SECOND-ORDER SOBOLEV INEQUALITIES ON A CLASS OF RIEMANNIAN MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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ABSTRACT. Let (M,g) be an $n-$dimensional complete open Riemannian manifold with nonnegative Ricci curvature verifying $\rho\Delta_g\rho \geq n-5 \geq 0$, where Δ_g is the Laplace-Beltrami operator on (M,g) and ρ is the distance function from a given point. If (M,g) supports a second-order Sobolev inequality with a constant $C>0$ close to the optimal constant K_0 in the second-order Sobolev inequality in $\mathbb{R}^n$, we show that a global volume non-collapsing property holds on (M,g). The latter property together with a Perelman-type construction established by Munn (J. Geom. Anal., 2010) provide several rigidity results in terms of the higher-order homotopy groups of (M,g). Furthermore, it turns out that (M,g) supports the second-order Sobolev inequality with the constant $C=K_0$ if and only if (M,g) is isometric to the Euclidean space $\mathbb{R}^n$.

1. INTRODUCTION

It is well known that the validity of first-order Sobolev inequalities on Riemannian manifolds strongly depend on the curvature; this is a rough conclusion of the famous AB-program initiated by Th. Aubin in the seventies, see the monograph of Hebey [13] for a systematic presentation. To be more precise, let (M,g) be an $n-$dimensional complete Riemannian manifold, $n\geq 3$, and consider for some $C>0$ the first-order Sobolev inequality

$$\left(\int_M |u|^{2^*}{\rm d}v_g\right)^{\frac{2}{2^*}} \leq C\int_M |\nabla_g u|^2{\rm d}v_g,\ \forall u\in C_0^\infty(M), \qquad (\mathbf{FSI})_C$$

where $2^*=\frac{2n}{n-2}$ is the first-order critical Sobolev exponent, and ${\rm d}v_g$ and ∇_g denote the canonical volume form and gradient on (M,g), respectively. On one hand, inequality $(\mathbf{FSI})_C$ holds on any $n-$dimensional Cartan-Hadamard manifold (M,g) (i.e., simply connected, complete Riemannian manifold with nonpositive sectional curvature) with the optimal Euclidean constant $C=c_0=[\pi n(n-2)]^{-1}\left(\Gamma(n)/\Gamma(\frac{n}{2})\right)^{2/n}$ whenever the Cartan-Hadamard conjecture holds on (M,g), e.g., $n\in\{3,4\}$. On the other hand, due to Ledoux [17], if (M,g) has nonnegative Ricci curvature, inequality $(\mathbf{FSI})_{c_0}$ holds if and only if (M,g) is isometric to the Euclidean space $\mathbb{R}^n$. Further first-order Sobolev-type inequalities on Riemannian/Finsler manifolds can be found in Bakry, Concordet and Ledoux [2], Druet, Hebey and Vaugon [8], do Carmo and Xia [6], Xia [24]-[26], Kristály [15]; moreover, similar Sobolev inequalities are also considered on 'nonnegatively' curved metric measure spaces formulated in terms of the Lott-Sturm-Villani-type curvature-dimension condition or the Bishop-Gromov-type doubling measure condition, see Kristály [14] and Kristály and Ohta [16].

With respect to first-order Sobolev inequalities, much less is know about higher-order Sobolev inequalities on curved spaces. The first studies concern the AB-program for Paneitz-type operators on compact Riemannian manifolds, see Djadli, Hebey and Ledoux [7], Hebey [12] and Biezuner and Montenegro [3]. Recently, Gursky and Malchiodi [11] studied strong maximum principles for Paneitz-type operators on complete Riemannian manifolds with semi-positive $Q-$curvature and nonnegative scalar curvature.

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The aim of the present paper is to establish rigidity results on Riemannian manifolds with nonnegative Ricci curvature supporting second-order Sobolev inequalities. In order to present our results, let (M,g) be an n-dimensional complete open Riemannian manifold, $n\geq 5$, $B(x,r)$ be the geodesic ball with center $x\in M$ and radius $r>0$, and $\mathrm{vol}_g[B(x,r)]$ be the volume of $B(x,r)$. We say that (M,g) supports the *second-order Sobolev inequality for* $C>0$ if

$$\left(\int_M |u|^{2^\sharp}\mathrm{d}v_g\right)^{\frac{2}{2^\sharp}} \leq C\int_M (\Delta_g u)^2\,\mathrm{d}v_g, \quad \forall u\in C_0^\infty(M), \qquad (\mathbf{SSI})_C$$

where $2^\sharp=\frac{2n}{n-4}$ is the second-order critical Sobolev exponent, and Δ_g is the Laplace-Beltrami operator on (M,g). Note that the Euclidean space $\mathbb{R}^n$ supports $(\mathbf{SSI})_{K_0}$ for

$$K_0=\left[\pi^2 n(n-4)(n^2-4)\right]^{-1}\left(\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\right)^{4/n}. \qquad (1)$$

