arXiv:1710.00086v1 [math.CO] 29 Sep 2017

On diregular digraphs with degree two and excess three

James Tuite

Open University, Walton Hall, Milton Keynes

Abstract

Moore digraphs, that is digraphs with out-degree d, diameter k and order equal to the Moore bound $M(d,k) = 1 + d + d^2 + \ldots + d^k$, arise in the study of optimal network topologies. In an attempt to find digraphs with a 'Moore-like' structure, attention has recently been devoted to the study of small digraphs with minimum out-degree d such that between any pair of vertices u, v there is at most one directed path of length $\leq k$ from u to v; such a digraph has order $M(d,k) + \epsilon$ for some small excess ϵ. Sillasen et al. have shown that there are no digraphs with out-degree two and excess one [26, 23]. The present author has classified all digraphs with out-degree two and excess two [27, 28]. In this paper it is proven that there are no diregular digraphs with out-degree two and excess three for $k \geq 3$, thereby providing the first classification of digraphs with order three away from the Moore bound for a fixed out-degree.

Keywords: Degree/diameter problem, Digraphs, Excess, k-geodetic, Extremal digraphs
2000 MSC: 05C35, 90C35

1. Introduction

The undirected degree/diameter problem asks for the largest possible order of a graph G with given maximum degree d and diameter k. This problem has applications in the design of efficient networks. A natural upper bound on the order of such a graph is

$$|V(G)| \leq 1 + d + d(d-1) + d(d-1)^2 + \ldots + d(d-1)^{k-1},$$

where the right-hand side of the inequality is the (undirected) *Moore bound*. A graph is *Moore* if it attains this upper bound. A graph is Moore if and only if it is regular with degree d, has diameter k and girth $2k+1$. The girth condition implies that a Moore graph is k-*geodetic*, i.e. any two vertices are connected by at most one path of length not exceeding k. In the classic paper [18] Hoffman and Singleton show that for diameter $k = 2$ the Moore bound is achieved only for degrees $d = 2, 3, 7$ and possibly 57. The unique Moore graphs for $k = 2$ and $d = 2, 3$ and 7 are the 5-cycle, the Petersen graph and the Hoffman-Singleton graph respectively. The existence of a Moore graph (or graphs) with diameter $k = 2$ and degree $d = 57$ is a famous open problem. It was later shown by other authors [11, 2] that for diameters $k \geq 3$ Moore graphs exist only in the trivial case $d = 2$.

Given the scarcity of Moore graphs, it is of great interest to find graphs with a 'Moore-like' structure. A survey of this problem is given in [22]. Graphs with maximum degree d, diameter k and order δ less than the Moore bound for some small *defect* δ have been studied intensively. In such graphs paths with length $\leq k$ between pairs of vertices are not necessarily unique; associated with each vertex u is a repeat multiset $R(u)$, such that $v \in V(G)$ appears t times in $R(u)$ if and

Email address: `james.tuite@open.ac.uk` (James Tuite)

only if there are $t+1$ distinct $\leq k$-paths between u and v. An important result in this direction is that the only graphs with defect one are cycles of length $2k$ [12, 3, 19].

Alternatively, one can preserve the k-geodecity condition and ask for the smallest d-regular graphs with girth $2k+1$. This is known as the degree/girth problem. A survey of this problem is given in [14]. A graph with minimal order subject to the above conditions is called a *cage*.

The directed version of the degree/diameter problem was posed in [7]. The Moore bound for a digraph with maximum out-degree d and diameter k is given by

$$M(d,k) = 1 + d + d^2 + \ldots + d^k.$$

Similarly to the undirected case, a digraph is Moore if and only if it is out-regular with degree d, has diameter k and is k-geodetic, i.e. for any (ordered) pair of vertices u, v there is at most one directed path from u to v with length $\leq k$. Using spectral analysis, it was shown in [7] that Moore digraphs exist only in the trivial cases $d = 1$ and $k = 1$, the Moore digraphs being directed cycles of length $k+1$ and complete digraphs of order $d+1$ respectively.

There is an extensive literature on digraphs with maximum out-degree d, diameter k and order $M(d,k) - \delta$ for small defects δ. Such digraphs arise from removing the k-geodecity condition in the requirements for a digraph to be Moore. As in the undirected case, each vertex u is associated with a repeat multiset $R(u)$, defined in the obvious manner. A digraph with defect $\delta = 1$ is an *almost Moore digraph*; for such a digraph, in place of a set-valued function R, we can think of a repeat function $r : V(G) \to V(G)$. In contrast to the undirected problem, for diameter $k = 2$ there exists an almost Moore digraph for every value of d [15]. It is known that there are no almost Moore digraphs with $d = 2$ and $k \geq 3$ [20], $d = 3$ and $k \geq 3$ [5] or diameters $k = 3$ and 4 [8, 9, 10]. It is also shown in [21] that there are no digraphs with degree $d = 2$ and defect $\delta = 2$ for diameters $k \geq 3$.

