NON-EMPTINESS OF BRILL-NOETHER LOCI OVER VERY GENERAL QUINTIC HYPERSURFACE

KRISHANU DAN AND SARBESWAR PAL

ABSTRACT. In this article we study Brill-Noether loci of moduli space of stable bundles over smooth surfaces. We define Petri map as an analogy with the case of curves. We show the non-emptiness of certain Brill-Noether loci over very general quintic hypersurface in $\mathbb{P}^3$, and use the Petri map to produce components of expected dimension.

1. INTRODUCTION

Let X be a smooth, irreducible, projective variety of dimension n over $\mathbb{C}$, H be an ample divisor on X, and let $\mathcal{M} := \mathcal{M}_{X,H}(r; c_1, \cdots, c_s)$ be the moduli space of rank r, H-stable vector bundles E over X with Chern classes $c_i(E) = c_i$, where $s := \min\{r, n\}$. A Brill-Noether locus $B^k_{r,X,H}$ is a closed subscheme of $\mathcal{M}$ whose support consists of points $E \in \mathcal{M}$ such that $h^0(X, E) \geq k + 1$. Göttsche et al ([6]) and M. He ([7]) studied the Brill-Noether loci of stable bundles over $\mathbb{P}^2$, and Yoshioka ([15], [16]), Markman ([4]), Leyenson ([10], [11]) studied it for $K3$ surfaces. In the case of smooth, projective, irreducible curves C over $\mathbb{C}$, the Brill-Noether loci of the moduli space, $\mathrm{Pic}^d(C)$, of degree d line bundles on C is well-studied. The questions like non-emptiness, connectedness, irreducibility, singular locus etc of Brill-Noether loci are known when C is a general curve in the sense of moduli (see e.g. [1]). This concept was generalized for vector bundles over curves by Newstead, Teixidor and others. For an account of the results and history in this case see [5] and the references therein.

Recently, in [2], authors have constructed a Brill-Noether loci over higher dimensional varieties under the additional cohomology vanishing assumptions: $H^i(X, E) = 0, \forall i \geq 2$ and for all $E \in \mathcal{M}$. This is a natural generalization of the Brill-Noether loci over the curves for higher dimensional varieties. In [2], [3], authors gave several examples of non-empty Brill-Noether locus, and examples of Brill-Noether locus where "expected dimension" is not same as the exact dimension. In all these examples, the surfaces and/or varieties chosen have the canonical bundle has no non-zero sections. In this article, we defined "Petri map" over a smooth projective variety with canonical bundle ample, as an analogue of that for curves. Similar to the case of curves, the injectivity of the "Petri map" implies the existence of smooth

points in Brill-Noether loci. We then use this fact to prove the existence of a smooth point and hence a component of expected dimension in the Brill-Noether loci over a very general quintic hypersurface in $\mathbb{P}^3$ where the canonical bundle is ample and globally generated.

Notation: We work throughout over the field $\mathbb{C}$ of complex numbers. If X is a smooth, projective variety, we denote by K_X the canonical bundle on X. For a coherent sheaf $\mathcal{F}$ on X, we denote by $H^i(X, \mathcal{F})$ the i-th cohomology group of $\mathcal{F}$ and by $h^i(X, \mathcal{F})$ its (complex) dimension. If V is a vector bundle on X, we denote by V^* the dual of V.

2. BRILL-NOETHER LOCI

In this section we will briefly recall the construction of Brill-Noether loci over higher dimensional varieties, following [2].

Let X be an irreducible, smooth, projective variety of dimension n, and let H be an ample divisor on X. For a torsion free sheaf F over X, let $c_i(F)$ denotes the i-th Chern class of F. Set $\mu(F) = \mu_H(F) := \frac{c_1(F) \cdot H^{n-1}}{\mathrm{rank}(F)}$.

Definition 2.1. A torsion-free sheaf F over X of rank r is called H-*semistable* if for all non-zero subsheaf G of F with $\mathrm{rank}(G) < \mathrm{rank}(F)$, we have

$$\mu(G) \leq \mu(F).$$

We say F is H-*stable* if the above inequality is strict.

