Gallai-Ramsey numbers of C_9 with multiple colors

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Abstract

We study Ramsey-type problems in Gallai-colorings. Given a graph G and an integer $k \geq 1$, the Gallai-Ramsey number $gr_k(K_3, G)$ is the least positive integer n such that every k-coloring of the edges of the complete graph on n vertices contains either a rainbow triangle or a monochromatic copy of G. It turns out that $gr_k(K_3, G)$ behaves more nicely than the classical Ramsey number $r_k(G)$. However, finding exact values of $gr_k(K_3, G)$ is far from trivial. In this paper, we prove that $gr_k(K_3, C_9) = 4 \cdot 2^k + 1$ for all $k \geq 1$. This new result provides partial evidence for the first open case of the Triple Odd Cycle Conjecture of Bondy and Erdős from 1973. Our technique relies heavily on the structural result of Gallai on edge-colorings of complete graphs without rainbow triangles. We believe the method we developed can be used to determine the exact values of $gr_k(K_3, C_n)$ for odd integers $n \geq 11$.

Key words: Gallai-coloring, Gallai-Ramsey number, Rainbow triangle **AMS subject classifications**: 05C15; 05C55

1 Introduction

In this paper, we only consider finite simple graphs. The complete graph and the cycle on n vertices are denoted K_n and C_n , respectively. We use |G| to denote the number of vertices of a graph G.

For an integer $k \geq 1$, let $c: E(G) \rightarrow [k]$ be a k-edge-coloring of a complete graph G, where $[k] := \{1, 2, ..., k\}$. Then c is a Gallai-coloring of G if G contains no rainbow triangle (that is, a triangle with all its edges colored differently) under c. Gallai-colorings naturally

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arise in several areas including in information theory [13], in the study of partially ordered sets, as in Gallai's original paper [8], and in the study of perfect graphs [3]. There are now a variety of papers which consider Ramsey-type problems in Gallai-colorings (see, e.g., [4, 7, 9, 10, 11]). These works mainly focus on finding various monochromatic subgraphs in such colorings. More information on this topic can be found in [5, 6].

For a graph G and a set $A \subseteq V(G)$, we use G[A] to denote the subgraph of G obtained from G by deleting all vertices in $V(G) \setminus A$. A graph H is an induced subgraph of G if H = G[A] for some $A \subseteq V(G)$. Recall that the classical Ramsey number $r_k(H)$ of a graph H is the least positive integer n such that every k-edge-coloring of K_n contains a monochromatic copy of H. Ramsey numbers are notoriously difficult to compute in general. In this paper, we consider Gallai-Ramsey problems. Given a graph H and an integer $k \geq 1$, the Gallai-Ramsey number $gr_k(K_3, H)$ is the least positive integer n such that every k-edge-coloring of K_n contains either a rainbow triangle or a monochromatic copy of H. Clearly, $gr_k(K_3, H) \leq r_k(H)$. The following is a result on the general behavior of $gr_k(K_3, H)$.

Theorem 1.1 ([10]) Let H be a fixed graph with no isolated vertices and let $k \geq 1$ be an integer. If H is not bipartite, then $gr_k(K_3, H)$ is exponential in k. If H is bipartite, then $gr_k(K_3, H)$ is linear in k.

It turns out that for some graphs H (e.g., when $H = C_3$), $gr_k(K_3, H)$ behaves nicely, while the order of magnitude of $r_k(H)$ seems hopelessly difficult to determine. It is worth noting that finding exact values of $gr_k(K_3, H)$ is far from trivial, even when |H| is small. We will utilize the following important structural result of Gallai [8] on Gallai-colorings of complete graphs.

Theorem 1.2 ([8]) For any Gallai-coloring c of a complete graph G, V(G) can be partitioned into nonempty sets V_1, V_2, \ldots, V_p with p > 1 so that at most two colors are used on the edges in $E(G) \setminus (E(V_1) \cup \cdots \cup E(V_p))$ and only one color is used on the edges between any fixed pair (V_i, V_j) under c, where $E(V_i)$ denotes the set of edges in $G[V_i]$ for all $i \in [p]$.

The partition given in Theorem 1.2 is a Gallai-partition of G under c. Given a Gallai-partition V_1, V_2, \ldots, V_p of the complete graph G under c, let $v_i \in V_i$ for all $i \in [p]$ and let $\mathcal{R} := G[\{v_1, v_2, \ldots, v_p\}]$. Then \mathcal{R} is the reduced graph of G corresponding to the given Gallai-partition under c. Clearly, \mathcal{R} is isomorphic to K_p . By Theorem 1.2, all edges in \mathcal{R} are colored by at most two colors under c. One can see that any monochromatic H in \mathcal{R} under c will result in a monochromatic H in G under c. It is not a surprise then that Gallai-Ramsey

numbers $gr_k(K_3, H)$ are related to the classical Ramsey numbers $r_2(H)$. Recently, Fox, Grinshpun and Pach posed the following conjecture on $gr_k(K_3, H)$ when H is a complete graph.

Conjecture 1.3 ([5]) For integers $k \ge 1$ and $t \ge 3$,

$$gr_k(K_3, K_t) = \begin{cases} (r_2(K_t) - 1)^{k/2} + 1 & \text{if } k \text{ is even} \\ (t - 1)(r_2(K_t) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

The first case of Conjecture 1.3 was verified in 1983 due to Chung and Graham [4]. The next open case when t = 4 was recently settled in [14]. A simpler proof of Theorem 1.4 can be found in [10].

Theorem 1.4 ([4]) For any integer
$$k \ge 1$$
, $gr_k(K_3, C_3) = \begin{cases} 5^{k/2} + 1 & \text{if } k \text{ is even} \\ 2 \cdot 5^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$

Theorem 1.5 below is a result of Fujita and Magnant [7], which provides a lower bound for $gr_k(K_3, C_{2n+1})$.

Theorem 1.5 ([7]) For integers
$$k \geq 2$$
 and $n \geq 2$, $gr_k(K_3, C_{2n+1}) \geq n \cdot 2^k + 1$.

The exact values of $gr_k(K_3, C_5)$ for any integer $k \geq 1$ were also determined in [7].

