quad \cdots \quad, A^{n-1,1}_{\xi(p_1), \xi(q_1)},$$

the coeffecients do not depend on $[\rho]$. This is an easy consequence of the way the $(\mathcal{T}, \mathcal{J})$-parallel vector fields are defined. $\qquad\square$

Lemma 5.10. *Let $T \in \Theta$ and let V_1, V_2 be a pair of $(\mathcal{T}, \mathcal{J})$-parallel vector fields on $\mathrm{Hit}_V(S)$. For each $\rho \in \widetilde{\mathrm{Hit}}_V(S)$, let $L_1, L_2 \in \mathcal{A}(\rho)$ be the admissible labellings corresponding to $V_1[\rho], V_2[\rho]$ respectively. Then the function $\mathrm{Hit}_V(S) \to \mathbb{R}$ given by*

$$[\rho] \mapsto \mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\delta(T),1}) \cdot \widetilde{\mu}_{L_2}(e_{\delta(T),2})\right)$$

is constant.

Proof. Let r_1, r_2, p be the vertices of T so that for $m = 1, 2$, the endpoints of $e_{\delta(T),m}$ lie on the edges $\{r_m, p\}$ and $\{r_1, r_2\}$ (see Figure 11). The three barriers that intersect $e_{\delta(T),m}$ have r_m as a common vertex, so by (2) of Proposition 3.2 and (2) of Proposition 3.4, we see that

$$\widetilde{\mu}_{L_m}(e_{\delta(T),m}) = \sum_{i=1}^{n-1}\left(F_{m,i} A^{i,n-i}_{\xi(p_m),\xi(q)} + G_{m,i} A^{i,n-i}_{\xi(p_m),\xi(p_{3-m})}\right.$$
$$\left. + \sum_{j+k=n-i} H_{m,i,j,k} A^{i,j,k}_{\xi(p_m),\xi(q),\xi(p_{3-m})}\right)$$

for some $F_{m,i}, G_{m,i}, H_{m,i,j,k} \in \mathbb{R}$ that do not depend on ρ. It follows from linearity and Lemma 5.7 that $\mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\delta(T),1}) \cdot \widetilde{\mu}_{L_2}(e_{\delta(T),2})\right)$ does not depend on ρ. $\qquad\square$

By (5.1), Lemma 5.10 and Lemma 5.8 immediately imply the Theorem 5.3.

6. A SYMPLECTIC BASIS OF VECTOR FIELDS ON $\mathrm{Hit}_V(S)$.

By Corollary 5.4, we already know that every $(\mathcal{T}, \mathcal{J})$-parallel vector field is Hamiltonian. In fact, by choosing a particular ideal triangulation $\mathcal{T}$, we can systematically produce an explicit collection of $(n^2-1)(2g-2)$ vector fields on $\mathrm{Hit}_V(S)$ that gives a symplectic basis at every point in $\mathrm{Hit}_V(S)$. In [SWZ17], these vector fields are used to compute an explicit set of global Darboux coordinates for $\mathrm{Hit}_V(S)$.

6.1. **Choice of oriented ideal triangulation and compatible bridge system.**
In this section, we will choose an oriented ideal triangulation $(r, \mathcal{T})$ on S and a compatible bridge system $\mathcal{J}$. These choices will be used in the rest of Section 6.

First, we describe $\mathcal{T}$. Choose a pants decomposition on S, i.e. a maximal collection $\mathcal{P} = \{c_1, \ldots, c_{3g-3}\}$ of pairwise disjoint, pairwise non-homotopic, non-contractible, simple closed curves on S. These cut the surface into $2g-2$ pairs of pants $\mathbb{P} = \{P_1, \cdots, P_{2g-2}\}$. For each $P \in \mathbb{P}$, choose peripheral group elements $\alpha_P, \beta_P, \gamma_P \in \pi_1(P)$ so that $\alpha_P \beta_P \gamma_P = \mathrm{id}$, and P lies to the right of its boundary components, oriented according to α_P, β_P and γ_P (see Figure 14).

By choosing base points, the inclusion $P \subset S$ induces an inclusion $\pi_1(P) \subset \Gamma$, so we can view $\alpha_P, \beta_P, \gamma_P$ as group elements in Γ. For any $\gamma \in \Gamma$, denote the repelling and attracting fixed points on $\pi_1(S)$ by γ^- and γ^+ respectively. Then define

$$\widetilde{\mathcal{T}} := \bigcup_{P \in \mathbb{P}} \Gamma \cdot \big\{ \{\alpha_P^-, \alpha_P^+\}, \{\beta_P^-, \beta_P^+\}, \{\gamma_P^-, \gamma_P^+\}, \{\alpha_P^-, \beta_P^-\}, \{\beta_P^-, \gamma_P^-\}, \{\gamma_P^-, \alpha_P^-\} \big\}.$$

It is easy to see that $\widetilde{\mathcal{T}}$ is an ideal triangulation, does not depend on the choice of base points, and is Γ-invariant. Define $\mathcal{T} := \widetilde{\mathcal{T}}/\Gamma$, and note that

$$\Theta = \bigcup_{P \in \mathbb{P}} \big\{ [\alpha_P^-, \beta_P^-, \gamma_P^-], [\alpha_P^-, \gamma_P^-, \gamma_P \cdot \beta_P^-] \big\}.$$

Let $\widehat{T}_P := [\alpha_P^-, \beta_P^-, \gamma_P^-]$ and $\widehat{T}_P' := [\alpha_P^-, \gamma_P^-, \gamma_P \cdot \beta_P^-]$.

Next we choose r on $\mathcal{T}$ to be any orientation so that $\widetilde{r}\{\beta_P^-, \alpha_P^-\} = (\beta_P^-, \alpha_P^-)$, $\widetilde{r}\{\gamma_P^-, \beta_P^-\} = (\gamma_P^-, \beta_P^-)$ and $\widetilde{r}\{\alpha_P^-, \gamma_P^-\} = (\alpha_P^-, \gamma_P^-)$ for any $P \in \mathbb{P}$. The orientation r evaluated on the closed edges in $\mathcal{P}$ can be arbitrary.

Finally, we choose the bridge system $\mathcal{J}$ compatible with $\mathcal{T}$ in the following way. Note that the pants decomposition we chose is naturally in one-to-one correspondence with the closed edges in $\mathcal{P}$, so we will identify $\mathcal{P}$ and the pants decomposition without further comment. Let P_1 and P_2 be the pairs of pants (possibly $P_1 = P_2$) that share some $\hat{e} \in \mathcal{P}$ as a common boundary component. Choose a representative $e \in \widetilde{\mathcal{P}}$ of $\hat{e}$. For $l = 1, 2$, choose a representative $T_{P_l} \in \widetilde{\Theta}$ of $\widehat{T}_{P_l}$ so that T_{P_l} and e share a vertex. Let $J_e := \{T_{P_1}, T_{P_2}\}$, let

$$\widetilde{\mathcal{J}} := \bigcup_{\hat{e} \in \mathcal{P}} \Gamma \cdot J_e$$

and define $\mathcal{J} := \widetilde{\mathcal{J}}/\Gamma$.

Let $\mathcal{B}$ be the barrier system associated to $\mathcal{T}$. Henceforth, we will choose a hyperbolic metric Σ on S to realize the elements in $(r, \mathcal{B})$ and $\mathcal{J}$ as oriented gedesics, geodesic rays, or geodesic segments on Σ. With this choice, every $\widehat{T} \in \Theta$ is realized as a hyperbolic ideal triangle on Σ, and every $P \in \mathbb{P}$ is realized as an embedded pair of pants in Σ with geodesic boundary. Note then that the interior of P is $\widehat{T}_P \cup \widehat{T}_P'$.
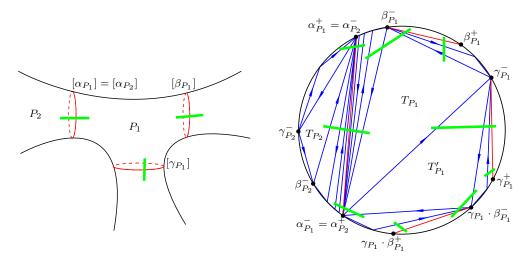
FIGURE 14. The two pairs of pants P_1 and P_2 drawn on the left share a common boundary edge $[\alpha_{P_1}] = [\alpha_{P_2}] \in \mathcal{P}$. On the right, the closed and isolated edges of $\widetilde{\mathcal{T}}$ are drawn in red and blue respectively, while the bridges in $\widetilde{\mathcal{J}}$ are drawn in green. The arrows on the isolated edges indicate the orientation r. T_{P_1}, T'_{P_1} and T_{P_2} are lifts of $\widehat{T}_{P_1}$, $\widehat{T}'_{P_1}$ and $\widehat{T}_{P_2}$ respectively.

The choice of $\mathcal{T}$, r, and $\mathcal{J}$ and Σ that we made in this section will be used in the rest of Section 6 without further comment.

6.2. **Special admissible labellings.** For the rest of Section 6, we will fix a choice of $\rho \in \widetilde{\mathrm{Hit}}_V(S)$, and let ξ be the ρ-equivariant Frenet curve. The goal of this section is to define the admissible labellings in $\mathcal{A}(\rho)$ that give a symplectic basis of $T_{[\rho]} \mathrm{Hit}_V(S)$ via the linear isomorphism Φ_ρ.

Since ρ and ξ are fixed, we will henceforth denote the endomorphisms $A^{i,j,k}_{\xi(x),\xi(y),\xi(z)}$ and $A^{i,n-i}_{\xi(x),\xi(y)}$ simply by $A^{i,j,k}_{x,y,z}$ and $A^{i,n-i}_{x,y}$ respectively. Using this, we specify the following notation.

Notation 6.1.
   (1) For any ideal triangle $\widehat{T} = [x,y,z] \in \Theta = \Theta_{\mathcal{T}}$ so that $x < y < z < x$, and any $i,j,k \in \mathbb{Z}^+$ so that $i+j+k = n$, let $L^{i,j,k}_{x,y,z} \in \mathcal{L}(\rho,\mathcal{T})$ denote the legal labelling so that
       • for any representative $T = \{x,y,z\} \in \widetilde{\Theta}$ of $\widehat{T}$, $L^{i,j,k}_{x,y,z}(x,T) = A^{i,j,k}_{x,y,z}$, $L^{i,j,k}_{x,y,z}(y,T) = A^{j,k,i}_{y,z,x}$, and $L^{i,j,k}_{x,y,z}(z,T) = A^{k,i,j}_{z,x,y}$.
       • for any other barrier $b \in \widetilde{\mathcal{B}}$, $L^{i,j,k}_{x,y,z}(b) = 0$.
   (2) For any edge $\hat{e} = [x,y] \in \mathcal{T}$ and $i \in \{1,\dots,n-1\}$, let $L^{i,n-i}_{x,y} \in \mathcal{L}(\rho,\mathcal{T})$ denote the legal labelling so that
       • for any representative $e = \{x,y\} \in \widetilde{\mathcal{T}}$ of $\hat{e}$, $L^{i,n-i}_{x,y}(e) = \alpha(y,x)A^{i,n-i}_{x,y}$
       • for any other barrier $b \in \widetilde{\mathcal{B}}$, $L^{i,n-i}_{x,y}(b) = 0$.

In other words, $L^{i,j,k}_{x,y,z}$ is the legal labelling where the only non-zero coefficients are $a_{i,j,k}\left(x,T,L^{i,j,k}_{x,y,z}\right) = a_{j,k,i}\left(y,T,L^{i,j,k}_{x,y,z}\right) = a_{k,i,j}\left(z,T,L^{i,j,k}_{x,y,z}\right) = 1$. Similarly,

$L_{x,y}^{i,n-i}$ is the legal labelling where the only non-zero coeffecients are $a_i(y,x,L_{x,y,z}^{i,j,k}) = \alpha(y,x)$. We will also use the notation

$$L_{x,y,z}^{0,j,k} := L_{y,z}^{j,k}, \quad L_{x,y,z}^{i,0,k} := L_{x,z}^{i,k} \quad \text{and} \quad L_{x,y,z}^{i,j,0} := L_{x,y}^{i,j}.$$

Note that for all $i,j,k \in \{1,\ldots,n-1\}$ so that $i+j+k = n$, we have $L_{x,y,z}^{i,j,k} = L_{y,z,x}^{j,k,i} = L_{z,x,y}^{k,i,j}$, and $L_{\gamma\cdot x,\gamma\cdot y,\gamma\cdot z}^{i,j,k} = L_{x,y,z}^{i,j,k}$ for all $\gamma \in \Gamma$.

Using this notation, we will define four families of admissible labellings in $\mathcal{A}(\rho)$. The first two families of admissible labellings are associated to pairs of pants in $\mathbb{P}$.

Definition 6.2. Let $P \in \mathbb{P}$, let $[x,y,z] = \widehat{T}_P$, and let $[x',y',z'] = \widehat{T}'_P$ so that

- $x < y < z < x$,
- $x = \gamma_x \cdot x'$, $y = \gamma_y \cdot y'$, and $z = \gamma_z \cdot z'$ for some $\gamma_x, \gamma_y, \gamma_z \in \Gamma$.

(Recall that $\widehat{T}_P$ and $\widehat{T}'_P$ were defined in Section 6.1). Also, let $i,j,k \in \mathbb{Z}^+$ such that $i+j+k=n$ (see Figure 15).

(1) The (i,j,k)-*eruption labelling* associated to P is

$$E_{x,y,z}^{i,j,k} = E_{y,z,x}^{j,k,i} = E_{z,x,y}^{k,i,j} := \frac{1}{2}\left(L_{x,y,z}^{i,j,k} - L_{x',z',y'}^{i,k,j}\right).$$

(2) The (i,j,k)-*hexagon labelling* associated to P is

$$H_{x,y,z}^{i,j,k} = H_{y,z,x}^{j,k,i} = H_{z,x,y}^{k,i,j} :=$$
$$L_{x,y,z}^{i,j+1,k-1} - L_{x,y,z}^{i-1,j+1,k} + L_{x,y,z}^{i-1,j,k+1} - L_{x,y,z}^{i,j-1,k+1} + L_{x,y,z}^{i+1,j-1,k} - L_{x,y,z}^{i+1,j,k-1}$$
$$+ L_{x',z',y'}^{i,k-1,j+1} - L_{x',z',y'}^{i-1,k,j+1} + L_{x',z',y'}^{i-1,k+1,j} - L_{x',z',y'}^{i,k+1,j-1} + L_{x',z',y'}^{i+1,k,j-1} - L_{x',z',y'}^{i+1,k-1,j}.$$

In the above definition, note that $x' < z' < y' < x'$, so $L_{x',z',y'}^{i,k,j}$ is defined. Also, every $P \in \mathbb{P}$ has $\frac{(n-1)(n-2)}{2}$ eruption labellings and $\frac{(n-1)(n-2)}{2}$ hexagon labellings, and one can check that they are indeed admissible. One should think of these as infinitesimal deformations of ρ that changes ρ restricted to P while keeping the "rest of the representation unchanged". When $n=3$, there is one eruption labelling and one hexagon labelling, and they are (multiples of) the tangent vectors to the eruption and internal bulging flows described by Wienhard-Zhang [WZ17].

The other two families of admissible labellings correspond to closed edges in $\mathcal{P}$.

