On m-Kropina Finsler Metrics of Scalar Flag Curvature

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Abstract

In this paper, we consider a special class of singular Finsler metrics: m-Kropina metrics which are defined by a Riemannian metric and a 1-form. We show that an m-Kropina metric ($m \neq -1$) of scalar flag curvature must be locally Minkowskian in dimension $n \geq 3$. We characterize by some PDEs a Kropina metric ($m = -1$) which is respectively of scalar flag curvature and locally projectively flat in dimension $n \geq 3$, and obtain some principles and approaches of constructing non-trivial examples of Kropina metrics of scalar flag curvature.

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1 Introduction

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, and every two-dimensional Finsler metric is of scalar flag curvature. It is the Hilbert's Fourth Problem to study and classify projectively flat metrics. The Beltrami Theorem states that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. It is known that every locally projectively flat Finsler metric is of scalar flag curvature. However, the converse is not true. There are regular or singular Finsler metrics of constant flag curvature which are not locally projectively flat ([1] [18]). Therefore, it is an interesting point to study and classify Finsler metrics of scalar flag curvature. This problem is far from being solved for general Finsler metrics. Thus we shall investigate some special classes of Finsler metrics. Recent studies on this problem are concentrated on Randers metrics, square metrics and some other special (α, β)-metrics.

A Randers metric is defined by $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form with $b = \|\beta\|_\alpha < 1$. After many mathematicians' efforts, Bao-Robles-Shen finally classify Randers metrics of constant flag curvature by using the navigation method ([1]). Further, Shen-Yildirim characterize Randers metrics of scalar flag curvature and classify Randers metrics of weakly isotropic flag curvature ([11]). There are Randers metrics of scalar flag curvature which are neither of weakly isotropic flag curvature nor locally projectively flat ([2] [7]). So far, the problem of classifying Randers metrics of scalar flag curvature still remains open.

A square metric is written as $F = (\alpha + \beta)^2/\alpha$, where α is a Riemannian metric and β is a 1-form with $b = \|\beta\|_\alpha < 1$. In [10], Shen-Yildirim determine the local structure of locally projectively flat square metrics of constant flag curvature. Zhou shows that a square metric of constant flag curvature is locally projectively flat ([19]). Later on, we prove that a square metric in dimension $n \geq 3$ is of scalar flag curvature iff. it is locally projectively flat ([8]).

In [14], we consider an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ with $\phi(s)$ satisfying

$$\big\{1 + (k_1 + k_3)s^2 + k_2 s^4\big\}\phi''(s) = (k_1 + k_2 s^2)\big\{\phi(s) - s\phi'(s)\big\},$$

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where k_1, k_2, k_3 are constant with $k_2 \neq k_1 k_3$. We prove that if β is closed and the dimension $n \geq 3$, then F is of scalar flag curvature if and only if F is locally projectively flat, and for a special case given by $\phi(s) = 1 + a_1 s + \epsilon s^2$ with a_1 and $\epsilon \neq 0$ being constant, we show that F is of scalar flag curvature if and only if F is locally projectively flat.

The Finsler metrics mentioned above are regular. It seems hard to characterize a general regular (α, β)-metric of scalar flag curvature in dimension $n \geq 3$. On the other hand, singular Finsler metrics, such as Kropina metrics and m-Kropina metrics, have a lot of applications in the real word. In this paper, we will study m-Kropina metrics of scalar flag curvature. An m-Kropina metric has the following form

$$F = \alpha^{1-m} \beta^m, \quad m \neq 0, 1.$$

When $m = -1$, F is called a Kropina metric ([4]). There have been a few research papers on Kropina metrics ([6] [9] [13], [15]–[18]). m-Kropina metrics naturally appear in characterizing a class of singular (α, β)-metrics which are locally projectively flat ([15] [16]) and locally projectively flat with constant flag curvature ([17]). Note that due to the deformation (6) below for an m-Kropina metric, we can always assume $b = ||\beta||_\alpha = 1$ without loss of generality.

Theorem 1.1 *Let $F = \alpha^{1-m}\beta^m$ be an $n(\geq 3)$-dimensional m-Kropina metric ($m \neq -1$) with $||\beta||_\alpha = 1$. Then F is of scalar flag curvature iff. F is locally Minkowskian, or more precisely, F is flat-parallel (α is locally flat and β is parallel with respect to α).*

In [9], we show that an $n(\geq 2)$-dimensional m-Kropina metric ($m \neq -1$) of constant flag curvature is locally Minkowskian. In [16], we prove that an $n(\geq 3)$-dimensional locally projectively flat m-Kropina metric ($m \neq -1$) is locally Minkowskian. Therefore, Theorem 1.1 generalizes the corresponding results in [9] [16]. Besides, we indicate that a two-dimensional Douglas m-Kropina metric ($m \neq -1$) is locally Minkowskian ([15]).

The case $m = -1$ will be much more complicated. In Section 4 below, we give respective characterizations by some PDEs for a Kropina metric to be of scalar flag curvature and locally projectively flat in dimension $n \geq 3$ (see Theorem 4.1 and Theorem 4.2 below). In Section 5, we use Theorem 4.1 to prove the known local classification for a Kropina metric of constant flag curvature (see Corollary 5.1). However, it is difficult to determine the local structure of a Kropina metric of scalar flag curvature, even if it is locally projectively flat (cf. [15] [16]). Here we will show some methods (including using Corollary 4.3 below) of constructing non-trivial Kropina metrics of scalar flag curvature.

Kropina metrics are related to Randers metrics to some extent. Every Kropina metric is the limit of a family of Randers metrics $F = \alpha + \beta$ as the norm $b = ||\beta||_\alpha \to 1^-$ (see Remark 7.2 below). Further, we have the following result.

Theorem 1.2 *Let $F = \alpha + \beta$ be a Randers metric and (h, ρ) be the navigation data of F. Suppose $\widetilde{\alpha} := \lim_{b\to 1^-} h$ is a Riemann metric and $\widetilde{\beta} := \lim_{b\to 1^-} \rho$ is a non-zero 1-form. Let $\widetilde{F} = \widetilde{\alpha}^2/\widetilde{\beta}$ be the Kropina metric derived from F. Then we have*

(i) *$\widetilde{F} = \lim_{b\to 1^-} 2F$, and $||\widetilde{\beta}||_{\widetilde{\alpha}} = 1$.*

(ii) *If F is of scalar flag curvature (resp. locally projectively flat, or Douglassian), then $\widetilde{F}$ is also of scalar flag curvature (resp. locally projectively flat, or Douglassian). If F is of weakly isotropic flag curvature, then $\widetilde{F}$ is of constant flag curvature.*

For a given Randers metric $F = \alpha + \beta$ in Theorem 1.2, to obtain the Kropina metric $\widetilde{F}$, we only need to require that $\lim_{b\to 1^-}(1-b^2)(\alpha^2-\beta^2)$ is a Riemann metric and $\widetilde{\beta} := \lim_{b\to 1^-}(1-b^2)\beta$ is a non-zero 1-form, since $h = \sqrt{(1-b^2)(\alpha^2-\beta^2)}$ and $\rho = -(1-b^2)\beta$. Equivalently, we can obtain $\widetilde{\alpha}$ and $\widetilde{\beta}$ by letting $h^{ij}\rho_i\rho_j = 1$, where we put $h = \sqrt{h_{ij}y^iy^j}$ and $\rho = \rho_i y^i$. By Theorem 1.2, to construct non-trivial Kropina metrics of scalar flag curvature in dimension $n \geq 3$, we can use the known examples of Randers metrics of scalar flag curvature (see [2] [7]).

Next we give another principle of constructing Kropina metrics of scalar flag curvature.

Theorem 1.3 *Let $F = \alpha^2/\beta$ be a Kropina metric with $||\beta||_\alpha = 1$ and define $\widetilde{F} = F + \eta$, where η is a closed 1-form with $||\eta||_\alpha$ sufficiently small.*

(i) *If F is of scalar flag curvature, then $\widetilde{F}$ is also a Kropina metric of scalar flag curvature.*

(ii) *Let F be of constant flag curvature. Then $\widetilde{F}$ is locally projectively flat if and only if F is flat-parallel, or equivalently, $\widetilde{F}$ can be locally written in the form*

$$\widetilde{F} = \frac{|y|}{y^1} + \eta. \tag{1}$$

Theorem 1.3 (ii) easily follows from a result in [17], since therein we prove that a locally projectively flat Kropina metric with constant flag curvature is flat-parallel. By Theorem 1.3 (ii), we can easily obtain a family of Kropina metrics which are of scalar flag curvature but are neither locally projectively flat nor of constant flag curvature in general. Take $\eta = \langle x, y\rangle$ with x close to origin, and then $\widetilde{F}$ in (1) is a projectively flat Kropina metric with the flag curvature given by

$$K = \frac{3}{4}\frac{|y|^4(y^1)^4}{(\eta y^1 + |y|^2)^4}.$$

Additionally, using Corollary 4.3 below and a warped product method, we obtain a family of Kropina metrics which are locally projectively flat (see Proposition 6.2 below).

