LOCI IN STRATA OF MEROMORPHIC DIFFERENTIALS WITH FULLY DEGENERATE LYAPUNOV SPECTRUM

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ABSTRACT. We construct explicit closed GL(2; ℝ)-invariant loci in strata of meromorphic differentials of arbitrary large dimension with fully degenerate Lyapunov spectrum. This answers a question of Forni-Matheus-Zorich.

CONTENTS

1. INTRODUCTION

Lyapunov exponents of the Teichmüller flow have been studied a lot since the work of Zorich ([Zor97], [Zor99]) and Forni [For02]. Their understanding is important for applications to the dynamics of interval exchange transformations and polygonal billiards. A big breakthrough is the Eskin-Kontsevich-Zorich formula for the sum of positive Lyapunov exponents [EKZ11b]. Given a SL(2; ℝ) invariant suborbifold of a stratum of quadratic differentials, they relate the sum $\lambda_1 + \cdots + \lambda_g$ to the Siegel-Veech constant of the invariant locus[1].

By a theorem of Kontsevich and Forni, the sum $\lambda_1 + \cdots + \lambda_g$ is also the integral over the invariant locus of the curvature of the Hodge bundle along Teichmüller discs ([For02], [EKZ11b]). Using this interpretation, every Lyapunov exponent is computed for cyclic covers of the sphere branched over 4 points ([EKZ11a],

[1]For quadratic differentials, two bundles can be considered. In this article, we will only be interested in the bundle with fiber $H^1(X, \mathbb{R})$ over a Riemann surface X. The Lyapunov exponents of this bundle are often denoted by $\lambda_1^+, \ldots, \lambda_g^+$.

[FMZ11], see also [BM10], and [Wri12] for abelian covers). For some cyclic covers, Forni-Matheus-Zorich remarked that the sum $\lambda_1 + \cdots + \lambda_g$ is equal to zero [FMZ11, Thm. 35]. This surprizing fact means that the complex structure of the underlying Riemann surface is constant along the Teichmüller disc. Forni-Matheus-Zorich ask whether it is possible to construct other invariant loci with this property (see [FMZ11, p. 312]). The content of this article is to give a simple explanation of the phenomenon discovered by Forni-Matheus-Zorich. We prove:

Theorem 1. *There exist closed* $\mathrm{GL}(2;\mathbb{R})$ *invariant loci of quadratic differentials of arbitrarily large dimension with zero Lyapunov exponents.*

This result can be interpreted in the following way: the projection of such a locus to the moduli space of *compact* Riemann surfaces is constant. Remark that the situation for strata of *abelian* differentials is completely different: there are finitely many invariant suborbifolds with fully degenerate Lyapunov spectrum (meaning in this setting that all exponents are zero except λ_1 which is 1), and they are arithmetic Teichmüller curves (see [Möl11], [For06], [FMZ11] and [Aul13]).

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2. BACKGROUND MATERIAL

2.1. **The Teichmüller flow for translation surfaces.**
A translation surface is a pair (X, ω) where X is a compact Riemann surface and ω is a holomorphic one-form on X. If $S(\omega)$ if the set of the zeroes of ω, there exists an open covering of $\tilde{X} = X \setminus S(\omega)$ and holomorphic charts $\varphi_i \colon U_i \to X$ such that $\varphi_i^* \omega = dz$ for all i. For such an atlas, the transition functions are translations. The form ω induces a flat metric $|\omega^2|$ on the open surface $\tilde{X}$, whose area is the integral $\frac{i}{2} \int_X \omega \wedge \bar{\omega}$. We could have taken meromorphic 1-forms instead of holomorphic ones, but in that case the area of the surface is never finite.

There is a natural action of $\mathrm{GL}(2, \mathbb{R})$ on translation surfaces given as follows: first we pick an atlas of charts of $\tilde{X}$ where all transitions functions are translations by some complex vectors v_{ij} which we will consider as vectors in $\mathbb{R}^2$. Then, for any M is $\mathrm{GL}(2, \mathbb{R})$, we get an open surface $\tilde{X}_M$ defined by an atlas whose transition functions are translations by $M.v_{ij}$. This surface is diffeomorphic to $\tilde{X}_M$. Therefore, we can fill the holes and extend the complex structure in a unique way: the result is a compact Riemann surface X_M diffeomorphic to X endowed with a meromorphic differential ω_M of finite volume, hence holomorphic. The action of $\mathrm{GL}(2; \mathbb{R})$ is defined by the formula $M.(X, \omega) = (X_M, \omega_M)$. The action of $\mathrm{SL}(2; \mathbb{R})$ preserves the volume of translation surfaces.

The subgroup of $\mathrm{SL}(2; \mathbb{R})$ of matrices M such that $M.(X, \omega) = (X, \omega)$ up to diffeomorphism is called the Veech group of (X, ω). If $M_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ then the

curve $(X_t, \omega_t) := M_t.(X, \omega)$ is called the orbit under the Teichmüller flow of (X, ω). If M_θ is a rotation in $SO(2, \mathbb{R})$, then $M_\theta.(X, \omega) = (X, e^{i\theta}\omega)$.

