

ON A GENERALIZATION OF $\text{Lie}(k)$: A CATALANKE THEOREM

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ABSTRACT. We define a generalization of the free Lie algebra based on an n -ary commutator and call it the free LAnKe. We show that the action of the symmetric group S_{2n-1} on the multilinear component with $2n - 1$ generators is given by the representation $S^{2^{n-1}-1}$, whose dimension is the n th Catalan number. An application involving Specht modules of staircase shape is presented. We also introduce a conjecture that extends the relation between the Whitehouse representation and $\text{Lie}(k)$.

1. INTRODUCTION

Lie algebras are defined as vector spaces equipped with an antisymmetric commutator and a Jacobi identity. They are a cornerstone of mathematics and have applications in a wide variety of areas of mathematics as well as physics. Also of fundamental importance is the free Lie algebra, a natural mathematical construction central in the field of algebraic combinatorics. The free Lie algebra has beautiful dimension formulas; an elegant basis in terms of binary trees; connections to the shuffle algebra, Lyndon words, necklaces, Witt vectors, the descent algebra of S_n , quasisymmetric functions, noncommutative symmetric functions, as well as the lattice of set partitions.

In this paper we consider a generalization of the free Lie algebra to n -fold commutators, and the representation of the symmetric group on its multilinear component. This representation is a direct generalization of the well-known representation $\text{Lie}(k)$.

The generalization of the free Lie algebra that we will consider is based on the following definition. Throughout this paper, all vector spaces are taken over the field \mathbb{C} .

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Definition 1.1. A Lie algebra \mathcal{L} of the n th kind (a “LAnKe,” or “LATKe” for $n = 3$) is a vector space equipped with an n -linear bracket

$$[\cdot, \cdot, \dots, \cdot] : \times^n L \rightarrow L$$

that satisfies the following antisymmetry relation for all σ in the symmetric group S_n :

$$(1.1) \quad [x_1, \dots, x_n] = \text{sgn}(\sigma)[x_{\sigma(1)}, \dots, x_{\sigma(n)}]$$

and the following generalization of the Jacobi identity:

$$(1.2) \quad [[x_1, x_2, \dots, x_n], x_{n+1}, \dots, x_{2n-1}] \\ = \sum_{i=1}^n [x_1, x_2, \dots, x_{i-1}, [x_i, x_{n+1}, \dots, x_{2n-1}], x_{i+1}, \dots, x_n],$$

for all $x_1, x_2, \dots, x_{2n-1} \in \mathcal{L}$.

The above definition arose in [Fr] from generalizing a relation between ADE singularities and ADE Lie algebras as a tool to solve a string-theoretic problem; it also arose in other contexts, see [Fi, Ta, DT, Ka, Li, BL, Gu]. A different generalization of Lie algebras that also involves n -ary brackets appeared in the 1990’s in work of Hanlon and Wachs [HW].

Homomorphisms, isomorphisms, and subalgebras of LAnKes are defined in the natural way; an ideal \mathcal{I} of a LAnKe \mathcal{L} is defined as a subalgebra satisfying $[\mathcal{L}, \mathcal{L}, \dots, \mathcal{L}, \mathcal{I}] \subset \mathcal{I}$.

The following generalizes the standard definition of a free Lie algebra.

Definition 1.2. Given a set X , a **free LAnKe on X** is a LAnKe L together with a mapping $i : X \rightarrow L$ with the following universal property: for each LAnKe \mathcal{K} and each mapping $f : X \rightarrow \mathcal{K}$, there is a unique LAnKe homomorphism $F : L \rightarrow \mathcal{K}$ such that $f = F \circ i$.

Similar to a free Lie algebra, a LAnKe is free on X if it is generated by all possible n -bracketings of elements of X , and if the only possible relations existing among these bracketings are consequences of n -linearity of the bracketing, the antisymmetry of the bracketing (1.1), and the generalized Jacobi identity (1.2).

We consider two variables: n , the number of entries in a given bracket (so $n = 2$ for Lie algebras), and k , the number of internal brackets plus 1. The number of generators for the multilinear component is then $kn - n - k + 2$, an expression symmetric in n and k . For example, an element of the form $[\cdot \cdot [\cdot \cdot \cdot]]$ has $n = 3$, $k = 3$, and $3 \cdot 3 - 3 - 3 + 2 = 5$ generators; $[\cdot \cdot [\cdot \cdot [\cdot \cdot \cdot]]]$ has $n = 3$, $k = 4$, and $4 \cdot 3 - 4 - 3 + 2 = 7$ generators.