Definition 6.3. Let $\hat{e} \in \mathcal{P}$ and let $P_1, P_2 \in \mathbb{P}$ be the pairs of pants that share $\hat{e}$ as a common boundary component (it is possible that $P_1 = P_2$), so that P_1 and P_2 lie to the right and left of $r(\hat{e})$ respectively. For $l = 1,2$, let $[x_l, y_l, z_l] = \widehat{T}_{P_l}$, $[x'_l, y'_l, z'_l] = \widehat{T}'_{P_l}$ so that

- $x_l < y_l < z_l < x_l$,
- $x_l = \gamma_{x_l} \cdot x'_l$, $y_l = \gamma_{y_l} \cdot y'_l$, and $z_l = \gamma_{z_l} \cdot z'_l$ for some $\gamma_{x_l}, \gamma_{y_l}, \gamma_{z_l} \in \Gamma$,
- there is a representative $e \in \widetilde{\mathcal{P}}$ of $\hat{e}$ so that $e = \{x_1, x_2\}$.

Also, let $i = 1,\ldots,n-1$ (see Figure 15).

(1) The i-*twist labelling* associated to $\hat{e}$ is

$$S_{x_1,x_2}^i = \frac{1}{2}L_{x_1,x_2}^{i,n-i}.$$

(2) The i-*length labelling* associated to $\hat{e}$ is

$$Y_{x_1,x_2}^i := Z_{x_1,x_2}^i + E_{x_1,y_1,z_1}^{i,n-i,1} - E_{x_1,y_1,z_1}^{i-1,n-i+1,1} - E_{x_2,y_2,z_2}^{n-i,i,1} + E_{x_2,y_2,z_2}^{n-i-1,i+1,1},$$
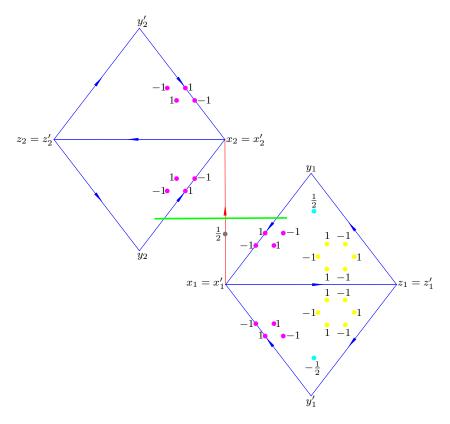
FIGURE 15. A closed edge of $\widetilde{\mathcal{P}}$ is drawn in red, a bridge in $\widetilde{\mathcal{J}}$ across the closed edge is drawn in green, and isolated edges in $\widetilde{\mathcal{Q}}$ are draw in blue. Each colored dot in an ideal triangle represents a triple of integers $i, j, k \in \mathbb{Z}^+$ so that $i + j + k = n$. Similarly, each dot along an edge represents an integer $i \in \{1, \ldots, n - 1\}$. In the picture above, we have drawn a diagramatic representative for the eruption, hexagon, twist and lozenge labellings in turquoise, yellow, grey, and purple respectively. The numbers above each of the colored dots are the corresponding coeffecients. For example, if the turquoise dot in the ideal triangle $\{x_1, y_1, z_1\}$ represents (i, j, k), then the pair of turquoise dots with the numbers they are labelled with represent $E^{i,j,k}_{x_1,y_1,z_1}$.

where

$$
\begin{aligned}
Z^i_{x_1,x_2} := & - L^{i+1,n-i-1,0}_{x_1,y_1,z_1} + L^{i,n-i,0}_{x_1,y_1,z_1} + L^{i,n-i-1,1}_{x_1,y_1,z_1} - L^{i-1,n-i,1}_{x_1,y_1,z_1} \\
& - L^{i+1,0,n-i-1}_{x'_1,z'_1,y'_1} + L^{i,0,n-i}_{x'_1,z'_1,y'_1} + L^{i,1,n-i-1}_{x'_1,z'_1,y'_1} - L^{i-1,1,n-i}_{x'_1,z'_1,y'_1} \\
& - L^{n-i+1,i-1,0}_{x_2,y_2,z_2} + L^{n-i,i,0}_{x_2,y_2,z_2} + L^{n-i,i-1,1}_{x_2,y_2,z_2} - L^{n-i-1,i,1}_{x_2,y_2,z_2} \\
& - L^{n-i+1,0,i-1}_{x'_2,z'_2,y'_2} + L^{n-i,0,i}_{x'_2,z'_2,y'_2} + L^{n-i,1,i-1}_{x'_2,z'_2,y'_2} - L^{n-i-1,1,i}_{x'_2,z'_2,y'_2}
\end{aligned}
$$

We will also refer to $Z^i_{x_1,x_2}$ as the i-lozenge labelling.

One can verify that the length and twist labellings are indeed admissible labellings, and that every $\hat{e} \in \mathcal{P}$ has $n-1$ twist labellings, and $n-1$ length labellings. The twist labellings are tangent vectors to certain generalized twist flows described previously by Goldman [Gol86], and one should think of the length labellings as infinitesimal deformations that change certain lengths of $\hat{e}$ while keeping the "rest of the representation unchanged".

Definition 6.4. Any eruption, hexagon, twist, or length labelling is a *special admissible labelling* at ρ. Denote the set of special admissible labellings at ρ by $\mathcal{SA}(\rho)$.

Since $|\mathbb{P}| = 2g - 2$ and $|\mathcal{P}| = 3g - 3$, it is immediate $|\mathcal{SA}(\rho)| = (n^2 - 1)(2g - 2)$. The purpose of Section 6 is to prove the following theorem.

Theorem 6.5. *Let $\rho \in \widetilde{\text{Hit}}_V(S)$, and let $L_1, L_2 \in \mathcal{A}(\rho)$ be any pair of special admissible labellings. Also, let $P \in \mathbb{P}$, let $\hat{e} \in \mathcal{P}$, let $[x, y, z] = \hat{T}_P$, let $e \in \widetilde{\mathcal{P}}$ be a representative of $\hat{e}$, and let $\widetilde{r}(e) = (x_1, x_2)$.*

- *If $L_1 = S^i_{x_1, x_2}$, then*

$$\omega([\mu_{L_1}], [\mu_{L_2}]) = \begin{cases} 1 & \text{if } L_2 = Z^i_{x_1, x_2}; \\ 0 & \text{otherwise.} \end{cases}$$

- *If $L_1 = E^{i,j,k}_{x,y,z}$, then*

$$\omega([\mu_{L_1}], [\mu_{L_2}]) = \begin{cases} -1 & \text{if } L_2 = H^{i,j,k}_{x,y,z}; \\ 0 & \text{otherwise.} \end{cases}$$

- *If $L_1 = H^{i,j,k}_{x,y,z}$, then*

$$\omega([\mu_{L_1}], [\mu_{L_2}]) = \begin{cases} 1 & \text{if } L_2 = E^{i,j,k}_{x,y,z}; \\ 0 & \text{otherwise.} \end{cases}$$

- *If $L_1 = Y^i_{x_1, x_2}$, then*

$$\omega([\mu_{L_1}], [\mu_{L_2}]) = \begin{cases} -1 & \text{if } L_2 = S^i_{x_1, x_2}; \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Theorem 6.5 involves an elementary but long computation. In Section 6.4, we state and prove some observations about special admissible labellings. Repeated applications of these observations will explicitly compute for us the symplectic pairing between any pair of special admissible labellings. We perform the explicit computations using these observations in Section 6.5.

Theorem 6.5 states that the image of the set of special admissible labellings in $\mathcal{A}(\rho)$ under Φ_ρ give a symplectic basis of $T_{[\rho]} \text{Hit}_V(S)$. In particular, $\mathcal{SA}(\rho)$ is a basis of $\mathcal{A}(\rho)$. Theorem 6.5 and Theorem 5.3 together imply the following corollary.

Corollary 6.6. *The set of $(\mathcal{T}, \mathcal{J})$-parallel vector fields associated to the special admissible labellings in $\mathcal{A}(\rho)$ give a symplectic basis of vector fields for $T \text{Hit}_V(S)$.*

Using Notation 4.14, we can explicitly describe the $(\mathcal{T}, \mathcal{J})$-parallel flows ϕ^μ_t whose tangent vector at $[\rho]$ correspond to the special admissible labellings at ρ via Φ_ρ. We will now give this explicit description, which will be used in [SWZ17] to compute an explicit Darboux basis for $\text{Hit}_V(S)$. It is sufficient to describe $\mu(L)$ for any special admissible labelling L.

Let $\mu \in W$ be an arbitrary vector. To simplify notation, we will denote $\mu_{x,y,z}^{i,n-i,0} = \mu_{z,x,y}^{0,i,n-i} = \mu_{y,z,x}^{n-i,0,i} := \mu_{x,y}^{i,n-i}$ for all $[x,y,z] \in \Theta$. With this, an arbitrary vector $\mu \in \mathbb{R}^{(n^2-1)(2g-2)+|\mathcal{P}|(n-1)}$ can be written in coordinates as

$$\mu = \left(\left(\mu_{x,y}^{i,n-i} = \mu_{y,x}^{n-i,i} \right)_{i \in \{1,\ldots,n-1\};[x,y] \in \mathcal{P}}, \right.$$
$$\left. \left(\mu_{x,y,z}^{i,j,k} = \mu_{y,z,x}^{j,k,i} = \mu_{z,x,y}^{k,i,j} \right)_{i,j,k \in \{0,\ldots,n-1\};i+j+k=n;[x,y,z] \in \Theta} \right).$$

Proposition 6.7. *Let $P \in \mathbb{P}$, and let $x,y,z,x',y',z' \in \partial\Gamma$ be as defined in Definition 6.2. Also, let $[x_1,x_2] \in \mathcal{P}$, let $P_1, P_2 \in \mathbb{P}$ be the pairs of pants that share $[x_1,x_2]$ as a common boundary (it is possible that $P_1 = P_2$), and for $l = 1,2$, let $x_l, y_l, z_l, x'_l, y'_l, z'_l$ be as defined in Definition 6.2 (see Figure 15).*

- *For all $i,j,k \in \{0,\ldots,n-1\}$ so that $i+j+k=n$, $\mu = \mu\left(E_{x,y,z}^{i,j,k}\right) \in W$ is the vector so that $\mu_{a,b}^{i',n-i'} = 0$ for all $[a,b] \in \mathcal{P}$ and $i' \in \{1,\ldots,n-1\}$, and*

$$\mu_{a,b,c}^{i',j',k'} = \begin{cases} \dfrac{1}{2} & \text{if } (i',j',k') = (i,j,k) \text{ and } (a,b,c) = (x,y,z); \\ -\dfrac{1}{2} & \text{if } (i',j',k') = (i,k,j) \text{ and } (a,b,c) = (x',z',y'); \\ 0 & \text{otherwise.} \end{cases}$$

- *For all $i,j,k \in \{0,\ldots,n-1\}$ so that $i+j+k=n$, $\mu = \mu\left(H_{x,y,z}^{i,j,k}\right) \in W$ is the vector so that $\mu_{a,b}^{i',n-i'} = 0$ for all $[a,b] \in \mathcal{P}$ and $i' \in \{1,\ldots,n-1\}$, and*

$$\mu_{a,b,c}^{i',j',k'} = \begin{cases} 1 & \begin{array}{l} \text{if } (i',j',k') = (i,j+1,k-1),\ (i-1,j,k+1) \\ \text{or } (i+1,j-1,k),\ \text{and } (a,b,c) = (x,y,z); \end{array} \\ 1 & \begin{array}{l} \text{if } (i',j',k') = (i,k-1,j+1),\ (i-1,k+1,j) \\ \text{or } (i+1,k,j-1),\ \text{and } (a,b,c) = (x',z',y'); \end{array} \\ -1 & \begin{array}{l} \text{if } (i',j',k') = (i-1,j+1,k),\ (i,j-1,k+1) \\ \text{or } (i+1,j,k-1),\ \text{and } (a,b,c) = (x,y,z); \end{array} \\ -1 & \begin{array}{l} \text{if } (i',j',k') = (i-1,k,j+1),\ (i,k+1,j-1) \\ \text{or }\ (i+1,k-1,j),\ \text{and } (a,b,c) = (x',z',y'); \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

- *For all $i \in \{1,\ldots,n-1\}$, $\mu = \mu\left(S_{x,x'}^i\right) \in W$ is the vector so that $\mu_{a,b,c}^{i',j',k'} = 0$ for all $[a,b,c] \in \Theta$ and $i',j',k' \in \{0,\ldots,n-1\}$ so that $i+j+k=n$, and*

$$\mu_{a,b}^{i',n-i'} = \begin{cases} \dfrac{1}{2} & \text{if } i' = i \text{ and } (a,b) = (x,x'); \\ 0 & \text{otherwise.} \end{cases}$$

- *For all $i \in \{1,\ldots,n-1\}$,*

$$\mu\left(Z_{x,x'}^i\right) = \mu\left(Y_{x,x'}^i\right) + \mu\left(E_{x,y,z}^{i,n-i-1,1}\right) - \mu\left(E_{x,y,z}^{i-1,n-i,1}\right) + \mu\left(E_{x,y,z}^{n-i,i-1,1}\right) - \mu\left(E_{x,y,z}^{n-i-1,i,1}\right),$$

where $\mu = \mu\left(Y_{x,x'}^i\right) \in W$ is the vector so that $\mu_{a,b}^{i',n-i'} = 0$ for all $[a,b] \in \mathcal{P}$ and $i' \in \{1,\ldots,n-1\}$, and

$$
\mu_{a,b,c}^{i',j',k'} = \begin{cases}
1 & \begin{aligned}&\text{if } (i',j',k') = (i,n-i,0) \text{ or } (i,n-i-1,1), \\ &\text{and } (a,b,c) = (x_1,y_1,z_1);\end{aligned} \\[6pt]
1 & \begin{aligned}&\text{if } (i',j',k') = (i,0,n-i) \text{ or } (i,1,n-i-1), \\ &\text{and } (a,b,c) = (x_1',z_1',y_1');\end{aligned} \\[6pt]
1 & \begin{aligned}&\text{if } (i',j',k') = (n-i,i,0) \text{ or } (n-i,i-1,1), \\ &\text{and } (a,b,c) = (x_2,y_2,z_2);\end{aligned} \\[6pt]
1 & \begin{aligned}&\text{if } (i',j',k') = (n-i,0,i) \text{ or } (n-i,1,i-1), \\ &\text{and } (a,b,c) = (x_2',z_2',y_2');\end{aligned} \\[6pt]
-1 & \begin{aligned}&\text{if } (i',j',k') = (i+1,n-i-1,0) \text{ or } (i-1,n-i,1), \\ &\text{and } (a,b,c) = (x_1,y_1,z_1);\end{aligned} \\[6pt]
-1 & \begin{aligned}&\text{if } (i',j',k') = (i+1,0,n-i-1) \text{ or } (i-1,1,n-i), \\ &\text{and } (a,b,c) = (x_1',z_1',y_1');\end{aligned} \\[6pt]
-1 & \begin{aligned}&\text{if } (i',j',k') = (n-i+1,i-1,0) \text{ or } (n-i-1,i,1), \\ &\text{and } (a,b,c) = (x_2,y_2,z_2);\end{aligned} \\[6pt]
-1 & \begin{aligned}&\text{if } (i',j',k') = (n-i+1,0,i-1) \text{ or } (n-i-1,1,i), \\ &\text{and } (a,b,c) = (x_2',z_2',y_2');\end{aligned} \\[6pt]
0 & \text{otherwise.}
\end{cases}
$$

The above proposition is an immediate consequence of Notation 4.14, Definition 6.2 and Definition 6.3.