2 Preliminaries

For a Finsler metric F, the Riemann curvature $R_y = R^i{}_k(y)\frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}, \tag{2}$$

where the spray coefficients G^i are given by

$$G^i := \frac{1}{4}g^{il}\big\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\big\}. \tag{3}$$

The Ricci curvature Ric is the trace of the Riemann curvature, that is, $Ric := R^m_{\ m}$. A Finsler metric is said to be of scalar flag curvature if there is a function $K = K(x,y)$ such that

$$R^i{}_k = KF^2(\delta^i_k - F^{-2}y^iy_k), \quad y_k := (F^2/2)_{y^iy^k}y^i. \tag{4}$$

If K is a constant, F is said to be of constant flag curvature. A Finsler metric F is said to be projectively flat in U, if there is a local coordinate system (U, x^i) such that $G^i = Py^i$, where $P = P(x,y)$ is called the projective factor satisfying $P(x, \lambda y) = \lambda P(x,y)$ for $\lambda > 0$.

The Weyl curvature $W^i{}_k$ and the Douglas curvature $D_h{}^i{}_{jk}$ are two important projectively invariant tensors and they are defined respectively by

$$W^i{}_k := R^i{}_k - \frac{R^m_{\ m}}{n-1}\delta^i_k - \frac{1}{n+1}\frac{\partial}{\partial y^m}\big(R^m_{\ k} - \frac{R^h_{\ h}}{n-1}\delta^m_k\big)y^i, \tag{5}$$

$$D_h{}^i{}_{jk} := \frac{\partial^3}{\partial y^h \partial y^j \partial y^k}\big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\big).$$

In two-dimensional case, there is a projectively invariant tensor W^o called Berwald-Weyl tensor. A Finsler metric is called a Douglas metric if $D_h{}^i{}_{jk} = 0$. A Finsler metric is of scalar flag curvature if and only if $W^i{}_k = 0$. An n-dimensional Finsler metric is locally projectively flat if and only if: $W^i{}_k = 0$ and $D_h{}^i{}_{jk} = 0$ for $n \geq 3$, and $W^o = 0$ and $D_h{}^i{}_{jk} = 0$ for $n = 2$ ([5]).

An (α, β)-metric F is a Finsler metric defined by a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a 1-form $\beta = b_i(x)y^i$ on a manifold M, which is expressed in the following form:

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where $\phi(s)$ is a suitable function. If we take $\phi(s) = 1 + s$, then we get the well-known Randers metric $F = \alpha + \beta$. In applications, there are a lot of singular Finsler metrics. In this paper, we will discuss a class of singular (α, β) Finsler metrics—m-Kropina metrics.

An m-Kropina metric is in the form $F = \alpha^{1-m}\beta^m$, where $m \neq 0, 1$ is real. In particular, it is called a Kropina metric when $m = -1$. For an m-Kropina metric $F = \alpha^{1-m}\beta^m$, we introduce a special deformation on α and β. Define a new pair $(\widetilde{\alpha}, \widetilde{\beta})$ by

$$\widetilde{\alpha} := b^m\alpha, \quad \widetilde{\beta} := b^{m-1}\beta, \tag{6}$$

which appears first in [9]. It is interesting that under the deformation (6), the m-Kropina metric $F = \alpha^{1-m}\beta^m$ keeps formally unchanged, that is,

$$F = \widetilde{\alpha}^{1-m}\widetilde{\beta}^m, \qquad (||\widetilde{\beta}||_{\widetilde{\alpha}} = 1). \tag{7}$$

It has been shown that the deformation (6) plays an important role on the study of m-Kropina metrics ([9] [15]–[17]). Due to (7), we can always assume $||\beta||_\alpha = 1$ for an m-Kropina metric $F = \alpha^{1-m}\beta^m$ without loss of generality.

For a Riemann metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and a 1-form $\beta = b_iy^i$, define

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r^i{}_j := a^{ik}r_{kj}, \quad s^i{}_j := a^{ik}s_{kj},$$

$$p_{ij} := r_{ik}r^k{}_j, \quad q_{ij} := r_{ik}s^k{}_j, \quad t_{ij} := s_{ik}s^k{}_j, \quad r_j := b^ir_{ij}, \quad s_j := b^is_{ij},$$

$$p_j := b^ip_{ij}, \quad q_j := b^iq_{ij}, \quad r_j := b^ir_{ij}, \quad t_j := b^ir_{ij}, \quad r := b^ir_i,$$

where b^i is defined by $b^i := a^{ij}b_j$, (a^{ij}) is the inverse of (a_{ij}), and $\nabla\beta = b_{i|j}y^idx^j$ denotes the covariant derivatives of β with respect to α. We use a_{ij} to raise or lower the indices of a tensor. For a tensor T_{ij} as an example, define $T_{i0} := T_{ij}y^j$ and $T_{00} := T_{ij}y^iy^j$, etc.

Lemma 2.1 *Here we list some identities as follows:*

$$q_{ik} + q_{ki} = r_{i|k} + r_{k|i} - 2p_{ik} - b^m(r_{mi|k} + r_{mk|i}), \qquad q_{ik} = t_{ik} + s_{k|i} - b^ms_{mk|i}, \tag{8}$$

$$s_{ij|k} = r_{ik|j} - r_{jk|i} - b_l\bar{R}_k{}^l{}_{ij}, \quad b^mb^v(r_{mv|k} - r_{mk|v}) = t_k - q_k + b^ms_{k|m}, \tag{9}$$

$$b^mq_{km} = -r_{km}s^m = b^ms_{m|k} + t_k, \qquad q^k{}_k = 0, \tag{10}$$

where $\bar{R}$ denotes the Riemann curvature tensor of α. If $||\beta||_\alpha = constant$, we have

$$r_k + s_k = 0, \quad b^mq_m = s_ms^m, \quad b^lb^ks_{k|l} = 2s^ls_l, \quad b^ib^jb^kr_{ij|k} = -4s_ls^l, \tag{11}$$

$$b^ms_{i|m} = 2s^mr_{im} - b^kb^lr_{ki|l} = -2b^mq_{im} - b^kb^lr_{ki|l}. \tag{12}$$

For an m-Kropina metric $F = \alpha^{1-m}\beta^m$, by (3) we get

$$G^i = G^i_\alpha - \frac{m}{(m-1)s}\alpha s^i_0 + \frac{m}{2(m-1)}\frac{(m-1)sr_{00} + 2m\alpha s_0}{s\big[mb^2 - (m+1)s^2\big]}(b^i - 2\alpha^{-1}sy^i). \tag{13}$$

Then by (5) and (13), we can get the expressions of the Weyl curvature tensor $W^i{}_k$ for an m-Kropina metric $F = \alpha^{1-m}\beta^m$. We have given a Maple program in [8] to compute the Weyl curvature for any (α, β)-metric. In this paper, we will write out the whole expression of the Weyl curvature for a Kropina metric ($m = -1$); while for $m \neq -1$, we will not write out the expression since it is very long, but some key terms will be given, similarly like what we have done in studying square metrics in [8].

For readers to verify the expression of $W^i{}_k$ for an m-Kropina metric $F = \alpha^{1-m}\beta^m$, we give the expression of a leading term. We see that $W^i{}_k \times (n^2-1)(m-1)^2\beta^3\big[mb^2\alpha^2-(m+1)\beta^2\big]^5 = 0$ can be written as

$$(n+1)m^7b^8A_{14}\alpha^{14} + A_{12}\alpha^{12} + A_{10}\alpha^{10} + \cdots + A_2\alpha^2 + A_0 = 0, \tag{14}$$

where $A_0, A_2, \cdots, A_{14}$ are polynomials in (y^i), and A_{14} is given by

$$\begin{aligned} A_{14} = &-(n-1)\big[(b^2t^i{}_0 + s_0s^i)b_k - (t_0b_k + \beta t_k)b_i + \beta(b^2t^i{}_k + s^is_k)\big] \\ &+(2s_js^j + b^2t^j{}_j)(\beta\delta^i_k + y^ib_k). \end{aligned}$$

When $m = -1$, eliminating the factor $-b^6\alpha^{10}$ from (14) we obtain

$$(n+1)b^2B_4\alpha^4 + 2(n+1)\beta B_2\alpha^2 + 4\beta^2B_0 = 0, \tag{15}$$

where $B_4 = A_{14}$, and B_2, B_0 are given by (denote by $\bar{W}^i{}_k$ the Weyl curvature of α)

$$\begin{aligned} B_2 = &(b^4s^j{}_{0|j} + b^2q_0 - b^2b^js_{0|j} - b^2r^j{}_js_0 - b^2b^jq_{0j} + rs_0)(2\beta\delta^i_k + y^ib_k) \\ &-y^i\big[b^2(2s_js^j + b^2t^j{}_j)y_k + \beta(r - b^2r^j{}_j)s_k + b^2\beta(q_k + b^2s^j{}_{k|j} - b^jq_{kj} - b^js_{k|j})\big] \\ &+(n-1)\big[b^2(b^2t^i{}_0 + s^is_0 - t_0b^i)y_k + (b^2s_{0|0} - b^2q_{00} - r_0s_0 - s_0^2)b^ib_k + \beta(r_0 - s_0)b^is_k \\ &+\beta b^i(b^2q_{0k} - 2b^2q_{k0} + 2b^2s_{0|k} - b^2s_{k|0} - 2s_0r_k) - (b^2s^i{}_{0|0} + r_{00}s^i - r^i{}_0s_0)b_k \\ &-b^2\beta r^i{}_0s_k - b^2\beta(r_{k0}s^i - 2r^i{}_ks_0 + 2b^2s^i{}_{0|k} - b^2s^i{}_{k|0})\big], \\ B_0 = &(n+1)\beta(b^2r_{0|0} - b^2r^j{}_jr_{00} + b^2s_{0|0} - 2b^2q_{00} - b^2b^jr_{00|j} - r_0^2 - s_0^2 + rr_{00} - 2r_0s_0)\delta^i_k \\ &-(n+1)(b^4s^j{}_{0|j} + b^2q_0 - b^2b^js_{0|j} - b^2r^j{}_js_0 - b^2b^jq_{0j} + rs_0)y^iy_k \\ &+\beta y^i\big[(n+1)(r_0 + s_0)(r_k + s_k) - (n+1)(r - b^2r^j{}_j)r_{k0} + (n-2)b^2r_{k|0} - (2n-1)b^2r_{0|k} \\ &+(n+1)b^2(q_{k0} + q_{0k}) + (n-2)b^2s_{k|0} - (2n-1)b^2s_{0|k} + (n+1)b^2b^jr_{k0|j}\big] \\ &+(n^2-1)\big\{y_k[(r_0s_0 + s_0^2 + b^2q_{00} - b^2s_{0|0})b^i + b^2r_{00}s^i + b^2(b^2s^i{}_{0|0} - s_0r^i{}_0)] \\ &+\beta b^i[(r_0 + s_0)r_{k0} - (r_k + s_k)r_{00} - b^2(r_{k0|0} - r_{00|k})] + b^2\beta(b^2\bar{W}^i{}_k + r_{00}r^i{}_k - r^i{}_0r_{k0})\big\}. \end{aligned}$$

