CONSTRUCTING THE VIRTUAL FUNDAMENTAL CLASS OF A KURANISHI ATLAS

DUSA MCDUFF

Abstract. Consider a space X, such as a compact space of J-holomorphic stable maps, that is the zero set of a Kuranishi atlas. This note explains how to define the virtual fundamental class of X by representing X via the zero set of a map $\mathscr{S}_M : M \to E$, where E is a finite dimensional vector space and the domain M is an oriented, weighted branched topological manifold. Moreover, $\mathscr{S}_M$ is equivariant under the action of the global isotropy group Γ on M and E. This tuple $(M, E, \Gamma, \mathscr{S}_M)$ together with a homeomorphism from $\mathscr{S}_M^{-1}(0)/\Gamma$ to X forms a single finite dimensional model (or chart) for X. The construction assumes only that the atlas satisfies a topological version of the index condition that can be obtained from a standard, rather than a smooth, gluing theorem. However if X is presented as the zero set of an sc-Fredholm operator on a strong polyfold bundle, we outline a much more direct construction of the branched manifold M that uses an sc-smooth partition of unity.

Contents

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1. Introduction

1.1. **Statement of main results.** Let X be a compact space that is locally the zero set of a Fredholm operator $\mathcal{F}$ of index d, such as a moduli space of J-holomorphic stable curves. The question of how to define its fundamental class is central to symplectic geometry, since so much information about the properties of this geometry depends on the ability to 'count' the number of elements in X. There are many possible approaches to this problem, e.g. [FO, FF, HWZ1, H]. In this note we develop the work of McDuff–Wehrheim [MW1, MW2, MW3] and Pardon [P] that uses atlases, in an attempt to clarify the passage from atlas to virtual fundamental class.

A d-dimensional atlas consists of a family of charts $\mathbf{K}_I$ indexed by subsets $I \subset \{1, \dots, N\} =: A$, together with coordinate changes $\widehat{\Phi}_{IJ}$ for $I \subset J$, where the chart $\mathbf{K}_I$ is a tuple

$$\mathbf{K}_I = (U_I, E_I, \Gamma_I, s_I, \psi_I),$$

consisting of a manifold U_I of dimension $d + \dim E_I$, a vector space E_I, actions of the group Γ_I on U_I and on E_I, a Γ_I-equivariant map $s_I : U_I \to E_I$, and finally the footprint map $\psi_I : s_I^{-1}(0) \to X$ that induces a homeomorphism from $(s_I^{-1}(0))/\Gamma_I$ onto an open subset F_I of X. The charts $\mathbf{K}_i$ that are indexed by sets $\{i\}$ of length one are called basic charts, and we assume that their footprints $(F_i)_{1 \le i \le N}$ cover X, while the other charts $\mathbf{K}_I$ with $|I| > 1$ form transition data. In applications, the corresponding vector spaces E_i cover the cokernel of the Fredholm operator $\mathcal{F}$ at the points in the footprint $F_i \subset X$, and are called obstruction spaces because they obstruct the existence of solutions when $\mathcal{F}$ is deformed. The essence of the problem lies in trying to assemble these local finite dimensional models for X into one structure that retains enough information to determine its fundamental class, which (when $d = 0$) one can think of as the number of solutions of a "generic" perturbation of $\mathcal{F}$.

The paper [MW3] explains one way to use a d-dimensional oriented atlas to define a Çech homology class $[X]^{vir}_{\mathcal{K}} \in \check{H}_d(X; \mathbb{Q})$. Roughly speaking, the idea is this. Using the coordinate changes to identify different domains, one constructs a metrizable, Hausdorff space $|\mathcal{K}| = \bigcup_I U_I/{\sim}$ that supports a (generalized) orbibundle $|\mathbf{E}_{\mathcal{K}}| \to |\mathcal{K}|$ with a canonical section $|\mathfrak{s}| : |\mathcal{K}| \to |\mathbf{E}_{\mathcal{K}}|$ together with a natural identification

$$\iota_X : X \overset{\cong}{\to} |\mathfrak{s}|^{-1}(0).$$

With some difficulty, one then defines a multi-valued perturbation section $|\nu| : |\mathcal{V}| \to |\mathbf{E}_{\mathcal{K}}|$ on a subset $|\mathcal{V}| \subset |\mathcal{K}|$, such that $|\mathfrak{s} + \nu|$ is transverse to 0. Finally, one shows that the perturbed zero set $|\mathfrak{s} + \nu|^{-1}(0)$ represents a unique element in $\check{H}_d(X; \mathbb{Q})$.

Because it uses the notion of transversality, the above construction requires that the atlas have some smoothness properties.[1] In particular, the transition maps between charts must satisfy the so-called tangent bundle (or index) condition. On the other hand, Pardon [P] introduces a new way to extract topological information from an atlas that satisfies a topological version of this condition that he calls the submersion

[1] See [C1, C2] for a weak form of these requirements.

axiom. Instead of gluing the chart domains together to form a topological space $|\mathcal{K}|$, Pardon works with K-homotopy sheaves of (co)chain complexes defined on homotopy colimits of spaces that are obtained from the chart domains. This gives a flexible way of assembling local homological information into a global object. Though this approach may be useful in many contexts, it is hard for a nonexpert in sheaf theory to understand where the technical difficulties are, and what actually has to be checked to ensure that the method works in any particular case. This becomes an issue if one wants to extend the method to cases (such as Hamiltonian Floer theory, or symplectic field theory) in which one must deal with a family of related moduli spaces and so should work on the chain level. The current paper was prompted by the desire to develop a different approach, that would replace Pardon's sophisticated sheaf theory by more elementary arguments that yet do not require smoothness.

This note only considers the simplest case, appropriate to Gromov–Witten theory, in which the aim is to construct a homology class $[X]^{vir}_{\mathcal{K}} \in \check{H}_d(X;\mathbb{Q})$. Working with Pardon's submersion axiom, we define a consistent thickening of the domains of the atlas charts to make them all have the same dimension. In the case with trivial isotropy, one thereby constructs an oriented topological manifold M of dimension $D := d + \dim E_A$, together with a map $\mathscr{S}_M : M \to E_A$ whose zero set can be identified with X. If the isotropy is nontrivial, M is a weighted branched manifold with a global action of the total isotropy group Γ_A and there is a homeomorphism $\mathscr{S}_M^{-1}(0)/\Gamma_A \stackrel{\cong}{\to} X$.[2] (A typical example of such a manifold (M,Λ) is the union of two circles, each of weight $\frac{1}{2}$, identified along a closed subarc A, so that the points $x \in A$ have weight $\Lambda(x) = 1$, while the others all have weight $\Lambda(x) = \frac{1}{2}$. See also §3.4.)

Here is the main result. (See Theorem 1.3.4 for a more precise statement.)

Theorem 1.1.1. *Let $\mathcal{K}$ be an oriented d-dimensional Kuranishi atlas on a compact space X that satisfies the submersion condition* (1.2.3) *and has total obstruction space $E_A := \prod_{i\in A} E_i$ and total isotropy group $\Gamma_A := \prod_{i\in A}\Gamma_i$. Let $D = d + \dim E_A$. Then there is an associated oriented weighted branched D-dimensional manifold (M,Λ) with an action of Γ_A, and a Γ_A-equivariant map $\mathscr{S}_M : M \to E_A$ with a compact zero set $\mathscr{S}_M^{-1}(0)$. Moreover, there is a map $\psi : \mathscr{S}_M^{-1}(0) \to X$ that induces a homeomorphism $\mathscr{S}_M^{-1}(0)/\Gamma_A \stackrel{\cong}{\to} X$.*

It is immediate from the construction that the oriented bordism class of a neighborhood of $\mathscr{S}_M^{-1}(0)$ in M depends only on the concordance class of $\mathcal{K}$.[3] Further, we show in Lemma 3.3.2 that (M,Λ) carries a fundamental class μ_M in rational Çech homology $\check{H}_*$. Hence we have the following.

[2] Another way to say this is that $M := |\widehat{\mathbf{M}}|_{\mathcal{H}}$ is the Hausdorff realization of a topological groupoid $\widehat{\mathbf{M}}$ that is étale but not proper: see §1.2, §1.3 for relevant definitions. However, just as in the case of the construction of the zero set in [MW3], it is most natural to construct a topological category $\mathbf{M}$ in which not all morphisms are invertible, i.e. it is a monoid, rather than a groupoid.

[3] Two atlases $\mathcal{K}^0, \mathcal{K}^1$ on X are said to be concordant if there is an atlas $\mathcal{K}^{01}$ on $[0,1]\times X$ whose restriction to $\{\alpha\}\times X$ is $\mathcal{K}^\alpha$, for $\alpha = 0,1$: see [MW1, Def. 4.1.6].

Corollary 1.1.2. *In the above situation there is a unique element* $[X]^{vir}_{\mathcal{K}} \in \check{H}_d(X;\mathbb{Q})$ *that is defined as follows. For* $\beta \in \check{H}^d(X;\mathbb{Q})$ *and* $D = d + \dim E_A$*, we have*

$$\langle \beta, [X]^{vir}_{\mathcal{K}} \rangle := (\mathscr{S}_M)_*(\widehat{\beta}) \in \check{H}_{\dim E_A}(E_A, E_A \smallsetminus \{0\};\mathbb{Q}) \cong \mathbb{Q}, \tag{1.1.1}$$

where $\widehat{\beta}$ *is the image of* β *under the composite*

$$\check{H}^d(X;\mathbb{Q}) \xrightarrow{\psi^*} \check{H}^d(\mathscr{S}_M^{-1}(0);\mathbb{Q}) \xrightarrow{\mathcal{D}} \check{H}_{\dim E_A}(M, M \smallsetminus \mathscr{S}_M^{-1}(0);\mathbb{Q}),$$

and $\mathcal{D}$ *is given by cap product with the fundamental class* μ_M*. Moreover,* $[X]^{vir}_{\mathcal{K}}$ *depends only on the oriented concordance class of* $\mathcal{K}$*, and in the smooth case agrees with the class defined in* [MW3].

A key element of the proof of Theorem 1.1.1 is Pardon's notion of *deformation to the normal cone*, which allows one to assemble different chart domains into a family of topological manifolds Y_J, albeit ones of the wrong dimension: see Proposition 2.1.1. The second key point is the existence of compatible collars for these manifolds Y_J. Remark 1.3.6 outlines the proof in more detail.

As we explain in Remark 2.2.3, if we start with a smooth atlas then the proofs of the above results can be somewhat simplified. In particular, by [M1] we can construct M to be a simplicial complex so that there is no need to use so much rational Čech homology when proving Corollary 1.1.2. Further, if one works with polyfolds, then the proof can be radically simplified. Indeed, it is not difficult to define a smooth Kuranishi atlas on any space X that appears as the (compact) zero set of a polyfold bundle [HWZ1, H, Y, MW4]. Because the polyfolds of Gromov–Witten theory support sc-smooth partitions of unity, if the isotropy is trivial, one can even define such an atlas with just one chart. In other words, one obtains a finite dimensional model

$$(U, \mathbb{R}^N, s, \psi), \quad \psi : s^{-1}(0) \xrightarrow{\cong} X,$$

for the whole of X, in which U is a smooth manifold of dimension $d+N$ and $s : U \to \mathbb{R}^N$ is a smooth map. As we show in Remark 1.3.8 this construction can be adapted in the presence of isotropy. However, the domain of the single chart is no longer a manifold, but a branched manifold with action of the total isotropy group Γ_A.

Another simple example is the calculation of the Euler class of an oriented vector bundle $\pi : \mathcal{E} \to X$ over a compact Hausdorff space X. If $\mathcal{E}' \to X$ is an oriented complement to $\mathcal{E}$ so that there is a vector bundle isomorphism $\phi : \mathcal{E} \oplus \mathcal{E}' \cong \mathbb{R}^N \times X$, let

$$M = \mathcal{E}', \qquad \mathscr{S} : M \to \mathbb{R}^N, \;\; (e', x) \mapsto \mathrm{pr}_{\mathbb{R}^N}\big(\phi(e', x)\big). \tag{1.1.2}$$

Then $\mathscr{S}^{-1}(0) \cong X$, and it is easy to check that the class $[X]^{vir}_{\mathcal{K}}$ defined by (1.1.1) is the Euler class of $\mathcal{E} \to X$: see Lemma 3.4.1. This is an instance of the construction in Pardon [P, Defn. 5.3.1] for the bundle $\pi : \mathcal{E} \to X$ with section $\mathfrak{s} \equiv 0$ in which the thickening $\lambda : \mathbb{R}^N \times X \to \mathcal{E}'$ is given by the projection.

Finally note that the methods of this paper should extend, e.g. to a more general notion of atlas, or to spaces more general than topological manifolds: see Remark 1.3.7.

1.2. **Basic definitions and facts about atlases.** A **weak Kuranishi atlas** $\mathcal{K}$ of dimension d on a compact metrizable space X consists of the following data.[4]

- **(footprint cover)** a finite open cover of X by nonempty sets $(F_i)_{i\in A}$;
- a **poset** $\mathcal{I}_{\mathcal{K}} = \big\{ I \subset A \;\big|\; F_I := \bigcap_{i\in I} F_i \neq \emptyset \big\}$ that indexes the charts;
- **(charts)** $\forall I \in \mathcal{I}_{\mathcal{K}}$, F_I is the footprint of a chart $\mathbf{K}_I := (U_I, \Gamma_I, E_I, s_I, \psi_I)$, where
  - U_I is a finite dimensional topological manifold of dimension $d + \dim E_I$;
  - $E_I := \prod_{i\in I} E_i$ is a product of even dimensional vector spaces such that $\dim U_I - \dim E_I = d$;
  - $\Gamma_I = \prod_{i\in I} \Gamma_i$ is a product of finite groups that acts on U_I, and acts by a product of linear actions on E_I;
  - $s_I : U_I \to E_I$ is a Γ_I-equivariant map;
  - the footprint map $\psi_I : s_I^{-1}(0) \to X$ induces a homeomorphism

$$
{}^{s_I^{-1}(0)}\!/_{\Gamma_I} \xrightarrow{\cong} F_I; \tag{1.2.1}
$$

- **(coordinate changes)** if $I \subset J$ there is a coordinate change $\widehat{\Phi}_{IJ} : \mathbf{K}_I \to \mathbf{K}_J$ given by the following data, where we identify E_I as a subspace of E_J in the obvious way:
  - a relatively open, Γ_J-invariant subset $\widetilde{U}_{IJ}$ of $s_J^{-1}(E_I) \subset U_J$ containing $s_J^{-1}(0)$ and with a free action of $\Gamma_{J\smallsetminus I}$,
  - a covering map $\rho_{IJ} : \widetilde{U}_{IJ} \to U_I$ that quotients out by the (free) action of $\Gamma_{J\smallsetminus I}$ and is equivariant with respect to the projection $\Gamma_J \to \Gamma_I$, further
  $$s_I \circ \rho_{IJ} = s_J, \quad \psi_J = \psi_I \circ \rho_{IJ} \text{ on } s_J^{-1}(0) \subset \widetilde{U}_{IJ},$$
  - if $I \subset J \subset K$, then
  $$\rho_{IK} = \rho_{IJ} \circ \rho_{JK} \text{ whenever both sides are defined} \tag{1.2.2}$$
  - in an **atlas** (rather than a weak atlas) we require in addition that the domain $\rho_{JK}^{-1}(\widetilde{U}_{IJ}) \cap \widetilde{U}_{JK}$ of $\rho_{JK} \circ \rho_{IJ}$ is a subset of the domain $\widetilde{U}_{IK}$ of ρ_{IK}.
  - in a **tame atlas** we require that both sides of (1.2.2) have the same domain and that $\widetilde{U}_{IJ} = s_J^{-1}(E_I)$.
- **(equivariant submersion condition)** for each $I \subset J$, each point $x \in \widetilde{U}_{IJ} \subset U_J$ has a product neighborhood that is compatible with the section s_J; more precisely for each such x with stabilizer subgroup $\Gamma_x \subset \Gamma_I$, there is a Γ_x-equivariant local homeomorphism of the form

$$
\phi_x : (B_{J\smallsetminus I}(0) \times W_x, \{0\} \times W_x) \to (V_J, \widetilde{V}_{IJ}) \tag{1.2.3}
$$

[4] These are essentially the same definitions as in [MW3], except that the smoothness requirements mentioned in Remark 1.2.1 (ii) below have been replaced by an equivariant version of Pardon's submersion axiom. The notion of topological atlas introduced in [MW1] is somewhat different; in particular the domains there need not be manifolds. For more details on all topics mentioned in this section, see the original papers [MW1, MW2, MW3] or [M2].

where $B_{J\smallsetminus I}(0)$ is a neighborhood of 0 in $E_{J\smallsetminus I}$ and W_x is a Γ_x-invariant neighborhood of x in $\widetilde{V}_{IJ}$, such that

$$s_{J\smallsetminus I}\circ\phi_x(e,y)=e,\qquad e\in E_{J\smallsetminus I}. \tag{1.2.4}$$

Remark 1.2.1. (i) Although the submersion axiom in [P] does not assume equivariance, this is needed in our set-up in order that M support an action of Γ_A. Notice that because $\Gamma_{J\smallsetminus I}$ acts freely on $\widetilde{U}_{IJ}$, the stabilizer Γ_x of $x\in\widetilde{U}_{IJ}$ lies in the subgroup Γ_I of $\Gamma_J\cong\Gamma_I\times\Gamma_{J\smallsetminus I}$. The standard proof of the submersion axiom for Gromov–Witten moduli spaces adapts easily to yield Γ_x-equivariance because it is an application of the gluing theorem at the stable map x. The process of gluing depends on various choices, for example of Riemannian metrics and of the complement to the image of the linearized Cauchy–Riemann operator at x, and these can always be chosen invariant under the finite stabilizer subgroup of x. This equivariance is built into the smooth index condition, since the latter is expressed in terms of the equivariant section maps $s_{J\smallsetminus I}$.

(ii) **(The smooth case)** In this case the manifolds U_I are assumed to be smooth, all structural maps (the group action on U_I, the section s_I, and coordinate changes ρ_{IJ}) are smooth, and the submersion axiom is replaced by the requirement that $\widetilde{U}_{IJ}$ be a submanifold of U_J such that

$$\begin{aligned}&\text{the derivative of } s_{J\smallsetminus I}: U_J\to E_{J\smallsetminus I} \text{ induce an isomorphism}\\ &\text{from the normal bundle of } \widetilde{U}_{IJ} \text{ in } U_J \text{ to } E_{J\smallsetminus I}\times\widetilde{U}_{IJ}.\end{aligned} \tag{1.2.5}$$

In this case we claim that each of the maps τ_{IJ} in Proposition 1.3.3 can be chosen to be a local diffeomorphism onto its image, so that M is a smooth manifold if the isotropy is trivial, and otherwise is a smooth branched manifold. The construction of such M is sketched in Remark 2.2.3.

(iii) **(Orientations)** We will consider an atlas to be oriented if each domain U_I (resp. each obstruction space E_I) has a Γ_I-invariant orientation that is respected by the coordinate changes. In fact, in the current situation, since we have assumed that the E_i are all even dimensional and invariantly oriented (e.g. that they are all complex vector spaces), then the E_I inherit natural orientations, and the local product structure given by the submersion condition permits the transfer of an orientation between charts. In the smooth case, a slightly more general notion of orientation is discussed extensively in [MW2, MW3]. $\diamond$

We now briefly recall some other terminology that will be useful later. An atlas $\mathcal{K}'=\big(\mathbf{K}'_I,\widehat{\Phi}'_{IJ}\big)$ is a **shrinking** of $\mathcal{K}=\big(\mathbf{K}'_I,\widehat{\Phi}'_{IJ}\big)$ if

- it has the same index set $\mathcal{I}_{\mathcal{K}}$, obstruction spaces E_I and groups Γ_I,
- each chart domain U'_I is a precompact subset of U_I, denoted $U'_I\sqsubset U_I$,
- the coordinate changes are given by restriction.

For short, in this situation we write

$$\mathcal{U}'\sqsubset\mathcal{U},\quad\text{where}\quad \mathcal{U}':=\textstyle\bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}}U'_I,\ \ \mathcal{U}:=\bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}}U_I. \tag{1.2.6}$$

It is shown in [MW1, §3.3] that every weak atlas has a tame shrinking $\mathcal{K}' \sqsubset \mathcal{K}$ that is unique up to a natural equivalence relation called concordance.

Each atlas[5] $\mathcal{K}$ determines a topological category $\mathbf{B}_{\mathcal{K}}$ with

$$\begin{aligned} \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}} &= \textstyle\bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} U_I, \quad \mathrm{Mor}_{\mathbf{B}_{\mathcal{K}}} = \bigsqcup_{I\subset J} \widetilde{U}_{IJ}\times\Gamma_I, \\ & s\times t : \mathrm{Mor}_{\mathbf{B}_{\mathcal{K}}} \to \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}} \times \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}, \\ & (I,J,y,\gamma) \mapsto \big((I,\gamma^{-1}(\rho_{IJ}(x)),\, (J,y)\big). \end{aligned} \tag{1.2.7}$$

We denote by $|\mathcal{K}| := |\mathbf{B}_{\mathcal{K}}|$ its (geometric or naive) realization. Thus

$$|\mathbf{B}_{\mathcal{K}}| := \textstyle\bigsqcup_I U_I/\sim,$$

where $\sim$ is the equivalence relation on $\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$ that is generated by the morphisms, i.e. $(I,x)\sim(J,y)$ if and only if there is a chain of morphisms

$$(I,x) = (I_0,x_1) \to (I_1,x_1) \leftarrow (I_2,x_2) \to \cdots \leftarrow (I_k,x_k) = (J,y).$$

Though for a general atlas the quotient topology is nonHausdorff, it is shown in [MW1, Thm 3.1.9] that if $\mathcal{K}$ is tame the quotient topology is Hausdorff and the natural maps

$$\pi_{\mathcal{K}} : U_I \to |\mathcal{K}| \tag{1.2.8}$$

induce homeomorphisms from U_I/Γ_I onto their images. Further, if $\mathcal{K}$ is also **preshrunk** (i.e. there is a double shrinking $\mathcal{K} \sqsubset \mathcal{K}' \sqsubset \mathcal{K}''$ where both $\mathcal{K}$ and $\mathcal{K}'$ are tame), the quotient topology on $|\mathcal{K}'|$ restricts to a metrizable topology on $|\mathcal{K}|$ that agrees with the quotient topology on each set $\pi_{\mathcal{K}}(U_I)$. We will say that $\mathcal{K}$ is **good** if its realization $|\mathcal{K}|$ has these properties.[6]

From now on we assume that $\mathcal{K}$ is good in this sense, e.g. preshrunk and tame.

There is a similar category $\mathbf{E}_{\mathcal{K}}$ formed by the obstruction bundles with

$$\begin{aligned} \mathrm{Obj}_{\mathbf{E}_{\mathcal{K}}} &= \textstyle\bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} U_I\times E_I, \quad \mathrm{Mor}_{\mathbf{B}_{\mathcal{K}}} = \bigsqcup_{I\subset J} \widetilde{U}_{IJ}\times E_I\times\Gamma_I, \\ & s\times t : \mathrm{Mor}_{\mathbf{E}_{\mathcal{K}}} \to \mathrm{Obj}_{\mathbf{E}_{\mathcal{K}}} \times \mathrm{Obj}_{\mathbf{E}_{\mathcal{K}}}, \\ & \big(I,J,y,e,\gamma\big) \mapsto \Big(\big(I,\gamma^{-1}(\rho_{IJ}(x),\gamma^{-1}(e)\big),\, \big(J,y,e\big)\Big). \end{aligned} \tag{1.2.9}$$

The projections $\mathrm{pr}_I : U_1\times E_I \to U_I$, sections s_I and footprint maps ψ_I fit together to give functors

$$\mathrm{pr} : \mathbf{E}_{\mathcal{K}} \to \mathbf{B}_{\mathcal{K}}, \quad \mathfrak{s} : \mathbf{B}_{\mathcal{K}} \to \mathbf{E}_{\mathcal{K}}, \quad \psi : \mathfrak{s}^{-1}(0) \to \mathbf{X},$$

where $\mathbf{X}$ is the category with objects X and only identity morphisms, and one can show that ψ induces a homeomorphism $|\psi| : |\mathfrak{s}|^{-1}(0)| \to X$.

Reductions and zero sets

[5] The extra assumption in the definition of atlas stated just after (1.2.2) implies that the set $\mathrm{Mor}_{\mathbf{B}_{\mathcal{K}}}$ defined below is closed under composition.

[6] The proof given in [MW1] that preshrunk and tame atlases are good is abstract, i.e. the argument only uses properties of the objects and maps in the category $\mathbf{B}_{\mathcal{K}}$. However, because the atlas domains are often constructed as subsets of an ambient Hausdorff metrizable space $\mathcal{S}$ (such as a space of stable maps), one can sometimes use the existence of $\mathcal{S}$ to bypass some of the arguments in [MW1].

The situation when all the obstruction spaces E_I vanish is considered in [M3]. In this case, the category $\mathbf{B}_{\mathcal{K}}$ is

- **étale**, i.e. its source and target maps are local homeomorphisms, and
- **proper**, i.e. the map $s \times t : \mathrm{Mor}_{\mathbf{B}_{\mathcal{K}}} \to \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}} \times \mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}}$ is proper.

Moreover it has a natural completion to a ep (étale, proper) groupoid (i.e. a category in which all morphisms are invertible) that also has realization $|\mathcal{K}|$. Thus $\mathbf{B}_{\mathcal{K}}$ provides an orbifold structure on $|\mathcal{K}|$.

If the obstruction spaces do not vanish, then the manifolds U_I have varying dimensions. However, if $\nu_I : U_I \to E_I$ is a perturbation section such that $s_I + \nu_I : U_I \to E_I$ is transverse to 0, then the perturbed zero set $Z_I := (s_I + \nu_I)^{-1}(0)$ has fixed dimension d. Hence, as is shown in [MW2], if the isotropy groups vanish and if we can choose the ν_I compatibly, i.e. they form a functor

$$\nu : \mathbf{B}_{\mathcal{K}} \to \mathbf{E}_{\mathcal{K}},$$

then these zero sets fit together to form a manifold. However, in general the domains U_I overlap too much for there to be such a functor.[7]

We deal with this by passing to a **reduction** $\mathcal{V}$, i.e. a family of Γ_I-invariant, precompact open subsets $V_I \sqsubset U_I$ with the following properties:

(1.2.10)
- the footprints $\big(G_I := \psi_I(V_I \cap s_I^{-1}(0))\big)_{I \in \mathcal{I}_{\mathcal{K}}}$ cover X,
- $\pi_{\mathcal{K}}(\overline{V}_I) \cap \pi_{\mathcal{K}}(\overline{V}_J) \neq \emptyset$ only if $I \subset J$ or $J \subset I$,

where $\pi_{\mathcal{K}} : U_I \to |\mathcal{K}|$ is the projection in (1.2.8). In the construction given in [MW2] for the trivial isotropy case, we define the perturbation section as a functor

$$\nu : \mathbf{B}_{\mathcal{K}}\big|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}\big|_{\mathcal{V}} \tag{1.2.11}$$

on the full subcategory $\mathbf{B}_{\mathcal{K}}\big|_{\mathcal{V}}$ of $\mathbf{B}_{\mathcal{K}}$ with objects $\bigsqcup_I V_I$.

If the isotropy groups are nontrivial then it is (in general) no longer possible to choose a transverse equivariant section ν, even on a reduction $\mathcal{V}$. However, because the morphisms in $\mathbf{B}_{\mathcal{K}}\big|_{\mathcal{V}}$ are described so explicitly, we show in [MW3] that we may construct the perturbation section as a (single valued) functor

$$\nu : \mathbf{B}_{\mathcal{K}}\big|_{\mathcal{V}}^{\setminus\Gamma} \to \mathbf{E}_{\mathcal{K}}\big|_{\mathcal{V}}^{\setminus\Gamma},$$

where $\mathbf{B}_{\mathcal{K}}\big|_{\mathcal{V}}^{\setminus\Gamma}$ is the (non full) subcategory of $\mathbf{B}_{\mathcal{K}}\big|_{\mathcal{V}}$ obtained by discarding the morphisms coming explicitly from the group actions. Thus

$$\mathrm{Mor}_{\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}} = \textstyle\bigsqcup_{I \subset J} \widetilde{V}_{IJ} \quad \text{where} \quad s \times t : \ (I, J, y) \mapsto \big((I, \rho_{IJ}(y)), (J, y)\big) \tag{1.2.12}$$
$$\text{and } \widetilde{V}_{IJ} = V_J \cap \rho_{IJ}^{-1}(V_I) \subset \widetilde{U}_{IJ}.$$

[7] See [MW1, §5.1]. The relation between $\mathcal{U}$ and its reduction $\mathcal{V}$ is similar to that between the cover of a simplicial space by the stars of its vertices and the cover by the stars of its first barycentric subdivision.

We show in [MW3] that if $(s+\nu) \pitchfork 0$, the full subcategory of $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}$ with objects

$$\bigsqcup_I \big(Z_I := (s_I + \nu_I)^{-1}(0) \big), \qquad \text{and} \qquad \text{weight}(Z_I) = 1/|\Gamma_I|,$$

can be completed to a weighted étale groupoid whose realization is therefore a weighted branched manifold as defined in §1.3. We will see below that in the current context the branched manifold structure of M appears in a similar way.

1.3. **The weighted branched manifold** (M, Λ)**.** We will construct M from the realization of an oriented étale category $\mathbf{M}$ whose objects are thickened versions of the domains V_I of a reduction $\mathcal{V}$ of the atlas $\mathcal{K}$, and whose morphisms have exactly the same structure as those in the category $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}$ defined in (1.2.12).

We begin with some relevant definitions from [M1]. (See also [MW3, App. A] that gives succinct proofs of the results we use.) If $\mathbf{G}$ is a wnb groupoid as described below, its realization $|\mathbf{G}|$ with the quotient topology is in general not Hausdorff. Hence we consider its maximal Hausdorff quotient $|\mathbf{G}|_{\mathcal{H}}$, which has the following universal property: any continuous map from $|\mathbf{G}|$ to a Hausdorff space factors through $|\mathbf{G}|_{\mathcal{H}}$. In the following we write $|\mathbf{G}|$ for the realization $\mathrm{Obj}_{\mathbf{G}}/\sim$ of an étale groupoid $\mathbf{G}$, and $|\mathbf{G}|_{\mathcal{H}}$ for its maximal Hausdorff quotient. We denote the natural maps by

$$\pi_{\mathbf{G}} : \mathrm{Obj}_{\mathbf{G}} \to |\mathbf{G}|, \qquad \pi^{\mathcal{H}}_{|\mathbf{G}|} : |\mathbf{G}| \longrightarrow |\mathbf{G}|_{\mathcal{H}}, \qquad \pi^{\mathcal{H}}_{\mathbf{G}} := \pi^{\mathcal{H}}_{|\mathbf{G}|} \circ \pi_{\mathbf{G}} : \mathrm{Obj}_{\mathbf{G}} \to |\mathbf{G}|_{\mathcal{H}}.$$

Definition 1.3.1 ([M1],Def. 3.2)**.** *A* **weighted nonsingular branched groupoid** *(or* **wnb groupoid**) *of dimension d is a pair $(\mathbf{G}, \Lambda_{\mathbf{G}})$ consisting of an oriented, nonsingular*[8]*, étale groupoid $\mathbf{G}$ of dimension d, together with a rational weighting function $\Lambda_{\mathbf{G}} : |\mathbf{G}|_{\mathcal{H}} \to \mathbb{Q}^+ := \mathbb{Q} \cap (0,\infty)$ that satisfies the following compatibility conditions. For each $p \in |\mathbf{G}|_{\mathcal{H}}$ there is an open neighborhood $N \subset |\mathbf{G}|_{\mathcal{H}}$ of p, a collection $U_1, \dots, U_\ell$ of disjoint open subsets of $(\pi^{\mathcal{H}}_{\mathbf{G}})^{-1}(N) \subset \mathrm{Obj}_{\mathbf{G}}$ (called* **local branches***), and a set of positive rational weights $m_1, \dots, m_\ell$ such that the following holds:*

(Cover) $(\pi^{\mathcal{H}}_{|\mathbf{G}|})^{-1}(N) = |U_1| \cup \cdots \cup |U_\ell| \subset |\mathbf{G}|$*;*

(Local Regularity) *for each $i = 1, \dots, \ell$ the projection $\pi^{\mathcal{H}}_{\mathbf{G}}|_{U_i} : U_i \to |\mathbf{G}|_{\mathcal{H}}$ is a homeomorphism onto a relatively closed subset of N;*

(Weighting) *for all $q \in N$, the number $\Lambda_{\mathbf{G}}(q)$ is the sum of the weights of the local branches whose image contains q:*

$$\Lambda_{\mathbf{G}}(q) = \textstyle\sum_{i : q \in |U_i|_{\mathcal{H}}} m_i.$$

Now we can formulate the notion of weighted branched manifold.[9] Analogous definitions for cobordisms may be found in [MW3, App. A] .

