

Strong-Feller property for Navier-Stokes equations driven by space-time white noise ^{*}

Rongchan Zhu^{a,c}, Xiangchan Zhu^{b,c,† ‡}

^aDepartment of Mathematics, Beijing Institute of Technology, Beijing 100081, China

^bSchool of Science, Beijing Jiaotong University, Beijing 100044, China

^cDepartment of Mathematics, University of Bielefeld, D-33615 Bielefeld, Germany

Abstract

In this paper we prove strong Feller property for the Markov semigroups associated to the two or three dimensional Navier-Stokes (N-S) equations driven by space-time white noise using the theory of regularity structures introduced by Martin Hairer in [14]. This implies global well-posedness of 2D N-S equation driven by space-time white noise starting from every initial point in C^η for $\eta \in (-\frac{1}{2}, 0)$.

2000 Mathematics Subject Classification AMS: 60H15, 82C28

Keywords: stochastic Navier-Stokes equation, regularity structure, space-time white noise, renormalisation

1 Introduction

In this paper, we consider the two dimensional (2D) and three dimensional (3D) Navier-Stokes equation driven by space-time white noise: Recall that the Navier-Stokes equations describe the time evolution of an incompressible fluid (see [23]) and are given by

$$\begin{aligned} \partial_t u + u \cdot \nabla u &= \nu \Delta u - \nabla p + \xi \\ u(0) &= u_0, \quad \operatorname{div} u = 0 \end{aligned} \tag{1.1}$$

where $u(t, x) \in \mathbb{R}^d$, $d = 2, 3$, denotes the value of the velocity field at time t and position x , $p(t, x)$ denotes the pressure, and $\xi(t, x)$ is an external force field acting on the fluid. We will consider the case when $x \in \mathbb{T}^d$, $d = 2, 3$, the d -dimensional torus. Our mathematical model for the driving force ξ is a Gaussian field which is white in time and space.

Random Navier-Stokes equations, especially the stochastic 2D Navier-Stokes equation driven by trace-class noise, have been studied in many articles (see e.g. [8], [15], [4], [20] and the

^{*}Research supported in part by NSFC (No.11671035, No.11401019) and DFG through IRTG 1132 and CRC 701

[†]Corresponding author

[‡]E-mail address: zhurongchan@126.com(R. C. Zhu), zhuxiangchan@126.com(X. C. Zhu)

reference therein). In the two dimensional case, existence and uniqueness of strong solutions have been obtained if the noisy forcing term is white in time and colored in space. In the three dimensional case, existence of martingale (=probabilistic weak) solutions, which form a Markov selection, have been constructed for the stochastic 3D Navier-Stokes equation driven by trace-class noise in [9], [6], [12]. Furthermore, the ergodicity has been obtained for every Markov selection of the martingale solutions if driven by non-degenerate trace-class noise (see [9]).

This paper aims at proving the strong Feller property to the equation (1.1) when ξ is space-time white noise. Such a noise might not be relevant for the study of turbulence. However, in other cases, when a flow is subjected to an external forcing with a very small time and space correlation length, a space-time white noise may be appropriate to model in this situation.

In the two dimensional case, the Navier-Stokes equation driven by space-time white noise has been studied in [5], where a unique global solution starting from almost every point has been obtained by using the Gaussian invariant measure for the equation.

In the three dimensional case, we use the theory of regularity structures introduced by Martin Hairer in [14] and the paracontrolled distribution proposed by Gubinelli, Imkeller and Perkowski in [11] and obtain existence and uniqueness of local solutions to the stochastic 3D Navier-Stokes equations driven by space-time white noise in [25]. Recently, these two approaches have been successful in giving a meaning to a lot of ill-posed stochastic PDEs like the Kardar-Parisi-Zhang (KPZ) equation ([19], [1], [13]), the dynamical Φ_3^4 model ([14], [2]) and so on. By these theories Markov semigroups generated by the solutions can be constructed.

Recently, there are some papers studying the long time behavior of the Markov semigroup associated to singular stochastic PDEs. In [22] Michael Röckner and the authors of this paper show that the Markov semigroup for the dynamical Φ_2^4 model converges to the Φ_2^4 measure by using asymptotic coupling method developed in [18]. By this and [21] we can give some characterization of the Φ_2^4 field. In [24] the authors obtain the strong Feller property for the dynamical $P(\Phi)_2$ model and also the exponential ergodicity of the dynamical Φ_2^4 model. In [17], strong Feller property for the Markov semigroups generated by a large class of singular stochastic PDEs has been obtained, which is a very useful ingredient to establish the ergodicity of a given Markov process.

It is very natural to ask whether the solutions to the stochastic Navier-Stokes equations driven by space-time white noise in dimensions 2 and 3 as constructed in [5], [25] also satisfy the strong Feller property. Because of the presence of the Leray projection, the results in [17] cannot be applied in this case directly. In this paper we obtain strong Feller property of the solutions to the stochastic Navier-Stokes equations driven by space-time white noise in dimensions 2 and 3 by using the abstract result developed in [17, Section 2].

This paper is organized as follows. In Section 2, we recall the regularity structure theory and the framework to obtain local existence and uniqueness of solutions to the 3D N-S equations driven by space-time white noise. In Section 3, we apply the abstract result developed in [17, Section 2] to deduce the strong Feller property of the semigroup associated to the 3D N-S equation driven by space-time white noise. 2D case has been considered in Section 4. By using the strong Feller property we obtain a unique global solution starting from every initial point in $\mathcal{C}^\eta, \eta \in (-\frac{1}{2}, 0)$ in Section 4. In Appendix we recall the abstract assumptions in [17].

2 N-S equation by regularity structure theory

2.1 Preliminary on regularity structure theory

In this subsection we recall some preliminaries for the theory of regularity structures from [14] and [16]. From this section we fixed a scaling $\mathfrak{s} = (\mathfrak{s}_0, 1, \dots, 1)$ of \mathbb{R}^{d+1} . We call $|\mathfrak{s}| = \mathfrak{s}_0 + d$ scaling dimension. We define the associate metric on \mathbb{R}^{d+1} by

$$\|z - z'\|_{\mathfrak{s}} := \sum_{i=0}^d |z_i - z'_i|^{1/\mathfrak{s}_i}.$$

For $k = (k_0, \dots, k_d)$ we define $|k|_{\mathfrak{s}} = \sum_{i=0}^d \mathfrak{s}_i k_i$.

Definition 2.1 A regularity structure $\mathfrak{T} = (A, T, G)$ consists of the following elements:

- (i) An index set $A \subset \mathbb{R}$ such that $0 \in A$, A is bounded from below and locally finite.
- (ii) A model space T , which is a graded vector space $T = \bigoplus_{\alpha \in A} T_{\alpha}$, with each T_{α} a Banach space. Furthermore, T_0 is one-dimensional and has a basis vector $\mathbf{1}$. Given $\tau \in T$ we write $\|\tau\|_{\alpha}$ for the norm of its component in T_{α} .
- (iii) A structure group G of (continuous) linear operators acting on T such that for every $\Gamma \in G$, every $\alpha \in A$ and every $\tau_{\alpha} \in T_{\alpha}$ one has

$$\Gamma \tau_{\alpha} - \tau_{\alpha} \in T_{<\alpha} := \bigoplus_{\beta < \alpha} T_{\beta}.$$

Furthermore, $\Gamma \mathbf{1} = \mathbf{1}$ for every $\Gamma \in G$.

Now we have the usual polynomial regularity structure $\bar{T} = \bigoplus_{n \in \mathbb{N}} \bar{T}_n$ given by all polynomials in $d + 1$ indeterminates, let us call them X_0, \dots, X_d , which denote the time and space directions respectively. Denote $X^k = X_0^{k_0} \cdots X_d^{k_d}$ with k a multi-index. In this case, $A = \mathbb{N}$ and \bar{T}_n denote the space of monomials that are homogeneous of degree n . The structure group can be defined by $\Gamma_h X^k = (X - h)^k$, $h \in \mathbb{R}^{d+1}$.

Given a smooth compactly supported test function φ , $x, y \in \mathbb{R}^d$, $\lambda > 0$, we define

$$\varphi_x^{\lambda}(y) = \lambda^{-d} \varphi\left(\frac{y - x}{\lambda}\right).$$

Denote by \mathcal{B}_r the set of smooth test functions $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ that are supported in the centred ball of radius 1 and such that their derivatives of order up to r are uniformly bounded by 1. We denote by \mathcal{S}' the space of all distributions on \mathbb{R}^d .

Now we give the definition of a model, which is a concrete way of associating every element in the abstract regularity structure to the actual Taylor polynomial at every point. For the Navier-Stokes equation we need to consider heat kernel composed with the Leray projection, which is not smooth on $\mathbb{R}^{d+1} \setminus \{0\}$. So we cannot apply [14, Lemma 5.5] directly. Instead we use the inhomogeneous modelled distribution introduced in [16].

Definition 2.2 Given a regularity structure \mathfrak{T} , an inhomogeneous model (Π, Γ, Σ) consists of the following three elements:

- A collection of maps $\Gamma^t : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{G}$ parametrized by $t \in \mathbb{R}$, such that

$$\Gamma_{xx}^t = 1, \quad \Gamma_{xy}^t \Gamma_{yz}^t = \Gamma_{xz}^t,$$

for any $x, y, z \in \mathbb{R}^d$ and $t \in \mathbb{R}$, and the action of Γ_{xy}^t on polynomials is given as above with $h = (0, y - x)$.

- A collection of maps $\Sigma_x : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{G}$, parametrized by $x \in \mathbb{R}^d$, such that, for any $x \in \mathbb{R}^d$ and $s, r, t \in \mathbb{R}$, one has

$$\Sigma_x^{tt} = 1, \quad \Sigma_x^{sr} \Sigma_x^{rt} = \Sigma_x^{st}, \quad \Sigma_x^{st} \Gamma_{xy}^t = \Gamma_{xy}^s \Sigma_y^{st},$$

and the action of Σ_x^{st} on polynomials is given as above with $h = (t - s, 0)$.

- A collection of linear maps $\Pi_x^t : T \rightarrow \mathcal{S}'$, such that

$$\Pi_y^t = \Pi_x^t \Gamma_{xy}^t, \quad (\Pi_x^t X^{(0, \bar{k})})(y) = (y - x)^{\bar{k}}, \quad (\Pi_x^t X^{(k_0, \bar{k})})(y) = 0,$$

for all $x, y \in \mathbb{R}^d, t \in \mathbb{R}, \bar{k} \in \mathbb{N}^d, k_0 \in \mathbb{N}$ such that $k_0 > 0$.