Theorem 1.6 ([7])
$$gr_k(K_3, C_5) = 2 \cdot 2^k + 1$$
 for all $k \ge 1$.

Recently, Bruce and Song [2] considered the next step and determined the exact values of $gr_k(K_3, C_7)$ for any integer $k \geq 1$.

Theorem 1.7 ([2])
$$gr_k(K_3, C_7) = 3 \cdot 2^k + 1$$
 for all $k \ge 1$.

In this paper, we continue to study the Gallai-Ramsey numbers of odd cycles. We determine the exact values of $gr_k(K_3, C_9)$ for all $k \geq 1$. We believe the method we developed will be helpful in determining the exact values of $gr_k(K_3, C_n)$ for odd integers $n \geq 11$. We prove the following main result.

Theorem 1.8
$$gr_k(K_3, C_9) = 4 \cdot 2^k + 1 \text{ for all } k \ge 1.$$

It is worth mentioning that Theorem 1.8 also provides partial evidence for the first open case of the Triple Odd Cycle Conjecture due to Bondy and Erdős [1], which states that $r_3(C_n) = 4n - 3$ for any odd integer n > 3. Luczak [15] showed that if n is odd, then

 $r_3(C_n) = 4n + o(n)$, as $n \to \infty$, and Kohayakawa, Simonovits and Skokan [12] proved that the Triple Odd Cycle Conjecture holds when n is sufficiently large. We will make use of the following result of Bondy and Erdős [1].

Theorem 1.9 ([1])
$$r_2(C_{2n+1}) = 4n + 1$$
 for all $n \ge 2$.

Finally, we need to introduce more notation. For positive integers n, k and $G = K_n$, let c be any k-edge-coloring of G with color classes E_1, \ldots, E_k . Then c is bad if G contains neither a rainbow K_3 nor a monochromatic C_9 under c. For any $E \subset E(G)$, let G[E] denote the subgraph of G with vertex set V(E) and edge set E. Let H be an induced subgraph of G and let $E = E_i \cap E(H)$ for some $i \in [k]$. Then G[E] is an induced matching in H if E is a matching in H. For two disjoint sets $A, B \subseteq V(G)$, if all the edges between A and B in G are colored the same color under c, say, blue, we say that A is blue-complete to B.

2 Proof of Theorem 1.8

By Theorem 1.5, $gr_k(K_3, C_9) \ge 4 \cdot 2^k + 1$ for all $k \ge 1$. We next show that $gr_k(K_3, C_9) \le 4 \cdot 2^k + 1$ for all $k \ge 1$. This is trivially true for k = 1. By Theorem 1.9, we may assume that $k \ge 3$. Let $G = K_{4 \cdot 2^k + 1}$ and let c be any k-edge-coloring of G such that G admits no rainbow triangle. We next show that G contains a monochromatic C_9 under the coloring c.

Suppose that G does not contain a monochromatic C_9 under c. Then c is bad. Among all complete graphs on $4 \cdot 2^k + 1$ vertices with a bad k-edge-coloring, we choose G with k minimum. We next prove several claims.

Claim 2.1 Let H be an induced subgraph of G. If there exist three distinct vertices $u, v, w \in V(G \setminus H)$ such that all edges between $\{u, v, w\}$ and V(H) are colored, say blue, under c, then

- (i) there exists $V_H \subseteq V(H)$ with $|V_H| \le 4$ such that $H \setminus V_H$ has no blue edges, and
- (ii) $|H| \le 4 \cdot 2^{k-1-q} + 4$, where $q \in \{0, 1, ..., k-1\}$ is the number of colors missing on E(H) under c, other than blue.

Proof. To prove (i), suppose that for any $V_H \subseteq V(H)$ satisfying that $H \setminus V_H$ has no blue edges, $|V_H| \geq 5$. Then H must contain three blue edges u_1v_1, u_2v_2, u_3v_3 such that $u_1, u_2, u_3, v_1, v_2, v_3$ are all distinct. Thus we obtain a blue C_9 with vertices $u, u_1, v_1, v, u_2, v_2, w, u_3, v_3$ in order, a contradiction.

By (i), $H \setminus V_H$ has no blue edges. By minimality of k, $H \setminus V_H \leq 4 \cdot 2^{k-1-q}$. Then $|H| = |H \setminus V_H| + |V_H| \leq 4 \cdot 2^{k-1-q} + 4$. This proves (ii).

Let X_1, X_2, \ldots, X_m be a maximum sequence of disjoint subsets of V(G) such that for all $j \in [m]$, $1 \le |X_j| \le 2$, and all edges between X_j and $V(G) \setminus \bigcup_{i \in [j]} X_i$ are colored the same color. Let $X := \bigcup_{j \in [m]} X_j$. For each $x \in X$, let c(x) be the unique color on the edges between x and $V(G) \setminus X$. For all $i \in [k]$, let $X_i^* := \{x \in X : c(x) = \text{color } i\}$. Notice that $X = \bigcup_{i \in [k]} X_i^*$, and for any $i \in [k]$, X_i^* is possibly empty.

Claim 2.2 For all $i \in [k], |X_i^*| \le 2$.

Proof. Suppose not. Then $m \geq 2$. When choosing X_1, X_2, \ldots, X_m , let $\ell \in [m-1]$ be the largest index such that $|X_p^* \cap (X_1 \cup \cdots \cup X_\ell)| \leq 2$ for all $p \in [k]$; and let $j \in \{\ell+1, \ldots, m\}$ be the smallest index such that $3 \leq |X_i^* \cap (X_1 \cup \cdots \cup X_j)| \leq 4$ for some $i \in [k]$. Such a color i exists due to the assumption that the statement is not true; and such an index j exists because $1 \leq |X_p| \leq 2$ for all $p \in [m]$. We may assume that color i is blue. Let $B := \{x \in X_1 \cup \cdots \cup X_j : x \text{ is blue-complete to } V(G) \setminus X\}$, and let $A := X_1 \cup \cdots \cup X_j$. By the choice of j, $3 \leq |B| \leq 4$, and $|X_p^* \cap A| \leq 2$ for any $p \in [k] \setminus i$. Thus $|A \setminus B| \leq 2(k-1)$. By Claim 2.1 applied to any three vertices in B and the induced subgraph $G \setminus A$, we see that $|G \setminus A| \leq 4 \cdot 2^{k-1} + 4$. Thus,

$$|G| = |A \setminus B| + |B| + |G \setminus A| \le 2(k-1) + 4 + 4 \cdot 2^{k-1} + 4 < 4 \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction.

