ON THE STRUCTURAL THEOREM OF PERSISTENT HOMOLOGY

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ABSTRACT. We study the categorical framework for the computation of persistent homology, without reliance on a particular computational algorithm. The computation of persistent homology is commonly summarized as a matrix theorem, which we call the Matrix Structural Theorem. Any of the various algorithms for computing persistent homology constitutes a *constructive* proof of the Matrix Structural Theorem. We show that the Matrix Structural Theorem is equivalent to the Krull-Schmidt property of the category of filtered chain complexes. We separately establish the Krull-Schmidt property by abstract categorical methods, yielding a novel *nonconstructive* proof of the Matrix Structural Theorem.

These results provide the foundation for an alternate categorical framework for decomposition in persient homology, bypassing the usual persistence vector spaces and quiver representations. Persistent Homology and Matrix Reduction and Krull-Schmidt Category

> *But the power of homology is seldom of much efficacy, except in those happy dispositions where it is almost superfluous.*
>
> *with apologies to Edward Gibbon*

CONTENTS
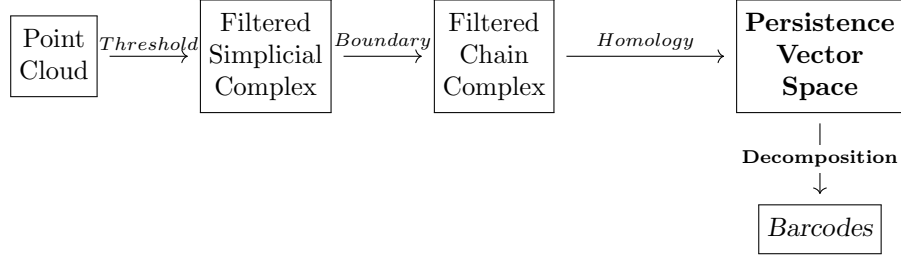
1. Manifestations of the Structural Theorem

1.1. **Introduction.** During the last decade, persistent homology [13, 6] has achieved great success as a powerful and versatile tool, particularly for topological analysis of point cloud data. The term *point cloud* usually means a finite subset of points in a Euclidean space $\mathbb{R}^k$, where the dimension k can be large, and the number of points is often very large. Many excellent surveys and introductions are available in the literature [7, 22, 12, 26, 15, 25]. *Decomposition* plays a central role both in the theory and in the applications of persistent homology. The ubiquitous "barcode diagrams" encode a decomposition in terms of the types and multiplicities of indecomposable summands. This data is an invariant, independent of the choice of decomposition. The summands represented by long barcodes contain important characteristic information, while the summands represented by short barcodes only contain random "noise" and may be disregarded. A number of "stability theorems" [8, 22] provide a firm foundation for this intuitively appealing interpretation of the long and short barcode invariants. In this paper we consider the interplay between the algorithmic and the categorical underpinnings for decomposition in persistent homology.

It is helpful to first review analogous decomposition issues for the much more familiar context of finite-dimensional vector spaces (over a fixed field $\mathbb{F}$). The ordinary Gaussian elimination algorithm can construct a basis for a vector space. Any choice of basis then constitutes a decomposition of the vector space, wherein the linear span of each basis element is a one-dimensional vector space. The direct sum of these one-dimensional summands is canonically identified (naturally isomorphic) to the original vector space. A one-dimensional vector space cannot be further decomposed as a sum of nonzero (dimensional) summands. This means that one-dimensional vector spaces are the *indecomposable objects*, in the the category of vector spaces. Since all one-dimensional vector spaces are mutually isomorphic, there is just one type of indecomposable (object) in the category of vector spaces. The familiar dimension of a vector space is just the multiplicity of the indecomposable (one-dimensional) summands in a decomposition, and this multiplicity is an invariant independent of the choice of decomposition.

Now setting aside what we know about Gaussian elimination, we ask more abstractly *why* is it that any vector space is actually decomposable? We first observe that decomposability is a categorical property, since it involves both objects (vector spaces) and morphisms (linear maps). The theory of *Krull-Schmidt categories* [17] provides an appropriate, albeit abstract, categorical setting for questions of decomposability. The axioms of a Krull-Schmidt category guarantee that every object admits an essentially unique decomposition as a finite sum of indecomposable objects. For the category of vector spaces, this essential uniqueness encodes the familiar fact that the dimension is an invariant independent of the choice of decomposition. The goal then becomes to verify (and understand) the Krull-Schmidt property for the category of vector spaces. A concrete constructive verification of the Krull-Schmidt axioms for the category of vector spaces follows easily from basic properties of Gaussian elimination and linearity, but this is more in line with describing *how* to perform a decomposition rather than *why* vector spaces are decomposable. Fortunately there is a complementary abstract tool available. A theorem of Atiyah [3] dating back to the early years of category theory provides a very useful criterion for verifying the Krull-Schmidt property of a category. For the category of vector
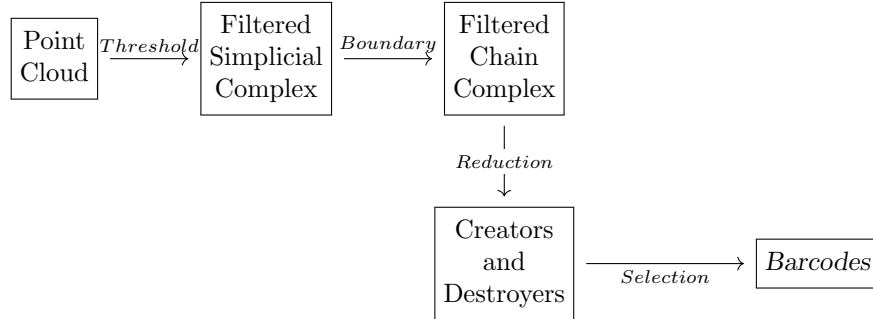
spaces, Atiyah's criterion reduces to checking certain elementary properties of linear maps. So Atiyah's theorem nonconstructively answers the abstract question of *why* any vector space admits a decomposition, complementing our understanding of *how* to constructively decompose a given vector space via Gaussian elimination.

In this paper we consider analogous questions of *how* and *why* decomposition works in persistent homology. The following picture summarizes one common description of the transformation from point cloud data to barcodes invariants:

Point Cloud $\xrightarrow{Threshold}$ Filtered Simplicial Complex $\xrightarrow{Boundary}$ Filtered Chain Complex $\xrightarrow{Homology}$ **Persistence Vector Space**

Persistence Vector Space $\xrightarrow{\text{Decomposition}}$ *Barcodes*

The initial stages, going from a point cloud to a filtered chain complex, will be briefly reviewed at the end of this section. Our primary focus will be the final stages, going from filtered chain complexes to barcodes. At the homology step, the homology functor H_n of the chosen dimension/degree n takes a *filtered chain complex* (which is a diagram of chain complexes) to a *persistence vector space* (which is a diagram of vector spaces). The key final step is to compute barcode invariants by decomposition of a persistence vector space. An important insight [6] is that persistence vector spaces are quiver representations. A concrete consequence is the applicability of decomposition algorithms from quiver representation theory, showing how to decompose a persistence vector space and compute the barcodes. An abstract consequence is that the appropriate category of quiver representations is Krull-Schmidt by Atiyah's theorem, showing *why* all of this works. So the Krull-Schmidt property of persistence vector spaces nicely ties together the theoretical and computational aspects. But there is one problem with this picture.

The standard computational algorithms for persistent homology [13, 27, 28] do *not* work by decomposing a persistence vector space. The following picture summarizes *how* barcodes are normally computed:

Point Cloud $\xrightarrow{Threshold}$ Filtered Simplicial Complex $\xrightarrow{Boundary}$ Filtered Chain Complex

Filtered Chain Complex $\xrightarrow{Reduction}$ Creators and Destroyers $\xrightarrow{Selection}$ *Barcodes*

The initial stages of the picture, going from a point cloud to a filtered chain complex, are unchanged. The key *reduction* step [13, 27, 28] is the construction of a

special type of basis. Each basis element is interpreted as either a *creator* or as a *destroyer* of a homology class. The *selection* step consists of keeping those creators and destroyers that correspond to nonzero barcodes of the desired homology dimension/degree n, and discarding the remaining basis elements. The question remains of *why* there should exist such algorithms operating on filtered complexes, rather than on persistence vector spaces.

In this paper we provide a categorical framework for the standard persistent homology algorithms, using an equivalence of categories to unify the two pictures above:

$$
\begin{array}{c}
\boxed{\text{Point Cloud}} \xrightarrow{\textit{Threshold}} \boxed{\begin{array}{c}\text{Filtered}\\\text{Simplicial}\\\text{Complex}\end{array}} \xrightarrow{\textit{Boundary}} \boxed{\begin{array}{c}\textbf{Filtered}\\\textbf{Chain}\\\textbf{Complex}\end{array}} \xrightarrow{\textit{Homology}} \boxed{\begin{array}{c}\textbf{Persistence}\\\textbf{Vector}\\\textbf{Space}\end{array}}
\end{array}
$$

with vertical maps: *Congruence* from **Filtered Chain Complex** to **Quotient Object**, **Decomposition** from **Persistence Vector Space** to *Barcodes*, **Equivalence** between **Quotient Object** and **Persistence Vector Space**, and **Decomposition** from **Quotient Object** to *Barcodes*.

The foundation is the *Categorical Structural Theorem* (Theorem 1.6), which asserts that the category of filtered chain complexes is Krull-Schmidt. This gives an alternate framework for persistent homology, where the barcodes describe the Krull-Schmidt decomposition of an object in a quotient of the category of filtered chain complexes. These are still the same barcodes as in the standard framework, because the quotient category is equivalent to the category of persistence vector spaces. This gives a unified answer for *why* and *how* decomposition actually works in persistent homology. We no longer need to rely on the Krull-Schmidt property of the category of persistence vector spaces as an indirect theoretical foundation for decomposition and barcodes, since we can directly appeal to the Krull-Schmidt property already in the category of filtered chain complexes.

The Categorical Structural Theorem is the abstract version of what we call the *Structural Theorem of Persistent Homology*. We give a nonconstructive categorical proof of the Categorical Structural Theorem, indirectly using Atiyah's criterion. Combining the Categorical Sructural Theorem with a classification of indecomposable filtered chain complexes then yields a novel *nonconstructive* proof of what we call the *Matrix Structural Theorem* (Theorem 1.4). The Matrix Structural Theorem characterizes the output of any of the various standard persistent homology algorithms in terms of a matrix factorization rather than the more common description in terms of creators and destroyers for homology. In this sense, any of the standard algorithms can be thought of as constituting a *constructive* proof of the Matrix Structural Theorem. In Appendix B, we give a self-contained description of one such algorithm in terms of elementary matrix operations.

This paper focuses on the final stages of topological data analysis, going from a filtered chain complex to barcode invariants. We give below a simple example to illustrate the first stage, going from a point cloud to a filtered simplicial complex. In our example, we use the α-complex construction [12, 11], which is suitable for low
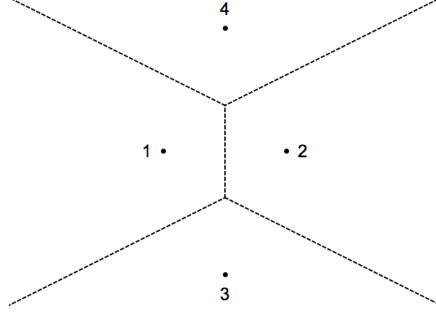
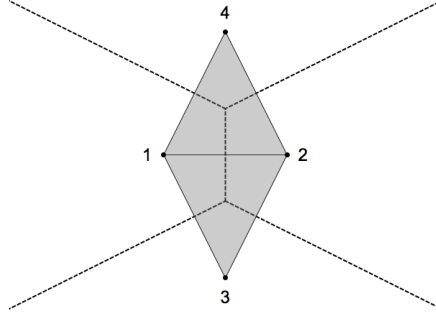**Figure 1.** A point cloud with its Voronoi cells.

Figure 2. Delaunay Complex.

dimensions. We note that for large point clouds in high dimensions, the Vietoris-Rips construction [7, 22] is often preferable.

