Nonuniqueness of weak solutions to the Navier-Stokes equation

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Abstract

For initial datum of finite kinetic energy, Leray has proven in 1934 that there exists at least one global in time finite energy weak solution of the 3D Navier-Stokes equations. In this paper we prove that weak solutions of the 3D Navier-Stokes equations are not unique in the class of weak solutions with finite kinetic energy. Moreover, we prove that Hölder continuous dissipative weak solutions of the 3D Euler equations may be obtained as a strong vanishing viscosity limit of a sequence of finite energy weak solutions of the 3D Navier-Stokes equations.

1 Introduction

In this paper we consider the 3D incompressible Navier-Stokes equation

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p - \nu \Delta v = 0 \tag{1.1a}$$

$$\operatorname{div} v = 0 \tag{1.1b}$$

posed on $\mathbb{T}^3 \times \mathbb{R}$, with periodic boundary conditions in $x \in \mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$. We consider solutions normalized to have zero spatial mean, i.e., $\int_{\mathbb{T}^3} v(x,t)dx = 0$. The constant $\nu \in (0,1]$ is the kinematic viscosity. We define weak solutions to the Navier-Stokes equations [47, Definition 1], [17, pp. 226]:

Definition 1.1. *We say $v \in C^0(\mathbb{R}; L^2(\mathbb{T}^3))$ is a weak solution of (1.1) if for any $t \in \mathbb{R}$ the vector field $v(\cdot,t)$ is weakly divergence free, has zero mean, and (1.1a) is satisfied in $\mathcal{D}'(\mathbb{T}^3 \times \mathbb{R})$, i.e.,*

$$\int_{\mathbb{R}}\int_{\mathbb{T}^3} v \cdot (\partial_t \varphi + (v\cdot\nabla)\varphi + \nu\Delta\varphi) dx dt = 0$$

holds for any test function $\varphi \in C_0^\infty(\mathbb{T}^3 \times \mathbb{R})$ such that $\varphi(\cdot,t)$ is divergence-free for all t.

As a direct result of the work of Fabes-Jones-Riviere [17], since the weak solutions defined above lie in $C^0(\mathbb{R}; L^2(\mathbb{T}^3))$, they are in fact solutions of the integral form of the Navier-Stokes equations

$$v(\cdot,t) = e^{\nu t\Delta} v(\cdot,0) + \int_0^t e^{\nu(t-s)\Delta} \mathbb{P}\operatorname{div}(v(\cdot,s)\otimes v(\cdot,s)) ds\,, \tag{1.2}$$

and are sometimes called *mild* or *Oseen* solutions (cf. [17] and [36, Definition 6.5]). Here $\mathbb{P}$ is the Leray projector and $e^{t\Delta}$ denotes convolution with the heat kernel.

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1.1 Previous works

In [37], Leray considered the Cauchy-problem for (1.1) for initial datum of finite kinetic energy, $v_0 \in L^2$. Leray proved that for any such datum, there exists a global in time weak solution $v \in L^\infty_t L^2_x$, which additionally has the regularity $L^2_t \dot{H}^1_x$, and obeys the energy inequality $\|v(t)\|^2_{L^2} + 2\nu \int_0^t \|\nabla u(s)\|^2_{L^2}\, ds \leq \|v_0\|^2_{L^2}$. Hopf [21] established a similar result for the equations posed in a smooth bounded domain, with Dirichlet boundary conditions. To date, the question of uniqueness of Leray-Hopf weak solutions for the 3D Navier-Stokes equations remains however open.

Based on the natural scaling of the equations $v(x,t) \mapsto v_\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t)$, a number of partial regularity results have been established [43, 7, 38, 34, 51, 32]; the local existence for the Cauchy problem has been proven in scaling-invariant spaces [27, 29, 25]; and conditional regularity has been established under geometric structure assumptions [10] or assuming a signed pressure [45]. The conditional regularity and weak-strong uniqueness results known under the umbrella of Ladyzhenskaya-Prodi-Serrin conditions [28, 41, 46], state that if a Leray-Hopf weak solution also lies in $L^p_t L^q_x$, with $2/p + 3/q \leq 1$, then the solution is unique and smooth in positive time. These conditions and their generalizations have culminated with the work of Escauriaza-Seregin-Šverák [24] who proved the $L^\infty_t L^3_x$ endpoint. The uniqueness of mild/Oseen solutions is also known under the Ladyzhenskaya-Prodi-Serrin conditions, cf. [17] for $p > 3$, and [19, 39, 35, 30] for $p = 3$. Note that the regularity of Leray-Hopf weak solutions, or of bounded energy weak solutions, is consistent with the scaling $2/p + 3/q = 3/2$. In contrast, the additional regularity required to ensure that the energy equality holds in the Navier-Stokes equations is consistent with $2/4 + 3/4 = 5/4$ for $p = q = 4$ [48, 31]. See [11, 50, 35, 42, 36] for surveys of results on the Navier-Stokes equations.

The gap between the scaling of the kinetic energy and the natural scaling of the equations leaves open the possibility of nonuniqueness of weak solutions to (1.1). In [25, 26] Jia-Šverák proved that non-uniqueness of Leray-Hopf weak solutions in the regularity class $L^\infty_t L^{3,\infty}_x$ holds if a certain spectral assumption holds for a linearized Navier-Stokes operator. While a rigorous proof of this spectral condition remains open, very recently Guillod-Šverák [20] have provided compelling numerical evidence of it, using a scenario related to the example of Ladyzhenskaya [33]. Thus, the works [26, 20] strongly suggest that the Ladyzhenskaya-Prodi-Serrin regularity criteria are sharp.

1.2 Main results

In this paper we prove that weak solutions to (1.1) (in the sense of Definition 1.1) are not unique within the class of weak solutions with bounded kinetic energy. We establish the stronger result[1]:

Theorem 1.2 (Nonuniqueness of weak solutions). *There exists $\beta > 0$, such that for any nonnegative smooth function $e(t)\colon [0,T] \to \mathbb{R}_{\geq 0}$, there exists $v \in C^0_t([0,T]; H^\beta_x(\mathbb{T}^3))$ a weak solution of the Navier-Stokes equations, such that $\int_{\mathbb{T}^3} |v(x,t)|^2\, dx = e(t)$ for all $t \in [0,T]$. Moreover, the associated vorticity $\nabla \times v$ lies in $C^0_t([0,T]; L^1_x(\mathbb{T}^3))$.*

In particular, the above theorem shows that $v \equiv 0$ is not the only weak solution which vanishes at a time slice, thereby implying the nonuniqueness of weak solutions. Theorem 1.2 shows that weak solutions may come to rest in finite time, a question posed by Serrin [47, pp. 88]. Moreover, by considering $e_1(t), e_2(t) > 0$ which are nonincreasing, such that $e_1(t) = e_2(t)$ for $t \in [0, T/2]$, and $e_1(T) < e_2(T)$, the construction used to prove Theorem 1.2 also proves the nonuniqueness of dissipative weak solutions.

From the proof of Theorem 1.2 it is clear that the constructed weak solutions v also have regularity in time, i.e. there exists $\gamma > 0$ such that $v \in C^\gamma_t([0,T]; L^2_x(\mathbb{T}^3))$. Thus, $v \otimes v$ lies in

[1] We denote by H^β the L^2-based Sobolev space with regularity index β. Clearly $C^0_t H^\beta_x \subset C^0_t L^2_x$.

$C_t^\gamma L_x^1 \cap C_t^0 L_x^{1+\gamma}$, and the fact that $\nabla v \in C_t^0 L_x^1$ follows from (1.2) and the maximal regularity of the heat equation.

We note that while the weak solutions Theorem 1.2 may attain any smooth energy profile, at the moment we do not prove that they are Leray-Hopf weak solutions, i.e., they do not obey the energy inequality or have $L_t^2 \dot{H}_x^1$ integrability. Moreover, the regularity parameter $\beta > 0$ cannot be expected to be too large, since at $\beta = 1/2$ one has weak-strong uniqueness [11]. We expect that the ideas used to prove Theorem 1.2 will in the future lead to a proof of nonuniqueness of weak solutions in $C_t^0 L_x^p$, for any $2 \le p < 3$, and the nonuniqueness of Leray-Hopf weak solutions.

The proof of Theorem 1.2 builds on several of the fundamental ideas pioneered by De Lellis-Székelyhidi Jr. [14, 15]. These ideas were used to tackle the Onsager conjecture for the Euler equation (set $\nu = 0$ in (1.1)) via convex integration methods [44, 49, 2, 16, 1, 3, 13], leading to the resolution of the conjecture by Isett [22, 23] and by the authors of this paper jointly with De Lellis-Székelyhidi Jr. in [4]. In order to treat the dissipative term $-\nu\Delta$, not present in the Euler system, we cannot proceed as in [6, 8], since in these works Hölder continuous weak solutions are constructed, which is possible only by using building blocks which are sparse in the frequency variable and for small fractional powers of the Laplacian. Instead, the main idea, which is also used in [5] in order to build nonconservative weak solutions of the Euler equations in the regularity class $C_t^0 H_x^{5/14-\varepsilon}$, is to use building blocks for the convex integration scheme which are "intermittent". That is, the building blocks we use are spatially inhomogenous, and have different scaling in different L^p norms. At high frequency, these building blocks attempt to saturate the Bernstein inequalities from Littlewood-Paley theory. Since they are built by adding eigenfunctions of curl in a certain geometric manner, we call these building blocks *intermittent Beltrami flows*. In particular, the proof of Theorem 1.2 breaks down in $2D$, as is expected, since there are not enough spatial directions to oscillate in. The proof of Theorem 1.2 is given in Section 2 below.

The idea of using intermittent building blocks can be traced back to classical observations in hydrodynamic turbulence, see for instance [18]. Moreover, in view of the aforementioned works on the Onsager conjecture for the Euler equations, we are naturally led to consider the set of accumulation points in the vanishing viscosity limit $\nu \to 0$ of the family of weak solutions to the Navier-Stokes equations which we constructed in Theorem 1.2. We prove in this paper that this set of accumulation points, in the $C_t^0 L_x^2$ topology, contains all the Hölder continuous weak solutions of the 3D Euler equations:

Theorem 1.3 (**Dissipative Euler solutions arise in the vanishing viscosity limit**)**.** *For $\bar{\beta} > 0$ let $u \in C_{t,x}^{\bar{\beta}}(\mathbb{T}^3 \times [-2T, 2T])$ be a zero-mean weak solution of the Euler equations. Then there exists $\beta > 0$, a sequence $\nu_n \to 0$, and a uniformly bounded sequence $v^{(\nu_n)} \in C_t^0([0,T]; H_x^\beta(\mathbb{T}^3))$ of weak solutions to the Navier-Stokes equations, with $v^{(\nu_n)} \to u$ strongly in $C_t^0([0,T]; L_x^2(\mathbb{T}^3))$.*

In particular, Theorem 1.3 shows that the nonconservative weak solutions to the Euler equations obtained in [22, 4] arise in the vanishing viscosity limit of weak solutions to the Navier-Stokes equations. Thus, being a strong limit of weak solutions to the Navier-Stokes equations (in the sense of Definition 1.1) cannot serve as a selection criterion for weak solutions of the Euler equation. We expect that a similar result holds for Leray-Hopf weak solutions. The proof of Theorem 1.3 is closely related to that of Theorem 1.2, and is also given in Section 2 below.

2 Outline of the convex integration scheme

In this section we sketch the proof of Theorem 1.2. For every integer $q \geq 0$ we will construct a solution $(v_q, p_q, \mathring{R}_q)$ to the Navier-Stokes-Reynolds system

$$\partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q - \nu \Delta v_q = \operatorname{div} \mathring{R}_q \tag{2.1a}$$

$$\operatorname{div} v_q = 0\,. \tag{2.1b}$$

where the Reynolds stress $\mathring{R}_q$ is assumed to be a trace-free symmetric matrix.

2.1 Parameters

Throughout the proof we fix a sufficiently large, universal constant $b \in 8\mathbb{N}$, and depending on b we fix a regularity parameter $\beta > 0$ such that $\beta b^2 \leq 4$ and $\beta b \leq 1/40$.[2]

The relative size of the approximate solution v_q and the Reynolds stress error $\mathring{R}_q$ will be measured in terms of a frequency parameter λ_q and an amplitude parameter δ_q defined as

$$\lambda_q = a^{(b^q)}$$

$$\delta_q = \lambda_1^{3\beta} \lambda_q^{-2\beta}$$

for some integer $a \gg 1$ to be chosen suitably.

2.2 Inductive estimates

By induction, we will assume the following estimates[3] on the solution of (2.1) at level q:

$$\|v_q\|_{C^1_{x,t}} \leq \lambda_q^3 \tag{2.2}$$

$$\left\|\mathring{R}_q\right\|_{L^1} \leq \lambda_q^{-\varepsilon_R} \delta_{q+1} \tag{2.3}$$

$$\left\|\mathring{R}_q\right\|_{C^1_{x,t}} \leq \lambda_q^{10}\,. \tag{2.4}$$

We additionally assume

$$0 \leq e(t) - \int_{\mathbb{T}^3} |v_q|^2 \;dx \leq \delta_{q+1} \tag{2.5}$$

and

$$e(t) - \int_{\mathbb{T}^3} |v_q|^2 \;dx \leq \frac{\delta_{q+1}}{100} \;\Rightarrow \mathring{R}_q(\cdot, t) \equiv 0 \tag{2.6}$$

for all $t \in [0, T]$.

2.3 The main proposition and iterative procedure

In addition to the sufficiently large universal constant b, and the regularity parameter $\beta = \beta(b) > 0$ fixed earlier, we fix the constant $M_e = \|e\|_{C^1_t}$. The following iteration lemma states the existence of a solution of (2.1) at level $q+1$, which obeys suitable bounds.

[2]It is sufficient to take $b = 2^9$ and $\beta = 2^{-16}$.

[3]Here and throughout the paper we use the notation: $\|f\|_{L^p} = \|f\|_{L^\infty_t L^p_x}$, for $1 \leq p \leq \infty$, $\|f\|_{C^N} = \|f\|_{L^\infty_t C^N_x} = \sum_{0 \leq |\alpha| \leq N} \|D^\alpha f\|_{L^\infty}$, $\|f\|_{C^N_{x,t}} = \sum_{0 \leq n + |\alpha| \leq N} \|\partial_t^n D^\alpha f\|_{L^\infty}$, and $\|f\|_{W^{s,p}} = \|f\|_{L^\infty_t W^{s,p}_x}$, for $s > 0$, and $1 \leq p \leq \infty$.

Proposition 2.1. *There exists a universal constant $M > 0$, a sufficiently small parameter $\varepsilon_R = \varepsilon_R(b, \beta) > 0$ and a sufficiently large parameter $a_0 = a_0(b, \beta, \varepsilon_R, M, M_e) > 0$ such that for any integer $a \geq a_0$ the following holds: Let $(v_q, p_q, \mathring{R}_q)$ be a triple solving the Navier-Stokes-Reynolds system (2.1) in $\mathbb{T}^3 \times [0, T]$ satisfying the inductive estimates (2.2)–(2.6). Then there exists a second triple $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ solving (2.1) and satisfying the (2.2)–(2.6) with q replaced by $q+1$. In addition we have that*

$$\|v_{q+1} - v_q\|_{L^2} \leq M \delta_{q+1}^{1/2} . \tag{2.7}$$

The principal new idea in the proof of Proposition 2.1 is to construct the perturbation $w_q := v_{q+1} - v_q$ as a sum of terms of the form

$$a_{(\overline{\xi})} \mathbb{W}_{(\overline{\xi})} \tag{2.8}$$

where $\mathbb{W}_{(\overline{\xi})}$ is an intermittent Beltrami wave (cf. (3.7) below) with frequency support centered at frequency $\overline{\xi}\lambda_{q+1}$ for $\overline{\xi} \in \mathbb{S}^2$. While these intermittent Beltrami waves have similar properties (cf. Propositions 3.3 and 3.4) to the usual Beltrami flows used in the previous convex integration constructions [15, 2, 16, 1, 3] for the Euler equations, they are fundamentally different since their L^1 norm is much smaller than their L^2 norm (cf. Proposition 3.5). The gain comes from the fact that the Reynolds stress has to be estimated in L^1 rather than L^2, and that the term $\nu\Delta v$ is linear in v. At the technical level, on difference with respect to [22, 4] is the usage of very large gaps between consecutive frequency parameters (i.e., $b \gg 1$), which is consistent with a small regularity parameter β. Next, we show that Proposition 2.1 implies the main theorems of the paper.

