Invariant character varieties of hyperbolic knots with symmetries

Luisa Paoluzzi* and Joan Porti†

October 10, 2017

Abstract

We study character varieties of symmetric knots and their reductions mod p. We observe that the varieties present a different behaviour according to whether the knots admit a free or periodic symmetry.

AMS classification: Primary 57M25; Secondary 20C99; 57M50.

Keywords: Character varieties, hyperbolic knots, symmetries.

1 Introduction

Character varieties of 3-manifold groups provide a useful tool in understanding the geometric structures of manifolds and notably the presence of essential surfaces. In this paper we wish to investigate SL_2-character varieties of symmetric hyperbolic knots in order to pinpoint specific behaviours related to the presence of free or periodic symmetries. We will be mostly concerned with symmetries of odd prime order and we will concentrate our attention to a subvariety of the character variety, i.e. the *invariant subvariety* in the sense of algebraic geometry, which is pointwise fixed by the action of the symmetry (see Section 4 for a precise definition of this action and of the invariant subvariety).

As already observed in [9], the excellent component of the character variety containing the character of the holonomy representation is fixed pointwise by the symmetry, since the symmetry can be chosen to act as a hyperbolic isometry of the complement of the knot. Hilden, Lozano, and Montesinos also observed that the invariant subvariety of a hyperbolic symmetric (more specifically, periodic) knot can be sometimes easier to determine than the whole variety. This follows from the fact that the invariant subvariety can be computed using the character variety of a two-component hyperbolic link. Such link is obtained as the quotient of the knot and the axis of its periodic symmetry by the action of the symmetry itself. Indeed, the link is sometimes much "simpler" than the original knot, in the sense that its fundamental group has a smaller number of generators and relations, making the computation of its character variety feasible. This is, for instance, the case when the quotient link is a 2-bridge link: Hilden, Lozano, and Montesinos studied precisely this situation and were able to recover

*Partially supported by ANR project 12-BS01-0003-01
†Partially supported by the Spanish Mineco through grant MTM2015-66165-P

a defining equation for the excellent components of several periodic knots up to ten crossings.

In what follows we will be interested in the structure of the invariant subvariety itself and we will consider not only knots admitting periodic symmetries but also free symmetries. Our main result shows that the invariant subvariety has in general a different structure according to whether the knot admits a free or periodic symmetry.

Theorem 1. *If K has a periodic symmetry of prime order $p \geq 3$, then $X(K)$ contains at least $(p-1)/2$ components that are curves and that are pointwise fixed by the symmetry.*

On the other hand, for each prime $p \geq 3$, there is a knot K_p with a free symmetry of order p such that the number of components of the invariant character variety of K_p is bounded, independently of p.

The main observation here is that the invariant subvariety for a hyperbolic symmetric knot, or more precisely the Zariski-open set of its irreducible characters, can be seen as a subvariety of the character variety of a well-chosen two-component hyperbolic link, even when the symmetry is free.

To make the second part of our result more concrete, in Section 7 we study an infinite family of examples all arising from the two-component 2-bridge link 6^2_2 in Rolfsen's notation (with 2-bridge invariant $10/3$). Our construction provides infinitely many knots with free symmetries such that the number of irreducible components of the invariant subvarieties of the knots is universally bounded.

As for the rest of the character variety, that is the components that are either invariant but not pointwise fixed, or equivariant, it is, in general, much harder to determine. In Section 8 we study a family of knots having either free or periodic symmetry (with both situations occurring) whose character varieties contain equivariant components. This suggests that one cannot distinguish the nature of the symmetry of the knot by looking at this part of the variety.

The invariant subvarieties of periodic knots over fields of positive characteristic exhibit another peculiar behaviour. It is well-known that for almost all odd primes p the character variety of a finitely presented group resembles the character variety over $\mathbb{C}$. For a finite set of primes, though, the character variety over p may differ from the one over $\mathbb{C}$, in the sense that there may be "jumps" either in the dimension of its irreducible components or in their number. In this case we say that *the variety ramifies at p*. The character varieties of the knots studied in [15] provide the first examples in which the dimension of a well-defined subvariety of the character variety is larger for certain primes. Here we give an infinite family of periodic knots for which the invariant character variety ramifies at p, where p is the order of the period. In this case, the ramification means that the number of 1-dimensional components of the invariant subvariety decreases in characteristic p. This gives some more insight in the relationship between the geometry of a knot and the algebra of its character variety, namely the primes that ramify.

The paper is organised as follows: Section 2 is purely topological and describes how one can construct any symmetric knot starting from a well-chosen two-component link. Section 3 provides basic facts on character varieties and establishes the setting in which we will work. In Section 4 we introduce and study invariant character varieties of symmetric knots. The first part of Theorem 1 on periodic knots is proved in Section 5 while in Section 6 we study properties of invariant character varieties of knots with free symmetries. The proof of Theorem 1 is achieved in Section 7, where an infinite family of free periodic knots with the desired properties is constructed. In Section 8 we discuss the non invariant part of the variety for a family of Montesinos knots: these have either a free or a periodic symmetry and their character varieties contain equivariant components. Finally, in Section 9 we describe how the character varieties of knots with period p may ramify $\mod p$.

2 Symmetric knots and two-component links

Let K be a knot in $\mathbf{S}^3$ and let $\psi : (\mathbf{S}^3, K) \longrightarrow (\mathbf{S}^3, K)$ be a finite order diffeomorphism of the pair which preserves the orientation of $\mathbf{S}^3$.

Definition 1. If the group $\langle\psi\rangle$ acts freely we say that ψ is a *free symmetry of* K. If ψ has a global fixed point then, according to the positive solution to Smith's conjecture [14], the fixed-point set of ψ is an unknotted circle and two situations can arise: either the fixed-point set of ψ is disjoint from K, and we say that ψ is a *periodic symmetry of* K, or it is not. In the latter case ψ has order 2, its fixed-point set meets K in two points, and ψ is called a *strong inversion of* K. In all other cases ψ is called a *semi-periodic symmetry of* K.

Remark 1. Note that if the order of ψ is an odd prime, then ψ can only be a *free* or *periodic symmetry* of K.

We start by recalling some well-known facts and a construction that will be central in the paper.

Let $L = A \sqcup K_0$ be a hyperbolic two-component link in the 3-sphere such that A is the trivial knot. Let $n \geq 2$ be an integer and assume that n and the linking number of A and K_0 are coprime. We can consider the n-fold cyclic cover $V = \mathbf{D}^2 \times \mathbf{S}^1 \longrightarrow E(A)$ of the solid torus $E(A)$ which is the exterior of A and contains K_0. The lift of K_0 in V is a (connected) simple closed curve C.

Let μ, λ be a meridian-longitude system for A on $\partial E(A)$ and let $\tilde{\mu}, \tilde{\lambda}$ be its lift on ∂V. The slopes $\gamma_k = \tilde{\mu} + k\tilde{\lambda}$, for $k = 0, \ldots, n-1$, on ∂V are equivariant by the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of deck transformations of the covering $V \longrightarrow E(A)$. More precisely, since the group acts by translation in the direction $\tilde{\mu}$, γ_k is invariant if $k = 0$ and has an orbit consisting of $d = n/(n \wedge k)$ disjoint images otherwise. Note that the manifold V_k obtained by Dehn filling V along γ_k is $\mathbf{S}^3$. The action of the group of deck transformations $\mathbb{Z}/n\mathbb{Z}$ on V extends to an action on V_k which is free if $k \neq 0$ is prime with n and has a circle of fixed points if $k = 0$. For all other values of k, the action is semi-periodic, that is a proper subgroup of $\mathbb{Z}/n\mathbb{Z}$ of order n/d acts with a circle of fixed points.

