On the intersection motive of certain Shimura varieties: the case of Siegel threefolds

by

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Abstract

In this article, we construct a Hecke-equivariant Chow motive whose realizations equal intersection cohomology of Siegel threefolds with regular algebraic coefficients. As a consequence, we are able to define Grothendieck motives for Siegel modular forms.

Keywords: Siegel threefolds, weight structures, intersection motive, motives for Siegel modular forms.

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Contents

0 Introduction

The purpose of this paper is the construction and analysis of the *intersection motive* of Kuga–Sato families over a Siegel threefold relative to its Satake–(Baily–Borel) compactification. As in earlier work on Hilbert–Blumenthal varieties [Wi5], Picard surfaces [Wi7], and more generally, Picard varieties of arbitrary dimension [C], the use of the formalism of *weight structures* [B] proves to be successful for dealing with a problem, for which explicit geometrical methods seem inefficient.

However, Siegel threefolds present a characteristic feature different from the cases treated so far: the dimension of the boundary of their Satake–(Baily–Borel) compactification is equal to one. In particular, it is strictly positive.

As a consequence, the context of *geometrical motives*, *i.e.*, motives over a point, is no longer adapted to the problem. Let us explain why.

The present construction, as the preceding ones, depends on *absence of weights* -1 *and* 0 in the *boundary motive*. To prove absence of weights, the idea remains, as previously, to employ *realizations*. But then, realizations need to detect weights (and therefore, their absence). One may expect this to be true in general; let us agree to refer to that principle as *weight conservativity*. To date, weight conservativity is *proved* for the restriction of the (generic) ℓ-adic realization to the category of *motives of Abelian type* of characteristic zero [Wi10].

However, unless the boundary of the Baily–Borel compactification of a given Shimura variety M is of dimension zero, its boundary motive, as well as the boundary motive of any Kuga–Sato family B over M, is in general not of Abelian type; this is in any case true if M is a Siegel threefold. Concretely, this means that even if the realization of the boundary motive were proved to avoid weights -1 and 0, we could not formally conclude that the same is true for the boundary motive itself.

This is where *relative motives*, together with the *formalism of six operations* enter. Denoting by j the open immersion of M into its Baily–Borel compactification M^*, by i its closed complement, and by $\mathbb{1}_M$ the structural motive over M, there is an exact triangle

$$i_*i^*j_*\mathbb{1}_M[-1] \longrightarrow j_!\mathbb{1}_M \longrightarrow j_*\mathbb{1}_M \longrightarrow i_*i^*j_*\mathbb{1}_M$$

of motives over M^*. The boundary motive of M is isomorphic to the dual of the direct image of $i_*i^*j_*\mathbb{1}_M$ under the structure morphism of M^*. More generally, the boundary motive of B is isomorphic to the dual of the direct image of $i_*i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}})$, where $\pi : \mathrm{B} \to M$ denotes the projection of the Kuga–Sato family B to its base.

It is then true that the relative motive $i_*i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}})$ over M^* *is* of Abelian type.

Whence our strategy of proof. First, identify the ℓ-adic realization of $i_*i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}})$, or more generally, of $i_*i^*j_*\mathcal{V}$, for direct factors $\mathcal{V}$ of $\pi_*(\mathbb{1}_{\mathrm{B}})$; in the cases where weights 0 and 1 are avoided, weight conservativity tells us that $i_*i^*j_*\mathcal{V}$ itself avoids weights 0 and 1. Second, apply the direct image d_* associated to the structure morphism d of M^*. It is proper, therefore, the functor d_* is *weight exact*. In particular, if $i_*i^*j_*\mathcal{V}$ avoids weights 0 and 1, then so does $d_*i_*i^*j_*\mathcal{V}$. The corresponding direct factor of the boundary motive of B thus avoids weights -1 and 0.

It may be useful to remark that if M is a Hilbert–Blumenthal or Picard variety, then there is essentially no difference between $i_*i^*j_*\mathcal{V}$ and its direct image under d, since the latter is of relative dimension zero on the boundary of M^*.

The passage from geometrical motives to relative motives necessitates a certain number of technical adjustments. For better legibility, we decided to separate these from the present text. The result is [Wi9]; it contains in particular the identification of the boundary motive and the dual of $d_*i_*i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}})$ mentioned above.

Compared to the cases treated earlier, another feature of the boundary of Siegel threefolds is new: its canonical stratification is not reduced to a single type of strata. Indeed, in the boundary, one finds a closed stratum of dimension zero, the so-called *Siegel stratum*, and its complement, the so-called *Klingen stratum*, which is a disjoint union of (open) modular curves. Control of the weights avoided by the restrictions of the ℓ-adic realization $R_\ell(i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}}))$ of $i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}})$ to the two strata is related to, but does not *a priori* determine the weights avoided by $R_\ell(i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}}))$. In fact, the precise relation is given by a long exact localization sequence. Its control it not obvious. In an earlier attempt, we succeeded to identify sufficiently many terms in this sequence, and (above all) certain morphisms, to prove absence of weights 0 and 1. This approach is technically difficult; moreover, it does not use the auto-duality property of the coefficients. Indeed, the device dual to the localization sequence is the co-localization sequence; even when the coefficients are auto-dual, the two sequences cannot be related. It turns out that both problems admit the same solution. Namely, the theory of *intermediate extensions* allows to represent $R_\ell(i^* j_* \pi_*(\mathbb{1}_{\rm B}))$ as an extension of two "halfs", one of which dual to the other, and both related to the intermediate extension $j_{!*}\, \pi_*(\mathbb{1}_{\rm B})$. This observation is equally integrated in [Wi9]; for our purposes, its concrete interest is to divide by two the number of cohomological degrees for which absence of weights has to be tested, and to reduce the number of morphism in the localization sequence, which need to be identified, to zero.

The rôle of the intermediate extension is not only technical. It turns out that the dual of its direct image under d is canonically isomorphic to the *interior motive*, which according to [Wi3] can be defined as soon as the boundary motive avoids weights -1 and 0. This motivates the slight change of terminology in the title, as compared to the earlier work mentioned above [Wi5, Wi7, C].

Let us now give a more detailed account of the content of the present article. Section 1 contains the statement of our main result, Theorem 1.6. Denote by $GSp_{4,\mathbb{Q}}$ the group of symplectic similitudes of a fixed four-dimensional $\mathbb{Q}$-vector space V. As will be recalled, irreducible representations of $GSp_{4,\mathbb{Q}}$ are indexed by weights $\underline{\alpha}$ depending on three integral parameters: $\underline{\alpha} = \alpha(k_1, k_2, r)$. The weight $\underline{\alpha}$ is dominant if and only if $k_1 \geq k_2 \geq 0$; it is regular if and only if $k_1 > k_2 > 0$. Denote by $V_{\underline{\alpha}}$ the irreducible representation of highest weight $\underline{\alpha}$. According to the main result from [A] (which will be recalled in Theorem 1.4), there is a Chow motive ${}^{\underline{\alpha}}\mathcal{V}$ over the Siegel threefold M, whose cohomological (Hodge theoretic or ℓ-adic) realizations equal the classical *canonical construction* $\mu(V_{\underline{\alpha}})$. Part (a) of Theorem 1.6 then states that $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ is of Abelian type. Part (b) asserts that if $\underline{\alpha}$ is regular, then $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ avoids weights 0 and 1. It has recently become increasingly important to determine the precise interval containing $[0, 1]$ of weights avoided by $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$. Theorem 1.6 (b) gives a complete answer: putting $k := \min(k_1 - k_2, k_2)$, the motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ avoids all the weights between $-k + 1$ and k, while both weights $-k$ and $k + 1$ *do* occur. Interestingly, this result does not depend on the level of the Siegel threefold. We then list the main consequences of this result (Corollaries 1.7, 1.8, 1.9, 1.11, 1.13), applying the theory developed in [Wi9].

Section 2 is devoted to the proof of Theorem 1.6. As in previous cases, our control of smooth *toroidal compactifications* of M is sufficiently explicit to verify that, as stated in Theorem 1.6 (a), the motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ is indeed of Abelian type. Given this result, and weight conservativity of the restriction of the ℓ-adic realization R_ℓ, part (b) of Theorem 1.6 may be checked on the image of $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ under R_ℓ. Given that ${}^{\underline{\alpha}}\mathcal{V}$ realizes to give $\mu(V_{\underline{\alpha}})$, the restriction of $R_\ell(i^* j_* {}^{\underline{\alpha}}\mathcal{V})$ to the (Siegel and Klingen) strata can be computed following a standard pattern, employing Pink's and Kostant's Theorems. This computation (Theorem 2.3) is considerably simplified by results of Lemma's [Lm]. It remains to glue the information coming from the strata, in order to get control of the weights on the whole boundary. The part of Theorem 1.6 (b) asserting that weights $-k$ and $k+1$ occur in $R_\ell(i^* j_* {}^{\underline{\alpha}}\mathcal{V})$ (Proposition 2.9) is the single ingredient requiring a proof longer than any other.

In the final Section 3, we give the necessary ingredients to perform the construction of the Grothendieck motive associated to a (Siegel) automorphic form with coefficients in an irreducible representation with regular highest weight (Definition 3.5). This is the analogue for Siegel threefolds of the main result from [Sc]. On the level of Galois representations, our definition coincides with Weissauer's [We, Thm. I]. We also recover Urban's result [U, Thm. 1] on characteristic polynomials associated to Frobenii (Corollary 3.7).

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Conventions: We use the triangulated, $\mathbb{Q}$-linear categories $DM_{\mathrm{B},c}(X)$ of *constructible Beilinson motives* over X [CD1, Def. 15.1.1], indexed by schemes X over $\mathbf{Spec}\,\mathbb{Q}$, which are separated and of finite type. As in [CD1], the symbol $\mathbb{1}_X$ is used to denote the unit for the tensor product in $DM_{\mathrm{B},c}(X)$. We shall employ the full formalism of six operations developed in [loc. cit.]. The reader may choose to consult [Hé, Sect. 2] or [Wi4, Sect. 1] for concise presentations of this formalism.

Beilinson motives can be endowed with a canonical weight structure, thanks to the main results from [Hé] (see [B, Prop. 6.5.3] for the case $X = \mathbf{Spec}\,k$, for a field k of characteristic zero). We refer to it as the *motivic weight structure*. Following [Wi4, Def. 1.5], the category $CHM(X)_\mathbb{Q}$ of *Chow motives* over X is defined as the heart $DM_{\mathrm{B},c}(X)_{w=0}$ of the motivic weight structure on $DM_{\mathrm{B},c}(X)$.

1 Statement of the main result

In order to state our main result (Theorem 1.6), let us introduce the situation we are going to consider. The $\mathbb{Q}$-scheme M^K is a *Siegel threefold*, and the Chow motive ${}^{\underline{\alpha}}\mathcal{V}$ over M^K is associated to a *dominant weight* $\underline{\alpha} = (k_1, k_2, r) \in \mathbb{Z}^3$, $k_1 \geq k_2 \geq 0$ (see below for the precise normalizations). Denote by j the open immersion of M^K into its *Satake–(Baily–Borel) compactification* $(M^K)^*$, and by $i : \partial(M^K)^* \hookrightarrow (M^K)^*$ the immersion of the complement of M^K in $(M^K)^*$ (with the reduced scheme structure, say). Recall the following.

Definition 1.1 (cmp. [Wi9, Def. 1.1 (a)]). Denote by $CHM(M^K)_{\mathbb{Q},\partial w \neq 0,1}$ the full sub-category of $CHM(M^K)_{\mathbb{Q}}$ of objects V such that $i^* j_* V$ is without weights 0 and 1.

Theorem 1.6 implies that in our setting, the motive ${}^{\underline{\alpha}}\mathcal{V} \in CHM(M^K)_{\mathbb{Q}}$ belongs to $CHM(M^K)_{\mathbb{Q},\partial w \neq 0,1}$ if and only if $\underline{\alpha}$ is *regular*: $k_1 > k_2 > 0$. More precisely, putting $k := \min(k_1 - k_2, k_2)$, the motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ is without weights $-k+1, -k+2, \ldots, k$. The proof of Theorem 1.6 will be given in Section 2. It is an application of [Wi9, Thm. 3.4]; in order to verify the hypotheses of the latter, we heavily rely on results from [Lm].

Fix a four-dimensional $\mathbb{Q}$-vector space V, together with a $\mathbb{Q}$-valued non-degenerate symplectic bilinear form J.

