REFINEMENTS OF A REVERSED AM–GM OPERATOR INEQUALITY

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ABSTRACT. We prove some refinements of a reverse AM-GM operator inequality due to M. Lin [Studia Math. 2013;215:187–194]. In particular, we show the operator inequality

$$\Phi^p\left(A\nabla_{\nu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right) \leq \alpha^p\Phi^p\left(A\sharp_{\nu}B\right),$$

where A,B are positive operators on a Hilbert space such that $0 < m \le A, B \le M$ for some positive numbers $m,M,~\Phi$ is a positive unital linear map, $\nu \in [0,1],$ $r=\min\{\nu,1-\nu\},~p>0$ and $\alpha=\max\left\{\frac{(M+m)^2}{4Mm},\frac{(M+m)^2}{4\frac{2}{p}Mm}\right\}.$

1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ denote the C^* -algebra of all bounded linear operators on a complex Hilbert space \mathscr{H} , with the identity I. In the case when $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra \mathbb{M}_n of all $n \times n$ matrices with entries in the complex field. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathscr{H}$, and we then write $A \geq 0$. We write A > 0 if A is a positive invertible operator. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ we say that $A \leq B$ if $B - A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical *-isomorphism between the C^* -algebra $C(\operatorname{sp}(A))$ of continuous functions on the spectrum $\operatorname{sp}(A)$ of a self-adjoint operator A and the C^* -algebra generated by A and $I_{\mathscr{H}}$. If $f, g \in C(\operatorname{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \operatorname{sp}(A))$ implies that $f(A) \geq g(A)$.

Let $A, B \in \mathbb{B}(\mathcal{H})$ be two positive invertible operators and $\nu \in [0, 1]$. The operator weighted arithmetic, geometric and harmonic means are defined by $A\nabla_{\nu}B = (1-\nu)A + \nu B$, $A\sharp_{\nu}B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\nu}A^{\frac{1}{2}}$ and $A!_{\nu}B = ((1-\nu)A^{-1} + \nu B^{-1})^{-1}$, respectively. In particular, for $\nu = \frac{1}{2}$ we get the usual operator arithmetic mean ∇ ,

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the geometric mean # and the harmonic mean!. The AM-GM inequality reads

$$\frac{A+B}{2} \ge A \sharp B,$$

for all positive operators A, B. It is shown in [10] the following reverse of AM–GM inequality involving positive linear maps

$$\Phi\left(\frac{A+B}{2}\right) \le \frac{(M+m)^2}{4Mm}\Phi(A\sharp B),\tag{1.1}$$

where $0 < m \le A, B \le M$ and Φ is a positive unital linear map.

For two positive operators $A, B \in \mathbb{B}(\mathcal{H})$, the Löwner–Heinz inequality states that, if $A \leq B$, then

$$A^p \le B^p, \qquad (0 \le p \le 1). \tag{1.2}$$

In general (1.2) is not true for p > 1. Lin [10, Theorem 2.1] showed however a squaring of (1.1), namely that the inequality

$$\Phi^2\left(\frac{A+B}{2}\right) \le \left(\frac{(M+m)^2}{4Mm}\right)^2 \Phi^2(A\sharp B) \tag{1.3}$$

as well as

$$\Phi^2\left(\frac{A+B}{2}\right) \le \left(\frac{(M+m)^2}{4Mm}\right)^2 (\Phi(A)\sharp\Phi(B))^2 \tag{1.4}$$

hold. Using inequality (1.2) we therefore get

$$\Phi^p\left(\frac{A+B}{2}\right) \le \left(\frac{(M+m)^2}{4Mm}\right)^p \Phi^p(A\sharp B) \qquad (0
(1.5)$$

and

$$\Phi^p \left(\frac{A+B}{2} \right) \le \left(\frac{(M+m)^2}{4Mm} \right)^p (\Phi(A) \sharp \Phi(B))^p \qquad (0$$

where $0 < m \le A, B \le M$ and Φ is a positive unital linear map.

