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CATEGORIES WITH NEGATION

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Abstract. We continue the theory of $\mathcal{T}$-systems from the work of the second author, developing their categorical aspects, including a categorical theory with formal negation, focusing on tensor products, ground systems, and module systems over a ground system (paralleling the theory of modules over an algebra). Here abelian categories are replaced by semi-abelian categories (where $\mathrm{Hom}(A, B)$ is not a group) with a negation morphism. The theory is broad enough to encapsulate general algebraic structures lacking negation but possessing a map resembling negation, such as tropical algebras, hyperfields and fuzzy rings.

Contents

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1. Introduction

This paper is the continuation of a project started in [56, 57] where, using [19] as a prototype, we developed a generalization of classical algebraic theory to include applications in related algebraic theories. In [57, Definitions 2.11, 2.25], inspired by the "symmetrized" theory of matrices of [2, 19, 20, 53], which also can be viewed as 2-graded structures, we framed the notion of **negation map** in the language of universal algebra. It was motivated by an attempt to understand whether or not it is coincidental that basic algebraic theorems are mirrored in supertropical algebra, and was spurred by the realization that some of the same results are obtained in parallel research on hypergroups and hyperfields [6, 22, 36, 37, 38, 63] and fuzzy rings [15, 16, 22], which lack negatives although each has an operation resembling negation [1].

The underlying idea was to take a set $\mathcal{T}$ that we want to study, and embed it into a "better" structure $\mathcal{A}$ possessing a formal negation map $(-)$, resembling negation, such that $\mathcal{T} \cup (-)\mathcal{T}$ generates $\mathcal{A}$ additively. Together they comprise a **$\mathcal{T}$-triple** $(\mathcal{A}, \mathcal{T}, (-))$. This is viewed categorically in Definition 3.6.

We write $a(-)b$ for $a + ((-)b)$, and a° for $a(-)a$, called a **quasi-zero**. Thus, in classical algebra, the only quasi-zero is 0 itself if we use the negation map as a usual negation. To generalize the uniqueness of additive inverses in our setting, we introduce the following definition.

Definition 1.1. *A (pseudo) triple $\mathcal{A} = (\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-))$ is **uniquely negated** if $a_1 + a_2 \in \mathcal{A}^\circ = \{a^\circ \mid a \in \mathcal{A}\}$ for $a_1, a_2 \in \mathcal{T}_{\mathcal{A}}$ implies $a_2 = (-)a_1$.*

Our main examples are uniquely negated. Having generalized negation to the negation map, the next step is to find a workable substitute for equality, which turns out to be a partial order (PO), not an equality.

Definition 1.2. *Define $\preceq_\circ$ by $a \preceq_\circ b$ if $b = a + c^\circ$ for some c.*

The relation $\preceq_\circ$ restricts to equality on $\mathcal{T}$ for uniquely negated triples, by [57, Proposition 4.4]. In [57, Definition 2.65], we define a **$\mathcal{T}$-surpassing PO** denoted by $\preceq$; this usually turns out to be $\preceq_\circ$, but for power sets of hypergroups, $\preceq$ is just $\subseteq$ as to be explained shortly.

A motivating example from classical algebra (in which $\preceq$ is just equality) is for $\mathcal{A}$ to be an associative algebra graded by a monoid; $\mathcal{T}$ could be the subset of homogeneous elements, in particular, a submonoid of $(\mathcal{A}, \cdot)$. A transparent example from classical mathematics is when $\mathcal{T}$ is the multiplicative subgroup of a field $\mathcal{A}$.

We are most interested in the non-classical situation, involving semirings which are not rings. Our structure of choice, a **$\mathcal{T}$-system**, is a quadruple $(\mathcal{A}, \mathcal{T}, (-), \preceq)$, where $(\mathcal{A}, \mathcal{T}, (-))$ is a $\mathcal{T}$-triple and $\preceq$ is a $\mathcal{T}$-surpassing PO. $\mathcal{T}$-Systems are the main subject in [57], employed there to establish basic connections with tropical structures, hypergroups, and fuzzy rings, by means of the following examples:

[1] In a hypergroup, for each element a there is an element called $-a$, such that $0 \in a \boxplus (-a)$; a fuzzy ring A has an element ε such $1 + \varepsilon$ is in a distinguished ideal A_0, but is not necessarily 0.

- (The standard supertropical triple) $(\mathcal{A}, \mathcal{T}, (-))$ where $\mathcal{A} = \mathcal{T} \cup \mathcal{A}^\circ$ and $(-)$ is the identity map. (In this case we call $\mathcal{A}^\circ$ the "ghost elements" $\mathcal{G}$.) We get the system by taking $\preceq$ to be $\preceq_\circ$.
- (The symmetrized triple) $(\hat{\mathcal{A}}, \widehat{\mathcal{T}}, (-))$ where $\hat{\mathcal{A}} = \mathcal{A} \times \mathcal{A}$ with componentwise addition, $\widehat{\mathcal{T}} = (\mathcal{T} \times \{\mathbb{0}\}) \cup (\{\mathbb{0}\} \times \mathcal{T})$ and with multiplication $\widehat{\mathcal{A}} \times \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ given by
$$(a_0, a_1)(b_0, b_1) = (a_0 b_0 + a_1 b_1, a_0 b_1 + a_0 b_1).$$
Here we take $(-)$ to be the "switch map" $((-)(a_0, a_1) = (a_1, a_0))$. Again $\preceq$ is $\preceq_\circ$.
- (The hyperfield triple [57, §2.4.1]) Let $\mathcal{T}$ be a hyperfield. Then one can associate a triple $(\mathcal{S}(\mathcal{T}), \mathcal{T}, (-))$, where $\mathcal{P}(\mathcal{T})$ is its power set (with componentwise operations), $\mathcal{S}(\mathcal{T})$ is the additive sub-semigroup of $\mathcal{P}(\mathcal{T})$ spanned by the singletons, and $(-)$ on the power set is induced from the hypernegation. Here $\preceq$ is $\subseteq$.
- For any $\mathcal{T}$-monoid module $\mathcal{A}$ with an element $\mathbb{1}' \in \mathcal{T}$ satisfying $(\mathbb{1}')^2 = \mathbb{1} \in \mathcal{A}$, we can define a negation map $(-)$ on $\mathcal{T}$ and $\mathcal{A}$ given by $a \mapsto \mathbb{1}'a$. In particular, $(-)\mathbb{1} = \mathbb{1}'$. This enables us to view fuzzy rings as systems, as shown in [57, Appendix A].

Structure theory involving several sets can be described in universal algebra in terms of operations and identities, which we review briefly in §4. But immediately the "surpassing PO" $\preceq$ takes us to the fringe of universal algebra, since it involves a non-symmetric relation. The triple together with $\preceq$ is called a **$\mathcal{T}$-system)**.

One can apply familiar concepts from classical algebra (direct sums [57, Definition 2.10], matrices [57, §4.5], involutions [57, §4.6], polynomials [57, §4.7], localization [57, §4.8], and tensor products [57, Remark 6.34]) to produce new triples and systems.

In this paper, categorical aspects of systems functor categories are treated in much greater depth, including (§3.4), tensor products (§6.1), and Hom (§9). This wealth of examples motivates a further development of the algebraic theory of $\mathcal{T}$-systems, which is the rationale for this paper. Here we continue the study of $\mathcal{T}$-systems, employing the customary methods of classical algebra, e.g., structure theory and representation theory.

Classical structure theory involves the investigation of algebraic structures as small categories (for example, viewing a monoid or a ring as a category with a single object whose morphisms are its elements), and homomorphisms then are functors between two of these small categories. On the other hand, representation theory often is obtained via an abelian category, such as the class of modules over a given ring.

Analogously, there are two main ways of applying triples. We call a triple (resp. $\mathcal{T}$-system) a **ground triple (resp. ground $\mathcal{T}$-system)** when we study it as a small category with a single object in its own right, often a semidomain. The emphasis in [57] was on these "ground" $\mathcal{T}$-triples for the theory, often **meta-tangible** $\mathcal{T}$-triple, by which we mean $a_1 + a_2 \in \mathcal{T}_\mathcal{A}$ for any $a_2 \neq (-)a_1$ in $\mathcal{T}_\mathcal{A}$ (see Definition 5.14). This leads to a structure theory based on functors of ground $\mathcal{T}$-systems, and translates into homomorphic images of $\mathcal{T}$-systems via congruences, in §7. As in classical algebra, the "prime" systems [57, Definitions 2.11, 2.25] play an important role in affine geometry, so it is significant that a polynomial system over a prime system is prime (Theorem 7.29 and Corollary 7.30).

A very important special case: $(\mathcal{A}, \mathcal{T}, (-))$ is $(-)$-**bipotent** if $a + b \in \{a, b\}$ whenever $a, b \in \mathcal{T}$ with $b \neq (-)a$. In other words, $a + b \in \{a, b, a^\circ\}$ for all $a, b \in \mathcal{T}$. (We also say that $\mathcal{T}$ is $(-)$-bipotent.)

We aim for a representation theory of $\mathcal{T}$-systems via category theory. We take a category of **module systems** over our ground triples, cf. §8, in analogy to modules over algebras. Since we lack the classical negative, the trivial subcategory of $\mathbb{0}$ morphisms and $\mathbb{0}$ objects is replaced by a more extensive subcategory of quasi-zero morphisms and quasi-zero objects, inferred from the "diodes" of [19], Izhakian's thesis [28], and the "ghost ideal" of [32] and studied explicitly in the symmetrized case in [56, §7.2.1] and [57, §2.3.7], and [11, §4] (over the Boolean semifield $\mathbb{B}$). Here symmetrization (§2.2) plays a key role, since it has a built-in negation map. The quasi-zero morphisms have been treated formally in [24, §1.3] under the name of **N-category** and **homological category**, with the terminology "null morphisms" and "null objects," and with the null subcategory designated as (E, Null)[2]. In this paper we make some brief references to [24], to be elaborated in [39].

[2] [24, §1.3] uses the notation $(E, \mathcal{N})$, but we prefer to reserve $\mathcal{N}$ for later use.

One key point here is that in the theory of semirings and their modules, homomorphisms are described in terms of congruences, and the null congruences contain the diagonal, and not necessarily zero, so these should be the focus of the theory.

There is a very well-developed category theory built on abelian categories, which one would like to utilize by generalizing to "semi-abelian" categories. This has already been done for a large part in [24], so one main task should be to arrange for the category of systems to fit into Grandis' hypotheses. This turns out to be trickier than one might expect.

An issue that we must confront at the outset is the proper definition of morphisms. In categories arising from universal algebra, one's intuition would be to take the homomorphisms, i.e., those maps which preserve equality in the operators. We call these morphisms "strict." However, restricting our attention to strict morphisms overlooks some major examples in hypergroups. Applications in tropical mathematics and hypergroups (cf. [36, Definition 2.3]) tend to depend on the "surpassing relation" $\preceq$ of Definition 4.8, so we are led to a broader definition called $\preceq$-category, in which we replace equality by $\preceq$ in the definition of morphisms, cf. Definition 5.1.

The tensor product and its abstraction to monoidal categories, one of the main tools in algebra, is exposed in [17] for monoidal abelian categories. But here, lacking negatives, we must make do with "semi-abelian" categories and, to our dismay, semi-abelian categories need not be monoidal. This complication impinges on the functoriality of the tensor product, which runs into stumbling blocks because of the asymmetry involved in $\preceq$-morphisms, cf. Example 6.7. So we have a give and play between morphisms and strict morphisms. These issues are treated in §6.1.

$\mathrm{Mor}(A, B)$ denotes the set of morphisms from A to B. We assume throughout that for a category $\mathcal{C}$, the set $\mathrm{Mor}(A, B)$ is endowed with addition. $(\mathrm{Mor}(A, B), +)$ will no longer necessarily be a group, but rather a semigroup. Accordingly, one needs to weaken the notion of additive and abelian categories, respectively to **semi-additive categories** and **semi-abelian categories** [24, §1.2.7]. By Theorem 8.16, the category of module systems whose morphisms are strict is a semi-abelian category, and we explore their categorical properties, where kernels of morphisms now are congruences rather than objects, and are not described explicitly as categorical morphisms.

In short, our overall strategy is to fix a ground $\mathcal{T}$-system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$, defined by means of techniques from universal algebra, and then consider its "prime" homomorphic images, as well as the module systems over this ground $\mathcal{T}$-system, which can be expressed in terms of universal algebra and thus comprise a semi-abelian category.

En route we also hit a technical glitch in applying universal algebra, which historically appeared before tensor categories. The tensor product, which exists for systems, cannot be described directly in universal algebra, since the length t of a sum $\sum_{i=1}^{t} a_i \otimes b_i$ of simple tensors need not be bounded. So we would need a "monoidal" universal algebra, where the signature contains the tensor products of the original structures. This would permit us to define comultiplication and co-semialgebras, as considered in [39].

Our objectives in this paper are as follows:

(1) Define negation morphisms and negation functors in categories, together with a surpassing relation.
(2) Develop the notion of negation map and "ground" $\mathcal{T}$-triples and $\mathcal{T}$-systems, which should parallel the classical structure theory of algebras.
(3) Adapt semiring theory to the theory of $\mathcal{T}$-systems. Lay out the theory of prime ground $\mathcal{T}$-triples.
(4) Generalize abelian categories to semi-abelian categories with negation, developing the theory of $\mathcal{A}$-module triples over $\mathcal{T}$-triples. This should parallel the theory of modules over semirings, which has been developed in the last few years by Katsov [40, 41, 42], Patchkoria [51], Macpherson [47], Takahashi [61].
(5) See which classical module-theoretic concepts (such as direct sums and exact sequences) have analogs for $\mathcal{A}$-module triples over ground $\mathcal{T}$-triples, viewed in terms of their congruences.

In §10.3 we provide the functorial context for the main ideas of this paper. Ultimately, in [39], we intend to use the negation morphism to compute homology and cohomology.

2. Basic notions

We review a few notions from semigroups and semirings, following [23], and categories [24], and introduce a few others. As customary, $\mathbb{N}$ denotes the nonnegative integers, $\mathbb{Q}$ denotes the rational numbers, and $\mathbb{R}$ denotes the real numbers, all ordered monoids under addition.

A **semiring**† $(\mathcal{A}, +, \cdot, 1)$ is an additive abelian semigroup $(\mathcal{A}, +)$ and a multiplicative monoid $(\mathcal{A}, \cdot, \mathbb{1})$ satisfying the usual distributive laws. A **semiring** is a semiring† which contains a $\mathbb{0}$ element. A **semidomain** is a semiring $\mathcal{A}$ such that $\mathcal{A} \setminus \{\mathbb{0}\}$ is closed under multiplication, i.e., $ab = \mathbb{0}$ only when $a = \mathbb{0}$ or $b = \mathbb{0}$ for all $a, b \in \mathcal{A}$. In contrast to [57], we do not assume commutativity of multiplication, since we want to consider matrices and other noncommutative structures more formally. But at times (for example in considering prime spectra) we specialize to the commutative situation when the proofs are simpler.

A **semialgebra** is defined analogously, but need not have associative multiplication.

2.1. $\mathcal{T}$-modules.

Our specific objective is to put $\mathcal{T}$ into the limelight, starting with the following basic notion.

Definition 2.1. *A left $\mathcal{T}$-**module** over a set $\mathcal{T}$ is an additive monoid $(\mathcal{A}, +, \mathbb{0}_M)$ with a scalar multiplication $\mathcal{T} \times \mathcal{A} \to \mathcal{A}$ satisfying the following axioms, $\forall u \in \mathbb{N}$, $a, a_i \in \mathcal{T}$, $b, b_i \in \mathcal{A}$:*

(i) *(Distributivity over $\mathcal{T}$): $a(\sum_{j=1}^{u} b_j) = \sum_{j=1}^{u}(ab_j)$.*

(ii) *$a\mathbb{0}_{\mathcal{A}} = \mathbb{0}_{\mathcal{A}}$.*

*A **right** $\mathcal{T}'$-**module** over a set $\mathcal{T}'$ is defined analogously on the right. A $(\mathcal{T}, \mathcal{T}')$-**bimodule** over sets $\mathcal{T}, \mathcal{T}'$ is a left $\mathcal{T}$-module which is also a right $\mathcal{T}'$-module satisfying $a(ba') = (ab)a'$ for $a \in \mathcal{T}$, $a' \in \mathcal{T}$.*

In other words, $\mathcal{T}$ acts on $\mathcal{A}$. (In Definition 5.52, we discuss the possibility of distributivity failing over $\mathcal{T}$.) When $\mathcal{T}$ is a monoid, we require some extra universal relations:

Definition 2.2. *A left $\mathcal{T}$-**monoid module** over a monoid $(\mathcal{T}, \cdot, \mathbb{1})$ is a left $\mathcal{T}$-module that also respects the monoid structure, i.e., satisfies the axioms, $\forall a_i \in \mathcal{T}$, $b \in \mathcal{A}$:*

(i) *$(\mathbb{1})b = b$.*

(ii) *$(a_1a_2)b = a_1(a_2b)$.*

*A right $\mathcal{T}$-monoid module is defined analogously on the right. When $(\mathcal{T}, \cdot)$ is a group, we call $\mathcal{A}$ a $\mathcal{T}$-**Gmodule** to emphasize this fact.*

Likewise, one defines a **module** over a semiring (called **semimodule** in [23]), necessarily satisfying distributivity. A **left ideal** of a semiring $\mathcal{A}$ is a proper left submodule of $\mathcal{A}$. A **right ideal** is defined analogously. An **ideal** is a left and right ideal. In what follows, we assume that all modules are left modules unless otherwise stated.

2.2. Symmetrization and the twist action.

Definition 2.3. *Let $\mathcal{A}$ be a $\mathcal{T}$-module with $\mathcal{T} \subseteq \mathcal{A}$. Then $\mathcal{A}$ is said to be a $\mathcal{T}$-**super module** if $\mathcal{A}$ is a $\mathbb{Z}_2$-graded semigroup $\mathcal{M}_0 \oplus \mathcal{M}_1$, satisfying $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ where $\mathcal{T}_i = \mathcal{T} \cap \mathcal{M}_i$ and $\mathcal{T}_i\mathcal{M}_j \subseteq \mathcal{M}_{i+j}$, subscripts modulo 2.*

From now on, $\widehat{\mathcal{M}}$ denotes $\mathcal{M} \times \mathcal{M}$, i.e., $\mathcal{M}_0 = \mathcal{M}_1 = \mathcal{M}$.

Definition 2.4. *The **switch map** on $\widehat{\mathcal{M}}$ is given by $(b_0, b_1) \mapsto (b_1, b_0)$.*

In order to identify the second component as the negation of the first, we employ an idea utilized in Bourbaki [10] and the Grothendieck group completion, as well as [8, 11, 19, 35, 37, 57] (but for $\mathcal{T}$-modules instead of the more special case of semirings†), which comes from the familiar construction of $\mathbb{Z}$ from $\mathbb{N}$. The idea arises from the elementary computation:

$$(a_0 - a_1)(b_0 - b_1) = (a_0b_0 + a_1b_1) - (a_0b_1 + a_1b_0).$$

Definition 2.5. *$\widehat{\mathcal{M}}$ is called the **symmetrization** of $\mathcal{M}$. For any $\mathcal{T}$-module $\mathcal{M}$, the **twist action** on $\widehat{\mathcal{M}}$ over $\widehat{\mathcal{T}}$ is given by the super-action, namely*

$$(a_0, a_1) \cdot_{\mathrm{tw}} (b_0, b_1) = (a_0b_0 + a_1b_1, a_0b_1 + a_1b_0). \tag{2.1}$$

Lemma 2.6. *$\widehat{\mathcal{M}}$ is a $\widehat{\mathcal{T}}$-module.*

Proof. To see that the twist action is associative, we note that

$$\begin{aligned}((a_0,a_1)\cdot_{\mathrm{tw}}(b_0,b_1))\cdot_{\mathrm{tw}}(c_0,c_1) &= (a_0b_0+a_1b_1, a_0b_1+a_1b_0)\cdot_{\mathrm{tw}}(c_0,c_1)\\ &= (a_0b_0c_0+a_1b_1c_0+a_0b_1c_1+a_1b_0c_1, a_0b_0c_1+a_1b_1c_1+a_0b_1c_0+a_1b_0c_0),\end{aligned} \tag{2.2}$$

in which we see that the subscript $_0$ appears an odd number of times on the left and an even number of times on the right, independently of the original placement of parentheses. □

Symmetrization plays an especially important role in homology theory, to be seen in [39].

2.2.1. *The symmetrized semiring*†.

Here is an important application of the twist, whose role in tropical geometry is featured in [35].

Definition 2.7. *The* ***symmetrized semiring***† *$\widehat{\mathcal{A}} := \mathcal{A}\times\mathcal{A}$ of a semiring $\mathcal{A}$ is $\widehat{\mathcal{A}}$ viewed as a twisted $\mathcal{T}$-module over $\mathcal{A}$, and made into a semiring via the "twisted" multiplication*

$$(a_0,a_1)\cdot_{\mathrm{tw}}(b_0,b_1) = (a_0b_0+a_1b_1, a_0b_1+a_1b_0)$$

for $a_i,b_i\in\mathcal{A}$. We identify $\mathcal{A}$ inside $\widehat{\mathcal{A}}$ via the injection $a\mapsto(a,\mathbb{0})$.

In particular, we have

$$(a_0,a_1)^2 = (a_0,a_1)\cdot_{\mathrm{tw}}(a_0,a_1) = (a_0^2+a_1^2, 2a_0a_1),$$

and inductively

$$(a_0,a_1)^n = (a_0,a_1)^{n-1}\cdot_{\mathrm{tw}}(a_0,a_1) = (\sum_{i\text{ even}}\binom{n}{i}a_0^ia_1^{n-i}, \sum_{i\text{ odd}}\binom{n}{i}a_0^ia_1^{n-i}).$$

Remark 2.8. *When a semiring*† *$\mathcal{A}$ is idempotent, the above formula simplifies to:*

$$(a_0,a_1)^n = \left(\sum_{i\ even}a_0^ia_1^{n-i}, \sum_{i\ odd}a_0^ia_1^{n-i}\right).$$

3. Some abstract category theory

We use the more general categorical definitions, as long as we can. Recall the usual categorical definitions of **monic** and **epic**, called mono and epi in [24, § 1.1.5]. Throughout the paper, we always assume that our category $\mathcal{C}$ is concrete unless otherwise stated.

3.1. N-categories.

Since we lack negatives in semirings and their modules, $\mathrm{Mor}(A,B)$ is not a group under addition, but rather a semigroup, and the zero morphism loses its special role, to be supplanted by a more general notion.

Definition 3.1. *Let $\mathcal{C}$ be a category.*

- *A* ***left absorbing set of morphisms*** *of $\mathcal{C}$ is a collection of sets of morphisms I such that if f belongs to I, then any composite gf (if defined) belongs to I.*
- *A* ***right absorbing set of morphisms*** *of $\mathcal{C}$ is a collection of sets of morphisms I such that if f belongs to I, then any composite fg (if defined) belongs to I.*
- *An* ***absorbing set of morphisms*** *is a left and right absorbing set of morphisms. (*[24, § 1.3.1] *calls this an "ideal" but we prefer to reserve this terminology for semirings.)*

For any given absorbing set N of morphisms of a category $\mathcal{C}$, one can associate the set of objects $O(N)$ as follows:

$$O(N) := \{A\in\mathrm{Obj}(\mathcal{C}) \mid 1_A\in N\}.$$

Conversely, for any set O of objects in $\mathcal{C}$, one can associate the absorbing set of morphisms $N(O)$ as follows:

$$N(O) := \{f\in\mathrm{Mor}(\mathcal{C}) \mid \ f \text{ factors through some object in } O\}.$$

Then one clearly has the following:

$$O \subseteq ON(O), \quad NO(N) \subseteq N.$$

Definition 3.2 ([24, §1.3]). *Let $\mathcal{C}$ be a category.*

- *An absorbing set N of $\mathcal{C}$ is **closed** if $N = N(O)$ for some set of objects O.*
- *We fix a class of null objects, and define the **null morphisms** to be the compositions of morphisms to and/or from null objects.*

The null morphisms Null (depending on a fixed class of null objects) are an example of an absorbing set of morphisms; the null morphisms in $\mathrm{Mor}(A, B)$ are designated as $\mathrm{Null}_{A,B}$, which will take the role of $\{0\}$.

