Stability and uniqueness for a two-dimensional inverse boundary value problem for less regular potentials

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Abstract

We consider inverse boundary value problems for the Schrödinger equations in two dimensions. Within less regular classes of potentials, we establish a conditional stability estimate of logarithmic order. Moreover we prove the uniqueness within L^p-class of potentials with $p > 2$.

In this paper, we prove stability estimates and the uniqueness for an inverse boundary value problem for the two-dimensional Schrödinger equation within a class of less regular unknown potentials. We refer to the first result Sylvester and Uhlmann [18] in the case where dimensions are higher than or equal to three, and since then many remarkable works concerning the uniqueness have been published. Here we do not intend to create a complete list of publications and see e.g., a survey by Uhlmann [19]. In particular, the arguments in two dimensions are different from higher dimensions and we refer to the uniqueness result by Nachman [14], and a stability estimate by Alessandrini [2]. Also see Liu [11], and as survey on the uniqueness mainly in two dimensions, see Imanuvilov and Yamamoto [8]. So far all these estimates have had a logarithmic modulus of continuity, which is no surprise because

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Mandache showed that this is the best one could expect [12]. The other fact is that most of the above mentioned work was done for the conductivity equation, and so there were not many papers on inverse boundary value problems for the Schrödinger equation with a potential in two dimensions. The result on uniqueness in this paper (Theorem 2.2) was announced by a pioneering contribution (Bukhgeim [3]) that has led to many developments in the study of two dimensional inverse boundary value problems. However, his proof only gives uniqueness for potentials in the class W^1_p as pointed out in Blåsten's licentiate thesis [5]. See also Novikov and Santacesaria [15], which proved stability assuming some smoothness and [16] which showed also a reconstruction formula. Santacesaria [17] continued working on stability, and showed that the smoother it is, the better exponent there will be on the logarithm.

There are not many results about stability and uniqueness for less regular potentials and we refer to Blåsten [6], and Imanuvilov and Yamamoto [9]. The former is the doctoral thesis of the first named author and proved conditional stability under some a priori boundedness of unknown potentials, and the latter proved the uniqueness in determining L^p-potentials with $p > 2$.

In this paper we prove the uniqueness result announced by Bukhgeim for L^p potentials, $p > 2$, and in addition give logarithmic type stability estimates for potentials in the class W^s_2, $s \in (0, 1] \setminus \{\frac{1}{2}\}$. After [6] and [9], the authors recognized that an improvement and simplification of the proofs are possible. That is, the main purpose of this paper is to improve the stability estimates obtained in [6] and simplify the proof of [9] by using a unified method.

The paper is composed of six sections. In Section 2, we formulate our inverse problem and in Section 3 we state two main results Theorems 2.1 on the conditional stability and Theorem 2.2 on the uniqueness and compare them with the results in [6] and [9]. Sections 3-6 are devoted for completing the proofs of Theorems 2.1 and 2.2.

1 Formulation

Let $X \subset \mathbb{R}^2$ be a bounded domain with boundary ∂X of C^∞-class. Although it is possible to relax the regularity of the boundary for example to a Lipschitz domain, we assume C^∞-boundary for simplicity. Moreover let $q \in L^p(X)$, $p > 2$, be a potential function. Consider the Schrödinger operator with the potential q in the domain X

$$L_q(x, D)u := \Delta u + qu.$$

We define define the *Cauchy data* $\mathcal{C}_q$ by

Definition 1.1. *Let $X \subset \mathbb{R}^2$ be a bounded domain with smooth boundary ∂X and $q \in L^p(X)$ with $p > 1$. Then*

$$\mathcal{C}_q = \{(u, \partial_\nu u) \in W_2^{1/2}(\partial X) \times W_2^{-1/2}(\partial X);\ L_q(x, D)u = 0, u \in W_2^1(X)\}.$$

If zero is not an eigenvalue of the operator $L_q(x, D)$ with the zero Dirichlet boundary conditions, then the Cauchy data are equivalent to the Dirichlet-to-Neumann map Λ_q defined by

$$\Lambda_q f = \frac{\partial u}{\partial \nu}|_{\partial X}, \quad f \in W_2^{1/2}(\partial X),$$

where $u \in W_2^1(X)$ is a unique solution to $L_q(x, D)u = 0$ in X and $u|_{\partial X} = f$.

The paper is concerned with a variant of the classical Calderón problem: *Suppose that for two potentials q_1 and q_2 the corresponding Cauchy data are equal. Does that imply the uniqueness of the potentials?*

The inverse problem asks whether the mapping $q \mapsto \mathcal{C}_q$ is invertible. The uniqueness means that no two different potentials q have the same Cauchy data $\mathcal{C}_q$. The stability means that the mapping inverse to $q \mapsto \mathcal{C}_q$ is continuous in some topologies. For formulating the stability, we define the difference of Cauchy data by

$$d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) := \sup_{(u_1,u_2)\in \mathcal{X}_{q_1} \times \mathcal{X}_{q_2}} \left| \int_X u_1(q_1 - q_2)u_2 dx \right|,$$

where

$$\mathcal{X}_q = \{u \in W_2^1(X);\ L_q(x, D)u = 0,\ \|u\|_{W_2^1(X)} = 1\}.$$

The difference $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})$ is not a metric, but if $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ then $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) = 0$. Moreover if zero is not an eigenvalue of the operator $L_{q_j}(x, D)$, $j = 1, 2$ with the zero Dirichlet boundary condition, then

$$d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \le C \left\| \Lambda_{q_1} - \Lambda_{q_2} \right\|_{\mathcal{L}(W_2^{1/2}(\partial X); W_2^{-1/2}(\partial X))}$$

by Lemma 3.2 proved below. Here the right-hand side denotes the operator norm. This inequality means that for given $\mathcal{C}_{q_1}$ and $\mathcal{C}_{q_2}$, without knowing q_1, q_2 in X, it is possible to calculate an upper bound for $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})$.

Usually one can show only *conditional stability*, which means stability under some assumptions on norms of unknown potentials q's. Other important topic is the reconstruction of a potential. That is, given a Cauchy data, reconstruct the potential using an explicit algorithm, and an even more valuable goal is to reconstruct q in a stable way by given noisy data about $\mathcal{C}_q$. As for the reconstruction of less regular potentials, see Astala, Faraco and

Rogers [4], which shows a reconstruction formula for potentials in $W_2^{1/2}$, and proves that there exists a set of positive measure where the reconstruction does not converge pointwise for less regular potentials. Our proof suggests that the reconstruction converges in the L^2-norm and we here do not discuss details.