In [13] the authors extended (1.3) and (1.4) to p > 2. They proved that the inequalities

$$\Phi^p\left(\frac{A+B}{2}\right) \le \left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^p \Phi^p(A\sharp B) \qquad (p>2) \tag{1.7}$$

and

$$\Phi^p\left(\frac{A+B}{2}\right) \le \left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^p (\Phi(A)\sharp\Phi(B))^p \qquad (p>2),\tag{1.8}$$

where $0 < m \le A, B \le M$. In [4] and [12] the authors showed that

$$\Phi^{p}(A\sigma B) \le \alpha^{p}\Phi^{p}(A\tau B), \qquad (1.9)$$

and

$$\Phi^{p}(A\sigma B) \le \alpha^{p} \left(\Phi(A)\tau\Phi(B)\right)^{p}, \tag{1.10}$$

where $0 < m \le A, B \le M$, Φ be a positive unital linear map, σ , τ be two arbitrary means between harmonic and arithmetic means, $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right\}$ and p > 0. Choi's inequality (see e.g. [1, p. 41]) reads

$$\Phi(A)^{-1} \le \Phi(A^{-1}),\tag{1.11}$$

for any positive unital linear map Φ and operator A > 0. Choi's inequality cannot be squared [10], but a reverse of Choi's inequality (known as the operator Kantorovich inequality) can be squared, see e.g. [11].

In this paper, we present some refinements of inequalities (1.5) and (1.6) under some mild conditions for 0 and some refinements of inequalities (1.7) and (1.8) for the operator norm and <math>p > 2. We also show a refinement of the operator Pólya–Szegö inequality.

2. Main results

We need the following lemmas to prove our results.

Lemma 2.1. [3] Let A, B > 0. Then

$$||AB|| \le \frac{1}{4}||A + B||^2.$$

Lemma 2.2. [8] Let $A, B \ge 0$ and p > 1. Then

$$||A^p + B^p|| \le ||(A+B)^p||.$$

Lemma 2.3. Let A, B > 0 and $\alpha > 0$. Then $A \le \alpha B$ if and only if $||A^{\frac{1}{2}}B^{\frac{-1}{2}}|| \le \alpha^{\frac{1}{2}}$.

Proof. Obviously, $A \leq \alpha B$ if and only if $B^{\frac{-1}{2}}AB^{\frac{-1}{2}} \leq \alpha$. By definition, this holds if and only if $\|A^{\frac{1}{2}}B^{\frac{-1}{2}}\|^2 \leq \alpha$ and the proof is complete.

Lemma 2.4. [4] Let $0 < m \le A, B \le M$, Φ be a positive unital linear map and σ , τ be two arbitrary means between harmonic and arithmetic means. Then

$$\Phi(A\sigma B) + Mm\Phi^{-1}(A\tau B) \le M + m.$$

In the next proposition we extend the inequalities (1.9) and (1.10) to p > 2 and the inequalities (1.7) and (1.8) to arbitrary means between harmonic and arithmetic means.

Proposition 2.5. Let $0 < m \le A, B \le M$, Φ be a positive unital linear map, σ , τ be two arbitrary means between harmonic and arithmetic means and p > 0. Then

$$\Phi^{p}(A\sigma B)\Phi^{-p}(A\tau B) + \Phi^{-p}(A\tau B)\Phi^{p}(A\sigma B) < 2\alpha^{p}$$

where $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{\frac{1}{4^p}Mm}\right\}$.

