On bifibrations of model categories

Pierre Cagne
IRIF, Université Paris 7
cagne@irif.fr

Paul-André Melliès
IRIF, Université Paris 7
mellies@irif.fr

Last updated on October 2, 2017

Abstract

In this article, we develop a notion of Quillen bifibration which combines the two notions of Grothendieck bifibration and of Quillen model structure. In particular, given a bifibration $p : \mathcal{E} \to \mathcal{B}$, we describe when a family of model structures on the fibers $\mathcal{E}_A$ and on the basis category $\mathcal{B}$ combines into a model structure on the total category $\mathcal{E}$, such that the functor p preserves cofibrations, fibrations and weak equivalences. Using this Grothendieck construction for model structures, we revisit the traditional definition of Reedy model structures, and possible generalizations, and exhibit their bifibrational nature.

Contents

1 Introduction **2**
1.1 Related works . . . 6

2 Liminaries **8**
2.1 Grothendieck bifibrations . . . 8
2.2 Weak factorization systems . . . 13
2.3 Intertwined weak factorization system and model categories . . . 15

3 Quillen bifibrations **18**

4 A Grothendieck construction for Quillen bifibrations **22**
4.1 Main theorem . . . 22
4.2 Proof, part I: necessity . . . 24
4.3 Proof, part II: sufficiency . . . 25

5 Illustrations **30**
5.1 A bifibrational view on Reedy model structures . . . 30
5.2 Notions of generalized Reedy categories . . . 36
5.3 Related works on Quillen bifibrations . . . 39

References **41**

1 Introduction

In this paper, we investigate how the two notions of *Grothendieck bifibration* and of *Quillen model category* may be suitably combined together. We are specifically interested in the situation of a Grothendieck bifibration $p : \mathcal{E} \to \mathcal{B}$ where the basis category $\mathcal{B}$ as well as each fiber $\mathcal{E}_A$ for an object A of the basis category $\mathcal{B}$ is equipped with a Quillen model structure. Our main purpose will be to identify necessary and sufficient conditions on the Grothendieck bifibration $p : \mathcal{E} \to \mathcal{B}$ to ensure that the total category $\mathcal{E}$ inherits a model structure from the model structures assigned to the basis $\mathcal{B}$ and to the fibers $\mathcal{E}_A$'s. We start our inquiry by recalling the fundamental relationship between bifibrations and adjunctions. This connection will guide us all along the paper. Our plan is indeed to proceed by analogy, and to carve out a notion of *Quillen bifibration* playing the same role for Grothendieck bifibrations as the notion of *Quillen adjunction* plays today for the notion of adjunction.

Grothendieck bifibrations and adjunctions. We will generally work with *cloven* bifibrations. Recall that a *cleavage on a Grothendieck fibration* is a choice, for every morphism $u : A \to B$ and for every object Y above B, of a cartesian morphism $\rho_{u,Y} : u^*Y \to Y$ above u. Dually, a *cleavage on a Grothendieck opfibration* is a choice, for every morphism $u : A \to B$ and for every object X above A, of a left cartesian morphism $\lambda_{u,X} : X \to u_!X$ above u. In a cloven Grothendieck fibration, every morphism $u : A \to B$ in the basis category $\mathcal{B}$ induces a functor

$$u^* \quad : \quad \mathcal{E}_B \longrightarrow \mathcal{E}_A \tag{1}$$

Symmetrically, in a cloven Grothendieck opfibration, every morphism $u : A \to B$ in the basis category $\mathcal{B}$ induces a functor

$$u_! \quad : \quad \mathcal{E}_A \longrightarrow \mathcal{E}_B \tag{2}$$

A *cloven bifibration* (or more simply a bifibration) is a left and right Grothendieck fibration $p : \mathcal{E} \to \mathcal{B}$ equipped with a cleavage on both sides.

Formulated in this way, a bifibration $p : \mathcal{E} \to \mathcal{B}$ is simply the "juxtaposition" of a left and of a right Grothendieck fibration, with no apparent connection between the two structures. Hence, a remarkable phenomenon is that the two fibrational structures are in fact strongly interdependent. Indeed, it appears that in a bifibration $p : \mathcal{E} \to \mathcal{B}$, the pair of functors (1) and (2) associated to a morphism $u : A \to B$ defines an adjunction between the fiber categories

$$u_! \quad : \quad \mathcal{E}_A \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathcal{E}_B \quad : \quad u^*$$

where the functor $u_!$ is left adjoint to the functor u^*. The bond between bifibrations and adjunctions is even tighter when one looks at it from the point of view of indexed categories. Recall that a (covariantly) *indexed category* of basis $\mathcal{B}$ is defined as a pseudofunctor

$$\mathcal{P} \quad : \quad \mathcal{B} \longrightarrow \mathsf{Cat} \tag{3}$$

where Cat denotes the 2-category of categories, functors and natural transformations. Every cloven Grothendieck opfibration $p : \mathcal{E} \to \mathcal{B}$ induces an indexed category $\mathcal{P}$ which transports every object A of the basis $\mathcal{B}$ to the fiber category $\mathcal{E}_A$, and every morphism $u : A \to B$ of the basis to the functor $u_! : \mathcal{E}_A \to \mathcal{E}_B$. Conversely, the Grothendieck construction enables one to construct a cloven Grothendieck opfibration $p : \mathcal{E} \to \mathcal{B}$ from an indexed category $\mathcal{P}$. This back-and-forth translation defines an equivalence of categories between

$$\begin{array}{c}\text{the category of}\\ \text{cloven Grothendieck opfibrations}\\ \text{with basis category } \mathcal{B}\end{array} \quad \rightleftharpoons \quad \begin{array}{c}\text{the category of}\\ \text{indexed categories}\\ \text{with basis category } \mathcal{B}\end{array}$$

All this is well-known. What is a little bit less familiar (possibly) and which matters to us here is that this correspondence may be adapted to Grothendieck bifibrations, in the following way. Consider the 2-category Adj with categories as objects, with adjunctions

$$L \quad : \quad \mathcal{M} \rightleftarrows \mathcal{N} \quad : \quad R \tag{4}$$

as morphisms from $\mathcal{M}$ to $\mathcal{N}$, and with natural transformations

$$\theta \quad : \quad L_1 \Longrightarrow L_2 \quad : \quad \mathcal{M} \longrightarrow \mathcal{N}$$

between the left adjoint functors as 2-dimensional cells $\theta : (L_1, R_1) \Rightarrow (L_2, R_2)$. In the same way as we have done earlier, an *indexed category-with-adjunctions* with basis category $\mathcal{B}$ is defined as a pseudofunctor

$$\mathcal{P} \quad : \quad \mathcal{B} \longrightarrow \mathsf{Adj} \tag{5}$$

For the same reasons as in the case of Grothendieck opfibrations, there is an equivalence of category between

$$\begin{array}{c}\text{the category of}\\ \text{cloven bifibrations}\\ \text{with basis category } \mathcal{B}\end{array} \quad \rightleftharpoons \quad \begin{array}{c}\text{the category of}\\ \text{indexed categories-with-adjunctions}\\ \text{with basis category } \mathcal{B}\end{array}$$

From this follows, among other consequences, that a cloven bifibration $p : \mathcal{E} \to \mathcal{B}$ is the same thing as a cloven right fibration where the functor $u^* : \mathcal{E}_B \to \mathcal{E}_A$ comes equipped with a left adjoint $u_! : \mathcal{E}_A \to \mathcal{E}_B$ for every morphism $u : A \to B$ of the basis category $\mathcal{B}$.

By way of illustration, consider the ordinal category $\mathbf{2}$ with two objects 0 and 1 and a unique non-identity morphism $u : 0 \to 1$. By the discussion above, a Grothendieck bifibration $p : \mathcal{E} \to \mathcal{B}$ on the basis category $\mathcal{B} = \mathbf{2}$ is the same thing as an adjunction (4). The correspondence relies on the observation that every adjunction (4) can be turned into a bifibration $p : \mathcal{E} \to \mathcal{B}$ where the category $\mathcal{E}$ is defined as the category of *collage* associated to the adjunction (L, R), with fibers $\mathcal{E}_0 = \mathcal{M}$, $\mathcal{E}_1 = \mathcal{N}$ and mediating functors $u^* = R$ and $u_! = L$, see [Str80] for the notion of collage. For that reason, the Grothendieck construction for bifibrations may be seen as a generalized and fibrational notion of collage.

Model structures and Quillen adjunctions. Seen from that angle, the notion of Grothendieck bifibration provides a fibrational counterpart (and also a far-reaching generalization) of the fundamental notion of adjunction between categories. This perspective opens a firm connection with modern homotopy theory, thanks to the notion of *Quillen adjunction* between model categories. Recall that a *model structure* on a category $\mathcal{M}$ delineates three classes $\mathfrak{C}$, $\mathfrak{W}$, $\mathfrak{F}$ of maps called *cofibrations*, *weak equivalences* and *fibrations* respectively ; these classes of maps are moreover required to satisfy a number of properties recalled in definition 2.7. A fibration or a cofibration which is at the same time a weak equivalence is called *acyclic*.

Remark 1.1. By extension, we find sometimes convenient to call *model structure* a category $\mathcal{M}$ *together* with its model structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$. The appropriate name for that notion would be *model category* but the terminology is already used in the literature for a *finitely complete* and *finitely cocomplete* category $\mathcal{C}$ equipped with a model structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$. The extra completeness assumptions play a role in the construction of the homotopy category $\mathbf{Ho}(\mathcal{C})$, and it is thus integrated in the accepted definition of a "model category". We prefer to work with "model structures" for two reasons. On the one hand, the construction of $\mathbf{Ho}(\mathcal{C})$ can be performed using the weaker assumption that the category $\mathcal{C}$ has finite products and finite coproducts, as noticed by Egger [Egg16]. On the other hand, the extra completeness assumptions are independent of the relationship between Grothendieck bifibrations and model structures, and may be treated separately.

We recall below the notions of left and right Quillen functor between model structures.

Definition 1.2 (Quillen functors). A functor $F : \mathcal{M} \to \mathcal{N}$ between two model structures $\mathcal{M}$ and $\mathcal{N}$ is called a *left Quillen functor* when it transports every cofibration of $\mathcal{M}$ to a cofibration of $\mathcal{N}$ and every acyclic cofibration of $\mathcal{M}$ to an acyclic cofibration of $\mathcal{N}$. Dually, a functor $F : \mathcal{M} \to \mathcal{N}$ is called a *right Quillen functor* when it transports every fibration of $\mathcal{M}$ to a fibration of $\mathcal{N}$ and every acyclic fibration of $\mathcal{M}$ to an acyclic fibration of $\mathcal{N}$. A functor $F : \mathcal{M} \to \mathcal{N}$ which is at the same time a left and a right Quillen functor is called a *Quillen functor*.

A simple argument shows that a Quillen functor $F : \mathcal{M} \to \mathcal{N}$ transports every weak equivalence of $\mathcal{M}$ to a weak equivalence of $\mathcal{N}$. For that reason, a Quillen functor is the same thing as a functor which transports every cofibration, weak equivalence or fibration $f : A \to B$ of $\mathcal{M}$ to a map $Ff : FA \to FB$ with the same status in the model structure of $\mathcal{N}$.

The notion of *Quillen adjunction* relies on the following observation.

Proposition 1.3. *Suppose given an adjunction*

$$L \quad : \quad \mathcal{M} \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathcal{N} \quad : \quad R \tag{6}$$

between two model categories $\mathcal{M}$ and $\mathcal{N}$. The following assertions are equivalent:

- *the left adjoint functor $L : \mathcal{M} \to \mathcal{N}$ is a left Quillen functor,*
- *the right adjoint functor $R : \mathcal{N} \to \mathcal{M}$ is a right Quillen functor.*

Definition 1.4 (Quillen adjunctions). An adjunction $L : \mathcal{M} \leftrightarrows \mathcal{N} : R$ between two model categories $\mathcal{M}$ and $\mathcal{N}$ is called a *Quillen adjunction* when the equivalent assertions of Prop. 1.3 hold.

Quillen bifibrations. At this stage, we are ready to introduce the notion of *Quillen bifibration* which we will study in the paper. We start by observing that whenever the total category $\mathcal{E}$ of a functor $p : \mathcal{E} \to \mathcal{B}$ is equipped with a model structure $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$, every fiber $\mathcal{E}_A$ associated to an object A of the basis category $\mathcal{B}$ comes equipped with three classes of maps noted $\mathfrak{C}_A$, $\mathfrak{W}_A$, $\mathfrak{F}_A$ called cofibrations, weak equivalences and fibrations above the object A, respectively. The classes are defined in the expected way:

$$\mathfrak{C}_A = \mathfrak{C}_{\mathcal{E}} \cap \mathfrak{Hom}_A \qquad \mathfrak{W}_A = \mathfrak{W}_{\mathcal{E}} \cap \mathfrak{Hom}_A \qquad \mathfrak{F}_A = \mathfrak{F}_{\mathcal{E}} \cap \mathfrak{Hom}_A$$

where $\mathfrak{Hom}_A$ denotes the class of maps f of the category $\mathcal{E}$ above the object A, that is, such that $p(f) = \mathbf{1}_A$. We declare that the model structure $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ on the total category $\mathcal{E}$ *restricts* to a model structure on the fiber $\mathcal{E}_A$ when the three classes $\mathfrak{C}_A$, $\mathfrak{W}_A$, $\mathfrak{F}_A$ satisfy the properties required of a model structure on the category $\mathcal{E}_A$.

This leads us to the main concept of the paper:

Definition 1.5 (Quillen bifibrations)**.** A Quillen bifibration $p : \mathcal{E} \to \mathcal{B}$ is a Grothendieck bifibration where the basis category $\mathcal{B}$ and the total category $\mathcal{E}$ are equipped with a model structure, in such a way that

- the functor $p : \mathcal{E} \to \mathcal{B}$ is a Quillen functor,
- the model structure of $\mathcal{E}$ restricts to a model structure on the fiber $\mathcal{E}_A$, for every object A of the basis category $\mathcal{B}$.

This definition of Quillen bifibration deserves to be commented. The first requirement that $p : \mathcal{E} \to \mathcal{B}$ is a Quillen functor means that every cofibration, weak equivalence and fibration $f : X \to Y$ of the total category $\mathcal{E}$ lies above a map $u : A \to B$ of the same status in the model category $\mathcal{B}$. This condition makes sense, and we will see in section 3 that it is satisfied in a number of important examples. The second requirement means that the model structure $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ combines into a single model structure on the total category $\mathcal{E}$ the family of model structures $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$ on the fiber categories $\mathcal{E}_A$.

A Grothendieck construction for Quillen bifibrations. The notion of Quillen bifibration is tightly connected to the notion of Quillen adjunction, thanks to the following observation established in section 3.

Proposition 1.6. *In a Quillen bifibration $p : \mathcal{E} \to \mathcal{B}$, the adjunction*

$$u_! \quad : \quad \mathcal{E}_A \rightleftarrows \mathcal{E}_B \quad : \quad u^*$$

is a Quillen adjunction, for every morphism $u : A \to B$ of the basis category $\mathcal{B}$.

From this follows that a Quillen bifibration induces an *indexed model structure*

$$\mathcal{P} \quad : \quad \mathcal{B} \longrightarrow \mathsf{Quil} \tag{7}$$

defined as a pseudofunctor from a model structure $\mathcal{B}$ to the 2-category Quil of model structures, Quillen adjunctions, and natural transformations. Our main contribution in this paper is to formulate necessary and sufficient conditions for a Grothendieck construction to hold in this situation. More specifically, we resolve the following problem.

A. Hypothesis of the problem. We suppose given an indexed Quillen category as we have just defined in (7) or equivalently, a Grothendieck bifibration $p : \mathcal{E} \to \mathcal{B}$ where

- the basis category $\mathcal{B}$ is equipped with a model structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$,
- every fiber $\mathcal{E}_A$ is equipped with a model structure $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$,
- the adjunction $(u_!, u^*)$ is a Quillen adjunction, for every morphism $u : A \to B$ of the basis category $\mathcal{B}$.

B. Resolution of the problem. We find necessary and sufficient conditions to ensure that there exists a model structure $(\mathfrak{C}_\mathcal{E}, \mathfrak{W}_\mathcal{E}, \mathfrak{F}_\mathcal{E})$ on the total category $\mathcal{E}$ such that

- the Grothendieck bifibration $p : \mathcal{E} \to \mathcal{B}$ defines a Quillen bifibration,
- for every object A of the basis category, the model structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ of the total category $\mathcal{E}$ restricts to the model structure $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$ of the fiber $\mathcal{E}_A$.

We establish in the course of the paper (see section 3) that there exists at most one solution to the problem, which is obtained by defining the cofibrations and fibrations of the total category $\mathcal{E}$ in the following way:

- a morphism $f : X \to Y$ of the total category $\mathcal{E}$ is a *total cofibration* when it factors as $X \to Z \to Y$ where $X \to Z$ is a cocartesian map above a cofibration $u : A \to B$ of $\mathcal{B}$, and $Z \to Y$ is a cofibration in the fiber $\mathcal{E}_B$,
- a morphism $f : X \to Y$ of the total category $\mathcal{E}$ is a *total fibration* when it factors as $X \to Z \to Y$ where $Z \to Y$ is a cartesian map above a fibration $u : A \to B$ of $\mathcal{B}$, and $X \to Z$ is a fibration in the fiber $\mathcal{E}_A$.

Proposition 1.7 (Uniqueness of the solution)**.** *When the solution $(\mathfrak{C}_\mathcal{E}, \mathfrak{W}_\mathcal{E}, \mathfrak{F}_\mathcal{E})$ exists, it is uniquely determined by the fact that its fibrations and cofibrations are the total cofibrations and total fibrations of the total category $\mathcal{E}$, respectively.*

Besides the formulation of Quillen bifibrations, our main contribution is to devise two conditions called (hCon) for *homotopical conservativity* and (hBC) for *homotopical Beck-Chevalley*, and to show (see theorem 4.2) that they are sufficient and necessary for the solution to exist.