Moreover, for any $\gamma > 0$ and every $T > 0$, there is a constant C for which the analytic bounds

$$|\langle \Pi_x^t \tau, \varphi_x^\lambda \rangle| \leq C \|\tau\|_l \lambda^l, \quad \|\Gamma_{xy}^t \tau\|_m \leq C \|\tau\|_l |x - y|^{l-m},$$

$$\|\Sigma_x^{st} \tau\|_m \leq C \|\tau\|_l |t - s|^{(l-m)/s_0},$$

holds uniformly over all $\tau \in T_l$, $l \in A$ with $l < \gamma$, all $m \in A$ such that $m < l$, and all test functions $\varphi \in \mathcal{B}_r$ with $r > -\inf A$, and all $t, s \in [-T, T]$ and $x, y \in \mathbb{R}^d$ such that $|t - s| \leq 1$ and $|x - y| \leq 1$.

For a model $Z = (\Pi, \Gamma, \Sigma)$ we denote by $\|\Pi\|_{\gamma; T}$, $\|\Gamma\|_{\gamma; T}$ and $\|\Sigma\|_{\gamma; T}$ the smallest constants C such that the bounds on Π, Γ and Σ in the above analytic bounds hold. Furthermore, we define

$$\|Z\|_{\gamma; T} := \|\Pi\|_{\gamma; T} + \|\Gamma\|_{\gamma; T} + \|\Sigma\|_{\gamma; T}.$$

If $\bar{Z} = (\bar{\Pi}, \bar{\Gamma}, \bar{\Sigma})$ is another model we define

$$\|Z; \bar{Z}\|_{\gamma; T} := \|\Pi - \bar{\Pi}\|_{\gamma; T} + \|\Gamma - \bar{\Gamma}\|_{\gamma; T} + \|\Sigma - \bar{\Sigma}\|_{\gamma; T},$$

This gives a natural topology for the space of all models for a given regularity structure. In the following we consider the models are periodic in space, which allows us to require the bounds to hold globally.

Now we have the following definition for the spaces of distributions \mathcal{C}^α , $\alpha < 0$, which is an extension of the definition of Hölder space to include $\alpha < 0$.

Definition 2.3 For $\alpha < 0$, \mathcal{C}^α consists of the closure of smooth compact functions under the norm

$$\|\eta\|_\alpha := \sup_{x \in \mathbb{R}^d} \sup_{\varphi \in \mathcal{B}_r} \sup_{\lambda \leq 1} \lambda^{-\alpha} |\eta(\varphi_x^\lambda)| < \infty.$$

We also have the following definition of spaces of inhomogeneous modelled distributions, which are the Hölder spaces on the regularity structure.

Definition 2.4 Given a model $Z = (\Pi, \Gamma, \Sigma)$ for a regularity structure \mathfrak{T} as above. Then for any $\gamma > 0$ and $\eta \in \mathbb{R}$, the space $\mathcal{D}^{\gamma, \eta}$ consists of all functions $H : (0, T] \times \mathbb{R}^d \rightarrow \bigoplus_{\alpha < \gamma} T_\alpha$ such that

$$|||H|||_{\gamma, \eta; T} := \|H\|_{\gamma, \eta; T} + \sup_{\substack{s \neq t \in (0, T] \\ |t-s| \leq |t|_0^{\frac{s_0}{\eta_0}}}} \sup_{x \in \mathbb{R}^d} \sup_{l < \gamma} \frac{\|H_t(x) - \Sigma_x^{ts} H_s(x)\|_l}{|t-s|^{(\gamma-l)/s_0} |t, s|_0^{\eta-\gamma}} < \infty,$$

with

$$\|H\|_{\gamma, \eta; T} := \sup_{t \in (0, T]} \sup_{x \in \mathbb{R}^d} \sup_{l < \gamma} |t|_0^{(l-\eta) \vee 0} \|H_t(x)\|_l + \sup_{t \in (0, T]} \sup_{\substack{x \neq y \in \mathbb{R}^d \\ |x-y| \leq 1}} \sup_{l < \gamma} \frac{\|H_t(x) - \Gamma_{xy}^t H_t(y)\|_l}{|x-y|^{\gamma-l} |t|_0^{\eta-\gamma}},$$

Here we wrote $\|\tau\|_l$ for the norm of the component of τ in T_l and $|t|_0 := |t|_0^{\frac{1}{s_0}} \wedge 1$ and $|t, s|_0 := |t|_0 \wedge |s|_0$.

For $H \in \mathcal{D}^{\gamma, \eta}$ and $\bar{H} \in \bar{\mathcal{D}}^{\gamma, \eta}$ (denoting by $\bar{\mathcal{D}}^{\gamma, \eta}$ the space built over another model $(\bar{\Pi}, \bar{\Gamma}, \bar{\Sigma})$), we also set

$$\begin{aligned} \|H; \bar{H}\|_{\gamma, \eta; T} &:= \sup_{t \in (0, T]} \sup_{x \in \mathbb{R}^d} \sup_{l < \gamma} |t|_0^{(l-\eta) \vee 0} \|H_t(x) - \bar{H}_t(x)\|_l \\ &+ \sup_{t \in (0, T]} \sup_{\substack{x \neq y \in \mathbb{R}^d \\ |x-y| \leq 1}} \sup_{l < \gamma} \frac{\|H_t(x) - \Gamma_{xy}^t H_t(y) - \bar{H}_t(x) + \bar{\Gamma}_{xy}^t \bar{H}_t(y)\|_l}{|x-y|^{\gamma-l} |t|_0^{\eta-\gamma}}, \end{aligned}$$

$$|||H; \bar{H}|||_{\gamma, \eta; T} := \|H; \bar{H}\|_{\gamma, \eta; T} + \sup_{\substack{s \neq t \in (0, T] \\ |t-s| \leq |t, s|_0}} \sup_{x \in \mathbb{R}^d} \sup_{l < \gamma} \frac{\|H_t(x) - \Sigma_x^{ts} H_s(x) - \bar{H}_t(x) + \bar{\Sigma}_x^{ts} \bar{H}_s(x)\|_l}{|t-s|^{(\gamma-l)/s_0} |t, s|_0^{\eta-\gamma}},$$

which gives a natural distance between elements $H \in \mathcal{D}^{\gamma, \eta}$ and $\bar{H} \in \bar{\mathcal{D}}^{\gamma, \eta}$.

Given a regularity structure, we say that a subspace $V \subset T$ is a sector of regularity α if it is invariant under the action of the structure group G and it can be written as $V = \bigoplus_{\beta \in A} V_\beta$ with $V_\beta \subset T_\beta$, and $V_\beta = \{0\}$ for $\beta < \alpha$. We will use $\mathcal{D}^{\gamma, \eta}(V)$ to denote all functions in $\mathcal{D}^{\gamma, \eta}$ taking values in V .

The reconstruction theorem, which defines the so-called reconstruction operator, is one of the most fundamental result in the regularity structures theory.

Theorem 2.5 (cf. [16, Theorem 2.11]) Given a model $Z = (\Pi, \Gamma, \Sigma)$ for a regularity structure \mathfrak{T} with $\alpha := \min A$. Then for every $\eta \in \mathbb{R}, \gamma > 0$ and $T > 0$, there is a unique family of linear operators $\mathcal{R}_t : \mathcal{D}^{\gamma, \eta} \rightarrow \mathcal{C}^\alpha(\mathbb{R}^d)$, parametrised by $t \in (0, T]$, such that the bound

$$|\langle \mathcal{R}_t H_t - \Pi_x^t H_t(x), \varphi_x^\lambda \rangle| \lesssim \lambda^\gamma |t|_0^{\eta-\gamma} \|H\|_{\gamma, \eta; T} \|\Pi\|_{\gamma; T},$$

holds uniformly in $H \in \mathcal{D}^{\gamma, \eta}, t \in (0, T], x \in \mathbb{R}^d, \lambda \in (0, 1]$ and $\varphi \in \mathcal{B}_r$ with $r > -\alpha + 1$.

Suppose that K is a 2-regularising kernel in the sense of [14, Section 5] and we will write $K_t(x) = K(z)$, for $z = (t, x)$. We say that a model $Z = (\Pi, \Gamma, \Sigma)$ realises K for an abstract integration map \mathcal{I} if, for every $\alpha \in A$, every $\tau \in T_\alpha$ and every $x \in \mathbb{R}^d$, one has

$$\Pi_x^t(\mathcal{I}\tau + \mathcal{J}_{t,x}\tau)(y) = \int_{\mathbb{R}} \langle \Pi_x^s \Sigma_x^{st} \tau, K_{t-s}(y - \cdot) \rangle ds. \quad (2.1)$$

Here

$$\mathcal{J}_{t,x}\tau = \sum_{|k|_s < \alpha + \beta} \frac{X^k}{k!} \int_{\mathbb{R}} \langle \Pi_x^s \Sigma_x^{st} \tau, D^k K_{t-s}(x - \cdot) \rangle ds,$$

where $k \in \mathbb{N}^{d+1}$ and the derivative D^k is in time-space. Moreover, we require that

$$\Gamma_{xy}^t(\mathcal{I} + \mathcal{J}_{t,y}) = (\mathcal{I} + \mathcal{J}_{t,x})\Gamma_{xy}^t, \quad \Sigma_x^{st}(\mathcal{I} + \mathcal{J}_{t,x}) = (\mathcal{I} + \mathcal{J}_{s,x})\Sigma_x^{st}, \quad (2.2)$$

for all $s, t \in \mathbb{R}$, and $x, y \in \mathbb{R}^d$.

Using \mathcal{I} we can also introduce the operator \mathcal{K}_γ acting on modelled distribution similarly as in [16] and [25].

In order to deal with the Leray Projection, we have to consider convolution with the singular kernel for space variable. As in [14] we introduce an abstract integration map $\mathcal{I}_0 : T \rightarrow T$ to provide an "abstract" representation of the Leray Projection operating at the level of the regularity structure. In the regularity structure theory \mathcal{I}_0 is a linear map from T to T such that $\mathcal{I}_0 T_\alpha \subset T_\alpha$ and $\mathcal{I}_0 \bar{T} = 0$ and for every $\Gamma \in G, \tau \in T$ one has $\Gamma \mathcal{I}_0 \tau - \mathcal{I}_0 \Gamma \tau \in \bar{T}$.

We say that P is a 0-regularising kernel on \mathbb{R}^d if one can write $P = \sum_{n \geq 0} P_n$, where each $P_n : \mathbb{R}^d \rightarrow \mathbb{R}$ is smooth and compactly supported in a ball of radius 2^{-n} around the origin. Furthermore, we assume that for every multi-index k , one has a constant C such that

$$\sup_x |D^k P_n(x)| \leq C 2^{n(d+|k|)},$$

holds uniformly in n . Finally, we assume that $\int P_n(x) E(x) dz = 0$ for every polynomial E of degree at most r for some sufficiently large value of r . A model $Z = (\Pi, \Gamma, \Sigma)$ realises a 0-regularising kernel P on \mathbb{R}^d for an abstract integration map \mathcal{I}_0 if, for every $\alpha \in A$, every $\tau \in T_\alpha$ and every $x \in \mathbb{R}^d$, one has

$$\Pi_x^t(\mathcal{I}_0 \tau + \mathcal{J}_{t,x}^0 \tau)(y) = \langle \Pi_x^t \tau, P(y - \cdot) \rangle, \quad (2.3)$$

where

$$\mathcal{J}_{t,x}^0 \tau = \sum_{|k| < \alpha} \frac{X^k}{k!} \langle \Pi_x^t \tau, D^k P(x - \cdot) \rangle,$$

with $k \in \mathbb{N}^d$ and the derivative D^k being the space derivative. Moreover, we require that

$$\Gamma_{xy}^t(\mathcal{I}_0 + \mathcal{J}_{t,y}^0) = (\mathcal{I}_0 + \mathcal{J}_{t,x}^0)\Gamma_{xy}^t, \quad \Sigma_x^{st}(\mathcal{I}_0 + \mathcal{J}_{t,x}^0) = (\mathcal{I}_0 + \mathcal{J}_{s,x}^0)\Sigma_x^{st}, \quad (2.4)$$

for all $s, t \in \mathbb{R}$, and $x, y \in \mathbb{R}^d$.

