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# ALGEBRAIC APPROXIMATIONS OF COMPACT KÄHLER THREEFOLDS OF KODAIRA DIMENSION 0 OR 1

by

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**Résumé.** — We prove that every compact Kähler threefold  $X$  of Kodaira dimension  $\kappa = 0$  or  $1$  has a  $\mathbf{Q}$ -factorial bimeromorphic model  $X'$  with at worst terminal singularities such that for each curve  $C \subset X'$ , the pair  $(X', C)$  admits a locally trivial algebraic approximation such that the restriction of the deformation of  $X'$  to some neighborhood of  $C$  is a trivial deformation. As an application, we prove that every compact Kähler threefold with  $\kappa = 0$  or  $1$  has an algebraic approximation. We also point out that in order to prove the existence of algebraic approximations of a compact Kähler threefold with  $\kappa = 2$ , it suffices to prove that of an elliptic fibration over a surface.

## 1 Introduction

From the point of view of the Hodge theory, compact Kähler manifolds can be considered as a natural generalization of smooth complex projective varieties. While an arbitrarily small deformation as a complex variety of a smooth complex projective variety might no longer be projective, a sufficiently small deformation of a Kähler manifold remains Kähler. The so-called Kodaira problem asks whether it is possible to obtain all compact Kähler manifolds through (arbitrarily small) deformations of projective varieties.

**Problem 1.1 (Kodaira problem).** — *Given a compact Kähler manifold  $X$ , does  $X$  always admit an (arbitrarily small) deformation to some projective variety?*

In dimension 1, compact complex curves are already projective. For surfaces, Problem 1.1 is known to have a positive answer, first due to Kodaira using the classification of compact complex surfaces [13], then to N. Buchdahl [2] proving that any compact Kähler surface has an algebraic approximation using M. Green's density criterion (cf. Theorem 4.3). We refer to [6, 9, 16, 7, 19] for other positive results.

As for negative answers, C. Voisin constructed in each dimension  $\geq 4$  examples of compact Kähler manifolds which do not have the homotopy type of a smooth projective variety [20], thus answered in particular negatively the Kodaira problem. Later on, she constructed in each even dimension  $\geq 8$  examples of compact Kähler manifolds all of whose smooth bimeromorphic models are homotopically obstructed to being a projective variety [21].

For threefolds, the Kodaira problem remains open at present. There are nevertheless positive results concerning a bimeromorphic variant of the Kodaira problem.

**Theorem 1.2** ( $\kappa = 0$  : [9],  $\kappa = 1$  : [16]). — *Let  $X$  be a compact Kähler threefold of Kodaira dimension  $\kappa = 0$  or  $1$ . There exists a  $\mathbf{Q}$ -factorial bimeromorphic model  $X'$  of  $X$  with at worst terminal singularities such that  $X'$  has a locally trivial algebraic approximation.*

In order to prove Theorem 1.2, thanks to the minimal model program (MMP) for Kähler threefolds [10], we can choose  $X'$  to be a minimal model of  $X$ , and this is what we did in most of the cases. Geometric descriptions of these varieties  $X'$  can be obtained as an output of the abundance conjecture [5] applied to  $X'$ , which is enough to prove the existence of a locally trivial algebraic approximation for  $X'$ .

The aim of this article is to prove the following stronger version of Theorem 1.2 by further exploiting the geometry of  $X'$ . We refer to Section 2.1 for the terminologies used in the statement of Theorem 1.3.

**Theorem 1.3.** — *Let  $X$  be a compact Kähler threefold of Kodaira dimension  $\kappa = 0$  or  $1$ . There exists a  $\mathbf{Q}$ -factorial bimeromorphic model  $X'$  with at worst terminal singularities such that whenever  $C \subset X'$  is a curve or empty, the pair  $(X', C)$  has a locally trivial and  $C$ -locally trivial algebraic approximation.*

We will also prove a result relating the type of algebraic approximation that  $X'$  has in Theorem 1.3 and the algebraic approximation of  $X$ .

**Proposition 1.4.** — *Let  $X$  be a compact Kähler threefold and  $X'$  a normal bimeromorphic model of  $X$ . If  $(X', C)$  has a locally trivial and  $C$ -locally trivial algebraic approximation whenever  $C \subset X'$  is a curve or empty, then  $X$  has an algebraic approximation.*

We refer to Corollary 2.4 for a more general statement. Since  $\mathbf{Q}$ -factorial varieties are normal by definition, putting Proposition 1.4 together with Theorem 1.3 yields immediately the following result.

**Theorem 1.5.** — *Every compact Kähler threefold of Kodaira dimension 0 or 1 has an algebraic approximation.*

As for threefolds of Kodaira dimension 2, since minimal models of such varieties are elliptic fibrations, the existence of algebraic approximations of these varieties is related to the following question.

**Question 1.6.** — *Let  $f : Y \rightarrow B$  be an elliptic fibration where  $Y$  is a compact Kähler and the base  $B$  is smooth and projective. Assume that the locus  $D \subset B$  parameterizing singular fibers of  $f$  is normal crossing, does  $Y$  have an algebraic approximation?*

We will see that a positive solution of Question 1.6 will eventually solve the Kodaira problem for threefolds of Kodaira dimension 2.

**Proposition 1.7.** — *If Question 1.6 has a positive answer in the case where  $B$  is a surface, then every compact Kähler threefold of Kodaira dimension 2 has an algebraic approximation.*

In view of [7, Theorem 1.1] and [16, Theorem 1.6], it is plausible that Question 1.6 would have a positive answer. It is a work in progress of Claudon and Höring toward an answer to Question 1.6.

The article is organized as follows. We will first introduce in Section 2 some deformation-theoretic terminologies including those appearing in Theorem 1.3 then prove some general results. In particular, we will prove Corollary 2.4 and deduce Proposition 1.4 from it. Next, we will turn to describing minimal models of a compact Kähler threefold of Kodaira dimension 0 or 1 in Section 3. According to these descriptions, we will choose some threefolds  $X$  and prove in Section 4 that whenever  $C \subset X$  is a curve or empty, the pair  $(X, C)$  always has a  $C$ -locally trivial algebraic approximation. Based on these results, the proof of Theorem 1.3 will be concluded in Section 5, where we also prove Proposition 1.7.

## 2 Deformations

### 2.1 Terminologies

Let  $X$  be a complex variety. A *deformation* of  $X$  is a surjective flat holomorphic map  $\pi : \mathcal{X} \rightarrow \Delta$  containing  $X$  as a fiber. We say that a deformation  $\pi : \mathcal{X} \rightarrow \Delta$  is *locally trivial* if for every  $x \in \mathcal{X}$ , there exists a neighborhood  $x \in \mathcal{U} \subset \mathcal{X}$  of  $x$  such that if  $U := \pi^{-1}(\pi(x)) \cap \mathcal{U}$ , then  $\mathcal{U}$  is isomorphic to  $U \times \pi(\mathcal{U})$  over  $\pi(\mathcal{U})$ .

In this article, a *fibration* is a surjective holomorphic map  $f : X \rightarrow B$  with connected fibers. A deformation  $\pi : \mathcal{X} \rightarrow \Delta$  of  $X$  is called *strongly locally trivial* with respect to the fibration structure  $f : X \rightarrow B$  if  $\pi$  has a factorization of the form

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{q} & \Delta \times B \\ \pi \downarrow & \swarrow \text{pr}_1 & \\ \Delta & & \end{array}$$

such that the restriction of  $q$  to  $X$  onto its image coincides with  $f$ , and that for every  $(t, b) \in \Delta \times B$ , there exist neighborhoods  $b \in U \subset B$  and  $t \in V \subset \Delta$  such that  $q^{-1}(V \times U)$  is isomorphic to  $q^{-1}(\{t\} \times U) \times V$  over  $V$ .

Let  $X$  be a complex variety and  $C \subset X$  a subvariety of  $X$ . A *C-locally trivial deformation* of  $(X, C)$  is a deformation  $(\mathcal{X}, \mathcal{C}) \rightarrow \Delta$  of the pair  $(X, C)$  such that the deformation  $(\mathcal{U}, \mathcal{C}) \rightarrow \Delta$  restricted to some neighborhood  $\mathcal{U} \subset \mathcal{X}$  of  $\mathcal{C}$  is isomorphic to the trivial deformation  $(U \times \Delta, C \times \Delta) \rightarrow \Delta$  with  $U := \mathcal{U} \cap X$ . An *algebraic approximation* of the pair  $(X, C)$  is a deformation  $(\mathcal{X}, \mathcal{C}) \rightarrow \Delta$  of  $(X, C)$  such that there exists a sequence of points  $(t_i)_{i \in \mathbb{N}}$  in  $\Delta$  parameterizing algebraic members and converging to  $o$ , the point which parameterizes  $(X, C)$ .

If  $X$  is endowed with a  $G$ -action where  $G$  is a group and  $C$  is a  $G$ -invariant subvariety, then a *G-equivariant deformation of the pair  $(X, C)$*  is a deformation  $(\mathcal{X}, \mathcal{C}) \rightarrow \Delta$  of  $(X, C)$  such that the  $G$ -action on  $X$  extends to an action on  $\mathcal{X}$  preserving each fiber of  $\mathcal{X} \rightarrow \Delta$  and  $\mathcal{C}$ .

### 2.2 Locally trivial deformations and bimeromorphic transformations

The following lemma concerns the behaviour of  $C$ -locally trivial deformations of a pair  $(X, C)$  under bimeromorphic transformations.

