FOURIER COEFFICIENTS OF $\times p$-INVARIANT MEASURES

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Abstract. We consider densities $D_\Sigma(A)$, $\overline{D}_\Sigma(A)$ and $\underline{D}_\Sigma(A)$ for a subset A of $\mathbb{N}$ with respect to a sequence Σ of finite subsets of $\mathbb{N}$ and study Fourier coefficients of ergodic, weakly mixing and strongly mixing $\times p$-invariant measures on the unit circle $\mathbb{T}$. Combining these, we prove the following measure rigidity results: on $\mathbb{T}$, the Lebesgue measure is the only non-atomic $\times p$-invariant measure satisfying one of the following: (1) μ is ergodic and there exist a Følner sequence Σ in $\mathbb{N}$ and a nonzero integer l such that μ is $\times(p^j + l)$-invariant for all j in a subset A of $\mathbb{N}$ with $D_\Sigma(A) = 1$; (2) μ is weakly mixing and there exist a Følner sequence Σ in $\mathbb{N}$ and a nonzero integer l such that μ is $\times(p^j + l)$-invariant for all j in a subset A of $\mathbb{N}$ with $\overline{D}_\Sigma(A) > 0$; (3) μ is strongly mixing and there exists a nonzero integer l such that μ is $\times(p^j + l)$-invariant for infinitely many j. Moreover, a $\times p$-invariant measure satisfying (2) or (3) is either a Dirac measure or the Lebesgue measure.

As an application we prove that for every increasing function τ defined on positive integers with $\lim_{n\to\infty} \tau(n) = \infty$, there exists a multiplicative semigroup S_τ of $\mathbb{Z}^+$ containing p such that $|S_\tau \cap [1, n]| \le (\log_p n)^{\tau(n)}$ and the Lebesgue measure is the only non-atomic ergodic $\times p$-invariant measure which is $\times q$-invariant for all q in S_τ.

1. Introduction

There are two motivations for this paper. Both are related to the celebrated $\times p, \times q$ conjecture by H. Furstenberg. The first motivation is Lyons' Theorem and Rudolph-Johnson's Theorem, and the second is a theorem due to E. A. Sataev and later independently discovered by M. Einsiedler and A. Fish.

For an integer p, consider the group homomorphism T_p (called the $\times p$ map) on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ given by $T_p(x) = px \mod \mathbb{Z}$ for all x in $\mathbb{R}/\mathbb{Z}$.

When p and q are positive integers greater than 1 with $\frac{\log p}{\log q} \notin \mathbb{Q}$, H. Furstenberg gave a classification of $\times p, \times q$-invariant closed subsets in $\mathbb{T}$ [Fur67, Thm. IV.1].

Theorem 1.1. *[Furstenberg, 1967]*

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A $\times p, \times q$-invariant closed subset in $\mathbb{T}$ is either finite or $\mathbb{T}$.

Motivated by this, H. Furstenberg conjectured a classification of $\times p, \times q$-invariant measures.

Conjecture. *[Furstenberg's $\times p, \times q$ conjecture]*

For two positive integers $p, q \geq 2$ with $\frac{\log p}{\log q} \notin \mathbb{Q}$, an ergodic $\times p, \times q$-invariant measure on $\mathbb{T}$ is either finitely supported or the Lebesgue measure. That is, the only non-atomic $\times p, \times q$-invariant measure on $\mathbb{T}$ is the Lebesgue measure.

The first progress was made by R. Lyons in 1988 [Lyo88, Thm. 1].

Theorem 1.2. *[Lyons' theorem]*

Suppose p, q are relatively prime. The Lebesgue measure is the only non-atomic $\times p, \times q$-invariant measure which is T_p-exact.

A measure μ is T_p-exact means $h_\mu(T_p, \xi) > 0$ for any nontrivial finite partition ξ of $\mathbb{T}$, where $h_\mu(T_p, \xi)$ stands for the measure entropy of T_p with respect to a finite partition ξ.

In 1990, D. Rudolph improved Lyons' theorem to the following [Rud90, Thm. 4.9].

Theorem 1.3. *[Rudolph's theorem]*

Suppose p, q are relatively prime. A $\times p, \times q$-invariant measure μ with $h_\mu(T_p) = \sup_\xi h_\mu(T_p, \xi) > 0$ must be the Lebesgue measure.

Rudolph's theorem was strengthened by A. S. A. Johnson [Joh82, Thm. A].

Theorem 1.4. *[Rudolph-Johnson's theorem]*

Suppose that $\frac{\log p}{\log q} \notin \mathbb{Q}$. Then a $\times p, \times q$-invariant measure with $h_\mu(T_p) > 0$ is the Lebesgue measure.