Such a category with a designated class of null objects and of null morphisms is called a **closed** N-**category**; we delete the word "closed" for brevity.

Definition 3.3. [24, § 1.3.1]. *Let $\mathcal{C}$ be an N-category with a fixed class of null objects O and the corresponding null morphisms N.*

- *The **kernel with respect to** N of a morphism $f : A \to B$, denoted by $\ker f$, is a monic which satisfies the universal property of a categorical kernel with respect to N, i.e.,*
  (1) *$f(\ker f)$ is null.*
  (2) *If fg is null then g uniquely factors through $\ker f$ in the sense that $g = (\ker f)h$ for some h.*
- *The **cokernel with respect to** N of a morphism $f : A \to B$, denoted by $\operatorname{coker} f$, is defined dually, i.e., $\operatorname{coker} f$ is an epic satisfying*
  (1) *$(\operatorname{coker} f)f$ is null.*
  (2) *If gf is null then g uniquely factors through $\operatorname{coker} f$ in the sense that $g = h(\operatorname{coker} f)$ for some h.*
- *Products and coproducts (direct sums) of morphisms also are defined in the usual way, cf.* [17, Definition 1.2.1(A2)].

3.2. **Categories with a negation functor.**

We introduce another categorical notion, to compensate for lack of negatives.

Definition 3.4. *Let $\mathcal{C}$ be a category. A **negation functor** $(-) : \mathcal{C} \to \mathcal{C}$ is an endofunctor satisfying $(-)A = A$ for each object A, and, for all compatible morphisms f, g:*

(i) *$(-)((-)f) = f$.*
(ii) *$(-)(fg) = ((-)f)g = f((-)g)$, i.e., a composite of morphisms (if defined) commutes with $(-)$.*

We write $f(-)g$ for $f + ((-)g)$, and f° for $f(-)f$.

Remark 3.5. *The identity functor obviously is a negation functor, since these conditions becomes tautological. But categories may fail to have a natural non-identity negation functor. For example, a nontrivial negation functor for the usual category* **Ring** *might be expected to contain "negated homomorphisms" $-f$ where $(-f)(a) := f(-a)$. Note then that $(-f)(a_1a_2) = -(-f)(a_1)(-f)(a_2)$, so $-f$ is not a homomorphism unless $f = -f$. In such a situation we must expand the set of morphisms to contain "negated homomorphisms." Also $(-f) \cdot (-g) = f \cdot g$, where $\cdot$ denotes pointwise multiplication.*

3.3. **Categorical triples.**

We are ready to formulate the main notion in categorical terms.

Definition 3.6. *A **categorical triple** $(\mathcal{C}, \mathcal{C}_{\mathcal{T}}, (-))$ is a category $\mathcal{C}$ with negation, together with a subcategory $\mathcal{C}_{\mathcal{T}}$ (i.e., there is a faithful functor $F : \mathcal{C}_{\mathcal{T}} \to \mathcal{C}$) for which $F((-)f) = (-)(F(f))$ for every morphism f, i.e., F commutes with $(-)$. We write $(-)F$ for the functor sending f to $(-)F(f)$.*

In other words, there is a set of morphisms $\mathrm{Mor}(A, B)_{\mathcal{T}} \subseteq \mathrm{Mor}(A, B)$, for each pair of objects A, B, which is closed under composition.

3.4. **Functor categories.**

Here is a wide-ranging example needed for geometry and linear algebra, cf. [6, Example 2.19], [23], [32, §3.5].

Definition 3.7. *Let S be a small category, i.e., Obj(S) and* $\mathrm{Mor}(S)$ *are sets. Let $\mathcal{C}$ be a category. We define $\mathcal{C}^S$ to be the category whose objects are the sets of functors from S to $\mathcal{C}$ and morphisms are natural transformations.*

We are particularly interested in the case that the category S is discrete, i.e., the only morphisms are the identity morphisms. In this case, we view $\mathcal{C}^S$ as $\mathrm{Fun}(S, \mathcal{C})$, the set of functions from S to $\mathcal{C}$. Many interesting examples of functor categories are given in a general setting in [24, §1.4.4]. Let us fit this in with triples.

Lemma 3.8. *If $\mathcal{C}$ is a categorical triple and S is a small and discrete category, then $\mathcal{C}^S$ also is a categorical triple, where $(-)F(s) = (-)F(s)$ for $F \in \mathcal{C}^S$ and $s \in S$.*

Proof. This is clear. $\square$

Definition 3.9. *Let $\mathcal{C}$ be a category. The* ***symmetrized category*** *$\widehat{\mathcal{C}}$ is defined to be $\mathcal{C}^{\mathbb{Z}_2}$, where we take the graded composition, i.e., $(f_0, f_1)(g_0, g_1) = (f_0g_0 + f_1g_1, f_0g_1 + f_1g_0)$, the negation functor is the switch map $(-)(f_0, f_1) = (f_1, f_0)$. We define $\hat{f} := (f, f)$.*

Remark 3.10. *The morphisms of $\widehat{\mathcal{C}}$ can be viewed as 2×2 matrices $(f_{i,j} : i, j = 0, 1)$, where $f_{i,j}$ sends some object of $\mathcal{C}_i$ to an object of $\mathcal{C}_j$.*

3.4.1. *$\mathcal{T}$-linear categories with negation.*

Since most of this paper involves $\mathcal{T}$-monoid modules, let us pause to see in what direction we want to proceed. We need to weaken the usual notion of additive and abelian categories.

Definition 3.11.

- *A category $\mathcal{C}$ is* ***$\mathcal{T}$-linear*** *over a monoid $\mathcal{T}$ if it satisfies the following two properties:*
  - (i) *Composition is bi-additive, i.e.,*
  $$(g + h)f = gf + hf, \qquad k(g + h) = kg + kh$$
  for all morphisms $f : A \to B$, $g, h : B \to C$, and $k : C \to D$. In particular, $\mathrm{Mor}(A, A)$ *is a semiring†, where multiplication is given by composition.*
  - (ii) *$\mathcal{T}$ acts naturally on* $\mathrm{Mor}(A, B)$ *in the sense that the action commutes with morphisms. (In practice, the objects will be $\mathcal{T}$-modules, and the action will be by left multiplication.)*
- *A $\mathcal{T}$-linear N-category $\mathcal{C}$ with categorical sums is called* ***semi-additive****. (These will be direct sums of $\mathcal{T}$-modules, to be considered more extensively in* [39]*.)*

Definition 3.12.

- *A* ***$\mathcal{T}$-linear category with negation*** *is a $\mathcal{T}$-linear category with a negation functor $(-)$.*
- *A* ***semi-additive*** category ***with negation*** *is a semi-additive category with a negation functor.*

Example 3.13. *Any additive category (in the classical sense) is $\mathbb{Z}$-linear and semi-additive with negation.*

Proposition 3.14. *For any $\mathcal{T}$-linear category with negation, the "quasi-zero" morphisms (of the form $f^\circ := f + ((-)f)$) comprise an absorbing set.*

Proof. One can easily observe the following: $(f(-)f)g = fg(-)fg$ and $g(f(-)f) = gf(-)gf$ and this shows the desired result. $\square$

Example 3.15. *Let $\mathcal{C}$ be a category with negation. We can impose an N-category structure on $\mathcal{C}$ by defining the null morphisms to be the morphisms of the form f°; this may or may not be a closed N-category. On the other hand, later in §8.1, we will introduce a closed N-category structure by means of elements of the form a°.*

Definition 3.16. *A* ***semi-abelian category (resp. with negation)*** *is a semi-additive category (resp. with negation) $\mathcal{C}$ satisfying the following extra property:*

Let N be the set of null morphisms of $\mathcal{C}$. Every morphism of $\mathcal{C}$ can be written as the composite of a cokernel with respect to N and a kernel with respect N.

(This property is called "semiexact" in [24, § 1.3.3]*.)*

3.4.2. *The categorical surpassing relation for N-categories with a negation functor.*
Here is our last categorical ingredient.

Definition 3.17. *The* ***categorical surpassing relation*** $\preceq_\circ$ *is defined by putting* $f \preceq_\circ g$ *if* $g = f + h^\circ$ *for some morphism* h.

4. The role of universal algebra

As in [57, §2], universal algebra provides a guide for the definitions, especially with regard to the roles of possible multiplication on $\mathcal{T}$ and the negation map. The notions of universal algebra are particularly appropriate here since we have a simultaneous double structure, of $\mathcal{A}$ and its designated subset (tangible elements) $\mathcal{T}$.

4.1. The basic notions in universal algebra.

We refer the reader to [34] for an excellent brief treatment of universal algebra and their congruences; more details can be found in [48], an easily readable book that deals with general algebraic structures.

We recall briefly that a **carrier**, called a **universe** in [48], is a t-tuple of sets $\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_t\}$ for some given t. A set of **operators** is a set $\Omega := \cup_{m\in\mathbb{N}} \Omega(m)$ where each $\Omega(m)$ in turn is a set of formal m-ary symbols $\{\omega_{m,j} = \omega_{m,j}(x_{1,j}, \dots, x_{m,j}) : j \in J_{\Omega(m)}\}$, interpreted as maps $\omega_{m,j} : \mathcal{A}_{i_{j,1}} \times \cdots \times \mathcal{A}_{i_{j,m}} \to \mathcal{A}_{i_j}$. (Here $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}_2 = \mathcal{T}$, and the $\omega_{m,j}$ include the various operations needed to define $\mathcal{T}$-modules.)

We write ω for a typical operator $\omega_{m,j}$. The 0-ary operators are just distinguished elements, that we call **constants**. For the purposes of this paper, we say that an operator ω of $\mathcal{A}$ is **$\mathbb{0}$-compatible** if $\omega(a_1, \dots, a_m) = \mathbb{0}$ whenever some $a_i = \mathbb{0}$. Thus, addition and inverses are not $\mathbb{0}$-compatible, but many other operators are (such as involutions and module multiplication).

The algebraic structure has **universal relations** (otherwise called **identities** in the literature), such as associativity, distributivity, which are expressed in terms of the operators, and this package of the carriers, operators and universal relations is called the **signature** of the carrier. For example, in classical algebra one might take the signature to be a ring or a module, endowed with various operations, together with identities written as universal relations.

The class of carriers of a given structure is called a **variety**. These are well known to be characterized by being closed under sub-algebras, homomorphic images, and direct products.

4.1.1. *The category of a variety in universal algebra.*
To view universal algebra more categorically, we work with a given variety (of a given signature). The objects of our category $\mathcal{C}$ are the carriers $\mathcal{A}$ of that signature (which clearly are sets), and the morphisms are the homomorphisms, which are maps $f : \mathcal{A} \to \mathcal{A}'$ satisfying, for all operators $\omega_{m,j} : \mathcal{A}_{i_{j,1}} \times \cdots \times \mathcal{A}_{i_{j,m}} \to \mathcal{A}_{i_{m,j}}$.

$$f(\omega_{m,j}(a_1, \dots, a_m)) = \omega_{m,j}(f(a_1), \dots, f(a_m)), \quad \forall a_k \in \mathcal{A}_{i_{j,k}}.$$ [3]

4.2. The functor category in universal algebra.

Let S be a small category. When $\mathcal{A}$ is a carrier, we view the carrier in the functor category $\mathcal{A}^S$ ($\mathcal{A}$ viewed as a small category) as the carrier $\{\mathcal{A}_1^S, \mathcal{A}_2^S, \dots, \mathcal{A}_t^S\}$, where $\mathcal{A}_i^S$ denotes the morphisms from S to $\mathcal{A}_i$ and, given an operator $\omega_{m,j} : \mathcal{A}_{i_{j,1}} \times \cdots \times \mathcal{A}_{i_{j,m}} \to \mathcal{A}_{i_j}$, we define the operator $\tilde{\omega} : \mathcal{A}^S_{i_{j,1}} \times \cdots \times \mathcal{A}^S_{i_{j,m}} \to \mathcal{A}^S_{i_{m,j}}$ "componentwise," by

$$\tilde{\omega}(f_1, \dots, f_t)(s) = \omega(f_1(s), \dots f_t(s)), \quad \forall s \in S.$$

For example, for $c \in \mathcal{A}$, the **constant function** $\tilde{c}$ is given by $\tilde{c}(s) = c$ for all $s \in S$. In particular, when relevant, the **zero function** $\tilde{\mathbb{0}}$ is given by $\tilde{\mathbb{0}}(s) = \mathbb{0}_{\mathcal{A}}$ for all $s \in S$. We can define $\mathcal{T}_{\mathcal{A}^S}$ to be the functions from S to $\mathcal{T}$.

Universal relations clearly pass from $\mathcal{A}$ to $\mathcal{A}^S$, verified componentwise. For convenience, we assume that the signature includes the operation $+$ together with the distinguished zero element $\mathbb{0}$.

Remark 4.1. *The following properties pass from the category* $\mathcal{C}$ *to* $\mathcal{C}^S$*, seen componentwise:*
$\mathcal{T}$*-linear, semi-additive, semi-additive with negation, semi-abelian.*

[3] Morphisms will be generalized in the discussion of systems, in which these will be called the "strict" morphisms.

Definition 4.2. *Given* $f \in \mathcal{A}^S$ *we write* $\mathrm{supp}(f)$ *for its* ***support*** $\{s \in S : f(s) \neq \mathbb{0}\}$, *and* $\mathrm{supp}(\mathcal{A}^S)$ *for* $\{\mathrm{supp}(f) : f \in \mathcal{A}^S\}$.

Lemma 4.3. *For any* $f, g \in \mathcal{A}^S$, *we have the following:*

(i) $\mathrm{supp}(f+g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$.
(ii) *(Under componentwise multiplication)* $\mathrm{supp}(fg) \subseteq \mathrm{supp}(f) \cap \mathrm{supp}(g)$.

Proof. For the first statement, one can see that $f(s) = \mathbb{0} = g(s)$ implies $f(s) + g(s) = \mathbb{0}$. The second statement is clear; $f(s) = \mathbb{0}$ or $g(s) = \mathbb{0}$ implies $f(s)g(s) = \mathbb{0}$. $\square$

Those morphisms having finite support will play a special role.

4.2.1. *The convolution product and polynomials.*

If the signature has multiplication and $(S, +)$ is a monoid, we often define instead the **convolution product** $f * g$ by

$$(f * g)(s) = \sum_{u+v=s} f(u)g(v),$$

but this only makes sense when there are only finitely many u, v with $u + v = s$.

This works for the morphisms of finite support.

Definition 4.4. *Let* $\mathcal{A}$ *and* S *be as above. We define the following notations:*

- $\mathcal{A}^{(S)} := \{f \in \mathcal{A}^S : \mathrm{supp}(f)$ *is finite*$\}$.
- $\mathcal{T}^{(S)} := \{f \in \mathcal{A}^{(S)} : f(S) \subseteq \mathcal{T}\}$.
- $\mathcal{T}_{\mathcal{A}^{(S)}} := \{f \in \mathcal{T}^S : |\mathrm{supp}(f)| = 1\}$.

Lemma 4.5. *Under convolution,* $\mathrm{supp}(f * g) \subseteq \mathrm{supp}(f) + \mathrm{supp}(g)$.

Proof. $f(u)g(v) \neq \mathbb{0}$ requires $u \in \mathrm{supp}\, f$ and $v \in \mathrm{supp}\, g$, which is necessary for $u + v \in \mathrm{supp}(f * g)$. $\square$

Lemma 4.6. *If* $\mathcal{T}$ *is a monoid then* $\mathcal{T}_{\mathcal{A}^{(S)}}$ *also is a monoid with the convolution product.*

Proof. Suppose $\mathrm{supp}(f_i) = \{s_i\}$ for $i = 1, 2$. Then Lemma 4.5 yields $\mathrm{supp}(f_1 * f_2) = \{s_1 + s_2\}$. $\square$

The convolution product unifies polynomial-type constructions.

Definition 4.7. *The* ***(functional) polynomial system*** $(\mathcal{A}^{(\mathbb{N})}, \mathcal{T}_{\mathcal{A}^{(\mathbb{N})}}, (-), \preceq)$ *is taken with* $\mathcal{A}^{(\mathbb{N})}$ *the convolution product,* $\mathcal{T}_{\mathcal{A}^{(\mathbb{N})}}$ *the set of monomials, and* $(-)$ *and* $\preceq$ *componentwise (i.e., according to the corresponding monomials).*

4.2.2. *The exterior product.*

Another intriguing example of functor categories (which we do not pursue here) is the **exterior product** where the functions in $\mathcal{T}^{(\mathbb{N})}$ satisfy $f(u)g(v) = (-1)^{uv} g(v)f(u)$.

4.3. **Abstract surpassing relations.**

One proposed innovation is to replace universal relations by something somewhat more general. We already did this in Definition 3.17, but now want to formulate it slightly more generally, in the context of universal algebra.

Definition 4.8. *An* ***abstract surpassing relation*** *is a relation* $\preceq$ *on* $\mathcal{A}$ *satisfying the following properties: for* $a, b \in \mathcal{A}$,

- $\mathbb{0} \preceq a$.
- *If* $a_i \preceq b_i$ *for* $1 \leq i \leq m$ *then* $\omega_{m,j}(a_{1,j}, \dots, a_{m,j}) \preceq \omega_{m,j}(b_{1,j}, \dots, b_{m,j})$.

An ***abstract surpassing PO*** *is an abstract surpassing relation that is a PO (partial order).*

In particular, if $a \preceq \mathbb{0}$ for an abstract surpassing PO, then $a = \mathbb{0}$.

Of course, equality is trivially an example of an abstract surpassing PO, but we will be interested in others specifically coming from triples, cf. Definition 5.26 below.

Surpassing relations can play a role in modifying universal algebra. Normally one uses equality to define sets, but now we use $\preceq$, as exemplified in the next definition.

Definition 4.9. *A $\preceq$-**root** of a polynomial $f(\lambda_1, \dots, \lambda_n)$ is defined as an n-tuple $(a_1, \dots, a_n)$ satisfying $\mathbb{0} \preceq f(a_1, \dots, a_n)$.*

This definition was the underlying approach to supertropical affine varieties in [32], and is to be used extensively in [39].

5. The $\preceq$-category in universal algebra

Having $\preceq$ at our disposal, we want to use it in our definition of category and variety. We proceed as in §4.1.1 but with a modification taking into account the surpassing relation $\preceq$. From now on, we will assume that $\mathcal{T}$ is a subset of $\mathcal{A}$ unless otherwise stated.

5.1. $\preceq$-varieties.

We work with a given signature $\mathcal{S}$ of universal algebras. The objects of our category $\mathcal{C}$ are the carriers of that signature $\mathcal{S}$ (which clearly are sets), and the morphisms are the $\preceq$-morphisms, which are defined as follows:

Definition 5.1. *A $\preceq$-**carrier** in universal algebra is a carrier with a given abstract surpassing PO denoted $\preceq$. Let $A = (\mathcal{A}_1, \dots \mathcal{A}_t)$ and $B = (\mathcal{A}'_1, \dots \mathcal{A}'_t)$ be $\preceq$-carriers (of $\mathcal{S}$) with abstract surpassing POs $\preceq$ and $\preceq'$, respectively. A $\preceq$-**morphism** $f : A \to B$ is a set of maps $f_j : \mathcal{A}_j \to \mathcal{A}'_j$, $1 \le j \le m$, satisfying the properties:*

(i) *If $a \preceq b$ then $f_j(a) \preceq' f_j(b)$.*
(ii) *$f(\omega(a_1, \dots, a_m)) \preceq' \omega(f_1(a_1), \dots, f_m(a_m))$, for every operator $\omega : \mathcal{A}_{i_1} \times \cdots \times \mathcal{A}_{i_m} \to \mathcal{A}_{i_{m+1}}$, with $a_j \in \mathcal{A}_{i_j}$.*
(iii) *For every operator $\omega : \mathcal{A}_{i_1} \times \cdots \times \mathcal{A}_{i_m} \to \mathcal{A}_{i_{m+1}}$, if $a_j \preceq b_j$ in $\mathcal{A}_{i_j}$ for each $1 \le j \le m$, then*
$$\omega(f_1(a_1), \dots, f_m(a_m)) \preceq' \omega(f_1(b_1), \dots, f_m(b_m)).$$

*A $\preceq$-**variety** in universal algebra is a class of $\preceq$-carriers, whose homomorphisms are the $\preceq$-morphisms.*

Often f is determined by f_1, so slightly abusing notation we write f for f_1.

Lemma 5.2. *Any $\preceq$-morphism f with respect to a surpassing PO satisfies the following convexity condition: If $f(a_0) = f(a_1)$ and $a_0 \preceq a \preceq a_1$, then $f(a_0) = f(a)$.*

Proof. $f(a_1) = f(a_0) \preceq f(a) \preceq f(a_1)$, so equality holds at each stage. $\square$

It follows that every $\preceq$-morphism "collapses" intervals. We also want the customary universal algebra approach in the following way:

Definition 5.3. *A $\preceq$-morphism $f : A \to B$ is ω-**strict** if $f(\omega(a_1, \dots, a_m)) = \omega(f_1(a_1), \dots, f_m(a_m))$.*

*A $\preceq$-morphism $f : A \to B$ is **strict** if f is ω-strict for all operators ω. (These are the homomorphisms described earlier.)*

Another way of viewing strict morphisms is as $\preceq$-morphisms when we take $\preceq$ to be the identity map. The following easy result shows that some $\preceq$-morphisms are necessarily ω-strict for suitable ω, with several useful applications to triples to be given in §5.8. We say that a unary operator $\phi : \mathcal{A}_i \to \mathcal{A}_i$ is **invertible** if there is some operator $\psi : \mathcal{A}_i \to \mathcal{A}_i$ such that $\psi\phi = 1_{\mathcal{A}_i} = \phi\psi$.

Proposition 5.4. *Suppose $\phi_i : \mathcal{A}_i \to \mathcal{A}_i$ is an invertible unary operator, and $\preceq$ is an abstract surpassing PO. Then $f_i(\phi_i(a)) = \phi_i(f_i(a))$, $\forall a \in \mathcal{A}_i$, for any $\preceq$-morphism $f_i : A_i \to B_i$.*

Proof. We are given
$$f_i(\phi_i(a)) \preceq \phi_i(f_i(a)), \ \forall a \in \mathcal{A}_j.$$
But then $\phi_i(f_i(a)) = \phi_i(f_i(\psi_i\phi_i(a))) \preceq \phi_i\psi_i(f_i(\phi_i(a))) = f_i(\phi_i(a))$, so we get equality. $\square$

5.2. Negation maps revisited.

Our next task is to apply the categorical concepts to define triples and $\mathcal{T}$-systems and their categories. We take a category of universal algebras, whose signature includes $\mathcal{A}_2 = \mathcal{T}$, a multiplicative monoid, and $\mathcal{A}_1 = \mathcal{A}$, a $\mathcal{T}$-module. We need some formalism to get around the lack of negation, which we pick up from Definition 2.4.

