Maximal equicontinuous generic factors and weak model sets

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Abstract

Regular model sets generated from a cut-and-project scheme given by a co-compact lattice $\mathcal{L} \subset G \times H$ and compact and aperiodic window $W \subseteq H$, have the maximal equicontinuous factor (MEF) $(G \times H)/\mathcal{L}$, if the window is toplogically regular. This picture breaks down completely, when the window has empty interior, in which case the MEF is always trivial, although $(G \times H)/\mathcal{L}$ continues to be the Kronecker factor for the Mirsky measure. As this situation occurs for many interesting examples like the square-free numbers or the visible lattice points, there is some need for a slightly weaker concept of topological factors that is still strong enough to capture basic properties of the system. Here we propose to use the concept of a generic factor [16] for this purpose. For so called ergodic topological dynamical systems we prove the existence of a maximal equicontinuous generic factor (MEGF) and characterize it in terms of the regional proximal relation. For such systems we also show that the MEGF is trivial if and only if the system is topologically weakly mixing. This part of the paper profits strongly from previous work by McMahon [24] and Auslander [1]. In the second part we show that $(G \times H)/\mathcal{L}$ is indeed the MEGF of each weak model.

1 Introduction

Let G and H be locally compact second countable groups. In many examples, $G = \mathbb{Z}^d$ or $\mathbb{R}^d$, whereas H will often be a more general group. Each pair $(\mathcal{L}, W)$, consisting of a cocompact lattice $\mathcal{L} \subset G \times H$ and a compact subset W of H, also called the window, defines a weak model set $\Lambda(\mathcal{L}, W)$ as the set of all points $x_G \in G$, for which there exists a point $x_H \in W$ such that $(x_G, x_H) \in \mathcal{L}$. There is an abundant literature on model sets, see e.g. the collection of references cited in [3]. Many of these sets exemplify *aperiodic order*, a concept which, so far, is mostly defined by a wealth of examples [4, 2]. The following seems to be a common feature of all of them: No $g \in G \setminus \{0\}$ satisfies $g + \Lambda(\mathcal{L}, W) = \Lambda(\mathcal{L}, W)$, but the orbit closure $\overline{\{g + \Lambda(\mathcal{L}, W) : g \in G\}}$ as a G-dynamical system has a nontrivial *maximal equicontinuous factor (MEF)* and/or a nontrivial *Kronecker factor (KF)* capturing the quasiperiodic aspects of the dynamics.

Many of the simpler examples are uniquely ergodic, so that one can talk unambigously about their KF, and quite often this KF is just the MEF equipped with its Haar measure. But, more recently,

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dynamically richer examples, like the set of square-free numbers [10, 27], the set of visible lattice points [5] and their generalizations [7], have attracted much attention. They share the common feature that the orbit closure $\overline{\{g + \Lambda(\mathcal{L}, W) : g \in G\}}$ has a fixed point, so that the MEF must be trivial, whereas there are plenty of invariant measures that have non-trivial KFs. Equipped with a very natural invariant measure (called Mirsky measure in some cases) these systems are actually isomorphic to their KFs. The approach to the dynamics of weak model sets from [19] encompasses all these examples. It suggests that the notion of *maximal equicontinuous generic factor (MEGF)* might be an appropriate unifying concept. Equicontinuous generic factors for abstract topological dynamical systems were studied in [16], see also [11] for generic eigenfunctions.

In the first part of this note we review some facts on equicontinuous generic factors and prove the existence of a unique MEGF for *ergodic topological dynamical systems*, i.e. for continuous actions of infinite (abstract) groups on compact metric spaces, which admit an ergodic invariant probability measure with full topological support (Theorem 2.1). The proof is inspired by work of McMahon [24] and Auslander [1]. If the acting group of such a system is abelian, we also show that the MEGF is trivial if and only if the system is topologically weakly mixing.

In the second part we test the potential usefulness of this construction on weak model sets, where there are good reasons to expect that this construction leads to the canonical G-action on the group $(G \times H)/\mathcal{L}$, see e.g. [19, 7]. We will see that $((G \times H)/\mathcal{L}, G)$ is indeed both, the MEGF and the KF of the orbit closure of any weak model set that is generic for its Mirsky measure (Theorem 3.1). It should be noted, however, that in general ergodic topological dynamical systems the MEGF and the KF need not coincide (Example 3.1).

2 Equicontinuous generic factors

Let (X, G) be a topological dynamical system with an abstract group G acting by homeomorphisms on a compact metrizable space X. For this action we adopt the short cut notation $x \mapsto gx$. By Gx we denote $\{gx : g \in G\}$.

Definition 2.1. *[14, 13] The system* (X, G) *is* ergodic*, if there exists an ergodic G-invariant Borel probability measure λ on X with* $\mathrm{supp}(\lambda) = X$.

Remark 2.1. a) Observe that the following properties are equivalent:

(i) (X, G) is ergodic.

(ii) There exists a G-invariant Borel probability measure λ on X with $\mathrm{supp}(\lambda) = X$ and $\lambda(X_t) = 1$.

The implication (i) ⇒ (ii) follows immediately from [21, Lemma 2.4]. For the reverse implication note that [21, Lemma 2.4] guarantees that λ is “closed ergodic” so that the ergodicity of (X, G) follows from [21, Proposition 2.6].

b) In particular, each ergodic (X, G) is topologically transitive.

c) The mere existence of a (non-ergodic) G-invariant probability measure with full support (i.e. being an E-system in the terminology of [14, 13]) is a weaker property than ergodicty of (X, G) [28].

From now on we assume that (X, G) is *topologically transitive*. (Ergodicity will be required only when it is needed.) Then the set

$$X_t := \{x \in X : \overline{Gx} = X\}$$

of transitive points is G-invariant and residual, see e.g. [8, Proposition 1].

Remark 2.2. As a topological space (endowed with the induced topology from the compact metric space X), X_t is second countable and in particular separable [26, Theorem 30.3]. The same is true for each G-invariant subset $X_0 \subseteq X_t$ with the induced topology.

For topological dynamical systems with $G = \mathbb{Z}$ (even with $G = \mathbb{N}$), Huang and Ye [16] introduced the notion of an equicontinuous generic factor, that we adapt here to general G:

Definition 2.2. *a) The system* (Y, G) *is an* equicontinuous generic factor *of* (X, G)*, if* (Y, G) *is an equicontinuous, transitive (and hence minimal) system, and if there is a continuous map* $\pi : X_t \to Y$ *equivariant under the action of* G*. (As* (Y, G) *is minimal, the image* $\pi(X_t)$ *is dense in* Y*.) We also write* $\pi : (X, G) \overset{gen}{\to} (Y, G)$*.*
b) An equicontinuous generic factor $\pi : (X, G) \overset{gen}{\to} (Z, G)$ *is* maximal*, if the following holds:*
If $\pi_Y : (X, G) \overset{gen}{\to} (Y, G)$ *is another equicontinuous generic factor, then there is a factor map* $\pi' : Z \to Y$ *such that* $\pi_Y = \pi' \circ \pi$*.*

The following theorem is the main result of this paper:

Theorem 2.1. *An ergodic system* (X, G) *has a unique maximal equicontinuous generic factor (MEGF). This factor is in particular independent of the ergodic invariant measure* λ *from Definition 2.1.*

A slight generalization of this theorem is stated and proved below in Subsection 2.3.

Remark 2.3. An equicontinuous generic factor map π is in particular a continuous map from a dense subset X_t of X to Y. If X and Y are both compact, such a map can always be extended to a measurable map from X to Y, continuous at each point of X_t. Indeed, denote by $\overline{\Pi} := \overline{\{(x, \pi(x)) : x \in X_t\}}$ the closure of the graph of π in $X \times Y$. The multivalued map $\phi : x \mapsto \overline{\Pi}_x$ that associates to each point $x \in X$ the (compact) x-section of $\overline{\Pi}$ is upper semi-continuous and hence measurable [9, Corollary III.3] so that there is a measurable selector $\tilde{\pi} : X \to Y$ such that $\tilde{\pi}(x) \in \overline{\Pi}_x$ for each $x \in X$ [9, Theorem III.6]. As $\overline{\Pi}_x = \{\pi(x)\}$ at all $x \in X_t$ (π is continuous at all these points!), $\tilde{\pi}$ extends π, and as the graph of $\tilde{\pi}$ is contained in $\overline{\Pi}$, $\tilde{\pi}$ is continuous at all $x \in X_t$.