The object we study in this paper is the representation of the symmetric group $S_{kn-n-k+2}$ on the multilinear component of the free LAnKe on $kn - n - k + 2$ generators. We denote this representation by $\rho_{n,k}$, and view $(\rho_{n,k})$ as an array of representations with $n, k \geq 2$. The well-known representations $\text{Lie}(k)$ correspond to the first row of the array, that is, $\rho_{2,k} = \text{Lie}(k)$. Hence, this paper constitutes a study of the LAnKe analog of $\text{Lie}(k)$.

We begin with a brief review of $\text{Lie}(k)$ (see [Re]). Let $X := \{x_1, x_2, \dots, x_k\}$ be a set of generators. Then the multilinear component of the free Lie algebra on X is the subspace spanned by bracketed “words” where each generator in X appears exactly once. For example, $[[x_1, x_3], [[x_4, x_5], x_2]]$ is such a bracketed word when $k = 5$, while $[[x_1, x_3], [[x_1, x_5], x_3]]$ is not. A certain type of bracketed word in the multilinear component has the form

$$(1.3) \quad [\cdots [[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}], x_{\sigma(4)}], \dots, x_{\sigma(k)}], \quad \sigma \in S_k.$$

Bracketed words that are not of this type, such as $[[x_1, x_3], [[x_4, x_5], x_2]]$, can be shown to be linear combinations of the bracketed words of the form in equation (1.3) using iterations of the Jacobi identity. Furthermore, if we restrict to the permutations that satisfy $\sigma(1) = 1$, then the words in equation (1.3) form a basis for the vector space. This vector space admits a natural $(k-1)!$ -dimensional representation of S_k denoted $\text{Lie}(k)$.

The representation $\text{Lie}(k)$ has several equivalent descriptions; we mention two of them here, as well as a further property of $\text{Lie}(k)$.

- Theorem 1.3.** *a. (Klyachko [Kl]) The representation $\text{Lie}(k)$ is equivalent to the representation induced to S_k by any faithful representation of a cyclic subgroup of order k generated by a k -cycle.*
- b. (Kraskiewicz and Weyman [KW]) Let i and k be relatively prime, and let λ be a partition of k . The multiplicity of the irreducible representation indexed by λ in $\text{Lie}(k)$ is equal to the number of standard Young tableaux of shape λ and of major index congruent to $i \pmod k$.*
- c. (Whitehouse [Wh]) The representation $\text{Lie}(k)$ is the restriction from S_{k+1} to S_k of the Whitehouse module*

$$W_{k+1} := \text{Lie}(k) \uparrow_{S_k}^{S_{k+1}} / \text{Lie}(k+1).$$

Interestingly, $\text{Lie}(k)$ appears in a variety of other contexts, such as the top homology of the lattice of set partitions in work of Stanley [St], Hanlon [Ha], Barcelo [Ba], and Wachs [Wa], homology of configuration spaces of k -tuples of distinct points in Euclidean space in work of Cohen [Co], and scattering amplitudes in gauge theories in work of Kol and Shir [KS].

Now we turn to the LAnKe analog of $\text{Lie}(k)$.

The central theorem in this paper provides the representations $\rho_{n,3}$, $n \geq 2$, of the $\{\rho_{n,k}\}$ array; the representations $\rho_{n,2}$, $n \geq 2$, are simply the sign representation of S_n given by the Specht module S^{1^n} . (We denote the Specht module corresponding to a Young diagram of shape λ by S^λ .)

Theorem 1.4. The CataLAnKe Theorem *The representation $\rho_{n,3}$, $n \geq 2$, is given by $S^{2^{n-1}1}$, whose dimension is the n^{th} Catalan number $\frac{1}{n+1} \binom{2n}{n}$.*

This theorem means – among other things – that the Specht module S^{2^1} , which previously could be viewed as the first interesting representation in the sequence $\text{Lie}(k) = \rho_{2,k}$, $k = 2, 3, \dots$, can now also be seen as the first interesting representation in the sequence $\rho_{n,3}$, $n = 2, 3, \dots$.

We may relate Theorem 1.4 to part (c) of Theorem 1.3 by making the following observation.

