

ON THE RELATIONSHIP BETWEEN IDEAL CLUSTER POINTS AND IDEAL LIMIT POINTS

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ABSTRACT. Let X be a first countable space which admits a non-trivial convergent sequence and let \mathcal{I} be an analytic P-ideal. First, it is shown that the sets of \mathcal{I} -limit points of all sequences in X are closed if and only if \mathcal{I} is also an F_σ -ideal.

Moreover, let (x_n) be a sequence taking values in a Polish space without isolated points. It is known that the set A of its statistical limit points is an F_σ -set, the set B of its statistical cluster points is closed, and that the set C of its ordinary limit points is closed, with $A \subseteq B \subseteq C$. It is proved the sets A and B own some additional relationship: indeed, the set S of isolated points of B is contained also in A .

Conversely, if A is an F_σ -set, B is a closed set with a subset S of isolated points such that $B \setminus S \neq \emptyset$ is regular closed, and C is a closed set with $S \subseteq A \subseteq B \subseteq C$, then there exists a sequence (x_n) for which: A is the set of its statistical limit points, B is the set of its statistical cluster points, and C is the set of its ordinary limit points.

Lastly, we discuss topological nature of the set of \mathcal{I} -limit points when \mathcal{I} is neither F_σ - nor analytic P-ideal.

1. INTRODUCTION

The aim of this article is to establish some relationship between the set of ideal cluster points and the set of ideal limit points of a given sequence.

To this aim, let \mathcal{I} be an ideal on the positive integers \mathbf{N} , i.e., a collection of subsets of \mathbf{N} closed under taking finite unions and subsets. It is assumed that \mathcal{I} contains the collection Fin of finite subsets of \mathbf{N} and it is different from the whole power set $\mathcal{P}(\mathbf{N})$. Note that the family \mathcal{I}_0 of subsets with zero asymptotic density, that is,

$$\mathcal{I}_0 := \left\{ S \subseteq \mathbf{N} : \lim_{n \rightarrow \infty} \frac{|S \cap \{1, \dots, n\}|}{n} = 0 \right\}$$

is an ideal. Let also $x = (x_n)$ be a sequence taking values in a topological space X , which will be always assumed hereafter to be Hausdorff. We denote by $\Lambda_x(\mathcal{I})$ the set of \mathcal{I} -limit points of x , that is, the set of all $\ell \in X$ for which $\lim_{k \rightarrow \infty} x_{n_k} = \ell$, for some subsequence (x_{n_k}) such that $\{n_k : k \in \mathbf{N}\} \notin \mathcal{I}$. In addition, let $\Gamma_x(\mathcal{I})$ be the set of \mathcal{I} -cluster points of x , that is, the set of all $\ell \in X$ such that $\{n : x_n \in U\} \notin \mathcal{I}$ for every neighborhood U of ℓ . Note that $L_x := \Lambda_x(\text{Fin}) = \Gamma_x(\text{Fin})$ is the set of ordinary limit points of x ; we also shorten $\Lambda_x := \Lambda_x(\mathcal{I}_0)$ and $\Gamma_x := \Gamma_x(\mathcal{I}_0)$.

Statistical limit points and statistical cluster points (i.e., \mathcal{I}_0 -limit points and \mathcal{I}_0 -cluster points, resp.) of real sequences were introduced by Fridy [10], cf. also [2, 5, 11, 13, 15, 17].

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We are going to provide in Section 2, under suitable assumptions on X and \mathcal{I} , a characterization of the set of \mathcal{I} -limit points. Recall that $\Gamma_x(\mathcal{I})$ is closed and contains $\Lambda_x(\mathcal{I})$, see e.g. [4, Section 5]. Then, it is shown that:

- (i) $\Lambda_x(\mathcal{I})$ is an F_σ -set, provided that \mathcal{I} is an analytic P-ideal (Theorem 2.2);
- (ii) $\Lambda_x(\mathcal{I})$ is closed, provided that \mathcal{I} is an F_σ -ideal (Theorem 2.3);
- (iii) $\Lambda_x(\mathcal{I})$ is closed for all x if and only if $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I})$ for all x if and only if \mathcal{I} is an F_σ -ideal, provided that \mathcal{I} is an analytic P-ideal (Theorem 2.5);
- (iv) For every F_σ -set A , there exists a sequence x such that $\Lambda_x(\mathcal{I}) = A$, provided that \mathcal{I} is an analytic P-ideal which is not F_σ (Theorem 2.6);
- (v) Each of isolated point \mathcal{I} -cluster point is also an \mathcal{I} -limit point (Theorem 2.7).

In addition, we provide in Section 3 some joint converse results:

- (vi) Given $A \subseteq B \subseteq C \subseteq \mathbf{R}$ where A is an F_σ -set, B is non-empty regular closed, and C is closed, then there exists a real sequence x such that $\Lambda_x = A$, $\Gamma_x = B$, and $L_x = C$ (Theorem 3.1 and Corollary 3.3);
- (vii) Given non-empty closed sets $B \subseteq C \subseteq \mathbf{R}$, there exists a real sequence x such that $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I}) = B$ and $L_x = C$, provided \mathcal{I} is an F_σ -ideal different from Fin (Theorem 3.4).

Lastly, it is shown in Section 4 that:

- (viii) $\Lambda_x(\mathcal{I})$ is analytic, provided that \mathcal{I} is a co-analytic ideal (Proposition 4.1);
- (ix) An ideal \mathcal{I} is maximal if and only if each real sequence x admits at most one \mathcal{I} -limit point (Proposition 4.2 and Corollary 4.3).

We conclude by showing that there exists an ideal \mathcal{I} and a real sequence x such that $\Lambda_x(\mathcal{I})$ is not an F_σ -set (Example 4.4).

2. TOPOLOGICAL STRUCTURE OF \mathcal{I} -LIMIT POINTS

We recall that an ideal \mathcal{I} is said to be a *P-ideal* if it is σ -directed modulo finite, i.e., for every sequence (A_n) of sets in \mathcal{I} there exists $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all n ; equivalent definitions were given, e.g., in [1, Proposition 1].

By identifying sets of integers with their characteristic function, we equip $\mathcal{P}(\mathbf{N})$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals on \mathbf{N} . In particular, an ideal \mathcal{I} is *analytic* if it is a continuous image of a G_δ -subset of the Cantor space. Moreover, a map $\varphi : \mathcal{P}(\mathbf{N}) \rightarrow [0, \infty]$ is a *lower semicontinuous submeasure* provided that: (i) $\varphi(\emptyset) = 0$; (ii) $\varphi(A) \leq \varphi(B)$ whenever $A \subseteq B$; (iii) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ for all A, B ; and (iv) $\varphi(A) = \lim_n \varphi(A \cap \{1, \dots, n\})$ for all A .

By a classical result of Solecki, an ideal \mathcal{I} is an analytic P-ideal if and only if there exists a lower semicontinuous submeasure φ such that

$$\mathcal{I} = \mathcal{I}_\varphi := \{A \subseteq \mathbf{N} : \|A\|_\varphi = 0\} \quad (1)$$

and $\varphi(\mathbf{N}) < \infty$, where $\|A\|_\varphi := \lim_n \varphi(A \setminus \{1, \dots, n\})$ for all $A \subseteq \mathbf{N}$, see [19, Theorem 3.1]. Note, in particular, that for every $n \in \mathbf{N}$ it holds

$$\|A\|_\varphi = \|A \setminus \{1, \dots, n\}\|_\varphi. \quad (2)$$

Hereafter, unless otherwise stated, an analytic P-ideal will be always denoted by \mathcal{I}_φ , where φ stands for the associated lower semicontinuous submeasure as in (1).

Given a sequence $x = (x_n)$ taking values in a first countable space X and an analytic \mathcal{P} -ideal \mathcal{I}_φ , define

$$\mathbf{u}(\ell) := \lim_{k \rightarrow \infty} \|\{n : x_n \in U_k\}\|_\varphi \quad (3)$$

for each $\ell \in X$, where (U_k) is a decreasing local base of neighborhoods at ℓ . It is easy to see that the limit in (3) exists and its value is independent from the choice of (U_k) .