Note, however, that in general $\tilde{\pi}$ cannot be chosen to be equivariant under the actions of G, because fixed points are always mapped to fixed points under equivariant maps.

For the case $G = \mathbb{Z}$ (and also $G = \mathbb{N}$) and without assuming that the system (X, G) is ergodic, Huang and Ye [16, Theorem 3.8] proved that a system (X, G) has a trivial MEGF if and only if it is weakly scattering, and if (X, G) is a E-system, then this happens if and only if the system is weakly mixing. Here we prove, for general acting groups G and ergodic (X, G), that such a system is weakly mixing if and only if it has a trivial MEGF.

Theorem 2.2. *Assume that* (X, G) *is topologically transitive.*
a) If (X, G) *is weakly scattering, i.e. if* $(X \times Z, G)$ *(the product action) is topologically transitive for each minimal equicontinuous* (Z, G)*, then the maximal equicontinuous generic factor of* (X, G) *is trivial (i.e. a singleton).*
b) If (X, G) *is ergodic and has a trivial maximal equicontinuous generic factor, then* (X, G) *is topologically weakly mixing.*

For the proof see Subsection 2.4. As weak mixing implies weak scatering, we have the following corollary:

Corollary 2.1. *Assume that* (X, G) *is an ergodic topologically transitive system. Then* (X, G) *is weakly scattering if and only if it is topologically weakly mixing if and only if its maximal equicontinuous generic factor is trivial.*

2.1 The regional proximal and the equicontinuous structure relation

The equicontinuous structure relation During all of this note, X_0 denotes a dense, G-invariant subset of X_t. We endow X_t and X_0 with the induced topology inherited from X, and we denote by $\mathcal{O}, \mathcal{O}_t$ and $\mathcal{O}_0$ the topologies on X, X_t and X_0, respectively. The corresponding product topologies on X^2, X_t^2 and X_0^2 are denoted by $\mathcal{O}^2, \mathcal{O}_t^2$ and $\mathcal{O}_0^2$, respectively. We recall the following elementary observation:

Remark 2.4. For $U \in \mathcal{O}$ and $V \in \mathcal{O}^2$ we have

$$\overline{U \cap X_0}^{\mathcal{O}_0} = \overline{U}^{\mathcal{O}} \cap X_0 \quad \text{and} \quad \overline{V \cap X_0^2}^{\mathcal{O}_0^2} = \overline{V}^{\mathcal{O}^2} \cap X_0^2 .$$

In particular, $\overline{X_0}^{\mathcal{O}_t} = \overline{X_0 \cap X_t}^{\mathcal{O}_t} = \overline{X_0}^{\mathcal{O}} \cap X_t$.

For a homomorphism (i.e. a continuous G-equivariant map) $\pi : X_0 \to Y$, where Y is any compact metric group on which G acts minimally by translation (so that $\pi(X_0)$ is dense in Y [1]), denote

$$S_0^\pi := \{(x, x') \in X_0^2 : \pi(x) = \pi(x')\} , \tag{1}$$

and let

$$S_0^{eq} := \bigcap_{\pi: X_0 \to Y} S_0^\pi . \tag{2}$$

(The intersection is taken over all such spaces Y and all homomorphisms $\pi : X_0 \to Y$.) We call S_0^{eq} the equicontinuous structure relation on X_0. It is obviously an $\mathcal{O}_0$-closed, G-invariant equivalence relation on X_0.

Remark 2.5. Recall that each topologically transitive, equicontinuous G-action on a compact metric space Y induces the structure of an abelian group on Y, such that Y (with its original topology) becomes a topological group, on which G acts by translation. In particular, using an equivalent metric on Y, if necessary, we can always assume that the action of G on Y is isometric. [2]

The regional proximal relation Denote

$$\Delta_0 := \{(x, x) : x \in X_0\} ,$$

$$\mathcal{U}_0 := \left\{ U_0 := U \cap X_0^2 : U \in \mathcal{O}^2 \ G\text{-invariant}, \Delta_0 \subset U \right\} ,$$

let

$$Q_0 := \bigcap_{U_0 \in \mathcal{U}_0} \overline{U_0} \cap X_0^2 ,$$

and denote by S_0^* the smallest $\mathcal{O}_0^2$-closed, G-invariant equivalence relation on X_0^2 containing Q_0. Because of Remark 2.4, it does not matter how the topological hull operation on $U_0 \subseteq X_0^2$ is interpreted.

Lemma 2.1. *Q_0 is the regional proximal relation of the non-compact dynamical system (X_0, G), i.e.*

$$Q_0 = \Big\{ (x, y) \in X_0^2 : \textit{for all neighbourhoods } A \in \mathcal{O} \textit{ of } x,\ B \in \mathcal{O} \textit{ of } y \textit{ and } V \in \mathcal{O}^2 \textit{ of } \Delta_0 \textit{ there exist } x' \in A \cap X_0,\ y' \in B \cap X_0 \textit{ and } g \in G \textit{ s.t. } (gx', gy') \in V \Big\} . \tag{3}$$

[1]Observe that, vice versa, denseness of $\pi(X_0)$ in Y implies minimality of the action of G on Y.

[2]Denote by d the given metric on Y. Then $d'(y_1, y_2) := \sup\{d(gy_1, gy_2) : g \in G\}$ has the desired properties.

Proof. Suppose first that $(x,y) \in Q_0 \subseteq X_0^2$ and that A, B and V are neighbourhoods as in (3). Define $U := \bigcup_{g\in G} gV$. Then $U \in O^2$ is a G-invariant neighbourhood of Δ_0, i.e. $U_0 \in \mathcal{U}_0$. Hence $(x,y) \in \overline{U_0} \cap X_0^2$, and there exists $(x',y') \in U_0 \cap (A\times B)$. So there is $g \in G$ such that $(x',y') \in gV \cap (A\times B) \cap (X_0 \times X_0)$, which means that $x' \in A \cap X_0$, $y' \in B \cap X_0$ and $(g^{-1}x', g^{-1}y') \in V$.

Conversely, suppose that (x,y) belongs to the set on the r.h.s. of (3), and consider any $U_0 = U \cap X_0^2 \in \mathcal{U}_0$ with a G-invariant neighbourhood $U \in O^2$ of Δ_0. Given any neighbourhood $O \in O^2$ of (x,y), fix neighbourhoods $A \in O$ of x and $B \in O$ of y such that $A \times B \subseteq O$. Then there are $x' \in A \cap X_0$, $y' \in B \times X_0$ and $g \in G$ such that $(gx', gy') \in U$, i.e. $(x',y') \in g^{-1}U \cap (X_0 \times X_0) = U_0$. Hence $(x,y) \in \overline{U_0} \cap X_0^2$, and as this holds for each $U_0 \in \mathcal{U}_0$, we have $(x,y) \in Q_0$. □

Remark 2.6. This lemma shows that (3) is not exactly the relation $Q_m(\varphi)$ of McMahon [24], even when his setting is specialized to the situation treated here, which is the special case where McMahon's Z is the trivial one-point system and where his X and Y coincide. In that case his notion of a section collapses to that of a Borel probability measure on X, and his set $R_m(\varphi)$ coincides with our $X_t \times X_t$. The definition of his $Q_m(\varphi)$, however, does not coincide with that of our Q_t, because he requires (3) for any neighbourhood V of any given point $(x_0, x_0) \in \Delta_t$. Hence also his $S_m(\varphi)$ may differ from our S_t^*.

Note also that his set X_m is our X_t, and that X_t is a Borel set under our assumptions. This setting, for the special case of minimal dynamics, is reproduced in Auslander's book [1].

Inclusions between the various relations

Lemma 2.2. *$Q_0 \subseteq S_0^{eq}$ and hence also $S_0^* \subseteq S_0^{eq}$.*

Proof. Let $\pi : X_0 \to Y$ be as in (2), and recall that w.l.o.g. we can assume that the action of G on Y is isometric. Let $(x,y) \in Q_0$. Suppose for a contradiction that $\pi(x) \neq \pi(y)$. Let $\delta := d(\pi(x), \pi(y)) > 0$. There are neighbourhoods $A \in O$ of x and $B \in O$ of y such that

$$d(\pi(gx'), \pi(gy')) = d(g\pi(x'), g\pi(y')) = d(\pi(x'), \pi(y')) > \delta/2$$

for all $x' \in A \cap X_0$, $y' \in B \cap X_0$ and $g \in G$.