Corollary 1.5. *The representation $\rho_{n,3}$ is the restriction to S_{2n-1} of the representation S^{2^n} of S_{2n} .*

Now, we may view the representation S^{2^n} mentioned in the above Corollary as one obtained by adding $n - 2$ rows of length $2 = k - 1$ to the top of the Young diagram of the Whitehouse module $W_4 = S^{2^2}$. The Whitehouse module is central to part (c) of Theorem 1.3. More generally, we have the following conjecture which applies to the entire array of the representations $\rho_{n,k}$.

Conjecture 1.6. *The representation $\rho_{n,k}$ is the restriction to $S_{kn-n-k+2}$ of the representation $W_{n,k}$ of $S_{kn-n-k+3}$, where $W_{n,k}$ is obtained by adding $n - 2$ rows of length $k - 1$ to the top of the Young diagram of each irreducible submodule of the Whitehouse module W_{k+1} .*

The conjecture generalizes part (c) of Theorem 1.3, to which it reduces for $n = 2$; for $k = 3$, the conjecture reduces to our Corollary 1.5 and for $k \geq 4$ it reduces to additional results on $\rho_{n,k}$ that will be presented separately ([FHSW]). The conjecture is further supported by a computer program that gives $\dim \rho_{3,5} = 1077$.

Finding analogs of parts (a) and (b) of Theorem 1.3 remains an open problem.

The proof of Theorem 1.4 is given in the next section. We also give an explicit basis for the vector space $\rho_{n,3}$. In Section 3 we show that Theorem 1.4 can be generalized in another direction. Theorem 1.4 gives a presentation of $S^{2^{n-1}}$ in terms of generators and relations. We observe that Theorem 1.4 can be used to generalize this presentation to all staircase partitions.

2. THE CATALANKE REPRESENTATION $\rho_{n,3}$

Recall from Section 1 that the *free LAnKe* on $[m] := \{1, 2, \dots, m\}$ is the vector space generated by the elements of $[m]$ and all possible n -bracketings involving these elements, subject only to the antisymmetry and generalized Jacobi relations given in Definition 1.1. Let $\text{Lie}_n(m)$ denote the multilinear component of the free LAnKe on $[m]$, that is, the subspace generated by bracketed words that contain each letter of $[m]$ exactly once. We call these bracketed words, *bracketed permutations*. The symmetric group S_m acts naturally on $\text{Lie}_n(m)$. Indeed, let $\sigma \in S_m$ act by replacing letter x of a bracketed permutation with $\sigma(x)$.

Note that $\text{Lie}_n(m) = 0$ unless $m = kn - n - k + 2$ for some positive integer k . Indeed, $k - 1$ is the number of brackets in the bracketed permutation. Let $\rho_{n,k}$ denote the representation of $S_{kn-n-k+2}$ on $\text{Lie}_n(kn-n-k+2)$. Here we are concerned with the case $k = 3$. So $\rho_{n,3}$ denotes the representation of S_{2n-1} on $\text{Lie}_n(2n-1)$. From Theorem 1.4, we see that this special case turns out to be quite interesting.

The proof of Theorem 1.4 involves viewing $\rho_{n,3}$ as a quotient of another S_{2n-1} -module. Let $V_{n,3}$ be the multilinear component of the vector space generated by all possible n -bracketings of elements of $[2n-1]$, subject only to antisymmetry of the brackets given in (1.1) (but not to generalized Jacobi, (1.2)). That is, $V_{n,3}$ is the subspace generated by

$$u_\tau := [[\tau_1, \dots, \tau_n], \tau_{n+1}, \dots, \tau_{2n-1}],$$

where $\tau \in S_{2n-1}$, $\tau_i = \tau(i)$ for each i , and $[\cdot, \dots, \cdot]$ is the antisymmetric n -linear bracket (that does not satisfy the generalized Jacobi relation).

The symmetric group S_{2n-1} acts on generators of $V_{n,3}$ by the following action: for $\sigma, \tau \in S_{2n-1}$

$$\sigma u_\tau = u_{\sigma\tau}.$$

This induces a representation of S_{2n-1} on $V_{n,3}$ since the action respects the antisymmetry relation.

For each n -element subset $S := \{a_1, \dots, a_n\}$ of $[2n-1]$, let

$$v_S = [[a_1, \dots, a_n], b_1, \dots, b_{n-1}],$$

where $\{b_1, \dots, b_{n-1}\} = [2n-1] \setminus S$, and the a_i 's and b_i 's are in increasing order. Clearly,

$$(2.1) \quad \left\{ v_S : S \in \binom{[2n-1]}{n} \right\}$$

is a basis for $V_{n,3}$. Thus $V_{n,3}$ has dimension $\binom{2n-1}{n}$.