Lemma 2.1. *The map \mathbf{u} is upper semi-continuous. In particular, the set*

$$\Lambda_x(\mathcal{I}_\varphi, q) := \{\ell \in X : \mathbf{u}(\ell) \geq q\}.$$

is closed for every $q > 0$.

Proof. We need to prove that $\mathcal{U}_y := \{\ell \in X : \mathbf{u}(\ell) < y\}$ is open for all $y \in \mathbf{R}$ (hence \mathcal{U}_∞ is open too). Clearly, $\mathcal{U}_y = \emptyset$ if $y \leq 0$. Hence, let us suppose hereafter $y > 0$ and $\mathcal{U}_y \neq \emptyset$. Fix $\ell \in \mathcal{U}_y$ and let (U_k) be a decreasing local base of neighborhoods at ℓ . Then, there exists $k_0 \in \mathbf{N}$ such that $\|\{n : x_n \in U_k\}\|_\varphi < y$ for every $k \geq k_0$. Fix $\ell' \in U_{k_0}$ and let (V_k) be a decreasing local base of neighborhoods at ℓ' . Fix also $k_1 \in \mathbf{N}$ such that $V_{k_1} \subseteq U_{k_0}$. It follows by the monotonicity of φ that

$$\|\{n : x_n \in V_k\}\|_\varphi \leq \|\{n : x_n \in U_{k_0}\}\|_\varphi < y$$

for every $k \geq k_1$. In particular, $\mathbf{u}(\ell') < y$ and, by the arbitrariness of ℓ' , $U_{k_0} \subseteq \mathcal{U}_y$. \square

At this point, we provide a useful characterization of the set $\Lambda_x(\mathcal{I}_\varphi)$ (without using limits of subsequences) and we obtain, as a by-product, that it is an F_σ -set.

Theorem 2.2. *Let x be a sequence taking values in a first countable space X and \mathcal{I}_φ be an analytic \mathcal{P} -ideal. Then*

$$\Lambda_x(\mathcal{I}_\varphi) = \{\ell \in X : \mathbf{u}(\ell) > 0\}. \quad (4)$$

In particular, $\Lambda_x(\mathcal{I}_\varphi)$ is an F_σ -set.

Proof. Let us suppose that there exists $\ell \in \Lambda_x(\mathcal{I}_\varphi)$ and let (U_k) be a decreasing local base of neighborhoods at ℓ . Then, there exists $A \subseteq \mathbf{N}$ such that $\lim_{n \rightarrow \infty, n \in A} x_n = \ell$ and $\|A\|_\varphi > 0$. At this point, note that, for each $k \in \mathbf{N}$, the set $\{n \in A : x_n \notin U_k\}$ is finite, hence it follows by (2) that $\mathbf{u}(\ell) \geq \|A\|_\varphi > 0$.

On the other hand, suppose that there exists $\ell \in X$ such that $\mathbf{u}(\ell) > 0$. Let (U_k) be a decreasing local base of neighborhoods at ℓ and define $\mathcal{A}_k := \{n : x_n \in U_k\}$ for each $k \in \mathbf{N}$; note that \mathcal{A}_k is infinite since $\|\mathcal{A}_k\|_\varphi \downarrow \mathbf{u}(\ell) > 0$ implies $\mathcal{A}_k \notin \mathcal{I}_\varphi$ for all k . Set for convenience $\theta_0 := 0$ and define recursively the increasing sequence of integers (θ_k) so that θ_k is the smallest integer greater than both θ_{k-1} and $\min \mathcal{A}_{k+1}$ such that

$$\varphi(\mathcal{A}_k \cap (\theta_{k-1}, \theta_k]) \geq \mathbf{u}(\ell) (1 - 1/k).$$

Finally, define $\mathcal{A} := \bigcup_k (\mathcal{A}_k \cap (\theta_{k-1}, \theta_k])$. Since $\theta_k \geq k$ for all k , we obtain

$$\varphi(\mathcal{A} \setminus \{1, \dots, n\}) \geq \varphi(\mathcal{A}_{n+1} \cap (\theta_n, \theta_{n+1}]) > \mathbf{u}(\ell) (1 - 1/n)$$

for all n , hence $\|\mathcal{A}\|_\varphi \geq \mathbf{u}(\ell) > 0$. In addition, we have by construction $\lim_{n \rightarrow \infty, n \in \mathcal{A}} x_n = \ell$. Therefore ℓ is an \mathcal{I}_φ -limit point of x . To sum up, this proves (4).

Lastly, rewriting (4) as $\Lambda_x(\mathcal{I}_\varphi) = \bigcup_n \Lambda_x(\mathcal{I}_\varphi, 1/n)$ and considering that each $\Lambda_x(\mathcal{I}_\varphi, 1/n)$ is closed by Lemma 2.1, we conclude that $\Lambda_x(\mathcal{I}_\varphi)$ is an F_σ -set. \square

The fact that $\Lambda_x(\mathcal{I}_\varphi)$ is an F_σ -set already appeared in [3, Theorem 2], although with a different argument. The first result of this type was given in [13, Theorem 1.1] for the case $\mathcal{I}_\varphi = \mathcal{I}_0$ and $X = \mathbf{R}$. Later, it was extended in [5, Theorem 2.6] for first countable spaces. However, in the proofs contained in [3, 5] it is unclear why the constructed subsequence $(x_n : n \in \mathcal{A})$ converges to ℓ . Lastly, Theorem 2.2 generalizes, again with a different argument, [14, Theorem 3.1] for the case X metrizable.

A stronger result holds in the case that the ideal is F_σ . We recall that, by a classical result of Mazur, an ideal \mathcal{I} is F_σ if and only if there exists a lower semicontinuous submeasure φ such that

$$\mathcal{I} = \{A \subseteq \mathbf{N} : \varphi(A) < \infty\}, \quad (5)$$

with $\varphi(\mathbf{N}) = \infty$, see [16, Lemma 1.2].

Theorem 2.3. *Let $x = (x_n)$ be a sequence taking values in a first countable space X and let \mathcal{I} be an F_σ -ideal. Then $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I})$. In particular, $\Lambda_x(\mathcal{I})$ is closed.*

Proof. Since it is known that $\Lambda_x(\mathcal{I}) \subseteq \Gamma_x(\mathcal{I})$, the claim is clear if $\Gamma_x(\mathcal{I}) = \emptyset$. Hence, let us suppose hereafter that $\Gamma_x(\mathcal{I})$ is non-empty. Fix $\ell \in \Gamma_x(\mathcal{I})$ and let (U_k) be a decreasing local base of neighborhoods at ℓ . Letting φ be a lower semicontinuous submeasure associated with \mathcal{I} as in (5) and considering that ℓ is an \mathcal{I} -cluster point, we have $\varphi(A_k) = \infty$ for all $k \in \mathbf{N}$, where $A_k := \{n : x_n \in U_k\}$.