For a fixed k, the image of C in V_k is a knot that we will denote by K admitting a periodic or free symmetry of order n according to whether $k = 0$ or prime with n. For n large enough, the resulting knot K is hyperbolic because of Thurston's hyperbolic Dehn surgery theorem [19], e.g. [1, App. B].

Remark 2. Of course, the above construction can be carried out for arbitrary integer values of k. However, it is not restrictive to require the value of k to be ≥ 0 and $< n$. Indeed, assume that $k' = k + an$ where $0 \leq k < n$. The knot K' resulting from $1/k'$ surgery along V coincides with the knot K obtained in the same manner but starting from a different link L' and choosing $\gamma_k = \tilde{\mu} + k\tilde{\lambda}$ as

Dehn filling slope. The link L' is obtained from L by Dehn surgery of slope $1/a$ along A.

The following proposition shows that periodic and free-symmetric knots can always be obtained this way.

Proposition 2. *Let K be a hyperbolic knot admitting a free or periodic symmetry of order n. Then there exists a two-component hyperbolic link $L = A \sqcup K_0 \subset \mathbf{S}^3$ with A the trivial knot, and an integer $0 \leq k < n$ such that the knot K can be obtained by the above construction.*

Proof. The statement is obvious if the symmetry is periodic: in this case the link L consists of the image A of the axis of the symmetry and the image K_0 of the knot K in the quotient of $\mathbf{S}^3$ by the action of the symmetry. Hyperbolicity of the link is a straightforward consequence of the hyperbolicity of K and the orbifold theorem.

If the symmetry is free, some extra work is necessary. The quotient of $\mathbf{S}^3$ by the action of the free symmetry is a lens space containing a hyperbolic knot K_0', image of K.

Consider the cores of the two solid tori of a genus-1 Heegaard splitting for the lens space induced by an invariant genus-1 splitting of $\mathbf{S}^3$. Up to small isotopy one can assume that K_0' misses one of them, say α. Note that the free homotopy class of α is non-trivial both in the lens space and in the complement of K_0'. Observe, moreover, that the exterior of α is a solid torus.

Let $\tilde{\alpha} \subset \mathbf{S}^3 - K$ denote the lift of α. If $K \sqcup \tilde{\alpha}$ is a hyperbolic link, then we are done by choosing $L = K_0 \sqcup A$ to be any link in $\mathbf{S}^3$ such that the exterior of A coincides with the exterior of α and $(E(A), K_0) = (E(\alpha), K_0')$. In other words, K_0 is the image of K_0' in a chosen Dehn surgery along α resulting in $\mathbf{S}^3$, and A is the core of the surgery.

If $K \sqcup \tilde{\alpha}$ is not hyperbolic we will modify the choice of $\tilde{\alpha}$. For basic terminology in 3-dimensional topology the reader is referred to [8] where the JSJ-decomposition is also discussed (for the latter see also [10, 11])

First of all, note that the link $K \sqcup \tilde{\alpha}$ is not split. This is a consequence of the equivariant sphere theorem and the fact that $\tilde{\alpha}$ is invariant, hence $E(K \sqcup \tilde{\alpha})$ is irreducible and boundary irreducible. Indeed, note that a compressing disc for the link would have a regular neighbourhood whose boundary would be a splitting sphere. In addition $E(K \sqcup \tilde{\alpha})$ is not Seifert fibred, because a Dehn filling on $\tilde{\alpha}$ yields $E(K)$, which is hyperbolic. Thus the only obstruction to hyperbolicity is that $E(K \sqcup \tilde{\alpha})$ could be toroidal. Observe that if $E(K \sqcup \tilde{\alpha})$ is annular, for instance if K and $\tilde{\alpha}$ are parallel, then either $E(K \sqcup \tilde{\alpha})$ is toroidal or Seifert fibred.

Assume that its JSJ-decomposition is nontrivial and let M be the piece of the splitting that is closest to K. Note that the JSJ-decomposition can be chosen to be invariant by the action of the symmetry, so that, in particular, M is invariant. The boundary of M consists of $T_0^2 = \partial\mathcal{N}(K)$, some tori $T_1^2, \ldots, T_k^2$, $k \geq 0$ that do not separate $\tilde{\alpha}$ from K, and possibly a torus T_{k+1}^2 that separates M from $\tilde{\alpha}$. We shall modify $\tilde{\alpha}$ so that $k = 0$ and $T_{k+1}^2 = \partial\mathcal{N}(\tilde{\alpha})$, which will yield hyperbolicity.

By hyperbolicity of K, for $i \geq 1$, each T_i^2 either bounds a solid torus in $E(K)$ or it is contained in a ball in $E(K)$. Notice that T_{k+1}^2 must bound a solid torus in $E(K)$, because $\tilde{\alpha}$ is not contained in a ball else the link $K \sqcup \tilde{\alpha}$ would be split. In addition, none of the $T_1^2, \ldots, T_k^2$ can bound a solid torus in $E(K)$, by nontriviality of the JSJ-decomposition. Note that, for each $1 \le i \le k$, $\tilde{\alpha}$ must pass through the ball in $E(K)$ that contains T_i^2.

We start by getting rid of the tori $T_1^2, \ldots, T_k^2$. Let $B_i^3 \subset E(K)$ denote the 3-ball containing T_i^2, for $i = 1, \ldots, k$. Note that T_i^2 cuts out a knot exterior inside B_i^3 whose complement in the ball is a solid cylinder contained in M. As a consequence, in each ball there is a proper arc $\beta_i \subset B_i^3$ such that $N_i = B_i^3 \backslash \mathcal{N}(\beta_i)$ is a knot exterior with boundary (parallel to) T_i^2. Moreover $N(\beta_i)$ contains the intersection of $\tilde{\alpha}$ with B_i^3; we stress again that B_i^3 does not meet K. This is schematically illustrated in Figure 1. We now replace equivariantly each N_i with a solid torus, that is the exterior of a trivial knot. Observe that on each T_i^2 there is a well-defined longitude-meridian system of curves corresponding to the longitude-meridian system of the knot exterior N_i. Replacing N_i with a solid torus corresponds to performing a Dehn filling of M along T_i^2 with slope the longitude. The N_i that are permuted by the symmetry are exteriors of the same knot and the symmetry must preserve longitudes. This ensures that the filling on the T_i^2s can be carried out in an equivariant way. The effect of the surgery on a T_i^2 is shown again in Figure 1. Note that by construction there is a degree-one map from N_i to the added solid torus which is the identity on the boundary T_i^2: the surgery is obtained by pinching a Seifert surface for N_i to a disc. Note that the only effect of this surgery is to modify $\tilde{\alpha}$ inside the B_i^3, in particular the resulting manifold is again $\mathbf{S}^3$. Moreover the surgery did not change K, because it just consisted in replacing the balls B_i^3 that are disjoint from K again with balls. Denote by $\tilde{\alpha}'$ the image of $\tilde{\alpha}$ after the surgery. Obviously $\tilde{\alpha}'$ is still invariant by construction. We want to show that $\tilde{\alpha}'$ is a unknotted. For this observe that there is a degree-one map from $(\mathbf{S}^3, \tilde{\alpha})$ to $(\mathbf{S}^3, \tilde{\alpha}')$ which is the identity on the complement of the union of the N_i and coincides with the degree-one maps defined above on each N_i. This map clearly induces a degree-one map from the exterior of $\tilde{\alpha}$ to that of $\tilde{\alpha}'$. The conclusion follows for $\tilde{\alpha}$ is the trivial knot and its exterior cannot have a degree-one map on the exterior of a non-trivial knot.