If we fix multiplicities $(m_1, \ldots, m_r)$ such that $\sum_{i=1}^r m_i = 2g - 2$, the stratum of translation surfaces $\mathcal{H}(m_1, \ldots, m_r)$ is the set of translations surfaces (X, ω) where ω has r distinct zeroes of multiplicities $m_1, \ldots, m_r$ modulo diffeomorphism. The normalized stratum $\mathcal{H}_1(m_1, \ldots, m_r)$ is the locus of flat surfaces with unit area in $\mathcal{H}(m_1, \ldots, m_r)$, and the projective stratum $P\mathcal{H}(m_1, \ldots, m_r)$ is obtained by taking the quotient of $\mathcal{H}(m_1, \ldots, m_r)$ under the natural $\mathbb{C}^\times$-action on forms. Strata and projective strata are complex orbifolds of respective dimensions dimensions $2g + r - 1$ and $2g + r - 2$ if $g \geq 2$.

There are standard coordinates on the stratum $\mathcal{H}(m_1, \ldots, m_r)$, called period coordinates. Fix (X, ω) in this stratum, and let $A_1, \ldots, A_g, B_1, \ldots, B_g$ be a symplectic basis of $H_1(X, \mathbb{Z})$ and $C_1, \ldots, C_{r-1}$ be $r - 1$ paths joining a zero of ω to all the $r - 1$ other zeroes. The map

$$(X, \omega) \to \left(\int_{A_1} \omega, \ldots, \int_{A_g} \omega, \int_{B_1} \omega, \ldots, \int_{B_g} \omega, \int_{C_1} \omega, \ldots, \int_{C_{r-1}} \omega \right)$$

yields an orbifold chart on $\mathcal{H}(m_1, \ldots, m_r)$. These charts allow to define a canonical volume element on $\mathcal{H}(m_1, \ldots, m_r)$, $\mathcal{H}_1(m_1, \ldots, m_r)$, and $P\mathcal{H}(m_1, \ldots, m_r)$. By classical results of Masur and Veech, connected components of projective strata have finite volume.

Let $\mathbb{H} = SL(2; \mathbb{R})/SO(2)$ denote the Poincaré upper-half plane. For any (X, ω) in a projective stratum, the $SL(2; \mathbb{R})$-action factorizes to a holomorphic map

$$\mathbb{H} \to P\mathcal{H}(m_1, \ldots, m_r)$$

which is an immersion. The image of this map is called a Teichmüller disc, it is stable under the Teichmüller flow. Besides, Teichmüller discs induce a smooth foliation with holomorphic leaves on $P\mathcal{H}(m_1, \ldots, m_r)$.

Assume that the Veech group Γ of (X, ω) is a lattice in $SL(2; \mathbb{R})$. Then the image $\mathbb{H}/\Gamma$ of the corresponding Teichmüller disc in the projective stratum is called a Teichmüller curve.

All these considerations generalize to the so-called half-translation surfaces, which are pairs (X, q) where q is a quadratic holomorphic (for the time being) differential on X. The transitions functions of a well-choosen atlas of charts on the open surface are half translations, that is either translations or flips. The area of the flat metric on $\tilde{X}$ is $\frac{1}{2} \int_X |q|$, and we still have an action of $GL(2, \mathbb{R})$ as well as a Teichmüller flow. The period coordinates on strata of quadratic differentials are obtained as follows: for any (X, q) in a stratum, we take the twofold branched covering $\mathfrak{p} \colon \widehat{X} \to X$ given by the holonomy representation of q, which is given by a morphism from $\pi_1(X)$ to $\mathbb{Z}/2\mathbb{Z}$. Let j be the corresponding involution acting on $\widehat{X}$. The quadratic differential $\mathfrak{p}^* q$ is the square of an abelian differential ω. The period coordinates of (X, q) are obtained by taking J-anti-invariant absolute and relative periods of (X, q).

However, a major difference happens for quadratic differentials: it is possible to take meromorphic quadratic differentials and still get finite volume for the corresponding flat surface. More precisely, (X, q) has finite volume if and only if q has poles of order at most one. Therefore we have strata, normalized strata and projective strata $Q(m_1, \ldots, m_r)$, $Q_1(m_1, \ldots, m_r)$ and $PQ(m_1, \ldots, m_r)$, where $\sum_{i=1}^{r} m_i = 4g - 4$ and each m_i is either positive or equal to -1.

Let S be a finite subset of X with cardinal n, so that (X, S) gives a point in the marked Teichmüller space $\mathcal{T}_{g,n}$ (genus g with n marked points). The cotangent space of $\mathcal{T}_{g,n}$ at X is exactly the space $Q_S(X)$ of holomorphic quadratic differentials on $X \setminus S$ with poles of order at most one on S. There is a norm on $Q_S(X)$ given by $\|q\| = \int_X |q|$, as well as a dual norm on $Q_S(X)^*$. The corresponding distance on $\mathcal{T}_{g,n}$ is the Teichmüller metric.

