Some noteworthy alternating trilinear forms

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Abstract

Given an alternating trilinear form $T \in \mathrm{Alt}(\times^3 V_n)$ on $V_n = V(n, \mathbb{F})$ let $\mathcal{L}_T$ denote the set of T-singular lines in $\mathrm{PG}(n-1) = \mathbb{P}V_n$, consisting that is of those lines $\langle a, b\rangle$ of $\mathrm{PG}(n-1)$ such that $T(a, b, x) = 0$ for all $x \in V_n$. Amongst the immense profusion of different kinds of T we single out a few which we deem noteworthy by virtue of the special nature of their set $\mathcal{L}_T$.

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1 Introduction

We will deal with a finite-dimensional vector space $V_n = V(n, \mathbb{F})$ and the associated projective space $\mathrm{PG}(n-1, \mathbb{F}) = \mathbb{P}V_n$. The $\binom{n}{2}$-dimensional space $\mathrm{Alt}(\times^2 V_n)$ consisting of alternating bilinear forms on V_n is of course very well understood. If $n = 2m$, or if $n = 2m+1$, then the nonzero elements $B \in \mathrm{Alt}(\times^2 V_n)$ fall into m $\mathrm{GL}(n, \mathbb{F})$-orbits $\{\Omega_k\}_{k=1,2,\ldots m}$, where Ω_k consists of those B which have rank $2k$. For a given $B \in \mathrm{Alt}(\times^2 V_n)$ a point $\langle a\rangle \in \mathbb{P}V_n$ is said to be (B-)*singular* whenever $B(a, x) = 0$ holds for all $x \in V_n$. Consequently if n is odd then B-singular points exist for any B, while if $n = 2m$ is even then only when B is on the orbit Ω_m do B-singular points not exist.

In the present paper we consider instead the $\binom{n}{3}$-dimensional space $\mathrm{Alt}(\times^3 V_n)$ consisting of alternating trilinear forms on V_n. In contrast with $\mathrm{Alt}(\times^2 V_n)$ the mathematics of the space $\mathrm{Alt}(\times^3 V_n)$ is much more complicated (and interesting!). In particular the orbit structure of $\mathrm{Alt}(\times^3 V_n)$ is only known in certain low-dimensional cases. Alternating trilinear forms have been classified in dimension $n \leq 7$ over an arbitrary field, see [3, 11], and also in dimension 8 over $\mathbb{C}$ and $\mathbb{R}$, see [4, 6]. Over $\mathbb{C}$ there are 23 orbits in dimension $n = 8$, but in dimension $n = 9$ the number of orbits is known to be infinite.

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Over a finite field $\mathrm{GF}(q)$ there are of course, in any finite dimension n, "only" a finite number of $\mathrm{GL}(n,q)$-orbits. But in fact the number of orbits increases extremely rapidly with increasing n. To demonstrate this it will suffice to use a crude upper bound on the order of the group $\mathrm{GL}(n,q)$, namely $|\,\mathrm{GL}(n,q)| \ll q^{n^2}$,which holds on account of the inclusion $\mathrm{GL}(V_n) \subset L(V_n, V_n)$. Since $\wedge^3 V_{n,q}$ is of size $q^{n(n-1)(n-2)/6}$ it follows that $|\wedge^3 V_{n,q}|/|\,\mathrm{GL}(n,q)| \gg q^{n(n^2-9n+2)/6}$. In particular for $n = 10$ we have $|\wedge^3 V_{10}|/|\,\mathrm{GL}(10,q)| \gg q^{20}$, and so even on the ridiculous assumption that the stabilizer group of every $T \in \wedge^3 V_{10}$ is the whole of $\mathrm{GL}(10,q)$ the number of $\mathrm{GL}(10,q)$-orbits would be substantially more than q^{20}. And for $n = 20$ the number of $\mathrm{GL}(20,q)$-orbits in $\wedge^3 V_{20}$ is much more than q^{740}.

The violence of the combinatorial explosion which takes place for $n > 8$ is really quite startling! This occurs even over the smallest fields. For on setting $N(n,q) = |\wedge^3 V_{n,q}|/|\,\mathrm{GL}(n,q)|$ we find for $q = 2$ the following approximate values

$n =$	5	6	7	8	9	10	11
$N(n,2)$	0.00010	0.000053	0.00021	0.0135	27.6	3.6×10^6	6.1×10^{13}

.

