Construction of Local Regular Dirichlet Form on the Sierpiński Gasket using Γ-Convergence

Meng Yang

Abstract

We construct a self-similar local regular Dirichlet form on the Sierpiński gasket using Γ-convergence of stable-like non-local closed forms. Such a Dirichlet form was constructed previously by Kigami [14], but our construction has the advantage that it is a realization of a more general method of construction of a local regular Dirichlet form that works also on the Sierpiński carpet [8]. A direct consequence of this construction is the fact that the domain of the local Dirichlet form is some Besov space.

1 Introduction

Sierpiński gasket (SG) is the simplest self-similar set in some sense, see Figure 1.

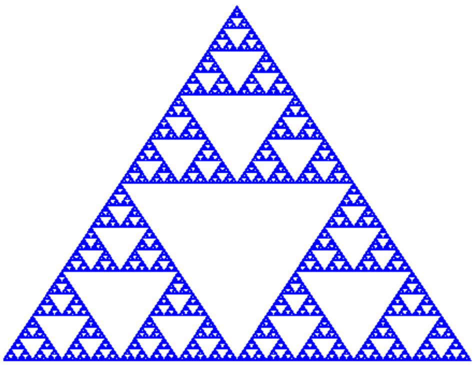


Figure 1: Sierpiński Gasket

SG can be obtained as follows. Given an equilateral triangle with sides of length 1. Divide the triangle into four congruent small triangles, each with sides of length $1/2$, remove the central one. Divide each of the three remaining small triangles into four congruent triangles, each with sides of length $1/4$, remove the central ones, see Figure 2. Repeat above procedure infinitely many times, SG is the compact connect set K that remains.

SG usually serves as a basic example of singular spaces for analysis and probability. Dirichlet form theory is a powerful tool in this approach. In general, local regular Dirichlet forms are in one-to-one correspondence to Brownian motions (BM). The construction of a BM on SG was given by Barlow and Perkins [2]. The construction of a local regular Dirichlet form on SG was given by Kigami [12] using difference quotients method which

Date: October 6, 2017

MSC2010: 28A80

Keywords: Sierpiński gasket, local regular Dirichlet form, Γ-convergence, stable-like non-local closed form

The author was supported by SFB701 of the German Research Council (DFG). The author is very grateful to Professor Alexander Grigor'yan for very helpful discussions.

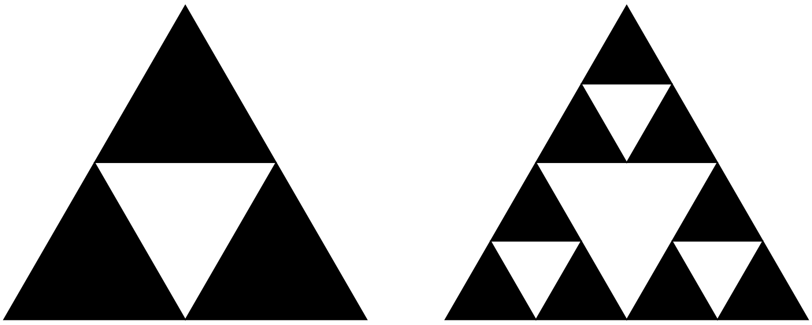


Figure 2: The Construction of Sierpiński Gasket

was generalized to p.c.f. (post critically finite) self-similar sets in [13, 14]. Subsequently, Strichartz [21] gave the characterization of the Dirichlet form and the Laplacian using the averaging method.

The local regular Dirichlet form $\mathcal{E}_{\text{loc}}$ on SG admits a heat kernel $p_t(x,y)$ satisfying

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta^*}} \exp\left(-c\left(\frac{|x-y|}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right) \tag{1}$$

for all $x,y\in K, t\in(0,1)$, where $\alpha=\log 3/\log 2$ is the Hausdorff dimension of SG and

$$\beta^* := \frac{\log 5}{\log 2}$$

is the *walk dimension of BM* which is frequently denoted also by d_w. The estimates (1) were obtained by Barlow and Perkins [2].

The domain $\mathcal{F}_{\text{loc}}$ of $\mathcal{E}_{\text{loc}}$ is some Besov space. This was given by Jonsson [11]. Later on, this kind of characterization was generalized to simple nested fractals by Pietruska-Pałuba [17] and p.c.f. self-similar sets by Hu and Wang [10]. This kind of characterization was also given by Pietruska-Pałuba [18], Grigor'yan, Hu and Lau [7], Kumagai and Sturm [15] if local regular Dirichlet forms on metric measure spaces admit sub-Gaussian heat kernel estimates. Here, we reprove this characterization as a direct corollary of our construction.

Consider the following stable-like non-local quadratic form

$$\begin{aligned}
&\mathfrak{E}_\beta(u,u) = \int_K\int_K \frac{(u(x)-u(y))^2}{|x-y|^{\alpha+\beta}}\nu(\mathrm{d}x)\nu(\mathrm{d}y),\\
&\mathcal{F}_\beta = \left\{u\in L^2(K;\nu) : \mathfrak{E}_\beta(u,u)<+\infty\right\},
\end{aligned} \tag{2}$$

where $\alpha=\dim_{\mathcal{H}}K$ as above, ν is the normalized Hausdorff measure on K of dimension α and $\beta>0$ is so far arbitrary.

Using the estimates (1) and subordination technique, it was proved by Pietruska-Pałuba [19] that

$$\varliminf_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathfrak{E}_\beta(u,u) \asymp \mathcal{E}_{\text{loc}}(u,u) \asymp \varlimsup_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathfrak{E}_\beta(u,u) \tag{3}$$

for all $u\in\mathcal{F}_{\text{loc}}$. This is similar to the following classical result

$$\lim_{\beta\uparrow 2}(2-\beta)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))^2}{|x-y|^{n+\beta}}\mathrm{d}x\mathrm{d}y = C(n)\int_{\mathbb{R}^n}|\nabla u(x)|^2\mathrm{d}x$$

for all $u\in W^{1,2}(\mathbb{R}^n)$, where $C(n)$ is some positive constant (see [6, Example 1.4.1]). Recently, the author [22] gave an alternative proof of (3) using discretization method. Here, we reprove (3) as a direct corollary of our construction.

The main purpose of this paper is to give a construction of a local regular Dirichlet form $\mathcal{E}_{\text{loc}}$ on SG using Γ-convergence of stable-like non-local closed forms of type (2) as $\beta\uparrow\beta^*$. This is our main result Theorem 2.1. The local regular Dirichlet form given here coincides with that given by Kigami due to the uniqueness result given by Sabot [20]. Kusuoka and

Zhou [16] gave a construction using the averaging method and approximation of Markov chains.

The idea of our construction of $\mathcal{E}_{\mathrm{loc}}$ is as follows. First, we use the averaging method to construct another quadratic form $\mathcal{E}_\beta$, equivalent to $\mathfrak{E}_\beta$, which turns out to be a regular closed form for all $\beta \in (\alpha, \beta^*)$. Second, we construct a regular closed form $\mathcal{E}$ as a Γ-limit of a sequence $\{(\beta^* - \beta_n)\mathcal{E}_{\beta_n}\}$ with $\beta_n \uparrow \beta^*$. However, $\mathcal{E}$ is not necessarily Markovian, local or self-similar. Third, we use a standard method from [16] to construct $\mathcal{E}_{\mathrm{loc}}$ from $\mathcal{E}$.

The main difficulty in our construction is that we do not have monotonicity property as in Kigami's construction. Nevertheless we have weak monotonicity that allows to obtain the characterization of the Γ-limit. To prove the non-triviality and the regularity of the Γ-limit, we construct on SG functions with controlled energy and with separation property that are called *good* functions.

A similar method was used in [8] to give a purely analytic construction of a local regular Dirichlet form on the Sierpiński carpet. The current paper can be regarded as a realization of this method on SG. The ultimate purpose of the current paper and [8] is to provide a new unified method of construction of local regular Dirichlet forms on a wide class of fractals that uses only self-similar property and ideally should be independent of other specific properties, in particular, p.c.f. property.

This paper is organized as follows. In Section 2, we give statement of the main results. In Section 3, we give resistance estimates and introduce good functions. In Section 4, we give weak monotonicity result. In Section 5, we prove Theorem 2.1.

2 Statement of the Main Results

Consider the following points in $\mathbb{R}^2$: $p_0 = (0,0)$, $p_1 = (1,0)$, $p_2 = (1/2, \sqrt{3}/2)$. Let $f_i(x) = (x + p_i)/2$, $x \in \mathbb{R}^2$. Then the Sierpiński gasket (SG) is the unique non-empty compact set K satisfying $K = f_0(K) \cup f_1(K) \cup f_2(K)$. Let ν be the normalized Hausdorff measure on K of dimension $\alpha = \log 3/\log 2$. Let

$$V_0 = \{p_0, p_1, p_2\}, V_{n+1} = f_0(V_n) \cup f_1(V_n) \cup f_2(V_n) \text{ for all } n \geq 0.$$

Then $\{V_n\}$ is an increasing sequence of finite sets and K is the closure of $V^* = \cup_{n=0}^\infty V_n$.

The classical construction of a self-similar local regular Dirichlet form on SG is as follows. Let

$$\mathfrak{E}_n(u,u) = \left(\frac{5}{3}\right)^n \sum_{\substack{x,y \in V_n \\ |x-y| = 2^{-n}}} (u(x) - u(y))^2, n \geq 0, u \in l(K),$$

where $l(S)$ is the set of all real-valued functions on the set S. Then $\mathfrak{E}_n(u,u)$ is monotone increasing in n for all $u \in l(K)$. Let

$$\mathfrak{E}_{\mathrm{loc}}(u,u) = \lim_{n \to +\infty} \mathfrak{E}_n(u,u) = \lim_{n \to +\infty} \left(\frac{5}{3}\right)^n \sum_{\substack{x,y \in V_n \\ |x-y| = 2^{-n}}} (u(x) - u(y))^2,$$

$$\mathfrak{F}_{\mathrm{loc}} = \{u \in C(K) : \mathfrak{E}_{\mathrm{loc}}(u,u) < +\infty\},$$

then $(\mathfrak{E}_{\mathrm{loc}}, \mathfrak{F}_{\mathrm{loc}})$ is a self-similar local regular Dirichlet form on $L^2(K;\nu)$, see [12, 13, 14].

Let $W_0 = \{\emptyset\}$ and

$$W_n = \{w = w_1 \ldots w_n : w_i = 0,1,2, i = 1,\ldots,n\} \text{ for all } n \geq 1.$$

For all $w^{(1)} = w^{(1)}_1 \ldots w^{(1)}_m \in W_m, w^{(2)} = w^{(2)}_1 \ldots w^{(2)}_n \in W_n$, denote $w^{(1)}w^{(2)}$ as $w = w_1 \ldots w_{m+n} \in W_{m+n}$ with $w_i = w^{(1)}_i$ for all $i = 1,\ldots,m$ and $w_{m+i} = w^{(2)}_i$ for all $i = 1,\ldots,n$. For all $i = 0,1,2$, denote i^n as $w = w_1 \ldots w_n \in W_n$ with $w_k = i$ for all $k = 1,\ldots,n$. For all $w = w_1 \ldots w_{n-1}w_n \in W_n$, denote $w^- = w_1 \ldots w_{n-1} \in W_{n-1}$.