6.3. **Choice of triangulation on Σ.** To prove Theorem 6.5, we will have to compute $\omega\left([\mu_{L_1}],[\mu_{L_2}]\right)$ for every pair of special admissible labellings $L_1, L_2 \in \mathcal{A}(\rho)$. This computation will be done using the cup product formula from simplicial cohomology, just like we did in Section 5.3. To do so, we use the triangulation $\mathbb{T}$ which we constructed from $\mathcal{T}$ and $\mathcal{J}$, as described in Section 5.2. In this section, we will describe $\mathbb{T}$ for our particular choice of $\mathcal{T}$ and $\mathcal{J}$, as well as describe the orientation on $\mathbb{T}$ which we will use to compute $\omega\left([\mu_{L_1}],[\mu_{L_2}]\right)$. The point of doing so is to set up notation that will be used in Section 6.4 and Section 6.5.

Recall that we have chosen a hyperbolic metric Σ on S, so the elements in $(r,\mathcal{B})$ and $\mathcal{J}$ are realized as oriented geodesics, geodesic rays, and geodesic segments in Σ. For each isolated edge $e \in \widetilde{\mathcal{Q}}$, choose a point d_e on each isolated edge $e \in \widetilde{\mathcal{Q}}$ so that $\gamma \cdot d_e = d_{\gamma \cdot e}$ for all $\gamma \in \Gamma$. With these choices, let $\mathbb{T}$ be the triangulation of Σ constructed in Section 5.2. Every isolated edge in $\mathcal{Q}$ contains a vertex of $\mathbb{T}$ and every closed edge in $\mathcal{P}$ has contains two vertices of $\mathbb{T}$, so $\mathbb{T}$ has $12(g-1)$ vertices.

Observe that for each pair of pants $P \in \mathbb{P}$, there are two triangles of $\mathbb{T}$ with the same vertices that lie in the interior of P. One of them, denoted δ_P, lies in the ideal triangle $\widehat{T}_P$, and the other, denoted δ_P', lies in the ideal triangle $\widehat{T}_P'$. Let

$$
\mathbb{T}(P) := \{\delta_P, \delta_P'\} \quad \text{and} \quad \mathbb{T}(\mathbb{P}) := \bigcup_{P \in \mathbb{P}} \mathbb{T}(P),
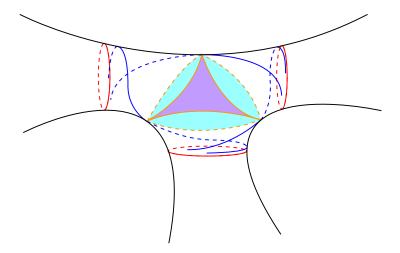$$

FIGURE 16. P is the pair of pants whose three boundary components are drawn in red. The blue lines are the isolated edges in P. The triangles δ_P, δ'_P are shaded in turquoise and purple. The boundary cylinders of P each have a red boundary component consisting of external boundary edges, and a yellow boundary component consisting of internal boundary edges.

and note that $|\mathbb{T}(\mathbb{P})| = 4(g-1)$. The edges of $\mathbb{T}(P)$ are called the *internal edges* of P. Also, each boundary component of P is a union of two edges of $\mathbb{T}$, which we refer to as the *external edges* of P (see Figure 16).

Clearly, $P \setminus (\delta_P \cup \delta'_P)$ is a union of three cylinders $C_{[\alpha_P]}, C_{[\beta_P]}, C_{[\gamma_P]}$ that contain $[\alpha_P^-, \alpha_P^+], [\beta_P^-, \beta_P^+], [\gamma_P^-, \gamma_P^+]$ respectively as boundary components. We refer to these three cylinders as the *boundary cylinders of P*. For any $\eta = \alpha_P, \beta_P$ or γ_P, one of the two boundary components of $C_{[\eta]}$ is the union of two internal edges of P, which we refer to as the *internal boundary edges* of $C_{[\eta]}$. The other component of $\partial C_{[\eta]}$ is a boundary component of P, and is thus a union of two external edges of P. We refer to these as the *external boundary edges* of $C_{[\eta]}$. The triangulation $\mathbb{T}$ restricted to $C_{[\eta]}$ has four edges that do not lie in $\partial C_{[\eta]}$, which we call the *crossing edges* of $C_{[\eta]}$. Note also that $C_{[\eta]}$ is a union of four triangles in $\mathbb{T}$, which we denote by $\mathbb{T}(C_{[\eta]})$.

An elementary count gives that $\mathbb{T}$ restricted to P consists of twelve crossing edges, six external edges, and six internal edges. Hence, $\mathbb{T}$ has a total of $42(g-1)$ edges and $28(g-1)$ triangles. Let $\mathbb{T}(\mathcal{P})$ denote the set of triangles in $\mathbb{T}$ that have a vertex in a closed edge in $\mathcal{P}$, and note that $\mathbb{T} = \mathbb{T}(\mathbb{P}) \cup \mathbb{T}(\mathcal{P})$. It is easy to see that, $\mathbb{T}(\mathcal{P})$ consists of $24(g-1)$ triangles.

Let P_1 and P_2 be two pairs of pants that share a common boundary component $\hat{e} \in \mathcal{P}$. For $l = 1, 2$, let $C_l \subset P_l$ be the boundary cylinder containing $\hat{e}$ as a boundary component. Note that there is a unique crossing edge $\hat{h}_l$ of C_l so that $\hat{h}_1 \cup \hat{h}_2$ is a geodesic segment whose end points that lie in the boundary of a pair of fragmented ideal triangles that are connected by a bridge. We refer to $\hat{h}_l$ as the *bridge parallel edge* of C_l (see Figure 17).

Choose an enumeration $\{r_1, \ldots, r_{12g-12}\}$ of the vertices of $\mathbb{T}$ with the following property:

- If r_a, r_b, r_c are the vertices of δ_P that lie on the geodesics $\{\alpha_P^-, \beta_P^-\}$, $\{\beta_P^-, \gamma_P^-\}$, $\{\gamma_P^-, \alpha_P^-\}$ respectively, then $b < a < c$.
- If r_a is a vertex in the interior of a pair of pants in $\mathbb{P}$ and r_b is a vertex in a pants curve in $\mathcal{P}$, then $a < b$.
- If r_a and r_b are the two vertices on the same pants curve in $\mathcal{P}$ and r_a is the vertex that lies on the bridge-parallel edge, then $a < b$.

This enumeration induces an orientation on the edges of $\mathbb{T}$, and hence an orientation on each triangle given by $\mathbb{T}$ as described in Section 5.2. Note that for all $P \in \mathbb{P}$, the orientation on δ_P is anti-clockwise with respect to the orientation on Σ, while the orientation on δ_P' is clockwise. Also, the crossing edges of $\mathbb{T}$ are oriented so that the forward endpoint lies in a closed edge in $\mathcal{P}$. Furthermore, for any closed edge $\hat{e} \in \mathcal{P}$, the two edges of $\mathbb{T}$ that lie in $\hat{e}$ are oriented so that their backward vertex lies in the bridge-parallel edge (see Figure 17).
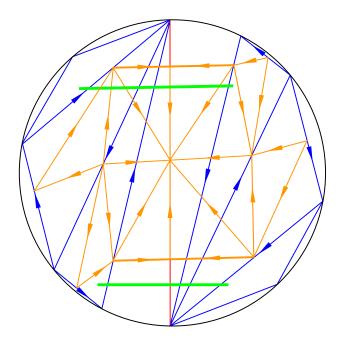

FIGURE 17. A closed edge in $\widetilde{\mathcal{P}}$ is drawn in red, the isolated edges in $\widetilde{\mathcal{Q}}$ are drawn in blue, and the bridges in $\widetilde{\mathcal{J}}$ are drawn in green. The edges of the triangulation $\mathbb{T}$ are drawn in orange. The bridge parallel edges of $\mathbb{T}$ are the thicker orange lines.

The computation $\omega\big([\mu_{L_1}], [\mu_{L_2}]\big)$ often involves some fine calculations on the boundary cylinders. To that end, we develop the following notation describing the edges and triangles in the boundary cylinders.

Let $C = C_{[\eta]}$ be a boundary cylinder of $P \in \mathbb{P}$ ($\eta = \alpha_P, \beta_P$ or γ_P) and let $\hat{h}$ be the bridge-parallel edge of C. Then let $\delta_1, \ldots, \delta_4$ be the triangles in $\mathbb{T}(C)$ so that δ_1 and δ_4 lie to the left and right of $\hat{h}$ (equipped with its orientation), δ_2 shares a crossing edge of C with δ_1, and δ_3 shares a crossing edge with δ_4. For each $l = 1, \ldots, 4$, choose a lift $\widetilde{\delta}_l$ of δ_l so that (see Figure 18)

- for $l = 1, 2, 3$, $\widetilde{\delta}_l$ and $\widetilde{\delta}_{l+1}$ share a common edge,
- $\widetilde{\delta}_4$ and $\eta^{-1} \cdot \widetilde{\delta}_1$ share a common edge.

Let $h_1 = h_{1,\eta}$ be the (oriented) lift of $\hat{h}$ that lies in the boundary of $\widetilde{\delta}_1$. Then

- for $l = 1, \ldots, 4$, let $\hat{k}_l$ be the internal or external boundary edge of C that is an edge of δ_l, oriented so that the backward vertex of k_l is a vertex of $\hat{h}$. Let $k_l = k_{l,\eta}$ be the lift of $\hat{k}_l$ that is an edge of $\widetilde{\delta}_l$.
- for $l = 1, 2, 3$, let $h_{l+1} = h_{l+1,\eta}$ be the common edge of $\widetilde{\delta}_l$ and $\widetilde{\delta}_{l+1}$, oriented so that the forward endpoint is a vertex of an external boundary edge of C. Also, let $h_5 := \eta^{-1} \cdot h_1$.
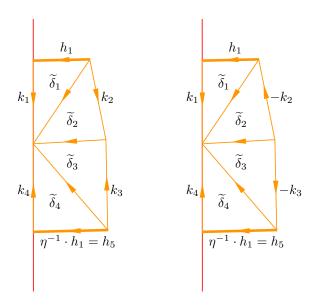

FIGURE 18. A lift of $C_{[\alpha]}$ or $C_{[\beta]}$ is drawn on the left, while a lift of $C_{[\gamma]}$ is drawn on the right.

The orientation on the edges of $\mathbb{T}$ ensures that $e_{\widetilde{\delta}_1,1} = h_1$, $e_{\widetilde{\delta}_1,2} = k_1$, $e_{\widetilde{\delta}_4,1} = h_5$, and $e_{\widetilde{\delta}_4,2} = k_4$ (as chains in $C_1(\widetilde{S}, \mathbb{Z})$). Similarly,

- $e_{\widetilde{\delta}_2,1} = k_2$, $e_{\widetilde{\delta}_3,1} = k_3$, and $e_{\widetilde{\delta}_2,2} = e_{\widetilde{\delta}_3,2} = h_3$ if $\eta = \alpha_P$ or β_P.
- $e_{\widetilde{\delta}_2,1} = -k_2$, $e_{\widetilde{\delta}_2,2} = h_2$, $e_{\widetilde{\delta}_3,1} = -k_3$, $e_{\widetilde{\delta}_3,2} = h_4$ if $\eta = \gamma_P$.

Thus, for any arbitrary pair $L_1, L_2 \in \mathcal{A}(\rho)$,

$$\sum_{\delta \in \mathbb{T}(C_{[\eta]})} \mathrm{sgn}(\delta) \, \mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right)$$

$$(6.1) \qquad = \quad \mathrm{tr}\left(\widetilde{\mu}_{L_1}(h_1) \cdot \widetilde{\mu}_{L_2}(k_4 - k_1)\right) + \mathrm{tr}\left(\widetilde{\mu}_{L_1}(k_2 - k_3) \cdot \widetilde{\mu}_{L_2}(h_3)\right),$$

if $\eta = \alpha_P$ or β_P, and

$$\sum_{\delta \in \mathbb{T}(C_{[\eta]})} \mathrm{sgn}(\delta) \, \mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right)$$

$$(6.2) \qquad = \quad \mathrm{tr}\left(\widetilde{\mu}_{L_1}(h_1) \cdot \widetilde{\mu}_{L_2}(k_4 - k_1)\right) + \mathrm{tr}\left(\widetilde{\mu}_{L_1}(k_2 - k_3) \cdot \widetilde{\mu}_{L_2}(h_3)\right)$$
$$\qquad\qquad - \mathrm{tr}\left(\widetilde{\mu}_{L_1}(k_3) \cdot \widetilde{\mu}_{L_2}(k_3)\right) + \mathrm{tr}\left(\widetilde{\mu}_{L_1}(k_2) \cdot \widetilde{\mu}_{L_2}(k_2)\right)$$

if $\eta = \gamma_P$.

6.4. **Computational tools.** In this section, we will describe general computational tools we need to compute $\omega\big([\mu_{L_1}], [\mu_{L_2}]\big)$ for any pair of special admissible labellings $L_1, L_2 \in \mathcal{A}(\rho)$. Repeated applications of the lemmas in this section will allow us to explicitly compute the symplectic pairing between any pair of special admissible labellings required for the proof of Theorem 6.5. This explicit computation is done in Section 6.5.

First, since ω is skew-symmetric, it is sufficient to compute $\omega\big([\mu_{L_1}], [\mu_{L_2}]\big)$ in the following cases.

(1) L_1 is a twist labelling and L_2 is any special admissible labelling;
(2) L_1 is an eruption labelling, L_2 is an eruption, hexagon or length labelling;
(3) L_1 is a hexagon labelling, L_2 is a hexagon or length labelling;
(4) L_1 and L_2 are both length labellings.

Definition 6.8. A pair $(L_1, L_2) \in \mathcal{A}(\rho)^2$ of special admissible labellings is a *considered pair* if it is one of the pairs described by the cases (1) to (4) above. The numbers (1) to (4) above is the *type* of the considered pair.

For computational purposes, it is useful to have the following definition.

Definition 6.9. For any admissible labelling $L \in \mathcal{A}(\rho)$, the *support* of L is the set

$$\{b \in \widetilde{\mathcal{B}} : L(b) \neq 0\}/\Gamma.$$

Since L is $\mathrm{Ad} \circ \rho$-equivariant, Γ acts on $\{b \in \widetilde{\mathcal{B}} : L(b) \neq 0\}$, so the support of L is well-defined. Note that for every eruption and hexagonal labelling L, there is a unique pair of pants in $\mathbb{P}$ that contains the support of L. On the other hand, there are at most two (necessarily adjacent) pairs of pants in $\mathbb{P}$ that contain the support of any twist labelling, while the support of every lozenge and length labelling lies in the union of a pair of adjacent pairs of pants in $\mathbb{P}$. Observe that if there is no $P \in \mathbb{P}$ that intersects both the supports of L_1 and L_2, then $\omega([\mu_{L_1}], [\mu_{L_2}]) = 0$.

To reduce the number of explicit computations we need to do, we exploit certain symmetries in the coeffecients of the special admissible labellings. It is thus useful to have the following definition.