Moreover, K_0 is optimal, see Edmunds, Fortunato and Janelli [9], Lieb [19] and Lions [20], and the unique class of extremal functions is

$$u_{\lambda,x_0}(x)=\left(\lambda+|x-x_0|^2\right)^{\frac{4-n}{2}},\ x\in\mathbb{R}^n,$$

where $\lambda>0$ and $x_0\in\mathbb{R}^n$ are arbitrarily fixed.

To state our results, we need a technical assumption on the manifold (M,g); namely, if ρ is the distance function on M from a given point $x_0\in M$, we say that (M,g) satisfies the *distance Laplacian growth condition* if

$$\rho\Delta_g\rho\geq n-5\,.$$

Now, our main result reads as follows.

Theorem 1.1. *Let $n\geq 5$ and (M,g) be an n-dimensional complete open Riemannian manifold with nonnegative Ricci curvature which satisfies the distance Laplacian growth condition. Assume that (M,g) supports the second-order Sobolev inequality $(\mathbf{SSI})_C$ for some $C>0$. Then the following properties hold:*

(i) $C\geq K_0$;

(ii) *if in addition $C\leq \frac{n+2}{n-2}K_0$, then we have the global volume non-collapsing property*

$$\mathrm{vol}_g[B(x,r)]\geq (C^{-1}K_0)^{\frac{n}{4}}\omega_n r^n \quad \textit{for all } r>0, x\in M\,,$$

where ω_n is the volume of the $n-$dimensional Euclidean unit ball.

Remark 1.1. The distance Laplacian growth condition on (M,g) is indispensable in our argument which shows the genuine second-order character of the studied problem. We notice that the counterpart of this condition in the first-order Sobolev inequality $(\mathbf{FSI})_C$ is the validity of an eikonal inequality $|\nabla_g\rho|\leq 1$ a.e. on M, which trivially holds on any complete Riemannian manifold (and any metric measure space with a suitable derivative notion). Further comments on this condition will be given in Section 3.

Having the global volume non-collapsing property of geodesic balls of (M,g) in Theorem 1.1 (ii), we shall prove that once $C>0$ in $(\mathbf{SSI})_C$ is closer and closer to the optimal Euclidean constant K_0, the Riemannian manifold (M,g) approaches topologically more and more to the Euclidean space $\mathbb{R}^n$. To describe quantitatively this phenomenon, we recall the construction of Munn [22] based on the double induction argument of Perelman [23]. In fact, Munn determined explicit lower bounds for the volume growth of the geodesic balls in terms of certain constants which guarantee the triviality of the k-th homotopy group $\pi_k(M)$ of (M,g). More precisely, let $n\geq 5$ and for $k\in\{1,...,n\}$, let us denote by $\delta_{k,n}>0$ the smallest positive solution to the equation

$$10^{k+2}C_{k,n}(k)s\left(1+\frac{s}{2k}\right)^k=1$$

in the variable $s>0$, where

$$C_{k,n}(i) = \begin{cases} 1 & \text{if} \quad i=0, \\ 3+10C_{k,n}(i-1)+(16k)^{n-1}(1+10C_{k,n}(i-1))^n & \text{if} \quad i\in\{1,...,k\}. \end{cases}$$

We now consider the smooth, bijective and increasing function $h_{k,n}:(0,\delta_{k,n})\to(1,\infty)$ defined by

$$h_{k,n}(s) = \left[1-10^{k+2}C_{k,n}(k)s\left(1+\frac{s}{2k}\right)^k\right]^{-1}.$$

For every $k\in\{1,...,n\}$, let

$$\alpha_{MP}(k,n) = \begin{cases} 1-\left[1+\frac{2}{h_{1,n}^{-1}(2)}\right]^{-1} & \text{if} \quad k=1, \\ 1-\left[1+\left(\frac{1+...+\frac{h_{k-1,n}^{-1}(1+\frac{\delta_{k,n}}{2k})}{2(k-1)}}{h_{1,n}^{-1}\left(1+...+\frac{h_{k-1,n}^{-1}(1+\frac{\delta_{k,n}}{2k})}{2(k-1)}\right)}\right)^n\right]^{-1} & \text{if} \quad k\in\{2,...,n\}, \end{cases}$$

be the so-called *Munn-Perelman constant*, see Munn [22, Tables 4 and 5, p. 749-750].