Definition 5.5. *A **negation map** on a $\mathcal{T}$-monoid module with negation $\mathcal{A}$ is a semigroup isomorphism $(-) : \mathcal{A} \to \mathcal{A}$ of order ≤ 2, written $a \mapsto (-)a$, which also respects the $\mathcal{T}$-action in the sense that*

$$(-)b = ((-)\mathbb{1})b$$

for $b \in \mathcal{A}$.

Lemma 5.6. *For any $a \in \mathcal{T}$ and $b \in \mathcal{A}$, we have that $(-)(a+b) = (-)a + (-)b$ and*

$$(-)(ab) = ((-)a)b = a((-)b). \tag{5.1}$$

Proof. The first assertion is clear. The second assertion is obtained as follows:

$$(-)(ab) = a((-)\mathbb{1})b = a((-)b) = (-)\mathbb{1}ab = ((-)a)b.$$

□

Definition 5.7. *A **negation map** on a semialgebra $(\mathcal{A}, \cdot, +)$ is simultaneously a negation map on the $\mathcal{T}$-monoid module $(\mathcal{A}, +)$, as well as satisfying* (5.1) *for all $a, b \in \mathcal{A}$.*

In each situation the negation map gives rise to a negation functor in the category arising from universal algebra, defining $(-)f$ for any morphism f to be given by $((-)f)(a) = (-)(f(a))$.

Recall that we write $a(-)a$ for $a + ((-)a)$, and a° for $a(-)a$, called a **quasi-zero**.

Example 5.8. *There are three main instances of negation maps.*

(i) *Equality (since one just erases all the $(-)$ appearing in the definition).*
(ii) *The switch map in symmetrization, in Definition 2.4.*
(iii) *The negation map in a hypergroup.*

Of special interest are the sets $\mathcal{A}^\circ := \{a^\circ : a \in \mathcal{A}\}$ and $\mathcal{T}^\circ := \{a^\circ : a \in \mathcal{T}\}$. Inductively, we define $\mathbf{1}a = a$ and $\mathbf{m}a = (\mathbf{m} - \mathbf{1})a + a$.

Lemma 5.9. *$a^\circ b^\circ = \mathbf{2}(ab)^\circ$ for all $a \in \mathcal{T}$.*

Proof. $a^\circ b^\circ = (a(-)a)(b(-)b) = \mathbf{2}ab(-)\mathbf{2}ab = \mathbf{2}(ab)^\circ$. □

Lemma 5.10. *The map $a \mapsto a^\circ$ is a $\preceq$-morphism of $\mathcal{T}$-systems when $\mathbf{3}a = \mathbf{2}a$ for all $a \in \mathcal{A}$.*

Proof. By Lemma 5.9, $a^\circ b^\circ = \mathbf{2}(ab)^\circ = (ab)^\circ + c^\circ$ where $c = (ab)^\circ$, so $(ab)^\circ \preceq a^\circ b^\circ$. Likewise $(a+b)^\circ = a^\circ + b^\circ$. □

5.3. $\mathcal{T}$-triples: Ground triples and module triples.

Definition 5.11. *A $\mathcal{T}$-**pseudo-triple** $(\mathcal{A}, \mathcal{T}_\mathcal{A}, (-))$ is a $\mathcal{T}$-module $\mathcal{A}$, with $\mathcal{T}_\mathcal{A}$ designated as a distinguished subset, and a negation map $(-)$ satisfying $(-)\mathcal{T}_\mathcal{A} = \mathcal{T}_\mathcal{A}$. A $\mathcal{T}$-**triple** $(\mathcal{A}, \mathcal{T}_\mathcal{A}, (-))$ (called a **triple** when $\mathcal{T}$ is understood) is a $\mathcal{T}$-pseudo-triple for which $\mathcal{T}_\mathcal{A}$ generates $(\mathcal{A}, +)$. (If $\mathcal{A}$ has a zero element $\mathbb{0}$ we only require that $\mathcal{T}_\mathcal{A}$ generates $(\mathcal{A} \setminus \{\mathbb{0}\}, +)$.)*

(Strictly speaking, one might not have the same multiplication for $\mathcal{T}_\mathcal{A}$ and $\mathcal{A}$; for example, $\mathcal{T}_\mathcal{A}$ could be a Lie semialgebra and $\mathcal{A}$ its associative enveloping algebra. But this paper focuses on the associative situation.)

Although $\mathcal{T}_\mathcal{A}$ is not necessarily $\mathcal{T}$ in the general definition, this will normally be the case if $\mathbb{1} \in \mathcal{A}$. We also will require that $\mathcal{T}_\mathcal{A} \cap \mathcal{A}^\circ = \emptyset$. This condition fails in the max-plus algebra, but holds in the other examples of interest to us.

Remark 5.12. *The condition that $\mathcal{T}_\mathcal{A}$ generates $(\mathcal{A}, +)$ is crucial to much of the theory, and an important issue will be to determine when it holds. The most straightforward way of ensuring this is to restrict $(\mathcal{A}, +)$ to the sub-monoid generated by $\mathcal{T}_\mathcal{A}$. (This does not affect $\mathcal{T}_\mathcal{A}$.)*

In this case we define the **height** of an element $c \in \mathcal{A}$ as the minimal t such that $c = \sum_{i=1}^{t} a_i$ with each $a_i \in \mathcal{T}_{\mathcal{A}}$. ($\mathbb{0}$ has height 0.) The **height** of $\mathcal{A}$ is the maximal height of its elements (which is said to be ∞ if these heights are not bounded).

Often the set $\mathcal{T}_{\mathcal{A}}$ is a multiplicative group. For example, $\mathcal{T}_{\mathcal{A}}$ might be a hyperfield generating $\mathcal{A}$ inside the power set. Or $\mathcal{T}_{\mathcal{A}}$ might be a modification of the max-plus algebra, and $\mathcal{A}$ its supertropical algebra (or symmetrized algebra). A more esoteric example: $\mathcal{A}$ might be a fuzzy ring, and $\mathcal{T}_{\mathcal{A}}$ its subset of invertible elements, as described in [57, Appendix B].

Let us describe various sorts of triples that come up.

Definition 5.13. *A $\mathcal{T}$-**semiring triple** is a semiring $(\mathcal{A}, +, \cdot, \mathbb{1}_{\mathcal{A}})$ that is also a triple with $\mathcal{T}_{\mathcal{A}} = \mathcal{T}$.*

The last definition is included for the sake of Lie theory, although we do not pursue it here.

Definition 5.14. *A triple $\mathcal{A} = (\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-))$ is **meta-tangible** if $\mathcal{A} = \mathcal{T}_{\mathcal{A}} + \mathcal{A}^{\circ}$, which by* [57, Lemma 5.1] *means $a_1 + a_2 \in \mathcal{T}_{\mathcal{A}}$ for any $a_2 \neq (-)a_1$ in $\mathcal{T}_{\mathcal{A}}$.*

The uniquely negated property is needed to get started in the theory, and meta-tangibility was the main property studied in [57]. When $\mathcal{A}$ is a semiring with a negation map, it can be viewed as a small semi-additive category with a negation functor. Now $\preceq$-morphisms as defined earlier become functors between two small categories.

Definition 5.15. *A **ground pseudo-triple** is a pseudo-triple $\mathcal{A} = (\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-))$ for which $\mathcal{A}$ and $\mathcal{T}_{\mathcal{A}}$ are multiplicative sets that stand by themselves in their own right, in the sense that they can be viewed as categories with a single object, whose respective morphisms are the elements of $\mathcal{A}$ and $\mathcal{T}_{\mathcal{A}}$.*

Just as one studies modules over a base ring, we will want a secondary notion of triples over a ground triple, for use in §9

Definition 5.16. *Given a ground $\mathcal{T}$-triple $\mathcal{A} = (\mathcal{A}, \mathcal{T}, (-))$, a **left $\mathcal{A}$-module triple** $\mathcal{M} := (\mathcal{M}, \mathcal{T}_{\mathcal{M}}, (-))$ is a $\mathcal{T}_{\mathcal{M}}$-triple which is a left $\mathcal{A}$-module satisfying $\mathcal{T}_{\mathcal{M}} \subseteq \mathcal{T}_{\mathcal{M}}$ and*

$$((-)a)b = a((-)b) = (-)(ab), \quad \forall a \in \mathcal{T}, \; b \in \mathcal{T}_{\mathcal{M}}.$$

*Analogously, we define a **right $\mathcal{A}$-module triple** from the other side, and an $(\mathcal{A}, \mathcal{A}')$-**bimodule triple** when $\mathcal{M}$ is an $\mathcal{A}, \mathcal{A}'$-bimodule.*

*A **sub-pseudo-triple** of a pseudo-triple $(\mathcal{M}, \mathcal{T}_{\mathcal{M}}, (-))$ is a pseudo-triple $(\mathcal{M}', \mathcal{T}_{\mathcal{M}'}, (-))$ where $\mathcal{T}_{\mathcal{M}'}$ is a subset of $\mathcal{T}_{\mathcal{M}}$ (with the relevant structure) and $\mathcal{M}'$ is the sub-semigroup of $\mathcal{M}$ generated by $\mathcal{T}_{\mathcal{M}'}$.*

*A **sub-triple** is a sub-pseudo-triple which is a triple. Thus, the infimum $\mathcal{M}_1 \wedge \mathcal{M}_2$ for $\mathcal{M}_i = (\mathcal{M}, \mathcal{T}_{\mathcal{M}_i}, (-))$ is the sub-triple generated by $\mathcal{T}_{\mathcal{M}_1} \cap \mathcal{T}_{\mathcal{M}_2}$.*

Example 5.17. *Suppose $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-))$ is a triple. The **characteristic sub-triple** $\mathcal{A}_{\mathbb{1}}$ is the sub-triple generated by $\mathbb{1}$, which is $\{\mathbb{1}, (-)\mathbb{1}, e := \mathbb{1}(-)\mathbb{1}, \dots\}$. If $\mathcal{A}$ is a semiring† then clearly $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-))$ is a module over $\mathcal{A}_{\mathbb{1}}$. This ties in with other approaches to tropical algebra, and to some fundamental hyperfields, as follows. If $e = \mathbb{1}$ then we have the Boolean semifield, so assume $e \neq \mathbb{1}$. If $e + \mathbb{1} = \mathbb{1}$ we can identify e with 0, and we wind up with $\mathbb{Z}$ or $\mathbb{Z}/n$ for some $n > 1$. So assume that $e + \mathbb{1} \neq \mathbb{1}$. In height 2, $e + \mathbb{1} \in \{(-)\mathbb{1}, e\}$.*

(i) *$(-)$ is of the first kind. Then $e + \mathbb{1} = e$. $\mathcal{A}_{\mathbb{1}}$ is $\{\mathbb{1}, e\}$ in (the supertropical case). This corresponds to the Krasner hyperfield $K = \{0, 1\}$ with the usual operations of Boolean algebra, except that now $1 \boxplus 1 = \{0, 1\}$, and we can identify $\{0, 1\}$ with 1^{ν}.*

(ii) *$(-)$ is of the second kind.*
   (a) *In the $(-)$-bipotent case $\mathcal{A}_{\mathbb{1}} = \{\mathbb{1}, (-)\mathbb{1}, e\}$, with $\mathbb{1} + \mathbb{1} = \mathbb{1}$, which is the symmetrized triple of the trivial idempotent triple $\{\mathbb{1}\}$. This corresponds to the hyperfield of signs $S := \{0, 1, -1\}$ with the usual multiplication law and hyperaddition defined by $1 \boxplus 1 = \{1\}$, $-1 \boxplus -1 = \{-1\}$, $x \boxplus 0 = 0 \boxplus x = \{x\}$, and $1 \boxplus -1 = -1 \boxplus 1 = \{0, 1, -1\} = S$.*
   (b) *$(-)$ is uniquely negated of the second kind but non-$(-)$-bipotent. Then $e + \mathbb{1} = (-)\mathbb{1}$, which leads to a strange structure of characteristic 4 since $\mathbf{2} = (e(-)\mathbb{1}) + \mathbb{1} = e + e = (-)\mathbb{1}(-)\mathbb{1} = (-)\mathbf{2}$, $e + \mathbf{2} = \mathbb{1}(-)\mathbb{1} = e$, and $e + e + e = (e + \mathbb{1}) + e + (-)\mathbb{1} = e$.*

In either case one could adjoin $\{zero\}$, of course. Height > 2 is more intricate, involving layered structures.

5.4. **Direct sums of triples and pseudo-triples.**

Definition 5.18. *The* ***direct sum*** $\oplus_{i\in I}(\mathcal{A}_i, \mathcal{T}_{\mathcal{A}_i}, (-))$ *of pseudo-triples is defined as* $(\oplus\mathcal{A}_i, \mathcal{T}_{\oplus\mathcal{A}_i}, (-))$ *where there are two natural possibilities for* $\mathcal{T}_{\oplus\mathcal{A}_i}$*, viewed in* $\oplus\mathcal{A}_i$ *via the same* $\nu_i : \mathcal{A}_i \to \oplus\mathcal{A}_i$ *being the canonical morphism:*

(i) $\mathcal{T}_{\oplus\mathcal{A}_i} = \sum \mathcal{T}_{\mathcal{A}_i}$*, viewed in* $\oplus\mathcal{A}_i$ *via* $\nu_i : \mathcal{A}_i \to \oplus\mathcal{A}_i$ *being the canonical morphism.*
(ii) $\mathcal{T}_{\oplus\mathcal{A}_i} = \cup\mathcal{T}_{\mathcal{A}_i}$*, viewed in* $\oplus\mathcal{T}_{\mathcal{A}_i}$*.*

Note that (i) and (ii) are essentially the same, since $\sum_i \mathcal{T}_{\mathcal{A}_i}$ is generated by $\cup_i\mathcal{T}_{\mathcal{A}_i}$.

Proposition 5.19. *In both instances, the direct sum* $\oplus(\mathcal{A}_i\mathcal{T}_i, (-))$ *of (uniquely negated) triples is a (uniquely negated) triple.*

Proof. $\mathcal{T}_{\oplus\mathcal{A}_i}$ generates $\oplus\mathcal{A}_i$, and unique negation is obtained componentwise. ☐

5.5. **Functor triples and pseudo-triples.**
When all $\mathcal{A}_i = \mathcal{A}$, this takes us back to $\mathcal{A}^{(S)}$, now viewed as a triple via the componentwise negation map.

Remark 5.20.

(i) *We must cope with a delicate issue. We have required for triples that* $\mathcal{T}_{\mathcal{A}}$ *generates* $(\mathcal{A}, +)$*. Thus,* $\mathcal{A}^S$ *a priori is only a pseudo-triple.* $\mathcal{A}^S$ *is a triple when* $\mathcal{A}$ *is a triple of finite height. Also see Remark 5.21(ii).*
(ii) *When* $\mathcal{M}$ *is an* $\mathcal{A}$*-module triple, then* $\mathcal{M}^S$ *is an* $\mathcal{A}$*-module triple, under the action* $(af)(s) = af(s)$*.*

When a small category S is discrete (i.e., only morphisms are the identity morphisms), we often write $I = S$ and call $(\mathcal{A}^{(I)}, \mathcal{T}_{\mathcal{A}}^{(I)}, (-))$ the **free $\mathcal{A}$-module triple**, to stress the role of I as an index set and the analogy with the free module.

Remark 5.21.

(i) *As customary, we could rewrite any* $f \in \mathcal{A}^S$ *as a vector* $v = (v_i)$ *where* $v_i = f(i)$*. Then* $f \in \mathcal{A}^{(I)}$ *means that almost all components* v_i *are* $\mathbb{0}$*.*
(ii) $\mathcal{A}^{(I)}$ *is a triple, since it is generated by* $\mathcal{T}^{(I)}$*. (Indeed, any element of* $\mathcal{A}^{(I)}$ *has finite support, so its components have bounded height.)*
(iii) *Suppose that* $\mathcal{T}$ *is a group.* $\mathcal{T}_{\mathcal{A}^{(I)}}$ *is not a group for* $|I| > 1$*, since we can have several nontrivial components. But* $\mathcal{T}_{\mathcal{A}^{(I)}}$ *is closed under multiplication and thus a group when* $\mathcal{A}^{(I)}$ *is endowed with the convolution product, which is one reason why we use convolution products for ground* $\mathcal{T}$*-group triples and ground* $\mathcal{T}$*-module triples.*
(iv) $\mathcal{A}^{(I)}$ *is not meta-tangible when* $|I| > 1$*, and the theory of module triples is quite different from that of meta-tangible triples, much as module theory differs from the structure theory of rings.*

5.5.1. *Triples* $(-)$*-layered by a semiring* L*.*
Here is an example paralleling graded algebras, which both relates to the symmetrized structure and is needed in differential calculus of $\mathcal{T}$-systems.

Definition 5.22. *A pseudo-triple* $\mathcal{A}$ *is* $(-)$***-layered*** *by a monoid* $(L, \cdot)$ *if*

$$\mathcal{A} = \cup_{\ell\in L}\mathcal{A}_\ell; \qquad (-)\mathcal{A}_\ell = \mathcal{A}_{(-)\ell}$$

as in Definition 5.18(ii) where the union of the A_ℓ *is disjoint, and* $\mathcal{A}_{\ell_1}\mathcal{A}_{\ell_2} \subseteq \mathcal{A}_{\ell_1\ell_2}$*.*

Example 5.23. $(-)$*-layering also provides "layered" structures, as described in* [3, §2.1]*. Namely, let us view a* $\mathcal{T}$*-monoid triple layered by a semiring*† L *as a special case of Definition 5.22, where all the* $\mathcal{A}_\ell$ *are the same (taking* $(-)$ *on* $\mathcal{A}_\ell$ *to be the identity map). This can be viewed as* $\mathcal{A}_1^L$*, viewing a monoid* L *as a small category, with* $\mathcal{T}_{\mathcal{A}} = \mathcal{A}_1$*. Addition in the layered triple* $\mathcal{A}'$ *of* $\mathcal{A}$ *by* L *is given in* [57, Example 2.42]*. This is treated in* [3, §2.1]*, which also considers other subtleties concerning layering of triples in general. We also note that there is a homomorphism* $\mathcal{A}_1^{(L)} \to \mathcal{A}'$ *given by* $(a_\ell) \mapsto (\ell, \sum a_\ell)$*.*

5.5.2. *Symmetrized triples.*

When L is taken to be (additively) idempotent, we also can recover the symmetrized triple as follows:

Proposition 5.24. *Define $L = \{\pm 1, \infty\}$ to be idempotent, with addition given by $a + \infty = \infty$ and $1 + (-1) = \infty$. Multiplication is as usual, again with ∞ absorbing. Then the layered triple $\mathcal{A}'$ of $\mathcal{A}$ by L (*[57, Example 2.42]*) is isomorphic to the symmetrized layered triple of* [57, Definition 2.48]*. The ∞ layer of $\mathcal{A}'$ is precisely the set $(\mathcal{A}')^\circ$ of quasi-zeros.*

Proof. First we show that L is a semiring†. Addition is associative since if ∞ appears as a summand we always get the sum ∞, and likewise if $+1, -1$ appear, so we are left with all the summands being the same, and we have associativity by idempotence. Multiplication is associative by a similar argument. For distributivity in L, to show that $a(b+c) = ab+ac$ we again may assume that these all are ± 1, and have $a(b(-)b) = a \cdot \infty = (ab)(-)ab$, whereas $a(b+b) = ab = ab+ab$.

Now we define the map φ from the symmetrized triple $\widehat{\mathcal{A}}$ to the layered triple by

$$(a_0, a_1) \mapsto (1, a_0) + (-1, a_1).$$

Then

$$\varphi((a_0, a_1) + (a_0', a_1')) = \varphi(a_0 + a_0', a_1 + a_1') = (1, a_0 + a_0') + (-1, a_1 + a_1') = \varphi((a_0, a_1) + \varphi((a_0', a_1')),$$

whereas

$$\varphi((a_0, a_1)(a_0', a_1')) = \varphi(a_0 a_0' + a_1 a_1', a_0 a_1' + a_1 a_0') = (1, a_0 a_0' + a_1 a_1') + (-1, a_0 a_1' + a_1 a_0') = \varphi(a_0, a_1)\varphi(a_0', a_1'),$$

so φ is a homomorphism which by inspection is 1:1. It is onto since $(\infty, a) = (1, a) + (-1, a) = \varphi(a, a)$, which also identifies the ∞ layer with $(\mathcal{A}')^\circ$. □

For $\mathcal{A}$ idempotent, this result leads us to the $\mathbb{B}$-mod theory of [11].

Lemma 5.25.

(i) *There is a strict morphism of triples $\mathcal{A} \to \widehat{\mathcal{A}}$ given by $a \mapsto (a, (-)a)$.*

(ii) *In the reverse direction, there is a strict morphism of triples $\widehat{\mathcal{A}} \to \mathcal{A}$ given by $(a_0, a_1) \mapsto a_0(-)a_1$.*

Proof.

(i) $(-)a \mapsto ((-)a, a) = (-)(a, (-)a)$, and $a + a' \mapsto (a, (-)a) + (a', (-)a') = (a + a', (-)(a + a'))$.

(ii) $(-)(a_0, a_1) = (a_1, a_0) \mapsto a_1(-)a_0 = (-)(a_0(-)a_1)$, and

$(a_0, a_1) + (a_0', a_1') = (a_0 + a_0', a_1 + a_1') \mapsto (a_0 + a_0')(-)(a_1 + a_1') = (a_1(-)a_0) + (a_1'(-)a_0')$. □

Note that the composition of these strict morphisms sends $a \mapsto a^\circ \in \mathcal{A}^\circ$, so is the identity only when $\mathcal{A}$ is idempotent. Nevertheless, they play an important role in [39].

5.6. $\mathcal{T}$-systems.

To obtain our specific structure of choice, we enrich the triple with a surpassing relation having specific properties relating to $\mathcal{T}$.

Definition 5.26. *A **$\mathcal{T}$-surpassing relation** on a $\mathcal{T}$-triple $\mathcal{A}$ is an abstract surpassing relation also satisfying the following: $b^\circ \not\preceq a$ for any $a \in \mathcal{T},\ b \in \mathcal{A}$.*

Definition 5.27. *A **surpassing PO** on $\mathcal{A}$ is a $\mathcal{T}$-surpassing relation $\preceq$ which is a PO.*

Here is our main example. Recall, from Definition 1.2, that $a \preceq_\circ b$ if and only if $b = a + c^\circ$ for some $c \in \mathcal{A}$. This canonical choice of a surpassing PO leads us to the following definition.

Definition 5.28. *Any triple has the **default $\mathcal{T}$-system** $(\mathcal{A}, \mathcal{T}, (-), \preceq_\circ)$.*

Lemma 5.29. *If $a \preceq_\circ c$ then $c(-)a \in \mathcal{A}^\circ$.*

Proof. Write $c = a + b^\circ$. Then $c(-)a = a^\circ + b^\circ \in \mathcal{A}^\circ$. □

We have two other motivating examples:

Example 5.30.

(i) *Equality (=) is the classical surpassing PO.*

(ii) *In the power set of a hypergroup, $\preceq$ is just $\subseteq$, cf.* [57, Definition 4.45].

Definition 5.31. *A $\mathcal{T}$-**system** (resp. $\mathcal{T}$-**pseudo-system**) is a quadruple $(\mathcal{A}, \mathcal{T}, (-), \preceq)$, where $\preceq$ is a surpassing PO on the uniquely negated triple (resp. pseudo-triple) $(\mathcal{A}, \mathcal{T}, (-))$. A $\mathcal{T}$-system is **meta-tangible** if its underlying triple is uniquely negated and meta-tangible.*

From now on, we will omit $\mathcal{T}$ to simplify the notation when it is understood.