Let $\mathcal{M} := \mathcal{M}_{X,H}(r; c_1, \cdots, c_s)$ be the moduli space of rank r, H-stable vector bundles E over X with Chern classes $c_i(E) = c_i$, where $s := \min\{r, n\}$. Assume that $\mathcal{M}$ is a fine moduli space, and let $\mathcal{E} \to \mathcal{M} \times X$ be an universal family such that for any $t \in \mathcal{M}, E_t := \mathcal{E}|_{t \times X}$ is a rank r, H-stable bundle over X with Chern classes $c_i(E_t) = c_i$. Choose an effective divisor D on X such that $H^i(X, E_t(D)) = 0, \forall i \geq 1$ and $\forall t \in M$. Let $\mathcal{D} := \mathcal{M} \times D$ be the product divisor, and $\phi : \mathcal{M} \times X \to \mathcal{M}$ be the projection map. From the exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}(\mathcal{D}) \to \mathcal{E}(\mathcal{D})/\mathcal{E} \to 0$$

on $\mathcal{M} \times X$, we get an exact sequence on $\mathcal{M}$:

$$0 \to \phi_*(\mathcal{E}) \to \phi_*(\mathcal{E}(\mathcal{D})) \xrightarrow{\gamma} \phi_*(\mathcal{E}(\mathcal{D})/\mathcal{E}) \to R^1\phi_*(\mathcal{E}) \to 0.$$

Note that, γ is a map between two locally free sheaves of ranks $\chi(E_t(D))$ and $\chi(E_t(D)) - \chi(E_t)$ respectively, on $\mathcal{M}$. For an integer $k \geq -1$, let $B^k_{r,X,H} \subset \mathcal{M}$ be the $(\chi(E_t(D)) - (k+1))$-th determinantal variety associated to the map γ. Now assume $H^i(X, E_t) = 0, \forall i \geq 2$ and $\forall t \in \mathcal{M}$. Then we have

$$\mathrm{Support}(B^k_{r,X,H}) = \{E \in \mathcal{M} : h^0(X, E) \geq k + 1\}.$$

When $\mathcal{M}$ is not a fine moduli space, it is possible to carry out this construction locally and then can be glued together to get a global algebraic object. We summarize the above construction as

Theorem 2.2. ([2], *Theorem 2.3) Let X be a smooth, irreducible, projective variety of dimension n, H be a fixed ample divisor on X, and $\mathcal{M} := \mathcal{M}_{X,H}(r; c_1, \cdots, c_s)$ be a moduli space of rank r, H-stable vector bundles E on X with fixed Chern classes $c_i(E) = c_i, s = min\{r, n\}$. Assume that for any $E \in \mathcal{M}, H^i(X, E) = 0$ for $i \geq 2$. Then for any $k \geq -1$, there exists a determinantal variety $B_{r,X,H}^k \subset \mathcal{M}$ such that*

$$Support(B_{r,X,H}^k) = \{E \in \mathcal{M} \ : \ h^0(X, E) \geq k + 1\}.$$

Moreover, each non-empty irreducible component of $B_{r,X,H}^k$ has dimension at least

$$dim(\mathcal{M}) - (k + 1)(k + 1 - \chi(r; c_1, \cdots, c_s))$$

where $\chi(r; c_1, \cdots, c_s) := \chi(E_t)$ for any $t \in \mathcal{M}$ and

$$B_{r,X,H}^{k+1} \subset Sing(B_{r,X,H}^k)$$

whenever $B_{r,X,H}^k \neq \mathcal{M}$.

Definition 2.3. The variety $B_{r,X,H}^k$ is called the *k-th Brill-Noether locus* of the moduli space $\mathcal{M}$ and the number

$$\rho_{r,X,H}^k := dim(\mathcal{M}) - (k + 1)(k + 1 - \chi(r; c_1, \cdots, c_s))$$

is called the *generalized Brill-Noether number.*

By the above theorem, the dimension of $B_{r,X,H}^k$ is at least $\rho_{r,X,H}^k$. We call $\rho_{r,X,H}^k$, the *expected dimension* of the Brill-Noether locus $B_{r,X,H}^k$. When there is no confusion about X and H, we will simply denote these by B_r^k and ρ_r^k.