**Lemma 2.1.** — *Let  $f : X \rightarrow Y$  be a map between complex varieties and assume that there exists a subvariety  $C \subset Y$  such that  $f$  maps  $X \setminus D$  isomorphically onto  $Y \setminus C$  where  $D := f^{-1}(C)$ . Then for every  $C$ -locally trivial deformation  $\pi : (\mathcal{Y}, \mathcal{C}) \rightarrow \Delta$  of  $Y$ , there exists a  $D$ -locally trivial deformation  $(\mathcal{X}, \mathcal{D}) \rightarrow \Delta$  of the pair  $(X, D)$  together with a map  $F : \mathcal{X} \rightarrow \mathcal{Y}$  over  $\Delta$  such that  $F^{-1}(\mathcal{C}) = \mathcal{D}$  and that  $F|_{\mathcal{X} \setminus \mathcal{D}}$  is an isomorphism onto  $\mathcal{Y} \setminus \mathcal{C}$ .*

*Proof.* — Let  $\mathcal{U} \subset \mathcal{Y}$  be a neighborhood of  $\mathcal{C}$  such that there exists an isomorphism over  $\Delta$  of the pairs

$$(\mathcal{U}, \mathcal{C}) \simeq (U \times \Delta, C \times \Delta) \tag{2.1}$$

where  $U := Y \cap \mathcal{U}$ . So we can write

$$\mathcal{Y} \simeq ((\mathcal{Y} \setminus \mathcal{C}) \sqcup (U \times \Delta)) / \sim$$

where  $\sim$  glues the two pieces of the union using isomorphism (2.1). Isomorphism (2.1) also implies that since  $f$  maps  $X \setminus D$  isomorphically onto  $Y \setminus C$ , we have over  $\Delta$

$$\mathcal{U} \setminus \mathcal{C} \simeq f^{-1}(U \setminus C) \times \Delta. \tag{2.2}$$

We define

$$\mathcal{X} := \left( (\mathcal{Y} \setminus \mathcal{C}) \sqcup (f^{-1}(U) \times \Delta) \right) / \sim$$

and

$$\mathcal{D} := D \times \Delta \subset \mathcal{X}$$

where  $\sim$  glues the two pieces of the union using isomorphism (2.2). One easily checks that  $\mathcal{X}$  is Hausdorff so that  $\mathcal{X}$  is a complex variety. The map  $\pi' : \mathcal{Y} \setminus \mathcal{C} \rightarrow \Delta$  and the projection  $\pi'' : f^{-1}(U) \times \Delta \rightarrow \Delta$  give rise to a map

$$\pi_X : (\mathcal{X}, \mathcal{D}) \rightarrow \Delta$$

which, by construction, is a  $D$ -locally trivial deformation of the pair  $(X, D)$ . Finally the restriction of  $f$  to  $f^{-1}(U)$  defines an obvious map  $F : \mathcal{X} \rightarrow \mathcal{Y}$  satisfying the property that  $F^{-1}(\mathcal{C}) = \mathcal{D}$  and that  $F|_{\mathcal{X} \setminus \mathcal{D}} : \mathcal{X} \setminus \mathcal{D} \simeq \mathcal{Y} \setminus \mathcal{C}$ .  $\square$

**Remark 2.2.** — We can also show that given a  $D$ -locally trivial deformation  $(\mathcal{X}, \mathcal{D}) \rightarrow \Delta$  of the pair  $(X, D)$ , there exists a  $C$ -locally trivial deformation  $(\mathcal{X}, \mathcal{C}) \rightarrow \Delta$  of the pair  $(X, C)$  together with a map  $F : \mathcal{X} \rightarrow \mathcal{Y}$  over  $\Delta$  such that  $F^{-1}(\mathcal{C}) = \mathcal{D}$  and that  $F|_{\mathcal{X} \setminus \mathcal{D}}$  is an isomorphism onto  $\mathcal{Y} \setminus \mathcal{C}$ . This can be proven by exchanging the role of  $C$  and  $D$  in the proof of Lemma 2.1.

Let  $X$  be a compact Kähler manifold. Assume that  $X$  is bimeromorphic to a compact Kähler variety  $Y$ . After a sequence of blow-ups of  $X$  along smooth centers, we obtain a resolution

$$X \xleftarrow{\eta} Z \xrightarrow{\nu} Y \tag{2.3}$$

of the bimeromorphic map  $X \dashrightarrow Y$ . Let  $C \subset Y$  be the image of the exceptional set of  $\nu$ . The following lemma shows in particular that a  $C$ -locally trivial deformation of the pair  $(Y, C)$  always induces a deformation of  $X$ .

**Lemma 2.3.** — Suppose that  $\pi : (\mathcal{Y}, \mathcal{C}) \rightarrow \Delta$  is a  $C$ -locally trivial deformation of the pair  $(Y, C)$ . Then up to shrinking  $\Delta$ , the deformation  $\pi$  induces a deformation

$$\mathcal{X} \longleftarrow \mathcal{Z} \longrightarrow \mathcal{Y}$$

of (5.1).

*Proof.* — Since  $\nu$  maps  $\nu^{-1}(Y \setminus C)$  isomorphically onto  $Y \setminus C$  and since  $(\mathcal{Y}, \mathcal{C}) \rightarrow \Delta$  is a  $C$ -locally trivial deformation of the pair  $(Y, C)$ , by Lemma 2.1 there exists a deformation  $\mathcal{Z} \rightarrow \Delta$  of  $Z$  and a map  $F : \mathcal{Z} \rightarrow \mathcal{Y}$  over  $\Delta$  whose restriction to the central fiber is  $\nu : Z \rightarrow Y$ .

As  $\eta_* \mathcal{O}_Z \simeq \mathcal{O}_X$  and  $R^1 \eta_* \mathcal{O}_Z = 0$  since  $\eta$  is a composition of blow-ups along smooth centers, by [17, Theorem 2.1] the deformation  $\mathcal{Z} \rightarrow \Delta$  of  $Z$  induces a deformation  $\mathcal{X} \rightarrow \Delta$  of the morphism  $Z \rightarrow X$  over  $\Delta$  up to shrinking  $\Delta$ .  $\square$

The following is an immediate consequence of Lemma 2.3.

**Corollary 2.4.** — With the same notation as above, if  $Y$  has a  $C$ -locally trivial algebraic approximation, then  $X$  also has an algebraic approximation. In particular, if  $Y$  is normal and satisfies the property that for every subvariety  $C \subset Y$  whose irreducible components are all of codimension  $\geq 2$ , the pair  $(Y, C)$  has a  $C$ -locally trivial algebraic approximation, then  $X$  also has an algebraic approximation.

*Proof.* — Let  $\mathcal{Y} \rightarrow \Delta$  be a  $C$ -locally trivial algebraic approximation of  $Y$  and let

$$\mathcal{X} \longleftarrow \mathcal{Z} \longrightarrow \mathcal{Y}$$

be the induced deformation of (5.1) as in Lemma 2.3. Up to shrinking  $\Delta$  we can suppose that for each  $t \in \Delta$ , the fibers  $\mathcal{Z}_t \rightarrow \mathcal{X}_t$  and  $\mathcal{Z}_t \rightarrow \mathcal{Y}_t$  of the maps  $\mathcal{Z} \rightarrow \mathcal{X}$  and  $\mathcal{Z} \rightarrow \mathcal{Y}$  over  $t$  are both bimeromorphic. Therefore if over a point  $t \in \Delta$  the variety  $\mathcal{Y}_t$  is algebraic, then  $\mathcal{X}_t$  is also algebraic.

For the last statement of Corollary 2.4, the normality of  $Y$  implies that each irreducible component of the image in  $Y$  of the exceptional set  $E$  of  $\nu$  is of codimension  $\geq 2$ . Thus  $(Y, \nu(E))$  has a  $\nu(E)$ -locally trivial algebraic approximation by assumption. We conclude by the first part of Corollary 2.4 that  $X$  has an algebraic approximation.  $\square$

*Proof of Proposition 1.4.* — Assume that  $X'$  satisfies the hypothesis made in the proposition. Let  $C := (C_0 \sqcup C_1) \subset X'$  be a subvariety of dimension  $\leq 1$  where  $C_i$  denotes the union of the irreducible components of  $C$  of dimension  $i$ . Since  $\dim C_0 = 0$ , a locally trivial deformation of  $X'$  induces in particular a  $C_0$ -locally trivial deformation of  $(X', C_0)$ . Hence by assumption, the pair  $(X', C)$  has a  $C$ -locally trivial algebraic approximation. It follows from the second part of Corollary 2.4 that  $X$  has an algebraic approximation.  $\square$

### 2.3 $G$ -equivariant locally trivial deformations

The following lemma shows that given a  $G$ -equivariant  $C$ -locally trivial deformation  $(\mathcal{X}, \mathcal{C}) \rightarrow \Delta$  of  $(X, C)$ , there always exists a  $G$ -equivariant trivialization of some neighborhood of  $\mathcal{C}$ . This will imply that the quotient  $(\mathcal{X}/G, \mathcal{C}/G) \rightarrow \Delta$  is a  $C/G$ -locally trivial deformation of  $(X/G, C/G)$ .

**Lemma 2.5.** — *Let  $X$  be a smooth complex variety and  $G$  a finite group acting on  $X$ . Let  $C$  be a  $G$ -invariant subvariety of  $X$  and assume that there exists a  $G$ -equivariant deformation of  $\pi : \mathcal{X} \rightarrow \Delta$  of  $X$  over a one-dimensional base  $\Delta$ . Assume also that there exists an open subset  $\mathcal{V} \subset \mathcal{X}$  and an isomorphism  $\mathcal{V} \simeq V \times \Delta$  over  $\Delta$  where  $V := \mathcal{V} \cap X$  such that  $V$  contains  $C$  (this hypothesis holds for instance, when  $\pi$  induces a  $G$ -equivariant  $C$ -locally trivial deformation of  $(X, C)$ ), then up to shrinking  $\Delta$ , there exist  $\mathcal{C} \subset \mathcal{X}$ , a  $G$ -invariant neighborhood  $\mathcal{U}$  of  $\mathcal{C}$ , and a  $G$ -equivariant isomorphism*

$$(\mathcal{U}, \mathcal{C}) \simeq (U \times \Delta, C \times \Delta)$$

over  $\Delta$  where  $U := \mathcal{U} \cap X$ .