Let us fix (X, S) as well as an element q in $Q_S(X)$. There is a complex linear form μ_q on $Q_S(X)$ given by scalar product with the L^∞ Beltrami differential $\dfrac{|q|}{q}$:

$$\mu_q(\tilde{q}) = \int_X \tilde{q}\, \frac{|q|}{q}.$$

Note that $\mu_q(q) = \int_X |q| > 0$ so that μ_q is nonzero. Besides, we have $\|\mu_q\| = 1$. The map $q \to \mu_q$ gives a non-linear isomorphism between the unit spheres of $Q_S(X)$ and $Q_S(X)^*$, hence between the unitary cotangent space $U^*\mathcal{T}_{g,n}$ and the unitary tangent space $U\mathcal{T}_{g,n}$.

If (X, q) is given and S is the set of poles of q, the Teichmüller flow of (X, q) introduced formerly is the geodesic flow (for the Teichmüller metric) on $\mathcal{T}_{g,n}$ starting from X in the direction μ_q.

2.2. **The period mapping.**

For any compact Riemann surface X, $H^1(X, \mathbb{C})$ is the orthogonal sum (for the intersection form) of $\Omega(X)$ and $\overline{\Omega(X)}$. Besides, the composition

$$\psi : H^1(X, \mathbb{R}) \to H^1(X, \mathbb{C}) \xrightarrow{\mathrm{pr}_1} \Omega(X)$$

is an isomorphism. The Hodge norm $\| \ \|_{\mathrm{Hodge}}$ is the unique norm on $H^1(X, \mathbb{R})$ making ψ an isometry.

Let us now consider a local holomorphic family of curves, that is a proper holomorphic submersion $\pi \colon \mathfrak{X} \to B$ whose fibers are compact Riemann surfaces of some genus g, where B is a small ball in $\mathbb{C}^n$. The Hodge bundle is a holomorphic vector bundle on B of rang g whose fiber at each point b is the vector space $\Omega(X_b)$. The local system $R^1\pi_* \mathbb{R}_{\mathfrak{X}}$ is trivial, which means that we can canonically identify all the vector spaces $H^1(\mathfrak{X}_b, \mathbb{R})$ to some fixed real vector space $\mathbb{V}$ of dimension $2g$. The local period map

$$\xi \colon B \to \mathrm{Gr}(g, \mathbb{V}^{\mathbb{C}})$$

associates to any b the subspace $\mathcal{H}_b$ in the Grassmannian of g-dimensional complex subspaces of $\mathbb{V}^\mathbb{C}$. The derivative of ξ at a point b in B is a linear map from $T_b^{1,0}B$ to $\mathrm{End}\,(\mathcal{H}_b, \mathbb{V}^C/\mathcal{H}_b)$, which is isomorphic to $\mathrm{End}\,(\mathcal{H}_b, \overline{\mathcal{H}}_b)$.

The differential of ξ can be explicitly computed: ξ induces a classifying map $\xi_{\mathrm{Teich}} : B \to \mathcal{T}_g$. Then we have the following formula due to Ahlfors: for any vector v in T_bB and any elements α and β in $\Omega(X_b)$,

$$(\beta, \overline{\xi'_v(\alpha)}) = \int_X \alpha \otimes \beta.\, \xi'_{\mathrm{Teich}}(v).$$

In this formula, $\xi'_{\mathrm{Teich}}(v)$ is a tangent vector to $\mathcal{T}_g$, hence represented by a Beltrami differential which is a tensor field on X of type $(-1, 1)$. Thus, the integrand in the above formula of type $(2, 0) + (-1, 1) = (1, 1)$. We can also think of $\xi'_{\mathrm{Teich}}(v)$ as a linear form on $Q(X)$; in this case the above formula reads

$$(\beta, \overline{\xi'_v(\alpha)}) = \xi'_{\mathrm{Teich}}(v)\,\{\alpha \otimes \beta\}.$$

It is also possible to give another interpretation on ξ'. For this we consider the exact sequence of holomorphic vector bundles

$$0 \to \mathcal{H} \to \mathbb{V} \otimes O_B \to \overline{\mathcal{H}} \to 0.$$

The bundle $\mathbb{V} \otimes O_B$ carries a natural flat connection (the Gauß-Manin connection), but $\mathcal{H}$ is not in general a flat sub-bundle of $\mathbb{V} \otimes O_B$. A precise way to measure this (see formula (2) below) is the second fundamental form σ associated with this exact sequence and the Gauß-Manin connexion; it is a $(1, 0)$-form with values in $\mathrm{Hom}\,(\mathcal{H}, \overline{\mathcal{H}})$. A simple calculation shows that

$$(1) \qquad\qquad\qquad\qquad \sigma = \xi'.$$

The Hodge bundle $\mathcal{H}$ carries a natural metric given by the intersection form, its curvature form is given by the formula

$$(2) \qquad\qquad\qquad\qquad \Theta_{\mathcal{H}} = \sigma^* \wedge \sigma.$$

By "$\wedge$" we mean composition on the fiber and wedge-product on the base. In particular, $i\,\mathrm{Tr}\Theta_{\mathcal{H}}$ is a positive $(1, 1)$-form on B.

