GENERIC NEWTON POLYGON FOR EXPONENTIAL SUMS IN n VARIABLES WITH PARALLELOTOPE BASE

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Abstract. Let p be a prime number. Every n-variable polynomial $f(\underline{x})$ over a finite field of characteristic p defines an Artin–Schreier–Witt tower of varieties whose Galois group is isomorphic to $\mathbb{Z}_p$. Our goal of this paper is to study the Newton polygon of the L-function associated to a finite character of $\mathbb{Z}_p$ and a generic polynomial whose convex hull is an n-dimensional paralleltope Δ. We denote this polygon by $\mathrm{GNP}(\Delta)$. We prove a lower bound of $\mathrm{GNP}(\Delta)$, which is called the improved Hodge polygon $\mathrm{IHP}(\Delta)$. We show that $\mathrm{IHP}(\Delta)$ lies above the usual Hodge polygon $\mathrm{HP}(\Delta)$ at certain infinitely many points, and when p is larger than a fixed number determined by Δ, it coincides with $\mathrm{GNP}(\Delta)$ at these points. As a corollary, we roughly determine the distribution of the slopes of $\mathrm{GNP}(\Delta)$.

Contents

1. Introduction

We shall state our main results and their motivation after recalling the notion of L-functions for Witt coverings. Let p be a prime number, and

$$f(\underline{x}) := \sum_{P\in\mathbb{Z}^n_{\geq 0}} a_P x^P$$

be an n-variable polynomial in $\overline{\mathbb{F}}_p[x_1,\dots,x_n]$. We denote by $\mathbb{F}_p(f)$ the splitting field of $f(\underline{x})$ and set $m(f) := [\mathbb{F}_p(f) : \mathbb{F}_p]$. Then we put $\hat{a}_P \in \mathbb{Z}_{p^{m(f)}}$ to be the Teichmuller lift of a_P, where $\mathbb{Z}_{p^{m(f)}}$ is the unramified extension of $\mathbb{Z}_p$ of degree $m(f)$, and call

$$\hat{f}(\underline{x}) := \sum_{P\in\mathbb{Z}^n_{\geq 0}} \hat{a}_P x^P$$

the *Teichmüller lift* of $f(\underline{x})$.

Viewing $\mathbb{Z}^n$ as the lattice points in $\mathbb{R}^n$ with origin denoted by $\mathcal{O}$, we call the convex hull of $\mathcal{O}\cup\{P \mid a_P \neq 0\}$ *the polytope* of f and denote it by Δ_f.

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The *Artin–Schreier–Witt tower* associated to f is the sequence of varieties $\mathcal{V}_i$ over $\mathbb{F}_{p^{m(f)}}$ defined by the following equations:

$$\mathcal{V}_i : y_i^F - y_i = \sum_{P \in \Delta_f} (a_P x^P, 0, 0, \dots),$$

where $y_i = (y_i^{(1)}, y_i^{(2)}, \dots, y_i^{(i)})$ are Witt vectors of length i, and $\bullet^F$ means raising each Witt coordinate to the pth power. The Artin–Schreier–Witt tower $\cdots \to \mathcal{V}_i \to \cdots \to \mathcal{V}_0 := \mathbb{A}^n$ is a tower of Galois covers of $\mathbb{A}^n$ with total Galois group $\mathbb{Z}_p$, and consequently the study of zeta function of the tower can be reduced to the study of the L-functions associated to (finite) characters of the Galois group $\mathbb{Z}_p$. For more details we refer the readers to [DWX, §1].

Let $(\mathbb{G}_m)^n$ be the n-dimensional torus over $\mathbb{F}_{p^{m(f)}}$. The main subject of our study is the L-function associated to a finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ of conductor p^{m_χ} which is given by

$$L_f^*(\chi, s) := \prod_{x \in |(\mathbb{G}_m)^n|} \frac{1}{1 - \chi\Big(\mathrm{Tr}_{\mathbb{Q}_{p^{m(f)\deg(x)}}/\mathbb{Q}_p}(\hat{f}(\hat{x}))\Big) s^{\deg(x)}},$$

where $|(\mathbb{G}_m)^n|$ is the set of closed points of $(\mathbb{G}_m)^n$, $\hat{x}$ is the Teichmuller lift of a geometric point at x, and $\deg(x)$ stands for the degree of x over $\mathbb{F}_{p^{m(f)}}$.

It is proved in [LWei, Theorem 1.3] that when f is a non-degenerate polynomial with convex hull Δ_f, the fucntion

$$L_f^*(\chi, s)^{(-1)^{n-1}} := \sum_{i=0}^{n! p^{n(m_\chi - 1)} \mathrm{Vol}(\Delta_f)} v_i s^i \in \mathbb{Z}_p[\zeta_{p^{m_\chi}}][s]$$

is a polynomial of degree $n!\, p^{n(m_\chi - 1)} \mathrm{Vol}(\Delta_f)$, where $\zeta_{p^{m_\chi}}$ is a primitive p^{m_χ}-th root of unity.

In this paper, we confine ourselves to studying these f whose polytopes are of the simpler shape. Let Δ be an n-dimensional paralleltope generated by linearly independent vectors $\overrightarrow{\mathcal{O}\mathbf{V}_1}, \overrightarrow{\mathcal{O}\mathbf{V}_2}, \dots, \overrightarrow{\mathcal{O}\mathbf{V}_n}$, where $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_n$ are integral points.

Hypothesis 1.1. We assume that p is a prime such that $p \nmid \mathrm{Vol}(\Delta)$ and $p > (n+4)D$, where D is a positive integer depending only on Δ (See Definition 2.2).

In particular, when Δ is an n-dimensional cube with side length d, the integer $D = d$.

Notation 1.2. Let $k\Delta$ denote a scaling of Δ. We write

$$\Delta_k^+ := k\Delta \cap \mathbb{Z}^n$$

for the set consisting of the lattice points in $k\Delta$ and put

$$\mathtt{x}_k^+ := \#\Delta_k^+.$$

Let Δ° denote the paralleltope Δ with all faces not containing $\mathcal{O}$ removed. We put

$$\Delta_k^- := k\Delta^\circ \cap \mathbb{Z}^n \quad \text{and} \quad \mathtt{x}_k^- := \#\Delta_k^-.$$

For simplicity, we write $\Delta^\pm$ for $\Delta_1^\pm$ when no confusion can rise.

Notation 1.3. One may naturally identify it as an open subscheme of $\overline{\mathbb{F}}_p^{\Delta^+}$ by recording the coefficients a_P of f.

Definition 1.4. If f satisfies the non-degenerate condition in [LWei], we call the lower convex hull of the set of points $\Big(i, p^{m_\chi - 1}(p-1)\mathrm{val}_{p^{m(f)}}(v_i)\Big)$ the *normalized Newton polygon of* $L_f^*(\chi, s)^{(-1)^{n-1}}$ which is denoted by $\mathrm{NP}(f, \chi)_L$. Here, $\mathrm{val}_{p^{m(f)}}(-)$ is the p-adic valuation normalized so that $\mathrm{val}_{p^{m(f)}}(p^{m(f)}) = 1$.

The following Theorem 1.5 and Theorem 1.7 are the main results of this paper.

Theorem 1.5. *Assume Hypothesis 1.1. Let $\mathcal{F}(\Delta)$ denote the set of all non-degenerate polynomials $f(\underline{x}) = \sum_{P\in\Delta^+} a_P x^P \in \overline{\mathbb{F}}_p[\underline{x}]$ with $\Delta_f = \Delta$. Then there is a Zariski open subset $O_{\mathrm{Zar}} \subset \mathcal{F}(\Delta)$ such that for any $f \in O_{\mathrm{Zar}}$ and any finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ of conductor p^{m_χ}, if we put $\{\alpha_1, \dots, \alpha_{p^{n(m_\chi-1)}n!\mathrm{Vol}(\Delta)}\}$ to be the set of $p^{m(f)}$-adic Newton slopes of $L_f^*(\chi, s)^{(-1)^{n-1}}$, then for any $0 \le i_1 \le n-1$ and $0 \le i_2 \le p^{m_\chi-1} - 1$ we have*

$$\text{(1)}\ \#\Big\{\alpha_j \;\Big|\; \alpha_j \in \big(i_1+\tfrac{i_2}{p^{m_\chi-1}}, i_1+\tfrac{i_2+1}{p^{m_\chi-1}}\big)\Big\} = \sum_{t=0}^{i_1}(-1)^t\binom{n}{t}\Big(\mathbb{x}^-_{(i_1-t)p^{m_\chi-1}+i_2+1} - \mathbb{x}^+_{(i_1-t)p^{m_\chi-1}+i_2}\Big),$$

$$\text{(2)}\ \#\Big\{\alpha_j \;\Big|\; \alpha_j = i_1+\tfrac{i_2}{p^{m_\chi-1}}\Big\} = \sum_{t=0}^{i_1}(-1)^t\binom{n}{t}\Big(\mathbb{x}^+_{(i_1-t)p^{m_\chi-1}+i_2} - \mathbb{x}^-_{(i_1-t)p^{m_\chi-1}+i_2}\Big).$$

To study L-function it is more convenient to work with the so-called *characteristic power series*

$$C_f^*(\chi, s) := \Big(\prod_{j=0}^{\infty} L_f^*(\chi, p^{m(f)j}s)^{\binom{n+j-1}{n-1}}\Big)^{(-1)^{n-1}}, \tag{1.1}$$

whose normalized p-adic Newton polygon we denote by $\mathrm{NP}(f,\chi)_C$.

Definition 1.6. The *generic Newton polygon* of Δ is defined by

$$\mathrm{GNP}(\Delta) := \inf_{\substack{\chi:\mathbb{Z}_p/p^{m_\chi}\mathbb{Z}_p\to\mathbb{C}_p^\times\\ \Delta_f=\Delta}} \Big(\mathrm{NP}(f,\chi)_C\Big),$$

where $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ runs over all finite characters, and f runs over all non-degenerate polynomials in $\overline{\mathbb{F}}_p[x]$ such that $\Delta_f = \Delta$.

Theorem 1.7. *The generic Newton polygon* $\mathrm{GNP}(\Delta)$ *passes through points* $(\mathbb{x}_k^\pm(\Delta), h(\Delta_k^\pm))$ *for any $k \ge 0$, where $\mathbb{x}_k^\pm$ and $h(\Delta_k^\pm)$ are defined in Notation 2.1 and Definition 3.11 respectively.*

In [DWX], Davis, Wan, and Xiao studied the p-adic Newton slopes of $L_f^*(\chi, s)$ and $C_f^*(\chi, s)$ when f is a one-variable polynomial whose degree d is coprime to p. They concluded that, for each character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ of a relatively large conductor, $\mathrm{NP}(f,\chi)_L$ depends only on f and its conductor. Their proof strongly inspired the proof of spectral halo conjecture by Liu, Wan, and Xiao in [LWX]; we refer to [RWXY, §1.5] for the discussion on the analogy of the two proofs.

Motivated by the attempt of extending spectral halo type results beyond the case of modular forms, it is natural to ask whether one can generalize the main results of [DWX] to more general cases:

(1) changing the tower to $\mathbb{Z}_{p^\ell}$ for $\ell \ge 2$, and
(2) making the base to higher dimensional.

The first case is examined in a joint work with Wan, Xiao, and Yu see [RWXY]. The goal of this current paper is to investigate the second case.

From the Iwasawa theory point of view, it is important to have access to the Newton polygon $\mathrm{NP}(f,\chi)_C$ associated to this Artin-Schreier-Witt tower. When p is "ordinary", namely $p \equiv 1 \pmod D$, this polygon was explicitly computed by Adolphson–Sperber [AS], Berndt–Evans [BE], and Wan [Wan] in many special cases, and by Liu-Wan [LWan] in the general case (and in the T-adic setup).

Going beyond the ordinary case, there has been many researches on understanding the generic Newton polygon of $L_f(\chi, s)$ when f is a polynomial of a single variable. Here is an incomplete list.

- In [DWX], Davis, Wan, and Xiao prove that the Newton slopes of $L_f(\chi, s)$ form a finite union of arithmetic progressions, when f is a one-variable polynomial and χ is a finite character of a relatively large conductor.
- When p is large enough, Zhu [Zhu1] and Scholten–Zhu [SZ] showed that for a non-degenerate one-variable polynomial f and a finite character χ_0 of conductor p, the Newton polygon $\mathrm{NP}(f, \chi_0)_L$ coincides $\mathrm{GNP}(\Delta)$.
- Later, Blache, Ferard, and Zhu in [BFZ] proved a lower bound for the Newton polygon of $f(x) \in \mathbb{F}_q[x, \frac{1}{x}]$ of degree (d_1, d_2), which is called a Hodge-Stickelberger polygon. They also showed that when p approaches to infinity, the Newton polygon $\mathrm{NP}(\chi, f)_L$ coincides with the Hodge-Stickelberger polygon.
- In [BF], Blache and Ferard worked on the generic Newton polygon associated to characters of large conductors.
- In [OY], Ouyang and Yang studied the one-variable polynomial $f(x) = x^d + a_1 x$. A similar result can be found in [OZ], where Ouyang and Zhang studied the family of polynomials of the form $f(x) = x^d + a_{d-1}x^{d-1}$.
- In [KW], Koster and Wan studied a more general $\mathbb{Z}_p$-tower, and they proved the genus stability of all such $\mathbb{Z}_p$-tower.

However, for technical reasons, it is difficult to prove that the slopes of the L-function form a union of arithmetic progression when f is a multi-variable polynomial. Zhu in [Zhu2] shows that $\mathrm{GNP}(\Delta)$ and $\mathrm{IHP}(\Delta)$ coincide for characters of $\mathbb{Z}_p$ of conductor p when Δ is a rectangular and p is large enough. A similar result is obtained by the author in [Ren2] when the polytope of f is an isosceles right triangle.

In this paper, we focus on the generic Newton polygon of an n-dimensional parallelotope Δ. Our main contribution in this paper is to prove the distribution of slopes of the generic Newton polygon $\mathrm{GNP}(\Delta)$, when p is not necessary to be ordinary with respect to Δ. We refer reader to Theorem 1.5 for the statement.

Now we list the key steps of the proof of our main theorems.

Step 1: Instead of working with the L-function itself, it is more convenient (from the point of view of Dwork trace formula) to work with the characteristic power series $C_f^*(\chi, s)$, which recovers the L-function by

$$L_f^*(\chi, s)^{(-1)^{n-1}} = \prod_{j=0}^{n} C_f^*(\chi, p^{m(f)j}s)^{(-1)^j\binom{n}{j}}. \tag{1.2}$$

The power series $C_f^*(\chi, s)$ is genuinely the characteristic power series of a nuclear operator (or equivalently an infinite matrix N with respect to some canonical basis). Moreover, we can do this for the universal character (as opposed to just finite characters) of the Galois group of the tower $\mathbb{Z}_p$.

Step 2: We construct the improved Hodge polygon $\mathrm{IHP}(\Delta)$ for Δ in Definition 3.1, and prove that it is a lower bound of $\mathrm{NP}(f, \chi)_C$ for any finite character χ. The polygon $\mathrm{IHP}(\Delta)$ lies above the usual Hodge polygon at $x = \mathbb{x}_k^{\pm}$ for any $k \geq 1$. In fact, we show in Proposition 3.3 the condition that $\mathrm{IHP}(\Delta)$ lies strictly above the usual Hodge polygon at $x = \mathbb{x}_k^{\pm}$.

The key point of this step lies in: the usual way of obtaining Hodge polygon is to conjugate N by an appropriate diagonal matrix, and observe that each row is entirely divisible by a certain power of p. In this paper, we dig into the definition of characteristic power series as the sum over permutations, which allows us to slightly but crucially improve the usual Hodge polygon.

Step 3: We show in Proposition 4.3 that for any polynomial $f \in \mathcal{F}(\Delta)$, if there is a finite character χ_0 of conductor p such that $\mathrm{NP}(f,\chi_0)_C$ coincides with $\mathrm{IHP}(\Delta)$ at

$$x = \mathtt{x}_k^{\pm} \quad \text{for} \quad 1 \leq k \leq n+2,$$

then for every finite character χ, $\mathrm{NP}(f,\chi)_C$ and $\mathrm{IHP}(\Delta)$ coincide at

$$x = \mathtt{x}_k^{\pm} \quad \text{for all} \quad k \geq 0.$$

This proposition reduces the problem to show that all the polynomials $f \in \mathcal{F}(\Delta)$ such that $\mathrm{NP}(f,\chi_0)_C$ and $\mathrm{IHP}(\Delta)$ coincide at $x = \mathtt{x}_k^{\pm}$ for $1 \leq k \leq n+2$ form a Zariski open subset of $\mathcal{F}(\Delta)$.

For this, one may consider the characteristic power series for a "universal" polynomial $\widetilde{f}$, namely all coefficients of $\widetilde{f}$ are viewed as determinants. We need to show that when we write $\widetilde{u}_{\mathtt{x}_k^{\pm}} = \widetilde{u}_{\mathtt{x}_k^{\pm},h(\Delta_k^{\pm})} T^{h(\Delta_k^{\pm})} + O(T^{h(\Delta_k^{\pm})+1})$, the coefficients

$$\widetilde{u}_{\mathtt{x}_k^{\pm},h(\Delta_k^{\pm})} \not\equiv 0 \pmod{p} \qquad \text{for any} \quad 1 \leq k \leq n+2. \tag{1.3}$$

Step 4: We show (1.3) in §5. The technical core of this paper lies in proving (1.3). Roughly speaking, the key is to show that for each $0 \leq k \leq n+2$, a certain monomial of $\widetilde{u}_{\mathtt{x}_k^{\pm}}$ is nonzero. Tracing back to the definition of $\widetilde{u}_{\mathtt{x}_k^{\pm}}$, we see the contribution to such *leading* monomial must come from a unique special permutation that appears in the definition of the characteristic power series. Computing explicitly the contribution of this special permutation to the leading term, which itself was subdivided into simpler cases, allows us to prove (1.3).

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2. Dwork's trace formula

In this section, we will introduce Dwork trace formula to express $C_f^*(\chi,s)$as the characteristic power series of some infinite matrix and deduce a natural lower bound of $\mathrm{NP}(T,f)_C$ called the improved Hodge polygon.