Definition 1.2. The group scheme G over $\mathbb{Q}$ is defined as the group of symplectic similitudes

$$G := GSp(V, J) \subset GL(V) \ .$$

Thus, G is reductive, and for any $\mathbb{Q}$-algebra R, the group $G(R)$ equals

$$\{g \in \mathrm{GL}(V \otimes_{\mathbb{Q}} R) \ , \ \exists\, \lambda(g) \in R^* \ , \ J(g \bullet , g \bullet) = \lambda(g) \cdot J(\bullet \, , \bullet)\} \ .$$

In particular, the similitude norm $\lambda(g)$ defines a canonical morphism

$$\lambda : G \longrightarrow \mathbb{G}_{m,\mathbb{Q}} \ .$$

The group G is split over $\mathbb{Q}$, and its center $Z(G)$ equals $\mathbb{G}_{m,\mathbb{Q}} \subset \mathrm{GL}(V)$ (inclusion of scalar automorphisms). Maximal $\mathbb{Q}$-split tori, together with an inclusion into a Borel sub-group of G, are in bijection with symplectic $\mathbb{Q}$-bases of V, in which J acquires the 4×4-matrix

$$\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} ,$$

equally denoted by J. Here as in the sequel, we denote by I_2 the 2×2-matrix representing the identity. Fix one such basis (e_1, e_2, e_3, e_4), use it to identify G with the sub-group $GSp_{4,\mathbb{Q}}$ of $GL_{4,\mathbb{Q}}$ of matrices g satisfying the relation

$${}^t g J g = \lambda(g) \cdot J \ ,$$

the maximal split torus with the sub-group T of diagonal matrices

$$\{\mathrm{diag}(a, b, a^{-1}q, b^{-1}q) \in GL_{4,\mathbb{Q}}\} ,$$

and the Borel sub-group with the sub-group of matrices stabilizing the flag of totally isotropic sub-spaces $(e_1)_\mathbb{Q} \subset (e_1, e_2)_\mathbb{Q}$ of V. We consider triplets $(k_1, k_2, r) \in \mathbb{Z}^3$ satisfying the congruence relation

$$r \equiv k_1 + k_2 \mod 2 .$$

To such a triplet, let us associate the (representation-theoretic) weight

$$\alpha(k_1, k_2, r) : T \longrightarrow \mathbb{G}_{m,\mathbb{Q}} \ , \ \mathrm{diag}(a, b, a^{-1}q, b^{-1}q) \longmapsto a^{k_1} b^{k_2} q^{-\frac{r+k_1+k_2}{2}} \ .$$

Note that restriction of $\alpha(k_1, k_2, r)$ to $T \cap Sp(V, J)$ corresponds to the projection onto (k_1, k_2). In particular, the weight $\alpha(k_1, k_2, r)$ is dominant if and only if $k_1 \geq k_2 \geq 0$; it is regular if and only if $k_1 > k_2 > 0$. Note also that the composition of $\alpha(k_1, k_2, r)$ with the cocharacter

$$\mathbb{G}_{m,\mathbb{Q}} \longrightarrow T \ , \ x \longmapsto \mathrm{diag}(x, x, x, x)$$

equals

$$\mathbb{G}_{m,\mathbb{Q}} \longrightarrow \mathbb{G}_{m,\mathbb{Q}} \ , \ x \mapsto x^{-r} \ .$$

The character λ on T equals $\alpha(0, 0, -2)$, and $\det = \lambda^2$.

Definition 1.3. The analytic space $\mathcal{H}$ is defined as the sub-space of $M_2(\mathbb{C})$ of those complex 2×2-matrices, which are symmetrical, and whose imaginary part is (positive or negative) definite:

$$\mathcal{H} := \{\tau \in M_2(\mathbb{C}) \ , \ {}^t\tau = \tau \text{ and } Im(\tau) \text{ definite} \} .$$

The group of real points $G(\mathbb{R})$ acts on $\mathcal{H}$ by analytical automorphisms [P1, Ex. 2.7]. In fact, $(G, \mathcal{H})$ are *pure Shimura data* [P1, Def. 2.1]. Their *reflex field* [P1, Sect. 11.1] equals $\mathbb{Q}$. Given that $Z(G) = \mathbb{G}_{m,\mathbb{Q}}$, the Shimura data $(G, \mathcal{H})$ satisfy condition (+) from [Wi2, Sect. 5].

Let us now fix additional data: (A) an open compact sub-group K of $G(\mathbb{A}_f)$ which is neat [P1, Sect. 0.6], (B) a triplet $(k_1, k_2, r) \in \mathbb{Z}^3$ satisfying the above congruence

$$r \equiv k_1 + k_2 \mod 2 ,$$

and in addition,

$$k_1 \geq k_2 \geq 0 .$$

In other words, the character $\underline{\alpha} := \alpha(k_1, k_2, r)$ is dominant.

These data are used as follows. The Shimura variety $M^K := M^K(G, \mathcal{H})$ is smooth over $\mathbb{Q}$. This is the Siegel threefold of level K. According to [P1, Thm. 11.16], it admits an interpretation as modular space of Abelian surfaces with additional structures. In particular, there is a universal family B of Abelian surfaces over M^K.

The following result holds in the general context of (smooth) Shimura varieties of PEL-type.

Theorem 1.4 ([A, Thm. 8.6])**.** *There is a* $\mathbb{Q}$*-linear tensor functor*

$$\widetilde{\mu} : \mathrm{Rep}(G) \longrightarrow CHM^s\big(M^K\big)_{\mathbb{Q}}$$

from the Tannakian category $\mathrm{Rep}(G)$ *of algebraic representations of* G *in finite dimensional* $\mathbb{Q}$*-vector spaces to the* $\mathbb{Q}$*-linear category* $CHM^s(M^K)_{\mathbb{Q}}$ *of* smooth Chow motives *over* M^K *[Lv, Def. 5.16]. It has the following properties.*

(a) The composition of $\widetilde{\mu}$ *with the cohomological Hodge theoretic realization is isomorphic to the* canonical construction *functor* $\mu_{\mathbf{H}}$ *(e.g. [Wi1, Thm. 2.2]) to the category of admissible graded-polarizable variations of Hodge structure on* $M^K_{\mathbb{C}}$*.*

(b) The composition of $\widetilde{\mu}$ *with the cohomological* ℓ*-adic realization is isomorphic to the* canonical construction *functor* μ_ℓ *(e.g. [Wi1, Chap. 4]) to the category of lisse* ℓ*-adic sheaves on* M^K*.*

(c) The functor $\widetilde{\mu}$ *commutes with Tate twists.*

(d) The functor $\widetilde{\mu}$ *maps the representation* V *to the dual of the Chow motive* $\pi^1_*\mathbb{1}_{\mathrm{B}}$ *over* M^K*.*

Here, we denote by $\pi^m_*\mathbb{1}_{\mathrm{B}}$ the m-th *Chow-Künneth component* of the Chow motive $\pi_*\mathbb{1}_{\mathrm{B}}$ over M^K [DM, Thm. 3.1].

Proof of Theorem 1.4. Parts (a), (c) and (d) are identical to [A, Thm. 8.6].

As for part (b), repeat the proof of [loc. cit.], observing that the ℓ-adic analogue of [A, Prop. 8.5] holds (the base change to $\mathbb{Q}_\ell$ of the sub-group G_1 of G coincides with the Lefschetz group). **q.e.d.**

Given that the representation on V is faithful, it follows that any object in the image of $\widetilde{\mu}$ is isomorphic to a direct sum of direct factors of Tate twists of the Chow motive $\pi_{n_i,*}\mathbb{1}_{\mathrm{B}^{n_i}}$ associated to B^{n_i}, for suitable $n_i \in \mathbb{N}$, where $\pi_{n_i} : \mathrm{B}^{n_i} \to M^K$ denotes the n_i-fold fibre product of B over M^K.

Definition 1.5. (a) Denote by $V_{\underline{\alpha}} \in \mathrm{Rep}(G)$ the irreducible representation of highest weight $\underline{\alpha}$.
(b) Define ${}^{\underline{\alpha}}\mathcal{V} \in CHM^s(M^K)_{\mathbb{Q}} \subset CHM(M^K)_{\mathbb{Q}}$ as

$${}^{\underline{\alpha}}\mathcal{V} := \widetilde{\mu}(V_{\underline{\alpha}}) .$$

Given that $V_{\underline{\alpha}}$ is of weight r, the cohomological realizations of ${}^{\underline{\alpha}}\mathcal{V}$ equal zero in (classical, *i.e.*, non-perverse) degrees $\neq r$, and $\mu_{\mathbf{H}}(V_{\underline{\alpha}})$ (in the Hodge theoretic setting) resp. $\mu_\ell(V_{\underline{\alpha}})$ (in the ℓ-adic setting) in degree r.

Denote by $j : M^K \hookrightarrow (M^K)^*$ the open immersion of M^K into its Satake–(Baily–Borel) compactification, and by $i : \partial(M^K)^* \hookrightarrow (M^K)^*$ its complement. Here is our main result.

Theorem 1.6. *(a) The motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V} \in DM_{\mathrm{B},c}(\partial(M^K)^*)$ is* of Abelian type *[Wi9, Def. 3.1].*
(b) The motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ is without weights

$$-k+1, -k+2, \ldots, k,$$

where $k := \min(k_1 - k_2, k_2)$. Both weights $-k$ and $k+1$ do *occur in $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$. In particular, ${}^{\underline{\alpha}}\mathcal{V}$ belongs to the sub-category $CHM(M^K)_{\mathbb{Q},\partial w \neq 0,1}$ of $CHM(M^K)_{\mathbb{Q}}$ if and only if $\underline{\alpha}$ is regular.*

Theorem 1.6 should be compared to [Wi5, Thm. 3.5], [Wi7, Thm. 3.8], and [C, Thm. 4.10, Prop. 4.14, Prop. 4.16], which treat the cases of Hilbert–Blumenthal varieties, of Picard surfaces, and of Picard varieties of arbitrary dimension, respectively.

Theorem 1.6 will be proved in Section 2. For the rest of the present section, assume that $k = \min(k_1 - k_2, k_2) \geq 1$, *i.e.*, $k_1 > k_2 > 0$. Given that according to Theorem 1.6 (b), the motive ${}^{\underline{\alpha}}\mathcal{V}$ belongs to $CHM(M^K)_{\mathbb{Q},\partial w \neq 0,1}$, [Wi9, Def. 2.7] can be applied. Therefore, the *intersection motive of M^K relative to $(M^K)^*$ with coefficients in ${}^{\underline{\alpha}}\mathcal{V}$* is at our disposal: it equals

$$d_* \, j_{!*} \, {}^{\underline{\alpha}}\mathcal{V} \in CHM(\mathbb{Q})_{\mathbb{Q}} \, ,$$

where $d : (M^K)^* \to \mathbf{Spec}\, \mathbb{Q}$ is the structure morphism of $(M^K)^*$. By abuse of language, let us abbreviate, and refer to $d_* \, j_{!*} \, {}^{\underline{\alpha}}\mathcal{V}$ as the *intersection motive with coefficients in* ${}^{\underline{\alpha}}\mathcal{V}$. Let us list the main corollaries of Theorem 1.6.

Corollary 1.7. *Denote by d and $\tilde{d}$ the structure morphisms of $(M^K)^*$ and M^K, respectively. Assume $k_1 > k_2 > 0$,* i.e., *$k \geq 1$.*
(a) The motive $\tilde{d}_! {}^{\underline{\alpha}}\mathcal{V} \in DM_{\mathrm{B},c}(\mathbb{Q})$ is without weights $-k, -k+1, \ldots, -1$, and the motive $\tilde{d}_* {}^{\underline{\alpha}}\mathcal{V} \in DM_{\mathrm{B},c}(\mathbb{Q})$ is without weights $1, 2, \ldots, k$. More precisely, the exact triangles

$$d_* i_* i^* \, j_{!*} \, {}^{\underline{\alpha}}\mathcal{V}[-1] \longrightarrow \tilde{d}_! {}^{\underline{\alpha}}\mathcal{V} \longrightarrow d_* \, j_{!*} \, {}^{\underline{\alpha}}\mathcal{V} \longrightarrow d_* i_* i^* \, j_{!*} \, {}^{\underline{\alpha}}\mathcal{V}$$

and

$$d_* \, j_{!*} \, {}^{\underline{\alpha}}\mathcal{V} \longrightarrow \tilde{d}_* {}^{\underline{\alpha}}\mathcal{V} \longrightarrow d_* i_* i^! \, j_{!*} \, {}^{\underline{\alpha}}\mathcal{V}[1] \longrightarrow d_* \, j_{!*} \, {}^{\underline{\alpha}}\mathcal{V}[1]$$

are weight filtrations (of $\tilde{d}_! {}^{\underline{\alpha}}\mathcal{V}$) avoiding weights $-k, -k+1, \ldots, -1$, and (of $\tilde{d}_* {}^{\underline{\alpha}}\mathcal{V}$) avoiding weights $1, 2, \ldots, k$, respectively.

