# AUTOMORPHISMS AND DEFORMATIONS OF CONFORMALLY KÄHLER, EINSTEIN–MAXWELL METRICS

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Abstract. We obtain a structure theorem for the group of holomorphic automorphisms of a conformally Kähler, Einstein–Maxwell metric, extending the classical results of Matsushima [25], Licherowicz [20] and Calabi [6] in the Kähler–Einstein, cscK, and extremal Kähler cases. Combined with previous results of LeBrun [19], Apostolov–Maschler [4] and Futaki–Ono [12], this completes the classification of the conformally Kähler, Einstein–Maxwell metrics on $\mathbb{CP}^1 \times \mathbb{CP}^1$. We also use our result in order to introduce a (relative) Mabuchi energy in the more general context of $(K, q, a)$-extremal Kähler metrics in a given Kähler class, and show that the existence of $(K, q, a)$-extremal Kähler metrics is stable under small deformation of the Kähler class, the Killing vector field $K$ and the normalization constant $a$.

## 1. Introduction

Let $(M, J)$ be a compact Kähler manifold of complex dimension $m$ and $g$ a $J$-compatible Kähler metric. Following [4], the Hermitian metric $\tilde{g} = \frac{1}{f^2} g$ is said to be *conformally Kähler, Einstein–Maxwell* if $\tilde{g}$ has

(a) $J$-invariant Ricci tensor, i.e.

$$\mathrm{Ric}_{\tilde{g}}(\cdot, \cdot) = \mathrm{Ric}_{\tilde{g}}(J\cdot, J\cdot), \tag{1}$$

(b) constant scalar curvature, i.e.

$$\mathrm{Scal}_{\tilde{g}} = const. \tag{2}$$

These conditions extend to higher dimensions a 4-dimensional riemannian signature analogue of the Einstein–Maxwell equations in General Relativity, see [18, 1].

In [4], Apostolov–Maschler initiated a study of conformally Kähler, Einstein–Maxwell Kähler metrics in a framework similar to the famous Calabi problem [6] of finding extremal Kähler metrics in a given Kähler class, and set the existence problem of the conformally Kähler, Einstein–Maxwell Kähler metrics in a formal GIT picture, extending the work of Donaldson and Fujiki [7, 8] characterizing the Calabi extremal metrics as critical points of the norm of the corresponding moment map. In particular, fixing a Kähler class $\Omega$ on $(M, J)$, a quasi-periodic real holomorphic vector field $K$ with zeroes, and a real positive constant $a > 0$, it was shown in [4] that there is a natural obstruction to the existence of conformally Kähler, Einstein–Maxwell Kähler metrics associated to the above data, similar to the Futaki invariant [10, 11] in the Kähler–Einstein and the constant scalar curvature Kähler (cscK) cases. More recently, Futaki–Ono [12] have characterized the latter obstruction in terms of a *volume-minimizing* condition on $K$, reminiscent to the constant scalar curvature Sasaki case [24, 14, 22].

The purpose of this paper is to extend two fundamental results in the theory of extremal Kähler metrics to a more general context relevant the conformally Kähler, Einstein–Maxwell metrics described above. The first result is a suitable extension of Calabi's Theorem [6] on the structure of the group of holomorphic automorhisms of a compact extremal Kähler manifold. To state it, let $g$ be a Kähler metric on $(M, J)$ endowed with a Killing vector field $K$ with zeroes. Hodge theory implies (see e.g. [16]) that $K$ is hamiltonian with respect to the Kähler form $\omega = gJ$, i.e. $\imath_K \omega = -df$ for a smooth function on $M$, called a *Killing potential* of $K$. We normalize $f = f_{(K,\omega,a)}$ by requiring $\int_M f_{(K,\omega,a)} dv_g = a > 0$, where the positive real constant $a$ is such that $f_{(K,\omega,a)} > 0$ on $M$. Then, for any fixed real number $q$ we define the $(K, q, a)$-*scalar curvature* of $g$ to be

$$\mathrm{S}_{(K,q,a)}(g) := f^2_{(K,\omega,a)}\mathrm{Scal}_g + 2q f_{(K,\omega,a)}\Delta_g\left(f_{(K,\omega,a)}\right) - q(q-1)|K|^2_g, \tag{3}$$

where $\mathrm{Scal}_g$ denotes the usual scalar curvature, $|\cdot|_g$ is the tensor norm induced by $g$, and $\Delta_g$ stands for the riemannian Laplacian on functions.

The point of this definition is that the condition (1) above yields that the conformal factor $f$ is a positive Killing potential of a Killing vector field $K$ for $g$, whereas the scalar curvature $\mathrm{Scal}_{\tilde{g}}$ of $\tilde{g} = \frac{1}{f^2}g$ is given by the formula (3) with $q = -(2m-1)$. Other choices of the *weight* $q$ lead to other interesting geometric problems, as it was observed in [2]. We also notice that the GIT framework of [4] makes sense for any choice of the weight $q$ as above, see Section 2 below.

**Definition 1.** Let $g$ be a Kähler metric on $(M, J)$ endowed with a Killing vector field $K$ as above, and $a > 0$ a real constant such the corresponding Killing potential $f_{(K,\omega,a)} > 0$ on $M$. We say that $g$ is $(K, q, a)$-extremal if its $(K, q, a)$-scalar curvature given by (3) is a Killing potential, i.e. $\Xi = J\mathrm{grad}_g(\mathrm{Scal}_{(K,q,a)}(g))$ is a Killing vector field for $g$.

The definition above incorporates the case when $\mathrm{S}_{(K,q,a)}(g)$ is constant, which in turn links to the initial motivation of studying conformally Kähler, Einstein–Maxwell metrics. We denote by $\mathfrak{h}^K$ (resp. $\mathfrak{k}^K$) the centralizer of $K$ in the Lie algebra of holomorphic vector fields (resp. Killing vector fields) of $(M, J)$ (resp. $(M, g)$) and $\mathrm{Aut}^K_0(M, J)$ (resp. $\mathrm{Isom}^K_0(M, g)$) the corresponding closed connected Lie groups. We then have:

**Theorem 1.** *Suppose $(M, g, J)$ is a compact $(K, q, a)$-extremal Kähler manifold. Then the group $\mathrm{Isom}^K_0(M, g)$ is a maximal compact connected subgroup of $\mathrm{Aut}^K_0(M, J)$. Furthermore, if $\mathrm{S}_{(K,q,a)}(g) = const$, then $\mathrm{Aut}^K_0(M, J)$ is a reductive complex Lie group.*

This basic result yields that each compact $(K, q, a)$-extremal Kähler manifold $(M, J, g)$ is invariant under a *maximal* torus $\mathbb{T}$ in the connected component of the identity of the reduced automorphism group $\mathrm{Aut}_{\mathrm{red}}(M, J)$, with $K \in \mathrm{Lie}(\mathbb{T})$, thus linking to the point of view of [4]. In particular, we can deduce from Theorem 1 and [4, Theorem 3] that if $(M, J)$ is toric, i.e. $\mathbb{T}$ is $m$-dimensional, then the $(K, q, a)$-extremal Kähler metrics are unique up to isometries in their Kähler classes (see Corollary 4 below). Concerning the existence of conformally Kähler, Einstein–Maxwell metrics, Theorem 1 and [4, Theorem 5] yield together a complete classification of the latter on the toric complex surfaces $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the Hirzebruch surfaces $\mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)) \to \mathbb{CP}^1$ in terms of *explicit* constructions given by either the Calabi Ansatz [18, 19, 17] or by the hyperbolic ambitoric ansatz [1] (a riemannian analogue of the Plebanski-Damianski explicit solutions [26]). In practice, however, the algorithm of [4, Theorem 5] allowing one to decide whether or not for a given Kähler class, a quasi-periodic holomorphic vector field $K$ and a constant $a > 0$ there exists a compatible conformally-Kähler, Einstein–Maxwell metric is of considerable complexity, see [12]. The case $\mathbb{CP}^1 \times \mathbb{CP}^1$ has been successfully resolved by [18, 4] (see also [12]):

**Corollary 1.** *Any conformally-Kähler, Einstein–Maxwell metric on $\mathbb{CP}^1 \times \mathbb{CP}^1$, must be toric, and if it is not a product of Fubini-Study metrics on each factor, it must be homothetically isometric to one of the metrics constructed in* [18].

We also notice that similarily to the Kähler–Einstein and cscK cases [25, 20], Theorem 1 places an obstruction in terms of $\mathrm{Aut}^K_0(M, J)$ for $(M, J)$ to admit a Kähler metric of constant $(K, q, a)$-scalar curvature, in particular a conformally Kähler, Einstein–Maxwell metric.

**Corollary 2.** *Let* $(M,J) = \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1)_E) \to \mathbb{F}_n$ *where* $E = (\mathcal{O}\oplus\mathcal{O}(n)) \to \mathbb{CP}^1$ *and* $\mathbb{F}_n = \mathbb{P}(E)$ *is the* $n$*-th Hirzebruch complex surface. Denote by* $K$ *the generator of the* $S^1$*-action on* $M$*, corresponding to diagonal multiplications on the* $\mathcal{O}_E(1)$*-factor. Then* $(M,J)$ *admits no Kähler metric of constant* $(bK,q,a)$*-scalar curvature for any values of* $b$ *and* $q$*.*

We now describe our second result, which is a suitable modification of the stability of the existence of extremal Kähler metrics under deformation of the Kähler class, proved by LeBrun–Simanca in [21], see also [9]. In our extended context, and without loss of generality by using Theorem 1 above, we fix a maximal real torus $\mathbb{T}\subset \mathrm{Aut}_{\mathrm{red}}(M,J)$, a real weight $q$, and study the existence of a $\mathbb{T}$-invariant $(K,q,a)$-extremal Kähler metric as a function of the Kähler class $\Omega\in H^2_{\mathrm{dR}}(M,\mathbb{R})$, the vector field $K\in \mathrm{Lie}(\mathbb{T})$, and the real constant $a>0$. We prove the following:

**Theorem 2.** *Suppose (without loss of generality by Theorem 1) that* $\mathbb{T}\subset\mathrm{Aut}_{\mathrm{red}}(M,J)$ *is a maximal real torus and* $(M,J)$ *admits a* $\mathbb{T}$*-invariant* $(K,q,a)$*-extremal Kähler metric* $(g,\omega)$*. Then, for any* $g$*-harmonic,* $\mathbb{T}-$*invariant,* $(1,1)$*-form* $\alpha$*, and any* $H\in\mathrm{Lie}(\mathbb{T})$*, there exist* $\varepsilon>0$*, such that for any real numbers* $|s|<\varepsilon, |t|<\varepsilon$ *and* $|u|<\varepsilon$*, there exists a* $(K+uH,q,a+s)$*-extremal Kähler metric in the Kähler class* $[\omega+t\alpha]$*.*

This result provides an efficient way to obtain many new examples from known ones.

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## 2. A familly of variational problems in Kähler geometry

In this section we recall the Apostolov–Maschler [4] moment map interpretation of the $(K,q,a)$-scalar curvature. In [4], the case $q=-2m+1$ is considered, but their argument works for any weight $q$ (see [2]).

Let $(M,J,\omega)$ be a compact Kähler manifold of real dimension $2m\geq 4$. We denote by $\mathfrak{h}_{\mathrm{red}}$ the Lie algebra of the reduced automorphism group $\mathrm{Aut}_{\mathrm{red}}(M,J)$, given by the real holomorphic vector field with zeros (see [16]). Let $K\in\mathfrak{h}_{\mathrm{red}}$ be a quasi-periodic Killing vector field generating a torus $G\subset\mathrm{Aut}_{\mathrm{red}}(M,J)$. It is well known that $G$ acts in a isometric hamiltonian way on $(M,\omega)$. Let $f_{(K,\omega,a)}\in C^\infty(M,\mathbb{R})$ be the normalized positive Killing potential of $K$, defined by the condition $\int_M f_{(K,\omega,a)}v_\omega = a$.

