Uniform finite-dimensional approximation of basic capacities of energy-constrained channels

M.E. Shirokov[*]

Abstract

We consider energy-constrained infinite-dimensional quantum channels from a given system (satisfying a certain condition) to any other systems. We show that dealing with basic capacities of these channels we may assume (accepting arbitrarily small error ε) that all channels have the same finite-dimensional input space – the subspace corresponding to the $m(\varepsilon)$ minimal eigenvalues of the input Hamiltonian.

We also show that for the class of energy-limited channels (mapping energy-bounded states to energy-bounded states) the above result is valid with substantially smaller dimension $m(\varepsilon)$.

The uniform finite-dimensional approximation allows to prove the uniform continuity of the basic capacities on the set of all quantum channels with respect to the strong (pointwise) convergence topology. For all the capacities we obtain continuity bounds depending only on the input energy bound and the energy-constrained-diamond-norm distance between quantum channels (generating the strong convergence on the set of quantum channels).

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[*]Steklov Mathematical Institute, RAS, Moscow, email:msh@mi.ras.ru

1 Introduction

When we consider transmission of classical or quantum information over infinite-dimensional quantum channels we have to impose energy constraints on states used for encoding information to be consistent with the physical implementation of the process [5, 6, 24].

The energy constraint for a single channel $\Phi : A \to B$ is expressed by the inequality

$$\mathrm{Tr} H_A \rho \leq E, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \tag{1}$$

where H_A is the Hamiltonian of the input system A.

We will assume that H_A is a positive operator having discrete spectrum $\{E_k^A\}_{k \geq 0}$ of finite multiplicity such that $E_k^A \to +\infty$ as $k \to +\infty$.[1] In this case H_A determines the special family $\{\mathcal{H}_A^m\}_{m=1}^{+\infty}$ of finite-dimensional subspaces of the input space $\mathcal{H}_A$, where $\mathcal{H}_A^m$ is the linear hull of the eigenvectors of H_A corresponding to its m minimal eigenvalues. The subspace $\mathcal{H}_A^m$ can be treated as the minimal energy m-dimensional subspace of $\mathcal{H}_A$. So, the states supported by $\mathcal{H}_A^m$ are more relevant to the constraint (1) than states supported by other m-dimensional subspaces of $\mathcal{H}_A$. At the same time, it is easy to show that all the states satisfying (1) can be uniformly approximated by states in $\mathfrak{S}(\mathcal{H}_A^m)$ satisfying (1) for large m.

So, it is reasonable to ask what happens if we will use for encoding information only states in $\mathfrak{S}(\mathcal{H}_A^m)$ satisfying (1) for sufficiently large m (for block encoding this means the use of the states in $\mathfrak{S}([\mathcal{H}_A^m]^{\otimes n})$ satisfying (1) with H_A replaced by the Hamiltonian of n copies of A and E replaced by nE).

It is clear that this additional restriction on the choice of codes-states (we will call it the *m-restriction*) can not increase the ultimate rate of information

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[1]This assumption holds for quantum systems used in applications, in particular, for a system of quantum oscillators.

transmission through a channel. On the other hand, the above observations give a reason to conjecture that the loss of the information transmission rate caused by the m-restriction can be made arbitrarily small by increasing m.

For a *fixed* channel Φ this conjecture can be easily verified for each of the basic capacities either by using operational definition of the capacity or by exploiting expressions of this capacity via entropic characteristics of a channel. In the paper we prove the channel-independent version of this assertion: the loss of each of the basic capacities caused by the m-restriction tends to zero as $m \to +\infty$ *uniformly on the set of all channels* from the system A to any other systems provided the Hamiltonian H_A satisfies the condition[2]

$$\lim_{\lambda \to +0} \left[\mathrm{Tr}\, e^{-\lambda H_A} \right]^{\lambda} = 1,$$

which holds, in particular, for a system of quantum oscillators playing a central role in continuous variable quantum information theory [5, 21].

We also show that the vanishing rate of the loss of the basic capacities caused by the m-restriction can be increased substantially by restricting attention to the class of quantum channels mapping energy-bounded states to energy-bounded states (called energy-limited channels in [27]).

The uniform finite-dimensional approximation allows to prove the uniform continuity of the basic capacities on the set of all quantum channels with respect to the strong (pointwise) convergence topology.[3]

2   Preliminaries

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded operators in $\mathcal{H}$ with the operator norm $\| \cdot \|$ and $\mathfrak{T}(\mathcal{H})$ the Banach space of all trace-class operators in $\mathcal{H}$ with the trace norm $\|\cdot\|_1$. Let $\mathfrak{S}(\mathcal{H})$ be the set of quantum states (positive operators in $\mathfrak{T}(\mathcal{H})$ with unit trace) [5, 23].

Denote by $I_{\mathcal{H}}$ the unit operator in a Hilbert space $\mathcal{H}$ and by $\mathrm{Id}_{\mathcal{H}}$ the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$.

We will repeatedly use the inequality

$$\| (I_{\mathcal{H}} - P)\, \rho\, P \|_1 \leq \sqrt{\mathrm{Tr}(I_{\mathcal{H}} - P)\rho} \tag{2}$$

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[2]I would be grateful for any comments concerning physical sense of this condition.

[3]I would be grateful for any comments about other applications of the uniform finite-dimensional approximation of energy-constrained channel capacities.

valid for any state $\rho \in \mathfrak{S}(\mathcal{H})$ and any orthogonal projector $P \in \mathfrak{B}(\mathcal{H})$, which can be easily proved via the operator Cauchy-Schwarz inequality (see the proof of Lemma 11.1 in [5]).

If quantum systems A and B are described by Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ then the bipartite system AB is described by the tensor product of these spaces, i.e. $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. A state in $\mathfrak{S}(\mathcal{H}_{AB})$ is denoted ρ_{AB}, its marginal states $\mathrm{Tr}_B \rho_{AB}$ and $\mathrm{Tr}_A \rho_{AB}$ are denoted ρ_A and ρ_B respectively.[4]

A *quantum channel* Φ from a system A to a system B is a completely positive trace preserving linear map from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_B)$ [5, 23].

For any quantum channel $\Phi : A \to B$ the Stinespring theorem implies existence of a Hilbert space $\mathcal{H}_E$ and of an isometry $V_\Phi : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = \mathrm{Tr}_E V_\Phi \rho V_\Phi^*, \quad \rho \in \mathfrak{T}(\mathcal{H}_A). \tag{3}$$

The quantum channel

$$\mathfrak{T}(\mathcal{H}_A) \ni \rho \mapsto \widehat{\Phi}(\rho) = \mathrm{Tr}_B V_\Phi \rho V_\Phi^* \in \mathfrak{T}(\mathcal{H}_E) \tag{4}$$

is called *complementary* to the channel Φ [5, Ch.6].

In finite dimensions (i.e. when $\dim \mathcal{H}_A$ and $\dim \mathcal{H}_B$ are finite) the distance between quantum channels from A to B generated by the diamond norm

$$\|\Phi\|_\diamond \doteq \sup_{\rho \in \mathfrak{S}(\mathcal{H}_{AR})} \|\Phi \otimes \mathrm{Id}_R(\rho)\|_1 \tag{5}$$

of a Hermitian-preserving superoperator $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ is widely used [1, 15, 23]. But this metric becomes singular in the case $\dim \mathcal{H}_A = \dim \mathcal{H}_B = +\infty$: there are infinite-dimensional channels with close physical parameters such that the diamond-norm distance between them equals to 2 [27]. In this case it is natural to use the distance between quantum channels generated by the energy-constrained diamond norm

$$\|\Phi\|_\diamond^E \doteq \sup_{\rho \in \mathfrak{S}(\mathcal{H}_{AR}), \mathrm{Tr} H_A \rho \leq E} \|\Phi \otimes \mathrm{Id}_R(\rho)\|_1, \quad E > E_0^A, \tag{6}$$

of a Hermitian-preserving superoperator $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$, where H_A is the Hamiltonian of the input system A and $E_0^A \doteq \inf_{\|\varphi\|=1} \langle \varphi | H_A | \varphi \rangle$ [20, 27].

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[4]Here and in what follows Tr_X means $\mathrm{Tr}_{\mathcal{H}_X}$.

The *von Neumann entropy* $H(\rho) = \text{Tr}\eta(\rho)$ of a state $\rho \in \mathfrak{S}(\mathcal{H})$, where $\eta(x) = -x\log x$ if $x > 0$ and $\eta(0) = 0$, is a concave nonnegative lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H})$ [5, 14, 22]. The concavity of the von Neumann entropy is supplemented by the inequality

$$H(p\rho + (1-p)\sigma) \leq pH(\rho) + (1-p)H(\sigma) + h_2(p), \tag{7}$$

where $h_2(p) = \eta(p) + \eta(1-p)$ is the binary entropy, valid for any states $\rho, \sigma \in \mathfrak{S}(\mathcal{H})$ and $p \in (0,1)$ [5, 23].

The *quantum conditional entropy*

$$H(A|B)_\rho = H(\rho_{AB}) - H(\rho_B) \tag{8}$$

of a bipartite state ρ_{AB} with finite marginal entropies is essentially used in analysis of quantum systems [5, 23]. It is concave and satisfies the following inequality

$$H(A|B)_{p\rho+(1-p)\sigma} \leq pH(A|B)_\rho + (1-p)H(A|B)_\sigma + h_2(p) \tag{9}$$

for any states $\rho, \sigma \in \mathfrak{S}(\mathcal{H}_{AB})$ and $p \in (0,1)$. Inequality (9) follows from concavity of the entropy and inequality (7).

The *quantum relative entropy* for two states ρ and σ in $\mathfrak{S}(\mathcal{H})$ is defined as

$$H(\rho\|\sigma) = \sum_i \langle i|\, \rho\log\rho - \rho\log\sigma\, |i\rangle,$$

where $\{|i\rangle\}$ is the orthonormal basis of eigenvectors of the state ρ and it is assumed that $H(\rho\|\sigma) = +\infty$ if the support of ρ is not contained in the support of σ [5, 14, 22].[5]

The *quantum mutual information* of a state ρ_{AB} of a bipartite quantum system is defined as

$$I(A\!:\!B)_\rho = H(\rho_{AB}\|\rho_A \otimes \rho_B) = H(\rho_A) + H(\rho_B) - H(\rho_{AB}), \tag{10}$$

where the second expression is valid if $H(\rho_{AB})$ is finite [13, 23].

Basic properties of the relative entropy show that $\rho \mapsto I(A:B)_\rho$ is a lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H}_{AB})$ taking values in $[0, +\infty]$. It is well known that

$$I(A\!:\!B)_\rho \leq 2\min\{H(\rho_A), H(\rho_B)\} \tag{11}$$

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[5]The support of a positive operator is the orthogonal complement of its kernel.

for any state ρ_{AB} [13, 23].

By using the quantum mutual information the conditional entropy (8) can be extended the set of all bipartite states ρ_{AB} with finite $H(\rho_A)$ as follows

$$H(A|B)_\rho = H(\rho_A) - I(A\!:\!B)_\rho. \tag{12}$$

This extension preserves all the basic properties of the conditional entropy (including concavity and inequality (9)) [10],[16, Sect.5].

A finite or countable collection $\{\rho_i\}$ of states with the corresponding probability distribution $\{p_i\}$ is conventionally called *ensemble* and denoted $\{p_i, \rho_i\}$. The state $\bar\rho = \sum_i p_i \rho_i$ is called the *average state* of this ensemble.

The Holevo quantity of an ensemble $\{p_i, \rho_i\}_{i=1}^m$ of $m \le +\infty$ quantum states is defined as

$$\chi\left(\{p_i, \rho_i\}_{i=1}^m\right) \doteq \sum_{i=1}^m p_i H(\rho_i \| \bar\rho) = H(\bar\rho) - \sum_{i=1}^m p_i H(\rho_i),$$

where the second formula is valid if $H(\bar\rho) < +\infty$. This quantity plays important role in analysis of information properties of quantum systems and channels [5, 23].

Let $\mathcal{H}_A = \mathcal{H}$ and $\{|i\rangle\}_{i=1}^m$ be an orthonormal basis in a m-dimensional Hilbert space $\mathcal{H}_B$. Then

$$\chi(\{p_i, \rho_i\}_{i=1}^m) = I(A\!:\!B)_{\hat\rho}, \text{ where } \hat\rho_{AB} = \sum_{i=1}^m p_i \rho_i \otimes |i\rangle\langle i|. \tag{13}$$

The *quantum conditional mutual information (QCMI)* of a state ρ_{ABC} of a tripartite finite-dimensional system is defined as

$$I(A\!:\!B|C)_\rho \doteq H(\rho_{AC}) + H(\rho_{BC}) - H(\rho_{ABC}) - H(\rho_C). \tag{14}$$

This quantity plays important role in quantum information theory [3, 23], its nonnegativity is a basic result well known as *strong subadditivity of von Neumann entropy* [12]. If system C is trivial then (14) coincides with (10).