By Claim 2.2, $|X| \leq 2k$. Let $X' \subseteq X$ be such that for all $i \in [k]$, $|X' \cap X_i^*| = 1$ when $X_i^* \neq \emptyset$, and $|X' \cap X_i^*| = 0$ when $X_i^* = \emptyset$. Let $X'' := X \setminus X'$. Now consider a Gallai partition A_1, \ldots, A_p of $G \setminus X$ with $p \geq 2$. We may assume that $1 \leq |A_1| \leq \cdots \leq |A_s| < 3 \leq |A_{s+1}| \leq \cdots \leq |A_p|$, where $0 \leq s \leq p$. Let \mathcal{R} be the reduced graph of $G \setminus X$ with vertices a_1, a_2, \ldots, a_p , where $a_i \in A_i$ for all $i \in [p]$. By Theorem 1.2, we may assume that the edges of \mathcal{R} are colored red and blue. Notice that any monochromatic C_9 in \mathcal{R} would yield a monochromatic C_9 in G. Thus \mathcal{R} has neither red nor blue C_9 . By Theorem 1.9, $p \leq 16$. Then $|A_p| \geq 3$ because $|G| \geq 33$. Thus $p - s \geq 1$. Let

 $B := \{a_i \in V(\mathcal{R}) \mid a_i a_p \text{ is colored blue }\}$ and $R := \{a_i \in V(\mathcal{R}) \mid a_i a_p \text{ is colored red }\}.$

Then |B| + |R| = p - 1. Let $B_G := \bigcup_{a_i \in B} A_i$, $R_G := \bigcup_{a_j \in R} A_j$, $B_G^* := B_G \cup \{x \in X : x \text{ is blue-complete to } V(G) \setminus X\}$, and $R_G^* := R_G \cup \{x \in X : x \text{ is red-complete to } V(G) \setminus X\}$.

Claim 2.3 For any two disjoint sets $Y, Z \subseteq V(G)$ with $|Y|, |Z| \ge 4$, if all edges between Y and Z are colored by the same color, say blue, then no vertex in $V(G) \setminus (Y \cup Z)$ can be blue-complete to $Y \cup Z$ in G. Moreover, if $|Z| \ge 5$, then G[Z] has no blue edges.

Proof. Suppose that there exists a vertex $x \in V(G) \setminus (Y \cup Z)$ such that x is blue-complete to $Y \cup Z$ in G. Let $Y = \{y_1, \ldots, y_{|Y|}\}$ and $Z = \{z_1, \ldots, z_{|Z|}\}$. We may further assume that z_1z_2 is colored blue under c if G[Z] has a blue edge. We then obtain a blue C_9 with vertices $y_1, x, z_1, y_2, z_2, y_3, z_3, y_4, z_4$ in order when $|Y|, |Z| \ge 4$ or vertices $y_1, z_1, z_2, y_2, z_3, y_3, z_4, y_4, z_5$ in order when $|Z| \ge 5$ and G[Z] has a blue edge, a contradiction.

Claim 2.4 If $|A_p| \ge 4$ and $|B| \ge 5$ (resp. $|R| \ge 5$), then $|B_G| \le 8$ (resp. $|R_G| \le 8$).

Proof. Suppose $|A_p| \ge 4$ and $|B| \ge 5$ but $|B_G| \ge 9$. By Claim 2.3, $G[B_G]$ has no blue edges and no vertex in X is blue-complete to $V(G) \setminus X$. Thus all edges of $\mathcal{R}[B]$ are colored red in \mathcal{R} . Since $|B| \ge 5$, we can partition B_G into two subsets B_1 and B_2 such that $|B_1| \ge 5$ and $|B_2| \ge 4$ and B_1 is red-complete to B_2 in G. Then $G[B_1]$ has no red edges and no vertex in $X \cup R_G$ is red-complete to B_1 in G, else we obtain a red C_9 , a contradiction. Thus |B| = 5, $|B_2| = 4$, $|B_1| = A_i$ for some $i \in \{s+1, \ldots, p-1\}$, and $|B_G|$ is blue-complete to $|B_1|$ in |G|. Then $|A_p| \ge |B_1| \ge 5$ and $|A_p \cup R_G|$ is blue-complete to $|B_1|$ in |G|. By Claim 2.3, |G| has no blue edges. Then |G| has neither blue nor red edges, and |G| and $|A_p \cup R_G| \cup |X'|$ has no blue edges. By minimality of $|A_1| \cup |X'| \le 4 \cdot 2^{k-2}$ and $|A_2| \cup |X'| \le 4 \cdot 2^{k-1}$. But then,

$$|G| = |B_2| + |B_1 \cup X'| + |A_p \cup R_G \cup X''| \le 4 + 4 \cdot 2^{k-2} + 4 \cdot 2^{k-1} < 4 \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. Hence, $|B_G| \leq 8$. Similarly, if $|A_p| \geq 4$ and $|R| \geq 5$, then $|R_G| \leq 8$.

Claim 2.5 $p \le 9$.