We denote by $\mathcal{K}^G(M,\omega)$ the space of all $\omega$-compatible, $G$-invariant Kähler structures on $(M,\omega)$, and consider the natural action of the infinite dimensional group $\mathrm{Ham}^G(M,\omega)$, of $G$-equivariant Hamiltonian transformations of $(M,\omega)$. We have the identification

$$\mathrm{Lie}\left(\mathrm{Ham}^G(M,\omega)\right)\cong C_0^\infty(M,\mathbb{R})^G$$

where $C_0^\infty(M,\mathbb{R})^G$ denote the space of smooth $G$-invariant functions with zero mean value with respect to $f^{q-2}_{(K,\omega,a)}v_\omega$, ($v_\omega=\frac{\omega^m}{m!}$ being the Riemanian volum form) endowed with the Poisson bracket.

For any $q\in\mathbb{R}$, the space $\mathcal{K}^G(M,\omega)$ carries a $q$-weighted formal Kähler structure $(\mathbf{J},\mathbf{\Omega}^{(K,q,a)})$ given by ([7, 8, 4])

$$\begin{aligned}\mathbf{\Omega}^{(K,q,a)}_J(\dot J_1,\dot J_2) &= \frac{1}{2}\int_M \mathrm{Tr}(J\dot J_1\dot J_2)f^q_{(K,\omega,a)}v_\omega,\\ \mathbf{J}_J(\dot J) &= J\dot J,\end{aligned}$$

where the tangent space of $\mathcal{K}^G(M,\omega)$ at $J$ is identified with the space of smooth $G$-invariant sections $\dot{J}$ of $\mathrm{End}(TM)$ satisfying

$$\dot{J}J + J\dot{J} = 0, \quad \omega(\dot{J}.,.) + \omega(.,\dot{J}.) = 0.$$

In what follows we denote by $g_J := \omega(.,J.)$ the Kähler metric corresponding to $J \in \mathcal{K}^G(M,\omega)$, and index all objects calculated with respect to $J$ similarly. On $C_0^\infty(M,\mathbb{R})^G$, we consider the scalar product given by,

$$\langle \phi, \psi \rangle_{(K,q,a)} = \int_M \phi\psi f_{(K,\omega,a)}^{q-2} v_\omega.$$

**Theorem 3.** [4] *The action of* $\mathrm{Ham}^G(M,\omega)$ *on* $\left(\mathcal{K}^G(M,\omega), \mathbf{J}, \mathbf{\Omega}^{(K,q,a)}\right)$ *is Hamiltonian with a momentum map given by the* $\langle .,. \rangle_{(K,q,a)}$*-dual of the* $(K,q,a)-$*scalar curvature given by* (3).

**Remark 1.**

(i) The weight $q = -2m+1$ corresponds to the conformally Kähler, Einstein–Maxwell case studied in [4], and $\mathrm{S}_{(K,q,a)}$ computes the scalar curvature of the hermitian metric $\tilde{g}_J := f_{(K,\omega,a)}^{-2} g_J$.

(ii) If $q = 0$, $\mathrm{S}_{(K,q,a)}(J)$ computes the so-called *conformal scalar curvature* $\tilde{\kappa}_J$ of the hermitian metric $\tilde{g}_J$ given by (see e.g. [15]),

$$\tilde{\kappa}_J = (2m-1)\left\langle \tilde{W}\left(\tilde{F}_J\right), \tilde{F}_J \right\rangle_{\tilde{g}_J},$$

where $\tilde{F}_J = \tilde{g}_J(J.,.)$ is its fundamental 2-form of $(\tilde{g}_J, J)$ and $\tilde{W}$ is the corresponding Weil tensor.

(iii) The weight $q = -m-1$ appears in the study of Levi-Kähler quotients (see e.g. [2]).

(iv) For a real number $p$, one can define,

$$\mathrm{S}_{(K,p,q,a)}(J) := f_{(K,\omega,a)}^{p-2} \mathrm{S}_{(K,q,a)}(g_J). \tag{4}$$

Then the $\langle \cdot,\cdot \rangle_{(K,q-p+2,a)}$-dual of (4) is a momentum map for the action of $\mathrm{Ham}^G(M,\omega)$ on $\left(\mathcal{K}^G(M,\omega), \mathbf{J}, \mathbf{\Omega}^{(K,q,a)}\right)$. Taking $(p,q) = (2,q)$ we obtain Theorem 3, wheres the value $(p,q) = (\frac{2}{m}, -1)$ corresponds to the Lejmi–Upmeier moment map given by the *hermitian scalar curvature* of $\tilde{g}_J$ (see [23]).

## 3. The extended Calabi problem

3.1. **The $(K,q,a)$-constant scalar curvature Kähler metrics.** Following [4] we now fix the complex manifold $(M,J)$ and vary the Kähler form $\omega$ within a fixed Kähler class $\Omega \in H^2(M,\mathbb{R}) \cap H^{1,1}(M,\mathbb{C})$. We also fix the compact torus $G \subset \mathrm{Aut}_{\mathrm{red}}(M,J)$ generated by a quasi periodic vector field $K \in \mathrm{Lie}(G)$, and denote by $\mathcal{K}_\Omega^G(M,J)$ the space of $G$-invariant Kähler forms $\omega \in \Omega$. Let $f_{(K,\omega,a)}$ be the normaliezed Killing potential of $K$ with respect to $\omega$, with normalization constant $a > 0$, such that $f_{(K,\omega,a)} > 0$. As shown in [4, Lemma 1] we have $f_{(K,\omega',a)} > 0$ on $M$ for all $\omega' \in \mathcal{K}_\Omega^G(M,J)$.

The space $\mathcal{K}_\Omega^G(M,J)$ is a Frechet manifold given near $\omega \in \mathcal{K}_\Omega^G(M,J)$ by the open subset of elements $\phi \in C^\infty(M,\mathbb{R})^G/\mathbb{R}$ such that $\omega + dd^c\phi > 0$ is positive definite. The tangent space of $\mathcal{K}_\Omega^G(M,J)$ at $\omega$ is identified with $C_0^\infty(M,\mathbb{R})^G$, the space of $G$-invariant smooth functions with mean value 0 with respect to $f_{(K,\omega,a)}^{q-2} v_\omega$.

We then consider the following generalized Calabi problem on $\mathcal{K}_\Omega^G(M,J)$ (see [4]):

**Problem.** For a weight $q \in \mathbb{R}$, a quasi-periodic vector field $K$ generating a torus $G$ in $\mathrm{Aut}_{\mathrm{red}}(M,J)$, $\omega \in \mathcal{K}_\Omega^G(M,J)$ and $a > 0$ such that $f_{(K,\omega,a)} > 0$, does there exist $\phi \in C_0^\infty(M,\mathbb{R})^G$ such that $\omega + dd^c\phi$ is $(K,q,a)$-extremal?

In what follows we calculate the first variation of the $(K, q, a)$-scalar curvature along $\omega \in \mathcal{K}^G_\Omega(M, J)$. We denote by $D$ the Levi-Civita connection and by $\delta = D^\star$ the co-differential of $(M, \omega, g)$. For a 1-form $\alpha$ on $M$, let $D^\pm\alpha$ be the $J$-invariant (resp. $J$-anti-invariant) part of $D\alpha$, i.e.

$$(D^\pm\alpha)_{X,Y} = \frac{1}{2}\left((D\alpha)_{JX,JY} \pm (D\alpha)_{X,Y}\right).$$

There is a natural action on $p$-forms $\psi$ induced by $J$ as follows,

$$(J\psi)(X_1, \cdots, X_p) = (-1)^p\psi(JX_1, \cdots, JX_p).$$

The twisted differential and the twisted codifferential on $p$-forms are defined by,

$$\begin{aligned} d^c =& JdJ^{-1}, \\ \delta^c =& J\delta J^{-1}. \end{aligned}$$

To simplify notation we omit below the index $(K, \omega, a)$ of $f_{(K,\omega,a)}$,

**Lemma 1.** *For any $G$-invariant 1-form $\alpha$ we have*

$$\begin{aligned} 2f^{2-q}\delta\delta\left(f^q D^-\alpha\right) =& 2f^2\delta\delta\left(D^-\alpha\right) - 2qf\left(\Delta\alpha, df\right) \\ &+ 2qf\left(\Delta df, \alpha\right) + 2qf(\delta d\alpha, df) \\ &- q(q-1)(\alpha, d(df, df)) + q(q-1)(df, d(\alpha, df)), \end{aligned} \tag{5}$$

*where $(\cdot,\cdot)$ stand for the inner product of tensors induced by $g$.*

*Proof.* Indeed,

$$\begin{aligned} 2f^{2-q}\delta\delta\left(f^q D^-\alpha\right) \;=\;& 2f^2\delta\delta(D^-\alpha) + 2qf(\delta D^-\alpha)(JK) \\ &+2qf\delta\left((D^-\alpha)(JK,\cdot)\right) - 2q(q-1)(D^-\alpha)(K,K). \end{aligned}$$

We consider the decomposition of the tensor $D^-\alpha$ in symmetric and skew-symmetric parts $\Psi$ and $\Phi$, respectively,

$$D^-\alpha = \Psi + \Phi.$$

For any vector field $X$ on $M$ we have

$$\begin{aligned} \delta\left(\Psi(X,.)\right) &= -(\Psi, DX^\flat) + (\delta\Psi)(X), \\ \delta\left(\Phi(X,.)\right) &= (\Phi, DX^\flat) - (\delta\Phi)(X). \end{aligned} \tag{6}$$

Using (6) for $X = JK$ we get

$$\begin{aligned} \delta\left(\Psi(JK,.)\right) &= (\delta\Psi)(JK), \\ \delta\left(\Phi(JK,.)\right) &= -(\delta\Phi)(JK). \end{aligned}$$

Thus,

$$2f^{2-q}\delta\delta\left(f^q D^-\alpha\right) = 2f^2\delta\delta(D^-\alpha) + 4\nu f(\delta\Psi)(JK) - 2q(q-1)(D^-\alpha)(K,K). \tag{7}$$

Using [16, Lemma 1.23.4] and $2\Phi = d\alpha - Jd\alpha$ we have

$$\begin{aligned} (\delta\Psi)(JK) &= -(\delta D^-\alpha, df) + (\delta\Phi)(df^\sharp) \\ &= -\frac{1}{2}(\Delta\alpha, df) + \mathrm{Ric}(\mathrm{grad}_g f, \alpha^\sharp) + \frac{1}{2}(\delta d\alpha, df) - \frac{1}{2}(\delta Jd\alpha, df) \\ &= -\frac{1}{2}(\Delta\alpha, df) + (\Delta df, \alpha) - \left(\delta D^+ df, \alpha\right) + \frac{1}{2}(\delta d\alpha, df) \\ &= -\frac{1}{2}(\Delta\alpha, df) + \frac{1}{2}(\Delta df, \alpha) + \frac{1}{2}(\delta d\alpha, df) \end{aligned} \tag{8}$$

where we have used the identity $(\delta J d\alpha, df) = -(\delta^c d\alpha)(K) = \mathcal{L}_K \delta^c \alpha = 0$ which holds since $K$ is Killing. Furthermore,

$$
\begin{aligned}
2(D^-\alpha)(K,K) &= (D_K\alpha)(K) - (D_{JK}\alpha)(JK) \\
&= -(df, d(\alpha, df)) + (d^c f, d(\alpha, d^c f)) - 2\alpha(D_K K) \\
&= -(df, d(\alpha, df)) + (d^c f, d(\alpha, d^c f)) + (\alpha, d(df, df)) \\
&= -(df, d(\alpha, df)) + (\alpha, d(df, df)),
\end{aligned}
\tag{9}
$$

since $(d^c f, d(\alpha, d^c f)) = \mathcal{L}_K(\alpha, d^c f) = 0$ by the $G$-invariance of $\alpha$. The result follows by substituting (8) and (9) in (7). This completes the proof. $\square$

**Definition 2.** We define the $(K,q,a)$-Lichnerowicz operator

$$\mathbb{L}^g_{(K,q,a)} : C^\infty(M,\mathbb{R})^G \to C^\infty(M,\mathbb{R})^G,$$

with respect to a metric $g$ in $\mathcal{K}^G_\Omega(M,J)$ by

$$\mathbb{L}^g_{(K,q,a)}(\phi) = f^{2-q}_{(K,\omega,a)} \delta\delta\left(f^q_{(K,\omega,a)} D^-(d\phi)\right),$$

for $\phi \in C^\infty(M,\mathbb{R})^G$.