Definition 5.32. *A $\mathcal{T}$-**subsystem** of a $\mathcal{T}$-system $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-), \preceq)$ is a sub-triple $\mathcal{N} = (\mathcal{N}, \mathcal{T}_{\mathcal{N}}, (-), \preceq)$ of $\mathcal{A}$, satisfying the following condition, where $a \in \mathcal{T}_{\mathcal{A}}$ and $b \in \mathcal{A}$:*

- *If $a \preceq b + v$, for $v \in \mathcal{N}$, then there is $w \in \mathcal{T}_{\mathcal{N}}$ for which $a \preceq b + w$.*

(In particular, if $\mathbb{0} \in \mathcal{A}$ then $\mathbb{0} \in \mathcal{N}$.)

Remark 5.33. *Given a triple $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-))$, there are two main ways that we want to define $\preceq$ to get the $\mathcal{T}_{\mathcal{A}}$-system $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-), \preceq)$. The first is to take $\preceq$ to be $\preceq_\circ$. A special case is the symmetrized $\mathcal{T}_{\widehat{\mathcal{A}}}$-system $(\widehat{\mathcal{A}}, \mathcal{T}_{\widehat{\mathcal{A}}}, (-), \preceq_\circ)$, where $(-)$ is the switch map.*

But one also can take $\preceq$ to be equality $(=)$. This covers the classical situation, and can be useful when we want a theory closer to more traditional universal algebra.

Here is a $\mathcal{T}$-system with a different flavor.

Example 5.34. *The $\mathcal{T}$-system of a hypergroup $\mathcal{T}$ is $(\mathcal{P}(\mathcal{T}), \mathcal{T}, (-), \subseteq)$, cf.* [57, Definition 4.45]. *We get a $\mathcal{T}$-system (contained in $\mathcal{P}(\mathcal{T})$ by means of Remark 5.12).*

5.7. **Ground $\mathcal{T}$-systems and module systems.**

We put all of these conditions together:

Definition 5.35.

- *A **ground $\mathcal{T}$-system** is a ground $\mathcal{T}$-triple which is a $\mathcal{T}$-system.*
- *An $\mathcal{A}$-**Gmodule system** is an $\mathcal{A}$-module system for which $\mathcal{A}$ is a $\mathcal{T}$-Gmodule. (This involves the implicit assumption that $\mathcal{T}$ is a group.)*

When $\mathcal{A}$ does contain $\mathbb{0}$ we write $\mathcal{T}_{\mathbb{0}}$ for $\mathcal{T} \cup \{\mathbb{0}\}$. In a semiring system over a monoid $\mathcal{T}$, we normally identify $\mathcal{T}_{\mathcal{A}}$ with $\mathcal{T} \cdot \mathbb{1}_{\mathcal{A}}$.

On the other hand, we have:

Definition 5.36.

- *A **left $\mathcal{A}$-module system** over a ground $\mathcal{T}$-system $\mathcal{A} = (\mathcal{A}, \mathcal{T}, (-), \preceq)$, is a left $\mathcal{A}$-module triple $(\mathcal{M}, \mathcal{T}_{\mathcal{M}}, (-))$ with a surpassing PO satisfying $a_1 b_1 \preceq a_2 b_2$ whenever $a_1 \preceq a_2$ in $\mathcal{T}_{\mathcal{M}}$ and $b_1 \preceq b_2$ in $\mathcal{M}$.*
- *A **right $\mathcal{A}$-module system** over $\mathcal{A} = (\mathcal{A}, \mathcal{T}, (-), \preceq)$, is defined analogously on the right, i.e., a right $\mathcal{A}$-module triple $(\mathcal{M}, \mathcal{T}_{\mathcal{M}}, (-))$ with a surpassing PO satisfying $b_1 a_1 \preceq b_2 a_2$ whenever $a_1 \preceq a_2$ in $\mathcal{T}_{\mathcal{M}}$ and $b_1 \preceq b_2$ in $\mathcal{M}$.*
- *An $(\mathcal{A}, \mathcal{A}')$-**bimodule system** is a left $\mathcal{A}$- and right $\mathcal{A}'$-module system $(\mathcal{M}, \mathcal{T}_{\mathcal{M}}, (-), \preceq)$ for which $\mathcal{M}$ is an $\mathcal{A}, \mathcal{A}'$-bimodule triple.*

Example 5.37. *For any ground $\mathcal{T}$-system $\mathcal{A} = (\mathcal{A}, \mathcal{T}, (-), \preceq)$, we define $\preceq$ in a componentwise way on $\mathcal{A}^S$ by putting $f \preceq g$ when $f(s) \preceq g(s)$ for each s in S. This is a surpassing PO whenever $\preceq$ is a surpassing PO on $\mathcal{A}$, and provides the **free module system** $(\mathcal{A}^{(I)}, \mathcal{T}^{(I)}, (-), \preceq)$,* [57, Definition 2.10].

A common property of $\mathcal{T}$-module systems:

Definition 5.38. *A $\mathcal{T}$-module system $\mathcal{M} = (\mathcal{M}, \mathcal{T}_{\mathcal{M}}, (-), \preceq)$ is $\mathcal{T}_{\mathcal{M}}$-**reversible** if $a_1 \preceq a_2 + b$ implies $a_2 \preceq a_1(-)b$ for $a_1, a_2 \in \mathcal{T}_{\mathcal{M}}$ and $b \in \mathcal{M}$.*

5.8. **Viewing systems categorically.**

Expressing uniquely negated systems categorically enables us to relate to the work [36, §2] of the first author on hyperrings and also appeal to the categorical literature. As a special case of Definition 5.1, we have:

Definition 5.39 (cf. [57, Definition 6.3 and Example 6.5]). *Let*

$$A = (\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-), \preceq), \qquad B = (\mathcal{A}', \mathcal{T}'_{\mathcal{A}'}, (-)', \preceq')$$

*be systems (of signature $\mathcal{S}$). A **morphism** $f : A \to B$ of systems is a $\preceq$-morphism in the sense of Definition 5.1. In particular, $f(\mathcal{T}_{\mathcal{A}}) \subset \mathcal{T}_{\mathcal{A}'}$.*

Remark 5.40. *By a "strict" morphism (cf. Definition 5.3) we mean "strict on addition," i.e., $f(a_1 + a_2) = f(a_1) + f(a_2)$. We automatically get strict morphisms by taking $\preceq$ to be equality, cf. Remark 5.33.*

Lemma 5.41. *A morphism f is strict on $\mathcal{T}_{\mathcal{A}}$ if and only if f is strict on $\mathcal{A}$.*

Proof. Let f be a strict morphism on $\mathcal{T}_{\mathcal{A}}$. For any $b_i \in \mathcal{A}$, write $b_i = \sum a_{i,j}$ for $a_{i,j} \in \mathcal{T}$. Then $f\left(\sum b_i\right) = f\left(\sum a_{i,j}\right) = \sum f(a_{i,j}) = \sum f(b_i)$ showing that f is strict on $\mathcal{A}$. The converse is trivial. □

Definition 5.42. *A category of systems (of signature $\mathcal{S}$) is **strict** if its morphisms are strict.*

We usually want categories of ground $\mathcal{T}$-systems to be strict, in order for extra requirements to be satisfied. The situation for morphisms of module systems is a touchy point which still remains to be resolved completely. Here is a compromise which is not so satisfying.

Definition 5.43. *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{A}$-module systems. A morphism $f : \mathcal{M} \to \mathcal{N}$ is $\mathcal{T}$-**admissible** when it satisfies the condition that if $\sum_{i=1}^t a_i = \sum_{j=1}^u a'_j$ for $a_i, a'_j \in \mathcal{T}_{\mathcal{M}}$, then $\sum f(a_i) = \sum f(a'_j)$.*

The following lemma shows that the class of $\mathcal{T}$-admissible morphisms is larger than the class of strict morphisms.

Lemma 5.44. *Every strict morphism is $\mathcal{T}$-admissible.*

Proof. If $\sum_{i=1}^t a_i = \sum_{j=1}^u a'_j$, then

$$\sum f(a_i) = f\left(\sum_i a_i\right) = f\left(\sum_j a'_j\right) = \sum f(a'_j).$$

□

Remark 5.45. *For $\mathcal{A}$-module systems $\mathcal{M}$ and $\mathcal{N}$, we say that a morphism $f : \mathcal{M} \to \mathcal{N}$ is an $\mathcal{A}$-module morphism if $f(bx) = bf(x)$ for all $b \in \mathcal{A}$ and x in $\mathcal{M}$. Suppose that $\mathcal{F}$ is a free $\mathcal{A}$-module system.*

(i) *Any $\mathcal{A}$-module morphism from $\mathcal{F}$ is $\mathcal{T}$-admissible, by uniqueness of the way of writing an element of $\mathcal{F}$ in terms of the canonical base elements.*
(ii) *Any map from a base of $\mathcal{F}$ to $\mathcal{N}$ defines a unique strict $\mathcal{A}$-module morphism in the natural way.*

Remark 5.46. *$\mathcal{T}$-admissibility sometimes is enough to enable one to carry out the theory, but it is rather close to strictness. Given a $\mathcal{T}$-admissible morphism f we can define its **strict closure** $\tilde{f}$ given by*

$$\tilde{f}\left(\sum_i a_i\right) = \sum f(a_i).$$

Passing to the strict closure sometimes enables one to extend our results.

Our motivating example is the category $\mathcal{A}$-**Mod** of $\mathcal{A}$-module systems over the same $\mathcal{T}$-system $\mathcal{A}$, since modules are so basic to representation theory.

One might object to $\preceq$ replacing equality in Definition 5.39, but we want the natural definition of $\mathcal{T}$-morphism paralleling [36]. The modification in Definition 5.39 comes about since one wants left multiplication by an element of $\mathcal{T}$ to be a morphism, even in cases where distributivity fails.

Remark 5.47. *In our situation, this means that $f(b_1+b_2) \preceq f(b_1)+f(b_2)$ for $b_i \in \mathcal{A}$, and, for $\mathcal{T}$-module systems, $f(ab) \preceq af(b)$ for $a \in \mathcal{T}$. In particular, this implies by induction that $f\left(\sum_{i=1}^t a_i\right) \preceq \sum_{i=1}^t f(a_i)$ for $a_i \in \mathcal{T}$.*

Example 5.48. *The left multiplication map ℓ_a given by $\ell_a(b) = ab$ is strict when $a(\sum b_i) = \sum ab_i$ for $b_i \in \mathcal{T}$, and thus for all $b_i \in \mathcal{A}$. This holds for the $\mathcal{T}$-systems obtained from hyperfields when $a \in \mathcal{T}$, but not for general $a \in \mathcal{A}$.*

We have the following consequences of Proposition 5.4 at our disposal, unifying several ad hoc observations in [57].

Proposition 5.49. *Any morphism f satisfies $f((-)a) = (-)f(a)$.*

Proof. $(-)$ is an invertible unary morphism of additive semigroups, so Proposition 5.4 is applicable. □

Proposition 5.50. *To prove that a $\mathcal{T}$-module is a $\mathcal{T}$-Gmodule for a group $\mathcal{T}$, it suffices to show that $a^{-1}(ab) = b$ for all $a, b \in \mathcal{T}$.*

Proof. The left multiplication map $\ell_a : \mathcal{A} \to \mathcal{A}$ by a is invertible, having the inverse $\ell_{a^{-1}}$. □

Proposition 5.51. *Any morphism f of $\mathcal{T}$-Gmodules satisfies $f(ab) = af(b)$ for all $a \in \mathcal{T}$ and $b \in \mathcal{A}$.*

Proof. The left multiplication map ℓ_a by $a \in \mathcal{T}$ is invertible on $\mathcal{T}$, having the inverse $\ell_{a^{-1}}$, and thus is invertible on $\mathcal{A}$. □

5.8.1. *$\preceq$-systems and weak distributivity.*

Definition 5.52 ([57, Definition 5.51])**.** *A $\mathcal{T}$-**weak module** over a set $\mathcal{T}$ is as in Definition 2.1, except that since now we have the surpassing relation $\preceq$ at our disposal, (i) can be weakened to*

(i) *(Weak distributivity over $\mathcal{T}$): $a(\sum_{j=1}^{u} b_j) \preceq \sum_{j=1}^{u}(ab_j)$.*

*An $\mathcal{A}$-**module** $\preceq$**-triple** is defined in the same way as in Definition 5.16, except that now $\mathcal{M}$ is just a weak $\mathcal{T}$-monoid module. Similarly for a $\preceq$**-system**.*

There often is a way to recover distributivity by "improving" multiplication in $\mathcal{A}$ (while not affecting multiplication over $\mathcal{T}$), as in [57, Theorem 2.9].

In order to deal with weak distributivity, it is convenient to introduce a weaker version of generation and to make an analogous version of a $\mathcal{T}$-system.

Definition 5.53. *An element $b \in \mathcal{A}$ is $\preceq$**-generated** by $\{a_i : i \in I\} \subseteq \mathcal{A}$ if $b \preceq \sum_i a_i$.*

*A $\preceq$**-system** is a quadruple $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-), \preceq)$, where $\mathcal{A}$ is a pseudo-triple and $\preceq$ is a surpassing relation with $\mathcal{T}_{\mathcal{A}}$ designated as a distinguished subset which $\preceq$-generates $(\mathcal{A}, +)$, and a negation map $(-)$ satisfying $(-)\mathcal{T}_{\mathcal{A}} = \mathcal{T}_{\mathcal{A}}$.*

Remark 5.54. *"Weak distributivity" in a $\preceq$**-system** means that left multiplication by any element of $\mathcal{T}$ is a $\preceq$-morphism.*

In other words, "weak distributivity" means that $a(\sum_{j=1}^{u} b_j)$ is $\preceq$-generated by the ab_j.

6. Tensored universal algebra and tensor products of systems

Two of the most important functors in the category theory of modules are the tensor product $\otimes$ and Hom. (In these situations we write Hom instead of Mor.) We turn to triples and systems emerging over a given ground triple $(\mathcal{A}, \mathcal{T}, (-))$, and the categories ($\otimes$ and Hom) that arise from them. Both appertain to systems, but each with somewhat unexpected difficulties. We start with tensor products. Hom will be studied in §9.

The tensor product is a very well-known process in general category theory [25, 40], as well as over semirings [41, 61], and has been studied formally in the context of **monoidal categories**, for example in [17, Chapter 2].

6.1. Tensor products of systems.

Here we need the tensor product of systems over a ground $\mathcal{T}$-system. These are described (for semirings) in terms of congruences, as given for example in [41, Definition 3] or, in our notation, [42, §3]. This material also is a special case of [24, § 1.4.5], but we present details which are specific to our situation, especially since we want to see just how far we can go with $\preceq$-morphisms. We also need to consider the negation map.

Let us work with a right $\mathcal{A}$-module system $\mathcal{M}_1$ and left $\mathcal{A}$-module system $\mathcal{M}_2$ over a given ground $\mathcal{T}$-system $\mathcal{A}$. One can define the tensor product $\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ in the usual way, to be

$(\mathcal{F}_1 \oplus \mathcal{F}_2)/\Phi$, where $\mathcal{F}_i$ is the free system (respectively right or left) with base $\mathcal{M}_i$ (and $\mathcal{T}_{\mathcal{F}_i} = \mathcal{M}_i$), and Φ is the congruence generated by all

$$\Big((\sum_j x_{1,j}, \sum_k x_{2,k}), \sum_{j,k} (x_{1,j}, x_{2,k}) \Big), \quad \Big((x_1 a, x_2), (x_1, a x_2) \Big) \tag{6.1}$$

$\forall x_{i,j} \in \mathcal{M}_i$, $a \in \mathcal{A}$.

Definition 6.1. *A map $\Phi : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{N}$ is **bilinear** if*

$$\Phi\Big((\sum_j x_{1,j}, \sum_k x_{2,k}) \Big) = \sum_{j,k} \Phi(x_{1,j}, x_{2,k}), \quad \Phi(x_1 a, x_1') = \Phi(x_1, a x_1'), \tag{6.2}$$

$\forall x_{i,j} \in \mathcal{M}_i$, $a \in \mathcal{A}$.

Lemma 6.2. *Any bilinear map $\Psi : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{N}$ induces a map $\bar{\Psi} : \mathcal{M}_1 \otimes \mathcal{M}_2 \to \mathcal{N}$ given by $\bar{\Psi}(a_1 \otimes a_2) = \Psi(a_1, a_2)$.*

Proof. Ψ passes through the defining congruence Φ of (6.1). $\square$

To handle negation maps we start again, with a slightly more technical version emphasizing $\mathcal{T}_{\mathcal{A}}$.

Definition 6.3. *The $\mathcal{T}_{\mathcal{A}}$**-tensor product** $\mathcal{M}_1 \otimes_{\mathcal{T}_{\mathcal{A}}} \mathcal{M}_2$ of a right $\mathcal{T}_{\mathcal{A}}$-module system $\mathcal{M}_1$ and a left $\mathcal{T}_{\mathcal{A}}$-module system $\mathcal{M}_2$ to be $(\mathcal{F}_1 \oplus \mathcal{F}_2)/\Phi$, where $\mathcal{F}_i$ is the free system with base $\mathcal{M}_i$ (and $\mathcal{T}_{\mathcal{F}_i} = \mathcal{M}_i$), and Φ is the congruence generated as in (6.1), but now with $a \in \mathcal{T}_{\mathcal{A}}$.*

*If $\mathcal{M}_1, \mathcal{M}_2$ have negation maps $(-)$, then we define a **negated tensor product** by further imposing the extra axiom*

$$((-)x) \otimes y = x \otimes ((-)y).$$

Note that this is done by modding out by the congruence generated by all elements $((-)x \otimes y,\ x \otimes (-)y)$, $x, y \in \mathcal{T}_{\mathcal{M}}$, in the congruence defining the tensor product in the universal algebra framework. From now on, the notation $\mathcal{M}_1 \otimes \mathcal{M}_2$ includes this negated tensor product stipulation, and $\mathcal{A}$ and $\mathcal{T}_{\mathcal{A}}$ are understood.

We can incorporate the negation map into the tensor product, defining a negation map on $\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2$ by $(-)(v \otimes w) = ((-)v) \otimes w$.

Remark 6.4. *As in the classical theory, if $\mathcal{M}_1$ is an $(\mathcal{A}, \mathcal{A}')$-bimodule system, then $\mathcal{M}_1 \otimes \mathcal{M}_2$ is an $\mathcal{A}$-module system. In particular this happens when $\mathcal{A}$ is commutative and the right and left actions on $\mathcal{M}_1$ are the same; then we take $\mathcal{A}' = \mathcal{A}$.*[4]

Note that there are fewer relations in the defining congruence, so the $\mathcal{T}_{\mathcal{A}}$-tensor product maps down onto the tensor product. In the classical case where $\mathcal{T}_{\mathcal{A}} = \mathcal{A} \setminus \{\mathbb{0}\}$ we have nothing new, but the general case is quite different.

Definition 6.5. *The **tensor product of triples** $(\mathcal{A}, \mathcal{T}, (-))$ and $(\mathcal{A}', \mathcal{T}', (-)')$ is the triple*

$$(\mathcal{A} \otimes \mathcal{A}', \{a_1 \otimes a_2 : a_1 \in \mathcal{T}, a_2 \in \mathcal{T}'\}, (-) \otimes 1_{\mathcal{A}'}).$$

When $\mathcal{A}$ is commutative, it can be viewed as a $\mathcal{T}$-bimodule triple, so the tensor product becomes a $\mathcal{T}$-triple in the usual way.

Remark 6.6. *If $\mathcal{S}$ is generated by "multilinear" operators (i.e.,*

$$\omega(a_1, \dots, a_i + a_i', \dots, a_m) = \omega(a_1, \dots, a_i, \dots, a_m) + \omega(a_1, \dots, a_i', \dots, a_m),$$

$$\omega(a_1, \dots, a a_i, \dots, a_m) = a\omega(a_1, \dots, a_i, \dots, a_m),$$

and $\mathcal{A}'$ is a commutative associative $\mathcal{T}$-bimodule triple over $\mathcal{A}$, then $\_ \otimes_{\mathcal{A}} \mathcal{A}'$ yields a functor from $\mathcal{T}_{\mathcal{A}}$-triples to $\mathcal{T}_{\mathcal{A}'}$-triples.

[4] There is some universal algebra lurking beneath the surface, since one must define an abelian carrier. We are indebted to D. Stanovsky and M. Bonato for providing the formal definition that $\mathcal{A}$ is abelian when $\omega(a_1, a_2, \dots, a_m) = \omega(a_1', a_2, \dots, a_m)$ for all operators ω and all a_1, a_1', and for pointing out the references [18] and [49]; also see [48, Definition 4.146].

When one repeats this construction, it does not matter in which order one builds the tensor products.

In order to be able to apply the theory of monoidal categories, we need to be able to show that the tensor product is functorial, i.e., given morphisms $f_i : \mathcal{M}_i \to \mathcal{N}_i$ for $i = 1, 2$, we want a well-defined morphism $f_1 \otimes f_2 : \mathcal{M}_1 \otimes \mathcal{M}_2 \to \mathcal{N}_1 \otimes \mathcal{N}_2$.

We would want to define the tensor product $f \otimes g$ of morphisms by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$. But this breaks down with our broad definition of $\preceq$-morphism. If the f_i are strict morphisms in universal algebra, then the theory flows rather smoothly, but otherwise we run into immediate difficulties. It falls apart even over free modules, over which all $\mathcal{A}$-morphisms are $\mathcal{T}$-admissible, cf. Remark 5.45(i).

Example 6.7. *Consider the polynomial system $\mathcal{A}[\lambda_1, \lambda_2]$ and the morphism $f : \mathcal{A}[\lambda_1, \lambda_2] \to \mathcal{A}[\lambda_1, \lambda_2]$ given by taking f to be the identity on all monomials and $f(q) = \mathbb{0}$ whenever q is a sum of at least two nonconstant monomials.*

Then $\lambda_1 \otimes \lambda_1 + \lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_2 = \lambda_1 \otimes (\lambda_1 + \lambda_2) + \lambda_2 \otimes \lambda_2 = \lambda_1 \otimes \lambda_1 + (\lambda_1 + \lambda_2) \otimes \lambda_2$, so

$$\lambda_2 \otimes \lambda_2 = (f \otimes f)(\lambda_1 \otimes \lambda_1 + \lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_2) = \lambda_1 \otimes \lambda_1,$$

which is absurd.

Also, $\lambda_1 \otimes \lambda_1 + \lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_2 = (f \otimes 1)(\lambda_1 \otimes \lambda_1 + \lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_2) = \lambda_1 \otimes \lambda_1$, so even $f \otimes 1$ is not well-defined.

Proposition 6.8. *Suppose that $f_i : \mathcal{M}_i \to \mathcal{N}_i$ are strict morphisms. Then the map*

$$f_1 \otimes f_2 : \mathcal{M}_1 \otimes \mathcal{M}_2 \to \mathcal{N}_1 \otimes \mathcal{N}_2$$

given by $(f_1 \otimes f_2)(\sum_i a_{1,i} \otimes a_{2,i}) = \sum f_1(a_{1,i}) \otimes f_2(a_{2,i})$ is a well-defined strict morphism.