For each $S \in \binom{[2n-1]}{n}$, use the generalized Jacobi Identity (1.2), to define the relation

(2.2)

$$R_S = v_S - \sum_{i=1}^n [a_1, \dots, a_{i-1}, [a_i, b_1, \dots, b_{n-1}], a_{i+1}, \dots, a_n],$$

where $a_1 < \dots < a_n$ and $b_1 < \dots < b_{n-1}$ are as in the previous paragraph. Let $R_{n,3}$ be the subspace of $V_{n,3}$ generated by the R_S . Then as S_{2n-1} -modules

$$(2.3) \quad V_{n,3}/R_{n,3} \cong \rho_{n,3}.$$

Lemma 2.1. *The linear map $\varphi : V_{n,3} \rightarrow V_{n,3}$ defined on basis elements by*

$$\varphi(v_S) = R_S,$$

is an S_{2n-1} -module homomorphism.

Proof. It is not difficult to check using the antisymmetry relations that for each $\tau \in S_{2n-1}$,

$$\begin{aligned} \varphi([\tau_1, \dots, \tau_n, \tau_{n+1}, \dots, \tau_{2n-1}]) &= [[\tau_1, \dots, \tau_n], \tau_{n+1}, \dots, \tau_{2n-1}] \\ &\quad - \sum_{i=1}^n [\tau_1, \dots, \tau_{i-1}, [\tau_i, \tau_{n+1}, \dots, \tau_{2n-1}], \tau_{i+1}, \dots, \tau_n]. \end{aligned}$$

From this it is clear that the map φ commutes with the action of S_{2n+1} . \square

Lemma 2.2. (a) *As S_{2n-1} -modules,*

$$V_{n,3} \cong \bigoplus_{i=0}^{n-1} S^{2^i 1^{2n-1-2^i}}.$$

(b) *Since $V_{n,3}$ is multiplicity-free, φ acts as a scalar on each irreducible submodule.*

(c) *As S_{2n-1} -modules,*

$$(2.4) \quad \ker \varphi \cong V_{n,3}/R_{n,3}.$$

Proof. Observe that, due to the antisymmetry of the bracket, the space $V_{n,3}$ constitutes the representation of S_{2n-1} induced from the sign representation on the Young subgroup $S_n \times S_{n-1}$:

$$V_{n,3} \cong (\text{sgn}_n \times \text{sgn}_{n-1}) \uparrow_{S_n \times S_{n-1}}^{S_{2n-1}}.$$

Part (a) then follows from Young's rule twisted by the sign representation, Part (b) from Schur's lemma, and Part (c) from Part (b). \square

We now compute the action of φ on each irreducible submodule of $V_{n,3}$ in order to find $\ker \varphi$, which by (2.3) and (2.4) is isomorphic to $\rho_{n,3}$. First we make the following observation.

Lemma 2.3. *For all $v \in V_{n,3}$, let $\langle v, v_S \rangle$ denote the coefficient of v_S in the expansion of v in the basis given in (2.1). Then for all $S, T \in \binom{[2n-1]}{n}$,*

$$\langle \varphi(v_S), v_T \rangle = \begin{cases} 1 & \text{if } S = T \\ (-1)^d & \text{if } S \cap T = \{d\} \\ 0 & \text{if } S \neq T \text{ but } |S \cap T| > 1 \end{cases}$$

Theorem 2.4. *On the irreducible submodule of $V_{n,3}$ isomorphic to $S^{2^i 1^{(2n-1)-2i}}$, the operator φ acts like the scalar w_i , where*

$$(2.5) \quad w_i := 1 + (n - i)(-1)^{n-i}.$$

Proof. By Lemma 2.2, φ acts as a scalar on each irreducible submodule. To compute the scalar, we start by letting t be the Young tableau of shape $2^i 1^{2n-1-2i}$ given by

$$t = \begin{array}{|c|c|} \hline 1 & n+1 \\ \hline 2 & n+2 \\ \hline \vdots & \vdots \\ \hline i & n+i \\ \hline i+1 & \\ \hline \vdots & \\ \hline n & \\ \hline n+i+1 & \\ \hline \vdots & \\ \hline 2n-1 & \\ \hline \end{array}$$

Let C_t be the column stabilizer of t and let R_t be the row stabilizer. Recall that the Young symmetrizer associated with t is defined by

$$e_t := \sum_{\alpha \in R_t} \alpha \sum_{\beta \in C_t} \text{sgn}(\beta) \beta$$

and that the Specht module $S^{2^i 1^{2n-1-2i}}$ is the submodule of the regular representation $\mathbb{C}S_{2n-1}$ spanned by $\{\tau e_t : \tau \in S_{2n-1}\}$.