**Proposition 1.** *For any variation $\dot\omega = dd^c\dot\phi$ of $\omega$ in $\mathcal{K}^G_\Omega(M,J)$, the first variation of the $(K,q,a)$–scalar curvature is given by*

$$\boldsymbol{\delta}\mathrm{S}_{(K,q,a)}(\dot\phi) = -2\mathbb{L}^g_{(K,q,a)}(\dot\phi) + \left(d\mathrm{S}_{(K,q,a)}(\omega), d\dot\phi\right). \tag{10}$$

*Proof.* For a variation $\dot\omega = dd^c\dot\phi$ in $\mathcal{K}^G_\Omega(M,J)$, the corresponding variations of $f_{(K,\omega,a)}$, $\Delta_\omega$, $\mathrm{Scal}_\omega$ are given by (see e.g. [16, 5]):

$$
\begin{aligned}
\dot v_\omega &= -(\Delta_\omega\dot\phi)v_\omega \\
\dot f &= (df, d\dot\phi) \\
\dot\Delta_\omega &= (dd^c\dot\phi, dd^c\cdot) \\
\dot{\mathrm{Scal}}_\omega &= -2\mathbb{L}^g(\dot\phi) + (d\,\mathrm{Scal}(\omega), d\dot\phi),
\end{aligned}
\tag{11}
$$

where $\mathbb{L}^g(\dot\phi) = \delta\delta(D^- d\dot\phi)$ is the usual Lichnerowicz operator. Then the first variation of the $(K,q,a)$–scalar curvature is given by:

$$
\begin{aligned}
\boldsymbol{\delta}\mathrm{S}_{(K,q,a)}(\dot\phi) \;=\;& -2f^2\mathbb{L}^g(\dot\phi) + f^2(d\mathrm{Scal}(\omega), d\dot\phi) + \mathrm{Scal}(\omega)(df^2, d\dot\phi) + 2qf\Delta_\omega(df, d\dot\phi) \\
& + 2q(df, d\dot\phi)\Delta_\omega f + 2qf(dd^c f, dd^c\dot\phi) - q(q-1)(df, d(df, d\dot\phi)).
\end{aligned}
$$

By (6) and the $G$-invariance of $\phi$ we have

$$(dd^c\dot\phi, dd^c f) = -\Delta(df, d\dot\phi) + (d\Delta\dot\phi, df).$$

Thus,

$$
\begin{aligned}
\boldsymbol{\delta}\mathrm{S}_{(K,q,a)}(\dot\phi) =& -2f^2\mathbb{L}^g(\dot\phi) + f^2(d\mathrm{Scal}(\omega), d\dot\phi) \\
& + \mathrm{Scal}_\omega(df^2, d\dot\phi) + 2qf(df, d\Delta\dot\phi) \\
& + 2q(df, d\dot\phi)\Delta_\omega f - q(q-1)(df, d(df, d\dot\phi)).
\end{aligned}
\tag{12}
$$

On the other hand we have

$$
\begin{aligned}
(d\mathrm{S}_{(K,q,a)}(\omega), d\dot\phi) =& f^2(d\mathrm{Scal}(\omega), d\dot\phi) + (df^2, d\dot\phi)\mathrm{Scal}(\omega) \\
& + 2q(\Delta f)(df, d\dot\phi) + 2qf(d\Delta f, d\dot\phi) \\
& - q(q-1)(d\dot\phi, d(df, df)).
\end{aligned}
\tag{13}
$$

By taking the difference (12)-(13) we get exactly (5) for $\alpha = d\dot\phi$, which, in turn, is equal to $-2\mathbb{L}^g_{(K,q,a)}(\dot\phi)$. $\square$

Let $\mathfrak{h}_{\mathrm{red}}^K$ denote the centralizer of $K$ in $\mathfrak{h}_{\mathrm{red}}$ (i.e. the space of vector fields $H \in \mathfrak{h}_{\mathrm{red}}$ such that $[H, K] = 0$) and $\mathrm{Aut}_{\mathrm{red}}^K(M, J)$ the closed connected Lie subgroup of $\mathrm{Aut}_{\mathrm{red}}(M, J)$ with Lie algebra $\mathfrak{h}_{\mathrm{red}}^K$.

Let $\omega \in \mathcal{K}_\Omega^G(M, J)$. To each element $\phi \in C_0^\infty(M, \mathbb{R})^G$ we associate a vector field $\hat{\phi}$ on $\mathcal{K}_\Omega^G(M, J)$, equal to $\phi$ at any point of $\mathcal{K}_\Omega^G(M, J)$. We then have $[\hat{\phi}, \hat{\psi}] = 0$ for any $\phi, \psi \in C_0^\infty(M, \mathbb{R})^G$. We will consider the natural action of $\mathrm{Aut}_{\mathrm{red}}^K(M, J)$ on $\mathcal{K}_\Omega^G(M, J)$ defined by:

$$\gamma \cdot \omega = \gamma^\star \omega.$$

Consider the 1-form $\sigma$ on $\mathcal{K}_\Omega^G(M, J)$ given by:

$$\sigma_\omega(\hat{\phi}) = \int_M \mathrm{S}_{(K,q,a)}(\omega) \phi f_{(K,\omega,a)}^{q-2} v_\omega$$

where $\omega \in \mathcal{K}_\Omega^G(M, J)$ and $\phi \in C_0^\infty(M, \mathbb{R})^G$.

**Proposition 2.** *The* 1*-form* $\sigma$ *is* $\mathrm{Aut}_{\mathrm{red}}^K(M, J)-$*invariant and we have the following expression for its first variation,*

$$\begin{aligned} \boldsymbol{\delta}\left(\sigma(\hat{\phi})\right)_\omega(\hat{\psi}) = &-2\int_M (D^- d\phi, D^- d\psi) f_{(K,\omega,a)}^q v_\omega \\ &- \int_M \mathrm{S}_{(K,q,a)}(\omega)(d\psi, d\phi) f_{(K,\omega,a)}^{q-2} v_\omega. \end{aligned} \tag{14}$$

*In particular* $\sigma$ *is closed.*

*Proof.* Since $\mathrm{Aut}_{\mathrm{red}}^K(M, J)$ preserves the complex structure $J$ and $K$, the invariance of $\sigma$ under the action of $\mathrm{Aut}_{\mathrm{red}}^K(M, J)$ follows. Now we will calculate the first variation of the functional $\omega \mapsto \sigma_\omega(\phi)$. By (11) we have for each $\psi \in C_0^\infty(M, \mathbb{R})^G$,

$$\begin{aligned} \boldsymbol{\delta}\left(\sigma(\hat{\phi})\right)_\omega(\hat{\psi}) &= \int_M \boldsymbol{\delta}\mathrm{S}_{(K,q,a)}(\dot{\omega}) \phi f^{q-2} v_\omega + \int_M \mathrm{S}_{(K,q,a)}(\omega) \phi (df^{q-2}, d\psi) v_\omega \\ &\quad - \int_M \mathrm{S}_{(K,q,a)}(\omega) \phi f^{q-2} (\Delta_\omega \psi) v_\omega \\ &= \int_M \boldsymbol{\delta}\mathrm{S}_{(K,q,a)}(\dot{\omega}) \phi f^{q-2} v_\omega - \int_M (d\mathrm{S}_{(K,q,a)}(\omega), d\psi) \phi f^{q-2} v_\omega \\ &\quad - \int_M \mathrm{S}_{(K,q,a)}(\omega)(d\psi, d\phi) f^{q-2} v_\omega. \end{aligned}$$

From (10) and the above formula we readily get (14). Thus,

$$(\mathbf{d}\sigma)_\omega\left(\hat{\phi}, \hat{\psi}\right) = \boldsymbol{\delta}\left(\sigma(\hat{\psi})\right)_\omega(\hat{\phi}) - \boldsymbol{\delta}\left(\sigma(\hat{\phi})\right)_\omega(\hat{\psi}) - \sigma_\omega([\hat{\phi}, \hat{\psi}]) = 0.$$

i.e. $\sigma$ is a closed $1-$form. $\square$

**Remark 2.** One can alternatively elaborate along the lines of [4]. For $\omega \in \mathcal{K}_\Omega^G(M, J)$ and $J \in \mathcal{K}^G(M, \omega)$ fixed, we consider the path of Kähler metrics $\omega_t = \omega + dd^c \phi_t$ with $\phi_t \in C_0^\infty(M, \mathbb{R})^G$, $\phi_0 = 0$ and $\dot{\phi}_t = \phi$. Using the equivariant Moser Lemma (see [4, Lemma 1]) there exists a family of $G$-equivariant diffeomorphisms $\Phi_t \in \mathrm{Diff}_0^G(M)$ such that $\Phi_0 = id_M$ and $\Phi_t \cdot \omega = \omega_t$. Then we have a path $J_t = \Phi_t \cdot J$ in $\mathcal{K}^G(M, \omega)$. Note that if $g_t = \omega_t(., J_t.)$

then $f_{(K,\omega_t,a)} = f_{(K,\omega,a)} \circ \Phi_t$. We have,

$$\begin{aligned}
\boldsymbol{\delta}\left(\sigma(\hat{\psi})\right)_{\omega}(\hat{\phi}) &= \left.\frac{d}{dt}\right|_{t=0} \int_M \mathrm{S}_{(K,q,a)}(\omega_t)\psi f^{q-2}_{(K,\omega_t,a)} v_{\omega_t} \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_M \Phi_t^{\star}\left(\mathrm{S}_{(K,q,a)}(J_t)\left(\psi\circ\Phi_t^{-1}\right) f^{q-2}_{(K,\omega,a)} v_{\omega}\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_M \mathrm{S}_{(K,q,a)}(J_t)\left(\psi\circ\Phi_t^{-1}\right) f^{q-2}_{(K,\omega,a)} v_{\omega} \\
&= \int_M \left(\dot{J}, DJd\psi\right) f^{q}_{(K,\omega,a)} v_{\omega} + \int_M \mathrm{S}_{(K,q,a)}(\omega) d\psi(Z) f^{q-2}_{(K,\omega,a)} v_{\omega},
\end{aligned}$$

where we used [4, Eq(9)] and that $Z$, $\dot{J}$ are given by:

$$\begin{aligned}
Z &= \left.\frac{d}{dt}\right|_{t=0} \Phi_t^{-1}\circ\Phi = -\mathrm{grad}_g\phi, \\
\dot{J} &= \left.\frac{d}{dt}\right|_{t=0} \Phi_t.J = -\mathcal{L}_Z J.
\end{aligned}$$

It follows that,

$$\begin{aligned}
d\psi(Z) &= -(d\psi, d\phi), \\
\left(\dot{J}, DJd\psi\right) &= -2\left(D^{-}d\phi, D^{-}d\psi\right).
\end{aligned}$$