For all $w = w_1 \ldots w_n \in W_n$, let

$$\begin{aligned} f_w &= f_{w_1} \circ \ldots \circ f_{w_n}, \\ V_w &= f_{w_1} \circ \ldots \circ f_{w_n}(V_0), \\ K_w &= f_{w_1} \circ \ldots \circ f_{w_n}(K), \\ P_w &= f_{w_1} \circ \ldots \circ f_{w_{n-1}}(p_{w_n}), \end{aligned}$$

where $f_\emptyset = \mathrm{id}$ is the identity map.

For all $n \geq 1$, let X_n be the graph with vertex set W_n and edge set H_n given by

$$H_n = \left\{(w^{(1)}, w^{(2)}) : w^{(1)}, w^{(2)} \in W_n, w^{(1)} \neq w^{(2)}, K_{w^{(1)}} \cap K_{w^{(2)}} \neq \emptyset\right\}.$$

For example, we have the figure of X_3 in Figure 3. Denote $w^{(1)} \sim_n w^{(2)}$ if $(w^{(1)}, w^{(2)}) \in H_n$. If $w^{(1)} \sim_n w^{(2)}$ satisfies $P_{w^{(1)}} \neq P_{w^{(2)}}$, we say that $w^{(1)} \sim_n w^{(2)}$ is of type I. If $w^{(1)} \sim_n w^{(2)}$ satisfies $P_{w^{(1)}} = P_{w^{(2)}}$, we say that $w^{(1)} \sim_n w^{(2)}$ is of type II. For example, $000 \sim_3 001$ is of type I, $001 \sim_3 010$ is of type II.

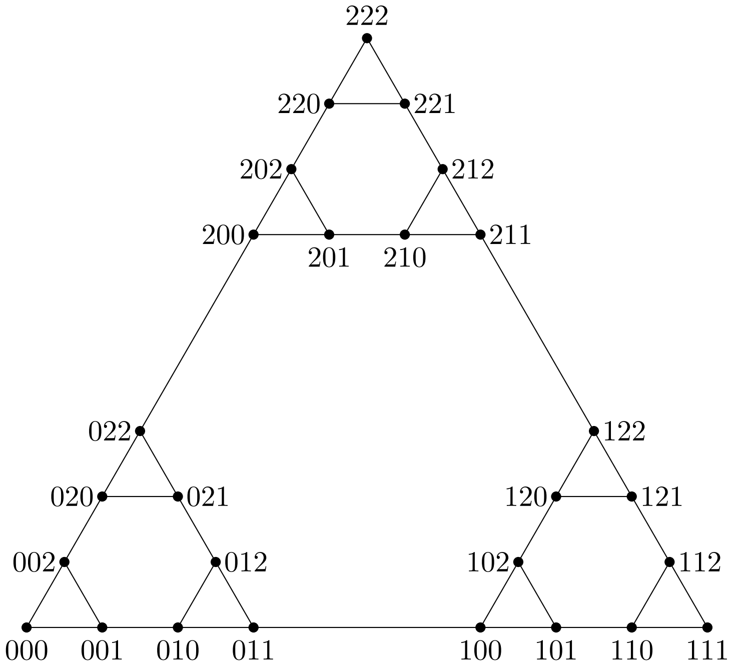


Figure 3: X_3

For all $n \geq 1, u \in L^2(K; \nu)$, let $P_n u : W_n \to \mathbb{R}$ be given by

$$P_n u(w) = \frac{1}{\nu(K_w)} \int_{K_w} u(x)\nu(\mathrm{d}x) = \int_K (u \circ f_w)(x)\nu(\mathrm{d}x), w \in W_n.$$

Our main result is as follows.

Theorem 2.1. *There exists a self-similar strongly local regular Dirichlet form $(\mathcal{E}_{\mathrm{loc}}, \mathcal{F}_{\mathrm{loc}})$ on $L^2(K; \nu)$ satisfying*

$$\mathcal{E}_{\mathrm{loc}}(u, u) \asymp \sup_{n \geq 1} \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2,$$

$$\mathcal{F}_{\mathrm{loc}} = \left\{u \in L^2(K; \nu) : \sup_{n \geq 1} \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2 < +\infty\right\}.$$

We give the proof of above theorem in Section 5.

Remark 2.2. *Above theorem was also proved by Kusuoka and Zhou [16, Theorem 7.19, Example 8.4] using approximation of Markov chains. Here we use Γ-convergence of stable-like non-local closed forms.*

Let us introduce the notion of Besov spaces. Let (M, d, μ) be a metric measure space and $\alpha,\beta>0$ two parameters. Define

$$[u]_{B^{2,2}_{\alpha,\beta}(M)}=\sum_{n=1}^{\infty}2^{(\alpha+\beta)n}\int_M\int_{d(x,y)<2^{-n}}(u(x)-u(y))^2\mu(\mathrm{d}y)\mu(\mathrm{d}x),$$

$$[u]_{B^{2,\infty}_{\alpha,\beta}(M)}=\sup_{n\geq1}2^{(\alpha+\beta)n}\int_M\int_{d(x,y)<2^{-n}}(u(x)-u(y))^2\mu(\mathrm{d}y)\mu(\mathrm{d}x),$$

and

$$B^{2,2}_{\alpha,\beta}(M)=\left\{u\in L^2(M;\mu):[u]_{B^{2,2}_{\alpha,\beta}(M)}<+\infty\right\},$$
$$B^{2,\infty}_{\alpha,\beta}(M)=\left\{u\in L^2(M;\mu):[u]_{B^{2,\infty}_{\alpha,\beta}(M)}<+\infty\right\}.$$

It is easily proved that $B^{2,2}_{\alpha,\beta}(K)$ and $B^{2,\infty}_{\alpha,\beta}(K)$ have the following equivalent semi-norms.

Lemma 2.3. *([9, Lemma 3.1],[22, Lemma 2.1]) For all $\beta\in(0,+\infty),u\in L^2(K;\nu)$*

$$\mathcal{E}_\beta(u,u)\asymp\mathfrak{E}_\beta(u,u)\asymp[u]_{B^{2,2}_{\alpha,\beta}(K)},$$

$$\sup_{n\geq1}2^{(\beta-\alpha)n}\sum_{w^{(1)}\sim_n w^{(2)}}\left(P_nu(w^{(1)})-P_nu(w^{(2)})\right)^2\asymp[u]_{B^{2,\infty}_{\alpha,\beta}(K)},$$

where

$$\mathcal{E}_\beta(u,u)=\sum_{n=1}^{\infty}2^{(\beta-\alpha)n}\sum_{w^{(1)}\sim_n w^{(2)}}\left(P_nu(w^{(1)})-P_nu(w^{(2)})\right)^2,$$

$$\mathfrak{E}_\beta(u,u)=\int_K\int_K\frac{(u(x)-u(y))^2}{|x-y|^{\alpha+\beta}}\nu(\mathrm{d}x)\nu(\mathrm{d}y).$$

We have the following two corollaries whose proofs are obvious by Lemma 2.3 and the proof of Theorem 2.1.

First, we have the characterization of the domain of the local Dirichlet form.

Corollary 2.4. *$\mathcal{F}_{\mathrm{loc}}=B^{2,\infty}_{\alpha,\beta^*}(K)$ and $\mathcal{E}_{\mathrm{loc}}(u,u)\asymp[u]_{B^{2,\infty}_{\alpha,\beta^*}(K)}$ for all $u\in\mathcal{F}_{\mathrm{loc}}$, where $\alpha=\log3/\log2$ is the Hausdorff dimension and $\beta^*=\log5/\log2$ is the walk dimension of BM.*

Second, we have the approximation of non-local Dirichlet forms to the local Dirichlet form.

Corollary 2.5. *There exists some positive constant C such that for all $u\in\mathcal{F}_{\mathrm{loc}}$*

$$\frac{1}{C}\mathcal{E}_{\mathrm{loc}}(u,u)\leq\varliminf_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathcal{E}_\beta(u,u)\leq\varlimsup_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathcal{E}_\beta(u,u)\leq C\mathcal{E}_{\mathrm{loc}}(u,u),$$

$$\frac{1}{C}\mathcal{E}_{\mathrm{loc}}(u,u)\leq\varliminf_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathfrak{E}_\beta(u,u)\leq\varlimsup_{\beta\uparrow\beta^*}(\beta^*-\beta)\mathfrak{E}_\beta(u,u)\leq C\mathcal{E}_{\mathrm{loc}}(u,u),$$

$$\frac{1}{C}\mathcal{E}_{\mathrm{loc}}(u,u)\leq\varliminf_{\beta\uparrow\beta^*}(\beta^*-\beta)[u]_{B^{2,2}_{\alpha,\beta}(K)}\leq\varlimsup_{\beta\uparrow\beta^*}(\beta^*-\beta)[u]_{B^{2,2}_{\alpha,\beta}(K)}\leq C\mathcal{E}_{\mathrm{loc}}(u,u).$$

3 Resistance Estimates and Good Functions

First, we give resistance estimates. We need two techniques from electrical networks.

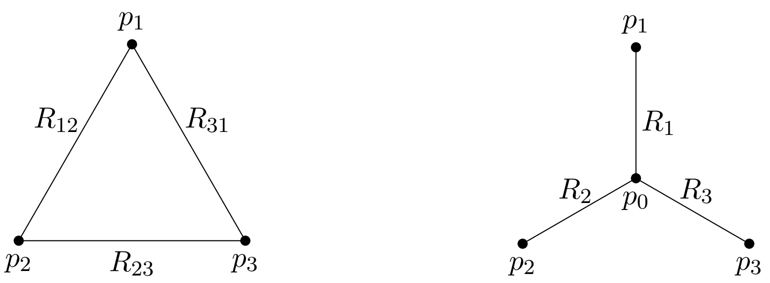
The first is Δ-Y transform.

Lemma 3.1. *The electrical networks in Figure 4 are equivalent, where*

$$R_1=\frac{R_{12}R_{31}}{R_{12}+R_{23}+R_{31}},R_2=\frac{R_{12}R_{23}}{R_{12}+R_{23}+R_{31}},R_3=\frac{R_{23}R_{31}}{R_{12}+R_{23}+R_{31}},$$

and

$$R_{12}=\frac{R_1R_2+R_2R_3+R_3R_1}{R_3},R_{23}=\frac{R_1R_2+R_2R_3+R_3R_1}{R_1},R_{31}=\frac{R_1R_2+R_2R_3+R_3R_1}{R_2}.$$



(a) Δ-circuit (b) Y-circuit

Figure 4: Δ-Y Transform

The second is shorting and cutting technique (see [5]). Shorting certain sets of vertices will decrease the resistance between arbitrary two vertices. Cutting certain sets of vertices will increase the resistance between arbitrary two vertices.

For all $n \geq 1$, let us introduce an energy on X_n given by

$$E_n(u,u) = \sum_{w^{(1)} \sim_n w^{(2)}} \left(u(w^{(1)}) - u(w^{(2)}) \right)^2, u \in l(W_n).$$

For all $w^{(1)}, w^{(2)} \in W_n$ with $w^{(1)} \neq w^{(2)}$, we define the resistance

$$\begin{aligned} R_n(w^{(1)}, w^{(2)}) &= \inf \left\{ E_n(u,u) : u(w^{(1)}) = 1, u(w^{(2)}) = 0, u \in l(W_n) \right\}^{-1} \\ &= \sup \left\{ \frac{\left(u(w^{(1)}) - u(w^{(2)}) \right)^2}{E_n(u,u)} : E_n(u,u) \neq 0, u \in l(W_n) \right\}. \end{aligned}$$

It is obvious that

$$\left(u(w^{(1)}) - u(w^{(2)}) \right)^2 \leq R_n(w^{(1)}, w^{(2)}) E_n(u,u) \text{ for all } w^{(1)}, w^{(2)} \in W_n, u \in l(W_n),$$

and R_n is a metric on W_n, hence

$$R_n(w^{(1)}, w^{(2)}) \leq R_n(w^{(1)}, w^{(3)}) + R_n(w^{(3)}, w^{(2)}) \text{ for all } w^{(1)}, w^{(2)}, w^{(3)} \in W_n.$$

We calculate the resistance of X_n as follows.