In particular,  $\pi : (\mathcal{X}, \mathcal{C}) \rightarrow \Delta$  is a  $G$ -equivariant  $C$ -locally trivial deformation of  $(X, C)$  and the quotient  $(\mathcal{X}/G, \mathcal{C}/G) \rightarrow \Delta$  is a locally trivial and  $C/G$ -locally trivial deformation of  $(X/G, C/G)$ .

Before proving Lemma 2.5, let us first prove a technical lemma.

**Lemma 2.6.** — *Let  $G$  be a finite group acting on a variety  $X$  and let  $\pi : \mathcal{X} \rightarrow \Delta$  be a  $G$ -equivariant deformation of  $X$  over a one-dimensional base. Let  $\mathcal{V} \subset \mathcal{X}$  be an open subset such that there exists an isomorphism  $\mathcal{V} \simeq V \times \Delta$  over  $\Delta$  where  $V := \mathcal{V} \cap X$ . Let*

$$\mathcal{V}^G := \bigcap_{g \in G} g(\mathcal{V}).$$

Then for every  $G$ -invariant relatively compact subset  $U \subset \mathcal{V}^G := \mathcal{V}^G \cap X$ , up to shrinking  $\Delta$  there exists a  $G$ -invariant subset  $\mathcal{U}$  of  $\mathcal{V}^G$  and a  $G$ -equivariant isomorphism  $\mathcal{U} \simeq U \times \Delta$  over  $\Delta$ .

*Proof.* — We may assume that  $\mathcal{V}^G \neq \emptyset$ . Since  $\mathcal{V}^G$  is open by finiteness of  $G$ , after shrinking  $\Delta$  we can also assume that the restriction of  $\pi$  to  $\mathcal{V}^G$  is surjective and that  $\Delta$  is isomorphic to the open unit disc

$B(0, 1) \subset \mathbf{C}$  such that 0 parameterizes the central fiber  $X$ . Fix a generator  $\frac{\partial}{\partial t}$  of the space of constant vector fields  $\Gamma(\Delta, T_\Delta)_{\text{const}} \simeq \mathbf{C}$  on  $\Delta$ . For  $z \in \mathbf{C}$ , let  $z\frac{\partial}{\partial t} \in \Gamma(\Delta, T_\Delta)_{\text{const}}$  denote the corresponding vector field.

By identifying  $\mathcal{V}^G$  with a subset of  $V \times \Delta$  through the isomorphism  $\mathcal{V} \simeq V \times \Delta$ , we can define the homomorphism of Lie algebras

$$\begin{aligned} \xi : \mathbf{C} &\rightarrow \Gamma(\mathcal{V}^G, T_{\mathcal{V}^G}) \\ z &\mapsto \sum_{g \in G} g^* \left( \chi(z)|_{\mathcal{V}^G} \right), \end{aligned} \quad (2.4)$$

where  $\chi(z)$  is the vector field on  $V \times \Delta$  which projects to  $z\frac{\partial}{\partial t}$  in  $\Delta$  and to 0 in  $V$ . By [12, Satz 3] (see also [9, Theorem 5.3]), there exists a local group action

$$\Phi : \Theta \rightarrow \mathcal{V}^G$$

of  $\mathbf{C}$  on  $\mathcal{V}^G$  inducing  $\xi$ , where  $\Theta \subset \mathbf{C} \times \mathcal{V}^G$  is a neighborhood of  $\{0\} \times \mathcal{V}^G$ . We recall that the meaning of a local group action is the following.

- i) For all  $x \in \mathcal{V}^G$ , the subset  $\Theta \cap (\mathbf{C} \times \{x\})$  is connected.
- ii)  $\Phi(0, \bullet)$  is the identity map on  $\mathcal{V}^G$ .
- iii)  $\Phi(gh, x) = \Phi(g, \Phi(h, x))$  whenever it is well-defined.
- iv) The morphism of Lie algebras  $\mathbf{C} \rightarrow \Gamma(\mathcal{V}^G, T_{\mathcal{V}^G})$  induced by  $\Phi$  coincides with  $\xi$ .

Since the vector field  $\xi(z)$  is  $G$ -invariant for all  $z \in \mathbf{C}$  by construction, the map  $\Phi$  is also  $G$ -equivariant (where  $G$  acts trivially on  $\mathbf{C}$ ). Also since  $G$  acts on  $\mathcal{V}^G \rightarrow \Delta$  in a fiber-preserving way, the projection of  $\xi(z)$  in  $\Gamma(\mathcal{V}^G, \pi^* T_\Delta)$  equals  $|G| \cdot p_2^* \left( z\frac{\partial}{\partial t} \right)$ . Hence if  $\Phi_\Delta$  denotes the local group action on  $\Delta$  defined by

$$\begin{aligned} \Phi_\Delta : (\text{Id}_\mathbf{C} \times \pi)(\Theta) &\rightarrow \Delta \\ (x, b) &\mapsto b + |G| \cdot x \end{aligned}$$

then we have the following commutative diagram.

$$\begin{array}{ccc} \Theta & \xrightarrow{\Phi} & \mathcal{V}^G \\ \downarrow & & \downarrow \pi \\ (\text{Id}_\mathbf{C} \times \pi)(\Theta) & \xrightarrow{\Phi_\Delta} & \Delta \end{array} \quad (2.5)$$

By the relative compactness of  $U$  inside  $V^G$ , there exists  $\varepsilon > 0$  such that

$$\mathfrak{U} := B(0, \varepsilon) \times U \subset \Theta.$$

The restriction of  $\Phi$  to  $\mathfrak{U}$  is isomorphic onto its image. We verify easily with the help of (2.5) and the properties ii) and iii) that the inverse of  $\Phi : \mathfrak{U} \rightarrow \Phi(\mathfrak{U})$  is

$$\begin{aligned} \Psi : \Phi(\mathfrak{U}) &\rightarrow \mathfrak{U} \\ v &\mapsto \left( \frac{\pi(v)}{|G|}, \Phi \left( -\frac{\pi(v)}{|G|}, v \right) \right). \end{aligned}$$

Let  $\mathcal{U} := \Phi\left(B\left(0, \frac{\varepsilon}{|\mathbf{G}|}\right) \times U\right) \subset \mathcal{V}^G$ . We have  $U := \mathcal{U} \cap X$  by ii) and up to replacing  $\Delta$  by  $B(0, \varepsilon)$ , we have thus by construction an isomorphism

$$U \times \Delta \xrightarrow{\sim} \mathcal{U}$$

$$(x, t) \mapsto \Phi\left(\frac{t}{|\mathbf{G}|}, x\right),$$

over  $\Delta$ , which is moreover  $G$ -equivariant since  $\Phi$  is  $G$ -equivariant.  $\square$

*Proof of Lemma 2.5.* — Since  $C$  is  $G$ -invariant and since the subset  $\mathcal{V}^G := \bigcap_{g \in G} g(\mathcal{V})$  is a finite intersection so is an open subset,  $V^G := \mathcal{V}^G \cap X$  is a  $G$ -invariant neighborhood of  $C$ . Let  $U \subset V^G$  be a  $G$ -invariant neighborhood of  $Y$  which is relatively compact in  $V^G$ . By applying Lemma 2.6 to  $\mathcal{V}$  and to  $U$ , we deduce that up to shrinking  $\Delta$ , there exists a  $G$ -invariant subset  $\mathcal{U} \subset \mathcal{V}^G$  together with a  $G$ -equivariant isomorphism

$$U \times \Delta \simeq \mathcal{U}$$

over  $\Delta$ . As  $C$  is a  $G$ -invariant subset of  $U$ , the image  $\mathcal{C} \subset \mathcal{U}$  of  $C \times \Delta$  under the above isomorphism is also  $G$ -invariant. This proves that the  $G$ -equivariant isomorphism  $\mathcal{U} \simeq U \times \Delta$  induces a  $G$ -equivariant isomorphism of the pairs  $(\mathcal{U}, \mathcal{C}) \simeq (U \times \Delta, C \times \Delta)$ , which is the main statement of the lemma.

It follows by definition that  $\pi : (\mathcal{X}, \mathcal{C}) \rightarrow \Delta$  is a  $G$ -equivariant  $C$ -locally trivial deformation of  $(X, C)$ . Since  $X$  is smooth, up to further shrinking  $\Delta$  we can assume that  $\mathcal{X} \rightarrow \Delta$  is a smooth deformation, so that the quotient  $\mathcal{X}/G \rightarrow \Delta$  is a locally trivial deformation [9, Proposition 8.2]. As

$$(\mathcal{X}/G, \mathcal{C}/G) \simeq ((U/G) \times \Delta, (C/G) \times \Delta)$$

over  $\Delta$ , the deformation  $(\mathcal{X}/G, \mathcal{C}/G) \rightarrow \Delta$  of the pair  $(X/G, C/G)$  is  $C/G$ -trivial.  $\square$

The following lemma is a special case of Lemma 2.5.

**Lemma 2.7.** — *Let  $f : X \rightarrow B$  be a  $G$ -equivariant fibration where  $G$  is a finite group. Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{q} & \Delta \times B \\ \pi \downarrow & \swarrow \text{pr}_1 & \\ \Delta & & \end{array}$$

*be a  $G$ -equivariant strongly locally trivial deformation of  $f$  over a one-dimensional base  $\Delta$ . Suppose that  $C$  is a  $G$ -invariant subvariety of  $X$  and that  $f(C)$  is a finite set of points, then the deformation  $\pi : \mathcal{X} \rightarrow \Delta$  induces a  $G$ -equivariant  $C$ -locally trivial deformation  $(\mathcal{X}, \mathcal{C}) \rightarrow \Delta$  of the pair  $(X, C)$ .*

*Proof.* — Let  $\{p_1, \dots, p_n\} := f(C) \subset B$ . By definition, up to shrinking  $\Delta$ , for each  $i$  there exists a neighborhood  $p_i \in V_i \subset B$  of  $p_i$  such that the restriction of  $\pi : \mathcal{X} \rightarrow \Delta$  to  $\mathcal{V}_i := q^{-1}(\Delta \times V_i)$  is isomorphic to  $(\mathcal{V}_i \cap X) \times \Delta$  over  $\Delta$ . Up to shrinking the  $V_i$ 's, we can assume that they are pairwise disjoint, so that  $\mathcal{V} := \sqcup_{i=1}^n \mathcal{V}_i$  is isomorphic to  $V \times \Delta$  over  $\Delta$  where  $V := \mathcal{V} \cap X$ . Applying Lemma 2.5 to the  $G$ -equivariant deformation  $\pi : \mathcal{X} \rightarrow \Delta$ , the  $G$ -invariant subvariety  $C$ , and  $\mathcal{V}$  yields Lemma 2.7.  $\square$

**Remark 2.8.** — For simplicity, Lemma 2.6 is stated and proven under the assumption that  $\dim \Delta = 1$  and so are Lemma 2.5 and Lemma 2.7, which will be enough for the purpose of this article. All these lemmata could have been stated without assuming that  $\dim \Delta = 1$ .



