

The degrees of a system of parameters of the ring of invariants of a binary form

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Abstract

We consider the degrees of the elements of a homogeneous system of parameters for the ring of invariants of a binary form, give a divisibility condition, and a complete classification for forms of degree at most 8.

1 The degrees of a system of parameters

Let R be a graded \mathbf{C} -algebra. A *homogeneous system of parameters* (hsop) of R is an algebraically independent set S of homogeneous elements of R such that R is module-finite over the subalgebra generated by S . By the Noether normalization lemma, a hsop always exists. The size $|S|$ of S equals the Krull dimension of R .

In this note we consider the special case where R is the ring I of invariants of binary forms of degree n under the action of $\mathrm{SL}(2, \mathbf{C})$. This ring is Cohen-Macaulay, that is, I is free over the subring generated by any hsop S . Its Krull dimension is $n - 2$.

One cannot expect to classify all hsops of I . Indeed, any generic subset with the right degrees will be a hsop (cf. Dixmier's criterion below). But one can expect to classify the sets of degrees of hsops. In this note we give a divisibility restriction on the set of degrees for the elements of a hsop, and conjecture that when all degrees are large this restriction also suffices for the existence of a hsop with these given degrees. For small degrees there are further restrictions. We give a complete classification for $n \leq 8$.

2 Hilbert's criterion

Hilbert's criterion gives a characterization of homogeneous systems of parameters as sets that define the nullcone.

Denote by V_n the set of binary forms of degree n . The *nullcone* of V_n , denoted $\mathcal{N}(V_n)$, is the set of binary forms of degree n on which all invariants vanish. By the Hilbert-Mumford numerical criterion (see [6] and [7, Chapter 2]) this is precisely the set of binary forms of degree n with a root of multiplicity $> \frac{n}{2}$. Moreover, the binary forms with no root of multiplicity $\geq \frac{n}{2}$ have closed $\mathrm{SL}(2, \mathbf{C})$ -orbits. The elements of $\mathcal{N}(V_n)$ are called *nullforms*. Another result from [6] that we will use is the following.

Proposition 2.1. *For $n \geq 3$, consider i_1, \dots, i_{n-2} homogeneous invariants of V_n . The following two conditions are equivalent:*

- (i) $\mathcal{N}(V_n) = \mathcal{V}(i_1, \dots, i_{n-2})$,
- (ii) $\{i_1, \dots, i_{n-2}\}$ is a hsop of the invariant ring of V_n .

3 A divisibility condition

Assume $n \geq 3$.

Lemma 3.1. *Fix integers j, t with $t > 0$. If an invariant of degree d is nonzero on a form $\sum a_i x^{n-i} y^i$ with the property that all nonzero a_i have $i \equiv j \pmod{t}$, then $d(n-2j)/2 \equiv 0 \pmod{t}$.*

Proof For an invariant of degree d with nonzero term $\prod a_i^{m_i}$ we have $\sum m_i = d$ and $\sum i m_i = nd/2$. If $i \equiv j \pmod{t}$ when $a_i \neq 0$, then $nd/2 = \sum i m_i \equiv j \sum m_i = jd \pmod{t}$. \square

For odd n we recover the well-known fact that all degrees are even (take $t = 1$).

Lemma 3.2. *Fix integers j, t with $t > 1$ and $0 \leq j \leq n$. Among the degrees d of a hsop, at least $\lfloor (n-j)/t \rfloor$ satisfy $d(n-2j)/2 \equiv 0 \pmod{t}$.*

Proof Subtracting a multiple of t from j results in a stronger statement, so it suffices to prove the lemma for $0 \leq j < t$. There are $1 + \lfloor (n-j)/t \rfloor =: 1 + N$ coefficients a_i with $i \equiv j \pmod{t}$, so the subspace U of V_n defined by $a_i = 0$ for $i \not\equiv j \pmod{t}$ has dimension $1 + N$. If $N = 0$ there is nothing to prove, so we assume that $N > 0$. We claim that a general form $f \in U$ has only zeroes of multiplicity strictly less than $n/2$. Indeed, write

$$f = a_j x^{n-j} y^j + a_{j+t} x^{n-j-t} y^{j+t} + \dots + a_{j+mt} x^{n-j-mt} y^{j+mt}$$

where $j + (m+1)t > n$ and $m > 0$. So f has a factor y of multiplicity j and a factor x of multiplicity $n-j-mt$. If j were at least $n/2$, then $j+mt \geq j+t > 2j \geq n$, a contradiction. If $n-j-mt$ were at least $n/2$, then $j+mt \leq n/2$ and hence $t \leq n/2$ and hence $j + (m+1)t \leq n$, a contradiction. The remaining roots of f are roots of

$$a_j x^{mt} + a_{j+t} x^{(m-1)t} y^t + \dots + a_{j+mt} y^{mt},$$

which is a general binary form of degree m in x^t, y^t and hence has mt distinct roots.

Let $\pi : V_n \rightarrow V_n // \mathrm{SL}(2, \mathbf{C})$ be the quotient map; so the right-hand side is the spectrum of the invariant ring I . Set $X := \pi(U)$. We claim that X has dimension N . It certainly cannot have dimension larger than N , since acting with the one-dimensional torus of diagonal matrices on an element of U gives another element of U . To show that $\dim X = N$ we need to show that for general $f \in U$ the fibre $\pi^{-1}(\pi(f))$ intersects U in a one-dimensional variety. By the above and the Hilbert-Mumford criterion, the $\mathrm{SL}(2, \mathbf{C})$ -orbit of f is closed. Moreover, its stabiliser is zero-dimensional. So by properties of the quotient map we have $\pi^{-1}(\pi(f)) = \mathrm{SL}(2, \mathbf{C}) \cdot f$. Hence it suffices that the intersection of this orbit with U is one-dimensional. For this a Lie algebra argument suffices,

in which we may ignore the Lie algebra of the torus: if $(bx\frac{\partial}{\partial y} + cy\frac{\partial}{\partial x})f$ lies in U , then we find that $b = c = 0$ if $t > 2$ (so that the contribution of one term from f cannot cancel the contribution from the next term); and $b = 0$ if $j > 0$ (look at the first term), and then also $c = 0$; and $c = 0$ if $j + mt < n$ (look at the last term), and then also $b = 0$. Hence the only case that remains is $t = 2, j = 0$, and $n \geq 4$ even. Then the equations $ca_0n + ba_22 = 0$ and $ca_2(n-2) + ba_44 = 0$ are independent and force $b = c = 0$.