We thus get,

$$\boldsymbol{\delta}\left(\sigma(\hat{\psi})\right)_{\omega}(\hat{\phi}) = -2\int_M \left(D^{-}d\phi, D^{-}d\psi\right) f^{q}_{(K,\omega,a)} v_{\omega} - \int_M \mathrm{S}_{(K,q,a)}(\omega)(d\psi, d\phi) f^{q-2}_{(K,\omega,a)} v_{\omega}.$$

As shown in [4], and as it easily follows from (14), the following expression:

$$c_{(\Omega,K,q,a)} := \frac{\int_M \mathrm{S}_{(K,q,a)}(\omega) f^{q-2}_{(K,\omega,a)} v_{\omega}}{\int_M f^{q-2}_{(K,\omega,a)} v_{\omega}}.$$

is a topological constant (i.e. independent of the choice of $\omega$ in the Kähler class $\Omega$). We consider the following 1-form, on $\mathcal{K}^G_{\Omega}(M,J)$ given by:

$$\tilde{\sigma}_{\omega}(\hat{\phi}) := \int_M \left(\mathrm{S}_{(K,q,a)}(\omega) - c_{(\Omega,K,q,a)}\right)\phi f^{q-2}_{(K,\omega,a)} v_{\omega}.$$

**Lemma 2.** *The* 1*-form* $\tilde{\sigma}$ *is closed.*

*Proof.* We consider the 1-form $\theta$ defined on $\mathcal{K}^G_{\Omega}(M,J)$ by

$$\theta_{\omega}(\hat{\phi}) = \int_M \phi f^{q-2}_{(K,\omega,a)} v_{\omega}.$$

We have using (11),

$$\begin{aligned}
\boldsymbol{\delta}(\theta_{\omega}(\hat{\phi}))(\hat{\psi}) &= \int_M \phi(df^{q-2}, d\psi) v_{\omega} - \int_M \phi f^{q-2}(\Delta_{\omega}\psi) v_{\omega} \\
&= -\int_M (d\psi, d\phi) f^{q-2} v_{\omega}.
\end{aligned}$$

Thus,

$$(\mathbf{d}\theta)_{\omega}\left(\hat{\phi},\hat{\psi}\right) = \boldsymbol{\delta}\left(\theta(\hat{\psi})\right)_{\omega}(\hat{\phi}) - \boldsymbol{\delta}\left(\theta(\hat{\phi})\right)_{\omega}(\hat{\psi}) - \theta_{\omega}([\hat{\phi},\hat{\psi}]) = 0,$$

i.e. $\theta$ is closed. By Proposition 2, $\tilde{\sigma} = \sigma - c_{(\Omega,K,q,a)}\cdot\theta$ is closed. $\square$

Since $\mathcal{K}^G_{\Omega}(M,J)$ is contractible, $\tilde{\sigma}$ is an exact form, so it admits a primitive functional.

**Definition 3.** We define the $(\Omega, K, q, a)$-Mabuchi energy

$$\mathcal{M}_{(\Omega,K,q,a)} : \mathcal{K}^G_\Omega(M,J) \to \mathbb{R}$$

as minus the primitive of the one form $\tilde{\sigma}$, i.e.

$$\tilde{\sigma} = -\mathbf{d}\mathcal{M}_{(\Omega,K,q,a)},$$

normalized by $\mathcal{M}_{(\Omega,K,q,a)}(\omega_0) = 0$ for some base point $\omega_0 \in \mathcal{K}^G_\Omega(M,J)$.

**Remark 3.** By its very definition, the Kähler metrics in $\mathcal{K}^G_\Omega(M,J)$ of constant $(\Omega, K, q, a)$-scalar curvature are critical points of the $(\Omega, K, q, a)-$Mabuchi functional.

3.2. **The** $(\Omega, K, q, a)$**-Futaki invariant.** For $\omega \in \mathcal{K}^G_\Omega(M,J)$ and $H \in \mathfrak{h}^K_{\mathrm{red}}$, we denote by $h'_{(H,\omega)} + \sqrt{-1}h_{(H,\omega)} \in C^\infty_0(M,\mathbb{C})$ the normalized holomorphy potential of $H$, i.e. $h'_{(H,\omega)}$ and $h_{(H,\omega)}$ are the normalized smooth functions such that,

$$H = \mathrm{grad}_g(h'_{(H,\omega)}) + J\mathrm{grad}_g(h_{(H,\omega)}).$$

Using the identification $T_\omega\mathcal{K}^G_\Omega(M,J) \cong C^\infty_0(M,\mathbb{R})^G$, the vector field $JH$ defines a vector field $\widehat{JH}$ on $\mathcal{K}^G_\Omega(M,J)$, given by:

$$\omega \mapsto \mathcal{L}_{JH}\omega = dd^c h_{(H,\omega)},$$

so that $\widehat{JH}_\omega = h_{(H,\omega)}$. By the invariance of $\tilde{\sigma}$ under the $\mathrm{Aut}^K_{\mathrm{red}}(M,J)$-action and Cartan's formula we get,

$$\mathcal{L}_{\widehat{JH}}\tilde{\sigma} = \mathbf{d}\left(\tilde{\sigma}_\omega(\widehat{JH})\right) = 0.$$

Then $\omega \mapsto \tilde{\sigma}(\widehat{JH})$ is constant on $\mathcal{K}^G_\Omega(M,J)$. We will use the following notation,

$$\mathcal{F}_{(\Omega,K,q,a)}(H) := \tilde{\sigma}(\widehat{JH}) = \int_M \left(\mathrm{S}_{(K,q,a)}(\omega) - c_{(\Omega,K,q,a)}\right) h_{(H,\omega)} f^{q-2}_{(K,\omega,a)} v_\omega.$$

**Definition 4.** The linear map $\mathcal{F}_{(\Omega,K,q,a)} : \mathfrak{h}^K_{red} \to \mathbb{R}$ will be called the $(\Omega, K, q, a)$-Futaki invariant associated to the data $(\Omega, K, q, a)$.

**Remark 4.**

(i) Definition 4 is consistant with the one given in [4] for $q = -2m + 1$, but it has the advantage to show that the $(\Omega, K, q, a)-$Futaki invariant extends to the whole of $\mathfrak{h}^K_{\mathrm{red}}$, not just $\mathrm{Lie}(G)$.

(ii) For a Kähler class $\Omega$ which admits $(K, q, a)$-extremal metric $\omega$, $\mathcal{F}_{\Omega,K,q,a} = 0$ if and only if $\omega$ is a $(K, q, a)$-constant scalar curvature Kähler metric. In fact, for $\Xi = J\mathrm{grad}_g\left(\mathrm{S}_{(K,q,a)}(\omega)\right) \in \mathfrak{h}^K_{\mathrm{red}}$ we have

$$\mathcal{F}_{(\Omega,K,q,a)}(\Xi) = \int_M \left(\mathrm{S}_{(K,q,a)}(\omega) - c_{(\Omega,K,q,a)}\right)^2 f^{q-2}_{(K,\omega,a)} v_\omega = 0,$$

thus $\mathrm{S}_{(K,q,a)}(\omega) = c_{(\Omega,K,q,a)}$.

## 4. Proof of Theorem 1

In this section we shall prove Theorem 1 from the introduction. We thus assume that $(g,\omega)$ is an $(K, q, a)$-extremal metric on a compact, connected Kähler manifold $(M,J)$, and $G \subset \mathrm{Aut}_{\mathrm{red}}(M,J)$ the torus generated by the quasi-periodic Killing vector field $K$.

**Lemma 3.** *For any $G$-invariant function $\phi \in C^\infty(M,\mathbb{R})^G$ we have*

$$\mathcal{L}_\Xi\phi = -2f^{2-q}_{(K,\omega,a)}\delta\delta\left(f^q_{(K,\omega,a)}D^-(d^c\phi)\right),$$

*where $\mathcal{L}_\Xi$ denotes the Lie derivative along the vector field $\Xi = J\mathrm{grad}\left(\mathrm{S}_{(K,q,a)}(\omega)\right)$.*

*Proof.* We have,

$$\begin{aligned}\mathcal{L}_{\Xi}\phi &= -(d\mathrm{S}_{(K,q,a)}(\omega), d^c\phi)\\ &= -f^2(d\mathrm{Scal}(\omega), d^c\phi) - 2qf(d\Delta f, d^c\phi) + q(q-1)(d^c\phi, d(df, df)).\end{aligned}$$

By taking $\alpha = d^c\phi$ in (5) we get,

$$\begin{aligned}2f^{2-q}(D^-d)^\star f^q(D^-d^c)\phi &= 2f^2(D^-d)^\star(D^-d^c)\phi + 2qf(d\Delta f, d^c\phi)\\ &\quad -q(q-1)(d^c\phi, d(df, df))\\ &= f^2(d\mathrm{S}_\omega, d^c\phi) + 2qf(d\Delta f, d^c\phi)\\ &\quad -q(q-1)(d^c\phi, d(df, df)),\end{aligned}$$

where we have used (see [16, p.63, Eq.(1.23.15)]),

$$(D^-d)^\star(D^-d^c)\phi = (d\mathrm{S}_\omega, d^c\phi),$$

and the identity,

$$(\delta dd^c\phi, df) = -(\delta^c d^c d\phi)(K) = -\mathcal{L}_K\delta^c d^c\phi = 0.$$

□

For a 1-forme $\alpha$ we denote by $D^{2,0}\alpha$ (resp. $D^{0,2}\alpha$) the $(2,0)$-part (resp. $(0,2)$-part) of the tensor $D\alpha$. We define the $(K,q,a)$-Calabi's operators $\mathbb{L}^{g,\pm}_{(K,q,a)}$ on $C^\infty(M,\mathbb{C})^G$ by

$$\begin{aligned}\mathbb{L}^{g,+}_{(K,q,a)}(F) &= 2f^{2-q}_{(K,\omega,a)}(D^{0,2}d)^\star f^q_{(K,\omega,a)}D^{0,2}dF\\ \mathbb{L}^{g,-}_{(K,q,a)}(F) &= 2f^{2-q}_{(K,\omega,a)}(D^{2,0}d)^\star f^q_{(K,\omega,a)}D^{2,0}dF.\end{aligned}$$

Recall that the space of hamiltonian Killing vector fields is given by (see [16])

$$\mathfrak{k}_{\mathrm{ham}} = \mathfrak{h}_{\mathrm{red}} \cap \mathfrak{k}.$$

The following Proposition is straifhtforword (see [16, Chapter 2]).