Theorem 3.2. *The electrical network X_n is equivalent to the electrical network in Figure 5, where*

$$r_n = \frac{1}{2} \left(\frac{5}{3} \right)^n - \frac{1}{2}.$$

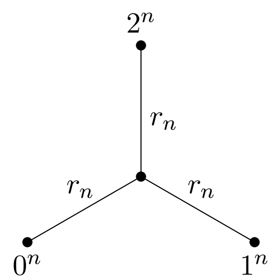


Figure 5: An Equivalent Electrical Network

Proof. Using Δ-Y transform directly, we have

$$r_1 = \frac{1 \cdot 1}{1+1+1} = \frac{1}{3} = \frac{1}{2}\left(\frac{5}{3}\right)^1 - \frac{1}{2}.$$

For $n+1$, using Δ-Y transform again, we have the electrical network X_{n+1} is equivalent to the electrical networks in Figure 6.

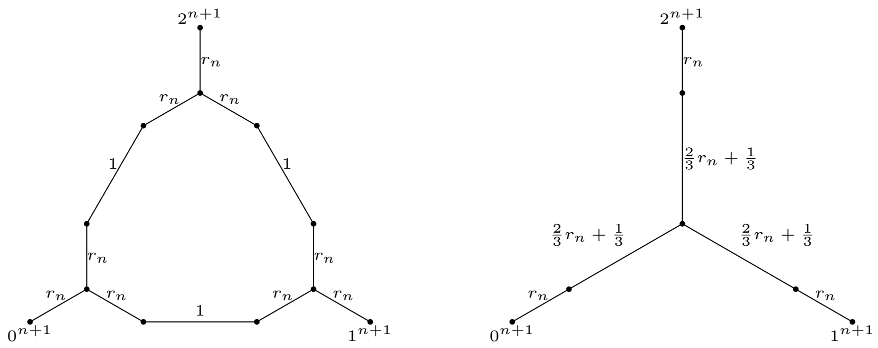


Figure 6: Equivalent Electrical Networks

Hence

$$r_{n+1} = \frac{5}{3}r_n + \frac{1}{3}.$$

By elementary calculation, we have

$$r_n = \frac{1}{2}\left(\frac{5}{3}\right)^n - \frac{1}{2} \text{ for all } n \geq 1.$$

□

Remark 3.3. *For all $n \geq 1$, we have*

$$R_n(0^n, 1^n) = R_n(1^n, 2^n) = R_n(0^n, 2^n) = 2r_n = \left(\frac{5}{3}\right)^n - 1.$$

Proposition 3.4. *For all $n \geq 1, w \in W_n$, we have*

$$R_n(w, 0^n), R_n(w, 1^n,), R_n(w, 2^n) \leq \frac{5}{2}\left(\frac{5}{3}\right)^n.$$

Proof. By symmetry, we only need to consider $R_n(w, 0^n)$. Letting $w = w_1 \ldots w_{n-2}w_{n-1}w_n$, we construct a finite sequence in W_n as follows.

$$\begin{aligned}
w^{(1)} &= w_1 \ldots w_{n-2}w_{n-1}w_n = w,\\
w^{(2)} &= w_1 \ldots w_{n-2}w_{n-1}w_{n-1},\\
w^{(3)} &= w_1 \ldots w_{n-2}w_{n-2}w_{n-2},\\
&\ldots\\
w^{(n)} &= w_1 \ldots w_1w_1w_1,\\
w^{(n+1)} &= 0 \ldots 000.
\end{aligned}$$

For all $i = 1, \ldots, n-1$, by cutting technique, we have

$$\begin{aligned}
&R_n(w^{(i)}, w^{(i+1)})\\
&= R_n(w_1 \ldots w_{n-i-1}w_{n-i}w_{n-i+1} \ldots w_{n-i+1}, w_1 \ldots w_{n-i-1}w_{n-i}w_{n-i} \ldots w_{n-i})\\
&\leq R_i(w_{n-i+1} \ldots w_{n-i+1}, w_{n-i} \ldots w_{n-i}) \leq R_i(0^i, 1^i) = \left(\frac{5}{3}\right)^i - 1 \leq \left(\frac{5}{3}\right)^i.
\end{aligned}$$

Since

$$R_n(w^{(n)}, w^{(n+1)}) = R_n(w_1^n, 0^n) \le R_n(0^n, 1^n) = \left(\frac{5}{3}\right)^n - 1 \le \left(\frac{5}{3}\right)^n,$$

we have

$$\begin{aligned} R_n(w, 0^n) = R_n(w^{(1)}, w^{(n+1)}) &\le \sum_{i=1}^n R_n(w^{(i)}, w^{(i+1)}) \\ &\le \sum_{i=1}^n \left(\frac{5}{3}\right)^i = \frac{\frac{5}{3}\left(1 - \left(\frac{5}{3}\right)^n\right)}{1 - \frac{5}{3}} \le \frac{5}{2}\left(\frac{5}{3}\right)^n. \end{aligned}$$

□

Second, we introduce good functions with energy property and separation property. For all $n \ge 1$, let

$$A_n(u) = E_n(P_n u, P_n u) = \sum_{w^{(1)} \sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2, u \in L^2(K; \nu).$$

For all $n \ge 0$, let

$$B_n(u) = \sum_{w \in W_n} \sum_{p,q \in V_w} (u(p) - u(q))^2, u \in l(K).$$

Given $x_0, x_1, x_2 \in \mathbb{R}$, we define $U = U^{(x_0,x_1,x_2)} : K \to \mathbb{R}$ as follows. We define U on V^* by induction. Let $U(p_i) = x_i, i = 0, 1, 2$. Assume that we have defined U on P_w for all $w \in W_{n+1}$. Then for all $w \in W_n$, note that $P_{wii} = P_{wi}$ and $P_{wij} = P_{wji}$ for all $i, j = 0, 1, 2$, define

$$\begin{aligned} U(P_{w01}) = U(P_{w10}) &= \frac{2U(P_{w0}) + 2U(P_{w1}) + U(P_{w2})}{5}, \\ U(P_{w12}) = U(P_{w21}) &= \frac{U(P_{w0}) + 2U(P_{w1}) + 2U(P_{w2})}{5}, \\ U(P_{w02}) = U(P_{w20}) &= \frac{2U(P_{w0}) + U(P_{w1}) + 2U(P_{w2})}{5}. \end{aligned} \tag{4}$$

Hence we have the definition of U on P_w for all $w \in W_{n+2}$. Then U is well-defined and uniformly continuous on V^*. We extend U on V^* to obtain a continuous function U on K. It is obvious that

$$B_n(U) = \left(\frac{3}{5}\right)^n \left((x_0 - x_1)^2 + (x_1 - x_2)^2 + (x_0 - x_2)^2\right), n \ge 0.$$

Remark 3.5. *In Kigami's construction, B_n is the energy of difference quotients type and $U^{(x_0,x_1,x_2)}$ is the standard harmonic function with boundary values x_0, x_1, x_2 on V_0, see [14]. Here we use $B_n(U)$ only to calculate $A_n(U)$.*

We calculate $A_n(U)$ as follows.

Theorem 3.6. *For all $n \ge 1$, we have*

$$A_n(U) = \frac{2}{3}\left[\left(\frac{3}{5}\right)^n - \left(\frac{3}{5}\right)^{2n}\right]\left((x_0 - x_1)^2 + (x_1 - x_2)^2 + (x_0 - x_2)^2\right).$$

Remark 3.7. *Above result was also obtained in [21, Theorem 3.1].*

Proof. We observe the following facts.

- For all $w^{(1)} \sim_n w^{(2)}$ of type I, $w^{(1)}, w^{(2)}$ are of the form wi, wj for some $w \in W_{n-1}$ and $i, j = 0, 1, 2$ with $i \ne j$. On the other hand, for all $w \in W_{n-1}$ and $i, j = 0, 1, 2$ with $i \ne j$, $wi \sim_n wj$ is of type I.
- For all $w^{(1)} = w_1^{(1)} \ldots w_n^{(1)} \sim_n w^{(2)} = w_1^{(2)} \ldots w_n^{(2)}$ of type II, there exists $k = 1, \ldots, n-1$ such that $w_1^{(1)} \ldots w_k^{(1)} \sim_k w_1^{(2)} \ldots w_k^{(2)}$ is of type I and $w_k^{(2)} = w_{k+1}^{(1)} = \ldots = w_n^{(1)}$, $w_k^{(1)} = w_{k+1}^{(2)} = \ldots = w_n^{(2)}$. On the other hand, for all $w_1^{(1)} \ldots w_k^{(1)} \sim_k w_1^{(2)} \ldots w_k^{(2)}$ of type I, $w_1^{(1)} \ldots w_k^{(1)} w_k^{(2)} \ldots w_k^{(2)} \sim_n w_1^{(2)} \ldots w_k^{(2)} w_k^{(1)} \ldots w_k^{(1)}$ is of type II for all $n = k+1, k+2, \ldots$.

It is obvious that for all $n \geq 1, w \in W_n$, we have $V_w = \{P_{w0}, P_{w1}, P_{w2}\}$ and

$$P_n U(w) = \frac{1}{\nu(K_w)} \int_{K_w} U(x) \nu(\mathrm{d}x) = \frac{1}{3} \left(U(P_{w0}) + U(P_{w1}) + U(P_{w2}) \right).$$

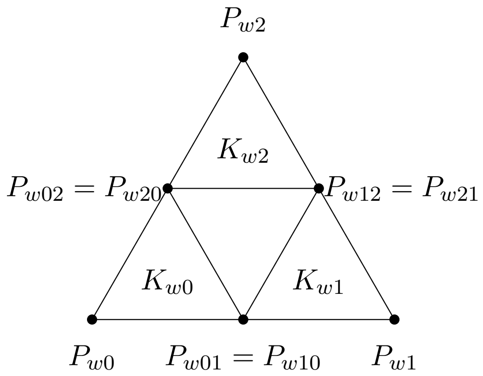


Figure 7: Cells and Nodes

For all $n \geq 1, w \in W_{n-1}$, we have

$$\begin{aligned}
&P_n U(w0) = \frac{1}{3} \left(U(P_{w00}) + U(P_{w01}) + U(P_{w02}) \right) \\
&= \frac{1}{3} \left(U(P_{w0}) + \frac{2U(P_{w0}) + 2U(P_{w1}) + U(P_{w2})}{5} + \frac{2U(P_{w0}) + U(P_{w1}) + 2U(P_{w2})}{5} \right) \\
&= \frac{1}{3} \frac{9U(P_{w0}) + 3U(P_{w1}) + 3U(P_{w2})}{5} = \frac{3U(P_{w0}) + U(P_{w1}) + U(P_{w2})}{5}.
\end{aligned}$$

Similarly

$$\begin{aligned}
P_n U(w1) &= \frac{U(P_{w0}) + 3U(P_{w1}) + U(P_{w2})}{5}, \\
P_n U(w2) &= \frac{U(P_{w0}) + U(P_{w1}) + 3U(P_{w2})}{5}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(P_n U(w0) - P_n U(w1) \right)^2 + \left(P_n U(w1) - P_n U(w2) \right)^2 + \left(P_n U(w0) - P_n U(w2) \right)^2 \\
&= \frac{4}{25} \left[\left(U(P_{w0}) - U(P_{w1}) \right)^2 + \left(U(P_{w1}) - U(P_{w2}) \right)^2 + \left(U(P_{w0}) - U(P_{w2}) \right)^2 \right].
\end{aligned}$$