In infinite dimensions formula (14) may contain the uncertainty "$\infty-\infty$". Nevertheless the conditional mutual information can be defined for any state ρ_{ABC} by one of the equivalent expressions

$$I(A\!:\!B|C)_\rho = \sup_{P_A} \left[I(A\!:\!BC)_{Q_A \rho Q_A} - I(A\!:\!C)_{Q_A \rho Q_A}\right],\ Q_A = P_A \otimes I_{BC}, \tag{15}$$

$$I(A\!:\!B|C)_\rho = \sup_{P_B}\left[I(B\!:\!AC)_{Q_B\rho Q_B} - I(B\!:\!C)_{Q_B\rho Q_B}\right],\ Q_B = P_B \otimes I_{AC},\ (16)$$

where the suprema are over all finite rank projectors $P_A \in \mathfrak{B}(\mathcal{H}_A)$ and $P_B \in \mathfrak{B}(\mathcal{H}_B)$ correspondingly and it is assumed that $I(X : Y)_{Q_X\rho Q_X} = \lambda I(X\!:\!Y)_{\lambda^{-1}Q_X\rho Q_X}$, where $\lambda = \mathrm{Tr}\,Q_X\rho_{ABC}$ [16].

Expressions (15) and (16) define the same lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H}_{ABC})$ possessing all basic properties of the quantum conditional mutual information valid in finite dimensions [16, Th.2]. In particular, the following relation (chain rule)

$$I(X\!:\!YZ|C)_\rho = I(X\!:\!Y|C)_\rho + I(X\!:\!Z|YC)_\rho \qquad (17)$$

holds for any state ρ in $\mathfrak{S}(\mathcal{H}_{XYZC})$ (with possible values $+\infty$ in both sides). To prove (17) is suffices to note that it holds if the systems X, Y, Z and C are finite-dimensional and to apply Corollary 9 in [16].

We will use the upper bound

$$I(A\!:\!B|C)_\rho \le 2\min\left\{H(\rho_A), H(\rho_B), H(\rho_{AC}), H(\rho_{BC})\right\} \qquad (18)$$

valid for any state ρ_{ABC}. It directly follows from upper bound (11) and the expression $I(X\!:\!Y|C)_\rho = I(X\!:\!YC)_\rho - I(X\!:\!C)_\rho$, $X, Y = A, B$, which is a partial case of (17).

The quantum conditional mutual information is not concave or convex but the following relation

$$\left|pI(A\!:\!B|C)_\rho + (1-p)I(A\!:\!B|C)_\sigma - I(A\!:\!B|C)_{p\rho+(1-p)\sigma}\right| \le h_2(p) \qquad (19)$$

holds for $p \in (0,1)$ and any states $\rho, \sigma \in \mathfrak{S}(\mathcal{H}_{ABC})$ with finite QCMI. If ρ and σ are states with finite marginal entropies then (19) can be easily proved by noting that

$$I(A\!:\!B|C)_\rho = H(A|C)_\rho - H(A|BC)_\rho,$$

and by using concavity of the conditional entropy and inequality (9). The validity of inequality (19) for any states ρ and σ with finite QCMI is proved by approximation (using Theorem 2B in [16]).

Let H_A be a positive operator in a Hilbert space $\mathcal{H}_A$ treated as a Hamiltonian of quantum system A. Then $\mathrm{Tr}H_A\rho$ is the (mean) energy of a state $\rho \in \mathfrak{S}(\mathcal{H}_A)$.[6] So,

$$\mathfrak{C}_{H_A,E} = \{\rho \in \mathfrak{S}(\mathcal{H}_A)\,|\,\mathrm{Tr}H_A\rho \le E\}, \quad E \ge E_0^A \doteq \inf_{\|\varphi\|=1}\langle\varphi|H_A|\varphi\rangle,$$

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[6]The value of $\mathrm{Tr}H_A\rho$ (finite or infinite) is defined as $\sup_n \mathrm{Tr}P_nH_A\rho$, where P_n is the spectral projector of H_A corresponding to the interval $[0, n]$.

is a closed convex subset of $\mathfrak{S}(\mathcal{H}_A)$ consisting of states with mean energy not exceeding E.

It is well known that the von Neumann entropy is continuous on the set $\mathfrak{C}_{H_A,E}$ for any $E \geq E_0^A$ if (and only if) the Hamiltonian H_A satisfies the condition

$$\mathrm{Tr} e^{-\lambda H_A} < +\infty \quad \text{for all } \lambda > 0 \tag{20}$$

and that it achieves the maximal value on this set at the *Gibbs state* $\gamma_A(E) = e^{-\lambda(E)H_A}/\mathrm{Tr} e^{-\lambda(E)H_A}$, where the parameter $\lambda(E)$ is determined by the equality $\mathrm{Tr} H_A e^{-\lambda(E)H_A} = E \mathrm{Tr} e^{-\lambda(E)H_A}$ [22].

Condition (20) implies that H_A is an unbounded operator having a discrete spectrum of finite multiplicity, i.e. it can be represented as follows

$$H_A = \sum_{k=0}^{+\infty} E_k^A |\tau_k\rangle\langle\tau_k|, \tag{21}$$

where $\{E_k^A\}$ is the nondecreasing sequence of eigenvalues of H_A tending to $+\infty$ and $\{|\tau_k\rangle\}$ – the corresponding basis of eigenvectors.

In what follows we will use the function

$$F_{H_A}(E) \doteq \sup_{\rho \in \mathfrak{C}_{H_A,E}} H(\rho) = H(\gamma_A(E)).$$

It is easy to show that F_{H_A} is a strictly increasing concave function on $[E_0^A, +\infty)$ such that $F_{H_A}(E_0) = \log d_0$, where d_0 is the multiplicity of the eigenvalue E_0^A [17, 26].

In this paper we will use the modification of the Alicki-Fannes-Winter method[7] adapted for the set of states with bounded energy [19]. This modification makes it possible to prove uniform continuity of any locally almost affine function[8] f on the set

$$\mathfrak{C}_{H_A,E}^{\mathrm{ext}} \doteq \{\rho \in \mathfrak{S}(\mathcal{H}_{AB}) \,|\, \rho_A \in \mathfrak{C}_{H_A,E}\} \qquad (B \text{ is any given system})$$

such that $|f(\rho_{AB})| \leq CH(\rho_A)$ for some $C \in \mathbb{R}_+$ provided that

$$F_{H_A}(E) = o(\sqrt{E}) \quad \text{as} \quad E \to +\infty. \tag{22}$$

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[7] This method is widely used in finite-dimensions for proving uniform continuity of functions on the set of quantum states [2, 26].

[8] This means that $|f(p\rho + (1-p)\sigma) - pf(\rho) - (1-p)f(\sigma)| \leq r(p) = o(1)$ as $p \to +0$.

By Lemma 1 in [19] condition (22) holds if and only if

$$\lim_{\lambda \to +0} \left[\mathrm{Tr} e^{-\lambda H_A}\right]^{\lambda} = 1. \tag{23}$$

Condition (23) is stronger than condition (20) (equivalent to $F_{H_A}(E) = o(E)$) but the difference between these conditions is not too large. In terms of the sequence $\{E_k^A\}$ of eigenvalues of H_A condition (20) means that $\lim_{k \to \infty} E_k^A / \log k = +\infty$, while (23) is valid if $\liminf_{k \to \infty} E_k^A / \log^q k > 0$ for some $q > 2$ [19, Pr.1].

It is essential that condition (23) holds for the Hamiltonian of the multimode quantum oscillator playing central role in continuous variable quantum information theory [5, 21].

If A is the ℓ-mode quantum oscillator with frequencies $\omega_1, ..., \omega_\ell$ then

$$F_{H_A}(E) = \max_{\{E_i\}} \sum_{i=1}^{\ell} g(E_i / \hbar\omega_i - 1/2), \quad E \geq E_0 \doteq \frac{1}{2} \sum_{i=1}^{\ell} \hbar\omega_i,$$

where $g(x) = (x+1)\log(x+1) - x\log x$ and the maximum is over all ℓ-tuples $E_1, ..., E_\ell$ such that $\sum_{i=1}^{\ell} E_i = E$ and $E_i \geq \frac{1}{2}\hbar\omega_i$ [5, Ch.12][26]. The exact value of $F_{H_A}(E)$ can be calculated by applying the Lagrange multiplier method which leads to a transcendental equation. But following [26] one can obtain ε-sharp upper bound for $F_{H_A}(E)$ by using the inequality $g(x) \leq \log(x+1) + 1$ valid for all $x > 0$. It implies

$$F_{H_A}(E) \leq \max_{\sum_{i=1}^{\ell} E_i = E} \sum_{i=1}^{\ell} \log(E_i / \hbar\omega_i + 1/2) + \ell.$$

By calculating this maximum via the Lagrange multiplier method we obtain

$$F_{H_A}(E) \leq \widehat{F}_{\ell,\omega}(E) \doteq \ell \log \frac{E + E_0}{\ell E_*} + \ell, \quad E_* = \left[\prod_{i=1}^{\ell} \hbar\omega_i\right]^{1/\ell}. \tag{24}$$

It is clear that the function $\widehat{F}_{\ell,\omega}$ satisfies condition (22). So, it can be used in the role of F_{H_A} in all the results obtained by the modified Alicki-Fannes-Winter method (in particular, in the below Lemmas 2 and 3).

We will use the following simple lemma (see Corollary 12 in [26]).

Lemma 1. *If f is a concave nonnegative function on $[0, +\infty)$ then for any positive $x < y$ and any $z \geq 0$ the following inequality holds*

$$xf(z/x) \leq yf(z/y).$$

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3 Basic lemmas

In the following two lemmas essentially used in the paper we will employ the function $g(x) \doteq (1+x)h_2\left(\frac{x}{1+x}\right) = (x+1)\log(x+1) - x\log x$, $x > 0$.

By applying the modification of the Alicki-Fannes-Winter method mentioned in Section 2 to the QCMI defined in (15),(16) we obtain the following

Lemma 2. *Let ρ and σ be states in $\mathfrak{S}(\mathcal{H}_{ABCD})$ s.t. $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon < \frac{1}{2}$. Let $\mathcal{H}_*$ be a subspace of $\mathcal{H}_{AD}$ containing the supports of ρ_{AD} and σ_{AD}. If $\mathrm{Tr}H_*\rho_{AD}$, $\mathrm{Tr}H_*\sigma_{AD} \leq E < +\infty$ for some positive operator H_* in $\mathcal{H}_*$ satisfying condition (23) then $I(A\!:\!B|C)_\rho$ and $I(A\!:\!B|C)_\sigma$ are finite and*

$$|I(A\!:\!B|C)_\rho - I(A\!:\!B|C)_\sigma| \leq 2\sqrt{2\varepsilon}F_{H_*}(E/\varepsilon) + 2g(\sqrt{2\varepsilon}), \qquad (25)$$

where $F_{H_*}(E) \doteq \sup\{H(\rho)\,|\,\mathrm{supp}\rho \subseteq \mathcal{H}_*, \mathrm{Tr}H_*\rho \leq E\}$.

If $\rho_{BC} = \sigma_{BC}$ then (25) holds with $2g(\sqrt{2\varepsilon})$ replaced by $g(\sqrt{2\varepsilon})$.

If ρ and σ are pure states then (25) and its specification for the case $\rho_{BC} = \sigma_{BC}$ hold with ε replaced by $\varepsilon^2/2$.

Since condition (23) implies that $F_{H_*}(E) = o(\sqrt{E})$ as $E \to +\infty$, the right hand side of (25) tends to zero as $\varepsilon \to 0^+$.

Proof. We may consider $I(A:B|C)$ as a function on $\mathfrak{S}(\mathcal{H}_{BC} \otimes \mathcal{H}_*)$. Continuity bound (25) and its specification for pure states ρ and σ can be directly obtained from Proposition 1 in [19] by using inequality (19) and the inequalities

$$0 \leq I(A\!:\!B|C)_\omega \leq I(A\!:\!BC)_\omega \leq I(AD\!:\!BC)_\omega \leq 2H(\omega_{AD})$$

valid for any state ω in $\mathfrak{S}(\mathcal{H}_{ABCD})$, which follow from the basic properties of QCMI and upper bound (11).

To prove the specification of (25) for the case $\rho_{BC} = \sigma_{BC}$ we have to repeat several steps from the proof of Theorem 1 in [19].

Assume first that $\mathrm{rank}\rho_B = \mathrm{rank}\sigma_B < +\infty$. Then

$$I(A\!:\!B|C)_\omega = H(B|C)_\omega - H(B|AC)_\omega, \quad \omega = \rho, \sigma, \qquad (26)$$

where $H(X|Y)$ is the extended conditional entropy defined in (12).