3. Petri Map

In this section we will define "Petri Map" for higher dimensional varieties, as an analogue of the one defined for curves. We note that the description of Petri map over curves as given in [5] works for higher dimensional varieties also. For convenience, we recall this description.

Let X be an irreducible, smooth, projective variety of dimension n, H be an ample divisor on X, and let $\mathcal{M} := \mathcal{M}_{X,H}(r; c_1, \cdots, c_s)$ be the moduli space of rank r, H-stable vector bundles E over X with Chern classes $c_i(E) = c_i$, where $s := \min\{r, n\}$. The tangent space to $\mathcal{M}$ at a point $E \in \mathcal{M}$ is given by $H^1(X, E \otimes E^*)$, where E^* is the dual of E. A tangent vector to $\mathcal{M}$ at E can be identified with a vector bundle E_ϵ on $X_\epsilon := X \times \operatorname{Spec}(k[\epsilon]/\epsilon^2)$ whose restriction on X is E and it fits into the exact sequence

$$0 \to E \to E_\epsilon \to E \to 0.$$

We call it a first order deformation of E.

One can give an explicit description of the bundle E_ϵ as follows: Let $\{U_i\}$ be an open cover of X such that $E_i := E|_{U_i}$ is the trivial bundle. Set $U_{ij} := U_i \cap U_j$, and let $\phi_{ij} \in H^0(U_{ij}, E \otimes E^*)$ be the co-boundary map

corresponding to $\phi \in H^1(X, E \otimes E^*)$. Consider the trivial extension of E_i to $U_i \times \mathrm{Spec}(k[\epsilon]/\epsilon^2)$ given by $E_i \oplus \epsilon E_i$. Then the matrix

$$\begin{bmatrix} \mathrm{Id} & 0 \\ \phi_{ij} & \mathrm{Id} \end{bmatrix}$$

will give the gluing data for the bundle E_ϵ.

Assume that a section s of E can be extended to a section of E_ϵ. Then we have local sections $(s_i') \in H^0(U_i, E_i)$ such that $(s|_{U_i}, s_i')$ defines a section of E_ϵ. If this is the case, then we have

$$\begin{bmatrix} \mathrm{Id} & 0 \\ \phi_{ij} & \mathrm{Id} \end{bmatrix} \begin{bmatrix} s|_{U_i} \\ s_i' \end{bmatrix} = \begin{bmatrix} s|_{U_j} \\ s_j' \end{bmatrix}.$$

This gives two conditions: $(s|_{U_i})|_{U_{ij}} = (s|_{U_j})|_{U_{ij}}$ and $\phi_{ij}(s) = s_j' - s_i'$. The first condition is automatically satisfied, since s is a global section and from the second condition we see that, in this case, $(\phi_{ij}(s))$ satisfies the co-cycle condition. In other words, $(\phi_{ij}(s))$ is in the kernel of the map

$$H^1(X, E \otimes E^*) \longrightarrow H^1(X, E), \quad (\nu_{ij}) \mapsto (\nu_{ij}(s)).$$

Let $E \in B_r^k, k \geq 0$ and let $s \in H^0(X, E)$. Then the first order deformation of E, as an element of B_r^k, is the subset $\{E_\epsilon :$ the section s can be extended to a section of $E_\epsilon\}$ of $H^1(X, E \otimes E^*)$, i.e.

$$E_\epsilon \in \mathrm{Ker}\big(H^1(X, E \otimes E^*) \longrightarrow H^1(X, E), \quad (\nu_{ij}) \mapsto (\nu_{ij}(s))\big).$$

Now assume $E \in B_r^k - B_r^{k+1}$ and let T be the tangent space to B_r^k at the point E. From the discussion above, we have a map

$$\alpha : H^0(X, E) \otimes H^1(X, E \otimes E^*) \longrightarrow H^1(X, E).$$

This induces the map

$$\mu : H^0(X, E) \otimes H^{n-1}(X, K_X \otimes E^*) \longrightarrow H^{n-1}(X, K_X \otimes E \otimes E^*).$$

Note that, T can be identified with $(\mathrm{Im}(\mu))^\perp$. We call the map μ, the *Petri map*.