Approaching the problem of approximating Moore digraphs from a different perspective, there are several different ways to adapt the undirected degree/girth problem to the directed case, as the connection between k-geodecity and the girth does not hold in the directed setting. The directed degree/girth problem, which concerns the minimisation of the order of out-regular digraphs with given girth, is well developed (see [24] for an introduction). A related problem is considered in [1]. However, the extremal digraphs considered in these problems are in general not k-geodetic; in fact, in the directed degree/girth problem, it is conjectured that extremal orders are achieved by circulant digraphs [6].

If we wish to retain the k-geodecity condition, but relax the requirement that the diameter should equal k, we obtain the following problem: *What is the smallest possible order of a k-geodetic digraph with minimum degree d?* A k-geodetic digraph G with minimum out-degree d and order $M(d,k) + \epsilon$ is called a $(d,k,+\epsilon)$-digraph, where $\epsilon > 0$ is the *excess* of G. With each vertex u of a $(d,k,+\epsilon)$-digraph we can associate the set $O(u) = \{v \in V(G) : d(u,v) \geq k+1\}$ of vertices that cannot be reached by $\leq k$-paths from u; any element of this set is an *outlier* of u. It is known that $(d,k,+1)$-digraphs are out-regular with degree d [26]. For digraph G with excess $\epsilon = 1$, the set-valued function O can be construed as an outlier function o, where for each vertex u of G the outlier $o(u)$ of u is the unique vertex of G with $d(u, o(u)) \geq k+1$. We will refer to a $(d,k,+\epsilon)$-digraph with smallest possible excess as a (d,k)-geodetic-cage.

The first paper to consider this problem was [26], in which Sillasen proves that there are no diregular $(2,k,+1)$-digraphs for $k \geq 2$. Strong conditions on non-diregular digraphs with excess one were also derived in this paper. These results were later strengthened [23] to show that any digraph with excess $\epsilon = 1$ must be diregular, thereby completing the proof of the nonexistence of $(2,k,+1)$-digraphs. It is also known that $(d,k,+1)$-digraphs do not exist for $k = 2$ and $d > 7$ [23]

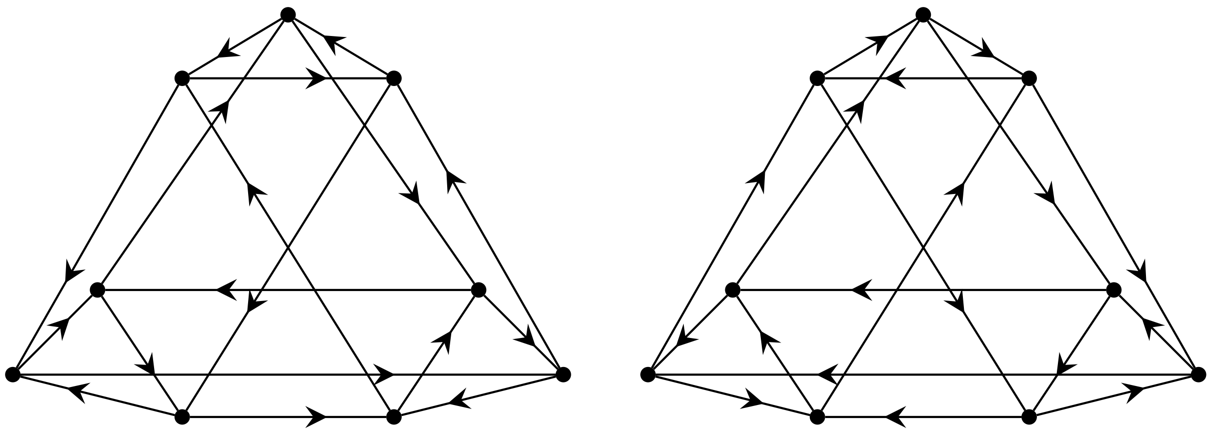

Figure 1: The two $(2,2)$-geodetic-cages

or $k = 3, 4$ for $d > 1$. In [13] it is shown that for all d and k there exists a diregular k-geodetic digraph with degree d, so that geodetic cages exist for all values of d and k, and that for fixed k the Moore bound can be approached asymptotically by arc-transitive k-geodetic digraphs as $d \to \infty$. Some small k-geodetic digraphs are constructed in the same paper and lower bounds for particular values of d and k established.

In [27] the present author has proven that for $k \geq 2$ any $(2, k, +2)$-digraphs must be diregular. Using an approach similar to that of [21] this analysis was completed in [28] by showing that there are no diregular $(2, k, +2)$-digraphs for $k \geq 3$ and classifying the diregular $(2, 2, +2)$-digraphs up to isomorphism. There are exactly two $(2, 2, +2)$-digraphs, which are displayed in Figure 1; these represent the only known non-trivial geodetic cages. New results have allowed the method of [28] to be extended to excess $\epsilon = 3$. In this paper, we therefore present a complete classification of diregular $(2, k, +3)$-digraphs for $k \geq 3$.