Then, set $a_0 := 0$ and define an increasing sequence of integers (a_k) which satisfies

$$\varphi(A_k \cap (a_{k-1}, a_k]) \geq k$$

for all k (note that this is possible since $\varphi(A_k \setminus S) = \infty$ whenever S is finite). At this point, set $A := \bigcup_k A_k \cap (a_{k-1}, a_k]$. It follows by the monotonicity of φ that $\varphi(A) = \infty$, hence $A \notin \mathcal{I}$. Moreover, for each $k \in \mathbf{N}$, we have that $\{n \in A : x_n \notin U_k\}$ is finite: indeed, if $n \in A_j \cap (a_{j-1}, a_j]$ for some $j \geq k$, then by construction $x_n \in U_j$, which is contained in U_k . Therefore $\lim_{n \rightarrow \infty, n \in A} x_n = \ell$, that is, $\ell \in \Lambda_x(\mathcal{I})$. \square

Since summable ideals are F_σ P-ideals, see e.g. [7, Example 1.2.3], we obtain the following corollary which was proved in [14, Theorem 3.4]:

Corollary 2.4. *Let x be a real sequence and let \mathcal{I} be a summable ideal. Then $\Lambda_x(\mathcal{I})$ is closed.*

It turns out that, within the class of analytic P-ideals, the property that the set of \mathcal{I} -limit points is always closed characterizes the subclass of F_σ -ideals:

Theorem 2.5. *Let X be a first countable space which admits a non-trivial convergent sequence. Let also \mathcal{I}_φ be an analytic P-ideal. Then the following are equivalent:*

- (i) \mathcal{I}_φ is also an F_σ -ideal;
- (ii) $\Lambda_x(\mathcal{I}_\varphi) = \Gamma_x(\mathcal{I}_\varphi)$ for all sequences x ;
- (iii) $\Lambda_x(\mathcal{I}_\varphi)$ is closed for all sequences x ;
- (iv) there does not exist a partition $\{A_n : n \in \mathbf{N}\}$ of \mathbf{N} such that $\|A_n\|_\varphi > 0$ for all n and $\lim_n \|\bigcup_{k>n} A_k\|_\varphi = 0$.

Proof. (i) \implies (ii) follows by Theorem 2.3 and (ii) \implies (iii) is clear.

(iii) \implies (iv) By hypothesis, there exists a sequence (ℓ_n) converging to $\ell \in X$ such that $\ell_n \neq \ell$ for all n . Let us suppose that there exists a partition $\{A_n : n \in \mathbf{N}\}$ of \mathbf{N}

such that $\|A_n\|_\varphi > 0$ for all n and $\lim_k \|\bigcup_{n \geq k} A_n\|_\varphi = 0$. Then, define the sequence $x = (x_n)$ by $x_n = \ell_i$ for all $n \in A_i$. Then, we have that $\{\ell_n : n \in \mathbf{N}\} \subseteq \Lambda_x(\mathcal{I}_\varphi)$. On the other hand, since X is first countable Hausdorff, it follows that for all $k \in \mathbf{N}$ there exists a neighborhood U_k of ℓ such that

$$\{n : x_n \in U_k\} \subseteq \{n : x_n = \ell_i \text{ for some } i \geq k\} = \bigcup_{n \geq k} A_n.$$

Hence, by the monotonicity of φ , we obtain $0 < \|\{n : x_n \in U_k\}\|_\varphi \downarrow 0$, i.e., $u(\ell) = 0$, which implies, thanks to Theorem 2.2, that $\ell \notin \Lambda_x(\mathcal{I}_\varphi)$. In particular, \mathcal{I}_φ is not closed.

(iv) \implies (i) Lastly, assume that the ideal \mathcal{I}_φ is not an F_σ -ideal. According to the proof of [19, Theorem 3.4], cf. also [18, pp. 342–343], this is equivalent to the existence, for each given $\varepsilon > 0$, of some set $M \subseteq \mathbf{N}$ such that $0 < \|M\|_\varphi \leq \varphi(M) < \varepsilon$. This allows to define recursively a sequence of sets (M_n) such that

$$\|M_n\|_\varphi > \sum_{k \geq n+1} \varphi(M_k) > 0. \quad (6)$$

for all n and, in addition, $\sum_k \varphi(M_k) < \varphi(\mathbf{N})$. Then, it is claimed that there exists a partition $\{A_n : n \in \mathbf{N}\}$ of \mathbf{N} such that $\|A_n\|_\varphi > 0$ for all n and $\lim_n \|\bigcup_{k > n} A_k\|_\varphi = 0$. To this aim, set $M_0 := \mathbf{N}$ and define $A_n := M_{n-1} \setminus \bigcup_{k \geq n} M_k$ for all $n \in \mathbf{N}$. It follows by the subadditivity and monotonicity of φ that

$$\varphi(M_{n-1} \setminus \{1, \dots, k\}) \leq \varphi(A_n \setminus \{1, \dots, k\}) + \varphi\left(\bigcup_{k \geq n} M_k\right)$$

for all $n, k \in \mathbf{N}$; hence, by the lower semicontinuity of φ and (6),

$$\|A_n\|_\varphi \geq \|M_{n-1}\|_\varphi - \varphi\left(\bigcup_{k \geq n} M_k\right) \geq \|M_{n-1}\|_\varphi - \sum_{k \geq n} \varphi(M_k) > 0$$

for all $n \in \mathbf{N}$. Finally, again by the lower semicontinuity of φ , we get

$$\|\bigcup_{k > n} A_k\|_\varphi = \|\bigcup_{k \geq n} M_k\|_\varphi \leq \varphi\left(\bigcup_{k \geq n} M_k\right) \leq \sum_{k \geq n} \varphi(M_k)$$

which goes to 0 as $n \rightarrow \infty$. This concludes the proof. \square

At this point, thanks to Theorem 2.2 and Theorem 2.5, observe that, if X is a first countable space which admits a non-trivial convergent sequence and \mathcal{I}_φ is an analytic P-ideal which is not F_σ , then there exists a sequence x such that $\Lambda_x(\mathcal{I}_\varphi)$ is a non-closed F_σ -set. In this case, indeed, all the F_σ -sets can be obtained:

Theorem 2.6. *Let X be a first countable space where all closed sets are separable and assume that there exists a non-trivial convergent sequence. Fix also an analytic P-ideal \mathcal{I}_φ which is not F_σ and let $B \subseteq X$ be a non-empty F_σ -set. Then, there exists a sequence x such that $\Lambda_x(\mathcal{I}_\varphi) = B$.*

Proof. Let (B_k) be a sequence of non-empty closed sets such that $\bigcup_k B_k = B$. Let also $\{b_{k,n} : n \in \mathbf{N}\}$ be a countable dense subset of B_k . Thanks to Theorem 2.5, there exists a partition $\{A_n : n \in \mathbf{N}\}$ of \mathbf{N} such that $\|A_n\|_\varphi > 0$ for all n and $\lim_n \|\bigcup_{k > n} A_k\|_\varphi = 0$. Moreover, for each $k \in \mathbf{N}$, set $\theta_{k,0} := 0$ and it is easily seen that there exists an increasing sequence of positive integers $(\theta_{k,n})$ such that

$$\varphi(A_k \cap (\theta_{k,n-1}, \theta_{k,n}]) \geq \frac{1}{2} \|A_k \setminus \{1, \dots, \theta_{k,n-1}\}\|_\varphi = \frac{1}{2} \|A_k\|_\varphi$$

for all n . Hence, setting $A_{k,n} := A_k \cap \bigcup_{m \in A_n} (\theta_{k,m-1}, \theta_{k,m}]$, we obtain that $\{A_{k,n} : n \in \mathbf{N}\}$ is a partition of A_k such that $\frac{1}{2}\|A_k\|_\varphi \leq \|A_{k,n}\|_\varphi \leq \|A_k\|_\varphi$ for all n, k .

At this point, let $x = (x_n)$ defined by $x_n = b_{k,m}$ whenever $n \in A_{k,m}$. Fix $\ell \in B$, then there exists $k \in \mathbf{N}$ such that $\ell \in B_k$. Let (b_{k,r_m}) be a sequence in B_k converging to ℓ . Thus, set $\tau_0 := 0$ and let (τ_m) be an increasing sequence of positive integers such that $\varphi(A_{k,r_m} \cap (\tau_{m-1}, \tau_m]) \geq \frac{1}{2}\|A_{k,r_m}\|_\varphi$ for each m . Setting $A := \bigcup_m A_{k,r_m} \cap (\tau_{m-1}, \tau_m]$, it follows by construction that $\lim_{n \rightarrow \infty, n \in A} x_n = \ell$ and $\|A\|_\varphi \geq \frac{1}{4}\|A_k\|_\varphi > 0$. This shows that $B \subseteq \Lambda_x(\mathcal{I}_\varphi)$.