Proof. Suppose $p \ge 10$. We may assume that $|B| \ge |R|$. Then $|B| \ge 5$ because |B| + |R| = p-1. Thus $|B_G| \ge |B| \ge 5$. We claim that $|A_p| \ge 4$. Suppose that $|A_p| = 3$. Then k = 3 and so |G| = 33. If $|A_{p-4}| = 3$ or $|A_{p-8}| \ge 2$, then either $\mathcal{R}[\{a_{p-4}, a_{p-3}, a_{p-2}, a_{p-1}, a_p\}]$ has a monochromatic triangle or C_5 , or $\mathcal{R}[\{a_{p-8}, a_{p-7}, \ldots, a_{p-1}, a_p\}]$ has a monochromatic C_5 because $r_2(C_5) = 9$. In either case, we see that G has a monochromatic C_9 , a contradiction. Thus $|A_{p-4}| \le 2$ and $|A_{p-8}| = 1$. Then $|A_{p-5}| = 2$, else $|G| \le 14 + 11 + 6 < 33$. Since $r_2(C_4) = 6$, we see that $\mathcal{R}[\{a_{p-5}, a_{p-4}, a_{p-3}, a_{p-2}, a_{p-1}, a_p\}]$ has a monochromatic, say blue,

 C_4 , and so $G \setminus X$ has a blue C_8 because $|A_{p-5}| = 2$. Thus no vertex in X is blue-complete to $G \setminus X$ and so $|X| \le 2(k-1) = 4$. But then $|G| \le 12 + 8 + 8 + 4 < 33$, a contradiction. Hence $|A_p| \ge 4$, as claimed.

By Claim 2.3, $G[B_G]$ has no blue edges and no vertex in X is blue-complete to $V(G)\setminus X$. Thus $|X|\leq 2(k-1)$ and all edges of $\mathcal{R}[B]$ are colored red in \mathcal{R} . Since $|A_p|\geq 4$ and $|B|\geq 5$, by Claim 2.4, $|B_G|\leq 8$. If $|A_p|=4$, then $|R_G|=|G|-|B_G|-|A_p|-|X|\geq 4\cdot 2^k+1-8-4-2(k-1)>5$ and $|R_G\cup X'|=|G|-|B_G|-|A_p|-|X''|\geq 4\cdot 2^k+1-8-4-(k-1)>4\cdot 2^{k-1}+1$. By Claim 2.3, $G[R_G]$ has no red edges and no vertex in X is red-complete to $V(G)\setminus X$. Thus $G[R_G\cup X']$ has no red edges and so $G[R_G\cup X']$ has a monochromatic C_9 by the choice of k, a contradiction. This proves that $|A_p|\geq 5$. By Claim 2.3, $G[A_p]$ has no blue edges. We next claim that $|R_G|\leq 4$. Suppose $|R_G|\geq 5$. By Claim 2.3, neither $G[R_G]$ nor $G[A_p]$ has red edges and no vertex in X is red-complete to $V(G)\setminus X$. Since $G[A_p\cup X']$ has neither red nor blue edges, we see that $|A_p\cup X'|\leq 4\cdot 2^{k-2}$ by the choice of k. Then $|R_G\cup X''|=4\cdot 2^k+1-|B_G|-|A_p\cup X'|\geq 4\cdot 2^k+1-8-4\cdot 2^{k-2}\geq 4\cdot 2^{k-1}+1$. Since $G[R_G\cup X'']$ has no red edges, we see that $|A_B\cup X'|=|G|-|B_G|-|R_G|-|X''|\geq (4\cdot 2^k+1)-8-4-(k-1)>4\cdot 2^{k-1}+1$. Since $G[A_p\cup X']$ has no blue edges, by the choice of k, $G[A_p\cup X']$ has a monochromatic C_9 , a contradiction. Therefore, $p\leq 9$.

Claim 2.6 $|A_p| \ge 5$.

Proof. By Claim 2.5, $p \leq 9$ and so $|A_p| \geq 4$. If $|A_p| = 4$, then $|A_{p-4}| \geq 3$, else $|G| \leq 16 + 10 + 2k < 4 \cdot 2^k + 1$ for all $k \geq 3$. Then $\mathcal{R}[\{a_{p-4}, a_{p-3}, a_{p-2}, a_{p-1}, a_p\}]$ has a blue triangle or a red C_5 . Then G contains a blue or red C_9 , a contradiction.

Claim 2.7 $2 \le p - s \le 4$.

Proof. Clearly, $p-s \leq 4$, else $\mathcal{R}[\{a_p, a_{p-1}, a_{p-2}, a_{p-3}, a_{p-4}\}]$ has a monochromatic K_3 or C_5 , which would yield a blue or red C_9 in G. Next suppose $p-s \leq 1$. Then p-s=1 because $p-s \geq 1$. Thus $|A_i| \leq 2$ for all $i \in [p-1]$. We may assume that $|R| \leq |B|$. We claim that $|B| \leq 3$. Suppose that $|B| \geq 4$. Then $|R| \leq 4$ because $|B| + |R| = p - 1 \leq 8$. Thus $|R_G| \leq 2|R| \leq 8$. Then $|B_G| \leq 2|B| \leq 8$ when |B| = 4. If $|B| \geq 5$, then $|B_G| \leq 8$ by Claim 2.4. Thus $4 \leq |B| \leq |B_G| \leq 8$. By Claim 2.3, $G[A_p]$ has no blue edges and no vertex in X is blue-complete to $V(G) \setminus X$. Thus $|X| \leq 2(k-1)$ and $G[A_p \cup X']$ has no blue edges.

By minimality of k, $|A_p \cup X'| \leq 4 \cdot 2^{k-1}$. Then

$$|R_G| = |G| - |B_G| - |A_p \cup X'| - |X''| \ge 4 \cdot 2^k + 1 - 8 - 4 \cdot 2^{k-1} - (k-1) > 5,$$

since $k \geq 3$. By Claim 2.3, $G[A_p]$ has no red edges and no vertex in X is red-complete to $V(G) \setminus X$. Thus $|X''| \leq k - 2$ and by minimality of k, $|A_p \cup X'| \leq 4 \cdot 2^{k-2}$. But then

$$|G| = |R_G| + |B_G| + |A_p \cup X'| + |X''| \le 8 + 8 + 4 \cdot 2^{k-2} + (k-2) < 4 \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. Thus $|B| \leq 3$, as claimed. Then $|R| \leq |B| \leq 3$. Thus $|B_G|, |R_G| \leq 2|B| \leq 6$. If $|B_G| \geq 4$, then by Claim 2.3, $G[A_p]$ has no blue edges and no vertex in X is blue-complete to $V(G) \setminus X$. Thus $|X''| \leq k - 1$ and $G[A_p \cup X']$ has no blue edges. By minimality of $k, |A_p \cup X'| \leq 4 \cdot 2^{k-1}$. But then

