Large-time asymptotic behavior of the infinite system of harmonic oscillators on the half-line

T.V. Dudnikova [1]

Keldysh institute of Applied Mathematics RAS
Miuskaya sq. 4, Moscow 125047, Russia

E-mail: tdudnikov@mail.ru

Abstract

The mixed initial-boundary value problem for infinite one-dimensional chain of harmonic oscillators on the half-line is considered. We study the large time behavior of solutions and derive the dispersive bounds.

Key words and phrases: one-dimensional system of harmonic oscillators on the half-line, mixed initial-boundary value problem, Fourier–Laplace transform, Puiseux expansion, dispersive estimates

1 Introduction

We consider the infinite system of harmonic oscillators on the half–line:

$$\ddot{u}(x,t) = (\nu^2\Delta_L - m^2)u(x,t), \quad x \in \mathbb{N}, \quad t > 0, \tag{1.1}$$

with the boundary condition (as $x = 0$)

$$\ddot{u}(0,t) = \nu^2(u(1,t) - u(0,t)) - m^2u(0,t) - \kappa u(0,t) - \gamma\dot{u}(0,t), \quad t > 0, \tag{1.2}$$

and with the initial condition (as $t = 0$)

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), \quad x \geq 0. \tag{1.3}$$

Here $u(x,t) \in \mathbb{R}$, $\nu > 0$, $m, \kappa, \gamma \geq 0$, Δ_L denotes the second derivative on $\mathbb{Z}$:

$$\Delta_L u(x) = u(x+1) - 2u(x) + u(x-1), \quad x \in \mathbb{Z}.$$

If $\gamma = 0$, then formally the system (1.1)–(1.2) is Hamiltonian with the Hamiltonian functional

$$\mathrm{H}(u,\dot{u}) := \frac{1}{2}\sum_{x\geq 0}\Big(|\dot{u}(x,t)|^2 + \nu^2|u(x+1,t) - u(x,t)|^2 + m^2|u(x,t)|^2\Big) + \frac{1}{2}\kappa|u(0,t)|^2. \tag{1.4}$$

We assume that the initial data $Y_0(x) = (u_0(x), v_0(x))$ belong to the Hilbert space $\mathcal{H}_{\alpha,+}$, $\alpha \in \mathbb{R}$, defined below.

[1] The work was supported partly by the research grants of Russian Science Fondation (Grant No. 14-21-00025) and of RFBR (Grant No. 15-01-03587)

Definition 1.1. *(i)* $\ell^2_{\alpha,+} \equiv \ell^2_{\alpha,+}(\mathbb{Z}_+)$, $\alpha \in \mathbb{R}$, *is the Hilbert space of sequences* $u(x)$, $x \geq 0$, *with norm* $\|u\|^2_{\alpha,+} = \sum\limits_{x\geq 0} |u(x)|^2 \langle x\rangle^{2\alpha} < \infty$, $\langle x\rangle := (1+x^2)^{1/2}$.
(ii) $\mathcal{H}_{\alpha,+} = \ell^2_{\alpha,+} \otimes \ell^2_{\alpha,+}$ *is the Hilbert space of pairs* $Y = (u,v)$ *of sequences equipped with norm* $\|Y\|^2_{\alpha,+} = \|u\|^2_{\alpha,+} + \|v\|^2_{\alpha,+} < \infty$.

On the coefficients m, κ, ν, γ of the system we impose condition **C** or $\mathbf{C}_0$.

C If $\gamma \neq 0$, then m or κ is not zero.
In addition, if $\gamma \in (0,\nu)$ and $m = 0$, then $\kappa \neq 2(\nu^2 - \gamma^2)$;
if $\gamma \in \left(0, \left(\sqrt{m^2+4\nu^2} - m\right)/2\right]$ and $m \neq 0$, then $\kappa \neq \nu^2 - \gamma^2 \pm \sqrt{(\nu^2-\gamma^2)^2 - m^2\gamma^2}$.
If $\gamma = 0$, then $\kappa \in (0, 2\nu^2)$.

$\mathbf{C}_0$ $\gamma = 0$ and $\kappa = 2\nu^2$ or $\gamma = \kappa = 0$ and $m \neq 0$.

The main objective of the paper is to prove that for any initial data $Y_0 \in \mathcal{H}_{\alpha,+}$ with $\alpha > 3/2$, the solution $Y(t) = (u(\cdot,t), \dot u(\cdot,t))$ of the system obeys the following bound

$$\|Y(t)\|_{-\alpha,+} \leq C\langle t\rangle^{-\beta/2}\|Y_0\|_{\alpha,+}, \quad t \in \mathbb{R}, \tag{1.5}$$

where $\beta = 3$ if **C** holds, and $\beta = 1$ if $\mathbf{C}_0$ holds. We specify the behavior of the solutions as $t \to \infty$ in Theorem 2.4.

For the solutions of the linear discrete Schrodinger and Klein–Gordon equations in the whole space, the dispersive estimates of the type (1.5) were obtained by Shaban and Vainberg [10], Komech, Kopylova and Kunze [8] and Pelinosky and Stefanov [9]. The wave operators for the discrete Schrodinger operators were studied by Cuccagna [1]. In [4], we considered the linear Hamiltonian system consisting of the discrete Klein–Gordon field coupled to a particle and obtained the similar results on the long–time behavior for the solutions. In [5], the considered model (1.1)–(1.3) was studied with *random* initial data $Y_0 \in \mathcal{H}_{\alpha,+}$ with $\alpha < -3/2$. In this paper, the model is studied with initial data from the space $\mathcal{H}_{\alpha,+}$ with $\alpha > 3/2$, and the long time asymptotics of the solutions are constructed.

2 Main Results

The existence and uniqueness of the solutions to the problem (1.1)–(1.3) was proved in [5].

Theorem 2.1. *Let* $\gamma, \kappa, m \geq 0$, $\nu > 0$, *and let* $Y_0 \in \mathcal{H}_{\alpha,+}$, $\alpha \in \mathbb{R}$. *Then the problem (1.1)–(1.3) has a unique solution* $Y(t) \in C(\mathbb{R}, \mathcal{H}_{\alpha,+})$. *The operator* $U(t) : Y_0 \to Y(t)$ *is continuous on* $\mathcal{H}_{\alpha,+}$. *Moreover, there exist constants* $C, B < \infty$ *such that* $\|U(t)Y_0\|_{\alpha,+} \leq Ce^{B|t|}\|Y_0\|_{\alpha,+}$, $t \in \mathbb{R}$. *For* $Y_0 \in \mathcal{H}_{0,+}$, *the following bound holds,*

$$H(Y(t)) + \gamma \int_0^t |\dot u(0,s)|^2\, ds = H(Y_0), \quad t \in \mathbb{R}, \tag{2.1}$$

where $H(Y(t))$ *is defined in (1.4).*

The proof is based on the following representation for the solution $u(x,t)$ of the problem (1.1)–(1.3):

$$u(x,t) = z(x,t) + q(x,t), \quad x \ge 0, \quad t > 0, \tag{2.2}$$

where $z(x,t)$ is a solution of the mixed problem with zero boundary condition,

$$\ddot{z}(x,t) = (\nu^2\Delta_L - m^2)z(x,t), \quad x \in \mathbb{N}, \quad t > 0, \tag{2.3}$$
$$z(0,t) = 0, \quad t \ge 0, \tag{2.4}$$
$$z(x,0) = u_0(x), \quad \dot{z}(x,0) = v_0(x), \quad x \in \mathbb{N}. \tag{2.5}$$

Therefore, $q(x,t)$ is a solution of the following mixed problem

$$\ddot{q}(x,t) = (\nu^2\Delta_L - m^2)q(x,t), \quad x \in \mathbb{N}, \quad t > 0, \tag{2.6}$$
$$\ddot{q}(0,t) = \nu^2(q(1,t) - q(0,t)) - (m^2+\kappa)q(0,t) - \gamma\dot{q}(0,t) + \nu^2 z(1,t), \tag{2.7}$$
$$q(x,0) = 0, \quad \dot{q}(x,0) = 0, \quad x \in \mathbb{N}, \tag{2.8}$$
$$q(0,0) = u_0(0), \quad \dot{q}(0,0) = v_0(0). \tag{2.9}$$

We state the results concerning the solutions of the problem (2.3)–(2.5).

Lemma 2.2. *(see Lemma 2.7 in [3]) Assume that $\alpha \in \mathbb{R}$. Then for any $Y_0 \in \mathcal{H}_{\alpha,+}$, there exists a unique solution $Z(t) \equiv (z(\cdot,t), \dot{z}(\cdot,t)) \in C(\mathbb{R}, \mathcal{H}_{\alpha,+})$ to the mixed problem (2.3)–(2.5); the operator $U_0(t) : Y_0 \mapsto Z(t)$ is continuous on $\mathcal{H}_{\alpha,+}$. Furthermore, the following bound holds,*

$$\|U_0(t)Y_0\|_{\alpha,+} \le C\langle t\rangle^{\sigma}\|Y_0\|_{\alpha,+}, \tag{2.10}$$

with some constants $C = C(\alpha), \sigma = \sigma(\alpha) < \infty$.