To complete the proof, fix $\ell \notin B$ and let us suppose for the sake of contradiction that there exists $A \subseteq \mathbf{N}$ such that $\lim_{n \rightarrow \infty, n \in A} x_n = \ell$ and $\|A\|_\varphi > 0$. For each $m \in \mathbf{N}$, let U_m be an open neighborhood of ℓ which is disjoint from the closed set $B_1 \cup \dots \cup B_m$. It follows by the subadditivity and the monotonicity of φ that there exists a finite set Y such that

$$\|A\|_\varphi \leq \|Y\|_\varphi + \|\{n \in A : x_n \notin B_1 \cup \dots \cup B_m\}\|_\varphi \leq \|\bigcup_{k>m} A_k\|_\varphi.$$

The claim follows by the arbitrariness of m and the fact that $\lim_m \|\bigcup_{k>m} A_k\|_\varphi = 0$. \square

Note that every analytic P-ideal without the Bolzano-Weierstrass property cannot be F_σ , see [8, Theorem 4.2]. Hence Theorem 2.6 applies to this class of ideals.

It was shown in [5, Theorem 2.8 and Theorem 2.10] that if X is a topological space where all closed sets are separable, then for each F_σ -set A and closed set B there exist sequences $a = (a_n)$ and $b = (b_n)$ with values in X such that $\Lambda_a = A$ and $\Gamma_b = B$.

As an application of Theorem 2.2, we prove that, in general, its stronger version with $a = b$ fails (e.g., there are no real sequences x such that $\Lambda_x = \{0\}$ and $\Gamma_x = \{0, 1\}$).

Here, a topological space X is said to be *locally compact* if for every $x \in X$ there exists a neighborhood U of x such that its closure \overline{U} is compact, cf. [6, Section 3.3].

Theorem 2.7. *Let $x = (x_n)$ be a sequence taking values in a locally compact first countable space and fix an analytic P-ideal \mathcal{I}_φ . Then each isolated \mathcal{I}_φ -cluster point is also an \mathcal{I}_φ -limit point.*

Proof. Let us suppose for the sake of contradiction that there exists an isolated \mathcal{I}_φ -cluster point, let us say ℓ , which is not an \mathcal{I}_φ -limit point. Let (U_k) be a decreasing local base of open neighborhoods at ℓ such that $\overline{U_1}$ is compact. Let also m be a sufficiently large integer such that $U_m \cap \Gamma_x(\mathcal{I}_\varphi) = \{\ell\}$. Thanks to [6, Theorem 3.3.1] the underlying space is, in particular, regular, hence there exists an integer $r > m$ such that $\overline{U_r}$ is a compact contained in U_m . In addition, since ℓ is an \mathcal{I}_φ -cluster point and it is not an \mathcal{I}_φ -limit point, it follows by Theorem 2.2 that

$$0 < \|\{n : x_n \in U_k\}\|_\varphi \downarrow \mathbf{u}(\ell) = 0.$$

In particular, there exists $s \in \mathbf{N}$ such that $0 < \|\{n : x_n \in U_s\}\|_\varphi < \|\{n : x_n \in U_r\}\|_\varphi$.

Observe that $K := \overline{U_r} \setminus U_s$ is a closed set contained in $\overline{U_1}$, hence it is compact. By construction we have that $K \cap \Gamma_x(\mathcal{I}_\varphi) = \emptyset$. Hence, for each $z \in K$, there exists an open neighborhood V_z of z such that $V_z \subseteq U_m$ and $\{n : x_n \in V_z\} \in \mathcal{I}_\varphi$, i.e., $\|\{n : x_n \in V_z\}\|_\varphi = 0$. It follows that $\bigcup_{z \in K} V_z$ is an open cover of K which is contained in U_m . Since K is compact, there exists a finite set $\{z_1, \dots, z_t\} \subseteq K$ for which

$$K \subseteq V_{z_1} \cup \dots \cup V_{z_t} \subseteq U_m. \quad (7)$$

At this point, by the subadditivity of φ , it easily follows that $\|A \cup B\|_\varphi \leq \|A\|_\varphi + \|B\|_\varphi$ for all $A, B \subseteq \mathbf{N}$. Hence we have

$$\begin{aligned} \|\{n : x_n \in K\}\|_\varphi &\geq \|\{n : x_n \in \overline{U_r}\}\|_\varphi - \|\{n : x_n \in U_s\}\|_\varphi \\ &\geq \|\{n : x_n \in U_r\}\|_\varphi - \|\{n : x_n \in U_s\}\|_\varphi > 0. \end{aligned}$$

On the other hand, it follows by (7) that

$$\|\{n : x_n \in K\}\|_\varphi \leq \|\{n : x_n \in \bigcup_{i=1}^t V_{z_i}\}\|_\varphi \leq \sum_{i=1}^t \|\{n : x_n \in V_{z_i}\}\|_\varphi = 0.$$

This contradiction concludes the proof. \square

The following corollary is immediate (we omit details):

Corollary 2.8. *Let x be a real sequence for which Γ_x is a discrete set. Then $\Lambda_x = \Gamma_x$.*

3. JOINT CONVERSE RESULTS

We provide now a kind of converse of Theorem 2.7, specializing to the case of the ideal \mathcal{I}_0 : informally, if B is a sufficiently smooth closed set and A is an F_σ -set containing the isolated points of B , then there exists a sequence x such that $\Lambda_x = A$ and $\Gamma_x = B$.

To this aim, we need some additional notation: let d^* , d_* , and d be the upper asymptotic density, lower asymptotic density, and asymptotic density on \mathbf{N} , resp.; in particular, $\mathcal{I}_0 = \{S \subseteq \mathbf{N} : d^*(S) = 0\}$.

Given a topological space X , the interior and the closure of a subset $S \subseteq X$ are denoted by S° and \overline{S} , respectively; S is said to be *regular closed* if $S = \overline{S^\circ}$. We let the Borel σ -algebra on X be $\mathcal{B}(X)$. A Borel probability measure $\mu : \mathcal{B}(X) \rightarrow [0, 1]$ is said to be *strictly positive* whenever $\mu(U) > 0$ for all non-empty open sets U . Moreover, μ is *atomless* if, for each measurable set A with $\mu(A) > 0$, there exists a measurable subset $B \subseteq A$ such that $0 < \mu(B) < \mu(A)$. Then, a sequence (x_n) taking values in X is said to be μ -uniformly distributed whenever

$$\mu(F) \geq d^*(\{n : x_n \in F\}) \quad (8)$$

for all closed sets F , cf. [9, Section 491B].

Theorem 3.1. *Let X be a separable metric space and $\mu : \mathcal{B}(X) \rightarrow [0, 1]$ be an atomless strictly positive Borel probability measure. Fix also sets $A \subseteq B \subseteq C \subseteq X$ such that A is an F_σ -set, and B, C are closed sets such that: (i) $\mu(B) > 0$, (ii) the set S of isolated points of B is contained in A , and (iii) $B \setminus S$ is regular closed. Then there exists a sequence x taking values in X such that*

$$\Lambda_x = A, \Gamma_x = B, \text{ and } L_x = C. \quad (9)$$

Proof. Set $F := B \setminus S$ and note that, by the separability of X , S at most countable. In particular, $\mu(S) = 0$, hence $\mu(F) = \mu(B) > 0$.

Let us assume for now that A is non-empty. Since A is an F_σ -set, there exists a sequence (A_k) of non-empty closed sets such that $\bigcup_k A_k = A$. Considering that X is (hereditarily) second countable, then every closed set is separable. Hence, for each $k \in \mathbf{N}$, there exists a countable set $\{a_{k,n} : n \in \mathbf{N}\} \subseteq A_k$ with closure A_k . Considering

that F is a separable metric space on its own right and that the (normalized) restriction μ_F of μ on F , that is,

$$\mu_F : \mathcal{B}(F) \rightarrow [0, 1] : Y \mapsto \frac{1}{\mu(F)} \mu(Y) \quad (10)$$

is a Borel probability measure, it follows by [9, Exercise 491Xw] that there exists a μ_F -uniformly distributed sequence (b_n) which takes values in F and satisfies (8). Lastly, let $\{c_n : n \in \mathbf{N}\}$ be a countable dense subset of C .

