ON THE STABLE HOM RELATION AND STABLE DEGENERATIONS OF COHEN-MACAULAY MODULES

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Dedicated to Professor Yuji Yoshino on the occasion of his sixtieth birthday.

Abstract. We study the stable hom relation for Cohen-Macaulay modules over Gorenstein local algebras. We give the sufficient condition to make the stable hom relation a partial order when the base algebra is of finite representation type. As an application, we give the description of stable degenerations of Cohen-Macaulay modules over simple singularities of several types by using the stable hom relation.

1. Introduction

In ring theory, the hom relation is a basic relation for finitely generated modules over finite dimensional algebras [1, 2, 12, 18, 15, 9]. Let k be a field and R a k-algebra. The relation is defined by a dimension of a hom-set between finitely generated modules as a k-module, that is, we define the relation $M \leq_{hom} N$ by a relation $\dim_k \mathrm{Hom}_R(X, M) \leq \dim_k \mathrm{Hom}_R(X, N)$ for each finitely generated modules X. Auslander-Reiten [2] use the relation to investigate when indecomposable modules are determined by the composition factors. In [18], Zwara gave a characterization of degenerations of modules over representation finite algebras in relation with the hom relation. We remark that the hom relation is not always a partial order. It has been studied by many authors [1, 2, 3, 15] when the relation is actually a partial order.

In the paper, we investigate the hom relation on a stable category of Cohen-Macaulay modules $\underline{\mathrm{CM}}(R)$ over (not necessary Artinian) Gorenstein k-algebra. First we compare the Auslander-Reiten theory on $\mathrm{CM}(R)$ with that on $\underline{\mathrm{CM}}(R)$. We look into the relation between AR sequences and AR triangles of Cohen-Macaulay modules (Proposition 2.2). We consider a relation on $\underline{\mathrm{CM}}(R)$ which is the stable analogue of the hom relation (Definition 2.5) and shall show that it is actually a partial order if the algebra is of finite representation type with certain assumptions (Theorem 2.9). In Section 4, we attempt to characterize the stable degenerations of Cohen-Macaulay modules by using the stable hom relation. The concept of stable degenerations of Cohen-Macaulay modules was introduced by Yoshino [17]. It is closely related to ordinary degenerations of modules [7, 13]. We shall show that the stable degenerations over several simple singularities can be controlled by the stable hom relation (Theorem 4.6). To show this, we use the stable analogue of the argument over finite dimensional algebras in [18]. As a conclusion, we give the description of stable degenerations of Cohen-Macaulay modules over simple singularities of type (A_n) (Theorem 4.15).

The stable hom relation has been studied by Auslander-Reiten [2] and they also considered when the relation is a partial order. But the techniques in this paper are different from them because they used the fact that the ordinary (not stable) hom relation is a partial order. In our setting, the hom-set does not always have finite dimension, so that we can not apply their argument.

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2. Stable hom relation on Cohen-Macaulay modules

Throughout the paper R is a commutative complete Gorenstein local k-algebra where k is an algebraically closed field of characteristic 0. For a finitely generated R-module M, we say that M is a Cohen-Macaulay R-module if

$$\mathrm{Ext}^i_R(M,R)=0 \quad \text{for any } i>0.$$

We denote by $\mathrm{CM}(R)$ the category of Cohen-Macaulay R-modules with all R-homomorphisms. We also denote by $\underline{\mathrm{CM}}(R)$ the stable category of $\mathrm{CM}(R)$. The objects of $\underline{\mathrm{CM}}(R)$ are the same as those of $\mathrm{CM}(R)$, and the morphisms of $\underline{\mathrm{CM}}(R)$ are elements of $\underline{\mathrm{Hom}}_R(M,N)=\mathrm{Hom}_R(M,N)/P(M,N)$ for $M,N\in\underline{\mathrm{CM}}(R)$, where $P(M,N)$ denote the set of morphisms from M to N factoring through free R-modules. We write $\underline{\mathrm{Hom}}_R(M,N)$ for $\mathrm{Hom}_{\underline{\mathrm{CM}}(R)}(M,N)$. For a Cohen-Macaulay module M, denote by $\underline{M}$ to indicate that it is an object of $\underline{\mathrm{CM}}(R)$. For a finitely generated R-module M, take a free resolution

$$\cdots\to F_1\xrightarrow{d}F_0\to M\to 0.$$

We denote $\mathrm{Im}(d)$ by ΩM. We also denote $\mathrm{Coker}(\mathrm{Hom}_R(d,R))$ by $\mathrm{Tr}M$, which is called an Auslander transpose of M. We note that the functor Ω defines a functor giving an auto-equivalence on $\underline{\mathrm{CM}}(R)$. It is known that $\underline{\mathrm{CM}}(R)$ has a structure of a triangulated category with the suspension functor defined by the functor Ω. See [6, Chapter 1] for details. Since R is Gorenstein, by the definition of a triangle, $\underline{L}\to\underline{M}\to\underline{N}\to\underline{L}[1]$ is a triangle in $\underline{\mathrm{CM}}(R)$ if and only if there is an exact sequence $0\to L\to M'\to N\to 0$ in $\mathrm{CM}(R)$ with $\underline{M'}\cong\underline{M}$ in $\underline{\mathrm{CM}}(R)$, that is, M' is isomorphic to M up to free summand. Since R is complete, $\mathrm{CM}(R)$, hence $\underline{\mathrm{CM}}(R)$, is a Krull-Schmidt category, namely each object can be decomposed into indecomposable objects up to isomorphism uniquely.

In the paper we use the theory of Auslander-Reiten (abbr. AR) sequences and triangles of Cohen-Macaulay modules. Let us recall the definitions of those notions. See [14] for AR sequences and [6, 11] for AR triangles.

Definition 2.1. Let X, Y and Z be Cohen-Macaulay R-modules.

(1) A short exact sequence $\Sigma_X:0\to Z\to Y\xrightarrow{f}X\to 0$ is said to be an AR sequence ending in X (or starting from Z) if it satisfies
 (AR1) X and Z are indecomposable.
 (AR2) Σ_X is not split.
 (AR3) If $g:W\to X$ is not a split epimorphism, then there exists $h:W\to Y$ such that $g=f\circ h$.

(2) We also say that a triangle $\underline{\Sigma_X}:\underline{Z}\to\underline{Y}\xrightarrow{\underline{f}}\underline{X}\xrightarrow{\underline{w}}\underline{Z}[1]$ is an AR triangle ending in $\underline{X}$ (or starting from $\underline{Z}$) if it satisfies
 (ART1) $\underline{X}$ and $\underline{Z}$ are indecomposable.
 (ART2) $\underline{w}\neq 0$.
 (ART3) If $\underline{g}:\underline{W}\to\underline{X}$ is not a split epimorphism, then there exists $\underline{h}:\underline{W}\to\underline{Y}$ such that $\underline{g}=\underline{f}\circ\underline{h}$.

Proposition 2.2. *Let $\Sigma_X:0\to Z\to Y\xrightarrow{f}X\to 0$ be an AR sequence ending in X. Then $\underline{\Sigma_X}:\underline{Z}\to\underline{Y}\xrightarrow{\underline{f}}\underline{X}\xrightarrow{\underline{w}}\underline{Z}[1]$ is an AR triangle ending in $\underline{X}$.*

Proof. We shall show that $\underline{\Sigma}$ satisfies (ART1), (ART2) and (ART3).
(ART1) It is obvious.
(ART2) If $\underline{w}$ is zero, then $\underline{\Sigma}$ is split. Thus there exists $\underline{g}:\underline{X}\to\underline{Y}$ such that $\underline{f}\circ\underline{g}=1_{\underline{X}}$. Note that $f\circ g\in\mathrm{radEnd}_R(X)$ since Σ is not split. It yields that $\underline{f}\circ\underline{g}=1_{\underline{X}}\in\mathrm{rad}\underline{\mathrm{End}}_R(X)$. This is a contradiction and $\underline{w}$ must be non zero.

(ART3) Let $\underline{g} : \underline{W} \to \underline{X}$ be not a split epimorphism. Then $g : W \to X$ is also not a split epimorphism. By (AR3), we have a morphism $h : W \to Y$ such that $f \circ h = g$. We conclude that $\underline{f} \circ \underline{h} = \underline{g}$. □

We say that $\mathrm{CM}(R)$ (resp. $\underline{\mathrm{CM}}(R)$) admits AR sequences (resp. AR triangles) if there exists an AR sequence (resp. AR triangle) ending in X (resp. $\underline{X}$) for each indecomposable Cohen-Macaulay R-module X which is not free. We also say that $(R, \mathfrak{m})$ is an isolated singularity if each localization $R_{\mathfrak{p}}$ is regular for each prime ideal $\mathfrak{p}$ with $\mathfrak{p} \neq \mathfrak{m}$. If R is an isolated singularity, $\mathrm{CM}(R)$ admits AR sequences (cf. [14, Theorem 3.2]). As a corollary of Proposition 2.2, $\underline{\mathrm{CM}}(R)$ admits AR triangles if R is an isolated singularity.

Corollary 2.3. *If R is an isolated singularity, we have an 1-1 correspondence between the set of isomorphism classes of AR sequences in* $\mathrm{CM}(R)$ *and that of AR triangles in* $\underline{\mathrm{CM}}(R)$.

Proof. According to Proposition 2.2, we can define the mapping from the set of AR sequences to the set of AR triangles. Note that AR triangles (resp. AR sequences) ending in $\underline{X}$ (resp. X) are unique up to isomorphism of triangles (resp. sequences) for a given indecomposable $\underline{X}$ (resp. X) (see [6, 11]). Hence it follows from Proposition 2.2 that the mapping is surjective. The injectivity of the mapping is clear. □

By virtue of the lemma below, we see that $\underline{\mathrm{Hom}}_R(M, N)$ has finite dimension over k for M, $N \in \mathrm{CM}(R)$ if R is an isolated singularity.