Faced with this great multitude of orbits for alternating trilinear forms one naturally hopes that there are a few orbits which are singled out by having some special property and which thus deserve further attention. In the case of alternating bilinear forms the outstanding $\mathrm{GL}(n,\mathbb{F})$-orbit occurs of course in even dimension $n = 2m$ and consists of those $B \in \mathrm{Alt}(\times^2 V_n)$ which have no singular points, the stabilizer groups being $\cong \mathrm{Sp}(2m,\mathbb{F})$. Now in the case of $T \in \mathrm{Alt}(\times^3 V_n)$ one may define a point $\langle a \rangle \in \mathbb{P}V_n$ to be T-*singular* whenever $T(a,x,y) = 0$ holds for all $x, y \in V_n$. Also one may define a subspace $\mathrm{rad}\, T$ of V_n by

$$\mathrm{rad}\, T = (a \in V_n : T(a,x,y) = 0 \quad \text{for all } x, y \in V_n\} \tag{1}$$

and call T *non-degenerate* whenever $\mathrm{rad}\, T = \{0\}$. But, just as in the bilinear case, there is not much interest in degenerate T, since one naturally switches one's attention to the non-degenerate trilinear form induced in the lower-dimensional quotient space $V_n/\,\mathrm{rad}\, T$. However of crucial importance in the case of $T \in \mathrm{Alt}(\times^3 V_n)$ are those projective lines $\langle a, b \rangle$ in $\mathrm{PG}(n-1) = \mathbb{P}V_n$ which are T-*singular*, satisfying that is

$$T(a,b,x) = 0 \quad \text{for all } x \in V_n. \tag{2}$$

For a given $T \in \mathrm{Alt}(\times^3 V_n)$ we will denote by $\mathcal{L}_T$ the set consisting of all the T-singular lines in $\mathrm{PG}(n-1,\mathbb{F}) = \mathbb{P}V_n$.

Remark 1 *The space* $\mathrm{Alt}(\times^3 V_n)$ *of alternating 3-forms is naturally isomorphic to the space* $\wedge^3 V_n^*$ *of dual trivectors, and sometimes statements concerning an element* $T \in \mathrm{Alt}(\times^3 V_n)$ *will be phrased in terms of its isomorphic image* $t \in$

$\wedge^3 V_n^*$. *If* $\{f_i\}_{1\leq i\leq n}$ *is the basis for* V_n^* *dual to the basis* $\{e_i\}_{1\leq i\leq n}$ *for* V_n *then, on writing* $f_{ijk} := f_i \wedge f_j \wedge f_k$, *we have*

$$t = \sum_{1\leq i<j<k\leq n} c_{ijk} f_{ijk}, \quad \textit{where } c_{ijk} := T(e_i, e_j, e_k). \tag{3}$$

Equivalently expressed, each $t \in \wedge^3 V_n^*$ *gives rise to an element* $T \in \mathrm{Alt}(\times^3 V_n)$ *by way of*

$$T(x, y, z) =< t|x \wedge y \wedge z >, \tag{4}$$

where $< \cdot|\cdot >$ *is the standard determinantal pairing of* $\wedge^3 V_n^*$ *with* $\wedge^3 V_n$ *given by* $< f_1 \wedge f_2 \wedge f_3 | v_1 \wedge v_2 \wedge v_3 >= \det[f_i(v_j)]$.

2 Alternating 3-forms having no singular lines?

A question immediately arises: *at least for some* $(n, \mathbb{F})$, *does there exist* $T \in \mathrm{Alt}(\times^3 V_n)$ *such that* $\mathcal{L}_T$ *is empty?*

Well, for any field $\mathbb{F}$, certainly not in even dimension $n = 2m$. For given $T \in \mathrm{Alt}(\times^3 V_{2m})$ choose any direct sum decomposition $V_{2m} = \prec a \succ \oplus V_{2m-1}$ and consider the element $B_a \in \mathrm{Alt}(\times^2 V_{2m-1})$ defined by $B_a(x, y) = T(a, x, y)$. Since the dimension of V_{2m-1} is odd there exists at least one B_a-singular point $\langle b\rangle$, whence $\langle a, b\rangle$ is a T-singular line. It follows that *through each point* $\langle a\rangle \in \mathbb{P}V_{2m}$ *there passes at least one* T*-singular line.*