For any compact half-translation surface (X, q), Forni's B-form is a bilinear form on $\Omega(X)$ defined by

$$B_q(\alpha, \beta) = \int_X \frac{\alpha \otimes \beta}{q}\,|q|.$$

If $\xi'_{\mathrm{Teich}}(v)$ has unit norm, we can write it as μ_q for some holomorphic quadratic differential on X. Then we have $(\beta, \overline{\xi'_v(\alpha)}) = B_q(\alpha, \beta)$. In case of a Teichmüller orbit (X_t, q_t), if we differentiate along the vector field $\frac{\partial}{\partial_t}$, we get the formula

$$(3) \qquad\qquad\qquad\qquad (\beta, \overline{\xi'_{\partial_t}(\alpha)}) = B_{q_t}(\alpha, \beta).$$

Applying Cauchy-Schwarz inequality, $|B_q(\alpha, \beta)| \le \|\alpha\| \times \|\beta\|$ with equality if and only if there exists a holomorphic one-form ω and two complex constants c and

c' such that $q = \omega^2$, $\alpha = c\omega$ and $\beta = c'\omega$. In particular, if q is meromorphic with simples poles, $\||B_q\|| < 1$.

We recall now Forni's inequality: let (X, q) be a half-translation surface, (X_t, q_t) be its orbit under the Teichmüller flow, v be in $H^1(X, \mathbb{R})$ and $t \to v_t$ be its parallel transport under the Teichmüller flow for the Gauß-Manin connection. We write $v_t = \chi_t + \overline{\chi_t}$ where χ_t is in $\Omega(X_t)$. Then a simple calculation gives

$$\partial_t \|v_t\|_{\text{Hodge}} = B_{q_t}(\chi_t, \chi_t).$$

Combined with the inequality $\||B\|| \leq 1$, this gives Forni's inequality

(4) $\left| \partial_t \{\log \|v_t\|_{\text{Hodge}}\} \right| \leq 1.$

2.3. **Lyapunov exponents of the KZ cocycle.**

The parallel transport for the Gauß-Manin connection of vectors of $H^1(X, \mathbb{R})$ under the Teichmüller flow is called the Kontsevich-Zorich cocycle. Recall that the Teichmüller flow is ergodic on every connected component $\mathfrak{D}_1$ of the normalized stratum $\mathcal{Q}_1(m_1, \ldots, m_r)$. By Osseledet's theorem, it is possible to associate $2g$ Lyapunov exposants to this cocycle.

Forni's inequality (4) implies that the KZ cocycle is log-integrable, so that the Lyapunov exponents are well-defined. Since the cocycle is symplectic, the Lyapunov spectrum is of the form $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_g, \lambda_g, \ldots, \lambda_{g-1}, \ldots, \lambda_1\}$ where $\lambda_1 \geq \ldots \geq \lambda_g$.

Note that the exponents λ_i are called λ_i^+ in numerous papers (e.g. in [EKZ11b], [FMZ11]). The exponents λ_i^- will never be considered in the article.

By (4), all λ_i's are at most one. If the component $\mathfrak{Q}$ is orientable, which means that every quadratic differential occuring in the stratum is the square of an abelian differential, then the top Lyapunov exponent λ_1 equals one. If not, the norm of Forni's B form is strictly smaller than one so that $\lambda_1 < 1$.

For any (X, q) in a stratum $P\mathcal{Q}(m_1, \ldots, m_r)$, the Poincaré metric on $\mathbb{H}$ induces a metric on the Teichmüller disc passing through (X, q). The corresponding volume element defines a relative $(1, 1)$ form dV_{Teich}, where by "relative" we mean relative with respect to the foliation by Teichmüller discs. If Θ is the curvature of the Hodge bundle on $P\mathcal{Q}(m_1, \ldots, m_r)$, its trace is also a relative $(1, 1)$ form on the projective stratum. Let $\Lambda : P\mathcal{Q}(m_1, \ldots, m_r) \to \mathbb{R}$ be defined by the formula

$$\Lambda = \frac{\text{Tr}\,\Theta}{dV_{\text{Teich}}}.$$

Then Kontsevich-Forni's main formula for the Lyapunov exponents is

$$\lambda_1 + \ldots + \lambda_g = \int_{\mathfrak{D}} \Lambda(X, q)\, dV$$

where $\mathfrak{D}$ is the projection of $\mathfrak{D}_1$ in the projective stratum and dV is the normalized volume element on $\mathfrak{D}$ of total mass one. For any (X, q) in $\mathfrak{D}$, let $\theta_1, \ldots, \theta_g$ be the eigenvalues of Forni's B-form in the direction of the Teichmüller flow when

diagonalized in an orthonormal basis for the intersection form. Using formulæ (2), (1) and (3), we see that

$$(5) \qquad \lambda_1 + \ldots + \lambda_g = \int_{\mathfrak{D}} \left\{ \theta_1(X, q) + \ldots + \theta_g(X, q) \right\} \, \mathrm{dV}$$

Forni's inequality implies that $\theta_i(X, q) \le 1$ for all i so that $\lambda_1 + \ldots + \lambda_g \le g$.