Let $M := \{(y,y') : y, y' \in Y,\ d(y,y') < \delta/2\}$. The set M is an open neighbourhood of the diagonal in $Y \times Y$, and as the metric is translation invariant, the set M is G-invariant. Furthermore, $(\pi(gx'), \pi(gy')) \notin M$ for all $x' \in A \cap X_0$, $y' \in B \cap X_0$ and $g \in G$. Let $\tilde{V} := \bigcup_{g\in G} g((\pi\times\pi)^{-1}M)$. Then $\tilde{V} \subseteq X_0^2$ and $(A \times B) \cap \tilde{V} = \emptyset$.

As $\pi : X_0 \to Y$ is continuous, $\tilde{V} \in O_0^2$ is a G-invariant neighbourhood of Δ_0. Hence $\tilde{V} = V \cap X_0^2$ for some $V \in O^2$. Let $U := \bigcup_{g\in G} gV$. This set is clearly O^2-open and G-invariant, and it contains Δ_0. Note also that

$$U_0 = \left(\bigcup_{g\in G} gV\right) \cap X_0^2 = \bigcup_{g\in G} g\left(V \cap X_0^2\right) = \bigcup_{g\in G} g\tilde{V} = \tilde{V}\,.$$

Hence $U_0 \in \mathcal{U}_0$ and $(A \times B) \cap U_0 = (A \times B) \cap \tilde{V} = \emptyset$. Therefore $(x,y) \notin \overline{U_0}$, which contradicts $(x,y) \in Q_0$. □

2.2 The role of invariant measures supported by the transitive points

We follow McMahon [24] and Auslander [1] in order to study the relation between Q_0 and S_0^*. Although some parts of the proofs carry over directly, we prefer to give full details here.

General assumptions and notations

- $\mathcal{N}$ denotes the family of all closed G-invariant subsets of X^2.
- For any $N \in \mathcal{N}$ and $x \in X$ denote by $N_x := \{y \in X : (x, y) \in N\}$ the x-section of N.
- We fix a G-invariant Borel probability measure λ on X. As X is compact metrizable, λ is regular.

Lemma 2.3 ([24, 1]). *Let $N \in \mathcal{N}$. Then*
a) $N_{gx} = gN_x$ and $\lambda(N_{gx}) = \lambda(N_x)$ for all $x \in X$ and $g \in G$.
b) The map $x \mapsto \lambda(N_x)$ $(x \in X)$ is upper semicontinuous.
c) $\lambda(N_x) \leqslant \lambda(N_{x'})$ for all $x \in X$ and $x' \in \overline{Gx}$.
d) $\lambda(N_x) \leqslant \lambda(N_{x'})$ for all $x \in X_t$ and $x' \in X$.
e) $\lambda(N_x) = \lambda(N_{x'})$ for all $x, x' \in X_t$.
f) The map $x \mapsto \lambda(N_x)$ $(x \in X)$ is continuous at each $x \in X_t$.
g) $\lambda(N_{gx} \triangle N_{gx'}) = \lambda(N_x \triangle N_{x'})$ for all $x, x' \in X$ and $g \in G$.
h) The map $(x, x') \mapsto \lambda(N_x \triangle N_{x'})$ $(x, x' \in X)$ is continuous at each $(x, y) \in X_t \times X_t$.

Proof. a) $N_{gx} = \{y \in X : (gx, y) \in N\} = \{y \in X : (x, g^{-1}y) \in N\} = \{gy' \in X : (x, y') \in N\} = gN_x$.
b) Let $\epsilon > 0$. There is an open neighbourhood of $V \supseteq N_x$ such that $\lambda(V) < \lambda(N_x) + \epsilon$. By compactness of N, there is a neighbourhood $U \subseteq X$ of x such that $N_{x'} \subseteq V$ for all $x' \in U$. Hence $\lambda(N_{x'}) \leqslant \lambda(V) < \lambda(N_x) + \epsilon$ for all $x' \in U$.
c) This follows from a) and b).
d) This is a special case of c).
e) This follows from d).
f) Let $\lambda_{min} := \inf_{x' \in X} \lambda(N_{x'})$, let $x \in X_t$ and $\epsilon > 0$. In view of b) and d), there is a neighbourhood U of x such that $\lambda(N_x) = \lambda_{min} \leqslant \lambda(N_{x'}) < \lambda(N_x) + \epsilon$ for each $x \in U$.
g) This follows from a): $\lambda(N_{gx} \triangle N_{gx'}) = \lambda(gN_x \triangle gN_{x'}) = \lambda(g(N_x \triangle N_{x'})) = \lambda(N_x \triangle N_{x'})$
h) This follows from f).

□

Following [1] we define pseudometrics d_N on X_t:[3] given $N \in \mathcal{N}$ let

$$d_N(x, x') := \lambda(N_x \triangle N_{x'}) .$$

Their restrictions to X_0^2 yields pseudo-metrics on X_0. For $X_0 \subseteq X_t$ let

$$K_0(N) = \left\{(x, x') \in X_0^2 : d_N(x, x') = 0\right\} .$$

If $X_0 = X_t$, we denote the this set by $K_t(N)$. Observe that

$$K_0(N) = K_t(N) \cap X_0^2 .$$

By Lemma 2.3, d_N is G-invariant and continuous, so that $K_0 = K_0(N)$ is a G-invariant O_0-closed equivalence relation on X_0. Let $Z^* := X_0/K_0$ and define $d_N^*([x], [y]) := d_N(x, y)$ for $x, y \in X_0$. Then (Z^*, d_N^*) is a metric space, and the canonical projection $\pi_{N,d} : X_0 \to Z^*$ is continuous. As K_0 is G-invariant, G acts in a canonical way on Z^*, and this action is isometric. Hence it extends isometrically to the completion of Z^*, which we denote by $X_{N,d}$. As Z^* is the continuous image of a separable space, it is separable, and so is its completion $X_{N,d}$. Finally, as Z^* is the continuous image of a subset X_0 of the set of transitive points, also the action of G on Z^* is topologically transitive, and as that action is equicontinuous, the action of G on $X_{N,d}$ is in fact minimal. In order to conclude that $S_0^{eq} \subseteq S_0^{\pi_{N,d}} = K_0(N)$, we would need to know that $(X_{N,d}.d_N)$ is compact. As this space is complete by construction, all that remains to be proved is that it is totally bounded. In order to prove that, we need the following lemma:

[3] In [1] this is written down for minimal (X, G).

Lemma 2.4. *Let Z be a separable metric space on which G acts isometrically and transitively. If there exists a finite, non-trivial, G-invariant Borel measure μ on Z, then Z is totally bounded.*

Proof. Suppose for a contradiction that Z is not totally bounded. Then there is $\epsilon > 0$ such that Z cannot be covered by finitely many 4ϵ-balls. We construct inductively an infinite sequence $z_1, z_2, \dots$ of points in Z such that the 2ϵ-balls $B_{2\epsilon}(z_i)$ are pairwise disjoint: Fix $z_1 \in Z$ arbitrary. Suppose that z_i are chosen for $i = 1, \dots, k$. By choice of ϵ, there is $z_{k+1} \notin \bigcup_{i=1}^k B_{4\epsilon}(z_i)$. Hence $B_{2\epsilon}(z_{k+1})$ is disjoint from all $B_{2\epsilon}(z_i)$ for $i = 1, \dots, k$. As G acts transitively on Z, there are $g_1, g_2, \dots \in G$ such that $g_i B_\epsilon(z_1) \subseteq B_{2\epsilon}(z_i)$. Hence $\mu(B_\epsilon(z_1)) = \mu(g_i B_\epsilon(z_1)) \leqslant \mu(B_{2\epsilon}(z_i))$, and this tends to 0 as $i \to \infty$, because the $B_{2\epsilon}(z_i)$ are pairwise disjoint. This argument applies to each ball $B_\epsilon(z_1)$, $z_1 \in Z$, and as Z is separable, this would imply that μ is the zero-measure. $\square$

Now we can finish the discussion started after Lemma 2.3 with the following lemma:

Lemma 2.5. *If $\lambda(X_0) = 1$, then $S_0^{eq} \subseteq \bigcap_{N\in\mathcal{N}} K_0(N)$.*

Proof. It suffices to show that $S_0^{eq} \subseteq S_0^{\pi_{N,d}} = K_0(N)$ for each $N \in \mathcal{N}$. In view of the discussion after Lemma 2.3 this will follow, once we have proved that $(X_{N,d}, d_N)$ is totally bounded. As (X_0, d) is separable by Remark 2.2, also $(X_0/K_0(N), d_N)$ is separable, and this separability carries over to its completion $(X_{N,d}, d_N)$. Hence we can apply Lemma 2.4 to the non-trivial G-invariant Borel probability measure $\mu := \lambda \circ \pi_{N,d}^{-1}$ on $X_{N,d}$ and conclude that $(X_{N,d}, d_N)$ is indeed totally bounded. $\square$

