DERIVED EQUIVALENCES INDUCED BY GOOD SILTING COMPLEXES

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ABSTRACT. Consider a (possibly big) silting object U in a derived category over a (dg-)algebra A. Under some fairly general appropriate hypotheses, we show that it induces derived equivalences between the derived category over A and a localization of the derived category of dg-endomorphism algebra B of U. If, in addition, U is small then this localization is the whole derived category over B.

Introduction

Let A be a k-algebra, where k is a commutative ring with one. Recall that tilting theory generalizes Morita theory, in the sense that if $T \in \operatorname{Mod}(A)$ is a classical 1-tilting module with endomorphism $E = \operatorname{End}_A(T)$, then the functors $\operatorname{Hom}_A(T,-), -\otimes_E T$ and their derived functors induce mutually inverse equivalences between some subcategories of respective module categories. We note that in this case T has to be finitely presented, and the above mentioned subcategories are the torsion or the torsion-free classes induced by $\operatorname{Hom}_A(T,-)$ and $-\otimes_E T$. A result which makes precise these things was formulated first by Brenner and Butler in [6] and was afterwards called the Tilting Theorem. Note that a version of the Tilting Theorem can be formulated at the level of derived category and functors, [9]. This leads to a Morita Theory for derived categories which has as a culminating point the theory developed by Rickard in [24] and [25]. We refer to [16] for a recent survey.

All above mention equivalences are induced by compact objects (i.e. objects such that the induced Hom-covariant functor commutes with respect to direct sums). On the other side, infinitely generated tilting modules play an important role in the study of module categories, and a tilting theorem is also valid for this case. But it is not easy to compute the categories of right E-modules which are involved in the equivalences induced by such a module, and approaches which use derived categories are useful. One of the papers which pursues such an approach in the so called 1-tilting case is [3]. Continuing the same way of thinking, in [4] it is shown that a good n-tilting module $T \in \text{Mod}(A)$ induces an equivalence between the derived category $\mathbf{D}(A)$ and a subcategory of the derived category $\mathbf{D}(B)$, where B is the endomorphism algebra of T. This equivalence is realized by the derived hom functor $\mathbb{R}\text{Hom}_A(T, -)$, and its adjoint, namely the derived tensor product.

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On the other side, since the class of classical tilting modules over finitely generated algebras is not closed under mutations, it was extended to the class of support τ -tilting modules in [1]. These correspond to the compact silting complexes of length 2, introduced in [17] in order to describe t-structures in derived categories. These modules also induce the first equivalence from the Tilting Theorem, [11], but it was proved in [7] that in order to obtain a pair of counter equivalences we have to replace the endomorphism ring of the module with the endomorphism ring of the corresponding compact silting complex as object in the corresponding derived category. We refer to [8] and [20] for recent results on equivalences induced by tilting or compact silting complexes.

As in the case of tilting theory, the theory of support τ -tilting modules was extended to a theory developed for infinitely generated modules, called semi-tilting or silting modules, in [27] and [2]. These modules correspond to non-compact silting complexes concentrated in -1 and 0. The equivalences induced by such complexes are described in [5]. It would be useful to have a Silting Theorem for non-compact general silting complexes. In this paper we will prove such a theorem, and we will describe the family of equivalences induced by a silting complex. In order to do this we will use one of the main ideas from [3]: to replace the initial silting complex by a so called good silting complex. The main difference is that a silting complex is not quasi-isomorphic (that is not isomorphic in the derived category) to the corresponding silting module, forcing us to consider the differential graded endomorphism algebra instead of the simple endomorphism algebra.

Therefore we will start in Section 1 with some generalities about the dgalgebras and their derived categories. If U is a cofibrant object in the derived category $\mathbf{D}(A,d)$ and put $B = \mathrm{DgEnd}_A(U)$, the dg-endomorphism algebra of U then induced Hom and tensor derived functors

$$\mathbb{R}\mathrm{Hom}_A(U,-): \mathbf{D}(A,d) \leftrightarrows \mathbf{D}(B,d): -\otimes_B^{\mathbb{L}} U$$

form an adjoint pair. We will use the notation

$$\mathcal{K}^{\perp} = \{ Y \mid \operatorname{Hom}_{\mathbf{D}(B,d)}(\operatorname{Ker}(-\otimes_{B}^{\mathbb{L}} U), Y) = 0 \}.$$

The main result of this Section says that if $\operatorname{codim}_{\operatorname{add}(U)} A$ is finite, i.e. there exist a positive integer n and a family of triangles

$$X_i \to C_i \to X_{i+1} \stackrel{+}{\to} \text{ with } 0 \le i \le n$$

in $\mathbf{D}(A,d)$, such that $C_i \in \mathrm{add}(U)$, $X_0 = A$ and $X_{n+1} = 0$, then we have an equivalence

$$\mathbb{R}\mathrm{Hom}_A(U,-):\mathbf{D}(A,d) \leftrightarrows \mathcal{K}^{\perp}:-\otimes_B^{\mathbb{L}}U.$$

In Section 2 where we define big, small and good silting objects in $\mathbf{D}(A,d)$, good lying somewhere between big and small. Further we show every silting object is equivalent to a good one and we will use the above result to obtain a Silting Theorem at the level of derived categories (Theorem 2.4). This states that every good silting object U induces the equivalence from the main result of the previous Section, and that the dg-endomorphism algebra of U is equivalent to a non-positive dg-algebra such that the heart of the standard t-structure is equivalent to the category of right E-modules, where E is the endomorphism ring of U in the $\mathbf{D}(A,d)$. As a consequence of this

theorem in Corollary 2.6 we restrict the derived functors in order to find equivalences between some subcategories of the ordinary module categories, expressing the Silting Theorem at this level.

In this paper all rings/algebras are unital, and $\operatorname{Mod}(R)$ means the category of right R-modules, and $\mathbf{D}(R)$ is the corresponding derived category. If $\mathcal C$ is a category then the groups of homomorphisms are denoted by $\operatorname{Hom}_{\mathcal C}(X,Y)$. If we are inside a derived category, we will use the word isomorphism instead of quasi-isomorphism. We recall that in our cases the hearts associated to t-structures are abelian categories, and that two objects in such a heart are isomorphic if and only if they are quasi-isomorphic in the corresponding triangulated category. We refer to [21] for other unexplained notions.

A very general result about equivalences between derived categories of dg categories is proved in [19]. The authors were informed that Theorem 1.4 and Theorem 2.4 are contained in [19, Theorem 6.4]. Therefore our work should be seen only as a survey of known results, made from the perspective of silting theory.

