Strong convergence for reduced free products

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Abstract

Using an inequality due to Ricard and Xu, we give a different proof of Paul Skoufranis's recent result showing that the strong convergence of possibly non-commutative random variables $X^{(k)} \to X$ is stable under reduced free product with a fixed non-commutative random variable Y. In fact we obtain a more general fact: assuming that the families $X^{(k)} = \{X_i^{(k)}\}$ and $Y^{(k)} = \{Y_j^{(k)}\}$ are $*$-free as well as their limits (in moments) $X = \{X_i\}$ and $Y = \{Y_j\}$, the strong convergences $X^{(k)} \to X$ and $Y^{(k)} \to Y$ imply that of $\{X^{(k)}, Y^{(k)}\}$ to $\{X, Y\}$. Phrased in more striking language: the reduced free product is "continuous" with respect to strong convergence. The analogue for weak convergence (i.e. convergence of all moments) is obvious. Our approach extends to the amalgamated free product, left open by Skoufranis.

By a faithful C^*-probability space, we mean a unital C^*-algebra equipped with a state for which the GNS representation is faithful (it suffices for this that the state be faithful). We say that a family $\{X_m \mid m \in \mathcal{I}\} \subset A$ generates A if A is the smallest unital C^*-subalgebra of A containing this family.
Let $(A^{(k)}, \phi^{(k)})$ and (A, ϕ) be faithful C^*-probability spaces ($k \geq 1$). Let $\{X_m^{(k)} \mid m \in \mathcal{I}\} \subset A^{(k)}$ (resp. $\{X_m \mid m \in \mathcal{I}\} \subset A$) be families generating $A^{(k)}$ (resp. A).
We say that $\{X_m^{(k)} \mid m \in \mathcal{I}\}$ tends strongly to $\{X_m \mid m \in \mathcal{I}\}$ and we write

$$\{X_m^{(k)} \mid m \in \mathcal{I}\} \xrightarrow{s} \{X_m \mid m \in \mathcal{I}\}$$

if for any polynomial P in the non-commutative variables $\{x_m, x_m^* \mid m \in \mathcal{I}\}$ (these are called $*$-polynomials) we have $\phi^{(k)}(P(X_m^{(k)})) \to \phi(P(X_m))$ (this is called the convergence in $*$-moments) and moreover

$$\|P(X_m^{(k)})\| \to \|P(X_m)\|.$$

For Hermitian random matrices, when $\mathcal{I}$ is a singleton, this was called the phenomenon "no eigenvalues outside (a small neighbourhood of) the support of the limiting distribution" in [2], where Bai and Silverstein obtained the case of single random covariance (Hermitian) $k \times k$-matrix; this was continued in [12]. See §5 below for a clarification of the meaning of strong convergence for a single Hermitian $k \times k$-matrix or more generally when the families $\{X_m^{(k)} \mid m \in \mathcal{I}\}$ are formed of commuting Hermitian (or normal) operators.

The notion of strong convergence, which was formally introduced in [11], was inspired by Haagerup and Thorbjørnsen's paper [7]. They prove there that if $\{X_m^{(k)}(\omega) \mid m \in \mathcal{I}\}$ are independent random $k \times k$-matrices the entries of which are all independent complex Gaussian $N(0, k^{-1})$,

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then for almost all ω, the random matrices $\{X^{(k)}(\omega) \mid m \in \mathcal{I}\}$ tend strongly to a $*$-free circular family $\{X_m \mid m \in \mathcal{I}\}$. More recent examples of strong convergence for infinite families (i.e. when $\mathcal{I}$ is infinite) are obtained by Collins and Male in [5]: In particular, strong convergence to the normalized Haar measure on the unit circle also holds almost surely for i.i.d. families of unitary random matrices of size k when $k \to \infty$. Additional examples of strong convergence for families can be found in Schultz's [15] (real and symplectic Gaussian random matrices, in other words GOE and GSE), and in Anderson's [1] (Wigner matrices). Related error estimates appear in [8].

Strong convergence is also connected to operator space theory, via the so-called "linearization tricks" from [13] and [7]. We briefly describe this link in §6.

The results below are motivated by work by Camille Male [11], who first considered the question of the stability of strong convergence, and by D. Shlyakhtenko's proof (see the appendix of [11]) that the reduced free product with the C^*-algebra generated by free creators on the Fock space satisfies the desired stability property. Very recently, this was generalized by P. Skoufranis [16] to essentially all reduced free products. In this note we give a different more direct proof based on an inequality due to É. Ricard and Q. Xu, which is a generalization to arbitrary reduced free products of results proved previously by Voiculescu, Haagerup and Buchholz (see [14]) for free products of groups. Our proof yields actually a stronger stability than the one appearing in [16], involving two limits as described in the abstract, but P. Skoufranis informed us that the original proof of [16] also yields that improvement. In the final section, we extend our approach to the amalgamated free product, in answer to a question raised in [16].

1. A rough outline

The main point to prove the result stated in the abstract is this: if we are dealing with P that is a polynomial in X's and Y's that are $*$-free and we want to compute its norm, we observe that if P is of (joint) degree at most d then $Q = (P^*P)^m$ will be of degree at most $2md$.

The Ricard-Xu non-commutative Khintchine inequality ([14]) is

$$(4d)^{-1} kh(P) \leq \|P\| \leq (2d+1)^2 \ kh(P). \tag{1.1}$$

This gives us

$$(8md)^{-1} kh(Q) \leq \|Q\| \leq (4md+1)^2 \ kh(Q)$$

where $kh(Q)$ is a certain expression (actually a norm depending on md) that we will need to analyse below.

Fix $\varepsilon > 0$. The last inequality gives us that if $m = m(d, \varepsilon)$ is fixed but chosen large enough so that $(\max\{8md, (4md+1)^2\})^{1/2m} < 1 + \varepsilon$ then we have

$$(1+\varepsilon)^{-1}[kh((P^*P)^m)]^{1/2m} \leq \|P\| \leq (1+\varepsilon)[kh((P^*P)^m)]^{1/2m}.$$

Thus to show the strong convergence of $X^{(k)}, Y^{(k)}$ to X, Y it suffices to show that for m fixed and $Q = (P^*P)^m$ we have

$$[kh(Q(X^{(k)}, Y^{(k)})]^{1/2m} \to [kh(Q(X, Y)]^{1/2m},$$

or merely

$$kh(Q(X^{(k)}, Y^{(k)})) \to kh(Q(X, Y)). \tag{1.2}$$

But now a closer look at $kh(Q(X, Y))$ in §4 will show that this holds.

2. GNS construction and a specific notation

We choose the convention to have all inner products $\langle y, x\rangle$ linear x and antilinear in y.
In the sequel, we denote by $X \otimes Y$ the algebraic tensor product of two Banach spaces.
Given Hilbert spaces $H, \mathcal{H}$ and C^*-subalgebras $A \subset B(H)$ and $B \subset B(\mathcal{H})$ we denote as usual by $A \otimes_{\min} B$ the closure of $A \otimes B$ in the space $B(H \otimes_2 \mathcal{H})$, and by $\|\cdot\|_{\min}$ the induced norm.

2.1. Let $\mathcal{A}$ be a unital $*$-algebra, assumed sitting inside some ambient unital C^*-algebra. In the sequel, we always make this assumption for our unital $*$-algebras. By a state on $\mathcal{A}$ we mean a linear functional such that $\phi(1) = 1$ and $\phi(x^*x) \geq 0$ for all $x \in \mathcal{A}$. Given this, the classical GNS construction produces a Hilbert space denoted by $L_2(\phi)$ and a $*$-homomorphism $\pi_\phi : \mathcal{A} \to B(L_2(\phi))$ equipped with a distinguished cyclic unit vector $\xi_\phi \in L_2(\phi)$, such that $\phi(c) = \langle \xi_\phi, \pi_\phi(c)\xi_\phi\rangle$ for any $c \in \mathcal{A}$. Let $\pi = \pi_\phi$ and $\xi = \xi_\phi$ for simplicity. Let $A = \overline{\pi(\mathcal{A})} \subset B(L_2(\phi))$. Then A is a unital C^*-algebra. Let $\hat\phi(a) = \langle \xi, \pi(a)\xi\rangle$ for any $a \in A$. Then $\hat\phi$ is a state on A, $L_2(\hat\phi) \simeq L_2(\phi)$ and the representation $A \subset B(L_2(\phi))$ can be identified with the result of the GNS construction applied to $(A, \hat\phi)$. We view the original algebra $\mathcal{A}$ as acting on $L_2(\phi)$ by the correspondence $a \mapsto \pi(a)$. Although this action may be non injective on $\mathcal{A}$, the resulting GNS representation $A \subset B(L_2(\phi))$ is (by definition) faithful on A.

2.2. Let $H = L_2(\phi)$. For any $x \in B(H)$ we denote by ${}^t x \in B(H^*)$ the adjoint operator. Let A^{op} denote the opposite of A i.e. the same as A but with reverse multiplication (i.e. we set $a \cdot b = ba$). We then define $\pi^{op} : A^{op} \to B(H^*)$ by $\pi^{op}(a) = {}^t\pi(a) \in B(H^*)$. Note that π^{op} is a $*$-homomorphism on A^{op}. Let $\xi^{op} \in H^*$ denote the linear form on H defined by

$$\forall h \in H \quad \xi^{op}(h) = \langle \xi, h\rangle.$$

Note that $\xi^{op} \in H^*$ could be identified with $\bar\xi \in \bar H$. Moreover, π^{op} can be viewed as the GNS representation associated to ϕ viewed as a state on A^{op}, with cyclic vector ξ^{op}.
Notation. In the sequel we will work with a dense $*$-subalgebra $\mathcal{A} \subset A$. It will be convenient to use the following notation valid for all $x \in A$, but used mostly for all x in $\mathcal{A}$:

$$\pi(x)\xi = x\xi \text{ and } \pi^{op}(x)\xi^{op} = \xi^{op}x. \tag{2.1}$$

2.3. (A notation for further reference) Fix an integer $d \geq 1$. Consider a linear subspace $X \subset A$. Let $X^{\otimes d}$ denote the algebraic tensor product. Let $0 \leq r \leq d$. We define a linear mapping

$$t_r : X^{\otimes d} \to H \otimes H^*$$

by

$$\forall x = a_1 \otimes \cdots \otimes a_d \in X^{\otimes d} \quad t_r(x) = a_1 \cdots a_r\xi \otimes \xi^{op}a_{r+1} \cdots a_d.$$

In the extreme cases $r = 0$ or $r = d$, we mean

$$t_0(x) = \xi \otimes \xi^{op}a_1 \cdots a_d \quad and \quad t_d(x) = a_1 \cdots a_d\xi \otimes \xi^{op}.$$

This definition is extended to the whole of $X^{\otimes d}$ by linearity.