Notations. Let $i = \sqrt{-1}$, $x = (x_1, x_2), x_1, x_2 \in \mathbb{R}^1$, $z = x_1 + ix_2$ and $\overline{z}$ denote the complex conjugate of $z \in \mathbb{C}$. We identify $x \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$ and $\xi = (\xi_1, \xi_2)$ with $\zeta = \xi_1 + i\xi_2$. We set $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$. By $\mathcal{L}(Y_1, Y_2)$ we denote the space of linear continuous operators from a Banach space Y_1 into a Banach space Y_2. Let $B(0, \delta)$ be a ball in $\mathbb{R}^2$ of radius δ centered at 0. We define the Fourier transform by $(\mathscr{F}u)(\xi) = \int_{\mathbb{R}^2} u(x)e^{-i(x,\xi)}dx$.

2 Main results

Henceforth $C > 0$ denotes generic constants which are dependent on X and constants s, M, but independent of parameters τ, where s, M, τ are given later.

We here state our two main results.

Theorem 2.1. *Let $X \subset \mathbb{R}^2$ be a bounded domain with smooth boundary ∂X and $s \in (0, 1] \setminus \{\frac{1}{2}\}$. We assume that $q_1, q_2 \in W_2^s(X)$ satisfy an a priori estimate $\|q_j\|_{W_2^s(X)} \le M$ with $M < \infty$ and $q_1 - q_2 \in \mathring{W}_2^s(X)$. Then there exists a constant $C > 0$ such that*

$$\|q_1 - q_2\|_{L^2(X)} \le \begin{cases} C\left(1 + \ln \frac{1}{d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})}\right)^{-s/2}, & \text{if } d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) < 1, \\ Cd(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}), & \text{if } d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \ge 1. \end{cases}$$

Note that when $s < \frac{1}{2}$ no boundary behaviour is required from the two potentials (e.g., Adams and Fournier [1], Lions and Magenes [10]).

In our stability result, we estimate the norm $\|q_1 - q_2\|_{L^2(X)}$ under the a priori boundedness of the norm in $\mathring{W}_2^s(X)$, while the work [6] uses different norms for $q_1 - q_2$ and a priori boundedness and for the norm. As for the exponent in the estimate, our result asserts $-s/2$ which is better than $-s/4$ in [6], but it is still controlled by a logarithmic rate.

By the theorem 2.1, we see that

$$\|q_1 - q_2\|_{L^2(X)} = O\left(\left(\ln \frac{1}{d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})}\right)^{-s/2}\right)$$

as $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \longrightarrow 0$. Thus the rate of the conditional stability is logarithmic.

By Lemma 3.2 below, from Theorem 2.1, we can derive

Corollary. *Under the same assumptions of Theorem 2.1, we further assume that zero is not an eigenvalue of $L_{q_j}(x, D)$ with the zero Dirichlet boundary condition. Let $s \in (0, 1]$, and let $q_1, q_2 \in W_2^s(X)$ satisfy $\|q_j\|_{W_2^s(X)} \leq M$ with $M < \infty$ and $q_1 - q_2 \in \mathring{W}_2^s(X)$. Then there exists a constant $C > 0$ such that*

$$\|q_1 - q_2\|_{L^2(X)} \leq \begin{cases} C\left(1 + \ln \frac{1}{\|\Lambda_{q_1} - \Lambda_{q_2}\|}\right)^{-s/2}, & \text{if} \|\Lambda_{q_1} - \Lambda_{q_2}\| < 1, \\ C\|\Lambda_{q_1} - \Lambda_{q_2}\|, & \text{if} \|\Lambda_{q_1} - \Lambda_{q_2}\| \geq 1. \end{cases}$$

where $\|\Lambda_{q_1} - \Lambda_{q_2}\|$ is the norm in $\mathcal{L}(W_2^{1/2}(\partial X); W_2^{-1/2}(\partial X))$.

Our second main result is the uniqueness in the recovery of the potential for the Schrödinger operator :

Theorem 2.2. *Let $X \subset \mathbb{R}^2$ be a bounded smooth domain and $q_1, q_2 \in L^p(X)$ with $p > 2$. If $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, then $q_1 = q_2$.*

The merits for the proof of our unified method are as follows.

1. The proofs of both stability and uniqueness are simplified. Blåsten [6] used Sobolev spaces where the L^p-norm has been replaced by a Lorentz-norm. We can avoid using the Lorentz-norm by showing a Carleman estimate formulated using conventional L^p-spaces.

2. Comparing with Imanuvilov and Yamamoto [9], we use a simpler L^2-convergent stationary-phase argument which avoids approximating the potentials by test functions and using Egorov's theorem.

3 Key lemmas and definitions

We start this section with the following Lemma:

Lemma 3.1. *Let $X \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $q_1, q_2 \in L^p(X)$, $p > 1$. If $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, then*

$$\int_X u_1(q_1 - q_2)u_2 dx = 0$$

for all $(u_1, u_2) \in \mathcal{X}_{q_1} \times \mathcal{X}_{q_2}$.

Lemma 3.2. *Let $X \subset \mathbb{R}^2$ be a bounded smooth domain and $q_1, q_2 \in L^p(X)$, $p > 1$ be potentials. We assume that 0 is not an eigenvalue of the operator $L_{q_j}(x, D)$, $j = 1, 2$, with the zero Dirichlet boundary condition. Then*

$$d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \le \|\mathrm{Tr}\|^2_{\mathcal{L}(W_2^1(X); W_2^{1/2}(\partial X))} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathcal{L}(W_2^{1/2}(\partial X); W_2^{-1/2}(\partial X))}.$$

Proof. Let $u_1, U \in W_2^1(X)$ satisfy $L_{q_1}(x, D)U = L_{q_1}(x, D)u_1 = 0$ in X and $U = u_2$ on ∂X. Then

$$\Delta(U - u_2) + q_1(U - u_2) + (q_1 - q_2)u_2 = 0 \quad \text{in } X$$

and $U - u_2 = 0$ on ∂X. Multiplying by u_1, integrating by parts and using $\Delta u_1 + q_1 u_1 = 0$ in X and $U - u_2 = 0$ on ∂X, we have

$$\int_X u_1(q_1 - q_2)u_2 dx = \int_{\partial X} \partial_\nu(u_2 - U)u_1 d\sigma.$$