Let $\hat{\rho}$ and $\hat{\sigma}$ be purifications of the states ρ and σ such that $\frac{1}{2}\|\hat{\rho} - \hat{\sigma}\|_1 = \delta \doteq \sqrt{2\varepsilon}$ and $\hat{\tau}_\pm = \delta^{-1}[\hat{\rho} - \hat{\sigma}]_\pm$. Then

$$\frac{1}{1+\delta}\rho + \frac{\delta}{1+\delta}\tau_- = \omega_* = \frac{1}{1+\delta}\sigma + \frac{\delta}{1+\delta}\tau_+, \qquad (27)$$

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where $\tau_{\pm} = [\hat\tau_{\pm}]_{ABCD}$ (see [19]). It is easy to see that $\mathrm{rank}[\tau_{\pm}]_B < +\infty$. So, representation (26) holds for $\omega = \tau_{\pm}$ as well. Since the assumption $\rho_{BC} = \sigma_{BC}$ and (27) imply $[\tau_+]_{BC} = [\tau_-]_{BC}$, we obtain from (26) that

$$I(A\!:\!B|C)_{\omega_1} - I(A\!:\!B|C)_{\omega_2} = H(B|AC)_{\omega_2} - H(B|AC)_{\omega_1} \qquad (28)$$

for $(\omega_1, \omega_2) = (\rho, \sigma), (\tau_+, \tau_-)$.

By applying concavity of the conditional entropy and inequality (9) to the convex decompositions (27) of ω_* and taking (28) into account we obtain

$$(1-p)\,[I(A\!:\!B|C)_\rho - I(A\!:\!B|C)_\sigma] = (1-p)\,[H(B|AC)_\sigma - H(B|AC)_\rho]$$

$$\leq p\,[H(B|AC)_{\tau_-} - H(B|AC)_{\tau_+}] + h_2(p)$$

$$= p\,[I(A\!:\!B|C)_{\tau_+} - I(A\!:\!B|C)_{\tau_-}] + h_2(p),$$

where $p = \frac{\delta}{1+\delta}$. Similarly,

$$(1-p)\,[I(A\!:\!B|C)_\sigma - I(A\!:\!B|C)_\rho] \leq p\,[I(A\!:\!B|C)_{\tau_-} - I(A\!:\!B|C)_{\tau_+}] + h_2(p).$$

Since $0 \leq I(A:B|C) \leq I(AD:B|C)$, these inequalities show that the left hand side of (25) does not exceed

$$\delta \max\left\{I(AD\!:\!B|C)_{\tau_-}, I(AD\!:\!B|C)_{\tau_+}\right\} + g(\delta). \qquad (29)$$

By the proof of Theorem 1 in [19] the assumption $\mathrm{Tr}H_*\rho_{AD}, \mathrm{Tr}H_*\sigma_{AD} \leq E$ implies $\mathrm{Tr}H_*[\tau_{\pm}]_{AD} \leq E/\varepsilon$. So, by using (18) we obtain

$$I(AD\!:\!B|C)_{\tau_{\pm}} \leq 2H([\tau_{\pm}]_{AD}) \leq 2F_{H_*}(E/\varepsilon)$$

and hence the quantity in (29) does not exceed the right hand side of (25) with $2g(\sqrt{2\varepsilon})$ replaced by $g(\sqrt{2\varepsilon})$.

Assume now that ρ and σ are arbitrary states such that $\rho_{BC} = \sigma_{BC}$. Let $\{P_B^n\}$ be a sequence of finite rank projectors in $\mathcal{H}_B$ strongly converging to the unit operator I_B. Consider two sequences consisting of the states

$$\rho^n = r_n^{-1}P_B^n \otimes I_{ADC}\,\rho P_B^n \otimes I_{ADC} \quad \text{and} \quad \sigma^n = r_n^{-1}P_B^n \otimes I_{ADC}\,\sigma P_B^n \otimes I_{ADC},$$

where $r_n = \mathrm{Tr}P_B^n\rho_B = \mathrm{Tr}P_B^n\sigma_B$ (here and in what follows we assume that n is sufficiently large). It is easy to see that $r_n\rho_{AD}^n \leq \rho_{AD}$ and $r_n\sigma_{AD}^n \leq \sigma_{AD}$ for all n. So, we have

$$\mathrm{Tr}H_*\rho_{AD}^n,\ \mathrm{Tr}H_*\sigma_{AD}^n \leq r_n^{-1}E.$$

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Take any sequence $\{\varepsilon_n\}$ tending to ε such that $\frac{1}{2}\|\rho^n - \sigma^n\|_1 \le \varepsilon_n < \frac{1}{2}$ for all n. Since $\rho^n_{BC} = \sigma^n_{BC}$ and $\mathrm{rank}\rho^n_B = \mathrm{rank}\sigma^n_B < +\infty$, the above part of the proof implies that

$$|I(A\!:\!B|C)_{\rho^n} - I(A\!:\!B|C)_{\sigma^n}| \le 2\sqrt{2\varepsilon_n}F_{H_*}(E/(r_n\varepsilon_n)) + g(\sqrt{2\varepsilon_n}). \qquad (30)$$

By using the lower semicontinuity of the function $\omega \to I(A\!:\!B|C)_\omega$ and its monotonicity under local operations (Th.2 in [16]) it is easy to show that

$$\lim_{n\to\infty} I(A\!:\!B|C)_{\omega^n} = I(A\!:\!B|C)_\omega, \quad \omega = \rho, \sigma.$$

So, passing to the limit in (30) implies (25) with $2g(\sqrt{2\varepsilon})$ replaced by $g(\sqrt{2\varepsilon})$.

If ρ and σ are pure states then we can take pure states $\hat{\rho} = \rho \otimes \varrho$ and $\hat{\sigma} = \sigma \otimes \varsigma$ such that $\frac{1}{2}\|\hat{\rho} - \hat{\sigma}\|_1 = \varepsilon$ and repeat the above arguments. $\square$

By using Lemma 2 and the Leung-Smith telescopic trick from [11] one can prove the following lemma in which we will assume that H_A is the Hamiltonian of system A having form (21). We will use the function

$$\bar{F}_{H_A}(E) = F_{H_A}(E + E_0^A), \quad \text{where} \quad F_{H_A}(E) \doteq \sup_{\mathrm{Tr}H_A\rho \le E} H(\rho),$$

and the notations $\bar{E} = E - E_0^A$, $\bar{E}_m^A = E_m^A - E_0^A$ for all $m > 0$.

Lemma 3. *Let* $\Pi_m(\rho) = P_m\rho P_m + [\mathrm{Tr}(I_A - P_m)\rho]|\tau_0\rangle\langle\tau_0|$, *where* P_m *is the projector on the subspace* $\mathcal{H}_A^m$ *corresponding to the minimal* m *eigenvalues* $E_0^A, .., E_{m-1}^A$ *of* H_A *and* τ_0 *is any eigenvector corresponding to the eigenvalue* E_0^A. *Let* ρ *be a state in* $\mathfrak{S}(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_R)$ *such that* $\sum_{k=1}^n \mathrm{Tr}H_A\rho_{A_k} \le nE$. *If* H_A *satisfies condition (23) then*

$$\left|I(B^n\!:\!R)_{\Phi^{\otimes n}\otimes\mathrm{Id}_R(\rho)} - I(B^n\!:\!R)_{\Psi_m^{\otimes n}\otimes\mathrm{Id}_R(\rho)}\right| \le nf(E, m), \qquad (31)$$

for any channel $\Phi : A \to B$, *where* $\Psi_m = \Phi \circ \Pi_m$ *and*

$$f(E, m) \doteq 4\sqrt[4]{\frac{\bar{E}}{\bar{E}_m^A}}\bar{F}_{H_A}\left(\frac{1}{2}\sqrt{\bar{E}\bar{E}_m^A}\right) + g\left(2\sqrt[4]{\frac{\bar{E}}{\bar{E}_m^A}}\right) + \frac{32\bar{E}}{\bar{E}_m^A}\bar{F}_{H_A}\left(\frac{\bar{E}_m^A}{16}\right). \qquad (32)$$

is a quantity tending to zero as $m \to +\infty$ *for each* $E > E_0^A$.

If $\bar{E} < \bar{E}_m^A/16$ and $\mathrm{Tr} H_A \rho_{A_k} \le E$ for all $k = \overline{1,n}$ then the last term in (32) can be removed. If $n = 1$ and $s \doteq \bar{E}/\bar{E}_m^A + \sqrt{\bar{E}/\bar{E}_m^A} < 1/2$ then $f(E, m)$ in (31) can be replaced by the quantity

$$2\sqrt{2s}\bar{F}_{H_A}\left(\frac{\bar{E}}{s}\right) + g\left(\sqrt{2s}\right).$$

If A is the ℓ-mode quantum oscillator with frequencies $\omega_1, ..., \omega_\ell$ then the function $\bar{F}_{H_A}(E)$ in all the above formulas can be replaced by its upper bound $\widehat{F}_{\ell,\omega}(E + E_0^A)$, where $\widehat{F}_{\ell,\omega}(E)$ is defined in (24). In this case the sequence $\{E_k^A\}_{k\ge 0}$ consists of the numbers $\sum_{i=1}^{\ell} \hbar\omega_i(n_i - 1/2), n_1, ..., n_\ell \in \mathbb{N}$ arranged in the nondecreasing order.

Remark 1. The below proof of Lemma 3 shows that its assertion can be generalized by replacing the quantum mutual information $I(B^n : R)$ in (31) by the (extended) quantum conditional mutual information $I(B^n : R|C)$ defined by the equivalent expressions (15) and (16).

Proof. The assumption of the lemma implies $H(\rho_{A_k}) < +\infty$ for $k = \overline{1,n}$.

Let E be an environment for the channel Φ, so that the Stinespring representations (3) holds with some isometry V_Φ from $\mathcal{H}_A$ into $\mathcal{H}_{BE}$.

Following the Leung-Smith telescopic method from [11] consider the states

$$\sigma_k = \Phi^{\otimes k} \otimes \Psi_m^{\otimes(n-k)} \otimes \mathrm{Id}_R(\rho), \quad k = 0, 1, ..., n.$$

We have

$$
\begin{aligned}
|I(B^n : R)_{\sigma_n} - I(B^n : R)_{\sigma_0}| &= \left| \sum_{k=1}^{n} I(B^n : R)_{\sigma_k} - I(B^n : R)_{\sigma_{k-1}} \right| \\
&\le \sum_{k=1}^{n} \left| I(B^n : R)_{\sigma_k} - I(B^n : R)_{\sigma_{k-1}} \right|.
\end{aligned}
\tag{33}
$$

By using the chain rule (17) we obtain for each k

$$
\begin{aligned}
I(B^n\!:\!R)_{\sigma_k} - I(B^n\!:\!R)_{\sigma_{k-1}} &= I(B_1...B_{k-1}B_{k+1}...B_n\!:\!R)_{\sigma_k} \\
&\quad + I(B_k\!:\!R|B_1...B_{k-1}B_{k+1}...B_n)_{\sigma_k} \\
&\quad - I(B_1...B_{k-1}B_{k+1}...B_n\!:\!R)_{\sigma_{k-1}} \\
&\quad - I(B_k\!:\!R|B_1...B_{k-1}B_{k+1}...B_n)_{\sigma_{k-1}} \\
&= I(B_k\!:\!R|B_1...B_{k-1}B_{k+1}...B_n)_{\sigma_k} \\
&\quad - I(B_k\!:\!R|B_1...B_{k-1}B_{k+1}...B_n)_{\sigma_{k-1}},
\end{aligned}
\tag{34}
$$

where it was used that $\mathrm{Tr}_{B_k}\sigma_k = \mathrm{Tr}_{B_k}\sigma_{k-1}$. Note that the finite entropy of the states $\rho_{A_1},...,\rho_{A_n}$, upper bound (18) and monotonicity of the QCMI under local channels guarantee finiteness of all the terms in (33) and (34).