Lemma 2.6. *Assume that λ has full topological support in X. Then $\bigcap_{N\in\mathcal{N}} K_0(N) \subseteq Q_0$.*

Proof. We follow the arguments in the proof of [1, Theorem 8] for the minimal case: Let $(x, y) \in \bigcap_{N\in\mathcal{N}} K_0(N) \subseteq X_0^2$, let $V \in \mathcal{O}^2$ be a neighbourhood of Δ_0, and let $A \in \mathcal{O}$ be a neighbourhood of x. Without loss we can assume that $A \times A \subseteq V$, because $(x, x) \in \Delta_0$. Define

$$N := \overline{\bigcup_{g\in G} g\left(\{y\} \times A\right)}^{\mathcal{O}^2} .$$

Obviously, N is $\mathcal{O}^2$-closed and G-invariant, i.e. $N \in \mathcal{N}$, and $\{y\} \times A \subseteq N$ (consider $g = e$), so that $A \subseteq N_y$. As $(x, y) \in K_0(N)$, it follows that $\lambda(A \setminus N_x) \leqslant \lambda(N_y \setminus N_x) \leqslant d_N(x, y) = 0$. As A is open and N_x is closed, this implies $A \subseteq N_x$ and hence $(x, x) \in \{x\} \times A \subseteq N$. Therefore, there are $g \in G$ and $x' \in A$ such that $(gy, gx') \in A \times A$. Hence $(gx', gy) \in A \times A \subseteq V$. As X_0 is dense in X by assumption and as $A \in \mathcal{O}$, one can choose $x' \in A \cap X_0$. As $y \in X_0$, this proves that $(x, y) \in Q_0$. (Observe that this proves a bit more, namely that in (3) of Lemma 2.1 the point y' can be chosen to be equal to y. Interchanging the roles of x and y, one could instead choose $x' = x$.) $\square$

Theorem 2.3. *Suppose there exists a G-invariant Borel probability measure λ on X with full topological support and with $\lambda(X_0) = 1$. Then*

$$Q_0 = S_0^* = S_0^{eq} = \bigcap_{N\in\mathcal{N}} K_0(N) = \bigcap_{N\in\mathcal{N}} K_t(N) \cap X_0^2 ,$$

where the equivalence relations $K_0(N)$ are determined as above using λ. In particular, $S_0^{eq} = S_t^{eq} \cap X_0^2$. (Observe that these identities hold for each such measure λ.)

Proof. We have $Q_0 \subseteq S_0^*$ by definition, $S_0^* \subseteq S_0^{eq}$ by Lemma 2.2, $S_0^{eq} \subseteq \bigcap_{N\in\mathcal{N}} K_0(N) = \bigcap_{N\in\mathcal{N}} K_t(N) \cap X_0^2$ by Lemma 2.5, and $\bigcap_{N\in\mathcal{N}} K_0(N) \subseteq Q_0$ by Lemma 2.6. $\square$

2.3 Existence of maximal equicontinuous generic factors

The natural question that arises now is whether there actually exists a maximal equicontinuous factor of (X_t, G), or, in the terminology of [16], a maximal equicontinuous generic factor of (X, G). We precede the proof of this fact with a more technical lemma.

Lemma 2.7. *There is an at most countable family $\mathcal{N}_c \subseteq \mathcal{N}$ such that $\bigcap_{N\in\mathcal{N}} K_0(N) = \bigcap_{N\in\mathcal{N}_c} K_0(N)$ for all invariant $X_0 \subseteq X_t$ with $\lambda(X_0) = 1$.*

Proof. As X is second countable, there is a countable base $O_1, O_2, \ldots$ for the topology of X^2. Hence, for each $N \in \mathcal{N}$, there is an index set $J_N \subseteq \mathbb{N}$ such that $X_t^2 \setminus K_t(N) = \bigcup_{j\in J_N} O_j \cap X_t^2$. Let $J = \bigcup_{N\in\mathcal{N}} J_N$. Using the axiom of choice, we can associate with each $j \in J$ a set $N_j \in \mathcal{N}$ such that $j \in J_{N_j}$, i.e. such that $O_j \cap X_t^2 \subseteq X_t^2 \setminus K_t(N_j)$. Let $\mathcal{N}_c = \{N_j : j \in J\}$. Then

$$
\begin{aligned}
X_t^2 \setminus \left(\bigcap_{N\in\mathcal{N}_c} K_t(N) \right) &\subseteq X_t^2 \setminus \left(\bigcap_{N\in\mathcal{N}} K_t(N) \right) = \bigcup_{N\in\mathcal{N}} X_t^2 \setminus K_t(N) = \bigcup_{j\in J} O_j \cap X_t^2 \\
&\subseteq \bigcup_{j\in J} X_t^2 \setminus K_t(N_j) = \bigcup_{N\in\mathcal{N}_c} X_t^2 \setminus K_t(N) = X_t^2 \setminus \left(\bigcap_{N\in\mathcal{N}_c} K_t(N) \right).
\end{aligned}
$$

Hence we have equalities everywhere in this chain of inclusions. As $K_t(N) \subseteq X_t^2$ for all $N \in \mathcal{N}$, this implies $\bigcap_{N\in\mathcal{N}_c} K_t(N) = \bigcap_{N\in\mathcal{N}} K_t(N)$ and hence

$$
\bigcap_{N\in\mathcal{N}} K_0(N) = X_0^2 \cap \bigcap_{N\in\mathcal{N}} K_t(N) = X_0^2 \cap \bigcap_{N\in\mathcal{N}_c} K_t(N) = \bigcap_{N\in\mathcal{N}_c} K_0(N)\,.
$$

□

The following theorem is the announced more detailed version of Theorem 2.1. The additional information is in part c).

Theorem 2.4. *Suppose that (X, G) is ergodic, i.e. there exists an ergodic G-invariant Borel probability measure λ on X with full topological support.*

a) (X, G) has a maximal equicontinuous generic factor $\pi : (X, G) \overset{gen}{\to} (Z, G)$, where (Z, G) is a compact, metrizable, equicontinuous system, unique up to isomorphism, and one can choose (Z, G) as a minimal group rotation.

b) The construction of π and (Z, G) does not depend on the particular ergodic measure λ with full topological support.

c) If X_0 is a G-invariant subset of X_t with $\lambda(X_0) = 1$, if $\pi_Y : X_0 \to Y$ is another homomorphism to a minimal equicontinuous compact system (Y, G), then there is a factor map $\pi' : (Z, G) \to (Y, G)$ such that $\pi_Y = \pi' \circ \pi|_{X_0}$. In particular, π_Y extends continuously to X_t.

Proof. Note first that $\lambda(X_t) = 1$ by Remark 2.1.

a) Enumerate the countable set $\mathcal{N}_c$ from the previous lemma as $\mathcal{N}_c = \{N_n : n \in \mathbb{N}\}$. Define $D : X_t \times X_t \to \mathbb{R}$ as $D(x, y) = \sum_{n\in\mathbb{N}} 2^{-n} d_{N_n}(x, y)$, where d_{N_n} is the G-invariant continuous pseudo-metric on X_t associated with N_n. Then also D is a G-invariant continuous pseudo-metric on X_t and

$$
\mathcal{X}_t^D := \left\{(x, y) \in X_t^2 : D(x, y) = 0\right\} = \bigcap_{n\in\mathbb{N}} K_t(N_n) = \bigcap_{N\in\mathcal{N}} K_t(N) \tag{4}
$$

is an O_t^2-closed G-invariant equivalence relation on X_t. Let $Z^* := X_t/\mathcal{X}_t^D$ and define $d_Z([x],[y]) := D(x,y)$ for $x,y \in X_t$. Then (Z^*, d_Z) is a metric space, and the canonical projection $\pi : X_t \to Z^*$ is continuous. As $\mathcal{X}_t^D$ is G-invariant, G acts in a canonical way on Z^*, and this action is isometric. Hence it extends isometrically to the completion Z of Z^*. As Z^* is the continuous image of a separable space, it is separable, and so is its completion Z. Finally, as Z^* is the continuous image of the set X_t of transitive points, also the action of G on Z is topologically transitive, and as that action is equicontinuous, the action of G on Z is in fact minimal. In order prove that Z is compact it suffices to note that it is totally bounded, see Lemma 2.4 with $\mu = \lambda \circ \pi^{-1}$. As $\mathcal{X}_t^D = S_t^{eq}$ by Theorem 2.3, and as $d_Z([x],[y]) = D(x,y)$ for $x,y \in X_t$, the system (Z,G) is a MEGF for (X,G).

