GENERALIZED MINIMUM DISTANCE FUNCTIONS

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Abstract. Using commutative algebra methods we study the generalized minimum distance function (gmd function) and the corresponding generalized footprint function of a graded ideal in a polynomial ring over a field. If $\mathbb{X}$ is a set of projective points over a finite field and I is its vanishing ideal, we show that the gmd function and the Vasconcelos function of I are equal to the r-th generalized Hamming weight of the corresponding Reed-Muller-type code $C_{\mathbb{X}}(d)$ of degree d. We show that the generalized footprint function of I gives a lower bound for the r-generalized Hamming weight of $C_{\mathbb{X}}(d)$. As an application to coding theory we show an explicit formula and a combinatorial formula for the second generalized Hamming weight of an affine cartesian code.

1. Introduction

Let $S=K[t_1,\dots,t_s]=\oplus_{d=0}^{\infty}S_d$ be a polynomial ring over a field K with the standard grading and let $I\neq(0)$ be a graded ideal of S. In this work we extend the scope of [17] by considering generalized footprint and minimum distance functions. Given $d,r\in\mathbb{N}_+$, let $\mathcal{F}_{d,r}$ be the set:

$$\mathcal{F}_{d,r}:=\{\,\{f_1,\dots,f_r\}\subset S_d\,|\,\overline{f}_1,\dots,\overline{f}_r \text{ are linearly independent over } K, (I\colon(f_1,\dots,f_r))\neq I\},$$

where $\overline{f}=f+I$ is the class of f modulo I, and $(I\colon J)=\{h\in S|\,hJ\subset I\}$ is a quotient ideal.

We denote the *degree* of S/I by $\deg(S/I)$. The function $\delta_I\colon\mathbb{N}_+\times\mathbb{N}_+\to\mathbb{Z}$ given by

$$\delta_I(d,r):=\begin{cases}\deg(S/I)-\max\{\deg(S/(I,F))\,|\,F\in\mathcal{F}_{d,r}\} & \text{if } \mathcal{F}_{d,r}\neq\emptyset,\\ \deg(S/I) & \text{if } \mathcal{F}_{d,r}=\emptyset,\end{cases}$$

is called the *generalized minimum distance function* of I, or simply the *gmd function* of I. To compute $\delta_I(d,r)$ is a difficult problem. One of our aims is to introduce lower bounds for $\delta_I(d,r)$ which are easier to compute.

Fix a monomial order $\prec$ on S. Let $\mathrm{in}_\prec(I)$ be the initial ideal of I and let $\Delta_\prec(I)$ be the *footprint* of S/I consisting of all the *standard monomials* of S/I with respect to $\prec$. The footprint of S/I is also called the *Gröbner éscalier* of I. Given integers $d,r\geq 1$, let $\mathcal{M}_{\prec,d,r}$ be the set of all subsets M of $\Delta_\prec(I)_d:=\Delta_\prec(I)\cap S_d$ with r distinct elements such that $(\mathrm{in}_\prec(I)\colon(M))\neq\mathrm{in}_\prec(I)$. The *generalized footprint function* of I, denoted fp_I, is the function $\mathrm{fp}_I\colon\mathbb{N}_+\times\mathbb{N}_+\to\mathbb{Z}$ given by

$$\mathrm{fp}_I(d,r):=\begin{cases}\deg(S/I)-\max\{\deg(S/(\mathrm{in}_\prec(I),M))\,|\,M\in\mathcal{M}_{\prec,d,r}\} & \text{if } \mathcal{M}_{\prec,d,r}\neq\emptyset,\\ \deg(S/I) & \text{if } \mathcal{M}_{\prec,d,r}=\emptyset.\end{cases}$$

The definition of $\delta_I(d,r)$ was motivated by the notion of generalized Hamming weight of a linear code [13, 24]. For convenience we recall this notion. Let $K=\mathbb{F}_q$ be a finite field and let C be a $[m,k]$ *linear code* of *length* m and *dimension* k, that is, C is a linear subspace of K^m

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with $k = \dim_K(C)$. Let $1 \leq r \leq k$ be an integer. Given a subcode D of C (that is, D is a linear subspace of C), the *support* $\chi(D)$ of D is the set of non-zero positions of D, that is,

$$\chi(D) := \{i \mid \exists\, (a_1, \dots, a_m) \in D,\, a_i \neq 0\}.$$

The r-th *generalized Hamming weight* of C, denoted $\delta_r(C)$, is the size of the smallest support of an r-dimensional subcode. Generalized Hamming weights have received a lot of attention; see [3, 6, 9, 20, 24] and the references therein. The study of these weights is related to trellis coding, t–resilient functions, and was motivated by some applications from cryptography [24].

The minimum distance of projective Reed-Muller-type codes has been studied using Gröbner bases and commutative algebra techniques; see [3, 4, 9, 10, 17, 19] and the references therein. In this work we extend these techniques to study the r-th generalized Hamming weights of projective Reed-Muller-type codes. These linear codes are constructed as follows.

Let $K = \mathbb{F}_q$ be a finite field with q elements, let $\mathbb{P}^{s-1}$ be a projective space over K, and let $\mathbb{X}$ be a subset of $\mathbb{P}^{s-1}$. The *vanishing ideal* of $\mathbb{X}$, denoted $I(\mathbb{X})$, is the ideal of S generated by the homogeneous polynomials that vanish at all points of $\mathbb{X}$. The Hilbert function of $S/I(\mathbb{X})$ is denoted by $H_{\mathbb{X}}(d)$. We can write $\mathbb{X} = \{[P_1], \dots, [P_m]\} \subset \mathbb{P}^{s-1}$ with $m = |\mathbb{X}|$. Here we assume that the first non-zero entry of each $[P_i]$ is 1. In the special case that $\mathbb{X}$ has the form $[X \times \{1\}]$ for some $X \subset \mathbb{F}_q^{s-1}$, we assume that the s-th entry of each $[P_i]$ is 1.

Fix a degree $d \geq 1$. There is a K-linear map given by

$$\mathrm{ev}_d \colon S_d \to K^m, \qquad f \mapsto (f(P_1), \dots, f(P_m)).$$

The image of S_d under ev_d, denoted by $C_{\mathbb{X}}(d)$, is called a *projective Reed-Muller-type code* of degree d on $\mathbb{X}$ [7, 12]. The *parameters* of the linear code $C_{\mathbb{X}}(d)$ are:

(a) *length*: $|\mathbb{X}|$,
(b) *dimension*: $\dim_K C_{\mathbb{X}}(d)$,
(c) r-th *generalized Hamming weight*: $\delta_{\mathbb{X}}(d, r) := \delta_r(C_{\mathbb{X}}(d))$.

The contents of this paper are as follows. In Section 2 we present some of the results and terminology that will be needed throughout the paper.

If $\mathbb{X}$ is a finite set of projective points over a finite field and $I(\mathbb{X})$ is its vanishing ideal, we show that $\delta_{I(\mathbb{X})}(d, r)$ is the r-th generalized Hamming weight $\delta_{\mathbb{X}}(d, r)$ of the corresponding Reed-Muller-type code $C_{\mathbb{X}}(d)$ (Theorem 4.5). We introduce the *Vasconcelos* function $\vartheta_I(d, r)$ of a graded ideal I (Definition 4.4) and show that $\vartheta_{I(\mathbb{X})}(d, r)$ is also equal to $\delta_{\mathbb{X}}(d, r)$ (Theorem 4.5). These two abstract algebraic formulations of $\delta_{\mathbb{X}}(d, r)$ gives us a new tool to study generalized Hamming weights. One of our results shows that $\mathrm{fp}_{I(\mathbb{X})}(d, r)$ is a lower bound for $\delta_{\mathbb{X}}(d, r)$ (Theorem 4.9). As is seen in this paper, in certain cases $\mathrm{fp}_{I(\mathbb{X})}(d, r)$ is equal to $\delta_{\mathbb{X}}(d, r)$.

To show some of our applications we prove the following interesting and non-trivial inequality.

Theorem 5.5 *Let $d \geq 1$ and $1 \leq e_1 \leq \cdots \leq e_m$ be integers. Suppose $1 \leq a_i \leq e_i$ and $1 \leq b_i \leq e_i$, for $i = 1, \dots, m$, are integers such that $d = \sum_i a_i = \sum_i b_i$ and $a \neq b$. Then*

$$\pi(a, b) \geq \left(\sum_{i=1}^{m} a_i - \sum_{i=k+1}^{m} e_i - (k-2) \right) e_{k+1} \cdots e_m - e_{k+2} \cdots e_m$$

for $k = 1, \dots, m-1$, where $\pi(a, b) = \prod_{i=1}^{m} a_i + \prod_{i=1}^{m} b_i - \prod_{i=1}^{m} \min(a_i, b_i)$.

We give two applications to coding theory. The first is the following explicit formula for the second generalized Hamming weight of an affine cartesian code.

Theorem 6.3 *Let A_i, $i = 1, \ldots, s-1$, be subsets of $\mathbb{F}_q$ and let $\mathbb{X} \subset \mathbb{P}^{s-1}$ be the projective set $\mathbb{X} = [A_1 \times \cdots \times A_{s-1} \times \{1\}]$. If $d_i = |A_i|$ for $i = 1, \ldots, s-1$ and $2 \leq d_1 \leq \cdots \leq d_{s-1}$, then*

$$\delta_{\mathbb{X}}(d,2) = \begin{cases} (d_{k+1} - \ell + 1)\, d_{k+2} \cdots d_{s-1} - d_{k+3} \cdots d_{s-1} & \text{if } k < s-3, \\ (d_{k+1} - \ell + 1)\, d_{k+2} \cdots d_{s-1} - 1 & \text{if } k = s-3, \\ \quad d_{s-1} - \ell + 1 & \text{if } k = s-2, \end{cases}$$

where $0 \leq k \leq s-2$ and ℓ are integers, $d = \sum_{i=1}^{k} (d_i - 1) + \ell$, and $1 \leq \ell \leq d_{k+1} - 1$.

Using this result one can recover the case when $\mathbb{X}$ is a projective torus in $\mathbb{P}^{s-1}$ [11, Theorem 17]. The second applications of this paper gives a combinatorial formula for the second generalized Hamming weight of an affine cartesian code, which is quite different from the corresponding formula of [1, Theorem 5.4].