2.4 Proof of Theorem 1.2

Choose all the parameters from the statement of Proposition 2.1, except for a, which we may need to larger (so that it is still larger than a_0).

For $q = 0$ we note that identically zero solution trivially satisfies (2.1) with $\mathring{R}_0 = 0$, and the inductive assumptions (2.2), (2.3), and (2.4) hold. Moreover, by taking a sufficiently large such that it is in the range of Proposition 2.1 (i.e. $a \geq a_0$) we may ensure that

$$|e(t)| \leq \|e\|_{C^1_t} = M_e \leq \frac{\lambda_1^\beta}{100} = \frac{\delta_1}{100}.$$

Then the zero solution also satisfies (2.5) and (2.6).

For $q \geq 1$ we inductively apply Proposition 2.1. The bound (2.7) and interpolation implies[4]

$$\begin{aligned}
\sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_{H^{\beta'}} &\lesssim \sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_{L^2}^{1-\beta'} \left(\|v_{q+1}\|_{C^1} + \|v_q\|_{C^1}\right)^{\beta'} \\
&\lesssim \sum_{q=0}^{\infty} \lambda_{q+1}^{-\beta\frac{1-\beta'}{2}} \lambda_{q+1}^{3\beta'} \\
&\lesssim 1 ,
\end{aligned} \tag{2.9}$$

for $\beta' < \beta/(6+\beta)$, and hence the sequence $\{v_q\}_{q\geq 0}$ is uniformly bounded $C^0_t H^{\beta'}_x$, for such β'. From (2.1), (2.3), the previously established uniform boundedness in $C^0_t L^2_x$, and the embedding $W^{2,1}_x \subset L^2_x$

[4] Throughout this paper, we we will write $A \lesssim B$ to denote that there exists a sufficiently large constant C, which is independent of q, such that $A \leq CB$.

we obtain that

$$\begin{aligned}\|\partial_t v_q\|_{H^{-3}} &\lesssim \left\|\mathbb{P}\mathrm{div}\,(v_q\otimes v_q) - \nu\Delta v_q - \mathbb{P}\mathrm{div}\,\mathring{R}_q\right\|_{H^{-3}}\\ &\lesssim \|v_q\otimes v_q\|_{L^1} + \|v_q\|_{L^2} + \left\|\mathring{R}_q\right\|_{L^1}\\ &\lesssim 1\end{aligned}$$

where $\mathbb{P}$ is the Leray projector. Thus, the sequence $\{v_q\}_{q\geq 0}$ is uniformly bounded in $C^1_t H^{-3}_x$. It follows that for any $0<\beta''<\beta'$ the sum

$$\sum_{q\geq 0}(v_{q+1}-v_q) =: v$$

converges in $C^0_t H^{\beta''}_x$, and since $\left\|\mathring{R}_q\right\|_{L^1}\to 0$ as $q\to\infty$, v is a $C^0_t H^{\beta''}_x$ weak solution of the Navier-Stokes equation. Lastly, in view of (2.5) we have that the kinetic energy of $v(\cdot,t)$ is given by $e(t)$ for all $t\in[0,T]$, concluding the proof of the theorem.

2.5 Proof of Theorem 1.3

Fix $\bar\beta>0$ and a weak solution $u\in C^{\bar\beta}_{t,x}$ to the Euler equation on $[-2T,2T]$. The existence of such solutions is guaranteed in view of the results of [22, 4] for $\bar\beta<1/3$, and for $\bar\beta>1$ from the classical local existence results. Let $M_u=\|u\|_{C^{\bar\beta}}$. Pick an integer $n\geq 1$.

Choose all the parameters as in Proposition 2.1, except for $a\geq a_0$, which we may take even larger, depending also on M_u and β' which obeys $0<\beta'<\min(\bar\beta/2, \beta/(6+\beta))$. We make a even larger, depending also on β', so that in view of (2.9) we may ensure that

$$\sum_{q=n}^{\infty}\lambda_{q+1}^{-\beta\frac{1-\beta'}{2}}\lambda_{q+1}^{3\beta'}\leq\frac{1}{2Cn} \tag{2.10}$$

where C is the implicit constant in (2.9).

Let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be a family of standard compact support (of width 2) Friedrichs mollifiers on $\mathbb{R}^3$ (space), and $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be a family of standard compact support (of width 2) Friedrichs mollifiers on $\mathbb{R}$ (time). We define

$$v_n = (u *_x \phi_{\lambda_n^{-1}}) *_t \varphi_{\lambda_n^{-1}}$$

to be a mollification of u in space and time, at length scale and time scale λ_n^{-1}, restricted to the temporal range $[0,T]$. Also, on $[0,T]$ define the energy function

$$e(t)=\int_{\mathbb{T}^3}|v_n(x,t)|^2\,dx+\frac{\delta_n}{2}$$

which ensures that (2.5) and (2.6) hold for $q=n$.

Since u is a solution of the Euler equations, there exists a mean-free p_n such that

$$\partial_t v_n + \mathrm{div}\,(v_n\otimes v_n)+\nabla p_n - \lambda_n^{-2}\Delta v_n = \mathrm{div}\,(\mathring{R}_n)$$

where $\mathring{R}_n$ is the traceless symmetric part of the tensor

$$(v_n\otimes v_n) - ((u\otimes u) *_x \phi_{\lambda_n^{-1}}) *_t \varphi_{\lambda_n^{-1}} - \lambda_n^{-2}\nabla v_n.$$

Using a version of the commutator estimate introduced in [9], which may for instance be found in [12, Lemma 1], we obtain that

$$\left\|\mathring{R}_n\right\|_{L^1} \lesssim \left\|\mathring{R}_n\right\|_{C^0} \lesssim \lambda_n^{-1} M_u + \lambda_n^{-2\bar{\beta}} M_u^2. \tag{2.11}$$

In addition, from a similar argument it follows that

$$\left\|\mathring{R}_n\right\|_{C^1_{t,x}} \lesssim M_u + \lambda_n^{1-2\bar{\beta}} M_u^2 \tag{2.12}$$

$$\|v_n\|_{C^1_{t,x}} \lesssim \lambda_n^{1-\bar{\beta}} M_u. \tag{2.13}$$

Setting

$$\nu := \nu_n := \lambda_n^{-1},$$

then with a sufficiently large, depending on M_u and $\bar{\beta}$, we may ensure the pair $(v_n, \mathring{R}_n)$ obey the inductive assumptions (2.2)–(2.4) for $q = n$. Additionally, we may also choose a sufficiently large, depending on M_u and $\bar{\beta}$, so that

$$\lambda_n^{\bar{\beta}-\beta'} M_u \leq \frac{1}{2n|\mathbb{T}^3|^{1/2}}. \tag{2.14}$$

At this stage we may start the inductive Proposition 2.1, and as in the proof of Theorem 1.2, we obtain a weak solution $u^{(\nu_n)}$ of the Navier-Stokes equations, with the desired regularity, such that

$$\left\|v^{(\nu_n)} - u\right\|_{H^{\beta'}} \leq \left\|v^{(\nu_n)} - v_n\right\|_{H^{\beta'}} + |\mathbb{T}^3|^{1/2} \|u - v_n\|_{C^{\beta'}} \leq \frac{1}{n}.$$

in view of (2.10) and (2.14). Since n was arbitrary, this concludes the proof of the theorem.

3 Intermittent Beltrami Waves

In this section we will describe in detail the construction of the *intermittent Beltrami waves* which will form the building blocks of our convex integration scheme. Very roughly, intermittent Betrami waves will be approximate Beltrami waves (approximate eigenfunctions to the curl operator) whose L^1 norm are significantly smaller than their L^2 norm.

3.1 Beltrami waves

Let us begin by recalling the two main propositions used in e.g. [2] for the construction of the Beltrami waves. In order to better suit our later goal of defining intermittent Beltrami waves, the statements of these propositions will be slightly modified from the form they appear in [2].

Proposition 3.1. *Given $\overline{\xi} \in \mathbb{S}^2 \cap \mathbb{Q}^3$, let $A_{\overline{\xi}} \in \mathbb{S}^2$ be such that*

$$A_{\overline{\xi}} \cdot \overline{\xi} = 0,\ |A_{\overline{\xi}}| = \tfrac{1}{\sqrt{2}},\ A_{-\overline{\xi}} = A_{\overline{\xi}}.$$

Furthermore, let

$$B_{\overline{\xi}} = A_{\overline{\xi}} + i\xi \times A_{\overline{\xi}}.$$

Let Λ be a given finite subset of $\mathbb{S}^2 \cap \mathbb{Q}^3$ and $\lambda \in \mathbb{Z}$ be such that $\lambda\Lambda \subset \mathbb{Z}^3$. Then for any choice of coefficients $a_{\overline{\xi}} \in \mathbb{C}$ with $\overline{a}_{\overline{\xi}} = a_{-\overline{\xi}}$ the vector field

$$W(x) = \sum_{\overline{\xi}\in\Lambda} a_{\overline{\xi}} B_{\overline{\xi}} e^{i\lambda\overline{\xi}\cdot x} \tag{3.1}$$

is real-valued, divergence-free and satisfies

$$\operatorname{div}(W \otimes W) = \nabla \frac{|W|^2}{2}. \tag{3.2}$$

Furthermore

$$\langle W \otimes W \rangle := \fint_{\mathbb{T}^3} W \otimes W \, d\xi = \frac{1}{2} \sum_{\overline{\xi} \in \Lambda} |a_{\overline{\xi}}|^2 \left(\mathrm{Id} - \overline{\xi} \otimes \overline{\xi} \right) . \tag{3.3}$$

Proposition 3.2. *For every $N \in \mathbb{N}$ we can choose $\varepsilon > 0$ and $\lambda > 1$ with the following property. Let $B_\varepsilon(\mathrm{Id})$ denote the ball of symmetric 3×3 matrices, centered at* Id*, of radius ε. Then, there exist pairwise disjoint subsets*

$$\Lambda_\alpha \subset \mathbb{S}^2 \cap \mathbb{Q}^3 \qquad \alpha \in \{1, \dots, N\} ,$$

with $\lambda \Lambda_\alpha \in \mathbb{Z}^3$, and smooth positive functions

$$\widetilde{\gamma}^{(\alpha)}_{\overline{\xi}} \in C^\infty(B_\varepsilon(\mathrm{Id})) \qquad \alpha \in \{1, \dots, N\}, \overline{\xi} \in \Lambda_\alpha ,$$

such that

(a) $\overline{\xi} \in \Lambda_\alpha$ implies $-\overline{\xi} \in \Lambda_\alpha$ and $\widetilde{\gamma}^{(\alpha)}_{\overline{\xi}} = \widetilde{\gamma}^{(\alpha)}_{-\overline{\xi}}$;

(b) For each $R \in B_\varepsilon(\mathrm{Id})$ we have the identity

$$R = \frac{1}{2} \sum_{\overline{\xi} \in \Lambda_\alpha} \left(\widetilde{\gamma}^{(\alpha)}_{\overline{\xi}}(R) \right)^2 \left(\mathrm{Id} - \overline{\xi} \otimes \overline{\xi} \right) \qquad \forall R \in B_\varepsilon(\mathrm{Id}) . \tag{3.4}$$

3.2 Intermittent Beltrami waves

Recall the Dirichlet kernel D_n is defined as

$$D_n(x) = \sum_{\xi=-n}^{n} e^{ix\xi} = \frac{\sin((n + 1/2)x}{\sin(x/2)} \tag{3.5}$$

and has the property that for any $p > 1$

$$\|D_n\|_{L^p} \sim n^{1-1/p} .$$

Replacing the sum in (3.5) of a sequence of integers by a sum of frequencies in a 3D cube

$$\Omega_r := \{(i, j, k) \colon i, j, k \in \{-r, \dots, r\}\}$$

and normalizing to unit size in L^2, we obtain a kernel

$$D_r(x) := \frac{1}{r^{3/2}} \sum_{\xi \in \Omega_r} e^{i\xi \cdot x}$$

such that for $p > 1$ we have

$$\|D_r\|_{L^p} \lesssim r^{3/2 - 3/p}, \qquad \|D_r\|_{L^2} \sim 1, \qquad \text{and} \qquad \|D_r\|_{L^\infty} \lesssim r^{3/2} .$$

The principle idea in the construction of intermittent Beltrami waves is to modify the Beltrami waves of the previous section by adding oscillations that mimic the structure of the kernels D_r in order to construct approximate Beltrami waves with small L^p norm for p close to 1. As such, we start by replacing the single frequency $\overline{\xi}$ by a cube of frequencies centered at $\overline{\xi}$ which we will denote $\Omega_{\overline{\xi},\sigma,r}$. The parameter σ *parameterizes the spacing between frequencies.* The parameter r *will parameterize the number of frequencies along edges of the cube.* We will assume

$$\sigma r \leq {}^1\!/_{10}\,. \tag{3.6}$$

For the unit vector $e_1 = (1, 0, 0)$ we define $\Omega_{e_1,\sigma,r}$ to be the cube grid of wave vectors centered at e_1 of length r:

$$\Omega_{e_1,\sigma,r} := \{(1 + k\sigma, i\sigma, j\sigma)|i, j, k \in \{-r, \dots, r\}\}\ .$$

For $\overline{\xi} \in \mathbb{S}^2 \cap \mathbb{Q}^3$, we define $\Omega_{\overline{\xi},\sigma,r}$ using rational rotation matrices $\widetilde{M_{\overline{\xi}}}$ that map e_1 to $\overline{\xi}$:

$$\Omega_{\overline{\xi},\sigma,r} := \widetilde{M_{\overline{\xi}}}\Omega_{e_1,\sigma,r}$$

for $\overline{\xi} \neq e_1$. To do so, for a given α we split Λ_α into to disjoint sets Λ_α^+ and Λ_α^- such that $\Lambda_\alpha^- = -\Lambda_\alpha^+$. Now suppose $\overline{\xi} \in \Lambda_\alpha^+$. Define

$$M_{\overline{\xi}} := \begin{pmatrix} 0 & e_2 \cdot \overline{\xi} & -e_3 \cdot \overline{\xi} \\ -e_2 \cdot \overline{\xi} & 0 & 0 \\ e_3 \cdot \overline{\xi} & 0 & 0 \end{pmatrix}.$$

Then the rotational matrix $\widetilde{M_{\overline{\xi}}}$, defined by

$$\widetilde{M_{\overline{\xi}}} := \mathrm{Id} + M_{\overline{\xi}} + \frac{1 - e_1 \cdot \overline{\xi}}{\left|e_1 \times \overline{\xi}\right|^2} M_{\overline{\xi}}^2$$

maps e_1 to ξ. Finally for $\overline{\xi} \in \Lambda_\alpha^-$ we set $\Omega_{\overline{\xi},\sigma,r} := -\Omega_{-\overline{\xi},\sigma,r}$. In order to reduce indices we denote

$$\Omega_{(\overline{\xi})} = \Omega_{\overline{\xi},\sigma,r}$$

when the meaning of σ and r are clear from the context.