We turn to the proof of the uniqueness (up to conjugacy) of the MEGF: Suppose that $\pi_Y : (X,G) \overset{gen}{\to} (Y,G)$ is a further MEGF of (X,G). Then there are factor maps $\pi' : Z \to Y$ and $\pi'' : Y \to Z$ such that $\pi_Y = \pi' \circ \pi$ and $\pi = \pi'' \circ \pi_Y$. Hence $\pi = \pi'' \circ \pi' \circ \pi$, which implies that $(\pi'' \circ \pi')|_{\pi(X_t)} = \mathrm{id}_{\pi(X_t)}$. As $\pi(X_t)$ is dense in Z and as $\pi'' \circ \pi' : Z \to Z$ is continuous, this implies that $\pi'' \circ \pi' = \mathrm{id}_Z$. In particular, π' is 1-1, i.e. tha factor map $\pi' : Z \to Y$ is a homeomorphism.

b) π is defined in (4) by the equivalence relation $\mathcal{X}_t^D$, whose definition, via the relations $K_t(N)$, is clearly based on a particular choice of the measure λ. But Theorem 2.3 shows that $\mathcal{X}_t^D = S_t^{eq}$ for any choice of λ.

c) We have to prove that (Z,G) is maximal among all compact metrizable equicontinuous generic factors of (X,G) (in the strengthened sense of part c) of the theorem). So suppose there are a G-invariant subset $X_0 \subseteq X_t$ with $\lambda(X_0) = 1$ and a homomorphism $\pi_Y : X_0 \to Y$ to a compact metrizable equicontinuous minimal system (Y,G). Then $\mathcal{X}_t^D \cap X_0^2 = \bigcap_{N \in \mathcal{N}} K_0(N) = S_0^{eq} \subseteq S_0^{\pi_Y}$ by Theorem 2.3. It follows that π_Y factorizes over $\pi|_{X_0} : X_0 \to Z$. □

2.4 Weak mixing and maximal equicontinuous generic factors

Proof of Theorem 2.2. a) We follow the proof of Theorem 2.7 in [16]. As (X,G) is weakly scattering, there is $(x_0, z_0) \in X_t \times Z$ such that $\overline{G(x_0, z_0)} = X \times Z$. Denote by $\pi : X_t \to Z$ the MEGF of (X,G), and define $\phi : X_t \times Z \to Z$, $\phi(x,z) = z^{-1}\pi(x)$. Then ϕ is continuous, and $\phi(gx_0, gz_0) = (gz_0)^{-1}\pi(gx_0) = z_0^{-1}g^{-1}g\pi(x_0) = \phi(x_0, z_0)$ for all $g \in G$. Hence $\phi(x,z) = \phi(x_0,z_0)$ for all $(x,z) \in X_t \times Z$. In particular, $\pi(x) = \phi(x,e) = \phi(x_0,z_0)$ for all $x \in X_t$, so that $Z = \overline{\pi(X_t)} = \{\phi(x_0,z_0)\}$ is a singleton.
b) Let (X,G) be an ergodic topological dynamical system, and denote by λ any G-invariant Borel probability measure on X with $\lambda(X_t) = 1$. Assume that the MEGF of (X,G) is trivial. That means that the equicontinuous structure relation S_t^{eq} defined in (2) is maximal, i.e. $S_t^{eq} = X_t^2$. Hence $K_t(N) = X_t^2$ for all $N \in \mathcal{N}$ by Lemma 2.5, where, as before, $\mathcal{N}$ denotes the family of all closed G-invariant subsets of X^2. Therefore $\lambda(N_{x'} \triangle N_x) = 0$ for all $x, x' \in X_t$.

In order to prove that (X,G) is weakly mixing we must show that each $N \in \mathcal{N}$ is either nowhere dense in or equal to X^2. So assume that $N \in \mathcal{N}$ is not nowhere dense, i.e. that $\mathrm{int}(N) \neq \emptyset$. Then there are open sets $U, V \subseteq X$ such that $U \times V \subseteq N$. Fix any $x_0 \in X_t \cap U$. Then $V \subseteq N_{x_0}$, and $\lambda(N_x \triangle N_{x_0}) = 0$ for all $x \in X_t$. As λ has also full topological support, this implies $V \subseteq N_x$ for all $x \in X_t$, i.e. $X_t \times V \subseteq N$. Let $W := \bigcup_{g \in G} gV$. Then W is open and G-invariant, and W is dense in X, because (X,G) is topologically transitive. It follows that

$$X^2 = \overline{X_t \times W} = \overline{\bigcup_{g \in G} g(X_t \times V)} \subseteq \overline{\bigcup_{g \in G} gN} = \overline{N} = N \,.$$

□

3 Maximal equicontinuous generic factors and weak model sets

The dynamics of weak model sets are an excellent testing ground for the relevance of MEGFs. We start by summarizing some essential notations and results from [19].

3.1 Some recollections on weak model sets

Assumptions and notations

(1) G and H are *locally compact second countable abelian groups* with Haar measures m_G and m_H. Then the product group $G\times H$ is locally compact second countable abelian as well, and we choose $m_{G\times H} = m_G \times m_H$ as Haar measure on $G \times H$.

(2) $\mathcal{L} \subseteq G \times H$ is a *cocompact lattice*, i.e., a discrete subgroup whose quotient space $(G \times H)/\mathcal{L}$ is compact. Thus $\hat{X} := (G \times H)/\mathcal{L}$ is a compact second countable abelian group. Denote by $\pi^G : G \times H \to G$ and $\pi^H : G \times H \to H$ the canonical projections. We assume that $\pi^G|_{\mathcal{L}}$ is 1-1 and that $\pi^H(\mathcal{L})$ is dense in H.[4]

(3) G acts on $G \times H$ by translation: $gx := (g, 0) + x$.

(4) Elements of $G \times H$ are denoted as $x = (x_G, x_H)$, elements of $\hat{X}$ as $\hat{x}$ or as $x + \mathcal{L} = (x_G, x_H) + \mathcal{L}$, when a representative x of $\hat{x}$ is to be stressed. We normalise the Haar measure $m_{\hat{X}}$ on $\hat{X}$ such that $m_{\hat{X}}(\hat{X}) = 1$. Thus $m_{\hat{X}}$ is a probability measure.

(5) The *window* W is a compact subset of H. We assume that $m_H(W) > 0$.

Consequences of the assumptions We list a few facts from topology and measure theory that follow from the above assumptions. We will call any neighbourhood of the neutral element in an abelian topological group a *zero neighbourhood*.

(1) Being locally compact second countable abelian groups, G, H and $G \times H$ are metrizable with a translation invariant metric with respect to which they are complete metric spaces. In particular they have the Baire property. As such groups are σ-compact, m_G, m_H and $m_{G\times H}$ are σ-finite.

(2) As $G \times H$ is σ-compact, the lattice $\mathcal{L} \subseteq G \times H$ is at most countable. Note that $G \times H$ can be partitioned by shifted copies of the relatively compact fundamental domain X. This means that $\mathcal{L}$ has a positive finite point density $\mathrm{dens}(\mathcal{L}) = 1/m_{G\times H}(X)$. We thus have $m_{\hat{X}}(\hat{A}) = \mathrm{dens}(\mathcal{L}) \cdot m_{G\times H}(X \cap (\pi^{\hat{X}})^{-1}(\hat{A}))$ for any measurable $\hat{A} \subseteq \hat{X}$, where $\pi^{\hat{X}} : G \times H \to \hat{X}$ denotes the quotient map. As a factor map between topological groups, $\pi^{\hat{X}}$ is open.

(3) $\mathcal{L}$ acts on (H, m_H) by $h \mapsto \ell_H + h$ metrically transitively, i.e., for every measurable $A \subseteq H$ such that $m_H(A) > 0$ there exist at most countably many $\ell^i \in \mathcal{L}$ such that $m_H((\bigcup_i(\ell^i_H + A))^c) = 0$, see [22, Ch. 16, Ex. 1].