$$|R_G| = |G| - |B_G| - |A_p \cup X'| - |X''| \ge 4 \cdot 2^k + 1 - 6 - 4 \cdot 2^{k-1} - (k-1) > 6,$$

a contradiction. Thus $|B_G| \leq 3$. Similarly, $|R_G| \leq 3$. Thus $|B_G^*| \leq 5$ and $|X \setminus B_G^*| \leq 2(k-1)$. If $|B_G^*| \geq 3$, by Claim 2.1 applied to any three vertices in B_G^* and the induced subgraph $G[A_p]$, $|A_p| \leq 4 \cdot 2^{k-1} + 4$. But then

$$|G| = |B_G^*| + |R_G| + |A_p| + |X \setminus B_G^*| \le 5 + 3 + (4 \cdot 2^{k-1} + 4) + 2(k-1) < 4 \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. Thus $|B_G^*| \leq 2$. Similarly, $|R_G^*| \leq 2$. Since $p \geq 2$, we see that $|B| \geq 1$. Then $1 \leq |B| \leq 2$. By maximality of $m, R \neq \emptyset, |B| = 2$, and B is neither blue-nor red-complete to R in R. Thus $|B| = |B_G^*| = 2$ (and so no vertex in X is blue-complete to $V(G) \setminus X$), $1 \leq |R| \leq |R_G^*| \leq 2$, $|X| \leq 2(k-2)+1$, $|X'| \leq k-1$ and $|X''| \leq k-2$.

Let Z be a minimal set of vertices in A_p such that $G[A_p \setminus Z]$ has no blue edges. Then $G[(A_p \setminus Z) \cup X']$ has no blue edges. By minimality of k, $|(A_p \setminus Z) \cup X'| \le 4 \cdot 2^{k-1}$ and so

$$|Z| = |G| - |(A_p \setminus Z) \cup X'| - |X'' \cup B_G \cup R_G| \ge (4 \cdot 2^k + 1) - 4 \cdot 2^{k-1} - (k-2+4) \ge (2 \cdot 2^{k-1} + 4) \ge 12,$$

since $k \geq 3$. Let P be a longest blue path in $G[A_p]$ with vertices $v_1, \ldots, v_{|P|}$ in order and let $B_G = \{b_1, b_2\}$. Then $|P| \leq 5$ else we obtain a blue C_9 with vertices $b_1, v_1, \ldots, v_6, b_2, u$ in order, where $u \in A_p \setminus \{v_1, \ldots, v_6\}$, a contradiction. Suppose $|P| \geq 4$. Then $G[A_p \setminus V(P)]$ has no blue path, say P^* , on 7 - |P| vertices, else we obtain a blue C_9 via the vertices in B_G , V(P) and $V(P^*)$, a contradiction. Since $|Z| \geq 12$, we see that |P| = 4 and all blue edges in $G[A_p \setminus V(P)]$ induce a blue matching. Notice that $Z \setminus V(P)$ contains exactly one vertex from each blue edge in $G[A_p \setminus V(P)]$ by the choice of Z. Thus $G[Z \setminus V(P - v_1)]$ has no blue edges

and $|Z \setminus V(P-v_1)| \ge (2 \cdot 2^{k-1}+4) - (|P|-1) \ge 2 \cdot 2^{k-1}+1$. By Theorem 1.6, $G[Z \setminus V(P-v_1)]$ has a monochromatic, say green (possibly red), C_5 . Let u_1, u_2, u_3, u_4, u_5 be the vertices of the C_5 in order. Let u_1u_1', \ldots, u_5u_5' be a blue matching in $G[A_p]$. This is possible because all blue edges in $G[A_p \setminus V(P)]$ induce a blue matching. Since G has no rainbow triangles under the coloring c, we see that for any $i \in \{1,3\}$, $\{u_i, u_i'\}$ is green-complete to $\{u_{i+1}, u_{i+1}'\}$. Thus we obtain a green C_9 with vertices $u_1, u_2', u_1', u_2, u_3, u_4', u_3', u_4, u_5$ in order, a contradiction. This proves that $|P| \le 3$.

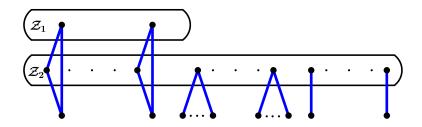


Figure 2.1: Partition of V(H)

Let H be the subgraph of $G[A_p]$ induced by all blue edges in $G[A_p]$. Since $|P| \leq 3$, we see that each component of H is isomorphic to a triangle, a star, or a K_2 . Let \mathcal{Z}_1 denote the set consisting of one vertex from each K_3 in H, and let \mathcal{Z}_2 be the set constructed by selecting: one vertex from each $K_3 \setminus \mathcal{Z}_1$ in H; the center vertex in each star in H; and one vertex in each K_2 component in H, as shown in Figure 2.1. Finally, let $\mathcal{Z}_3 := A_p \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$. Since no vertex in X is blue-complete to $V(G) \setminus X$, neither $G[\mathcal{Z}_2 \cup X']$ nor $G[\mathcal{Z}_3 \cup X'']$ has blue edges. Suppose $|\mathcal{Z}_2 \cup X'| \geq 2 \cdot 2^{k-1} + 1$. By Theorem 1.6, $G[\mathcal{Z}_2 \cup X']$ has a monochromatic, say green (possibly red), C_5 with vertices, say b_1, b_2, b_3, b_4, b_5 , in order. By the choice of X', $|X' \cap V(C_5)| \leq 1$. We may assume that $b_1, b_2, b_3, b_4 \notin X'$. Let $b'_1, b'_2, b'_3, b'_4 \in V(H)$ be such that $b_1b'_1, b_2b'_2, b_3b'_3, b_4b'_4 \in E(H)$. By the choice of $\mathcal{Z}_2, b'_1, b'_2, b'_3, b'_4$ are all distinct. Since G has no rainbow triangle under c, $\{b_i, b'_i\}$ is green-complete to $\{b_{i+1}, b'_{i+1}\}$ in G for all $i \in \{1, 3\}$. We then obtain a green C_9 in G with vertices $b_1, b'_2, b'_1, b_2, b_3, b'_4, b'_3, b_4, b_5$ in order, a contradiction. Thus $|\mathcal{Z}_2 \cup X'| \leq 2 \cdot 2^{k-1}$. Since $G[\mathcal{Z}_3 \cup X'']$ has no blue edges, by minimality of k, $|\mathcal{Z}_3 \cup X''| \leq 4 \cdot 2^{k-1}$. Hence,

$$|\mathcal{Z}_{1}| = |G| - |\mathcal{Z}_{2} \cup X'| - |\mathcal{Z}_{3} \cup X''| - |B_{G} \cup R_{G}|$$

$$\geq (4 \cdot 2^{k} + 1) - (2 \cdot 2^{k-1}) - (4 \cdot 2^{k-1}) - |B_{G} \cup R_{G}|$$

$$= 2^{k} + 1 - |B_{G} \cup R_{G}|.$$
(1)