Example 1.1. *The first step is to construct a Delaunay complex, the second step is to construct a filtration of the Delaunay complex. We illustrate the construction of the Delaunay simplicial complex associated to a point cloud. Figure 1 shows a point cloud consisting four planar points labeled by $n \in \{1, 2, 3, 4\}$, together with the Voronoi cell $V(n)$ of each labeled point. We recall [12] that a Voronoi cell $V(n)$ contains all the points $x \in \mathbb{R}^2$ such that n is the closest labeled point to x (or one of the closest if several are equidistant). Figure 2 shows the Delaunay simplicial complex encoding the intersections of the Voronoi cells. We recall that the simplex $[n_0, \ldots, n_k]$, where $n_i \in \{1, 2, 3, 4\}$ and $n_0 < \cdots < n_k$, is included in the Delaunay complex iff $V(n_0) \cap \cdots \cap V(n_k) \neq \emptyset$. For example, the simplex $[1, 2]$ is included because $V(1) \cap V(2) \neq \emptyset$, but the simplex $[3, 4]$ is not included because $V(3) \cap V(4) = \emptyset$.*
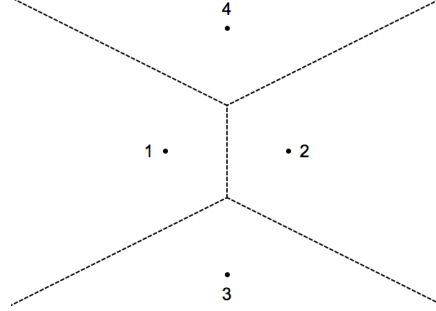
Figure 3. Level $p = 1$ is the threshold $r = 0 = b([1]) = b([2]) = b([3]) = b([4])$.
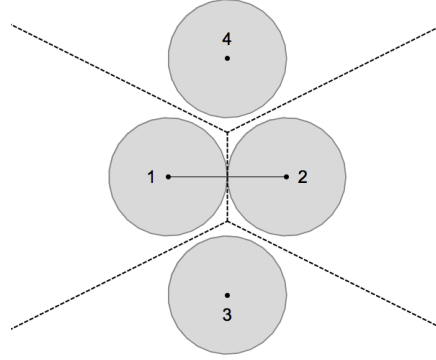

Figure 4. Level $p = 2$ is the threshold $r = 1 = b([1, 2])$.

The α construction assigns to each Delaunay simplex $[n_0, \ldots, n_k]$ a real nonnegative "birth parameter" $b([n_0, \ldots, n_k])$. Let $B_r(n)$ denote the closed ball of radius r centered at the labeled point n, and consider the subset $A_r(n) = B_r(n) \cap V(n)$ of the Voronoi cell $V(n)$. The birth parameter of the Delaunay simplex $[n_0, \ldots, n_k]$ is defined to be the smallest value of r such that $A_r(n_0) \cap \cdots \cap A_r(n_k) \neq \emptyset$. A value of r is called a "threshold' if it is the birth parameter for some Delaunay simplex. The integer "level" p indexes the thresholds in increasing order, as illustrated in Figures 3 through 6:

The α construction produces a filtration of the Delaunay complex, and the simplicial homology [16] of this filtered complex is described in terms of the barcode invariants [22, 12, 26]. Conventionally the filtration and the corresponding barcodes are indexed by the real-valued threshold parameter r, which for our example yields the H_1 barcode diagram of Figure 7. The diagram indicates that the first homology H_1 detects two one-dimensional "holes" that appear at $r = 1.12$ and are filled in at $r = 1.25$. In this paper we will index filtrations and the corresponding
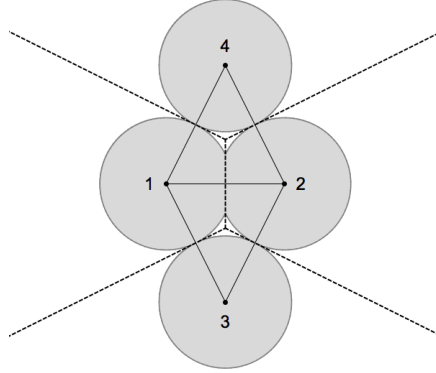
Figure 5. Level $p = 3$ is the threshold $r = 1.12 = b([1,3]) = b([1,4]) = b([2,3]) = b([2,4])$.
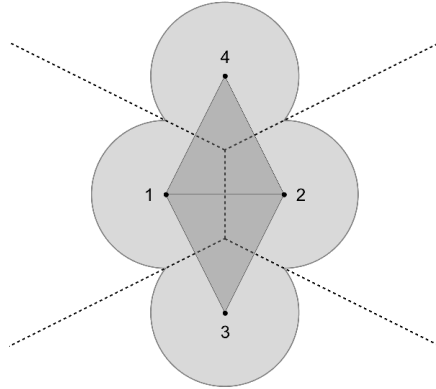
Figure 6. Level $p = 4$ is the threshold $r = 1.25 = b([1,2,3]) = b([1,2,4])$.
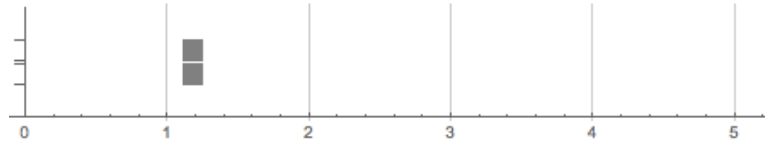
Figure 7. H_1 barcodes indexed by the real threshold parameter r.

barcodes by the integer-valued level p, which for our example yields the H_1 barcode diagram of Figure 8. This diagram indicates the same information, namely that the first homology H_1 detects two one-dimensional "holes" that appear at $p = 3$ (which corresponds to $r = 1.12$) and are filled in at $p = 4$ (which corresponds to $r = 1.25$).

Figure 8. H_1 barcodes indexed by the integer level p.

1.2. **Matrix Structural Theorem.** For simplicity, we start with the ungraded version of the structural theorem. A *differential matrix* is a square matrix D satisfying $D^2 = 0$. We'll say a differential matrix is *Jordan* 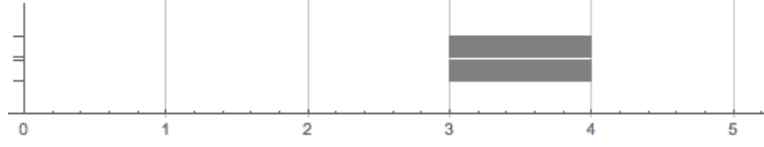if it is in Jordan normal form, meaning it decomposes as a block-diagonal matrix built from copies of the two differential Jordan block matrices

$$J = \begin{bmatrix} 0 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We'll say a differential matrix $\underline{D}$ is *almost-Jordan* if there exists a permutation matrix P such that the differential matrix $P^{-1}\underline{D}P$ is Jordan. Given an almost-Jordan differential matrix $\underline{D}$, it is trivial to construct such a permutation matrix P. We will say a square matrix B is *triangular* if it is upper-triangular and invertible.

The standard algorithm for computing persistent homology is based on the papers [13, 27, 28]. The result of a persistent homology computation, not depending on a choice of algorithm, is conveniently summarized [10, 22] as a matrix factorization:

Theorem 1.2. *(Ungraded Matrix Structural Theorem) Any differential matrix D factors as $D = B\underline{D}B^{-1}$ where $\underline{D}$ is an almost-Jordan differential matrix and B is a triangular matrix.*

It is the triangular condition that makes this interesting: without the triangular condition, this would follow immediately from the ordinary Jordan normal form. Furthermore, the matrix $\underline{D}$ is unique by a standard result from Bruhat factorization, as reviewed in Appendix A. We'll call $\underline{D}$ the *persistence canonical form* of the differential matrix D. A column of the triangular matrix B is in $\ker D$ iff the corresponding column of $\underline{D}$ is zero. We will say that B is *normalized* if each such column has diagonal entry equal to 1. It is always possible to normalize B by scalar multiplication of columns, but even with normalization B is not unique in general. A constructive proof of Theorem 1.2 follows from any of the algorithms for computing persistent homology. In Appendix B we discuss a simple version of the standard algorithm, with complete proofs using ordinary linear algebra of matrices.

Example 1.3. *Consider the filtered simplicial complex shown in Figure 9. With the usual convention for an adapted basis, the ordering of basis elements prioritizes the level of the filtration over the degree/dimension of the simplex. The initial basis of simplices is then ordered so the level (denoted by prescript) is nondecreasing, and within each level the degree (denoted by postscript) is nondecreasing. Using lexicographic order to break any remaining ties, the initial adapted basis is $_1a_0, _1b_0, _2ab_1, _3c_0, _4bc_1, _5ac_1, _6abc_2$, and the boundary operator over the field $\mathbb{F} = \mathbb{Q}$*
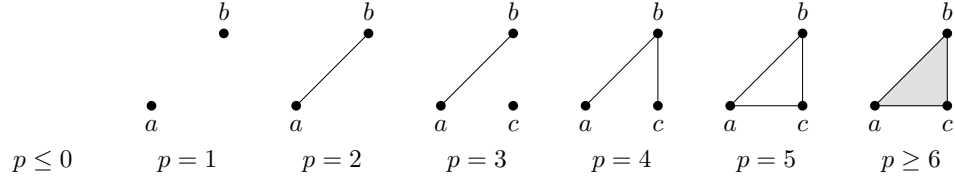
Figure 9. A filtered Simplicial Complex.

of rationals is represented by the differential matrix

$$
D = \begin{array}{c} \\ {}_1a_0 \\ {}_1b_0 \\ {}_2ab_1 \\ {}_3c_0 \\ {}_4bc_1 \\ {}_5ac_1 \\ {}_6abc_2 \end{array}
\begin{array}{c} {}_1a_0 \;\; {}_1b_0 \;\; {}_2ab_1 \;\; {}_3c_0 \;\; {}_4bc_1 \;\; {}_5ac_1 \;\; {}_6abc_2 \\
\left[\begin{array}{ccccccc}
0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

The persistence canonical form is

$$
\underline{D} = \begin{array}{c} \\ {}_1\underline{a}_0 \\ {}_1\underline{b}_0 \\ {}_2\underline{ab}_1 \\ {}_3\underline{c}_0 \\ {}_4\underline{bc}_1 \\ {}_5\underline{ac}_1 \\ {}_6\underline{abc}_2 \end{array}
\begin{array}{c} {}_1\underline{a}_0 \;\; {}_1\underline{b}_0 \;\; {}_2\underline{ab}_1 \;\; {}_3\underline{c}_0 \;\; {}_4\underline{bc}_1 \;\; {}_5\underline{ac}_1 \;\; {}_6\underline{abc}_2 \\
\left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array} ,
$$

as verified by checking that $\underline{D} = B^{-1}DB$ for the triangular (and normalized) matrix

$$
B = \begin{array}{c} \\ {}_1a_0 \\ {}_1b_0 \\ {}_2ab_1 \\ {}_3c_0 \\ {}_4bc_1 \\ {}_5ac_1 \\ {}_6abc_2 \end{array}
\begin{array}{c} {}_1\underline{a}_0 \;\; {}_1\underline{b}_0 \;\; {}_2\underline{ab}_1 \;\; {}_3\underline{c}_0 \;\; {}_4\underline{bc}_1 \;\; {}_5\underline{ac}_1 \;\; {}_6\underline{abc}_2 \\
\left[\begin{array}{ccccccc}
1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]
\end{array} .
$$

The persistence canonical form $\underline{D}$ is almost-Jordan in general, and in this example it happens to be actually Jordan. The matrix B represents the basis change to

the new adapted basis $_1\underline{a}_0, _1\underline{b}_0, _2\underline{ab}_1, _3\underline{c}_0, _4\underline{bc}_1, _5\underline{ac}_1, _6\underline{abc}_2$. *The level remains nonde-creasing because B is triangular. Each basis element retains pure degree, although Theorem 1.2 does not explicitly address issues of degree. The matrix $\underline{D}$ represents the boundary operator relative to the new adapted basis.*

We prefer to prioritize degree over level in ordering the elements of an adapted basis. This has the advantage of encoding the degree in the matrix block structure. The following version of the structural theorem is then manifestly compatible with the grading by degree:

Theorem 1.4. *(Matrix Structural Theorem) Any block-superdiagonal differential matrix D factors as $D = B\underline{D}B^{-1}$ where $\underline{D}$ is a block-superdiagonal almost-Jordan differential matrix and B is a block-diagonal triangular matrix.*

The block-diagonal structure of B ensures that the transformed basis elements retain pure degree. The persistence canonical form $\underline{D}$ inherits the block-superdiagonal structure of the differential D. It is always possible to normalize B by scalar multiplication of columns as in the ungraded case. Any of the algorithmic proofs of Theorem 1.2 [13, 27, 28] can be used to prove Theorem 1.4 by keeping track of degrees. We discuss this point for the standard algorithm in Appendix B.2.