2.4. (More notation) Assume now that the linear subspace $X \subset A$ is included in the direct sum of two subspaces $X_1, X_2 \subset A$, so that $X \subset X_1 + X_2$ and we have a linear embedding $J : X \to X_1 \oplus X_2$. For further reference, we define for any $1 \le r \le d$ a linear mapping

$$s_r : X^{\otimes d} \to H \otimes H^* \otimes (X_1 \oplus X_2)$$

by setting

$$\forall x = a_1 \otimes \cdots \otimes a_d \in X^{\otimes d} \quad s_r(x) = a_1 \cdots a_{r-1}\xi \otimes \xi^{op} a_{r+1} \cdots a_d \otimes J(a_r).$$

In the extreme cases $r = 1$ or $r = d$, we mean

$$s_1(x) = \xi \otimes \xi^{op} a_2 \cdots a_d \otimes J(a_1) \quad and \quad s_d(x) = a_1 \cdots a_{d-1}\xi \otimes \xi^{op} \otimes J(a_d).$$

This definition is extended to the whole of $X^{\otimes d}$ by linearity.

3. Background on ultraproducts

It will be convenient to use ultraproducts, but we only need the very basic and elementary facts that are recalled below.

3.1. Let $\mathcal{U}$ be a non trivial ultrafilter on $\mathbb{N}$. Given a sequence $(X^{(k)})$ of Banach spaces, their ultraproduct is usually denoted by

$$\prod_{k \in \mathbb{N}} X^{(k)}/\mathcal{U}.$$

We will more often denote it by $X^{\mathcal{U}}$ (see e.g. [9] for more background information). The elements $x \in X^{\mathcal{U}}$ are equivalence classes of bounded sequences $(x^{(k)})$ with $x^{(k)} \in X^{(k)}$ for all k. By definition, two such sequences $(x^{(k)})$, $(y^{(k)})$ are equivalent if $\lim_{\mathcal{U}} \|x^{(k)} - y^{(k)}\| = 0$. We will sometimes write $x = [x^{(k)}]_{\mathcal{U}}$ to denote that $(x^{(k)})$ is a representative of x. Whenever this holds we have $\|x\|_{X^{\mathcal{U}}} = \lim_{\mathcal{U}} \|x^{(k)}\|$.

3.2. Let $(X^{(k)}), (Y^{(k)})$ be sequences of Banach spaces (resp. unital C^*-algebras). Let $T^{(k)} : X^{(k)} \to Y^{(k)}$ be a bounded sequence of linear mappings (resp. unital $*$-homomorphisms) then the mapping $T^{\mathcal{U}} : X^{\mathcal{U}} \to Y^{\mathcal{U}}$ defined whenever $x = [x^{(k)}]_{\mathcal{U}}$ by $T^{\mathcal{U}}(x) = [T^{(k)}(x^{(k)})]_{\mathcal{U}}$ is bounded (resp. a unital $*$-homomorphism) with $\|T^{\mathcal{U}}\| = \lim_{\mathcal{U}} \|T^{(k)}\|$.

3.3. As is well known, when all the spaces in $(X^{(k)})$ are Hilbert spaces $X^{\mathcal{U}}$ is also a Hilbert space. It may be worthwhile to remind the reader that if $(Y^{(k)})$ is another family of Hilbert spaces, we have a canonical isometric embedding

$$X^{\mathcal{U}} \otimes_2 Y^{\mathcal{U}} \subset \prod_{k \in \mathbb{N}} X^{(k)} \otimes_2 Y^{(k)}/\mathcal{U},$$

taking $[x^{(k)}]_{\mathcal{U}} \otimes [y^{(k)}]_{\mathcal{U}}$ to $[x^{(k)} \otimes y^{(k)}]_{\mathcal{U}}$. Of course this extends to an arbitrary finite number of factors. Moreover, if $\sup_k \dim(X^{(k)}) < \infty$, then this embedding is an isomorphism.

3.4. Let $(H^{(k)})$ be a sequence of Hilbert spaces. We have then an isometric identification

$$H^{\mathcal{U}*} = \prod\nolimits_{k\in\mathbb{N}} H^{(k)*}/\mathcal{U}.$$

Let $H^{(k)} \otimes H^{(k)*} \subset B(H^{(k)})$ be the usual embedding (taking $x \otimes f$ to the mapping $h \mapsto xf(h)$). Then, for any sequence $t^{(k)} \in H^{(k)} \otimes H^{(k)*}$ with ranks uniformly bounded by some number N, associated, as in 3.3, to some $t \in H^{\mathcal{U}} \otimes H^{\mathcal{U}*}$ (of rank at most N) we have

$$\lim\nolimits_{\mathcal{U}} \|t^{(k)}\|_{B(H^{(k)})} = \|t\|_{B(H^{\mathcal{U}})}. \tag{3.1}$$

More explicitly, let $t = \sum_1^N h(\alpha) \otimes \xi(\alpha) \in H^{\mathcal{U}} \otimes H^{\mathcal{U}*}$. Assume $h(\alpha) = [h^{(k)}(\alpha)]_{\mathcal{U}}$ and $\xi(\alpha) = [\xi^{(k)}(\alpha)]_{\mathcal{U}}$. Let $t^{(k)} = \sum_1^N h^{(k)}(\alpha) \otimes \xi^{(k)}(\alpha) \in H^{(k)} \otimes H^{(k)*}$. Then (3.1) holds.

For the convenience of the reader, let us sketch a quick and instructive verification of (3.1). Let $E \subset H^{\mathcal{U}}$ and $F \subset H^{\mathcal{U}*}$ be N-dimensional subspaces such that $t \in E \otimes F$. Let $(e_i) = (e_i^{(k)})$ and $(f_j) = (f_j^{(k)})$ be orthonormal bases of E and F. We can then write $t = \sum a_{ij} e_i \otimes f_j$ and $t^{(k)} = \sum a_{ij} e_i^{(k)} \otimes f_j^{(k)}$. By an elementary perturbation, we may assume that $(e_i^{(k)})$ and $(f_j^{(k)})$ are orthonormal in $H^{(k)}$ and $H^{(k)*}$ for all k large enough. Then $\|t^{(k)}\|_{B(H^{(k)})} = \|t\|_{B(H^{\mathcal{U}})} = \|[a_{ij}]\|_{M_N}$ for all k large enough.

3.5. Let $(\mathcal{H}^{(k)})$ be a sequence of Hilbert spaces. Let $S^{(k)} \in B(\mathcal{H}^{(k)})$ be a bounded sequence. Let $S^{\mathcal{U}} \in B(\mathcal{H}^{\mathcal{U}})$ be the associated operator. By 3.2, we already know that $\|S^{\mathcal{U}}\| = \lim_{\mathcal{U}} \|S^{(k)}\|$.
More generally, let N be a fixed integer and H a Hilbert space. We denote by $M_N(B(H))$ the space of $N \times N$ matrices with entries in $B(H)$ with the usual norm.
Let $[a_{ij}^{(k)}] \in M_N(B(\mathcal{H}^{(k)}))$ be a bounded sequence. Note that if $K^{(k)} = \mathcal{H}^{(k)} \oplus \cdots \oplus \mathcal{H}^{(k)}$ (N-times), there is a natural identification $K^{\mathcal{U}} = \mathcal{H}^{\mathcal{U}} \oplus \cdots \oplus \mathcal{H}^{\mathcal{U}}$ (N-times). Then clearly

$$\|[a_{ij}^{\mathcal{U}}]\|_{M_N(B(\mathcal{H}^{\mathcal{U}}))} = \lim\nolimits_{\mathcal{U}} \|[a_{ij}^{(k)}]\|_{M_N(B(\mathcal{H}^{(k)}))}. \tag{3.2}$$

Consider now C^*-subalgebras $B^{(k)} \subset B(\mathcal{H}^{(k)})$, and their ultraproducts $B^{\mathcal{U}} \subset B(\mathcal{H}^{\mathcal{U}})$. Let

$$s = \sum\nolimits_1^N h(\alpha) \otimes \xi(\alpha) \otimes \beta(\alpha) \in H^{\mathcal{U}} \otimes H^{\mathcal{U}*} \otimes B^{\mathcal{U}}.$$

Assume $h(\alpha) = [h^{(k)}(\alpha)]_{\mathcal{U}}$, $\xi(\alpha) = [\xi^{(k)}(\alpha)]_{\mathcal{U}}$ and $\beta(\alpha) = [\beta^{(k)}(\alpha)]_{\mathcal{U}}$. Then let

$$s^{(k)} = \sum\nolimits_1^N h^{(k)}(\alpha) \otimes \xi^{(k)}(\alpha) \otimes \beta^{(k)}(\alpha) \in H^{(k)} \otimes H^{(k)*} \otimes B^{(k)}.$$

We claim that

$$\|s\|_{B(H^{\mathcal{U}}) \otimes_{\min} B^{\mathcal{U}}} = \lim\nolimits_{\mathcal{U}} \|s^{(k)}\|_{B(H^{(k)}) \otimes_{\min} B^{(k)}}. \tag{3.3}$$

Arguing as in 3.4 we can find orthonormal systems $(e_i^{(k)})$ and $(f_j^{(k)})$ of length N in $H^{(k)}$ and $H^{(k)*}$ with respect to which we may write $s^{(k)} = \sum_{ij} e_i^{(k)} \otimes f_j^{(k)} \otimes a_{ij}^{(k)}$. We have then

$$\|s^{(k)}\|_{B(H^{(k)}) \otimes_{\min} B^{(k)}} = \|s^{(k)}\|_{B(H^{(k)}) \otimes_{\min} B(\mathcal{H}^{(k)})} = \|[a_{ij}^{(k)}]\|_{M_N(B(\mathcal{H}^{(k)}))},$$

and similarly $\|s\|_{B(H^{\mathcal{U}}) \otimes_{\min} B^{\mathcal{U}}} = \|s\|_{B(H^{\mathcal{U}}) \otimes_{\min} B(\mathcal{H}^{\mathcal{U}})} = \|[a_{ij}^{\mathcal{U}}]\|_{M_N(B(\mathcal{H}^{\mathcal{U}}))}$. Now the claim follows from (3.2).

