On the Battery Recharge Time in a Stochastic Energy Harvesting System

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***Abstract*—Systems with energy harvesting capability from stochastic sources have been widely studied in the literature. However, the determination of the recharge time of such systems has not received as much attention as it deserves. Here, we examine the recharge time of a battery/super-capacitor when the energy arrival is discrete stochastic. We consider the cases when the energy storage system is modeled as a linear and a nonlinear system. The energy arrival is assumed to be a Poisson process, or more generally, a renewal process; while the energy packet size may assume any distribution with finite mean and variance. We obtain formulas for the distribution and the expected value of the recharge time. Monte-Carlo simulations verify the obtained formulas.**

***Index Terms*—Energy harvesting, recharge time**

I. Introduction

In recent years, ambient energy harvesting and its applications have become a topic to great interest. Basically, a device is assumed to harvest energy from random energy source for its future use. The readers are referred to [1], [2, and reference therein] for a general survey of this field. From the system theoretical perspective, the harvested energy can be modeled as a deterministic or a stochastic process. Furthermore, we can also distinguish the harvested energy as being continuous or discrete. Renewable energy sources such as the sun can be modeled as a deterministic, continuous source of energy, assuming no cloud coverage. Sources such as wind can be modeled as a stochastic, continuous source of energy. Non-traditional sources of energy such as bodily motion can be modeled as a discrete, stochastic source of energy. Some aspects of the recharge time were investigated in [3], [4] where the input power is fixed and continuous, making the system deterministic. The authors obtained the time it takes the system to recharge up to a certain voltage for ideal as well as non-ideal super-capacitor models [5]. Recharge time for systems with stochastic, intermittent energy availability has yet to be studied.

In this letter, we will focus on discrete, stochastic energy sources, where energy arrives as random stream of impulses; and by *battery* we refer to any generic energy storage device, such as an electro-chemical battery or a super-capacitor. *We will model such energy sources as a renewal process and examine the time it takes to recharge a battery up to a certain fixed level. We believe that this work can help in further modeling and optimization of energy harvesting systems.*

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II. System Model, Assumptions, and Definitions

Consider an energy harvesting system where the harvested energy is stored in a battery before being consumed. Let the battery be initially empty[1] and the external energy consumption be turned off. If the system relies only on its energy harvesting capability to recharge itself, then we can model the energy accumulated in the battery, $U(t)$, at any given time $t \geq 0$ after energy outage, as

$$U(t) = \sum_{i=1}^{N_A(t)} \eta_i h(t - t_i; X_i) - \int_0^t p(t)\mathrm{d}t, \tag{1}$$

where X_i is the energy packet size, $p(t)$ is the battery's (possibly) time varying self-discharge rate, $\eta_i \in (0, 1)$ is the recharge efficiency, and $h(t; X)$ is the transient of charging process of the battery given X. Here, $h(t; X)$ is any function such that $h(t; X) = 0$ for $t < 0$ and $\lim_{t\to\infty} h(t; X) = X$. Based on the behavior of η_i, we will refer to the energy storage system as *linear* if η_i is constant and *nonlinear* if η_i depends on the amount of energy stored [6]. Lastly, the $N_A(t) = \min\{k : A_0 + A_1 + \cdots + A_k \leq t\}$ is the counting process of the arriving energy, where $A_{i\geq 1}$ is the inter-arrival time and A_0 is the residual time.

For an *ideal system*, we will assume that: (i) there is no self-discharge, $p(t) = 0$; (ii) $h(t; X) = X\delta(t)$, where $\delta(t)$ is a unit step function; and (iii) recharge efficiency is linear, $\eta_i = \mathrm{constant}$, which we take as unity. In a *realistic system*, the self-discharge rate of a battery is very small and can be neglected. Likewise, for impulsive energy arrivals, which last for very short durations, assuming $h(t; X)$ to be a step function is also a reasonable approximation. As such, the non-idealness of the system can be largely attributed to the nonlinearity of η_i. Therefore, we will focus our subsequent analysis on linear and nonlinear systems.