This concludes the proof that $\dim X = N$. Intersecting X with the hypersurfaces corresponding to elements of an hsop reduces X to the single point in X representing the null-cone. In the process, $\dim X$ drops by N . But the only invariants that contribute to this dimension drop, i.e., the only invariants that do not vanish identically on X (hence on U) are those considered in Lemma 3.1. Hence there must be at least N of these among the hsop. \square

Lemma 3.3. *Let t be an integer with $t > 1$.*

- (i) *If n is odd, and j is minimal such that $0 \leq j \leq n$ and $(n - 2j, t) = 1$, then among the degrees of any hsop at least $\lfloor (n - j)/t \rfloor$ are divisible by $2t$.*
- (ii) *If n is even, and j is minimal with $0 \leq j \leq \frac{1}{2}n$ and $(\frac{1}{2}n - j, t) = 1$, then among the degrees of any hsop at least $\lfloor (n - j)/t \rfloor$ are divisible by t .* \square

Theorem 3.4. *Let t be an integer with $t > 1$.*

- (i) *If n is odd, then among the degrees of any hsop at least $\lfloor (n - 1)/t \rfloor$ are divisible by $2t$ (and all degrees are even).*
- (ii) *If n is even, then among the degrees of any hsop at least $\lfloor (n - 1)/t \rfloor$ are divisible by t , and if $n \equiv 2 \pmod{4}$ then at least $n/2$ by 2 .*

Proof (i) By part (i) of Lemma 3.3 we find a lower bound $\lfloor (n - j)/t \rfloor$ for a j as described there. If that is smaller than $\lfloor (n - 1)/t \rfloor$, then there is some multiple at of t with $n - j + 1 \leq at \leq n - 1$. Put $n = at + b$, where $1 \leq b \leq j - 1$. By definition of j we have $(b - 2i, t) > 1$ for $i = 0, 1, \dots, j - 1$. If b is odd, say $b = 2i + 1$, we find a contradiction. If b is even, say $b = 2i + 2$, then t is even and n is even, contradiction.

(ii) By part (ii) of Lemma 3.3 we find a lower bound $\lfloor (n - j)/t \rfloor$ for a j as described there. For $t = 2$ our claim follows. Now let $t > 2$. If $\lfloor (n - j)/t \rfloor$ is smaller than $\lfloor (n - 1)/t \rfloor$, then there is some multiple at of t with $n - j + 1 \leq at \leq n - 1$. Put $n = at + b$, where $1 \leq b \leq j - 1$. By definition of j we have $(b - 2i, 2t) > 2$ for $i = 0, 1, \dots, j - 1$, impossible. \square

For example, it is known that there exist homogeneous systems of parameters with degree sequences 4 ($n = 3$); 2, 3 ($n = 4$); 4, 8, 12 ($n = 5$); 2, 4, 6, 10 ($n = 6$); 4, 8, 12, 12, 20 and 4, 8, 8, 12, 30 ($n = 7$) [3]; 2, 3, 4, 5, 6, 7 ($n = 8$) [10]; 4, 8, 10, 12, 12, 14, 16 and 4, 4, 10, 12, 14, 16, 24 and 4, 4, 8, 12, 14, 16, 30 and 4, 4, 8, 10, 12, 16, 42 and 4, 4, 8, 10, 12, 14, 48 ($n = 9$) [1]; 2, 4, 6, 6, 8, 9, 10, 14 ($n = 10$) [2].

Conjecture 3.5. *Any sequence d_1, \dots, d_{n-2} of sufficiently large integers satisfying the divisibility conditions of Theorem 3.4 is the sequence of degrees of a hsop.*

This can be compared to the conjecture

Conjecture 3.6. (Dixmier[4])

- (i) If n is odd, $n \geq 15$, then $4, 6, 8, \dots, 2n - 2$ is the sequence of degrees of a hsop.
- (ii) If $n \equiv 2 \pmod{4}$, $n \geq 18$, then $2, 4, 5, 6, 6, 7, 8, 9, \dots, n - 1$ is the sequence of degrees of a hsop.
- (iii) If $n \equiv 0 \pmod{4}$, then $2, 3, 4, \dots, n - 1$ is the sequence of degrees of a hsop.

4 Poincaré series

If there exists a hsop with degrees d_1, \dots, d_{n-2} , then the Poincaré series can be written as a quotient $P(t) = a(t) / \prod (t^{d_i} - 1)$ for some polynomial $a(t)$ with nonnegative coefficients. If one does not have a hsop, but only a sequence of degrees, the conditions of Theorem 3.4 above are strong enough to guarantee that $P(t)$ can be written in this way, but without the condition that the numerator has nonnegative coefficients.

Proposition 4.1. *Let d_1, \dots, d_{n-2} be a sequence of positive integers satisfying the conditions of Theorem 3.4. Then $P(t) \prod (t^{d_i} - 1)$ is a polynomial.*

Proof Dixmier [4] proves that $P(t)B(t)$ is a polynomial, where $B(t)$ is defined by

$$B(t) = \begin{cases} \prod_{i=2}^{n-1} (1 - t^{2i}) & \text{if } n \text{ is odd} \\ \prod_{i=2}^{n-1} (1 - t^i) \cdot (1 + t) & \text{if } n \equiv 2 \pmod{4} \\ \prod_{i=2}^{n-3} (1 - t^i) \cdot (1 + t)(1 - t^{(n-2)/2})(1 - t^{n-1}) & \text{if } n \equiv 0 \pmod{4} \end{cases}$$

Consider a primitive t -th root of unity ζ . We have to show that if $B(t)$ has root ζ with multiplicity m , then at least m of the d_i are divisible by t , but this follows immediately from Theorem 3.4. Note that in case $n \equiv 0 \pmod{4}$ the factor $(1 + t)(1 - t^{(n-2)/2})$ divides $(1 - t^{n-2})$. \square

We see that if $n \equiv 0 \pmod{4}$, $n > 4$, then $P(t)$ can be written with a smaller denominator than corresponds to the degrees of a hsop.

We shall need the first few coefficients of $P(t)$. Messy details arise for small n because there are too few invariants of certain small degrees. Let I be the ring of invariants of a binary form of degree (order) n , let I_m be the graded part of I of degree m , and put $h_m = h_m^n = \dim_{\mathbf{C}} I_m$, so that $P(t) = \sum_m h_m t^m$.

The coefficients h_m^n can be computed by the Cayley-Sylvester formula: The dimension of the space of covariants of degree m and order a is zero when $mn - a$ is odd, and equals $N(n, m, t) - N(n, m, t - 1)$ if $nm - a = 2t$, where $N(n, m, t)$ is the number of ways t can be written as sum of m integers in the range $0..n$, that is, the number of Ferrers diagrams of size t that fit into a $m \times n$ rectangle.

We have Hermite reciprocity $h_m^n = h_n^m$, as follows immediately since reflection in the main diagonal shows $N(n, m, t) = N(m, n, t)$. That means that Table 1 is symmetric.

Dixmier [4] gives the cases in which $h_m = 0$. Since his statement is not precisely accurate, we repeat his proof.

h_m^n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1
2	.	1	.	1	.	1	.	1	.	1	.	1	.	1	.
3	.	.	.	1	.	.	.	1	.	.	.	1	.	.	.
4	.	1	1	1	1	2	1	2	2	2	2	3	2	3	3
5	.	.	.	1	.	.	.	2	.	.	.	3	.	.	.
6	.	1	.	2	.	3	.	4	.	6	.	8	.	10	1
7	.	.	.	1	.	.	.	4	.	.	.	10	.	4	.
8	.	1	1	2	2	4	4	7	8	12	13	20	22	31	36
9	.	.	.	2	.	.	.	8	.	5	.	28	.	27	.
10	.	1	.	2	.	6	.	12	5	24	13	52	33	97	80
11	.	.	.	2	.	.	.	13	.	13	.	73	.	110	.
12	.	1	1	3	3	8	10	20	28	52	73	127	181	291	418
13	.	.	.	2	.	.	.	22	.	33	.	181	.	375	.
14	.	1	.	3	.	10	4	31	27	97	110	291	375	802	1111
15	.	.	.	3	.	1	.	36	.	80	.	418	.	1111	.
16	.	1	1	3	4	13	18	47	84	177	320	639	1120	2077	3581
17	.	.	.	3	.	1	.	54	.	160	.	902	.	2930	.
18	.	1	.	4	1	16	13	71	99	319	529	1330	2342	5034	8899

Table 1: Values of $h_m^n = \dim_{\mathbb{C}} I_m$ with I the ring of invariants of a binary form of degree n . Here . denotes 0. One has $h_m^n = h_n^m$ and $P(t) = \sum_m h_m^n t^m$.