Lemma 2.2 *Let $F = \alpha^2/\beta$ be an n-dimensional Kropina metric. Then $W^i{}_k = 0$ is equivalent to (15), and the Ricci curvature Ric of F is given by*

$$\begin{aligned} Ric = &\bar{R}ic - \frac{1}{4b^4\alpha^2s^2}\Big\{b^2(b^2t^l{}_l + 2s_ls^l)\alpha^4 + 4s\big[b^4s^l{}_{0|l} - (n-1)b^2t_0 + (r - b^2r^l{}_l)s_0 \\ &+b^2(q_0 - b^ls_{0|l} - b^lq_{0l})\big]\alpha^3 + 4s^2\big[(r - b^2r^l{}_l)r_{00} + (n-2)s_0^2 + 2(2n-3)r_0s_0 \\ &-(n-2)b^2s_{0|0} + b^2r_{0|0} - r_0^2 - 2nb^2q_{00} - b^lr_{00|l}\big]\alpha^2 \\ &+4(n-1)s^3\big[2r_{00}(2r_0 - s_0) - b^2r_{00|0}\big]\alpha - 12(n-1)s^4r_{00}^2\Big\}, \end{aligned} \tag{16}$$

where $\bar{R}ic$ denotes the Ricci curvature of α.

3 Proof of Theorem 1.1

Lemma 3.1 *β is closed $\Longleftrightarrow t_{ij}=0 \Longleftrightarrow t^k{}_k=0$.*

Lemma 3.2 *Let $F=\alpha^{1-m}\beta^m$ be an m-Kropina metric ($m\neq -1$) of scalar flag curvature on an $n(\geq 3)$-dimensional manifold M. Then r_{00} satisfies*

$$r_{00}=2\tau\big[mb^2\alpha^2-(m+1)\beta^2\big]-\frac{2(m+1)}{(m-1)b^2}\beta s_0, \tag{17}$$

where $\tau=\tau(x)$ is a scalar function.

Proof : Since $F=\alpha^{1-m}\beta^m$ is of scalar flag curvature, we have $W^i{}_k=0$. Then we have (14). Now $\alpha^2\times(14)$ can be written as

$$C^i_k\big[mb^2\alpha^2-(m+1)\beta^2\big]-24(n-2)(m+1)^3\beta^8y^i(\alpha^2b_k-\beta y_k)\big[(m-1)\beta r_{00}+2m\alpha^2s_0\big]^2=0, \tag{18}$$

where C^i_k are polynomials in (y^i). It is easy to see that $mb^2\alpha^2-(m+1)\beta^2$ is an irreducible polynomial in (y^i) since $m\neq 0$ and $n>2$. Further, if $\alpha^2b_k-\beta y_k$ is divisible by $mb^2\alpha^2-(m+1)\beta^2$ for all k, then there are scalar functions $\tau_k=\tau_k(x)$ such that

$$\alpha^2b_k-\beta y_k=\tau_k\big[mb^2\alpha^2-(m+1)\beta^2\big].$$

Contracting the above by y^k we have $\tau_0=0$ and hence $\alpha^2b_k-\beta y_k=0$. This is a contradiction. Now since $n>2$ and $m\neq -1$, it follows from (18) that $(m-1)\beta r_{00}+2m\alpha^2s_0$ is divisible by $mb^2\alpha^2-(m+1)\beta^2$, which implies

$$(m-1)\beta r_{00}+2m\alpha^2s_0=\theta\big[mb^2\alpha^2-(m+1)\beta^2\big], \tag{19}$$

where θ is a 1-form. Eq. (19) is equivalent to

$$m(2s_0-b^2\theta)\alpha^2+\beta\big[(m-1)r_{00}+(m+1)\theta\beta\big]=0. \tag{20}$$

By (20), there is a scalar function $\tau=\tau(x)$ such that

$$2s_0-b^2\theta=-2(m-1)b^2\tau\beta. \tag{21}$$

Now plugging (21) into (20) immediately yields (17). Q.E.D.

Lemma 3.3 *Let $F=\alpha^{1-m}\beta^m$ be an m-Kropina metric ($m\neq -1$) of scalar flag curvature. Then we have*

$$t^k{}_k=-\frac{2s_ks^k}{b^2}. \tag{22}$$

Proof : Since $F=\alpha^{1-m}\beta^m$ is of scalar flag curvature, we have (14), and further we can rewrite (14) as

$$D^i_k\beta+m^6(n+1)b^8\alpha^{12}b_kT^i=0, \tag{23}$$

where D^i_k are polynomial in (y^i) and T^i are defined by

$$T^i:=m\big[(n-1)(b^it_0-s^is_0-b^2t^i{}_0)+y^i(b^2t^j{}_j+2s_js^j)\big]\alpha^2+2(m+1)(b^2t_{00}+s_0^2)y^i.$$

Now it follows from (23) that there are polynomials f^i in (y^i) of degree two such that

$$T_i-f_i\beta=0. \tag{24}$$

Contracting (24) by y^i we get

$$m(2s_ks^k+b^2t^k{}_k)\alpha^4+\big[(2+3m-nm)(b^2t_{00}+s_0^2)+m(n-1)\beta t_0\big]\alpha^2-f_0\beta=0. \tag{25}$$

Then by (25), we have $f_0=\theta\alpha^2$ for some 1-form $\theta=\theta_i(x)y^i$. Plugging it into (25) gives

$$\begin{aligned} 0 &= 2m(2s_ks^k+b^2t^k{}_k)a_{ij}+2(2+3m-nm)(b^2t_{ij}+s_is_j)+ \\ & \quad m(n-1)(b_it_j+b_jt_i)-(b_i\theta_j+b_j\theta_i). \end{aligned} \tag{26}$$

Contracting (26) by a^{ij} yields

$$(2+3m)b^2t^k{}_k+2(1+2m)s_ks^k-b^k\theta_k=0. \tag{27}$$

Further contracting (26) by b^ib^j gives

$$mb^2t^k{}_k-2s_ks^k-b^k\theta_k=0. \tag{28}$$

Now it is easy to follow from (27) and (28) that (22) holds. Q.E.D.

Proof of Theorem 1.1 : Let $F=\alpha^{1-m}\beta^m$ be an $n(\geq 3)$-dimensional m-Kropina metric ($m\neq -1$) of scalar flag curvature. Then under the deformation (6), $F=\widetilde{\alpha}^{1-m}\widetilde{\beta}^m$ is also an m-Kropina metric of scalar flag curvature. So we obtain Lemma 3.2 and Lemma 3.3 under $\widetilde{\alpha}$ and $\widetilde{\beta}$.

Note that $\widetilde{b}^2=1$, and then by (17) we have

$$\widetilde{r}_{ij}=2\widetilde{\tau}\big[m\widetilde{a}_{ij}-(m+1)\widetilde{b}_i\widetilde{b}_j\big]-\frac{m+1}{m-1}(\widetilde{b}_i\widetilde{s}_j+\widetilde{b}_j\widetilde{s}_i), \tag{29}$$

We will prove $\widetilde{r}_{ij}=0$ by (29). This fact is essentially proved in [9] [16]. For convenience, we give the proof here. Contracting (29) by $\widetilde{b}^i$ and using $||\widetilde{\beta}||_{\widetilde{\alpha}}=constant=1$ we have

$$\widetilde{r}_j+\widetilde{s}_j=-2\widetilde{\tau}\widetilde{b}_j-\frac{2}{m-1}\widetilde{s}_j=0. \tag{30}$$

Contracting (30) by $\widetilde{b}^j$ we get $\widetilde{\tau}=0$ and then by (30) again we have $\widetilde{s}_j=0$. Thus by (29) again we have

$$\widetilde{r}_{ij}=0.$$

Next by (22) we have

$$\widetilde{t}^k{}_k=-2\widetilde{s}_k\widetilde{s}^k. \tag{31}$$

Since we have proved $\widetilde{s}_k=0$, we have $\widetilde{t}^k{}_k=0$ by (31). Thus Lemma 3.1 implies that $\widetilde{\beta}$ is closed. Thus by this fact and $\widetilde{r}_{ij}=0$, we obtain that $\widetilde{\beta}$ is parallel with respect to $\widetilde{\alpha}$. Q.E.D.

