TIAN'S INVARIANT OF THE GRASSMANN MANIFOLD

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Abstract. — We prove that Tian's invariant on the complex Grassmann manifold $G_{p,q}(\mathbb{C})$ is equal to $1/(p+q)$. The method introduced here uses a Lie group of holomorphic isometries which operates transitively on the considered manifolds and a natural imbedding of $\big(\mathbb{P}^1(\mathbb{C})\big)^p$ in $G_{p,q}(\mathbb{C})$.

Résumé. — On prouve que l'invariant de Tian sur la grassmannienne $G_{p,q}(\mathbb{C})$ est $1/(p+q)$. La méthode présentée dans cet article utilise un groupe de Lie d'isométries holomorphes qui opère transitivement sur les variétés considérées ainsi qu'un plongement naturel de $\big(\mathbb{P}^1(\mathbb{C})\big)^p$ dans $G_{p,q}(\mathbb{C})$.

1. Introduction

On a complex manifold, an hermitian metric h is characterized by the *1-1 symplectic* form ω defined by $\omega = i\, g_{\lambda\bar{\mu}}\, dz^\lambda \wedge d\bar{z}^\mu$, where $g_{\lambda\bar{\mu}} = h_{\lambda\bar{\mu}}/2$.

The metric is a *Kähler metric* if ω is closed, i. e. $d\omega = 0$; then M is a *Kähler manifold*.

On a Kähler manifold, we can define the *Ricci form* by $R = i\, R_{\lambda\bar{\mu}}\, dz^\lambda \wedge d\bar{z}^\mu$, where $R_{\lambda\bar{\mu}} = -\partial_{\lambda\bar{\mu}} \log |g|$.

A Kähler manifold is *Einstein with factor* k if $R = k\omega$. For instance, choosing a local coordinate system $Z = (z_1, \ldots, z_m)$, the projective space $\mathbb{P}_m(\mathbb{C})$ with the Fubini-Study metric $\omega = i\partial\bar{\partial}\log\big(1 + ||Z||^2\big)$ is Einstein with factor $m+1$.

On a Kähler manifold M, the *first Chern class* $C^1(M)$ is the cohomology class of the Ricci tensor, that is the set of the forms $R + i\partial\bar{\partial}\varphi$, where φ is C^∞ on M. If there is a form in $C^1(M)$ which is positive (resp. negative, zero), then $C^1(M)$ is *positive* (resp. *negative*, *zero*). If a Kähler manifold is Einstein, then $C^1(M)$ and k are both positive (resp. negative, zero). In the negative case, it was proved by Aubin ([Au1], see also [Au4]), that there exists a unique Einstein-Kähler metric (E.K. metric) on M . It is so for the zero case too ([Au1], [Ya]). The question for the positive case is still open: some manifolds, such as the complex projective space blown up at one point, do not admit an E.K. metric (for obstructions, see [Li] and [Fu]). Aubin [Au2] and Tian [Ti] have shown that for suitable values of holomorphic invariants of the metric, there exists an E.K. metric on M.

1991 *Mathematics Subject Classification.* 53C55, 32M10.

Key words and phrases. Kähler manifold, Einstein-Kähler metric, first Chern class, admissible functions, Tian's invariant, grassmannian.

For $\omega/2\pi$ in $C^1(M)$, *Tian's invariant* $\alpha(M)$ is the supremum of the set of the real numbers α satisfying the following: there exists a constant C such that the inequality $\displaystyle\int_M e^{-\alpha\varphi} \leq C$ holds for all the C^∞ functions φ with $\omega + i\partial\overline{\partial}\varphi > 0$ and $\sup\varphi \geq 0$, where $\omega = i\, g_{\lambda\overline{\mu}}\, dz^\lambda \wedge d\bar{z}^\mu$ is the metric form. Such functions φ are said *ω-admissible.*
In [Ti], Tian established that if $\alpha(M) > m/(m+1)$, m being the dimension of M, there exists an E.K. metric on M. This condition is not necessary: it does not hold on the projective space, where Tian's invariant is $1/(m+1)$.
In the same paper, Tian introduces a more restrictive invariant $\alpha_G(M)$, considering only the admissible functions φ invariant by the action of a compact group G of holomorphic isometries. The sufficient condition for the existence of an E.K. metric on M remains $\alpha_G(M) > m/(m+1)$; it is more easily satisfied if the group G is rich enough.
In many cases, the group G is a non-discrete Lie group. The invariant $\alpha_G(M)$ can be computed using subharmonic functions methods and the maximum principle (for effective examples, see [Be1], [Be2], [Be-Ch1], [Be-Ch2], [Re]).
In this paper, we prove the following theorem:

Theorem 1.1. *Tian's invariant on $G_{p,q}(\mathbb{C})$ is given by $\alpha\left(G_{p,q}(\mathbb{C})\right) = 1/(p+q)$.*

This generalizes the known result on $\mathbb{P}^m(\mathbb{C})$ ([Ti], see also [Au3]). Let us also mention that Tian's invariant has been computed on $\mathbb{P}^m(\mathbb{C})$ blown up at one point and on certain Fermat hypersurfaces using Hörmander L^2 estimates for the $\overline{\partial}$-equation ([Ti]).
We first compute the volume element of the metric $\mathscr{G}_{p,q}$; then we will establish some general preliminary results concerning Tian's invariant as well as imbeddings of $\left\{\mathbb{P}^1(\mathbb{C})\right\}^p$ in $G_{p,q}(\mathbb{C})$ which allow us to deduce $\alpha\left(G_{p,q}(\mathbb{C})\right)$ from $\alpha\left(\mathbb{P}^1(\mathbb{C})\right)$.