Now set $T := [n]$, $r_t := \sum_{\alpha \in R_t} \alpha$ and factor

$$\sum_{\beta \in C_t} \text{sgn}(\beta) \beta = f_t d_t,$$

where d_t is the signed sum of permutations in C_t that stabilize $\{1, 2, \dots, n\}$, $\{n+1, \dots, n+i\}$, $\{n+i+1, \dots, 2n-1\}$ and f_t is the signed sum of permutations in C_t that maintain the vertical order of these sets. So $e_t v_T = r_t f_t d_t v_T$. Because of the antisymmetry of the bracket, we have

$$\begin{aligned} d_t v_T &= n! ((n+i) - (n+1) + 1)! ((2n-1) - (n+i+1) + 1)! v_T \\ &= n! i! (n-i-1)! v_T. \end{aligned}$$

Hence $r_t f_t v_T$ is a scalar multiple of $e_t v_T$. Since the coefficient of v_T in the expansion of $r_t f_t v_T$ is 1, we have $e_t v_T \neq 0$.

Let $\psi : \mathbb{C}S_{2n-1} \rightarrow V_{n,3}$ be the S_{2n-1} -module homomorphism defined by $\psi(\sigma) = \sigma v_T$, where $\sigma \in S_{2n-1}$ and $T := [n]$. Now consider the restriction of ψ to the Specht module $S^{2^i 1^{2n-1-2i}}$. By the irreducibility of the Specht module and the fact that $e_t v_T \neq 0$, this restriction is an isomorphism from $S^{2^i 1^{2n-1-2i}}$ to the subspace of $V_{n,3}$ spanned by $\{\tau e_t v_T : \tau \in S_{2n-1}\}$. This subspace is therefore the unique subspace of $V_{n,3}$ isomorphic to $S^{2^i 1^{(2n-1)-2i}}$. From here on, we will abuse notation by letting $S^{2^i 1^{(2n-1)-2i}}$ denote the subspace of $V_{n,3}$ spanned by $\{\tau e_t v_T : \tau \in S_{2n-1}\}$.

Since $r_t f_t v_T$ is a scalar multiple of $e_t v_T$, it is in $S^{2^i 1^{(2n-1)-2i}}$. It follows that

$$\varphi(r_t f_t v_T) = c r_t f_t v_T,$$

for some scalar c , which we want to show equals w_i . Using the fact that the coefficient of v_T in $r_t f_t v_T$ is 1, we conclude that c is the coefficient of v_T in $\varphi(r_t f_t v_T)$. Hence to complete the proof we need only show that

$$(2.6) \quad \langle \varphi(r_t f_t v_T), v_T \rangle = w_i := 1 + (n-i)(-1)^{n-i}.$$

Consider the expansion,

$$r_t f_t v_T = \sum_{S \in \binom{[2n-1]}{n}} \langle r_t f_t v_T, v_S \rangle v_S,$$

which by linearity yields,

$$\varphi(r_t f_t v_T) = \sum_{S \in \binom{[2n-1]}{n}} \langle r_t f_t v_T, v_S \rangle \varphi(v_S).$$

Hence the coefficient of v_T is given by

$$\langle \varphi(r_t f_t v_T), v_T \rangle = \sum_{S \in \binom{[2n-1]}{n}} \langle r_t f_t v_T, v_S \rangle \langle \varphi(v_S), v_T \rangle.$$

Looking back at Lemma 2.3, we see that the $S = T$ term is 1, which yields,

$$\langle \varphi(r_t f_t v_T), v_T \rangle = 1 + \sum_{S \in \binom{[2n-1]}{n} \setminus \{T\}} \langle r_t f_t v_T, v_S \rangle \langle \varphi(v_S), v_T \rangle.$$