Definition 6.10. Let $P \in \mathbb{P}$, and let $T_P, T'_P \in \widetilde{\Theta}$ be representatives of $\widehat{T}_P, \widehat{T}'_P \in \Theta$ respectively. An admissible labelling L is *symmetric (resp. skew-symmetric)* in P if for any $i, j, k \in \mathbb{Z}^+$ with $i + j + k = n$, $a_{i,j,k}(x, T_P, L) = a_{i,k,j}(x, T'_P, L)$ (resp. $a_{i,j,k}(x, T_P, L) = -a_{i,k,j}(x, T'_P, L)$). (Recall that $a_{i,j,k}(x, T, L)$ are coeffecients of L that were defined in Definition 3.5.)

For example, the lozenge and hexagon labellings are symmetric in every $P \in \mathbb{P}$, while the eruption labellings are skew-symmetric in every $P \in \mathbb{P}$. The twist labellings are both symmetric and skew-symmetric in every $P \in \mathbb{P}$, while the length labellings are neither symmetric nor skew-symmetric in any $P \in \mathbb{P}$ that intersects its support.

Using the triangulation $\mathbb{T}$, the formula for $\omega\big([\mu_{L_1}], [\mu_{L_2}]\big)$ given by (2.1) can be decomposed into

$$\sum_{\delta \in \mathbb{T}(\mathbb{P})} \mathrm{sgn}(\delta) \, \mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right) + \sum_{\delta \in \mathbb{T}(\mathcal{P})} \mathrm{sgn}(\delta) \, \mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right).$$

We refer to the first and second sum above respectively as the *internal part* and the *cylinder part* of $\omega([\mu_{L_1}], [\mu_{L_2}])$. The next lemma gives an easily verified condition for when the internal part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ is zero.

Lemma 6.11. *Let $L_1, L_2 \in \mathcal{A}(\rho)$ be admissible labellings so that for all $P \in \mathbb{P}$, L_1 and L_2 are either both symmetric in P or both skew-symmetric in P. Then*

$$\sum_{\delta \in \mathbb{T}(\mathbb{P})} \mathrm{sgn}(\delta) \, \mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right) = 0.$$

Proof. Note that

$$\sum_{\delta \in \mathbb{T}(\mathbb{P})} \mathrm{sgn}(\delta) \, \mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right)$$

$$= \sum_{P \in \mathbb{P}} \left(\mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta_P},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta_P},2})\right) - \mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta'_P},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta'_P},2})\right) \right)$$

where $\widetilde{\delta_P}$ and $\widetilde{\delta'_P}$ are lifts of δ_P and δ'_P respectively. The hypothesis and Lemma 5.7 implies that

$$\mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta_P},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta_P},2})\right) = \mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta'_P},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta'_P},2})\right).$$

$\square$

The next pair of lemmas give us an easy way to calculate the cylinder part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ in certain cases. The key observation to prove these pair of lemmas is recorded as the following remark, which we will also repeatedly use later.

Remark 6.12. Let $P \in \mathbb{P}$ and let $C = C_{[\eta]}$ be a boundary cylinder of P. Let $(f_{1,\eta}, \ldots, f_{n,\eta})$ be an ordered basis of V so that $\mathrm{Span}_{\mathbb{R}}(f_{m,\eta}) = \xi^{(m)}(\eta^-) \cap \xi^{(n-m+1)}(\eta^+)$ for any $m = 1, \ldots, n$. For $l = 1, \ldots, 4$ and any $L \in \mathcal{A}(\rho)$, $\widetilde{\mu}_L(h_l)$ and $\widetilde{\mu}_L(k_l)$ are represented in this basis by an upper-triangular matrix. (Recall that h_l and k_l were defined at the end of Section 6.3.)

Lemma 6.13. *Let $P \in \mathbb{P}$ and let $C = C_{[\alpha_P]}$ or $C_{[\beta_P]}$. Also, let $L_1, L_2 \in \mathcal{A}(\rho)$ be admissible labellings so that one of the following hold.*

   (1) $\widetilde{\mu}_{L_m}(k_2 - k_3)$ is a nilpotent endomoprhism for $m = 1, 2$,
   (2) $\widetilde{\mu}_{L_1}(k_2 - k_3)$ and $\widetilde{\mu}_{L_1}(h_1)$ are nilpotent endomoprhisms.
Then

$$\sum_{\delta \in \mathbb{T}(\mathbb{C})} \mathrm{sgn}(\delta) \, \mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right) = 0.$$

Proof. By the definition of μ_{L_2}, note that if $\widetilde{\mu}_{L_2}(k_2 - k_3)$ is nilpotent, then $\widetilde{\mu}_{L_2}(k_4 - k_1) = 0$. The lemma then follows immediately from Remark 6.12 and (6.1). $\square$

Lemma 6.14. *Let $P \in \mathbb{P}$ and let $C = C_{[\gamma_P]}$. Also, let $L_1, L_2 \in \mathcal{A}(\rho)$ be admissible labellings so that $\mathrm{tr}\left(\widetilde{\mu}_{L_1}(k_3) \cdot \widetilde{\mu}_{L_2}(k_3)\right) = \mathrm{tr}\left(\widetilde{\mu}_{L_1}(k_2) \cdot \widetilde{\mu}_{L_2}(k_2)\right)$, and one of the following hold.*

   (1) $\widetilde{\mu}_{L_m}(k_2 - k_3)$ is a nilpotent endomoprhism for $m = 1, 2$,
   (2) $\widetilde{\mu}_{L_1}(k_2 - k_3)$ and $\widetilde{\mu}_{L_1}(h_1)$ are nilpotent endomoprhisms.
Then

$$\sum_{\delta \in \mathbb{T}(\mathbb{C})} \mathrm{sgn}(\delta) \, \mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right) = 0.$$

Proof. Use the proof of Lemma 6.13, but replace (6.1) with (6.2).   □

Lemma 6.13 and Lemma 6.14 are especially useful because for all cylinders, $\widetilde{\mu}_L(k_2 - k_3)$ is nilpotent if L is an eruption, hexagon or twist labelling, $\widetilde{\mu}_L(k_2)$ and $\widetilde{\mu}_L(k_3)$ are both nilpotent if L is a twist or hexagon labellings, and $\widetilde{\mu}_L(h_1)$ is nilpotent if L is an eruption or twist labelling.

6.5. **Explicit computations for the proof of Theorem 6.5.** In this section, we compute $\omega([\mu_{L_1}], [\mu_{L_2}])$ for any considered pair (L_1, L_2) of special admissible labellings. There are four propositions in this appendix, each of which proves Theorem 6.5 for one of the four types of considered pairs described in Definition 6.8. Together, they finish the proof of Theorem 6.5.

Proposition 6.15 (TYPE (1) CONSIDERED PAIRS). *Let $L_1 = S^i_{x_1,x_2}$ be a twist labelling for some $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$ so that $\widetilde{r}\{x_1, x_2\} = (x_1, x_2)$, and let L_2 is any special admissible labelling. Then*

$$\omega([\mu_{L_1}], [\mu_{L_2}]) = \begin{cases} 1 & \text{if } L_2 = Y^i_{x_1,x_2}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since L_1 is a twist labelling, it is clear that $\widetilde{\mu}_{L_1}$ evaluates every internal edge of every $P \in \mathbb{P}$ to 0. It follows immediately that the internal part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ is zero for any special admissible labelling L_2.

L_2 **is a twist, eruption or hexagon labelling.** Since L_1 is a twist labelling, one sees that for any $P \in \mathbb{P}$ and any boundary cylinder C of P, $\widetilde{\mu}_{L_1}(k_2) = 0 = \widetilde{\mu}_{L_1}(k_3)$. Also, if L_2 is a twist, eruption or hexagon labelling, then $\widetilde{\mu}_{L_2}(k_2 - k_3)$ is also a nilpotent endomorphism. Lemma 6.13 and Lemma 6.14 then imply that the cylinder part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ is also zero in these cases.

L_2 **is a length labelling.** Let $\hat{e} \in \mathcal{T}$ be the support of L_1, and let $P_1, P_2 \in \mathbb{P}$ be the pairs of pants that lie to the left and right of $\hat{e}$. (It is possible that $P_1 = P_2$.) If P_1 and P_2 do not contain the support of L_2, then it is clear that $\omega([\mu_{L_1}], [\mu_{L_2}]) = 0$. Hence, we may assume that L_2 is associated to a boundary component of P_1 or P_2. By relabelling α_{P_m}, β_{P_m} and γ_{P_m} if necessary for $m = 1, 2$, we may assume by without loss of generality that the support of L_1 is $[\alpha_{P_1}] = [\alpha_{P_2}] \in \mathcal{P}$. For the rest of this proof, we will simplify notation by writing $\alpha_1 := \alpha_{P_1}$, $\alpha_2 := \alpha_{P_2}$, $x_1 := \alpha^-_{P_1} = \alpha^+_{P_2}$ and $x_2 := \alpha^+_{P_1} = \alpha^-_{P_2}$.

Observe that for all the boundary cylinders of P_1 and P_2 (there are three of these if $P_1 = P_2$ and six of these if $P_1 \neq P_2$), $\widetilde{\mu}_{L_1}(k_2) = 0 = \widetilde{\mu}_{L_1}(k_3)$. Among these, for the boundary cylinders that are not $C_{[\alpha_1]}$ or $C_{[\alpha_2]}$ (there is one such cylinder if $P_1 = P_2$ and four such cylinders if $P_1 \neq P_2$), $\widetilde{\mu}_{L_1}(h_1) = 0$ as well. Lemma 6.13 and Lemma 6.14 then imply that when C is a boundary cylinder of P_1 or P_2 that is neither $C_{[\alpha_1]}$ nor $C_{[\alpha_2]}$,

$$\sum_{\delta \in \mathbb{T}(C)} \text{sgn}(\delta) \, \text{tr} \left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2}) \right) = 0.$$

Finally, we will consider the cylinders $C = C_{[\alpha_1]}$ and $C_{[\alpha_2]}$. If L_2 and L_1 are associated to different closed edges, then from the definition of L_2, one sees that $\widetilde{\mu}_{L_2}(k_2 - k_3) = 0$, so Lemma 6.13 implies that

$$\sum_{\delta \in \mathbb{T}(C)} \text{sgn}(\delta) \, \text{tr} \left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2}) \right) = 0.$$

On the other hand, if L_1 and L_2 are associated to the same closed edge, then $L_2 = Y^{i'}_{x_1,x_2}$ for some $i' = 1, \ldots, n-1$. Recall that $Y^{i'}_{x_1,x_2}$ is the sum of $X := Z^{i'}_{x_1,x_2}$ with some eruption labellings. Since we have already proven that the ω pairing between the L_1 and any eruption labelling is zero, we can use (6.1) to compute

$$\sum_{\eta=\alpha_1,\alpha_2} \sum_{\delta \in \mathbb{T}(C_{[\eta]})} \operatorname{sgn}(\delta) \operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right)$$

$$= \operatorname{tr}\left(\widetilde{\mu}_{L_1}(h_{1,\alpha_1}) \cdot \widetilde{\mu}_X(k_{4,\alpha_1} - k_{1,\alpha_1})\right) + \operatorname{tr}\left(\widetilde{\mu}_{L_1}(h_{1,\alpha_2}) \cdot \widetilde{\mu}_X(k_{4,\alpha_2} - k_{1,\alpha_2})\right)$$

$$= \operatorname{tr}\left(\left(\frac{1}{4} A^{i,n-i}_{x_1,x_2}\right) \cdot \left(2E^{i',i'}_{\alpha_1} - 2E^{i'+1,i'+1}_{\alpha_1}\right)\right)$$

$$\quad + \operatorname{tr}\left(\left(-\frac{1}{4} A^{i,n-i}_{x_1,x_2}\right) \cdot \left(2E^{n-i',n-i'}_{\alpha_2} - 2E^{n-i'+1,n-i'+1}_{\alpha_2}\right)\right)$$

$$= \begin{cases} 1 & \text{if } i' = i; \\ 0 & \text{otherwise.} \end{cases}$$

Here, $E^{l,l}_\eta : V \to V$ is the endomorphism that acts as the identity on $\operatorname{Span}_{\mathbb{R}}(f_{l,\eta})$, and whose kernel is $\operatorname{Span}_{\mathbb{R}}(f_{1,\eta}, \ldots, \hat{f}_{l,\eta}, \ldots, f_{n,\eta})$. (Recall that $(f_{1,\eta}, \ldots, f_{n,\eta})$ is the ordered basis of V defined in Remark 6.12.)

We have thus proven that when L_2 is a length labelling, the cylinder part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ satisfies

$$\sum_{\delta \in \mathbb{T}(\mathcal{P})} \operatorname{sgn}(\delta) \operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right) = \begin{cases} 1 & \text{if } i' = i; \\ 0 & \text{otherwise.} \end{cases}$$

This proves the proposition. $\square$

Proposition 6.16 (TYPE (2) CONSIDERED PAIRS). *Suppose that $L_1 = E^{i,j,k}_{x,y,z}$ is an eruption labelling for some $P \in \mathbb{P}$ and L_2 is any eruption, hexagon or length labelling. Then*

$$\omega([\mu_{L_1}], [\mu_{L_2}]) = \begin{cases} -1 & \text{if } L_2 = H^{i,j,k}_{x,y,z}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will prove the lemma separately for the three cases when L_2 is an eruption labelling, a hexagon labelling or a length labelling.

L_2 **is an eruption labelling.** Since both L_1 and L_2 are skew-symmetric, Lemma 6.11 implies that the internal part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ is zero. Also, for any $P \in \mathbb{P}$ and any boundary cylinder C of P, both $\widetilde{\mu}_{L_1}(k_2 - k_3)$ and $\widetilde{\mu}_{L_2}(k_2 - k_3)$ are nilpotent. Furthermore, the fact that both L_1 and L_2 are skew-symmetric together with Remark 6.12 implies that $\operatorname{tr}\left(\widetilde{\mu}_{L_1}(k_3) \cdot \widetilde{\mu}_{L_2}(k_3)\right) = \operatorname{tr}\left(\widetilde{\mu}_{L_1}(k_2) \cdot \widetilde{\mu}_{L_2}(k_2)\right)$. We can thus use Lemma 6.13 and Lemma 6.14 to conclude that the cylinder part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ is zero.

L_2 **is a length labelling.** If P does not intersect the support of L_2, then $\omega([\mu_{L_1}], [\mu_{L_2}]) = 0$, so we may assume that L_2 is associated to one of the boundary curves of P. By relabelling $\alpha_P, \beta_P, \gamma_P$ if necessary, we may assume that $x = \alpha_P^-$, $y = \beta_P^-$, $z = \gamma_P^-$ and $L_2 = Y^{i'}_{\alpha_P^-, \alpha_P^+}$. Furthermore, by the previous paragraph, the linearity of Φ_ρ and the definition of $Y^{i'}_{\alpha_P^-, \alpha_P^+}$, we may assume that $L_2 = Z^{i'}_{\alpha_P^-, \alpha_P^+}$. Thus, we need to prove that $\omega([\mu_{L_1}], [\mu_{L_2}]) = 0$ when $L_1 = E^{i,j,k}_{\alpha_P^-, \beta_P^-, \gamma_P^-}$ and $L_2 =$

$Z^{i'}_{\alpha_P^-,\alpha_P^+}$. We will prove this by showing that the cylinder part and the internal part of $\omega([\mu_{L_1}],[\mu_{L_2}])$ are both zero.