Remark 3.1. Let X be an irreducible, smooth, projective surface and H be an ample divisor on X. In this case, Petri map, as defined above, is the cup product map

(1) $\mu : H^0(X, E) \otimes H^1(X, K_X \otimes E^*) \longrightarrow H^1(X, K_X \otimes E \otimes E^*).$

If E is a smooth point in the moduli space $\mathcal{M}$, we have

$$\dim(T) = \dim(\mathcal{M}) - h^0(X, E)h^1(X, E) + \dim \mathrm{Ker}(\mu).$$

Thus, if the Petri map is injective, then E is a smooth point of B_r^k and and the component of B_r^k through E has the expected dimension.

Remark 3.2. The Petri map can also be derived from [7]. Indeed, by taking $\Lambda = \Lambda' = (H^0(X, E), Id, E)$ in [7, Corollary 1.6], we see that the above Petri map is dual of the map $Ext^1(E, E) \to Hom(H^0(X, E), H^1(X, E))$ in the given exact sequence.

4. Brill-Noether loci over quintic hypersurface

Let X be a very general quintic hypersurface in $\mathbb{P}^3$. Then we have $\text{Pic}(X) \simeq \text{Pic}(\mathbb{P}^3) \simeq \mathbb{Z}$, $K_X \simeq \mathcal{O}_X(1)$, $\chi(X, \mathcal{O}_X) = 5$. Let H be a hyperplane class, and $\mathcal{M}(c_2) := \mathcal{M}_{X,H}(2; 3H, c_2)$ be the moduli space of rank two H-stable bundles on X with first Chern class $3H$, and second Chern class c_2. It is known ([13]) that $\mathcal{M}(c_2)$ is irreducible for $c_2 \geq 14$, generically smooth for $c_2 \geq 21$, and has the expected dimension $4c_2 - 60$. Also note that $H^2(X, E) = 0, \forall E \in \mathcal{M}$. Thus the hypothesis of the Theorem 2.2 is satisfied.

Let C be a smooth, irreducible, projective curve in the complete linear system $|3H|$. Then the genus of C, $g := 31$.

Proposition 4.1. *With the notations as above, assume that there is a base point free line bundle of degree ≥ 16 on C with exactly two sections, then $B_2^1 \subset \mathcal{M}(c_2)$ is non-empty.*

Proof. Let A be a base point free line bundle on C with $h^0(C, A) = 2$. Consider the elementary transformation

$$(2) \qquad 0 \to F \to H^0(C, A) \otimes \mathcal{O}_X \to A \to 0.$$

Then F is a rank two vector bundle on X with $c_1(F) = -3H$, $c_2(F) = \deg(A)$, $h^0(X, F) = 0 = h^1(X, F)$ ([8, Chapter 5, Proposition 5.2.2]). Dualizing the above exact sequence we get

$$(3) \qquad 0 \to H^0(C, A)^* \otimes \mathcal{O}_X \to F^* \to \mathcal{O}_C(C) \otimes A^* \to 0.$$

Thus $h^0(X, F^*) = 2 + h^0(C, \mathcal{O}_C(C) \otimes A^*) \geq 2$.
Claim: $F^* \in B_2^1$.
It is sufficient to show that F is H-stable. Let $\mathcal{O}_X(m)$ destabilizes F. Then $m \geq -1$. On the other hand, from (2), we have $m \leq 0$. Since $h^0(X, F) = 0$, $m \neq 0$. Thus we are reduced to show that $h^0(X, F \otimes \mathcal{O}_X(1)) = 0$. Note that $F \otimes \mathcal{O}_X(1) \simeq F^* \otimes \mathcal{O}_X(-2)$. Tensoring the exact sequence (3) by $\mathcal{O}_X(-2)$ we get

$$0 \to \mathcal{O}_X(-2)^{\oplus 2} \to F^* \otimes \mathcal{O}_X(-2) \to \mathcal{O}_C(H) \otimes A^* \to 0.$$