Recall from introduction that Δ is an n-dimensional paralleltope generated by linearly independent vectors $\overrightarrow{\mathcal{O}\mathbf{V}_1}, \overrightarrow{\mathcal{O}\mathbf{V}_2}, \ldots, \overrightarrow{\mathcal{O}\mathbf{V}_n}$.

Notation 2.1. We denote the *cone* of Δ by

$$\mathrm{Cone}(\Delta) := \Big\{ Q \in \mathbb{R}^n \mid kQ \in \Delta \text{ for some } k > 0 \Big\},$$

and put

$$\mathbb{M}(\Delta) := \mathrm{Cone}(\Delta) \cap \mathbb{Z}^n$$

to be the set of lattice points in $\mathrm{Cone}(\Delta)$.

Definition 2.2. Let D be the smallest positive integer such that

$$\mathbb{M}(\Delta) \subset \Big\{ z_1\mathbf{V}_1 + z_2\mathbf{V}_2 + \cdots + z_n\mathbf{V}_n \;\Big|\; z_i \in \frac{1}{D}\mathbb{Z}_{\geq 0} \Big\}. \tag{2.1}$$

In particular, when Δ is an n-cube with side length d, the number $D = d$.

Notation 2.3. Let

$$\Lambda_\Delta := \bigoplus_{i=1}^{n} \mathbb{Z}_{\geq 0} \cdot \mathbf{V}_i \subset \mathbb{M}(\Delta).$$

From now on, let p be a prime satisfying Hypothesis 1.1, and let $\mathcal{F}(\Delta)$ denote the set of all non-degenerate polynomials $f(\underline{x}) = \sum\limits_{P\in\Delta^+} a_P x^P \in \overline{\mathbb{F}}_p[\underline{x}]$ with $\Delta_f = \Delta$. We denote by $\mathbb{F}_p(f)$ the *splitting field* of f which is the finite field generated by the coefficients of f.

We will fix such a polynomial $f \in \mathcal{F}(\Delta)$ in §2 and §3. We put $\mathbb{F}_q = \mathbb{F}_p(f)$ and $m = [\mathbb{F}_q : \mathbb{F}_p]$. Let $\hat{a}_P \in \mathbb{Z}_q$ be the Teichmüller lift of a_P. We call $\hat{f}(\underline{x}) := \sum\limits_{P\in\Delta^+} \hat{a}_P x^P$ the *Teichmüller lift* of $f(\underline{x})$.

2.1. T-adic exponential sums.

Notation 2.4. We recall that the *Artin–Hasse exponential series* is defined by

$$E(\pi) := \sum_{i=0}^{\infty} c_i \pi^i = \exp\big(\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i}\big) \in 1 + \pi + \pi^2 \mathbb{Z}_p[\![\pi]\!]. \tag{2.2}$$

Setting $E(\pi) = 1 + T$ gives an isomorphism $\mathbb{Z}_p[\![\pi]\!] \cong \mathbb{Z}_p[\![T]\!]$.

Definition 2.5. For a ring R and a power series $g \in R[\![T]\!]$, we define its *T-adic valuation*, denoted by $\mathrm{val}_T(g)$, as the largest k such that $g \in T^k R[\![T]\!]$.

Definition 2.6. For each $k \geq 1$, the *T-adic exponential sum* of f over $\mathbb{F}_{q^k}^\times$ is

$$S_f^*(k, T) := \sum_{\underline{x} \in (\mathbb{F}_{q^k}^\times)^n} (1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(\hat{f}(\hat{\underline{x}}))} \in \mathbb{Z}_p[\![T]\!].$$

Definition 2.7. The *T-adic characteristic power series* of f is defined by

$$\begin{aligned} C_f^*(T, s) &:= \exp\Big(\sum_{k=1}^{\infty} -(q^k-1)^{-n} S_f^*(k, T) \frac{s^k}{k}\Big) \\ &= \sum_{k=0}^{\infty} u_k(T) s^k \in \mathbb{Z}_p[\![T, s]\!]. \end{aligned} \tag{2.3}$$

It is not difficult to check that any finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ satisfies

$$C_f^*(\chi, s) = C_f^*(T, s)|_{T=\chi(1)-1}, \tag{2.4}$$

where $C_f^*(\chi, s)$ is defined in §1. We refer the readers to [DWX, §2] for the proof.

Notation 2.8. We put

$$E_f(\underline{x}) := \prod_{P\in\Delta^+} E(\hat{a}_P \pi x^P) \in \mathbb{Z}_q[\![T]\!][\![\underline{x}]\!] \tag{2.5}$$

and

$$\prod_{P\in\Delta^+\backslash\{\mathcal{O}\}} E(\hat{a}_P \pi x^P) = \sum_{Q\in\mathbb{Z}_{\geq 0}^n} e_Q(T) x^Q \in \mathbb{Z}_q[\![T]\!][\![\underline{x}]\!]. \tag{2.6}$$

We shall later in §4 and §5 need a version of $E_f(\underline{x})$ for a universal polynomial $\widetilde{f}(\underline{x})$. Namely, we consider the universal polynomial

$$\widetilde{f}(\underline{x}) = \sum_{P\in\Delta^+} \widetilde{a}_P x^P \in \mathbb{F}_p[\widetilde{a}_P; P \in \Delta^+][\underline{x}],$$

where $\widetilde{a}_P$ are treated as variables. Then we put

$$\prod_{P\in\Delta^+\backslash\{\mathcal{O}\}} E(\widetilde{a}_P \pi x^P) = \sum_{Q\in\mathbb{Z}_{\geq 0}^n} \widetilde{e}_Q(T) x^Q \in \mathbb{Z}_p[\widetilde{a}_P; P \in \Delta^+\backslash\{\mathcal{O}\}][\![T]\!][\![\underline{x}]\!]. \tag{2.7}$$

2.2. **Dwork's trace formula.**

Definition 2.9. We fix a D-th root $T^{1/D}$ of T. Define

$$\mathbf{B}=\Big\{\sum_{Q\in\mathbb{M}(\Delta)} b_Q x^Q \;\Big|\; b_Q\in\mathbb{Z}_q[\![T^{1/D}]\!], \mathrm{val}_T(b_Q)\to+\infty, \text{when } Q\to\infty\Big\}.$$

Let ψ_p denote the operator on $\mathbf{B}$ such that

$$\psi_p\Big(\sum_{Q\in\mathbb{M}(\Delta)} b_Q x^Q\Big):=\sum_{Q\in\mathbb{M}(\Delta)} b_{pQ}x^Q.$$

Definition 2.10. Define

$$\psi:=\sigma_{\mathrm{Frob}}\circ\psi_p\circ E_f(\underline{x}):\mathbf{B}\longrightarrow\mathbf{B}, \tag{2.8}$$

and its k-th iterate

$$\psi^k=\psi_p^k\circ\prod_{i=0}^{k-1} E_f^{\sigma^i_{\mathrm{Frob}}}(x_1^{p^i},x_2^{p^i},\dots,x_n^{p^i}),$$

where σ_{Frob} represents the arithmetic Frobenius acting on the coefficients, and $E_f(\underline{x})(g):=E_f(\underline{x})\cdot g$ for any $g\in\mathbf{B}$.

Lemma 2.11. *Let N be the matrix of ψ acting on $\mathbf{B}$ with respect to the basis x^Q, then the entries*

$$N_{Q',Q}=E(\hat{a}_{\mathcal{O}}\pi)e_{pQ'-Q}(T),$$

where $e_{pQ'-Q}(T)$ is defined in (2.6)*.*

Proof. One checks easily that

$$\begin{aligned}\psi_p\circ E_f(\underline{x})(x^Q)=&\psi_p\Big(E(\hat{a}_{\mathcal{O}}\pi)\sum_{Q''\in\mathbb{M}(\Delta)} e_{Q''}(T)x^{Q''+Q}\Big)\\ =&E(\hat{a}_{\mathcal{O}}\pi)\sum_{\substack{Q''\in\mathbb{M}(\Delta)\\ Q+Q''=pQ'}} e_{Q''}(T)x^{Q'}\\ =&\sum_{Q'\in\mathbb{M}(\Delta)} E(\hat{a}_{\mathcal{O}}\pi)e_{pQ'-Q}(T)x^{Q'},\end{aligned}$$

which completes the proof. □

Recall that $m=[\mathbb{F}_q:\mathbb{F}_p]$ and $C_f^*(T,s)=\sum_{k=0}^{\infty}u_k(T)s^k\in\mathbb{Z}_p[\![T,s]\!]$.

Theorem 2.12 (Analytic trace formula)**.** *The theorem above has an equivalent multiplicative form:*

$$C_f^*(T,s)=\det\big(I-s\psi^m \mid \mathbf{B}/\mathbb{Z}_q[\![\pi]\!]\big). \tag{2.9}$$

Proof. It follows from Dwork trace formula. For proof, see [LWei, Theorem 4.8]. □

Definition 2.13. The *normalized Newton polygon* of $C_f^*(T,s)$, denoted by $\mathrm{NP}(f,T)_C$, is the lower convex hull of the set of points $\Big\{\Big(i,\frac{\mathrm{val}_T(u_i(T))}{m}\Big)\Big\}$.

Now we recall the weight function and the usual Hodge polygon.

Definition 2.14. For each lattice point Q in $\mathbb{Z}^n$, assume that the line $\overline{OQ}$ intersects some surface of Δ at a point Q'. Then we call

$$w(Q) := \frac{\overrightarrow{OQ}}{\overrightarrow{OQ'}}$$

the weight of Q.

Definition 2.15. Let $\mathbb{W}_\ell$ denote the set consisting of ℓ elements of $\mathbb{M}(\Delta)$ with minimal weights. The usual Hodge polygon, denoted by $\mathrm{HP}(\Delta)$, is the low convex hull of

$$\Big\{\Big(\ell, \sum_{Q\in\mathbb{W}_\ell} (p-1)w(Q)\Big)\Big\}.$$

Proposition 2.16. *The Newton polygon* $\mathrm{NP}(f,T)_C$ *does not lie below* $\mathrm{HP}(\Delta)$.

Proof. See [LWei, Theorem 1.3]. □

Definition 2.17. We call p *ordinary with respect to* Δ if $p \equiv 1 \pmod D$.

3. The improved Hodge polygon

As we have explained in §2 that for $f \in \mathcal{F}(\Delta)$, the Newton polygon $\mathrm{NP}(f,T)_C$ lies above the Hodge polygon introduced in Notation 2.15. However, unless p is ordinary with respect to Δ, this Hodge polygon is not expected to be sharp. In this section, we introduce an improved Hodge polygon which is again a lower bound of $\mathrm{NP}(f,T)_C$, and we prove that for a generic f, our improved Hodge polygon coincides with $\mathrm{NP}(f,T)_C$ at infinitely many points.

Definition 3.1. The *improved Hodge polygon* of Δ, denoted by $\mathrm{IHP}(\Delta)$, is the lower convex hull of the set of points

$$\Big\{\Big(\ell, \sum_{Q\in\mathbb{W}_\ell} \big(\lfloor w(pQ)\rfloor - \lfloor w(Q)\rfloor\big)\Big)\Big\}, \tag{3.1}$$

where $\mathbb{W}_\ell$ consists of ℓ elements of $\mathbb{M}(\Delta)$ with minimal weight as in Definition 2.15.

We will later give a simplified expression of this polygon at some particular points.

Remark 3.2. Each point $Q \in \mathbb{M}(\Delta)$ can be written as a rational linear combination $\sum\limits_{i=1}^{n} z_i\mathbf{V}_i$ of the basis vectors. It is straightforward to see that

$$w(Q) = \max_{1\le i\le n}\{z_i\}. \tag{3.2}$$

In particular, since each $z_i \in \frac{1}{D}\mathbb{Z}$, we have $w(Q) \in \frac{1}{D}\mathbb{Z}$ for every $Q \in \mathbb{M}(\Delta)$.

Moreover, the weight function is subadditive, namely, for any two points $Q_1, Q_2 \in \mathbb{M}(\Delta)$, we have

$$w(Q_1) + w(Q_2) \ge w(Q_1 + Q_2). \tag{3.3}$$

Proposition 3.3.

(1) The improved Hodge polygon $\mathrm{IHP}(\Delta)$ *lies not below* $\mathrm{HP}(\Delta)$ *at* $x = \mathbbm{x}_k^{\pm}$ *for every* $k \ge 1$.

(2) If there are a point $P_0 = \sum\limits_{i=1}^{n} r_i\mathbf{V}_i \in \Delta^-$ *and* $j_1, j_2 \in \{1,2,\dots,n\}$ *such that*

(a) $r_{j_1} < r_{j_2}$, *and*

(b) $pr_{j_1} - \lfloor pr_{j_1}\rfloor > pr_{j_2} - \lfloor pr_{j_2}\rfloor$,

then $\mathrm{IHP}(\Delta)$ *lies strictly above* $\mathrm{HP}(\Delta)$ *at* $x = \mathbbm{x}_k^{\pm}$ *for every* $k \ge 2$.

(3) If there is a point $P_0 = \sum_{i=1}^{n} r_i \mathbf{V}_i \in \Delta^-$ *such that*

$$\left\{j \;\middle|\; r_j = \max_{1\le i\le n}(r_i)\right\} \cap \left\{j \;\middle|\; pr_j - \lfloor pr_j \rfloor = \max_{1\le i\le n}(pr_i - \lfloor pr_i \rfloor)\right\} = \emptyset,$$

then $\mathrm{IHP}(\Delta)$ *lies strictly above* $\mathrm{HP}(\Delta)$ *at* $x = \mathbb{x}_k^{\pm}$ *for every* $k \geq 1$.

Notation 3.4. For each point Q in $\mathbb{M}(\Delta)$ we write $Q\%$ for its residue in Δ^- modulo Λ_Δ, and put

$$\begin{aligned} \eta: \quad & \Delta^- \to \Delta^- \\ & Q \mapsto (pQ)\%. \end{aligned}$$

It is easy to show that η is a permutation of Δ^-.

Proof of Proposition 3.3. (1) Since any point $Q \in \Delta_k^+ \backslash \Delta_k^-$ has integer weight, we get

$$\lfloor w(pQ) \rfloor - \lfloor w(Q) \rfloor = w(pQ) - w(Q).$$

Therefore, we only need to prove this proposition for $x = \mathbb{x}_k^-$.

Since Δ_k^- can be decomposed into a disjoint union of shifts of Δ^- by points in Λ_Δ, we reduce the question to show that for any $Q_1 = \sum_{i=1}^{n} m_i \mathbf{V}_i \in \Lambda_\Delta$,

$$\sum_{Q-Q_1\in\Delta^-} \Big(\lfloor w(pQ) \rfloor - \lfloor w(Q) \rfloor - w(pQ) + w(Q)\Big) \geq 0. \tag{3.4}$$

Let $Q - Q_1 = \sum_{i=1}^{n} r_i \mathbf{V}_i \in \Delta^-$. We set

$$m_{\max} = \max_{1\le j\le n}(m_j) \quad \text{and} \quad S = \{1 \le i \le n \mid m_i = m_{\max}\}.$$

Take $j \in S$ such that $r_j = \max_{i\in S}(r_i)$. Then we have

$$\lfloor w(Q) \rfloor + w(pQ) = m_j + pr_j + pm_j = w\Big(pQ + Q_1\Big). \tag{3.5}$$

Since η is a permutation of Δ^-, we know that

$$\sum_{Q-Q_1\in\Delta^-} w(Q) = \sum_{Q-Q_1\in\Delta^-} w(Q_1 + \eta(Q - Q_1)). \tag{3.6}$$

Since $pQ \equiv \eta(Q - Q_1) \pmod{\Lambda_\Delta}$, we know

$$\begin{aligned} w\Big(pQ - \eta(Q - Q_1)\Big) &= (pr_j + pm_j) - (pr_j - \lfloor pr_j \rfloor) \\ &= pm_j + \lfloor pr_j \rfloor = \lfloor w(pQ) \rfloor. \end{aligned} \tag{3.7}$$

Combining (3.6), (3.7), (3.5), and (3.3), we get

$$\begin{aligned}
&\sum_{Q-Q_1\in\Delta^-}\Big(\lfloor w(pQ)\rfloor-\lfloor w(Q)\rfloor-w(pQ)+w(Q)\Big)\\
\overset{(3.6)}{=}&\sum_{Q-Q_1\in\Delta^-}\Big(\lfloor w(pQ)\rfloor-\lfloor w(Q)\rfloor-w(pQ)+w\Big(Q_1+\eta(Q-Q_1)\Big)\Big)\\
\overset{(3.7)}{=}&\sum_{Q-Q_1\in\Delta^-}\Big(w\Big(pQ-\eta(Q-Q_1)\Big)-\lfloor w(Q)\rfloor-w(pQ)+w\Big(Q_1+\eta(Q-Q_1)\Big)\Big)\\
\overset{(3.5)}{=}&\sum_{Q-Q_1\in\Delta^-}\Big(w\Big(pQ-\eta(Q-Q_1)\Big)+w\Big(Q_1+\eta(Q-Q_1)\Big)-w\Big(pQ+Q_1\Big)\Big)\\
\overset{(3.3)}{\geq}&\ 0.
\end{aligned}$$

(2) We put $Q_1=\mathbf{V}_{j_1}+\mathbf{V}_{j_2}$ and $Q=Q_1+P_0$. From the assumptions (a) and (b), we have

$$\begin{aligned}
&w\Big(pQ-\eta(Q-Q_1)\Big)+w\Big(Q_1+\eta(Q-Q_1)\Big)-w\Big(pQ+Q_1\Big)\\
=&p+\lfloor pr_{j_2}\rfloor+(1+pr_{j_1}-\lfloor pr_{j_1}\rfloor)-(p+pr_2+1)\\
=&pr_{j_1}-\lfloor pr_{j_1}\rfloor-(pr_{j_2}-\lfloor pr_{j_2}\rfloor)>0,
\end{aligned}$$

which completes the proof.

(3) By the similarity of the proofs of (2) and (3), we leave its proof to the readers. ☐

Example 3.5. Let $p=29$, $\mathbf{V}_1=(3,0)$ and $\mathbf{V}_2=(0,2)$. Then we have

$$\mathbb{W}_6=\Delta_1^-=\{(0,0),(0,1),(1,0),(1,1),(0,2),(1,2)\},$$

and the chart of weight of points in Δ^-:

P_0	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(0,2)$	$(1,2)$
$w(P_0)$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$
$w(pP_0)$	0	$\frac{29}{3}$	$\frac{29}{2}$	$\frac{29}{2}$	$\frac{58}{3}$	$\frac{58}{3}$

.

