OPTIMAL REGULARITY OF PLURISUBHARMONIC ENVELOPES ON COMPACT HERMITIAN MANIFOLDS

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ABSTRACT. In this paper, we prove the $C^{1,1}$-regularity of the plurisubharmonic envelope of a $C^{1,1}$ function on a compact Hermitian manifold. We also present examples to show this regularity is sharp.

1. INTRODUCTION

The subharmonic envelope is an important tool in the classical potential theory for Laplacian equation. This notion can be extended to the potential theory of nonlinear elliptic equations, and the issue of regularity of the envelope also arises naturally. For convex envelopes, the optimal $C^{1,1}$-regularity has been confirmed recently in [16]. For complex Monge-Ampère equations on a domain in $\mathbb{C}^n$, the Perron-Bremermann plurisubharmonic upper envelope has been studied in [2]. It is also interesting to establish the regularity for plurisubharmonic envelopes on complex manifolds. On a Kähler manifold, the plurisubharmonic envelopes have been studied for cohomology classes of great extent, including big classes [3, 8].

Let (M, ω) be a compact Hemitian manifold of complex dimension n and $PSH(M, \omega)$ be the set of ω-plurisubharmonic functions [17]. For any function f on M, following [3, 8], we define its *plurisubharmonic envelope (or extremal function)* by

$$(1.1) \qquad \varphi_f(x) = \sup\{\varphi(x) \mid \varphi \in PSH(M, \omega) \text{ and } \varphi \le f\}, \ x \in M.$$

Then $\varphi_f \in PSH(M, \omega)$. Moreover, it is shown in [3] that, when ω is Kähler and $f \in C^\infty(M)$, $\varphi_f \in C^{1,\alpha}(M)$ for any $\alpha \in (0,1)$. It is expected that the optimal regularity for this envelope is $C^{1,1}$, which has been realized when $[\omega]$ is an integral class [4, 22]. We prove the sharp regularity for general Hermitian manifold in this paper.

The idea of the proof is to consider the envelope as the solution to an obstacle problem for the complex Monge-Ampère equation [5]. The similar treatment for the real Monge-Ampère equations and other equations can be found in [20, 21, 13]. Then the regularity relies on the a priori estimates of the solutions to the following complex Monge-Ampère equations

$$(1.2) \qquad\qquad (\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\frac{1}{\varepsilon}(\varphi - f)}\omega^n$$

for small $\varepsilon > 0$. It is well-known that in Kähler case, the solution to the above equation has been established by [1, 26]. The solvability has been extended to the Hermitian case by [9, 18]. Let φ_ε be the solution to (1.2). Then we have

Theorem 1.1. *Let (M, ω) be a compact Hermitian manifold and $f \in C^{1,1}(M)$. Then we have φ_ε converges to φ_f and there is a constant C independent of ε such that $\|\varphi_\varepsilon\|_{C^2(M)} \leq C$. In particular, $\varphi_f \in C^{1,1}(M)$.*

It would be also interesting to study the regularity of envelopes with prescribed singularity as in [4, 22]. However, there are still difficulties in deriving the a priori estimates.

The paper is organized as follows. In Section 2, we establish the uniform a priori estimates for the Monge-Ampère equation (1.2). In particular, we apply the new techniques in [11] with a modification of the auxilary function to estimate the second order derivatives. Theorem 1.1 is proved in Section 3. In the last section, we give some examples showing that the $C^{1,1}$-regularity is optimal.

2. THE A PRIORI ESTIMATE

Let (M, ω) be a compact Hermitian manifold of complex dimension n. We use g and ∇ to denote the corresponding Riemannian metric and Levi-Civita connection (Note that we use Levi-Civita connection, not Chern connection). In this section, we study the a priori estimates of the following complex Monge-Ampère equation

$$(2.1) \qquad \begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{\frac{1}{\varepsilon}(\varphi - F)}\omega^n \\ \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \end{cases},$$

where $\varepsilon \in (0, 1)$ is a constant and F is a real-valued C^2 function on M. The solvability of the equation can be guaranteed by [9, 18]. For our purpose, stronger estimates are needed. We write $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ and $\tilde{g}$ be the corresponding Riemannian metric for convenience. We often use C to denote a uniform constant depending only on $\|F\|_{C^2}$ and (M, ω). All norms $\|\cdot\|_{C^k}$ in this paper are taken with respect to (M, ω). And all the following estimates are uniform with respect to ε.

Proposition 2.1. *Let φ be a smooth solution to (2.1). Then we have*

$$\max_M(\varphi - F) \leq C_0\varepsilon \ \text{and} \ \min_M \varphi \geq \min_M F,$$

where C_0 is a constant depending only on $\|F\|_{C^2}$ and (M, ω).