Following the idea from Kristály [14], our quantitative result reads as follows:

Theorem 1.2. *Under the same assumptions as in Theorem 1.1, we have*

(i) *if $C\le\frac{n+2}{n-2}K_0$, the order of the fundamental group $\pi_1(M)$ is bounded above by $\left(\frac{C}{K_0}\right)^{\frac{n}{4}}$ (in particular, if $C<2^{\frac{4}{n}}K_0$, then M is simply connected*);

(ii) *if $C<\alpha_{MP}(k_0,n)^{-\frac{4}{n}}K_0$ for some $k_0\in\{1,...,n\}$ then $\pi_1(M)=...=\pi_{k_0}(M)=0$;*

(iii) *if $C<\alpha_{MP}(n,n)^{-\frac{4}{n}}K_0$ then M is contractible*;

(iv) *$C=K_0$ if and only if (M,g) is isometric to the Euclidean space $\mathbb{R}^n$.*

2. Proof of Theorems 1.1&1.2

Throughout this section, we assume the hypotheses of Theorem 1.1 are verified, i.e., (M,g) is an n-dimensional complete open Riemannian manifold with nonnegative Ricci curvature which satisfies the distance Laplacian growth condition and supports the second-order Sobolev inequality $(\mathbf{SSI})_C$ for $C>0$.

(i) The inequality $C\ge K_0$ follows in a similar way as in Djadli, Hebey and Ledoux [7, Lemmas 1.1&1.2] by using a geodesic, normal coordinate system at a given point $x_0\in M$.

(ii) Before starting the proof explicitly, we notice that one can assume that $C>K_0$; otherwise, if $C=K_0$ then we can assume that $(\mathbf{SSI})_C$ holds with $C=K_0+\varepsilon$, where $\varepsilon>0$ is arbitrarily small, and then letting $\varepsilon\to0$. Now, we split the proof into five steps.

Step 1. *ODE via the Euclidean optimizer.* We consider the function $G:(0,\infty)\to\mathbb{R}$ defined by

$$G(\lambda)=\int_{\mathbb{R}^n}\frac{\mathrm{d}x}{(\lambda+|x|^2)^{n-2}}.$$

The layer cake representation shows that for every $\lambda>0$,

$$G(\lambda)=2(n-2)\omega_n\int_0^\infty\frac{t^{n+1}}{(\lambda+t^2)^{n-1}}\mathrm{d}t=\frac{2^{4-n}\pi^{\frac{n+1}{2}}}{(n-4)\Gamma(\frac{n-1}{2})}\lambda^{\frac{4-n}{2}}. \tag{2}$$

Clearly, G is smooth on $(0,\infty)$.

We recall now by (1) that

$$\left(\int_{\mathbb{R}^n}|u_\lambda|^{2^\sharp}\mathrm{d}x\right)^{\frac{2}{2^\sharp}}=K_0\int_{\mathbb{R}^n}(\Delta u_\lambda)^2\,\mathrm{d}x\,,$$

where

$$u_\lambda(x) = (\lambda + |x|^2)^{\frac{4-n}{2}},\ x \in \mathbb{R}^n,$$

and $\lambda > 0$ is arbitrarily fixed. In terms of the function G, the above equality can be rewritten as

$$\left(\frac{G''(\lambda)}{(n-2)(n-1)}\right)^{\frac{n-4}{n}} = K_0(n-4)^2 \left\{4G(\lambda) - 4\lambda G'(\lambda) + \frac{n-2}{n-1}\lambda^2 G''(\lambda)\right\}.$$

By introducing the function

$$G_0(\lambda) = \left(\frac{K_0}{C}\right)^{\frac{n}{4}} (G(\lambda) - \lambda G'(\lambda)),\ \lambda > 0,$$

the latter relation is equivalent to the ODE

$$\left(-\frac{G_0'(\lambda)}{\lambda(n-2)(n-1)}\right)^{\frac{n-4}{n}} = C(n-4)^2 \left\{4G_0(\lambda) - \frac{n-2}{n-1}\lambda G_0'(\lambda)\right\},\quad \lambda > 0. \tag{3}$$