Lemma 2.4. [14, Lemma 3.9] *Let M and N be finitely generated R-modules. Then we have a functorial isomorphism*

$$\underline{\mathrm{Hom}}_R(M, N) \cong \mathrm{Tor}_1^R(\mathrm{Tr}M, N).$$

In what follows, we always assume that R is an isolated singularity, and then the following definition makes sense.

Definition 2.5. For M, $N \in \mathrm{CM}(R)$ we define $\underline{M} \leq_{\underline{hom}} \underline{N}$ if $[\underline{X}, \underline{M}] \leq [\underline{X}, \underline{N}]$ for each $\underline{X} \in \underline{\mathrm{CM}}(R)$. Here $[\underline{X}, \underline{M}]$ is an abbreviation of $\dim_k \underline{\mathrm{Hom}}_R(X, M)$.

For an AR triangle $\underline{Z} \to \underline{Y} \to \underline{X} \to \underline{Z}[1]$, we denote $\underline{Z}$ (resp. $\underline{X}$) by $\tau\underline{X}$ (resp. $\tau^{-1}\underline{Z}$). For an AR sequence $0 \to Z \to Y \to X \to 0$, Z (resp. X) is also denoted by τX (resp. $\tau^{-1}Z$) (see [14, Definition 2.8].). By Proposition 2.2, $\tau\underline{X} \cong \underline{\tau X}$ for each indecomposable Cohen-Macaulay R-module X.

Remark 2.6. Reiten and Van den Bergh [11] show that a Hom-finite k-linear triangulated category $\mathcal{T}$ admits AR triangles if and only if $\mathcal{T}$ has a Serre functor. We can show that $\underline{\mathrm{CM}}(R)$ is also a Hom-finite triangulated category which has a Serre functor if R is an isolated singularity. Actually $\underline{\mathrm{CM}}(R)$ has a Serre functor $\underline{\tau(-)}[1]$ (cf. [14, Lemma 3.10]). Note that $\underline{\Omega X} \cong \underline{X}[-1]$ and $\underline{\tau X} \cong \underline{\Omega^{2-d} X} \cong \underline{X}[d-2]$ where $d = \dim R$. Hence we have $\underline{\tau(-)}[1] \cong (-)[d-1]$. See also [8, Corollary 2.5.].

Now let us consider the full subcategory of the functor category of $\mathrm{CM}(R)$ which is called the Auslander category. We give a brief review of the Auslander category (see [14, Chapter 4 and 13] for details). The Auslander category $\mathrm{mod}(\mathrm{CM}(R))$ is the category whose objects are finitely presented contravariant additive functors from $\mathrm{CM}(R)$ to the category of abelian groups and whose morphisms are natural transformations between functors. The following lemma is a key of our result in this section. For an additive subcategory $\mathcal{A}$ of an abelian category, which is skeletally small and closed under extensions, we denote by $K_0(\mathcal{A})$ the Grothendieck group of $\mathcal{A}$.

Lemma 2.7. [14, Theorem 13.7] *The group homomorphism*

$$\gamma : \mathrm{G}(\mathrm{CM}(R)) \to K_0(\mathrm{mod}(\mathrm{CM}(R))),$$

defined by $\gamma(M) = [\mathrm{Hom}_R(\ ,M)]$ *for* $M \in \mathrm{CM}(R)$*, is injective. Here* $\mathrm{G}(\mathrm{CM}(R))$ *is a free abelian group* $\bigoplus \mathbb{Z}\cdot X$*, where* X *runs through all isomorphism classes of indecomposable objects in* $\mathrm{CM}(R)$*.*

We denote by $\underline{\mathrm{mod}}(\mathrm{CM}(R))$ the full subcategory $\mathrm{mod}(\mathrm{CM}(R))$ consisting of functors F with $F(R) = 0$. Note that every object $F \in \underline{\mathrm{mod}}(\mathrm{CM}(R))$ is obtained from a short exact sequence in $\mathrm{CM}(R)$. Namely we have the short exact sequence $0 \to L \to M \to N \to 0$ such that

$$0 \to \mathrm{Hom}_R(\ ,L) \to \mathrm{Hom}_R(\ ,M) \to \mathrm{Hom}_R(\ ,N) \to F \to 0$$

is exact in $\mathrm{mod}(\mathrm{CM}(R))$. Since $F \in \underline{\mathrm{mod}}(\mathrm{CM}(R))$ is a subfunctor of $\mathrm{Ext}^1_R(\ ,L)$ for some $L \in \mathrm{CM}(R)$, $F(X)$ has finite length for each $X \in \mathrm{CM}(R)$ if R is an isolated singularity. Therefore we can define a group homomorphism associated with X in $\mathrm{CM}(R)$

$$\varphi_X : K_0(\underline{\mathrm{mod}}(\mathrm{CM}(R))) \to \mathbb{Z}\ ; \quad [F] \mapsto \dim_k F(X). \tag{2.1}$$

If $0 \to Z \to Y \to X \to 0$ is an AR sequence in $\mathrm{CM}(R)$, then the functor S_X defined by an exact sequence

$$0 \to \mathrm{Hom}_R(\ ,Z) \to \mathrm{Hom}_R(\ ,Y) \to \mathrm{Hom}_R(\ ,X) \to S_X \to 0$$

is a simple object in $\mathrm{mod}(\mathrm{CM}(R))$ and all the simple objects in $\mathrm{mod}(\mathrm{CM}(R))$ are obtained in this way from AR sequences.

We say that R is of finite representation type if there are only a finite number of isomorphism classes of indecomposable Cohen-Macaulay R-modules. We note that if R is of finite representation type, then R is an isolated singularity (cf. [14, Chapter 3.]). It is proved in [14, (13.7.4)] that

for each object F *in* $\underline{\mathrm{mod}}(\mathrm{CM}(R))$*, there is a filtration by subobjects* $0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F$ *such that each* F_i/F_{i-1} *is a simple object in* $\mathrm{mod}(\mathrm{CM}(R))$ *if* R *is of finite representation type.*

We also remark that, since $\mathrm{CM}(R)$ is a Krull-Schmidt category,

$$S_X(Y) = \begin{cases} k & \text{if } X \cong Y, \\ 0 & \text{if } X \ncong Y. \end{cases}$$

for an indecomposable module $Y \in \mathrm{CM}(R)$. See [14, (4.11)] for instance.

Lemma 2.8. *If* R *is of finite representation type, then we have the equality in* $K_0(\underline{\mathrm{mod}}(\mathrm{CM}(R)))$

$$[\underline{\mathrm{Hom}}_R(-,M)] = \sum_{X_i \in \mathrm{ind}\mathrm{CM}(R)} [\underline{X_i}, \underline{M}] \cdot [S_{X_i}]$$

for each $M \in \mathrm{CM}(R)$*.*

Proof. For $F = \underline{\mathrm{Hom}}_R(-,M)$, $F(R) = 0$, so that $F \in \underline{\mathrm{mod}}(\mathrm{CM}(R))$. Since R is of finite representation type, F has a filtration by simple objects S_{X_i}. Hence we have the equality in $K_0(\underline{\mathrm{mod}}(\mathrm{CM}(R)))$:

$$[F] = \sum_{X_i \in \mathrm{ind}\mathrm{CM}(R)} c_i \cdot [S_{X_i}].$$

By using the homomorphism in (2.1), we see that

$$[\underline{X_j}, \underline{M}] = \varphi_{X_j}([F]) = \dim_k \sum_{X_i \in \mathrm{ind}\mathrm{CM}(R)} c_i \cdot \dim_k S_{X_i}(X_j) = c_j.$$

Therefore we obtain the equation in the lemma. $\square$

Theorem 2.9. *Let* R *be of finite representation type and* M *and* N *be Cohen-Macaulay* R*-modules. Suppose that* $[\underline{X}, \underline{M}] = [\underline{X}, \underline{N}]$ *for each* $\underline{X} \in \underline{\mathrm{CM}}(R)$*. Then* $\underline{M} \oplus \underline{\Omega M} \cong \underline{N} \oplus \underline{\Omega N}$*.*

Proof. Under the circumstances, we see that $[\underline{\mathrm{Hom}}_R(-, M)] = [\underline{\mathrm{Hom}}_R(-, N)]$ in $K_0(\underline{\mathrm{mod}}(\mathrm{CM}(R)))$, hence in $K_0(\mathrm{mod}(\mathrm{CM}(R)))$. Note that $\underline{\mathrm{Hom}}_R(-, M) \cong \mathrm{Ext}^1_R(-, \Omega M)$ for each $M \in \mathrm{CM}(R)$ (cf. [4]). We have the resolution in $\mathrm{mod}(\mathrm{CM}(R))$:

$$0 \to \mathrm{Hom}_R(-, \Omega M) \to \mathrm{Hom}_R(-, P_M) \to \mathrm{Hom}_R(-, M) \to \underline{\mathrm{Hom}}_R(-, M) \to 0,$$

where P_M is a free R-module. Thus we have

$$[\mathrm{Hom}_R(-, M)]+[\mathrm{Hom}_R(-, \Omega M)]-[\mathrm{Hom}_R(-, P_M)] = [\mathrm{Hom}_R(-, N)]+[\mathrm{Hom}_R(-, \Omega N)]-[\mathrm{Hom}_R(-, P_N)].$$

Hence,

$$[\mathrm{Hom}_R(-, M)]+[\mathrm{Hom}_R(-, \Omega M)]+[\mathrm{Hom}_R(-, P_N)] = [\mathrm{Hom}_R(-, N)]+[\mathrm{Hom}_R(-, \Omega N)]+[\mathrm{Hom}_R(-, P_M)].$$

According to Lemma 2.7, we get

$$M \oplus \Omega M \oplus P_N \cong N \oplus \Omega N \oplus P_M.$$

Therefore $\underline{M} \oplus \underline{\Omega M} \cong \underline{N} \oplus \underline{\Omega N}$. ☐

As a corollary, we have the following.