(b) The intersection motive $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V} \in CHM(\mathbb{Q})_{\mathbb{Q}}$ behaves functorially with respect to both $\tilde{d}_! {}^{\underline{\alpha}}\mathcal{V}$ and $\tilde{d}_* {}^{\underline{\alpha}}\mathcal{V}$. In particular, any endomorphism of $\tilde{d}_! {}^{\underline{\alpha}}\mathcal{V}$ or of $\tilde{d}_* {}^{\underline{\alpha}}\mathcal{V}$ induces an endomorphism of $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}$.

(c) Let $\tilde{d}_! {}^{\underline{\alpha}}\mathcal{V} \to N \to \tilde{d}_* {}^{\underline{\alpha}}\mathcal{V}$ be a factorization of the morphism $d_* m : \tilde{d}_! {}^{\underline{\alpha}}\mathcal{V} \to N \to \tilde{d}_* {}^{\underline{\alpha}}\mathcal{V}$ through a Chow motive $N \in CHM(\mathbb{Q})_{\mathbb{Q}}$. Then the intersection motive $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}$ is canonically identified with a direct factor of N, with a canonical direct complement.

Proof. Given Theorem 1.6, parts (a), (b) and (c) follow from [Wi9, Thm. 2.4], [Wi9, Thm. 2.5] and [Wi9, Thm. 2.6], respectively. **q.e.d.**

The equivariance statement from Corollary 1.7 (b) applies in particular to endomorphisms coming from the *Hecke algebra* $\mathfrak{H}(K, G(\mathbb{A}_f))$ associated to the neat open compact sub-group K of $G(\mathbb{A}_f)$. Recall that by what was said earlier, the relative Chow motive ${}^{\underline{\alpha}}\mathcal{V}$ is a direct factor of a Tate twist of $\pi_{N,*}\mathbb{1}_{\mathrm{B}^N}$, where $\pi_N : \mathrm{B}^N \to M^K$ denotes the N-fold fibre product of the universal Abelian scheme B over M^K.

Corollary 1.8. *Assume $k \geq 1$. Then every element of the Hecke algebra $\mathfrak{H}(K, G(\mathbb{A}_f))$ acts naturally on the intersection motive $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}$.*

Proof. Let $T \in \mathfrak{H}(K, G(\mathbb{A}_f))$. According to Corollary 1.7 (b), it suffices to show that T acts on $\tilde{d}_* {}^{\underline{\alpha}}\mathcal{V}$.

To do so, we refer to [Wi8, pp. 591–592]. **q.e.d.**

Here is a particular feature of Corollary 1.7 (c).

Corollary 1.9. *Assume $k \geq 1$, and let $\widetilde{\mathrm{B}^N}$ be any smooth compactification of B^N. Then the intersection motive $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}$ is canonically a direct factor of a Tate twist of the Chow motive $b_*\mathbb{1}_{\widetilde{\mathrm{B}^N}}$ ($b :=$ the structure morphism of the $\mathbb{Q}$-scheme $\widetilde{\mathrm{B}^N}$), with a canonical direct complement.*

Remark 1.10. When $r \geq 0$, then according to [A, Lemma 4.13], the Chow motive ${}^{\underline{\alpha}}\mathcal{V}$ is a direct factor of $\pi_{N,*}\mathbb{1}_{\mathrm{B}^N}$ (no Tate twist needed).

In this context, let us recall [Wi9, Cor. 2.10]: the intersection motive $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}$ is canonically dual to the $e_{\underline{\alpha}}$-part of the *interior motive* of B^N, where $e_{\underline{\alpha}}$ is the idempotent endomorphism corresponding to the direct factor ${}^{\underline{\alpha}}\mathcal{V}$ of $\pi_{N,*}\mathbb{1}_{\mathrm{B}^N}$.

Corollary 1.11. *Assume $k \geq 1$,* i.e.*, that $\underline{\alpha}$ is regular. Then for all $n \in \mathbb{Z}$, the natural maps*

$$H^n\big((M^K)^*(\mathbb{C}), j_{!*}\, \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) \longrightarrow H^n\big(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big)$$

(in the Hodge theoretic setting) and

$$H^n\big((M^K)^* \times_{\mathbb{Q}} \bar{\mathbb{Q}}, j_{!*}\, \mu_\ell(V_{\underline{\alpha}})\big) \longrightarrow H^n\big(M^K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})\big)$$

(in the ℓ-adic setting) are injective. Dually,

$$H^n_c\bigl(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})\bigr) \longrightarrow H^n\bigl((M^K)^*(\mathbb{C}), j_{!*}\,\mu_{\mathbf{H}}(V_{\underline{\alpha}})\bigr)$$

and

$$H^n_c\bigl(M^K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})\bigr) \longrightarrow H^n\bigl((M^K)^* \times_{\mathbb{Q}} \bar{\mathbb{Q}}, j_{!*}\,\mu_\ell(V_{\underline{\alpha}})\bigr)$$

are surjective. In other words, the natural maps from intersection cohomology to interior cohomology with coefficients in $\mu_{\mathbf{H}}(V_{\underline{\alpha}})$, resp. $\mu_\ell(V_{\underline{\alpha}})$ are isomorphisms.

Proof. Given Theorem 1.6 (b), we may quote [Wi9, Rem. 2.13 (c)]. **q.e.d.**

Corollary 1.11 is already known. Indeed, according to [MT, Prop. 1], the result generalizes to Siegel varieties of arbitrary dimension. (However, the proof of [loc. cit.] is analytic.)

Remark 1.12. By [Wi3, Thm. 4.14], control of the reduction of *some* compactification of B^N implies control of certain properties of the ℓ-adic realization of the intersection motive $d_*\, j_{!*}\, {}^{\underline{\alpha}}\mathcal{V}$. According to [FC, Thm. VI.1.1], there exists a smooth compactification of B^N having good reduction at each prime number p not dividing the level n of K.

[Wi3, Thm. 4.14] then yields the following: (a) for all prime numbers p not dividing n, the p-adic realization of $d_*\, j_{!*}\, {}^{\underline{\alpha}}\mathcal{V}$ is crystalline, (b) if furthermore p and ℓ are different, then the ℓ-adic realization of $d_*\, j_{!*}\, {}^{\underline{\alpha}}\mathcal{V}$ is unramified at p.

Corollary 1.13. *Assume $k \geq 1$. Let p be a prime number not dividing the level of K. Let ℓ be different from p. Then the characteristic polynomials of the following coincide: (1) the action of Frobenius ϕ on the ϕ-filtered module associated to the (crystalline) p-adic realization of the intersection motive $d_*\, j_{!*}\, {}^{\underline{\alpha}}\mathcal{V}$, (2) the action of a geometrical Frobenius automorphism at p on the (unramified) ℓ-adic realization of $d_*\, j_{!*}\, {}^{\underline{\alpha}}\mathcal{V}$.*

Proof. According to our construction, and what was recalled in Remark 1.12, there is a smooth and proper scheme $\bar{\mathcal{B}}^N_{\mathbb{F}_p}$ over the field $\mathbb{F}_p$, and an endomorphism $e_{\underline{\alpha},p}$ of the Chow motive associated to $\bar{\mathcal{B}}^N_{\mathbb{F}_p}$, whose images in the endomorphism rings of the realizations of $\bar{\mathcal{B}}^N_{\mathbb{F}_p}$ become idempotent; furthermore, the latter are projectors onto the realizations of $d_*\, j_{!*}\, {}^{\underline{\alpha}}\mathcal{V}$. The claim thus follows from [KM, Thm. 2. 2)]. **q.e.d.**

2 Proof of the main result

We keep the notation of the preceding section. In order to prove Theorem 1.6, the idea is to apply the criterion from [Wi9, Cor. 3.6].

In order to check the hypotheses of [loc. cit.], we first need to fix a finite stratification Φ of $\partial(M^K)^*$ by locally closed sub-schemes. The canonical choice would be the restriction Φ' to $\partial(M^K)^*$ of the natural (finite) stratification of $(M^K)^*$ from [P1, Main Theorem 12.3 (c)], in other words, all the strata of $(M^K)^*$ except the open one, *i.e.*, except M^K. According to [Wi8, Lemma 8.2 (a)], Φ' is *good*, meaning that the closure of every stratum is a union of strata. Furthermore [Wi8, Lemma 8.2 (b)], all strata, denoted $i_g(M^{\pi_1(K_1)})$, are smooth over $\mathbb{Q}$ (recall that K is assumed neat, and that $(G,\mathcal{H})$ satisfy condition (+)), hence regular. The same is therefore true for the following coarser stratification $\Phi = \{0,1\}$ of $\partial(M^K)^*$: denote by $i_0 : Z_0 \hookrightarrow \partial(M^K)^*$ the disjoint union of all closed strata of Φ', and by $i_1 : Z_1 \hookrightarrow \partial(M^K)^*$ the disjoint union of all strata of Φ', which are open in $\partial(M^K)^*$. Indeed, according to [P1, Sect. 6.3, Ex. 4.25 (with $g = 2$)],

$$\partial(M^K)^* = Z_0 \coprod Z_1 \ ;$$

more precisely, Z_0 is of dimension zero, and Z_1 of dimension one (hence so is the whole of $\partial(M^K)^*$). Let us refer to Z_0 as the *Siegel stratum*, and to Z_1 as the *Klingen stratum* of $\partial(M^K)^*$.

Definition 2.1. An object $V \in DM_{\text{Б},c}(\partial(M^K)^*)$ is a *motive of Abelian type over* $\partial(M^K)^*$, and the stratification Φ is *adapted to* V if the following holds: the motive V belongs to the strict, full, dense, $\mathbb{Q}$-linear triangulated sub-category $DM^{Ab}_{\text{Б},c,\Phi}(\partial(M^K)^*)$ generated by the images under π_* of $\mathfrak{S}$*-constructible Tate motives over* $S(\mathfrak{S})$ [Wi8, Def. 4.6 (a)], where

$$\pi : S(\mathfrak{S}) \longrightarrow \partial(M^K)^*$$

runs through the morphisms *of Abelian type* [Wi8, p. 579] with target equal to $\partial(M^K)^*$.

Theorem 2.2. *Let $\underline{\alpha} = \alpha(k_1, k_2, r)$, with $(k_1, k_2, r) \in \mathbb{Z}^3$ such that*

$$r \equiv k_1 + k_2 \mod 2 \quad \textit{and} \quad k_1 \geq k_2 \geq 0 \ ,$$

and consider ${}^{\underline{\alpha}}\mathcal{V} = \widetilde{\mu}(V_{\underline{\alpha}}) \in CHM(M^K)_{\mathbb{Q}}$. The motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ belongs to the full sub-category $DM^{Ab}_{\text{Б},c,\Phi}(\partial(M^K)^*)$ of $DM_{\text{Б},c}(\partial(M^K)^*)$. In other words, it is of Abelian type, and Φ is adapted to $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$.

Proof. As recalled earlier, the relative Chow motive ${}^{\underline{\alpha}}\mathcal{V}$ belongs to the strict, full, dense, $\mathbb{Q}$-linear triangulated sub-category

$$\pi_{N,*} DMT\big(\mathrm{B}^N\big)^{\natural}_{\mathbb{Q}}$$

of $DM_{\text{Б},c}(M^K)$ generated by the images under $\pi_{N,*}$ of the category of Tate motives over B^N. Here as before, $\pi_N : \mathrm{B}^N \to M^K$ denotes the N-fold fibre product of the universal Abelian scheme B over M^K.

The latter equals the projection from a *mixed Shimura variety*: indeed [P1, Ex. 2.7], the representation V of G is of Hodge type $\{(-1,0),(0,-1)\}$.