Since $\deg(\mathcal{O}_C(H) \otimes A^*) < 0$, $h^0(C, \mathcal{O}_C(H) \otimes A^*) = 0$ and consequently $h^0(X, F^* \otimes \mathcal{O}_X(-2)) = 0$. $\qquad \square$

Since $C \in |3H|$, C is a smooth complete intersection of two smooth hypersurfaces of $\mathbb{P}^3$ of degrees 5 and 3. Thus by [1, Page 139, C-4], C does not have a g_5^1. In particular, C is not hyperelliptic, trigonal or tetragonal (i.e. C does not have a g_2^1, g_3^1, g_4^1 respectively). Also note that $K_C = \mathcal{O}_C(4)$. Now for $D = \mathcal{O}_C(3)$, $K_C \otimes \mathcal{O}_C(-D) = \mathcal{O}_C(1)$, and this gives an embedding of $C \hookrightarrow \mathbb{P}^3$. So by [1, Page 221, B-4], C is not a bi-elliptic.

Proposition 4.2. *With the notations as above, for $28 \leq d \leq 32$, there exists a base point free line bundle of degree d on C.*

Proof. Let us denote by $W_d^r(C)$ the Brill-Noether loci of degree d line bundles L on C with $h^0(C, L) \geq r + 1$.

Case I: $d = 32$.

In this case, $\dim W_{32}^1(C) = 31$. If $W_{32}^1(C) - W_{32}^2(C)(\neq \varnothing)$ does not contain any base point free line bundle, then tensoring by the ideal sheaf of the base locus, we obtain a family of base-point free line bundles with exactly two sections of dimension 31 and is contained in $\bigcup_{e \leq 31} W_e^1(C)$. Now $\dim W_e^0(C) = e \leq 31 = g$, and $W_e^1(C)$ is a proper closed subset of $W_e^0(C)$. So $\dim W_e^1(C) \leq e - 1 \leq 30$. Thus $\dim(\bigcup_{e \leq 31} W_e^1(C)) < 31$, a contradiction.

Case II: $d = 31$.

In this case, $\dim W_{31}^1(C) \geq 29$. Applying Martens' theorem [1, Chapter IV, Theorem 5.1], we see that $\dim(\bigcup_{2 \leq e \leq 30} W_e^1(C)) < 29$. Note that, for any $L \in \mathrm{Pic}(C)$ with $h^0(C, L) = 2$, if B is the base locus of $|L|$, then $\deg(B) \leq \deg(L) - 2$. Now arguing as above, we get a base point free line bundle on C of degree 31 with exactly two sections.

Case III: $d = 30$.

This follows from the existence of base point free complete g_{30}^1 on C [9, Theorem 1.4].

Case IV: $d = 29$.

We have $\dim W_{29}^1(C) \geq 25$. By Mumford's theorem [1, Chapter IV, Theorem 5.2], we get $\dim(\bigcup_{2 \leq e \leq 28} W_e^1(C)) < 25$. Now we argue as in Case I to conclude.

Case V: $d = 28$.

Here $\dim W_{28}^1(C) \geq 23$. Then using Keem's theorem [1, Page 200], we get $\dim(\bigcup_{e \leq 27} W_e^1(C)) < 23$. Arguing as in Case I, we conclude that there exists a base point free line bundle of degree 28 on C with exactly two global sections. $\square$

Now we are ready to prove

Theorem 4.3. *With the notations as in the beginning of this section, $B_2^1 - B_2^2 \subset \mathcal{M}(c_2)$ is non-empty for $28 \leq c_2 \leq 32$.*

Proof. Combining Propositions 4.1 and 4.2, we see that, for $28 \leq c_2 \leq 32$, $B_2^1 \subset \mathcal{M}(c_2)$ is non-empty. Moreover, for $28 \leq d \leq 32$, we can find base point free line bundles A on C with $h^0(C, A) = 2$ and $h^0(C, \mathcal{O}_C(C) \otimes A^*) = 0$. Now using (3), we conclude. $\square$