Theorem 6.4 *Let $\mathcal{P}_d$ be the set of all pairs (a,b), a, b in $\mathbb{N}^s$, $a = (a_i)$, $b = (b_i)$, such that $a \neq b$, $d = \sum_i a_i = \sum_i b_i$, $1 \leq a_i, b_i \leq d_i - 1$ for $i = 1, \ldots, n$, $n := s-1$, $a_i \neq 0$ and $b_j \neq 0$ for some $1 \leq i, j \leq n$. If $\mathbb{X} = [A_1 \times \cdots \times A_n \times \{1\}]$, with $A_i \subset \mathbb{F}_q$, $d_i = |A_i|$, and $2 \leq d_1 \leq \cdots \leq d_n$, then*

$$\mathrm{fp}_{I(\mathbb{X})}(d,2) = \delta_{\mathbb{X}}(d,2) = \min\{P(a,b) \mid (a,b) \in \mathcal{P}_d\} \quad \text{for } \ d \leq \textstyle\sum_{i=1}^{n} (d_i - 1),$$

where $P(a,b) = \prod_{i=1}^{n} (d_i - a_i) + \prod_{i=1}^{n} (d_i - b_i) - \prod_{i=1}^{n} \min\{d_i - a_i, d_i - b_i\}$.

It is an open problem to find an explicit formula for the r-th generalized Hamming weight of an affine cartesian code. The first evidence that this problem could have a positive answer is the explicit formula for the second generalized Hamming weight of a projective torus given in [11, Theorem 17]. A second evidence is the recent combinatorial expression for the r-th generalized Hamming weight of an affine cartesian code [1, Theorem 5.4], which depends on the r-th monomial in ascending lexicographic order of a certain family of monomials (see [1] and the proof of Theorem 7.1). Using this result we give an explicit formula to compute the r-th generalized Hamming weight for a family of affine cartesian codes (Theorem 7.1).

For all unexplained terminology and additional information we refer to [2, 5, 8] (for the theory of Gröbner bases, commutative algebra, and Hilbert functions), and [15, 22] (for the theory of error-correcting codes and linear codes).

2. Preliminaries

In this section we present some of the results that will be needed throughout the paper and introduce some more notation. All results of this section are well-known. To avoid repetitions, we continue to employ the notations and definitions used in Section 1.

Generalized Hamming weights. Let $K = \mathbb{F}_q$ be a finite field and let C be a $[m, k]$ *linear code* of *length* m and *dimension* k.

The r-th *generalized Hamming weight* of C, denoted $\delta_r(C)$, is the size of the smallest support of an r-dimensional subcode, that is,

$$\delta_r(C) := \min\{|\chi(D)| : D \text{ is a linear subcode of } C \text{ with } \dim_K(D) = r\}.$$

The *weight hierarchy* of C is the sequence $(\delta_1(C), \ldots, \delta_k(C))$. The integer $\delta_1(C)$ is called the *minimum distance* of C and is denoted by $\delta(C)$. According to [24, Theorem 1, Corollary 1] the weight hierarchy is an increasing sequence

$$1 \leq \delta_1(C) < \cdots < \delta_r(C) \leq m,$$

and $\delta_r(C) \leq m - k + r$ for $r = 1, \ldots, k$. For $r = 1$ this is the Singleton bound for the minimum distance. Notice that $\delta_r(C) \geq r$.

Recall that the *support* $\chi(\beta)$ of a vector $\beta \in K^m$ is $\chi(K\beta)$, that is, $\chi(\beta)$ is the set of non-zero entries of β.

Lemma 2.1. *Let D be a subcode of C of dimension $r \geq 1$. If $\beta_1, \ldots, \beta_r$ is a K-basis for D with $\beta_i = (\beta_{i,1}, \ldots, \beta_{i,m})$ for $i = 1, \ldots, r$, then $\chi(D) = \cup_{i=1}^r \chi(\beta_i)$ and the number of elements of $\chi(D)$ is the number of non-zero columns of the matrix:*

$$\begin{bmatrix} \beta_{1,1} & \cdots & \beta_{1,i} & \cdots & \beta_{1,m} \\ \beta_{2,1} & \cdots & \beta_{2,i} & \cdots & \beta_{2,m} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \beta_{r,1} & \cdots & \beta_{r,i} & \cdots & \beta_{r,m} \end{bmatrix}.$$

Commutative algebra. Let $S = K[t_1, \ldots, t_s] = \oplus_{d=0}^{\infty} S_d$ be a polynomial ring over a field K with the standard grading and let $I \neq (0)$ be a graded ideal of S of Krull dimension k. The *Hilbert function* of S/I is:

$$H_I(d) := \dim_K(S_d/I_d), \quad d = 0, 1, 2, \ldots,$$

where $I_d = I \cap S_d$. By a theorem of Hilbert [21, p. 58], there is a unique polynomial $h_I(x) \in \mathbb{Q}[x]$ of degree $k-1$ such that $H_I(d) = h_I(d)$ for $d \gg 0$. The degree of the zero polynomial is -1.

The *degree* or *multiplicity* of S/I is the positive integer

$$\deg(S/I) := \begin{cases} (k-1)! \lim_{d\to\infty} H_I(d)/d^{k-1} & \text{if } k \geq 1, \\ \dim_K(S/I) & \text{if } k = 0. \end{cases}$$

We will use the following multi-index notation: for $a = (a_1, \ldots, a_s) \in \mathbb{N}^s$, set $t^a := t_1^{a_1} \cdots t_s^{a_s}$. The multiplicative group of the field K is denoted by K^*. As usual $\mathrm{ht}(I)$ will denote the height of the ideal I. By the dimension of I (resp. S/I) we mean the Krull dimension of S/I. The Krull dimension of S/I is denoted by $\dim(S/I)$.

One of the most useful and well-known facts about the degree is its additivity:

Proposition 2.2. (Additivity of the degree [18, Proposition 2.5]) *If I is an ideal of S and $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ is an irredundant primary decomposition, then*

$$\deg(S/I) = \sum_{\mathrm{ht}(\mathfrak{q}_i)=\mathrm{ht}(I)} \deg(S/\mathfrak{q}_i).$$

If $F \subset S$, the *quotient ideal* of I with respect to (F) is given by $(I\colon (F)) = \{h \in S \mid hF \subset I\}$. An element f is called a *zero-divisor* of S/I if there is $\overline{0} \neq \overline{a} \in S/I$ such that $f\overline{a} = \overline{0}$, and f is called *regular* on S/I if f is not a zero-divisor. Thus f is a zero-divisor if and only if $(I\colon f) \neq I$. An associated prime of I is a prime ideal $\mathfrak{p}$ of S of the form $\mathfrak{p} = (I\colon f)$ for some f in S.

Theorem 2.3. [23, Lemma 2.1.19, Corollary 2.1.30] *If I is an ideal of S and $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ is an irredundant primary decomposition with $\mathrm{rad}(\mathfrak{q}_i) = \mathfrak{p}_i$, then the set of zero-divisors $\mathcal{Z}(S/I)$ of S/I is equal to $\bigcup_{i=1}^m \mathfrak{p}_i$, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are the associated primes of I.*

Definition 2.4. The *regularity* of S/I, denoted $\mathrm{reg}(S/I)$, is the least integer $r_0 \geq 0$ such that $H_I(d)$ is equal to $h_I(d)$ for $d \geq r_0$.

The footprint of an ideal. Let $\prec$ be a monomial order on S and let $(0) \neq I \subset S$ be an ideal. If f is a non-zero polynomial in S, then one can write

$$f = \lambda_1 t^{\alpha_1} + \cdots + \lambda_r t^{\alpha_r},$$

with $\lambda_i \in K^*$ for all i and $t^{\alpha_1} \succ \cdots \succ t^{\alpha_r}$. The *leading monomial* t^{α_1} of f is denoted by $\mathrm{in}_\prec(f)$. The *initial ideal* of I, denoted by $\mathrm{in}_\prec(I)$, is the monomial ideal given by

$$\mathrm{in}_\prec(I) = (\{\mathrm{in}_\prec(f) |\, f \in I\}).$$

A monomial t^a is called a *standard monomial* of S/I, with respect to $\prec$, if t^a is not the leading monomial of any polynomial in I. The set of standard monomials, denoted $\Delta_\prec(I)$, is called the *footprint* of S/I. The image of the standard polynomials of degree d, under the canonical map $S \mapsto S/I$, $x \mapsto \overline{x}$, is equal to S_d/I_d, and the image of $\Delta_\prec(I)$ is a basis of S/I as a K-vector space (see [23, Proposition 3.3.13]). In particular, if I is graded, then $H_I(d)$ is the number of standard monomials of degree d.

A subset $\mathcal{G} = \{g_1, \ldots, g_r\}$ of I is called a *Gröbner basis* of I if

$$\mathrm{in}_\prec(I) = (\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_r)).$$

Lemma 2.5. [3, p. 2] *Let $I \subset S$ be an ideal generated by $\mathcal{G} = \{g_1, \ldots, g_r\}$, then*

$$\Delta_\prec(I) \subset \Delta_\prec(\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_r)).$$

Vanishing ideal of a finite set. The *projective space* of dimension $s-1$ over the field K is denoted $\mathbb{P}^{s-1}$. It is usual to denote the equivalence class of α by $[\alpha]$.

For a given a subset $\mathbb{X} \subset \mathbb{P}^{s-1}$ define $I(\mathbb{X})$, the *vanishing ideal* of $\mathbb{X}$, as the ideal generated by the homogeneous polynomials in S that vanish at all points of $\mathbb{X}$, and given a graded ideal $I \subset S$ define its *zero set* relative to $\mathbb{X}$ as

$$V_{\mathbb{X}}(I) = \left\{[\alpha] \in \mathbb{X} |\, f(\alpha) = 0 \ \forall f \in I \ \text{ homogeneous}\right\}.$$

In particular, if $f \in S$ is homogeneous, the zero set $V_{\mathbb{X}}(f)$ of f is the set of all $[\alpha] \in \mathbb{X}$ such that $f(\alpha) = 0$, that is $V_{\mathbb{X}}(f)$ is the set of zeros of f in $\mathbb{X}$.

Lemma 2.6. *Let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$, let $[\alpha]$ be a point in $\mathbb{X}$ with $\alpha = (\alpha_1, \ldots, \alpha_s)$ and $\alpha_k \neq 0$ for some k, and let $I_{[\alpha]}$ be the vanishing ideal of $[\alpha]$. Then $I_{[\alpha]}$ is a prime ideal,*

$$I_{[\alpha]} = (\{\alpha_k t_i - \alpha_i t_k |\, k \neq i \in \{1, \ldots, s\}),\ \deg(S/I_{[\alpha]}) = 1,$$

$\mathrm{ht}(I_{[\alpha]}) = s-1$, *and $I(\mathbb{X}) = \bigcap_{[\beta]\in\mathbb{X}} I_{[\beta]}$ is the primary decomposition of $I(\mathbb{X})$.*

Definition 2.7. The set $\mathbb{T} = \{[(x_1, \ldots, x_s)] \in \mathbb{P}^{s-1} |\, x_i \in K^* \,\forall\, i\}$ is called a *projective torus*.

3. Computing the number of points of a variety

In this section we give a degree formula to compute the number of solutions of a system of homogeneous polynomials over any given finite set of points in a projective space over a field.