The proof of Lemma 2.2 is based on the following formula for the solution $Z(x,t) = (Z^0(x,t), Z^1(x,t)) \equiv (z(x,t), \dot{z}(x,t))$ of the problem (2.3)–(2.5):

$$Z^i(x,t) = \sum_{j=0,1}\sum_{x'\ge 1} \mathcal{G}^{ij}_{t,+}(x,x')Y^j_0(x'), \quad x \in \mathbb{Z}_+, \tag{2.11}$$

where the Green function $\mathcal{G}_{t,+}(x,x') = (\mathcal{G}^{ij}_{t,+}(x,x'))^1_{i,j=0}$ is

$$\mathcal{G}^{ij}_{t,+}(x,x') := \mathcal{G}^{ij}_t(x-x') - \mathcal{G}^{ij}_t(x+x'), \quad \mathcal{G}^{ij}_t(x) \equiv \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ix\theta}\hat{\mathcal{G}}^{ij}_t(\theta)\,d\theta, \tag{2.12}$$

$$(\hat{\mathcal{G}}^{ij}_t(\theta))^1_{i,j=0} = \left(\begin{array}{cc} \cos\phi(\theta)t & \frac{\sin\phi(\theta)t}{\phi(\theta)} \\ -\phi(\theta)\sin\phi(\theta)t & \cos\phi(\theta)t \end{array}\right), \quad \phi(\theta) = \sqrt{\nu^2(2-2\cos\theta)+m^2}. \tag{2.13}$$

In particular, $\phi(\theta) = 2\nu|\sin(\theta/2)|$ if $m = 0$. We see that $Z(0,t) \equiv 0$ for any t, since $\mathcal{G}^{ij}_t(-x) = \mathcal{G}^{ij}_t(x)$. For the solutions of the problem (2.3)–(2.5), the following bound is true.

Theorem 2.3. *Let $Y_0 \in \mathcal{H}_{\alpha,+}$ and $\alpha > 3/2$. Then*

$$\|U_0(t)Y_0\|_{-\alpha,+} \le C\langle t\rangle^{-3/2}\|Y_0\|_{\alpha,+}, \quad t \in \mathbb{R}. \tag{2.14}$$

This theorem is proved in Appendix B.

To formulate the main result, introduce the following notations.
(i) Denote by $\mathbf{G}_1^j(y,t)$, $j=0,1$, the following function

$$\begin{aligned}\mathbf{G}_1^j(y,t) &:= \Big(\mathcal{G}_{t,+}^{j0}(1,y),\mathcal{G}_{t,+}^{j1}(1,y)\Big)\\ &= \Big(\mathcal{G}_t^{j0}(1-y)-\mathcal{G}_t^{j0}(1+y),\mathcal{G}_t^{j1}(1-y)-\mathcal{G}_t^{j1}(1+y)\Big),\ y\in\mathbb{Z},\ \ t\in\mathbb{R}. \quad (2.15)\end{aligned}$$

(ii) Let $\mathbf{G}^j(y)$, $j=0,1$, stand for the vector valued function defined as

$$\mathbf{G}^j(y)=\int_0^{+\infty}N(s)\mathbf{G}_1^j(y,-s)\,ds=\int_0^{+\infty}N^{(j)}(s)\mathbf{G}_1^0(y,-s)\,ds,\quad y\in\mathbb{Z},\quad j=0,1,\quad (2.16)$$

where the function $N(s)$ is introduced in (3.11)–(3.13).
(iii) Denote by $U_0'(t)$ the operator adjoint to $U_0(t)$:

$$\langle Y,U_0'(t)\Psi\rangle_+=\langle U_0(t)Y,\Psi\rangle_+,\quad Y\in\mathcal{H}_{\alpha,+},\quad \Psi\in\mathcal{S}\equiv[S(\mathbb{Z}_+)]^2,\quad t\in\mathbb{R}, \qquad (2.17)$$

where $S(\mathbb{Z}_+)$ denotes the class of rapidly decreasing sequences on $\mathbb{Z}_+=\{x\in\mathbb{Z}:x\ge 0\}$. Using the Green function $\mathcal{G}_{t,+}$, we rewrite $U_0'(t)\Psi$ in the form

$$(U_0'(t)\Psi)^j(y)=\sum_{i=0,1}\sum_{x\ge0}\mathcal{G}_{t,+}^{ij}(x,y)\Psi^i(x),\quad t\in\mathbb{R},\quad y\in\mathbb{Z}_+,\quad j=0,1.$$

In particular, $\mathbf{G}_1^0(y,t)=(U_0'(t)Y_0)(y)$ with $Y_0(x)=(\delta_{1x},0)$ (see (2.15)), where δ_{1x} denotes the Kronekker symbol.
(iv) Denote by $\mathbf{K}^j(x,y)$ $j=0,1$, $x\in\mathbb{N}$, $y\in\mathbb{Z}$, vector-valued functions of a form

$$\begin{aligned}\mathbf{K}^j(x,y) &= \int_0^{+\infty}K(x,s)\Big(U_0'(-s)\mathbf{G}^j\Big)(y)\,ds\\ &= \int_0^{+\infty}\int_0^{+\infty}K(x,s)N^{(j)}(\tau)\mathbf{G}_1^0(y,-s-\tau)\,ds\,d\tau,\ \ x\in\mathbb{N},\ \ y\in\mathbb{Z},\quad (2.18)\end{aligned}$$

where $K(x,s)$ is defined in (3.5).
(v) Define an operator $\Omega:\mathcal{H}_{\alpha,+}\to\mathcal{H}_{-\alpha,+}$, $\alpha>3/2$, by the rule

$$\Omega: Y\to Y+\nu^2\Big(\langle Y(\cdot),\mathbf{K}^0(x,\cdot)\rangle_+,\langle Y(\cdot),\mathbf{K}^1(x,\cdot)\rangle_+\Big). \qquad (2.19)$$

Here we put $\mathbf{K}^j(x,y)|_{x=0}:=\mathbf{G}^j(y)$, $y\in\mathbb{Z}$. The properties of the functions $\mathbf{K}^j$ and the operator Ω are specified in Remark 4.4. The main result of the paper is the following theorem.

Theorem 2.4. *Let $Y_0\in\mathcal{H}_{\alpha,+}$, $\alpha>3/2$, and condition* **C** *or* $\mathbf{C}_0$ *hold. Then the following assertions are fulfilled.*
(i) $U(t)Y_0=\Omega(U_0(t)Y_0)+r(t)$, where $\|r(t)\|_{-\alpha,+}\le C\langle t\rangle^{-\beta/2}\|Y_0\|_{\alpha,+}$, $\beta=3$ if **C** *holds and $\beta=1$ if $\mathbf{C}_0$ holds, Ω is a bounded operator defined by (2.19).*
(ii) The solution of the problem (1.1)–(1.3) obeys the bound (1.5).

This theorem is proved in Section 4. The behavior of the solutions with the initial data from the space $\mathcal{H}_{0,+}$ is discussed in Remark 4.5. If conditions **C** and $\mathbf{C}_0$ are not fulfilled, then the bound (1.5) for *any* initial data from $\mathcal{H}_{\alpha,+}$ is incorrect, see Remark 4.6.

3 Fourier–Laplace transform

In this section, we study the properties of the solutions $q(x,t)$ to the problem (2.6)–(2.9) using the Fourier–Laplace transform.

Definition 3.1. *Let $|q(t)| \le Ce^{Bt}$. The Fourier–Laplace transform of $q(t)$ is given by the formula*

$$\tilde{q}(\omega) = \int_0^{+\infty} e^{i\omega t} q(t)\, dt, \quad \Im\omega > B. \tag{3.1}$$

The Gronwall inequality implies standard a priori estimate for the solutions $q(x,t)$, $x \ge 1$. In particular, there exist constants $A, B < \infty$ such that

$$\sum_{x\in\mathbb{N}} (|q(x,t)|^2 + |\dot{q}(x,t)|^2) \le Ce^{Bt} \quad \text{ as } \; t \to +\infty.$$

Hence, the Fourier–Laplace transform of the solutions $q(x,t)$ to the problem (2.6), (2.8) with respect to t-variable, $q(x,t) \to \tilde{q}(x,\omega)$, exists at least for $\Im\omega > B$ and satisfies the following equation

$$(-\nu^2\Delta_L + m^2 - \omega^2)\tilde{q}(x,\omega) = 0, \quad x \in \mathbb{N}, \quad \Im\omega > B. \tag{3.2}$$

We construct the solution of (3.2). We first note that the Fourier transform of the operator $-\nu^2\Delta_L + m^2$ is the operator of multiplication by the function $\phi^2(\theta) = \nu^2(2 - 2\cos\theta) + m^2$. Thus, $-\nu^2\Delta_L + m^2$ is a self-adjoint operator and its spectrum is absolutely continuous and coincides with the range of $\phi^2(\theta)$, i.e., with the segment $[m^2, m^2 + 4\nu^2]$.

Lemma 3.2. *(see Lemma 2.1 in [8]) Denote $\Lambda := [-\sqrt{4\nu^2 + m^2}, -m] \cup [m, \sqrt{4\nu^2 + m^2}]$. For given $\omega \in \mathbb{C} \setminus \Lambda$, the equation*

$$\nu^2(2 - 2\cos\theta) = \omega^2 - m^2 \tag{3.3}$$

has the unique solution $\theta(\omega)$ in the domain $\{\theta \in \mathbb{C} : \Im\theta > 0, -\pi < \Re\theta \le \pi\}$. Moreover, $\theta(\omega)$ is an analytic function in $\mathbb{C} \setminus \Lambda$.

Since we seek the solution $q(\cdot,t) \in \ell^2_{\alpha,+}$ with some α, $\tilde{q}(x,\omega)$ has a form

$$\tilde{q}(x,\omega) = \tilde{q}(0,\omega)e^{i\theta(\omega)x}, \quad x \ge 0.$$

Introduce a function $\tilde{K}(x,\omega) = e^{i\theta(\omega)x}$. Applying the inverse Fourier–Laplace transform with respect to ω-variable, we write the solution $q(x,t)$ of the problem (2.6), (2.8) in the form

$$(q(x,t), \dot{q}(x,t)) = \int_0^t K(x, t-s)\big(q(0,s), \dot{q}(0,s)\big)\, ds, \quad x \in \mathbb{N}, \quad t > 0, \tag{3.4}$$

where

$$K(x,t) = \frac{1}{2\pi} \int\limits_{-\infty+i\mu}^{+\infty+i\mu} e^{-i\omega t}\tilde{K}(x,\omega)\, d\omega, \quad \tilde{K}(x,\omega) = e^{i\theta(\omega)x}, \quad x \in \mathbb{N}, \quad t > 0, \tag{3.5}$$

with some $\mu > 0$. The following theorem was proved in [5].