Computing the left hand side of (3.4) for $Q_1=(0,0)$, we have

$$\sum_{P_0\in\Delta^-}\Big(\lfloor w(pP_0)\rfloor-\lfloor w(P_0)\rfloor-w(pP_0)+w(P_0)\Big)=\frac{1}{3}>0.$$

We next give an estimate of the T-adic valuation of each entry of the matrix N, for this, we need to control the T-adic valuation of each $e_Q(T)$.

Lemma 3.6.

(1) Recall that $\prod_{P\in\Delta^+\backslash\{\mathcal{O}\}}E(\hat{a}_P\pi x^P)$ *expands as* $\sum_{Q\in\mathbb{M}(\Delta)}e_Q(T)x^Q$. *Then we have*

$$e_O(T)=1\quad\textit{and}\quad\mathrm{val}_T(e_Q(T))\geq\lceil w(Q)\rceil\ \textit{for all}\ Q\in\mathbb{M}(\Delta).$$

(2) Recall that $\prod_{P\in\Delta^+\backslash\{\mathcal{O}\}}E(\tilde{a}_P\pi x^P)$ *expands as* $\sum_{Q\in\mathbb{M}(\Delta)}\tilde{e}_Q(T)x^Q$. *Then we have*

$$\tilde{e}_O(T)=1\quad\textit{and}\quad\mathrm{val}_T(\tilde{e}_Q(T))\geq\lceil w(Q)\rceil\ \textit{for all}\ Q\in\mathbb{M}(\Delta).$$

Proof. (1) We will only prove (1) and the proof of (2) is similar.

It follows from Definition 2.6 that $e_O(T)=1$.

For each $P \in \Delta^+\backslash\{\mathcal{O}\} = \Delta \cap \mathbb{Z}^n\backslash\{\mathcal{O}\}$, we expand $E(\hat{a}_P\pi x^P)$ to a power series in variables $x_1, x_2, \ldots, x_n$, and get

$$\prod_{P\in\Delta^+\backslash\{\mathcal{O}\}} E(\hat{a}_P\pi x^P) = \sum_{\vec{j}\in\mathbb{Z}_{\geq 0}^{\Delta^+\backslash\{\mathcal{O}\}}} \Big(\prod_{P\in\Delta^+\backslash\{\mathcal{O}\}} c_{j_P}(\hat{a}_P\pi x^P)^{j_P}\Big),$$

where $\{\hat{a}_P\}$ is the set of coefficients of $\hat{f}(\underline{x})$ and $c_i \in \mathbb{Z}_p$ is the π^i coefficient of $E(\pi)$.

Expanding this product and the sum, we deduce

$$\begin{aligned} e_Q(T) = & \sum_{\left\{\vec{j}\in\mathbb{Z}_{\geq 0}^{\Delta^+\backslash\{\mathcal{O}\}} \,\middle|\, \sum\limits_{P\in\Delta^+\backslash\{\mathcal{O}\}} j_P P = Q\right\}} \Big(\prod_{P\in\Delta^+\backslash\{\mathcal{O}\}} \Big(c_{j_P}(\hat{a}_P\pi)^{j_P}\Big)\Big) \\ = & \sum_{\left\{\vec{j}\in\mathbb{Z}_{\geq 0}^{\Delta^+\backslash\{\mathcal{O}\}} \,\middle|\, \sum\limits_{P\in\Delta^+\backslash\{\mathcal{O}\}} j_P P = Q\right\}} \Big(\prod_{P\in\Delta^+\backslash\{\mathcal{O}\}} \Big(c_{j_P}(\hat{a}_P)^{j_P}\Big)\pi^{\sum\limits_{P\in\Delta^+\backslash\{\mathcal{O}\}} j_P}\Big). \end{aligned}$$

Since each point in $\Delta^+\backslash\{\mathcal{O}\}$ has weight less or equal to 1, for each $\vec{j}$ such that $\sum\limits_{P\in\Delta^+\backslash\{\mathcal{O}\}} j_P P = Q$, we have

$$\begin{aligned} & \mathrm{val}_T\Big(\prod_{P\in\Delta^+\backslash\{\mathcal{O}\}} c_{j_P}(\hat{a}_P)^{j_P}\pi^{\sum\limits_{P\in\Delta^+\backslash\{\mathcal{O}\}} j_P}\Big) \\ = & \sum_{P\in\Delta^+\backslash\{\mathcal{O}\}} j_P \geq \sum_{P\in\Delta^+\backslash\{\mathcal{O}\}} j_P w(P) \geq w(Q) \end{aligned}$$

(note that $\mathrm{val}_T(\pi) = 1$). Therefore, we immediately obtain $\mathrm{val}_T(e_Q(T)) \geq w(Q)$. Since $\mathrm{val}_T(e_Q(T))$ is an integer, we have

$$\mathrm{val}_T(e_Q(T)) \geq \lceil w(Q) \rceil. \qquad \square$$

Corollary 3.7. *If both Q and Q' belong to $\mathbb{M}(\Delta)$, then*

$$\mathrm{val}_T(N_{Q',Q}) = \mathrm{val}_T(e_{pQ'-Q}) \geq \lfloor w(pQ') \rfloor - \lfloor w(Q) \rfloor, \tag{3.8}$$

where $N_{Q',Q}$ is the entry of matrix N as in Lemma 2.11.

Proof. Since $\mathrm{val}_T(E(\hat{a}_{\mathcal{O}})) = 0$, we have

$$\mathrm{val}_T(N_{Q',Q}) = \mathrm{val}_T(E(\hat{a}_{\mathcal{O}}) \cdot e_{pQ'-Q}) = \mathrm{val}_T(e_{pQ'-Q}).$$

We assume that $pQ' - Q \in \mathbb{M}(\Delta)$, for otherwise $\mathrm{val}(e_{pQ'-Q}) = \infty$, which leads to (3.8) directly.

By Lemma 3.6, we have

$$\begin{aligned} & \mathrm{val}(e_{pQ'-Q}) \geq \lceil w(pQ') - w(Q) \rceil \\ = & \Big\lceil w(pQ') - w(Q) - \Big(\lfloor w(pQ') \rfloor - \lfloor w(Q) \rfloor\Big)\Big\rceil + \lfloor w(pQ') \rfloor - \lfloor w(Q) \rfloor \\ \geq & \lfloor w(pQ') \rfloor - \lfloor w(Q) \rfloor. \end{aligned} \qquad \square$$

Notation 3.8.

(1) Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be two sets of the same cardinality ℓ. We denote by $\mathrm{Iso}(\mathbb{S}_1, \mathbb{S}_2)$ the set of bijections from $\mathbb{S}_1$ to $\mathbb{S}_2$. If the elements in these two sets are labeled as

$$\mathbb{S}_1 := \{Q_1, Q_2, \ldots, Q_\ell\} \quad \text{and} \quad \mathbb{S}_2 := \{Q'_1, Q'_2, \ldots, Q'_\ell\},$$

then for any $\tau \in \mathrm{Iso}(\mathbb{S}_1, \mathbb{S}_2)$ such that $\tau(Q_i) = Q'_{j_i}$ we put

$$\mathrm{sgn}(\tau) = \mathrm{sgn}(j_1, j_2, \ldots, j_\ell).$$

(2) Let G be a function on $\mathbb{S}_1 \times \mathbb{S}_2$. We put

$$\det\Big(G(Q,Q')\Big)_{Q,Q'\in\mathbb{S}_1\times\mathbb{S}_2} := \pm \sum_{\tau\in\mathrm{Iso}(\mathbb{S}_1,\mathbb{S}_2)} \mathrm{sgn}(\tau) \prod_{Q\in\mathbb{S}_1} G(Q,\tau(Q)).$$

It is easy to see that $\det\Big(G(Q,Q')\Big)_{Q,Q'\in\mathbb{S}_1\times\mathbb{S}_2}$ is independent of the order of elements in $\mathbb{S}_1$ and $\mathbb{S}_2$ up to a sign.

Notation 3.9.

(1) For the rest of this section, we shall consider multisets, i.e. sets of possibly repeating elements. They are marked with a superscript star to be distinguished from usual sets.
(2) The disjoint union of two multisets $\mathbb{S}^\star$ and $\mathbb{S}'^\star$ is denoted by $\mathbb{S}^\star \uplus \mathbb{S}'^\star$ as a multiset.
(3) For a set $\mathbb{S}$, we write $\mathbb{S}^{\star m}$ (resp. $\mathbb{S}^{\star\infty}$) for the union of m (resp. countably infinite) copies of $\mathbb{S}$ as a multiset.

Notation 3.10. For any multisets $\mathbb{S}_1^\star$ and $\mathbb{S}_2^\star$ of the same cardinality, we denote by $\mathrm{Iso}(\mathbb{S}_1^\star, \mathbb{S}_2^\star)$ the set of bijections (as multisets) from $\mathbb{S}_1^\star$ to $\mathbb{S}_2^\star$. When $\mathbb{S}_1^\star = \mathbb{S}_2^\star = \mathbb{S}^\star$, we put $\mathrm{Iso}(\mathbb{S}^\star) := \mathrm{Iso}(\mathbb{S}^\star, \mathbb{S}^\star)$.

Definition 3.11. Let $\mathbb{S}^\star$ be a subset of $\mathbb{M}(\Delta)^{\star\infty}$. We define

$$h(\mathbb{S}^\star) := \sum_{Q\in\mathbb{S}^\star} \lfloor w(pQ)\rfloor - \lfloor w(Q)\rfloor. \tag{3.9}$$

The expression on the right hand side will be related to the estimate in Corollary 3.7, and to the expression (3.1).

If $\mathbb{S}^\star$ belongs to $\mathbb{M}(\Delta)$, we suppress the star from the notation.

Notation 3.12. We denote $\mathscr{M}_\ell(k)$ to be the set consisting of all sub-multisets of $\mathbb{M}(\Delta)^{\star k}$ of cardinality $k\ell$. For simplicity, we put $\mathscr{M}_\ell := \mathscr{M}_\ell(1)$.

Remark 3.13. It is clear that $\mathrm{IHP}(\Delta)$ is the lower convex hull of the set of points $\Big\{\Big(\ell, \min_{\mathbb{S}\in\mathscr{M}_\ell} h(\mathbb{S})\Big)\Big\}$.

Now we gather more informations of $\mathrm{IHP}(\Delta)$. Recall that D is the smallest positive integer that satisfies (2.1).

Lemma 3.14. *Let $\mathbb{S}_1, \ldots, \mathbb{S}_m \in \mathscr{M}_\ell$ so that their disjoint union $\mathbb{S}^\star = \biguplus_{j=0}^{m-1} \mathbb{S}_j \in \mathscr{M}_\ell(m)$. Then the following two statements are equivalent.*

(1) The minimum of $h(\mathbb{S}'^\star)$ over all $\mathbb{S}'^\star \in \mathscr{M}_\ell(m)$ is achieved by $\mathbb{S}^\star$,
(2) For each i, $\sum_{Q\in\mathbb{S}_i} w(Q)$ achieves the minimum of $\sum_{Q\in\mathbb{S}'} w(Q)$ over all $\mathbb{S}' \in \mathscr{M}_\ell$.

Proof. It is obvious that h is additive. Hence, without loss of generality, we assume $m = 1$ in the proof.

First, we claim that for Q and Q' two points in $\mathbb{M}(\Delta)$, if $w(Q) > w(Q')$, then

$$\lfloor w(pQ)\rfloor - \lfloor w(Q)\rfloor - (\lfloor w(pQ')\rfloor - \lfloor w(Q')\rfloor) > 0. \tag{3.10}$$

Indeed, by Definition 2.2, if $w(Q) > w(Q')$, then

$$w(Q) \geq w(Q') + \frac{1}{D}.$$

Since we assume $p > D(n+4)$ in Hypothesis 1.1, we have

$$\begin{aligned}
&\lfloor w(pQ) \rfloor - \lfloor w(Q) \rfloor - (\lfloor w(pQ') \rfloor - \lfloor w(Q') \rfloor) \\
>& w(pQ) - w(pQ') - w(Q) + w(Q') - 2 \\
\geq& \frac{p-1}{D} - 2 \\
>& 0.
\end{aligned}$$

Therefore, if $\mathbb{S}' \in \mathscr{M}_\ell$ such that $h(\mathbb{S}')$ is minimal over all $\mathbb{S} \in \mathscr{M}_\ell$, then $\sum\limits_{Q \in \mathbb{S}'} w(Q)$ takes the minimum of $\sum\limits_{Q \in \mathbb{S}} w(Q)$ for all $\mathbb{S} \in \mathscr{M}_\ell$.

Let $\mathbb{S}'$ and $\mathbb{S}''$ be any two sets in $\mathscr{M}_\ell$ such that $\sum\limits_{Q \in \mathbb{S}'} w(Q)$ and $\sum\limits_{Q \in \mathbb{S}''} w(Q)$ reach the minimum of $\sum\limits_{Q \in \mathbb{S}} w(Q)$ over all $\mathbb{S} \in \mathscr{M}_\ell$. It is easy to check that the two multisets $\{w(Q) \mid Q \in \mathbb{S}'\}$ and $\{w(Q) \mid Q \in \mathbb{S}''\}$ are same, which implies

$$h(\mathbb{S}') = h(\mathbb{S}'').$$

This completes the proof. $\square$

Proposition 3.15. *For any $\ell \geq 1$, let $\mathbb{W}_\ell$ denote the set of ℓ elements in $\mathbb{M}(\Delta)$ with minimal weight.*

(1) We have

$$\min_{\mathbb{S}^\star \in \mathscr{M}_\ell(m)} h(\mathbb{S}^\star) = m h(\mathbb{W}_\ell). \tag{3.11}$$

(2) Order the elements in $\mathbb{M}(\Delta)$ by their weights (in the non-decreasing order): $P_1, P_2, \ldots$. Suppose that $n_1, n_2, \ldots$ are the exactly the indices such that $w(P_{n_i+1}) > w(P_{n_i})$. Then the $\mathrm{IHP}(\Delta)$ is given by the polygon with vertices $(n_i, h(\mathbb{W}_{n_i}))$.

Proof. (1) From the proof of Lemma 3.14, we now that the minimum on the left is achieved when $\mathbb{S}^\star = \biguplus\limits_{j=0}^{m-1} \mathbb{S}_j$ with $\mathbb{S}_j = \mathbb{W}_\ell$. Since h is additive, we have

$$h\Big(\biguplus_{j=0}^{m-1} \mathbb{S}_j\Big) = \sum_{j=0}^{m-1} h(\mathbb{S}_j) = m h(\mathbb{W}_\ell).$$

(2) Since the weight of P_i is non-decreasing with respect to i, by Lemma 3.14, the improved Hodge polygon $\mathrm{IHP}(\Delta)$ passes point $(\ell, \sum\limits_{i=1}^{\ell} (\lfloor w(pP_i) \rfloor - \lfloor w(P_i) \rfloor))$ for every $\ell \geq 1$. Since $w(P_{n_i+1}) > w(P_{n_i})$, by (3.10), we have

$$\lfloor w(pP_{n_i+1}) \rfloor - \lfloor w(P_{n_i+1}) \rfloor - (\lfloor w(pP_{n_i}) \rfloor - \lfloor w(P_{n_i}) \rfloor) > 0.$$

Therefore, (n_i, λ_{n_i}) is a vertex of $\mathrm{IHP}(\Delta)$ for every $i \geq 1$. $\square$

Corollary 3.16.

(1) The height of the improved Hodge polygon at $x = \mathtt{x}_k^{\pm}$ satisfies

$$\min_{\mathbb{S} \in \mathscr{M}_{\mathtt{x}_k^{\pm}}} h(\mathbb{S}) = h(\Delta_k^{\pm}).$$

(2) Every Newton slope of $\mathrm{IHP}(\Delta)$ before point $x = \mathtt{x}_k^-$ is strictly less than $k(p-1)$.

(3) Every slope after the point $x = \mathbb{x}_k^+$ is strictly greater than $k(p-1)$.
(4) Every Newton slope of $\mathrm{IHP}(\Delta)$ between points $x = \mathbb{x}_k^-$ and $x = \mathbb{x}_k^+$ is equal to $k(p-1)$.
(5) For every $k \geq 1$, the points $(\mathbb{x}_k^\pm, h(\Delta_k^\pm))$ are vertices of $\mathrm{IHP}(\Delta)$.

Proof. They are straightforward corollaries of Proposition 3.15. □

Notation 3.17. For any subset $I \subseteq \{1, 2, \dots, n\}$ we put

$$\Delta^-(I) = \left\{ P_0 = \sum_{i=1}^n r_i \mathbf{V}_i \in \Delta^- \,\middle|\, \begin{array}{ll} r_i = 0 & \text{if } i \in I, \\ r_i \in (0,1) & \text{otherwise} \end{array} \right\} \subseteq \Delta^-.$$

Lemma 3.18. *We have*
(1) $\mathbb{x}_k^- = k^n \mathrm{Vol}(\Delta)$, and
(2) $\mathbb{x}_k^+ = \sum\limits_{I \subseteq \{1,2,\dots,n\}} \#\Delta^-(I) k^{\#I}(k+1)^{n-\#I}$.

Proof. (1) Every point in $Q \in \mathbb{M}(\Delta)$ belongs to Δ_k^- if and only if $Q = P_0 + Q_1$ for some $P_0 \in \Delta^-$ and $Q_1 = \sum\limits_{i=1}^n m_i \mathbf{V}_i \in \Lambda_\Delta$ such that $0 \leq m_i \leq k-1$ for every $1 \leq i \leq n$. Hence, we get

$$\mathbb{x}_k^- = \#\Delta^- k^n = k^n \mathrm{Vol}(\Delta).$$

(2) Let $I \subset \{1, 2, \dots, n\}$, $P_0 \in \Delta^-(I)$, and $Q \in \mathbb{M}(\Delta)$ such that $Q \equiv P_0 \pmod{\Lambda_\Delta}$. Write $Q = P_0 + Q_1$ for some $Q_1 = \sum\limits_{i=1}^n m_i \mathbf{V}_i \in \Lambda_\Delta$. Then Q belongs to Δ_k^+ if and only if $0 \leq m_i \leq k$ for each $i \in I$ and $0 \leq m_i \leq k-1$ for each $i \notin I$. Therefore, there are $k^{\#I}(k+1)^{n-\#I}$ points in Δ_k^+ with the residue P_0 module Λ_Δ. It completes the proof. □

Lemma 3.19. *The function $h(\Delta_k^\pm)$ is a polynomial in k of degree $n+1$, i.e. for any $k \geq 1$,*

$$h(\Delta_k^\pm) = \sum_{i=0}^{n+1} A_i^\pm k^i,$$

where $A_i^\pm$ are integers which depend only on Δ.