In this paper by assuming that a non-atomic $\times p$-invariant measure μ satisfies weaker conditions than T_p-exactness or positive entropy, we prove that if μ is invariant under enough many $\times q$-maps of special forms, then μ is the Lebesgue measure.

Theorem 5.1. *The Lebesgue measure is the only non-atomic $\times p$-invariant measure on $\mathbb{T}$ satisfying one of the following:*

(1) *it is ergodic and there exist a nonzero integer l and a Følner sequence $\Sigma = \{F_n\}_{n=1}^\infty$ in $\mathbb{N}$ such that μ is $\times(p^j + l)$-invariant for all j in some $A \subseteq \mathbb{N}$ with $D_\Sigma(A) = 1$;*

(2) *it is weakly mixing and there exist a nonzero integer l and a Følner sequence $\Sigma = \{F_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ such that μ is $\times(p^j + l)$-invariant for all j in some $A \subseteq \mathbb{N}$ with $\overline{D}_{\Sigma}(A) > 0$;*

(3) *it is strongly mixing and there exist a nonzero integer l and an infinite set $A \subseteq \mathbb{N}$ such that μ is $\times(p^j + l)$-invariant for all j in A.*

Moreover, a $\times p$-invariant measure satisfying (2) or (3) is either a Dirac measure or the Lebesgue measure.

Here $D_{\Sigma}(A)$ and $\overline{D}_{\Sigma}(A)$ are density and upper density of A with respect to Σ respectively. See Section 2 for their definitions.

The second motivation is a theorem which is independently discovered by E. A. Sataev in 1975 [Sat75, Thm. 1] and M. Einsiedler and A. Fish in 2010 [EF10, Thm. 1.2].

Theorem 1.5. *For a multiplicative semigroup S of positive integers with*

$$\liminf_{n\to\infty} \frac{\log |S \cap [1, n]|}{\log n} > 0,$$

if a Borel probability measure on $\mathbb{T}$ is an ergodic $\times p$-invariant measure for some p in S and is $\times q$-invariant for every q in S, then it is either finitely supported or Lebesgue measure.

As an application of Theorem 5.1, we prove that there exists a multiplicative semigroup S of positive integers with $\lim_{n\to\infty} \dfrac{\log |S \cap [1, n]|}{\log n} = 0$ such that Theorem 1.5 still holds (see Theorem 5.3).

The paper is organized as follows.

Firstly we give definitions of density functions of a subset A of nonnegative integers with respect to a Følner sequence. In Section 3, we lay down some basic facts about Fourier coefficients of a measure on the unit circle. In Section 4, we give the characterizations of ergodic, weakly mixing and strongly mixing $\times p$-invariant measures via their Fourier coefficients. In the last section, we prove the main theorem, Theorem 5.1. Applying it, we prove Theorem 5.3 at the end.

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2. Preliminaries

Let $\mathbb{N}$ stand for the set of nonnegative integers and $\mathbb{Z}^+$ stand for the set of positive integers. Throughout this article, for two integers $a < b$, we denote the set $\{a, \cdots, b\}$ by $[a, b]$. Denote by $|F|$ the cardinality of a set F.

The following definition of Følner sequence in $\mathbb{N}$ is a special case of Følner sequences in an amenable semigroup [Bow71, p.2].

Definition 2.1. *A sequence $\Sigma = \{F_n\}_{n=1}^{\infty}$ of finite subsets in $\mathbb{N}$ is called a* **Følner sequence** *if*

$$\lim_{n\to\infty} \frac{|(F_n + m)\Delta F_n|}{|F_n|} = 0$$

for every m in $\mathbb{N}$.

The density $D(A)$ of a subset A of $\mathbb{N}$ is given by $D(A) = \lim\limits_{n\to\infty} \frac{|A \cap [0, n-1]|}{n}$. The upper density of A, $\overline{D}(A) := \limsup\limits_{n\to\infty} \frac{|A \cap [0, n-1]|}{n}$ and the lower density of A, $\underline{D}(A) := \liminf\limits_{n\to\infty} \frac{|A \cap [0, n-1]|}{n}$. These densities are defined via the sequence of finite subsets $\{[0, n-1]\}_{n=1}^{\infty}$ in $\mathbb{N}$. Generalizing these, one can define densities of A with respect to every sequence of finite subsets of $\mathbb{N}$.

Definition 2.2. *Let $\Sigma = \{F_n\}_{n=1}^{\infty}$ be a sequence of finite subsets of $\mathbb{N}$. The density $D_\Sigma(A)$ of a subset A of $\mathbb{N}$ with respect to Σ is given by*

$$D_\Sigma(A) = \lim_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}.$$

The upper density $\overline{D}_\Sigma(A) := \limsup\limits_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}$, and the lower density $\underline{D}_\Sigma(A) := \liminf\limits_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}$.