Proof. First, we assume that $(\varphi - F)$ attains its maximum at $p \in M$. By maximum principle, it is clear that

$$\sqrt{-1}\partial\overline{\partial}\varphi(p) - \sqrt{-1}\partial\overline{\partial}F(p) \le 0,$$

which implies

$$(\varphi - F)(p) = \varepsilon \log \frac{(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\omega^n}(p) \le \varepsilon \log \frac{(\omega + \sqrt{-1}\partial\overline{\partial}F)^n}{\omega^n}(p) \le C_0\varepsilon.$$

By the definition of p, we obtain

(2.2) $$\max_M(\varphi - F) \le C_0\varepsilon.$$

Next, we assume that $\varphi(q) = \min_M \varphi$ for $q \in M$. By a similar argument, we have

$$(\varphi - F)(q) = \varepsilon \log \frac{(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\omega^n}(q) \ge 0,$$

which implies

(2.3) $$\min_M \varphi = \varphi(q) \ge F(q) \ge \min_M F.$$

Combining (2.2) and (2.3), we complete the proof. $\square$

The following proposition is the gradient estimate of (2.1). It is established by Blocki [6] in Kähler case. For the Hermitian case, we use some calculations in [11] to prove Proposition 2.2, but the idea is similar with [6].

Proposition 2.2. *If φ is a smooth solution to (2.1), then there exists a constant C depending only on $\|F\|_{C^1}$ and (M, ω) such that*

$$\sup_M |\partial\varphi|_g \le C.$$

Proof. First, without loss of generality, we assume $\sup_M \varphi \le 0$. Otherwise, we consider the following functions

$$\psi = \varphi - \|\varphi\|_{L^\infty} \text{ and } \tilde{F} = F - \|\varphi\|_{L^\infty}.$$

It then follows that $\sup_M \psi \le 0$ and

$$(\omega + \sqrt{-1}\partial\overline{\partial}\psi)^n = e^{\frac{1}{\varepsilon}(\psi - \tilde{F})}\omega^n.$$

By the definition of $\tilde{F}$ and Proposition 2.1, it is clear that $\|\tilde{F}\|_{C^1} \le \|F\|_{C^1} + C$.

As in [11] (see Proposition 4.1 in [11]), we consider the following quantity

$$Q = e^{f(\varphi)}|\partial\varphi|_g^2,$$

where $f(t) = \frac{1}{A}e^{-A(t-1)}$ and A is a constant to be determined. We assume that Q attains its maximum at $p \in M$. Let $\{e_i\}_{i=1}^n$ be a local holomorphic frame for $T^{(1,0)}M$ near p, such that $\{e_i\}_{i=1}^n$ is unitary with respect to g and $\tilde{g}_{i\overline{j}}(p)$ is diagonal. For convenience, we write $\varphi_i = e_i(\varphi)$ and $\varphi_{\overline{i}} = \overline{e}_i(\varphi)$. By (4.13) in [11] (To avoid confusion of notations, we replace ε in (4.13) by δ

and F in (4.13) should be replaced by $\frac{1}{\varepsilon}(\varphi - F)$), at p, for any $\delta \in (0, \frac{1}{2}]$, we have

$$0 \geq e^f(f'' - 3\delta(f')^2)|\partial\varphi|_g^2|\partial\varphi|_{\tilde{g}}^2 + e^f(-f' - C_0\delta^{-1})|\partial\varphi|_g^2 \sum_i \tilde{g}^{i\bar{i}}$$

$$(2.4) \qquad + 2e^f\mathrm{Re}\left(\sum_i \frac{1}{\varepsilon}(\varphi_i - F_i)\varphi_i\right) + (2+n)e^f f'|\partial\varphi|_g^2 - 2e^f f'|\partial\varphi|_{\tilde{g}}^2,$$

where C_0 is a constant depending only on $\|F\|_{C^1}$ and (M, ω). Now, we choose $A = 12C_0$ and $\delta = \frac{A}{6}e^{A(\varphi(p)-1)}$. By direct calculations and Proposition 2.1, it is clear that

$$(2.5) \qquad f'' - 3\delta(f')^2 \geq C^{-1} \text{ and } -f' - C_0\delta^{-1} \geq C^{-1}.$$

Without loss of generality, we assume that $|\partial\varphi|_g \geq \sup_M |\partial F|_g$ at p, which implies

$$(2.6) \qquad \mathrm{Re}\left(\sum_i \frac{1}{\varepsilon}(\varphi_i - F_i)\varphi_i\right) \geq \frac{1}{\varepsilon}\left(|\partial\varphi|_g^2 - |\partial F|_g|\partial\varphi|_g\right) \geq 0.$$

Combining (2.4), (2.5) and (2.6), we get

$$0 \geq C^{-1}|\partial\varphi|_g^2|\partial\varphi|_{\tilde{g}}^2 + C^{-1}|\partial\varphi|_g^2 \sum_i \tilde{g}^{i\bar{i}} - C|\partial\varphi|_g^2 - C$$

at p. Then by the similar argument of [11], we obtain $|\partial\varphi|_g^2(p) \leq C$, which completes the proof of Proposition 2.2. $\qquad\qquad\qquad\qquad\qquad\square$

Proposition 2.3. *If φ is a smooth solution of (2.1), then there exists a constant C depending only on $\|F\|_{C^2}$ and (M, ω), such that*

$$\sup_M |\nabla^2\varphi|_g \leq C,$$

where ∇ is the Levi-Civita connection of (M, ω).