Concerning odd dimension n, certainly a non-zero $T \in \mathrm{Alt}(\times^3 V_n)$ has no T-singular lines in the special case $n = 3$. But *for* $n > 3$ *if* $\mathbb{F}$ *is quasi-algebraically closed then it is known that* T*-singular lines always exist:* see *[5, Theorem 1.1].*

An important field not covered in this last statement is the real field $\mathbb{R}$. And in the case of a real 7-dimensional space we have an affirmative answer to our query: *if* $V_7 = V(7, \mathbb{R})$ *there exists* $T \in \mathrm{Alt}(\times^3 V_7)$ such that $\mathcal{L}_T$ *is empty.* To see this, recall that in a real 7-dimensional Euclidean space V_7 there exist, see [2], bilinear vector cross products $a \times b$ which satisfy *(i)* $a \times b.a = 0 = a \times b.b$ and *(ii)* $a \times b.a \times b = (a.a)(b.b) - (a.b)^2$. Then upon defining $T(a, b, c) = a \times b, c$ it follows that $T \in \mathrm{Alt}(\times^3 V_7)$; moreover, since, by *(ii),* $a \times b \neq 0$ for linearly independent a, b, we see that $\mathcal{L}_T$ is empty. One such T has for its isomorphic image the dual trivector t given by

$$t = f_{124} + f_{235} + f_{346} + f_{457} + f_{561} + f_{672} + f_{713}, \tag{5}$$

where the presence of $+f_{ijk}$ in this expression for t goes along with the cross product relation $e_i \times e_j = +e_k$.

As is well known, the existence in real seven dimensions of these vector cross products is related to the exceptional existence of the real division algebra of the (non-split) octonions. The next theorem shows that the real dimension $n = 7$ is also exceptional since T-singular lines always exist for $T \in \mathrm{Alt}(\times^3 V_n)$ in any other real dimension $n > 3$.

Theorem 2 *Except when $n \in \{3,7\}$ every alternating trilinear form on a real vector space $V_n, n > 2$, possesses singular lines.*

Proof. Suppose that $T \in \mathrm{Alt}(\times^3 V_n)$ is such that $\mathcal{L}_T$ is empty. Set $V_{n+1} = \mathbb{R} \oplus V_n$, and equip V_n with $\mathrm{O}(n)$-geometry by making a(ny) choice $x.y$ of a positive definite scalar product on V_n. Extend this to a positive definite scalar product $a.b$ on V_{n+1} by defining

$$(\alpha, x).(\beta, y) = \alpha\beta + x.y \ , \quad \alpha, \beta \in \mathbb{R},\ x, y \in V_n. \tag{6}$$

Make V_n into a real algebra by defining the algebra product of x and y to be that element $x \times y \in V_n$ such that

$$x \times y.z = T(x, y, z), \qquad \text{for all } z \in V_n. \tag{7}$$

Since T is alternating it follows that

$$x \times y = -y \times x \in \prec x, y \succ^{\perp} . \tag{8}$$

Further, since we are supposing that there are no T-singular lines, if x and y are linearly independent then

$$x \times y \text{ is a } \textit{nonzero} \text{ element of } V_n \text{ which is perpendicular to the plane } \prec x, y \succ . \tag{9}$$

We now make V_{n+1} into a real algebra $\mathcal{A}$ by laying down that $1 \in \mathbb{R}$ is an identity element and that the $\mathcal{A}$-product of $x, y \in V_n$ is

$$xy = -x.y + x \times y, \qquad x, y \in V_n. \tag{10}$$

It follows from (8)-(10) that the algebra $\mathcal{A}$ has no zero divisors. Consequently, see [10, Section II.2], $\mathcal{A}$ is a division algebra over $\mathbb{R}$. The theorem now follows since real division algebras exist only in dimensions $1, 2, 4, 8$: see [1], [7]. ∎