Example 1.5. *We again consider the filtered chain complex of Example 1.3, but with basis order prioritizing degree over level. Now the degree of basis elements (denoted by postscript) is nondecreasing, and within a degree the level (denoted by prescript) of basis elements is nondecreasing. Using lexicographic order to break any remaining ties, the initial adapted basis is now $_1a_0, _1b_0, _3c_0, _2ab_1, _4bc_1, _5ac_1, _6abc_2$, and the boundary operator over the field $\mathbb{F} = \mathbb{Q}$ of rationals is now represented by the block-superdiagonal differential matrix*

$$
D = \begin{array}{c} \\ {}_1a_0 \\ {}_1b_0 \\ {}_3c_0 \\ {}_2ab_1 \\ {}_4bc_1 \\ {}_5ac_1 \\ {}_6abc_2 \end{array}
\begin{array}{c} {}_1a_0 \;\; {}_1b_0 \;\; {}_3c_0 \;\; {}_2ab_1 \; {}_4bc_1 \; {}_5ac_1 \, {}_6abc_2 \\
\left[\begin{array}{ccc|ccc|c}
0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

The persistence canonical form inherits the block-superdiagonal structure

$$
\underline{D} = \begin{array}{c} \\ {}_1\underline{a}_0 \\ {}_1\underline{b}_0 \\ {}_3\underline{c}_0 \\ {}_2\underline{ab}_1 \\ {}_4\underline{bc}_1 \\ {}_5\underline{ac}_1 \\ {}_6\underline{abc}_2 \end{array}
\begin{array}{c} {}_1\underline{a}_0 \;\; {}_1\underline{b}_0 \;\; {}_3\underline{c}_0 \;\; {}_2\underline{ab}_1 \; {}_4\underline{bc}_1 \; {}_5\underline{ac}_1 \, {}_6\underline{abc}_2 \\
\left[\begin{array}{ccc|ccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

as verified by checking that $\underline{D} = B^{-1}DB$ for the block-diagonal triangular (and normalized) matrix

$$
B = \begin{array}{c} \\ _1a_0 \\ _1b_0 \\ _3c_0 \\ _2ab_1 \\ _4bc_1 \\ _5ac_1 \\ _6abc_2 \end{array}
\begin{array}{c}
_1\underline{a}_0 \quad _1\underline{b}_0 \quad _3\underline{c}_0 \quad _2\underline{ab}_1 \quad _4\underline{bc}_1 \quad _5\underline{ac}_1 \quad _6\underline{abc}_2 \\
\left[\begin{array}{ccc|ccc|c}
1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]
\end{array}
$$

The persistence canonical form $\underline{D}$ is almost-Jordan, but not actually Jordan in this example. The matrix B represents the basis change to the new adapted basis $_1\underline{a}_0, {}_1\underline{b}_0, {}_3\underline{c}_0, {}_2\underline{ab}_1, {}_4\underline{bc}_1, {}_5\underline{ac}_1, {}_6\underline{abc}_2$. Since B is block-diagonal, each basis element retains pure degree, and the degree remains nondecreasing. Since B is triangular, the level remains nondecreasing within each degree. The computation of this matrix B via the standard algorithm is worked out in Appendix B.2.

1.3. **Categorical Structural Theorem and Structural Equivalence.** A Krull-Schmidt category is an additive category where objects decompose nicely as direct sums of indecomposable objects. In chapter 2, we study the additive category of filtered chain complexes in the setting of Krull-Schmidt categories, starting with a review of Krull-Schmidt categories in section 2.1. A filtered complex will be called *basic* if its boundary operator can be represented by differential matrix consisting of a single Jordan block. We will use nonconstructive categorical methods to prove the following structural theorem for the category of filtered complexes:

Theorem 1.6. *(Categorical Structural Theorem) The category of filtered complexes is Krull-Schmidt. A filtered complex is indecomposable iff it is basic.*

In chapter 3 we will prove the equivalence of the matrix and the categorical versions of the structural theorem. One direction is proved in section 3.1:

Proposition 1.7. *(Forward Structural Equivalence) The Matrix Structural Theorem implies the Categorical Structural Theorem.*

This is followed by a detailed example of a Krull-Schmidt decomposition. The other direction is proved in section 3.2:

Proposition 1.8. *(Reverse Structural Equivalence) The Categorical Structural Theorem implies the Matrix Structural Theorem.*

Combining the Categorical Structural Theorem 1.6 and the Reverse Structural Equivalence Proposition 1.8 yields a *nonconstructive* categorical proof of the Matrix Structural Theorem 1.4. This contrasts with the various *constructive* algorithmic proofs of Theorem 1.4, one of which is reviewed in Appendix B.2. The constructive algorithmic proofs explain *how* persistent homology works, the nonconstructive proof explains *why* persistent homology works.

2. Proving the Categorical Structural Theorem

2.1. **Additive and Krull-Schmidt Categories.** This section reviews the relevant background from category theory. General references for category theory include [20, 4, 1]. Additive categories are discussed in [20, 24]. Krull-Schmidt categories are discussed in [17, 21, 3].

Definition 2.1. *A category is additive if:*

(1) Each $Hom(X,Y)$ is an abelian group, and the morphism composition map $Hom(Y,Z) \times Hom(X,Y) \to Hom(X,Z)$ is biadditive/bilinear.
(2) There exists a zero object 0.
(3) Any finite collection of objects $X_1, X_2, \ldots, X_n$ has a direct sum $X_1 \oplus X_2 \oplus \cdots \oplus X_n$.

An additive category is *linear* over the field $\mathbb{F}$ if each $Hom(X,Y)$ is a finite-dimensional $\mathbb{F}$-vector space, and each map $Hom(Y,Z) \times Hom(X,Y) \to Hom(X,Z)$ describing composition of morphisms $(g,f) \mapsto g \circ f$ is $\mathbb{F}$-bilinear. All of the categories we will be studying are linear.

The *endomorphism ring* of an object X in an additive category is the Abelian group $Hom(X,X)$ of endomorphisms, with multiplicative structure given by composition of endomorphisms. In a linear category, the endomorphism ring is an $\mathbb{F}$-algebra.

Definition 2.2. *A ring is local if:*

(1) $1 \neq 0$.
(2) If an element f does not have a multiplicative inverse, then the element $1 - f$ has a multiplicative inverse.

The local property is important, because a finite direct sum decomposition into summands with a local endomorphism rings is essentially unique:

Theorem 2.3. *(e.g. [17] Theorem 4.2) Let X be an object in an additive category, and suppose there are two finite decompositions*

$$X_1 \oplus \cdots \oplus X_m = X = Y_1 \oplus \cdots \oplus Y_n$$

into (nonzero) objects with local endomorphism rings. Then $m = n$ and there exists a permutation π such that $X_i \simeq Y_{\pi(i)}$ for each $1 \leq i \leq m$.

An object X in an additive category is *decomposable* if it is the direct sum $X = Y \oplus Z$ of two nonzero objects Y and Z. An *indecomposable object*, often abbreviated as an *indecomposable*, is a nonzero object that is not decomposable.

Lemma 2.4. *An object is indecomposable if it has a local endomorphism ring.*

Proof. We will show that the endomorphism ring of a decomposable object X is not local. We may assume that $X = Y \oplus Z$ with Y and Z nonzero. Then neither $f = 1_Y \oplus 0_Z$ nor $1_X - f = 1_Y \oplus 1_Z - f = 0_Y \oplus 1_Z$ has a multiplicative inverse in the ring $Hom(X,X) = Hom(Y \oplus Z, Y \oplus Z)$. $\square$

A Krull-Schmidt category has properties that guarantee both the existence and essential uniqueness of finite direct sum decompositions of any object, see e.g. [17, 21] for more details:

Definition 2.5. *An additive category is Krull-Schmidt if:*

(1) Every object admits a finite decomposition as a sum of indecomposables.
(2) Every indecomposable has a local endomorphism ring.

Recall that an additive category is *Abelian* if every morphism has a kernel and a cokernel, every monic morphism is normal (is the kernel of some morphism), and every epic morphism is conormal (is the cokernel of some morphism). Note that Definition 2.5 of Krull-Schmidt category does not assume that the additive category is Abelian, or even the existence of kernels and cokernels. We are primarily interested in the the linear category of filtered chain complexes, which is not Abelian. But we will use Abelian categories and their subcategories to show that this category is nonetheless Krull-Schmidt. Atiyah's Criterion [3, 17] provides a very general sufficient condition for an Abelian category to be Krull-Schmidt. Since all of our categories are linear, we will only need the following special case:

Theorem 2.6. *(Atiyah's Criterion) A linear Abelian category is Krull-Schmidt.*

The proof of Atiyah's Criterion is nonconstructive. It neither provides an algorithm to decompose a given object as a direct sum of indecomposables, nor a classification of indecomposables.

2.2. **Persistence Objects and Filtered Objects.** Persistence objects [6] and filtered objects [24] are described by categorical diagrams. Suppose that $\mathcal{X}$ is a linear Abelian category (and therefore Krull-Schmidt by Theorem 2.6). We will study persistence indexed by an integer $p \in \mathbb{Z}$, with $\leq$ denoting the standard partial order. A *persistence object* in $\mathcal{X}$ is a diagram $_\bullet X$ in the category $\mathcal{X}$ of type

$$\cdots \longrightarrow {}_{p-1}X \longrightarrow {}_pX \longrightarrow {}_{p+1}X \longrightarrow \cdots .$$

A morphism of persistence objects $_\bullet f : {}_\bullet X \to {}_\bullet X'$ is a commutative diagram of "ladder" type

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & {}_{p-1}X & \longrightarrow & {}_pX & \longrightarrow & {}_{p+1}X & \longrightarrow \cdots \\
& {}_{p-1}f \downarrow & & {}_pf \downarrow & & {}_{p+1}f \downarrow & \\
\cdots \longrightarrow & {}_{p-1}X' & \longrightarrow & {}_pX' & \longrightarrow & {}_{p+1}X' & \longrightarrow \cdots
\end{array}
$$

The category of persistence objects in $\mathcal{X}$ is Abelian, with pointwise kernels, cokernels, and direct sums. The set of morphisms $_\bullet X \to {}_\bullet X'$ between two persistence objects is a vector space, but not finite-dimensional in general. We will say a categorical diagram is *tempered* if all but finitely many of its arrows are iso(morphisms). We remark that the term "tame" is often used for conditions of this type, but we prefer the specificity afforded by the uses of the nonstandard term "tempered". Since $\mathcal{X}$ is linear, the set of morphisms $_\bullet X \to {}_\bullet X'$ between two tempered persistence objects is a finite-dimensional vector space. The tempered persistence objects comprise a strictly full Abelian subcategory of the persistence objects. Theorem 2.6 now yields:

Proposition 2.7. *Let $\mathcal{X}$ be a linear Abelian category. The category of tempered persistence objects in $\mathcal{X}$ is Krull-Schmidt.*

We next discuss subobjects in a linear Abelian category $\mathcal{X}$. We make the additional assumption that the category $\mathcal{X}$ is concrete, meaning that an object in $\mathcal{X}$ is a set with some additional structure, and a morphism in $\mathcal{X}$ is a map of sets compatible with the additional structure. For example, the linear Abelian category $\mathcal{V}$ of (finite-dimensional) vector spaces is a concrete linear Abelian category. An *inclusion* $X \hookrightarrow X'$ in $\mathcal{X}$ is an arrow that is an inclusion of the underlying sets. We say X is a *subobject* of X' iff such an inclusion arrow exists. An inclusion arrow is monic [20, 4], and the composition of inclusion arrows is an inclusion arrow. Any object X' in $\mathcal{X}$ has a zero subobject $0 \hookrightarrow X'$, and is its own subobject $X' \hookrightarrow X'$. A subobject $X \hookrightarrow X'$ is *proper* if $X \neq X'$. A nonzero object is said to be *simple* if it does not have a proper nonzero subobject. A simple object is obviously indecomposable, but an indecomposable object need not be simple.