Now we introduce the following operator acting on modelled distribution $H \in \mathcal{D}^{\gamma, \eta}$ with $\gamma + \beta > 0$:

$$(\mathcal{P}_{\gamma,t} H_t)(x) := \mathcal{I}_0 H_t(x) + \mathcal{J}_{t,x}^0 H_t(x) + (\mathcal{N}_{\gamma,t}^0 H_t)(x).$$

Here

$$(\mathcal{N}_{\gamma,t}^0 H_t)(x) := \sum_{|k| < \gamma} \frac{X^k}{k!} \langle \mathcal{R}_t H_t - \Pi_x^t H_t(x), D^k P(x - \cdot) \rangle,$$

where $k \in \mathbb{N}^d$ and the derivative D^k is in space. By [25, Theorem 2.7] we have

$$\mathcal{R}_t(\mathcal{P}_{\gamma,t} H_t)(x) = \langle \mathcal{R}_t H_t, P(x - \cdot) \rangle.$$

In the following we extend [14, Thm 5.14] for the inhomogeneous model.

Proposition 2.6 Let $\mathfrak{T} = (A, T, G)$ be a regularity structure containing the canonical regularity structure. Let $V \subset T$ be a sector of order $\bar{\gamma}$. Let $W \subset V$ be a subsector of V and let K be a 2-regularising kernel and P be a 0-regularising kernel on \mathbb{R}^d . Let (Π, Γ, Σ) be a model for \mathfrak{T} , and let $\mathcal{I} : W \rightarrow T$ be an abstract integration map of order 2 such that Π realises K for \mathcal{I} and let $\mathcal{I}_0 : W \rightarrow T$ be an abstract integration map of order 0 such that Π realises P for \mathcal{I}_0 .

Then, there exists a regularity structure $\hat{\mathfrak{T}}$ containing \mathfrak{T} , a model $(\hat{\Pi}, \hat{\Gamma}, \hat{\Sigma})$ for $\hat{\mathfrak{T}}$ extending (Π, Γ, Σ) , and abstract integration maps $\hat{\mathcal{I}}$ of order 2, $\hat{\mathcal{I}}_0$ of order 0 acting on $\hat{V} = \iota V$ such that:

- The model $\hat{\Pi}$ realises K for $\hat{\mathcal{I}}$ and realises P for $\hat{\mathcal{I}}_0$.
- The map $\hat{\mathcal{I}}$ and $\hat{\mathcal{I}}_0$ extend \mathcal{I} and \mathcal{I}_0 in the sense that $\hat{\mathcal{I}}\iota a = \iota \mathcal{I}a$, $\hat{\mathcal{I}}_0 \iota a = \iota \mathcal{I}_0 a$ for every $a \in W$.

Furthermore, the map $(\Pi, \Gamma, \Sigma) \mapsto (\hat{\Pi}, \hat{\Gamma}, \hat{\Sigma})$ is locally bounded and Lipschitz continuous.

Proof As in [14, Theorem 5.14] we can assume without loss of generality that the sector V is given by a finite sum $V = V_{\alpha_1} \oplus V_{\alpha_2} \oplus \dots \oplus V_{\alpha_n}$, where the α_i are an increasing sequence of elements in A , and $W_{\alpha_k} = V_{\alpha_k}$ for $k < n$. We then denote by \bar{W} the complement of W_{α_n} in V_{α_n} so that $V_{\alpha_n} = W_{\alpha_n} \oplus \bar{W}_{\alpha_n}$. By similar arguments as in the proof of [14, Theorem 5.14] we can extend the regularity structure $\hat{T} = T \oplus \bar{W}$ with $\beta = 2$ and define $\hat{\mathcal{I}}$ similarly. For $a \in T, b \in \bar{W}$ we define $\hat{\Pi}_x^t$ to be given by

$$\hat{\Pi}_x^t(a, b) = \Pi_x^t a + \int_{\mathbb{R}} \langle \Pi_x^s \Sigma_x^{st} b(dy), K_{t-s}(\cdot - y) \rangle ds - \Pi_x^t \mathcal{J}_{t,x} b,$$

where $\mathcal{J}_{t,x}$ is given by (2.1). By [14, Lemma 5.19] $\hat{\Pi}_x^t$ satisfies the required bounds when tested against smooth test functions that are localized near x . We set

$$\hat{\Gamma}_{xy}^t = (\Gamma_{xy}^t, M_{xy}^t), \quad M_{xy}^t b = \mathcal{J}_{t,x} \Gamma_{xy}^t b - \Gamma_{xy}^t \mathcal{J}_{t,y} b.$$

$$\hat{\Sigma}_x^{ts} = (\Sigma_x^{ts}, M_x^{ts}), \quad M_x^{ts} b = \mathcal{J}_{t,x} \Sigma_x^{ts} b - \Sigma_x^{ts} \mathcal{J}_{s,x} b.$$

The corresponding algebraic identity and the corresponding analytic bounds can be obtained by similar arguments as in the proof of [14, Theorem 5.14]. Furthermore, similarly we can extend the regularity structure with $\beta = 0$ and define $\hat{\mathcal{I}}_0$ similarly. For $a \in \hat{T}, b \in \bar{W}$ we define $\hat{\Pi}_x^t$ to be given by

$$\hat{\Pi}_x^t(a, b) = \hat{\Pi}_x^t a + \langle \Pi_x^t b(dy), P(\cdot - y) \rangle - \Pi_x^t \mathcal{J}_{t,x}^0 b,$$

where $J_{t,x}^0$ is given by (2.3). By [14, Lemma 5.19] $\hat{\Pi}_x^t$ satisfies the required bounds when tested against smooth test functions that are localized near x . We set

$$\hat{\Gamma}_{xy}^t = (\hat{\Gamma}_{xy}^t, \hat{M}_{xy}^t), \quad \hat{M}_{xy}^t b = \mathcal{J}_{t,x}^0 \Gamma_{xy}^t b - \Gamma_{xy}^t \mathcal{J}_{t,y}^0 b.$$

$$\hat{\Sigma}_x^{ts} = (\hat{\Sigma}_x^{ts}, \hat{M}_x^{ts}), \quad \hat{M}_x^{ts} b = \mathcal{J}_{t,x}^0 \Sigma_x^{ts} b - \Sigma_x^{ts} \mathcal{J}_{s,x}^0 b.$$

The corresponding algebraic identity and the analytic bounds for $\hat{\Gamma}_{xy}^t$, $\hat{M}_{xy}^t b$ and $\hat{\Sigma}_x^{ts}$ can be checked by similar arguments as in the proof of [14, Theorem 5.14]. To obtain the analytic bounds for \hat{M}_x^{st} , we introduce the following model for T :

$$(\tilde{\Pi}_{(t,x)} \tau)(s, y) = (\Pi_x^s \Sigma_x^{st} \tau)(y), \quad \tilde{\Gamma}_{(t,x),(s,y)} = \Gamma_{xy}^t \Sigma_y^{ts} = \Sigma_x^{ts} \Gamma_{xy}^s,$$

which is a model in the original sense of [14, Def 2.17]. For $\tau \in T_l$, $k \in \mathbb{N}^3$, consider

$$((\mathcal{J}_{t,x}^0 \Sigma_x^{ts} - \Sigma_x^{ts} \mathcal{J}_{s,x}^0) \tau)_k,$$

with \mathcal{J}^0 defined in (2.3). We decompose \mathcal{J}^0 as $\mathcal{J}^0 = \sum_{n \geq 0} \mathcal{J}_{t,x}^{0,(n)}$, where the n th term in each sum is obtained by replacing P by P_n in the expressions for \mathcal{J}^0 . Moreover, for $\tau \in T_l$

$$(\mathcal{J}_{t,x}^{0,(n)} \Sigma_x^{ts} \tau)_k = \frac{1}{k!} \sum_{|k| < \zeta < l} \langle \Pi_x^t \mathcal{Q}_\zeta \Sigma_x^{ts} \tau, D^k P_n(x - \cdot) \rangle,$$

$$(\Sigma_x^{ts} \mathcal{J}_{s,x}^{0,(n)} \tau)_k = \frac{1}{k!} \langle \Pi_x^s \tau, D^k P_n(x - \cdot) \rangle,$$

where $\mathcal{Q}_\zeta a$ denotes the component of a in T_ζ . We first consider the case $2^{-n} \leq |t - s|^{\frac{1}{2}}$: by Definition 2.2 we have

$$|(\mathcal{J}_{t,x}^{0,(n)} \Sigma_x^{ts} \tau)_k| \lesssim \sum_{|k| < \zeta < l} 2^{n|k|} 2^{-n\zeta} |s - t|^{\frac{l-\zeta}{2}} \lesssim \sum_{\delta < 0} |t - s|^{\frac{l-|k|+\delta}{2}} 2^{\delta n},$$

and

$$|(\Sigma_x^{ts} \mathcal{J}_{s,x}^{0,(n)} \tau)_k| \lesssim 2^{n|k|} 2^{-nl} \lesssim \sum_{\delta < 0} |t - s|^{\frac{l-|k|+\delta}{2}} 2^{\delta n}.$$

For the case that $|t - s|^{\frac{1}{2}} \leq 2^{-n}$ we have

$$\begin{aligned} & (\mathcal{J}_{t,x}^{0,(n)} \Sigma_x^{ts} \tau)_k - (\Sigma_x^{ts} \mathcal{J}_{s,x}^{0,(n)} \tau)_k \\ &= -\frac{1}{k!} \sum_{\zeta \leq |k|} \langle \Pi_x^t \mathcal{Q}_\zeta \Sigma_x^{ts} \tau, D^k P_n(x - \cdot) \rangle + \frac{1}{k!} \langle \Pi_x^t \Sigma_x^{ts} \tau - \Pi_x^s \tau, D^k P_n(x - \cdot) \rangle \\ &=: T_1^k + T_2^k, \end{aligned}$$