**Proposition 3.**

*(i)* *Let $H = \mathrm{grad}_g P + J\mathrm{grad}_g Q$, where $P, Q$ are real valued functions with zero mean, such that $[H,K]=0$. Then $H \in \mathfrak{h}_{\mathrm{red}}$ if and only if $\mathbb{L}^{g,+}_{(K,q,a)}(P+\sqrt{-1}Q) = 0$ and $P,Q \in C^\infty(M,\mathbb{R})^G$ i.e. we have*

$$\mathfrak{h}^K_{\mathrm{red}} \cong \ker(\mathbb{L}^{g,+}_{(K,q,a)}) \cap C^\infty_0(M,\mathbb{C})^G.$$

*(ii)* *For any $F \in C^\infty(M,\mathbb{C})^G$ we have,*

$$\mathbb{L}^{g,\pm}_{(K\nu,a)}(F) = \mathbb{L}^g_{(K,q,a)}(F) \pm \frac{\sqrt{-1}}{2}\mathcal{L}_\Xi F.$$

*(iii)* *Let $X$ be real holomorphic vector field. Then $X \in \mathfrak{k}^K_{\mathrm{ham}}$ if and only if there exists $h \in C^\infty(M,\mathbb{R})^G$ such that $X = J\mathrm{grad}_g h$ and $\mathbb{L}^g_{(K,q,a)}(h) = 0$.*

**Theorem 4.** *Suppose $(M,g,J)$ is a compact $(K,q,a)$-extremal Kähler manifold. Then $\mathfrak{h}^K$ admits the following $\langle\cdot,\cdot\rangle_{(g,q,a)}$-orthogonal decomposition*

$$\mathfrak{h}^K = \mathfrak{h}^K_{(0)} \oplus \left(\bigoplus_{\lambda>0}\mathfrak{h}^K_{(\lambda)}\right), \tag{15}$$

*where $\mathfrak{h}^K_{(0)}$ is the centralizer of $\Xi$ in $\mathfrak{h}^K$ and for $\lambda > 0$, $\mathfrak{h}^K_{(\lambda)}$ denote the subspace of elements $X \in \mathfrak{h}^K$ such that $\mathcal{L}_\Xi X = \lambda JX$.*

*The subspace $\mathfrak{h}^K_{(0)}$ is a reductive complex Lie subalgebra of $\mathfrak{h}^K$; it contains $\mathfrak{a}$, $\mathfrak{k}^K_{\mathrm{ham}}$ and $J\mathfrak{k}^K_{\mathrm{ham}}$ and is given by the $\langle\cdot,\cdot\rangle_{(g,q,a)}$-orthogonal sum of these three spaces:*

$$\mathfrak{h}^K_{(0)} = \mathfrak{a} \oplus \mathfrak{k}^K_{\mathrm{ham}} \oplus J\mathfrak{k}^K_{\mathrm{ham}}. \tag{16}$$

*Moreover, each* $\mathfrak{h}^K_{(\lambda)}$, $\lambda > 0$, *is contained in the ideal* $\mathfrak{h}^K_{\mathrm{red}}$, *so that we also have the following* $\langle\cdot,\cdot\rangle_{(g,q,a)}$*-orthogonal decompositions:*

$$\begin{aligned}\mathfrak{h}^K &= \mathfrak{a} \oplus \mathfrak{h}^K_{\mathrm{red}},\\ \mathfrak{k}^K &= \mathfrak{a} \oplus \mathfrak{k}^K_{\mathrm{ham}}.\end{aligned} \tag{17}$$

*Proof.* Let $X = X_H + \mathrm{grad}_g P + J\mathrm{grad}_g Q \in \mathfrak{h}^K$, where $X_H$ is the dual of the harmonic part of $\xi = X^\flat$ denoted $\xi_H$, and $P, Q \in C^\infty(M,\mathbb{R})$ with zero mean value. Since $K = J\mathrm{grad}_g f$ is Killing we have

$$\mathcal{L}_K P = \mathcal{L}_K Q = 0.$$

By (5) in Lemma 1 we have

$$\begin{aligned} 2f^{2-q}(D^-d)^\star f^q D^-\xi_H &= 2f^2(D^-d)^\star D^-\xi_H\\ &= f^2(d\mathrm{Scal}(\omega),\xi_H) + 2qf(d\Delta f,\xi_H)\\ &\quad -q(q-1)(\xi_H, d(df,df))\\ &= J\mathcal{L}_\Xi \xi_H = 0,\end{aligned}$$

where we have used $(\xi_H, df) = 0$ and the fact that $\Xi$ is a Killing vector field. It follows that

$$0 = f^{2-q}(D^-d)^\star f^q D^-\xi = f^{2-q}(D^-d)^\star f^q D^-(dP + d^cQ) = \mathrm{Re}\left(\mathbb{L}_{(K,q,a)}(P+\sqrt{-1}Q)\right).$$

Starting from $JX$ instead of $X$ we similarly get

$$\mathrm{Im}\left(\mathbb{L}^+_{(K,q,a)}(P+\sqrt{-1}Q)\right) = 0.$$

It follows that $\mathbb{L}^+_{(K,q,a)}(P+\sqrt{-1}Q) = 0$, then by Proposition 3(i) we have that $X_H$ and $\mathrm{grad}_g P + J\mathrm{grad}_g Q$ are real holomorphic vector fields, which proves (17) (for the decomposition of $\mathfrak{k}^K$ we use the fact that $\mathfrak{k}_{\mathrm{ham}} := \mathfrak{k}\cap\mathfrak{h}_{\mathrm{red}}$ and $\mathfrak{k}\cap\mathfrak{a} = \mathfrak{a}$).
Since $\Xi$ is Killing and commutes with $K$, the operators $\mathbb{L}^{g,\pm}_{(K,q,a)}$ commute. Then $\mathbb{L}^{g,-}_{(K,q,a)}$ acts on $\mathfrak{h}^K_{red}$ and by Proposition 3(ii) this action is given by $-\sqrt{-1}\mathcal{L}_\Xi$. Since $\mathbb{L}^-_{(K,q,a)}$ is $\langle\cdot,\cdot\rangle_{(g,q,a)}$-self-adjoint and semi-positive, $\mathfrak{h}^K_{\mathrm{red}}$ splits as

$$\mathfrak{h}^K_{\mathrm{red}} = \mathfrak{h}^K_{\mathrm{red},(0)} \oplus \left(\bigoplus_{\lambda>0}\mathfrak{h}^K_{(\lambda)}\right),$$

where $\mathfrak{h}^K_{\mathrm{red},(0)}$ is the kernel of $\mathcal{L}_\Xi$ in $\mathfrak{h}^K_{\mathrm{red}}$ whereas, for each $\lambda > 0$, $\mathfrak{h}^K_{(\lambda)}$ is the subspace of elements $X \in \mathfrak{h}^K$ such that $\mathcal{L}_\Xi X = \lambda JX$. Using (17) we get (15)(Notice that $\mathfrak{h}^K_{(\lambda)} = \mathfrak{h}^K_{\mathrm{red},(\lambda)}$ since $\Xi$ is Killing and commutes with $K$).
We have $\mathfrak{a}\oplus\mathfrak{k}^K_{\mathrm{ham}}\oplus J\mathfrak{k}^K_{\mathrm{ham}} \subset \mathfrak{h}^K_{(0)}$. By Proposition 3(ii) the restriction of $\mathcal{L}_\Xi$ to $\ker\left(\mathbb{L}^{g,+}_{(K,q,a)}\right)\cap C^\infty_0(M,\mathbb{C})^G$ coincides with the restriction of $\mathbb{L}^g_{(K,q,a)}$ to the same space. Then, using Proposition 3 (iii), we obtain the converse inclusion, which proves (16). $\square$

Now we are in position to give a proof for Theorem 1.

**Proof of Theorem 1.** This is done as in the case where $(G = \{1\}, q = 0, a = 0)$ (see [16, 6]). Let $\mathfrak{s}$ be the Lie algebra of a connected, compact Lie subgroup, $S \subset \mathrm{Aut}^K_0(M,J)$ containing $\mathrm{Isom}^K_0(M,g)$. Suppose, for a contradiction, that there exists $X \in \mathfrak{s}$ that doesn't belong to $\mathfrak{k}^K$. By Theorem 4, (see (15), (17) and (16)) we have the splitting

$$\mathfrak{h}^K = \mathfrak{k}^K \oplus J\mathfrak{k}^K_{\mathrm{ham}} \oplus \left(\bigoplus_{\lambda>0}\mathfrak{h}^K_{(\lambda)}\right),$$

then we can assume that $X \in J\mathfrak{k}^K_{\mathrm{ham}} \oplus \left(\bigoplus_{\lambda>0} \mathfrak{h}^K_{(\lambda)}\right)$. Let $X = X_0 + \sum_{\lambda>0} X_\lambda$ be the corresponding decomposition of $X$, then for any positive integer $r$ we have

$$(\mathcal{L}_\Xi)^{2r} X = -\sum_{\lambda>0} \lambda^{2r} X_\lambda \in \mathfrak{s}.$$

It follows that each component $X_\lambda$ of $X$ is in $\mathfrak{s}$. We can therefore assume that $X \in \mathfrak{s}_\lambda := \mathfrak{s} \cap \mathfrak{h}^K_{(\lambda)}$ or $X \in J\mathfrak{k}^K_{\mathrm{ham}} \subset \mathfrak{s}_0$. Suppose that $X \in \mathfrak{s}_\lambda$ for some $\lambda > 0$. Let $B$ denote the Killing form of $\mathfrak{s}$. Since $S$ is a compact Lie group, $B$ is semi-negative and it's kernel coincides with the center of $\mathfrak{s}$. On the other hand $X$ belongs to the kernel of $B$, indeed for any $Y \in \mathfrak{s}_{\lambda_1}$ and $Z \in \mathfrak{s}_{\lambda_2}$, by Jacobi identity we can easily show that $[X,[Y,Z]] \in \mathfrak{s}_{\lambda+\lambda_1+\lambda_2} \neq \mathfrak{s}_{\lambda_2}$ then $\mathfrak{s}_{\lambda+\lambda_1+\lambda_2} = \{0\}$ and by consequence $[X,[Y,Z]] = 0$. It follows that for any $Y \in \mathfrak{s}$ we have $B(X,Y) = 0$. Hence $X$ belongs to the center of $\mathfrak{s}$, but we have $\Xi \in \mathfrak{k}^K \subset \mathfrak{s}$ and $[X,\Xi] = -\lambda JX \neq 0$, a contradiction.
It follows that $X \in J\mathfrak{k}^K_{\mathrm{ham}}$. Then $X = \mathrm{grad}_g(P)$ for some real function $P$. By the hypothesis the flow $\Phi^X_t$ of $X$ is contained in a compact connected subgroup of $\mathrm{Aut}^K_0(M,J)$. It follows that $X$ is quasi-periodic with a flow closure in $\mathrm{Aut}^K_0(M,J)$ given by a torus $T^k$ of dimension $k \geq 1$. Note that $k \neq 1$ since a gradient vector field does not admit any non-trivial closed integral curve, as $\frac{d}{dt}P\left(\Phi^X_t(x)\right) = |X|^2_{\Phi^X_t(x)} \geq 0$. It follows that $k > 1$. Let $x \in M$ such that $X_x \neq 0$. We have that $P(\Phi^X_t(x))$ is an increasing function of $t$, so that $P(\Phi^X_t(x)) - P(x) > c$, for $t > 1$, where $c > 0$. But by density of $\Phi^X_t$ in the torus $T^k$, $\Phi^X_t$ meets any small neighborhood $U$ of $x$, which is a contradiction. We conclude that $\mathfrak{s} = \mathfrak{k}^K$.

If the $(K,q,a)$-scalar curvature is constant then by Theorem 4, $\mathfrak{h}^K$ splits as

$$\mathfrak{h}^K = \mathfrak{a} \oplus \mathfrak{k}^K_{\mathrm{ham}} \oplus J\mathfrak{k}^K_{\mathrm{ham}},$$

since $\mathfrak{h}^K_{(\lambda)} = \{0\}$. In particular $\mathfrak{h}^K$ is a reductive complex Lie algebra. □

We have the following immediate consequences of Theorem 1.