Theorem 3.3. *For any* $\alpha < -3/2$ *, the following bound holds,*

$$\sum_{x\in\mathbb{N}} \langle x\rangle^{2\alpha} |K(x,t)|^2 \le C(1+t)^{-3} \quad for \quad t > 0. \tag{3.6}$$

In particular,

$$|K(1,t)| \le C(1+t)^{-3/2}, \quad t > 0. \tag{3.7}$$

To estimate $q(0,t)$, we use (3.4) and rewrite Eqn (2.7) in the form

$$\ddot q(0,t) = -(\kappa+\nu^2+m^2)q(0,t) - \gamma\dot q(0,t) + \nu^2\int_0^t K(1,t-s)q(0,s)\,ds + \nu^2 z(1,t), \quad t > 0. \tag{3.8}$$

At first, we study the solutions of the corresponding homogeneous equation

$$\ddot q(0,t) = -(\kappa+\nu^2+m^2)q(0,t) - \gamma\dot q(0,t) + \nu^2\int_0^t K(1,t-s)q(0,s)\,ds, \quad t > 0, \tag{3.9}$$

with the initial data

$$q(0,t)|_{t=0} = u_0(0) =: q_0, \quad \dot q(0,t)|_{t=0} = v_0(0) =: p_0. \tag{3.10}$$

Applying the Fourier–Laplace transform to the solutions $q(0,t)$ of (3.9), we obtain

$$\tilde q(0,\omega) = \tilde N(\omega)\,(-i\omega q_0 + q_0\gamma + p_0) \quad \text{for } \Im\omega > B, \tag{3.11}$$

where, by definition, $\tilde N(\omega) := [\tilde D(\omega)]^{-1}$ and

$$\tilde D(\omega) := -\omega^2 + \kappa + \nu^2 + m^2 - i\omega\gamma - \nu^2\tilde K(1,\omega), \quad \tilde K(1,\omega) = e^{i\theta(\omega)}, \quad \omega\in\mathbb{C}. \tag{3.12}$$

The properties of the functions $\tilde D(\omega)$ and $\tilde N(\omega)$ are studied in Appendix A. In particular, we prove that $\tilde N(\omega)$ is an analytic function in the upper half-space. Denote

$$N(t) = \frac{1}{2\pi}\int_{-\infty+i\mu}^{+\infty+i\mu} e^{-i\omega t}\tilde N(\omega)\,d\omega, \quad t \ge 0, \quad \text{with some } \mu > 0. \tag{3.13}$$

The following theorem is proved in Appendix A.

Theorem 3.4. *let condition* $\mathbf{C}$ *or* $\mathbf{C}_0$ *hold. Then*

$$|N^{(k)}(t)| \le C(1+t)^{-\beta/2}, \quad t\ge 0, \quad k = 0,1,2, \tag{3.14}$$

where $\beta = 3$ *if* $\mathbf{C}$ *holds and* $\beta = 1$ *if* $\mathbf{C}_0$ *holds.*

Corollary 3.5. *Denote by* $S(t)$ *a solving operator of the Cauchy problem (3.9), (3.10). Then the variation constants formula gives the following representation for the solution of the problem (3.8), (3.10):*

$$\begin{pmatrix} q(0,t) \\ \dot q(0,t) \end{pmatrix} = S(t)\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} + \int_0^t S(\tau)\begin{pmatrix} 0 \\ \nu^2 z(1,t-\tau) \end{pmatrix} d\tau, \quad t > 0.$$

Evidently, $S(0) = I$ *, and the matrix* $S(t)$ *has a form* $\begin{pmatrix} \dot N(t)+\gamma N(t) & N(t) \\ \ddot N(t)+\gamma\dot N(t) & \dot N(t) \end{pmatrix}$*. Moreover,* $|S(t)| \le C(1+t)^{-\beta/2}$ *, by Theorem 3.4.*

4 Asymptotic behavior of $Y(t)$ as $t \to \infty$

Set $q^{(0)}(x,t) = q(x,t)$, $q^{(1)}(x,t) = \dot{q}(x,t)$, $x \in \mathbb{Z}_+$.

Proposition 4.1. *Let $Y_0 \in \mathcal{H}_{\alpha,+}$, $\alpha > 3/2$, condition $\mathbf{C}$ or $\mathbf{C}_0$ hold, and $q(0,t)$ be a solution of the problem (3.8), (3.10). Then*

$$q^{(j)}(0,t) = \nu^2 \langle U_0(t)Y_0, \mathbf{G}^j\rangle_+ + r_j(t), \quad t > 0, \quad |r_j(t)| \le C\langle t\rangle^{-\beta/2}\|Y_0\|_{\alpha,+}, \ j = 0,1, \quad (4.1)$$

where the functions $\mathbf{G}^j$ are defined in (2.16), the number β is introduced in Theorem 2.4.

Proof Corollary 3.5 and the bound (3.14) imply that

$$q^{(j)}(0,t) = \nu^2 \int_0^t N^{(j)}(\tau) z(1, t-\tau)\, d\tau + O((1+t)^{-\beta/2}), \quad t > 0, \quad j = 0,1.$$

Moreover, the bounds (2.14) and (3.14) give

$$\Big| \int_t^{+\infty} N^{(j)}(\tau) z(1,t-\tau) d\tau \Big| \le C \int_t^{+\infty} \langle\tau\rangle^{-\beta/2} \langle t-\tau\rangle^{-3/2} d\tau \le C\langle t\rangle^{-\beta/2}\|Y_0\|_{\alpha,+}.$$

This implies the representation (4.1), since by (2.15) and (2.11), we have

$$z(1,t-\tau) = \langle U_0(t)Y_0(\cdot), \mathbf{G}^0_1(\cdot,-\tau)\rangle_+. \ \blacksquare$$

Remark 4.2. Now we list the properties of the functions $\mathbf{G}^j_1(y,t)$ and $\mathbf{G}^j(y)$, $j = 0,1$.
(i) By (2.12) and (2.13), the function $\mathbf{G}^j_1(y,t)$ is odd w.r.t. $y \in \mathbb{Z}$. Then the function $\mathbf{G}^j$ is also odd. Formulas (2.13) and the Parseval identity give

$$\|\mathbf{G}^0_1(\cdot,t)\|_0^2 = C \int_{-\pi}^{\pi} \Big(\cos^2(\phi(\theta)t) + \frac{\sin^2(\phi(\theta)t)}{\phi^2(\theta)} \Big) \sin^2(\theta)\, d\theta \le C < \infty. \quad (4.2)$$

Here $\|\cdot\|_0$ denotes norm in $\ell^2 \times \ell^2$.

(ii) Let condition $\mathbf{C}$ or $\mathbf{C}_0$ hold. Since $\mathbf{G}^0_1(y,t) = U_0'(t)(\delta_{1x},0)$, then for any $\alpha > 3/2$,

$$\|U_0'(t)\mathbf{G}^j\|_{-\alpha,+} \le \int_0^{+\infty} |N^{(j)}(s)| \|U_0'(t-s)(\delta_{1x},0)\|_{-\alpha,+}\, ds \le C\langle t\rangle^{-\beta/2}, \quad (4.3)$$

due to the bound (2.14), because the action of the group $U_0'(t)$ coincides with action of the group $U_0(t)$, up to order of the components. Therefore, for $\alpha > 3/2$,

$$|\langle U_0(t)Y_0, \mathbf{G}^j\rangle_+| \le \|Y_0\|_{\alpha,+} \|U_0'(t)\mathbf{G}^j\|_{-\alpha,+} \le C\langle t\rangle^{-\beta/2}\|Y_0\|_{\alpha,+}. \quad (4.4)$$

(iii) Let condition $\mathbf{C}$ hold. Since $U_0'(t)\mathbf{G}^0_1(y,-s) = \mathbf{G}^0_1(y,t-s)$, then the bounds (3.14) and (4.2) yield

$$\sup_{t\in\mathbb{R}} \|U_0'(t)\mathbf{G}^j(\cdot)\|_0 \le \sup_{t\in\mathbb{R}} \int_0^{+\infty} |N^{(j)}(s)| \|\mathbf{G}^0_1(\cdot,t-s)\|_0\, ds \le C \int_0^{+\infty} |N^{(j)}(s)|\, ds < \infty. \quad (4.5)$$

Lemma 4.3. *Let $Y_0 \in \mathcal{H}_{\alpha,+}$, $\alpha > 3/2$, and condition* **C** *or* $\mathbf{C}_0$ *hold. Then the solution $q(x,t)$ of the problem (2.6)–(2.9) with $x \geq 1$, admits the following representation*

$$q^{(j)}(x,t) = \nu^2 \langle U_0(t)Y_0, \mathbf{K}^j(x,\cdot)\rangle_+ + r_j(x,t), \quad j=0,1, \quad t>0, \tag{4.6}$$

where $\mathbf{K}^j$ is introduced in (2.18), $\|r_j(\cdot,t)\|_{-\alpha,+} \leq C\langle t\rangle^{-\beta/2}\|Y_0\|_{\alpha,+}$. Here, by definition, $\|r\|^2_{-\alpha,+} := \sum_{x\in\mathbb{N}} \langle x\rangle^{-2\alpha}|r(x)|^2$.