Remark 3 *In [5, Theorem 3.2] it was proved that, over any field $\mathbb{F}$, the union of all lines $L \in \mathcal{L}_T$, $T \in \mathrm{Alt}(\times^3 V_{2k+1})$, is either the whole of V_{2k+1} or is a hypersurface in $\mathrm{PG}(2k)$ with equation $f_T(x) = 0$, $x \in V_{2k+1}$, where f_T is a certain homogeneous polynomial of degree $k-1$. In the case of a space V_7 the polynomial f_T has degree 2, and if T has trivector t as in (5) then (up to an overall sign) one finds that*

$$f_T(x) = \sum\nolimits_{i=1}^{7} (x_i)^2. \tag{11}$$

In the case $\mathbb{F} = \mathbb{R}$ it follows that $f_T(x) \neq 0$ for all $x \neq 0$. Thus we have obtained another proof that $\mathcal{L}_T$ is empty if t is as in (5) and $V_7 = V(7, \mathbb{R})$.

Remark 4 *It is of some interest to consider t as in (5) in the cases when $V_7 = V(7, q)$.*

(i) *First suppose that* $q = 2^h$ *is even. In which case* $f_T(x) = (\sum_{i=1}^{7} x_i)^2$, *whence the union of all the* T*-singular lines is the hyperplane with equation* $\sum_{i=1}^{7} x_i = 0$.

(ii) *If* q *is odd then the union of all the* T*-singular lines is the parabolic quadric* $\mathcal{P}_6$ *in* $\mathrm{PG}(6,2)$ *having equation* $\sum_{i=1}^{7}(x_i)^2 = 0$.

That the cases of even q *and odd* q *are quite different is highlighted by the fact that the alternating trilinear form given by* $t = f_{124} + f_{235} + f_{346} + f_{457} + f_{561} + f_{672} + f_{713}$ *can be seen to belong to the same* $\mathrm{GL}(7,q)$*-orbit as* f_6 *in [3, Table 1] if* q *is even, but to belong to the same* $\mathrm{GL}(7,q)$*-orbit as* f_9 *in [3, Table 1] if* q *is odd. In particular, upon using Mathematica to compute the quadratic form* f_T *for all orbit representatives in [3, Table 1], we found that only* f_9 *gives rise to a nonsingular quadratic form.*

3 Alternating 3-forms yielding spreads in $\mathbb{P}V_{2m}$?

As noted in the preceding section, *if* T *is any alternating 3-form on an even-dimensional space* V_{2m}, *over any field* $\mathbb{F}$, *then through each point* $\langle a\rangle \in \mathbb{P}V_{2m}$ *there passes at least one* T*-singular line.* In our attempt to find alternating 3-forms T whose set $\mathcal{L}_T$ of singular lines is in some manner special, perhaps there exists $T \in \mathrm{Alt}(\times^3 V_{2m})$ such that *through each point* $\langle a\rangle \in \mathbb{P}V_{2m}$ *there passes precisely one* T*-singular line?* That is, for some $T \in \mathrm{Alt}(\times^3 V_{2m})$, *perhaps* $\mathcal{L}_T$ *is a spread of lines in* $\mathbb{P}V_{2m}$? Certainly, in finite geometry circles, line-spreads in $\mathrm{PG}(2m-1,q)$ are of continuing interest. They also exist in great profusion, even in low dimension; in particular in $\mathrm{PG}(5,2)$ there exist, see [9], 131,044 inequivalent line-spreads!

In the present section we will show that in the case of $V_6 = V(6,q)$ there exists $T \in \mathrm{Alt}(\times^3 V_6)$ such that $\mathcal{L}_T$ is a line-spread in $\mathrm{PG}(5,q)$. To this end consider a space $V(3,q^2)$ with basis $\prec e_1, e_2, e_3 \succ$. Choose any element $\rho \in \mathrm{GF}(q^2)\setminus\mathrm{GF}(q)$ and define

$$e_4 = \rho e_1, \quad e_5 = \rho e_2, \quad e_6 = \rho e_3. \tag{12}$$

Then we may view $V(3,q^2)$ as a 6-dimensional vector space over $\mathrm{GF}(q)$:

$$V_6 = V(6,q) = \prec e_1, e_2, e_3, e_4, e_5, e_6 \succ . \tag{13}$$

The $q^4 + q^2 + 1$ points $\langle a\rangle$ of $\mathrm{PG}(2,q^2) = \mathbb{P}(V(3,q^2))$ give rise over $\mathrm{GF}(q)$ to a Desarguesian spread $\mathcal{L}$ of $q^4 + q^2 + 1$ lines $\langle a, \rho a\rangle$ in $\mathrm{PG}(5,q) = \mathbb{P}(V(6,q))$. We aim to show that $\mathcal{L} = \mathcal{L}_T$ for some $T \in \mathrm{Alt}(\times^3 V_6)$.

