Non-Ergodic Delocalization in the Rosenzweig-Porter Model

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Abstract

We consider the Rosenzweig-Porter model $H = V + \sqrt{T}\,\Phi$, where V is a $N \times N$ diagonal matrix, Φ is drawn from the $N \times N$ Gaussian Orthogonal Ensemble, and $N^{-1} \ll T \ll 1$. We prove that the eigenfunctions of H are typically supported in a set of approximately NT sites, thereby confirming the existence of a previously conjectured non-ergodic delocalized phase. Our proof is based on martingale estimates along the characteristic curves of the stochastic advection equation satisfied by the local resolvent of the Brownian motion representation of H.

1 Introduction

This paper is concerned with the eigenfunctions of the Rosenzweig-Porter model [17], which is the simplest example of a random matrix with non-trivial spatial structure and provides a standard interpolation between integrability and chaos. Consisting of a rotationally invariant term and a potential, the model provides a highly simplified and analytically tractable toy model for the localization transition in disordered systems. Furthermore, the Rosenzweig-Porter model and its relatives, the Anderson models on the Bethe lattice and on the random regular graph, have recently received a renewed surge of interest related to the many-body localization transition [2, 11, 12]. In that context, they provide basic examples of phases in which eigenfunctions delocalize over a large number of sites, but not uniformly over the entire volume.

The Hamiltonian in question is defined on $\ell^2(\{1, ..., N\})$ by

$$H = V + \sqrt{T}\,\Phi, \tag{1.1}$$

where $T > 0$,

$$V = \sum_x V_x |\delta_x\rangle\langle\delta_x|$$

is a sufficiently regular diagonal matrix with real entries $\{V_x\}$, and the Hermitian matrix Φ has random entries with normal distribution

$$\langle \delta_y, \Phi \delta_x \rangle \sim \mathcal{N}\left(0, \sqrt{\frac{1+\delta_{xy}}{N}}\right)$$

which are independent up to the symmetry constraint. Here and throughout, $\delta_x \in \ell^2(\{1, ..., N\})$ denotes the site basis element

$$\delta_x(u) = \begin{cases} 1 & \text{if } u = x \\ 0 & \text{if } u \neq x \end{cases}$$

and $\delta_{xy} = \langle \delta_y, \delta_x \rangle$. We will follow Dyson's idea [9] and represent H as a stochastic process

$$H_t = V + \Phi_t,$$

where the entries of Φ_t are normalized Brownian motions

$$\langle \delta_y, \Phi_t \delta_x \rangle = \sqrt{\frac{1+\delta_{xy}}{N}}\, B_{xy}(t)$$

independent up to the symmetry constraint. Hence, H_T has the same distribution as (1.1) and we consider only H_T from now on.

One expects the spectral behavior of H_T to interpolate between V and Φ_T as T increases. In terms of local eigenvalue statistics, recent works have established rigorously that there is a sharp transition at $T = N^{-1}$. On the one hand, if $T \gg N^{-1}$, Landon, Sosoe, and Yau [13,14] proved that the local statistics fall into the Wigner-Dyson-Mehta universality class and agree asymptotically with those of the Gaussian Orthogonal Ensemble. On the other hand, if $T \ll N^{-1}$ and the $\{V_x\}$ are independent random variables, the local statistics converge to a Poisson point process and agree asymptotically with those of V [22]. Nevertheless, the present understanding of the transition is incomplete with regard to the eigenfunctions of H_T. While it is known that the eigenfunctions are completely extended when $T \geq 1$ [15], and slightly weakened localization bounds were proved in [22] when $T \ll N^{-1}$, there are no previous rigorous results concerning the behavior of the eigenfunctions in the intermediate regime $N^{-1} \ll T \ll 1$. Moreover, the nature of the transition in the eigenfunctions in the related Anderson models on the Bethe lattice and random regular graph has been widely disputed even in the physics literature [1,5,8,19]. In the main result of this paper, we confirm the picture conjectured by Facoetti, Vivo, and Biroli [11], by proving that in the intermediate regime a normalized eigenfunction ψ_λ corresponding to $\lambda \in \sigma(H_T)$ delocalizes across approximately those $NT \gg 1$ sites for which V_x is closest to λ. This means that the mass of each eigenfunction spreads to a large number of sites. These sites nevertheless form a vanishing fraction of the entire volume $\{1, ..., N\}$, indicating the existence of a non-ergodic delocalized phase.

Throughout this paper, we will fix a time $T = N^{-1+\delta}$ with $\delta \in (0,1)$ and a spectral domain of the form

$$D = W + i[\eta, 1]$$

where $W \subset \mathbb{R}$ is a bounded interval and $\eta = N^{-1+\alpha}$ is a spectral scale whose parameter $\alpha > 0$ is fixed but may be arbitrarily small. In what follows, we will require V to possess some regularity that will be expressed in terms of the resolvent-like functionals

$$F_I(z) = \frac{1}{N} \sum_{V_x \notin I} \frac{1}{V_x - z},$$

where $I \subset \mathbb{R}$ is a possibly empty interval. Notice that $F_\emptyset(z) = N^{-1} \sum_x (V_x - z)^{-1}$ coincides with the Stieltjes transform of the empirical eigenvalue measure of V.

Assumption 1.1. *There exist $\epsilon > 0$ and constants $K_m, K_\ell, K_i \in (0, \infty)$ such that*

1. $|F_\emptyset(z)| \leq K_m \log N$ *uniformly in* $\operatorname{Im} z > \eta$,
2. $\operatorname{Im} F_\emptyset(z) \geq K_l$ *uniformly in* $z \in \mathbb{C}_+$ *with* $\operatorname{dist}(z, D) \leq \epsilon$, *and,*
3. *if* $z \in D$ *with* $\operatorname{Im} z > K_l T/2$ *and* $I \subset W$ *is an interval with* $\operatorname{Re} z \in I$ *and* $\operatorname{dist}(\operatorname{Re} z, \partial I) > N^{-1+\kappa}$ *for some* $\kappa > \delta$, *then*

$$\operatorname{Im} F_I(z) \leq K_i \frac{\operatorname{Im} z}{N^{-1+\kappa}} + N^{-\delta/4}.$$

If the entries V_x are drawn independently from a compactly supported density $\rho \in L^\infty$, we will show in Section 4 that Assumption 1.1 is satisfied with asymptotically full probability for any interval W on which ρ is bounded below. The restriction of the conclusion of the following Theorem 1.2 to the states in W is then only a mild condition since ρ coincides with the asymptotic density of states of H_T (see, for example, [22]) and hence one expects that the majority of $\sigma(H_T)$ typically lies in $\operatorname{supp} \rho$.