The same is then true for the r-th power V^N of V. By [P1, Prop. 2.17], this allows for the construction of the *unipotent extension* $(P^N, \mathfrak{X}^N)$ of $(G, \mathcal{H})$ by V^N. The pair $(P^N, \mathfrak{X}^N)$ constitute *mixed Shimura data* [P1, Def. 2.1]. By construction, they come endowed with a morphism $\pi_N : (P^N, \mathfrak{X}^N) \to (G, \mathcal{H})$ of Shimura data, identifying $(G, \mathcal{H})$ with the pure Shimura data underlying $(P^N, \mathfrak{X}^N)$. In particular, $(P^N, \mathfrak{X}^N)$ also satisfy condition (+). Now [P1, Thm. 11.18] there is an open compact neat subgroup K_N of $P^N(\mathbb{A}_f)$, whose image under π_N equals K, such that B^N is identified with the *mixed Shimura variety* $M^{K_N} := M^{K_N}(P^N, \mathfrak{X}^N)$, and such the morphism $M^{K_N} \to M^K$ induced by the morphism π_N of Shimura data is identified with the structure morphism of B^N.

Choose a smooth *toroidal compactification* $M^{K_N}(\mathfrak{S}) := M^{K_N}(P^N, \mathfrak{X}^N, \mathfrak{S})$ of M^{K_N}, associated to a K_N*-admissible complete cone decomposition* $\mathfrak{S}$ [P1, Sect. 6.4]. Then [P1, proof of Thm. 9.21], modulo a suitable refinement of $\mathfrak{S}$, the natural stratification of $M^{K_N}(\mathfrak{S})$, also denoted $\mathfrak{S}$, satisfies the conclusions of [Wi8, Lemma 8.1], *i.e.*, it is good, and the closures of all strata are regular. Note that the unique open stratum equals M^{K_N}. According to [P1, Sect. 6.24, Main Theorem 12.4 (b)], the morphism $\pi_N : \mathrm{B}^N = M^{K_N} \to M^K$ extends to a proper, surjective morphism $M^{K_N}(\mathfrak{S}) \to (M^K)^*$, still denoted π_N. From the description given in [P1, Sect. 7.3], one sees that π_N is a morphism of stratifications.

According to [Wi8, Cor. 4.10 (b), Rem. 4.7], the category

$$\pi_{N,*} DMT_{\mathfrak{S}}\big(M^{K_N}(\mathfrak{S})\big)^{\natural}_{\mathbb{Q}}$$

is obtained by gluing $\pi_{N,*} DMT(\mathrm{B}^N)^{\natural}_{\mathbb{Q}}$ and $\pi_{N,*} DMT_{\mathfrak{S}}(\pi_N^{-1}(\partial(M^K)^*))^{\natural}_{\mathbb{Q}}$. In particular,

$$i^* j_* {}^{\alpha}\mathcal{V} \in \pi_{N,*} DMT_{\mathfrak{S}}\big(\pi_N^{-1}\big(\partial(M^K)^*\big)\big)^{\natural}_{\mathbb{Q}} \ .$$

But π_N is of Abelian type [Wi8, Lemma 8.4]; therefore,

$$\pi_{N,*} DMT_{\mathfrak{S}}\big(\pi_N^{-1}\big(\partial(M^K)^*\big)\big)^{\natural}_{\mathbb{Q}} \subset DM^{Ab}_{\mathrm{B},c,\Phi}(\partial(M^K)^*) \ .$$

q.e.d.

Next, we collect information on the restriction of $i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V})$ to the strata Z_0 and Z_1. The following is essentially due to Lemma [Lm, Sect. 4].

Theorem 2.3. *Let ℓ be a prime number.*
(a) For all integers $n \le r+2$, the perverse cohomology sheaf

$$H^n i_0^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V})$$

on Z_0 is of weights $\le n - (k_1 - k_2)$. The perverse cohomology sheaf

$$H^{r+2} i_0^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V})$$

is non-zero, and pure of weight $(r+2) - (k_1 - k_2)$.

(b) For all integers $n \le r+2$, the perverse cohomology sheaf

$$H^n i_1^* i^* j_* R_{\ell,M^K}({}^{\alpha}\mathcal{V})$$

on Z_1 is of weights $\le n - k_2$. The perverse cohomology sheaf

$$H^{r+2} i_1^* i^* j_* R_{\ell,M^K}({}^{\alpha}\mathcal{V})$$

is non-zero, and pure of weight $(r+2) - k_2$.

In order to prepare the proof of Theorem 2.3, recall [P1, Ex. 4.25] that Z_0 and Z_1 correspond bijectively to the $G(\mathbb{Q})$-conjugacy classes of proper *rational boundary components* [P1, Sect. 4.11] of $(G, \mathcal{H})$. Indeed, the group $G(\mathbb{Q})$ acts transitively on the set of totally isotropic sub-spaces of V of a given, strictly positive dimension.

We already fixed a basis (e_1, e_2, e_3, e_4) of V, in which our symplectic bilinear form J acquires the 4×4-matrix

$$\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} ,$$

which we equally denoted by J. The sub-spaces V_0' and V_1' generated by $\{e_1, e_2\}$ and $\{e_1\}$, respectively, are both totally isotropic.

Following [P1, Ex. 4.25], we put $Q_m := Stab_G(V_m')$, $m = 0, 1$. Let P_m denote the normal sub-group of Q_m underlying the rational boundary component $(P_m, \mathfrak{X}_m)$ giving rise to Z_m [P1, Sect. 4.7], and W_m its unipotent radical (which equals the unipotent radical of Q_m). Then, still according to [P1, Ex. 4.25], Q_0 equals

$$\left\{ \begin{pmatrix} q \cdot A & A \cdot M \\ 0 & {}^tA^{-1} \end{pmatrix} ,\ q \in \mathbb{G}_{m,\mathbb{Q}} , A \in GL_{2,\mathbb{Q}} , {}^tM = M \right\} ,$$

$$P_0 = \left\{ \begin{pmatrix} q \cdot I_2 & M \\ 0 & I_2 \end{pmatrix} ,\ q \in \mathbb{G}_{m,\mathbb{Q}} , {}^tM = M \right\} ,$$

$$W_0 = \left\{ \begin{pmatrix} I_2 & M \\ 0 & I_2 \end{pmatrix} ,\ {}^tM = M \right\} ,$$

while Q_1 equals

$$\left\{ \begin{pmatrix} a & aq^{-1}(bu+dw) & v & aq^{-1}(cu+ew) \\ 0 & b & w & c \\ 0 & 0 & a^{-1}q & 0 \\ 0 & d & -u & e \end{pmatrix} ,\ a , be - cd = q \in \mathbb{G}_{m,\mathbb{Q}} \right\} ,$$

$$P_1 = \left\{ \begin{pmatrix} be-cd & bu+dw & v & cu+ew \\ 0 & b & w & c \\ 0 & 0 & 1 & 0 \\ 0 & d & -u & e \end{pmatrix} \, , \; be-cd \in \mathbb{G}_{m,\mathbb{Q}} \right\} ,$$

$$W_1 = \left\{ \begin{pmatrix} 1 & u & v & w \\ 0 & 1 & w & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u & 1 \end{pmatrix} \right\} .$$

Observe that $Q_0 \cap Q_1$ equals the Borel sub-group of G stabilizing the flag $V_1' \subset V_0'$, and that both Q_0 and Q_1 contain the fixed maximal split torus

$$T = \{ \mathrm{diag}(a, b, a^{-1}q, b^{-1}q) \ , \ a, b, q \in \mathbb{G}_{m,\mathbb{Q}} \} \ .$$

In particular, T is canonically identified with a maximal $\mathbb{Q}$-split torus of the reductive group Q_m/W_m, for $m = 0, 1$. Given a (representation-theoretic) weight $\alpha : T \to \mathbb{G}_{m,\mathbb{Q}}$, let us denote by α_m the same application, but with T seen as a sub-group of Q_m/W_m, $m = 0, 1$.

Note that

$$R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V}) = \mu_\ell(V_{\underline{\alpha}})[-r] \ .$$

In order to determine the classical cohomology objects $R^n i_m^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$, for $m = 0, 1$, and $n \in \mathbb{Z}$, one applies the following standard strategy. (1) By Pink's Theorem [P2, Thm. (5.3.1)], the restriction of $R^n i_m^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ to any individual stratum Z' of Φ' contributing to Z_m equals

$$R^n i_m^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z'} = \bigoplus_{p+q=n} \mu_{\ell,Z'} \big(H^p \big(H_C/K_W, H^q \big(Lie(W_m), V_{\underline{\alpha}} \big) \big) \big) \ .$$

Here, H_C/K_W is an arithmetic sub-group (depending on Z') of C_m/W_m [P2, Sect. (5.2)], where C_m is the identity component of the Zariski closure of the centralizer in $Q_m(\mathbb{Q})$ of the rational boundary component $(P_m, \mathfrak{X}_m)$ [P2, Sect. (3.7)], and $\mu_{\ell,Z'}$ is the canonical construction functor to the category of lisse ℓ-adic sheaves on Z'. (2) Apply Kostant's Theorem [V, Thm. 3.2.3], in order to identify $H^q(Lie(W_m), V_{\underline{\alpha}})$ as a representation of the reductive group Q_m/W_m; this allows in particular to obtain its weights, and gives potential information concerning cohomology of H_C/K_W with coeffients in $H^q(Lie(W_m), V_{\underline{\alpha}})$.

The Hodge theoretic analogue of the above strategy yields the cohomology objects of $i_m^* i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}})_{|Z'}$; this was made explicit in [Lm, Sect. 4]. Since steps (2) of the ℓ-adic and the Hodge theoretic strategies are identical, we may use the computations from [loc. cit.] in our setting.

Proposition 2.4 ([Lm, Sect. 4.3]). *Let $\underline{\alpha} = \alpha(k_1, k_2, r)$, with $(k_1, k_2, r) \in \mathbb{Z}^3$ such that*

$$r \equiv k_1 + k_2 \mod 2 \quad \textit{and} \quad k_1 \geq k_2 \geq 0 \; .$$

(a) For $m = 0, 1$, we have

$$H^q(Lie(W_m), V_{\underline{\alpha}}) = 0$$

whenever $q < 0$ or $q > 3$. If $0 \leq q \leq 3$, then the Q_m/W_m-representation $H^q(Lie(W_1), V_{\underline{\alpha}})$ is (non-zero and) irreducible.
(b) The highest (representation-theoretic) weight of $H^q(Lie(W_0), V_{\underline{\alpha}})$, $0 \leq q \leq 3$, is

$$\alpha_0(k_1, k_2, r) \quad \textit{for} \quad q = 0 \; ,$$

$$\alpha_0(k_1, -k_2 - 2, r) \quad \textit{for} \quad q = 1 \; ,$$

$$\alpha_0(k_2 - 1, -k_1 - 3, r) \quad \textit{for} \quad q = 2 \; ,$$

$$\alpha_0(-k_2 - 3, -k_1 - 3, r) \quad \textit{for} \quad q = 3 \; .$$

(c) The highest (representation-theoretic) weight of $H^q(Lie(W_1), V_{\underline{\alpha}})$, $0 \leq q \leq 3$, is

$$\alpha_1(k_1, k_2, r) \quad \textit{for} \quad q = 0 \; ,$$

$$\alpha_1(k_2 - 1, k_1 + 1, r) \quad \textit{for} \quad q = 1 \; ,$$

$$\alpha_1(-k_2 - 3, k_1 + 1, r) \quad \textit{for} \quad q = 2 \; ,$$

$$\alpha_1(-k_1 - 4, k_2, r) \quad \textit{for} \quad q = 3 \; .$$

Proof. Note that given our normalization, we have

$$\alpha(k_1, k_2, r) = \lambda(k_1, k_2, -r)$$

in the notation of [Lm, top of p. 87].

Part (a) follows from Kostant's Theorem, and from the following fact (see [Lm, proof of Lemma 4.8] and [Lm, proof of Lemma 4.10]), valid for both $m = 0$ and $m = 1$: the set of Weyl representatives for Q_m contains no element of length < 0 or > 3, and exactly one element of respective lengths 0, 1, 2 and 3.

As for part (c), we refer to [Lm, proof of Lemma 4.10].