Thanks to the main result of [EM13], any closed $\mathrm{SL}(2; \mathbb{R})$-invariant locus in the projective stratum $\mathrm{P}Q(m_1, \ldots, m_r)$ is affine in period coordinates, hence carries a natural $\mathrm{SL}(2; \mathbb{R})$-invariant probability measure. It is also possible to define Lyapunov exponents for this measure, and formula (5) holds.

If (X, q) is any half-translation surface, the closure of its $\mathrm{SL}(2; \mathbb{R})$-orbit in the normalized stratum is affine in period coordinates. It follows from [CE13] that almost every direction θ, the real Teichmüller flow of $(X, e^{i\theta}q)$ is Osseledets-generic for the corresponding natural probability measure. Therefore it makes sense to consider Lyapunov exponents of (X, q), and formula (5) is still valid if we integrate on the closure of the $\mathrm{PGL}(2; \mathbb{R})$-orbit.

3. THE DETERMINANT LOCUS

3.1. **General properties.** Let $\mathfrak{D}$ be a connected component of the projective stratum $\mathrm{P}Q(m_1, \ldots, m_r)$.

Definition 1. *The determinant locus of $\mathfrak{D}$ is the set of elements (X, q) in $\mathfrak{D}$ such that for all holomorphic 1-forms α and β on X, $\mathrm{B}_q(\alpha, \beta) = 0$.*

Let us now recall Noether's theorem (see [FK92, p. 104 & 159]):

Proposition 1. *Let X be a compact Riemann surface of genus g and*

$$\tau \colon \mathrm{Sym}^2 \Omega^1(X) \to Q(X)$$

be the multiplication map.
  (i) *If X is not hyperelliptic or if $g \le 2$, τ is surjective.*
  (ii) *If X is hyperelliptic, $\mathrm{Im}(\tau)$ has codimension $g - 2$ in $Q(X)$ and consists of the quadratic differentials invariant by the hyperelliptic involution.*

Since τ is the transpose of the derivative of the period map, Noether's result has the following geometric interpretation:

Proposition 2 (Infinitesimal Torelli's theorem, [Voi07, Cor. 10.25])**.**
  Let $\xi \colon \mathcal{T}_g \to \mathbb{H}_g$ be the period map. Then ξ is an immersion outside the hyperelliptic locus or everywhere if $g \le 2$, and the restriction of ξ to the hyperelliptic locus is also an immersion.

Remark that Forni's B-form factors through $\mathrm{Im}\,\tau$, and can be extended naturally to $Q(X)$ by the formula $\mathrm{B}_q(\tilde{q}) = \int_X \tilde{q}\, \dfrac{|q|}{q}$.

The key proposition of this section is:

Proposition 3. *Let (X, q) be a half-translation surface, n the number of poles of q, and $\mathbb{D}$ be its Teichmüller disc. Then the following are equivalent:*

(i) *$\mathbb{D}$ lies in the determinant locus.*
(ii) *The forgetful map $\mathcal{T}_{g,n} \to \mathcal{T}_g$ maps $\mathbb{D}$ to a point.*
(iii) *For any (X_t, q_t) in $\mathbb{D}$, the extension of B_{q_t} to $\mathcal{Q}(X_t)$ vanishes.*
(iv) *All Lyapunov exponents of (X, q) are zero.*

Proof.

(i) $\Rightarrow$ (ii) Using (3), the composite map $\mathbb{D} \hookrightarrow \mathcal{T}_{g,n} \to \mathcal{T}_g \overset{\tau}{\to} \mathbb{H}_g$ has zero derivative. Assume that $\mathbb{D}$ is not contained in the hyperelliptic locus. Thanks to the infinitesimal Torelli theorem, $\mathbb{D}$ is mapped to a point via the forgetful map $\mathcal{T}_{g,n} \to \mathcal{T}_g$. Assume now that $\mathbb{D}$ is contained in the hyperelliptic locus. Then the restriction of τ to this locus is again an immersion, and we can apply the same argument.

(ii) $\Rightarrow$ (iii) If (X_t, q_t) is a point in $\mathbb{D}$, the derivative of projection of the Teichmüller flow of (X_t, q_t) on $\mathcal{T}_g$ is the linear form $\tilde{q} \to \mathrm{B}_{q_t}(\tilde{q})$ on $\mathcal{Q}(X_t)$.

(iii) $\Rightarrow$ (i) Obvious.

(i) $\Leftrightarrow$ (iv) Let $\mathcal{V}$ be the closure of the PSL$(2; \mathbb{R})$-orbit of X and ν the corresponding PSL$(2; \mathbb{R})$-invariant probability measure. If $\lambda_1, \ldots, \lambda_g$ are the Lyapunov exponents of (X, q), then

$$\lambda_1 + \ldots + \lambda_g = \int_{\mathcal{V}} \left\{ \theta_1(X, q) + \ldots + \theta_g(X, q) \right\} d\nu.$$

Since all θ_i's are nonnegative and continuous functions, $\lambda_1 = \ldots = \lambda_g = 0$ if and only if all θ_i's vanish on $\mathbb{D}$.