If τ is any element of the 1-dimensional $\mathrm{GF}(q^2)$-space $\mathrm{Alt}(\times^3 V(3,q^2))$ then we may define an element $T \in \mathrm{Alt}(\times^3 V_6)$ by

$$T(x,y,z) = \mathrm{Tr}(\tau(x,y,z)). \tag{14}$$

Here we use Tr to denote the trace $\mathrm{Tr}_{\mathrm{GF}(q^2)/\mathrm{GF}(q)}$ over the subfield $\mathrm{GF}(q)$ defined, see [8, Section 2.3], by

$$\mathrm{Tr}(\beta) = \beta + \beta^q, \quad \beta \in \mathrm{GF}(q^2). \tag{15}$$

(So Tr here is not the absolute trace over the prime subfield GF(p) except if $q = p$.) Thus defined, Tr is a GF(q)-linear mapping GF(q^2) $\to$ GF(q). whence T is indeed an element of Alt($\times^3 V_6$).

Let us fix τ by requiring $\tau(e_1, e_2, e_3) = \beta$ for some choice of nonzero element $\beta \in \mathrm{GF}(q^2)$. Upon defining $c_i \in \mathrm{GF}(q^2)$ by

$$c_0 = \mathrm{Tr}(\beta),\ c_1 = \mathrm{Tr}(\beta\rho),\ c_2 = \mathrm{Tr}(\beta\rho^2),\ c_3 = \mathrm{Tr}(\beta\rho^3) \tag{16}$$

it follows from (12), (14) that t in (3) is given by

$$c_{123} = c_0,\ c_{234} = -c_{135} = c_{126} = c_1,\ c_{156} = -c_{246} = c_{345} = c_2,\ c_{456} = c_3, \tag{17}$$

with $c_{ijk} = 0$ for other $i < j < k$. That is

$$t = c_0 t_0 + c_1 t_1 + c_2 t_2 + c_3 t_3, \tag{18}$$

where the trivectors t_0, t_1, t_2, t_3 are defined by

$$t_0 = f_{123},\ t_1 = f_{234} - f_{135} + f_{126},\ t_2 = f_{156} - f_{246} + f_{345},\ t_3 = f_{456}. \tag{19}$$

The multiplicative group $\mathrm{GF}(q^2)^\times$ of the field GF(q^2) is a cyclic group $\langle\zeta\rangle$ generated by an irreducible element $\zeta \in \mathrm{GF}(q^2)$ of order $q^2 - 1$: $\zeta^{(q-1)(q+1)} = 1$. The multiplicative group $\mathrm{GF}(q)^\times$ of the subfield GF(q) is the cyclic group $\langle\xi\rangle$ generated by $\xi = \zeta^{q+1}$, of order $q - 1$. In making a specific choice of the field elements β, ρ in (16), it will help to consider separately the cases of odd q and even q.

3.1 The case of odd q

Suppose that $q = 2k + 1$ is odd. Then

$$\mathrm{GF}(q^2)^\times = \langle\zeta\rangle, \quad \text{where} \quad \zeta^{4k(k+1)} = 1, \quad \zeta^{2k(k+1)} = -1. \tag{20}$$

In (12) let us make the following choice of $\rho \notin \mathrm{GF}(q)$:

$$\rho = \zeta^{k+1}, \quad \text{and so} \quad \rho^{4k} = 1, \quad \rho^{2k} = -1. \tag{21}$$

It follows that

$$\mathrm{Tr}(\rho) = \rho(1+\rho^{2k}) = 0, \quad \mathrm{Tr}(\rho^2) = \rho^2(1+\rho^{4k}) = 2\rho^2, \quad \mathrm{Tr}(\rho^3) = \rho^3(1+\rho^{6k}) = 0. \tag{22}$$