### 3 Bimeromorphic models of non-algebraic compact Kähler threefolds

The reader is referred to [11] for a survey of the minimal model program (MMP) for Kähler threefolds. Let  $X$  be a compact Kähler threefold with non-negative Kodaira dimension  $\kappa(X)$ . By running the MMP on  $X$ , we obtain a  $\mathbf{Q}$ -factorial bimeromorphic model  $X_{\min}$  of  $X$  with at worst terminal singularities (which are isolated, since  $\dim X = 3$ ) whose canonical line bundle  $K_{X_{\min}}$  is nef. Such a variety  $X_{\min}$  is called a *minimal model* of  $X$ . By the abundance conjecture, which is known to be true for Kähler threefolds, there exists  $m \in \mathbf{Z}_{>0}$  such that  $mK_{X_{\min}}$  is base-point free and that the surjective map  $f : X_{\min} \rightarrow B$  defined by the linear system  $|mK_{X_{\min}}|$  is a fibration satisfying  $\dim B = \kappa(B) = \kappa(X)$ . The fibration  $f : X_{\min} \rightarrow B$  is called the *canonical fibration* of  $X_{\min}$  and a general fiber  $F$  of  $f$  satisfies  $\mathcal{O}(mK_F) \simeq \mathcal{O}_F$  by the adjunction formula.

The aim of this section is to describe minimal models of non-algebraic compact Kähler threefolds of Kodaira dimension  $\kappa = 0$  or 1. Let us start from varieties with  $\kappa = 0$ .

**Proposition 3.1.** — *Let  $X$  be a non-algebraic compact Kähler threefold with  $\kappa(X) = 0$  and let  $X_{\min}$  be a minimal model of  $X$ . Then  $X_{\min}$  is isomorphic to a quotient  $\tilde{X}/G$  by a finite group  $G$  where  $\tilde{X}$  is either a 3-torus or a product of a K3 surface and an elliptic curve.*

*Proof.* — Since  $\kappa(X) = 0$ , there exists  $m \in \mathbf{Z}_{>0}$  such that  $\mathcal{O}(mK_{X_{\min}}) \simeq \mathcal{O}_{X_{\min}}$ . Let  $\pi : \tilde{X}_{\min} \rightarrow X_{\min}$  be the index 1 cover of  $X_{\min}$ : this is a finite cyclic cover étale over  $X \setminus \text{Sing}(X)$  such that  $K_{\tilde{X}_{\min}} \simeq \mathcal{O}_{\tilde{X}_{\min}}$  [14, p. 159]. As  $X_{\min}$  has at worst terminal singularities, by [14, Corollary 5.21 (2)], the variety  $\tilde{X}_{\min}$  has also at worst terminal singularities. Since  $X$  is assumed to be non-algebraic, by [9, Theorem 6.1]  $\tilde{X}_{\min}$  is smooth. Thus by the Beauville-Bogomolov decomposition theorem [1, Théorème 1], there exists a finite étale cover  $X' \rightarrow \tilde{X}_{\min}$  such that  $X'$  is either a 3-torus or a product of a K3 surface and an elliptic curve (as  $X'$  is non-algebraic,  $X'$  cannot be a Calabi-Yau threefold); let  $\tau : X' \rightarrow X_{\min}$  denote the composition of  $X' \rightarrow \tilde{X}_{\min}$  with  $\pi$ .

The finite map  $\tau$  is étale over  $X \setminus \text{Sing}(X)$ . Let  $\tilde{X}^\circ \rightarrow X' \setminus Z \rightarrow X \setminus \text{Sing}(X)$  be the Galois closure of  $\tau|_{X' \setminus Z}$  where  $Z := \tau^{-1}(\text{Sing}(X))$  and let

$$G := \text{Gal}(\tilde{X}^\circ / (X \setminus \text{Sing}(X))).$$

Since  $\text{Sing}(X)$  and hence  $Z$  are finite sets of points, we have  $\pi_1(X' \setminus Z) \simeq \pi_1(X')$ . It follows that  $\tilde{X}^\circ \rightarrow X' \setminus Z$  extends to  $\tilde{X} \rightarrow X'$  which is the finite étale cover associated to the subgroup  $\text{Gal}(\tilde{X}^\circ / (X' \setminus Z)) < \pi_1(X' \setminus Z) \simeq \pi_1(X')$ . The variety  $\tilde{X}$  is still a 3-torus or a product of a K3 surface and an elliptic curve. As  $\tilde{X} \setminus \tilde{X}^\circ$  is a set of isolated points, the  $G$ -action on  $\tilde{X}^\circ$  extends to a  $G$ -action on  $\tilde{X}$  whose quotient is  $X_{\min}$ .  $\square$

**Remark 3.2.** — The group  $G$  constructed in the proof of Proposition 3.1 acts freely outside of a finite set of points of  $\tilde{X}$ .

For quotients  $(S \times E)/G$  of the product of a non-algebraic K3 surface  $S$  and an elliptic curve  $E$ , we can show that the  $G$ -action is necessarily diagonal.

**Lemma 3.3.** — *Let  $G$  be a group acting on  $S \times E$  where  $S$  is a non-algebraic K3 surface and  $E$  is an elliptic curve. Then this  $G$ -action is the product of a  $G$ -action on  $S$  and a  $G$ -action on  $E$ .*

*Proof.* — For each  $g \in G$  and each fiber  $F$  of the second projection  $p_2 : S \times E \rightarrow E$ , since  $h^{0,1}(F) < h^{0,1}(E)$ , it follows that  $g(F)$  is still a fiber of  $p_2$ . So the  $G$ -action on  $S \times E$  induces a  $G$ -action on  $E$ . Suppose that there exist  $g \in G$  and a fiber  $E_t$  of the first projection  $p_1 : S \times E \rightarrow S$  such that  $g(E_t)$  is not contracted by  $p_1$ , then if we vary  $t \in S$ , we have a two-dimensional covering family of curves

$$\{E'_t := p_1(g(E_t))\}_{t \in S}$$



on  $S$  generically of geometric genus 1. Since algebraic equivalence coincides with linear equivalence for curves on a K3 surface and since there is only one-dimensional families of curves of geometric genus 1 in each linear system,  $\{E'_t\}_{t \in S}$  is in fact a one-dimensional family of curves, say parameterized by some proper curve  $T$ . As  $S$  is non-algebraic, the family  $\{E'_t\}_{t \in T}$  is an elliptic fibration and there exists  $t \in T$  such that the normalization  $\widetilde{E}'_t$  of  $E'_t$  is  $\mathbf{P}^1$ .

Let  $C \subset S$  be a curve such that for each  $p \in C$ , we have  $E'_p = E'_t$ . Since the curves  $g(E_p) \subset S \times E$  are mutually disjoint for  $p \in C$ , their strict transformations  $\widetilde{g(E_p)}$  in the normalization  $\widetilde{E}'_t \times E$  of  $E'_t \times E$  are also disjoint from each other. It follows that  $[g(E_p)]^2 = 0$  in  $H^4(\widetilde{E}'_t \times E, \mathbf{Z})$  and since  $\widetilde{E}'_t \simeq \mathbf{P}^1$ , the curve  $\widetilde{g(E_p)}$  has to be a fiber of  $\widetilde{E}'_t \times E \rightarrow \widetilde{E}'_t$ . The latter is in contradiction with the assumption that  $g(E_p)$  is not contracted by  $p_1$ .  $\square$

Next we turn to varieties with  $\kappa = 1$ .

**Theorem 3.4.** — *Let  $X$  be a non-algebraic compact Kähler threefold with  $\kappa(X) = 1$ . Let  $X_{\min}$  be a minimal model of  $X$  and  $X_{\min} \rightarrow B$  the canonical fibration of  $X_{\min}$ . Then  $X_{\min} \rightarrow B$  satisfies one of the following descriptions :*

- i) *If a general fiber  $F$  of  $X_{\min} \rightarrow B$  is algebraic, then  $F$  is either an abelian surface or a bielliptic surface ;*
- ii) *If  $F$  is not algebraic, then  $F$  is either a K3 surface or a 2-torus, and there exists a finite Galois cover  $\widetilde{B} \rightarrow B$  of  $B$  and a smooth fibration  $\widetilde{X} \rightarrow \widetilde{B}$  whose fibers are all isomorphic to  $F$ , such that  $\widetilde{X}$  is bimeromorphic to  $X_{\min} \times_B \widetilde{B}$  over  $\widetilde{B}$ . Moreover, the monodromy action of  $\pi_1(\widetilde{B})$  on  $F$  preserves the holomorphic symplectic form. Finally if either  $F$  is a K3 surface or  $X_{\min}$  contains a curve which dominates  $B$ , then there exists a finite Galois base change as above such that  $\widetilde{X} \rightarrow \widetilde{B}$  is isomorphic to the standard projection  $F \times \widetilde{B} \rightarrow \widetilde{B}$ .*

*Proof.* — Since  $X_{\min}$  has only isolated singularities, a general fiber  $F$  of  $X_{\min} \rightarrow B$  is a connected smooth surface. As  $K_F$  is torsion, the classification of surfaces shows that  $F$  is either a K3 surface, an Enriques surface, a 2-torus, or a bielliptic surface. Since  $X$ , and thus  $X_{\min}$  is non-algebraic, if  $F$  is algebraic then by Fujiki's result [8, Proposition 7]  $F$  is irregular, so  $F$  can only be an abelian surface or a bielliptic surface, which proves i).