By Lemma 5.7

$$\sum_{\delta\in\mathbb{T}(\mathbb{P})} \operatorname{sgn}(\delta)\operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1})\cdot\widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right)$$

$$= \sum_{\delta\in\mathbb{T}(P)} \operatorname{sgn}(\delta)\operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1})\cdot\widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right)$$

$$= -\operatorname{tr}\left(\left(-\frac{1}{2}A^{j,k,i}_{y,z,x}\right)\cdot\left(A^{i'+1,n-i'-1,0}_{x,y,z}-A^{i',n-i',0}_{x,y,z}-A^{i',n-i'-1,1}_{x,y,z}+A^{i'-1,n-i',1}_{x,y,z}\right)\right)$$

$$\quad +\operatorname{tr}\left(\left(-\frac{1}{2}A^{j,i,k}_{\gamma y,x,z}\right)\cdot\left(-A^{i'+1,0,n-i'-1}_{x,z,\gamma y}+A^{i',0.n-i'}_{x,z,\gamma y}+A^{i',1,n-i'-1}_{x,z,\gamma y}-A^{i'-1,1,n-i'}_{x,z,\gamma y}\right)\right)$$

$$= [\min(i'-i+1,j-(n-i'-1))]_+ - [\min(i'-i,j-(n-i'))]_+$$

$$\quad -[\min(i'-i,j-(n-i'-1))]_+ + [\min(i'-i-1,j-(n-i'))]_+$$

$$= [i'-i-k+1]_+ - [i'-i-k]_+ - [i'-i-k+1]_+ + [i'-i-k]_+$$

$$= 0.$$

This proves that the internal part of $\omega([\mu_{L_1}],[\mu_{L_2}])$ is zero.

Since L_1 is an eruption labelling, $\widetilde{\mu}_{L_1}(k_2-k_3)$ and $\widetilde{\mu}_{L_1}(h_1)$ are nilpotent for any boundary cylinder C of P. Lemma 6.13 then implies that

$$\sum_{\delta\in\mathbb{T}(C_{[\eta]})} \operatorname{sgn}(\delta)\operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1})\cdot\widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right) = 0$$

for $\eta=\alpha_P,\beta_P$. Also, when $C=C_{[\gamma_P]}$, the fact that eruption labellings are skew-symmetric in P and lozenge labellings are symmetric in P implies

$$\operatorname{tr}\left(\widetilde{\mu}_{L_1}(k_3)\cdot\widetilde{\mu}_{L_2}(k_3)\right) = -\operatorname{tr}\left(\widetilde{\mu}_{L_1}(k_2)\cdot\widetilde{\mu}_{L_2}(k_2)\right).$$

By Lemma 5.7, we can compute

$$\operatorname{tr}(\widetilde{\mu}_{L_1}(k_2)\cdot\widetilde{\mu}_{L_2}(k_2))$$

$$= \operatorname{tr}\left((A^{k',i',j'}_{z,x,y})\cdot\left(A^{1,i,n-i-1}_{z,x,y}-A^{1,i-1,n-i}_{z,x,y}\right)\right)$$

$$= 0$$

Thus, Lemma 6.14 allows us to conclude that

$$\sum_{\delta\in\mathbb{T}(C_{[\gamma_P]})} \operatorname{sgn}(\delta)\operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1})\cdot\widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right) = 0$$

as well. Hence, the cylinder part of $\omega([\mu_{L_1}],[\mu_{L_2}])$ is also zero.

L_2 **is a hexagon labelling.** If L_1 and L_2 are not associated to the same pair of pants in $\mathbb{P}$, then it is clear that $\omega([\mu_{L_1}],[\mu_{L_2}])=0$. Hence, we may assume that the supports of L_1 and L_2 lie in the same pair of pants $P\in\mathbb{P}$. Let C be any boundary cylinder of P. Observe that since L_2 is a hexagon labelling, $\widetilde{\mu}_{L_2}(k_2)$, $\widetilde{\mu}_{L_2}(k_3)$ and $\widetilde{\mu}_{L_2}(k_2-k_3)$ are all nilpotent. We have already seen that $\widetilde{\mu}_{L_1}(k_2-k_3)$ is nilpotent, so Remark 6.12, Lemma 6.13 and Lemma 6.14 imply that

$$\sum_{\delta\in\mathbb{T}(C)} \operatorname{sgn}(\delta)\operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1})\cdot\widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right) = 0$$

for all three boundary cylinders C of P. Thus, the cylinder part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ is zero.

To finish the proof, we now need to compute the internal part of $\omega([\mu_{L_1}], [\mu_{L_2}])$. By Lemma 5.7,

$$
\sum_{\delta \in \mathbb{T}(\mathbb{P})} \operatorname{sgn}(\delta) \operatorname{tr} \left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2}) \right)
$$

$$
= \sum_{\delta \in \mathbb{T}(P)} \operatorname{sgn}(\delta) \operatorname{tr} \left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2}) \right)
$$

$$
= -\operatorname{tr} \left(\left(-\frac{1}{2} A_{y,z,x}^{j_0,k_0,i_0} \right) \cdot \left(-A_{x,y,z}^{i,j+1,k-1} + A_{x,y,z}^{i-1,j+1,k} - A_{x,y,z}^{i-1,j,k+1} + A_{x,y,z}^{i,j-1,k+1} \right.\right.
$$

$$
\left.\left. -A_{x,y,z}^{i+1,j-1,k} + A_{x,y,z}^{i+1,j,k-1} \right) \right) + \operatorname{tr} \left(\left(-\frac{1}{2} A_{\gamma y,x,z}^{j_0,i_0,k_0} \right) \cdot \left(A_{x,z,\gamma y}^{i,k-1,j+1} - A_{x,z,\gamma y}^{i-1,k,j+1} \right.\right.
$$

$$
\left.\left. + A_{x,z,\gamma y}^{i-1,k+1,j} - A_{x,z,\gamma y}^{i,k+1,j-1} + A_{x,z,\gamma y}^{i+1,k,j-1} - A_{x,z,\gamma y}^{i+1,k-1,j} \right) \right)
$$

$$
= -[\min(i - i_0, j_0 - j - 1)]_+ + [\min(i - i_0 - 1, j_0 - j - 1)]_+
$$

$$
-[\min(i - i_0 - 1, j_0 - j)]_+ + [\min(i - i_0, j_0 - j + 1)]_+
$$

$$
-[\min(i - i_0 + 1, j_0 - j + 1)]_+ + [\min(i - i_0 + 1, j_0 - j)]_+
$$

$$
= \begin{cases} -1 & i_0 = i, j_0 = j, k_0 = k; \\ 0 & \text{otherwise.} \end{cases}
$$

$\square$

Proposition 6.17 (TYPE (3) CONSIDERED PAIRS). *Suppose that $L_1 = H_{x,y,z}^{i,j,k}$ is a hexagon labelling for some $P \in \mathbb{P}$ and L_2 is any hexagon or length labelling. Then*

$$
\omega([\mu_{L_1}], [\mu_{L_2}]) = 0.
$$

Proof. We prove the proposition separately for the case when L_2 is a hexagon labelling, and the case when L_2 is a length labelling.

L_2 **is a hexagon labelling.** Since L_1 and L_2 are both symmetric in every pair of pants in $\mathbb{P}$, Lemma 6.11 tells us that the internal part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ is zero. On the other hand, for any $P \in \mathbb{P}$ and any boundary cylinder C of P, $\widetilde{\mu}_{L_m}(k_2 - k_3)$, $\widetilde{\mu}_{L_m}(k_3)$ and $\widetilde{\mu}_{L_m}(k_2)$ are nilpotent for $m = 1, 2$. Lemma 6.13 and Lemma 6.14 imply that the cylinder part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ is also zero.

L_2 **is a length labelling.** It is clear that if P does not intersect the support of L_2, then $\omega([\mu_{L_1}], [\mu_{L_2}]) = 0$. Hence, we may assume that L_2 is associated to one of the boundary curves of P. Also, by relabelling $\alpha_P, \beta_P, \gamma_P$ if necessary, we may assume that $x = \alpha_P^-$, $y = \beta_P^-$, $z = \gamma_P^-$ and $L_2 = Y_{\alpha_P^-, \alpha_P^+}^{i'}$ for some $i' = 1, \dots, n-1$.

Let $X_1 := Y_{\alpha_P^-, \alpha_P^+}^{i'} - Z_{\alpha_P^-, \alpha_P^+}^{i'}$ and let $X_2 := Z_{\alpha_P^-, \alpha_P^+}^{i'}$. It is easy to verify that X_1 and X_2 are admissible labellings. Observe from the definition of $Y_{\alpha_P^-, \alpha_P^+}^{i'}$ that X_1 is a sum of eruption labellings. Hence, by Proposition 6.16, we know that

$$
(6.3) \qquad \omega([\widetilde{\mu}_{L_1}], [\widetilde{\mu}_{X_1}]) = \begin{cases} -1 & \text{if } k = 1 \text{ and } i' = i + 1; \\ 1 & \text{if } k = 1 \text{ and } i' = i; \\ 0 & \text{otherwise.} \end{cases}
$$

We will now compute $\omega([\widetilde{\mu}_{L_1}], [\widetilde{\mu}_{X_2}])$. Since both L_1 and X_2 are symmetric in every pair of pants in $\mathbb{P}$, Lemma 6.11 implies that the internal part of $\omega([\widetilde{\mu}_{L_1}], [\widetilde{\mu}_{X_2}])$

is zero. For the cylinders $C = C_{[\beta_P]}, C_{[\gamma_P]}, \widetilde{\mu}_{L_m}(k_2), \widetilde{\mu}_{L_m}(k_3)$ and $\widetilde{\mu}_{L_m}(k_2 - k_3)$ are nilpotent for $m = 1, 2$. Hence, Lemma 6.13 and Lemma 6.14 imply that

$$\sum_{\delta \in \mathbb{T}(\mathbb{C})} \operatorname{sgn}(\delta) \operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right) = 0.$$

Finally, consider $C = C_{[\alpha_P]}$. If $L_1 = H_{x,y,z}^{i,j,k}$ for $k > 1$, then $\widetilde{\mu}_{L_1}(k_2 - k_3)$ and $\widetilde{\mu}_{L_1}(h_1)$ are both nilpotent, so Lemma 6.13 implies that

$$\sum_{\delta \in \mathbb{T}(\mathbb{C})} \operatorname{sgn}(\delta) \operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right) = 0.$$

On the other hand, if $L_1 = H_{x,y,z}^{i,n-i,1}$, then $\widetilde{\mu}_{L_1}(k_2 - k_3)$ is still nilpotent, but

$$\widetilde{\mu}_{L_1}(h_1) = \frac{1}{2}\left(-A_{\alpha_P^-,\beta_P^-,\gamma_P^-}^{i,n-i,0} + A_{\alpha_P^-,\beta_P^-,\gamma_P^-}^{i+1,n-i-1,0}\right)$$

and

$$\widetilde{\mu}_{L_2}(k_4 - k_1) = 2\left(E^{i',i'} - E^{i'+1,i'+1}\right).$$

Here, $E^{l,l} : V \to V$ is the endomorphism that acts as the identity on $\operatorname{Span}_{\mathbb{R}}(f_{l,\alpha_P})$, and whose kernel is $\operatorname{Span}_{\mathbb{R}}(f_{1,\alpha_P}, \ldots, \hat{f}_{l,\alpha_P}, \ldots, f_{n,\alpha_P})$.

By (6.1), we can then compute

$$\sum_{\delta \in \mathbb{T}(C_{[\alpha_P]})} \operatorname{sgn}(\delta) \operatorname{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right)$$

$$= \operatorname{tr}\left(\widetilde{\mu}_{L_1}(h_1) \cdot \widetilde{\mu}_{L_2}(k_4 - k_1)\right) + \operatorname{tr}\left(\widetilde{\mu}_{L_1}(k_2 - k_3) \cdot \widetilde{\mu}_{L_2}(h_3)\right)$$

$$= \operatorname{tr}\left(\left(-A_{\alpha^-,\beta^-,\gamma^-}^{i,n-i,0} + A_{\alpha^-,\beta^-,\gamma^-}^{i+1,n-i-1,0}\right) \cdot \left(E_{i',i'} - E_{i'+1,i'+1}\right)\right)$$

$$= \begin{cases} 1 & \text{if } i' = i + 1; \\ -1 & \text{if } i' = i; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if $L_1 = H_{x,y,z}^{i,j,k}$ for any $i, j, k \in \mathbb{Z}^+$ so that $i + j + k = n$, we have

$$(6.4) \qquad \omega([\widetilde{\mu}_{L_1}], [\widetilde{\mu}_{X_2}]) = \begin{cases} 1 & \text{if } k = 1 \text{ and } i' = i + 1; \\ -1 & \text{if } k = 1 \text{ and } i' = i; \\ 0 & \text{otherwise.} \end{cases}$$

Adding (6.3) and (6.4) together proves the proposition. $\qquad\square$

Proposition 6.18 (TYPE (4) CONSIDERED PAIRS). *If L_1 and L_2 are both length labellings, then*

$$\omega([\mu_{L_1}], [\mu_{L_2}]) = 0.$$

Proof. Let $\hat{e} \in \mathcal{P}$ be the closed geodesic associated to L_1 and let $P_1, P_2 \in \mathbb{P}$ be the pairs of pants that lie to the right and left of $r(\hat{e})$ respectively. Also, we may assume that $[\alpha_{P_1}] = [\alpha_{P_2}] \in \mathcal{P}$. Simplify notation by denoting $\alpha_1 := \alpha_{P_1}, \alpha_2 := \alpha_{P_2}$. Recall that in the proof of Proposition 6.16, we showed that if X is an eruption labelling and Y is a lozenge labelling, then $\omega([\mu_X], [\mu_Y]) = 0$. Since the length labellings are a sum of a lozenge labelling with some eruption labellings, we may assume L_1 and L_2 are lozenge labellings. Clearly, $\omega([\mu_{L_1}], [\mu_{L_2}]) = 0$ if neither P_1 nor P_2 intersects the support of L_1. Hence, we may assume that L_2 is associated to a boundary component of P_1 or P_2.

For the boundary cylinders of P_1 and P_2 that are not $C_{[\alpha_1]}$ or $C_{[\alpha_2]}$, $\widetilde{\mu}_{L_1}(k_1) = 0 = \widetilde{\mu}_{L_1}(k_4)$, and both $\widetilde{\mu}_{L_1}(k_2 - k_3)$ are nilpotent. Lemma 6.13 and Lemma 6.14 then implies that for any such cylinder C,

$$\sum_{\delta \in \mathbb{T}(C)} \mathrm{sgn}(\delta) \, \mathrm{tr} \left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2}) \right) = 0.$$

Next, we consider the cylinders $C = C_{[\alpha_1]}$, $C_{[\alpha_2]}$. If L_2 is not associated to $\hat{e} = [\alpha_1] = [\alpha_2]$, then $\widetilde{\mu}_{L_2}(k_2 - k_3)$ is nilpotent. We can then use Lemma 6.13 to deduce that

$$\sum_{\delta \in \mathbb{T}(C)} \mathrm{sgn}(\delta) \, \mathrm{tr} \left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2}) \right) = 0.$$

This implies that the cylinder part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ is zero if L_1 and L_2 are not associated to the same closed edge in $\mathcal{P}$.