Proposition 4.2. *Let $m, n \geq 1$. One has $h_m = h_m^n = 0$ precisely in the following cases:*

- (i) if mn is odd,
- (ii) if $m = 1$; if $n = 1$,
- (iii) if $m = 2$ and n is odd; if $n = 2$ and m is odd,
- (iv) if $m = 3$ and $n \equiv 2 \pmod{4}$; if $n = 3$ and $m \equiv 2 \pmod{4}$,
- (v) if $m = 5$ and $n = 6, 10, 14$; if $n = 5$ and $m = 6, 10, 14$,
- (vi) if $m = 6$ and $n = 7, 9, 11, 13$; if $n = 6$ and $m = 7, 9, 11, 13$,
- (vii) if $m = 7$ and $n = 10$; if $n = 7$ and $m = 10$.

Proof (i) If n is odd, then all degrees are even. (ii) For $n = 1$ we have $P(t) = 1$. (iii) For $n = 2$ we have $P(t) = 1/(1 - t^2)$. (iv) For $n = 3$ we have $P(t) = 1/(1 - t^4)$. Now let $m, n \geq 4$. For $n = 4$ we have invariants of degrees 2, 3 and hence of all degrees $m \neq 1$. That means that $h_4^n \neq 0$. For $n = 6$ we have invariants of degrees 2, 15 and hence of all degrees $m \geq 14$. That means that $h_6^n \neq 0$ for $n \geq 14$. If n is odd this shows the presence of invariants of degrees 4, 6 and hence of all even degrees $m > 2$, provided $n \geq 15$. For $n = 5$ we have invariants of degrees 4, 18 and hence of all even degrees $m \geq 16$. That means that $h_5^n \neq 0$ for even $n \geq 16$. If n is even this shows the presence of invariants of degrees 2, 5 and hence of all degrees $m \geq 4$, provided $n \geq 16$. It remains only to inspect the table for $4 \leq m, n \leq 14$. \square

5 Dixmier's criterion

Dividing out the ideal spanned by p elements of a hsop diminishes the dimension by precisely (and hence at least) p . This means that the below gives a necessary and sufficient condition for a sequence of degrees to be the degree sequence of a hsop.

Proposition 5.1. (Dixmier [4]) *Let G be a reductive group over \mathbf{C} , with a rational representation in a vector space R of finite dimension over \mathbf{C} . Let $\mathbf{C}[R]$ be the algebra of complex polynomials on R , $\mathbf{C}[R]^G$ the subalgebra of G -invariants, and $\mathbf{C}[R]_d^G$ the subset of homogeneous polynomials of degree d in $\mathbf{C}[R]^G$. Let V be the affine variety such that $\mathbf{C}[V] = \mathbf{C}[R]^G$. Let $r = \dim V$. Let (d_1, \dots, d_r) be a sequence of positive integers. Assume that for each subsequence (j_1, \dots, j_p) of (d_1, \dots, d_r) the subset of points of V where all elements of all $\mathbf{C}[R]_{j_j}^G$ with $j \in \{j_1, \dots, j_p\}$ vanish has codimension not less than p in V . Then $\mathbf{C}[R]^G$ has a system of parameters of degrees d_1, \dots, d_r . \square*

This criterion is very convenient, it means that one can work with degrees only, without worrying about individual elements of a hsop.

6 Minimal degree sequences

If y_1, \dots, y_r is a hsop, then also $y_1^{e_1}, \dots, y_r^{e_r}$ for any sequence of positive integers e_1, \dots, e_r , not all 1. This means that if the degree sequence d_1, \dots, d_r occurs, also the sequence $d_1 e_1, \dots, d_r e_r$ occurs. We would like to describe the minimal sequences, where such multiples are discarded.

There are further reasons for non-minimality.

Lemma 6.1. *If there exist hsops with degree sequences d_1, \dots, d_{r-1}, d' and d_1, \dots, d_{r-1}, d'' , then there also exists a hsop with degree sequence $d_1, \dots, d_{r-1}, d' + d''$.*

Proof We verify Dixmier's criterion. Consider a finite basis f_1, \dots, f_s for the space of invariants of degree d' . Split the variety V in the s pieces defined by $f_i \neq 0$ ($1 \leq i \leq s$) together with the single piece defined by $f_1 = \dots = f_s = 0$. Given p elements of the sequence $d_1, \dots, d_{r-1}, d' + d''$ we have to show that the codimension in V obtained by requiring all invariants of such degrees to vanish is at least p , that is, that the dimension is at most $r - p$. This is true by assumption if $d' + d''$ is not among these p elements. Otherwise, consider the $s + 1$ pieces separately. We wish to show that each has dimension at most $r - p$, then the same will hold for their union. For the last piece, where all invariants of degree d' vanish, this is true by assumption. But if some invariant of degree d' does not vanish, and all invariants of degree $d' + d''$ vanish, then all invariants of degree d'' vanish, and we are done. \square

Note that taking multiples is a special case of (repeated application of) this lemma, used with $d' = d''$.

Let us call a sequence *minimal* if it occurs (as the degree sequence of the elements of a hsop), and its occurrence is not a consequence, via the above lemma or via taking multiples, of the occurrence of smaller sequences. We might try to classify all minimal sequences, at least in small cases.

Is it perhaps true that a hsop exists for any degree sequence that satisfies the conditions of Theorem 3.4 when there are sufficiently many invariants? E.g. when the coefficients of $P(t) \prod (1 - t^{d_i})$ are nonnegative?

Example Some caution is required. For example, look at $n = 6$. The conditions of Theorem 3.4 are: at least three factors 2, at least one factor of each of 3, 4, 5. The sequence 6, 6, 6, 20 satisfies this restriction. Moreover, $P(t)(1 - t^6)^3(1 - t^{20}) = 1 + t^2 + 2t^4 + t^8 + 2t^{12} + t^{14} + t^{15} + t^{16} + t^{17} + 2t^{19} + t^{23} + 2t^{27} + t^{29} + t^{31}$ has only nonnegative coefficients. But no hsop with these degrees exists: since $h_2 = 1$, $h_4 = 2$, $h_6 = 3$ it follows that there are invariants i_2, i_4, i_6 of degrees 2, 4, 6, and we have $I_4 = \langle i_2^2, i_4 \rangle$ and $I_6 = \langle i_2^3, i_2 i_4, i_6 \rangle$. Requiring all invariants of degree 6 to vanish is equivalent to the two conditions $i_2 = i_6 = 0$, and a hsop cannot contain three elements of degree 6.