To estimate the last difference in (34) consider the states

$$
\hat{\sigma}_k = V_\Phi^{\otimes n} \otimes I_R \, \varrho_k \, [V_\Phi^{\otimes n}]^* \otimes I_R
$$

in $\mathfrak{S}(\mathcal{H}_{B^n E^n R})$, where $\varrho_k = \mathrm{Id}_A^{\otimes k} \otimes \Pi_m^{\otimes(n-k)} \otimes \mathrm{Id}_R(\rho)$, $k = 0,1,2,...,n$. The state $\hat{\sigma}_k$ is an extension of the state σ_k for each k, i.e. $\mathrm{Tr}_{E^n}\hat{\sigma}_k = \sigma_k$. Note that $[\varrho_k]_{A_j} = \rho_{A_j}$ for $j \leq k$ and $[\varrho_k]_{A_j} = \Pi_m(\rho_{A_j})$ for $j > k$. Hence

$$
\mathrm{Tr}H_A[\varrho_k]_{A_j} \leq x_j \doteq \mathrm{Tr}H_A\rho_{A_j} \quad \text{for all } k \text{ and } j.
\tag{35}
$$

By using monotonicity of the trace norm under action of a channel and Lemmas 4,5 below we obtain

$$
\|\hat{\sigma}_k - \hat{\sigma}_{k-1}\|_1 = \|\varrho_k - \varrho_{k-1}\|_1
$$

$$
= \left\| \mathrm{Id}_A^{\otimes k} \otimes \Pi_m^{\otimes(n-k)} \otimes \mathrm{Id}_R \left(\rho - \mathrm{Id}_A^{\otimes(k-1)} \otimes \Pi_m \otimes \mathrm{Id}_A^{\otimes(n-k)} \otimes \mathrm{Id}_R(\rho) \right) \right\|_1
\tag{36}
$$

$$
\leq \|\rho - \mathrm{Id}_A^{\otimes(k-1)} \otimes \Pi_m \otimes \mathrm{Id}_A^{\otimes(n-k)} \otimes \mathrm{Id}_R(\rho)\|_1
$$

$$
\leq 2\mathrm{Tr}(I_A - P_m)\rho_{A_k} + 2\sqrt{\mathrm{Tr}(I_A - P_m)\rho_{A_k}} \leq 2\varepsilon_k,
$$

where $\varepsilon_k \doteq 2\sqrt{\bar{x}_k/\bar{E}_m^A}$, $\bar{x}_k = x_k - E_0^A$.

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Let N_1 be the set of all indexes k for which $\bar{x}_k < \bar{E}_m^A/16$ and $N_2 = \{1,..,n\} \setminus N_1$.[9] Let $n_i = \sharp(N_i)$ and $X_i = \frac{1}{n_i} \sum_{k \in N_i} x_k$ and $\bar{X}_i = X_i - E_0^A$, $i = 1, 2$. It follows from (33) and (34) that the left hand side of (31) do not exceed $S_1 + S_2$, where

$$S_i = \sum_{k \in N_i} |I(B_k\!:\!R|Y_k)_{\sigma_k} - I(B_k\!:\!R|Y_k)_{\sigma_{k-1}}|, \quad Y_k = B_1...B_{k-1}B_{k+1}...B_n.$$

For each $k \in N_1$ we have $\varepsilon_k < 1/2$. So, by using (35) and (36) and by noting that $\mathrm{Tr}_{B_k}\sigma_k = \mathrm{Tr}_{B_k}\sigma_{k-1}$ we obtain from Lemma 2 with $\mathcal{H}_* = V_\Phi \mathcal{H}_{A_k} \subseteq \mathcal{H}_{B_k E_k}$ and $H_* = V_\Phi H_A V_\Phi^* - E_0^A I_{\mathcal{H}_*}$ that

$$|I(B_k\!:\!R|Y_k)_{\sigma_k} - I(B_k\!:\!R|Y_k)_{\sigma_{k-1}}| \le 2\sqrt{2\varepsilon_k}\bar{F}_{H_A}(\bar{x}_k/\varepsilon_k) + g(\sqrt{2\varepsilon_k})$$

$$= 4\sqrt[4]{\bar{x}_k/\bar{E}_m^A}\,\bar{F}_{H_A}\!\left(\frac{1}{2}\sqrt{\bar{x}_k\bar{E}_m^A}\right) + g\!\left(2\sqrt[4]{\bar{x}_k/\bar{E}_m^A}\right).$$

Hence, by using the concavity[10] of the functions $\sqrt[4]{x}\,\bar{F}_{H_A}(\sqrt{x})$, $\sqrt[4]{x}$ and $g(x)$ along with the monotonicity of $g(x)$ we obtain

$$
\begin{aligned}
S_1 &\le \sum_{k \in N_1} 4\sqrt[4]{\bar{x}_k/\bar{E}_m^A}\,\bar{F}_{H_A}\!\left(\frac{1}{2}\sqrt{\bar{x}_k\bar{E}_m}\right) + \sum_{k \in N_1} g\!\left(2\sqrt[4]{\bar{x}_k/\bar{E}_m^A}\right) \\
&\le n_1 4\sqrt[4]{\bar{X}_1/\bar{E}_m^A}\,\bar{F}_{H_A}\!\left(\frac{1}{2}\sqrt{\bar{X}_1\bar{E}_m^A}\right) + n_1 g\!\left(2\sqrt[4]{\bar{X}_1/\bar{E}_m^A}\right).
\end{aligned}
\tag{37}
$$

For each $k \in N_2$ the inequality $I(B_k\!:\!R|Y_k) \le I(B_k E_k\!:\!R|Y_k)$ and upper bound (18) imply

$$|I(B_k\!:\!R|Y_k)_{\sigma_k} - I(B_k\!:\!R|Y_k)_{\sigma_{k-1}}| \le 2\max\{H([\hat{\sigma}_k]_{B_k E_k}), H([\hat{\sigma}_{k-1}]_{B_k E_k})\}$$

$$= 2\max\{H([\varrho_k]_{A_k}), H([\varrho_{k-1}]_{A_k})\} \le 2F_{H_A}(x_k),$$

where the last inequality follows from (35). Since $(n - n_2)X_1 + n_2 X_2 \le nE$ and $X_1 \ge E_0^A$, we have $X_2 \le n\bar{E}/n_2 + E_0^A$. So, by using concavity and monotonicity of the function F_{H_A} we obtain

$$S_2 \le \sum_{k \in N_2} 2F_{H_A}(x_k) \le 2n_2 F_{H_A}(X_2) \le 2n_2 \bar{F}_{H_A}(n\bar{E}/n_2). \tag{38}$$

---

[9]Similar splitting is used in the proof of Lemma 7 in [27].

[10]The concavity of the function $\sqrt[4]{x}\,\bar{F}_{H_A}(\sqrt{x})$ follows from the concavity and nonnegativity the function $\bar{F}_{H_A}(x)$. This can be shown by calculation of the second derivative.

It is easy to see that $\bar{X}_1 \le \bar{E}$. Since $\bar{x}_k > \bar{E}_m^A/16$ for all $k \in N_2$ and $(n - n_2)E_0^A + \sum_{k \in N_2} \bar{x}_k + n_2 E_0^A \le \sum_{k \in N_1} x_k + \sum_{k \in N_2} x_k \le nE$, we have $n_2/n \le 16\bar{E}/\bar{E}_m^A$. So, it follows from (37),(38) and Lemma 1 that

$$\frac{S_1 + S_2}{n} \le 4\sqrt[4]{\frac{\bar{E}}{\bar{E}_m^A}}\,\bar{F}_{H_A}\left(\frac{1}{2}\sqrt{\bar{E}\bar{E}_m^A}\right) + g\left(2\sqrt[4]{\frac{\bar{E}}{\bar{E}_m^A}}\right) + \frac{32\bar{E}}{\bar{E}_m^A}\bar{F}_{H_A}\left(\frac{\bar{E}_m^A}{16}\right).$$

The vanishing of the quantity $f(E,m)$ as $m \to +\infty$ follows from Lemma 1 in [19] stating the equivalence of (22) and (23).

The assertion concerning the case $\mathrm{Tr}H_A\rho_{A_k} \le E$ for all $k = \overline{1,n}$ follow from the above proof, since in this case the set N_2 is empty. In the case $n = 1$ one can directly apply Lemma 2 with trivial C, $\mathcal{H}_* = V_\Phi\mathcal{H}_A \subseteq \mathcal{H}_{BE}$ and $H_* = V_\Phi H_A V_\Phi^* - E_0^A I_{\mathcal{H}_*}$ by using (36) with $k = 1$. $\square$

Lemma 4. *Let $\Pi : A \to A$ be the channel defined by the formula $\Pi(\rho) = P\rho P + [\mathrm{Tr}(I_A - P)\rho]\tau$, $\rho \in \mathfrak{T}(\mathcal{H}_A)$, where P is an orthogonal projector and τ is any state in $\mathfrak{S}(\mathcal{H}_A)$. Then for arbitrary state $\omega \in \mathfrak{S}(\mathcal{H}_{AB})$, where B is any system, the following inequality holds*

$$\|\omega - \Pi \otimes \mathrm{Id}_B(\omega)\|_1 \le 2\mathrm{Tr}(I_A - P)\omega_A + 2\sqrt{\mathrm{Tr}(I_A - P)\omega_A}.$$

Proof. The required inequality is easily obtained from inequality (2).

Lemma 5. *Let H_A be a positive operator in $\mathcal{H}_A$ having form (21) and P_m the projector on the subspace $\mathcal{H}_A^m$ corresponding to the minimal m eigenvalues $E_0^A,..,E_{m-1}^A$ of H_A. Then for any state $\rho \in \mathfrak{S}(\mathcal{H}_A)$ such that $\mathrm{Tr}H_A\rho \le E$ the following inequality holds*

$$\mathrm{Tr}(I_A - P_m)\rho \le (E - E_0^A)/(E_m^A - E_0^A).$$

Proof. Since $\mathrm{Tr}(I_A - P_m)\rho = 1 - \mathrm{Tr}P_m\rho$, the required inequality follows directly from the inequalities

$$E_0^A\mathrm{Tr}P_m\rho \le \mathrm{Tr}P_m H_A\rho, \quad E_m^A\mathrm{Tr}(I_A - P_m)\rho \le \mathrm{Tr}(I_A - P_m)H_A\rho. \ \square$$

4 Capacities of energy-constrained infinite-dimensional channels and their approximation

In this section we show that dealing with basic capacities of energy constrained infinite-dimensional channels from a given system to any other sys-

tems we may consider (accepting arbitrarily small error $\varepsilon > 0$) that all these channels have *the same finite-dimensional input space* – the subspace corresponding to the minimal eigenvalues of the input Hamiltonian. For each of the capacities the dimension of this subspace is explicitly determined by ε.

4.1 Survey of basic capacities

When we consider transmission of classical or quantum information over infinite-dimensional quantum channels we have to impose constraints on states used for encoding information. A typical physically motivated constraint is the requirement of boundedness of states-codes average energy. For a single channel this constraint is expressed by the inequality

$$\mathrm{Tr} H_A \rho \leq E, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \tag{39}$$

where H_A is the Hamiltonian of the input quantum system A, for n-copies of a channel it can be written as follows

$$\mathrm{Tr} H_{A^n} \rho \leq nE, \quad \rho \in \mathfrak{S}(\mathcal{H}_A^{\otimes n}), \tag{40}$$

where $H_{A^n} = H_A \otimes I_A \otimes \ldots \otimes I_A + \ldots + I_A \otimes \ldots \otimes I_A \otimes H_A$ is the Hamiltonian of the system A^n (n copies of A) [5, 6, 24].

We will assume that the Hamiltonian H_A satisfies condition (20).

The Holevo capacity of a channel $\Phi : A \to B$ with the (input) energy constraint is defined as:

$$C_\chi(\Phi, H_A, E) = \sup_{\mathrm{Tr} H_A \bar{\rho} \leq E} \chi(\{p_i, \Phi(\rho_i)\}),$$

where the supremum is over all input ensembles $\{p_i, \rho_i\}$ with the average energy $\sum_i p_i \mathrm{Tr} H_A \rho_i = \mathrm{Tr} H_A \bar{\rho}$ not exceeding E. This quantity determines the ultimate rate of transmission of classical information through the channel Φ by using nonentangled block encoding, for many channels it coincides with the classical capacity under the energy constraint [4, 5, 6].

Operational definition of the classical capacity of energy-constrained infinite-dimensional channels is presented in [6]. By the Holevo-Schumacher-Westmoreland theorem adapted for constrained channels ([6, Proposition 3]) the classical capacity of any channel $\Phi : A \to B$ with constraint (40) is given by the regularized expression

$$C(\Phi, H_A, E) = \lim_{n \to +\infty} n^{-1} C_\chi(\Phi^{\otimes n}, H_{A^n}, nE).$$

The entanglement-assisted classical capacity of a quantum channel determines the ultimate rate of transmission of classical information when an entangled state between the input and the output of a channel is used as an additional resource (see details in [5, 23]). Operational definition of the entanglement-assisted classical capacity of energy-constrained infinite-dimensional channels is presented in [6]. By the most general version of the Bennett-Shor-Smolin-Thaplyal theorem for energy-constrained infinite-dimensional channels ([7, Theorem 1]) the classical entanglement-assisted capacity of any channel $\Phi : A \to B$ with constraint (40) determined by arbitrary positive operator H_A is given by the expression

$$C_{\mathrm{ea}}(\Phi, H_A, E) = \sup_{\mathrm{Tr} H_A \rho \leq E} I(\Phi, \rho),$$

in which $I(\Phi, \rho)$ is the quantum mutual information of a channel Φ at a state ρ defined as

$$I(\Phi, \rho) = I(B\!:\!R)_{\Phi \otimes \mathrm{Id}_R(\hat{\rho})},$$

where $\mathcal{H}_R \cong \mathcal{H}_A$ and $\hat{\rho}$ is a pure state in $\mathfrak{S}(\mathcal{H}_{AR})$ such that $\hat{\rho}_A = \rho$.

