# Strong squeezing limit in quantum stochastic models

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#### Abstract

In this paper we study quantum stochastic differential equations (QSDEs) [14] that are driven by strongly squeezed vacuum noise. We show that for strong squeezing such a QSDE can be approximated (via a limit in the strong sense) by a QSDE that is driven by a single commuting noise process. We find that the approximation has an additional Hamiltonian term.

## 1 Introduction

Quantum stochastic differential equations (QSDEs) [14] arise via a weak coupling limit from QED [1, 10] and are an important tool for modeling the time evolution of systems that interact with the electromagnetic field in a Markovian approximation (i.e. quantum optics). Many techniques have been developed for models that are based on unitaries that are given by a QSDE, e.g. quantum filtering [2, 4], adiabatic elimination [3, 5, 11] and input-output theory [9].

In this paper we will look at QSDEs that are driven by squeezed noise [8, 13]. The electromagnetic field acts on a system via two field quadrature processes that are both commuting processes (i.e. given one of these processes: the operators at different times of the process commute with each other), but the two quadrature processes do not commute with each other. In terms of the Hudson-Parthasarathy theory these quadrature processes are given by linear combinations of the annihilator process  $A_t$  and creation process  $A_t$ , namely  $A_t + A_t^*$  and  $i(A_t - A_t^*)$ . With respect to the vacuum state both of these quadrature process are Wiener processes (but these Wiener processes do not commute with each other). In the squeezed vacuum one of the processes has an increased variance and the other has a decreased variance.

In the case of strong squeezing we expect that we can neglect the noise with the small variance and can approximate the system as if it was driven only by the noise with the large variance. In this paper we are going to make this idea precise. As we will see in Theorem 2.1: in case of strong squeezing the time evolution can be approximated well by an equation that is driven only by one commutative noise process. We call such dynamics essentially commutative [16]. As observed in [12], in principle it is possible to completely undo the decoherence for essentially commutative dynamics. This was studied on the

level of the filter in [6, Chapter 4], where a control scheme was introduced that restores quantum information (i.e. completely freezes the time evolution of the filter estimates).

We prove that the difference of the essentially commutative approximation and the original system dynamics converges strongly to zero. The proof is heavily inspired by the proof of the adiabatic theorem in [3] and relies heavily on the Trotter-Kato Theorem [17, 15]. The essentially commutative approximation has an additional Hamiltonian term in its dynamics when compared to the original dynamics.

The remainder of this article is organized as follows: Section 2 introduces the system, its essentially commutative approximation and states the main theorem (Theorem 2.1). In Section 3 we apply the main Thm in example systems. We conclude the article with Section 4 in which we proof the main theorem.

### 2 The main result

Throughout this paper n is a positive real number,  $c = |c| \exp(i\theta)$  is a complex number such that  $|c| = \sqrt{n(n+1)}$ , a denotes the real part of c,  $\mathcal{H}$  is a separable Hilbert space (the *initial space*) and  $\mathcal{F}$  is the symmetric Fock space over  $L^2(\mathbb{R}^+)$ . We denote the vacuum vector in  $\mathcal{F}$  by  $\Phi$ . On the Fock space  $\mathcal{F}$  we have the usual annihilation process  $A_t$ , creation process  $A_t$  and gauge process  $\Lambda_t$  in the sense of Hudson and Parthasarathy [14]. These noises satisfy the following quantum Itô table [14]:

$$\begin{array}{c|cccc} & dA_t^* & d\Lambda_t & dA_t \\ \hline dA_t^* & 0 & 0 & 0 \\ d\Lambda_t & dA_t^* & d\Lambda_t & 0 \\ dA_t & dt & dA_t & 0 \\ \end{array}$$

We define the squeezed noise processes  $B_t$  and  $B_t^*$  on  $\mathcal{F}$  as the following linear combinations of  $A_t$  and  $A_t^*$ :

$$B_{t} := \frac{n+c}{\sqrt{2n+1+2a}} A_{t}^{*} + \frac{n+1+c}{\sqrt{2n+1+2a}} A_{t},$$

$$B_{t}^{*} := \frac{n+\overline{c}}{\sqrt{2n+1+2a}} A_{t} + \frac{n+1+\overline{c}}{\sqrt{2n+1+2a}} A_{t}^{*}.$$
(1)

Note that these noises obey the squeezed noise quantum Itô table [8]

$$\begin{array}{c|cc} & dB_t^* & dB_t \\ \hline dB_t^* & \overline{c}dt & ndt \\ dB_t & (n+1)dt & cdt \\ \end{array}$$

Note that if c=a is a real number, then  $B_t+B_t^*=\sqrt{2n+1+2a}(A_t+A_t^*)$  and

 $i(B_t - B_t^*) = i(A_t - A_t^*)/\sqrt{2n+1+2a}$ , i.e. with respect to the vacuum one is a Wiener process with an increased variance whereas the other has a decreased variance.