2. Basic properties of the Grassmann manifold

We propose here a short survey of the properties of Grassmann manifold (for more details, see [Ko-No]). We denote by $G_{p,q}(\mathbb{C})$ the set of the subspaces of dimension p in $\mathbb{C}^{p+q}$; in particular, $G_{1,m}(\mathbb{C})$ is the complex projective space of dimension m. It is known (see [Au3]) that on $\mathbb{P}_m(\mathbb{C})$, the Fubini-Study metric is Einstein with factor $m+1$ and that Tian's invariant is $1/(m+1)$. Now, let $M^*(p+q,p)$ be the set of the matrices of rank p in $M_{p+q,p}(\mathbb{C})$. The group $Gl_p(\mathbb{C})$ acts by multiplication on the right on $M^*(p+q,p)$. More precisely $\left(M^*(p+q,p),\ \pi,\ G_{p,q}(\mathbb{C})\right)$ is a principal fiber bundle with group $Gl_p(\mathbb{C})$. The group $Gl_{p+q}(\mathbb{C})$ acts by multiplication on the left on $M^*(p+q,p)$ and induces an action on $G_{p,q}(\mathbb{C})$; so does the unitary group $U(p+q)$. These groups act transitively on $G_{p,q}(\mathbb{C})$, which shows that $G_{p,q}(\mathbb{C})$ is compact.

We denote by $\mathcal{I}$ the set of all increasing-ordered subsets of p elements in $\{1,\ldots,p+q\}$. Let P be an element of $M^*(p+q,p)$, $P=\big(p_{ij}\big)_{\substack{1\le i\le p+q\\ 1\le j\le p}}$. By Cauchy-Binet formula we get: $\det\big({}^tP\overline{P}\big)=\sum_{I\in\mathcal{I}}|\det m_I(P)|^2$, where $m_I(P)$ is the matrix $\big(p_{ij}\big)_{\substack{i\in I\\ 1\le j\le p}}$. The form ω, where $\omega=i\,\partial\overline{\partial}\log\det\big({}^tP\overline{P}\big)$, is invariant by the action of $Gl_p(\mathbb{C})$ on $M^*(p+q,p)$, and so it projects onto a form $\mathscr{G}_{p,q}$. The metric $\mathscr{G}_{p,q}$ is a Kähler metric form on $G_{p,q}(\mathbb{C})$. For $p=1$, this metric on $G_{1,m}(\mathbb{C})$ is the Fubini-Study metric on the complex projective space. The action of the unitary group $U(p+q)$ on $G_{p,q}(\mathbb{C})$ preserves the metric $\mathscr{G}_{p,q}$ so that $U(p+q)$ is a group of holomorphic isometries which operates transitively on $G_{p,q}(\mathbb{C})$. For I in $\mathcal{I}$, let U_I be the set of the matrices P in $M^*(p+q,p)$ such that $\det(m_I(P))$ is non-zero. Then $\pi(U_I)$ is a coordinate open set on $G_{p,q}(\mathbb{C})$, the matrix Z_I in $M_{q,p}(\mathbb{C})$ is the coordinate, the inverse of the chart φ_I sends $M^*(p+q,p)$ onto $\pi(U_I)$ and we have $m_I\big(\varphi_I^{-1}(Z_I)\big)=I^{(p)}$ where $I^{(p)}$ is the $p\times p$ identity matrix, and $m_{I^c}\big(\varphi_I^{-1}(Z_I)\big)=Z_I$.

Lemma 2.1. *For I in $\mathcal{I}$, let λ_I be the map from $\pi(U_I)$ to $\mathbb{R}_+$ defined by*

$$\lambda_I(Z_I)=\big|\det(\mathrm{Id}+{}^tZ_I\overline{Z}_I)\big|^{-(p+q)}.$$

Then $\big(\lambda_I\big)_{I\in\mathcal{I}}$ are the components of a maximal differential form η on $G_{p,q}(\mathbb{C})$, namely:

$$\eta=\lambda_I\big(i/2\big)^{pq}\big(dZ\wedge d\overline{Z}\big)_I.$$

Proof. It suffices to show that the following transformation rule holds:

$$\text{for every } I,\,\widetilde{I}\text{ in }\mathcal{I},\ \lambda_I\text{ is equal to }\lambda_{\widetilde{I}}\times\left|\det\ \frac{\partial Z_{\widetilde{I}}}{\partial Z_I}\right|^2\ \text{ on }\pi(U_I)\cap\pi(U_{\widetilde{I}}).$$

Let P_I be the matrix $\varphi_I^{-1}(Z_I)$. Then $P_I\big\{m_{\widetilde{I}}(P_I)\big\}^{-1}=P_{\widetilde{I}}$, so $Z_{\widetilde{I}}=m_{\widetilde{I}^c}(P_I)\,\big\{m_{\widetilde{I}}(P_I)\big\}^{-1}$. The differential of the map which sends Z_I on P_I is the map which sends H on $\breve{H}$, where $m_{I^c}(\breve{H})=H$ and $m_I(\breve{H})=0$. The change of charts sending Z_I on $Z_{\widetilde{I}}$, we obtain

$$\begin{aligned}
D\,Z_{\widetilde{I}}(H)&=m_{\widetilde{I}^c}(\breve{H})\,\big\{m_{\widetilde{I}}(P_I)\big\}^{-1}-m_{\widetilde{I}^c}(P_I)\,\big\{m_{\widetilde{I}}(P_I)\big\}^{-1}m_{\widetilde{I}}(\breve{H})\,\big\{m_{\widetilde{I}}(P_I)\big\}^{-1}\\
&=\big(m_{\widetilde{I}^c}(\breve{H})-\gamma\,m_{\widetilde{I}}(\breve{H})\big)\,\alpha^{-1},\\
\text{where}\quad&\alpha=m_{\widetilde{I}}(P_I),\ \beta=m_{\widetilde{I}^c}(P_I)\text{ and }\gamma=\beta\,\alpha^{-1}.
\end{aligned}$$

Let us define a map u from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ by $u(H)=m_{\widetilde{I}^c}(\breve{H})-\gamma\,m_{\widetilde{I}}(\breve{H})$. We can choose $I=\{q+1,\ldots,q+p\}$ and $\widetilde{I}=\{1,\ldots,r\}\cup\{q+1+r,\ldots,q+p\}$, where $0\le r\le\inf(p,q)$. We define the $k\times l$ matrix $E_{i,j}^{(k\times l)}$ by $\big(E_{i,j}^{(k\times l)}\big)_{\lambda\mu}=\delta_{i\lambda}\,\delta_{j\mu}$. We have

$$\begin{aligned}
&m_{\widetilde{I}}\big(\breve{E}_{i,j}^{(q\times p)}\big)=E_{i,j}^{(p\times p)}\quad\text{if } i\le r,\ \text{and 0 if } i>r,\\
\text{and}\quad&m_{\widetilde{I}^c}\big(\breve{E}_{i,j}^{(q\times p)}\big)=E_{i-r,j}^{(q\times p)}\quad\text{if } i>r,\ \text{and 0 if } i\le r.\ \text{Hence}\\
&\big(\gamma\,m_{\widetilde{I}}(\breve{E}_{i,j}^{(q\times p)})\big)_{\alpha\beta}=\gamma_{\alpha i}\,m_{\widetilde{I}}(\breve{E}_{i,j}^{(q\times p)})_{ij}\,\delta_{j\beta}=\gamma_{\alpha i}\,\delta_{j\beta}\quad\text{if } i\le r,\ \text{and 0 elsewhere.}
\end{aligned}$$