Theorem 1.2. *Let $\kappa > \delta > \theta$ and set*

$$X_\lambda = \{x \in \{1, ..., N\} : |\lambda - V_x| > N^{-1+\kappa}\}.$$

Then, there exists $\gamma > 0$ such that for any $p > 0$ and all sufficiently large N

1. *The normalized eigenfunctions in W carry only negligible mass inside X_λ:*

$$\mathbb{P}\left(\sup_{\lambda \in \sigma(H_T) \cap W} \sum_{x \in X_\lambda} |\psi_\lambda(x)|^2 > N^{-\gamma} \right) \leq N^{-p}.$$

2. *The normalized eigenfunctions in W are maximally extended outside X_λ:*

$$\mathbb{P}\left(\sup_{\lambda \in \sigma(H_T) \cap W} \|\psi_\lambda\|_\infty > N^{-\theta/2} \right) \leq N^{-p}.$$

Theorem 1.2 becomes meaningful when κ and θ are chosen close to δ. Under Assumption 1.1, the number of sites outside X_λ is then bounded by

$$|\{V_x : |\lambda - V_x| \leq N^{-1+\kappa}\}| \leq 2NN^{-1+\kappa} \operatorname{Im} F_\emptyset(\lambda + iN^{-1+\kappa}) \approx N^\kappa \log N \approx NT.$$

Moreover, the fact that

$$|\psi_\lambda(x)|^2 \leq N^{-\theta} \approx (NT)^{-1}$$

shows that the eigenfunctions are maximally extended inside the subvolume $\{1, ..., N\} \backslash X_\lambda$.

The proof of Theorem 1.2 is based on controlling the local resolvent

$$G_t(x, z) = \langle \delta_x, R_t(z)\delta_x \rangle,$$

where $R_t(z) = (H_t - z)^{-1}$. As the Stieltjes transform of the spectral measure at x, $G_t(x, z)$ naturally encodes the average of $|\psi_\lambda(x)|^2$ over intervals of length $\operatorname{Im} z$ centered at $\operatorname{Re} z$. It is shown in [22] that these quantities evolve according to the stochastic differential equation

$$dG_t(x, z) = \left(S_t(z) \frac{\partial}{\partial z} G_t(x, z) + \frac{1}{2N} \frac{\partial^2}{\partial z^2} G_t(x, z) \right) dt + dM_t(x, z), \tag{1.2}$$

where

$$S_t(z) = \frac{1}{N} \operatorname{Tr} R_t(z)$$

is the normalized trace and $M_t(x, z)$ is a martingale given explicitly in terms of $R_t(z)$ below, cf. (2.2). If we retain only the leading term on the right hand side of (1.2), we obtain an advection equation transporting $G_0(x, z)$ along the characteristic curve defined by

$$\dot{z}_t = -S_t(z_t). \tag{1.3}$$

We will prove in this paper that the remaining terms of (1.2) are negligible, which, combined with the previous observation, yields

$$G_T(x, z_T) \approx G_0(x, z). \tag{1.4}$$

Our bounds are strong enough to conclude that for every $z \in \mathbb{C}_+$ with $\operatorname{Im} z \gg N^{-1}$ there exists $w \in \mathbb{C}_+$ with $|w - z| = \mathcal{O}(T)$ such that

$$G_T(x, z) \approx G_0(x, w). \tag{1.5}$$

This means that the effect on the eigenfunctions of perturbing V by Φ_T locally consists of a shift in the energy followed by a smearing of the scales below T. In essence, the change in the local resolvent on the given time scale is through an energy renormalization.

The relations (1.3)–(1.5) amount to strong finite volume versions of the famous semi-circular flow of Pastur [16] localized to a single site and energy. They also bear some similarity to a preliminary Schur complement relations in Erdős, Schlein, and Yau's proof of the local semi-circle law [10], although our results are clearly only valid for Gaussian ensembles. The fact that (1.5) merely changes the spectral parameter at which the local resolvent is evaluated can also be seen as a particular instance of the subordination relations in free probability [4, 20].

A powerful method for studying the eigenfunctions of H_t directly was devised by Bourgade and Yau [7] and developed further by Bourgade, Huang, and Yau [6], whose Theorem 2.1 may also be used to derive the second point of Theorem 1.2 above. The method was adapted to the present problem by Benigni [3]. Here, it yields the local eigenvector statistics even for mesoscopic Wigner perturbations, covering Theorem 1.2, albeit with lower probability.

The paper is organized as follows. In Section 2, we study the properties of the characteristic curves for (1.2). In Section 3, we bound the growth of the local resolvent along the characteristic curves and use this to prove Theorem 1.2. Finally, in Section 4, we prove Assumption 1.1 for random $\{V_x\}$.

2 Characteristic Curves

In this section, we study the properties of the characteristic curve

$$\dot{z}_t = -S_t(z_t), \qquad z_0 = z \tag{2.1}$$

of the transport equation (1.2). However, it is technically more convenient to consider instead the process

$$\xi_t(z) = z_{t\wedge\tau_z}$$

which is stopped at

$$\tau_z = \{\inf t > 0 : \operatorname{Im} z_t \le \eta/2\}.$$

Regarding $R_t(\xi_t(z))$ as a function of the processes $\{B_{uv}(t)\}$ and $\xi_t(z)$, Itô's lemma shows that

$$\begin{aligned} dG_t(x,\xi_t(z)) = &\frac{1}{N}\sum_{u\le v}\langle\delta_x, R_t(\xi_t(z))P_{uv}R_t(\xi_t(z))P_{uv}R_t(\xi_t(z))\delta_x\rangle\, dt \\ &-\frac{1}{\sqrt{N}}\sum_{u\le v}\langle\delta_x, R_t(\xi_t(z))P_{uv}R_t(\xi_t(z))\delta_x\rangle\, dB_{uv}(t) + \dot{\xi}_t(z)\frac{\partial}{\partial\xi}G_t(x,\xi_t(z))\, dt \end{aligned}$$

with

$$P_{uv} = \frac{1}{\sqrt{1+\delta_{uv}}}\left(|\delta_u\rangle\langle\delta_v| + |\delta_v\rangle\langle\delta_u|\right).$$