Having determined the set of frequencies that will make up our intermittent Beltrami waves, we also need to choose directions B_ξ. For $\xi \in \Omega_{(\overline{\xi})}$ we define

$$A_\xi := A_{\overline{\xi}} - (A_{\overline{\xi}} \cdot \xi)\xi$$

where $A_{\overline{\xi}}$ is as in Proposition 3.1, and let

$$B_\xi = A_\xi + i\xi \times A_\xi, \qquad B_{(\xi)} := B_{\xi,r} := \frac{1}{(2r+1)^{3/2}} B_\xi.$$

We are now in the position to define our intermittent Beltrami waves which will form the building blocks for the convex integration scheme presented in the paper. Set

$$\mathbb{W}_{(\overline{\xi})}(x) := \mathbb{W}_{\overline{\xi},\lambda,\sigma,r}(x) := \sum_{\xi \in \Omega_{(\overline{\xi})}} B_{\xi,r} e^{i\lambda\xi\cdot x}. \tag{3.7}$$

For notational convenience we define $W_{(\xi)}$ to be the individual Beltrami waves that make up $\mathbb{W}_{(\overline{\xi})}$, namely

$$W_{(\xi)} := W_{\xi,\lambda,r} := B_{\xi,r} e^{i\lambda\xi\cdot x}.$$

Thus, with this notation we may write

$$\mathbb{W}_{(\overline{\xi})}(x) = \sum_{\xi\in\Omega_{(\overline{\xi})}} W_{(\xi)}\,.$$

Proposition 3.3. *Let Λ_α be defined as in Proposition 3.2. If $\Omega_{(\overline{\xi})}$ and $\mathbb{W}_{(\overline{\xi})}$ are defined as above and $a_{\overline{\xi}} \in \mathbb{C}$ are constants chosen such that $\overline{a}_{\overline{\xi}} = a_{-\overline{\xi}}$, let W be the vector field:*

$$W(x) = \sum_\alpha \sum_{\overline{\xi}\in\Lambda_\alpha} a_{\overline{\xi}} \mathbb{W}_{(\overline{\xi})}(x).$$

Then $W(x)$ is real valued and divergence free. Moreover

$$\operatorname{div}(W\otimes W) = \frac{1}{2}\nabla |W|^2 - \lambda W \times \left(\sum_\alpha \sum_{\overline{\xi}\in\Lambda_\alpha} \sum_{\xi\in\Omega_{(\overline{\xi})}} a_{\overline{\xi}} (1-|\xi|) W_{(\xi)} \right). \tag{3.8}$$

Observe that since our intermittent Beltrami waves are not supported exactly on a given frequency shell, in contrast to the identity (3.2), a second term appears in (3.8). This term will be small as long as the frequency support of our intermittent Beltrami is suitably close to a shell, i.e. $|1-|\xi||$ is small for all $\xi \in \Omega_{(\overline{\xi})}$ and $\overline{\xi} \in \Lambda_\alpha$. In particular, for any such ξ we have that

$$|1-|\xi|| \lesssim \sigma r \tag{3.9}$$

when $\sigma r \leq 1/2$.

In the spirit of Proposition 3.2, we aim to construct functions $\gamma_{(\overline{\xi})}$ such that

$$\sum_{\overline{\xi}\in\Lambda_\alpha} \sum_{\xi\in\Omega_{(\overline{\xi})}} \left(\gamma_{(\overline{\xi})}(R)\right)^2 B_\xi \otimes B_{-\xi} = R\,. \tag{3.10}$$

where $\Omega_{(\overline{\xi})}$ is a family of vectors close to $\overline{\xi}$ to be defined below. This is the subject of the following proposition. For the sake of brevity, we omit the (α) upper index of $\gamma_{(\overline{\xi})}$.

Proposition 3.4. *Let Λ_α and $\widetilde{\gamma}_{\overline{\xi}}$ be defined as in Proposition 3.2. Furthermore assume that $B_{(\xi)}$ and $\Omega_{(\overline{\xi})}$ are defined as above. Then there exists an $\varepsilon_\gamma > 0$ such that if*

$$r\sigma < \varepsilon_\gamma,$$

there exists smooth positive functions (with derivatives that are bounded independently on r and σ)

$$\gamma_{(\overline{\xi})} := \gamma_{\overline{\xi},r,\sigma} \in C^\infty(B_{\varepsilon_\gamma}(\mathrm{Id})), \quad \overline{\xi} \in \Lambda_\alpha\,,$$

such that for each $R \in B_{\varepsilon_\gamma}(\mathrm{Id})$ we have the identity

$$\sum_{\overline{\xi}\in\Lambda_\alpha} \sum_{\xi\in\Omega_{(\overline{\xi})}} \left(\gamma_{(\overline{\xi})}(R)\right)^2 W_{(\xi)} \otimes W_{(-\xi)} = \sum_{\overline{\xi}\in\Lambda_\alpha} \sum_{\xi\in\Omega_{(\overline{\xi})}} \left(\gamma_{(\overline{\xi})}(R)\right)^2 B_{(\xi)} \otimes B_{(-\xi)} = R. \tag{3.11}$$

Proof of Proposition 3.4. We begin by defining the following auxiliary function f on $B_\varepsilon(\mathrm{Id})$, where ε is chosen as in Proposition 3.2. For $R \in B_\varepsilon(\mathrm{Id})$ define

$$\begin{aligned} f(R) &:= \sum_{\overline{\xi}\in\Lambda_\alpha}\sum_{\xi\in\Omega_{(\overline{\xi})}} \left(\widetilde{\gamma}_{\overline{\xi}}(R)\right)^2 B_{(\xi)}\otimes B_{(-\xi)} \\ &= \mathrm{Id} + (R-\mathrm{Id}) + \frac{1}{(2r+1)^3}\sum_{\overline{\xi}\in\Lambda_\alpha}\sum_{\xi\in\Omega_{(\overline{\xi})}} \left(\widetilde{\gamma}_{\overline{\xi}}(R)\right)^2 (B_\xi\otimes B_{-\xi} - B_{\overline{\xi}}\otimes B_{-\overline{\xi}}) \\ &= \mathrm{Id} + \underbrace{\frac{1}{(2r+1)^3}\sum_{\overline{\xi}\in\Lambda_\alpha}\sum_{\xi\in\Omega_{(\overline{\xi})}} \left(\gamma_{\overline{\xi}}(\mathrm{Id})\right)^2 (B_\xi\otimes B_{-\xi} - B_{\overline{\xi}}\otimes B_{-\overline{\xi}})}_{E} \\ &\quad + (R-\mathrm{Id}) + \underbrace{\frac{1}{(2r+1)^3}\sum_{\overline{\xi}\in\Lambda_\alpha}\sum_{\xi\in\Omega_{(\overline{\xi})}} \left[\left(\widetilde{\gamma}_{\overline{\xi}}(R)\right)^2 - \left(\gamma_{\overline{\xi}}(\mathrm{Id})\right)^2\right] (B_\xi\otimes B_{-\xi} - B_{\overline{\xi}}\otimes B_{-\overline{\xi}})}_{R'} \end{aligned}$$

Then it is easy to check that

$$\begin{aligned} |E| &\lesssim r\sigma \\ |R'| &\lesssim r\sigma\,|\mathrm{Id}-R| \end{aligned}$$

Then so long as $r\sigma \ll 1$, the Jacobian of f at $R=\mathrm{Id}$ is invertible. Moreover, if we assume $r\sigma$ to be sufficiently small then E can be made arbitrarily close to the zero matrix. Taking $\varepsilon_\gamma \le \varepsilon$ to be smaller if need be, by the inverse function theorem there exists a smooth inverse $f^{-1}: B_{\varepsilon_\gamma}(\mathrm{Id}) \to B_{\varepsilon_\gamma}(\mathrm{Id})$ such that $f^{-1}\circ f(R) = R$. To complete the proof we define $\gamma_{(\overline{\xi})} := \widetilde{\gamma}_{\overline{\xi}}\circ f^{-1}$. $\square$

3.3 Estimates involving intermittent Beltrami waves

Recall, the intermittent Beltrami waves were designed to include additional oscillations that cancel in order to minimize their L^1 norm, in a way that is analogous to the cancellations of the Dirichlet kernel. In this section we estimate our intermittent Beltrami waves using these cancellations.

Proposition 3.5. *Let $\Omega_{\overline{\xi},\sigma,r}$ and $\mathbb{W}_{(\overline{\xi})}$ be defined as above. The bound*

$$\left\|\mathbb{W}_{(\overline{\xi})}(x)\right\|_{L^p} \lesssim r^{3/2-3/p} \tag{3.12}$$

holds for any $1 < p \le \infty$.

Proof of Proposition 3.5. From the standard bounds of the Dirichlet kernel we have

$$\left\|\frac{1}{(2r+1)^{3/2}}\sum_{\xi\in\Omega_{e_1,\sigma,r}} e^{i\lambda\xi\cdot x}\right\|_{L^p} \lesssim r^{3/2-3/p}.$$

Then by rotational and scaling symmetries we obtain for each $\overline{\xi}\in\Lambda_\alpha$

$$\left\|\frac{1}{(2r+1)^{3/2}}\sum_{\xi\in\Omega_{\overline{\xi},\sigma,r}} e^{i\lambda\xi\cdot x}\right\|_{L^p} \lesssim r^{3/2-3/p}. \tag{3.13}$$

Now consider the Fourier multiplier $\mathrm{op}\,(m_{\overline{\xi}}) = \mathrm{op}\,(m_{\overline{\xi},1}) - \mathrm{op}\,(m_{\overline{\xi},2}) + \mathrm{op}\,(m_{\overline{\xi},3})$, whose symbol is given by

$$m_{\overline{\xi}}(\xi) := \underbrace{A_{\overline{\xi}}}_{m_{\overline{\xi},1}(\xi)} + \underbrace{\frac{i}{\lambda}\xi \times A_{\overline{\xi}}}_{m_{\overline{\xi},2}(\xi)} - \underbrace{\frac{1}{\lambda^2}(A_{\overline{\xi}} \cdot \xi)\xi}_{m_{\overline{\xi},3}(\xi)} . \tag{3.14}$$

Clearly, by definition $m_{\lambda\overline{\xi}}(\xi) = B_\xi$ and hence

$$\mathbb{W}_{(\overline{\xi})} = \mathrm{op}\,(m_{\overline{\xi}}) \left(\frac{1}{(2r+1)^{3/2}} \sum_{\xi \in \Omega_{\overline{\xi},\sigma,r}} e^{i\lambda\xi\cdot x} \right) . \tag{3.15}$$

Owing to (3.6) and (3.13), and the frequency support of $\mathbb{W}_{(\overline{\xi})}$ it suffices to bound the operator $\mathrm{op}\,(m_{\overline{\xi},i}(\xi))\mathbb{P}_{\approx\lambda}$.[5] as an operator on L^p, for $i \in \{1,2,3\}$. For $i = 1$ there is nothing to prove. From the standard Bernstein inequalities we obtain

$$\begin{aligned} \left\| \mathrm{op}\,(m_{\overline{\xi},2})\mathbb{P}_{\approx\lambda} \right\|_{L^p\to L^p} &\lesssim \frac{1}{\lambda} \left\| |\nabla| \, \mathbb{P}_{\approx\lambda} \right\|_{L^p\to L^p} \lesssim \frac{\lambda}{\lambda} = 1 \, , \\ \left\| \mathrm{op}\,(m_{\overline{\xi},3})\mathbb{P}_{\approx\lambda} \right\|_{L^p\to L^p} &\lesssim \frac{1}{\lambda^2} \left\| |\nabla|^2 \, \mathbb{P}_{\approx\lambda} \right\|_{L^p\to L^p} \lesssim \frac{\lambda^2}{\lambda^2} = 1 \, . \end{aligned}$$

Hence applying (3.13) and (3.15) we obtain

$$\left\| \mathbb{W}_{(\overline{\xi})} \right\|_{L^p} \lesssim \left\| \mathrm{op}\,(m_{\overline{\xi}})\mathbb{P}_{\approx\lambda} \right\|_{L^p\to L^p} \lesssim r^{3/2-3/p}$$

which concludes the proof. □

Owing to the second term in (3.8), it will be useful to estimate the object

$$\widetilde{\mathbb{W}}_{(\overline{\xi})} := \sum_{\xi \in \Omega_{(\overline{\xi})}} (1 - |\xi|) W_{(\xi)}$$

Proposition 3.6. *With $\mathbb{W}_{(\overline{\xi})}$ and $\widetilde{\mathbb{W}}_{(\overline{\xi})}$ defined as above, we have the following estimates*

$$\left\| \widetilde{\mathbb{W}}_{(\overline{\xi})} \right\|_{L^p} \lesssim \sigma r^{5/2-3/p} \tag{3.16}$$

$$\left\| |\nabla|^{-1} \left(\widetilde{\mathbb{W}}_{(\overline{\xi})} \times \mathbb{W}_{(-\overline{\xi})} \right) \right\|_{L^p} \lesssim \sigma r^{4-3/p} \lambda^{-1} \tag{3.17}$$

for any $1 < p \leq \infty$.

Proof of Proposition 3.6. To prove (3.16) we will use (3.12) and an estimate involving the Fourier multiplier $\mathrm{op}\,(m(\xi))$ with symbol

$$m(\xi) := 1 - \frac{\left|\lambda\overline{\xi} + \xi\right|}{\lambda^2} = -\frac{2\lambda\overline{\xi}\cdot\xi + |\xi|^2}{\lambda^2} .$$

By shifting the center of the cube $\Omega_{(\overline{\xi})}$ to the origin we see that

$$\widetilde{\mathbb{W}}_{(\overline{\xi})} e^{-i\lambda\overline{\xi}\cdot x} = \mathrm{op}\,(m) \left(\mathbb{W}_{(\overline{\xi})} e^{-i\lambda\overline{\xi}\cdot x} \right) .$$

[5]Throughout the paper we shall use the standard notation and bounds from Littlewood-Paley analysis [40]. For instance $\mathbb{P}_{\approx\lambda}$ denotes the Fourier multiplier operator which restricts the frequencies to the annulus $\{\lambda/2 \leq |\xi| \leq 2\lambda\}$.

Observe that due to the frequency support of $\widetilde{\mathbb{W}}_{(\bar{\xi})}$ we have

$$\widetilde{\mathbb{W}}_{(\bar{\xi})} e^{-i\lambda\bar{\xi}\cdot x} = \mathrm{op}\,(m)\left(\mathbb{W}_{(\bar{\xi})} e^{-i\lambda\bar{\xi}\cdot x}\right) = \mathrm{op}\,(m)\mathbb{P}_{\leq 2\lambda\sigma r}\left(\mathbb{W}_{(\bar{\xi})} e^{-i\lambda\bar{\xi}\cdot x}\right).$$

and in view of (3.12) it suffices to bound the operator $\mathrm{op}\,(m)\mathbb{P}_{\leq 2\lambda\sigma r}$ on L^p.

Let us write $\mathrm{op}\,(m)$ as the sum of two homogenous Fourier multipliers with symbols

$$m_1(\xi) := -\frac{2\bar{\xi}\cdot\xi}{\lambda} \quad \text{and} \quad m_2(\xi) := -\frac{|\xi|^2}{\lambda^2}$$

which obey the bounds

$$\begin{aligned} \|\mathrm{op}\,(m_1)\mathbb{P}_{\leq 2\lambda\sigma r}\|_{L^p\to L^p} &\lesssim \frac{\lambda\sigma r}{\lambda} = \sigma r \\ \|\mathrm{op}\,(m_2)\mathbb{P}_{\leq 2\lambda\sigma r}\|_{L^p\to L^p} &\lesssim \frac{\lambda^2\sigma^2 r^2}{\lambda^2} = \sigma^2 r^2 \leq \sigma r\,, \end{aligned}$$

where in the last inequality we used (3.6). Hence

$$\|\mathrm{op}\,(m)\mathbb{P}_{\leq 2\lambda\sigma r}\|_{L^p\to L^p} \lesssim \sigma r$$

and thus from (3.12) we conclude (3.16).