At this point, let \mathcal{C} be the set of non-zero integer squares and note that $d(\mathcal{C}) = 0$. For each $k \in \mathbf{N}$ define $\mathcal{A}_k := \{2^k n : n \in \mathbf{N} \setminus 2\mathbf{N}\} \setminus \mathcal{C}$ and $\mathcal{B} := \mathbf{N} \setminus (2\mathbf{N} \cup \mathcal{C})$. It follows by construction that $\{\mathcal{A}_k : k \in \mathbf{N}\} \cup \{\mathcal{B}, \mathcal{C}\}$ is a partition of \mathbf{N} . Moreover, each \mathcal{A}_k admits asymptotic density and

$$\lim_{n \rightarrow \infty} d\left(\bigcup_{k \geq n} \mathcal{A}_k\right) = 0. \quad (11)$$

Finally, for each positive integer k , let $\{\mathcal{A}_{k,m} : m \in \mathbf{N}\}$ be the partition of \mathcal{A}_k defined by $\mathcal{A}_{k,1} := \mathcal{A}_k \cap \bigcup_{n \in \mathcal{A}_1 \cup \mathcal{B} \cup \mathcal{C}} [n!, (n+1)!)$ and $\mathcal{A}_{k,m} := \mathcal{A}_k \cap \bigcup_{n \in \mathcal{A}_m} [n!, (n+1)!)$ for all integers $m \geq 2$. Then, it is easy to check that

$$d^*(\mathcal{A}_{k,1}) = d^*(\mathcal{A}_{k,2}) = \dots = d(\mathcal{A}_k) = 2^{-k-1}.$$

Hence, define the sequence $x = (x_n)$ by

$$x_n = \begin{cases} a_{k,m} & \text{if } n \in \mathcal{A}_{k,m}, \\ b_m & \text{if } n \text{ is the } m\text{-th term of } \mathcal{B}, \\ c_m & \text{if } n \text{ is the } m\text{-th term of } \mathcal{C}. \end{cases} \quad (12)$$

To complete the proof, let us verify that (9) holds true:

CLAIM (I): $L_x = C$. Note that $x_n \in C$ for all $n \in \mathbf{N}$. Since C is closed by hypothesis, then $L_x \subseteq C$. On the other hand, if $\ell \in C$, then there exists a sequence (c_n) taking values in C converging (in the ordinary sense) to ℓ . It follows by the definition of (x_n) that there exists a subsequence (x_{n_k}) converging to ℓ , i.e., $C \subseteq L_x$.

CLAIM (II): $\Gamma_x = B$. Fix $\ell \notin B$ and let U be an open neighborhood of ℓ disjoint from B (this is possible since, in the opposite, ℓ would belong to $\overline{B} = B$). Then, $\{n : x_n \in U\} \subseteq \mathcal{C}$, which implies that $\Gamma_x \subseteq B$.

Note that the Borel probability measure μ_F defined in (10) is clearly atomless. Moreover, given an open set $U \subseteq X$ with non-empty intersection with F , then $U \cap F^\circ \neq \emptyset$: indeed, in the opposite, we would have $F^\circ \subseteq U^c$, which is closed, hence $F = \overline{F^\circ} \subseteq U^c$, contradicting our hypothesis. This proves that every non-empty open set V (relative to F) contains a non-empty open set of X . Therefore μ_F is also strictly positive. With these premises, fix $\ell \in F$ and let V be a open neighborhood of ℓ (relative to F). Since (b_n) is μ_F -uniformly distributed and μ_F is strictly positive, it follows by (8) that

$$\begin{aligned} 0 < \mu_F(V) &= 1 - \mu_F(V^c) \leq 1 - d^*(\{n : b_n \in V^c\}) \\ &= d_*(\{n : b_n \in V\}) \leq d^*(\{n : b_n \in V\}). \end{aligned}$$

Since $d(\mathcal{B}) = 1/2$, we obtain by standard properties of d^* that

$$d^*(\{n : x_n \in V\}) \geq d^*(\{n \in \mathcal{B} : x_n \in V\}) = \frac{1}{2} d^*(\{n : b_n \in V\}) > 0.$$

We conclude by the arbitrariness of V and ℓ that $F \subseteq \Gamma_x$.

Hence, we miss only to show that $S \subseteq \Gamma_x$. To this aim, fix $\ell \in S$, thus ℓ is also an isolated point of A . Hence, there exist $k, m \in \mathbf{N}$ such that $a_{k,m} = \ell$. We conclude that $d^*(\{n : x_n \in U\}) \geq d^*(\{n : x_n = \ell\}) \geq d(\mathcal{A}_k) > 0$ for each neighborhood U of ℓ . Therefore $B = F \cup S \subseteq \Gamma_x$.

CLAIM (III): $\Lambda_x = A$. Fix $\ell \in A$, hence there exists $k \in \mathbf{N}$ for which ℓ belongs to the (non-empty) closed set A_k . Since $\{a_{k,n} : n \in \mathbf{N}\}$ is dense in A_k , there exists a sequence $(a_{k,r_m} : m \in \mathbf{N})$ converging to ℓ . Recall that $x_n = a_{k,r_m}$ whenever $n \in \mathcal{A}_{k,r_m}$ for each $m \in \mathbf{N}$. Set by convenience $\theta_0 := 0$ and define recursively an increasing sequence of positive integers (θ_m) such that θ_m is an integer greater than θ_{m-1} for which

$$d^*(\mathcal{A}_{k,r_m} \cap (\theta_{m-1}, \theta_m]) \geq \frac{d(\mathcal{A}_k)}{2} = 2^{-k-2}.$$

Then, setting $\mathcal{A} := \bigcup_m \mathcal{A}_{k,r_m} \cap (\theta_{m-1}, \theta_m]$, we obtain that the subsequence $(x_n : n \in \mathcal{A})$ converges to ℓ and $d^*(\mathcal{A}) > 0$. In particular, $A \subseteq \Lambda_x$.

On the other hand, it is known that $\Lambda_x \subseteq \Gamma_x$, see e.g. [10]. If $A = B$, it follows by Claim (II) that $\Lambda_x \subseteq A$ and we are done. Otherwise, fix $\ell \in B \setminus A = F \setminus A$ and let us suppose for the sake of contradiction that there exists a subsequence (x_{n_k}) such that $\lim_k x_{n_k} = \ell$ and $d^*(\{n_k : k \in \mathbf{N}\}) > 0$. Fix a real $\varepsilon > 0$. Then, thanks to (11), there exists a sufficiently large integer n_0 such that $d(\bigcup_{k > n_0} \mathcal{A}_k) \leq \varepsilon$. In addition, since F is a metric space and μ_F is atomless and strictly positive (see Claim (II)), we have

$$\lim_{n \rightarrow \infty} \mu_F(V_n) = \mu_F(\{\ell\}) = 0,$$

where V_n is the open ball (relative to F) with center ℓ and radius $1/n$. Hence, there exists a sufficiently large integer m' such that $0 < \mu_F(V_{m'}) \leq \varepsilon$. In addition, there exists an integer m'' such that $V_{m''}$ is disjoint from the closed set $A_1 \cup \dots \cup A_{n_0}$. Then, set $V := V_m$, where m is an integer greater than $\max(m', m'')$ such that $\mu_F(V) < \mu_F(V_{\max(m', m'')})$. In particular, by the monotonicity of μ_F , we have

$$0 < \mu_F(V) \leq \mu_F(\overline{V}) \leq \mu_F(V_{m'}) \leq \varepsilon. \quad (13)$$

At this point, observe there exists a finite set Y such that

$$\begin{aligned} \{n_k : k \in \mathbf{N}\} &= \{n_k : x_{n_k} \in V\} \cup Y \\ &\subseteq (\bigcup_{k > n_0} \mathcal{A}_k) \cup \{n \in \mathcal{B} : x_n \in V\} \cup \mathcal{C} \cup Y. \end{aligned}$$

Therefore, by the subadditivity of d^* , (8), and (13), we obtain

$$\begin{aligned} d^*(\{n_k : k \in \mathbf{N}\}) &\leq \varepsilon + d^*(\{n \in \mathcal{B} : x_n \in V\}) \leq \varepsilon + d^*(\{n \in \mathcal{B} : b_n \in V\}) \\ &\leq \varepsilon + d^*(\{n \in \mathcal{B} : b_n \in \overline{V}\}) \leq \varepsilon + \mu_F(\overline{V}) \leq 2\varepsilon. \end{aligned}$$

It follows by the arbitrariness of ε that $d(\{n_k : k \in \mathbf{N}\}) = 0$, i.e., $\Lambda_x \subseteq A$.