Still, the above conditions almost suffice. And for $n < 6$ they actually do suffice.

6.1 $n = 3$

For $n = 3$ we only have simple multiples of the minimal degree.

Proposition 6.2. *A positive integer d is the degree of a hsop in case $n = 3$ if and only if it is divisible by 4.* \square

If i_4 is an invariant of degree 4, then $\{i_4\}$ is a hsop.

6.2 $n = 4$

For $n = 4$ one has the sequence 2, 3, but for example also 5, 6.

Proposition 6.3. *A sequence d_1, d_2 of two positive integers is the sequence of degrees of a hsop for the quartic if and only if neither of them equals 1, at least one is divisible by 2, and at least one is divisible by 3.*

Proof Clearly the conditions are necessary. In order to show that they suffice apply induction and the known existence of a hsop with degrees 2, 3. If $d_2 > 7$, then apply Lemma 6.1 to the two sequences $d_1, 6$ and $d_1, d_2 - 6$ to conclude the existence of a hsop with degrees d_1, d_2 . If $2 \leq d_1, d_2 \leq 7$ and one is divisible by 2, the other by 3, then we have a multiple of the sequence 2, 3. Otherwise, one equals 6 and the other is 5 or 7. But 5, 6 is obtained from 2, 6 and 3, 6, and 7, 6 is obtained from 2, 6 and 5, 6. \square

If i_2 and i_3 are invariants of degrees 2 and 3, then $\{i_2, i_3\}$ is a hsop.

Proposition 6.4. *There is precisely one minimal degree sequence of hsops in case $n = 4$, namely 2, 3.* \square

6.3 $n = 5$

Proposition 6.5. *A sequence d_1, d_2, d_3 of three positive integers is the sequence of degrees of a hsop for the quintic if and only if all d_i are even, and distinct from 2, 6, 10, 14, and no two are 4, 4 or 4, 22 and at least two are divisible by 4, at least one is divisible by 6, and at least one is divisible by 8.*

Proof For $n = 5$ the Poincaré series is $P(t) = 1 + t^4 + 2t^8 + 3t^{12} + 4t^{16} + t^{18} + 5t^{20} + t^{22} + 7t^{24} + 2t^{26} + 8t^{28} + 3t^{30} + \dots$. The stated conditions are necessary: the divisibility conditions are seen from Theorem 3.4, and there are no invariants of degrees 2, 6, 10, 14. Finally, we have $h_4 = 1$ and $h_{18} = h_{22} = 1$, so that there are unique invariants i_4 and i_{18} of degrees 4 and 18, respectively, and $I_{22} = \langle i_4 i_{18} \rangle$, so that all invariants of degree 22 will vanish as soon as i_4 vanishes.

The stated conditions suffice: We use (and verify below) that there are hsops with degrees 4, 8, 12 and with degrees 4, 8, 18. If all d_i are divisible by 4, and we do not have a multiple of 4, 8, 12, then we have $4a, 4b, 24c$ where a and b have no factor 2 or 3, and not both are 1. It suffices to find 4, 4b, 24. Since 4, 8, 24 exists, we can decrease b by 2, and it suffices to find 4, 12, 24, which exists.

So, some d_i is not divisible by 4. We have one of the three cases $24a, 4b, 2c$ and $8a, 12b, 2c$ and $8a, 4b, 6c$, where c is odd. In the middle case we have $c \geq 9$ and it suffices to make 8, 12, 2c. Since 8, 12, 4 exists, we can reduce c by 2, and it suffices to make 8, 12, 18, which exists since 4, 8, 18 exists.

In the first case we have $c \geq 9$ and it suffices to make 24, 4, 2c. Since 12, 4, 8 exists, we can reduce c by 4, and it suffices to make 24, 4, 18 and 24, 4, 30. The former is a multiple of 4, 8, 18 and the latter follows from 24, 4, 18 and 24, 4, 12. Since 24, 4, 22 does not exist we still have to consider $24a, 4b, 22$. Since 8, 12, 22 exists we can reduce b by 2, and it suffices to make 24, 12, 22. But that is a multiple of 8, 12, 22.

Finally in the last case we have $c \geq 3$, and since 8, 4, 12 exists we can reduce c by 2. So it suffices to do 4, 8, 18, and that exists. \square

Proposition 6.6. *There are precisely two minimal degree sequences of hsops in case $n = 5$, namely 4, 8, 12 and 4, 8, 18.*

Proof By the proof of the previous proposition, all we have to do is show the existence of hsops with the indicated degree sequences. It is well-known (see, e.g., Schur [9], p.86) that the quintic has four invariants i_4, i_8, i_{12}, i_{18} (with degrees as indicated by the index) that generate the ring of invariants, and every invariant of degree divisible by 4 (in particular i_{18}^2) is a polynomial in the first three. Thus, when i_4, i_8, i_{12} vanish, all invariants vanish, and $\{i_4, i_8, i_{12}\}$ is a hsop. Knowing this, it is easy to see that also $\{i_4, i_8, i_{18}\}$ is a hsop: a simple Groebner computation shows that $i_{12}^3 \in (i_4, i_8, i_{18})$, hence $\mathcal{N}(V_5) = \mathcal{V}(i_4, i_8, i_{18})$. \square

6.4 $n = 6$

Similarly, we find for $n = 6$:

Proposition 6.7. *A sequence d_1, d_2, d_3, d_4 of four positive integers is the sequence of degrees of a hsop for the sextic if and only if all d_i are distinct from 1, 3, 5, 7, 9, 11, 13, and no two are in $\{2, 17\}$, and no three are in $\{2, 4, 8, 14, 17, 19, 23, 29\}$, and no three are in $\{2, 6, 17, 21\}$, and at least three are divisible by 2, at least one is divisible by 3, at least one by 4, and at least one by 5.*

Proof For $n = 6$ the Poincaré series is

$$\begin{aligned} P(t) = & 1 + t^2 + 2t^4 + 3t^6 + 4t^8 + 6t^{10} + 8t^{12} + 10t^{14} + t^{15} + 13t^{16} + t^{17} + \\ & 16t^{18} + 2t^{19} + 20t^{20} + 3t^{21} + 24t^{22} + 4t^{23} + 29t^{24} + 6t^{25} + 34t^{26} + \\ & 8t^{27} + 40t^{28} + 10t^{29} + 47t^{30} + \dots \end{aligned}$$