[Lm, proof of Lemma 4.8] contains the complete setting for the application of Kostant's Theorem for $m = 0$, but makes it explicit only for $H^2(Lie(W_0), V_{\underline{\alpha}})$ and $H^3(Lie(W_0), V_{\underline{\alpha}})$. The reader will have no difficulty to fill in the missing information needed for part (b). **q.e.d.**

Note that both Q_0/W_0 and Q_1/W_1 are isomorphic to the direct product $\mathbb{G}_{m,\mathbb{Q}} \times_{\mathbb{Q}} GL_{2,\mathbb{Q}}$. More precisely,

$$Q_0/W_0 = P_0/W_0 \times_{\mathbb{Q}} GL_{2,\mathbb{Q}} = \mathbb{G}_{m,\mathbb{Q}} \times_{\mathbb{Q}} GL_{2,\mathbb{Q}} \ ,$$

the identification given by sending the class of a matrix

$$\begin{pmatrix} q \cdot A & A \cdot M \\ 0 & {}^t A^{-1} \end{pmatrix}$$

to the pair (q, A), and

$$Q_1/W_1 = P_1/W_1 \times_{\mathbb{Q}} \mathbb{G}_{m,\mathbb{Q}} = GL_{2,\mathbb{Q}} \times_{\mathbb{Q}} \mathbb{G}_{m,\mathbb{Q}} \ ,$$

the identification given by sending the class of a matrix

$$\begin{pmatrix} a & aq^{-1}(bu + dw) & v & aq^{-1}(cu + ew) \\ 0 & b & w & c \\ 0 & 0 & a^{-1}q & 0 \\ 0 & d & -u & e \end{pmatrix}$$

to the pair

$$\left(\begin{pmatrix} b & c \\ d & e \end{pmatrix}, aq^{-1} \right) \ .$$

The restriction of the inverse identification to maximal split tori sends

$$\left(q, \begin{pmatrix} x & 0 \\ 0 & x^{-1}y \end{pmatrix} \right) \in P_0/W_0 \times_{\mathbb{Q}} GL_{2,\mathbb{Q}}$$

to

$$\mathrm{diag}(qx, qx^{-1}y, x^{-1}, xy^{-1}) \in T \subset Q_0/W_0$$

for $m = 0$, and

$$\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}q \end{pmatrix}, y \right) \in P_1/W_1 \times_{\mathbb{Q}} \mathbb{G}_{m,\mathbb{Q}}$$

to

$$\mathrm{diag}(yq, x, y^{-1}, x^{-1}q) \in T \subset Q_1/W_1$$

for $m = 1$.

In the following, the reader will be particularly careful not to confuse two notions of *weight* associated to representations of reductive groups: the highest weights in the sense of representation theory (*e.g.*, those occurring in Kostant's Theorem), when the representation is irreducible, and the weights as determined by the action of the weight cocharacter [P1, Sect. 1.3], when the group underlies Shimura data.

Corollary 2.5. *(a) The irreducible Q_0/W_0-representations*

$$H^0(Lie(W_0), V_{\underline{\alpha}}) \ , \ H^1(Lie(W_0), V_{\underline{\alpha}}) \ , \textit{and} \ H^2(Lie(W_0), V_{\underline{\alpha}})$$

are regular, except when $q = 0$ and $k_1 = k_2$, in which case $H^0(Lie(W_0), V_{\underline{\alpha}})$ factors through the quotient $\mathbb{G}_{m,\mathbb{Q}} \times_{\mathbb{Q}} \mathbb{G}_{m,\mathbb{Q}}$ of the group

$$Q_0/W_0 = \mathbb{G}_{m,\mathbb{Q}} \times_{\mathbb{Q}} GL_{2,\mathbb{Q}}$$

via *the determinant on the factor $GL_{2,\mathbb{Q}}$. The restriction to $SL_{2,\mathbb{Q}} \subset GL_{2,\mathbb{Q}}$ of $H^1(Lie(W_0), V_{\underline{\alpha}})$ is of highest (representation-theoretic) weight $k_1 + k_2 + 2$. The restriction to P_0/W_0 of $H^0(Lie(W_0), V_{\underline{\alpha}})$ is of weight $(r+1)-(k_1+k_2)-1$, and the restriction of $H^1(Lie(W_0), V_{\underline{\alpha}})$ is of weight $(r+2)-(k_1-k_2)$.*
(b) The restriction to P_1/W_1 of $H^0(Lie(W_1), V_{\underline{\alpha}})$ is of weight $(r+1)-k_1-1$, and the restriction of $H^1(Lie(W_1), V_{\underline{\alpha}})$ is of weight $(r+2)-k_2-1$.

Proof. (a): Given the above identifications, the weight $\alpha_0(n_1, n_2, r)$ on T maps

$$\left(q, \begin{pmatrix} x & 0 \\ 0 & x^{-1}y \end{pmatrix} \right) \in P_0/W_0 \times_{\mathbb{Q}} GL_{2,\mathbb{Q}}$$

to

$$\alpha_0(n_1, n_2, r)(\mathrm{diag}(qx, qx^{-1}y, x^{-1}, xy^{-1})) = x^{n_1-n_2} y^{n_2} q^{-\frac{r-n_1-n_2}{2}} \ .$$

In particular, the restriction of $\alpha_0(n_1, n_2, r)$ to $T \cap SL_{2,\mathbb{Q}}$ corresponds to the integer $n_1 - n_2$. The first and the second claim thus follow from Proposition 2.4 (b).

The weight cocharacter $\mathbb{G}_{m,\mathbb{Q}} \to P_0/W_0 = \mathbb{G}_{m,\mathbb{Q}}$ maps z to z^2 [P1, Ex. 4.25, Ex. 2.8]. Its composition with the inclusion into T, and with $\alpha_0(n_1, n_2, r)$ yields

$$\mathbb{G}_{m,\mathbb{Q}} \longrightarrow \mathbb{G}_{m,\mathbb{Q}} \ , \ z \longmapsto z^{-r+n_1+n_2} \ .$$

The third claim thus follows from Proposition 2.4 (b), and from the normalization of weights of representations [P1, Sect. 1.3].

(b): The weight cocharacter $\mathbb{G}_{m,\mathbb{Q}} \to P_1/W_1 = GL_{2,\mathbb{Q}}$ maps z to

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

[P1, Ex. 4.25, Ex. 2.8]. Given the above identifications, its composition with the inclusion into T maps z to $\mathrm{diag}(z^2, z, 1, z)$. Further composition with $\alpha_1(n_1, n_2, r)$ then yields

$$\mathbb{G}_{m,\mathbb{Q}} \longrightarrow \mathbb{G}_{m,\mathbb{Q}} \ , \ z \longmapsto z^{-r+n_1} \ .$$

The claim thus follows from Proposition 2.4 (c). **q.e.d.**

In order to complete the ingredients needed for the computation of the $R^n i_m^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ according to the strategy (1), (2) sketched ealier in this section, observe that the group H_C/K_W associated to a stratum Z' of Z_m is a neat arithmetic sub-group of $GL_2(\mathbb{Q})$ for $m = 0$ [Lm, proof of Lemma 4.8], hence of $SL_2(\mathbb{Q})$. In particular, it is of cohomological dimension one. For $m = 1$, the group H_C/K_W, being a neat arithmetic sub-group of $\mathbb{G}_m(\mathbb{Q})$, is trivial [Lm, proof of Lemma 4.10].

Remark 2.6. When $m = 0$, let V_2 denote the standard representation of $SL_{2,\mathbb{Q}}$, and $u \in \mathbb{N}$. Then $\mathrm{Sym}^u V_2 \in \mathrm{Rep}(SL_{2,\mathbb{Q}})$; in fact, $\mathrm{Sym}^u V_2$ is the irreductible representation of highest (representation-theoretic) weight u. Denote by g the genus of the quotient of the upper half space by H_C/K_W, and by $c \geq 1$ the number of its cusps. (Thus, $c \geq 3$ if $g = 0$ since H_C/K_W is neat.) Then $H^1(H_C/K_W, \mathrm{Sym}^u V_2)$ is of dimension $(u+1)(2g-2+c)$ if $u \geq 1$, and of dimension $2g-1+c$ if $u = 0$. In particular,

$$H^1\big(H_C/K_W, \mathrm{Sym}^u V_2\big) \neq 0 \ , \forall\, u \in \mathbb{N} \ .$$

Proof of Theorem 2.3. (a): According to Corollary 2.5 (a) and Proposition 2.4 (a),

(o) $0 \neq H^0(Lie(W_0), V_{\underline{\alpha}})$ is of weight $(r+1) - (k_1+k_2) - 1$,

(i) $0 \neq H^1(Lie(W_0), V_{\underline{\alpha}})$ is of weight $(r+2) - (k_1-k_2)$,

and $H^q(Lie(W_0), V_{\underline{\alpha}}) = 0$ whenever $q < 0$. The group H_C/K_W associated to a stratum Z' of Z_0 is a neat arithmetic sub-group of $SL_2(\mathbb{Q})$. It is therefore of cohomological dimension one, and admits no non-zero invariants on regular irreducible representations of $Q_0/W_0 = \mathbb{G}_{m,\mathbb{Q}} \times_{\mathbb{Q}} GL_{2,\mathbb{Q}}$.

According to Proposition 2.4 (a) and Corollary 2.5 (a), $H^q(Lie(W_0), V_{\underline{\alpha}})$, $0 \leq q \leq 2$, are irreducible as representations of Q_0/W_0, and regular unless $q = 0$ and $k_1 = k_2$, in which case $SL_{2,\mathbb{Q}}$, hence H_C/K_W acts trivially. Pink's Theorem and [P2, Prop. (5.5.4)] then tell us that

(o) $R^0 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ is non-zero if and only if $k_1 = k_2$, in which case it is of weight $r - (k_1+k_2)$,

(i) $0 \neq R^1 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ is of weight $(r+1) - (k_1+k_2) - 1$,

(ii) $0 \neq R^2 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ is of weight $(r+2) - (k_1-k_2)$,

and that $R^n i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = 0$ whenever $n < 0$ (for the non-vanishing statements in (i), (ii), see Remark 2.6).

The scheme Z_0 is of dimension zero; therefore,

$$H^n i_0^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V}) = H^{n-r} i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = R^{n-r} i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) \ .$$

From (o), (i), (ii) and the vanishing of $R^n i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = 0$ for $n < 0$, we conclude that

(r) $H^r i_0^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$ is zero if $k_1 > k_2$, and non-zero of weight $r-(k_1+k_2)$ if $k_1 = k_2$, (r+1) $0 \neq H^{r+1} i_0^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$ is of weight $(r+1) - (k_1 + k_2) - 1$,

(r+2) $0 \neq H^{r+2} i_0^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$ is of weight $(r+2) - (k_1 - k_2)$,

and that $H^n i_0^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V}) = 0$ whenever $n \leq r-1$.

(b): According to Corollary 2.5 (b) and Proposition 2.4 (a),

(o) $0 \neq H^0(Lie(W_1), V_{\underline{\alpha}})$ is of weight $(r+1) - k_1 - 1$,

(i) $0 \neq H^1(Lie(W_1), V_{\underline{\alpha}})$ is of weight $(r+2) - k_2 - 1$,

and $H^q(Lie(W_1), V_{\underline{\alpha}}) = 0$ whenever $q < 0$. The group H_C/K_W associated to a stratum Z' of Z_1 is trivial. Pink's Theorem and [P2, Lemma (5.6.6)] then tell us that

(o) $0 \neq R^0 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ is of weight $(r+1) - k_1 - 1$,

(i) $0 \neq R^1 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ is of weight $(r+2) - k_2 - 1$,

and that $R^n i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = 0$ whenever $n < 0$. Furthermore, Pink's Theorem tells us that all classical cohomology objects $R^n i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$, $n \in \mathbb{Z}$ are lisse. The formula

$$H^n i_1^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V}) = H^{n-r} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = \big(R^{n-r-1} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})\big)[1]$$

is valid: the first equation comes from

$$R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V}) = \mu_\ell(V_{\underline{\alpha}})[-r] \; .$$

As for the second, note that any lisse ℓ-adic sheaf $\mathcal{F}$ on a one-dimensional regular scheme is a perverse sheaf $\mathcal{F}'$ up to a shift by -1:

$$\mathcal{F} = \mathcal{F}'[-1] \quad \text{and} \quad \mathcal{F}' = \mathcal{F}[1] \; .$$

From (o), (i) and the vanishing of $R^n i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = 0$ for $n < 0$, we conclude that

(r+1) $0 \neq H^{r+1} i_1^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$ is of weight $(r+1) - k_1$,

(r+2) $0 \neq H^{r+2} i_1^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$ is of weight $(r+2) - k_2$,

and that $H^n i_1^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V}) = 0$ whenever $n \leq r$. **q.e.d.**

For the final step of the proof of Theorem 1.6, the following commutative diagram of immersions will be useful.

$$
\begin{tikzcd}
 & & & Z_1 \arrow[dll, hook, "i'"'] \arrow[dl, hook, "i_1", "\circ"'] \\
 & (M^K)^* - Z_0 \arrow[d, hook, "j''", "\circ"'] & & \\
M^K \arrow[ur, hook, "j'", "\circ"'] \arrow[r, hook, "j"'] & (M^K)^* & \partial(M^K)^* \arrow[l, hook', "i"] & \\
 & Z_0 \arrow[u, hook, "i''"] \arrow[ur, hook, "i_0"'] & &
\end{tikzcd}
$$

Immersions situated on the same line are complementary to each other (example: j'' and i''), the four immersions marked by "$\circ$" are open (example: i_1), and the other four are closed (example: i').