Step 2. *ODI via* $(\mathbf{SSI})_C$. Let $x_0 \in M$ be the point for which the distance Laplacian growth condition holds and let $F : (0,\infty) \to \mathbb{R}$ be defined by

$$F(\lambda) = \int_M \frac{\mathrm{d}v_g}{(\lambda + \rho^2)^{n-2}}.$$

Since (M, g) has nonnegative Ricci curvature, the Bishop-Gromov comparison theorem asserts that $\mathrm{vol}_g[B(x_0,t)] \leq \omega_n t^n$ for every $t > 0$; thus, by the layer cake representation and a change of variables, it turns out that

$$\begin{aligned} F(\lambda) &= \int_0^\infty \mathrm{vol}_g\left\{x \in M : \frac{1}{(\lambda + \rho(x)^2)^{n-2}} > s\right\} \mathrm{d}s \\ &= 2(n-2)\int_0^\infty \mathrm{vol}_g[B(x_0,t)] \frac{t}{(\lambda + t^2)^{n-1}} \mathrm{d}t \\ &\leq 2(n-2)\omega_n \int_0^\infty \frac{t^{n+1}}{(\lambda + t^2)^{n-1}} \mathrm{d}t \\ &= G(\lambda). \end{aligned} \tag{4}$$

Thus $0 < F(\lambda) < \infty$ for every $\lambda > 0$, and F is smooth. In a similar way,

$$F'(\lambda) = -2(n-2)(n-1)\int_0^\infty \mathrm{vol}_g[B(x_0,t)] \frac{t}{(\lambda + t^2)^n} \mathrm{d}t, \tag{5}$$

$$F''(\lambda) = 2(n-2)(n-1)n\int_0^\infty \mathrm{vol}_g[B(x_0,t)] \frac{t}{(\lambda + t^2)^{n-1}} \mathrm{d}t,$$

and for every $\lambda > 0$,

$$-\infty < G'(\lambda) \leq F'(\lambda) < 0 \ \text{ and } \ 0 < F''(\lambda) \leq G''(\lambda) < \infty. \tag{6}$$

Let $\lambda > 0$ be fixed; we observe that the function

$$w_\lambda = \left(\lambda + \rho^2\right)^{\frac{4-n}{2}}$$

can be approximated by elements from $C_0^\infty(M)$; in particular, by using an approximation procedure, one can use the function w_λ as a test-function in $(\mathbf{SSI})_C$. Accordingly,

$$\left(\int_M |w_\lambda|^{2^\sharp} \mathrm{d}v_g\right)^{\frac{2}{2^\sharp}} \leq C \int_M (\Delta_g w_\lambda)^2 \,\mathrm{d}v_g,\ \forall \lambda > 0. \tag{7}$$

A chain rule and the eikonal equation $|\nabla_g \rho| = 1$ shows that

$$(\Delta_g w_\lambda)^2 = (n-4)^2(\lambda + \rho^2)^{-n}\left(\lambda + (3-n)\rho^2 + (\lambda + \rho^2)\rho\Delta_g\rho\right)^2.$$

Since the Ricci curvature is nonnegative on (M,g), we first have the distance Laplacian comparison $\rho\Delta_g\rho \leq n-1$. Thus,

$$\lambda + (3-n)\rho^2 + (\lambda+\rho^2)\rho\Delta_g\rho \leq 2\rho^2 + n\lambda\,,\ \forall\lambda>0. \tag{8}$$

On the other hand, by the distance Laplacian growth condition, i.e., $\rho\Delta_g\rho \geq n-5$, we obtain that

$$-\left(2\rho^2+n\lambda\right) \leq \lambda + (3-n)\rho^2 + (\lambda+\rho^2)\rho\Delta_g\rho\,,\ \forall\lambda>0. \tag{9}$$

Consequently, by (8) and (9), we have that

$$|\lambda + (3-n)\rho^2 + (\lambda+\rho^2)\rho\Delta_g\rho| \leq 2\rho^2 + n\lambda\,,\ \forall\lambda>0.$$

Thus, it turns out that

$$(\Delta_g w_\lambda)^2 \leq (n-4)^2(\lambda+\rho^2)^{-n}\left(2\rho^2+n\lambda\right)^2\,.$$

According to the latter estimate, relation (7) can be written in terms of the function F as

$$\left(\frac{F''(\lambda)}{(n-2)(n-1)}\right)^{\frac{n-4}{n}} \leq C(n-4)^2\left\{4F(\lambda) - 4\lambda F'(\lambda) + \frac{n-2}{n-1}\lambda^2F''(\lambda)\right\}.$$

By defining the function

$$F_0(\lambda) = F(\lambda) - \lambda F'(\lambda),$$

the latter relation is equivalent to the ordinary differential inequality

$$\left(-\frac{F_0'(\lambda)}{\lambda(n-2)(n-1)}\right)^{\frac{n-4}{n}} \leq C(n-4)^2\left\{4F_0(\lambda) - \frac{n-2}{n-1}\lambda F_0'(\lambda)\right\},\quad \lambda>0. \tag{10}$$