2. The Neighbourhood Lemma

Let us first establish our notation. G will stand for a diregular $(d, k, +\epsilon)$-digraph, i.e. a diregular digraph with degree d and order $M(d, k) + \epsilon = 1 + d + \ldots + d^k + \epsilon$ that is *k-geodetic*, so that for all $u, v \in V(G)$ if there is a path P from u to v of length $\leq k$ then it is the unique such path. For vertices u, v we will write $u \to v$ to indicate that there is an arc from u to v in G. The set of out-neighbours of a vertex u of G is $N^+(u) = \{v \in V(G) : u \to v\}$; similarly $N^-(u) = \{v \in V(G) : v \to u\}$ is the set of in-neighbours of u. More generally, for $l > 0$ $N^l(u)$ will stand for the set of vertices that are end-points of paths of length l with initial point u and $N^{-l}(u)$ for the set of vertices that are the initial points of l-paths that terminate at u. Trivially $N^0(u) = \{u\}, N^1(u) = N^+(u)$ and $N^{-1}(u) = N^-(u)$. For $0 \leq l \leq k$ the set of vertices that lie within a distance l from a vertex u will be denoted by $T_l(u)$; hence $T_l(u) = \cup_{i=0}^{l} N^i(u)$. The set $T_{k-1}(u)$ will be written as $T(u)$ for short and will be indicated in diagrams by a triangle based at the vertex u. For each vertex u of G there are exactly ϵ vertices that lie at distance $\geq k + 1$ from u; the set $O(u)$ of these ϵ vertices is the *outlier set* of u and each element of $O(u)$ is an *outlier* of u. We have $O(u) = V(G) - T_k(u)$. If S is a set of vertices of G, then we define $N^+(S)$ to be the multiset $\cup_{v \in S} N^+(v)$ and $O(S)$ to be the multiset $\cup_{v \in S} O(v)$.

For digraphs with order close to the Moore bound there is a useful interplay between the combinatorial notions of repeat and outlier and the symmetries of the digraph. For digraphs with defect $\delta = 1$, the repeat function r was shown to be a digraph automorphism in [4] by a counting argument.

This can also be proven by a short matrix argument [16]. In her thesis [25] Sillasen extended this result for almost Moore digraphs to digraphs with larger defects, showing that for any vertex u in a diregular digraph with defect $\delta \geq 2$ the multiset equation $N^+(R(u)) = R(N^+(u))$ holds. This relationship is known as the Neighbourhood Lemma.

In [26] Sillasen demonstrated that there is a strong analogy between the structure of almost Moore digraphs and digraphs with excess $\epsilon = 1$ by proving, by an argument similar to that presented in [16], that the outlier function o of a diregular $(d, k, +1)$-digraph is an automorphism. We now complete this line of reasoning by showing that a Neighbourhood Lemma holds for digraphs with small excess $\epsilon \geq 2$.

Lemma 1 (Neighbourhood Lemma)**.** *Let G be a diregular $(d, k, +\epsilon)$-digraph for any $d, k \geq 2$ and $\epsilon \geq 1$. Then for any vertex u of G we have $O(N^+(u)) = N^+(O(u))$ as multisets.*

Proof. As G is diregular, any vertex can occur at most d times in either multiset. Suppose that a vertex v occurs t times in $N^+(O(u))$. Let $N^-(v) = \{v_1, v_2, \dots, v_t, v_{t+1}, \dots, v_d\}$ and $N^+(u) = \{u_1, u_2, \dots, u_d\}$, where $O(u) \cap N^-(v) = \{v_1, v_2, \dots, v_t\}$. Suppose that $u \notin N^-(v)$. Since no set $T(u_i)$ contains more than one in-neighbour of v by k-geodecity, there are exactly $d - t$ out-neighbours of u that can reach v by a $\leq k$-path, so that v occurs t times in $O(N^+(u))$. A similar argument deals with the case $u \in N^-(v)$. As both multisets have size $d\epsilon$, this implies the result. $\square$

It is pleasing to regard the Neighbourhood Lemma for diregular digraphs with small excess as a limiting case of Lemmas 2 and 3 of [27] for non-diregular digraphs.

3. Main Result

For the remainder of this paper G will be a diregular $(2, k, +3)$-digraph for some $k \geq 3$. Geodetic cages for degree $d = 2$ and $k = 2$ have been found to have excess two [27, 28]; for completeness, we mention that there are $(2, 2, +3)$-digraphs, both diregular and non-diregular. We will now complete the classification of diregular $(2, k, +3)$-digraphs by showing that for $k \geq 3$ diregular $(2, k, +\epsilon)$-digraphs have excess $\epsilon \geq 4$. Our argument for $k = 3$ is too lengthy to include here, so we will merely state the following Theorem.

Theorem 1. *There are no diregular $(2, 3, +3)$-digraphs.*

As a diregular $(2, 3, +5)$-digraph is constructed in [13], extremal diregular $(2, 3, +\epsilon)$-digraphs have excess 4 or 5.