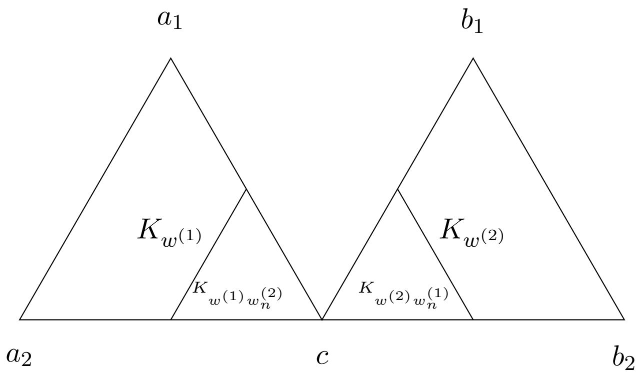


Figure 8: Adjacent Cells

For all $n \geq 1, w^{(1)} = w_1^{(1)} \ldots w_n^{(1)} \sim_n w^{(2)} = w_1^{(2)} \ldots w_n^{(2)}$. Assume that U takes values a_1, a_2, c and b_1, b_2, c on $V_{w^{(1)}}$ and $V_{w^{(2)}}$, respectively, see Figure 8. By above, we have

$$P_n U(w^{(1)}) = \frac{a_1 + a_2 + c}{3}, P_n U(w^{(2)}) = \frac{b_1 + b_2 + c}{3},$$

$$P_{n+1}U(w^{(1)}w_n^{(2)}) = \frac{a_1+a_2+3c}{5}, P_{n+1}U(w^{(2)}w_n^{(1)}) = \frac{b_1+b_2+3c}{5},$$

hence

$$P_nU(w^{(1)}) - P_nU(w^{(2)}) = \frac{1}{3}\left((a_1+a_2)-(b_1+b_2)\right),$$

$$P_{n+1}U(w^{(1)}w_n^{(2)}) - P_{n+1}U(w^{(2)}w_n^{(1)}) = \frac{1}{5}\left((a_1+a_2)-(b_1+b_2)\right).$$

Hence

$$P_{n+1}U(w^{(1)}w_n^{(2)}) - P_{n+1}U(w^{(2)}w_n^{(1)}) = \frac{3}{5}\left(P_nU(w^{(1)}) - P_nU(w^{(2)})\right).$$

Therefore

$$\begin{aligned} A_n(U) &= \frac{4}{25}B_{n-1}(U) + \left(\frac{3}{5}\right)^2\left[\frac{4}{25}B_{n-2}(U)\right] + \ldots + \left(\frac{3}{5}\right)^{2(n-1)}\left[\frac{4}{25}B_0(U)\right] \\ &= \frac{4}{25}\left((x_0-x_1)^2+(x_1-x_2)^2+(x_0-x_2)^2\right) \\ &\cdot\left[\left(\frac{9}{25}\right)^0\left(\frac{3}{5}\right)^{n-1} + \left(\frac{9}{25}\right)^1\left(\frac{3}{5}\right)^{n-2} + \ldots + \left(\frac{9}{25}\right)^{n-1}\left(\frac{3}{5}\right)^0\right] \\ &= \frac{2}{3}\left[\left(\frac{3}{5}\right)^n - \left(\frac{3}{5}\right)^{2n}\right]\left((x_0-x_1)^2+(x_1-x_2)^2+(x_0-x_2)^2\right). \end{aligned}$$

□

Let

$$\mathcal{U} = \left\{U^{(x_0,x_1,x_2)} : x_0,x_1,x_2 \in \mathbb{R}\right\}.$$

We show that $\mathcal{U}$ separates points as follows.

Proposition 3.8. *$\mathcal{U}$ separates points, that is, for all $x, y \in K$ with $x \neq y$, there exists $U \in \mathcal{U}$ such that $U(x) \neq U(y)$.*

Proof. First, we show that for all $n \geq 1, w \in W_n, u \in l(V_w)$, there exists $U \in \mathcal{U}$ such that $U|_{V_w} = u$. Indeed, we only need to show that there exists $v \in l(V_{w^-0} \cup V_{w^-1} \cup V_{w^-2})$ satisfying Equation (4) such that $v|_{V_w} = u$. We only need to consider the following three cases in Figure 9.

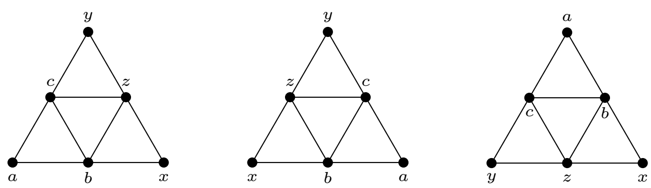


Figure 9: Separation Property

Assume that u takes a, b, c on V_w and v takes x, y, z on $(V_{w^-0} \cup V_{w^-1} \cup V_{w^-2})\backslash V_w$ as in Figure 9. Letting

$$b = \frac{2a+2x+y}{5}, c = \frac{2a+x+2y}{5}, z = \frac{a+2x+2y}{5},$$

we have

$$x = \frac{-2a+10b-5c}{3}, y = \frac{-2a-5b+10c}{3}, z = \frac{-a+2b+2c}{3}.$$

Second, without lose of generality, we may assume that $x \in K_0\backslash K_1$ and $y \in K_1\backslash K_0$. Take $U = U^{(1,0,0)} \in \mathcal{U}$, then $U(x) \in [\frac{2}{5}, 1]$ and $U(y) \in [0, \frac{2}{5})$, hence $U(x) > U(y)$. □

4 Weak Monotonicity Result

In this section, we give weak monotonicity result using resistance estimates.

For all $n \geq 1$, let

$$D_n(u) = \left(\frac{5}{3}\right)^n A_n(u) = \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2, u \in L^2(K;\nu).$$

The weak monotonicity result is as follows.

Theorem 4.1. *There exists some positive constant C such that*

$$D_n(u) \leq C D_{n+m}(u) \text{ for all } u \in L^2(K;\nu), n, m \geq 1.$$

Indeed, we can take $C = 36$.

Remark 4.2. *In Kigami's construction, the energies are monotone, that is, the constant $C = 1$. Hence, the above result is called weak monotonicity.*

Theorem 4.1 can be reduced as follows.

For all $n \geq 1$, let

$$G_n(u) = \left(\frac{5}{3}\right)^n E_n(u,u) = \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(u(w^{(1)}) - u(w^{(2)})\right)^2, u \in l(W_n).$$

For all $n, m \geq 1$, let $M_{n,m} : l(W_{n+m}) \to l(W_n)$ be a mean value operator given by

$$(M_{n,m}u)(w) = \frac{1}{3^m} \sum_{v \in W_m} u(wv), w \in W_n, u \in l(W_{n+m}).$$

Theorem 4.3. *There exists some positive constant C such that*

$$G_n(M_{n,m}u) \leq C G_{n+m}(u) \text{ for all } u \in l(W_{n+m}), n, m \geq 1.$$

Proof of Theorem 4.1 using Theorem 4.3. Note $P_n u = M_{n,m}(P_{n+m}u)$, hence

$$\begin{aligned} D_n(u) &= \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(P_n u(w^{(1)}) - P_n u(w^{(2)})\right)^2 = G_n(P_n u) \\ &= G_n(M_{n,m}(P_{n+m}u)) \leq C G_{n+m}(P_{n+m}u) \\ &= C \left(\frac{5}{3}\right)^{n+m} \sum_{w^{(1)} \sim_{n+m} w^{(2)}} \left(P_{n+m}u(w^{(1)}) - P_{n+m}u(w^{(2)})\right)^2 = C D_{n+m}(u). \end{aligned}$$

□

Remark 4.4. *The constant in Theorem 4.1 can be taken as the one in Theorem 4.3.*

Proof of Theorem 4.3. For all $n \geq 1$. Assume that $W \subseteq W_n$ is connected, that is, for all $w^{(1)}, w^{(2)} \in W$, there exists a finite sequence $\left\{v^{(1)}, \ldots, v^{(k)}\right\} \subseteq W$ with $v^{(1)} = w^{(1)}, v^{(k)} = w^{(2)}$ and $v^{(i)} \sim_n v^{(i+1)}$ for all $i = 1, \ldots, k-1$. Let

$$E_W(u,u) = \sum_{\substack{w^{(1)}, w^{(2)} \in W \\ w^{(1)} \sim_n w^{(2)}}} (u(w^{(1)}) - u(w^{(2)}))^2, u \in l(W).$$

For all $w^{(1)}, w^{(2)} \in W$, let

$$\begin{aligned} R_W(w^{(1)}, w^{(2)}) &= \inf \left\{E_W(u,u) : u(w^{(1)}) = 1, u(w^{(2)}) = 0, u \in l(W)\right\}^{-1} \\ &= \sup \left\{\frac{(u(w^{(1)}) - u(w^{(2)}))^2}{E_W(u,u)} : E_W(u,u) \neq 0, u \in l(W)\right\}. \end{aligned}$$

It is obvious that

$$\left(u(w^{(1)}) - u(w^{(2)})\right)^2 \le R_W(w^{(1)}, w^{(2)}) E_W(u,u) \text{ for all } w^{(1)}, w^{(2)} \in W, u \in l(W),$$

and R_W is a metric on W, hence

$$R_W(w^{(1)}, w^{(2)}) \le R_W(w^{(1)}, w^{(3)}) + R_W(w^{(3)}, w^{(2)}) \text{ for all } w^{(1)}, w^{(2)}, w^{(3)} \in W.$$

By definition, we have

$$\begin{aligned}
G_n(M_{n,m}u) &= \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(M_{n,m}u(w^{(1)}) - M_{n,m}u(w^{(2)})\right)^2 \\
&= \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(\frac{1}{3^m} \sum_{v \in W_m} \left(u(w^{(1)}v) - u(w^{(2)}v)\right)\right)^2 \\
&\le \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \frac{1}{3^m} \sum_{v \in W_m} \left(u(w^{(1)}v) - u(w^{(2)}v)\right)^2.
\end{aligned}$$

Fix $w^{(1)} \sim_n w^{(2)}$, there exist $i, j = 0, 1, 2$ with $i \ne j$ such that $w^{(1)}i^m \sim_{n+m} w^{(2)}j^m$, see Figure 10.