Remark 2.3. (1) *Denote $\cup F_n$ by F. Then $D_\Sigma(A) = D_\Sigma(A \cap F)$.*

(2) *The density $D_\Sigma(A)$ depends on choices of Σ. For instance, let $A = \cup_{n=1}^{\infty}[2^n, 2^n + n]$. For the Følner sequence $\Sigma = \{F_m\}_{m=1}^{\infty}$ with $F_m = [1, m]$, one has $D_\Sigma(A) = 0$. On the other hand $D_{\Sigma'}(A) = 1$ for the Følner sequence $\Sigma' = \{[2^n, 2^n + n]\}_{n=1}^{\infty}$.*

Within this paper, a measure on a compact metrizable X always means a Borel probability measure. A measure μ is called **non-atomic** if $\mu\{x\} = 0$ for every x in X.

A topological dynamical system consists of a compact metrizable space X and a continuous map $T : X \to X$.

A measure μ on X is called T-**invariant** if $\mu(B) = \mu(T^{-1}B)$ for any Borel subset B of X. A T-invariant measure μ is called **ergodic** if every Borel subset B with $T^{-1}B = B$ satisfies that $\mu(B)^2 = \mu(B)$, it is called **weakly mixing** if $\mu \times \mu$ is an ergodic $T \times T$-invariant measure on $X \times X$, and it is called **strongly mixing** if $\lim\limits_{j\to\infty} \mu(T^{-j}A \cap B) = \mu(A)\mu(B)$ for all Borel subsets A, B in X.

It's well-known that strongly mixing$\Longrightarrow$weakly mixing$\Longrightarrow$ergodic.

Within this paper, we only consider that $X = \mathbb{T}$ and $T = T_p$ is the $\times p$ map on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ defined by $T_p(x) = px \mod \mathbb{Z}$ for all x in $\mathbb{R}/\mathbb{Z}$ and p in $\mathbb{Z}$.

3. Some basic facts about Fourier coefficients

Denote the support of μ by $\mathrm{Supp}(\mu)$. For n in $\mathbb{Z}$, the **Fourier coefficient** $\hat{\mu}(n)$ of a measure μ on $\mathbb{T}$ is given by $\hat{\mu}(n) = \int_{\mathbb{T}} z^n \, d\mu(z)$ when taking $\mathbb{T} = \{z \in \mathbb{C} || z| = 1\}$.

Lemma 3.1. *For nonzero k in $\mathbb{Z}$ and c in $\mathbb{T}$, $\int_{\mathbb{T}} |z^k - c|^2 \, d\mu(z) = 0$ if and only if* $\mathrm{Supp}(\mu) \subseteq \{z | z^k = c\}$.

Proof. Obvious. □

Proposition 3.2. *For a nonzero integer k, one has $|\hat{\mu}(k)| < 1$ if and only if there is no c in $\mathbb{T}$ such that* $\mathrm{Supp}(\mu) \subseteq \{z | z^k = c\}$.

Proof. Let k be a nonzero integer. By Lemma 3.1, it suffices to show that $|\hat{\mu}(k)| < 1$ if and only if $\int_{\mathbb{T}} |z^k - c|^2 \, d\mu(z) > 0$ for all $c \in \mathbb{T}$.

If $|\hat{\mu}(k)| < 1$, then for any $c \in \mathbb{T}$, we have

$$\int_{\mathbb{T}} |z^k - c|^2 \, d\mu(z) = \int_{\mathbb{T}} (z^k - c)(\bar{z}^k - \bar{c}) d\mu(z)$$

$$= 2 - 2Re(\bar{c}\hat{\mu}(k)) \geqslant 2 - 2|\hat{\mu}(k)| > 0.$$

Conversely assume that $\int_{\mathbb{T}} |z^k - c|^2 \, d\mu(z) > 0$ for all $c \in \mathbb{T}$. When choosing $c \in \mathbb{T}$ such that $c\hat{\mu}(k) = |\hat{\mu}(k)|$, we get $0 < \int_{\mathbb{T}} |z^k - c|^2 \, d\mu(z) = 2 - 2|\hat{\mu}(k)|$, which implies that $|\hat{\mu}(k)| < 1$. $\square$

4. Fourier coefficients of ergodic, weakly mixing or strongly mixing $\times p$-invariant measures

In this section, we give characterizations of ergodic, weakly mixing and strongly mixing $\times p$-invariant measures via their Fourier coefficients.

Theorem 4.1. *The following are true.*

(1) *A measure μ on $\mathbb{T}$ is an ergodic $\times p$-invariant measure if and only if*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} \hat{\mu}(kp^j + l) = \hat{\mu}(k)\hat{\mu}(l)$$

for every Følner sequence $\Sigma = \{F_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ and all k, l in $\mathbb{Z}$.