**Corollary 3.** *Any $(K,q,a)$-extremal metric on a compact Kähler manifold $(M,J)$ belongs to $\mathcal{K}^{\mathbb{T}}_\Omega(M,J)$ for some maximal torus $\mathbb{T}$ of $\mathrm{Aut}_{\mathrm{red}}(M,J)$ such that $K \in \mathrm{Lie}(\mathbb{T})$.*

**Corollary 4.** *Let $g$ and $\tilde{g}$ be two $(K,q,a)-$extremal metrics on $(M,J)$. Then there is $\Phi \in \mathrm{Aut}^K_0(M,J)$ such that $\mathrm{Isom}^K_0(M,g) = \mathrm{Isom}^K_0(M,\Phi^\star\tilde{g})$. Furthermore if $(M,J)$ is a toric manifold and $g$ and $\tilde{g}$ are two $(K,q,a)$-extremal metrics in the same Kähler class $\Omega$, then they are isometric (see* [4]*).*

**Proof of Corollary 1.** This follows from Corollary 3 and [4, Proposition 6] □

**Proof of Corollary 2.** We have the following exact sequence (see [3, Proposition 1.3]):

$$0 \to \mathfrak{h}_B(M) \to \mathfrak{h}(M) \to \mathfrak{h}(B) \to 0$$

where $B = \mathbb{F}_n$ and $\mathfrak{h}_B(M)$ denote the Lie algebra of holomorphic vector fields on $M$ which are tangent to the fibers of $\pi$. The proof of [3, Proposition 1.3] also shows that,

$$0 \to \mathfrak{h}^K_B(M) \to \mathfrak{h}^K(M) \to \mathfrak{h}(B) \to 0$$

where $\mathfrak{h}^K_B(M) = \mathrm{span}_{\mathbb{C}}\{K, JK\}$ is the abelian sub-algebra generated by the vector fields $K$, $JK$. If $M$ admits a Kähler metric of constant $(bK,q,a)$-scalar curvature, then $\mathfrak{h}^K(M)$ must be reductive by Theorem 1. As $\mathfrak{h}^K_B(M)$ is in the center of $\mathfrak{h}^K(M)$, it would follow that $\mathfrak{h}(B)$ is reductive, which is not the case for $B = \mathbb{F}_n$ (see e.g. [5]). It follows that $M$ admits no Kähler metric of constant $(bK,q,a)$-scalar curvature. □

## 5. The $(K,q,a)$-extremal Kähler metrics relatively to a maximal torus $\mathbb{T}$, and $(\mathbb{T},K,q,a)$-extremal vector field

Using Corollaries 3 and 4, we assume from now on that $\mathbb{T}$ is a fixed maximal torus in $\mathrm{Aut}_{\mathrm{red}}(M,J)$ and $K\in\mathrm{Lie}(\mathbb{T})$. We denote by $\Pi_g^{\mathbb{T}}$ the orthogonal projection with respect to the $L^2-$scalar product

$$\langle\phi,\psi\rangle_{(g,q,a)}:=\int_M \phi\psi f_{(K,\omega,a)}^{q-2}v_\omega$$

defined on the Hilbert space $L^2_{\mathbb{T}}(M,\mathbb{R})$ onto the space $P_g^{\mathbb{T}}(M,\mathbb{R})$ of Killing potentials of the elements of $\mathrm{Lie}(\mathbb{T})$ relatively to $g$ which is isomorphic to $\mathbb{R}\oplus\mathrm{Lie}(\mathbb{T})$. Then we have the following decomposition of the $(K,q,a)$-scalar curvature,

$$\mathrm{S}_{(K,q,a)}(\omega)=\mathrm{S}^{\mathbb{T}}_{(K,q,a)}(\omega)+\Pi_g^{\mathbb{T}}\left(\mathrm{S}_{(K,q,a)}(\omega)\right),$$

**Definition 5.** We call $\mathrm{S}^{\mathbb{T}}_{(K,q,a)}(\omega)$ the reduced $(K,q,a)$-scalar curvature with respect to $\mathbb{T}$. We say that $\omega\in\mathcal{K}_\Omega^{\mathbb{T}}(M,J)$ is $(K,q,a)$-extremal relatively $\mathbb{T}$ if $\mathrm{S}^{\mathbb{T}}_{(K,q,a)}(\omega)$ is identically zero.

**Remark 5.** Notice that by Corollary 3, any $(K,q,a)$-extremal metric is extremal relatively to the maximal torus of $\mathrm{Aut}_{\mathrm{red}}(M,J)$ containing $K$.

Following [16, Proposition 4.11.1] we have,

**Definition 6.** For $X,Y\in\mathfrak{h}_{\mathrm{red}}$ with normalized complex potentials $F_\omega^X,F_\omega^Y$ we define the $(\Omega,K,q,a)$-Futaki-Mabuchi bilinear by the following expression

$$\mathcal{B}_{(\Omega,K,q,a)}(X,Y):=\int_M F_\omega^X F_\omega^Y f_{(K,\omega,a)}^{q-2}v_\omega$$

which is independent from the choice of $\omega\in\mathcal{K}_\Omega^{\mathbb{T}}(M,J)$.

We denote by $Z_\omega^{\mathbb{T}}(K,q,a)$ the vector field given by

$$Z_\omega^{\mathbb{T}}(K,q,a):=J\mathrm{grad}_g\left(\Pi_g^{\mathbb{T}}\left(\mathrm{S}_{(K,q,a)}(\omega)\right)\right).$$

Then for all $H\in\mathrm{Lie}(\mathbb{T})$, we have

$$\mathcal{F}_{(\Omega,K,q,a)}(H)=-\mathcal{B}_{(\Omega,K,q,a)}\left(H,Z_\omega^{\mathbb{T}}(K,q,a)\right). \tag{18}$$

From its very definition, the restriction of $\mathcal{B}_{(\Omega,K,q,a)}$ to $\mathrm{Lie}(\mathbb{T})$ is negative definite. Then $Z_\omega^{\mathbb{T}}(K,q,a)$ is well-defined by the above expression, so it is an element of $\mathrm{Lie}(\mathbb{T})$, independent of the choice of $\omega\in\mathcal{K}_\Omega^{\mathbb{T}}(M,J)$.

**Definition 7.** We call $Z^{\mathbb{T}}(\Omega,K,q,a)\in\mathrm{Lie}(\mathbb{T})$ the $(\Omega,K,q,a)$-extremal vector field.

Now we consider the 1-form $\zeta^{\mathbb{T}}$ defined on $\mathcal{K}_\Omega^{\mathbb{T}}(M,J)$ by,

$$\zeta_\omega^{\mathbb{T}}(\dot{\phi})=\int_M \Pi_g^{\mathbb{T}}\left(\mathrm{S}_{(K,q,a)}(\omega)\right)\phi f_{(K,\omega,a)}^{q-2}v_\omega.$$

**Lemma 4.** *The* 1*-form* $\zeta^{\mathbb{T}}$ *is closed.*

*Proof.* To simplify notations we denote $z(\omega):=\Pi_g^{\mathbb{T}}\left(\mathrm{S}_{(K,q,a)}(\omega)\right)$. Then $Z^{\mathbb{T}}(\Omega,K,q,a)=J\mathrm{grad}_g z(\omega)$. For a variation $\dot{\omega}=dd^c\phi$ in $\mathcal{K}_\Omega^{\mathbb{T}}(M,J)$, using (11) we have

$$\dot{z}(\phi)=(d\phi,dz(\omega))_\omega$$

and therefore,

$$\begin{aligned}\boldsymbol{\delta}\left(\zeta^{\mathbb{T}}(\hat{\psi})\right)_{\omega}(\hat{\phi}) &= \int_M (d\phi, dz(\omega))_{\omega}\psi f^{q-2}v_{\omega} + \int_M z(\omega)\psi(d\phi, df^{q-2})v_{\omega} \\ &\quad - \int_M z(\omega)\psi\Delta_{\omega}\phi f^{q-2}v_{\omega} \\ &= -\int_M z(\omega)(d\phi, d\psi)f^{q-2}v_{\omega}.\end{aligned}$$

It follows that,

$$\left(\mathbf{d}\zeta^{\mathbb{T}}\right)_{\omega}(\hat{\phi},\hat{\psi}) = \boldsymbol{\delta}\left(\zeta^{\mathbb{T}}(\hat{\psi})\right)_{\omega}(\hat{\phi}) - \boldsymbol{\delta}\left(\zeta^{\mathbb{T}}(\hat{\phi})\right)_{\omega}(\hat{\psi}) = 0.$$

□

Now we consider the 1-form $\sigma^{\mathbb{T}}$ on $\mathcal{K}^{\mathbb{T}}_{\Omega}(M,J)$ given by

$$\sigma^{\mathbb{T}} := \sigma - \zeta^{\mathbb{T}}$$

which is a closed 1-form by virtue of Proposition 2 and Lemma 4.

**Definition 8.** The relative Mabuchi energy $\mathcal{M}^{\mathbb{T}}_{(\Omega,K,q,a)}$ is defined by

$$\sigma^{\mathbb{T}} = -\mathbf{d}\mathcal{M}^{\mathbb{T}}_{(\Omega,K,q,a)},$$

where the primitive $\mathcal{M}^{\mathbb{T}}_{(\Omega,K,q,a)}$ is normalized by requiring $\mathcal{M}^{\mathbb{T}}_{(\Omega,K,q,a)}(\omega_0) = 0$ for some base point $\omega_0 \in \mathcal{K}^{\mathbb{T}}_{\Omega}(M,J)$.

**Remark 6.** By its very definition, the critical points of the relative Mabuchi energy are the $\mathbb{T}$–invariant $(K,q,a)$–extremal metrics.

## 6. Proof of Theorem 2

Let $(M,J)$ be a compact Kähler manifold. We fix $K \in \mathfrak{h}_{\mathrm{red}}$, and $q \in \mathbb{R}$. Suppose that $(g,\omega)$ is a $(K,q,a)$-extremal Kähler metric on $M$ with $\Omega = [\omega]$. Without loss of generality, by Corollary 3, we can assume that $(g,\omega)$ is invariant under the action of a maximal torus $\mathbb{T} \subseteq \mathrm{Aut}_{\mathrm{red}}(M,J)$. Let $\alpha$ be $\mathbb{T}-$invariant $g$-harmonic $(1,1)$-form. We take $(\omega,\alpha) = 0$ to avoid trivial deformations of the form $\alpha = \lambda\omega$. We denote by

$$\omega_{t,\phi} := \omega + t\alpha + dd^c\phi,$$

a $\mathbb{T}-$invariant deformations of $\omega$ for $t \in \mathbb{R}$ and $\phi \in C^{\infty}(M,\mathbb{R})^{\mathbb{T}}$. We consider the following map,

$$\mathrm{S} : \mathbb{R}^3 \times C^{\infty}(M,\mathbb{R})^{\mathbb{T}} \to C^{\infty}(M,\mathbb{R})^{\mathbb{T}}$$

defined by,

$$\mathrm{S}(s,t,u,\phi) := \mathrm{S}_{(K+uH,q,a+s)}(\omega_{t,\phi}),$$

so that $\mathrm{S}(0) = \mathrm{S}_{(K,q,a)}(\omega) := \mathrm{S}$. We denote $f_{(s,t,u,\phi)} = f_{(K+uH,\omega_{t,\phi},a+s)} > 0$ the hamiltonian function of $K+uH$ with respect to $\omega_{t,\phi}$, with normalization constant $a+s$, so that $f_0 = f_{(K,\omega,a)} := f$. We take $k > n$ such that the Sobolev space $L^2_k(M,\mathbb{R})^{\mathbb{T}}$ form an algebra for the usual multiplication of functions, embadded in $C^4(M,\mathbb{R})^{\mathbb{T}}$. Then S defines a map

$$\mathrm{S} : \mathbb{R}^3 \times L^2_{k+4}(M,\mathbb{R})^{\mathbb{T}} \to L^2_k(M,\mathbb{R})^{\mathbb{T}},$$

and we have:

**Lemma 5.** *The map* S *is* $C^1$ *with Fréchet derivative in* 0 *given by*

$$\mathrm{D}_0\mathrm{S} = \begin{pmatrix} A & B & C & D \end{pmatrix}$$

*with,*

$$\begin{aligned} A =& 2f\mathrm{Scal}(\omega) + 2q\Delta(f),\\ B =& -2\left(\rho_{(K,q,a)}(\omega), \alpha\right) + 2\lambda f\mathrm{S} + 2q\Delta(f) + 2qf\Delta\lambda - 2q(q-1)(d\lambda, df),\\ C =& 2f_{(H,\omega)}f\mathrm{Scal}(\omega) - 2q(q-1)(K,H) + 2q\left[f_{(H,\omega)}\Delta(f) + f\Delta(f_{(H,\omega)})\right],\\ D(\dot{\phi}) =& -2\mathbb{L}^g_{(K,q,a)}(\phi) + \left(d\mathrm{S}, d\dot{\phi}\right), \end{aligned}$$

*where* $(.,.)$, grad, $\Delta$, *the green operator* $\mathbb{G}$ *are calculated with respect to* $\omega$, *and* $\lambda := -\mathbb{G}(\alpha, dd^c f)$.