On the other hand, if L_2 is also associated to $\hat{e}$, then $\widetilde{\mu}_{L_m}(h_{l,\alpha_1}) = \widetilde{\mu}_{L_m}(h_{l,\alpha_2})$ and $\widetilde{\mu}_{L_m}(k_{l,\alpha_1}) = \widetilde{\mu}_{L_m}(k_{l,\alpha_2})$ for all $m = 1, 2$ and $l = 1, \ldots, 4$. The orientation on the triangles in $\mathbb{T}(C_{[\alpha_1]})$ and $\mathbb{T}(C_{[\alpha_2]})$ then imply

$$\sum_{\delta \in \mathbb{T}(C_{[\alpha_1]})} \mathrm{sgn}(\delta) \, \mathrm{tr} \left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2}) \right) = - \sum_{\delta \in \mathbb{T}(C_{[\alpha_2]})} \mathrm{sgn}(\delta) \, \mathrm{tr} \left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1}) \cdot \widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2}) \right).$$

Hence, the cylinder part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ is zero if L_1 and L_2 are associated to the same closed edge in $\mathcal{P}$.

Finally, since lozenge labellings are symmetric in every pair of pants in $\mathbb{P}$, Lemma 6.11, states that the internal part of $\omega([\mu_{L_1}], [\mu_{L_2}])$ is zero. This proves the proposition. $\qquad\square$

Appendix A. Proof of Proposition 2.13

In this appendix, we will prove Proposition 2.13, which we restate here for the reader's convenience.

Proposition A.1. *The map* $\mu_{\rho,[\nu]} : C_1(\widetilde{S}, \mathbb{Z}) \to \mathfrak{sl}(V)$ *is a well-defined,* $\mathrm{Ad} \circ \rho$*-equivariant 1-cocycle.*

We use the notation of Section 2.5 throughout this appendix. The proof of Proposition 2.13 is divided into two lemmas. The first says that $\mu_{\rho,[\nu]}$ is a cocycle if we disregard the 1-simplices with an endpoint in a closed edge in $\widetilde{\mathcal{P}}$.

Lemma A.2. *Let* $h, h' : [0, 1] \to \widetilde{\Sigma}$ *be 1-simplices so that* $h(1) = h'(0)$, *and let* h'' *is the concatenation of* h *and* h'. *Suppose that* $h(0)$, $h'(1)$, *and* $h(1) = h'(0)$ *do not lie in the closed edges in* $\widetilde{\mathcal{P}}$. *Then*

(A.1) $$\mu_{\rho,[\nu]}(h) + \mu_{\rho,[\nu]}(h') = \mu_{\rho,[\nu]}(h'').$$

Remark A.3. It is clear that under the assumptions of Lemma A.2, $\mu_{\rho,[\nu]}(h)$, $\mu_{\rho,[\nu]}(h')$ and $\mu_{\rho,[\nu]}(h'')$ are well-defined.

Proof. Choose a smooth path $[\rho_t] \in \mathrm{Hit}_V(S)$ so that $[\rho_0] = [\rho]$ and $\frac{d}{dt}|_{t=0}[\rho_t] = [\mu]$. Assume for now that the endpoints of h and h' do not lie on any barrier in $\widetilde{\mathcal{B}}$. Let $\rho_t, \rho'_t \in \widetilde{\mathrm{Hit}}_V(S)$ be smooth lifts of $[\rho_t]$ with $\rho_0 = \rho = \rho'_0$ so that the path of ρ_t-equivariant Frenet curves ξ_t satisfy $\xi_t(x_{h,0}) = \xi_0(x_{h,0})$, $\xi_t(y_{h,0}) = \xi_0(y_{h,0})$ and $\xi_t^{(1)}(z_{h,0}) = \xi_0^{(1)}(z_{h,0})$, and the path of ρ'_t-equivariant Frenet curves ξ'_t satisfy

$\xi'_t(x_{h',0}) = \xi'_0(x_{h',0})$, $\xi'_t(y_{h',0}) = \xi'_0(y_{h',0})$ and $\xi'^{(1)}_t(z_{h',0}) = \xi'^{(1)}_0(z_{h',0})$. Since $x_{h,0} = x_{h'',0}$, $x_{h,1} = x_{h',0}$, $x_{h',1} = x_{h'',1}$ and $\xi_0 = \xi'_0$, we see that

$$\xi_t(x_{h,1}) = g_{t,h} \cdot \xi_0(x_{h,1}) = g_{t,h} \cdot \xi'_0(x_{h,0}) = g_{t,h} \cdot \xi'_t(x_{h,1}).$$

Similarly, $\xi_t(y_{h,1}) = g_{t,h} \cdot \xi'_t(y_{h,1})$ and $\xi^{(1)}_t(z_{h,1}) = g_{t,h} \cdot \xi'^{(1)}_t(z_{h,1})$, so $\xi_t = g_{t,h} \cdot \xi'_t$.

This allows us to compute

$$
\begin{aligned}
g_{t,h''} \cdot \xi_0(x_{h'',1}) &= \xi_t(x_{h'',1}) \\
&= g_{t,h} \cdot \xi'_t(x_{h',1}) \\
&= g_{t,h}g_{t,h'} \cdot \xi'_0(x_{h',1}) \\
&= g_{t,h}g_{t,h'} \cdot \xi_0(x_{h'',1})
\end{aligned}
$$

Similarly, $g_{t,h''} \cdot \xi_0(y_{h'',1}) = g_{t,h}g_{t,h'} \cdot \xi_0(y_{h'',1})$ and $g_{t,h''} \cdot \xi^{(1)}_0(z_{h'',1}) = g_{t,h}g_{t,h'} \cdot \xi^{(1)}_0(z_{h'',1})$. This means that $g_{t,h''} = g_{t,h}g_{t,h'}$ by Remark 2.2(1). Taking derivatives with respect to t and evaluating at $t = 0$ proves (A.1) in the case when the endpoints of h and h' do not lie in any of the geodesics or geodesic rays in $\widetilde{\mathcal{B}}$.

Next, we will show that (A.1) holds in the setting where the endpoints of h and h' are not sources of non-edge barriers in $\widetilde{\mathcal{M}}$ and do not lie in closed edges in $\widetilde{\mathcal{P}}$. When $h(0)$, $h(1) = h'(0)$ or $h'(1)$ lie in an isolated edge in $\widetilde{\mathcal{Q}}$ or in the interior of a non-edge barrier in $\widetilde{\mathcal{M}}$, define the 1-simplices h''_1, h''_2, h'_1, h'_2, h_1, h_2 as we did above in Step 2. If $h(0)$ lies in an isolated edge in $\widetilde{\mathcal{Q}}$ or in the interior of a non-edge barrier in $\widetilde{\mathcal{M}}$, then

$$
\begin{aligned}
\mu_{\rho,[\nu]}(h'') &= \frac{1}{2}\left(\mu_{\rho,[\nu]}(h''_1) + \mu_{\rho,[\nu]}(h''_2)\right) \\
&= \frac{1}{2}\left(\mu_{\rho,[\nu]}(h_1) + \mu_{\rho,[\nu]}(h') + \mu_{\rho,[\nu]}(h_2) + \mu_{\rho,[\nu]}(h')\right) \\
&= \mu_{\rho,[\nu]}(h) + \mu_{\rho,[\nu]}(h')
\end{aligned}
$$

where the second equality holds by the previous case. For the same reasons (A.1) holds in the case when $h'(1)$ lies in an isolated edge in $\widetilde{\mathcal{Q}}$ or in the interior of a non-edge barrier in $\widetilde{\mathcal{M}}$. On the other hand, if $h(1) = h'(0)$ in an isolated edge in $\widetilde{\mathcal{Q}}$ or in the interior of a non-edge barrier in $\widetilde{\mathcal{M}}$, we also see that

$$
\begin{aligned}
\mu_{\rho,[\nu]}(h'') &= \frac{1}{2}(\mu_{\rho,[\nu]}(h'') + \mu_{\rho,[\nu]}(h'')) \\
&= \frac{1}{2}\left(\mu_{\rho,[\nu]}(h_1) + \mu_{\rho,[\nu]}(h'_1) + \mu_{\rho,[\nu]}(h_2) + \mu_{\rho,[\nu]}(h'_2)\right) \\
&= \mu_{\rho,[\nu]}(h) + \mu_{\rho,[\nu]}(h').
\end{aligned}
$$

Iterating these calculations then proves (A.1) in this setting.

Finally, suppose that none of $h(0)$, $h'(1)$ or $h(1) = h'(0)$ lie in a closed edge in $\widetilde{\mathcal{P}}$. When $h(0)$, $h'(1)$ or $h(1) = h'(0)$ is the source of a non-edge barrier in $\widetilde{\mathcal{M}}$, define $\bar{h}_m$, $\bar{h}'_m$ and $\bar{h}''_m$ as we did in Step 3 for $m = 0, 1$. If $h(0)$ is the source of a non-edge barrier in $\widetilde{\mathcal{M}}$, we have

$$
\begin{aligned}
\mu_{\rho,[\nu]}(h'') &= \mu_{\rho,[\nu]}(\bar{h}''_0) + \frac{1}{3}\left(\mu_{\rho,[\nu]}(l_{i-1,i,h''(0)}) + \mu_{\rho,[\nu]}(l_{i+1,i,h''(0)})\right) \\
&= \mu_{\rho,[\nu]}(\bar{h}_0) + \mu_{\rho,[\nu]}(h') + \frac{1}{3}\left(\mu_{\rho,[\nu]}(l_{i-1,i,h(0)}) + \mu_{\rho,[\nu]}(l_{i+1,i,h(0)})\right) \\
&= \mu_{\rho,[\nu]}(h) + \mu_{\rho,[\nu]}(h').
\end{aligned}
$$

Similarly for the case when $h'(1)$ is the source of a non-edge barrier in $\widetilde{\mathcal{M}}$. On the other hand, suppose that $h(1) = h'(0)$ is the source of a non-edge barrier in $\widetilde{\mathcal{M}}$, $h(0) \in U_{i,h(1)}$, and $h'(1) \in U_{j,h'(0)}$. Since the endpoints of each $l_{i,j,h(1)}$ do not lie in the barriers in $\widetilde{\mathcal{B}}$, we have already proven above that

$$\mu_{\rho,[\nu]}(l_{i,i-1,h(1)}) + \mu_{\rho,[\nu]}(l_{i-1,i+1,h(1)}) + \mu_{\rho,[\nu]}(l_{i+1,i,h(1)}) = 0.$$

Thus,

$$
\begin{aligned}
\mu_{\rho,[\nu]}(h) + \mu_{\rho,[\nu]}(h') &= \mu_{\rho,[\nu]}(\bar{h}_1) + \frac{1}{3}\left(\mu_{\rho,[\nu]}(l_{i,i+1,h(1)}) + \mu_{\rho,[\nu]}(l_{i,i-1,h(1)})\right) \\
&\quad + \mu_{\rho,[\nu]}(\bar{h}'_0) + \frac{1}{3}\left(\mu_{\rho,[\nu]}(l_{j-1,j,h'(0)}) + \mu_{\rho,[nu]}(l_{j+1,j,h'(0)})\right) \\
&= \mu_{\rho,[\nu]}(\bar{h}_1) + \mu_{\rho,[\nu]}(\bar{h}'_0) + \mu_{\rho,[\nu]}(l_{i,j,h(1)}) \\
&= \mu_{\rho,[\nu]}(h'').
\end{aligned}
$$

Iterating this calculation with the calculations in the previous cases proves the preliminary claim. $\qquad\square$

The next lemma ensures that the $\mu_{\rho,[\nu]}$ when restricted to some special 1-simplices associated to a closed edge satisfy the cocycle condition.
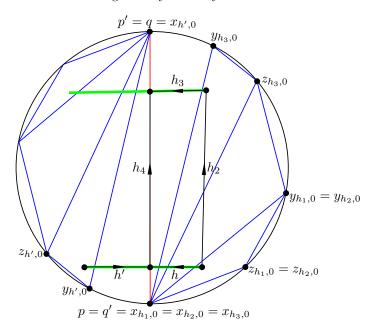
FIGURE 19.

Lemma A.4. *Let J be any bridge across a closed edge $e = \{x, y\} \in \widetilde{\mathcal{P}}$ and let $\gamma \in \Gamma$ be any primitive group element that fixes x and y. Let $h = h_1, h' : [0, 1] \to \widetilde{S}$ be two 1-simplices so that $h(1) = h'(1)$, and the endpoints of the geodesic segment J are $h(0)$ and $h'(0)$. Let $h_3 := \gamma \cdot h_1$ and let $h_2, h_4 : [0, 1] \to \widetilde{S}$ be 1-simplices so that $h_2(0) = h_1(0)$, $h_2(1) = h_3(0)$, $h_4(0) = h_1(1)$ and $h_4(1) = h_3(1)$ (see Figure 19). Also, let h'' be the concatenation of h and h'^{-1}. Then*

(1) $\mu_{\rho,[\nu]}(h_1) + \mu_{\rho,[\nu]}(h_4) = \mu_{\rho,[\nu]}(h_2) + \mu_{\rho,[\nu]}(h_3),$

(2) $\mu_{\rho,[\nu]}(h) - \mu_{\rho,[\nu]}(h') = \mu_{\rho,[\nu]}(h'')$.

Remark A.5. It is clear that $\mu_{\rho,[\nu]}(h_1)$, $\mu_{\rho,[\nu]}(h_2)$, $\mu_{\rho,[\nu]}(h_3)$, $\mu_{\rho,[\nu]}(h_4)$, $\mu_{\rho,[\nu]}(h_1')$ and $\mu_{\rho,[\nu]}(h_1'^{-1} \cdot h_1)$ in the statement of Lemma A.4 are well-defined.

Proof. Choose a smooth path $[\rho_t] \in \text{Hit}_V(S)$ so that $[\rho_0] = [\rho]$ and $\frac{d}{dt}|_{t=0}[\rho_t] = [\nu]$. Assume without loss of generality that for each $k = h_1, h_2, h_3$, $p := x_{k,0}$ is the common vertex of $T_{k,0}$ and e, and denote the other vertex of e by q. Also, let $x_{h',0} =: p'$ denote the common vertex of $T_{h',0}$ and e, and denote the other vertex of e by q' (see Figure 19). Choose a basis $\{f_1, \ldots, f_n\}$ of V so that $\text{Span}_{\mathbb{R}}(f_i) = \xi^{(i)}(p) \cap \xi^{(n-i+1)}(q)$, where ξ is the ρ-equivariant Frenet curve.

(1) Let ρ_t be the lift of $[\rho_t]$ so that $\rho_0 = \rho$, and the ρ_t-equivariant Frenet curves ξ_t satisfies

$$\xi_t(x_{h_2,0}) = \xi(x_{h_2,0}), \quad \xi_t(y_{h_2,0}) = \xi(y_{h_2,0}), \quad \xi_t^{(1)}(z_{h_2,0}) = \xi^{(1)}(z_{h_2,0}),$$

Also, let ρ_t' be a choice of lift of $[\rho_t]$ so that the ρ_t'-equivariant Frenet curve satisfies $\xi_t'(p) = \xi'(p)$ and $\xi_t'(q) = \xi'(q)$.

Observe that $g_{t,h_2} = \rho_t(\gamma)\rho(\gamma)^{-1}$ fixes $\xi(p)$, so it is represented by an upper triangular matrix in the basis $\{f_1, \ldots, f_n\}$. Also, it is clear that g_{t,h_2} is conjugate to $d_4 := \rho_t'(\gamma)\rho(\gamma)^{-1}$ which is represented by a diagonal matrix in the basis $\{f_1, \ldots, f_n\}$. Thus, if we write $g_{t,h_2} = u_2 d_2$ for some diagonal matrix d_2 and some unipotent upper triangular matrix u_2 in this basis, then $d_2 = d_4$.