The piecewise C^1 process $\xi_t(z)$ has vanishing covariation with all the $B_{uv}(t)$. The calculations in the proof of Theorem 2.1 in [22] then show that

$$\begin{aligned} dG_t(x,\xi_t(z)) = &\left(S_t(\xi_t(z))\frac{\partial}{\partial\xi}G_t(x,\xi_t(z)) + \frac{1}{2N}\frac{\partial^2}{\partial\xi^2}G_t(x,\xi_t(z))\right)\, dt \\ &+\dot{\xi}_t(z)\frac{\partial}{\partial\xi}G_t(x,\xi_t(z))\, dt + dM_t(x,z) \end{aligned}$$

with

$$dM_t(x,z) = -\frac{1}{\sqrt{N}}\sum_{u\le v}\langle \delta_x, R_t(\xi_t(z))P_{uv}R_t(\xi_t(z))\delta_x\rangle\, dB_{uv}(t). \tag{2.2}$$

If τ is any stopping time such that $\tau \le \tau_z$ almost surely, (2.1) yields

$$\begin{aligned} G_\tau(x,\xi_\tau(z)) - G_0(x,z) = &\int_0^\tau \frac{1}{2N}\frac{\partial^2}{\partial\xi^2}G_t(x,\xi_t(z))\,dt \\ &- \frac{1}{\sqrt{N}}\sum_{u\le v}\int_0^\tau \langle \delta_x, R_t(\xi_t(z))P_{uv}R_t(\xi_t(z))\delta_x\rangle\, dB_{uv}(t) \end{aligned} \tag{2.3}$$

for the change in the local resolvent along the characteristic curve.

Our next goal is to show that with high probability the change in S_t along the curve $\xi_t(z)$ is small for a sufficiently dense set of initial points z. Let

$$\tilde{D} \subset \{z \in \mathbb{C}_+ : \operatorname{Im} z > \eta\}$$

be some finite set. The next theorem bounds the probability of the event

$$\mathcal{A}_S = \left\{ \sup_{z\in\tilde{D}}\sup_{t\le\tau_z} |S_t(\xi_t(z)) - S_0(z)| > \frac{4}{\sqrt{N\eta}} \right\},$$

showing that with high probability $S_t(\xi_t(z))$ is approximately constant if $|\tilde{D}|$ grows only polynomially in N. In the statement of the theorem, and throughout this paper, $C, c \in (0,\infty)$ denote deterministic constants that are independent of N but whose value may change from instance to instance.

Theorem 2.1. *For every $z \in \mathbb{C}_+$ with $\operatorname{Im} z > \eta$ we have*

$$\mathbb{P}\left(\sup_{t\le\tau_z}|S_t(\xi_t(z)) - S_0(z)| > \frac{4}{\sqrt{N\eta}}\right) \le 2e^{-\frac{1}{2}N\eta} \tag{2.4}$$

and therefore $\mathbb{P}(\mathcal{A}_S) \le C|\tilde{D}|e^{-\frac{1}{2}N\eta}$.

Proof. By averaging (2.3) over all x, we see that the process

$$\tilde{S}_t = S_{t\wedge\tau_z}(\xi_t(z))$$

satisfies

$$\begin{aligned} \tilde{S}_t - \tilde{S}_0 = &\int_0^{t\wedge\tau_z} \frac{1}{2N}\frac{\partial^2}{\partial\xi^2}S_s(\xi_s(z))\,ds \\ &- \frac{1}{\sqrt{N^3}}\sum_x\sum_{u\le v}\int_0^{t\wedge\tau_z}\langle \delta_x, R_s(\xi_s(z))P_{uv}R_s(\xi_s(z))\delta_x\rangle\, dB_{uv}(s). \end{aligned}$$

The drift component of $\tilde{S}$ is bounded by

$$\begin{aligned}\int_0^{t\wedge\tau_z} \frac{1}{2N}\left|\frac{\partial^2}{\partial\xi^2}S_s(\xi_s(z))\right| ds &\le \frac{1}{N}\int_0^{t\wedge\tau_z} \frac{\operatorname{Im} S_s(\xi_s(z))}{(\operatorname{Im}\xi_s(z))^2}\, ds \\ &= \frac{1}{N}\int_0^{t\wedge\tau_z} \frac{-d(\operatorname{Im}\xi_s(z))}{(\operatorname{Im}\xi_s(z))^2} \\ &= \frac{1}{N\operatorname{Im}\xi_t(z)} - \frac{1}{N\operatorname{Im} z} \le \frac{2}{N\eta}.\end{aligned}$$

The martingale part of $\tilde{S}$ is given by

$$\begin{aligned} M_t &= -\frac{1}{\sqrt{N^3}}\sum_x\sum_{u\le v}\int_0^{t\wedge\tau_z} \langle \delta_x, R_s(\xi_s(z))P_{uv}R_s(\xi_s(z))\delta_x\rangle\, dB_{uv}(s) \\ &= -\frac{1}{\sqrt{N^3}}\sum_{u,v}\sqrt{1+\delta_{uv}}\int_0^{t\wedge\tau_z}\langle\delta_v, R_s(\xi_s(z))^2\delta_u\rangle\, dB_{uv}(s)\,.\end{aligned}$$

Its quadratic variation may be expressed as

$$\begin{aligned}[M]_t &\le \frac{2}{N^3}\int_0^{t\wedge\tau_z}\sum_{u,v}\left|\langle\delta_v, R_s(\xi_s(z))^2\delta_u\rangle\right|^2\, ds \\ &\le \frac{2}{N^2}\int_0^{t\wedge\tau_z}\frac{\operatorname{Im} S_s(\xi_s(z))}{(\operatorname{Im}\xi_s(z))^3}\, ds. \\ &= \frac{1}{N^2}\int_0^{t\wedge\tau_z}\frac{-2}{(\operatorname{Im}\xi_s(z))^3}\, d(\operatorname{Im}\xi_s(z)) \\ &= \frac{1}{(N\operatorname{Im}\xi_t(z))^2} - \frac{1}{(N\operatorname{Im} z)^2} \le \frac{4}{(N\eta)^2}.\end{aligned}$$