In this paper we study the following quantum stochastic differential equation (QSDE) on  $\mathcal{H} \otimes \mathcal{F}$  in the sense of Hudson and Parthasarathy [14]:

$$d\tilde{U}_{t}^{nc} = \left\{ LdB_{t}^{*} - L^{*}dB_{t} + \frac{1}{2} \left( LL\overline{c} - LL^{*}n - L^{*}L(n+1) + L^{*}L^{*}c \right) dt - iHdt \right\} \tilde{U}_{t}^{nc}, \quad \tilde{U}_{0}^{nc} = I.$$
(2)

Here L and H are assumed to be bounded operators on  $\mathcal{H}$  such that S is unitary and H is self-adjoint. We will make the definition a bit more general later and then we will drop de tilde in the notation. Note that the solution to the above equation is unitary [14].

We now define:

$$L_{nc} := \frac{n+1+\overline{c}}{\sqrt{2n+1+2a}}L - \frac{n+c}{\sqrt{2n+1+2a}}L^*,$$

$$F_{nc} := \frac{n+\overline{c}}{\sqrt{2n+1+2a}}L - \frac{n+c}{\sqrt{2n+1+2a}}L^*.$$
(3)

Note that  $F_{nc}$  is skew-selfadjoint.

Using the definition of  $L_{nc}$  in Eqn (3) and the definitions of  $B_t$  and  $B_t^*$  in Eqn (1) we find after some re-arranging:

$$d\tilde{U}_{t}^{nc} = \left\{ L_{nc} dA_{t}^{*} - L_{nc}^{*} dA_{t} - \frac{1}{2} L_{nc}^{*} L_{nc} dt - iH dt \right\} \tilde{U}_{t}^{nc}, \quad \tilde{U}_{0}^{nc} = I$$
 (4)

Note that we could re-write Eqn (4) in a way that makes it clear that the equation is driven by two in themselves commuting noise processes  $\{i(A_t - A_t^*)\}_{t\geq 0}$  and  $\{A_t + A_t^*\}_{g\geq 0}$ . However, these two noises do not commute with each other.

$$d\tilde{U}_{t}^{nc} = \left\{ \frac{L_{nc} + L_{nc}^{*}}{2} (dA_{t}^{*} - dA_{t}) + \frac{L_{nc} - L_{nc}^{*}}{2} (dA_{t} + dA_{t}^{*}) - \frac{1}{2} L_{nc}^{*} L_{nc} dt - iH dt \right\} \tilde{U}_{t}^{nc}.$$

Note that if n becomes very large (strong squeezing), then 1 is negligible with respect to n. This is why we expect that for large n we can replace the operators  $L_{nc}$  by  $F_{nc}$ . This can significantly reduce the complexity of the interaction between the system living on  $\mathcal{H}$  and the field that lives on  $\mathcal{F}$ . If we replace  $L_{nc}$  by  $F_{nc}$  in Eqn (4), then we see since  $F_{nc}^* = -F_{nc}$ , that the QSDE is now driven by only one classical noise process  $\{A_t + A_t^*\}_{t\geq 0}$ . QSDE's that are driven by noises that are commutative in themselves and also all commute with each other are called essentially commutative [16].

We can generalize Eqn (4) by introducing a gauge term in the equation. Often these terms appear after an adiabatic elimination procedure [3, 5].

$$dU_t^{nc} = \left\{ (S - I)d\Lambda_t + L_{nc}dA_t^* - L_{nc}^*SdA_t - \frac{1}{2}L_{nc}^*L_{nc}dt - iHdt \right\} U_t^{nc}, \quad U_0^{nc} = I. \quad (5)$$

Here S is a unitary operator on  $\mathcal{H}$ .