Remark 2.7. Denote by $\tau_{Z_m}^{t\leq\bullet}$ and $\tau_{Z_m}^{t\geq\bullet}$ the truncation functors with respect to the perverse t-structure on Z_m, $m = 0, 1$.

(a) The immersions j' and i' being complementary,

$$(i')^* \, j'_{!*} \, \mathcal{F}' = \tau_{Z_1}^{t\leq -1} (i')^* j'_* \mathcal{F}'$$

for any perverse sheaf $\mathcal{F}'$ on M^K [BBD, Prop. 1.4.23].

(b) The intermediate extension is transitive, *i.e.*,

$$j_{!*} = j''_{!*} \, j'_{!*}$$

[BBD, Cor. 1.4.24]. Application of the functor $(i'')^* j''_*$ to the exact triangle

$$i'_* \tau_{Z_1}^{t\geq 0} (i')^* j'_* [-1] \longrightarrow j'_{!*} \longrightarrow j'_* \longrightarrow i'_* \tau_{Z_1}^{t\geq 0} (i')^* j'_*$$

of functors on perverse sheaves on M^K (see (a)) yields the exact triangle

$$i_0^* i_{1,*} \tau_{Z_1}^{t\geq 0} (i')^* j'_* [-1] \longrightarrow (i'')^* j''_* \, j'_{!*} \longrightarrow i_0^* i^* j_* \longrightarrow i_0^* i_{1,*} \tau_{Z_1}^{t\geq 0} (i')^* j'_*$$

The immersions j'' and i'' being complementary, we have as in (a)

$$(i'')^* \, j''_{!*} \, \mathcal{F}'' = \tau_{Z_0}^{t\leq -1} (i'')^* j''_* \mathcal{F}''$$

for any perverse sheaf $\mathcal{F}''$ on $(M^K)^* - Z_0$. It follows that for any perverse sheaf $\mathcal{F}'$ on M^K, there are exact sequences of perverse cohomology objects

$$H^{n-1}\big(i_0^* i_{1,*} \tau_{Z_1}^{t\geq 0} i_1^* i^* j_* \mathcal{F}'\big) \longrightarrow H^n\big(i_0^* i^* \, j_{!*} \, \mathcal{F}'\big)$$

$$\longrightarrow H^n\big(i_0^* i^* j_* \mathcal{F}'\big) \longrightarrow H^n\big(i_0^* i_{1,*} \tau_{Z_1}^{t\geq 0} i_1^* i^* j_* \mathcal{F}'\big)$$

for $n \leq -1$, while $H^n(i_0^* i^* \, j_{!*} \, \mathcal{F}') = 0$ for all $n \geq 0$.

(c) Recall that $R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V}) = \mu_\ell(V_{\underline{\alpha}})[-r]$; the variety M^K being of dimension three, the complex $R_{\ell,M^K}({}^{\alpha}\mathcal{V})$ is therefore concentrated in perverse degree $r+3$. According to our conventions, $i_1^* i^* j_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V}) = (i')^* j'_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V})$ thus equals

$$\big((i')^* j'_{!*}(R_{\ell,M^K}({}^{\alpha}\mathcal{V})[r+3])\big)[-(r+3)]\ .$$

According to (a), we thus have

$$i_1^* i^* j_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V}) = \tau_{Z_1}^{t\le r+2}(i')^* j'_* R_{\ell,M^K}({}^{\alpha}\mathcal{V}) = \tau_{Z_1}^{t\le r+2} i_1^* i^* j_* R_{\ell,M^K}({}^{\alpha}\mathcal{V})\ .$$

Similarly, following (b),

$$H^n i_0^* i^* j_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V}) = 0$$

for all $n \ge r+3$, and there are exact sequences

$$H^{n-1} i_0^* i_{1,*} \tau_{Z_1}^{t\ge r+3} i_1^* i^* j_* R_{\ell,M^K}({}^{\alpha}\mathcal{V}) \longrightarrow H^n i_0^* i^* j_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V})$$

$$\longrightarrow H^n i_0^* i^* j_* R_{\ell,M^K}({}^{\alpha}\mathcal{V}) \longrightarrow H^n i_0^* i_{1,*} \tau_{Z_1}^{t\ge r+3} i_1^* i^* j_* R_{\ell,M^K}({}^{\alpha}\mathcal{V})$$

for $n \le r+2$.

(d) We claim that

$$H^n i_0^* i_{1,*} \tau_{Z_1}^{t\ge r+3} i_1^* i^* j_* R_{\ell,M^K}({}^{\alpha}\mathcal{V}) = 0$$

for all $n \le r+1$. Equivalently,

$$H^n i_0^* i_{1,*} \tau_{Z_1}^{t\ge 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = 0$$

for all $n \le 1$. Indeed, by Pink's Theorem, the classical cohomology objects of $i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ are all lisse. Applying $\tau_{Z_1}^{t\ge 3}$, we thus get a complex concentrated in classical degrees ≥ 2 (recall that Z_1 is of dimension one). The same is thus true after application of $i_0^* i_{1,*}$ (recall that inverse images are t-exact for the classical t-structure). In other words, the complex

$$i_0^* i_{1,*} \tau_{Z_1}^{t\ge 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$$

has trivial cohomology (classical or perverse; recall that Z_0 is of dimension zero) in degrees ≤ 1.

(e) From (c) and (d), we deduce that

$$H^n i_0^* i^* j_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V}) \xrightarrow{\ \sim\ } H^n i_0^* i^* j_* R_{\ell,M^K}({}^{\alpha}\mathcal{V})$$

for $n \le r+1$, and that $H^{r+2} i_0^* i^* j_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V})$ equals the kernel of

$$H^{r+2} i_0^* i^* j_* R_{\ell,M^K}({}^{\alpha}\mathcal{V}) \longrightarrow H^{r+2} i_0^* i_{1,*} \tau_{Z_1}^{t\ge r+3} i_1^* i^* j_* R_{\ell,M^K}({}^{\alpha}\mathcal{V})\ .$$

Corollary 2.8. *Let ℓ be a prime number.*
(a) For all $n \in \mathbb{Z}$,

$$H^n i_0^* i^* j_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V})$$

is of weights $\le n - (k_1 - k_2)$.

(b) For all $n \in \mathbb{Z}$,

$$H^n i_1^* i^* j_{!*} R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$$

is of weights $\leq n - k_2$. The perverse cohomology sheaf

$$H^{r+2} i_1^* i^* j_{!*} R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$$

is non-zero, and pure of weight $(r+2) - k_2$.

Proof. Part (a) follows from Remark 2.7 (c), (e), and from Theorem 2.3 (a).

Part (b) follows from Remark 2.7 (c), and from Theorem 2.3 (b). **q.e.d.**

Corollary 2.8 suffices to prove the part of Theorem 1.6 (b) asserting that regularity of $\underline{\alpha}$ is sufficient for weights 0 and 1 to be avoided by $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$. In order to prove that it is necessary, we need the following statement.

Proposition 2.9. *Let ℓ be a prime number. Then provided that $k_1 \geq 1$, the perverse cohomology sheaf*

$$H^{r+2} i_0^* i^* j_{!*} R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$$

is non-zero, and pure of weight $(r+2) - (k_1 - k_2)$.

Proof. According to Remark 2.7 (e),

$$H^{r+2} i_0^* i^* j_{!*} R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$$

equals the kernel of

$$ad : H^{r+2} i_0^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V}) \longrightarrow H^{r+2} i_0^* i_{1,*} \tau_{Z_1}^{t \geq r+3} i_1^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$$

— in particular, it is pure of weight $(r+2) - (k_1 - k_2)$ (Theorem 2.3 (a)) —, *i.e.*, it equals the kernel of

$$H^2 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) \longrightarrow H^2 i_0^* i_{1,*} \tau_{Z_1}^{t \geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) \ .$$

Thanks to Pink's Theorem, the regularity of $H^2(Lie(W_0), V_{\underline{\alpha}})$ as a representation of Q_0/W_0 (Corollary 2.5 (a)), and the fact that the group H_C/K_W is of cohomological dimension one, locally on Z_0, the (perverse or classical) sheaf

$$H^2 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = R^2 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$$

equals

$$\mu_{\ell,Z'}\big(H^1\big(H_C/K_W, H^1\big(Lie(W_0), V_{\underline{\alpha}}\big)\big)\big) \ .$$

Furthermore (Corollary 2.5 (a)), the restriction of $H^1(Lie(W_0), V_{\underline{\alpha}})$ to the group H_C/K_W is isomorphic to the $(k_1 + k_2 + 2)$-nd symmetric power of the standard representation of $SL_{2,\mathbb{Q}}$. By Remark 2.6, $H^2 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z'}$ is therefore of constant rank $(k_1 + k_2 + 3)(2g - 2 + c)$, where g denotes the genus of H_C/K_W, and c the number of cusps.

We claim that the restriction to the same Z' of

$$H^2 i_0^* i_{1,*} \tau_{Z_1}^{t \geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$$

is of constant rank c. Indeed, according to Remark 2.7 (d), the classical cohomology objects of $i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ are all lisse. Therefore, perverse truncation above degree three equals classical truncation above degree two (recall that Z_1 is of dimension one). The complex

$$i_0^* i_{1,*} \tau_{Z_1}^{t \geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$$

is concentrated in degrees ≥ 2, and we get

$$H^2 i_0^* i_{1,*} \tau_{Z_1}^{t \geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = R^0 i_0^* i_{1,*} R^2 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) \ .$$

Restriction to Z' yields

$$H^2 i_0^* i_{1,*} \tau_{Z_1}^{t \geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z'} = \bigoplus_{Z''} \bigl(R^0 i_0^* i_{1,*} \bigl(R^2 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z''} \bigr) \bigr)_{|Z'} \ ,$$

where the direct sum is indexed by all strata Z'' contributing to Z_1, and containing Z' in their closure. For every such Z'',

$$R^2 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z''} = \mu_{\ell, Z''} \bigl(H^2 \bigl(Lie(W_1), V_{\underline{\alpha}} \bigr) \bigr)$$

according to Pink's Theorem (since the group H_C / K_W (for $m = 1!$) is trivial).

Denote by $j_1 : Z_1 \hookrightarrow Z_1^*$ the Baily–Borel compactification, and by $i_{01} : \partial Z_1^* \hookrightarrow Z_1^*$ its complement. The immersion $i_1 : Z_1 \hookrightarrow (M^K)^*$ admits a natural extension $\bar{i}_1 : Z_1^* \to (M^K)^*$ [P1, Main Thm. 12.3 (c), Sect. 7.6], which is finite. The diagram

$$\begin{tikzcd} Z_1^* \arrow[d, "\bar{i}_1"'] & \partial Z_1^* \arrow[l, hook', "i_{01}"'] \arrow[d, "\bar{i}_1"] \\ (M^K)^* & Z_0 \arrow[l, hook', "i_0"'] \end{tikzcd}$$

is cartesian up to nilpotent elements. Proper base change therefore yields the formula

$$R^0 i_0^* i_{1,*} = R^0 \bar{i}_{1,*} i_{0,1}^* j_{1,*} \ .$$

The functors $\bar{i}_{1,*}$ and $i_{0,1}^*$ being exact on sheaves, we have

$$R^0 i_0^* i_{1,*} \bigl(R^2 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z''} \bigr) = \bar{i}_{1,*} i_{0,1}^* R^0 j_{1,*} \mu_{\ell, Z''} \bigl(H^2 \bigl(Lie(W_1), V_{\underline{\alpha}} \bigr) \bigr) \ .$$

According to Proposition 2.4 (a), $H^2(Lie(W_1), V_{\underline{\alpha}})$ is irreducible as a representation of Q_1/W_1, hence of $GL_{2,\mathbb{Q}}$. Yet another application of Pink's Theorem shows that

$$i_{0,1}^* R^0 j_{1,*} \mu_{\ell, Z''} \bigl(H^2 \bigl(Lie(W_1), V_{\underline{\alpha}} \bigr) \bigr)$$

is of constant rank one on the intersection of ∂Z_1^* with the closure of Z'' in $(Z_1)^*$.