We have

$$\begin{aligned} I_2 &= \langle i_2 \rangle, \quad I_4 = \langle i_2^2, i_4 \rangle, \quad I_6 = \langle i_2^3, i_2 i_4, i_6 \rangle, \quad I_8 = \langle i_2^4, i_2^2 i_4, i_2 i_6, i_4^2 \rangle, \\ I_{10} &= \langle i_2^5, i_2^3 i_4, i_2^2 i_6, i_2 i_4^2, i_4 i_6, i_{10} \rangle, \quad I_{12} = \langle i_2^6, i_2^4 i_4, i_2^3 i_6, i_2^2 i_4^2, i_2 i_4 i_6, i_2 i_{10}, i_4^3, i_6^2 \rangle, \\ I_{14} &= \langle i_2^7, i_2^5 i_4, i_2^4 i_6, i_2^3 i_4^2, i_2^2 i_4 i_6, i_2^2 i_{10}, i_2 i_4^3, i_2 i_6^2, i_4^2 i_6, i_4 i_{10} \rangle, \quad I_{15} = \langle i_{15} \rangle, \end{aligned}$$

and the invariants in degrees 17, 19, 23, 29 are i_{15} times the invariants in degrees 2, 4, 8, 14, respectively. Let us denote by $[i_1, \dots, i_t]$ the condition that all invariants of degrees i_1, \dots, i_t vanish. Then $[2] = [2, 17]$ and hence a hsop cannot have two element degrees among 2, 17. Also $[4] = [2, 4, 8, 14, 17, 19, 23, 29]$ and hence a hsop cannot have three element degrees among 2, 4, 8, 14, 17, 19, 23, 29. And $[6] = [2, 6, 17, 21]$ is the condition $i_2 = i_6 = 0$ so that a hsop cannot have three element degrees among 2, 6, 17, 21. It follows that the stated conditions are necessary.

The stated conditions suffice: We use (and verify below) that there are hsops with each of the degree sequences 2, 4, 6, 10 and 2, 4, 6, 15 and 2, 4, 10, 15. Prove by induction that any 4-tuple of degrees that satisfies the given conditions occurs as the degree sequence of a hsop. Given d_1, d_2, d_3, d_4 , if $d_i \geq 90$ then by induction we already have the 4-tuples obtained by replacing d_i by 60 and by $d_i - 60$. It remains to check the finitely many cases where all d_i are less than 90. A small computer check settles this. \square

Proposition 6.8. *There are precisely three minimal degree sequences of hsops in case $n = 6$, namely 2, 4, 6, 10 and 2, 4, 6, 15 and 2, 4, 10, 15.*

Proof By the proof of the previous proposition, all we have to do is show the existence of hsops with the indicated degree sequences. It is well-known (see, e.g., Schur [9], p.90) that the sextic has five invariants $i_2, i_4, i_6, i_{10}, i_{15}$ (with degrees as indicated by the index) that generate the ring of invariants, where i_{15}^2 is a polynomial in the first four. This implies that $\mathcal{N}(V_6) = \mathcal{V}(i_2, i_4, i_6, i_{10})$, so that $\{i_2, i_4, i_6, i_{10}\}$ is a hsop. Now $\{i_2, i_4, i_6, i_{15}\}$ and $\{i_2, i_4, i_{10}, i_{15}\}$ are also hsops: we verified by computer that $i_{10}^3 \in (i_2, i_4, i_6, i_{15})$ and $i_6^5 \in (i_2, i_4, i_{10}, i_{15})$, so that $\mathcal{N}(V_6) = \mathcal{V}(i_2, i_4, i_6, i_{15}) = \mathcal{V}(i_2, i_4, i_{10}, i_{15})$. \square

6.5 $n = 7$

For $n = 7$ we have to consider the invariants a bit more closely in order to decide which degree sequences are admissible for hsops.

Let f be our septic and let ψ be the covariant $\psi = (f, f)_6$. There are thirty basic invariants, of degrees 4, 8 ($3\times$), 12 ($6\times$), 14 ($4\times$), 16 ($2\times$), 18 ($9\times$), 20, 22 ($2\times$), 26, 30. These can all be taken to be transvectants with a power of ψ except for three basic invariants of degrees 12, 20 and 30 (that von Gall [5] calls R, A, B and Dixmier [3] q_{12}, p_{20}, p_{30}). This means that all invariants

of degrees not of the form $12a + 20b + 30c$ vanish on the set defined by $\psi = 0$. But ψ is a covariant of order 2, i.e., $\psi = Ax^2 + Bxy + Cy^2$ for certain A, B, C . It follows that no hsop degree sequence can have four elements in the set $\{4, 8, 14, 16, 18, 22, 26, 28, 34, 38, 46, 58\}$.

Proposition 6.9. *A sequence of five positive even integers is the sequence of degrees of a hsop for the septic if and only if all are distinct from 2, 6, 10, no two equal 4, no four are in $\{4, 8, 14, 16, 18, 22, 26, 28, 34, 38, 46, 58\}$ and at least three are divisible by 4, at least two by 6, at least one by 8, at least one by 10 and at least one by 12.*

Proof We already saw that these conditions are necessary. For sufficiency, use induction. The divisibility conditions concern moduli with l.c.m. 120, and the restrictions concern numbers smaller than 60, so if one of the degrees is not less than 180, we are done by induction. A small computer program checks all degree sequences with degrees at most 180, and finds that all can be reduced to the 23 sequences given in the following proposition. \square

Proposition 6.10. *There are precisely 23 minimal degree sequences of hsops in case $n = 7$, namely*

4, 8, 8, 12, 30	4, 12, 12, 12, 40	4, 12, 18, 18, 40	8, 12, 12, 14, 20
4, 8, 12, 12, 20	4, 12, 12, 14, 40	4, 14, 14, 24, 60	8, 12, 14, 14, 60
4, 8, 12, 12, 30	4, 12, 12, 18, 40	4, 14, 18, 20, 24	8, 12, 14, 18, 20
4, 8, 12, 14, 30	4, 12, 14, 14, 120	4, 14, 18, 32, 60	12, 12, 14, 14, 40
4, 8, 12, 18, 20	4, 12, 14, 18, 40	4, 18, 18, 20, 24	12, 14, 14, 20, 24
4, 8, 12, 18, 30	4, 12, 14, 20, 24	4, 18, 18, 32, 60	

Proof We only have to show existence. Apply Dixmier's criterion. Denote by $[d_1, \dots, d_p]$ the codimension in V of the subset of points of V where all elements of all $\mathbf{C}[R]_{d_j}^G$ vanish ($1 \leq j \leq p$). We have to show that for all p and each of these 23 sequences (d_i) the inequality $[d_1, \dots, d_p] \geq p$ holds.

For $p = 1$ that means that we need $[m] \geq 1$ for $m = 4, 8, 12, 14, 18, 20, 24, 30, 32, 40, 60, 120$, and that is true, for example by inspection of Table 1.

We can save some work by observing that Dixmier [3] already showed the existence of hsops with degree sequences 4, 8, 8, 12, 30 and 4, 8, 12, 12, 20. It follows that $[8] \geq 3$ and $[12] \geq 3$ and $[24] \geq [8, 12] \geq 4$ and $[20] \geq 2$ and $[60] \geq [12, 20] \geq 4$ and $[4, 30] \geq 2$ and $[8, 30] \geq 4$. Since there are several basic invariants of degree 14 or 18, no two of which can have a common factor, it follows that $[14] \geq 2$ and $[18] \geq 2$. This suffices to settle $p = 2$.

For $p = 3$ we must look at triples $[d, d', d'']$ without element 8 or 12 or multiple. First check that $[4, 14] \geq 3$ and $[4, 18] \geq 3$. We'll do this below. Now all the rest needed for $p = 3$ follows.