Assume that  $F$  is not algebraic, then  $F$  is either a K3 surface or a 2-torus and by [4], the fibration  $X_{\min} \rightarrow B$  is isotrivial. By [16, Lemma 4.2], there exists some finite map  $\widetilde{B} \rightarrow B$  of  $B$  and a smooth fibration  $\widetilde{X} \rightarrow \widetilde{B}$  all of whose fibers are isomorphic to  $F$ , such that  $\widetilde{X}$  is bimeromorphic to  $X_{\min} \times_B \widetilde{B}$  over  $\widetilde{B}$ . Up to taking the Galois closure of  $\widetilde{B} \rightarrow B$ , we can assume that  $\widetilde{B} \rightarrow B$  is Galois.

Since  $\widetilde{f}$  is smooth and isotrivial, the fundamental group  $\pi_1(\widetilde{B})$  acts on  $F$  by monodromy transformations. Since  $\widetilde{X}$  is assumed to be non-algebraic, we have  $H^0(X, \Omega_X^2) \neq 0$ . Hence by the global cycle invariant theorem, the  $\pi_1(\widetilde{B})$ -action on  $F$  is symplectic.

As  $\widetilde{X}$  is Kähler, again by the global cycle invariant theorem there exists a Kähler class on  $F$  fixed by the induced monodromy action on  $H^2(F, \mathbf{R})$ . It follows that the map  $\pi_1(\widetilde{B}) \rightarrow \text{Aut}(F)/\text{Aut}_0(F)$  has finite image where  $\text{Aut}_0(F)$  denotes the identity component of  $\text{Aut}(F)$  [15, Proposition 2.2].

In the case where  $F$  is a K3 surface,  $\text{Aut}_0(F)$  is trivial, so  $\pi_1(\widetilde{B})$  acts as a finite group on  $F$ . Accordingly after some finite base change of  $\widetilde{f} : \widetilde{X} \rightarrow \widetilde{B}$ , the fibration  $\widetilde{f}$  becomes a trivial. Now assume that  $F$  is a 2-torus and that  $X_{\min}$  contains a curve dominating  $B$ . After another finite base change of  $\widetilde{f} : \widetilde{X} \rightarrow \widetilde{B}$  we can assume that  $\widetilde{f}$  has a section  $\sigma : \widetilde{B} \rightarrow \widetilde{X}$ , namely  $\widetilde{f}$  is a Jacobian fibration. Recall that  $\pi_1(\widetilde{B}) \rightarrow \text{Aut}(F)/\text{Aut}_0(F)$  has finite image, so after a further finite base change of  $\widetilde{f} : \widetilde{X} \rightarrow \widetilde{B}$ , we can assume that the monodromy action of  $\pi_1(\widetilde{B})$  on  $H^1(F, \mathbf{Z})$  is trivial. As  $\widetilde{f} : \widetilde{X} \rightarrow \widetilde{B}$  is a Jacobian fibration, we conclude that  $\widetilde{X} \simeq F \times \widetilde{B}$  and that  $\widetilde{f}$  is isomorphic to the projection  $F \times \widetilde{B} \rightarrow \widetilde{B}$ .

As before, both in the case where  $F$  is a K3 surface or a 2-torus, up to taking the Galois closure of  $\tilde{B} \rightarrow B$  we can assume that  $\tilde{B} \rightarrow B$  is Galois.  $\square$

#### 4 Equivariant algebraic approximations of pairs

In this section, we will prove for some compact Kähler threefolds  $X$  endowed with a  $G$ -action that for every  $G$ -invariant curve  $C \subset X$ , there exists a  $G$ -equivariant  $C$ -locally trivial algebraic approximation of the pair  $(X, C)$ . Results in Section 3 show that the quotients  $X/G$  of these varieties cover all compact Kähler threefolds of Kodaira dimension 0 or 1 up to bimeromorphic transformations, hence will allow us to conclude the proof of Theorem 1.3 in Section 5.

Before dealing with threefolds, we start by proving analogue statements concerning the existence of a  $G$ -equivariant  $C$ -locally trivial algebraic approximation for fibrations admitting a strongly locally trivial algebraic approximation and for surfaces in the next two subsections.

##### 4.1 Fibrations admitting a strongly locally trivial algebraic approximation

**Lemma 4.1.** — *Let  $X$  be a non-algebraic compact Kähler variety and  $f : X \rightarrow B$  a surjective map onto a curve with algebraic fibers. Suppose that  $X$  has a strongly locally trivial algebraic approximation  $\pi : \mathcal{X} \rightarrow \Delta$  with respect to  $f$ , then for any subvariety  $C \subset X$ , up to shrinking  $\Delta$  the deformation  $\pi$  induces a  $C$ -locally trivial algebraic approximation of  $(X, C)$ .*

*If moreover there exists a finite group  $G$  acting  $f$ -equivariantly on  $X$  and on  $B$  and the algebraic approximation of  $X$  in the assumption above is  $G$ -equivariant, then the induced  $C$ -locally trivial algebraic approximation is also  $G$ -equivariant for every  $G$ -invariant subvariety  $C$ .*

*Proof.* — Since  $X$  is non-algebraic and since the base and the fibers of  $f$  are algebraic, by Campana's criterion [3, Corollaire in p.212] every subvariety of  $X$  (in particular  $C$ ) is contained in a finite number of fibers of  $f$ . We can thus apply Lemma 2.7 to conclude.  $\square$

**Corollary 4.2.** — *Let  $X$  be a non-algebraic compact Kähler variety and  $f : X \rightarrow B$  a surjective map onto a curve. Let  $G$  be a finite group acting  $f$ -equivariantly on  $X$  and on  $B$ . Assume that a general fiber of  $f$  is an abelian variety, then for every  $G$ -invariant subvariety  $C \subset X$ , the pair  $(X, C)$  has a  $G$ -equivariant  $C$ -locally trivial algebraic approximation.*

*Proof.* — By [16, Theorem 1.6], the fibration  $f$  has a  $G$ -equivariant strongly locally trivial algebraic approximation. Hence Corollary 4.2 follows from Lemma 4.1.  $\square$

##### 4.2 Surfaces with a finite group action

First we recall some Hodge-theoretical criteria for the existence of an algebraic approximation.

**Theorem 4.3 (Green's criterion [2, Proposition 1]).** — *Let  $\pi : \mathcal{X} \rightarrow B$  be a family of compact Kähler manifolds over a smooth base. If a fiber  $X = \pi^{-1}(b)$  satisfies the property that the composition of the Kodaira-Spencer map and the contraction with some Kähler class  $[\omega] \in H^1(X, \Omega_X^1)$*

$$\mu_{[\omega]} : T_{B,b} \xrightarrow{\text{KS}} H^1(X, T_X) \xrightarrow{\sim[\omega]} H^2(X, T_X \otimes \Omega_X) \longrightarrow H^2(X, \mathcal{O}_X)$$

*is surjective, then there exists a sequence of points in  $B$  parameterizing algebraic members which converges to  $b$ .*

The following is a variant of Theorem 4.3 when the variety  $X$  is endowed with a finite group action.

**Theorem 4.4** ([9, Theorem 9.1]). — *Let  $X$  be a compact Kähler manifold with an action of a finite group  $G$ . Suppose that the universal deformation space of  $X$  is smooth. If there exists a  $G$ -invariant Kähler class  $[\omega] \in H^1(X, \Omega_X^1)$  such that the following composition of maps*

$$\mu_{[\omega]} : H^1(X, T_X) \xrightarrow{\sim[\omega]} H^2(X, T_X \otimes \Omega_X) \longrightarrow H^2(X, \mathcal{O}_X)$$

*is surjective, then  $X$  has a  $G$ -equivariant algebraic approximation.*

The following is an easy application of Theorem 4.4.

**Lemma 4.5.** — *Let  $S$  be a non-algebraic compact Kähler surface and  $G$  a finite group acting on  $S$ . If  $K_S \simeq \mathcal{O}_S$ , namely if  $S$  is either a K3 surface or a 2-torus, then  $S$  has a  $G$ -equivariant algebraic approximation.*

*Proof.* — Since  $S$  is a surface with trivial  $K_S$ , the universal deformation space of  $S$  is smooth. Also, we have the isomorphism  $T_S \simeq \Omega_S^1$  defined by the contraction with a fixed holomorphic symplectic form. So for a  $G$ -invariant Kähler class  $[\omega]$ , the map  $\mu_{[\omega]}$  defined in Theorem 4.4 with  $\mathcal{X} \rightarrow B$  replaced by the family of K3 surfaces  $\mathcal{S}_U \rightarrow U$  has the factorization

$$\mu_{[\omega]} : H^1(S, T_S) \simeq H^1(S, \Omega_S^1) \xrightarrow{\sim[\omega]} H^2(S, \Omega_S^2) \simeq H^2(X, \mathcal{O}_X). \quad (4.1)$$

Since  $[\omega]^2 \neq 0$ , the map  $\mu_{[\omega]}$  is non-zero. Moreover since  $h^2(S, \mathcal{O}_S) = 1$ , the map  $\mu_{[\omega]}$  has to be surjective. Hence Lemma 4.5 is a consequence of Theorem 4.4.  $\square$

Lemma 4.6 and 4.7 concern  $C$ -locally trivial algebraic approximations of a pair  $(S, C)$  for  $K$ -trivial surfaces.