Corollary 2.10. *Let R be of finite representation type and M and N be Cohen-Macaulay R-modules. Suppose that $\underline{U} \cong \underline{U}[-1]$ for each indecomposable Cohen-Macaulay R-module U. Then $[\underline{X}, \underline{M}] = [\underline{X}, \underline{N}]$ for each $\underline{X} \in \underline{\mathrm{CM}}(R)$ if and only if $\underline{M} \cong \underline{N}$. In partiqular, $\le_{\underline{hom}}$ is a partial order on $\underline{\mathrm{CM}}(R)$.*

Example 2.11. Let R be a one dimensional simple singularity of type (A_n), that is $R = k[[x, y]]/(x^{n+1} + y^2)$. If n is an even integer, one can show that X is isomorphic to ΩX up to a free summand for each $X \in \mathrm{CM}(R)$, so that $\underline{X} \cong \underline{X}[-1]$. See [14, Proposition 5.11]. Thus $\le_{\underline{hom}}$ is a partial order on $\underline{\mathrm{CM}}(R)$ if n is an even integer.

On the above example, if n is an odd integer, we have indecomposable modules $X \in \mathrm{CM}(R)$ such that $\underline{X} \ncong \underline{X}[-1]$. In fact, let $N_{\pm} = R/(x^{(n+1)/2} \pm \sqrt{-1}y)$. Then N_+ (resp. N_-) is a Cohen-Macaulay R-module which is isomorphic to ΩN_- (resp. ΩN_+), so that $\underline{N_+} \ncong \underline{N_+}[-1]$ (resp. $\underline{N_-} \ncong \underline{N_-}[-1]$). Though we can also show that $\le_{\underline{hom}}$ is a partial order on $\underline{\mathrm{CM}}(R)$ if n is an odd integer.

Proposition 2.12. *Let $R = k[[x, y]]/(x^{n+1} + y^2)$. Then $[\underline{X}, \underline{M}] = [\underline{X}, \underline{N}]$ for each $\underline{X} \in \underline{\mathrm{CM}}(R)$ if and only if $\underline{M} \cong \underline{N}$.*

Proof. We show the case when n is an odd integer. Let $I_i = (x^i, y)$ be ideals of R for $1 \le i \le (n-1)/2$. Then $\{I_1, \cdots I_{(n-1)/2}, N_+, N_-\}$ is a complete list of non free indecomposable Cohen-Macaulay R-modules. Note that $I_i \cong \Omega I_i$ up to free summand for $i = 1, \cdots (n-1)/2$. See [14, Paragraph (9.9)]. Now let $M, N \in \mathrm{CM}(R)$ and suppose that $[\underline{X}, \underline{M}] = [\underline{X}, \underline{N}]$ for each $\underline{X} \in \underline{\mathrm{CM}}(R)$. Set $\underline{M} = \bigoplus_{i=1}^{(n-1)/2} \underline{I_i}^{m_i} \oplus \underline{N_+}^{m_+} \oplus \underline{N_-}^{m_-}$ and $\underline{N} = \bigoplus_{i=1}^{(n-1)/2} \underline{I_i}^{n_i} \oplus \underline{N_+}^{n_+} \oplus \underline{N_-}^{n_-}$. By Theorem 2.9, $\underline{M} \oplus \underline{\Omega M} \cong \underline{N} \oplus \underline{\Omega N}$. Thus

$$\bigoplus_{i=1}^{(n-1)/2} \underline{I_i}^{2m_i} \oplus \underline{N_+}^{m_+ + m_-} \oplus \underline{N_-}^{m_+ + m_-} \cong \bigoplus_{i=1}^{(n-1)/2} \underline{I_i}^{2n_i} \oplus \underline{N_+}^{n_+ + n_-} \oplus \underline{N_-}^{n_+ + n_-}.$$

Hence we have equalities:

$$m_i = n_i, \quad m_+ + m_- = n_+ + n_-.$$

Here we remark that

$$\underline{\mathrm{Hom}}_R(N_{\pm}, N_{\mp}) \cong \mathrm{Ext}^1_R(N_{\pm}, N_{\pm}) = 0.$$

Using this, we have

$$[\underline{N_+}, \underline{M}] = \sum_{i=1}^{(n-1)/2} m_i[\underline{N_+}, \underline{I_i}] + m_+[\underline{N_+}, \underline{N_+}] = \sum_{i=1}^{(n-1)/2} n_i[\underline{N_+}, \underline{I_i}] + n_+[\underline{N_+}, \underline{N_+}] = [\underline{N_+}, \underline{N}].$$

This equality show that $m_+ = n_+$, so that $m_- = n_-$. Consequently $\underline{M} \cong \underline{N}$. □

Remark 2.13. The stable hom relation $\leq_{\underline{hom}}$ is not always a partial order on $\underline{\mathrm{CM}}(R)$ even if the base ring R is a simple singularity of type (A_n). Let $R = k[[x, y, z]]/(x^3 + y^2 + z^2)$, that is, R is a two dimensional simple singularity of type (A_2). And let I (resp. J) be an ideal generated by (x, y) (resp. (x^2, y)). Note that the set $\{I, J\}$ is a complete list of non free indecomposable Cohen-Macaulay R-modules (see [14, Chapter 10] for instance). Then it is easy to see that $[\underline{I}, \underline{I}] = [\underline{I}, \underline{J}] = [\underline{J}, \underline{I}] = [\underline{J}, \underline{J}] = 1$. However $\underline{I} \ncong \underline{J}$. Thus $\leq_{\underline{hom}}$ is not a partial order on $\underline{\mathrm{CM}}(R)$.

When Z is indecomposable then denote by $\mu(\underline{Z}, \underline{M})$ the multiplicity of $\underline{Z}$ as a direct summand of $\underline{M}$. On an AR triangle, we have the following.

Proposition 2.14. *Let $\underline{\Sigma_X} : \underline{Z} \to \underline{Y} \to \underline{X} \to \underline{Z}[1]$ be an AR triangle in $\underline{\mathrm{CM}}(R)$. Then the following statements hold for each indecomposable $U \in \mathrm{CM}(R)$.*

(1) $[\underline{U}, \underline{X}] + [\underline{U}, \underline{Z}] - [\underline{U}, \underline{Y}] = \mu(\underline{U}, \underline{X}) + \mu(\underline{U}, \underline{X}[-1])$.
(2) *If $\underline{U}$ is periodic of period 2, that is, $\underline{U} \cong \underline{U}[2]$, then*

$$[\underline{U}, \underline{X}] + [\underline{U}, \underline{Z}] - [\underline{U}, \underline{Y}] = [\underline{U}[-1], \underline{X}] + [\underline{U}[-1], \underline{Z}] - [\underline{U}[-1], \underline{Y}]$$

Proof. (1) Apply $\underline{\mathrm{Hom}}_R(U, -)$ to the triangle $\underline{\Sigma_X}$, we have a long exact sequence as follows:

$$\begin{array}{ccccccc}
\underline{\mathrm{Hom}}_R(U, Z[-1]) & \longrightarrow & \underline{\mathrm{Hom}}_R(U, Y[-1]) & \xrightarrow{\underline{\mathrm{Hom}}_R(U, g[-1])} & \underline{\mathrm{Hom}}_R(U, X[-1]) & \longrightarrow \\
\underline{\mathrm{Hom}}_R(U, Z) & \xrightarrow{\underline{\mathrm{Hom}}_R(U, f)} & \underline{\mathrm{Hom}}_R(U, Y) & \xrightarrow{\underline{\mathrm{Hom}}_R(M, g)} & \underline{\mathrm{Hom}}_R(U, X) & \longrightarrow \\
\underline{\mathrm{Hom}}_R(U, Z[1]) & \xrightarrow{\underline{\mathrm{Hom}}_R(U, f[1])} & \underline{\mathrm{Hom}}_R(U, Y[1]) & \longrightarrow & \underline{\mathrm{Hom}}_R(U, X[1]) & \longrightarrow .
\end{array}$$

Since $\underline{\mathrm{End}}_R(X)/\mathrm{rad}\underline{\mathrm{End}}_R(X) \cong k$ for each non free indecomposable Cohen-Macaulay module X and by the property of an AR triangle (ART3), we have

$$\mathrm{Ker}\ \underline{\mathrm{Hom}}_R(U, f[i+1]) \cong \mathrm{Coker}\ \underline{\mathrm{Hom}}_R(U, g[i]) \cong k^{\mu(\underline{U}, \underline{X}[i])}$$

for all $i \in \mathbb{Z}$. In particular, the following sequence is exact.

$$0 \to k^{\mu(\underline{U}, \underline{X}[-1])} \to \underline{\mathrm{Hom}}_R(U, Z) \to \underline{\mathrm{Hom}}_R(U, Y) \to \underline{\mathrm{Hom}}_R(U, X) \to k^{\mu(\underline{U}, \underline{X})} \to 0.$$

Therefore we obtain the required equation.