4 Kropina metrics of scalar flag curvature

4.1 Main results

Theorem 4.1 *Let $F=\alpha^2/\beta$ be an $n(\geq 2)$-dimensional Kropina metric with $||\beta||_\alpha=1$. Denote by $\bar{R}^i{}_k$ the Riemann curvature tensor of α. Then F is of scalar flag curvature if and only if the following hold*

$$s_{ij|k} = \Big\{t_j-\frac{t^l{}_l-(n-3)s^ls_l}{n-1}b_j\Big\}a_{ik}+r_{ik}s_j+q^*_{ki}b_j+s_{j|k}b_i-(i/j), \tag{32}$$

$$\begin{aligned} \bar{R}^i{}_k &= \frac{(n-3)s^ls_l-t^l{}_l}{n-1}\big(\alpha^2\delta^i_k-y^iy_k\big)-B_{00}\delta^i_k-B^i{}_k\alpha^2+B_{0k}y^i+B^i{}_0y_k \\ & \quad +r^i{}_0r_{k0}-r_{00}r^i{}_k, \end{aligned} \tag{33}$$

where the symbol (i/j) above denotes the terms obtained from the proceeding terms by the interchange of the indices i and j, and q^*_{ik}, σ_i and $B^i{}_k$ are defined by

$$
\begin{aligned}
q^*_{ik}: &= \frac{1}{2}b^pb^l\big[(r_{lp|i}-r_{li|p})b_k-(i/k)\big]-\frac{1}{2}b^l(r_{lk|i}+r_{li|k})-p_{ik}-s_{i|k}, & (34)\\
\sigma_i: &= 2\big[(n-3)s^ls_l-t^l{}_l-(n-1)\lambda\big]b_i+2(n-1)b^pb^l(r_{lp|i}-r_{li|p}), & (35)\\
B_{ik}: &= \frac{1}{2}(r_{il}r^l{}_k+b^lr_{lk|i})+\frac{b_k\sigma_i}{4(n-1)}+s_{i|k}+(i/k), & (36)
\end{aligned}
$$

and $\lambda=\lambda(x)$ is a scalar function. In this case, the flag curvature K of F is given by

$$
\begin{aligned}
K &= \lambda s^2+\frac{s^2}{\alpha^2}\Big\{\frac{3s^2}{\alpha^2}r^2_{00}+\frac{s}{\alpha}(r_{00|0}+6r_{00}s_0)+3q_{00}+3s^2_0-b^l(r_{l0|0}-r_{00|l})\Big\}\\
&\quad+\frac{1}{4(n-1)}\big[(4s^2-1)t^l{}_l-2(1+2ns^2-6s^2)s^ls_l\big]. \qquad (37)
\end{aligned}
$$

In [15] [16], we give a way to characterize locally projectively flat Kropina metrics in dimension $n\geq 2$ by (38) and an equation on the spray G^i_α of α. Now using Theorem 4.1, we can obtain a different way to characterize locally projectively flat Kropina metrics by adding a Douglasian condition (38) in $n\geq 3$.

Theorem 4.2 *Let $F=\alpha^2/\beta$ be an $n(\geq 3)$-dimensional Kropina metric with $||\beta||_\alpha=1$. Then F is locally projectively flat if and only if (33) and the following hold*

$$s_{ij}=b_is_j-b_js_i. \qquad (38)$$

In this case, the flag curvature K of F is given by (37), and σ_i in (33) are given by

$$\sigma_i=2(n-1)\big[b^ls_{i|l}-(\lambda+s_ls^l)b_i\big]. \qquad (39)$$

In a special case, we have the following simple corollary. We will construct some examples in Section 6 below by Corollary 4.3.

Corollary 4.3 *Let $F=\alpha^2/\beta$ be an $n(\geq 3)$-dimensional Kropina metric with $||\beta||_\alpha=1$. Suppose*

$$b_{i|j}=\epsilon(a_{ij}-b_ib_j),\quad \epsilon_i=ub_i, \qquad (40)$$

where $u=u(x),\epsilon=\epsilon(x)$ are scalar functions and $\epsilon_i:=\epsilon_{x^i}$. Then F is locally projectively flat if and only if

$$\bar{R}^i{}_k=-\epsilon^2(\alpha^2\delta^i_k-y^iy_k)-u(\alpha^2b^ib_k+\beta^2\delta^i_k-\beta y^ib_k-\beta y_kb^i). \qquad (41)$$

In this case, the flag curvature K is given by

$$K=s^6\big[\epsilon^2(3s^2-4)-u\big]. \qquad (42)$$

Remark 4.4 *It is known in [11] that a Randers metric $F=\alpha+\beta$ in dimension $n\geq 2$ is of scalar flag curvature if and only if for some scalar $\lambda=\lambda(x)$,*

$$
\begin{aligned}
s_{ij|k} &= \frac{1}{n-1}(a_{ik}s^m{}_{j|m}-a_{jk}s^m{}_{i|m}), & (43)\\
\bar{R}^i{}_k &= \lambda(\alpha^2\delta^i_k-y^iy_k)+\alpha^2t^i{}_k+t_{00}\delta^i_k-t_{k0}y^i-t^i{}_0y_k-3s^i{}_0s_{k0}. & (44)
\end{aligned}
$$

In Theorem 4.1, for a Kropina metric of scalar flag curvature, we obtain the equations (32) and (33) similar to (43) and (44). However, the characterization and proof for a Kropina metric are much more complicated than that for a Randers metric.

4.2 Proof of Theorem 4.1

Proposition 4.5 *Let $F = \alpha^2/\beta$ be an $n(\geq 2)$-dimensional Kropina metric with $||\beta||_\alpha = 1$. Then F is of scalar flag curvature if and only if (33) and the following hold*

$$t_{ij} = b_i t_j + b_j t_i - s_i s_j + \frac{1}{n-1}\Big\{(t^l{}_l + 2s^l s_l)a_{ij} - \big[t^l{}_l - (n-3)s^l s_l\big]b_i b_j\Big\}, \tag{45}$$

$$s_{ij|k} = \Big\{t_j - \frac{t^l{}_l - (n-3)s^l s_l}{n-1}b_j\Big\}a_{ik} + r_{ik}s_j + q_{ki}b_j + s_{j|k}b_i - (i/j), \tag{46}$$

$$q_{ik} = \frac{1}{2}b^m b^l\big[(r_{lm|i} - r_{li|m})b_k - (i/k)\big] - \frac{1}{2}b^l(r_{lk|i} + r_{li|k}) - p_{ik} - s_{i|k}. \tag{47}$$

Proof : Assume $F = \alpha^2/\beta$ is of scalar flag curvature in dimension $n \geq 2$. By $W^i{}_k = 0$, we get (15). Here we put $b = ||\beta||_\alpha = 1$ and hence $r_k + s_k = 0 = r$ in (15). First, (15) can be written as

$$(\cdots)\beta - (n+1)\alpha^4 b_k\big[(n-1)(t_{i0} - t_0 b_i + s_0 s_i) - (t^l{}_l + 2s^l s_l)y_i\big] = 0, \tag{48}$$

where the omitted term is a homogeneous polynomial in (y^i). Then by (48) we have

$$t_{ij} = t_j b_i - s_i s_j - \frac{\rho_i b_j - (t^l{}_l + 2s^l s_l)a_{ij}}{n-1}, \tag{49}$$

where $\rho_i = \rho_i(x)$ are some scalar functions. By (49), using $t_{ij} = t_{ji}$ we get

$$\rho_i = \sigma b_i - (n-1)t_i, \tag{50}$$

where $\sigma = \sigma(x)$ is a scalar function. Plugging (50) into (49) and then contracting (49) by a^{ij}, we get

$$\sigma = t^l{}_l - (n-3)s^l s_l. \tag{51}$$

Now plugging (50) and (51) into (49) we obtain (45).

By t_{ik} and t_{i0} given by (45), we can write $(48)/\beta$ as

$$(\cdots)\beta + 2(+1)\alpha^2 b_k C_i = 0, \tag{52}$$

where C_i is a homogeneous polynomial of degree two in y (the expression is omitted here). It is easy to see from (52) that C_i is divisible by β. Hence we have $C_i = c_{i0}\beta$ for a 1-form $c_{i0} = c_{ij}y^j$, which is equivalent to

$$\begin{aligned} s_{i0|0} = {} & \frac{q_0 + s^l{}_{0|l} - b^l q_{0l} - b^l s_{0|l} - r^l{}_l s_0}{n-1}\, y_i + \Big\{\frac{t^l{}_l - (n-3)s^l s_l}{n-1}\alpha^2 - q_{00} + s_{0|0}\Big\}b_i \\ & -\alpha^2 t_i + s_0 r_{i0} - r_{00}s_i - \frac{c_{i0}\beta}{n-1}. \end{aligned} \tag{53}$$