The bound (3.17) requires a more delicate argument, which explores that for $\xi, -\xi' \in \Omega_{(\bar{\xi})}$, the vectors B_ξ and $B_{\xi'}$ are either parallel or nearly parallel, giving a good bound on $B_\xi \times B_{\xi'}$. Observe

$$|\nabla|^{-1}\left(\widetilde{\mathbb{W}}_{(\bar{\xi})} \times \mathbb{W}_{(-\bar{\xi})}\right) = \frac{1}{(2r+1)^3}\sum_{\xi,-\xi'\in\Omega_{(\bar{\xi})},\xi+\xi'\neq 0}(1-|\xi|)\frac{B_{(\xi)}\times B_{(\xi')}}{\lambda\,|\xi+\xi'|}e^{i\lambda(\xi+\xi')\cdot x}$$

and hence, using the multiplier argument used to prove (3.16) we have

$$\begin{aligned} \left\||\nabla|^{-1}\left(\widetilde{\mathbb{W}}_{(\bar{\xi})} \times \mathbb{W}_{(-\bar{\xi})}\right)\right\|_{L^p} &\lesssim \sup_{\xi\in\Omega_{(\bar{\xi})}}\left\|\sum_{-\xi'\in\Omega_{(\bar{\xi})},\xi+\xi'\neq 0}(1-|\xi|)\frac{B_{(\xi)}\times B_{(\xi')}}{\lambda|\xi+\xi'|}e^{i\lambda(\xi+\xi')\cdot x}\right\|_{L^p} \\ &\lesssim \sigma r \sup_{\xi\in\Omega_{(\bar{\xi})}}\left\|\sum_{-\xi'\in\Omega_{(\bar{\xi})},\xi+\xi'\neq 0}\frac{B_{(\xi)}\times B_{(\xi')}}{\lambda|\xi+\xi'|}e^{i\lambda(\xi+\xi')\cdot x}\right\|_{L^p}. \end{aligned}$$

Applying a phase shift $\xi' \mapsto \xi' - \xi$, and the fact that $\left\|\mathrm{op}\,(B_{(\xi)}\times)\mathbb{P}_{\geq|\sigma|}\mathbb{P}_{\leq\lambda\sigma r}\right\|_{L^p\to L^p} \lesssim 1$ for $\xi \in \Omega_{(\bar{\xi})}$, we obtain

$$\begin{aligned} \left\|\sum_{-\xi'\in\Omega_{(\bar{\xi})},\xi+\xi'\neq 0}\frac{B_{(\xi)}\times B_{(\xi')}}{\lambda\,|\xi+\xi'|}e^{i\lambda(\xi+\xi')\cdot x}\right\|_{L^p} &\lesssim \left\||\nabla|^{-1}\left(\sum_{\xi-\xi'\in\Omega_{(\bar{\xi})},\xi'\neq 0}B_{(\xi)}\times B_{(\xi'-\xi)}e^{i\lambda\xi'\cdot x}\right)\right\|_{L^p} \\ &\lesssim \left\||\nabla|^{-1}\left(\sum_{\xi-\xi'\in\Omega_{(\bar{\xi})},\xi'\neq 0}B_{(\xi)}\times(B_{(\xi'-\xi)}-B_{(-\xi)})e^{i\lambda\xi'\cdot x}\right)\right\|_{L^p} \\ &\lesssim \frac{1}{\lambda}\left\|\sum_{\xi-\xi'\in\Omega_{(\bar{\xi})},\xi'\neq 0}\frac{B_{(\xi'-\xi)}-B_{(-\xi)}}{|\xi'|}e^{i\lambda\xi'\cdot x}\right\|_{L^p}. \end{aligned}$$

By definition we have

$$\begin{aligned}\frac{B_{(\xi'-\xi)}-B_{(-\xi)}}{|\xi'|} &= \frac{i\xi'\times A_{\overline{\xi}}+(A_{\overline{\xi}}\cdot\xi)\xi-(A_{\overline{\xi}}\cdot(\xi-\xi'))(\xi-\xi')}{|\xi'|}\\ &= -iA_{\overline{\xi}}\times\frac{\xi'}{|\xi'|}+(A_{\overline{\xi}}\cdot\xi)\frac{\xi'}{|\xi'|}+\left(A_{\overline{\xi}}\cdot\frac{\xi'}{|\xi'|}\right)\xi-|\xi'|\left(A_{\overline{\xi}}\cdot\frac{\xi'}{|\xi'|}\right)\frac{\xi'}{|\xi'|}\end{aligned}$$

and that for $\xi-\xi'\in\Omega_{(\overline{\xi})}$ we have $|\xi'|\le 2\sigma r$. Thus, viewing ${(B_{(\xi'-\xi)}-B_{(-\xi)})}/{|\xi'|}$ as a sum of zero order and first order homogenous Fourier multiplier operators, which are bounded similarly to $\mathrm{op}\,(m)$, we obtain

$$\begin{aligned}&\left\|\sum_{\xi-\xi'\in\Omega_{(\overline{\xi})},\xi'\neq 0}\frac{B_{(\xi'-\xi)}-B_{(-\xi)}}{|\xi'|}e^{i\lambda\xi'\cdot x}\right\|_{L^p}\\ &\quad\lesssim\left\|\sum_{\xi-\xi'\in\Omega_{(\overline{\xi})},\xi'\neq 0}e^{i\lambda\xi'\cdot x}\right\|_{L^p}+\left\|\frac{|\nabla|}{\lambda}\left(\sum_{-\xi'-\xi\in\Omega_{(\overline{\xi})},\xi'\neq 0}e^{i\lambda\xi'\cdot x}\right)\right\|_{L^p}\lesssim r^{3-3/p}\end{aligned}$$

since $\sigma r\lesssim 1$. Combining the above estimates we conclude

$$\left\||\nabla|^{-1}\left(\widetilde{\mathbb{W}}_{(\xi)}\times\mathbb{W}_{(\xi)}\right)\right\|_{L^p}\lesssim\sigma r\frac{1}{\lambda}r^{3-3/p}=\frac{\sigma r^{4-3/p}}{\lambda}$$

which establishes (3.17). □

We now introduce a crucial lemma from [5] that will be used throughout the paper. Suppose we wish to estimate

$$\left\|f\,\mathbb{W}_{(\overline{\xi})}\right\|_{L^1}$$

for some arbitrary function $f:\mathbb{T}^3\to\mathbb{R}$. The trivial estimate is

$$\left\|f\,\mathbb{W}_{(\overline{\xi})}\right\|_{L^1}\lesssim\|f\|_{L^2}\left\|\mathbb{W}_{(\overline{\xi})}\right\|_{L^2}.$$

Such an estimate does not however take advantage of the special structure of the $2\pi\lambda\sigma)^{-1}$ periodic function $\mathbb{W}_{(\overline{\xi})}e^{-i\lambda\overline{\xi}\cdot x}$. It turns out that if say f has frequency contained in a ball of radius μ and $\lambda\sigma\gg\mu$ then one obtains the improved estimate

$$\left\|f\,\mathbb{W}_{(\overline{\xi})}\right\|_{L^1}\lesssim\|f\|_{L^1}\left\|\mathbb{W}_{(\overline{\xi})}\right\|_{L^1}$$

which gives us the needed gain because $\left\|\mathbb{W}_{(\overline{\xi})}\right\|_{L^1}\ll\left\|\mathbb{W}_{(\overline{\xi})}\right\|_{L^2}$. This idea is one of the key insights of [5] and is summarized in Lemma 3.7 below. For convenience we include the proof in Appendix A.1.

Lemma 3.7. *Let $p\in\{1,2\}$, and let f be a periodic function defined on $\mathbb{T}^3$ such that there exists a constants C_f and λ such that*

$$\|D^jf\|_{L^p}\le C_f\lambda^j$$

for all $1\le j\le M+4$. For instance, this is the case if f has frequency support in the ball of radius λ. In addition, let g be a $2\pi\mu^{-1}$-periodic function, where

$$\frac{2\pi\sqrt{3}\lambda}{\mu}\le\frac{1}{3}\qquad\textit{and}\qquad\lambda^4\frac{(2\pi\sqrt{3}\lambda)^M}{\mu^M}\le 1$$

Then we have that

$$\|fg\|_{L^p}\lesssim C_f\|g\|_{L^p}$$

holds, where the implicit constant is independent of μ and λ.

4 The perturbation

In this section we will construct the perturbation w_{q+1}. Because the Reynolds stress $\mathring{R}_q$ is not spatially homogenous, first we need to introduce stress cutoff functions.

4.1 Stress cutoffs

Let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be a family of standard Friedrichs mollifiers (of compact support of radius 2) on $\mathbb{R}^3$ (space), and $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be a family of standard Friedrichs mollifiers (of compact support of width 2) on $\mathbb{R}$ (time). We define

$$\mathring{R}_\ell = (\mathring{R}_q *_x \phi_\ell) *_t \varphi_\ell \tag{4.1}$$

to be a mollification of $\mathring{R}_q$ in space and time, at length scale and time scale ℓ. The parameter ℓ will be chosen (cf. (4.10) below) to satisfy

$$(\sigma\lambda_{q+1})^{-1/2} \ll \ell \ll \lambda_q^{-19}\delta_{q+1}\,. \tag{4.2}$$

In particular, in view (2.4) and the second inequality in (4.2), we have that

$$\left\|\mathring{R}_q\right\|_{C^1_{t,x}} \leq \ell^{-1}\,. \tag{4.3}$$

We let $0 \leq \widetilde{\chi}_0, \widetilde{\chi} \leq 1$ be bump functions adapted to the intervals $[0,4]$ and $[1/4,4]$ respectively, such that together they form a partition of unity:

$$\widetilde{\chi}_0^2(y) + \sum_{i\geq 1}\widetilde{\chi}_i^2(y) \equiv 1, \quad \text{where} \quad \widetilde{\chi}_i(y) = \widetilde{\chi}(4^{-i}y), \tag{4.4}$$

for any $y > 0$. We then define

$$\chi_{(i)}(x,t) = \chi_{i,q+1}(x,t) = \widetilde{\chi}_i\left(\left\langle \frac{\mathring{R}_\ell(x,t)}{100\lambda_q^{-\varepsilon_R}\delta_{q+1}}\right\rangle\right) \tag{4.5}$$

for all $i \geq 0$. By definition the cutoffs $\chi_{(i)}$ for a partition of unity

$$\sum_{i\geq 0}\chi_{(i)}^2 \equiv 1 \tag{4.6}$$

and we will show in Lemma 4.1 below that there exists an index $i_{\max} = i_{\max}(q)$, such that $\chi_{(i)} \equiv 0$ for all $i > i_{\max}$, and moreover that $4^{i_{\max}} \lesssim \ell^{-1}$.

4.2 The definition of the velocity increment

Define the coefficient function $a_{\overline{\xi},i,q+1}$ by

$$a_{(\overline{\xi})} := a_{\overline{\xi},i,q+1} := \rho_i^{1/2}(t)\chi_{i,q+1}\gamma_{(\overline{\xi})}\left(\mathrm{Id} - \frac{\mathring{R}_\ell}{\rho_i(t)}\right). \tag{4.7}$$

where for $i \geq 1$, the parameters ρ_i are defined by

$$\rho_i := \lambda_q^{-\varepsilon_R}\delta_{q+1}4^{i+c_0} \tag{4.8}$$

where $c_0 \in \mathbb{N}$ is a sufficiently large constant, which depends on the ε_γ in Proposition 3.4. The addition of the factor 4^{c_0} ensures that the argument of $\gamma_{(\overline{\xi})}$ is in the range of definition. The definition ρ_0 is slightly more complicated and as such its definition will be delayed to Section 4.3 below.

By a slight abuse of notation, let us now fix λ, σ, r for the short hand notation $\mathbb{W}_{(\overline{\xi})}, W_{(\xi)}$ and $\widetilde{\mathbb{W}}_{(\overline{\xi})}$ introduced in Section 3.2:

$$\mathbb{W}_{(\overline{\xi})} := \mathbb{W}_{\lambda_{q+1},\overline{\xi},\sigma,r}, \quad W_{(\xi)} := W_{\lambda_{q+1},\xi,\sigma,r}, \quad \text{and} \quad \widetilde{\mathbb{W}}_{(\overline{\xi})} := \widetilde{\mathbb{W}}_{\lambda_{q+1},\overline{\xi},\sigma,r}\,,$$

where the integer r and the parameter σ are defined by

$$r = \lambda_{q+1}^{3/4}, \qquad \text{and} \qquad \sigma = \lambda_{q+1}^{-7/8}. \tag{4.9}$$

Moreover, at this stage we fix

$$\ell = \lambda_q^{-20}\,, \tag{4.10}$$

which in view of the choice of σ in (4.9), ensures that (4.2) holds, upon taking λ_0 sufficiently large. In view of (4.10), throughout the rest of the paper we may use either $\ell^\varepsilon \leq \lambda_0^{-20\varepsilon}$ or $\lambda_q^{-\varepsilon} \leq \lambda_0^{-\varepsilon}$, with $\varepsilon > 0$ arbitrarily small, to absorb any of the constants (which are q-independent) appearing due to $\lesssim$ signs in the below inequalities. This is possible by choosing $\lambda_0 = a$, sufficiently large.

The principal part of w_{q+1} will be defined as

$$w_{q+1}^{(p)} := \sum_i \sum_{\overline{\xi} \in \Lambda_{(i)}} a_{(\overline{\xi})}\, \mathbb{W}_{(\overline{\xi})}\,, \tag{4.11}$$

where the sum is over $0 \leq i \leq i_{\max}(q)$. The sets $\Lambda_{(i)}$ are defined as follows. In Lemma 3.2 it suffices to take $N = 2$, so that $\alpha \in \{\alpha_0, \alpha_1\}$, and we define $\Lambda_{(i)} = \Lambda_{\alpha_{i \bmod 2}}$. This choice is allowable since $\chi_i \chi_j \equiv 0$ for $|i - j| \geq 2$. In order to fix the fact that $w_{q+1}^{(p)}$ is not divergence free, we define a corrector by

$$w_{q+1}^{(c)} := \sum_i \sum_{\overline{\xi} \in \Lambda_{(i)}} \sum_{\xi \in \Omega_{(\overline{\xi})}} \frac{i}{\lambda_{q+1}} \nabla a_{(\overline{\xi})} \times \frac{\xi \times W_{(\xi)}}{|\xi|^2}\,. \tag{4.12}$$

Using that $\xi \cdot W_{(\xi)} = 0$, we then have

$$w_{q+1}^{(p)} + w_{q+1}^{(c)} = \frac{i}{\lambda_{q+1}} \sum_i \sum_{\overline{\xi} \in \Lambda_{(i)}} \sum_{\xi \in \Omega_{(\overline{\xi})}} \operatorname{curl} \left(a_{(\overline{\xi})} \frac{\xi \times W_{(\xi)}}{|\xi|^2} \right)\,,$$

and thus

$$\operatorname{div} \left(w_{q+1}^{(p)} + w_{q+1}^{(c)} \right) = 0\,.$$

We note that as a consequence of (3.11) and (4.6) we have that

$$\sum_{i \geq 0} \sum_{\overline{\xi} \in \Lambda_{(i)}} \sum_{\xi \in \Omega_{(\overline{\xi})}} a_{(\overline{\xi})}^2 W_{(\xi)} \otimes W_{(-\xi)} = \sum_{i \geq 0} \rho_i \chi_{(i)}^2 \mathrm{Id} - \mathring{R}_\ell\,. \tag{4.13}$$

Finally, we define the velocity increment w_{q+1} by

$$w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)}. \tag{4.14}$$

4.3 The definition of ρ_0

It follows from (4.13) that with the ρ_i defined above we have

$$\begin{aligned}\sum_{i\geq 1}\int_{\mathbb{T}^3}\left|\sum_{\overline{\xi}\in\Lambda_{(i)}} a_{(\overline{\xi})}\mathbb{W}_{(\overline{\xi})}\right|^2 dx &= \sum_{i\geq 1}\sum_{\overline{\xi},\bar{\xi}'\in\Lambda_{(i)}}\sum_{\xi\in\Omega_{(\xi)},\xi'\in\Omega_{(\bar{\xi}')}}\int_{\mathbb{T}^3} a_{(\overline{\xi})}a_{(\bar{\xi}')}\mathrm{tr}\left(W_{(\xi)}\otimes W_{-(\xi')}\right)dx\\ &= \sum_{i\geq 1}\sum_{\overline{\xi}\in\Lambda_{(i)}}\sum_{\xi\in\Omega_{(\xi)}}\int_{\mathbb{T}^3} a^2_{(\overline{\xi})}\mathrm{tr}\left(W_{(\xi)}\otimes W_{(-\xi)}\right)dx + \mathrm{error}\\ &= 3\sum_{i\geq 1}\rho_i\int_{\mathbb{T}^3}\chi^2_{(i)}\,dx + \mathrm{error}\,,\end{aligned} \tag{4.15}$$

where the error term comes from $\xi'\neq -\xi$ and can be made arbitrarily small since the spatial frequency of the $a_{(\xi)}$'s is ℓ^{-1}, while that of the oscillatory phase it is $\lambda_{q+1}\gg\ell^{-1}$. We will show in the next section, Lemma 4.3 that

$$\sum_{i\geq 1}\rho_i\int_{\mathbb{T}^3}\chi^2_{(i)}\,dx \lesssim \delta_{q+1}\lambda_q^{-\varepsilon_R}\,. \tag{4.16}$$