To get a contribution from an $S \neq T$ term, by Lemma 2.3, we must have $S \cap T = \{d\}$ for some d , in which case $\langle \varphi(v_S), v_T \rangle = (-1)^d$. Hence

$$(2.7) \quad \langle \varphi(r_t f_t v_T), v_T \rangle = 1 + \sum_{d=1}^n (-1)^d \langle r_t f_t v_T, v_{S(d)} \rangle,$$

where

$$S(d) = \{d, n+1, n+2, \dots, 2n-1\}.$$

To compute $\langle r_t f_t v_T, v_{S(d)} \rangle$, we must consider how we get $v_{S(d)}$ from the action of permutations appearing in $r_t f_t$ on v_T . Recall that f_t is a sum of column permutations σ of t (with sign) that maintain the vertical order of $\{1, 2, \dots, n\}$, $\{n+1, \dots, n+i\}$, and $\{n+i+1, \dots, 2n-1\}$. In order to get $S(d)$, we have that σ fixes $1, 2, \dots, i$ and $n+1, \dots, n+i$ and then the row permutation α is

$$\alpha = (1, n+1)(2, n+2) \cdots (i, n+i)(i+1) \cdots (2n-1)$$

and σ interchanges $\{n+i+1, \dots, 2n-1\}$ with a subset of $\{i+1, \dots, n\}$, leaving one element d of $\{i+1, \dots, n\}$ in rows $i+1, \dots, n$.

Since σ maintains the vertical order of $1, 2, \dots, n$, it must be that $d = i+1$. Thus the summation in (2.7) is left only with the $d = i+1$ term. Suppose that $i+1$ goes to row j with $i+1 \leq j \leq n$. So

$$\begin{array}{c}
 t = \\
 \begin{array}{|c|c|}
 \hline
 1 & n+1 \\
 \hline
 2 & n+2 \\
 \hline
 \vdots & \vdots \\
 \hline
 i & n+i \\
 \hline
 i+1 & \\
 \hline
 \vdots & \\
 \hline
 j-1 & \\
 \hline
 j & \\
 \hline
 j+1 & \\
 \hline
 \vdots & \\
 \hline
 n & \\
 \hline
 n+i+1 & \\
 \hline
 \vdots & \\
 \hline
 2n-1 & \\
 \hline
 \end{array}
 \end{array}
 \longrightarrow
 \begin{array}{c}
 \alpha\sigma t = \\
 \begin{array}{|c|c|}
 \hline
 n+1 & 1 \\
 \hline
 n+2 & 2 \\
 \hline
 \vdots & \vdots \\
 \hline
 n+i & i \\
 \hline
 n+i+1 & \\
 \hline
 \vdots & \\
 \hline
 n+j-1 & \\
 \hline
 i+1 & \\
 \hline
 n+j & \\
 \hline
 \vdots & \\
 \hline
 2n-1 & \\
 \hline
 i+2 & \\
 \hline
 \vdots & \\
 \hline
 n & \\
 \hline
 \end{array}
 \end{array}$$

One can easily compute the sign of σ by counting inversions or writing σ in cycle form as

$$\begin{aligned}
 \sigma = & (1) \cdots (i)(n+1) \cdots (n+i)(i+1, n+i+1, i+2, n+i+2, \dots, j) \\
 & (n+j, j+1)(n+j+1, j+2) \cdots (2n-1, n),
 \end{aligned}$$

where the cycle involving $i+1$ is a $((2n-1-2i)-2(n-j))$ -cycle. So

$$\text{sgn}(\sigma) = (-1)^{(2j-2i-1)+1}(-1)^{n-j} = (-1)^{n-j}.$$

In terms of our basis,

$$\begin{aligned}
 \alpha\sigma v_T = & [[n+1, n+2, \dots, n+i, n+i+1, \dots, n+j-1, i+1, n+j, \dots, 2n-1], \\
 & 1, 2, \dots, i, i+2, \dots, n].
 \end{aligned}$$

To put this basis in canonical form, we need to move $i+1$ to the front of the inside bracket, which yields $\alpha\sigma v_T = (-1)^{j-1} v_{S(i+1)}$. Hence $\text{sgn}(\sigma)\alpha\sigma v_T = (-1)^{n-1} v_{S(i+1)}$. Since there are $n-i$ positions j where $i+1$ might land,

$$\langle r_t f_t v_T, v_{S(i+1)} \rangle = (n-i)(-1)^{n-1}.$$

Since all other terms in the summation in (2.7) vanish, by plugging this into (2.7), we obtain (2.6), which completes the proof. \square

Proof of Theorem 1.4. By Theorem 2.4 and Lemma 2.2, since $w_i = 0$ for $i = n - 1$ only, $S^{2^{n-1}-1}$ is the only irreducible submodule that maps to 0 under φ . It follows that $\ker \varphi = S^{2^{n-1}-1}$. The result now follows from (2.3) and (2.4). \square

We say that a bracketed permutation

$$[[a_1, a_2, \dots, a_n], b_1, b_2, \dots, b_{n-1}]$$

is *standard* if $a_1 < \dots < a_n$, $b_1 < \dots < b_{n-1}$ and $a_i < b_i$ for each $i \in [n - 1]$. Using Theorem 1.4, it can be shown that the standard bracketed permutations on $[2n - 1]$ form a basis for the vector space $\rho_{n,3}$; see [BF].

