Threefolds

B. Wang (汪 镔)

Abstract

This is an example on the cohomology of threefolds.

Theorem 0.1. Let X be a smooth projective variety over \mathbb{C} . If Hodge classes on X are algebraic, Hodge level 1 is geometric level 1.

Proof. In theorem 8, [1], Voisin proved that if the usual Hodge conjecture holds on $C \times X$ for all smooth projective curves C, then the generalized Hodge conjecture of level 1 holds on X. Let L be a sub-Hodge structure of $H^{2r+1}(X;\mathbb{Q})$ with coniveau r > 0. She showed that there is a smooth projective curve C, and a Hodge cycle

$$\tilde{\Psi} \in Hdg^{2r+2}(C \times X) \tag{1}$$

such that

$$\tilde{\Psi}_*(H^1(C;\mathbb{Q})) = L. \tag{2}$$

where $\tilde{\Psi}_*$ is defined as

$$P_!\Big(\tilde{\Psi}\cup(\alpha\otimes 1_X)\Big),\quad \alpha\in H^1(C;\mathbb{Q})$$

with the projection $P: C \times X \to X$. She then used the usual Hodge conjecture on the n+1-fold $C \times X$ to conclude $\tilde{\Psi}$ is a fundamental class of an algebraic cycle Z. Then the process of formula (2) can go through Z for the determination of the "level" (or coniveau). We claim that a CURRENT representing $\tilde{\Psi}$ plays the same role of Z, provided there is an intersection theory for currents. Notice $P_!(\tilde{\Psi})$ is a Hodge cycle in X. By the assumption

it is algebraic on X, i.e there is a closed current $T_{\tilde{\Psi}}$ on $C \times X$ representing the class $\tilde{\Psi}$ such that

$$P_*(T_{\tilde{\Psi}}) = S_a + bK \tag{3}$$

where S_a is a non-zero current of integration over an algebraic cycle S, and bK is an exact current of degree 2r in X. Then we let process (2) go through the currents. Specifically we consider another current in $C \times X$,

$$T := T_{\tilde{\Psi}} - [t] \otimes bK, \tag{4}$$

where [t] is a current of evaluation at a point $t \in C$. By adjusting the exact current on the right hand side of (4), we may assume the projection P satisfies

$$P(supp(T)) = supp(P_*(T)). \tag{5}$$

Let Θ be a collection of closed currents on C representing the classes in $H^1(C;\mathbb{Q})$. Applying the definition of "intersection of currents" ([2]), we obtain a family of currents

$$T_*(\Theta),$$
 (6)

defined as

$$P_* \left[T \wedge [\eta \otimes X] \right], \quad \eta \in \Theta,$$
 (7)

whose members are all supported on the support of the current,

$$P_*(T) = S_a. (8)$$

where \wedge is the intersection of currents. The property of the "intersection of currents" ([2]) says these currents in $T_*(\Theta)$ represent the classes in L. This shows that, on one hand, members in the family

$$T_*(\Theta)$$
 (9)

represent the classes in L, and on the other hand, they are all supported on the algebraic set |S|.

Corollary 0.2. On threefolds, Hodge level 1 is geometric level 1.

Appendix:

Intersection of currents

The following is a short description of "intersection of currents" used above. Let X be a compact differential manifold equipped with a de Rham data \mathcal{U} (the data is irreverent to the result). For any two closed currents T_1, T_2 , there exists another current, called the intersection of currents (depending on \mathcal{U}),

$$[T_1 \wedge T_2]. \tag{10}$$

which is a strong limit of a regularization. The intersection satisfies many basic properties. Two of them used above:

(1) $[T_1 \wedge T_2]$ is closed and represents the cup-product of the cohomology of T_1, T_2 .

(2)

$$supp([T_1 \wedge T_2]) \subset supp(T_1) \cap supp(T_2).$$

Now we consider two compact manifolds X, C. Let P be the projection $C \times X \to X$. Suppose there is a closed current T on $C \times X$ such that

$$P(supp(T)) = supp(P_*(T)). \tag{11}$$

(this condition is needed to control the supports). Then there is a well-defined operation induced from the intersection

$$T_*: C\mathcal{D}'(C) \to C\mathcal{D}'(X)$$

$$a \to P_*([T \wedge (a \times X)]), \tag{12}$$

where $C\mathcal{D}'$ stands for the space of closed real currents. This operation T_* does not only descend to cohomology, but also preserves the supports (by property (2) and (11)), i.e.

$$supp(image(T_*)) \subset supp(P_*(T)).$$
 (13)

Note: The formula (5) implies that T can be chosen algebraic. So the proof is identical to Voisin's.

References

- [1] C. Voisin, Lectures on the Hodge and Grothendieck-Hodge conjectures, Rend.Sem., Univ. Politec. Torino, Vol. 69, 2(2011), pp 149-198.
- [2] B. Wang, Intersection of currents, Preprint, 2016.