Proof. By Lemma 3.18, the function

$$\begin{aligned} h(\Delta_k^+) =& h(\Delta_k^-) + (p-1)k(\mathbb{x}_k^+ - \mathbb{x}_k^-) \\ =& h(\Delta_k^-) + (p-1)k \sum_{I \subseteq \{1,2,\dots,n\}} \left(\#\Delta^-(I) k^{\#I}(k+1)^{n-\#I} - \#\Delta^-(I) k^n \right). \end{aligned}$$

Hence, it is enough to prove the statement for $h(\Delta_k^-)$.

For any nonempty subset $I \subseteq \{1, 2, \dots, n\}$ and $k \geq 1$, we put

$$U(k; I) = \left\{ \sum_{i=1}^n m_i \mathbf{V}_i \in \Lambda_\Delta \,\middle|\, \begin{array}{ll} m_i = k & \text{if } i \in I, \\ m_i < k & \text{otherwise} \end{array} \right\}.$$

It is easy to check that every point $Q \in \mathbb{M}(\Delta)$ belongs to $\Delta_k^- \backslash \Delta^-$ if and only if there exist $P_0 \in \Delta^-$, $\ell \leq k-1$, and a nonempty subset $I \subseteq \{1, 2, \dots, n\}$ such that $Q \in P_0 + U(\ell; I)$. Therefore, we have the following decomposition:

$$\Delta_k^- = \Delta^- \sqcup \coprod_{P_0 \in \Delta^-} \coprod_{\substack{I \subseteq \{1,2,\dots,n\} \\ I \neq \emptyset}} \coprod_{\ell=1}^{k-1} P_0 + U(\ell; I).$$

Since h is additive, we obtain

$$h(\Delta_k^-) = h(\Delta^-) + \sum_{P_0\in\Delta^-}\sum_{\substack{I\subseteq\{1,2,\dots,n\}\\ I\neq\emptyset}}\sum_{\ell=1}^{k-1} h(P_0+U(\ell;I)).$$

It is enough to show that for a fixed P_0 and a nonempty subset $I\subseteq I(P_0)$, the sum $\sum_{\ell=1}^{k-1} h(P_0+U(\ell;I))$ is a polynomial in k of degree less or equal $n+1$.

Now we prove that every $P_1 = P_0+\sum_{i=1}^{n} m_{1,i}\mathbf{V}_i \in P_0+U(k_1;I)$ and $P_2 = P_0+\sum_{i=1}^{n} m_{2,i}\mathbf{V}_i \in P_0+U(k_2;I)$ satisfy

$$h(P_1) = h(P_2) + (k_1-k_2)(p-1), \tag{3.12}$$

where $h(P) := \lfloor w(pP)\rfloor - \lfloor w(P)\rfloor$.

Let $P_0 = \sum_{i=1}^{n} r_i\mathbf{V}_i$ and $r_s = \max_{i\in I}\{r_i\}$ for some $s\in I$. By (3.2), we have

$$\Big\lfloor w(p(P_0+\sum_{i=1}^{n} m_{j,i}V_i))\Big\rfloor - \Big\lfloor w(P_0+\sum_{i=1}^{n} m_{j,i}V_i)\Big\rfloor = \lfloor pr_s\rfloor + pk_j - k_j$$

for $j=1,2$, which implies (3.12).

We choose a representative $P'\in U(1;I)$, and put

$$h(P_0;I) := h(P_0+P').$$

By (3.12), $h(P_0;I)$ is well defined and for any point $P\in P_0+U(\ell,I)$,

$$h(P) = h(P_0;I) + (\ell-1)(p-1).$$

Therefore, we have

$$\begin{aligned} h(P_0+U(\ell,I)) =& \#U(\ell,I)\Big(h(P_0;I)+(\ell-1)(p-1)\Big)\\ =&\ell^{n-\#I}\Big(h(P_0;I)+(\ell-1)(p-1)\Big). \end{aligned}\tag{3.13}$$

For any $\ell\geq 1$ and $k\geq 0$ we put $G_{k,\ell} := \sum_{i=1}^{\ell} i^k$, which is well-known to be a polynomial in ℓ of degree $k+1$. Therefore, the function

$$\begin{aligned}&\sum_{\ell=1}^{k-1}\ell^{n-\#I}\Big(h(P_0;I)+(\ell-1)(p-1)\Big)\\ =&G_{n-\#I,k-1}(h(P_0;I)-p+1)+(p-1)G_{n+1-\#I,k-1}\end{aligned}$$

is a polynomial in k of degree in $n+1$, so is $h(\Delta_k^-)$ when combined with (3.13). □

Notation 3.20. We denote by

$$\begin{bmatrix} P_0 & P_1 & \cdots & P_{\ell-1}\\ Q_0 & Q_1 & \cdots & Q_{\ell-1}\end{bmatrix}_M$$

the $\ell\times\ell$-submatrix formed by elements of a matrix M whose row indices belong to $\{P_0,P_1,\dots,P_{\ell-1}\}$ and whose column indices belong to $\{Q_0,Q_1,\dots,Q_{\ell-1}\}$.

Recall that we defined the improved Hodge polygon $\mathrm{IHP}(\Delta)$ in Definition 3.1.

Proposition 3.21. *For any $f\in\mathcal{F}(\Delta)$, the normalized Newton polygon* $\mathrm{NP}(f,T)_C$ *lies above* $\mathrm{IHP}(\Delta)$.

Proof. We first recall the definition of u_ℓ in (2.3), and write N for the standard matrix of $\psi_p \circ E_f$ corresponding to the basis $\{x^Q\}_{Q\in\mathbb{M}(\Delta)}$ of the Banach space $\mathbf{B}$. By [RWXY, Corollary 3.9], we know that the standard matrix of ψ^n corresponding to the same basis is equal to $\sigma_{\mathrm{Frob}}^{m-1}(N) \circ \sigma_{\mathrm{Frob}}^{m-2}(N) \circ \cdots \circ N$. By [RWXY, Proposition 4.6], for every $\ell \in \mathbb{N}$ we have

$$u_\ell(T) = \sum_{\substack{\{Q_{0,0},Q_{0,1},\dots,Q_{0,\ell-1}\}\in\mathscr{M}_\ell \\ \{Q_{1,0},Q_{1,1},\dots,Q_{1,\ell-1}\}\in\mathscr{M}_\ell \\ \vdots \\ \{Q_{m-1,0},Q_{m-1,1},\dots,Q_{m-1,\ell-1}\}\in\mathscr{M}_\ell}} \det\left(\prod_{j=0}^{m-1}\begin{bmatrix} Q_{j+1,0} & Q_{j+1,1} & \cdots & Q_{j+1,\ell-1} \\ Q_{j,0} & Q_{j,1} & \cdots & Q_{j,\ell-1}\end{bmatrix}_{\sigma_{\mathrm{Frob}}^j(N)}\right), \tag{3.14}$$

where $Q_{m,i} := Q_{0,i}$ for each $0 \le i \le \ell-1$.

We set $\mathbb{S}_j = \{Q_{j,0}, Q_{j,1}, \dots, Q_{j,\ell-1}\}$, where $\mathbb{S}_m = \mathbb{S}_0$, then

$$\begin{aligned}
&\mathrm{val}_T\left(\det\left(\prod_{j=0}^{m-1}\begin{bmatrix} Q_{j+1,0} & Q_{j+1,1} & \cdots & Q_{j+1,\ell-1} \\ Q_{j,0} & Q_{j,1} & \cdots & Q_{j,\ell-1}\end{bmatrix}_{\sigma_{\mathrm{Frob}}^j(N)}\right)\right) \\
=&\mathrm{val}_T\left(\prod_{j=0}^{m-1}\det\left(\sigma_{\mathrm{Frob}}^j(e_{pQ'-Q})\right)_{Q,Q'\in\mathbb{S}_{j+1}\times\mathbb{S}_j}\right) \\
\ge&\sum_{j=0}^{m-1}\min_{\tau_j\in\mathrm{Iso}(\mathbb{S}_{j+1},\mathbb{S}_j)}\Big\{\sum_{Q\in\mathbb{S}_{j+1}}\mathrm{val}_T(\sigma_{\mathrm{Frob}}^j(e_{p\tau_j(Q)-Q}))\Big\} \\
\ge&\sum_{j=0}^{m-1}\min_{\tau_j\in\mathrm{Iso}(\mathbb{S}_{j+1},\mathbb{S}_j)}\Big\{\sum_{Q\in\mathbb{S}_{j+1}}\lfloor w(p\tau_j(Q))\rfloor - \lfloor w(Q)\rfloor\Big\} \\
=&\sum_{Q\in\biguplus_{j=0}^{m-1}\mathbb{S}_j}(\lfloor w(pQ)\rfloor - \lfloor w(Q)\rfloor) \\
=&h\Big(\biguplus_{j=0}^{m-1}\mathbb{S}_j\Big).
\end{aligned} \tag{3.15}$$

Therefore, we deduce

$$\frac{\mathrm{val}_T(u_\ell(T))}{m} \ge \frac{\min\limits_{\mathbb{S}_0,\dots,\mathbb{S}_{m-1}\in\mathscr{M}_\ell} h\big(\biguplus_{j=0}^{m-1}\mathbb{S}_j\big)}{m} = \min_{\mathbb{S}\in\mathscr{M}_\ell} h(\mathbb{S}) = h(\mathbb{W}_\ell). \tag{3.16}$$

So $\mathrm{NP}(T,f)_C$ is above $\mathrm{IHP}(\Delta)$. □

Corollary 3.22. *Let $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ be a finite character of conductor p^{m_χ}. The improved Hodge polygon is a lower bound of the normalized Newton polygon $\mathrm{NP}(f,\chi)_C$.*

Proof. Specializing will only increase the valuations. Therefore, for any finite character χ, the Newton polygon $\mathrm{NP}(f,\chi)_C$ lies above $\mathrm{NP}(f,T)_C$. Hence, $\mathrm{NP}(f,\chi)_C$ lies above $\mathrm{IHP}(\Delta)$, when combined with Proposition 3.21. □

Corollary 3.23. *We have*

$$u_{\mathbb{x}_k^\pm} \equiv \prod_{j=0}^{m-1}\sigma_{\mathrm{Frob}}^j\Big(\det(e_{pQ'-Q})_{Q,Q'\in\Delta_k^\pm}\Big) \pmod{T^{mh(\Delta_k^\pm)+1}}.$$

Proof. By Proposition 3.15, when $\ell = \mathbb{x}_k^{\pm}$, the equalities hold in (3.15) if and only if

$$\mathbb{S}_j = \Delta_k^{\pm} \text{ for every } 0 \le j \le m-1. \qquad \square$$

4. The generic Newton polygon

In this section, we will prove Theorem 1.7, which is given by a sequence of sufficient statements.

Recall that $\widetilde{e}_Q(T)$ is defined in (2.7), and satisfies $\mathrm{val}_T(\widetilde{e}_Q(T)) \ge \lfloor w(pQ) \rfloor - \lfloor w(Q) \rfloor$ for all $Q \in \mathbb{M}(\Delta)$ (see Lemma 3.6 (2)).

Notation 4.1. We put

$$\det(I - \widetilde{e}_{pQ'-Q}s)_{Q,Q' \in \mathbb{M}(\Delta)} = \sum_{i=0}^{\infty} \widetilde{u}_i(T) s^i \in \mathbb{Z}_p[\widetilde{a}_P; P \in \Delta^+ \backslash \{\mathcal{O}\}][\![s]\!]. \tag{4.1}$$

Similar to (3.16), we have

$$\mathrm{val}_T(\widetilde{u}_{\mathbb{x}_k^{\pm}}(T)) \ge h(\Delta_k^{\pm}), \tag{4.2}$$

which allows us to put

$$\widetilde{u}_{\mathbb{x}_k^{\pm}}(T) := \sum_{i=h(\Delta_k^{\pm})}^{\infty} \widetilde{u}_{\mathbb{x}_k^{\pm},i} T^i \quad \text{for } \widetilde{u}_{\mathbb{x}_k^{\pm},i} \in \mathbb{Z}_p[\widetilde{a}_P; P \in \Delta^+ \backslash \{\mathcal{O}\}]. \tag{4.3}$$

Proposition 4.2. *For any $1 \le k \le n+2$, we have*

$$\widetilde{u}_{\mathbb{x}_k^{\pm}, h(\Delta_k^{\pm})} \not\equiv 0 \pmod p. \tag{4.4}$$

We will give its proof in §5.

Proposition 4.3. *Proposition 4.2 $\Longrightarrow$ Theorem 1.7 and Theorem 1.5.*

We give its proof after several lemmas.

Lemma 4.4. *Assume Proposition 4.2. Then there exists a Zariski open subset $O_{\mathrm{Zar}} \subseteq \mathcal{F}(\Delta)$ such that for every $f \in O_{\mathrm{Zar}}$ and every finite character χ_0 of conductor p, $\mathrm{NP}(f, \chi_0)_C$ passes though $(\mathbb{x}_k^{\pm}, h(\Delta_k^{\pm}))$ for any $1 \le k \le n+2$.*

Proof. Since $\widetilde{u}_{\mathbb{x}_k^{\pm}, h(\Delta_k^{\pm})} \not\equiv 0 \pmod p$ holds for every $1 \le k \le n+2$, the subset

$$O_{\mathrm{Zar}}(\Delta) := \Big\{ f(\underline{x}) = \sum_{P \in \Delta^+} a_P x^P \in \mathcal{F}(\Delta) : \widetilde{u}_{\mathbb{x}_k^{\pm}, h(\Delta_k^{\pm})}\Big|_{\widetilde{a}_P = \hat{a}_P} \not\equiv 0 \pmod p \text{ for any } 1 \le k \le n+2 \Big\} \tag{4.5}$$

is a nonempty open subset of $\mathcal{F}(\Delta)$.

Similar to Corollary 3.23, the determinant

$$\det(\widetilde{e}_{pQ'-Q})_{Q,Q' \in \Delta_k^{\pm}} \equiv \widetilde{u}_{\mathbb{x}_k^{\pm}, h(\Delta_k^{\pm})} T^{h(\Delta_k^{\pm})} \mod (p, T^{h(\Delta_k^{\pm})+1}). \tag{4.6}$$

Hence, for any $f(\underline{x}) = \sum_{P\in\Delta^+} a_P x^P \in O_{\mathrm{Zar}}(\Delta)$ with $m = [\mathbb{F}_p(f):\mathbb{F}_p]$, the congruence

$$
\begin{aligned}
u_{\mathtt{x}_k^\pm} &\equiv \prod_{j=0}^{m-1} \sigma_{\mathrm{Frob}}^j\Big(\det(e_{pQ'-Q})_{Q,Q'\in\Delta_k^\pm}\Big)\\
&= \prod_{j=0}^{m-1} \sigma_{\mathrm{Frob}}^j\Big(\det(\widetilde{e}_{pQ'-Q})_{Q,Q'\in\Delta_k^\pm}\Big|_{\widetilde{a}_P=\hat{a}_P}\Big)\\
&\equiv \prod_{j=0}^{m-1} \sigma_{\mathrm{Frob}}^j(\widetilde{u}_{\mathtt{x}_k,h(\Delta_k^\pm)}\Big|_{\widetilde{a}_P=\hat{a}_P})T^{mh(\Delta_k^\pm)}\\
&\not\equiv 0 \pmod{p, T^{mh(\Delta_k^\pm)+1}}
\end{aligned}
\tag{4.7}
$$

holds for every $1\le k\le n+2$.

By (2.4) and Corollary 3.22, we get

$$
C_f^*(\chi_0,s) = C_f^*(T,s)\Big|_{T=\chi_0(1)-1} \quad \text{and} \quad u_{\mathtt{x}_k^\pm} = \sum_{\ell=T^{mh(\Delta_k^\pm)}}^{\infty} u_{\mathtt{x}_k^\pm,\ell}T^\ell.
$$

Combining these two equalities with (4.7), we have for any $1\le k\le n+2$ the p-adic valuation of the coefficient of $s^{\mathtt{x}_k^\pm}$ in $C_f^*(\chi_0,s)$ satisfies

$$
\mathrm{val}_p(u_{\mathtt{x}_k^\pm}\Big|_{T=\chi_0(1)-1}) = m\mathrm{val}_p(\xi_p-1)h(\Delta^\pm) = \frac{mh(\Delta_k^\pm)}{p-1}.
$$

It implies that $\mathrm{NP}(f,\chi_0)_C$ lies below $(\mathtt{x}_k^\pm, h(\Delta_k^\pm))$ for any $1\le k\le n+2$. Since $(\mathtt{x}_k^\pm, h(\Delta_k^\pm))$ are *vertices* of $\mathrm{IHP}(\Delta)$ which, by Corollary 3.22, is a lower bound for $\mathrm{NP}(f,\chi_0)_C$, we conclude that $\mathrm{NP}(f,\chi_0)_C$ must pass though the points $(\mathtt{x}_k^\pm, h(\Delta_k^\pm))$ for any $1\le k\le n+2$. □

Lemma 4.5. *For any $f\in\mathcal{F}(\Delta)$ and any finite character χ_0 of conductor p, if $\mathrm{NP}(f,\chi_0)_C$ passes though $(\mathtt{x}_k^\pm, h(\Delta_k^\pm))$ for any $1\le k\le n+2$, then it passes $(\mathtt{x}_k^\pm, h(\Delta_k^\pm))$ for all $k\ge 1$.*

Proof. Since for any $1\le k\le n+2$, the Newton polygon $\mathrm{NP}(f,\chi_0)_C$ passes though $(\mathtt{x}_k^\pm, h(\Delta_k^\pm))$ which are vertices of its lower bound $\mathrm{IHP}(\Delta)$, we have

(a) the points $(\mathtt{x}_k^-, h(\Delta_k^-))$ and $(\mathtt{x}_k^+, h(\Delta_k^+))$ are vertices of $\mathrm{NP}(f,\chi_0)_C$, and the segment connecting them coincides $\mathrm{NP}(f,\chi_0)_C$.
(b) There are $\mathtt{x}_k^-$ slopes of $\mathrm{NP}(f,\chi_0)_C$ strictly less than $k(p-1)$.
(c) There are $\mathtt{x}_k^+$ slopes of $\mathrm{NP}(f,\chi_0)_C$ less than or equal to $k(p-1)$.