Detailed analysis of the energy-constrained quantum and private capacities in the context of general-type infinite-dimensional channels[11] has been made recently by Wilde and Qi in [24]. The results in [24] and [25] give considerable reasons to conjecture validity of the following generalizations of the Lloyd-Devetak-Shor theorem and of the Devetak theorem to constrained infinite-dimensional channels:

- the quantum capacity of any channel $\Phi : A \to B$ with constraint (40) is given by the regularized expression

$$Q(\Phi, H_A, E) = \lim_{n \to +\infty} n^{-1} \bar{Q}(\Phi^{\otimes n}, H_{A^n}, nE),$$

  where $\bar{Q}(\Phi, H_A, E)$ is the supremum of the coherent information $I_c(\Phi, \rho) \doteq I(\Phi, \rho) - H(\rho)$ on the set of all input states $\rho \in \mathfrak{S}(\mathcal{H}_A)$ satisfying (39).

- the private capacity of any channel $\Phi : A \to B$ with constraint (40) is given by the regularized expression

$$C_{\mathrm{p}}(\Phi, H_A, E) = \lim_{n \to +\infty} n^{-1} \bar{C}_{\mathrm{p}}(\Phi^{\otimes n}, H_{A^n}, nE),$$

---

[11]There are many papers devoted to analysis of these capacities for Gaussian channels, see [8, 28] and the surveys in [21, 24].

where

$$\bar{C}_{\mathrm{p}}(\Phi, H_A, E) = \sup_{\mathrm{Tr} H_A \bar\rho \le E} \left[\chi(\{p_i, \Phi(\rho_i)\}) - \chi(\{p_i, \widehat{\Phi}(\rho_i)\}) \right] \qquad (41)$$

(the supremum is over all input ensembles $\{p_i, \rho_i\}$ with the average energy not exceeding E and $\widehat{\Phi}$ is the complementary channel to the channel Φ defined in (4)).

4.2 Uniform finite-dimensional approximation theorem.

Assume that H_A is an unbounded operator in $\mathcal{H}_A$ with dense domain having discrete spectrum of finite multiplicity, i.e. it can be represented as follows

$$H_A = \sum_{k=0}^{+\infty} E_k^A |\tau_k\rangle\langle\tau_k|,$$

where $\{E_k^A\}$ is the nondecreasing sequence of eigenvalues of H_A tending to $+\infty$ and $\{|\tau_k\rangle\}$ – the corresponding basis of eigenvectors. Denote by $\mathcal{H}_A^m$ the linear span of the vectors $|\tau_0\rangle, ..., |\tau_{m-1}\rangle$, i.e. $\mathcal{H}_A^m$ is the subspace corresponding to the minimal m eigenvalues of H_A (taking the multiplicity into account). Let P_m be the projector onto $\mathcal{H}_A^m$.

For a given channel $\Phi : A \to B$ denote by Φ_m the restriction of Φ to the Banach space $\mathfrak{T}(\mathcal{H}_A^m)$ of all operators in $\mathfrak{T}(\mathcal{H}_A)$ supported by $\mathcal{H}_A^m$. The channel Φ_m can be called the *subchannel of Φ corresponding to the subspace* $\mathcal{H}_A^m$. Since $H_A^m = P_m H_A$ is a positive (bounded) operator in $\mathfrak{B}(\mathcal{H}_A^m)$, we may consider the capacities

$$C_*(\Phi_m, H_A^m, E), \quad C_* = C_\chi, C, C_{\mathrm{ea}}, Q, C_{\mathrm{p}}.$$

These capacities can be treated as the corresponding capacities of Φ obtained by block encoding used only states supported by the tensor powers of the m-dimensional subspace $\mathcal{H}_A^m$. We will call they *m-restricted* capacities and will use the notations $C_*^m(\Phi, H_A, E) \doteq C_*(\Phi_m, H_A^m, E)$, $C_* = C_\chi, C, C_{\mathrm{ea}}, Q, C_{\mathrm{p}}$.

The following theorem states that any m-restricted capacity $C_*^m(\Phi, H_A, E)$ tends to the corresponding capacity $C_*(\Phi, H_A, E)$ as $m \to +\infty$ *uniformly on the set of all channels* from a given system A to any other systems and gives

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explicit estimates for the rate of this convergence. In this theorem we use the function

$$\bar{F}_{H_A}(E) = F_{H_A}(E + E_0^A), \quad \text{where} \quad F_{H_A}(E) \doteq \sup_{\mathrm{Tr} H_A \rho \leq E} H(\rho),$$

and the notations $\bar{E} = E - E_0^A$, $\bar{E}_m^A = E_m^A - E_0^A$ for all $m > 0$.

Theorem 1. *Let C_* be one of the capacities $C_\chi, C, C_{\mathrm{ea}}, Q$ and C_{p}. If the Hamiltonian H_A satisfies condition (23) and $E \geq E_0^A$ then for any $\varepsilon > 0$ there exists natural number $m_{C_*}(\varepsilon)$ such that*

$$|C_*(\Phi, H_A, E) - C_*^m(\Phi, H_A, E)| \leq \varepsilon \qquad \forall m \geq m_{C_*}(\varepsilon) \tag{42}$$

for arbitrary channel Φ from the system A to any system B.

The above $m_{C_*}(\varepsilon)$ is the minimal natural number such that $f_{C_*}(E, m) \leq \varepsilon$ and $\bar{E}_m^A \geq 16\bar{E}$, where[12]

$$f_{C_\chi}(E, m) = 2\sqrt{2s}\,\bar{F}_{H_A}\!\left(\frac{\bar{E}}{s}\right) + g\!\left(\sqrt{2s}\right), \quad s = \frac{\bar{E}}{\bar{E}_m^A} + \sqrt{\frac{\bar{E}}{\bar{E}_m^A}},$$

$$f_C(E, m) = 4\sqrt[4]{\frac{\bar{E}}{\bar{E}_m^A}}\,\bar{F}_{H_A}\!\left(\frac{1}{2}\sqrt{\bar{E}\bar{E}_m^A}\right) + g\!\left(2\sqrt[4]{\frac{\bar{E}}{\bar{E}_m^A}}\right),$$

$$f_{C_{\mathrm{ea}}}(E, m) = 2s\bar{F}_{H_A}\!\left(\frac{2\bar{E}}{s^2}\right) + 2g(s), \quad s = \frac{\bar{E}}{\bar{E}_m^A} + \sqrt{\frac{\bar{E}}{\bar{E}_m^A}},$$

$$f_Q(E, m) = f_C(E, m) + \frac{32\bar{E}}{\bar{E}_m^A}\,\bar{F}_{H_A}\!\left(\frac{\bar{E}_m^A}{16}\right), \quad f_{C_{\mathrm{p}}}(E, m) = 2f_Q(E, m).$$

If A is the ℓ-mode quantum oscillator with frequencies $\omega_1, ..., \omega_\ell$ then the function $\bar{F}_{H_A}$ in all the above formulas can be replaced by its upper bound $\widehat{F}_{\ell,\omega}(E + E_0^A)$, where $\widehat{F}_{\ell,\omega}(E)$ is defined in (24). In this case the sequence $\{E_k^A\}_{k \geq 0}$ consists of the numbers $\sum_{i=1}^\ell \hbar\omega_i(n_i - 1/2)$, $n_1, ..., n_\ell \in \mathbb{N}$ arranged in the nondecreasing order.

Remark 2. The existence of solutions of the inequalities $f_{C_*}(E, m) \leq \varepsilon$, $C_* = C_\chi, ..., C_{\mathrm{p}}$, for any $\varepsilon > 0$ is guaranteed by condition (23), since it implies that $\bar{F}_{H_A}(E) = o(\sqrt{E})$ as $E \to +\infty$ by Lemma 1 in [19].

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[12] $g(x) \doteq (1 + x)h_2\!\left(\frac{x}{1+x}\right) = (x + 1)\log(x + 1) - x\log x.$

The number $m_{C_*}(\varepsilon)$ will be called ε-*sufficient input dimension* for C_*.

Proof. Let $P_m = \sum_{k=0}^{m-1} |\tau_k\rangle\langle\tau_k|$ be the projector on the subspace $\mathcal{H}_A^m$ and $\Pi_m : A \to A$ the channel introduced in Lemma 3.

$C_* = C_\chi$. If $\{p_i, \rho_i\}$ is an ensemble of input states such that $\mathrm{Tr}H_A\bar\rho \le E$ then the ensemble $\{p_i, \rho_i^m\}$, where $\rho_i^m = \Pi_m(\rho_i)$ for all i, satisfies the same condition for all m. So, the last assertion of Lemma 3 and representation (13) show that

$$|\chi(\{p_i, \Phi(\rho_i)\}) - \chi(\{p_i, \Phi(\rho_i^m)\})| \le f_{C_\chi}(E, m).$$

This implies the assertion of the theorem for $C_* = C_\chi$, since all the states ρ_i^m are supported by the subspace $\mathcal{H}_A^m$.

$C_* = C$. Note that

$$C_\chi(\Phi^{\otimes n}, H_{A^n}, nE) = \sup \chi(\{p_i, \Phi^{\otimes n}(\rho_i)\}),$$

where the supremum is over all ensembles $\{p_i, \rho_i\}$ of states in $\mathfrak{S}(\mathcal{H}_A^{\otimes n})$ with the average state $\bar\rho$ such that $\mathrm{Tr}H_A\bar\rho_{A_j} \le E$ for all $j = \overline{1, n}$. This can be easily shown by using the symmetry arguments and the following well known property of the Holevo quantity:

$$\frac{1}{n}\sum_{j=1}^n \chi\left(\{q_i^j, \sigma_i^j\}_i\right) \le \chi\left(\left\{\frac{q_i^j}{n}, \sigma_i^j\right\}_{ij}\right)$$

for any collection $\{q_i^1, \sigma_i^1\}, ..., \{q_i^n, \sigma_i^n\}$ of discrete ensembles.

If $\{p_i, \rho_i\}$ is an ensemble of states in $\mathfrak{S}(\mathcal{H}_A^{\otimes n})$ satisfying the above condition then the ensemble $\{p_i, \rho_i^m\}$, where $\rho_i^m = \Pi_m^{\otimes n}(\rho_i)$ for all i, satisfies the same condition for all m. So, the last assertion of Lemma 3 and representation (13) show that

$$\left|\chi(\{p_i, \Phi^{\otimes n}(\rho_i)\}) - \chi(\{p_i, \Phi^{\otimes n}(\rho_i^m)\})\right| \le f_C(E, m).$$

This implies the assertion of the theorem for $C_* = C$, since all the states ρ_i^m are supported by the subspace $[\mathcal{H}_A^m]^{\otimes n}$.

$C_* = C_{\mathrm{ea}}$. Let ρ be any state in $\mathfrak{S}(\mathcal{H}_A)$ such that $\mathrm{Tr}H_A\rho \le E$ and $\hat\rho$ its purification in $\mathfrak{S}(\mathcal{H}_{AR})$. Then $\rho_m \doteq (1 - r_m)^{-1}P_m\rho P_m$, $r_m = 1 - \mathrm{Tr}P_m\rho$, is a state in $\mathfrak{S}(\mathcal{H}_A)$ satisfying the same condition for all m such that $E_m^A > E$

and $\hat{\rho}_m \doteq (1 - r_m)^{-1} P_m \otimes I_R \hat{\rho} P_m \otimes I_R$ is a purification of this state. It follows from inequality (2) that

$$\|\hat{\rho} - \hat{\rho}_m\|_1 \leq \|\hat{\rho} - P_m \otimes I_R \, \hat{\rho} \, P_m \otimes I_R\|_1 + \|P_m \otimes I_R \, \hat{\rho} \, P_m \otimes I_R - \hat{\rho}_m\|_1 \leq 2r_m + 2\sqrt{r_m}.$$

By Lemma 5 the condition $\mathrm{Tr} H_A \rho \leq E$ implies $r_m \leq \bar{E}/\bar{E}_m^A \leq 1/16$. So, by using the Stinespring representation (3) and the last assertion of Lemma 2 with trivial C, $\mathcal{H}_* = V_\Phi \mathcal{H}_A \subseteq \mathcal{H}_{BE}$ and $H_* = V_\Phi H_A V_\Phi^* - E_0^A I_{\mathcal{H}_*}$ one can show that

$$\left| I(B\!:\!R)_{\Phi \otimes \mathrm{Id}_R(\hat{\rho})} - I(B\!:\!R)_{\Phi \otimes \mathrm{Id}_R(\hat{\rho}_m)} \right| \leq 2s\bar{F}_{H_A}(2\bar{E}/s^2) + 2g(s).$$

This implies the assertion of the theorem for $C_* = C_{\mathrm{ea}}$, since the state ρ_m is supported by the subspace $\mathcal{H}_A^m$.