A filtered object in a concrete linear Abelian category $\mathcal{X}$ is a special type of tempered persistence object in $\mathcal{X}$. We say a tempered persistence object $_\bullet X$ in $\mathcal{X}$ is *bounded below* if there exists an integer j such that $_j X = 0$ whenever $p \leq j$. We say $_\bullet X$ is a *filtered object* if it is bounded below and if every arrow is an inclusion arrow:

$$\cdots \overset{\hookrightarrow}{\longrightarrow} {}_{p-1}X \overset{\hookrightarrow}{\longrightarrow} {}_p X \overset{\hookrightarrow}{\longrightarrow} {}_{p+1}X \overset{\hookrightarrow}{\longrightarrow} \cdots .$$

The filtered objects in $\mathcal{X}$ comprise a strictly full subcategory of the tempered persistence objects. The properties of monics have several consequences. A filtered object diagram has a categorical limit and a colimit [20, 4]. The limit is 0 since the diagram is bounded below. The colimit X is $_k X$ for k sufficiently large (satisfying $_p X = X$ whenever $k \leq p$). Finally, any summand of a filtered object is isomorphic to a filtered object. Combining these facts with Lemma 2.7 yields:

Lemma 2.8. *Let $\mathcal{X}$ be a linear Abelian category. The category of filtered objects in $\mathcal{X}$ is Krull-Schmidt. A filtered object $_\bullet X$ in $\mathcal{X}$ is indecomposable iff its colimit X is an indecomposable object in $\mathcal{X}$.*

We note that the filtered objects comprise a subcategory of the tempered persistence objects, but this subcategory is not Abelian because a morphism of filtered objects may have a kernel and/or cokernel that is not a filtered object. So Lemma 2.8 is not merely a corollary of Theorem 2.6. Finally we observe that the category of persistence objects in a (concrete) linear Abelian category is itself a (concrete) linear Abelian category, to which Lemma 2.8 applies.

2.3. **Chain Complexes and Filtered Chain Complexes.** A *persistence vector space* is a persistence object in the (concrete) linear Abelian category $\mathcal{X} = \mathcal{V}$ of (finite-dimensional) $\mathbb{F}$-vector spaces. Tempered persistence vector spaces are well-understood via the theory of quiver representations. A nonempty subset $I \subseteq \mathbb{Z}$ will be called an *interval* if $c \in I$ whenever $a \leq c \leq b$ with $a \in I$ and $b \in I$. We associate to an interval $I \subseteq \mathbb{Z}$ the *interval persistence vector space* $_\bullet I$ constructed as follows: $_p I = \mathbb{F}$ whenever $p \in I$, $_p I = 0$ whenever $p \notin I$, and every arrow $\mathbb{F} \to \mathbb{F}$ is the identity morphism 1. We will often omit the bullet prescript when context allows.

For example, the interval persistence vector space $[1,4) = {}_\bullet[1,4)$ is the diagram of vector spaces

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{F} \xrightarrow{\ 1\ } \mathbb{F} \xrightarrow{\ 1\ } \mathbb{F} \longrightarrow 0 \longrightarrow \cdots$$

$$p=0 \qquad p=1 \qquad p=2 \qquad p=3 \qquad p=4$$

associated to the interval $[1,4) = \{1,2,3\} \subseteq \mathbb{Z}$. Proposition 2.7 applies to the linear Abelian category of tempered persistence vector spaces. Furthermore, the well-studied representation theory of A_n quivers (see e.g. [23]) carries over by a limiting argument to prove the following structural theorem for the category of tempered persistence vector spaces:

Theorem 2.9. *The category of tempered persistence vector spaces is Krull-Schmidt. A tempered persistence vector space is indecomposable iff it is isomorphic to an interval.*

Theorem 2.9 can be applied to cochain complexes. A *cochain complex*, or *cocomplex* for short, is a tempered persistence vector space ${}_\bullet V$

$$\cdots \longrightarrow {}_{p-1}V \xrightarrow{\ \partial^p\ } {}_pV \xrightarrow{\ \partial^{p+1}\ } {}_{p+1}V \longrightarrow \cdots$$

with the property that the composition of successive arrow $\partial^{p+1} \circ \partial^p$ is zero. The kernel of a morphism between cocomplexes is a cocomplex, as is the cokernel, so the cocomplexes comprise a strictly full Abelian subcategory of the tempered persistence vector spaces. Theorem 2.6 and Theorem 2.9 now yield the structural result:

Proposition 2.10. *The category $\mathcal{C}^{op}$ of cocomplexes is linear and Abelian, and therefore Krull-Schmidt. A cocomplex is indecomposable iff it is isomorphic to an interval cocomplex.*

Chain complexes are dual to cochain complexes. A *complex* (short for chain complex) is a tempered diagram $V_\bullet$ in $\mathcal{V}$ of type

$$\cdots \longleftarrow V_{n-2} \xleftarrow{\ \partial_{n-1}\ } V_{n-1} \xleftarrow{\ \partial_n\ } V_n \longleftarrow \cdots$$

with the property that the composition of successive arrows $\partial_{n-1} \circ \partial_n$ is zero. A morphism of complexes $f_\bullet : V_\bullet \to V'_\bullet$ is a commutative ladder diagram. The category $\mathcal{V}$ of vector spaces is isomorphic to its opposite category $\mathcal{V}^{op}$ via the duality functor that takes a vector space to its dual and a linear map to its transpose/adjoint [20, 4]. Duality takes the category $\mathcal{C}^{op}$ of cocomplexes to the category of complexes $\mathcal{C}$. A complex is called an *interval complex* if its dual is an interval cocomplex, and Proposition 2.10 becomes:

Proposition 2.11. *The category $\mathcal{C}$ of complexes is linear and Abelian, and therefore Krull-Schmidt. A complex is indecomposable iff it is isomorphic to an interval complex.*

The interval complexes are easily classified. An interval complex $I_\bullet$ is associated to an interval $I \subseteq \mathbb{Z}$ as follows: $n \in I$ whenever $I_n = \mathbb{F}$, and $n \notin I$ whenever $I_n = 0$.

Since adjacent nonzero arrows in a complex cannot be iso(morphisms), the interval complexes are in bijective correspondence with the intervals $I \subseteq \mathbb{Z}$ of cardinality at most two. We will often omit the bullet postscript when the context allows. We denote by $J[n] = \{n\} \subseteq \mathbb{Z}$ the intervals of cardinality one. For example, the complex $J[1] = J[1]_\bullet$ is the diagram of vector spaces

$$\cdots \longleftarrow 0 \longleftarrow \mathbb{F} \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$
$$n = 0 \qquad n = 1 \qquad n = 2 \qquad n = 3$$

The indecomposable complex $J[n]$ is simple. We denote by $K[n] = [n, n+1] \subseteq \mathbb{Z}$ the intervals of cardinality two. For example, the complex $K[1] = K[1]_\bullet$ is the diagram of vector spaces

$$\cdots \longleftarrow 0 \longleftarrow \mathbb{F} \xleftarrow{\ \ 1\ \ } \mathbb{F} \longleftarrow 0 \longleftarrow \cdots$$
$$n = 0 \qquad n = 1 \qquad n = 2 \qquad n = 3$$

The indecomposable complex $K[n]$ has exactly one nonzero proper subobject $J[n] \hookrightarrow K[n]$. For example, the inclusion of complexes $J[1] \hookrightarrow K[1]$ is the commutative ladder diagram

$$\begin{array}{ccccccccc}
\cdots \longleftarrow & 0 & \longleftarrow & \mathbb{F} & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \cdots \\
& \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & \\
\cdots \longleftarrow & 0 & \longleftarrow & \mathbb{F} & \xleftarrow{\ 1\ } & \mathbb{F} & \longleftarrow & 0 & \longleftarrow \cdots
\end{array}$$
$$n = 0 \qquad n = 1 \qquad n = 3 \qquad n = 4$$

We now return to the the Categorical Structural Theorem 1.6. A filtered complex is a diagram in the category $\mathcal{C}$ of complexes

$$\cdots \overset{\hookrightarrow}{\longrightarrow}{}_{p-1} V_\bullet \overset{\hookrightarrow}{\longrightarrow}{}_p V_\bullet \overset{\hookrightarrow}{\longrightarrow}{}_{p+1} V_\bullet \overset{\hookrightarrow}{\longrightarrow} \cdots .$$

We will say a filtered complex is *basic* if its colimit $V_\bullet$ is isomorphic to an interval complex. The first statement of Proposition 2.11 tells us that the category $\mathcal{C}$ of complexes is linear and Abelian. Then Lemma 2.8 tells us that the category of filtered complexes is Krull-Schmidt. The second statement of Proposition 2.11 classifies the indecomposable filtered complexes, completing the proof of:

Theorem 1.6. *(Categorical Structural Theorem) The category of filtered complexes is Krull-Schmidt. A filtered complex is indecomposable iff it is basic.*

The basic filtered complexes are easily classified since we know all proper subobjects of interval complexes, namely $0 \hookrightarrow J[n]$, $0 \hookrightarrow K[n]$, and $J[n] \hookrightarrow K[n]$. Details and examples of basic filtered complexes appear in Chapter 4.

3. CATEGORICAL FRAMEWORKS FOR PERSISTENT HOMOLOGY

3.1. **Standard Framework using Persistence Vector Spaces.** The structural theorem for the category of tempered persistence vector spaces, Theorem 2.9, is the foundation for the standard framework for persistent homology.

For each integer n, the *homology* of degree n is a functor $H_n : \mathcal{C} \to \mathcal{V}$ from the category $\mathcal{C}$ of complexes to the category $\mathcal{V}$ of vector spaces. An object C in $\mathcal{C}$ is a diagram of vector spaces

$$\cdots \longleftarrow V_{n-1} \xleftarrow{\ \partial_n\ } V_n \xleftarrow{\ \partial_{n+1}\ } V_{n+1} \longleftarrow \cdots$$

where $\partial_n \circ \partial_{n+1} = 0$. Then $\operatorname{im} \partial_{n+1} \hookrightarrow \ker \partial_n$ is a subobject inclusion of vector spaces, and the corresponding quotient vector space is the homology

$$H_n(C) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

More generally, the homology functor H_n takes a diagram in $\mathcal{C}$ to a diagram in $\mathcal{V}$.

Denote by $\mathcal{F}$ the category of filtered complexes. An object F in $\mathcal{F}$ is a diagram of complexes

$$\cdots \longhookrightarrow {}_{p-1}V_\bullet \longhookrightarrow {}_pV_\bullet \longhookrightarrow {}_{p+1}V_\bullet \longhookrightarrow \cdots ,$$

which is tempered and bounded below, and which has monic arrows. Denote by $\mathcal{P}$ the category of tempered persistence vector spaces. The homology functor H_n takes the the diagram F to the diagram of vector spaces

$$\cdots \longrightarrow H_n({}_{p-1}V_\bullet) \longrightarrow H_n({}_pV_\bullet) \longrightarrow H_n({}_{p+1}V_\bullet) \longrightarrow \cdots ,$$

which is tempered and bounded below, but which need not have monic arrows in general. An object F in $\mathcal{F}$ then goes to an object $P_n(F)$ in $\mathcal{P}$. Similarly a morphism in $\mathcal{F}$, which is a commutative ladder diagram of complexes, goes to a morphism in $\mathcal{P}$, which is a commutative ladder diagram of vector spaces. The resulting functor $P_n : \mathcal{F} \to \mathcal{P}$ is the *persistent homology* of degree n.

The standard framework for studying the persistent homology functors $P_n : \mathcal{F} \to \mathcal{P}$ is based on the structural theorem for the category $\mathcal{P}$, Theorem 2.9. It suffices to work with an appropriate Krull-Schmidt subcategory of the Krull-Schmidt category $\mathcal{P}$. A filtered complex F is studied by decomposing the persistence vector space $P_n(F)$ as a sum of indecomposables. Since the diagram $P_n(F)$ is bounded below, all of its indecomposables are bounded below. The persistence vector spaces that are bounded below comprise the full Abelian subcategory of $\mathcal{P}$, which we will denote by $\operatorname{im} P_n$. Despite the notation, the category $\operatorname{im} P_n$ does not depend on n; it is always the same subcategory of $\mathcal{P}$. The isomorphism class of an indecomposables in the Krull-Schmidt category $\operatorname{im} P_n$ is described by the familar barcode. An interval $I \subseteq \mathbb{Z}$ will be called a *barcode* if it is bounded below. A *barcode persistence vector space* is a persistence vector space ${}_\bullet I$ corresponding to a barcode $I \subseteq \mathbb{Z}$.