Proof. This is standard. If f_1 and f_2 are strict then the map $\Phi : (a_1, a_2) \mapsto f_1(a_1) \otimes f_2(a_2)$ is bilinear, since

$$\begin{aligned} \Phi\left(\left(\sum_j x_{1,j}, \sum_k x_{2,k}\right)\right) &= \sum_j f_1(x_{1,j}) \otimes \sum_k f_2(x_{2,k}) = \sum_{j,k} \Phi(x_{1,j}, x_{2,k}), \\ \Phi(x_1 a, x_2) &= f(x_1 a) \otimes f(x_2) = f(x_1) \otimes f(a x_2) = \Phi(x_1, a x_2), \end{aligned} \tag{6.3}$$

so our assertion follows from Lemma 6.2. $\square$

Corollary 6.9. *Suppose that $f_1 : \mathcal{M}_1 \to \mathcal{N}_1$ is a strict morphism. Then the map*

$$f_1 \otimes 1 : \mathcal{M}_1 \otimes \mathcal{M}_2 \to \mathcal{N}_1 \otimes \mathcal{M}_2$$

given by $(f_1 \otimes 1)(a_1 \otimes a_2) = f_1(a_1) \otimes a_2$ is a well-defined strict morphism, which is $\circ$-onto when f_1 is $\circ$-onto (see, Definition 8.4 for the definition of $\circ$-onto).

Accordingly, we need a category to be strict, as in Definition 5.42, to get a monoidal category.

6.1.1. *The tensor semialgebra triple and the polynomial semialgebra.*

Next, as usual, given a bimodule V over $\mathcal{T}_{\mathcal{A}}$, one defines $V^{\otimes(1)} = V$, and inductively

$$V^{\otimes(k)} = V \otimes V^{\otimes(k-1)}.$$

From what we just described, if V has a negation map $(-)$ then $V^{\otimes(k)}$ also has a natural negation map, and often is uniquely negated.

Definition 6.10. *Define the **tensor semialgebra** $T(V) = \bigoplus_k V^{\otimes(k)}$ (adjoining a copy of $\mathcal{T}_{\mathcal{A}}$ if we want to have a unit element), with the usual multiplication.*

*Given a $\mathcal{T}_{\mathcal{A}}$-module triple $(\mathcal{M}, \mathcal{T}_{\mathcal{M}}, (-))$, the **tensor semialgebra triple** $(T(\mathcal{M}), \mathcal{T}_{T(\mathcal{M})}, (-))$ of $\mathcal{M}$ is defined by using the negated tensor products of Definition 6.3 to define $T(\mathcal{M})$, induced from the negation maps on $\mathcal{M}^{\otimes(k)}$; writing $\tilde{a}_k = a_{k,1} \otimes \cdots \otimes a_{k,k}$ for $a_{k,j} \in \mathcal{T}_{T(\mathcal{M})}$, we put*

$$(-)(\tilde{a}_k) = (-)(a_{k,1} \otimes \cdots \otimes a_{k,k}).$$

$\mathcal{T}_{T(\mathcal{M})}$ is $\cup\{\tilde{a}_k :\}$, the set of simple multi-tensors.

Lemma 6.11. *The identity map is a strict morphism of triples $(\mathcal{M}, \mathcal{T}_{T(\mathcal{M})}, (-)) \to (T(\mathcal{M}), \mathcal{T}_{T(\mathcal{M})}, (-))$.*

Proof. Immediate by the definition. $\square$

We can now view the polynomial semialgebra (Definition 4.7) in this context.

Example 6.12. *Suppose $\mathcal{A} = (\mathcal{A}, \mathcal{T}, (-))$ is a triple. The polynomial semialgebra $\mathcal{A}[\lambda]$ now is defined as a special case of the tensor semialgebra. $\mathcal{T}_{A[\lambda]}$ again is the set of monomials with coefficients in $\mathcal{T}$.*

The **polynomial triple** $\mathcal{A}[\lambda] := (\mathcal{A}[\lambda], \mathcal{T}_{A[\lambda]}, (-))$ where $(-)$ is again defined according to monomials: $(-)(a\lambda^j) = ((-)a)\lambda^j$. If $\mathcal{A}$ is uniquely negated then so is $\mathcal{A}[\lambda]$. Viewing $\mathcal{A}[\lambda]$ as the module triple $\mathcal{A}^{\mathbb{N}}$ (according to monomials) one extends a surpassing PO $\preceq$ on $\mathcal{A}$ according to Example 5.37. Then we recover the polynomial system $\mathcal{A}[\lambda]$.

One can iterate this construction to define $\mathcal{A}[\lambda_1, \dots \lambda_n]$ (and then take direct limits to handle an infinite number of indeterminates).

There is a subtlety here which should be addressed. When defining the module of monomials $\mathcal{A}\lambda$ from which we construct the tensor subalgebra, we could view λ either as a formal indeterminate, or as a placemark for a function $f : \lambda \to \mathcal{A}$ given by choosing a and defining $f : \lambda \to a$. These are not the same, since two formal polynomials could agree as functions. Our point of view will be the functional one.

7. The structure theory of ground triples via congruences

We are ready to embark on the structure theory of ground triples, in analogy to the structure theory of rings and integral domains. Our objective in this section is to modify the ideal-theory of homomorphisms (which clearly cannot work over semirings) and the corresponding factor-module theory to an analog which is robust enough to support the structure theory of ground $\mathcal{T}$-systems. To do this, we first ignore the issue of negatives, and then, as in [57], use "symmetry" (which is formal negation) instead of negatives, but presented here more categorically.

7.1. **The role of congruences.**

Unfortunately everything starts to unravel at once. For starters, [24, §1.6.2] is too restrictive for our purposes. Factor $\mathcal{T}_{\mathcal{A}}$-modules (or factor semirings†) are a serious obstacle, since cosets need not be disjoint (this fact relying on additive cancellation, which fails in the max-plus algebra, since $1 + 3 = 2 + 3 = 3$).

We also have the following problematic homomorphism, if we want to use the preimage of $\mathbb{0}$ in the theory.

Example 7.1. *Define the homomorphism $f : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ by $f(a_0, a_1) = (a_0 + a_1, a_0 + a_1)$. Then f is not 1:1 over the max-plus algebra, but $f^{-1}(\mathbb{0}, \mathbb{0}) = (\mathbb{0}, \mathbb{0})$. This example also blocks the naive definitions of kernels and cokernels. (In what follows, the null elements will take the place of the element $\mathbb{0}$ in $\mathcal{A}$.)*

This is solved in universal algebra via the use of congruences, which are equivalence relations respecting the given algebraic structure. Congruences can also be viewed as subalgebras of $\mathcal{A} \times \mathcal{A}$ which are also equivalence relations.

Definition 7.2. *A congruence Φ is $\mathcal{T}$-**trivial** if $\Phi|_{\mathcal{T}} = \{(a, a) : a \in \mathcal{T}\}$. Viewing a congruence of $\mathcal{A}$ as a substructure of $\widehat{\mathcal{A}}$, we define the **trivial** congruence*

$$\mathrm{Diag}_{\mathcal{A}} := \{(a, a) : a \in \mathcal{A}\}.$$

Clearly every congruence contains $\mathrm{Diag}_{\mathcal{A}}$. Furthermore, since we are viewing $(-)$ in the signature $\mathcal{S}$, we require that $((-)a_1, (-)a_2) \in \Phi$ whenever $(a_1, a_2) \in \Phi$. The trivial congruences are the ones on which we want to focus.

Definition 7.3. *Given a triple $\mathcal{A} = (\mathcal{A}, \mathcal{T}, (-))$ with $\mathbb{0} \in \mathcal{A}$, we write*

$$\widehat{\mathcal{A}} = (\widehat{\mathcal{A}}, \mathcal{T}_{\widehat{\mathcal{A}}}, (-)),$$

where other operators are defined componentwise, $\mathcal{T}_{\widehat{\mathcal{A}}} := (\mathcal{T} \times \{\mathbb{0}\}) \cup (\{\mathbb{0}\} \times \mathcal{T})$, and $(-)$ is taken to be the switch map $(a_0, a_1) \mapsto (a_1, a_0)$.

Put $\mathcal{T}_0 = \mathcal{T} \times \{\mathbb{0}\}$ and $\mathcal{T}_1 = \{\mathbb{0}\} \times \mathcal{T}$. Then $\mathcal{T}_{\widehat{\mathcal{A}}} = \mathcal{T}_0 \cup \mathcal{T}_1$ and we have:

Lemma 7.4. *$(\widehat{\mathcal{A}}, \mathcal{T}_{\widehat{\mathcal{A}}}, (-), \preceq_\circ)$ is a system, where $(-)$ is the switch map and $\preceq_\circ$ is given by $(a_0, a_1) \preceq_\circ (a_0', a_1')$ if there is $b \in \mathcal{A}$ such that $a_i + b = a_i'$ for $i = 0, 1$.*

Proof. Clearly $\mathcal{T}$ generates $\widehat{\mathcal{A}}$, and $\widehat{\mathcal{A}}^\circ = \{(b,b) : b \in \mathcal{A}\}$. It remains to show that $\preceq_\circ$ is as stated which is clear since the elements of $\widehat{\mathcal{A}}^\circ$ have the form (b,b). $\square$

Since we are incorporating the switch morphism into the signature, we require for an $\mathcal{A}$-module $\mathcal{M}$ that if $(a,b) \in \mathcal{M}$ then $(b,a) = (-)(a,b) \in \mathcal{M}$. This is a very mild condition; for example, if $\mathbb{0} \in \mathcal{A}$ then $(b,a) = (\mathbb{0},\mathbb{1})(a,b) \in \mathcal{M}$. Every congruence is invariant under the switch map.

Lemma 7.5.

(i) *The twist action (Definition 2.5) on the module $\widehat{\mathcal{M}}$ over $\widehat{\mathcal{A}}$ extends the $\mathcal{T}_{\mathcal{A}}$-module action on $\mathcal{M}$.*
(ii) *Any congruence is closed under the twist action.*

Proof. Suppose $(x_0,x_1) \in \widehat{\mathcal{M}}$ and $(a_0,\mathbb{0}) \in \widehat{\mathcal{T}_{\mathcal{A}}}$. Then $(a_0,\mathbb{0}) \cdot_{\mathrm{tw}} (x_0,x_1) = (a_0x_0, a_0x_1) \in \widehat{\mathcal{M}}$, yielding (i). For the second assertion, let Φ be a congruence. By applying (i) again, we obtain that

$$(\mathbb{0},a_1) \cdot_{\mathrm{tw}} (x_0,x_1) = (a_1x_0, a_1x_1) \in \Phi.$$

But, since Φ is a congruence (in particular symmetric), we have that $(a_1x_1, a_1x_0) \in \Phi$ and hence the sum $(a_0x_0 + a_1x_1, a_0x_1 + a_1x_0) \in \Phi$, yielding (ii). $\square$

7.1.1. *The twist action on congruences.*
In universal algebra, congruences are more important to us than ideals [5]. But $\mathcal{T}_{\mathcal{A}}$-module congruences are difficult to work with, since they are $\mathcal{T}_{\mathcal{A}}$-submodules of $\widehat{\mathcal{M}} = \mathcal{M} \times \mathcal{M}$ over $\mathcal{A}$. The next concept eases this difficulty by bringing in the twist action as a negation map on semirings[†].

For any $\mathcal{T}$-semiring system, the **twist action** on congruences Φ_1 and Φ_2 is given by

$$(a_0,a_1) \cdot_{\mathrm{tw}} (b_0,b_1) = (a_0b_0 + a_1b_1, a_0b_1 + a_1b_0) \tag{7.1}$$

for $(a_0,a_1) \in \Phi_1$ and $(b_0,b_1) \in \Phi_2$.

We write $\Phi_1 \cdot_{\mathrm{tw}} \Phi_2$ for the congruence generated by $\{(a_0,a_1) \cdot_{\mathrm{tw}} (b_0,b_1) : a_i \in \Phi_1,\ b_i \in \Phi_2\}$. From associativity, it makes sense to write Φ^n for the twist product of n copies of Φ.

Lemma 7.6. $(\widehat{\mathcal{A}}, \mathcal{T}_{\widehat{\mathcal{A}}}, (-))$ *is a uniquely negated triple and* $\widehat{\mathcal{T}}^\circ = \{(a,a) : a \in \mathcal{T} \setminus \{\mathbb{0}\}\}$.

Proof. $(-)(a,\mathbb{0}) = (\mathbb{0},a)$, its unique quasi-negative. $(a,a) = (a,\mathbb{0}) + (\mathbb{0},a) = (a,\mathbb{0})(-)(a,\mathbb{0})$. $\square$

The triple $\widehat{\mathcal{A}}$ is not meta-tangible. This could be rectified in order to obtain ground triples, by redefining addition on $\widehat{\mathcal{T}}$ to make it meta-tangible as done in [19] or [57, Example 2.53], but here we find it convenient to use the natural (componentwise) addition on $\widehat{\mathcal{A}}$, to make it applicable for congruences.

Proposition 7.7. *If $\mathcal{N}$ is a sub-triple of an $\mathcal{A}$-module triple $\mathcal{M}$ with negation map $(-)$, then $\mathcal{N}$ becomes a $\widehat{\mathcal{T}_{\mathcal{A}}}$-submodule of $\mathcal{M}$ under the action $(a_0,a_1)x = a_0x(-)a_1x$.*

Proof. $(a(a_0,a_1))x = aa_0x(-)aa_1x = a((a_0,a_1)x)$. Likewise for addition. $\square$

We note that there is a conflict between the switch map on $\widehat{\mathcal{A}}$ and the original negation map on $\mathcal{A}$, which do not match; $(a_1,a_0) = \widehat{(-)}(a_0,a_1) \neq ((-)a_0,(-)a_1)$ unless $a_1 = (-)a_0$. Fortunately this does not affect Proposition 7.7 since

$$(a_1,a_0)x = a_1x(-)a_0x = (-)(a_0x(-)a_1x) = (-)((a_0,a_1)x) = (((-)a_0,(-)a_1)x).$$

The analogous result holds for weak modules.

Corollary 7.8. *If $\mathcal{N}$ is a sub-triple of a $\mathcal{A}$-module triple $\mathcal{M}$ with negation map $(-)$, and Φ is a congruence on $\mathcal{M}$, then Φ becomes a sub-triple of $\widehat{\mathcal{M}}$ (multiplication as in Proposition 7.7).*

---

[5] We have not defined an ideal in universal algebra, but intuitively it should satisfy $\omega(a_1,\dots,a_{i-1},b,a_{i+1},\dots,a_m) \in \mathcal{I}$ whenever $b \in \mathcal{I}$, for any i and all "extra" operators ω; we say "extra" because the condition fails for addition!

7.1.2. *$\mathcal{T}$-congruences.*

We restrict our congruences somewhat, to give $\mathcal{T}$ its proper role.

Definition 7.9. *For any congruence Φ, we define $\Phi|_{\mathcal{T}} = \{(a,a') \in \Phi : a, a' \in \mathcal{T}_{\mathbb{0}}\}$. We say that a congruence Φ is a $\mathcal{T}$-**congruence** if and only if Φ is additively generated by $\Phi|_{\mathcal{T}}$. Thus $(c,d) \in \Phi$ when $c = \sum_j a_j$ and $d = \sum_j a'_j$ for $(a_j, a'_j) \in \Phi|_{\mathcal{T}}$.*

Obviously $\mathrm{Diag}_{\mathcal{A}}$ is a $\mathcal{T}$-congruence.

Lemma 7.10.

(i) *The union of a chain of $\mathcal{T}$-congruences is a $\mathcal{T}$-congruence.*
(ii) *The categorical sum of finitely many $\mathcal{T}$-congruences is a $\mathcal{T}$-congruence.*

Proof. (i) Any element is in some finite union, so is in one of the chain.
(ii) Clear by the definition of generation. $\square$

As with rings and modules, there are two ways of factoring out $\mathcal{T}$-congruences. If Φ is a $\mathcal{T}$-congruence on a triple $\mathcal{A}$, we can form the **factor triple** $\mathcal{A}/\Phi$, generated by

$$\mathcal{T}/\Phi|_{\mathcal{T}} := \{([a_1],[a_2]) : (a_1,a_2) \in \mathcal{T}\},$$

where the equivalence classes are taken with respect to Φ.

Lemma 7.11. *If $\Phi|_{\mathcal{T}}$ is a $\mathcal{T}$-congruence on a triple $(\mathcal{A}, \mathcal{T}, (-))$, then $(\mathcal{A}/\Phi|_{\mathcal{T}}, \mathcal{T}/\Phi|_{\mathcal{T}}, (-))$ is a triple, where one defines $(-)[a] = [(-)a]$, and there is a strict morphism $\mathcal{A} \to \mathcal{A}/\Phi$ (as $\mathcal{A}$-module triples) given by $a \mapsto [a]$.*

Proof. We need $(-)$ to be well-defined on $\mathcal{A}/\Phi|_{\mathcal{T}}$, which means that if $(\sum a_{1i}, \sum a_{2i}) \in \Phi|_{\mathcal{T}}$, then $(\sum (-)a_{1i}, \sum (-)a_{2i}) \in \Phi|_{\mathcal{T}}$. But this is patent. $\square$

On the other hand, if $\Phi_2 \subseteq \Phi_1$ are $\mathcal{T}$-congruences, we can define the **$\mathcal{T}$-factor congruence** Φ_1/Φ_2 generated by

$$\{([a_1],[a_2]) : (a_1,a_2) \in \Phi_1\}$$

where the equivalence classes are taken with respect to Φ_2. $\Phi/\mathrm{Diag}_{\mathcal{A}}$ is just Φ. If Φ_1 is a $\mathcal{T}$-congruence then so is Φ_1/Φ_2.

7.2. **Prime $\mathcal{T}$-systems and prime $\mathcal{T}$-congruences.**

Since classical algebra focuses on algebras over integral domains (i.e., prime commutative rings), we look for their systemic generalization. The idea is taken from Joó and Mincheva [35] as well as Berkovich [7]. In [7], Berkovich defined a notion of the prime spectrum $\mathrm{Spec}(\mathcal{A})$ when $\mathcal{A}$ is a commutative monoid. Berkovich's definition of a prime ideal of a commutative monoid $\mathcal{A}$ is a congruence relation $\sim$ on $\mathcal{A}$ such that $A/\sim$ is nontrivial and cancellative. On the other hand, for a semiring $\mathcal{A}$, Joó and Mincheva defined a prime congruence for $\mathcal{A}$ as a congruence P of $\mathcal{A}$ such that if $a \cdot_{\mathrm{tw}} b \in P$ then $a \in P$ or $b \in P$ for $a, b \in \widehat{\mathcal{A}}$. This definition implies the definition of Berkovich in the following sense.

Lemma 7.12. *Let $\mathcal{A}$ be an idempotent commutative $\mathcal{T}$-semiring triple. Then, for any prime $\mathcal{T}$-congruence P (defined as in* [35]*), the $\mathcal{T}/P$-semiring triple $\mathcal{A}/P$ is cancellative.*

Proof. The case when $\mathcal{A}$ is an idempotent semiring is proved in [35, Proposition 2.8] although the converse is not true in general (cf. [35, Theorem 2.12]). One may use the argument in [35] to prove the case for triples. $\square$

Let us modify this, to get both directions. We drop the assumption of commutativity whenever the proofs are essentially the same.

Definition 7.13. *Let $(\mathcal{A}, \mathcal{T}, (-))$ be a $\mathcal{T}$-semiring† triple.*

(i) *A $\mathcal{T}$-congruence $\Phi \neq \widehat{\mathcal{A}}$ of a semiring† triple $\mathcal{A}$ is **prime** if $\Phi' \cdot_{\mathrm{tw}} \Phi'' \subseteq \Phi$ for $\mathcal{T}$-congruences implies $\Phi' \subseteq \Phi$ or $\Phi'' \subseteq \Phi$.*
(ii) *The triple $(\mathcal{A}, \mathcal{T}, (-))$ is **prime** (i.e., integral domain) if the trivial congruence* $\mathrm{Diag}_{\mathcal{A}}$ *is a prime $\mathcal{T}$-congruence.*

(iii) *A $\mathcal{T}$-congruence Φ is **$\mathcal{T}$-semiprime** if $(\Phi')^2 \subseteq \Phi$ implies $\Phi' \subseteq \Phi$. Semiprime $\mathcal{T}$-congruences are called **radical** when $\mathcal{A}$ is commutative.*
(iv) *The triple $(\mathcal{A}, \mathcal{T}, (-))$ is **$\mathcal{T}$-semiprime** if the trivial congruence $\mathrm{Diag}_{\mathcal{A}}$ is a $\mathcal{T}$-semiprime $\mathcal{T}$-congruence. A commutative semiprime triple is called **reduced**, in analogy with the classical theory.*
(v) *The triple $(\mathcal{A}, \mathcal{T}, (-))$ is **$\mathcal{T}$-irreducible** if the intersection of nontrivial $\mathcal{T}$-congruences is nontrivial.*

Lemma 7.14. *A triple is prime if and only if it is $\mathcal{T}$-semiprime and $\mathcal{T}$-irreducible.*

Proof. ($\Rightarrow$) Semiprime is a fortiori. But if $\Phi \cap \Phi'$ is trivial then $\Phi \cdot_{\mathrm{tw}} \Phi' \subseteq \Phi \cap \Phi'$ is trivial, implying Φ or Φ' is trivial.

($\Leftarrow$) If $\Phi \cdot_{\mathrm{tw}} \Phi'$ is trivial, then $(\Phi \cap \Phi')^2$ is trivial, implying $\Phi \cap \Phi'$ is trivial, so Φ or Φ' is trivial. $\square$

This is done in the same way for congruences in general, but it is convenient for us to stay within the system and use $\mathcal{T}$-congruences. The following assertions are straightforward.

Proposition 7.15.
(i) *For a $\mathcal{T}$-congruence Φ', $\Phi|'_{\mathcal{T}} \subseteq \Phi$ if and only if $\Phi' \subseteq \Phi$.*
(ii) *The intersection of prime $\mathcal{T}$-congruences is radical.*
(iii) *$\Phi \cdot_{\mathrm{tw}} \mathcal{M} \subseteq \mathcal{M}$ is a $\mathcal{T}$-subsystem of a module system $\mathcal{M}$, for any $\mathcal{T}$-congruence Φ.*

We say that $\mathcal{A}$ satisfies the ACC on $\mathcal{T}$-congruences if for every ascending chain $\{\Phi_i : i \in I\}$ there is some i such that $\Phi_i = \Phi_{i'}$ for all $i' > i$.

Proposition 7.16. *If $\mathcal{A}$ satisfies the ACC on $\mathcal{T}$-congruences, then every $\mathcal{T}$-congruence Φ contains a finite product of prime $\mathcal{T}$-congruences, and in particular there is a finite set of prime $\mathcal{T}$-congruences whose product is trivial.*

Proof. A standard argument on Noetherian induction: We take a maximal counterexample Φ. If Φ is not already prime, then there are two $\mathcal{T}$-congruences Φ', Φ'' whose intersection with $\mathcal{T}$ is not in Φ, but whose product is in Φ. By Noetherian induction applied to Φ/Φ' in $\mathcal{A}/\Phi'$, and to Φ/Φ'' in $\mathcal{A}/\Phi''$, we get finite sets of prime $\mathcal{T}$-congruences whose respective products are in Φ' and Φ'', so the product of all of them is in Φ. $\square$

The following assertions can all be formulated in terms of the Baer-Levitzki-Amitsur theory of radicals [4, 5], but become much easier when our semirings are assumed commutative, and their proofs are standard, with (iii) replaced by the notions in the Baer-Levitzki-Amitsur theory.