We now introduce the following QSDE:

$$dV_t^{nc} = \left\{ (S - I)d\Lambda_t + F_{nc}dA_t^* - F_{nc}^*SdA_t - \frac{1}{2}F_{nc}^*F_{nc}dt - i(H + H_{nc})dt \right\} V_t^{nc},$$

$$V_0^{nc} = I.$$
(6)

Here the Hamiltonian  $H_{nc}$  is given by

$$H_{nc} = -\frac{i}{2} \left( \frac{n+\overline{c}}{2n+1+2a} L^2 - \frac{n+c}{2n+1+2a} L^{*2} + \frac{(\overline{c}-c)}{2n+1+2a} L^*L \right).$$

We can now state our main result:

**Theorem 2.1:** Let  $U_t^{nc}$  be given by Eqn (5) and  $V_t^{nc}$  by Eqn (6). We then have

$$\lim_{t \to \infty} \left\| \left( U_t^{nc} - V_t^{nc} \right) \psi \right\| = 0, \quad \forall \psi \in \mathcal{H} \otimes \mathcal{F}.$$

Note that  $|c| = \sqrt{n(n+1)}$  also goes to infinity as n goes to infinity. The phase  $\theta$  of  $c = |c| \exp(i\theta)$  stays constant.

Proof. See Section 4. 
$$\Box$$

**Remark 1:** If one studies the proof of Theorem 2.1 in Section 4, then one easily sees that the Theorem could be stated a little bit more general. It is possible to add in extra channels that do not scale with n, provided that they are are present both in Eqn (5) and Eqn (6) in the same way. It is even possible to have scattering  $S_{ij}$  between the channel that does scale with n and the other channels. We have not stated the Theorem in this way, because it is an obvious generalization and it would force us to carry a lot of notation around.

#### Remark 2: Define

$$Z_{t} := \frac{n+c}{\sqrt{2n+1+2a}} A_{t}^{*} + \frac{n+c}{\sqrt{2n+1+2a}} A_{t},$$

$$Z_{t}^{*} := \frac{n+\overline{c}}{\sqrt{2n+1+2a}} A_{t} + \frac{n+\overline{c}}{\sqrt{2n+1+2a}} A_{t}^{*}.$$
(7)

Note that it immediately follows that these noises satisfy the following quantum Itô table

Suppose that S = I in Eqn (5) and Eqn (6). Rewriting Eqn (5) in terms of the noises  $B_t$  and  $B_t^*$  and Eqn (6) in terms of the noises  $Z_t$  and  $Z_t^*$ , we find

$$dU_t^{nc} = \left\{ LdB_t^* - L^*dB_t + \frac{1}{2} \left( LL\overline{c} - LL^*n - L^*L(n+1) + L^*L^*c \right) dt - iHdt \right\} U_t^{nc}, \quad U_0^{nc} = I,$$

$$dV_t^{nc} = \left\{ LdZ_t^* - L^*dZ_t + \frac{1}{2} \left( LL \left( \overline{c} - \frac{n+\overline{c}}{2n+1+2a} \right) - LL^*n - L^*Ln + L^*L^* \left( c - \frac{n+c}{2n+1+2a} \right) \right) dt - i(H+H_{nc})dt \right\} V_t^{nc}, \quad V_0^{nc} = I.$$

This provides a second perspective on Thm 2.1: instead of replacing the coefficients  $L_{nc}$  by  $F_{nc}$ , we can equivalently replace the noises  $B_t, B_t^*$  by  $Z_t, Z_t^*$  (where in both cases we also have to add the extra Hamiltonian term  $H_{nc}$ ). Both procedures lead to the same approximation for the case of strong squeezing.

## 3 Examples

Example 1: (Two level atom coupled to strongly squeezed noise) Let  $\sigma_+$  and  $\sigma_-$  be the usual two level raising and lowering operators

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

A two-level atom driven by squeezed light can be described by the following QSDE

$$dU_t = \left\{ \kappa \sigma_{nc} dA_t^* - \kappa \sigma_{nc}^* dA_t - \frac{1}{2} \kappa^2 \sigma_{nc}^* \sigma_{nc} dt - iHdt \right\} U_t, \quad U_0 = I.$$
 (8)

Here  $\kappa^2$  is the decay rate of the two-level atom, H is an internal atom Hamiltonian and  $\sigma_{nc}$  is given by

$$\sigma_{nc} = \frac{n+1+\overline{c}}{\sqrt{2n+1+2a}}\sigma_{-} - \frac{n+c}{\sqrt{2n+1+2a}}\sigma_{+}.$$

This system was studied [6, Chapter 4] in the strong squeezing limit at the level of the quantum filter (see [4] for a review of quantum filtering theory). The aim was to control the decoherence. It turns out that with the control strategy proposed in [6] it is possible to freeze the system dynamics. That is: the estimates from the filter have no time evolution any more. The reason why the control strategy works, is because the system dynamics become essentially commutative [16] in the strong squeezing limit.