*Proof.* The expressions of $A$, $C$ and $D$ are straightforward. For the partial derivative with respect to $t$ we have $\mathrm{S}_{(K,q,a)}(\omega) = 2\Lambda_\omega\left(\rho_{(K,q,a)}(\omega)\right)$ where (see [2])

$$\rho_{(K,q,a)}(\omega) = f^2\rho(\omega) - qf dd^c f - \frac{1}{2}q(q-1)df \wedge d^c f,$$

with $\rho(\omega)$ is the Ricci form of $(g,\omega)$. By taking $X = -JK$ in [16, Lemma 5.2.4] we get,

$$\left.\frac{\partial}{\partial t}\right|_0 f_{(s,t,u,\phi)} = -\mathbb{G}\left(\delta(\alpha(K,))\right) = -\mathbb{G}\left(\alpha, dd^c f_{(K,\omega,a)}\right) = \lambda,$$

and we have $\left.\frac{\partial}{\partial t}\right|_0 \rho(\omega_{t,\phi}) = 0$ since we assumed $(\omega,\alpha) = 0$. Thus,

$$\begin{aligned} B &= 2\left(\left.\frac{\partial}{\partial t}\right|_0 \Lambda_{\omega_{t,\phi}}\right)\rho_{(K,q,a)}(\omega) + 2\Lambda_\omega\left(\left.\frac{\partial}{\partial t}\right|_0 \rho_{(K,q,a+s)}(\omega_{t,\phi})\right)\\ &= -2\left(\rho_{(K,q,a)}(\omega), \alpha\right) + 2\lambda f\mathrm{S} + 2q\Delta f + 2qf\Delta\lambda - 2q(q-1)(d\lambda, df). \end{aligned}$$

□

We consider the following maps,

$$\begin{aligned} \mathcal{F}(s,t,u) &:= \mathcal{F}_{([\omega+t\alpha],K+uH,q,a+s)},\\ \mathcal{B}(s,t,u) &:= \mathcal{B}_{([\omega+t\alpha],K+uH,q,a+s)},\\ Z(s,t,u) &:= Z^{\mathbb{T}}([\omega+t\alpha],K+uH,q,a+s). \end{aligned}$$

**Lemma 6.** *The* $t$*-derivative of the character* $\mathcal{F}(s,t,u)$ *and the bilinear for* $\mathcal{B}(s,t,u)$ *in the point* $(s,t,u) = 0$ *is given by*

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_0 \mathcal{F}(s,t,u)(X) =& \left\langle h_{(X,\omega)}, B\right\rangle_{(g,q,a)} + \left\langle h_{(X,\omega)}, (q-2)\lambda f\mathrm{S}\right\rangle_{(g,q,a)}\\ &- \left\langle h_{(X,\omega)}, f^{2-q}.\left(\alpha, dd^c\mathbb{G}\left(f^{q-2}\mathrm{S}\right)\right)\right\rangle_{(g,q,a)}, \end{aligned} \tag{19}$$

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_0 \mathcal{B}(s,t,u)(X,Y) =& \left\langle \alpha, f^{2-q}h_{(X,\omega)}dd^c\mathbb{G}\left(h_{(Y,\omega)}f^{q-2}\right)\right\rangle_{(g,q,a)}\\ &+ \left\langle \alpha, f^{2-q}\mathbb{G}\left(h_{(X,\omega)}f^{q-2}\right)dd^c h_{(Y,\omega)}\right\rangle_{(g,q,a)}\\ &+ (q-2)\left\langle \alpha, f^{2-q}\mathbb{G}\left(h_{(X,\omega)}h_{(Y,\omega)}f^{q-3}\right)dd^c f\right\rangle_{(g,q,a)}, \end{aligned} \tag{20}$$

*for any* $X = J\mathrm{grad}_g(h_{(X,\omega)})$ *and* $Y = J\mathrm{grad}_g(h_{(Y,\omega)})$ *in* $\mathrm{Lie}(\mathbb{T})$ *with* $h_{X,\omega}$, $h_{Y,\omega}$ *are the normalized real potential of* $-JX$, $-JY$ *respectively.*

*Proof.* For the derivative of $\mathcal{F}(s,t,u)$ we have,

$$\mathcal{F}(s,t,u)(X) = \int_M \mathrm{S}(s,t,u) h_{(X,\omega+t\alpha)} f^{q-2}_{(s,t,u)} v_{\omega+t\alpha},$$

then,

$$\left.\frac{\partial}{\partial t}\right|_0 \mathcal{F}(s,t,u)(X) = \int_M B h_{(X,\omega)} f^{q-2} v_\omega - (q-2)\int_M \mathrm{S}\lambda f^{q-3} h_{(X,\omega)} v_\omega \\ - \int_M \mathrm{S} f^{q-2}\mathbb{G}\left(\delta(\alpha(X,\cdot))\right) v_\omega.$$

On the other hand,

$$\begin{aligned}\int_M \mathrm{S} f^{q-2}\mathbb{G}\left(\delta(\alpha(X,.))\right) v_\omega &= \int_M \left(\alpha(X,.), d\mathbb{G}\left(f^{q-2}\mathrm{S}\right)\right) v_\omega \\ &= \int_M \left(\alpha, X^\flat \wedge d\mathbb{G}\left(f^{q-2}\mathrm{S}\right)\right) v_\omega \\ &= \int_M \left(\alpha, d^c h \wedge d\mathbb{G}\left(f^{q-2}\mathrm{S}\right)\right) v_\omega \\ &= \int_M h.\left(\alpha, dd^c\mathbb{G}\left(f^{q-2}\mathrm{S}\right)\right) v_\omega \\ &= \left\langle h_{(X,\omega)}, f^{2-q}\left(\alpha, dd^c\mathbb{G}\left(f^{q-2}\mathrm{S}\right)\right)\right\rangle_{(g,q,a)}.\end{aligned}$$

which gives the expression (19) for the $t$-derivative of $\mathcal{F}(s,t,u)$.

Now we calculate the $t$-derivative of $\mathcal{B}(s,t,u)$. We have

$$\mathcal{B}(s,t,u)(X,Y) = -\int_M h_{(X,\omega+t\alpha)} h_{(Y,\omega+t\alpha)} f^{q-2}_{(s,t,u)} v_{\omega+t\alpha}.$$

Then

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_0 \mathcal{B}(s,t,u)(X,Y) &= \int_M \mathbb{G}\left(\delta(\alpha(X,\cdot))\right) h_Y f^{q-2} v_\omega \\ &\quad + \int_M \mathbb{G}\left(\delta(\alpha(Y,\cdot))\right) h_X f^{q-2} v_\omega \\ &\quad + (q-2)\int_M \mathbb{G}\left(\delta(\alpha(K,\cdot))\right) h_X h_Y f^{q-3} v_\omega.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_M \mathbb{G}\left(\delta(\alpha(X,.))\right) h_Y f^{q-2} v_\omega &= \int_M \left(\alpha(X,\cdot), d\mathbb{G}\left(h_Y f^{q-2}\right)\right) v_\omega \\ &= \int_M \left(\alpha, d^c h_X \wedge d\mathbb{G}\left(h_Y f^{q-2}\right)\right) v_\omega \\ &= \left\langle \alpha, f^{2-q} h_X dd^c\mathbb{G}\left(h_Y f^{q-2}\right)\right\rangle_{(g,q,a)},\end{aligned}$$

$$\begin{aligned}\int_M \mathbb{G}\left(\delta(\alpha(Y,.))\right) h_X f^{q-2} v_\omega &= \int_M \left(\alpha(Y,\cdot), d\mathbb{G}\left(h_X f^{q-2}\right)\right) v_\omega \\ &= \int_M \left(\alpha, d^c h_Y \wedge d\mathbb{G}\left(h_X f^{q-2}\right)\right) v_\omega \\ &= \int_M \left(\alpha, \mathbb{G}\left(h_X f^{q-2}\right) dd^c h_Y\right) v_\omega \\ &= \left\langle \alpha, f^{2-q}\mathbb{G}\left(h_X f^{q-2}\right) dd^c h_Y\right\rangle_{(g,q,a)},\end{aligned}$$

and,

$$\int_M \mathbb{G}\left(\delta(\alpha(K,.))\right) h_X h_Y f^{q-3} v_\omega = \left\langle \alpha, f^{2-q}\mathbb{G}\left(h_X h_Y f^{q-3}\right) dd^c f \right\rangle_{(g,q,a)}.$$

Which proves (20).

□

In the following lemma we give the $s$ and $u$-derivatives of $\mathcal{F}(s,t,u)$ and $\mathcal{B}(s,t,u)$ in $(s,t,u)=(0,0,0)$. We omit the proof since it follows from straightforward calculations.

**Lemma 7.**

*(i) The $s$-derivative of $\mathcal{F}(s,t,u)$ is given by*

$$\left.\frac{\partial}{\partial s}\right|_0 \mathcal{F}(s,t,u)(X) = \left\langle q f^{-1}\mathrm{S}_{q-1}, h_{(X,\omega)}\right\rangle_{(g,q,a)}. \tag{21}$$

*where $\mathrm{S}_{q-1} := \mathrm{S}_{(K,q-1,a)}(\omega)$.*

*(ii) The $u$-derivative of $\mathcal{F}(s,t,u)$*

$$\left.\frac{\partial}{\partial u}\right|_0 \mathcal{F}(s,t,u)(X) = \left\langle C + (q-2)f^{-1}f_{(H,\omega)}\mathrm{S}, h_{(X,\omega)}\right\rangle_{(g,q,a)}. \tag{22}$$

*(iii) The $s$-derivative of $\mathcal{B}(s,t,u)$ is given by*

$$\left.\frac{\partial}{\partial s}\right|_0 \mathcal{B}(s,t,u) = (q-2)\mathcal{B}_{(\Omega,K,q-1,a)}. \tag{23}$$

*(iv) The $u$-derivative of $\mathcal{B}(s,t,u)$ is given by*

$$\left.\frac{\partial}{\partial u}\right|_0 \mathcal{B}(s,t,u)(X,Y) = (q-2)\int_M h_{(X,\omega)}h_{(Y,\omega)}f_{(H,\omega)}f^{q-3}v_\omega \tag{24}$$

*for any $X = J\mathrm{grad}_g(h_{(X,\omega)})$ and $Y = J\mathrm{grad}_g(h_{(Y,\omega)})$ in $\mathrm{Lie}(\mathbb{T})$.*

**Lemma 8.** *Let $\omega$ be a $(K,q,a)$-extremal metric, we have*

*(i) The $t$-derivative of $Z(s,t,u)$ is given by*

$$\left.\frac{\partial}{\partial t}\right|_0 Z(s,t,u) = J\mathrm{grad}_g\left(\Pi_g^{\mathbb{T}}\left[B + \mathbb{G}\left(\alpha, dd^c\mathrm{S}\right)\right]\right). \tag{25}$$

*(ii) The $s$-derivative of $Z(s,t,u)$ is given by*

$$\left.\frac{\partial}{\partial s}\right|_0 Z(s,t,u) = J\mathrm{grad}_g\left(\Pi_g^{\mathbb{T}}\left[f^{-1}\left(q\mathrm{S}_{q-1}+\mathrm{S}\right)\right]\right). \tag{26}$$