1.1 Related works

The interplay between bifibred categories and model structures was first explored by Roig in [Roi94], providing results in homological diffrentially graded algebra. Stanculescu then spotted a mistake in Roig's theorem and subsequently corrected it in [Sta12]. Finally, [HP15] tackles the problem of reflecting Lurie's Grothendieck construction for ∞-categories at the level of model categories, hence giving a model for lax colimits of diagrams of ∞-categories.

This work is directly in line with, and greatly inspired by, these papers. In our view, both Roig-Stanculescu's and Harpaz-Prasma's results suffer from flaws. The former introduces a very strong asymmetry, making natural expectations unmet. For example, for any Grothendieck bifibration $p : \mathcal{E} \to \mathcal{B}$, the opposite functor $p^\circ : \mathcal{E}^\circ \to \mathcal{B}^\circ$ is also a Grothendieck bifibration. So we shall expect that when it is possible to apply Roig-Stanculescu's result to the functor p, provinding this way a model structure on $\mathcal{E}$, it is also possible to apply it to p°, yielding on $\mathcal{E}^\circ$ the opposite model structure. This is not the case: for almost every such p for which the result applies, it does not for the functor p°. The latter result by Harpaz and Prasma on the contrary forces the symmetry by imposing a rather strong assumption: the adjoint pair $(u_!, u^*)$ associated to a morphism u of the base $\mathcal{B}$, already required to be a Quillen adjunction in [Roi94] and [Sta12], needs in addition to be a Quillen equivalence whenever u is a weak equivalence. While it is a key property for their applications, it put aside real world examples that nevertheless satisfy the conclusion of the result. The goal of this paper is to lay out a common framework fixing these flaws. This is achieved in theorem 4.2 by giving necessary and sufficient conditions for the resulting model structure on $\mathcal{E}$ to be the one described in both cited results.

Plan of the paper. Section 2 recalls the basic facts we will need latter about Grothendieck bifibrations and model categories. It also introduces *intertwined weak factorization systems*, a notion that pops here and there on forums and the n-Category Café, but does not appear in the literature to the best of our knowledge. Its interest mostly resides in that it singles out the 2-out-of-3 property of weak equivalences in a model category from the other more *combinatorial* properties. Finally we recall in that section a result of [Sta12] in order to make this paper self-contained.

Section 4 contains the main theorem 4.2 that we previously announced. Its proof is cut into two parts: first we prove the necessity of conditions hCon and hBC, and then we show that they are sufficient as well. The proof of necessity is the easy part and comes somehow as a bonus, while the proof of sufficiency is much harder and expose how conditions (hCon) and (hBC) play their role.

Section 5 illustrates 4.2 with some applications in usual homotopical algebra. First, it gives an original view on Kan's theorem about Reedy model structures by stating it in a bifibrationnal setting. Here should it be said that this was our motivating example. We realized that neither Roig-Stanculescu's or Harpaz-Prasma's theorem could be apply to the Reedy construction, although the conclusion of these results was giving Kan's theorem back. As in any of those *too good no to be true* situations, we took that as an incentive to strip down the previous results in order to only keep what makes them *tick*, which eventually has led to the equivalence of theorem 4.2. Section 5.3 gives more details about Roig-Stanculescu's and Harpaz-Prasma's theorem, and explains how their analysis started the process of this work.

Convention. All written diagrams commute if not said otherwise. When objects are missing and replaced by a dot, they can be parsed from other informations on the diagram. Gray parts help to understand the diagram's context.

Acknowledgments. The authors are grateful to Clemens Berger for making them aware of important references at the beginning of this work, and to Georges Maltsiniotis for an early review of theorem 4.2 and instructive discussions around possible weakenings of the notion of Quillen bifibration.

2 Liminaries

2.1 Grothendieck bifibrations

In this section, we recall a number of basic definitions and facts about Grothendieck bifibrations.

Given a functor $p : \mathcal{E} \to \mathcal{B}$, we shall use the following terminology. The categories $\mathcal{B}$ and $\mathcal{E}$ are called the *basis category* $\mathcal{B}$ and the *total category* $\mathcal{E}$ of the functor $p : \mathcal{E} \to \mathcal{B}$. We say that an object X of the total category $\mathcal{E}$ is above an object A of the basis category $\mathcal{B}$ when $p(X) = A$ and, similarly, that a morphism $f : X \to Y$ is above a morphism $u : A \to B$ when $p(f) = u$. The fiber of an object A in the basis category $\mathcal{B}$ with respect to p is defined as the subcategory of $\mathcal{E}$ whose objects are the objects X such that $p(X) = A$ and whose morphisms are the morphisms f such that $p(f) = \mathbf{1}_A$. In other words, the fiber of A is the category of objects above A, and of morphisms above the identity $\mathbf{1}_A$. The fiber is noted p_A or $\mathcal{E}_A$ when no confusion is possible.

A morphism $f : X \to Y$ in a category $\mathcal{E}$ is called *cartesian* with respect to the functor $p : \mathcal{E} \to \mathcal{B}$ when the commutative diagram

$$\begin{tikzcd}
\mathcal{E}(Z,X) \arrow[r, "f\circ-"] \arrow[d, "p"'] & \mathcal{E}(Z,Y) \arrow[d, "p"] \\
\mathcal{B}(C,A) \arrow[r, "u\circ-"] & \mathcal{B}(C,B)
\end{tikzcd}$$

is a pullback diagram for every object Z in the category $\mathcal{E}$. Here, we write $u : A \to B$ and C for the images $u = p(f)$ and $C = p(Z)$ of the morphism f and of the object Z, respectively. Unfolding the definition, this means that for every pair of morphisms $v : C \to A$ and $g : Z \to Y$ above $u \circ v : C \to B$, there exists a unique morphism $h : Z \to X$ above v such that $h \circ f = g$. The situation may be depicted as follows:

$$\begin{tikzcd}
Z \arrow[drr, bend left, "g"] \arrow[dr, dashed, "h"] \arrow[dd, dotted, no head] & & \\
& X \arrow[r, "f"] \arrow[dd, dotted, no head] & Y \arrow[dd, dotted, no head] \\
C \arrow[drr, bend left] \arrow[dr, "v"'] & & \\
& A \arrow[r, "u"'] & B
\end{tikzcd}$$

Dually, a morphism $f : X \to Y$ in a category $\mathcal{E}$ is called *cocartesian* with respect to the functor $p : \mathcal{E} \to \mathcal{B}$ when the commutative diagram

$$\begin{tikzcd}
\mathcal{E}(Y,Z) \arrow[r, "-\circ f"] \arrow[d, "p"'] & \mathcal{E}(X,Z) \arrow[d, "p"] \\
\mathcal{B}(B,C) \arrow[r, "-\circ u"] & \mathcal{B}(A,C)
\end{tikzcd}$$

is a pullback diagram for every object Z in the category $\mathcal{E}$. This means that for every pair of morphisms $v : A \to C$ and $g : X \to Z$ above $v \circ u : A \to C$, there exists a unique morphism $h : Z \to X$ above v such that $h \circ f = g$. Diagrammatically:

$$\begin{tikzcd}
 & & Z \arrow[dd, dash, dotted] \\
X \arrow[r, "f"] \arrow[urr, bend left, "g"] \arrow[dd, dash, dotted] & Y \arrow[ur, dashed, "h"] \arrow[dd, dash, dotted] & \\
 & & C \\
A \arrow[r, "u"'] \arrow[urr, bend left] & B \arrow[ur, "v"'] &
\end{tikzcd}$$

A functor $p : \mathcal{E} \to \mathcal{B}$ is called a *Grothendieck opfibration* when for every morphism $u : A \to B$ and for every object Y above B, there exists a cartesian morphism $f : X \to Y$ above u. Symmetrically, a functor $p : \mathcal{E} \to \mathcal{B}$ is called a *Grothendieck opfibration* when for every morphism $u : A \to B$ and for every object X above A, there exists a cocartesian morphism $f : X \to Y$ above u. Note that a functor $p : \mathcal{E} \to \mathcal{B}$ is a Grothendieck opfibration precisely when the functor $p^{op} : \mathcal{E}^{op} \to \mathcal{B}^{op}$ is a Grothendieck fibration. A *Grothendieck bifibration* is a functor $p : \mathcal{E} \to \mathcal{B}$ which is at the same time a Grothendieck fibration and opfibration.

Definition 2.1. A *cloven Grothendieck bifibration* is a functor $p : \mathcal{E} \to \mathcal{B}$ together with

- for any $Y \in \mathcal{E}$ and $u : A \to pY$, an object $u^*Y \in \mathcal{E}$ and a cartesian morphism $\rho^p_{u,Y} : u^*Y \to Y$ above u,
- for any $X \in \mathcal{E}$ and $u : pX \to B$, an object $u_!X \in \mathcal{E}$ and a cocartesian morphism $\lambda^p_{u,X} : X \to u_!X$ above u.

When the context is clear enough, we might omit the index p. The domain category $\mathcal{E}$ is often called the *total category* of p, and its codomain $\mathcal{B}$ the *base category*. We shall use this terminology when suited.

Remark 2.2. If $\mathcal{E}$ and $\mathcal{B}$ are small relatively to a universe $\mathbb{U}$ in which we suppose the axiom of choice, then a cloven Grothendieck bifibration is exactly the same as the original notion of Grothendieck bifibration. Hence, in this article, we treat the two names as synonym.

The data of such cartesian and cocartesian morphisms gives two factorizations of an arrow $f : X \to Y$ above some arrow $u : A \to B$, $f_{\triangleright}$ in the fiber $\mathcal{E}_B$ and $f^{\triangleleft}$ in the fiber $\mathcal{E}_A$: one goes through $\rho_{u,Y}$ and the other through $\lambda_{u,X}$. See the diagram below:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f^{\triangleleft}"'] \arrow[dr, "f"] & u_!X \arrow[d, "f_{\triangleright}"] \\
u^*Y \arrow[r] & Y
\end{tikzcd}$$

In turn, this allows $u_!$ and u^* to be extended as *adjoint functors*:

$$u_! : \mathcal{E}_A \rightleftarrows \mathcal{E}_B : u^*$$

where the action of $u_!$ on a morphism $k : X \to X'$ of $\mathcal{E}_A$ is given by $(\lambda_{u,X'} \circ k)_{\triangleright}$ and the action of u^* on a morphism $\ell : Y' \to Y$ is given by $(\ell \circ \rho_{u,Y'})^{\triangleleft}$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "k"'] & u_!X \arrow[d, "u_!(k)"] \\
X' \arrow[r] & u_!X'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
u^*Y' \arrow[r] \arrow[d, "u^*\ell"'] & Y' \arrow[d, "\ell"] \\
u^*Y \arrow[r] & Y
\end{tikzcd}$$

This gives a mapping $\mathcal{B} \to \mathsf{Adj}$ from the category $\mathcal{B}$ to the 2-category Adj of adjunctions: it maps an object A to the fiber $\mathcal{E}_A$, and a morphism u to the push-pull adjunction $(u_!, u^*)$. This mapping is even a pseudofunctor:

- For any $A \in \mathcal{B}$ and $X \in \mathcal{E}_A$, we can factor $\mathbf{1}_X : X \to X$ through $\lambda_{\mathbf{1}_A,X}$ and $\rho_{\mathbf{1}_A,X}$:

$$\begin{tikzcd} X \arrow[r, "\lambda_{\mathbf{1}_A,X}"] \arrow[dr, equal] & (\mathbf{1}_A)_! X \arrow[d, "(\mathbf{1}_X)_\triangleright"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "(\mathbf{1}_X)^\triangleleft"'] \arrow[dr, equal] & \\ (\mathbf{1}_A)^* X \arrow[r, "\rho_{\mathbf{1}_A,X}"'] & X \end{tikzcd}$$

  In particular by looking at the diagram on the left, both $\lambda_{\mathbf{1}_A,X} \circ (\mathbf{1}_X)_\triangleright$ and the identity of $(\mathbf{1}_A)_! X$ are solution to the problem of finding an arrow f above $\mathbf{1}_A$ such that $f\lambda_{\mathbf{1}_A,X} = \lambda_{\mathbf{1}_A,X}$: by the unicity condition of the cocartesian morphisms, it means that they are equal, or otherwise said that $(\mathbf{1}_X)_\triangleright$ is an isomorphism with inverse $\lambda_{\mathbf{1}_A,X}$. Dually, looking at the diagram on the right, we deduce that $(\mathbf{1}_X)^\triangleleft$ is an isomorphism with inverse $\rho_{\mathbf{1}_A,X}$. All is natural in X, so we end up with

$$(\mathbf{1}_A)_! \simeq \mathbf{1}_{\mathcal{E}_A} \simeq (\mathbf{1}_A)^*$$

- For any $u : A \to B$ and $v : B \to C$ in $\mathcal{B}$, and for any $X \in \mathcal{E}_A$, the cocartesian morphism $\lambda_{vu,X} : X \to (vu)_! X$ is above vu by definition hence should factorize as $h\lambda_{u,X}$ for some h above v, yielding $h_\triangleright$ as a morphism in $\mathcal{E}_C$ such that the following commutes:

$$\begin{tikzcd} X \arrow[r, "\lambda_{u,X}"] \arrow[drr, bend right, "\lambda_{vu,X}"'] & u_! X \arrow[r, "\lambda_{v,u_!X}"] \arrow[dr, "h"'] & v_! u_! X \arrow[d, "h_\triangleright"] \\ & & (vu)_! X \end{tikzcd}$$

  Writing simply k for the composite $\lambda_{v,u_!X} \circ \lambda_{u,X}$, the following commutes:

$$\begin{tikzcd} X \arrow[rr, "\lambda_{vu,X}"] \arrow[dr, "\lambda_{u,X}"'] & & (vu)_! X \arrow[dd, "k_\triangleright"] \\ & u_! X \arrow[dr, "\lambda_{v,u_!X}"'] & \\ & & v_! u_! X \end{tikzcd}$$

  Clearly $h_\triangleright k_\triangleright$ and $\mathbf{1}_{v_! u_! X}$ both are solution to the problem of finding f above $\mathbf{1}_C$ such that $f\lambda_{vu,X} = \lambda_{vu,X}$: the uniqueness condition in the definition of cocartesian morphisms forces them to be equal. Conversely, we use the cocartesianness of $\lambda_{u,X}$ and $\lambda_{v,u_!X}$ in two steps: first $k_\triangleright h_\triangleright \lambda_{v,u_!X} = \lambda_{v,u_!X}$ because they both answer the problem of finding f above v such that $f\lambda_{u,X} = \lambda_{v,u_!X} \circ \lambda_{u,X}$; from which we deduce $k_\triangleright h_\triangleright = \mathbf{1}_{v_! u_! X}$ as they both answer the problem of finding a map f above $\mathbf{1}_C$ such that $f\lambda_{v,u_!X} = \lambda_{v,u_!X}$. In the end, $h_\triangleright$ and $k_\triangleright$ are isomorphisms, inverse to each other. All we did was natural in X, hence we have

$$(vu)_! \simeq v_! u_!$$

- The dual argument shows that $(vu)^* \simeq u^*v^*$.
- To prove rigorously the pseudo functoriality of $\mathcal{B} \to \mathsf{Adj}$, we should show that the isomorphisms we have exhibited above are coherent. This is true, but irrelevant to this work, so we will skip it.

The pseudo functoriality relates through an isomorphism the chosen (co)cartesian morphism above a composite vu with the composite of the chosen (co)cartesian morphisms above u and v. The following lemma gives some kind of extension of this result.

Lemma 2.3. *Let $u : A \to B$, $v : B \to C$ and $w : C \to D$ in $\mathcal{B}$. Suppose $f : X \to Y$ in $\mathcal{E}$ is above the composite wvu. Then for the unique maps $h : v_!u_!X \to Y$ and $k : (vu)_!X \to Y$ above w that fill the commutative triangles*

$$
\begin{tikzcd}
X \arrow[r, "\lambda_{u,X}"] \arrow[rrd, bend right, "f"'] & u_!X \arrow[r, "\lambda_{v,u_!X}"] \arrow[rd] & v_!u_!X \arrow[d, "h"] \\
 & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\lambda_{vu,X}"] \arrow[rd, bend right, "f"'] & (vu)_!X \arrow[d, "k"] \\
 & Y
\end{tikzcd}
$$

there exists an isomorphism ϕ in the fiber $\mathcal{E}_C$ such that $h\phi = k$.

Proof. We know there is a isomorphism $\phi : (vu)_!X \to u_!v_!X$ above $\mathbf{1}_C$ such that $\phi\lambda_{vu,X} = \lambda_{v,u_!X} \circ \lambda_{u,X}$. But then $h\phi : (vu)_!X \to Y$ is above w and fills the same triangle k does in the statement: by unicity, $k = h\phi$.

$$
\begin{tikzcd}
 & & (vu)_!X \arrow[d, "\phi"] \arrow[rdd, bend left, "k"] & \\
X \arrow[rru, bend left, "\lambda_{vu,X}"] \arrow[r, "\lambda_{u,X}"] \arrow[rrrd, bend right, "f"'] & u_!X \arrow[r, "\lambda_{v,u_!X}"] & v_!u_!X \arrow[rd, "h"] & \\
 & & & Y
\end{tikzcd}
$$

$\square$

Of course, we have the dual statement, that accepts a dual proof.

Lemma 2.4. *Let $u : A \to B$, $v : B \to C$ and $w : C \to D$ in $\mathcal{B}$. Suppose $f : X \to Y$ in $\mathcal{E}$ is above the composite wvu. Then for the unique maps $h : X \to u^*v^*Y$ and $k : X \to (vu)^*Y$ above w that fill the commutative triangles*

$$
\begin{tikzcd}
X \arrow[rd, "h"'] \arrow[rrd] \arrow[rrrd, bend left, "f"] & & & \\
 & u^*v^*Y \arrow[r, "\rho_{u,v^*Y}"'] & v^*Y \arrow[r, "\rho_{v,Y}"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rd, "k"'] \arrow[rrd, bend left, "f"] & & \\
 & (vu)^*Y \arrow[r, "\rho_{vu,Y}"'] & Y
\end{tikzcd}
$$

there exists an isomorphism ϕ in the fiber $\mathcal{E}_C$ such that $\phi k = h$.