Since $\mathrm{Tr}(1) = 2$, if we make the choice $\beta = \frac{1}{2}$ then the trivector t in (18) is

$$t = f_{123} + \mu(f_{156} - f_{246} + f_{345}), \quad \text{where} \quad \mu = \rho^2. \tag{23}$$

Observe that μ is the square of an element ρ of GF(q^2), but *that μ is an element of* GF(q) *which is one of the non-squares in* GF(q). By making different choices of the irreducible element ζ in the definition (21) of ρ we can arrange for μ in (23) to be any of the non-square elements in GF(q).

Theorem 5 *If $V_6 = V(6, q)$ where q is odd, consider the 3-form $T \in \mathrm{Alt}(\times^3 V_6)$ given by the dual trivector*

$$t = f_{123} + \mu(f_{156} - f_{246} + f_{345}).$$

Then, provided only that $\mu \in \mathrm{GF}(q)$ is chosen to be a non-square, $\mathcal{L}_T$ is a Desarguesian line-spread in $\mathrm{PG}(5, q)$.

Proof. Because in (14) we have $\tau(a, \rho a, z) = 0$ for all z, the $q^4 + q^2 + 1$ lines $\langle a, \rho a\rangle$ in $\mathrm{PG}(5, q)$ are certainly all T-singular. To complete the proof we need to show that no other lines $\langle a, b\rangle$ in $\mathrm{PG}(5, q)$ are T-singular. Suppose to the contrary that $b \notin \langle a, \rho a\rangle$ yet $T(a, b, x) = 0$ for all $x \in V_6$. Now $\tau(Aa, Ab, Ax) = \tau(a, b, x)$ for any $A \in \mathrm{SL}(3, q^2)$. So if $b \notin \langle a, \rho a\rangle$ we can choose $A \in \mathrm{SL}(3, q^2)$ such that $Aa = e_1,\ Ab = e_2$, Since $\mathrm{Tr}(\tau(e_1, e_2, x)) \neq 0$ if $x = e_3$ it follows that $T(a, b, x) \neq 0$ for all $x \in V_6$. ■

3.2 The case of even q

If $q = 2^h$ then every element $\mu \in \mathrm{GF}(q)$ is a square and for no choice of μ in (23) is $\mathcal{L}_T$ a spread. For example, if $\mu = 1$ then every line in the plane $\langle e_1 + e_4, e_2 + e_5, e_3 + e_6\rangle$ is T-singular. However, as we now demonstrate, $\mathcal{L}_T$ is a line-spread for a different choice of T.

In proving this, in addition to the previous $\mathrm{GF}(q)$-linear mapping $\mathrm{Tr} : \mathrm{GF}(q^2) \to \mathrm{GF}(q)$, we will also make use of the $\mathrm{GF}(2)$-linear mapping $\mathrm{tr} : \mathrm{GF}(q) \to \mathrm{GF}(2)$, where $\mathrm{tr}(\mu) \in \mathrm{GF}(2)$ is the absolute trace of $\mu \in \mathrm{GF}(2^h)$:

$$\mathrm{tr}(\mu) = \mu + \mu^2 + \mu^4 + \ldots + \mu^{2^{h-1}}. \tag{24}$$

Observe that $\mathrm{GF}(q) = \mathcal{T}_0 \cup \mathcal{T}_1$ where

$$\mathcal{T}_i := \{\mu \in \mathrm{GF}(q) |\, \mathrm{tr}(\mu) = i\}, \quad i \in \{0, 1\}; \tag{25}$$

in particular $\mathcal{T}_0$ is the kernel of the linear mapping tr, and is a hyperplane in the $\mathrm{GF}(2)$-space $\mathrm{GF}(q)$. It is easy to see also that $\mathcal{T}_0 = \mathrm{im}\, F$, where F denotes the linear endomorphism of the $\mathrm{GF}(2)$-space $\mathrm{GF}(q)$ defined by $F(\lambda) = \lambda + \lambda^2$. Consequently $\mathcal{T}_1$ consists of those $\mu \in \mathrm{GF}(q)$ *not* expressible as $\mu = \lambda + \lambda^2$ for any $\lambda \in \mathrm{GF}(q)$.