For the recharge process of an ideal system, we have a simplification of (1) as

$$U(t) = \sum_{i=1}^{N_A(t)} X_i. \tag{2}$$

We now ask for the time required to recharge the battery beyond some desired level. This is essentially a *first passage*

[1] For convenience, the reference energy level, which may have any arbitrary non-zero value, is assumed to be the zero energy level. The reference energy should be taken at the corresponding voltage required for the device operation. Thus $U(t)$ represents the *change* in the battery's energy from this reference level, rather than the total battery energy.

time problem [8]. Let the first passage time for $U(t)$ given in (1) to cross some level $u > 0$ be $\tau(u) = \inf_t\{t : U(t) > u\}$. Obviously, $u \leq U_{\max}$, where $U_{\max}$ is the maximum battery capacity. Here τ is a random variable for a given u. For an ideal system, $U(t)$ in (2) is a pure jump process; hence we have the events $\{\tau(u) < t\} \equiv \{U(t) > u\}$ to be equivalent. Thus,

$$P(\tau(u) \leq t) = P(U(t) > u). \tag{3}$$

Definition 1 (Renewal process)**.** *A sequence of arrival times $\{t_n\}$ is a* ***renewal process*** *if $t_n = A_0 + A_1 + \cdots + A_{n-1}$, where the inter-arrival times $\{A_{i\geq 1}\}$ given by $A_i = t_{i+1} - t_i$ are mutually independent, non-negative random variables with common distribution F_A such that $F_A(0) = 0$. Here A_0 is the time for the first arrival and is known as* ***residual time****. If there is an arrival at the origin, that is $A_0 = 0$, then renewal process is said to be a* ***pure*** *renewal process. Otherwise, the renewal process is said to be a* ***delayed*** *renewal process.*

We can also represent $\tau(u)$ by the decomposition

$$\tau(u) = A_0 + \sum_{i=1}^{N_X(u)} A_i, \tag{4}$$

where $N_X(u) = \min\{k : X_1 + \cdots + X_k \leq u\}$ is again a counting process with X_i as the energy packet size.

Let the mean and variance of A be finite. Also, $\{X_i\}$ is assumed to be a sequence of non-negative random variables, with common distribution F_X such that the mean and variance are finite. We assume that $\{A_i\}$ and $\{X_i\}$ are independent of each other. Lastly, we assume that the random vectors $\{(A_i, X_i)\}$ are identically distributed as (A, X). For notational convenience, we will denote $\lambda = 1/\mathbb{E}[A]$ and $\bar{X} = \mathbb{E}[X]$.

Note that the renewal process becomes a Poisson process when the inter-arrival times are exponentially distributed. The Poisson process can also arise as a result of superposition of common renewal processes [9]. Thus, Poisson energy arrival can be used to model situations where multiple harvesters send energy, according to a common renewal process, to a common battery. Likewise, when the inter-arrival time is deterministic, the renewal process can model the slotted time models. Hence, the renewal process is a generalization of these special cases.

III. Analysis of Linear Storage System

The $N_A(t)$ in (2) will be a pure renewal process only when there is an energy arrival immediately after outage. Since this is an unrealistic expectation, $N_A(t)$ is a *delayed renewal process*. That is, for the epoch of n-th energy arrival t_n as given by $t_n = A_0 + A_1 + \cdots + A_n$, A_0 has a different distribution from the inter-arrival times A_i, $i \geq 1$. Only for Poisson process, due to its memoryless property, $A_0 \overset{d}{=} A_i \overset{d}{=} \mathrm{Exp}(\lambda)$.

Assuming that the outage occurs a long time after the start of the system[2], the residual time A_0 converges in distribution to $f_{A_0}(t) = \frac{1-F_A(t)}{\mu_A}$, where f_{A_0} is the density of A_0, F_A is the inter-arrival distribution, and μ_A is the mean of A. Thus, the resulting renewal process after the outage becomes an *equilibrium (or stationary) renewal process* (see [8, Theo. 4.1]). *Therefore, if the outage occurs a long time after the system is switched on, the subsequent renewal process after the outage behaves as a stationary renewal process.*

In the following subsections, we will first investigate the special case when the energy arrival follows a Poisson process, and then later, more generally, for the case when the energy arrival follows a renewal process.