Theorem 3.1. *The persistent homology functor $P_n : \mathcal{F} \to \mathcal{P}$ factors as*

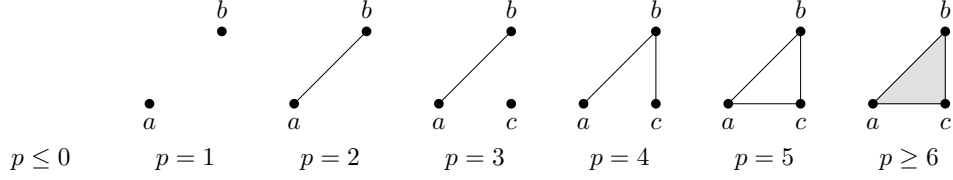$$\mathcal{F} \to \operatorname{im} P_n \to \mathcal{P}.$$
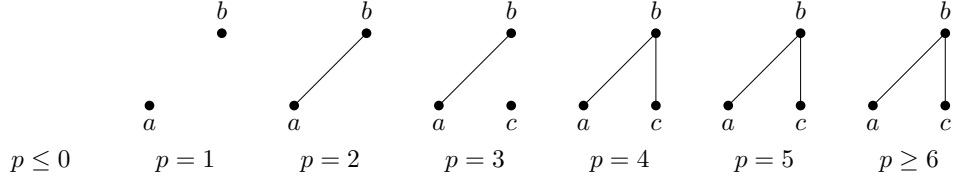
Figure 10. A filtered simplicial complex.

Figure 11. A subobject of the filtered simplicial complex.

The category $\operatorname{im} P_n$ *is Krull-Schmidt. An object in* $\operatorname{im} P_n$ *is indecomposable iff it is isomorphic to a barcode persistence vector space.*

We can now express the standard framework for persistent homology in terms of the functor $\mathcal{F} \to \operatorname{im} P_n$ which takes a filtered complex to a persistence vector space in $\operatorname{im} P_n$. The Krull-Schmidt property of $\operatorname{im} P_n$ then allows decomposition as a sum of indecomposables. Each indecomposable in $\operatorname{im} P_n$ is a barcode persistence vector space, which is specified up to isomorphism by its barcode $I \subseteq \mathbb{Z}$. An object in $\operatorname{im} P_n$ is determined up to isomorphism by its set of barcodes.

3.2. **Alternate Framework using Quotient Categories.**

The structural theorem for the category of filtered complexes, Theorem 1.6, is the foundation for an alternate framework for persistent homology.

We will work with an appropriate Krull-Schmidt quotient category of the Krull-Schmidt category $\mathcal{F}$. Recall in general [20] that an object of a quotient category of $\mathcal{F}$ is an object of $\mathcal{F}$, and a morphism is an equivalence class of morphisms of $\mathcal{F}$. Our quotient category $\operatorname{coim} P_n$ is defined via the following equivalence relation (congruence) on morphisms: two morphisms f and f' in $\mathcal{F}$ are equivalent iff the morphisms $P_n(f)$ and $P_n(f')$ in $\mathcal{P}$ are equal. Note that the category $\operatorname{coim} P_n$ now does depend on the integer n; each $\operatorname{coim} P_n$ is a different quotient category of $\mathcal{F}$.

Example 3.2. *We return to the filtered simplicial complex of Example 1.3, as shown in Figure 10.*

We first consider the subobject shown in Figure 11. In $\mathcal{F}$, this is a proper nonzero subobject. In the quotient category $\operatorname{coim} P_0$*, the subobject inclusion becomes an isomorphism between nonzero objects. In the quotient category* $\operatorname{coim} P_1$*, this becomes an a proper zero subobject.*

Now consider another subobject as shown in Figure 12. In $\mathcal{F}$, this is a proper nonzero subobject. In the quotient category $\operatorname{coim} P_0$*, this remains a proper nonzero subobject. In the quotient category* $\operatorname{coim} P_1$*, the inclusion morphism becomes an isomorphism between nonzero objects.*

$$p \leq 0 \qquad p = 1 \qquad p = 2 \qquad p = 3 \qquad p = 4 \qquad p = 5 \qquad p \geq 6$$
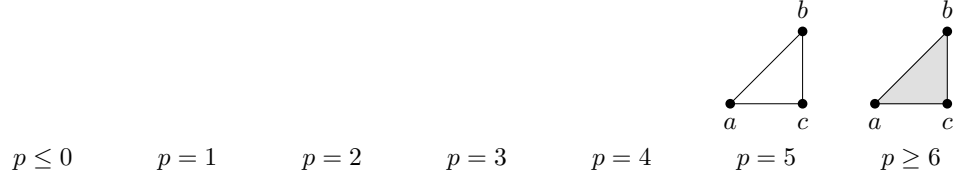
Figure 12. Another subobject of the filtered simplicial complex.

We recall that a quotient of a Krull-Schmidt category is Krull-Schmidt in general. This is because an indecomposable in $\mathcal{F}$ becomes either a zero object or an indecomposable with a local endomorphism ring in the quotient category (see e.g. [18] p. 431). The classification of indecomposables in the quotient category $\operatorname{coim} P_n$ is now easily obtained from Theorem 1.6. This is independent of the well-known classification of indecomposables in the category of persistence vector spaces (Theorem 2.9). Using the classification of indecomposables in each of the Krull-Schmidt categories $\operatorname{coim} P_n$ and $\operatorname{im} P_n$, it is now easy to verify that the functor $\operatorname{coim} P_n \to \operatorname{im} P_n$ is full, faithful, and essentially surjective. Recalling [20, 4] that a functor satisfying these conditions is an equivalence of categories, we have:

Theorem 3.3. *The persistent homology functor* $P_n : \mathcal{F} \to \mathcal{P}$ *factors as*

$$\mathcal{F} \to \operatorname{coim} P_n \to \operatorname{im} P_n \to \mathcal{P},$$

where the functor $\operatorname{coim} P_n \to \operatorname{im} P_n$ *is an equivalence of categories.*

The isomorphism class of an indecomposable in the Krull-Schmidt category $\operatorname{coim} P_n$ can be specified by I_n, where the integer n is the degree of the homology, and $I \subseteq \mathbb{Z}$ is a barcode specifying the isomorphism class of the corresponding indecomposable in $\operatorname{im} P_n$. The example in Chapter 4 will show how the barcode I can be understood directly in terms of the indecomposable filtered complex, without reference to homology.

We can now express the alternate framework for persistent homology in terms of the functor $\mathcal{F} \to \operatorname{coim} P_n$ which takes a filtered complex in $\mathcal{F}$ to the same filtered complex viewed as an object in $\operatorname{coim} P_n$. The Krull-Schmidt property of $\operatorname{coim} P_n$ then allows decomposition as a sum of indecomposables in $\operatorname{coim} P_n$. Each indecomposable is specified up to isomorphism by I_n. An object in $\operatorname{coim} P_n$ is determined up to isomorphism by the collection of of intervals $I_n \subseteq \mathbb{Z}$ indexing its decomposition. This framework obviates the need for auxiliary objects such as persistence vector spaces, while providing exactly the same information about filtered complexes as the standard framework.

4. Proving Structural Equivalence

4.1. **Forward Structural Equivalence.** We now consider in more detail matrix representations of a filtered complex and its automorphisms. The first step is to associate to a filtered complex a finite-dimensional vector space with an appropriately adapted basis. A filtered complex is a diagram of complexes indexed by the

integer level p, displayed below together with its colimit:

$$\cdots \hookrightarrow {}_{-1}V_\bullet \hookrightarrow {}_0 V_\bullet \hookrightarrow {}_1 V_\bullet \hookrightarrow {}_2 V_\bullet \hookrightarrow {}_3 V_\bullet \hookrightarrow \cdots \qquad V_\bullet$$

$$\text{colim}$$

A filtered complex becomes a "lattice" diagram of finite-dimensional vector spaces:

$$
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\cdots \hookrightarrow {}_{-1}V_2 \hookrightarrow {}_0 V_2 \hookrightarrow {}_1 V_2 \hookrightarrow {}_2 V_2 \hookrightarrow {}_3 V_2 \hookrightarrow \cdots & & V_2 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow & & \downarrow \partial_2 \\
\cdots \hookrightarrow {}_{-1}V_1 \hookrightarrow {}_0 V_1 \hookrightarrow {}_1 V_1 \hookrightarrow {}_2 V_1 \hookrightarrow {}_3 V_1 \hookrightarrow \cdots & & V_1 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow & & \downarrow \partial_1 \\
\cdots \hookrightarrow {}_{-1}V_0 \hookrightarrow {}_0 V_0 \hookrightarrow {}_1 V_0 \hookrightarrow {}_2 V_0 \hookrightarrow {}_3 V_0 \hookrightarrow \cdots & & V_0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow & & \downarrow \partial_0 \\
\cdots \hookrightarrow {}_{-1}V_{-1} \hookrightarrow {}_0 V_{-1} \hookrightarrow {}_1 V_{-1} \hookrightarrow {}_2 V_{-1} \hookrightarrow {}_3 V_{-1} \hookrightarrow \cdots & & V_{-1} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow & & \downarrow \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\end{array}
$$

$$\text{colim}$$

In the colimit complex, the composition $\partial_{n-1} \circ \partial_n : V_n \to V_{n-2}$ is zero for all n. Since the diagram is tempered, $\partial_{n-1} \circ \partial_n$ is an isomorphism for all but finitely many n. It follows that the complex is bounded, meaning that the vector space V_n is zero-dimensional for all but finitely many n. The direct sum $V = \oplus_{n \in \mathbb{Z}} V_n$ is then a finite-dimensional vector space associated to the filtered complex. A vector $v \in V$ is said to have *pure degree* iff $v \in V_n \subseteq V$ for some integer n. The integer n is then called the *degree* of the pure degree vector v, and is encoded by a postscript v_n. The *(filtration) level* of a degree n vector $v_n \in V$ is the smallest integer p such that $v_n \in {}_p V_n \subseteq V_n$. The level of the degree n vector v_n is encoded by a prescript ${}_p v_n$. An *adapted basis* of a filtered complex is a basis of the vector space V satisfying the three conditions:

- Every basis element has pure degree.
- For each n and p, the vector space ${}_p V_n$ is spanned by the basis vectors with degree equal to n and level less than or equal to p.
- The basis elements are ordered so that degree is nondecreasing, and within each degree the level is nondecreasing.

A block-diagonal triangular matrix B transforms an adapted basis to a new adapted basis, representing an automorphism of the filtered complex. Here we assume that the block structure of the matrix is compatible with degrees of the basis elements.

The *colimit boundary* $\partial = \oplus_{n \in \mathbb{Z}} \partial_n$ of a filtered complex is a linear endomorphism $\partial : V \to V$. The colimit boundary ∂ is represented by a matrix D relative to an adapted basis. The matrix representative D is block-superdiagonal because ∂ is homogeneous of degree -1, and $D^2 = 0$ because $\partial^2 = 0$. If additionally the matrix

representative is almost-Jordan, we will say the adapted basis is *special*. The Matrix Structural Theorem 1.4 yields:

Proposition 4.1. *A filtered complex admits a special adapted basis.*

Proof. Choose an adapted basis. Let D be the block-superdiagonal differential matrix representing ∂ relative to the adapted basis. Theorem 1.4 provides a block-diagonal triangular matrix B such that $\underline{D} = B^{-1}DB$ is almost-Jordan. So the matrix B transforms the original adapted basis to a special adapted basis. □

Corollary 4.2. *A filtered complex admits a finite decomposition as sum of basic filtered complexes.*

Proof. Choose a special adapated basis, and denote by $\underline{D}$ the corresponding almost-Jordan block-superdiagonal differential matrix representative. Let P be a permutation matrix such that the matrix $P^{-1}\underline{D}P$ is Jordan. Each Jordan block of this matrix represents a basic subobject of the filtered complex. The decomposition into Jordan blocks represents the decomposition of the filtered complex as a direct sum of basic filtered complexes. □

To verify the Krull-Schmidt property, we will also need:

Lemma 4.3. *A basic filtered complex has local endomorphism ring.*

Proof. We first show that the colimit complex of a basic filtered complex has local endomorphism ring. The colimit complex is isomorphic to an interval complex. An interval complex is an indecomposable in the linear Abelian category of complexes, so it has local endomorphism ring by Atiyah's Criterion 2.6. (Or less abstractly, it is easy to check that the endomorphism ring of an interval complex is isomorphic to the field $\mathbb{F}$.)