Theorem 2.1 shows that it is possible to approximate the system already at the level of the unitary evolution from which the filter needs to be derived. It follows from Thm 2.1

that in the case of strong squeezing (large n), the system can be approximated by the following unitary evolution

$$V_{t}^{nc} = \left\{ \kappa \gamma_{nc} (dA_{t}^{*} + dA_{t}) + \frac{1}{2} \kappa^{2} \gamma_{nc}^{2} dt - iHdt - iH_{nc} dt \right\} V_{t}^{nc}, \quad V_{0} = I,$$

$$\gamma_{nc} = \frac{n + \overline{c}}{\sqrt{2n + 1 + 2a}} \sigma_{-} - \frac{n + c}{\sqrt{2n + 1 + 2a}} \sigma_{+},$$

$$H_{nc} = -\frac{i}{2} \frac{(\overline{c} - c)\kappa^{2}}{2n + 1 + 2a} \sigma_{+} \sigma_{-}.$$

This equation is indeed only driven by one commuting noise process:  $A_t + A_t^*$ . That is: there is no  $i(A_t - A_t^*)$  term driving  $V_t^{nc}$ : the dynamics is essentially commutative.

Example 2: (A cavity coupled to strongly squeezed noise) We consider a cavity coupled to squeezed vacuum noise via one of its mirrors. The system lives on the Hilbert space  $\ell^2(\mathbb{N}) \otimes \mathcal{F}$  and is given by

$$dU_t^{nc} = \left\{ \kappa b_{nc} dA_t^* - \kappa b_{nc}^* dA_t - \frac{1}{2} \kappa^2 b_{nc}^* b_{nc} - i\hbar \omega b^* b \right\} U_t^{nc}, \quad U_0 = I,$$

$$b_{nc} = \frac{n+1+\overline{c}}{\sqrt{2n+1+2a}} b - \frac{n+c}{\sqrt{2n+1+2a}} b^*.$$

Here  $\kappa^2$  is the decay rate of the cavity,  $\omega$  is the cavity frequency and b is the standard lowering operator and  $b^*$  is the standard raising operator for the eigen functions of  $b^*b$ 

$$b\phi_i = \sqrt{i}\phi_{i-1}, \quad b^*\phi_i = \sqrt{i+1}\phi_{i+1}, \quad b^*b\phi_i = i\phi_i.$$

Note that  $[b, b^*] = 1$ . The operators b and  $b^*$  are unbounded which means that this example is technically out of the scope of Theorem 2.1. We fix this by simply truncating the operators at a very high level N.

We can now apply Theorem 2.1 and find that the time evolution of the cavity and its environment in the case of strong squeezing can be approximated by

$$dV_t^{nc} = \left\{ \kappa f_{nc} (dA_t^* + dA_t) + \frac{1}{2} \kappa^2 f_{nc}^2 - i(\hbar \omega b^* b + H_{nc}) \right\} V_t^{nc}, \quad V_0 = I,$$

$$f_{nc} = \frac{n + \overline{c}}{\sqrt{2n + 1 + 2a}} b - \frac{n + c}{\sqrt{2n + 1 + 2a}} b^*,$$

$$H_{nc} = -\frac{i\kappa^2}{2} \left( \frac{n + \overline{c}}{2n + 1 + 2a} b^2 - \frac{n + c}{2n + 1 + 2a} b^{*2} + \frac{(\overline{c} - c)}{2n + 1 + 2a} b^* b \right).$$

Notice that the dynamics given by  $V_t^{nc}$  is again essentially commutative.