1. Derived equivalences

We begin with some generalities about the total derived functors. For the sake of generality we need, we will work in the context of dg-algebras. We will follow [13] and [15] in these considerations. Fix a ground ring k (commutative, with one). Recall that a k-algebra is \mathbb{Z} -graded provided that it has a decomposition as a direct sum of k-submodules $A = \bigoplus_{i \in \mathbb{Z}} A_i$ such that $A_i A_j \subseteq A_{i+j}$, for all $i, j \in \mathbb{Z}$. Further a dg-algebra is a \mathbb{Z} -graded algebra A endowed with a k-linear differential $d: A \to A$ which is homogeneous of degree 1, that is $d(A^i) \subseteq A^{i+1}$ for all $i \in \mathbb{Z}$, and satisfies $d^2 = 0$ and the graded Leibniz rule:

$$d(ab) = d(a)b + (-1)^i ad(b)$$
, for all $a \in A^i$ and $b \in A$.

A (right) dg-module over A is a \mathbb{Z} -graded module

$$X = \bigoplus_{i \in \mathbb{Z}} X^i$$

endowed with a k-linear square-zero differential $d: M \to M$, which is homogeneous of degree 1 and satisfies the graded Leibnitz rule:

$$d(xa) = d(x)a + (-1)^i x d(a)$$
, for all $x \in X^i$ and $a \in A$.

Left dg-A-modules are defined similarly. A morphism of dg-A-modules is an A-linear map $f: M \to N$ such that $f(M^i) \subseteq N^i$ and f commutes with the differential. In this way we obtained the category $\operatorname{Mod}(A,d)$ of all dg-A-modules.

If A is a dg-algebra, then the dual dg-algebra A^{op} is defined as follows: $A^{op} = A$ as graded k-modules, $ab = (-1)^{ij}ba$ for all $a \in A^i$ and all $b \in A^j$ and the same differential d. It is clear than a left dg-A-module X is a right dg- A^{op} -module with the "opposite" multiplication $xa = (-1)^{ij}ax$, for all $a \in A^i$ and all $x \in M^j$, henceforth we denote by $\operatorname{Mod}(A^{op}, d)$ the category of left dg-A-modules.

It is not hard to see that an ordinary k-algebra A can be viewed as a dg-algebra concentrated in degree 0, case in which a dg-module is nothing else than a complex of ordinary (right) A-modules, that is Mod(A, d) is the category of all complexes (we will not use another special notation for the category of complexes).

For dg-module $X \in \operatorname{Mod}(A,d)$ one defines (functorially) the following k-modules $Z^n(X) = \operatorname{Ker}(X^n \xrightarrow{d} X^{n+1})$, $B^n(X) = \operatorname{Im}(X^{n-1} \xrightarrow{d} X^n)$, and $H^n(X) = Z^n(X)/B^n(X)$, for all $n \in \mathbb{Z}$. We call $H^n(X)$ the n-th cohomology group of X. A morphism of dg-modules is called quasi-isomorphism if it induces isomorphisms in cohomologies. A dg-module $X \in \operatorname{Mod}(A,d)$ is called acyclic if $H^n(X) = 0$ for all $n \in \mathbb{Z}$. A morphism of dg-A-modules $f: X \to Y$ is called null-homotopic provided that there is a graded homomorphism $s: X \to Y$ of degree -1 such that f = sd + ds. The homotopy category $\mathbf{K}(A,d)$ has the same objects as $\operatorname{Mod}(A,d)$ and the morphisms are equivalence classes of morphims of dg-modules, modulo the homotopy. It is well-known that the homotopy category is triangulated. Moreover a null-homotopic morphism is acyclic, therefore the functors H^n factor through $\mathbf{K}(A,d)$ for all $n \in \mathbb{Z}$.

The derived category $\mathbf{D}(A, d)$ is obtained from $\mathbf{K}(A, d)$, by formally inverting all quasi-isomorphisms.

Let now A and B be two dg-algebras and let U be a dg-B-A-bimodule (that is U is a dg- $B^{op} \otimes_k A$ -module). For every $X \in \text{Mod}(A, d)$ then we can consider the so called dg-Hom:

$$\operatorname{Hom}_A^{\bullet}(U,X) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_A^n(U,X)$$

whith $\operatorname{Hom}_A^n(U,X) = \prod_{i \in \mathbb{Z}} \operatorname{Hom}_{A^0}(U^i,X^{n+i})$, whose differentials are given by $d(f)(x) = d_Y f(x) - (-1^n) f d_X(x)$ for all $f \in \operatorname{Hom}_A^n(X,Y)$. Then $\operatorname{Hom}_A^{\bullet}(U,X)$ becomes a dg-B-module, so we get a functor

$$\operatorname{Hom}_A^{\bullet}(U,-):\operatorname{Mod}(A,d)\to\operatorname{Mod}(B,d).$$

It induces a triangle functor

$$\operatorname{Hom}_{A}^{\bullet}(U,-): \mathbf{K}(A,d) \to \mathbf{K}(B,d)$$

This last functor has a total right derived functor, which is defined as follows: A dg-A-module C it is called cofibrant if $\mathbf{K}(A,d)(C,N)=0$ for all acyclic dg-A-module N, or equivalently, $\mathbf{K}(A,d)(C,X)=\mathrm{Hom}_{\mathbf{D}(A,d)}(C,X)$ for every dg-A-module X. Then a cofibrant replacement for U is a cofibrant dg-module U' together with a quasi-isomorphism $U' \to U$. Dually we define the notions of fibrant object and fibrant replacement. It turns out that (co)fibrant replacements always exist in $\mathbf{K}(A,d)$ (see [13, Theorem 3.1]), hence we can define .

$$\mathbb{R}\mathrm{Hom}_A(U,-): \mathbf{D}(A,d) \to \mathbf{D}(B,d),$$

by $\mathbb{R}\mathrm{Hom}_A(U,X) = \mathrm{Hom}_A^{\bullet}(U',X) \cong \mathrm{Hom}_A^{\bullet}(U,X')$ where U' is a cofibrant replacement of U and X' is a fibrant replacement of X.