3.6. Given a sequence of states $\phi^{(k)}$ on a unital $*$-algebra $\mathcal{A}$, let $(\pi^{(k)}, H^{(k)}, \xi^{(k)})$ be the associated GNS construction and let $A^{(k)} = \overline{\pi^{(k)}(\mathcal{A})} \subset B(H^{(k)})$ be the associated C^*-algebra. Let $\pi^{\mathcal{U}} : \mathcal{A} \to B(H^{\mathcal{U}})$ be the representation defined for any $z = [z^{(k)}]_{\mathcal{U}} \in H^{\mathcal{U}}$ and $b \in \mathcal{A}$ by

$$\pi^{\mathcal{U}}(b)([z^{(k)}]_{\mathcal{U}}) = [\pi^{(k)}(b)(z^{(k)})]_{\mathcal{U}}.$$

Obviously,

$$\|\pi^{\mathcal{U}}(b)\| = \lim_{\mathcal{U}} \|\pi^{(k)}(b)\|.$$

Let $\phi = \lim_{\mathcal{U}} \phi^{(k)}$ relative to *pointwise* convergence on $\mathcal{A}$, let $\pi : \mathcal{A} \to B(L_2(\phi))$ be the associated GNS representation and let $\xi = \xi_\phi$. Let also $\xi^{\mathcal{U}} = [\xi^{(k)}]_{\mathcal{U}}$. Then

$$\|\pi^{\mathcal{U}}(b)\xi^{\mathcal{U}}\|^2 = \lim_{\mathcal{U}} \|\pi^{(k)}(b)\xi^{(k)}\|^2 = \lim_{\mathcal{U}} \phi^{(k)}(b^*b) = \phi(b^*b) = \|\pi(b)\xi\|^2.$$

Similarly, using the identity $H^{\mathcal{U}*} = \prod_{k\in\mathbb{N}} H^{(k)*}/\mathcal{U}$ (see 3.4), we have for any $b \in \mathcal{A}$

$$\|\pi^{\mathcal{U}\,op}(b)\xi^{\mathcal{U}\,op}\|^2_{H^{\mathcal{U}*}} = \lim_{\mathcal{U}} \|\pi^{(k)\,op}(b)\xi^{(k)\,op}\|^2_{H^{(k)*}} = \lim_{\mathcal{U}} \phi^{(k)}(bb^*) = \phi(bb^*) = \|\pi^{op}(b)\xi^{op}\|^2.$$

It is natural to extend the notation (2.1) by setting

$$\pi^{(k)}(b)\xi^{(k)} = b\xi^{(k)} \text{ and } \pi^{(k)\,op}(b)\xi^{(k)\,op} = \xi^{(k)}b, \tag{3.4}$$

$$\pi^{\mathcal{U}}(b)\xi^{\mathcal{U}} = b\xi^{\mathcal{U}} \text{ and } \pi^{\mathcal{U}\,op}(b)\xi^{\mathcal{U}\,op} = \xi^{\mathcal{U}}b, \tag{3.5}$$

3.7. Therefore, the correspondence $b\xi \mapsto b\xi^{\mathcal{U}}$ extends to an isometric isomorphism from $L_2(\phi)$ onto the subspace $K^{\mathcal{U}} \subset H^{\mathcal{U}}$ that is the closure of $\{b\xi^{\mathcal{U}} \mid b \in \mathcal{A}\}$. More precisely, the restriction of $\pi^{\mathcal{U}}$ to $K^{\mathcal{U}}$, i.e. $b \mapsto \pi^{\mathcal{U}}(b)_{|K^{\mathcal{U}}} \in B(K^{\mathcal{U}})$, is unitarily equivalent to the representation $\pi = \pi_\phi$. Similarly, $\xi^{op}b \mapsto \xi^{\mathcal{U}\,op}b$ extends to an isometric isomorphism from $L_2(\phi)^*$ onto a subspace of $H^{\mathcal{U}*}$, which can be identified isometrically, via $y \mapsto y_{|K^{\mathcal{U}}}$ with $K^{\mathcal{U}*}$.

3.8. Now let us assume moreover that $\phi = \lim_{\mathcal{U}} \phi^{(k)}$ *strongly*. This means (see below) that $\|\pi(b)\| = \lim_{\mathcal{U}} \|\pi^{(k)}(b)\| = \|\pi^{\mathcal{U}}(b)\|$ for any $b \in \mathcal{A}$. Then the mapping $\pi(b) \mapsto \pi^{\mathcal{U}}(b)$ defines an isometric (and automatically completely isometric) embedding of C^*-algebras

$$\psi : \ A = \overline{\pi(\mathcal{A})} \to B(H^{\mathcal{U}}).$$

3.9. Let $(H_i, \xi_i)_{i\in I}$ be a family of Hilbert spaces, each equipped with a distinguished unit vector. Let $(H, \xi) = *_{i\in I}(H_i, \xi_i)$ be their free product in the sense of [17]. This is defined as

$$(H, \xi) = (H_0 \oplus \oplus_{d\geq 1} H_d, \xi)$$

where $H_0 = \mathbb{C}$ with unit vector $\xi = 1_{\mathbb{C}}$ (viewed as sitting in H) and

$$H_d = \oplus_{i(1)\neq\cdots\neq i(d)} [H_{i(1)} \ominus \mathbb{C}\xi_{i(1)}] \otimes_2 \cdots \otimes_2 [H_{i(d)} \ominus \mathbb{C}\xi_{i(d)}].$$

It is natural to wonder whether this free product commutes with utraproducts. Let $(H_i^{(k)}, \xi_i^{(k)})_{i\in I}$ be a sequence of such families (indexed by $k \in \mathbb{N}$). Let $(H^{(k)}, \xi^{(k)}) = *_{i\in I}(H_i^{(k)}, \xi_i^{(k)})$. Going back to the definition of the free product, a moment of thought (recall 3.3) shows that we have a canonical isometric embedding

$$\chi : *_{i\in I} H_i^{\mathcal{U}} \subset H^{\mathcal{U}} \tag{3.6}$$

that respects the distinguished vectors.
Assuming $I = \{1, 2\}$, the mapping χ can be described like this: First we have $\chi(\xi) = \xi^{\mathcal{U}}$, then whenever we consider an element x_j in $H_1^{\mathcal{U}} \cap \{\xi_1^{\mathcal{U}}\}^\perp$ (resp. $H_2^{\mathcal{U}} \cap \{\xi_2^{\mathcal{U}}\}^\perp$) we can choose representatives $(x_j^{(k)})$ of x_j with $x_j^{(k)}$ in $H_1^{(k)} \cap \{\xi_1^{(k)}\}^\perp$ (resp. $H_2^{(k)} \cap \{\xi_2^{(k)}\}^\perp$), so that given an element $x = x_1 \otimes \cdots \otimes x_d$ of degree d in $*_{i\in I} H_i^{\mathcal{U}}$, with alternating factors in $H_1^{(k)} \cap \{\xi_1^{(k)}\}^\perp$ and $H_2^{(k)} \cap \{\xi_2^{(k)}\}^\perp$, we then define $\chi(x)$ as the element of $H^{\mathcal{U}}$ admitting as representative the sequence $(x^{(k)})$ with $x^{(k)} = x_1^{(k)} \otimes \cdots \otimes x_d^{(k)}$. However, it is easy to see that this embedding χ is not surjective.

4. Main result

We now turn to a more formal description of our main result.
A more abstract (but equivalent) version of the statement in the abstract can be given in terms of convergence of states. We use the notation in 2.1.

Definition 4.1. Let ϕ (resp. $\phi^{(k)}$, $(k \in \mathbb{N})$) be states on a unital $*$-algebra $\mathcal{A}$ (assumed included in some C^*-algebra) with associated GNS Hilbert spaces denoted by $L_2(\phi)$ (resp. $L_2(\phi^{(k)})$). Let π (resp. $\pi^{(k)}$) be the associated GNS representations of $\mathcal{A}$ on these Hilbert spaces. We say that $\phi^{(k)}$ tends to ϕ strongly and we write $\phi^{(k)} \xrightarrow{s} \phi$ if $\phi^{(k)}$ tends to ϕ pointwise on $\mathcal{A}$ and moreover if $\|\pi^{(k)}(c)\| \to \|\pi(c)\|$ for any c in $\mathcal{A}$.

In [17] the notion of free product of a family of states is defined. It can be described as follows. Consider a family of states $\{\phi_i \mid i \in I\}$ with GNS Hilbert space $H_i = L_2(\phi_i)$, GNS representation $\pi_i : \mathcal{A}_i \to B(L_2(\phi_i))$ and distinguished unit vector ξ_i. Let $A_i \subset B(L_2(\phi_i))$ be the associated C^*-algebra. We denote by $\hat{\pi}_i : A_i \to B(L_2(\phi_i))$ the inclusion map. Let $\mathcal{A} = *_{i\in I}\mathcal{A}_i$ be the (algebraic) free product of unital $*$-algebras. Following Voiculescu (see [17]) one defines a Hilbert space free product $(H, \xi) = *_{i\in I}(H_i, \xi_i)$ and a representation π of $\mathcal{A}$ acting on (H, ξ). Let $\hat{\phi}_i$ (resp. $\hat{\phi}$) be the vector state on A_i (resp. $\overline{\pi(\mathcal{A})}$) associated to ξ_i (resp. ξ). The unital C^*-subalgebra $A = \overline{\pi(\mathcal{A})} \subset B(H)$, equipped with $\hat{\phi}$, is called the reduced free product of $(A_i, \hat{\phi}_i)_{i\in I}$. Note that, by [6], $\hat{\phi}$ is faithful on A if each $\hat{\phi}_i$ is faithful on A_i.

We will denote by $\phi = *_{i\in I}\phi_i$ the vector state on $\mathcal{A}$ defined by $\phi(b) = \langle \xi, \pi(b)\xi \rangle$. We call it the free product of the states $\{\phi_i \mid i \in I\}$. Then we can reformulate the main result like this:

Theorem 4.2. *Let $\mathcal{A}_i$ $(i \in I)$ be a family of unital $*$-algebras. Let $\{\phi_i^{(k)} \mid i \in I\}$ $(k \in \mathbb{N})$ be a sequence of families of states, each $\phi_i^{(k)}$ being a state on $\mathcal{A}_i$. Assume that we have states ϕ_i on $\mathcal{A}_i$ such that, for each $i \in I$, when $k \to \infty$ we have*

$$\phi_i^{(k)} \xrightarrow{s} \phi_i.$$

Then

$$*_{i\in I}\phi_i^{(k)} \xrightarrow{s} *_{i\in I}\phi_i.$$

4.1. The analogue of the preceding statement for pointwise convergence of states is obvious from the definition of the reduced free product in [17].

4.2. Let A_i be associated to $(\mathcal{A}_i, \phi_i)$ by the GNS construction as above. We view each A_i as a subalgebra of the reduced free product $A = *_{i\in I}A_i$. Let

$$\overset{\circ}{A}_i = \{x \in A_i \mid \phi_i(x) = 0\}.$$

By a monomial of degree d we mean a product of the form $x_1 \cdots x_d$ with $x_j \in \overset{\circ}{A}_{i_j}$ such that $i_1 \neq i_2 \neq \cdots \neq i_d$. By a homogeneous element of degree d in $A = *_{i \in I} A_i$ we mean a finite sum of monomials of degree d. An element is called of degree $\leq d$ if it is a sum of homogeneous elements each of degree $\leq d$. Note that the elements of finite degree are dense in A.