Proof At first, by (3.4) and (4.1), we have

$$q^{(j)}(x,t) = \nu^2 \int_0^t K(x,t-s)\langle U_0(s)Y_0, \mathbf{G}^j(\cdot)\rangle_+ \, ds + r'_j(x,t), \quad x\in\mathbb{N}, \tag{4.7}$$

where $\|r'_j(\cdot,t)\|_{-\alpha,+} \leq C\langle t\rangle^{-\beta/2}$. Indeed, (4.1) and (3.6) give

$$\begin{aligned}\|r'_j(\cdot,t)\|_{-\alpha,+} &= \Big\| \int_0^t K(\cdot,t-s) r_j(s)\, ds \Big\|_{-\alpha,+} \leq \int_0^t \|K(\cdot,t-s)\|_{-\alpha,+} |r_j(s)|\, ds \\ &\leq C\int_0^t (1+t-s)^{-3/2}(1+s)^{-\beta/2}\, ds \leq C_1\langle t\rangle^{-\beta/2}.\end{aligned}$$

Second, the first term in the r.h.s. of (4.7) has a form (see (2.18))

$$\nu^2 \int_0^t K(x,s)\langle U_0(t-s)Y_0, \mathbf{G}^j(\cdot)\rangle_+ \, ds = \nu^2\langle U_0(t)Y_0, \mathbf{K}^j(x,\cdot)\rangle_+ + r''_j(x,t), \tag{4.8}$$

where, by definition, $r''_j(x,t) := -\nu^2 \displaystyle\int_t^{+\infty} K(x,s)\langle U_0(t-s)Y_0, \mathbf{G}^j\rangle_+ \, ds$. The bounds (3.6) and (4.4) yield

$$\|r''_j(\cdot,t)\|_{-\alpha,+} \leq \nu^2 \int_t^{+\infty} \|K(\cdot,s)\|_{-\alpha,+} \left|\langle U_0(t-s)Y_0, \mathbf{G}^j\rangle_+\right| \, ds \leq C\langle t\rangle^{-\beta/2}\|Y_0\|_{\alpha,+}. \tag{4.9}$$

Hence, the bounds (4.7)–(4.9) imply (4.6) with $r_j(x,t) = r'_j(x,t) + r''_j(x,t)$. ■

Remark 4.4. (i) Set $\tilde{K}(0,\omega) := 1$. Then, formally, $K(0,t) = \delta_{0t}$. Hence, we can put $\mathbf{K}^j(0,y) = \mathbf{G}^j(y)$, $y\in\mathbb{Z}$. Then, the representation (4.1) follows from (4.6).

(ii) By Remark 4.2 and definition (2.18), the function $\mathbf{K}^j(x,y)$ is odd w.r.t. $y\in\mathbb{Z}$. Furthermore, formulas (2.18), (3.6) and (4.3) imply that for $\alpha > 3/2$,

$$\Big\| \|U'_0(t)\mathbf{K}^j(x,\cdot)\|_{-\alpha,+} \Big\|_{-\alpha,+} \leq \int_0^{+\infty} \|K(x,s)\|_{-\alpha,+} \|U'_0(t-s)\mathbf{G}^j\|_{-\alpha,+} ds < C\langle t\rangle^{-\beta/2}.$$

Hence, for $\alpha > 3/2$,

$$\|\langle U_0(t)Y_0, \mathbf{K}^j(x,\cdot)\rangle_+\|_{-\alpha,+} = \|\langle Y_0(\cdot), U'_0(t)\mathbf{K}^j(x,\cdot)\rangle_+\|_{-\alpha,+} \leq C\langle t\rangle^{-\beta/2}\|Y_0\|_{\alpha,+}, \ t\in\mathbb{R}. \tag{4.10}$$

In particular, the operator Ω introduced in (2.19) is bounded.

(iii) If condition **C** holds, then $\|\mathbf{K}^j(x,\cdot)\|_0 \in \mathcal{H}_{-\alpha,+}$ with $\alpha > 3/2$, since

$$\Big\| \|\mathbf{K}^j(x,\cdot)\|_0 \Big\|_{-\alpha,+} \leq \int_0^{+\infty} \|K(x,s)\|_{-\alpha,+} \|U'_0(-s)\mathbf{G}^j\|_0 \, ds < \infty$$

by virtue of (2.18), (3.6) and (4.5). Therefore, (2.19) implies that for $Y\in\mathcal{H}_{0,+}$,

$$\|\Omega Y\|_{-\alpha,+} \leq \|Y\|_{-\alpha,+} + \nu^2 \sum_{j=0,1} \|\langle Y(\cdot), \mathbf{K}^j(x,\cdot)\rangle_+\|_{-\alpha,+} \leq C\|Y\|_{0,+}.$$

Proof of Theorem 2.4 The item (i) follows from the representations (2.2), (4.1) and (4.6). Further, definition (2.19), the bounds (2.14), (4.4) and (4.10) give

$$\begin{aligned}\|\Omega(U_0(t)Y_0)\|_{-\alpha,+} &\le \|U_0(t)Y_0\|_{-\alpha,+} + \nu^2 \textstyle\sum_{j=0,1} \|\langle U_0(t)Y_0, \mathbf{K}^j(x,\cdot)\rangle_+\|_{-\alpha,+} \\ &\le C\langle t\rangle^{-\beta/2}\|Y_0\|_{\alpha,+}, \quad \alpha > 3/2.\end{aligned} \tag{4.11}$$

Thus, the bound (1.5) follows from the part (i) of Theorem 2.4 and the bound (4.11). ■

Remark 4.5. Let condition **C** hold. From the proofs of Lemmas 4.1 and 4.3 we see that the remainders $r_j(t)$ and $r_j(x,t)$ in decompositions (4.1) and (4.6) are estimated by $z(1,t)$,

$$|r_j(t)| \le C_1|Y_0(0)|\langle t\rangle^{-3/2} + C_2\int_t^{+\infty}\langle\tau\rangle^{-3/2}|z(1,t-\tau)|\,d\tau.$$

$$\begin{aligned}\|r_j(\cdot,t)\|_{-\alpha,+} \le\ & C_1|Y_0(0)|\langle t\rangle^{-3/2} + C_2\int_0^t\langle t-s\rangle^{-3/2}\,ds\int_s^{+\infty}\langle\tau\rangle^{-3/2}|z(1,s-\tau)|\,d\tau \\ &+C_3\int_t^{+\infty}\langle s\rangle^{-3/2}ds\int_0^{+\infty}\langle\tau\rangle^{-3/2}|z(1,t-s-\tau)|\,d\tau.\end{aligned}$$

Hence, if $\sup_{t\in\mathbb{R}}|z(1,t)| =: M_0 < \infty$, then $U(t)Y_0 = \Omega(U_0(t)Y_0) + r(x,t)$, where

$$\|r(\cdot,t)\|_{-\alpha,+} \le C_1|Y_0(0)|\langle t\rangle^{-3/2} + CM_0\langle t\rangle^{-1/2} \le C\langle t\rangle^{-1/2}.$$

For instance, if initial data $Y_0(x)$ are such that $\hat{u}_{\rm odd}(\theta), \hat{v}_{\rm odd}(\theta)/\phi(\theta) \in L^1(\mathbb{T})$, where $Y_{\rm odd}(x) = (u_{\rm odd}(x), v_{\rm odd}(x))$ is defined in (B.2), then $|z(1,t)| \le C < \infty$. In particular, this is true if $m \ne 0$ and $Y_0 \in \mathcal{H}_{0,+}$.

Remark 4.6. If conditions **C** and $\mathbf{C}_0$ are not fulfilled, then the bound (1.5) for any initial data $Y_0 \in \mathcal{H}_{\alpha,+}$ is incorrect. Indeed, if $m = \kappa = 0$, then $\tilde{N}(\omega)$ has a simple pole at zero, and any constant is a solution of the system (1.1)–(1.2). If $\gamma = 0$ and $\kappa > 2\nu^2$, then there exists a number $\omega_0 > \sqrt{4\nu^2 + m^2}$ such that $\tilde{D}(\omega_0) = 0$, and $\tilde{N}(\omega)$ has simple poles at the points $\omega = \pm\omega_0$, see Remark A.3 below. Therefore, a function of the form $u(x,t) = e^{i\theta(\omega_0)x}\sin(\omega_0 t)$ is the solution of the system, where $\theta(\omega)$ is the solution of (3.3), $\Re\theta(\omega_0) = \pi$, $\Im\theta(\omega_0) > 0$. If one of the following two conditions holds: (1) $m = 0$, $\kappa = 2(\nu^2 - \gamma^2)$ and $\gamma \in (0,\nu)$, or (2) $m \ne 0$, $\kappa = \nu^2 - \gamma^2 \pm \sqrt{(\nu^2-\gamma^2)^2 - m^2\gamma^2}$ and $\gamma \in \left(0, \left(\sqrt{m^2+4\nu^2} - m\right)/2\right]$, then there exist points $\omega_* \in \Lambda\setminus\Lambda_0$ such that $\tilde{D}(\omega_* - i0) = 0$ (see item (iv) of Lemma A.2). We denote $\theta_+ := \lim_{\varepsilon\to+0}\theta(\omega_* + i\varepsilon)$, $\theta_+ \in \mathbb{R}$. Then the function of the form $u(x,t) = \sin(\theta_+ x + \omega_* t)$ is a solution of the system (1.1)–(1.2).

Appendix A: Properties of $\tilde{D}(\omega)$ and $\tilde{N}(\omega)$ for $\omega \in \mathbb{C}$

Let $\Lambda := [-\sqrt{4\nu^2+m^2}, -m] \cup [m, \sqrt{4\nu^2+m^2}]$. $\Lambda_0 = \{\pm m, \pm\sqrt{4\nu^2+m^2}\}$ denotes the set of the "spectral edges". We first list the properties of the function $e^{i\theta(\omega)}$ for $\omega \in \mathbb{C}\setminus\Lambda$, $\omega \in \Lambda\setminus\Lambda_0$, and $\omega \in \Lambda_0$.

Let $\omega \in \mathbb{C}\setminus\Lambda$. Then $\Im\theta(\omega) > 0$ and $e^{i\theta(\omega)}$ is an analytic function. Moreover, by (3.3) and the condition $\Im\theta(\omega) > 0$, we have

$$\left|e^{i\theta(\omega)}\right| \le C|\omega|^{-2} \quad \text{as} \quad |\omega| \to \infty. \tag{A.1}$$

For $\omega\in\Lambda\setminus\Lambda_0$, put $\theta(\omega\pm i0)=\lim\limits_{\varepsilon\to+0}\theta(\omega\pm i\varepsilon)$. Since $\overline{\theta(\omega)}=-\theta(\bar\omega)$ for $\omega\in\mathbb{C}\setminus\Lambda$, then $e^{i\theta(\omega-i0)}=\overline{e^{i\theta(\omega+i0)}}$ for $\omega\in\Lambda\setminus\Lambda_0$.

