A UNIVERSAL HOMOGENEOUS SIMPLE RANK 3 MATROID

GIANLUCA PAOLINI

Abstract. We construct a countably infinite simple rank 3 matroid M_* which $\wedge$-embeds every finite simple rank 3 matroid, and such that every isomorphism between finite $\wedge$-subgeometries of M_* extends to an automorphism of M_*. We prove that M_* is not $\aleph_0$-categorical, it has the independence property, it admits a stationary independence relation, and that $Aut(M_*)$ embeds the symmetric group $Sym(\omega)$. Finally, we use the free projective extension of M_* to conclude the existence of a countably infinite projective plane embedding all the finite simple rank 3 matroids and whose automorphism group contains $Sym(\omega)$.

1. Introduction

A countably infinite structure M is said to be *homogeneous* if every automorphism between finitely generated substructures of M extends to an automorphism of M. The study of homogeneous combinatorial structures such as graph, digraphs and hypergraphs is a very rich field of study (see e.g. [3], [4], [7], [8], [15], [16]). Despite this very little is known on homogeneous matroids. In this work we start a study of homogeneous matroids constructing a structure that is the analogous of the random graph ([23]) in the context of simple rank 3 matroids (a.k.a. planes).

As well-known (see also [19, Example 7.2.3]), the class of finite simple rank 3 matroids does not have the amalgamation property with respect to any choice of signature with respect to which substructures correspond to subgeometries. Thus, the construction of an homogeneous (with respect to subgeometries) simple rank 3 matroid containing all the finite simple rank 3 matroids as subgeometries is hopeless.

We show here that for a very natural choice of signature L the construction of an homogeneous simple rank 3 matroid containing all the simple rank 3 matroids as L-substructures is possible. In more geometric terms, we prove:

Theorem 1. *There exists a countably infinite locally finite simple rank* 3 *matroid* M_* *which* $\wedge$*-embeds every finite simple rank* 3 *matroid and such that every isomorphism between finite* $\wedge$*-subgeometries of* M_* *extends to an automorphism of* M_*.

We then establish various properties of interest of M_*:

Theorem 2. *(1)* M_* *is not* $\aleph_0$*-categorical;*
(2) M_* *has the independence property;*
(3) M_* *admits a stationary independence relation;*
(4) $Aut(M_*)$ *embeds the symmetric group* $Sym(\omega)$*;*
(5) if the age of M_* *has the extension property for partial automorphisms, then* $Aut(M_*)$ *has ample generics;*

Date: October 10, 2017.
Partially supported by European Research Council grant 338821.

(6) if $Aut(M_*)$ has the small index property, then it has the strong small index property;
(7) if $Aut(M_*)$ has the small index property, then it is complete.

Finally, we give an application to projective geometry proving:

Corollary 3. *Let $F(M_*)$ be the free projective extension of M_* (cf. [9]). Then:*
(1) $F(M_*)$ embeds all the finite simple rank 3 matroids as subgeometries;
(2) every $f \in Aut(M_*)$ extends to an $\hat{f} \in Aut(F(M_*))$;
(3) $f \mapsto \hat{f}$ is an isomorphism from $Aut(M_*)$ onto $Aut(F(M_*))$;
(4) $Aut(F(M_*))$ embeds the symmetric group $Sym(\omega)$.

Concerning M_*, we leave the following open questions:

Question 4. *(1) Does $Aut(M_*)$ have the small index property?*
(2) Does $Aut(M_*)$ have ample generics?
(3) Does the age of M_* have the extension property for partial automorphisms?

By Theorem 2(5), a positive answer to Question 4(3) would settle also Question 4(1-2). Conjecturally, the answer to Question 4(3) is in fact positive, but this does not follow from any known result on the extension property for partial automorphisms, and an answer may require some non-trivial combinatorics.

Concerning $F(M_*)$, in [13] Kalhoff constructs a projective plane of Lenz-Barlotti class V embedding all the finite simple rank 3 matroids. In [2] Baldwin constructs some almost strongly minimal projective planes of Lenz-Barlotti class I.1. We leave as an open problem the determination of the Lenz-Barlotti class of $F(M_*)$.

2. Preliminaries

For an introduction to matroids of finite rank see e.g. the canonical references [5] or [1]. For an introduction directed to model theorists see [20, Section 2].