Let $3\mathcal{Z}_1$ denote the set of vertices of all K_3 's in H. By (1),

$$|3\mathcal{Z}_1| \ge 3 \cdot 2^k + 3 - 3|B_G \cup R_G| = (2 \cdot 2^k + 1) + (2^k + 2 - 3|B_G \cup R_G|). \tag{2}$$

If $k \geq 4$ or $|B_G \cup R_G| \leq 3$, then by (2), $|3\mathcal{Z}_1| \geq 2 \cdot 2^k + 1$. By Theorem 1.6, $G[3\mathcal{Z}_1]$ has a monochromatic, say green (possibly red), C_5 with vertices, say b_1, b_2, b_3, b_4, b_5 , in order. Since $H[3\mathcal{Z}_1]$ consists of disjoint copies of K_3 's, we may assume that $N_H(b_3) \cap V(C_5) = \emptyset$. Let $b_3, b_3', b_3'' \in V(H)$ be the vertices of the K_3 in H containing b_3 . Let $b_1', b_2', b_4', b_5' \in V(H \setminus \{b_3, b_3', b_3''\})$ be such that $b_1b_1', b_2b_2', b_4b_4', b_5b_5' \in E(H)$. Note that b_1', b_2', b_4', b_5' are not necessarily distinct.

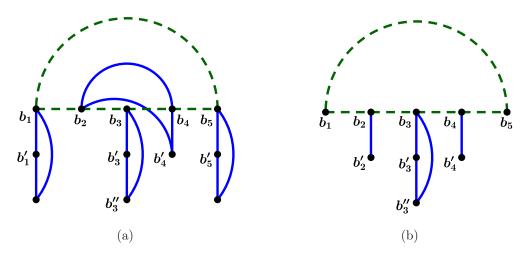


Figure 2.2: When $G[3\mathcal{Z}_1]$ has a green C_5 .

Suppose $b'_2 = b'_4$. Then b_1, b_3, b_4, b_5 each belong to different K_3 's in H, as shown in Figure 2(a). Since G has no rainbow triangle under c, $\{b_1, b'_1\}$ is green-complete to $\{b_5, b'_5\}$, and $\{b_3, b'_3\}$ is green-complete to $\{b_4, b'_4\}$ in G. We then obtain a green C_9 in G with vertices $b_1, b_2, b_3, b'_4, b'_5, b'_1, b_5$ in order, a contradiction. Thus $b'_2 \neq b'_4$. Then b'_2, b'_4, b'_3, b''_3 are all distinct and b_2, b_3, b_4 each belong to different K_3 's in H, as shown in Figure 2(b). It is possible that $H[\{b_1, b_4, b'_4\}] = K_3$ or $H[\{b_2, b'_2, b_5\}] = K_3$. Since G has no rainbow triangle under c, $\{b_3, b'_3, b''_3\}$ is green-complete to $\{b_2, b'_2, b_4, b'_4\}$ in G. We then obtain a green C_9 in G with vertices $b_1, b_2, b'_3, b'_2, b_3, b'_4, b_3'', b_4, b_5$ in order, a contradiction. Thus, k = 3 and $|B_G \cup R_G| = 4$. Let the third color of c be green. Thus all edges of $G[\mathcal{Z}_1]$ are colored red or green. By (1), $|\mathcal{Z}_1| \geq 8 + 1 - 4 = 5$. Let $b_1, b_2, b_3 \in \mathcal{Z}_1$ be distinct and let $R_G = \{u_1, u_2\}$. Let b_i, b'_i, b''_i be the vertices of the K_3 in H containing b_i for each $i \in [3]$. If b_1b_2 is colored red in G, then $\{b_1, b'_1, b''_1\}$ must be red-complete to $\{b_2, b'_2, b''_2\}$ in G because G has no rainbow triangle. Then we obtain a red C_9 in G with vertices $u_1, b_1, b'_2, b'_1, b''_2, b''_1, b_2, u_2, b_3$ in order, a

contradiction. Thus all edges of $G[\mathcal{Z}_1]$ are colored green. Since G has no rainbow triangle, $\{b_1, b_1', b_1''\}$ must be green-complete to $\{b_2, b_2', b_2'', b_3, b_3', b_3''\}$ and $\{b_2, b_2', b_2''\}$ must be green-complete to $\{b_3, b_3', b_3''\}$. We obtain a green C_9 with vertices $b_1, b_2, b_3, b_1', b_2', b_3', b_1'', b_2'', b_3''$ in order, a contradiction.

Claim 2.8 $|A_{p-2}| \le 3$.