Proof. First, without loss of generality, we assume that $\sup_M \varphi \leq 0$ as in the proof of Proposition 2.2.

For $\nabla^2\varphi$, we use $\lambda_1(\nabla^2\varphi)$ to denote its largest eigenvalue. Since $\omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$, it is clear that $|\nabla^2\varphi|_g \leq C\lambda_1(\nabla^2\varphi) + C$ for a uniform constant C (see (5.1) in [11]). Then we apply maximum principle to the following quantity

$$Q = \log\lambda_1(\nabla^2\varphi) + h_D(|\partial\varphi|_g^2) + e^{-A\varphi},$$

where

$$h_D(s) = -\frac{1}{2}\log(D + \sup_M |\partial\varphi|_g^2 - s)$$

and $A, D > 1$ are constants to be determined. It then follows that

$$(2.7) \qquad \frac{1}{2D} \geq h_D' \geq \frac{1}{2D + 2\sup_M |\partial\varphi|_g^2} \text{ and } h_D'' = 2(h_D')^2.$$

Here the definition of h_D is different from the definition of h in [11]. In fact, we will choose D suitably to deal with the bad terms arise from the right hand side of (2.1).

We assume the set $\{x \in M \mid \lambda_1(\nabla^2\varphi) > 0\}$ is nonempty, otherwise we get $Q \leq C$ directly. Let p be the maximum point of Q, i.e., $Q(p) = \max\limits_M Q$. As before, we can find local holomorphic frame $\{e_i\}_{i=1}^n$ for $T^{(1,0)}M$ near p, such that

(1) $\{e_i\}_{i=1}^n$ is unitary with respect to g.
(2) At p, we have $\tilde{g}_{i\bar{j}}$ is diagonal and

$$\tilde{g}_{1\bar{1}} \geq \tilde{g}_{2\bar{2}} \geq \cdots \geq \tilde{g}_{n\bar{n}}.$$

Since (M, ω) is a Hermitian manifold, there exists a real coordinates $\{x^\alpha\}_{\alpha=1}^{2n}$ near p, such that

(1) At p, for any $k = 1, 2, \cdots, n$, we have

$$e_k = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^{2k-1}} - \sqrt{-1}\frac{\partial}{\partial x^{2k}}\right).$$

(2) At p, for any $\alpha, \beta, \gamma = 1, 2, \cdots, 2n$, we have

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = 0.$$

Let $V_1, V_2, \cdots, V_{2n}$ be the unit eigenvectors of $\nabla^2\varphi$ (with respect to g) at p, corresponding to eigenvalues $\lambda_1(\nabla^2\varphi) \geq \lambda_2(\nabla^2\varphi) \geq \cdots \geq \lambda_{2n}(\nabla^2\varphi)$. Extend $\{V_\alpha\}_{\alpha=1}^{2n}$ to be vector fields near p by taking the components (in above local real coordinates) to be constant.

However, at p, Q is not smooth if $\lambda_1(\nabla^2\varphi) = \lambda_2(\nabla^2\varphi)$. To avoid this, we use a perturbation argument (see [10, 11, 12, 23, 24]). As in [11], near p, we consider the following perturbed smooth quantity

$$\hat{Q} = \log \lambda_1(\Phi) + h_D(|\partial\varphi|_g^2) + e^{-A\varphi},$$

where $\Phi = (\Phi^\alpha{}_\beta)$ is a local endomorphism of TM given by [11]

$$\begin{aligned}
\Phi^\alpha{}_\beta &= g^{\alpha\gamma}\nabla^2_{\gamma\beta}\varphi - g^{\alpha\gamma}B_{\gamma\beta}, \\
B_{\alpha\beta} &= \delta_{\alpha\beta} - V^\alpha{}_1 V^\beta{}_1.
\end{aligned}$$

Here $\{V^\alpha{}_1\}_{\alpha=1}^{2n}$ are the components of V_1 at p. Then p is still local maximum point of $\hat{Q}$. By the definition of Φ, at p, $V_1, V_2, \cdots, V_{2n}$ are eigenvectors of Φ corresponding to eigenvalues $\lambda_1(\Phi) > \lambda_2(\Phi) \geq \cdots \geq \lambda_{2n}(\Phi)$. For convenience, in the following argument, we use λ_α and $\varphi_{V_\alpha V_\beta}$ to denote $\lambda_\alpha(\Phi)$ and $\nabla^2\varphi(V_\alpha, V_\beta)$ respectively.