Suppose now that we have a chain of composable maps in $\mathcal{B}$:

$$
A_0 \xrightarrow{u_1} A_1 \xrightarrow{u_2} \dots \xrightarrow{u_n} A_n
$$

And let $f : X \to Y$ be a map above the composite $u_n \dots u_1 u_0$. Choose $0 \leq i, j \leq n$ such that $i + j \leq n$. Then, using (co)cartesian choices above maps in $\mathcal{B}$, one can construct two canonical maps associated to f: these are the unique maps

$$h : (u_i)_! \cdots (u_0)_! X \to (u_{n-j+1})^* \cdots (u_n)^* Y$$
and
$$k : (u_i \cdots u_0)_! X \to (u_n \cdots u_{n-j+1})^* Y$$

above $u_{n-j} \cdots u_{i+1} : A_i \to A_{n-j}$ (which is defined as $\mathbf{1}_{A_i}$ in case $i + j = n$) filling in the following commutative diagrams:

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[drrrrr, bend right, "f"'] & \dots \arrow[r, "\lambda"] & (u_i)_! \cdots (u_0)_! X \arrow[dr, "h"] & & & \\
& & & (u_{n-j+1})^* \cdots (u_n)^* Y \arrow[r, "\rho"] & \dots \arrow[r, "\rho"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[drr, bend right, "f"'] & (u_i \cdots u_0)_! X \arrow[d, "k"] & \\
& (u_n \cdots u_{n-j+1})^* Y \arrow[r, "\rho"] & Y
\end{tikzcd}$$

By applying the previous lemmas multiples times, we get the following useful corollary.

Corollary 2.5. *There is fiber isomorphisms ϕ and ψ such that the following commutes:*

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[ddrr, "\lambda"'] \arrow[drrrrr, bend right, "f"] & \dots \arrow[r, "\lambda"] & (u_i)_! \cdots (u_0)_! X \arrow[dr, "h"] & & & \\
& & & (u_{n-j+1})^* \cdots (u_n)^* Y \arrow[r, "\rho"] & \dots \arrow[r, "\rho"] & Y \\
& & (u_i \cdots u_0)_! X \arrow[uu, "\phi"] \arrow[dr, "k"] & & & \\
& & & (u_n \cdots u_{n-j+1})^* Y \arrow[uu, "\psi"] \arrow[uurr, "\rho"'] & &
\end{tikzcd}$$

We will extensively use this corollary when $i + j = n$. Indeed, in that case h, k, ϕ, ψ all are in the same fiber $\mathcal{E}_{A_i}$ and then h and k are isomorphic as arrows in that fiber. Every property on h that is invariant by isomorphism of arrows will still hold on k, and conversely.

2.2 Weak factorization systems

In any category $\mathcal{M}$, we denote $j \boxslash q$, and we say that j has the left lifting property relatively to q (or that q has the right lifting property relatively to j), when for any commutative square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & C \arrow[d, "q"] \\ B \arrow[r] & D \end{tikzcd}$$

there exists a morphism $h : B \to C$, making the two triangles commute in the following diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & C \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & D \end{tikzcd}$$

Such a morphism h is called a *lift* of the original commutative square.

A *weak factorization system* on a category $\mathcal{M}$ is the data of a couple $(\mathfrak{L}, \mathfrak{R})$ of classes of arrows in $\mathcal{M}$ such that

$$\mathfrak{L} = \{j : \forall q \in R, j \boxslash q\} \quad \text{and} \quad \mathfrak{R} = \{q : \forall j \in L, j \boxslash q\}$$

and such that every morphism f of $\mathcal{M}$ may be factored as $f = qj$ with $j \in \mathfrak{L}, q \in \mathfrak{R}$. The elements of $\mathfrak{L}$ are called the *left maps* and the elements of $\mathfrak{R}$ the *right maps* of the factorization system.

Let now $\mathcal{M}$ and $\mathcal{N}$ be categories with both a factorization system. Then an adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is said to be *wfs-preserving* if the left adjoint L preserves the left maps, or equivalently if the right adjoint R respects the right maps.

As a key ingredient in the proof of our main result, the following lemma deserves to be stated fully and independently. It explains how to construct a weak factorization system on the total category of a Grothendieck bifibration, given that the basis and fibers all have one in a way that the adjunctions arising from the bifibration are wfs-preserving.

Lemma 2.6 (Stanculescu). *Let $\pi : \mathcal{F} \to \mathcal{C}$ be a Grothendieck bifibration with weak factorization systems $(\mathfrak{L}_C, \mathfrak{R}_C)$ on each fiber $\mathcal{F}_C$ and $(\mathfrak{L}, \mathfrak{R})$ on $\mathcal{C}$. If the adjoint pair $(u_!, u^*)$ is a wfs-adjunction for every morphism u of $\mathcal{C}$, then there is a weak factorization system $(\mathfrak{L}_{\mathcal{F}}, \mathfrak{R}_{\mathcal{F}})$ on $\mathcal{F}$ defined by*

$$\mathfrak{L}_{\mathcal{F}} = \{f : X \to Y \in \mathcal{F} : \pi(f) \in \mathfrak{L}, f_{\triangleright} \in \mathfrak{L}_{\pi Y}\},$$
$$\mathfrak{R}_{\mathcal{F}} = \{f : X \to Y \in \mathcal{F} : \pi(f) \in \mathfrak{R}, f^{\triangleleft} \in \mathfrak{R}_{\pi X}\}$$

For the proof in [Sta12, 2.2] is based on a different (yet equivalent) definition of weak factorization systems, here is a proof in our language for readers's convinience.

Proof. Let us begin with the easy part, which is the factorization property. For a map $f : X \to Y$ of $\mathcal{F}$, one gets a factorization $\pi(f) = r\ell$ in $\mathcal{C}$ with $\ell : \pi X \to C \in \mathfrak{L}$

and $r : C \to \pi Y \in \mathfrak{R}$. It induces a fiber morphism $\ell_! X \to r^* Y$ in $\mathcal{F}_C$ that we can in turn factor as $r_C \ell_C$ with $\ell_C \in \mathfrak{L}_C$ and $r_C \in \mathfrak{R}_C$.

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, dashed, "\tilde{\ell}"'] & \ell_! X \arrow[d, "\ell_C"] & \\
& \cdot \arrow[d, "r_C"'] \arrow[dr, dashed, "\tilde{r}"] & \\
& r^* Y \arrow[r] & Y
\end{tikzcd}$$

Then the wanted factorization of f is $\tilde{r}\tilde{\ell}$ where $\tilde{r}$ is the morphism of $\mathcal{F}$ such that $\pi(\tilde{r}) = r$ and $\tilde{r}^{\triangleleft} = r_C$, and $\tilde{\ell}$ the one such that $\pi(\tilde{\ell}) = \ell$ and $\tilde{\ell}_{\triangleright} = \ell_C$. This is summed up in the previous diagram.

Lifting properties follow the same kind of pattern: take the image by π and do the job in $\mathcal{C}$, then push and pull in $\mathcal{F}$ so that you end up in a fiber when everything goes smoothly. Take a map $j : X \to Y \in \mathfrak{L}_{\mathcal{F}}$ and let us show that it lift against elements of $\mathfrak{R}_{\mathcal{F}}$. Consider in $\mathcal{F}$ a commutative square with the map q on the right in $\mathfrak{R}_{\mathcal{F}}$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "j"'] & V \arrow[d, "q"] \\
Y \arrow[r, "g"'] & W
\end{tikzcd}$$

By definition, $\pi(j) \in \mathfrak{L}$ has the left lifting property against $\pi(q)$, hence a lift h:

$$\begin{tikzcd}
\pi X \arrow[r, "\pi(f)"] \arrow[d, "\pi(j)"'] & \pi V \arrow[d, "\pi(q)"] \\
\pi Y \arrow[r, "\pi(g)"'] \arrow[ur, "h"] & \pi W
\end{tikzcd}$$

Now filling the original square with $\tilde{h} : Y \to V$ above h is equivalent to fill the following induced solid square in $\mathcal{F}_{\pi Y}$:

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "j"'] \arrow[rrr, bend left, "f"] & \cdot \arrow[r] \arrow[d, "j_{\triangleright}"'] & \cdot \arrow[r] \arrow[d, "h^*(q^{\triangleleft})"] & V \arrow[d, "q^{\triangleleft}"'] \arrow[dr, "q"] & \\
& Y \arrow[r] \arrow[rrr, bend right, "g"'] & \cdot \arrow[r] & \cdot \arrow[r] & W
\end{tikzcd}$$

But $j_{\triangleright} \in \mathfrak{L}_{\pi Y}$, and h^* is the right adjoint of a wfs-preserving adjunction, hence maps the right map $q^{\triangleleft}$ of $\mathcal{F}_{\pi V}$ to a right map in $\mathcal{F}_{\pi Y}$: so there is such a filler.

Conversely, if $j : X \to Y$ in $\mathcal{F}$ has the left lifting property relatively to all maps of $\mathfrak{R}_{\mathcal{F}}$, then one has to show that it is in $\mathfrak{L}_{\mathcal{F}}$. Consider in $\mathcal{F}_{\pi Y}$ a commutative square as

$$\begin{tikzcd}
X \arrow[r, "\lambda_{\pi(j),X}"] \arrow[dr, "j"'] & \cdot \arrow[r, "f"] \arrow[d, "j_{\triangleright}"] & Y' \arrow[d, "q"] \\
& Y \arrow[r, "g"'] & Y''
\end{tikzcd} \qquad q \in \mathfrak{R}_{\pi Y}$$

Then, because q also is in $\mathfrak{R}_{\mathcal{F}}$, there is an $h : Y \to Y'$ such that $g = qh$ and $hj = f\lambda_{\pi(j),X}$. But then, $hj_\triangleright$ and f both are solution to the factorization problem of j through the cocartesian arrow $\lambda_{\pi(j),X}$, hence should be equal. Meaning h is a filler of the original square in the fiber $\mathcal{F}_{\pi Y}$. We conclude that $j_\triangleright$ is a left map in its fiber.
Now consider a commutative square in $\mathcal{C}$:

$$\begin{tikzcd} \pi X \arrow[r, "f"] \arrow[d, "\pi(j)"'] & C \arrow[d, "q"] \\ \pi Y \arrow[r, "g"'] & D \end{tikzcd} \qquad q \in \mathfrak{R}$$

It induced a commutative square in $\mathcal{F}$:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "j"'] & q^* g_! Y \arrow[d, "\kappa"] \\ Y \arrow[r] & g_! Y \end{tikzcd}$$

Now the arrow on the right is cartesian above a right map, hence is in $\mathfrak{R}_{\mathcal{F}}$ by definition. So j lift against it, giving us a filler $h : Y \to q^* g_! Y$ whose image $\pi(h) : Y \to C$ fills the square in $\mathcal{C}$. We conclude that $\pi(j)$ is a left map of $\mathcal{C}$. In the end, $j \in \mathfrak{L}_{\mathcal{F}}$ as we wanted to show.

Similarly, we can show that $\mathfrak{R}_{\mathcal{F}}$ is exactly the class of maps that have the right lifting property against all maps of $\mathfrak{L}_{\mathcal{F}}$. □

2.3 Intertwined weak factorization system and model categories

Quillen introduced model categories in [Qui67] as categories with sufficient structural analogies with the category of topological spaces so that a sensible notion of *homotopy between maps* can be provided. Not necessarily obvious at first sight are the redundancies of Quillen's definition. Even though intentionally important in the conceptual understanding of a model category, the extra checkings required can make a simple proof into a painful process. To ease things a little bit, this part is dedicated to extract the minimal definition of a model category at the cost of trading topological intuition for combinatorial comfort.

Recall the definition of a model structure.

Definition 2.7. A model structure on a category $\mathcal{M}$ is the data of three classes of maps $\mathfrak{C}$, $\mathfrak{W}$, $\mathfrak{F}$ such that:

(i) $\mathfrak{W}$ has the 2-out-of-3 property, i.e. if two elements among $\{f, g, gf\}$ are in W for composable morphisms f and g, then so is the third,

(ii) $(\mathfrak{C}, \mathfrak{W} \cap \mathfrak{F})$ and $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$ both are weak fatorization systems.

The morphism in $\mathfrak{W}$ are called the *weak equivalences*, those in $\mathfrak{C}$ the *cofibrations* and those in $\mathfrak{F}$ the *fibrations*. Given the role played by the two classes $\mathfrak{C} \cap W$ and $\mathfrak{F} \cap \mathfrak{W}$, we also give names to their elements: a fibration (respectively a cofibration) which is also a weak equivalence is called an *acyclic fibration* (respectively an *acyclic cofibration*).

REMARK 2.8. It is crucial for the rest of the document to remark that there is some redundancy in the previous definition: in a model structure, any two of the three classes $\mathfrak{C}, \mathfrak{W}, \mathfrak{F}$ determine the last one. Indeed, knowing of $\mathfrak{C}$ and $\mathfrak{W}$ gives us $\mathfrak{F}$ as the class of morphisms having the right lifting property relatively to every element of $\mathfrak{C} \cap \mathfrak{W}$. Dually, $\mathfrak{C}$ is given as the class of morphism having the left lifting property relatively to every element of $\mathfrak{W} \cap \mathfrak{F}$.

Finally, and it is the relevant case for the purpose of this article, the weak equivalences are exactly those morphisms that we can write qj where j is an acyclic cofibration and q is an acyclic fibration. The first inclusion $\{qj : q \in \mathfrak{F} \cap \mathfrak{W}, j \in \mathfrak{C} \cap \mathfrak{W}\} \subseteq \mathfrak{W}$ is a direct consequence of the 2-out-of-3 property. The converse inclusion is given by applying one of the two weak factorization systems and then using the 2-out-of-3 property: if $w \in \mathfrak{W}$, it is writable as $w = qj$ with $q \in \mathfrak{F}$, $j \in \mathfrak{C} \cap \mathfrak{W}$; but then w and j being a weak equivalence, q also is. Hence the conclusion.

Recall also the notion of morphisms between model structures: a *Quillen adjunction* between two model structures $\mathcal{M}$ and $\mathcal{N}$ is an adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ which is wfs-preserving for both the weak factorization system (acyclic cofibrations, fibrations) and the (cofibrations, acyclic fibrations) one.

Finally, to conclude those remainders about model structures, let us introduce some new vocabulary.

Definition 2.9 (Homotopically conservative functor)**.** A functor $F : \mathcal{M} \to \mathcal{N}$ between model structures is said to be *homotopically conservative* if it preserves and reflects weak equivalences.

REMARK 2.10. To get one's head around this terminlogy, let us make two observations:

(1) If $\mathcal{M}$ and $\mathcal{N}$ are endowed with the trivial model structure, in which weak equivalences are isomorphisms and cofibrations and fibrations are all morphisms, then the notion boils down to the usual conservative functors.

(2) Every functor $F : \mathcal{M} \to \mathcal{N}$ preserving weak equivalences induces a functor $\mathbf{Ho}(F) : \mathbf{Ho}(\mathcal{M}) \to \mathbf{Ho}(\mathcal{N})$. Given that weak equivalences are saturated in a model category, homotopically conservative functors are exactly those F such that $\mathbf{Ho}(F)$ is conservative as a usual functor.

Let us pursue with the following definition, apparently absent from literature.

Definition 2.11. A weak factorization system $(\mathfrak{L}_1, \mathfrak{R}_1)$ on a category $\mathcal{C}$ is *intertwined* with another $(\mathfrak{L}_2, \mathfrak{R}_2)$ on the same category when:

$$\mathfrak{L}_1 \subseteq \mathfrak{L}_2 \qquad \text{and} \qquad \mathfrak{R}_2 \subseteq \mathfrak{R}_1.$$

The careful reader will notice that the properties $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ and $\mathfrak{R}_2 \subseteq \mathfrak{R}_1$ are actually equivalent to each other, but the definition is more naturally stated in this way. A similar notion is formulated by Shulman for orthogonal factorization systems, in a blog post on the n-Category Café [Shu10] with a brief mention at the end of a version for weak factorization systems. This is the only appearance of such objects known to us.

The similarity with the weak factorization systems of a model category is immediately noticeable and in fact it goes further than a mere resemblance, as indicated in the following two results.

Proposition 2.12. *Let $(\mathfrak{L}_1, \mathfrak{R}_1)$ together with $(\mathfrak{L}_2, \mathfrak{R}_2)$ form intertwined weak factorization systems on a category $\mathcal{C}$. Denoting $\mathfrak{W} = \mathfrak{R}_2 \circ \mathfrak{L}_1$, the following class identities hold:*

$$\mathfrak{L}_1 = \mathfrak{W} \cap \mathfrak{L}_2, \qquad \mathfrak{R}_2 = \mathfrak{W} \cap \mathfrak{R}_1.$$

Proof. Let us prove the first identity only, as the second one is strictly dual. Suppose $f : A \to B \in \mathfrak{L}_1$, then $f \in \mathfrak{L}_2$ by the very definition of intertwined weak factorization systems, and $f = \mathbf{1}_B f \in \mathfrak{W}$, hence the first inclusion: $\mathfrak{L}_1 \subseteq \mathfrak{W} \cap \mathfrak{L}_2$.