(2) *A measure μ on $\mathbb{T}$ is a weakly mixing $\times p$-invariant measure if and only if*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} |\hat{\mu}(kp^j + l) - \hat{\mu}(k)\hat{\mu}(l)|^2 = 0$$

for every Følner sequence $\Sigma = \{F_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ and all k, l in $\mathbb{Z}$.

(3) *A measure μ on $\mathbb{T}$ is a strongly mixing $\times p$-invariant measure if and only if*

$$\lim_{j\to\infty} \hat{\mu}(kp^j + l) = \hat{\mu}(k)\hat{\mu}(l)$$

for all k, l in $\mathbb{Z}$.

To prove (1) and (2) of Proposition 4.1, we need a preliminary result, which is a special case of von Neumann's mean ergodic theorem for amenable semigroups proved by Bowley [Bow71, Thm. 1].

Lemma 4.2. *For a topological dynamical system (X, T), if ν is an ergodic T-invariant measure on X, then for every Følner sequence $\{F_n\}_{n=1}^{\infty}$ in $\mathbb{N}$, one has*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} f(T^j x) = \int_X f \, d\nu$$

for every f in $L^2(X,\nu)$ (note that the identity holds with respect to L^2-norm). Consequently

$$\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{j\in F_n}\int_X f(T^j x)g(x)\,d\nu(x)=\int_X f\,d\nu\int_X g\,d\nu$$

for every f,g in $L^2(X,\nu)$

Proof. [Proof of Theorem 4.1]

(1) Suppose μ is an ergodic $\times p$-invariant measure on $\mathbb{T}$. Denote the $\times p$ map by T_p. Consider the measurable dynamical system $(\mathbb{T},T_p,\mu)$. Using Lemma 4.2, we get

$$\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{j\in F_n}\int_{\mathbb{T}} f(T_p^j(x))g(x)\,d\mu(x)=\int_{\mathbb{T}} f\,d\mu\int_{\mathbb{T}} g\,d\mu$$

for all continuous functions f,g on $\mathbb{T}$. By choosing $f=z^k$ and $g=z^l$, we prove the necessity.

Now assume that $\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{j\in F_n}\hat{\mu}(kp^j+l)=\hat{\mu}(k)\hat{\mu}(l)$ for every Følner sequence $\{F_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ and all k,l in $\mathbb{Z}$. Let $l=0$, we have

$$\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{j\in F_n}\hat{\mu}(kp^j)=\hat{\mu}(k)$$

for every k in $\mathbb{Z}$. Replacing k by kp, one has

$$\begin{aligned}\hat{\mu}(kp)&=\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{j\in F_n}\hat{\mu}(kp^{j+1})\\&=\lim_{n\to\infty}\frac{1}{|F_n+1|}\sum_{j\in F_n+1}\hat{\mu}(kp^j)\end{aligned}$$

($\{F_n+1\}_{n=1}^{\infty}$ is a Følner sequence in $\mathbb{N}$.)

$$=\hat{\mu}(k)$$

for every k in $\mathbb{N}$. Hence μ is $\times p$-invariant.

From $\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{j\in F_n}\hat{\mu}(kp^j+l)=\hat{\mu}(k)\hat{\mu}(l)$, we have $\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{j\in F_n}\int_{\mathbb{T}} f(T_p^j x)g(x)\,d\mu(x)=\int_{\mathbb{T}} f\int_{\mathbb{T}} g$ for all polynomials on $\mathbb{T}$. Polynomials are dense in $L^2(\mathbb{T},\mu)$, so $\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{j\in F_n}\int_{\mathbb{T}} f(T_p^j x)g(x)\,d\mu(x)=\int_{\mathbb{T}} f\,d\mu\int_{\mathbb{T}} g\,d\mu$ for all $f,g\in L^2(\mathbb{T},\mu)$. In particular, it is true for $f=g=1_A$ for a Borel subset A with $T_p^{-1}A=A$. Hence $\mu(A)=\mu(A)^2$. This proves that μ is ergodic.