Now note that $(U, \partial_\nu U) \in \mathcal{C}_{q_1}$ and $(u_2, \partial_\nu u_2) \in \mathcal{C}_{q_2}$. This observation allows us to switch to the Dirichlet-to-Neumann maps, and so

$$\begin{aligned}
&\left| \int_{\partial X} (\partial_\nu u_2 - \partial_\nu U)u_1 d\sigma \right| \\
&\qquad = \left| \int_{\partial X} (\Lambda_{q_2} u_2 - \Lambda_{q_1} U)u_1 d\sigma \right| = \left| \int_{\partial X} ((\Lambda_{q_2} - \Lambda_{q_1})u_2)u_1 d\sigma \right|
\end{aligned}$$

because $u_2 = U$ on ∂X. Now take the supremum over $(u_1, u_2) \in \mathcal{X}_{q_1} \times \mathcal{X}_{q_2}$, to obtain

$$\begin{aligned}
&\sup_{(u_1,u_2)\in \mathcal{X}_{q_1} \times \mathcal{X}_{q_2}} \left| \int_X u_1(q_1 - q_2)u_2 dx \right| \\
&\qquad = \sup_{(u_1,u_2)\in \mathcal{X}_{q_1} \times \mathcal{X}_{q_2}} \left| \int_{\partial X} ((\Lambda_{q_2} - \Lambda_{q_1})u_2)u_1 d\sigma \right| \\
&\qquad \le \|\mathrm{Tr}\|^2_{\mathcal{L}(W_2^1(X); W_2^{1/2}(\partial X))} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathcal{L}(W_2^{1/2}(\partial X); W_2^{-1/2}(\partial X))}.
\end{aligned}$$

The proof of Lemma 3.2 is complete. □

Henceforth we identify $z_0 = x_{01} + ix_{02} \in \mathbb{C}$ with $x_0 = (x_{01}, x_{02}) \in \mathbb{R}^2$.

The following lemma plays the important role in the proof of Theorems 2.1 and 2.2.

Lemma 3.3. *Let $\tau > 0$, $0 \le s \le 1$ and $Q \in W_2^s(\mathbb{R}^2)$, $z_0 \in \mathbb{C}$. Then*

$$\left\| Q - \int_{\mathbb{R}^2} \frac{2\tau}{\pi} e^{\pm i\tau\left((z-z_0)^2 + (\overline{z-z_0})^2\right)} Q dx \right\|_{L^2(\mathbb{R}^2; dx_0)} \le 2\tau^{-s/2} \|Q\|_{W_2^s(\mathbb{R}^2)}. \qquad (1)$$

If $s = 0$, then the left-hand side tends to 0 as $\tau \to \infty$.

Proof. First for $\delta > 0$, we have

$$\begin{aligned} &\theta_\delta(\xi) := \mathscr{F}(e^{\pm i\tau(z^2+\overline{z}^2)-\delta|z|^2})(\xi) \\ = \quad &\frac{\pi}{\sqrt{\delta^2+4\tau^2}} \exp\left(-\frac{\delta|\xi|^2}{16\tau^2+4\delta^2}\right) \exp\left(\frac{\mp i\tau(\xi_1^2-\xi_2^2)\tau}{8\tau^2+2\delta^2}\right). \end{aligned}$$

The calculations are direct and we refer to pp.210-211 in Evans [7] for example. Let $\mathcal{S}(\mathbb{R}^2)$ be the space rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^2)$ be the dual, that is, the space of tempered distributions. Since

$$\theta_\delta \longrightarrow \frac{\pi}{2\tau} \exp\left(\frac{\mp i(\xi_1^2-\xi_2^2)}{8\tau}\right) = \frac{\pi}{2\tau} \exp\left(\mp \frac{i(\zeta^2+\overline{\zeta}^2)}{16\tau}\right)$$

and

$$e^{\pm i\tau(z^2+\overline{z}^2)-\delta|z|^2} \longrightarrow e^{\pm i\tau(z^2+\overline{z}^2)}$$

as $\delta \downarrow 0$ in $\mathcal{S}'(\mathbb{R}^2)$ and $\mathscr{F}$ is continuous from $\mathcal{S}'(\mathbb{R}^2)$ to itself, we see

$$\mathscr{F}(e^{\pm i\tau(z^2+\overline{z}^2)})(\xi) = \frac{\pi}{2\tau} \exp\left(\mp \frac{i(\zeta^2+\overline{\zeta}^2)}{16\tau}\right)$$

in $\mathcal{S}'(\mathbb{R}^2)$. This equality holds for almost all $\xi \in \mathbb{R}^2$, because the right-hand side is in $L^\infty(\mathbb{R}^2)$.

Next let $Q \in C_0^\infty(\mathbb{R}^2)$ be arbitrarily chosen. Then

$$\mathscr{F}\left(\frac{2\tau}{\pi} e^{\pm i\tau(z^2+\overline{z}^2)} * Q\right) = \exp\left(\frac{\mp i(\zeta^2+\overline{\zeta}^2)}{16\tau}\right) \mathscr{F}(Q)(\xi).$$

Hence by the Plancherel theorem, we have

$$\begin{aligned} \left\| Q - \tfrac{2\tau}{\pi} e^{\pm i\tau(z^2+\overline{z}^2)} * Q \right\|_{L^2(\mathbb{R}^2)} &= \tfrac{1}{2\pi} \left\| \mathscr{F}Q - \mathscr{F}\left(\tfrac{2\tau}{\pi} e^{\pm i\tau(z^2+\overline{z}^2)} * Q\right) \right\|_{L^2(\mathbb{R}^2)} \\ &= \tfrac{1}{2\pi} \left\| \left(1 - e^{\mp i \frac{\xi^2+\overline{\xi}^2}{16\tau}}\right) \mathscr{F}Q \right\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

On the other hand, we can prove

$$|1 - e^{\mp i(\zeta^2+\overline{\zeta}^2)}| \le 2^{1+s/2}\,|\xi|^s$$

for $0 \le s \le 1$ and $\zeta \in \mathbb{C}$. In fact, if $|\xi| \ge 1$, then $|1 - e^{\mp i(\zeta^2+\overline{\zeta}^2)}| \le 2 \le 2^{1+s/2}$ and so the inequality is seen. Let $|\xi| \le 1$. Direct calculations yield $|1 - e^{\mp i(\zeta^2+\overline{\zeta}^2)}|^2 = 4\sin^2(\xi_1^2 - \xi_2^2)$. Therefore

$$|1 - e^{\mp i(\zeta^2+\overline{\zeta}^2)}|^2 \le 4|\xi_1^2 - \xi_2^2|^2 \le 4|\xi_1^2 + \xi_2^2|^2 \le 4 \times 2^s |\xi|^{2s},$$

where we used $0 \le s \le 1$ and $|\xi| \le 1$. Thus we have seen $|1 - e^{\mp i(\xi^2+\overline{\xi}^2)}| \le 2^{1+s/2}\,|\xi|^s$ for $0 \le s \le 1$ and $\xi \in \mathbb{C}$.