Now the map which sends H to $\gamma\, m_{\tilde{I}}(\breve{H})$ can be restricted if $1 \leq j \leq p$ to the span B_j of the $\big(E_{i,j}\big)_{1\leq i\leq q}$. The r first columns of its matrix are those of γ, the others are 0. The map which sends H to $\gamma\, m_{\tilde{I}^c}(\breve{H})$ maps also B_j into itself. The right upper block of its matrix is $I^{(q-r)}$, the other elements are 0. This allows us to compute the matrix of the restriction of u to B_j, whose determinant is $(-1)^{r\times(q-r)}\det\big(\gamma_{ij}\big)_{\substack{q-r+1\leq i\leq q\\ 1\leq j\leq r}}$. So $\det u = (-1)^{p\times r\times (q-r)}\Big[\det\big(\gamma_{ij}\big)_{\substack{q-r+1\leq i\leq q\\ 1\leq j\leq r}}\Big]^p$. For $1\leq i\leq q$, let C_i be the span of the $(E_{i,j})_{1\leq j\leq p}$. Each C_i is stable by the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to $H\,\alpha^{-1}$. The matrix of the restriction is α^{-1}, so the determinant of the map is $(\det\alpha)^{-q}$. Hence

$$\big|\det DZ_{\tilde{I}}(H)\big|^2 = \big|\det\left(\gamma_{i,j}\right)_{\substack{q-r+1\leq i\leq q\\ 1\leq j\leq r}}\big|^{2p}\times\big|\det\alpha\big|^{-2q}.$$

Let A be the right $r\times r$ upper block of α. The left $(p-r)\times(p-r)$ lower block of α is $I^{(p-r)}$ and the right $(p-r)\times r$ lower block is 0, so $\det\alpha = (-1)^{r(p-r)}\det A$. The left $r\times(p-r)$ lower block of β is 0, the right $r\times r$ block is $I^{(r)}$ so that the left $r\times r$ lower block of γ is A^{-1}.
From this we deduce $\big|\det\, DZ_{\tilde{I}}(H)\big|^2 = \big|\det\alpha\big|^{-2(p+q)}$. Since $P_I\,\alpha^{-1} = P_{\tilde{I}}$, we have

$$\lambda_{\tilde{I}} = \Big|\det\big({}^t P_{\tilde{I}}\overline{P}_{\tilde{I}}\big)\Big|^{-(p+q)} = \big|\det\alpha\big|^{2(p+q)}\lambda_I = \left|\det\frac{\partial Z_{\tilde{I}}}{\partial Z_I}\right|^{-2}\lambda_I.$$

□

Lemma 2.2. *The unitary group $U(p+q)$ preserves η.*

Proof. We call I the set $\{q+1,\ldots,q+p\}$. We define P_I in $\pi(U_I)$ by $P_I = \varphi_I^{-1}(Z_I)$. Let U be an element in $U(p+q)$ such that $m_I(UP_I)$ is invertible. Let $\tilde{P}_I = UP_I\left\{m_I(UP_I)\right\}^{-1}$ and $\tilde{Z}_I = m_{I^c}(\tilde{P}_I)$. We have $\tilde{Z}_I = m_{I^c}(U)\,P_I\left\{m_I(U)P_I\right\}^{-1}$. So

$$D\tilde{Z}_I(H) = m_{I^c}(U)\Big[\breve{H}\big\{m_I(U)P_I\big\}^{-1} - P_I\big\{m_I(U)P_I\big\}^{-1}m_I(U)\,\breve{H}\,\big\{m_I(U)P_I\big\}^{-1}\Big].$$

Thus $D\tilde{Z}_I(H) = X\breve{H}\delta^{-1}$, where $\delta = m_I(U)P_I$ and $X = m_{I^c}(U)\big[I^{(p+q)} - P_I\,\delta^{-1}\,m_I(U)\big]$. Let X_1 be the $q\times q$ matrix of the q first columns of X. Then, $X\breve{H} = X_1 H$ and we get $D\tilde{Z}_I(H) = X_1 H\delta^{-1}$. The determinant of the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to $H\delta^{-1}$ is $(\det\delta)^{-q}$. The determinant of the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to X_1H is $(\det X_1)^p$, so $\det D\tilde{Z}_I = (\det X_1)^p\,(\det\delta)^{-q}$. We divide U into four blocks:

$$U = \begin{pmatrix} U_q & U_{q,p}\\ U_{p,q} & U_p\end{pmatrix},\quad U_q\in M_q(\mathbb{C}),\ U_p\in M_p(\mathbb{C}),\ U_{p,q}\in M_{p,q}(\mathbb{C}),\ U_{q,p}\in M_{q,p}(\mathbb{C}).$$

Then $\delta = U_{p,q}\,Z_I + U_p$, so $X_1 = U_q - \big(U_q\,Z_I + U_{q,p}\big)\big(U_{p,q}\,Z_I + U_p\big)^{-1} U_{q,p}$. Let Z in $M_{p+q,p+q}(\mathbb{C})$ be the matrix with blocks $Z_q = I^{(q)}$, $Z_{p,q} = 0$, $Z_{q,p} = Z_I$, $Z_p = I^{(p)}$, the notations being the same as above. Writing $\det U = \det(UZ)$ and using the column transformation $C_1 \leftarrow C_1 - C_2\big(U_{p,q}\,Z_I + U_p\big)^{-1} U_{p,q}$ where C_1 is made of the first q columns and C_2 of the remaining ones, we get

$$\det U = \det\Big[U_q - \big(U_q\,Z_I + U_{q,p}\big)\big(U_{p,q}\,Z_I + U_p\big)^{-1} U_{p,q}\Big] \times \det\big(U_{p,q}Z_I + U_p\big).$$