Lemma 3.1. *Let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$ over a field K. If $F = \{f_1, \ldots, f_r\}$ is a set of homogeneous polynomials of $S \setminus \{0\}$, then $V_{\mathbb{X}}(F) = \emptyset$ if and only if $(I(\mathbb{X})\colon (F)) = I(\mathbb{X})$.*

Proof. $\Rightarrow$) We proceed by contradiction assuming that $I(\mathbb{X}) \subsetneq (I(\mathbb{X})\colon (F))$. Pick a homogeneous polynomial g such that $gf_i \in I(\mathbb{X})$ for all i and $g \notin I(\mathbb{X})$. Then there is $[\alpha]$ in $\mathbb{X}$ such that $g(\alpha) \neq 0$. Thus $f_i(\alpha) = 0$ for all i, that is, $[\alpha] \in V_{\mathbb{X}}(F)$, a contradiction.

$\Leftarrow$) We can write $\mathbb{X} = \{[P_1], \ldots, [P_m]\}$ and $I(\mathbb{X}) = \cap_{i=1}^m \mathfrak{p}_i$, where $\mathfrak{p}_i$ is equal to $I_{[P_i]}$, the vanishing ideal of $[P_i]$. We proceed by contradiction assuming that $V_{\mathbb{X}}(F) \neq \emptyset$. Pick $[P_i]$ in $V_{\mathbb{X}}(F)$. For simplicity of notation assume that $i = 1$. Notice that $(\mathfrak{p}_1\colon (F)) = (1)$. Therefore

$$\bigcap_{i=1}^m \mathfrak{p}_i = I(\mathbb{X}) = (I(\mathbb{X})\colon (F)) = \bigcap_{i=1}^m (\mathfrak{p}_i\colon (F)) = \bigcap_{i=2}^m (\mathfrak{p}_i\colon (F)) \subset \mathfrak{p}_1.$$

Hence $\mathfrak{p}_i \subset (\mathfrak{p}_i\colon (F)) \subset \mathfrak{p}_1$ for some $i \geq 2$, see [23, p. 74]. Thus $\mathfrak{p}_i = \mathfrak{p}_1$, a contradiction. $\square$

An ideal $I \subset S$ is called *unmixed* if all its associated primes have the same height, and I is called *radical* if I is equal to its radical. The radical of I is denoted by $\mathrm{rad}(I)$.

Lemma 3.2. *Let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$ over a field K and let $I(\mathbb{X}) \subset S$ be its vanishing ideal. If $F = \{f_1, \ldots, f_r\}$ is a set of homogeneous polynomials of $S \setminus \{0\}$, then*

$$|\mathbb{X} \setminus V_{\mathbb{X}}(F)| = \begin{cases} \deg(S/(I(\mathbb{X})\colon (F)) & \text{if } (I(\mathbb{X})\colon (F)) \neq I(\mathbb{X}), \\ \deg(S/I(\mathbb{X})) & \text{if } (I(\mathbb{X})\colon (F)) = I(\mathbb{X}). \end{cases}$$

Proof. Let $[P_1], \ldots, [P_m]$ be the points of $\mathbb{X}$ with $m = |\mathbb{X}|$, and let $[P]$ be a point in $\mathbb{X}$ with $P = (\alpha_1, \ldots, \alpha_s)$ and $\alpha_k \neq 0$ for some k. Then the vanishing ideal $I_{[P]}$ of $[P]$ is a prime ideal of height $s-1$,

$$I_{[P]} = (\{\alpha_k t_i - \alpha_i t_k \,|\, k \neq i \in \{1, \ldots, s\}),\ \deg(S/I_{[P]}) = 1,$$

and $I(\mathbb{X}) = \bigcap_{i=1}^m I_{[P_i]}$ is a primary decomposition (see Lemma 2.6).

Assume that $(I(\mathbb{X})\colon (F)) \neq I(\mathbb{X})$. We set $I = I(\mathbb{X})$ and $\mathfrak{p}_i = I_{[P_i]}$ for $i = 1, \ldots, m$. Notice that $(\mathfrak{p}_j\colon f_i) = (1)$ if and only if $f_i \in \mathfrak{p}_j$ if and only if $f_i(P_j) = 0$. Then

$$(I\colon (F)) = \bigcap_{i=1}^{r} (I\colon f_i) = \left(\bigcap_{f_1(P_j) \neq 0} \mathfrak{p}_j \right) \bigcap \cdots \bigcap \left(\bigcap_{f_r(P_j) \neq 0} \mathfrak{p}_j \right) = \bigcap_{[P_j] \notin V_{\mathbb{X}}(F)} \mathfrak{p}_j.$$

Therefore, by the additivity of the degree of Proposition 2.2, we get that $\deg(S/(I\colon (F)))$ is equal to $|\mathbb{X} \setminus V_{\mathbb{X}}(F)|$. If $(I(\mathbb{X})\colon (F)) = I(\mathbb{X})$, then $V_{\mathbb{X}}(F) = \emptyset$ (see Lemma 3.1). Thus $|V_{\mathbb{X}}(F)| = 0$ and the required formula follows because $|\mathbb{X}| = \deg(S/I(\mathbb{X}))$. $\square$

Lemma 3.3. *Let $I \subset S$ be a radical unmixed graded ideal. If $F = \{f_1, \ldots, f_r\}$ is a set of homogeneous polynomials of $S \setminus \{0\}$, $(I\colon (F)) \neq I$, and $\mathcal{A}$ is the set of all associated primes of S/I that contain F, then* $\mathrm{ht}(I) = \mathrm{ht}(I, F)$ *and*

$$\deg(S/(I, F)) = \sum_{\mathfrak{p} \in \mathcal{A}} \deg(S/\mathfrak{p}).$$

Proof. As $I \subsetneq (I\colon (F))$, there is $g \in S \setminus I$ such that $g(F) \subset I$. Hence the ideal (F) is contained in the set of zero-divisors of S/I. Thus, by Theorem 2.3 and since I is unmixed, (F) is contained in an associated prime ideal $\mathfrak{p}$ of S/I of height $\mathrm{ht}(I)$. Thus $I \subset (I, F) \subset \mathfrak{p}$, and consequently $\mathrm{ht}(I) = \mathrm{ht}(I, F)$. Therefore the set of associated primes of (I, F) of height equal to $\mathrm{ht}(I)$ is not empty and is equal to $\mathcal{A}$. There is an irredundant primary decomposition

$$(I, F) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r \cap \mathfrak{q}'_{r+1} \cap \cdots \cap \mathfrak{q}'_t, \tag{3.1}$$

where $\mathrm{rad}(\mathfrak{q}_i) = \mathfrak{p}_i$, $\mathcal{A} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, and $\mathrm{ht}(\mathfrak{q}'_i) > \mathrm{ht}(I)$ for $i > r$. We may assume that the associated primes of S/I are $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ with $r \leq m$. Since I is a radical ideal, we get that $I = \cap_{i=1}^m \mathfrak{p}_i$. Next we show the following equality:

$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r \cap \mathfrak{q}'_{r+1} \cap \cdots \cap \mathfrak{q}'_t \cap \mathfrak{p}_{r+1} \cap \cdots \cap \mathfrak{p}_m. \tag{3.2}$$

The inclusion "$\supset$" is clear because $\mathfrak{q}_i \subset \mathfrak{p}_i$ for $i = 1, \ldots, r$. The inclusion "$\subset$" follows by noticing that the right hand side of Eq. (3.2) is equal to $(I, f) \cap \mathfrak{p}_{r+1} \cap \cdots \cap \mathfrak{p}_m$, and consequently it contains $I = \cap_{i=1}^m \mathfrak{p}_i$. Notice that $\mathrm{rad}(\mathfrak{q}'_j) = \mathfrak{p}'_j \not\subset \mathfrak{p}_i$ for all i, j and $\mathfrak{p}_j \not\subset \mathfrak{p}_i$ for $i \neq j$. Hence localizing Eq. (3.2) at the prime ideal $\mathfrak{p}_i$ for $i = 1, \ldots, r$, we get that $\mathfrak{p}_i = I_{\mathfrak{p}_i} \cap S = (\mathfrak{q}_i)_{\mathfrak{p}_i} \cap S = \mathfrak{q}_i$ for $i = 1, \ldots, r$. Using Eq. (3.1), together with the additivity of the degree of Proposition 2.2, the required equality follows. $\square$

Lemma 3.4. *Let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$ over a field K and let $I(\mathbb{X}) \subset S$ be its vanishing ideal. If $F = \{f_1, \ldots, f_r\}$ is a set of homogeneous polynomials of $S \setminus \{0\}$, then the number of points of $V_{\mathbb{X}}(F)$ is given by*

$$|V_{\mathbb{X}}(F)| = \begin{cases} \deg(S/(I(\mathbb{X}), F)) & \text{if } (I(\mathbb{X})\colon (F)) \neq I(\mathbb{X}), \\ 0 & \text{if } (I(\mathbb{X})\colon (F)) = I(\mathbb{X}). \end{cases}$$

Proof. Let $[P_1], \ldots, [P_m]$ be the points of $\mathbb{X}$ with $m = |\mathbb{X}|$. The vanishing ideal $I_{[P_i]}$ of $[P_i]$ is a prime ideal of height $s-1$, $\deg(S/I_{[P_i]}) = 1$, and $I(\mathbb{X}) = \bigcap_{i=1}^{m} I_{[P_i]}$ (see Lemma 2.6).

Assume that $(I(\mathbb{X})\colon (F)) \neq I(\mathbb{X})$. Let $\mathcal{A}$ be the set of all $I_{[P_i]}$ that contain the set F. Notice that $f_j \in I_{[P_i]}$ if and only if $f_j(P_i) = 0$. Then $[P_i]$ is in $V_{\mathbb{X}}(F)$ if and only if $F \subset I_{[P_i]}$. Thus $[P_i]$ is in $V_{\mathbb{X}}(F)$ if and only if $I_{[P_i]}$ is in $\mathcal{A}$. Hence, by Lemma 3.3, we get

$$|V_{\mathbb{X}}(F)| = \sum_{[P_i] \in V_{\mathbb{X}}(F)} \deg(S/I_{[P_i]}) = \sum_{F \subset I_{[P_i]}} \deg(S/I_{[P_i]}) = \deg(S/(I(\mathbb{X}), F)).$$

Assume that $(I(\mathbb{X})\colon F) = I(\mathbb{X})$. Then, by Lemma 3.1, $V_{\mathbb{X}}(f) = \emptyset$ and $|V_{\mathbb{X}}(f)| = 0$. $\square$

Proposition 3.5. *If $\mathbb{X}$ is a finite subset of $\mathbb{P}^{s-1}$, then*

$$\deg(S/I(\mathbb{X})) = \deg(S/(I(\mathbb{X}), F)) + \deg(S/(I(\mathbb{X})\colon (F))).$$

Proof. It follows from Lemmas 3.2 and 3.4. $\square$

4. Generalized minimum distance function of a graded ideal

In this part we study the generalized minimum distance function of a graded ideal and show that it generalizes the generalized Hamming weight of a projective Reed-Muller-type code. To avoid repetitions, we continue to employ the notations and definitions used in Sections 1 and 2.