(4) The action $\hat{x} \mapsto (g, 0) + \hat{x}$ of G on $\hat{X}$ is minimal. Indeed: let $x + \mathcal{L}, y + \mathcal{L} \in \hat{X}$ be arbitrary. Choose a sequence $(\ell_n)_n$ from $\mathcal{L}$ such that $\ell_{n,H} \to y_H - x_H$ and let $g_n = y_G - \ell_{n,G} - x_G$. Then

$$g_n(x + \mathcal{L}) = (g_n, 0) + x + \mathcal{L} = (g_n, 0) + x + \ell_n + \mathcal{L} = (y_G, \ell_{n,H} + x_H) + \mathcal{L} \to y + \mathcal{L} .$$

[4] Denseness of $\pi^H(\mathcal{L})$ can be assumed without loss of generality by passing from H to the closure of $\pi^H(\mathcal{L})$. In that case m_H must be replaced by $m_{\overline{\pi^H(\mathcal{L})}}$.

This shows that the G-orbit of $x + \mathcal{L}$ is dense in $\hat{X}$. It follows that $\hat{X}$ with its natural action is uniquely ergodic, see e.g. [25, Prop. 1].

(5) Denote by $\mathcal{M}$ and $\mathcal{M}^G$ the spaces of all locally finite measures on $G \times H$ and G, respectively. They are endowed with the topology of vague convergence. As G and $G \times H$ are complete metric spaces, this is a Polish topology, see [17, Thm. A.2.3].

The objects of interest The pair $(\mathcal{L}, W)$ assigns to each point $\hat{x} \in \hat{X}$ a discrete point set in $G \times H$. Such point sets P are identified with the measures $\sum_{y \in P} \delta_y \in \mathcal{M}$ and called *configurations*. More precisely:

(1) For $\hat{x} = x + \mathcal{L} \in \hat{X}$ define the configuration

$$\nu_W(\hat{x}) := \sum_{y \in (x+\mathcal{L}) \cap (G \times W)} \delta_y \,. \tag{5}$$

It is important to understand ν_W as a map from $\hat{X}$ to $\mathcal{M}$. The canonical projection $\pi^G : G \times H \to G$ projects measures $\nu \in \mathcal{M}$ to measures $\pi^G_* \nu$ on G defined by $\pi^G_* \nu(A) := \nu((\pi^G)^{-1}(A))$. We abbreviate

$$\nu^G_W := \pi^G_* \circ \nu_W : \hat{X} \to \mathcal{M}^G \tag{6}$$

(2) Denote by

- $\mathcal{M}_W$ the vague closure of $\nu_W(\hat{X})$ in $\mathcal{M}$,
- $\mathcal{M}^G_W$ the vague closure of $\nu^G_W(\hat{X})$ in $\mathcal{M}^G$,

The group G acts continuously by translations on all these spaces: $(g\nu)(A) := \nu(g^{-1}A)$. As $\nu_W(\hat{x})(g^{-1}A) = (g\nu_W(\hat{x}))(A) = \nu_W(g\hat{x})(A)$, it is obvious that all $\nu_W(\hat{x})$ are uniformly translation bounded, and it follows from [6, Thm. 2] that both spaces are compact.

(3) $Q_{\mathcal{M}} := m_{\hat{X}} \circ {\nu_W}^{-1}$ and $Q_{\mathcal{M}^G} := m_{\hat{X}} \circ (\nu^G_W)^{-1}$ are the *Mirsky measures* on $\mathcal{M}_W$ and $\mathcal{M}^G_W$, respectively. Note that $Q_{\mathcal{M}^G} = Q_{\mathcal{M}} \circ (\pi^G_*)^{-1}$.

3.2 The MEGF of the Mirsky measure

The following facts are taken from [19] and [20]:
(1) Facts about the MEF [19, Theorem 1]:
 a) If $\operatorname{int}(W) \neq \emptyset$, then $(\hat{X}, G)$ is the MEF of $(\mathcal{M}_W, G)$.
 b) If $\operatorname{int}(W) = \emptyset$, then the MEFs of $(\mathcal{M}_W, G)$ and $(\mathcal{M}^G_W, G)$ are trivial.
(2) Facts about the KF [19, Theorems 2] and [20, Theorem 1]:
 a) $(\hat{X}, m_{\hat{X}}, G)$ is the KF of $(\mathcal{M}_W, Q_{\mathcal{M}}, G)$. Even more, both systems are isomorphic.
 b) If the window W is Haar aperiodic, then the same is true for the system $(\mathcal{M}^G_W, Q_{\mathcal{M}^G}, G)$.
 Here W is *Haar aperiodic*, if $m_H\,((h + W) \triangle W) = 0$ implies $h = 0$.

Denote by $X \subseteq \mathcal{M}_W$ the topological support of $Q_{\mathcal{M}}$ and by $X^G \subseteq \mathcal{M}^G_W$ that of $Q_{\mathcal{M}^G}$. Although both sets may be strictly contained in their ambient spaces, they capture the most important aspects of the dynamics. For example, $m_{\hat{X}}$-a.a. configurations $\nu_W(\hat{x})$ belong to these spaces [19, Theorem 5d].

Corollary 3.1. *Statements (1) and (2) above remain true for the subsystems* (X, G) *and* (X^G, G)*.*

Proof. For (2) this is trivial, because $Q_{\mathcal{M}}(X) = Q_{\mathcal{M}^G}(X^G) = 1$. For (1) observe first, that $(\mathcal{M}_W, G)$ has a unique minimal subsystem [19, Lemma 6.3], so this system is contained in X. In case a) it is an almost automorphic extension of $(\hat{X}, G)$ [19, Theorem 1a], so that $(\hat{X}, G)$ is its MEF. But then $(\hat{X}, G)$ is also the MEF of (X, G). In case b), the minimal system is a fixed point, so that the MEF of any subsystem containing this fixed point is trivial. □

The facts listed as (1) above, and also Corollary 3.1, provide no useful information on the MEF of the systems $(\mathcal{M}_W^G, G)$ and (X^G, G), respectively. This changes completely when the MEGF of (X^G, G) is considered. (Recall that X^G denotes the support of the Mirsky measure on $\mathcal{M}_W^G$.)

Theorem 3.1. *a) $(\hat{X}, G)$ is the MEGF of (X, G).*
b) If the window W is Haar aperiodic, then the same is true for the system (X^G, G).

For the proof of the theorem we need the following lemma.

Lemma 3.1. *Let A and B be compact metric spaces, $\alpha : A \to A$ and $\beta : B \to B$ homeomorphisms, and $f : A \to B$ a continuous map such that $f \circ \alpha = \beta \circ f$. Define $\rho : A \to [0, \infty)$ by $\rho(x) = \inf\left\{r > 0 : f^{-1}\{f(x)\} \subseteq B_r(x)\right\}$, and let $A_0 := \left\{x \in A : f^{-1}\{f(x)\} = \{x\}\right\}$.*
a) If α is an isometry, then $\rho \circ \alpha = \rho$.
b) $\alpha(A_0) = A_0$.
c) ρ is upper semicontinuous.
d) A_0 is a Borel subset of A, and $f|_{A_0} : A_0 \to f(A_0)$ is a homeomorphism w.r.t. the induced topologies.

Proof. a) and b) A chain of elementary identities shows that $\alpha(f^{-1}\{y\}) = f^{-1}\{\alpha(y)\}$ for each $y \in B$. Hence $\alpha(f^{-1}\{f(x)\}) = f^{-1}\{\alpha(f(x))\} = f^{-1}\{f(\alpha(x))\}$ for each $x \in A$. As α is continuous and A is compact, given $r > 0$ there is $\delta(r) > 0$ such that $\alpha(B_{\delta(r)}(x)) \subseteq B_r(\alpha(x))$, where $\delta(r) = r$ if α is isometric. Therefore,

$$
\begin{aligned}
\rho(\alpha(x)) &= \inf\left\{r > 0 : f^{-1}\{f(\alpha(x))\} \subseteq B_r(\alpha(x))\right\} \\
&\leqslant \inf\left\{r > 0 : \alpha(f^{-1}\{f(x)\}) \subseteq \alpha(B_{\delta(r)}(x))\right\} \\
&= \inf\left\{r > 0 : f^{-1}\{f(x)\} \subseteq B_{\delta(r)}(x)\right\}.
\end{aligned}
$$