Conversely, take $f \in \mathfrak{W} \cap \mathfrak{L}_2$. Then in particular there exists $j \in \mathfrak{L}_1$ and $q \in \mathfrak{R}_2$ such that $f = qj$. Put otherwise, the following square commutes:

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"'] & C \arrow[d, "q"] \\ B \arrow[r, equal] & B. \end{tikzcd}$$

But f is in $\mathfrak{L}_2$ and q is in $\mathfrak{R}_2 \subseteq \mathfrak{R}_1$, hence a lift $s : B \to C$ such that $qs = \mathbf{1}_B$ and $sf = j$. Now for any $p \in \mathfrak{R}_1$ and any commutative square

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "j"] \arrow[dd, bend right, "f"'] & D \arrow[dd, "p"] \\ C \arrow[d, "q"] & \\ B \arrow[r, "y"] & E \end{tikzcd}$$

there is a lift $h : C \to D$ taking advantage of j having the left lifting property against p. Then $hs : B \to D$ provides a lift showing that f has the left lifting property against p: indeed $phs = yqs = y$ and $hsf = hsqj = hj = x$. Having the left lifting property against any morphism in $\mathfrak{R}_1$, the morphism f ought to be in $\mathfrak{L}_1$, hence providing the reverse inclusion: $\mathfrak{W} \cap \mathfrak{L}_2 \subseteq \mathfrak{L}_1$. □

Corollary 2.13. *Let $(\mathfrak{L}_1, \mathfrak{R}_1)$ and $(\mathfrak{L}_2, \mathfrak{R}_2)$ form intertwined weak factorization systems on a category $\mathcal{M}$, and denote again $\mathfrak{W} = \mathfrak{R}_2 \circ \mathfrak{L}_1$. The category $\mathcal{M}$ has a model structure with weak equivalences $\mathfrak{W}$, fibrations $\mathfrak{R}_1$ and cofibrations $\mathfrak{L}_2$ if and only if $\mathfrak{W}$ has the 2-out-of-3 property.*

Of course in that case, we also get the class of acyclic cofibrations as $\mathfrak{L}_1$ and the class of acyclic fibrations as $\mathfrak{R}_2$.

So there it is: we shreded apart the notion of a model structure to the point that what remains is the pretty tame notion of intertwined factorization systems $(\mathfrak{L}_1, \mathfrak{R}_1)$ and $(\mathfrak{L}_2, \mathfrak{R}_2)$ such that $\mathfrak{R}_2 \circ \mathfrak{L}_1$ has the 2-out-of-3 property. But it has the neat advantage to be easily checkable, especially in the context of formal constructions, as it is the case in this paper. It also emphasizes the fact that Quillen adjunctions are really the right notion of morphisms for intertwined weak factorization systems and have *a priori* nothing to do with weak equivalences. We shall really put that on a stand because everything that follows in the main theorem can be restated with mere intertwined weak factorization systems in place of model structures and it still holds: in fact it represents the easy part of the theorem and all the hard core of the result resides in the 2-out-of-3 property, as usually encountered with model structures.

3 Quillen bifibrations

Recall from the introduction that a *Quillen bifibration* is a Grothendieck bifibration $p : \mathcal{E} \to \mathcal{B}$ between categories with model structures such that:

(i) the functor p is both a left and right Quillen functor,

(ii) the model structure on $\mathcal{E}$ restricts to a model structure on the fiber $\mathcal{E}_A$, for every object A of the category $\mathcal{B}$.

In this section, we show that in a Quillen bifibration the model structure on the basis $\mathcal{B}$ and on every fiber $\mathcal{E}_A$ determines the original model structure on the total category $\mathcal{E}$. In the remainder of this section, we fix a Quillen bifibration $p : \mathcal{E} \to \mathcal{B}$.

Lemma 3.1. *For every morphism $u : A \to B$ in $\mathcal{B}$, the adjunction $u_! : \mathcal{E}_A \rightleftarrows \mathcal{E}_B : u^*$ is a Quillen adjunction.*

Proof. Let $f : X \to Y$ be a cofibration in the fiber $\mathcal{E}_A$. We want to show that the morphism $u_!(f)$ of $\mathcal{E}_B$ is a cofibration. Take an arbitrary acyclic fibration $q : W \to Z$ in $\mathcal{E}_B$ and a commutative square in that fiber:

$$\begin{tikzcd}
u_!X \arrow[r, "g"] \arrow[d, "u_!(f)"'] & W \arrow[d, "q"] \\
u_!Y \arrow[r, "g'"'] & Z
\end{tikzcd}$$

We need to find a lift $h : u_!Y \to W$ making the diagram commutes, i.e. such that $qh = g'$ and $hu_!(f) = g$. Let us begin by precomposing with the square defining $u_!(f)$:

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[d, "f"'] & u_!X \arrow[r, "g"] \arrow[d, "u_!(f)"'] & W \arrow[d, "q"] \\
Y \arrow[r, "\lambda"'] & u_!Y \arrow[r, "g'"'] & Z
\end{tikzcd}$$

As a cofibration, f has the left lifting property against q, providing a map $k : Y \to W$ that makes the following commute:

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[d, "f"'] & u_!X \arrow[r, "g"] \arrow[d, "u_!(f)"'] & W \arrow[d, "q"] \\
Y \arrow[r, "\lambda"'] \arrow[urr, "k"] & u_!Y \arrow[r, "g'"'] & Z
\end{tikzcd}$$

Now we use the cocartesian property of $\lambda_{u,Y} : Y \to u_!Y$ on k, to find a map $h : u_!Y \to W$ above the identity $\mathbf{1}_B$ such that $h\lambda_{u,Y} = k$. All it remains to show is that $qh = g'$ and $hu_!(f) = g$. Notice that both qh and g' answer to the problem of finding a map $x : u_!Y \to Z$ above $\mathbf{1}_B$ such that $x\lambda_{u,Y} = qk$: hence, by the unicity condition in the cocartesian property of $\lambda_{u,Y}$, they must be equal. Similarly, $h \circ u_!(f)$ and g solve the problem of finding $x : u_!X \to W$ above $\mathbf{1}_B$ such that $x\lambda_{u,X} = kf$: the cocartesian property of $\lambda_{u,X}$ allows us to conclude that they are equal. In the end, $u_!(f)$ has the left lifting property against every acyclic fibration of $\mathcal{E}_B$, so it is a cofibration. We prove dually that the image u^*f of a fibration f in $\mathcal{E}_B$ is a fibration of the fiber $\mathcal{E}_A$. $\square$

Lemma 3.2. *A cocartesian morphism in $\mathcal{E}$ above a (acyclic) cofibration of $\mathcal{B}$ is a (acyclic) cofibration.*

Proof. Let $f : X \to Y$ be cocartesian above a cofibration $u : A \to B$ in $\mathcal{B}$. Given a commutative square of $\mathcal{E}$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & W \arrow[d, "q"] \\ Y \arrow[r, "g'"'] & Z \end{tikzcd} \tag{8}$$

with q an acyclic fibration, we can take its image in $\mathcal{B}$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & pW \arrow[d, "p(q)"] \\ B \arrow[r] & pZ \end{tikzcd}$$

Since u is a cofibration and $p(q)$ an acyclic fibration, there exists a morphism $h : B \to pW$ making the expected diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & pW \arrow[d, "p(q)"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & pZ \end{tikzcd}$$

Because f is cocartesian, we know that there exists a (unique) map $\tilde{h} : Y \to W$ above h making the diagram below commute:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & W \\ Y \arrow[ur, "\tilde{h}"'] & \end{tikzcd}$$

For the morphism $\tilde{h}$ to be a lift in the first commutative square (8), there remains to show that $q\tilde{h} = g'$. Because $\tilde{h}$ is above h and $p(q)h = p(g')$, we have that the composite $q\tilde{h}$ is above g'. Moreover $q\tilde{h}f = qg = g'f$. Using the uniqueness property in the universal definition of cocartesian maps, we deduce $q\tilde{h} = g'$. We have just shown that the cocartesian morphism f is weakly orthogonal to every acyclic fibration, and we thus conclude that f is a cofibration. The case of cocartesian morphisms above acyclic cofibrations is treated in a similar way. □

The same argument establishes the dual statement:

Lemma 3.3. *A cartesian morphism in $\mathcal{E}$ above a (acyclic) fibration of $\mathcal{B}$ is a (acyclic) fibration.*

Proposition 3.4. *A map $f : X \to Y$ in $\mathcal{E}$ is a (acyclic) cofibration if and only if $p(f)$ is a (acyclic) cofibration in $\mathcal{E}$ and $f_\triangleright$ is a (acyclic) cofibration in the fiber $\mathcal{E}_{pY}$.*

Proof. A direction of the equivalence is easy: if $p(f) = u : A \to B$ is a cofibration, then so is the cocartesian morphism $\lambda_{u,X}$ above it by lemma 3.2; if moreover $f_\triangleright$ is a cofibration in the fiber $\mathcal{E}_B$, then $f = f_\triangleright \lambda_{u,X}$ is a composite of cofibration, hence it is a cofibration itself.

Conversely, suppose that $f : X \to Y$ is a cofibration in $\mathcal{E}$. Then surely $p(f) = u : A \to B$ also is a cofibration in $\mathcal{B}$, since p is a left Quillen functor. Now we want to show that $f_{\triangleright} : u_!X \to Y$ is a cofibration in the fiber $\mathcal{E}_B$. Consider a commutative square in that fiber

$$\begin{tikzcd} u_!X \arrow[r, "g"] \arrow[d, "f_{\triangleright}"'] & W \arrow[d, "q"] \\ Y \arrow[r, "g'"'] & Z \end{tikzcd}$$

where q is an acyclic fibration of the fiber $\mathcal{E}_B$, and g, g' are arbitrary morphisms in that fiber. Since f itself is a cofibration in $\mathcal{E}$, we know that there exists a lift $h : Y \to W$ for the outer square (with four sides f, q, $g\lambda_{u,X}$ and g') of the following diagram:

$$\begin{tikzcd} \lambda_{u,X}X \arrow[r, "\lambda_{u,X}"] \arrow[dr, "f"'] & u_!X \arrow[r, "g"] \arrow[d, "f_{\triangleright}"] & W \arrow[d, "q"] \\ & Y \arrow[r, "g'"'] \arrow[ur, "h"] & Z \end{tikzcd}$$

Now, there remains to show that $hf_{\triangleright} = g$. We already know that $g\lambda_{u,X} = hf_{\triangleright}\lambda_{u,X}$, and taking advantage of the fact that the morphism $\lambda_{u,X}$ is cocartesian, we only need to show that $p(g) = p(hf_{\triangleright})$. Since g and $f_{\triangleright}$ are fiber morphisms, it means we need to show that h also. This follows from the fact that $qh = g'$ and that q and g' are fiber morphisms. ☐

In the same way, we get the dual statement:

Proposition 3.5. *A map $f : X \to Y$ in $\mathcal{E}$ is a (acyclic) cofibration if and only if $p(f)$ is a (acyclic) cofibration in $\mathcal{E}$ and $f_{\triangleright}$ is a (acyclic) cofibration in the fiber $\mathcal{E}_{pY}$.*

In particular, this means that the model structure on the total category $\mathcal{E}$ is entirely determined by the model structures on the basis $\mathcal{B}$ and on each fiber $\mathcal{E}_B$ of the bifibration. As these characterizations turn out to be important for what follows, we shall name them.

Definition 3.6. Let $p : \mathcal{E} \to \mathcal{B}$ be a Grothendieck bifibration such that its basis $\mathcal{B}$ and each fiber $\mathcal{E}_A$ ($A \in \mathcal{B}$) have a model structure.

- a *total cofibration* is a morphism $f : X \to Y$ of $\mathcal{E}$ above a cofibration $u : A \to B$ of $\mathcal{B}$ such that $f_{\triangleright}$ is a cofibration in the fiber $\mathcal{E}_B$,
- a *total fibration* is a morphism $f : X \to Y$ of $\mathcal{E}$ above a fibration $u : A \to B$ of $\mathcal{B}$ such that $f^{\triangleleft}$ is a fibration in the fiber $\mathcal{E}_A$,
- a *total acyclic cofibration* is a morphism $f : X \to Y$ of $\mathcal{E}$ above an acyclic cofibration $u : A \to B$ of $\mathcal{B}$ such that $f_{\triangleright}$ is an acyclic cofibration in the fiber $\mathcal{E}_B$,
- a *total acyclic fibration* is a morphism $f : X \to Y$ of $\mathcal{E}$ above an acyclic fibration $u : A \to B$ of $\mathcal{B}$ such that $f^{\triangleleft}$ is an acyclic fibration in the fiber $\mathcal{E}_A$.

Using this terminology, the two propositions 3.4 and 3.5 just established come together as: the cofibrations, fibrations, acyclic cofibrations and acyclic fibrations of a Quillen bifibration are necessarily the total ones. Note also that the definitions of total cofibration and total fibration given in definition 3.6 coincides with the definition given in the introduction.

We end this section by giving simple examples of Quillen bifibrations. They should serve as both a motivation and a guide for the reader to navigate into the following definitions and proofs: it surely has worked that way for us authors.

EXAMPLE(S) 3.7.

(1) One of the simplest instances of a Grothendieck bifibration other than the identity functor, is a projection from a product:

$$p : \mathcal{M} \times \mathcal{B} \to \mathcal{B}$$

Cartesian and cocartesian morphisms coincide and are those of the form $(\mathbf{1}_M, u)$ for $M \in \mathcal{M}$ and u a morphism of $\mathcal{B}$. In particular, one have $(f,u)^{\triangleleft} = (f, \mathbf{1}_A)$ and $(f,u)_{\triangleright} = (f, \mathbf{1}_B)$ for any $u : A \to B$ in $\mathcal{B}$ and any f in $\mathcal{M}$.

If $\mathcal{B}$ and $\mathcal{M}$ are model categories, each fiber $p_A \simeq \mathcal{M}$ inherits a model structure from $\mathcal{M}$ and the total fibrations and cofibrations coincide precisely with the one of the usual model structure on the product $\mathcal{M} \times \mathcal{B}$.

(2) For a category $\mathcal{B}$, one can consider the codomain functor:

$$\mathrm{cod} : [\mathbf{2}, \mathcal{B}] \to \mathcal{B},\ (X \xrightarrow{f} A) \mapsto A$$

Cocartesian morphisms above u relatively to cod are those commutative square of the form

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"] & X \arrow[d, "uf"] \\ A \arrow[r, "u"] & B \end{tikzcd}$$

whereas cartesian morphisms above u are the pullback squares along u. Hence cod is a Grothendieck bifibration whenever $\mathcal{B}$ admits pullbacks.

If moreover $\mathcal{B}$ is a model category, then each fiber $\mathrm{cod}_A \simeq \mathcal{B}_{/A}$ inherits a model structure (namely an arrow is a fibration or a cofibration if it is such as an arrow of $\mathcal{B}$), and the total fibrations and cofibrations coincide with the one in the injective model structure on $[\mathbf{2}, \mathcal{B}]$: i.e. a cofibration is a commutative square with the top and bottom arrows being cofibrations in $\mathcal{B}$, whereas fibrations are those commutative squares

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, dashed, "h"] \arrow[dd, "f"'] & & Y \arrow[dd, "g"] \\ & A \times_{u,g} Y \arrow[ur] \arrow[dl] & \\ A \arrow[rr, "u"] & & B \end{tikzcd}$$

where both u and h are fibrations in $\mathcal{B}$.

(3) Similarly, the total fibrations and cofibrations of the Grothendieck bifibration $\mathrm{dom} : [\mathbf{2}, \mathcal{B}] \to \mathcal{B}$ over a model category $\mathcal{B}$ are exactly those of the projective model structure on $[\mathbf{2}, \mathcal{B}]$.

(4) In both [Sta12] and [HP15], the authors prove a theorem similar to our, putting a model structure on the total category of a Grothendieck bifibration under specific hypothesis. In both case, fibrations and cofibrations of this model structure end up being the total ones. The following theorem encompasses in particular this two results.

4 A Grothendieck construction for Quillen bifibrations

Now we have the tools to move on to the main goal of this paper, which is to turn a Grothendieck bifibration $p : \mathcal{E} \to \mathcal{B}$ into a Quillen bifibration whenever both the basis category $\mathcal{B}$ and every fiber $\mathcal{E}_A$ ($A \in \mathcal{B}$) admit model structures in such a way that all the pairs of adjoint push and pull functors between fibers are "homotopically well-behaved". To be more precise, we now suppose $\mathcal{B}$ to be equipped with a model structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$, and each fiber $\mathcal{E}_A$ ($A \in \mathcal{B}$) to be equipped with a model structure $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$. We also make the following **fundamental assumption**:

$$\text{For all } u \text{ in } \mathcal{B}, \text{ the adjoint pair } (u_!, u^*) \text{ is a Quillen adjunction.} \tag{Q}$$

We defined in definition 3.6 notions of total cofibrations and total fibrations, as well as their acyclic counterparts. These are reminiscent of what happens with Quillen bifibrations, but they can be defined for any Grothendieck bifibration whose basis and fibers have model structures. We must insist that in that framework, *total cofibrations* and *total fibrations* are only names, and by no means are they giving the total category $\mathcal{E}$ a model structure. Indeed, the goal of this section, and to some extent even the goal of this paper, is to provide a complete characterization, under hypothesis (Q), of the Grothendieck bifibrations $p : \mathcal{E} \to \mathcal{B}$ for which the total cofibrations and total fibrations make p into a Quillen bifibration. For the rest of this section, we shall denote $\mathfrak{C}_\mathcal{E}$, $\mathfrak{F}_\mathcal{E}$, $\mathfrak{C}_\mathcal{E}^\sim$ and $\mathfrak{F}_\mathcal{E}^\sim$ for the respective classes of total cofibrations, total fibrations, total acyclic cofibrations, and total acyclic fibrations, that is:

$$\begin{aligned}
\mathfrak{C}_\mathcal{E} &= \{f : X \to Y \in \mathcal{E} : p(f) \in \mathfrak{C}, f_\triangleright \in \mathfrak{C}_{pY}\},\\
\mathfrak{F}_\mathcal{E} &= \{f : X \to Y \in \mathcal{E} : p(f) \in \mathfrak{F}, f^\triangleleft \in \mathfrak{F}_{pX}\},\\
\mathfrak{C}_\mathcal{E}^\sim &= \{f : X \to Y \in \mathcal{E} : p(f) \in \mathfrak{W} \cap \mathfrak{C}, f_\triangleright \in \mathfrak{W}_{pY} \cap \mathfrak{C}_{pY}\},\\
\mathfrak{F}_\mathcal{E}^\sim &= \{f : X \to Y \in \mathcal{E} : p(f) \in \mathfrak{W} \cap \mathfrak{F}, f^\triangleleft \in \mathfrak{W}_{pX} \cap \mathfrak{F}_{pX}\}
\end{aligned}$$

4.1 Main theorem

In order to state the theorem correctly, we will need some vocabulary. Recall that the *mate* $\mu\colon u'_! v^* \to v'^* u_!$ associated to a commutative square of $\mathcal{B}$

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "u'"'] & C \arrow[d, "u"] \\
C' \arrow[r, "v'"'] & B
\end{tikzcd}$$

is the natural transformation constructed at point $Z \in \mathcal{E}_C$ in two steps as follow: the composite

$$v^* Z \to Z \to u_! Z$$

which is above uv, factors through the cartesian arrow $\rho_{v',u_!Z} : v'^* u_! Z \to u_! Z$ (because $v'u' = uv$) into a morphism $v^*Z \to v'^* u_! Z$ above u', which in turn factors through the cocartesian arrow $\lambda_{u',v_!Z} : u'^* v_! Z \to v'^* u_! Z$ giving rise to μ_Z, as summarized in the diagram below.

$$\begin{tikzcd}
v^*Z \arrow[rr, "\rho"] \arrow[d, "\lambda"] & & Z \arrow[dd, "\lambda"] \\
u'_! v^* Z \arrow[dr, dashed, "\mu_Z"] & & \\
& v'^* u_! Z \arrow[r, "\rho"] & u_! Z
\end{tikzcd}$$

Definition 4.1. A commutative square of $\mathcal{B}$ is said to satisfy the *homotopical Beck-Chevalley condition* if its mate is pointwise a weak equivalence.