Lemma 4.1. *Let $I \subset S$ be an unmixed graded ideal and let $\prec$ be a monomial order. If F is a finite set of homogeneous polynomials of S and $(I\colon (F)) \neq I$, then*

$$\deg(S/(I, F)) \leq \deg(S/(\mathrm{in}_\prec(I), \mathrm{in}_\prec(F))) \leq \deg(S/I),$$

and $\deg(S/(I, F)) < \deg(S/I)$ if I is an unmixed radical ideal and $(F) \not\subset I$.

Proof. To simplify notation we set $J = (I, F)$, $L = (\mathrm{in}_\prec(I), \mathrm{in}_\prec(F))$, and $F = \{f_1, \ldots, f_r\}$. We denote the Krull dimension of S/I by $\dim(S/I)$. Recall that $\dim(S/I) = \dim(S) - \mathrm{ht}(I)$. First we show that S/J and S/L have Krull dimension equal to $\dim(S/I)$. As $I \subsetneq (I\colon F)$, all elements of F are zero divisors of S/I. Hence, as I is unmixed, there is an associated prime ideal $\mathfrak{p}$ of S/I such that $(F) \subset \mathfrak{p}$ and $\dim(S/I) = \dim(S/\mathfrak{p})$. Since $I \subset J \subset \mathfrak{p}$, we get that $\dim(S/J)$ is $\dim(S/I)$. Since S/I and $S/\mathrm{in}_\prec(I)$ have the same Hilbert function, and so does $S/\mathfrak{p}$ and $S/\mathrm{in}_\prec(\mathfrak{p})$, we obtain

$$\dim(S/\mathrm{in}_\prec(I)) = \dim(S/I) = \dim(S/\mathfrak{p}) = \dim(S/\mathrm{in}_\prec(\mathfrak{p})).$$

Hence, taking heights in the inclusions $\mathrm{in}_\prec(I) \subset L \subset \mathrm{in}_\prec(\mathfrak{p})$, we obtain $\mathrm{ht}(I) = \mathrm{ht}(L)$.

Pick a Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_r\}$ of I. Then J is generated by $\mathcal{G} \cup F$ and by Lemma 2.5 one has the inclusions

$$\Delta_\prec(J) = \Delta_\prec(I, F) \subset \Delta_\prec(\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_r), \mathrm{in}_\prec(F)) =$$
$$\Delta_\prec(\mathrm{in}_\prec(I), \mathrm{in}_\prec(F)) = \Delta_\prec(L) \subset \Delta_\prec(\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_r)) = \Delta_\prec(I).$$

Thus $\Delta_\prec(J)\subset\Delta_\prec(L)\subset\Delta_\prec(I)$. Recall that $H_I(d)$, the Hilbert function of I at d, is the number of standard monomials of degree d. Hence $H_J(d)\leq H_L(d)\leq H_I(d)$ for $d\geq 0$. If $\dim(S/I)$ is equal to 0, then

$$\deg(S/J)=\sum_{d\geq 0}H_J(d)\leq\deg(S/L)=\sum_{d\geq 0}H_L(d)\leq\deg(S/I)=\sum_{d\geq 0}H_I(d).$$

Assume now that $\dim(S/I)\geq 1$. By a theorem of Hilbert [21, p. 58], H_J, H_L, H_I are polynomial functions of degree equal to $k=\dim(S/I)-1$ (see [2, Theorem 4.1.3]). Thus

$$k!\lim_{d\to\infty}H_J(d)/d^k\leq k!\lim_{d\to\infty}H_L(d)/d^k\leq k!\lim_{d\to\infty}H_I(d)/d^k,$$

that is, $\deg(S/J)\leq\deg(S/L)\leq\deg(S/I)$.

If I is an unmixed radical ideal and $(F)\not\subset I$, then there is at least one minimal prime that does not contains (F). Hence, by Lemma 3.3, it follows that $\deg(S/(I,F))<\deg(S/I)$. $\square$

Corollary 4.2. *Let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$, let $I(\mathbb{X})\subset S$ be its vanishing ideal, and let $\prec$ be a monomial order. If F is a finite set of homogeneous polynomials of S and $(I(\mathbb{X})\colon(F))\neq I(\mathbb{X})$, then*

$$|V_{\mathbb{X}}(F)|=\deg(S/(I(\mathbb{X}),F))\leq\deg(S/(\mathrm{in}_\prec(I(\mathbb{X})),\mathrm{in}_\prec(F)))\leq\deg(S/I(\mathbb{X})),$$

and $\deg(S/(I(\mathbb{X}),F))<\deg(S/I(\mathbb{X}))$ *if* $(F)\not\subset I(\mathbb{X})$.

Proof. It follows from Lemmas 3.4 and 4.1. $\square$

Lemma 4.3. *Let $\mathbb{X}=\{[P_1],\dots,[P_m]\}$ be a finite subset of $\mathbb{P}^{s-1}$ and let D be a linear subspace of $C_{\mathbb{X}}(d)$ of dimension $r\geq 1$. The following hold.*

(i) *There are $\overline{f}_1,\dots,\overline{f}_r$ linearly independent elements of S_d/I_d such that $D=\oplus_{i=1}^r K\beta_i$, where β_i is $(f_i(P_1),\dots,f_i(P_m))$, and the support $\chi(D)$ of D is equal to $\cup_{i=1}^r\chi(\beta_i)$.*
(ii) $|\chi(D)|=|\mathbb{X}\setminus V_{\mathbb{X}}(f_1,\dots,f_r)|$.
(iii) $\delta_r(C_{\mathbb{X}}(d))=\min\{|\mathbb{X}\setminus V_{\mathbb{X}}(F)|:\ F=\{f_i\}_{i=1}^r\subset S_d,\ \{\overline{f}_i\}_{i=1}^r$ *linearly independent over* $K\}$.

Proof. (i): This part follows from Lemma 2.1 and using that the evaluation map ev_d induces an isomorphism between S_d/I_d and $C_{\mathbb{X}}(d)$.

(ii): Consider the matrix A with rows $\beta_1,\dots,\beta_r$. Notice that the i-th column of A is not zero if and only if $[P_i]$ is in $\mathbb{X}\setminus V_{\mathbb{X}}(f_1,\dots,f_r)$. It suffices to observe that the number of non-zero columns of A is $|\chi(D)|$ (see Lemma 2.1).

(iii): This follows from part (ii) and using the definition of the r-th generalized Hamming weight of $C_{\mathbb{X}}(d)$ (see Section 2). $\square$

Definition 4.4. If $I\subset S$ is a graded ideal, the *Vasconcelos function* of I is the function $\vartheta_I\colon\mathbb{N}_+\times\mathbb{N}_+\to\mathbb{N}$ given by

$$\vartheta_I(d,r):=\begin{cases}\min\{\deg(S/(I\colon(F)))\,|\,F\in\mathcal{F}_{d,r}\} & \text{if } \mathcal{F}_{d,r}\neq\emptyset,\\ \deg(S/I) & \text{if } \mathcal{F}_{d,r}=\emptyset.\end{cases}$$

Theorem 4.5. *Let K be a field and let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$. If $|\mathbb{X}|\geq 2$ and $\delta_{\mathbb{X}}(d,r)$ is the r-th generalized Hamming weight of $C_{\mathbb{X}}(d)$, then*

$$\delta_{\mathbb{X}}(d,r)=\delta_{I(\mathbb{X})}(d,r)=\vartheta_{I(\mathbb{X})}(d,r)\ \ \text{for } d\geq 1 \text{ and } 1\leq r\leq H_{I(\mathbb{X})}(d),$$

and $\delta_{\mathbb{X}}(d,r)=r$ *for* $d\geq\mathrm{reg}(S/I(\mathbb{X}))$.

Proof. If $\mathcal{F}_{d,r} = \emptyset$, then using Lemmas 3.2, 3.4, and 4.3 we get that $\delta_{\mathbb{X}}(d,r)$, $\delta_{I(\mathbb{X})}(d,r)$, and $\vartheta_{I(\mathbb{X})}(d,r)$ are equal to $\deg(S/I(\mathbb{X})) = |\mathbb{X}|$. Assume that $\mathcal{F}_{d,r} \neq \emptyset$ and set $I = I(\mathbb{X})$. Using Lemma 4.3 and the formula for $V_{\mathbb{X}}(f)$ of Lemma 3.4, we obtain

$$\begin{aligned}
\delta_{\mathbb{X}}(d,r) &\overset{(4.3)}{=} \min\{|\mathbb{X} \setminus V_{\mathbb{X}}(F)| \colon F \in \mathcal{F}_{d,r}\} \overset{(3.4)}{=} |\mathbb{X}| - \max\{\deg(S/(I,F))|\, F \in \mathcal{F}_{d,r}\} \\
&= \deg(S/I) - \max\{\deg(S/(I,F))|\, F \in \mathcal{F}_{d,r}\} = \delta_I(d,r), \text{ and} \\
\delta_{\mathbb{X}}(d,r) &\overset{(4.3)}{=} \min\{|\mathbb{X} \setminus V_{\mathbb{X}}(F)| \colon F \in \mathcal{F}_{d,r}\} \overset{(3.2)}{=} \min\{\deg(S/(I \colon (F)))|\, F \in \mathcal{F}_{d,r}\} = \vartheta_I(d,r).
\end{aligned}$$

In these equalities we used the fact that $\deg(S/I(\mathbb{X})) = |\mathbb{X}|$. As $H_I(d) = |\mathbb{X}|$ for $d \geq \operatorname{reg}(S/I)$, using the generalized Singleton bound for the generalized Hamming weight and the fact that the weight hierarchy is an increasing sequence we obtain that $\delta_{\mathbb{X}}(d,r) = r$ for $d \geq \operatorname{reg}(S/I(\mathbb{X}))$ (see [24, Theorem 1, Corollary 1]). $\square$

Remark 4.6. $r \leq \delta_{\mathbb{X}}(d,r) \leq |\mathbb{X}|$ for $d \geq 1$ and $1 \leq r \leq H_{I(\mathbb{X})}(d)$. This follows from the fact that the weight hierarchy is an increasing sequence (see [24, Theorem 1]).