Below we shall show that $[12] \geq 4$. For $p = 4$ we must look at quadruples $[d, d', d'', d''']$ without element 12 or 8, 30 or multiple. The minimal of these are (omitting implied elements) $[18, 20]$ and $[18, 32]$. However, $[18, 32] \geq \min([18, 12], [18, 20])$ and $[18, 20] \geq \min([18, 20, 8], [18, 20, 12])$.

Finally for $p = 5$ we have to show that each of these 23 sets determines the nullcone. But that follows immediately, since it is known already that $[8, 12, 20] = [8, 12, 30] = 5$.

Altogether, our obligations are: show that $[4, 14] \geq 3$, $[4, 18] \geq 3$, $[12] \geq 4$ and $[8, 18, 20] \geq 4$.

Consider the part of V defined by $\psi = 0$. Dixmier shows that if $\psi = q_{12} = p_{20} = 0$ (for certain invariants q_{12} and p_{20} of degrees 12 and 20, respectively), then f is a nullform. It follows that the subsets of V defined by $\psi = q_{12} = 0$ or by $\psi = p_{20} = 0$ have codimension at least 4 in V .

Now we have to do some actual computations. With $f = ax^7 + \binom{7}{1}bx^6y + \dots + \binom{7}{1}gxy^6 + hy^7$ (the two meanings of f , as form and as coefficient will not cause confusion), we find $\psi = (ag - 6bf + 15ce - 10d^2)x^2 + (ah - 5bg + 9cf - 5de)xy + (bh - 6cg + 15df - 10e^2)y^2$.

Assume that the invariant of degree 4 vanishes, as it does in all cases we still have to consider. Then ψ has zero discriminant. If $\psi \neq 0$, then w.l.o.g. $\psi \sim x^2$, and $ah - 5bg + 9cf - 5de = bh - 6cg + 15df - 10e^2 = 0$, $ag - 6bf + 15ce - 10d^2 \neq 0$.

Distinguish the four cases (i) $h \neq 0$, (ii) $h = 0$, $g \neq 0$, (iii) $h = g = 0$, $f \neq 0$, (iv) $h = g = f = 0$, $e \neq 0$. W.l.o.g. these become (i) $h = 1$, $g = 0$, $a + 9cf - 5de = 0$, $b + 15df - 10e^2 = 0$, (ii) $h = 0$, $g = 1$, $f = 0$, $b + de = 0$, $3c + 5e^2 = 0$, (iii) $h = g = 0$, $f = 1$, $e = 0$, $c = 0$, $d = 0$, $b \neq 0$, (iv) $h = g = f = 0$, $e = 1$, $d = 0$, contradiction.

Let us first show that $[12] \geq 4$. We may suppose $\psi \neq 0$. One of the invariants of degree 12 is $(\psi_1, \psi^5)_{10} \sim (\psi_1, x^{10})_{10} = fh - g^2$, where $\psi_1 = (f, f)_2$. If all invariants of degree 12 vanish, then in case (i) $f = 0$, and in case (ii) contradiction. Look at case (iii). The only invariant of degree 12 that does not vanish identically is $a^2b^2f^8$, and we find $a = 0$, a 1-dimensional set. Finally, in case (i), if all invariants of degree 12 vanish, but $ag - 6bf + 15ce - 10d^2 \neq 0$, then the remaining conditions define an ideal $(18e^3 - cd, 12de^2 - c^2, 2cd^2 - 3c^2e)$ in the three variables c, d, e and the quotient is 1-dimensional. This shows that $[12] \geq 4$.

Let us show next that $[8, 18] \geq 4$. We may suppose $\psi \neq 0$. One of the invariants of degree 8 is $(\psi_2, \psi^3)_6 \sim (\psi_2, x^6)_6 = dh - 4eg + 3f^2$ where $\psi_2 = (f, f)_4$. This gives a contradiction in case (iii). In case (ii) it gives $e = b = c = 0$, leaving only variables a, d . In case (i) it gives $d + 3f^2 = 0$, leaving only variables c, e, f .

An invariant of degree 18 is $((\psi_1, \psi_2)_1, \psi^7)_{14} \sim ((\psi_1, \psi_2)_1, x^{14})_{14} = -cfh^2 + cg^2h + deh^2 + 2dfgh - 3dg^3 - 4e^2gh + ef^2h + 6efg^2 - 3f^3g$. In case (ii) this says $d = 0$, leaving only variable a . In case (i) this says $f(2ef + c) = 0$. This gives us two subcases: (ia) with $f = 0$ and variables c, e , and (ib) with $c + 2ef = 0$ and variables e, f .

Another invariant of degree 8 is $(\psi_3, \psi^2)_4 \sim (\psi_3, x^4)_4$, where $\psi_3 = (\psi_2, \psi_2)_4$, which vanishes in case (ii) and says $c^2f + 4cef^2 + 76e^2f^3 + 9e^4 + 144f^6 = 0$ in case (i). In case (ia) this means $e = 0$ leaving only variable c . In case (ib) this means $(4f^3 + e^2)^2 = 0$, leaving the dimension 1. This proves $[8, 18] \geq 4$.

Let us show next that $[4, 14] \geq 3$. First consider the case $\psi = 0$. Now all invariants of degrees 4 or 14 (or 18) vanish, but the condition $\psi = 0$ itself yields the three equations $A = B = C = 0$ where $\psi = Ax^2 + Bxy + Cy^2$. Earlier, the choice $\psi \sim x^2$ used up some of the freedom given by the group, but here we are free to choose a zero for the form, and assume $h = 0$. Again consider the four cases, this time with $ag - 6bf + 15ce - 10d^2$ zero instead of nonzero. We have (iii) $f = 1$, $h = g = e = d = c = b = 0$, only variables a, f left. And (ii) $g = 1$,

$h = f = 0$, $b + de = 0$, $3c + 5e^2 = 0$, $a + 15ce - 10d^2 = 0$, only variables d, e left. And by assumption $h = 0$ we are not in case (i). That settles the case $\psi = 0$.

Now assume $\psi \neq 0$ and take $\psi \sim x^2$. In case (iii) only variables a, b are left, and we are done. In case (ii) only variables a, d, e are left. In case (i) only variables c, d, e, f are left. An invariant of degree 14 is $(f \cdot (f, \psi_2)_5, \psi^5)_{10} \sim (f \cdot (f, \psi_2)_5, x^{10})_{10} = -2afh^2 + 2ag^2h + 7beh^2 - 7bfg h - 5cdh^2 - 22cegh + 27cf^2h + 25d^2gh - 45defh + 20e^3h$. In case (ii) this vanishes. In case (i) this becomes (up to a constant) $18e^3 - 32def + 9cf^2 - cd$. Another invariant of degree 14 is $((\psi_2, \psi_3)_1, \psi^4)_8 \sim ((\psi_2, \psi_3)_1, x^8)_8$. In case (ii) this becomes $de(26e^3 - 35d^2 - 10a)$ and we are reduced to three pieces, each with only two variables. In case (i) this becomes (up to a constant) $70e^3f^4 - 120def^5 + 27cf^6 + 36e^5f - 60de^3f^2 + 6ce^2f^3 + 3cdf^4 + 6d^2e^3 + 18ce^4 - 8d^3ef - 54cde^2f + 33cd^2f^2 + 3c^2ef^2 + cd^3 - 3c^2de + 2c^3f$. Both polynomials found are irreducible and hence have no common factor, and we are reduced to a 2-dimensional situation. This proves $[4, 14] \geq 3$.