Consider then the following properties on the Grothendieck bifibration p:

Every commutative square of $\mathcal{B}$ of the form

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "u'"'] & C \arrow[d, "u"] \\
C' \arrow[r, "v'"'] & B
\end{tikzcd} \qquad \begin{aligned} &u, u' \in \mathfrak{C} \cap \mathfrak{W}, \\ &v, v' \in \mathfrak{F} \cap \mathfrak{W} \end{aligned} \tag{hBC}$$

satisfies the homotopical Beck-Chevalley condition.

and

The functors $u_!$ and v^* are homotopically conservative whenever u is an acyclic cofibration and v an acyclic fibration. (hCon)

The theorem states that this is exactly what it takes to make the names "total cofibrations" and "total fibrations" legitimate, and to turn $p : \mathcal{E} \to \mathcal{B}$ into a Quillen bifibration.

Theorem 4.2. *Under hypothesis* (Q)*, the total category $\mathcal{E}$ admits a model structure with $\mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}$ as cofibrations and fibrations respectively if and only if properties* (hBC) *and* (hCon) *are satisfied.*

In that case, the functor $p : \mathcal{E} \to \mathcal{B}$ is a Quillen bifibration.

The proof begin with a very candid remark that we promote as a proposition because we shall use it several times in the rest of the proof.

Proposition 4.3. $(\mathfrak{C}^{\sim}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ *and* $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{F}^{\sim}_{\mathcal{E}})$ *are intertwined weak factorization systems.*

Proof. Obviously $\mathfrak{C}^{\sim}_{\mathcal{E}} \subseteq \mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}^{\sim}_{\mathcal{E}} \subseteq \mathfrak{F}_{\mathcal{E}}$. Independently, a direct application of lemma 2.6 shows that $(\mathfrak{C}^{\sim}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ and $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{F}^{\sim}_{\mathcal{E}})$ are both weak factorization systems on $\mathcal{E}$. □

The strategy to prove theorem 4.2 then goes as follow:

- first we will show the necessity of conditions (hBC) and (hCon): if $\mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}$ are the cofibrations and fibrations of a model structure on $\mathcal{E}$, then hypothesis (hBC) and (hCon) are met,

- next, the harder part is the sufficiency: because of proposition 4.3, it is enough to show that the induced class $\mathfrak{W}_{\mathcal{E}} = \mathfrak{F}_{\mathcal{E}}^{\sim} \circ \mathfrak{C}_{\mathcal{E}}^{\sim}$ of *total weak equivalences* has the 2-out-of-3 property to conclude through corollary 2.13.

4.2 Proof, part I: necessity

In all this section, we suppose that $\mathfrak{C}_{\mathcal{E}}$ and $\mathfrak{F}_{\mathcal{E}}$ provide respectively the cofibrations and fibrations of a model structure on the total category $\mathcal{E}$. We will denote $\mathfrak{W}_{\mathcal{E}}$ the corresponding class of weak equivalences.

First, we prove a technical lemma, directly following from proposition 4.3, that will be extensively used in the following. Informally, it states that the name given to the members of $\mathfrak{C}_{\mathcal{E}}^{\sim}$ and $\mathfrak{F}_{\mathcal{E}}^{\sim}$ are not foolish.

Lemma 4.4. $\mathfrak{C}_{\mathcal{E}}^{\sim} = \mathfrak{W}_{\mathcal{E}} \cap \mathfrak{C}_{\mathcal{E}}$ *and* $\mathfrak{F}_{\mathcal{E}}^{\sim} = \mathfrak{W}_{\mathcal{E}} \cap \mathfrak{F}_{\mathcal{E}}$.

Proof. By proposition 4.3, we know that both $(\mathfrak{C}_{\mathcal{E}}^{\sim}, \mathfrak{F}_{\mathcal{E}})$ and $(\mathfrak{W}_{\mathcal{E}} \cap \mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}})$ are weak factorization systems with the same class of right maps, hence their class of left maps should coincide. Similarly the weak factorization systems $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{F}_{\mathcal{E}}^{\sim})$ and $(\mathfrak{C}_{\mathcal{E}}, \mathfrak{W}_{\mathcal{E}} \cap \mathfrak{F}_{\mathcal{E}})$ have the same class of left maps, hence their class of right maps coincide. □

Corollary 4.5. *For any object A of $\mathcal{B}$, the inclusion functor $\mathcal{E}_A \to \mathcal{E}$ is homotopically conservative.*

Proof. The preservation of weak equivalences comes from the fact that acyclic cofibrations and acyclic fibrations of $\mathcal{E}_A$ are elements of $\mathfrak{C}_{\mathcal{E}}^{\sim}$ and $\mathfrak{F}_{\mathcal{E}}^{\sim}$ respectively. Thus, by lemma 4.4, they are elements of $\mathfrak{W}_{\mathcal{E}}$.

Conversely, suppose that f is a map of $\mathcal{E}_A$ which is a weak equivalence of $\mathcal{E}$. We want to show that f is a weak equivalence of the fiber $\mathcal{E}_A$. The map f factors in the fiber $\mathcal{E}_A$ as $f = qj$ where $j \in \mathfrak{C}_A \cap \mathfrak{W}_A$ and $q \in \mathfrak{F}_A$. We just need to show that $q \in \mathfrak{W}_A$. By lemma 4.4, j is also a weak equivalence of $\mathcal{E}$. By the 2-out-of-3 property of $\mathfrak{W}_{\mathcal{E}}$, the map q is a weak equivalence of $\mathcal{E}$. As a fibration of $\mathcal{E}_A$, q is also a fibration of $\mathcal{E}$. This establishes that q is an acyclic fibration of $\mathcal{E}$. By lemma 4.4, q is thus an element of $\mathfrak{F}_{\mathcal{E}}^{\sim}$. This conludes the proof that $q = q^{\triangleleft}$ is an acyclic fibration, and thus a weak equivalence, in the fiber $\mathcal{E}_A$. □

Proposition 4.6 (Property (hCon))**.** *If $j : A \to B$ in an acyclic cofibration in $\mathcal{B}$, then $j_! : \mathcal{E}_A \to \mathcal{E}_B$ is homotopically conservative.*

If $q : A \to B$ in an acyclic fibration in $\mathcal{B}$, then $q^* : \mathcal{E}_B \to \mathcal{E}_A$ is homotopically conservative.

Proof. We only prove the first part of the proposition, as the second one is dual. Recall that the image $j_!(f)$ of a map $f : X \to Y$ of $\mathcal{E}_A$ is computed as the unique morphism of $\mathcal{E}_B$ making the following square commute:

$$\begin{array}{ccc} X & \longrightarrow & j_!X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j_!(f)} \\ Y & \longrightarrow & j_!Y \end{array}$$

The horizontal morphisms in the diagram are cocartesian above the acyclic cofibration j. As such they are elements $\mathfrak{C}_{\mathcal{E}}^{\sim}$, and thus weak equivalence in $\mathcal{E}$ by lemma 4.4.

By the 2-out-of-3 property of $\mathfrak{W}_\mathcal{E}$, f is a weak equivalence in $\mathcal{E}$ if and only if $j_!(f)$ is one also in $\mathcal{E}$. Corollary 4.5 allows then to conclude: f is a weak equivalence in the fiber $\mathcal{E}_A$ if and only if $j_!(f)$ is one in the fiber $\mathcal{E}_B$. ☐

Proposition 4.7 (Property (hBC)). *Commutative squares of $\mathcal{B}$ of the form*

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u'"'] & C \arrow[d, "u"] \\ C' \arrow[r, "v'"'] & B \end{tikzcd} \qquad u, u' \in \mathfrak{C} \cap \mathfrak{W} \quad v, v' \in \mathfrak{F} \cap \mathfrak{W}$$

satisfy the homotopical Beck-Chevalley condition.

Proof. Recall that for such a square in $\mathcal{B}$, the component of the mate $\mu : u'_! v^* \to v'^* u_!$ at $Z \in \mathcal{E}_C$ is defined as the unique map of $\mathcal{E}_{Z'}$ making the following diagram commute:

$$\begin{tikzcd} v^*Z \arrow[rr, "\rho"] \arrow[d, "\lambda"] & & Z \arrow[dd, "\lambda"] \\ u'_! v^* Z \arrow[dr, "\mu_Z"] & & \\ & v'^* u_! Z \arrow[r, "\rho"] & u_! Z \end{tikzcd}$$

Arrows labelled ρ and λ are respectively cartesian above acyclic fibrations and cocartesian above acyclic cofibrations, hence weak equivalences of $\mathcal{E}$ by lemma 4.4. By applying the 2-out-of-3 property of $\mathfrak{W}_\mathcal{E}$ three times in a row, we conclude that the fiber map μ_Z is a weak equivalence of $\mathcal{E}$, hence also of $\mathcal{E}_{C'}$ by corollary 4.5. ☐

4.3 Proof, part II: sufficiency

We have established the necessity of (hBC) and (hCon) in theorem 4.2. We now prove the sufficiency of these conditions. This is the hard part of the proof. Recall that every fiber $\mathcal{E}_A$ of the Grothendieck bifibration $p : \mathcal{E} \to \mathcal{B}$ is equipped with a model structure in such a way that (Q) is satisfied. From now on, we make the additional assumptions that (hBC) and (hCon) are satisfied.

We will use the notation $\mathfrak{W}_\mathcal{E} = \mathfrak{F}_\mathcal{E}^\sim \circ \mathfrak{C}_\mathcal{E}^\sim$ the class of maps that can be written as a total acyclic cofibration postcomposed with a total acyclic fibration. The overall goal of this section is to prove that

Claim. $(\mathfrak{C}_\mathcal{E}, \mathfrak{W}_\mathcal{E}, \mathfrak{F}_\mathcal{E})$ *defines a model structure on the total category* $\mathcal{E}$.

By proposition 4.3, we already know that $(\mathfrak{C}_\mathcal{E}^\sim, \mathfrak{F}_\mathcal{E})$ and $(\mathfrak{C}_\mathcal{E}, \mathfrak{F}_\mathcal{E}^\sim)$ are intertwined weak factorization systems. From this follows that, by corollary 2.13, we only need to show that the class $\mathfrak{W}_\mathcal{E}$ of *total weak equivalences* satisfies the 2-out-of-3 property.

A first step is to get a better understanding of the total weak equivalences. For $f : X \to Y$ in $\mathcal{E}$ such that $p(f) = vu$ for two composable morphisms $u : pX \to C$ and $v : C \to pY$ of $\mathcal{B}$, there is a unique morphism inside the fiber $\mathcal{E}_C$

$${}_u f^v : u_! X \to v^* Y$$

such that $f = \rho_{v,Y} \circ {}_u f^v \circ \lambda_{u,X}$. This morphism ${}_u f^v$ can be constructed as $k^{\triangleleft}$ where k is the unique morphism above v factorizing f through $\lambda_{u,X}$; or equivalently as $\ell_{\triangleright}$

where ℓ is the unique morphism above u factorizing f through $\rho_{v,Y}$. This is summed up in the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[dr, dashed, "\ell"'] & u_!X \arrow[d, "\ell_\triangleright = {}_u f^v = k^\triangleleft"] \arrow[dr, dashed, "k"] & \\
& v^*Y \arrow[r, "\rho"'] & Y
\end{tikzcd}$$

Notice that, in particular, a morphism f of $\mathfrak{W}_{\mathcal{E}}$ is exactly a morphism of $\mathcal{E}$ for which **there exists** a factorization $p(f) = qj$ with $j \in \mathfrak{W} \cap \mathfrak{C}$ and $q \in \mathfrak{W} \cap \mathfrak{F}$ such that ${}_jf^q$ is a weak equivalence in the corresponding fiber. We shall strive to show that, under our hypothesis (hCon) and (hBC), a morphism f of $\mathfrak{W}_{\mathcal{E}}$ satisfies the same property that ${}_jf^q$ is a weak equivalence **for all** such factorization $p(f) = qj$. This is the contain of proposition 4.10. We start by showing the property in the particular case where $p(f)$ is an acyclic cofibration (lemma 4.8) or an acyclic fibration (lemma 4.9).

Lemma 4.8. *Suppose that $f : X \to Y$ is a morphism of $\mathcal{E}$ such that $p(f)$ is an acyclic cofibration in $\mathcal{B}$. If $p(f) = qj$ with $q \in \mathfrak{W} \cap \mathfrak{C}$ and $j \in \mathfrak{W} \cap \mathfrak{F}$, then $f_\triangleright$ is a weak equivalence if and only if ${}_jf^q$ is a weak equivalence.*

Proof. Since $p(f) = qj$, lemma 2.3 provides an isomorphism ϕ in the fiber $\mathcal{E}_{pY}$ such that $f_\triangleright = \tilde{f}_\triangleright \phi$, where $\tilde{f}_\triangleright$ is the morphism obtained by pushing in two steps:

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[rrr, bend left, "\lambda"] \arrow[drr, "f"'] & j_!X \arrow[r, "\lambda"] & q_!j_!X \arrow[d, "\tilde{f}_\triangleright"] & (qj)_!X \arrow[l, "\phi"', "\simeq"] \arrow[dl, "f_\triangleright"] \\
& & Y &
\end{tikzcd}$$

By definition, ${}_jf^q$ is the image of $\tilde{f}_\triangleright$ under the natural bijection $\mathcal{E}_{pY}(q_!j_!X, Y) \xrightarrow{\simeq} \mathcal{E}_{pX}(j_!X, q^*Y)$. So it can be written ${}_jf^q = q^*(\tilde{f}_\triangleright) \circ \eta_{j_!X}$ using the unit η of the adjunction $(q_!, q^*)$. We can now complete the previous diagram as follow:

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[rrr, bend left, "\lambda"] \arrow[drr, gray, "f"'] & j_!X \arrow[r, "\lambda"] \arrow[d, "\eta"] \arrow[dd, bend right=60, "{}_jf^q"'] & q_!j_!X \arrow[d, "\tilde{f}_\triangleright"] & (qj)_!X \arrow[l, "\phi"', "\simeq"] \arrow[dl, "f_\triangleright"] \\
& q^*q_!j_!X \arrow[ur, "\rho"] \arrow[d, "q^*(\tilde{f}_\triangleright)"] & Y & \\
& q^*Y \arrow[ur, "\rho"'] & &
\end{tikzcd}$$

Proving that $\eta_{j_!X}$ is a weak equivalence is then enough to conclude: in that case ${}_jf^q$ is an weak equivalence if and only if $q^*(\tilde{f}_\triangleright)$ is such by the two-of-three property ; $q^*(\tilde{f}_\triangleright)$ is a weak equivalence if and only if $\tilde{f}_\triangleright$ is a weak equivalence in $\mathcal{E}_{pY}$ by (hCon) ; and finally $\tilde{f}_\triangleright$ is a weak equivalence if an only if $f_\triangleright$ is such because they are isomorphic as arrows in $\mathcal{E}_{pY}$.

So it remains to show that $\eta_{j_!X}$ is a weak equivalence in its fiber. Since $p(f) = qj$, the following square commutes in $\mathcal{B}$:

$$\begin{tikzcd} pX \arrow[r, equal, "\mathbf{1}_{pX}"] \arrow[d, "j"'] & pX \arrow[d, "qj"] \\ C \arrow[r, "q"'] & pY \end{tikzcd}$$

This is a square of the correct form to apply (hBC): hence the associated mate at component X

$$\mu_X : j_!(\mathbf{1}_{pX})^*X \to q^*(qj)_!X$$

is a weak equivalence in the fiber $\mathcal{E}_C$. Corollary 2.5 ensures that μ_X is isomorphic as arrow of $\mathcal{E}_C$ to the unique fiber morphism that factors $\rho_{q,q_!j_!X}$ through $\lambda_{q,j_!X}$:

$$\begin{tikzcd} j_!X \arrow[r, "\lambda"] \arrow[d] & q_!j_!X \\ q^*q_!j_!X \arrow[ur, "\rho"'] & \end{tikzcd}$$

This is exactly the definition of the unit η at $j_!X$. Isomorphic morphisms being weak equivalences together, $\eta_{j_!X}$ is also acyclic in $\mathcal{E}_C$. ☐

Of course, one gets the dual lemma by dualizing the proof that we let for the reader to write down.