(2) Note from (1) that the equations

$$[\underline{U}, \underline{X}] + [\underline{U}, \underline{Z}] - [\underline{U}, \underline{Y}] = \mu(\underline{U}, \underline{X}) + \mu(\underline{U}, \underline{X}[-1])$$

and

$$[\underline{U}[-1], \underline{X}] + [\underline{U}[-1], \underline{Z}] - [\underline{U}[-1], \underline{Y}] = \mu(\underline{U}[-1], \underline{X}) + \mu(\underline{U}[-1], \underline{X}[-1])$$

hold. Since the shift functor $(-)[1]$ (hence $(-)[-1]$) is an auto-functor, $\mu(\underline{U}, \underline{X}) = \mu(\underline{U}[-1], \underline{X}[-1])$. Moreover $\mu(\underline{U}[-1], \underline{X}) = \mu(\underline{U}, \underline{X}[-1])$ for $\underline{U}[-1][-1] \cong \underline{U}[-2] \cong \underline{U}$. Consequently we get the assertion. □

Remark 2.15. By using Proposition 2.14, one can show that $\leq_{\underline{hom}}$ is a partial order on $\underline{\mathrm{CM}}(R)$ if $\underline{U} \cong \underline{U}[-1]$ for each $U \in \mathrm{CM}(R)$ without the assumption that R is of finite representation type.

3. Stable hom relation and Grothendieck group of $\underline{\mathrm{CM}}(R)$

For later reference we state some results on the stable hom relation between modules which give the same class in the Grothendieck group of $\underline{\mathrm{CM}}(R)$. The Grothendieck group of $\underline{\mathrm{CM}}(R)$ (more generally a triangulated category) is defined by

$$K_0(\underline{\mathrm{CM}}(R)) = \mathrm{G}(\underline{\mathrm{CM}}(R))/ < \underline{X}+\underline{Z}-\underline{Y} \mid \text{There is a triangle } \underline{Z}\to\underline{Y}\to\underline{X}\to\underline{Z}[1] \text{ in } \underline{\mathrm{CM}}(R) >,$$

where $\mathrm{G}(\underline{\mathrm{CM}}(R))$ is a free abelian group $\bigoplus_{\underline{X}\in \mathrm{ind}\underline{\mathrm{CM}}(R)} \mathbb{Z}\cdot\underline{X}$. We refer the reader to [6, Chapter 3] for the details. Since R is Gorenstein, one can show that $[\underline{M}]=[\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$ if and only if $[M\oplus P]=[N\oplus Q]$ in $K_0(\mathrm{CM}(R))$ for some free R-modules P, Q.

Lemma 3.1. *If R is of finite representation type, we have the equality of subgroups of* $\mathrm{G}(\underline{\mathrm{CM}}(R))$*:*

$$\begin{aligned}&< \underline{X}+\underline{Z}-\underline{Y} \mid \textit{There is a triangle } \underline{Z}\to\underline{Y}\to\underline{X}\to\underline{Z}[1] \textit{ in } \underline{\mathrm{CM}}(R) >\\ =&< \underline{X}+\underline{Z}-\underline{Y} \mid \textit{There is an AR triangle } \underline{Z}\to\underline{Y}\to\underline{X}\to\underline{Z}[1] \textit{ in } \underline{\mathrm{CM}}(R) > .\end{aligned}$$

Proof. It follows from Corollary 2.3, [14, Theorem 13.7] and the definition of triangles in stable categories. □

Let R be of finite representation type. In the rest of the paper, we assume that the nonisomorphic indecomposable Cohen-Macaulay R-modules are indexed by $\{1,2,\dots,n\}$, say $X_1, \dots, X_n$. By Lemma 3.1, $[\underline{M}]=[\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$ if and only if there exist AR triangles $\underline{Z_k}\to\underline{Y_k}\to\underline{X_k}\to\underline{Z_k}[1]$ and non-zero integers b_k such that the equality

$$[\underline{N}]-[\underline{M}]=\sum_{k\in\mathcal{K}} b_k\cdot([\underline{X_k}]+[\underline{Z_k}]-[\underline{Y_k}])$$

holds for some $\mathcal{K}\subseteq\{1,2,\dots,n\}$. We write positive and negative coefficients separately. That is, we express the equality as

$$[\underline{N}]-[\underline{M}]=\sum_{i\in\mathcal{I}} c_i\cdot([\underline{X_i}]+[\underline{Z_i}]-[\underline{Y_i}])-\sum_{j\in\mathcal{J}} d_j\cdot([\underline{X_j}]+[\underline{Z_j}]-[\underline{Y_j}]), \tag{3.1}$$

where $\mathcal{I}$, $\mathcal{J}\subseteq\{1,2,\dots,n\}$ are disjoint sets and c_i, d_j are non-negative integers. Note that $X_i\ncong X_j$ if $i\in\mathcal{I}$ and $j\in\mathcal{J}$.

Now we consider the following condition:

($*$) For an AR triangle $\underline{Z}\to\underline{Y}\to\underline{X}\to\underline{Z}[1]$, $[\underline{X}]+[\underline{Z}]-[\underline{Y}]=[\underline{X}[-1]]+[\underline{Z}[-1]]-[\underline{Y}[-1]]$ in $\mathrm{G}(\underline{\mathrm{CM}}(R))$.

Remark 3.2. (1) The condition ($*$) says that, for AR triangles $\underline{Z}\to\underline{Y}\to\underline{X}\to\underline{Z}[1]$ and $\underline{Z}'\to\underline{Y}'\to\underline{X}'\to\underline{Z}'[1]$, $[\underline{X}]+[\underline{Z}]-[\underline{Y}]=[\underline{X}']+[\underline{Z}']-[\underline{Y}']$ in $\mathrm{G}(\underline{\mathrm{CM}}(R))$ if $[\underline{U},\underline{X}]+[\underline{U},\underline{Z}]-[\underline{U},\underline{Y}]=[\underline{U},\underline{X}']+[\underline{U},\underline{Z}']-[\underline{U},\underline{Y}']$ for each indecomposable $\underline{U}$. Since it follows from Proposition 2.14 that $\mu(\underline{X},\underline{X})+\mu(\underline{X},\underline{X}[-1])=\mu(\underline{X},\underline{X}')+\mu(\underline{X},\underline{X}'[-1])$, we see that $\underline{X}\cong\underline{X}'$ or $\underline{X}\cong\underline{X}'[-1]$. If $\underline{X}\cong\underline{X}'$ the equation is obvious and assume that $\underline{X}\cong\underline{X}'[-1]$. Then by ($*$),

$$\begin{aligned}[\underline{X}]+[\underline{Z}]-[\underline{Y}] &= [\underline{X}'[-1]]+[\underline{Z}'[-1]]-[\underline{Y}'[-1]]\\ &= [\underline{X}']+[\underline{Z}']-[\underline{Y}']\end{aligned}$$

in $\mathrm{G}(\underline{\mathrm{CM}}(R))$.

(2) We also remark that, under the condition ($*$), $\tau\underline{X}\cong\underline{X}[-1]$ holds if $\underline{X}\cong\underline{X}[2]$. Suppose that $\underline{X}\cong\underline{X}[-1]$. The claim follows from Remak 2.6. Suppose that $\underline{X}\ncong\underline{X}[-1]$. For the AR triangle $\underline{\Sigma_X}:\tau\underline{X}\to\underline{E_X}\to\underline{X}\to\tau\underline{X}[1]$, we have

$$\underline{X}\oplus\tau\underline{X}\oplus\underline{E_X}[-1]\cong\underline{X}[-1]\oplus\tau\underline{X}[-1]\oplus\underline{E_X}.$$

Assume that $\tau\underline{X} \ncong \underline{X}[-1]$. Then $\tau\underline{X}[-1] \ncong \underline{X}[-2] \cong \underline{X}$. Since $\underline{X} \ncong \underline{X}[-1]$, $\tau\underline{X} \ncong \tau\underline{X}[-1]$. Then $\underline{X} \oplus \tau\underline{X}$ is ismorphic to a direct summand of $\underline{E_X}$, so that $\underline{E_X} \cong \underline{X} \oplus \tau\underline{X} \oplus \underline{E}'$. Hence we have the equality;

$$[\underline{X}, \underline{X}] + [\underline{X}, \tau\underline{X}] - [\underline{X}, \underline{E_X}] = [\underline{X}, \underline{X}] + [\underline{X}, \tau\underline{X}] - [\underline{X}, \underline{X} \oplus \tau\underline{X} \oplus \underline{E}'] = -[\underline{X}, \underline{E}'].$$

However, by Proposition 2.14,

$$0 \geqq -[\underline{X}, \underline{E}'] = \mu(\underline{X}, \underline{X}) + \mu(\underline{X}, \underline{X}[-1]) = 1.$$

This makes a contradiction, such that $\tau\underline{X} \cong \underline{X}[-1]$.

(3) The condition $(*)$ holds when $R = k[[x, y]]/(x^{n+1}+y^2)$ (cf. [14, Proposition 5.11, Paragraph (9.9)]).

We say that R satisfies $(*)$ if each AR triangle satisfies the condition $(*)$.

Proposition 3.3. *Let R be of finite representation type which satisfies $(*)$ and M and N be Cohen-Macaulay R-modules with $[\underline{M}] = [\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$. Then $\underline{M} \leq_{\underline{hom}} \underline{N}$ if and only if there exist AR triangles $\underline{Z_i} \to \underline{Y_i} \to \underline{X_i} \to \underline{Z_i}[1]$ and non-negative integers c_i with the equation in $\mathrm{G}(\underline{\mathrm{CM}}(R))$;*

$$[\underline{N}] - [\underline{M}] = \sum_{i\in\mathcal{I}} c_i \cdot ([\underline{X_i}] + [\underline{Z_i}] - [\underline{Y_i}]), \tag{3.2}$$

where $\mathcal{I} \subseteq \{1, 2, \ldots, n\}$.