$C_* = Q$. Let $\Psi_m = \Phi \circ \Pi_m$, ρ be any state in $\mathfrak{S}(\mathcal{H}_A^{\otimes n})$ such that $\sum_{k=1}^n \mathrm{Tr} H_A \rho_{A_k} \leq nE$ and $\hat{\rho}$ its purification in $\mathfrak{S}(\mathcal{H}_{A^n R})$. Then Lemma 3 implies

$$\left| I_c(\Phi^{\otimes n}, \rho) - I_c(\Psi_m^{\otimes n}, \rho) \right| = \left| I(B^n\!:\!R)_{\Phi^{\otimes n} \otimes \mathrm{Id}_R(\hat{\rho})} - I(B^n\!:\!R)_{\Psi_m^{\otimes n} \otimes \mathrm{Id}_R(\hat{\rho})} \right| \leq f_Q(E, m)$$

This implies the assertion of the theorem for $C_* = Q$, since the operational definition of the quantum capacity with the energy constraint (see Section III in [24]) and the implication

$$\mathrm{Tr} H_{A^n} \rho \leq nE \quad \Rightarrow \quad \mathrm{Tr} H_{A^n} \Pi_m^{\otimes n}(\rho) \leq nE \tag{43}$$

valid for any state $\rho \in \mathfrak{S}(\mathcal{H}_A^{\otimes n})$ and $E \geq E_0^A$ show that

$$Q(\Psi_m, H_A, E) \leq Q_m(\Phi, H_A, E) \leq Q(\Phi, H_A, E).$$

$C_* = C_{\mathrm{p}}$. If $\{p_i, \rho_i\}$ is an ensemble of states in $\mathfrak{S}(\mathcal{H}_A^{\otimes n})$ such that $\sum_{k=1}^n \mathrm{Tr} H_A \bar{\rho}_{A_k} \leq nE$ then the ensemble $\{p_i, \rho_i^m\}$, where $\rho_i^m = \Pi_m^{\otimes n}(\rho_i)$ for all i, satisfies the same condition for all m. So, Lemma 3 and representation (13) show that

$$\left| \chi(\{p_i, \Phi^{\otimes n}(\rho_i)\}) - \chi(\{p_i, \Phi^{\otimes n}(\rho_i^m)\}) \right| \leq f_{C_{\mathrm{p}}}(E, m)/2.$$

and

$$\left| \chi(\{p_i, \widehat{\Phi}^{\otimes n}(\rho_i)\}) - \chi(\{p_i, \widehat{\Phi}^{\otimes n}(\rho_i^m)\}) \right| \leq f_{C_{\mathrm{p}}}(E, m)/2.$$

This implies the assertion of the theorem for $C_* = C_{\mathrm{p}}$, since all the states ρ_i^m are supported by the subspace $[\mathcal{H}_A^m]^{\otimes n}$. $\square$

Unfortunately, the values of $m_{C_*}(\varepsilon)$ given by Theorem 1 for real physical systems are extremely high.

Example 1. Let A be the one-mode quantum oscillator with the frequency ω. In this case the Hamiltonian H_A has the spectrum $\{E_k^A = (k + 1/2)\hbar\omega\}_{k \geq 0}$ and $F_{H_A}(E) = g(E/\hbar\omega - 1/2)$ [5, Ch.12]. The results of numerical calculations of $m_{C_*}(\varepsilon)$ for different values of the input energy bound E are presented in the following tables corresponding to two values of the relative error $\varepsilon/F_{H_A}(E)$ equal respectively to 0.1 and 0.01.[13]

Table 1. The approximate values of $m_{C_*}(\varepsilon)$ for $\varepsilon = 0.1F_{H_A}(E)$.

$E/\hbar\omega$	$m_{C_\chi}(\varepsilon)$	$m_C(\varepsilon)$	$m_{C_{\mathrm{ea}}}(\varepsilon)$	$m_Q(\varepsilon)$	$m_{C_p}(\varepsilon)$
3	$5.0 \cdot 10^9$	$2.0 \cdot 10^{10}$	$8.6 \cdot 10^4$	$2.0 \cdot 10^{10}$	$5.2 \cdot 10^{11}$
10	$3.2 \cdot 10^9$	$1.3 \cdot 10^{10}$	$1.3 \cdot 10^5$	$1.3 \cdot 10^{10}$	$3.4 \cdot 10^{11}$
100	$5.5 \cdot 10^9$	$2.2 \cdot 10^{10}$	$5.3 \cdot 10^5$	$2.2 \cdot 10^{10}$	$5.5 \cdot 10^{11}$

Table 2. The approximate values of $m_{C_*}(\varepsilon)$ for $\varepsilon = 0.01F_{H_A}(E)$.

$E/\hbar\omega$	$m_{C_\chi}(\varepsilon)$	$m_C(\varepsilon)$	$m_{C_{\mathrm{ea}}}(\varepsilon)$	$m_Q(\varepsilon)$	$m_{C_p}(\varepsilon)$
3	$2.1 \cdot 10^{14}$	$8.2 \cdot 10^{14}$	$1.7 \cdot 10^7$	$8.2 \cdot 10^{14}$	$1.8 \cdot 10^{16}$
10	$1.3 \cdot 10^{14}$	$5.3 \cdot 10^{14}$	$2.6 \cdot 10^7$	$5.3 \cdot 10^{14}$	$1.7 \cdot 10^{16}$
100	$2.0 \cdot 10^{14}$	$8.1 \cdot 10^{14}$	$1.0 \cdot 10^8$	$8.1 \cdot 10^{14}$	$1.8 \cdot 10^{16}$

We see that the values of the ε-sufficient input dimension $m_{C_*}(\varepsilon)$ are extremely high for all the capacities excepting C_{ea}. It is clear that this is explained by inaccuracy of the used estimates rather than physical reasons. In a sense, this is a cost of the universality of Theorem 1 in which the class of *all channels* from a given system A to arbitrary systems B are considered. In the next subsection we show that estimates of the ε-sufficient input dimension can be decreased substantially by restricting the class of channels Φ for which the validity of (42) is required.

4.3 Specifications for energy-limited channels

Theorem 1 gives estimates of the ε-sufficient input dimensions for all quantum channels from a given system A to arbitrary system B, which do not depend on system B at all. Unfortunately, for a real quantum system (quantum

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[13]The capacities $C_*(\Phi, H_A, E)$, $C_* = C_\chi, C, Q, C_{\mathrm{p}}$, take values in $[0, F_{H_A}(E)]$, the capacity $C_{\mathrm{ea}}(\Phi, H_A, E)$ takes values in $[0, 2F_{H_A}(E)]$.

oscillator) in the role of A these estimates are extremely hight (see Example 1 and the comments below). In this section we show that estimates of the ε-sufficient input dimensions can be decreased substantially by imposing constraints on the class of quantum channels used for communications.

Assume that B is a quantum system with the Hamiltonian H_B satisfying condition (20) while A is any quantum system with the Hamiltonian H_A having form (21). Consider quantum channels Φ from A to B such that

$$\mathrm{Tr}H_B\Phi(\rho) \leq \alpha\mathrm{Tr}H_A\rho + E_c \quad \text{for any } \rho \in \mathfrak{S}(\mathcal{H}_A), \tag{44}$$

where α and E_c are nonnegative parameters. Such channels are called *energy-limited* in [27], where it is mentioned that any quantum channel mapping energy-bounded states to energy-bounded states satisfies (44) with some α and E_c.

Let $\widehat{F}_{H_B}$ be any upper bound for the function

$$F_{H_B}(E) \doteq \sup_{\mathrm{Tr}H_B\rho \leq E} H(\rho) = H(\gamma_B(E)), \quad E \geq E_0^B \doteq \inf_{\|\varphi\|=1} \langle\varphi|H_B|\varphi\rangle,$$

defined on $[0, +\infty)$ such that

$$\widehat{F}_{H_B}(E) > 0, \quad \widehat{F}'_{H_B}(E) > 0, \quad \widehat{F}''_{H_B}(E) < 0 \quad \text{for all } E > 0. \tag{45}$$

and

$$\widehat{F}_{H_B}(E) = o(E) \quad \text{as} \quad E \to +\infty. \tag{46}$$

Since H_B satisfies condition (20), one can use the function $E \mapsto F_{H_B}(E+E_0^B)$ in the role of $\widehat{F}_{H_B}$ [17]. If B is the ℓ-mode quantum oscillator with the frequencies ω_i ([5, Ch.12]) then the function $\widehat{F}_{\ell,\omega}$ defined in (24) also satisfies the above requirements for $\widehat{F}_{H_B}$.

Denote by $\mathfrak{F}_{\alpha,E_c}(A,B)$ the class of all quantum channels from A to B satisfying (44). The following theorem is a version (specification) of Theorem 1 for energy-limited channels and all the basic capacities excepting C_{p}.

Theorem 2. *Let C_* be one of the capacities $C_\chi, C, C_{\mathrm{ea}}$ and Q. If the Hamiltonian H_B satisfies condition (20) and $E \geq E_0^A$ then for any $\alpha > 0$, $E_c \geq 0$ and $\varepsilon > 0$ there exists natural number $m_{C_*}(\varepsilon|\,\alpha, E_c)$ such that*

$$|C_*(\Phi, H_A, E) - C_*^m(\Phi, H_A, E)| \leq \varepsilon \qquad \forall m \geq m_{C_*}(\varepsilon|\,\alpha, E_c)$$

for arbitrary channel Φ from the class $\mathfrak{F}_{\alpha,E_c}(A,B)$.

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If $C_* = C_\chi, C, C_{ea}$ then $m_{C_*}(\varepsilon\,|\,\alpha, E_c)$ is the minimal natural number such that $f_{C_*}^{\alpha, E_c}(E, m\,|\,t) \le \varepsilon$ for some $t \in (0, \frac{1}{2}]$, where

$$f_{C_\chi}^{\alpha, E_c}(E, m\,|\,t) = (2t + s_m(t))\widehat{F}_{H_B}\left(\frac{\alpha E + E_c}{t}\right) + 2g(s_m(t)) + 2h_2(t),$$

$$f_C^{\alpha, E_c}(E, m\,|\,t) = f_{C_{ea}}^{\alpha, E_c}(E, m\,|\,t) = (4t + 2s_m(t))\widehat{F}_{H_B}\left(\frac{\alpha E + E_c}{t}\right) + 2g(s_m(t)) + 4h_2(t),$$

where $s_m(t) = \frac{\bar{E}/\bar{E}_m^A + \sqrt{\bar{E}/\bar{E}_m^A} + t/2}{1-t}$, $\bar{E} = E - E_0^A$, $\bar{E}_m^A = E_m^A - E_0^A$.

The above $m_Q(\varepsilon\,|\,\alpha, E_c)$ is the minimal natural number s.t. $f_Q^{\alpha, E_c}(E, m\,|\,p, t) \le \varepsilon$ for some $p > 1$ and $t \in (0, \frac{1}{2}]$, where

$$f_Q^{\alpha, E_c}(E, m\,|\,p, t) = (4t + 2s_m(t))\widehat{F}_{H_B}\left(\frac{E_p}{t}\right) + 2g(s_m(t)) + 4h_2(t) + \frac{2}{p}\widehat{F}_{H_B}(E_p),$$

where $E_p = \alpha p E + E_c$.

Remark 3. The existence of solutions of the inequalities determining $m_{C_*}(\varepsilon\,|\,\alpha, E_c)$, $C_* = C_\chi, ..., Q$, for any $\varepsilon > 0$ is guaranteed by condition (46).

Proof. Let $\Psi_m = \Phi \circ \Pi_m$, where $\Pi_m : A \to A$ is the channel defined in Lemma 3. By Lemmas 4,5 and definition (6) of the energy-constrained diamond norm we have

$$\begin{aligned}
\tfrac{1}{2}\|\Phi - \Psi_m\|_\diamond^E &\le \tfrac{1}{2}\|\mathrm{Id}_A - \Pi_m\|_\diamond^E \le \sup_{\mathrm{Tr}H_A\rho \le E}\left[\mathrm{Tr}P_m^\perp \rho + \sqrt{\mathrm{Tr}P_m^\perp \rho}\right] \\
&\le \bar{E}/\bar{E}_m + \sqrt{\bar{E}/\bar{E}_m}, \quad \text{where } P_m^\perp = I_A - P_m.
\end{aligned} \tag{47}$$

So, by using Proposition 6 in [20] and the change of variables $t \mapsto t/\varepsilon$ we obtain

$$|C_\chi(\Phi, H_A, E) - C_\chi(\Psi_m, H_A, E)| \le f_{C_\chi}^{\alpha, E_c}(E, m\,|\,t)$$

for any $t \in (0, \frac{1}{2}]$. This implies the assertion of the theorem for $C_* = C_\chi$, since the definition of the Holevo capacity and the implication (43) show that

$$C_\chi(\Psi_m, H_A, E) \le C_\chi^m(\Phi, H_A, E) \le C_\chi(\Phi, H_A, E).$$

The assertions of the theorem for $C_* = C$ and $C_* = C_{ea}$ are proved similarly by using Proposition 6 and 7B in [20].