To complete the proof, assume now that $A = \emptyset$. In this case, note that necessarily $S = \emptyset$, and it is enough to replace (12) with

$$x_n = \begin{cases} b_{n - \lfloor \sqrt{n} \rfloor} & \text{if } n \notin \mathcal{C}, \\ c_{\sqrt{n}} & \text{if } n \in \mathcal{C}. \end{cases}$$

Then, it can be shown with a similar argument that $\Lambda_x = \emptyset$, $\Gamma_x = B$, and $L_x = C$. \square

It is worth noting that Theorem 3.1 cannot be extended to the whole class of analytic P-ideals. Indeed, it follows by Theorem 2.3 that if \mathcal{I} is an F_σ ideal on \mathbf{N} then the set of \mathcal{I} -limit points is closed set, cf. also Theorem 3.4 below.

In addition, the regular closedness of $B \setminus S$ is essential in the proof of Theorem 3.1. On the other hand, there exist real sequences x such that Γ_x is the Cantor set \mathcal{C} (which is a perfect set but not regular closed):

Example 3.2. Given a real $r \in [0, 1)$ and an integer $b \geq 2$, we write r in base b as $\sum_n a_n/b^n$, where each a_n belongs to $\{0, 1, \dots, b-1\}$ and $a_n = \zeta$ for all sufficiently large n only if $\zeta = 0$. This representation is unique.

Let $x = (x_n)$ be the sequence $(0, 0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, \dots)$. This sequence is uniformly distributed in $[0, 1]$, i.e., $d(\{n : x_n \in [a, b]\}) = b - a$ for all $0 \leq a < b \leq 1$, and $\Gamma_x = [0, 1]$, see e.g. [10, Example 4]. Let also $T : [0, 1] \rightarrow \mathcal{C}$ be the injection defined by $r \mapsto T(r)$, where if $r = \sum_n a_n/2^n \in [0, 1)$ in base 2 then $T(r) = \sum_n 2a_n/3^n$ in base 3, and $1 \mapsto 1$. Observe that $\mathcal{C} \setminus T([0, 1))$ is the set of points of the type $2(1/3^{n_1} + \dots + 1/3^{n_k-1}) + 1/3^{n_k}$, for some non-negative integers $n_1 < \dots < n_k$; in particular, $\overline{T([0, 1))} = \mathcal{C}$.

Since the sequence $T(x) := (T(x_n))$ takes values in the closed set \mathcal{C} , it is clear that $\Gamma_{T(x)} \subseteq \mathcal{C}$. On the other hand, fix $\ell \in T([0, 1))$ with representation $\sum_n 2a_n/3^n$ in base 3, where $a_n \in \{0, 1\}$ for all n . For each k , let U_k be the open ball with center ℓ and radius $1/3^k$. It follows that

$$\begin{aligned} \{n : T(x_n) \in U_k\} &\supseteq \left\{ n : T(x_n) \in \left[\frac{2a_1}{3} + \dots + \frac{2a_k}{3^k}, \frac{2a_1}{3} + \dots + \frac{2a_k}{3^k} + \frac{1}{3^k} \right) \right\} \\ &= \left\{ n : x_n \in \left[\frac{a_1}{2} + \dots + \frac{a_k}{2^k}, \frac{a_1}{2} + \dots + \frac{a_k}{2^k} + \frac{1}{2^k} \right) \right\}. \end{aligned}$$

Since (x_n) is equidistributed, then $d^*(\{n : T(x_n) \in U_k\}) \geq 1/2^k$ for all k . In particular, $\Gamma_{T(x)}$ is a closed set containing $T([0, 1))$, therefore $\Gamma_{T(x)} = \mathcal{C}$.

Finally, we provide a sufficient condition for the existence of an atomless strictly positive Borel probability measure:

Corollary 3.3. *Let X be a Polish space without isolated points and fix sets $A \subseteq B \subseteq C \subseteq X$ such that A is an F_σ -set, $B \neq \emptyset$ is regular closed, and C is closed. Then, there exists a sequence x taking values in X which satisfies (9).*

Proof. First, observe that the restriction $\tilde{\lambda}$ of the Lebesgue measure λ on the set $\mathcal{J} := (0, 1) \setminus \mathbf{Q}$ is an atomless strictly positive Borel probability measure. Thanks to [6, Exercise 6.2.A(e)], X contains a dense subspace D which is homeomorphic to $\mathbf{R} \setminus \mathbf{Q}$, which in turn is homeomorphic to \mathcal{J} , let us say through $\eta : D \rightarrow \mathcal{J}$. This embedding can be used to transfer the measure $\tilde{\lambda}$ to the target space by setting

$$\mu : \mathcal{B}(X) \rightarrow [0, 1] : Y \mapsto \tilde{\lambda}(\eta(Y \cap D)). \quad (14)$$

Lastly, since B is non-empty closed regular, then it has no isolated points and contains an open set U of X . In particular, considering that η is an open map, we get by (14) that $\mu(B) \geq \mu(U) = \tilde{\lambda}(\eta(U \cap D)) > 0$. The claim follows by Theorem 3.1. \square

Note that, in general, the condition $B \neq \emptyset$ cannot be dropped: indeed, it follows by [5, Theorem 2.14] that, if X is compact, then every sequence (x_n) admits at least one statistical cluster point.

We conclude with another converse result related to ideals \mathcal{I} of the type F_σ (recall that, thanks to Theorem 2.3, every \mathcal{I} -limit point is also an \mathcal{I} -cluster point):

Theorem 3.4. *Let X be a first countable space where all closed sets are separable and let $\mathcal{I} \neq \text{Fin}$ be an F_σ -ideal. Fix also closed sets $B, C \subseteq X$ such that $\emptyset \neq B \subseteq C$. Then there exists a sequence x such that $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I}) = B$ and $L_x = C$.*

Proof. By hypothesis, there exists an infinite set $I \in \mathcal{I}$. Let φ be a lower semicontinuous submeasures associated to \mathcal{I} as in (5). Let $\{b_n : n \in \mathbb{N}\}$ and $\{c_n : n \in \mathbb{N}\}$ be countable dense subsets of B and C , respectively. In addition, set $m_0 := 0$ and let (m_k) be an increasing sequence of positive integers such that $\varphi((\mathbb{N} \setminus I) \cap (m_{k-1}, m_k]) \geq k$ for all k (note that this is possible since $\varphi(\mathbb{N} \setminus I) = \infty$ and φ is a lower semicontinuous submeasure). At this point, given a partition $\{H_n : n \in \mathbb{N}\}$ of $\mathbb{N} \setminus I$, where each H_n is infinite, we set

$$M_k := (\mathbb{N} \setminus I) \cap \bigcup_{n \in H_k} (m_{n-1}, m_n]$$

for all $k \in \mathbb{N}$. Then, it is easily checked that $\{M_k : k \in \mathbb{N}\}$ is a partition of $\mathbb{N} \setminus I$ with $M_k \notin \mathcal{I}$ for all k , and that the sequence (x_n) defined by

$$x_n = \begin{cases} b_k & \text{if } n \in M_k, \\ c_k & \text{if } n \text{ is the } k\text{-th term of } I. \end{cases}$$

satisfies the claimed conditions. \square

In particular, Theorem 2.6 and Theorem 3.4 fix a gap in a result of Das [3, Theorem 3] and provide its correct version.