Lemma 2.4. *There exists a uniform $C > 0$ such that if $\lambda_1 \geq C \sup\limits_{M} |\partial\varphi|_g + C\|F\|_{C^2}$ at p, then we have*

$$
L(|\partial\varphi|_g^2) \geq \frac{1}{2}\sum_k \tilde{g}^{i\bar{i}}(|e_i e_k(\varphi)|^2 + |e_i \bar{e}_k(\varphi)|^2) - C\sum_i \tilde{g}^{i\bar{i}}
$$

$$
(2.8) \qquad\qquad -\frac{1}{\varepsilon}\left(3\sup_M |\partial\varphi|_g^2 + \|F\|_{C^1}^2\right)
$$

and

$$
L(\lambda_1) \geq 2\sum_{\alpha>1} \tilde{g}^{i\bar{i}}\frac{|e_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1 - \lambda_\alpha} + \tilde{g}^{p\bar{p}}\tilde{g}^{q\bar{q}}|V_1(\tilde{g}_{p\bar{q}})|^2 - 2\tilde{g}^{i\bar{i}}[V_1, e_i]V_1\bar{e}_i(\varphi)
$$

$$
(2.9) \qquad\qquad -2\tilde{g}^{i\bar{i}}[V_1, \bar{e}_i]V_1 e_i(\varphi) - C\lambda_1\sum_i \tilde{g}^{i\bar{i}} + \frac{1}{2\varepsilon}\lambda_1.
$$

where $L = \tilde{g}^{i\bar{j}}(e_i \bar{e}_j - [e_i, \bar{e}_j]^{(1,0)})$ is the operator defined in [11, p.12].

Proof. For (2.8), by (4.8) in [11] (as before, we replace ε and F in (4.13) by δ and $\frac{1}{\varepsilon}(\varphi - F)$ to avoid confusion of notations), we have

$$
L(|\partial\varphi|_g^2) \geq (1-\delta)\sum_k \tilde{g}^{i\bar{i}}(|e_i e_k(\varphi)|^2 + |e_i \bar{e}_k(\varphi)|^2) - C\delta^{-1}|\partial\varphi|_g^2 \sum_i \tilde{g}^{i\bar{i}}
$$

$$
+2\text{Re}\left(\sum_i \frac{1}{\varepsilon}(\varphi_i - F_i)\varphi_{\bar{i}}\right)
$$

at p. Now we take $\delta = \frac{1}{2}$. By Proposition 2.2, we obtain

$$
L(|\partial\varphi|_g^2) \geq \frac{1}{2}\sum_k \tilde{g}^{i\bar{i}}(|e_i e_k(\varphi)|^2 + |e_i \bar{e}_k(\varphi)|^2) - C\sum_i \tilde{g}^{i\bar{i}}
$$

$$
(2.10) \qquad\qquad +2\text{Re}\left(\sum_i \frac{1}{\varepsilon}(\varphi_i - F_i)\varphi_{\bar{i}}\right).
$$

By Cauchy inequality, it is clear that

$$
(2.11) \qquad 2\text{Re}\left(\sum_i \frac{1}{\varepsilon}(\varphi_i - F_i)\varphi_{\bar{i}}\right) \geq -\frac{1}{\varepsilon}\left(3\sup_M |\partial\varphi|_g^2 + \|F\|_{C^1}^2\right).
$$

Combining (2.10) and (2.11), we obtain (2.8).

For (2.9), by (5.11) and (5.12) in [11], we have

$$
L(\lambda_1) \geq 2\sum_{\alpha>1} \tilde{g}^{i\bar{i}}\frac{|e_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1 - \lambda_\alpha} + \tilde{g}^{i\bar{i}}V_1 V_1(\tilde{g}_{i\bar{i}}) - 2\tilde{g}^{i\bar{i}}[V_1, e_i]V_1\bar{e}_i(\varphi)
$$

$$
(2.12) \qquad\qquad -2\tilde{g}^{i\bar{i}}[V_1, \bar{e}_i]V_1 e_i(\varphi) - C\lambda_1\sum_i \tilde{g}^{i\bar{i}}
$$

at p. In the local frame $\{e_i\}_{i=1}^n$, the complex Monge-Ampère equation (2.1) can be written as

$$\log \det \tilde{g} = \frac{1}{\varepsilon}(\varphi - F).$$

Differentiating the equation twice with V_1, we obtain

$$(2.13) \qquad \tilde{g}^{i\bar{i}}V_1 V_1(\tilde{g}_{i\bar{i}}) = \tilde{g}^{p\bar{p}}\tilde{g}^{q\bar{q}}|V_1(g_{p\bar{q}})|^2 + V_1 V_1\left(\frac{1}{\varepsilon}(\varphi - F)\right).$$