A. When Energy Arrival Follows a Poisson Process

If energy arrival follows a Poisson process, then the distribution of $U(t)$ in (2) can be obtained by conditioning on $N_A(t) = n$, and using total probability theorem as

$$P(U(t) \leq u) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} F_X^{(n)}(u),$$

where $F_X^{(n)}$ is the n-fold convolution of F_X defined recursively as $F_X^{(i)}(x) = \int F_X^{(i-1)}(x-t)\, \mathrm{d}F(t)$ where $i = 1, 2, 3, \ldots, n$; and $F_X^{(0)}(x)$ is a unit step function at the origin. Thus, from (3) the distribution of the level crossing time is

$$P(\tau(u) < t) = 1 - e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} F_X^{(n)}(u). \tag{5}$$

Except for few packet size distributions (like exponential, gamma, and inverse Gaussian), it is difficult to evaluate (5) exactly for general F_X distribution. However, if we make normal approximation for $F_X^{(n)}$, which is the distribution for the sum of n independent random variables, as per the central limit theorem, as $F_X^{(n)}(x) \approx \Phi(\frac{x - n\bar{X}}{\sigma_X \sqrt{n}})$, where $\Phi(\cdot)$ is the CDF of standard normal distribution, we have

$$P(\tau(u) < t) = 1 - e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \Phi\left(\frac{u - n\bar{X}}{\sigma_X \sqrt{n}}\right). \tag{6}$$

Since τ is a non-negative random variable, the k-th moment of recharge time can then be obtained as $\mathbb{E}[\tau^k] = \int_0^\infty k t^{k-1} P(\tau \geq t) \mathrm{d}t$. Using this identity, the expected recharge time is

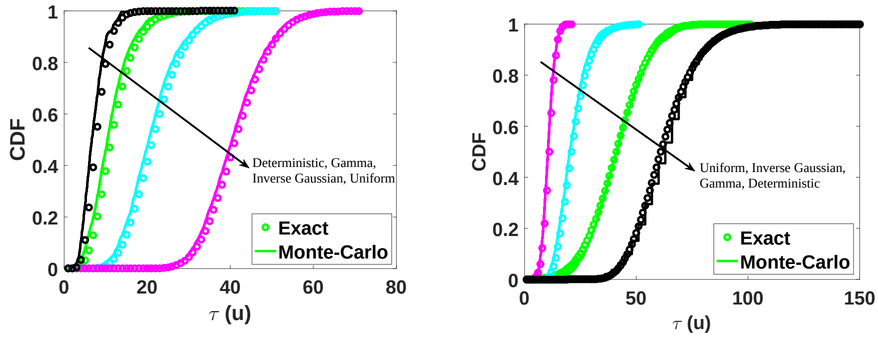
$$\begin{aligned}
\mathbb{E}[\tau(u)] &= \int_0^\infty P(\tau(u) \geq t)\, \mathrm{d}t \\
&= \sum_{n=0}^{\infty} \Phi\left(\frac{u - n\bar{X}}{\sigma_X \sqrt{n}}\right) \cdot \frac{1}{n!} \int_0^\infty (\lambda t)^n e^{-\lambda t}\, \mathrm{d}t \\
&= \frac{1}{\lambda} \sum_{n=0}^{\infty} \Phi\left(\frac{u - n\bar{X}}{\sigma_X \sqrt{n}}\right),
\end{aligned} \tag{7}$$

where the last step is because $n! = \lambda \int_0^\infty (\lambda t)^n e^{-\lambda t}\, \mathrm{d}t$.

B. When Energy Arrival Follows a Renewal Process

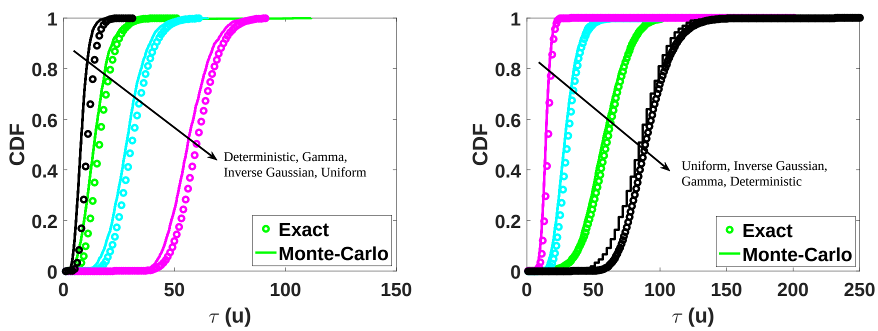
In full generality, since both $\{A_i\}$ and $\{X_i\}$ define a renewal process, here we will directly work with (4), where A_0 is differently distributed from A_i, $i \geq 1$. The sum $\sum_{i=1}^{N_X(u)} A_i$ is in itself a pure renewal process, while $\tau(u)$ is a stationary, delayed renewal process.