Proof. By [5, Lemma 3.5.12] we have that $||X|| \le t$ if and only if $\begin{pmatrix} tI & X \\ X^* & tI \end{pmatrix} \ge 0$, for any $X \in \mathbb{B}(\mathcal{H})$. If $0 , then <math>\alpha = \frac{(M+m)^2}{4Mm}$. Applying inequality (1.9) and Lemma 2.3 we get

$$\|\Phi^p(A\sigma B)\Phi^{-p}(A\tau B)\| \le \alpha^p.$$

Hence

$$\left(\begin{array}{cc} \alpha^{p}I & \Phi^{p}\left(A\sigma B\right)\Phi^{-p}\left(A\tau B\right) \\ \Phi^{-p}\left(A\tau B\right)\Phi^{p}\left(A\sigma B\right) & \alpha^{p}I \end{array}\right) \geq 0$$

and

$$\begin{pmatrix} \alpha^{p}I & \Phi^{-p}(A\tau B) \Phi^{p}(A\sigma B) \\ \Phi^{p}(A\sigma B) \Phi^{-p}(A\tau B) & \alpha^{p}I \end{pmatrix} \geq 0.$$

Hence

$$\begin{pmatrix}
2\alpha^{p}I & \Phi^{-p}(A\tau B) \Phi^{p}(A\sigma B) + \Phi^{p}(A\sigma B) \Phi^{-p}(A\tau B) \\
\Phi^{p}(A\sigma B) \Phi^{-p}(A\tau B) + \Phi^{-p}(A\tau B) \Phi^{p}(A\sigma B) & 2\alpha^{p}I
\end{pmatrix}$$

is positive and the desired inequality for 0 . Using inequality (1.9) with the same argument, we get the desired inequality for <math>p > 1.

Now, we are ready to present our main result. We need the following lemma, proved in [7]; (see also [2]).

Lemma 2.6. [7] Let a, b > 0 and $\nu \in [0, 1]$. Then

$$a^{1-\nu}b^{\nu} + r(\sqrt{a} - \sqrt{b})^2 \le (1-\nu)a + \nu b, \tag{2.1}$$

where $r = \min\{\nu, 1 - \nu\}.$

Theorem 2.7. Let $0 < m \le A, B \le M$, Φ be a positive unital linear map, $\nu \in [0, 1]$ and p > 0. Then

$$\Phi^{p}\left(A\nabla_{\nu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right) \le \alpha^{p}\Phi^{p}\left(A\sharp_{\nu}B\right)$$
 (2.2)

and

$$\Phi^{p}\left(A\nabla_{\nu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right) \leq \alpha^{p}\left(\Phi\left(A\right)\sharp_{\nu}\Phi\left(B\right)\right)^{p}, \qquad (2.3)$$
where $r = \min\{\nu, 1 - \nu\}$ and $\alpha = \max\left\{\frac{(M+m)^{2}}{4Mm}, \frac{(M+m)^{2}}{4\frac{2}{2}Mm}\right\}.$

Proof. We prove first the inequalities (2.2) and (2.3) for $0 . Since <math>0 < m \le A, B \le M$ we get that

$$A + MmA^{-1} \le M + m$$
 and $B + MmB^{-1} \le M + m$.

Therefore, for a positive unital linear map Φ we have

$$\Phi(A) + Mm\Phi(A^{-1}) \le M + m$$

and

$$\Phi(B) + Mm\Phi(B^{-1}) \le M + m.$$

Obviously we have also the inequalities

$$\Phi((1-\nu)A) + Mm\Phi((1-\nu)A^{-1}) \le (1-\nu)M + (1-\nu)m$$

and

$$\Phi(\nu B) + Mm\Phi(\nu B^{-1}) \le \nu M + \nu m.$$

for any $\nu \in [0,1]$. Summing up, we therefore get

$$\Phi(A\nabla_{\nu}B) + Mm\Phi((1-\nu)A^{-1} + \nu B^{-1}) \le M + m. \tag{2.4}$$

Moreover, by using the inequality (2.1) and functional calculus for the positive operator $A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ we have

$$\left(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right)^{\nu} + r\left(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}} + 1 - 2\left(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right)^{\frac{1}{2}}\right) \le (1 - \nu) + \nu A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}.$$

Multiplying both sides of the above inequality both to the left and to the right by $A^{-\frac{1}{2}}$ we get that

$$A^{-1}\sharp_{\nu}B^{-1} + 2r\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right) \le (1-\nu)A^{-1} + \nu B^{-1}.$$
 (2.5)