Let us denote by W_d (resp. $W_{\leq d}$) the space of homogeneous elements of degree d (resp. $\leq d$) in the preceding sense. We also set $W_0 = \mathbb{C}1$ and denote by $\overline{W_d}$ the closure of W_d in A. Then the Ricard-Xu inequality we will use is this (note that, by our convention in 2.1, the assumption in [14] that all the GNS constructions are faithful is here automatic): There are constants $c' > 0$ and $\beta > 0$ such that

$$\forall d \ \forall x \in W_d \quad (c' d^{\beta})^{-1} kh(x) \leq \|x\| \leq c' d^{\beta} kh(x), \tag{4.1}$$

where we set

$$kh(x) = \max\{\max_{0 \leq r \leq d} \|t_r(x)\|, \max_{1 \leq r \leq d} \|s_r(x)\|\}, \tag{4.2}$$

and where $t_r(x), s_r(x)$ are defined as follows: We assume $I = \{1, 2\}$ for notational simplicity. Let $X = \overset{\circ}{A}_1 + \overset{\circ}{A}_2 \subset A$. We have obviously an embedding denoted by $x \mapsto [x]$ of W_d into $X^{\otimes d}$. So following 2.2 we may set for any $x \in W_d$

$$t_r(x) = t_r([x]).$$

We will identify an element $\sum x_j \otimes y_j \in H \otimes H^*$ with the linear map $T \in B(H)$ defined by $T(z) = \sum x_j \otimes y_j(z)$. In this way we will view $t_r(x)$ as an element of $B(H)$, and we denote by $\|t_r(x)\|$ its norm.

Let $X = \overset{\circ}{A}_1 + \overset{\circ}{A}_2$, $X_i = A_i$ and $J : \overset{\circ}{A}_1 + \overset{\circ}{A}_2 \to A_1 \oplus A_2$ be the canonical embedding. Following 2.4, we set

$$s_r(x) = s_r([x]) \in H \otimes H^* \otimes (A_1 \oplus A_2).$$

We then denote by $\|s_r(x)\|$ its norm in the minimal tensor product $B(H) \otimes_{\min} [A_1 \oplus A_2]$. Equivalently, if we are given isometric representations $\psi_j : \ A_j \to B(H_j)$ $(j = 1, 2)$ this is the maximum of two norms, one in $B(H) \otimes_{\min} A_1$ (induced by $B(H \otimes H_1)$) and one in $B(H) \otimes_{\min} A_2$ (induced by $B(H \otimes H_2)$).

4.3. (On recentering) For any $i \in I$, let $\overset{\circ}{\mathcal{A}}_i = \{x \in \mathcal{A}_i \mid \phi_i(x) = 0\}$. The elements of $\overset{\circ}{\mathcal{A}}_i$ are sometimes called "centered" (with respect to ϕ_i). Let $\mathcal{A}_0 = \mathbb{C}1$. By elementary (free) algebra, one can show that the algebraic free product $\mathcal{A}$ is linearly isomorphic to the direct sum

$$\mathcal{A}_0 \oplus \oplus_{d \geq 1, i_1 \neq \cdots \neq i_d} \mathcal{A}(i_1, \cdots, i_d) \tag{4.3}$$

where the subspaces $\mathcal{A}(i_1, \cdots, i_d)$ $(d \geq 1,\ i_i \neq i_2 \neq \cdots)$ are formed of all products of the form

$$x_{i_1} \cdots x_{i_d} \quad \text{with} \quad x_{i_j} \in \overset{\circ}{\mathcal{A}}_{i_j} \quad \forall 1 \leq j \leq d. \tag{4.4}$$

Recall that, for each k, we are given a state $\phi_i^{(k)}$ on $\mathcal{A}_i$. We will denote by $v_i^{(k)} : \mathcal{A}_i \to \mathcal{A}_i$ the linear mapping that transforms centering with respect to ϕ into centering with respect to $\phi_i^{(k)}$. More precisely, $v_i^{(k)}$ is defined by $v_i^{(k)}(1) = 1$ and $\forall a \in \overset{\circ}{\mathcal{A}}_i \ \ v_i^{(k)}(a) = a - \phi_i^{(k)}(a)1$. Using the above direct sum decomposition of $\mathcal{A}$, and viewing $\mathcal{A}_i \subset \mathcal{A}$ we can extend the mappings $v_i^{(k)}$ to a single mapping on $\mathcal{A}$. More precisely, using the freeness of the product, there is a unique linear map $v^{(k)} : \mathcal{A} \to \mathcal{A}$ that coincides with $v_i^{(k)}$ on $\mathcal{A}_i$ for each $i \in I$ and is such that for any element of the form (4.4) we have

$$v^{(k)}(x_{i_1} \cdots x_{i_d}) = v_{i_1}^{(k)}(x_{i_1}) \cdots v_{i_d}^{(k)}(x_{i_d}). \tag{4.5}$$

We note that if we equip $\mathcal{A}$ with the maximal C^*-norm then, since $\phi_i^{(k)} \to \phi_i$, we clearly have

$$\forall b \in \mathcal{A} \quad \|v^{(k)}(b) - b\| \to 0. \tag{4.6}$$

Let us denote by $\mathcal{P}_d$ the linear projection (relative to (4.3)) from $\mathcal{A}$ to the subspace $\mathcal{W}_d \subset \mathcal{A}$ defined by

$$\mathcal{W}_d = \oplus_{i_1 \neq \cdots \neq i_d} \mathcal{A}(i_1, \cdots, i_d).$$

Fix k. Again let $\mathcal{W}_0 = \mathbb{C}1$ and $\mathcal{W}_{\leq d} = \mathcal{W}_0 + \cdots + \mathcal{W}_d$.
Suppose now that we replace ϕ by $\phi_i^{(k)}$ so that we have a direct sum decomposition as above but now associated to $\phi_i^{(k)}$. This leads to subspaces $\mathcal{W}_d^{(k)} \subset \mathcal{A}$ defined exactly like $\mathcal{W}_d$ but with respect to $\phi_i^{(k)}$. Let $\mathcal{P}_d^{(k)}$ denote the linear projection from $\mathcal{A}$ to the subspace $\mathcal{W}_d^{(k)} \subset \mathcal{A}$ in the said direct sum decomposition relative to $\phi_i^{(k)}$. It is easy to check that we have for any k

$$v^{(k)} \mathcal{P}_d = \mathcal{P}_d^{(k)} v^{(k)}. \tag{4.7}$$

4.4. By [14, Cor. 3.3] $\mathcal{P}_d$ extends by density to a completely bounded projection from $*_{i \in I} A_i$ to $\overline{W_d}$, that we will still denote abusively by $\mathcal{P}_d$ satisfying:

$$\|\mathcal{P}_d\|_{cb} \leq \max\{1, 4d\}.$$

Obviously this implies

$$(\max\{1, 4d\})^{-1} \max_{0 \leq d \leq D} \{\|\mathcal{P}_d(x)\|\} \leq \|x\| \leq (d+1) \max_{0 \leq d \leq D} \{\|\mathcal{P}_d(x)\|\}. \tag{4.8}$$

It will be convenient for us to extend the above definition of kh as follows: for any D and any $x \in W_{\leq D}$ we define

$$kh(x) = \sup_{0 \leq d \leq D} kh(\mathcal{P}_d(x)). \tag{4.9}$$

Combining (4.8) with (4.1) we now obtain that there are constants $c > 0$ and $\alpha > 0$ such that

$$\forall d \; \forall x \in W_{\leq d} \quad (cd^\alpha)^{-1} kh(x) \leq \|x\| \leq cd^\alpha kh(x). \tag{4.10}$$

4.5. Returning to the situation of Theorem 4.2, let $H_i^{(k)} = L_2(\phi_i^{(k)})$ with distinguished vector $\xi_i^{(k)}$, GNS representation $\pi_i^{(k)} : \mathcal{A}_i \to A_i^{(k)} \subset B(H_i^{(k)})$ with $A_i^{(k)} = \overline{\pi_i^{(k)}(\mathcal{A}_i)}$. We define $A^{(k)} = *_{i \in I} A_i^{(k)}$, $H^{(k)} = *_{i \in I} H_i^{(k)}$ and let $\pi^{(k)} : \ \mathcal{A} \to *_{i \in I} A_i^{(k)} \subset B(H^{(k)})$ be the corresponding representation. Let $H^{\mathcal{U}}$ (resp. $H_i^{\mathcal{U}}$) denote the ultraproduct of $(H^{(k)})$ (resp. $(H_i^{(k)})$).
We will use 3.6 when $\phi = *_{i \in I} \phi_i$ and $\phi^{(k)} = *_{i \in I} \phi_i^{(k)}$. We denote by $\pi : \ \mathcal{A} = *_{i \in I} \mathcal{A}_i \to B(H)$ the GNS representation relative to ϕ. We have natural identifications

$$(L_2(\phi), \xi_\phi) = *_{i \in I}(H_i, \xi_i) \quad \text{and} \quad (L_2(\phi^{(k)}), \xi_{\phi^{(k)}}) = *_{i \in I}(H_i^{(k)}, \xi_i^{(k)}).$$

If we assume that $\phi_i^{(k)} \to \phi_i$ pointwise, then $\phi^{(k)} \to \phi$ pointwise on $\mathcal{A}$ and of course $\phi = \phi^{\mathcal{U}}$ on $\mathcal{A}$. Therefore the correspondence $b\xi \mapsto b\xi^{\mathcal{U}} = [b\xi^{(k)}]_{\mathcal{U}}$ is isometric from H to $H^{\mathcal{U}}$ (see 3.7). Similarly, the correspondence $\xi^{op}b \mapsto \xi^{\mathcal{U}}b = [\xi^{(k)}b]_{\mathcal{U}}$ is isometric from from H^* to $H^{\mathcal{U}*}$ (see 3.7). We will denote respectively by $V:\ H \to H^{\mathcal{U}}$ and $W:\ H^* \to H^{\mathcal{U}*}$ these *isometric embeddings*, so that we have for any $b \in \mathcal{A}$

$$V(b\xi) = [b\xi^{(k)}]_{\mathcal{U}} \in H^{\mathcal{U}} \quad \text{and} \quad W(\xi b) = [\xi^{(k)}b]_{\mathcal{U}} \in H^{\mathcal{U}*}. \tag{4.11}$$

Assume now that $\phi_i^{(k)} \xrightarrow{s} \phi_i$. Then (see 3.8) we also have an *isometric embedding*

$$\psi_i:\ A_i \to A_i^{\mathcal{U}} \subset B(H_i^{\mathcal{U}})$$

such that $\psi_i(\pi_i(b)) = \pi_i^{\mathcal{U}}(b)$ for any $b \in \mathcal{A}_i$.