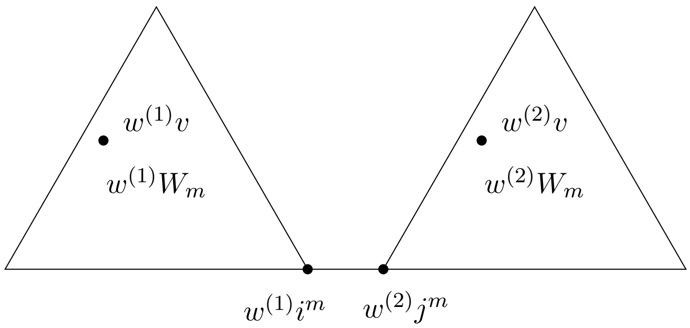


Figure 10: $w^{(1)}W_m$ and $w^{(2)}W_m$

Fix $v \in W_m$, we have

$$\left(u(w^{(1)}v) - u(w^{(2)}v)\right)^2 \le R_{w^{(1)}W_m \cup w^{(2)}W_m}(w^{(1)}v, w^{(2)}v) E_{w^{(1)}W_m \cup w^{(2)}W_m}(u,u).$$

By cutting technique and Proposition 3.4, we have

$$\begin{aligned}
&R_{w^{(1)}W_m \cup w^{(2)}W_m}(w^{(1)}v, w^{(2)}v) \\
&\le R_{w^{(1)}W_m \cup w^{(2)}W_m}(w^{(1)}v, w^{(1)}i^m) + R_{w^{(1)}W_m \cup w^{(2)}W_m}(w^{(1)}i^m, w^{(2)}j^m) \\
&+ R_{w^{(1)}W_m \cup w^{(2)}W_m}(w^{(2)}j^m, w^{(2)}v) \\
&\le R_m(v, i^m) + 1 + R_m(v, j^m) \le 5\left(\frac{5}{3}\right)^m + 1 \le 6\left(\frac{5}{3}\right)^m,
\end{aligned}$$

hence

$$\begin{aligned}
&(u(w^{(1)}v) - u(w^{(2)}v))^2 \le 6\left(\frac{5}{3}\right)^m E_{w^{(1)}W_m \cup w^{(2)}W_m}(u,u) \\
&= 6\left(\frac{5}{3}\right)^m \left(E_{w^{(1)}W_m}(u,u) + E_{w^{(2)}W_m}(u,u) + \left(u(w^{(1)}i^m) - u(w^{(2)}j^m)\right)^2\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{3^m} \sum_{v \in W_m} \left(u(w^{(1)}v) - u(w^{(2)}v)\right)^2 \\
&\le 6\left(\frac{5}{3}\right)^m \left(E_{w^{(1)}W_m}(u,u) + E_{w^{(2)}W_m}(u,u) + \left(u(w^{(1)}i^m) - u(w^{(2)}j^m)\right)^2\right).
\end{aligned}$$

In the summation with respect to $w^{(1)}\sim_n w^{(2)}$, the terms $E_{w^{(1)}W_m}(u,u), E_{w^{(2)}W_m}(u,u)$ are summed at most 6 times, hence

$$
\begin{aligned}
G_n(M_{n,m}u) &\le \left(\frac{5}{3}\right)^n \sum_{w^{(1)}\sim_n w^{(2)}} \frac{1}{3^m} \sum_{v\in W_m} \left(u(w^{(1)}v)-u(w^{(2)}v)\right)^2\\
&\le 6\left(\frac{5}{3}\right)^n 6\left(\frac{5}{3}\right)^m E_{n+m}(u,u) = 36\left(\frac{5}{3}\right)^{n+m} E_{n+m}(u,u) = 36 G_{n+m}(u).
\end{aligned}
$$

□

5 Proof of Theorem 2.1

We need the following theorem for preparation.

Theorem 5.1. *For all $u\in L^2(K;\nu)$, let*

$$
\begin{aligned}
E(u) &= \sum_{n=1}^{\infty} 2^{(\beta-\alpha)n} \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_nu(w^{(1)})-P_nu(w^{(2)})\right)^2,\\
F(u) &= \sup_{n\ge1} 2^{(\beta-\alpha)n} \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_nu(w^{(1)})-P_nu(w^{(2)})\right)^2.
\end{aligned}
$$

Then for all $\beta\in(\alpha,+\infty)$, there exists some positive constant c such that

$$|u(x)-u(y)|\le c\sqrt{E(u)}|x-y|^{\frac{\beta-\alpha}{2}}, \tag{5}$$

$$|u(x)-u(y)|\le c\sqrt{F(u)}|x-y|^{\frac{\beta-\alpha}{2}}, \tag{6}$$

for ν-almost every $x,y\in K$, for all $u\in L^2(K;\nu)$.

Therefore, if $u\in L^2(K;\nu)$ satisfies $E(u)<+\infty$ or $F(u)<+\infty$, then u has a continuous version in $C^{\frac{\beta-\alpha}{2}}(K)$.

Remark 5.2. *Theorem 5.1 was also proved in [7, Theorem 4.11 (iii)]. The proof here is direct since E,F are defined using the averaging method.*

Proof. For all $n\ge1, w\in W_n$, we have

$$
\begin{aligned}
&|P_nu(w)-P_{n+1}u(w0)| = |\frac{1}{3}\left(P_{n+1}u(w0)+P_{n+1}u(w1)+P_{n+1}u(w2)\right)-P_{n+1}u(w0)|\\
&= \frac{1}{3}|(P_{n+1}u(w1)-P_{n+1}u(w0))+(P_{n+1}u(w2)-P_{n+1}u(w0))|\\
&\le \frac{\sqrt{2}}{3}\left[(P_{n+1}u(w1)-P_{n+1}u(w0))^2+(P_{n+1}u(w2)-P_{n+1}u(w0))^2\right]^{1/2}\\
&\le \frac{\sqrt{2}}{3}\left(\frac{E(u)}{2^{(\beta-\alpha)(n+1)}}\right)^{1/2} = \frac{\sqrt{2}}{3}\frac{\sqrt{E(u)}}{2^{\frac{\beta-\alpha}{2}(n+1)}}.
\end{aligned}
$$

Similarly

$$|P_nu(w)-P_{n+1}u(w1)|, |P_nu(w)-P_{n+1}u(w2)| \le \frac{\sqrt{2}}{3}\frac{\sqrt{E(u)}}{2^{\frac{\beta-\alpha}{2}(n+1)}}.$$

Since $u\in L^2(K;\nu)$, we have ν-almost every point in K is a Lebesgue point of u. For all Lebesgue points $x,y\in K$ with $x\ne y$, there exist $n\ge1$, $w^{(1)}=w^{(1)}_1\ldots w^{(1)}_n\sim_n w^{(2)}=w^{(2)}_1\ldots w^{(2)}_n$ such that $x\in K_{w^{(1)}}$, $y\in K_{w^{(2)}}$ but $x\notin K_{w^{(1)}w^{(2)}_n}$ or $y\notin K_{w^{(2)}w^{(1)}_n}$, see Figure 11.

Then

$$\frac{\sqrt{3}}{2}\frac{1}{2^{n+1}}\le|x-y|\le\frac{1}{2^{n-1}}.$$

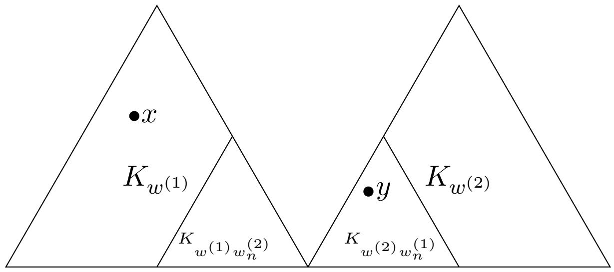


Figure 11: Positions of x and y

There exist $v_1^{(1)}, v_2^{(1)}, \ldots, v_1^{(2)}, v_2^{(2)}, \ldots = 0, 1, 2$ such that

$$\{x\} = \bigcap_{k=1}^{\infty} K_{w^{(1)}v_1^{(1)}\ldots v_k^{(1)}}, \{y\} = \bigcap_{k=1}^{\infty} K_{w^{(2)}v_1^{(2)}\ldots v_k^{(2)}}.$$

Since x, y are Lebesgue points of u, we have

$$\lim_{k\to+\infty} P_{n+k}u(w^{(1)}v_1^{(1)}\ldots v_k^{(1)}) = u(x), \lim_{k\to+\infty} P_{n+k}u(w^{(2)}v_1^{(2)}\ldots v_k^{(2)}) = u(y).$$

Note that

$$|P_nu(w^{(1)}) - P_nu(w^{(2)})| \le \left(\frac{E(u)}{2^{(\beta-\alpha)n}}\right)^{1/2} = \frac{\sqrt{E(u)}}{2^{\frac{\beta-\alpha}{2}n}},$$

hence

$$\begin{aligned}
&|u(x) - u(y)| \le |u(x) - P_nu(w^{(1)})| + |P_nu(w^{(1)}) - P_nu(w^{(2)})| + |u(y) - P_nu(w^{(2)})| \\
&\le \sum_{k=0}^{\infty} |P_{n+k}u(w^{(1)}v_1^{(1)}\ldots v_k^{(1)}) - P_{n+k+1}u(w^{(1)}v_1^{(1)}\ldots v_k^{(1)}v_{k+1}^{(1)})| + |P_nu(w^{(1)}) - P_nu(w^{(2)})| \\
&+ \sum_{k=0}^{\infty} |P_{n+k}u(w^{(2)}v_1^{(2)}\ldots v_k^{(2)}) - P_{n+k+1}u(w^{(2)}v_1^{(2)}\ldots v_k^{(2)}v_{k+1}^{(2)})| \\
&\le 2\sum_{k=0}^{\infty} \frac{\sqrt{2}}{3} \frac{\sqrt{E(u)}}{2^{\frac{\beta-\alpha}{2}(n+k+1)}} + \frac{\sqrt{E(u)}}{2^{\frac{\beta-\alpha}{2}n}} = \sqrt{E(u)} \frac{1}{2^{\frac{\beta-\alpha}{2}(n+1)}} \left(\frac{2\sqrt{2}}{3} \frac{2^{\frac{\beta-\alpha}{2}}}{2^{\frac{\beta-\alpha}{2}} - 1} + 2^{\frac{\beta-\alpha}{2}}\right) \\
&\le \sqrt{E(u)}|x-y|^{\frac{\beta-\alpha}{2}} \left[\left(\frac{2}{\sqrt{3}}\right)^{\frac{\beta-\alpha}{2}} \left(\frac{2\sqrt{2}}{3} \frac{2^{\frac{\beta-\alpha}{2}}}{2^{\frac{\beta-\alpha}{2}} - 1} + 2^{\frac{\beta-\alpha}{2}}\right)\right] = c\sqrt{E(u)}|x-y|^{\frac{\beta-\alpha}{2}},
\end{aligned}$$

where

$$c = \left(\frac{2}{\sqrt{3}}\right)^{\frac{\beta-\alpha}{2}} \left(\frac{2\sqrt{2}}{3} \frac{2^{\frac{\beta-\alpha}{2}}}{2^{\frac{\beta-\alpha}{2}} - 1} + 2^{\frac{\beta-\alpha}{2}}\right).$$

Hence we obtain Equation (5). Replacing $E(u)$ by $F(u)$ in above proof, we obtain Equation (6) with the same constant c. □

For all $\beta > 0$, let

$$\mathcal{E}_\beta(u,u) = \sum_{n=1}^{\infty} 2^{(\beta-\alpha)n} \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_nu(w^{(1)}) - P_nu(w^{(2)})\right)^2,$$

$$\mathcal{F}_\beta = \left\{u \in L^2(K;\nu) : \sum_{n=1}^{\infty} 2^{(\beta-\alpha)n} \sum_{w^{(1)}\sim_n w^{(2)}} \left(P_nu(w^{(1)}) - P_nu(w^{(2)})\right)^2 < +\infty\right\},$$

denote $\mathscr{E}_\beta(u,u) = [u]_{B^{2,2}_{\alpha,\beta}(K)}$ for simplicity.