**Lemma 4.6.** — *Let  $S$  be a non-algebraic 2-torus and let  $G$  be a finite group acting on  $S$ . Let  $C \subset S$  be a  $G$ -invariant curve. Then the pair  $(S, C)$  has a  $G$ -equivariant  $C$ -locally trivial algebraic approximation.*

*Proof.* — Since  $S$  is a non-algebraic 2-torus containing a curve, it is a smooth isotrivial elliptic fibration  $f : S \rightarrow B$  and the only curves of  $S$  are fibers of  $f$ . As the  $G$ -action sends curves to curves, the fibration  $f$  is  $G$ -equivariant. We thus conclude by Corollary 4.2 that  $(S, C)$  admits a  $G$ -equivariant  $C$ -trivial algebraic approximation.  $\square$

**Lemma 4.7.** — *Let  $S$  be a non-algebraic K3 surface and let  $G$  be a finite group acting on  $S$ . Let  $C \subset S$  be a  $G$ -invariant curve. Then  $(S, C)$  has a  $G$ -equivariant  $C$ -locally trivial algebraic approximation. When the algebraic dimension  $a(S)$  of  $S$  is zero, more precisely the deformation  $\mathcal{S} \rightarrow \Delta$  of  $S$  over the Noether-Lefschetz locus preserving the classes of each irreducible component of  $C$  in the universal deformation of  $S$  preserving the  $G$ -action is a  $G$ -equivariant  $C$ -locally trivial algebraic approximation.*

*Proof.* — First we note that since  $H^2(S, \mathbf{Z})^G$  is a sub- $\mathbf{Z}$ -Hodge structure of  $H^2(S, \mathbf{Z})$  of weight 2, if the  $G$ -action does not preserve the holomorphic symplectic form, then  $H^2(S, \mathbf{Z})^G$  is concentrated in bi-degree  $(1, 1)$ . As the intersection of  $H^{1,1}(S)^G$  with the Kähler cone  $\mathcal{K}_S \subset H^2(S, \mathbf{C})$  is not 0, we deduce that  $H^2(S, \mathbf{Z})^G$  contains a Kähler class, which is in contradiction with the hypothesis that  $S$  is non-algebraic. We deduce that the  $G$ -action preserves the holomorphic symplectic form of  $S$ .

Since  $S$  is assumed to be non-algebraic, according to whether  $a(S) = 0$  or 1 only two situations can happen :

1.  $a(S) = 0$  : every curve in  $S$  is a disjoint union of trees of smooth  $(-2)$ -curves intersecting transversally ;
2.  $a(S) = 1$  :  $S$  is an elliptic fibration  $f : S \rightarrow B$  and the  $G$ -action sends fibers to fibers.

In the second situation, we can apply Corollary 4.2 to get a  $G$ -equivariant  $C$ -locally trivial algebraic approximation of  $(S, C)$  as we did in the proof of Lemma 4.6.

In the first situation, let us write  $C = \cup_{i \in I} C_i$  where the  $C_i$ 's are irreducible components of  $C$ . Since the universal deformation space of  $S$  is smooth, its locus preserving the  $G$ -action can be identified with an open subset of  $H^1(S, T_S)^G$ . As the group action  $G$  on  $S$  is symplectic, the isomorphism  $T_S \simeq \Omega_S^1$  defined by the contraction with a fixed holomorphic symplectic form induces an isomorphism

$$H^1(S, T_S)^G \simeq H^1(S, \Omega_S^1)^G.$$

Under this identification, the universal deformation space  $\Delta$  of  $S$  preserving the  $G$ -action and the curve classes  $[C_i]$  can be identified with an open subset  $U$  of

$$V := H^1(S, \Omega_S^1)^G \cap \langle [C_i] \rangle_{i \in I}^\perp$$

where  $\langle [C_i] \rangle_{i \in I}$  denotes the linear subspace of  $H^1(S, \Omega_S^1)$  spanned by the classes  $[C_i]$  and the orthogonality is defined with respect to the cup product. Since  $\langle [C_i] \rangle_{i \in I}$  is  $G$ -invariant and since the  $G$ -action preserves the cup product, the orthogonal  $\langle [C_i] \rangle_{i \in I}^\perp$  is also  $G$ -invariant. Therefore  $V = \langle [C_i] \rangle_{i \in I}^\perp$ .

Since  $S$  is not algebraic, the curve classes  $[C_i]$  cannot generate the whole  $H^1(S, \Omega_S^1)$ , hence  $V \neq 0$  and let  $v$  be a non-zero element in  $V$ . As  $C_i^2 < 0$  for all  $i$ , by the Hodge index theorem  $v^2 > 0$ . If  $[\omega]$  is a Kähler class, then again by the Hodge index theorem we have  $v \cdot [\omega] \neq 0$ . Using the factorization (4.1), we see again that since  $h^2(S, \mathcal{O}_S) = 1$ , the map  $\mu_{[\omega]}$  defined in Theorem 4.3 with  $\mathcal{X} \rightarrow B$  replaced by the  $G$ -equivariant deformation  $\mathcal{S} \rightarrow \Delta$  of  $S$  over the Noether-Lefschetz locus  $\Delta$ , is surjective. Therefore by Theorem 4.3,  $\mathcal{S} \rightarrow \Delta$  is an algebraic approximation of  $S$ . Since the curve classes  $[C_i] \in H^2(S, \mathbb{C})$  remains of type  $(1, 1)$ ,  $\mathcal{S} \rightarrow \Delta$  induces for each  $i$ , a deformation  $(\mathcal{S}, \mathcal{C}_i)$  of the pair  $(S, C_i)$ . It remains to show that  $(\mathcal{S}, \mathcal{C} := \cup_{i \in I} \mathcal{C}_i) \rightarrow \Delta$  is a  $C$ -locally trivial deformation.

Let us decompose  $C = \sqcup_{i=1}^m C'_i$  into its connected components. As we mentioned before, each  $C'_i$  is a tree of smooth  $(-2)$ -curves intersecting transversally. Therefore up to shrinking  $\Delta$ , if  $\mathcal{C} = \sqcup_{i=1}^m \mathcal{C}'_i$  denotes the decomposition of  $\mathcal{C}$  into its connected components, then up to reordering the indices  $i$ , each fiber of  $\mathcal{C}'_i \rightarrow \Delta$  is still a tree of  $(-2)$ -curves isomorphic to  $C'_i$ .

Since a tree of smooth  $(-2)$ -curve on a surface can be contracted to a rational double point, there exists a bimeromorphic morphism  $\nu : \mathcal{S} \rightarrow \mathcal{S}'$  over  $\Delta$  such that for each fiber  $\mathcal{S}_t$  of  $\mathcal{S} \rightarrow \Delta$ , the restriction of  $\nu$  to  $\mathcal{S}_t$  is the contraction of  $\mathcal{C}'_i \cap \mathcal{S}_t$  to a rational double point [18, Theorem 2]. Since fibers of  $\mathcal{C}'_i \rightarrow \Delta$  are all isomorphic, the singularity type of  $\nu(\mathcal{C}'_i \cap \mathcal{S}_t) \subset \mathcal{S}_t$  does not depend on  $t \in \Delta$ . As the germs of a rational double point of a fixed type on a surface are all isomorphic, up to shrinking  $\Delta$  there exists a neighborhood  $\mathcal{U}_i \subset \mathcal{S}'$  of  $\nu(\mathcal{C}'_i)$  such that the pair  $(\mathcal{U}_i, \nu(\mathcal{C}'_i))$  is isomorphic over  $\Delta$  to the trivial product  $(U_i \times \Delta, \nu(C'_i))$  with  $U_i := \mathcal{U}_i \cap \nu(S)$ . It follows that  $(\mathcal{S}', \nu(\mathcal{C})) \rightarrow \Delta$  is a  $\nu(C)$ -locally trivial deformation of  $(\nu(S), \nu(C))$ , hence  $(\mathcal{S}, \mathcal{C}) \rightarrow \Delta$  is  $C$ -locally trivial by Lemma 2.1.  $\square$

For the sake of completeness, we conclude the present subsection by the following proposition which will not be used latter in the article. It is the generalization of [19, Lemma 5.1] in the  $G$ -equivariant setting.

**Proposition 4.8.** — *Let  $S$  be a compact Kähler surface and  $G$  a finite group acting on  $S$ . Whenever  $C \subset S$  is a curve or empty, the pair  $(S, C)$  has a  $G$ -equivariant  $C$ -locally trivial algebraic approximation.*

*Proof.* — We may assume that  $S$  is non-algebraic. If the algebraic dimension  $a(S)$  of  $S$  is 1, then  $S$  is an elliptic fibration and we can use Corollary 4.2 to conclude. If  $a(S) = 0$ , then the minimal model  $S'$  of  $S$  is either a 2-torus or a K3 surface and the map  $\nu : S \rightarrow S'$  is  $G$ -equivariant. By Lemma 4.5, 4.6, and 4.7, the

pair  $(S', \nu(C))$  has a  $G$ -equivariant  $\nu(C)$ -locally trivial algebraic approximation. Hence by Lemma 2.1,  $(S, C)$  has a  $G$ -equivariant  $C$ -locally trivial algebraic approximation.  $\square$

### 4.3 K3 fibrations

**Lemma 4.9.** — *Let  $X := S \times B$  where  $S$  is a non-algebraic K3 surface and  $B$  is a smooth projective curve. Let  $G$  be a finite group acting on  $B$  and on  $S$  and let  $G$  act on  $X$  by the product action. Whenever  $C \subset X$  is a  $G$ -invariant curve or empty, the pair  $(X, C)$  has a  $G$ -equivariant  $C$ -locally trivial algebraic approximation.*

*Proof.* — Let  $p_1 : S \times B \rightarrow S$  denote the first projection. As the  $G$ -action on  $S \times B$  is a product action, the image  $C' := p_1(C)$  is a  $G$ -invariant curve. By Lemma 4.5 and 4.7, there exists a  $G$ -equivariant  $C'$ -locally trivial algebraic approximation  $\pi : (\mathcal{S}, \mathcal{C}') \rightarrow \Delta$  of the pair  $(S, C')$ . Let  $\mathcal{U} \subset \mathcal{S}$  be a neighborhood of  $\mathcal{C}'$  such that there exists an isomorphism  $\mathcal{U} \simeq U \times \Delta$  over  $\Delta$ , so  $U \times B \times \Delta \simeq \mathcal{U} \times B$  over  $\Delta$ . Since  $U \times B$  is a neighborhood of  $C$  and since  $C$  is  $G$ -invariant, Lemma 2.5 implies that the algebraic approximation  $\Pi : \mathcal{X} := \mathcal{S} \times B \rightarrow \Delta$  of  $X$  defined by the composition of  $\pi$  with the first projection  $\mathcal{S} \times B \rightarrow \mathcal{S}$  induces a  $C$ -locally trivial algebraic approximation of  $(X, C)$ .  $\square$

### 4.4 2-torus fibrations

Before we study the existence of  $(C)$ -locally trivial algebraic approximations of a pair  $(X, C)$  in the case of 2-torus fibrations, let us first prove a statement concerning the existence of multisections of a torus fibration via strongly locally trivial perturbation.