Definition 5. *By a simple matroid $M = (M, cl_M)$ we mean a combinatorial geometry of finite rank (cf. [5]), i.e. (M, cl_M) satisfies the following:*
(1) (M, cl_M) is a closure operator;
(2) $cl_M(\emptyset) = \emptyset$, and $cl_M(\{a\}) = \{a\}$, for every $a \in M$;
(3) if $a \in \mathrm{cl}(A \cup \{b\}) - \mathrm{cl}(A)$, then $b \in \mathrm{cl}(A \cup \{a\})$;
(4) there exists finite $A \subseteq M$ such that $\mathrm{cl}(A) = M$.

Definition 6. *We say that a simple matroid M has rank $\leqslant 3$ if:*
(5') there exists $A \subseteq M$ with $|A| \leqslant 3$ such that $\mathrm{cl}(A) = M$.

When considering simple matroids M of finite rank we will freely refer to the canonically associated geometric lattice $L(M)$, see e.g. [1, Chapter VI] or [20, Theorem 2.7]. Also, we will use freely the theory of one-point extensions of matroids of [6] (see [20, Section 2] for a thorough introduction to this notion, and examples).

Definition 7. *(1) We say that a matroid M is a $\wedge$-subgeometry of the matroid N if M is a subgeometry of N and the inclusion map $i_M : M \to N$ induces an embedding (with respect to both $\vee$ and $\wedge$) of $L(M)$ into $L(N)$ (on this notion see also [20, Section 2] and [11, Definition 10]).*
(2) We say that a matroid M is locally finite if the corresponding geometric lattice $L(M)$ is locally finite, i.e. for every finite set of points A from M the sublattice generated by A in $L(M)$ (with respect to both $\vee$ and $\wedge$) is finite.

Definition 8. *Let $L = \{0, R, \wedge\}$ be the following language: 0 is a constant, R is a ternary irreflexive predicate and $\wedge$ is a quaternary function symbol. Every simple matroid M of rank $\leqslant 3$ can be naturally seen as an L-structure expanding its domain with a new element $x_\emptyset$ and letting:*
(1) $0^M = x_\emptyset$;
(2) $R^M(a, b, c)$ *if and only if* $|\{a, b, c\}| = 3$ *and* $\{a, b, c\}$ *is dependent;*
(3)

$$\wedge_M(a, b, c, d) = \begin{cases} (a \vee b) \wedge (c \vee d) & \text{if } (a \vee b) \wedge (c \vee d) \notin \{\emptyset, a, b, c, d, a \vee b, c \vee d\}, \\ x_\emptyset & \text{otherwise.} \end{cases}$$

To make explicit the choice of signature, when we consider simple matroids of rank $\leqslant 3$ as L-structures we talk about L-matroids. We also establish that the the size of an L-matroid is the size of the associated matroid. Given an L-matroid M we refer to to the domain of the associated matroid as the proper domain of M. When convenient we will be sloppy in distinguishing between matroids and L-matroids.

Remark 9. *Notice that if N and M are simple L-matroid of rank $\leqslant 3$, then N is a substructure of M if and only if N is a $\wedge$-subgeometry of M (cf. Definition 7(1)). Notice also that a simple rank 3 L-matroid is locally finite as an L-structure if and only if it is locally finite in the sense of Definition 7(2).*

For background on Fraïssé theory and homogeneous structures we refer to [10, Chapter 6]. In particular, given an homogeneous structure M we refer to the collection of finitely generated substructures of M as the *age* of M and denote it by $\mathbf{K}(M)$. For background on the notions on automorphism groups occurring in Theorem 2 see e.g. [14]. Concerning free projective extensions see [9] and [12, Chapter XI]. Concerning the notion of stationary independence relation:

Definition 10 ([24] and [18])**.** *Let M be an homogeneous structure. We say that a ternary relation $A \mathop{\smile\!\!\!\!\!\!|}_C B$ between finitely generated substructures of M is a* stationary independence relation *if the following axioms are satisfied:*
(*A*) *(Invariance) if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $f \in Aut(M)$, then $f(A) \mathop{\smile\!\!\!\!\!\!|}_{f(C)} f(B)$;*
(*B*) *(Symmetry) if $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $B \mathop{\smile\!\!\!\!\!\!|}_C A$;*
(*C*) *(Monotonicity) if $A \mathop{\smile\!\!\!\!\!\!|}_C \langle BD \rangle$ and $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $A \mathop{\smile\!\!\!\!\!\!|}_{\langle BC \rangle} D$;*
(*D*) *(Existence) there exists $A' \equiv_B A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_B C$;*
(*E*) *(Stationarity) if $A \equiv_C A'$, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A' \mathop{\smile\!\!\!\!\!\!|}_C B$, then $A \equiv_{\langle BC \rangle} A'$.*

3. Proofs

We will prove a series of claims from which Theorems 1 and 2 follow.