Proposition 4.4. *With the notations as in the beginning of this section, let $E \in B_2^1 - B_2^2 \subset \mathcal{M}(c_2), 28 \leq c_2 \leq 32$ be a general element constructed as in Proposition 4.1. Then E is a smooth point in $\mathcal{M}(c_2)$.*

Proof. Since the morphism $\mathcal{M}_{X,H}(2; 3H, c_2) \xrightarrow{\otimes \mathcal{O}_X(-1)} \mathcal{M}_{X,H}(2; H, c_2 - 10)$ is an isomorphism, it is sufficient to prove that $E \otimes \mathcal{O}_X(-1)$ is a smooth point of $\mathcal{M}_{X,H}(2; H, c_2 - 10)$. Set $F := E \otimes \mathcal{O}_X(-1)$. If F is not a smooth point of $\mathcal{M}_{X,H}(2; H, c_2 - 10)$, we have $H^0(X, AdF \otimes K_X) \neq 0$. Thus there is a non-zero map $\phi : F \to F \otimes K_X$. Two cases can occur:

Case I: ϕ drops rank every where.

Case II: ϕ does not drop rank every where.

Following [12], we call the Case I as singularity of first kind, and the Case II as singularity of second kind. If Case I occurs, we will have $h^0(X, F) \neq 0$ (see [12, Proposition 5.1]), which is impossible. Also, from [13, Lemma 10.1], we see that the dimension of the singular locus of the second kind is at most 8 [1]. But the dimension of the family of globally generated line bundles A on C with $28 \leq \deg(A) \leq 32, h^0(C, A) = 2$ and $h^0(C, \mathcal{O}_C(C) \otimes A^*) = 0$ is greater than 8. Hence a general E, as constructed in Proposition 4.1, will be a smooth point in $\mathcal{M}_{X,H}(2; 3H, c_2)$. $\qquad\square$

Let $E \in B_2^1 - B_2^2$ be a smooth point in $\mathcal{M}(c_2), 28 \leq c_2 \leq 32$, as constructed in Proposition 4.1. Existence of such a smooth point is assured by Proposition 4.4. Then E fits into the following exact sequence

$$(4) \qquad\qquad 0 \to \mathcal{O}_X^{\oplus 2} \to E \to \mathcal{O}_C(C) \otimes A^* \to 0$$

(see equation (3)). Tensoring this exact sequence by $E^* \otimes K_X$ we get

$$(5) \quad 0 \to (E^* \otimes K_X)^{\oplus 2} \to \mathcal{E}nd(E) \otimes K_X \to \mathcal{O}_C(C) \otimes A^* \otimes (E^* \otimes K_X)|_C \to 0.$$

Note that $h^0(X, E^* \otimes K_X) = h^2(X, E) = 0$, by our assumption, and since E is a smooth point in the moduli space, $H^0(X, \mathcal{E}nd(E) \otimes K_X) \simeq H^0(X, K_X)$. Thus, by taking cohomology long exact sequence corresponding to the exact sequence (5), we get

$$(6) \quad \begin{aligned} &0 \to H^0(X, K_X) \to H^0(C, \mathcal{O}_C(C) \otimes A^* \otimes (E^* \otimes K_X)|_C) \\ &\qquad \to H^1(X, (E^* \otimes K_X)^{\oplus 2}) \xrightarrow{\eta} H^1(X, \mathcal{E}nd(E) \otimes K_X) \to \dots \end{aligned}$$

Note that the Petri map μ in (1) is same as the map η above. Now we will show that the map η is injective, and this will in turn imply that the bundle E is a smooth point in B_2^1.