It follows that there exists a Brownian motion $\tilde{B}$ such that

$$\sup_t|\tilde{S}_t - \tilde{S}_0| \le \sup_t\left(\frac{2}{N\eta} + \left|\tilde{B}_{[M]_t}\right|\right) \le \frac{2}{\sqrt{N\eta}} + \sup_{t\le 4/(N\eta)^2}|\tilde{B}_t|.$$

Applying the reflection principle to $\tilde{B}$ we obtain (2.4) and the second assertion follows from the union bound. ◻

Once we know that $S_t(\xi_t(z))$ is approximately constant, this term can be inserted into integrals involving $\xi_t(z)$ more or less at will, and the substitution trick from Theorem 2.1 gives bounds improving on the trivial bound by a factor of η. We illustrate this in the following corollary, which will prove useful in extending our method to the local resolvents.

Corollary 2.2. *If $\mathcal{A}_S$ does not occur, then*

$$\int_0^{t\wedge\tau_z}\frac{1}{(\operatorname{Im}\xi_s(z))^2}\, ds \le \frac{4}{K_l\eta}$$

for all $t > 0$ and $z \in D\cap\tilde{D}$.

Proof. If $\mathcal{A}_S$ does not occur and $z \in D \cap \tilde{D}$, then for sufficiently large N

$$\inf_{s \le t \wedge \tau_z} \operatorname{Im} S_s(\xi_s(z)) \ge \operatorname{Im} S_0(z) - \frac{4}{\sqrt{N\eta}} \ge \frac{K_l}{2}$$

where K_l is the lower bound from Assumption 1.1. Hence,

$$\begin{aligned}\int_0^{t\wedge\tau_z} \frac{1}{(\operatorname{Im}\xi_s(z))^2}\,ds &\le \frac{2}{K_l}\int_0^{t\wedge\tau_z} \frac{\operatorname{Im} S_s(\xi_s(z))}{(\operatorname{Im}\xi_s(z))^2}\,ds \\ &= \frac{2}{K_l}\int_0^{t\wedge\tau_z} \frac{d(\operatorname{Im}\xi_s(z))}{(\operatorname{Im}\xi_s(z))^2} \le \frac{4}{K_l\eta}.\end{aligned}$$

□

The Picard-Lindelöf theorem and the Herglotz property of S_s imply that, almost surely, for every $z \in D$ there exists a $w \in \mathbb{C}_+$ with $\xi_T(w) = z$ satisfying the a-priori deterministic bound

$$|w - z| \le \int_0^T |S_s(\xi_s(w))|\,ds \le \frac{T}{\eta}.$$

In order for Theorem 2.1 to be useful in the study of the function S_T, we need to guarantee that a sufficiently dense subset of D is of the form $\xi_T(w)$ with $w \in \tilde{D}$. To this end, we define the distance

$$r = \min\left\{KT\eta^2, N^{-2\theta}\eta^3, N^{-(1+2\gamma)}\eta^3\right\}, \tag{2.5}$$

where

$$K = \sup_{\operatorname{Im} z > \eta} |S_0(z)| + \frac{4}{\sqrt{N\eta}} \le C\log N, \tag{2.6}$$

and $\gamma, \theta > 0$ are the parameters from the statement of Theorem 1.2. We now require $\tilde{D}$ to be such that

$$\operatorname{dist}(z, \tilde{D}) \le r$$

for all $z \in \mathbb{C}_+$ with $\operatorname{Im} z > \eta$ and $\operatorname{dist}(z, D) \le T/\eta$. The grid $\tilde{D}$ can hence be chosen such that its cardinality is bounded by

$$|\tilde{D}| \le C(\eta r)^{-2}.$$

The following theorem provides a Lipschitz constant for the characteristic flow which grows only polynomially in η. The resulting bound is a significant improvement on the exponential bound provided by the direct application of Grönwall's inequality and enables us to keep the cardinality of $\tilde{D}$ polynomial in N.

Theorem 2.3. *Suppose $\mathcal{A}_S$ does not occur and N is sufficiently large. Then for every $z \in D$ there exists $w \in \tilde{D}$ such that:*

1. $|\xi_T(w) - z| \le C\eta^{-2}r$,

2. $|w - z| \le CKT$ *with* K *as in* (2.6)*, and*

3. $\operatorname{Im} w \ge \frac{1}{2} K_l T$ *with* K_l *defined in Assumption 1.1.*

Proof. By the construction of $\tilde{D}$, for any $z \in D$ there exist $w_0 \in \mathbb{C}_+$ with $\xi_T(w_0) = z$ and $w \in \tilde{D}$ with $|w - w_0| \le r$. If $t \le \tau_{w_0} \wedge \tau_w$, the evolution (2.1) yields

$$\begin{aligned}
|\xi_t(w_0) - \xi_t(w)| &\le |w_0 - w| + \int_0^t |S_s(\xi_s(w_0)) - S_s(\xi_s(w))|\, ds \\
&\le |w_0 - w| + \frac{1}{N} \int_0^t \sum_i \frac{|\xi_s(w_0) - \xi_s(w)|}{|\lambda_i(s) - \xi_s(w_0)||\lambda_i(s) - \xi_s(w)|}\, ds.
\end{aligned}$$