Note that since the surgeries we performed are "small", we cannot be sure that the resulting link $K \sqcup \tilde{\alpha}'$ is hyperbolic yet. If $K \sqcup \tilde{\alpha}'$ admits again essential tori that do not separate K from $\tilde{\alpha}'$, we repeat the procedure just seen. To be able to conclude we need to make sure that we will find an atoroidal link in a finite number of steps. This follows from the fact that at each step the simplicial volume of the link strictly decreases and from the structure of the set of hyperbolic volumes. One can also use the fact that at each step we have degree-one maps from the old link to the new one and use the fact that there are no infinite chains of such maps by [18].

By the above argument we can thus assume that for $K \sqcup \tilde{\alpha}$ $k = 0$. We will modify $\tilde{\alpha}$ so that $\partial T_{k+1}^2 = \partial \mathcal{N}(\tilde{\alpha})$. Let V be the solid torus bounded by T_{k+1}^2. Then $\tilde{\alpha} \subset V$ and V must be equivariant. In addition V is not knotted, because $\tilde{\alpha}$ is the trivial knot but also a satellite with companion $\mathbf{S}^3 \setminus V$. Then the modification consists in replacing $\tilde{\alpha}$ by the core of V. This makes ∂T_{k+1}^2 boundary parallel, and hence inessential.

We can now quotient the resulting hyperbolic link by the symmetry to obtain a new $K_0' \sqcup \alpha'$ in a lens space. The link L is then obtained as in the previous case. $\square$

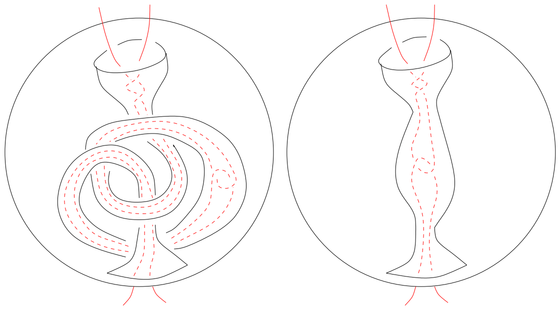

Figure 1: A schematic picture of one of the balls B_i^3 with $\tilde{\alpha}$ passing through it before the surgery (on the left), and the effect of surgery (on the right).

Note that for a given K the choice of L is not unique. Indeed, links are not determined by their complements, and there are infinitely many slopes on the boundary of a solid torus such that performing Dehn filling along them gives the 3-sphere (see also Remark 2).

Remark 3. Note that if K admits a semi-periodic symmetry, then either all powers of the symmetry that act as periods have the same fixed-point set or the union of their fixed-point sets consists of two circles forming a Hopf link. In the first situation a hyperbolic link L can be constructed as in the case of periodic knots. In the second situation, one can construct L by choosing one of the two components of the Hopf link, but L will not be hyperbolic in general. Since we only consider symmetries of odd prime order in the following, we are not going to analyse this situation further.

3 Character varieties

Let G be a finitely presented group. Given a representation $\rho : G \longrightarrow \mathrm{SL}_2(\mathbb{C})$, its character is the map $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma))$, $\forall \gamma \in G$. The set of all characters is denoted by $\mathbf{X}(G)$.

Given an element $\gamma \in G$, we define the map

$$\begin{array}{rcl} \tau_\gamma : \mathbf{X}(G) & \to & \mathbb{C} \\ \chi & \mapsto & \chi(\gamma) \end{array}.$$

Proposition 3 ([5, 6])**.** *The set of characters $\mathbf{X}(G)$ is an affine algebraic set defined over $\mathbb{Z}$, which embeds in $\mathbb{C}^N$ with coordinate functions $(\tau_{\gamma_1}, \ldots, \tau_{\gamma_N})$ for some $\gamma_1, \ldots, \gamma_N \in G$.*

The affine algebraic set $\mathbf{X}(G)$ is called the *character variety* of G: it can be interpreted as the algebraic quotient of the variety of representations of G by the conjugacy action of $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\mathcal{Z}(\mathrm{SL}_2(\mathbb{C}))$.

Note that the set $\{\gamma_1, \ldots, \gamma_N\}$ in the above proposition can be chosen to contain a generating set of G. For G the fundamental group of a knot exterior, we will then assume that it always contains a representative of the meridian.

A careful analysis of the arguments in [6] shows that Proposition 3 still holds if $\mathbb{C}$ is replaced by any algebraically closed field, provided that its characteristic is different from 2. Let $\mathbb{F}_p$ denote the field with p elements and $\bar{\mathbb{F}}_p$ its algebraic closure. We have:

Proposition 4 ([6]). *Let $p > 2$ be an odd prime number. The set of characters $\mathbf{X}(G)_{\bar{\mathbb{F}}_p}$ associated to representations of G over the field $\bar{\mathbb{F}}_p$ is an algebraic set which embed in $\bar{\mathbb{F}}_p^N$ with the same coordinate functions $(\tau_{\gamma_1}, \ldots, \tau_{\gamma_N})$ seen in Proposition 3. Moreover, $\mathbf{X}(G)_{\bar{\mathbb{F}}_p}$ is defined by the reductions mod p of the polynomials over $\mathbb{Z}$ which define $\mathbf{X}(G)_{\mathbb{C}}$.*

One of the steps of the proof of Proposition 3 in [6] is the fact that, for any $g \in G$, $\tau_g \in \mathbb{Z}[\tau_{\gamma_1}, \ldots, \tau_{\gamma_N}]$. This yields:

Remark 4. For any group homomorphism $\phi\colon G \to H$, the induced map $\phi^*\colon \mathbf{X}(H)_{\mathbb{C}} \to \mathbf{X}(G)_{\mathbb{C}}$ is polynomial with coefficients in $\mathbb{Z}$, because its coordinates are obtained by writing $\tau_{\phi(\gamma_1)}, \ldots, \tau_{\phi(\gamma_N)}$ as polynomials in the traces of the elements in H provided by Proposition 3.

In addition, the reduction mod p of $\phi^*\colon \mathbf{X}(H)_{\mathbb{C}} \to \mathbf{X}(G)_{\mathbb{C}}$ is the induced map $\mathbf{X}(H)_{\bar{\mathbb{F}}_p} \to \mathbf{X}(G)_{\bar{\mathbb{F}}_p}$.

Let $\mathbb{K}$ be an algebraically closed field of characteristic different from 2. A representation ρ of G in $\mathrm{SL}_2(\mathbb{K})$ is called *reducible* if there is a 1-dimensional subspace of $\mathbb{K}^2$ that is $\rho(G)$-invariant; otherwise ρ is called *irreducible*. The character of a representation ρ is called *reducible* (respectively *irreducible*) if so is ρ.

The set of reducible characters coincides with the set of characters of abelian representations. Such set is Zariski closed and moreover is a union of irreducible components of $\mathbf{X}(G)$ that we will denote $\mathbf{X}_{ab}(G)$ [5].

Assume now that G is the fundamental group of a link in the 3-sphere with r components. In this case, $\mathbf{X}_{ab}(G)$ is an r-dimensional variety that coincides with the character variety of $\mathbb{Z}^r$, i.e. the homology of the link. In the case where $r = 1$, that is the link is a knot, $\mathbf{X}_{ab}(G)$ is a line parametrised by the trace of the meridian. When $r = 2$, that is the link has two components, $\mathbf{X}_{ab}(G)$ is parametrised by the traces x, y of the two meridians and that, z, of their product subject to the equation $x^2 + y^2 + z^2 - xyz - 4 = 0$.