The assertions of the theorem for $C_* = Q$ is proved by repeating the corresponding arguments from the proof of Theorem 1 with the use of Lemma 6 below instead of Lemma 3. $\square$

The following lemma is a version of Lemma 3 in Section 3 adapted for energy limited channels.

Lemma 6. *Let* $\Pi_m : A \to A$ *be the channel defined in Lemma 3 and* ρ *a state in* $\mathfrak{S}(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_R)$ *such that* $\sum_{k=1}^n \mathrm{Tr} H_A \rho_{A_k} \leq nE < +\infty$. *If the Hamiltonian* H_B *of system* B *satisfies condition (20) then*

$$\left| I(B^n:R)_{\Phi^{\otimes n} \otimes \mathrm{Id}_R(\rho)} - I(B^n:R)_{\Psi_m^{\otimes n} \otimes \mathrm{Id}_R(\rho)} \right| \leq n f_Q^{\alpha, E_c}(E, m \mid p, t) \qquad (48)$$

for any $p > 1$, $t \in (0, \frac{1}{2}]$ *and any channel* $\Phi \in \mathfrak{F}_{\alpha, E_c}(A, B)$, *where* $\Psi_m = \Phi \circ \Pi_m$, $f_Q^{\alpha, E_c}(E, m \mid p, t)$ *is the quantity defined in Theorem 2 and* $\widehat{F}_{H_B}$ *is any upper bound for the function* F_{H_B} *with properties (45) and (46).*

If $\mathrm{Tr} H_A \rho_{A_k} \leq E$ *for all* $k = \overline{1, n}$ *then (48) holds with* $f_Q^{\alpha, E_c}(E, m \mid p, t)$ *replaced by the quantity* $f_C^{\alpha, E_c}(E, m \mid t)$ *defined in Theorem 2 for all* $t \in (0, \frac{1}{2}]$.

Proof. All the assertions of the lemma are easily derived from Lemma 7 in the Appendix with trivial C by using (47). $\square$

Example 2. Let $A = B$ be the one-mode quantum oscillator with the frequency ω. In this case $\{E_k^A = E_k^B = (k + 1/2)\hbar\omega\}_{k \geq 0}$ and $F_{H_A}(E) = F_{H_B}(E) = g(E/\hbar\omega - 1/2)$ [5, Ch.12]. The function defined in (24) with $\ell = 1$, i.e. $\widehat{F}_{1, \omega}(E) \doteq \log(E/\hbar\omega + 1/2) + 1$ can be used in the role of the upper bound $\widehat{F}_{H_B}$.

Consider first the case $\alpha = 1$, $E_c = 0$. The set $\mathfrak{F}_{1,0}(A, B)$ consists of channels not increasing the energy of a state, which can be called energy attenuators. The results of numerical calculations of $m_{C_*}(\varepsilon \mid 1, 0)$ for different values of the input energy bound E are presented in the following tables corresponding to different values of the relative error $\varepsilon / F_{H_A}(E)$.

Table 3. The approximate values of $m_{C_*}(\varepsilon \mid \alpha, E_c)$ for $\varepsilon = 0.1 F_{H_A}(E)$, $\alpha = 1$, $E_c = 0$.

$E/\hbar\omega$	$m_{C_\chi}(\varepsilon \mid \alpha, E_c)$	$m_C(\varepsilon \mid \alpha, E_c)$	$m_{C_{\mathrm{ea}}}(\varepsilon \mid \alpha, E_c)$	$m_Q(\varepsilon \mid \alpha, E_c)$
3	$3.1 \cdot 10^4$	$7.8 \cdot 10^4$	$7.8 \cdot 10^4$	$1.9 \cdot 10^5$
10	$4.8 \cdot 10^4$	$1.3 \cdot 10^5$	$1.3 \cdot 10^5$	$2.9 \cdot 10^5$
100	$1.9 \cdot 10^5$	$5.3 \cdot 10^5$	$5.3 \cdot 10^5$	$1.1 \cdot 10^6$

Table 4. The approximate values of $m_{C_*}(\varepsilon|\,\alpha, E_c)$ for $\varepsilon = 0.01 F_{H_A}(E)$,
$\alpha = 1$, $E_c = 0$.

$E/\hbar\omega$	$m_{C_\chi}(\varepsilon|\,\alpha, E_c)$	$m_C(\varepsilon|\,\alpha, E_c)$	$m_{C_{\mathrm{ea}}}(\varepsilon|\,\alpha, E_c)$	$m_Q(\varepsilon|\,\alpha, E_c)$
3	$5.6 \cdot 10^6$	$1.3 \cdot 10^7$	$1.3 \cdot 10^7$	$3.1 \cdot 10^7$
10	$8.5 \cdot 10^6$	$2.0 \cdot 10^7$	$2.0 \cdot 10^7$	$4.7 \cdot 10^7$
100	$3.3 \cdot 10^7$	$8.3 \cdot 10^7$	$8.3 \cdot 10^7$	$1.8 \cdot 10^8$

Comparing these results with the approximate values of $m_{C_*}(\varepsilon)$ presented in Tables 1 and 2 we see that the estimates of the ε-sufficient input dimensions given by Theorem 2 for the class $\mathfrak{F}_{1,0}(A, B)$ of energy attenuators are significantly less than the estimates of the ε-sufficient input dimensions given by Theorem 1 for the class of all channel from the one-mode quantum oscillator to any other systems.

It is clear that $m_{C_*}(\varepsilon|\,\alpha, E_c)$ increases to $+\infty$ as either α or E_c tends to $+\infty$. But numerical calculations show that (in the case when $A = B$ is the one-mode quantum oscillator) the rate of increasing of $m_{C_*}(\varepsilon|\,\alpha, E_c)$ is quite low for all the capacities. This is illustrated by the following tables corresponding to different values of the relative error $\varepsilon/F_{H_A}(E)$.

Table 5. The approximate values of $m_{C_*}(\varepsilon|\,\alpha, E_c)$ for $\varepsilon = 0.1 F_{H_A}(E)$,
$\alpha = 10^6$, $E_c = 10^6 \hbar\omega$.

$E/\hbar\omega$	$m_{C_\chi}(\varepsilon|\,\alpha, E_c)$	$m_C(\varepsilon|\,\alpha, E_c)$	$m_{C_{\mathrm{ea}}}(\varepsilon|\,\alpha, E_c)$	$m_Q(\varepsilon|\,\alpha, E_c)$
3	$9.0 \cdot 10^4$	$2.7 \cdot 10^5$	$2.7 \cdot 10^5$	$4.7 \cdot 10^5$
10	$1.4 \cdot 10^5$	$4.2 \cdot 10^5$	$4.2 \cdot 10^5$	$7.1 \cdot 10^5$
100	$5.2 \cdot 10^5$	$1.7 \cdot 10^6$	$1.7 \cdot 10^6$	$2.7 \cdot 10^6$

Table 6. The approximate values of $m_{C_*}(\varepsilon|\,\alpha, E_c)$ for $\varepsilon = 0.01 F_{H_A}(E)$,
$\alpha = 10^6$, $E_c = 10^6 \hbar\omega$.

$E/\hbar\omega$	$m_{C_\chi}(\varepsilon|\,\alpha, E_c)$	$m_C(\varepsilon|\,\alpha, E_c)$	$m_{C_{\mathrm{ea}}}(\varepsilon|\,\alpha, E_c)$	$m_Q(\varepsilon|\,\alpha, E_c)$
3	$1.3 \cdot 10^7$	$3.6 \cdot 10^7$	$3.6 \cdot 10^7$	$6.4 \cdot 10^7$
10	$1.9 \cdot 10^7$	$5.4 \cdot 10^7$	$5.4 \cdot 10^7$	$9.7 \cdot 10^7$
100	$7.2 \cdot 10^7$	$2.1 \cdot 10^8$	$2.1 \cdot 10^8$	$3.6 \cdot 10^8$

Comparing Tables 5 and 6 with the Tables 3 and 4 shows that the change of the parameters $\alpha : 1 \rightarrow 10^6$ and $E_c : 0 \rightarrow 10^6 \hbar\omega$ does not lead to significant growth of the ε-sufficient input dimensions for all the capacities.

5 Uniform continuity of basic capacities of energy-constrained channels with respect to the strong convergence topology

Real physical channels are always prepared with a finite accuracy. So, in study of their capacities we should be able to estimate variations of the capacities caused by all possible perturbations of a channel. In other words, we have to quantitatively analyse continuity of quantum channel capacities as functions of a channel with respect to appropriate topology (convergence) on the set of all channels.

In finite dimensions this problem is solved by Leung and Smith who obtained in [11] (uniform) continuity bounds for basic capacities of quantum channels with finite-dimensional output with respect to the distance between quantum channels generated by the diamond norm (5).

Speaking about generalizations of the Leung-Smith results to energy-constrained infinite-dimensional channels we have to choose appropriate metric on the set of quantum channels, since the diamond-norm distance can not properly describe all physical perturbations of infinite-dimensional channels (this is illustrated by the examples of channels with close physical parameters having the diamond-norm distance equal to 2 [27]).

Mathematically, the drawback of the diamond-norm distance in infinite-dimensions follows from Theorem 1 in [9] stating that the closeness of two quantum channels in the diamond-norm distance means *the operator norm* closeness of the corresponding Stinespring isometries. To take into account deformations of the Stinespring isometry in the strong operator topology one can consider the *strong convergence topology* on the set of quantum channels defined by the family of seminorms $\Phi \mapsto \|\Phi(\rho)\|_1, \rho \in \mathfrak{S}(\mathcal{H}_A)$ [20]. The strong convergence of a sequence of channels Φ_n to a channel Φ_0 means that

$$\lim_{n \to \infty} \Phi_n(\rho) = \Phi_0(\rho) \text{ for all } \rho \in \mathfrak{S}(\mathcal{H}_A).$$

The separability of the set $\mathfrak{S}(\mathcal{H}_A)$ implies that the strong convergence topology on the set of quantum channels is metrisable (can be defined by some metric). Moreover, it is shown in [20] that this topology is generated by any of the energy-constrained diamond norms (6) provided the operator H_A has discrete spectrum $\{E_k^A\}_{k \geq 0}$ of finite multiplicity such that $E_k^A \to +\infty$ as $k \to +\infty$.

In [20, 27] continuity bounds for basic capacities of infinite-dimensional energy-constrained channels with respect to the energy-constrained diamond norms (6) are obtained under the condition of boundedness of the energy amplification factor of these channels. The continuity bound for the entanglement-assisted capacity C_{ea} obtained in [20] holds for arbitrary quantum channels and hence implies uniform continuity of this capacity on the set of *all quantum channels* with respect to the strong convergence topology provided the input Hamiltonian H_A satisfies condition (20). The finite-dimensional approximation theorem makes it possible to obtain similar result for other basic capacities under slightly stronger condition on H_A.