In order to ensure (2.5) is satisfied for $q+1$, we design ρ_0 such that

$$\int_{\mathbb{T}^3}\left|\sum_{\overline{\xi}\in\Lambda_{(0)}} a_{(\overline{\xi})}\mathbb{W}_{(\overline{\xi})}\right|^2 dx \approx \widetilde{e}(t) := e(t) - \int_{\mathbb{T}^3}|v_q|^2\,dx - 3\sum_{i\geq 1}\rho_i\int_{\mathbb{T}^3}\chi_i^2\,dx\,.$$

We thus define

$$\rho(t) := \frac{1}{3\,|\mathbb{T}^3|}\left(\int_{\mathbb{T}^2}\chi_0^2\,dx\right)^{-1}\max\left(\widetilde{e}(t)-\frac{\delta_{q+2}}{2},\,0\right)\,.$$

The term $-\delta_{q+2}/2$ is added to ensure that we leave room for future corrections and the max is in place to ensure that we do not correct the energy when the energy of v_q is already sufficiently close to the prescribed energy profile. This later property will allow us to take energy profiles with compact support. Finally, in order to ensure ρ_0 is sufficiently smooth, we define ρ_0 as the mollification of ρ at time scale ℓ

$$\rho_0 = \rho *_t \varphi_\ell\,.$$

We note that (2.5) and (4.24) below imply that

$$\rho_0 \leq 2\delta_{q+1} \quad \text{and} \quad \|\rho_0\|_{C_t^N}\lesssim \delta_{q+1}\ell^{-N} \tag{4.17}$$

for $N\geq 1$. By a slight abuse of notation we will denote

$$\frac{\mathring{R}_\ell}{\rho_0(t)} = \begin{cases}\frac{\mathring{R}_\ell}{\rho_0(t)} & \text{if } \mathring{R}_\ell\neq 0\\ 0 & \text{otherwise}\,.\end{cases}$$

In order to ensure that $\mathrm{Id} - \frac{\mathring{R}_\ell}{\rho_0(t)}$ is in the domain of the functions $\gamma_{(\bar{\xi})}$ from Proposition 3.4, we will need to ensure that

$$\left\| \frac{\mathring{R}_\ell}{\rho_0(t)} \right\|_{L^\infty(\operatorname{supp}\chi_{(0)})} \leq \varepsilon_\gamma \,. \tag{4.18}$$

We give the proof of (4.18) next. Owing to the estimate

$$\left| e(t) - \int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx - e(t') - \int_{\mathbb{T}^3} |v_q(x,t')|^2 \right| dx \lesssim \ell^{1/2}$$

for $t' \in (t-\ell, t+\ell)$ which follows from Lemma 6.1 in Section 6, and the inequality $\ell^{1/2} \ll \delta_{q+1}$, we may apply (2.6) to conclude that it is sufficient to check the above condition when

$$e(t) - \int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx \geq \frac{\delta_{q+1}}{200} \,.$$

Then by (4.16), the above lower bound implies

$$\widetilde{e}(t) \geq \frac{\delta_{q+1}}{400} \,.$$

and thus

$$\rho(t) \geq \frac{1}{|\mathbb{T}^3|} \left(\frac{\delta_{q+1}}{400} - \frac{\delta_{q+2}}{2} \right) \geq \frac{\delta_{q+1}}{500}$$

where we used (4.24) from Lemma 4.3 below to bound the integral. Finally, using the estimate (6.4) from Section 6 we obtain

$$\rho_0 \geq \frac{\delta_{q+1}}{600}$$

Since on the support of χ_0 we have $\left| \mathring{R}_\ell \right| \lesssim \lambda_q^{-\varepsilon_R} \delta_{q+1}$ we obtain (4.18).

4.4 Estimates of the perturbation

We first collect a number of estimates concerning the cutoffs $\chi_{(i)}$ defined in (4.5).

Lemma 4.1. *For $q \geq 0$, there exists $i_{\max}(q) \geq 0$, determined by (4.21) below, such that*

$$\chi_{(i)} \equiv 0 \quad \textit{for all} \quad i > i_{\max}.$$

Moreover, we have that for all $0 \leq i \leq i_{\max}$

$$\rho_i \lesssim 4^{i_{\max}} \lesssim \ell^{-1} \tag{4.19}$$

where the implicit constants is independent of q. Moreover, we have

$$\sum_{i \geq 0} \rho_i^{1/2} 2^{-i} \leq 3 \delta_{q+1}^{1/2} \,. \tag{4.20}$$

Proof of Lemma 4.1. By the definition of $\widetilde{\chi}_i$ we have that $\chi_{(i)} = 0$ for all (x,t) such that

$$\langle 100^{-1}\lambda_q^{\varepsilon_R}\delta_{q+1}^{-1}\mathring{R}_\ell(x,t)\rangle > 4^{i+1}$$

and in particular, for all (x,t) such that

$$|\mathring{R}_\ell(x,t)| > 100\lambda_q^{-\varepsilon_R}4^{i+1}\delta_{q+1}.$$

Using the inductive assumption (2.4), we have that

$$\left\|\mathring{R}_\ell\right\|_{L^\infty} \lesssim \left\|\mathring{R}_\ell\right\|_{C^1} \lesssim \left\|\mathring{R}_q\right\|_{C^1} \le C_{\max}\lambda_q^{10} \le \lambda_q^{10+\varepsilon_R}$$

since the implicit constant $C_{\max}$ is independent of q (it only depends on norms of the mollifier ϕ used to define $\mathring{R}_\ell$), and thus we have $C_{\max} \le \lambda_q^{\varepsilon_R}$. Therefore, as $\varepsilon_R \le 1/4$, we may define $i_{\max}$ by

$$i_{\max}(q) = \max\left\{i \ge 0\colon 4^{i+1} \le \lambda_q^{11}\delta_{q+1}^{-1}\right\}. \tag{4.21}$$

Observe that, the first inequality of (4.19) follows trivially from the definition of ρ_i for $i \ge 1$ and (4.17). The second inequality follows from the fact that $\lambda_q^{11}\delta_{q+1}^{-1} \le \ell^{-1}$, which is a consequence of $b\beta$ being small. Finally, from the definition (4.8) and the bound (4.17) we have

$$\sum_{i\ge 0}\rho_i^{1/2}2^{-i} \le 2\delta_{q+1}^{1/2} + \sum_{i\ge 1}\lambda_q^{-\varepsilon_R/2}\delta_{q+1}^{1/2} \le \delta_{q+1}^{1/2}(2 + \lambda_q^{-\varepsilon_R/2}\log_4(\ell^{-1})).$$

By (4.19), we can bound the second term as

$$\lambda_q^{-\varepsilon_R/2}\log_4(\ell^{-1}) \le \lambda_q^{-\varepsilon_R/2}20\log_4(\lambda_q) \le 1$$

as soon as a is taken to be sufficiently large, depending on ε_R. This concludes the proof of (4.20). $\square$

The size and derivative estimate for the $\chi_{(i)}$ are summarized in the following lemma

Lemma 4.2. *Let $0 \le i \le i_{\max}$. Then we have*

$$\left\|\chi_{(i)}\right\|_{L^2} \lesssim 2^{-i} \tag{4.22}$$

$$\left\|\chi_{(i)}\right\|_{C^N_{x,t}} \lesssim \lambda_q^{10}\ell^{1-N} \lesssim \ell^{-N} \tag{4.23}$$

for all $N \ge 1$.

Proof of Lemma 4.2. We prove that

$$\left\|\chi_{(i)}\right\|_{L^1} \lesssim 4^{-i},$$

so that the bound (4.22) follows since $\chi_{(i)} \le 1$, upon interpolating the L^2 norm between the L^1 and the L^∞ norms.

When $i = 0, 1$, we have that $\left\|\chi_{(i)}\right\|_{L^1} \le |\mathbb{T}^3|\left\|\chi_{(0)}\right\|_{L^\infty} \lesssim 1 \lesssim 4^{-i}$. For $i \ge 2$, we use the Chebyshev's inequality and the inductive assumption (2.3) to conclude

$$\begin{aligned}
\left\|\chi_{(i)}\right\|_{L^1} &\le \sup_t\left|\left\{x\colon 4^{i-1} \le \langle\lambda_q^{\varepsilon_R}\delta_{q+1}^{-1}\mathring{R}_\ell(x,t)/100\rangle \le 4^{i+1}\right\}\right| \\
&\le \sup_t\left|\left\{x\colon 4^{i-1} \le \lambda_q^{\varepsilon_R}\delta_{q+1}^{-1}|\mathring{R}_\ell(x,t)|/100 + 1\right\}\right| \\
&\le \sup_t\left|\left\{x\colon 100\lambda_q^{-\varepsilon_R}\delta_{q+1}4^{i-2} \le |\mathring{R}_\ell(x,t)|\right\}\right| \\
&\lesssim \lambda_q^{\varepsilon_R}\delta_{q+1}^{-1}4^{-i}\left\|\mathring{R}_\ell\right\|_{L^\infty_t L^1_x} \\
&\lesssim \lambda_q^{\varepsilon_R}\delta_{q+1}^{-1}4^{-i}\left\|\mathring{R}_q\right\|_{L^\infty_t L^1_x} \lesssim 4^{-i}
\end{aligned}$$

proving the desired L^1 bound. In order to prove the estimate (4.23) we appeal to [2, Proposition C.1] which yields

$$\begin{aligned}
\left\|\chi_{(i)}\right\|_{C^N_{t,x}} &\lesssim \left\|\langle \lambda_q^{\varepsilon_R}\delta_{q+1}^{-1}\mathring{R}_\ell/100\rangle\right\|_{C^N_{t,x}} + \left\|\langle \lambda_q^{\varepsilon_R}\delta_{q+1}^{-1}\mathring{R}_\ell/100\rangle\right\|^N_{C^1_{t,x}} \\
&\lesssim \ell^{-N+1}\left\|\mathring{R}_\ell\right\|_{C^1_{t,x}} + \left\|\mathring{R}_\ell\right\|^N_{C^1_{t,x}} \\
&\lesssim \ell^{-N+1}\left\|\mathring{R}_q\right\|_{C^1_{t,x}} + \left\|\mathring{R}_q\right\|^N_{C^1_{t,x}} \\
&\lesssim \lambda_q^{10}\ell^{-N+1} + \lambda_q^{10N} \lesssim \lambda_q^{10}\ell^{1-N}
\end{aligned}$$

where we have used that $\delta_{q+1} \lesssim 1$ and (2.4). ☐

Lemma 4.3. *We have that the following lower and upper bounds hold:*

$$\int_{\mathbb{T}^3} \chi^2_{(0)}\,dx \geq \frac{|\mathbb{T}^3|}{2} \tag{4.24}$$

$$\sum_{i\geq 1}\rho_i\int_{\mathbb{T}^3}\chi^2_{(i)}(x,t)\,dx \lesssim \lambda_q^{-\varepsilon_R}\delta_{q+1} \tag{4.25}$$

Proof of Lemma 4.3. By Chebyshev's inequality we have

$$\left|\left\{x\middle|\left|\mathring{R}_\ell\right| \geq 2\lambda_q^{-\varepsilon_R}\delta_{q+1}\left|\mathbb{T}^3\right|^{-1}\right\}\right| \leq \frac{\left|\mathbb{T}^3\right|\left\|\mathring{R}_\ell\right\|_{L^1}}{2\lambda_q^{-\varepsilon_R}\delta_{q+1}} \leq \frac{\left|\mathbb{T}^3\right|\left\|\mathring{R}_q\right\|_{L^1}}{2\lambda_q^{-\varepsilon_R}\delta_{q+1}} \leq \frac{\left|\mathbb{T}^3\right|}{2}$$

where we have used (2.3). Then from the definition of $\chi_{(0)}$ we obtain (4.24).

Observe that by definition,

$$\begin{aligned}
\sum_{i\geq 1}\rho_i\int_{\mathbb{T}^3}\chi^2_{(i)}dx &\lesssim \sum_{i\geq 1}(4^i\lambda_q^{-\varepsilon_R}\delta_{q+1})\widetilde{\chi}^2\left(\frac{1}{4^i}\left\langle\frac{\mathring{R}_\ell}{100\lambda_q^{-\varepsilon_R}\delta_{q+1}}\right\rangle\right)dx \\
&\lesssim \left\|\mathring{R}_\ell\right\|_{L^1} \lesssim \left\|\mathring{R}_q\right\|_{L^1} \lesssim \lambda_q^{-\varepsilon_R}\delta_{q+1}
\end{aligned}$$

from which we conclude (4.25). ☐

Lemma 4.4. *The bounds*

$$\left\|a_{(\overline{\xi})}\right\|_{L^2} \lesssim \rho_i^{1/2}2^{-i} \lesssim \delta_{q+1}^{1/2}, \tag{4.26}$$

$$\left\|a_{(\overline{\xi})}\right\|_{L^\infty} \lesssim \rho_i^{1/2} \lesssim \delta_{q+1}^{1/2}2^i, \tag{4.27}$$

$$\left\|a_{(\overline{\xi})}\right\|_{C^N_{x,t}} \lesssim \ell^{-N} \tag{4.28}$$

hold for all $0 \leq i \leq i_{\max}$ *and* $N \geq 1$.

Proof of Lemma 4.4. The bound (4.27) follows directly from the definitions (4.7), (4.8) together with the boundedness of the functions $\gamma_{(\overline{\xi})}$ and ρ_0 given in (4.17). Using additionally (4.22), the estimate (4.26) follows similarly. For (4.28), we apply derivatives to (4.7), use [2, Proposition C.1], estimate (4.23), Lemma 4.1, the bound (2.4) for $\mathring{R}_\ell$, and the restriction (4.2) on ℓ, to obtain for the case $i \geq 1$ the estimate

$$\begin{aligned}
\left\|a_{(\bar{\xi})}\right\|_{C^N_{x,t}} &\lesssim \rho_i^{1/2}\left(\left\|\chi_{(i)}\right\|_{L^\infty}\left\|\gamma_{(\bar{\xi})}\left(\rho_i^{-1}\mathring{R}_\ell\right)\right\|_{C^N_{x,t}} + \left\|\chi_{(i)}\right\|_{C^N_{x,t}}\left\|\gamma_{(\bar{\xi})}\left(\rho_i^{-1}\mathring{R}_\ell\right)\right\|_{L^\infty}\right)\\
&\lesssim \rho_i^{1/2}\left(\rho_i^{-1}\left\|\mathring{R}_\ell\right\|_{C^N_{x,t}} + \rho_i^{-1}\left\|\mathring{R}_\ell\right\|^N_{C^1_{t,x}} + \lambda_q^{10}\ell^{1-N}\right)\\
&\lesssim \rho_i^{1/2}\left(\rho_i^{-1}\ell^{-N+1}\left\|\mathring{R}_\ell\right\|_{C^1_{x,t}} + \rho_i^{-1}\left\|\mathring{R}_\ell\right\|^N_{C^1_{t,x}} + \rho_i^{-1/2}\ell^{-N}\right)\\
&\lesssim \ell^{-N},
\end{aligned}$$

For $i = 0$, time derivative may land on ρ_0, and we use in addition (4.17) to estimate similarly. ☐

Proposition 4.5. *The principal part and of the velocity perturbation, and the corresponding corrector, obey the bounds*

$$\left\|w^{(p)}_{q+1}\right\|_{L^2} \leq \frac{M}{2}\delta^{1/2}_{q+1} \tag{4.29}$$

$$\left\|w^{(c)}_{q+1}\right\|_{L^2} \lesssim \ell^{-1}\lambda^{-1}_{q+1} \tag{4.30}$$

$$\left\|w^{(p)}_{q+1}\right\|_{W^{1,p}} + \left\|\partial_t w^{(p)}_{q+1}\right\|_{L^p} \lesssim \ell^{-2}\lambda_{q+1}r^{3/2-3/p} \tag{4.31}$$

$$\left\|w^{(c)}_{q+1}\right\|_{W^{1,p}} + \left\|\partial_t w^{(c)}_{q+1}\right\|_{L^p} \lesssim \ell^{-3}r^{3/2-3/p} \tag{4.32}$$

$$\left\|w^{(p)}_{q+1}\right\|_{C^N_{x,t}} + \left\|w^{(c)}_{q+1}\right\|_{C^N_{x,t}} \leq \lambda^{3N}_{q+1} \tag{4.33}$$

for $N \geq 1$ and $p > 1$.