For T_1^k we have

$$|T_1^k| \lesssim \sum_{\zeta < |k|} 2^{n|k|} 2^{-n\zeta} |s - t|^{\frac{l-\zeta}{2}} \lesssim \sum_{\delta > 0} |t - s|^{\frac{l-|k|+\delta}{2}} 2^{\delta n},$$

where the sum runs over a finite number of exponents. In the following we consider T_2^k :

$$\begin{aligned} |T_2^k| &= \frac{1}{k!} |\langle \tilde{\Pi}_{(s,x)} \tau(t, \cdot) - \tilde{\Pi}_{(s,x)} \tau(s, \cdot), D^k P_n(x - \cdot) \rangle| \\ &= \lim_{m \rightarrow \infty} \frac{1}{k!} \left| \sum_{(s_0, y) \in \Lambda_m^s} \langle \tilde{\Pi}_{(s,x)} \tau, \varphi_{(s_0, y)}^{m, s} \rangle \langle \varphi_{(s_0, y)}^{m, s}(t) - \varphi_{(s_0, y)}^{m, s}(s), D^k P_n(x - \cdot) \rangle \right|, \end{aligned}$$

where $\{\varphi_{(s_0, y)}^{m, s}\} \subset \mathcal{C}^r$ is the wavelet basis introduced in [14, Section 3], $\Lambda_m^s = \{\sum_{j=0}^d 2^{-ms_j} k_j e_j : k_j \in \mathbb{Z}\}$, with e_j denoting the j th element of the canonical basis of \mathbb{R}^{d+1} . By the definition of the model we have

$$\begin{aligned} |\langle \tilde{\Pi}_{(s,x)} \tau, \varphi_{(s_0, y)}^{m, s} \rangle| &= |\langle \tilde{\Pi}_{(s_0, y)} \tilde{\Gamma}_{(s_0, y), (s, x)} \tau, \varphi_{(s_0, y)}^{m, s} \rangle| \\ &\lesssim \sum_{l_0 < l} \|(s, x) - (s_0, y)\|_s^{l-l_0} 2^{-\frac{m|s|}{2} - l_0 m}. \end{aligned}$$

For $\langle \varphi_{(s_0, y)}^{m, s}(t) - \varphi_{(s_0, y)}^{m, s}(s), D^k P_n(x - \cdot) \rangle$ we choose m large enough such that $2^{-m} \leq |t-s|^{\frac{1}{s_0}} \leq 2^{-n}$. In this case by a similar calculation as in the proof of [14, Theorem 3.10] we know that

$$|\langle \varphi_{(s_0, y)}^{m, s}(t) - \varphi_{(s_0, y)}^{m, s}(s), D^k P_n(x - \cdot) \rangle| \lesssim 2^{n|k|} 2^{-\frac{3m}{2} - rm} 2^{n(3+r)} 2^{\frac{s_0}{2} m}.$$

Furthermore, $|\langle \varphi_{(s_0, y)}^{m, s}(t) - \varphi_{(s_0, y)}^{m, s}(s), D^k P_n(x - \cdot) \rangle| = 0$ unless $|x - y| \lesssim 2^{-n}$ and $|s - s_0|^{\frac{1}{s_0}} \wedge |t - s_0|^{\frac{1}{s_0}} \lesssim 2^{-m}$. Thus we have

$$\begin{aligned} |T_2^k| &\lesssim \lim_{m \rightarrow \infty} \sum_{l_0 < l} 2^{3m} 2^{-3n} 2^{-\frac{m|s|}{2} - l_0 m} 2^{-n(l-l_0)} 2^{n|k|} 2^{-\frac{3m}{2} - rm} 2^{n(3+r)} 2^m \\ &\lesssim \sum_{l_0 < l} \lim_{m \rightarrow \infty} 2^{-lm} 2^{n|k|} 2^{(l-l_0-r)(m-n)} \\ &\lesssim \sum_{\delta > 0} |t - s|^{\frac{l-|k|+\delta}{2}} 2^{\delta n}, \end{aligned}$$

where the sum runs over a finite number of exponents and in the first inequality we used the factor $2^{3m} 2^{-3n}$ counts the number of non-zero terms appearing in the sum over (s_0, y) and in the last inequality we choose m large enough such that $2^{-m} < |t-s|^{\frac{1}{2}}$ and r large enough such that $r > l - l_0$. Taking the sum over n we obtain the desired bounds for $\|\hat{M}_x^{ts} \tau\|_k$ and the result follows. \square

2.2 N-S equation on \mathbb{T}^3

In this subsection we recall the regularity structure theory for the 3D Navier-Stokes equations on \mathbb{T}^3 driven by space-time white noise in [25]. In this case we have the scaling $\mathfrak{s} = (2, 1, 1, 1)$, so that the scaling dimension of space-time is 5. Since the heat kernel G is smooth on $\mathbb{R}^4 \setminus \{0\}$ and has the scaling property $G(\frac{t}{\delta^2}, \frac{x}{\delta}) = \delta^3 G(t, x)$ for $\delta > 0$, by [14, Lemma 5.5] it can be decomposed into $K + R$ where K is a 2-regularising kernel and $R \in \mathcal{C}^\infty$

We know that the kernel P^{ij} , $i, j = 1, 2, 3$, for the Leray projection is smooth on $\mathbb{R}^3 \setminus \{0\}$ and has the scaling property $P^{ij}(\frac{x}{\delta}) = \delta^3 P^{ij}(x)$ for $\delta > 0$, by [14, Lemma 5.5] it can be decomposed into $\bar{P}^{ij} + R_0^{ij}$, $i, j = 1, 2, 3$, where \bar{P}^{ij} is a 0-regularising kernel on \mathbb{R}^3 and $R_0^{ij} \in \mathcal{C}^\infty$. Define

$$K^{ij} := K * \bar{P}^{ij}.$$

Consider the regularity structure generated by the stochastic N-S equation with $\beta = 2$. In the regularity structure we use symbol the Ξ_i to replace the driving noise ξ^i . We introduce the integration map \mathcal{I} associated with K and the integration map \mathcal{I}_0^{ij} associated with \bar{P}^{ij} , which helps us to define \mathcal{K}_γ and $\bar{\mathcal{P}}_\gamma^{ij}$. We also need the integration maps $\mathcal{I}_{0,k}^{i_1}$, $i, i_1 = 1, 2, 3$, \mathcal{I}_k for a multiindex k , which represents integration against $D^k P^{i_1}$, $i, i_1 = 1, 2, 3$, $D^k K$ respectively.

To apply the regularity structure theory we write the equation as follows: for $i = 1, 2, 3$

$$\begin{aligned} \partial_t v_1^i &= \nu \Delta v_1^i + \sum_{i_1=1}^3 P^{i i_1} \xi^{i_1}, \quad \operatorname{div} v_1 = 0, \\ \partial_t v^i &= \nu \Delta v^i - \sum_{i_1, j=1}^3 P^{i i_1} \frac{1}{2} D_j [(v^{i_1} + v_1^{i_1})(v^j + v_1^j)], \quad \operatorname{div} v = 0. \end{aligned} \tag{2.5}$$

Then $v_1 + v$ is the solution to the 3D Navier-Stokes equations driven by space-time white noise. Now we consider the second equation in (2.5). Define for $i, j, i_1 = 1, 2, 3$,

$$\mathcal{I}^{ij} := \mathcal{I}_0^{ij} \mathcal{I}, \quad \mathcal{I}_{i_1}^{ij} := \mathcal{I}_0^{ij} \mathcal{I}_{i_1}$$

$$\mathfrak{M}_F^{ij} = \{1, \mathcal{I}^{i i_1}(\Xi_{i_1}), \mathcal{I}^{j j_1}(\Xi_{j_1}), \mathcal{I}^{i i_1}(\Xi_{i_1}) \mathcal{I}^{j j_1}(\Xi_{j_1}), U_i, U_j, U_i U_j, \mathcal{I}^{i i_1}(\Xi_{i_1}) U_j, U_i \mathcal{I}^{j j_1}(\Xi_{j_1}), i_1, j_1 = 1, 2, 3\},$$

where the product is commutative and associative. Then we build subsets $\{\mathcal{P}_n^i\}_{n \geq 0}$, $\{\mathcal{U}_n\}_{n \geq 0}$ and $\{\mathcal{W}_n\}_{n \geq 0}$ by the following algorithm: For $i, j = 1, 2, 3$, set $\mathcal{W}_0^{ij} = \mathcal{P}_0^i = \mathcal{U}_0 = \emptyset$ and

$$\mathcal{W}_n^{ij} = \mathcal{W}_{n-1}^{ij} \cup \bigcup_{\mathcal{Q} \in \mathfrak{M}_F^{ij}} \mathcal{Q}(\mathcal{P}_{n-1}^i, \mathcal{P}_{n-1}^j),$$

$$\mathcal{P}_n^i = \{X^k\} \cup \{\mathcal{I}_{i_2}^{i i_1}(\tau) : \tau \in \mathcal{W}_{n-1}^{i_1 i_2}, i_1, i_2 = 1, 2, 3\},$$

$$\mathcal{U}_n = \{\mathcal{I}_{i_2}(\tau) : \tau \in \mathcal{W}_{n-1}^{i_1 i_2}, i_1, i_2 = 1, 2, 3\},$$

and

$$\mathcal{F}_F := \bigcup_{n \geq 0} \left(\bigcup_{i, j=1}^3 \mathcal{W}_n^{ij} \cup \mathcal{U}_n \right).$$

Then \mathcal{F}_F contains the elements required to describe both the solution and the terms in the equation (2.5). We denote by \mathcal{H}_F the set of finite linear combinations of elements in \mathcal{F}_F . For each $\tau \in \mathcal{F}$ a degree $|\tau|_s$ is obtained by setting $|\mathbf{1}|_s = 0$,

$$|\tau \bar{\tau}|_s = |\tau|_s + |\bar{\tau}|_s,$$

for any two formal expressions τ and $\bar{\tau}$ in \mathcal{F} such that

$$|\Xi_i|_s = \alpha, \quad |X_i|_s = \mathfrak{s}_i, \quad |\mathcal{I}_k(\tau)|_s = |\tau|_s + 2 - |k|_s, \quad |\mathcal{I}_{0,k}^{i_1}(\tau)|_s = |\tau|_s - |k|_s,$$

for $\alpha \in (-\frac{13}{5}, -\frac{5}{2})$. Let $T = \mathcal{H}_F$ with $T_\gamma = \langle \{\tau \in \mathcal{F}_F : |\tau|_s = \gamma\} \rangle$, $A = \{|\tau|_s : \tau \in \mathcal{F}_F\}$ and let the structure group G_F be as in [25]. Then by [25, Theorem 2.8] we know that $\mathfrak{T}_F = (A, \mathcal{H}_F, G_F)$ defines a regularity structure \mathfrak{T} . We also recall the following definition from [25, Definition 2.10].