We study the behavior of $e^{i\theta(\omega)}$ near the points in the set Λ_0. From Eqn (3.3) we have

$$e^{i\theta(\omega)}=\cos\theta(\omega)+i\sin\theta(\omega)=1-\frac{1}{2\nu^2}(\omega^2-m^2)+\frac{i}{2\nu^2}\sqrt{(\omega^2-m^2)(4\nu^2+m^2-\omega^2)}\quad\text{(A.2)}$$

for $\omega\in\mathbb{C}\setminus\Lambda$. The Taylor expansion implies

$$e^{i\theta(\omega)}=1+\frac{i}{\nu}\sqrt{\omega^2-m^2}-\frac{1}{2\nu^2}(\omega^2-m^2)-\frac{i}{8\nu^3}(\omega^2-m^2)^{3/2}+\ldots\quad\text{as }\omega\to\pm m+i0,\quad\text{(A.3)}$$

where $\omega\in\mathbb{C}_+:=\{\omega\in\mathbb{C}:\Im\omega>0\}$, $\Im\sqrt{\omega^2-m^2}>0$. Here $\mathrm{sgn}(\Re\sqrt{\omega^2-m^2})=\mathrm{sgn}(\Re\omega)$ for $\omega\in\mathbb{C}_+$. This choice of the branch of the complex root $\sqrt{\omega^2-m^2}$ follows from the condition $\Im\theta(\omega)>0$. Similarly,

$$e^{i\theta(\omega)}=-1+\frac{i}{\nu}\sqrt{m^2+4\nu^2-\omega^2}+\frac{1}{2\nu^2}(m^2+4\nu^2-\omega^2)-\frac{i}{8\nu^3}(m^2+4\nu^2-\omega^2)^{3/2}+\ldots\quad\text{(A.4)}$$

as $\omega\to\pm\sqrt{m^2+4\nu^2}$, $\omega\in\mathbb{C}_+$. Here the branch of the complex root $\sqrt{m^2+4\nu^2-\omega^2}$ is chosen so that $\mathrm{sgn}(\Re\sqrt{m^2+4\nu^2-\omega^2})=\mathrm{sgn}(\Re\omega)$ that follows from the condition $\Im\theta(\omega)>0$. If $m=0$, then (A.2) and the Taylor expansion imply

$$e^{i\theta(\omega)}=1+\frac{i\omega}{\nu}-\frac{\omega^2}{2\nu^2}-\frac{i\omega^3}{8\nu^3}+\ldots\quad\text{as }\omega\to0,\quad\text{(A.5)}$$

and $e^{i\theta(\omega)}=-1+i\sqrt{4\nu^2-\omega^2}/\nu+\ldots$ as $\omega\to\pm2\nu$, $\omega\in\mathbb{C}_+$.

Lemma A.1. *(i)* $\tilde N(\omega)$ *is meromorphic for* $\omega\in\mathbb{C}\setminus\Lambda$.
(ii) $|\tilde N(\omega)|=O(|\omega|^{-2})$ *as* $|\omega|\to\infty$.
(iii) $\tilde D(\omega)\ne0$ *for all* $\omega\in\mathbb{C}_+=\{\omega\in\mathbb{C}:\,\Im\omega>0\}$.
(iv) If $\gamma=0$, *then* $\tilde D(\omega)\ne0$ *for any* $\omega\in\mathbb{C}_-=\{\omega\in\mathbb{C}:\,\Im\omega<0\}$.

Proof The first assertion of the lemma follows from the analyticity of $\tilde D(\omega)$ for $\omega\in\mathbb{C}\setminus\Lambda$. The assertion (ii) follows from (3.12) and (A.1). To prove the third assertion, we assume opposite that $\tilde D(\omega_0)=0$ for some $\omega_0\in\mathbb{C}_+$. Hence, the function $u_*(x,t)=e^{i\theta(\omega_0)x}e^{-i\omega_0t}$, $x\ge0$, $t\ge0$, is a solution of the problem (1.1)–(1.2) with the initial data $Y_*=e^{i\theta(\omega_0)x}(1,-i\omega_0)$. Therefore, the Hamiltonian (1.4) is

$$H(u_*(\cdot,t),\dot u_*(\cdot,t))=e^{2t\,\Im\omega_0}H(Y_*)\quad\text{for any }t>0,\quad\text{where }\ H(Y_*)>0.$$

Since $\Im\omega_0>0$ and $Y_*\in\mathcal{H}_{0,+}$, this exponential growth contradicts the energy estimate (2.1). Hence, $\tilde D(\omega)\ne0$ for any $\omega\in\mathbb{C}_+$.

If $\gamma=0$, then $\overline{\tilde D(\omega)}=\tilde D(\bar\omega)$, since $\overline{\theta(\omega)}=-\theta(\bar\omega)$ for $\omega\in\mathbb{C}\setminus\Lambda$. Therefore, item (iv) of the lemma follows from item (iii). ■

Lemma A.2. *Let the condition* $\mathbf{C}$ *or* $\mathbf{C}_0$ *hold. Then* $\tilde D(\omega)\ne0$ *for* $\omega\in\mathbb{R}\setminus\Lambda$, $\tilde D(\omega\pm i0)\ne0$ *for* $\omega\in\Lambda\setminus\Lambda_0$.

Proof (i) Let $\omega \in \mathbb{R}$ and $|\omega| > \sqrt{4\nu^2+m^2}$. Then $\Re\theta(\omega) = \pm\pi$. Therefore,

$$\tilde{D}(\omega) = -\omega^2 + \kappa + \nu^2 + m^2 - i\omega\gamma + \nu^2 e^{-\Im\theta(\omega)} \quad \text{with} \quad \Im\theta(\omega) > 0.$$

Hence, $\Im\tilde{D}(\omega) \neq 0$ iff $\gamma \neq 0$. On the other hand, $\Re\tilde{D}(\omega) = \kappa - 2\nu^2$ for $\omega = \pm\sqrt{4\nu^2+m^2}$, and $\Re\tilde{D}(\omega_1) < \Re\tilde{D}(\omega_2)$ if $|\omega_1| > |\omega_2| \geq \sqrt{4\nu^2+m^2}$. In particular, $\Re\tilde{D}(\omega) \to -\infty$ as $|\omega| \to \infty$. Hence, for $|\omega| > \sqrt{4\nu^2+m^2}$, $\Re\tilde{D}(\omega) \neq 0$ iff $\kappa \leq 2\nu^2$. Therefore, for such values of ω, $\tilde{D}(\omega) \neq 0$ iff either $\gamma \neq 0$ or $\gamma = 0$ and $\kappa \leq 2\nu^2$.

(ii) Let $m \neq 0$ and $\omega \in (-m, m)$. Then, $\Re\theta(\omega) = 0$. Hence,

$$\Re\tilde{D}(\omega) = -\omega^2 + \kappa + \nu^2 + m^2 - \nu^2 e^{i\theta(\omega)} > \kappa \quad \text{for} \quad |\omega| < m,$$

and $\Re\tilde{D}(\pm m) = \kappa$. Therefore, $\tilde{D}(\omega) \neq 0$ for any $|\omega| < m$, since $\kappa \geq 0$.

(iii) Let $\omega \in (-\sqrt{4\nu^2+m^2}, -m) \cup (m, \sqrt{4\nu^2+m^2})$. Then $\Re\theta(\omega + i0) \in (-\pi, 0) \cup (0, \pi)$ and $\Im\theta(\omega + i0) = 0$. Moreover, $\text{sign}(\sin\theta(\omega + i0)) = \text{sign}\,\omega$. Hence, for $m \neq 0$,

$$\begin{aligned}
\Im\tilde{D}(\omega + i0) &= -\omega\gamma - \nu^2 \sin\theta(\omega + i0) \\
&= -\text{sign}(\omega)\left(|\omega|\gamma + \sqrt{\omega^2 - m^2}\sqrt{\nu^2 - (\omega^2 - m^2)/4}\right) \neq 0.
\end{aligned}$$

If $m = 0$, then $\tilde{D}(\omega + i0) = \kappa - \omega^2/2 - i\omega\left(\gamma + \sqrt{\nu^2 - \omega^2/4}\right) \neq 0$ for any $\kappa, \gamma \geq 0$.

(iv) Since $\tilde{D}(\omega - i0) = \overline{\tilde{D}(\omega + i0)} - 2i\omega\gamma$ for $\omega \in \Lambda \setminus \Lambda_0$, then

$$\begin{aligned}
\tilde{D}(\omega - i0) &= -\omega^2 + \kappa + \nu^2 + m^2 - \nu^2\cos\theta(\omega + i0) + i\nu^2 \sin\theta(\omega + i0) - i\omega\gamma \\
&= \kappa - (\omega^2 - m^2)/2 + i\left(\text{sign}(\omega)\frac{1}{2}\sqrt{\omega^2 - m^2}\sqrt{4\nu^2 + m^2 - \omega^2} - \omega\gamma\right)
\end{aligned}$$

for $\omega \in \Lambda \setminus \Lambda_0$. Hence, $\tilde{D}(\omega - i0) = 0$ for $\omega \in \Lambda \setminus \Lambda_0$ iff

$$\kappa = (\omega^2 - m^2)/2 \quad \text{and} \quad \sqrt{\omega^2 - m^2}\sqrt{4\nu^2 + m^2 - \omega^2} = 2|\omega|\gamma, \quad \omega^2 \in (m^2, m^2 + 4\nu^2). \quad \text{(A.6)}$$

Then, $\gamma \neq 0$. Put $P := \omega^2 - m^2$. Hence, P is a solution of the following equation

$$P^2 + 4P(\gamma^2 - \nu^2) + 4m^2\gamma^2 = 0, \quad P \in (0, 4\nu^2). \quad \text{(A.7)}$$

If $m = 0$, then Eqn (A.7) has a unique solution $P = 4(\nu^2 - \gamma^2) \in (0, 4\nu^2)$ iff $\gamma < \nu$. Then, $\kappa = (\omega^2 - m^2)/2 = P/2 = 2(\nu^2 - \gamma^2)$ by the first equation in (A.6). Thus, if $m = 0$, $\kappa = 2(\nu^2 - \gamma^2)$ and $\gamma \in (0, \nu)$, then there exist two points $\omega = \pm\omega_* = \pm 2\sqrt{\nu^2 - \gamma^2} \in \Lambda \setminus \Lambda_0$ such that $\tilde{D}(\omega_* - i0) = 0$.