Proof. Suppose $|A_{p-2}| \geq 4$. Then $4 \leq |A_{p-2}| \leq |A_{p-1}| \leq |A_p|$ and so $\mathcal{R}[\{a_{p-2}, a_{p-1}, a_p\}]$ is not a monochromatic triangle in \mathcal{R} . Let B_1 , B_2 , B_3 be a permutation of A_{p-2} , A_{p-1} , A_p such that B_2 is, say blue-complete, to $B_1 \cup B_3$ in G. Then B_1 must be red-complete to B_3 in G. We may assume that $|B_1| \geq |B_3|$. By Claim 2.3, no vertex in X is blue- or red-complete to $V(G) \setminus X$. Let $A := V(G) \setminus (B_1 \cup B_2 \cup B_3 \cup X)$. Then by Claim 2.3, no vertex in A is red-complete to $B_1 \cup B_3$ in G, and no vertex in A is blue-complete to $B_1 \cup B_2$ or $B_2 \cup B_3$ in G. This implies that A must be red-complete to B_2 in G. We next show that G[A] has no blue edges. Suppose that G[A] has a blue edge, say, uv. Let

 $B_1^* := \{b \in A \mid b \text{ is blue-complete to } B_1 \text{ only}\}$ $B_2^* := \{b \in A \mid b \text{ is blue-complete to both } B_1 \text{ and } B_3\}$ $B_3^* := \{b \in A \mid b \text{ is blue-complete to } B_3 \text{ only}\}.$

Then $A = B_1^* \cup B_2^* \cup B_3^*$. Notice that B_1^*, B_2^*, B_3^* are possibly empty and pairwise disjoint. Let $b_1, b_2, b_3 \in B_1$, $b_4, b_5, b_6 \in B_2$, and $b_7 \in B_3$. If uv is an edge in $G[B_1^* \cup B_2^*]$, then we obtain a blue C_9 with vertices $b_1, u, v, b_2, b_4, b_7, b_5, b_3, b_6$ in order, a contradiction. Similarly, uv is not an edge in $G[B_2^* \cup B_3^*]$. Thus uv must be an edge in $G[B_1^* \cup B_3^*]$ with one end in B_1^* and the other in B_3^* . We may assume that $u \in B_1^*$ and $v \in B_3^*$. Then we obtain a blue C_9 with vertices $b_1, u, v, b_7, b_4, b_2, b_5, b_3, b_6$ in order, a contradiction. This proves that G[A] has no blue edges.

We next show that $|B_2 \cup A \cup X'| \le 4 \cdot 2^{k-1}$. If $|B_2| \ge 5$, then by Claim 2.3, $G[B_2]$ has no blue edges. Since G[A] has no blue edges, A is red-complete to B_2 , and no vertex in X is blue-complete to B_2 , we see that $G[B_2 \cup A \cup X']$ has no blue edges. By minimality of k, $|B_2 \cup A \cup X'| \le 4 \cdot 2^{k-1}$. So we may assume that $|B_2| = 4$. Then $|B_2 \cup A \cup X'| \le 4 + 4 + (k-1) < 4 \cdot 2^{k-1}$ when $|A| \le 4$. So we may assume that $|A| \ge 5$. By Claim 2.3, G[A] has no red edges and no vertex in X is red-complete to $V(G) \setminus X$. Then $G[A \cup X']$ has neither blue nor red edges. By minimality of k, $|A \cup X'| \le 4 \cdot 2^{k-2}$ and so $|B_2 \cup A \cup X'| \le 4 + 4 \cdot 2^{k-2} < 4 \cdot 2^{k-1}$. This proves that $|B_2 \cup A \cup X'| \le 4 \cdot 2^{k-1}$.

Since $|B_1| \ge |B_3|$ and $|B_1| + |B_3| = |G| - |B_2 \cup A \cup X'| - |X''| \ge 4 \cdot 2^{k-1} + 1 - (k-1) \ge 15$, we see that $|B_1| > 4$. Note that $|B_2| \ge 4$ and $|B_3| \ge 4$. By Claim 2.3, $G[B_1]$ has neither red nor blue edges. Since each vertex in X is neither red- nor blue-complete to B_1 , $G[B_1 \cup X'']$ has neither red nor blue edges. By minimality of k, $|B_1 \cup X''| \le 4 \cdot 2^{k-2}$ and so $|B_3| \le |B_1| \le 4 \cdot 2^{k-2}$. But then

$$|G| = |B_2 \cup A \cup X'| + |B_1 \cup X''| + |B_3| \le 4 \cdot 2^{k-1} + 4 \cdot 2^{k-2} + 4 \cdot 2^{k-2} = 4 \cdot 2^k$$

a contradiction.

By Claim 2.7, $2 \le p - s \le 4$ and so $|A_{p-1}| \ge 3$. We may now assume that $a_p a_{p-1}$ is colored blue in \mathcal{R} . Then $a_{p-1} \in B$ and so $A_{p-1} \subseteq B_G$. Thus $|B_G| \ge |A_{p-1}| \ge 3$.

Claim 2.9 $|R_G^*| \le 8$.

Proof. Suppose that $|R_G^*| \geq 9$. By Claim 2.6, $|A_p| \geq 5$. By Claim 2.3, $G[R_G^*]$ has no red edges. Thus $|R_G^*| = |R_G| \geq 9$ and so no vertex in X is red-complete to $V(G) \setminus X$. By Claim 2.4, $|R| \leq 4$. By Claim 2.8, $|A_{p-2}| \leq 3$. Since $A_{p-1} \cap R_G = \emptyset$ and $p-s \leq 4$, we see that $|A_{p-2}| = 3$ and |R| = 4. Then all edges in $\mathcal{R}[R]$ are colored blue because $G[R_G]$ has no red edges. It can be easily checked that $G[R_G]$ has a blue C_9 , a contradiction.

Claim 2.10 $|A_{p-1}| \le 4$.

Proof. Suppose $|A_{p-1}| \geq 5$. Then $5 \leq |A_{p-1}| \leq |A_p|$. Thus $|B_G| \geq |A_{p-1}| \geq 5$. By Claim 2.3, neither $G[A_p]$ nor $G[B_G]$ has blue edges, and no vertex in X is blue-complete to $V(G) \setminus X$. Thus $|X| \leq 2(k-1)$. By the choice of k, $|B_G \cup X''| \leq 4 \cdot 2^{k-1}$ and $|A_p \cup X'| \leq 4 \cdot 2^{k-1}$. We claim that $G[R_G]$ has blue edges. Suppose that $G[R_G]$ has no blue edges. Then $G[A_p \cup R_G \cup X']$ has no blue edges. By the choice of k, $|A_p \cup R_G \cup X'| \leq 4 \cdot 2^{k-1}$. But then $|B_G \cup X''| = |G| - |A_p \cup R_G \cup X'| \geq 4 \cdot 2^{k-1} + 1$, a contradiction. Thus $G[R_G]$ has blue edges, as claimed. Then $|R_G| \geq 2$. Suppose that $|R_G^*| \geq 4$. By Claim 2.3, $G[A_p]$ has no red edges. By the choice of k, $|A_p \cup (X' \setminus R_G^*)| \leq 4 \cdot 2^{k-2}$. By Claim 2.9, $|R_G^*| \leq 8$. But then