Proof of Proposition 4.5. For $i \geq 0$, from (4.26) and (4.28) we may estimate

$$\left\|D^N a_{(\bar{\xi})}\right\|_{L^2} \lesssim \delta^{1/2}_{q+1}\ell^{-2N},$$

where we have used that $\ell\delta^{-1/2}_{q+1} = \lambda_q^{-20+\beta b} \lesssim 1$, which follows from the restriction imposed on the smallness of βb. Since $\mathbb{W}_{(\bar{\xi})}$ is $2\pi(\lambda_{q+1}\sigma)^{-1}$ periodic, and the condition (4.2) gives that $\ell^{-2} \ll \lambda_{q+1}\sigma$, we may apply Lemma 3.7 to conclude

$$\left\|a_{(\bar{\xi})}\mathbb{W}_{(\bar{\xi})}\right\|_{L^2} \lesssim \delta^{1/2}_{q+1}\left\|\mathbb{W}_{(\bar{\xi})}\right\|_{L^2} \lesssim \delta^{1/2}_{q+1}.$$

Hence we obtain (4.29) for some fixed constant M independent of any parameters.

In order to bound the L^2 norm of $w^{(c)}_{q+1}$ we use (4.28) to estimate

$$\begin{aligned}
\left\|\sum_{\xi\in\Omega_{(\bar{\xi})}} \frac{i}{\lambda_{q+1}}\nabla a_{(\bar{\xi})} \times \frac{\xi \times W_{(\xi)}}{|\xi|^2}\right\|_{L^2} &\lesssim \frac{1}{\lambda_{q+1}}\left\|\nabla a_{(\bar{\xi})}\right\|_{L^\infty}\left\|\sum_{\xi\in\Omega_{(\bar{\xi})}} \frac{\xi \times W_{(\xi)}}{|\xi|^2}\right\|_{L^2}\\
&\lesssim \frac{\ell^{-1}}{\lambda_{q+1}}\left\|\sum_{\xi\in\Omega_{(\bar{\xi})}} \frac{\xi \times W_{(\xi)}}{|\xi|^2}\right\|_{L^2}.
\end{aligned}$$

We then note that

$$\left\| \sum_{\xi \in \Omega_{(\overline{\xi})}} \frac{\xi \times W_{(\xi)}}{|\xi|^2} \right\|_{L^2} \lesssim \lambda_{q+1} \left\| \operatorname{op}\left(\frac{\xi \times}{|\xi|^2} \right) \mathbb{P}_{\approx \lambda_{q+1}} \right\|_{L^2 \to L^2} \left\| \mathbb{W}_{(\overline{\xi})} \right\|_{L^2} \lesssim \left\| \mathbb{W}_{(\overline{\xi})} \right\|_{L^2} \lesssim 1$$

from which we conclude

$$\left\| w_{q+1}^{(c)} \right\|_{L^2} \lesssim \frac{\ell^{-1}}{\lambda_{q+1}} .$$

Now consider (4.31). Observe that by definition (4.11) we have

$$\begin{aligned} \left\| w_{q+1}^{(p)} \right\|_{W^{1,p}} + \left\| \partial_t w_{q+1}^{(p)} \right\|_{L^p} &\lesssim \sum_i \sum_{\overline{\xi} \in \Lambda_\alpha} \left\| a_{\overline{\xi}} \right\|_{C^1_{x,t}} \left\| \mathbb{W}_{(\overline{\xi})} \right\|_{W^{1,p}} \\ &\lesssim \sum_i \sum_{\overline{\xi} \in \Lambda_\alpha} \ell^{-1} \lambda_{q+1} r^{3/2 - 3/p} \\ &\lesssim \ell^{-2} \lambda_{q+1} r^{3/2 - 3/p} \end{aligned} \tag{4.34}$$

For the analogous bound on $w_{q+1}^{(c)}$ we have

$$\begin{aligned} &\left\| \sum_{\xi \in \Omega_{(\overline{\xi})}} \frac{i}{\lambda_{q+1}} \nabla a_{(\overline{\xi})} \times \frac{\xi \times W_{(\xi)}}{|\xi|^2} \right\|_{W^{1,p}} + \left\| \sum_{\xi \in \Omega_{(\overline{\xi})}} \frac{i}{\lambda_{q+1}} \partial_t \nabla a_{(\overline{\xi})} \times \frac{\xi \times W_{(\xi)}}{|\xi|^2} \right\|_{L^p} \\ &\lesssim \frac{1}{\lambda_{q+1}} \left\| \nabla a_{(\overline{\xi})} \right\|_{C^2_{x,t}} \left\| \sum_{\xi \in \Omega_{(\overline{\xi})}} \frac{\xi \times W_{(\xi)}}{|\xi|^2} \right\|_{W^{1,p}} \\ &\lesssim \ell^{-2} r^{3/2 - 3/p} \end{aligned}$$

from which it follows that

$$\left\| w_{q+1}^{(c)} \right\|_{W^{1,p}} + \left\| \partial_t w_{q+1}^{(c)} \right\|_{L^p} \lesssim \ell^{-3} r^{3/2 - 3/p}.$$

For the derivative bounds of $w_{q+1}^{(p)}$, we use and (4.28) to conclude

$$\left\| a_{(\overline{\xi})} \mathbb{W}_{(\overline{\xi})} \right\|_{C^N_{x,t}} \lesssim \left\| a_{(\overline{\xi})} \right\|_{C^N_{x,t}} \left\| \mathbb{W}_{(\overline{\xi})} \right\|_{C^N} \lesssim \ell^{-N} \lambda_{q+1}^N r^{3/2}$$

from which the first part of (4.33) immediately follows in view of our parameter choices. The bound for the C^N norm of $w_{q+1}^{(c)}$ follows mutatis mutandis. ☐

In view of the definition of w_{q+1} in (4.14), and the bound $\ell^{-1} \lambda_{q+1}^{-1} \leq \frac{1}{2} \delta_{q+1}^{1/2}$, which holds since b was taken to be sufficiently large, the previous lemma trivially implies:

Corollary 4.6.

$$\| w_{q+1} \|_{L^2} \leq M \delta_{q+1}^{1/2} \tag{4.35}$$

$$\| w_{q+1} \|_{W^{1,p}} + \| \partial_t w_{q+1} \|_{L^p} \lesssim \ell^{-2} \lambda_{q+1} r^{3/2 - 3/p} \tag{4.36}$$

$$\| w_{q+1} \|_{C^N_{x,t}} \leq \lambda_{q+1}^{3N} \tag{4.37}$$

$$\| v_{q+1} \|_{C^N_{x,t}} \leq \lambda_{q+1}^{3N} \tag{4.38}$$

for all $N \geq 1$ and $p > 1$.

5 Reynolds Stress

The main result of this section may be summarized as:

Proposition 5.1. *For $p > 1$ sufficiently close to 1, there exists a traceless symmetric 2 tensor $\widetilde{R}$, defined implicitly in (5.4) below, such that*

$$\partial_t v_{q+1} + \operatorname{div}\,(v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1} - \nu \Delta v_{q+1} = \operatorname{div} \widetilde{R}$$

and the bound

$$\left\| \widetilde{R} \right\|_{L^p} \lesssim \lambda_{q+1}^{-2\varepsilon_R} \delta_{q+2} \tag{5.1}$$

holds for all $p > 1$ which are sufficiently close to 1.

An immediate consequence of Proposition 5.1 is that the desired inductive estimates (2.3)–(2.4) hold for the Reynolds stress $\mathring{R}_{q+1}$.

Corollary 5.2. *There exists $\mathring{R}_{q+1}$ such that*

$$\partial_t v_{q+1} + \operatorname{div}\,(v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1} - \nu \Delta v_{q+1} = \operatorname{div} \mathring{R}_{q+1}$$

and

$$\left\| \mathring{R}_{q+1} \right\|_{L^1} \le \lambda_{q+1}^{-\varepsilon_R} \delta_{q+2} \tag{5.2}$$

$$\left\| \mathring{R}_{q+1} \right\|_{C^1_{x,t}} \le \lambda_{q+1}^{10} \tag{5.3}$$

Before giving the proof of the corollary, we recall from [2, Definition 1.4] the 2-tensor valued operator $\mathcal{R}$ which has the property that $\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$, and $\mathcal{R}$ is an right inverse of the div operator, i.e.

$$\operatorname{div} \mathcal{R} v = v - \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} v(x)\, dx$$

for any smooth v. Moreover, we have the classical Calderón-Zygmund bound $\| |\nabla| \mathcal{R} \|_{L^p \to L^p} \lesssim 1$, and the Schauder estimates $\| \mathcal{R} \|_{L^p \to L^p} \lesssim 1$ and $\| \mathcal{R} \|_{C^0 \to C^0} \lesssim 1$.

Proof of Corollary 5.2. Define $\mathring{R}_{q+1}$ by

$$\mathring{R}_{q+1} = \mathcal{R}(\operatorname{div} \widetilde{R})$$

Using the $p > 1$ in the proposition, we use $\| \mathcal{R} \operatorname{div} \|_{L^p \to L^p} \lesssim 1$ to directly bound

$$\left\| \mathring{R}_{q+1} \right\|_{L^1} \lesssim \left\| \mathring{R}_{q+1} \right\|_{L^p} \lesssim \left\| \widetilde{R} \right\|_{L^p} \lesssim \lambda_{q+1}^{-2\varepsilon_R} \delta_{q+2}\,.$$

The estimate (5.2) then follows since the factor $\lambda_{q+1}^{-\varepsilon_R}$ can absorb any constant if we assume a is sufficiently large.

Now consider (5.3). Observe that using the bounds of Corollary 4.6, we obtain

$$\begin{aligned}
\left\| \mathring{R}_{q+1} \right\|_{C^1} &= \left\| \mathcal{R}(\operatorname{div} \widetilde{R}) \right\|_{C^1} \\
&= \| \mathcal{R}(\partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) - \nu \Delta v_{q+1}) \|_{C^1} \\
&\lesssim \| \partial_t v_{q+1} \|_{C^1} + \| v_{q+1} \otimes v_{q+1} \|_{C^2} + \| v_{q+1} \|_{C^3} \\
&\lesssim \lambda_{q+1}^{9}
\end{aligned}$$

by using the Schauder estimates $\|\mathcal{R}\|_{C^0\to C^0} \lesssim 1$ and $\|D\mathcal{R}\|_{C^1\to C^0} \lesssim 1$. Similarly, we have that

$$
\begin{aligned}
\left\|\partial_t \mathring{R}_{q+1}\right\|_{L^\infty} &= \|\partial_t \mathcal{R}(\partial_t v_{q+1} + \operatorname{div}(v_{q+1}\otimes v_{q+1}) - \nu\Delta v_{q+1})\|_{C^0} \\
&\lesssim \left\|\partial_t^2 v_{q+1}\right\|_{C^0} + \|\partial_t v_{q+1}\otimes v_{q+1}\|_{C^1} + \|\partial_t v_{q+1}\|_{C^2} \\
&\lesssim \lambda_{q+1}^9
\end{aligned}
$$

which concludes the proof of (5.3) upon using the leftover power of λ_{q+1} to absorb all q independent constants. ☐

5.1 Proof of Proposition 5.1

Letting $v_{q+1} = w_{q+1} + v_q$ and using (2.1), we obtain

$$
\begin{aligned}
\operatorname{div}\widetilde{R} &= (-\nu\Delta w_{q+1} + \partial_t w_{q+1}) + \operatorname{div}(v_q\otimes w_{q+1} + w_{q+1}\otimes v_q) \\
&\quad + \operatorname{div}(w_{q+1}^{(c)}\otimes w_{q+1}^{(c)} + w_{q+1}^{(c)}\otimes w_{q+1}^{(p)} + w_{q+1}^{(p)}\otimes w_{q+1}^{(c)}) \\
&\quad + \operatorname{div}(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)} - \mathring{R}_\ell) + \operatorname{div}(\mathring{R}_\ell - \mathring{R}_q) + \nabla(p_{q+1}-p_q) \\
&=: \operatorname{div}\left(\widetilde{R}_{\text{linear}} + \widetilde{R}_{\text{quadratic}} + \widetilde{R}_{\text{corrector}} + \widetilde{R}_{\text{oscillation}} + \widetilde{R}_{\text{approximation}}\right) + \nabla(p_{q+1}-p_q). \quad (5.4)
\end{aligned}
$$

Besides the already used inequalities between the parameters, ℓ, r, σ, and λ_{q+1}, we shall use the following bound in order to achieve (5.1):

$$
\ell^{-2}\lambda_{q+1} r^{3/2-3/p} + \ell^{-2}\lambda_{q+1}^{6(p-1)-(2-p)} + \lambda_q^{-10} + \frac{\ell^{-2} r^{3-3/p}}{\lambda_{q+1}\sigma} + \ell^{-2}\sigma r^{4-3/p} \lesssim \lambda_{q+1}^{-2\varepsilon_R}\delta_{q+2} \quad (5.5)
$$

In view of (4.9)–(4.10), the above inequality holds for b sufficiently large, β sufficiently small depending on b, parameters $\varepsilon_R, p-1>0$ sufficiently small depending on b and β, and for $\lambda_0 = a$ sufficiently large depending on all these parameters and on M.

5.2 The linear, quadratic, corrector, and approximation errors

In view of (5.1), we estimate contributions to the $\widetilde{R}$ coming from the first line in (5.4) as

$$
\begin{aligned}
\left\|\widetilde{R}_{\text{linear}}\right\|_{L^p} + \left\|\widetilde{R}_{\text{quadratic}}\right\|_{L^p} &\lesssim \|\mathcal{R}(-\nu\Delta w_{q+1} + \partial_t w_{q+1})\|_{L^p} + \|\mathcal{R}\operatorname{div}(v_q\otimes w_{q+1} + w_{q+1}\otimes v_q)\|_{L^p} \\
&\lesssim \|w_{q+1}\|_{W^{1,p}} + \|\partial_t w_{q+1}\|_{L^p} + \|v_q\|_{L^\infty}\|w_{q+1}\|_{L^p} \\
&\lesssim \lambda_q^3\|w_{q+1}\|_{W^{1,p}} + \|\partial_t w_{q+1}\|_{L^p} \\
&\lesssim \ell^{-2}\lambda_{q+1} r^{3/2-3/p} \quad (5.6)
\end{aligned}
$$

where we have used $\nu \le 1$, the inductive estimate (2.2) to bound $\|v_q\|_{L^\infty} \le \|v_q\|_{C^1}$, and (4.36). Next we turn to the quadratic errors involving $w_{q+1}^{(c)}$, for which we appeal to L^p interpolation, the Poincaré inequality, and Proposition 4.5:

$$
\begin{aligned}
\left\|\widetilde{R}_{\text{corrector}}\right\|_{L^p} &\le \left\|\mathcal{R}\operatorname{div}\left(w_{q+1}^{(c)}\otimes w_{q+1}^{(c)} + w_{q+1}^{(p)}\otimes w_{q+1}^{(c)} + w_{q+1}^{(c)}\otimes w_{q+1}^{(p)}\right)\right\|_{L^p} \\
&\lesssim \left\|w_{q+1}^{(c)}\otimes w_{q+1}^{(p)}\right\|_{L^1}^{2-p}\left\|w_{q+1}^{(c)}\otimes w_{q+1}^{(p)}\right\|_{L^\infty}^{p-1} + \left\|w_{q+1}^{(c)}\otimes w_{q+1}^{(c)}\right\|_{L^1}^{2-p}\left\|w_{q+1}^{(c)}\otimes w_{q+1}^{(c)}\right\|_{L^\infty}^{p-1} \\
&\lesssim (\ell^{-1}\lambda_{q+1}^{-1})^{2-p}\delta_{q+1}^{1-p/2}\lambda_{q+1}^{6(p-1)} \\
&\lesssim \ell^{-2}\lambda_{q+1}^{6(p-1)-(2-p)}
\end{aligned}
$$

which is more than sufficient. Lastly, in view of (2.4) we have that

$$\left\|\widetilde{R}_{\text{approximation}}\right\|_{L^p} \lesssim \left\|\mathring{R}_q - \mathring{R}_\ell\right\|_{L^p} \lesssim \left\|\mathring{R}_q - \mathring{R}_\ell\right\|_{L^\infty} \lesssim \ell \left\|\mathring{R}_q\right\|_{C^1} \lesssim \ell\lambda_q^{10} = \lambda_q^{-10} \tag{5.7}$$

In view of (5.5), all the bounds in this section are consistent with (5.1).