Let $Y \in \text{Mod}(B,d)$. On the usual tensor product $Y \otimes_B$ there is a natural grading:

$$Y \otimes_B^{\bullet} U = \bigoplus_{n \in \mathbb{Z}} Y \otimes_B^n U,$$

where $Y \otimes_B^n U$ is the quotient of $\bigoplus_{i \in \mathbb{Z}} Y^i \otimes_{B^0} U^{n-i}$ by the submodule generated by $y \otimes bu - yb \otimes u$ where $y \in Y^i$, $u \in U^j$ and $b \in B^{n-i-j}$, for all $i, j \in \mathbb{Z}$. Together with the differential

$$d(y \otimes u) = d(y) \otimes u + (-1)^{i} y d(u)$$
, for all $y \in Y^{i}$, $u \in U$,

we get a dg-A-module inducing a functor $-\otimes_B^{\bullet}U: \operatorname{Mod}(B,d) \to \operatorname{Mod}(A,d)$ and further a triangle functor $-\otimes_B^{\bullet}U: \mathbf{K}(B,d) \to \mathbf{K}(A,d)$. The left derived tensor product is defined by $Y \otimes_B^{\mathbb{L}} U = Y' \otimes_B^{\bullet} U \cong Y \otimes_B^{\bullet} U'$ where Y' and U' are cofibrant replacements for Y and U in $\mathbf{K}(B,d)$ and $\mathbf{K}(B^{op},d)$ respectively. It induces a triangle functor

$$-\otimes_B^{\mathbb{L}} U : \mathbf{D}(B,d) \to \mathbf{D}(A,d)$$

which is the left adjoint of $\mathbb{R}\mathrm{Hom}_A(U,-)$.

Let A be a dg-algebra and an object $U \in Mod(A, d)$. Then

$$\operatorname{DgEnd}_A(U) = \operatorname{Hom}_A^{\bullet}(U, U)$$

is a dg-algebra, called the endomorphism dg-algebra of U. We will denote $B = \mathrm{DgEnd}_A(U)$. Then U becomes a dg-B-A-bimodule and consequently the total derived functors:

$$\mathbb{R}\mathrm{Hom}_A(U,-): \mathbf{D}(A,d) \leftrightarrows \mathbf{D}(B,d): -\otimes_B^{\mathbb{L}} U.$$

form an adjoint pair. Moreover the total derived functors:

$$\mathbb{R}\mathrm{Hom}_A(-,U): \mathbf{D}(A,d) \leftrightarrows \mathbf{D}(B^{op},d): \mathbb{R}\mathrm{Hom}_{B^{op}}(-,U)$$

form a right adjoint pair (see [14, Lemma 13.6]). For both these adjunctions we consider canonical morphisms:

 $\phi: \mathbb{R}\mathrm{Hom}_A(U, -) \otimes_B^{\mathbb{L}} U \to 1_{\mathbf{D}(A,d)} \text{ and } \psi: \mathbb{R}\mathrm{Hom}_A(U, -\otimes_B^{\mathbb{L}} U) \to 1_{\mathbf{D}(B,d)}$ respectively

$$\delta: 1_{\mathbf{D}(A,d)} \to \mathbb{R}\mathrm{Hom}_{B^{op}}(\mathbb{R}\mathrm{Hom}_A(-,U),U)$$
 and

$$\mu: 1_{\mathbf{D}(B^{op},d)} \to \mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(-,U),U).$$

Let \mathcal{D} be a triangulated category with the shift functor denoted by [1], and let \mathcal{C} be a subcategory of \mathcal{D} . We define the full subcategories of \mathcal{D}

$$\mathcal{C}^{\perp_I} = \{X \mid \mathcal{D}(C, X[i]) = 0 \text{ for all } i \in I, C \in \mathcal{C}\},\$$

$${}^{\perp_I}\mathcal{C} = \{X \mid \mathcal{D}(X, C[i]) = 0 \text{ for all } i \in I, C \in \mathcal{C}\}$$

and $\mathcal{C}[\mathbb{Z}] = \bigcup_{i \in \mathbb{Z}} \mathcal{C}[i]$. If $\mathcal{C} = \mathcal{C}[1]$ (that is, \mathcal{C} is closed under all shifts) then clearly $\mathcal{C}^{\perp_{\mathbb{Z}}} = \mathcal{C}^{\perp_{0}}$ and we write simply \mathcal{C}^{\perp} instead of both.

Consider an object $X \in \mathcal{D}$. Following [27], we say that X has the C-resolution dimension (respectively C-coresolution dimension) $\leq n$, and we write $\dim_{\mathcal{C}} X \leq n$, ($\operatorname{codim}_{\mathcal{C}} X \leq n$) provided that there is a sequence of triangles

$$X_{i+1} \to C_i \to X_i \xrightarrow{+} \text{ with } 0 \le i \le n$$
 (respectively $X_i \to C_i \to X_{i+1} \xrightarrow{+} \text{ with } 0 \le i \le n$)

in \mathcal{D} , such that $C_i \in \mathcal{C}$, $X_0 = X$ and $X_{n+1} = 0$. We will write $\dim_{\mathcal{C}} X < \infty$ (codim_{\mathcal{C}} $X < \infty$) if we can find a positive integer n such that $\dim_{\mathcal{C}} X \leq n$ (respectively, codim_{\mathcal{C}} $X \leq n$)

Denote also by $\langle \mathcal{C} \rangle$ the smallest triangulated category which contains \mathcal{C} . Then by [27, Proposition 2.5]) we have

$$\langle \mathcal{C} \rangle = \{ X \in \mathcal{D} \mid \dim_{\mathcal{C}} X < \infty \} [\mathbb{Z}] = \{ X \in \mathcal{D} \mid \operatorname{codim}_{\mathcal{C}} X < \infty \} [\mathbb{Z}].$$

It is not hard to see that $\langle \operatorname{add}(\mathcal{C}) \rangle = \operatorname{add}(\langle \mathcal{C} \rangle)$, hence this is the smallest thick subcategory (that is triangulated and closed under direct summands) containing \mathcal{C} .

Lemma 1.1. Let $U \in \mathbf{D}(A,d)$ with $\operatorname{codim}_{\operatorname{add}(U)} A \leq n$, and let $B = \operatorname{DgEnd}_A(U)$. Then $\operatorname{dim}_{\operatorname{add}(B)} U \leq n$. Similarly, if $\operatorname{dim}_{\operatorname{add}(A)} U \leq n$ then $\operatorname{codim}_{\operatorname{add}(U)} B \leq n$.

Proof. By hypothesis there is a sequence of triangles in $\mathbf{D}(A,d)$

(†)
$$A_i \to U_i \to A_{i+1} \stackrel{+}{\to} \text{ with } 0 \le i \le n$$

such that $U_i \in \text{add}(U)$, $A_0 = A$ and $A_{n+1} = 0$. Applying the exact functor $\mathbb{R}\text{Hom}_A(-,U)$ we get triangles in $\mathbf{D}(B^{op},d)$ of the form

(†)
$$V_{i+1} \to B_i \to V_i \stackrel{+}{\to} \text{ with } 0 < i < n$$

such that $B_i \in \operatorname{add}(B)$, $V_0 = U$ and $V_{n+1} = 0$. The conclusion follows by observing that $\mathbb{R}\operatorname{Hom}_A(A_0, U) = \mathbb{R}\operatorname{Hom}_A(A, U) \cong U$, $\mathbb{R}\operatorname{Hom}_A(U, U) = B$, and by Lemma 1.2 we have

$$B_i = \mathbb{R} \operatorname{Hom}_A(U_i, U) \in \operatorname{add}(B), \text{ for } 0 \leq i \leq n.$$

The last statement is proved in a similar manner.