(2) Suppose μ is a weakly mixing $\times p$-invariant measure on $\mathbb{T}$, which means, $\mu \times \mu$ is an ergodic $T_p \times T_p$-invariant measure on $\mathbb{T}^2$. Applying the second identity of Lemma 4.2 to $X = \mathbb{T}^2$, $\nu = \mu \times \mu$ and letting $f(z_1, z_2) = z_1^k z_2^{-k}$ and $g(z_1, z_2) = z_1^l z_2^{-l}$ for any k, l in $\mathbb{Z}$, we have

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} |\hat{\mu}(kp^j + l)|^2 = |\hat{\mu}(k)|^2 |\hat{\mu}(l)|^2.$$

Note that

$$\begin{aligned}&|\hat{\mu}(kp^j + l) - \hat{\mu}(k)\hat{\mu}(l)|^2 \\ =&|\hat{\mu}(kp^j + l)|^2 + |\hat{\mu}(k)|^2|\hat{\mu}(l)|^2 - \hat{\mu}(kp^j + l)\hat{\mu}(-k)\hat{\mu}(-l) - \hat{\mu}(-kp^j - l)\hat{\mu}(k)\hat{\mu}(l).\end{aligned}$$

So we get

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} |\hat{\mu}(kp^j + l) - \hat{\mu}(k)\hat{\mu}(l)|^2 \\ =&\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} [|\hat{\mu}(kp^j + l)|^2 + |\hat{\mu}(k)|^2|\hat{\mu}(l)|^2 - \hat{\mu}(kp^j + l)\hat{\mu}(-k)\hat{\mu}(-l) - \hat{\mu}(-kp^j - l)\hat{\mu}(k)\hat{\mu}(l)]\end{aligned}$$

(Use (1) since that μ is weakly mixing implies ergodicity of μ.)

$$=|\hat{\mu}(k)|^2|\hat{\mu}(l)|^2 + |\hat{\mu}(k)|^2|\hat{\mu}(l)|^2 - |\hat{\mu}(k)|^2|\hat{\mu}(l)|^2 - |\hat{\mu}(k)|^2|\hat{\mu}(l)|^2 = 0$$

for all k, l in $\mathbb{Z}$.

On the other hand, suppose

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} |\hat{\mu}(kp^j + l) - \hat{\mu}(k)\hat{\mu}(l)|^2 = 0$$

for every Følner sequence $\{F_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ and all k, l in $\mathbb{Z}$.

Firstly we have

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} \hat{\mu}(kp^j + l) = \hat{\mu}(k)\hat{\mu}(l).$$

So by (1) μ is an ergodic $\times p$-invariant measure. To prove $\mu \times \mu$ is an ergodic $T_p \times T_p$-invariant measure on $\mathbb{T}^2$, it suffices to prove that

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} \int_{\mathbb{T}^2} f((T_p \times T_p)^j(z_1, z_2))g(z_1, z_2)\, d\mu(z_1)\, d\mu(z_2) = \int_{\mathbb{T}^2} f\, d\mu\, d\mu \int_{\mathbb{T}^2} g\, d\mu\, d\mu$$

for all continuous functions f and g on $\mathbb{T}^2$, which is equivalent to that

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} \hat{\mu}(k_1 p^j + l_1)\hat{\mu}(k_2 p^j + l_2) = \hat{\mu}(k_1)\hat{\mu}(k_2)\hat{\mu}(l_1)\hat{\mu}(l_2)$$

for all k_1, k_2, l_1, l_2 in $\mathbb{Z}$ by letting $f = z_1^{k_1} z_2^{k_2}$ and $g = z_1^{l_1} z_2^{l_2}$ whose linear spans are dense in $C(\mathbb{T}^2)$.

Note that

$$\begin{aligned}
&|\hat{\mu}(k_1p^j + l_1)\hat{\mu}(k_2p^j + l_2) - \hat{\mu}(k_1)\hat{\mu}(k_2)\hat{\mu}(l_1)\hat{\mu}(l_2)| \\
\leq &|\hat{\mu}(k_1p^j + l_1)[\hat{\mu}(k_2p^j + l_2) - \hat{\mu}(k_2)\hat{\mu}(l_2)]| + |[\hat{\mu}(k_1p^j + l_1) - \hat{\mu}(k_1)\hat{\mu}(l_1)]\hat{\mu}(k_2)\hat{\mu}(l_2)| \\
\leq &|\hat{\mu}(k_2p^j + l_2) - \hat{\mu}(k_2)\hat{\mu}(l_2)| + |\hat{\mu}(k_1p^j + l_1) - \hat{\mu}(k_1)\hat{\mu}(l_1)|
\end{aligned}$$

for all k_1, k_2, l_1, l_2 in $\mathbb{Z}$.