We employ the following labelling convention for vertices at distance $\leq k$ from a vertex u of G. The out-neighbours of u will be labelled according to $N^+(u) = \{u_1, u_2\}$ and vertices at a greater distance from u are labelled inductively as follows: $N^+(u_1) = \{u_3, u_4\}, N^+(u_2) = \{u_5, u_6\}, N^+(u_3) = \{u_7, u_8\}$ and so on. Since the vertex u_{45} will play a part in our argument, the reader is urged to familiarise themselves with this scheme. See Figure 2 for an example.

A first step in previous studies [20, 28, 21] of digraphs with degree two and order close to the Moore bound has been to establish the existence of a pair of vertices with exactly one out-neighbour in common. The argument of [28] can be generalised to show that for degree two such a pair exists for any even excess ϵ. For $\epsilon = 3$, we can establish the existence of the necessary pair as follows.

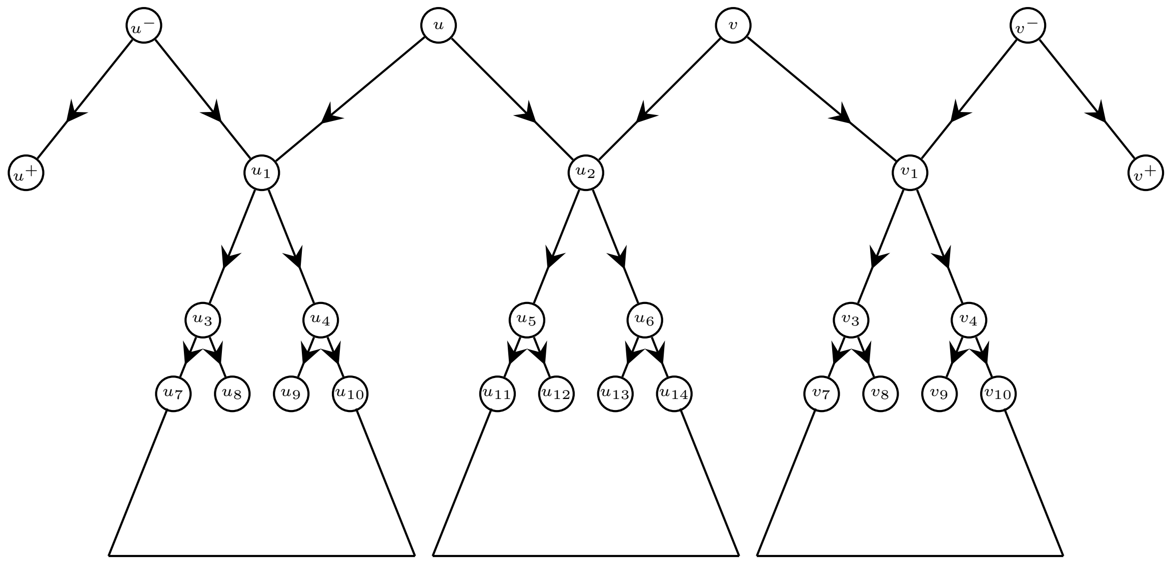

Figure 2: Configuration for $k \geq 3$

Theorem 2. *For $k \geq 3$, any diregular $(2, k, +3)$-digraph G contains a pair of vertices u, v with exactly one common out-neighbour.*

Proof. Let G be a diregular $(2, k, +3)$-digraph without the required pair of vertices. Then all out-neighbourhoods are either disjoint or identical. Then by Heuchenne's condition G is the line digraph of a digraph H with degree two [17]. H must be at least $(k-1)$-geodetic. As $2|V(H)| = |V(G)|$, H must be a $(2, k-1, +2)$-digraph. Since the line digraphs of the $(2, 2)$-geodetic-cages are not 3-geodetic and there are no $(2, k, +2)$-digraphs for $k \geq 3$ [28], we have a contradiction. $\square$

There is no guarantee that distinct vertices do not have identical out-neighbourhoods; witness the geodetic-cage on the left of Figure 1. However, we can say a great deal about the outlier sets of such vertices. The proof of the following lemma is practically identical to that of the corresponding result for $\epsilon = 2$ in [28] and is omitted.

Lemma 2. *Let z, z' be vertices of a $(d, k, +\epsilon)$-digraph H for some $\epsilon \geq 1$. If $N^+(z) = N^+(z')$, then there exists a set X of $\epsilon - 1$ vertices of H such that $O(z) = \{z'\} \cup X, O(z') = \{z\} \cup X$.*

We now fix an arbitrary pair of vertices u, v of G with a unique out-neighbour in common. We will assume that $u_2 = v_2$, so that, following the vertex labelling convention established earlier, we have the situation shown in Figure 2. We will also write $N^-(u_1) = \{u, u^-\}, N^-(v_1) = \{v, v^-\}, N^+(u^-) = \{u_1, u^+\}$ and $N^+(v^-) = \{v_1, v^+\}$. It is easily seen that $u^- \neq v, v^- \neq u$.