Using the inequality $ab \le \frac{1}{2}(a^2 + b^2)$, the integral in the last term is bounded by

$$\begin{aligned}
&\frac{1}{2} \int_0^t |\xi_s(w_0) - \xi_s(w)| \frac{1}{N} \sum_i \left(\frac{1}{|\lambda_i(s) - \xi_s(w_0)|^2} + \frac{1}{|\lambda_i(s) - \xi_s(w)|^2} \right) ds \\
&\le \frac{1}{2} \int_0^t |\xi_s(w_0) - \xi_s(w)| \left(\frac{\operatorname{Im} S_s(\xi_s(w_0))}{\operatorname{Im} \xi_s(w_0)} + \frac{\operatorname{Im} S_s(\xi_s(w))}{\operatorname{Im} \xi_s(w)} \right) ds
\end{aligned}$$

so Grönwall's inequality shows that

$$\begin{aligned}
\log \frac{|\xi_t(w_0) - \xi_t(w)|}{|w_0 - w|} &\le \frac{1}{2} \int_0^t \frac{\operatorname{Im} S_s(\xi_s(w_0))}{\operatorname{Im} \xi_s(w_0)} + \frac{\operatorname{Im} S_s(\xi_s(w))}{\operatorname{Im} \xi_s(w)}\, ds \\
&= -\frac{1}{2} \int_0^t \frac{d(\operatorname{Im} \xi_s(w_0))}{\operatorname{Im} \xi_s(w_0)} - \frac{1}{2} \int_0^t \frac{d(\operatorname{Im} \xi_s(w))}{\operatorname{Im} \xi_s(w)} \\
&= \log \sqrt{\frac{\operatorname{Im} w_0}{\operatorname{Im} \xi_t(w_0)} \frac{\operatorname{Im} w}{\operatorname{Im} \xi_t(w)}}.
\end{aligned}$$

Thus, using $\operatorname{Im} w \le \operatorname{Im} w_0 + r$, $\operatorname{Im} w_0 \le 1 + T\eta^{-1} \le c\eta^{-1}$, and the stopping rules, we obtain

$$|\xi_t(w_0) - \xi_t(w)| \le C\eta^{-2} |w_0 - w| \le C\eta^{-2} r \tag{2.7}$$

for all $t \le \tau_{w_0} \wedge \tau_w$. Since $\operatorname{Im} \xi_T(w_0) = z$ and $\operatorname{Im} \xi_t(w_0)$ is decreasing, $\tau_{w_0} > T$, so (2.7) and the definition of r shows that for sufficiently large N we have $|\xi_t(w_0) - \xi_t(w)| \le \eta/4$ for all $t \le T \wedge \tau_w$. If it were true that $\tau_w < T$, we would obtain the contradiction

$$\frac{\eta}{2} = \operatorname{Im} \xi_{\tau_w}(w) \ge \operatorname{Im} \xi_{\tau_w}(w_0) - \frac{\eta}{4} \ge \eta - \frac{\eta}{4}.$$

Hence (2.7) is valid for $t = T$, establishing the first claim of the theorem. If $\mathcal{A}_S$ does not occur, then

$$|\xi_T(w) - w| \le \int_0^T |S_s(\xi_s(w))|\, ds \le \int_0^T |S_0(w)| + \frac{4}{\sqrt{N\eta}}\, ds \le KT$$

since $w \in \tilde{D}$. Hence the definition of r and (2.7) yield

$$|w - z| \leq |w - \xi_T(w)| + |\xi_T(w) - z| \leq KT + CKT = CKT.$$

proving the second claim of the theorem. The second claim also implies that $\operatorname{dist}(w, D) \leq \epsilon$ for sufficiently large N so that Assumption 1.1 guarantees $\operatorname{Im} S_0(w) \geq K_l$. On the complement of $\mathcal{A}_S$ this yields

$$\operatorname{Im} w = \operatorname{Im} \xi_T(w) + \int_0^T \operatorname{Im} S_s(\xi_s(w))\, ds \geq T\left(\operatorname{Im} S_0(w) - \frac{4}{\sqrt{N\eta}}\right) \geq \frac{K_l T}{2}$$

for sufficiently large N. □

3 Local Resolvent Bounds

Since $S_t(z)$ is entirely featureless regarding a possible localization transition when $\operatorname{Im} z \gg N^{-1}$, we now turn our attention to controlling the local resolvents $G_t(x, z)$ along the characteristic $\xi_t(z)$. Unlike S_t, the function $G_t(x, \cdot)$ may be heavily concentrated around certain energies in non-ergodic regimes. Therefore, its derivative may be large in all directions and we cannot expect an exact analogue of Theorem 2.1 to hold true for all energies. However, one may hope that the change in $G_t(x, z)$ along the characteristic is small in those regions where $G_t(x, z)$ itself is small. We encode this phenomenon in the event

$$\mathcal{A}_G(\ell) = \left\{ \sup_x \sup_{z \in D \cap \tilde{D}} \sup_{s \leq \tau_z} \frac{\operatorname{Im} G_s(x, \xi_s(z))}{\operatorname{Im} G_0(x, z)} > N^\ell \right\},$$

whose probability does in fact decay as $N \to \infty$. The proof is somewhat reminiscent of a Grönwall-type argument for martingales, which is greatly facilitated by the built-in control of the running maximum. Still, the basic mechanism behind the following argument is somewhat different from the stochastic Grönwall lemmas that previously appeared in [18, 21].

Theorem 3.1. *For every $\ell > 0$ and $p > 0$ we have*

$$\mathbb{P}\left(\mathcal{A}_G(\ell)\right) \leq N^{-p}$$

for all sufficiently large N.

Proof. Fix $z \in D \cap \tilde{D}$ and consider the stopping time

$$\tau = \tau_z \wedge \inf\left\{ t \geq 0 : \int_0^{t \wedge \tau_z} \frac{1}{(\operatorname{Im} \xi_s(z))^2}\, ds \geq \frac{5}{K_l \eta} \right\},$$

where K_l is the lower bound from Assumption 1.1. As in Theorem 2.1, the stopped process $\tilde{G}_t = G_{t\wedge\tau}(x, \xi_{t\wedge\tau}(z))$ satisfies

$$\begin{aligned}\tilde{G}_t - \tilde{G}_0 = &\int_0^{t\wedge\tau} \frac{1}{2N}\frac{\partial^2}{\partial\xi^2} G_s(x, \xi_s(z))\, ds \\ &- \frac{1}{\sqrt{N}} \sum_{u\le v} \int_0^{t\wedge\tau} \langle \delta_x, R_s(\xi_s(z)) P_{uv} R_s(\xi_s(z))\delta_x\rangle \, dB_{uv}(s).\end{aligned}$$