The subvariety of abelian characters is well-understood for the groups that we will be considering. Hence, in the rest of the paper, we will only consider the irreducible components of $\mathbf{X}(G)$ that are not contained in the subvariety of abelian characters.

Notation 1. We will denote by $X(G)$ the Zariski closed set which is the union of of the irreducible components of $\mathbf{X}(G)$ that are not contained in the subvariety of abelian characters. If G is the fundamental group of a manifold or orbifold M we will write for short $X(M)$ instead of $X(G)$. Similarly if G is the fundamental group of the exterior of a link L we shall write $X(L)$ instead of $X(G)$. Notice that if G is the fundamental group of a finite volume hyperbolic manifold then $X(G)$ is non empty for it contains the character of the hyperbolic holonomy.

Assume now that f is in $Aut(G)$. The automorphism f induces an action on both $\mathbf{X}(G)$ and $X(G)$ defined by $\chi \mapsto \chi \circ f$. This action only depends on the class of f in $Out(G)$ since traces are invariant by conjugacy. According to Remark 4 applied to $G = H$ and $\phi = f$, we see that this action on the character varieties is realised by an algebraic morphism defined over $\mathbb{Z}$. It follows readily that the set of fixed points of the action is Zariski closed and itself defined over $\mathbb{Z}$. As a consequence, the defining relations of the variety of characters that are fixed by the action considered over a field of characteristic p, an odd prime number, are just the reduction $\mod p$ of the given equations with integral coefficients.

4 The character variety of L and the invariant subvariety of K

In this section we define and study the invariant subvariety of K, where K is a hyperbolic knot admitting a free or periodic symmetry of order an odd prime p.

Let ψ denote the symmetry of K of order p and let $L = K_0 \sqcup A$ be the associated link as defined Section 2. Denote by $E(K)/\psi$ the space of orbits of the action of ψ on the exterior $E(K)$ of the knot K. The space $E(K)/\psi$ is either a manifold or an orbifold according to whether ψ is free or periodic. Recall that $E(K)/\psi$ is obtained by a (possibly orbifold) Dehn filling on the component A of the link L. We have

$$1 \longrightarrow \pi_1(K) \longrightarrow \pi_1(E(K)/\psi) \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 1$$

where $\pi_1(E(K)/\psi)$ denotes the orbifold fundamental group if ψ is periodic. We have that the sequence splits if and only if ψ is periodic. Note that if ψ is free then the quotient group $\mathbb{Z}/p\mathbb{Z}$ can also be seen as the fundamental group of the lens space quotient. In any case, we see that ψ defines an element ψ_* of the outer automorphism group of $\pi_1(K)$. Remark now that, since $E(K)/\psi$ is obtained by Dehn filling a component of L, the exterior $E(L)$ of the link L is naturally embedded into $E(K)/\psi$. Let μ be an element of $\pi_1(E(K)/\psi)$ corresponding to the image of a meridian of A via this natural inclusion: it maps to a generator of $\mathbb{Z}/p\mathbb{Z}$. Let $f \in Aut(\pi_1(K))$ be the automorphism of $\pi_1(K)$ induced by conjugacy by μ. Note that f is a representative of ψ_*. Thus the symmetry ψ induces an action on the character variety $X(K)$ of the exterior of K as defined in the previous section.

We have seen that the fixed-point set of this action is an algebraic subvariety of $X(K)$. We will denote by $X(K)^\psi$ the union of its irreducible components that are not contained in $\mathbf{X}_{ab}(K)$. Note that $X(K)^\psi$ is non empty for the character of the holonomy is fixed by the action. Remark also that each irreducible component of $X(K)^\psi$ contains at least one irreducible character by definition. Indeed, each irreducible component of $X(K)^\psi$ contains a whole Zariski-open set of irreducible characters. We shall call $X(K)^\psi$ *the invariant subvariety of* K.

Let us now consider how the different character varieties of K and L are related.

It is straightforward to see that the character variety $X(E(K)/\psi)$ of the quotient of the exterior $E(K)$ of K by the action of the symmetry injects into the character variety $X(L)$ of the exterior of L. Indeed the (orbifold) fundamental group of $E(K)/\psi$ is a quotient of the fundamental group of L, induced by the Dehn filling along the A component of L.

On the other hand, there is a natural map from $X(E(K)/\psi)$ to the invariant submanifold $X(K)^\psi$ of K, induced by restriction in the short exact sequence above.

Assume now that χ is a character in $X(K)^\psi$ associated to an irreducible representation ρ of K. We will show that ρ extends in a unique way to a (necessarily irreducible) representation of $E(K)/\psi$ giving a character in $X(E(K)/\psi)$ (observe that here we only use that p is odd). This proves that the above natural map is one-to-one and onto when restricted to the Zariski-open set of irreducible characters.

Note that if ρ is a representation of $\pi_1(K)$ that extends to a representation of $\pi_1(E(K)/\psi)$ then, necessarily, its character must be fixed by the symmetry ψ, for the action of μ on $\pi_1(K)$ is by conjugacy and cannot change the character of a representation.

The idea is to extend ρ to $\pi_1(E(K)/\psi)$ by defining $\rho(\mu)$ in such a way that the action of μ by conjugacy on the normal subgroup $\pi_1(K)$ coincides with the action of the automorphism f. We know that $\chi = [\rho] = [\rho \circ f]$. Since ρ is irreducible, $\mathrm{SL}_2(\mathbb{C})$ acts transitively on the fibre of χ so that there exists an element $M \in \mathrm{SL}_2(\mathbb{C})$ such that $\rho \circ f = M\rho M^{-1}$ [13, Theorem 1.28]. The element M is well-defined, up to multiplication times ± 1, i.e. up to an element in the centre of $\mathrm{SL}_2(\mathbb{C})$. The fact that ψ has odd order implies that there is a unique way to choose the sign and so that $\rho(\mu) = M$ is well-defined. Note that in some instances $\rho(\mu)$ can be the identity.

We have thus proved the following fact.

Proposition 5. *Let K be a hyperbolic knot admitting a symmetry ψ of prime odd order. The restriction map from the character variety of $E(K)/\psi$ to the ψ-invariant subvariety of K induces a bijection between the Zariski-open sets consisting of their irreducible characters.*

Remark 5. Proposition 5 holds more generally for hyperbolic knots admitting either a free or a periodic symmetry of odd order and for character varieties over fields of positive odd characteristic.

5 Knots with periodic symmetries

Let K be a hyperbolic knot admitting a periodic symmetry ψ of odd prime order p. Let $L = A \sqcup K_0$ be the associated quotient link. Denote by t_μ the coordinate of the variety $X(L)$ corresponding to the trace of μ. Proposition 5 implies at once that $X(K)^\psi$ is birationally equivalent to a subvariety $Z \cup Z_0$, where $Z \subset X(L) \cap (\cup_{\ell=1}^{p-1}\{t_\mu = 2\cos(2\ell\pi/p)\})$ and $Z_0 \subset X(L) \cap \{t_\mu = 2\}$.

Note that since p is odd, the set $\{2\cos(2\ell\pi/p) \mid \ell = 1, \dots, p-1\}$ equals $\{-2\cos(\ell\pi/p) \mid \ell = 1, \dots, (p-1)/2\}$. In particular this includes a lift to $\mathrm{SL}_2(\mathbb{C})$ of the holonomy of $E(K)/\psi$, when $t_\mu = -2\cos(\pi/p)$; observe that this means that the image of the meridian is conjugate to

$$-\begin{pmatrix} e^{i\frac{\pi}{p}} & 0 \\ 0 & e^{-i\frac{\pi}{p}} \end{pmatrix},$$

a rotation of angle $\frac{2\pi}{p}$ that has order p in $\mathrm{SL}_2(\mathbb{C})$.