Assume $\lambda_1 \geq C \sup_M |\partial\varphi|_g + C\|F\|_{C^2}$ at p. When C is sufficiently large,

$$V_1 V_1\left(\frac{1}{\varepsilon}(\varphi - F)\right) = \frac{1}{\varepsilon}(\lambda_1 + (\nabla_{V_1}V_1)\varphi - V_1 V_1(F))$$

$$(2.14) \hspace{5cm} \geq \frac{1}{2\varepsilon}\lambda_1.$$

Then (2.9) follows from (2.12), (2.13) and (2.14). $\qquad\square$

Lemma 2.5. *There exists a uniform $C > 0$ such that if $\lambda_1 \geq C \sup_M |\partial\varphi|_g + C\|F\|_{C^2}$ at p, then for any $\delta \in (0, \frac{1}{2}]$, we have*

$$0 \geq (2-\delta)\sum_{\alpha>1}\tilde{g}^{i\bar{i}}\frac{|e_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1(\lambda_1 - \lambda_\alpha)} + \frac{\tilde{g}^{p\bar{p}}\tilde{g}^{q\bar{q}}|V_1(\tilde{g}_{p\bar{q}})|^2}{\lambda_1} - (1+\delta)\frac{\tilde{g}^{i\bar{i}}|e_i(\varphi_{V_1 V_1})|^2}{\lambda_1^2}$$

$$+\frac{h'_D}{2}\sum_k \tilde{g}^{i\bar{i}}(|e_i e_k(\varphi)|^2 + |e_i \bar{e}_k(\varphi)|^2) + h''_D \tilde{g}^{i\bar{i}}|\partial_i|\partial\varphi|_g^2|^2$$

$$+(Ae^{-A\varphi} - \frac{C}{\delta})\sum_i \tilde{g}^{i\bar{i}} + A^2 e^{-A\varphi}\tilde{g}^{i\bar{i}}|e_i(\varphi)|^2 - Ane^{-A\varphi}.$$

Proof. First, by direct calculations, at p, we have

$$L(\hat{Q}) = \frac{L(\lambda_1)}{\lambda_1} - \frac{\tilde{g}^{i\bar{i}}|e_i(\lambda_1)|^2}{\lambda_1^2} + h'_D L(|\partial\varphi|_g^2) + h''_D \tilde{g}^{i\bar{i}}|e_i|\partial\varphi|_g^2|^2$$

$$(2.15) \hspace{3cm} -Ae^{-A\varphi}L(\varphi) + A^2 e^{-A\varphi}\tilde{g}^{i\bar{i}}|e_i(\varphi)|^2.$$

By the proof of Lemma 5.4 in [11], for any $\delta \in (0, \frac{1}{2}]$, we get

$$2\frac{\tilde{g}^{i\bar{i}}[V_1, e_i]V_1\bar{e}_i(\varphi) + \tilde{g}^{i\bar{i}}[V_1, \bar{e}_i]V_1 e_i(\varphi)}{\lambda_1}$$

$$(2.16) \qquad \leq \delta\frac{\tilde{g}^{i\bar{i}}|e_i(\varphi_{V_1 V_1})|^2}{\lambda_1^2} + \delta\sum_{\alpha>1}\tilde{g}^{i\bar{i}}\frac{|e_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1(\lambda_1 - \lambda_\alpha)} + \frac{C}{\delta}\sum_i \tilde{g}^{i\bar{i}}.$$

For the first term of (2.15), by (2.9) and (2.16),

$$\frac{L(\lambda_1)}{\lambda_1} \geq (2-\delta)\sum_{\alpha>1}\tilde{g}^{i\bar{i}}\frac{|e_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1(\lambda_1 - \lambda_\alpha)} + \frac{\tilde{g}^{p\bar{p}}\tilde{g}^{q\bar{q}}|V_1(\tilde{g}_{p\bar{q}})|^2}{\lambda_1}$$

$$(2.17) \hspace{3cm} -\delta\frac{\tilde{g}^{i\bar{i}}|e_i(\varphi_{V_1 V_1})|^2}{\lambda_1^2} - \frac{C}{\delta}\sum_i \tilde{g}^{i\bar{i}} + \frac{1}{2\varepsilon}.$$

For the second term of (2.15), by Lemma 5.2 in [11], we obtain

$$(2.18) \qquad -\frac{\tilde{g}^{i\bar{i}}|e_i(\lambda_1)|^2}{\lambda_1^2} = -\frac{\tilde{g}^{i\bar{i}}|e_i(\varphi_{V_1 V_1})|^2}{\lambda_1^2}.$$

For the third term of (2.15), by (2.7) and (2.8), we have

$$h'_D L(|\partial\varphi|_g^2) \geq \frac{h'_D}{2}\sum_k \tilde{g}^{i\bar{i}}(|e_i e_k(\varphi)|^2 + |e_i \bar{e}_k(\varphi)|^2) - \frac{C}{2D}\sum_i \tilde{g}^{i\bar{i}}$$

$$(2.19) \qquad\qquad -\frac{h'_D}{\varepsilon}(3\sup_M |\partial\varphi|_g^2 + \|F\|_{C^1}^2).$$

Now we choose $D = 3\sup_M |\partial\varphi|_g^2 + \|F\|_{C^1}^2$. Then by (2.7),

$$(2.20) \qquad \frac{1}{2\varepsilon} - \frac{h'_D}{\varepsilon}(3\sup_M |\partial\varphi|_g^2 + \|F\|_{C^1}^2) \geq 0.$$

For the fifth term of (2.15), we have

$$(2.21) \qquad -Ae^{-A\varphi}L(\varphi) = Ae^{-A\varphi}\sum_i \tilde{g}^{i\bar{i}} - Ane^{-A\varphi}.$$

Therefore, combining $L(\hat{Q})(p) \leq 0$, (2.15), (2.16), (2.17), (2.18), (2.19), (2.20) and (2.21), we complete the proof. $\qquad\qquad\square$

Lemma 2.5 is just the analogue of [11, Lemma 5.4]. Finally, by Proposition 2.1, 2.2, Lemma 2.5 and the similar argument of [11, Proposition 5.1], we obtain the uniform upper bound of λ_1 at p, which completes the proof of Proposition 2.3. There are two problems that need to be explained.