### 4 Proof of Theorem 2.1

Let  $\alpha$  be a complex number. We define the Weyl operator  $W_t(\alpha)$  as the solution to the following QSDE

$$dW_t(\alpha) = \left\{ \alpha dA_t^* - \overline{\alpha} dA_t - \frac{1}{2} |\alpha|^2 dt \right\} W_t(\alpha), \quad W_0(\alpha) = I.$$

Now we define  $U_t^{nc}(\alpha) := U_t^{nc}W_t(\alpha)$  and  $V_t^{nc}(\alpha) := V_t^{nc}W_t(\alpha)$ . It then follows from the quantum Itô rules that

$$dV_{t}^{nc}(\alpha)^{*} = V_{t}^{nc}(\alpha)^{*} \left\{ (S^{*} - I)d\Lambda_{t} + (F_{nc} + \alpha)^{*}dA_{t} - S^{*}(F_{nc} + \alpha)dA_{t}^{*} - \frac{1}{2}(F_{nc} + \alpha)^{*}(F_{nc} + \alpha)dt - \frac{1}{2}(\overline{\alpha}F_{nc} - \alpha F_{nc}^{*})dt + i(H + H_{nc})dt \right\},$$
(9)

$$dU_t^{nc}(\beta) = \left\{ (S - I)d\Lambda_t + (L_{nc} + \alpha)dA_t^* - (L_{nc} + \alpha)^*SdA_t - \frac{1}{2}(L_{nc} + \alpha)^*(L_{nc} + \alpha)dt + \frac{1}{2}(\overline{\alpha}L_{nc} - \alpha L_{nc}^*)dt - iHdt \right\} U_t^{nc}(\alpha).$$

$$(10)$$

**Definition 1:** We denote by  $id : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$  the identity map id(X) = X. We write  $\phi$  for the state on  $\mathcal{B}(\mathcal{H})$  given by taking the inner product with the vacuum vector  $\Phi$ . We let  $\mathcal{B}_0$  be the Banach subalgebra of  $\mathcal{B}(\mathcal{H})$  generated by the identity element I in  $\mathcal{B}(\mathcal{H})$ . We now define:

$$T_t^{(\alpha nc)}(X) := \mathrm{id} \otimes \phi \Big( V_t^{nc}(\alpha)^* X U_t^{nc}(\alpha) \Big), \quad \text{for all} \quad X \in \mathcal{B}(\mathcal{H}), \ t \ge 0,$$
  
$$T_t(X) := X, \quad \text{for all} \quad X \in \mathcal{B}_0, \ t \ge 0.$$

**Lemma 1:** For every  $n \ge 0$ , the families of bounded linear maps  $T_t^{(\alpha nc)}$   $t \ge 0$  and  $T_t$   $(t \ge 0)$  given by Definition 1 are norm-continuous one-parameter semigroups.

*Proof.* The semigroup property of  $T_t^{(\alpha nc)}$  follows immediately from the cocycle property (wrt the shift) of  $V_t^{nc}(\alpha)$  and  $U_t^{nc}(\alpha)$ . Since the conditional expectation id  $\otimes \phi$  is norm-contractive and  $V_t^{nc}(\alpha)$  and  $U_t^{nc}(\alpha)$  are unitary, we have

$$\left\|T_t^{(\alpha nc)}(X)\right\| \leq \left\|V_t^{nc}(\alpha)^*XU_t^{nc}(\alpha)\right\| \leq \left\|V_t^{nc}(\alpha)^*\right\| \left\|X\right\| \left\|U_t^{nc}(\alpha)\right\| \leq \left\|X\right\|,$$

i.e.  $T_t^{(\alpha nc)}$  is norm-contractive. Note that due to the boundedness of all coefficients in the QSDEs for  $V_t^{nc}(\alpha)$  and  $U_t^{nc}(\alpha)$  (Eqns (9) and (10)), it immediately follows that the generator of  $T_t^{(\alpha nc)}$  is bounded. This means that  $T_t^{(\alpha nc)}$  is norm-continuous. Note that the statements about  $T_t$  are trivially true.