If $m \neq 0$, then (A.7) has a solution iff $(\gamma^2 - \nu^2)^2 - m^2\gamma^2 \geq 0$ and $\gamma \in (0, \nu)$. This is equivalent to the conditions $\gamma^2 + m\gamma - \nu^2 \leq 0$ and $\gamma \in (0, \nu)$, that coincides with the inequality $\gamma \in \left(0, \left(\sqrt{m^2 + 4\nu^2} - m\right)/2\right]$. Therefore, if $m \neq 0$ and $\gamma \in \left(0, \left(\sqrt{m^2 + 4\nu^2} - m\right)/2\right]$, then Eqn (A.7) has solutions

$$P = 2(\nu^2 - \gamma^2) \pm 2\sqrt{(\nu^2 - \gamma^2)^2 - m^2\gamma^2} \in (0, 4\nu^2).$$

Hence, $\kappa = (\omega^2 - m^2)/2 = P/2 = \nu^2 - \gamma^2 \pm \sqrt{(\nu^2 - \gamma^2)^2 - m^2\gamma^2}$.

Thus, there are points $\omega_* \in \Lambda \setminus \Lambda_0$, in which $\tilde{D}(\omega_* - i0) = 0$, iff $\gamma \neq 0$ and one of the following conditions is fulfilled: (1) $m = 0$, $\kappa = 2(\nu^2 - \gamma^2)$ and $\gamma \in (0, \nu)$; (2) $m \neq 0$, $\kappa = \nu^2 - \gamma^2 \pm \sqrt{(\nu^2 - \gamma^2)^2 - m^2\gamma^2}$ and $\gamma \in \left(0, \left(\sqrt{m^2 + 4\nu^2} - m\right)/2\right]$. These values of κ, m, γ are eliminated by the condition **C**. ∎

Remark A.3. If condition **C** holds, then $\tilde{D}(\omega) \neq 0$ for $\omega \in \Lambda_0$, because $\tilde{D}(\pm\sqrt{4\nu^2+m^2}) = \kappa - 2\nu^2 \mp i\gamma\sqrt{4\nu^2+m^2}$ and $\tilde{D}(\pm m) = \kappa \mp i\gamma m$. If condition **C** is not satisfied, then there are points $\omega \in \mathbb{R}$, in which $\tilde{D}(\omega) = 0$. For example, $\tilde{D}(0) = 0$ in the case $m = \kappa = 0$. If $\gamma = \kappa = 0$, then $\tilde{D}(\pm m) = 0$. If $\gamma = 0$ and $\kappa = 2\nu^2$, then $\tilde{D}(\pm\sqrt{m^2+4\nu^2}) = 0$. If $\gamma = 0$ and $\kappa > 2\nu^2$, then $\exists\, \omega_0 > \sqrt{4\nu^2+m^2}$ such that $\tilde{D}(\pm\omega_0) = 0$, and $\tilde{D}'(\omega_0) = -2\omega_0(\kappa - \nu^2)/(2\kappa + m^2 - \omega_0^2) < 0$.

Now we study the asymptotic behavior of $\tilde{D}(\omega)$ and $\tilde{N}(\omega) = (\tilde{D}(\omega))^{-1}$ near the points $\omega \in \Lambda_0$. In the neighborhood of the points $\omega = \pm\sqrt{4\nu^2+m^2}$ we use the representation (A.4) and obtain

$$\tilde{D}(\omega) = \kappa - 2\nu^2 \mp i\gamma\sqrt{4\nu^2+m^2} - i\nu(4\nu^2+m^2-\omega^2)^{1/2} + \ldots \tag{A.8}$$

as $\omega \to \pm\sqrt{4\nu^2+m^2}$, $\omega \in \mathbb{C}_+$. Therefore, if $\gamma \neq 0$ or $\gamma = 0$ and $\kappa \neq 2\nu^2$, then

$$\tilde{N}(\omega) = (\tilde{D}(\omega))^{-1} = C_1 + i\, C_2\sqrt{4\nu^2+m^2-\omega^2} + \ldots, \quad \omega \to \pm\sqrt{4\nu^2+m^2}, \quad \omega \in \mathbb{C}_+,$$

where $C_1 = (\kappa - 2\nu^2 \mp i\gamma\sqrt{4\nu^2+m^2})^{-1}$ and $C_2 = \nu C_1^2$. If $\gamma = 0$ and $\kappa = 2\nu^2$, then

$$(\tilde{D}(\omega))^{-1} = \frac{i}{\nu}(4\nu^2+m^2-\omega^2)^{-1/2} + \frac{1}{2\nu^2} + \ldots, \quad \omega \to \pm\sqrt{4\nu^2+m^2}.$$

In the neighborhood of the points $\omega = \pm m$ we apply (A.3) (if $m \neq 0$) and obtain

$$\tilde{D}(\omega) = \kappa \mp im\gamma - i\nu\sqrt{\omega^2-m^2} - i(\omega \mp m)\gamma - \frac{1}{2}(\omega^2-m^2) + \ldots, \ \omega \to \pm m, \ \omega \in \mathbb{C}_+. \tag{A.9}$$

In the case when $m = 0$, (A.5) yields

$$\tilde{D}(\omega) = \kappa - i\omega(\gamma+\nu) - \frac{1}{2}\omega^2 + \frac{i}{8\nu}\omega^3 + .., \quad \omega \to 0. \tag{A.10}$$

Suppose that $m\gamma \neq 0$ or $\kappa \neq 0$. Then, by virtue of (A.9) and (A.10), we obtain

$$\tilde{N}(\omega) = (\tilde{D}(\omega))^{-1} = \begin{cases} 1/\kappa + i\,\omega(\gamma+\nu)/\kappa^2 + \ldots, & \omega \to 0, \quad m = 0, \\ C_3 + i\, C_4(\omega^2-m^2)^{1/2} + \ldots, & \omega \to \pm m, \quad m \neq 0, \end{cases} \quad \omega \in \mathbb{C}_+,$$

where $C_3 = (\kappa \mp im\gamma)^{-1}$ and $C_4 = \nu C_3^2$. If $\gamma = \kappa = 0$ and $m \neq 0$, then

$$\tilde{N}(\omega) = \frac{i}{\nu}(\omega^2-m^2)^{-1/2} - \frac{1}{2\nu^2} + \ldots, \quad \omega \to \pm m, \quad \omega \in \mathbb{C}_+.$$

If $\kappa = m = 0$, then $\tilde{N}(\omega) = i\omega^{-1}/(\gamma+\nu) - 1/(2(\gamma+\nu)^2) + \ldots$ as $\omega \to 0$.

Since $\tilde{N}(\omega) = (\overline{\tilde{D}(\bar{\omega})} - 2i\omega\gamma)^{-1}$ for $\omega \in \mathbb{C}_-$, then the expansion for $\tilde{N}(\omega)$ as $\omega \to \omega_0$ ($\omega_0 \in \Lambda_0$, $\omega \in \mathbb{C}_-$) can be constructed using (A.8) and (A.9). In particular,

$$\tilde{N}(\omega + i0) - \tilde{N}(\omega - i0) = O(|\omega^2 - \omega_0^2|^{j/2}) \quad \text{as } \omega \to \omega_0, \ \omega_0 \in \Lambda_0, \tag{A.11}$$

where $j = 1$ if the condition **C** is satisfied, and $j = -1$ if the condition $\mathbf{C}_0$ is satisfied.

Proof of Theorem 3.4 Using Lemma A.1, we vary the integration contour in the right hand side of (3.13):

$$N(t) = -\frac{1}{2\pi}\int_{|\omega|=R} e^{-i\omega t}\tilde N(\omega)\,d\omega, \quad t>0, \tag{A.12}$$

where R is chosen enough large such that $\tilde N(\omega)$ has no poles in the region $\mathbb{C}_-\cap\{|\omega|\ge R\}$. Note that if $\gamma=0$, then $\tilde N(\omega)$ has no poles in $\mathbb{C}_-$ by Lemma A.1 (iv). Denote by σ_j the poles of $\tilde N(\omega)$ in $\mathbb{C}_-$ (if they exist). By Lemmas A.1 and A.2, there exists a $\delta>0$ such that $\tilde N(\omega)$ has no poles in the region $\Im\omega\in[-\delta,0)$. Hence, we can rewrite $N(t)$ as

$$N(t) = -i\sum_{j=1}^{K}\mathrm{Res}_{\omega=\sigma_j}\left[e^{-i\omega t}\tilde N(\omega)\right] - \frac{1}{2\pi}\int_{\Lambda_\varepsilon} e^{-i\omega t}\tilde N(\omega)\,d\omega, \quad t>0,$$

where $\varepsilon\in(0,\delta)$, the contour Λ_ε surrounds segments of Λ and belongs to an ε-neighborhood of Λ (Λ_ε is oriented anticlockwise). Passing to a limit as $\varepsilon\to 0$, we obtain

$$\begin{aligned} N(t) &= \frac{1}{2\pi}\int_{\Lambda} e^{-i\omega t}\Big(\tilde N(\omega+i0)-\tilde N(\omega-i0)\Big)\,d\omega + o(t^{-N}) \\ &= \sum_{\pm}\sum_{j=1}^{2}\frac{1}{2\pi}\int_{\Lambda} e^{-i\omega t}P_j^{\pm}(\omega)\,d\omega + o(t^{-N}), \quad t\to+\infty, \quad \text{with any } N>0. \end{aligned}$$