$\square$

Corollary 1. *If q is a holomorphic quadratic differential on X, the Teichmüller disc of (X, q) is not included in the determinant locus.*

Proof. If q is holomorphic, $\mathrm{B}_q(q) > 0$ and we apply Proposition 3. $\square$

Remark 1. *In the hyperelliptic case, it can happen that q is holomorphic but that (X, q) lies in the determinant locus. Let X be an hyperelliptic surface of genus at least 3, let j be the hyperelliptic involution, and let q be an anti-invariant holomorphic quadratic differential (if X is the Riemann surface of a polynomial $w^2 - P(z)$, we can take $q = w^{-1} dz^{\otimes 2}$). Since any holomorphic 1-form on X is anti-invariant under j^*, $\mathrm{B}_q = 0$. Hence (X, q) lies in the determinant locus, but the Teichmüller disc of (X, q) goes outside of the hyperelliptic locus.*

We can give an explicit lower bound on the number n.

Proposition 4. *Let (X, q) be a half-translation surface of genus at least 1 satisfying the equivalent conditions of Proposition 3. Then q has at least $\max(2g - 2, 2)$ poles.*

Proof. The fact that the number n of poles of q must be at least one follows from [Kra81, Thm 4']. To get the lower bound $2g - 2$ in the proposition, we use [EKZ11b, Thm 2] for the closure of the SL$(2; \mathbb{R})$-orbit O of (X, q), which is contained in a stratum $PQ((-1)^n, m_1, \ldots, m_r)$: we get

$$\lambda_1 + \ldots + \lambda_g = \frac{1}{24} \sum_{j=1}^{r} \frac{m_j(m_j + 4)}{m_j + 2} - \frac{n}{8} + \frac{\pi^2}{3} C_{\text{area}}(\overline{O})$$

where $C_{\text{area}}(\overline{O})$ is a Siegel-Veech constant of the locus $\overline{O}$ which is nonnegative. Thus, if $\lambda_1 + \ldots + \lambda_g = 0$,

$$\sum_{j=1}^{r} \frac{m_j(m_j + 4)}{m_j + 2} \le 3n.$$

Since $(\sum m_j) - n = 4g - 4$,

$$2g - 2 \le \sum_{j=1}^{r} \frac{m_j}{m_j + 2} + 2g - 2 \le n$$

and we get the required estimate.

Remark 2. *We will see that this bound is asymptotically sharp in §3.3.*

$\square$

3.2. **Pillow-tiled surfaces.** In this section, we give constraints on pillow-tiled surfaces whose Teichmüller disc lies in the determinant locus. Let us start with a technical result:

Proposition 5. *Let X be a Riemann surface of genus g, B(t_0, ε) a small ball in $\mathbb{C} \setminus \{0, 1, \infty\}$, and $\varphi \colon X \times B(t_0, \varepsilon) \to \mathbb{P}_1$ be a holomorphic map satisfying the following conditions:*

(1) *For any t in B(t_0, ε), φ_t is non-constant and B$(\varphi_t) = \{0, 1, \infty, t\}$.*
(2) *The configuration of the ramification points of φ_t remains constant with t.*

If d is the degree of the branched coverings φ_t, then $3(g - 1) \le d$.

Proof. For any x in X, let $s(x) = \frac{\partial}{\partial t}\big|_{t=t_0} \varphi_t(x) \in T_{\varphi_{t_0}(x)} \mathbb{P}^1$. Then s is a holomorphic section of the holomorphic line bundle $\varphi_{t_0}^* T\mathbb{P}^1$. Let x_0 be a ramification point of φ_{t_0} such that $\varphi_{t_0}(x_0) = 0$. Let us assume that $s(x_0) \ne 0$. By the implicit function theorem, the equation $\varphi_t(x) = 0$ has a unique solution $(x, t(x))$ depending holomorphically on x for (x, t) near (x_0, t_0). Since $\varphi_{t(x)}(x) = 0$, we get

$$\partial_t \varphi(x, t(x)) \, t'(x) + (\varphi_{t(x)})'(x) = 0.$$

By hypothesis, x is a ramification point of $\varphi_{t(x)}$, i.e. $(\varphi_{t(x)})'(x) = 0$. Besides, since $\partial_t \varphi(t(x), x) \to s(x_0)$ as $x \to x_0$, t' vanishes. Hence $\varphi_{t_0}(x)$ vanishes for x near x_0, so that φ_{t_0} is constant and we get a contradiction. It follows that s vanishes at x_0. The same result also holds over any ramification point of φ_{t_0} lying over 1 and ∞. Lastly, if $\psi_t(x) = \varphi_t(x) - t$, the argument we used proves that for any ramification

point x of ψ_t lying over 0, $\frac{\partial}{\partial t}_{|t=t_0}\psi_t(x) = 0$, which means that $s(x) = 1$. In particular s in nonzero.