Applying (1.11), (2.4), (2.5) and taking into account the properties of Φ we have

$$\|\Phi\left(A\nabla_{\nu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right)Mm\Phi^{-1}(A\sharp_{\nu}B)\|$$

$$\leq \frac{1}{4}\|\Phi(A\nabla_{\nu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + Mm\Phi^{-1}(A\sharp_{\nu}B)\|^{2}$$
(by Lemma 2.1)
$$\leq \frac{1}{4}\|\Phi(A\nabla_{\nu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) + Mm\Phi(A^{-1}\sharp_{\nu}B^{-1})\|^{2}$$
(by inequality (1.11))
$$= \frac{1}{4}\|\Phi(A\nabla_{\nu}B) + Mm\Phi(A^{-1}\sharp_{\nu}B^{-1} + 2r(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}))\|^{2}$$

$$\leq \frac{1}{4}\|\Phi(A\nabla_{\nu}B) + Mm\Phi((1-\nu)A^{-1} + \nu B^{-1})\|^{2} \quad \text{(by inequality (2.5))}$$

$$\leq \frac{1}{4}(M+m)^{2} \quad \text{(by inequality (2.4))}.$$

Therefore

$$\left\| \Phi \left(A \nabla_{\nu} B + 2r M m (A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) \Phi^{-1} (A \sharp_{\nu} B) \right\| \le \frac{(M+m)^2}{4Mm}. \tag{2.6}$$

Hence

$$\Phi^{2}\left(A\nabla_{\nu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right) \leq \left(\frac{(M+m)^{2}}{4Mm}\right)^{2}\Phi^{2}\left(A\sharp_{\nu}B\right).$$

Since $0 < p/2 \le 1$, by inequality (1.2) we have

$$\Phi^{p} \left(A \nabla_{\nu} B + 2r M m (A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) \le \left(\frac{(M+m)^{2}}{4Mm} \right)^{p} \Phi^{p} \left(A \sharp_{\nu} B \right).$$

Thus we get the inequality (2.2) for 0 . We prove now (2.3) for <math>0 . Applying Lemma 2.1 and then inequality (2.2) we have

$$\begin{split} & \left\| \Phi \left(A \nabla_{\nu} B + 2r M m (A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) M m (\Phi(A) \sharp_{\nu} \Phi(B))^{-1} \right\| \\ & \leq \frac{1}{4} \left\| \Phi (A \nabla_{\nu} B + 2r M m (A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1})) + M m (\Phi(A) \sharp_{\nu} \Phi(B))^{-1} \right\|^{2} \\ & \qquad \qquad \text{(by Lemma 2.1)} \\ & \leq \frac{1}{4} \left\| \Phi (A \nabla_{\nu} B + 2r M m (A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1})) + M m \Phi^{-1} (A \sharp_{\nu} B) \right\|^{2} \\ & \leq \frac{1}{4} (M + m)^{2} \qquad \text{(by inequality (2.6))}. \end{split}$$

Hence the inequality (2.3) for 0 .

Now, we prove the inequalities (2.2) and (2.3) for p > 2. Then, by Lemma 2.1 and

2.2 we get

$$\begin{split} M^{\frac{p}{2}}m^{\frac{p}{2}} & \left\| \Phi^{\frac{p}{2}} \left(A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) \Phi^{\frac{-p}{2}} (A \sharp_{\nu} B) \right\| \\ & = \left\| \Phi^{\frac{p}{2}} \left(A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) M^{\frac{p}{2}} m^{\frac{p}{2}} \Phi^{\frac{-p}{2}} (A \sharp_{\nu} B) \right\| \\ & \leq \frac{1}{4} \left\| \Phi^{\frac{p}{2}} (A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1})) + M^{\frac{p}{2}} m^{\frac{p}{2}} \Phi^{\frac{-p}{2}} (A \sharp_{\nu} B) \right\|^{2} \\ & \leq \frac{1}{4} \left\| \left(\Phi(A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1})) + Mm \Phi^{-1} (A \sharp_{\nu} B) \right)^{\frac{p}{2}} \right\|^{2} \\ & = \frac{1}{4} \left\| \Phi(A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1})) + Mm \Phi^{-1} (A \sharp_{\nu} B) \right\|^{p} \\ & \leq \frac{1}{4} (M + m)^{p}. \end{split}$$