The proof is completed by checking that that the endomorphism ring of a basic filtered complex maps isomorphically to the endomorphism ring of its colimit interval complex. In general, the endomorphism ring of a filtered object maps *injectively* to the endomorphism ring of its colimit. We need to show that the endomorphism ring of a basic filtered complex maps *surjectively* to the endomorphism ring of its colimit. It suffices to show that an endomorphism of an interval complex restricts to an endomorphism of any subobject. There are two types of interval complexes to consider. If the interval complex is isomorphic to $J[n]$, then the subobjects are 0 and $J[n]$, and any endomorphism restricts. If the interval complex is isomorphic to $K[n]$, then the subobjects are 0, $J[n]$, and $K[n]$, and any endmorphism restricts. □

Assembling the pieces proves the main result of this section:

Proposition 1.7. *(Forward Structural Equivalence) The Matrix Structural Theorem implies the Categorical Structural Theorem.*

Proof. We first prove that a filtered complex is indecomposable iff it is basic. A basic filtered complex has a local endomorphism ring by Lemma 4.3, so it is indecomposable by Lemma 2.4. An indecomposable filtered complex is a finite direct sum of basic filtered complexes by Corollary 4.2. The direct sum cannot have more than one summand, because that would contradict the indecomposability. So an indecomposable filtered complex is basic.

Now it remains to check the two conditions of Definition 2.5. Since a basic filtered complex is indecomposable, Corollary 4.2 asserts that every filtered complex admits a finite decomposition as a sum of indecomposables. Since an indecomposable filtered complex is basic, Lemma 4.3 asserts that every indecomposable has a local endomorphism ring. $\square$

Example 4.4. *Let F be the filtered complex of Example 1.5. The initial adapted basis consists of appropriately ordered simplices:* $_1\underline{a}_0, {}_1\underline{b}_0, {}_3\underline{c}_0, {}_2\underline{ab}_1, {}_4\underline{bc}_1, {}_5\underline{ac}_1, {}_6\underline{abc}_2$. *The block-superdiagonal differential matrix D represents the colimit boundary operator relative to the initial adapted basis.*

The triangular block-diagonal matrix B represents an automorphism of the filtered complex. This automorphism takes the initial adapted basis to the transformed adapted basis $_1\underline{a}_0, {}_1\underline{b}_0, {}_3\underline{c}_0, {}_2\underline{ab}_1, {}_4\underline{bc}_1, {}_5\underline{ac}_1, {}_6\underline{abc}_2$. *This transformed adapted basis is special, because the block-superdiagonal differential matrix representative $\underline{D} = B^{-1}DB$ is almost-Jordan:*

$$
\underline{D} = \begin{array}{c} \\ {}_1\underline{a}_0 \\ {}_1\underline{b}_0 \\ {}_3\underline{c}_0 \\ {}_2\underline{ab}_1 \\ {}_4\underline{bc}_1 \\ {}_5\underline{ac}_1 \\ {}_6\underline{abc}_2 \end{array}
\begin{array}{ccccccc}
{}_1\underline{a}_0 & {}_1\underline{b}_0 & {}_3\underline{c}_0 & {}_2\underline{ab}_1 & {}_4\underline{bc}_1 & {}_5\underline{ac}_1 & {}_6\underline{abc}_2 \\
\end{array}
\left[\begin{array}{ccc|cccc}
\mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{0} & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & \mathbf{0} & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\
0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

We have retained the shading denoting the super-diagonal blocks, and we have also boldfaced the nonzero entries and the diagonal entries of zero columns. An almost-Jordan differential matrix $P^{-1}\underline{D}P$ is Jordan iff the matrix $\underline{D}P$, which is related to $\underline{D}$ by a permutation of columns, has each boldfaced $\mathbf{1}$ immediately following the boldfaced $\mathbf{0}$ in the same row. Permuting columns 3 and 4 suffices for this example:

$$
P = \begin{array}{c} \\ {}_1\underline{a}_0 \\ {}_1\underline{b}_0 \\ {}_3\underline{c}_0 \\ {}_2\underline{ab}_1 \\ {}_4\underline{bc}_1 \\ {}_5\underline{ac}_1 \\ {}_6\underline{abc}_2 \end{array}
\begin{array}{ccccccc}
{}_1\underline{a}_0 & {}_1\underline{b}_0 & {}_2\underline{ab}_1 & {}_3\underline{c}_0 & {}_4\underline{bc}_1 & {}_5\underline{ac}_1 & {}_6\underline{abc}_2 \\
\end{array}
\left[\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

produces the Jordan matrix

$$P^{-1}\underline{D}P = \begin{array}{c} \\ {}_1\underline{a}_0 \\ {}_1\underline{b}_0 \\ {}_2\underline{ab}_1 \\ {}_3\underline{c}_0 \\ {}_4\underline{bc}_1 \\ {}_5\underline{ac}_1 \\ {}_6\underline{abc}_2 \end{array} \begin{array}{cccccccc} {}_1\underline{a}_0 & {}_1\underline{b}_0 & {}_2\underline{ab}_1 & {}_3\underline{c}_0 & {}_4\underline{bc}_1 & {}_5\underline{ac}_1 & {}_6\underline{abc}_2 \\ \left[\begin{array}{ccccccc} \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

The decomposition of the Jordan matrix into its Jordan blocks represents the decomposition of the filtered complex into indecomposable/basic summands. We now list the indecomposable summands, denoting by $\langle v \rangle$ the linear span of a vector $v \in V$:

- *The Jordan block matrix* ${}_1\underline{a}_0 \begin{array}{c} {}^{{}_1\underline{a}_0} \\ \left[\; \mathbf{0} \; \right] \end{array}$ *represents the filtered complex*

$$
\begin{array}{ccccccccccccc}
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\cdots \hookrightarrow & 0 & \hookrightarrow & 0 & \longrightarrow & 0 & \hookrightarrow & 0 & \longrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow \cdots & 0 \\
\cdots \hookrightarrow & 0 & \hookrightarrow & \langle {}_1\underline{a}_0 \rangle & \hookrightarrow & \langle {}_1\underline{a}_0 \rangle & \hookrightarrow & \langle {}_1\underline{a}_0 \rangle & \hookrightarrow & \langle {}_1\underline{a}_0 \rangle & \hookrightarrow & \langle {}_1\underline{a}_0 \rangle & \hookrightarrow \cdots & \langle {}_1\underline{a}_0 \rangle. \\
\cdots \hookrightarrow & 0 & \hookrightarrow & 0 & \longrightarrow & 0 & \hookrightarrow & 0 & \longrightarrow & 0 & \hookrightarrow & 0 & \hookrightarrow \cdots & 0 \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
& p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & \mathrm{colim}
\end{array}
$$

This filtered complex is basic because in $\mathcal{F}$ it is isomorphic to the filtered complex

$$
\begin{array}{ccccccccccccc}
\cdots \hookrightarrow & 0 & \hookrightarrow & J[0] & \hookrightarrow & J[0] & \hookrightarrow & J[0] & \hookrightarrow & J[0] & \hookrightarrow & J[0] & \hookrightarrow \cdots & J[0], \\
& p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & \mathrm{colim}
\end{array}
$$

which has the interval complex $J[0]$ as colimit. The quotient functor $\mathcal{F} \to \mathrm{coim}\, P_0$ takes the filtered complex to an indecomposable in the isomorphism class $[1, \infty)_0$. This corresponds under the equivalence $\mathrm{coim}\, P_0 \to \mathrm{im}\, P_0$ to an indecomposable in the isomorphism class of the barcode persistence vector space $[1, \infty)$,

$$
\begin{array}{ccccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow \cdots . \\
& p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5
\end{array}
$$

For any $n \neq 0$, the quotient functor $\mathcal{F} \to \mathrm{coim}\, P_n$ takes the filtered complex to a zero object.

- *The Jordan block matrix* $\begin{array}{cc} & {}_1\underline{b}_0 \quad {}_2\underline{ab}_1 \\ \begin{array}{c} {}_1\underline{b}_0 \\ {}_2\underline{ab}_1 \end{array} & \left[\begin{array}{cc} \mathbf{0} & \boxed{\mathbf{1}} \\ 0 & 0 \end{array}\right] \end{array}$ *represents the filtered complex*

$$
\begin{array}{ccccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \hookrightarrow \cdots & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 & \longrightarrow & 0 & \longrightarrow & \langle {}_2\underline{ab}_1\rangle & \hookrightarrow & \langle {}_2\underline{ab}_1\rangle & \hookrightarrow & \langle {}_2\underline{ab}_1\rangle & \hookrightarrow & \langle {}_2\underline{ab}_1\rangle & \hookrightarrow \cdots & \langle {}_2\underline{ab}_1\rangle \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 & \longrightarrow & \langle {}_1\underline{b}_0\rangle & \hookrightarrow & \langle {}_1\underline{b}_0\rangle & \hookrightarrow & \langle {}_1\underline{b}_0\rangle & \hookrightarrow & \langle {}_1\underline{b}_0\rangle & \hookrightarrow & \langle {}_1\underline{b}_0\rangle & \hookrightarrow \cdots & \langle {}_1\underline{b}_0\rangle \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \hookrightarrow \cdots & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & \text{colim}
\end{array}
$$

This filtered complex is basic because in $\mathcal{F}$ it is isomorphic to the filtered complex

$$
\begin{array}{ccccccccccccc}
\cdots \hookrightarrow 0 & \longrightarrow & J[0] & \hookrightarrow & K[0] & \hookrightarrow & K[0] & \hookrightarrow & K[0] & \hookrightarrow & K[0] & \hookrightarrow \cdots & K[0], \\
p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5 & & \text{colim}
\end{array}
$$

which has the interval complex $K[0]$ as colimit. The quotient functor $\mathcal{F} \to \operatorname{coim} P_0$ takes the filtered complex to an indecomposable in the isomorphism class $[1,2)_0$. This corresponds under the equivalence $\operatorname{coim} P_0 \to \operatorname{im} P_0$ to an indecomposable in the isomorphism class of the barcode persistence vector space $[1,2)$,

$$
\begin{array}{ccccccccccc}
\cdots \longrightarrow 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow 0 \longrightarrow \cdots . \\
p=0 & & p=1 & & p=2 & & p=3 & & p=4 & & p=5
\end{array}
$$

For any $n \neq 0$, the quotient functor $\mathcal{F} \to \operatorname{coim} P_n$ takes the filtered complex to a zero object.