[8] i.e. there is at most one morphism between any two objects. Further, we restrict here to rational weights, but clearly this condition could be generalized.

[9] Note that a weighted branched manifold must be oriented, since otherwise one cannot define a consistent weighting function.

Definition 1.3.2. *A* **weighted branched manifold** *of dimension d is a pair (Z, Λ_Z) consisting of a topological space Z together with a function $\Lambda_Z : Z \to \mathbb{Q}^+$ and an equivalence class*[10] *of tuples $(\mathbf{G}, \Lambda_{\mathbf{G}}, f)$, where $(\mathbf{G}, \Lambda_{\mathbf{G}})$ is a d-dimensional wnb groupoid and $f : |\mathbf{G}|_{\mathcal{H}} \to Z$ is a homeomorphism that induces the function $\Lambda_Z := \Lambda_{\mathbf{G}} \circ f^{-1}$.*

We define the weighted branched manifold (M, Λ) of Theorem 1.1.1 as the realization of a category $\mathbf{M}$ constructed as follows. First choose a Γ_i-invariant norm $\|\cdot\|$ on each E_i, and for any $J \subset A$ give the vector space $E_J := \prod_{i\in J} E_i$ the sup norm

$$\|e_J\| = \sup_{i\in J}\|e_i\|.$$

Further, denote

$$E_{J,\varepsilon} := \{e_J \in E_J \mid \|e_J\| < \varepsilon\}, \tag{1.3.1}$$

and

$$\underline{\varepsilon} := (\varepsilon_I)_{I\in\mathcal{I}_{\mathcal{K}}}, \quad \text{where } I \subsetneq J \Longrightarrow 0 < \kappa\,\varepsilon_I < \varepsilon_J, \quad \text{for } \kappa := \max\{|J| \mid J \in \mathcal{I}_{\mathcal{K}}\}. \tag{1.3.2}$$

Given a reduction $\mathcal{V}$ of an atlas $\mathcal{K}$ as in (1.2.10), for each $I \subset J$ we denote

$$V_{IJ} = V_I \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(V_J)) \subset V_I, \quad \widetilde{V}_{IJ} = V_J \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(V_I)) \subset V_J, \tag{1.3.3}$$

where $\pi_{\mathcal{K}} : V_I \to |\mathcal{K}|$ is the obvious projection. Thus $\rho_{IJ}(\widetilde{V}_{IJ}) = V_{IJ}$. Observe also that the group Γ_A acts on $E_{A\smallsetminus J,\varepsilon_J} \times V_J$ by

$$\gamma \cdot (e, x) = \big(\gamma|_{A\smallsetminus J}(e), \gamma|_J(x)\big), \quad \gamma \in \Gamma_A, \tag{1.3.4}$$

where $\gamma|_J$ denotes the projection of $\gamma \in \Gamma_A := \prod_{i\in A}\Gamma_i$ to $\Gamma_J := \prod_{i\in J}\Gamma_i$.

The following result is proved in Proposition 2.2.2 below; compare with (1.2.12) above.

Proposition 1.3.3. *Let $\mathcal{K}$ be a tame (or good) oriented atlas on X of dimension d. Then there is a reduction $\mathcal{V}$ and choice of constants $\underline{\delta} > 0$ such that the following holds.*

(i) *There is an oriented étale category $\mathbf{M}$ of dimension $D := d + \dim E_A$ with*

$$\begin{aligned} &\mathrm{Obj}_{\mathbf{M}} = \textstyle\bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} M_J := E_{A\smallsetminus J,\delta_J} \times V_J, \qquad \mathrm{Mor}_{\mathbf{M}} = \textstyle\bigsqcup_{I\subset J,\, I,J\in\mathcal{I}_{\mathcal{K}}} \widetilde{M}_{IJ} \\ &s \times t :\ \mathrm{Mor}_{\mathbf{M}} \to \mathrm{Obj}_{\mathbf{M}} \times \mathrm{Obj}_{\mathbf{M}}, \quad (I, J, y) \mapsto \Big(\big(I, \tau_{IJ}(y)\big), \big(J, y\big)\Big), \end{aligned} \tag{1.3.5}$$

where $\widetilde{M}_{IJ} \subset M_J$ is an open Γ_A-invariant subset, and

$$\tau_{IJ} : \widetilde{M}_{IJ} \to M_{IJ} := E_{A\smallsetminus I,\delta_I} \times V_{IJ} \subset M_I \tag{1.3.6}$$

is a Γ_A-equivariant covering map onto $M_{IJ} \subset M_I$ that quotients out by a free action of $\Gamma_{J\smallsetminus I}$ that extends to a neighborhood of the closure of $\widetilde{M}_{IJ}$ in M_J.

---

[10] The precise notion of equivalence is given in [M1, Definition 3.12]. In particular it ensures that the induced function $\Lambda_Z := \Lambda_{\mathbf{G}} \circ f^{-1}$ and the dimension of $\mathrm{Obj}_{\mathbf{G}}$ is the same for equivalent structures $(\mathbf{G}, \Lambda_{\mathbf{G}}, f)$.

(ii) $\mathbf{M}$ *supports an action of* Γ_A *by* (1.3.4) *on objects, and by*

$$\big(I,J,y\big)\mapsto \gamma\cdot(I,J,y) := \big(I,J,\gamma^{-1}y\big),\quad \gamma\in\Gamma_A,\ y\in \widetilde{M}_{IJ}, \tag{1.3.7}$$

on morphisms.

(iii) *There is a* Γ_A*-equivariant functor* $\mathscr{S}:\mathbf{M}\to\mathbf{E}_A$*, where the category* $\mathbf{E}_A$ *has objects* E_A *and only identity morphisms, that is given on objects by maps* $\mathscr{S}_J: M_J\to E_A$ *such that*

$$\mathscr{S}_J(0,x) = s_J(x),\quad \mathscr{S}_J^{-1}(E_J)\subset\{0\}\times V_J \tag{1.3.8}$$

so that

$$(\mathscr{S}_J)^{-1}(0) = \big\{(0,x)\in E_{A\smallsetminus J}\times V_J : s_J(x)=0\big\}.$$

Here is a precise statement of Theorem 1.1.1. Note that $\mathscr{S}$ denotes a functor $\mathbf{M}\to\mathbf{E}_A$, while $\mathscr{S}_M: M\to E_A$ is the corresponding function on M.

Theorem 1.3.4. (i) *The category* $\mathbf{M}$ *can be completed to an oriented wnb groupoid* $\widehat{\mathbf{M}}$ *with the same objects as* $\mathbf{M}$ *and the same realization* $|\widehat{\mathbf{M}}| = |\mathbf{M}|$.

(ii) *If we denote the composite* $\mathrm{Obj}_{\mathbf{M}}\to|\mathbf{M}|\to|\widehat{\mathbf{M}}|_{\mathcal{H}}$ *by* $y\mapsto|y|\mapsto\pi^{\mathcal{H}}_{\mathbf{M}}(|y|)$*, the function* $\Lambda: M:=|\widehat{\mathbf{M}}|_{\mathcal{H}}\to\mathbb{Q}^+$ *defined by*

$$\Lambda(p) := \tfrac{1}{|\Gamma_I|}\cdot\#\big\{y\in M_I\,\big|\,\pi^{\mathcal{H}}_{\mathbf{M}}(|y|)=p\big\}\qquad \text{for } p\in|M_I|_{\mathcal{H}}$$

is a weighting function that gives (M,Λ) *the structure of a weighted branched manifold.*

(iii) *The group action by* Γ_A *and functor* $\mathscr{S}$ *extend to* $\widehat{\mathbf{M}}$*, so that there is a* Γ_A*-equivariant map* $\mathscr{S}_M: M\to E_A$*. Moreover, the zero set* $\mathscr{S}_M^{-1}(0)$ *is a compact subset of* M*, and the footprint maps* ψ_I *induce a homeomorphism*

$$\underline{\psi}:\ \mathscr{S}_M^{-1}(0)/\Gamma_A\xrightarrow{\cong}X.$$

Proof of Theorem 1.3.4 assuming Proposition 1.3.3. Part (i) holds by [M3, Prop 2.3], where we show that a category such as $\mathbf{M}$ has a groupoid completion with the same realization. That proposition assumes that the initial category $\mathbf{M}$ is proper (see [M3, Def 2.1]). However, that assumption plays no role in the proof, which is purely algebraic. Part (ii) holds by [M3, Prop 4.5]: see also [MW3, Thm.3.2.8]. The Hausdorff quotient $|\mathbf{M}|_{\mathcal{H}}$ is simply given by quotienting by the closure of the morphism relation on $\mathrm{Obj}_{\mathbf{M}}$. The fact that the $\Gamma_{J\smallsetminus I}$ action extends to the closure of $\widetilde{M}_{IJ}$ in M_J implies that the local regularity condition in Definition 1.3.1 holds; i.e. that the maps $M_J\to|\mathbf{M}|_{\mathcal{H}}$ are local homeomorphisms along the frontier $cl_{M_J}(\widetilde{M}_{IJ})\smallsetminus\widetilde{M}_{IJ}$ of $\widetilde{M}_{IJ}$ in M_J. Finally, the first statement in (iii) follows immediately from Proposition 1.3.3 (iii). The only part of the second statement that requires care is the proof of compactness, since it is then an immediate consequence of (1.2.1) that $\mathscr{S}_M^{-1}(0)/\Gamma_A\cong X$. Because the category $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}$ of (1.2.12) embeds in $\mathbf{M}$ as the full subcategory with objects $\bigsqcup_I\{0\}\times V_J$, it suffices to prove compactness for the zero set $|\widehat{\mathbf{Z}}_{\mathcal{H}}|$ of $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}$. But as in the proof of [MW3, Thm. 3.2.8], this follows from [MW1, Thm 5.2.2]. $\square$

Remark 1.3.5. Instead of taking M to be a weighted branched manifold with action of Γ_A, one could add the morphisms in Γ_A to the completed category $\widehat{\mathbf{M}}$ to obtain an étale groupoid $\widehat{\mathbf{M}}\times\Gamma_A$. In general, this groupoid is not proper. However, it does inherit a weighting function and so its realization is a weighted branched orbifold M/Γ_A: for an explicit example see §3.4 (VI). Note also that the action of the group Γ_A on $\mathbf{M}$ only affects the fundamental class μ_M (and hence $[X]^{vir}_{\mathcal{K}}$) via the weighting function Λ whose values depend on the groups Γ_I as well as on the category $\mathbf{M}$. $\diamond$

Remark 1.3.6. (Outline of the argument) We will explain the main points of the proof of Proposition 1.3.3 in §2. The first step is to use 'deformation to the normal cone' (see [P]) to construct manifolds $(Y_{\mathcal{U},J,\underline{\varepsilon}})_{J\in\mathcal{I}_{\mathcal{K}}}$ of dimension $d+\dim E_A+|J|-1$, that each have a boundary collar with 'corner control': see Proposition 2.1.3. Then we use the collar to construct the covering maps $\tau_{IJ}:\widetilde{M}_{IJ}\to M_I$. Since the general definition of these maps is quite complicated, we explain in Example 2.2.1 how this works for an atlas with just three basic charts. Proposition 2.2.2 gives the general construction. §3.1 contains technical details about compatible shrinkings, and the proof that each $Y_{\mathcal{V},J,\underline{\varepsilon}}$ is a manifold. The argument here is based on the existence of the local product structures provided by the submersion axiom. As we show in Step 1 of the proof of Proposition 2.1.3 in §3.2, this axiom also allows one to construct local collars that are compatible with the covering maps ρ_{IJ} and with projection to the vector spaces $E_{J\setminus I}$. In Step 2 of this proof we explain a standard method (described in Hatcher [Hat]) for assembling these local collars into a global collar for each $Y_{\mathcal{V},J,\underline{\varepsilon}}$, and show in Step 3 how to arrange that these collars have the consistency properties listed in Proposition 2.1.3 that are needed in the definition of the maps τ_{IJ}. This last step works under the assumption that the domains of the local collars are compatible with the reduction $\mathcal{V}$ and choice $\underline{\varepsilon}$ of thickening constants in a rather subtle way, which is summarized in the notion of compatible reduction $(\mathcal{V},\underline{\varepsilon})$ in Definition 3.1.8. $\diamond$

Remark 1.3.7. (Generalizations) The construction of $\mathbf{M}$ could be generalized in various ways. The argument relies in an essential way on the submersion property in order to construct the collars in Proposition 2.1.3, i.e. on the fact that along $\widetilde{U}_{IJ}$ the space U_J is locally the product of the vector space $E_{J\setminus I}$ with the domain U_I. However, it does not use the fact that the domains U_I themselves are topological manifolds: for example, since all we want in the end is information on homology, it would no doubt suffice if they were (locally compact, metrizable) homology manifolds of dimension $\dim E_I+d$. One could also consider atlases (or equivalently categories $\mathbf{B}_{\mathcal{K}}$) whose charts are indexed by a poset more general than that given by the subsets of A. However, one does need to be able to restrict attention to a subcategory such as $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}$ in which there are morphisms between the elements of two components of the domain only if the indices of those components are comparable in the given poset. Some possible generalizations of this kind are discussed in the last section of [M2]. $\diamond$

Remark 1.3.8. (The polyfold approach) If X is the zero set of a Fredholm section $\mathfrak{s}$ of a polyfold bundle $\mathcal{E}\to\mathcal{S}$ of index d, then one can use the fact that the realization $|\mathcal{S}|$ supports partitions of unity to give a very simple construction for the branched manifold $\mathbf{M}$ and section $\mathscr{S}$. (In the applications of interest to us $\mathcal{S}$ is a category[11] whose realization is a space of stable maps with the Gromov topology: see [H, HWZ2].) Here is a very brief outline: for details see [MW4]. Given $x \in X$ with stabilizer subgroup Γ_x, choose a lift $q_x \in \mathrm{Obj}_{\mathcal{S}}$, and a Γ_x-invariant open neighborhood $\mathcal{O} \subset \mathrm{Obj}_{\mathcal{S}}$ of q_x such that the map $\mathcal{O} \to |\mathcal{O}| \subset |\mathcal{S}|$ factors through a homeomorphism $\mathcal{O}/\Gamma_x \stackrel{\cong}{\to} |\mathcal{O}|$. Because $\mathcal{S}$ is Fredholm, there is a Γ_x-equivariant linear map $\lambda : E \to \mathrm{Sect}\,(\mathcal{E}|_{\mathcal{O}})$ from a Γ_x-invariant normed linear space E to a subspace of sc^+-smooth sections that covers the cokernel of the linearization of $\mathfrak{s}$ at x. It follows that there is $\varepsilon > 0$ such that the set

$$U := \big\{(e,q) \in E \times \mathcal{O} \;\big|\; \mathfrak{s}(q) = \lambda(e), \|e\| < \varepsilon\big\}$$

is a manifold of dimension $d + \dim E$. (The proof involves a nontrivial amount of analytic detail that will appear in [MW4].) Choose a finite covering of the compact set $X := |\mathfrak{s}^{-1}(0)|$ by the footprints $\psi_i(s_i^{-1}(0))$ of such charts

$$\mathbf{K}_i := \big(U_i, E_i, \Gamma_i, s_i, \psi_i\big)_{i\in A}, \qquad s_i(e,q) = e,$$

and let $(|\mathcal{O}_i|)_{i\in A}$ be the associated open cover of a neighborhood of X in the ambient space $|\mathcal{S}|$. Just as in [M3], one can use the groupoid structure of $\mathcal{S}$ to show that the $\mathbf{K}_i$ form the basic charts for a tame Kuranishi atlas whose transition charts are given by tuples of composable morphisms. Instead of giving more detail about this construction, we will outline how to modify these definitions so that the domains of the charts all have the same dimension $d + \dim E_A$.

First choose a reduction $\big(|\mathcal{W}_I|\big)_{I\in\mathcal{I}_{\mathcal{K}}}$ of the covering $\big(|\mathcal{O}_i|\big)_{i\in A}$ of a neighborhood of $X = |\mathfrak{s}^{-1}(0)|$ in $|\mathcal{S}|$. Thus, as in (1.2.10) we assume that

- for each $I \in \mathcal{I}_{\mathcal{K}}$, $|\mathcal{W}_I| \subset |\mathcal{O}_I| := \bigcap_{i\in I} |\mathcal{O}_i|$,
- $X \subset \bigcup_{i\in I} |\mathcal{W}_I|$;
- $|\mathcal{W}_I| \cap |\mathcal{W}_J| \neq \emptyset \implies I \subset J$ or $J \subset I$.

Next choose an ordering of the elements $i \in A$ and a partition of unity $(\rho_i)_{i\in A}$ subordinate to the cover $\big(|\mathcal{W}_i|\big)_{i\in A}$. Then, given $I = \{i_0, \dots, i_k\}$ where $i_0 < i_1 < \cdots < i_k$, the space $M_I^{\mathcal{W}}$ consists of all tuples

$$\Big\{\big(e_A, q_{i_0}, \Psi_{q_{i_0}q_{i_1}}, \cdots, q_{i_k}\big) \;\big|\; |q_{i_0}| \in |\mathcal{W}_I|,\ \Psi_{q\,q'} \in \mathrm{Mor}(q,q'),\ \|e_A\| < \varepsilon,$$
$$\mathfrak{s}(q_{i_0}) = \textstyle\sum_j \rho_{i_j}(|q_{i_0}|)\ \Psi^*\big(\lambda_{i_j}(e_{i_j})(q_{i_j})\big) \in \mathcal{E}_{q_0}\Big\},$$

where $(q_{i_0}, \Psi_{q_{i_0}q_{i_1}}, q_{1_i}, \Psi_{q_{i_1}q_{i_2}}, \cdots, q_{i_k})$ is a composable k-tuple of morphisms from a point $q_0 \in \mathcal{O}_{i_0}$ to $q_k \in \mathcal{O}_{i_k}$. By [HWZ2, Thm 7.4], we may choose the ρ_j so that for each $i, j \in A$ the function

$$\mathcal{O}_i \to [0,1], \qquad q \mapsto \rho_j(|q|)$$

[11] One can think of $\mathcal{S}$ as an infinite dimensional version of an ep groupoid, where the objects $\mathrm{Obj}_{\mathcal{S}}$ do not form a set but nevertheless the quotient $|\mathcal{S}| = \mathrm{Obj}_{\mathcal{S}}/{\sim}$ is a topological space, where $\sim$ is defined by setting $x \sim y \iff \mathrm{Mor}_{\mathcal{S}}(x,y) \neq \emptyset$.

is sc-smooth. It follows that if $\varepsilon > 0$ is suitably small, then, for each I, $M_I^{\mathcal{W}}$ is a manifold of dimension $d + \dim E_A$ with action of Γ_A. Moreover, much as in [M3, Prop.2.3], for each $I \subset J$ one can define a Γ_A-equivariant covering map

$$\tau_{IJ} : M_J^{\mathcal{W}} \supset \widetilde{M}_{IJ}^{\mathcal{W}} \ \to\ M_{IJ}^{\mathcal{W}} \subset M_I^{\mathcal{W}}$$

by taking an appropriate combination of the structural maps in $\mathcal{S}$ (such as compositions and source/target maps), where $M_{IJ}^{\mathcal{W}}$ (resp. $\widetilde{M}_{IJ}^{\mathcal{W}}$) consists of all elements in $M_I^{\mathcal{W}}$ (resp. $M_J^{\mathcal{W}}$) with $|q_{i_0}| \in |\mathcal{W}_I| \cap |\mathcal{W}_J|$. This gives a category $\mathbf{M}$ whose structure is precisely as described in Proposition 1.3.3. $\diamond$

2. Outline of the construction

The key element in our construction is the manifold $Y_{\mathcal{U},J,\underline{\varepsilon}}$, which lies over the $(|J|-1)$-dimensional simplex Δ_J. Its submanifold $Y_{\mathcal{V},J,\underline{\varepsilon}}$, corresponding to a choice of reduction $\mathcal{V} \subset \mathcal{U}$, has a boundary collar that is compatible both with shrinking of chart domains and with projection to Δ_J. We will define the attaching maps $\widetilde{M}_{IJ} \to M_{IJ}$ of the different components of $\mathrm{Obj}_{\mathbf{M}}$ by thinking of M_J as a subset of $Y_{\mathcal{V},J,\underline{\varepsilon}}$. In this section we state the main results about $Y_{\mathcal{V},J,\underline{\varepsilon}}$ and its boundary collar, and then explain the construction of $\mathbf{M}$, first by example (see Example 2.2.1) and then in general (see Proposition 2.2.2).

2.1. **The collared manifold Y.** Suppose given a tame atlas $\mathcal{K}$ with set of chart domains $\mathcal{U} := (U_I)_{I \in \mathcal{I}_\mathcal{K}}$. The next definition uses a choice of constants $\underline{\varepsilon} = (\varepsilon_I)$ as in (1.3.2), and the following notation:

- $\Delta_J := \bigl\{ t = (t_i)_{i \in J} \bigm| |t| := \sum_{i \in J} t_i = 1 \bigr\}$ is the $(|J|-1)$-simplex;
- for $\emptyset \ne I \subsetneq J$, we denote by $\iota_{IJ} : \Delta_I \to \Delta_J$ the natural inclusion with image
$$\partial_{J \smallsetminus I} \Delta_J := \{ t \in \Delta_J \bigm| t_j = 0, j \in J \smallsetminus I \} \subset \Delta_J;$$
(we often omit ι_{IJ} if there is no danger of confusion)
- $t \cdot e := \sum_{i \in J} t_i e_i, \quad$ where $t \in \Delta_J, e \in E_A$;
- $\kappa := \max\{|J| : J \in \mathcal{I}_\mathcal{K}\}$;
- for $x \in U_J$, $I(x) := \{ j : s_j(x) \ne 0\} \subset J$ and
- $\underline{\varepsilon} := (\varepsilon_I)_{I \in \mathcal{I}_\mathcal{K}}$ is a set of positive constants such that $\kappa\, \varepsilon_I \le \varepsilon_J$ whenever $I \subsetneq J$.

Given $J \in \mathcal{I}_\mathcal{K}$, consider the set

$$\begin{aligned} Y_J :&= Y_{\mathcal{U},J,\underline{\varepsilon}} \\ &= \left\{ (e, x; t) \in E_A \times U_J \times \Delta_J \;\middle|\; \begin{array}{ll} s_J(x) = t \cdot e, & \|e\| < \kappa\, \varepsilon_{I(x)}, \\ \|s_i(x)\| < \varepsilon_{I(x)} & \forall i \in J \end{array} \right\}. \end{aligned} \tag{2.1.1}$$

Here are some properties of this definition.

- Γ_A acts on $Y_{\mathcal{U},J,\underline{\varepsilon}}$ by
$$\gamma \cdot (e, x; t) = (\gamma^{-1} \cdot e, \gamma^{-1}(x); t).$$

- The condition $s_J(x) = t \cdot e$ implies that

$$I(x) := \{j : s_j(x) \neq 0\} \ \subset \ I(t) := \{i : t_i > 0\}. \tag{2.1.2}$$

  In particular, if $(e, x; t) \in Y_{\mathcal{U},J,\underline{\varepsilon}}$ we must have

$$x \in s_J^{-1}(E_{I(x)}) = \widetilde{U}_{I(x)J} \subset \widetilde{U}_{I(t)J},$$

  where the equality holds because $\mathcal{K}$ is tame (see (1.2.2)). Further, the components of e in $E_{I(t)}$ are determined by the pair (x, t), while those in $E_{A \setminus I(t)}$ can vary freely.
- For each element of the form $\bigl(e, x; \iota_{IJ}(t)\bigr) \in Y_{\mathcal{U},J,\underline{\varepsilon}}$ there is a corresponding element $(e, \rho_{IJ}(x); t) \in Y_{\mathcal{U},I,\underline{\varepsilon}}$, where $\rho_{IJ} : \widetilde{U}_{IJ} \to U_{IJ}$ is part of the atlas coordinate change. Thus there is a Γ_A-equivariant covering map

$$\partial_{J \setminus I} Y_J \to Y_I \cap (E_A \times U_{IJ} \times \Delta_I) \subset Y_I. \tag{2.1.3}$$

  If the isotropy is trivial, we can therefore identify $\partial_{J \setminus I} Y_J$ with an open subset of Y_I.
- The relevance of the conditions involving the constants $\underline{\varepsilon}$ are explained by the following remark. For each $x \in U_J$ such that $\|s_i(x)\| < \varepsilon_{I(x)}, \forall i \in J$, and every H satisfying $I(x) \subset H \subset J$, there is a corresponding element

$$\bigl(e, x; \iota_{HJ}(b_H)\bigr) \ \in \ Y_{\mathcal{U},J,\underline{\varepsilon}}, \tag{2.1.4}$$

  where b_H is the barycenter of Δ_H. Indeed, since b_H has components $\frac{1}{|H|} \geq \frac{1}{\kappa}$, we can take $e = \bigl(|H| s_i(x)\bigr)_{i \in A}$.

There are three Γ_A-equivariant projections of $Y_{\mathcal{U},J,\underline{\varepsilon}}$ onto the factors of its domain.

- $\mathrm{pr}_E : Y_{\mathcal{U},J,\underline{\varepsilon}} \to E_A, \quad (e, x; t) \mapsto e$. For $I \subset A$, we denote by e_I the elements of E_I, and denote by pr_{E_I} the projection to E_I.
- The projection $\mathrm{pr}_U : (e, x; t) \mapsto x \in U_J$ has contractible fibers that vary with $x \in U_J$.
- The fibers of $\mathrm{pr}_\Delta : Y_{\mathcal{U},J,\underline{\varepsilon}} \to \Delta_J, \quad (e, x; t) \mapsto t$ also depend on the image $t \in \Delta_J$. In particular, if for some $I \subsetneq J$ we have $t \in \operatorname{int} \Delta_I := \Delta_I \setminus \partial \Delta_I \subset \partial \Delta_J$ then for any $(e, x; t) \in \mathrm{pr}_\Delta^{-1}(t)$, we must have $x \in \widetilde{U}_{IJ}$ while the restriction $\mathrm{pr}_{E_{A \setminus I}}(e)$ can vary freely.

The following result is proved in Lemma 3.1.3.

Proposition 2.1.1. *One can choose $\mathcal{U}$ and the constants $\underline{\varepsilon} > 0$ so that the space $Y_J := Y_{\mathcal{U},J,\underline{\varepsilon}}$ is a manifold of dimension $D + |J| - 1$ where $D := \dim E_A + d$, with boundary equal to*

$$\begin{aligned} Y_J \cap \mathrm{pr}_\Delta^{-1}(\partial \Delta_J) &= \textstyle\bigcup_{I \subsetneq J} \partial_{J \setminus I} Y_J \\ &= \textstyle\bigcup_{I \subsetneq J} \bigl\{(e, x; t) \in Y_J : x \in \widetilde{U}_{IJ}, \, t \in \partial_{J \setminus I} \Delta_J\bigr\}. \end{aligned}$$

Proposition 2.1.1 shows that the boundary of Y_J lies over that of Δ_J. It is well known that the boundary of every topological manifold can be collared. The next step is to show that we can construct this collar to have a special form, with control over the components in $E_{A \setminus I}$ near the 'corner' $\mathrm{pr}_\Delta^{-1}(\partial_{J \setminus I} \Delta_J)$. However, to establish this we need to pass to a **reduction** $\mathcal{V} = (V_I)_{I\in\mathcal{I}_{\mathcal{K}}}$ of the atlas (see (1.2.10)), since this severely restricts the overlaps $\pi_{\mathcal{K}}(V_I) \cap \pi_{\mathcal{K}}(V_J)$ in $|\mathcal{K}|$ of the different chart domains. We define

$$Y_{\mathcal{V},J,\underline{\varepsilon}} := Y_{\mathcal{U},J,\underline{\varepsilon}} \cap (E_A \times V_J \times \Delta_J). \tag{2.1.5}$$

Since $Y_{\mathcal{V},J,\underline{\varepsilon}}$ is an open subset of $Y_{\mathcal{U},J,\underline{\varepsilon}}$, it is a manifold of dimension $d+N+|J|-1$ with boundary

$$\partial Y_{\mathcal{V},J,\underline{\varepsilon}} = Y_{\mathcal{V},J,\underline{\varepsilon}} \cap \partial Y_{\mathcal{U},J,\underline{\varepsilon}} \subset \bigcup_{I\subsetneq J}(E_A \times (V_J \cap \widetilde{U}_{IJ}) \times \partial_{J\smallsetminus I}\Delta_J).$$

We denote

$$\begin{aligned} \mathrm{pr}_V &: Y_{\mathcal{V},J,\underline{\varepsilon}} \to V_J, \quad (e,x;t) \mapsto x,\\ \mathrm{pr}_{|V|} &: Y_{\mathcal{V},J,\underline{\varepsilon}} \to |V_J|, \quad (e,x;t) \mapsto |x| := \pi_{\mathcal{K}}(x), \end{aligned}$$

where $\pi_{\mathcal{K}}$ is as in (1.2.8).

There is a corresponding category with objects $\bigsqcup_{J\in\mathcal{I}_{\mathcal{K}}} Y_{\mathcal{V},J,\underline{\varepsilon}}$ and morphisms given by the covering maps

(2.1.6)

$$(\rho_{IJ})_* : Y_{\mathcal{V},J,\underline{\varepsilon}} \cap \big(E_A \times \widetilde{V}_{IJ} \times \iota_{IJ}(\Delta_I)\big) \to Y_{\mathcal{V},I,\underline{\varepsilon}}, \quad \big(e,x;\iota_{IJ}(t)\big) \mapsto (e,\rho_{IJ}(x);t).$$

This category has realization

$$\underline{Y}_{\mathcal{V}} := \bigcup_{J\in\mathcal{I}_{\mathcal{K}}} Y_{\mathcal{V},J,\underline{\varepsilon}}/{\sim} \tag{2.1.7}$$

where $(e,x;t)_I \sim (e',x';t')_J$ for $|I|\le|J|$ if $I\subset J$, $e'=e,\ t'=\iota_{IJ}(t)$, and $\rho_{IJ}(x')=x$. Notice that the projection to Δ_J induce a map

$$\mathrm{pr}_\Delta :\ \underline{Y}_{\mathcal{V}} \ \to\ \Delta_{\mathcal{K}} = \bigcup_{J\in\mathcal{I}_{\mathcal{K}}} \Delta_J/{\sim} \tag{2.1.8}$$

where the simplicial complex $\Delta_{\mathcal{K}}$ (with boundary identifications induced by the face inclusions ι_{IJ}) is the topological realization of the poset $\mathcal{I}_{\mathcal{K}}$.[12] There is also a projection

$$\mathrm{pr}_{|\mathcal{V}|} : \underline{Y}_{\mathcal{V}} \to |\mathcal{V}| \sqsubset |\mathcal{K}|, \qquad [e,x;t] \mapsto |x|.$$

Remark 2.1.2. (i) The projection $\mathrm{pr}_{|\mathcal{V}|}\times\mathrm{pr}_\Delta$ induces a map

$$\underline{Y}_{\mathcal{V}} \to \|\mathcal{V}\|' \ \subset\ |\mathcal{V}|\times\Delta_{\mathcal{K}},$$

whose image $\|\mathcal{V}\|'$ is closely related to, but not the same as, the topological realization $\|\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}\|$ of the category $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}$. For example, if $x\in V_J$ is such that its image $|x| := \pi_{\mathcal{K}}(x)$ in $|\mathcal{K}|$ lies outside all the other sets $\mathrm{pr}_{\mathcal{K}}(V_I), I\ne J$, then it gives rise to a single point in $\|\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}\|$ while it corresponds to a whole simplex $x\times\Delta_J$ in $\|\mathcal{V}\|'$.[13] The partial boundary $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \subset \partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ that we consider below could be understood in terms of

[12] The topological realization of a topological category has one k-simplex for each length-k composable string of morphisms, with the 'obvious' boundary identifications. Observe that as the associated footprint covering $(F_I)_{i\in\mathcal{I}_{\mathcal{K}}}$ of the zero set X is refined, the space $\Delta_{\mathcal{K}}$ gives better and better approximations to the topology of X: indeed the Čech cohomology of $\Delta_{\mathcal{K}}$ converges to that of X.