*(iii)*

$$\left.\frac{\partial}{\partial u}\right|_0 Z(s,t,u) = J\mathrm{grad}_g\left(\Pi_g^{\mathbb{T}}\left[C + 2(q-2)f^{-1}f_{(H,\omega)}\mathrm{S}\right]\right). \tag{27}$$

*Proof.*

(i) We have $\left.\frac{\partial}{\partial t}\right|_0 Z(s,t,u) = J\mathrm{grad}_g(P_g)$ for some function $P_g \in P_g^{\mathbb{T}}(M,\mathbb{R})$, since $Z(s,t,u)\in \mathrm{Lie}(\mathbb{T})$ for all $(s,t,u)$. By (18), for all $X\in\mathrm{Lie}(\mathbb{T})$ we have,

$$\mathcal{B}(s,t,u)(Z(s,t,u),X) = -\mathcal{F}(s,t,u)(X) \tag{28}$$

then

$$\begin{aligned}\mathcal{B}(0)\left(\left.\frac{\partial}{\partial t}\right|_0 Z(s,t,u),X\right) &= -\langle P_g, h_{(X,\omega)}\rangle_{(g,q,a)}\\ &= -\left.\frac{\partial}{\partial t}\right|_0 \mathcal{F}(s,t,u)(X) - \left(\left.\frac{\partial}{\partial t}\right|_0 \mathcal{B}(s,t,u)\right)(Z(0),X).\end{aligned}$$

Using (19), (20) and the fact that $\omega$ is $(K,q,a)-$extremal we get,

$$P_g = \Pi_g^{\mathbb{T}}\left(B + \mathbb{G}\left(\alpha, dd^c\mathrm{S}\right)\right).$$

(ii) We have $\frac{\partial}{\partial s}\big|_0 Z(s,t,u) = J\mathrm{grad}_g(Q_g)$ for some function $Q_g \in P_g^{\mathbb{T}}(M,\mathbb{R})$, since $Z(s,t,u) \in \mathrm{Lie}(\mathbb{T})$ for all $(s,t,u)$. Taking the derivative of (28) with respect to $s$ we get,

$$\begin{aligned}
-\mathcal{B}(0)\left(\frac{\partial}{\partial s}\bigg|_0 Z(s,t,u), X\right) &= \langle Q_g, h_{(X,\omega)}\rangle_{(g,q,a)} \\
&= \frac{\partial}{\partial s}\bigg|_0 \mathcal{F}(s,t,u)(X) + \left(\frac{\partial}{\partial s}\bigg|_0 \mathcal{B}(s,t,u)\right)(Z(0),X) \\
&= \left\langle f^{-1}\left(q\mathrm{S}_{(q-1} + \mathrm{S}\right), h_{(X,\omega)}\right\rangle_{(g,q,a)}
\end{aligned}$$

where we used the fact that $\omega$ is $(K,q,a)$-extremal and (21), (23). Thus

$$Q_g = f^{-1}\left(q\mathrm{S}_{(K,q-1,a)}(\omega) + \mathrm{S}_{(K,q,a)}(\omega)\right)$$

which proves the result.

(iii) This is done similarly to (25) and (26) by using (28), (22) and (24).

□

We denote by $\Pi^{\mathbb{T}}_{(s,t,u,\phi)}$ the orthogonal projection on $\mathcal{P}^{\mathbb{T}}_{g_{t,\phi}}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{g_{t,\phi},q,a+s}$.

**Lemma 9.** *For a $(K,q,a)$-extremal metric $\omega$ we have,*

$$\frac{\partial}{\partial t}\bigg|_0 \Pi^{\mathbb{T}}_{(s,t,u,\phi)}\mathrm{S}(s,t,u,\phi) = \Pi_g^{\mathbb{T}}B + \left(\Pi_g^{\mathbb{T}} - Id\right)\left(\mathbb{G}(\alpha, dd^c\mathrm{S}\right). \tag{29}$$

$$\frac{\partial}{\partial s}\bigg|_0 \Pi^{\mathbb{T}}_{(s,t,u,\phi)}\mathrm{S}(s,t,u,\phi) = \Pi_g^{\mathbb{T}}\left[f^{-1}\left(q\mathrm{S}_{q-1} + \mathrm{S}\right)\right]. \tag{30}$$

$$\frac{\partial}{\partial u}\bigg|_0 \Pi^{\mathbb{T}}_{(s,t,u,\phi)}\mathrm{S}(s,t,u,\phi) = \Pi_g^{\mathbb{T}}\left[C + 2(q-2)f^{-1}f_{(H,\omega)}\mathrm{S}\right]. \tag{31}$$

*Proof.* We have $Z(s,t,u) = J\mathrm{grad}_{g_{t,\phi}}(\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\mathrm{S}(s,t,u,\phi))$ then

$$J\mathrm{grad}_g\left(\frac{\partial}{\partial t}\bigg|_0 (\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\mathrm{S}(s,t,u,\phi)\right) = \frac{\partial}{\partial t}\bigg|_0 Z(s,t,u) - J\left(\frac{\partial}{\partial t}\bigg|_0 \mathrm{grad}_{g_{t,\phi}}\right)(\Pi_g^{\mathbb{T}}\mathrm{S}).$$

On the other hand we have,

$$\left(\frac{\partial}{\partial t}\bigg|_0 \mathrm{grad}_{g_{t,\phi}}\right)(\Pi_g^{\mathbb{T}}\mathrm{S}) = (\alpha(Z(0),.))^{\sharp} = \mathrm{grad}_g\left(\mathbb{G}\left(\alpha, dd^c\mathrm{S}\right)\right).$$

By (25) it follows that

$$\frac{\partial}{\partial t}\bigg|_0 \Pi^{\mathbb{T}}_{(s,t,u,\phi)}\mathrm{S}(s,t,u,\phi) = \Pi_g^{\mathbb{T}}B + \left(\Pi_g^{\mathbb{T}} - I\right)\left(\mathbb{G}(\alpha, dd^c\mathrm{S})\right) + c.$$

By differentiating in $t=0$ the equality,

$$\int_M (\Pi^{\mathbb{T}}_{g_t}\mathrm{S}(\omega_t))f^{q-2}_{(K,\omega_t,a)} = \int_M \mathrm{S}(\omega_t)f^{q-2}_{(K,\omega_t,a)}$$

we get $c=0$, which proofs (29). Similarly we can show (30) and (31). □

Following LeBrun-Simanca's arguments [21] we give a proof of Theorem 2.

*Proof.* Let $L^2_k(M,\mathbb{R})^{\mathbb{T},\perp}$ be the orthogonal complement of $P^{\mathbb{T}}_g(M,\mathbb{R})$ with respect to $\langle\cdot,\cdot\rangle_{(g,q,a)}$ in $L^2_k(M,\mathbb{R})^{\mathbb{T}}$. For $t\in(-\epsilon,\epsilon)$ and $\phi\in U$ where $U$ is a small neighborhood of the origin in $L^2_{k+4}(M,\mathbb{R})^{\mathbb{T}}$. As in [21] by taking a smaller open set $U$ and smaller $\epsilon$ we may assume that,

$$\ker\left(Id-\Pi^{\mathbb{T}}_g\right)\circ\left(Id-\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\right)=\ker\left(Id-\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\right).$$

Now we consider the LeBrun-Simanca map

$$\Psi:(-\epsilon,\epsilon)^3\times U\to(-\epsilon,\epsilon)^3\times L^2_k(M,\mathbb{R})^{\mathbb{T},\perp}$$

defined by

$$\Psi(s,t,u,\phi):=\left(s,t,\left(Id-\Pi^{\mathbb{T}}_g\right)\circ\left(Id-\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\right)\mathrm{S}(s,t,u,\phi)\right).$$

Note that $\Psi(0)=0$ and if $\Psi(s,t,u,\phi)=(s,t,u,0)$ then $\omega_{t,\phi}$ is $(K+uH,q,a+s)$-extremal.

The map $\Psi$ is $C^1$ and its Fréchet derivative at the origin is given by:

$$\mathrm{D}_0\Psi \;=\; \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&Id-\Pi^{\mathbb{T}}_g\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\A&B+\mathbb{G}(\alpha,dd^c\mathrm{S})&C&-2\mathbb{L}^g_{(K,q,a)}\end{pmatrix}$$

where $A$, $B$, and $C$ are given in Lemma 5. Indeed, by Lemma 9 we have,

$$\begin{aligned}\left.\frac{\partial}{\partial\phi}\right|_0\left(Id-\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\right)\mathrm{S}(s,t,u,\phi).\dot\phi &= D(\dot\phi)-\left.\frac{\partial}{\partial\phi}\right|_0\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\mathrm{S}(s,t,u,\phi)\\ &= D(\dot\phi)-(d\mathrm{S},d\dot\phi)\\ &= -2f^{2-q}(D^-d)^\star f^q(D^-d)\dot\phi.\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_0\left(Id-\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\right)\mathrm{S}(s,t,u,\phi) &= B-\left.\frac{\partial}{\partial t}\right|_0\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\mathrm{S}(s,t,u,\phi)\\ &= B-\Pi^{\mathbb{T}}_gB+\left(Id-\Pi^{\mathbb{T}}_g\right)(\mathbb{G}(\alpha,dd^c\mathrm{S}))\\ &= \left(Id-\Pi^{\mathbb{T}}_{(g,a)}\right)(B+\mathbb{G}(\alpha,dd^c\mathrm{S})).\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial}{\partial s}\right|_0\left(Id-\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\right)\mathrm{S}(s,t,u,\phi) &= A-\left.\frac{\partial}{\partial s}\right|_0\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\mathrm{S}(s,t,u,\phi)\\ &= A-\Pi^{\mathbb{T}}_g\left[f^{-1}\left(q\mathrm{S}_{q-1}+\mathrm{S}\right)\right].\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial}{\partial u}\right|_0\left(Id-\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\right)\mathrm{S}(s,t,u,\phi) &= C-\left.\frac{\partial}{\partial s}\right|_0\Pi^{\mathbb{T}}_{(s,t,u,\phi)}\mathrm{S}(s,t,u,\phi)\\ &= \left(Id-\Pi^{\mathbb{T}}_g\right)C-\Pi^{\mathbb{T}}_g\left[2(q-2)f^{-1}f_{(H,\omega)}\mathrm{S}\right].\end{aligned}$$

The operator $\mathbb{L}^g_{(K,q,a)}$ is a formally $\langle\cdot,\cdot\rangle_{(g,q,a)}$-self-adjoint, $\mathbb{T}-$invariant, elliptic fourth-order differential operator and extends to a continuous linear operator,

$$\mathbb{L}^g_{(K,q,a)}:L^2_{k+4}(M,\mathbb{R})^{\mathbb{T},\perp}\to L^2_k(M,\mathbb{R})^{\mathbb{T},\perp}$$

which is an isomorphism (since $\mathbb{T}$ is a maximal torus of $\mathrm{Aut}_{\mathrm{red}}(M,J)$). Thus

$$D\Psi_0:\mathbb{R}^3\times L^2_{k+4}(M,\mathbb{R})^{\mathbb{T},\perp}\to\mathbb{R}^3\times L^2_k(M,\mathbb{R})^{\mathbb{T},\perp}$$

is an isomorphisme. It follows from the inverse function theorem that $\Psi$ is an isomorphisme in a neighborhood $(-\epsilon,\epsilon)^2\times U$ of 0. Using the Sobolev embeding theorem, we can assume that the solution is of regularity at least $C^4$. We conclude using a similar bootstraping argument as in the case of extremal metrics [21, Proposition 4]. □

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