- *The Jordan block matrix* $\begin{array}{c} {}_3\underline{c}_0 \\ {}_4\underline{bc}_1 \end{array}\begin{bmatrix} \mathbf{0} & \boxed{\mathbf{1}} \\ 0 & 0 \end{bmatrix}$ *represents the filtered complex*

$$
\begin{array}{ccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \hookleftarrow 0 \hookleftarrow \cdots & & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 \longrightarrow 0 \longrightarrow 0 \hookrightarrow \langle {}_4\underline{bc}_1\rangle \hookrightarrow \langle {}_4\underline{bc}_1\rangle \hookrightarrow \langle {}_4\underline{bc}_1\rangle \hookrightarrow \cdots & & \langle {}_4\underline{bc}_1\rangle \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 \longrightarrow 0 \longrightarrow \langle {}_3\underline{c}_0\rangle \hookrightarrow \langle {}_3\underline{c}_0\rangle \hookrightarrow \langle {}_3\underline{c}_0\rangle \hookrightarrow \langle {}_3\underline{c}_0\rangle \hookrightarrow \cdots & & \langle {}_3\underline{c}_0\rangle \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \hookleftarrow 0 \hookleftarrow \cdots & & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
p=1 & p=2 & p=3 & p=4 & p=5 & p=6 & & \text{colim}
\end{array}
$$

This filtered complex is basic because in $\mathcal{F}$ *it is isomorphic to the filtered complex*

$$
\cdots \hookrightarrow 0 \longrightarrow 0 \longrightarrow J[0] \hookrightarrow K[0] \hookrightarrow K[0] \hookrightarrow K[0] \hookrightarrow \cdots \qquad K[0]
$$

$$
p=1 \qquad p=2 \qquad p=3 \qquad p=4 \qquad p=5 \qquad p=6 \qquad\qquad \text{colim}
$$

which has the interval complex $K[0]$ *as colimit. The quotient functor* $\mathcal{F} \to$ $\operatorname{coim} P_0$ *takes the filtered complex to an indecomposable in the isomorphism class* $[3,4)_0$. *This corresponds under the equivalence* $\operatorname{coim} P_0 \to \operatorname{im} P_0$ *to an indecomposable in the isomorphism class of the barcode persistence vector space* $[3,4)$,

$$
\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Q} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots .
$$

$$
p=1 \qquad p=2 \qquad p=3 \qquad p=4 \qquad p=5 \qquad p=6
$$

For any $n \neq 0$, *the quotient functor* $\mathcal{F} \to \operatorname{coim} P_n$ *takes the filtered complex to a zero object.*

- *The Jordan block matrix*
$$\begin{array}{c} {}_5\underline{ac}_1 \quad {}_6\underline{abc}_2 \\ \begin{array}{c} {}_5\underline{ac}_1 \\ {}_6\underline{abc}_2 \end{array} \left[\begin{array}{cc} \mathbf{0} & \mathbf{1} \\ 0 & 0 \end{array} \right] \end{array}$$
represents the filtered complex

$$
\begin{array}{ccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 \hookrightarrow 0 \longrightarrow 0 \longrightarrow 0 \hookrightarrow 0 \hookrightarrow 0 \hookrightarrow \cdots & & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 \hookrightarrow 0 \longrightarrow 0 \hookrightarrow 0 \longrightarrow \langle {}_6\underline{abc}_2 \rangle \hookrightarrow \cdots & & \langle {}_6\underline{abc}_2 \rangle \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 \hookrightarrow 0 \longrightarrow 0 \hookrightarrow 0 \longrightarrow \langle {}_5\underline{ac}_1 \rangle \hookrightarrow \langle {}_5\underline{ac}_1 \rangle \hookrightarrow \cdots & & \langle {}_5\underline{ac}_1 \rangle \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\cdots \hookrightarrow 0 \hookrightarrow 0 \longrightarrow 0 \hookrightarrow 0 \hookrightarrow 0 \hookrightarrow 0 \hookrightarrow \cdots & & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
p=1 & p=2 & p=3 & p=4 & p=5 & p=6 & & \text{colim}
\end{array}
$$

This filtered complex is basic because in $\mathcal{F}$ it is isomorphic to the filtered complex

$$
\begin{array}{ccccccccc}
\cdots \hookrightarrow 0 \hookrightarrow 0 \longrightarrow 0 \hookrightarrow J[1] \hookrightarrow K[1] \hookrightarrow \cdots & & K[1] \\
p=1 & p=2 & p=3 & p=4 & p=5 & p=6 & & \text{colim}
\end{array}
$$

which has the interval complex $K[0]$ as colimit. The quotient functor $\mathcal{F} \to$ coim P_1 takes the filtered complex to an indecomposable in the isomorphism class $[5,6)_1$. This corresponds under the equivalence coim $P_1 \to$ im P_1 to an indecomposable in the isomorphism class of the barcode persistence vector space $[5,6)$,

$$
\begin{array}{cccccc}
\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Q} \longrightarrow 0 \longrightarrow \cdots \\
p=1 \quad p=2 \quad p=3 \quad p=4 \quad p=5 \quad p=6
\end{array}
$$

For any $n \neq 1$, the quotient functor $\mathcal{F} \to$ coim P_n takes the filtered complex to a zero object.

This completes the decomposition of the filtered complex F in the category $\mathcal{F}$. As an object in the quotient category coim P_0, the filtered complex F is isomorphic to $[0,\infty)_0 \oplus [1,2)_0 \oplus [3,4)_0$. As an object in the quotient category coim P_1, the filtered complex F is isomorphic to $[5,6)_1$. For any other value of n, the filtered complex F is a zero object in the quotient category coim P_n.

4.2. **Reverse Structural Equivalence.** Special adapted bases help to intermediate between the Matrix Structural Theorem and Categorical Structural Theorem. In Proposition 4.1, we established the existence of a special adapted basis using the Matrix Structural Theorem 1.4. Now in the reverse direction, we establish the existence of a special adapted basis using the Categorical Structural Theorem 1.6:

Proposition 4.5. *A filtered complex admits a special adapted basis.*

Proof. The Categorical Structural Theorem decomposes the filtered complex as a finite direct sum of indecomposables. Each indecomposable summand is a basic filtered complex, so it admits a special adapted basis. With appropriate ordering, the union over the summands of these basis elements is a special adapted basis for the direct sum filtered complex. □

An automorphism of a filtered complex transforms an adapted basis to another adapted basis. The change of basis is represented by a matrix B, which is block-diagonal because an automorphism preserves the degree of basis elements. But the matrix B need not be triangular in general. We call a filtered complex *nondegenerate* if $\dim_{p+1} V_n \leq 1 + \dim_p V_n$ for any p and any n.

Lemma 4.6. *If a filtered complex is nondegenerate, then any change of adapted basis is represented by a triangular matrix B.*

Proof. An automorphism takes a basis element of degree n and level p to a linear combination of basis elements of degree n and level at most p. A filtered complex is nondegenerate iff an adapted basis contains no pair of elements with the same degree and same level. In this case the linear combination does not contain any basis elements that appear later in the ordering of the basis. The matrix B is then triangular, since it has no nonzero entries below the diagonal. □

We will construct nondegenerate filtered complexes by using the upper-left submatrices of a differential matrix. We illustrate submatrices with an example:

Example 4.7. *The upper-left submatrices are indicated below for a block-superdiagonal differential matrix $D : \mathbb{Q}^7 \to \mathbb{Q}^7$.*

$$
D = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Note that for each integer $0 < p < 7$, the upper-left submatrix $_pD : \mathbb{Q}^p \to \mathbb{Q}^p$ is itself a block-superdiagonal differential matrix. We remark that the matrix D had appeared previously in Example 1.5, representing the degenerate (not nondegenerate) filtered complex of Example 1.3.

Lemma 4.8. *Any block-superdiagonal differential matrix D represents the colimit boundary of some nondegenerate filtered complex.*

Proof. Let $D : \mathbb{F}^m \to \mathbb{F}^m$ be a block-superdiagonal differential matrix. We construct a filtered complex

$$\cdots \hookrightarrow {}_{-1}V_\bullet \hookrightarrow {}_0V_\bullet \hookrightarrow {}_1V_\bullet \hookrightarrow {}_2V_\bullet \hookrightarrow {}_3V_\bullet \hookrightarrow \cdots \qquad V_\bullet$$

colim

by specifying for each integer p the complex $_p V_\bullet$ at level p:

- For $p \leq 0$, the complex is the zero complex.
- For $1 < p < m$, the complex is specified by the block-superdiagonal differential submatrix $_p D : \mathbb{F}^p \to \mathbb{F}^p$.
- For $m \leq p$, the complex is specified by the initial block-superdiagonal differential matrix $D : \mathbb{F}^m \to \mathbb{F}^m$.

The arrows are the subobject inclusions $_p V_\bullet \hookrightarrow _{p+1} V_\bullet$. Then the diagram is a filtered complex since the zero complex is a limit and the complex $D : \mathbb{F}^m \to \mathbb{F}^m$ is a colimit. It only remains to observe that the filtered complex is nondegenerate, and that the matrix D represents its colimit boundary. $\qquad \square$

Note that the block structure of the differential matrix D is important in the preceding proof. If a differential matrix does not have block-superdiagonal structure, then an upper-left submatrix need not be a differential matrix in general.

Now we have assembled the ingredients to prove:

Proposition 1.8. *(Reverse Structural Equivalence) The Categorical Structural Theorem implies the Matrix Structural Theorem.*

Proof. Let D be a block-superdiagonal differential matrix. Lemma 4.8 lets us choose a nondegenerate filtered complex that is represented by D. Proposition 4.5 lets us make a change of basis to a special adapted basis. The block-diagonal matrix B representing the basis change is triangular by Lemma 4.6. Finally, the block-superdiagonal differential $\underline{D} = B^{-1} D B$ is almost-Jordan because the adapted basis is special. $\qquad \square$

Appendix A. Bruhat Uniqueness Lemma

Here we establish the uniqueness of the persistence canonical form $\underline{D}$ appearing in the Matrix Structural Theorem 1.4, as well as in the ungraded version Theorem 1.2. Our result generalizes the uniqueness statement for the usual Bruhat factorization of an invertible matrix [2, 14].

It is convenient to make the following definitions. We call an (upper) triangular matrix U *unitriangular* if it is unipotent, meaning that each diagonal entry is 1. We call a matrix M *quasi-monomial* if each row has at most one nonzero entry and each column has at most one nonzero entry. We remark that a unitriangular matrix is always square, but a quasi-monomial matrix need not be square. The key to proving uniqueness is:

Lemma A.1. *Suppose $M_1 U = V M_2$, where M_1 and M_2 are quasi-monomial and U and V are unitriangular. Then $M_2 = M_1$.*

In the following proof, the term *row-pivot* denotes a matrix entry that is the leftmost nonzero entry in its row, and *column-pivot* denotes a matrix entry that is the bottommost nonzero entry in its column.

Proof. The first half of the proof consists of showing that every nonzero entry of M_2 is also an entry of M_1. A nonzero entry of the quasi-monomial matrix M_2 is a column-pivot. Similarly a nonzero entry of the quasi-monomial matrix M_1 is a row-pivot. It now suffices to show that a column-pivot of M_2 is a row-pivot of M_1. Since V is unitriangular, $V M_2$ has the same column-pivots as M_2. Similarly since U is unitriangular, $M_1 U$ has the same row-pivots as M_1. It now suffices to prove

that a column-pivot of $S = VM_2$ is a row-pivot of $S = M_1U$. Suppose to the contrary that some column-pivot of S is not a row-pivot of S. Let x be the leftmost such column-pivot. Since x is not a row-pivot, there exists a row-pivot y to the left of x in the same row. If y were a column-pivot of $S = VM_2$, then it would be a column-pivot of M_2. But the quasi-monomial matrix M_2 cannot have two nonzero entries y and x in the same row. So y is not a column-pivot of S, and there exists a column-pivot z below y in the same column. If z were a row-pivot of $S = UM_1$, then it would be a row-pivot of M_1. But the quasi-monomial matrix M_1 cannot have two nonzero entries z and y in the same column. So z is a column-pivot of S that is not a row-pivot of S, and z is to the left of (and below) x. This is a contradiction, because x is the leftmost such column-pivot.

The second half of the proof consists of showing that every nonzero entry of M_1 is also an entry of M_2. This is analogous to the first half, and we omit the details. The two matrices then have the same nonzero entries, so they must also have the same zero entries. Since all the entries of the two matrices are the same, we have proved $M_2 = M_1$. $\qquad\square$

Recall that a matrix M is *Boolean* if every non-zero entry is 1. An almost-Jordan differential matrix $\underline{D}$ is Boolean and quasi-monomial.

Proposition A.2. *Suppose $P_1A = BP_2$ where P_1 and P_2 are Boolean quasi-monomial and A and B are invertible triangular. Then $P_2 = P_1$.*

Proof. Factor $A = T_1U$ as the product of an invertible diagonal matrix T_1 and a unitriangular matrix U. Factor $B = VT_2$ as the product of a unitriangular matrix V and an invertible diagonal matrix T_2. Then $(P_1T_1)U = V(T_2P_2)$, with (P_1T_1) and (T_2P_2) quasi-monomial. Lemma A.1 then gives the $P_1T_1 = T_2P_2$. Since the quasi-monomial matrices P_1 and P_2 are Boolean, the conclusion follows. $\qquad\square$

We remark that a permutation matrix P is also Boolean and quasi-monomial, so Proposition A.2 generalizes the standard uniqueness result for Bruhat factorization of an invertible matrix [2, 14].

The uniqueness of the persistence canonical form $\underline{D}$ appearing in Theorem 1.2 and in the Matrix Structural Theorem 1.4 now follows easily:

Corollary A.3. *Suppose D is a differential matrix and B_1 and B_2 are invertible triangular matrices. If both differential matrices $\underline{D}_1 = B_1^{-1}DB_1$ and $\underline{D}_2 = B_2^{-1}DB_2$ are almost-Jordan, then $\underline{D}_2 = \underline{D}_1$.*

Proof. $\underline{D}_1(B_1^{-1}B_2) = (B_1^{-1}B_2)\underline{D}_2$, and the result follows from Proposition A.2. $\quad\square$

Appendix B. Constructively Proving the Matrix Structural Theorem

B.1. **Linear Algebra of Reduction.** In this section we discuss column-reduction of a matrix $M : \mathbb{F}^m \to \mathbb{F}^n$, including its application to describing the kernel and image of the matrix. Column-reduction of a differential matrix D is a standard tool in the computation of persistent homology, where it is usually just called *reduction* [13, 27, 28, 9]. We prefer the more precise terminology in order to maintain the distinction with row-reduction, since both are used for Bruhat factorization [2, 14, 19].