Hence

$$\left\| Q - \frac{2\tau}{\pi} e^{\pm i\tau(z^2+\overline{z}^2)} * Q \right\|_{L^2(\mathbb{R}^2)} \le \frac{1}{\pi} 2^{s/2} \left\| \left(\frac{\xi}{4\sqrt{|\tau|}} \right)^s \mathscr{F} Q \right\|_{L^2(\mathbb{R}^2)}$$

$$\le \quad \frac{1}{\pi} 2^{s/2} 2^{-2s} |\tau|^{-s/2} \|(1+|\xi|^2)^{s/2} \mathscr{F} Q\|_{L^2(\mathbb{R}^2)}. \tag{2}$$

for each $Q \in C_0^\infty(\mathbb{R}^2)$. Since $C_0^\infty(\mathbb{R}^2)$ is dense in $W_2^s(\mathbb{R}^2)$, passing to the limits, we complete the proof of Lemma 3.3 for $s > 0$. If $s = 0$ and $Q \in L^2(\mathbb{R}^2)$ for any positive ϵ we take a function $Q_\epsilon \in C_0^\infty(\mathbb{R}^2)$ such that $\|Q - Q_\epsilon\|_{L^2(\mathbb{R}^2)} \le \epsilon$. Then (2) implies that for any positive τ

$$\left\| Q - Q_\epsilon - \frac{2\tau}{\pi} e^{\pm i\tau(z^2+\overline{z}^2)} * (Q - Q_\epsilon) \right\|_{L^2(\mathbb{R}^2)} \le \frac{1}{\pi} \|Q - Q_\epsilon\|_{L^2(\mathbb{R}^2)} \le \epsilon.$$

Then applying to the function Q_ϵ estimate (1), we obtain the statement of our lemma for $s = 0$. $\square$

4 Preliminary estimates

Let us introduce the operators:

$$\bar{\partial}^{-1} g = -\frac{1}{\pi} \int_X \frac{g(\xi_1,\xi_2)}{\zeta - z} d\xi_1 d\xi_2, \quad \partial^{-1} g = -\frac{1}{\pi} \int_X \frac{g(\xi_1,\xi_2)}{\overline{\zeta} - \overline{z}} d\xi_1 d\xi_2,$$

where $X \subset \mathbb{R}^2$ is a bounded domain with the smooth boundary.

We have

Proposition 4.1. A) *Let $1 \le p \le 2$ and $1 < \gamma < \frac{2p}{2-p}$. Then $\bar{\partial}^{-1}, \partial^{-1} \in \mathcal{L}(L^p(X), L^\gamma(X))$.*
B)*Let $1 < p < \infty$. Then $\bar{\partial}^{-1}, \partial^{-1} \in \mathcal{L}(L^p(X), W_p^1(X))$.*

A) is proved on p.47 in [20] and B) can be verified by using Theorem 1.32 (p.56) in [20]. ■

Henceforth for arbitrarily fixed $z_0 \in \mathbb{C}$, we set

$$\Phi(z) = \Phi(z; z_0) := (z - z_0)^2$$

and introduce the operator:

$$\widetilde{\mathcal{R}}_\tau g = \frac{1}{2} e^{-i\tau(\overline{\Phi}+\Phi)} \partial^{-1}(g e^{i\tau(\Phi+\overline{\Phi})}).$$

We set

$$U_0 = 1, \quad U_1 = \widetilde{\mathcal{R}}_\tau(\frac{1}{2}(\bar{\partial}^{-1}q - \bar{\partial}^{-1}q(x_0))), \tag{3}$$

$$U_j = \widetilde{\mathcal{R}}_\tau(\frac{1}{2}\bar{\partial}^{-1}(qU_{j-1})) \quad \forall j \geq 2. \tag{4}$$

We construct a solution to the Schrödinger equation in the form

$$u_1 = \sum_{j=0}^{\infty} e^{i\tau\Phi}(-1)^j U_j. \tag{5}$$

Henceforth $C(\epsilon)$ denotes generic constants which are dependent on not only s, M, X but also ϵ.

We will prove that the infinite series is convergent in $L^r(X)$ with some $r > 2$. For it, we show the following propositions.

Proposition 4.2. *Let $u \in W_p^1(X)$ for any $p > 2$. Then for any $\epsilon \in (0,1)$ there exists a constant $C(\epsilon)$ independent of $x_0 \in X$ and τ such that*

$$\tau^{1-\epsilon}\|\widetilde{\mathcal{R}}_\tau u\|_{L^2(X)} + \tau^{1/p}\|\widetilde{\mathcal{R}}_\tau u\|_{L^\infty(X)} \leq C(\epsilon)\|u\|_{W_p^1(X)} \quad \forall \tau > 0. \tag{6}$$

Proof. Let $\rho \in C_0^\infty(B(0,1))$ and $\rho|_{B(0,\frac{1}{2})} = 1$. We set $\rho_\tau = \rho(\sqrt{\tau}(x - x_0))$. Since $\widetilde{\mathcal{R}}_\tau u = \widetilde{\mathcal{R}}_\tau(\rho_\tau u) + \widetilde{\mathcal{R}}_\tau((1-\rho_\tau)u)$ for any positive ϵ, there exists $p_0(\epsilon) > 1$ such that $\|e^{i\tau(\Phi+\bar{\Phi})}\rho_\tau u\|_{L^{p_0(\epsilon)}(X)} \leq C(\epsilon)\|u\|_{W_p^1(X)}/\tau^{1-\epsilon}$. Moreover since $\|e^{i\tau(\Phi+\bar{\Phi})}u\|_{L^\infty(X)} \leq C\|u\|_{W_p^1(X)}$ we have

$$\|e^{i\tau(\Phi+\bar{\Phi})}\rho_\tau u\|_{L^\infty(X)} \leq C(\epsilon)\|u\|_{W_p^1(X)}/\tau^{1-\epsilon}.$$

Hence applying Proposition 4.1 and the Sobolev embedding theorem, we have

$$\tau^{1-\epsilon}\|\widetilde{\mathcal{R}}_\tau(\rho_\tau u)\|_{L^2(X)} + \tau^{1/p}\|\widetilde{\mathcal{R}}_\tau(\rho_\tau u)\|_{L^\infty(X)} \leq C(\epsilon)\|u\|_{W_p^1(X)}, \quad \forall \epsilon \in (0,1). \tag{7}$$