The drift component of $\tilde{G}$ is bounded by

$$\begin{aligned}\int_0^{t\wedge\tau} \frac{1}{2N}\left|\frac{\partial^2}{\partial\xi^2} G_s(x, \xi_s(z))\right| \, ds &\le \left(\sup_{s\le T} \operatorname{Im} \tilde{G}_s\right) \int_0^{T\wedge\tau} \frac{1}{N(\operatorname{Im}\xi_s(z))^2}\, ds \\ &\le \frac{5}{K_l N\eta}\left(\sup_{s\le T} \operatorname{Im} \tilde{G}_s\right),\end{aligned}$$

and, letting M denote the martingale part of $\tilde{G}$, its quadratic variation is bounded as follows,

$$\begin{aligned}[M]_T &\le \frac{2}{N}\int_0^{T\wedge\tau} \sum_{u,v} |\langle\delta_x, R_s(\xi_s(z))\delta_u\rangle\langle\delta_v, R_s(\xi_s(z))\delta_x\rangle|^2 \; ds \\ &= \frac{2}{N}\int_0^{T\wedge\tau} \left(\sum_u |\langle\delta_x, R_s(\xi_s(z))\delta_u\rangle|^2\right)\left(\sum_v |\langle\delta_v, R_s(\xi_s(z))\delta_x\rangle|^2\right) ds \\ &= \frac{2}{N}\int_0^{T\wedge\tau} \left(\frac{\operatorname{Im} G_s(x, \xi_s(z))}{\operatorname{Im}\xi_s(z)}\right)^2 ds \\ &\le \left(\sup_{s\le T\wedge\tau} \operatorname{Im} G_s(x,\xi_s(z))\right)^2 \int_0^{T\wedge\tau} \frac{2}{N(\operatorname{Im}\xi_s(z))^2}\, ds \\ &\le \frac{10}{K_l N\eta}\left(\sup_{s\le T}\operatorname{Im}\tilde{G}_s\right)^2.\end{aligned}$$

Hence,

$$\sup_{s\le T} \operatorname{Im}\tilde{G}_s \le \operatorname{Im}\tilde{G}_0 + \frac{5}{K_l N\eta} \sup_{s\le T}\operatorname{Im}\tilde{G}_s + \sup_{s\le T}|M_s|$$

so the Burkholder-Davis-Gundy inequality (with exponent $q > 0$ and constant C_q) yields

$$
\begin{aligned}
\left(1-\frac{5}{K_l N\eta}\right)\left(\mathbb{E}\left|\sup_{s\le T}\operatorname{Im}\tilde G_s\right|^q\right)^{1/q} &\le \operatorname{Im}\tilde G_0+\left(\mathbb{E}\left|\sup_{s\le T}|M_s|\right|^q\right)^{1/q}\\
&\le \operatorname{Im}\tilde G_0+C_q\left(\mathbb{E}\,[M]_T^{q/2}\right)^{1/q}\\
&\le \operatorname{Im}\tilde G_0+C_q\sqrt{\frac{10}{K_l N\eta}}\left(\mathbb{E}\left|\sup_{s\le T}\operatorname{Im}\tilde G_s\right|^q\right)^{1/q}.
\end{aligned}
$$

Since $N\eta\to\infty$, we can choose N large enough such that $(1+C_q)\sqrt{\frac{10}{K_l N\eta}} < 1/2$. Rearranging and applying Markov's inequality shows

$$
\mathbb{P}\left(\sup_{s\le T\wedge\tau}\operatorname{Im}G_s(x,\xi_s(z)) > 4N^{\ell}\operatorname{Im}G_0(x,z)\right)\le N^{-\ell q}.
$$

By Corollary 2.2, $\tau=\tau_z$ on the event $\mathcal{A}_S$ and we conclude that

$$
\mathbb{P}\left(\sup_{s\le T\wedge\tau_z}\operatorname{Im}G_s(x,\xi_s(z)) > 4N^{\ell}\operatorname{Im}G_0(x,z)\right)\le N^{-\ell q}+\mathbb{P}(\mathcal{A}_S),
$$

so, choosing q large enough, the theorem follows from the union bound. □

To prove Theorem 1.2, it remains only to combine the previous results with the fact that $G_T(x,\cdot)$ is the Stieltjes transform of the spectral measure at x.

Proof of Theorem 1.2. We now specify the parameters $\alpha,\gamma,\ell>0$ occuring in the spectral scale $\eta=N^{-1+\alpha}$, the definition of r in (2.5), and the event $\mathcal{A}_G(\ell)$ of Theorem 3.1 by requiring that

$$
\alpha+\ell+\delta<\kappa,\quad \alpha+\ell<\delta/4,\quad \gamma<\kappa-(\alpha+\ell+\delta).
$$

Suppose that neither of the events $\mathcal{A}_S,\mathcal{A}_G(\ell)$ of Theorems 2.1 and 3.1 occur, which is the case with probability $1-N^{-p}$ provided N is sufficiently large. For every $\lambda\in\sigma(H_T)\cap W$,

$$
\sum_{x\in X_\lambda}|\psi_\lambda(x)|^2\le\sum_{x\in X_\lambda}\sum_{E\in\sigma(H_T)}\frac{\eta^2}{(E-\lambda)^2+\eta^2}|\psi_E(x)|^2=\eta\sum_{x\in X_\lambda}\operatorname{Im}G_T(x,z)
$$

with $z=\lambda+i\eta$. By Theorem 2.3, there exists $w\in\tilde D$ is such that $|w-z|\le CKT$, $\operatorname{Im}w>K_lT/2$, and $|\xi_T(w)-z|\le C\eta N^{-(1+2\gamma)}$. Hence, for sufficiently large N,

$$
\operatorname{Re}w\in I:=1\{|V_x-\lambda|>N^{-1+\kappa}\}
$$

and $\operatorname{dist}(\operatorname{Re} w, \partial I) > \frac{1}{2}N^{-1+\kappa}$. Using Assumption 1.1 and the η^{-2}-Lipschitz continuity of $G_T(x,z)$, this yields

$$\begin{aligned}
\sum_{x\in X_\lambda} |\psi_\lambda(x)|^2 &\le \eta \sum_{x\in X_\lambda} \operatorname{Im} G_T(x, \xi_T(w)) + CN^{-2\gamma} \\
&\le \eta N^\ell \sum_{x\in X_\lambda} \operatorname{Im} G_0(x,w) + CN^{-2\gamma} \\
&= N^{\alpha+\ell} \operatorname{Im} F_I(w) + CN^{-2\gamma} \\
&\le CN^{\alpha+\ell}\left(\frac{\operatorname{Im} w}{N^{-1+\kappa}} + N^{-\delta/4}\right) + CN^{-2\gamma}.
\end{aligned}$$

Since $\operatorname{Im} w \le \eta + KT \le CN^{-1+\delta}\log N$, the last term is bounded by $N^{-\gamma}$ if and N is large enough, proving the first claim of the theorem.