We will also mention two lemmas which will be useful in the proof of Theorem 1.4.

Lemma 1.2. Let $F, F' : A \to \mathcal{B}$ two additive functors between preadditive categories, let $\varphi : F \to F'$ be a natural transformation and let $U \in A$. If $X \in \operatorname{add}(U)$, then $F(X) \in \operatorname{add}(F(G))$. If φ_U is an isomorphism, then φ_X is an isomorphism for all $X \in \operatorname{add}(U)$.

Lemma 1.3. If $U \in \mathbf{D}(A, d)$ is a cofibrant object and $B = \mathrm{DgEnd}_A(U)$ then there is a morphism of dg k-modules

$$\Gamma_{X,Y}: \mathbb{R}\mathrm{Hom}_A(U,X) \otimes_B^{\mathbb{L}} Y \to \mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(Y,U),X)$$

which is natural in both $X \in \mathbf{D}(A, d)$ and $Y \in \mathbf{D}(B^{op}, d)$.

Proof. In the first step we want to define a natural map

$$\alpha: \mathbb{R}\mathrm{Hom}_A(U,X) \to \mathbb{R}\mathrm{Hom}_k(Y,\mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(Y,U),X),$$

in $\mathbf{D}(k,d)$ as follows: Since U is cofibrant, we obtain (e.g. by [26, Theorem 21.4]):

$$\mathbb{R}\mathrm{Hom}_A(U,X) = \mathrm{Hom}_A^{\bullet}(U,X) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_A(U,X).$$

Fix a map $f \in \text{Hom}^{\bullet}(U, X)$. By definition of the right derived Hom functor, we have

$$\mathbb{R}\mathrm{Hom}_{A}(\mathbb{R}\mathrm{Hom}_{B^{op}}(Y,U),X)=\mathrm{Hom}_{A}^{\bullet}(C,X)=\bigoplus_{n\in\mathbb{Z}}\mathrm{Hom}^{n}(C,X),$$

where $C \to \mathbb{R}\mathrm{Hom}_{B^{op}}(Y,U)$ is a cofibrant replacement. It is straightforward to check that the assignment $h \mapsto [g \mapsto h \circ g]$ defines a map from $\mathrm{Hom}_A^{\bullet}(U,X)$ to $\mathrm{Hom}_A^{\bullet}(H\mathrm{om}_A^{\bullet}(C,U),\mathrm{Hom}_A^{\bullet}(C,X))$. Therefore, since $\mathrm{Hom}_A^{\bullet}(C,-)$ and $\mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(Y,U),-)$ are natural isomorphic, the fixed morphism f induces further a natural map of dg k-modules

$$f^*: \mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(Y,U),U) \to \mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(Y,U),X).$$

Thus we define $\alpha(f) = f^*\mu_Y$, that is $\alpha(f)$ is the composite map

$$Y \xrightarrow{\mu_Y} \mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(Y,U),U) \xrightarrow{f^*} \mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(Y,U),X),$$

where μ_Y is the canonical morphisms of the adjunction.

For the second step, observe that the left dg-B-module Y can be regarded as a dg-B-k-bimodule, so there is a pair of adjoint functors

$$\mathbb{R}\mathrm{Hom}_k(Y,-): \mathbf{D}(k,d) \leftrightarrows \mathbf{D}(B,d): -\otimes_B^{\mathbb{L}} Y.$$

We denote the adjunction isomorphism by

$$\omega: \operatorname{Hom}_{\mathbf{D}(B,d)}(M, \mathbb{R} \operatorname{Hom}_k(Y,N)) \to \operatorname{Hom}_{\mathbf{D}(k,d)}(M \otimes_B^{\mathbb{L}} Y, N)$$

where $M \in \mathbf{D}(B, d)$ and $N \in \mathbf{D}(k, d)$. We consider $M = \mathbb{R}\mathrm{Hom}_A(U, X)$ and $N = \mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(Y, U), X)$. Therefore the map

$$\Gamma_{X,Y} = \omega(\alpha) : \mathbb{R}\mathrm{Hom}_A(U,X) \otimes_B^{\mathbb{L}} Y \to \mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(Y,U),X)$$

is the desired natural map in $\mathbf{D}(k,d)$, where α in constructed in the first step of this proof.

After finishing the first version of this work, the authors learned that the following Theorem is contained in [19, Theorem 6.4]. Therefore our proof should be regarded only as an alternative argument for a known result.

Theorem 1.4. Consider a cofibrant object $U \in \mathbf{D}(A,d)$ and put $B = \mathrm{DgEnd}_A(U)$. If there exists a positive integer n such that $\mathrm{codim}_{\mathrm{add}(U)} A \leq n$ then $\delta_A : A \to \mathbb{R}\mathrm{Hom}_{B^{op}}(U,U)$ is a quasi-isomorphism (that is, an isomorphism in $\mathbf{D}(A,d)$). Moreover the counit of the adjunction

$$\phi_X : \mathbb{R} \mathrm{Hom}_A(U, X) \otimes_B^{\mathbb{L}} U \to X$$

is an isomorphism for all $X \in \mathbf{D}(A,d)$, or equivalently, the functor

$$\mathbb{R}\mathrm{Hom}_A(U,-): \mathbf{D}(A,d) \to \mathbf{D}(B,d)$$

is fully faithful and it induces an equivalence of categories

$$\mathbf{D}(A,d) \xrightarrow{\sim} \mathrm{Ker}(-\otimes_B^{\mathbb{L}} U)^{\perp}.$$

whose inverse is $-\otimes_B^{\mathbb{L}} U$.

If, in addition, we have $\dim_{\mathrm{add}(A)} U < \infty$ then $\mathrm{Ker}(-\otimes_B^{\mathbb{L}} U) = 0$, hence the categories $\mathbf{D}(A,d)$ and $\mathbf{D}(B,d)$ are equivalent.