Step 3. *Comparison of G and F near the origin.* We claim that

$$\liminf_{\lambda\to0}\frac{F(\lambda)-\lambda F'(\lambda)}{G(\lambda)-\lambda G'(\lambda)} \geq 1\,.$$

To see this, fix $\varepsilon>0$ arbitrarily small. Since

$$\lim_{t\to0}\frac{\mathrm{vol}_g[B(x_0,t)]}{\omega_n t^n} = 1\,,$$

there exists a $\delta>0$ such that $\mathrm{vol}_g[B(x_0,t)] \geq (1-\varepsilon)\omega_n t^n$ for all $t\in(0,\delta]$. Thus, by (4) and (5), we have

$$\begin{aligned} F(\lambda) &\geq 2(n-2)\int_0^\delta \mathrm{vol}_g[B(x_0,t)]\frac{t}{(\lambda+t^2)^{n-1}}\mathrm{d}t \\ &\geq 2(n-2)\omega_n(1-\varepsilon)\int_0^\delta\frac{t^{n+1}}{(\lambda+t^2)^{n-1}}\mathrm{d}t \\ &= 2(n-2)\omega_n\lambda^{\frac{4-n}{2}}(1-\varepsilon)\int_0^{\delta\lambda^{-\frac12}}\frac{s^{n+1}}{(1+s^2)^{n-1}}\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} -\lambda F'(\lambda) &\geq 2(n-2)(n-1)\lambda\int_0^\delta \mathrm{vol}_g[B(x_0,t)]\frac{t}{(\lambda+t^2)^n}\mathrm{d}t \\ &\geq 2(n-2)(n-1)\omega_n\lambda(1-\varepsilon)\int_0^\delta\frac{t^{n+1}}{(\lambda+t^2)^n}\mathrm{d}t \\ &= 2(n-2)(n-1)\omega_n\lambda^{\frac{4-n}{2}}(1-\varepsilon)\int_0^{\delta\lambda^{-\frac12}}\frac{s^{n+1}}{(1+s^2)^n}\mathrm{d}s. \end{aligned}$$

Combining this estimates with relation (2), we obtain

$$\liminf_{\lambda\to 0}\frac{F(\lambda)-\lambda F'(\lambda)}{G(\lambda)-\lambda G'(\lambda)}\geq 1-\varepsilon\,.$$

Letting $\varepsilon\to 0$, we get the required claim.

Step 4. *Global comparison of G_0 and F_0.* We claim that

$$F_0(\lambda)\geq G_0(\lambda),\ \forall\lambda>0. \tag{11}$$

First of all, by Step 3 and the fact that $C>K_0$, we have

$$\begin{aligned}\liminf_{\lambda\to 0}\frac{F_0(\lambda)}{G_0(\lambda)} &= \left(\frac{C}{K_0}\right)^{\frac{n}{4}}\liminf_{\lambda\to 0}\frac{F(\lambda)-\lambda F'(\lambda)}{G(\lambda)-\lambda G'(\lambda)}\\ &\geq \left(\frac{C}{K_0}\right)^{\frac{n}{4}}\\ &> 1\,.\end{aligned}$$

Thus, for sufficiently small $\delta_0>0$, one has

$$F_0(\lambda)\geq G_0(\lambda),\ \forall\lambda\in(0,\delta_0). \tag{12}$$

In fact, we shall prove that δ_0 can be arbitrarily large in (12) which ends the proof of (11). By contradiction, let us assume that $F_0(\lambda_0)<G_0(\lambda_0)$ for some $\lambda_0>0$; clearly, $\lambda_0>\delta_0$. Due to (12), we may set

$$\lambda_s=\sup\{\lambda<\lambda_0;\ F_0(\lambda)=G_0(\lambda)\}\,.$$

Then, $\lambda_s<\lambda_0$ and for any $\lambda\in[\lambda_s,\lambda_0]$, one has $F_0(\lambda)\leq G_0(\lambda)$. For $\lambda>0$, we define the function $\varphi_\lambda:(0,\infty)\to\mathbb{R}$ by

$$\varphi_\lambda(t)=t^{\frac{n-4}{n}}-C(n-2)^2(n-4)^2\lambda^2 t\,.$$

We notice that φ_λ is non-decreasing in $(0,t_\lambda]$, where

$$t_\lambda=\frac{\lambda^{-\frac{n}{2}}}{(Cn(n-4)(n-2)^2)^{\frac{n}{4}}}\,.$$

On one hand, a straightforward computation shows that for every $\lambda>0$, one has

$$0<-\frac{G_0'(\lambda)}{\lambda(n-2)(n-1)}=\left(\frac{K_0}{C}\right)^{\frac{n}{4}}\frac{G''(\lambda)}{(n-2)(n-1)}<t_\lambda.$$