If, in addition, we require that $\alpha+\ell < \delta-\theta$, the second claim follows by the same token. Combining the Lipschitz continuity of $G_T(x,z)$ with $|\xi_T(w)-z| \le C\eta N^{-2\theta}$, we obtain

$$\begin{aligned}
|\psi_\lambda(x)|^2 &\le \eta \operatorname{Im} G_T(x, \lambda+i\eta) \\
&\le \eta \operatorname{Im} G_T(x, \xi_T(w)) + CN^{-2\theta} \\
&\le \eta N^\ell G_0(x,w) + CN^{-2\theta} \\
&\le CN^{\alpha+\ell-\delta} + CN^{-2\theta} \le N^{-\theta}
\end{aligned}$$

since $\operatorname{Im} w > K_l T/2$. □

4 Regularity Estimates for Random $\{V_x\}$

This section is devoted to the verification of Assumption 1.1 in the case that the $\{V_x\}$ are drawn independently from a compactly supported density $\rho \in L^\infty$. We will assume that ρ is bounded below in a neighborhood of W, i.e. there exists $\epsilon > 0$ such that

$$\inf_{v\in W(\epsilon)} \rho(v) > 0$$

with $W(\epsilon) = W + [-\epsilon, \epsilon]$. We start by proving a concentration inequality in the spirit of Cramér's theorem for F_I, which is uniform in spectral domains of the form

$$D(J,\zeta) = \{z \in \mathbb{C}_+ : \operatorname{Re} z \in J, \quad \zeta \le \operatorname{Im} z \le 1\}.$$

Theorem 4.1. *Let $I \subset \mathbb{R}$ and let $J \subset \mathbb{R}$ be bounded. Then*

$$\mathbb{P}\left(\sup_{z\in D(J,\zeta)} |\operatorname{Im} F_I(z) - \mathbb{E}\operatorname{Im} F_I(z)| > \mu\right) \le C|J|\mu^{-2}\zeta^{-4}e^{-c\mu\sqrt{N\zeta}}$$

for all $\mu > 0$.

Proof. Let $z = \alpha + i\beta$. Performing the substitution $v = (\tilde{v} - \alpha)/\beta$ and denoting the indicator of $\mathbb{R} \setminus I$ by χ, we obtain

$$\begin{aligned}
\mathbb{E}\, e^{t \mathrm{Im}\, F_I(z)} &= \left(\beta \int \rho(\alpha + \beta v) \exp\left(\frac{t}{N\beta} \chi(\alpha + \beta v) \frac{1}{1+v^2} \right) dv \right)^N \\
&\leq \left(1 + \frac{t \mathbb{E}\, \mathrm{Im}\, F_I(z)}{N} + \frac{t^2 \|\rho\|_\infty}{N^2 \beta} \int \left(\frac{1}{1+v^2} \right)^2 \exp\left(\frac{t}{N\beta} \frac{1}{1+v^2} \right) dv \right)^N
\end{aligned}$$

by Taylor's theorem. We choose $t = \sqrt{N\beta}$. Since $(1+v^2)^{-2} \in L^1$ and

$$\frac{t}{N\beta} \frac{1}{1+v^2} \leq \sqrt{2},$$

there exists an absolute constant $C < \infty$ such that

$$\begin{aligned}
\mathbb{E}\, e^{t \mathrm{Im}\, F_I(z)} &\leq \left(1 + \frac{t \mathbb{E}\, \mathrm{Im}\, F_I(z)}{N} + C \left(\frac{t}{N\beta} \right)^2 \beta \right)^N \\
&\leq \exp\left(N \left(\frac{t \mathbb{E}\, \mathrm{Im}\, F_I(z)}{N} + C \left(\frac{t}{N\beta} \right)^2 \beta \right) \right) \\
&= \exp(t \mathbb{E}\, \mathrm{Im}\, F_I(z)) \exp\left(\frac{C t^2}{N\beta} \right).
\end{aligned}$$

Using an exponential Chebyshev argument, we conclude that

$$\begin{aligned}
\mathbb{P}\left(\mathrm{Im}\, F_I(z) \geq \mathbb{E}\, \mathrm{Im}\, F_I(z) + \mu \right) &\leq e^{-t(\mathbb{E}\, \mathrm{Im}\, F_I(z) + \mu)} \mathbb{E}\, e^{t \mathrm{Im}\, F_I(z)} \\
&\leq e^{-t\mu} \exp\left(\frac{C t^2}{N\beta} \right) \\
&\leq C\, e^{-c\mu\sqrt{N\zeta}}.
\end{aligned}$$

The proof of the lower bound works the same way. Replacing the previous Chebyshev bound with

$$\mathbb{P}\left(\mathrm{Im}\, F_I(z) \leq \mathbb{E}\, \mathrm{Im}\, F_I(z) - \mu \right) \leq e^{-t(\mathbb{E}\, \mathrm{Im}\, F_I(z) - \mu)} \mathbb{E}\, e^{-t \mathrm{Im}\, F_I(z)},$$

yields that for every fixed $z \in D(J, \zeta)$

$$\mathbb{P}\left(|\mathrm{Im}\, F_I(z) - \mathbb{E}\, \mathrm{Im}\, F_I(z)| > \mu \right) \leq C\, e^{-c\mu\sqrt{N\zeta}}.$$