**Proposition 1:** The generator  $\mathcal{L}^{(\alpha nc)}$  of the semigroup  $T_t^{(\alpha nc)} = \exp(t\mathcal{L}^{(\alpha nc)})$   $(t \ge 0)$  evaluated at the identity element I of  $\mathcal{B}(\mathcal{H})$  is given by

$$\mathcal{L}^{(\alpha nc)}(I) = -\frac{1}{2}(L_{nc} - F_{nc})^* (L_{nc} - F_{nc}) + \overline{\alpha}(L_{nc} - F_{nc}) - \alpha(L_{nc} - F_{nc})^* =$$

$$= -\frac{1}{2} \frac{L^* L}{2n + 1 + 2a} + \frac{\overline{\alpha}L - \alpha L^*}{\sqrt{2n + 1 + 2a}}.$$
(11)

*Proof.* Note that  $dT_t^{\alpha nc}(I) = \mathrm{id} \otimes \phi(d(V_t^{nc}(\alpha)^*U_t^{nc}(\alpha))) = T_t^{(\alpha nc)}(\mathcal{L}^{(\alpha nc)}(I))dt$ . Using Eqns (9) and (10), the quantum Itô rule [14] and the fact that vacuum expectations of stochastic integrals vanish [14], we find

$$\mathcal{L}^{(\alpha nc)}(I) = -\frac{1}{2} (F_{nc} + \alpha)^* (F_{nc} + \alpha) - \frac{1}{2} (\overline{\alpha} F_{nc} - \alpha F_{nc}^*) + i(H + H_{nc})$$
$$-\frac{1}{2} (L_{nc} + \alpha)^* (L_{nc} + \alpha) + \frac{1}{2} (\overline{\alpha} L_{nc} - \alpha L_{nc}^*) - iH$$
$$+ (F_{nc} + \alpha)^* (L_{nc} + \alpha).$$

We can easily re-write this to obtain

$$\mathcal{L}^{(\alpha nc)}(I) = -\frac{1}{2} \Big( (L_{nc} + \alpha)^* (L_{nc} + \alpha) + (F_{nc} + \alpha)^* (F_{nc} + \alpha) - 2(F_{nc} + \alpha)^* (L_{nc} + \alpha) \Big) + \frac{1}{2} (\overline{\alpha} (L_{nc} - F_{nc}) - \alpha (L_{nc} - F_{nc})^*) + iH_{nc}.$$

Now we complete the squares and obtain

$$\mathcal{L}^{(\alpha nc)}(I) = -\frac{1}{2} \Big( (L_{nc} - F_{nc})^* (L_{nc} - F_{nc}) - (F_{nc} + \alpha)^* (L_{nc} + \alpha) + (L_{nc} + \alpha)^* (F_{nc} + \alpha) \Big) + \frac{1}{2} (\overline{\alpha} (L_{nc} - F_{nc}) - \alpha (L_{nc} - F_{nc})^*) + iH_{nc}.$$

Taking all  $\alpha$  and  $\overline{\alpha}$  terms together, we find

$$\mathcal{L}^{(\alpha nc)}(I) = -\frac{1}{2}(L_{nc} - F_{nc})^* (L_{nc} - F_{nc}) + \overline{\alpha}(L_{nc} - F_{nc}) - \alpha(L_{nc} - F_{nc})^* + \frac{1}{2}(F_{nc}^* L_{nc} - L_{nc}^* F_{nc}) + iH_{nc}.$$

The proposition now follows from the definition of  $L_{nc}$ ,  $F_{nc}$  and  $H_{nc}$ .

The proof of our main result (Theorem 2.1) relies heavily on the Trotter-Kato theorem [17, 15]. We have taken the formulation of the Trotter-Kato theorem from [7, Thm 3.17, page 80].

**Theorem 4.1: Trotter-Kato Theorem** Let  $\mathcal{B}$  be a Banach space and let  $\mathcal{B}_0$  be a closed subspace of  $\mathcal{B}$ . For each  $n \geq 0$ , let  $T_t^{(n)}$  be a strongly continuous one-parameter contraction semigroup on  $\mathcal{B}$  with generator  $\mathcal{L}^{(n)}$ . Moreover, let  $T_t$  be a strongly continuous one-parameter contraction semigroup on  $\mathcal{B}_0$  with generator  $\mathcal{L}$ . Let  $\mathcal{D}$  be a core for  $\mathcal{L}$ . The following conditions are equivalent:

1. For all  $X \in \mathcal{D}$  there exist  $X^{(n)} \in \text{Dom}(\mathcal{L}^{(n)})$  such that

$$\lim_{n \to \infty} X^{(n)} = X, \qquad \lim_{n \to \infty} \mathscr{L}^{(n)} \left( X^{(n)} \right) = \mathscr{L}(X).$$