We can make some immediate deductions concerning the position of the vertices u, v and u_2 in the diagram in Figure 2.

Lemma 3. *$v \in N^{k-1}(u_1) \cup O(u)$ and $u \in N^{k-1}(v_1) \cup O(v)$. If $v \in O(u)$, then $u_2 \in O(u_1)$ and if $u \in O(v)$, then $u_2 \in O(v_1)$.*

Proof. v cannot lie in $T(u)$, or the vertex u_2 would be repeated in $T_k(u)$. Also, $v \notin T(u_2)$, or there would be a $\leq k$-cycle through v. Therefore, if $v \notin O(u)$, then $v \in N^{k-1}(u_1)$. Likewise for the other result. If $v \in O(u)$, then neither in-neighbour of u_2 lies in $T(u_1)$, so that $u_2 \in O(u_1)$. $\square$

The following lemma is the main tool in our analysis.

Lemma 4 (Contraction Lemma)**.** *Let $w \in T(v_1)$, with $d(v_1, w) = l$. Suppose that $w \in T(u_1)$, with $d(u_1, w) = m$. Then either $m \leq l$ or $w \in N^{k-1}(u_1)$. A similar result holds for $w \in T(u_1)$.*

Proof. Let w be as described and suppose that $m > l$. Consider the set $N^{k-m}(w)$. By construction, $N^{k-m}(w) \subseteq N^k(u_1)$, so by k-geodecity $N^{k-m}(w) \cap T(u_1) = \varnothing$. At the same time, we have $l+k-m \leq k-1$, so $N^{k-m}(w) \subseteq T(v_1)$. This implies that $N^{k-m}(w) \cap T(v_2) = N^{k-m}(w) \cap T(u_2) = \varnothing$. As $V(G) = \{u\} \cup T(u_1) \cup T(u_2) \cup O(u)$, it follows that $N^{k-m}(w) \subseteq \{u\} \cup O(u)$. Therefore $|N^{k-m}(w)| = 2^{k-m} \leq 4$, so either $m = k-1$ or $m = k-2$. Suppose that $m = k-2$; then $N^2(w) = \{u\} \cup O(u)$. Neither v nor v_1 lies in $N^2(w)$, so that neither v nor v_1 lies in $O(u)$. By k-geodecity and Lemma 3, $v \in N^{k-1}(u_1)$ and $v_1 \in T(u_1)$, so that v_1 appears twice in $T_k(u_1)$. Thus $m = k-1$. □

Corollary 1. *If $w \in T(v_1)$, then either $w \in \{u\} \cup O(u)$ or $w \in T(u_1)$ with $d(u_1, w) = k-1$ or $d(u_1, w) \leq d(v_1, w)$.*

This allows us to restrict the possible positions of u_1 and v_1 in Figure 2.

Corollary 2. *$v_1 \in N^{k-1}(u_1) \cup O(u)$ and $u_1 \in N^{k-1}(v_1) \cup O(v)$.*

Proof. We prove the first inclusion. By Corollary 1, $v_1 \in \{u\} \cup O(u) \cup \{u_1\} \cup N^{k-1}(u_1)$. By k-geodecity $v_1 \neq u$ and by construction $v_1 \neq u_1$. □

Corollary 3. *If $v_1 \notin O(u)$, then $O(u) = \{v, v_3, v_4\}$, with a similar result for v.*

Proof. Similar to the corresponding result in [28]. □

Lemma 5. *For $k \geq 3$, either $v_1 \in O(u)$ or $u_1 \in O(v)$.*

Proof. Suppose that $O(u) = \{v, v_3, v_4\}, O(v) = \{u, u_3, u_4\}$. By the Neighbourhood Lemma,

$$O(\{u_1, u_2\}) = O(N^+(u)) = N^+(O(u)) = \{u_2, v_1, v_7, v_8, v_9, v_{10}\}$$

and

$$O(\{v_1, u_2\}) = O(N^+(v)) = N^+(O(v)) = \{u_2, u_1, u_7, u_8, u_9, u_{10}\}.$$

By Corollary 2, $v_1 \in N^{k-1}(u_1)$ and $u_1 \in N^{k-1}(v_1)$, so we must have $u_1, v_1 \in O(u_2)$. As $O(u_2) \subset N^+(O(u))$, it follows that $u_1 \in N^2(v_1)$, so $k \leq 3$. Now set $k = 3$. We can put $u_9 = v_1, v_9 = u_1$. As $N^2(v_1) \cap O(u) = \varnothing$ and $u \notin N^2(v_1)$, $\{v_7, v_8, v_{10}\} = \{u_7, u_8, u_{10}\}$. $u_{10} \in N^2(v_1)$ implies that there are two distinct ≤ 3-paths from u_4 to u_{10}, contradicting 3-geodecity. □

We will now identify an outlier of u and v using the Neighbourhood Lemma.