**Lemma 4.10.** — *Let  $f : X \rightarrow B$  be a smooth torus fibration whose total space  $X$  is compact Kähler. There exists an arbitrarily small strongly locally trivial deformation  $f' : X' \rightarrow B$  of  $f$  such that  $f'$  has a multisection. Moreover if  $f$  is endowed with an  $f$ -equivariant  $G$ -action for some finite group  $G$ , then one can choose the above deformation to be  $G$ -equivariant.*

*Proof.* — The construction of an arbitrarily small deformation of  $f$  possessing a multi-section already appeared in [7]. We will recall how this deformation is constructed and prove that it is strongly locally trivial along the way.

Let  $J \rightarrow B$  be the Jacobian fibration associated to  $f$  and  $\mathcal{J}$  its sheaf of sections. The sheaf can be defined by the exact sequence

$$0 \longrightarrow \mathbf{H}_{\mathbf{Z}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{J} \longrightarrow 0$$

where  $\mathbf{H}_{\mathbf{Z}} := R^{2g-1}f_*\mathbf{Z}$  and  $\mathcal{E} := \mathcal{H}/\mathcal{H}^{g,g-1} := R^{2g-1}f_*\mathbf{C}/R^{g-1}f_*\Omega_{X/B}^g$ . To each isomorphism class of  $J$ -torsor  $g : Y \rightarrow B$ , one can associate in a biunivocal way, an element  $\eta(g) \in H^1(B, \mathcal{J})$  satisfying the property that  $g$  has a multisection if and only if  $\eta(g)$  is torsion (cf. [7, Section 2.2]). Moreover, if  $\exp : V := H^1(B, \mathcal{E}) \rightarrow H^1(B, \mathcal{J})$  denotes the morphism induced by the quotient  $\mathcal{E} \rightarrow \mathcal{J}$ , then there exists a family

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{q} & V \times B \\ \pi \downarrow & \swarrow \text{pr}_1 & \\ V & & \end{array} \quad (4.2)$$

of  $J$ -torsor such that for each  $v \in V$ , the element in  $V$  associated to the  $J$ -torsor  $\pi^{-1}(v) \rightarrow B$  is  $\eta(f) + \exp(v)$ .

Concretely, the above family is constructed as follows. The map

$$\begin{aligned} V &\rightarrow H^1(B, \mathcal{J}) \\ v &\mapsto \eta(f) + \exp(v), \end{aligned}$$

defines an element

$$\eta^V \in \text{Map}(V, H^1(B, \mathcal{J})) \simeq H^0(V, \mathcal{O}_V) \otimes H^1(B, \mathcal{J}) \simeq H^1(V \times B, \text{pr}_2^* \mathcal{J})$$

where  $\text{Map}(V, H^1(B, \mathcal{J}))$  denotes the space of holomorphic maps between  $V$  and  $H^1(B, \mathcal{J})$ . So one can find a covering  $\cup_{i=1}^n U_i = B$  of  $B$  by open subsets such that  $\eta^V$  represents a Čech 1-cocycle

$$\eta_{ij}^V \in \Gamma(V \times U_{ij}, \text{pr}_2^* \mathcal{J}) \simeq \text{Map}(V, \Gamma(U_{ij}, \mathcal{J}))$$

where  $U_{ij} := U_i \cap U_j$ . Let us write  $X_i := f^{-1}(U_i)$  and  $X_{ij} := f^{-1}(U_{ij})$  for all  $i$  and  $j$ . The 1-cocycle  $(\eta_{ij}^V)_{i,j}$  defines the transition maps  $V \times X_{ij} \rightarrow V \times X_{ij}$  which are translations by  $\eta_{ij}^V$  and the family  $\mathcal{X} \rightarrow V \times B$  is obtained by glueing  $(V \times X_i \rightarrow V \times U_i)_i$  together using these transition maps. Since  $q^{-1}(V \times U_i) \simeq V \times X_i$  over  $V$  for all  $i$ , the family  $\pi : \mathcal{X} \rightarrow V$  is strongly locally trivial.

If  $f : X \rightarrow B$  is endowed with an  $f$ -equivariant  $G$ -action for some finite group  $G$ , then this  $G$ -action induces an action on  $\mathcal{E}$  and on  $\mathcal{J}$ . The restriction to the  $G$ -invariant subspace  $V^G \subset V$  of (4.2) is a deformation of the  $J$ -torsor  $f : X \rightarrow B$  preserving the equivariant  $G$ -action [7, Proposition 2.10]. The proof that  $V^G$  contains a dense subset of points parameterizing  $J$ -torsors having a multi-section is contained in the proof of [7, Theorem 1.1], which we sketch now and provide necessary references for the detail.

By Deligne's theorem,  $W := H^1(B, \mathbf{H}_{\mathbf{Z}})$  is a pure Hodge structure of degree  $2g$  and concentrated in bi-degrees  $(g-1, g+1)$ ,  $(g, g)$ , and  $(g+1, g-1)$  [22, Section 2]. Let  $W_K := W \otimes K$  for any field  $K$ . If  $F^\bullet W_{\mathbf{C}}$  denotes the Hodge filtration, then  $V$  is isomorphic to  $W_{\mathbf{C}}/F^g W_{\mathbf{C}}$  [22, Section 2]. Let  $\mu : W_{\mathbf{R}} \rightarrow V$  denote the composition

$$\mu : W_{\mathbf{R}} \hookrightarrow W_{\mathbf{C}} \rightarrow V.$$

Using the Hodge theory we see easily that  $\mu$  is surjective, so  $\mu(W_{\mathbf{Q}})$  is dense in  $V$ . Since  $G$  is finite, we have

$$\mu(W_{\mathbf{Q}}^G) \otimes \mathbf{R} = \mu(W_{\mathbf{Q}})^G \otimes \mathbf{R} = V^G.$$

Therefore  $\mu(W_{\mathbf{Q}}^G)$  is dense in  $V^G$ .

Using the assumption that  $X$  is Kähler, one can prove that the image of the  $G$ -equivariant class  $\eta_G(f) \in H_G^1(B, \mathcal{J})$  associated to  $X$  (which is a refinement of  $\eta(f)$ , cf. [7, Section 2.4]) under the connection morphism

$$H_G^1(B, \mathcal{J}) \rightarrow H_G^2(B, \mathbf{H}_{\mathbf{Z}})$$

is torsion [7, Proposition 2.11]. It follows that there exists  $m \in \mathbf{Z}_{>0}$  and  $v_0 \in V^G$  such that  $m\eta(f) = \exp(v_0)$ . Therefore  $\eta(f) + \exp\left(v - \frac{v_0}{m}\right)$  is torsion for each  $v \in \mu(W_{\mathbf{Q}}^G)$ , so each of the fibrations  $\mathcal{X}_v \rightarrow B$  parameterized by the subset

$$\mu(W_{\mathbf{Q}}^G) - \frac{v_0}{m} \subset V^G$$

in the family (4.2) has a multisection. As we saw that  $\mu(W_{\mathbf{Q}}^G) \subset V^G$  is dense, we conclude that the restriction of (4.2) to  $V^G$  is a deformation of  $f : X \rightarrow B$  containing a dense subset of members having a multisection.  $\square$

**Lemma 4.11.** — *Let  $f : X \rightarrow B$  be a smooth isotrivial 2-torus fibration over a smooth projective curve  $B$ . Let  $G$  be a finite group acting  $f$ -equivariantly on  $X$  and on  $B$  such that  $X \rightarrow B$  coincides with the base change of  $X/G \rightarrow B/G$  by*

$B \rightarrow B/G$ . Whenever  $C \subset X$  is a  $G$ -invariant curve or empty, the pair  $(X/G, C/G)$  has a  $C/G$ -locally trivial algebraic approximation.

*Proof.* — First we assume that  $f$  does not have any multisection. In particular, the curve  $C$  is contained in a finite union of fibers of  $f$ . Using Lemma 4.10 there exists an arbitrarily small strongly locally trivial, so in particular  $C$ -locally trivial, deformation of  $f$  to some fibration which has a multisection. Thus up to replacing  $f$  by this arbitrarily small deformation, we can assume that  $f$  has a multisection.

Since  $f$  has a multisection, there exists a finite base change  $\tilde{f} : \tilde{X} \rightarrow \tilde{B}$  of  $X \rightarrow B$  such that  $\tilde{X} \simeq S \times \tilde{B}$  where  $S$  is a fiber of  $f$  and that  $\tilde{f}$  is the second projection. After base changing with the Galois closure of  $\tilde{B} \rightarrow B/G$ , we can assume that  $\tilde{B} \rightarrow B/G$  is Galois whose Galois group is denoted by  $\tilde{G}$  acting on  $\tilde{B}$  and on  $S$  by monodromy transformations.