Here $P_j^{\pm}(\omega):=\zeta_j^{\pm}(\omega)(\tilde N(\omega+i0)-\tilde N(\omega-i0))$, $j=1,2$, where $\zeta_j^{\pm}(\omega)$ are smooth functions such that $\sum\limits_{\pm,j}\zeta_j^{\pm}(\omega)=1$, $\omega\in\mathbb{R}$, $\mathrm{supp}\,\zeta_1^{\pm}\subset\mathcal{O}(\pm m)$, $\mathrm{supp}\,\zeta_2^{\pm}\subset\mathcal{O}(\pm\sqrt{4\nu^2+m^2})$ ($\mathcal{O}(a)$ denotes a neighborhood of the point $\omega=a$). In the case $m=0$, instead of $\zeta_1^{\pm}$ ($P_1^{\pm}$) we introduce the function ζ_1 (respectively, P_1) with $\mathrm{supp}\,\zeta_1\subset\mathcal{O}(0)$. Then, (A.11) implies the bound (3.14) with $k=0$. Here we use the following estimate (with $j=\pm1$)

$$\Big|\int_{\mathbb{R}}\zeta(\omega)e^{-i\omega t}(a^2-\omega^2)^{j/2}\,d\omega\Big| \le C(1+t)^{-1-j/2} \quad \text{as } t\to+\infty, \quad j \text{ is odd}, \tag{A.13}$$

where $\zeta(\omega)$ is a smooth function, and $\zeta(\omega)=1$ for $|\omega-a|\le\delta$ with some $\delta>0$ (see, for example, [11, Lemma 2]). The bound (3.14) with $k=1,2$ can be proved by a similar way. ■

Remark A.4. If conditions **C** and $\mathbf{C}_0$ are not fulfilled, then $N(t)$ does not decay as $t\to\infty$. For example, if $\kappa=m=0$, then $\tilde N(\omega)$ has a simple pole at zero. Calculating the residue of $\tilde N(\omega)$ at the point $\omega=0$, we obtain $N(t)=(\gamma+\nu)^{-1}+O(t^{-3/2})$, $t\to\infty$.

If $\gamma=0$ and $\kappa>2\nu^2$, then there exists a number $\omega_0>\sqrt{4\nu^2+m^2}$ such that $\tilde N(\omega)$ has simple poles at the points $\omega=\pm\omega_0$. Calculating the residue of $e^{-i\omega t}\tilde N(\omega)$ at these points, we obtain $N(t)\sim C\sin(\omega_0 t)+O(t^{-3/2})$ as $t\to\infty$.

Appendix B: Proof of Theorem 2.4

Consider the mixed initial-boundary value problem (2.3)–(2.5). Without loss of generality, we assume that $u_0(0) = v_0(0) = 0$. Write $Z(x,t) = (Z^0(x,t), Z^1(x,t)) \equiv (z(x,t), \dot z(x,t))$, $Y_0(x) = (u_0(x), v_0(x))$. The solution of problem (2.3)–(2.5) can be represented as the restriction of the solution to the Cauchy problem with odd initial data on the half-line,

$$Z^i(x,t) = \sum_{y\in\mathbb{Z}} \mathcal{G}_t^{ij}(x-y) Y^j_{\rm odd}(y), \quad x \ge 0, \quad i = 0,1, \tag{B.1}$$

where $\mathcal{G}_t(x)$ is defined in (2.12) and (2.13), and, by definition,

$$Y_{\rm odd}(x) = Y_0(x) \quad \text{for } x > 0, \quad Y_{\rm odd}(0) = 0, \quad Y_{\rm odd}(x) = -Y_0(-x) \quad \text{for } x < 0. \tag{B.2}$$

To prove Theorem 2.3 we first consider the following Cauchy problem for the discrete Klein–Gordon equation in the whole line,

$$\begin{cases} \ddot u(x,t) = (\nu^2\Delta_L - m^2) u(x,t), & t \in \mathbb{R}, \quad x \in \mathbb{Z}, \\ u(x,t)|_{t=0} = u_0(x), & \dot u(x,t)|_{t=0} = v_0(x). \end{cases} \tag{B.3}$$

By $\ell^2_\alpha \equiv \ell^2_\alpha(\mathbb{Z})$, $\alpha \in \mathbb{R}$, we denote the Hilbert space of sequences with the norm $\|u\|^2_\alpha = \sum\limits_{x\in\mathbb{Z}} |u(x)|^2 \langle x\rangle^{2\alpha} < \infty$. Let $\mathcal{H}_\alpha := \ell^2_\alpha \otimes \ell^2_\alpha$ be the Hilbert space of pairs $Y = (u,v)$ with the norm $\|Y\|^2_\alpha = \sum\limits_{x\in\mathbb{Z}} \langle x\rangle^{2\alpha}(|u(x)|^2 + |v(x)|^2) < \infty$.

It is well-known (see for instance, [3]), that for any $Z_0 \equiv (u_0, v_0) \in \mathcal{H}_\alpha$, there exists a unique solution $W(t)Z_0 \in C(\mathbb{R}, \mathcal{H}_\alpha)$ to the problem (B.3). Moreover, there exist constants $C, \sigma = \sigma(\alpha) < \infty$ such that the following bound holds,

$$\|W(t)Z_0\|_\alpha \le C\langle t\rangle^\sigma \|Z_0\|_\alpha, \quad t \in \mathbb{R}, \quad \alpha \in \mathbb{R}. \tag{B.4}$$

Lemma B.1. *Let $Z_0 \equiv (u_0, v_0) \in \mathcal{H}_\alpha$ with $\alpha > 5/2$. If $\hat Z_0(0) = \hat Z_0(\pi) = 0$, then*

$$\|W(t)Z_0\|_{-\alpha} \le C\langle t\rangle^{-3/2} \|Z_0\|_\alpha, \quad t \in \mathbb{R}. \tag{B.5}$$

Otherwise, $\|W(t)Z_0\|_{-\alpha} \le C\langle t\rangle^{-1/2}\|Z_0\|_\alpha$, $t \in \mathbb{R}$.

Below we outline the proof of this lemma.

By the bound (B.4), the Laplace–Fourier transform of the solution $u(x,t)$ with respect to t-variable exists at least for $\Im\omega > 0$ and satisfies equation (3.2) for $x \in \mathbb{Z}$, $\Im\omega > 0$. Let u be a solution of the equation $(-\nu^2\Delta_L + m^2 - \omega^2)u = f$ with $f \in \ell^2$. Define the resolvent operator R_ω as $u = R_\omega f = (-\nu^2\Delta_L + m^2 - \omega^2)^{-1} f$.

Applying the inverse Fourier–Laplace transform with respect to ω-variable, we write the solution $u(x,t)$ of the problem (B.3) in the form

$$u(x,t) = \frac{1}{2\pi}\int_{\Im\omega=\mu} e^{-i\omega t} R_\omega(v_0(x) - i\omega u_0(x))\, d\omega, \quad x \in \mathbb{Z}, \quad t > 0, \quad \mu > 0. \tag{B.6}$$

To derive the asymptotic behavior of $u(x,t)$, we first study the properties of the operator R_ω for $\omega \in \mathbb{C}$, see [6, 10, 8]. To formulate them, we denote by $B(\alpha,\alpha') = \mathcal{L}(\ell^2_\alpha, \ell^2_{\alpha'})$ the space of bounded linear operators from ℓ^2_α to $\ell^2_{-\alpha}$.

I. For $\omega\in\mathbb{C}\setminus\Lambda$, the resolvent R_ω is the integral operator with the kernel $R_\omega(x,y)$, $x,y\in\mathbb{Z}$, and by the Cauchy Residue Theorem, we have

$$R_\omega(x,y)=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{e^{-i\theta(x-y)}}{\nu^2(2-2\cos\theta)+m^2-\omega^2}\,d\theta=i\frac{e^{i\theta(\omega)|x-y|}}{2\nu^2\sin(\theta(\omega))},\quad \omega\in\mathbb{C}\setminus\Lambda, \tag{B.7}$$

where $\theta(\omega)$ is defined in Lemma 3.2. Therefore, for $\omega\in\mathbb{C}\setminus\Lambda$, the resolvent R_ω is an analytic operator-valued function in the complex ω-plane with the cut along the intervals in Λ. Moreover, the sequence $\{e^{-i\theta(\omega)|x|}\}$, $x\in\mathbb{Z}$, is exponentially decaying as $|x|\to\infty$. Hence for $\omega\in\mathbb{C}\setminus\Lambda$, R_ω is a bounded operator in $\ell^2(\mathbb{Z})$.

II. Write $\theta(\omega\pm i0):=\lim\limits_{\varepsilon\to+0}\theta(\omega\pm i\varepsilon)$. For $\omega\in\Lambda\setminus\Lambda_0$ and $x,y\in\mathbb{Z}$, the following pointwise limit exists $R_{\omega\pm i\varepsilon}(x,y)\to R_{\omega\pm i0}(x,y)$ as $\varepsilon\to+0$. Moreover, $|\theta(\omega\pm i\varepsilon)|\le C(\omega)$ and $|\sin\theta(\omega\pm i\varepsilon)|>0$ for $\omega\in\Lambda\setminus\Lambda_0$. Hence, $|R_{\omega\pm i\varepsilon}(x,y)|\le C(\omega)$ for $\omega\in\Lambda\setminus\Lambda_0$. Therefore, for any $\alpha>1/2$ and $\omega\notin\Lambda_0$, we have

$$\sum_{x,y\in\mathbb{Z}}|R_{\omega\pm i\varepsilon}(x,y)-R_{\omega\pm i0}(x,y)|^2\,\langle x\rangle^{-2\alpha}\langle y\rangle^{-2\alpha}\to 0,\quad \varepsilon\to+0,$$

by the Lebesgue dominated convergence theorem. Thus, for $\omega\in\Lambda\setminus\Lambda_0$, the resolvent $R_{\omega\pm i\varepsilon}$ converges to $R_{\omega\pm i0}$ ($\varepsilon\to+0$) as Hilbert–Schmidt operator in the space $B(\alpha,-\alpha)$, $\alpha>1/2$. Moreover, $\overline{\theta(\omega)}=-\theta(\bar\omega)$ for $\omega\in\mathbb{C}\setminus\Lambda$. Hence, $R_{\omega-i0}(x,y)=\overline{R_{\omega+i0}(x,y)}$ for $\omega\in\Lambda\setminus\Lambda_0$, $x,y\in\mathbb{Z}$.