Definition 2.7 A model (Π, Γ, Σ) for \mathfrak{T} is admissible if it satisfies

$$(\Sigma_x^{st} \tau \bar{\tau}) = (\Sigma_x^{st} \tau)(\Sigma_x^{st} \bar{\tau}), \quad (\Gamma_{xy}^t \tau \bar{\tau}) = (\Gamma_{xy}^t \tau)(\Gamma_{xy}^t \bar{\tau}), \quad (2.6)$$

and furthermore realizes $K, \bar{P}^{ij}, i, j = 1, 2, 3$, for $\mathcal{I}, \mathcal{I}_0^{ij}$ respectively. We denote by \mathcal{M}_F the set of admissible models.

Using the same tree notation from [25] we can denote $\mathcal{F}_0 := \mathcal{T}_{<0}$ as follows:

For Ξ we simply draw a dot. The integration map \mathcal{I}^{ij} is then represented by a downfacing line while the integration map $\mathcal{I}_0 \mathcal{I}_j$ is then represented by a downfacing dotted line. The integration map \mathcal{I}_j is represented by \searrow . The multiplication of symbols is obtained by joining them at the root.

$$\mathcal{F}_0 = \{\mathbf{1}, \uparrow, \vee, \vee^\cdot, \searrow, \searrow^\cdot, \searrow \searrow, \searrow \searrow^\cdot, \searrow \searrow \searrow, \searrow \searrow \searrow^\cdot\},$$

Here the solution to the stochastic N-S equation is vector valued and there are a lot of superscripts and subscripts for the elements in \mathcal{F}_0 , which will not be noticeable in the tree notation. To see more details for \mathcal{F}_0 we refer to [25].

Similarly as in [14], the renormalization group \mathfrak{R} for the regularity structure \mathfrak{T}_F is also introduced in [25]. For $g \in \mathfrak{R} := \mathbb{R}^n, n = 3^4 + 3 \cdot 3^{10}$, $g = (C_{ii_1 j j_1}^1, C_{ii_1 i_2 j j_1 j_2 k k_1 l l_1}^2, C_{ii_1 i_2 i_3 k k_1 l l_1 j j_1}^3, C_{ii_1 i_2 k k_1 l l_1 l_2 j j_1}^4)$, $i, j, k, l, i_1, i_2, i_3, j_1, k_1, l_1, l_2 = 1, 2, 3$, we define a linear map M_g on \mathcal{F}_0 by

$$\begin{aligned} M_g \vee &= \vee - C_{ii_1 j j_1}^1 \mathbf{1}, \\ M_g \searrow \searrow &= \searrow \searrow - C_{ii_1 i_2 j j_1 j_2 k k_1 l l_1}^2 \mathbf{1}, \\ M_g \searrow \searrow^\cdot &= \searrow \searrow^\cdot - C_{ii_1 i_2 i_3 k k_1 l l_1 j j_1}^p \mathbf{1}, \quad p = 3, 4 \end{aligned}$$

as well as $M_g(\tau) = \tau$ for the remaining basis vectors in \mathcal{F}_0 . Here the choice that $p = 3$ or 4 depends on the explicit formula of $\searrow \searrow^\cdot$. For more details, we refer to [25].

For given $\xi \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}^3)$ by [25, Section 2] we can construct a canonical model $\mathcal{L}\xi \in \mathcal{M}_F$. For $g \in \mathfrak{R}$, we can define M_g on \mathcal{F}_0 as above. Since τ satisfies $M_g \tau = \tau - C \mathbf{1}$ for any $\tau \in \mathcal{F}_0$, we can easily lift this action M_g on the space \mathcal{M}_F of admissible models via the construction of [14, Section 8] and [25, Section 2]. We also use the following definition from [17]. We choose $\eta \in (-1, \alpha + 2]$ and $|\alpha + 2| < \gamma < \eta - \alpha$ for $\alpha \in (-\frac{13}{5}, -\frac{5}{2})$. Here α is the degree of Ξ_i and the initial value belongs to \mathcal{C}^η .

Definition 2.8 An admissible model $\Pi = (\Pi, \Gamma, \Sigma) \in \mathcal{M}_F$ is *nice* if there exist $\xi_n \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^d)$ and $g_n \in \mathfrak{R}$ such that $\Pi = \lim_{n \rightarrow \infty} M_{g_n} \mathcal{L}(\xi_n)$ and furthermore the distribution $K^{ii_1} * \xi^{i_1} := \mathcal{R}_t \mathcal{I}^{ii_1}(\Xi_{i_1}) \in C(\mathbb{R}, \mathcal{C}^\eta)$, $D_j K^{i_0 i} * (K^{ii_1} * \xi^{i_1} \diamond K^{jj_1} * \xi^{j_1}) \in C(\mathbb{R}, \mathcal{C}^{2\alpha+5})$, $i, i_0, i_1, j, j_1 = 1, 2, 3$. Here \mathcal{R}_t is the family of reconstruction operators in Theorem 2.5.

Write \mathcal{M} for the closure of all smooth and nice models in the space of nice admissible models for the regularity structure \mathfrak{T}_F . Now we consider the following equations on the regularity

structure:

$$\begin{aligned} u^i &= -\frac{1}{2} \sum_{i_1, j=1}^3 \mathcal{T}_j^{ii_1} \mathbf{R}^+(u^{i_1} \star u^j) + \sum_{i_1=1}^3 \bar{\mathcal{T}}_j^{ii_1} \mathbf{R}^+ \Xi_{i_1} + \sum_{i_1=1}^3 \mathcal{G}^{ii_1} u_0^{i_1} \\ J^i &= -\frac{1}{2} \sum_{i_1, j=1}^3 \mathcal{T}_j^{ii_1} \mathbf{R}^+(u^{i_1} \star J^j + J^{i_1} \star u^j) + \sum_{i_1=1}^3 \mathcal{G}^{ii_1} J_0^{i_1}, \end{aligned} \tag{2.7}$$

with the convolution operators $\mathcal{T}_j^{ii_1}$ and $\bar{\mathcal{T}}_j^{ii_1}$ satisfying

$$\begin{aligned} \mathcal{R}_t(\mathcal{T}_j^{ii_1} H)_t(x) &= \int_0^t \langle (P^{ii_1} * D_j G)_{t-s}(x - \cdot), \mathcal{R}_s H_s \rangle ds, \\ \mathcal{R}_t(\bar{\mathcal{T}}_j^{ii_1} H)_t(x) &= \int_0^t \langle (P^{ii_1} * G)_{t-s}(x - \cdot), \mathcal{R}_s H_s \rangle ds, \end{aligned}$$

for $H \in \mathcal{D}^{\gamma, \eta}$, and \mathcal{G}^{ii_1} denotes the solution to the linearized problem, viewed via its truncated Taylor expansion as an element in $\mathcal{D}^{\gamma, \eta}$. Here $\mathbf{R}^+ : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is given by $\mathbf{R}^+(t, x) = 1$ for $t > 0$ and $\mathbf{R}^+(t, x) = 0$ otherwise and for more details on the convolution operators $\mathcal{T}_j^{ii_1}$ and $\bar{\mathcal{T}}_j^{ii_1}$, we refer to [25]. By similar arguments as in the proof of [25, Proposition 2.13], we have the following results:

Proposition 2.9 Let \mathfrak{T}_F be the regularity structure constructed above associated to the stochastic N-S equation driven by space-time white noise. Suppose that $\eta \in (-1, \alpha + 2]$ and $|\alpha + 2| < \gamma < \eta - \alpha$ for $\alpha \in (-\frac{13}{5}, -\frac{5}{2})$. Let $u_0^i, J_0^i \in \mathcal{C}^\eta(\mathbb{R}^3)$, $i = 1, 2, 3$, periodic and let $\mathbf{\Pi} = (\Pi, \Gamma, \Sigma) \in \mathcal{M}$ be a nice model for \mathfrak{T}_F . Then there exists a maximal solution to equations (2.7).

Proof This result for the first equation in (2.7) has been obtained in [25, Proposition 2.13]. It is sufficient to consider the second one in (2.7). We have that J takes values in a sector $V \subset \bar{T} \oplus T_{\geq \zeta}$ with $\zeta = \alpha + 3$ and $u^i J^j$, $i, j = 1, 2, 3$, takes value in a sector $V \subset \bar{T} \oplus T_{\geq \bar{\zeta}}$ with $\bar{\zeta} = 2\alpha + 5$ satisfying $\zeta < \bar{\zeta} + 1$. For η and γ we have $\bar{\eta} = 2\eta$ and $\gamma > \bar{\gamma} = \gamma + \alpha + 2 > 0$, $\bar{\gamma} < \eta + 2$ and $\bar{\gamma} + 1 > \gamma$. Then by similar arguments as in the proof of [25, Proposition 2.13] the results follow. \square

3 Strong Feller Property

We consider a Gaussian probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is a separable Banach space and $L^2(\mathbb{R}; L^2)$ is the Cameron-Martin space for the Gaussian measure \mathbf{P} . The canonical random variable ω induces the usual two-sided continuous filtration $\{\mathcal{F}_{s,t}, s < t\}$. $\mathcal{F}_t := \mathcal{F}_{-\infty, t}$. Let $\xi = (\xi^1, \xi^2, \xi^3)$, with $\xi^i, i = 1, 2, 3$ being independent white noises on $\mathbb{R} \times \mathbb{T}^3$ on $(\Omega, \mathcal{F}, \mathbf{P})$, which we extend periodically to \mathbb{R}^4 . By [25, Theorem 2.17] we know that there exists a random variable $\mathbf{Z} : \Omega \rightarrow \mathcal{M}$ associated with space-time white noise ξ .

Choose $U = (\mathcal{C}^\eta)^3$ with the usual product norm still denoted by $\|\cdot\|_\eta$ and define $\bar{U} = U \cup \{\Delta\}$. \bar{U} is a separable metric space by setting $d(\Delta, u)^2 = 1 + \|u\|_\eta^2$ for all $u \in U$. For $u_0 \in U, \mathbf{\Pi} \in \mathcal{M}$ we use $\bar{u}_{s,t}(u_0, \mathbf{\Pi})$ to denote the maximal solution to the first equation in (2.7) obtained in

Proposition 2.9 with initial condition $\bar{u}_{s,s}(u_0, \mathbf{\Pi}) = u_0 \in U$. Define

$$u_{s,t}(u_0, \mathbf{\Pi}) = \begin{cases} \mathcal{R}_t \bar{u}_{s,t}(u_0, \mathbf{\Pi}), & \text{for } t \in [s, \zeta) \\ \Delta & \text{for } t \in [\zeta, \infty), \end{cases}$$

and $u_{s,t}(\Delta, \mathbf{\Pi}) = \Delta$, where ζ is the explosion time for s, u_0 and $\mathbf{\Pi}$. We also use the shorthands $u_t = u_{0,t}$ and $u = u_{0,1}$. It follows from the locality of the reconstruction map and the locality of the construction of the model that $u_{s,t}(u_0, \mathbf{\Pi})$ depends on the underlying white noise only on the time interval $[s, t]$. Moreover, as a consequence of [14, Prop. 7.11], one has for $u_0 \in U, \mathbf{\Pi} \in \mathcal{M}$

$$u_{0,s+t}(u_0, \mathbf{\Pi}) = u_{s,s+t}(u_{0,s}(u_0, \mathbf{\Pi}), \mathbf{\Pi}).$$

By this we can conclude that $u.(u_0, \mathbf{Z})$ is a Markov process. For any bounded measurable map $\Psi : \bar{U} \rightarrow \mathbf{R}$ we set for $t \geq 0$

$$P_t \Psi(u_0) := \mathbf{E} \Psi(u_t(u_0, \mathbf{Z})),$$

which forms a Markov semigroup.