[2]By this, we mean that the energy arrival process begins at $t = -\infty$, while the recharging process begins at $t = 0$.



(a) (b)

Fig. 1: CDF of recharge time for linear system with $u = 20$ (a) When energy arrival is a Poisson process and packet sizes are deterministic, gamma, inverse Gaussian, and uniform distributed. (b) When energy arrival is a renewal process where packet sizes are exponentially distributed and energy arrivals are deterministic, gamma, inverse Gaussian, and uniform distributed.



(a) (b)

Fig. 2: CDF of recharge time for nonlinear system with u = 20, $U_{\max}$ = 25, β = 1.1 (a) When energy arrival is a Poisson process and packet sizes are deterministic, gamma, inverse Gaussian, and uniform distributed. (b) When energy arrival is a renewal process where packet sizes are exponentially distributed and energy arrivals are deterministic, gamma, inverse Gaussian, and uniform distributed.

By the linearity of expectation and Wald's identity, $\mathbb{E}[\tau(u)] = \mathbb{E}[A_0] + \mathbb{E}[N_X(u)]\mathbb{E}[A]$. Here for the stationary renewal process, the expected value of the residual time A_0 is $\mathbb{E}[A_0] = \frac{\mu_A^2+\sigma_A^2}{2\mu_A} = \frac{1+\lambda^2\sigma_A^2}{2\lambda}$. Also, asymptotically[3] for pure renewal process [8, Theo. 4.5] [10] $\mathbb{E}[N_X(u)] \sim \frac{1}{2}(\bar{X}^{-2}\sigma_X^2 - 1) + \bar{X}^{-1}u$. Hence, asymptotically $\mathbb{E}[N_X(u)]\mathbb{E}[A] \sim \frac{1}{2\lambda}(\bar{X}^{-2}\sigma_X^2 - 1) + (\lambda\bar{X})^{-1}u$. Therefore, we have the mean value of $\tau(u)$ asymptotically as

$$\mathbb{E}[\tau(u)] \sim \frac{\lambda\gamma^2}{2\bar{X}^2} + \frac{u}{\lambda\bar{X}}, \tag{8}$$

where $\gamma^2 = (\lambda^{-2}\sigma_X^2 + \sigma_A^2\bar{X}^2)$. For large u, the constant term may be neglected, and we obtain $\mathbb{E}[\tau(u)] \sim (\lambda\bar{X})^{-1}u$.

Similarly, we can analyze the variance of $\tau(u)$ using the total variance theorem as

$$\begin{aligned}\mathbb{V}[\tau(u)] &= \mathbb{V}[A_0] + \mathbb{E}[\mathbb{V}[\tau|N_X]] + \mathbb{V}[\mathbb{E}[\tau|N_X]]\\ &= \mathbb{V}[A_0] + \mathbb{E}[N_X\mathbb{V}[A]] + \mathbb{V}[N_X\mathbb{E}[A]]\\ &= \mathbb{V}[A_0] + \mathbb{E}[N_X]\mathbb{V}[A] + \mathbb{V}[N_X](\mathbb{E}[A])^2.\end{aligned}$$

[3]Here $f(x) \sim g(x)$ if and only if $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$.