Proposition 6. *The variety Z contains at least $(p-1)/2$ irreducible curves $Z_1, \dots, Z_{(p-1)/2}$, each of which contains at least one irreducible character. As a consequence, all these components are birationally equivalent to a subvariety $\tilde{Z}_1, \dots, \tilde{Z}_{(p-1)/2}$ of $X(K)^\psi$.*

Furthermore, the curves $\tilde{Z}_1, \ldots, \tilde{Z}_{(p-1)/2}$ are irreducible components of the whole $X(K)$, not only the invariant part.

Proof. First of all, remark that the intersection of $X(L)$ with the hyperplane $\{t_\mu = -2\cos(\pi/p)\}$ contains the holonomy character χ_1 of the hyperbolic orbifold structure of $E(K)/\psi$. In particular, a component of $X(E(K)/\psi) \cap \{t_\mu = -2\cos(\pi/p)\}$ is an irreducible curve Z_1 containing χ_1, the so called excellent or distinguished component of $E(K)/\psi$. This is the curve that, viewed as a deformation space, allows to prove Thurston's hyperbolic Dehn filling theorem [19], e.g. [1, App. B].

The character χ_1 takes values in a number field $\mathbf{K}$ containing the subfield $\mathbb{Q}(\cos\frac{\pi}{p})$ of degree $\frac{p-1}{2}$. The Galois conjugates of χ_1 are contained in $X(L) \cap \{t_\mu = -2\cos(\ell\pi/p)\})$ for some $\ell = 1, \ldots, (p-1)/2$. As $\{-2\cos(\ell\pi/p) \mid \ell = 1, \ldots, (p-1)/2\}$ is precisely the set of Galois conjugates of $t_\mu(\chi_1)$, this yields the $\frac{p-1}{2}$ components defined by $t_\mu = -2\cos(\ell\pi/p)$, $\ell = 1, \ldots, \frac{p-1}{2}$ (though the number of conjugates may be larger, depending on the degree of the number field $\mathbf{K}$).

To prove the assertion that these curves are irreducible components of $X(K)$, notice that the restriction $\chi_1|_{E(K)}$ is the holonomy of the hyperbolic structure of $E(K)$. Therefore, by Calabi-Weil rigidity, the Zariski tangent space of $\tilde{Z}_1$ at $\chi_1|_{E(K)}$ is one dimensional. This space is isomorphic to the first cohomology group of $\pi_1(E(K))$ with coefficients in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ twisted by the adjoint of the holonomy, cf. [13, 20]. We aim to show that this cohomology group does not change under Galois conjugation of the representation. This cohomology group can be computed as Z^1/B^1, where Z^1 is the space of crossed morphisms, or maps $f\colon \pi_1(E(K)) \to \mathfrak{sl}(2,\mathbb{C})$ satisfying $f(\gamma_1\gamma_2) = f(\gamma_1) + \mathrm{Ad}_{\mathrm{hol}(\gamma_1)} f(\gamma_2)$, $\forall \gamma_1, \gamma_2 \in \pi_1(E(K))$, and B^1 denotes the subspace of inner crossed morphisms, or maps $f_a(\gamma) = a - \mathrm{Ad}_{\mathrm{hol}(\gamma)}(a)$, $\forall \gamma \in \pi_1(E(K))$ and for some $a \in \mathfrak{sl}(2,\mathbb{C})$, cf. [3, §II.5]. Let us check that the dimension of Z^1 and B^1 does not change under Galois conjugation. Firstly, a crossed morphism is determined by the image of a generating set for $\pi_1(E(K))$ of cardinality k, thus we view $Z^1 \subset \mathfrak{sl}(2,\mathbb{C})^k$. In addition, k elements in $\mathfrak{sl}(2,\mathbb{C})$ determine a crossed morphism iff they satisfy linear relations given by a presentation of $\pi_1(E(K))$. More precisely, Z^1 is isomorphic to the kernel of a linear map $\mathfrak{sl}(2,\mathbb{C})^k \to \mathfrak{sl}(2,\mathbb{C})^r$, where r is the number of relaions, with coefficients that are $\mathbb{Z}$-polynomials on the holonomy of the generators. Therefore its dimension does not change under Galois conjugation. Secondly, B^1 can be seen as the image of the linear map from $\mathfrak{sl}(2,\mathbb{C})$ to $Z^1 \subset \mathfrak{sl}(2,\mathbb{C})^k$ that maps each $a \in \mathfrak{sl}(2,\mathbb{C})$ to the crossed morphism $\gamma \mapsto a - \mathrm{Ad}_{\mathrm{hol}(\gamma)}(a)$. By considering a $\mathbb{C}$-basis for $\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{sl}(2,\mathbb{Q})$, the dimension of B^1 is independent of the Galois conjugate. Thus the Zariski tangent space of the curves we are looking at is one dimensional, which gives an upper bound on the dimension that establishes the final claim. □

Remark that $X(K)^\psi$ may contain other components than the ones described above. In particular, if K_0 is itself hyperbolic, there is at least one extra component whose characters correspond to representations that map μ to the trivial element, that is the lift of the excellent component of $E(K_0)$.

Corollary 7. *Let K be a hyperbolic knot which is periodic of prime order $p \neq 2$. Then $X(K)$ contains at least $\frac{p-1}{2}$ irreducible components which are curves.*

In addition there is an extra irreducible component when K_0 itself is hyperbolic.

Remark 6. By considering the abelianisation $\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ of the fundamental group of the orbifold $E(K)/\psi$, it is not difficult to prove that $\mathbf{X}_{ab}(E(K)/\psi)$ consists of $(p+1)/2$ lines. On the other hand, the abelianisation of the fundamental group of the exterior of K consists in a unique line which is fixed pointwise by the action induced by ψ on $\mathbf{X}(K)$. It follows that, in general, the fixed subvariety of the whole character variety of K is not birationally equivalent to the whole character variety of the orbifold. For this reason we have restricted our attention to $X(K)^\psi$.

6 Knots with free symmetries

Let K be a hyperbolic knot admitting a free symmetry ψ of odd prime order p. Let $L = A \sqcup K_0$ be the associated link as defined in Section 2 (see in particular Proposition 2).

In this case, the irreducible characters of $X(K)^\psi$ are mapped inside the subvariety of $X(L)$ obtained by intersection with the hypersurface defined by the condition that its characters correspond to representations that send $\tilde{\mu} + k\tilde{\lambda}$ to the trivial element, where $0 < k < p$ is the integer coprime with p prescribed by Proposition 2.

Note that in $\pi_1(E(K)/\psi)$ one has $\tilde{\mu} + k\tilde{\lambda} = p\mu + k\lambda$. We write:

$$\rho(\mu) = \begin{pmatrix} m_A & * \\ 0 & m_A^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\lambda) = \begin{pmatrix} l_A & * \\ 0 & l_A^{-1} \end{pmatrix} \tag{1}$$

Thus the representations of $E(K)/\psi$ must satisfy $m_A^p l_A^k = 1$. This provides a motivation to look at the restriction to the peripheral subgroup $\pi_1(\partial\mathcal{N}(A))$ generated by μ and λ:

$$\mathrm{res} : X(L) \to \mathbf{X}(\partial\mathcal{N}(A)). \tag{2}$$

When this restriction has finiteness properties, we are able to find uniform bounds on the number of components of $X(K)^\psi$:

Proposition 8. *Assume that* (2) *is a dominant morphism and that the dimension of $X(L)$ is at most 2. Then there is a constant C depending only on $X(L)$ such that the number of components of $X(K)^\psi$ is $\leq C$.*

Here by dominant morphism we mean a morphism whose restriction to each irreducible component of $X(L)$ is dominant.