On the other hand, relation (6) and the assumption $C\leq\frac{n+2}{n-2}K_0$ imply that for every $\lambda>0$,

$$0<-\frac{F_0'(\lambda)}{\lambda(n-2)(n-1)}=\frac{F''(\lambda)}{(n-2)(n-1)}\leq\frac{G''(\lambda)}{(n-2)(n-1)}\leq t_\lambda.$$

We claim that

$$F_0'(\lambda)\geq G_0'(\lambda)\,,\ \forall\lambda\in[\lambda_s,\lambda_0]. \tag{13}$$

Since $F_0(\lambda)\leq G_0(\lambda)$ for every $\lambda\in[\lambda_s,\lambda_0]$, by relations (10) and (3) we have that

$$\begin{aligned}\varphi_\lambda\left(-\frac{F_0'(\lambda)}{\lambda(n-2)(n-1)}\right) &= \left(-\frac{F_0'(\lambda)}{\lambda(n-2)(n-1)}\right)^{\frac{n-4}{n}}+C(n-4)^2\frac{n-2}{n-1}\lambda F_0'(\lambda)\\ &\leq 4C(n-4)^2F_0(\lambda)\\ &\leq 4C(n-4)^2G_0(\lambda)\\ &= \left(-\frac{G_0'(\lambda)}{\lambda(n-2)(n-1)}\right)^{\frac{n-4}{n}}+C(n-4)^2\frac{n-2}{n-1}\lambda G_0'(\lambda)\\ &= \varphi_\lambda\left(-\frac{G_0'(\lambda)}{\lambda(n-2)(n-1)}\right),\qquad\forall\lambda\in[\lambda_s,\lambda_0].\end{aligned}$$

By the monotonicity of φ_λ on $(0,t_\lambda]$, relation (13) follows at once. In particular, the function F_0-G_0 is non-decreasing on the interval $[\lambda_s,\lambda_0]$. Consequently, we have

$$0=F_0(\lambda_s)-G_0(\lambda_s)\leq F_0(\lambda_0)-G_0(\lambda_0)<0,$$

a contradiction, which shows the validity of (11).

Step 5. *Global volume non-collapsing property concluded.* Inequality (11) can be rewritten into

$$\int_0^\infty \left(\mathrm{vol}_g[B(x_0,t)]-b\omega_n t^n\right)\frac{((n-1)\lambda+t^2)t}{(\lambda+t^2)^n}\mathrm{d}t\geq 0\,,\ \forall\lambda>0, \tag{14}$$

where

$$b=(C^{-1}K_0)^{\frac{n}{4}}\,.$$

The Bishop-Gromov comparison theorem implies that the function $t\mapsto \frac{\mathrm{vol}_g[B(x_0,t)]}{\omega_n t^n}$ is non-increasing on $(0,\infty)$; thus, the asymptotic volume growth

$$\limsup_{t\to\infty}\frac{\mathrm{vol}_g[B(x_0,t)]}{\omega_n t^n}=b_0$$

is finite (and independent of the base point x_0).

We shall prove that $b_0\geq b$. By contradiction, let us suppose that $b_0=b-\varepsilon_0$ for some $\varepsilon_0>0$. Thus, there exists a number $N_0>0$ such that

$$\frac{\mathrm{vol}_g[B(x_0,t)]}{\omega_n t^n}\leq b-\frac{\varepsilon_0}{2}\,,\ \forall t\geq N_0\,. \tag{15}$$

For simplicity of notation, let

$$f(\lambda,t)=\frac{((n-1)\lambda+t^2)t}{(\lambda+t^2)^n},\quad \lambda,t>0.$$

Substituting (15) into (14) and by using the Bishop-Gromov comparison theorem, we obtain for every $\lambda>0$ that

$$\begin{aligned}
0 &\leq \int_0^\infty \left(\mathrm{vol}_g[B(x_0,t)]-b\omega_n t^n\right)f(\lambda,t)\mathrm{d}t\\
&\leq \int_0^{N_0}\mathrm{vol}_g[B(x_0,t)]f(\lambda,t)\mathrm{d}t+(b-\frac{\varepsilon_0}{2})\omega_n\int_{N_0}^\infty t^nf(\lambda,t)\mathrm{d}t-b\omega_n\int_0^\infty t^nf(\lambda,t)\mathrm{d}t\\
&\leq \omega_n\int_0^{N_0}t^nf(\lambda,t)\mathrm{d}t-b\omega_n\int_0^{N_0}t^nf(\lambda,t)\mathrm{d}t-\frac{\varepsilon_0}{2}\omega_n\int_{N_0}^\infty t^nf(\lambda,t)\mathrm{d}t\\
&= \omega_n(1-b+\frac{\varepsilon_0}{2})\int_0^{N_0}t^nf(\lambda,t)\mathrm{d}-\frac{\varepsilon_0}{2}\omega_n\int_0^\infty t^nf(\lambda,t)\mathrm{d}t.
\end{aligned}$$