[13] If the isotropy is trivial, there is an embedding $\|\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}\| \to \|\mathcal{V}\|'$, whose image can be described using versions of the sets $\overline{\mathrm{st}}_J^{\Delta}(|x|)$ in (2.1.12) below.

an embedding of $\|\mathbf{B}_{\mathcal K}|_{\mathcal V}^{\smallsetminus\Gamma}\|$ into $\|\mathcal V\|'$. However, we will take a more naive, geometric point of view.

(ii) We saw in Remark 1.3.8 that one can use sc-smooth partitions of unity in the polyfold setting to construct a finite dimensional manifold M with section $\mathscr{S}: M\to E_A$ that is a global chart for X. One can think of the extra coordinates $t\in\Delta_J$ (with $\sum t_i=1$) as a kind of 'external' partition of unity that gives a more indirect way to patch the different coordinate charts together. $\diamond$

The boundary collar. We now consider lifts to $Y_{\mathcal V,J,\underline\varepsilon}$ of the following collar on $\partial\Delta_J$

$$c_J^\Delta: \partial\Delta_J\times[0,w)\to\Delta_J,\quad (t,r)\mapsto(1-r|J|)\,t+r|J|\,b_J, \tag{2.1.9}$$

where $b_J=(\frac{1}{|J|},\cdots,\frac{1}{|J|})$ is the barycenter of Δ_J and $w<\frac{1}{4|J|}$; see Figure 3.2. Note that any $t\in\Delta_J$ with at least one component $t_i<w$ is in the image of this collar. In order to get maximal control over the collar we will not define it on all of $\partial Y_{\mathcal V,J,\underline\varepsilon}$ since much of $\partial Y_{\mathcal V,J,\underline\varepsilon}$ is irrelevant to the task at hand. Indeed, we are only interested in boundary points $(e,x;t)$ with $x\in\widetilde V_{IJ}$ for $I\subsetneq J$ while, by Proposition 2.1.1, a general boundary point has

$$x\in V_J\cap s_J^{-1}(E_I)=V_J\cap\widetilde U_{IJ},$$

a set that is usually strictly larger than the overlap $\widetilde V_{IJ}$ (which is defined in (1.3.3)). Although the submersion axiom (1.2.3) implies that each $\widetilde V_{IJ}$ is a submanifold in V_J of codimension $\dim(E_{J\smallsetminus I})$, we will make the following definition of the 'boundary' of V_J:

$$\partial V_J=\textstyle\bigcup_{H\subsetneq J}\widetilde V_{HJ}, \tag{2.1.10}$$

which lies over the 'boundary' $\partial|V_J|=\bigcup_{H\subsetneq J}|V_{HJ}|$ of $|V_J|$.

We will define the collar

$$c_J^Y:\partial' Y_{\mathcal V,J,\underline\varepsilon}\times[0,w_J)\to Y_{\mathcal V,J,\underline\varepsilon},$$

over a subset $\partial' Y_{\mathcal V,J,\underline\varepsilon}$ of points $(e,x;t)\in\partial Y_{\mathcal V,J,\underline\varepsilon}$ such that $x\in\partial V_J$ and t is restricted to lie in the set $\overline{\mathrm{st}}_J^\Delta(|x|)$ defined as follows. Recall that for each $x\in V_J$ the sets H such that $|x|:=\pi_{\mathcal K}(x)\in\pi_{\mathcal K}(V_H)$ (where $\pi_{\mathcal K}:V_J\to|\mathcal K|$ is the projection (1.2.8)) form a chain

$$I:=I_{\min}(|x|)=I_0(|x|)\subsetneq I_1(|x|)\subsetneq\cdots\subsetneq I_k(|x|)=I_{\max}(|x|)=:K. \tag{2.1.11}$$

If $J=I_n(|x|),\ n\le k$ we will write

$$\overline{\mathrm{st}}_J^\Delta(|x|):=\mathrm{conv}(b_{I_0},b_{I_1},\dots,b_{I_{n-1}})\subset\partial_{J\smallsetminus I_{n-1}(|x|)}\Delta_J, \tag{2.1.12}$$

for the convex hull of the barycenters of the simplices corresponding to the elements of this chain: see Figure 2.1. Note that $\overline{\mathrm{st}}_J^\Delta(|x|)$ lies in the boundary of Δ_J.

By (2.1.4), when $I\subsetneq J$ there is an embedding $\iota_{EV}:E_{A\smallsetminus I,\varepsilon_I}\times\widetilde V_{IJ}\to Y_{\mathcal V,J,\underline\varepsilon}$ given by

$$\iota_{EV}:\ (e_{A\smallsetminus I},x)\mapsto(e_{A\smallsetminus I}+b_I^{-1}\cdot s_I(x),x;b_I). \tag{2.1.13}$$

The domain $\partial' Y_{\mathcal V,J,\underline\varepsilon}\subset\partial Y_{\mathcal V,J,\underline\varepsilon}$ of the collar map c_J^Y contains all such points, as well as the lifts to $Y_{\mathcal V,J,\underline\varepsilon}$ of all points in $\mathrm{im}\,(c_H^Y)$ where $I\subsetneq H\subsetneq J$. All these points have

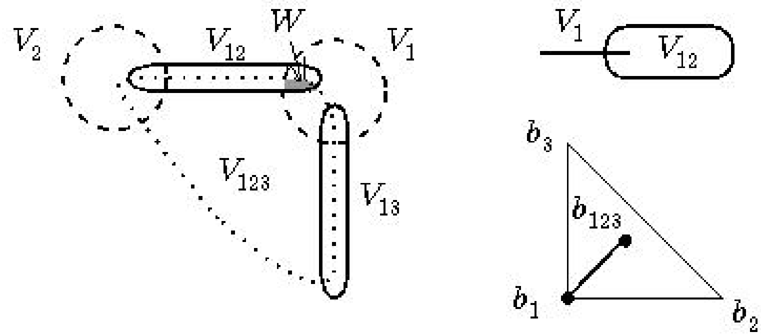

FIGURE 2.1. For x in the shaded set W, $I_{\min}(|x|) = \{1\}, I_1(|x|) = \{1,2\}, I_{\max}(|x|) = \{1,2,3\}$. The figure on the left is schematic, the top right illustrates the change in dimension from V_1 and V_{12}, while the bottom right shows $\overline{\mathrm{st}}_J^\Delta(|x|)$ for $x \in \widetilde{V}_{1,123}$.

t-coordinate close to b_I. To obtain points with more general t-coordinate we consider the following rescaling operation. Suppose given $t \in \Delta_J$ and a tuple $\mu_J = (\mu_j > 0)_{j\in A}$ such that $\mu_j = 1, j \notin J$ and $\mu_J \cdot t \in \Delta_J$. Then for any element $(e, x;\, t) \in Y_{\mathcal{V},J,\underline{\varepsilon}}$, there is a commutative diagram

$$
\begin{tikzcd}
(e, x;\, t) \arrow[r, mapsto, "\mu_J\,\cdot"] \arrow[d, mapsto, "\mathrm{pr}_{E_{A\smallsetminus J}}\times \mathrm{pr}_V"'] & \big((\mu_J)^{-1}\cdot e, x;\, \mu_J\cdot t\big) \arrow[d, mapsto, "\mathrm{pr}_{E_{A\smallsetminus J}}\times \mathrm{pr}_V"] \\
(e_{A\smallsetminus J}, x) \arrow[r, mapsto] & (e_{A\smallsetminus J}, x),
\end{tikzcd}
\tag{2.1.14}
$$

where we assume $\|(\mu_J)^{-1}\cdot e\| < \kappa\, \varepsilon_{I(x)}$ so that the top arrow has target in $Y_{\mathcal{V},J,\underline{\varepsilon}}$.

The following result concerns a reduction $\mathcal{V}$ plus choice of constants $\underline{\varepsilon}$ that are **compatible** in the sense of Definition 3.1.8. In particular this means that $(\mathcal{V},\underline{\varepsilon})$ is compatible with a fixed choice of local product structures as in (1.2.3) that are used to construct the collar maps c_J^Y.

Proposition 2.1.3. *Let $(\mathcal{V},\, \underline{\varepsilon})$ be a compatible reduction. Then for each $J \in \mathcal{I}_{\mathcal{K}}$ there is a subset $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \subset \partial Y_{\mathcal{V},J,\underline{\varepsilon}}$, a constant $w_J > 0$, and a Γ_A-equivariant embedding*

$$
c_J^Y : \partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \times [0, w_J) \to Y_{\mathcal{V},J,\underline{\varepsilon}}, \quad \big((e,x;t), r\big) \mapsto (e', x'; c_J^\Delta(t,r)),
\tag{2.1.15}
$$

with the following properties:

- $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \subset \big\{(e,x;t) : \exists I \subsetneq J, x^0 \in \widetilde{V}_{IJ},\ s.t.\ x \approx x^0,\, t \in \overline{\mathrm{st}}_J^\Delta(|x|^0)\big\}$.
- c_J^Y *is* **compatible with the projections** *to $E_{A\smallsetminus\bullet}$ as follows: for each $I \subsetneq J$, we have $\iota_{EV}(E_{A\smallsetminus I,\varepsilon_I} \times \widetilde{V}_{IJ}) \ \subset\ \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ and*

$$
\begin{aligned}
& c_J^Y\big((e,x;t), 0\big) = (e,x;t), \quad \forall (e,x;t) \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}, \quad \textit{and} \\
& \mathrm{pr}_{E_{J\smallsetminus I}}(e) = 0 \implies c_J^Y\big((e,x;t), r\big) = \big(e, x; c_J^\Delta(t,r)\big).
\end{aligned}
\tag{2.1.16}
$$

and

$$\mathrm{pr}_{E_{A\smallsetminus I}} \circ c^Y_J\big(\iota_{EV}(e,x),r\big) = \mathrm{pr}_{E_{A\smallsetminus I}}(e), \quad \forall (e,x) \in E_{A\smallsetminus I,\varepsilon_I} \times \widetilde{V}_{IJ}, \tag{2.1.17}$$

- *the sets* $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *are* **compatible with covering maps** *as follows: if* $I \subsetneq H \subsetneq J$, *then the relevant part of the image of* c^Y_H *lifts to the domain* $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *of* c^Y_J. *More precisely, if* $(e,x;t) \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *has* $x \in \widetilde{V}_{IH} \cap \widetilde{V}_{HJ}$, *then* $(e, \rho_{HJ}(x), t)$ *is in the domain* $\partial' Y_{\mathcal{V},H,\underline{\varepsilon}}$ *of* c^Y_H *and for all* $r \in [0, w_H)$ *there is* $(e',x',t') \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *with* $x' \in \widetilde{V}_{HJ}$ *such that*

$$c^Y_H((e,x;t),r) = \big(e', \rho_{HJ}(x'), t'\big). \tag{2.1.18}$$

  Further, the restriction of c^Y_H *to* $Y_{\mathcal{V},H,\underline{\varepsilon}} \cap \mathrm{pr}_V^{-1}(\widetilde{V}_{IH} \cap V_{HJ})$ *has a well defined lift (also called* c^Y_H*) to* $Y_{\mathcal{V},J,\underline{\varepsilon}}$ *such that*

$$(\mathrm{pr}_{HJ})_*\big(c^Y_H(e,x;t),r\big) = \big(c^Y_H(e,\rho_{HJ}(x),t),r\big), \quad r \in [0,w_H),\ x \in \widetilde{V}_{IJ} \cap \widetilde{V}_{HJ}, \tag{2.1.19}$$

  where $(\mathrm{pr}_{HJ})_*$ *is as in* (2.1.6).
- *each* $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *is* **invariant under rescaling** *as follows: if* $(e,\, x;\, t) \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *where* $t \in \overline{\mathrm{st}}^{\Delta}_H(|x|)$ *then for all* μ_H *such that* $\mu_H \cdot t \in \overline{\mathrm{st}}^{\Delta}_H(|x|)$ *we have*

$$\mu_H \cdot \big(e, x;\, t\big) := \big(\mu_H^{-1} \cdot e, x;\, \mu_H \cdot t\big) \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$$

  and

$$\begin{aligned}\mathrm{pr}_{E_{A\smallsetminus H}\times V} \circ c^Y_J\big((e,\, x;\, t),r\big) &= \\ &\mathrm{pr}_{E_{A\smallsetminus H}\times V} \circ c^Y_J\big((\mu_H^{-1}\cdot e, x;\, \mu_H \cdot t),r\big) \in E_{A\smallsetminus H} \times V_J;\end{aligned} \tag{2.1.20}$$

- *the collar maps* c^Y_J *are* **compatible with shrinkings** *as follows: if* $(\mathcal{V}', \underline{\varepsilon}') \sqsubset (\mathcal{V}, \underline{\varepsilon})$ *is another compatible reduction, then there are constants* $0 < w'_J < w_J$ *such that the restrictions of the maps* c^Y_J *to* $\partial' Y_{\mathcal{V}',J,\underline{\varepsilon}'} := \partial Y_{\mathcal{V}',J,\underline{\varepsilon}'} \cap \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *have all the above properties with respect to the constants* w'_J.

By Lemma 3.1.10, any reduction $\mathcal{V}''$ has a shrinking $\mathcal{V} \sqsubset \mathcal{V}''$ that is compatible with respect to some choice of constants $\underline{\varepsilon}$ and hence supports a collar $\big(c^Y_J\big)_{J\in\mathcal{I}_\mathcal{K}}$ as in Proposition 2.1.3. Further, we show in Corollary 3.2.3 that $(\mathcal{V}^\infty, \underline{\varepsilon}^\infty)$ has a further nested shrinkings that are collar compatible in the following sense.

Definition 2.1.4. *Let* $(\mathcal{V}^\infty, \underline{\varepsilon}^\infty)$ *be a compatible reduction, with collars* $\big(c^{Y,\infty}_J\big)_{J\in\mathcal{I}_\mathcal{K}}$. *We say that a shrinking* $(\mathcal{V}, \underline{\varepsilon}) \sqsubset (\mathcal{V}^\infty, \underline{\varepsilon}^\infty)$ *is* **collar compatible** *if it is compatible as in Definition 3.1.8 and if for all* $J \in \mathcal{I}_\mathcal{K}$ *the collar map* $c^{Y,\infty}_J$ *restricts to a collar* $(c^Y_J)_J$ *on* $(\mathcal{V}, \underline{\varepsilon})$ *whose widths* w_J *satisfy* $\sqrt{\varepsilon_I} \le w_J$ *for all* $I \subsetneq J$.

2.2. **Construction of the category $\mathbf{M}$ and functor $\mathscr{S} : \mathbf{M} \to \mathbf{E}_A$.** In (1.3.5), the component M_J of $\mathrm{Obj}_{\mathbf{M}}$ was defined as

$$M_J = E_{A\smallsetminus J,\varepsilon_J} \times V_J, \tag{2.2.1}$$

which is a manifold of dimension $d+\dim E_A$. We take $M_{IJ} := E_{A\smallsetminus J,\varepsilon_J}\times V_{IJ}$, and define the map $\tau_{IJ}:\widetilde{M}_{IJ}\to M_{IJ}$ that attaches M_J to M_I to have domain a suitable open subset $\widetilde{M}_{IJ}\subset M_J$ and to extend the atlas structural map

$$\rho_{IJ}:\{0\}\times\widetilde{V}_{IJ}\to\{0\}\times V_{IJ}\subset M_{IJ}\subset M_I.$$

We require that τ_{IJ} is a Γ_A-equivariant covering map, induced by a free action of $\Gamma_{J\smallsetminus I}$. Further, to obtain a category, they must be compatible with composition: i.e. for $I\subset H\subset J$

$$\tau_{HJ}\circ\tau_{IH}=\tau_{IJ}\ \text{on}\ \widetilde{M}_{IJ}\cap\widetilde{M}_{HJ}\cap\tau_{HJ}^{-1}(\widetilde{M}_{IH})=\widetilde{M}_{IJ}\cap\widetilde{M}_{HJ}. \tag{2.2.2}$$

(Note that by (1.3.3) any two of the sets $\widetilde{M}_{IJ},\widetilde{M}_{HJ},\tau_{HJ}^{-1}(\widetilde{M}_{IH})$ determine the third.) For maximal elements J of $\mathcal{I}_{\mathcal{K}}$, we then define $\mathscr{S}_J:M_J\to E_A$ as the projection

$$\mathscr{S}_J:M_J\to E_A,\quad (e_{J\smallsetminus A},x)\mapsto(e_{J\smallsetminus A},s_J(x)). \tag{2.2.3}$$

The above should be considered as the default formula for $\mathscr{S}_J$, that holds at points $(e_{J\smallsetminus A},x)\in M_J$ where x is far from any overlap V_{JK} with $J\subsetneq K$. However, in general it must be modified in ways explained in Example 2.2.1 below.

Before giving the general formulas for $\mu_J,\tau_{IJ},\mathscr{S}_J$, we discuss an example. Part (i) shows the role of the collar in constructing τ_{IJ}, while part (ii) explains the relevance of the collar's compatibility with projections and rescaling to the proof of the composition rule(2.2.2) . The usefulness of considering multiple collar compatible shrinkings $(\mathcal{V}^n,\underline{\varepsilon}^n)$ will also become apparent. We will use cutoff functions $\big(\beta_{IJ}:V_I\to[0,1]\big)_{I\subsetneq J}$ of the following form: if $\mathcal{V}\sqsubset\mathcal{V}'$ we have

$$\operatorname{supp}(\beta_{IJ})\subset\textstyle\bigcup_{I\subsetneq H\subset J}V'_{IH},\qquad\text{and}\qquad\bigcup_{I\subsetneq H\subset J}\overline{V}_{IH}\subset\operatorname{int}(\beta_{IJ}^{-1}(1)). \tag{2.2.4}$$

Example 2.2.1. (Attaching the M_J). We begin by considering the case when the isotropy groups are trivial, so that $\tau_{IJ}:\widetilde{M}_{IJ}\to M_{IJ}$ is a homeomorphism. It is then easiest to define its inverse

$$\alpha_{IJ}:=\tau_{IJ}^{-1}:M_{IJ}\to\widetilde{M}_{IJ},$$

since $M_{IJ}\subset M_I$ is defined to be the product $E_{A\smallsetminus I,\delta_I}\times V_{IJ}$ (where V_{IJ} is defined in (1.3.3)) while $\widetilde{M}_{IJ}$ will simply be defined as the image $\alpha_{IJ}(M_{IJ})$. As in [MW2], we use the notation $\phi_{IJ}:=\rho_{IJ}^{-1}:V_{IJ}\to\widetilde{V}_{IJ}$ for the inverse of the atlas structural map ρ_{IJ}.

(i) Consider the case when there are two basic charts with labels $1,2$. Then $\mathbf{M}$ has three components:[14]

$$M_1=E_{2,\delta_2}\times V_1,\quad M_2=E_{1,\delta_1}\times V_2,\quad M_{12}:=V_{12},$$

[14] Here we simplify notation by writing $M_{12}:=M_{\{1,2\}},M_{1,12}:=M_{\{1\}\{1,2\}}$ and so on. For an example of this construction, see §3.4..

where we assume $(\mathcal{V}, \underline{\delta})$ is collar compatible as in Definition 2.1.4. In particular, this means that for $i = 1, 2$ we have $\delta_i \le w_{12}^2$, where w_{12} is the width of the collar c_{12}^Y. We first define the attaching maps $\alpha_{1,12}$ and $\alpha_{2,12}$, and then the sections $\mathscr{S}_I$.

We define $\alpha_{1,12}$ as a composite $E_{2,\varepsilon_2} \times V_{1,12} \to Y_{\mathcal{V},12,\underline{\varepsilon}} \to M_{12}$:

$$
\begin{aligned}
\alpha_{1,12}\big((e_2, x)\big) &= \mathrm{pr}_V\Big(c_{12}^Y\big((s_1(x), e_2, \phi_{IJ}(x); b_1), r\big)\Big) \quad \text{with } r := \sqrt{\|e_2\|} \\
&= \mathrm{pr}_V\Big(\big(e_1', e_2, x'; (1- \ r, r)\big)\Big), \\
&= x' \in V_{12} = M_{12},
\end{aligned}
\tag{2.2.5}
$$

where $b_1 = (1, 0)$ is the barycenter of Δ_1 considered as a point in Δ_2, we have used formula (2.1.9) for c_{12}^{Δ}, and we have used the fact from (2.1.17) that e_2 is unchanged by c_{12}^Y. We note the following.

- Because $(\mathcal{V}, \delta)$ is collar compatible, the collar width satisfies $w_{12} > \sqrt{\|\varepsilon_2\|}$, and this map is defined for all elements in $M_{1,12}$ by (2.1.17).
- Because the collar variable $r := \sqrt{\|e_2\|}$ vanishes for the points $(0, x) \in M_{1,12}$, the map $\alpha_{1,12}$ does extend the inclusion $\phi_{IJ} : V_{IJ} \to \widetilde{V}_{IJ}$ by (2.1.16). Hence $\alpha_{1,12}$ is well defined, and for small enough δ_i has image disjoint from the similarly defined map $\alpha_{2,12}$.
- Because the points $(e, x; t) \in Y_{\mathcal{V},J,\underline{\varepsilon}}$ satisfy $s_J(x) = t \cdot e_J$ and we chose $r = \sqrt{\|e_2\|}$, we have
$$r\,\|e_2\| = (\|e_2\|)^{3/2} = \|s_2(x')\|,$$
so that $r = \|s_2(x')\|^{1/3}$ is determined by x'.
- To see that $\alpha_{1,12}$ is injective, notice that because c_J^Y is injective it suffices to check that the other elements, e_1', e_2, r that appear in the tuple $\big(e_1', e_2, x'; (1 - r, r)\big) \in Y_{\mathcal{V},12,\underline{\varepsilon}}$ are determined by $x' \in V_2$. But we saw above that $r = \|s_2(x')\|^{1/3}$, so that the equations $s_1(x) = (1 - t)e_1',\ s_2(x) = te_2$ determine e_1', e_2.

We now define $\mathscr{S}_{12} := s_{12} : M_2 = V_2 \to E_{12}$, and define $\mathscr{S}_i$ on $\alpha_{i,12}^{-1}(\widetilde{M}_{i,12})$ by pullback: thus on this set

$$\mathscr{S}_i(e_j, x) = \big(\|e_j\|^{1/2} e_j,\ s_i(\alpha_{i,12}(e_j, x)\big), \quad i \ne j,$$

has the form claimed in (1.3.8). We then extend $\mathscr{S}_i$ to the rest of M_i by patching it to the default map $(e_j, x) \mapsto (e_j\, s_i(x))$ via a cutoff function β_i as in (2.2.4):

$$\mathscr{S}_i(e_j, x) = \beta_{i,12}(x)\big(\|e_j\|^{1/2} e_j,\ s_i(\alpha_{i,12}(e_j, x)\big) + (1 - \beta_{i,12}(x))\big(e_j,\ s_i(x)\big) \in E_{12}. \tag{2.2.6}$$

For this to be well defined, we need $\alpha_{i,12}$ to extend to a neighborhood of $M_{i,12}$ in M_i. But we can always assume that $\mathcal{V}$ is a shrinking of some other reduction $\mathcal{V}'$. Then because the collar extends over $\mathcal{V}'$ we may extend $\alpha_{i,12}$ over the corresponding set $M_{i,12}'$ by using the above formula (2.2.5). It is then clear that $\mathscr{S}_i^{-1}(0) = \{0\} \times s_i^{-1}(0)$.

(ii) Now suppose that there are three basic charts with labels $1,2,3$ that all intersect as in Figure 2.1. We assume that all $E_i \neq 0$, and again explain how to choose the constants δ_i, and define the attaching maps α_{IJ} and sections $\mathscr{S}_I$ that involve the vertex 1, namely those with labels $1, 12, 13$, and 123. It is now convenient to assume that we have four nested collar compatible shrinkings $(\mathcal{V}^1, \underline{\varepsilon}^1) \sqsubset (\mathcal{V}^2, \underline{\varepsilon}^2) \sqsubset (\mathcal{V}^3, \underline{\varepsilon}^3) \sqsubset (\mathcal{V}^4, \underline{\varepsilon}^4)$ of $\mathcal{V}'$. Correspondingly, with $I \subset \{1,2,3\}$ and $k \le \ell \le 4$ we define

$$M_I^k := E_{I,\varepsilon_i^k} \times V_I^k, \quad M_{IH}^{k,\ell} = E_{I,\varepsilon_i^k} \times V_{IH}^{k,\ell} \subset M_I^k,$$

where

$$V_{IH}^{k,\ell} = V_I^k \cap V_{IH}^\ell = V_I^k \cap \pi_{\mathcal{K}}^{-1}\big(\pi_{\mathcal{K}}(V_H^\ell)\big)$$

We aim to define a category with basic domains of the form $M_I^{|I|}$ and compatible morphisms $\alpha_{IH} : M_{IH}^{|I|,|H|} \to M_H^{|H|}$. However, to make these continuous and to define the corresponding maps $\mathscr{S}_I$ we have to define transition functions on larger sets such as $M_{IH}^{|I|+1,|H|}$.

The methods in (i) easily adapt to define the maps α_{IH} for $|I| \ge 2$ and $\mathscr{S}_I : M_I \to E_A$ for $|I| \ge 2$. Indeed, if $J := \{1,2,3\}$ then

$$\alpha_{1i,J} : M_{1i,J}^{2,3} \to M_J^3$$

can be defined much as in (2.2.5). The only new point is that because Δ_{1i} is a 1-simplex, we have to decide how to lift $V_{1i,J}$ to $\partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ in order to use the collar. For now, we use the default choice given by the embedding ι_{EV} in (2.1.13), i.e. we embed it over the barycenter b_{1i} of Δ_{1i} which we identify with the corresponding point $\iota_{1i,J}(b_{1i})$ in Δ_J. Thus with $i \ne j,\ i,j \in \{2,3\}$, we define

$$\alpha_{1i,J} : M_{1i,J}^{2,3} \to M_J^3 :\ (e_j, x) \mapsto x' \quad \text{as follows:} \tag{2.2.7}$$

$$\begin{aligned} E_{3,\varepsilon_i^2} \times V_{1i}^{2,3} \ni (e_j, x) &\longmapsto c_J^Y\Big(\big(\iota_{EV}(e,x)\big), r\Big), \quad r = \sqrt{\|e_j\|} \\ &= \big(e'_{1i}, e_j, x'; c_J^\Delta(b_{1i}, r)\big) \in Y_{\mathcal{V}^3,J,\underline{\varepsilon}^3} \\ &\longmapsto x' \in M_J^3. \end{aligned}$$

Since r depends on e_3 and hence on $s_3(x')$ as above, it follows as before that $\alpha_{1i,J}$ is injective. Notice also that if $x \in V_{1i,J}^{2,\ell}$ the point $\phi_{1i,J}(x)$ would lie in $\widetilde{V}_{1i,J}^\ell$ as would its image x' under the collar map since the collar maps preserve the shrinkings by Proposition 2.1.3. Taking $\ell = 4$ here, we may therefore define $\mathscr{S}_{12}$ by pullback from $\mathscr{S}_J$ on $M_{12,J}^{2,3}$, tapering it off to the product $s_{12} \times \mathrm{pr}_{E_j}$ outside the larger set $E_{j,\varepsilon_{1i}} \times V_{1i,J}^{2,4}$ by using the cutoff functions $\beta_{1i,J}$ as in (2.2.6).

The main new task is to define

$$\alpha_{1,J} : M_{1,J}^{1,3} \to M_J^3, \text{ so that } \alpha_{1,J} := \alpha_{1i,J} \circ \alpha_{1,1i} \text{ in } M_{1,J}^{1,3} \cap M_{1,1i}^{1,2}.$$

If $x \in V_{1,J}^{1,3} \smallsetminus \bigcup_{i=2,3} V_{1,1i}^{1,3}$, (i.e. x is "far" from $V_{1,1i}^{1,2}$) then we may define

$$\alpha_{1,J}(e_{23}, x) = \mathrm{pr}_{E_3 \times V}\Big(c_J^Y\big((s_1(x), e_{23},\ \phi_{1,J}(x),\ b_1), r\big)\Big), \quad r = \sqrt{\|e_{23}\|}, \tag{2.2.8}$$

as in (2.2.5). Hence the lift of $\alpha_{1,J}(e_{23},x)$ to $Y_{\mathcal{V}^3,J,\underline{\varepsilon}^3}$ lies over the ray $c^\Delta_J\bigl(b_1\times[0,w_0]\bigr)\subset\Delta_J$. On the other hand, the composite $\alpha_{1i,J}\circ\alpha_{1,1i}$ first uses the collar c^Y_{1i} for b_1 in Δ_{1i} and then the collar c^Y_J of b_{1i} in Δ_J, and hence its natural lift to $Y_{\mathcal{V}^3,J,\underline{\varepsilon}^3}$ is rather different. We interpolate between these two maps as follows, where we take $i=2$ for clarity, and use cut-off functions $\beta_{1,12}$ as in (2.2.4), with support in $V^{1,3}_{1,12}$ and that equal 1 near the closed set $\overline{V}^{1,2}_{1,12}\sqsubset V^{1,3}_{1,12}$. Thus with $x\in V^{1,3}_{1,12}\cap V^{1,3}_{1,J}$, we define

$$
\begin{aligned}
(e_{23},x)\longmapsto\;& c^Y_{12}\Bigl(\bigl(s_1(x),e_{23},\phi_{1,12}(x);b_1\bigr),\ r\Bigr)\ \text{where } r:=\beta_{1,12}(x)\sqrt{\|e_2\|}\\
&\qquad =:(e_1',e_{23},x';1-r,r)\ \in\ Y_{\mathcal{V}^3,12,\underline{\varepsilon}^3}\\
\longmapsto\;& c^Y_J\bigl((e_1',e_{23},\phi_{12,J}(x');\,1-r,r),r'\bigr)=:\bigl(e_1'',e_{23},x'';t''\bigr)\in Y_{\mathcal{V}^3,J,\underline{\varepsilon}^3}\\
&\qquad \text{where}\quad r':=\max\bigl((1-\beta_{1,12}(x))\sqrt{\|e_2\|},\sqrt{\|e_3\|}\bigr)\\
\longmapsto\;& x''=:\alpha_{1,J}\bigl((e_{23},x)\bigr)\ \in\ M^3_J=V^3_J
\end{aligned}
\tag{2.2.9}
$$

Note the following.

- Here (as in (2.1.19)) we consider c^Y_{12} to be the lift to $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ of the collar for $\partial' Y_{\mathcal{V},12,\underline{\varepsilon}}$, and the composite $c^Y_J\circ c^Y_{12}$ is defined by (2.1.18).
- The above map $(e_{23},x)\mapsto x''$ is continuous, and equals that given in (2.2.8) when $\beta_{1,12}(x)=0$ because $\|e_{23}\|=\max\{\|e_i\|: i=2,3\}$ by definition.
- If $x\in V^{1,3}_{1,J}\cap V^{1,2}_{1,12}\subset V^{1,3}_{1,J}\cap(\beta^{-1}_{1,12}(1))$, then
$$\alpha_{1,12}(e_{23},x)=\alpha_{12,J}\circ\alpha_{1,12}(e_{23},x).$$
  Indeed, the invariance of the collar under rescaling in (2.1.20) shows that applying the second collar map at $(1-r,r)$ with $r'=\sqrt{\|e_3\|}$ and then projecting to M^3_J gives the same result as rescaling, then applying the second collar at b_{12} with the same r', and then projecting to M^3_J. Note that by (2.1.17) this last claim holds even if $e_3=0$, so that the second collar map has $r'=0$ when $\beta_{1,12}(x)=1$.
- It remains to check that this map $(e_{23},x)\mapsto x''$ is injective. Since the first two maps in (2.2.8) are injective, it suffices to check that the projection $(e_1'',e_{23},x'';t'')\to x''$ is injective. But both collar maps preserve e_2,e_3 by the extended corner control in (2.1.17). Hence, for $i=2,3$ we know $\|e_i\|$ and therefore t_i'' from $s_i(x'')=t_i''e_i$. Since $\sum_i t_i''=1$, we therefore know t'' and hence also $e''=(e_1'',e_{23})$.