Notice that the components of $X(L)$ have dimension at least two [19], the hypothesis in Proposition 8 implies in particular that they are always surfaces. We give in the next section an example of a link for which (2) is a dominant morphism. As a consequence we have:

Corollary 9. *There exists a sequence of hyperbolic knots K_p parametrised by infinitely many prime numbers p such that K_p has a free symmetry ψ of order p but $X(K_p)^\psi$ is bounded, uniformly on p.*

Proof of the proposition. Since p and k are coprime, there exist q and h such that the elements $p\mu + k\lambda$ and $q\mu + h\lambda$ generate the fundamental group of

$\partial\mathcal{N}(A)$. The character variety $\mathbf{X}(\partial\mathcal{N}(A))$ is a surface in $\mathbb{C}^3$ with coordinates $x = tr(p\mu + k\lambda)$, $y = tr(q\mu + h\lambda)$, and $z = tr((p+q)\mu + (k+h)\lambda)$, defined by the equation $x^2 + y^2 + z^2 - xyz - 4 = 0$. The equations $x = 2$ and $y = z$ determine a line D contained in the surface $\mathbf{X}(\partial\mathcal{N}(A))$ which corresponds to the subvariety of characters of representations that are trivial on $p\mu + k\lambda$.

To count the components of $X(K)^\psi$ it is enough to count the components of $\mathrm{res}^{-1}(D)$. The map res being a dominant morphism, there is a Zariski open subset of each irreducible component of $X(L)$ on which the map is finite to one. As a consequence, by dimensional reasons, there is a finite number N of curves in $X(L)$ which are mapped to points of $\mathbf{X}(\partial\mathcal{N}(A))$. It follows that the number of irreducible components $\mathrm{res}^{-1}(D)$ is bounded above by $d + N$ where d is the cardinality of the generic fibre of res. $\square$

7 A family of examples

Consider the two-component 2-bridge link 6^2_2 pictured in Figure 2.

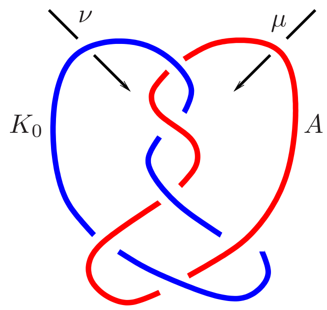

Figure 2: The 2-bridge link 6^2_2 and the generators of its fundamental group.

For each prime $p > 4$ and each $0 \leq k < p$ one can construct a symmetric knot K as described in Section 2. Since the absolute value of the linking number of the two components of L is 3, the construction does not give a knot for $p = 3$, which must thus be excluded.

Using Wirtinger's method one can compute a presentation of its fundamental group:

$$\langle \mu, \nu \mid \mu(\nu\mu^{-1}\nu\mu\nu^{-1}\mu\nu\mu^{-1}\nu) = (\nu\mu^{-1}\nu\mu\nu^{-1}\mu\nu\mu^{-1}\nu)\mu \rangle$$

where the generators μ and ν are shown in Figure 2. Having chosen the meridian μ, the corresponding longitude is $\lambda = \nu\mu^{-1}\nu\mu\nu^{-1}\mu\nu\mu^{-1}\nu$.

An involved but elementary computation gives the following defining equation for $X(L)$

$$(\alpha\beta\gamma - \alpha^2 - \beta^2 - \gamma^2 + 4)(-\gamma^4 + \alpha\beta\gamma^3 - (\alpha^2 + \beta^2 - 3)\gamma^2 + \alpha\beta\gamma - 1)$$

where α, β, and γ represent the traces of μ, ν, and $\mu^{-1}\nu$ respectively. The equation can also be found in [7].

Note that the variety consists of two irreducible components, the first one being that of the abelian characters.

A similar computation gives an expression for the trace of λ in terms of α, β, and γ:

$$\mathrm{tr}(\lambda) = (\alpha\beta - (\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 3)\gamma)(\beta\gamma - \alpha) - (\alpha\gamma - \beta)$$

We want to understand the generic fibre of the restriction map res : $X(L) \to \mathbf{X}(\partial\mathcal{N}(A))$, where $X(L)$ is a surface contained in $\mathbb{C}^3$ with coordinates α, β, and γ and $\mathbf{X}(\partial\mathcal{N}(A))$ is also a surface contained in $\mathbb{C}^3$ but with coordinates $\mathrm{tr}(\mu)$, $\mathrm{tr}(\lambda)$, and $\mathrm{tr}(\mu\lambda)$. For each fixed point $(\mathrm{tr}(\mu), \mathrm{tr}(\lambda), \mathrm{tr}(\mu\lambda))$ in $\mathbf{X}(\partial\mathcal{N}(A))$, the fibre of res consists of the points (α, β, γ) which satisfy

$$\begin{cases} \alpha = \mathrm{tr}(\mu) \\ (\alpha\beta - (\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 3)\gamma)(\beta\gamma - \alpha) - (\alpha\gamma - \beta) = \mathrm{tr}(\lambda) \\ -\gamma^4 + \alpha\beta\gamma^3 - (\alpha^2 + \beta^2 - 3)\gamma^2 + \alpha\beta\gamma - 1 = 0 \end{cases}$$

Once α is replaced by its value $\mathrm{tr}(\mu)$, the points we are interested in correspond to the intersection of two curves in $\mathbb{C}^2$ with coordinates β, γ. We see immediately that, for generic values of $\mathrm{tr}(\lambda)$, each point of $\mathbf{X}(\partial\mathcal{N}(A))$ is the image of at most a finite number of points in $X(L)$ and such finite number is bounded above by the product of the degrees of the two polynomials in β and γ, i.e. 20.

This shows that Proposition 8 applies to this link and Corollary 9 holds.

8 Non invariant examples

Here we construct examples of knots with free or periodic symmetries with characters that are not invariant. Firstly, we describe a component of the set of characters fixed by the symmetry that is strictly contained in an irreducible component (i.e. fixed and non-fixed characters lie in the same irreducible component). Secondly, we show an example where some pairwise disjoint irreducible components of the variety of characters are permuted (in particular no character in the irreducible component is fixed).

We consider a special family of Montesinos knots, though our considerations can be easily adapted to other Montesinos knots. We use the description of symmetries of Montesinos links in [2]. Following the notation there, consider the Montesinos link

$$K = M(e, \frac{1}{\alpha}, \overset{(q)}{\ldots}, \frac{1}{\alpha}),$$

with $q \geq 4$ and $\alpha \geq 5$. Since we are only interested in knots we require moreover that α and $q + e$ are odd. To describe a finite order diffeomorphism, view $\mathbf{S}^3$ as the join of two circles (i.e. the components of the Hopf link), and describe it as the join action obtained by composing rotations around each circle. By [2, §3, Figure 5], K can be arranged so that it is invariant by a diffeomorphism ψ that is the composition of a rotation of angle $\frac{2\pi}{q}$ around one of the circles, and $\frac{\pi}{q}e$ around the other circle. In particular, ψ has order q if e is even, and order $2q$ if e odd. In addition, ψ is periodic if $e \in 2q\mathbb{Z}$, and ψ is free if e is even, and $e/2$ and q are coprime. Note that we are mainly concerned with symmetries of odd prime order, so in our case we can choose q to be an odd prime number and e must be even.