Let $\mathbb{L}^\star = \{\alpha_1,\alpha_2,\ldots,\alpha_{n!\mathrm{Vol}(\Delta)}\}$ denote the Newton slopes of $\mathrm{NP}(f,\chi_0)_L$ in a non-descending order. By Weil conjecture, $\alpha\in[0,n(p-1)]$ for any $\alpha\in\mathbb{L}^\star$. We denote by $\mathfrak{t}_i$ the index such that

$$
\mathfrak{t}_i = \#\{\alpha\in\mathbb{L}^\star \mid \alpha < i(p-1)\} \quad \text{for} \quad 0\le i\le n.
$$

Set

$$
\mathbf{X}_i := \sum_{j=1}^{\mathfrak{t}_i} \alpha_j \quad \text{and} \quad C_f^*(\chi_0,s) := \sum_{i=0}^{\infty} u_{\chi_0,i}s^i \in \mathbb{Z}_p[\![s]\!].
$$

Consider the relation between the characteristic function and L-function associated to χ_0 and f, we have

$$
C_f^*(\chi_0,s) = \Big(\prod_{j=0}^{\infty} L_f^*(\chi_0,q^js)^{\binom{n+j-1}{n-1}}\Big)^{(-1)^{n-1}}. \tag{4.8}
$$

A simple computation shows that there are

$$\sum_{j=\ell-n}^{\ell-1}\binom{n+j-1}{n-1}(\mathbb{t}_{\ell-j+1}-\mathbb{t}_{\ell-j})$$

slopes of $\mathrm{NP}(f,\chi_0)_C$ belonging to $[(\ell-1)(p-1),\ell(p-1))$, where $\mathbb{t}_{<0}=0$ and

$$\binom{n_1}{n_2}:=\frac{n_1(n_1-1)\cdots(n_1-n_2+1)}{n_2!}.$$

Therefore, the slopes of $\mathrm{NP}(f,\chi_0)_C$ strictly less than $k(p-1)$ is equal to

$$\sum_{\ell=1}^{k}\sum_{j=\ell-n}^{\ell-1}\binom{n+j-1}{n-1}(\mathbb{t}_{\ell-j+1}-\mathbb{t}_{\ell-j}),$$

and $\mathrm{NP}(f,\chi_0)_C$ passes through points

$$\Big(\sum_{\ell=1}^{k}\sum_{j=\ell-n}^{\ell-1}\binom{n+j-1}{n-1}(\mathbb{t}_{\ell-j+1}-\mathbb{t}_{\ell-j}),\sum_{\ell=1}^{k}\sum_{j=\ell-n}^{\ell-1}\binom{n+j-1}{n-1}(\mathbf{X}_{\ell-j+1}-\mathbf{X}_{\ell-j}+j(p-1)(\mathbb{t}_{\ell-j+1}-\mathbb{t}_{\ell-j})\Big)$$

for all $k\geq 1$.

Combining it with (b), we obtain

$$\sum_{\ell=1}^{k}\sum_{j=\ell-n}^{\ell-1}\binom{n+j-1}{n-1}(\mathbb{t}_{\ell-j+1}-\mathbb{t}_{\ell-j})=\mathbb{x}_k^-=k^n\mathrm{Vol}(\Delta),\tag{4.9}$$

for every $1\leq k\leq n+2$.

The left hand side of (4.9) is equal to

$$\sum_{i=1}^{n}\Big((\mathbb{t}_{i+1}-\mathbb{t}_i)\sum_{\ell=1}^{k}\binom{n+\ell-i-1}{n-1}\Big)$$

which is a polynomial of degree n.

Since the both sides of (4.9) are polynomials in k of degree n and they have the same values for any $1\leq k\leq n+2$, they must be identical as polynomials. Namely, the equality (4.9) holds for all $k\geq 1$.

Combining (4.9) with (a), we get

$$\sum_{\ell=1}^{k}\sum_{j=\ell-n}^{\ell-1}\binom{n+j-1}{n-1}(\mathbf{X}_{\ell-j+1}-\mathbf{X}_{\ell-j}+j(p-1)(\mathbb{t}_{\ell-j+1}-\mathbb{t}_{\ell-j}))=h(\Delta_k^-)\tag{4.10}$$

for every $1\leq k\leq n+2$.

It is easy to check that the left hand side of (4.10) is a polynomial in k of degree $n+1$. By Lemma 3.19, the right hand side of (4.10) is also a polynomial in k of degree $n+1$. Running the similar argument as above, we conclude that the equality (4.10) holds for every $k\geq 1$. Therefore, the Newton polygon $\mathrm{NP}(f,\chi_0)_C$ passes through $(\mathbb{x}_k^-,h(\Delta_k^-))$ for every $k\geq 1$.

A similar argument shows that $\mathrm{NP}(f,\chi_0)_C$ passes through $(\mathbb{x}_k^+,h(\Delta_k^+))$ for every $k\geq 1$. We leave the details to the readers. ☐

Lemma 4.6. *For any $f\in\mathcal{F}(\Delta)$ with $m=[\mathbb{F}_p(f):\mathbb{F}_p]$, if there exists a finite character χ_0 of conductor p such that $\mathrm{NP}(f,\chi_0)_C$ passes though $(\mathbb{x}_k^\pm,h(\Delta_k^\pm))$ for all $k\geq 1$, then for any finite character χ of conductor p^{m_χ}, $\mathrm{NP}(f,\chi)_C$ passes though $(\mathbb{x}_k^\pm,h(\Delta_k^\pm))$ for every $k\geq 0$.*

Proof. Recall that $C_f^*(T,s)=\sum_{i=0}^{\infty}u_is^i$. It is easy to see that

$$u_{\mathbb{x}_0^-}=1 \quad \text{and} \quad u_{\mathbb{x}_0^+}=-\mathrm{Norm}_{\mathbb{F}_p(f)/\mathbb{F}_p}(E(\hat{a}_{\mathcal{O}}\pi))\equiv -1 \pmod{T},$$

which implies that $\mathrm{NP}(f,\chi)_C$ passes thought point $(\mathbb{x}_0^{\pm},h(\Delta_0^{\pm}))$.

Then we show that

$$u_{\mathbb{x}_k^{\pm}}\not\equiv 0 \pmod{p,T^{mh(\Delta_k^{\pm})+1}} \tag{4.11}$$

for any $k\geq 1$.

Suppose it is false. Without loss of generality, we may assume that there exists $k_0\geq 1$ such that

$$u_{\mathbb{x}_{k_0}^-}\equiv 0 \pmod{p,T^{mh(\Delta_{k_0}^-)+1}},$$

which implies that the p-adic valuation of the coefficient of $s^{\mathbb{x}_{k_0}^-}$ in $C_f^*(\chi_0,s)$ satisfies

$$\mathrm{val}_p(u_{\mathbb{x}_{k_0}^-}\Big|_{T=\chi_0(1)-1})>\frac{mh(\Delta_{k_0}^-)}{p-1}.$$

It shows that the Newton polygon $\mathrm{NP}(f,\chi_0)_C$ can never pass this point $(\mathbb{x}_{k_0}^-,h(\Delta_{k_0}^-))$, a vertex of its lower bound $\mathrm{IHP}(\Delta)$, a contradiction to the assumption in this lemma.

Therefore, (4.11) holds and we have

$$\mathrm{val}_p(u_{\mathbb{x}_k^{\pm}}\Big|_{T=\chi(1)-1})=\frac{mh(\Delta_k^{\pm})}{(p-1)p^{m_\chi-1}},$$

which implies that $\mathrm{NP}(f,\chi)_C$ passes through $(\mathbb{x}_k^{\pm},h(\Delta_k^{\pm}))$ for every $k\geq 1$ when combined with Corollary 3.22. □

Proof of Theorem 1.7. We know already from Corollary 3.22 that $\mathrm{IHP}(\Delta)$ is a lower bound for $\mathrm{GNP}(\Delta)$. Now we show that it is sharp at points $(\mathbb{x}_k^{\pm},h(\Delta_k^{\pm}))$ for every $k\geq 0$.

If we choose a polynomial $f\in O_{\mathrm{Zar}}$, by Lemmas 4.4 and 4.5, for any finite character χ_0 of conductor $p^{m_{\chi_0}}$, $\mathrm{NP}(f,\chi_0)_C$ passes though $(\mathbb{x}_k^{\pm},h(\Delta_k^{\pm}))$ for every $k\geq 0$. □

Proof of Theorem 1.5. Let χ be a finite character and $f\in O_{\mathrm{Zar}}$ where O_{Zar} is defined as in Lemma 4.4. By Lemmas 4.4, 4.5, and 4.6, $\mathrm{NP}(f,\chi)_C$ passes through $(\mathbb{x}_k^{\pm},h(\Delta_k^{\pm}))$ for every $k\geq 0$. Since these points are vertices of $\mathrm{IHP}(\Delta)$ which is a lower bound of $\mathrm{NP}(f,\chi)_C$, they are also vertices of $\mathrm{NP}(f,\chi)_C$.

Therefore, by (1.2), for any $0\leq i_1\leq n-1$ and $0\leq i_2\leq p^{m_\chi-1}-1$ there are

$$\sum_{t=0}^{i_1}(-1)^i\binom{n}{i}\mathbb{x}^+_{(i_1-t)p^{m_\chi-1}+i_2}(\text{resp. }\sum_{t=0}^{i_1}(-1)^i\binom{n}{i}\mathbb{x}^-_{(i_1-t)p^{m_\chi-1}+i_2})$$

slopes of $\mathrm{NP}(f,\chi)_L$ less than or equal to (resp. strictly less than) $i_1(p-1)p^{m_\chi-1}+(p-1)i_2$. Since the slopes of $\mathrm{NP}(f,\chi)_L$ divided by $(p-1)p^{m_\chi}$ are the $p^{m(f)}$-adic Newton slopes of $L_f^*(\chi,s)$, we finish the proof. □

5. Nonvanishing of leading terms in universal coefficients.

To show (4.4) for $1\leq k\leq n+2$, it is enough to show the coefficient of one special term in $\widetilde{u}_{\mathbb{x}_k^{\pm}}$ is not zero. Computing explicitly the contribution of this permutations of $\Delta_k^{\pm}$ to this special term, which itself was subdivided into simpler cases, allows us to prove (4.4).

5.1. **Proof of Proposition 4.2 with assuming Propositions 5.13 and 5.14.** Recall that in Notation 3.4, we define the function η which maps $Q \in \Delta^-$ to the residue of pQ module Λ_Δ in Δ^-.

Notation 5.1.

(1) For every non-empty subset $S \subseteq \{1, 2, \ldots, n\}$, we put $\mathbf{V}_S = \sum_{i \in S} \mathbf{V}_i$, and denote by

$$\mathrm{Ver}(\Delta) := \{\mathbf{V}_S \mid S \subseteq \{1, 2, \ldots, n\} \text{ and } S \neq \emptyset\}$$

the set consisting of all vertices of Δ except the origin $\mathcal{O}$.

(2) For any $g(\widetilde{a}_P) \in \mathbb{Z}_p[\![\widetilde{a}_P, P \in \Delta^+ \backslash \{\mathcal{O}\}]\!]$, we put

$$g^{\mathrm{res}} = g\Big|_{\substack{\widetilde{a}_P = 0 \text{ if } P \notin \mathrm{Ver}(\Delta) \\ \widetilde{a}_{\mathbf{V}_S} = \widetilde{a}_{\# S}}} \in \mathbb{Z}_p[\![\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n]\!].$$

(3) For any $P_0 \in \Delta^-$, we put

$$\Delta_k^{\pm}(P_0) = \{Q \in \Delta_k^{\pm} \mid Q \equiv P_0 \pmod{\Lambda_\Delta}\}.$$

Lemma 5.2. *For any $1 \leq k \leq n+2$, we have*

$$\det(\widetilde{e}^{\mathrm{res}}_{pQ-Q'})_{Q,Q' \in \Delta_k^{\pm}} \not\equiv 0 \mod (p, T^{h(\Delta_k^{\pm})+1}). \tag{5.1}$$

Lemma 5.3. *Lemma 5.2 $\Rightarrow$ Proposition 4.2.*

Proof. It follows from (5.1) that

$$\det(\widetilde{e}_{pQ'-Q})_{Q,Q' \in \Delta_k^{\pm}} \not\equiv 0 \mod (p, T^{h(\Delta_k^{\pm})+1})$$

for every $1 \leq k \leq n+2$. Therefore, by (4.6), we have

$$\widetilde{u}_{\mathbb{x}_k^{\pm}, h(\Delta_k^{\pm})} \not\equiv 0 \pmod{p}$$

for every $1 \leq k \leq n+2$. $\square$

Lemma 5.4. *For any $\tau \in \mathrm{Iso}(\Delta_k^{\pm})$, if there exist $P_0 \in \Delta^-$ and $Q_0 \in \Delta_k^{\pm}(P_0)$ such that $\tau(Q_0) \notin \Delta_k^{\pm}(\eta(P_0))$, then*

$$\prod_{Q \in \Delta_k^{\pm}(P_0)} \widetilde{e}^{\mathrm{res}}_{pQ-\tau(Q)} = 0. \tag{5.2}$$

Proof. Recall that for any $Q \in \mathbb{M}(\Delta)$, we have

$$\widetilde{e}_Q(T) = \sum_{\left\{\vec{j} \in \mathbb{Z}_{\geq 0}^{\Delta^+ \backslash \{\mathcal{O}\}} \,\middle|\, \sum_{P \in \Delta^+ \backslash \{\mathcal{O}\}} j_P P = Q\right\}} \Big(\prod_{P \in \Delta^+ \backslash \{\mathcal{O}\}} c_{j_P} (\widetilde{a}_P \pi)^{j_P} \Big). \tag{5.3}$$

If $\tau(Q_0) \notin \Delta_k^{\pm}(\eta(P_0))$, then $pQ_0 - \tau(Q_0) \not\equiv \mathcal{O} \mod \Lambda_\Delta$. Therefore, any linear decomposition

$$pQ_0 - \tau(Q_0) = \sum_{P \in \Delta^+ \backslash \{\mathcal{O}\}} j_P P$$

contains $P \notin \mathrm{Ver}(\Delta)$ such that $j_P \neq 0$, which implies

$$\widetilde{e}^{\mathrm{res}}_{pQ_0 - \tau(Q_0)} = 0$$

and consequently to compute $\det(\widetilde{e}^{\mathrm{res}}_{pQ-Q'})_{Q,Q' \in \Delta_k^{\pm}}$, it suffices to take sum over these permutations τ such that $\tau(Q) \equiv pQ \pmod{\Lambda_\Delta}$ for all $Q \in \Delta_k^{\pm}$. $\square$

Notation 5.5. For a point $Q = \sum_{i=1}^{n} z_i \mathbf{V}_i$, we set

$$I(Q) := \{i \mid z_i = 0\}.$$

For any $I \subseteq \{1, 2, \ldots, n\}$, generalizing Notation 3.17 and Notation 5.1 (3), we put

- $\Delta_k^{\pm}(I) := \left\{ Q = \sum_{i=1}^{n} z_i \mathbf{V}_i \in \Delta_k^{\pm} \;\middle|\; \begin{array}{ll} z_i = 0 & \text{if } i \in I \\ z_i > 0 & \text{otherwise} \end{array} \right\}$, and
- $\Delta_k^{+}(I, P_0) := \Delta_k^{+}(I) \cap \Delta_k^{+}(P_0)$ and $\Delta_k^{-}(I, P_0) := \Delta_k^{-}(I) \cap \Delta_k^{-}(P_0)$.

Lemma 5.6.

(1) For each $P_0 \in \Delta^-$, if $I \not\subseteq I(P_0)$, then $\Delta_k^{\pm}(I, P_0) = \emptyset$.
(2) The set $\Delta_k^{\pm}$ is a disjoint union of $\Delta_k^{\pm}(P_0, I)$ for all $P_0 \in \Delta^-$ and $I \subseteq I(P_0)$.

Proof. Both (1) and (2) are straightforward. □

Lemma 5.7. *For any $\tau \in \mathrm{Iso}(\Delta_k^{\pm})$, if there exists a subset $I \subseteq \{1, 2, \ldots, n\}$ such that $\tau(\Delta_k^{\pm}(I)) \neq \Delta_k^{\pm}(I)$, then*

$$\prod_{Q \in \Delta_k^{\pm}(P_0)} \widetilde{e}_{pQ - \tau(Q)} = 0. \tag{5.4}$$

Proof. We put $I_0 \subseteq \{1, 2, \ldots, n\}$ to be one of the smallest subset such that

$$\tau(\Delta_k^{\pm}(I_0)) \neq \Delta_k^{\pm}(I_0).$$

Namely, every set I with fewer elements than I_0 satisfies

$$\tau(\Delta_k^{\pm}(I)) = \Delta_k^{\pm}(I).$$

It is easy to see that there exists $Q_0 \in \Delta_k^{\pm}(I_0)$ such that $pQ_0 - \tau(Q_0) \notin \mathrm{Cone}(\Delta)$, which implies $\widetilde{e}_{pQ_0 - \tau(Q_0)} = 0$, and consequently (5.4). □

Proposition 5.8. *The determinant*

$$\det(\widetilde{e}^{\mathrm{res}}_{pQ - Q'})_{Q, Q' \in \Delta_k^{\pm}} = \prod_{P_0 \in \Delta^-} \prod_{I \subseteq I(P_0)} \det(\widetilde{e}^{\mathrm{res}}_{P_0 - \eta(P_0) + pQ - Q'})_{Q, Q' \in \Delta_k^{\pm}(I, P_0)}. \tag{5.5}$$

Proof. By Lemma 5.4 and Lemma 5.7, for every $\tau \in \mathrm{Iso}(\Delta_k^{\pm})$ if

$$\prod_{Q \in \Delta^{\pm}(P_0)} \widetilde{e}^{\mathrm{res}}_{pQ - \tau(Q)} \neq 0,$$

then τ must map $\Delta_k^{\pm}(I, P_0)$ onto $\Delta_k^{\pm}(I, \eta(P_0))$ for every $P_0 \in \Delta^-$ and $I \in I(P_0)$. On the other hand, we know easily that

$$\Delta_k^{\pm}(I, \eta(P_0)) = \Delta_k^{\pm}(I, P_0) - P_0 + \eta(P_0). \tag{5.6}$$

Therefore, we have

$$\begin{aligned} \det(\widetilde{e}^{\mathrm{res}}_{pQ - Q'})_{Q, Q' \in \Delta_k^{\pm}} &= \prod_{P_0 \in \Delta^-} \prod_{I \subseteq I(P_0)} \det(\widetilde{e}^{\mathrm{res}}_{pQ - Q'})_{Q' \in \Delta_k^{\pm}(I, P_0), Q' \in \Delta_k^{\pm}(I, \eta(P_0))} \\ &= \prod_{P_0 \in \Delta^-} \prod_{I \subseteq I(P_0)} \det(\widetilde{e}^{\mathrm{res}}_{P_0 - \eta(P_0) + pQ - Q'})_{Q, Q' \in \Delta_k^{\pm}(I, P_0)}. \end{aligned}$$

□

Definition 5.9.