Theorem 3.1 The Markov semigroup $(P_t)_{t \geq 0}$ satisfies the strong Feller property, i.e. $P_t \Psi$ is continuous for all Ψ that are only bounded and measurable.

Proof It is sufficient to check that Assumptions 1-5 in [17] hold. Here for the reader's convenience, we summarize them in appendix. We first check Assumption 1: By [25, Propositions 2.13, 2.15] we know that the preimage of U under the map $(s, t, u_0, \mathbf{\Pi}) \mapsto u_{s,t}(u_0, \mathbf{\Pi})$ is open and the map is jointly continuous on the preimage of U . The continuity in time for u follows from [25, Theorem 1.1]. Now we consider Frechét differentiability with respect to the initial condition. By the implicit function theorem in [7, Theorem 19.28] and similar arguments as in the proof of [3, Proposition 4.7], it follows that the solution u_t is differentiable in the initial condition and its derivative in the direction $J_0 \in (\mathcal{C}^\eta)^3$ is given by J . Here $J_t = \mathcal{R}_t \bar{J}_t$ with \bar{J} being the maximal solution to the second equation in (2.7) obtained in Proposition 2.9.

Now we check Assumption 2 in [17]. We set

$$r_t(u_0, \mathbf{\Pi}) = \begin{cases} +\infty, & \text{if } u_t(u_0, \mathbf{\Pi}) = \Delta \\ \|\bar{u}\|_{\gamma, \eta; t} & \text{otherwise.} \end{cases}$$

The \mathcal{F}_t -measurability of r_t is an immediate consequence of the adaptedness of u . It is obvious that $r_0 < \infty$ and that the map $t \mapsto r_t(u_0, \mathbf{\Pi})$ is increasing and is continuous in t , except at the explosion time when it has to diverge to $+\infty$. By [25, Proposition 2.13] we know that r is locally Lipschitz continuous on \mathcal{N}_t as a function of both the initial condition and the underlying model.

Now we check Assumption 3 in [17]. With M_g and \mathfrak{L} given in Section 2, we first check Assumption 10 in [17] as in [17, Section 5.1]. For $p > 1$ sufficiently large, we choose $E := L^p([0, 1], \mathcal{C}^3)$ as a suitable space of shifts with \mathcal{C} the space of periodic continuous functions. We prove that E generates a continuous action on our space \mathcal{M} of nice models. We introduce an auxiliary regularity structure (\hat{T}, \hat{G}) as follows. We introduce $\hat{\Xi}_i, i = 1, 2, 3$, for the shifts. Define for $i, j = 1, 2, 3$,

$$\begin{aligned} \hat{\mathfrak{M}}_F^{ij} = & \{1, \mathcal{I}^{ii_1}(\bar{\Xi}_{i_1}), \mathcal{I}^{jj_1}(\bar{\Xi}_{j_1}), \mathcal{I}^{ii_1}(\bar{\Xi}_{i_1})\mathcal{I}^{jj_1}(\bar{\Xi}_{j_1}), U_i, U_j, U_i U_j, \mathcal{I}^{ii_1}(\bar{\Xi}_{i_1})U_j, U_i \mathcal{I}^{jj_1}(\bar{\Xi}_{j_1}), \\ & \bar{\Xi}_{i_1} = \Xi_{i_1} \text{ or } \hat{\Xi}_{i_1}, \bar{\Xi}_{j_1} = \Xi_{j_1} \text{ or } \hat{\Xi}_{j_1}, i_1, j_1 = 1, 2, 3\}. \end{aligned}$$

Then we build $\hat{\mathcal{F}}_F$ by a similar argument as that for \mathcal{F}_F , but with \mathfrak{M}_F^{ij} replaced by $\hat{\mathfrak{M}}_F^{ij}$. Graphically, if we denote $\hat{\Xi}$ be a circle, the symbols appearing in $\hat{\mathcal{F}}_F$ are the same as those appearing in \mathcal{F}_F but with any occurrence of a bullet possibly replaced by a circle.

We set $|\hat{\Xi}|_s = -\kappa$ for κ sufficiently small, the degrees of the remaining basis vectors being obtained by using the same rules as above. The structure group \hat{G} is also defined similarly as in [25] and by imposing that $\Gamma\hat{\Xi}_j = \hat{\Xi}_j$ for $j = 1, 2, 3$. Let $\hat{T} = \langle \hat{\mathcal{F}}_F \rangle$ with $\hat{T}_\gamma = \langle \{\tau \in \hat{\mathcal{F}}_F : |\tau|_s = \gamma\} \rangle$, $\hat{A} = \{|\tau|_s : \tau \in \hat{\mathcal{F}}_F\}$ and let \hat{G} be as above. Then $\hat{\mathfrak{T}}_F = (\hat{A}, \hat{T}, \hat{G})$ defines a regularity structure $\hat{\mathfrak{T}}$.

Define $\hat{\mathcal{F}}_0 = \mathcal{F}_0 \cup \{\mathcal{I}^{i_1}(\hat{\Xi}_{i_1}), i, i_1 = 1, 2, 3\}$. For $g \in \mathfrak{R}$, we define a linear map \hat{M}_g on $\langle \hat{\mathcal{F}}_0 \rangle$ as in Section 2.2 and for $\tau \in \hat{\mathcal{F}}_0 \setminus \mathcal{F}_0$, $\hat{M}_g \tau = \tau$. As a consequence of the calculation in [25], \hat{M}_g belongs to the renormalisation group defined in [14, Definition 8.43]. Then similarly as in [14, Theorem 8.46] we can define the related renormalised model $(\hat{\Pi}^{\hat{M}_g}, \hat{\Gamma}^{\hat{M}_g}, \hat{\Sigma}^{\hat{M}_g})$ and it is an admissible model for $\hat{\mathfrak{T}}_F$ on $\langle \hat{\mathcal{F}}_0 \rangle$. Furthermore, by Proposition 2.6 and [14, Prop. 3.31] it extends uniquely to an admissible model for all of $\hat{\mathfrak{T}}_F$. Furthermore, \mathcal{I} is an abstract integration map of order 2 on $\langle \hat{\mathcal{F}}_0 \rangle$ and for every $i, i_1 = 1, 2, 3$, $\mathcal{I}_0^{i_1}$ is an abstract integration map of order 0 on $\langle \hat{\mathcal{F}}_0 \rangle$. We also have a space of "nice models" $\hat{\mathcal{M}}$ for this large regularity structure $\hat{\mathfrak{T}}_F$.

Since κ is sufficiently small, all of the elements of $\hat{\mathcal{F}}_F \setminus \mathcal{F}_F$ are of strictly positive degree. By repeatedly applying Proposition 2.6 and writing $\hat{\mathcal{M}}_F$ as the space of admissible models for (\hat{T}, \hat{G}) , there exists a unique locally Lipschitz continuous map $\mathcal{Y} : E \times \mathcal{M}_F \rightarrow \hat{\mathcal{M}}_F$ such that for every $h \in E$ and $\Pi \in \mathcal{M}_F$, the model $\hat{\Pi} = \mathcal{Y}(h, \Pi) \in \hat{\mathcal{M}}_F$ agrees with Π on \mathfrak{T}_F and for $(\hat{\Pi}, \hat{\Gamma}, \hat{\Sigma}) := \mathcal{Y}(h, \Pi)$ we have

$$\hat{\Pi}_x \mathcal{I}^{i_1}(\hat{\Xi}_{i_1})(y) = K^{i_1} * h^{i_1}(t, y) - K^{i_1} * h^{i_1}(t, x) - \sum_j (y_j - x_j) D_j K^{i_1} * h^{i_1}(t, x).$$

Since $\hat{M}_g \mathcal{Y}(h, \Pi)$ and $\mathcal{Y}(h, M_g \Pi)$ agree on $\hat{\mathcal{F}}_0$ and any admissible model is uniquely determined by this, we have for every $g \in \mathfrak{R}$, every $\Pi \in \mathcal{M}$ and every $h \in E$, $\hat{M}_g \mathcal{Y}(h, \Pi) = \mathcal{Y}(h, M_g \Pi)$. Now as in [17] we introduce a map $\mathcal{Z} : \hat{\mathcal{M}}_F \rightarrow \mathcal{M}_F$, also commuting with the action of \mathfrak{R} , so that we can define $\tau = \mathcal{Z} \circ \mathcal{Y}$. For this, given a model $\hat{\Pi} \in \hat{\mathcal{M}}_F$ we introduce the following notion of a $\hat{\Pi}$ -polynomial as in [17]:

Definition 3.2 A $\hat{\Pi}$ -polynomial f is a map $f : \mathbb{R}^{d+1} \rightarrow \hat{\mathcal{M}}_F$ such that $f(t, x) = \Gamma_{xy}^t f(t, y)$ and $f(t, x) = \Sigma_x^{ts} f(s, x)$ for every s, t, x, y .