Note that for every $\lambda>0$, one has

$$\begin{aligned}
I_1(\lambda) &= \int_0^\infty t^nf(\lambda,t)\mathrm{d}t=\lambda^{\frac{4-n}{2}}\int_0^\infty s^nf(1,s)\mathrm{d}s\\
&= \frac{2^{1-n}\pi^{\frac12}(n^2-4n+6)\Gamma(\frac{n}{2}+1)}{(n-2)(n-4)\Gamma(\frac{n+1}{2})}\lambda^{\frac{4-n}{2}},
\end{aligned}$$

and

$$\begin{aligned}
I_2(\lambda) &= \int_0^{N_0}t^nf(\lambda,t)\mathrm{d}t=\int_0^{N_0}t^{n+1}\frac{(n-1)\lambda+t^2}{(\lambda+t^2)^n}\mathrm{d}t\\
&\leq (n-1)N_0^{n+1}\lambda^{-n+1}+N_0^{n+3}\lambda^{-n}.
\end{aligned}$$

Consequently, the above estimates show that for every $\lambda>0$,

$$M_0\lambda^{\frac{4-n}{2}}\leq M_1\lambda^{-n+1}+M_2\lambda^{-n},$$

where $M_0, M_1, M_2 > 0$ are independent on $\lambda > 0$. It is clear that the latter inequality is not valid for large values of $\lambda > 0$, i.e., we arrived to a contradiction. Accordingly, for every $r > 0$,

$$\frac{\mathrm{vol}_g[B(x_0,r)]}{\omega_n r^n} \geq \limsup_{t\to\infty} \frac{\mathrm{vol}_g[B(x_0,t)]}{\omega_n t^n} = b_0 \geq b = (C^{-1}K_0)^{\frac{n}{4}}.$$

Since the asymptotic volume growth of (M, g) is independent of the point x_0, we obtain the desired property, which completes the proof of Theorem 1.1. □

Remark 2.1. Note that relation (9) is equivalent to the distance Laplacian growth condition. Indeed, a simple computation in Step 2 led us to relation (9) through the distance Laplacian growth condition. Conversely, if $\lambda \to 0$ in (9), we obtain precisely that $\rho\Delta_g\rho \geq n - 5$.

Proof of Theorem 1.2. (i) Due to Anderson [1] and Li [18], if $\mathrm{vol}_g[B(x,r)] \geq k_0\omega_n r^n$ for every $r > 0$, then (M, g) has finite fundamental group $\pi_1(M)$ and its order is bounded above by ${k_0}^{-1}$. By Theorem 1.1 (ii) the property follows directly. In particular, if $C < 2^{\frac{4}{n}} K_0$, then the order of $\pi_1(M)$ is strictly less than 2, thus M is simply connected.

(ii) First of all, due to Munn [22, Table 5] and a direct computation, for every $n \geq 5$ one has

$$\alpha_{MP}(1,n)^{-\frac{4}{n}} = 2^{\frac{4}{n}} < \frac{n+2}{n-2}.$$

Thus, since $\alpha_{MP}(\cdot, n)$ is increasing, the values $\alpha_{MP}(k,n)^{-\frac{4}{n}} K_0$ are within the range where Theorem 1.1 (ii) applies, $k \in \{1, ..., n\}$.

Now, let us assume that $C < \alpha_{MP}(k_0,n)^{-\frac{4}{n}} K_0$ for some $k_0 \in \{1, ..., n\}$. By Theorem 1.1 (ii) we have the following estimate for the asymptotic volume growth of (M, g):

$$\lim_{t\to\infty} \frac{\mathrm{vol}_g[B(x,t)]}{\omega_n t^n} \geq \left(\frac{K_0}{C}\right)^{\frac{n}{4}} > \alpha_{MP}(k_0,n) \geq ... \geq \alpha_{MP}(1,n).$$

Therefore, due to Munn [22, Theorem 1.2], one has that $\pi_1(M) = ... = \pi_{k_0}(M) = 0$.

(iii) If $C < \alpha_{MP}(n,n)^{-\frac{4}{n}} K_0$, then $\pi_1(M) = ... = \pi_n(M) = 0$. Standard topological argument implies -based on Hurewicz's isomorphism theorem,- that M is contractible.