In what follows, K is a locally compact separable metric space and ν is a Radon measure on K with full support. We say that $(\mathcal{E}, \mathcal{F})$ is a *closed form on $L^2(K;\nu)$ in the wide sense* if $\mathcal{F}$ is complete under the inner product $\mathcal{E}_1$ but $\mathcal{F}$ is not necessary to be dense in $L^2(K;\nu)$. If $(\mathcal{E},\mathcal{F})$ is a closed form on $L^2(K;\nu)$ in the wide sense, we extend $\mathcal{E}$ to be $+\infty$ outside $\mathcal{F}$, hence the information of $\mathcal{F}$ is encoded in $\mathcal{E}$.

In what follows, K is SG in $\mathbb{R}^2$ and ν is the normalized Hausdorff measure on K.

We obtain non-local regular closed forms and Dirichlet forms as follows.

Theorem 5.3. *For all $\beta\in(\alpha,\beta^*)$, $(\mathcal{E}_\beta,\mathcal{F}_\beta)$ is a non-local regular closed form on $L^2(K;\nu)$, $(\mathscr{E}_\beta,\mathcal{F}_\beta),(\mathfrak{E}_\beta,\mathcal{F}_\beta)$ are non-local regular Dirichlet forms on $L^2(K;\nu)$. For all $\beta\in[\beta^*,+\infty)$, $\mathcal{F}_\beta$ consists only of constant functions.*

Remark 5.4. *$\mathcal{E}_\beta$ does not have Markovian property but $\mathscr{E}_\beta,\mathfrak{E}_\beta$ do have Markovian property. An interesting problem in non-local analysis is for which value $\beta>0$, $(\mathfrak{E}_\beta,\mathcal{F}_\beta)$ is a regular Dirichlet form on $L^2(K;\nu)$. The critical value*

$$\beta_*:=\sup\left\{\beta>0:(\mathfrak{E}_\beta,\mathcal{F}_\beta)\text{ is a regular Dirichlet form on }L^2(K;\nu)\right\}$$

is called the walk dimension of SG *with Euclidean metric and Hausdorff measure. A classical approach to determine β_* is using the estimates (1) and subordination technique to have*

$$\beta_*=\beta^*=\frac{\log 5}{\log 2},$$

see [18]. The proof of above theorem provides an alternative approach without using BM.

Proof. By Fatou's lemma, it is obvious that $(\mathcal{E}_\beta,\mathcal{F}_\beta)$ is a closed form on $L^2(K;\nu)$ in the wide sense.

For all $\beta\in(\alpha,\beta^*)$. By Theorem 5.1, $\mathcal{F}_\beta\subseteq C(K)$. We only need to show that $\mathcal{F}_\beta$ is uniformly dense in $C(K)$, then $\mathcal{F}_\beta$ is dense in $L^2(K;\nu)$, hence $(\mathcal{E}_\beta,\mathcal{F}_\beta)$ is a regular closed form on $L^2(K;\nu)$.

Indeed, by Theorem 3.6, for all $U=U^{(x_0,x_1,x_2)}\in\mathcal{U}$, we have

$$\begin{aligned}
&\mathcal{E}_\beta(U,U)=\sum_{n=1}^\infty 2^{(\beta-\alpha)n}A_n(U)\\
&=\sum_{n=1}^\infty 2^{(\beta-\alpha)n}\frac{2}{3}\left[\left(\frac{3}{5}\right)^n-\left(\frac{3}{5}\right)^{2n}\right]\left((x_0-x_1)^2+(x_1-x_2)^2+(x_0-x_2)^2\right)\\
&\le\frac{2}{3}\left((x_0-x_1)^2+(x_1-x_2)^2+(x_0-x_2)^2\right)\sum_{n=1}^\infty 2^{(\beta-\alpha)n}\left(\frac{3}{5}\right)^n<+\infty,
\end{aligned}$$

hence $U\in\mathcal{F}_\beta$, $\mathcal{U}\subseteq\mathcal{F}_\beta$. By Proposition 3.8, $\mathcal{F}_\beta$ separates points. It is obvious that $\mathcal{F}_\beta$ is a sub-algebra of $C(K)$, that is, for all $u,v\in\mathcal{F}_\beta,c\in\mathbb{R}$, we have $u+v,uv,cu\in\mathcal{F}_\beta$. By Stone-Weierstrass theorem, $\mathcal{F}_\beta$ is uniformly dense in $C(K)$.

Since $\mathscr{E}_\beta,\mathfrak{E}_\beta$ do have Markovian property, by above, $(\mathscr{E}_\beta,\mathcal{F}_\beta),(\mathfrak{E}_\beta,\mathcal{F}_\beta)$ are non-local regular Dirichlet forms on $L^2(K;\nu)$.

For all $\beta\in[\beta^*,+\infty)$. Assume that $u\in\mathcal{F}_\beta$ is not constant, then there exists some integer $N\ge1$ such that $D_N(u)>0$. By Theorem 4.1, we have

$$\begin{aligned}
\mathcal{E}_\beta(u,u)&=\sum_{n=1}^\infty 2^{(\beta-\alpha)n}\left(\frac{3}{5}\right)^n D_n(u)\ge\sum_{n=N+1}^\infty 2^{(\beta-\alpha)n}\left(\frac{3}{5}\right)^n D_n(u)\\
&\ge\frac{1}{C}\sum_{n=N+1}^\infty 2^{(\beta-\alpha)n}\left(\frac{3}{5}\right)^n D_N(u)=+\infty,
\end{aligned}$$

contradiction! Hence $\mathcal{F}_\beta$ consists of constant functions. □

We need some preparation about Γ-convergence.

In what follows, K is a locally compact separable metric space and ν is a Radon measure on K with full support.

Definition 5.5. *Let $\mathcal{E}^n,\mathcal{E}$ be closed forms on $L^2(K;\nu)$ in the wide sense. We say that $\mathcal{E}^n$ is Γ-convergent to $\mathcal{E}$ if the following conditions are satisfied.*

(1) For all $\{u_n\}\subseteq L^2(K;\nu)$ that converges strongly to $u\in L^2(K;\nu)$, we have

$$\varliminf_{n\to+\infty}\mathcal{E}^n(u_n,u_n)\ge\mathcal{E}(u,u).$$

(2) For all $u\in L^2(K;\nu)$, there exists a sequence $\{u_n\}\subseteq L^2(K;\nu)$ converging strongly to u in $L^2(K;\nu)$ such that

$$\varlimsup_{n\to+\infty}\mathcal{E}^n(u_n,u_n)\le\mathcal{E}(u,u).$$

We have the following result about Γ-convergence.

Proposition 5.6. *([4, Proposition 6.8, Theorem 8.5, Theorem 11.10, Proposition 12.16]) Let $\{(\mathcal{E}^n,\mathcal{F}^n)\}$ be a sequence of closed forms on $L^2(K;\nu)$ in the wide sense, then there exist some subsequence $\{(\mathcal{E}^{n_k},\mathcal{F}^{n_k})\}$ and some closed form $(\mathcal{E},\mathcal{F})$ on $L^2(K;\nu)$ in the wide sense such that $\mathcal{E}^{n_k}$ is Γ-convergent to $\mathcal{E}$.*

We need an elementary result as follows.

Proposition 5.7. *Let $\{x_n\}$ be a sequence of nonnegative real numbers.*

(1)

$$\varliminf_{n\to+\infty}x_n\le\varliminf_{\lambda\uparrow1}(1-\lambda)\sum_{n=1}^{\infty}\lambda^nx_n\le\varlimsup_{\lambda\uparrow1}(1-\lambda)\sum_{n=1}^{\infty}\lambda^nx_n\le\varlimsup_{n\to+\infty}x_n\le\sup_{n\ge1}x_n.$$

(2) If there exists some positive constant C such that

$$x_n\le Cx_{n+m}\text{ for all }n,m\ge1,$$

then

$$\sup_{n\ge1}x_n\le C\varliminf_{n\to+\infty}x_n.$$

Proof. The proof is elementary using ε-N argument. □

In what follows, K is SG in $\mathbb{R}^2$ and ν is the normalized Hausdorff measure on K.

Take $\{\beta_n\}\subseteq(\alpha,\beta^*)$ with $\beta_n\uparrow\beta^*$. By Proposition 5.6, there exist some subsequence still denoted by $\{\beta_n\}$ and some closed form $(\mathcal{E},\mathcal{F})$ on $L^2(K;\nu)$ in the wide sense such that $(\beta^*-\beta_n)\mathcal{E}_{\beta_n}$ is Γ-convergent to $\mathcal{E}$. Without lose of generality, we may assume that

$$0<\beta^*-\beta_n<\frac{1}{n+1}\text{ for all }n\ge1.$$

We have the characterization of $(\mathcal{E},\mathcal{F})$ on $L^2(K;\nu)$ as follows.

Theorem 5.8.

$$\mathcal{E}(u,u)\asymp\sup_{n\ge1}D_n(u)=\sup_{n\ge1}\left(\frac{5}{3}\right)^n\sum_{w^{(1)}\sim_nw^{(2)}}\left(P_nu(w^{(1)})-P_nu(w^{(2)})\right)^2,$$

$$\mathcal{F}=\left\{u\in L^2(K;\nu):\sup_{n\ge1}\left(\frac{5}{3}\right)^n\sum_{w^{(1)}\sim_nw^{(2)}}\left(P_nu(w^{(1)})-P_nu(w^{(2)})\right)^2<+\infty\right\}.$$

Moreover, $(\mathcal{E},\mathcal{F})$ is a regular closed form on $L^2(K;\nu)$ and

$$\frac{1}{2(\log2)C^2}\sup_{n\ge1}D_n(u)\le\mathcal{E}(u,u)\le\frac{1}{\log2}\sup_{n\ge1}D_n(u).$$

Proof. Recall that

$$\mathcal{E}_\beta(u,u)=\sum_{n=1}^{\infty}2^{(\beta-\alpha)n}A_n(u)=\sum_{n=1}^{\infty}2^{(\beta-\beta^*)n}D_n(u).$$

We use weak monotonicity result Theorem 4.1 and elementary result Proposition 5.7.

On the one hand, for all $u \in L^2(K;\nu)$

$$\begin{aligned}\mathcal{E}(u,u) &\le \varliminf_{n\to+\infty}(\beta^*-\beta_n)\mathcal{E}_{\beta_n}(u,u) = \varliminf_{n\to+\infty}(\beta^*-\beta_n)\sum_{k=1}^{\infty}2^{(\beta_n-\beta^*)k}D_k(u)\\ &= \frac{1}{\log 2}\varliminf_{n\to+\infty}(1-2^{\beta_n-\beta^*})\sum_{k=1}^{\infty}2^{(\beta_n-\beta^*)k}D_k(u) \le \frac{1}{\log 2}\sup_{k\ge1}D_k(u).\end{aligned}$$

On the other hand, for all $u \in L^2(K;\nu)$, there exists $\{u_n\} \subseteq L^2(K;\nu)$ converging strongly to u in $L^2(K;\nu)$ such that

$$\begin{aligned}&\mathcal{E}(u,u) \ge \varlimsup_{n\to+\infty}(\beta^*-\beta_n)\mathcal{E}_{\beta_n}(u_n,u_n) = \varlimsup_{n\to+\infty}(\beta^*-\beta_n)\sum_{k=1}^{\infty}2^{(\beta_n-\beta^*)k}D_k(u_n)\\ &\ge \varlimsup_{n\to+\infty}(\beta^*-\beta_n)\sum_{k=n+1}^{\infty}2^{(\beta_n-\beta^*)k}D_k(u_n) \ge \frac{1}{C}\varlimsup_{n\to+\infty}(\beta^*-\beta_n)\sum_{k=n+1}^{\infty}2^{(\beta_n-\beta^*)k}D_n(u_n)\\ &= \frac{1}{C}\varlimsup_{n\to+\infty}\left[(\beta^*-\beta_n)\frac{2^{(\beta_n-\beta^*)(n+1)}}{1-2^{\beta_n-\beta^*}}D_n(u_n)\right].\end{aligned}$$