Hence $\big|\det\, D\tilde{Z}_I\big|^2 = \big|\det\delta\big|^{-2(p+q)}$. We have $\tilde{P}_I = AP_I\delta^{-1}$, so

$$\lambda_{\tilde{I}} = \det\big({}^t\tilde{P}_I\overline{\tilde{P}}_I\big)^{-(p+q)} = \det\big({}^tP_I\overline{P}_I\big)^{-(p+q)} \times \big|\det\delta\big|^{2(p+q)} = \lambda_I\,\big|\det D\tilde{Z}_I\big|^{-2},$$

which proves the result. $\square$

Proposition 2.3. 1. *$dV\left(\mathscr{G}_{p,q}\right) = \eta$.*

2. *If $I \in \mathcal{I}$, $\left|\mathscr{G}_{p,q}\right|_I = \Big\{\det\big(I^{(p)} + {}^tZ_I\overline{Z}_I\big)\Big\}^{-(p+q)}$.*
3. *$\mathcal{R}\left(\mathscr{G}_{p,q}\right) = (p+q)\mathscr{G}_{p,q}$.*

Proof. 1. Let I in $\mathcal{I}$. It is easy to compute $\mathscr{G}_{p,q}$ at the point $Z_I = 0$: $\mathscr{G}_{p,q}(H,K) = Tr(H\overline{K})$. Then $dV(\mathscr{G}_{p,q})\big|_{Z_I=0} = \big(i/2\big)^{pq}\big(dZ \wedge d\overline{Z}\big)_I = \eta\big|_{Z_I=0}$. Since $dV(\mathscr{G}_{p,q})$ and η are invariant by the transitive action of $U(p+q)$, we have $dV(\mathscr{G}_{p,q}) = \eta$.

2. Since $dV(\mathscr{G}_{p,q}) = \big|\mathscr{G}_{p,q}\big|_I\,(i/2)^{pq}\big(dZ \wedge d\overline{Z}\big)_I$, property 1 gives the result.

3. Remark that $\mathscr{G}_{p,q} = i\,\partial\overline{\partial}\,\log\big\{\det(I^{(p)} + {}^tZ_I\overline{Z}_I)\big\}$. Since $\mathcal{R}\left(\mathscr{G}_{p,q}\right) = -i\,\partial\overline{\partial}\,\log\left|\mathscr{G}_{p,q}\right|_I$, we obtain $\mathcal{R}\left(\mathscr{G}_{p,q}\right) = (p+q)\mathscr{G}_{p,q}$, which expresses that $\mathscr{G}_{p,q}$ is Einstein, with factor $p+q$. $\square$

3. Some general results about Tian's invariant

3.1. **Tian's invariant with a normalization on a finite set.** If X is a manifold, we will denote by μ_X a measure on X compatible with the manifold structure.

Theorem 3.1. *Let M be a compact Kähler manifold. We suppose that there exists a compact Lie group G of holomorphic isometries. Let $\Delta_n = \{P_1, \ldots, P_n\}$ be a finite subset of M. Let $\alpha(\omega)$ (resp. $\alpha_{\Delta_n}(\omega)$) be the supremum of the set of the nonnegative real numbers α satisfying the condition: there exists a constant C such that the inequality $\displaystyle\int_M e^{-\alpha\varphi} \leq C$ holds for all the ω-admissible functions φ with $\sup\varphi \geq 0$ (resp. with $\varphi(P_i) \geq 0$ for $1 \leq i \leq n$). Suppose in addition that the orbit of each P_i under the action of G has positive measure. Then $\alpha(\omega) = \alpha_{\Delta_n}(\omega)$.*

We first establish a few lemmas which will be useful for the proof.

Lemma 3.2. *Let $\left(\varphi_n\right)_{n\geq 0}$ be a sequence of admissible functions with nonnegative maxima. Then there exists a subset Ω of M, with $\mu_{_M}(\Omega)=\mu_{_M}(M)$, and a subsequence φ_{n_k} of φ_n, such that for every p in Ω, the sequence $\left(\varphi_{n_k}(p)\right)_{k\geq 0}$ has a finite lower bound (depending on p).*

Proof. It is sufficient to assume that φ_n has null maxima. Let Q_n be a point such that $\varphi_n(Q_n)$ vanishes. Green's formula runs as follows:

$$\varphi_n(Q_n)=\frac{1}{V}\int_M \varphi_n+\int_M G(Q_n,R)\,\Delta\varphi_n(R)\,dV(R),$$

with $G(Q,R)\geq 0$ and $\displaystyle\int_M G(Q,R)\,dV(R)=C$, where C is a positive constant (see [Au4]). Since φ_n is admissible, $\Delta\varphi_n$ is less than m, m being the dimension of M. Thus $\displaystyle\int_M |\varphi_n|\leq C\,m\,V$. Furthermore, $\displaystyle\int_M \Delta\varphi_n=0$, so $\displaystyle\int_M |\Delta\varphi_n|=2\int_{\{\Delta\varphi_n>0\}}\Delta\varphi_n\leq 2mV$. For every Q in M, we have $\nabla\varphi_n(Q)=\displaystyle\int_M \nabla_Q G(Q,R)\Delta\varphi_n(R)dv(R)$, so that

$$\int_M |\nabla\varphi_n|\leq \int_M\Big[\int_M |\nabla_Q G(Q,R)|dv(Q)\Big]|\Delta\varphi_n(R)|dv(R)\leq 2m\widetilde{C}V,$$

since $\displaystyle\int_M |\nabla_Q G(Q,R)|dv(Q)$ is a continuous, hence a bounded function on M. Thus $(\varphi_n)_{n\geq 0}$ is bounded in the Sobolev space $H^{1,1}(M)$. By Kondrakov's theorem, we can extract from $(\varphi_n)_{n\geq 0}$ a subsequence which converges in $L^1(M)$, and after an other extraction we can suppose that this sequence converges almost everywhere to a function φ of $L^1(M)$. Since φ is finite almost everywhere, we get the result. □