(1) The *partial degree* of a monomial $\prod_{i=1}^{n}\widetilde{a}_i^{t_i}$, denoted by $\mathrm{Deg}(\prod_{i=1}^{n}\widetilde{a}_i^{t_i})$, is defined to be the vector $(t_n, t_{n-1}, \ldots, t_1) \in \mathbb{Z}_{\geq 0}^n$, where $\mathbb{Z}_{\geq 0}^n$ equips with a reverse lexicographic order. Namely, for two vectors $\vec{v} = (v_1, v_2, \ldots, v_n)$ and $\vec{u} = (u_1, u_2, \ldots, u_n)$ in $\mathbb{R}^n$, we call $\vec{v}$ *strictly bigger than* $\vec{u}$, denoted by $\vec{v} \succ \vec{u}$, if there is $1 \leq k \leq n$ such that
   - $v_i = u_i$ for every $1 \leq i \leq k-1$, and
   - $v_k > u_k$.

   In particular, the partial degree of every nonzero constant is just $(0, 0, \ldots, 0)$.

(2) The *partial degree* of a polynomial $g \in \mathbb{Z}_p[\![T]\!][\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n]$, denoted by $\mathrm{Deg}(g)$, is the maximal partial degree of nonzero monomials in g.

(3) For any point $Q = \sum_{j=1}^{\ell} z_j \mathbf{V}_{S_j}$ such that $0 = z_0 < z_1 < \cdots < z_\ell$, the *degree* of Q, denoted by $\mathrm{Deg}(Q)$, is the n-dimensional vector $(v_1, v_2, \ldots, v_n)$ such that
$$v_{n+1-\sum_{j=i}^{\ell}\#S_j} = z_i - z_{i-1} \quad \text{for any} \quad 1 \leq i \leq \ell,$$
and all other components are zeros.

(4) The leading term of g, denoted by $\mathrm{LD}(g)$, is the sum of monomials in g of the maximal partial degree.

Property 5.10.

(1) Every nonzero $g \in \mathbb{Z}_p[\![T]\!][\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n]$ satisfies
$$\mathrm{Deg}(g - \mathrm{LD}(g)) \prec \mathrm{Deg}(g).$$

(2) For any $g_1, g_2 \in \mathbb{Z}_p[\![T]\!][\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n]$, we have
   (I) $\mathrm{Deg}(g_1 - g_2) \preceq \max(\mathrm{Deg}(g_1), \mathrm{Deg}(g_2))$.
   (II) $\mathrm{Deg}(g_1 g_2) = \mathrm{Deg}(g_1) + \mathrm{Deg}(g_2)$.
   (III) $\mathrm{LD}(g_1 g_2) = \mathrm{LD}(g_1)\mathrm{LD}(g_2)$.
   (IV) $\mathrm{LD}(g_1) = \mathrm{LD}(g_2)$ if and only if $\mathrm{Deg}(g_1) \succ \mathrm{Deg}(g_1 - g_2)$.

Lemma 5.11. *Let* $\coprod_{i=1}^{\ell} S_i \subseteq \{1, 2, \ldots, n'\}$ *and* $Q_1 = \sum_{i=1}^{\ell} m_i \mathbf{V}_{S_i} \in \Lambda_\Delta$ *such that* $0 = m_0 < m_1 < \cdots < m_\ell$. *Then*

(1) the leading term
$$\mathrm{LD}(\widetilde{e}_{Q_1}^{\mathrm{res}}) = \prod_{i=1}^{\ell} c_{m_i - m_{i-1}} (\widetilde{a}_{\sum_{j=i}^{n}\#S_j}\pi)^{m_i - m_{i-1}}. \tag{5.7}$$

(2) $\mathrm{Deg}(\widetilde{e}_{Q_1}^{\mathrm{res}}) = \mathrm{Deg}(Q_1)$.

Proof. (1) By (5.3) and Notation 5.1 (3), the polynomial $\widetilde{e}_{Q_1}$ can be written explicitly as
$$\widetilde{e}_{Q_1}^{\mathrm{res}} = \sum_{\left\{\vec{j} \,\middle|\, \sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ S \neq \emptyset}} j_S \mathbf{V}_S = Q_1\right\}} \Big(\prod_{\substack{S \subseteq \{1,2,\ldots,n\} \\ S \neq \emptyset}} c_{j_S} (\widetilde{a}_{\#S}\pi)^{j_S} \Big).$$

The monomial
$$\prod_{i=1}^{n} c_{m_i - m_{i-1}} (\widetilde{a}_{\sum_{j=i}^{n}\#S_j}\pi)^{m_i - m_{i-1}}$$
is the unique term in $\widetilde{e}_{Q_1}^{\mathrm{res}}$ of the maximal degree, which completes the proof.

(2) It follows directly from (1). $\square$

Notation 5.12.

(1) Let $P_0 \in \Delta^-$ and $I \subseteq I(P_0)$. By relabeling indices, we may assume that $I = \{n'+1, n'+2, \dots, n\}$. Let $S_1, S_2, \dots, S_\ell$ be an ordered disjoint subsets of $\{1, 2, \dots, n'\}$ such that $\coprod_{i=1}^{\ell} S_i \subseteq \{1, 2, \dots, n'\}$. We put

$$\Delta_k^{\pm}(I, P_0; S_1, \dots, S_\ell) := \Big\{ \sum_{j=1}^{\ell} z_j \mathbf{V}_{S_j} \in \Delta_k^{\pm}(I, P_0) \,\Big|\, 0 < z_1 < \cdots < z_\ell \Big\}.$$

(2) For any $1 \le \ell \le n'$, we put

$$\mathscr{S}_\ell(\Delta_k^{\pm}(I, P_0)) := \{\Delta_k^{\pm}(I, P_0; S_1, \dots, S_\ell)\},$$

where $S_1, \dots, S_\ell$ runs over all ordered disjoint subsets of $\{1, 2, \dots, n'\}$.

(3) Let

$$\mathscr{S}(\Delta_k^{\pm}(I, P_0)) = \coprod_{\ell=0}^{n'} \mathscr{S}_\ell(\Delta_k^{\pm}(I, P_0)).$$

Apparently, we have

$$\Delta_k^{\pm}(I, P_0) = \coprod_{\mathbb{S} \in \mathscr{S}(\Delta_k^{\pm}(I, P_0))} \mathbb{S}.$$

We next show that Lemma 5.2 follows from Proposition 5.13 and Proposition 5.14 whose proofs are given later in §5.2 and §5.3 respectively.

Proposition 5.13. *For every subset $I \subseteq I(P_0)$, we have*

$$\mathrm{LD}\Big(\det(\widetilde{e}^{\mathrm{res}}_{P_0-\eta(P_0)+pQ-Q'})_{Q,Q' \in \Delta_k^{\pm}(I,P_0)}\Big) = \prod_{\mathbb{S} \in \mathscr{S}(\Delta_k^{\pm}(I,P_0))} \det\Big(\mathrm{LD}(\widetilde{e}^{\mathrm{res}}_{P_0-\eta(P_0)+pQ-Q'})\Big)_{Q,Q' \in \mathbb{S}}. \tag{5.8}$$

Proposition 5.14. *For every $P_0 \in \Delta^-$ and $\mathbb{S} \in \mathscr{S}(\Delta_k^{\pm}(I, P_0))$ we have*

$$\det\Big(\mathrm{LD}(\widetilde{e}^{\mathrm{res}}_{P_0-\eta(P_0)+pQ-Q'})\Big)_{Q,Q' \in \mathbb{S}} = b_{P_0,\mathbb{S}} \times g_{P_0,\mathbb{S}}(\underline{\widetilde{a}})\pi^{\sum_{Q\in\mathbb{S}}(\lfloor pw(Q) \rfloor - \lfloor w(Q-P_0+\eta(P_0)) \rfloor)}, \tag{5.9}$$

where $b_{P_0,\mathbb{S}}$ is a p-adic unit in $\mathbb{Z}_p$ and $g_{P_0,\mathbb{S}}(\underline{\widetilde{a}})$ is a monomial in $\mathbb{Z}_p[\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n]$.

Proof of Lemma 5.2(Assuming Proposition 5.13 and Proposition 5.14). By Property 5.10, Proposition 5.8, Proposition 5.13 and Proposition 5.14, we obtain

$$\begin{aligned}
&\mathrm{LD}\Big(\det(\widetilde{e}^{\mathrm{res}}_{pQ-Q'})_{Q,Q' \in \Delta_k^{\pm}}\Big)\\
=&\prod_{P_0 \in \Delta^-} \prod_{I \subseteq I(P_0)} \mathrm{LD}\Big(\det(\widetilde{e}^{\mathrm{res}}_{P_0-\eta(P_0)+pQ-Q'})_{Q,Q' \in \Delta_k^{\pm}(I,P_0)}\Big) && \text{(Prop 5.8)}\\
=&\prod_{P_0 \in \Delta^-} \prod_{I \subseteq I(P_0)} \prod_{\mathbb{S} \in \mathscr{S}(\Delta_k^{\pm}(I,P_0))} \det\Big(\mathrm{LD}(\widetilde{e}^{\mathrm{res}}_{P_0-\eta(P_0)+pQ-Q'})\Big)_{Q,Q' \in \mathbb{S}} && \text{(Prop 5.13)}\\
=&\prod_{P_0 \in \Delta^-} \prod_{I \subseteq I(P_0)} \prod_{\mathbb{S} \in \mathscr{S}(\Delta_k^{\pm}(I,P_0))} b_{P_0,\mathbb{S}} \times g_{P_0,\mathbb{S}}(\underline{\widetilde{a}})\pi^{\sum_{Q\in\mathbb{S}}(\lfloor pw(Q) \rfloor - \lfloor w(Q-P_0+\eta(P_0)) \rfloor)} && \text{(Prop 5.14)}\\
=&\pi^{h(\Delta_k^{\pm})} \prod_{P_0 \in \Delta^-} \prod_{I \subseteq I(P_0)} \prod_{\mathbb{S} \in \mathscr{S}(\Delta_k^{\pm}(I,P_0))} b_{P_0,\mathbb{S}} \times g_{P_0,\mathbb{S}}(\underline{\widetilde{a}}).
\end{aligned}$$

The last equality comes from

$$\begin{aligned} h(\Delta_k^{\pm}) &= \coprod_{P_0\in\Delta^-}\coprod_{I\subseteq I(P_0)}\coprod_{\mathbb{S}\in\mathscr{S}(\Delta_k^{\pm}(I,P_0))}\mathbb{S} \\ &= \coprod_{P_0\in\Delta^-}\coprod_{I\subseteq I(P_0)}\coprod_{\mathbb{S}\in\mathscr{S}(\Delta_k^{\pm}(I,P_0))}\{Q-P_0+\eta(P_0)\mid Q\in\mathbb{S}\}. \end{aligned}$$ □

Now we prove Proposition 5.13 and Proposition 5.14.

5.2. **Proof of Proposition 5.13 assuming Proposition 5.14.** The essence of the proof is the only terms that contribute to the left hand side of (5.8) come from the determinant of the leading terms on its right hand side.

Lemma 5.15. *Let*

$$\coprod_{j=1}^{\ell'} S'_j \subseteq \coprod_{i=1}^{\ell} S_i \subseteq \{1,2,\dots,n'\}.$$

Assume

$$Q=\sum_{i=1}^{\ell} z_i\mathbf{V}_{S_i} \text{ and } Q'=\sum_{j=1}^{\ell'} z'_j\mathbf{V}_{S'_j}$$

are two points in Λ_Δ such that

(1) $0<z_1<z_2<\cdots<z_\ell$ *and* $0<z'_1<z'_2<\cdots<z'_{\ell'}$.

(2) $z_i-z_{i-1}>z'_{\ell'}-z'_1$ *for every* $1\le i\le\ell$.

Then

$$\mathrm{Deg}(Q-Q')\preceq \mathrm{Deg}(Q)-\mathrm{Deg}(Q'), \tag{5.10}$$

where the equality holds if and only if for any $1\le i\le\ell$, there is $1\le j_i\le\ell'$ such that

(I) $S_i\subseteq S'_{j_i}$, *and*

(II) $j_{i_1}\le j_{i_2}$ *for any* $1\le i_1<i_2\le\ell$.

Proof. When $\ell=1$, we assume that $\ell'>1$, for otherwise it is easy to show that

$$\mathrm{Deg}(Q-Q')=\mathrm{Deg}(Q)-\mathrm{Deg}(Q').$$

Now by condition (2) and Definition 5.9 (3), we obtain

$$\mathrm{Deg}(Q-Q')=(\overbrace{0,\dots,0}^{n-\sum_{j=1}^{\ell}\#S_j},(z_1-z'_{\ell'}),\overbrace{\cdots}^{\sum_{j=1}^{\ell}\#S_j-1}) \tag{5.11}$$

and

$$\mathrm{Deg}(Q)-\mathrm{Deg}(Q')=(\overbrace{0,\dots,0}^{n-\sum_{j=1}^{\ell}\#S_j},(z_1-z'_1),\overbrace{\cdots}^{\sum_{j=1}^{\ell}\#S_j-1}), \tag{5.12}$$

which directly imply $\mathrm{Deg}(Q-Q')\prec\mathrm{Deg}(Q)-\mathrm{Deg}(Q')$.

Assume (5.10) and its equality condition hold for $\ell>1$; we will prove them for $\ell+1$.

We set $j'=\max\{j\mid S_1\cap S'_j\neq\emptyset\}$. By condition (2) and Definition 5.9 (3) again, we have

$$\mathrm{Deg}(Q-Q')=(\overbrace{0,\dots,0}^{n-\ell-1},(z_1-z'_{j'}),\dots) \tag{5.13}$$

and

$$\mathrm{Deg}(Q)-\mathrm{Deg}(Q')=(\overbrace{0,\dots,0}^{n-\ell-1},(z_1-z'_1),\dots). \tag{5.14}$$

If $j' > 1$, then

$$\mathrm{Deg}(Q - Q') \prec \mathrm{Deg}(Q) - \mathrm{Deg}(Q').$$

Then we show that in this case S'_j's can not satisfy both (I) and (II). Otherwise, from (I), we have $S_1 \subseteq S_{j'}$. Since $j' > 1$, there exists $1 < i \leq \ell$ such that $S_i \subseteq S'_1$. Therefore, we obtain $j_1 = j' > 1 = j_i$, which contradicts to (II).

If $j' = 1$, we have $S_1 \subseteq S'_1$, hence

$$\coprod_{j=2}^{\ell'} S'_j \subseteq \coprod_{i=2}^{\ell} S_i \subseteq \{1, 2, \dots, n'\}.$$

We put

$$Q_0 = \sum_{i=2}^{\ell+1}(z_i - z_1)\mathbf{V}_{S_i} \text{ and } Q'_0 = \sum_{j=2}^{\ell'}(z'_j - z'_1)\mathbf{V}_{S'_j}.$$

It is easy to check that Q_0 and Q'_0 also satisfy the conditions (1) and (2). By condition (2) and Lemma 5.11, we get

$$\mathrm{Deg}(Q) - \mathrm{Deg}(Q') - \mathrm{Deg}(Q - Q') = \mathrm{Deg}(Q_0) - \mathrm{Deg}(Q'_0) - \mathrm{Deg}(Q_0 - Q'_0).$$

Therefore, we are left to show that this result holds for Q_0 and Q'_0, which follows directly from the induction. □

Lemma 5.16. *Let $\coprod_{i=1}^{\ell} S_i \subseteq \{1, 2, \dots, n'\}$ and $Q = \sum_{i=1}^{\ell} z_i \mathbf{V}_{S_i} \in \mathrm{Cone}(\Delta)$ such that $0 < z_1 < \cdots < z_\ell$. Then*

$$(p-1)Q - \eta(Q\%) = \sum_{i=1}^{\ell} t_i \mathbf{V}_{S_i},$$

and $t_i - t_{i-1} > n + 2$ for every $1 \leq i \leq \ell$, where $t_0 = 0$.