Our claim on the rank of

$$H^2 i_0^* i_{1,*} \tau_{Z_1}^{t\ge 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z'} = \bar{i}_{1,*} \bigoplus_{Z''} \big(i_{0,1}^* R^0 j_{1,*} \mu_{\ell,Z''} \big(H^2 \big(Lie(W_1), V_{\underline{\alpha}} \big) \big) \big)_{|Z'}$$

is therefore proven as soon as we establish that the number of points in the geometrical fibres of the morphism $\bar{i}_1 : \partial Z_1^* \to Z_0$ above $Z' \subset Z_0$ equals c. This verification can be done on the level of $\mathbb{C}$-valued points, where the adelic description of the situation is at our disposal. More precisely, write $(G_m, \mathcal{H}_m) := (P_m, \mathfrak{X}_m)/W_m$ [P1, Prop. 2.9], $m = 0, 1$, for the Shimura data contributing to $\partial(M^K)^*$, and Q_{01} for the Borel sub-group $Q_0 \cap Q_1$ of G. According to [P1, Sect. 6.3], the diagram of $\mathbb{C}$-valued points corresponding to the diagram

$$\begin{tikzcd} Z_1^* \arrow[d, "\bar{i}_1"'] & \partial Z_1^* \arrow[l, hook', "i_{01}"'] \arrow[d, "\bar{i}_1"] \\ (M^K)^* & Z_0 \arrow[l, hook', "i_0"'] \end{tikzcd}$$

equals

$$\begin{tikzcd} Q_1(\mathbb{Q}) \backslash \big(\mathcal{H}_1^* \times G(\mathbb{A}_f)/K \big) \arrow[d, "\bar{i}_1"'] & Q_{01}(\mathbb{Q}) \backslash \big(\mathcal{H}_0 \times G(\mathbb{A}_f)/K \big) \arrow[l, hook', "i_{01}"'] \arrow[d, "\bar{i}_1"] \\ G(\mathbb{Q}) \backslash \big(\mathcal{H}^* \times G(\mathbb{A}_f)/K \big) & Q_0(\mathbb{Q}) \backslash \big(\mathcal{H}_0 \times G(\mathbb{A}_f)/K \big) \arrow[l, hook', "i_0"'] \end{tikzcd}$$

where all maps are induced by canonical inclusions of groups and spaces. Indeed, the full group $Q_m(\mathbb{Q})$ (and not only a sub-group of finite index) stabilizes $\mathcal{H}_m$, $m = 0, 1$, and two rational boundary components of $(G_1, \mathcal{H}_1)$ are conjugate under $G_1(\mathbb{Q})$ if and only if they are conjugate under $G(\mathbb{Q})$ (explicit computation, or [P1, (iii) of Remark on p. 91]). The sub-scheme $Z' \subset Z_0$ equals the image of a Shimura variety associated to $(G_0, \mathcal{H}_0)$ under a morphism i_g associated to an element $g \in G(\mathbb{A}_f)$ [P1, Main Theorem 12.3 (c)]; given the adelic description of i_g from [P1, Sect. 6.3], we see that under the above identification, any $z \in Z'(\mathbb{C})$ equals the class $[h_0, p_0 g]$ in

$$Q_0(\mathbb{Q}) \backslash \big(\mathcal{H}_0 \times G(\mathbb{A}_f)/K \big)$$

of a pair of the form $(h_0, p_0 g)$, with $h_0 \in \mathcal{H}_0$ and $p_0 \in P_0(\mathbb{A}_f)$. Put

$$Q_0^+(\mathbb{Q}) := \{ q_0 \in Q_0(\mathbb{Q}) \ , \ \lambda(q_0) > 0 \} \ ;$$

this group equals the centralizer in $Q_0(\mathbb{Q})$ of h_0, and indeed, of the whole of $\mathcal{H}_0$. Putting

$$H'_C := Q_0^+(\mathbb{Q}) \cap p_0 g K g^{-1} p_0^{-1} \ ,$$

we leave it to the reader to verify that the map

$$Q_{01}(\mathbb{Q}) \backslash Q_0(\mathbb{Q}) / H'_C \longrightarrow \bar{i}_1^{-1}(z) \ , \ [q_0] \longmapsto q_0 [h_0, p_0 g] = [q_0 h_0, q_0 p_0 g]$$

is well-defined, and bijective. By strong approximation,

$$W_0(\mathbb{Q}) \cdot H'_C = Q_0^+(\mathbb{Q}) \cap W_0(\mathbb{A}_f) \cdot p_0 g K g^{-1} p_0^{-1} \; .$$

But

$$Q_0 / W_0 = P_0 / W_0 \times_{\mathbb{Q}} GL_{2,\mathbb{Q}} \; ,$$

meaning that modulo W_0, elements in P_0 and in Q_0 commute with each other. Thus,

$$W_0(\mathbb{Q}) \cdot H'_C = Q_0^+(\mathbb{Q}) \cap W_0(\mathbb{A}_f) \cdot g K g^{-1} \; .$$

The image of $W_0(\mathbb{Q}) \cdot H'_C$ under the projection $\pi_0 : Q_0 \longrightarrow\!\!\!\!\!\rightarrow Q_0/W_0$ coincides with the image of

$$W_0(\mathbb{A}_f) \cdot Q_0^+(\mathbb{Q}) \cap g K g^{-1}$$

(both images equals $\pi_0(Q_0^+(\mathbb{Q})) \cap \pi(gKg^{-1})$). But by definition [P2, (3.7.4)], $W_0(\mathbb{A}_f) \cdot Q_0^+(\mathbb{Q}) \cap gKg^{-1}$ equals H_C. We thus showed that

$$\pi_0(H'_C) = \pi_0(H_C) \; .$$

Now the quotient morphism $Q_0 \longrightarrow\!\!\!\!\!\rightarrow Q_0/P_0, q_0 \mapsto \overline{q_0}$ induces an isomorphism

$$Q_{01}(\mathbb{Q}) \backslash Q_0(\mathbb{Q}) / H'_C \stackrel{\sim}{\longrightarrow} \overline{Q_{01}(\mathbb{Q})} \backslash \overline{Q_0(\mathbb{Q})} / \overline{H'_C} = \overline{Q_{01}(\mathbb{Q})} \backslash \overline{Q_0(\mathbb{Q})} / \overline{H_C}$$

But $\overline{Q_0(\mathbb{Q})} = GL_2(\mathbb{Q})$, and under this identification, $\overline{Q_{01}(\mathbb{Q})}$ equals the subgroup of upper triangular matrices, while $\overline{H_C} = H_C / K_W$. In other words,

$$Q_{01}(\mathbb{Q}) \backslash Q_0(\mathbb{Q}) / H'_C$$

is identified with the set up cusps of H_C / K_W.

The formula

$$(k_1 + k_2 + 3)(2g - 2 + c) \geq 4(2g - 2 + c) > c$$

(recall that c is greater or equal to 1, and that $c \geq 3$ if $g = 0$) implies that the rank of the source of ad is strictly greater than the rank of its target; the kernel of ad is therefore non-trivial. **q.e.d.**

Remark 2.10. (a) As the reader may verify,

$$H^{r+2} i_0^* i_{1,*} \tau_{Z_1}^{t \geq r+3} i_1^* i^* j_* R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V})$$

is pure of weight $(r+2) - (k_1 - k_2)$, *i.e.*, of the same weight as

$$H^{r+2} i_0^* i^* j_* R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V}) \; .$$

Weight considerations alone do therefore not imply non-triviality of the kernel of the map ad from the proof of Proposition 2.9.

(b) A more conceptual proof of Proposition 2.9 would consist in showing that locally on Z_0, the map ad equals the direct sum over all cusps of H_C / K_W of the residue maps. Identify $H^1(H_C/K_W, H^1(Lie(W_0), V_{\underline{\alpha}})) \otimes_{\mathbb{Q}} \mathbb{C}$ with the direct sum of the space of modular forms and (the conjugate of) the space

of cusp forms for H_C/K_W of weight $k_1 + k_2 + 4 \geq 5$. The kernel of the residues contains the space of cusp forms. Its dimension is computed in [Sh, Thm. 2.24 and Thm. 2.25]; thanks to [Sh, Prop. 1.40] (always remember that H_C/K_W is neat), this dimension can be seen to be strictly positive.
(c) On the level of geometry of Baily–Borel compactifications, a "strange duality" seems to be involved in the proof of Proposition 2.9: we need to know how many modular curves in the boundary of $(M^K)^*$ contain a given cusp Z' in their closure. The response yields the number of cusps of a "modular curve", which does not explicitely occur in $(M^K)^*$, namely the quotient of the upper half space by H_C/K_W. It would be interesting to see how this phenomenon generalizes to higher dimensional Siegel varieties.
(d) Our computation of the fibres of the morphism $\bar{i}_1 : Z_1^* \to (M^K)^*$ over points of Z_0 is a quantitative version of a classical non-injectivity result of Satake [Sat, Exemple on p. 13-06].

Remark 2.11. The Hodge theoretic analogues of Theorem 2.3, Corollary 2.8 and Proposition 2.9 hold. The proofs are identical up to the use of Pink's Theorem, which is replaced by [BW, Thm. 2.9].

Proof of Theorem 1.6. According to Theorem 2.2, the motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ is of Abelian type, and Φ is adapted to $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$; this proves part (a) of our claim.

By [P1, Summ. 1.18 (d)], there is a perfect pairing

$$V_{\underline{\alpha}} \otimes_{\mathbb{Q}} V_{\underline{\alpha}} \longrightarrow \mathbb{Q}(-r)$$

in $\mathrm{Rep}(G)$.

Fix a prime ℓ. Applying μ_ℓ, we get a perfect pairing

$$\mu_\ell(V_{\underline{\alpha}}) \otimes_{\mathbb{Q}_\ell} \mu_\ell(V_{\underline{\alpha}}) \longrightarrow \mathbb{Q}_\ell(-r)$$

of ℓ-adic lisse sheaves on M^K. In terms of local duality, the pairing induces an isomorphism

$$\mathbb{D}_{\ell,M^K}\big(\mu_\ell(V_{\underline{\alpha}})\big) \cong \mu_\ell(V_{\underline{\alpha}})(r+3)[6]$$

(M^K is smooth of dimension three). Given $R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V}) = \mu_\ell(V_{\underline{\alpha}})[-r]$, we find that

$$\mathbb{D}_{\ell,M^K}\big(R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})\big) \cong R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})(s)[2s] \ ,$$

where $s = r + 3$.

Corollary 2.8 tells us that for all $n \in \mathbb{Z}$, and $m = 0, 1$,

$$H^n i_m^* i^* j_{!*} R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$$

is of weights $\leq n - k$. According to [Wi9, Cor. 3.6 (b)], the motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ therefore avoids weights $-k+1, -k+2, \ldots, k$.