3. GARNIR RELATIONS AND GENERALIZED JACOBI RELATIONS

Theorem 1.4 gives a new presentation of the Specht module $S^{2^{n-1}-1}$ in which the generators are Young tableaux and the relations correspond to the antisymmetry relations and the generalized Jacobi relations. In this section, we use Theorem 1.4 to prove a generalization giving a new presentation for a wider class of shapes, which includes staircase shapes.

For each partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_l)$ of m , let \mathcal{T}_λ be the set of standard Young tableaux of shape λ . Let M^λ be the vector space generated by \mathcal{T}_λ subject only to column relations, which are of the form $t + s$, where s is obtained from t by switching two entries in the same column. Given $t \in \mathcal{T}_\lambda$, let \bar{t} denote the coset of t in M^λ . The symmetric group S_m acts on \mathcal{T}_λ by replacing each entry of a tableau by its image under the permutation in S_m . This induces a representation of S_m on M^λ .

There are various different presentations of S^λ in the literature, which involve the column relations and additional relations called *Garnir relations*. Here we are interested in a presentation of S^λ discussed in Fulton [Fu, Section 7.4]. The Garnir relations used by Fulton are of the form $\bar{t} - \sum \bar{s}$, where the sum is over all $s \in \mathcal{T}_\lambda$ obtained from $t \in \mathcal{T}_\lambda$ by exchanging any k entries of any column with the top k entries of the next column, while maintaining the vertical order of each of the exchanged sets. There is a Garnir relation $g_{c,k}^t$ for every $t \in \mathcal{T}_\lambda$, every column $c \in [\lambda_1 - 1]$, and every k from 1 to the length of the column $c + 1$. Let G^λ be the subspace of M^λ generated by these Garnir relations. Clearly G^λ is invariant under the action of S_m . The following presentation of S^λ is proved in [Fu, Section 7.4]:

$$(3.1) \quad M^\lambda / G^\lambda \cong S^\lambda.$$

Suppose the length of column c of the Young tableau t is n and the length of column $c + 1$ is $n - 1$. One of the Garnir relations for column c is $g_{c,n-1}^t$, which is $\bar{t} - \sum \bar{s}$, where the sum is over all s obtained from t by exchanging the entire column $c + 1$ with all but one element of column c . There will be one s for each entry of column c that remains behind in the exchange.

Suppose column c of t has entries a_1, a_2, \dots, a_n reading from top down and column $c + 1$ has entries b_1, \dots, b_{n-1} , also reading from top down. We can associate \bar{t} with the bracketed permutation,

$$[[a_1, a_2, \dots, a_n], b_1, \dots, b_{n-1}],$$

where the bracket is antisymmetric. The Garnir relation $g_{c,n-1}^t$ corresponds to the relation

$$[[a_1, a_2, \dots, a_n], b_1, \dots, b_{n-1}] - \sum_{i=1}^n [[b_1, \dots, b_{i-1}, a_i, b_i, \dots, b_{n-1}], a_1, \dots, \hat{a}_i, \dots, a_n],$$

where $\hat{}$ denotes deletion. If we move the a_i to the front of the inner bracket and move the inner bracket to the place where the a_i was deleted, the signs will cancel each other, and we will get the generalized Jacobi relation (1.2). It therefore follows from Theorem 1.4 that $\{g_{c,n-1}^t : t \in \mathcal{T}_\lambda\}$ generates all the other Garnir relations in $\{g_{c,k}^t : t \in \mathcal{T}_\lambda, k \in [n-1]\}$ for fixed column c . This allows us to reduce the number of relations in the presentation of S^λ given in (3.1). We express this in the following result.