Finally we define $\mathscr{S}_1$ by pullback via $\alpha_{1,*}$ over $E_{23,\varepsilon_1}\times\bigcup_{\{1\}\subsetneq J}V_{1,J}$, extending to the rest of M_1 via a cutoff function $\beta_{1,J}$. However, to do this we need the pullback of $\mathscr{S}_1$ to be compatibly defined on a set that is larger than that on which we ultimately want $\mathscr{S}_1$ to equal the pullback. But we can arrange that the identity $\alpha_{1,J}=\alpha_{12,J}\circ\alpha_{1,12}$ actually holds on a neighborhood of the closure of $V^{1,2}_{1,12}\cap V^{1,3}_{1,J}$, since in (2.2.9) $\beta_{1,12}=1$ on a neighborhood of $\overline{V}^{1,3}_{1,J}$, and we can always extend the domain of $\alpha_{1,12}$ to $V^{1,3}_{1,2}$. Therefore we can imitate the formula in (2.2.6).

(iii) If the isotropy is nontrivial, then we can still adopt the above approach, but now must interpret α_{IJ} as a local Γ_x-invariant inverse to τ_{IJ} and then define $\widetilde{M}_{IJ}$ to be the Γ_A-orbit of its image. Further, we must make equivariant constructions, but this is possible since the collar is equivariant, so that all the above formulas are appropriately equivariant. $\diamond$

The following result proves Proposition 1.3.3.

Proposition 2.2.2. *Suppose given a good atlas $\mathcal{K}$ on X. Then there is a reduction $\mathcal{V}$ and set of constants $\underline{\delta} = (\delta_I)_{I\in\mathcal{I}_\mathcal{K}} > 0$, such that the following properties hold with*

$$M_I := E_{A\smallsetminus I,\delta_I} \times V_I, \quad M_{IJ} := E_{A\smallsetminus I,\delta_I} \times V_{IJ}.$$

For each $I \subset J$ there are open sets $\widetilde{M}_{IJ} \subset M_J$ and Γ_A-equivariant maps

$$\tau_{IJ} : \widetilde{M}_{IJ} \to M_{IJ}$$

such that

(i) *$\widetilde{M}_{IJ}$ is a product $E_{A\smallsetminus J,\delta_I} \times \widetilde{M}^0_{IJ}$ where $\widetilde{V}_{IJ} \subset \widetilde{M}^0_{IJ} \subset V_J$, and $\tau_{IJ} = \mathrm{id}_E \times \tau^0_{IJ}$ where $\tau^0_{IJ} : \widetilde{M}_{IJ} \to E_{J\smallsetminus I,\delta_I} \times V_{IJ}$ quotients out by a free action of $\Gamma_{J\smallsetminus I}$ that extends to a neighborhood of $\widetilde{M}^0_{IJ}$ in V_J. Moreover*

$$\tau^0_{IJ}|_{\widetilde{V}_{IJ}} = \rho_{IJ}. \tag{2.2.10}$$

(ii) *for $I \subsetneq J \subsetneq K$ we have*

$$\tau_{JK}(\widetilde{M}_{IK} \cap \widetilde{M}_{JK}) = \widetilde{M}_{IJ} \cap M_{JK}, \quad \text{and} \quad \tau_{IK} = \tau_{IJ} \circ \tau_{JK}. \tag{2.2.11}$$

(iii) *For each J there is $\mathscr{S}_J : M_J \to E_A$ such that for all $J \subset K$, we have*

$$\mathscr{S}_J \circ \tau_{JK} = \mathscr{S}_K|_{\widetilde{M}_{JK}}, \quad \mathscr{S}_J^{-1}(E_J) \subset \{0\} \times V_J \ \text{ and} \tag{2.2.12}$$
$$\mathscr{S}_J(0, x) = (0, s_J(x)).$$

Further, if the initial atlas $\mathcal{K}$ is oriented, then so is $\mathbf{M}$.

Proof. Fix a shrinking $\mathcal{G}^0 = (G^0_I)_{I\in\mathcal{I}_\mathcal{K}}$ of the footprint cover. By Corollary 3.2.3 we may choose a family of nested collar compatible shrinkings as above

$$\psi^{-1}(\mathcal{G}^0) \sqsubset (\mathcal{V}^1, \underline{\varepsilon}^1) \sqsubset \cdots \sqsubset (\mathcal{V}^{\kappa+1}, \underline{\varepsilon}^{\kappa+1}) \sqsubset \mathcal{U}^\infty,$$

with collar widths that increase with m. The projection $\pi_\mathcal{K} : U^\infty_I \to |\mathcal{K}|$ quotients out by Γ_I and its restrictions to the $\mathcal{V}^m$ have the property that

$$\pi_\mathcal{K}(\overline{V^k_I}) \cap \pi_\mathcal{K}(\overline{V^\ell_J}) \neq \emptyset \iff I \subset J \text{ or } J \subset I.$$

For $m \le \ell$ we denote $V^{m,\ell}_{IJ} := V^m_I \cap \pi_\mathcal{K}^{-1}(\pi_\mathcal{K}(V^\ell_J))$, and for $m \le |I|,\ m \le \ell \le |J|$ define

$$M^m_I := E_{A\smallsetminus I,\varepsilon^m_I} \times V^m_I, \qquad M^{m,\ell}_{IJ} = E_{A\smallsetminus I,\varepsilon^m_I} \times V^{m,\ell}_{IJ}. \tag{2.2.13}$$

For each $I \subsetneq J$ and $m \le |I|$ we will define $\mathscr{S}_J : M^{|J|}_J \to E_A$, a subset $\widetilde{M}^{m,\ell}_{IJ} \subset M^\ell_J$ and a Γ_A-equivariant covering map

$$\tau^{m,\ell}_{IJ} : \widetilde{M}^{m,\ell}_{IJ} \to M^{m,\ell}_{IJ}$$

with the following properties:

(a) $\tau_{IJ}^{m,\ell}$ has product form as in (i) and quotients out by a free action of $\Gamma_{J\smallsetminus I}$ on $(\widetilde{M}_{IJ}^{m,\ell})^0$;

(b) for all $m \le m' \le |I|, \ell \le \ell' \le |J|$, $\widetilde{M}_{IJ}^{m,\ell} \subset \widetilde{M}_{IJ}^{m',\ell'}$ and $\tau_{IJ}^{m',\ell'}|_{\widetilde{M}_{IJ}^{m,\ell}} = \tau_{IJ}^{m,\ell}$;

(c) if $I \subsetneq H \subsetneq J$ then $\tau_{IJ}^{|I|,|J|} = \tau_{HJ}^{|H|,|J|} \circ \tau_{IH}^{|I|,|H|}$ on their common domain; moreover this domain maps onto

$$E_{A\smallsetminus I,\varepsilon_I} \times \Big(V_{IJ}^{|I|,|J|} \cap \rho_{IJ}(V_{HJ}^{|H|,|J|})\Big) \subset M_I^{|I|}$$

(d) if $I \subsetneq J$ then $\mathscr{S}_I \circ \tau_{IJ} = \mathscr{S}_J$ on $\widetilde{M}_{IJ}^{|I|,|J|}$;

(e) $\mathscr{S}_J^{-1}(0) = \{0\} \times s_J^{-1}(0) \subset M_J^{|J|}$.

In the end we will take $M_I := M_I^{|I|}, M_{IJ} := M_{IJ}^{|I|,|J|}$ with the corresponding sets $\widetilde{M}_{IJ}^{|I|,|J|}$, and the restrictions of the maps τ_{IJ} and $\mathscr{S}_I$. In particular, $\delta_I = \varepsilon_I^{|I|}$.

For simplicity, we first assume that the isotropy groups are trivial. As in Example 2.2.1 (see in particular (2.2.9)) for $I \subsetneq J$ we will define a family of injective maps

$$\alpha_{IJ} : M_{IJ}^{|I|+1,|J|+1} \cap \{(e,x) \,\big|\, \|e_{J\smallsetminus I}\| < \varepsilon_I^{|I|}\} \;\;\to\;\; M_J^{|J|+1}, \quad \lambda \ge 1$$

(where $e_{J\smallsetminus I} := \mathrm{pr}_{E_{J\smallsetminus I}}(e)$) with well defined restrictions

$$\alpha_{IJ} := \alpha_{IJ}\big|_{M_{IJ}^{m,k}} : M_{IJ}^{m,k} \to M_J^k, \quad m \le |I|+1, k \le |J|+1, m \le k, \tag{2.2.14}$$

such that

$$\alpha_{IJ} = \alpha_{HJ} \circ \alpha_{IH} \text{ on } M_{IJ}^{|I|,|J|} \cap \alpha_{IH}^{-1}(M_{HJ}^{|H|,|J|}), \quad \forall\, I \subsetneq H \subsetneq J. \tag{2.2.15}$$

Then we define

$$\widetilde{M}_{IJ}^{m,\ell} = \alpha_{IJ}(M_{IJ}^{m,\ell}), \quad \tau_{IJ} = \alpha_{IJ}^{-1}.$$

With this, conditions (b), (c) will hold, and (a) is trivial.

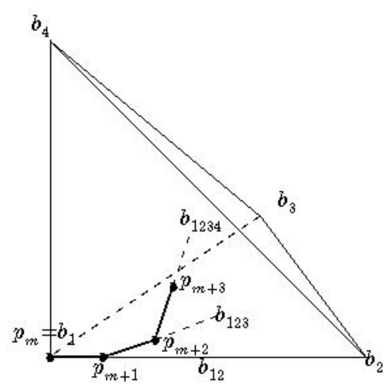

FIGURE 2.2. The path $\mathcal{P}(x)$ with $I_m = \{1\}, \ldots, I_{m+3} = \{1,2,3,4\}$.

To define $\alpha_{IJ}(e,x)$ we consider the chain of length $k(|x|)$ formed by the sets H such that $|x|\in |V_H|^{|H|+1}$

$$I_{\min}(|x|)=I_0(|x|)\subsetneq I_1(|x|)\subsetneq\cdots\subsetneq I_k(|x|)=I_{\max}(|x|), \tag{2.2.16}$$

modifying the definition of $\overline{\mathrm{st}}_J^{\Delta}(|x|)$ from (2.1.12) accordingly. Extending the procedure in (2.2.9), if $I=I_m(|x|)$ we define $\alpha_{IJ}(e,x)$ by applying collar maps in $Y_{\mathcal{V}^{\kappa+1},I_k,\underline{\varepsilon}^{\kappa+1}}$ a total of $k(|x|)-m$ times with initial points $p_{n-1}\in \mathrm{pr}_\Delta^{-1}(\Delta_{I_n})$ and collar lengths r_n for $n=m+1,\dots,k(|x|)$. In fact, it is useful to think of applying the iterated collar map that lies over the path $\mathcal{P}(x)$ in $\overline{\mathrm{st}}^D e_J(|x|)$ with the following vertices:

$$p_m=b_{I_m},\;\; p_n=(1-r_n)p_{I_{n-1}}+r_n b_{I_n}=c^{\Delta}_{I_n}(p_{n-1},r_n),\quad m<n\le k(|x|), \tag{2.2.17}$$

(see Figure 2.2) where the r_n are described below. Note that by the collar compatibility with covering maps in (2.1.19) it makes no difference whether at the nth step we apply the collar map over the segment p_{n-1},p_n in $Y_{\mathcal{V},I_n,\underline{\varepsilon}}$ (where $\mathcal{V}:=\mathcal{V}^\kappa$) and then lift to the next level $Y_{\mathcal{V},I_{n+1},\underline{\varepsilon}}$, or whether we first lift all the way to $Y_{\mathcal{V},I_{\max},\underline{\varepsilon}}$, and then apply the collar maps. We take the second approach, first lifting the initial point $(e_{A\smallsetminus I},x)$ to

$$(e_{A\smallsetminus I}, b_{I_m}^{-1}\cdot s_{I_m},\, \phi_{I_m i_k}(x), b_{I_m})\in \partial_{I_{\max}\smallsetminus I_m} Y_{\mathcal{V},I_{\max},\underline{\varepsilon}^{\kappa+1}}\cap \partial'_{I_{\max}\smallsetminus I_0} Y_{\mathcal{V},I_{\max},\underline{\varepsilon}^{\kappa+1}}$$

and then applying successive collar maps that remain in the boundary $\partial Y_{\mathcal{V},I_{\max},\underline{\varepsilon}^{\kappa+1}}$ until the very last step. Similarly, by the collar compatibility with shrinkings we can work in $\mathcal{V}:=\mathcal{V}^\kappa$ rather than in the different $\mathcal{V}^m$.

To complete this definition of $\alpha_{IJ}(e_{A\smallsetminus I},x)$ it remains to define the lengths $r_n=r_n(x)$ for $m+1\le n\le k$. To achieve consistency with coordinate changes, for each $I\in\mathcal{I}_{\mathcal{K}}$, we choose a cutoff function $\chi_I:|\mathcal{K}|\to[0,1]$ such that

$$\mathrm{supp}\,(\chi_I)\subset \pi_{\mathcal{K}}(V_I^{|I|+1}),\qquad \chi_I^{-1}(1)\subset \pi_{\mathcal{K}}(V_I^{|I|}), \tag{2.2.18}$$

and for each J denote its pullback to the set $V_J^{|J|+1}$ by the same letter. Then, writing $a_n:=\sqrt{\|e_{I_n\smallsetminus I_{n-1}}\|}$ and $\chi_i:=\chi_{I_i}$, we define

(2.2.19)

$$\begin{aligned} r_{m+1}(x) &:= \chi_{m+1}(x)a_{m+1},\;\; r_{m+2}(x)=\chi_{m+2}(x)\max\big((1-\chi_{m+1}(x))a_{m+1},a_{m+2}\big),\dots\\ r_n(x) &:= \chi_n(x)\Big(\max_{m<j\le n}\lambda_j a_j\Big),\quad \lambda_j:=\prod_{i=j}^{n-1}(1-\chi_i(x)),\;\; j<n,\;\; \lambda_n:=1. \end{aligned}$$

Note the following.

- In order for the collar maps to be defined over $\mathcal{P}(x)$, we must have $r_n(x)<w_{I_n}$ for all n. But
$$r_n\le \max_{m<j\le n}\sqrt{\|e_{I_n\smallsetminus I_{n-1}}\|}<\sqrt{\|e_{A\smallsetminus I}\|}<\sqrt{\varepsilon_{I_m}}<w_{I_n}$$
for all $m>n$ because $(\mathcal{V},\underline{\varepsilon})$ is collar compatible: see Definition 2.1.4.

- To see that the path $\mathcal{P}(x)$ varies continuously with x, it suffices to check continuity for a sequence of points $x^\nu \to x^\infty$ for which just one of the functions χ — say χ_s — changes from a positive value to zero. But in this case (assuming that e is fixed) the functions $r_i(x)$ are continuous for $i < s$, while for $i \ge s$ we have
$$\begin{aligned} &a_i^\nu = a_i^\infty, i \neq s, s+1, \quad a_s^\infty = \max(a_s^\nu, a_{s+1}^\nu), \\ &\lim_\nu r_i(x^\nu) = r_i(x^\infty), i < s, \quad \lim_\nu r_s(x^\nu) = 0, \quad \lim_\nu r_i(x^\nu) = r_{i-1}(x^\infty), i > s. \end{aligned}$$
- If $x \in V_{IH}^{|I|+1,|H|}$ for $H = I_s$ where $m < s < n$, then $\chi_s(x) = 1$. In this case, we can divide $\mathcal{P}(x)$ into two independent segments at the point p_s, because the lengths $r_n(x), n > s$ no longer depend on $a_i, i \le s$ since $\lambda_i = 0$ for $i \le s$. Further, the second part of $\mathcal{P}(x)$ projects to the path $\mathcal{P}(\phi_{IH}(x))$ under the natural projection
$$\big(\operatorname{conv}(b_{I_0}, \dots, b_{I_{\max(|x|)}})\big) \smallsetminus \big(\operatorname{conv}(b_{I_0}, \dots, b_{I_s})\big) \;\to\; \operatorname{conv}(b_{I_s}, \dots, b_{I_{\max}}).$$

With these formulas in hand, we now define the maps α_{IJ} and sections $\mathscr{S}_I$ by downwards recursion on $|I|$. For $|J| = \kappa := \max\{|J| : J \in \mathcal{I}_{\mathcal{K}}\}$, we define
$$\begin{aligned} &\mathscr{S}_J := \mathscr{S}'_J, \quad \text{where} \\ &\mathscr{S}'_J : M_J \to E_A, \quad (e_{A-J}, x) \mapsto (e_{A \smallsetminus J}, s_J(x)). \end{aligned}$$
If $|I| = \kappa - 1$, for $x \in V_{IJ}^{|I|+1,|J|}$ the path $\mathcal{P}(|x|)$ has one segment of length $\chi_I a_k := \chi_I \sqrt{\|e_{J\smallsetminus I}\|}$, and we define $\alpha_{IJ} : M_{IJ}^{|I|,|J|} \to M_J^{|J|}$ by applying the collar map as in (2.2.7). For these values of x we have $\chi_I(x) = 1$. However the fact that we have defined α_{IJ} over the larger set $V_{IJ}^{|I|+1,|J|}$ means that the function
$$\mathscr{S}_I := \prod_{J : I \subsetneq J} (1 - \chi_J) \mathscr{S}'_I + \sum_{J : I \subsetneq J} \chi_J \alpha_{IJ}^*(\mathscr{S}_J) : V_I^{|I|} \to E_I. \tag{2.2.20}$$
is well defined and is compatible under pullback from V_J.

Let us now suppose that maps $\alpha_{IJ} : V_{IJ}^{|I|+1,|J|+1} \to V_J^{|J|+1}$, and functions $\mathscr{S}_I : V_I^{|I|} \to E_A$ have been defined for all $I \subsetneq J$ with $|I| > k$ so as to satisfy conditions (2.2.14),(2.2.15), and consider I with $|I| = k$. Because there are no transition functions $\alpha_{II'}$ between these sets V_I we can work separately with each such I. Then define $\alpha_{IJ}(x)$ for $x \in V_{IJ}^{|I|+1,|J|+1}$ by applying the collar maps c_{HJ}^Y over the part, called $\mathcal{P}_{IJ}(x)$ below, of the path $\mathcal{P}(x)$ from $p_m = b_I$ (where $I = I_m(|x|)$) to p_q, where $J = I_q(|x|)$.

We check the properties of α_{IJ} as follows.

- The map α_{IJ} depends continuously on x because we saw above that the path $\mathcal{P}(x)$ depends continuously on x, and because by (2.1.17) the collar map along a path segment of length 0 is the identity.
- Both $\widetilde{M}_{IJ}$ and α_{IJ} have the product form required by (a) because the collar map c_J^Y does not change the components of $e_{A\smallsetminus I}$ that lie in $E_{A\smallsetminus J}$; cf. (2.1.17).
- We repeatedly use the fact that the collar is compatible with all the shrinkings to show that (b) holds.

- To prove the composition formula (c), we use the fact proved above that when $x \in M_{IH}^{|I|+1,|H|}$, the path $\mathcal{P}_{IJ}(x)$ divides into two independent segments, the first of which is simply $\mathcal{P}_{IH}(x)$, while the second projects onto $\mathcal{P}_{HJ}(\phi_{IJ}(x))$. Now use the invariance of the collar map under rescaling (2.1.20).
- To see that α_{IJ} is injective, notice first that the path $\mathcal{P}_{IJ}(x)$ is determined by x. Hence the collar maps applied to the lift $(e', \phi_{IJ}(x), b_I)$ of $(e,x) \in M_{II}$ to $Y = Y_{\mathcal{V},J,\underline{\varepsilon}}$ give a point in Y that lies over a point $t_x \in \Delta_J$, that is determined by $\mathcal{P}(x)$ because the collar c_J^Y lifts c_J^Δ by (2.1.15). But the collar maps are injective, as is the projection $Y_{\mathcal{V},J,\underline{\varepsilon}} \cap \mathrm{pr}_\Delta^{-1}(t_x)$ to M_J.

Finally, we define $\mathscr{S}_I$ as in (2.2.20). This clearly has the properties required in (iii).

This completes the proof when the isotropy groups are trivial. In general, we argue as above, taking ϕ_{IJ} to be a local inverse to the covering map ρ_{IJ} in the atlas, and then defining τ_{IJ} to be the equivariant extension of α_{IJ}^{-1}. For this to make sense we just need to ensure that α_{IJ} is invariant under the appropriate stabilizer subgroup. But this holds because of the equivariance of the collar map, and the fact that once the shrinkings $(\mathcal{V}^k, \underline{\varepsilon}^k)$ are chosen, the only other choice in the above construction is that of the cutoff functions χ_I in (2.2.18) whose pullbacks to the sets V_I are equivariant.

It remains to check that (i) holds, i.e. that $\widetilde{M}_{IJ}$ is a product of the form $E_{A\smallsetminus J,\delta_I} \times \widetilde{M}_{IJ}^0$, that τ_{IJ}^0 extends ρ_{IJ}, and finally that τ_{IJ}^0 quotients out by a free action of $\Gamma_{J\smallsetminus I}$ on $\widetilde{M}_{IJ}^0$. The first two claims are clear.

To establish the third, we must define an appropriate action of $\Gamma_{J\smallsetminus I}$ on $\widetilde{M}_{IJ}^0$. If $\Gamma_{J\smallsetminus I}$ acts trivially on $E_{J\smallsetminus I}$, then this action is simply the restriction of the given action of $\Gamma_{J\smallsetminus I}$ on V_J. However, in general this is not the case, and the new action

$$\Gamma_{J\smallsetminus I} \times \widetilde{M}_{IJ}^0 \to \widetilde{M}_{IJ}^0, \quad (\gamma, x) \mapsto \gamma * x$$

is described as follows. Notice first that because the collar c_J^Y is Γ_A-equivariant and injective, each point $x_0 \in \widetilde{V}_{IJ} \subset \widetilde{M}_{IJ}^0$ with $\tau_{IJ}^0(x_0) = (0, x_0') \in E_{J\smallsetminus I,\delta_I} \times V_{IJ}$ has a neighborhood $\mathcal{N}(x_0)$ on which τ_{IJ}^0 is injective and has image $\mathcal{N}'$ of product form, namely $\mathcal{N}' = E_{J\smallsetminus I,\delta_I} \times \mathcal{O}' \subset E_{J\smallsetminus I,\delta_I} \times V_{IJ}$. Further, $\Gamma_{J\smallsetminus I}$ acts on $\mathcal{N}'$ via its action on $E_{J\smallsetminus I,\delta_I}$, since it fixes the points of V_{IJ}. If $\tau_{IJ,x_0}^{-1} : \mathcal{N}' \to \mathcal{N}(x_0)$ is the local inverse to τ_{IJ}, we now define

$$\gamma * x = \gamma \cdot_J \tau_{IJ,x_0}^{-1}(\gamma^{-1} \cdot_I \tau_{IJ}(x)), \quad x \in \mathcal{N}. \tag{2.2.21}$$

where for clarity we have written $x \mapsto \gamma \cdot_I x$ (resp. $x \mapsto \gamma \cdot_J x$ for the standard action of $\gamma \in \Gamma_{J\smallsetminus I}$ on $E_{J\smallsetminus I,\delta_I} \times V_{IJ}$ (resp. on M_{IJ}^0). Then

$$\begin{aligned} \tau_{IJ}(\gamma * x) &= \tau_{IJ}\big(\gamma \cdot_J \tau_{IJ,x_0}^{-1}(\gamma^{-1} \cdot_I \tau_{IJ}(x))\big) \\ &= \gamma \cdot_I \big(\tau_{IJ} \circ \tau_{IJ,x_0}^{-1}(\gamma^{-1} \cdot_I \tau_{IJ}(x)\big) \;=\; \tau_{IJ}(x), \end{aligned}$$

as required. This action extends to a neighborhood of the closure of $\widetilde{M}_{IJ}^0$ since it is determined by τ_{IJ}, and hence by the collar, both of which can be extended.

Finally, if $\mathbf{K}$ is oriented, then so are all the manifolds $Y_{\mathcal{U},J,\underline{\varepsilon}}$. Therefore the charts M_I inherit natural orientations that are preserved by the structural maps. □

Remark 2.2.3. (The smooth case.) Note first that if we apply the above construction to a smooth atlas (i.e. one that satisfies the smooth submersions condition in (1.2.5)), then the manifold $Y_{\mathcal{V},J,\underline{\varepsilon}}$ is not smooth because its defining equation $s_J(x) = t\cdot e$ is not smooth at points $t \in \partial\Delta$. Further the attaching map $\alpha_{1,12}$ in (2.2.5) is given by the collar, which by (3.2.3) has the form $(e_2, x) \mapsto x' := \phi(\|e_2\|^{1/2}e_2, x)$, where ϕ is the local product structure along $\widetilde{V}_{IJ}$ in (1.2.3). Thus, even when ϕ is a diffeomorphism, $\alpha_{1,12}$ does not have a smooth inverse along the submanifold $e_2 = 0$. Thus, just as in standard blow-up constructions, in order to obtain a smooth category $\mathbf{M}$ from a smooth atlas one needs to choose a smoothing of $Y_{\mathcal{V},J,\underline{\varepsilon}}$ along its boundary.

Alternatively, one could use a different construction that avoids introducing the manifold Y. Instead, one can construct the all important collar structure used to define the maps τ_{IJ} by using the exponential map with respect to a suitable family of metrics on the sets V_J. Indeed, recall that by the smooth tangent bundle condition (1.2.5) the derivative $\mathrm{d}s_{J\smallsetminus I}$ induces an isomorphism from the normal bundle $T^\perp(\widetilde{V}_{IJ})$ of $\widetilde{V}_{IJ}$ in V_J to the product $E_{J\smallsetminus I,\varepsilon_I} \times \widetilde{V}_{IJ}$. To explain the idea, let us suppose for simplicity that that the cover $\mathcal{V}$ is refined so that the group $\Gamma_{J\smallsetminus I}$ acts freely on the components on $\widetilde{V}_{IJ}$, so that the restriction of τ_{IJ} to each component is a diffeomorphism onto V_{IJ} . Then we can think of V_{IJ} as a subset of $\widetilde{V}_{IJ}$ and the task is to define a consistent family of injections $\alpha_{IJ} : E_{J\smallsetminus I,\varepsilon_I} \times V_{IJ} \to V_J$. To this end, choose a family of Γ_I-invariant Riemannian metrics g_I on V_I and constants ε_I that are compatible in the following sense:

- for each $I \subsetneq J$, $\widetilde{V}_{IJ}$ is a totally geodesic submanifold of (V_J, g_J) and
$$(\rho_{IJ})_*(g_J|_{\widetilde{V}_{IJ}}) = g_I|_{V_{IJ}}.$$
- $0 < \varepsilon_I < \varepsilon_J$ if $I \subsetneq J$;
- for each $I \subsetneq J$, the g_J-exponential map along directions perpendicular to $\widetilde{V}_{IJ}$ defines an embedding $\alpha_{IJ} : E_{J\smallsetminus I,\varepsilon_I} \times V_{IJ} \to V_J$;
- the corners are locally flat, i.e. if $x \in \widetilde{V}_{IJ} \cap \widetilde{V}_{HJ}$ for $I \subsetneq H \subsetneq J$ then
$$\alpha_{IJ}\big(e_{J\smallsetminus H} + e_{H\smallsetminus I},\ x\big) = \alpha_{HJ}\big(e_{J\smallsetminus H},\ \alpha_{IH}(e_{H\smallsetminus I},\ x)\big).$$

The last condition means that the composition rule holds directly, without having to introduce analogs of the paths $\mathcal{P}(x)$. Of course, the choice of the g_I, ε_I requires some attention to detail as in the proof of Lemma 3.1.10 below; see also the construction of the perturbation section in [MW2]. Thus one begins with a family of shrinkings $\mathcal{V}^\kappa \sqsubset \cdots \sqsubset \mathcal{V}^1 \sqsubset \mathcal{V}^0$ of an initial reduction $\mathcal{V}^0$, where $\kappa := \max\{|J| \mid J \in \mathcal{I}_{\mathcal{K}}\}$ and then chooses metrics g_J on $V_J^{|J|}$, starting with J of length $|J| = 1$, that satisfy the above conditions for the submanifolds $\widetilde{V}_{IJ}^{|J|}$ of $V_J^{|J|}$ for some constant $\varepsilon'_I > 0$. Finally, once g_J is defined on V_J^κ for all J, one chooses suitable constants ε_J, now starting with maximal $|J|$ and working down. Further details are left to the reader. ◇

3. Further details and constructions

In §3.1 we first define the notion of a compatible shrinking $(\mathcal{U}, \underline{\varepsilon})$ and prove Proposition 2.1.1. We then introduce the more intricate notion of a compatible reduction $(\mathcal{V}, \underline{\varepsilon})$, which involves not only the compatibility of $\mathcal{V}$ with a set of constants $\underline{\varepsilon}$ but also its compatibility with a suitable cover of the set of overlaps in $|\mathcal{V}|$, properties that are essential for the proof in §3.2 that $Y_{\mathcal{V},J,\underline{\varepsilon}}$ has a collar that satisfies the conditions listed in Proposition 2.1.3.

3.1. **Shrinkings and the manifold Y.** We assume given an ambient preshrunk tame[15] atlas $\mathcal{K}^\Omega$ with chart domains $\mathcal{U}^\Omega$, together with a tame shrinking $\mathcal{U}^\infty \sqsubset \mathcal{U}^\Omega$, and then choose a further shrinking $\mathcal{F}^0$ of the footprints $\mathcal{F}^\infty$ of $\mathcal{U}^\infty$. For short we write $\psi^{-1}(\mathcal{F}^0) \sqsubset \mathcal{U}^\infty \sqsubset \mathcal{U}^\Omega$. By the submersion axiom and the precompactness of $\widetilde{U}^\infty_{IK}$ in $\widetilde{U}^\Omega_{IK}$ for each $I \subsetneq K$, we may choose a finite set of points $z_\alpha \in \widetilde{U}^\Omega_{IK}$, constants $\varepsilon_\alpha > 0$, and Γ_{z_α}-equivariant local homeomorphisms

$$\phi^E_{IK,z_\alpha} : E_{K\smallsetminus I,\varepsilon_\alpha} \times \widetilde{W}_{IK,z_\alpha} \to U^\Omega_K, \qquad 1 \le \alpha \le A_{IK}, \tag{3.1.1}$$

where $\widetilde{W}_{IK,z_\alpha} \subset \widetilde{U}^\Omega_{IK}$ such that

$$\begin{aligned} &s_{K\smallsetminus I} \circ \phi^E_{IK,z_\alpha}(e,y) = e, \quad \text{and} \\ &\widetilde{U}^\infty_{IK} \;\subset \bigcup_{1\le\alpha\le A_{IK}} \widetilde{W}_{IK,z_\alpha} \subset \widetilde{U}^\Omega_{IK}, \qquad \forall\, I \subsetneq K. \end{aligned} \tag{3.1.2}$$

We may and will assume that each ϕ^E_{IK,z_α} is Γ_K-equivariant. (To do this, first shrink the $\widetilde{W}_{IK,z_\alpha}$ so that they have disjoint images under the group $\Gamma_K/\Gamma_{z_\alpha}$, and then replace them by their orbit under $\Gamma_K/\Gamma_{z_\alpha}$.)