As in Appendix A, the term *column-pivot* denotes a matrix entry that is the bottommost nonzero entry in its column. A matrix R is said to be *column-reduced* if each row has at most one column-pivot.

Definition B.1. *A column-reduction of a matrix M is an invertible triangular matrix V such that $R = MV$ is column-reduced.*

A column-reduction V exists for any matrix M, but is not unique in general. Column-reduction algorithms used for persistent homology [13, 27, 28, 10] usually prioritize computational efficiency. For our computational examples, we will use a column-reduction algorithm that is popular for Bruhat factorization [2, 14]. This algorithm is easy to implement, but is not very efficient computationally. The algorithm starts at the leftmost column of M and proceeds rightward by successive columns as follows:

- If the current column is zero, do nothing.
- If the current column is nonzero, add an appropriate multiple of the current column to each column to the right in order to zero the entries to the right of the column-pivot (in the same row).

Stop if the current column is the rightmost column, otherwise proceed to the column immediately to the right and repeat. By design, the resulting matrix R has the property that any column-pivot has only zeros to the right of it (in the same row). So a row of R cannot contain more than one column-pivot, implying that R is column-reduced. The invertible triangular column-reduction matrix V is constructed by performing the same column operations on the identity matrix I, where I has same number of columns as M.

We digress briefly to discuss some linear-algebraic properties of column-reduction. A column-reduction easily yields a basis for the kernel of a matrix as well as a basis for the image. By contrast, Gaussian elimination easily yields a basis for the image a matrix, but requires additional back-substitution to produce a basis for the kernel. Column-reduction algorithms are therefore a convenient alternative to Gaussian elimination for matrix computations in general, and this fact seems to be underappreciated. We use a variant of the usual adapted basis for a filtered vector space, disregarding the ordering of basis elements. We'll say that a basis of a finite-dimensional vector space X is *almost-adapted* to a subspace $Y \subseteq X$ if Y is spanned by the set of basis elements that are contained in Y. Proposition B.1 yields:

Corollary B.2. *Let $V : \mathbb{F}^m \to \mathbb{F}^m$ be a column-reduction of a matrix $M : \mathbb{F}^m \to \mathbb{F}^n$. Then:*

(1) The nonzero columns of the column-reduced matrix $R = MV$ are a basis of $\operatorname{im} M$.

(2) The columns of the invertible triangular matrix V are a basis of $\mathbb{F}^m$, and this basis is almost-adapted to $\ker M$.

Proof.

(1) The nonzero columns of R span $\operatorname{im} M$. The nonzero columns of R are linearly independent because R is column-reduced.

(2) The columns of V are a basis of $\mathbb{F}^m$ because V is invertible. This basis is almost-adapted to $\ker M$ because the nonzero columns of $R = MV$ are linearly independent.

$\square$

Example B.3. *We compute in detail a column-reduction of the matrix $M : \mathbb{Q}^4 \to \mathbb{Q}^3$, which is presented below with a column augmentation by the identity matrix I.*

$$
\frac{M}{I} =
\left[
\begin{array}{cccc}
1 & -2 & 0 & -8 \\
2 & -4 & 6 & 2 \\
1 & -2 & 2 & -2 \\
\hline
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}
\right]
\mapsto
\left[
\begin{array}{cccc}
1 & 0 & -2 & -6 \\
2 & 0 & 2 & 6 \\
\mathbf{1} & 0 & 0 & 0 \\
\hline
1 & 2 & -2 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}
\right]
\mapsto
\left[
\begin{array}{cccc}
1 & 0 & -2 & 0 \\
2 & 0 & \mathbf{2} & 0 \\
\mathbf{1} & 0 & 0 & 0 \\
\hline
1 & 2 & -2 & 8 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1
\end{array}
\right]
= \frac{R}{V}
$$

The result of the computation is a factorization $R = MV$, where R is column-reduced and V is invertible triangular (and unipotent). We describe each step of the computation:

(1) *The first column of M is nonzero, so it has a column-pivot. At the next processing step, boldface the column-pivot for clarity, and add an appropriate multiple of the first column to each column to the right in order to zero the entries to the right of the column-pivot (in the same row).*

(2) *At this point the second column is zero, so requires no processing step.*

(3) *At this point the third column is nonzero, so it has a column-pivot. At the next processing step, boldface the column pivot, and add an appropriate multiple of the third column to each column to the right in order to zero the entries to the right of the column-pivot (in the same row).*

(4) *At this point the fourth column is zero, so requires no processing step.*

The columns of V are a basis of $\mathbb{Q}^4$ that is amost-adapted to $\ker M$. Columns 2 and 4 of V are the columns corresponding to zero columns of R, so they are a basis of $\ker M$. Columns 1 and 3 of R are the nonzero columns, so they are a basis of $\operatorname{im} M$.

B.2. **Matrix Structural Theorem via Reduction.** The standard algorithm of persistent homology [13, 27, 28] starts with a differential matrix D and constructs a matrix B satisfying the conditions of:

Theorem 1.2. *(Ungraded Matrix Structural Theorem) Any differential matrix D factors as $D = B\underline{D}B^{-1}$ where $\underline{D}$ is an almost-Jordan differential matrix and B is a triangular matrix.*

The matrix formulation of the *standard algorithm* constructs a matrix $B = \hat{V}$ from a column-reduction V of a differential matrix D, as discussed in [10, 5] for $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Since $R = DV$ is column-reduced, there exists at most one nonzero column of R that has its column-pivot in row k. Here $1 \le k \le m$ where m is the number of rows of the square matrix D. $\hat{V}$ is constucted one column at a time using the following rule:

- If there exists a nonzero column of R that has its column-pivot in row k, then column k of $\hat{V}$ is equal to this column of R.
- If there does not exist a nonzero column of R that has its column-pivot in row k, then column k of $\hat{V}$ is equal to column k of V.

The matrix $\hat{V}$ is invertible triangular, because each column is nonzero and has its column-pivot on the diagonal.

We will now prove that $B = \hat{V}$ satisfies the conditions of Theorem 1.2. The key is to recognize when a differential matrix $\underline{D} : \mathbb{F}^m \to \mathbb{F}^m$ is almost-Jordan. The columns of any invertible matrix $G : \mathbb{F}^m \to \mathbb{F}^m$ comprise a basis of $\mathbb{F}^m$. When $G = I$ is the identity matrix, we have:

Lemma B.4. *Let $\underline{D}$ be a differential matrix, and let I be the identity matrix of the same size. The differential matrix $I^{-1}\underline{D}I = \underline{D}$ is almost-Jordan iff the following two conditions hold:*

(1) Every nonzero column of $\underline{D}I = \underline{D}$ is equal to some column of I.
(2) The nonzero columns of $\underline{D}I = \underline{D}$ are distinct (meaning no two are equal).

Proof. Suppose the differential matrix $\underline{D}$ is almost-Jordan. Then for some permutation P the differential matrix $P^{-1}\underline{D}P$ is Jordan. Each of the two conditions holds for a Jordan differential matrix. Each of the two conditions is preserved by conjugation with a permutation, so each of the two conditions holds for the differential matrix $\underline{D}$.

Suppose the two conditions hold. Any permutation of the columns of $\underline{D}$ is expressed as the matrix product $\underline{D}P$ where P is the associated permutation matrix. It is possible to choose P so that any column of $\underline{D}P$ that is not in ker $\underline{D}$ immediately follows the column that is its image under $\underline{D}$. Then the differential matrix $P^{-1}\underline{D}P$ is Jordan, so $\underline{D}$ is almost-Jordan.   □

Now we can recognize basis changes that make a differential matrix almost-Jordan:

Corollary B.5. *Let D be a differential matrix, and let G be an invertible matrix of the same size. The differential matrix $G^{-1}DG$ is almost-Jordan iff the following two conditions hold:*

(1) Every nonzero column of DG is equal to some column of G.
(2) The nonzero columns of DG are distinct.

We can now give a constructive proof of Theorem 1.2:

Proof. Let $B = \hat{V}$ be an invertible triangular matrix constructed from the differential matrix D by the standard algorithm. A column of $B = \hat{V}$ is either equal to the corresponding column of V or to some nonzero column of $R = DV$. Then a column of $D\hat{V}$ is either equal to the corresponding column of $DV = R$ or to some column of $DR = D^2 V = 0$. So a nonzero column of $D\hat{V}$ is equal to the corresponding column of R. We can use this fact to verify two conditions of Corollary B.5:

(1) We know that a nonzero column of $D\hat{V}$ is equal to the corresponding column of R. Since R is column-reduced, the standard algorithm ensures that every nonzero column of R is equal to some column of $\hat{V}$. So any nonzero column of $D\hat{V}$ is equal to some column of $\hat{V}$.

(2) We know that a nonzero column of $D\hat{V}$ is equal to the corresponding column of R. The nonzero columns of R are distinct since R is column-reduced. So the nonzero columns of $D\hat{V}$ are distinct.

                              □

Note that an invertible triangular matrix $\hat{V}$ produced by the standard algorithm is not normalized in general. But it is easy to construct a diagonal matrix T such

that the invertible diagonal matrix $\hat{V}T$ is normalized. This will be illustrated in the example at the end of the section.

We now consider the graded case:

Theorem 1.4. *(Matrix Structural Theorem) Any block-superdiagonal differential matrix D factors as $D = B\underline{D}B^{-1}$ where $\underline{D}$ is a block-superdiagonal almost-Jordan differential matrix and B is a block-diagonal triangular matrix.*

Proof. Let D be a block-superdiagonal differential matrix D. Then the invertible triangular column-reduction marix V produced by a reduction algorithm, such as [13, 27, 28] or our Appendix B.1, is block-diagonal. If V is block-diagonal, then so is the invertible triangular matrix $B = \hat{V}$ constructed by the standard algorithm from V and $R = DV$. $\qquad\square$

The following example of a standard algorithm computation illustrates both block-structure and normalization.

Example B.6. *We work with block-superdiagonal differential $D : \mathbb{Q}^7 \to \mathbb{Q}^7$ of Example 1.5, which is presented below with a column augmentation by the identity matrix I. The identity matrix is block-diagonal with respect to the grading structure inherited from D. We first compute a column-reduction of D.*

$$\frac{D}{I} = \left[\begin{array}{ccccccc} 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \mapsto \cdots \mapsto \left[\begin{array}{ccccccc} 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \frac{R}{V}$$

The result of the computation is a factorization $R = DV$, where R is column-reduced and V is invertible triangular (and unipotent). The intervening steps are omitted for brevity.

Next we use the standard algorithm to construct $\hat{V}$ as a modification of V. Each nonzero column of R replaces the column of V that has its column-pivot in the same row. $\hat{V}$ inherits the block-diagonal structure of V:

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \cdots \mapsto \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \hat{V}$$

We list the columns of $\hat{V}$ that are equal to columns of R:

- Column 2 of $\hat{V}$ is equal to column 4 of R.
- Column 3 of $\hat{V}$ is equal to column 5 of R.
- Column 6 of $\hat{V}$ is equal to colum 7 of R.

Each of the remaining columns of $\hat{V}$ is equal to the corresponding column of V. Now by Corollary B.5, the block-superdiagonal differential matrix $\underline{D} = \hat{V}^{-1} D \, \hat{V}$ is almost-Jordan:

$$\underline{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The invertible triangular matrix $\hat{V}$ is not normalized: column 6 of $\hat{V}$ corresponds to a zero column of $\underline{D}$, but its diagonal entry is not equal to 1. We can normalize by scalar multiplication of the appropriate columns. Let T be the diagonal matrix with 1 in the first five diagonal entries and -1 in the last two. Then the invertible triangular matrix $B = \hat{V}T$ is normalized, and this is the matrix that appears in Example 1.5. Note that $B^{-1}DB = \underline{D} = \hat{V}^{-1} D \, \hat{V}$ by Corollary A.3.

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