Observe that

$$\begin{aligned}
\int_X \frac{(1-\rho_\tau)ue^{i\tau(\Phi+\overline{\Phi})}}{\overline{z}-\overline{\zeta}}d\xi &= \int_X \frac{(1-\rho_\tau)u\partial e^{i\tau(\Phi+\overline{\Phi})}}{\tau(\overline{z}-\overline{\zeta})i\partial\Phi}d\xi \\
&= \int_{\partial X} \frac{(\nu_1 - i\nu_2)(1-\rho_\tau)ue^{i\tau(\Phi+\overline{\Phi})}}{2i\tau(\overline{z}-\overline{\zeta})\partial\Phi}d\sigma \\
-\int_X \frac{1}{\tau(\overline{z}-\overline{\zeta})}\partial\left(\frac{(1-\rho_\tau)u}{i\partial\Phi}\right)e^{i\tau(\Phi+\overline{\Phi})}d\xi &+ \frac{(1-\rho_\tau)ue^{i\tau(\Phi+\overline{\Phi})}}{i\tau\partial\Phi}.
\end{aligned} \tag{8}$$

Obviously, by the Sobolev embedding theorem, for any positive ϵ, there exists a constant $C(\epsilon)$ such that

$$\tau^{1-\epsilon}\left\|\frac{(1-\rho_\tau)u}{\tau\partial\Phi}\right\|_{L^2(X)}+\tau^{1/2}\left\|\frac{(1-\rho_\tau)u}{\tau\partial\Phi}\right\|_{L^\infty(X)}\le C(\epsilon)\|u\|_{W^1_p(X)}. \quad (9)$$

For the second term on the right-hand side of (8), we have

$$\left|\int_X \frac{1}{\tau(\overline{z}-\overline{\zeta})}\partial\left(\frac{(1-\rho_\tau)u}{\partial\Phi}\right)e^{i\tau(\Phi+\overline{\Phi})}d\xi\right|\le\int_X\left|\frac{1}{\tau^{\frac12}(\overline{z}-\overline{\zeta})}\left(\frac{\partial\rho(\sqrt{\tau}\xi)u}{\partial\Phi}\right)\right|d\xi$$
$$+\int_X\left|\frac{1}{\tau(\overline{z}-\overline{\zeta})}\left(\frac{(1-\rho_\tau)\partial u}{\partial\Phi}\right)\right|d\xi+\int_X\left|\frac{2}{\tau(\overline{z}-\overline{\zeta})}\left(\frac{(1-\rho_\tau)u}{(\partial\Phi)^2}\right)\right|d\xi.$$

The functions $\frac{(1-\rho_\tau)\partial u}{\partial\Phi}$ are uniformly bounded in τ in $L^{p_1}(X)$ for some $p_1\in(1,2)$. Moreover, since $\|(1-\rho_\tau)/\partial\Phi\|_{L^\infty(X)}\le C\sqrt{\tau}$, the functions$\sqrt{\tau}\frac{(1-\rho_\tau)\partial u}{\partial\Phi}$ are uniformly bounded in τ in functions $\frac{(1-\rho_\tau)\partial u}{\sqrt{\tau}\partial\Phi}$ are uniformly bounded in τ in $L^p(X)$. Applying Proposition 4.1, we have

$$\tau\left\|\partial^{-1}\left(\frac{(1-\rho_\tau)\partial_z u}{\tau\partial\Phi}\right)\right\|_{L^2(X)}+\tau^{1/p}\left\|\partial^{-1}\left(\frac{(1-\rho_\tau)\partial_z u}{\tau\partial\Phi}\right)\right\|_{L^\infty(X)}\le C\|u\|_{W^1_p(X)}. \quad (10)$$

On the other hand, for any $p_2>1$ we have

$$\left\|\frac{\partial\rho(\sqrt{\tau}\cdot)u}{\partial\Phi}\right\|_{L^{p_2}(X)}\le C\|u\|_{C^0(\overline{X})}\left\|\frac{1}{\partial\Phi}\right\|_{L^{p_2}(B(0,\frac{1}{\sqrt{\tau}}))}\le C\tau^{(2-p_2)/2p_2}\|u\|_{W^1_p(X)}.$$

Thanks to this inequality, applying Proposition 4.1 again, we have:

$$\tau^{1-\epsilon}\left\|\frac{1}{\tau^{\frac12}}\partial^{-1}\left(\frac{\partial\rho(\sqrt{\tau}\cdot)u}{\partial\Phi}\right)\right\|_{L^2(X)}+\tau^{1/p}\left\|\frac{1}{\tau^{\frac12}}\partial^{-1}\left(\frac{\partial\rho(\sqrt{\tau}\cdot)u}{\partial\Phi}\right)\right\|_{L^\infty(X)}$$
$$\le C(\epsilon)\|u\|_{W^1_p(X)}. \quad (11)$$

For any $p_3>1$, we have

$$\left\|\frac{(1-\rho_\tau)u}{(\partial\Phi)^2}\right\|_{L^{p_3}(X)}\le C\|u\|_{C^0(\overline{X})}\left\|\frac{1}{(\partial\Phi)^2}\right\|_{L^{p_3}(X\setminus B(0,\frac{1}{2\sqrt{\tau}}))}$$
$$\le C(p_3)\|u\|_{W^1_p(X)}\tau^{(2p_3-2)/2p_3}.$$

Therefore

$$\tau^{1-\epsilon}\left\|\partial^{-1}\left(\frac{(1-\rho_\tau)u}{\tau(\partial\Phi)^2}\right)\right\|_{L^2(X)}+\tau^{1/p}\left\|\partial^{-1}\left(\frac{(1-\rho_\tau)u}{\tau(\partial\Phi)^2}\right)\right\|_{L^\infty(X)}$$
$$\le C(\epsilon)\|u\|_{W^1_p(X)}. \quad (12)$$

From the classical representation of the Cauchy integral (see e.g. [13] p.27) we obtain

$$\left\|\int_{\partial X}\frac{(\nu_1-i\nu_2)(1-\rho_\tau)ue^{i\tau(\Phi+\overline{\Phi})}}{2i\tau(\overline{z}-\overline{\zeta})\partial\Phi}d\sigma\right\|_{L^2(X)}$$
$$\le C\left\|\frac{(\nu_1-i\nu_2)(1-\rho_\tau)ue^{i\tau(\Phi+\overline{\Phi})}}{2i\tau\partial\Phi}\right\|_{L^1(\partial X)}$$
$$\le C\left\|\frac{(1-\rho_\tau)}{\partial\Phi}\right\|_{L^1(\partial X)}\|u\|_{W^1_p(X)}/\tau\le C\|u\|_{W^1_p(X)}\ln\tau/\tau. \qquad (13)$$