Since $D(J, \zeta)$ is bounded, there exists a set of at most $C|J|\mu^{-2}\zeta^{-4}$ points $\{z_k\} \subset D(J, \zeta)$ such that for every $z \in D(J, \zeta)$ there exists k with $|z - z_k| \leq \frac{\mu\zeta^2}{12}$. By the union bound,

$$\mathbb{P}\left(\sup_k |\mathrm{Im}\, F_I(z_k) - \mathbb{E}\, \mathrm{Im}\, F_I(z_k)| > \frac{\mu}{3} \right) \leq C|J|\mu^{-2}\zeta^{-4} e^{-c\mu\sqrt{N\zeta}}.$$

But $\operatorname{Im} F_I$ and $\mathbb{E} \operatorname{Im} F_I$ are $(2/\zeta)^2$-Lipschitz continuous in $D(J,\zeta)$ and thus

$$\begin{aligned}|\operatorname{Im} F_I(z) - \mathbb{E}\operatorname{Im} F_I(z)| &\le |\operatorname{Im} F_I(z) - \operatorname{Im} F_I(z_k)| + |\operatorname{Im} F_I(z_k) - \mathbb{E}\operatorname{Im} F_I(z_k)| \\ &\quad + |\mathbb{E}\operatorname{Im} F_I(z_k) - \mathbb{E}\operatorname{Im} F_I(z)| \le \mu,\end{aligned}$$

extending the bound to all $z \in D(J,\zeta)$. □

Since ρ was assumed to be bounded below in $W(\epsilon)$, the corresponding lower bound for $\operatorname{Im} F_\emptyset$ in the ϵ-fattening of the original spectral domain D follows immediately, proving the second point in Assumption 1.1.

Corollary 4.2. *There exists $K_l \in (0,\infty)$ such that*

$$\mathbb{P}\left(\inf_{\operatorname{dist}(z,D)\le\epsilon} \operatorname{Im} F_\emptyset(z) < K_l\right) \le C\eta^{-4} e^{-c\sqrt{N\eta}}.$$

Next, we combine the previous estimates with a standard argument for the Hilbert transform to produce a logarithmic bound for $|F_\emptyset|$, proving the first point in Assumption 1.1.

Corollary 4.3. *There exists $K_m \in (0,\infty)$ such that*

$$\mathbb{P}\left(\sup_{\operatorname{Im} z>\eta} |F_\emptyset(z)| > K_m + K_m \log\left(1+\eta^{-2}\right)\right) \le C\eta^{-4} e^{-c\sqrt{N\eta}}.$$

Proof. Using that ρ is compactly supported and the trivial estimate

$$\operatorname{Im} S_0(z) \le \frac{1}{\operatorname{dist}(z, \operatorname{supp}\rho)}$$

Theorem 4.1 with $J = \operatorname{supp}\rho + [-1,1]$ and $\zeta = \eta/2$ shows that there exist $C, c, K_m \in (0,\infty)$ such that

$$\mathbb{P}\left(\sup_{\operatorname{Im} z > \frac{\eta}{2}} \operatorname{Im} S_0(z) > K_m\right) \le C\eta^{-4} e^{-c\sqrt{N\eta}}.$$

Letting

$$Q_z = \frac{1}{\pi}\frac{t - \operatorname{Re} z}{(t-\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$$

be the conjugate Poisson kernel and writing $z = \alpha + i(\beta/2)$, we see that on the complement of this event

$$\begin{aligned}\operatorname{Re} S_0(\alpha+i\beta) &= \int \operatorname{Im} S_0\left(t-z\right) Q_{i\frac{\beta}{2}}(t)\,dt \\ &= \int_{[-1,1]} \operatorname{Im} S_0\left(t-z\right) Q_{i\frac{\beta}{2}}(t)\,dt + \int_{\mathbb{R}\setminus[-1,1]} \operatorname{Im} S_0\left(t-z\right) Q_{i\frac{\beta}{2}}(t)\,dt \\ &\le K_m \frac{1}{\pi}\int_{-1}^{1} \frac{|t|}{t^2+\beta^2}\,dt + \frac{1}{\pi}\int \operatorname{Im} S_0\left(t + i\frac{\beta}{2}\right)\,dt \\ &\le K_m \log\left(1+\beta^{-2}\right) + 1.\end{aligned}$$

□

Finally, we use an entropy argument to prove the third point of Assumption 1.1. Let $\{I_k\}$ be a collection of at most CN adjacent intervals covering $W(\epsilon)$ with $|I_k| = \frac{1}{4}N^{-1+\kappa}$ and set

$$J_k = I_{k-1} \cup I_k \cup I_{k+1}.$$

We prove the desired inequality on the event

$$\bigcup_k \left\{ \sup_{z \in D(J,\zeta)} \operatorname{Im} F_{J_k}(z) - \mathbb{E} \operatorname{Im} F_{J_k}(z) \leq N^{-\delta/4} \right\}$$

with $J = W(\epsilon)$ and $\zeta = K_l T/2$. By Theorem 4.1, the probability of this event is close to one since $N^{\delta/4} \ll N^{\delta/2} = \sqrt{NT}$. Now let $I \subset W$ and $z \in D$ be such that $\operatorname{Re} z \in I$ and $\operatorname{dist}(\operatorname{Re} z, \partial I) > \frac{1}{2}N^{-1+\kappa}$. Then there exists k such that $\operatorname{Re} z \in I_k$ and $J_k \subset I$, so that

$$\operatorname{Im} F_I(z) \leq \operatorname{Im} F_{J_k}(z) \leq \mathbb{E} \operatorname{Im} F_{J_k}(z) + N^{-\delta/4}.$$

But $\operatorname{dist}(\operatorname{Re} z, \partial J_k) \geq \frac{1}{4}N^{-1+\kappa}$ and hence

$$\mathbb{E}\, F_{J_k}(z) = \int_{\mathbb{R}\setminus J_k} \frac{\operatorname{Im} z}{(v - \operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \rho(v)\, dv \leq K_i \frac{\operatorname{Im} z}{N^{-1+\kappa}}.$$

Acknowledgment

It is a pleasure to thank L. Benigni for interesting discussions about the Rosenzweig-Porter model during the PCMI Summer Session (funded by NSF grant DMS:1441467). This work was supported by the DFG (WA 1699/2-1).

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