Plug (53) into (52) and then $(52)/(2\beta)$ can be written as

$$(\cdots)\beta + (n+1)\alpha^2 D_{ik} = 0, \tag{54}$$

where D_{ik} is a 1-form (the expression is omitted here). It is easy to see from (54) that D_{ik} is divisible by β. Hence we have $D_{ik} = f_{ik}\beta$ for a scalar function f_{ik}, which is equivalent to

$$(n-1)s_{ik|j} - 2(n-1)s_{ij|k} + \cdots = f_{ik}b_j. \tag{55}$$

Interchanging j, k in (55) we have

$$(n-1)s_{ij|k} - 2(n-1)s_{ik|j} + \cdots = f_{ij}b_k, \tag{56}$$

Then $2\times(55)+(56)$ gives

$$\begin{aligned} s_{ij|k} &= \frac{q_j+s^l{}_{j|l}-b^lq_{jl}-b^ls_{j|l}-r^l{}_ls_j}{n-1}\ a_{ik}+\Big\{\frac{t^l{}_l-(n-3)s^ls_l}{n-1}b_i-t_i\Big\}a_{jk} \\ &\quad +\frac{2b_kc_{ij}+b_jc_{ik}-b_kf_{ij}-2b_jf_{ik}}{3(n-1)}-b_iq_{kj}+b_is_{j|k}+s_jr_{ik}-s_ir_{jk}. \end{aligned} \tag{57}$$

By (53) and (57) we get

$$f_{ij}=2c_{ij}. \tag{58}$$

By (58) and $s_{ij|k}+s_{ji|k}=0$, it follows from (57) that

$$\begin{aligned} 0 &= \Big\{\frac{q_j+s^l{}_{j|l}-b^lq_{jl}-b^ls_{j|l}-r^l{}_ls_j}{n-1}+\frac{t^l{}_l-(n-3)s^ls_l}{n-1}b_j-t_j\Big\}a_{ik} \\ &\quad -b_iq_{kj}+b_is_{j|k}-\frac{b_ic_{jk}}{n-1}+(i/j). \end{aligned} \tag{59}$$

Contracting (59) by b^ib^j we can first get the expression of b^lc_{lk}, and then using b^lc_{lk} and contracting (59) by b^j we can get the expression of c_{ik}. Now plugging c_{ik} into (59) yields

$$0=\Big\{\big[\frac{b^m(b^ls_{m|l}-s^l{}_{m|l})}{n-1}-s^ls_l\big]b_j+\frac{q_j+s^l{}_{j|l}-b^lq_{jl}-b^ls_{j|l}-r^l{}_ls_j}{n-1}-t_j\Big\}(a_{ik}-b_ib_k)+(i/j). \tag{60}$$

Contracting (60) by a^{ik} we obtain

$$s^l{}_{j|l}=b^l(q_{jl}+s_{j|l})+(n-1)t_j+r^l{}_ls_j-q_j+\big[(n-1)s^ls_l-b^m(b^ls_{m|l}-s^l{}_{m|l})\big]b_j. \tag{61}$$

Finally, plugging (58), c_{ij} and (61) into (57) we obtain (46).

By (46), we can determine the expressions of the following quantities

$$s_{ik|0},\ s^l{}_{0|l},\ s^l{}_{k|l},\ s_{i0|k},\ s_{i0|0},\ b^ms^l{}_{m|l}.$$

Plug these quantities into (52) and then (52) is equivalent to ($\bar W_{ik}:=a_{il}\bar W^l{}_k$)

$$\begin{aligned} \bar W_{ik} &= \frac{1}{n-1}\big\{s^l_{|l}(\alpha^2a_{ik}-y_iy_k)+(2q_{00}+b^lr_{00|l}+r^l{}_lr_{00})a_{ik}- \\ &\quad (r^l{}_lr_{k0}+b^lr_{k0|l}+q_{k0}+q_{0k})y_i\big\}+(s_{i|0}-q_{0i})y_k+(r_{k0|0}-r_{00|k})b_i+ \\ &\quad (q_{ki}-s_{i|k})\alpha^2+r_{k0}r_{i0}-r_{00}r_{ik}. \end{aligned} \tag{62}$$

Lemma 4.6 *(62) is equivalent to the following equation*

$$\begin{aligned} \bar R^i{}_k &= \lambda(\alpha^2\delta^i_k-y^iy_k)+\big[b^l(r_{00|l}-r_{l0|0})+q_{00}-s_{0|0}\big]\delta^i_k+r^i{}_0r_{k0}-r_{00}r^i{}_k \\ &\quad +(q_k{}^i-s^i_{|k})\alpha^2+\frac12\big[b^l(r_{l0|k}+r_{lk|0}-2r_{k0|l})-q_{k0}-q_{0k}+s_{k|0}+s_{0|k}\big]y^i \\ &\quad +(s^i_{|0}-q_0{}^i)y_k+(r_{k0|0}-r_{00|k})b^i, \end{aligned} \tag{63}$$

where $\lambda=\lambda(x)$ is a scalar function, and $\bar R^i{}_k$ denotes the Riemann curvature of α.

Proof : $\Longrightarrow$ **:** By the definition of the Weyl curvature $\bar W_{ik}$ of α we have

$$\bar W_{ik}=\bar R_{ik}-\frac{1}{n-1}\bar Ric_{00}a_{ik}+\frac{1}{n-1}\bar Ric_{k0}y_i, \tag{64}$$

where $\bar{R}_{ik} := a_{im}\bar{R}^m_{\ k}$ and $\bar{R}ic_{ik}$ denote the Ricci tensor of α. By $\bar{R}_{ik} = \bar{R}_{ki}$ and (64) we get

$$\bar{W}_{ik} - \bar{W}_{ki} = \frac{1}{n-1}\big(\bar{R}ic_{k0}y_i - \bar{R}ic_{i0}y_k\big). \tag{65}$$

Plugging (62) into (65) yields

$$T_k y_i - T_i y_k + (n-1)\big[(s_{k|i} - s_{i|k} + q_{ki} - q_{ik})\alpha^2 + (r_{k0|0} - r_{00|k})b_i - (r_{i0|0} - r_{00|i})b_k\big] = 0, \tag{66}$$

where we define

$$T_k := (n-2)q_{0k} - q_{k0} - (n-1)s_{k|0} - b^l r_{k0|l} - r_{k0}r^l_{\ l} - \bar{R}ic_{k0}.$$

Contracting (66) by $y^k b^i$ we get

$$(\cdots)\alpha^2 + \beta[T_0 + (n-1)b^l(r_{00|l} - r_{l0|0})] = 0. \tag{67}$$

By (67) we obtain

$$T_0 + (n-1)b^l(r_{00|l} - r_{l0|0}) = (n+1)\eta\alpha^2, \tag{68}$$

where $\eta = \eta(x)$ is a scalar function. Then it follows from the definition of T_i and (68) that

$$\bar{R}ic_{00} = (n-3)q_{00} - (n-1)s_{0|0} - (n+1)\eta\alpha^2 + (n-2)b^l r_{00|l} - (n-1)b^l r_{l0|0} - r^l_{\ l}r_{00}. \tag{69}$$

By (69) we can get $\bar{R}ic_{k0}$. Plugging (69) and $\bar{R}ic_{k0}$ into (64) we get $\bar{W}_{ik}$, and then by (64) and (62) we obtain (63), where λ is defined by

$$\lambda := -\frac{(n+1)\eta - s^l_{\ |l}}{n-1}. \tag{70}$$

$\Longleftarrow$ **:** Suppose that (63) holds. Using the first formula in (8) and $b = constant$ we have

$$q_{00} = -s_{0|0} - p_{00} - b^l r_{l0|0}. \tag{71}$$

Contracting (63) over i, k we get $\bar{R}ic_{00}$, and then using (71) we obtain (69) with η defined by (70). Now plugging (69) and (63) into (64), we immediately obtain (62). Q.E.D.

It is clear that no obvious way shows that $\bar{R}_{ik} = \bar{R}_{ki}$ in (63). It follows from (63) that the symmetric condition $\bar{R}_{ik} = \bar{R}_{ki}$ is equivalent to

$$\begin{aligned} 0 \ &= \ \big[b^l(r_{l0|k} + r_{lk|0} - 2r_{k0|l}) - q_{k0} + q_{0k} - s_{k|0} + s_{0|k}\big]y_i + 2(r_{k0|0} - r_{00|k})b_i \\ &\quad +2(q_{ki} + s_{k|i})\alpha^2 - (i/k). \end{aligned} \tag{72}$$

Lemma 4.7 *(46) and (63)* $\Longleftrightarrow$ *(46), (47) and (33).*

Proof : $\Longrightarrow$ **:** To simplify (72), we first give two formulas as follows by (46), (71) and (63):

$$b^l(r_{l0|0} - r_{00|l}) \ = \ \Big[\lambda - \frac{(n-3)s^l s_l - t^l_{\ l}}{n-1}\Big](\alpha^2 - \beta^2) + (t_0 - q_0 + b^l s_{0|l})\beta, \tag{73}$$

$$\begin{aligned} q_{0i} - q_{i0} \ &= \ 2\Big[\lambda - \frac{(n-3)s^l s_l - t^l_{\ l}}{n-1}\Big](y_i - \beta b_i) + 2(t_0 - q_0 + b^l s_{0|l})b_i + s_{i|0} - s_{0|i} \\ &\quad -b^l(r_{li|0} + r_{l0|i} - 2r_{i0|l}). \end{aligned} \tag{74}$$

To show (73) and (74), by the first formula in (9) we have

$$b^l(r_{l0|0} - r_{00|l}) = b^l(s_{l0|0} + b^k\bar{R}_{kl}), \qquad r_{i0|0} - r_{00|i} = s_{i0|0} + b^k\bar{R}_{ki}. \tag{75}$$

Contracting (63) by $b_i b^k$, and then using (71), the second formula of (9), the first formula of (10) and the third formula of (11), we have

$$b^m b^l \bar{R}_{ml} = (\lambda - s_l s^l)\alpha^2 - \lambda\beta^2 + (b^l s_{0|l} - t_0 - q_0)\beta - 2s_{0|0} + s_0^2 - b^l r_{l0|0} - p_{00}. \qquad (76)$$

Similarly, by (46), (71) and the first formula of (10), we have

$$b^l s_{l0|0} = \frac{2s_l s^l + t^l{}_l}{n-1}\alpha^2 + \frac{(n-3)s_l s^l - t^l{}_l}{n-1}\beta^2 + 2\beta t_0 - s_0^2 + 2s_{0|0} + b^l r_{l0|0} + p_{00}. \qquad (77)$$

Then by the first formula in (75), we obtain (73) from (76) and (77). Now by a contraction of (63) we get $b^k \bar{R}_{ki}$, and then using the obtained $b^k \bar{R}_{ki}$, (46), (73) and the first formula of (10), we obtain (74) from the second formula in (75).