Since $0 < \beta^*-\beta_n < 1/(n+1)$, we have $2^{(\beta_n-\beta^*)(n+1)} > 1/2$. Since

$$\lim_{n\to+\infty}\frac{\beta^*-\beta_n}{1-2^{\beta_n-\beta^*}} = \frac{1}{\log 2},$$

we have

$$\mathcal{E}(u,u) \ge \frac{1}{2C}\varlimsup_{n\to+\infty}\frac{\beta^*-\beta_n}{1-2^{\beta_n-\beta^*}}D_n(u_n) \ge \frac{1}{2(\log 2)C}\varlimsup_{n\to+\infty}D_n(u_n).$$

Since $u_n \to u$ in $L^2(K;\nu)$, for all $k \ge 1$, we have

$$D_k(u) = \lim_{n\to+\infty}D_k(u_n) = \lim_{k\le n\to+\infty}D_k(u_n) \le C\varliminf_{n\to+\infty}D_n(u_n).$$

Taking supremum with respect to $k \ge 1$, we have

$$\sup_{k\ge1}D_k(u) \le C\varliminf_{n\to+\infty}D_n(u_n) \le C\varlimsup_{n\to+\infty}D_n(u_n) \le 2(\log 2)C^2\mathcal{E}(u,u).$$

By Theorem 5.1, $\mathcal{F} \subseteq C(K)$. We only need to show that $\mathcal{F}$ is uniformly dense in $C(K)$, then $\mathcal{F}$ is dense in $L^2(K;\nu)$, hence $(\mathcal{E},\mathcal{F})$ is a regular closed form on $L^2(K;\nu)$.

Indeed, by Theorem 3.6, for all $U = U^{(x_0,x_1,x_2)} \in \mathcal{U}$, we have

$$\begin{aligned}&\sup_{n\ge1}D_n(U) = \sup_{n\ge1}\left(\frac{5}{3}\right)^n A_n(U)\\ &= \sup_{n\ge1}\left(\frac{5}{3}\right)^n\frac{2}{3}\left[\left(\frac{3}{5}\right)^n - \left(\frac{3}{5}\right)^{2n}\right]\left((x_0-x_1)^2+(x_1-x_2)^2+(x_0-x_2)^2\right)\\ &\le \frac{2}{3}\left((x_0-x_1)^2+(x_1-x_2)^2+(x_0-x_2)^2\right) < +\infty,\end{aligned}$$

hence $U \in \mathcal{F}$, $\mathcal{U} \subseteq \mathcal{F}$. By Proposition 3.8, $\mathcal{F}$ separates points. It is obvious that $\mathcal{F}$ is a sub-algebra of $C(K)$. By Stone-Weierstrass theorem, $\mathcal{F}$ is uniformly dense in $C(K)$. □

Now we prove Theorem 2.1 using a standard approach as follows.

Proof of Theorem 2.1. For all $u \in L^2(K;\nu), n, k \ge 1$, we have

$$\begin{aligned}&\sum_{w^{(1)}\sim_{n+k}w^{(2)}}\left(P_{n+k}u(w^{(1)}) - P_{n+k}u(w^{(2)})\right)^2\\ &= \sum_{w\in W_n}\sum_{w^{(1)}\sim_k w^{(2)}}\left(P_{n+k}u(ww^{(1)}) - P_{n+k}u(ww^{(2)})\right)^2\\ &+ \sum_{w^{(1)}=w_1^{(1)}\ldots w_n^{(1)}\sim_n w^{(2)}=w_1^{(2)}\ldots w_n^{(2)}}\left(P_{n+k}u(w^{(1)}w_n^{(2)}\ldots w_n^{(2)}) - P_{n+k}u(w^{(2)}w_n^{(1)}\ldots w_n^{(1)})\right)^2,\end{aligned}$$

where for all $i = 1, 2$

$$P_{n+k}u(ww^{(i)}) = \int_K (u \circ f_{ww^{(i)}})(x)\nu(\mathrm{d}x) = \int_K (u \circ f_w \circ f_{w^{(i)}})(x)\nu(\mathrm{d}x) = P_k(u \circ f_w)(w^{(i)}),$$

hence

$$\begin{aligned}
\sum_{w \in W_n} A_k(u \circ f_w) &= \sum_{w \in W_n} \sum_{w^{(1)} \sim_k w^{(2)}} \left(P_k(u \circ f_w)(w^{(1)}) - P_k(u \circ f_w)(w^{(2)})\right)^2 \\
&\le \sum_{w^{(1)} \sim_{n+k} w^{(2)}} \left(P_{n+k}u(w^{(1)}) - P_{n+k}u(w^{(2)})\right)^2 = A_{n+k}(u),
\end{aligned}$$

and

$$\left(\frac{5}{3}\right)^n \sum_{w \in W_n} D_k(u \circ f_w) = \left(\frac{5}{3}\right)^{n+k} \sum_{w \in W_n} A_k(u \circ f_w) \le \left(\frac{5}{3}\right)^{n+k} A_{n+k}(u) = D_{n+k}(u).$$

For all $u \in \mathcal{F}, n \ge 1, w \in W_n$, we have

$$\sup_{k \ge 1} D_k(u \circ f_w) \le \sup_{k \ge 1} \sum_{w \in W_n} D_k(u \circ f_w) \le \left(\frac{3}{5}\right)^n \sup_{k \ge 1} D_{n+k}(u) \le \left(\frac{3}{5}\right)^n \sup_{k \ge 1} D_k(u) < +\infty,$$

hence $u \circ f_w \in \mathcal{F}$.

For all $u \in L^2(K;\nu), n \ge 1$, let

$$\begin{aligned}
\overline{\mathcal{E}}(u,u) &= \sum_{i=0}^{2} \left(u(p_i) - \int_K u(x)\nu(\mathrm{d}x)\right)^2, \\
\overline{\mathcal{E}}^{(n)}(u,u) &= \left(\frac{5}{3}\right)^n \sum_{w \in W_n} \overline{\mathcal{E}}(u \circ f_w, u \circ f_w).
\end{aligned}$$

By Theorem 5.1, we have

$$\begin{aligned}
\overline{\mathcal{E}}(u,u) &= \sum_{i=0}^{2} \left(\int_K (u(p_i) - u(x))\,\nu(\mathrm{d}x)\right)^2 \le \sum_{i=0}^{2} \int_K (u(p_i) - u(x))^2\,\nu(\mathrm{d}x) \\
&\le \sum_{i=0}^{2} \int_K c^2 |p_i - x|^{\beta^* - \alpha} \left(\sup_{k \ge 1} D_k(u)\right) \nu(\mathrm{d}x) \le 3c^2 \sup_{k \ge 1} D_k(u),
\end{aligned}$$

hence

$$\begin{aligned}
\overline{\mathcal{E}}^{(n)}(u,u) &\le \left(\frac{5}{3}\right)^n \sum_{w \in W_n} 3c^2 \sup_{k \ge 1} D_k(u \circ f_w) \le 3c^2 C \left(\frac{5}{3}\right)^n \sum_{w \in W_n} \varliminf_{k \to +\infty} D_k(u \circ f_w) \\
&\le 3c^2 C \left(\frac{5}{3}\right)^n \varliminf_{k \to +\infty} \sum_{w \in W_n} D_k(u \circ f_w) \le 3c^2 C \varliminf_{k \to +\infty} D_{n+k}(u) \le 3c^2 C \sup_{k \ge 1} D_k(u).
\end{aligned} \tag{7}$$

On the other hand, for all $u \in L^2(K;\nu), n \ge 1$, we have

$$D_n(u) = \left(\frac{5}{3}\right)^n \sum_{w^{(1)} \sim_n w^{(2)}} \left(\int_K (u \circ f_{w^{(1)}})(x)\nu(\mathrm{d}x) - \int_K (u \circ f_{w^{(2)}})(x)\nu(\mathrm{d}x)\right)^2.$$

For all $w^{(1)} \sim_n w^{(2)}$, there exist $i, j = 0, 1, 2$ such that

$$K_{w^{(1)}} \cap K_{w^{(2)}} = \left\{f_{w^{(1)}}(p_i)\right\} = \left\{f_{w^{(2)}}(p_j)\right\}.$$

Hence

$$
\begin{aligned}
&D_n(u)\\
&=\left(\frac{5}{3}\right)^n\sum_{w^{(1)}\sim_n w^{(2)}}\left[\left((u\circ f_{w^{(1)}})(p_i)-\int_K(u\circ f_{w^{(1)}})(x)\nu(\mathrm{d}x)\right)\right.\\
&\left.-\left((u\circ f_{w^{(2)}})(p_j)-\int_K(u\circ f_{w^{(2)}})(x)\nu(\mathrm{d}x)\right)\right]^2\\
&\le 2\left(\frac{5}{3}\right)^n\sum_{w^{(1)}\sim_n w^{(2)}}\left[\left((u\circ f_{w^{(1)}})(p_i)-\int_K(u\circ f_{w^{(1)}})(x)\nu(\mathrm{d}x)\right)^2\right.\\
&\left.+\left((u\circ f_{w^{(2)}})(p_j)-\int_K(u\circ f_{w^{(2)}})(x)\nu(\mathrm{d}x)\right)^2\right]\\
&\le 6\left(\frac{5}{3}\right)^n\sum_{w\in W_n}\sum_{i=0}^2\left((u\circ f_w)(p_i)-\int_K(u\circ f_w)(x)\nu(\mathrm{d}x)\right)^2\\
&=6\left(\frac{5}{3}\right)^n\sum_{w\in W_n}\overline{\mathcal{E}}(u\circ f_w,u\circ f_w)=6\overline{\mathcal{E}}^{(n)}(u,u).
\end{aligned}
\tag{8}
$$

For all $u\in L^2(K;\nu),n\ge 1$, we have

$$
\begin{aligned}
&\overline{\mathcal{E}}^{(n+1)}(u,u)=\left(\frac{5}{3}\right)^{n+1}\sum_{w\in W_{n+1}}\overline{\mathcal{E}}(u\circ f_w,u\circ f_w)\\
&=\left(\frac{5}{3}\right)^{n+1}\sum_{i=0}^2\sum_{w\in W_n}\overline{\mathcal{E}}(u\circ f_i\circ f_w,u\circ f_i\circ f_w)=\frac{5}{3}\sum_{i=0}^2\overline{\mathcal{E}}^{(n)}(u\circ f_i,u\circ f_i).
\end{aligned}
\tag{9}
$$

Let

$$\tilde{\mathcal{E}}^{(n)}(u,u)=\frac{1}{n}\sum_{l=1}^n\overline{\mathcal{E}}^{(l)}(u,u),u\in L^2(K;\nu),n\ge 1.$$

By Equation (7), we have

$$\tilde{\mathcal{E}}^{(n)}(u,u)\le 3c^2C\sup_{k\ge 1}D_k(u)\asymp\mathcal{E}(u,u)\text{ for all }u\in\mathcal{F},n\ge 1.$$