Proof. According to the equality (3.1), we see that

$$\begin{aligned} [\underline{U}, \underline{N}] - [\underline{U}, \underline{M}] &= \textstyle\sum_{i\in\mathcal{I}} c_i \cdot ([\underline{U}, \underline{X_i}] + [\underline{U}, \underline{Z_i}] - [\underline{U}, \underline{Y_i}]) - \sum_{j\in\mathcal{J}} d_j \cdot ([\underline{U}, \underline{X_j}] + [\underline{U}, \underline{Z_j}] - [\underline{U}, \underline{Y_j}]) \\ &= \textstyle\sum_{i\in\mathcal{I}} c_i \cdot \{\mu(\underline{U}, \underline{X_i}) + \mu(\underline{U}, \underline{X_i}[-1])\} - \sum_{j\in\mathcal{J}} d_j \cdot \{\mu(\underline{U}, \underline{X_j}) + \mu(\underline{U}, \underline{X_j}[-1])\} \end{aligned}$$

for each indecomposable module $U \in \mathrm{CM}(R)$. Assume that $\underline{M} \leq_{\underline{hom}} \underline{N}$. If $d_j \neq 0$, there exist i such that $\underline{X_j}[-1] \cong \underline{X_i}$ with $d_j \leq c_i$. By the condition $(*)$, we can omit such constituents in the expression. Repeating this procedure, we obtain the equation. □

Corollary 3.4. *Let R be of finite representation type which satisfies $(*)$. Then the stable hom relation is a partial order between modules which give the same class in the Grothendieck group of $\underline{\mathrm{CM}}(R)$.*

Proof. Let M and N be Cohen-Macaulay R-modules with $[\underline{M}] = [\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$. It follows from Proposition 2.14 and the expression in Proposition 3.3 that

$$\begin{aligned} [\underline{X}, \underline{N}] - [\underline{X}, \underline{M}] &= \textstyle\sum_{i\in\mathcal{I}} c_i \cdot ([\underline{X}, \underline{X_i}] + [\underline{X}, \underline{Z_i}] - [\underline{X}, \underline{Y_i}]) \\ &= \textstyle\sum_{i\in\mathcal{I}} c_i (\mu(\underline{X}, \underline{X_i}) + \mu(\underline{X}, \underline{X_i}[-1])). \end{aligned}$$

Suppose that $[\underline{X}, \underline{M}] = [\underline{X}, \underline{N}]$ for each $\underline{X} \in \underline{\mathrm{CM}}(R)$ and then $c_i = 0$. Thus $[\underline{M}] - [\underline{N}] = 0$ in $\mathrm{G}(\underline{\mathrm{CM}}(R))$, so that $\underline{M} \cong \underline{N}$. □

Under the circumstance of Proposition 3.3, we say that the expression (3.2) is irredundant if $\underline{X_i} \ncong \underline{X_j}[-1]$ for each i and j with $i \neq j$. Since R is of finite representation type, we can always take the expression irredundant.

Lemma 3.5. *Let R be of finite representation type which satisfies $(*)$ and M and N be Cohen-Macaulay R-modules with $\underline{M} \leq_{\underline{hom}} \underline{N}$ and $[\underline{M}] = [\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$. Let U be a non free indecomposable direct summand of N. Suppose that $[\underline{U}, \underline{M}] = [\underline{U}, \underline{N}]$ and $\underline{U} \cong \underline{U}[2]$. Then U is also a direct summand of M.*

Proof. By virtue of Proposition 3.3, since $\underline{M} \leq_{\underline{hom}} \underline{N}$, there exist AR triangles $\underline{Z_i} \to \underline{Y_i} \to \underline{X_i} \to \underline{Z_i}[1]$ with

$$[\underline{N}] - [\underline{M}] = \sum_{i\in\mathcal{I}} c_i \cdot ([\underline{X_i}] + [\underline{Z_i}] - [\underline{Y_i}])$$

in $\mathrm{G}(\underline{\mathrm{CM}}(R))$. Thus we have

$$\underline{N} \oplus \bigoplus_{i \in \mathcal{I}} \underline{Y_i}^{c_i} \cong \underline{M} \oplus \bigoplus_{i \in \mathcal{I}} (\underline{X_i} \oplus \underline{Z_i})^{c_i}.$$

Now we assume that U is not a direct summand of M. Then there exists i such that $\underline{U} \cong \underline{X_i}$ or $\underline{U} \cong \underline{Z_i}$ and we can show the inequality

$$[\underline{U}, \underline{X_i}] + [\underline{U}, \underline{Z_i}] - [\underline{U}, \underline{Y_i}] = \mu(\underline{U}, \underline{X}) + \mu(\underline{U}, \underline{X}[-1]) > 0$$

holds. If $\underline{U} \cong \underline{X_i}$, it is clear. If $\underline{U} \cong \underline{Z_i}$, since $\underline{U}$ is periodic of period at most 2,

$$\underline{X_i} \cong \tau^{-1}\underline{U} \cong \underline{U}[2-d] \cong \underline{U}[-d] \cong \underline{U} \text{ or } \underline{U}[1].$$

See Remark 2.6. Hence we have the inequality above. However the inequality never happens since

$$[\underline{U}, \bigoplus_{i \in \mathcal{I}} \underline{Y_i}^{c_i}] = [\underline{U}, \bigoplus_{i \in \mathcal{I}} (\underline{X_i} \oplus \underline{Z_i})^{c_i}].$$

Therefore, U is a direct summand of M. $\square$

Proposition 3.6. *Let R be of finite representation type which satisfies $(*)$ and M and N be Cohen-Macaulay R-modules with $\underline{M} \le_{\underline{hom}} \underline{N}$ and $[\underline{M}] = [\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$. Suppose that $[\underline{U}, \underline{M}] = [\underline{U}, \underline{N}]$ and $\underline{U} \cong \underline{U}[2]$ for each indecomposable direct summand U of N. Then $\underline{M} \cong \underline{N}$.*

Proof. For each indecomposable direct summand U of N, it follows from Lemma 3.5 that $\underline{U}$ is isomorphic to a direct summand of $\underline{M}$, so that $\underline{M} \cong \underline{M}' \oplus \underline{U}$. Set $\underline{N} \cong \underline{N}' \oplus \underline{U}$. Then $\underline{N}' \le_{\underline{hom}} \underline{M}'$ since $\underline{N}' \oplus \underline{U} \le_{\underline{hom}} \underline{M}' \oplus \underline{U}$. Note that $[\underline{M}'] = [\underline{N}']$ in $K_0(\underline{\mathrm{CM}}(R))$ and $[\underline{U}', \underline{M}'] = [\underline{U}', \underline{N}']$ for each indecomposable direct summand U' of N'. It also follows that $\underline{U}'$ is isomorphic to a direct summand of $\underline{M}'$ by Lemma 3.5. Hence, repeating the procedure, we see that $\underline{N}$ is isomorphic to a direct summand of $\underline{M}$. Let $\underline{M} \cong \underline{M}'' \oplus \underline{N}$. Since $\underline{M} \le_{\underline{hom}} \underline{N}$, $\underline{M}'' \le_{\underline{hom}} \underline{0}$. Thus $\underline{M}' \cong \underline{0}$. Hence, we have $\underline{M} \cong \underline{N}$. $\square$

4. Stable degenerations of Cohen-Macaulay modules

In this section, we attempt to describe the stable degenerations of Cohen-Macaulay modules in relation with the stable hom relation. First let us recall the definition of stable degenerations of Cohen-Macaulay modules. For the detail, we refer the reader to [17]. See also [13, 16].

Definition 4.1. [17, Definition 4.1] Let $V = k[t]_{(t)}$ and $K = k(t)$. For $\underline{M}, \underline{N} \in \underline{\mathrm{CM}}(R)$, we say that $\underline{M}$ stably degenerates to $\underline{N}$ if there exists a Cohen-Macaulay module $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ such that $\underline{Q[1/t]} \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\underline{Q \otimes_V V/tV} \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$.

If a ring is an isolated singularity, there is a nice characterization of stable degenerations.

Theorem 4.2. [17, Theorem 5.1, 6.1] *Consider the following three conditions for Cohen-Macaulay R-modules M and N:*

1. *$M \oplus P$ degenerates to $N \oplus Q$ for some free R-modules P, Q.*
2. *$\underline{M}$ stably degenerates to $\underline{N}$.*
3. *There is a triangle*

$$\underline{Z} \longrightarrow \underline{M} \oplus \underline{Z} \longrightarrow \underline{N} \longrightarrow \underline{Z}[1]$$

in $\underline{\mathrm{CM}}(R)$.

If R is an isolated singularity, then (2) and (3) are equivalent. Moreover, if R is Artinian, the conditions (1), (2) and (3) are equivalent.

Remark 4.3. In general, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold and it is required that the endmorphism of $\underline{Z}$ in the triangle in (3) is nilpotent. However if R is an isolated singularity, we do not need the nilpotency assumption (cf. [17, Lemma 6.5.]). It follows from the theorem that $\underline{M}$ and $\underline{N}$ give the same class in the Grothendieck group of $\underline{\mathrm{CM}}(R)$ if $\underline{M}$ stably degenerates to $\underline{N}$.

We state order relations with respect to stable degenerations and triangles.

Definition 4.4. [7, Definition 3.2., 3.3.] Let M and N be Cohen-Macaulay R-modules.

(1) We denote by $\underline{M} \leq_{st} \underline{N}$ if $\underline{N}$ is obtained from $\underline{M}$ by iterative stable degenerations, i.e. there is a sequence of Cohen-Macaulay R-modules $\underline{L_0}, \underline{L_1}, \dots, \underline{L_r}$ such that $\underline{M} \cong \underline{L_0}$, $\underline{N} \cong \underline{L_r}$ and each $\underline{L_i}$ stably degenerates to $\underline{L_{i+1}}$ for $0 \leq i < r$.
(2) We say that $\underline{M}$ stably degenerates by a triangle to $\underline{N}$, if there is a triangle of the form $\underline{U} \to \underline{M} \to \underline{V} \to \underline{U}[1]$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{U} \oplus \underline{V} \cong \underline{N}$. We also denote by $\underline{M} \leq_{tri} \underline{N}$ if $\underline{N}$ is obtain from $\underline{M}$ by iterative stable degenerations by a triangle.