To construct representations of $E(K)$, use the orbifold Seifert fibration induced by the double branched cover of K, as in Montesinos's original description. Namely, consider $\mathcal{O}^3$ the orbifold with underlying space $\mathbf{S}^3$, branching locus K, and branching order 2, so that $\pi_1(\mathcal{O}^3) = \pi_1(E(K))/\langle\mu^2\rangle$, where μ denotes a meridian. Since K is a Montesinos knot, $\mathcal{O}^3$ is orbifold-Seifert fibred, with basis a Coxeter hyperbolic orbifold $\mathcal{B}^2$. Namely $\mathcal{B}^2 = \mathbf{H}^2/\pi_1(\mathcal{B}^2)$, where $\pi_1(\mathcal{B}^2)$ is viewed as the Coxeter group corresponding to a hyperbolic polygon $P \subset \mathbf{H}^2$ with q edges and angles π/α. In particular $\pi_1(\mathcal{B}^2)$ is generated by reflections r_i along the edges of P, and it admits a presentation

$$\pi_1(\mathcal{B}^2) = \langle r_1, \ldots, r_q \mid r_i^2 = (r_i r_{i+1})^\alpha = 1\rangle.$$

The hyperbolic structure of P is not unique: the angles being π/α, the length of the edges of P can be deformed in a $(q-3)$-dimensional space [16], this yields a $(q-3)$-dimensional space of representations of Γ into the isometry group of $\mathbf{H}^2$, which is in fact the Teichmüller space of the Fuchsian orbifold $\pi_1(\mathcal{B}^2)$, that we denote by $\mathcal{T}$. Any isometry of $\mathbf{H}^2$ (orientable or not) extends uniquely to an orientable isometry of $\mathbf{H}^3$, in particular reflections along geodesics extend to rotations of angle π. Thus we get a space of characters of $\pi_1(\mathcal{B}^2)$ into $\mathrm{PSL}_2(\mathbb{C})$. Let T denote the component of the variety of $\mathrm{PSL}_2(\mathbb{C})$-characters of $\pi_1(\mathcal{B}^2)$ that contains the holonomies of the Teichmüller space $\mathcal{T}$. Since we have a surjection $\pi_1(\mathbf{S}^3 - K) \to \pi_1(\mathcal{B}^2)$, this yields a family of representations of $\pi_1(E(K))$ to $\mathrm{PSL}_2(\mathbf{C})$, that lift to $\mathrm{SL}_2(\mathbf{C})$ by [6].

Lemma 10. *There is a component Z of $X(E(L))$ of dimension $q-3$ consisting of lifts of representations induced by T.*

Proof. The Teichmüller space of $\mathcal{B}^2$ has real dimension $q-3$, as this is the dimension of the space of lengths of $P \subset \mathbf{H}^2$. If we view $P \subset \mathbf{H}^3$, and we allow P not to be in a plane anymore, instead of lengths, to each edge we associate complex lengths (the real part is the usual length and the imaginary part a twist parameter). The very same argument as in [16] tells that the complex dimension of the space of possible complex lengths is $q-3$, and this yields complex dimension $q-3$ for T.

Each representation in T induces a representation of $\pi_1(E(K))$ to $\mathrm{PSL}_2(\mathbb{C})$, and it lifts to precisely two representations in $\mathrm{SL}_2(\mathbb{C})$, one for each choice of sign for the meridian. Thus we obtain at most two components in $\mathbf{X}(E(K))$ and we choose an irreducible component Z that contains one of them. We claim that all characters in Z are lifts of characters induced by representations of $\mathcal{B}^2$. Indeed, when a generic character in T is viewed in $\mathbf{X}(E(K))$, the trace of the meridian remains constant in a (usual) neighborhood [17, Theorem 2.10 and Lemma 3.6], so it remains constant in Z. Hence all characters in Z come from representations of $\mathcal{O}^3$ that, by irreducibility, must map the fibre of the fibration to a trivial element in $\mathrm{PSL}_2(\mathbb{C})$, and so they come from a representation of $\mathcal{B}^2$. □

Proposition 11. *The component Z is preserved by ψ and contains characters that are fixed by ψ and characters that are not. Hence $\emptyset \neq Z^\psi \neq Z$.*

Proof. We follow the discussion in the previous paragraphs. A polygon in $\mathbf{H}^2$ is invariant by rotations iff it is regular, thus with given angles π/α, the polygon $P \subset \mathbf{H}^2$ is fixed iff all edge lengths are the same. Therefore, when we consider the corresponding characters in Z, we find fixed and non-fixed characters. □

For the next example, we consider components that generalise Z. Fix natural numbers $0 < \epsilon_1, \dots, \epsilon_q < \alpha/2$, so that there exists a hyperbolic polygon $P_{\epsilon_1,\dots,\epsilon_q} \subset \mathbf{H}^2$ with angles $\pi\epsilon_1/\alpha, \dots, \pi\epsilon_q/\alpha$ (it exists by the bounds on the ϵ_i and Gauss-Bonnet). This polygon does not need to span a Coxeter group, but still it can be used to construct representations of $\pi_1(\mathcal{B}^2)$ to the isometry group of $\mathbf{H}^2$, just map r_i to a reflection on the i-th edge, so that $r_i r_{i+1}$ is mapped to a rotation of angle $\pi\epsilon_i/\alpha$ instead of angle π/α. Now the previous construction for $\epsilon_i = 1$ applies verbatim and in this way we obtain components $Z_{\epsilon_1,\dots,\epsilon_q}$ of $X(E(L))$ of dimension $q-3$. In particular $Z_{1,\dots,1} = Z$. Moreover, for different values of the ϵ_i the components are disjoint, as the trace of the element of $\pi_1(E(K)$ mapped to $r_i r_{i+1}$ is $\pm 2\cos(\pi\epsilon_i/\alpha)$. As the symmetry ψ cyclically permutes the rational tangles of L, it acts on the indices $\{\epsilon_1, \dots, \epsilon_q\}$ by cyclic permutation. Summarising:

Proposition 12. *The components $Z_{\epsilon_1,\dots,\epsilon_q}$ of $X(E(L))$ are pairwise disjoint for different values of $0 < \epsilon_1, \dots, \epsilon_q < \alpha/2$, $\epsilon_i \in \mathbb{N}$. The symmetry ψ maps $Z_{\epsilon_1,\dots,\epsilon_q}$ to $Z_{\epsilon_2,\dots,\epsilon_q,\epsilon_1}$.*

Remark 7. For certain symmetric Montesinos knots, one can find other components of the character variety that are not contained in the invariant subvariety. For instance, according to [12, Theorem 1 & Remark 1], the knot $K = M(e, \frac{1}{\alpha}, \frac{-1}{\alpha}, \overset{(2q)}{\dots}, \frac{1}{\alpha}, \frac{-1}{\alpha})$, with α and e odd, admits a degree-one map on $K_s = M(e, \frac{1}{\alpha}, \frac{-1}{\alpha}, \overset{(2s)}{\dots}, \frac{1}{\alpha}, \frac{-1}{\alpha})$, provided $3 \leq s \leq q$. As a consequence, the character variety of K contains that of K_s. Assume q is an odd prime: K admits a symmetry of order q either periodic if e is a multiple of q or free otherwise. Such symmetry does not descend to a symmetry of K_s if $s < q$. As a consequence, the characters of K induced by irreducible representations of K_s cannot be fixed by the action of the symmetry and must lie outside the invariant subvariety. In addition, the degree-one map preserves meridians (it sends tangles to tangles by either preserving or reversing their orientation). Thus, the same argument as in Lemma 10 yields that the Teichmüller component of K_s induces genuine irreducible components of $\mathbf{X}(E(K))$ of dimension $2s-3$, which are permuted by the symmetry.