Proof. It follows from fact that

$$(p-1)Q - \eta(Q\%) = \sum_{i=1}^{\ell} t_i \mathbf{V}_{S_i},$$

where $t_i = \lfloor p z_i \rfloor - z_i$. For any $1 \leq i \leq \ell$,

$$t_{i+1} - t_i \geq (p-1)(z_{i+1} - z_i) - 1 \geq \frac{p-1}{D} - 1 > n + 2 \geq 0, \tag{5.15}$$

where the last but one inequality follows from the assumption $p > D(n+4)$. □

Lemma 5.17. *Let $1 \leq k \leq n+2$, $\coprod_{i=1}^{\ell} S_i \subseteq \{1, 2, \dots, n'\}$, and $\coprod_{j=1}^{\ell'} S'_j \subseteq \{1, 2, \dots, n'\}$. For $Q \in \Delta_k^{\pm}(I, P_0; S_1, \dots, S_\ell)$ and $Q' \in \Delta_k^{\pm}(I, P_0; S'_1, S'_2, \dots, S'_{\ell'})$, we have*

$$\mathrm{Deg}(pQ - \eta(P_0) + P_0 - Q') \preceq \mathrm{Deg}(pQ - \eta(P_0) + P_0) - \mathrm{Deg}(Q'). \tag{5.16}$$

Moreover, the equality holds if and only if S_i's and S'_j's satisfies that

- *for any $1 \leq i \leq \ell'$ there exists $1 \leq j_i \leq \ell$ such that $S_i \subseteq S'_{j_i}$, and*
- *for any $1 \leq i_1 \leq \ell'$ and $1 \leq i_2 \leq \ell'$ if $i_1 < i_2$, then $j_{i_1} \leq j_{i_2}$.*

Proof. Let

$$Q = \sum_{i=1}^{\ell} z_i \mathbf{V}_{S_i}, \quad pQ - \eta(P_0) + P_0 = \sum_{i=1}^{\ell} t_i \mathbf{V}_{S_i} \quad \text{and} \quad Q' = \sum_{j=1}^{\ell'} z'_j \mathbf{V}_{S'_j}$$

such that $0 < z_1 < \cdots < z_\ell$ and $0 < z'_1 < \cdots < z'_{\ell'}$.

By Lemma 5.16, we have

$$t_i - t_{i-1} > n + 2 \geq z'_{\ell'} - z'_1, \tag{5.17}$$

for any $1 \leq i \leq \ell$. Then it follows directly from Lemma 5.15. $\square$

Notation 5.18. We put

$$K = \min\{i \mid \mathscr{S}_i(\Delta_k^{\pm}(I, P_0)) \neq \emptyset\}.$$

Corollary 5.19. *Let $\mathbb{S} \in \mathscr{S}_\ell(\Delta_k^{\pm}(I, P_0))$. If $Q \in \mathbb{S}$ and $Q' \in \Delta_k(I, P_0)$ satisfying*

$$\mathrm{Deg}(pQ - \eta(P_0) + P_0 - Q') = \mathrm{Deg}(pQ - \eta(P_0) + P_0) - \mathrm{Deg}(Q'), \tag{5.18}$$

then

$$Q' \in \coprod_{j=K}^{\ell} \coprod_{\mathbb{S}' \in \mathscr{S}_j(\Delta_k^{\pm}(I, P_0))} \mathbb{S}'. \tag{5.19}$$

Moreover, if

$$Q' \in \coprod_{\mathbb{S}' \in \mathscr{S}_\ell(\Delta_k^{\pm}(I, P_0))} \mathbb{S}',$$

then $Q' \in \mathbb{S}$.

Proof. It follows directly from Lemma 5.17. $\square$

Proposition 5.20. *For a permutation $\tau \in \mathrm{Iso}(\Delta_k^{\pm}(I, P_0))$, the equality*

$$\mathrm{Deg}(pQ - \eta(P_0) + P_0 - \tau(Q)) = \mathrm{Deg}(pQ - \eta(P_0) + P_0) - \mathrm{Deg}(\tau(Q)) \tag{5.20}$$

holds for every $Q \in \Delta_k^{\pm}(I, P_0)$ if and only if

$$\tau(\mathbb{S}) = \mathbb{S}$$

for every $\mathbb{S} \in \mathscr{S}(\Delta_k^{\pm}(I, P_0))$.

Proof. "$\Longleftarrow$"

It follows from Lemma 5.17.

"$\Longrightarrow$"

Assume τ is a permutation of $\mathscr{S}(\Delta_k^{\pm}(I, P_0))$ such that every $Q \in \Delta_k^{\pm}(I, P_0)$ satisfies (5.20). Combining two statements in Corollary 5.19 gives that $\tau(\mathbb{S}_K) = \mathbb{S}_K$.

Assume that any $K \leq \ell' < \ell$ and any $\mathbb{S}_{\ell'} \in \mathscr{S}_{\ell'}(\Delta_k^{\pm}(I, P_0))$ satisfies $\tau(\mathbb{S}_{\ell'}) = \mathbb{S}_{\ell'}$. Let $\mathbb{S}_\ell$ be any subset of $\Delta_k^{\pm}(I, P_0)$ in $\mathscr{S}_\ell(\Delta_k^{\pm}(I, P_0))$. Combining the induction assumption with Corollary 5.19, we know

$$\tau(\mathbb{S}_\ell) \subseteq \coprod_{\mathbb{S}' \in \mathscr{S}_\ell(\Delta_k^{\pm}(I, P_0))} \mathbb{S}',$$

and hence $\tau(\mathbb{S}_\ell) = \mathbb{S}_\ell$. $\square$

Notation 5.21. Let

$$\mathrm{Iso}^{\mathrm{sp}}(\Delta_k^{\pm}(I, P_0)) := \{\tau \in \mathrm{Iso}(\Delta_k^{\pm}(I, P_0)) \mid \tau(\mathbb{S}) = \mathbb{S} \text{ for any } \mathbb{S} \in \mathscr{S}(\Delta_k^{\pm}(I, P_0))\}.$$

Lemma 5.22. *Let τ be a permutation of $\Delta_k^{\pm}(I, P_0))$. We have*

$$\mathrm{Deg}(\prod_{Q \in \Delta_k^{\pm}(I, P_0)} \widetilde{e}^{\mathrm{res}}_{pQ - \eta(P_0) + P_0 - \tau(Q)}) = \max_{\tau' \in \mathrm{Iso}(\Delta_k^{\pm}(I, P_0))} \Big(\mathrm{Deg}(\prod_{Q \in \Delta_k^{\pm}(I, P_0)} \widetilde{e}^{\mathrm{res}}_{pQ - \eta(P_0) + P_0 - \tau'(Q)})\Big)$$

if and only if

$$\tau \in \mathrm{Iso}^{\mathrm{sp}}(\Delta_k^{\pm}(I, P_0)).$$

Proof. By Property 5.10 (2) II, Lemma 5.11 (2), and Lemma 5.17, we know that

$$\begin{aligned}
&\mathrm{Deg}\Big(\prod_{Q\in\Delta_k^{\pm}(I,P_0)} \widetilde{e}^{\mathrm{res}}_{pQ-\eta(P_0)+P_0-\tau(Q)}\Big)\\
=&\sum_{Q\in\Delta_k^{\pm}(I,P_0)} \mathrm{Deg}(\widetilde{e}^{\mathrm{res}}_{pQ-\eta(P_0)+P_0-\tau(Q)}) && \text{(Property 5.10 (2) II)}\\
=&\sum_{Q\in\Delta_k^{\pm}(I,P_0)} \mathrm{Deg}\Big(pQ-\eta(P_0)+P_0-\tau(Q)\Big) && \text{(Lemma 5.11 (2))}\\
\preceq&\sum_{Q\in\Delta_k^{\pm}(I,P_0)} \Big(\mathrm{Deg}\Big(pQ-\eta(P_0)+P_0\Big)-\mathrm{Deg}\Big(\tau(Q)\Big)\Big) && \text{(Lemma 5.17)}\\
=&\sum_{Q\in\Delta_k^{\pm}(I,P_0)} \Big(\mathrm{Deg}\Big(pQ-\eta(P_0)+P_0\Big)-\mathrm{Deg}(Q)\Big).
\end{aligned}$$

By Proposition 5.20, the equality holds in the above inequality if and only if

$$\tau\in\mathrm{Iso}^{\mathrm{sp}}(\Delta_k^{\pm}(I,P_0)).$$ ☐

Lemma 5.23. *Let J be an index set and $\{g_j \mid j\in J\}$ be a set of polynomials in $\mathbb{Z}_p[\![T]\!][\![\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n]\!]$. Put J_0 be the subset of J such that for any $j\in J$, we have*

$$\mathrm{Deg}(g_j)=\max_{i\in J}\Big(\mathrm{Deg}(g_i)\Big)\quad \textit{if and only if}\quad j\in J_0.$$

Then we have either

(1) $\sum_{j\in J_0}\mathrm{LD}(g_j)=\mathrm{LD}(\sum_{j\in J} g_j)$ *or*
(2) $\sum_{j\in J_0}\mathrm{LD}(g_j)=0.$

Proof. It is straightforward, we leave the proof to the readers. ☐

We prove Proposition 5.13 by assuming Proposition 5.14.

Proof of Proposition 5.13. By Property 5.10 (2) III and the definition of $\mathrm{Iso}^{\mathrm{sp}}(\Delta_k^{\pm}(I,P_0))$, we have

$$\begin{aligned}
&\sum_{\tau\in\mathrm{Iso}^{\mathrm{sp}}(\Delta_k^{\pm}(I,P_0))}\mathrm{LD}\Big(\mathrm{sgn}(\tau)\prod_{P\in\Delta_k^{\pm}(I,P_0)}\widetilde{e}^{\mathrm{res}}_{P_0-\eta(P_0)+pQ-\tau(Q)}\Big)\\
=&\prod_{\mathbb{S}\in\mathscr{S}(\Delta_k^{\pm}(I,P_0))}\det\Big(\mathrm{LD}(\widetilde{e}^{\mathrm{res}}_{P_0-\eta(P_0)+pQ-Q'})\Big)_{Q,Q'\in\mathbb{S}}.
\end{aligned}\tag{5.21}$$

Assuming Proposition 5.14, we know that

$$\prod_{\mathbb{S}\in\mathscr{S}(\Delta_k^{\pm}(I,P_0))}\det\Big(\mathrm{LD}(\widetilde{e}^{\mathrm{res}}_{P_0-\eta(P_0)+pQ-Q'})\Big)_{Q,Q'\in\mathbb{S}}\neq 0.\tag{5.22}$$

Combining (5.22) with Lemma 5.23 and Lemma 5.22, we obtain (5.8). ☐

5.3. **Proof of Proposition 5.14.**

Notation 5.24. We put

$$Y_k(S_1,\dots,S_\ell) := \Big\{ \sum_{i=1}^{\ell} m_i \mathbf{V}_{S_i} \,\Big|\, m_i \in \mathbb{Z} \quad \text{and} \quad 0 \le m_1 \le m_2 \le \cdots \le m_\ell \le k \Big\}.$$

As in Notation 5.12, we assume that $I = \{n'+1, n'+2, \dots, n\}$.

Lemma 5.25. *For each nonempty set $\Delta_k^{\pm}(I, P_0; S_1, \dots, S_\ell)$, there exist*

$$Q_{\min} = \sum_{i=1}^{\ell} z_{\min,i} \mathbf{V}_{S_i} \in \Delta_k^{\pm}(I, P_0; S_1, \dots, S_\ell),$$

where $0 < z_{\min,i} - z_{\min,i-1} \le 1$ for all $1 \le i \le \ell$, and integers $K^{\pm}$ such that

$$\Delta_k^{\pm}(I, P_0; S_1, \dots, S_\ell) = Q_{\min} + Y_{K^{\pm}}(S_1, \dots, S_\ell). \tag{5.23}$$

Proof. Let $Q_{\min}$ denote the point in $\Delta_k^{\pm}(I, P_0; S_1, \dots, S_\ell)$ with the minimal degree. Now we show that $0 < z_{\min,i} - z_{\min,i-1} \le 1$ for all $1 \le i \le \ell$. Suppose that it is false, then there exists an integer j such that $z_{\min,j} - z_{\min,j-1} > 1$. It is easy to check that $\sum_{i=1}^{\ell} z_{\min,i} \mathbf{V}_{S_i} - \mathbf{V}_{S_j}$ is a point in $\Delta_k^{\pm}(I, P_0; S_1, \dots, S_\ell)$ with smaller degree than $Q_{\min}$, a contradiction.

The rest part of this lemma is obvious. We show it by an example. □

Example 5.26. When Δ is a cube generated by $(3,0,0), (0,3,0), (0,0,3)$, $p = 29$, and $P_0 = (1,0,0)$.

(1) For $\Delta_3^{\pm}(\{2\}, P_0; \{1\}, \{3\})$, it is easy to show that

- $\Delta_3^{-}(\{2\}, P_0; \{1\}, \{3\}) = \{(1,0,3), (1,0,6), (4,0,6)\}$ and
- $\Delta_3^{+}(\{2\}, P_0; \{1\}, \{3\}) = \{(1,0,3), (1,0,6), (4,0,6), (1,0,9), (4,0,9), (7,0,9)\}$,

hence $Q_{\min} = (1,0,3)$.

It is easy to check that

$$Q_{\min} + Y_1(\{1\}, \{3\}) = \Delta_3^{-}(\{2\}, P_0; \{1\}, \{3\})$$

and

$$Q_{\min} + Y_2(\{1\}, \{3\}) = \Delta_3^{+}(\{2\}, P_0; \{1\}, \{3\}).$$

(2) For $\Delta_3^{\pm}(\{2\}, P_0; \{3\}, \{1\})$, we get

$$\Delta_3^{\pm}(\{2\}, P_0; \{3\}, \{1\}) = \{(4,0,3), (7,0,3), (7,0,6)\},$$

hence $Q_{\min} = (4,0,3)$.

It is easy to check that

$$Q_{\min} + Y_1(\{3\}, \{1\}) = \Delta_3^{\pm}(\{2\}, P_0; \{3\}, \{1\}).$$

Recall that in Notation 2.4, we denote by $c_i \in \mathbb{Z}_p$ the coefficients of π^i in $E(\pi)$.

Definition 5.27. Let $Q_1 = \sum_{i=1}^{\ell} m_i \mathbf{V}_{S_i} \in \Lambda_\Delta$ such that $0 \le m_1 \le \cdots \le m_\ell$. We put

$$\gamma(Q_1) = \pi^{m_\ell} \prod_{i=1}^{\ell} \Big(\widetilde{a}_{\sum_{j=i}^{\ell} \# S_j(Q_1)} \Big)^{m_i - m_{i-1}}. \tag{5.24}$$

Remark 5.28. Although the method of writing Q_1 as a sum like that is not unique, but $\gamma(Q_1)$ is well-defined.

Lemma 5.29. *Let $k \in \mathbb{Z}_{>0}$ and $Q = \sum_{i=1}^{\ell} z_i \mathbf{V}_{S_i}$. Assume*

$$z_i - z_{i-1} > k \quad \text{for every} \quad 1 \le i \le \ell.$$

For any $Q_1 = \sum_{i=1}^{\ell} m_i \mathbf{V}_{S_i} \in Y_k(S_1, \dots, S_\ell)$ and $Q_2 = \sum_{i=1}^{\ell} m_i' \mathbf{V}_{S_i} \in Y_k(S_1, \dots, S_\ell)$, the leading term

$$\mathrm{LD}(\tilde{e}^{\mathrm{res}}_{Q+pQ_1-Q_2}) = \frac{\gamma(Q)\gamma(pQ_1)}{\gamma(Q_2)} \prod_{i=1}^{\ell} c_{z_i+pm_i-m_i'}.$$

Proof. By Lemma 5.11, we have

$$\mathrm{LD}(\tilde{e}^{\mathrm{res}}_{Q+pQ_1-Q_2}) = \gamma(Q+pQ_1-Q_2) \prod_{i=1}^{\ell} c_{z_i+pm_i-m_i'}.$$

By Definition 5.27, we know that

$$\gamma(Q+pQ_1-Q_2) = \frac{\gamma(Q)\gamma(pQ_1)}{\gamma(Q_2)},$$

which completes the proof. □

Notation 5.30.

(1) For any vector $\vec{w} = (w_1, w_2, \dots, w_\ell) \in \mathbb{Z}^\ell$ such that $w_0 < w_1 < \cdots < w_\ell$, we denote

$$\xi(\vec{w}) := \prod_{i=1}^{\ell-1} c_{w_{i+1}-w_i}.$$

(2) Let

$$V_{\ell,k} := \{(z_1, z_2, \dots, z_\ell) \in \mathbb{Z}^\ell \mid 0 \le z_1 \le \cdots \le z_\ell \le k\}.$$

Rearranging the vectors in $V_{\ell,k}$ with increasing partial order as defined in Definition 5.9 (1), we obtain a sequence of vectors as $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{\binom{\ell+k}{k}}$. For any $\vec{w} = (w_1, w_2, \dots, w_\ell) \in \mathbb{Z}^\ell$ such that $k < w_i - w_{i-1}$ for any $1 \le i \le \ell$, we set

$$M(\vec{w}, k) := \Big(\xi(\vec{w} + p\vec{v}_i - \vec{v}_j)\Big)_{1 \le i,j \le \binom{\ell+k}{k}}.$$

Proposition 5.31. *Let $k \in \mathbb{Z}_{>0}$ and $Q = \sum_{i=1}^{\ell} z_i \mathbf{V}_{S_i}$ such that $0 = z_0 < z_1 < \cdots < z_\ell$. If*

$$z_i - z_{i-1} > k \quad \text{for every} \quad 1 \le i \le \ell,$$

then

$$\det\Big(\mathrm{LD}(\tilde{e}^{\mathrm{res}}_{Q+pQ_1-Q_2})\Big)_{Q_1,Q_2 \in Y_k(S_1,\dots,S_\ell)} = \det\Big(M\Big((z_1,\dots,z_\ell),k\Big)\Big) \times \prod_{Q_1 \in Y_k(S_1,\dots,S_\ell)} \gamma(Q+(p-1)Q_1). \tag{5.25}$$

Proof. By Lemma 5.29, we simplify $\det\Big(\mathrm{LD}(\widetilde{e}^{\mathrm{res}}_{Q+pQ_1-Q_2})\Big)_{Q_1,Q_2\in Y_k(S_1,\dots,S_\ell)}$ as follows:

$$\begin{aligned}
&\det\Big(\mathrm{LD}(\widetilde{e}^{\mathrm{res}}_{Q+pQ_1-Q_2})\Big)_{Q_1,Q_2\in Y_k(S_1,\dots,S_\ell)}\\
=&\det\left(\mathrm{diag}\Big(\gamma(Q_1)^{-1}\Big)_{Q_1\in Y_k(S_1,\dots,S_\ell)}\Big(\mathrm{LD}(\widetilde{e}^{\mathrm{res}}_{Q+pQ_1-Q_2})\Big)_{Q_1,Q_2\in Y_k(S_1,\dots,S_\ell)}\mathrm{diag}\Big(\gamma(Q_2)\Big)_{Q_2\in Y_k(S_1,\dots,S_\ell)}\right)\\
=&\det\Big(\mathrm{LD}\Big(\widetilde{e}^{\mathrm{res}}_{Q+pQ_1-Q_2}\Big)\gamma(Q_2)\gamma^{-1}(Q_1)\Big)_{Q_1,Q_2\in Y_k(S_1,\dots,S_\ell)}\\
=&\det\left(\mathrm{diag}\Big(\gamma(Q+pQ_1-Q_1)\Big)_{Q_1\in Y_k(S_1,\dots,S_\ell)}M\Big((z_1,\dots,z_\ell),k\Big)\right) \qquad \text{(Lemma 5.29)}\\
=&\det\Big(M\Big((z_1,\dots,z_\ell),k\Big)\Big)\times\prod_{Q_1\in Y_k(S_1,\dots,S_\ell)}\Big(\gamma(Q+pQ_1-Q_1)\Big),
\end{aligned}$$

where $\mathrm{diag}\Big(\gamma(Q_1)\Big)_{Q_1\in Y_k(S_1,\dots,S_\ell)}$ is a diagonal matrix whose rows and columns are indexed by the points in $Y_k(S_1,\dots,S_\ell)$. ☐

Lemma 5.32. *Let $\vec{w}=(w_1,w_2,\dots,w_\ell)$ be an ℓ-dimensional vector. If*

$$k\le w_i-w_{i-1}\le p-k \tag{5.26}$$

for every $1\le i\le\ell$, then

$$\det\Big(M(\vec{w},k)\Big)\not\equiv 0 \pmod p.$$

Proof. We prove it by induction. When $\ell=1$, we have $\vec{w}=(w_1)$. For any $k\ge1$ we write the determinant explicitly as

$$\det\Big(M(\vec{w},k)\Big)=\det\begin{pmatrix} c_{w_1} & c_{w_1+p} & \cdots & c_{w_1+kp}\\ c_{w_1-1} & c_{w_1+p-1} & \cdots & c_{w_1+kp-1}\\ c_{w_1-2} & c_{w_1+p-2} & \cdots & c_{w_1+kp-2}\\ \vdots & \vdots & \ddots & \vdots\\ c_{w_1-k} & c_{w_1+p-k} & \cdots & c_{w_1+kp-k}\end{pmatrix}. \tag{5.27}$$

By [RWXY][Lemma 5.2] and (5.26), the congruence relation

$$(w_1-j)c_{w_1+ip-j}-c_{w_1+ip-j-1}\equiv c_{w_1+(i-1)p-j} \pmod p \tag{5.28}$$

holds for every $0\le i\le k$ and $0\le j\le k-1$, where $c_s=0$ for all $s<0$.