Lemma 11. *The class of finite simple L-matroids of rank $\leqslant 3$ is a Fraïssé class.*

Proof. The hereditary property is clear. The joint embedding property is easy and the amalgamation property is proved in [20, Theorem 4.2] (the fact that one of the factors to be amalgamated might be of rank < 3 is not a problem). ■

Definition 12 ([22, Definition 6])**.** *Let M be an homogeneous structure and $\mathbf{K} = \mathbf{K}(M)$ its age. We say that M has* canonical amalgamation *if there exists an operation $B_1 \oplus_A B_2$ on $\mathbf{K}^3$ satisfying the following conditions:*
(a) $B_1 \oplus_A B_2$ is defined when $A \leqslant B_i$ ($i = 1, 2$) and $B_1 \cap B_2 = A$;
(b) $B_1 \oplus_A B_2$ is an amalgam of B_1 and B_2 over A (in the usual sense of Fraïssé);

(c) if $B_1 \oplus_A B_2$ and $B'_1 \oplus_{A'} B'_2$ are defined, and there exist $f_i : B_i \cong B'_i$ $(i = 1, 2)$ with $f_1 \restriction A = f_2 \restriction A$, then there is:

$$f : B_1 \oplus_A B_2 \cong B'_1 \oplus_{A'} B'_2$$

such that $f \restriction B_1 = f_1$ and $f \restriction B_2 = f_2$.

Remark 13. *Notice that the amalgamation from* [20, Theorem 4.2] *is canonical in the sense of Definition 12. We will denote the canonical amalgam of M_1 and M_2 over M_0 from* [20, Theorem 4.2] *as $M_1 \oplus_{M_0} M_2$ (when we use this notation we tacitly assume that $M_0 \subseteq M_1$, $M_0 \subseteq M_2$ and $M_1 \cap M_2 = M_0$). Notice that the amalgam $M_3 := M_1 \oplus_{M_0} M_2$ can be characterized as the following L-structure:*

(1) $R^{M_3} = R^{M_1} \cup R^{M_2} \cup \{\{a, b, c\} : a \vee b \vee c = a' \vee b' \text{ and } \{a', b'\} \subseteq M_0\}$;

(2) $\wedge_{M_3}(a, b, c, d) = 0^{M_0}$, unless $a \vee b = a' \vee b'$ and $c \vee d = c' \vee d'$, for some $\{a', b', c', d'\} \subseteq M_\ell$ and $\ell = 1, 2$, and $\wedge_{M_3}(a, b, c, d) = \wedge_{M_\ell}(a', b', c', d') \neq 0^{M_0}$.

The intuition behind (2) is that the value of the function symbol $\wedge_{M_3}(a, b, c, d)$ is trivial unless $a \vee b$ and $c \vee d$ are two intersecting lines from one of the M_ℓ $(\ell = 1, 2)$.

Proof of Theorem 1. This follows from Lemma 11 and Remark 9. ■

Lemma 14. *For every $n < \omega$ there exists a finite simple rank 3 L-matroid $M(n)$ of size $6 + (n + 1)$, and 6 distinct points $p_1, .., p_6 \in M(n)$ such that $\langle p_1, ..., p_6 \rangle_{M(n)} = M(n)$, where $\langle A \rangle_B$ denotes the L-substructure generated by A in B.*

Proof. By induction on $n < \omega$, we construct a finite simple rank 3 L-matroid $M(n)$ such that:

(a) the proper domain of $M(n)$ is $\{p^-_1, p^-_2, p^+_1, p^+_2, p^*_1, p^*_2, q_0, ..., q_n\}$;
(b) $|\{p^-_1, p^-_2, p^+_1, p^+_2, p^*_1, p^*_2, q_0, ..., q_n\}| = 6 + (n + 1)$;
(c) if n is even, then $p^-_1 \vee q_n$ and $p^-_2 \vee p^*_1$ are parallel in $M(n)$;
(d) if n is odd, then $p^+_1 \vee q_n$ and $p^+_2 \vee p^*_2$ are parallel in $M(n)$;
(e) $\langle p^-_1, p^-_2, p^+_1, p^+_2, p_*, p_{**} \rangle_{M(n)} = M(n)$.

$n = 0$. Let M be the simple rank 3 L-matroid with proper domain $\{p^-_1, p^-_2, p^+_1\}$. Add to M the points p^+_2, p^*_1 and p^*_2 freely, i.e. using the non-empty modular cut containing no lines (cf. also [20, Section 2] on one-point extensions of matroids). Finally, add the point q_0 under the parallel lines $p^+_1 \vee p^+_2$ and $p^*_1 \vee p^*_2$ (using the modular cut determined by the two parallel lines $p^-_1 \vee p^-_2$ and $p^*_1 \vee p^*_2$), and let $M(0)$ be the resulting L-matroid.