As before, for the pure renewal process [10], $\lim_{u\to\infty}\frac{1}{u}\mathbb{V}[N_X(u)] = \bar{X}^{-3}\sigma_X^2$, so we have the asymptotic $\mathbb{V}[N_X(u)] \sim \bar{X}^{-3}\sigma_X^2 u$. Similarly, the variance of residual time is $\mathbb{V}[A_0] = \frac{\mu_A^{(3)}}{3\mu_A} - \left(\frac{\mu_A^2+\sigma_A^2}{2\mu_A}\right)^2$, where $\mu_A^{(3)}$ is the third moment of A. Note that $\mathbb{V}[A_0]$ is a constant and is independent of u. We have the asymptotic for $\mathbb{V}[\tau(u)]$ as

$$\mathbb{V}[\tau(u)] \sim \mathbb{V}[A_0] + \frac{\gamma^2 u}{\bar{X}^3}, \tag{9}$$

where $\gamma^2 = (\lambda^{-2}\sigma_X^2 + \sigma_A^2\bar{X}^2)$. For large value of u, we can neglect the first constant term, so that $\mathbb{V}[\tau(u)] \sim \gamma^2\bar{X}^{-3}u$.

From the central limit theorem, we have for large u

$$P(\tau(u) \le t) \approx \Phi\left(\frac{t - (\lambda\bar{X})^{-1}u}{\gamma\bar{X}^{-3/2}\sqrt{u}}\right). \tag{10}$$

Greater accuracy can be obtained by including the constant terms in the mean and variance in (10).

IV. Analysis of Nonlinear Storage System

In a nonlinear energy storage system, the efficiency of the charging process is dependent on the amount of energy stored in the system. Here we still assume that $p(t) = 0$ and $h(t)$ is unit step function in (1). In the model suggested by [6], [7], we have the nonlinear efficiency given by

$$\eta(U) = 1 - \left(\frac{U-a}{b}\right)^2, \tag{11}$$

where $a = \frac{1}{2}U_{\max}$ and $b = \beta(\frac{U_{\max}}{2})$. Here, $U_{\max}$ is the battery capacity and $\beta > 1$ is the nonlinearity parameter. Note that $\eta \to 1$ as $\beta \to \infty$.

To find the instantaneous relationship between the input energy X and the stored energy U, we solve the differential equation $\frac{\mathrm{d}U}{\mathrm{d}X} = \eta(U)$ with initial condition $U = 0$ when $X = 0$. Hence, we have the integral $\int \mathrm{d}X = \int \frac{\mathrm{d}U}{\eta(U)}$, whose solution is given by the arc hyperbolic tangent function

$$X = b\tanh^{-1}\left(\frac{U-a}{b}\right) + C, \tag{12}$$

where C is the constant of integration. Using the initial condition, we obtain $C = b\tanh^{-1}\left(\frac{a}{b}\right)$. Substituting the expressions for a and b, we get $C = \frac{\beta U_{\max}}{2}\tanh^{-1}\left(\frac{1}{\beta}\right)$. Rearranging (12), we have

$$U = a + b\tanh\left(\frac{X-C}{b}\right). \tag{13}$$

Equation (13) tells us how to transform a linear system into a non-linear system. Thus, given the aggregate harvested energy, $\sum_{i=1}^{N_A(t)} X_i$, we can now find the distribution of the stored energy as

$$P(U(t) \le u) = P\left(\sum_{i=1}^{N_A(t)} X_i \le C + b\tanh^{-1}\left(\frac{u-a}{b}\right)\right).$$

We see that effect of nonlinearity is equivalent to changing the threshold energy level of the linear system. Since we have assumed no self-discharge, the $U(t)$ for nonlinear system is also a pure jump process. Thus, from (3) we have

$$P(\tau(u) \le t) = P(\tau_\ell(u') \le t), \tag{14}$$

where τ_ℓ is the level-crossing time for linear system and $u' = C + b\tanh^{-1}\left(\frac{u-a}{b}\right)$. For the general renewal process, using (10) for the linear system, we have the equivalent formula for nonlinear system as

$$P(\tau(u) \le t) \approx \Phi\left(\frac{t - (\lambda\bar{X})^{-1}u'}{\gamma\bar{X}^{-3/2}\sqrt{u'}}\right). \tag{15}$$

V. Numerical Verification

Fig. 1 and Fig. 2 plot the cumulative distribution function (CDF) for the linear and nonlinear storage systems, respectively. In Fig. 1a and Fig. 2a, we assume energy arrival as the Poisson process; while in Fig. 1b and Fig. 2b, the energy arrival is a renewal process. The theoretical expressions for first passage time $\tau(u)$ for the Poisson energy arrival is given by (6) while for renewal process is given by (10). In the evaluation of (6), we truncate after $n = 100$. We consider $u = 20$ energy-units and $U_{\max} = 25$ energy-units. For a given distribution and a fixed value of u, 2000 simulations are run to construct the empirical CDF of the first passage time.