Given an admissible model $\hat{\Pi} = (\Pi, \Gamma, \Sigma) \in \hat{\mathcal{M}}_F$ we define a collection of $\hat{\Pi}$ -polynomial $\{f_{t,x}^\tau\}$ for $\tau \in \mathcal{F}_F$ and $(t, x) \in \mathbb{R}^{d+1}$ recursively as follows. Define $\deg \tau := |\tau|_s$ for $\tau \in \mathcal{F}_F$, but then setting $\deg \hat{\Xi} := |\Xi|_s$ and defining it on the rest of $\hat{\mathcal{F}}_F$ by using the same rules as for $|\cdot|_s$. This allows us to define operators $\bar{\mathcal{J}}_{t,x} : \hat{T} \rightarrow \bar{T}$, $\bar{\mathcal{J}}_{t,x}^{0,ij} : \hat{T} \rightarrow \bar{T}$ by setting

$$\bar{\mathcal{J}}_{t,x} \tau = \mathcal{Q}_{<\deg \tau + 2} \mathcal{J}_{t,x} \tau, \quad \bar{\mathcal{J}}_{t,x}^{0,ij} \tau = \mathcal{Q}_{<\deg \tau} \mathcal{J}_{t,x}^{0,ij} \tau,$$

where $\mathcal{J}_{t,x}^{0,ij}$ is defined in (2.3) associated with \bar{P}^{ij} . With these definitions at hand we set for $z = (t, x)$, $\bar{z} = (\bar{t}, \bar{x})$

$$f_z^{X^k}(\bar{z}) = \Gamma_{\bar{x}x}^{\bar{t}} \Sigma_x^{\bar{t}t} X^k, \quad f_z^{\Xi_j}(\bar{z}) = \Xi_j + \hat{\Xi}_j.$$

Then we set recursively

$$f_z^{\tau\bar{\tau}}(\bar{z}) = f_z^\tau(\bar{z}) f_z^{\bar{\tau}}(\bar{z}), \quad f_z^{\mathcal{I}(\tau)}(\bar{z}) = (\mathcal{I} + \mathcal{J}_{\bar{t},\bar{x}} - \Gamma_{\bar{x}x}^{\bar{t}} \Sigma_x^{\bar{t}t} \bar{\mathcal{J}}_{t,x} \Gamma_{x\bar{x}}^t \Sigma_{\bar{x}}^{\bar{t}\bar{t}}) f_z^\tau(\bar{z}),$$

$$f_z^{\mathcal{D}_{j\tau}}(\bar{z}) = \mathcal{D}_j f_z^\tau(\bar{z}), \quad f_z^{\mathcal{I}_0^{ij}(\tau)}(\bar{z}) = (\mathcal{I}_0^{ij} + \mathcal{J}_{t,\bar{x}}^{0,ij} - \Gamma_{\bar{x}x}^t \Sigma_x^{t\bar{t}} \bar{\mathcal{J}}_{t,x}^{0,ij} \Gamma_{x\bar{x}}^t \Sigma_{\bar{x}}^{t\bar{t}}) f_z^\tau(\bar{z}).$$

It is easy to verify that for $\tau \in \mathcal{F}_F$, f_z^τ is indeed a $\hat{\Pi}$ -polynomial and for $\tau \in \mathcal{F}_F$

$$\hat{\mathcal{Q}}_{<|\tau|_s} f_z^\tau(z) = 0, \forall z \in \mathbb{R}^{d+1}, \quad (3.1)$$

where $\hat{\mathcal{Q}}_{<\gamma}$ is the projection onto $\hat{T}_{<\gamma}$. Set $\mathcal{T} := \mathcal{H}_F \cup \{\Xi_j, j = 1, 2, 3\}$. Define operators $\check{\Gamma}_{x\bar{x}}^t : \mathcal{T} \rightarrow \mathcal{T}, \check{\Sigma}_x^{t\bar{t}} : \mathcal{T} \rightarrow \mathcal{T}$ by setting

$$\check{\Gamma}_{x\bar{x}}^t \Xi_j = \Xi_j, \quad \check{\Gamma}_{x\bar{x}}^t X^k = \Gamma_{x\bar{x}}^t X^k, \quad \check{\Sigma}_x^{t\bar{t}} \Xi_j = \Xi_j, \quad \check{\Sigma}_x^{t\bar{t}} X^k = \Sigma_x^{t\bar{t}} X^k,$$

and then recursively by

$$\check{\Gamma}_{x\bar{x}}^t(\tau\bar{\tau}) = (\check{\Gamma}_{x\bar{x}}^t \tau)(\check{\Gamma}_{x\bar{x}}^t \bar{\tau}), \quad \check{\Sigma}_x^{t\bar{t}}(\tau\bar{\tau}) = (\check{\Sigma}_x^{t\bar{t}} \tau)(\check{\Sigma}_x^{t\bar{t}} \bar{\tau}),$$

as well as

$$\begin{aligned} \check{\Gamma}_{x\bar{x}}^t(\mathcal{I}\tau) &= \mathcal{I}\check{\Gamma}_{x\bar{x}}^t \tau + (\bar{\mathcal{J}}_{t,x} \Gamma_{x\bar{x}}^t - \Gamma_{x\bar{x}}^t \bar{\mathcal{J}}_{t,\bar{x}}) f_{t,\bar{x}}^\tau(t, \bar{x}), \\ \check{\Gamma}_{x\bar{x}}^t(\mathcal{I}_0^{ij} \tau) &= \mathcal{I}_0^{ij} \check{\Gamma}_{x\bar{x}}^t \tau + (\bar{\mathcal{J}}_{t,x}^{0,ij} \Gamma_{x\bar{x}}^t - \Gamma_{x\bar{x}}^t \bar{\mathcal{J}}_{t,\bar{x}}^{0,ij}) f_{t,\bar{x}}^\tau(t, \bar{x}), \\ \check{\Sigma}_x^{t\bar{t}}(\mathcal{I}\tau) &= \mathcal{I}\check{\Sigma}_x^{t\bar{t}} \tau + (\bar{\mathcal{J}}_{t,x} \check{\Sigma}_x^{t\bar{t}} - \check{\Sigma}_x^{t\bar{t}} \bar{\mathcal{J}}_{t,x}) f_{t,x}^\tau(\bar{t}, x), \\ \check{\Sigma}_x^{t\bar{t}}(\mathcal{I}_0^{ij} \tau) &= \mathcal{I}_0^{ij} \check{\Sigma}_x^{t\bar{t}} \tau + (\bar{\mathcal{J}}_{t,x}^{0,ij} \check{\Sigma}_x^{t\bar{t}} - \check{\Sigma}_x^{t\bar{t}} \bar{\mathcal{J}}_{t,x}^{0,ij}) f_{t,x}^\tau(\bar{t}, x). \end{aligned}$$

We also extend the definition of f_z^τ to all of $\tau \in \mathcal{T}$ by linearity. It is straightforward to verify that we have

$$f_{t,x}^{\check{\Gamma}_{x\bar{x}}^t \tau} = f_{t,\bar{x}}^\tau, \quad f_{t,x}^{\check{\Sigma}_x^{t\bar{t}} \tau} = f_{t,x}^\tau. \quad (3.2)$$

The map \mathcal{Z} is defined as follows: Given a model $\hat{\Pi} = (\Pi, \Gamma, \Sigma) \in \hat{\mathcal{M}}_F$, we define a new model $\mathcal{Z}\hat{\Pi} = (\check{\Pi}, \check{\Gamma}, \check{\Sigma})$ on \mathfrak{T}_F with

$$\check{\Pi}_x^t \tau = \mathcal{R}_t f_{t,x}^\tau,$$

and $\check{\Gamma}, \check{\Sigma}$ being defined as above. It follows from (3.2) that $\check{\Pi}_y^t = \check{\Pi}_x^t \check{\Gamma}_{xy}^t$. By definition of $\check{\Gamma}, \check{\Sigma}$ we can easily verify by recursion that $\check{\Gamma}_{xx}^t = 1, \check{\Sigma}_x^{t\bar{t}} = 1$. Since the map $\tau \mapsto f_x^\tau$ is injective, (3.2) also implies that $\Sigma_x^{sr} \Sigma_x^{rt} = \Sigma_x^{st}, \Sigma_x^{st} \Gamma_{xy}^t = \Gamma_{xy}^s \Sigma_y^{st}, \Gamma_{xy}^t \Gamma_{yz}^t = \Gamma_{xz}^t$. It remains to verify that the required analytical bounds hold. The bounds of $\check{\Pi}$ follow from the corresponding on Π and the fact that $\mathcal{R}_t f_{t,x}^\tau = \Pi_x^t f_{t,x}^\tau(t, x)$ combined with (3.1). The bounds of $\check{\Gamma}$ and the bounds of $\check{\Sigma}$ on $\mathcal{I}\tau$ follow inductively from the corresponding bounds on (Γ, Σ) combined with [14, Lem. 5.21]. The bounds of $\check{\Sigma}$ on $\mathcal{I}_0^{ij} \tau$ follow from the proof in Proposition 2.6 and the bound of Σ . The fact that the new model is admissible follows from

$$\check{\Pi}_x^t \mathcal{I}(\tau) = \Pi_x^t (\mathcal{I} + \mathcal{J}_{t,x} - \bar{\mathcal{J}}_{t,x}) f_{t,x}^\tau(t, x),$$

$$\check{\Pi}_x^t \mathcal{I}_0^{ij}(\tau) = \Pi_x^t (\mathcal{I}_0 + \mathcal{J}_{t,x}^{0,ij} - \bar{\mathcal{J}}_{t,x}^{0,ij}) f_{t,x}^\tau(t, x),$$

and the definition of $\bar{\mathcal{J}}_{t,x}, \bar{\mathcal{J}}_{t,x}^{0,ij}$. Setting $\tau(h, \Pi) = \mathcal{Z}(\mathcal{Y}(h, \Pi))$, we have

$$\tau(h, \mathfrak{L}(\xi)) = \mathfrak{L}(\xi + h),$$

for every smooth ξ and h . Moreover, by similar arguments as in [17, Section 5.1] we have

$$\tau(h, M_g \Pi) = M_g \tau(h, \Pi),$$

for every smooth h and every smooth $\mathbf{\Pi} \in \mathcal{M}_F$ and every $g \in \mathfrak{R}$. Now by similar arguments as in the proof of [17, Prop. 4.7], Assumptions 3 in [17] holds.

Furthermore, for any $h \in E$ supported in $(0, 1) \times \mathbb{R}^d$ and any model $\mathbf{\Pi} \in \mathcal{M}$, the solution u^h to the first component in (2.7) with model $\mathbf{\Pi}^h = \tau(h, \mathbf{\Pi})$ is related to the solution \hat{u}^h to

$$u^{h,i} = -\frac{1}{2} \sum_{i_1, j=1}^3 \mathcal{T}_j^{ii_1} \mathbf{R}^+(u^{h,i_1} \star u^{h,j}) + \sum_{i_1=1}^3 \bar{\mathcal{T}}_j^{ii_1} \mathbf{R}^+ \Xi_{i_1} + \sum_{i_1=1}^3 \mathcal{G}^{ii_1} (u_0^{i_1} + h^{i_1}).$$

with model $\mathbf{\Pi}$ by $\mathcal{R}^h u^h = \mathcal{R} \hat{u}^h$. Here, \mathcal{R}^h and \mathcal{R} denote the reconstruction operators for the model $\mathbf{\Pi}^h$ and $\mathbf{\Pi}$, respectively. By the implicit function theorem in [7, Theorem 19.28] and similar arguments as in the proof of [3, Proposition 4.7], Assumption 4 follows.