Lemma 4.7. *Let $\prec$ be a monomial order, let $I \subset S$ be an ideal, let $F = \{f_1, \ldots, f_r\}$ be a set of polynomial of S of positive degree, and let $\operatorname{in}_\prec(F) = \{\operatorname{in}_\prec(f_1), \ldots, \operatorname{in}_\prec(f_r)\}$ be the set of initial terms of F. If $(\operatorname{in}_\prec(I) \colon (\operatorname{in}_\prec(F))) = \operatorname{in}_\prec(I)$, then $(I \colon (F)) = I$.*

Proof. Let g be a polynomial of $(I \colon (F))$, that is, $gf_i \in I$ for $i = 1, \ldots, r$. It suffices to show that $g \in I$. Pick a Gröbner basis $g_1, \ldots, g_n$ of I. Then, by the division algorithm [5, Theorem 3, p. 63], we can write $g = \sum_{i=1}^n h_i g_i + h$, where $h = 0$ or h is a finite sum of monomials not in $\operatorname{in}_\prec(I) = (\operatorname{in}_\prec(g_1), \ldots, \operatorname{in}_\prec(g_n))$. We need only show that $h = 0$. If $h \neq 0$, then hf_i is in I and $\operatorname{in}_\prec(h)\operatorname{in}_\prec(f_i)$ is in the ideal $\operatorname{in}_\prec(I)$ for $i = 1, \ldots, r$. Hence $\operatorname{in}_\prec(h)$ is in $(\operatorname{in}_\prec(I) \colon (\operatorname{in}_\prec(F)))$. Therefore, by hypothesis, $\operatorname{in}_\prec(h)$ is in the ideal $\operatorname{in}_\prec(I)$, a contradiction. $\square$

Let $\mathcal{F}_{\prec,d,r}$ be the set consisting of all subsets $F = \{f_1, \ldots, f_r\}$ of S_d such that $(I \colon (F)) \neq I$, f_i is a standard polynomial for all i, $\overline{f}_1, \ldots, \overline{f}_r$ are linearly independent over the field K, and $\operatorname{in}_\prec(f_1), \ldots, \operatorname{in}_\prec(f_r)$ are distinct monomials.

Proposition 4.8. *The generalized minimum distance function of I is given by*

$$\delta_I(d,r) = \begin{cases} \deg(S/I) - \max\{\deg(S/(I,F))|\, F \in \mathcal{F}_{\prec,d,r}\} & \text{if } \mathcal{F}_{\prec,d,r} \neq \emptyset, \\ \deg(S/I) & \text{if } \mathcal{F}_{\prec,d,r} = \emptyset. \end{cases}$$

Proof. Take $F = \{f_1, \ldots, f_r\}$ in $\mathcal{F}_{d,r}$. By the division algorithm any f_i can be written as $f_i = p_i + h_i$, where p_i is in I_d and h_i is a K-linear combination of standard monomials of degree d. Setting $H = \{h_1, \ldots, h_r\}$, notice that $(I \colon (F)) = (I \colon (H))$, $(I, F) = (I, H)$, $\overline{f}_i = \overline{h}_i$ for $i = 1, \ldots, r$. Thus $H \in \mathcal{F}_{d,r}$, that is, we may assume that $f_1, \ldots, f_r$ are standard polynomials. Setting $KF = Kf_1 + \cdots + Kf_r$, we claim that there is a set $G = \{g_1, \ldots, g_r\}$ consisting of homogeneous standard polynomials of S/I of degree d such that $KF = KG$, $\operatorname{in}_\prec(g_1), \ldots, \operatorname{in}_\prec(g_r)$ distinct monomials, and $\operatorname{in}_\prec(f_i) \succeq \operatorname{in}_\prec(g_i)$ for all i. We proceed by induction on r. The case $r = 1$ is clear. Assume that $r > 1$. Permuting the f_i's if necessary we may assume that $\operatorname{in}_\prec(f_1) \succeq \cdots \succeq \operatorname{in}_\prec(f_r)$. If $\operatorname{in}_\prec(f_1) \succ \operatorname{in}_\prec(f_2)$, the claim follows applying the induction hypothesis to $f_2, \ldots, f_r$. If $\operatorname{in}_\prec(f_1) = \operatorname{in}_\prec(f_2)$, there is $k \geq 2$ such that $\operatorname{in}_\prec(f_1) = \operatorname{in}_\prec(f_i)$ for $i \leq k$ and $\operatorname{in}_\prec(f_1) \succ \operatorname{in}_\prec(f_i)$ for $i > k$. We set $h_i = f_1 - f_i$ for $i = 2, \ldots, k$ and $h_i = f_i$ for $i = k+1, \ldots, r$. Notice that $\operatorname{in}_\prec(f_1) \succ h_i$ for $i \geq 2$ and that $h_2, \ldots, h_r$ are standard monomials of degree d which are linearly independent over K. Hence the claim follows applying the induction hypothesis to $H = \{h_2, \ldots, h_r\}$. The required expression for $\delta_I(d,r)$ follows readily using Theorem 4.5. $\square$

Theorem 4.9. *Let K be a field, let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$, and let $\prec$ be a monomial order. If $|\mathbb{X}|\geq 2$ and $\delta_{\mathbb{X}}(d,r)$ is the r-th generalized Hamming weight of $C_{\mathbb{X}}(d)$, then*

$$\mathrm{fp}_{I(\mathbb{X})}(d,r)\leq\delta_{\mathbb{X}}(d,r)\quad \text{for } d\geq 1 \text{ and } 1\leq r\leq H_{I(\mathbb{X})}(d).$$

Proof. This follows from Theorem 4.5, Lemma 4.7, and Proposition 4.8. □

5. An integer inequality

For $a:a_1,\ldots,a_m$ and $b:b_1,\ldots,b_m$ sequences in $\mathbb{Z}^+=\{1,2,\ldots\}$ we define

$$\pi(a,b):=\prod_{i=1}^m a_i+\prod_{i=1}^m b_i-\prod_{i=1}^m\min(a_i,b_i).$$

Lemma 5.1. *Let $a_1,a_2,b_1,b_2\in\mathbb{Z}^+$. Set $a_1'=\min(a_1,a_2)$ and $a_2'=\max(a_1,a_2)$. Then*

$$\min(a_1,b_1)\min(a_2,b_2)\leq\min(a_1',b_1')\,\min(a_2',b_2').$$

Proof. It is an easy case-by-case verification of 4! possible cases. □

Lemma 5.2. *Let $a:a_1,\ldots,a_m$ and $b:b_1,\ldots,b_m$ be sequences in $\mathbb{Z}^+$. Suppose:*

(i) $r<s$, $a_r>a_s$. *Set* $a_r'=a_s$, $a_s'=a_r$, $a_i'=a_i$ *for* $i\neq r,s$; *and* $b_r'=\min(b_r,b_s)$, $b_s'=\max(b_r,b_s)$, $b_i'=b_i$ *for* $i\neq r,s$. *Then* $\pi(a,b)\geq\pi(a',b')$.

(ii) $r<s$, $b_r=a_r\leq a_s<b_s$. *Set* $a_r'=a_r-1$, $a_s'=a_s+1$, $a_i'=a_i$ *for* $i\neq r,s$. *Then* $\pi(a,b)\geq\pi(a',b)$.

(iii) $r<s$, $b_r<a_r\leq a_s$. *Set* $a_r'=a_r-1$, $a_s'=a_s+1$, $a_i'=a_i$ *for* $i\neq r,s$. *Then* $\pi(a,b)\geq\pi(a',b)$.

(iv) $r<s$, $a_r<a_s$, $b_r=a_s, b_s=a_r$, $b_i=a_i$ *for* $i\neq r,s$, $h:=a_s-a_r\geq 2$. *Set* $b_r'=a_r+1$, $b_s'=a_s-1$, $b_i'=a_i$ *for* $i\neq r,s$. *Then* $\pi(a,b)\geq\pi(a,b')$.

Proof. We verify all cases by direct substitution of a' and b' into $\pi(a,b)$.

$$\begin{aligned}
&(i)\ \ \pi(a,b)-\pi(a',b')=\prod a_i+\prod b_i-\prod a_i'-\prod b_i'+\prod\min(a_i',b_i')-\prod\min(a_i,b_i)\\
&=\ \ (\min(a_r',b_r')\min(a_s',b_s')-\min(a_r,b_r)\min(a_s,b_s))\prod_{i\neq r,s}\min(a_i,b_i)\geq 0.\ \ (\textit{Lemma } 5.1)
\end{aligned}$$

$$\begin{aligned}
(ii)\ \ \pi(a,b)-\pi(a',b)&=\prod a_i-\prod a_i'+\prod\min(a_i',b_i)-\prod\min(a_i,b_i)\\
&=\ (a_ra_s-(a_r-1)(a_s+1))\prod_{i\neq r,s}a_i\\
&\quad+(\min(a_r',b_r)\min(a_s',b_s)-\min(a_r,b_r)\min(a_s,b_s))\prod_{i\neq r,s}\min(a_i,b_i)\\
&=\ (a_s-a_r+1)\prod_{i\neq r,s}a_i+((a_r-1)(a_s+1)-a_ra_s)\prod_{i\neq r,s}\min(a_i,b_i)\\
&=\ (a_s-a_r+1)\left(\prod_{i\neq r,s}a_i-\prod_{i\neq r,s}\min(a_i,b_i)\right)\geq 0.
\end{aligned}$$

$$(iii)\quad \pi(a,b)-\pi(a',b) = \prod a_i - \prod a_i' + \prod \min(a_i',b_i) - \prod \min(a_i,b_i)$$
$$\begin{aligned} &= (a_ra_s-(a_r-1)(a_s+1))\prod_{i\neq r,s} a_i \\ &\quad +(\min(a_r',b_r)\min(a_s',b_s)-\min(a_r,b_r)\min(a_s,b_s))\prod_{i\neq r,s}\min(a_i,b_i) \\ &= (a_s-a_r+1)\prod_{i\neq r,s} a_i + b_r(\min(a_s+1,b_s)-\min(a_s,b_s))\prod_{i\neq r,s}\min(a_i,b_i)\geq 0. \end{aligned}$$

For the last inequality note that $\min(a_s+1,b_s)-\min(a_s,b_s)=0$ or 1.

$$\begin{aligned} (iv)\quad \pi(a,b)-\pi(a,b') &= \prod b_i - \prod b_i' + \prod \min(a_i,b_i') - \prod \min(a_i,b_i) \\ &= (a_ra_s-(a_r+1)(a_s-1)+a_r(a_s-1)-a_r^2)\prod_{i\neq r,s} a_i \\ &= (a_r-1)(h-1)\prod_{i\neq r,s} a_i \geq 0. \quad \square \end{aligned}$$

Lemma 5.3. *If $a_1,\ldots,a_r$ are positive integers, then $a_1\cdots a_r \geq (a_1+\cdots+a_r)-(r-1)$.*

Proof. It follows by induction on r. $\square$

Lemma 5.4. *Let $1\leq e_1\leq\cdots\leq e_m$ and $1\leq a_i,b_i\leq e_i$, for $i=1,\ldots,m$ be integers. Suppose $a_i=b_i=1$ for $i<r$, $a_i=b_i=e_i$ for $i>r+1:=s$, $1\leq a_i,b_i\leq e_i$ for $i=r,s$, with $a_r+a_s=b_r+b_s$ and $(a_r,a_s)\neq(b_r,b_s)$. If $b_r\leq a_s$ and $b_s=a_s-1$, then*

$$\pi(a,b)\geq\left(\sum_{i=1}^m a_i - \sum_{i=k+1}^m e_i-(k-2)\right)e_{k+1}\cdots e_m - e_{k+2}\cdots e_m \tag{5.1}$$

for $i=1,\ldots,m-1$, where $e_{k+2}\cdots e_m=1$ when $k=m-1$.