III. The operator $R_{\omega\pm i0}$ diverges near points $\omega\in\Lambda_0$ because $\sin\theta(\omega+i0)$ vanishes in these points. Using formula (B.7) and decompositions (A.3)–(A.5), we obtain a formal Puiseux expansion of R_ω as $\omega\to\omega_0$, $\omega\in\mathbb{C}\setminus\Lambda$, $\omega_0\in\Lambda_0$. Indeed, for $\omega\to\pm m$ ($m\neq0$, $\omega\in\mathbb{C}_+$), we have

$$R_\omega(x,y)=\frac{i}{2\nu}(\omega^2-m^2)^{-1/2}-\frac{1}{2\nu^2}|x-y|-\frac{i}{16\nu^3}(4|x-y|^2-1)(\omega^2-m^2)^{1/2}+\dots, \tag{B.8}$$

where $\Im\sqrt{\omega^2-m^2}>0$. In particular, if $m=0$, then

$$R_\omega(x,y)=\frac{i}{2\nu\omega}-\frac{1}{2\nu^2}|x-y|-\frac{i\omega}{16\nu^3}(4|x-y|^2-1)+\dots,\quad \omega\to0.$$

For $\omega\to\pm\sqrt{4\nu^2+m^2}$, $\omega\in\mathbb{C}_+$,

$$\begin{aligned}R_\omega(x,y)\;&=\;(-1)^{|x-y|}\Big(\frac{i}{2\nu}(4\nu^2+m^2-\omega^2)^{-1/2}+\frac{1}{2\nu^2}|x-y|\\ &\quad-\frac{i}{16\nu^3}(4|x-y|^2-1)\sqrt{4\nu^2+m^2-\omega^2}+\dots\Big).\end{aligned} \tag{B.9}$$

Since $\sum\limits_{x,y\in\mathbb{Z}}\langle x\rangle^{-2\alpha}|x-y|^{2p}\langle y\rangle^{-2\alpha}<\infty$ for $\alpha>\frac12+p$, with any $p=0,1,2,\dots$,

$$\||x-y|^pf(y)\|_{-\alpha}\le C\|f\|_\alpha,\quad f\in\ell^2_\alpha,\quad \alpha>\frac12+p,\quad p=0,1,2,\dots. \tag{B.10}$$

Applying these estimates to the terms in the expansions (B.8) and (B.9), we come to the following result.

Lemma B.2. *(see [8, Lemma 3.2]) Let* $f\in\ell^2_\alpha$, $\alpha>5/2$. *Then for* $\omega\to\pm m$, $\omega\in\mathbb{C}\setminus\Lambda$, *we have*

$$(R_\omega f)(x)=\frac{i\hat f(0)}{2\nu\sqrt{\omega^2-m^2}}-\frac{1}{2\nu^2}\sum_{y\in\mathbb{Z}}|x-y|f(y)+\sqrt{\omega^2-m^2}\;r^1_\omega f,$$

and for $\omega\to\pm\sqrt{4\nu^2+m^2}$, $\omega\in\mathbb{C}\setminus\Lambda$,

$$(R_\omega f)(x)=\frac{i(-1)^x\hat f(\pi)}{2\nu\sqrt{4\nu^2+m^2-\omega^2}}+\frac{1}{2\nu^2}\sum_{y\in\mathbb{Z}}(-1)^{|x-y|}|x-y|f(y)+\sqrt{4\nu^2+m^2-\omega^2}\;r^2_\omega f,$$

where the remainder terms have the form $r^j_\omega f=\sum_{k=0}^2 b^j_k(\omega)\sum_{y\in\mathbb{Z}}|x-y|^kf(y)$, $b^1_k(\omega)=O(1)$ *as* $\omega\to\pm m$ *and* $b^2_k(\omega)=O(1)$ *as* $\omega\to\pm\sqrt{4\nu^2+m^2}$. *In particular,* $\|r^j_\omega f\|_{-\alpha}\le C\|f\|_\alpha$.

Now Lemma B.1 follows from the equality (B.6) and Lemma B.2, using arguments similar to the proof Theorem 3.4 and technique of the paper [8].

Proof of the bound (2.14). Using the representation (B.1) and formula (B.6), we rewrite the solution of the problem (2.3)–(2.5) in the form

$$z(x,t)=\frac{1}{2\pi}\int_{\Im\omega=\mu}e^{-i\omega t}R_\omega f_{\rm odd}\,d\omega,\quad x\in\mathbb{Z}_+,\quad t>0,\quad \mu>0,$$

where $f_{\rm odd}(x):=v_{\rm odd}(x)-i\omega\,u_{\rm odd}(x)$ (see (B.2)). Applying arguments similar to the proof of Theorem 3.4, we obtain

$$z(x,t)=\frac{1}{2\pi}\int_\Lambda e^{-i\omega t}\Big(R_{\omega+i0}-R_{\omega+i0}\Big)f_{\rm odd}\,d\omega=\frac{1}{\pi}\int_\Lambda e^{-i\omega t}\,\Im\Big(R_{\omega+i0}f_{\rm odd}\Big)\,d\omega.\tag{B.11}$$

Let $Y_0\in\mathcal{H}_{\alpha,+}$ with $\alpha>3/2$. Then, $Y_{\rm odd}\in\mathcal{H}_\alpha$ and $\hat f_{\rm odd}(0)=\hat f_{\rm odd}(\pi)=0$. We want to apply Lemma B.2 to the function $f_{\rm odd}(x)$, but with $\alpha>3/2$ instead of $\alpha>5/2$, using the oddness of $f_{\rm odd}(x)$. Note that for $k=1,2$,

$$\Big|\sum_{y\in\mathbb{Z}}|x-y|^kf_{\rm odd}(y)\Big|\le 2|x|\sum_{y\ge1}y|f_0(y)|,$$

where $f_0(x):=v_0(x)-i\omega\,u_0(x)$, $x\in\mathbb{Z}_+$. Therefore, applying the Cauchy–Bunyakovskii inequality, we obtain for $\alpha>3/2$,

$$\begin{aligned}\|r^j_\omega f_{\rm odd}\|_{-\alpha,+}\;&\le\;C\sum_{k=0}^2\Big\|\sum_{y\in\mathbb{Z}}|x-y|^kf_{\rm odd}(y)\Big\|_{-\alpha,+}\le C_1\sqrt{\sum_{x\in\mathbb{Z}_+}\langle x\rangle^{-2\alpha}x^2\Big(\sum_{y\in\mathbb{Z}_+}y|f_0(y)|\Big)^2}\\&\le\;C_2\sum_{y\in\mathbb{Z}_+}\langle y\rangle^{-\alpha}|y|\cdot\langle y\rangle^{\alpha}|f_0(y)|\le C\|f_0\|_{\alpha,+}.\end{aligned}$$

Thus, in the neighborhood of the singular points $\omega_0\in\Lambda_0$ the following estimate holds

$$\|\Im R_{\omega+i0}f_{\rm odd}\|_{-\alpha,+}\le C|\omega^2-\omega_0^2|^{1/2}\|f_0\|_{\alpha,+},\quad \omega\to\omega_0,\tag{B.12}$$

where $\alpha>3/2$, $\omega_0\in\Lambda_0$, $\omega\in\mathbb{R}$. Now the estimate (2.14) follows from the equality (B.11), estimate (B.12) and Lemma 10.2 from [7], which is a generalization of the estimate (A.13). ■

References

[1] *Cuccagna S.* L^p continuity of wave operators in $\mathbb{Z}$ // J. Math. Anal. Appl. 2009. V. 354. P. 594–605.

[2] *Dudnikova T.V., Komech A.I., Spohn H.* On the convergence to statistical equilibrium for harmonic crystals // J. Math. Phys. 2003. V. 44. P. 2596–2620.

[3] *Dudnikova T.V.* On the asymptotical normality of statistical solutions for harmonic crystals in half-space // Russian J. Math. Phys. 2008. V. 15, N 4. P. 460–472.

[4] *Dudnikova T.V.* Long-time asymptotics of solutions to a Hamiltonian system on a lattice // Problems in Mathematical Analysis 2016. V. 85. P. 69–82.

[5] *Dudnikova T.V.* On convergence to equilibrium for one-dimensional chain of harmonic oscillators on the half-line // J. Math. Phys. 2017. V. 58, N 4. P. 043301.

[6] *Islami H., Vainberg B.* Large time behavoir of the solutions to difference wave operators // Commun. in Partial Dif. Eq. 2006. V. 31, N 3. P. 397–416.

[7] *Jensen A., Kato T.* Spectral properties of Schrodinger operators and time-decay of the wave functions // Duke Math. J. 1979. V.46. P. 583–611.

[8] *Komech A.I., Kopylova E.A., Kunze M.* Dispersive estimates for 1D discrete Schrodinger and Klein–Gordon equations // Applicable Anal. 2006. V. 85, N 12. P. 1487–1508.

[9] *Pelinosky D.E., Stefanov A.* On the spectral theory and dispersive estimates for a discrete Schrödinger equation in one dimension // J. Math. Phys. 2008. V. 49, N 11. P. 113501.

[10] *Shaban W., Vainberg B.R.* Radiation conditions for the difference Schrödinger operators // Applicable Anal. 2001. V. 80. P. 525–556.

[11] *Vainberg B.R.* Behavior for large time of solutions of the Klein–Gordon equation // Trudi Mosk. Mat. Obs. 1974. V. 30. P. 139–158.