Definition 3.1.1. *Given $\psi^{-1}(\mathcal{F}_0) \sqsubset \mathcal{U}^\infty \sqsubset \mathcal{U}^\Omega$ as above, we will say that a shrinking $\mathcal{U}$ and set of positive constants $\underline{\varepsilon} := (\varepsilon_K)_{K\in\mathcal{I}_\mathcal{K}}$ are $(\mathcal{G}_0,\mathcal{U}^\infty)$-**compatible** if the following holds:*

(a) $0 < \kappa\,\varepsilon_I < \varepsilon_K$ *if* $I \subsetneq K$ *(see* (1.3.2)*);*

(b) $\psi^{-1}(\mathcal{F}^0) \sqsubset \mathcal{U} \sqsubset \mathcal{U}^\infty$*;*

(c) $s_I(\overline{U_I}) \subset E_{I,\varepsilon_I}$ *for all* I*;*

(d) *for all* $I \subsetneq K$*, each* $z \in \widetilde{U}_{IK} \subset U_K$ *has a neighborhood* $\widetilde{\mathcal{O}}_{IK} \subset \widetilde{U}_{IK}$ *such that one of the homeomorphisms* ϕ^E_{IK,z_α} *in* (3.1.1) *restricts to give a map*

$$\phi^E_{IK} : E_{K\smallsetminus I,(\kappa+1)\varepsilon_I} \times \widetilde{\mathcal{O}}_{IK} \to U^\Omega_K \tag{3.1.3}$$

that is a homeomorphism to its image, where $\kappa := \max\{|K| : K \in \mathcal{I}_\mathcal{K}\}$.

For simplicity we call the pair $(\mathcal{U}, \underline{\varepsilon})$ a **compatible shrinking**.

Lemma 3.1.2. *Suppose given $\psi^{-1}(\mathcal{G}_0) \sqsubset \psi^{-1}(\mathcal{F}_0) \sqsubset \mathcal{U}^\infty \sqsubset \mathcal{U}^\Omega$ as above. Then there is an $(\mathcal{F}_0,\mathcal{U}^\infty)$-compatible shrinking $(\mathcal{U}, \underline{\varepsilon})$.*

[15] For terminology see §1.2.

Proof. First choose any tame shrinking $\mathcal{U}'$ such that $\psi^{-1}(\mathcal{F}^0) \sqsubset \mathcal{U}' \sqsubset \mathcal{U}^\infty$, which is possible by [MW1, Prop. 3.3.5]. Then each set U'_I is covered by a finite number of the sets $\widetilde{W}_{IK,z_\alpha}$ in (3.1.1) and we choose any set of constants $\underline{\varepsilon}$ satisfying (a), and also so that $\varepsilon_I < \frac{\varepsilon_\alpha}{\kappa+1}$ for all relevant α. Then, if we define $U_I := U'_I \cap s_I^{-1}(E_{I,\varepsilon_I})$, property (d) holds. Further, $\mathcal{U} := (U_I)$ is a tame shrinking of $\mathcal{U}^\infty$ because the coordinate changes commute with the section maps s_I and preserve the norms $\|\cdot\|$ on E_A. (More precisely

$$\widehat{\phi}_{IK} \circ s_I \circ \rho_{IK} = s_K : \widetilde{U}_{IK} \to E_K,$$

where the canonical inclusion $\widehat{\phi}_{IK} : E_I \to E_K$ preserves $\|\cdot\|$, i.e. $\|\widehat{\phi}_{IK}(e)\| = \|e\|$.) Hence $\mathcal{U}$ satisfies (c) and (b), as required. $\square$

From now on, we fix $(\mathcal{F}^0,\mathcal{U}^\infty)$, and hence cease to refer to them explicitly. The following lemma provides a proof of Proposition 2.1.1.

Lemma 3.1.3. *If $(\mathcal{U}, \underline{\varepsilon})$ is compatible, then for each J, $Y_{\mathcal{U},J,\underline{\varepsilon}}$ is a manifold of dimension $D := N + d + |J| - 1$, where $N = \dim E_A$, with boundary equal to*

$$\begin{aligned} Y_{\mathcal{U},J,\underline{\varepsilon}} \cap \mathrm{pr}_\Delta^{-1}(\partial\Delta) &= \textstyle\bigcup_{I \subsetneq J} \partial_{J \smallsetminus I} Y_{\mathcal{U},J,\underline{\varepsilon}} \\ &= \textstyle\bigcup_{I \subsetneq J} \big\{(e,x;t) : x \in \widetilde{U}_{IJ}, t \in \partial_{J\smallsetminus I}\Delta_J\big\}. \end{aligned}$$

Proof. We show that each point $(e,x;t) \in Y_{\mathcal{U},J,\underline{\varepsilon}}$ has a neighborhood homeomorphic to an open subset of $(\mathbb{R}_{\ge 0})^k \times \mathbb{R}^{D-k}$, where $k = \#\{j \in J \mid t_j = 0\}$. Thus, the projection $\mathrm{pr}_\Delta : Y_{\mathcal{U},J,\underline{\varepsilon}} \to \Delta_J$ is compatible with the boundary structure of Δ_J.

First consider a point $(e,x;t) \in Y_{\mathcal{U},J,\underline{\varepsilon}}$ with $t_i \neq 0$ for all $i \in J$. Then the coordinates $e_j, j \in J$, are determined by (x,t) via the requirement $s_J(x) = t \cdot e|_J$ while the components of $e|_{A\smallsetminus J} := (e_i)_{i\in A\smallsetminus J}$ can vary freely. Hence the tuple $(e,x;t)$ is uniquely determined by the point $(e|_{A\smallsetminus J}, x; t) \in E_{A\smallsetminus J} \times U_J \times \operatorname{int}\Delta_J$, and so has a manifold neighborhood of dimension $N + d + |J| - 1$.

It remains to define boundary charts at the points $(e^0, x^0, t^0) \in Y_{\mathcal{U},J,\varepsilon}$ with

$$t^0 \in \partial\Delta_J = \textstyle\bigcup_{I\subsetneq J} \partial_{J\smallsetminus I}\Delta_J =: \bigcup_{I\subsetneq J} \operatorname{int}\Delta_I.$$

Suppose first that

$$I(x^0) := \{i : s_i(x^0) \neq 0\} = \{i : t_i^0 > 0\} =: I(t^0) =: I,$$

so that $x^0 \in \widetilde{U}_{IJ}$. By (3.1.2), there is a neighborhood $\widetilde{\mathcal{O}}$ of x^0 in $\widetilde{U}_{IJ}$ that is contained in one of the sets $\widetilde{W}_{IJ,z_\alpha}$ in (3.1.1), and below we denote by ϕ the associated map ϕ^E_{IJ,z_α}. There is a corresponding neighborhood of (e^0, x^0, t^0) in

$$\partial_{J\smallsetminus I} Y_{\mathcal{U},J,\underline{\varepsilon}} \cap \{(e,x;t) : e|_{J\smallsetminus I} = 0\}$$

given by

$$\widetilde{\mathcal{O}}'_{I,J,\underline{\varepsilon}} := \big\{(e_{A\smallsetminus J} + t_I^{-1}\cdot s_I(x), x; t_I) \;\big|\; x \in \widetilde{\mathcal{O}}, t_I \approx t_I^0, \|e_{A\smallsetminus J}\| < \kappa\varepsilon_I\big\} \subset\; \partial_{J\smallsetminus I} Y_{\mathcal{U},J,\underline{\varepsilon}}.$$

Now consider the map

$$
\begin{aligned}
(3.1.4)\qquad \psi : E_{J\smallsetminus I,(\kappa+1)\varepsilon_I} \times [0,\delta)^{|J\smallsetminus I|} \times \widetilde{\mathcal{O}}'_{I,J,\underline{\varepsilon}} &\longrightarrow Y_{\mathcal{U}^\Omega,J,\underline{\varepsilon}},\\
\big(e_{J\smallsetminus I}, r_{J\smallsetminus I}, (e_{A\smallsetminus J} + t_I^{-1}\cdot s_I(x), x; t_I)\big) &\longmapsto \\
\big(e_{A\smallsetminus J} + e_{J\smallsetminus I} + (\lambda t_I)^{-1}\cdot s_I(x'),\ x';\ \lambda t_I + r_{J\smallsetminus I}\big),\\
\text{where } x' := \phi\big(r_{J\smallsetminus I}\cdot e_{J\smallsetminus I}, x\big) \text{ for } \phi := \phi^E_{IJ,z_\alpha},\\
\text{and } \lambda := 1 - |r_{J\smallsetminus I}| = 1 - \textstyle\sum_{j\in J\smallsetminus I} r_j.
\end{aligned}
$$

To see that ψ does have image in $Y_{\mathcal{U}^\Omega,J,\underline{\varepsilon}}$ for sufficiently small $\delta > 0$ and $\widetilde{\mathcal{O}}$, we check the conditions in (2.1.1) as follows.

- By (3.1.2)
$$r_{J\smallsetminus I}\cdot e_{J\smallsetminus I} = s_{J\smallsetminus I}\circ\phi\big(r_{J\smallsetminus I}\cdot e_{J\smallsetminus I}, x\big) = s_{J\smallsetminus I}(x'),$$
so that the image $(e,x;t)$ of ψ does satisfy the equation $s_J(x) = t\cdot e$ if $x' \approx x^0$ and $\delta > 0$ is sufficiently small.
- Next, we check that $s_J(x') \in E_{A,\varepsilon_{I(x')}}$. To this end, note first that because we started by assuming $I(x^0) = I$, the definition of $Y_{\mathcal{U},J,\underline{\varepsilon}}$ implies that $\|s_I(x^0)\| < \varepsilon_I$. Second, because $s_I(x') \approx s_I(x^0)$, we have $s_I(x') < \varepsilon_I$ for sufficiently small $\delta, \widetilde{\mathcal{O}}$. But if $r_{J\smallsetminus I} \ne 0$ we have $I(x') \supsetneq I(x^0)$ so that $\varepsilon_I < \frac{1}{\kappa}\varepsilon_{I(x')}$ by (1.3.2). Therefore because $\lambda \approx 1$ and we use the sup norm on the product E_A, we have
$$s_J(x') = e_{J\smallsetminus I} + (\lambda t_I)^{-1}\cdot s_{J\smallsetminus I}\circ\phi\big(r_{J\smallsetminus I}\cdot e_{J\smallsetminus I}, x\big) \in E_{A,\varepsilon_{I(x')}}$$
for sufficiently small $\delta > 0$.
- Since elements in the domain of ψ have $\|e_{A\smallsetminus J}\| < \kappa\varepsilon_I < \varepsilon_{I(x')}$, elements in its image also satisfy this condition.

It is now easy to check that ψ is a local homeomorphism that equals the identity map when $r_{J\smallsetminus I} = 0$ since $\phi(0,x) = x$. Hence its restriction to a suitable open subset of its domain provides a local boundary chart for $Y_{\mathcal{U},J,\underline{\varepsilon}}$ at (e^0, x^0, t^0).

It remains to consider the case when $I = I(x^0) \subsetneq H = I(t^0)$. In this case, write $t^0 = t^0_I + t^0_{H\smallsetminus I}$.Then the above formula for ψ must be modified as follows: Denote the elements of $I(t^0)$ by $t'_H = t'_I + t'_{H\smallsetminus I}$. Then, for $r_{J\smallsetminus I} \approx 0$, we define

$$
\begin{aligned}
(3.1.5)\qquad \psi\Big(e_{J\smallsetminus I}, r_{J\smallsetminus I}, \big(e_{A\smallsetminus J} + (t'_H)^{-1}\cdot s_I(x), x;\ t'_H\big)\Big) &= \\
\Big(e_{A\smallsetminus J} + e_{J\smallsetminus I} + (t''_I)^{-1}\cdot s_I(x''),\ x'';\ t''_J\Big)
\end{aligned}
$$

where

- x varies in a neighborhood $\widetilde{\mathcal{O}} \subset \widetilde{U}_{HJ}$ of x^0;
- $\lambda < 1$ is chosen so that $t''_J := \big((t_i)''\big)_{i\in J}$ has $|t''_J| := \sum_{i\in J} t''_i = 1$, where
$$t''_i = \lambda t'_i,\ \text{if } i \in I, \qquad t''_h = \lambda t'_h + r_h \text{ if } i \in H\smallsetminus I, \qquad t''_j = r_j \text{ if } j \in J\smallsetminus H$$
- $x'' = \phi\big(t''_{J\smallsetminus I}\cdot e_{J\smallsetminus I}, x\big) \in V_J$.

Then one can check as above that im ψ is a neighborhood of (e^0, x^0, t^0) in $Y_{\mathcal{U},J,\underline{\varepsilon}}$. This completes the proof. $\square$

Remark 3.1.4. Notice that in (3.1.5) the coordinates $r_{H\smallsetminus I} \in \mathbb{R}^{H\smallsetminus I}$ parametrize directions tangent to $\partial_{J\smallsetminus H} Y_{\mathcal{U},J,\underline{\varepsilon}}$, while the coordinates $r_{J\smallsetminus H} \in \mathbb{R}^{J\smallsetminus H}$ parametrize the directions normal to the codimension $|J\smallsetminus H|$-face $\partial_{J\smallsetminus H} Y_{\mathcal{U},J,\underline{\varepsilon}}$. $\diamond$

We now define and construct **compatible reductions** $(\mathcal{V}, \underline{\varepsilon})$. In order to prove Proposition 2.1.3, it turns out that we need more control over the sets $\mathcal{O}_{IK}$ in Definition 3.1.1 (d). Because of the consistency requirements on the collar, it is not sufficient to choose the $\mathcal{O}_{IK}$ separately for each pair $I \subsetneq K$; rather they must be chosen consistently for all pairs as we now describe. Further, because the collar has fixed width and image in $Y_{\mathcal{V},J,\underline{\varepsilon}}$, the product maps in (d) must have image in V_K rather than in V_K^Ω.

Note first that because the local product structures

$$\phi^E_{IK,z_\alpha} : E_{K\smallsetminus I,\varepsilon_\alpha} \times \widetilde{W}_{IK,z_\alpha} \to U_K^\Omega, \qquad 1 \le \alpha \le A_{IK}, \tag{3.1.6}$$

in (3.1.1) are equivariant and satisfy $s_{K\smallsetminus I} \circ \phi^E_{IK,z_\alpha}(e, y) = e$, they descend via ρ_{HK} whenever $I \subsetneq H \subsetneq K$. More precisely, for such H

$$\phi^E_{IK,z_\alpha} : E_{H\smallsetminus I,\varepsilon_\alpha} \times (\widetilde{U}_{HK} \cap \widetilde{W}_{IK,z_\alpha}) \to \widetilde{U}^\Omega_{HK} = s_K^{-1}(E_H)$$

is the lift of a well-defined map

$$\phi^E_{IH,\rho_{HK}(z_\alpha)} : E_{H\smallsetminus I,\varepsilon_\alpha} \times \rho_{HK}(\widetilde{U}_{HK} \cap \widetilde{W}_{IK,z_\alpha}) \to U_H^\Omega. \tag{3.1.7}$$

Before defining the notion of compatible reduction, we describe certain covers of the set $\overline{\mathcal{OL}}(|\mathcal{V}|)$ of ‘overlaps’ in $|\mathcal{V}|$, which is the image in $|\mathcal{V}|$ of the relevant part of the boundary of $\bigcup_J Y_{\mathcal{V},J,\underline{\varepsilon}}$. See Figure 3.1.

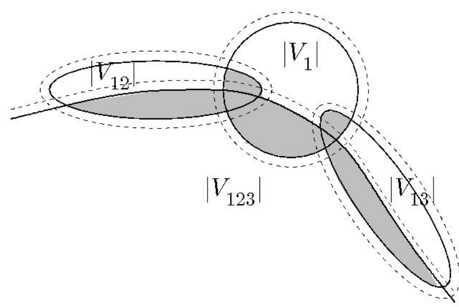

FIGURE 3.1. Here $\overline{\mathcal{OL}}(|\mathcal{V}|)$ is shaded, and the sets V_I^Ω are given by dotted lines; no $|W_\alpha|$ meets both $|V_{12}|$ and $|V_{13}|$.

Definition 3.1.5. *Given a subset $|W| \subset |\mathcal{V}^\Omega|$ we say that $W \subset V_I^\Omega$ is a* **lift** *of $|W|$ if*

$$\pi_{\mathcal{K}}(W) = |W|, \quad W = V_I^\Omega \cap \pi_| K k^{-1}(|W|),$$

i.e. W is a ‘full’ inverse image of $|W|$ in $\mathcal{V}^\Omega$.

Lemma 3.1.6. *If $(\mathcal{U},\underline{\varepsilon})$ is compatible, and $\mathcal{V}\sqsubset\mathcal{V}^\Omega\sqsubset\mathcal{U}$ is any nested reduction, denote by*

$$\overline{\mathcal{OL}}(|\mathcal{V}|):=\bigcup_{I\subsetneq K}|\overline{V}_{IK}|\ \subset\ |\mathcal{V}^\Omega|, \tag{3.1.8}$$

the closure of the set of overlaps in $|\mathcal{V}|$. Then we may cover $\overline{\mathcal{OL}}(|\mathcal{V}|)$ by a finite number of sets $(|W_\alpha|)_{1\le\alpha\le N}$, where for each α there is $\widetilde{W}_{IK,z_\alpha}$ as in (3.1.6) *such that*

$$W_\alpha:=\widetilde{V}^\Omega_{IK}\cap\pi_K^{-1}(|W_\alpha|)\ \subset\ \widetilde{W}_{IK,z_\alpha}$$

is a lift of $|W_\alpha|$. Moreover, we require that I is minimal and K is maximal in the sense that

(i) *W_α is an open subset of $\widetilde{V}^\Omega_{IK}$,*
(ii) *$|V_H|\cap|W_\alpha|\ne\emptyset\Longrightarrow I\subset H\subset K$.*

In this situation, we say that $\mathcal{V}$ is **adapted to the cover** $(|W_\alpha|)_{1\le\alpha\le N}$.

Proof. Choose compatible shrinkings $\mathcal{V}\sqsubset\mathcal{V}^1\sqsubset\cdots\sqsubset\mathcal{V}^\kappa\sqsubset\mathcal{V}^\Omega\sqsubset\mathcal{U}$. Work by downwards induction on $|I|=\ell\le\kappa-1$ so that at the ℓth stage we have a covering $(|W^\ell_\alpha|)_{\alpha\in B_\ell}$ of

$$\bigcup_{I\subsetneq K,\ell\le|I|}|V_{IK}|$$

with lifts W^ℓ_α satisfying (i) and such that (ii) holds if $|H|\ge\ell$. When $\ell=\kappa-1$, the existence of the finite covering holds by the precompactness of $|\mathcal{V}|$ in $|\mathcal{U}|$ while (ii) is easy to arrange because the sets $|V_H|$ with $|H|=\ell$ are disjoint. Now let us suppose that this holds for $\ell+1$ with the sets $(|W^{\ell+1}_\alpha|)_{\alpha\in B_{\ell+1}}$ and consider the statement for ℓ.

The covering $(|W^\ell_\alpha|)$ will consist of sets of two kinds:

- If $|W^{\ell+1}_\alpha|$ lifts to $W^{\ell+1}_\alpha\subset V^\Omega_I$ where $|I|\ge\ell+1$ is as in (i) then we take the set $|W'_\alpha|$ where
$$W'_\alpha:=W^{\ell+1}_\alpha\smallsetminus\bigcup_{|H|=\ell}cl(\widetilde{V}^\ell_{HI}).$$
This is open in V^Ω_I since we have removed a closed set, and satisfies (ii) for ℓ. These sets cover
$$\big(\bigcup_{I\subsetneq K,\ell+1\le|I|}|V^\ell_{IK}|\big)\smallsetminus\big(\bigcup_{H\subsetneq K,|H|=\ell}cl(|V^\ell_{HK}|)\big).$$
- Next add a finite cover of the compact set $\bigcup_{H\subsetneq K,|H|=\ell}cl(|V^\ell_{HK}|)$ by sets $|W_\alpha|$ whose lifts lie in V^Ω_{HK} where $|H|=\ell$. These obviously satisfy (ii).

This completes the proof. □

Remark 3.1.7. (i) If $\mathcal{V}$ is adapted to the cover $(|W_\alpha|)_{1\le\alpha\le N}$, and $\mathcal{V}'\sqsubset\mathcal{V}$ is any shrinking, then $\mathcal{V}'$ is also adapted to the cover $(|W_\alpha|)_{1\le\alpha\le N}$.
(ii) If $I\subsetneq H$ then in general $\widetilde{V}_{IH}$ is not closed in V_H. Therefore, in order to cover $\overline{\mathcal{OL}}(|\mathcal{V}|)$ by sets $|W_\alpha|$ that satisfy condition (ii) in Lemma 3.1.6 one cannot insist that each set $|W_\alpha|$ lift to an open subset of some V_I, but rather as in Lemma 3.1.6 (i) that it have a lift to an open subset of some $V^\Omega_I\sqsupset V_I$. ◇

Definition 3.1.8. *Suppose that $\psi^{-1}\mathcal{G}^0 \sqsubset \mathcal{V}^\Omega \sqsubset \mathcal{U}$ where $\mathcal{G}^0 \sqsubset \mathcal{F}$ is a reduction of the footprint cover (i.e. $G^0_I \sqsubset F_I$, $\forall I$ and $\bigcup_I G^0_I = X$), and choose a shrinking $\mathcal{V}^\infty \sqsubset \mathcal{V}^\omega$ that is adapted to the cover $(|W_\alpha|)_{1\le\alpha\le N}$ where $|W_\alpha| \subset \mathcal{V}^\Omega$. With these choices fixed, we then say that the pair $(\mathcal{V}, \underline{\varepsilon})$ is* **precompatible** *if the following conditions hold.*

(a′) $0 < \kappa\varepsilon_I < \varepsilon_J$ *for all* $I \subsetneq J$*,*

(b′) $\psi^{-1}(\mathcal{G}^0) \sqsubset \mathcal{V} \sqsubset \mathcal{V}^\infty$*;*

(c′) $s_I(\overline{V_I}) \subset E_{I,\varepsilon_I}$ *for all* I*;*

(d′) *for all* α *with* $W_\alpha \subset \widetilde{V}^\Omega_{IK}$ *and* $I \subsetneq H \subset K$

$$\phi^E_{IH,\alpha}\big(E_{H\smallsetminus I,(\kappa+1)\varepsilon_I} \times (\widetilde{V}_{IH} \cap \rho_{HK}(W_\alpha \cap \widetilde{V}_{HK})\big) \subset V_H. \tag{3.1.9}$$

Further, we say that $(\mathcal{V}, \underline{\varepsilon})$ is **compatible** *if it is precompatible and if $(\mathcal{V}, \underline{\varepsilon}) \sqsubset (\mathcal{V}', \underline{\varepsilon}')$ where $(\mathcal{V}', \underline{\varepsilon}')$ is also precompatible and $\underline{\varepsilon} \le \underline{\varepsilon}'$, i.e. $\varepsilon_J \le \varepsilon'_J$ for all $J \in \mathcal{I}_\mathcal{K}$.*

Remark 3.1.9. If $(\mathcal{V}, \underline{\varepsilon})$ is compatible, so that it is a shrinking of the precompatible $(\mathcal{V}', \underline{\varepsilon}')$, then we may assume that each set $|W_\alpha|$ of the associated covering of $|\mathcal{V}|$ lifts to some subset $\widetilde{V}'_{IK}$. In other words, we can equivalently define $(\mathcal{V}, \underline{\varepsilon})$ to be compatible if $(\mathcal{V}, \underline{\varepsilon}) \sqsubset \mathcal{V}^\infty$ is precompatible as above, for some reduction $\mathcal{V}^\infty$ that is provided with constants $\underline{\varepsilon}^\infty \ge \underline{\varepsilon}$ such that (a′) and (c′) hold. ◇

Lemma 3.1.10. *Suppose given $\psi^{-1}(\mathcal{G}_0) \sqsubset \psi^{-1}(\mathcal{F}_0) \sqsubset \mathcal{V}^\infty \sqsubset \mathcal{V}^\Omega$ such that $\mathcal{V}^\infty$ is adapted to the covering $(|W_\alpha|)_{1\le\alpha\le N}$, where $|W_\alpha| \subset |\mathcal{V}^\Omega|$. Then:*

(i) *There is a precompatible shrinking $(\mathcal{V}, \underline{\varepsilon}) \sqsubset \mathcal{V}^\infty$.*

(ii) *Any precompatible $(\mathcal{V}', \underline{\varepsilon}')$ has a compatible shrinking $(\mathcal{V}, \underline{\varepsilon}) \sqsubset (\mathcal{V}', \underline{\varepsilon}')$.*

Proof. The proof of (i) is somewhat similar to that of Lemmas 3.1.2 and 3.1.6, except that now we have to make sure that (d′) holds, i.e. that we can choose $\mathcal{V}$ so that the image of $\phi^E_{IH,\alpha}$ lies in V_H for all $I \subsetneq H \subset K$ rather than just in the fixed ambient space U^Ω_J as in (3.1.3). Claim (ii) then follows by the same argument, with $\mathcal{V}^\infty$ replaced by $\mathcal{V}'$.

To prove (i), we first choose any reduction $\mathcal{V}^\kappa$ of $\mathcal{U}$, where $(\mathcal{U}, \underline{\varepsilon}^\kappa)$ is compatible, so that $(\mathcal{V}, \underline{\varepsilon})$ satisfies (a),(b′),(c′). We then work by downwards induction on $\ell := |J|$, so that after the ℓth stage we have chosen a reduction $(\mathcal{V}^\ell, \underline{\varepsilon}^\ell)$ with

$$\psi^{-1}(\mathcal{G}^0) \sqsubset \mathcal{V}^\ell \subset \mathcal{V}^\kappa, \quad \underline{\varepsilon}^\ell \le \underline{\varepsilon}^\kappa$$

that satisfies (a′), (b′), (c′) for all I, K, and satisfies (d′) for all I with $|I| \ge \ell$. Since (d′) is vacuous when $\ell = \kappa$, it suffices to suppose that we have found suitable $(\mathcal{V}^{\ell+1}, \underline{\varepsilon}^{\ell+1})$ for some $1 < \ell + 1 \le \kappa$, and consider the construction of $(\mathcal{V}^\ell, \underline{\varepsilon}^\ell)$. Our method gives $\underline{\varepsilon}^\ell$ where $\varepsilon^\ell_J = \varepsilon^{\ell+1}_J$ if $|J| > \ell$ and $\varepsilon^\ell_I \le \varepsilon^{\ell+1}_I$ if $|I| \le \ell$. Further, for $|J| > \ell$ we construct V^ℓ_J by removing some points in $\widetilde{V}^{\ell+1}_{IJ}$ from $V^{\ell+1}_J$ for $|I| = \ell$. Note that removing these points does not affect the validity of (d′) for pairs $I \subsetneq K$ with $|I| \ge \ell + 1$.

Choose an intermediate reduction $\mathcal{V}'$ such that $\mathcal{V}^0 \sqsubset \mathcal{V}' \sqsubset \mathcal{V}^{\ell+1}$. Because the subsets $\pi_\mathcal{K}(V^\infty_I) \subset |\mathcal{K}|$ with $|I| = \ell$ are disjoint, we may work separately with each such I. Given $x \in V_I$ with $I \subsetneq K = I_{\max}(|x|)$ the set $\widetilde{V}'_{IK} = V'_K \cap \pi^{-1}_\mathcal{K}(\pi_\mathcal{K}(V'_I))$ is precompact in $\widetilde{V}_{IK}^{\ell+1} = V_K^{\ell+1} \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(V_I^{\ell+1}))$ and hence there is $0 < \varepsilon_I^\ell \le \varepsilon_I^{\ell+1}$ so that for each α with $W_\alpha \subset V_I^\Omega$ and each $I \subsetneq H \subset K$ we have

$$\phi_{IJ}^E\Big(E_{H\smallsetminus I,(\kappa+1)\varepsilon_I} \times (\widetilde{V}'_{IH} \cap \rho_{IH}^{-1}(W_\alpha))\Big) \subset V_H^{\ell+1}. \tag{3.1.10}$$

For J with $|J| > \ell$ we now define

$$V_J^\ell := V_J^{\ell+1} \smallsetminus \bigcup_{I\subset J,|I|=\ell}\big(s_J^{-1}(E_I) \cap (V_J^{\ell+1} \smallsetminus V'_J)\big).$$

Then V_J^ℓ is an open subset of $V_J^{\ell+1}$, since we have removed a closed subset. Now choose ε_J^ℓ for $|J| < \ell$ so as to satisfy (a$'$) and then define

$$V_J^\ell := \big\{x \in V_J^{\ell+1} \mid s_H(x) < \tfrac{1}{2}\varepsilon_J^\ell\big\}, \quad |J| \le \ell.$$

Then (c$'$) holds, and (b$'$) still holds for J with $|J| > \ell$ because it holds for $\mathcal{V}'$, and it holds when $|J| \le \ell$ because we did not change the zero sets $s_J^{-1}(0)$. Moreover (d$'$) holds because when $|J| > \ell$ the only points in $V_J^{\ell+1}$ that were removed to form V_J^ℓ lie in $s_J^{-1}(E_I)$ for $I = \ell$. But this does not affect the validity of (3.1.10) (and hence (3.1.9)) because

$$\phi_{IJ}^E\big((E_{J\smallsetminus I,(\kappa+1)\varepsilon_I} \smallsetminus \{0\}) \times \{z\}\big) \cap s_J^{-1}(E_I) = \emptyset$$

by the first equation in (3.1.2). This completes the proof. □

3.2. **Construction of the boundary collar.** It remains to establish the existence of a collar with the properties stated in Proposition 2.1.3. Recall from (2.1.9) that Δ_J has a collar of the following form[16]

$$c_J^\Delta : \partial\Delta_J \times [0,\delta] \to \Delta_J, \quad (t^\partial, r) \mapsto (1 - r|J|)\, t^\partial + r|J|\, b_J,$$

where b_J is the barycenter of Δ_J and $0 < \delta < \frac{1}{4}$: see Figure 3.2. It is convenient to write

$$\mathcal{N}_\delta^\Delta(\partial_{J\smallsetminus I}\Delta) := \{t \in \Delta_J \bigm| t_j < \delta,\ \forall j \in J\smallsetminus I\}. \tag{3.2.1}$$

Notice that

$$c_J^\Delta\Big((\partial\Delta \cap \mathcal{N}_\delta^\Delta(\partial_{J\smallsetminus I}\Delta)) \times [0,\delta)\Big) \subset \mathcal{N}_{2\delta}(\partial_{J\smallsetminus I}\Delta); \tag{3.2.2}$$

i.e. the width-δ collar of the corner $\partial\Delta \cap \mathcal{N}_\delta^\Delta(\partial_{J\smallsetminus I}\Delta)$ lies in $\mathcal{N}_{2\delta}^\Delta(\partial_{J\smallsetminus I}\Delta)$. We now show that for each J this collar lifts to a (partial) collar for $\partial Y_{\mathcal{V},J\,\underline{\varepsilon}}$ with the properties stated in Proposition 2.1.3.

Lemma 3.2.1. *Suppose that $(\mathcal{V}, \underline{\varepsilon})$ is a compatible reduction. Then for each $J \in \mathcal{I}_{\mathcal{K}}$ there is a constant $w_J > 0$, subset $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \subset \partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ and map c_J^Y as in* (2.1.15) *with the properties detailed in Proposition 2.1.3.*

[16] Here for the sake of clarity we write t^∂ for the coordinate of a general point in $\partial\Delta_J$, while t could be any point in Δ_J.

Proof. The proof has three steps.

Step I: *Construction of local collars.* As in Remark 3.1.9 we will assume that $(\mathcal{V}, \underline{\varepsilon}) \sqsubset (\mathcal{V}^\infty, \underline{\varepsilon}^\infty)$ is precompatible, where each set $|W_\alpha|$ lifts to some $\widetilde{V}^\infty_{IK}$. In this step, we fix $\alpha, I = I_\alpha$, and $K = K_\alpha$, and define a local collar of width w_α over a subset $\mathcal{O}^\infty_{K,\alpha}$ of $\partial Y_{\mathcal{V}^\infty, K, \underline{\varepsilon}^\infty}$. This subset is determined by the set $W_\alpha \subset \widetilde{V}^\infty_{IK}$, and is the inverse image of an open subset $|\mathcal{O}^\infty_{K,\alpha}|$ of the set of overlaps $\overline{\mathcal{OL}}(|\mathcal{V}^\infty|)$ in (3.1.8).