Hence we get the inequality (2.2) for p > 2. Further, we have

$$\begin{split} M^{\frac{p}{2}}m^{\frac{p}{2}} & \left\| \Phi^{\frac{p}{2}} \left(A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) \left(\Phi \left(A \right) \sharp_{\nu} \Phi \left(B \right) \right)^{\frac{-p}{2}} \right\| \\ & = \left\| \Phi^{\frac{p}{2}} \left(A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) M^{\frac{p}{2}}m^{\frac{p}{2}} \left(\Phi \left(A \right) \sharp_{\nu} \Phi \left(B \right) \right)^{\frac{-p}{2}} \right\| \\ & \leq \frac{1}{4} \left\| \Phi^{\frac{p}{2}} \left(A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) + M^{\frac{p}{2}}m^{\frac{p}{2}} \left(\Phi \left(A \right) \sharp_{\nu} \Phi \left(B \right) \right)^{\frac{-p}{2}} \right\|^{2} \\ & \leq \frac{1}{4} \left\| \left(\Phi \left(A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) + Mm \left(\Phi \left(A \right) \sharp_{\nu} \Phi \left(B \right) \right)^{-1} \right\|^{\frac{p}{2}} \\ & = \frac{1}{4} \left\| \Phi \left(A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) + Mm \left(\Phi \left(A \right) \sharp_{\nu} \Phi \left(B \right) \right)^{-1} \right\|^{p} \\ & \leq \frac{1}{4} \left\| \Phi \left(A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) + Mm \Phi^{-1} \left(A \sharp_{\nu} B \right) \right\|^{p} \\ & \leq \frac{1}{4} \left\| \Phi \left(A \nabla_{\nu} B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \right) + Mm \Phi^{-1} \left(A \sharp_{\nu} B \right) \right\|^{p} \\ & \leq \frac{1}{4} (M + m)^{p}. \end{split}$$

Thus we get the inequality (2.3) for p > 2 and this completes the proof of the theorem.

Remark 2.8. Let $0 < m \le A, B \le M$, Φ be a positive unital linear map. If 0 , then, obviously,

$$\Phi^{p}(A\nabla_{\nu}B) \le \left(\Phi(A\nabla_{\nu}B) + 2rMm\Phi\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right)^{p}.$$
 (2.7)

Hence the inequality (2.7) shows that Theorem 2.7 is a refinement of inequalities (1.5) and (1.6) for 0 .

We also have

$$\Phi^{p}(A\nabla_{\nu}B) \leq \Phi^{p}(A\nabla_{\nu}B) + (2rMm)^{p}\Phi^{p}(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}),$$

where $p \ge 1$, $\nu \in [0, 1]$ and $r = \min\{\nu, 1 - \nu\}$.

Hence

$$\|\Phi^{p}(A\nabla_{\nu}B)\| \leq \|\Phi^{p}(A\nabla_{\nu}B) + (2rMm)^{p}\Phi^{p}(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\|$$

$$\leq \|\Phi^{p}(A\nabla_{\nu}B + 2rMmA^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\| \quad \text{(by Lemma 2.2)}.$$

Therefore, Theorem 2.7 is a refinement of the inequalities, (1.7) and (1.8) for the operator norm and $p \ge 2$.

The following examples show that inequality (2.2) is a refinement of (1.5) and (1.7).