Since $(\mathcal{E},\mathcal{F})$ is a regular closed form on $L^2(K;\nu)$, by [3, Definition 1.3.8, Remark 1.3.9, Definition 1.3.10, Remark 1.3.11], we have $(\mathcal{F},\mathcal{E}_1)$ is a separable Hilbert space. Let $\{u_i\}_{i\ge 1}$ be a dense subset of $(\mathcal{F},\mathcal{E}_1)$. For all $i\ge 1$, $\left\{\tilde{\mathcal{E}}^{(n)}(u_i,u_i)\right\}_{n\ge 1}$ is a bounded sequence. By diagonal argument, there exists a subsequence $\{n_k\}_{k\ge 1}$ such that $\left\{\tilde{\mathcal{E}}^{(n_k)}(u_i,u_i)\right\}_{k\ge 1}$ converges for all $i\ge 1$. Hence $\left\{\tilde{\mathcal{E}}^{(n_k)}(u,u)\right\}_{k\ge 1}$ converges for all $u\in\mathcal{F}$. Let

$$\mathcal{E}_{\mathrm{loc}}(u,u)=\lim_{k\to+\infty}\tilde{\mathcal{E}}^{(n_k)}(u,u)\text{ for all }u\in\mathcal{F}_{\mathrm{loc}}:=\mathcal{F}.$$

Then

$$\mathcal{E}_{\mathrm{loc}}(u,u)\le 3c^2C\sup_{k\ge 1}D_k(u)\text{ for all }u\in\mathcal{F}_{\mathrm{loc}}=\mathcal{F}.$$

By Equation (8), for all $u\in\mathcal{F}_{\mathrm{loc}}=\mathcal{F}$, we have

$$\mathcal{E}_{\mathrm{loc}}(u,u)=\lim_{k\to+\infty}\tilde{\mathcal{E}}^{(n_k)}(u,u)\ge\varliminf_{n\to+\infty}\overline{\mathcal{E}}^{(n)}(u,u)\ge\frac{1}{6}\varliminf_{k\to+\infty}D_k(u)\ge\frac{1}{6C}\sup_{k\ge 1}D_k(u).$$

Hence

$$\mathcal{E}_{\mathrm{loc}}(u,u)\asymp\sup_{k\ge 1}D_k(u)\text{ for all }u\in\mathcal{F}_{\mathrm{loc}}=\mathcal{F}.$$

Hence $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ is a regular closed form on $L^2(K;\nu)$. Since $1\in\mathcal{F}_{\mathrm{loc}}$ and $\mathcal{E}_{\mathrm{loc}}(1,1)=0$, by [6, Lemma 1.6.5, Theorem 1.6.3], $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ on $L^2(K;\nu)$ is conservative.

For all $u\in\mathcal{F}_{\mathrm{loc}}=\mathcal{F}$, we have $u\circ f_i\in\mathcal{F}=\mathcal{F}_{\mathrm{loc}}$ for all $i=0,1,2$. Moreover, by Equation (9), we have

$$
\begin{aligned}
&\frac{5}{3}\sum_{i=0}^{2}\mathcal{E}_{\mathrm{loc}}(u\circ f_i,u\circ f_i)=\frac{5}{3}\sum_{i=0}^{2}\lim_{k\to+\infty}\tilde{\mathcal{E}}^{(n_k)}(u\circ f_i,u\circ f_i)\\
&=\frac{5}{3}\sum_{i=0}^{2}\lim_{k\to+\infty}\frac{1}{n_k}\sum_{l=1}^{n_k}\overline{\mathcal{E}}^{(l)}(u\circ f_i,u\circ f_i)=\lim_{k\to+\infty}\frac{1}{n_k}\sum_{l=1}^{n_k}\left[\frac{5}{3}\sum_{i=0}^{2}\overline{\mathcal{E}}^{(l)}(u\circ f_i,u\circ f_i)\right]\\
&=\lim_{k\to+\infty}\frac{1}{n_k}\sum_{l=1}^{n_k}\overline{\mathcal{E}}^{(l+1)}(u,u)=\lim_{k\to+\infty}\frac{1}{n_k}\sum_{l=2}^{n_k+1}\overline{\mathcal{E}}^{(l)}(u,u)\\
&=\lim_{k\to+\infty}\left[\frac{1}{n_k}\sum_{l=1}^{n_k}\overline{\mathcal{E}}^{(l)}(u,u)+\frac{1}{n_k}\overline{\mathcal{E}}^{(n_k+1)}(u,u)-\frac{1}{n_k}\overline{\mathcal{E}}^{(1)}(u,u)\right]\\
&=\lim_{k\to+\infty}\tilde{\mathcal{E}}^{(n_k)}(u,u)=\mathcal{E}_{\mathrm{loc}}(u,u).
\end{aligned}
$$

Hence $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ on $L^2(K;\nu)$ is self-similar.

For all $u,v\in\mathcal{F}_{\mathrm{loc}}$ satisfying $\mathrm{supp}(u),\mathrm{supp}(v)$ are compact and v is constant in an open neighborhood U of $\mathrm{supp}(u)$, we have $K\backslash U$ is compact and $\mathrm{supp}(u)\cap(K\backslash U)=\emptyset$, hence $\delta=\mathrm{dist}(\mathrm{supp}(u),K\backslash U)>0$. Taking sufficiently large $n\ge1$ such that $2^{1-n}<\delta$, by self-similarity, we have

$$
\mathcal{E}_{\mathrm{loc}}(u,v)=\left(\frac{5}{3}\right)^n\sum_{w\in W_n}\mathcal{E}_{\mathrm{loc}}(u\circ f_w,v\circ f_w).
$$

For all $w\in W_n$, we have $u\circ f_w=0$ or $v\circ f_w$ is constant, hence $\mathcal{E}_{\mathrm{loc}}(u\circ f_w,v\circ f_w)=0$, hence $\mathcal{E}_{\mathrm{loc}}(u,v)=0$, that is, $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ on $L^2(K;\nu)$ is strongly local.

For all $u\in\mathcal{F}_{\mathrm{loc}}$, it is obvious that $u^+,u^-,1-u,\overline{u}=(0\vee u)\wedge1\in\mathcal{F}_{\mathrm{loc}}$ and

$$
\mathcal{E}_{\mathrm{loc}}(u,u)=\mathcal{E}_{\mathrm{loc}}(1-u,1-u).
$$

Since $u^+u^-=0$ and $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ on $L^2(K;\nu)$ is strongly local, we have $\mathcal{E}_{\mathrm{loc}}(u^+,u^-)=0$. Hence

$$
\begin{aligned}
\mathcal{E}_{\mathrm{loc}}(u,u)&=\mathcal{E}_{\mathrm{loc}}(u^+-u^-,u^+-u^-)=\mathcal{E}_{\mathrm{loc}}(u^+,u^+)+\mathcal{E}_{\mathrm{loc}}(u^-,u^-)-2\mathcal{E}_{\mathrm{loc}}(u^+,u^-)\\
&=\mathcal{E}_{\mathrm{loc}}(u^+,u^+)+\mathcal{E}_{\mathrm{loc}}(u^-,u^-)\ge\mathcal{E}_{\mathrm{loc}}(u^+,u^+)=\mathcal{E}_{\mathrm{loc}}(1-u^+,1-u^+)\\
&\ge\mathcal{E}_{\mathrm{loc}}((1-u^+)^+,(1-u^+)^+)=\mathcal{E}_{\mathrm{loc}}(1-(1-u^+)^+,1-(1-u^+)^+)=\mathcal{E}_{\mathrm{loc}}(\overline{u},\overline{u}),
\end{aligned}
$$

that is, $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ on $L^2(K;\nu)$ is Markovian. Hence $(\mathcal{E}_{\mathrm{loc}},\mathcal{F}_{\mathrm{loc}})$ is a self-similar strongly local regular Dirichlet form on $L^2(K;\nu)$. □

Remark 5.9. *The idea of the standard approach is from [16, Section 6]. The proof of Markovian property is from the proof of [1, Theorem 2.1].*

References

[1] M. T. Barlow, R. F. Bass, T. Kumagai, and A. Teplyaev, *Uniqueness of Brownian motion on Sierpiński carpets*, J. Eur. Math. Soc. (JEMS), 12 (2010), pp. 655–701.

[2] M. T. Barlow and E. A. Perkins, *Brownian motion on the Sierpiński gasket*, Probab. Theory Related Fields, 79 (1988), pp. 543–623.

[3] Z.-Q. Chen and M. Fukushima, *Symmetric Markov processes, time change, and boundary theory*, vol. 35 of London Mathematical Society Monographs Series, Princeton University Press, Princeton, NJ, 2012.

[4] G. Dal Maso, *An introduction to Γ-convergence*, Progress in Nonlinear Differential Equations and their Applications, 8, Birkhäuser Boston, Inc., Boston, MA, 1993.

[5] P. G. Doyle and J. L. Snell, *Random walks and electric networks*, vol. 22 of Carus Mathematical Monographs, Mathematical Association of America, Washington, DC, 1984.

[6] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, vol. 19 of De Gruyter studies in mathematics ; 19, de Gruyter, Berlin [u.a.], 2., rev. and extended ed. ed., 2011.

[7] A. Grigor'yan, J. Hu, and K.-S. Lau, *Heat kernels on metric measure spaces and an application to semilinear elliptic equations*, Trans. Amer. Math. Soc., 355 (2003), pp. 2065–2095 (electronic).

[8] A. Grigor'yan and M. Yang, *Local and Non-Local Dirichlet Forms on the Sierpiński Carpet*, ArXiv e-prints, (2017).

[9] M. Hino and T. Kumagai, *A trace theorem for Dirichlet forms on fractals*, J. Funct. Anal., 238 (2006), pp. 578–611.

[10] J. Hu and X. Wang, *Domains of Dirichlet forms and effective resistance estimates on p.c.f. fractals*, Studia Math., 177 (2006), pp. 153–172.

[11] A. Jonsson, *Brownian motion on fractals and function spaces*, Math. Z., 222 (1996), pp. 495–504.

[12] J. Kigami, *A harmonic calculus on the Sierpiński spaces*, Japan J. Appl. Math., 6 (1989), pp. 259–290.

[13] ——, *Harmonic calculus on p.c.f. self-similar sets*, Trans. Amer. Math. Soc., 335 (1993), pp. 721–755.

[14] ——, *Analysis on fractals*, vol. 143 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2001.

[15] T. Kumagai and K.-T. Sturm, *Construction of diffusion processes on fractals, d-sets, and general metric measure spaces*, J. Math. Kyoto Univ., 45 (2005), pp. 307–327.

[16] S. Kusuoka and Z. X. Yin, *Dirichlet forms on fractals: Poincaré constant and resistance*, Probab. Theory Related Fields, 93 (1992), pp. 169–196.

[17] K. Pietruska-Pałuba, *Some function spaces related to the Brownian motion on simple nested fractals*, Stochastics Stochastics Rep., 67 (1999), pp. 267–285.

[18] ——, *On function spaces related to fractional diffusions on d-sets*, Stochastics Stochastics Rep., 70 (2000), pp. 153–164.

[19] ——, *Limiting behaviour of Dirichlet forms for stable processes on metric spaces*, Bull. Pol. Acad. Sci. Math., 56 (2008), pp. 257–266.

[20] C. Sabot, *Existence and uniqueness of diffusions on finitely ramified self-similar fractals*, Ann. Sci. École Norm. Sup. (4), 30 (1997), pp. 605–673.

[21] R. S. Strichartz, *The Laplacian on the Sierpinski gasket via the method of averages*, Pacific J. Math., 201 (2001), pp. 241–256.

[22] M. Yang, *Equivalent semi-norms of non-local dirichlet forms on the Sierpiński gasket and applications*, Potential Anal., (2017, to appear).

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany.

E-mail address: `ymeng@math.uni-bielefeld.de`

and

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.

E-mail address: `meng-yang13@mails.tsinghua.edu.cn`