Proof of Theorem 4.2. Let $\phi = *_{i\in I}\phi_i$ and $\phi^{(k)} = *_{i\in I}\phi_i^{(k)}$. Recall $\mathcal{A} = *_{i\in I}\mathcal{A}_i$ (algebraic free product). Note that $\mathcal{A} = \cup_K \mathcal{W}_{\le K}$. The pointwise convergence on $\mathcal{A}$ of $\phi^{(k)}$ to ϕ is obvious by definition of the free product of states. To show the strong convergence it suffices to show that $\lim_{\mathcal{U}} \|\pi^{(k)}(b)\| = \|\pi(b)\|$ for any $b \in \mathcal{A}$ and any non trivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

The main point is that, by the Ricard-Xu inequality, there are constants $c > 0$ and $\alpha > 0$ such that

$$\forall b \in \mathcal{W}_{\le d} \quad (cd^\alpha)^{-1} \lim\nolimits_{\mathcal{U}} \|\pi^{(k)}(b)\| \le \|\pi(b)\| \le cd^\alpha \lim\nolimits_{\mathcal{U}} \|\pi^{(k)}(b)\|. \tag{4.12}$$

If we accept this result, the proof is immediate: we just note that $(b^*b)^m$ is of degree at most $2md$, therefore $(c(2md)^\alpha)^{-1} \lim_{\mathcal{U}} \|\pi^{(k)}((b^*b)^m)\| \le \|\pi((b^*b)^m)\| = \|\pi(b)\|^{2m} \le c(2md)^\alpha \lim_{\mathcal{U}} \|\pi^{(k)}((b^*b)^m)\|$ and $\|\pi^{(k)}((b^*b)^m)\| = \|\pi^{(k)}(b)\|^{2m}$. So we find

$$(c(2md)^\alpha)^{-1/2m} \lim\nolimits_{\mathcal{U}} \|\pi^{(k)}(b)\| \le \|\pi(b)\| \le (c(2md)^\alpha)^{1/2m} \lim\nolimits_{\mathcal{U}} \|\pi^{(k)}(b)\|$$

and letting $m \to \infty$ yields the equality $\lim_{\mathcal{U}} \|\pi^{(k)}(b)\| = \|\pi(b)\|$.

We now turn to the proof of (4.12). By the Ricard-Xu inequality (4.10), we have

$$\forall K \ge 1, \forall b \in \mathcal{W}_{\le K} \quad (cd^\alpha)^{-1} kh(\pi(b)) \le \|\pi(b)\| \le cd^\alpha kh(\pi(b)). \tag{4.13}$$

and

$$(cd^\alpha)^{-1} \lim\nolimits_{\mathcal{U}} kh(\pi^{(k)}(b)) \le \lim\nolimits_{\mathcal{U}} \|\pi^{(k)}(b)\| \le cd^\alpha \lim\nolimits_{\mathcal{U}} kh(\pi^{(k)}(b)).$$

Thus to conclude, it suffices to show that $\lim_{\mathcal{U}} kh(\pi^{(k)}(b)) = kh(\pi(b))$ for any $b \in \mathcal{W}_{\le K}$ and any $K \ge 1$.

Let $b = \sum_0^K b_d$ and $b = \sum_0^K b_d^{(k)}$ be the decomposition of $\mathcal{W}_{\le K}$ into its homogeneous parts relative respectively to ϕ and $\phi^{(k)}$. More precisely, $b_d = \mathcal{P}_d(b)$ and $b_d^{(k)} = \mathcal{P}_d^{(k)}(b)$. By (4.7) we have

$$v^{(k)} b_d = \mathcal{P}_d^{(k)}(v^{(k)} b)$$

and hence by (4.6)

$$\|v^{(k)} b_d - b_d^{(k)}\| \to 0 \tag{4.14}$$

with respect to the maximal C^*-norm on $\mathcal{A}$.

Let us denote by $t_r^{(k)}, s_r^{(k)}$ the mappings t_r, s_r relative to the free product $\phi^{(k)}$.

By (4.2) and (4.9) to conclude it obviously suffices to show

$$\lim_{k\to\infty} \|t_r^{(k)}(b_d^{(k)})\| = \|t_r(b_d)\| \text{ and } \lim_{k\to\infty} \|s_r^{(k)}(b_d^{(k)})\| = \|s_r(b_d)\|.$$

By (4.14), it actually suffices to show

$$\lim_{k\to\infty} \|t_r^{(k)}(v^{(k)}b_d)\| = \|t_r(b_d)\| \text{ and } \lim_{k\to\infty} \|s_r^{(k)}(v^{(k)}b_d)\| = \|s_r(b_d)\|. \tag{4.15}$$

Fix d. We may assume that b_d is a finite sum of the form $b_d = \sum_{\alpha=1}^N b(\alpha)$ with $b(\alpha)$ of the form $b(\alpha) = x_1(\alpha)\cdots x_d(\alpha)$ where $x_1(\alpha) \in \overset{\circ}{\mathcal{A}}_{i_1}, \cdots, x_d(\alpha) \in \overset{\circ}{\mathcal{A}}_{i_d}$ and $i_1 \neq i_2 \neq \cdots \neq i_d$.

We remind the reader that ξ denotes the distinguished unit vector in $H = *_{i\in I} H_i$, and

$$t_r(b_d) = \sum\nolimits_\alpha x_1(\alpha)\cdots x_r(\alpha)\xi \otimes \xi x_{r+1}(\alpha)\cdots x_d(\alpha) \in H \otimes H^*$$

is viewed as an element of $B(H)$. Moreover, in the case $I = \{1,2\}$, $s_r(b_d) \in B(H) \otimes [A_1 \oplus A_2]$.
By (4.5), we have

$$v^{(k)}(b_d) = \sum\nolimits_\alpha v^{(k)}(x_1(\alpha)\cdots x_d(\alpha)) = v^{(k)}(x_1(\alpha))\cdots v^{(k)}(x_d(\alpha)) \tag{4.16}$$

and also for any $1 \le j \le d$ we have $v^{(k)}(x_j(\alpha)) = x_j(\alpha) - \phi^{(k)}(x_j(\alpha))1$ and hence

$$\|v^{(k)}(x_j(\alpha)) - x_j(\alpha)\| \to 0 \tag{4.17}$$

where the norm is (say) the maximal C^*-norm on $\mathcal{A}$.

Recall that $H^{\mathcal{U}}$ denotes the Hilbert space ultraproduct of the free products defined by $(H^{(k)}, \xi^{(k)}) = *_{i\in I}(H_i^{(k)}, \xi_i^{(k)})$. We should compare $t_r(b_d)$ with $t_r^{(k)}(v^{(k)}b_d) \in H^{(k)} \otimes H^{(k)*}$. By (4.16) and (4.17) we have

$$\|t_r^{(k)}(v^{(k)}b_d) - \sum\nolimits_\alpha x_1(\alpha)\cdots x_r(\alpha)\xi^{(k)} \otimes \xi^{(k)} x_{r+1}(\alpha)\cdots x_d(\alpha)\|_{B(H^{(k)})} \to 0. \tag{4.18}$$

Let

$$T_r^{(k)} = \sum\nolimits_\alpha x_1(\alpha)\cdots x_r(\alpha)\xi^{(k)} \otimes \xi^{(k)} x_{r+1}(\alpha)\cdots x_d(\alpha),$$
$$T_r = \sum\nolimits_\alpha [x_1(\alpha)\cdots x_r(\alpha)\xi^{(k)}]_{\mathcal{U}} \otimes [\xi^{(k)} x_{r+1}(\alpha)\cdots x_d(\alpha)]_{\mathcal{U}}.$$

By (3.1) we have

$$\|T_r\|_{B(H^{\mathcal{U}})} = \lim\nolimits_{\mathcal{U}} \|T_r^{(k)}\|_{B(H^{(k)})}. \tag{4.19}$$

Consider now (see 4.5) the *isometries* $V:\ H \to H^{\mathcal{U}}$ and $W:\ H^* \to H^{\mathcal{U}*}$. We have then, by (4.11)

$$(V \otimes W)(t_r(b_d)) = T_r$$

from which $\|T_r\|_{B(H^{\mathcal{U}})} = \|t_r(b_d)\|_{B(H)}$ follows. Now by (4.18) and (4.19) we conclude that

$$\|t_r(b_d)\|_{B(H)} = \lim\nolimits_{\mathcal{U}} \|t_r^{(k)}(v^{(k)}b_d)\|_{B(H^{(k)})}.$$

So far we used only the pointwise convergence.

Similarly, assuming $I = \{1,2\}$ for simplicity, we should compare $s_r(b_d)$ with $s_r^{(k)}(v^{(k)}b_d)$. We set

$$S_r^{(k)} = \sum\nolimits_\alpha x_1(\alpha)\cdots x_{r-1}(\alpha)\xi^{(k)} \otimes \xi^{(k)} x_{r+1}(\alpha)\cdots x_d(\alpha) \otimes J^{(k)}\pi^{(k)}v^{(k)}(x_r(\alpha)),$$

$$S_r = \sum\nolimits_\alpha [x_1(\alpha)\cdots x_{r-1}(\alpha)\xi^{(k)}]_{\mathcal{U}} \otimes [\xi^{(k)} x_{r+1}(\alpha)\cdots x_d(\alpha)]_{\mathcal{U}} \otimes [J^{(k)}\pi^{(k)}v^{(k)}(x_r(\alpha))]_{\mathcal{U}}.$$

By (4.16) and (4.17) we have

$$\|s_r^{(k)}(v^{(k)}b_d) - S_r^{(k)}\|_{B(H^{(k)})\otimes_{\min}(A_1^{(k)}\oplus A_2^{(k)})} \to 0,$$

and hence

$$\lim\nolimits_{\mathcal{U}} \|s_r^{(k)}(v^{(k)}b_d)\|_{B(H^{(k)})\otimes_{\min}(A_1^{(k)}\oplus A_2^{(k)})} = \lim\nolimits_{\mathcal{U}} \|S_r^{(k)}\|_{B(H^{(k)})\otimes_{\min}(A_1^{(k)}\oplus A_2^{(k)})}. \tag{4.20}$$

By (3.3)

$$\|S_r\|_{B(H^{\mathcal{U}})\otimes_{\min}(A_1^{\mathcal{U}}\oplus A_2^{\mathcal{U}})} = \lim\nolimits_{\mathcal{U}} \|S_r^{(k)}\|_{B(H^{(k)})\otimes_{\min}(A_1^{(k)}\oplus A_2^{(k)})}. \tag{4.21}$$

Now since we assume *strong* convergence, as explained in 4.5 we have *isometric embeddings*

$$\psi_1 : \ A_1 \to A_1^{\mathcal{U}} \text{ and } \psi_2 : \ A_2 \to A_2^{\mathcal{U}}$$

such that

$$S_r = (V \otimes W) \otimes [\psi_1 \oplus \psi_2](s_r(b_d))$$

from which $\|S_r\| = \|s_r(b_d)\|$ follows. Thus, by (4.20) and (4.21), we obtain (4.15), and this concludes the proof. $\square$

We now turn to the situation considered in the abstract.