By the trace theorem and the Sobolev embedding theorem, for any $p>2$ there exists a positive $\alpha=\alpha(p)$ such that the trace operator is continuous from $W^1_p(X)$ into $C^\alpha(\partial X)$. Using Theorem 1.11 (see p. 22 of [20]), for any $\delta\in(0,\alpha(p))$, there exists a constant $C(\delta)>0$ such that

$$\left\|\int_{\partial X}\frac{(\nu_1-i\nu_2)(1-\rho_\tau)ue^{i\tau(\Phi+\overline{\Phi})}}{2i\tau(\overline{z}-\overline{\zeta})\partial\Phi}d\sigma\right\|_{L^\infty(X)}$$
$$\le C(\delta)\left\|\frac{(\nu_1-i\nu_2)(1-\rho_\tau)ue^{i\tau(\Phi+\overline{\Phi})}}{2i\tau\partial\Phi}\right\|_{C^\delta(\partial X)}$$
$$\le C(\delta)\|\frac{(1-\rho_\tau)}{\partial\Phi}e^{i\tau(\Phi+\overline{\Phi})}\|_{C^\delta(\partial X)}\|u\|_{W^1_p(X)}/\tau.$$

Denote $\mu_\tau(x)=\frac{(1-\rho_\tau)}{\partial\Phi}e^{i\tau(\Phi+\overline{\Phi})}$. Then by the definitions of the functions Φ and ρ_τ (noting that we identify z_0 with x_0), we estimate

$$\|\mu_\tau(\cdot)\|_{C^0(\partial X)}\le C\sqrt{\tau}\quad\text{and}\quad\|\nabla\mu_\tau(\cdot)\|_{C^0(\partial X)}\le C\tau\quad\forall\tau>1.$$

Since in view of the mean value theorem, we can estimate

$$|\mu_\tau(x)-\mu_\tau(x')|=|\mu_\tau(x)-\mu_\tau(x')|^{1-\delta}|\mu_\tau(x)-\mu_\tau(x')|^\delta\le C\tau^{\frac{1-\delta}{2}}\tau^\delta|x-x'|^\delta \qquad (14)$$

and we obtain

$$\left\|\int_{\partial X}\frac{(\nu_1-i\nu_2)(1-\rho_\tau)ue^{i\tau(\Phi+\overline{\Phi})}}{2i\tau(\overline{z}-\overline{\zeta})\partial\Phi}d\sigma\right\|_{L^\infty(X)}\le C(\delta)\|u\|_{W^1_p(X)}/\tau^{(1-\delta)/2}. \qquad (15)$$

From (7)-(15) we have (6). ■

Now we proceed to the proof that the infinite series (5) is convergent in $L^r(X)$ for all sufficiently large τ. Let $\tilde{p} \in (2,p)$. By (6) and Proposition 4.1 and the Hölder inequality, there exists a positive constant $\delta(\tilde{p})$ such that

$$\|\widetilde{\mathcal{R}}_\tau u\|_{L^{\frac{p\tilde{p}}{p-\tilde{p}}}(X)} \le C\|u\|_{W^1_{\tilde{p}}(X)}/\tau^\delta. \tag{16}$$

Using (16) we have

$$\begin{aligned}
\|U_j\|_{L^{\frac{p\tilde{p}}{p-\tilde{p}}}(X)} &\le \frac{C}{\tau^\delta}\|\frac{1}{2}\bar{\partial}^{-1}(qU_{j-1})\|_{W^1_{\tilde{p}}(X)} \\
&\le \frac{C}{2\tau^\delta}\|\bar{\partial}^{-1}\|_{\mathcal{L}(L^{\tilde{p}}(X);W^1_{\tilde{p}}(X))}\|qU_{j-1}\|_{L^{\tilde{p}}(X)} \\
&\le \frac{C}{2\tau^\delta}\|\bar{\partial}^{-1}\|_{\mathcal{L}(L^{\tilde{p}}(X);W^1_{\tilde{p}}(X))}\|q\|_{L^p(X)}\|U_{j-1}\|_{L^{\frac{\tilde{p}p}{p-\tilde{p}}}(X)} \\
&\le \left(\frac{C\|\bar{\partial}^{-1}\|_{\mathcal{L}(L^{\tilde{p}}(X);W^1_{\tilde{p}}(X))}\|q\|_{L^p(X)}}{2\tau^\delta}\right)^{j-1}\|U_1\|_{L^{\frac{p\tilde{p}}{p-\tilde{p}}}(X)}.
\end{aligned} \tag{17}$$

Therefore there exists τ_0 such that for all $\tau > \tau_0$

$$\|U_j\|_{L^{\frac{p\tilde{p}}{p-\tilde{p}}}(X)} \le \frac{1}{2^j}\|U_1\|_{L^{\frac{p\tilde{p}}{p-\tilde{p}}}(X)} \quad \forall j \ge 2.$$

Hence the convergence of the series is proved.

Since

$$\begin{aligned}
& L_q(x,D)(U_je^{i\tau\Phi}) = 4\bar{\partial}\partial(e^{i\tau\Phi}\widetilde{R}_\tau(\frac{1}{2}\bar{\partial}^{-1}(qU_{j-1}))) + qU_je^{i\tau\Phi} \\
= \quad & 2\bar{\partial}(e^{i\tau\Phi}\frac{1}{2}\bar{\partial}^{-1}(qU_{j-1})) + q_1U_je^{i\tau\Phi} = qU_{j-1}e^{i\tau\Phi} + qU_je^{i\tau\Phi},
\end{aligned}$$

the infinite series (5) represents the solution to the Schrödinger equation. By Proposition 4.2, we have

$$\left\|\sum_{j=2}^\infty(-1)^jU_j\right\|_{L^2(X)} = O\left(\frac{1}{\tau^{\frac{3}{2}}}\right) \quad \text{as } \tau \to +\infty. \tag{18}$$

Besides the estimate (18) we need the estimate of the infinite series $\sum_{j=2}^\infty(-1)^jU_j$ in the space $L^\infty(X)$.