Now contracting (72) by y^k and using (73), we can write (72) as

$$A_i\alpha^2 + \beta B_i = 0, \qquad (78)$$

where A_i, B_i are polynomials in y. By (78) we have $B_i = \sigma_i \alpha^2$, which is expressed as follows

$$r_{00|i} - r_{i0|0} = \Big[\lambda - \frac{(n-3)s^l s_l - t^l{}_l}{n-1}\Big]\beta y_i - (t_0 - q_0 + b^l s_{0|l})y_i + \frac{\alpha^2}{2(n-1)}\sigma_i. \qquad (79)$$

Plugging (79) into (78) yields

$$b^l(r_{li|0} + r_{l0|i} - 2r_{i0|l}) = \Big[\lambda - \frac{(n-3)s^l s_l - t^l{}_l}{n-1}\Big](2y_i - \beta b_i) + (t_0 - q_0 + b^l s_{0|l})b_i + \frac{\beta}{2(n-1)}\sigma_i. \qquad (80)$$

Now by (74), (79) and (80), we see that (72) is equivalent to

$$q_{ik} - q_{ki} = s_{k|i} - s_{i|k} + \frac{b_k\sigma_i - b_i\sigma_k}{2(n-1)}. \qquad (81)$$

By a contraction on (80), we easily obtain (35) for the expression of σ_i by the second formula of (9) and the third formula of (11). Now using (35), we can easily obtain (47) by (71) and (81) since we can write (71) as $q_{ik} + q_{ki} = \cdots$.

Finally, by (35), (79) and (80) are respectively reduced to

$$r_{00|i} - r_{i0|0} = \frac{\alpha^2\sigma_i - \sigma_0 y_i}{2(n-1)}, \qquad (82)$$

$$b^l(r_{li|0} + r_{l0|i} - 2r_{i0|l}) = 2\Big[\lambda - \frac{(n-3)s^l s_l - t^l{}_l}{n-1}\Big]y_i + \frac{b_i\sigma_0 + \beta\sigma_i}{2(n-1)}. \qquad (83)$$

Now under the formulas (47), (82) and (83), we can easily show that (63) is equivalent to (33) with B_{ik} defined by (36), where we have used (by (35))

$$b^l\sigma_l = -2t^l{}_l - 2(n-1)\lambda + 2(n-3)s_l s^l. \qquad (84)$$

$\Longleftarrow$ **:** To verify (63), by the last argument above, we only need to verify (82) and (83), and then we get (63) following from (33).

Contracting (33) by $b_i b^k$ and using (35), (71), (84), and the second formula of (9), the first formula of (10) and the third and fourth formulas of (11), we obtain

$$\begin{aligned} b^m b^l \bar{R}_{ml} &= \Big[2\lambda - \frac{2(n-2)s_l s^l - t^l{}_l}{n-1}\Big](\alpha^2 - \beta^2) - s_l s^l\beta^2 + 2(b^m s_{0|m} - q_0)\beta \\ &\quad -2s_{0|0} + s_0^2 - p_{00} - b^l(2r_{l0|0} - r_{00|l}). \end{aligned} \qquad (85)$$

Similarly, by (46), (71) and the first formula of (10), we have (77). Then by the first formula in (75), we also obtain (73) by (85) and (77). Next we prove (82). First , by (47) and (35) we have

$$q_{0k} = \frac{\sigma_0 b_k - \beta\sigma_k}{4(n-1)} - \frac{1}{2}b^l(r_{l0|k} + r_{lk|0}) - p_{k0} - s_{0|k}. \tag{86}$$

Now by (46) and (33), we can get $s_{k0|0}$ and $b^l\bar{R}_{lk}$ respectively. Then we can obtain (82) from the second formula in (75), by using (35), (73), (84), (86), the second formula of (9), the first formula of (10) and (12). For (83), it follows from (82) by (84). Q.E.D.

Conversely, let (46), (33), (45) and (47) be satisfied. Then F is of scalar flag curvature, since it is easy to see from the above proof that the Weyl curvature of F vanishes if (46), (33), (45) and (47) are satisfied. This completes the proof of Proposition 4.5. Q.E.D.

Proof of Theorem 4.1 : $\Longrightarrow$ **:** Let F be of scalar flag curvature. Then we have (33), and (45)–(47) by Proposition 4.5. It is obvious that (32) follows from (46) and (47).

$\Longleftarrow$ **:** By Proposition 4.5, we only need to show that (45) and (47) automatically hold, provided that (32) and (33) hold. In fact, we can show that (32) directly implies (45) and (47). By (8) in Lemma 2.1 and $r_i = -s_i$, we have

$$t_{ik} = -s_{i|k} - s_{k|i} - p_{ik} + \frac{1}{2}b^m(s_{mi|k} + s_{mk|i} - r_{mi|k} - r_{mk|i}), \tag{87}$$

$$q_{ik} = -s_{i|k} - p_{ik} + \frac{1}{2}b^m(s_{mi|k} - s_{mk|i} - r_{mi|k} - r_{mk|i}). \tag{88}$$

A direct computation from (32) gives

$$\begin{aligned} b^m s_{mi|k} \ &= \frac{t^l{}_l + 2s_l s^l}{n-1}a_{ik} + b_i\Big\{\frac{(n-3)s_l s^l - t^l{}_l}{n-1}b_k - \frac{1}{2}b^m b^v(r_{mk|v} + r_{mv|k}) + s^m r_{mk} \\ &\quad -b^m(s_{k|m} + s_{m|k})\Big\} + b_k\big[t_i + \frac{1}{2}b^m b^v(r_{mv,i} - r_{mi,v})\big] + \frac{1}{2}b^m(r_{mi|k} + r_{mk|i}) \\ &\quad +p_{ik} + s_{i|k} + s_{k|i} - s_i s_k. \end{aligned} \tag{89}$$

Now plugging (89) into (87) and (88) respectively and using the first formula of (10) and (12), we obtain (45) and (47) respectively.

For the proof of (37), we first get $\bar{Ric}_{00}$ by (33) and $s^l{}_{0|l}$ by (32), and then plugging them into $K = Ric/((n-1)F^2)$ yields (37), where Ric is given by (16) with $b = 1$. Q.E.D.

4.3 Proofs of Theorem 4.2 and Corollary 4.3

Proof of Theorem 4.2 : It is shown in [16] that a Kropina metric $F = \alpha^2/\beta$ with $||\beta||_\alpha = 1$ is a Douglas metric if and only if (38) holds. Therefore, by Theorem 4.1, we only need to use (38) to show that (32) holds. By (38) and definitions, we easily get

$$t_{ij} = -s^l s_l b_i b_j - s_i s_j, \quad t_i = -s^l s_l b_i, \quad t^l{}_l = -2s^l s_l, \quad q_{ik} = -s_i s_k - s^m r_{mi} b_k, \quad q_i = s_l s^l b_i. \tag{90}$$

Now for the left hand side of (32), we have

$$s_{ij|k} \overset{(38)}{=} (r_{ik} + s_{ik})s_j + b_i s_{j|k} - (r_{jk} + s_{jk})s_i - b_j s_{i|k} \overset{(38)}{=} s_j(r_{ik} + b_i s_k) + b_i s_{j|k} - (i/j),$$

and for the right hand side of (32), we also obtain the same result as above by using $t_i, t^l{}_l$ and $q^*_{ik} = q_{ik}$ in (90). Thus we have verified (32). For (39), it directly follows from using the second formula in (9) and then plugging $t^l{}_l, q_i, t_i$ of (90) into (35). Q.E.D.

Proof of Corollary 4.3 : Since β is closed by (40), we see that (38) automatically holds. Plug (40) into (39) and (36) we get

$$\sigma_i = -2(n-1)\lambda b_i, \quad B_{ik} = -\lambda b_i b_k. \tag{91}$$

Now plugging (40) and (91) into (33) we obtain

$$\bar{R}^i{}_k = -\epsilon^2(\alpha^2\delta^i_k - y^i y_k) + (\lambda+\epsilon^2)(\alpha^2 b^i b_k + \beta^2\delta^i_k - \beta y^i b_k - \beta y_k b^i). \tag{92}$$

By (40) and (91), it follows from (82) that

$$\lambda + u + \epsilon^2 = 0. \tag{93}$$

Then (92) and (93) imply (41), and we get (42) from (37), (40) and (93). Q.E.D.

5 Kropina metrics of constant flag curvature

It has been solved for the local structure of Kropina metrics of constant flag curvature (cf. [9] [18]). In this section, we will use Theorem 4.1 to investigate it.