5.3 Oscillation error

In this section we estimate the remaining error, $\widetilde{R}_{\text{oscillation}}$ which obeys

$$\operatorname{div} \widetilde{R}_{\text{oscillation}} + \nabla(p_{q+1} - p_q) = \operatorname{div} \left(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}\right) - \operatorname{div} \mathring{R}_\ell + \nabla P\,.$$

In the below computation the pressure P may change from line to line. Recall from the definition of $w_{q+1}^{(p)}$ and of the coefficients $a_{(\overline{\xi})}$, via (4.13) we have

$$\begin{aligned}
&\operatorname{div} \left(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}\right) - \operatorname{div} \mathring{R}_\ell + \nabla P \\
&\quad= \sum_{i,j} \sum_{\overline{\xi}\in\Lambda_{(i)},\overline{\xi}'\in\Lambda_{(j)}} \operatorname{div} \left(a_{(\overline{\xi})} a_{(\overline{\xi}')} \mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(\overline{\xi}')}\right) - \operatorname{div} \mathring{R}_\ell + \nabla P \\
&\quad= \sum_{i,j} \sum_{\overline{\xi}\in\Lambda_{(i)},\overline{\xi}'\in\Lambda_{(j)}} \operatorname{div} \left(a_{(\overline{\xi})} a_{(\overline{\xi}')} \left(\mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(\overline{\xi}')} - \fint_{\mathbb{T}^3} \mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(\overline{\xi}')}\, dx\right)\right) + \nabla P \\
&\quad= \sum_{i,j} \sum_{\overline{\xi}\in\Lambda_{(i)},\overline{\xi}'\in\Lambda_{(j)}} \underbrace{\operatorname{div} \left(a_{(\overline{\xi})} a_{(\overline{\xi}')} \mathbb{P}_{\geq \lambda_{q+1}\sigma/2} \left(\mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(\overline{\xi}')}\right)\right)}_{E_{(\overline{\xi},\overline{\xi}')}} + \nabla P\,.
\end{aligned}$$

Here we use that the minimal separation between active frequencies of $\mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(\overline{\xi}')}$ and the 0 frequency is given by $\lambda_{q+1}\sigma$. We proceed to estimate each summand $E_{(\overline{\xi},\overline{\xi}')}$ individually. We split

$$\begin{aligned}
E_{(\overline{\xi},\overline{\xi}')} =& \nabla \left(a_{(\overline{\xi})} a_{(\overline{\xi}')}\right) \mathbb{P}_{\geq \lambda_{q+1}\sigma/2} \left(\mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(\overline{\xi}')}\right) \\
&+ a_{(\overline{\xi})} a_{(\overline{\xi}')} \operatorname{div} \left(\mathbb{P}_{\geq \lambda_{q+1}\sigma/2} \left(\mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(\overline{\xi}')}\right)\right) = E_{(\overline{\xi},\overline{\xi}',1)} + E_{(\overline{\xi},\overline{\xi}',2)}\,.
\end{aligned}$$

The term $E_{(\overline{\xi},\overline{\xi}',1)}$ can easily be estimated using Lemma 4.4 and Lemma A.1, estimate (A.2), with $\lambda = \ell^{-1}$, $C_a = \ell^{-2}$, $\mu = \lambda_{q+1}\sigma/2$, and L sufficiently large, as

$$\begin{aligned}
\left\|\mathcal{R} E_{(\overline{\xi},\overline{\xi}',1)}\right\|_{L^p} &\lesssim \left\||\nabla|^{-1} E_{(\overline{\xi},\overline{\xi}',1)}\right\|_{L^p} \\
&\lesssim \left\||\nabla|^{-1} \left(\nabla \left(a_{(\overline{\xi})} a_{(\overline{\xi}')}\right) \mathbb{P}_{\geq \lambda_{q+1}\sigma/2} \left(\mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(\overline{\xi}')}\right)\right)\right\|_{L^p} \\
&\lesssim \frac{\ell^{-2}}{\lambda_{q+1}\sigma} \left\|\mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(\overline{\xi}')}\right\|_{L^p} \\
&\lesssim \frac{\ell^{-2}}{\lambda_{q+1}\sigma} \left\|\mathbb{W}_{(\overline{\xi})}\right\|_{L^{2p}} \left\|\mathbb{W}_{(\overline{\xi}')}\right\|_{L^{2p}} \\
&\lesssim \frac{\ell^{-2} r^{3-3/p}}{\lambda_{q+1}\sigma}\,.
\end{aligned}$$

In the last inequality above we have used estimate (3.12), and have used that for b sufficiently large and L sufficiently large we have $\ell^{-L}(\lambda_{q+1}\sigma)^{2-L} \lesssim 1$. By (5.5), this bound is consistent with (5.1).

For $E_{(\overline{\xi},\overline{\xi}',2)}$ we recall

$$a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq \lambda_{q+1}\sigma/2}\mathrm{div}\left(\mathbb{W}_{(\overline{\xi})}\otimes \mathbb{W}_{(\overline{\xi}')}\right) = a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq \lambda_{q+1}\sigma/2}\sum_{\xi\in\Omega_{(\overline{\xi})},\xi'\in\Omega_{(\overline{\xi}')}}\mathrm{div}\left(W_{(\xi)}\otimes W_{(\xi')}\right).$$

Symmetrizing and applying the identity

$$\frac{1}{2}\nabla|F|^2 = F\times(\nabla\times F) + F\cdot\nabla F$$

we can rewrite $E_{(\overline{\xi},\overline{\xi}',2)} + E_{(\overline{\xi}',\overline{\xi},2)}$ as

$$\begin{aligned}E_{(\overline{\xi},\overline{\xi}',2)} + E_{(\overline{\xi}',\overline{\xi},2)} =& -a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq \lambda_{q+1}\sigma/2}\sum_{\xi\in\Omega_{(\overline{\xi})},\xi'\in\Omega_{(\overline{\xi}')}}\left(\left(W_{(\xi)}+W_{(\xi')}\right)\times\mathrm{curl}\left(W_{(\xi)}+W_{(\xi')}\right)\right)\\ &+ a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq \lambda_{q+1}\sigma/2}\nabla\left|\mathbb{W}_{(\xi)}+\mathbb{W}_{(\xi')}\right|^2 = E_{(\overline{\xi},\overline{\xi}',3)} + E_{(\overline{\xi},\overline{\xi}',4)}.\end{aligned}$$

The second term is an approximate pressure term, its difference from a gradient can be estimated in the same manner as $E_{(\overline{\xi},\overline{\xi}',1)}$. It remains to estimate $E_{(\overline{\xi},\overline{\xi}',3)}$. Since $\mathrm{curl}\, W_{(\xi)} = \lambda_{q+1}|\xi|W_{(\xi)}$, the term $E_{(\overline{\xi},\overline{\xi}',3)}$ becomes

$$\lambda_{q+1}a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq \lambda_{q+1}\sigma/2}\sum_{\xi\in\Omega_{(\overline{\xi})},\xi'\in\Omega_{(\overline{\xi}')}}(|\xi'|-|\xi|)W_{(\xi)}\times W_{(\xi')}$$

Since $(|\xi'|-|\xi|) = (|\xi'|-1)-(1-|\xi|)$, it suffices by symmetry to estimate

$$\lambda_{q+1}a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq \lambda_{q+1}\sigma/2}\sum_{\xi\in\Omega_{(\overline{\xi})},\xi'\in\Omega_{(\overline{\xi}')}}(1-|\xi|)W_{(\overline{\xi})}\times W_{(\overline{\xi}')} = \lambda_{q+1}a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq \lambda_{q+1}\sigma/2}\left(\widetilde{\mathbb{W}}_{(\overline{\xi})}\times\mathbb{W}_{(\overline{\xi}')}\right).$$

Again, we may use Lemma 4.4, and Lemma A.1, estimate (A.3), with $\lambda = \ell^{-1}$, $C_a = \ell^{-2}$, and $\mu = {}^{\lambda_{q+1}\sigma}/2$ to arrive at

$$\begin{aligned}&\lambda_{q+1}\left\|\mathcal{R}\left(a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq \lambda_{q+1}\sigma/2}\left(\widetilde{\mathbb{W}}_{(\overline{\xi})}\times\mathbb{W}_{(\overline{\xi}')}\right)\right)\right\|_{L^p}\\ &\lesssim \lambda_{q+1}\left\||\nabla|^{-1}\left(a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq \lambda_{q+1}\sigma/2}\left(\widetilde{\mathbb{W}}_{(\overline{\xi})}\times\mathbb{W}_{(\overline{\xi}')}\right)\right)\right\|_{L^p}\\ &\lesssim \lambda_{q+1}\ell^{-2}\left\||\nabla|^{-1}\left(\widetilde{\mathbb{W}}_{(\overline{\xi})}\times\mathbb{W}_{(\overline{\xi}')}\right)\right\|_{L^p} + \lambda_{q+1}\ell^{-2}\frac{\ell^{-L}}{(\lambda_{q+1}\sigma)^{L-1}}\left\|\widetilde{\mathbb{W}}_{(\overline{\xi})}\times\mathbb{W}_{(\overline{\xi}')}\right\|_{L^p}\\ &\lesssim \lambda_{q+1}\ell^{-2}\left\||\nabla|^{-1}\left(\widetilde{\mathbb{W}}_{(\overline{\xi})}\times\mathbb{W}_{(-\overline{\xi})}\right)\right\|_{L^p} + \ell^{-2}\mathbf{1}_{\xi+\xi'\neq 0}\left\|\widetilde{\mathbb{W}}_{(\overline{\xi})}\right\|_{L^{2p}}\left\|\mathbb{W}_{(\overline{\xi}')}\right\|_{L^{2p}}\\ &\qquad + \lambda_{q+1}\ell^{-2}\frac{\ell^{-L}}{(\lambda_{q+1}\sigma)^{L-1}}\left\|\widetilde{\mathbb{W}}_{(\overline{\xi})}\right\|_{L^{2p}}\left\|\mathbb{W}_{(\overline{\xi}')}\right\|_{L^{2p}}\\ &\lesssim \lambda_{q+1}\ell^{-2}\frac{\sigma r^{4-3/p}}{\lambda_{q+1}} + \ell^{-2}\mathbf{1}_{\xi+\xi'\neq 0}\sigma r^{4-3/p} + \lambda_{q+1}\ell^{-2}\frac{\ell^{-L}}{(\lambda_{q+1}\sigma)^{L-1}}\sigma r^{4-3/p}\\ &\lesssim \ell^{-2}\sigma r^{4-3/p}\end{aligned}$$

for $L\in\mathbb{N}$, sufficiently large. In the above inequality we have essentially used (3.12), (3.16), (3.17), and have used that for b sufficiently large, and L sufficiently large, we have $\lambda_{q+1}\ell^{-L}(\lambda_{q+1}\sigma)^{1-L}\leq 1$. In view of (5.5), this bound is consistent with (5.1), and concludes the proof of the proposition.

6 The energy iterate

Lemma 6.1. *For all t and t' satisfying $|t-t'| \le 2\ell$, and all $i \ge 0$, we have*

$$\left|e(t') - e(t'')\right| \lesssim \ell^{1/2} \tag{6.1}$$

$$\left|\int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx - \int_{\mathbb{T}^3} |v_q(x,t')|^2 \, dx\right| \lesssim \ell^{1/2} \tag{6.2}$$

$$\left|\int_{\mathbb{T}^3} \left(\chi_i^2(x,t) - \chi_i^2(x,t')\right) dx\right| \lesssim \ell^{1/6} \tag{6.3}$$

$$\left|\rho(t) - \rho(t')\right| \lesssim \ell^{1/6} \tag{6.4}$$

Proof of Lemma 6.1. In the proof of the lemma, we crudely use a factor of λ_q to absorb constants. First note that (6.1) follows immediately from the assumed estimate $\|e\|_{C^1_t} \le M_e$. Using (2.2) we have

$$\left|\int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx - \int_{\mathbb{T}^3} |v_q(x,t')|^2 \, dx\right| \lesssim \ell \, \|v_q\|^2_{C^1_{t,x}} \lesssim \lambda_q^6 \ell \,,$$

which implies (6.2). The estimate (6.3) follows in a similar fashion, from Lemma 4.2. Finally, (6.4) follows directly the definition of $\rho(t)$, (4.24) and the bounds (6.1)–(6.3) above. ◻

Lemma 6.2. *If $\rho_0(t) \neq 0$ then the energy of v_{q+1} satisfies the following estimate:*

$$\left|e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x,t)|^2 \, dx - \frac{\delta_{q+2}}{2}\right| \le \frac{\delta_{q+2}}{4} \,. \tag{6.5}$$

Proof of Lemma 6.2. By definition we have

$$\int_{\mathbb{T}^3} |v_{q+1}(x,t)|^2 \, dx = \int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx + 2\int_{\mathbb{T}^3} w_{q+1}(x,t) \cdot v_q(x,t) \, dx + \int_{\mathbb{T}^3} |w_{q+1}(x,t)|^2 \, dx \,. \tag{6.6}$$

Using (4.13), similarly to (4.15), we have that

$$\begin{aligned}
&\int_{\mathbb{T}^3} \left|w^{(p)}_{q+1}(x,t)\right|^2 dx - 3\sum_{i\ge 0} \rho_i \int_{\mathbb{T}^3} \chi^2_{(i)}(x,t) \, dx \\
&= \sum_{i,j\ge 0} \sum_{\overline{\xi}\in\Lambda_{(i)}, \overline{\xi}'\in\Lambda_{(j)}} \sum_{\xi\in\Omega_{(\overline{\xi})}, \xi'\in\Omega_{(\overline{\xi}')}, \xi\neq\xi'} \int_{\mathbb{T}^3} a_{(\overline{\xi})} a_{(\overline{\xi}')} \mathbb{P}_{\ge\lambda_{q+1}\sigma/2}(W_{(\xi)} \cdot W_{(-\xi')}) dx =: E_\rho(t).
\end{aligned}$$

Using the standard integration by parts argument $|\int_{\mathbb{T}^3} f \mathbb{P}_{\ge\mu} g dx| = |\int_{\mathbb{T}^3} |\nabla|^L f \, |\nabla|^{-L} \mathbb{P}_{\ge\mu} g dx| \lesssim \|g\|_{L^2} \mu^{-L} \|f\|_{C^L}$, with L sufficiently large, we obtain from (4.28), since $\ell^{-1} \ll \lambda_{q+1}\sigma$, that

$$\left|\int_{\mathbb{T}^3} \left|w^{(p)}_{q+1}(x,t)\right|^2 dx - 3\sum_{i\ge 0} \rho_i \int_{\mathbb{T}^3} \chi^2_{(i)}(x,t) \, dx\right| = |E_\rho(t)| \le \ell^{1/2}. \tag{6.7}$$

We consider two sub-cases: $\rho(t) \neq 0$ and $\rho(t) = 0$. First consider the case $\rho(t) \neq 0$, then using the definition of ρ we obtain

$$\begin{aligned}
&3\sum_{i\ge 0} \rho_i \int_{\mathbb{T}^3} \chi^2_{(i)}(x,t) \, dx \\
&= 3\rho(t) \int_{\mathbb{T}^3} \chi^2_{(0)}(x,t) \, dx + 3\left(\rho_0(t) - \rho(t)\right) \int_{\mathbb{T}^3} \chi^2_{(0)}(x,t) \, dx + 3\sum_{i\ge 1} \rho_i \int_{\mathbb{T}^3} \chi^2_{(i)}(x,t) \, dx \\
&= e(t) - \int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx + 3\left(\rho_0(t) - \rho(t)\right) \int_{\mathbb{T}^3} \chi^2_0(x,t) \, dx - \frac{\delta_{q+2}}{2} \,.
\end{aligned}$$

For the case that $\rho(t) = 0$ we have that by continuity for some $t' \in (t-\ell, t+\ell)$

$$e(t') - \int_{\mathbb{T}^3} \left|v_q(x,t')\right|^2 \, dx - 3\sum_{i\geq 1} \rho_i \int_{\mathbb{T}^3} \chi_i^2(x,t')\, dx - \frac{\delta_{q+2}}{2} = 0$$