Hence we have

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} |\hat{\mu}(k_1p^j + l_1)\hat{\mu}(k_2p^j + l_2) - \hat{\mu}(k_1)\hat{\mu}(k_2)\hat{\mu}(l_1)\hat{\mu}(l_2)|^2$$

$$\leq \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} [|\hat{\mu}(k_2p^j + l_2) - \hat{\mu}(k_2)\hat{\mu}(l_2)| + |\hat{\mu}(k_1p^j + l_1) - \hat{\mu}(k_1)\hat{\mu}(l_1)|]^2$$

(Cauchy-Schwarz inequality)

$$\leq 2 \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} [|\hat{\mu}(k_2p^j + l_2) - \hat{\mu}(k_2)\hat{\mu}(l_2)|^2 + |\hat{\mu}(k_1p^j + l_1) - \hat{\mu}(k_1)\hat{\mu}(l_1)|^2] = 0.$$

Using the inequality $(\frac{|x_1|+\cdots+|x_n|}{n})^2 \leq \frac{|x_1|^2+\cdots+|x_n|^2}{n}$, we have

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} |\hat{\mu}(k_1p^j + l_1)\hat{\mu}(k_2p^j + l_2) - \hat{\mu}(k_1)\hat{\mu}(k_2)\hat{\mu}(l_1)\hat{\mu}(l_2)| = 0.$$

This completes the proof.

(3) If μ is strongly mixing, then $\lim\limits_{j\to\infty} \mu(T_p^{-j}A \cap B) = \mu(A)\mu(B)$ for all Borel subsets A and B in $\mathbb{T}$. This means $\lim\limits_{j\to\infty} \int_{\mathbb{T}} 1_A(T_p^j x) 1_B(x)\, d\mu(x) = \int_{\mathbb{T}} 1_A\, d\mu \int_{\mathbb{T}} 1_B\, d\mu$ for all Borel subsets A and B, where 1_A stands for the characteristic function of A.

Note that linear combinations of characteristic functions are dense in $L^2(\mathbb{T}, \mu)$, so $\lim_{j\to\infty} f(T_p^j x) g(x)\, d\mu(x) = \int_{\mathbb{T}} f\, d\mu \int_{\mathbb{T}} g\, d\mu$ for all f, g in $C(\mathbb{T})$. In particular, this holds for $f = z^k$ and $g = z^l$ for all k, l in $\mathbb{Z}$, which means

$$\lim_{j\to\infty} \hat{\mu}(kp^j + l) = \hat{\mu}(k)\hat{\mu}(l)$$

for all $k, l \in \mathbb{Z}$.

On the other hand, if a measure μ satisfies that

$$\lim_{j\to\infty}\hat{\mu}(kp^j+l)=\hat{\mu}(k)\hat{\mu}(l)$$

for all $k,l\in\mathbb{Z}$. Let $l=0$ and replace k by kp. Then we have

$$\hat{\mu}(kp)=\lim_{j\to\infty}\hat{\mu}(kp^{j+1})=\hat{\mu}(k)$$

for all k in $\mathbb{Z}$.

Linear combinations of z^k and z^l are polynomials on $\mathbb{T}$, which is dense in $L^2(\mathbb{T},\mu)$. Hence

$$\lim_{j\to\infty}\mu(f(T_p^j)g)=\mu(f)\mu(g)$$

for all f,g in $L^2(\mathbb{T},\mu)$. In particular, it holds for $f=1_A$ and $g=1_B$ for any Borel subsets A,B of $\mathbb{T}$, which completes the proof.

□

Remark 4.3. (1) *As shown in [Lyo88], a measure μ is T_p-exact iff*

$$\lim_{j\to\infty}\sup_{k\in\mathbb{Z}}|\hat{\mu}(kp^j+l)-\hat{\mu}(k)\hat{\mu}(l)|=0$$

for every l in $\mathbb{Z}$. Hence T_p-exactness is much stronger than being strongly mixing.

(2) *So far it is unknown how to characterize that $h_\mu(T_p)>0$ via Fourier coefficients of μ.*

5. Rigidity of $\times p$-invariant measures

With the above preliminaries, we are ready to prove the main theorem.

Theorem 5.1. *The Lebesgue measure is the only non-atomic $\times p$-invariant measure on $\mathbb{T}$ satisfying one of the following:*

(1) *it is ergodic and there exist a nonzero integer l and a Følner sequence $\Sigma=\{F_n\}_{n=1}^\infty$ in $\mathbb{N}$ such that μ is $\times(p^j+l)$-invariant for all j in some $A\subseteq\mathbb{N}$ with $D_\Sigma(A)=1$;*

(2) *it is weakly mixing and there exist a nonzero integer l and a Følner sequence $\Sigma=\{F_n\}_{n=1}^\infty$ in $\mathbb{N}$ such that μ is $\times(p^j+l)$-invariant for all j in some $A\subseteq\mathbb{N}$ with $\overline{D}_\Sigma(A)>0$;*

(3) *it is strongly mixing and there exist a nonzero integer l and an infinite set $A\subseteq\mathbb{N}$ such that μ is $\times(p^j+l)$-invariant for all j in A.*

Moreover, a $\times p$-invariant measures satisfying (2) or (3) is either a Dirac measure or the Lebesgue measure.