Lemma 3.3. *Let $\left(\varphi_n\right)_{n\geq 0}$ be a sequence of admissible functions with nonnegative maxima and suppose that there exists a compact group G of holomorphic isometries of M such that the orbit of each P_i has positive measure. Let $\Phi: G\to\mathbb{R}\cup\{-\infty\}$ be the map defined by $\Phi(g)=\inf\limits_{\Delta_n}\inf\limits_{k\geq 0}(\varphi_k\circ g)$. Then there exists g in G such that $\Phi(g)$ is finite.*

Proof. Suppose that $\Phi\equiv-\infty$. For $i=1,\ldots,n$, let A_i be the set of the g in G such that $\inf\limits_{k\geq 0}(\varphi_k\circ g)(P_i)=-\infty$. The sets A_i are measurable and $\cup_{i=1}^n A_i=G$, so there exists i such that A_i has positive measure. From Lemma (3.2), $A_i.P_i$ is a subset of Ω^c. Since Ω and M have the same measure, the measure of $A_i.P_i$ vanishes. Let u_i be the map from G to M which sends g to $g(P_i)$. Then u_i has constant rank on G. Indeed, $u_i\circ L(g)=\sigma_g\circ u_i$, where $L(g)$ is the left translation by g and σ_g the map from M to M which sends x to $g.x$. Since $G.P_i$ has positive measure, u_i is a submersion on G, so that $u_i(A_i)$ has positive measure. This is a contradiction since $u_i(A_i)=A_i.P_i$. □

We can now prove Theorem (3.1).

Proof. It is clear that $\alpha(\omega) \leq \alpha_{\Delta_n}(\omega)$. Conversely, let $\varepsilon > 0$. There exists a sequence $\big(\varphi_n\big)_{n\geq 0}$ of admissible functions with positive maxima such that $\displaystyle\int_M e^{-(\alpha(\omega)+\varepsilon)\varphi_k}$ goes to infinity as k goes to infinity. Replacing φ_n by $\varphi_n - \sup\varphi_n$, we can take $\sup\varphi_n = 0$. First we apply Lemma(3.2). For the sake of simplicity, we take $\varphi_{n_k} = \varphi_k$. From Lemma (3.3), there exists an element g in G such that $\Phi(g)$ is finite; we define Ψ_k by $\Psi_k = \varphi_k \circ g - \Phi(g)$. Since g is an isometry, Ψ_k is ω-admissible, and from the very definition of Φ, $\Psi_k(P_i)$ is nonnegative. Furthermore, $\displaystyle\int_M e^{-(\alpha(\omega)+\varepsilon)\Psi_k} = e^{(\alpha(\omega)+\varepsilon)\Phi(g)}\int_M e^{-(\alpha(\omega)+\varepsilon)\varphi_k}$. This proves that $\displaystyle\int_M e^{-(\alpha(\omega)+\varepsilon)\Psi_k}$ goes to infinity as k goes to infinity. Then, $\alpha_{\Delta_n}(\omega) \leq \alpha(\omega)+\varepsilon$. This inequality holds for every positive ε, and so $\alpha_{\Delta_n}(\omega) \leq \alpha(\omega)$. $\square$

3.2. **Tian's invariant on a product.** For a Kähler form ω on a compact Kähler manifold M, $\alpha(\omega)$ is defined as in Theorem (3.1).

Proposition 3.4. *Let $\big(M_i\big)_{1\leq i\leq n}$ be compact Kähler manifolds with metric forms $\big(\omega_i\big)_{1\leq i\leq n}$. We endow the product $M_1\times\cdots\times M_n$ with the metric $\omega_1\oplus\cdots\oplus\omega_n$. Then $\alpha(\omega_1\oplus\cdots\oplus\omega_n) = \inf\limits_{1\leq i\leq n}\alpha(\omega_i)$.*

Proof. It suffices to make the proof when $n=2$, the general result will follow by induction.

(1) Suppose that $\alpha(\omega_1)\leq\alpha(\omega_2)$, and let $\varepsilon>0$. There exists a sequence $\big(\varphi_n\big)_{n\geq 0}$ of ω_1-admissible functions on M_1 with positive maxima such that $\displaystyle\int_{M_1} e^{-\big(\alpha(\omega_1)+\varepsilon\big)\varphi_n}$ goes to infinity when n goes to infinity. We define ψ_n on $M_1\times M_2$ by $\psi_n(m_1,m_2)=\varphi_n(m_1)$. Thus ψ_n is $(\omega_1\oplus\omega_2)$-admissible on $M_1\times M_2$, with positive maximum, and $\displaystyle\int_{M_1\times M_2} e^{-\big(\alpha(\omega_1)+\varepsilon\big)\psi_n} = V(M_2)\int_{M_1} e^{-\big(\alpha(\omega_1)+\varepsilon\big)\varphi_n}$, so that $\displaystyle\int_{M_1\times M_2} e^{-\big(\alpha(\omega_1)+\varepsilon\big)\psi_n}$ goes to infinity when n goes to infinity. We have therefore $\alpha(\omega_1\oplus\omega_2)\leq\alpha(\omega_1)+\varepsilon$. This yields $\alpha(\omega_1\oplus\omega_2)\leq\alpha(\omega_1)$.

(2) Let us now prove the opposite inequality. Let α be a real number such that $\alpha < \inf\big(\alpha(\omega_1),\alpha(\omega_2)\big)$ and φ an $(\omega_1\oplus\omega_2)$-admissible function on $M_1\times M_2$. If m_2 is in M_2, the function which sends m_1 to $\varphi(m_1,m_2)$ is ω_1-admissible. The same holds for M_1. Let (u,v) in $M_1\times M_2$ be such that $\varphi(u,v)\geq 0$. Then

$$\begin{aligned}\int_{M_1\times M_2} e^{-\alpha\varphi(m_1,m_2)}dV_1\,dV_2 &= \int_{M_1} e^{-\alpha\varphi(m_1,v)}\bigg(\int_{M_2} e^{-\alpha\big[\varphi(m_1,m_2)-\varphi(m_1,v)\big]}dV_2\bigg)dV_1\\ &\leq C_2\int_{M_1} e^{-\alpha\varphi(m_1,v)}dV_1 \leq C_1\,C_2.\end{aligned}$$

Thus, $\alpha \leq \alpha\big(\omega_1 \oplus \omega_2\big)$ and we get $\inf\big(\alpha(\omega_1), \alpha(\omega_2)\big) \leq \alpha\big(\omega_1 \oplus \omega_2\big)$. $\square$

3.3. **Tian's invariant on $G_{p,q}(\mathbb{C})$.** Since there is a natural duality isomorphism between $G_{p,q}(\mathbb{C})$ and $G_{q,p}(\mathbb{C})$, we can assume that $p \leq q$ without loss of generality.