Corollary 4.3. *Let $(A_1^{(k)}, \phi_1^{(k)})$, $(A_2^{(k)}, \phi_2^{(k)})$, (A_1, ϕ_1) and (A_2, ϕ_2) be faithful C^*-probability spaces. Let $\{X_m^{(k)} \mid m \in \mathcal{I}\} \subset A_1^{(k)}$, $\{Y_n^{(k)} \mid n \in \mathcal{J}\} \subset A_2^{(k)}$ and $\{X_m \mid m \in \mathcal{I}\} \subset A_1$, $\{Y_n \mid n \in \mathcal{J}\} \subset A_2$ be families of non-commutative random variables, generating respectively $A_1^{(k)}, A_2^{(k)}, A_1, A_2$. Assume $\{X_m^{(k)} \mid m \in \mathcal{I}\} \xrightarrow{s} \{X_m \mid m \in \mathcal{I}\}$ and $\{Y_n^{(k)} \mid n \in \mathcal{J}\} \xrightarrow{s} \{Y_n \mid n \in \mathcal{J}\}$ when $k \to \infty$. Assume moreover that, for each k, $\{X_m^{(k)} \mid m \in \mathcal{I}\}$ and $\{Y_n^{(k)} \mid n \in \mathcal{J}\}$ are $*$-free and also that $\{X_m \mid m \in \mathcal{I}\}$ and $\{Y_n \mid n \in \mathcal{J}\}$ are $*$-free. Then the joint family $\{X_m^{(k)}, Y_n^{(k)} \mid m \in \mathcal{I}, n \in \mathcal{J}\}$ viewed as sitting in the free product $(A_1^{(k)}, \phi_1^{(k)}) * (A_2^{(k)}, \phi_2^{(k)})$ tends strongly to $\{X_m, Y_n \mid m \in \mathcal{I}, n \in \mathcal{J}\}$, viewed as sitting in the free product $(A_1, \phi_1) * (A_2, \phi_2)$.*

Proof. Let $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) be the unital $*$-algebra generated by non-commutative (i.e. algebraically free) variables $\{x_m, x_m^* \mid m \in \mathcal{I}\}$ (resp. $\{y_n, y_n^* \mid n \in \mathcal{J}\}$). We define the associated state ϕ_1 on $\mathcal{A}_1$ by $\phi_1(P) = \phi(P(X_m, X_m^*))$ and similarly for ϕ_2 on $\mathcal{A}_2$. Repeating this for each k, this leads to states $\phi_i^{(k)}$ on $\mathcal{A}_i$ for $i = 1, 2$. We may identify the free product $\mathcal{A}_1 * \mathcal{A}_2$ with the unital $*$-algebra generated by non-commutative variables $\{x_m, x_m^*, y_n, y_n^* \mid m \in \mathcal{I}, n \in \mathcal{J}\}$. Since the GNS representations are assumed isometric, the Corollary appears as a particular case of the preceding Theorem. $\square$

Remark 4.4. If a state ϕ is faithful on A, then its associated GNS representation is faithful. Moreover, the GNS representation associated to the restriction of ϕ to any unital C^*-subalgebra of A is still faithful, so the requirement that the variables generate the C^*-algebras can be dispensed with if we assume all states faithful. Moreover, if ϕ is faithful on A, then for any $x \in A$ we have

$$\|\pi_\phi(x)\| = \lim_{p\to\infty} \uparrow (\phi((x^*x)^p))^{1/2p}.$$

4.6. With the notation of the Corollary, whenever $X^{(k)} = \{X_m^{(k)}, X_m^{(k)*} \mid m \in \mathcal{I}\}$ on $(A^{(k)}, \phi^{(k)})$ converges in moments to $\{X_m, X_m{}^* \mid m \in \mathcal{I}\}$ on (A, ϕ), it is well known that for any non-trivial ultrafilter $\mathcal{U}$ and any polynomial P, we have

$$\|P(X)\| \leq \lim_{\mathcal{U}} \|P(X^{(k)})\|. \tag{4.22}$$

Indeed, for any $\varepsilon > 0$ there are polynomials Q, R with unit norm in $L_2(\phi)$ so that

$$\|P(X)\| - \varepsilon < |\phi(P(X)Q(X)R(X))| = \lim_{\mathcal{U}} |\phi^{(k)}(P(X^{(k)})Q(X^{(k)})R(X^{(k)}))| \leq \lim_{\mathcal{U}} \|P(X^{(k)})\|,$$

from which (4.22) follows. The converse inequality is the essence of strong convergence.

5. Commutative case. Random matrices

With the same notation as 4.6, if the variables $X = \{X_m, X_m{}^* \mid m \in \mathcal{I}\}$ all commute i.e. we are dealing with a family $\{X_m \mid m \in \mathcal{I}\}$ of commuting normal operators in $B(H)$, then the (commutative) C^*-algebra A they generate in $B(H)$ is isometric to the C^*-algebra $C(K)$ of all continuous functions on a compact set K, namely the spectrum of A. Moreover, ϕ corresponds to a probability measure on K. Assume $\mathcal{I}$ finite for simplicity. Then the situation reduces to the following: We have a probability measure μ with support a compact set $K \subset \mathbb{C}^{\mathcal{I}}$ and $X_m \in C(K)$ is defined by $X_m(\lambda) = \lambda_m$ for all $\lambda = (\lambda_m) \in K$. For any $a \in \mathcal{A}$, $\phi(a) = \int a d\mu$ and $\pi_\phi(a)$ is the operator of multiplication by a on $L_2(\mu)$. When the family (X_m) is reduced to a single normal operator X, K is the spectrum of X.

Similarly, if all $\{X_m^{(k)}, X_m^{(k)*} \mid m \in \mathcal{I}\}$ commute, we may reduce consideration to $A^{(k)} = C(K^{(k)})$ with $\phi^{(k)} = \mu^{(k)}$ for some $K^{(k)} \subset \mathbb{C}^{\mathcal{I}}$ and some probability $\mu^{(k)}$ with support $K^{(k)}$, and again $X_m^{(k)}(\lambda) = \lambda_m$ for all $\lambda \in K^{(k)}$.

Then $X^{(k)} \xrightarrow{s} X$ implies that for any polynomial f in $\lambda_m, \bar{\lambda}_m$ we have

$$\sup_{\lambda \in K^{(k)}} |f(\lambda)| \to \sup_{\lambda \in K} |f(\lambda)|. \tag{5.1}$$

Taking $f(\lambda) = \lambda_m$ we see that the sets $\cup_k K^{(k)} \subset \mathbb{C}$ and K are all included in some compact set $L \subset \mathbb{C}^{\mathcal{I}}$. By the Stone-Weierstrass theorem on L, (5.1) remains valid for any continuous function on L. Applying this to $f(\lambda) = d(\lambda, K)$ we find

$$\lim_{k \to \infty} \sup_{\lambda \in K^{(k)}} d(\lambda, K) = 0. \tag{5.2}$$

The mere convergence in moments $X^{(k)} \to X$ is equivalent (by Stone-Weierstrass on L) to the weak convergence $\mu^{(k)} \to \mu$. In the language of probability, the latter means that $X^{(k)} \to X$ "in distribution", or "in law". Thus $X^{(k)} \xrightarrow{s} X$ iff $\mu^{(k)} \to \mu$ weakly and (5.2) holds.

Wigner's classical theorem about the convergence of the eigenvalues of Gaussian random matrices was strengthened in [7] as follows: let $X^{(k)}(\omega)$ be a random $k \times k$-matrix the entries of which are independent complex Gaussian $N(0, k^{-1})$, then for almost all ω, the *nonnormal* random matrices $X^{(k)}(\omega)$ tend strongly to a circular random variable. A similar result is valid for the classical Gaussian Wigner Hermitian (and hence normal) matrices (model for the so-called GUE) now with a semi-circular limit. In the latter case, if $\mu^{(k)}(\omega)$ is the spectral probability distribution of the eigenvalues of $X^{(k)}(\omega)$, with support $K^{(k)}(\omega)$, then $\mu^{(k)}(\omega)$ tends weakly to the "circular" probability measure on $K = [-2, 2]$ and (5.2) holds for almost all ω. A similar result holds for random unitary $k \times k$-matrices. In that case the limit μ is the uniform Haar probability on the unit circle.

In Random Matrix Theory, (5.2) for the spectra of random matrices, is viewed as a result on "the edge of the spectrum", while mere weak convergence deals with "the bulk of the spectrum". See [7, 5, 8, 15, 1] for more general results.

6. Strong convergence and operator spaces

Let (A,ϕ) and $(A^{(k)},\phi^{(k)})$ be C^*-probability spaces, with ϕ and $\phi^{(k)}$ all faithful. For any $N \geq 1$, we equip $M_N(A)$ with the faithful state $\phi \otimes \tau_N$ where τ_N is the normalized trace on M_N (and similarly for $M_N(A^{(k)})$). Let $\{X_m \mid m \in \mathcal{I}\} \subset A$ and $\{X_m^{(k)} \mid m \in \mathcal{I}\} \subset A^{(k)}$.
If $\{X_m^{(k)} \mid m \in \mathcal{I}\} \xrightarrow{s} \{X_m \mid m \in \mathcal{I}\}$, then for any N, any $a \in M_N$ and any finitely supported family $\{a_m \mid m \in \mathcal{I}\} \subset M_N$, we have

$$1 \otimes a + \sum\nolimits_m X_m^{(k)} \otimes a_m \xrightarrow{s} 1 \otimes a + \sum\nolimits_m X_m \otimes a_m. \tag{6.1}$$

In particular we have for any such N, a, a_m and for any non trivial $\mathcal{U}$

$$\lim_{\mathcal{U}} \|1 \otimes a + \sum\nolimits_m X_m^{(k)} \otimes a_m\|_{M_N(A^{(k)})} = \|1 \otimes a + \sum\nolimits_m X_m \otimes a_m\|_{M_N(A)}. \tag{6.2}$$

Indeed, the strong convergence implies that the mapping $u: \ P(X_m) \mapsto P(X_m^{\mathcal{U}})$ is an isometric, and hence completely isometric, $*$-homomorphism. Therefore if we restrict u to the linear span, denoted by E, of the unit and $\{X_m \mid m \in \mathcal{I}\}$ then we obtain (6.2). Let $S = 1 \otimes a + \sum_m X_m \otimes a_m$, and let P be a polynomial in noncommuting variables x, x^*. Then $P(S) = 1 \otimes a' + \sum_m P_m(X_m) \otimes a'_m$ for some $*$-polynomials (P_m) and some $a', a'_m \in M_N$. Clearly if $(X_m^{(k)}) \xrightarrow{s} (X_m)$ then $(P_m(X_m^{(k)})) \xrightarrow{s} (P_m(X_m))$. Therefore, (6.2) applied with $(P_m(X_m^{(k)}))$ and $(P_m(X_m))$ in place of $X_m^{(k)}$ and X_m gives us (6.1).