Thus applying Lemma 6.1 we conclude that for either case $\rho(t) \neq 0$ or $\rho(t) = 0$

$$\left| 3\sum_{i\geq 0} \rho_i \int_{\mathbb{T}^3} \chi_{(i)}^2(x,t)\, dx - e(t) + \int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx + \frac{\delta_{q+2}}{2} \right| \lesssim \ell^{1/2} . \tag{6.8}$$

Observe that if we define $v_\ell := v_q *_x \phi_\ell$ then using (2.2) and (4.28) we obtain

$$\begin{aligned}
\left|\int_{\mathbb{T}^3} w_{q+1}(x,t)\cdot v_q(x,t)\, dx\right| &\leq \left|\int_{\mathbb{T}^3} w_{q+1}(x,t)\cdot (v_q(x,t) - v_\ell(x,t))\, dx\right| + \left|\int_{\mathbb{T}^3} w_{q+1}(x,t)\cdot v_\ell(x,t)\, dx\right| \\
&\lesssim \ell \left\|w_{q+1}\right\|_{L^2} \left\|v_q\right\|_{C^1} + \sum_i \sum_{\overline{\xi}\in\Lambda_{(i)}} \sum_{\xi\in\Omega_{(\overline{\xi})}} \left|\int a_{(\overline{\xi})} W_{(\xi)}(x,t)\cdot v_\ell(x,t)\, dx\right| \\
&\lesssim \delta_{q+1}^{1/2}\lambda_q^3 \ell + \sum_i \sum_{\overline{\xi}\in\Lambda_{(i)}} \sum_{\xi\in\Omega_{(\overline{\xi})}} \lambda_{q+1}^{-N} \left\|a_{(\overline{\xi})} v_\ell\right\|_{C^N} \\
&\lesssim \delta_{q+1}^{1/2}\lambda_q^3 \ell + \sum_i \sum_{\overline{\xi}\in\Lambda_{(i)}} \sum_{\xi\in\Omega_{(\overline{\xi})}} \lambda_{q+1}^{-N} \ell^{-N-1}
\end{aligned}$$

where we applied repeated integrations by part. Taking N sufficiently large we obtain

$$\left|\int_{\mathbb{T}^3} w_{q+1}(x,t)\cdot v_q(x,t)\, dx\right| \lesssim \delta_{q+1}^{1/2}\lambda_q^3 \ell \lesssim \ell^{1/2} . \tag{6.9}$$

Using (4.29) and (4.30) yields

$$\left|\int_{\mathbb{T}^3} 2 w_{q+1}^{(p)}(x,t)\cdot w_{q+1}^{(c)}(x,t) + \left|w_{q+1}^{(c)}(x,t)\right|^2 \, dx\right| \lesssim \delta_{q+1}^{1/2}\ell^{-1}\lambda_{q+1}^{-2} \lesssim \ell^{1/2} . \tag{6.10}$$

Thus we conclude from (6.6), (6.7), (6.8), (6.9) and (6.10)

$$\left| e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x,t)|^2 \, dx - \frac{\delta_{q+2}}{2}\right| \lesssim \ell^{1/2}$$

from which (6.5) immediately follows. ☐

Lemma 6.3. *If $\rho_0(t) = 0$ then $w_{q+1}(\cdot,t) \equiv 0$, $\mathring{R}_q(\cdot,t) \equiv 0$, $\mathring{R}_{q+1}(\cdot,t) \equiv 0$ and*

$$e(t) - \int_{\mathbb{T}} |v_{q+1}(x,t)|^2 \leq \frac{3\delta_{q+2}}{4} . \tag{6.11}$$

Proof of Lemma 6.3. Since $\rho_0(t) = 0$, it follows from the definition of ρ_0 and ρ that for all $t' \in (t-\ell, t+\ell)$ we have

$$e(t') - \int_{\mathbb{T}^3} \left|v_q(x,t')\right|^2 \, dx - 3\sum_{i\geq 1} \rho_i \int_{\mathbb{T}^3} \chi_i^2(x,t')\, dx \leq \frac{\delta_{q+2}}{2} .$$

Using (4.25), this implies that

$$e(t') - \int_{\mathbb{T}^3} \left|v_q(x,t')\right|^2 \, dx - \frac{\delta_{q+2}}{2} \lesssim \lambda_q^{-\varepsilon_R}\delta_{q+1}\,. \tag{6.12}$$

Using that $\lambda_q^{-\varepsilon_R}$ and the ratio $\delta_{q+2}\delta_{q+1}^{-1}$ can absorb any constant, from (2.6) we conclude that $\mathring{R}_q(\cdot,t') \equiv 0$ and hence $\mathring{R}_\ell(\cdot,t) \equiv 0$. This in turn implies that $\operatorname{supp}\chi_i = \emptyset$ for all $i \geq 1$. Since in addition, $\rho_0(t) = 0$, it follows that $w_{q+1}(\cdot,t) \equiv 0$. Hence using (6.12) we obtain

$$e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x,t)|^2 = e(t) - \int_{\mathbb{T}} |v_q(x,t)|^2 \leq \frac{3\delta_{q+2}}{4}\,.$$

Hence we obtain (6.11). □

We conclude this section by using Lemmas 6.2 and 6.3 to conclude (2.5) and (2.6) for $q+1$. Observe that the estimates (6.5) and (6.11), together imply (2.5) for $q+1$. From (6.5), if

$$e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x,t)|^2 \, dx \leq \frac{\delta_{q+2}}{100}$$

then $\rho_0(t) = 0$. Hence from Lemma 6.3 we obtain the $\mathring{R}_{q+1} \equiv 0$ from which we conclude (2.6).

A Appendix

A.1 L^p product estimate

Proof of Lemma 3.7. For convenience we give here the proof from [5]. We first consider the case $p = 1$. With these assumptions we have

$$\|fg\|_{L^1} \leq \sum_j \int_{T_j} |fg|$$

where T_j are cubes of side-length $\frac{2\pi}{\mu}$. For any function h, let $\overline{h}_j$ denote its mean on the cube T_j. Observe that for $x \in T_j$ we have

$$\begin{aligned}
|f(x)| = \left|\overline{f}_j + f(x) - \overline{f}_j\right| &\leq \left|\overline{f}_j\right| + \sup_{T_j}\left|f(x) - \overline{f}_j\right| \\
&\leq \left|\overline{f}_j\right| + \frac{2\pi\sqrt{3}}{\mu}\sup_{T_j}|Df| \\
&\leq \left|\overline{f}_j\right| + \frac{2\pi\sqrt{3}}{\mu}\left|\overline{Df}_j\right| + \frac{2\pi\sqrt{3}}{\mu}\sup_{T_j}\left|Df - \overline{Df}_j\right| \\
&\leq \left|\overline{f}_j\right| + \frac{2\pi\sqrt{3}}{\mu}\left|\overline{Df}_j\right| + \frac{6\pi}{\mu^2}\sup_{T_j}\left|D^2 f\right| \\
&\leq \left|\overline{f}_j\right| + \frac{2\pi\sqrt{3}}{\mu}\left|\overline{Df}_j\right| + \frac{6\pi}{\mu^2}\sup_{T_j}\left|\overline{D^2 f}_j\right| + \frac{6\pi}{\mu^2}\sup_{T_j}\left|D^2 f - \overline{D^2 f}_j\right|.
\end{aligned}$$

Iterating this procedure M times we see that on T_j we have the pointwise estimate

$$|f| \leq \sum_{m=0}^{M} (2\pi\sqrt{3}\mu^{-1})^m \left|\overline{D^m f}_j\right| + (2\pi\sqrt{3}\mu^{-1})^M \|D^M f\|_{L^\infty}.$$

Upon multiplying the above by $|g|$ and integrating over T_j, and then summing over j, we obtain

$$
\begin{aligned}
\|fg\|_{L^1(\mathbb{T}^3)} &\le \sum_j \int_{T_j} \left(|g| \sum_{m=0}^{M} (2\pi\sqrt{3}\mu^{-1})^m \left|\overline{D^m f}_j\right| \right) dx + (2\pi\sqrt{3}\mu^{-1})^M \|D^M f\|_{L^\infty} \|g\|_{L^1} \\
&\le \sum_{m=0}^{M} (2\pi\sqrt{3}\mu^{-1})^m \left(\sum_j \frac{1}{|T_j|} \|D^m f\|_{L^1(T_j)} \|g\|_{L^1(T_j)} \right) + (2\pi\sqrt{3}\mu^{-1})^M \|D^M f\|_{L^\infty} \|g\|_{L^1}.
\end{aligned}
$$

Since g is a T_j-periodic function, we have

$$
\|g\|_{L^1(\mathbb{T}^3)} = \frac{|\mathbb{T}^3|}{|T_j|} \|g\|_{L^1(T_j)}
$$

for any value of j, and since the interiors of the $\{T_j\}$ are mutually disjoint, based on the assumption on the L^1 cost of a derivative acting on f and the Sobolev embedding, we conclude from the above that (here we used the Sobolev embedding of $W^{d+1,1} \subset L^\infty$)

$$
\begin{aligned}
\|fg\|_{L^1(\mathbb{T}^3)} &\le \frac{1}{|\mathbb{T}^3|} \|g\|_{L^1(\mathbb{T}^3)} \sum_{m=0}^{M} (2\pi\sqrt{3}\mu^{-1})^m \|D^m f\|_{L^1(\mathbb{T}^3)} + (2\pi\sqrt{3}\mu^{-1})^M \|D^{M+4} f\|_{L^1} \|g\|_{L^1} \\
&\le \frac{1}{|\mathbb{T}^3|} \|g\|_{L^1(\mathbb{T}^3)} \sum_{m=0}^{M} (2\pi\sqrt{3}\mu^{-1})^m \lambda^m C_f + (2\pi\sqrt{3}\mu^{-1})^M \lambda^{M+4} C_f \|g\|_{L^1} \\
&\le (1 + 2|\mathbb{T}^3|) C_f \|g\|_{L^1(\mathbb{T}^3)}.
\end{aligned}
$$

The case $p = 2$, follows from the case $p = 1$ applied to the functions f^2 and g^2, and from the bound

$$
\|D^m(f^2)\|_{L^1} \le \sum_{k=0}^{m} \binom{m}{k} \|D^k f\|_{L^2} \|D^{m-k} f\|_{L^2} \le \sum_{k=0}^{m} \binom{m}{k} \lambda^m C_f^2 = (2\lambda)^m C_f^2.
$$

Here we are thus using that $4\pi\sqrt{3}\lambda\mu^{-1} \le 2/3 < 1$ so that we have a geometric sum. $\square$

A.2 Commutator estimate

Lemma A.1. *Fix $\mu \ge 1$, $p \in (1,2]$. and a sufficiently large $L \in \mathbb{N}$. Let $a \in C^L(\mathbb{T}^3)$ be such that there exists $1 \le \lambda \le \mu$, and $C_a > 0$ with*

$$
\left\|D^j a\right\|_{L^\infty} \le C_a \lambda^j \tag{A.1}
$$

for all $0 \le j \le L$. Then we have

$$
\left\||\nabla|^{-1}(a\, \mathbb{P}_{\ge\mu} f)\right\|_{L^p} \lesssim C_a \left(1 + \frac{\lambda^L}{\mu^{L-2}}\right) \frac{\|f\|_{L^p}}{\mu} \tag{A.2}
$$

and

$$
\left\||\nabla|^{-1}(a\, \mathbb{P}_{\ge\mu} f)\right\|_{L^p} \lesssim C_a \left\||\nabla|^{-1}\mathbb{P}_{\ge\mu} f\right\|_{L^p} + C_a \frac{\lambda^L}{\mu^{L-1}} \|f\|_{L^p} \tag{A.3}
$$

for any $f \in L^p(\mathbb{T}^3)$.

Proof of Lemma A.1. We have that

$$
\begin{aligned}
|\nabla|^{-1}(a\,\mathbb{P}_{\geq\mu}f) &= |\nabla|^{-1}(\mathbb{P}_{\leq\mu/2}a\,\mathbb{P}_{\geq\mu}f) + |\nabla|^{-1}(\mathbb{P}_{\geq\mu/2}a\,\mathbb{P}_{\geq\mu}f) \\
&= (\mathbb{P}_{\geq\mu/2}|\nabla|^{-1})(\mathbb{P}_{\leq\mu/2}a\,\mathbb{P}_{\geq\mu}f) + |\nabla|^{-1}(\mathbb{P}_{\geq\mu/2}a\,\mathbb{P}_{\geq\mu}f)
\end{aligned}
\tag{A.4}
$$

We then use

$$
\left\|\mathbb{P}_{\geq\mu/2}|\nabla|^{-1}\right\|_{L^p\to L^p} \lesssim \frac{1}{\mu}
$$

which is a direct consequence of the Littlewood-Paley decomposition, and when acting of functions of zero mean, we have

$$
\left\||\nabla|^{-1}\right\|_{L^p\to L^p} \lesssim 1
$$

which is a direct consequence of Schauder estimates/Hardy-Littlewood-Sobolev. Combining these facts, we obtain

$$
\begin{aligned}
\left\||\nabla|^{-1}(a\,\mathbb{P}_{\geq\mu}f)\right\|_{L^p} &\lesssim \frac{1}{\mu}\left\|\mathbb{P}_{\leq\mu/2}a\,\mathbb{P}_{\geq\mu}f\right\|_{L^p} + \left\|\mathbb{P}_{\geq\mu/2}a\,\mathbb{P}_{\geq\mu}f\right\|_{L^p} \\
&\lesssim \left(\|a\|_{L^\infty} + \mu\left\|D\mathbb{P}_{\geq\mu/2}a\right\|_{L^4}\right)\frac{\|f\|_{L^p}}{\mu} \\
&\lesssim \left(\|a\|_{L^\infty} + \mu^{2-L}\left\|D^L\mathbb{P}_{\geq\mu/2}a\right\|_{L^4}\right)\frac{\|f\|_{L^p}}{\mu} \\
&\lesssim \left(\|a\|_{L^\infty} + \mu^2\frac{\left\|D^L a\right\|_{L^\infty}}{\mu^L}\right)\frac{\|f\|_{L^p}}{\mu}
\end{aligned}
$$

and the proof of (A.2) is concluded in view of assumption (A.1).

In order to prove (A.3), we use (A.4) and the previous argument, to first obtain

$$
\begin{aligned}
\left\||\nabla|^{-1}(a\,\mathbb{P}_{\geq\mu}f)\right\|_{L^p} &\lesssim C_a\frac{\lambda^L}{\mu^{L-1}}\|f\|_{L^p} + \left\|(\mathbb{P}_{\geq\mu/2}|\nabla|^{-1})\left(\mathbb{P}_{\leq\mu/2}a\,|\nabla|(|\nabla|^{-1}\mathbb{P}_{\geq\mu}f)\right)\right\|_{L^p} \\
&\lesssim C_a\frac{\lambda^L}{\mu^{L-1}}\|f\|_{L^p} + C_a\left\||\nabla|^{-1}\mathbb{P}_{\geq\mu}f\right\|_{L^p} + \frac{1}{\mu}\left\|\left[\mathbb{P}_{\leq\mu/2}a\,,|\nabla|\right](|\nabla|^{-1}\mathbb{P}_{\geq\mu}f)\right\|_{L^p} \\
&\lesssim C_a\frac{\lambda^L}{\mu^{L-1}}\|f\|_{L^p} + \left(C_a + \frac{1}{\mu}\left\|\left[\mathbb{P}_{\leq\mu/2}a\,,|\nabla|\right]\right\|_{L^p\to L^p}\right)\left\||\nabla|^{-1}\mathbb{P}_{\geq\mu}f\right\|_{L^p}.
\end{aligned}
$$

From the L^p boundedness theorem for the first Calderón commutator [40], and (A.1) we have that

$$
\frac{1}{\mu}\left\|\left[\mathbb{P}_{\leq\mu/2}a\,,|\nabla|\right]\right\|_{L^p\to L^p} \lesssim \frac{\|Da\|_{L^\infty}}{\mu} \lesssim \frac{C_a\lambda}{\mu} \lesssim C_a,
$$

which concludes the proof of the lemma since $\lambda\leq\mu$. ∎

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