Remark 4.5. It was shown in [17] that the stable degeneration relation is a partial order. Moreover if there is a triangle $\underline{U} \to \underline{M} \to \underline{V} \to \underline{U}[1]$, then we can show that $\underline{M}$ stably degenerates to $\underline{U} \oplus \underline{V}$ (cf. [7, Remark 3.4. (2)]). Hence $\underline{M} \leq_{tri} \underline{N}$ implies $\underline{M} \leq_{st} \underline{N}$. It also follows from Theorem 4.2 that $\underline{M} \leq_{st} \underline{N}$ induces that $\underline{M} \leq_{\underline{hom}} \underline{N}$.

In this section, we shall show

Theorem 4.6. *Let R be a hypersurface which is of finite representation type and satisfies $(*)$. Then $\underline{M} \leq_{\underline{hom}} \underline{N}$ if and only if $\underline{M} \leq_{st} \underline{N}$ for Cohen-Macaulay R-modules M and N with $[\underline{M}] = [\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$.*

To show this, we use the stable analogue of arguments in [18].

The lemma below is well known for the case in abelian categories (cf. [18, Lemma 2.6]). The same statement follows in an arbitrary k-linear triangulated category, not necessary $\underline{\mathrm{CM}}(R)$. (The author thanks Yuji Yoshino for telling him this argument.)

Lemma 4.7. *Let*

$$\Sigma_1 : N_1 \xrightarrow{\binom{f_1}{v}} L_1 \oplus N_2 \xrightarrow{(u, g_1)} L_2 \longrightarrow N_1[1]$$

and

$$\Sigma_2 : M_1 \xrightarrow{\binom{f_2}{w}} N_1 \oplus M_2 \xrightarrow{(v, g_2)} N_2 \longrightarrow M_1[1]$$

be triangles in a k-linear triangulated category. Then we also have the following triangle.

$$M_1 \to L_1 \oplus M_2 \to L_2 \to M_1[1].$$

Proof. We consider the following triangle associated with Σ_2:

$$M_1 \xrightarrow{\begin{pmatrix} -f_1 \circ f_2 \\ f_2 \\ w \end{pmatrix}} L_1 \oplus N_1 \oplus M_2 \xrightarrow{\begin{pmatrix} 1 & f_1 & 0 \\ 0 & v & g_2 \end{pmatrix}} N_2 \longrightarrow M_1[1].$$

We remark that the left morphism is given by

$$\begin{pmatrix} -f_1 \circ f_2 \\ f_2 \\ w \end{pmatrix} = \begin{pmatrix} 1 & -f_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ f_2 \\ w \end{pmatrix},$$

to make the diagram below:

$$\begin{tikzcd}
N_1 \arrow[r, "\binom{f_1}{v}"] & L_1 \oplus N_2 \arrow[r, "(u,g_1)"] & L_2 \arrow[r] & N_1[1] \\
N_1 \arrow[u, equal] \arrow[r, "\left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)"'] & L_1 \oplus N_1 \oplus M_2 \arrow[u, "\left(\begin{smallmatrix}1 & f_1 & 0\\0 & v & g_2\end{smallmatrix}\right)"] \arrow[r] & L_1 \oplus M_2 \arrow[u] \arrow[r] & N_1[1] \arrow[u, equal] \\
& M_1 \arrow[u, "\left(\begin{smallmatrix}-f_1\circ f_2\\f_2\\w\end{smallmatrix}\right)"] \arrow[r, equal] & M_1. \arrow[u] &
\end{tikzcd}$$

By the octahedral axiom, we obtain the required triangle. □

Remark 4.8. Combining (the abelian version of) Lemma 4.7 with Lemma 2.8, the dimension of $\underline{\mathrm{Hom}}$ can be calculated easily from the datum of AR sequences. For instance, let $R = k[[x,y]]/(x^{n+1}+y^2)$ where n is even. As stated in [14, Proposition 5.11], the set of ideals of R $\{\ I_i = (x^i, y) \mid 1 \le i \le n/2\ \}$ is a complete list of isomorphic classes of indecomposable non free Cohen-Macaulay R-modules. The AR sequences are

$$0 \to I_i \to I_{i-1} \oplus I_{i+1} \to I_i \to 0$$

for $i = 1, \cdots, n/2$ where $I_0 = R$ and $I_{n/2+1} \cong I_{n/2}$. Then we have

$$\begin{tikzcd}
I_1 \arrow[r] \arrow[d] & I_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & I_{n/2} \arrow[r] \arrow[d] & I_{n/2} \arrow[r] \arrow[d] & \cdots \arrow[r] & I_1 \arrow[r] \arrow[d] & R \arrow[d] \\
R \arrow[r] & I_1 \arrow[r] & \cdots \arrow[r] & I_{n/2-1} \arrow[r] & I_{n/2+1} \cong I_{n/2} \arrow[r] & \cdots \arrow[r] & I_2 \arrow[r] & I_1.
\end{tikzcd}$$

The diagram shows that

$$[\underline{\mathrm{Hom}}_R(-, I_1)] = \sum_{i=1}^{n/2} 2[S_{I_i}]$$

in $K_0(\underline{\mathrm{mod}}(\mathrm{CM}(R)))$. Thus we obtain $[\underline{I_i}, \underline{I_1}] = 2$ for $i = 1, \cdots, n/2$.

Definition 4.9. Let M and N be Cohen-Macaulay R-modules. We define a function $\delta_{\underline{M},\underline{N}}(-)$ on $\underline{\mathrm{CM}}(R)$ by

$$\delta_{\underline{M},\underline{N}}(-) = [-, \underline{N}] - [-, \underline{M}].$$

For a triangle $\underline{\Sigma} : \underline{L} \to \underline{M} \to \underline{N} \to \underline{L}[1]$, we also define a function $\delta_{\underline{\Sigma}}(-)$ on $\underline{\mathrm{CM}}(R)$ by

$$\delta_{\underline{\Sigma}}(-) = [-, \underline{L}] + [-, \underline{N}] - [-, \underline{M}].$$

Remark 4.10. As shown in Proposition 3.3, for modules M, $N \in \mathrm{CM}(R)$ with $\underline{M} \le_{\underline{hom}} \underline{N}$ and $[\underline{M}] = [\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$, we have the irredundant expression

$$[\underline{N}] - [\underline{M}] = \sum_{i \in \mathcal{I}} c_i \cdot ([\underline{X_i}] + [\underline{Z_i}] - [\underline{Y_i}]),$$

where $\underline{\Sigma_{X_i}} : \underline{Z_i} \to \underline{Y_i} \to \underline{X_i} \to \underline{Z_i}[1]$ are AR triangles. On the number c_i, we have

$$c_i = \delta_{\underline{M},\underline{N}}(\underline{X_i}) / \delta_{\underline{\Sigma_{X_i}}}(\underline{X_i}).$$

In the remaining results of the paper, we assume that each indecomposable Cohen-Macaulay module is periodic of period at most 2. Then, by [5, Corollary 6.2], we see that R must be a hypersurface since R is complete. Therefore, in the rest of the paper, we always assume that R is a hypersurface.

We say that a triangle $\underline{Z} \xrightarrow{\underline{g}} \underline{Y} \xrightarrow{\underline{f}} \underline{X} \to \underline{Z}[1]$ is without isomorphisms if $\underline{f} \in \underline{\mathrm{rad}}(\underline{Y}, \underline{X})$ and $\underline{g} \in \underline{\mathrm{rad}}(\underline{Z}, \underline{Y})$. Let $\underline{Z} \to \underline{Y} \to \underline{X} \to \underline{Z}[1]$ be any triangle. As in the case of a sequence, there is a triangle without isomorphisms $\underline{Z}' \to \underline{Y}' \to \underline{X}' \to \underline{Z}'[1]$ such that $\underline{Z} \cong \underline{Z}' \oplus \underline{U}$, $\underline{Y} \cong \underline{Y}' \oplus \underline{U} \oplus \underline{V}$ and $\underline{X} \cong \underline{X}' \oplus \underline{V}$ for some $\underline{U}$, $\underline{V} \in \underline{\mathrm{CM}}(R)$. (Cf. [18, Paragraph (2.7)]).