Corollary 5.1 *Let $F = \alpha^2/\beta$ be an n-dimensional Kropina metric with $||\beta||_\alpha = 1$. Then F is of constant flag curvature if and only if α is of constant sectional curvature μ and β satisfies $r_{00} = 0$. In this case, we have $\mu \geq 0$, and F is flat-parallel (α is flat and β is parallel), or up to a scaling on F, α and β can be locally written as*

$$\alpha = \frac{\sqrt{(1+|x|^2)|y|^2 - \langle x, y\rangle^2}}{1+|x|^2}, \qquad \beta = \frac{\langle Ux + e, y\rangle}{1+|x|^2}, \tag{94}$$

where $U = (u^i_j)$ is a skew-symmetric matrix, $e = (e^i)$ is a constant vector satisfying

$$|e| = 1, \quad Ue = 0, \quad \delta^{ij} - e^i e^j = \delta^{kl} u^i_k u^j_l. \tag{95}$$

Proof : For $n = 2$, it has been proved in [9] that F is flat-parallel. Now assume that F is of constant flag curvature K. Then it follows from Theorem 4.1 that its flag curvature K is given by (37). First we can write (37) as

$$(\cdots)\alpha^2 + 12(n-1)\beta^4 r_{00}^2 = 0, \tag{96}$$

which implies $r_{00} = c\alpha^2$ for some scalar function $c = c(x)$. Since $||\beta||_\alpha = 1$, we have $r_i + s_i = 0$. Then it is easily shown that $c = 0$ and hence $r_{00} = 0$. Now plug $r_{ij} = 0, r_{ij|k} = 0, q_{ij} = 0, s_i = 0$ into (96) we have

$$(4K - 4nK - t^l{}_l)\alpha^2 + 4(n\lambda - \lambda + t^l{}_l)\beta^2 = 0. \tag{97}$$

By (97) we easily get

$$K = -\frac{t^l{}_l}{4(n-1)} = \frac{\lambda}{4} \geq 0, \quad (\text{since } t^l{}_l \leq 0). \tag{98}$$

Hence we have

$$t^l{}_l = -(n-1)\lambda \ (= constant). \tag{99}$$

By $r_{ij} = 0, s_i = 0$, (98) and (99), it follows from (33) that

$$\bar{R}^i{}_k = \lambda(\alpha^2\delta^i_k - y^i y_k),$$

which shows that α is of constant sectional curvature $\lambda \geq 0$. If $\lambda = 0$, then it is easy to show that F is flat-parallel since by (98), we have $t^l{}_l = 0$ (this implies that β is closed and then parallel by $r_{00} = 0$). If $\lambda > 0$, since $r_{00} = 0$ and α is of constant sectional curvature, by solving Killing fields on a Riemannian space of constant sectional curvature, it follows that, up to a scaling on F, α and β can be locally given by (94) with U, e satisfying (95) (cf. [9]).

Conversely, assume $r_{00} = 0$ and α is of constant sectional curvature μ with $||\beta||_\alpha = 1$. First by assumption we have

$$r_{ij} = 0, \quad r_i = 0, \quad s_i = 0, \quad q_{ij} = r_{im}s^m{}_j = 0, \quad t_i = s_m s^m{}_i = 0. \tag{100}$$

Then by (100), it follows from the first formulas of (9) and the second formula of (8) that

$$s_{ij|k} = -b_l \bar{R}_k{}^l{}_{ij} = -\mu(b_i a_{jk} - b_j a_{ik}), \qquad t_{ij} = b^l s_{li|j} = -\mu(a_{ij} - b_i b_j), \tag{101}$$

It is clear that the second formula implies $t^l{}_l = -(n-1)\mu$. Now we use Theorem 4.1 to verify that F is of constant flag curvature, namely, we show that (32) and (33) hold for some scalar function $\lambda = \lambda(x)$ and K in (37) is a constant. Now define $\lambda := -t^l{}_l/(n-1) = \mu$, and then (33) naturally holds since $B_{ij} = 0$. Finally we verify (32). By (100), the first formula of (101) and $t^l{}_l = -(n-1)\mu$, we see that (32) also holds automatically. Therefore, F is of scalar flag curvature by Theorem 4.1, and its flag curvature is given by (37). Now by (37) we have

$$K = \lambda s^2 + \frac{4s^2 - 1}{4(n-1)} t^l{}_l. \tag{102}$$

Since $t^l{}_l = -\mu(n-1)$ as shown above, we have $K = \mu/4 = \textit{constant}$ by (102). Q.E.D.

6 Construction by warped product method

In this section, we will use Corollary 4.3 to show a family of examples of projectively flat Kropina metrics with α in warped product.

Let $M = \mathcal{R} \times \widetilde{M}$ be a product manifold, where $\widetilde{M}$ is an $(n-1)$-dimensional manifold. Let $\{x^A\}_{A=2}^n$ be a local coordinate system on $\widetilde{M}$. A Riemann metric α of warped product type is defined as

$$\alpha^2 = (y^1)^2 + h^2(x^1)\widetilde{\alpha}^2, \tag{103}$$

where $\widetilde{\alpha}^2 = \widetilde{a}_{AC} y^A y^C$ is a Riemann metric on $\widetilde{M}$. The Riemann curvature tensors $\bar{R}$ of α and $\widetilde{R}$ of $\widetilde{\alpha}$ in (103) are related by

$$\bar{R}^1{}_k = \frac{h''}{h}(y^1 y_k - \alpha^2 \delta^1_k), \tag{104}$$

$$\bar{R}^A{}_C = \widetilde{R}^A{}_C - (h')^2(\widetilde{\alpha}^2 \delta^A_C - y^A \widetilde{y}_C) - \frac{h''}{h}(y^1)^2 \delta^A_C, \tag{105}$$

where $y_k := a_{kl} y^l, \widetilde{y}_C := \widetilde{a}_{CA} y^A$. Define $\eta = \eta(x^1) := \int h(x^1) dx^1$, and then a direct computation shows that

$$\eta_{i|j} = \eta'' \alpha^2, \quad (\eta_i := \eta_{x^i}),$$

where the covariant derivative is taken with respect to α. The converse is proved in the following.

Lemma 6.1 *([3] [12]) Let α be a Riemann metric on M. Suppose there are two functions η and ξ on M with $d\eta \neq 0$ such that*

$$\eta_{i|j} = \xi a_{ij}, \quad (\eta_i := \eta_{x^i}).$$

Then α is a warped product on $M = \mathcal{R} \times \widetilde{M}$, namely, locally η depends only on the the parameter x^1 of $\mathcal{R}$, $\xi = \eta''(x^1)$ and α can be expressed as

$$\alpha^2 = (y^1)^2 + (\eta'(x^1))^2 \widetilde{\alpha}^2.$$

Now we show a construction of examples of Kropina metrics of scalar flag curvature.

Proposition 6.2 *Let $F = \alpha^2/\beta$ be an $n(\geq 2)$-dimensional Kropina metric on a product manifold $M = \mathcal{R} \times \widetilde{M}$, where*

$$\alpha^2 := (y^1)^2 + h^2(x^1)\widetilde{\alpha}^2, \quad \beta := y^1, \tag{106}$$

where $h \neq 0$ is a smooth function on $\mathcal{R}$ and $\widetilde{\alpha}$ is an $(n-1)$-dimensional Riemann metric on $\widetilde{M}$. Then F is locally projectively flat if and only if $\widetilde{\alpha}$ is locally flat. In this case, the flag curvature K is given by

$$K = -\Big(\frac{\beta}{\alpha}\Big)^6 \Big\{ \frac{h''}{h} + 3(h')^2 \Big(\frac{\widetilde{\alpha}}{\alpha}\Big)^2 \Big\}. \tag{107}$$

Proof : For $n = 2$, we can directly verify that $F = \alpha^2/\beta$ defined by (106) is projectively flat (we may put $\widetilde{\alpha} = c(x^2)y^2$). We consider $n \geq 3$. For the α and β defined by (106), a direct computation shows that $||\beta||_\alpha = 1$ and (40) holds with

$$\epsilon = \frac{h'}{h}, \quad u = \Big(\frac{h'}{h}\Big)'. \tag{108}$$

So F is locally projectively flat if and only if (41) holds by Corollary 4.3.

It can be easily verified that (41) is equivalent to

$$\bar{R}^1{}_k = \big[-(u+\epsilon^2)(y^1)^2 - \epsilon^2 h^2 \widetilde{\alpha}^2 \big] \delta^1_k + (u+\epsilon^2) y^1 y_k - u h^2 \widetilde{\alpha}^2 b_k, \tag{109}$$

and

$$\bar{R}^A{}_C = \big[-(u+\epsilon^2)(y^1)^2 - \epsilon^2 h^2 \widetilde{\alpha}^2 \big] \delta^A_C + \epsilon^2 h^2 y^A \widetilde{y}_C, \tag{110}$$

where $\widetilde{y}_C := \widetilde{a}_{CA} y^A$. By (104), we see that (109) is equivalent to

$$u + \epsilon^2 = \frac{h''}{h}, \tag{111}$$

which automatically holds by (108). By the first equation in (108), it follows from (105) that (110) is equivalent to

$$\widetilde{R}^A{}_C = \big[-\epsilon^2 h^2 + (h')^2 \big] (\widetilde{\alpha}^2 \delta^A_C - y^A \widetilde{y}_C) = 0. \tag{112}$$

Now suppose F is locally projectively flat. Then we have (112), namely, $\widetilde{\alpha}$ is locally flat. Conversely, if $\widetilde{\alpha}$ is locally flat, then by the above proof, we can easily get (41).