Proof. Set $\sigma=\sum_{i=1}^m a_i-\sum_{i=k+1}^m e_i-(k-2)$. Since $b_s(b_r-a_r)=a_s-1$, one has the equality

$$\pi(a,b)=(a_ra_s+b_rb_s-a_rb_s)\prod_{i=r+2}^m e_i=(a_ra_s+a_s-1)\prod_{i=r+2}^m e_i. \tag{5.2}$$

Case $k+1<r$: The integer σ can be rewritten as

$$\sigma=k+(1-e_{k+1})+\cdots+(1-e_{r-1})+(a_r-e_r)+(a_s-e_s)-(k-2).$$

Since $a_r<b_r\leq e_r$, it holds that $a_r-e_r\leq -1$, and hence $\sigma\leq 1$. If $\sigma\leq 0$, Eq. (5.1) trivially follows (because the left hand side is positive and the right hand side would be negative). So we may assume $\sigma=1$. This assumption implies that $e_{k+1}=1$ because $a_r<b_r\leq e_r$. Then the right hand side of Eq. (5.1) is

$$(\sigma)e_{k+1}\cdots e_m-e_{k+2}\cdots e_m=(e_{k+1}-1)e_{k+2}\cdots e_r=0.$$

Case $k+1=r$: The integer σ can be rewritten as

$$\sigma=k+(a_r-e_r)+(a_s-e_s)-(k-2).$$

By the same reason as above, we may assume $\sigma=1$. This assumption implies $a_r=e_r-1$ and $a_s=e_s$. Then, by Eq. (5.2), we obtain that Eq. (5.1) is equivalent to

$$(e_r e_s - 1)\prod_{i=r+2}^{m} e_i \geq (\sigma)\, e_r \cdots e_m - e_{r+1}\cdots e_m,$$

which reduces to $e_r e_s - 1 \geq (1)\, e_r e_s - e_s$, or equivalently, $e_s \geq 1$.

Case $k+1 = r+1$: We can rewrite σ as

$$\sigma = (k-1) + a_r + (a_s - e_s) - (k-2) = a_r + (a_s - e_s) + 1.$$

Then, using Eq. (5.2), we obtain that Eq. (5.1) is equivalent to

$$(a_r a_s + a_s - 1)\prod_{i=r+2}^{m} e_i \geq (\sigma)\, e_{r+1}\cdots e_m - e_{r+2}\cdots e_m,$$

which reduces to $a_r a_s + a_s \geq (a_r + a_s - e_s + 1)\, e_s$, or equivalently,

$$(e_s - a_s)(e_s - a_r - 1) \geq 0.$$

Case $k+1 > r+1$: One can rewrite σ as

$$\sigma = (r-1) + a_r + \cdots + a_k - (k-2).$$

Then, using Eq. (5.2), we obtain that Eq. (5.1) reduces to

$$(a_r a_s + a_s - 1) e_{r+2}\cdots e_{k+1} \geq (\sigma)\, e_{k+1} - 1.$$

But, as $a_s \geq 2$, using Lemma 5.3, we get

$$\begin{aligned}(a_r a_s + a_s - 1) e_{r+2}\cdots e_k &\geq (a_r a_s + a_s - 1) + e_{r+2} + \cdots + e_k - (k-r-1)\\ &\geq (a_r + a_s) + a_{r+2} + \cdots + a_k + r - k + 1 = \sigma.\end{aligned}$$

So, multiplying by e_{k+1}, the required inequality follows. □

Theorem 5.5. *Let $d \geq 1$ and $1 \leq e_1 \leq \cdots \leq e_m$ be integers. Suppose $1 \leq a_i \leq e_i$ and $1 \leq b_i \leq e_i$, for $i = 1, \ldots, m$, are integers such that $d = \sum_{i=1}^m a_i = \sum_{i=1}^m b_i$ and $a \neq b$. Then*

$$\pi(a,b) \geq \left(\sum_{i=1}^{m} a_i - \sum_{i=k+1}^{m} e_i - (k-2)\right) e_{k+1}\cdots e_m - e_{k+2}\cdots e_m$$

for $k = 1, \ldots, m-1$, where $e_{k+2}\cdots e_m = 1$ when $k = m-1$.

Proof. Apply to (a,b) any of the four "operations" described in Lemma 5.2, and let (a',b') be the new obtained pair. These operations should be applied in such a way that $1 \leq a_i', b_i' \leq e_i$ for $i = 1, \ldots, m$ and $a' \neq b'$; this is called a *valid* operation. One can order the set of all pairs (a,b) that satisfy the hypothesis of the proposition using the GRevLex order defined by $(a,b) \succ (a',b')$ if and only if the last non-zero entry of $(a,b) - (a',b')$ is negative. Note that by construction $d = \sum a_i' = \sum b_i' = \sum a_i$. Repeat this step as many times as possible (which is a finite number because the result (a',b') of any valid operation applied to (a,b) satisfies $(a,b) \succ (a',b')$). Permitting an abuse of notation, let a and b be the resulting sequences at the end of that process. We will show that these a and b satisfy the hypothesis of Lemma 5.4.

Set $r = \min(i : a_i \neq b_i)$. By symmetry we may assume $a_r < b_r$. Pick the first $s > r$ such that $a_s > b_s$ (the case $a_r > b_r$ and $a_s < b_s$ can be shown similarly).

Claim (a): For $p < r$, $a_p = 1$. Assume $a_p > 1$. If $a_p > a_r$, we can apply Lemma 5.2$(i)[p,r]$, which is assumed not possible; (this last notation means that we are applying Lemma 5.2(i) with the indexes p and r). Otherwise apply Lemma 5.2$(ii)[p,r]$. So $a_p = b_p = 1$ for $p < r$.

Claim (b): $s=r+1$. Suppose $r<p<s$. To obtain a contradiction it suffices to show that we can apply a valid operation to a, b. By the choice of s, $a_p \leq b_p$. If $b_r > b_p$, we can apply Lemma 5.2$(i)[r,p]$. If $b_p > b_s$, we can apply Lemma 5.2$(i)[p,s]$. Hence $b_r \leq b_p \leq b_s$. Notice that $b_p \geq 2$ because $a_r < b_r \leq b_p$. If $a_p = b_p$, we can apply Lemma 5.2$(ii)[p,s]$. If $a_p < b_p$, we can apply Lemma 5.2$(iii)[p,s]$ because $a_p < b_p \leq b_s < a_s$.

Claim (c): For $p > s$, $a_p = b_p = e_p$. If $b_p < a_p$, applying Claim (b) to r and p we get a contradiction. Thus we may assume $b_p \geq a_p$. It suffices to show that $a_p = e_p$. If $a_s > a_p$, then by Lemma 5.2$(i)[s,p]$ one can apply a valid operation to a, b, a contradiction. Thus $a_s \leq a_p$. If $a_p < e_p$, then $b_s < a_s \leq a_p < e_p$, and by Lemma 5.2$(iii)[s,p]$ we can apply a valid operation to a, b, a contradiction. Hence $a_p = e_p$.

Claim (d): $b_r \leq a_s$ and $b_s = a_s - 1$. By the previous claims one has the equalities $s = r+1$ and $a_r + a_s = b_r + b_s$. If $a_s < b_r$. Then $b_s < a_s < b_r$, and by Lemma 5.2$(i)[r,s]$ we can apply a valid operation to a, b, a contradiction. Hence $a_s \geq b_r$. Suppose $a_s = b_r$, then $a_r = b_s$. If $a_s - a_r \geq 2$, by Lemma 5.2$(iv)[r,s]$ we can apply a valid operation to a, b, a contradiction. Hence, in this case, $a_s - b_s = a_s - a_r = 1$. Suppose $a_s > b_r$. If $b_r > b_s$, then $a_s > b_r > b_s$, and we can use Lemma 5.2$(i)[r,s]$ to apply a valid operation to a, b, a contradiction. Hence $b_r \leq b_s$. If $a_s - b_s = b_r - a_r \geq 2$, then $a_r < b_r \leq b_s < a_s$, and by Lemma 5.2$(iii)[r,s]$ we can apply a valid operation to a, b, a contradiction. So, in this other case, also $a_s - b_s = 1$. In conclusion, we have that $b_r \leq a_s$ and $b_s = a_s - 1$, as claimed.

From Claims (a)–(d), we obtain that a, b satisfy the hypothesis of Lemma 5.4. Hence the required inequality follows from Lemmas 5.2 and 5.4. $\square$

For $\alpha : \alpha_1, \ldots, \alpha_n$ and $\beta : \beta_1, \ldots, \beta_n$ sequences in $\mathbb{Z}^+$ we define

$$P(\alpha, \beta) = \prod_{i=1}^{n}(d_i - \alpha_i) + \prod_{i=1}^{n}(d_i - \beta_i) - \prod_{i=1}^{n}\min\{d_i - \alpha_i, d_i - \beta_i\}.$$

Lemma 5.6. *Let $1 \leq d_1 \leq \cdots \leq d_n$, $0 \leq \alpha_i, \beta_i \leq d_i - 1$ for $i = 1, \ldots, n$, $n \geq 2$, be integers such that $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$ and $(\alpha_1, \ldots, \alpha_n) \neq (\beta_1, \ldots, \beta_n)$. Then*

$$P(\alpha, \beta) \geq \left(\sum_{i=1}^{k+1}(d_i - \alpha_i) - (k-1) - \sum_{i=k+2}^{n} \alpha_i\right) d_{k+2} \cdots d_n - d_{k+3} \cdots d_n \tag{5.3}$$

for $k = 0, \ldots, n-2$, where $d_{k+3} \cdots d_n = 1$ if $k = n-2$.