Proof. Fix $n \in \mathbb{N}$ such that $\operatorname{codim}_{\operatorname{add}(U)} U \leq n$. Compare the triangles in (†) with the corresponding triangles obtained by applying the functor $\mathbb{R}\operatorname{Hom}_{B^{op}}(-,U)$ to (‡):

$$A_{i} \xrightarrow{\qquad} U_{i} \xrightarrow{\qquad} A_{i+1} \xrightarrow{\qquad} \\ \downarrow^{\delta_{A_{i}}} \qquad \downarrow^{\delta_{U_{i}}} \qquad \downarrow^{\delta_{A_{i+1}}} \\ \mathbb{R}\mathrm{Hom}_{B^{op}}(V_{i}, U) \xrightarrow{\rightarrow} \mathbb{R}\mathrm{Hom}_{B^{op}}(B_{i}, U) \xrightarrow{\rightarrow} \mathbb{R}\mathrm{Hom}_{B^{op}}(V_{i+1}, U) \xrightarrow{+} \\ \downarrow^{\delta_{A_{i+1}}} \qquad .$$

It is clear that δ_U is the isomorphism $U \cong \mathbb{R} \text{Hom}_{B^{op}}(B, U)$, therefore δ_{U_i} , $0 \leq i \leq n$ are isomorphisms too by Lemma 1.2. Moreover $A_{n+1} = 0 = \mathbb{R} \text{Hom}_A(V_{n+1}, U)$, hence we deduce δ_{A_i} are isomorphisms in $\mathbf{D}(A, d)$ for all $0 \leq i \leq n$. For i = 0 we obtain exactly the first part of the conclusion, namely the (quasi-)isomorphism $\mathbb{R} \text{Hom}_{B^{op}}(U, U) \cong A$.

Let $X \in \mathbf{D}(A,d)$. Applying the functors $\mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(-,U),X)$ and $\mathbb{R}\mathrm{Hom}_A(U,X)\otimes_B^{\mathbb{L}}$ – to each triangle in (\ddagger) we obtain two triangles in $\mathbf{D}(A,d)$:

$$\mathbb{R}\mathrm{Hom}_{A}(\mathbb{R}\mathrm{Hom}_{B^{op}}(V_{i+1},U),X) \to \mathbb{R}\mathrm{Hom}_{A}(\mathbb{R}\mathrm{Hom}_{B^{op}}(B_{i},U),X)$$

 $\to \mathbb{R}\mathrm{Hom}_{A}(\mathbb{R}\mathrm{Hom}_{B^{op}}(V_{i},U),X) \xrightarrow{+}$

and

$$\mathbb{R}\mathrm{Hom}_{A}(U,X)\otimes_{B}^{\mathbb{L}}V_{i+1}\to\mathbb{R}\mathrm{Hom}_{A}(U,X)\otimes_{B}^{\mathbb{L}}B_{i}$$
$$\to\mathbb{R}\mathrm{Hom}_{A}(U,X)\otimes_{B}^{\mathbb{L}}V_{i}\overset{+}{\to}.$$

In order to compare the triangles above, we need a natural morphism between their terms, whose existence is proved in Lemma 1.3. Note than that this morphism evaluated at B, i.e. $\Gamma_{X,B}$, can be identified to the natural isomorphism

$$\mathbb{R}\mathrm{Hom}_A(U,X)\otimes_B^{\mathbb{L}}B\stackrel{\cong}{\to}\mathbb{R}\mathrm{Hom}_A(U,X)\stackrel{\cong}{\to}\mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(B,U),X).$$

Consequently Lemma 1.2 implies that Γ_{X,B_i} are isomorphisms. We deduce the existence of natural isomorphisms in $\mathbf{D}(A,d)$:

$$\mathbb{R}\mathrm{Hom}_A(U,X)\otimes_B^{\mathbb{L}}V_i\cong\mathbb{R}\mathrm{Hom}_A(\mathbb{R}\mathrm{Hom}_{B^{op}}(V_i,U),X)$$
 for all $0\leq i\leq n$.

For i=0 we use the isomorphism $A \cong \mathbb{R}\mathrm{Hom}_{B^{op}}(U,U)$ in order to write ϕ_X as a composition of the natural isomorphisms:

$$\mathbb{R}\mathrm{Hom}_{A}(U,X)\otimes_{B}^{\mathbb{L}}U\overset{\Gamma_{X,U}}{\cong}\mathbb{R}\mathrm{Hom}_{A}(\mathbb{R}\mathrm{Hom}_{B^{op}}(U,U),X)$$
$$\cong \mathbb{R}\mathrm{Hom}_{A}(A,X)\cong X,$$

showing that ϕ_X is an isomorphism.

By formal arguments, since the functor $-\otimes_B^{\mathbb{L}} U$ has a fully faithful right adjoint triangle functor $\mathbb{R}\mathrm{Hom}_A(U,-)$, it follows that $\mathbb{R}\mathrm{Hom}_A(U,\mathbf{D}(A,d))\subseteq \mathrm{Ker}(-\otimes_B^{\mathbb{L}} U)^{\perp}$. Conversely, if $Y\in \mathrm{Ker}(-\otimes_B^{\mathbb{L}} U)^{\perp}$, we observe that $(\psi_Y\otimes_B^{\mathbb{L}} U)\phi_{Y\otimes_B^{\mathbb{L}} U}=1_{Y\otimes_B^{\mathbb{L}} U}$, hence $\psi_Y\otimes_B^{\mathbb{L}} U$ is an isomorphism. Therefore, if we look at the triangle

$$Z \to Y \xrightarrow{\psi_Y} \mathbb{R} \operatorname{Hom}_A(U, Y \otimes_R^{\mathbb{L}} U) \xrightarrow{+}$$

induced by ψ_Y , we obtain $Z \otimes_B^{\mathbb{L}} U = 0$, and this is possible only if Z = 0. It follows that $\mathbb{R}\mathrm{Hom}_A(U,\mathbf{D}(A,d)) = \mathrm{Ker}(-\otimes_B^{\mathbb{L}} U)^{\perp}$, hence we have the equivalence.

Finally suppose, in addition, that $\dim_{\mathrm{add}(A)} U < \infty$. By Lemma 1.1 above we have $m = \mathrm{codim}_{\mathrm{add}(U)} B < \infty$, hence there is a sequence of triangles

$$B_i \to W_i \to B_{i+1} \stackrel{+}{\to} \quad (0 \le i \le m)$$

in $\mathbf{D}(B,d)$ with $B_0 = B$, $B_{m+1} = 0$ and $W_i \in \operatorname{add}(U)$. For $Y \in \operatorname{Ker}(-\otimes_U^{\mathbb{L}} B)$ we have $Y \otimes_B^{\mathbb{L}} W_i = 0$, for $0 \leq i \leq m$, and by induction, $Y \otimes_B^{\mathbb{L}} B_i = 0$ too for $0 \leq i \leq m+1$. Therefore we have

$$Y \cong Y \otimes_B^{\mathbb{L}} B = Y \otimes_B^{\mathbb{L}} B_0 = 0,$$

and this shows that $\operatorname{Ker}(-\otimes_B^{\mathbb{L}} U) = 0$, or equivalently $\operatorname{Ker}(-\otimes_B^{\mathbb{L}} U)^{\perp} = \mathbf{D}(B,d)$.