To this end, consider the coordinate chart for $Y_{\mathcal{V}^\infty, K, \underline{\varepsilon}^\infty}$ given much as in (3.1.4) by

$$\begin{aligned}(3.2.3)\qquad & \psi: \ E_{A\smallsetminus I,(\kappa+1)\varepsilon_I} \times W_\alpha \times [0,\delta_\alpha]^{|K\smallsetminus I|} \ \longrightarrow \ Y_{\mathcal{V}^\infty, K, \underline{\varepsilon}^\infty}, \\ & \big(e_{A\smallsetminus I}, x, r_{K\smallsetminus I}\big) \longmapsto \big(e_{A\smallsetminus I} + (\lambda b_I)^{-1}\cdot s_I(x'),\ x';\ \lambda b_I + r_{K\smallsetminus I}\big), \quad \text{where} \\ & \qquad x' = \phi^E_{IK,z_\alpha}(r_{K\smallsetminus I}\cdot e_{K\smallsetminus I}, x), \quad \lambda := 1 - |r_{K\smallsetminus I}| =: 1 - \textstyle\sum_{j\in K\smallsetminus I} r_j.\end{aligned}$$

For each $x \in W_\alpha := W_\alpha \cap \widetilde{V}^\infty_{IK}$, restrict to those $r^\partial_{K\smallsetminus I}$ such that

$$\lambda^\partial b_I + r^\partial_{K\smallsetminus I} \in \overline{\mathrm{st}}^\Delta_K(|x|) \subset \partial\Delta_K,$$

where the superscript ∂ indicates that the corresponding point lies in the boundary. The above map provides coordinates

$$(3.2.4)\qquad \mathcal{C}^\delta : \big(e_{A\smallsetminus I}, x, r^\partial_{K\smallsetminus I}\big) \mapsto \psi\big(e_{A\smallsetminus I}, x, r^\partial_{K\smallsetminus I}\big) = (e_{A\smallsetminus I} + e''_I, x''; t^\partial)$$

for an open subset

$$(3.2.5)\qquad \mathcal{O}^\infty_{K,\alpha} \subset \big\{(e,x;t^\partial) : t^\partial \in \overline{\mathrm{st}}^\Delta_K(|x|),\ t^\partial \approx 0\big\}$$

of the boundary $\partial Y_{\mathcal{V}^\infty, K, \underline{\varepsilon}}$. We will assume, as we may, that $\mathcal{O}^\infty_{K,\alpha} = \mathrm{pr}_V^{-1}(|\mathcal{O}^\infty_{K,\alpha}|)$, where $|\mathcal{O}^\infty_{K,\alpha}|$ is open in $\overline{\mathcal{OL}}(|\mathcal{V}^\infty|) \subset |\mathcal{V}|$.

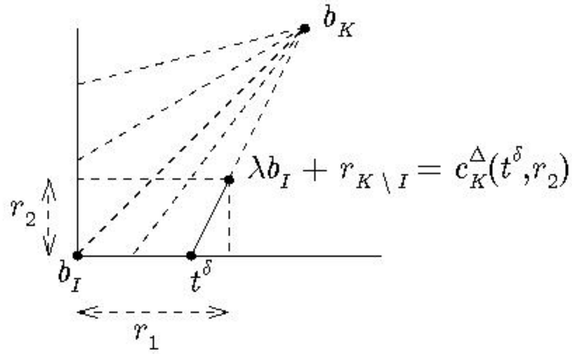

FIGURE 3.2. Here $K = I \cup \{1,2\}$ and t^δ lies on the boundary with $t_2 = 0$. Hence $r_{K\smallsetminus I} = (r_1, r_2)$ where r_2 is the collar coordinate along the ray from t^δ to b_K, while $t^\delta = c^\Delta_K(b_I, r)$ for $r = (t^\delta)_1$.

We now define a collar over $\mathcal{O}^\infty_{K,\alpha}$ of width $w_\alpha < \frac{1}{2}\delta_\alpha$ (see (3.2.2)) as follows. Given

$$(t^\partial, r) \in \overline{\mathrm{st}}^\Delta_K(|x|) \times [0,\delta), \quad \text{where} \ \ (t^\partial, r) \approx (b_I, 0),$$

choose $r^\partial_{K\smallsetminus I}, r_{K\smallsetminus I}$ (both ≈ 0) so that

$$t^\partial := \lambda^\partial b_I + r^\partial_{K\smallsetminus I}, \quad c^\Delta_K(t^\partial, r) = \lambda\, b_I + r_{K\smallsetminus I}, \tag{3.2.6}$$

where $\lambda^\partial := 1 - |r^\partial_{K-1}|$, $\lambda := 1 - |r_{K-1}|$; see Figure 3.2. Then, with $\mathcal{C}^\delta$ as in (3.2.4), define

$$\begin{aligned} c^Y_{K,\alpha} \ &: \ \mathcal{O}^\infty_{K,\alpha} \times [0, w_\alpha) \to Y_{\mathcal{V}^\infty, K, \underline{\varepsilon}}, \\ &\big((e_{A\smallsetminus I} + e''_I, x''; t^\partial), r\big) \overset{(\mathcal{C}^\delta)^{-1}\times \mathrm{id}}{\mapsto} \big((e_{A\smallsetminus I}, x, r^\partial_{K\smallsetminus I}), r\big) \ \mapsto \ \psi\big(e_{A\smallsetminus I}, x, r_{K\smallsetminus I}\big), \end{aligned} \tag{3.2.7}$$

where $r_{K\smallsetminus I} \in [0,\delta)^{K\smallsetminus I}$ is the function of $r^\delta_{K\smallsetminus I}$ and δ defined in (3.2.6). In particular, if $|K\smallsetminus I| = 1$ then $r_{K\smallsetminus I}$ has only one component, and so is the same as the collar variable r, while $t^\delta = b_I$. Therefore the collar is simply given by ψ:

$$\begin{aligned} c^Y_{I\cup\{j\},\alpha} \ &: \ \mathcal{O}^\infty_{I\cup\{j\},\alpha} \times [0, w_\alpha) \to Y_{\mathcal{V}^\infty, I\cup\{j\}, \underline{\varepsilon}}, \\ &\big((e_{A\smallsetminus I} + e_I, x; b_I), r\big) \mapsto \ \psi\big(e_{A\smallsetminus I}, x, r\big). \end{aligned} \tag{3.2.8}$$

The next task is to extend the domain of this collar to

$$\overline{\mathrm{st}}\big(\mathcal{O}^\infty_{K,\alpha}\big) := \big\{(\mu_H \cdot (e, x; t) \ |(e, x; t) \in \mathcal{O}_\alpha, \ \mu_H \cdot t \in \overline{\mathrm{st}}^\Delta_K(|x|)\big\} \tag{3.2.9}$$

by rescaling as follows. Consider a tuple μ_H (as in (2.1.20)),where $I \subset H \subsetneq K$, and point $t^\delta \in \overline{\mathrm{st}}^\Delta_K(|x|) \cap (\{b_I\} \times [0,\delta]^{|K\smallsetminus I|})$ such that

$$\mu_H \cdot t^\partial \in \overline{\mathrm{st}}^\Delta_K(|x|) \cap (\{b_I\} \times [0,\delta]^{|K\smallsetminus I|}),$$

and let $\mu'_H \cdot$ with $(\mu'_H)_i = 1$ for $i \notin H$ give the corresponding rescaling in the coordinates $\Delta_I \times [0, \delta_\alpha]^{|K\smallsetminus I|}$. Thus if $c^\Delta(t^\partial, r) = (1 - |r_{K\smallsetminus I}|)\, b_I + r_{K\smallsetminus I}$ as in (3.2.6), we have

$$c^\Delta_K(\mu_H \cdot t^\partial, r) = \mu'_H \cdot (\lambda\, b_I + r_{K\smallsetminus I}). \tag{3.2.10}$$

Note that this rescaling in the boundary $\partial_{K\smallsetminus H}\Delta_K$ does not affect the collar variable r along this part of the boundary. Then the following diagram commutes, where we write $e'_I = (t_I)^{-1} \cdot s_I(x')$, $y := (e_I, x;\, t_I) \in \partial Y$:

(3.2.11)

$$\begin{tikzcd} \big(e_{A\smallsetminus I}, y, r_{K\smallsetminus I}\big) \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "\mu'_H\cdot"'] & \big(e_{A\smallsetminus I} + e'_I,\, x' = \phi(r_{K\smallsetminus I} \cdot e_{K\smallsetminus I}, x);\, \lambda b_I + r_{K\smallsetminus I}\big) \arrow[d, mapsto, "\mu'_H\cdot"'] \\ \big((\mu'_H)^{-1} \cdot e_{A\smallsetminus I}, y, \mu'_H \cdot r_{K\smallsetminus I}\big) \arrow[r, mapsto, "\psi"] & \big((\mu'_H)^{-1} \cdot (e_{A\smallsetminus I} + e'_I),\, x' \,;\, \mu'_H \cdot (\lambda b_I + r_{K\smallsetminus I})\big), \end{tikzcd}$$

because the rescaling on the left does not affect the image point $x' = \phi(r_{K\smallsetminus I} \cdot e_{K\smallsetminus I}, x) \in V_K$ on the right. Therefore, because $c^Y_{K,\alpha}$ is a composite of ψ^{-1} (at $r = 0$) with ψ, and because rescaling does not affect the collar variable r, the following diagram commutes:

$$
\begin{tikzcd}
\big((e_{A\smallsetminus I}+e''_I,x'';\,t^\partial),r\big) \arrow[r, mapsto, "c^Y_{K,\alpha}"] \arrow[d, mapsto, "\mu_H\cdot"'] & (e',x';t') \arrow[d, mapsto, "\mu_H\cdot"'] \\
\big((\mu_H^{-1}\cdot(e_{A\smallsetminus I}+e''_I),x'';\,\mu_H\cdot t^\partial),r\big) \arrow[r, mapsto, "c^Y_{K,\alpha}"] & \big((\mu_H)^{-1}\cdot e',x',\mu_H\cdot t'\big).
\end{tikzcd}
\tag{3.2.12}
$$

In other words, if we apply the collar and then rescale (a little) by μ_H, we get the same result as rescaling by μ_H and then applying the collar. It follows that we can unambiguously extend the domain of the local collar to $\overline{\mathrm{st}}(\mathcal{O}^\infty_{K,\alpha})$ by defining

$$c^Y_{K,\alpha}\big((e_{A\smallsetminus I}+e''_I,x'';\,t),r\big) := \mu_H^{-1}\cdot c^Y_{K,\alpha}\big(\mu_H\cdot(e',x',t')\big),$$

where μ_H is chosen so that $\mu_H\cdot(e',x',t')$ lies in the domain of the map in (3.2.7). Note that $c^Y_{K,\alpha}$ is equivariant because the maps in (3.1.6) and (3.2.3) used to construct it are equivariant.

Although we assumed in the above construction that K was maximal, so that $W_\alpha\subset V^\infty_{IK}$ this condition was not used in any essential way in the above construction. Thus for any J such that $I\subsetneq J\subset K$, by using the map in (3.1.7) instead of (3.1.6) we can define a collar $c^Y_{J,\alpha}$ over

$$
\begin{aligned}
&c^Y_{J,\alpha}:\overline{\mathrm{st}}(\mathcal{O}^\infty_{J,\alpha})\times[0,w_\alpha)\to Y_{\mathcal{V}^\infty,J,\underline{\varepsilon}^\infty}\quad\text{where}\\
&\quad\overline{\mathrm{st}}(\mathcal{O}^\infty_{J,\alpha}):=\big\{(e,\rho_{JK}(x),t^\partial)\in\partial Y_{\mathcal{V}^\infty,J,\underline{\varepsilon}^\infty}\;\big|\;x\in\widetilde{V}_{JK}\cap W_\alpha,(e,x;t^\partial)\in\overline{\mathrm{st}}(\mathcal{O}^\infty_{K,\alpha})\big\},
\end{aligned}
\tag{3.2.13}
$$

and $\overline{\mathrm{st}}(\mathcal{O}^\infty_{K,\alpha})$ is defined in (3.2.9).

Further we can restrict these collars to the corresponding subsets $\overline{\mathrm{st}}(\mathcal{O}_{J,\alpha})$ of $\partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ for all $I\subsetneq J\subset K$, obtaining a set of locally defined collars of width w_α. Note that this collar still has width w_α because we used the constant ε_I in (3.2.3) rather than ε^∞_I. Hence although $\underline{\varepsilon}<\underline{\varepsilon}^\infty$ in general, when we restrict the domain of ϕ in (3.2.3) to the points in $\partial Y_{\mathcal{V},K,\underline{\varepsilon}}$ the image of ϕ lies in $Y_{\mathcal{V},K,\underline{\varepsilon}}$ by condition (d$'$) in Definition 3.1.8.

We claim that these collars satisfy all the conditions in Proposition 2.1.3. In particular, if $I\subsetneq H\subsetneq K$ the domain of $c^Y_{K,\alpha}$ contains the image of the collar $c^Y_{H,\alpha}$ by (3.2.5). They are compatible with projections and invariant under rescaling by construction.

The domains $\overline{\mathrm{st}}(\mathcal{O}_{J,\alpha})$ of these collars are not open in $\partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ because of the restriction $t\in\overline{\mathrm{st}}^\Delta_J(|x|)$, and because the condition that $(e,x;t^\partial)\in\overline{\mathrm{st}}(\mathcal{O}_{K,\alpha})$ places certain extra (but unimportant) restrictions on $\|\mathrm{pr}_{E_{K\smallsetminus I}}e\|$ when t^∂ has been rescaled far from b_I. However, modulo these provisos, for each such J they consist of the full inverse image in $\partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ of the following open subset $|\mathcal{O}_\alpha|:=|\mathcal{O}^\infty_{K,\alpha}|$ of the 'boundary' $\partial|V^\infty_K|$ of $|V^\infty_K|$:

$$|\mathcal{O}_\alpha|:=|\mathcal{O}^\infty_{K,\alpha}|\;\subset\;\partial|V^\infty_K|:=\textstyle\bigcup_{H\subsetneq K}|V^\infty_{HK}|\subset\overline{\mathcal{OL}}(|\mathcal{V}^\infty|),\tag{3.2.14}$$

where $\mathcal{O}^\infty_{K,\alpha}$ is defined in (3.2.5).

Step 2: *Construction of a global collar from a covering by local collars*

We now explain a method from [Hat, Prop. 3.42] that combines local collars

$$\big(c_\alpha : \mathcal{U}_\alpha \times [0, w_\alpha) \to Y\big)_{1\le\alpha\le N}$$

defined over open subsets $\mathcal{U}_\alpha \subset \partial Y$ of the boundary of a manifold Y into a global collar over $\partial' Y$ of width w, where $\partial' Y$ is any precompact subset of $\bigcup_\alpha \mathcal{U}_\alpha$ and $w < \min_\alpha w_\alpha/2$.

To this end, choose a partition of unity $(\lambda_\alpha)_\alpha$ subordinate to the covering of $\partial' Y$ by the sets $(U_\alpha)_\alpha$, and define

$$Y' := Y \cup_\theta \big(\partial' Y \times [-w, 0]\big),$$

where θ identifies $\partial' Y \times \{0\}$ with $\partial' Y$ in the obvious way. We claim that there is a homeomorphism

$$\Psi : \big(Y', \partial' Y \times [-w, 0]\big) \longrightarrow \Big(Y,\ \bigcup_\alpha c_\alpha\big(\mathcal{U}_\alpha \times [0, 2w)\big)\Big).$$

Granted this, we define the collar by

$$c^Y : \partial' Y \times [0, w) \longrightarrow Y, \quad \big(y, r\big) \mapsto \Psi_J\big(y,\ r - w\big).$$

The homeomorphism Ψ is a composite

$$\Psi = \Psi_N \circ \cdots \circ \Psi_1,$$

of homeomorphisms,

$$\Psi_\ell : Y'(-1 + \sum_{\alpha<\ell} \lambda_\alpha) \to Y'(-1 + \sum_{\alpha\le\ell} \lambda_\alpha),$$

where for any function $\sigma : \partial' Y \to [0, 1]$ we define

$$Y'(-1 + \sigma) := Y \cup_\theta \big\{(y, r) \ \big|\ y \in \partial' Y,\ (-1 + \sigma(y))w \le r \le 0\big\}.$$

To define Ψ_ℓ, first extend the product structure of the external collar $\partial Y \times [-w, 0]$ via the local collar c_ℓ to obtain an extended collar neighborhood

$$\widehat{c}_\ell : \mathcal{U}_\ell \times [-w, w_\ell) \to Y'.$$

Then define

$$\Psi_\ell(\widehat{c}_\ell(y, r)) = \widehat{c}_\ell(y, f_{y,\ell}(r))$$

where

$$f_{y,\ell} : \big[(-1 + \textstyle\sum_{\alpha<\ell} \lambda_\alpha(y))w,\ 2w\big] \to \big[(-1 + \textstyle\sum_{\alpha\le\ell} \lambda_\alpha(y))w, 2w\big]$$

is a homeomorphism that translates by $\lambda_\ell(y)$ if $r \le \sum_{\alpha<\ell} \lambda_\alpha(y))w$. This completes the construction.

Remark 3.2.2. Notice that if each local collar c_α lifts a map $\mathrm{pr}_\Delta : (Y, \partial Y) \to \big([0, 1), \{0\}\big)$, then the global collar does as well; i.e. we have

$$\mathrm{pr}_\Delta \circ c(y, r) = r.$$

This holds because each $f_{y,\ell}$ is a translation by $\lambda_\ell(y)w$ on the relevant part of its domain, where $\sum_\ell \lambda_\ell(y) = 1$. Further, if for some map $\mathrm{pr}_E : Y \to E$ we have $c_\alpha(y, r) = \mathrm{pr}_E(y)$, then the global collar also satisfies $c^Y(y, r) = \mathrm{pr}_E(y)$. $\diamond$

Step 3: *Completion of the proof.*

Once the cover and partition of unity are chosen, the construction in Step 2 depends only on the ordering of the sets in the cover. Even though we saw in Step 1 that the local covers satisfy all the compatibility conditions required in Proposition 2.1.3, we will have to organize the construction rather carefully in order to achieve this for the global collars.

Recall from the discussion of (1.2.8) that because the atlas $\mathcal{K}$ is assumed tame and preshrunk and hence good, the subspace topology on $|\mathcal{V}^\infty|$ (considered as a subset of $|\mathcal{K}|$) is metrizable, and so we may fix a metric on $|\mathcal{V}^\infty|$. Since the sets $|V_I|, |V_J|$ have disjoint closures unless $I \subset J$ or $J \subset I$, we may choose

$$\delta_0 > 0 \text{ smaller than half the distance between any two such sets.} \tag{3.2.15}$$

We next order the sets $|W_\alpha|_{1\le\alpha\le N}$ of the cover of $\overline{\mathcal{OL}}(\mathcal{V})$ so that as α increases the cardinality $|I_\alpha|$ of the minimal set I in Lemma 3.1.6 (i) increases. Thus we assume that there are numbers $0 = n_0 \le n_1 \le n_2 \le \cdots \le n_{\kappa-1} = N$ so that

$$N_{k-1} < \alpha \le N_k \Longrightarrow |I_\alpha| = k.$$

By (3.2.14) the sets $\big(|\mathcal{O}_\alpha|\big)_{1\le\alpha\le N}$ cover a neighborhood of the compact subset $\overline{\mathcal{OL}}(|\mathcal{V}|)$ in $|\mathcal{V}^\infty|$. Further by condition (ii) in Lemma 3.1.6 and our choice of N_k, if $\alpha > N_k$ the set $|\mathcal{O}_\alpha|$ does not meet any $|V_I|$ with $|I| \le k$. Hence we may choose $\delta_0 > \delta_1 > 0$ so that for each k, the sets $\big(|\mathcal{O}_\alpha|\big)_{1\le\alpha\le N_k}$ cover the closed δ_1-neighborhood

$$\overline{\mathcal{N}}_{\delta_1}(k) := \overline{\mathcal{N}}_{\delta_1}\big(\textstyle\bigcup_{|I|\le k, L\in\mathcal{I}_\mathcal{K}} |\overline{V_{IL}}|\big) \subset \overline{\mathcal{OL}}(|\mathcal{V}|)$$

of the compact subset $\bigcup_{|I|\le k, L\in\mathcal{I}_\mathcal{K}} |\overline{V_{IK}}|$. By shrinking the sets $\mathcal{O}_\alpha$ to $\mathcal{O}'_\alpha$, we may then assume in addition that for some $0 < \delta_2 < \delta_1$ we have

$$\big(\alpha > N_k\big) \Longrightarrow |\mathcal{O}'_\alpha| \cap \overline{\mathcal{N}_{\delta_2}}(k) = \emptyset, \quad \forall k. \tag{3.2.16}$$

For each $k \le \kappa$, choose a partition of unity $(\lambda^k_\alpha)_{1\le\alpha\le N_k}$ for $\overline{\mathcal{N}}_{\delta_2}(k)$ with respect to the covering by $(|\mathcal{O}'_\alpha|)_{1\le\alpha\le N_k}$, such that

$$1 \le \alpha \le N_{k-1} \Longrightarrow \lambda^k_\alpha = \lambda^{k-1}_\alpha \tag{3.2.17}$$

Finally, choose $w' > 0$ such that

$$2w' < \min_\alpha w_\alpha. \tag{3.2.18}$$

Now define

$$\partial^k Y_{\mathcal{V},J,\underline{\varepsilon}} = \textstyle\bigcup_{1\le\alpha\le N_k} \big\{(e,x;t) \mid (e,x;t) \in \overline{\mathrm{st}}(\mathcal{O}'_{J,\alpha})\big\}, \tag{3.2.19}$$

where $\overline{\mathrm{st}}(\mathcal{O}'_{J,\alpha})$ is defined just as in (3.2.13) but with $\mathcal{O}^\infty_{K,\alpha}$ replaced by $\mathcal{O}^\infty_{K,\alpha} \cap \pi_\mathcal{K}^{-1}(|\mathcal{O}'_\alpha|)$.

Then for each $I \subsetneq J$ with $|I| = k$, we may use the local collars $c^Y_{J,\alpha}$ together with the partition of unity on $\partial^k Y_{\mathcal{V},J,\underline{\varepsilon}}$ obtained by pulling back (λ^k_α) to construct a collar

$$c^Y_{J,k} : \partial^k Y_{\mathcal{V},J,\underline{\varepsilon}} \times [0, w'_J) \to Y_{\mathcal{V},J,\underline{\varepsilon}}$$

as in Step 2. Condition 3.2.17 implies that $c^Y_{J,k}$ agrees with $c^Y_{J,k-1}$ on their common domain of definition. Hence the collars fit together to give a well defined collar

$$\begin{aligned} c^Y_J : \partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \times [0, w'_J) \to Y_{\mathcal{V},J,\underline{\varepsilon}}, \quad \text{where} \\ \partial' Y_{\mathcal{V},J,\underline{\varepsilon}} := \bigcup_{k<|J|} \partial^k Y_{\mathcal{V},J,\underline{\varepsilon}}. \end{aligned} \tag{3.2.20}$$

Note that c^Y_J lifts c^Δ_J by Remark 3.2.2. Thus it does have the form required by (2.1.15).

It remains to check that that we can choose collar widths $w_J \le w'_J$ so that the resulting collars have all the required properties.

- The maps c^Y_J are equivariant, because the local collars are, and the partition of unity is pulled back from $|\mathcal{V}^\infty|$.
- To see that the c^Y_J are compatible with projection to $E_{A\setminus\bullet}$, suppose that $I \subsetneq J$ has $|I| = k < |J|$. Then c^Y_J has the properties in (2.1.16) because all the local collars do. Further the points $\iota_{EV}(e, x) = (b_I^{-1} \cdot e, x, b_I)$ mentioned in (2.1.17) lie in $\partial^k Y_{\mathcal{V},J,\underline{\varepsilon}}$. Therefore $c^Y_J(\iota_{EV}(e, x), r)$ is made by combining the local collars $(c^Y_{J,\alpha})_{\alpha\le N_k}$. But we saw in Step 1 that all these local collars satisfy (2.1.17) for $E_{A\setminus I}$. It follows that the combined collar formed in Step 2 must also satisfy (2.1.17) for $E_{A\setminus I}$.
- Similarly, the fact that the relevant local collars that form c^Y_J are invariant under rescaling as in (2.1.20) implies that c^Y_J also satisfies (2.1.20).
- To prove that the pairs (c^Y_J, w_J) are compatible with covering maps we need to check two things:
  (a) that their domains are large enough (i.e. that (2.1.18) holds for all $I \subsetneq H \subsetneq J$) and
  (b) that when $H \subsetneq J$ the collar c^Y_H has a natural lift to $Y_{\mathcal{V},J,\underline{\varepsilon}}$.

  Claim (b) again follows because the local collars used to form c^Y_H (as well as the partition of unity) can be lifted in this way. (This is just a consequence of equivariance.) Claim (a) has two parts. The first claims that if $(e, x; t) \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ has $x \in \widetilde{V}_{IH} \cap \widetilde{V}_{HJ}$ where $I \subsetneq H \subsetneq J$, then $(e, \rho_{HJ}(x), t)$ is in the domain $\partial' Y_{\mathcal{V},H,\underline{\varepsilon}}$ of c^Y_H. To see this, note that $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ is the union over k of the sets $\partial^k Y_{\mathcal{V},J,\underline{\varepsilon}}$ of (3.2.19). But we have

$$\begin{aligned} \partial^k Y_{\mathcal{V},J,\underline{\varepsilon}} \cap \bigl\{(e, x; t) \mid x \in \widetilde{V}_{IH} \cap \widetilde{V}_{HJ}\bigr\} &= \partial^{|H|} Y_{\mathcal{V},J,\underline{\varepsilon}} \cap \bigl\{(e, x; t) \mid x \in \widetilde{V}_{IH} \cap \widetilde{V}_{HJ}\bigr\}, \\ &= \bigl\{(e, x; t) \mid (e, \rho_{HK}(x), t) \in \partial^{|H|} Y_{\mathcal{V},H,\underline{\varepsilon}}\bigr\}, \end{aligned}$$

  where the first equality holds by (3.2.16), while the second holds because the sets $\overline{\mathrm{st}}(\mathcal{O}^\infty_{J,\alpha}$ are compatible with the covering maps ρ_{HJ} by (3.2.13).

  The second part of (a) concerns the choice of suitable widths $w_H \le w'_H$ for all $H \in \mathcal{I}_{\mathcal{K}}$. Since the domains of the collars are by now fixed, we can choose each w_H independently: its choice depends only on the domains of the collars c^Y_J for $J \supsetneq H$. Notice that by the definition of the set $\mathcal{O}^\infty_{K,\alpha}$ in (3.2.5), it holds (with $w_H = \frac{1}{2}\delta_\alpha$ for example) for the original domains $\mathcal{O}^\infty_{K,\alpha}$ of the local collars. Moreover, because $\delta_2 < \delta_0$ (where δ_0 is the separation distance in (3.2.15)), this property is not affected by the shrinking from $|\mathcal{O}^{\infty}_{K,\alpha}|$ to $|\mathcal{O}'_{\alpha}|$ in (3.2.16). Hence it is easy to see that one can choose suitable w_H for the global collars.

- Finally we must check that this collar restricts to any compatible shrinking $(\mathcal{V}', \underline{\varepsilon}') \sqsubset (\mathcal{V}, \underline{\varepsilon})$. But this is immediate since the above construction depends only on the choice of coordinate charts in (3.2.3) which restrict to $(\mathcal{V}', \underline{\varepsilon}')$ by the definition of compatibility, and the choice of an appropriate partition of unity that we can also restrict to $\mathcal{V}'$.

This completes the proof of Lemma 3.2.1. □

Corollary 3.2.3. *Any reduction $\mathcal{V}'$ has a collar compatible shrinking $(\mathcal{V}, \underline{\varepsilon})$.*

Proof. By Definition 2.1.4, it suffices to construct a compatible (V_J, ε_J) such that

(e) for all pairs $I \subsetneq J$ we have $\varepsilon_I \le w_J^2$, where w_J is the collar width for V_J.

Without loss of generality, let us suppose that $(\mathcal{V}', \underline{\varepsilon}')$ is compatible, with collars c^Y_J of widths w'_J. As in the proof of Lemma 3.1.10 we work by downwards induction on $|J|$. Hence at the kth stage, we assume that we have compatible $(\mathcal{V}^{k+1}, \underline{\varepsilon}^{k+1})$ such that condition (e) holds for all $I \subsetneq J$ with $|I| \ge k+1$, and aim to construct compatible $(\mathcal{V}^k, \underline{\varepsilon}^k, w^k_J)$ so that (e) holds whenever $|I| \ge k$. As before we take $(V^k_J, \varepsilon^k_J, w^k_J) = (V^{k+1}_J, \varepsilon^{k+1}_J, w^{k+1}_J)$ if $|J| \ge k+1$. The key point is this: if we shrink the set $(V^{k+1}_I, \varepsilon^{k+1}_I)$ where $|I| \le k$ by decreasing ε^{k+1}_I and hence V^{k+1}_I (because of condition (c) in Definition 3.1.1), then this does not decrease the collar width $c^Y_{J,k+1}$ of any V^{k+1}_J with $I \subsetneq J$, since this change only affects points that either lie in the boundary of $Y_{\mathcal{V}^{k+1}, J, \underline{\varepsilon}^{k+1}}$ or are interior points with $I(x) = \{i | s_i(x) \ne 0\} \subset I$ that do not occur in $\operatorname{im}(c^Y_{J,k+1})$ because of its construction. Hence it makes sense to choose $\varepsilon^k_I \le \varepsilon^{k+1}_I$ for the elements $|I| = k$ so that condition (e) holds at level k, and then shrink V^{k+1}_I to a set V^k_I that satisfies (a,b,c). As usual, this can be done independently for each I at level k. To complete the inductive step, we then make appropriate choices for lower level I as in Lemma 3.1.10 to obtain a compatible shrinking $(\mathcal{V}^k, \underline{\varepsilon}^k)$ that satisfies (e) at levels $\ge k$. This completes the proof. □

3.3. **Construction of the VFC.** We now turn to the Corollary 1.1.2 which is based on the assertion that the weighted branched manifold (M, Λ) carries a natural fundamental class. This was proven in [M1] in the case when $\mathbf{M}$ is smooth and compact, with or without boundary. Although smoothness is assumed throughout [M1], the only place where this condition is essential is in the construction of the fundamental class in the proof of [M1, Proposition 3.25]. In this case, we may replace $\mathbf{M}$ by an equivalent wnb groupoid that is tame in the sense that its branching loci are piecewise smooth and hence triangulable, which allows us to work with singular homology. In the present case, we must use rational Çech cohomology, and the appropriate dual homology theory for noncompact manifolds as described in §A.

We begin with a lemma that describes the properties of $M = |\mathbf{M}|_{\mathcal{H}}$ as a topological space. Recall that we define the realization $|\mathbf{M}|$ to be the quotient space

$$|\mathbf{M}| := \operatorname{Obj}_{\mathbf{M}}/\!\sim\, = \textstyle\bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} M_I/\!\sim$$

where $\sim$ is the equivalence relation generated by the morphisms in $\mathbf{M}$. Further, $|\mathbf{M}|_{\mathcal{H}}$ is its maximal Hausdorff quotient, in other words, there is a quotient map $\pi^{\mathcal{H}}_{|\mathbf{M}|} : |\mathbf{M}| \to |\mathbf{M}|_{\mathcal{H}}$ with the property that any continuous map $|\mathbf{M}| \to Z$, where Z is Hausdorff, factors through $\pi^{\mathcal{H}}_{|\mathbf{M}|}$. (Such a space exists by [MW3, Lemma A.2].) In the case at hand, it is very easy to describe $|\mathbf{M}|_{\mathcal{H}}$.