Therefore, by (5.28), the equality (5.27) can be simplified as

$$
\begin{aligned}
&\det\Big(M(\vec{w},k)\Big)\\
=&\det\begin{pmatrix}
c_{w_1} & c_{w_1+p} & \cdots & c_{w_1+kp}\\
c_{w_1-1}-w_1c_{w_1} & c_{w_1+p-1}-w_1c_{w_1+p} & \cdots & c_{w_1+kp-1}-w_1c_{w_1+kp}\\
c_{w_1-2}-(w_1-1)c_{w_1-1} & c_{w_1+p-2}-(w_1-1)c_{w_1+p-1} & \cdots & c_{w_1+kp-2}-(w_1-1)c_{w_1+kp-1}\\
\vdots & \vdots & \ddots & \vdots\\
c_{w_1-k}-(w_1-k+1)c_{w_1-k+1} & c_{w_1+p-k}-(w_1-k+1)c_{w_1+p-k+1} & \cdots & c_{w_1+kp-k}-(w_1-k+1)c_{w_1+kp-k+1}
\end{pmatrix}\\
\equiv&\det\begin{pmatrix}
c_{w_1} & c_{w_1+p} & \cdots & -c_{w_1+kp}\\
0 & -c_{w_1} & \cdots & -c_{w_1+(k-1)p}\\
0 & -c_{w_1-1} & \cdots & -c_{w_1+(k-1)p-1}\\
\vdots & \vdots & \ddots & \vdots\\
0 & -c_{w_1-k+1} & \cdots & -c_{w_1+(k-1)p-k+1}
\end{pmatrix}\\
\equiv&(-1)^k c_{w_1}\det\Big(M(\vec{w},k-1)\Big) \pmod p.
\end{aligned}
\tag{5.29}
$$

Since

$$k-1\le w_{i+1}-w_i\le p-k+1$$

for every $0\le i\le k-1$, by simply taking induction on k, we know that

$$\det\Big(M(\vec{w},k)\Big)\equiv(-1)^{\frac{k(k+1)}{2}}c_{w_1}^{k-1} \pmod p. \tag{5.30}$$

Condition (5.26) shows

$$c_{w_1}=\frac{1}{w_1!}\not\equiv 0 \pmod p. \tag{5.31}$$

Combining (5.31) and (5.30), we complete the proof of the case when $\ell=1$.

For $\ell=2$, by definition, we have

$$M(\vec{w},k)=\begin{pmatrix}
c_{w_1}M_{00} & c_{w_1+p}M_{01} & \cdots & c_{w_1+kp}M_{0k}\\
c_{w_1-1}M_{10} & c_{w_1-1+p}M_{11} & \cdots & c_{w_1-1+kp}M_{1k}\\
c_{w_1-2}M_{20} & c_{w_1-2+p}M_{21} & \cdots & c_{w_1-2+kp}M_{2k}\\
\vdots & \vdots & \ddots & \vdots\\
c_{w_1-k}M_{k0} & c_{w_1-k+p}M_{k1} & \cdots & c_{w_1-k+kp}M_{kk}
\end{pmatrix},$$

where for any $0\le i\le k$ and $0\le j\le k$,

$$M_{ij}=\begin{pmatrix}
c_{w_2-w_1} & c_{w_2-w_1+p} & \cdots & c_{w_2-w_1+jp}\\
c_{w_2-w_1-1} & c_{w_2-w_1+p-1} & \cdots & c_{w_2-w_1+jp-1}\\
c_{w_2-w_1-2} & c_{w_2-w_1+p-2} & \cdots & c_{w_2-w_1+jp-2}\\
\vdots & \vdots & \ddots & \vdots\\
c_{w_2-w_1-i} & c_{w_2-w_1+p-i} & \cdots & c_{w_2-w_1+jp-i}
\end{pmatrix}.$$

Let $A_i=[I_{(n-i)\times(n-i)},0_{(n-i)\times 1}]$ for $1\le i\le k$. It is easy to see that $M_{i,j}=A_iM_{i-1,j}$ for any $0\le j\le k$ and $1\le i\le k$. Therefore, imitating the row operations used in (5.29), we modify $M(\vec{w},k)$ by "block row operations" as follows:

$$\begin{pmatrix} I_0 & & & \\ -w_1A_1 & I_1 & & \\ & & \ddots & \\ & & & I_k \end{pmatrix}\begin{pmatrix} I_0 & & & & \\ & I_1 & & & \\ & -(w_1-1)A_2 & I_2 & & \\ & & & \ddots & \\ & & & & I_k \end{pmatrix}\begin{pmatrix} I_0 & & & & \\ & I_1 & & & \\ & & \ddots & & \\ & & & I_{k-1} & \\ & & & -(w_1-k+1)A_k & I_k \end{pmatrix} M(\vec{w},k)$$

$$= \begin{pmatrix} c_{w_1}M_{00} & c_{w_1+p}M_{01} & \cdots & c_{w_1+kp}M_{0k} \\ (c_{w_1-1}-w_1c_{w_1})M_{10} & (c_{w_1-1+p}-w_1c_{w_1+p})M_{11}) & \cdots & (c_{w_1-1+kp}-w_1c_{w_1+kp})M_{1k} \\ (c_{w_1-2}-(w_1-1)c_{w_1-1})M_{20} & (c_{w_1+p-2}-(w_1-1)c_{w_1+p-1})M_{21} & \cdots & (c_{w_1+k-2}-(w_1-1)c_{w_1+k-1})M_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ (c_{w_1-k}-(w_1-k+1)c_{w_1-k+1})M_{k0} & (c_{w_1-k+p}-(w_1-k+1)c_{w_1-k+1+p})M_{k1} & \cdots & (c_{w_1-k+pk}-(w_1-k+1)c_{w_1-k+1+kp})M_{kk} \end{pmatrix}$$

$$\equiv \begin{pmatrix} c_{w_1}M_{00} & c_{w_1+p}M_{01} & \cdots & -c_{w_1+kp}M_{0k} \\ 0 & -c_{w_1}M_{11} & \cdots & -c_{w_1+(k-1)p}M_{1k} \\ 0 & -c_{w_1-1}M_{21} & \cdots & -c_{w_1+(k-1)p-1}M_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -c_{w_1-k+1}M_{k1} & \cdots & -c_{w_1+(k-1)p-k+1}M_{kk} \end{pmatrix} \pmod p. \tag{5.32}$$

It is not hard to show that

$$M(\vec{w},k-1) = \begin{pmatrix} c_{w_1}M_{11} & \cdots & c_{w_1+(k-1)p}M_{1k} \\ c_{w_1-1}M_{21} & \cdots & c_{w_1+(k-1)p-1}M_{2k} \\ \vdots & \ddots & \vdots \\ c_{w_1-k+1}M_{k1} & \cdots & c_{w_1+(k-1)(p-1)}M_{kk} \end{pmatrix}. \tag{5.33}$$

Combining it with (5.32) gives

$$\det\Big(M(\vec{w},k)\Big) \equiv \det\begin{pmatrix} c_{w_1}M(\vec{w}',k) & * \\ 0 & -M(\vec{w},k-1) \end{pmatrix} \pmod p,$$

where $\vec{w}'=(w_1)$ and $*$ represents for a $\binom{k+\ell-1}{k}\times\binom{k+\ell-1}{k-1}$ matrix.

Since $\vec{w}'$ is a one-dimensional vector, as the argument above, the determinant

$$\det\Big(M(\vec{w}',k)\Big)\not\equiv 0 \pmod p.$$

Combining it with (5.31) shows that

$$\det\Big(M(\vec{w},k)\Big)\not\equiv 0 \pmod p \quad \text{if and only if} \quad \det\Big(M(\vec{w},k-1)\Big)\not\equiv 0 \pmod p.$$

Since

$$M(\vec{w},0)=c_{w_2-w_1}=\frac{1}{(w_2-w_1)!}\not\equiv 0 \pmod p,$$

we show that

$$\det\Big(M(\vec{w},k)\Big)\not\equiv 0 \pmod p.$$

Assume this statement holds for all $t\le \ell-1$; we will prove it for ℓ.

We first put

$$M(\vec{w},k)=\begin{pmatrix} c_{w_1}M_{00} & c_{w_1+p}M_{01} & \cdots & c_{w_1+kp}M_{0k}\\ c_{w_1-1}M_{10} & c_{w_1-1+p}M_{11} & \cdots & c_{w_1-1+kp}M_{1k}\\ c_{w_1-2}M_{20} & c_{w_1-2+p}M_{21} & \cdots & c_{w_1-2+kp}M_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ c_{w_1-k}M_{k0} & c_{w_1-k+p}M_{k1} & \cdots & c_{w_1-k+kp}M_{kk}\end{pmatrix},$$

where M_{ij} is a $\binom{k-1+\ell-i}{k-1}\times\binom{k-1+\ell-j}{k-1}$ matrix; and

$$M_{00}=M(\vec{w}',k) \tag{5.34}$$

for $\vec{w}'=(w_2-w_1,w_3-w_1,\ldots,w_\ell-w_1)$.

Similar to the case $\ell=2$ there exists a set of matrices $\{A_i\}_{1\le i\le k}$, such that for every $1\le i\le k$ and $0\le j\le k$, A_i is a $\binom{k-1+\ell-i}{k-1}\times\binom{k-1+\ell-i+1}{k-1}$ reduced echelon matrix and

$$M_{i,j}=A_iM_{i-1,j}.$$

Similar to (5.32) and (5.33), we have

$$\begin{aligned}&\begin{pmatrix} I_0 &&&\\ -w_1A_1 & I_1 &&\\ && \ddots &\\ &&& I_k\end{pmatrix}\begin{pmatrix} I_0 &&&&\\ & I_1 &&&\\ & -(w_1-1)A_2 & I_2 &&\\ &&& \ddots &\\ &&&& I_k\end{pmatrix}\begin{pmatrix} I_0 &&&&\\ & I_1 &&&\\ && \ddots &&\\ &&& I_{k-1} &\\ &&& -(w_1-k+1)A_k & I_k\end{pmatrix}M(\vec{w}',k)\\ &\equiv\begin{pmatrix} c_{w_1}M_{00} & c_{w_1+p}M_{01} & \cdots & c_{w_1+kp}M_{0k}\\ 0 & -c_{w_1}M_{11} & \cdots & -c_{w_1+(k-1)p}M_{1k}\\ 0 & -c_{w_1-1}M_{21} & \cdots & -c_{w_1+(k-1)p-1}M_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & -c_{w_1-k+1}M_{k1} & \cdots & -c_{w_1+(k-1)(p-1)}M_{kk}\end{pmatrix}\\ &=\begin{pmatrix} c_{w_1}M(\vec{w}',k) & *\\ 0 & -M(\vec{w},k-1)\end{pmatrix}\pmod p,\end{aligned} \tag{5.35}$$

where $*$ represents for a $\binom{k+\ell-1}{k}\times\binom{k+\ell-1}{k-1}$ matrix, and

$$M(\vec{w},k-1)=\begin{pmatrix} c_{w_1}M_{11} & \cdots & c_{w_1+(k-1)p}M_{1k}\\ c_{w_1-1}M_{21} & \cdots & c_{w_1+(k-1)p-1}M_{2k}\\ \vdots & \ddots & \vdots\\ c_{w_1-k+1}M_{k1} & \cdots & c_{w_1+(k-1)(p-1)}M_{kk}\end{pmatrix}.$$

By induction and (5.31), the determinant

$$\det\Big(c_{w_1}M(\vec{w}',k)\Big)\not\equiv 0 \pmod p,$$

which implies

$$\det\Big(M(\vec{w},k)\Big)\not\equiv 0 \pmod p \quad\text{if and only if}\quad \det\Big(M(\vec{w},k-1)\Big)\not\equiv 0 \pmod p.$$

Since

$$M(\vec{w},0)=\prod_{i=0}^{\ell}c_{w_{i+1}-w_i}=\prod_{i=0}^{\ell}\frac{1}{(w_{i+1}-w_i)!}\not\equiv 0 \pmod p,$$

we show that

$$\det\Big(M(\vec{w},k)\Big) \not\equiv 0 \pmod p. \qquad \square$$

Proof of Proposition 5.14. Let $\mathbb{S} = \Delta_k^{\pm}(P_0, I; S_1, S_2, \dots, S_\ell)$. By Lemma 5.25, there exists

$$Q_{\min} = \sum_{i=1}^{\ell} z_{\min,i}\mathbf{V}_{S_i} \in \Delta_k^{\pm}(I, P_0; S_1, \dots, S_\ell),$$

where $0 < z_{\min,i} - z_{\min,i-1} \le 1$ for all $1 \le i \le \ell$, and integers $K^{\pm}$ such that

$$\Delta_k^{\pm}(I, P_0; S_1, \dots, S_\ell) = Q_{\min} + Y_{K^{\pm}}(S_1, \dots, S_\ell). \tag{5.36}$$

It is easy to see that we can put

$$Q := (p-1)Q_{\min} - \eta(P_0) + P_0 = \sum_{i=1}^{\ell} z_i \mathbf{V}_{S_i} \in \Lambda_\Delta.$$

Since P_0 and $\eta(P_0)$ are both in Δ^- and $z_i \in \mathbb{Z}$ for every $1 \le i \le \ell$, we know that

$$\Big| z_i - z_{i-1} - (p-1)(z_{\min,i} - z_{\min,i-1}) \Big| \le 1.$$

Therefore, by Hypothesis 1.1, we get

$$n+2 < z_i - z_{i-1} < p - (n+2) \tag{5.37}$$

for every $1 \le i \le \ell$.

Since $K^{\pm} < n+2$, we know that Q and $Y_{K^{\pm}}(S_1, S_2, \dots, S_\ell)$ satisfy the condition in Proposition 5.31, hence

$$\begin{aligned}
&\det\Big(\mathrm{LD}(\widetilde{e}^{\mathrm{res}}_{P_0-\eta(P_0)+pP-Q})\Big)_{P,Q\in\mathbb{S}} \\
&= \det\Big(\mathrm{LD}(\widetilde{e}^{\mathrm{res}}_{Q+pQ_1-Q_2})\Big)_{Q_1,Q_2\in Y_{K^{\pm}}(S_1,\dots,S_\ell)} \\
&= \det\Big(M\Big((z_1,\dots,z_\ell),K^{\pm}\Big)\Big) \times \prod_{Q_1\in Y_{K^{\pm}}(S_1,\dots,S_\ell)} (\gamma(Q+pQ_1-Q_1)) \\
&= \det\Big(M\Big((z_1,\dots,z_\ell),K^{\pm}\Big)\Big) \times g_{P_0,\mathbb{S}}(\underline{\widetilde{a}})\pi^{\sum\limits_{Q_1\in Y_{K^{\pm}}(S_1,\dots,S_\ell)} w(Q+pQ_1-Q_1)} \\
&= \det\Big(M\Big((z_1,\dots,z_\ell),K^{\pm}\Big)\Big) \times g_{P_0,\mathbb{S}}(\underline{\widetilde{a}})\pi^{\sum\limits_{Q'\in\mathbb{S}} w(P_0-\eta(P_0)+pQ'-Q')} \\
&= \det\Big(M\Big((z_1,\dots,z_\ell),K^{\pm}\Big)\Big) \times g_{P_0,\mathbb{S}}(\underline{\widetilde{a}})\pi^{\sum\limits_{Q'\in\mathbb{S}}\Big(\lfloor pw(Q')\rfloor - \lfloor w(\eta(P_0)-P_0+Q')\rfloor\Big)}.
\end{aligned} \tag{5.38}$$

By (5.37), the vector $(z_1, z_2, \dots, z_\ell)$ satisfies the conditions in Lemma 5.32. Therefore, we obtain

$$\det\Big(M\Big((z_1,\dots,z_\ell),K^{\pm}\Big)\Big) \not\equiv 0 \pmod p,$$

which completes the proof. $\square$

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