Example 2.9. If
$$A = \begin{pmatrix} 1.75 & 0.433 \\ 0.433 & 1.25 \end{pmatrix}$$
, $B = \begin{pmatrix} 2.5 & 0.5 \\ 0.5 & 2.5 \end{pmatrix}$, $\Phi(X) = \frac{1}{2} \operatorname{tr}(X)$ $(X \in \mathbb{M}_2)$, $m = 1$, $M = 3$, $\nu = \frac{1}{2}$ and $p = 3$, then $A\nabla_{\nu}B = \begin{pmatrix} 2.1250 & 0.4665 \\ 0.4665 & 1.8750 \end{pmatrix}$ and $A\nabla_{\nu}B + 2rMm \left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right) = \begin{pmatrix} 2.1601 & 0.4260 \\ 0.4260 & 2.0016 \end{pmatrix}$. Hence $\Phi^3 \left(A\nabla_{\nu}B + 2rMm \left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) - \Phi^3 \left(A\nabla_{\nu}B\right) = 9.0095 - 8 = 1.0095 > 0$.

Example 2.10. Let
$$\Phi(X) = T^*XT$$
 $(X \in \mathbb{M}_2)$, where $T = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$. If $A = \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 4.75 & 0.433 \\ 0.433 & 4.25 \end{pmatrix}$, $m = 3$, $M = 7$, $\nu = \frac{1}{2}$ and $p = \frac{5}{3}$, then $A\nabla_{\nu}B = \begin{pmatrix} 4.8750 & -0.7835 \\ -0.7835 & 4.6250 \end{pmatrix}$ and $A\nabla_{\nu}B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}) = \begin{pmatrix} 5.0283 & -0.7730 \\ -0.7730 & 4.7909 \end{pmatrix}$. Hence

$$\Phi^{\frac{5}{3}}\left(A\nabla_{\nu}B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) - \Phi^{\frac{5}{3}}(A\nabla_{\nu}B) = \begin{pmatrix} 0.7838 & -1.0172 \\ -1.0172 & 0.7199 \end{pmatrix} > 0.$$

Corollary 2.11. Let $0 < m \le A, B \le M$ and Φ be a positive unital linear map. Then

$$\Phi^p\left(\frac{A+B}{2} + Mm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right) \le \alpha^p \Phi^p\left(A\sharp B\right)$$

and

$$\Phi^{p}\left(\frac{A+B}{2} + Mm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right) \leq \alpha^{p} \left(\Phi(A) \sharp \Phi(B)\right)^{p}.$$

Proof. Take $r = \nu = \frac{1}{2}$ in Theorem 2.7.

If the positive unital linear map $\Phi(A) = A$ ($A \in \mathbb{B}(\mathcal{H})$), then we get from Theorem 2.7 the following reverse AM-GM inequalities, which improve the reversed AM-GM inequality (1.1).

Corollary 2.12. Let $0 < m \le A, B \le M$. Then, the inequalities

$$\left(\frac{A+B}{2} + Mm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right)^p \le \left(\frac{(M+m)^2}{4Mm}\right)^p (A\sharp B)^p \quad (0$$

and

$$\left(\frac{A+B}{2} + Mm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right)^p \le \left(\frac{(M+m)^2}{4^{2/p}Mm}\right)^p (A\sharp B)^p \quad (p > 2).$$

hold.

The operator Pólya–Szegö inequality states that

$$\Phi(A)\sharp\Phi(B) \le \frac{M+m}{2\sqrt{mM}}\Phi(A\sharp B). \tag{2.8}$$

where $0 < m_1^2 \le A \le M_1^2$, $0 < m_2^2 \le B \le M_2^2$, $m = \frac{m_2}{M_1}$ and $M = \frac{M_1}{m_2}$. Also the operator Kantorovich inequality says that

$$\Phi(A)\sharp\Phi(A^{-1}) \le \frac{M^2 + m^2}{2mM},\tag{2.9}$$

where $0 < m_1^2 \le A \le M_1^2$, $0 < m_2^2 \le B \le M_2^2$, $m = \frac{m_2}{M_1}$, $M = \frac{M_1}{m_2}$; see [6]. In the following result we show some refinements of (2.8) and (2.9).