Proof. Making the substitutions $m = n$, $k = k-1$, $d_i - \alpha_i = a_i$, $d_i - \beta_i = b_i$, and $d_i = e_i$, the inequality follows at once from Theorem 5.5. $\square$

6. Second generalized Hamming weight

Let $A_1, \ldots, A_{s-1}$ be subsets of $\mathbb{F}_q$ and let $\mathbb{X} := [A_1 \times \cdots \times A_{s-1} \times \{1\}] \subset \mathbb{P}^{s-1}$ be a projective cartesian set, where $d_i = |A_i|$ for all $i = 1, \ldots, s-1$ and $2 \leq d_1 \leq \cdots \leq d_{s-1}$. The Reed–Muller-type code $C_{\mathbb{X}}(d)$ is called an *affine cartesian code* [14]. If $\mathbb{X}^* = A_1 \times \cdots \times A_{s-1}$, then $C_{\mathbb{X}}(d) = C_{\mathbb{X}^*}(d)$ [14]. Assume $d = \sum_{i=1}^{k}(d_i - 1) + \ell$, where k, ℓ are integers such that $0 \leq k \leq s-2$ and $1 \leq \ell \leq d_{k+1} - 1$.

Lemma 6.1. *We can find two linearly independent polynomials F and $G \in S_{\leq d}$ such that*

$$|V_{\mathbb{X}^*}(F) \cap V_{\mathbb{X}^*}(G)| = \begin{cases} d_1 \cdots d_{s-1} - (d_{k+1} - \ell + 1)d_{k+2} \cdots d_{s-1} + d_{k+3} \cdots d_{s-1} & \text{if } \; k < s-3, \\ d_1 \cdots d_{s-1} - (d_{k+1} - \ell + 1)d_{k+2} \cdots d_{s-1} + 1 & \text{if } \; k = s-3, \\ d_1 \cdots d_{s-1} - d_{s-1} + \ell - 1 & \text{if } \; k = s-2. \end{cases}$$

Proof. Case (I): $k \leq s-3$. Similarly to [14] we take $A_i = \{\beta_{i,1}, \ldots, \beta_{i,d_i}\}$, for $i = 1, \ldots, s-1$. Also, for $i = 1, \ldots, k$, let

$$f_i := (\beta_{i,1} - t_i)(\beta_{i,2} - t_i) \cdots (\beta_{i,d_{i-1}} - t_i),$$
$$g := (\beta_{k+1,1} - t_{k+1})(\beta_{k+1,2} - t_{k+1}) \cdots (\beta_{k+1,\ell-1} - t_{k+1}).$$

Setting $h_1 := \beta_{k+1,\ell} - t_{k+1}$ and $h_2 := \beta_{k+2,\ell} - t_{k+2}$. We define $F := f_1 \cdots f_k \cdot g \cdot h_1$ and $G := f_1 \cdots f_k \cdot g \cdot h_2$. Notice that $\deg F = \deg G = \sum_{i=1}^{k}(d_i - 1) + \ell = d$ and that they are linearly independent over $\mathbb{F}_q$. Let

$$V_1 := (A_1 \times \cdots \times A_{s-1}) \setminus (V_{\mathbb{X}^*}(F) \cap V_{\mathbb{X}^*}(G)),$$
$$V_2 := \{\beta_{1,d_1}\} \times \cdots \times \{\beta_{k,d_k}\} \times \{\beta_{k+1,i}\}_{i=\ell}^{d_{k+1}} \times A_{k+2} \times \cdots \times A_{s-1}.$$

It is easy to see that $V_1 \subset V_2$ and $(V_2 \setminus V_1) \cap (V_{\mathbb{X}^*}(F) \cap V_{\mathbb{X}^*}(G)) = V_3$, where

$$V_3 = \begin{cases} \{\beta_{1,d_1}\} \times \cdots \times \{\beta_{k,d_k}\} \times \{\beta_{k+1,\ell}\} \times \{\beta_{k+2,\ell}\} \times A_{k+3} \times \cdots \times A_{s-1} & \text{if } \ k < s-3, \\ \{\beta_{1,d_1}\} \times \cdots \times \{\beta_{k,d_k}\} \times \{\beta_{k+1,\ell}\} \times \{\beta_{k+2,\ell}\} & \text{if } \ k = s-3. \end{cases}$$

Therefore

$$|V_1| = |V_2| - |V_3| = \begin{cases} (d_{k+1} - \ell + 1)d_{k+2} \cdots d_{s-1} - d_{k+3} \cdots d_{s-1} & \text{if } \ k < s-3, \\ (d_{k+1} - \ell + 1)d_{k+2} \cdots d_{s-1} - 1 & \text{if } \ k = s-3, \end{cases}$$

and the claim follows because $|V_{\mathbb{X}^*}(F) \cap V_{\mathbb{X}^*}(G)| = d_1 \cdots d_{s-1} - |V_1|$.

Case (II): $k = s-2$. As $\ell \leq d_{k+1} - 1$ then $\ell + 1 \leq d_{k+1}$. Let $h_3 := \beta_{k+1,\ell+1} - t_{k+1}$, and F, f_i, g, h_1 as in Case (I). Let $G' := f_1 \cdots f_k \cdot g \cdot h_3$. If

$$V_1' := (A_1 \times \cdots \times A_{s-1}) \setminus (V_{\mathbb{X}^*}(F) \cap V_{\mathbb{X}^*}(G')),$$
$$V_2' := \{\beta_{1,d_1}\} \times \cdots \times \{\beta_{k,d_k}\} \times \{\beta_{k+1,i}\}_{i=\ell}^{d_{k+1}},$$

then (because h_1 and h_3 do not have common zeros) $V_1' = V_2'$ and thus

$$|V_1'| = d_{k+1} - \ell + 1 = d_{s-1} - \ell + 1.$$

The result follows because $|V_{\mathbb{X}^*}(F) \cap V_{\mathbb{X}^*}(G')| = d_1 \cdots d_{s-1} - |V_1'|$. $\square$

Lemma 6.2. [16, Lemma 3.3] *Let $L \subset S$ be the ideal $(t_1^{d_1}, \ldots, t_{s-1}^{d_{s-1}})$, where $d_1, \ldots, d_{s-1}$ are in $\mathbb{N}_+$. If $t^a = t_1^{a_1} \cdots t_s^{a_s}$, $a_j \geq 1$ for some $1 \leq j \leq s-1$, and $a_i \leq d_i - 1$ for $i \leq s-1$, then*

$$\deg(S/(L, t^a)) = \deg(S/(L, t_1^{a_1} \cdots t_{s-1}^{a_{s-1}})) = d_1 \cdots d_{s-1} - \prod_{i=1}^{s-1}(d_i - a_i).$$

We come to one of our applications to coding theory.

Theorem 6.3. *Let A_i, $i = 1, \ldots, s-1$, be subsets of $\mathbb{F}_q$ and let $\mathbb{X} \subset \mathbb{P}^{s-1}$ be the projective cartesian set given by $\mathbb{X} = [A_1 \times \cdots \times A_{s-1} \times \{1\}]$. If $d_i = |A_i|$ for $i = 1, \ldots, s-1$ and $2 \leq d_1 \leq \cdots \leq d_{s-1}$, then*

$$\delta_{\mathbb{X}}(d, 2) = \begin{cases} (d_{k+1} - \ell + 1)\, d_{k+2} \cdots d_{s-1} - d_{k+3} \cdots d_{s-1} & \text{if } k < s-3, \\ (d_{k+1} - \ell + 1)\, d_{k+2} \cdots d_{s-1} - 1 & \text{if } k = s-3, \\ d_{s-1} - \ell + 1 & \text{if } k = s-2, \\ 2 & \text{if } d \geq \sum\limits_{i=1}^{s-1} (d_i - 1)\,, \end{cases}$$

where $0 \leq k \leq s-2$ and ℓ are integers such that $d = \sum_{i=1}^{k} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+1} - 1$.

Proof. We set $n=s-1$, $I=I(\mathbb{X})$, and $L=(t_1^{d_1},\ldots,t_n^{d_n})$. By [24, Theorem 1, Corollary 1], we get $\delta_{\mathbb{X}}(d,2)=2$ for $d\geq\sum_{i=1}^{s-1}(d_i-1)$. Thus we may assume $d<\sum_{i=1}^{s-1}(d_i-1)$. First we show the inequality "$\geq$". Let $\prec$ be a graded monomial order with $t_1\succ\cdots\succ t_s$. The initial ideal $\mathrm{in}_\prec(I)$ of I is equal to $L=(t_1^{d_1},\ldots,t_n^{d_n})$; see [14]. Let $F=\{t^a,t^b\}$ be an element of $\mathcal{M}_{\prec,d,2}$, that is, $t^a=t_1^{a_1}\cdots t_s^{a_s}$, $t^b=t_1^{b_1}\cdots t_s^{b_s}$, $d=\sum_{i=1}^s a_i=\sum_{i=1}^s b_i$, $a\neq b$, $a_i\leq d_i-1$ and $b_i\leq d_i-1$ for $i=1,\ldots,n$, and $(L\colon(F))\neq L$. In particular, from the last condition it follows readily that $a_i\neq 0$ and $b_j\neq 0$ for some $1\leq i,j\leq n$. There are exact sequences of graded S-modules

$$0\to(S/((L,t^a)\colon t^b))[-|b|]\xrightarrow{t^b}S/(L,t^a)\to S/(L,t^a,t^b)\to 0,$$
$$0\to(S/((L,t^b)\colon t^a))[-|a|]\xrightarrow{t^a}S/(L,t^b)\to S/(L,t^a,t^b)\to 0,$$

where $|a|=\sum_{i=1}^s a_i$. From the equalities

$$((L,t^a)\colon t^b)=(L\colon t^b)+(t^a\colon t^b)=(t_1^{d_1-b_1},\ldots,t_n^{d_n-b_n},\textstyle\prod_{i=1}^s t_i^{\max\{a_i,b_i\}-b_i}),$$
$$((L,t^b)\colon t^a)=(L\colon t^a)+(t^b\colon t^a)=(t_1^{d_1-a_1},\ldots,t_n^{d_n-a_n},\textstyle\prod_{i=1}^s t_i^{\max\{a_i,b_i\}-a_i}),$$

it follows that either $((L,t^a)\colon t^b)$ or $((L,t^b)\colon t^a)$ is contained in $(t_1,\ldots,t_n)$. Hence at least one of these ideals has height n. Therefore, setting

$$P(a,b)=\prod_{i=1}^n(d_i-a_i)+\prod_{i=1}^n(d_i-b_i)-\prod_{i=1}^n\min\{d_i-a_i,d_i-b_i\},$$

and using Lemma 6.2 it is not hard to see that the degree of $S/(L,t^a,t^b)$ is

$$\deg(S/(L,t^a,t^b))=\prod_{i=1}^n d_i-P(a,b),$$

and the second generalized footprint function of I is

$$\mathrm{fp}_I(d,2)=\min\left\{P(a,b)\,|\,\{t^a,t^b\}\in\mathcal{M}_{\prec,d,2}\right\}. \tag{6.1}$$