For $\mathcal{A}$ commutative with a $\mathcal{T}$-congruence Φ, define $\sqrt{\Phi}$ to be the $\mathcal{T}$-congruence generated by the set

$$\{(a_0, a_1) \in \widehat{\mathcal{T}} : (a_0, a_1)^n \in \Phi \text{ for some } n\}.$$

Lemma 7.17. *In a commutative $\mathcal{T}$-semiring system,*
(i) *Φ is prime if and only if it satisfies the condition that $(a_0, a_1) \cdot_{\mathrm{tw}} (b_0, b_1) \in \Phi$ implies $(a_0, a_1) \in \Phi$ or $(b_0, b_1) \in \Phi$ for $(a_0, a_1), (b_0, b_1) \in \widehat{\mathcal{T}}$.*
(ii) *Every maximal $\mathcal{T}$-congruence is prime.*
(iii) *Φ is $\mathcal{T}$-radical if and only if $(a_0, a_1)^2 \in \Phi$ implies $(a_0, a_1) \in \Phi$ for $(a_0, a_1) \in \widehat{\mathcal{T}}$.*
(iv) *If Φ is a $\mathcal{T}$-congruence, then $\sqrt{\Phi}$ is a radical $\mathcal{T}$-congruence.*
(v) *For any $(a_0, a_1) \in \widehat{\mathcal{T}}$, $\{(a_0, a_1)\} \cdot_{\mathrm{tw}} \mathcal{M} \subseteq \mathcal{M}$ is a $\mathcal{T}$-subsystem of $\mathcal{M}$.*

Joó and Mincheva [35] showed that any irreducible, cancellative commutative $\mathbb{B}$-algebra is prime, and it follows that the polynomial system $\mathcal{A}[\lambda]$ is prime. In general, we need the following property based on Proposition 7.17:

Lemma 7.18. *Any $\mathcal{T}$-congruence P on a commutative $\mathcal{T}$-semiring system is prime if and only if the equality $a_0b_0 + a_1b_1 = a_0b_1 + a_1b_0$ implies $a_0 = a_1$ or $b_0 = b_1$.*

Proof. This is a translation of the twisted product. $\square$

The following proposition is a $\mathcal{T}$-congruence analogue of the weak Nullstellensatz.

Proposition 7.19. *For every $\mathcal{T}$-congruence Φ on a commutative $\mathcal{T}$-semiring system, $\sqrt{\Phi}$ is an intersection of prime $\mathcal{T}$-congruences.*

Proof. For any given $\mathbf{a} = (a_0, a_1) \in \widehat{\mathcal{T}}$, let $S_{\mathbf{a}} = \{(a_0, a_1)^n : n \in \mathbb{N}\}$. We say that $\mathbf{a}$ is **trivial-potent with respect to** Φ if $S_{\mathbf{a}} \cap \Phi \neq \emptyset$. For any non-trivial-potent $\mathbf{a}$, Zorn's lemma gives us a $\mathcal{T}$-congruence containing $\sqrt{\Phi}$, maximal with respect to being disjoint from $S_{\mathbf{a}}$, and easily seen to be prime, so their intersection is precisely $\sqrt{\Phi}$. □

7.3. **Annihilators.**

Definition 7.20. *Suppose $\mathcal{M}$ is a $\mathcal{T}$-module system over a $\mathcal{T}$-semiring† system $\mathcal{A}$. For any $S \subseteq \mathcal{M}$, define the* ***annihilator*** $\mathrm{Ann}_{\mathcal{T}}(S)$ *to be the $\mathcal{T}$-congruence generated by $\{(a_0, a_1) \in \widehat{\mathcal{T}_{\mathbb{0}}} : a_0 s = a_1 s, \ \forall s \in S\}$.*

Likewise, suppose Φ is a $\mathcal{T}$-congruence on the $\mathcal{T}$-module system $\mathcal{M}$. For $S \subseteq \Phi$, define the ***annihilator*** $\mathrm{Ann}_{\mathcal{T}}(S)$ *to be the $\mathcal{T}$-congruence generated by $\{(a_0, a_1) \in \widehat{\mathcal{T}} : a_0 s_0 + a_1 s_1 = a_0 s_1 + a_1 s_0, \ \forall s \in S\}$.*

In other words, $\mathrm{Ann}_{\mathcal{T}}(S)S$ is trivial, under the twist multiplication.

Definition 7.21. *An $\mathcal{A}$-module pseudo-triple $\mathcal{M}$ is* ***$\mathcal{T}$-simple*** *if proper sub-pseudo-triples are all in $\mathcal{M}^\circ$.*

As in classical algebra, these are the building blocks in the structure theory of triples that are monoid modules.

Proposition 7.22. *If $\mathcal{M}$ is a $\mathcal{T}$-simple $\mathcal{T}$-module pseudo-triple, then $\mathrm{Ann}_{\mathcal{T}}(\mathcal{M})$ is a $\mathcal{T}$-prime $\mathcal{T}$-congruence of $\mathcal{A}$.*

Proof. If $\Phi\Phi' \subseteq \mathrm{Ann}_{\mathcal{T}}(\mathcal{M})$ for $\mathcal{T}$-congruences $\Phi\Phi'$, then $\Phi\Phi'\mathcal{M}$ is trivial. Thus either $\Phi' \subseteq \mathrm{Ann}_{\mathcal{A}}(\mathcal{M})$ or $\Phi'\mathcal{M} = \mathcal{M}$, in which case $\Phi\mathcal{M}$ is trivial and hence $\Phi \subseteq \mathrm{Ann}_{\mathcal{A}}(\mathcal{M})$. □

Proposition 7.23. *For $\mathcal{A}$ commutative, $\mathcal{M}$ is a $\mathcal{T}$-simple $\mathcal{T}$-module system if and only if $\mathrm{Ann}_{\mathcal{T}}(\mathcal{M})$ is a maximal $\mathcal{T}$-congruence of $\mathcal{A}$.*

Proof. As in the usual commutative theory, $\mathrm{Ann}_{\mathcal{T}}(\mathcal{M}) = \mathrm{Ann}_{\mathcal{T}}(\{x\})$ for any nonzero $x \in \mathcal{M}$, since $\mathcal{A}x = \mathcal{M}$. □

In the noncommutative case, one could go on to define primitive $\mathcal{T}$-congruences of a ground $\mathcal{T}$-system to be the annihilators of simple module systems, but that is outside the scope of this work.

7.4. **Primeness of the polynomial system $\mathcal{A}[\lambda]$.**

Let us consider when $\mathcal{A}[\lambda]$ is prime. We need a slightly different definition of "trivial."

Definition 7.24. *Let Φ be a $\mathcal{T}$-congruence on $\mathcal{A}$ and S be a small category which is also discrete. We define*

$$\mathcal{A}_\Phi^S := \{(f, g) \in \mathcal{A}^S \times \mathcal{A}^S \mid (f(s), g(s)) \in \Phi, \ \forall s \in S\}.$$

- *An element $(f, g) \in \mathcal{A}_\Phi^S$ is* ***functionally trivial*** *if $(f(s), g(s)) \in \mathrm{Diag}_{\mathcal{A}}, \ \forall s \in S$.*
- *A $\mathcal{T}$-congruence Φ is* ***functionally trivial*** *if any pair $(f, g) \in \mathcal{A}_\Phi^S$ is functionally trivial.*

Proposition 7.25. *Let Φ be a $\mathcal{T}$-congruence on $\mathcal{A}$. If Φ is radical then so are the congruences $\mathcal{A}_\Phi^S$ on $\mathcal{A}^S$, and $\mathcal{A}_\Phi^{(S)}$ on $\mathcal{A}^{(S)}$, for any small category S, and in particular so is $\mathcal{A}[\lambda_1, \dots, \lambda_n]_\Phi$.*

Proof. Let $(f, g) \in \mathcal{A}_\Phi^S$ and suppose that $(f, g)^2 = (f^2, g^2) \in \mathcal{A}_\Phi^S$, i.e., one has

$$(f^2(s), g^2(s)) \in \Phi^2, \ \forall s \in S.$$

It follows that $(f(s), g(s)) \in \Phi$ since Φ is radical, and hence $(f, g) \in \mathcal{A}_\Phi^S$. This proves that $\mathcal{A}_\Phi^S$ is radical. The case of $\mathcal{A}_\Phi^{(S)}$ is similar. □

On the other hand, the other ingredient, irreducibility, is harder to attain. Given a $\mathcal{T}$-congruence Φ on $\mathcal{A}^S$ (in particular, Φ is a subset of $\mathcal{A}^S \times \mathcal{A}^S$), we define

$$S_\Phi := \{s \in S \mid (f(s), g(s)) \notin \mathrm{Diag}_{\mathcal{A}} \ \forall (f, g) \in \Phi\}.$$

This leads to a kind of consideration of density.

Lemma 7.26. *Suppose that $\mathcal{A}$ has $\mathcal{T}$-congruences Φ, Φ' with $\Phi \cap \Phi'$ functionally trivial. If $\mathcal{A}$ is irreducible, then $S_{\mathcal{A}_\Phi^S} \cap S_{\mathcal{A}_{\Phi'}^S} = \emptyset$.*

Proof. For each $s \in S$, since $\Phi \cap \Phi'$ is functionally trivial, either $s \notin S_{\mathcal{A}^S_\Phi}$ or $s \notin S_{\mathcal{A}^S_{\Phi'}}$. This implies that $S_{\mathcal{A}^S_\Phi} \cap S_{\mathcal{A}^S_{\Phi'}} = \emptyset$. □

It is well-known by means of a Vandermonde determinant argument that over an integral domain, any nonzero polynomial of degree n cannot have $n+1$ distinct zeros. The analog for semirings also holds for triples, using ideas from [2]. Namely, we recall [57, Definition 4.23]:

Definition 7.27. *Suppose $\mathcal{A}$ has a negation map $(-)$. For a permutation π, write*

$$(-)^\pi a = \begin{cases} a : \pi \ \textit{even}; \\ (-)a : \pi \ \textit{odd}. \end{cases}$$

(i) *The $(-)$-**determinant** $|A|$ of a matrix A is*

$$\sum_{\pi \in S_n} (-)^\pi \left(\prod_i a_{i,\pi(i)} \right).$$

(ii) *The $n \times n$ **Vandermonde matrix** $V(a_1, \dots, a_n)$ is defined to be* $\begin{pmatrix} \mathbb{1} & a_1 & a_1^2 & \dots & a_1^{n-1} \\ \mathbb{1} & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{1} & a_n & a_n^2 & \dots & a_n^{n-1} \end{pmatrix}$.

(iii) *Write $a'_{i,j}$ for the $(-)$-determinant of the j,i minor of a matrix A. The $(-)$-**adjoint** matrix* $\operatorname{adj}(A)$ *is* $(a'_{i,j})$.

Lemma 7.28. (i) $|A| = \sum_{j=1}^n (-)^{i+j} a'_{i,j} a_{i,j}$, *for any given i.*
(ii) $|V(a_1, \dots, a_n)| = \prod_{i>j} (a_j (-) a_i)$.

Proof. This is well-known for rings, so is an application of the transfer principle in [2]. (Put another way, one could view the a_i as indeterminates, so the assertion holds formally.) □

We say b is a $\circ$-root of $f \in \mathcal{T}[\lambda]$ if $f(b) \in \mathcal{A}^\circ$, i.e., $f(b) \succeq \mathbb{0}$.

Theorem 7.29. *Over a commutative prime triple $\mathcal{A}$, any nonzero polynomial $f \in \mathcal{T}[\lambda]$ of degree n cannot have $n+1$ distinct $\circ$-roots in $\mathcal{T}$.*

Proof. Write $f = \sum_{i=0}^n b_i \lambda^i$ for $b_i \in \mathcal{T}$. Suppose on the contrary that $a_1, \dots, a_{n+1}$ are distinct $\circ$-roots. Write v for the column vector $(a_0, \dots, a_n)$. Then Av is the column vector $(f(a_1), \dots, f(a_n))$ which is a quasi-zero, so

$$\prod_{i>j} (a_j (-) a_i) v = |A| v = \operatorname{adj}(A) A v \in \operatorname{adj}(A) \mathcal{A}^\circ \preceq \mathcal{A}^\circ,$$

implying $\prod_{i>j} (a_j (-) a_i) \in \mathcal{A}^\circ$, contrary to $a_1, \dots, a_{n+1}$ being distinct. □

Corollary 7.30. *If $(\mathcal{A}, \mathcal{T}, (-))$ is a prime commutative triple with $\mathcal{T}$ infinite, then so is $(\mathcal{A}[\lambda], \mathcal{T}, (-))$.*

Proof. Follows from Lemma 7.26, since any finite set of functions cannot have infinitely many common roots. □

$\mathcal{A}[\lambda_1, \dots, \lambda_n]$ enables us to define affine geometry over $\mathcal{T}$-systems and $\mathcal{T}$-congruences, to be handled in [39].

8. The category of $\mathcal{T}$-module systems over a ground $\mathcal{T}$-system

We turn to a representation theory of $\mathcal{T}$-systems, starting with a general observation. There are two approaches, the first via chains of composites of morphisms, and the second using $\mathcal{T}$-congruences.

8.1. The N-category of $\mathcal{T}$-module systems.

Throughout, we take $\mathcal{T}$-module systems with a fixed signature. To view systems $\mathcal{A} = (\mathcal{A}, \mathcal{T}, (-), \preceq)$ in the context of N-categories, we must identify the null objects $\mathcal{A}_{\mathrm{Null}}$. The intuitive choice might be $\mathcal{A}^\circ$, but the following modification often seems to be more inclusive:

Definition 8.1. *For any system $\mathcal{A}$,*

$$\mathcal{A}_{\mathrm{Null}} := \{c \in \mathcal{A} : a^\circ \preceq c \text{ for some } a \in \mathcal{T}_{\mathbb{0}}\}.$$

Remark 8.2.

(i) $\mathcal{A}_{\mathrm{Null}} = \mathcal{A}^\circ$ *when* $\preceq$ *is* $\preceq_\circ$ *as given in Definition 1.2. Using the default system, this enables us to define* $\mathcal{A}_{\mathrm{Null}}$ *for a triple* $\mathcal{A}$.
(ii) *The symmetrized system is a special case of (i).*
(iii) *In the hypergroup setting of* [57, Definition 4.45], $\mathcal{A}_{\mathrm{Null}}$ *consists of those sets containing* $\mathbb{0}$, *which is the version usually considered in the hypergroup literature, for instance, in* [22].
(iv) $\mathcal{A}_{\mathrm{Null}} = \mathcal{A}\backslash\{\mathbb{0}\}$ *for the Green relation* [57, Example 2.61(i)], *so in this case everything degenerates.*

We turn to $\mathcal{T}$-module systems $(\mathcal{M}, \mathcal{T}_{\mathcal{M}}, (-), \preceq)$, to which we refer merely as $\mathcal{M}$ for shorthand. From now on we take $\preceq = \preceq_\circ$ in order to simplify the exposition.

Definition 8.3.

(i) *A* ***chain*** *of* $\mathcal{T}$*-morphisms of* $\mathcal{T}$*-module systems is a sequence*

$$\cdots \to \mathcal{K} \xrightarrow{g} \mathcal{M} \xrightarrow{f} \mathcal{N} \to \cdots$$

such that $(fg)(k) \in \mathcal{N}_{\mathrm{Null}}$ *for all* $k \in \mathcal{K}$.
(ii) *(Compare with* [1]*) The chain is* ***exact*** *at* $\mathcal{M}$ *if* $g(\mathcal{K}) = \{b \in \mathcal{M} : f(b) \in \mathcal{N}_{\mathrm{Null}}\}$.

One can easily see from the above definitions that if $\mathcal{K}$ and $\mathcal{N}$ are null (i.e., $\mathcal{K} = \mathcal{K}_{\mathrm{Null}}$ and $\mathcal{N} = \mathcal{N}_{\mathrm{Null}}$) and the chain is exact, then $\mathcal{M}$ is null as well.

Definition 8.4. *Let* $f : \mathcal{M} \to \mathcal{N}$ *be a morphism of* $\mathcal{T}$*-modules.*

- *The* $\mathcal{T}$***-module kernel*** $\mathcal{T}\text{-}\ker f$ *of* f *is* $\{a \in \mathcal{T} : f(a) \in \mathcal{N}_{\mathrm{Null}}\}$.
- f *is* ***null*** *if* $f(a) \in \mathcal{N}_{\mathrm{Null}}$ *for all* $a \in \mathcal{T}_{\mathcal{M}}$, *i.e.* $\mathcal{T}\text{-}\ker f = \mathcal{T}_{\mathcal{M}}$.
- *The* $\mathcal{T}$***-module image*** $f_{\mathcal{T}_{\mathcal{M}}}(\mathcal{M})$ *is the* $\mathcal{T}$*-submodule spanned by* $\{f(a) : a \in \mathcal{T}_{\mathcal{M}}\}$.
- f *is* $\circ$***-monic*** *if* $f(a_0) = f(a_1)$ *implies that* $a_0(-)a_1 \in \mathcal{M}_{\mathrm{Null}}$.
- f *is* $\circ$***-onto*** *if* $f_{\mathcal{T}_{\mathcal{M}}}(\mathcal{M}) + \mathcal{N}_{\mathrm{Null}} = \mathcal{N}$.

Thus, the null morphisms are closed null in the categorical sense, and take the place of the $\mathbb{0}$ morphism. Since $\mathbb{0} \notin \mathcal{T}_{\mathcal{M}}$, we might take $f(\mathcal{T}_{\mathcal{M}})$ to be any tangible constant z, i.e., $z \in \mathcal{T}_{\mathcal{N}}$.

Lemma 8.5. *Suppose* $\mathcal{T}$ *has* t *elements whose sum is in* $\mathcal{T}$, *and assume that such* t *is finite and is as maximal as possible. Then a morphism* $f : \mathcal{M} \to \mathcal{N}$ *is trivial if and only if* $f(a)$ *is a constant* $c \in \mathcal{T}_N$ *for all* $a \in \mathcal{T}_{\mathcal{M}}$, *such that* $(-)c = c$ *and* $c \preceq \mathbf{t}c$.

Proof. Clearly $f(a)$ is a constant $c \in \mathcal{T}_N$ since f should be trivial on $\mathcal{T}_{\mathcal{M}}$. But then if $\sum_{i=1}^t a_i$ is tangible then $c = f(\sum a_i) \preceq \sum f(a_i) = \mathbf{t}c$. Likewise $(-)c = f((-)a) = f(a) = c$, cf. Proposition 5.49. The converse is also clear, by definition. □

Definition 8.6. *A functor of categories of systems is* ***strict-exact*** *if it preserves exactness of exact strict sequences.*

8.1.1. $\mathcal{T}$-*ideals.*

Definition 8.7. *A* $\mathcal{T}$***-ideal*** *of a* $\mathcal{T}$*-bimodule triple* $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ *is an ideal* $\mathcal{I}$ *of* $\mathcal{A}$ *as an* $(\Omega; \mathcal{I})$*-algebra with negation, which is a subsystem of* $\mathcal{A}$ *satisfying the following properties where* $\mathcal{T}_{\mathcal{I}} = \mathcal{T}_{\mathbb{0}} \cap \mathcal{I}$:

(i) $\mathcal{T}_{\mathcal{I}}\mathcal{T} = \mathcal{T}\mathcal{T}_{\mathcal{I}} = \mathcal{T}_{\mathcal{I}}$.
(ii) $\omega(a_1, \ldots, b, \ldots, a_m) \in \mathcal{T}_{\mathcal{I}}$ *for all* $a_k \in \mathcal{T}$ *and* $b \in \mathcal{T}_{\mathcal{I}}$, *and every operator* ω *other than addition.*

Lemma 8.8. *If* $\mathcal{I}$ *is a* $\mathcal{T}$*-ideal of a triple* $\mathcal{A}$ *and* $\mathcal{M}$ *is an* $\mathcal{A}$*-module triple, then* $\mathcal{I}\mathcal{M}$ *is a sub-triple of* $\mathcal{M}$.

Proof. Define $\mathcal{T}_{\mathcal{I}\mathcal{M}} = \{ax : a \in \mathcal{T}_{\mathcal{I}},\ x \in \mathcal{T}_{\mathcal{M}}\}$, and $(-)$ to be the restriction. □

8.2. **The category of congruences of $\mathcal{T}$-module systems.**

We have the analogous results for $\mathcal{T}$-congruences. Recall the definition of strict morphism (Definition 5.3), corresponding to $\preceq$ being equality. The next definition takes into account that any $\mathcal{T}$-congruence contains the diagonal.

Definition 8.9. *For any $\mathcal{T}$-congruence Φ and any morphism $f : \mathcal{M} \to \mathcal{N}$, define the* ***congruence image*** *$f(\mathcal{M})_{\mathrm{cong}}$ to be the $\mathcal{T}$-congruence*

$$\mathrm{Diag}_{\mathcal{N}} + \left\{ \left(\sum_i f(a_i), \sum_j f(a'_j) \right) : \ \left(\sum a_i, \sum a'_j \right) \in \Phi \right\}. \tag{8.1}$$

The ***$\mathcal{T}$-congruence kernel*** *$\mathcal{T}_{\mathrm{cong}}\text{-}\ker(f)$ is the $\mathcal{T}$-congruence of $\mathcal{M}$, generated by*

$$\{(a_0, a_1) \in \mathcal{T}_{\mathcal{M}} : f(a_0) = f(a_1)\}.$$

For f strict, the term in the summation in (8.1) is $\{(f(x_0), f(x_1)) : x_i \in \mathcal{M},\ (x_0, x_1) \in \Phi\}$.

In particular, if the morphism f is strict, then $f(\mathcal{M})_{\mathrm{cong}}$ is $\mathrm{Diag}_{\mathcal{N}} + \left\{ \left(\sum_i f(a_i), \sum_j f(a'_j) \right) : \sum a_i = \sum a'_j \right\}$, the $\mathcal{T}$-congruence of $\mathcal{N}$ generated by $\mathrm{Diag}_{\mathcal{N}}$ and $\{(f(a_1), f(a_2)) : a_i \in \mathcal{T}_{\mathbb{0},\mathcal{M}}\}$.

The fact that $f(\Phi)$ is more complicated for non-strict morphisms presents serious obstacles later on.

Definition 8.10. *A congruence morphism $f : \Phi \to \Phi'$ is* ***trivial*** *if $f(\mathbb{0}) = \mathbb{0}$ and $f(\Phi) \subseteq \mathrm{Diag}_{\Phi'}$.*

Lemma 8.11. *For any strict morphism $f : \mathcal{M} \to \mathcal{N}$, $\mathcal{T}_{\mathrm{cong}}\text{-}\ker(f) = \{(x, x') \in \mathcal{M} : f(x) = f(x')\}$.*

Proof. Write $x = \sum x_i$ and $x' = \sum x'_j$ and $y = \sum y_i$ and $y' = \sum y'_j$. If $f(x) = f(x')$ and $f(y) = f(y')$, then $f(x + y) = \sum f(x_i) + \sum f(y_i) = \sum f(x'_j) + \sum f(y'_j) = f(x' + y')$. □

Lemma 8.12. *For any strict $\mathcal{T}_{\mathcal{M}}$-morphism $f : \mathcal{M} \to \mathcal{N}$, the induced morphism $\widehat{f} : \widehat{\mathcal{M}} \to \widehat{\mathcal{N}}$ is $\circ$-onto if and only if f is epic.*

Proof. If $\widehat{f} : \widehat{\mathcal{M}} \to \widehat{\mathcal{N}}$ is $\circ$-onto and $\widehat{gf}$ is trivial, then $\widehat{gf}(\mathcal{M}) = \widehat{g}(\widehat{\mathcal{M}})$, implying $\widehat{g}$ is trivial.
Conversely, assume that $\widehat{f} : \widehat{\mathcal{M}} \to \widehat{\mathcal{N}}$ is not $\circ$-onto. Then take the canonical map $g : \mathcal{N} \to \mathcal{N}/f(\mathcal{M})_{\mathrm{cong}}$ to get gf trivial, but g is not trivial, so f is not epic. □

To construct cokernels, we define $\mathcal{N} \to \mathcal{N}/f(\mathcal{M})_{\mathrm{cong}}$, which will turn out to be the categorical cokernel of f.