Let  $\tilde{C}$  be the pre-image of  $C$  under the map  $\tilde{X} \rightarrow X$ , which is  $\tilde{G}$ -invariant by assumption. By Lemma 2.5, it suffices to show that the pair  $(\tilde{X}, \tilde{C})$  has a  $\tilde{G}$ -equivariant  $\tilde{C}$ -locally trivial algebraic approximation. Since  $S \times \tilde{B} \rightarrow \tilde{B}$  is isomorphic to the base change of  $\tilde{X}/G \rightarrow \tilde{B}/G$  by  $\tilde{B} \rightarrow B/G$ , the  $\tilde{G}$ -action on  $S \times \tilde{B}$  induces a  $\tilde{G}$ -action on  $S$  such that the first projection  $p_1 : S \times \tilde{B} \rightarrow S$  is  $\tilde{G}$ -equivariant. As  $C$  is  $\tilde{G}$ -invariant, the curve  $C' := p_1(C)$  is also  $\tilde{G}$ -invariant. By Lemma 4.5 and 4.6, there exists a  $\tilde{G}$ -equivariant  $C'$ -locally trivial algebraic approximation  $(\mathcal{S}, \mathcal{C}') \rightarrow \Delta$  of the pair  $(S, C')$ . By repeating the same argument as in the proof of Lemma 4.9, we conclude that the deformation  $\Pi : \mathcal{S} \times \tilde{B} \rightarrow \Delta$  induces a  $\tilde{G}$ -equivariant  $\tilde{C}$ -locally trivial algebraic approximation of the pair  $(\tilde{X}, \tilde{C})$ .  $\square$

#### 4.5 Non-algebraic 3-tori

**Lemma 4.12.** — *Let  $X$  be a non-algebraic 3-torus and  $G$  a finite group acting on  $X$ . Then for every  $G$ -invariant curve  $C \subset X$ , the pair  $(X, C)$  has a  $G$ -equivariant algebraic approximation.*

*Proof.* — First we assume that there exists a generically injective morphism  $\nu : C' \rightarrow X$  from a smooth curve of geometric genus  $\geq 2$  to  $X$ . Since  $\nu$  factorizes through  $C' \rightarrow J(C') \xrightarrow{j} X$  where  $J(C')$  denotes the Jacobian associated to  $C'$ , the 3-torus  $X$  contains an abelian variety of dimension  $\geq 2$  which is  $j(J(C')) \subset X$ . As  $X$  is non-algebraic, we have  $\dim j(J(C')) = 2$  and hence  $X$  is a smooth isotrivial fibration  $f : X \rightarrow B$  in abelian surfaces. As  $X$  is assumed to be non-algebraic, the  $G$ -action on  $X$  preserves the fibers of  $f$ . Hence we can apply Corollary 4.2 to conclude that  $(X, C)$  has a  $G$ -equivariant algebraic approximation.

Now assume that  $X$  does not contain any curve of geometric genus  $\geq 2$ , then  $C$  is a union of smooth elliptic curves. It follows that  $X$  is a smooth isotrivial elliptic fibration  $f : X \rightarrow S$ . Moreover, the fibration  $f$  does not have any proper curve other than the fibers of  $f$ . Indeed, if such a curve  $C'$  exists, then for any fiber  $F$  of  $f$  the image of  $\alpha : C' \times F \rightarrow X$  defined by  $\alpha(x, y) := x + y$  is an algebraic surface, so necessarily contains a curve of geometric genus  $\geq 2$  which is in contradiction with our assumption.

Since the only curves of  $X$  are fibers of  $f$ , the curve  $C$  is a union of fibers of  $f$ . It also follows that the  $G$ -action preserves the fibers of  $f$ , so induces a  $G$ -action on  $S$ . By Lemma 4.10, there exists an arbitrarily small  $G$ -equivariant strongly locally trivial, hence  $C$ -locally trivial, deformation  $f' : (X', C) \rightarrow B$  of  $f$  having a multisection. For such an  $X'$ , we already saw that  $X'$  contains at least one curve of geometric genus 2, so that  $(X', C)$ , and hence  $(X, C)$ , have a  $G$ -equivariant  $C$ -locally trivial algebraic approximation.  $\square$

### 5 Algebraic approximations of compact Kähler threefolds

We can now conclude the proof of Theorem 1.3.



*Proof of Theorem 1.3.* — Let  $X$  be a non-algebraic compact Kähler threefold and let  $X'$  be a bimeromorphic model of  $X$  for which we wish to prove that whenever  $C \subset X'$  is a curve or empty, the pair  $(X', C)$  has a locally trivial and  $C$ -locally trivial algebraic approximation. If the choice of  $X'$  is isomorphic to the quotient  $\tilde{X}/G$  of some smooth variety  $\tilde{X}$  by a finite group  $G$ , then first of all  $\tilde{X}/G$  is  $\mathbf{Q}$ -factorial. To prove that  $(\tilde{X}/G, C)$  has a locally trivial and  $C$ -locally trivial algebraic approximation, it suffices by Lemma 2.5 to prove that the pair  $(\tilde{X}, \tilde{C})$  has a  $G$ -equivariant  $\tilde{C}$ -locally trivial algebraic approximation  $(\tilde{\mathcal{X}}, \tilde{\mathcal{C}}) \rightarrow \Delta$  where  $\tilde{C}$  is the pre-image of  $C$  under the quotient map  $\tilde{X} \rightarrow \tilde{X}/G$ .

If  $\kappa(X) = 0$ , we choose  $X'$  to be a minimal model of  $X$ . In particular,  $X'$  is  $\mathbf{Q}$ -factorial and has at worst terminal singularities. By Proposition 3.1, the variety  $X'$  is a quotient  $\tilde{X}/G$  by a finite group where  $\tilde{X}$  is either a non-algebraic 3-torus or the product of a non-algebraic K3 surface and an elliptic curve. If  $C = \emptyset$ , since  $X'$  is minimal, the existence of a locally trivial algebraic approximation of  $X'$  results from [9, Theorem 1.4]. If  $C$  is a curve, the existence of a  $G$ -equivariant  $\tilde{C}$ -locally trivial algebraic approximation of  $(\tilde{X}, \tilde{C})$  is a consequence of Lemma 4.12 or of Lemma 4.9 together with Lemma 3.3, according to whether  $\tilde{X}$  is a 3-torus or the product of a non-algebraic K3 surface and an elliptic curve.

If  $\kappa(X) = 1$ , then by Theorem 3.4 there are two cases to be distinguished. If we are in the first case of Theorem 3.4, with the same notation therein we take  $X' = X_{\min}$ , so in particular  $X'$  is  $\mathbf{Q}$ -factorial with at worst terminal singularities. Since the canonical fibration  $X_{\min} \rightarrow B$  has a strongly locally trivial algebraic approximation [16, Theorem 1.6, Corollary 6.2], we can apply Lemma 4.1 and deduce that  $(X', C)$  has a locally trivial and  $C$ -locally trivial algebraic approximation for every curve  $C \subset X'$ . If we are in the second case of Theorem 3.4, with the same notation therein we take  $X' = \tilde{X}/G$  where  $G := \text{Gal}(\tilde{B}/B)$ . By [16, Proposition 4.7], the variety  $X'$  has at worst terminal singularities. The existence of a  $G$ -equivariant  $\tilde{C}$ -locally trivial algebraic approximation of  $(\tilde{X}, \tilde{C})$  is a consequence of Lemma 4.9 or Lemma 4.11, according to whether  $\tilde{X} \rightarrow \tilde{B}$  is a fibration in K3 surfaces or 2-tori.  $\square$

As was mentioned in the introduction, the combination of Proposition 1.4 and Theorem 1.3 proves Theorem 1.5, the existence of an algebraic approximation of any compact Kähler threefold of Kodaira dimension 0 or 1.

Finally we prove Proposition 1.7, which concerns threefold of Kodaira dimension 2.

*Proof of Proposition 1.7.* — As an output of the Kähler MMP for threefolds and the abundance theorem (cf. the beginning of Section 3), a compact Kähler threefold  $X$  with  $\kappa(X) = 2$  is bimeromorphic to an elliptic fibration  $X' \rightarrow B'$  with  $X'$  being normal and  $B'$  a projective surface. Let

$$X \xleftarrow{\mu} Y' \xrightarrow{\nu} X' \quad (5.1)$$

be a resolution of the bimeromorphic map  $X \dashrightarrow X'$  where  $\mu$  is bimeromorphic. Since  $X'$  is normal, there exists a subvariety  $C \subset X'$  of dimension at most 1 such that the restriction of  $\nu$  to  $Y' \setminus \nu^{-1}(C)$  is an isomorphism onto  $X' \setminus C$ . Accordingly  $f' : Y' \rightarrow B'$  is still an elliptic fibration. Let  $D' \subset B'$  denote the locus parameterizing singular fibers of  $f'$  and let  $(B, D) \rightarrow (B', D')$  be a log-resolution of the pair  $(B', D')$ . Let  $\nu' : Y \rightarrow Y' \times_{B'} B$  be a desingularization of  $Y' \times_{B'} B$ . As  $U := Y' \times_{B'} (B \setminus D) \rightarrow B \setminus D$  and hence  $U$  are smooth, we can assume that the restriction of  $\nu'$  to the Zariski open  $\nu'^{-1}(U)$  is an isomorphism onto  $U$ . It follows that  $Y \rightarrow B$  is an elliptic fibration whose locus of singular fibers is contained in the normal crossing divisor  $D$ .

Let  $\eta : Y \rightarrow Y' \times_{B'} B \rightarrow Y' \rightarrow X$  denote the composition, which is bimeromorphic. Since both  $Y$  and  $X$  are smooth, we have  $\eta_* \mathcal{O}_Y = \mathcal{O}_X$  and  $R^1 \eta_* \mathcal{O}_Y = 0$ . We can therefore apply [17, Theorem 2.1] as in the proof

of Lemma 2.3 to conclude that if Question 1.6 has a positive answer for the elliptic fibration  $Y \rightarrow B$ , then  $X$  has an algebraic approximation by [17, Theorem 2.1].

□

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