Lemma 4.11. [18, Lemma 3.1.] *Let R be a hypersurface which is of finite representation type and satisfies $(*)$. Let M and N be Cohen-Macaulay R-modules with $\underline{M} \leq_{\underline{hom}} \underline{N}$ and $[\underline{M}] = [\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$ and let $\underline{\Sigma} : \underline{U} \to \underline{W} \to \underline{V} \to \underline{U}[1]$ be a triangle without isomorphisms such that $\delta_{\underline{\Sigma}} \leq \delta_{\underline{M},\underline{N}}$. Then there exists a triangle $\underline{\Phi} : \underline{Z} \to \underline{Y} \to \underline{V} \to \underline{Z}[1]$ without isomorphisms such that $\delta_{\underline{\Phi}}(\underline{Y}) = \delta_{\underline{M},\underline{N}}(\underline{Y})$.*

Proof. If $\delta_{\underline{\Sigma}}(\underline{W}) = \delta_{\underline{M},\underline{N}}(\underline{W})$, we have nothing to prove. Otherwise, we assume that there exists an indecomposable direct summand W_1 of W $(= W_1 \oplus W_2)$ such that $\delta_{\underline{\Sigma}}(\underline{W_1}) < \delta_{\underline{M},\underline{N}}(\underline{W_1})$. Under the assumptions, we have AR triangles $\underline{\Sigma_{X_i}} : \tau\underline{X_i} \to \underline{E_{X_i}} \to \underline{X_i} \to \tau\underline{X_i}[1]$ such that $\delta_{\underline{M},\underline{N}} = \sum_i c_i \cdot \delta_{\underline{\Sigma_{X_i}}}$. (See Remark 4.10.) Thus there exists i such that $\delta_{\underline{\Sigma_{X_i}}}(\underline{W_1}) > 0$. This yields that $\underline{W_1} \cong \underline{X_i}$ or $\underline{W_1} \cong \underline{X_i}[-1]$. Since each $\underline{X} \in \underline{\mathrm{CM}}(R)$ is periodic of period at most 2, by Proposition 2.14, $\delta_{\underline{\Sigma_{X_i}}} = \delta_{\underline{\Sigma_{X_i[-1]}}}$. Hence we have $\delta_{\underline{\Sigma_{X_i}}} = \delta_{\underline{\Sigma_{W_1}}}$ for the AR triangle $\underline{\Sigma_{W_1}}$ of $\underline{W_1}$.

Let $\underline{f}$ be the morphism $\underline{U} \to \underline{W_1}$ in the triangle $\underline{\Sigma}$. Take the AR triangle $\underline{\Sigma_{W_1}}$ of $\underline{W_1}$ and construct a pullback diagram:

$$\begin{tikzcd}
\tau\underline{W_1} \arrow[r] \arrow[d, equal] & \underline{E_{W_1}} \arrow[r] & \underline{W_1} \arrow[r] & \tau\underline{W_1}[1] \arrow[d, equal] \\
\tau\underline{W_1} \arrow[r] & \underline{E} \arrow[r] \arrow[u] & \underline{U} \arrow[r] \arrow[u, "\underline{f}"'] & \tau\underline{W_1}[1].
\end{tikzcd}$$

Since $\underline{\Sigma}$ is without isomorphisms, $\underline{f}$ is not isomorphism. By the property of an AR triangle (ART3), the bottom triangle splits, so that $\underline{E} \cong \underline{U} \oplus \tau\underline{W_1}$. Then we have a new triangle:

$$\underline{\Psi} : \underline{U} \oplus \tau\underline{W_1} \to \underline{E_{W_1}} \oplus \underline{U} \to \underline{W_1} \to (\underline{U} \oplus \tau\underline{W_1})[1].$$

Applying Lemma 4.7 to the triangles $\underline{\Sigma}$ and $\underline{\Psi}$, we get

$$\underline{\Theta} : \underline{U} \oplus \tau\underline{W_1} \to \underline{W_2} \oplus \underline{E_{W_1}} \to \underline{V} \to (\underline{U} \oplus \tau\underline{W_1})[1].$$

It is easy to see that we have the following equality

$$\delta_{\underline{\Theta}}(\underline{X}) = \delta_{\underline{\Sigma}}(\underline{X}) + \delta_{\underline{\Psi}}(\underline{X}) = \delta_{\underline{\Sigma}}(\underline{X}) + \delta_{\underline{\Sigma_{W_1}}}(\underline{X}) = \delta_{\underline{\Sigma}}(\underline{X}) + \delta_{\underline{\Sigma_{X_i}}}(\underline{X})$$

for each $\underline{X} \in \underline{\mathrm{CM}}(R)$. Therefore $\delta_{\underline{\Sigma}} < \delta_{\underline{\Theta}} \leq \delta_{\underline{M},\underline{N}}$. Repeating this procedure, we obtain the required triangle noting that this process stops since R is of finite representation type. □

Proof of Theorem 4.6. As mentioned in Remark 4.5, $\underline{M} \leq_{st} \underline{N}$ implies that $\underline{M} \leq_{\underline{hom}} \underline{N}$.

To show the converse, we assume that $\underline{M} \leq_{\underline{hom}} \underline{N}$ and $\underline{M}$ and $\underline{N}$ have no common non-zero direct summand. For each indecomposable Cohen-Macaulay module X, we set $r(\underline{X}) = min\{\delta'_{\underline{M},\underline{N}}(\underline{X}), \mu(\underline{X}, \underline{N})\}$ where $\delta'_{\underline{M},\underline{N}}(\underline{X}) := \delta_{\underline{M},\underline{N}}(\underline{X})/\delta_{\underline{\Sigma_X}}(\underline{X})$. We consider the following set of isomorphism classes of (non free) indecomposable Cohen-Macaulay modules:

$$\mathcal{F} = \{\underline{X} \mid r(\underline{X}) > 0\}/\cong.$$

Now we consider a subset $\mathcal{G}$ of $\mathcal{F}$ consisting of modules chosen by the following manner.

- If $\underline{X} \in \mathcal{F}$ and $\underline{X}[-1] \notin \mathcal{F}$, $\underline{X}$ belongs to $\mathcal{G}$.
- Assume that $\underline{X}$ is such that both $\underline{X}$ and $\underline{X}[-1]$ belong to $\mathcal{F}$.
  - If $r(\underline{X}[-1]) \leq r(\underline{X})$, $\underline{X}$ belongs to $\mathcal{G}$.
  - If $r(\underline{X}) < r(\underline{X}[-1])$, $\underline{X}[-1]$ belongs to $\mathcal{G}$.

Let $N_1 = \bigoplus_{\underline{X}\in\mathcal{G}} X^{r(\underline{X})}$, $N_2 = \bigoplus_{\underline{X}\in\mathcal{G}} X^{\mu(\underline{X},\underline{N})-r(\underline{X})}$ and $N_3 = \bigoplus_{\underline{X}\notin\mathcal{G}} X^{\mu(\underline{X},\underline{N})}$. Then $\underline{N_1}\oplus\underline{N_2}\oplus\underline{N_3}\cong\underline{N}$ in $\underline{\mathrm{CM}}(R)$. Since R is an isolated singularity, we can take an Auslander-Reiten triangle:

$$\underline{\Sigma_X} : \tau\underline{X} \to \underline{E_X} \to \underline{X} \to \tau\underline{X}[1].$$

Consider a triangle $\underline{\Sigma}$ which is a direct sum of $r(\underline{X})$ copies of triangles $\underline{\Sigma_X}$ where X runs through all modules in $\mathcal{G}$:

$$\underline{\Sigma} : \bigoplus_{\underline{X}\in\mathcal{G}} (\tau\underline{X})^{r(\underline{X})} \to \bigoplus_{\underline{X}\in\mathcal{G}} \underline{E_X}^{r(\underline{X})} \to \bigoplus_{\underline{X}\in\mathcal{G}} \underline{X}^{r(\underline{X})} \to \bigoplus_{\underline{X}\in\mathcal{G}} (\tau\underline{X}[1])^{r(\underline{X})}.$$

Here we note that $\bigoplus_{\underline{X}\in\mathcal{G}} \underline{X}^{r(\underline{X})} \cong \underline{N_1}$. First we claim that

Claim 1: $\underline{N}$ is isomorphic to a direct summand of $\bigoplus_{\underline{X}\in\mathcal{G}}(\underline{X}\oplus\tau\underline{X})^{r(\underline{X})}$.

According to Proposition 3.3, we have the following irredundant expression in $\mathrm{G}(\underline{\mathrm{CM}}(R))$:

$$[\underline{N}] - [\underline{M}] = \sum_{i\in\mathcal{I}} \delta'_{\underline{M},\underline{N}}(\underline{X_i})\cdot([\underline{X_i}] + [\tau\underline{X_i}] - [\underline{E_{X_i}}]). \tag{4.1}$$

Since M and N have no common direct summand, $\underline{N}$ is isomorphic to a direct summand of $\bigoplus_{i\in\mathcal{I}}(\underline{X_i}\oplus\tau\underline{X_i})^{\delta'_{\underline{M},\underline{N}}(\underline{X_i})}$. Thus, for each indecomposable direct summand $\underline{N'}$ of $\underline{N}$, there exists i such that $\underline{N'}\cong\underline{X_i}$ or $\tau\underline{X_i}$. Since each indecomposable Cohen-Macaulay R-module is periodic of period at most 2, $\tau\underline{X_i}\cong\underline{X_i}$ or $\underline{X_i}[-1]$ (Remark 2.6). Combining Proposition 2.14 with Remark 4.10, we also have $\delta'_{\underline{M},\underline{N}}(\underline{X_i}) = \delta'_{\underline{M},\underline{N}}(\underline{X_i}[-1])$. This yields that $r(\underline{N'})>0$. Hence $\underline{N'}\in\mathcal{F}$.