Lemma 4.9. *Let $f : X \to Y$ a morphism of $\mathcal{E}$ such that $p(f)$ is an acyclic fibration in $\mathcal{B}$. If $p(f) = qj$ with $q \in \mathfrak{W} \cap \mathfrak{C}$ and $j \in \mathfrak{W} \cap \mathfrak{F}$, then $f^{\triangleleft}$ is a weak equivalence if and only if ${}_jf^q$ is a weak equivalence.*

We shall now prove the key proposition of this section.

Proposition 4.10. *Let $f : X \to Y$ in $\mathcal{E}$. If $p(f) = qj = q'j'$ for some $j, j \in \mathfrak{W} \cap \mathfrak{C}$ and $q, q' \in \mathfrak{W} \cap \mathfrak{F}$, then ${}_jf^q$ is a weak equivalence if and only if ${}_{j'}f^{q'}$ is a weak equivalence.*

Proof. By hypothesis the following square commutes in $\mathcal{B}$:

$$\begin{tikzcd} pX \arrow[r, "j'"] \arrow[d, "j"'] & C' \arrow[d, "q'"] \\ C \arrow[r, "q"'] & pY \end{tikzcd}$$

Since j is an acyclic cofibration and q' a (acyclic) fibration, there is a filler $h : C \to C'$ of the previous square, that is a weak equivalence by the 2-out-of-3 property. Hence it can be factored $h = h_f h_c$ as an acyclic cofibration followed by an acyclic fibration in $\mathcal{B}$. Write $j'' = h_c j$ and $q'' = q'h_f$ which are respectively an acyclic cofibration and an acyclic fibration as composite of such, and produce a new factorization of

$p(f) = q''j''$.

$$\begin{tikzcd}
pX \arrow[rr, "j'"] \arrow[dd, "j"'] \arrow[dr, "j''"] & & C' \arrow[dd, "q'"] \\
& C'' \arrow[ur, "h_f"] \arrow[dr, "q''"] & \\
C \arrow[ur, "h_c"] \arrow[rr, "q"'] & & pY
\end{tikzcd}$$

Write r for the composite ${}_{j'}f^{q'} \circ \lambda_{X,j'} : X \to j'_!X \to q'^*Y$. Then r is above the acyclic cofibration $j' = h_f j''$ and lemma 4.8 can be applied: $r_{\triangleright}$ is a weak equivalence in $\mathcal{E}_{C'}$ if and only if ${}_{j''}r^{h_f} : j''_!X \to (h_f)^* q'^*Y$ is a weak equivalence in $\mathcal{E}_{C''}$. And by very definition $r_{\triangleright} = {}_f j'^{q'}$. So ${}_f j'^{q'}$ is a weak equivalence in $\mathcal{E}_{C'}$ if and only if ${}_{j''}r^{h_f}$ is such in $\mathcal{E}_{C''}$.

Similarly write s for the composite $\rho_{q,Y} \circ {}_j f^q : j_!X \to q^*Y \to Y$. Then s is above the acyclic fibration $q = q''h_c$ and lemma 4.9 can be applied: $s^{\triangleleft}$ is a weak equivalence in $\mathcal{E}_C$ if and only if ${}_{h_c}s^{q''} : (h_c)_! j_! X \to q''^*Y$ is a weak equivalence (in $\mathcal{E}_{C''}$). And by very definition $s^{\triangleleft} = {}_f j^q$. So ${}_f j^q$ is a weak equivalence in $\mathcal{E}_C$ if and only if ${}_{h_c}s^{q''}$ is such in $\mathcal{E}_{C''}$.

Now recall that $j'' = h_c j$ and $q'' = q'h_f$. By lemmas 2.3 and 2.4, there exists isomorphisms $j''_!X \simeq h_{c!}j_!X$ and $q''^*Y \simeq h_f{}^*q'^*Y$ in fiber $\mathcal{E}_{C''}$ making the following commute :

$$\begin{tikzcd}
h_{c!}j_!X \arrow[r, "\simeq"] \arrow[dr, "{}_{h_c}s^{q''}"'] & j''_!X \arrow[d, "{}_{j''}f^{q''}"] \arrow[dr, "{}_{j''}r^{h_f}"] & \\
& q''^*Y \arrow[r, "\simeq"] & h_f{}^*q'^*Y
\end{tikzcd}$$

In particular, the morphisms ${}_{j''}r^{h_f}$ and ${}_{h_c}s^{q''}$ are weak equivalences together. We conclude the argument: ${}_{j'}f^{q'}$ is a weak equivalence in $\mathcal{E}_{C'}$ if and only if ${}_{j''}r^{h_f}$ is such in $\mathcal{E}_{C''}$ if and only if ${}_{h_c}s^{q''}$ is such in $\mathcal{E}_{C''}$ if and only if ${}_j f^q$ is a weak equivalence in $\mathcal{E}_C$. ☐

The previous result allow the following "trick": to prove that a map f of $\mathcal{E}$ is in $\mathfrak{W}_{\mathcal{E}}$, you just need to find **some** factorization $p(f) = qj$ as an acyclic cofibration followed by an acyclic fibration such that ${}_j f^q$ is acyclic inside its fiber (this is just the definition of $\mathfrak{W}_{\mathcal{E}}$ after all); but if given that $f \in \mathfrak{W}_{\mathcal{E}}$, you can use that ${}_j f^q$ is a weak equivalence for **every** admissible factorization of $p(f)$!

We shall use that extensively in the proof of the two-out-of-three property for $\mathfrak{W}_{\mathcal{E}}$. This will conclude the proof of sufficiency in theorem 4.2.

Proposition 4.11. *The class $\mathfrak{W}_{\mathcal{E}}$ has the 2-out-of-3 property.*

Proof. We suppose given a commutative triangle $h = gf$ in the total category $\mathcal{E}$, and we proceed by case analysis.

First case: suppose that $f, g \in \mathfrak{W}_{\mathcal{E}}$, and we want to show that $h \in \mathfrak{W}_{\mathcal{E}}$. Since f and g are elements of $\mathfrak{W}_{\mathcal{E}}$, there exists a pair of factorizations $p(f) = qj$ and $p(g) = q'j'$ with j, j' acyclic cofibrations and q, q' acyclic fibrations of $\mathcal{B}$ such that both ${}_j f^q$ and ${}_{q'}g^{j'}$ are weak equivalences in their respective fibers. The weak equivalence $j'q$

of $\mathcal{B}$ can be factorized as $q''j''$ with j'' acyclic cofibration and q'' acyclic fibration. We write $i = j''j$ and $r = q'q''$ and we notice that $p(h) = ri$, as depicted below.

$$
\begin{tikzcd}
pX \arrow[r, "j"] \arrow[rr, bend left, "i"] \arrow[dr, "p(f)"'] \arrow[ddrr, bend left, "p(h)"] & A \arrow[r, "j''"] \arrow[d, "q"] & C \arrow[d, "q''"] \arrow[dd, bend left, "r"] \\
& pY \arrow[r, "j'"] \arrow[dr, "p(g)"'] & B \arrow[d, "q'"] \\
& & pZ
\end{tikzcd}
\tag{9}
$$

Since i is an acyclic cofibration and r is an acyclic fibration, it is enough to show that ${}_ih^r : i_!X \to r^*Y$ is a weak equivalence in $\mathcal{E}_C$ in order to conclude that $h \in \mathfrak{W}_{\mathcal{E}}$. Since $i = j''j$ and $r = q'q''$, corollary 2.5 states that it is equivalent to show that the isomorphic arrow $\tilde{h} : j''_!j_!X \to q''^*q'^*$ is a weak equivalence, where $\tilde{h}$ is defined as the unique arrow in fiber $\mathcal{E}_C$ making the following commute:

$$
\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[drrrr, bend right] & j_!X \arrow[r, "\lambda"] & j''_!j_!X \arrow[d, "\tilde{h}"] & & \\
& & q''^*q'^*Z \arrow[r, "\rho"'] & q'^*Z \arrow[r, "\rho"'] & Z
\end{tikzcd}
$$

Since $h = gf$, such an arrow $\tilde{h}$ is given by the composite

$$
j''_!j'_!X \xrightarrow{j''_!({}_jf^q)} j''_!q^*Y \xrightarrow{\mu_Y} q''^*j'_!Y \xrightarrow{q''^*({}_{j'}g^{q'})} q''^*q'^*Z
$$

where μ_Y is the component at Y of the mate $\mu : j''_!q^* \to q''^*j'_!$ of the commutative square $q''j'' = j'q$ of $\mathcal{B}$ (see diagram (9) above).

$$
\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[rrrr, bend left, "f"] & j_!X \arrow[r, "\lambda"] \arrow[d, "{}_jf^q"] & j''_!j_!X \arrow[d, "j''_!({}_jf^q)"] & & Y \arrow[ddl, "\lambda"'] \arrow[ddd, bend left, "g"] \\
& q^*Y \arrow[r, "\lambda"] \arrow[urrr, "\rho"] & j''_!q^*Y \arrow[d, "\mu_Y"] & & \\
& & q''^*j'_!Y \arrow[r, "\rho"'] \arrow[d, "q''^*({}_{j'}g^{q'})"'] & j'_!Y \arrow[d, "{}_{j'}g^{q'}"'] & \\
& & q''^*q'^*Z \arrow[r, "\rho"'] & q'^*Z \arrow[r, "\rho"'] & Z
\end{tikzcd}
$$

We can conclude that $\tilde{h}$ is a weak equivalence in $\mathcal{E}_C$ because it is a composite of such. Indeed:

- hypothesis (hBC) can be applied to the square $q''j'' = j'q$, and so μ_Y is a weak equivalence in $\mathcal{E}_C$,

- and by hypothesis (hCon), the functors $j''_!$ and q''^* maps the weak equivalences ${}_jf^q$ and ${}_{j'}g^{q'}$ to weak equivalences in $\mathcal{E}_C$.

Suppose now that f and h are in $\mathfrak{W}_\mathcal{E}$ and we will show that g also is. Since $p(f)$ and $p(h)$ are weak equivalences in $\mathcal{B}$, we can use the two-out-of-three property of $\mathfrak{W}$ to deduce that also $p(g)$ is. By hypothesis, $p(f) = qj$ with $j \in \mathfrak{C} \cap \mathfrak{W}$ and $q \in \mathfrak{F} \cap \mathfrak{W}$ and ${}_jf^q$ a weak equivalence. Also write $p(g) = q'j'$ for some $j' \in \mathfrak{C} \cap \mathfrak{W}$ and $q' \in \mathfrak{F} \cap \mathfrak{W}$. We are done if we show that ${}_{j'}g^{q'}$ is a weak equivalence. But in that situation, one can define j'', q'', i, r, and $\tilde{h}$ as before. So we end up with the same big diagram, except that this time $j''_!({}_jf^q)$, μ_Y and the composite $\tilde{h}$ are weak equivalences of $\mathcal{E}_C$, yielding $q''^*({}_{j'}g^{q'})$ as a weak equivalence by the 2-out-of-3 property. But q''^* being homotopically conservative by (hCon), this shows that ${}_{j'}g^{q'}$ is a weak equivalence in $\mathcal{E}_B$.

The last case, where g and h are in $\mathfrak{W}_\mathcal{E}$ is strictly dual. □

5 Illustrations

Since the very start, our work is motivated by the idea that the Reedy model structure can be reconstructed by applying a series of Grothendieck constructions of model categories. The key observation is that the notion of latching and matching functors define a bifibration at each step of the construction of the model structure. We explain in 5.1 how the Reedy construction can be reunderstood from our bifibrational point of view. In section 5.2, we describe how to adapt to express generalized Reedy constructions in a similar fashion. In section 5.3, we recall the previous notions of bifibration of model categories appearing in the literature and, although all of them are special cases of Quillen bifibrations, we indicate why they do not fit the purpose.

5.1 A bifibrational view on Reedy model structures

Recall that a *Reedy category* is a small category $\mathcal{R}$ together with two subcategories $\mathcal{R}^+$ and $\mathcal{R}^-$ and a degree function $d : \mathrm{Ob}\,\mathcal{R} \to \lambda$ for some ordinal λ such that

- every morphism f admits a unique factorization $f = f^+f^-$ with $f^- \in \mathcal{R}^-$ and $f^+ \in \mathcal{R}^+$,
- non-identity morphisms of $\mathcal{R}^+$ strictly raise the degree and those of $\mathcal{R}^-$ strictly lower it.

For such a Reedy category, let $\mathcal{R}_\mu$ denote the full subcategory spanned by objects of degree strictly less than μ. In particular, $\mathcal{R} = \mathcal{R}_\lambda$. Remark also that every $\mathcal{R}_\mu$ inherits a structure of Reedy category from $\mathcal{R}$.

We are interested in the structure of the category of diagrams of shape $\mathcal{R}$ in a complete and cocomplete category $\mathcal{C}$. The category $\mathcal{C}$ is in particular tensored and cotensored over Set, those being respectively given by

$$S \odot C = \coprod_{s\in S} C, \qquad S \pitchfork C = \prod_{s\in S} C, \qquad S \in \mathsf{Set}, C \in \mathcal{C}.$$

For every $r \in \mathcal{R}$ of degree μ, a diagram $X : \mathcal{R}_\mu \to \mathcal{C}$ induces two objects in $\mathcal{C}$, called the *latching* and *matching* objects of X at r, and respectively defined as:

$$\mathrm{L}_r X = \int^{s \in \mathcal{R}_\mu} \mathcal{R}(s,r) \odot X_s, \qquad \mathrm{M}_r X = \int_{s \in \mathcal{R}_\mu} \mathcal{R}(r,s) \pitchfork X_s$$

By abuse, we also denote $\mathrm{L}_r X$ and $\mathrm{M}_r X$ for the latching and matching objects of the restriction to $\mathcal{R}_\mu$ of some $X : \mathcal{R}_\kappa \to \mathcal{C}$ with $\kappa \geq \mu$. In particular, when $\kappa = \lambda$, X is a diagram of shape the entire category $\mathcal{R}$ and we retrieve the textbook notion of latching and matching objects (see for instance [Hov99]). Universal properties of limits and colimits induce a family of canonical morphisms $\alpha_r : \mathrm{L}_r X \to \mathrm{M}_r X$, which can also be understood in the following way. First, one notices that the two functors defined as $\mathcal{R}_{\mu+1} \to \mathcal{C}$

$$r \mapsto \begin{cases} X_r \text{ if } d(r) < \mu \\ \mathrm{L}_r X \text{ if } d(r) = \mu \end{cases}, \qquad r \mapsto \begin{cases} X_r \text{ if } d(r) < \mu \\ \mathrm{M}_r X \text{ if } d(r) = \mu \end{cases}$$

are the skeleton and coskeleton X, which provide a left and a right Kan extensions X along the inclusion $i_\mu : \mathcal{R}_\mu \to \mathcal{R}_{\mu+1}$. We will write these two functors $\mathrm{L}_\mu X$ and $\mathrm{M}_\mu X$ respectively. The family of morphisms α_r then describes the unique natural transformation $\alpha : \mathrm{L}_\mu X \to \mathrm{M}_\mu X$ that restrict to the identity on $\mathcal{R}_\mu$.

The following property is, in our opinion, the key feature of Reedy categories.

Proposition 5.1. *Extensions of a diagram $X : \mathcal{R}_\mu \to \mathcal{C}$ to $\mathcal{R}_{\mu+1}$ are in one-to-one correspondence with families of factorizations of the α_r's*

$$(\mathrm{L}_r X \to \bullet \to \mathrm{M}_r X)_{r \in \mathcal{R}, d(r) = \mu}$$

Proof. One direction is easy. Every extension $\hat{X} : \mathcal{R}_{\mu+1} \to \mathcal{C}$ of X produces such a family of factorizations, but it has nothing to do with the structure of Reedy category: for every r of degree μ in $\mathcal{R}$, the functoriality of $\hat{X}$ ensures that there is a coherent family of morphisms $X_s = \hat{X}_s \to \hat{X}_r$ for each arrow $s \to r$, and symetrically a coherent family of morphisms $\hat{X}_r \to \hat{X}_{s'} = X_{s'}$ for each arrow $r \to s'$. Hence the factorization of α_r given by the universal properties of limits and colimits

$$\mathrm{L}_r X \to \hat{X}_r \to \mathrm{M}_r X$$

The useful feature is the converse: when usually, to construct an extension of X, one should define images for arrows $r \to r'$ between objects of degree μ in a functorial way, here every family automatically induces such arrows! This is a fortunate effect of the unique factorization property. Given factorizations $\mathrm{L}_r X \to X_r \to \mathrm{M}_r X$, one can define $X(f)$ for $f : r \to r'$ as follow: factor $f = f^+ f^-$ with $f^- : r \to s$ lowering the degree and $f^+ : s \to r'$ raising it, so that in particular $s \in \mathcal{R}_\mu$; f^- then gives rise to a canonical projection $\mathrm{M}_r X \to X_s$ and f^+ to a canonical injection $X_s \to \mathrm{L}_{r'} X$; the wanted arrow $X(f)$ is given by the composite

$$X_r \to \mathrm{M}_r X \to X_s \to \mathrm{L}_{r'} X \to X_{r'}$$

Well-definition and functoriality of the said extension are following from uniqueness in the factorization property of the Reedy category $\mathcal{R}$. $\square$

From now on, we fix a model category $\mathcal{M}$, that is a complete and cocomplete category $\mathcal{M}$ with a model structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$. The motivation behind Kan's notion of Reedy categories is to gives sufficient conditions on $\mathcal{R}$ to equip $[\mathcal{R}, \mathcal{M}]$ with a model structure where weak equivalences are pointwise.