By Proposition 4.2, we have

$$\left\|\sum_{j=2}^\infty(-1)^jU_j\right\|_{L^\infty(X)} = O\left(\frac{1}{\tau^{\frac{1}{p}}}\right) \quad \text{as } \tau \to +\infty. \tag{19}$$

Proposition 4.3. *Let $q \in L^p(X)$ and $2 < p < \infty$. Then there exists a positive constant $\widehat{C}(\|q\|_{L^p(X)})$ independent of τ and x_0 such that if $\tau > \widehat{C}(\|q\|_{L^p(X)})$ and $x_0 \in X$, then there exists $u \in W_2^1(X)$ such that $L_q(x,D)u = 0$ in X and*

$$u(x,x_0) = e^{i\tau\Phi}(1-\frac{1}{4}e^{-i\tau(\bar{\Phi}+\Phi)}\partial^{-1}(e^{i\tau(\bar{\Phi}+\Phi)}(\bar{\partial}^{-1}q-\bar{\partial}^{-1}q(x_0)))+r(x,x_0)), \quad (20)$$

and there exists a positive constant C_1, independent of τ and $x_0 \in X$, such that

$$\tau^{\frac{3}{2}} \sup_{x_0\in X} \|r(\cdot,x_0)\|_{L^2(X)} + \tau^{\frac{1}{2}+\frac{1}{2p}} \sup_{x_0\in X} \|r(\cdot,x_0)\|_{L^4(X)} \le C_1\|q\|_{L^p(X)}, \quad (21)$$

$$\|u\|_{W_2^1(X)} \le C_1 e^{4R^2\tau}, \quad (22)$$

whenever $|x_0| < R$ where $R > 0$ is large enough that $\overline{X} \subset B(0,R)$.

Proof. Above we proved that the infinite series (5) for all sufficiently large τ is the solution to the equation $L_q(x,D)u = 0$. We set $r(x,x_0) = \sum_{j=2}^{\infty}(-1)^j U_j$. Thanks to (3) we have (20). The estimate of the first term in (21) follows from (18). By (18) and (19), we have

$$\sup_{x_0\in X} \|r(\cdot,x_0)\|_{L^4(X)} \le \sup_{x_0\in X} \|r(\cdot,x_0)\|^{\frac{1}{2}}_{L^2(X)} \sup_{x_0\in X} \|r(\cdot,x_0)\|^{\frac{1}{2}}_{L^\infty(X)}$$
$$\le C\frac{\|q\|_{L^p(X)}}{\tau^{\frac{1}{2}-\frac{1}{2p}}}\tau^{-1/p} \le C\frac{\|q\|_{L^p(X)}}{\tau^{\frac{1}{2}+\frac{1}{2p}}}. \quad (23)$$

Finally estimate (22) follows from (20), (21) and the classical estimate for elliptic equations. ■

5 Proof of Theorem 2.1.

We set $\tau_0 = \max\{\widehat{C}(\|q_1\|_{L^p(X)})\widehat{C}(\|q_2\|_{L^p(X)})\}$, where $\widehat{C}(\|q_k\|_{L^p(X)})$ are determined in Proposition 4.3 and let $\tau \ge \tau_0$ such that it is larger than τ_0 from Proposition 4.3. For point $x_0 \in X$ and $\tau \ge \tau_0$ let $u_1 \in W_2^1(X)$ be the solution to $L_{q_1}(x,D)u_1 = 0$ given by Proposition 4.3. In particular we have

$$u_1(x,x_0) = e^{i\tau\Phi}(1-\frac{1}{4}e^{-i\tau(\bar{\Phi}+\Phi)}\partial^{-1}(e^{i\tau(\bar{\Phi}+\Phi)}(\bar{\partial}^{-1}q_1-\bar{\partial}^{-1}q_1(x_0)))+r_1(x,x_0)), \quad (24)$$

$$\sup_{x_0\in X} \|r_1(\cdot,x_0)\|_{L^2(X)}\,\tau^{\frac{3}{2}} + \sup_{x_0\in X} \|r_1(\cdot,x_0)\|_{L^4(X)}\,\tau^{\frac{1}{2}+\frac{1}{2p}} \le C\,\|q_1\|_{L^p(X)}\,, \quad (25)$$

$$\sup_{x_0\in X}\|u_1(\cdot,x_0)\|_{W_2^1(X)}\le Ce^{4R^2\tau}, \tag{26}$$

and there exists a solution $u_2\in W_2^1(X)$ for $L_{q_2}(x,D)u_2=0$ with

$$u_2(x,x_0)=e^{i\tau\overline{\Phi}}(1-\frac{1}{4}e^{-i\tau(\bar\Phi+\Phi)}\bar\partial^{-1}(e^{i\tau(\bar\Phi+\Phi)}(\partial^{-1}q_2-\partial^{-1}q_2(x_0)))+r_2(x,x_0)), \tag{27}$$

$$\sup_{x_0\in X}\|r_2(\cdot,x_0)\|_{L^2(X)}\,\tau^{\frac{3}{2}}+\sup_{x_0\in X}\|r_2(\cdot,x_0)\|_{L^4(X)}\,\tau^{\frac12+\frac{1}{2p}}\le C\|q_2\|_{L^p(X)}, \tag{28}$$

$$\sup_{x_0\in X}\|u_2(\cdot,x_0)\|_{W_2^1(X)}\le Ce^{4R^2\tau}, \tag{29}$$

where constant C is independent of τ and x_0. Substituting (24) and (27) into $\int_X u_1(q_1-q_2)u_2dx$ and using the Fubini theorem on the Cauchy-operators, we obtain

$$\begin{aligned}(q_1-q_2)(x_0)=&\left((q_1-q_2)(x_0)-\int_X\frac{2\tau}{\pi}e^{i\tau(\Phi+\overline{\Phi})}(q_1-q_2)(x)dx\right)\\&+\frac{2\tau}{\pi}\int_X u_1(q_1-q_2)u_2dx\\&-\frac{2\tau}{\pi}\int_X\bar\partial^{-1}(q_1-q_2)(\partial^{-1}q_2-\partial^{-1}q_2(x_0))e^{i\tau(\bar\Phi+\Phi)}dx\\&-\frac{2\tau}{\pi}\int_X\partial^{-1}(q_1-q_2)(\bar\partial^{-1}q_1-\bar\partial^{-1}q_1(x_0))e^{i\tau(\bar\Phi+\Phi)}dx\\&-\frac{2\tau}{\pi}\int_X e^{i\tau(\Phi+\overline{\Phi})}(q_1-q_2)(x)(p_1p_2+r_1+r_2)(x,x_0)dx,\end{aligned} \tag{30}$$

where

$$p_1=r_1-\frac14e^{-i\tau(\overline{\Phi}+\Phi)}\partial^{-1}(e^{i\tau(\overline{\Phi}+\Phi)}(\overline{\partial}^{-1}q_1-\overline{\partial}^{-1}q_1(x_0))), \tag{31}$$

$$p_2=r_2-\frac14e^{-i\tau(\overline{\Phi}+\Phi)}\bar\partial^{-1}(e^{i\tau(\overline{\Phi}+\Phi)}(\partial^{-1}q_2-\partial^{-1}q_2(x_0))). \tag{32}$$

We recall that $q_1-q_2\in\mathring{W}_2^s(X)$ by the assumptions of the theorem. For $s\in(0,1]\setminus\left\{\frac12\right\}$ and $q\in\mathring{W}_2^s(X)$, let E_0q be the extension in $\mathbb{R}^2$ by the zero extension outside X. Then $E_0q\in W_2^s(\mathbb{R}^2)$.