If $\underline{N'}\ncong\tau\underline{N'}$ then $\mu(\underline{N'},\underline{N})\le\delta'_{\underline{M},\underline{N}}(\underline{N'})$, so that $\mu(\underline{N'},\underline{N}) = r(\underline{N'})$. If $\underline{N'}\cong\tau\underline{N'}$ then $\mu(\underline{N'},\underline{N})\le 2\delta'_{\underline{M},\underline{N}}(\underline{N'})$, so that $\mu(\underline{N'},\underline{N})\le 2r(\underline{N'})$. Hence, if $\underline{N'}\in\mathcal{G}$, $\underline{N'}^{\mu(\underline{N'},\underline{N})}$ is isomorphic to a direct summand of $\bigoplus_{\underline{X}\in\mathcal{G}}(\underline{X}\oplus\tau\underline{X})^{r(\underline{X})}$. Suppose that $\underline{N'}\notin\mathcal{G}$. Then $\underline{N'}[-1]$ belongs to $\mathcal{G}$ by the definition of $\mathcal{G}$. In this case $r(\underline{N'}) < r(\underline{N'}[-1])$. As mentioned in Remark 3.2(2), $\underline{\tau}(\underline{N'}[-1])\cong(\underline{N'}[-1])[-1]\cong\underline{N'}[-2]\cong\underline{N'}$. Thus $\underline{N'}^{\mu(\underline{N'},\underline{N})}$ is isomorphic to a direct summand of $(\underline{N'}[-1]\oplus\underline{\tau N'}[-1])^{r(\underline{N'}[-1])}$. Hence it is also isomorphic to a direct summand of $\bigoplus_{\underline{X}\in\mathcal{G}}(\underline{X}\oplus\tau\underline{X})^{r(\underline{X})}$. Consequently, the claim holds. $\square$

For the triangle $\underline{\Sigma}$, we have an inequality $\delta_{\underline{\Sigma}}(\underline{X}) = \sum_{\underline{X}\in\mathcal{G}} r(\underline{X})\delta_{\underline{\Sigma_X}}(\underline{X}) \le \delta_{\underline{M},\underline{N}}(\underline{X})$ for each $X\in\mathrm{CM}(R)$. By virtue of Lemma 4.11, we have a triangle $\underline{\Phi} : \underline{Z}\to\underline{Y}\to\underline{N_1}\to\underline{Z}[1]$ such that $\delta_{\underline{\Sigma}}\le\delta_{\underline{\phi}}\le\delta_{\underline{M},\underline{N}}$ and $\delta_{\underline{\Phi}}(\underline{Y}) = \delta_{\underline{M},\underline{N}}(\underline{Y})$. Next we claim that

Claim 2: $\delta_{\underline{\Phi}}(\underline{U}) = \delta_{\underline{M},\underline{N}}(\underline{U})$ for each $U\in\mathrm{CM}(R)$.

Note that $\sum_{\underline{X}\in\mathcal{G}} r(\underline{X})([\underline{X}]+[\tau\underline{X}]-[\underline{E_X}])$ is a constituent of (4.1) since $\delta'_{\underline{M},\underline{N}}(\underline{X})\ge r(\underline{X})>0$ for each $\underline{X}\in\mathcal{G}$. Seeing the proof of Lemma 4.11, $[\underline{N_1}]+[\underline{Z}]-[\underline{Y}]$ can be also taken as one, say

$$[\underline{N}] - [\underline{M}] = [\underline{N_1}] + [\underline{Z}] - [\underline{Y}] + \sum_{i'\in\mathcal{I}'} \delta'_{\underline{M},\underline{N}}(X_{i'})\cdot([\underline{X_{i'}}] + [\tau\underline{X_{i'}}] - [\underline{E_{X_{i'}}}])$$

in $\mathrm{G}(\underline{\mathrm{CM}}(R))$. Since $\delta_{\underline{\Phi}}(\underline{Y}) = \delta_{\underline{M},\underline{N}}(\underline{Y})$ for each indecomposable direct summand Y' of Y, $\underline{Y'}\ncong\underline{X_{i'}}$ and $\underline{X_{i'}}[-1]$. Moreover $\underline{Y'}\ncong\tau\underline{X_{i'}}$ and $\tau\underline{X_{i'}}[-1]$ since $\tau\underline{X_{i'}}\cong\underline{X_{i'}}$ or $\underline{X_{i'}}[-1]$. This implies that $\underline{Y}$ and $\bigoplus_{i'\in\mathcal{I}'}(\underline{X_{i'}}\oplus\tau\underline{X_{i'}})^{\delta'_{\underline{M},\underline{N}}(\underline{X_{i'}})}$ have no common direct summand. We remark that

$$\underline{N}\oplus\underline{Y}\oplus\bigoplus_{i'\in\mathcal{I}'}\underline{E_{X_{i'}}}^{\delta'_{\underline{M},\underline{N}}(\underline{X_{i'}})} \cong \underline{M}\oplus\underline{N_1}\oplus\underline{Z}\oplus\bigoplus_{i'\in\mathcal{I}'}(\underline{X_{i'}}\oplus\tau\underline{X_{i'}})^{\delta'_{\underline{M},\underline{N}}(\underline{X_{i'}})}.$$

By the construction of $\underline{\Phi}$, $\underline{Z}$ contains $\bigoplus_{\underline{X}\in\mathcal{G}}\tau\underline{X}$ as a direct summand. Thus, by *Claim 1*, one can show that $\underline{N}$ is isomorphic to a direct summand of $\underline{N_1}\oplus\underline{Z}$. Hence, $\bigoplus_{i'\in\mathcal{I}'}(\underline{X_{i'}}\oplus\tau\underline{X_{i'}})^{\delta'_{\underline{M},\underline{N}}(\underline{X_{i'}})}$ is isomorphic to a direct summand of $\bigoplus_{i'\in\mathcal{I}'}\underline{E_{X_{i'}}}^{\delta'_{\underline{M},\underline{N}}(\underline{X_{i'}})}$. By using the same arguments in Remark 3.2(2), we have

$$\bigoplus_{i'\in\mathcal{I}'}\underline{E_{X_{i'}}}^{\delta'_{\underline{M},\underline{N}}(\underline{X_{i'}})}\cong\bigoplus_{i'\in\mathcal{I}'}(\underline{X_{i'}}\oplus\tau\underline{X_{i'}})^{\delta'_{\underline{M},\underline{N}}(\underline{X_{i'}})}.$$

Hence $\underline{N}\oplus\underline{Y}\cong\underline{M}\oplus\underline{N_1}\oplus\underline{Z}$, so that $\delta_{\underline{\Phi}}=\delta_{\underline{M},\underline{N}}$. □

Since $0\le\delta_{\underline{M},\underline{N}}-\delta_{\underline{\Phi}}=\delta_{\underline{M}\oplus\underline{Z},\underline{N_2}\oplus\underline{N_3}\oplus\underline{Y}}$, we see that $\underline{M}\oplus\underline{Z}\le_{\underline{hom}}\underline{N_2}\oplus\underline{N_3}\oplus\underline{Y}$. Moreover *Claim 2* implies that $\delta_{\underline{M}\oplus\underline{Z},\underline{N_2}\oplus\underline{N_3}\oplus\underline{Y}}(\underline{N_2}\oplus\underline{N_3}\oplus\underline{Y})=\delta_{\underline{M},\underline{N}}(\underline{N_2}\oplus\underline{N_3}\oplus\underline{Y})-\delta_{\underline{\Phi}}(\underline{N_2}\oplus\underline{N_3}\oplus\underline{Y})=0$. By Proposition 3.6, we have $\underline{M}\oplus\underline{Z}\cong\underline{N_2}\oplus\underline{N_3}\oplus\underline{Y}$. Therefore, the triangle $\underline{\Phi}:\underline{Z}\to\underline{Y}\to\underline{N_1}\to\underline{Z}[1]$ induces a triangle

$$\underline{Z}\to\underline{N_2}\oplus\underline{N_3}\oplus\underline{Y}\cong\underline{M}\oplus\underline{Z}\to\underline{N_1}\oplus\underline{N_2}\oplus\underline{N_3}\cong\underline{N}\to\underline{Z}[1].$$

This makes the stable degeneration $\underline{M}\le_{st}\underline{N}$. □

Remark 4.12. It is known that hypersurfaces which are of finite representation type are simple singularities of type (A_n), (D_n), (E_6), (E_7) or (E_8). The Cohen-Macaulay modules are classified and the AR quivers are also described (cf. [14, 10]). By the classification theorem, we obtain the list of singularities which satisfy the assumption of Theorem 4.6:

dimension	singularities
odd	A_n
even	D_{2n}, E_7, E_8

We end this paper by giving the description of stable degenerations of Cohen-Macaulay modules over simple singularities of type (A_n).

Example 4.13. Let $R=k[[x,y]]/(x^{n+1}+y^2)$. Since the stable hom order coincides with the stable degeneration order, we have the following.

(1) If n is an even integer,

$$\underline{0}\le_{st}\underline{I_1}\le_{st}\underline{I_2}\le_{st}\cdots\le_{st}\underline{I_{n/2}}.$$

(2) If n is an odd integer,

$$\underline{0}\le_{st}\underline{I_1}\le_{st}\underline{I_2}\le_{st}\cdots\le_{st}\underline{I_{(n-1)/2}}\le_{st}\underline{N_+}\oplus\underline{N_-}.$$

and

$$\underline{N_\pm}\le_{st}\underline{N_\pm}\oplus\underline{I_1}\le_{st}\cdots\le_{st}\underline{N_\pm}\oplus\underline{I_{(n-1)/2}}\le_{st}\underline{N_\pm}\oplus\underline{N_+}\oplus\underline{N_-}\quad\text{(double sign corresponds).}$$

See also Proposition 2.12 and Remark 4.8.

On Example 4.13, the author also investigate the case that the dimension is even in [7]. The essential part is the following.

Proposition 4.14. [7, Corollary 2.12., Proposition 3.10.] *Let $R=k[[x]]/(x^{n+1})$. Then the stable degeneration order coincides with the triangle order on $\underline{\mathrm{CM}}(R)$.*

Theorem 4.15. *Let R be a simple singularity of type (A_n). For Cohen-Macaulay R-modules M and N with $[\underline{M}]=[\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$, the following statements hold.*

(1) *If R is of odd dimension then the stable degeneration order coincides with the stable hom order.*
(2) *If R is of even dimension, then the stable degeneration order coincides with the triangle order.*

Proof. By virtue of Knörrer's periodicity (cf. [14, Theorem 12.10]), we have only to deal with the case $\dim R=1$ to show (1) and the case $\dim R=0$ to show (2). Hence, by Theorem 4.6 and Proposition 4.14, we obtain the assertion. □

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