One is that h_D in this paper is different from h in [11] (the definition of h corresponds to the definition of h_D when $D = 1$, i.e., $h = h_1$). However, the reader can verify, this minor difference does not influence the argument.

The other is that in the proof of [11, Proposition 5.1] the lower bound of $\frac{\tilde{\omega}^n}{\omega^n}$ is used, while in our case Proposition 2.1 only guarantees the upper bound of $\sup_M \frac{\tilde{\omega}^n}{\omega^n}$. However, we point out that $\inf_M \frac{\tilde{\omega}^n}{\omega^n}$ is not needed in the proof of [11, Proposition 5.1]. In fact, the proof of [11, Proposition 5.1] is split up into different cases. In Case 1(a), we have (see [11, p.23])

$$0 \geq \sum_k \tilde{g}^{i\bar{i}}(|e_i e_k(\varphi)|^2 + |e_i \bar{e}_k(\varphi)|^2) - C\sum_i \tilde{g}^{i\bar{i}}.$$

By $\tilde{g}_{1\bar{1}} \geq \tilde{g}_{2\bar{2}} \geq \cdots \geq \tilde{g}_{n\bar{n}}$, we obtain

$$0 \geq \sum_{i,k} \tilde{g}^{1\bar{1}}(|e_i e_k(\varphi)|^2 + |e_i \bar{e}_k(\varphi)|^2) - Cn\tilde{g}^{n\bar{n}}.$$

Combining this and assumption $\tilde{g}_{1\bar{1}} \leq A^3 e^{-2A\varphi}\tilde{g}_{n\bar{n}}$, it is clear that

$$0 \geq \sum_{i,k}(|e_i e_k(\varphi)|^2 + |e_i \bar{e}_k(\varphi)|^2) - C,$$

which is enough for the following proof. In Case 1(b) and Case 2, we have $\sum_i \tilde{g}^{i\bar{i}} \leq C$, where C is independent of $\inf_M \dfrac{\tilde{\omega}^n}{\omega^n}$ (see [11, p.12 and p.33]).

Combining this and $\prod_i \tilde{g}_{i\bar{i}} = \dfrac{\tilde{\omega}^n}{\omega^n}$, for each i, we obtain

$$\tilde{g}_{i\bar{i}} \leq C \sup_M \frac{\tilde{\omega}^n}{\omega^n},$$

which implies $\tilde{g}^{i\bar{i}} \geq C^{-1}$ (independent of $\inf_M \dfrac{\tilde{\omega}^n}{\omega^n}$). And this is also enough for the following proof. Therefore, the estimate in [11, Proposition 5.1] is independent of $\inf_M \dfrac{\tilde{\omega}^n}{\omega^n}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Regularity of envelope

In this section, we prove the $C^{1,1}$-regularity of envelopes. First, we recall the regularization theorem of plurisubharmonic functions on Hermitian manifolds.

Theorem 3.1. [7, 14, 15, 19] *Let (M, ω) be a compact Hermitian manifold. For any $\varphi \in PSH(M, \omega)$, there exists a sequence $\varphi_i \in PSH(M, \omega) \cap C^\infty(M)$ such that φ_i converges decreasingly to φ.*

Next lemma can be regarded as a special case of Theorem 1.1.

Lemma 3.2. *Let (M, ω) be a compact Hermitian manifold. For any $f \in C^\infty(M)$, we have $\varphi_f \in C^{1,1}(M)$ and*

$$\|\varphi_f\|_{C^{1,1}} \leq C,$$

where C is a constant depending only on $\|f\|_{C^2}$ and (M, ω).

Proof. We consider the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\frac{1}{\varepsilon}(\varphi - f)}\omega^n.$$

We use φ_ε to denote its unique solution, i.e.,

(3.1) $$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi_\varepsilon)^n = e^{\frac{1}{\varepsilon}(\varphi_\varepsilon - f)}\omega^n.$$

Next, for any $u \in PSH(M, \omega) \cap C^\infty(M)$ such that $u \leq f$, we define

$$u_\varepsilon = (1 - \varepsilon)u + \varepsilon(\log \varepsilon^n + \min_M f).$$

By direct calculation, we have

(3.2) $$(\omega + \sqrt{-1}\partial\overline{\partial}u_\varepsilon)^n \geq \varepsilon^n\omega^n \geq e^{\frac{1}{\varepsilon}(u_\varepsilon - f)}\omega^n.$$