9 Invariant character varieties over fields of positive characteristic

Let $L = K_0 \sqcup A$ be a hyperbolic link with two components such that A is trivial. Assume that $\mathrm{lk}(K_0, A) \neq 0$. For each odd prime number p that does not divide the linking number $\mathrm{lk}(K_0, A)$, the knot K_0 lifts to a knot K_p in the p-fold cyclic cover of $\mathbf{S}^3$ branched along A.

By construction (see Section 2), K_p admits a periodic symmetry ψ of order p, and the invariant subvariety $X(K_p)^\psi$ contains at least $(p-1)/2$ irreducible components of dimension 1. These components of $X(K_p)^\psi$ are constructed in Proposition 6 as the intersection of the character variety $X(L)$ with a family of $(p-1)/2$ parallel hyperplanes. The union of these parallel hyperplanes corresponds to a hypersurface which is the vanishing locus of the minimal polynomial for $2\cos(2\pi/p)$ in the variable $tr(\mu)$. Such polynomial can be easily computed from the pth cyclotomic polynomial and is defined over $\mathbb{Z}$.

The characters of $X(K_p)^\psi$ correspond to representations of the orbifold $E(K_p)/\psi$. Note that $X(E(K_p)/\psi)$ may have further components besides those provided by Proposition 6, since the orbifold may admit irreducible representations that are trivial on μ. These irreducible representations correspond to characters for which $\mathrm{tr}(\mu) = 2$. In any case, $X(E(K_p)/\psi)$ contains at least $(p-1)/2$ components of dimension 1.

If we consider the character variety of $E(K_p)/\psi$ in characteristic p, we have that, since the only elements of order p are parabolic, the entire character variety must be contained in the hyperplane defined by $\mathrm{tr}(\mu) = 2$. We note that if p is not a ramified prime for $E(K_p)/\psi$, then it must contain as many 1-dimensional irreducible components as the one over $\mathbb{C}$, that is at least $(p-1)/2$.

Let us now turn our attention to the subvariety of $X(L)$ which consists in the intersection of $X(L)$ with the hyperplane $\mathrm{tr}(\mu) = 2$. We remark that it is non-empty since it must contain the character of the holonomy representation of L. We are interested in its irreducible components of dimension 1. These are in finite number, say N, depending on L only, and constitute an affine variety of dimension 1 that we shall denote Y.

Standard arguments of algebraic geometry show that for almost all (odd) primes q, the character variety $X(L)$ as well as its subvariety Y have the same properties (like dimension and irreducible components) over an algebraically closed field of characteristic q they have over the complex numbers. This follows basically from the fact that the dimension of an affine variety (that is, the maximal dimension of its irreducible components) and its irreducible components can be computed algorithmically (see, for instance, [4, Chapter 9] for the dimension, and [4, page 209] for the decomposition into irreducible components). In our specific situation we have that Y has N irreducible components.

Proposition 13. *For infinitely many periodic knots K_p as above, the character variety $X(K_p)$ ramifies at p.*

Proof. We start by considering the invariant variety $X(K_p)^\psi$ and show that this variety ramifies at p if p is large enough. Indeed, if this were not the case, the above discussion implies that the number of irreducible curves of $X(K_p)^\psi$ should be at least $(p-1)/2$ on one hand and at most N on the other. It follows readily that $X(K_p)^\psi$ ramifies at p.

Now, since $(p-1)/2$ curves of the invariant variety $X(K_p)^\psi$ are also irreducible components of $X(K_p)$ and since $X(K_p)^\psi$ is defined over $\mathbb{Z}$, the character variety of K_p ramifies at p, too. $\square$

Remark 8. The polynomial equations defined over $\mathbb{Z}$ of the character variety of the orbifold $E(K_p)/\psi$ generate a non radical ideal when considered $\mod p$, since the minimal polynomial of $2\cos(\pi/p)$ is not reduced when considered $\mod p$.

Acknowledgement The authors are thankful to María Teresa Lozano for valuable discussions.

References

[1] Michel Boileau and Joan Porti. Geometrization of 3-orbifolds of cyclic type. *Astérisque*, 272:208, 2001. Appendix A by Michael Heusener and Porti.

[2] Michel Boileau and Bruno Zimmermann. Symmetries of nonelliptic Montesinos links. *Math. Ann.*, 277(3):563–584, 1987.

[3] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[4] David Cox, John Little, and Donal O'Shea An introduction to computational algebraic geometry and commutative algebra *Undergraduate Texts in Mathematics* Springer, New York, xvi+551, 2007.

[5] Marc Culler and Peter B. Shalen. Varieties of group representations and splittings of 3-manifolds. *Ann. of Math. (2)*, 117(1):109–146, 1983.

[6] F. González-Acuña and José María Montesinos-Amilibia. On the character variety of group representations in $\mathrm{SL}(2, \mathbf{C})$ and $\mathrm{PSL}(2, \mathbf{C})$. *Math. Z.*, 214(4):627–652, 1993.

[7] Shinya Harada. Canonical components of character varieties of arithmetic two bridge link complements. arXiv:1112.3441, 2011.

[8] Allen Hatcher. Notes on basic 3-manifold topology. Available at https://www.math.cornell.edu/ hatcher/3M/3Mdownloads.html

[9] Hugh M. Hilden, María Teresa Lozano, and José María Montesinos-Amilibia. On the character variety of periodic knots and links. *Math. Proc. Cambridge Philos. Soc.*, 129(3):477–490, 2000.

[10] William H. Jaco and Peter B. Shalen. Seifert fibred spaces in 3-manifolds. *Mem. Amer. Math. Soc.* 220, 1979.

[11] Klaus Johannson. Homotopy equivalence of 3-manifolds with boundary. *Lecture Notes in Math. 761*, Springer-Verlag, Berlin, 1979.

[12] Youlin Li. 1-dominations between Montesinos knots with at least 6 rational tangles. *Topology Appl.* 157:1033–1043, 2010.

[13] Alexander Lubotzky and Andy R. Magid. Varieties of representations of finitely generated groups. *Mem. Amer. Math. Soc.*, 58(336):xi+117, 1985.

[14] John W. Morgan and Hyman Bass, editors. *The Smith conjecture*, volume 112 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1984. Papers presented at the symposium held at Columbia University, New York, 1979.

[15] Luisa Paoluzzi and Joan Porti. Non-standard components of the character variety for a family of Montesinos knots. *Proc. Lond. Math. Soc. (3)*, 107(3):655–679, 2013.

[16] Joan Porti. Hyperbolic polygons of minimal perimeter with given angles. *Geom. Dedicata*, 156:165–170, 2012.

[17] Joan Porti. Regenerating hyperbolic cone 3-manifolds from dimension 2. *Ann. Inst. Fourier (Grenoble)*, 63(5):1971–2015, 2013.

[18] Yongwu Rong. Degree one maps between geometric 3-manifolds. *Trans. Amer. Math. Soc.*, 332(1):411–436, 1992.

[19] William P. Thurston. The Geometry and Topology of Three-Manifolds. (Electronic version from 2002 available at http://www.msri.org/publications/books/gt3m/), 1980.

[20] André Weil. Remarks on the cohomology of groups. *Ann. of Math. (2)*, 80:149–157, 1964.

Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373,
13453 Marseille, France
luisa.paoluzzi@univ-amu.fr

Departament de Matemàtiques, Universitat Autònoma de Barcelona.
08193 Bellaterra, Spain
porti@mat.uab.cat