Remark 1.5. Every functor making invertible the morphisms in $\mathbf{D}(B,d)$ with the cone in $\mathrm{Ker}(-\otimes_B^{\mathbb{L}} U)$ factors uniquely through $-\otimes_B^{\mathbb{L}} U$. Thus $-\otimes_B^{\mathbb{L}} U$ satisfies the universal property which the canonical quotient functor into $\mathbf{D}(B,d)/\mathrm{Ker}(-\otimes_B^{\mathbb{L}} U)$ is required to do, hence we obtain the equivalence of categories

$$\mathbf{D}(A,d) \xrightarrow{\sim} \mathbf{D}(B,d) / \operatorname{Ker}(-\otimes_B^{\mathbb{L}} U),$$

as an application of Bousfiled localization theory presented in [21, Chapter 9].

2. Good silting complexes

An object $U \in \mathbf{D}(A, d)$ is called *(pre)silting* provided that it satisfies (the first two of) the following conditions:

- S1. $\dim_{\mathrm{Add}(A)} U < \infty$;
- S2. $U^{(I)} \in U^{\perp_{>0}}$, for every set I;
- S3. $\operatorname{codim}_{\operatorname{Add}(U)} A < \infty$.

An object $U \in \mathbf{D}(A, d)$ is called *small (pre)silting* provided that it satisfies (the first two of) the following conditions:

- s1. $\dim_{\mathrm{add}(A)} U < \infty$;
- s2. $U \in U^{\perp_{>0}}$;
- s3. $\operatorname{codim}_{\operatorname{add}(U)} A < \infty$.

Recall that an object in $X \in \mathbf{D}(A,d)$ is called *small* (or *compact*) if $\operatorname{Hom}_{\mathbf{D}(A,d)}(X,-)$ commutes with coproducts. Small objects can be characterized by the property that they are in the smallest thick subcategory of $\mathbf{D}(A,d)$ containing A. Henceforth a small silting object is an object which is both silting and small.

A silting object is called *good* if the condition S3 above can be replaced by the strongest condition s3. An object U is called *shifted silting* if there is an integer $i \in \mathbb{Z}$ such that U[i] is silting.

For a silting object U and an $n \in \mathbb{N}$ the conditions $\operatorname{codim}_{\operatorname{Add}(U)} A \leq n$ and $\operatorname{dim}_{\operatorname{Add}(A)} U \leq n$ are equivalent, according to [27, Proposition 3.9]. Call n-silting a silting object satisfying these equivalent conditions.

Remark 2.1. Observe that we can consider the category $\mathbf{K}^b(\mathrm{Add}(A))$ as a subcategory of $\mathbf{D}(A,d)$, more precisely, $\mathbf{K}^b(\mathrm{Add}(A))$ coincides to the category of all objects U satisfying S1. More precisely if $\dim_{\mathrm{Add}(A)} U \leq n$ then $\mathrm{H}^i(U) = 0$ for $i \notin \{-n+1,\ldots,-1,0\}$. Consequently, if we assume $U \in \mathbf{D}(A,d)^{\leq 0}$ then by [27, Theorem 3.5] U is silting in the sense of the our definition if and only if it satisfies the conditions:

S'1. $U \in \mathbf{K}^b(\operatorname{Add}(A));$

S'2. $U^{(I)} \in U^{\perp_{>0}}$, for every set I;

S'3. $\langle \operatorname{Add}(U) \rangle = \mathbf{K}^b(\operatorname{Add}(A)).$

that is U is silting in the sense of [2]. We have to warn the reader that our notion of an n-silting object agrees to the notion n-semitilting complex in [27] and to those of an (n+1)-silting complex in [2]. We made this choice because it also agrees to the older one used in tilting theory. Moreover, a silting complex in the sense of [2] is what we call a shifted silting complex. Observe that the most of the results in [2] uses silting complexes in our sense.

The next lemma shows that a good silting object U is cofibrant both as an A and a B^{op} module, allowing us to compute the functors $\mathbb{R}\text{Hom}(U, -)$ and $-\otimes_B^{\mathbb{L}} U$, for a silting object $U \in \mathbf{D}(A, d)$.

Lemma 2.2. Let $U \in \mathbf{D}(A, d)$ be a good silting object with $B = \mathrm{DgEnd}_A(U)$. Then for all $X \in \mathbf{D}(A, d)$ and all $Y \in \mathbf{D}(B, d)$ we have:

$$\mathbb{R}\mathrm{Hom}_A(U,X) = \mathrm{Hom}_A^{\bullet}(U,X) \ \ and \ Y \otimes_B^{\mathbb{L}} U = Y \otimes_B^{\bullet} U.$$

Proof. By the very definition of a silting object and by Lemma 1.1, we infer that $\dim_{\operatorname{Add}(A)} U < \infty$ in $\mathbf{D}(A,d)$ and $\dim_{\operatorname{add}(B)} U < \infty$ in $\mathbf{D}(B^{op},d)$. Thus in both cases U lies in the smallest triangulated subcategory containing A, respectively B, implying that U cofibrant viewed both as a dg-A-module and a dg- B^{op} -module, hence the conclusion.

We introduce an equivalence relation on the class of silting objects, by declaring two such objects U and U' equivalent if Add(U) = Add(U'). Remark that a silting object $U \in \mathbf{D}(A,d)$ induce a t-structure in $\mathbf{D}(A,d)$ (see [23, 4.1]) and the equivalence between two silting objects can be characterized by the fact that they induce the same t-structure (see [23, Definition 4.6]).

Lemma 2.3. If U is an n-silting object then there is a good n-silting complex U', such that U and U' are equivalent.

Proof. Let U be an n-term silting complex. Since $\operatorname{codim}_{\operatorname{Add}(U)} A \leq n$ there are triangles

$$A_i \to U_i \to A_{i+1} \stackrel{+}{\to} \text{ with } 0 \le i \le n$$

such that $U_i \in \operatorname{add}(U)$, $A_0 = A$ and $A_{n+1} = 0$. By [27, Proposition 3.9], $U' = \bigoplus_{i=0}^n U_i$ is an *n*-silting object and $\operatorname{Add}(U') = \operatorname{Add}(U)$. Hence U' is equivalent to U. It is clear that U' is good since all U_i are direct summand of U'.

Recall that a dg algebra $B = \bigoplus_{i \in \mathbb{Z}} B^i$ is called *non-positive* (see [12, 2.4]) if $B^i = 0$ for all i > 0. We will call *weak non-positive* a dg algebra B for

which $\mathrm{H}^i(B)=0$ for all i>0. It is clear that a weak non-positive dg-algebra is quasi-isomorphic to its smart truncation at 0 which is a non-positive one (by [12, 2.5]), so the corresponding derived categories are equivalent (see also [12, 2.2]). Consequently, if B is a weak non-positive dg-algebra, then there is a (standard) t-structure $(\mathbf{D}(B,d)^{\leq 0},\mathbf{D}(B,d)^{\geq 0})$ in $\mathbf{D}(B,d)$ whose heart is equivalent to $\mathrm{Mod}(\mathrm{H}^0(B))$ (see [12, Proposition 2.1]).