In the converse direction, we have the following two "linearization tricks" :

Proposition 6.1 ([13])**.** *If the operators $\{X_m^{(k)} \mid m \in \mathcal{I}\}$ converge in moments and are all unitary, then* (6.2) *implies $\{X_m^{(k)} \mid m \in \mathcal{I}\} \xrightarrow{s} \{X_m \mid m \in \mathcal{I}\}$.*

Proof. Indeed, by [13, Prop.1.7], if $u_{|E}$ is completely contractive, then $u: \ P(X_m) \mapsto P(X_m^{\mathcal{U}})$ is contractive. By 4.6, since we assume convergence in moments, u is isometric. $\square$

Proposition 6.2 ([7])**.** *If the operators $\{X_m^{(k)} \mid m \in \mathcal{I}\}$ converge in moments and are all Hermitian, then* (6.1) *restricted to Hermitian matrices a, a_m implies that $\{X_m^{(k)} \mid m \in \mathcal{I}\} \xrightarrow{s} \{X_m \mid m \in \mathcal{I}\}$.*

Proof. Indeed, let $S^{(k)} = 1 \otimes a + \sum_m X_m^{(k)} \otimes a_m$. If $S^{(k)}$ is Hermitian, and $S^{(k)} \xrightarrow{s} S$ then (5.2) holds, $K^{(k)}$ and K being the spectra of $S^{(k)}$ and S. This implies that the spectrum of $S^{\mathcal{U}}$ is included in that of S. Thus the Proposition follows from [7, Th.2.2] (recalling 4.6). $\square$

7. Amalgamated free products

Since the Ricard-Xu version of the Khintchine inequalities extends to reduced free products with amalgamation (see [14, §5]), it is not surprising that the preceding approach also does, as was kindly pointed out to the author independently by Éric Ricard and Paul Skoufranis.

We should first define what is meant by strong convergence in this framework.

Let A be a unital C^*-algebra, with a unital C^*-subalgebra $D \subset A$. Let $\phi: \ A \to D$ be a conditional expectation. As in [4, p. 138] we denote by $L_2(\phi)$ the right Hilbert D-module obtained by the classical GNS construction. We refer to [10, 3] for more information on Hilbert C^*-modules. The D-valued inner product of two elements $\dot{a}, \dot{b}$ of $L_2(\phi)$ associated to $a, b \in A$ is defined by $\langle \dot{a}, \dot{b} \rangle = \phi(a^*b)$. We have $\|\dot{a}\|_{L_2(\phi)} = \|\langle \dot{a}, \dot{a} \rangle\|_D^{1/2}$. We will denote by $\pi_\phi : A \to \mathbb{B}(L_2(\phi))$ the associated representation of A in the C^*-algebra $\mathbb{B}(L_2(\phi))$ of adjointable maps on $L_2(\phi)$, so that $\pi_\phi(a)\dot{b} = \dot{ab}$. Let $\xi_\phi = \dot{1} \in L_2(\phi)$ so that $\dot{a} = \pi_\phi(a)\xi_\phi$. Then $\pi_\phi(A)\xi_\phi$ is dense in $L_2(\phi)$, and $\phi(a) = \langle \xi_\phi, \pi_\phi(a)\xi_\phi \rangle$. We will denote simply $\pi_\phi(a)\xi_\phi = a\xi_\phi$.

We say that a conditional expectation ϕ is nondegenerate if π_ϕ is faithful (i.e. isometric). We should warn the reader that although this (well established) notation does not reflect it, the preceding notions obviously all depend on D.

Let $H = L_2(\phi)$. By definition any adjointable $T : H \to H$ admits an adjoint $T^* : H \to H$ such that, as usual, $\langle T^*y, x \rangle = \langle y, Tx \rangle$.
Let $H^* = \mathbb{B}(H, D)$. Clearly, H^* is antilinearly isomorphic to H, via the correspondence $h \in H \mapsto t_h \in H^*$ defined by $t_h(y) = \langle h, y \rangle$. The space H^* is equipped with the left Hilbert D-module structure defined by $d \cdot t_h = t_{hd^*}$, and D-valued inner product $\langle t_k, t_h \rangle = \langle h, k \rangle$ $(h, k \in H)$. Let $\xi^{op} \in H^*$ be defined by $\xi^{op}(h) = \langle \xi, h \rangle = \phi(h)$. We have a natural embedding $A^{op} \subset H^*$ that we will write as $a \mapsto \xi^{op}a$, with the notation $(fa)(x) = f(ax)$ when $f \in H^*$, $x \in H$, $a \in A$. Then

$$\|\xi^{op}a\|_{H^*} = \|\phi(aa^*)\|_D^{1/2},$$

and $\xi^{op}A$ is dense in H^*.
We have a natural representation $\pi_\phi^{op} : A^{op} \to \mathbb{B}(H^*)$, defined as before: $\pi_\phi^{op}(a)$ is the transpose of the adjointable map $\pi_\phi(a)$, i.e. $\pi_\phi^{op}(a) = {}^t\pi_\phi(a)$, in the following sense: The operator ${}^tT : H^* \to H^*$ is characterized by the identity ${}^tT(\xi^{op}a)(x) = \phi(aT(x))$ $(a \in A, x \in H)$.
Moreover $\|{}^tT\|_{\mathbb{B}(H^*)} = \|T\|_{\mathbb{B}(H)}$. Equivalently, if we define $\bar{h} \in H^*$ by $\bar{h}(x) = \langle h, x \rangle$, i.e. we set $\bar{h} = t_h$, then we may write

$${}^tT(\bar{h}) = \overline{T^*(h)}.$$

Let $\mathcal{A}$ be a dense unital $*$-subalgebra of a C^*-algebra B such that $D \subset \mathcal{A} \subset B$. We will say that a mapping $\phi : \mathcal{A} \to D$ is a conditional expectation if it extends to a conditional expectation on B with range D. The GNS construction produces a Hilbert D-module $H = L_2(\phi)$ with distinguished unit vector ξ_ϕ and a $*$-homomorphism $\pi_\phi : \mathcal{A} \to \mathbb{B}(H)$ such that $\phi(x) = \langle \xi_\phi, \pi_\phi(x)\xi_\phi \rangle$ for any $x \in \mathcal{A}$.
Let $A = \overline{\pi_\phi(\mathcal{A})}$. Then for any $d \in D$, $\pi_\phi(d)$ is the left action of d on H. Thus $\pi_\phi(D)$ is a copy of D in A and $\psi : x \mapsto \pi_\phi(\langle \xi_\phi, x\xi_\phi \rangle)$ is a conditional expectation from A onto $\pi_\phi(D) \simeq D$.

We will say that a sequence $\phi^{(k)} : \mathcal{A} \to D$ $(k \in \mathbb{N})$ of conditional expectations tends strongly to $\phi : \mathcal{A} \to D$, if it converges pointwise to ϕ and if moreover we have $\|\pi_{\phi^{(k)}}(a)\| \to \|\pi_\phi(a)\|$ for any $a \in \mathcal{A}$. We denote this again by $\phi^{(k)} \xrightarrow{s} \phi$.

Let B_i $(i \in I)$ be a family of unital C^*-algebras. Let $\mathcal{A}_i \subset B_i$ be dense $*$-subalgebras, each $\mathcal{A}_i$ containing a unital copy of a fixed C^*-algebra D, given with conditional expectations $\phi_i : \mathcal{A}_i \to D$. Let $\mathcal{A}$ denote the $*$-algebra that is the algebraic free product, amalgamated over D, of the family $(\mathcal{A}_i)_{i \in I}$. The free product of the family of conditional expectations $\phi_i : \mathcal{A}_i \to D$ will be defined below as a conditional expectation $\phi : \mathcal{A} \to D$, but we need more notation to make this clear, in part because the free product requires nondegenerate conditional expectations.

Let $H_i = L_2(\phi_i)$ and $\xi_i = \xi_{\phi_i}$. Let $A_i = \overline{\pi_{\phi_i}(\mathcal{A})} \subset \mathbb{B}(H_i)$. Then D can be identified with $\pi_{\phi_i}(D) \subset A_i$ isometrically, and $\psi_i(x) = \pi_{\phi_i}(\langle \xi_i, x\xi_i \rangle)$ is a *nondegenerate* conditional expectation from A_i onto $\pi_{\phi_i}(D) \simeq D$.

We refer the reader to [4, p. 138] for the precise definition of the reduced free product $A = *_D(A_i, \psi_i)$ of the family $(A_i)_{i \in I}$ with respect to *nondegenerate* conditional expectations $(\psi_i)_{i \in I}$: One first introduces the right Hilbert D module that is the free product (H,ξ) of the family (H_i,ξ_i), and the $*$-homomorphism $\pi:\mathcal{A}\to\mathbb{B}(H)$ associated to the algebraic free product of the family of morphisms $\pi_{\phi_i}:\mathcal{A}_i\to A_i$ (acting on H on the left). Let $A=\overline{\pi(\mathcal{A})}\subset\mathbb{B}(H)$. Lastly $*_{i\in I}\phi_i:\ \mathcal{A}\to D$ is defined by

$$(*_{i\in I}\phi_i)(a)=\langle\xi,\pi(a)\xi\rangle.$$

This is a conditional expectation extending each ϕ_i.

Theorem 7.1. *Let $D\subset\mathcal{A}_i$ $(i\in I)$ be a family of unital C^*-algebras, as above. Let $\{\phi_i^{(k)}\mid i\in I\}$ $(k\in\mathbb{N})$ be a sequence of families of mappings, each $\phi_i^{(k)}$ being a conditional expectation from $\mathcal{A}_i$ onto D. Assume that we have conditional expectations $\phi_i:\ \mathcal{A}_i\to D$ such that, for each $i\in I$, $\phi_i^{(k)}\xrightarrow{s}\phi_i$ when $k\to\infty$. Then*

$$*_{i\in I}\phi_i^{(k)}\xrightarrow{s}*_{i\in I}\phi_i.$$

In the amalgamated case, (4.2) remains valid provided the norms of s_r and t_r are interpreted as follows. Assume $I=\{1,2\}$ for simplicity. As before $\mathcal{A}=\cup_d\mathcal{W}_{\le d}$ with the same meaning of $\mathcal{W}_{\le d}$ or $\mathcal{W}_d$.

Let $b\in\mathcal{A}$. Let $b=\sum b_d$ be its decomposition into homogeneous terms. Fix d. We may assume that b_d is a finite sum of the form $b_d=\sum_{\alpha=1}^N b(\alpha)$ with $b(\alpha)$ of the form $b(\alpha)=x_1(\alpha)\cdots x_d(\alpha)$ We then set as before

$$t_r(b_d)=\sum\nolimits_\alpha x_1(\alpha)\cdots x_r(\alpha)\xi\otimes\xi^{op}x_{r+1}(\alpha)\cdots x_d(\alpha)\in H\otimes H^* \tag{7.1}$$

and

$$s_r(b_d)=\sum\nolimits_\alpha x_1(\alpha)\cdots x_{r-1}(\alpha)\xi\otimes\xi^{op}x_{r+1}(\alpha)\cdots x_d(\alpha)\otimes J(x_r)\in H\otimes H^*\otimes[A_1\oplus A_2]. \tag{7.2}$$

Then $\|t_r(b_d)\|$ is the norm in $\mathbb{B}(H)$ and $\|s_r(b_d)\|$ is its norm in $\mathbb{B}(H)\otimes_{\min}[A_1\oplus A_2]$.