We can now decompose the ramification divisor $\mathcal{R}$ of the branched covering φ_{t_0} as the sum $\mathcal{R}_0 + \mathcal{R}_1 + \mathcal{R}_\infty + \mathcal{R}_t$. Besides, we can assume that $\deg \mathcal{R}_t$ is smaller than $\deg \mathcal{R}_0$, $\deg \mathcal{R}_1$ and $\deg \mathcal{R}_\infty$, otherwise we move the points $0, 1, \infty$ and t by a suitable homographic transformation. Besides, thanks to the Riemann-Hurwitz formula, we have

$$\deg \mathcal{R} = 2(g + d - 1)$$

Now s is a nonzero section of the line bundle $\mathcal{L} = \varphi_{t_0}^* T\mathbb{P}^1(-\mathcal{R}_0 - \mathcal{R}_1 - \mathcal{R}_\infty)$, and

$$0 \leq \deg \mathcal{L} = 2d - \deg \mathcal{R} + \deg \mathcal{R}_t \leq 2d - \frac{3}{4} \deg \mathcal{R} = \frac{d}{2} - \frac{3g}{2} + \frac{3}{2}.$$

The result follows.   □

Corollary 2. *Let (X, q, π) be a pillow-tiled surface of genus g, and let d be the degree of π. If the Teichmüller disc of (X, q) lies in the determinant locus, then $d \geq 3(g - 1)$.*

Remark 3. *It is not possible to find an upper bound on the primitive degree d in a given connected component of strata since there are infinitely many pillow-tiled surfaces with arbitrary large primitive degree.*

Let (X, q) be a half-translation surface and (Y, π) be an arbitrary finite covering of X with branching locus S. Assume that for any point y in Y above S, the ramification index of π at y is at least 2. Then $\pi^* q$ is holomorphic, so that $B_{\pi^* q}$ is non zero on $\mathcal{Q}(Y)$. Thanks to Corollary 1, the Teichmüller disc of $(Y, \pi^* q)$ doesn't belong to the determinant locus. Using this observation, we can prove the following:

Corollary 3. *Let (X, q) be a half-translation surface and (Y, π) be a finite Galois covering of X with branch locus S. If the Teichmüller disc of $(Y, \pi^* q)$ lies in the determinant locus, then at least one pole of q does not belong to S.*

As a particular by-product, we get:

Proposition 6. *Let (X, q, π) be a pillow-tiled surface such that π is Galois. Then the Teichmüller disc of (X, q) lies in the determinant locus if and only the branching locus of π contains at most three points.*

Proof. Let q_{st} be the standard meromorphic differential on $\mathbb{P}^1$ with four simple poles such that $q = \pi^* q_{\text{st}}$. Then the branching locus of π lies in the set of poles of q_{st}. If X is in the determinant locus, according to Corollary 3, one of the poles of q_{st} is not a branching point of π.

Conversely, assume that the branching locus of π has less than four points. If $\{z_1, z_2, z_3, z_4\}$ are the four poles of q_{st}, let us assume that z_4 is not a branch point of π. The complex Teichmüller flow of $(\mathbb{P}^1, q_{\text{st}})$ is of the form $(\mathbb{P}^1, q_t)$ where q_t has poles at z_1, z_2, z_3 and another point $z_4(t)$ such that $[z_1, z_2, z_3, z_4(t)] = t$. Let

$\overset{\circ}{X}$ be the *open* Riemann surface obtained by removing $\pi^{-1}\{z_1, z_2, z_3\}$. Then $\overset{\circ}{X}$ is an unramified covering of $\mathbb{P}^1 \setminus \{z_1, z_2, z_3\}$. It follows that $(\overset{\circ}{X} \setminus \pi^{-1}(z_4(t)), \pi^* q_t)$ parametrizes the Teichmüller disc of (X, q) in $\mathrm{T}^* \mathcal{T}_{g,n}$ (where n is the number of poles of q). This disc maps to $\{X\}$ via the forgetful map $\mathcal{T}_{g,n} \to \mathcal{T}_g$. Thanks to Proposition 3, the Teichmüller disc of (X, q) lies in the determinant locus. $\qquad\square$

Let us now consider pillow-tiled surfaces arising as *cyclic* coverings of the projective line. They are given by a combinatorial datum (N, a_1, a_2, a_3, a_4) where $0 < a_i \le N$, $\gcd(a_1, a_2, a_3, a_4, N) = 1$ and $\sum_{i=1}^4 a_i \equiv 0 \,(N)$: the associated cyclic covering is the Riemann surface of the polynomial

$$w^N - (z - z_1)^{a_1}(z - z_2)^{a_2}(z - z_3)^{a_3}(z - z_4)^{a_4}.$$

In topological terms, if $(\gamma_i)_{1 \le i \le 4}$ are small loops around the z_i's for $1 \le i \le 4$, then the kernel of the group morphism

$$\pi_1(\mathbb{P}^1 \setminus \{z_1, z_2, z_3, z_4\}) \to \mathbb{Z}/N\mathbb{Z}$$

given by $\gamma_i \to a_i$ defines a true cyclic covering of $\mathbb{P}^1 \setminus \{z_1, z_2, z_3, z_4\}$ of degree N, which extends to a branched cyclic covering of the projective line.

In [FMZ11, Thm. 35], the authors prove that all Lyapunov exponents of the Teichmüller curve corresponding to a cyclic covering are 0 if one of the integers a_i equals N.