Theorem 3. *For $k \geq 3$, $v_1 \in O(u)$ and $u_1 \in O(v)$.*

Proof. Assume for a contradiction that $O(v) = \{u, u_3, u_4\}$ and $v_1 \in O(u)$. Let $k \geq 4$. v can reach u_1 by a $\leq k$-path, so by Corollary 2 $u_1 \in N^{k-1}(v_1)$. Suppose that $x \in (T_{k-2}(u_1) - \{u_1\}) \cap N^{k-1}(v_1)$ and write $N^+(x) = \{x_1, x_2\}$. Clearly $x_1, x_2 \notin \{u, u_3, u_4\}$, so $x_1, x_2 \in T_k(v)$. However, by k-geodecity $x_1, x_2 \notin T(u_2) \cup T(v_1)$, so we are forced to conclude that $x_1 = x_2 = v$, which is absurd. It follows from the Contraction Lemma that for any vertex $w \in T_{k-2}(u_1) - \{u_1, u_3, u_4\}$ we have $d(u_1, w) = d(v_1, w)$. In particular, $N^2(u_1) = N^2(v_1)$. However, as $u_1 \in N^{k-1}(v_1)$, this implies the existence of a $(k-1)$-cycle through u_1.

Now set $k = 3$. We can put $v_9 = u_1$. $N^2(u_1) \cap O(v) = \varnothing$, so $N^2(u_1) \subset \{v, v_3, v_4, v_7, v_8, v_{10}\}$. v_4 has paths of length 3 to every vertex in $N^2(u_1)$, so $v_4, v_{10} \notin N^2(u_1)$, yielding $N^2(u_1) = \{v, v_3, v_7, v_8\}$. Without loss of generality, $u_7 = v_3$. $u_7 \not\to u_8$, so $u_8 = v$ and $N^+(v_3) = N^+(u_7) = N^+(u_4)$, which is impossible. □

The next stage of our approach is to show that exactly one member of $N^+(v_1)$ is also an outlier of u and similary for v. This will be accomplished by analysing the possible positions of u_3, u_4, v_3, v_4 in Figure 2. The possibilities are described in the following lemma.

Lemma 6. *For $k \geq 4$, $\{u_3, u_4\} \subset \{v_3, v_4\} \cup O(v)$ and $\{v_3, v_4\} \subset \{u_3, u_4\} \cup O(u)$.*

Proof. Let $u_3 \notin N^+(v_1) \cup O(v)$. By Corollary 1 and Theorem 3, $u_3 \in N^{k-1}(v_1)$. By k-geodecity, $u_7, u_8 \notin T(u_2) \cup T(v_1)$. Also for $k \geq 4$ we cannot have $v \in N^+(u_3)$. Therefore $O(v) = \{u_1, u_7, u_8\}$. Hence v can reach u_4 by a $\leq k$-path. We cannot have $u_4 \in N^{k-1}(v_1)$, or the same argument would imply that $N^+(u_4) \subset O(v) = \{u_1, u_7, u_8\}$. By Corollary 1 we can assume that $u_4 = v_4$. As $u \notin O(v)$, $u \in N^{k-1}(v_1)$. Since $u_4 = v_4$, to avoid k-cycles we must conclude that $u \in N^{k-2}(v_3)$. Likewise $u_3 \in N^{k-2}(v_3)$. However, as there is a path $u \to u_1 \to u_3$, v_3 has a $(k-2)$-path and a k-path to u_3, which violates k-geodecity. □

Firstly, we show using the Neighbourhood Lemma that $O(u)$ does not contain both out-neighbours of v_1 and vice versa.

Lemma 7. *For $k \geq 4$, $N^+(u_1) \cap N^+(v_1) \neq \varnothing$.*

Proof. Suppose that $\{u_3, u_4\}$ and $\{v_3, v_4\}$ are disjoint. Then by Theorem 3 and Lemma 6 we have $O(u) = \{v_1, v_3, v_4\}, O(v) = \{u_1, u_3, u_4\}$. The Neighbourhood Lemma yields

$$N^+(O(v)) = \{u_3, u_4, u_7, u_8, u_9, u_{10}\} = O(v_1) \cup O(u_2).$$

Recall that $N^-(u_1) = \{u^-, u\}, N^-(v_1) = \{v^-, v\}, N^+(u^-) = \{u_1, u^+\}, N^+(v^-) = \{v_1, v^+\}$. Then as $u_2 \neq u^+, v^+$, it follows by Theorem 3 that $u^+ \in O(u)$ and $v^+ \in O(v)$. If $u^+ = v_1$, then, as $T(u_2) \cap (T(u_1) \cup T(v_1)) = \varnothing$, examining $T_k(u^-)$ we see that we would have $T(u_2) \subseteq \{u^-\} \cup O(u^-)$, so that $M(2, k-1) \leq 4$, which is impossible. Without loss of generality, $u^+ = v_3, v^+ = u_3$. Then v_1 and u^- have v_3 as a unique common out-neighbour, so by Theorem 3

$$u_1 \in O(v_1) \subset \{u_3, u_4, u_7, u_8, u_9, u_{10}\},$$

which contradicts k-geodecity. □

It will now be demonstrated that u cannot reach both out-neighbours of v_1 by $\leq k$-paths, so that $O(u)$ contains exactly one out-neighbour of v_1, again with a similar result for v.