Definition 5.2. Let $\mathcal{R}$ be Reedy. The *Reedy triple* on the functor category $[\mathcal{R}, \mathcal{M}]$ is the data of the three following classes

- Reedy cofibrations : those $f : X \to Y$ such that for all $r \in \mathcal{R}$, the map $\mathrm{L}_r Y \sqcup_{\mathrm{L}_r X} X_r \to Y_r$ is a cofibration,
- Reedy weak equivalences : those $f : X \to Y$ such that for $r \in \mathcal{R}$, $f_r : X_r \to Y_r$ is a weak equivalence,
- Reedy fibrations : those $f : X \to Y$ such that for all $r \in \mathcal{R}$, the map $X_r \to \mathrm{M}_r X \times_{\mathrm{M}_r Y} Y_r$ is a fibration.

Kan's theorem about Reedy categories, whose our main result gives a slick proof, then states as follow: the Reedy triple makes $[\mathcal{R}, \mathcal{M}]$ into a model category. A first reading of this definition/theorem is quite astonishing: the distinguished morphisms are defined through those latching and matching objects, and it is not clear, apart from being driven by the proof, why we should emphasize those construction that much. We shall say a word about that later.

REMARK 5.3. Before going into proposition 5.4 below, we need to make a quick remark about extensions of diagrams up to isomorphism. Suppose given a injective-on-objects functor $i : \mathcal{A} \to \mathcal{B}$ between small categories and a category $\mathcal{C}$, then for every diagram $D : \mathcal{A} \to \mathcal{C}$, every diagram $D' : \mathcal{B} \to \mathcal{C}$ and every isomorphism $\alpha : D \to D'i$, there exists a diagram $D'' : \mathcal{B} \to \mathcal{C}$ isomorphic to D' such that $D''i = D$ (and the isomorphism $\beta : D'' \to D'$ can be chosen so that $\beta i = \alpha$). Informally it says that every "up to isomorphism" extension of D can be rectified into a strict extension of D.

Put formally, we are claiming that the restriction functor $i^* : [\mathcal{B}, \mathcal{C}] \to [\mathcal{A}, \mathcal{C}]$ is an isofibration. Although it can be shown easily by hand, we would like to present an alternate proof based on homotopical algebra. Taking a universe $\mathbb{U}$ big enough for $\mathcal{C}$ to be small relatively to $\mathbb{U}$, we can consider the folk model structure on the category Cat of $\mathbb{U}$-small categories. With its usual cartesian product, Cat is a closed monoidal model category in which every object is fibrant. It follows that $[-, \mathcal{C}]$ maps cofibrations to fibrations (see [Hov99, Remark 4.2.3]). Then, the injective-on-objects functor $i : \mathcal{A} \to \mathcal{B}$ is a cofibration, so it is mapped to a fibration $i^* : [\mathcal{B}, \mathcal{C}] \to [\mathcal{A}, \mathcal{C}]$. Recall that fibrations in Cat are precisely the isofibrations and we obtain the result.

Proposition 5.4. *Let $\mathcal{R}$ be Reedy. The restriction functor ${i_\mu}^* : [\mathcal{R}_{\mu+1}, \mathcal{M}] \to [\mathcal{R}_\mu, \mathcal{M}]$ is a Grothendieck bifibration.*

Proof. The claim is that a morphism $f : X \to Y$ is cartesian precisely when the following diagram is a pullback square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \mathrm{M}_\mu pX \arrow[r, "\mathrm{M}_\mu p(f)"'] & \mathrm{M}_\mu pY \end{tikzcd} \tag{10}$$

where the vertical arrows are the component at X and Y of the unit η of the adjunction (p, M_μ). Indeed, such a diagram is a pullback square if and only if the following square is a pullback for all Z:

$$\begin{tikzcd}
{[\mathcal{R}_{\mu+1}, \mathcal{M}](Z, X)} \arrow[r, "f\circ-"] \arrow[d, "\eta_X \circ -"] & {[\mathcal{R}_{\mu+1}, \mathcal{M}](Z, Y)} \arrow[d, "\eta_Y \circ -"] \\
{[\mathcal{R}_{\mu+1}, \mathcal{M}]\big(Z, \mathrm{M}_\mu pX\big)} \arrow[r, "\mathrm{M}_\mu p(f)\circ-"'] & {[\mathcal{R}_{\mu+1}, \mathcal{M}]\big(Z, \mathrm{M}_\mu pY\big)}
\end{tikzcd}$$

We can take advantage of the adjunction (p, M_μ) and its natural isomorphism

$$\phi_{Z,A} : [\mathcal{R}_{\mu+1}, \mathcal{M}]\big(Z, \mathrm{M}_\mu A\big) \simeq [\mathcal{R}_\mu, \mathcal{M}](pZ, A)$$

As in any adjunction, this isomorphism is related to the unit by the following identity: for any $g : Z \to X$, $p(g) = \phi(\eta_X g)$. So in the end, the square in (10) is a pullback if and only if for every Z the outer square of the following diagram is a pullback:

$$\begin{tikzcd}
{[\mathcal{R}_{\mu+1}, \mathcal{M}](Z, X)} \arrow[r, "f\circ-"] \arrow[d, "\eta_X \circ -"] \arrow[dd, bend right, "p"'] & {[\mathcal{R}_{\mu+1}, \mathcal{M}](Z, Y)} \arrow[d, "\eta_Y \circ -"] \arrow[dd, bend left, "p"] \\
{[\mathcal{R}_{\mu+1}, \mathcal{M}]\big(Z, \mathrm{M}_\mu pX\big)} \arrow[r, "\mathrm{M}_\mu p(f)\circ-"'] \arrow[d, "\phi"', "\wr"] & {[\mathcal{R}_{\mu+1}, \mathcal{M}]\big(Z, \mathrm{M}_\mu pY\big)} \arrow[d, "\phi"', "\wr"] \\
{[\mathcal{R}_\mu, \mathcal{M}](pZ, pX)} \arrow[r, "p(f)\circ-"'] & {[\mathcal{R}_\mu, \mathcal{M}](pZ, pY)}
\end{tikzcd}$$

This is exactly the definition of a cartesian morphism. Dually, we can prove that cocartesian morphisms are those $f : X \to Y$ such that the following is a pushout square:

$$\begin{tikzcd}
\mathrm{L}_\mu pX \arrow[r, "\mathrm{L}_\mu p(f)"] \arrow[d] & \mathrm{L}_\mu pY \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Now for $u : A \to pY$ in $[\mathcal{R}_\mu, \mathcal{M}]$, one should construct a cartesian morphism $f : X \to Y$ above u. First notice that we constructed M_μ in such a way that $p\,\mathrm{M}_\mu = \mathbf{1}$ (even more, the counit $p\,\mathrm{M}_\mu \to \mathbf{1}$ is the identity natural transformation). So $\mathrm{M}_\mu A$ is above A and we could be tempted to take, for the wanted f, the morphism $\kappa : \mathrm{M}_\mu A \times_{\mathrm{M}_\mu pY} Y \to Y$ appearing in the following pullback square:

$$\begin{tikzcd}
\bullet \arrow[r, "\kappa"] \arrow[d] & Y \arrow[d] \\
\mathrm{M}_\mu A \arrow[r, "\mathrm{M}_\mu u"'] & \mathrm{M}_\mu pY
\end{tikzcd} \tag{11}$$

But κ is not necessarily above u. Indeed, as a right adjoint, p preserves pullbacks.

So we get that the following is a pullback in $[\mathcal{R}_\mu, \mathcal{M}]$:

$$\begin{tikzcd} p(\bullet) \arrow[r, "p(\kappa)"] \arrow[d] & pY \arrow[d, "\mathbf{1}_Y"] \\ A \arrow[r, "u"'] & pY \end{tikzcd}$$

We certainly know another pullback square of the same diagram, namely

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\mathbf{1}_A"'] & pY \arrow[d, "\mathbf{1}_Y"] \\ A \arrow[r, "u"'] & pY \end{tikzcd}$$

So, by universal property, we obtain an isomorphism $\alpha : A \to p(\mathrm{M}_\mu A \times_{\mathrm{M}_\mu pY} Y)$. Now we summon remark 5.3 to get an extension X of A and an isomorphism $\beta : X \to \mathrm{M}_\mu A \times_{\mathrm{M}_\mu pY} Y$ above α. The wanted $f : X \to Y$ is then just the composite $\kappa\beta$, which is cartesian because the outer square in the following is a pullback (as we chose (11) to be one):

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\beta"] \arrow[dd] & & Y \arrow[dd] \\ & \bullet \arrow[ur, "\kappa"] \arrow[dl] & \\ \mathrm{M}_\mu A \arrow[rr, "\mathrm{M}_\mu u"'] & & \mathrm{M}_\mu pY \end{tikzcd}$$

The fact that the vertical map $X \to \mathrm{M}_\mu A = \mathrm{M}_\mu pX$ is indeed the unit η of the adjunction at component X comes directly from the fact that its image by p is $\mathbf{1}_A$. The existence of cocartesian morphism above any $u : pX \to B$ is strictly dual, using this time the cocontinuity of p as a left adjoint. $\square$

Remark 5.5. First, we should notice that proposition 5.1 make the following multievaluation functor an equivalence:

$$[\mathcal{R}_{\mu+1}, \mathcal{M}]_A \xrightarrow{\sim} \prod_{r \in \mathcal{R}, d(r) = \mu} {}_{\mathrm{L}_r A \backslash}\mathcal{M}_{/\mathrm{M}_r A} \tag{I}$$

The notation ${}_{\mathrm{L}_r A \backslash}\mathcal{M}_{/\mathrm{M}_r A}$ is slightly abusive and means the coslice category of $\mathcal{M}_{/\mathrm{M}_r A}$ by α_r, or equivalently the slice category of ${}_{\mathrm{L}_r A \backslash}\mathcal{M}$ by α_r.

Secondly, we can draw from the previous proof that for a morphism $f : X \to Y$, the fiber morphisms $f^{\triangleleft}$ and $f_{\triangleright}$ are, modulo identification (I), the respective induced families defining the Reedy triple:

$$(X_r \to \mathrm{M}_r X \times_{\mathrm{M}_r Y} Y_r)_{r, d(r) = \mu}, \qquad (X_r \sqcup_{\mathrm{L}_r X} \mathrm{L}_r Y \to Y_r)_{r, d(r) = \mu}$$

So here it is: the reason behind those *a priori* mysterious morphisms, involving latching an matching, are nothing else but the witness of a hidden bifibrational structure. Putting this into light was a tremendous leap in our conceptual understanding of Reedy model structures and their generalizations.

The following proposition is the induction step for successor ordinals in the usual proof of the existence of Reedy model structures. Our main theorem 4.2 allows a very smooth argument.

Proposition 5.6. *If the Reedy triple on $[\mathcal{R}_\mu, \mathcal{M}]$ forms a model structure, then it is also the case on $[\mathcal{R}_{\mu+1}, \mathcal{M}]$.*

Proof. Our course, the goal is to use theorem 4.2 on the Grothendieck bifibration ${i_\mu}^* : [\mathcal{R}_{\mu+1}, \mathcal{M}] \to [\mathcal{R}_\mu, \mathcal{M}]$. By hypothesis, the base $[\mathcal{R}_\mu, \mathcal{M}]$ has a model structure given by the Reedy triple. Each fiber $({i_\mu}^*)_A$ above a diagram A is endowed, via identification (I), with the product model structure: indeed, if $\mathcal{N}$ is a model category, so is its slices $\mathcal{N}_{/N}$ and coslices ${}_{N\backslash}\mathcal{N}$ categories, just defining a morphism to be a cofibration, a fibration or a weak equivalence if it is in $\mathcal{N}$; products of model categories are model categories by taking the pointwise defined structure. All in all, it means that the following makes the fiber $({i_\mu}^*)_A$ into a model category: a fiber map $f : X \to X'$ in $({i_\mu}^*)_A$ is a cofibration, a fibration or a weak equivalence if and only if $f_r : X_r \to X'_r$ is one for every $r \in \mathcal{R}$ of degree μ.

Now the proof amounts to show that hypothesis (Q), (hCon) and (hBC) are satisfied in this framework. Let us first tackle (Q). Suppose $u : A \to B$ in $[\mathcal{R}_\mu, \mathcal{M}]$ and $f : Y \to Y'$ a fiber morphism at B. Then by definition of the cartesian morphisms in $[\mathcal{R}_{\mu+1}, \mathcal{M}]$, u^*f is the unique map above A making the following diagram commute for all r of degree μ:

$$\begin{tikzcd}
(u^*Y)_r \arrow[r] \arrow[d, "(u^*f)_r"'] & Y_r \arrow[d, "f_r"] \\
(u^*Y')_r \arrow[r] \arrow[d] & Y'_r \arrow[d] \\
\mathrm{M}_r A \arrow[r, "\mathrm{M}_r u"'] & \mathrm{M}_r B
\end{tikzcd} \tag{12}$$

where the lower square and outer square are pullback diagrams. By the pasting lemma, so is the upper square. Hence $(u^*f)_r$ is a pullback of f_r, and as such is a (acyclic) fibration whenever f_r is one. This proves that u^* is right Quillen for any u, that is (Q).

Goals (hCon) and (hBC) will be handle pretty much the same way one another and it lies on the following well know fact about Reedy model structures [Hir03, lemma 15.3.9]: for $r \in \mathcal{R}$ of degree μ, the functor $\mathrm{M}_r : [\mathcal{R}_\mu, \mathcal{M}] \to \mathcal{M}$ preserves acyclic fibrations[1]. This has a wonderful consequence: if u is an acyclic fibration of $[\mathcal{R}_\mu, \mathcal{M}]$, any pullback of $\mathrm{M}_r\, u$ is an acyclic fibration hence a weak equivalence. So the upper square of diagram (12) has acyclic horizontal arrows. By the 2-out-of-3 property, f_r on the right is a weak equivalence if an only if $(u^*f)_r$ is one. This being true for each $r \in \mathcal{R}$ of degree μ makes u^* homotopically conservative whenever u is an acyclic fibration. This validates half of the property (hCon). The other half is proven dually, resting on the dual lemma: for any $r \in \mathcal{R}$ of degree μ, the latching functor $\mathrm{L}_r : [\mathcal{R}_\mu, \mathcal{M}] \to \mathcal{M}$ preserves acyclic cofibrations; then deducing that pushouts of $\mathrm{L}_r\, u$ are weak equivalences whenever u is an acyclic cofibration.

It remains to show (hBC). Everything is already in place and it is just a matter of expressing it. For a commutative square of $[\mathcal{R}_\mu, \mathcal{M}]$

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "u'"'] & C \arrow[d, "u"] \\
C' \arrow[r, "v'"'] & B
\end{tikzcd}$$

[1]Actually it is right Quillen, but we will not need that much here.

with u, u' Reedy acyclic cofibrations and v, v' Reedy acyclic fibrations, the mate at an extension Z of C is the unique fiber morphism $v_Z : (u'_! v^* Z) \to (v'^* u_! Z)$ making the following commute for every $r \in \mathcal{R}$ of degree μ:

$$\begin{tikzcd}
 & \mathrm{M}_r A \arrow[rr, "\mathrm{M}_r v"] & & \mathrm{M}_r C & \\
\mathrm{L}_r A \arrow[r] \arrow[d, "\mathrm{L}_r u'"'] & (v^* Z)_r \arrow[u] \arrow[rr] \arrow[d] & & Z_r \arrow[u] \arrow[dd] & \mathrm{L}_r C \arrow[l] \arrow[dd, "\mathrm{L}_r v"] \\
\mathrm{L}_r C' \arrow[r] & (u'_! v^* Z)_r \arrow[dr, "(v_Z)_r"] & & & \\
 & & (v'^* u_! Z)_r \arrow[r] \arrow[d] & (u_! Z)_r \arrow[d] & \mathrm{L}_r B \arrow[l] \\
 & & \mathrm{M}_r C' \arrow[r, "\mathrm{M}_r v'"'] & \mathrm{M}_r B &
\end{tikzcd}$$

where grayscaled square are either pullbacks (when involving matching objects) or pushouts (when involving latching objects). So by the same argument as above, the horizontal and vertical arrows of the pentagone are weak equivalences, making the r-component of the mate $(v_Z)_r$ a weak equivalence also by the 2-out-of-3 property.

Theorem 4.2 now applies, and yield a model structure on $[\mathcal{R}_{\mu+1}, \mathcal{M}]$ which is readily the Reedy triple. □

5.2 Notions of generalized Reedy categories

From time to time, people stumble accross almost Reedy categories and build *ad hoc* workarounds to end up with a structure "à la Reedy". The most popular such generalizations are probably Cisinski's [Cis06] and Berger-Moerdijk's [BM11], allowing for non trivial automorphisms. In [Shu15], Shulman establishes a common framework for every such known generalization of Reedy categories (including enriched ones, which go behind the scope of this paper). Roughly put, Shulman defines *almost-Reedy categories* to be those small categories $\mathcal{C}$ with a degree function on the objects that satisfy the following property: taking x of degree μ and denoting $\mathcal{C}_\mu$ the full subcategory of $\mathcal{C}$ of objects of degree strictly less than μ, and $\mathcal{C}_x$ the full subcategory of $\mathcal{C}$ spanned by $\mathcal{C}_\mu$ and x, then the diagram category $[\mathcal{C}_x, \mathcal{M}]$ is obtained as the *bigluing* (to be defined below) of two nicely behaved functors $[\mathcal{C}_\mu, \mathcal{M}] \to \mathcal{M}$, namely the weighted colimit and weigthed limit functors, respectively weighted by $\mathcal{C}(-, x)$ and $\mathcal{C}(x, -)$. In particular, usual Reedy categories are recovered when realizing that the given formulas of latching and matching objects are precisely these weighted colimits and limits.