Theorem 2.4. Consider a good silting object $U \in \mathbf{D}(A,d)$, denote $B = \mathrm{DgEnd}_A(U)$, $\mathcal{K} = \mathrm{Ker}(-\otimes_B^{\mathbb{L}} U)$, and let $E = \mathrm{Hom}_{\mathbf{D}(A,d)}(U,U)$ be the endomorphism ring of $U \in \mathbf{D}(A,d)$. Then there is an equivalence of categories

$$\mathbb{R}\mathrm{Hom}_A(U,-): \mathbf{D}(A,d) \leftrightarrows \mathcal{K}^{\perp}: -\otimes_B^{\mathbb{L}} U,$$

and the dg-algebra B is weak non-positive. Consequently, the heart of the standard t-structure on $\mathbf{D}(B,d)$ is equivalent to the category $\mathrm{Mod}(E)$. Finally, if U is a small silting object, then $\mathcal{K}^{\perp} = \mathbf{D}(B,d)$.

Proof. The equivalence follows at once from Theorem 1.4. Further we have

$$\mathrm{H}^{i}(B) \cong \mathrm{Hom}_{\mathbf{D}(B,d)}(B,B[i]) = \mathrm{Hom}_{\mathbf{D}(B,d)}(\mathbb{R}\mathrm{Hom}_{A}(U,U),\mathbb{R}\mathrm{Hom}_{A}(U,U)[i])$$

$$\cong \operatorname{Hom}_{\mathbf{D}(B,d)}(\mathbb{R}\operatorname{Hom}_A(U,U),\mathbb{R}\operatorname{Hom}_A(U,U[i]))$$

$$\cong \operatorname{Hom}_{\mathbf{D}(A,d)}(U,U[i]) = 0$$

for all i > 0. Moreover we have $E \cong H^0(B)$. The last statement also follows by (the last part of) Theorem 1.4.

Notation 2.5. Suppose A is an ordinary algebra (that is, A is a dgalgebra concentrated in degree 0), and consider an n-silting complex $U \in \mathbf{D}(A)$, whose endomorphism ring is denoted by E. By Theorem above $B = \mathrm{DgEnd}_A(U)$ is weakly non-positive and we can identify the category $\mathrm{Mod}(E)$ with the heart of the standard t-structure in $\mathbf{D}(B,d)$. For $0 \le i \le n$ we denote

$$\mathcal{X}_i = \{ X \in \operatorname{Mod}(A) \mid \operatorname{H}^j(\mathbb{R} \operatorname{Hom}_A(U, X)) = 0 \text{ for all } j \neq i \},$$
$$\mathcal{Y}_i = \{ Y \in \operatorname{Mod}(E) \mid \operatorname{H}^j(Y \otimes_B^{\mathbb{L}} U) = 0 \text{ for all } j \neq -i \}.$$

The following Corollary is the correspondent in the silting case of the Tilting Theorem, as it appears in [18, Theorem 1.16]:

Corollary 2.6. Let A be an ordinary algebra, and consider a good n-silting complex $U \in \mathbf{D}(A)$, with the endomorphism ring denoted by E. For the subcategories \mathcal{X}_i and \mathcal{Y}_i defined in Notation 2.5, there are equivalences of categories

$$\mathbb{R}\mathrm{Hom}_A(U,-)[i]:\mathcal{X}_i \leftrightarrows \mathcal{Y}_i \cap \mathcal{K}^{\perp}:(-\otimes_B^{\mathbb{L}}U)[-i],$$

for all $1 \leq i \leq n$, where $\mathcal{K} = \operatorname{Ker}(-\otimes_B^{\mathbb{L}} U)$. If, in addition, U is a small silting complex, then there are equivalences $\mathcal{X}_i \leftrightarrows \mathcal{Y}_i$, for all $1 \leq i \leq n$.

Proof. For $X \in \mathcal{X}_i$ we put $Y = \mathbb{R}\mathrm{Hom}_A(U, X[i])$. Then for all $j \neq 0$ we have

$$H^{j}(Y) = H^{j}(\mathbb{R}Hom_{A}(U, X[i])) \cong H^{j}(\mathbb{R}Hom_{A}(U, X)[i])$$

$$\cong H^{j+i}(\mathbb{R}Hom_{A}(U, X)) = 0,$$

hence $Y \in \text{Mod}(E)$.

According to Theorem 2.4 the natural map $\mathbb{R}\mathrm{Hom}_A(U,X[i])\otimes_B^{\mathbb{L}}U\to X[i]$ is a quasi-isomorphism, hence for all $j\neq -i$ we have

$$\mathrm{H}^{j}(Y \otimes_{B}^{\mathbb{L}} U) = \mathrm{H}^{j}(\mathbb{R}\mathrm{Hom}_{A}(U, X[i]) \otimes_{B}^{\mathbb{L}} U) \cong \mathrm{H}^{j}(X[i]) = \mathrm{H}^{j+i}(X) = 0,$$

showing that $Y \in \mathcal{Y}_i$. On the other hand it follows by Theorem 2.4 that $Y = \mathbb{R}\mathrm{Hom}_A(U,X[i]) \in \mathcal{K}^\perp$, thus $Y \in \mathcal{Y}_i \cap \mathcal{K}^\perp$.

For $Y \in \mathcal{Y}_i \cap \mathcal{K}^{\perp}$ we put $X = Y[-i] \otimes_B^{\mathbb{L}} U$. Then for all $j \neq 0$ we have

$$\mathrm{H}^{j}(X) = \mathrm{H}^{j}(Y[i] \otimes_{B}^{\mathbb{L}} U) \cong \mathrm{H}^{j}((Y \otimes_{B}^{\mathbb{L}} U)[i]) = \mathrm{H}^{j-i}(Y \otimes_{B}^{\mathbb{L}} U) = 0,$$

implying that $X \in \operatorname{Mod}(A)$. On the other hand since $Y \in \mathcal{K}^{\perp}$ and \mathcal{K}^{\perp} is closed under all shifts, we obtain $Y[-i] \in \mathcal{K}^{\perp}$ and Theorem 2.4 provides a quasi-isomorphism $Y[-i] \to \mathbb{R}\operatorname{Hom}_A(U, Y[-i] \otimes_B^{\mathbb{L}} U)$. Therefore $X \in \mathcal{X}_i$ since for all $j \neq i$ we have

$$H^{j}(\mathbb{R}\operatorname{Hom}_{A}(U,X)) = H^{j}(\mathbb{R}\operatorname{Hom}_{A}(U,Y[-i] \otimes_{B}^{\mathbb{L}} U)$$

$$\cong H^{j}(Y[-i]) = H^{j-i}(Y) = 0.$$

If
$$U$$
 small tilting, then $\mathcal{Y}_i \cap \mathcal{K}^{\perp} = \mathcal{Y}_i \cap \mathbf{D}(B, d) = \mathcal{Y}_i$.