3.3.1. *Imbedding of $\big\{\mathbb{P}^1(\mathbb{C})\big\}^p$ into $G_{p,q}(\mathbb{C})$ when $p \leq q$.* For w in $\mathbb{C}^{p(q-1)}$, $w = \big(w_{i,j}\big)_{\substack{1\leq i\leq q\\ 1\leq j\leq p\\ i\neq j}}$, we define the map $\tilde{\rho}_w$ from $\Big\{\mathbb{C}^2 \setminus (0,0)\Big\}^p$ to $M_{p+q,p}(\mathbb{C})$ by

$$\tilde{\rho}_w\Big((\lambda_i, \mu_i)_{1\leq i\leq p}\Big) = \begin{cases} \lambda_i\, \delta_{ij} & \text{if} \quad i \leq p \\ w_{i-p,j}\, \lambda_j & \text{if} \quad i > p \text{ and } i \neq j+p \\ \mu_i & \text{if} \quad i > p \text{ and } i = j+p \end{cases}$$

We make, for $p+1 \leq i \leq p+q$, the following row transformations: $L_i \leftarrow L_i - \sum\limits_{\substack{1\leq j\leq p\\ i\neq j+p}} w_{i-p,j}\, L_j$.

We get a matrix $\big(c_{ij}\big)_{\substack{1\leq i\leq p+q\\ 1\leq j\leq p}}$ with $c_{ij} = \delta_{ij}\,\lambda_i$ if $1 \leq i \leq p$ and $c_{ij} = \delta_{i-p,j}\,\mu_j$ if $p+1 \leq i \leq p+q$, which has rank p. $\tilde{\rho}_w$ induces a map from $\big\{\mathbb{P}^1(\mathbb{C})\big\}^p$ into $G_{p,q}(\mathbb{C})$ as shown on the following diagram, where γ is the projection of the principal fiber bundle $\Big\{\mathbb{C}^2 \setminus (0,0)\Big\}^p$ onto $\big\{\mathbb{P}^1(\mathbb{C})\big\}^p$. Remark that $\tilde{\rho}_w$ sends $[0,1] \times \cdots \times [0,1]$ onto $\pi(A)$, where $m_{\{p+1,\dots,2p\}}(A) = I^{(p)}$ and $m_{\{p+1,\dots,2p\}^c}(A) = 0^{(q\times p)}$.

$$\begin{tikzcd} \big\{\mathbb{C}^2 \setminus (0,0)\big\}^p \arrow[r, "\tilde{\rho}_w"] \arrow[d, "\gamma"'] & M^*(p+q,p) \arrow[d, "\pi"] \\ \big\{\mathbb{P}^1(\mathbb{C})\big\}^p \arrow[r, "\rho_w"'] & G_{p,q}(\mathbb{C}) \end{tikzcd}$$

We have

$$\begin{aligned} \big(\pi \circ \tilde{\rho}_w\big)^*\big(\mathscr{G}_{p,q}\big) &= i\,\partial\overline{\partial}\,\log\Big(\det\big({}^t\tilde{\rho}_w \bar{\tilde{\rho}}_w\big)\Big) \\ &= i\,\partial\overline{\partial}\,\log\left(\frac{\det\big({}^t\tilde{\rho}_w \bar{\tilde{\rho}}_w\big)}{\prod\limits_{k=1}^{p}\big(|\lambda_k|^2 + |\mu_k|^2\big)}\right) + \sum_{k=1}^{p} i\,\partial\overline{\partial}\,\log\big(|\lambda_k|^2 + |\mu_k|^2\big) \\ &= i\,\partial\overline{\partial}\,\log\tilde{\Phi} + \gamma^*\big(FS_1 \oplus \cdots \oplus FS_1\big), \end{aligned}$$

where FS_1 is the Fubini-Study metric on $\mathbb{P}^1(\mathbb{C})$. $\tilde{\Phi}$ is invariant by the action of the structural group $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$, so it induces a map Φ from $\big\{\mathbb{P}^1(\mathbb{C})\big\}^p$ into $\mathbb{C}$. Note that $\Phi\big([0,1] \times \cdots \times [0,1]\big) = 1$. Then $\big(\pi \circ \tilde{\rho}_w\big)^*\big(\mathscr{G}_{p,q}\big) = \pi^*\big(i\,\partial\overline{\partial}\,\log\Phi + FS_1 \oplus \cdots \oplus FS_1\big)$, so that $\rho_w^*\big(\mathscr{G}_{p,q}\big) = i\,\partial\overline{\partial}\,\log\Phi + FS_1 \oplus \cdots \oplus FS_1$.

3.3.2. *Lower bound of* $\alpha(\mathscr{G}_{p,q})$. For I in $\mathcal{I}$, we define P_I by $m_I(P_I) = I^{(p)}$ and $m_{I^c}(P_I) = 0^{(q\times p)}$. If $n = \binom{p+q}{p}$, we set $\Delta_n = \big\{P_I\big\}_{I\in\mathcal{I}}$. Since $U(p+q)$ is a transitive group of holomorphic isometries of $G_{p,q}(\mathbb{C})$, we know from proposition (3.1), that $\alpha\big(\mathscr{G}_{p,q}\big) = \alpha_{\Delta_n}\big(\mathscr{G}_{p,q}\big)$. We set $I = \{p+1,\dots,2p\}$. Let φ be an admissible function on $G_{p,q}(\mathbb{C})$, nonnegative on Δ_n. The last equality of the precedent section shows that the function $\varphi\circ\rho_w + \log\Phi$ is $\big(FS_1\oplus\cdots\oplus FS_1\big)$-admissible for every w in $\mathbb{C}^{p(q-1)}$. Furthermore, $\big(\varphi\circ\rho_w + \log\Phi\big)$ sends $[0,1]\times\cdots\times[0,1]$ to the nonnegative number $\varphi(P_I)$. It is known that $\alpha(FS_1) = 1$ (see [Au3]). Proposition (3.4) yields $\alpha\big(FS_1\oplus\cdots\oplus FS_1\big) = 1$.