Making the substitution $-\ell=\sum_{i=1}^k(d_i-1)-\sum_{i=1}^s a_i$ and using the fact that $\mathrm{fp}_{I(\mathbb{X})}(d,r)$ is less than or equal to $\delta_{\mathbb{X}}(d,r)$ (see Theorem 4.9) it suffices to show the inequalities

$$P(a,b)\geq\left(\sum_{i=1}^{k+1}(d_i-a_i)-(k-1)-a_s-\sum_{i=k+2}^{n}a_i\right)d_{k+2}\cdots d_n-d_{k+3}\cdots d_n, \tag{6.2}$$

for $\{t^a,t^b\}\in\mathcal{M}_{\prec,d,2}$ if $0\leq k\leq s-3$, where $d_{k+3}\cdots d_n=1$ if $k=s-3$, and

$$P(a,b)\geq\sum_{i=1}^{s-1}(d_i-a_i)-(s-3)-a_s, \tag{6.3}$$

for $\{t^a,t^b\}\in\mathcal{M}_{\prec,d,2}$ if $k=s-2$. As $(a_1,\ldots,a_n)$ is not equal to $(b_1,\ldots,b_n)$, one has that either $\prod_{i=1}^n d_i-\prod_{i=1}^n(d_i-a_i)\geq 1$ or $\prod_{i=1}^n d_i-\prod_{i=1}^n(d_i-b_i)\geq 1$. If $a_s\geq 1$ or $b_s\geq 1$ (resp. $a_s=b_s=0$), the inequality of Eq. (6.2) follows at once from [17, Proposition 5.7] (resp. Lemma 5.6). If $a_s\geq 1$ or $b_s\geq 1$ (resp. $a_s=b_s=0$), the inequality of Eq. (6.3) follows at once from [17, Proposition 5.7] (resp. Lemma 5.3). This completes the proof of the inequality "$\geq$".

The inequality "$\leq$" follows directly from Lemma 6.1. $\square$

Another of our applications to coding theory is the following purely combinatorial formula for the second generalized Hamming weight of an affine cartesian code which is quite different from the corresponding formula of [1, Theorem 5.4].

Theorem 6.4. *Let $\mathcal{P}_d$ be the set of all pairs (a,b), a,b in $\mathbb{N}^s$, $a=(a_1,\ldots,a_s)$, $b=(b_1,\ldots,b_s)$, such that $a\neq b$, $d=\sum_{i=1}^s a_i=\sum_{i=1}^s b_i$, $1\leq a_i,b_i\leq d_i-1$ for $i=1,\ldots,n$, $n:=s-1$, $a_i\neq 0$ and $b_j\neq 0$ for some $1\leq i,j\leq n$. If $\mathbb{X}=[A_1\times\cdots\times A_n\times\{1\}]\subset\mathbb{P}^n$, with $A_i\subset\mathbb{F}_q$, $d_i=|A_i|$, and $2\leq d_1\leq\cdots\leq d_n$, then*

$$\mathrm{fp}_{I(\mathbb{X})}(d,2)=\delta_{\mathbb{X}}(d,2)=\min\{P(a,b)|\,(a,b)\in\mathcal{P}_d\}\quad \textit{for}\quad d\leq \textstyle\sum_{i=1}^n(d_i-1),$$

where $P(a,b)=\prod_{i=1}^n(d_i-a_i)+\prod_{i=1}^n(d_i-b_i)-\prod_{i=1}^n\min\{d_i-a_i,d_i-b_i\}$.

Proof. Let $\psi(d)$ be the formula for $\delta_{\mathbb{X}}(d,2)$ given in Theorem 6.3. Then using Eqs. (6.2) and Eqs. (6.3) one has $\psi(d)\leq \mathrm{fp}_{I(\mathbb{X})}(d,2)$. By Theorem 4.9 one has $\mathrm{fp}_{I(\mathbb{X})}(d,r)\leq\delta_{\mathbb{X}}(d,r)$, and by Lemma 6.1 one has $\delta_{\mathbb{X}}(d,r)\leq\psi(d)$. Therefore

$$\psi(d)\leq \mathrm{fp}_{I(\mathbb{X})}(d,2)\leq\delta_{\mathbb{X}}(d,r)\leq\psi(d).$$

Thus we have equality everywhere and the result follows from Eq. (6.1). □

Remark 6.5. Let $\psi(d)$ be the formula for $\delta_{\mathbb{X}}(d,2)$ given in Theorem 6.3. Then

$$\psi(d)=\min\{P(a,b)|\,(a,b)\in\mathcal{P}_d\}$$

for $d\leq\sum_{i=1}^n(d_i-1)$. This equality is interesting in its own right.

7. Generalized Hamming weights of affine cartesian codes

There is a nice combinatorial formula for the r-th generalized Hamming weight of an affine cartesian code [1, Theorem 5.4]. Using this combinatorial formula we give an explicit formula to compute the r-th generalized Hamming weight for a family of cartesian codes.

Theorem 7.1. *Let $\mathbb{X}:=[A_1\times\cdots\times A_{s-1}\times\{1\}]$ be a subset of $\mathbb{P}^{s-1}$, where $A_i\subset\mathbb{F}_q$ and $d_i=|A_i|$ for $i=1,\ldots,s-1$. If $2\leq d_1\leq\cdots\leq d_{s-1}$, then*

$$\delta_{\mathbb{X}}(d,r)=\begin{cases} d_{k+r+1}\cdots d_{s-1}[(d_{k+1}-\ell+1)d_{k+2}\cdots d_{k+r}-1] & \textit{if}\quad k<s-r-1,\\ (d_{k+1}-\ell+1)d_{k+2}\cdots d_{s-1}-1 & \textit{if}\quad k=s-r-1,\end{cases}\tag{7.1}$$

where we set $d_i\cdots d_j=1$ if $i>j$ or $i<1$, and $r\geq 1$, $k\geq 0$, ℓ are integers such that $d=\sum_{i=1}^k(d_i-1)+\ell$, and $1\leq\ell\leq d_{k+1}-1$.

Proof. Setting $n=s-1$, $R=K[t_1,\ldots,t_n]$ a polynomial ring with coefficients in K, and $L=(t_1^{d_1},\ldots,t_n^{d_n})$, we order the set $M_{\leq d}:=\Delta_\prec(L)\cap R_{\leq d}$ of all standard monomials of R/L of degree at most d with the lexicographic order (lex order for short), that is, $t^a\succ t^b$ if and only if the first non-zero entry of $a-b$ is positive. For $r>1$, $0\leq k\leq n-r$, the r-th monomial $t_1^{b_{r,1}}\cdots t_n^{b_{r,n}}$ of $M_{\leq d}$ in decreasing lex order is

$$t_1^{d_1-1}\cdots t_k^{d_k-1}t_{k+1}^{\ell-1}t_{k+r}$$

and the r-th monomial $t_1^{a_{r,1}}\cdots t_n^{a_{r,n}}$ of $M_{\geq c_0-d}:=\Delta_\prec(L)\cap R_{\geq c_0-d}$ in ascending lex order, where $c_0=\sum_{i=1}^n(d_i-1)$, is

$$t_{k+1}^{d_{k+1}-\ell}t_{k+2}^{d_{k+2}-1}\cdots t_{k+r-1}^{d_{k+r-1}-1}t_{k+r}^{d_{k+r}-2}t_{k+r+1}^{d_{k+r+1}-1}\cdots t_n^{d_n-1}.$$

Case (I): $0 \le k < n-r$. The case $r=1$ was proved in [14, Theorem 3.8]. Thus we may also assume $r \ge 2$. Therefore, applying [1, Theorem 5.4], we obtain that $\delta_{\mathbb{X}}(d,r)$ is given by

$$
\begin{aligned}
&1+\sum_{i=1}^{n} a_{r,i}\prod_{j=i+1}^{n} d_j = 1+(d_{k+1}-\ell)d_{k+2}\cdots d_n + \sum_{i=k+2,i\neq k+r}^{n}(d_i-1)\prod_{j=i+1}^{n} d_j\\
&+(d_{k+r}-2)d_{k+r+1}\cdots d_n\\
&=(d_{k+1}-\ell)d_{k+2}\cdots d_n+\left(1+\sum_{i=k+2}^{n}(d_i-1)\prod_{j=i+1}^{n} d_j\right)-d_{k+r+1}\cdots d_n\\
&=(d_{k+1}-\ell)d_{k+2}\cdots d_n+(d_{k+2}\cdots d_n)-d_{k+r+1}\cdots d_n\\
&=(d_{k+1}-\ell+1)d_{k+2}\cdots d_n-d_{k+r+1}\cdots d_n=d_{k+r+1}\cdots d_n[(d_{k+1}-\ell+1)d_{k+2}\cdots d_{k+r}-1].
\end{aligned}
$$

Case (II): $k=n-r$. In this case the r-th monomial $t_1^{a_{r,1}}\cdots t_n^{a_{r,n}}$ of $M_{\ge c_0-d}$ in ascending lex order is

$$t_{k+1}^{d_{k+1}-\ell}t_{k+2}^{d_{k+2}-1}\cdots t_{k+r-1}^{d_{k+r-1}-1}t_{k+r}^{d_{k+r}-2}t_{k+r+1}^{d_{k+r+1}-1}\cdots t_n^{d_n-1}.$$

Therefore, applying [1, Theorem 5.4], we obtain that $\delta_{\mathbb{X}}(d,r)$ is given by

$$
\begin{aligned}
&1+\sum_{i=1}^{n} a_{r,i}\prod_{j=i+1}^{n} d_j = 1+(d_{k+1}-\ell)d_{k+2}\cdots d_n + \sum_{i=k+2}^{n-1}(d_i-1)\prod_{j=i+1}^{n} d_j+(d_n-2)\\
&=(d_{k+1}-\ell)d_{k+2}\cdots d_n+\left(1+\sum_{i=k+2}^{n}(d_i-1)\prod_{j=i+1}^{n} d_j\right)-1\\
&=(d_{k+1}-\ell)d_{k+2}\cdots d_n+(d_{k+2}\cdots d_n)-1=(d_{k+1}-\ell+1)d_{k+2}\cdots d_n-1. \quad \square
\end{aligned}
$$

Corollary 7.2. *Let $\mathbb{T}$ be a projective torus in $\mathbb{P}^{s-1}$ and let $\delta_{\mathbb{T}}(d,r)$ be the r-th generalized Hamming weight of $C_{\mathbb{T}}(d)$. Then*

$$\delta_{\mathbb{T}}(d,r)=\left[(q-1)^{r-1}(q-\ell)-1\right](q-1)^{s-k-r-1}$$

for $1\le r\le s-k-1$, $1\le r\le H_{\mathbb{T}}(d)$, where $d=k(q-2)+\ell$, $0\le k\le s-2$, $1\le\ell\le q-2$.

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