In the end of this section we want to show that our results generalize those of [4]. In order to do that, let A be an ordinary k-algebra. As mentioned, A may be seen as a dg-algebra concentrated in degree 0. As usual Mod(A) is the category of ordinary (right) modules over A. We keep the notation Mod(A, d) for the category of complexes of A-modules but we denote simply $\mathbf{K}(A)$ and $\mathbf{D}(A)$ the respective homotopy and derived category.

An (n-)silting object $U \in \mathbf{D}(A)$ is called (n-)tilting if instead condition S2 it satisfies the strongest condition $U^{(I)} \in U^{\perp_{\neq 0}}$ for all sets I.

According to [27, Corollary 3.6 and Corollary 3.7] $U \in \mathbf{D}(A)$ is tilting exactly if it is isomorphic to its 0-th cohomology, that is to $T = \mathrm{H}^0(U) \in \mathrm{Mod}(A)$ (actually U is a projective resolution of T), and the A-module T satisfies the usual properties defining a tilting module, namely

- T1. T is of finite projective dimension.
- T2. $\operatorname{Ext}(T^{(I)},T)=0$ for all sets I.
- T3. There is an exact sequence $0 \to A \to T_0 \to \ldots \to T_n \to 0$ with $T_i \in Add(T)$ for all $0 \le i \le n$.

Moreover in this case $\operatorname{Hom}_{\mathbf{D}(A)}(U,U) \cong \operatorname{Hom}_{\mathbf{D}(A)}(T,T) = \operatorname{Hom}_A(T,T)$. We note that the tilting module T is good in the sense of [4] exactly if the corresponding tilting object $U \in \mathbf{D}(A)$ is good in the sense of definition given in Section 2. A tilting module in $\operatorname{Mod}(A)$ is called *classical* (or small), provided that it has a projective resolution U which is a small, good tilting object in $\mathbf{D}(A)$.

Theorem 2.7. [4, Theorem 2.2 and Corollary 2.4]. Let $T \in \text{Mod}(A)$ be a good tilting module and let $B = \text{End}_A(T)$. Then T is a B-A-bimodule and there is an isomorphism of k-algebras $A \cong \text{End}_{B^{op}}(T)$. Further the adjunction morphism $\phi : \mathbb{R}\text{Hom}_A(T,X) \otimes_A^{\mathbb{L}} U \to X$ is an isomorphism for all $X \in \mathbf{D}(A)$, hence the functor

$$\mathbb{R}\mathrm{Hom}_A(T,-):\mathbf{D}(A)\to\mathbf{D}(B)$$

is fully faithful. Consequently it induces equivalences of categories

$$\mathbf{D}(A) \xrightarrow{\sim} \mathbf{D}(B)/\mathcal{K} \xrightarrow{\sim} \mathcal{K}^{\perp},$$

where $K = \text{Ker}(-\otimes_B^{\mathbb{L}} U)$. Moreover $K^{\perp} = \mathbf{D}(B)$, provided that T is classical tilting.

Proof. In order to apply above results, denote by U a projective resolution of $T = H^0(U)$, that is $U \in \mathbf{D}(A)$ is a good tilting object. Then

$$\mathrm{H}^n(\mathrm{DgEnd}_A(U)) = \mathrm{H}^n(\mathbb{R}\mathrm{Hom}_A(U,U)) \cong \mathrm{Hom}_{\mathbf{D}(A)}(U,U[n]) = 0$$

for $n \neq 0$, hence DgEnd_A(U) is concentrated in degree 0. Since

$$\mathrm{H}^{0}(\mathrm{DgEnd}_{A}(U)) = \mathrm{Hom}_{\mathbf{D}(A)}(U, U) \cong \mathrm{Hom}_{A}(T, T) = B,$$

the quasi-isomorphism $A \to \mathbb{R}\mathrm{Hom}_{B^{op}}(U,U)$ of Theorem 1.4 becomes an isomorphism of k-algebras $A \to \mathrm{End}_{B^{op}}(T)$. Finally

$$Mod(DgEnd_A(U), d) = Mod(B, d)$$

is the category of differential complexes of B-modules,

$$\mathbf{K}(\mathrm{DgEnd}_A(U), d) = \mathbf{K}(B) \text{ and } \mathbf{D}(\mathrm{DgEnd}_A(U), d) = \mathbf{D}(B)$$

and the conclusion follows by Theorem 2.4.

Corollary 2.8. [4, Corollary 2.5], [18, Theorem 1.16] With the assumptions and notations made in Theorem 2.7, we have the equivalences of categories

$$\operatorname{Ext}_A^i(T,-): \mathcal{X}_i \leftrightarrows \mathcal{Y}_i \cap \mathcal{K}^{\perp}: \operatorname{Tor}_i^B(Y,-),$$

where $\mathcal{X}_i = \{X \in \operatorname{Mod}(A) \mid \operatorname{Ext}_A^j(T, X) = 0 \text{ for all } j \geq 0, j \neq i\}$ and $\mathcal{Y}_i = \{Y \in \operatorname{Mod}(B) \mid \operatorname{Tor}_j^B(T, Y) = 0 \text{ for all } j \geq 0, j \neq i\}.$

Proof. In order to apply Corollary 2.6 we note that in our case $B = \operatorname{End}_A(T)$ is isomorphic to the endomorphism ring of the projective resolution U of T in the category $\mathbf{D}(A)$. Moreover by definitions of \mathcal{X}_i and \mathcal{Y}_i we deduce that $\mathbb{R}\operatorname{Hom}_A(T,X[i])$ and $Y[-i] \otimes_B^{\mathbb{L}} T$ are both concentrated in degree 0, for all $X \in \mathcal{X}_i$ and all $Y \in \mathcal{Y}_i$.

Therefore we have

$$\mathbb{R}\mathrm{Hom}_A(T,X[i]) \cong \mathrm{H}^i(\mathbb{R}\mathrm{Hom}_A(T,X)) \cong \mathrm{Ext}_A^i(T,X)$$

and

$$Y[-i] \otimes_B^{\mathbb{L}} T \cong H^{-i}(Y \otimes_B^{\mathbb{L}} T) \cong Tor_i^B(T, Y),$$

so the proof is complete.

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