Dualizing the exact sequence (4) and restricting to the curve C, we obtain the exact sequence (on C)

$$0 \to \mathcal{O}_C(-C) \otimes A \to E^*|_C \to \mathcal{O}_C^{\oplus 2} \to A \to 0$$

which induces the following short exact sequence (on C)

$$(7) \qquad\qquad 0 \to \mathcal{O}_C(-C) \otimes A \to E^*|_C \to A^* \to 0.$$

Tensoring the sequence (7) by $\mathcal{O}_C(C) \otimes A^* \otimes K_X|_C$ and taking cohomology long exact sequence we get

$$\begin{aligned} &0 \to H^0(C, K_X|_C) \to H^0(C, \mathcal{O}_C(C) \otimes A^* \otimes (E^* \otimes K_X)|_C) \\ &\qquad \to H^0(C, \mathcal{O}_C(C) \otimes A^{*\otimes 2} \otimes K_X|_C) \to \dots \end{aligned}$$

By our assumption on A, $H^0(C, \mathcal{O}_C(C) \otimes A^*) = 0$. Since degree$(A^* \otimes K_X|_C) < 0$, we have $H^0(C, \mathcal{O}_C(C) \otimes A^{*\otimes 2} \otimes K_X|_C) = 0$. Thus from the

---

[1] In [12, Corollary 5.1], the authors have estimated the bound for singularity of second kind as ≤ 13. But in [13, Lemma 10.1], the authors have improved the above bound and shown that it is ≤ 8

above exact sequence $H^0(C, K_X|_C) \simeq H^0(C, \mathcal{O}_C(C) \otimes A^* \otimes (E^* \otimes K_X)|_C)$. It is easy to see that $H^0(X, K_X) \simeq H^0(C, K_X|_C)$. Consequently, the map η in (6) is injective.

We summarize the above discussion as

Theorem 4.5. $B_2^1 \subset \mathcal{M}(c_2), 28 \leq c_2 \leq 32$, *contains a smooth point, and hence the irreducible component containing it has the expected dimension.*

5. NON-EMPTINESS OF B_2^0

Let X and $\mathcal{M}(c_2)$ be as in the previous section. Let $\mathrm{Hilb}^{c_2}(X)^{\mathrm{lci}}$ denotes the open subscheme of the Hilbert scheme $\mathrm{Hilb}^{c_2}(X)$ consisting of length c_2 subschemes of X which are locally complete intersections. Given any point $Z \in \mathrm{Hilb}^{c_2}(X)^{\mathrm{lci}}$, we have an exact sequence

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_Z \otimes \mathcal{O}_X(3) \to 0$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z and E is a rank two torsion-free sheaf on X. The space of such (isomorphic classes of) extensions is parametrized by $\mathbb{P}\mathrm{Ext}^1(\mathcal{I}_Z \otimes \mathcal{O}_X(3), \mathcal{O}_X)$. By duality, $\mathrm{Ext}^1(\mathcal{I}_Z \otimes \mathcal{O}_X(3), \mathcal{O}_X) = H^1(X, \mathcal{I}_Z \otimes \mathcal{O}_X(4))^*$. Since $h^0(X, \mathcal{O}_X(4)) = 35$, for a general element of length $c_2 \geq 36$ in $\mathrm{Hilb}^{c_2}(X)^{\mathrm{lci}}$, we have $h^0(X, \mathcal{I}_Z \otimes \mathcal{O}_X(4)) = 0$. Thus a general element in $\mathrm{Hilb}^{c_2}(X)^{\mathrm{lci}}, c_2 \geq 36$, satisfies the Cayley-Bacharach property for $\mathcal{O}_X(4)$, and consequently, we get that the corresponding extension E is locally free. Also any such vector bundle E is H-stable. Thus for $c_2 \geq 36, B_2^0 \subset \mathcal{M}(c_2)$ is non-empty.

The following two results give a bound for $\dim B_2^0$.

Lemma 5.1. *([14, Proposition 1.1][2]) With the notations as above, for $c_2 \geq 36, \dim B_2^0 \leq 3c_2 - 36$.*

Lemma 5.2. *([12, Corollary 3.1]) With the notations as above, every irreducible component of B_2^0 has dimension $\geq 3c_2 - h^0(X, \mathcal{O}_X(3) \otimes K_X) - 1$.*

Since $h^0(X, \mathcal{O}_X(3) \otimes K_X) = 35$, we see that, for $c_2 \geq 36$, every irreducible component of B_2^0 has dimension exactly $3c_2 - 36$. On the other hand, the Brill-Noether number $\rho_2^0 = 3c_2 - 36$. We summarize the above discussion as