Combining (3.1), (3.2) and maximum principle, we obtain $u_\varepsilon \leq \varphi_\varepsilon$, which implies

$$(1 - \varepsilon)u + \varepsilon(\log \varepsilon^n - \|f\|_{L^\infty}) \leq \varphi_\varepsilon.$$

Theorem 3.1 implies

$$\varphi_f(x) = \sup\{\varphi(x) \mid \varphi \in PSH(M,\omega) \cap C^\infty(M) \text{ and } \varphi \le f\}.$$

Since u is arbitrary, by $u \le f$, it is clear that

$$(1-\varepsilon)\varphi_f + \varepsilon(\log \varepsilon^n - \|f\|_{L^\infty}) \le \varphi_\varepsilon,$$

which implies

$$(3.3) \qquad \varphi_f + \varepsilon(\log \varepsilon^n - 2\|f\|_{L^\infty}) \le \varphi_\varepsilon,$$

where we used $\varphi_f \le f$. By Proposition 2.1 and the definition of φ_ε, we obtain

$$(3.4) \qquad \varphi_\varepsilon - C_0\varepsilon \le f \text{ and } \varphi_\varepsilon - C_0\varepsilon \in PSH(M,\omega).$$

Combining (3.3) and (3.4), it is clear that

$$(3.5) \qquad \lim_{\varepsilon \to 0} \|\varphi_\varepsilon - \varphi_f\|_{L^\infty} = 0.$$

On the other hand, by Proposition 2.1, 2.2 and 2.3, we have

$$(3.6) \qquad \|\varphi_\varepsilon\|_{C^2} \le C.$$

Combining (3.5) and (3.6), we have φ_ε converges in $C^{1,1}$ to φ_f. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Since $f \in C^{1,1}(M)$, by smooth approximation, there exists a sequence of smooth function f_i on M such that

$$(3.7) \qquad \lim_{i \to \infty} \|f_i - f\|_{L^\infty} = 0 \text{ and } \|f_i\|_{C^2} \le C_0,$$

where C_0 is a constant depending only on $\|f\|_{C^{1,1}}$ and (M,ω). On the other hand, for any $u \in PSH(M,\omega)$ with $u \le f$, we have

$$u - \|f_i - f\|_{L^\infty} \le f_i.$$

By the definition of φ_{f_i}, it is clear that

$$u - \|f_i - f\|_{L^\infty} \le \varphi_{f_i}.$$

Since u is arbitrary, we obtain

$$\varphi_f - \|f_i - f\|_{L^\infty} \le \varphi_{f_i}.$$

Similarly, we get

$$\varphi_{f_i} - \|f - f_i\|_{L^\infty} \le \varphi_f.$$

It then follows that

$$(3.8) \qquad \|\varphi_{f_i} - \varphi_f\|_{L^\infty} \le \|f_i - f\|_{L^\infty}.$$

Combining (3.7) and (3.8), we complete the proof. □

4. EXAMPLES

In this section, we give some examples to explain the regularity result in Theorem 1.1 is optimal.

4.1. **Example of complex dimension one.** In this subsection, we construct a smooth function f on the complex projective space $(\mathbb{CP}^1, \omega_{FS})$, such that $\varphi_f \notin C^2(\mathbb{CP}^1)$, where ω_{FS} is the Fubini-Study metric.

First, we define a function $h(t)$ on $[0, 2]$ by

$$h(t) = \begin{cases} \left(\frac{1}{\sqrt{3}-1} - 1\right)^2 + \log(\sqrt{3} - 1), & t \in [0, \sqrt{3} - 1], \\ \left((\frac{1}{t} - 1)_+\right)^2 + \log t, & t \in [\sqrt{3} - 1, 2], \end{cases}$$

where $(\frac{1}{t} - 1)_+ = \max\{\frac{1}{t} - 1, 0\}$. It is clear that h is a convex function on $[0, 1]$. Let $\tilde{h}$ be a smooth function on $[0, 2]$ such that

$$\tilde{h}(t) = \left(\frac{1}{t} - 1\right)^2 + \log t \text{ in } [\frac{4}{5}, 2] \text{ and } \tilde{h}(t) \geq h(t) \text{ in } [0, 2].$$

Denote by $[z_0, z_1]$ the homogeneous coordinates on $\mathbb{CP}^1$. Let

$$U = \{[1, z_1] \mid |z_1|^2 \leq \frac{5}{4}\} \text{ and } V = \{[z_0, 1] \mid |z_0|^2 \leq 2\}$$

be two subsets of $\mathbb{CP}^1$ such that $\mathbb{CP}^1 = U \cup V$. We define a function f on $\mathbb{CP}^1$ by

$$f = \begin{cases} (|z_1|^2 - 1)^2 - \log(1 + |z_1|^2) \text{ in } U, \\ \tilde{h}(|z_0|^2) - \log(1 + |z_0|^2) \text{ in } V. \end{cases}$$

Since $\tilde{h} \in C^\infty([0, 2])$, we obtain that $f \in C^\infty(\mathbb{CP}^1)$. Then we prove

Proposition 4.1. $\varphi_f \in C^{1,1}(\mathbb{CP}^1) \setminus C^2(\mathbb{CP}^1)$.