Lemma 3.3.1. *The space $M = |\mathbf{M}|_{\mathcal{H}}$ is the quotient $\mathrm{Obj}_{\mathbf{M}}/{\sim_{cl}}$ where $\sim_{cl}$ is the closure of the relation on $\mathrm{Obj}_{\mathbf{M}}$ generated by the morphisms in $\mathbf{M}$. In particular, for each I, the map $\pi^{\mathcal{H}}_I : M_I \to |\mathbf{M}|_{\mathcal{H}}$ is a local homeomorphism with open image, and in particular is a proper map onto its image.*

Proof. The first statement may be proved as in [MW3, Lemma 3.2.10]; see also [M1, Lemma 3.5]. To prove the second, first consider the projection $\pi_I : M_I \to |\mathbf{M}|$. If $\pi_I(x) = \pi_I(y)$ for $x, y \in M_I$, then there is $H \subsetneq I$ and an element $\gamma \in \Gamma_{I\smallsetminus H}$ such that $x, y \in \widetilde{M}_{HI}$ and $y = \gamma * x$, where $*$ now denotes the action on $\widetilde{M}_{HI} \subset E_{A\smallsetminus I} \times \widetilde{M}^0_{HI}$ obtained from (2.2.21) by multiplying by id_E. This action of $\Gamma_{H\smallsetminus I}$ evidently extends to a free action on the closure of $\widetilde{M}_{HI}$ in M_I. It follows easily that $\pi^{\mathcal{H}}_I(x) = \pi^{\mathcal{H}}_I(y)$ exactly if there is $H \subsetneq I$ such that $x, y \in cl(\widetilde{M}_{HI})$ and $\gamma \in \Gamma_{I\smallsetminus H}$ with $y = \gamma * x$. Further, since the set of such H is nested (cf. (2.1.11)), if we choose a minimal such $H = H_{min,x}$, then $\pi^{\mathcal{H}}_I$ is injective (and hence a local homeomorphism) on any neighborhood $\mathcal{N}^H_x$ of x that is disjoint from its translates under the $*$ action of $\Gamma_{I\smallsetminus H}$ and also disjoint from all $\widetilde{M}_{LI}$ with $L \subsetneq H$. $\square$

This lemma implies that M is locally compact and Hausdorff. Moreover, with $H = H_{\min,x}$ as above, the local branching structure is given near $|x| = \pi^{\mathcal{H}}_I(x)$ by the translates of $\mathcal{N}^H_x$ under $\Gamma_{I\smallsetminus H}$. Thus the weighting function is given by

$$\Lambda(|x|) = \frac{|\Gamma_{I\smallsetminus H}|}{|\Gamma_I|} = \frac{1}{|\Gamma_H|}.$$

With these preliminaries in hand, it is easy to show that M has a fundamental class.

Lemma 3.3.2. *Let $M = |\mathbf{M}|_{\mathcal{H}}$ be constructed from the oriented wnb groupoid in Theorem 1.1.1. Then there is a class $\mu_M \in \check{H}^{\infty}_N(M)$ with the following property: if $U := \pi^{\mathcal{H}}_I(M_I)$ for some $I \in \mathcal{I}_{\mathcal{K}}$, then*

$$\rho_{M,U}(\mu_M) = \frac{1}{|\Gamma_I|}\ (\pi^{\mathcal{H}}_I)_*(\mu_I)\ \in\ \check{H}^{\infty}_N(\pi^{\mathcal{H}}_I(M_I)), \tag{3.3.1}$$

where $\mu_I \in \check{H}^{\infty}_N(M_I)$ is the fundamental class in (A.3) *and $\rho_{M,U}$ is as in* (A.4).

Proof. It follows from Lemma 3.3.1 that the statement of the lemma makes sense: the class μ_I exists by property (a′) in §A, the restriction exists by (b′) because U is open, and the pushforward exists by (c′). We prove the lemma by showing that for $k = 1, 2, \dots$, there is a class μ_k on $W_k := \bigcup_{I:|I|\le k} \pi^{\mathcal{H}}_I(M_I)$ such that

$$\mu_k|_{\pi^{\mathcal{H}}_I(M_I)}\ =\ (\pi^{\mathcal{H}}_I)_*(\frac{1}{|\Gamma_I|}\mu_I), \qquad \forall I, |I| \le k.$$

When $k = 1$, W_1 is a disjoint union of sets $\pi_I^{\mathcal H}(M_I)$, where $|I| = 1$, and we simply define μ_1 to be the given pushforward. Let us suppose that μ_k is constructed, and consider the definition of μ_{k+1}. Since the sets $\big(\pi_J^{\mathcal H}(M_J)\big)_{|J|=k+1}$ are disjoint, it follows from (e′) that we can consider each of them separately. Further, by applying Mayer–Vietoris with $U = W_k, V = \pi_I^{\mathcal H}(M_J)$ it suffices to show that the classes $\mu_k \in \check H_N^\infty(W_k)$ and $(\pi_J^{\mathcal H})_*(\frac{1}{|\Gamma_J|}\mu_J) \in \check H_N^\infty(V)$ have the same restriction to $W_k \cap V = \cup_{I\subsetneq J}\pi_I^{\mathcal H}(M_{IJ})$. But because restriction commutes with pushforward by (d′), it suffices to prove the corresponding statement for the fundamental classes of the spaces M_J. Namely, we must check that

$$\frac{1}{|\Gamma_J|}(\tau_{IJ})_*(\mu_J|_{\widetilde{M}_{IJ}}) = \frac{1}{|\Gamma_I|}\ (\mu_I|_{M_{IJ}}).$$

But on manifolds the homology theory $\check H^\infty$ agrees with the usual locally compact singular homology. Hence the above property holds because the maps $\tau_{IJ} : \widetilde{M}_{IJ} \to M_{IJ}$ are covering maps of degree $|\Gamma_J|/|\Gamma_I|$. □

We are now in a position to prove Corollary 1.1.2, which we restate for the convenience of the reader.

Proposition 3.3.3. *There is a unique element $[X]^{vir}_{\mathcal K} \in \check H_d(X;\mathbb Q)$ that is defined as follows. For $\beta \in \check H^d(X;\mathbb Q)$ and $D = d + \dim E_A$, we have*

$$\langle\beta,[X]^{vir}_{\mathcal K}\rangle := (\mathscr S_M)_*(\widehat\beta) \in \check H^c_{\dim E_A}(E_A, E_A\setminus\{0\};\mathbb Q) \cong \mathbb Q, \tag{3.3.2}$$

where $\widehat\beta$ is the image of β under the composite

$$\check H^d(X;\mathbb Q) \xrightarrow{\psi^*} \check H^d(\mathscr S_M^{-1}(0);\mathbb Q) \xrightarrow{\mathcal D} \check H^c_{\dim E_A}(M, M\setminus\mathscr S_M^{-1}(0);\mathbb Q),$$

and $\mathcal D$ is given by cap product with the fundamental class $\mu_M \in H^{d+\dim E_A}(M)$. Moreover, $[X]^{vir}_{\mathcal K}$ depends only on the oriented concordance class of $\mathcal K$, and in the smooth case agrees with the class defined in [MW3].

Proof. **Step 1:** *Definition of $[X]^{vir}_{\mathcal K}$.*

Since the fundamental class μ_M exists by Lemma 3.3.2, and an appropriate cap product exists by point (f′) in the appendix, in order to see that $\langle\beta,[X]^{vir}_{\mathcal K}\rangle$ is well defined it remains to note that the map

$$(\mathscr S_M)_* : \check H^c_{\dim E_A}(M, M\setminus\mathscr S_M^{-1}(0);\mathbb Q) \to \check H^c_{\dim E_A}(E_A, E_A\setminus\{0\};\mathbb Q)\ \cong\ \mathbb Q$$

is well defined. Further, it takes values in $\mathbb Q$, because E_A is oriented by the definition in Remark 1.2.1 (iii) and the theory $\check H^c_*$ coincides with singular homology theory on simplicial spaces. ◇

Step 2: *Proof of uniqueness.*

To prove the uniqueness claim in Corollary 1.1.2, one must state and prove the analog of Proposition 1.3.3 for cobordism atlases, and also prove that all choices made in the construction are unique modulo cobordism. For the constructions that involve atlases, such results are proved in [MW1, MW2, MW3]: see [MW1, §4] for different choices of tame shrinkings and metrics, [MW1, §5] for a discussion of reductions, [MW2,

§8] for orientations and [MW3, Appendix] for weighted branched cobordisms. The present construction also requires a choice of local product structures (as in (1.2.3)) and partition of unity (as in (3.2.17)) in order to define the collar. However, in distinction to the smooth case, it is not necessary to arrange that cobordism atlases have specified collars (i.e. local product structures) near the two boundary components because the VFC $[X]^{vir}_{\mathcal{K}}$ is now defined via diagram (A.7) which involves restriction to the boundary rather than via a perturbation section that must be extended from the boundary to the interior.

Thus we define a cobordism atlas $\mathcal{K}^{01}$ over $[0,1]\times X$ between two d-dimensional atlases $\mathcal{K}^0, \mathcal{K}^1$ on X to be an atlas $\mathcal{K}^{01}$ over $[0,1]\times X$ of dimension $d+1$ such that

(i) the charts whose footprints intersect $\partial([0,1]\times X) = \sqcup_\alpha \alpha\times X$ are manifolds with boundary;

(ii) for $\alpha = 0,1$ there are functorial inclusions
$$\iota_\alpha : \mathcal{K}^\alpha \to \mathcal{K}^{01}, \quad \iota^{\mathcal{I}}_\alpha : \mathcal{I}_{\mathcal{K}^\alpha} \to \mathcal{I}_{\mathcal{K}^{01}}, \qquad \alpha = 0,1$$
that (for simplicity) we assume to have disjoint images, and for each $I \in \mathcal{I}_{\mathcal{K}^\alpha}$ take the chart domain U^α_I onto the boundary $\partial U^{01}_{I'}$ of the corresponding chart in $\mathcal{K}^{01}$, where $I' := \iota^{\mathcal{I}}_\alpha(I)$, preserving orientation for $\alpha = 1$ and reversing it for $\alpha = 0$;

(iii) we further require that the local product structures in (1.2.3) for the chart domains in $\mathcal{K}^\alpha$ extend to local product structures near the boundary points of the corresponding chart domains in $\mathcal{K}^{01}$;

We show in [MW2, Thm. 7.1.5] that any pair of reductions $\mathcal{V}^\alpha$ of $\mathcal{K}^\alpha$ may be extended to a reduction $\mathcal{V}^{01}$ of $\mathcal{K}^{01}$ such that there are natural inclusions $\iota^V_\alpha : |\mathcal{V}^\alpha| \to |\mathcal{V}^{01}|$ that are homeomorphisms to their image. Further, if $J \in \mathcal{I}_{\mathcal{K}^\alpha}$ for $\alpha = 0,1$, then there is a commutative diagram

$$\begin{array}{ccc} E_{A^{01}\smallsetminus A^\alpha} \times Y_{\mathcal{V}^\alpha, J, \underline{\varepsilon}} & \xrightarrow{\iota^Y_\alpha} & Y_{\mathcal{V}^{01}, \iota^\alpha(J), \underline{\varepsilon}} \\ \downarrow{\scriptstyle \mathrm{pr}_V} & & \downarrow{\scriptstyle \mathrm{pr}_V} \\ V^\alpha_J & \xrightarrow{\iota^V_\alpha} & V^{01}_{\iota^\alpha(J)}. \end{array}$$

Notice here that we take the product of $Y_{\mathcal{V}^\alpha, J, \underline{\varepsilon}}$ with the extra obstruction spaces $E_{A^{01}\smallsetminus A^\alpha}$ in order to increase its dimension to that of $Y_{\mathcal{V}^{01}, \iota^\alpha(J), \underline{\varepsilon}}$. Because the maps (1.2.3) in the submersion axiom for $\mathcal{V}^{01}$ extend those for $\mathcal{V}^\alpha$, we can choose the covering and partition of unity in Step 2 of the proof of Lemma 3.2.1 for $\mathcal{V}^{01}$ to extend those already chosen for $\mathcal{V}^\alpha$. Therefore, we can construct the collars on $Y_{\mathcal{V}^{01}, \iota^\alpha(J), \underline{\varepsilon}}$ to extend already constructed collars on the sets $Y_{\mathcal{V}^\alpha, J, \underline{\varepsilon}}$. Hence, after possibly shrinking $\underline{\varepsilon} > 0$, we can arrange that there are embeddings

$$\iota^M_\alpha : E_{A^{01}\smallsetminus A^\alpha} \times M^\alpha \to M^{01}, \quad s.t. \quad \sqcup_\alpha \operatorname{im}(\iota^M_\alpha) = \partial M^{01}; \tag{3.3.3}$$

and also that the map $\mathscr{S}_M^{01}: M^{01} \to E_A$ satisfies

$$\mathscr{S}_M^{01} \circ \iota_\alpha^M = \mathscr{S}_M^\alpha \circ \mathrm{pr}_M^\alpha : \quad E_{A^{01} \smallsetminus A^\alpha} \times M^\alpha \to E_A, \tag{3.3.4}$$

where $\mathrm{pr}_M^\alpha : E_{A^{01} \smallsetminus A^\alpha} \times M^\alpha \to M^\alpha$ is the projection.

Because M^{01} is constructed from an atlas for the product $[0,1] \times X$, the natural projection $(\mathscr{S}_M^{01})^{-1}(0) \to [0,1] \times X$ factors through a homeomorphism.

$$(\mathscr{S}_M^{01})^{-1}(0)/\Gamma^{01} \xrightarrow{\cong} [0,1] \times X$$

Notice here that for $\alpha = 0,1$, the group Γ_{01} decomposes as a product that we will write $\Gamma'_{01\smallsetminus\alpha} \times \Gamma_\alpha$, where $\Gamma'_{01\smallsetminus\alpha}$ acts trivially on $(\mathscr{S}_M^{01})^{-1}(0) \cap (\mathrm{im}\, \iota_\alpha^M)$. Therefore there are natural identifications

$$\big((\mathscr{S}_M^{01})^{-1}(0) \cap (\mathrm{im}\, \iota_M^\alpha)\big)/\Gamma^{01} \;\cong\; \big((\mathscr{S}_M^{\alpha})^{-1}(0)\big)/\Gamma^\alpha \cong\; I_\varepsilon^\alpha \times X.$$

Thus, M^{01} is a branched manifold of dimension $N^{01}+1$, where $N^{01} = d + \dim E_{A^{01}}$, with boundary that decomposes as a union

$$\partial M^{01} = \sqcup_{\alpha=0,1} EM^\alpha \quad \text{ where } \quad EM^\alpha := \iota_M^\alpha(E_{A^{01} \smallsetminus A_\alpha} \times M^\alpha). \tag{3.3.5}$$

For $\alpha = 0,1$, the branched manifold EM^α carries a fundamental class

$$\mu_{EM^\alpha} := \mu_{E_{A^{01} \smallsetminus A_\alpha}} \times \mu_{M^\alpha},$$

and the proof of Lemma 3.3.2 adapts to show that the interior of M^{01} also carries a fundamental class

$$\mu_{M^{01}} \in \check{H}^\infty_{N^{01}+1}(M^{01} \smallsetminus \partial M^{01}) \tag{3.3.6}$$

such that

$$\partial\big(\mu_{M^{01}}\big) = (\mu_{EM^1}, -\mu_{EM^0}) \in \check{H}^\infty_{N^{01}}(EM^0) \oplus \check{H}^\infty_{N^{01}}(EM^1) \cong \check{H}_{N^{01}}(\partial M^{01}), \tag{3.3.7}$$

where ∂ is the boundary map in the long exact sequence in (A.5).

We now apply the cap product in (A.7) with

$$Y = M^{01}, \quad U = (\mathscr{S}_M^{01})^{-1}(E_{A^{01}} \smallsetminus \{0\}) \;\subset\; M^{01}, \quad A = \sqcup_{\alpha=0,1} EM^\alpha. \tag{3.3.8}$$

Then $Y \smallsetminus U = (\mathscr{S}_M^{01})^{-1}(0)$ is compact with a natural projection to $[0,1] \times X$ and hence to X. Since these maps are proper, any class $\beta \in \check{H}^d(X)$ pulls back to a class $\beta_Y \in \check{H}^d(Y \smallsetminus U)$ such that $\iota^*(\beta_Y) = \beta_A$ where $\iota : A \to Y$ is the inclusion, and $\beta_A = (\beta_0, \beta_1)$, where β_α can be identified with the pullback of β to $(\mathscr{S}_M^\alpha)^{-1}(0) \subset M_\alpha$. Hence the cap product

$$(\partial \mu_{M^{01}}) \cap \beta_A \in \check{H}^c_{N^{01}}(A, U \cap A)$$

is in the image of the map ∂' in (A.7) and hence vanishes when pushed forward to $\check{H}^c_{N^{01}}(Y,U)$. But there is a commutative diagram

$$\begin{array}{ccccc}
(\partial\mu_{M^{01}})\cap\beta_A & \in & \check{H}^c_{N^{01}}(A,U\cap A) & \xrightarrow{\mathscr{S}_M} & \check{H}^c_{N^{01}}(E_{A^{01}},E_{A^{01}}\setminus\{0\}) \\
\downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle =} \\
0 & \in & \check{H}^c_{N^{01}}(Y,U) & \xrightarrow{\mathscr{S}_M} & \check{H}^c_{N^{01}}(E_{A^{01}},E_{A^{01}}\setminus\{0\}).
\end{array}$$

Hence $(\mathscr{S}_M)_*\big((\partial\mu_{M^{01}})\cap\beta_A\big)=0$. Since $(\partial\mu_{M^{01}})\cap\beta_A$ measures the difference between the two classes $\mu_{EM^\alpha}\cap\beta_\alpha$, these classes have the same image in $\check{H}^c_{N^{01}}(E_{A^{01}},E_{A^{01}}\setminus\{0\})$, as claimed.

Step 3: *Agreement with previous definition in the smooth case.* It remains to show that in the smooth case the class $[X]^{vir}_{\mathcal{K}}$ constructed here agrees with that constructed in [MW3, §3]. The idea there was to construct a small smooth perturbation functor[17]

$$\nu=(\nu_I):\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}\to\mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$$

such that $s_I+\nu_I$ is transverse to zero for all I, and then assemble the resulting zero sets $Z_I^\nu:=(s_1+\nu_I)^{-1}(0)\subset V_I$ into a weighted branched manifold $Z^\nu:=|\widehat{Z}^\nu_{\mathcal{H}}|$. Note that Z^ν is oriented and has a natural inclusion into M. Now choose a sequence ν_k of perturbation sections with $\|\nu_k\|\to 0$. There is a corresponding nested sequence of neighborhoods $B_{\varepsilon_k}(\iota_{\mathcal{K}}(X))$ of the zero set $X\cong\iota_{\mathcal{K}}(X)\subset|\mathcal{V}|$ with intersection equal to $\iota_{\mathcal{K}}(X)$. Then the zero sets Z^{ν_k} map to $B_{\varepsilon_k}(\iota_{\mathcal{K}}(X))\subset|\mathcal{V}|$, and we showed that for all $\ell>k$ the two branched manifolds Z^{ν_ℓ},Z^{ν_k} are cobordant in $B_{\varepsilon_k}(\iota_{\mathcal{K}}(X))$ and hence represent the same homology class in $B_{\varepsilon_k}(\iota_{\mathcal{K}}(X))$. It follows from the tautness property of rational Čech homology (see §A(f′)) that the inverse limit of this sequence of classes in $B_{\varepsilon_k}(\iota_{\mathcal{K}}(X))$ determines a unique element of $\check{H}_d(\iota_{\mathcal{K}}(X);\mathbb{Q})\cong\check{H}_d(X;\mathbb{Q})$ that we called $[X]^{vir}_{\mathcal{K}}$ and showed to be independent of all choices.

We now interpret this construction in the current setting. As above, fix a compact neighborhood[18] $\overline{\mathcal{N}_0}$ of $\mathscr{S}_M^{-1}(0)$, so that

$$\delta_0:=\inf\big\{\|\mathscr{S}_M(x)\|:x\in\mathrm{Fr}\mathcal{N}_0:=\overline{\mathcal{N}_0}\setminus\mathcal{N}_0\big\}>0,$$

and choose a nested sequence $\overline{\mathcal{N}_k}$ of compact neighborhoods of $\mathscr{S}_M^{-1}(0)$ such that

$$\textstyle\bigcap_k\overline{\mathcal{N}_k}=\mathscr{S}_M^{-1}(0),\quad \mathscr{S}_M(\overline{\mathcal{N}}_k)\subset E_{A,\delta_k},\quad \text{where } \delta_{k+1}<\delta_k<\delta_0.$$

Choose a corresponding sequence of transverse perturbation sections $\nu_k=(\nu_{k,I})$ such that the perturbed zero set $(s_I+\nu_{k,I})^{-1}(0)$ is contained in $V_I\cap\pi_{\mathcal{K}}^{-1}(\mathcal{N}_k)$ for all I, and for each k, consider the map

$$\widehat{\nu}_k:M\to E_A,\quad \widehat{\nu}_k\big(\pi_I(e_{A\setminus I},x)\big)=\nu_k(x)\in E_I\subset E_A.$$

[17] for notation see (1.2.12)

[18] One important difference between $|\mathcal{V}|$ and M is that the zero set $|\mathfrak{s}|^{-1}(0)$ does *not* have compact neighborhoods in $|\mathcal{V}|$ by [MW2, Ex. 6.1.11], while it does in the branched manifold M.

This is well defined because $\nu_k : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}$ is a functor. Then

$$\mathrm{pr}_{E_{A\smallsetminus I}}\big((\mathscr{S}_M + \widehat{\nu}_k)(\pi_I(e_{A\smallsetminus I}, x))\big) = \mathrm{pr}_{E_{A\smallsetminus I}}\big(\mathscr{S}_M(\pi_I(e_{A\smallsetminus I}, x))\big) \neq 0 \text{ if } e_{A\smallsetminus I} \neq 0, \quad \text{while}$$
$$\mathrm{pr}_{E_I}\big((\mathscr{S}_M + \widehat{\nu}_k)(\pi_I(e_{A\smallsetminus I}, x))\big) = (s_I + \nu_{k,I})(x).$$

Therefore we may identify the weighted branched manifold Z_{ν_k} with the perturbed zero set

$$(\mathscr{S}_M + \widehat{\nu}_k)^{-1}(0) \;\subset\; \mathcal{N}_k \;\subset\; \mathscr{S}_M^{-1}(E_{A,\delta_k}).$$

Given $\beta \in \check{H}^d(X;\mathbb{Q})$, choose a sequence $\beta_k \in \check{H}^d(\overline{\mathcal{N}_k};\mathbb{Q})$ such that $\varprojlim \beta_k = \psi^*(\beta)$, where $\psi : \mathscr{S}_M^{-1}(0) \to X$ is the footprint map, and let $\iota_k : Z_{\nu^k} \to \mathcal{N}_k$ be the inclusion. We must show that

$$\lim_k \; \langle \iota_k^*(\beta_k), \mu_{Z_{\nu_k}} \rangle = (\mathscr{S}_M)_*(\mu_M \cap \psi^*(\beta)) \in \mathbb{Q}.$$

Consider the diagram below, in which the top and bottom square commute while the middle homotopy commutes:

$$\begin{tikzcd}
(M, M\smallsetminus\mathscr{S}_M^{-1}(0)) \arrow[r, "\mathscr{S}_M"] & (E_A, E_A\smallsetminus\{0\}) \arrow[d, leftrightarrow, "="] \\
(M, M\smallsetminus\overline{\mathcal{N}}_k) \arrow[u, "\iota"] \arrow[r, "\mathscr{S}_M"] \arrow[d, leftrightarrow, "="'] & (E_A, E_A\smallsetminus\{0\}) \arrow[d, leftrightarrow, "="] \\
(M, M\smallsetminus\overline{\mathcal{N}}_k) \arrow[r, "\mathscr{S}_M+\nu_k"] \arrow[d, "\iota"'] & (E_A, E_A\smallsetminus\{0\}) \arrow[d, leftrightarrow, "="] \\
(M, M\smallsetminus Z_{\nu_k}) \arrow[r, "\mathscr{S}_M+\nu_k"] & (E_A, E_A\smallsetminus\{0\}).
\end{tikzcd}$$

Because Z_{ν_k} is a weighted branched smooth submanifold of M with orientation compatible with that of E_A and M, its fundamental class $\mu_{Z_{\nu_k}}$ satisfies

$$\mu_{Z_{\nu_k}} = \mu_M \cap \big((\mathscr{S}_M + \nu_k)^*(\lambda_E)\big) \in H_d(Z_{\nu_k}, \mathbb{Q}), \tag{3.3.9}$$

where $\lambda_E \in H^{\dim E_A}(E_A, E_A\smallsetminus\{0\})$ is the natural generator.[19] This immediately implies that

$$\big\langle \iota_k^*(\beta_k), \mu_{Z_{\nu_k}} \big\rangle = \big\langle (\mathscr{S}_M + \nu_k)_*(\mu_M \cap \iota_k^*(\beta_k)), \lambda_E \big\rangle \;\in \mathbb{Q}.$$

Now note that the commutativity of the above diagram implies that

$$\varprojlim \, (\mathscr{S}_M + \nu_k)_*(\mu_M \cap \iota_k^*(\beta_k)) = (\mathscr{S}_M)_*(\mu_M \cap \psi^*(\beta)) \;\in\; H_{\dim E_A}(M, M\smallsetminus\mathscr{S}_M^{-1}(0)).$$

The result follows. □

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[19]Note that we can use singular homology since we can assume that Z_{ν_k} and M are simplicial complexes by [M1].

With a little more work, we can prove that our construction extends to atlases for compact pairs (W, X) as in [P, Lemma 5.2.4]. The following lemma defines

$$[W]^{vir}_{\mathcal{K}} \;\in\; \check{H}^{\infty}_{d+1}(W{\smallsetminus}X) = \mathrm{Hom}(\check{H}^{d+1}(W{\smallsetminus}X);\mathbb{Q}),$$

Note that $\check{H}^{\infty}_{d+1}(W{\smallsetminus}X) = \check{H}^{c}_{d+1}(W, X)$ by §A property (g′).

Lemma 3.3.4. *Given an oriented $(d+1)$-dimensional Kuranishi atlas $\mathcal{K}^W$ with boundary on a compact pair $(W, X := \partial W)$, there is an associated virtual fundamental class $[W]^{vir}_{\mathcal{K}} \in \check{H}^{\infty}_{d+1}(W{\smallsetminus}X)$ such that*

$$\partial([W]^{vir}_{\mathcal{K}}) = [X]^{vir}_{\mathcal{K}} \;\in\; \check{H}^{\infty}_{d}(X) = \check{H}^{c}_{d}(X). \tag{3.3.10}$$

where ∂ is the differential in the long exact sequence (A.5)*. In particular, the image of $[X]^{vir}_{\mathcal{K}}$ in $\check{H}^{\infty}_{d}(W) = \check{H}^{c}_{d}(W)$ is zero.*

Proof. We define the notion of an oriented $(d+1)$-dimensional Kuranishi atlas $(\mathcal{K}, \partial\mathcal{K})$ for the pair $(W, \partial W)$ by replacing $[0,1]\times X$ by W in the above definition of a cobordism atlas. Thus we take $\mathcal{K}^{01} =: \mathcal{K}^W$ to be an atlas for W, $\mathcal{K}^1 =: \mathcal{K}^X$ an atlas for X and $\mathcal{K}^0$ to be empty, and assume the obvious analogs of (i) –(iii) above. Then, given a branched manifold (M^X, Λ^X) constructed from $\mathcal{K}^X$, we may construct a branched manifold (M^W, Λ^W) with boundary

$$\partial(M^W) = E_{A^W\smallsetminus A^X} \times M^X,$$

and extend $\mathrm{id}\times \mathscr{S}_X$ from $\partial(M^W)$ to a map $\mathscr{S}_W : M^W \to E_{A^W}$ that satisfies the analogs of equations (3.3.4) and (3.3.5) above. Further, using the fundamental class $\mu^W_M \in H^{\infty}_{N^W}(M^W{\smallsetminus}\partial M^W)$ defined as in (3.3.6), we define an element

$$\begin{aligned}&[W]^{vir}_{\mathcal{K}} \in \check{H}^{\infty}_{d+1}(W{\smallsetminus}X) \quad \text{by setting}\\ &\langle \beta, [W]^{vir}_{\mathcal{K}}\rangle := (\mathscr{S}_{M^W})_*(\widehat{\beta}) \in \check{H}_{\dim E_{A^W}}(E_{A^W}, E_{A^W}{\smallsetminus}\{0\};\mathbb{Q}) \cong \mathbb{Q},\end{aligned}$$

where $\widehat{\beta}$ is defined as follows. Let

$$Y = M^W, \quad A = \partial M^W, \quad U = \mathscr{S}_W^{-1}(E_{A^W}{\smallsetminus}\{0\}).$$

Then the pullback $\psi^*\beta \in \check{H}^{d+1}(Y{\smallsetminus}(U\cup A);\mathbb{Q})$ of $\beta \in \check{H}^{d+1}(W{\smallsetminus}X;\mathbb{Q})$ determines

$$\widehat{\beta} := \mu^W_M \cap \psi^*\beta \;\in \check{H}^{c}_{\dim E_{A^W}}\big(Y{\smallsetminus}A, U{\smallsetminus}A;\mathbb{Q}\big)$$

where $\cap : \check{H}^{\infty}_{p+q}(Y{\smallsetminus}A)\otimes \check{H}^{p}(Y{\smallsetminus}U) \to \check{H}^{c}_{q}(Y, U\cup A)$ is as in (A.6) with $A = \emptyset$.

To prove (3.3.10), note that in the following diagram (with the same Y, U, A)

$$\begin{tikzcd} \check{H}^{\infty}_{p+q+1}(Y{\smallsetminus}A)\otimes \check{H}^{p+1}(Y{\smallsetminus}(A\cup U)) \arrow[r, "\cap"] \arrow[d, "\partial"'] & \check{H}^{c}_{q}(Y{\smallsetminus}A, U{\smallsetminus}A) \arrow[r, "\iota_*"] & \check{H}^{c}_{q}(Y, U) \\ \check{H}^{\infty}_{p+q}(A) \otimes \check{H}^{p}(A{\smallsetminus}U) \arrow[u, "\delta"] \arrow[r, "\cap"] & \check{H}^{c}_{q}(A, A\cap U) \arrow[ur, "j_*"'] & \end{tikzcd} \tag{3.3.11}$$

we have

$$j_*\big((\partial\mu^W_M)\cap\beta'\big) = \iota_*\big(\mu^W_M \cap (\delta\beta')\big) \;\in\; \check{H}^{c}_{q}(Y, U),$$

for all $\mu \in \check{H}^\infty_{p+q+1}(Y \smallsetminus A)$ and $\beta' \in \check{H}^p(A \smallsetminus U)$, where δ is as in (A.1).[20] Since ψ^* commutes with δ, this implies

$$\big\langle \partial([W]^{vir}_{\mathcal{K}}), \beta \big\rangle = \big\langle [W]^{vir}_{\mathcal{K}}, \delta\beta \big\rangle, \quad \forall\, \beta \in \check{H}^d(X).$$

The result follows. □

3.4. **Examples.** We begin by discussing the definition of the relative Euler class of an oriented vector bundle $\pi : \mathcal{E} \to X$ over a manifold that is equipped with a section $\mathfrak{s} : X \to \mathcal{E}$ whose zero set $\mathfrak{s}^{-1}(0)$ is compact. In particuar, we explain why the method outlined in equation (1.1.2) does compute the Euler class when X is compact and $\mathfrak{s} \equiv 0$. In Remark 3.4.2, we describe how to extend the construction to orbibundles. Finally, we show in detail how our main construction works to calculate the Euler class of the tangent bundle of S^2, starting from the atlas defined in [MW3]. Our approach easily generalizes to the football orbifold $S^2_{p,q}$, which is S^2 with orbifold points of orders p, q at the two poles.

Let $\pi : \mathcal{E} \to W$ be an oriented, vector bundle over the manifold W, together with a section $\mathfrak{s} : W \to \mathcal{E}$ with compact zero set $X \subset W$. As always (see Remark 1.2.1 (iii)), we suppose that $\mathcal{E}$ has even rank to avoid problems with orientation.[21] We build a (Kuranishi) atlas whose charts are defined using tuples

$$(\mathcal{O}, E, \tau, s),$$

where

- $\mathcal{O} \subset W$ is open,
- E is an even dimensional, oriented vector space,
- $\lambda : E \times \mathcal{O} \to \mathcal{E}|_{\mathcal{O}}$ is a surjective orientation-preserving bundle homomorphism over $\mathrm{id}_{\mathcal{O}}$, and
- $s : \mathcal{O} \to E$ pushes forward to $\mathfrak{s}|_{\mathcal{O}}$, i.e. $\lambda(s(x), x)) = \mathfrak{s}(x) \in \mathcal{E}|_x, \ \forall x \in \mathcal{O}$.