In particular, $\rho(\alpha(x)) = 0$ whenever $\rho(x) = 0$, so that $\alpha(A_0) \subseteq A_0$. If α is isometric, then $\rho \circ \alpha \leqslant \rho$. The same argument with α^{-1} instead of α yields the reverse inclusion and inequality.
c) Let $x \in A$, $r > 0$, and assume that there are $x_n \in A$ with $x_n \to x$ and $\rho(x_n) > r$ ($n \in \mathbb{N}$). Then there are $x_n' \in f^{-1}\{f(x_n)\}$ ($n \in \mathbb{N}$) with $d(x_n', x_n) \geqslant r$. Passing to a subsequence we can assume w.l.o.g. that the x_n' converge to some $x' \in A$. Then $d(x', x) \geqslant r$ and $f(x') = \lim_n f(x_n') = \lim_n f(x_n) = f(x)$, i.e. $\rho(x) \geqslant r$. This proves the upper semi-continuity of ρ.
d) Because of c), $A_0 = \{x \in A : \rho(x) = 0\} = \bigcap_n \{x \in A : \rho(x) < 1/n\}$ is G_δ-set, in particular a Borel set. Next observe that $f|_{A_0} : A_0 \to f(A_0)$ is bijective and continuous by assumption. As A is a compact Hausdorff space, it remains to show that $f(K \cap A_0) = f(K) \cap f(A_0)$ for each closed subset K of A. The $\subseteq$-inclusion is trivial. So let $y \in f(K) \cap f(A_0)$. There are $x' \in K$ and $x \in A_0$ such that $f(x') = y = f(x)$. As $x \in A_0$, this implies $x' = x$ and hence $y = f(x) \in f(K \cap A_0)$. □

Proof of Theorem 3.1. (X, G) and (X^G, G) are ergodic systems in the sense of Definition 2.1, where the respective Mirsky measures play the role of the measure λ. Hence Theorem 2.4 applies, and both systems have compact abelian groups as MEGFs, call them (Z, G) and (Z^G, G) with factor maps π^Z and π^{Z^G}, respectively.
a) The system $(\mathcal{M}_W, G)$ can be extended to the *graph system* $(\mathcal{G}\mathcal{M}_W, G)$, where $\mathcal{G}\mathcal{M}_W \subseteq \hat{X} \times \mathcal{M}$ is the

closure of the graph of $\nu_W : \hat{X} \to \mathcal{M}$. This system is a topological joining of $(\hat{X}, G)$ and $(\mathcal{M}_W, G)$. The natural projection $\pi_*^{G\times H} : \mathcal{GM}_W \to \mathcal{M}_W$ is continuous, and its restriction to $(\pi_*^{G\times H})^{-1}(X \setminus \{\underline{0}\})$ is 1-1, where $\underline{0}$ denotes the zero measure (in other words, the empty configuration) [19, Proposition 3.5b]. Hence this restriction is a homeomorphism between the (possibly non-compact) spaces $(\pi_*^{G\times H})^{-1}(X \setminus \{\underline{0}\})$ and $X \setminus \{\underline{0}\}$.[5] As the natural projection $\pi_*^{\hat{X}} : \mathcal{GM}_W \to \hat{X}$ is continuous, $\pi_*^{\hat{X}} \circ (\pi_*^{G\times H})^{-1}|_{X_t} : X_t \to \hat{X}$ describes an equicontinuous generic factor $(X, G) \overset{gen}{\to} (\hat{X}, G)$.

By Theorem 2.4c, there is a factor map $\pi' : Z \to \hat{X}$ such that $\pi_*^{\hat{X}} \circ (\pi_*^{G\times H})^{-1} = \pi' \circ \pi^Z$ on X_t. For the corresponding *measure dynamical systems* this implies:

- $\pi' : (Z, Q_{\mathcal{M}} \circ (\pi^Z)^{-1}, G) \to (\hat{X}, m_{\hat{X}}, G)$ is a factor map, where $Q_{\mathcal{M}} \circ (\pi^Z)^{-1}$ necessarily is the Haar measure on Z.
- $\pi^Z \circ \nu_W : (\hat{X}, m_{\hat{X}}, G) \to (Z, Q_{\mathcal{M}} \circ (\pi^Z)^{-1}, G)$ is a $m_{\hat{X}}$-a.e. defined factor map, because $\nu_W : (\hat{X}, m_{\hat{X}}, G) \to (X, Q_{\mathcal{M}}, G)$ is one. (Observe that $m_{\hat{X}}(\nu_W^{-1}(X)) = Q_{\mathcal{M}}(X) = 1$.)
- $\pi' \circ (\pi^Z \circ \nu_W) = \pi_*^{\hat{X}} \circ (\pi_*^{G\times H})^{-1} \circ \nu_W = \mathrm{id}_{\nu_W^{-1}(X_t)}$ on $\nu_W^{-1}(X_t)$. where $m_{\hat{X}}(\nu_W^{-1}(X_t)) = Q_{\mathcal{M}}(X_t) = 1$, because X is the support of the ergodic invariant measure $Q_{\mathcal{M}}$.

Hence the topological factor map $\pi' : (Z, Q_{\mathcal{M}} \circ (\pi^Z)^{-1}, G) \to (\hat{X}, m_{\hat{X}}, G)$ is a measure theoretic isomorphism[6], in particular is $\rho(z) = 0$ for m_Z-a.e. $z \in Z$, where ρ is defined as in Lemma 3.1 (for $f = \pi'$). As ρ is upper semicontinuous and G-invariant (Lemma 3.1), and as the action of G on Z is minimal, ρ is constant everywhere, so that $\rho(z) = 0$ for all $z \in Z$, which means that π' is everywhere 1-1. As $\pi' : (Z, G) \to (\hat{X}, G)$ is a factor map, it is necessarily a homeomorphism, i.e. $(\hat{X}, G)$ is (topologically conjugate to) the MEGF of (X, G).[7]

b) Let us first assume that the window is *Haar regular*, i.e. that the support of the restriction of m_H to W is all of W (and not only a subset of W), see [20] for a discussion of this notion. Then Theorem 1 of [20] asserts that $\pi_*^G : \mathcal{M}_W \to \mathcal{M}_W^G$ is 1-1 at $Q_{\mathcal{M}}$-a.a. $\nu \in \mathcal{M}_W$. More precisely, the set $X_0 := \left\{\nu \in \mathcal{M}_W : (\pi_*^G)^{-1}\{\pi_*^G(\nu)\} = \{\nu\}\right\}$ has $Q_{\mathcal{M}}(X_0) = 1$. By Lemma 3.1, X_0 is indeed a Borel-measurable, G-invariant subset of $\mathcal{M}_W$, and $\pi_*^G|_{X_0} : X_0 \to \pi_*^G(X_0)$ is a homeomorphism (w.r.t. the induced topologies). Hence, also $X_0' := X_0 \cap X_t \cap (\pi_*^G)^{-1}(X_t^G)$ is a G-invariant set of full $Q_{\mathcal{M}}$-measure (observe that $Q_{\mathcal{M}}(X_t) = 1 = Q_{\mathcal{M}^G}(X_t^G)$), and $\pi_*^G|_{X_0'} : X_0' \to \pi_*^G(X_0')$ is a homeomorphism. Observe that $Q_{\mathcal{M}^G}(\pi_*^G(X_0')) = Q_{\mathcal{M}}(X_0') = 1$.

It follows that $\pi^{Z^G} \circ \pi_*^G : (X_0', G) \to (Z^G, G)$ is an equicontinuous factor. As $(\hat{X}, G)$ is the MEGF of (X, G) by part a) of the present theorem, Theorem 2.4c guarantees the existence of1 a factor map $\pi' : (\hat{X}, G) \to (Z^G, G)$ such that $\pi^{Z^G} \circ \pi_*^G = \pi' \circ \pi_*^{\hat{X}}$ on X_0'. On the other hand, Theorem 2.4c applies as well to the factor map $\pi_*^{\hat{X}} \circ (\pi_*^G)^{-1} : \pi_*^G(X_0') \to \hat{X}$, i.e. there is a factor map $\pi'' : (Z^G, G) \to (\hat{X}, G)$ such that $\pi_*^{\hat{X}} \circ (\pi_*^G)^{-1} = \pi'' \circ \pi^{Z^G}$ on $\pi_*^G(X_0')$. Hence $\pi_*^{\hat{X}} = \pi'' \circ \pi^{Z^G} \circ \pi_*^G = \pi'' \circ \pi' \circ \pi_*^{\hat{X}}$ on X_0', so that $\pi'' \circ \pi' = \mathrm{id}_{\hat{X}}$ on $\pi_*^{\hat{X}}(X_0')$, where $m_H(\pi_*^{\hat{X}}(X_0')) \geqslant m_H(\nu_W^{-1}(X_0')) = Q_{\mathcal{M}}(X_0') = 1$. As $\pi'' \circ \pi'$ is continuous, this shows that $\pi'' \circ \pi' = \mathrm{id}_{\hat{X}}$, in particular π' is 1-1. Being a factor map onto Z^G, π' thus is a homeomorphism, so that $(\hat{X}, G)$ is (isomorphic to) the MEGF (Z^G, G) of (X^G, G).