In Fig. 1a, while the energy packet size can have any distribution with finite mean and variance, we consider the case when the energy packet sizes are given by uniform, deterministic, inverse Gaussian, and gamma distributions. Here the inter-arrival time A_i is exponentially distributed $\mathrm{Exp}(1)$ and packet sizes X_i are uniform, $\mathrm{U}(0,1)$, deterministic $\delta(x-3)$, inverse Gaussian, $\mathrm{IG}(1,2)$ and gamma, $\mathrm{Gamma}(1,2)$, distributed. For the more general case, when the energy arrival is a renewal process, there can be any distribution for energy inter-arrival time and packet size. In Fig. 1b, we plot the CDF for the cases when the packet size is exponentially distributed while the energy inter-arrival times have uniform, deterministic, inverse Gaussian and gamma distributions, parametrized as before. The results from the simulations match closely with the theoretical prediction given by (6) and (10).

Fig. 2 shows a more realistic scenario, when the system is non-linear. During Monte-Carlo simulations, the non-linear efficiency is included for every energy packet arrival using (11). For theoretical analysis, expression given by (15) is used. In the evaluation, the value of the non-linearity parameter is taken to be $\beta = 1.1$. The results for non-linear system, when energy arrival follows a Poisson process and a renewal process, are shown in Fig. 2a and Fig. 2b, respectively, parameterized as in the linear case. Here too the results from the simulations match closely with the theoretical prediction.

VI. Conclusion

We have studied the time it takes for a battery to recharge up to a given level, when the energy source is discrete stochastic. We have examined the cases when the energy arrival is a Poisson process, and more generally, a renewal process, for both the linear as well as nonlinear charging processes. Formulas for the distributions of the recharge time and the expected recharge time have been obtained, which have been verified via Monte Carlo simulations.

References

[1] A.R. El-Sayed, *et al.*, "A survey on recent energy harvesting mechanisms," in Proc. *2016 IEEE Canadian Conf. on Electrical & Computer Eng. (CCECE'16)*, pp. 1–5, 2016.

[2] M.-L. Ku, *et al.*, "Advances in energy harvesting communications: Past, present, and future challenges," *IEEE Commun. Surveys & Tutorials*, no. 2, vol. 18, pp. 1384–1412, 2016.

[3] D. Mishra, S. De, and K. R. Chowdhury, "Charging time characterization for wireless RF energy transfer," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 62, no. 4, pp. 362–366, Apr. 2015.

[4] D. Mishra and S. De, "Effects of practical rechargeability constraints on perpetual RF harvesting sensor network operation," *IEEE Access*, vol. 4, pp. 750–765, 2016.

[5] L. Zubieta and R. Bonert, "Characterization of double-layer capacitors for power electronics applications," *IEEE Trans. Ind. Appl.*, vol. 36, no. 1, pp. 199–205, Jan./Feb. 2000.

[6] M. Gorlatova, A. Wallwater, and G. Zussman, "Networking low-power energy harvesting devices: Measurements and algorithms," *IEEE Trans. Mobile Comput.*, vol. 12, no. 9, pp. 1853–1865, Sept. 2013.

[7] A. Biason and M. Zorzi, "On the effects of battery imperfections in an energy harvesting device, *2016 Int. Conf. Comput. Netw. Commun. (ICNC)*, pp. 1–7, Feb. 2016.

[8] F.E. Beichelt and L.P. Fatti, *Stochastic Processes and Their Applications*. CRC Press, 2002.

[9] D.R. Cox and W.L. Smith, "On the superposition of renewal processes," *Biometrika*, vol. 41, no. 1/2, pp. 91–99, Jun. 1954.

[10] W.L. Smith, "On the Cumulants of Renewal Processes," *Biometrika*, vol. 46, no. 1/2, pp. 1–29, Jun., 1959.