Lemma 6 *There exists $\rho \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q),\ q = 2^h$, such that*

(i) h odd: $\quad \mathrm{Tr}(\rho) = 1,\ \mathrm{Tr}(\rho^2) = 1,\ \mathrm{Tr}(\rho^3) = 0; \quad$ (26)

(ii) h even: $\quad \mathrm{Tr}(\rho) = 1,\ \mathrm{Tr}(\rho^2) = 1,\ \mathrm{Tr}(\rho^3) = \mu,$ *where* $\mathrm{tr}(\mu) = 1. \quad$ (27)

Moreover in (ii) we can choose ρ so that μ is any pre-assigned element of $\mathcal{T}_1$.

Proof. For any $\zeta \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$ we have $\mathrm{Tr}(\zeta) \neq 0$. So, since $\mathrm{Tr}(\alpha\zeta) = \alpha\,\mathrm{Tr}(\zeta)$ for $\alpha \in \mathrm{GF}(q)$, $\zeta \in \mathrm{GF}(q^2)$, then $\rho = (\mathrm{Tr}\,\zeta)^{-1}\zeta$ achieves $\mathrm{Tr}(\rho) = 1$. It then follows that $\mathrm{Tr}(\rho^2) = \rho^2 + \rho^{2q} = (\rho + \rho^q)^2 = 1$. It further follows that $\rho^{3q} = \rho^q\rho^{2q} = (1+\rho)(1+\rho^2)$, whence

$$\mu := \mathrm{Tr}(\rho^3) = \rho^3 + \rho^{3q} = 1 + \rho + \rho^2. \tag{28}$$

Now from $\mu + 1 = \rho + \rho^2$ we obtain

$$\begin{aligned} \mathrm{tr}(\mu+1) &= (\rho+\rho^2) + (\rho^2+\rho^4) + \ldots (\rho^{2^{h-1}} + \rho^{2^h}) \\ &= \rho + \rho^q = \mathrm{Tr}(\rho) = 1. \end{aligned} \tag{29}$$

Suppose first that h is odd. Then $\mathrm{tr}(1) = 1$ and so $\mu \in \mathcal{T}_0$. Now for any $\alpha \in \mathrm{GF}(q)$ consider $\rho' := \rho + \alpha$. Then $\mathrm{Tr}(\rho') = 1$, and hence $\mathrm{Tr}((\rho')^2) = 1$. Further if $\mu' := \mathrm{Tr}((\rho')^3)$ then

$$\mu' = \mathrm{Tr}(\rho^3) + \alpha\,\mathrm{Tr}(\rho^2) + \alpha^2\,\mathrm{Tr}(\rho) + \alpha^3\,\mathrm{Tr}(1) = \mu + \alpha + \alpha^2, \tag{30}$$

whence $\mu' = 0$ for a suitable choice of α, thus achieving (26).

If instead h is even and so $\mu \in \mathcal{T}_1$, then we see from (30) that we can achieve (27) for any pre-assigned $\mu \in \mathcal{T}_1$. ■

Theorem 7 *If $V_6 = V(6,q)$ where $q = 2^h$ is even, then if h is odd the 3-form $T \in \mathrm{Alt}(\times^3 V_6)$ given by the trivector*

$$t = f_{234} + f_{135} + f_{126} + f_{156} + f_{246} + f_{345} \tag{31}$$

is such that $\mathcal{L}_T$ is a Desarguesian line-spread in $\mathrm{PG}(5,q)$. *If h is even then, for any $\mu \in \mathrm{GF}(q)$ satisfying* $\mathrm{tr}(\mu) = 1$, *the 3-form $T \in \mathrm{Alt}(\times^3 V_6)$ given by the trivector*

$$t = f_{234} + f_{135} + f_{126} + f_{156} + f_{246} + f_{345} + \mu f_{456} \tag{32}$$

is such that $\mathcal{L}_T$ is a Desarguesian line-spread in $\mathrm{PG}(5,q)$.