Proposition 5.3. *With the notations as above, $B_2^0 \subset \mathcal{M}(c_2)$ is non-empty for $c_2 \geq 36$, and every irreducible component of B_2^0 has the expected dimension.*

---

[2]In [14], Proposition 1.1 was proved with the assumption that $c_1(E) = H$, and the bound author got there is $\dim B_2^0 \leq 3c_2 - 11$, for $c_2 \geq 10$. The same calculation, in this case, gives us the stated bound

References

[1] Arbarello, E.; Cornalba, M.; Griffiths, P. A.; Harris, J.: *Geometry of algebraic curves. Vol. I.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 267. Springer-Verlag, New York, 1985.

[2] Costa, Laura; Miró-Roig, Rosa Maria: *Brill-Noether theory for moduli spaces of sheaves on algebraic varieties.* Forum Math. 22 (2010), no. 3, 411-432.

[3] Costa, L.; Miró-Roig, R. M: *Brill-Noether theory on Hirzebruch surfaces.* J. Pure Appl. Algebra 214 (2010), no. 9, 1612-1622.

[4] Markman, Eyal: *Brill-Noether duality for moduli spaces of sheaves on K3 surfaces.* J. Algebraic Geom. 10 (2001), no. 4, 623-694.

[5] I. Grzegorczyk; M. Teixidor i Bigas: *Brill-Noether theory for stable vector bundles.* arXiv:0801.4740

[6] Göttsche, L; Hirschowitz, A: *Weak Brill-Noether for vector bundles on the projective plane.* Algebraic geometry (Catania, 1993/Barcelona, 1994), 63-74, Lecture Notes in Pure and Appl. Math., 200, Dekker, New York, 1998.

[7] He, Min: *Espaces de modules de systèmes cohérents.* Internat. J. Math. 9 (1998), no. 5, 545–598.

[8] Huybrechts, Daniel; Lehn, Manfred: *The geometry of moduli spaces of sheaves.* Second edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010.

[9] Keem, Changho: *Double coverings of smooth algebraic curves.* Algebraic geometry in East Asia (Kyoto, 2001), 75-111, World Sci. Publ., River Edge, NJ, 2002.

[10] Leyenson, M: *On the Brill-Noether theory for K3 surfaces.* Cent. Eur. J. Math. 10 (2012), no. 4, 1486-1540.

[11] Leyenson M.: *On the Brill-Noether theory for K3 surfaces, II.* https://arxiv.org/abs/math/0602358

[12] Mestrano, Nicole; Simpson, Carlos: *Obstructed bundles of rank two on a quintic surface.* Internat. J. Math. 22 (2011), no. 6, 789-836.

[13] N. Mestrano, C. Simpson: *Irreducibility of the moduli space of stable vector bundles of rank two and odd degree on a very general quintic surface.* arXiv. https://arxiv.org/abs/1302.3736

[14] P. Nijsse: *The irreducibility of the moduli space of stable vector bundles of rank 2 on a quintic in $\mathbb{P}^3$.* arXiv. https://arxiv.org/abs/alg-geom/9503012

[15] Yoshioka, K.: *Some examples of Mukai's reflections on K3 surfaces.* J. Reine Angew. Math. 515 (1999), 97-123.

[16] Yoshioka, K: *Brill-Noether problem for sheaves on K3 surfaces.* Proceedings of the Workshop "Algebraic Geometry and Integrable Systems related to String Theory" (Kyoto, 2000). Surikaisekikenkyusho Kokyuroku No. 1232 (2001), 109-124.

CHENNAI MATHEMATICAL INSTITUTE, H1 SIPCOT IT PARK, SIRUSERI, KELAMBAKKAM - 603103, INDIA.

E-mail address: krishanud@cmi.ac.in

IISER - THIRUVANANTHAPURAM, COMPUTER SCIENCE BUILDING, COLLEGE OF ENGINEERING TRIVANDRUM CAMPUS, TRIVANDRUM - 695016, KERALA, INDIA

E-mail address: spal@iisertvm.ac.in