With this reinterpretation, (1.1) remains valid (see [14, §5]). Thus, with the obvious extension of the previous notation, the proof of Theorem 7.1 boils down to check that we still have (4.15). One difficulty is that Hilbert modules do not necessarily admit orthonormal bases, so that we cannot argue as we did above to check (3.1). Instead, following Ricard and Xu in [14], we restrict without loss of generality to the case of separable C^*-algebras and a countable set of indices I. To justify this reduction to the separable case, observe that for any countable subset $S\subset A$ in a unital C^*-algebra A equipped with a conditional expectation $\phi:\ A\to D$, there are separable unital C^*-subalgebras $\widetilde{A}\subset A$ and $\widetilde{D}\subset D\cap\widetilde{A}$ such that $\phi_{|\widetilde{A}}$ is a conditional expectation onto $\widetilde{D}$. Using this, we may assume all our Hilbert modules countably generated and, as in [14], we can use Kasparov's absorption Theorem, for which we refer to [10, p. 60]. The latter says that for any countably generated Hilbert module H there exists a (contractive and D-modular) factorization of the identity of H through the standard (column) module formed of sequences $x=(x_n)\in D^{\mathbb{N}_*}$ such that the series $\sum x_n^*x_n$ converges in norm in D, equipped with the D-valued inner product such that $\langle x,x\rangle=\sum x_n^*x_n$. We denote this module by $C(D)$. Let $C_n(D)$ be the submodule formed of all $x=(x_n)\in D^{\mathbb{N}_*}$ supported in $[1,n]$. Then the union $\cup_nC_n(D)$ is norm dense in $C(D)$. It follows that there is a sequence of contractive module mappings $F_n:\ H\to C_n(D)$ and $G_n:\ C_n(D)\to H$ such that

$$G_nF_n(x)\to x \text{ for any } x\in H \text{ and } (G_nF_n)^{op}(y)\to y \text{ for any } y\in H^*. \tag{7.3}$$

We will apply this to $H=L_2(\phi)$. Recall that the algebraic free product $\mathcal{A}$ is such that $\mathcal{A}\xi$ is dense in H. We may assume that $\|F_n\|<1$ and $\|G_n\|<1$. Then by an elementary approximation argument we may assume that, for each n, there are $\{p_j^n \mid 1 \le j \le n\} \subset \mathcal{A}$ and $\{q_j^n \mid 1 \le j \le n\} \subset \mathcal{A}$ such that

$$\forall x \in H \ \forall (c_j) \in C_n(D) \quad F_n(x) = (\phi(p_j^{n*}x)) \text{ and } G_n((c_j)) = \sum\nolimits_j q_j^n c_j \xi.$$

Note that

$$\|F_n\| = \|\sum\nolimits_j \phi(p_j^{n*}p_j^n)\|^{1/2} \text{ and } \|G_n\| = \|\sum\nolimits_j \phi(q_j^n q_j^{n*})\|^{1/2}.$$

By the pointwise convergence $\phi_k \to \phi$, for any fixed n we have

$$\lim\nolimits_{k,\mathcal{U}} \|\sum\nolimits_j \phi^{(k)}(p_j^{n*}p_j^n)\|^{1/2} < 1 \text{ and } \lim\nolimits_{k,\mathcal{U}} \|\sum\nolimits_j \phi^{(k)}(q_j^n q_j^{n*})\|^{1/2} < 1.$$

Equivalently, if we define $F_n^{(k)} : \ H^{(k)} \to C_n(D)$ and $G_n^{(k)} : \ C_n(D) \to H^{(k)}$ by $F_n^{(k)}(x\xi^{(k)}) = (\phi^{(k)}(p_j^{n*}x))$ and $G_n^{(k)}((c_j)) = \sum_j q_j^n c_j \xi^{(k)}$. Then for all k large enough we have

$$\|F_n^{(k)}\|_{\mathbb{B}(H^{(k)},C_n(D))} < 1 \text{ and } \|G_n^{(k)}\|_{\mathbb{B}(C_n(D),H^{(k)})} = \|G_n^{(k)op}\|_{\mathbb{B}(H^{(k)*},C_n(D)^*)} < 1. \tag{7.4}$$

Lemma 7.2. *Fix $\varepsilon > 0$ and $N \ge 1$. Let $x(\alpha), y(\alpha) \in \mathcal{A}$ $(1 \le \alpha \le N)$, Then there is an n such that*

$$\forall \alpha \le N \quad \|x(\alpha)\xi - G_n F_n x(\alpha)\xi\|_H < \varepsilon \text{ and } \|\xi^{op}y(\alpha) - (G_n F_n)^{op}\xi^{op}y(\alpha)\|_{H^*} < \varepsilon \tag{7.5}$$

and moreover such that for all k large enough we have

$$\|x(\alpha)\xi^{(k)}\|_{H^{(k)}} \le (1+\varepsilon)\|x(\alpha)\xi\|_H \ , \ \|\xi^{(k)op}y(\alpha)\|_{H^{(k)*}} \le (1+\varepsilon)\|\xi^{op}y(\alpha)\|_{H^*}$$

$$\|x(\alpha)\xi^{(k)} - G_n^{(k)}F_n^{(k)}x(\alpha)\xi^{(k)}\|_{H^{(k)}} < \varepsilon \text{ and } \|\xi^{(k)op}y(\alpha) - (G_n^{(k)}F_n^{(k)})^{op}\xi^{(k)op}y(\alpha)\|_{H^{(k)*}} < \varepsilon.$$

Proof. For any $x \in \mathcal{A}$, and any n we have

$$G_n F_n x\xi = \sum_j q_j^n \phi(p_j^{n*}x) \text{ and } G_n^{(k)}F_n^{(k)}x\xi^{(k)} = \sum_j q_j^n \phi^{(k)}(p_j^{n*}x).$$

Therefore

$$\lim_{\mathcal{U}} \|x\xi^{(k)} - G_n^{(k)}F_n^{(k)}x\xi^{(k)}\|_{H^{(k)}} = \|x\xi - G_n F_n x\xi\|_H \text{ and } \lim_{\mathcal{U}} \|x\xi^{(k)}\|_{H^{(k)}} = \|x\xi\|_H. \tag{7.6}$$

By (7.3), we can choose an n large enough such that (7.5) holds. Then the other conditions can be achieved using (7.6). $\square$

Proof of Theorem 7.1. By the above observations, it suffices to check that we still have (4.15). We may assume $t_r(b_d) = \sum_\alpha x(\alpha)\xi \otimes \xi^{op}y(\alpha)$, so that

$$(F_n \otimes G_n^{op})[t_r(b_d)] = \sum\nolimits_\alpha F_n(x(\alpha)\xi) \otimes G_n^{op}(\xi^{op}y(\alpha)) \in C_n(D) \otimes C_n(D)^*.$$

The latter defines a mapping in $\mathbb{B}(C_n(D))$ defined by

$$(d_j) \mapsto (\sum\nolimits_j a_{ij}d_j)_i$$

where

$$a_{ij} = \sum\nolimits_\alpha \phi(p_i^{n*}x(\alpha))\phi(y(\alpha)q_j^n). \tag{7.7}$$

By Lemma 7.2 there is an n such that

$$\|t_r(b_d)\|_{\mathbb{B}(H)} \leq (1+\varepsilon)\|(G_nF_n)\otimes(G_nF_n)^{op}[t_r(b_d)]\|_{\mathbb{B}(H)}$$

and such that for all k large enough (recall that, by (4.6), $v^{(k)}b_d$ is just a perturbation of b_d)

$$\|t_r^{(k)}(v^{(k)}b_d)\|_{\mathbb{B}(H^{(k)})} \leq (1+\varepsilon)\|(G_n^{(k)}F_n^{(k)})\otimes(G_n^{(k)}F_n^{(k)})^{op}[t_r^{(k)}(v^{(k)}b_d)]\|_{\mathbb{B}(H^{(k)})}.$$

Since F_n, G_n are contractions

$$\|t_r(b_d)\|_{\mathbb{B}(H)} \leq (1+\varepsilon)\|(F_n\otimes G_n^{op})[t_r(b_d)]\|_{\mathbb{B}(C_n(D))} \leq (1+\varepsilon)\|t_r(b_d)\|_{\mathbb{B}(H)}. \tag{7.8}$$

and using (7.4)

$$\|t_r^{(k)}(v^{(k)}b_d)\|_{\mathbb{B}(H^{(k)})} \leq (1+\varepsilon)\|(F_n^{(k)}\otimes G_n^{(k)op})[t_r^{(k)}(v^{(k)}b_d)]\|_{\mathbb{B}(C_n(D))} \leq (1+\varepsilon)\|t_r^{(k)}(v^{(k)}b_d)\|_{\mathbb{B}(H^{(k)})}. \tag{7.9}$$

But now assuming $t_r(b_d) = \sum_\alpha x(\alpha)\xi\otimes\xi^{op}y(\alpha)$, then $(F_n\otimes G_n^{op})[t_r(b_d)]$ acts on $C_n(D)$ as the matrix $a = [a_{ij}] \in M_n(D)$ defined by (7.7). Thus we have by (7.8)

$$\|t_r(b_d)\|_{\mathbb{B}(H)} \leq (1+\varepsilon)\|[a_{ij}]\|_{M_n(D)} \leq (1+\varepsilon)\|t_r(b_d)\|_{\mathbb{B}(H)}.$$

Let

$$a_{ij}^{(k)} = \sum\nolimits_\alpha \phi^{(k)}(p_i^{n*}[v^{(k)}x(\alpha)])\phi^{(k)}([v^{(k)}y(\alpha)]q_j^n).$$

Then by (7.9)

$$\|t_r^{(k)}(v^{(k)}b_d)\|_{\mathbb{B}(H^{(k)})} \leq (1+\varepsilon)\|[a_{ij}^{(k)}]\|_{M_n(D)} \leq (1+\varepsilon)\|t_r^{(k)}(v^{(k)}b_d)\|_{\mathbb{B}(H^{(k)})}.$$

But since $\phi^{(k)} \to \phi$ pointwise on $\mathcal{A}$, we have $\lim_{\mathcal{U}} a_{ij}^{(k)} = a_{ij}$ and hence passing to the limit in k in the last two equivalences we obtain

$$\|t_r(b_d)\|_{\mathbb{B}(H)} \leq (1+\varepsilon)\lim\nolimits_{\mathcal{U}}\|t_r^{(k)}(v^{(k)}b_d)\|_{\mathbb{B}(H^{(k)})} \leq (1+\varepsilon)^2\|t_r(b_d)\|_{\mathbb{B}(H)}.$$

Since $\varepsilon > 0$ is arbitrary, this establishes the first part of (4.15).
The second part can be checked by a similar argument: Using F_n, G_n we are led to compare the norm of $s_r(b_d)$ in $H_C \otimes_{hD} [A_1 \oplus A_2] \otimes_{hD} (H^*)_R$ (as described in [14, Prop. 5.1]) with the norm in $C_n(D) \otimes_{hD} [A_1 \oplus A_2] \otimes_{hD} C_n(D)^*$, but the latter is isometric to $M_n(A_1 \oplus A_2)$. We skip the remaining details. □

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