Theorem 3. *If the Hamiltonian H_A of input system A satisfies condition (23) then for any $E > E_0^A$ all the functions*

$$\Phi \mapsto C_*(\Phi, H_A, E), \quad C_* = C_\chi, C, Q, C_{\mathrm{p}},$$

are uniformly continuous on the set of all channels from A to arbitrary system B with respect to the strong convergence topology. Quantitatively, if Φ and Ψ are any channels from A to B such that $\frac{1}{2}\|\Phi - \Psi\|_\diamond^E \leq \varepsilon$ then

$$|C_*(\Phi, H_A, E) - C_*(\Psi, H_A, E)| \leq v_{C_*}(\varepsilon, E), \quad C_* = C_\chi, C, Q, C_{\mathrm{p}}, \qquad (49)$$

where $v_{C_*}(\varepsilon, E)$ is a function vanishing as $\varepsilon \to 0^+$ for any $E > E_0^A$ defined for each of the capacities by the formulas

$$v_{C_\chi}(\varepsilon, E) = \min_{m \in \mathbb{N}_*} \left[\sqrt{k(m)\varepsilon} \log(2m) + 2g\left(\sqrt{k(m)\varepsilon}\right) + 2f_{C_\chi}(E, m) \right],$$

$$v_C(\varepsilon, E) = \min_{m \in \mathbb{N}_*} \left[2\sqrt{k(m)\varepsilon} \log(2m) + 2g\left(\sqrt{k(m)\varepsilon}\right) + 2f_C(E, m) \right],$$

$$v_Q(\varepsilon, E) = \min_{m \in \mathbb{N}_*} \left[2\sqrt{k(m)\varepsilon} \log(2m) + 2g\left(\sqrt{k(m)\varepsilon}\right) + 2f_Q(E, m) \right],$$

$$v_{C_{\mathrm{p}}}(\varepsilon, E) = \min_{m \in \mathbb{N}_*} \left[4\sqrt{k(m)\varepsilon} \log(2m) + 4g\left(\sqrt{k(m)\varepsilon}\right) + 2f_{C_{\mathrm{p}}}(E, m) \right],$$

where $\mathbb{N}_* \doteq \{m \in \mathbb{N} \,|\, E_m^A \geq 16E - 15E_0^A\}$, the functions f_{C_χ}, f_C, f_Q and $f_{C_{\mathrm{p}}}$ are defined in Theorem 1 and $k(m) = 2(E_m^A - E_0^A)/(E - E_0^A)$.[14]

Proof. The first assertion of the theorem follows from continuity bounds (49), since the energy-constrained diamond norm $\|\cdot\|_\diamond^E$ with any $E > E_0^A$

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[14]E_m^A is the m-th eigenvalue of H_A (taking the multiplicity into account).

generates the strong convergence topology on the set of all quantum channels from A to B by Proposition 3 in [20].

For given natural m let Φ_m and Ψ_m be the restrictions of the channels Φ and Ψ to the set $\mathfrak{S}(\mathcal{H}_A^m)$. By repeating the arguments from the proof of Proposition 5 in [18] one can show that

$$|C_\chi(\Phi_m, H_A^m, E) - C_\chi(\Psi_m, H_A^m, E)| \leq \epsilon \log(2m) + 2g(\epsilon),$$

$$|C(\Phi_m, H_A^m, E) - C(\Psi_m, H_A^m, E)| \leq 2\epsilon \log(2m) + 2g(\epsilon),$$

$$(50)$$

$$|Q(\Phi_m, H_A^m, E) - Q(\Psi_m, H_A^m, E)| \leq 2\epsilon \log(2m) + 2g(\epsilon),$$

$$|C_{\mathrm{p}}(\Phi_m, H_A^m, E) - C_{\mathrm{p}}(\Psi_m, H_A^m, E)| \leq 4\epsilon \log(2m) + 4g(\epsilon),$$

where $\epsilon = \|\Phi_m - \Psi_m\|_\diamond^{1/2}$ and $H_A^m = H_A P_m$ (here P_m is the projector onto $\mathcal{H}_A^m$). Since $\|\Phi_m - \Psi_m\|_\diamond = \sup\limits_{\omega_A \in \mathfrak{S}(\mathcal{H}_A^m)} \|(\Phi - \Psi) \otimes \mathrm{Id}_R(\omega)\|_1$, by noting that $\mathrm{Tr}H_A\rho \leq E_m^A$ for any $\rho \in \mathfrak{S}(\mathcal{H}_A^m)$ and by using monotonicity and concavity of the function $E \mapsto \|\Phi\|_\diamond^E$ (proved in [27]) we obtain

$$\epsilon^2 \leq \|\Phi - \Psi\|_\diamond^{E_m^A} \leq \tfrac{1}{2}k(m)\|\Phi - \Psi\|_\diamond^E \leq k(m)\varepsilon.$$

If C_* is one of the capacities C_χ, C, Q and C_{p} then

$$|C_*(\Phi, H_A, E) - C_*(\Psi, H_A, E)| \leq |C_*(\Phi_m, H_A, E) - C_*(\Psi_m, H_A^m, E)|$$

$$+|C_*(\Phi, H_A, E) - C_*(\Phi_m, H_A^m, E)| + |C_*(\Psi, H_A^m, E) - C_*(\Psi_m, H_A^m, E)|.$$

Thus, the continuity bounds in the theorem follow from continuity bounds (50) and Theorem 1.

The vanishing of all the functions $v_{C_*}(\varepsilon, E)$ as $\varepsilon \to 0^+$ follows from the vanishing of the functions $f_{C_*}(E, m)$ as $m \to +\infty$. $\square$

Remark 4. Continuity bounds (49) are universal (valid for any channels) but they give too rough estimates for variations of the capacities because of the low decreasing rate of the functions $f_{C_*}(E, m)$ as $m \to +\infty$. So, dealing with quantum channels produced in a physical experiment it is reasonable to use the continuity bounds for basic capacities depending on the energy-constrained diamond norm distance obtained in [20, 27] for classes of channels with bounded energy amplification factor.

Appendix

The following lemma is the QCMI-version of Lemma 7 in [27].[15]

Lemma 7. *Let Φ and Ψ be channels from A to B satisfying condition (44), C, D any systems, $n \in \mathbb{N}$ and ρ a state in $\mathfrak{S}(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{CD})$ such that $\sum_{k=1}^n \mathrm{Tr} H_A \rho_{A_k} \leq nE < +\infty$. If $\frac{1}{2}\|\Phi - \Psi\|_\diamond^E \leq \varepsilon$ then[16]*

$$(1/n)\left|I(B^n:D|C)_{\Phi^{\otimes n} \otimes \mathrm{Id}_{CD}(\rho)} - I(B^n:D|C)_{\Psi^{\otimes n} \otimes \mathrm{Id}_{CD}(\rho)}\right|$$

$$\tag{51}$$

$$\leq (4t + 2r(t,\varepsilon))\widehat{F}_{H_B}\left(\frac{E_p}{t}\right) + 2g(r(t,\varepsilon)) + 4h_2(t) + \frac{2}{p}\widehat{F}_{H_B}(E_p)$$

for any $p > 1$ and $t \in (0, \frac{1}{2}]$, where $E_p = \alpha pE + E_c$, $r(t,\varepsilon) = \frac{\varepsilon + t/2}{1-t}$ and $\widehat{F}_{H_B}$ is any upper bound for the function F_{H_B} with properties (45) and (46).

If $\mathrm{Tr} H_A \rho_{A_k} \leq E$ for all $k = \overline{1,n}$ then (51) holds with $p = 1$ without the last term in the right hand side.

Proof. Denote by $\Delta^n(\Phi, \Psi, \rho)$ the left hand side of (51). By the proof of Proposition 3B in [17] (based on the Leung-Smith telescopic method), we have

$$n\Delta^n(\Phi, \Psi, \rho) \leq \sum_{k=1}^n |I(B_k:D|X)_{\sigma_k} - I(B_k:D|X)_{\sigma_{k-1}}|,$$

where $X = B_1...B_{k-1}B_{k+1}...B_nC$ and $\sigma_k = \Phi^{\otimes k} \otimes \Psi^{\otimes(n-k)} \otimes \mathrm{Id}_{CD}(\rho)$, $k = 0, 1, ..., n$. The proof of Proposition 3B in [17] also implies

$$\|\sigma_k - \sigma_{k-1}\|_1 \leq \sup\left\{\|(\Phi - \Psi) \otimes \mathrm{Id}_R(\omega)\|_1 \mid \omega_A = \rho_{A_k}\right\} \leq \|\Phi - \Psi\|_\diamond^{x_k}, \quad (52)$$

where $x_k = \mathrm{Tr} H_A \rho_{A_k}$.

Since $[\sigma_k]_{B_k} = \Phi(\rho_{A_k})$ and $[\sigma_{k-1}]_{B_k} = \Psi(\rho_{A_k})$, we have

$$\mathrm{Tr} H_B[\sigma_k]_{B_k}, \mathrm{Tr} H_B[\sigma_{k-1}]_{B_k} \leq \alpha x_k + E_c. \tag{53}$$

Let N_1 be the set of indexes k for which $x_k \leq pE$ and $N_2 = \{1,..,n\} \setminus N_1$. Thus,

$$n\Delta^n(\Phi, \Psi, \rho) \leq \sum_{k \in N_1} D_k + \sum_{k \in N_2} D_k, \quad D_k = |I(B_k:D|X)_{\sigma_k} - I(B_k:D|X)_{\sigma_{k-1}}|.$$

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[15] The proof of this lemma differs from the proof of Lemma 7 in [27] (containing similar continuity bound for the conditional entropy) by the way of splitting of $\{1,2,...,n\}$ into the sets N_1 and N_2. This makes the resulting continuity bound more accurate in the case of logarithmic growth of F_{H_B} (in particular, when B is a multi-mode quantum oscillator).

[16] $\|\cdot\|_\diamond^E$ is the energy-constrained diamond norm defined in (6).

For each $k \in N_1$ Proposition 2 in [17] along with (52) and (53) imply

$$D_k \leq \left(4\varepsilon_k t_k + \frac{2\varepsilon_k + \varepsilon_k t_k}{1 - \varepsilon_k t_k}\right)\widehat{F}_{H_B}\left(\frac{\alpha x_k + E_c}{\varepsilon_k t_k}\right) + 2g\left(\frac{\varepsilon_k + \varepsilon_k t_k/2}{1 - \varepsilon_k t_k}\right) + 4h_2(\varepsilon_k t_k),$$

for any $t_k \in (0, \frac{1}{2\varepsilon_k}]$, where $\varepsilon_k = \frac{1}{2}\|\Phi - \Psi\|_\diamond^{x_k}$. By choosing free parameters t_k such that $\varepsilon_k t_k = t$ for all $k \in N_1$ we obtain

$$\sum_{k \in N_1} D_k \leq \sum_{k \in N_1}\left(\left(4t + \frac{2\varepsilon_k + t}{1 - t}\right)\widehat{F}_{H_B}\left(\frac{\alpha x_k + E_c}{t}\right) + 2g\left(\frac{\varepsilon_k + t/2}{1 - t}\right)\right) + 4n_1 h_2(t)$$

$$\leq n_1\left(4t + \frac{2\bar{\varepsilon}_1 + t}{1 - t}\right)\widehat{F}_{H_B}\left(\frac{\alpha pE + E_c}{t}\right) + 2n_1 g\left(\frac{\bar{\varepsilon}_1 + t/2}{1 - t}\right) + 4n_1 h_2(t),$$

where $n_1 = \sharp(N_1)$ and $\bar{\varepsilon}_1 \doteq n_1^{-1}\sum_{k \in N_1}\varepsilon_k$. The last inequality follows from monotonicity of the function $\widehat{F}_{H_B}$ (since $x_k \leq pE$ for all $k \in N_1$) and concavity of the function $g(x)$.

By using monotonicity and concavity of the function $E \mapsto \|\Phi\|_\diamond^E$ (proved in [27]) it is easy to show that $\bar{\varepsilon}_1 \leq \frac{1}{2}\|\Phi - \Psi\|_\diamond^E \leq \varepsilon$. So, by monotonicity of $g(x)$ we have

$$n^{-1}\sum_{k \in N_1} D_k \leq \left(4t + \frac{2\varepsilon + t}{1 - t}\right)\widehat{F}_{H_B}\left(\frac{\alpha pE + E_c}{t}\right) + 2g\left(\frac{\varepsilon + t/2}{1 - t}\right) + 4h_2(t).$$

For each $k \in N_2$ upper bound (18), nonnegativity of QCMI and inequalities (53) imply

$$D_k \leq 2\max\{H([\sigma_k]_{B_k}), H([\sigma_{k-1}]_{B_k})\} \leq 2\widehat{F}_{H_B}(\alpha x_k + E_c).$$

So, by concavity of $\widehat{F}_{H_B}$ we have

$$\sum_{k \in N_2} D_k \leq 2\sum_{k \in N_2}\widehat{F}_{H_B}(\alpha x_k + E_c) \leq 2n_2\widehat{F}_{H_B}(\alpha X_2 + E_c),$$

where $n_2 = \sharp(N_2)$ and $X_2 = n_2^{-1}\sum_{k \in N_2}x_2$. Since $\sum_{k \in N_2}x_k \leq nE$ and $x_k > pE$ for all $k \in N_2$, we have $X_2 \leq nE/n_2$ and $n_2/n \leq 1/p$. By using monotonicity of $\widehat{F}_{H_B}$ and applying Lemma 1 to the concave nonnegative function $x \mapsto \widehat{F}_{H_B}(\alpha x + E_c)$ on $\mathbb{R}_+$ we obtain

$$n^{-1}\sum_{k \in N_2} D_k \leq 2(n_2/n)\widehat{F}_{H_B}(\alpha(n/n_2)E + E_c) \leq (2/p)\widehat{F}_{H_B}(\alpha pE + E_c).$$

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This and the above estimate for $n^{-1}\sum_{k\in N_1} D_k$ imply (51).

The last assertion of the lemma follows from the above arguments with $p = 1$, since in this case the set N_2 is empty. $\square$

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