It remains to treat the case where W is not Haar regular. Then, what we proved so far, applies to the system defined by the *Haar regularization* $W_{reg} := \mathrm{supp}(m_H|_W) \subseteq W$ of W. As $m_H(W \setminus W_{reg}) = 0$, both windows define the same Mirsky measures and, in particular, the systems X^G associated with both windows are identical. □

[5]Indeed, if $A \subseteq (\pi_*^{G\times H})^{-1}(X \setminus \{\underline{0}\})$ is closed, then there is a closed (and hence compact) subset K of $\mathcal{GM}_W$ such that $A = K \cap (\pi_*^{G\times H})^{-1}(X \setminus \{\underline{0}\})$. Hence $\pi_*^{G\times H}(A) = \pi_*^{G\times H}(K) \cap (X \setminus \{\underline{0}\})$ is closed in $X \setminus \{\underline{0}\}$.

[6]Let $z_1, z_2 \in Z$ and assume that $\pi'(z_1) = \pi'(z_2)$. As the Haar measure on Z can be written as $m_{\hat{X}} \circ (\pi^Z \circ \nu_W)^{-1}$, almost all z_1, z_2 have the form $z_i = \pi^Z(\nu_W(\hat{x}_i))$ for some $\hat{x}_i \in \hat{X}$. Hence $\hat{x}_1 = \pi'(z_1) = \pi'(z_2) = \hat{x}_2$, so that also $z_1 = z_2$.

[7]As (Z, G) is equicontinuous and $(\hat{X}, G)$ is minimal, the invertibility of π' (as a consequence of $\rho(z) = 0$ for some z) would also follows from [19, Lemma 7.2].

3.3 The special case of $\mathcal{B}$-free dynamics

For any given set $\mathcal{B} \subseteq \mathbb{N}$ one can define its set of multiples $\mathcal{M}_{\mathcal{B}} = \bigcup_{b\in\mathcal{B}} b\mathbb{Z}$ and the set of $\mathcal{B}$-free numbers $\mathcal{F}_{\mathcal{B}} = \mathbb{Z} \setminus \mathcal{M}_{\mathcal{B}}$. Investigations into the structure of $\mathcal{M}_{\mathcal{B}}$ or, equivalently, of $\mathcal{F}_{\mathcal{B}}$ have a long history (see [7] for references), and dynamical systems theory provides some useful tools for such studies. One way to see this is to interpret such systems as weak model sets, where $G = \mathbb{Z}$, H is a closed subgroup of $\tilde{H} := \prod_{b\in\mathcal{B}} \mathbb{Z}/b\mathbb{Z}$, namely the closure of the canonically embedded integers: $H = \overline{\Delta(\mathbb{Z})}$ where $\Delta(n) = (n, n, n, \dots) \in \tilde{H}$. Finally, $\mathcal{L} = \{(n, \Delta(n)) : N \in \mathbb{Z}\}$ and $W = \{h \in H : h_b \neq 0 \ \forall b \in \mathcal{B}\}$. It is easy to see that in this case $\hat{X}$ is isomorphic to H, so that instead of $\nu_W^G : \hat{X} \to \mathcal{M}^G$ one simply looks at $\nu_W^G : H \to \mathcal{M}^G$, where $\nu_W^G(h) = \sum_{n\in\mathbb{Z}} \delta_n \cdot 1_W(h + \Delta(n))$ $(h \in H)$. Then $\nu_W^G(0) = \sum_{n\in\mathbb{Z}} \delta_n \cdot 1_W(\Delta(n)) = \sum_{n\in\mathcal{F}_{\mathcal{B}}} \delta_n$. In the literature on $\mathcal{B}$-free dynamics these point measures on $\mathbb{Z}$ are represented as 0-1-sequences, so that $\nu_W^G(0)$ is interpreted as $\eta \in \{0, 1\}^{\mathbb{Z}}$, where $\eta_n = 1$ if and only if $\nu_W^G(0)\{n\} = 1$.

If the set $\mathcal{B} \subseteq \mathbb{Z}$ is *taut* (a basic regularity property, see e.g. [15, 7]), then the window is always Haar aperiodic [18]. Hence $(H, \mathbb{Z})$ is the MEGF of $(\mathcal{M}_W^G, \mathbb{Z})$ by Theorem 3.1, where $\mathbb{Z}$ acts on H by $h \mapsto h + \Delta(n)$ and on $\mathcal{M}_W^G$ through the left shift by n positions. ($\mathcal{M}_W^G$ can be canonically identified with a closed subshift of $\{0, 1\}^{\mathbb{Z}}$.) In any case, $(H, m_H, \mathbb{Z})$ is also the Kronecker factor of $(\mathcal{M}_W^G, Q_{\mathcal{M}^G}, \mathbb{Z})$, and there are many other invariant measures with the same KF. That this need not be the case for all ergodic invariant probability measures on $(\mathcal{M}_W^G, \mathbb{Z})$ is demonstrated in the following example.

Example 3.1. Consider the setting studied in [23], where the elements of $\mathcal{B}$ are pairwise co-prime and $\sum_{b\in\mathcal{B}} 1/b < \infty$. Even for this rather special class of systems (which includes the square-free numbers) one can find ergodic invariant measures with full topological support, for which the KF is not supported by the MEGF. The following construction of such examples uses a result from [23] whose proof relies on [12, Theorem 2].

Let κ be an ergodic shift-invariant probability measure on $\{0, 1\}^{\mathbb{Z}}$, and denote by $Q_{\mathcal{M}^G} * \kappa := M_*(Q_{\mathcal{M}^G} \otimes \kappa)$ the "convolution" of $Q_{\mathcal{M}^G}$ and κ, where $M : (\{0, 1\}^{\mathbb{Z}})^2 \to \{0, 1\}^{\mathbb{Z}}$ is the coordinate-wise multiplication [23, Section 2]. Corollary 2.2.4 together with Remark 2.2.5 of that reference shows that the system $(\mathcal{M}_W^G, Q_{\mathcal{M}^G} * \kappa, \mathbb{Z})$ is isomorphic to the direct product of the systems system $(\mathcal{M}_W^G, Q_{\mathcal{M}^G}, \mathbb{Z})$ and $(\{0, 1\}^{\mathbb{Z}}, \kappa, \mathbb{Z})$, whenever the latter system is (isomorphic to) an irrational rotation. In order to make sure that $Q_{\mathcal{M}^G} * \kappa$ has the same topological support as $Q_{\mathcal{M}^G}$ itself, it suffices to produce a 0-1-coding of an irrational rotation for which each block of 1's (of arbitrary length) has positive probability.

To this end fix any irrational number $\alpha \in (0, 1]$ and a sequence $(J_n)_{n>0}$ of open subintervals of $\mathbb{R}/\mathbb{Z}$ with length $|J_n| = (2n2^n)^{-1}$. Define

$$E := \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{n-1} (J_n + k\alpha) .$$

Denote the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ by λ. Then $0 < \lambda(E) \leqslant 1/2$. The coding map $\varphi_E : \mathbb{R}/\mathbb{Z} \to \{0, 1\}^{\mathbb{Z}}$, $x \mapsto (1_E(x + k\alpha))_{k\in\mathbb{Z}}$, is λ-almost surely 1-1, and the probability, that this coding produces only 1's at positions $-n, \dots, n$ is at least $(2(2n + 1)2^{2n+1})^{-1}$.

Formally this construction can be written as a kind of model set, where the internal space H is replaced by $H \times (\mathbb{R}/\mathbb{Z})$ and where the set $W \times E$ is taken as a window. Observe however, that E is an open dense subset of $\mathbb{R}/\mathbb{Z}$, so that the closure of this window would be $W \times (\mathbb{R}/\mathbb{Z})$, which by itself is a window that reproduces precisely the original system. Hence this construction is far from any weak model set.

Remark 3.1. Example 2 in [23, Section 2.2.2] shows that many $\mathcal{B}$-free systems support invariant ergodic measures P with the following two properties:

i) The KF of P is bigger than the KF of the Mirsky measure and is hence not supported by the MEGF of the system.
ii) P is obtained as the Mirsky measure of a compact sub-window of the original window.

These measures do not have full topological support, however.

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