4. CONCLUDING REMARKS

In this last section, we are interested in the topological nature of the set of \mathcal{I} -limit points when \mathcal{I} is neither F_σ - nor analytic P-ideal.

Let \mathcal{N} be the set of strictly increasing sequences of positive integers. Then \mathcal{N} is a Polish space, since it is closed subspace of the Polish space $\mathbb{N}^{\mathbb{N}}$ (equipped with the product topology of the discrete topology on \mathbb{N}). Let also $x = (x_n)$ be a sequence taking values in a first countable regular space X and fix an arbitrary ideal \mathcal{I} on \mathbb{N} . For each $\ell \in X$, let $(U_{\ell,m})$ be a decreasing local base of open neighborhoods at ℓ . Then, ℓ is an \mathcal{I} -limit point of x if and only if there exists a sequence $(n_k) \in \mathcal{N}$ such that

$$\{n_k : k \in \mathbb{N}\} \notin \mathcal{I} \text{ and } \{n : x_n \notin U_{\ell,m}\} \in \text{Fin} \text{ for all } m. \quad (15)$$

Set $\mathcal{I}^c := \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$ and define the continuous function $\psi : \mathcal{N} \rightarrow \{0,1\}^{\mathbb{N}} : (n_k) \mapsto \chi_{\{n_k : k \in \mathbb{N}\}}$, where χ_S is the characteristic function of a set $S \subseteq \mathbb{N}$. Moreover, define

$$\zeta_m : \mathcal{N} \times X \rightarrow \{0,1\}^{\mathbb{N}} : (n_k) \times \ell \mapsto \chi_{\{n : x_n \notin U_{\ell,m}\}}$$

for each m . Hence, it easily follows by (15) that

$$\Lambda_x(\mathcal{I}) = \pi_X \left(\bigcap_m (\psi^{-1}(\mathcal{I}^c) \times X \cap \zeta_m^{-1}(\text{Fin})) \right),$$

where $\pi_X : \mathcal{N} \times X \rightarrow X$ stands for the projection on X .

Proposition 4.1. *Let $x = (x_n)$ be a sequence taking values in a first countable regular space X and let \mathcal{I} be a co-analytic ideal. Then $\Lambda_x(\mathcal{I})$ is analytic.*

Proof. For each $(n_k) \in \mathcal{N}$ and $\ell \in X$, the sections $\zeta_m((n_k), \cdot)$ and $\zeta_m(\cdot, \ell)$ are continuous. Hence, thanks to [20, Theorem 3.1.30], each function ζ_m is Borel measurable. Since Fin is an F_σ -set, we obtain that each $\zeta_m^{-1}(\text{Fin})$ is Borel. Moreover, since \mathcal{I} is a co-analytic ideal and ψ is continuous, it follows that $\psi^{-1}(\mathcal{I}^c) \times X$ is an analytic subset of $\mathcal{N} \times X$. Therefore $\Lambda_x(\mathcal{I})$ is the projection on X of the analytic set $\bigcap_m (\psi^{-1}(\mathcal{I}^c) \times X \cap \zeta_m^{-1}(\text{Fin}))$, which proves the claim. \square

The situation is much different for *maximal ideals*, i.e., ideals which are maximal with respect to inclusion. In this regard, we recall if \mathcal{I} is a maximal ideal then every bounded real sequence x is \mathcal{I} -convergent, i.e., there exists $\ell \in \mathbf{R}$ such that $\{n : |x_n - \ell| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$, cf. [12, Theorem 2.2].

Let $B(a, r)$ the open ball with center a and radius r in a given metric space (X, d) , and denote by $\text{diam } S$ the diameter of a non-empty set $S \subseteq X$, namely, $\sup_{a, b \in S} d(a, b)$. Then, the metric space is said to be *smooth* if

$$\lim_{k \rightarrow \infty} \sup_{a \in X} \text{diam } \overline{B(a, 1/k)} = 0. \quad (16)$$

Note that (16) holds if, e.g., the closure of each open ball $B(a, r)$ coincides with the corresponding closed ball $\{b \in X : d(a, b) \leq r\}$.

Proposition 4.2. *Let x be a sequence taking values in a smooth compact metric space X and let \mathcal{I} be a maximal ideal. Then x has exactly one \mathcal{I} -cluster point. In particular, $\Lambda_x(\mathcal{I})$ is closed.*

Proof. Since X is a compact metric space, then X is totally bounded, i.e., for each $\varepsilon > 0$ there exist finitely many open balls with radius ε covering X . Moreover, it is well known that an ideal \mathcal{I} is maximal if and only if either $A \in \mathcal{I}$ or $A^c \in \mathcal{I}$ for every $A \subseteq \mathbf{N}$.

Hence, fix $k \in \mathbf{N}$, let $\{B_{k,1}, \dots, B_{k,m_k}\}$ be a cover of X of open balls with radius $1/k$, and define $\mathcal{C}_{k,i} := \{n : x_n \in C_{k,i}\}$ for each $i \leq m_k$, where $C_{k,i} := B_{k,i} \setminus (B_{k,1} \cup \dots \cup B_{k,i-1})$ and $B_{k,0} := \emptyset$. Considering that $\{\mathcal{C}_{k,1}, \dots, \mathcal{C}_{k,m_k}\}$ is a partition of \mathbf{N} , it follows by the above observations that there exists a unique $i_k \in \{1, \dots, m_k\}$ for which $\mathcal{C}_{k,i_k} \notin \mathcal{I}$.

At this point, let (G_k) be the decreasing sequence of closed sets defined by

$$G_k := \overline{C_{1,i_1} \cap \dots \cap C_{k,i_k}}$$

for all k . Note that each G_k is non-empty, the diameter of G_k (which is contained in $\overline{B_{k,i_k}}$) goes to 0 as $k \rightarrow \infty$, and $\{n : x_n \in G_k\} \notin \mathcal{I}$ for all k . Since X is a compact metric space, then $\bigcap_k G_k$ is a singleton $\{\ell\}$. Considering that every open ball with center ℓ contains some G_k with k sufficiently large, it easily follows that $\Gamma_x(\mathcal{I}) = \{\ell\}$. In particular, since each \mathcal{I} -limit point is also an \mathcal{I} -cluster point, we conclude that $\Lambda_x(\mathcal{I})$ is either empty or the singleton $\{\ell\}$. \square

Corollary 4.3. *An ideal \mathcal{I} is maximal if and only if every real sequence x has at most one \mathcal{I} -limit point.*

Proof. First, let us assume that \mathcal{I} is a maximal ideal. Let us suppose that there exists $k > 0$ such that $A_k := \{n : |x_n| > k\} \in \mathcal{I}$ and define a sequence $y = (y_n)$ by $y_n = k$ if $n \in A_k$ and $y_n = x_n$ otherwise. Then, it follows by [3, Theorem 4] and Proposition 4.2 that there exists $\ell \in \mathbf{R}$ such that $\Lambda_x(\mathcal{I}) = \Lambda_y(\mathcal{I}) \subseteq \Gamma_y(\mathcal{I}) = \{\ell\}$.

Now, assume that $A_k^c \in \mathcal{I}$ for all $k \in \mathbf{N}$. Hence, letting $z = (z_n)$ be the sequence defined by $z_n = x_n$ if $n \in A_k$ and $z_n = k$ otherwise, we obtain

$$\Lambda_x(\mathcal{I}) = \Lambda_z(\mathcal{I}) \subseteq L_z \subseteq \mathbf{R} \setminus (-k, k).$$

Therefore, it follows by the arbitrariness of k that $\Lambda_x(\mathcal{I}) = \emptyset$.