Finally we come to Assumption 5: the map $u_{s,t}$ is Fréchet differentiable w.r.t. the initial value. Denote its derivative in the direction v by $J_{s,t}v$. Since the solution to the second component of the solution to the fixed point (2.7) takes values in a sector $V \subset \bar{T} \oplus T_{\geq \zeta}$ with $\zeta = \alpha + 3$, $J_{s,t}$ is a bounded linear operator from \mathcal{C}^η to \mathcal{C}^ζ . Applying [17, Corollary A.3], the derivative of u_t w.r.t. $h \in L^p([0, 1], X_0)$ is given by

$$\mathcal{D}u_t(u_0, \mathbf{\Pi})h = \int_0^t J_{s,t}h(s)ds,$$

for $(\Phi_0, \mathbf{\Pi}) \in \mathcal{N}_t$. By this and the same arguments as in the proof of Theorem 4.8 in [17], the results follow. \square

4 Two dimensional case

We consider (1.1) on \mathbb{T}^2 . The strong Feller property in this case can be obtained in the same way as for the three dimensional case. More precisely, we choose $U = (\mathcal{C}^\eta)^2$ for $\eta \in (-\kappa, 0)$ for κ small enough, and

$$\mathcal{F}_0 = \{\mathbf{1}, \mathcal{I}^{ii_1}(\Xi_{i_1}), \mathcal{I}^{ii_1}(\Xi_{i_1})\mathcal{I}^{jj_1}(\Xi_{j_1}), i, j, i_1, j_1 = 1, 2, 3\}.$$

All the arguments in Section 3 can be applied in this case. It is well-known that the invariant measure of the equation is given by a Gaussian measure μ (cf. [5]). By using the invariant measure in this case we obtain the following result:

Theorem 4.1 For the N-S equation driven by space-time white noise on \mathbb{T}^2 , its solutions are almost surely global in time for every initial condition $u_0 \in \mathcal{C}^\eta$ for $\eta \in (-\frac{1}{2}, 0)$.

Proof We use ζ_x to denote the explosion time for $u(x, \mathbf{\Pi})$, $x \in \mathcal{C}^\eta$. The invariant measure for the N-S equation driven by space-time white noise on \mathbb{T}^2 is given by Gaussian measure μ (cf. [5]), which has full support. By [5] we know that $P(\zeta_x = \infty) = 1$ for μ -almost every starting point x . By strong Feller property we know that for every $t \geq 0$, $x \mapsto P(t < \zeta_x)$ is continuous, which implies that $P(\zeta_x = \infty) = 1$ for every starting point x in \mathcal{C}^η .

Appendix

In this section we are in the setting of Section 3 and we recall the following assumptions from [17]. In [17], it has been proved that the Markov semigroup satisfies the strong Feller property under the Assumptions 1-5.

Assumption 1 The preimage of U under the map

$$\mathbf{u} : (s, t, u_0, \mathbf{\Pi}) \mapsto u_{s,t}(u_0, \mathbf{\Pi}),$$

is open and \mathbf{u} is jointly continuous on $\mathbf{u}^{-1}(U)$. Furthermore, \mathbf{u} is Fréchet differentiable in u_0 at every point of $\mathbf{u}^{-1}(U)$.

Define the sets

$$\mathcal{N}_t = \{(u_0, \mathbf{\Pi}) : u_t(u_0, \mathbf{\Pi}) \neq \Delta\}.$$

We will denote the Fréchet derivative of u_t in the direction $v \in U$ by $Du_t(u_0, \mathbf{\Pi})v$, with the understanding that Du_t is only defined on \mathcal{N}_t .

Assumption 2 We are given a lower semi-continuous map $r : [0, 1] \times U \times \mathcal{M} \rightarrow [0, \infty]$ with the following properties.

1. For every $u_0 \in U$ and every $t \in [0, 1]$, the map $\omega \mapsto r_t(u_0, \mathbf{Z}(\omega))$ is \mathcal{F}_t -measurable.
2. For every $(u_0, \mathbf{\Pi}) \in U \times \mathcal{M}$, one has $r_0(u_0, \mathbf{\Pi}) < \infty$ and the map $t \mapsto r_t(u_0, \mathbf{\Pi})$ is continuous and increasing.
3. One has $\{(u_0, \mathbf{\Pi}) : u_t(u_0, \mathbf{\Pi}) = \Delta\} = \{(u_0, \mathbf{\Pi}) : r_t(u_0, \mathbf{\Pi}) = \infty\}$.
4. For every $t \in (0, 1]$, the map r_t is locally Lipschitz continuous on \mathcal{N}_t .

We consider a space $E = L^p([0, 1], X_0) \subset L^2(\mathbb{R}, L^2)$, where X_0 is some separable Banach subspace of L^2 and $p \in (2, \infty)$. Define $E_s := L^p([0, s], X_0)$.

Assumption 3 We are given a continuous action $\tau : E \times \mathcal{M} \rightarrow \mathcal{M}$ of E onto \mathcal{M} such that, for every $\mathbf{\Pi} \in \mathcal{M}$, the map $h \mapsto \tau(h, \mathbf{\Pi})$ is locally Lipschitz continuous and such that, for every $h \in E$, the identity

$$\mathbf{Z}(\omega + h) = \tau(h, \mathbf{Z}(\omega)),$$

holds \mathbf{P} -almost surely. Furthermore, the action τ is compatible with the maps $u_{s,t}$ in the sense that if h is such that $h(r) = 0$ for $r \in [s, t]$, then

$$u_{s,t}(u_0, \tau(h, \mathbf{\Pi})) = u_{s,t}(u_0, \mathbf{\Pi}),$$

for every $u_0 \in \bar{U}$.

Assumption 4 For every $(u_0, \mathbf{\Pi}) \in \mathcal{N}_t$, the map $h \mapsto u_t(u_0, \tau(h, \mathbf{\Pi}))$ is Fréchet differentiable at $h = 0$.

We denote this Fréchet derivative by $\mathcal{D}u_t(u_0, \mathbf{\Pi})$. Let $L(U, E_s)$ to denote the space of bounded linear operators $U \rightarrow E_s$.

Assumption 5 For every $s \leq t$ with $t \in (0, 1]$, we are given a map $A_t^{(s)} : \mathcal{N}_s \rightarrow L(U, E_s)$, and these maps are compatible in the sense that, for any $0 < s < r \leq t$, any $(u_0, \mathbf{\Pi}) \in \mathcal{N}_r$, and any $v \in U$, one has

$$A_t^{(r)}(u_0, \mathbf{\Pi})v|_{[0,s]} = A_t^{(s)}(u_0, \mathbf{\Pi})v.$$

Furthermore, for every $u_0 \in U$ and $s \leq t$, the map $\omega \mapsto A_t^{(s)}(u_0, \mathbf{Z}(\omega))$ is \mathcal{F}_s -measurable and one has the identity

$$Du_t(u_0, \mathbf{\Pi})v + \mathcal{D}u_t(u_0, \mathbf{\Pi})(A_t^{(s)}(u_0, \mathbf{\Pi})v) = 0,$$

for all $v \in U$ and all $(u_0, \mathbf{\Pi}) \in \mathcal{N}_t$. Furthermore, for every $0 < s \leq t \leq 1$, the map $A_t^{(s)}$ is locally Lipschitz continuous from \mathcal{N}_s to $L(U, E_s)$ and bounded on every set of the form $\{(u_0, \mathbf{\Pi}) : r_s(u_0, \mathbf{\Pi}) \leq R\}$ with $R > 0$.

References

- [1] L. Bertini, G. Giacomin, Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.* 183, no. 3, (1997), 571-607.
- [2] Rémi Catellier, Khalil Chouk, Paracontrolled Distributions and the 3-dimensional Stochastic Quantization Equation, arXiv:1310.6869
- [3] G. Cannizzaro, P. K. Friz, and P. Gassiat, Malliavin calculus for regularity structures: the case of gPAM. *ArXiv e-prints* (2015). arXiv: 1511.08888.
- [4] A. Debussche, Ergodicity Results for the Stochastic Navier-Stokes Equations: An Introduction, *Topics in Mathematical Fluid Mechanics Lecture Notes in Mathematics* 2013, 23-108
- [5] G. Da Prato, A. Debussche, 2D Navier-Stokes equations driven by a space-time white noise. *J. Funct. Anal.* 2002, 196 (1), 180-210.
- [6] G. Da Prato and A. Debussche, Ergodicity for the 3D stochastic Navier-Stokes equations, *J. Math. Pures Appl.* (9) 82 (2003), no. 8, 877-947.
- [7] B. K. Driver, *Analysis Tools with Applications*, Springer, 2003.
- [8] F. Flandoli, D. Gatarek, Martingale and stationary solutions for stochastic Navier-Stokes equations, *Probability Theory and Related Fields* **102** (1995), 367-391
- [9] F. Flandoli, M. Romito, Markov selections for the 3D stochastic Navier-Stokes equations, *Probab. Theory Relat. Fields* **140** (2008), 407-458
- [10] M. Gubinelli, Controlling rough paths. *J. Funct. Anal.* 216, no. 1, (2004), 86-140.
- [11] M. Gubinelli, P. Imkeller, N. Perkowski, Paracontrolled distributions and singular PDEs, arXiv:1210.2684
- [12] B. Goldys, M. Röckner and X.C. Zhang, Martingale solutions and Markov selections for stochastic partial differential equations, *Stochastic Processes and their Applications* **119** (2009) 1725-1764
- [13] M. Hairer, Solving the KPZ equation. *Ann. of Math.* (2) 178, no. 2, (2013), 559-664.
- [14] M. Hairer, A theory of regularity structures. *Invent. Math.* (2014).
- [15] M. Hairer, J. C. Mattingly, Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing *Annals of Math.*, **164** (2006), 993-1032

- [16] M. Hairer, K. Mateski, Discretisations of rough stochastic pdes, <https://arxiv.org/abs/1511.06937>, (2015)
- [17] M. Hairer, J. C. Mattingly, The strong Feller property for singular stochastic PDEs, [arXiv:1610.03415v1](https://arxiv.org/abs/1610.03415), (2016)
- [18] M. Hairer, J. C. Mattingly, M. Scheutzow, Asymptotic coupling and a general form of Harris theorem with applications to stochastic delay equations, *Probability Theory and Related Fields* 149, 1-2, 223-259, (2011)
- [19] M. Kardar, G. Parisi, Y.-C. Zhang, Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* 56, no. 9, (1986), 889-892.
- [20] M. Röckner, R.-C. Zhu, X.-C. Zhu, Local existence and non-explosion of solutions for stochastic fractional partial differential equations driven by multiplicative noise, *Stochastic Processes and their Applications* 124 (2014) 1974-2002
- [21] M. Röckner, R. Zhu, X. Zhu, Restricted Markov uniqueness for the stochastic quantization of $P(\phi)_2$ and its applications, [arXiv:1511.08030](https://arxiv.org/abs/1511.08030) (2015), to appear in *Journal of functional analysis*
- [22] M. Röckner, R. Zhu, X. Zhu, Ergodicity for the stochastic quantization problems on the 2D-torus, *Communication in Mathematical physics*, [arXiv:1606.02102](https://arxiv.org/abs/1606.02102), (2016)
- [23] R. Temam, *Navier-Stokes Equations*, North-Holland, Amsterdam, (1984)
- [24] P. Tsatsoulis, H. Weber, Spectral Gap for the Stochastic Quantization Equation on the 2-dimensional Torus. (2016) [arXiv:1609.08447](https://arxiv.org/abs/1609.08447).
- [25] R. Zhu and X. Zhu. Three-dimensional Navier-Stokes equations driven by space-time white noise. *J. Differential Equations* 259, no. 9, (2015), 4443-4508, [http://arxiv.org/abs/1406.0047](https://arxiv.org/abs/1406.0047)