In order to conclude the proof of part (b), it remains to show, again thanks to [Wi9, Cor. 3.6 (b)], that for some $n \in \mathbb{Z}$, and $m = 0$ or $m = 1$, weight $n - k$ *does* occur in

$$H^n i_m^* i^* j_{!*} R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V}) .$$

We take $n = r+2$, and distinguish two cases. If $k = k_2$, *i.e.*, $k_2 \le k_1 - k_2$, then take $m = 1$; the claim then follows from Corollary 2.8 (b). Else, $k_2 > k_1 - k_2$ and $k = k_1 - k_2$. Since $k_1 \ge k_2$, we necessarily have $k_1 \ge 1$. Take $m = 0$ and apply Proposition 2.9. **q.e.d.**

Remark 2.12. (a) An element of $H^n(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}}))$ is called a *ghost class* if it lies in the image of

$$H^n\big(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) \longrightarrow H^n\big(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big)$$

and in the kernel of both restriction maps

$$H^n\big(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) \longrightarrow H^n\big(Z_m(\mathbb{C}), i_m^* i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) ,$$

$m = 0, 1$. One of the main results of [M] implies that if $\underline{\alpha}$ is regular, then there are no non-zero ghost classes [M, Thm. 16]. This result does not formally imply, nor is it implied by our Theorem 1.6. Nonetheless, it might be worthwhile to note that the weight arguments that occur in the proofs are quite similar. The most relevant information from Theorem 1.6, as far as [M, Thm. 16] is concerned, comes from the weight filtration

$$d_* j_{!*} {}^{\underline{\alpha}}\mathcal{V} \longrightarrow \tilde{d}_* {}^{\underline{\alpha}}\mathcal{V} \longrightarrow d_* i_* i^! j_{!*} {}^{\underline{\alpha}}\mathcal{V}[1] \longrightarrow d_* j_{!*} {}^{\underline{\alpha}}\mathcal{V}[1]$$

avoiding weights $1, 2, \ldots, k$ (Corollary 1.7 (a)), hence avoiding weight 1 if $\underline{\alpha}$ is regular, which we assume in the sequel. This implies that any element of $H^n(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}}))$ not mapping to zero in $H^n(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}}))$, remains non-zero in

$$H^n\big(\partial(M^K)^*(\mathbb{C}), i^! j_{!*} \mu_{\mathbf{H}}(V_{\underline{\alpha}})[1]\big) = H^n\big(\partial(M^K)^*(\mathbb{C}), \tau^{t \ge 3}_{\partial(M^K)^*} i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) .$$

In other words, a ghost class vanishing in $H^n(\partial(M^K)^*(\mathbb{C}), \tau^{t \ge 3}_{\partial(M^K)^*} i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}}))$ is zero. The Hodge structure $H^n(\partial(M^K)^*(\mathbb{C}), \tau^{t \ge 3}_{\partial(M^K)^*} i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}}))$ has weights $\ge (r+n)+2$; the same type of considerations as those leading to Corollary 2.8 then imply that the direct sum of the restriction maps

$$H^n\big(\partial(M^K)^*(\mathbb{C}), \tau^{t \ge 3}_{\partial(M^K)^*} i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) \longrightarrow H^n\big(Z_m(\mathbb{C}), i_m^* \tau^{t \ge 3}_{\partial(M^K)^*} i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) ,$$

$m = 0, 1$, is injective.

(b) The above illustrates the Remark made on [M, middle of p. 7]: *for a class in the cohomology of the boundary whose weight is not the middle weight nor the middle weight plus one we can determine exactly whether it is or not in the image of the morphism*

$$H^n\big(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) \longrightarrow H^n\big(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) .$$

In fact, it appears amusing to note that the "middle weights" are relevant in another context than the one studied in the present paper. According to [M, middle of p. 16], the representation $V_{\underline{\alpha}}$ *satisfies the weak middle weight property* if the weights occurring in the space of ghost classes in $H^n(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}}))$ are contained in $\{r+n, r+n+1\}$. [M, Thm. 16] implies in particular that for all $\underline{\alpha}$ (regular or not), the representation $V_{\underline{\alpha}}$ does satisfy the weak middle weight property, while our Theorem 1.6 implies that weights $\{r+n, r+n+1\}$ do not occur at all in $H^n(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\mathbf{H}}(V_{\underline{\alpha}}))$, as soon as $\underline{\alpha}$ is regular.

Remark 2.13. Saper's vanishing theorem [Sap, Thm. 5] says that if $\underline{\alpha}$ is regular, then the groups $H^n(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}}))$, hence (by comparison) $H^n(M^K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}}))$ vanish for $n < 3 = \dim M^K$. By duality, one obtains that $H^n_c(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})) = 0$ and $H^n_c(M^K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})) = 0$ for $n > 3$. It follows that interior cohomology with coefficients in $\mu_{\mathbf{H}}(V_{\underline{\alpha}})$, denoted

$$H^n_!\big(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) ,$$

and interior cohomology with coefficients in $\mu_\ell(V_{\underline{\alpha}})$, denoted

$$H^n_!\big(M^K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})\big) ,$$

both vanish for $n \neq 3$, provided that $\underline{\alpha}$ is regular.

3 The motive for an automorphic form

This final section contains the analogues for Siegel threefolds of the main results from [Sc]. Since we shall not restrict ourselves to the case of Hecke eigenforms, our notation becomes a little more technical than in [loc. cit.].

We continue to consider the situation of Sections 1 and 2. In particular, we fix a dominant $\underline{\alpha} = \alpha(k_1, k_2, r)$, which we assume to be regular, *i.e.*, $k_1 > k_2 > 0$. Consider the intersection motive $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V} \in CHM(\mathbb{Q})_{\mathbb{Q}}$. According to [Wi9, Rem. 2.13 (a)] and Remark 2.13, its Hodge theoretic realization equals

$$H^3_!\big(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big)[-(r+3)] ,$$

and its ℓ-adic realization equals

$$H^3_!\big(M^K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})\big)[-(r+3)] .$$

By Corollary 1.8, every element of the Hecke algebra $\mathfrak{H}(K, G(\mathbb{A}_f))$ acts on $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}$.

Theorem 3.1 ([Ha, Chap. 2, Thm. 2, p. 50])**.** *Let L be any field of characteristic zero. Then the $\mathfrak{H}(K, G(\mathbb{A}_f)) \otimes_{\mathbb{Q}} L$-module $H^3_!(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})) \otimes_{\mathbb{Q}} L$ is semi-simple.*

Note that [Ha, Chap. 3, Sect. 4.3.5] gives a proof of Theorem 3.1, while the statement in [Ha, Chap. 2, Thm. 2, p. 50] is "non-adelic". Denote by $R(\mathfrak{H}) := R(\mathfrak{H}(K, G(\mathbb{A}_f)))$ the image of the Hecke algebra in the endomorphism algebra of $H^3_!(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}}))$.

Corollary 3.2. *Let L be any field of characteristic zero. Then the L-algebra $R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$ is semi-simple.*

In particular, the isomorphism classes of simple right $R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$-modules correspond bijectively to isomorphism classes of minimal right ideals.

Fix L, and let Y_{π_f} be such a minimal right ideal of $R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$. There is a (primitive) idempotent $e_{\pi_f} \in R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$ generating Y_{π_f}.

Definition 3.3. (a) The *Hodge structure $W(\pi_f)$ associated to Y_{π_f}* is defined as

$$W(\pi_f) := \mathrm{Hom}_{R(\mathfrak{H}) \otimes_{\mathbb{Q}} L}\big(Y_{\pi_f}, H^3_!\big(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) \otimes_{\mathbb{Q}} L\big) \ .$$

(b) Let ℓ be a prime number. The *Galois module $W(\pi_f)_\ell$ associated to Y_{π_f}* is defined as

$$W(\pi_f)_\ell := \mathrm{Hom}_{R(\mathfrak{H}) \otimes_{\mathbb{Q}} L}\big(Y_{\pi_f}, H^3_!\big(M^K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})\big) \otimes_{\mathbb{Q}} L\big) \ .$$

Definition 3.3 (b) should be compared to [We, Thm. I].

In order to define a motivic object whose realizations equal $W(\pi_f)$ and $W(\pi_f)_\ell$, respectively, one uses the idempotent generator e_{π_f} of Y_{π_f}.

Proposition 3.4. *There is canonical isomorphism of Hodge structures*

$$W(\pi_f) \xrightarrow{\ \sim\ } \big(H^3_!\big(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) \otimes_{\mathbb{Q}} L\big) \cdot e_{\pi_f} \ ,$$

and a canonical isomorphism of Galois modules

$$W(\pi_f)_\ell \xrightarrow{\ \sim\ } \big(H^3_!\big(M^K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})\big) \otimes_{\mathbb{Q}} L\big) \cdot e_{\pi_f} \ .$$

Proof. We shall perform the proof for Hodge structures; the one for Galois modules is formally identical. Obviously,

$$\mathrm{Hom}_{R(\mathfrak{H}) \otimes_{\mathbb{Q}} L}\big(R(\mathfrak{H}) \otimes_{\mathbb{Q}} L, H^3_!\big(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) \otimes_{\mathbb{Q}} L\big)$$

is canonically identified with

$$H^3_!\big(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) \otimes_{\mathbb{Q}} L$$

by mapping an morphism g to the image of $1 = 1_{R(\mathfrak{H})}$ under g. Inside

$$\mathrm{Hom}_{R(\mathfrak{H}) \otimes_{\mathbb{Q}} L}\big(R(\mathfrak{H}) \otimes_{\mathbb{Q}} L, H^3_!\big(M^K(\mathbb{C}), \mu_{\mathbf{H}}(V_{\underline{\alpha}})\big) \otimes_{\mathbb{Q}} L\big) \ ,$$

the object $W(\pi_f)$ contains precisely those morphisms g vanishing on $1 - e_{\pi_f}$, in other words, satisfying the relation $g(1) = g(e_{\pi_f}) = g(1) \cdot e_{\pi_f}$. **q.e.d.**

Since we do not know whether the Chow motive $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}$ is finite dimensional, we cannot apply [K, Cor. 7.8], and therefore do not know whether e_{π_f} can be lifted *idempotently* to the Hecke algebra $\mathfrak{H}(K, G(\mathbb{A}_f))$. This is why we need to descend to the level of *Grothendieck motives*. Denote by $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}'$ the Grothendieck motive underlying $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}$.

Definition 3.5. Assume $\underline{\alpha} = \alpha(k_1, k_2, r)$ to be regular. Let L be a field of characteristic zero, and Y_{π_f} a minimal right ideal of $R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$. The *motive associated to* Y_{π_f} is defined as

$$\mathcal{W}(\pi_f) := d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}' \cdot e_{\pi_f} \; .$$

Definition 3.5 should be compared to [Sc, Sect. 4.2.0]. Given our construction, the following is obvious.

Theorem 3.6. *Assume* $\underline{\alpha} = \alpha(k_1, k_2, r)$ *to be regular,* i.e., $k_1 > k_2 > 0$. *Let* L *be a field of characteristic zero, and* Y_{π_f} *a minimal right ideal of* $R(\mathfrak{H}) \otimes_F L$. *The realizations of the motive* $\mathcal{W}(\pi_f)$ *associated to* Y_{π_f} *are concentrated in the single cohomological degree* $r+3$*, and they take the values* $W(\pi_f)$ *(in the Hodge theoretic setting) resp.* $W(\pi_f)_\ell$ *(in the* ℓ*-adic setting).*

A special case occurs when Y_{π_f} is of dimension one over L, *i.e.*, corresponds to a non-trivial character of $R(\mathfrak{H})$ with values in L. The automorphic form is then an eigenform for the Hecke algebra. This is the analogue of the situation considered in [Sc] for elliptic cusp forms.

The motive $\mathcal{W}(\pi_f)$ being a direct factor of $d_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}'$, our results on the latter from Section 1 have obvious consequences for the realizations of $\mathcal{W}(\pi_f)$.

Corollary 3.7. *Assume* $\underline{\alpha} = \alpha(k_1, k_2, r)$ *to be regular. Let* L *be a field of characteristic zero, and* Y_{π_f} *a minimal right ideal of* $R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$. *Let* p *be a prime number not dividing the level of* K. *Let* ℓ *be different from* p.

(a) The p*-adic realization* $W(\pi_f)_p$ *of* $\mathcal{W}(\pi_f)$ *is crystalline.*

(b) The ℓ*-adic realization* $W(\pi_f)_\ell$ *of* $\mathcal{W}(\pi_f)$ *is unramified at* p.

(c) The characteristic polynomials of the following coincide: (1) the action of Frobenius ϕ *on the* ϕ*-filtered module associated to* $W(\pi_f)_p$*, (2) the action of a geometrical Frobenius automorphism at* p *on* $W(\pi_f)_\ell$.

Proof. Parts (a) and (b) follow from Remark 1.12.

As for (c), in order to apply [KM, Thm. 2. 2)], use that both realizations are cut out by the *same* cycle from the cohomology of a smooth and proper scheme over the field $\mathbb{F}_p$ (cmp. the proof of Corollary 1.13). **q.e.d.**

Corollary 3.7 should be compared to [Sc, Thm. 1.2.4].

Remark 3.8. Part (c) of Corollary 3.7 is already contained in [U, Thm. 1].

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