In order to understand completely the generalization proposed in [Shu15], we propose an alternative view on the Reedy construction that we exposed in detail in the previous section. For starter, here is a nice consequence of theorem 4.2:

Lemma 5.7. *Suppose there is a strict pullback square of categories*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "q"] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{B} \end{tikzcd}$$

in which $\mathcal{C}$ has a model structure and p is a Quillen bifibration. If

(i) *$F(u)_!$ and $F(v)^*$ are homotopically conservative whenever u is an acyclic cofibration and v an acyclic fibration in $\mathcal{C}$,*

(ii) *F maps squares of the form*

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u'"] & C \arrow[d, "u"] \\ C' \arrow[r, "v'"] & B \end{tikzcd}$$

with u, u' acyclic cofibrations and v, v' acyclic fibrations in $\mathcal{C}$ to squares in $\mathcal{B}$ that satisfy the homotopical Beck-Chevalley condition,

then q is also a Quillen bifibration.

Proof. Denote $p' : \mathcal{B} \to \mathsf{Adj}$ the pseudo functor $A \mapsto \mathcal{E}_A$ associated to p. Then it is widely known that the pullback q of p along F is the bifibration obtained by Grothendieck construction of the pseudo functor $p'F : \mathcal{C} \to \mathsf{Adj}$. It has fiber $\mathcal{F}_C = \mathcal{E}_{FC}$ at $C \in \mathcal{C}$, which has a model structure; and for any $u : C \to D$ in $\mathcal{C}$, the adjunction

$$u_! : \mathcal{F}_C \rightleftarrows \mathcal{F}_D : u^*$$

is given by the pair $(F(u)_!, F(u)^*)$ defined by p. Hence theorem 4.2 asserts that q is a Quillen bifibration as soon as (hBC) and (hCon) are satisfied. The conditions of the lemma are precisely there to ensure that this is the case. $\square$

Now recall that $\Delta[1]$ and $\Delta[2]$ are the posetal categories associated to $\{0 < 1\}$ and $\{0 < 1 < 2\}$ respectively, and write $c : \Delta[1] \to \Delta[2]$ for the functor associated with the mapping $0 \mapsto 0, 1 \mapsto 2$. Given a Reedy category $\mathcal{R}$ and an object r of degree μ, denote $i_r : \mathcal{R}_\mu \to \mathcal{R}_r$ the inclusion of the full subcategory of $\mathcal{R}$ spanned by the object of degree strictly less than μ into the one spanned by the same objects plus r. Then proposition 5.1 asserts that the following is a strict pullback square of categories:

$$\begin{tikzcd} {[\mathcal{R}_r, \mathcal{M}]} \arrow[r] \arrow[d, "{i_r}^*"'] & {[\Delta[2], \mathcal{M}]} \arrow[d, "c^*"] \\ {[\mathcal{R}_\mu, \mathcal{M}]} \arrow[r, "\alpha_r"'] & {[\Delta[1], \mathcal{M}]} \end{tikzcd}$$

where the bottom functor maps every diagram $X : \mathcal{R}_\mu \to \mathcal{M}$ to the canonical arrow $\alpha_r : \mathrm{L}_r X \to \mathrm{M}_r X$. Moreover the functor c^* is a Grothendieck bifibration: one can easily verify that an arrow in $[\Delta[2], \mathcal{M}]$

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r, "g"] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r, "h"] & \bullet \end{tikzcd}$$

is cartesian if and only if the bottom square is a pullback, and is cocartesian if and only if the top square is a pushout. In particular, for each object $k : A \to B$ of $[\Delta[1], \mathcal{M}]$ we have a model structure on its fiber $(c^*)_k \simeq {}_{A\backslash}\mathcal{M}_{/B}$. Stability of cofibrations by pushout and of fibrations by pullback in the model category $\mathcal{M}$ translates to say that hypothesis Q is satisfied by c^*. In other word, by equipping the basis category $[\Delta[1], \mathcal{M}]$ with the trivial model structure, theorem 4.2 applies ((hBC) and (hCon) are vacuously met) and makes c^* a Quillen bifibration. The content of the proof of proposition 5.6 is precisely showing conditions (i) and (ii) of lemma 5.7. We can then conclude that $i^*r : [\mathcal{R}_r, \mathcal{M}] \to [\mathcal{R}_r, \mathcal{M}]$ is a Quillen bifibration as in proposition 5.6.

The result of [Shu15, Theorem 3.11] fall within this view. Shulman defines the *bigluing* of a natural transformation $\alpha : F \to G$ between two functors $F, G : \mathcal{M} \to \mathcal{N}$ as the category $\mathcal{G}\ell(\alpha)$ whose:

- objects are factorizations

$$\alpha_M : FM \xrightarrow{f} N \xrightarrow{g} GM$$

- morphisms $(f, g) \xrightarrow{(h,k)} (f', g')$ are commutative diagrams of the form

$$\begin{tikzcd}
FM \arrow[r, "f"] \arrow[d, "F(h)"'] & N \arrow[r, "g"] \arrow[d, "k"] & GM \arrow[d, "G(h)"] \\
FM' \arrow[r, "f'"'] & N' \arrow[r, "g'"'] & GM'
\end{tikzcd}$$

Otherwise put, the category $\mathcal{G}\ell(\alpha)$ is a pullback as in:

$$\begin{tikzcd}
\mathcal{G}\ell(\alpha) \arrow[r] \arrow[d] & {[\Delta[2], \mathcal{N}]} \arrow[d, "c^*"] \\
\mathcal{M} \arrow[r, "\alpha"'] & {[\Delta[1], \mathcal{N}]}
\end{tikzcd}$$

In the same fashion as in the proof of proposition 5.6, we can show that conditions (i) and (ii) are satisfied for the bottom functor (that we named abusively α) when F maps acyclic cofibrations to *couniversal weak equivalences* and G maps acyclic fibrations to *universal weak equivalences*. By a couniversal weak equivalence is meant a map every pushout of which is a weak equivalence; and by a universal weak equivalence is meant a map every pullback of which is a weak equivalence. Now lemma 5.7 directly proves Shulman's theorem.

Theorem 5.8 (Shulman)**.** *Suppose $\mathcal{N}$ and $\mathcal{M}$ are both model categories. Let $\alpha : F \to G$ between $F, G : \mathcal{M} \to \mathcal{N}$ satisfying that:*

- *F is cocontinuous and maps acyclic cofibrations to couniversal weak equivalences,*
- *G is continuous and maps acyclic fibrations to universal weak equivalence.*

Then $\mathcal{G}\ell(\alpha)$ is a model category whose:

- *cofibrations are the maps (h, k) such that both h and the map $FM' \sqcup_{FM} N \to N'$ induced by k are cofibrations in $\mathcal{M}$ and $\mathcal{N}$ respectively,*

- *fibrations are the maps (h,k) such that both h and the map $N \to GM \times_{GM'} N'$ induced by k are fibrations in $\mathcal{M}$ and $\mathcal{N}$ respectively,*
- *weak equivalences are the maps (h,k) where both h and k are weak equivalences in $\mathcal{M}$ and $\mathcal{N}$ respectively.*

Maybe the best way to understand this theorem is to see it at play. Recall that a generalized Reedy category in the sense of Berger and Moerdijk is a kind of Reedy category with degree preserving isomorphism: precisely it is a category $\mathcal{R}$ with a degree function $d : \mathrm{Ob}\,\mathcal{R} \to \lambda$ and wide subcategories $\mathcal{R}^+$ and $\mathcal{R}^-$ such that:

- **non-invertible** morphisms of $\mathcal{R}^+$ strictly raise the degree while those of $\mathcal{R}^-$ striclty lower it,
- isomorphisms all preserve the degree,
- $\mathcal{R}^+ \cap \mathcal{R}^-$ contains exactly the isomorphisms as morphisms,
- every morphism f can be factorized as $f = f^+ f^-$ with $f^+ \in \mathcal{R}^+$ and $f^- \in \mathcal{R}^-$, and such a factorization is unique up to isomorphism,
- if θ is an isomorphism and $\theta f = f$ for some $f \in \mathcal{R}^-$, then θ is an identity.

The central result in [BM11] goes as follow:

(1) the latching and matching objects at $r \in \mathcal{R}$ of some $X : \mathcal{R} \to \mathcal{M}$ are defined as in the classical case, but now the automorphism group $\mathrm{Aut}(r)$ acts on them, so that $\mathrm{L}_r X$ and $\mathrm{M}_r X$ are objects of $[\mathrm{Aut}(r), \mathcal{M}]$ rather than mere objects of $\mathcal{M}$.

(2) suppose $\mathcal{M}$ such that every $[\mathrm{Aut}(r), \mathcal{M}]$ bears the projective model structure, and define Reedy cofibrations, Reedy fibrations and Reedy weak equivalences as usual but considering the usual induced maps $X_r \sqcup_{\mathrm{L}_r X} \mathrm{L}_r Y \to Y_r$ and $X_r \to Y_r \times_{\mathrm{M}_r Y} \mathrm{M}_r X$ in $[\mathrm{Aut}(r), \mathcal{M}]$, not in $\mathcal{M}$.

(3) then Reedy cofibrations, Reedy fibrations and Reedy weak equivalences give $[\mathcal{R}, \mathcal{M}]$ a model structure.

In that framework, theorem 5.8 is applied repeatedly with α being the canonical natural transformation between $\mathrm{L}_r, \mathrm{M}_r : [\mathcal{R}_\mu, \mathcal{M}] \to [\mathrm{Aut}(r), \mathcal{M}]$ whenever r is of degree μ. In particular, here we see the importance to be able to vary the codomain category $\mathcal{N}$ of Shulman's result in each successor step, and not to work with an homogeneous $\mathcal{N}$ all along.

5.3 Related works on Quillen bifibrations

Our work builds on the papers [Roi94], [Sta12] on the one hand, and [HP15] on the other hand, whose results can be seen as special instances of our main theorem 4.2. In these two lines of work, a number of sufficient conditions are given in order to construct a Quillen bifibration. The fact that their conditions and constructions are special cases of ours follows from the equivalence established in theorem 4.2. As a matter of fact, it is quite instructive to review and to point out the divergences between the two approaches and ours, since it also provides a way to appreciate the subtle aspects of our construction.

Let us state the two results and comment them.

Theorem 5.9 (Roig, Stanculescu)**.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a Grothendieck bifibration. Suppose that $\mathcal{B}$ is a model category with structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ and that each fiber $\mathcal{E}_A$ also with structure $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$. Suppose also assumption (Q). Then $\mathcal{E}$ is a model category with*

- *cofibrations the total ones,*
- *weak equivalences those $f : X \to Y$ such that $p(f) \in \mathfrak{W}$ and $f^{\triangleleft} \in \mathfrak{W}_{pX}$,*
- *fibrations the total ones,*

provided that

(i) u^* is homotopically conservative for all $u \in \mathfrak{W}$,

(ii) for $u : A \to B$ an acyclic cofibration in $\mathcal{B}$, the unit of the adjoint pair $(u_!, u^*)$ is pointwise a weak equivalence in $\mathcal{E}_A$.

The formulation of the theorem is not symmetric, since it emphasizes the cartesian morphisms over the cocartesian ones in the definition of weak equivalences. This lack of symmetry in the definition of the weak equivalences has the unfortunate effect of giving a similar bias to the sufficient conditions: in order to obtain the weak factorization systems, cocartesian morphisms above acyclic cofibrations should be acyclic, which is the meaning of this apparently weird condition (ii); at the same time, cartesian morphisms above acyclic fibrations should also be acyclic but this is vacuously true with the definition of weak equivalences in theorem 5.9. Condition (i) is only here for the 2-out-of-3 property, which boils down to it.

Theorem 5.10 (Harpaz, Prasma)**.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a Grothendieck bifibration. Suppose that $\mathcal{B}$ is a model category with structure $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ and that each fiber $\mathcal{E}_A$ also with structure $(\mathfrak{C}_A, \mathfrak{W}_A, \mathfrak{F}_A)$. Suppose also assumption (Q). Then $\mathcal{E}$ is a model category with*

- *cofibrations the total ones,*
- *weak equivalences those $f : X \to Y$ such that $u = p(f) \in \mathfrak{W}$ and $u^*(r) \circ f^{\triangleleft} \in \mathfrak{W}_{pX}$, where $r : Y \to Y^{\mathrm{fib}}$ is a fibrant replacement of Y in $\mathcal{E}_{pY}$,*
- *fibrations the total ones,*

provided that

(i') the adjoint pair $(u_!, u^*)$ is a Quillen equivalence for all $u \in \mathfrak{W}$,

(ii') $u_!$ and v^* preserves weak equivalences whenever u is an acyclic cofibration and v an acyclic fibration.

At first glance, Harpaz and Prasma introduces the same asymmetry that Roig and Stanculescu in the definition of weak equivalences. They show however that, under condition (i'), weak equivalences can be equivalently described as those $f : X \to Y$ such that $u = p(f) \in \mathfrak{W}$ and

$$u_! X^{\mathrm{cof}} \to u_! X \to Y \in \mathfrak{W}_{pY}$$

where the first arrow is the image by $u_!$ of a cofibrant replacement $X^{\mathrm{cof}} \to X$. Hence, they manage to adapt Roig-Stanculescu's result and to make it self dual. There is a cost however, namely condition (i'). Informally, it says that weakly equivalent objects of $\mathcal{B}$ should have fibers with the same homotopy theory. Harpaz and Prasma observe moreover that under (i'), (i) and (ii) implies (ii'). The condition is quite strong: in particular for the simple Grothendieck bifibration cod : $[\mathbf{2}, \mathcal{B}] \to \mathcal{B}$ of example 3.7, it is equivalent to the fact that the model category $\mathcal{B}$ is right proper. This explains why condition (i') has to be weakened in order to recover the Reedy construction, as we do in this paper.

It is possible to understand our work as a reflection on these results, in the following way. A common pattern in the train of thoughts developped in the three papers [Roi94, Sta12, HP15] is their strong focus on cartesian and cocartesian morphisms above *weak equivalences*. Looking at what it takes to construct weak factorization systems using Stancuslescu's lemma (cf. lemma 2.6), it is quite unavoidable to *push* along (acyclic) cofibrations and *pull* along (acyclic) fibrations in order to put everything in a common fiber, and then to use the fiberwise model structure. On the other hand, *nothing* compels us apparently to push or to pull along weak equivalences of $\mathcal{B}$ in order to define a model structure on $\mathcal{E}$. This is precisely the Ariadne's thread which we followed in the paper: organize everything so that cocartesian morphisms above (acyclic) cofibrations are (acyclic) cofibrations, and cartesian morphisms above (acyclic) fibrations are (acyclic) fibrations. This line of thought requires in particular to see every weak equivalences of the basis category $\mathcal{B}$ as the *composite* of an acyclic cofibration followed by an acyclic fibration. One hidden source of inspiration for this divide comes from the dualities of proof theory, and the intuition that pushing along an (acyclic) cofibration should be seen as a *positive operation* (or a constructor) while pulling along an (acyclic) fibration should be seen as a *negative operation* (or a deconstructor), see [MZ16, MZ17] for details. All the rest, and in particular hypothesis (hCon) and (hBC), follows from that perspective, together with the idea of applying the framework to reunderstand the Reedy construction from a bifibrational point of view.

Let us finally mention that we are currently preparing a companion paper [CM17] where we carefully analyze the relationship between the functor $\mathbf{Ho}(p) : \mathbf{Ho}(\mathcal{E}) \to \mathbf{Ho}(\mathcal{B})$ between the homotopy categories $\mathbf{Ho}(\mathcal{E})$ and $\mathbf{Ho}(\mathcal{B})$ obtained from a Quillen bifibration $p : \mathcal{E} \to \mathcal{B}$ by Quillen localisation, and the Grothendieck bifibration $q : \mathcal{F} \to \mathcal{B}$ obtained by localising each fiber $\mathcal{E}_A$ of the Quillen bifibration p independently as $\mathcal{F}_A = \mathbf{Ho}(\mathcal{E}_A)$.

References

[BM11] Clemens Berger and Ieke Moerdijk. On an extension of the notion of reedy category. *Mathematische Zeitschrift*, 269(3-4):977–1004, 2011.

[Cis06] Denis-Charles Cisinski. *Les préfaisceaux comme modèles des types d'homotopie*. Société mathématique de France, 2006.

[CM17] Pierre Cagne and Paul-André Melliès. Homotopy categories of Quillen bifibrations. In preparation, 2017.

[Egg16] Jeffrey M Egger. Quillen model categories without equalisers or coequalisers, September 2016. arxiv:0609808.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2003.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1999.

[HP15] Yonatan Harpaz and Matan Prasma. The Grothendieck construction for model categories. *Advances in Mathematics*, 281:1306–1363, August 2015.

[MZ16] Paul-André Melliès and Noam Zeilberger. A bifibrational reconstruction of Lawvere's presheaf hyperdoctrine. In *Proceedings of the Thirty first Annual IEEE Symposium on Logic in Computer Science (LICS 2016)*, pages 555–564. IEEE Computer Society Press, July 2016.

[MZ17] Paul-André Melliès and Noam Zeilberger. An Isbell duality theorem for type refinement systems. *Mathematical Structure in Computer Science*, pages 1–39, 2017.

[Qui67] Daniel Gray Quillen. *Homotopical Algebra*, volume 43 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, 1967.

[Roi94] Agustí Roig. Model category structures in bifibred categories. *Journal of Pure and Applied Algebra*, 95(2):203–223, August 1994.

[Shu10] Michael Shulman. Ternary factorization systems, July 2010. The n-Category Café.

[Shu15] Michael Shulman. Reedy categories and their generalizations, July 2015. arxiv:1507.01065v1.

[Sta12] Alexandru Emil Stanculescu. Bifibrations and weak factorisation systems. *Applied Categorical Structures*, 20(1):19–30, 2012.

[Str80] Ross Street. Fibrations in bicategories. *Cahiers de toplogie et géométrie différentielle catégoriques*, 21(2):111–160, 1980.