2. For all  $0 \le s < \infty$  and all  $X \in \mathcal{B}_0$ 

$$\lim_{n \to \infty} \left\{ \sup_{0 < t < s} \left\| T_t^{(n)}(X) - T_t(X) \right\| \right\} = 0.$$

**Proposition 2:** Let  $T_t^{(\alpha nc)}$  and  $T_t$  be the one-parameter semigroups on  $\mathcal{B}(\mathcal{H})$  and  $\mathcal{B}_0$  defined in Definition 1, respectively. We now have:

$$\lim_{n \to \infty} \left\{ \sup_{0 < t < s} \left\| T_t^{(\alpha nc)}(I) - I \right\| \right\} = 0,$$

for all  $0 \le s < \infty$ .

*Proof.* Note that both semigroups are norm-continuous and therefore also strongly continuous. Note that the generator  $\mathscr{L}$  of  $T_t = \exp(t\mathscr{L})$  is equal to 0. We are now going to apply the Trotter-Kato theorem (Thm 4.1) with  $\mathcal{D} = \mathcal{B}_0$ . If we take  $X^{(n)} = I$  for all n, then obviously we have  $\lim_{n\to\infty} X^{(n)} = I$ . It follows from Propostion 1 that

$$\lim_{n \to \infty} \mathcal{L}^{(\alpha nc)}(X^{(n)}) = \lim_{n \to \infty} \mathcal{L}^{(\alpha nc)}(I) = 0 = \mathcal{L}(I).$$

Since all elements in  $\mathcal{B}_0$  are multiples of I, we have the above result for all elements in  $\mathcal{B}_0$ . The proposition then follows from the Trotter-Kato Theorem and the fact that  $T_t(I) = I$ .

Let f be a function in  $L^2(\mathbb{R})$ . We define the Weyl operator  $W_t(f) = W(f\chi_{[0,t]})$  (where  $\chi_{[0,t]}$  is the indicator function of the interval [0,t]), by the following QSDE

$$dW_t(f) = \left\{ f(t)dA_t^* - \overline{f(t)}dA_t - \frac{1}{2}|f(t)|^2 dt \right\} W_t(f), \quad W_0(f) = I.$$

If we act with W(f) on the vacuum  $\Phi$ , then we get the coherent vector  $\psi(f)$ . The coherent vectors form a dense set in  $\mathcal{F}$ .

**Proof of Theorem 2.1** Let  $t \ge 0$ . Let f be a step function in  $L^2([0,t])$ , i.e. there exists an  $m \in \mathbb{N}$  and  $0 = t_0 < t_1 < \ldots < t_m = t$  and  $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$  such that

$$s \in [t_{i-1}, t_i) \Longrightarrow f(s) = \alpha_i, \quad \forall i \in \{1, \dots, m\}.$$

Let  $\psi(f)$  be the coherent vector with respect to f. Let v be an element in  $\mathcal{H}$ . The cocycle property of solutions to QSDE's and the exponential property of the symmetric Fock space lead to

$$\left\langle v \otimes \psi(f), V_t^{nc*} U_t^{nc} v \otimes \psi(f) \right\rangle = \left\langle v \otimes \Phi, \left( V_t^{nc} W(f) \right)^* U_t^{nc} W(f) v \otimes \Phi \right\rangle =$$

$$= \left\langle v, T_{t_1}^{(\alpha_1 nc)} \cdots T_{t-t_m}^{(\alpha_m nc)} (I) v \right\rangle$$

Now we have due to Proposition 2

$$\lim_{n \to \infty} \left\| (U_t^{nc} - V_t^{nc})v \otimes \psi(f) \right\|^2 =$$

$$\lim_{n \to \infty} \left\langle v, \left( 2I - T_{t_1}^{(\alpha_1 nc)} \cdots T_{t-t_m}^{(\alpha_m nc)}(I) - T_{t_1}^{(\alpha_1 nc)} \cdots T_{t-t_m}^{(\alpha_m nc)}(I)^* \right) v \right\rangle = 0.$$

The Thm now follows because the step functions are dense in  $L^2(\mathbb{R})$  and the span of all coherent vectors, i.e. span $\{\Psi(f), f \in L^2(\mathbb{R})\}$ , is dense in  $\mathcal{F}$ .

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