Let α be a real number such that $\alpha < 1$. There exists a constant C, independent of φ, such that $\displaystyle\int_{\{\mathbb{P}^1(\mathbb{C})\}^p} e^{-\alpha\varphi\circ\rho_w}\,\Phi^{-\alpha} \le C$. We define the map F_I from $\pi(U_I)$ to $\mathbb{R}_+$ by $F_I(Z_I) = \det\big(Id + {}^tZ_I\overline{Z}_I\big)$. On $\big\{\mathbb{P}^1(\mathbb{C})\big\}^p$, we work with the coordinates $\mu_1,\dots,\mu_p$ in the chart $\lambda_1 = \cdots = \lambda_p = 1$. Thus

$$\Phi(\mu) = \frac{F_I\circ\rho_w(\mu)}{\prod_{k=1}^{p}\big(1+|\mu_k|^2\big)}, \quad \text{so that} \quad \int_{\mu\in\mathbb{C}^p} e^{-\alpha\varphi\circ\rho_w(\mu)}\,\frac{dV_\mu\big(\mathbb{C}^p\big)}{\prod_{k=1}^{p}\big(1+|\mu_k|^2\big)^{2-\alpha}\big(F_I\circ\rho_w(\mu)\big)^{\alpha}} \le C.$$

We have the inequality $\displaystyle\sum_{i=1}^{q}\sum_{j=1}^{p}\big|Z_{ij}\big|^2 \le F_I\big(P_I\big)$. In particular, for every k in $\{1,\dots,p\}$, $1+|\mu_k|^2 \le F_I\circ\rho_w(\mu)$, and $f_I\circ\rho_w(\mu) \ge 1 + \displaystyle\sum_{\substack{1\le i\le q\\ 1\le j\le p\\ i\ne j}}\big|w_{ij}\big|^2$. Thus, for $\kappa > 0$ and $w\in\mathbb{C}^{p(q-1)}$,

$$\frac{\prod_{k=1}^{p}\big(1+|\mu_k|^2\big)^{2-\alpha}}{\big(F_I\circ\rho_w(\mu)\big)^{\kappa+p+q-\alpha}} \le \frac{1}{\big(F_I\circ\rho_w(\mu)\big)^{\kappa-p+q+\alpha(p-1)}} \le \frac{1}{\Big(1+\sum_{\substack{1\le i\le q\\ 1\le j\le p\\ i\ne j}}\big|w_{ij}\big|^2\Big)^{\kappa}} = \frac{1}{\Big(1+\big\|w\big\|^2\Big)^{\kappa}}.$$

We have, according to Proposition (2.3),

$$\int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^{\kappa}} = \int_{w\in\mathbb{C}^{p(q-1)}} \int_{\mu\in\mathbb{C}^p} \frac{e^{-\alpha\varphi\circ\rho_w(\mu)}}{\big(F_I\circ\rho_w(\mu)\big)^{\kappa+p+q}}\, dV_\mu\big(\mathbb{C}^p\big)\, dV_w\big(\mathbb{C}^{p(q-1)}\big)$$

$$= \int_{w\in\mathbb{C}^{p(q-1)}} \int_{\mu\in\mathbb{C}^p} \left(\frac{e^{-\alpha\varphi\circ\rho_w(\mu)}}{\prod_{k=1}^{p}\big(1+|\mu_k|^2\big)^{2-\alpha}\big(F_I\circ\rho_w(\mu)\big)^{\alpha}} \right) \times \frac{\prod_{k=1}^{p}\big(1+|\mu_k|^2\big)^{2-\alpha}}{\big(F_I\circ\rho_w(\mu)\big)^{\kappa+p+q-\alpha}}\, dV_\mu\big(\mathbb{C}^p\big)\, dV_w\big(\mathbb{C}^{p(q-1)}\big)$$

$$= \int_{w\in\mathbb{C}^{p(q-1)}} \left(\int_{\mu\in\mathbb{C}^p} \frac{e^{-\alpha\varphi\circ\rho_w(\mu)}}{\prod_{k=1}^{p}\big(1+|\mu_k|^2\big)^{2-\alpha}\big(F_{_I}\circ\rho_w(\mu)\big)^{\alpha}}\, dV_\mu\big(\mathbb{C}^p\big) \right) \times \frac{dV_w\big(\mathbb{C}^{p(q-1)}\big)}{\big(1+\|w\|^2\big)^{\kappa}}$$

$$\leq C \int_{w\in\mathbb{C}^{p(q-1)}} \frac{dV_w\big(\mathbb{C}^{p(q-1)}\big)}{\big(1+\|w\|^2\big)^{\kappa}} \leq C' \qquad \text{if } \kappa > p(q-1).$$

Thus, we obtain that for all I in $\mathcal{I}$, $\displaystyle\int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^{\kappa}} \leq C$, where C is independent of φ.

Since $G_{p,q}(\mathbb{C})$ is compact, there exists a family $\big(V_I\big)_{I\in\mathcal{I}}$ of open sets of $G_{p,q}(\mathbb{C})$ such that V_I is relatively compact in $\pi(U_I)$ for every $I\in\mathcal{I}$, and $\bigcup_{I\in\mathcal{I}} V_I = G_{p,q}(\mathbb{C})$. There exists $M>0$ such that $F_I \leq M$ on V_I for every $I\in\mathcal{I}$. Thus

$$\int_{G_{p,q}(\mathbb{C})} e^{-\alpha\varphi} \leq \sum_{I\in\mathcal{I}} \int_{V_I} e^{-\alpha\varphi} \leq \sum_{I\in\mathcal{I}} M^{\kappa} \int_{V_I} \frac{e^{-\alpha\varphi}}{F_I^{\kappa}} \leq M^{\kappa} \sum_{I\in\mathcal{I}} \int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^{\kappa}} \leq C\, M^{\kappa} \binom{p+q}{p}.$$

We deduce that $\alpha\big(\mathscr{G}_{p,q}\big) \geq 1$.