$$|G| = |A_p \cup (X' \setminus R_G^*)| + |B_G \cup (X'' \setminus R_G^*)| + |R_G^*| \le 4 \cdot 2^{k-2} + 4 \cdot 2^{k-1} + 8 < 4 \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. Next suppose that $|R_G^*| = 3$. By Claim 2.1 applied to the three vertices in R_G^* and the induced subgraph $G[A_p]$, $|A_p| \leq 4 \cdot 2^{k-2} + 4$. But then

$$|G| \le |A_p| + |B_G \cup X''| + |R_G^*| + |X' \setminus R_G^*| \le (4 \cdot 2^{k-2} + 4) + 4 \cdot 2^{k-1} + 3 + (k-2) < 4 \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. Thus $|R_G^*| = |R_G| = 2$. Then $|X''| \leq k - 2$. Let $R_G = \{a, b\}$. Then ab must be colored blue under c because $G[R_G]$ has blue edges. If a or b, say b, is red-complete to B_G in G, then neither $G[A_p \cup \{a\} \cup X']$ nor $G[B_G \cup \{b\} \cup X'']$ has blue edges. By minimality of k, $|A_p \cup \{a\} \cup X'| \leq 4 \cdot 2^{k-1}$ and $|B_G \cup \{b\} \cup X''| \leq 4 \cdot 2^{k-1}$. But then $|G| = |A_p \cup \{a\} \cup X'| + |B_G \cup \{b\} \cup X''| \leq 4 \cdot 2^{k-1} + 4 \cdot 2^{k-1} < 4 \cdot 2^k + 1$ for all $k \geq 3$, a contradiction. Let $a', b' \in B_G$ be such that aa' and bb' are colored blue under c. Then a' = b', else we obtain a blue C_9 in G with vertices $a', a, b, b', x_1, y_1, x_2, y_2, x_3$ in order, where $x_1, x_2, x_3 \in A_p$ and $y_1, y_2, y_3 \in B_G \setminus \{a', b'\}$, a contradiction. Thus $\{a, b\}$ is red-complete to $B_G \setminus a'$ in G. Then there exists $i \in [s]$ such that $A_i = \{a'\}$. Since $G[B_G]$ has no blue edges, we see that $\{a, b, a'\}$ must be red-complete to $B_G \setminus a'$ in G. By Claim 2.1 applied to the three vertices a, b, a' and the induced subgraph $G[B_G \setminus a']$, we see that $|B_G \setminus a'| \leq 4 \cdot 2^{k-2} + 4$. But then

$$|G| = |A_p \cup X'| + |B_G \setminus a'| + |\{a, b, a'\}| + |X''| \le 4 \cdot 2^{k-1} + (4 \cdot 2^{k-2} + 4) + 3 + (k-2) < 4 \cdot 2^k + 1,$$
 for all $k \ge 3$, a contradiction. Hence, $|A_{p-1}| \le 4$.

By Claim 2.8, $|A_{p-2}| \leq 3$. Thus $|A_p| \geq 6$ because $|G| \geq 33$ and $p \leq 9$. By Claim 2.7, $2 \leq p - s \leq 4$ and so $|A_{p-1}| \geq 3$. By Claim 2.10, $3 \leq |A_{p-1}| \leq 4$. Then $|B_G \cup R_G| \leq 4 + 6 + 10 = 20$. Clearly, $|B_G| \leq 12$ by Claim 2.4 (when $|B| \geq 5$) and the fact $p - s \leq 4$ (when $|B| \leq 4$). By Claim 2.9, $|R_G| \leq |R_G^*| \leq 8$. Note that $|B_G| \geq |A_{p-1}| \geq 3$. We first consider the case when $|R_G^*| \geq 4$. Since $|A_p| \geq 6$, by Claim 2.3, $G[A_p]$ has no red edges. If $|B_G| = 3$, then by Claim 2.1 applied to the three vertices in B_G and the induced subgraph $G[A_p]$, $|A_p| \leq 4 \cdot 2^{k-2} + 4$. Clearly, $|X| \leq 2k$. But then

$$|G| = |A_p| + |B_G| + |R_G| + |X| \le (4 \cdot 2^{k-2} + 4) + 3 + 8 + 2k < 4 \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. Thus $|B_G| \geq 4$. By Claim 2.3, $G[A_p]$ has no blue edges and no vertex in X is blue-complete to A_p in G. Since $G[A_p]$ has neither red nor blue edges, and no vertex in X is blue-complete to A_p in G, it follows that $|X| \leq 2(k-1)$ and $|A_p| \leq 4 \cdot 2^{k-2}$ by minimality of k. But then

$$|G| = |A_p| + |X| + (|B_G| + |R_G|) \le 4 \cdot 2^{k-2} + 2(k-1) + 20 < 4 \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction.

It remains to consider the case when $|R_G^*| \leq 3$. If $|B_G| = 3$, then by Claim 2.1 applied to the three vertices in B_G and the induced subgraph $G[A_p]$, $|A_p| \leq 4 \cdot 2^{k-1} + 4$. But then

$$|G| = |A_p| + |B_G| + |R_G| + |X| \le (4 \cdot 2^{k-1} + 4) + 3 + 3 + 2k < 4 \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. Thus $|B_G| \geq 4$. By Claim 2.3, $G[A_p]$ has no blue edges and no vertex in X is blue-complete to A_p in G. Thus $|X| \leq 2(k-1)$ and $|X'' \setminus R_G^*| \leq k-2$. By minimality of k, $|A_p \cup X'| \leq 4 \cdot 2^{k-1}$. But then

$$|G| \le |A_p \cup X'| + |B_G| + |R_G^*| + |X'' \setminus R_G^*| \le 4 \cdot 2^{k-1} + 12 + 3 + (k-2) < 4 \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction.

This completes the proof of Theorem 1.8.

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