Proof. Define

$$\varphi = \begin{cases} \left((|z_1|^2 - 1)_+\right)^2 - \log(1 + |z_1|^2) \text{ in } U, \\ h(|z_0|^2) - \log(1 + |z_0|^2) \text{ in } V. \end{cases}$$

It is clear that $\varphi \in C^{1,1}(U) \setminus C^2(U)$. Since $\tilde{h} \geq h$, we have $\varphi \leq f$.

Next, we verify that $\varphi \in PSH(\mathbb{CP}^1, \omega_{FS})$. On U, we compute

$$\begin{aligned} & \omega_{FS} + \sqrt{-1}\partial\bar{\partial}\varphi \\ = \ & \sqrt{-1}\partial\bar{\partial}\left[\log(1 + |z_1|^2) + \left((|z_1|^2 - 1)_+\right)^2 - \log(1 + |z_1|^2)\right] \\ \geq \ & 0. \end{aligned}$$

Similarly, on $V_0 = \{[z_0, 1] \mid |z_0|^2 \leq 1\}$, we have

$$\begin{aligned} & \omega_{FS} + \sqrt{-1}\partial\bar{\partial}\varphi \\ = \ & \sqrt{-1}\partial\bar{\partial}\left[\log(1 + |z_1|^2) + h(|z_0|^2) - \log(1 + |z_1|^2)\right] \\ = \ & \sqrt{-1}\partial\bar{\partial}\left(h(|z_0|^2)\right) \\ \geq \ & 0, \end{aligned}$$

where we used the fact that h is a convex function on $[0, 1]$. Since $\mathbb{CP}^1 = U \cup V_0$, we obtain $\varphi \in PSH(\mathbb{CP}^1, \omega_{FS})$.

Now we show φ_f is not C^2. For convenience, we denote

$$U_0 = \{[1, z_1] \mid |z_1|^2 \leq 1\}.$$

Then for any $u \in PSH(\mathbb{CP}^1, \omega_{FS})$ such that $u \leq f$, we have

$$\omega_{FS} + \sqrt{-1}\partial\overline{\partial}u = \sqrt{-1}\partial\overline{\partial}\left[\log(1 + |z_1|^2) + u)\right] \geq 0 \text{ in } U_0$$

and

$$\log(1 + |z_1|^2) + u \leq \log(1 + |z_1|^2) + f = 0 \text{ on } \partial U_0.$$

By maximum principle, it is clear that

$$(4.1) \qquad\qquad u \leq -\log(1 + |z_1|^2) = \varphi \text{ in } U_0.$$

Since $\varphi = f$ on $U \setminus U_0$ and $u \leq f$, we have

$$(4.2) \qquad\qquad u \leq f = \varphi \text{ in } U \setminus U_0.$$

Combining (4.1) and (4.2), we have $u \leq \varphi$ on U, which implies $\varphi_f = \varphi$ on U. The proposition is proved. $\qquad\square$

4.2. **Examples of higher dimensions.** In this subsection, we give more examples on compact Hermitian manifolds of higher dimensions. First, we have the following lemma.

Lemma 4.2. *Let (M, ω_M) and (N, ω_N) be Hermitian manifolds and let $\pi : M \times N \to M$ be the projection map. For any $f \in C^{1,1}(M)$, we have*

$$\pi^*\varphi_f = \varphi_{\pi^*f},$$

where π^* is the pullback map.

Proof. First, since $\varphi_f \in PSH(M, \omega_M)$ and $\varphi_f \leq f$, we obtain $\pi^*\varphi_f \in PSH(M \times N, \omega_M + \omega_N)$ and $\pi^*\varphi_f \leq \pi^*f$, which implies

$$(4.3) \qquad\qquad \pi^*\varphi_f \leq \varphi_{\pi^*f}.$$

Next, for any $(p.q) \in M \times N$ and $u \in PSH(M \times N, \omega_M + \omega_N)$ with $u \leq \pi^*f$, we have $u(\cdot, q) \in PSH(M, \omega_M)$ and $u(\cdot, q) \leq f$ on M. It then follows that $u(p, q) \leq \varphi_f(p)$, i.e.,

$$u(p, q) \leq \pi^*\varphi_f(p, q).$$

Since (p, q) and u are arbitrary, by the definition of φ_{π^*f}, we have

$$(4.4) \qquad\qquad \varphi_{\pi^*f} \leq \pi^*\varphi_f.$$

Combining (4.3) and (4.4), we complete the proof. $\qquad\square$

Now, let (M, ω) be a compact Hermitian manifold and let $\pi : \mathbb{CP}^1 \times M \to \mathbb{CP}^1$ be the projection map. Then π^*f is a smooth function on $\mathbb{CP}^1 \times M$, where f is defined in subsection 4.1. However, by Proposition 4.1 and Lemma 4.2, $\varphi_{\pi^*f} = \pi^*\varphi_f$ is not C^2.

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