3.3.3. *Upper bound of* $\alpha(\mathscr{G}_{p,q})$. We use here a method which can be found in [Re] for the complex projective space. Let I in $\mathcal{I}$. We define $\tilde{K}$ from $M^*(p+q,p)$ to $\mathbb{P}^1(\mathbb{R})$ by the relation $\tilde{K}(M) = \big[\,|\det m_{_I}(H)\,|^2, \det {}^t\!M\overline{M}\,\big]$. $\tilde{K}$ is invariant by the action of the structural group $G_p(\mathbb{C})$, so it induces a C^∞ map K from $G_{p,q}(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{R})$. Remark that $\psi = \log K$ is a Kähler potential on U_I for the metric $\mathscr{G}_{p,q}$.

Lemma 3.5. *There exists a decreasing sequence* $\big(\varphi_n\big)_{n\geq 0}$ *of admissible functions with positive maxima which converges pointwise to* $-\psi$ *on* $\pi\big(U_I\big)$.

Proof. We construct a decreasing sequence $\big(f_n\big)_{n\geq 0}$ of C^∞ convex functions on $\mathbb{R}_+$ satisfying the conditions $1+f_n'>0$, $f_n(x) = -\big(1-1/n\big)x$ for x in $[0,n]$ and $f_n(x) = -n$ for

$x \geq 2n$. Let y be an element of $\pi(U_I)^c$ and Ω_n the set of the elements x in $\pi(U_I)$ such that $\psi(x) > 2n$. Since $F_I(y) = [0,1]$, there exists a neighborhood V of y such that the inequality $z > e^{2n}$ holds for every point $[1,z]$ in $F_I(V)$. Thus $V \cap \pi\big(U_I\big)$ is included in Ω_n. We have proved that $W_n = \Omega_n \cup \pi\big(U_I\big)^c$, so that W_n is an open neighborhood of $\pi\big(U_I\big)^c$. We define φ_n by $\varphi_n = f_n \circ \psi$ on $\pi\big(U_I\big)$ and $\varphi_n = -n$ on W_n. Thus φ_n is well defined and $\varphi_n(0) = 0$. It remains to show that φ_n is admissible on $\pi\big(U_I\big)$. We have

$$\Big(\mathscr{G}_{p,q} + i\,\partial\overline{\partial}\,\varphi_n\Big)_{\lambda\overline{\mu}} = \partial_{\lambda\overline{\mu}}\psi + \partial_\lambda\big(f_n'\circ\psi\big)\partial_{\overline{\mu}}\psi = \big(1 + f_n'\circ\psi\big)\partial_{\lambda\overline{\mu}}\psi + f_n''\circ\psi\,\partial_\lambda\psi\,\partial_{\overline{\mu}}\psi.$$

Hence the matrix of the metric $\mathscr{G}_{p,q} + i\,\partial\overline{\partial}\,\varphi_n$ is of the form $A+T$ where A is positive definite and T has rank one and positive trace. So $A+T$ is positive definite and we get the result. □

Lemma 3.6. *Let n in $\mathbb{N}^*$ and r a positive real number. Then*

$$\int_{||X||\leq r} \frac{dV_X(M_n(\mathbb{C}))}{\big|\det X\big|^2} = +\infty.$$

Proof. We can write

$$\int_{||X||\leq r} \frac{dV_X(M_n(\mathbb{C}))}{\big|\det X\big|^2} = \sum_{k=0}^{\infty} \int_{r/2^{k+1}\leq||X||\leq r/2^k} \frac{dV_X(M_n(\mathbb{C}))}{\big|\det X\big|^2}.$$

We put $Y = 2^k X$, so

$$\int_{r/2^{k+1}\leq||X||\leq r/2^k} \frac{dV_X(M_n(\mathbb{C}))}{\big|\det X\big|^2} = \int_{1/2\leq||Y||\leq 1} \frac{dV_Y(M_n(\mathbb{C}))}{\big|\det Y\big|^2}.$$

The terms in the series are strictly positive and independent of k. The sum is therefore infinite. □

We can now prove that $\alpha\big(\mathscr{G}_{p,q}\big)$ is upper bounded by 1. Suppose that $\alpha\big(\mathscr{G}_{p,q}\big) > 1$. Then there exists a positive C such that for every integer n, $\displaystyle\int_{\pi(U_I)} e^{-\varphi_n} \leq C$. Using Lemma (3.5) and monotonous convergence, $\displaystyle\int_{\pi(U_I)} F_I \leq C$. Since $\pi(U_I)^c$ has zero measure, $\displaystyle\int_{G_{p,q}(\mathbb{C})} F_I \leq C$. Let $\tilde{I}$ in $\mathcal{I}$ be such that $I \cap \tilde{I} = \emptyset$ (this is possible since $p \leq q$). We have $P_{\tilde{I}}\big\{m_I\left(P_{\tilde{I}}\right)\big\}^{-1} = P_I$. Remark that $m_I\big(P_{\tilde{I}}\big) = m_I\big(Z_{\tilde{I}}\big)$. Thus $\det\big(Id +^t Z_I \overline{Z}_I\big) = \det\big({}^t P_{\tilde{I}}\overline{P}_{\tilde{I}}\big)\,\big|\det m_I\big(Z_{\tilde{I}}\big)\big|^{-2}$. For $\big\|Z_{\tilde{I}}\big\| \leq r$, $\det\big({}^t P_{\tilde{I}}\overline{P}_{\tilde{I}}\big) \leq M$, so that $\displaystyle\int_{||Z_{\tilde{I}}||\leq r} \frac{dV_{Z_{\tilde{I}}}(M_{q,p}(\mathbb{C}))}{\big|\det m_I\big(Z_{\tilde{I}}\big)\big|^2} < +\infty$. Integrating over the remaining variables $\big(Z_{ij}\big)_{i\in\tilde{I}^c\cap I^c}$ yields

$\displaystyle\int_{||Z||\leq r} \frac{dV_Z(M_p(\mathbb{C}))}{\left|\det Z\right|^2} < +\infty$, which is in contradiction with the result of Lemma (3.6). Thus we obtain $\alpha\big(\mathscr{G}_{p,q}\big) \leq 1$.

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