Lemma 8.13. *For any strict morphism $f : \mathcal{M} \to \mathcal{N}$, there is a $\mathcal{T}$-monic $\bar{f} : \mathcal{M}/\mathcal{T}_{\mathrm{cong}}\text{-}\ker(f) \xrightarrow{\bar{f}} \mathcal{N}$, given by $\overline{f}([a]) = f(a)$, where $[a]$ is the equivalence class of $a \in \mathcal{T}$ under the congruence kernel $\mathcal{T}_{\mathrm{cong}}\text{-}\ker(f)$.*

Proof. If $(a, a') \in \mathcal{T}_{\mathrm{cong}}\text{-}\ker(f)$, then $f(a) = f(a')$, showing that $\overline{f}$ is well-defined. Clearly $\overline{f}$ is a $\mathcal{T}$-morphism since $\mathcal{T}\text{-}\ker(f)$ is a $\mathcal{T}$-congruence. Finally, one can easily check that $\overline{f}$ is a monic as it is injective. □

Lemma 8.14. *A strict $\mathcal{T}_{\mathcal{M}}$-morphism $f : \mathcal{M} \to \mathcal{N}$ is $\mathcal{T}$-monic if and only if $\mathcal{T}_{\mathrm{cong}}\text{-}\ker(f)$ is diagonal.*

Proof. Assume first that the $\mathcal{T}$-congruence $\mathcal{T}_{\mathrm{cong}}\text{-}\ker(f)$ is not diagonal, i.e., $b = \sum \alpha_i \neq \sum a'_j = b'$ although $\sum f(a_i) = \sum f(a'_j)$. Define the cokernel $\bar{f} : \mathcal{T}_{\mathrm{cong}}\text{-}\ker(f) \to \mathcal{M}$ by $\bar{f}(a) = \sum f(a_i)$ where $a = \sum a_i$. $f\bar{f}$ is trivial, implying $\mathcal{T}_{\mathrm{cong}}\text{-}\ker(f)$ is trivial.

On the other hand, if $\mathcal{T}_{\mathrm{cong}}\text{-}\ker(f)$ is diagonal, and fg is trivial, then $g(\mathcal{M})$ is diagonal, i.e., g is trivial. □

Lemma 8.15. *Any strict morphism $f : \mathcal{M} \to \mathcal{N}$ is composed as $\mathcal{M} \to \mathcal{M}/\mathcal{T}_{\mathrm{cong}}\text{-}\ker(f) \xrightarrow{\bar{f}} \mathcal{N}$, where the first map is the canonical (strict) morphism, and the second map is given in Lemma 8.13.*

Proof. The first map sends $a \in \mathcal{M}$ to $\bar{a}$, where $\bar{a}$ is the equivalence class of a under $\mathcal{T}_{\mathrm{cong}}\text{-}\ker(f)$. This is clearly a $\mathcal{T}$-morphism and strict since $\mathcal{T}_{\mathrm{cong}}\text{-}\ker(f)$ is a congruence relation. The second map is just Lemma 8.13. □

We call $\bar{f}$ the monic **associated** to the $\mathcal{T}$-congruence kernel.

Theorem 8.16. *The category of module systems whose morphisms are strict is a semi-abelian category.*

Proof. Immediate from Lemma 8.15. □

8.3. **Submodules versus subcongruences.**

Definition 8.17. *A* $\mathcal{T}$***-submodule system*** *of* $(\mathcal{M}, \mathcal{T}, (-), \preceq)$ *is a submodule* $\mathcal{N}$ *of* $\mathcal{M}$ *as an* $(\Omega; \mathcal{I})$*-algebra, satisfying the following conditions, where* $a \in \mathcal{T}$*:*

(i) *Write* $\mathcal{T}_{\mathcal{N},\mathbb{0}}$ *for* $\mathcal{T}_{\mathcal{N}} \cup \{\mathbb{0}\}$*.* $(\mathcal{N}, \mathcal{T}_{\mathcal{N}}, (-), \preceq)$ *is a subsystem of* $(\mathcal{M}, \mathcal{T}_{\mathcal{M}}, (-), \preceq)$*. (In particular,* $\mathcal{T}_{\mathcal{N},\mathbb{0}}$ *generates* $(\mathcal{N}, +)$*.)*
(ii) $\mathcal{T}^\circ \subseteq \mathcal{N}$ *(and thus* $\mathcal{M}^\circ \subseteq \mathcal{N}$*).*
(iii) *If* $a \preceq b + v$*, for* $v \in \mathcal{N}$*, then there is* $w \in \mathcal{T}_{\mathcal{N},\mathbb{0}}$ *for which* $a \preceq b + w$*.*

Note that (ii) implies that $\{\mathbb{0}\}$ is not a submodule, for we want submodules to contain the null elements. In what follows, $\mathcal{N}$ always denotes a $\mathcal{T}$-submodule system of $\mathcal{M}$.

Remark 8.18. *The definition implicitly includes the condition that* $(-)\mathcal{T}_{\mathcal{N},\mathbb{0}} = \mathcal{T}_{\mathcal{N},\mathbb{0}}$*, since* $(-)a = ((-)\mathbb{1})a$*.*

Lemma 8.19. *If* $a \in \mathcal{N}$ *and* $a \preceq_\circ b$*, then* $b \in \mathcal{N}$*.*

Proof. Just write $b = a + c^\circ$, noting that $c^\circ \in \mathcal{N}$. □

The first stab at defining a $\mathcal{T}$-module of a $\mathcal{T}$-congruence Φ might be to take $\{a(-)b : (a,b) \in \Phi\}$, which works in classical algebra. We will modify this slightly, but the real difficulty lies in the other direction. The natural candidate for the $\mathcal{T}$-congruence of a $\mathcal{T}$-submodule $\mathcal{N}$ might be $\{(a,b) : a(-)b \in \mathcal{N}\}$, but it fails to satisfy transitivity!

Definition 8.20.

(1) *Given a* $\mathcal{T}$*-submodule* $\mathcal{N}$ *of* $\mathcal{M}$*, define the* $\mathcal{T}$*-congruence* $\Phi_{\mathcal{N}}$ *on* $\mathcal{N}$ *by* $a \equiv b$ *if and only if we can write* $a = \sum_j a_j$ *and* $b = \sum_j b_j$ *for* $a_j, b_j \in \mathcal{T}_{\mathcal{N},\mathbb{0}}$ *such that* $a_j \preceq b_j + v_j$ *for* $v_j \in \mathcal{T}_{\mathcal{N},\mathbb{0}}$*, each* j*.*
(2) *Given a* $\mathcal{T}$*-congruence* Φ*, define* $\mathcal{N}_\Phi$ *to be the additive sub-semigroup of* $\mathcal{M}$ *generated by all* $c \in \mathcal{T}_{\mathcal{M},\mathbb{0}}$ *such that* $c = a(-)b$ *for some* $a, b \in \mathcal{T}_{\mathcal{N},\mathbb{0}}$ *such that* $(a,b) \in \Phi$*.*

Example 8.21. *When the system* $\mathcal{M}$ *is meta-tangible, then in the definition of* $\Phi_{\mathcal{N}}$*, either* $b_j = (-)v_j$ *in which case* $a_j \preceq v_j^\circ \in \mathcal{T}_{\mathcal{N}}^\circ$*, or* $a_j = b_j$ *(yielding the diagonal) or* $a_j = v_j \in \mathcal{T}_{\mathcal{N}}$*.*

The results from [57] pass over, with the same proofs, to module systems.

Lemma 8.22 ([57, Lemma 6.28])**.** *In a* $\mathcal{T}$*-reversible system,* $a \equiv b$ *(with respect to* $\Phi_{\mathcal{N}}$*) for* $a, b \in \mathcal{T}_{\mathcal{N}}$*, if and only if either* $a = b$ *or* $\mathcal{T}_{\mathcal{N}}$ *contains an element* v *such that* $v \preceq a(-)b$*.*

Remark 8.23 ([57, Remark 6.29])**.** *In a* $\mathcal{T}$*-reversible system, Condition (iii) of Definition 8.17 yields* $w \preceq a(-)b$*. Likewise, in Definition 8.20,* $a_j \preceq b_j + v_j$ *implies* $b_j \preceq a_j(-)v_j$*.*

Proposition 8.24 ([57, Proposition 6.30])**.** *In a* $\mathcal{T}$*-reversible system,* $\Phi_{\mathcal{N}}$ *is a* $\mathcal{T}$*-congruence for any* $\mathcal{T}$*-submodule* $\mathcal{N}$*. For any* $\mathcal{T}$*-congruence* Φ*,* $\mathcal{N}_\Phi$ *is a* $\mathcal{T}$*-submodule. Furthermore,* $\Phi_{\mathcal{N}_\Phi} \supseteq \Phi$ *and* $\mathcal{N}_{\Phi_{\mathcal{N}}} = \mathcal{N}$*.*

Remark 8.25 ([57, Remark 6.31])**.** *The inclusion* $\Phi_{\mathcal{N}_\Phi} \subseteq \Phi$ *holds if and only if* Φ *satisfies the condition that if* $(a,b) \in \mathcal{T}_\Phi$ *and* $a(-)b = a'(-)b'$*, then* $(a',b') \in \mathcal{T}_\Phi$*. This leads to the notion of the* ***closure*** $\bar{\Phi}$ *of a* $\mathcal{T}$*-congruence* Φ*, to be generated by*

$$\{(a',b') \in \mathcal{T}_{\mathcal{M}} \times \mathcal{T}_{\mathcal{M}},\ a(-)b = a'(-)b' \text{ for } (a,b) \in \mathcal{T}_\Phi\}.$$

But even without this, we could mod out $\mathcal{T}$ *by* $\mathcal{T}_{\mathcal{N},\mathbb{0}}$*, namely take the* $\mathcal{T}$*-congruence* $\Phi := \Phi_{\mathcal{N}}$*, and pass to* $(\mathcal{M}/\Phi,\ \mathcal{T}_{\mathcal{M}}/\mathcal{T}_\Phi,\ (-), \preceq)$ *where* $(-)[a] = [(-)a]$*.*

In [39] we will see the usefulness in passing back and forth from modules to congruences, so the following observation is handy.

Lemma 8.26. *If* $\mathcal{N}$ *is an* $\mathcal{A}$*-submodule of* $\mathcal{M}$ *then* $\widehat{\mathcal{N}}$ *is an* $\widehat{\mathcal{A}}$*-submodule of* $\widehat{\mathcal{M}}$*.*

Proof. If $(a_0, a_1), (a_0', a_1') \in \widehat{\mathcal{A}}$ and $(b_0, b_1) \in \widehat{\mathcal{N}}$ then

$$(a_0, a_1)(b_0, b_1) = (a_0 b_0 + a_1 b_1, a_0 b_1 + a_1 b_0) \in \widehat{\mathcal{N}}$$

and

$$((a_0, a_1)(a_0', a_1'))(b_0, b_1) = (a_0 a_0' + a_1 a_1', a_0 a_1' + a_1 a_0')(b_0, b_1) = (a_0, a_1)(a_0' b_0 + a_1' b_1, a_0' b_1 + a_1' b_0) = (a_0, a_1)((a_0', a_1')(b_0, b_1)).$$

□

8.4. **Passing from the category of $\mathcal{T}$-congruences to transitive $\mathcal{T}$-submodules of $\widehat{\mathcal{M}}$.**

Definition 8.27. *A submodule $\mathcal{N}$ of a $\mathcal{T}$-module with negation $\mathcal{M}$ is **transitive** if $a_1(-)a_2 \in \mathcal{N}$ and $a_2(-)a_3 \in \mathcal{N}$ imply $a_1(-)a_3 \in \mathcal{N}$.*

This ties in with transitive relations and the transitive closure in [48, p. 33].

Proposition 8.28. *A subset Φ of $\mathcal{M} \times \mathcal{M}$ is a $\mathcal{T}$-congruence on $\mathcal{M}$ if and only if it is a transitive $\mathcal{T}$-submodule of $\widehat{\mathcal{M}}$ containing the diagonal $\{(a,a) : a \in \mathcal{M}\}$.*

Proof. ($\Rightarrow$) By definition of $\mathcal{T}$-congruence. ($\Leftarrow$) We need to show that Φ is an equivalence relation. $(a,a) \in \Phi$ by hypothesis, and symmetry holds because $(a,b) = (-)(b,a) \in \Phi$. $\square$

Lemma 8.29. *Any transitive $\mathcal{T}$-submodule $\mathcal{N}$ of $\mathcal{M}$ over a ground triple $\mathcal{A}$ yields a transitive $\mathcal{T}$-submodule of $\widehat{\mathcal{M}}$ over $\widehat{\mathcal{A}}$, under the twist action $(r_0, r_1) \cdot_{\mathrm{tw}} (a_0, a_1) = (r_0 a_0 + r_1 a_1, r_0 a_1 + r_1 a_0)$.*

Proof. We apply the switch map to Proposition 7.7. Transitivity carries over by definition. $\square$

Lemma 8.29 enables us to prove results about morphisms of $\mathcal{T}$-modules, by appealing to transitive $\mathcal{T}$-modules (over $\widehat{\mathcal{A}}$). Towards this end, the following observation is useful.

Lemma 8.30. *For any $\mathcal{T}$-module morphism $f : \mathcal{M} \to \mathcal{N}$, we let $\widehat{f} : \widehat{\mathcal{M}} \to \widehat{\mathcal{N}}$ be the induced morphism. If $\widehat{\mathcal{N}}$ is transitive then $(\widehat{f})^{-1}(\widehat{\mathcal{N}})$ is a transitive $\mathcal{T}$-submodule of $\widehat{\mathcal{M}}$.*

Proof. If $(a_1, a_2), (a_2, a_3) \in (\widehat{f})^{-1}(\widehat{\mathcal{N}})$, then $(f(a_1), f(a_2)), (f(a_2), f(a_3)) \in \widehat{\mathcal{N}}$, implying $(f(a_1), f(a_3)) \in \widehat{\mathcal{N}}$, and thus $(a_1, a_3) \in (\widehat{f})^{-1}(\widehat{\mathcal{N}})$. $\square$

9. The Hom triple

Let $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ denote $\mathrm{Mor}(\mathcal{M}, \mathcal{N})$, the set of morphisms from $\mathcal{M}$ to $\mathcal{N}$ and $\mathrm{Hom}(\mathcal{M}, \mathcal{N})_{\mathcal{T}-\mathrm{str}}$ be the subset of $\mathcal{T}$-strict morphisms. We use the given module negation map to define a negation map on Hom, and get a pseudo-triple.

Remark 9.1. *Suppose $\mathcal{M} := (\mathcal{M}, \mathcal{T}_{\mathcal{M}}, (-), \preceq)$ and $\mathcal{N} := (N, \mathcal{T}_{\mathcal{N}}, (-) \preceq)$ are module systems over a ground $\mathcal{T}$-system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$. If $\mathcal{M}, \mathcal{N}$ are $\mathcal{T}$-monoid modules then so is $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$, and, for $\mathcal{A}$ commutative, $\mathcal{T}$ acts on $\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{N})$ via the left multiplication map ℓ_a given by $\ell_a(f) = af$. The action is elementwise: $(af)(x) := af(x)$. We view $\mathcal{T}_{\mathrm{Hom}(\mathcal{M},\mathcal{N})} := \{f \in \mathrm{Hom}(M,N) : f(\mathcal{T}_{\mathcal{M}}) \subseteq f(\mathcal{T}_N)\}$ in $\mathrm{Hom}(M,N)$ via these left multiplication maps.*

Proposition 9.2.

(i) *The triple $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ is uniquely negated if $(\mathcal{N}, \mathcal{T}_{\mathcal{N}}, (-) \preceq)$ is uniquely negated.*

(ii) *Taking $\preceq_\circ$ as in Definition 1.2, $\mathrm{Hom}(\mathcal{M}, \mathcal{N}) := (\mathrm{Hom}(\mathcal{M}, \mathcal{N}), \mathcal{T}_{\mathrm{Hom}(\mathcal{M},\mathcal{N})}, (-), \preceq)$ is a pseudo-system, where all operators (including $(-)$) are defined elementwise, i.e., $((-)f)(x) = (-)(f(x))$, and $f \preceq g$ if and only if $f(x) \preceq g(x)$ for all $x \in M$.*

(iii) *If $\mathcal{A}$ is a $\mathcal{T}$-semiring system, then $\mathrm{End}\mathcal{M} := \mathrm{Hom}(\mathcal{M}, \mathcal{M})$ is a semiring system.*

Proof. (i) Suppose $f + g = h^\circ$. Then $f(x) + g(x) = h(x)^\circ$ for each $x \in \mathcal{T}_{\mathcal{M}}$, implying $g(x) = (-)f(x)$, and thus $g = (-)f$.

(ii) Check all properties elementwise, including the conditions of Definition 5.26.

(iii) $fg(x) = f(g(x))$.

$\square$

Proposition 9.3. *If $\mathcal{A} = (\mathcal{A}, \mathcal{T}, (-), \preceq)$ is a $\mathcal{T}$-system, and $\mathcal{M} = \mathcal{A}^{(I)}$, and $\mathcal{N} = \mathcal{A}^{(J)}$, then*

$$(\mathrm{Hom}(\mathcal{M}, \mathcal{N}), \mathcal{T}_{\mathrm{Hom}(\mathcal{M},\mathcal{N})}, (-))$$

is a pseudo-triple, where $(-)$is defined elementwise. $(\mathrm{Hom}(\mathcal{M}, \mathcal{N}), \mathcal{T}_{\mathrm{Hom}(\mathcal{M},\mathcal{N})}, (-), \preceq)$, where $\preceq$ is defined elementwise, is a system if J is finite.

Proof. Let $\mathbb{1}_i$ denote the vector whose only component $\neq \mathbb{0}$ is $\mathbb{1}$ in the i-th position. Given a morphism $f : \mathcal{A}^{(I)} \to \mathcal{A}^{(J)}$, we define $f_{ij} \in \mathcal{T}_{\mathrm{Hom}(\mathcal{T}_{\mathcal{M}},\mathcal{T}_{\mathcal{N}})}$ to be the map sending $\mathbb{1}_i$ to the j-th component of $f(\mathbb{1}_i)$, and $\mathcal{T}_{i'}$ to $\mathbb{0}$ for all other components $i' \neq i$. It is easy to see that f is generated by the f_{ij}. $\square$

Remark 9.4. *In the interest of being able to utilize category theory we would like*

$$(\mathrm{Hom}(\mathcal{M},\mathcal{N}), \mathcal{T}_{\mathrm{Hom}(\mathcal{M},\mathcal{N})}, (-), \preceq)$$

to be a system, i.e., for $\mathcal{T}_{\mathrm{Hom}(\mathcal{M},\mathcal{N})}$ *to generate* $\mathrm{Hom}(\mathcal{M},\mathcal{N})$*, but this is elusive since we do not have a hold on* $\mathcal{T}_{\mathrm{Hom}(\mathcal{M},\mathcal{N})}$ *in the absence of some freeness assumption. But we can always bypass this difficulty by cutting down on our morphisms by means of Remark 5.12.*

9.1. **The dual system.**

Definition 9.5. *For a system* $\mathcal{S} = (\mathcal{S}, \mathcal{T}_{\mathcal{S}}, (-), \preceq)$ *write* $\mathcal{S}^*$ *for* $\mathrm{Hom}(\mathcal{S},\mathcal{A})$*, and* $\mathcal{T}^*$ *for* $\{f|_{\mathcal{T}_{\mathcal{S}}} : f \in \mathcal{S}^*$ *with* $f(\mathcal{T}_{\mathcal{S}}) \subseteq \mathcal{T}_{\mathcal{A}}\}$*.*

Corollary 9.6. *Let* $\mathcal{S} = \mathcal{A}^{(I)}$*, where* $\mathcal{A} = (\mathcal{A}, \mathcal{T}, (-), \preceq)$ *is a system. For I finite,* $(\mathcal{S}^*, \mathcal{T}^*, (-), \preceq)$ *also is a system, where* $(-)$ *and* $\preceq$ *are defined elementwise.*

Proof. Take $|J| = 1$ in Proposition 9.3. (We need the hypothesis that I is finite in order for $\mathcal{T}^*$ to generate $\mathcal{S}^*$.) ☐

As in usual linear algebra, when $\mathcal{A}$ is commutative as well as associative, we can embed $\mathcal{S}$ into $\mathcal{S}^*$. Write $\mathbf{a}$ for $(a_i) \in \mathcal{T}^{(I)}$, and define $\mathbf{a}^* \in \mathcal{S}^*$ by $\mathbf{a}^*((b_i)) = \mathbf{a} \cdot (b_i) = \sum a_i b_i$. Let e_i denote the vector with $\mathbb{1}$ in the i position and $\mathbb{0}$ elsewhere.

Proposition 9.7. *Suppose* $\mathbf{a} = (a_i) \in \mathcal{S} = \mathcal{A}^{(I)}$*. Then* $\mathbf{a}^* = \sum_i a_i e_i^* \in \mathcal{S}^*$ *is spanned over* $\mathcal{T}$ *by the* e_i^**. There is an injection* $(\mathcal{S}, \mathcal{T}_{\mathcal{S}}, (-)) \to (\mathcal{S}^*, \mathcal{T}_{\mathcal{S}}^*, (-))$ *given by* $\mathbf{a} \to \mathbf{a}^*$*, which is onto when* I *is finite.*

Proof. Just as in the classical case, noting that negation is not used in its proof. ☐

9.1.1. *The adjoint isomorphism.*

In view of [41], also cf. [23, Proposition 17.15], one has the adjoint isomorphism of the tensor functor over Hom for modules over semirings. More generally, the adjoint isomorphism holds for "rigid" monoidal categories, cf. [17, Proposition 2.10.8]. Here is the version for systems (with respect to strict morphisms):

Lemma 9.8. *The natural map*

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}_1 \otimes_{\mathcal{B}} \mathcal{M}_2, \mathcal{M}_3)_{\mathcal{T}-\mathrm{str}} \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{M}_1, \mathrm{Hom}_{\mathcal{A}}(M_2, M_3)_{\mathcal{T}-\mathrm{str}})_{\mathcal{T}-\mathrm{str}},$$

is a natural isomorphism.

Proof. The standard proof given for algebras does not involve negation, cf. [55, Proposition 18.44] for example. The restriction to the tangible part sends a homomorphism $f : a_1 \otimes a_2 \mapsto a_3$ to the homomorphism sending a_1 to the map $a_2 \mapsto f(a_1 \otimes a_2)$. ☐

One can define the monoidal property in terms of the adjoint isomorphism, cf. [46]. In short, our discussion fits into the well-known theory of monoidal categories as described in [17], and makes the strict system category amenable to [62]. Thus, $\mathcal{A}$-module triples with negated $\mathcal{T}$-tensor products yield a monoidal (semi-abelian) category with respect to strict morphisms.

10. Functors among ground triples and systems

In this section we recapitulate the previous connections among the notions of systems, viewed categorically. We focus on ground triples, but at the end indicate some of the important functors for module systems, in prelude to [39].