By a similar argument as the one given in the first two paragraphs of the proof of Lemma A.2, we see that $g_{t,h_2} u_{t,h_3} \cdot \xi(p) = u_{t,h_1} d_4 \cdot \xi(p) = \xi(p)$ and $g_{t,h_2} u_{t,h_3} \cdot \xi(q) = u_{t,h_1} d_4 \cdot \xi(q)$. This means that $g_{t,h_2} u_{t,h_3} = u_2 d_2 u_{t,h_3} = d u_{t,h_1} d_4$ for some projective transformation d that is diagonal in the basis $\{f_1, \ldots, f_n\}$. However, since $d_2 = d_4$, this means that $d = \text{id}$, so

$$g_{t,h_2} u_{t,h_3} = u_{t,h_1} d_4.$$

Taking derivative with respect to t, evaluating at $t = 0$, and then adding

$$\frac{1}{2} \cdot \frac{d}{dt}\Big|_{t=0} d_{t,h_3} = \frac{1}{2} \cdot \frac{d}{dt}\Big|_{t=0} \rho(\gamma) d_{t,h_1} \rho(\gamma^{-1}) = \frac{1}{2} \cdot \frac{d}{dt}\Big|_{t=0} d_{t,h_1}$$

to both sides proves (1).

(2) Let ρ_t, ρ_t' be the lift of $[\rho_t]$ so that $\rho_0 = \rho_0' = \rho$ and the corresponding equivariant Frenet curves ξ_t and ξ_t' satisfy

$$\xi_t(x_{h,0}) = \xi(x_{h,0}), \quad \xi_t(y_{h,0}) = \xi(z_{h,0}), \quad \xi_t^{(1)}(z_{h,0}) = \xi^{(1)}(z_{h,0}),$$

$$\xi_t'(x_{h',0}) = \xi'(x_{h',0}), \quad \xi_t'(y_{h',0}) = \xi'(z_{h',0}), \quad \xi_t'^{(1)}(z_{h',0}) = \xi'^{(1)}(z_{h',0}).$$

Recall that by definition,

$$\big(\xi_t(p), \xi_t(y_{h,0}), \xi_t^{(1)}(z_{h,0}), \xi_t(q)\big) = \big(\xi(p), \xi(y_{h,0}), \xi^{(1)}(z_{h,0}), u_{t,h} \cdot \xi(q)\big),$$

$$\big(\xi_t'(p'), \xi_t'(y_{h',0}), \xi_t'^{(1)}(z_{h',0}), \xi_t'(q')\big) = \big(\xi(p'), \xi(y_{h',0}), \xi^{(1)}(z_{h',0}), u_{t,h'} \cdot \xi(q')\big),$$

and $d_{t,h} \in \text{PGL}(V)$ is the projective transformation that sends $u_{t,h'}^{-1} \cdot \xi_t'$ to $u_{t,h}^{-1} \cdot \xi_t$.

From this, it is clear that $d_{t,h}$ fixes $\xi(x)$ and $\xi(y)$, and $d_{t,h} = d_{t,h'}^{-1}$. Also, this implies that $\xi_t = u_{t,h} d_{t,h} u_{t,h'}^{-1} \xi_t'$, so in particular,

$$g_{t,h''} \cdot \xi(x_{h'',1}) = \xi_t(x_{h'',1}) = u_{t,h} d_{t,h} u_{t,h'}^{-1} \cdot \xi_t'(x_{h',0}) = u_{t,h} d_{t,h} u_{t,h'}^{-1} \cdot \xi(x_{h'',1}).$$

Similarly, $g_{t,h''} \cdot \xi(y_{h'',1}) = u_{t,h} d_{t,h} u_{t,h'}^{-1} \cdot \xi(y_{h'',1})$ and $g_{t,h''} \cdot \xi^{(1)}(z_{h'',1}) = u_{t,h} d_{t,h} u_{t,h'}^{-1} \cdot \xi^{(1)}(z_{h'',1})$. Remark 2.2(1) then implies that $g_{t,h''} = u_{t,h} d_{t,h} u_{t,h'}^{-1}$, so

$$
\begin{aligned}
\mu_{\rho,[\nu]}(h'') &= \left.\frac{d}{dt}\right|_{t=0} g_{t,h''} \\
&= \left.\frac{d}{dt}\right|_{t=0} u_{t,h} + \left.\frac{d}{dt}\right|_{t=0} d_{t,h} + \left.\frac{d}{dt}\right|_{t=0} u_{t,h'}^{-1} \\
&= \left.\frac{d}{dt}\right|_{t=0} u_{t,h} + \frac{1}{2}\cdot\left.\frac{d}{dt}\right|_{t=0} d_{t,h} - \frac{1}{2}\cdot\left.\frac{d}{dt}\right|_{t=0} d_{t,h'} - \left.\frac{d}{dt}\right|_{t=0} u_{t,h'} \\
&= \mu_{\rho,[\nu]}(h) - \mu_{\rho,[\nu]}(h')
\end{aligned}
$$

$\square$

Using Lemma A.2 and Lemma A.4, we can now prove Proposition 2.13.

Proof of Proposition 2.13. Once we have proven that $\mu_{\rho,[\nu]}$ is well-defined, the ρ-equivariance of $\mu_{\rho,[\nu]}$ is an immediate consequence of the fact that ξ is ρ-equivariant.

First, we will prove that $\mu_{\rho,[\nu]}$ is well-defined. Let $h : [0,1] \to \widetilde{\Sigma}$ be a 1-simplex so that $h(1)$ lies in a closed edge $e \in \widetilde{\mathcal{P}}$. It is sufficient to show that $\mu_{\rho,[\nu]}(h)$ is well-defined, i.e. if h is homotopic to the concatenations $h_1 \cdot h_2 \cdot h_3$ and $h_1' \cdot h_2' \cdot h_3'$ so that

- $h_1(0) = h(0) = h_1'(0)$,
- $h_1(1) = h_2(0)$ and $h_1'(1) = h_2'(0)$ are respectively the endpoints of bridges $J, J' \in \widetilde{\mathcal{J}}$ that intersect e,
- $h_2(1) = h_3(0)$ and $h_2'(1) = h_3'(0)$ are respectively the intersections of J and J' with e
- $h_3(1) = h(1) = h_3'(1)$,

then

$$
\mu_{\rho,[\nu]}(h_1') + \mu_{\rho,[\nu]}(h_2') + \mu_{\rho,[\nu]}(h_3') = \mu_{\rho,[\nu]}(h_1) + \mu_{\rho,[\nu]}(h_2) + \mu_{\rho,[\nu]}(h_3).
$$

Let $k_1, k_2 : [0,1] \to \widetilde{\Sigma}$ be 1-simplices so that $k_1(0) = h_1'(1)$, $k_1(1) = h_1(1)$, $k_2(0) = h_2'(1)$ and $k_2(1) = h_2(1)$. Since the endpoints of k_2, h_3 and h_3' all lie in the closed edge e, it is easy to see from the definition that

$$
\mu_{\rho,[\nu]}(h_3') = \mu_{\rho,[\nu]}(k_2) + \mu_{\rho,[\nu]}(h_3).
$$

On the other hand, Lemma A.2 implies that

$$
\mu_{\rho,[\nu]}(h_1') + \mu_{\rho,[\nu]}(k_1) = \mu_{\rho,[\nu]}(h_1)
$$

and Lemma A.4 implies that

$$
\mu_{\rho,[\nu]}(h_2') + \mu_{\rho,[\nu]}(k_2) = \mu_{\rho,[\nu]}(k_1) + \mu_{\rho,[\nu]}(h_2).
$$

Combining these together, we have

$$
\begin{aligned}
&\mu_{\rho,[\nu]}(h_1') + \mu_{\rho,[\nu]}(h_2') + \mu_{\rho,[\nu]}(h_3') \\
=\ &\mu_{\rho,[\nu]}(h_1) - \mu_{\rho,[\nu]}(k_1) + \mu_{\rho,[\nu]}(k_1) - \mu_{\rho,[\nu]}(k_2) + \mu_{\rho,[\nu]}(h_2) + \mu_{\rho,[\nu]}(k_2) + \mu_{\rho,[\nu]}(h_3) \\
=\ &\mu_{\rho,[\nu]}(h_1) + \mu_{\rho,[\nu]}(h_2) + \mu_{\rho,[\nu]}(h_3).
\end{aligned}
$$

Next, we will argue that $\mu_{\rho,[\nu]}$ is a cocycle. Let $h, h', h'' : [0,1] \to \widetilde{\Sigma}$ be 1-simplices so that $h(1) = h'(0)$ and h'' is the concatenation of h with h'. We need to show that

$$\mu_{\rho,[\nu]}(h) + \mu_{\rho,[\nu]}(h') = \mu_{\rho,[\nu]}(h'').$$

If the endpoints of h and h' do not lie in any closed edge in $\widetilde{\mathcal{P}}$, this follows from Lemma A.2. If one of h or h' lies entirely in a closed edge, then this follows using the same proof of the well-definedness of $\mu_{\rho,[\nu]}$ given above. Suppose that $h(1) = h'(0)$ lies in a closed edge in $\widetilde{\mathcal{P}}$. Let $h_1 \cdot h_2 \cdot h_3$ and $h'_1 \cdot h'_2 \cdot h'_3$ be concatenations that are homotopic to h and h' respectively so that

- $h_1(0) = h(0)$ and $h'_3(1) = h'(1)$
- $h_1(1) = h_2(0)$ and $h'_2(1) = h'_3(0)$ are respectively the endpoints of bridges $J, J' \in \widetilde{\mathcal{J}}$ that intersect e,
- $h_2(1) = h_3(0)$ and $h'_1(1) = h'_2(0)$ are respectively the intersections of J and J' with e
- $h_3(1) = h(1) = h'(0) = h'_1(0)$.

Let $k : [0,1] \to \widetilde{\Sigma}$ be the 1-simplex so that $k(0) = h_2(0)$ and $k(1) = h'_2(1)$. Then

$$
\begin{aligned}
&\mu_{\rho,[\nu]}(h) + \mu_{\rho,[\nu]}(h') \\
=\ &\mu_{\rho,[\nu]}(h_1) + \mu_{\rho,[\nu]}(h_2) + \mu_{\rho,[\nu]}(h_3) + \mu_{\rho,[\nu]}(h'_1) + \mu_{\rho,[\nu]}(h'_2) + \mu_{\rho,[\nu]}(h'_3) \\
=\ &\mu_{\rho,[\nu]}(h_1) + \mu_{\rho,[\nu]}(k) + \mu_{\rho,[\nu]}(h'_3) \\
=\ &\mu_{\rho,[\nu]}(h'')
\end{aligned}
$$

where the second equality is a consequence of Lemma A.4 while the last equality is a consequence of Lemma A.2. $\qquad\square$

Appendix B. Proof of Proposition 4.6

In this section, we will prove Proposition 4.6, which we restate here for the reader's convenience. Recall that L is an admissible labelling in $\mathcal{A}(\rho, r, \mathcal{T})$ and $\widetilde{\mu}_L$ is the map constructed in Section 4.2 using a compatible bridge system $\mathcal{J}$.

Proposition B.1. *For any simplex $h : [0,1] \to \widetilde{\Sigma}$, $\widetilde{\mu}_L(h)$ is well-defined. Furthermore, if h_1, h_2, h_3 are 1-simplices so that $h_{i+1}(0) = h_i(1)$ for $i = 1, 2, 3$, then $\widetilde{\mu}_L(h_1 \cdot h_2 \cdot h_3) = 0$. In particular, $\widetilde{\mu}_L \in C^1(\widetilde{S}, \mathfrak{sl}(V))$ is a cocycle.*

Proof. For any 1-simplex h, let h^{-1} be the 1-simplex with the same image as h, but equipped with the opposite orientation. It is clear that $\widetilde{\mu}_L(h) = -\widetilde{\mu}_L(h^{-1})$. Thus, it is sufficient to prove that if $h_1, \ldots, h_k : [0,1] \to \widetilde{S}$ are elementary simplices so that $h_k(1) = h_1(0)$ and $h_i(1) = h_{i+1}(0)$ for $i = 1, \ldots, k-1$, then

$$(B.1) \qquad\qquad \sum_{i=1}^{k} \widetilde{\mu}_L(h_i) = 0.$$

We will do so by induction on k.

For the base case when $k = 2$, observe that h_1 and h_2 have to be elementary 1-simplices of the same type. Also, if they are Type A or Type B 1-simplices, then $h_1^{-1} = h_2$ necessarily, so (B.1) holds. Now suppose that h_1 and h_2 are Type C

1-simplices. We will prove (B.1) holds in this situation by showing that

$$\text{(B.2)} \qquad \sum_{j=1}^{3} \hat{i}(h_1, b_j) L(b_j) = -\sum_{j=1}^{3} \hat{i}(h_2, b_j) L(b_j)$$

for any triple of non-edge barriers b_1, b_2, b_3 in any ideal triangle $T \in \widetilde{\Theta}$, and

$$\text{(B.3)} \qquad \hat{i}(h_1, \widetilde{r}(e)) L(e) = -\hat{i}(h_2, \widetilde{r}(e)) L(e)$$

for any edge $e \in \widetilde{\mathcal{T}}$.

To prove (B.2), assume without loss of generality that if $b_i = (T, v_i)$, then $v_1 < v_2 < v_3 < v_1$ lies in $\partial\Gamma$ in this cyclic order. Let $p \in \widetilde{\Sigma}$ be the common source for b_1, b_2 and b_3, and let $U_{1,p}$, $U_{2,p}$ and $U_{3,p}$ be the three connected components of $\widetilde{\Sigma} \setminus (b_1 \cup b_2 \cup b_3)$, so that $b_{j-1}, b_{j+1} \subset \overline{U_{j,p}}$ (see Figure 20).
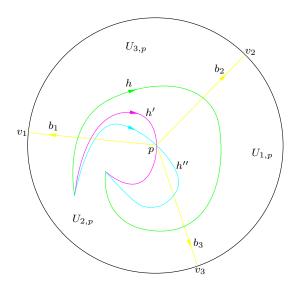

FIGURE 20. $\hat{i}(h, b_m) - \frac{1}{3} = \hat{i}(h'', b_m)$ and $\hat{i}(h, b_m) - \frac{2}{3} = \hat{i}(h', b_m)$. for all $m = 1, 2, 3$.

Let $h : [0,1] \to \widetilde{\Sigma}$ be any 1-simplex so that $h(0) \in U_{j,p}$. By a straightforward winding number argument, we see that if $h[0,1]$ does not contain p, then $h(1) \in U_{j,p}$ if and only if $\hat{i}(h, b_1) = \hat{i}(h, b_2) = \hat{i}(h, b_3)$. By our definition of $\hat{i}$, one can then verify that if we homotope h to a Type C 1-simplex $h' : [0,1] \to \widetilde{\Sigma}$ with $h'(0) = h(0)$ and $h'(1) = h(1)$ so that $h'(t) = p$ for a unique $t \in [0,1]$, then each of $\hat{i}(h, b_1)$, $\hat{i}(h, b_2)$ and $\hat{i}(h, b_3)$ either all change by $\frac{1}{3}$ or all change by $\frac{2}{3}$ (see Figure 20). By using similar arguments iteratively, one observes the following statements.