Theorem 2.13. Let Φ be a unital positive linear map, $0 < m_1^2 \le A \le M_1^2$, $0 < m_2^2 \le B \le M_2^2$, $m = \frac{m_2}{M_1}$, $M = \frac{M_1}{m_2}$.

$$\Phi(A)\sharp\Phi(B) + \frac{1}{2}\left(\sqrt{Mm}\Phi(A) + \frac{1}{\sqrt{Mm}}\Phi(B) - 2\left(\Phi(A)\sharp\Phi(B)\right)\right) \le \frac{M+m}{2\sqrt{mM}}\Phi(A\sharp B). \tag{2.10}$$

In particular, if $B = A^{-1}$, then

$$\Phi(A)\sharp\Phi(A^{-1}) + \frac{1}{2}\left(Mm\Phi(A) + \frac{1}{Mm}\Phi(A^{-1}) - 2\left(\Phi(A)\sharp\Phi(A^{-1})\right)\right) \le \frac{M^2 + m^2}{2mM}.$$

Proof. If $0 < m_1^2 \le A \le M_1^2$ and $0 < m_2^2 \le B \le M_2^2$, then

$$m^2 = \frac{m_2^2}{M_1^2} \le A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \le \frac{M_1^2}{m_2^2} = M^2,$$

whence

$$\left(M - \left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\frac{1}{2}}\right) \left(\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\frac{1}{2}} - m\right) \ge 0.$$

Hence

$$MmA + B \le (M+m)A\sharp B,$$

whence

$$Mm\Phi(A) + \Phi(B) \le (M+m)\Phi(A\sharp B). \tag{2.11}$$

Using lemma 2.1 for the operators $Mm\Phi(A)$, $\Phi(B)$ and $\nu=\frac{1}{2}$ we get

$$\sqrt{Mm}\left(\Phi(A)\sharp\Phi(B)\right) + \frac{1}{2}\left(Mm\Phi(A) + \Phi(B) - 2\sqrt{Mm}\left(\Phi(A)\sharp\Phi(B)\right)\right) \le \frac{1}{2}\left(Mm\Phi(A) + \Phi(B)\right). \tag{2.12}$$

Applying inequalities (2.11) and (2.12) we get the first inequality. In particular, if we consider $m_1^2 = m^2 \le A \le M^2 = M_1^2$, then by putting $m_2^2 = \frac{1}{M^2} \le A^{-1} \le \frac{1}{m^2} = M_2^2$ in (2.10) we reach the desired inequality.

If we take Φ in (2.10) to be the positive linear map defined on the diagonal blocks of operators by $\Phi(\operatorname{diag}(A_1, \dots, A_n)) = \frac{1}{n} \sum_{j=1}^n A_j$, then we get the following refinements of a reversed Cauchy-Schwarz operator inequality.

Corollary 2.14. Let $0 < m_1^2 \le A_j \le M_1^2$, $0 < m_2^2 \le B_j \le M_2^2$ $(1 \le j \le n)$, $m = \frac{m_2}{M_1}$, $M = \frac{M_1}{m_2}$. Then

$$\left(\sum_{j=1}^{n} A_j \sharp \sum_{j=1}^{n} B_j\right) + \frac{1}{2} \left(\sqrt{Mm} \sum_{j=1}^{n} A_j \frac{1}{\sqrt{Mm}} \sum_{j=1}^{n} B_j - 2 \left(\sum_{j=1}^{n} A_j \sharp \sum_{j=1}^{n} B_j\right)\right)$$

$$\leq \frac{M+m}{2\sqrt{mM}} \left(\sum_{j=1}^{n} A_j \sharp B_j\right).$$

Proposition 2.15. Let $0 < m \le A \le M$ and $x \in \mathcal{H}$. Then

$$\langle Ax, x \rangle^{\frac{1}{2}} \langle A^{-1}x, x \rangle^{\frac{1}{2}} + \frac{1}{2} \left(\sqrt[4]{Mm} \langle Ax, x \rangle^{\frac{1}{2}} - \frac{1}{\sqrt[4]{Mm}} \langle A^{-1}x, x \rangle^{\frac{1}{2}} \right)^2 \leq \frac{M+m}{2\sqrt{Mm}} \langle x, x \rangle^2.$$

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