10.1. **Functors among ground triples and systems.**

Notation. *Let us introduce the following notations:*

- $\mathfrak{Rings}$= *the category of commutative rings.*
- $\mathfrak{Doms}$= *the category of integral domains.*
- $\mathfrak{SRings}$= *the category of commutative semirings.*
- $\mathfrak{SDoms}$= *the category of commutative semidomains.*

- $\mathfrak{SDM}$= *the category whose objects are pairs* (S, M) *consisting of a semiring* S *and a multiplicative submonoid* M *of* S *which (additively) generates* S *and does not contain* 0_S*. A morphism from* (S_1, M_1) *to* (S_2, M_2) *is a semiring homomorphism* $f : S_1 \to S_2$ *such that* $f(M_1) \subseteq M_2$*.*
- $\mathfrak{SRT}_{\mathcal{T}}$= *the category of* $\mathcal{T}$*-triples, with* $\preceq_\circ$*-morphisms, whose objects are semirings.*
- $\mathfrak{HDoms}$= *the category of hyperrings without multiplicative zero-divisors.*
- $\mathfrak{HFields}$= *the category of hyperfields with only strict morphisms.*
- $\mathfrak{FRings}_w$= *the category of fuzzy rings with weak morphisms (cf.* [22]*).*
- $\mathfrak{FRings}_{str}$= *the category of coherent fuzzy rings with strict morphisms (cf.* [22] *and* [57] *for the notion of coherence).*

The following diagram illustrates how various categories are related (note that this is not a commutative diagram, for instance, $\mathbf{e} \circ \mathbf{t} \circ \mathbf{i} \neq \mathbf{c} \circ \mathbf{j}$):

$$\begin{tikzcd}
\mathfrak{Doms} \arrow[r, "\mathbf{i}"] \arrow[dr, "\mathbf{j}"] & \mathfrak{SDoms} \arrow[r, "\mathbf{t}"] & \mathfrak{SDM} \arrow[r, "\mathbf{e}"] & \mathfrak{SRT}_{\mathcal{T}} \\
\mathfrak{HFields} \arrow[r, "\mathbf{k}"] \arrow[urr, "\mathbf{a}"] & \mathfrak{HDoms} \arrow[r, "\mathbf{g}"] \arrow[urr, "\mathbf{c}"] & \mathfrak{FRings}_w \arrow[ur, "\mathbf{d}"] &
\end{tikzcd} \tag{10.1}$$

(If we are willing to bypass $\mathfrak{SDM}$ in this diagram, then we could generalize the first two terms of the top row to $\mathfrak{Rings}$ and $\mathfrak{SRings}$ and accordingly, in the second row, $\mathfrak{HDoms}$ can be generalized to the category of hyperrings.)

In the following propositions, we explain the functors in the above diagram. All of these functors are stipulated to preserve the negation map. First, one can easily see that the functors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are simply embeddings. To be precise:

Proposition 10.1. *(The functors* $\boldsymbol{i}$*,* $\boldsymbol{j}$*, and* $\boldsymbol{k}$*)*

1. *The functor* $\boldsymbol{i} : \mathfrak{Doms} \to \mathfrak{SDoms}$*, sending an object* A *to* A *and a morphism* $f \in \mathrm{Hom}_{\mathfrak{Rings}}(A, B)$ *to* $f \in \mathrm{Hom}_{\mathfrak{SRings}}(\boldsymbol{i}(A), \boldsymbol{i}(B))$*, is fully faithful.*
2. *The functor* $\boldsymbol{j} : \mathfrak{Doms} \to \mathfrak{HDoms}$*, sending an object* A *to* A *and a morphism* $f \in \mathrm{Hom}_{\mathfrak{Rings}}(A, B)$ *to* $f \in \mathrm{Hom}_{\mathfrak{HRings}}(\boldsymbol{j}(A), \boldsymbol{j}(B))$*, is fully faithful.*
3. *The functor* $\boldsymbol{k} : \mathfrak{Hfields} \to \mathfrak{HDoms}$*, sending an object* A *to* A *and a morphism* $f \in \mathrm{Hom}_{\mathfrak{Hfields}}(A, B)$ *to* $f \in \mathrm{Hom}_{\mathfrak{HRings}}(\boldsymbol{k}(A), \boldsymbol{k}(B))$*, is faithful.*

Proof. This is straightforward. □

Remark 10.2. *The functor* $\boldsymbol{k} : \mathfrak{Hfields} \to \mathfrak{HDoms}$ *cannot be full since we require morphisms in* $\mathfrak{Hfields}$ *to be strict, whereas non-strict morphisms exist, cf.* [36].

Now, for a semidomain A, we let $\mathbf{t}(A) = (A, A - \{0_A\})$. It is crucial that A is a semidomain for $A - \{0_A\}$ to be a monoid. For a semiring homomorphism $f : A_1 \to A_2$, we define $\mathbf{t}(f) : (A_1, A_1 - \{0_{A_1}\}) \to (A_2, A_2 - \{0_{A_2}\})$ which is induced by f. Then, clearly $\mathbf{t}$ is a functor from $\mathfrak{SDoms}$ to $\mathfrak{SDM}$. In fact, we have the following:

Proposition 10.3. *The functor* $\boldsymbol{t} : \mathfrak{SDoms} \to \mathfrak{SDM}$ *is fully faithful.*

Proof. This is clear from the definition of $\mathbf{t}$. □

Remark 10.4. *One may notice that our construction of* $\boldsymbol{t}$ *is not canonical since any semiring may have different sets of monoid generators. For instance, the coordinate ring of an affine tropical scheme may have different sets of monoid generators depending on torus embeddings, cf.* [21].

The functors **a** and **g** are already constructed in [22]. For the sake of completeness, we recall the construction.

Let H be a hyperring. Then one can define the following set:

$$S(H) := \{\sum_{i=1}^{n} h_i \mid h_i \in H, \quad n \in \mathbb{N}\} \subseteq \mathcal{P}(H), \tag{10.2}$$

where $\mathcal{P}(H)$ is the power set of H. By [57, Theorem 2.9], [22], $S(H)$ is a semiring with multiplication and addition as follows:

$$(\sum_{i=1}^{n} h_i)(\sum_{j=1}^{m} h_j) = \sum_{i,j} h_i h_j \in S(H), \quad (\sum_{i=1}^{n} h_i) + (\sum_{i=1j}^{m} h_j) = \sum_{i,j}(h_i + h_i) \in S(H). \tag{10.3}$$

Now, the functor $\mathbf{a} : \mathfrak{Hfields} \to \mathfrak{SDM}$ sends any hyperfield H to $(S(H), H^\times)$, i.e., $\mathbf{a}(H) = (S(H), H^\times)$. Also, if $f : H_1 \to H_2$ is a strict homomorphism of hyperfields, then f canonically induces a morphism

$$\mathbf{a}(f) : (S(H_1), H_1^\times, -) \to (S(H_1), H_1^\times, (-)), \quad \mathbf{a}(f)(\sum_{i=1}^{n} h_i) = \sum_{i=1}^{n} f(h_i).$$

We emphasize that since the subcategory $\mathfrak{Hfields}$ of $\mathfrak{HDoms}$ only has strict morphisms, **a** becomes a functor.

The construction of the functor **g** is similar to **a**; we use the powerset $\mathcal{P}(H)$ of H instead of $S(H)$ in this case. For details, we refer the readers to [22]. It is proved in [22] that when one restricts the functors to hyperfields, the functors **a** and **g** are faithful, but not full.

Remark 10.5. *Since a fuzzy ring assumes weaker axioms than semirings, the functor* ***g*** *can be defined for all hyperrings, whereas the functor* ***a*** *can be only defined for hyperfields with strict morphisms.*

Next, we construct the functor $\mathbf{e} : \mathfrak{SDM} \to \mathfrak{SYS}_{\mathcal{T}}$. To this end, we need to fix a negation map of interest, so a priori the functor **e** is not canonical. For an object (S, M) of $\mathfrak{SDM}$, we let $\mathbf{e}(S, M)$ be the $\mathcal{T}$-system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$, where $\mathcal{A} = S$, $\mathcal{T} = M$, and $(-)$ is the identity map and $\preceq = \preceq_\circ$. Since, we choose $(-)$ to be equality, any morphism $f : (S_1, M_1) \to (S_2, M_2)$ induces a morphism $\mathbf{e}(f)$.

The functor **c** is defined as follows: For a hyperring R without zero-divisors, we associate $S(R)$ as in (10.2) and also impose addition and multiplication as in (10.3). Now, the $\mathcal{T}$-system $\mathbf{c}(R) = (\mathcal{A}, \mathcal{T}, (-), \preceq)$ consists of $\mathcal{A} = S(R)$, $\mathcal{T} = R$, $(-) : S(R) \to S(R)$ sending A to $-A := \{-a \mid a \in A\}$, where $-$ is the negation in R, and $\preceq$ is set inclusion $\subseteq$. One checks easily that $\mathbf{c}(R)$ is indeed a $\mathcal{T}$-system and any homomorphism $f : R_1 \to R_2$ of hyperrings induces a morphism $\mathbf{c}(f)$ of $\mathcal{T}$-systems.

Finally, we review the functor $\mathbf{d} : \mathfrak{FRings}_w \to \mathfrak{SYS}_{\mathcal{T}}$, defining $(-)a$ to be εa. [57, Lemma 12.5] shows how this can be retracted at times. One can easily see that the definition of coherent fuzzy rings is similar to $\mathcal{T}$-systems; we only have to specify a negation map $(-)$ and a surpassing relation $\preceq$. To be precise, let F be a fuzzy ring. The system $\mathbf{d}(F) = (\mathcal{A}, \mathcal{T}, (-), \preceq)$ consists of $\mathcal{A} = F$, $\mathcal{T} = \mathcal{A}^\times$, $(-) : \mathcal{A} \to \mathcal{A}$ sending a to $\varepsilon \cdot a$, and $\preceq$ is defined to be the equality. One can easily check that any weak morphism $f : F_1 \to F_2$ of fuzzy rings induces a morphism $\mathbf{d}(f)$ of the corresponding $\mathcal{T}$-systems.

Remark 10.6. *In the commutative diagram, one can think of forgetful functors in opposite directions. For instance, the functor* ***t*** *has a forgetful functor (forgetting $\mathcal{T}$) as an adjoint functor.*

Remark 10.7. *Although we do not pursue them in this paper, we point out two possible links to partial fields, first introduced by C. Semple and G. Whittle* [58] *(see, also* [52] *to study representability of matroids). Recall that a commutative partial field $\mathbb{P} = (R, G)$ is a commutative ring R together with a subgroup $G \leq R^\times$ of the group of multiplicative units of R such that $-1 \in G$.*

1. *If one considers the subring R' of R which is generated by G, then the pair $(R', G, -, =)$ becomes a system.*
2. *Any commutative partial field $\mathbb{P} = (R, G)$ gives rise to a quotient hyperring R/G. This defines a functor from the category of commutative partial fields to the category of hyperrings.*

Remark 10.8. *Since any semiring*† *containing* $\mathcal{T}$ *is a* $\mathcal{T}$*-module, we have forgetful functors from* $\mathcal{T}$*-semiring systems to* $\mathcal{T}$*-module systems. We also have the tensor functor and Hom functors, which are to be studied in* [39].

10.2. **Valuations of semirings via systems.**

We briefly mention one potential application of systems. In [38], the first author introduced the notion of valuations for semirings by implementing the idea of hyperrings, and this was put in the context of systems in [57, Definition 6.8].

Definition 10.9. *Let* $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$*, where* $\mathbb{R}$ *is the set of real numbers. The multiplication* $\boxdot$ *of* $\mathbb{T}$ *is the usual addition of real numbers such that* $a \boxdot (-\infty) = (-\infty)$ *for all* $a \in \mathbb{T}$*. Hyperaddition is defined as follows:*

$$a \boxplus b = \begin{cases} \max\{a, b\} & \text{if } a \neq b \\ [-\infty, a] & \text{if } a = b, \end{cases}$$

For a commutative ring A, a homomorphism from A to $\mathbb{T}$ is what sometimes is called a "semivaluation," (but in a different context from which we have used the prefix "semi") i.e., which could have a nontrivial kernel. Inspired by this observation, in [38], the first author proposed the following definition to study tropical curves by means of valuations; this is analogous to the classical construction of abstract nonsingular curve via discrete valuations.

Definition 10.10. *Let* S *be an idempotent semiring and* $\mathbb{T}$ *be the tropical hyperfield. A valuation on* S *is a function* $\nu : S \to \mathbb{T}$ *such that*

$$\nu(a \cdot b) = \nu(a) \boxdot \nu(b), \quad \nu(0_S) = -\infty, \quad \nu(a + b) \in \nu(a) \boxplus \nu(b), \quad \nu(S) \neq \{-\infty\}.$$

The implementation of systems, through the aforementioned functors, allows one to reinterpret Definition 10.10 as a morphism in the category of systems. To be precise, let S be a semiring which is additively generated by a multiplicative submonoid M. This gives rise to the $\mathcal{T}$-system $\mathbf{e}(S, M) = (S, M, id, =)$ via the functor $\mathbf{e}$. Also, for the hyperfield $\mathbb{T}$, via the functor $\mathbf{e} \circ \mathbf{a}$, we obtain a $\mathcal{T}$-system, say $\mathcal{A}_{\mathbb{T}}$. Then a semiring valuation on S is simply a morphism $\nu : (S, M, id, =) \to \mathcal{A}_{\mathbb{T}}$ of $\mathcal{T}$-systems. This matter is to be discussed in the context of tropicalization of systems in [39].

10.3. **Functors among module triples and systems.**

We conclude with some important functors needed to study module triples and systems in [39]. Given a $\mathcal{T}$-system $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-), \preceq)$, define $\mathcal{A}$-$\mathfrak{Mod}$ to be the category of $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}, (-), \preceq)$-module systems.

Remark 10.11. *The* ***symmetrizing*** *functor injecting* $\mathcal{A}$-$\mathfrak{Mod}$ *into* $(\widehat{\mathcal{A}}, \mathcal{T}_{\widehat{\mathcal{A}}}, (-), \preceq)$-$\mathfrak{Mod}$ *is given by* $f \mapsto (f, 0)$*, with the reverse direction* $(\widehat{\mathcal{A}}, \mathcal{T}_{\widehat{\mathcal{A}}}, (-), \preceq)$-$\mathfrak{Mod}$ *to* $\mathcal{A}$-$\mathfrak{Mod}$ *given by* $(f_0, f_1) \mapsto f_0(-)f_1$*. This functor respects the universal algebra approach since* $\mathcal{T}_{\mathcal{M}} \mapsto \mathcal{T}_{\widehat{\mathcal{M}}}$*.*

An alternative tack, less in line with universal algebra, would be $f \mapsto \hat{f} = (f, f)$*.*

Other important functors in this vein are the tensor functor and the Hom functor.

References

[1] J. Abuhlail, Exact sequences of semimodules over semirings, arXiv preprint arXiv:1210.4566, 2012.
[2] M. Akian, S. Gaubert, and A. Guterman. Linear independence over tropical semirings and beyond. In *Tropical and Idempotent Mathematics*, G.L. Litvinov and S.N. Sergeev, (eds.), *Contemp. Math.* 495 (2009), 1–38.
[3] M. Akian, S. Gaubert, and L. Rowen, Linear algebra over systems, preprint (2017).
[4] S.A. Amitsur, A general theory of radicals. I: Radicals in complete lattices, Amer. J. Math. 74 (1952), 774–786.
[5] S.A. Amitsur, A general theory of radicals. II: Radicals in rings and bicategories, Amer. J. Math. 75 (1954), 100–125.
[6] M. Baker and N. Bowler. Matroids over hyperfields, arXiv:1601.01204v4, 24 Aug 2016.
[7] V. Berkovich. Algebraic and analytic geometry over the field with one element, preprint (2014).
[8] A. Bertram and R. Easton, The tropical nullstellensatz for congruences, Advances in Mathematics 308 (2017), 36–82.
[9] G. Blachar, Exploded Layered Tropical Lie Algebras, Masters Thesis, Bar-Ilan University (2015).
[10] N. Bourbaki. *Commutative Algebra.* Paris and Reading, 1972.
[11] A. Connes and C. Consani, Homological algebra in characteristic one, arXiv:1703.02325v1, 7 Mar 2017.
[12] G. Cortinas, C. Haesemeyer, M. Walker, and C. Weibel, *Toric varieties, monoid schemes, and cdh-descent*, J. Reine Angew. Math. 698 (2015), 1-54.
[13] A.A. Costa. Sur la thêorie générale des demi-anneaux, *Publ. Math. Decebren* 10:14–29, 1963.

[14] D. Dolzan, P. Oblak, Invertible and nilpotent matrices over antirings, Linear Algebra and its Applications 430 (2009) 271-278.

[15] A. Dress, Duality theory for finite and infinite matroids with coefficients, Advances in Mathematics 93(2) (1986), 214–250.

[16] A. Dress and W. Wenzel, Algebraic, tropical, and fuzzy geometry, Beitrage zur Algebra und Geometrie/ Contributions to Algebra and Geometry 52 (2), 431–461, 2011.

[17] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, Tensor Categories. Mathematical Surveys and Monographs Volume 205 American Mathematical Society.

[18] R. Freese and R. McKenzie, Commutator theory for Congruence Modular Varieties, London Mathematical Society Lecture Notes 125, 1987.

[19] S. Gaubert. *Théorie des systèmes linéaires dans les diodes.* Thèse, École des Mines de Paris, 1992.

[20] S. Gaubert and M. Plus. Methods and applications of (max,+) linear algebra. In R. Reischuk and M. Morvan, editors, STACS97, number 1200 in LNCS, Lubeck, March 1997. Springer.

[21] J. Giansiracusa and N. Giansiracusa, Equations of tropical varieties, Duke Mathematical Journal 165.18 (2016): 3379-3433.

[22] J. Giansiracusa, J. Jun, and O. Lorscheid, On the relation between hyperrings and fuzzy rings, arXiv preprint arXiv:1607.01973 (2016), to appear in Beitrage zur Algebra und Geometrie/ Contributions to Algebra and Geometry.

[23] J. Golan, *The theory of semirings with applications in mathematics and theoretical computer science*, volume 54. Longman Sci & Tech., 1992.

[24] M. Grandis, *Homological Algebra: In Strongly Non-abelian Settings*, World scientific, 2013.

[25] A. Grothendieck, *Produits tensoriels topologiques et espaces nucleaires*, Memoirs of the Amer. Math. Soc., no. 16, 1955.

[26] S. Henry, *Symmetrization of monoïds as hypergroups*, arXiv preprint arXiv:1309.1963 (2013).

[27] S.N. Il'in, *Direct sums of injective semimodules and direct products of injective modules*, Russian Mathematics 54 (2010), 17–37 .

[28] Z. Izhakian, Tropical arithmetic & algebra of tropical matrices, Preprint at arXiv:math.AG/0505458, 2005.

[29] Z. Izhakian, M. Knebusch, and L. Rowen, Supertropical linear algebra, *Pacific Journal of Mathematics*, 266(1) (2013), 43–75.

[30] Z. Izhakian, M. Knebusch, and L. Rowen, Decompositions of Modules lacking zero sums (2015), arXiv:1511.04041, to appear in Israel J. Math.

[31] Z. Izhakian, A. Niv, and L. Rowen, Supertropical SL, to appear in Linear and multilinear algebra.

[32] Z. Izhakian and L. Rowen, Supertropical algebra, *Advances in Mathematics*, 225(4)(2010), 2222–2286.

[33] Z. Izhakian and L. Rowen, Supertropical matrix algebra, *Israel J. Math.*, 182(1) (2011), 383–424.

[34] N. Jacobson, *Basic algebra II*, Freeman, 1980.

[35] D. Joó and K. Mincheva, *Prime congruences of idempotent semirings and a Nullstellensatz for tropical polynomials*, arXiv:1408.3817v1, to appear in Selecta Mathematica.

[36] J. Jun, Algebraic geometry over hyperrings, arXiv preprint arXiv:1512.04837 (2015).

[37] J. Jun, Čech cohomology of semiring schemes, Journal of Algebra 483 (2017): 306-328.

[38] J. Jun, Valuations of Semirings, arXiv preprint arXiv:1503.01392 (2015), to appear in Journal of Pure and Applied Algebra.

[39] J. Jun and L. Rowen, Geometry and homology of systems, in preparation.

[40] Y. Katsov, *Tensor products of functors*, Siberian J. Math. 19 (1978), 222-229, trans. from Sirbiskii Mathematischekii Zhurnal 19 no. 2 (1978), 318-327.

[41] Y. Katsov, *Tensor products and injective envelopes of semimodules over additively regular semirings*, Algebra Colloquium 4 no. 2, (1997), 121-131.

[42] Y. Katsov, *Toward homological characterization of semirings: Serres conjecture and Bass perfectness in a semiring context*, Algebra Universalis 52 (2004), 197-214.

[43] T.Y. Lam, Ring Theory, Lectures on Modules and Rings. Springer (1999).

[44] O. Lorscheid, A blueprinted view on $\mathbb{F}_1$-geometry, in Absolute Arithmetic and $\mathbb{F}_1$-geometry (edited by Koen Thas), European Mathematical Society Publishing House, 2016.

[45] S. MacLane, Natural associativity and commutativity. Rice Univ. Studies 49 (1963), no. 4, 28-46.

[46] S. MacLane, Categories for the working mathematician (2nd edition), AMS Graduate texts in mathematics 5 (1998).

[47] A.W. Macpherson, Projective modules over polyhedral semirings, arXiv preprint arXiv:1507.07213 (2015).

[48] R. McKenzie, G. McNulty, and W. Taylor, Algebras, Lattices and Varieties, Vol. 1 Wadsworth and Brooks (1987).

[49] P. Ouwehand, Commutator Theory and abelian Algebras, arXiv:1309.0662v1.

[50] B. Parker, Exploded manifolds, Adv. Math 229, 3256-3319, (2012).

[51] A. Patchkoria, On exactness of long sequences of homology semimodules, Journal of Homotopy and Related Structures, vol. 1(1), (2006), 229-243.

[52] R. Pendavingh, and S. H. M. van Zwam, Skew partial fields, multilinear representations of matroids, and a matrix tree theorem, Advances in Applied Mathematics 50.1 (2013): 201-227.

[53] M. Plus (M. Akian, G. Cohen, S. Gaubert, R. Nikoukhah, and J.P. Quadrat), Linear systems in (max,+)-algebra. In Proceedings of the 29th Conference on Decision and Control, Honolulu, Dec. 1990.

[54] L.H. Rowen, *Ring Theory, Vol. I*, Academic Press, Pure and Applied Mathematics **127**, 1988.

[55] L.H. Rowen, *Graduate algebra: Noncommutative view*, AMS Graduate Studies in Mathematics **91** (2008).

[56] L.H. Rowen, "Symmetries in tropical algebra" in arXiv:1602.00353 [math.RA]
[57] L.H. Rowen, *Algebras with a negation map*, 78 pages, arXiv:1602.00353 [math.RA]
[58] C. Semple, and G. Whittle, Partial fields and matroid representation, Advances in Applied Mathematics 17.2 (1996): 184-208.
[59] E. Sheiner, Doctoral dissertation, Bar-Ilan University, 2015.
[60] Y. Tan, On invertible matrices over antirings, Linear Algebra and its Applications 423 (2007) 428-444.
[61] M. Takahashi, On the bordism categories III, Math. Seminar Notes Kobe University 10 (1982), 211–236.
[62] B. Toën and M. Vaquié, Au-dessous de Spec $\mathbb{Z}$, Journal of K-theory 3.3 (2009): 437-500.
[63] O.Y. Viro, Hyperfields for tropical geometry I. Hyperfields and dequantization (2010). ArXiv AG/1006.3034.
[64] H. Wang, Injective hulls of semimodules over additively-idempotent semirings, Semigroup Forum 48 (1994), 377–379.