Given such a tuple the corresponding chart

$$\mathbf{K} := (U, E, s, \psi), \qquad \text{with footprint } F,$$

is defined by setting

$$U = \big\{(e, x) \in E \times \mathcal{O} \mid \lambda(e, x) = \mathfrak{s}(x)\big\}, \quad s(e, x) = e, \quad \psi(0, x) \mapsto x \in X.$$

One obtains an atlas as defined in §1.2 by taking the basic charts to be a finite family $\big(\mathbf{K}_i\big)_{i=1,\dots A}$ of charts of this form whose footprints (F_i) cover the compact set $X = \mathfrak{s}^{-1}(0)$, and the transition charts $\big(\mathbf{K}_I\big)_{I \in \mathcal{I}_{\mathcal{K}}}$ to be the corresponding charts (U_I, E_I, s_I, ψ_I) with footprints $F_I := \bigcap_i F_i$ that are formed just as above but now with $E_I = \prod_{i \in I} E_i, \lambda_I = \sum_{i \in I} \lambda_i$. In particular,

$$U_I = \big\{\big((e_i), x\big) \in E_I \times \mathcal{O}_I \;\big|\; \sum \lambda_i(e_i, x) = \mathfrak{s}(x)\big\}, \qquad \text{where } \mathcal{O}_I := \textstyle\bigcap_{i \in I} \mathcal{O}_i.$$

[20] This extension to property (B5) on [Ma, p.336] holds by combining Properties (B4) and (B6).

[21] Of course, over $\mathbb{Q}$ the Euler class vanishes for bundles of odd rank anyway.

This gives an atlas in which the coordinate changes $\mathbf{K}_I \to \mathbf{K}_J$ are given by the obvious identifications

$$\widetilde{U}_{IJ} := \bigl\{(e,x) \in U_J \bigm| e \in E_I,\ x \in \mathcal{O}_J\bigr\} \xrightarrow{\cong} U_{IJ} = \bigl\{(e,x) \in U_I \bigm| x \in \mathcal{O}_J\bigr\}.$$

To see that the submersion condition holds, choose for each I a right inverse $\sigma_I : \mathcal{E}|_{\mathcal{O}_I} \to E_I \times \mathcal{O}_I$ to λ_I, so that $\lambda_I \circ \sigma_I = \mathrm{id}$, and define

$$\mathcal{E}'_{J\smallsetminus I} = \bigl\{\bigl(e' - \sigma_I(\mathfrak{s}(x)), x\bigr) \bigm| e' \in E_{J\smallsetminus I}, x \in \mathcal{O}_{IJ}\bigr\} \subset E_J \times \mathcal{O}_J.$$

Then $\mathcal{E}'_{J\smallsetminus I}$ is an affine subbundle of $E_J \times \mathcal{O}_J \to \mathcal{O}_J$, and we may identify U_J with the pullback of $\mathcal{E}'_{J\smallsetminus I}$ to $\widetilde{U}_{IJ}$ by the projection $\widetilde{U}_{IJ} \to U_J,\ (e,x) \mapsto x$.

Since there is such an atlas for every collection of charts $\mathbf{K}$ whose footprints cover X, any two such atlases $\mathcal{K}^0, \mathcal{K}^1$ are **directly commensurate**, i.e. there is an atlas $\mathcal{K}$ whose charts include those of $\mathcal{K}^0$ and $\mathcal{K}^1$. Therefore $\mathcal{K}^0, \mathcal{K}^1$ are cobordant by [MW2, §6.2]. Hence, they define cobordant manifold models $(M, E_A, \mathscr{S})$ by Theorem 1.1.1 and the same class $[X]^{vir}_{\mathcal{K}}$ by Corollary 1.1.2.

If the bundle $\mathcal{E} \to W$ is smooth, then we can define the VFC either as we did above or via an inverse limit of the homology classes of the zero sets of a family of perturbed sections $\mathfrak{s} + \nu_k$ of $\mathcal{E} \to W$. As explained in the proof of Corollary 1.1.2 these two approaches give the same answer. In the general case, it is of course easiest to represent the Euler class by starting with an atlas with just one basic chart (and hence just one chart). In this case, our general method of building an atlas gives the tuple described in (1.1.2). We now show that if $\mathfrak{s} \equiv 0$ so that X is a manifold, then $[X]^{vir}_{\mathcal{K}}$ is Poincaré dual to the usual Euler class $\chi(\mathcal{E}) \in H^k(X;\mathbb{Z})$, where $k = \operatorname{rank} \mathcal{E}$. We will use standard homology/cohomology since X is simplicial, and take coefficients $\mathbb{Z}$ since the isotropy is trivial.

Lemma 3.4.1. *If $\mathcal{E} \to M$ is an oriented $2k$-dimensional vector bundle over a $(2k+d)$-dimensional manifold X with $\mathfrak{s} \equiv 0$ and atlas $\mathcal{K}$ as above, then*

$$[X]^{vir}_{\mathcal{K}} = \mu_X \cap \chi(\mathcal{E}) \in \operatorname{Hom}\bigl(H_d(X), \mathbb{Z}\bigr),$$

where μ_X is the fundamental class of X and $\chi(\mathcal{E}) \in H^{2k}(X;\mathbb{Z})$ is the Euler class of $\mathcal{E}$.

Proof. By Corollary 1.1.2 and the above remarks, it suffices to calculate $[X]^{vir}_{\mathcal{K}}$ using an atlas with one chart as in (1.1.2). Thus we may take

$$M = \mathcal{E}', \qquad \mathscr{S} : M \to \mathbb{R}^N,\ \ (e', x) \mapsto \mathrm{pr}_{\mathbb{R}^N}\bigl(\phi(e', x)\bigr),$$

where

$$\phi : \mathcal{E} \oplus \mathcal{E}' \cong \mathcal{O}^N_X := \mathbb{R}^N \times X, \qquad N \text{ even}.$$

If we denote the pullbacks of the Thom classes of $\mathcal{E}, \mathcal{E}'$ by $\tau_{\mathcal{E}} \in H^*(\mathcal{O}^N_X, \mathcal{O}^N_X \smallsetminus \mathcal{E}')$, and $\tau'_{\mathcal{E}} \in H^*(\mathcal{O}^N_X, \mathcal{O}^N_X \smallsetminus \mathcal{E})$, we have

$$\mathscr{S}^*(\tau_{\mathbb{R}^N}) = \tau_{\mathcal{O}^N_X} = \tau_{\mathcal{E}} \cup \tau_{\mathcal{E}'} \ \in\ H^N\bigl(\mathcal{O}^N_X, (\mathcal{O}^N_X \smallsetminus \{0\}\bigr).$$

We can also identify $\mu_M \cap \tau_{\mathcal{E}'}$ with the fundamental class $\mu_X \in H_{k+d}(X)$. Hence for any class $\beta \in H^d(X)$, we have

$$\begin{aligned}\big\langle \beta, [X]^{vir}_{\mathcal{K}} \big\rangle &= \big\langle \tau_{\mathbb{R}^N}, \mathscr{S}_*(\mu_M \cap \beta) \big\rangle \\ &= \big\langle \tau_{\mathcal{E}} \cup \tau_{\mathcal{E}'}, \mu_M \cap \beta \big\rangle = \big\langle \tau_{\mathcal{E}} \cup \beta, \mu_M \cap \tau_{\mathcal{E}'} \big\rangle \\ &= \big\langle \pi_*(\tau_{\mathcal{E}}) \cup \beta, \mu_X \big\rangle = \big\langle \chi(\mathcal{E}) \cup \beta, \mu_X \big\rangle\end{aligned}$$

where $\pi : \mathcal{O}_X^N \to X$ is the projection. The result follows. □

Remark 3.4.2. (i) The above construction easily adapts to the case of an oriented orbifold bundle $\mathcal{E} \to W$ over an oriented orbifold W, where now we should think of the spaces $\mathcal{E}, W$ as the realizations of suitable ep categories $\mathbf{E}, \mathbf{W}$. Thus, one can build an atlas whose basic charts are as above with the addition of a group action, while the transition charts are made using composable tuples of morphisms in $\mathbf{E}$. For details, see [M2, §5.2]. One can then piece the corresponding fattened charts together by the method explained in §2,3 above to obtain a tuple $(M, E_A, \mathscr{S})$ as in Theorem 1.1.1. However, we can also build the category $\mathbf{M}$ directly from the set of basic charts $(U_i, E_i, \Gamma_i, s_i, \psi_i)$ by using a partition of unity subordinate to the associated covering of X by the sets $|\mathcal{O}_i|$. This should be thought of a baby example of the construction explained in Remark 1.3.8.

(ii) In Gromov–Witten theory it sometimes happens that the space of J-holomorphic maps in class A does form a compact manifold (or orbifold) X such that the rank of the cokernel of the linearized Cauchy–Riemann operator D_x at $x \in X$ is independent of x. In this case, these cokernels fit together to form a bundle $\mathcal{E} \to X$ such that the map $\mathfrak{s}$ induced by the Cauchy–Riemann operator is zero. We explain in [M2, Remark 5.2.4] why one can choose a Gromov–Witten type atlas (constructed as in [M2, §4] or [P]) with precisely the structure considered above.

(iii) In [P, Prop. 5.3.4], Pardon proves the analog of Lemma 3.4.1 in the smooth case, using a transverse perturbation of $\mathfrak{s}$ as in Step 3 of the proof of Lemma 3.3.2. ◇

Example 3.4.3. (The tangent bundle of the 2-sphere and the football) We now illustrate the construction in the proof of Theorem 1.1.1 in the case of $\mathrm{T}S^2$, starting from the atlas with two basic charts that was constructed in [MW3, Example 3.4.2]. We organize the details into several steps.

(I) *Atlas for the tangent bundle of the 2-sphere.* To build a Kuranishi atlas whose associated 'bundle' $\mathrm{pr} : |\mathbf{E}_{\mathcal{K}}| \to |\mathcal{K}|$ models $\mathrm{T}S^2$, cover S^2 by two copies D_1, D_2 of the unit disc in $\mathbb{C}$, whose intersection $D_1 \cap D_2 =: D_{12} =: A \cong [0,1] \times S^1$ is an annulus, and for $i = 1, 2$ define

$$\mathbf{K}_i := (U_i := D_i,\ E_i := \mathbb{C},\ s_i := 0,\ \psi_i := \mathrm{id}).$$

For $i = 1, 2$, choose unitary trivializations $\tau_i : D_i \times \mathbb{C} \to \mathrm{T}S^2|_{D_i}, (x, e) \mapsto \tau_{i,x}(e)$ that depend only on the absolute value $|x|$ of $x \in D_i$, and then define the transition chart

$$\mathbf{K}_{12} := \big(U_{12} \subset E_1 \times E_2 \times A,\ E_1 \times E_2,\ s_{12} = \mathrm{pr}_{E_1 \times E_2},\ \psi_{12} = \mathrm{pr}_A|_{0 \times 0 \times A}\big)$$

by setting

$$U_{12} := \big\{(e_1, e_2, x) \;\big|\; x \in A, \tau_{1,x}(e_1) + \tau_{2,x}(e_2) = 0\big\}.$$

The coordinate changes $\widehat{\Psi}_{i,12}$ are given by taking $U_{i,12} = A$ and $\psi_{i,12}(x) = (0,0,x)$. To justify this choice of Kuranishi atlas note that one can construct a commutative diagram

$$\begin{array}{ccc} |\mathbf{E}_{\mathcal K}| & \longrightarrow & \mathrm{T}S^2 \\ \downarrow & & \downarrow \\ |\mathbf{B}_{\mathcal K}| & \longrightarrow & S^2, \end{array}$$

where the top horizontal map restricts on $U_{12} \times E_{12}$ to the map

$$\big((e_1, e_2, x), e_1', e_2'\big) \mapsto \tau_{1,x}(e_1') + \tau_{2,x}(e_2') \in \mathrm{T}_x S^2 \subset \mathrm{T}S^2|_A.$$

Thus it takes

$$\operatorname{graph} s_{12} = \big\{\big((e_1, e_2, x),\ e_1, e_2\big) \;\big|\; (e_1, e_2, x) \in U_{12}\big\} \ \subset\ U_{12} \times E_{12}$$

to the zero section of $\mathrm{T}S^2$.

This construction is generalized to other (orbi)bundles in [M2]. $\diamond$

(II) *Calculating the Euler class.* In order to calculate the Euler class of $\mathrm{T}S^2$ it is convenient to identify A with $[0,1] \times S^1$, and then consider the corresponding trivialization $\mathrm{T}S^2|_A \equiv A \times \mathbb{R}_t \times \mathbb{R}_\theta$ where $t \in [0,1]$ and $\theta \in S^1$ are coordinates. Then for $i = 1,2$ there is a section $\nu_i : U_i \to E_i$ with one transverse zero such that

$$\tau_{i,x}(\nu_i(x)) = (x, 1, 0) \in A \times \mathbb{R}_t \times \mathbb{R}_\theta \equiv \mathrm{T}S^2|_A, \quad x \in A$$

(Take suitably modified versions of the sections $\nu_1(z) = z, \nu_2(z) = -z$ where $D_i \subset \mathbb{C}$.) Therefore the ν_i fit together to give a global section of $\mathrm{T}S^2$ with two transverse zeros, and it follows that the Poincaré dual of $\chi(\mathrm{T}S^2)$ is represented by $2[pt] \in H_0(S^2)$.

To see how $\chi(\mathrm{T}S^2)$ is calculated via the atlas, we start by choosing a reduction $\mathcal{G}$ of the footprint covering. For example, we may take $G_{12} = (\varepsilon, 1-\varepsilon) \times S^1 \sqsubset A$ for some $\varepsilon \in (0, \frac14)$ and choose $G_i \sqsubset D_i$ so that

$$\widetilde{V}_{1,12} = (0,0) \times (\varepsilon, \tfrac14) \times S^1 \subset U_{12}, \quad \widetilde{V}_{2,12} = (0,0) \times (\tfrac34, 1-\varepsilon) \times S^1 \subset U_{12}.$$

Choose a cutoff function $\beta : [0,1] \to [0,1]$ that equals 1 in $[0, \frac14]$ and 0 in $[\frac34, 1]$. Then the map $\nu_{12} : V_{12} \to E_1 \times E_2$ given by

$$\nu_{12}(e_1, e_2, x) \ =\ \big(\beta(x)\nu_1(x), (1-\beta(x))\nu_2(x)\big) \ \in\ E_1 \times E_2$$

restricts to ν_i on $V_{i,12} \subset (0,0) \times A$ for $i = 1,2$, so that the tuple (ν_1, ν_2, ν_{12}) is an admissible perturbation section in the sense of [MW3]. Moreover $s_{12} + \nu_{12}$ does not vanish at any point $(e_1, e_2, x_0) \in V_{12}$ because the equations

$$\tau_{1,x_0}(e_1) + \tau_{2,x_0}(e_2) = 0,$$
$$\tau_{1,x_0}(e_1) + \beta(x_0)(1,0) = \tau_{2,x_0}(e_2) + (1-\beta(x_0))(1,0) = 0 \ \in\ \{x_0\} \times \mathbb{R}_t \times \mathbb{R}_\theta$$

together imply that the vector $(1,0) \in \mathbb{R}_t \times \mathbb{R}_\theta$ is zero, a contradiction. Hence, as before, the perturbed zero set consists of two points, each with weight one. $\diamond$

(III) *Construction of the corresponding manifold M and section $\mathscr{S}_M : M \to E_{12}$.* When, as in the case at hand, the isotropy groups are trivial, the current paper constructs from the above reduction $\mathcal{V}$ of $\mathcal{K}$ a manifold M that is the union of three charts

$$M = \Big((M_1 = E_{2,\varepsilon} \times V_1) \sqcup (M_2 = E_{1,\varepsilon} \times V_2) \sqcup (M_{12} = V_{12}) \Big) / \sim,$$

where $\sim$ identifies $(e_j, x) \in M_{i,12}$ with $\alpha_{i,12}(e_j, x) \in \widetilde{M}_{i,12} \subset M_{12}$, where $\alpha_{i,12} = \tau_{i,12}^{-1}$ as in Example 2.2.1. With $i = 1, j = 2$, we may take the local product structure of (3.1.1) along

$$\widetilde{V}_{1,12} = \big\{ (0,0,x) \in V_{12} \subset E_1 \times E_2 \times A \big\}$$

to be given by the map

$$\phi = \phi^E_{IK,z_\alpha} : (E_{2,\varepsilon} \times \widetilde{V}_{1,12}, \{0\} \times \widetilde{V}_{1,12}) \to (V_{12}, \widetilde{V}_{1,12}), \quad (e_2, x) \mapsto \big(-\tau_{1,x}^{-1}(\tau_{2,x}(e_2)), e_2, x \big).$$

Since $\widetilde{V}_{1,12}$ is covered by one chart, the boundary collar is given by the map ψ of (3.2.8), and as in (2.2.5) the attaching map $\alpha_{1,12}$ has the form

$$\begin{aligned} &\alpha_{1,12} : \ E_{2,\varepsilon} \times V_{1,12} \to V_{12}, \\ &(e_2, x) \ \mapsto \ x' = \phi(\lambda \cdot e_2, x) = \big(-\tau_{1,x}^{-1}(\tau_{2,x}(\lambda e_2)), \lambda e_2, \, x \big), \quad \lambda := \sqrt{\|e_2\|}. \end{aligned}$$

Further, we take

$$\mathscr{S}_{12} = s_{12}, \quad \big(-\tau_{1,x}^{-1}(\tau_{2,x}(e_2)), e_2, x \big) \mapsto \big(-\tau_{1,x}^{-1}(\tau_{2,x}(\lambda e_2)), \lambda e_2 \big),$$

so that as in (2.2.6)

$$\mathscr{S}_1(e_2, x) = \beta_{1,12}(x) \big(-\tau_{1,x}^{-1}(\tau_{2,x}(\lambda e_2)), \lambda e_2 \big) + (1 - \beta_{1,12}(x)(0, e_2)$$

where $\beta_{1,12} : V_1 \to [0,1]$ equals 0 near $x = 0$ and 1 on $V_{1,12}$. There are similar formulas for $\alpha_{2,12}$ and $\mathcal{S}_2$. This construction gives a 4-manifold M together with a map $\mathscr{S}_M : M \to E_{12}$ whose zero set is homeomorphic to S^2. $\diamond$

(IV) *The normal bundle of $\mathscr{S}_M^{-1}(0) \cong S^2$ in M is isomorphic to* $\mathrm{T}S^2$. To see this, note that there is an embedding

$$M_1 \cup_{\alpha_{1,12}} \widetilde{M}_{1,12} \to \mathbb{C} \times D_1$$

given on $M_1 = E_{2,\varepsilon} \times V_1$ by the obvious inclusion (where we identify $E_2 \equiv \mathbb{C}$) and on $\widetilde{M}_{1,12}$ by

$$\big(-\tau_{1,x}^{-1}(\tau_{2,x}(e_2)), e_2, x \big) \mapsto (\lambda^{-1} e_2, x) \in E_2 \times A \subset \mathbb{C} \times D_1, \quad \lambda = \sqrt{\|e_2\|}.$$

Identifying A with $(\varepsilon, 1 - \varepsilon) \times S^1$ as above, we may extend this embedding over a neighborhood $\mathcal{N}_1 \subset M_{12}$ of the set $\{(0,0)\} \times (\varepsilon, \frac{1}{2}] \times S^1$ so that it equals

$$\big(-\tau_{1,x}^{-1}(\tau_{2,x}(e_2)), e_2, x \big) \mapsto (e_2, x), \qquad \forall\, x \in (\tfrac{1}{2} - \delta, \tfrac{1}{2}] \times S^1.$$

The similar embedding

$$(E_{1,\varepsilon} \times V_2) \cup_{\alpha_{1,12}} \mathcal{N}_2 \to \mathbb{C} \times D_2$$

is given near the circle $\{\frac{1}{2}\}\times S^1$ by the map $(e_1, -\tau_{2,x}^{-1}(\tau_{1,x}(e_1)), x) \mapsto (e_1, x)$. Therefore this bundle over S^2 is determined by the clutching map $x \mapsto -\tau_{2,x}^{-1}(\tau_{1,x})$, which is homotopic to the map $x \mapsto \tau_{2,x}^{-1}(\tau_{1,x})$ that determines $\mathrm{T}S^2$. $\diamond$

(V) *The case of the football orbifold* $S^2_{p,q}$. This orbifold is topologically S^2, but has orbifold points of orders p, q at the two poles. Thus it is again covered by two charts as above, with $\Gamma_1 = \mathbb{Z}/p\mathbb{Z}$ acting by rotations on D_1, E_1 and with $\Gamma_2 = \mathbb{Z}/q\mathbb{Z}$ acting by rotations on D_2, E_2. Since the trivializations $\tau_{i,x}$ are unitary and depend only on the absolute value $|x|$, they commute with the group actions. Hence we may describe the domain U_{12} of the transition chart $\mathbf{K}_{12}$ as before, with the obvious action of $\Gamma_{12} = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$. The most significant change is that the maps

$$\rho_{j,12} : \widetilde{U}_{j,12} \to U_{j,12} \subset D_j$$

are now nontrivial covering maps that quotient out by a free action of Γ_i, where $i \neq j$. We may calculate the Euler class by using essentially the same perturbation section as before, because $\nu_{12} : V_{12} \to E_{12}$ was chosen to be invariant under the rotation action of Γ_i, so that its pushforward to V_i has the same formulas as before. But now the two zeros of the section count with weights, $\frac{1}{p}$ for the zero in V_1 and $\frac{1}{q}$ for the zero in V_2.

The corresponding category $\mathbf{M}$ has three charts that are given by the same formulas as before, where Γ_{12} acts on U_{12} by

$$(\gamma_1, \gamma_2) \cdot (e_1, e_2, x) = \big(\gamma_1\gamma_2 \cdot e_1,\ \gamma_1\gamma_2 \cdot e_2,\ \gamma_1\gamma_2 \cdot x\big).$$

(This preserves the equation $\tau_{1,x}(e_1) + \tau_{2,x}(e_2) = 0$ because we assumed that the trivializations $\tau_{i,x}$ are unitary and depend only on the absolute value $|x|$.) Again, the attaching maps $\tau_{i,12} : \widetilde{M}_{i,12} \to M_{i,12} \subset M_i$ are nontrivial covering maps. However, now they do *not* quotient by the induced action of Γ_j on $\widetilde{M}_{i,ij}$ since they are constructed to be Γ_{12} equivariant, and Γ_{12} acts effectively on M_i, via

$$(\gamma_1, \gamma_2) \cdot (e_j, x_i) = (\gamma_j \cdot e_j,\ \gamma_i \cdot x_i).$$

However, as explained at the end of the proof of Proposition 2.2.2 (see for example (2.2.21)), they do quotient out by *some* action of Γ_j on $\widetilde{M}_{12}$ that extends its free action on $\widetilde{V}_{i,12} \subset \widetilde{M}_{i,12}$. For example, the map $\tau_{1,12}$ quotients out by the free action of Γ_q on $\widetilde{M}_{1,12} \subset E_1 \times E_2 \times (\varepsilon, \frac{1}{4}) \times S^1$ given by

$$\gamma \cdot (e_1, e_2, x) \mapsto (e_1, e_2, \gamma \cdot x).$$

Therefore, in the quotient space $M = |\mathbf{M}|$ there are q branches of M_{12} that come together over the 3-dimensional branching locus

$$Br_1 := \big\{(e_1, e_2, x) \in M_{12} \ \big|\ x \in \tfrac{1}{4} \times S^1\big\}.$$

This is consistent with the requirements of Definition 1.3.1 since the component M_{12} has weight $1/pq$ while M_1 has weight $1/p$.

The construction of $\mathscr{S}_M : M \to E_{12}$ is as before. Moreover, one can identify a neighborhood of its zero set $S^2_{p,q}$ with a neighborhood of the zero section of the tangent orbibundle to $S^2_{p,q}$. Hence the Poincaré dual of $\chi(\mathrm{T}S^2_{p,q})$ is represented by

$$(1/p+1/q)[pt] \in H_0(S^2_{p,q}).$$

(VI) *The quotient space M/Γ for $\mathrm{T}S^2_{p,q}$.* The only morphisms in the category $\mathbf{M}$ come from the covering maps $\tau_{j,12}$. Since these are Γ_{12}-equivariant, we can add the action $\Gamma_{12}\times \mathrm{Obj}_{\mathbf{M}} \to \mathrm{Obj}_{\mathbf{M}}$ to the morphisms in $\mathbf{M}$. The resulting quotient space M/Γ_{12} has the following structure.

- It is covered by three branches M_1, M_2, M_{12} with weights $1/p^2q$, $1/pq^2$ and $1/p^2q^2$;
- the two poles $[(0,0)] \in M_i/\Gamma_{12}$ have stabilizer subgroup Γ_{12};
- the other points with nontrivial stabilizers lie on the two closed discs
$$\{0\}\times(\overline{V_i}\setminus\{0\})/\Gamma_{12} \subset M_i/\Gamma_{12}, \quad i=1,2$$
with isotropy subgroups $\Gamma_j, j\neq i$;
- for $i=1,2$ there is branching of order $|\Gamma_j|$ over the image of the 3-manifolds Br_i.

We do not consider this space further, since it plays no role in the definition of the fundamental class.

Appendix A. Rational Çech cohomology and homology

We briefly describe the properties of the (co)homology theories in [Ma] that are based on the properties of Alexander–Spanier cochains. We do not need the full generality of this theory because the space $M=|\mathbf{M}|_{\mathcal{H}}$ is locally compact and Hausdorff. Throughout we assume that Y is locally compact and Hausdorff, with $A\subset Y$ closed and $U\subset Y$ open, and take coefficients in $\mathbb{Q}$. Further, we denote these theories by $\check{H}$ to distinguish them from singular (co)cohomology. [22]

We need the following properties of the cohomology theory.

(a) ([Ma, Thm 3.21]) If Y is a connected orientable n-dimensional manifold then $\check{H}^i(Y)=0$ unless $i=n$ in which case $\check{H}^n(Y)=\mathbb{Q}$, i.e. $\check{H}^*$ is like rational singular cohomology with compact supports;

(b) ([Ma, §1.2]) If $f: A\to Y$ is proper, there is an induced map $f^*:\check{H}^i(Y)\to\check{H}^i(A)$;

(c) ([Ma, §1.3]) if $U\subset Y$ is open, there is an induced map $f_*:\check{H}^i(U)\to\check{H}^i(Y)$. Further, if Y is as in (a), and U is an open n-disc, then f_* is an isomorphism.

(d) ([Ma, Thm 1.6]) if $A\subset Y$ is closed then there is an exact sequence

$$\cdots\to\check{H}^i(Y\setminus A)\to\check{H}^i(Y)\to\check{H}^i(A)\xrightarrow{\delta}\check{H}^{i+1}(Y\setminus A)\to\cdots, \tag{A.1}$$

i.e. the group $\check{H}^i(A)$ plays the role of the relative group $H^i(Y, Y\setminus A)$.

[22] In [Ma, Ch 10] the theory we call $\check{H}^*$ below is denoted by H^*_c to distinguish it from another theory that does not concern us here.

The dual homology theory developed in [Ma, Ch 4] is denoted H_*^∞ in [Ma, Ch 10] to emphasize that it is analogous to locally finite singular homology; we shall call it $\check{H}_*^\infty$. It follows from the universal coefficient theorem [Ma, Thm. 4.17] that

$$\check{H}_k^\infty(X) = \operatorname{Hom}\big(\check{H}^k(X);\mathbb{Q}\big). \tag{A.2}$$

As shown by the following, the functorial properties of H_*^∞ are different from the usual singular theory.

(a$'$) If Y is a connected orientable n-manifold, then $\check{H}_i^\infty(Y) = 0$ unless $i = n$ in which case $\check{H}_n^\infty(Y) = \mathbb{Q}$; more generally, any orientable n-manifold has a fundamental class

$$\mu_Y \in \check{H}_n^\infty(Y). \tag{A.3}$$

(b$'$) ([Ma, §4.6]) if $U \subset Y$ is open, there is an induced restriction

$$\rho_{Y,U} : \check{H}_i^\infty(Y) \to \check{H}_i^\infty(U); \tag{A.4}$$

moreover for $U_1 \subset U_2 \subset Y$ we have $\rho_{Y,U_1} = \rho_{U_2,U_1} \circ \rho_{Y,U_2}$.

(c$'$) ([Ma, §4.6]) If $f : A \to Y$ is continuous and proper, then there is an induced pushforward $f_* : \check{H}_i^\infty(A) \to \check{H}_i^\infty(Y)$; moreover, given a proper inclusion $\iota : A \to Y$, there is a functorial long exact sequence

$$\cdots \to \check{H}_i^\infty(A) \xrightarrow{\iota_*} \check{H}_i^\infty(Y) \xrightarrow{\rho_{Y,Y\smallsetminus A}} \check{H}_i^\infty(Y\smallsetminus A) \xrightarrow{\partial} \check{H}_{i-1}^\infty(A) \to \cdots \tag{A.5}$$

(d$'$) ([Ma, §4.3 (3c)]) If $f : A \to Y$ is proper and U is open in Y, then the following diagram commutes:

$$\begin{tikzcd} \check{H}_i^\infty(A) \arrow[r, "f_*"] \arrow[d, "\rho_{A,A\cap f^{-1}(U)}"] & \check{H}_i^\infty(Y) \arrow[d, "\rho_{Y,U}"] \\ \check{H}_i^\infty(A\cap f^{-1}(U)) \arrow[r, "f_*"] & \check{H}_i^\infty(U). \end{tikzcd}$$

(e$'$) ([Ma, §4.9 (6)]) if $Y = U \cup V$ where U, V are open then there is an exact Mayer–Vietoris sequence of the form:

$$\cdots \to \check{H}_{i+1}^\infty(U\cap V) \to \check{H}_i^\infty(Y) \to \check{H}_i^\infty(U) \oplus \check{H}_i^\infty(V) \to \check{H}_i^\infty(U\cap V) \to \cdots$$

In particular, if U is the disjoint union of a finite number of sets of U_i, then

$$\check{H}_*^\infty(U) \cong \oplus_i \check{H}_*^\infty(U_i).$$

(f$'$) ([Ma, p.334]) if $U \subset Y$ is open while $A \subset Y$ is closed, there is a cap product

$$\cap : \check{H}_{p+q}^\infty(Y\smallsetminus A) \otimes \check{H}^p(Y\smallsetminus U) \to \check{H}_q^c(Y, U\cup A). \tag{A.6}$$

This takes values in compactly supported Čech homology, a theory whose functorial properties are analogous to those of the usual singular homology. In particular, if the triple $(U \cup A; U, A)$ is *excisive* for $\check{H}^c$ (i.e. $\check{H}^c_q(A, U \cap A) \cong \check{H}^c_q(U \cup A, U)$), then there is a commutative diagram

$$\begin{CD} \check{H}^\infty_{p+q+1}(Y \setminus A) \otimes \check{H}^p(Y \setminus U) @>{\cap}>> \check{H}^c_{q+1}(Y, U \cup A) \\ @V{\partial \otimes i^*}VV @VV{\delta}V \\ \check{H}^\infty_{p+q}(A) \otimes \check{H}^p(A \setminus U) @>{\cap}>> \check{H}^c_q(A, U \cap A). \end{CD} \tag{A.7}$$

Note that the above diagram exists when Y is locally compact, A is closed and $Y \setminus U$ is compact. To see this, choose a nested sequence $\mathcal{N}_k$ of precompact open neighborhoods of $Y \setminus U$ in Y with

$$Y \setminus U = \textstyle\bigcap_k \mathcal{N}_k, \quad U = \textstyle\bigcup_k (Y \setminus \mathcal{N}_k).$$

Since by definition

$$\check{H}^c_*(Y, U \cup A) = \lim_{\leftarrow} \check{H}^c_*(Y, (Y \setminus \mathcal{N}_k) \cup A), \qquad \check{H}^c_*(A, U \cap A) = \lim_{\leftarrow} \check{H}^c_*(A, A \setminus \mathcal{N}_k),$$

and the triple of closed sets $(Y, Y \setminus \mathcal{N}_k, A)$ is excisive by [Ma, Cor.9.5], it follows that $(Y, Y \setminus U, A)$ is excisive as required.

(g′) (Exercise 5 on p 272 of [Ma]) If X is Hausdorff and $X \setminus A$ is a precompact open subset of X, then $\check{H}^\infty_*(X \setminus A) = \check{H}^c_*(X, A)$.

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Department of Mathematics, Barnard College, Columbia University
E-mail address: `dusa@math.columbia.edu`