Conversely, let us assume that \mathcal{I} is not a maximal ideal. Then there exists $A \subseteq \mathbf{N}$ such that $A \notin \mathcal{I}$ and $A^c \notin \mathcal{I}$. Then, the sequence (x_n) defined by $x_n = \chi_A(n)$ for each n has two \mathcal{I} -limit points. \square

We conclude by showing that there exists an ideal \mathcal{I} and a real sequence x such that $\Lambda_x(\mathcal{I})$ is not an F_σ -set.

Example 4.4. Fix a partition $\{P_m : m \in \mathbf{N}\}$ of \mathbf{N} such that each P_m is infinite. Then, define the ideal

$$\mathcal{I} := \{A \subseteq \mathbf{N} : \{m : A \cap P_m \notin \text{Fin}\} \in \text{Fin}\},$$

which corresponds to the Fubini product $\text{Fin} \times \text{Fin}$ on \mathbf{N}^2 (it is known that \mathcal{I} is a $F_{\sigma\delta\sigma}$ -ideal and it is not a P-ideal). Given a real sequence $x = (x_n)$, let us denote by $x \upharpoonright P_m$ the subsequence $(x_n : n \in P_m)$. Hence, a real ℓ is an \mathcal{I} -limit point of x if and only if there exists a subsequence (x_{n_k}) converging to ℓ such that $\{n_k : k \in \mathbf{N}\} \cap P_m$ is infinite for infinitely many m . Moreover, for each m of this type, the subsequence $(x_{n_k}) \upharpoonright P_m$ converges to ℓ . It easily follows that

$$\Lambda_x(\mathcal{I}) = \bigcap_k \bigcup_{m \geq k} L_{x \upharpoonright P_m}. \quad (17)$$

(In particular, since each $L_{x \upharpoonright P_m}$ is closed, then $\Lambda_x(\mathcal{I})$ is an $F_{\sigma\delta}$ -set.)

At this point, let $(q_t : t \in \mathbf{N})$ be the sequence $(0/1, 1/1, 0/2, 1/2, 2/2, 0/3, 1/3, 2/3, 3/3, \dots)$, where $q_t := a_t/b_t$ for each t , and note that $\{q_t : t \in \mathbf{N}\} = \mathbf{Q} \cap [0, 1]$. It follows by construction that $a_t \leq b_t$ for all t and $b_t = \sqrt{2t}(1 + o(1))$ as $t \rightarrow \infty$. In particular, if m is a sufficiently large integer, then

$$\min_{i \leq m: q_i \neq q_m} |q_i - q_m| \geq \left(\frac{1}{\sqrt{2m}(1 + o(1))} \right)^2 > \frac{1}{3m}. \quad (18)$$

Lastly, for each $m \in \mathbf{N}$, define the closed set

$$C_m := [0, 1] \cap \bigcap_{t \leq m} \left(q_t - \frac{1}{2^m}, q_t + \frac{1}{2^m} \right)^c.$$

We obtain by (18) that, if m is sufficiently large, let us say $\geq k_0$, then

$$C_m \cup C_{m+1} = [0, 1] \cap \bigcap_{t \leq m} \left(q_t - \frac{1}{2^{m+1}}, q_t + \frac{1}{2^{m+1}} \right)^c.$$

It follows by induction that

$$C_m \cup C_{m+1} \cup \dots \cup C_{m+n} = [0, 1] \cap \bigcap_{t \leq m} \left(q_t - \frac{1}{2^{m+n}}, q_t + \frac{1}{2^{m+n}} \right)^c.$$

for all $n \in \mathbf{N}$. In particular, $\bigcup_{m \geq k} C_m = [0, 1] \setminus \{q_1, \dots, q_k\}$ whenever $k \geq k_0$.

Let x be a real sequence such that each $\{x_n : n \in P_m\}$ is a dense subset of C_m . Therefore, it follows by (17) that

$$\Lambda_x(\mathcal{I}) = \bigcap_k \bigcup_{m \geq k} C_m \subseteq \bigcap_{k \geq k_0} \bigcup_{m \geq k} C_m = \bigcap_{k \geq k_0} [0, 1] \setminus \{q_1, \dots, q_k\} = [0, 1] \setminus \mathbf{Q}.$$

On the other hand, if a rational q_t belongs to $\Lambda_x(\mathcal{I})$, then $q_t \in \bigcup_{m \geq k} C_m$ for all $k \in \mathbf{N}$, which is impossible whenever $k \geq t$. This proves that $\Lambda_x(\mathcal{I}) = [0, 1] \setminus \mathbf{Q}$, which is not an F_σ -set.

We leave as an open question to determine whether there exists a real sequence x and an ideal \mathcal{I} such that $\Lambda_x(\mathcal{I})$ is not Borel measurable.

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REFERENCES

- [1] M. Balcerzak, K. Dems, and A. Komisarski, *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl. **328** (2007), no. 1, 715–729.
- [2] J. Connor and J. Kline, *On statistical limit points and the consistency of statistical convergence*, J. Math. Anal. Appl. **197** (1996), no. 2, 392–399.
- [3] P. Das, *Some further results on ideal convergence in topological spaces*, Topology Appl. **159** (2012), no. 10–11, 2621–2626.
- [4] P. Das and B. K. Lahiri, *I and I*-convergence in topological spaces*, Math. Bohem. **130** (2005), no. 2, 153–160.
- [5] G. Di Maio and L. D. R. Kočinac, *Statistical convergence in topology*, Topology Appl. **156** (2008), no. 1, 28–45.
- [6] R. Engelking, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from the Polish by the author.
- [7] I. Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc. **148** (2000), no. 702, xvi+177.
- [8] R. Filipów, N. Mrozek, I. Reclaw, and P. Szuca, *Ideal convergence of bounded sequences*, J. Symbolic Logic **72** (2007), no. 2, 501–512.
- [9] D. H. Fremlin, *Measure theory. Vol. 4*, Torres Fremlin, Colchester, 2006, Topological measure spaces. Part I, II, Corrected second printing of the 2003 original.
- [10] J. A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc. **118** (1993), no. 4, 1187–1192.
- [11] J. A. Fridy and C. Orhan, *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. **125** (1997), no. 12, 3625–3631.
- [12] P. Kostyrko, M. Mačaj, T. Šalát, and M. Szeziak, *\mathcal{I} -convergence and extremal \mathcal{I} -limit points*, Math. Slovaca **55** (2005), no. 4, 443–464.
- [13] P. Kostyrko, M. Mačaj, T. Šalát, and O. Strauch, *On statistical limit points*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2647–2654.
- [14] P. Leonetti, *Invariance of ideal limit points*, preprint, last updated: Jul 24, 2017 ([arXiv:1707.07475](https://arxiv.org/abs/1707.07475)).
- [15] M. A. Mamedov and S. Pehlivan, *Statistical cluster points and turnpike theorem in nonconvex problems*, J. Math. Anal. Appl. **256** (2001), no. 2, 686–693.
- [16] K. Mazur, *F_σ -ideals and $\omega_1\omega_1^*$ -gaps in the Boolean algebras $P(\omega)/I$* , Fund. Math. **138** (1991), no. 2, 103–111.
- [17] H. I. Miller, *A measure theoretical subsequence characterization of statistical convergence*, Trans. Amer. Math. Soc. **347** (1995), no. 5, 1811–1819.
- [18] S. Solecki, *Analytic ideals*, Bull. Symbolic Logic **2** (1996), no. 3, 339–348.

-
- [19] ———, *Analytic ideals and their applications*, Ann. Pure Appl. Logic **99** (1999), no. 1-3, 51–72.
[20] S. M. Srivastava, *A course on Borel sets*, Graduate Texts in Mathematics, vol. 180, Springer-Verlag, New York, 1998.

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