Proof. In (16) choose ρ as in Lemma 6 and choose $\beta = 1$, and so $c_0 = \mathrm{Tr}(1) = 0$. Then we obtain (31) and (32) from (18) (19). The rest of the proof is as in the proof of Theorem 5. ■

Remark 8 *The canonical form (31) was obtained previously in [12], albeit only in the special case $q = 2$. In [12] two alternative canonical forms were also found, namely t' and t'' as given by*

$$\begin{aligned} t' &= f_{156} + f_{246} + f_{345} + f_{123} + f_{456} \\ t'' &= f_{234} + f_{135} + f_{126} + f_{123} + f_{456}\ . \end{aligned} \tag{33}$$

These are also alternatives to (31) for any $q = 2^h$, h odd. *One way to obtain these alternatives is by use of the choice $\rho = \zeta^{(q-1)(q+1)/3}$.in (16). For if h is odd then $3|(q+1)$ and so $\zeta^{(q+1)/3} \in$* $\mathrm{GF}(q)$, *whence $\rho \notin$* $\mathrm{GF}(q)$. *Since $\rho^3 = 1$, we have* $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho)$ *and* $\mathrm{Tr}(\rho^3) = 0$. *So in (16) the choices $\beta = 1$, $\beta = \rho$, $\beta = \rho^2$ give rise respectively to the trivectors t (as in (31)), t', t''.*

Remark 9 *For even n one might wonder whether $\mathcal{L}_T$ can be a line spread for fields other than finite fields. Indeed, suppose that $\mathbb{F}$ is algebraically closed; is it possible that $\mathcal{L}_T$ is a spread? The T's for which this is the case can be shown to form a Zariski-open subset of* $\mathrm{Alt}(\times^3 V_n)$*, but this subset may be empty. Indeed, we have performed Gröbner basis calculations which show that for $n = 4$ and $n = 6$ and for algebraically closed $\mathbb{F}$ of characteristic zero there are no trilinear forms T for which $\mathcal{L}_T$ is a spread. For $n = 8$ our computational approach is not feasible, and new ideas will be needed.*

4 Some other noteworthy alternating 3-forms

So far we have been looking at alternating 3-forms T for which $\mathcal{L}_T$ is as small a set as possible. In contrast we now give some examples of interesting alternating 3-forms $T \in \mathrm{Alt}(\times^3 V_n)$ which are non-degenerate yet for which some sizeable subspace V_r of V_n is such that *every line in $\mathbb{P}V_r$ is T-singular.* Let us term such a subspace V_r *totally T-singular.* Our first example is in dimension $n = 6$, where

$$t = f_{156} + f_{246} + f_{345} \tag{34}$$

is the trivector of a non-degenerate $T \in \mathrm{Alt}(\times^3 V_6)$ for which, for any field $\mathbb{F}$, the 3-space $V_3 = \prec e_1, e_2, e_3 \succ$ is totally T-singular. It is easy to check that no subspace of dimension > 3 is totally singular. A second example is in dimension $n = 10$, where, writing $\mathtt{x} = 10$,

$$t = f_{17\mathtt{x}} + f_{28\mathtt{x}} + f_{39\mathtt{x}} + f_{489} + f_{579} + f_{678} \tag{35}$$

is the trivector of a non-degenerate $T \in \mathrm{Alt}(\times^3 V_{10})$ for which the 6-space $V_6 = \prec e_1, \ldots, e_6 \succ$ is totally T-singular.

Theorem 10 *If $n = \frac{1}{2}s(s+1)$, $s > 2$, then, for any field $\mathbb{F}$, there exists a single* $\mathrm{GL}(n, \mathbb{F})$*-orbit, say Ω, of non-degenerate alternating 3-forms T on V_n with the property that there is a unique totally T-singular subspace V_r of V_n of dimension $r = \frac{1}{2}s(s-1)$.*

Proof. See [5, Section 4]. ■

Remark 11 *In dimension $n = 15$ the subspace V_r in the theorem is of dimension 10. By use of the crude inequality $|\,\mathrm{GL}(n, q)| \ll q^{n^2}$, as in Section 1, we get the even cruder lower bound q^{230} for the number of $\mathrm{GL}(15, q)$ orbits of alternating 3-forms T on $V(15, q)$. Clearly there is no possibility of studying all of these zillions of orbits! But perhaps the orbit Ω does deserve further study?*

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