Finally, by (42), we obtain the flag curvature K given by (107). Q.E.D.

By Proposition 6.2, $F = \alpha^2/\beta$ in dimension $n \geq 2$ is locally projectively flat, where α and β are defined by (106) with $h \neq 0$ being arbitrary and $\widetilde{\alpha}$ being locally flat.

Proposition 6.3 *Let $F = \alpha^2/\beta$ be an $n(\geq 2)$-dimensional Kropina metric, where α and β satisfy (40) with $||\beta||_\alpha = 1$, $d\epsilon \neq 0$ and $u = f(\epsilon) \neq 0$ for some function f. Then F is locally projectively flat if and only if α and β can be locally written as*

$$\alpha^2 = (y^1)^2 + h^2(x^1)\widetilde{\alpha}^2, \quad \beta = y^1, \tag{113}$$

where $\widetilde{\alpha}$ is a locally flat Riemann metric. Further, h can be actually determined by f.

Proof : We firstly show (113) by (40). Define

$$\varphi := \int \frac{1}{f(\epsilon)}\ e^{\int \frac{\epsilon}{f(\epsilon)}d\epsilon} d\epsilon. \tag{114}$$

Then by (40) with $u = f(\epsilon) \neq 0$, we can easily verify that

$$\varphi_{i|j} = \epsilon\ e^{\int \frac{\epsilon}{f(\epsilon)}d\epsilon} a_{ij}, \quad (\epsilon_i := \epsilon_{x^i}). \tag{115}$$

Obviously we have $d\varphi \neq 0$. Then by (115) and Lemma 6.1, α is a warped product which can be locally written as the first expression in (113) with $h(x^1) = \varphi'(x^1)$. By (114), we can define

$$g(\varphi) := \int \frac{1}{f(\epsilon)} d\epsilon.$$

Further by (40) we have

$$\beta = \frac{\epsilon_i}{f(\epsilon)} dx^i = \frac{d\epsilon}{f(\epsilon)} = d\Big(\int \frac{1}{f(\epsilon)} d\epsilon\Big) = d(g(\varphi)) = g'(\varphi)\varphi'(x^1)dx^1. \tag{116}$$

Then by $||\beta||_\alpha = 1$, α in (113), and (116), we must have $g'(\varphi)\varphi'(x^1) = 1$ and $\beta = y^1$.

Therefore, by Proposition 6.2, we conclude that F is locally projectively flat if and only if $\widetilde{\alpha}$ in (113) is locally flat. Q.E.D.

7 Proof of Theorem 1.2

Let $F = \alpha + \beta$ be a Randers metric and (h, ρ) be its navigation data. It is known that

$$\alpha^2 = \frac{(1-b^2)h^2 + \rho^2}{(1-b^2)^2}, \quad \beta = -\frac{\rho}{1-b^2}, \quad (b = ||\beta||_\alpha = ||\rho||_h).$$

By assumption there hold $\lim_{b\to 1^-} h = \widetilde{\alpha}$ and $\lim_{b\to 1^-} \rho = \widetilde{\beta}$. Therefore we have

$$\begin{aligned} \lim_{b\to 1^-} F &= \lim_{b\to 1^-} \Big(\sqrt{\frac{(1-b^2)h^2 + \rho^2}{(1-b^2)^2}} - \frac{\rho}{1-b^2}\Big) = \lim_{b\to 1^-} \frac{h^2}{\sqrt{(1-b^2)h^2 + \rho^2} + \rho} \\ &= \frac{\widetilde{\alpha}^2}{2\widetilde{\beta}} = \frac{1}{2}\widetilde{F}, \quad (let\ \widetilde{\beta} > 0\ by\ \widetilde{F} > 0). \end{aligned}$$

This proves Theorem 1.2 (i).

Since $\widetilde{F} = \lim_{b\to 1^-} 2F$, we have $\widetilde{G}^i = \lim_{b\to 1^-} G^i$. So for the curvatures $W^i{}_k, D_h{}^i{}_{jk}, W^o$ of F and corresponding $\widetilde{W}^i{}_k, \widetilde{D}_h{}^i{}_{jk}, \widetilde{W}^o$ of $\widetilde{F}$, we obtain

$$\lim_{b\to 1^-} \widetilde{W}^i{}_k = W^i{}_k, \quad \lim_{b\to 1^-} \widetilde{D}_h{}^i{}_{jk} = D_h{}^i{}_{jk}, \quad \lim_{b\to 1^-} \widetilde{W}^o = W^o.$$

Therefore, if F is of scalar flag curvature (resp. locally projectively flat, or Douglassian), then $\widetilde{F}$ is also of scalar flag curvature (resp. locally projectively flat, or Douglassian).

Now assume F is of weakly isotropic flag curvature with the flag curvature K in the form

$$K = \frac{3\theta}{F} + \sigma,$$

where θ is a 1-form and $\sigma = \sigma(x)$ is a scalar function. Since θ and σ are uniquely determined by F, by taking the limit $b \to 1^-$ on both sides of the above, we see that $\widetilde{F}$ is also of weakly isotropic flag curvature. Thus by [9], F is of constant flag curvature. This fact can also be proved in another way. Let (h, ρ) be the navigation data of the Randers metric $F = \alpha+\beta$. Since F is of weakly isotropic flag curvature, it is known that $h = \sqrt{h_{ij}y^iy^j}$ is of isotropic sectional curvature $\mu = \mu(x)$ (a constant in dimension $n \geq 3$) and $\rho = \rho_i y^i$ satisfies $\rho_{i|j} + \rho_{j|i} = ch_{ij}$ for some scalar function $c = c(x)$ ([11]). Since $\lim_{b\to 1^-} h = \widetilde{\alpha}$ and $\lim_{b\to 1^-} \rho = \widetilde{\beta}$, by taking the limit $b \to 1^-$, we have $\widetilde{r}_{00} = \widetilde{c}\widetilde{\alpha}^2$ from $\rho_{i|j} + \rho_{j|i} = ch_{ij}$, and $\widetilde{\alpha}$ is of isotropic sectional curvature $\widetilde{\mu}$, where $\widetilde{c} := \lim_{b\to 1^-} c$ and $\widetilde{\mu} := \lim_{b\to 1^-} \mu$. We have $\widetilde{c} = 0$ from $\widetilde{r}_{00} = \widetilde{c}\widetilde{\alpha}^2$ and $||\widetilde{\beta}||_{\widetilde{\alpha}} = 1$. Further, we have $\widetilde{\mu} = \mu = constant$ in dimension $n \geq 3$ and in particular $\widetilde{\mu} = 0$ in dimension $n = 2$. So $\widetilde{r}_{00} = 0$ and $\widetilde{\alpha}$ is of constant sectional curvature. Thus by Corollary 5.1, $\widetilde{F} = \widetilde{\alpha}^2/\widetilde{\beta}$ is of constant flag curvature. Q.E.D.

Remark 7.1 *In Theorem 1.2, let $F = \alpha + \beta$ be a Randers metric and (h, ρ) be its navigation data. Suppose that $h = \sqrt{h_{ij}y^iy^j}$ and $\rho = \rho_i y^i$ are locally given by ($\rho^i := h^{ij}\rho_j$)*

$$\begin{aligned} h &= \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1+\mu|x|^2}, \\ \rho^i &= -2(\lambda\sqrt{1+\mu|x|^2} + \langle d, x\rangle)x^i + \frac{2|x|^2 d_i}{1+\sqrt{1+\mu|x|^2}} + u^i_k x^k + e^i + \mu\langle e, x\rangle x^i, \end{aligned}$$

where λ, μ are constants, $U = (u^i_k)$ is a skew-symmetric matrix and $d, e \in \mathcal{R}^n$ are constant vectors. To take $b = ||\beta||_\alpha \to 1^-$, we only require $h_{ij}\rho^i\rho^j = 1$. A direct computation gives a Kropina metric $\widetilde{F} = \widetilde{\alpha}^2/\widetilde{\beta}$ in two cases: (A). $\widetilde{\alpha} = |y|$, $\widetilde{\beta} = \langle e, y\rangle$; (B). $\widetilde{\alpha} = h$ and $\widetilde{\beta}$ is given by

$$\widetilde{\beta} = \frac{\langle Ux + e, y\rangle}{1+|x|^2},$$

where U and e satisfy

$$|e| = 1, \quad Ue = 0, \quad \mu(\delta^{ij} - e^ie^j) = \delta^{kl}u^i_k u^j_l.$$

Remark 7.2 *For a given Kropina metric $F = \alpha^2/\beta$ with $||\beta||_\alpha = 1$, we can construct a family of Randers metrics $\bar{F} = \bar{\alpha} + \bar{\beta}$ with F as the limit of $\bar{F}$ as $\bar{b} = ||\bar{\beta}||_{\bar{\alpha}} \to 1^-$. Define*

$$\bar{\alpha}^2 = \frac{(1-\bar{b}^2)\alpha^2 + \bar{b}^2\beta^2}{(1-\bar{b}^2)^2}, \quad \bar{\beta} = -\frac{\bar{b}\beta}{1-\bar{b}^2}, \quad (|\bar{b}| < 1).$$

Then it can be easily verified that $\bar{F} = \bar{\alpha} + \bar{\beta}$ is a Randes metric with $\bar{b} = ||\bar{\beta}||_{\bar{\alpha}}$ and $F = \lim_{\bar{b}\to 1^-} 2\bar{F}$.

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