Finally, let us show that $[4, 18] \geq 3$. The subcase $\psi = 0$ was handled already, so we can assume that $\psi \neq 0$ and take $\psi \sim x^2$. Again only cases (i) and (ii) need to be considered. Above we already considered the invariant $((\psi_1, \psi_2)_1, \psi^7)_{14}$ of degree 18. In case (ii) this yields $d = 0$, leaving only the two variables a, e . In case (i) we find $ef^2 + de - cf = 0$. Another invariant of degree 18 is $(f \cdot ((f, \psi_2)_5, \psi_2)_2, \psi^6)_{12}$. In case (i) this yields $70e^3f^3 - 120def^4 + 27cf^5 - 54e^5 + 210de^3f - 200d^2ef^2 - 15ce^2f^2 + 30cdf^3 + 15cde^2 - 25cd^2f - c^3 = 0$. Both polynomials found are irreducible and hence have no common factor, and we are reduced to a 2-dimensional situation. This proves $[4, 18] \geq 3$. \square

6.6 $n = 8$

For the octavic there are nine basic invariants i_d ($2 \leq d \leq 10$). There is a hsop with degrees 2, 3, 4, 5, 6, 7. The Poincaré series is

$$\begin{aligned} P(t) &= 1 + t^2 + t^3 + 2t^4 + 2t^5 + 4t^6 + 4t^7 + 7t^8 + 8t^9 + \\ &\quad 12t^{10} + 13t^{11} + 20t^{12} + 22t^{13} + 31t^{14} + \dots = \\ &= (1 + t^8 + t^9 + t^{10} + t^{18}) / \prod_{d=2}^7 (1 - t^d). \end{aligned}$$

Given a finite sequence (d_i) , the *numerator* of $P(t)$ corresponding to this sequence is by definition $P(t) \prod (1 - t^{d_i})$. If (d_i) is a subsequence of the sequence of degrees of a hsop, then the corresponding numerator has nonnegative coefficients. This rules out, e.g., the following sequences (d_i) .

$$\begin{array}{cccccc} 2, 2 & 2, 4, 4 & 3, 5, 5 & 5, 5, 5 \\ 3, 3 & 2, 5, 5 & 4, 4, 4 & 2, 3, 7, 7 \end{array}$$

What is wrong with these sequences is that there just aren't enough invariants of these degrees. More interesting are the cases where there are enough invariants, but they cannot be chosen algebraically independent.

Proposition 6.11. *A sequence of six integers larger than 1 is the sequence of degrees of a hsop for the octavic if and only if*

(i) ('divisibility') *at least three of them are even, at least two are divisible by 3, at least one has a factor 4, at least one a factor 5, at least one a factor 6, and at least one a factor 7, and moreover*

(ii) ('nonnegativity') none of the eight sequences in the above table occur as a subsequence, and moreover

(iii) ('algebraic independence') there are no four elements in any of $\{2, 3, 6\}$, $\{2, 4, 5\}$, $\{2, 4, 7\}$, and no five elements in any of $\{2, 3, 4, 5, 11\}$, $\{2, 3, 4, 6, 11\}$, $\{2, 3, 4, 7\}$, $\{2, 3, 4, 8\}$, $\{2, 3, 4, 9\}$, $\{2, 3, 5, 6\}$, $\{2, 3, 6, 7, 11\}$.

Proof We have

$$I_2 = \langle i_2 \rangle, \quad I_3 = \langle i_3 \rangle, \quad I_4 = \langle i_2^2, i_4 \rangle, \quad I_5 = \langle i_2 i_3, i_5 \rangle, \quad I_6 = \langle i_2^3, i_2 i_4, i_3^2, i_6 \rangle,$$

$$I_7 = \langle i_2^2 i_3, i_2 i_5, i_3 i_4, i_7 \rangle, \quad I_8 = \langle i_2^4, i_2^2 i_4, i_2 i_3^2, i_2 i_6, i_3 i_5, i_4^2, i_8 \rangle,$$

$$I_9 = \langle i_2^3 i_3, i_2^2 i_5, i_2 i_3 i_4, i_2 i_7, i_3^3, i_3 i_6, i_4 i_5, i_9 \rangle,$$

$$I_{11} = \langle i_2^4 i_3, i_2^3 i_5, i_2^2 i_3 i_4, i_2^2 i_7, i_2 i_3^3, i_2 i_3 i_6, i_2 i_4 i_5, i_2 i_9, i_3^2 i_5, i_3 i_4^2, i_3 i_8, i_4 i_7, i_5 i_6 \rangle.$$

We see that $V(\cup_{a \in A} I_a) = V(\{i_b \mid b \in B\})$ for A and B as in the table below.

A	B	A	B	A	B
2,3,6	2,3,6	2,3,4,6,11	2,3,4,6	2,3,5,6	2,3,5,6
2,4,5	2,4,5	2,3,4,7	2,3,4,7	2,3,6,7,11	2,3,6,7
2,4,7	2,4,7	2,3,4,8	2,3,4,8		
2,3,4,5,11	2,3,4,5	2,3,4,9	2,3,4,9		

This shows that the given conditions are necessary. For sufficiency, use induction. The basis of the induction is provided by the 13 hsops constructed in the next proposition. Given a sequence of six numbers satisfying the conditions, order the numbers in such a way that the last is divisible by 7 and at least one of the last two is divisible by 5. All restrictions concern numbers at most 11, so if we split a number from the sequence into two parts each at least 12, such that the divisibility conditions remain true for the two resulting sequences, then by Lemma 6.1 and induction there exists a hsop with the given sequence as degree sequence. This means that one can reduce the first four numbers modulo 12, the fifth modulo 60, and the last modulo 420. It remains to check a $24 \times 24 \times 24 \times 24 \times 72 \times 432$ box, and this is done by a small computer program. \square

Proposition 6.12. *There are precisely 13 minimal degree sequences of hsops in case $n = 8$, namely*

$$\begin{array}{lll} 2, 3, 4, 5, 6, 7 & 2, 3, 4, 6, 9, 35 & 2, 3, 5, 6, 10, 28 \\ 2, 3, 4, 5, 8, 42 & 2, 3, 4, 7, 8, 30 & 2, 3, 5, 9, 12, 14 \\ 2, 3, 4, 5, 9, 42 & 2, 3, 4, 7, 9, 30 & 2, 4, 5, 6, 8, 21 \\ 2, 3, 4, 5, 10, 42 & 2, 3, 4, 8, 9, 210 & \\ 2, 3, 4, 6, 8, 35 & 2, 3, 5, 6, 9, 28 & \end{array}$$

Proof Minimality is immediately clear, so we only have to show existence. Apply Dixmier's criterion. As before we have to show that for all p and each subsequence d_1, \dots, d_p of one of these 13 sequences the inequality $[d_1, \dots, d_p] \geq p$ holds.