Lemma 8. *For $k \geq 4$, $N^+(u_1) \neq N^+(v_1)$.*

Proof. Let $N^+(u_1) = N^+(v_1) = \{u_3, u_4\}$. If u can reach v by a $\leq k$-path, so that $v \in N^{k-1}(u_1)$, then there would be a k-cycle through v, so $v \in O(u)$ and $u \in O(v)$. Hence by Lemmas 2 and 3, there exists a vertex x such that $O(u_1) = \{v_1, u_2, x\}$ and $O(v_1) = \{u_1, u_2, x\}$. Since $u_1, v_1 \notin T(u_2)$, $u_3, u_4 \in O(u_2)$. Applying Theorem 3 to the pairs (u, u^-) and (u, v), we see that $u^+, v_1 \in O(u)$. As $N^+(u_1) = N^+(v_1)$, we cannot have $u^+ \in \{v, v_1\}$. Therefore $O(u) = \{v, v_1, u^+\}$ and similarly $O(v) = \{u, u_1, v^+\}$.

Suppose that $u^+ = v^+$. Then u^- and v^- have a single common out-neighbour, so that $v_1 \in O(u^-), u_1 \in O(v^-)$. Hence $u_1 \in O(v) \cap O(v_1) \cap O(v^-)$. As G is diregular, a simple counting argument shows that every vertex is an outlier of exactly three distinct vertices. As $u_2 \notin \{v, v_1, v^-\}$, it follows that u_2 can reach u_1 by a k-path; likewise u_2 can reach v_1. Therefore $u^-, v^- \in N^{k-1}(u_2)$; however, as $u^+ = v^+$, this is impossible. Hence $u^+ \neq v^+$.

The Neighbourhood Lemma gives

$$N^+(O(u)) = \{v_1, u_2, u_3, u_4\} \cup N^+(u^+) = O(u_1) \cup O(u_2)$$

and

$$N^+(O(v)) = \{u_1, u_2, u_3, u_4\} \cup N^+(v^+) = O(v_1) \cup O(u_2).$$

It follows that $O(u_2)$ contains a vertex $z \in N^+(u^+) \cap N^+(v^+)$. Therefore $u^+, v^+ \notin T(u_2)$. Examining $T_k(u^-)$, we see that u^+ does not lie in $T(u_1) - \{u_1\} = T(v_1) - \{v_1\}$. As already mentioned, $u^+ \neq v, v_1$. Therefore v cannot reach u^+ by a $\leq k$-path, so $u^+ \in O(v) = \{u, u_1, v^+\}$, a contradiction. □

Since u, v was an arbitrary pair of vertices with a unique common out-neighbour, Lemmas 6, 7 and 8 imply the following result.

Corollary 4. *For $k \geq 4$, if u, v are vertices with a single out-neighbour u_2 in common, then $v_1 \in O(u), u_1 \in O(v)$ and $|O(u) \cap N^+(v_1)| = |O(v) \cap N^+(u_1)| = 1$.*

Thanks to Corollary 4 we can assume that $u_3 = v_3$, $u_4 \neq v_4$, $v_1, v_4 \in O(u)$ and $u_1, u_4 \in O(v)$. Repeated applications of Corollary 4 allow us to prove that there are no diregular $(2, k, +3)$-digraphs for $k \geq 4$ by inductively identifying outliers of u_2.

Theorem 4. *There are no diregular $(2, k, +3)$-digraphs for $k \geq 4$.*

Proof. Let $k \geq 5$. As $u_3 \in N^+(u_1) \cap N^+(v_1)$, $u_3 \in O(u_2)$. The pair (u_1, v_1) have u_3 as a unique common out-neighbour, so by Corollary 4 we can assume that $u_9 = v_9$, $u_{10} \neq v_{10}$. $u_4, v_4, u_9 \notin T(u_2)$, so $u_9 \in O(u_2)$. The pair (u_4, v_4) have u_9 as a unique common out-neighbour, so we can assume that $u_{21} = v_{21}, u_{22} \neq v_{22}$. As $u_{10}, v_{10}, u_{21} \notin T(u_2)$, $u_{21} \in O(u_2)$. Continuing further we see that $u_{45} \in O(u_2)$. In fact, it follows inductively that $O(u_2)$ contains at least $k-1$ distinct vertices, which is impossible, as G has excess $\epsilon = 3$.