(iv) If $C = K_0$ then by Theorem 1.1 (ii) and the Bishop-Gromov volume comparison theorem follows that $\mathrm{vol}_g[B(x,r)] = \omega_n r^n$ for every $x \in M$ and $r > 0$. Now, the equality in Bishop-Gromov theorem implies that (M, g) is isometric to the Euclidean space $\mathbb{R}^n$. The converse is trivial. □

3. Final remarks

We conclude the paper with some remarks and further questions:

(a) If (M, g) is a complete $n-$dimensional Riemannian manifold and $x_0 \in M$ is arbitrarily fixed, we notice that

$$\rho\Delta_g\rho = n - 1 + \rho\frac{J'(u,\rho)}{J(u,\rho)} \text{ a.e. on } M,$$

where $\rho(x) = \rho(x, x_0)$, $x = \exp_{x_0}(\rho(x)u)$ for some $u \in T_{x_0}M$ with $|u| = 1$, and J is the density of the volume form in normal coordinates, see Gallot, Hulin and Lafontaine [10, Proposition 4.16]. On one hand, if the Ricci curvature on (M, g) is nonnegative, one has $J'(u, \rho) \leq 0$. On the other hand, the distance Laplacian growth condition $\rho\Delta_g\rho \geq n - 5$ is equivalent to

$$\frac{J'(u,\rho)}{J(u,\rho)} \geq -\frac{4}{\rho},$$

which is a curvature restriction on the manifold (M, g). We are wondering if the latter condition can be removed from our results, which plays a crucial role in our arguments; see also Remark 2.1. Examples of Riemannian manifolds verifying the distance Laplacian growth condition (that are isometrically immersed into $\mathbb{R}^N$ with N large enough) can be found in Carron [4].

(b) The requirement $C \leq \frac{n+2}{n-2}K_0$ is needed to explore the monotonicity of the function φ_λ on $(0, t_\lambda]$, see Step 4 in the proof of Theorem 1.1. Although this condition is widely enough to obtain quantitative results, cf. Theorem 1.2, we still believe that it can be somehow removed.

(c) Let (M, g) be an n-dimensional complete open Riemannian manifold with nonnegative Ricci curvature and fix $k \in \mathbb{N}$ such that $n > 2k$. Let us consider for some $C > 0$ the k-th order Sobolev inequality

$$\left(\int_M |u|^{\frac{2n}{n-2k}} \mathrm{d}v_g\right)^{\frac{n-2k}{n}} \leq C \int_M (\Delta_g^{k/2} u)^2 \mathrm{d}v_g, \ \forall u \in C_0^\infty(M), \tag{$\mathbf{SI}$}_C^k$$

where

$$\Delta_g^{k/2} u = \begin{cases} \Delta_g^{k/2} u & \text{if} \quad k \text{ is even,} \\ |\nabla_g(\Delta_g^{(k-1)/2} u)| & \text{if} \quad k \text{ is odd.} \end{cases}$$

Clearly, $(\mathbf{SI})_C^1 = (\mathbf{FSI})_C$ and $(\mathbf{SI})_C^2 = (\mathbf{SSI})_C$. It would be interesting to establish k-th order counterparts of Theorems 1.1&1.2 with $k \geq 3$, noticing that the optimal Euclidean k-th order Sobolev inequalities are well known with the optimal constant

$$\Lambda_k = \left[\pi^k n(n-2k)\Pi_{i=1}^{k-1}(n^2 - 4i^2)\right]^{-1} \left(\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\right)^{2k/n},$$

and the unique class of extremal functions (up to translations and multiplications)

$$u_\lambda(x) = (\lambda + |x^2|)^{\frac{2k-n}{2}}, \ x \in \mathbb{R}^n,$$

see Cotsiolis and Tavoularis [5], Liu [21]. Once we use $w_\lambda = (\lambda + \rho^2)^{\frac{2k-n}{2}}$ as a test-function in $(\mathbf{SI})_C^k$, after a multiple application of the chain rule we have to estimate in a sharp way the terms appearing in $\Delta_g^{k/2} w_\lambda$, similar to the eikonal equation $|\nabla_g \rho| = 1$ and the distance Laplacian comparison $\rho\Delta_g\rho \leq n-1$, respectively. In the second-order case this fact is highlighted in relation (8). Furthermore, higher-order counterparts of the distance Laplacian growth condition $\rho\Delta_g\rho \geq n-5$ should be found, (see relation (9) for the second order case), assuming this condition cannot be removed, see (a).

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