We can save some work by observing that Shioda [10] already showed the existence of a hsop with degree sequence 2, 3, 4, 5, 6, 7. It follows that $[d_1, \dots, d_p] \geq p$ when (at least) p of the numbers 2, 3, 4, 5, 6, 7 divide some of the d_i .

For $p = 1$, nothing remains to check.

For $p = 2$, there only remains to show $[9] \geq 2$, and this follows since there are two invariants of degree 9 without common factor, for example i_3i_6 and i_4i_5 .

For $p = 3$, we have to show $[8] \geq 3$, $[2, 9] \geq 3$, $[5, 9] \geq 3$, $[7, 9] \geq 3$, $[10] \geq 3$.

For $p = 4$, we have to show $[3, 8] \geq 4$, $[5, 8] \geq 4$, $[7, 8] \geq 4$, $[4, 9] \geq 4$, $[2, 5, 9] \geq 4$, $[6, 9] \geq 4$, $[2, 7, 9] \geq 4$, $[8, 9] \geq 4$, $[3, 10] \geq 4$, $[4, 10] \geq 4$, $[9, 14] \geq 4$.

For $p = 5$, we have to show $[3, 5, 8] \geq 5$, $[6, 8] \geq 5$, $[3, 7, 8] \geq 5$, $[4, 5, 9] \geq 5$, $[4, 6, 9] \geq 5$, $[5, 6, 9] \geq 5$, $[4, 7, 9] \geq 5$, $[8, 9] \geq 5$, $[3, 4, 10] \geq 5$, $[6, 10] \geq 5$, $[5, 9, 14] \geq 5$.

There are no conditions left to check for $p = 6$.

Remain 27 conditions to check. Let $V[d_1, \dots, d_p]$ denote the variety defined by all invariants of degrees d_i . Split $V[9]$ into two parts depending on whether i_2 vanishes or not. Where it does not vanish, all invariants of degrees 3, 5, 7 must vanish. Hence $[5, 9], [7, 9] \geq [9] \geq \min([2, 9], [3, 5, 7, 9])$. Split $[2, 9]$ into two parts depending on whether i_4 vanishes or not. The first part has $[2, 3, 4, 9] \geq 3$, the second $[2, 3, 5, 9] \geq 3$. Hence $[9] \geq 3$. Similarly, $[8] = [2, 4, 8] \geq \min([2, 3, 4, 8], [2, 4, 5, 8]) \geq 3$. Finally, $[10] = [2, 5, 10] \geq \min([2, 3, 5, 10], [2, 3, 7, 10]) \geq 3$. This settles $p = 3$.

The same argument shows that $[7, 8], [2, 7, 9], [6, 9], [3, 10], [4, 10], [9, 14] \geq 4$ and $[5, 9, 14] \geq 5$.

Since adding a single condition diminishes the dimension by at most one, $[3, 8] \geq 4$ follows from $[3, 5, 8] \geq 5$. (Given that i_2 vanishes since i_2^4 has degree 8, the condition that all invariants of degree 5 vanish is equivalent to the requirement that i_5 vanishes.) Similarly $[5, 8] \geq 4$ and $[4, 9] \geq 4$ and $[2, 5, 9] \geq 4$ follow from $[3, 5, 8] \geq 5$ and $[4, 5, 9] \geq 5$. Trivially, $[8, 9] \geq 4$ follows from $[8, 9] \geq 5$. This settles $p = 4$, assuming the inequalities for $p = 5$.

Remain 10 conditions to check: $[3, 5, 8] \geq 5$, $[6, 8] \geq 5$, $[3, 7, 8] \geq 5$, $[4, 5, 9] \geq 5$, $[4, 6, 9] \geq 5$, $[5, 6, 9] \geq 5$, $[4, 7, 9] \geq 5$, $[8, 9] \geq 5$, $[3, 4, 10] \geq 5$, $[6, 10] \geq 5$.

Equivalently, for each of the sets A , where A is one of

$$\begin{aligned} &\{2, 3, 4, 5, 8\}, \quad \{2, 3, 4, 6, 8\}, \quad \{2, 3, 4, 7, 8\}, \quad \{2, 3, 4, 5, 9\}, \quad \{2, 3, 4, 6, 9\}, \\ &\{2, 3, 5, 6, 9\}, \quad \{2, 3, 4, 7, 9\}, \quad \{2, 3, 4, 8, 9\}, \quad \{2, 3, 4, 5, 10\}, \quad \{2, 3, 5, 6, 10\}, \end{aligned}$$

we must have $\dim V(\{i_a \mid a \in A\}) = 1$.

For example, we want $\dim V(i_2, i_3, i_4, i_5, i_8) = 1$. Now i_2, i_3, i_4, i_5 form part of a hsop, so $V(i_2, i_3, i_4, i_5)$ is irreducible and has dimension 2. Moreover i_8 does not vanish identically on $V(i_2, i_3, i_4, i_5)$ as we shall see, and it follows that $\dim V(i_2, i_3, i_4, i_5, i_8) = 1$.

This argument works in all cases except that of $V(i_2, i_3, i_4, i_8, i_9)$ and shows that each of the claimed sequences of degrees with the possible exception of 2, 3, 4, 8, 9, 210, is that of a hsop. In particular, e.g. 2, 3, 4, 5, 8, 42 is the sequences of degrees of a hsop. But now this argument also applies to $V(i_2, i_3, i_4, i_8, i_9)$: $V(i_2, i_3, i_4, i_8)$ is irreducible of dimension 2 and i_9 does not vanish identically on it, and it follows that $V(i_2, i_3, i_4, i_8, i_9)$ has dimension 1.

It remains to check the ten conditions that say that i_8 does not vanish on any of $V(i_2, i_3, i_4, i_5)$, $V(i_2, i_3, i_4, i_6)$, $V(i_2, i_3, i_4, i_7)$, that i_9 does not vanish on any of $V(i_2, i_3, i_4, i_5)$, $V(i_2, i_3, i_4, i_6)$, $V(i_2, i_3, i_5, i_6)$, $V(i_2, i_3, i_4, i_7)$, $V(i_2, i_3, i_4, i_8)$, and that i_{10} does not vanish on $V(i_2, i_3, i_4, i_5)$ or $V(i_2, i_3, i_5, i_6)$. Using Singular

we computed the radical of the ideals (i_2, i_3, i_4, i_5) , (i_2, i_3, i_4, i_6) , (i_2, i_3, i_4, i_7) , (i_2, i_3, i_5, i_6) and (i_2, i_3, i_4, i_8) and checked the required facts.

(This shows that i_8 , i_9 and i_{10} do not vanish on the 2-dimensional pieces mentioned. Note that these invariants do vanish on various 1-dimensional pieces. For example, $i_8^2 \in (i_2, i_3, i_4, i_6, i_7)$, so that i_8 vanishes on $V(i_2, i_3, i_4, i_6, i_7)$, and $i_8^5 \in (i_2, i_3, i_4, i_5, i_6)$, and $i_{10}^2 \in (i_2, i_3, i_4, i_5, i_6)$ and $i_9^3 \in (i_2, i_3, i_4, i_5, i_6) \cap (i_2, i_3, i_4, i_6, i_7) \cap (i_2, i_3, i_5, i_6, i_7)$.)

□

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