$n = 2k+1$. Let $M(n-1)$ be constructed, then $M(n-1)$ contains the lines $p^-_1 \vee q_{2k}$ and $p^-_2 \vee q_{2k}$, and, by induction hypothesis, they are parallel in $M(n - 1)$. Add the point q_n under the lines $p^-_1 \vee q_{2k}$ and $p^-_2 \vee q_{2k}$, and let $M(n)$ be the resulting L-matroid.

$n = 2k > 0$. Let $M(n-1)$ be constructed, then $M(n-1)$ contains the lines $p^+_1 \vee q_{2k-1}$ and $p^+_2 \vee q_{2k-1}$, and, by induction hypothesis, they are parallel in $M(n - 1)$. Add the point q_n under the lines $p^+_1 \vee q_{2k-1}$ and $p^+_2 \vee q_{2k-1}$, and let $M(n)$ be the resulting L-matroid. ■

Lemma 15. *M_* has the independence property.*

Proof. As in [20, Theorem 4.6]. ■

Lemma 16. *For every finite substructures A, B, C of M_*, define $A \perp_C B$ if and only if $\langle A, B, C \rangle_{M_*} \cong \langle A, C \rangle_{M_*} \oplus_C \langle B, C \rangle_{M_*}$. Then $A \perp_C B$ is a stationary independence relation.*

Proof. Easy to see using Remark 13. ■

Lemma 17. *If $f \in Sym(M_*)$ induces an automorphism of $Aut(M_*)$ (i.e. $g \mapsto fgf^{-1} \in Aut(Aut(M_*))$), then $f \in Aut(M_*)$.*

Proof. First of all notice that if M is a simple rank 3 L-matroid and M^- is the reduct of M to the language $L = \{R\}$ then we have that $f \in Aut(M)$ if and only if $f \in Aut(M^-)$ (modulo ignoring 0^M). Thus if $f \notin Aut(M_*)$, then $f \notin Aut(M_*^-)$, i.e. there exists a set $\{a, b, c\} \subseteq M_*$ such that either $\{a, b, c\}$ is dependent in M_* and $\{f(a), f(b), f(c)\}$ is independent in M_*, or $\{a, b, c\}$ is independent in M_* and $\{f(a), f(b), f(c)\}$ is dependent in M_*. Modulo replacing f with f^{-1}, we can assume, that $\{a, b, c\}$ is independent in M_* and $\{f(a), f(b), f(c)\}$ is dependent in M_*. Suppose now that in addition f induces an automorphism of $Aut(M_*)$. Since M_* is homogeneous, it is easy to see that f has to send dependent sets of size 3 to independent sets of size 3, i.e. f has to send circuits of of size 3 to bases of M_*. Now, as well known, if C_1 and C_2 are circuits of a matroid, $x \in C_1 \cap C_2$ and $C_1 \neq C_2$, then $(C_1 \cup C_2) - \{x\}$ contains a circuit of the matroid. On the other hand, trivially in M_* we can find two bases $B_1 \neq B_2$ such that $x \in B_1 \cap B_2$, $B_1 \neq B_2$ and $(B_1 \cup B_2) - \{x\}$ does not contain a basis of M_*. Hence, we reach a contradiction. ■

Lemma 18. *There exists a locally finite simple rank 3 L-matroid M such that $Aut(M) \cong Sym(\omega)$.*

Proof. Let N be a simple rank 3 L-matroid of size 3. Add ω many free points to N (using the non-empty modular cut containing no lines), and let the resulting L-matroid be M. Then obviously M is locally finite and $Aut(M) \cong Sym(\omega)$. ■

Proof of Theorem 2. Item (1) follows from Lemma 14. Item (2) is Lemma 15. Item (3) follows from Lemma 16. Item (4) follows from item (3), the main result of [18] and Lemma 18. Item (5) follows from Remark 13 (JEP for partial automorphisms is easy to see) and [14, Theorem 6.2]. Item (6) follows from [22, Theorem 1] and Remark 13. Item (7) follows from item (6), [21, Theorem 1] and Lemma 17. ■

Proof of Corollary 3. Notice that every point and every line of M_* is contained in a copy of the Fano plane (which is a confined configuration, in the terminology of [12, pg. 220]). Thus, the result follows from [12, Theorem 11.18] or [17, Lemma 1]. ■

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Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Israel