Let $h : [0,1] \to \widetilde{\Sigma}$ be any 1-simplex so that $h(0) \in U_{j,p}$, then $h(1)$

- lies in $U_{j,p}$ if and only if $\hat{i}(h, b_1) = \hat{i}(h, b_2) = \hat{i}(h, b_3)$,
- lies in $U_{j+1,p}$ if and only if $\hat{i}(h, b_{j-1}) - 1 = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1})$,
- lies in $U_{j-1,p}$ if and only if $\hat{i}(h, b_{j-1}) = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1}) + 1$.
- lies in the interior of b_{j-1} if and only if $\hat{i}(h, b_{j-1}) - \frac{1}{2} = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1})$,
- lies in the interior of b_j if and only if $\hat{i}(h, b_{j-1}) - \frac{1}{2} = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1}) + \frac{1}{2}$,

- lies in the interior of b_{j+1} if and only if $\hat{i}(h, b_{j-1}) = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1}) + \frac{1}{2}$,
- is p if and only if $\hat{i}(h, b_{j-1}) - \frac{1}{3} = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1}) + \frac{1}{3}$.

Similarly, if $h(0)$ lies in the interior of b_j, then $h(1)$

- lies in $U_{j-1,p}$ if and only if $\hat{i}(h, b_{j-1}) = \hat{i}(h, b_j) - \frac{1}{2} = \hat{i}(h, b_{j+1})$,
- lies in $U_{j,p}$ if and only if $\hat{i}(h, b_{j-1}) + \frac{1}{2} = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1}) - \frac{1}{2}$,
- lies in $U_{j+1,p}$ if and only if $\hat{i}(h, b_{j-1}) = \hat{i}(h, b_j) + \frac{1}{2} = \hat{i}(h, b_{j+1})$,
- lies in the interior of b_j if and only if $\hat{i}(h, b_1) = \hat{i}(h, b_2) = \hat{i}(h, b_3)$,
- lies in the interior of b_{j+1} if and only if $\hat{i}(h, b_{j-1}) + \frac{1}{2} = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1})$,
- lies in the interior of b_{j-1} if and only if $\hat{i}(h, b_{j-1}) = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1}) - \frac{1}{2}$.
- is p if and only if $\hat{i}(h, b_{j-1}) + \frac{1}{6} = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1}) - \frac{1}{6}$.

Finally, if $h(0) = p$, then the $h(1)$

- lies in $U_{j,p}$ if and only if $\hat{i}(h, b_{j-1}) + \frac{1}{3} = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1}) - \frac{1}{3}$,
- lies in the interior of b_j if and only if $\hat{i}(h, b_{j-1}) - \frac{1}{6} = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1}) + \frac{1}{6}$,
- is p if and only if $\hat{i}(h, b_{j-1}) = \hat{i}(h, b_j) = \hat{i}(h, b_{j+1})$.

Since L is a legal labelling, $L(b_1) + L(b_2) + L(b_3) = 0$. Using this, we see that in all possible cases, $\sum_{k=1}^{3} \hat{i}(h, b_k) L(b_k)$ depends only on the position of the endpoints of h. Since the backward and forward endpoints of h_1 are respectively the forward and backward endpoints of h_2, one can compute

$$\sum_{k=1}^{3} \hat{i}(h_1, b_k) L(b_k) = \sum_{k=1}^{3} \hat{i}(h_2^{-1}, b_k) L(b_k) = -\sum_{k=1}^{3} \hat{i}(h_2, b_k) L(b_k).$$

This proves (B.2).

To prove (B.3), let V_1 and V_2 be the connected components of $\widetilde{S} \setminus e$, that lie to the right and left of $\widetilde{r}(e)$ respectively. Observe that for any Type C 1-simplex $h : [0, 1] \to \widetilde{S}$, we have that

$$\hat{i}\big(h, \widetilde{r}(e)\big) = \begin{cases} -1 & \text{if } h(0) \in V_1 \text{ and } h(1) \in V_2; \\ 1 & \text{if } h(0) \in V_2 \text{ and } h(1) \in V_1; \\ -\frac{1}{2} & \text{if } h(0) \in e \text{ and } h(1) \in V_2, \text{ or } h(0) \in V_1 \text{ and } h(1) \in e; \\ \frac{1}{2} & \text{if } h(0) \in V_2 \text{ and } h(1) \in e, \text{ or } h(0) \in e \text{ and } h(1) \in V_1; \\ 0 & \text{otherwise.} \end{cases}$$

which immediately implies that (B.3) holds because backward and forward endpoints of h_1 are the forward and backward endpoints of h_2 respectively. Thus, (B.1) also holds in the case when $k = 2$ and h_1, h_2 are Type C 1-simplices. This proves the base case for our induction.

Next, assume the inductive hypothesis that for $k' > 2$, (B.1) holds for all $k < k'$. For the rest of this proof, assume that arithmetic in the subscripts of $h_1, \ldots, h_{k'}$ are modulo k'.

If $h := h_1 \cdot \ldots \cdot h_{k'}$ lies in some $e \in \widetilde{\mathcal{P}}$, then let $h_{1,2} : [0, 1] \to \widetilde{S}$ be the geodesic segment so that $h_{1,2}(0) = h_1(0)$ and $h_{1,2}(1) = h_2(1)$. Note that $h_{1,2}$ is also a Type A 1-simplex, and $\widetilde{\mu}_L(h_{1,2}) = \widetilde{\mu}_L(h_1) + \widetilde{\mu}_L(h_2)$. Hence,

$$\sum_{i=1}^{k'} \widetilde{\mu}_L(h_i) = \widetilde{\mu}_L(h_{1,2}) + \sum_{i=3}^{k'} \widetilde{\mu}_L(h_i) = 0$$

by the inductive hypothesis. Also, if there is some $j = 1, \ldots, k'$ so that $h_j = h_{j+1}^{-1}$, then $\widetilde{\mu}_L(h_j) + \widetilde{\mu}_L(h_{j+1}) = 0$, and $h' := h_1 \cdot \ldots \cdot h_{j-1} \cdot h_{j+2} \cdot \ldots \cdot h_{k'}$ is still a closed curve in $\widetilde{S}$. The inductive hypothesis then implies that

$$\sum_{i=1}^{k'} \widetilde{\mu}_L(h_i) = \sum_{i=1}^{j-1} \widetilde{\mu}_L(h_i) + \sum_{i=j+2}^{k'} \widetilde{\mu}_L(h_i) = 0.$$

Thus, for the rest of the proof, we will assume that h intersects some connected component U of $\widetilde{S} \setminus \widetilde{\mathcal{P}}$, and $h_j \neq h_{j+1}^{-1}$ for all $j = 1, \ldots, k'$.

We say two connected components U_1, U_2 of $\widetilde{S} \setminus \widetilde{\mathcal{P}}$ are *neighbors* if there is some $e \in \widetilde{\mathcal{P}}$ so that $U_1 \cup U_2 \cup e \subset \widetilde{S}$ is connected. Since $h := h_1 \cdot \ldots \cdot h_{k'}$ is a closed loop in $\widetilde{S}$ and each $e \in \widetilde{\mathcal{P}}$ cuts $\widetilde{S}$ into two connected components, there is some connected component U of $\widetilde{S} \setminus \widetilde{\mathcal{P}}$ so that the image of h intersects at most one neighbor of U. Then one of the following must hold:

(1) There is some $j = 1, \ldots, k'$ so that h_j and h_{j+1} are Type C 1-simplices whose images lie in U.

(2) There is some $j = 1, \ldots, k'$ so that h_{j-1} and h_{j+1} are Type B 1-simplices that lie in U so that $h_{j-1}(0)$ and $h_{j+1}(1)$ lie in the same closed edge $e \in \widetilde{P}$, while h_j is a Type C 1-simplex whose image lies in U.

(3) There is some $j = 1, \ldots, k'$ so that h_{j-1} and h_{j+1} are Type B 1-simplices that lie in U, while h_j is a Type A 1-simplex whose image lies in the closure $\overline{U}$ of U in $\widetilde{S}$.

Suppose that (1) holds, then let $h_{j,j+1} := h_j \cdot h_{j+1}$. By the base case, $h_{j,j+1}$ is also a Type C 1-simplex, and $\widetilde{\mu}_L(h_{j,j+1}) = \widetilde{\mu}_L(h_j) + \widetilde{\mu}_L(h_{j+1})$. Hence,

$$\sum_{i=1}^{k'} \widetilde{\mu}_L(h_i) = \sum_{i=1}^{j-1} \widetilde{\mu}_L(h_i) + \widetilde{\mu}_L(h_{j,j+1}) + \sum_{i=j+2}^{k'} \widetilde{\mu}_L(h_i) = 0$$

by the inductive hypothesis.

Suppose that (2) holds. Let $h_{j-1,j+1} : [0,1] \to \widetilde{S}$ be the oriented geodesic segment with $h_{j-1,j+1}(0) = h_{j-1}(0)$ and $h_{j-1,j+1}(1) = h_{j+1}(1)$. Note that $h_{j-1,j+1}$ is a Type A 1-simplex that lies in the closed edge e. We will now show that

(B.4) $\widetilde{\mu}_L(h_{j-1,j+1}) = \widetilde{\mu}_L(h_{j-1}) + \widetilde{\mu}_L(h_j) + \widetilde{\mu}_L(h_{j+1})$.

Since $h_1 \cdot \ldots \cdot h_{j-2} \cdot h_{j-1,j+1} \cdot h_{j+2} \cdot \ldots \cdot h_{k'}$ is a closed loop, (B.4) and the inductive hypothesis imply that

$$\sum_{i=1}^{k'} \widetilde{\mu}_L(h_i) = \sum_{i=1}^{j-2} \widetilde{\mu}_L(h_i) + \widetilde{\mu}_L(h_{j-1,j+1}) + \sum_{i=j+2}^{k'} \widetilde{\mu}_L(h_i) = 0.$$

Let J_1 and J_2 be the bridges that contain h_{j-1} and h_{j+1}, and let $\overrightarrow{J_1}$ and $\overrightarrow{J_2}$ be the orientations on J_1 and J_2 with backward endpoints $h_{j-1}(1)$ and $h_{j+1}(0)$ respectively. If $h_{j-1}(1)$ lies to the right of $\widetilde{r}(e)$, denote the triangle containing $h_{j-1}(1)$ by T_1, and if $h_{j-1}(1)$ lies to the left of $\widetilde{r}(e)$, denote the triangle containing $h_{j-1}(1)$ by T_2. For $m = 1, 2$, we can define $B(J_1, T_m)$, $N(J_1, T_m)$ and $D_m(e)$ as we did in Section 3.2, and let $\gamma_m \in \Gamma$ be as defined in Notation 3.6. Note that since $\gamma_m^l \cdot \overrightarrow{J_1} = \overrightarrow{J_2}$ for some $l \in \mathbb{Z} \setminus \{0\}$, $\gamma_m^l \cdot T_m$ contains $h_{j+1}(0)$.

From the definition of K_0, we see that

$$\widetilde{\mu}_L(h_{j-1,j+1}) = l \cdot D_m(e)$$

Also, the $\mathrm{Ad} \circ \rho$-equivariance of K_1 implies that $\rho(\gamma_m)^l \cdot K_1(\overrightarrow{J_1}) \cdot \rho(\gamma_m)^{-l} = K_1(\overrightarrow{J_2})$. This then implies that

$$
\begin{aligned}
\widetilde{\mu}_L(h_{j-1}) + \widetilde{\mu}_L(h_{j+1}) &= -K_1(\overrightarrow{J_1}) + K_1(\overrightarrow{J_2}) \\
&= \begin{cases} -\displaystyle\sum_{i=0}^{l-1} \rho(\gamma_m)^i \cdot N(J_1, T_m) \cdot \rho(\gamma_m)^{-i} & \text{if } l > 0; \\ \displaystyle\sum_{i=1}^{-l} \rho(\gamma_m)^{-i} \cdot N(J_1, T_m) \cdot \rho(\gamma_m)^{i} & \text{if } l < 0. \end{cases}
\end{aligned}
$$

On the other hand, since h_j is a Type C 1-simplex, we see by the base case that if h_j' is the oriented geodesic segment with the same backward and forward endpoint as h_j, then $\widetilde{\mu}_L(h_j) = \widetilde{\mu}_L(h_j')$. Thus, we may assume that h_j is a geodesic segment. In that case, it is easy to identify the barriers that h_j intersect transversely, and we can also compute that

$$
\widetilde{\mu}_L(h_j) = \begin{cases} \displaystyle\sum_{i=0}^{l-1} \rho(\gamma_m)^i \cdot B(J_1, T_m) \cdot \rho(\gamma_m)^{-i} & \text{if } l > 0; \\ -\displaystyle\sum_{i=1}^{-l} \rho(\gamma_m)^{-i} \cdot B(J_1, T_m) \cdot \rho(\gamma_m)^{i} & \text{if } l < 0. \end{cases}
$$

From the definition of $B(J_1, T_m)$ and $N(J_1, T_m)$, we see that

$$
\begin{aligned}
\rho(\gamma_m)^i \cdot B(J_1, T_m) \cdot \rho(\gamma_m)^{-i} &= B(J_1, \gamma_m^i \cdot T_m) \\
&= N(J_1, \gamma^i \cdot T_m) + D_m(e) \\
&= \rho(\gamma)^i \cdot N(J_1, T_m) \cdot \rho(\gamma)^{-i} + D_m(e).
\end{aligned}
$$

for $m = 1, 2$ and $i \in \mathbb{Z}$. This allows us to conclude that

$$\widetilde{\mu}_L(h_{j-1}) + \widetilde{\mu}_L(h_j) + \widetilde{\mu}_L(h_{j+1}) = l \cdot D_m(e) = \widetilde{\mu}_L(h_{j-1,j+1}),$$

which is exactly (B.4).

Finally, suppose that (3) holds. Let $h_{j-1,j+1} : [0,1] \to \widetilde{S}$ be the oriented geodesic segment so that $h_{j-1,j+1}(0) = h_{j-1}(0)$ and $h_{j-1,j+1}(1) = h_{j+1}(1)$. Then $h_{j-1,j+1}$ is a Type C 1-simplex that lies in U. In the proof of (2), we showed that

$$\widetilde{\mu}_L(h_{j-1}^{-1}) + \widetilde{\mu}_L(h_{j-1,j+1}) + \widetilde{\mu}_L(h_{j+1}^{-1}) = \widetilde{\mu}_L(h_j),$$

which is equivalent to

$$\widetilde{\mu}_L(h_{j-1}) + \widetilde{\mu}_L(h_j) + \widetilde{\mu}_L(h_{j+1}) = \widetilde{\mu}_L(h_{j-1,j+1}).$$

We can then apply the inductive hypothesis to conclude that

$$\sum_{i=1}^{k'} \widetilde{\mu}_L(h_i) = \sum_{i=1}^{j-2} \widetilde{\mu}_L(h_i) + \widetilde{\mu}_L(h_{j-1,j+1}) + \sum_{i=j+2}^{k'} \widetilde{\mu}_L(h_i) = 0.$$

$\square$

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Yau Mathematical Sciences Center, Tsinghua University, Jingzhai 204, Haidian district, Beijing, 100084, China

E-mail address: sunzhe1985@gmail.com

Mathematics Department, California Institute of Technology, 1200 East California Boulevard, Mail Code 253-37, Pasadena, CA 91125

E-mail address: tengren@caltech.edu