Theorem 3.1. *Let $(\lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_j)$ be the conjugate of $\lambda \vdash m$. Let \tilde{G}^λ be the subspace of M^λ generated the union of the sets $\{g_{c,\lambda'_{c+1}}^t : t \in \mathcal{T}_\lambda\}$ for each column c for which $\lambda'_{c+1} = \lambda'_c - 1$ and the sets $\{g_{c,k}^t : t \in \mathcal{T}_\lambda, k \in [\lambda'_{c+1}]\}$ for the other columns. Then*

$$S^\lambda \cong M^\lambda / \tilde{G}^\lambda.$$

We will say that λ is a *staircase partition* if its conjugate has the form $(n, n-1, n-2, \dots, n-r)$. Note that the partition $2^{n-1}1$ is a staircase partition. The following result reduces to Theorem 1.4 for the shape $2^{n-1}1$.

Corollary 3.2. *Let λ be a staircase partition of m and let \tilde{G}^λ be the subspace of M^λ generated by*

$$\{g_{c,\lambda'_{c+1}}^t : c \in [\lambda_1 - 1], t \in \mathcal{T}_\lambda^*\},$$

where \mathcal{T}_λ^* is the set of Young tableaux of shape λ in which each element of $[m]$ appears once and the columns increase. Then

$$S^\lambda \cong M^\lambda / \tilde{G}^\lambda.$$

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REFERENCES

- [BL] J. Bagger and N. Lambert, *Three-Algebras and N=6 Chern-Simons Gauge Theories*, Phys. Rev. D **79** (2009), 025002, 8 pp., [arXiv:0807.0163 [hep-th]].
- [Ba] H. Barcelo, *On the action of the symmetric group on the free Lie algebra and the partition lattice*, J. Combin. Theory Ser. A, **55**(1) (1990), 93–129.
- [BF] S. Brauner and T. Friedmann, in progress.
- [Co] F. Cohen, *On configuration spaces, their homology, and Lie algebras*, J. Pure and Applied Algebra **100** (1995) 19–42.
- [DT] Yu.L. Daletskii and L.A. Takhtajan, *Leibniz and Lie algebra structures for Nambu algebra*, Lett. Math. Phys. **39** (1997), 127–141.
- [Fi] V.T. Filippov, *n-Lie algebras*, Sibirsk. Math. Zh. **26** (1985), 126–140.
- [Fr] T. Friedmann, *Orbifold Singularities, Lie Algebras of the Third Kind (LATKes), and Pure Yang-Mills with Matter*, J. Math. Phys. **52** (2011), 022–304, [arXiv:0806.0024 [math.AG]].
- [FHSW] T. Friedmann, P. Hanlon, R.P. Stanley, M.L. Wachs, in progress.
- [Fu] W. Fulton, *Young Tableaux*, London Mathematical Society, Student Texts **35**, Cambridge University Press, Cambridge, 1996.
- [Gu] A. Gustavsson, *Algebraic structures on parallel M2-branes*, Nucl. Phys. B **811** (2009), 66–76, [arXiv:0709.1260 [hep-th]].
- [Ha] P. Hanlon, *The fixed-point partition lattices*, Pacific J. Math. **96** (1981), 319–341.
- [HW] P. Hanlon and M.L. Wachs, *On Lie k-algebras*, Adv. Math. **113** (1995), 206–236.
- [Ka] S. M. Kasymov, *On a theory of n-Lie algebras*, Algebra i Logika **26** (1987), 277–297.
- [Kl] A.A. Klyachko, *Lie elements in the tensor algebra*, Sibirsk. Math. Zh. (translation) **15** (1974), 1296–1304.
- [KW] W. Kraskiewicz and J. Weyman, *Algebra of coinvariants and the action of a Coxeter element*, Bayreuth. Math. Schr. **63** (2001), 265–284.
- [KS] B. Kol and R. Shir, *Color Structures and Permutations*, J. High Energ. Phys. (2014) 2014: 20.
- [Li] W. X. Ling, *On the structure of n-Lie algebras*, PhD Thesis, Siegen, 1993.
- [Re] C. Reutenauer, *Free Lie Algebras*, London Mathematical Society Monographs, New Series **7**, Clarendon Press, Oxford University Press, New York, 1993.
- [St] R.P. Stanley, *Some aspects of groups acting on finite posets*, J. Combin. Theory, Ser. A **32** (1982), 132–161.

- [Ta] L.A. Takhtajan, *On foundation of the generalized Nambu mechanics*, Comm. Math. Phys. **160** (1994), 295–315.
- [Wa] M.L. Wachs, *On the (co)homology of the partition lattice and the free Lie algebra*, Discrete Math. **93** (1998), 287–319.
- [Wh] S. Whitehouse, *The Eulerian representations of Σ_n as restrictions of representations of Σ_{n+1}* , J. Pure Appl. Algebra **115** (1997), 309–320.

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