Proof. [Proof of the first part of Theorem 5.1]

(1) Suppose μ is an ergodic $\times p$-invariant measure and there exist a nonzero integer l and a Følner sequence $\Sigma = \{F_n\}_{n=1}^{\infty}$ such that μ is $\times(p^j + l)$-invariant for all j in some $A \subseteq \mathbb{N}$ with $D_\Sigma(A) = 1$.

If μ is not Lebesgue measure, then there exists nonzero k in $\mathbb{Z}$ such that $0\hat{\mu}(k)$ is nonzero.

By Theorem 4.1(1), one has

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} \hat{\mu}(kp^j + kl) = \hat{\mu}(k)\hat{\mu}(kl).$$

Note that

$$\frac{1}{|F_n|} \sum_{j\in F_n} \hat{\mu}(kp^j + kl) = \frac{1}{|F_n|}[\sum_{j\in F_n\cap A} + \sum_{j\in F_n\setminus A}]\hat{\mu}(kp^j + kl)$$

$$=\frac{|F_n\cap A|}{|F_n|}\hat{\mu}(k) + \frac{1}{|F_n|}\sum_{j\in F_n\setminus A}]\hat{\mu}(kp^j + kl) \to \hat{\mu}(k)$$

as $n \to \infty$. Hence $\hat{\mu}(k) = \hat{\mu}(k)\hat{\mu}(kl)$. This implies that $\hat{\mu}(kl) = 1$ which contradict that μ is non-atomic according to Proposition 3.2.

(2) Suppose μ is a weakly mixing $\times p$-invariant measure and there exist a nonzero integer l and a Følner sequence $\Sigma = \{F_n\}_{n=1}^{\infty}$ such that μ is $\times(p^j+l)$-invariant for all j in some $A \subseteq \mathbb{N}$ with $\overline{D}_\Sigma(A) > 0$.

If μ is not Lebesgue measure, then there exists nonzero k in $\mathbb{Z}$ such that $\hat{\mu}(k)$ is nonzero.

By Theorem 4.1(2), one has

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} |\hat{\mu}(kp^j + kl) - \hat{\mu}(k)\hat{\mu}(kl)|^2 = 0.$$

It follows that

$$0 = \limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n} |\hat{\mu}(kp^j + kl) - \hat{\mu}(k)\hat{\mu}(kl)|^2$$

$$\geq \limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n\cap A} |\hat{\mu}(kp^j + kl) - \hat{\mu}(k)\hat{\mu}(kl)|^2$$

$$= \limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{j\in F_n\cap A} |\hat{\mu}(k) - \hat{\mu}(k)\hat{\mu}(kl)|^2$$

$$=|\hat{\mu}(k) - \hat{\mu}(k)\hat{\mu}(kl)|^2\overline{D}_\Sigma(A).$$

Hence $\hat{\mu}(k) - \hat{\mu}(k)\hat{\mu}(kl) = 0$ which implies that $\hat{\mu}(kl) = 1$. This again leads to a contradiction.

(3) Assume that μ is a strongly mixing $\times p$-invariant measure and there exist a nonzero integer l and an infinite $A \subseteq \mathbb{N}$ such that μ is $\times(p^j + l)$-invariant for all j in A.

If μ is not Lebesgue measure, then there exists nonzero k in $\mathbb{Z}$ such that $\hat{\mu}(k)$ is nonzero.

By Theorem 4.1(3), we have

$$\lim_{\substack{j\to\infty \\ j\in A}} \hat{\mu}(kp^j + kl) = \hat{\mu}(k)\hat{\mu}(kl).$$

On the other hand, for all $j \in A$, one has $\hat{\mu}(kp^j + kl) = \hat{\mu}(k)$. So $\hat{\mu}(k) = \hat{\mu}(k)\hat{\mu}(kl)$. Again this leads to a contradiction.

We finish the proof the first part of Theorem 5.1. □

Before proceeding to the proof of the second part of Theorem 5.1, we need a lemma.

An **atom** for a measure μ on a compact metrizable space X is a point x in X such that $\mu\{x\} > 0$.

Lemma 5.2. *Let $T : X \to X$ be a continuous map on a compact metrizable space X. If a T-invariant measure μ has an atom x with $\mu\{x\} < 1$, then μ is not weakly mixing.*

Proof. Suppose μ is weakly mixing and has an atom x with $\lambda = \mu\{x\} < 1$.

Note that μ is weakly mixing if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\mu(T^{-j}A \cap B) - \mu(A)\mu(B)|^2 = 0$$

for all Borel subsets A, B of X [Wal82, Defn. 1.5(i)&Thm. 1.24].