Now set $k = 4$. By the foregoing reasoning, we can write $O(u_2) = \{u_3, u_9, u_{21}\}, O(u) = \{v_1, v_4, z\}, O(v) = \{u_1, u_4, z'\}$ for some vertices z, z' and assume that $u_3 = v_3, u_9 = v_9, u_{21} = v_{21}$ and that u_{22} and v_{22} have a single common out-neighbour. Trivially $u, v, u_1, v_1, u_4, v_4 \notin O(u_2)$. Taking into account adjacencies among u, v, u_1 and v_1, we can assume that $u_{23} \to u, u_{25} \to v_1, u_{27} \to v$ and $u_{29} \to u_1$. As $u_1 \to u_4$, $u_4 \notin N^3(u_6)$. If $u_4 \in N^2(u_{11})$, then u_{11} has two distinct ≤ 4-paths to u_4. Thus $u_4 \in N^2(u_{12})$. However, now there are distinct ≤ 4-paths from u_{12} to u_9, violating 4-geodecity. □

References

[1] G. Araujo-Pardo, C. Balbuena, M. Olsen, On $(k, g; l)$-dicages, Ars Combinatoria 92 (2009), 289-301

[2] E. Bannai, T. Ito, On finite Moore graphs, J. Fac. Sci. Tokyo Univ. 20 (1973), 191-208

[3] E. Bannai, T. Ito, Regular graphs with excess one, Discrete Math. 77 (1997), 99-118

[4] E. T. Baskoro, M. Miller, J. Plesník, S. Znám, Digraphs of degree 3 and order close to the Moore bound, J. Graph Theory 20 (1995), 339-349

[5] E. T. Baskoro, M. Miller, J. Širáň, M. Sutton, Complete characterization of almost Moore digraphs of degree three, J. Graph Theory 48 (2005), 112-126

[6] M. Behzad, G. Chartrand, C. Wall, On minimal regular digraphs with given girth, Fundam. Math. 69 (1970), 227-231

[7] W. G. Bridges, S. Toueg, On the impossibility of directed Moore graphs, J. Combinatorial Theory B29 (1980), 339-341

[8] J. Conde, J. Gimbert, J. Gonzàlez, M. Miller, J. M. Miret, On the nonexistence of almost Moore digraphs, Europ. Jour. Combin. 39 (2014), 170-177

[9] J. Conde, J. Gimbert, J. Gonzàlez, J. M. Miret, R. Moreno, Nonexistence of almost Moore digraphs of diameter three, Electron. J. Combin. 15 (2008)

[10] J. Conde, J. Gimbert, J. Gonzàlez, J. M. Miret, R. Moreno, Nonexistence of almost Moore digraphs of diameter four, Electron. J. Combin. 20(1) (2013)

[11] R. M. Damerell, On Moore graphs, Proc. Cambridge Phil. Soc. 74 (1973), 227-236

[12] P. Erdős, S. Fajtlowicz, A. J. Hoffman, Maximum degree in graphs of diameter 2, Networks 10 (1980), 87-90

[13] G. Erskine, J. Tuite, Searching for digraphs with small excess, in preparation (2017)

[14] G. Exoo, R. Jajcay, Dynamic cage survey, Electron. Jour. Combin. 1000 (2011), DS16-May

[15] M. A. Fiol, I. Alegre, J. L. A. Yebra, Line digraph iterations and the (d, k)-digraph problem for directed graphs, Proc. 10th Int. Symp. Comput. Architecture (1983), 174-177

[16] J. Gimbert, On the existence of (d, k)-digraphs, 16th British Combinatorial Conference (1997), Discrete Math. 197/198 (1999), 375-391

[17] C. Heuchenne, Sur une certaine correspondance entre graphes, Bull. Soc. Roy. Sci. Liege 33 (1964), 743-753

[18] A. J. Hoffman, R. R. Singleton, On Moore graphs with diameter 2 and 3, IBM J. Res. Develop. 4 (1960), 497-504

[19] K. Kurosawa, S. Tsujii, Considerations on diameter of communication networks, Electronics and Communications in Japan 64A (4) (1981), 37-45

[20] M. Miller, I. Fris, Maximum order digraphs for diameter 2 or degree 2, Pullman Volume of Graphs and Matrices, Lecture Notes in Pure and Applied Math. 139 (1992), 269-298

[21] M. Miller, J. Širáň, Digraphs of degree two which miss the Moore bound by two, Discrete Math. 226, No. 1-3 (2001), 269-280

[22] M. Miller, J. Širáň, Moore graphs and beyond: A survey of the degree/diameter problem, Electron. J. Combin., Dynamic Survey DS14 (2005)

[23] M. Miller, J.M. Miret, A. Sillasen, On digraphs of excess one, Discrete Applied Math. (2017)

[24] J. Shen, On the girth of digraphs, Discrete Math. 211 (1) (2000), 167-181

[25] A. Sillasen, Digraphs of small defect or excess, Dep. of Math. Sciences, Aalborg Uni. (Ph.D. Report Series No.26) (2014)

[26] A. Sillasen, On k-geodetic digraphs with excess one, Electron. J. Graph Theory and Applications 2 (2) (2014), 150-159

[27] J. Tuite, Digraphs with degree two and excess two are diregular, submitted to Discrete Math. (2016)

[28] J. Tuite, On diregular digraphs with degree two and excess two, Discrete Applied Math. (2016), to appear