Proposition 7. *If (X, q) is a pillow-tiled surface obtained by a cyclic covering of $\mathbb{P}^1$ with combinatorial datum (N, a_1, a_2, a_3, a_4), then the Teichmüller disc of (X, q) lies in the determinant locus if and only if one of the $a_i's$ equals N.*

Proof. Thanks to Proposition 6, it suffices to prove that the projection π of the covering is branched at three points or less if and only if one of the $a_i's$ equals N. If $\{z_1, z_2, z_3, z_4\}$ are the four points defining the cyclic cover, the ramification index of π at any point of $\pi^{-1}(z_i)$ is $\frac{N}{\mathrm{pgcd}\,(N, a_i)}$ qed. $\qquad\square$

3.3. Construction of invariant subvarieties.

In this section, we provide the precise statement underlying Theorem 1 as well as its proof.

Let $m_1, \ldots, m_r$ and k be positive integers such that $(\sum_{i=1}^r m_i) - k = -4$, and let $\mathcal{S}$ be the set of couples $(q, x_1, \ldots, x_{k-3})$ such that such that q is a meromorphic differential on $\mathbb{P}^1$ with simple poles at 0, 1 and ∞ and the x_i's, and q has r zeroes of order $m_1, \ldots, m_r$. It is a smooth $\mathrm{GL}(2; \mathbb{R})$-invariant submanifold of $\mathrm{T}^* \mathcal{M}_{0,[k]}$ (where the bracket means that the points are ordered).

Let us fix a covering (Y, π) of $\mathbb{P}^1$ ramified over 0, 1 and ∞, and let g be the genus of Y. Put

$$n = \#\Big\{y \in \pi^{-1}\{0, 1, \infty\} \text{ such that } \pi \text{ is unramified at } y\Big\} + \deg(\pi) \times (k - 3)$$

We have a natural map

$$\chi \colon \mathcal{S} \to \mathrm{T}^*_{\mathrm{orb}} \mathcal{M}_{g,n}$$

given by $\chi(q) = (Y, \pi^* q)$, where $\mathrm{T}^*_{\mathrm{orb}}$ denotes the *orbifold* cotangent bundle.

Theorem 2. *Let $\mathcal{W}$ be the image of χ.*

(1) *The map $\chi \colon \mathcal{S} \to \mathcal{W}$ is a holomorphic orbifold map, which is a local immersion. Besides, $\mathcal{W}$ is a suborbifold[2] of the orbifold cotangent bundle of $\mathcal{M}_{g,n}$ of dimension $r + k - 2$.*

(2) *$\mathcal{W}$ is $\mathrm{GL}(2; \mathbb{R})$ invariant and lies in the determinant locus, and the projection of $\mathcal{W}$ by the map $\mathrm{T}^*_{\mathrm{orb}}\mathcal{M}_{g,n} \to \mathcal{M}_{g,n} \to \mathcal{M}_g$ is $\{Y\}$.*

(3) *The Lyapunov spectrum of $\mathcal{W}$ is fully degenerate.*

Proof. Let q be a point in $\mathcal{S}$, and U be a small neighborhood of q in $\mathcal{S}$. It is possible to lift locally χ to a smooth map $\widehat{\chi}$ from U to $\mathrm{T}^*\mathcal{T}_{g,n}$, so that χ is a smooth orbifold map.

If q_1, q_2 are two elements in U such that $\chi(q_1) = \chi(q_2)$, then there exists φ in $\mathrm{Aut}\,(Y)$ such that $\varphi^*(Y, \pi^*q_1) = (Y, \pi^*q_2)$. Thus the fibers of $\widehat{\chi}_{|U}$ are finite. But $\widehat{\chi}$ is affine in period coordinates, so that it is an immersion on U.

The $\mathrm{GL}(2; \mathbb{R})$-invariance of $\mathcal{W}$ is proved using the same argument as in Proposition 6, which corresponds to the particular case $r = 0$.

Lastly, the fact that the Lyapunov spectrum of $\mathcal{W}$ is totally degenerate results from the implication (ii) $\Rightarrow$ (iv) in Proposition 3.

$\square$

We end this section by an example proving that the estimate $n \geq 2g-2$ obtained in Proposition 4 is asymptotically sharp. Let p be a prime number, and let Y be a cyclic covering of order p of the projective line fully ramified at three distinct points z_1, z_2 and z_3 obtained by taking the Riemann surface of the polynomial

$$w^p - (z - z_1)^{a_1}(z - z_2)^{a_2}(z - z_3)^{a_3}$$

where $1 \leq a_1, a_2, a_3 \leq p - 1$ and $a_1 + a_2 + a_3 = p$. Then the genus of Y is $\frac{p-1}{2}$. If q is the pull-back of a meromorphic differential on the sphere with four simple poles at z_1, z_2, z_3 and another point z_4, then π^*q has exactly p poles, and three zeros of order $p - 2$. Therefore, we have

$$\underbrace{n}_{p} = \underbrace{2g - 2}_{p-3} + \underbrace{\sum_j \frac{m_j}{m_j + 2}}_{3-\frac{6}{p}} + \underbrace{\frac{\pi^2}{3}C_{\mathrm{area}}}_{\frac{6}{p}}$$

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[2]By suborbifold, we mean as usually done in this theory "locally finite union of suborbifolds".

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