Choose $A = X \setminus \{x\}$ and $B = \{x\}$. Note that $\mu(T^{-j}A \cap B)$ can only have two possible values: 0 or λ, hence for all j, we have

$$|\mu(T^{-j}A \cap B) - \mu(A)\mu(B)| \geq \min\{\lambda(1-\lambda), \lambda - \lambda(1-\lambda)\} \geq c$$

for some constant $c > 0$. This leads to a contradiction. □

Now we are ready to finish the proof of Theorem 5.1.

Proof. [Proof of the second part of Theorem 5.1]

Suppose μ is a measure satisfying (2) or (3). By the first part of Theorem 5.1, if μ is not a Lebesgue measure, then μ has an atom. By Lemma 5.2, we obtain that μ is a Dirac measure at some point z in $\mathbb{T}$. $\square$

Next we prove the following.

Theorem 5.3. *Let $\tau : \mathbb{Z}^+ \to \mathbb{R}$ be an arbitrary increasing function with $\lim\limits_{n\to\infty} \tau(n) = \infty$. Then there exists a multiplicative semigroup S_τ of $\mathbb{Z}^+$ containing p and satisfying:*

1. $|S_\tau \cap [1, n]| \le (\log_p n)^{\tau(n)}$*;*
2. *the Lebesgue measure is the only non-atomic ergodic $\times p$-invariant measure which is $\times q$-invariant for all q in S_τ.*

In particular, there exists a multiplicative semigroup S of $\mathbb{Z}^+$ containing p and satisfying:

1. $\lim\limits_{n\to\infty} \dfrac{\log |S \cap [1, n]|}{\log n} = 0$*;*
2. *the Lebesgue measure is the only non-atomic ergodic $\times p$-invariant measure which is $\times q$-invariant for all q in S.*

Proof. Let $\{l_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $\lim_{n\to\infty} l_n = \infty$ and define $f(m) = \sum_{n=1}^{m} l_n$ for every positive integer m.

Define $g(m) = \min\{\log_p N \,|\, \tau(N) \ge 1 + f(m)\}$ for every positive integer m.

Define $F_n = [p^{g(n)}, p^{g(n)} + l_n]$ for every positive integer n. Then $\Sigma = \{F_n\}_{n=1}^{\infty}$ is a Følner sequence in $\mathbb{N}$. Denote $\cup F_n$ by A.

Let S_τ be the multiplicative semigroup generated by p and $p^j + 1$ for all $j \in A$.

Since $D_\Sigma(A) = 1$, by (1) of Theorem 5.1, a non-atomic ergodic $\times p$-invariant measure which is $\times q$-invariant for all q in S_f must be the Lebesgue measure.

The remaining thing is to prove that $|S_\tau \cap [1, n]| \le (\log_p n)^{\tau(n)}$.

Every positive integer n locates in $[p^{g(m)}, p^{g(m+1)})$ for some nonnegative integer m.

Consider $S_\tau \cap [1, n]$.

Firstly $|\{j \in A | p^j + 1 \le n\}| \le l_1 + l_2 + \cdots + l_m = f(m)$. This means that $S_f \cap [1, n]$ has at most $1 + f(m)$ generators.

Note that $|\{k\,|p^k \leq n\}| \leq \log_p n$. So for each generator, there are at most $\log_p n$ choices for its powers.

Since n is in $[p^{g(m)}, p^{g(m+1)})$, we have $g(m) \leq \log_p n$. Then $1 + f(m) \leq \tau(n)$ by the definition of g.

Hence

$$|S_\tau \cap [1, n]| \leq (\log_p n)^{1+f(m)} \leq (\log_p n)^{\tau(n)}.$$

This proves the first half of the theorem.

For the second half, choose $\tau(n) = \log\log(n+3)$ for every n in $\mathbb{Z}^+$. Then for $S = S_\tau$, we obtain that

$$\lim_{n\to\infty} \frac{\log|S \cap [1, n]|}{\log n} \leq \lim_{n\to\infty} \frac{[\log\log(n + 3)](\log\log_p n)}{\log n} = 0.$$

$\square$

Remark 5.4. *Furstenberg's conjecture asks for measure rigidity of a non-lacunary semigroup generated by two positive integers p, q and this semigroup has asymptotically $(\log n)^2$ elements in $[1, n]$. Sataev, Einsiedler and Fish prove measure rigidity of a semigroup containing asymptotically n^α elements in $[1, n]$ for some $0 < \alpha < 1$. Theorem 5.3 says that for an arbitrary increasing function $\tau(n)$ with $\lim_{n\to\infty} \tau(n) = \infty$, there is a semigroup with asymptotically $(\log n)^{\tau(n)}$ elements in $[1, n]$ for which measure rigidity still holds. One can choose τ such that the semigroup S_τ is sparsely scattered in $\mathbb{Z}^+$.*

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