We can now deal with the first term. Take the $L^2(X)$-norm with respect to x_0 to obtain

$$\begin{aligned}&\left\|q_1-q_2-\int_X\frac{2\tau}{\pi}e^{i\tau(\Phi+\overline{\Phi})}(q_1-q_2)(x)dx\right\|_{L^2(X:,dx_0)}\\=&\left\|E_0(q_1-q_2)-\int_{\mathbb{R}^2}\frac{2\tau}{\pi}e^{i\tau(\Phi+\overline{\Phi})}E_0(q_1-q_2)(x)dx\right\|_{L^2(\mathbb{R}^2;dx_0)}.\end{aligned}$$

Applying Lemma 3.3 we have

$$\left\| q_1 - q_2 - \int_X \frac{2\tau}{\pi} e^{i\tau(\Phi+\overline{\Phi})}(q_1-q_2)(x)dx \right\|_{L^2(X;dx_0)}$$
$$\le 2\tau^{-s/2} \left\| E_0(q_1-q_2) \right\|_{W_2^s(\mathbb{R}^2)} \le C\tau^{-s/2} \left\| q_1-q_2 \right\|_{W_2^s(X)} \le 2CM\tau^{-s/2}. \quad (33)$$

The second term on the right-hand side of (30) is estimated by the difference of the boundary data and the definition of $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})$:

$$\left\| \frac{2\tau}{\pi} \int_X u_1(q_1-q_2)u_2 dx \right\|_{L^2(X;dx_0)} \le C \sup_{x_0\in X} \left| \frac{2\tau}{\pi} \int_X u_1(q_1-q_2)u_2 dx \right|$$
$$\le C\tau d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \sup_{x_0\in X} (\|u_1\|_{W_2^1(X)} \|u_2\|_{W_2^1(X)}) \le C_M e^{\tau(8R^2+1)} d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}). \quad (34)$$

Here in order to obtain the last estimate, we used (26) and (29). Applying Lemma 3.3 again, we obtain that there exists $\tilde{s} > 0$ such that

$$\left\| \frac{2\tau}{\pi} \int_X \bar{\partial}^{-1}(q_1-q_2)(\partial^{-1}q_2 - \partial^{-1}q_2(x_0)) e^{i\tau(\bar{\Phi}+\Phi)} dx \right\|_{L^2(X;dx_0)} \quad (35)$$

$$\begin{aligned}
&\le \left\| \bar{\partial}^{-1}(q_1-q_2)\partial^{-1}q_2 - \frac{2\tau}{\pi}\int_X \bar{\partial}^{-1}(q_1-q_2)\partial^{-1}q_2 e^{i\tau(\bar{\Phi}+\Phi)}dx \right\|_{L^2(X;dx_0)} \\
&+ \left\| \bar{\partial}^{-1}(q_1-q_2)\partial^{-1}q_2 - \partial^{-1}q_2 \frac{2\tau}{\pi}\int_X \bar{\partial}^{-1}(q_1-q_2) e^{i\tau(\bar{\Phi}+\Phi)}dx \right\|_{L^2(X;dx_0)} \\
&\le \left\| E_0\bar{\partial}^{-1}(q_1-q_2)E_0\partial^{-1}q_2 - \frac{2\tau}{\pi}\int_{\mathbb{R}^2} E_0\bar{\partial}^{-1}(q_1-q_2)E_0\partial^{-1}q_2 e^{i\tau(\bar{\Phi}+\Phi)}dx \right\|_{L^2(\mathbb{R}^2;dx_0)} \\
&+ \left\| E_0\bar{\partial}^{-1}(q_1-q_2)E_0\partial^{-1}q_2 - E_0\partial^{-1}q_2 \frac{2\tau}{\pi}\int_{\mathbb{R}^2} E_0\bar{\partial}^{-1}(q_1-q_2) e^{i\tau(\bar{\Phi}+\Phi)}dx \right\|_{L^2(\mathbb{R}^2;dx_0)} \\
&\le \frac{C}{\tau^{\tilde{s}}} \|E_0\bar{\partial}^{-1}(q_1-q_2)E_0\partial^{-1}q_2\|_{W_2^1(\mathbb{R}^2)} + \frac{C}{\tau^{\tilde{s}}} \|E_0\bar{\partial}^{-1}(q_1-q_2)\|_{W_2^1(\mathbb{R}^2)} \|\partial^{-1}q_2\|_{L^\infty(X)} \\
&\le \frac{C'}{\tau^{\tilde{s}}} \|q_1-q_2\|_{L^2(X)}.
\end{aligned}$$

In a similar way we obtain

$$\left\| \frac{2\tau}{\pi} \int_X \partial^{-1}(q_1-q_2)(\bar{\partial}^{-1}q_1 - \bar{\partial}^{-1}q_1(x_0)) e^{i\tau(\bar{\Phi}+\Phi)} dx \right\|_{L^2(X;dx_0)}$$
$$\le \frac{C'}{\tau^{\tilde{s}}} \|q_1-q_2\|_{L^2(X)}. \quad (36)$$

Estimating the L^2-norm of the last term on the righ-hand side of (30), we have

$$\mathcal{I} = \left\| \frac{2\tau}{\pi} \int_X e^{i\tau(\Phi+\overline{\Phi})}(q_1 - q_2)(x)(p_1p_2 + r_1 + r_2)(x, x_0)dx \right\|_{L^2(X;dx_0)}$$
$$\leq C \sup_{x_0\in X} \frac{2\tau}{\pi} \int_X |(q_1 - q_2)(x)||(p_1p_2 + r_1 + r_2)(x, x_0)|dx.$$

Thanks to (25) and (28), we obtain

$$\mathcal{I} \leq C\tau\|q_1 - q_2\|_{L^2(X)} \sup_{x_0\in X} \|(p_1p_2 + r_1 + r_2)(\cdot, x_0)\|_{L^2(X)}$$
$$\leq C\tau\|q_1 - q_2\|_{L^2(X)} \sup_{x_0\in X} (\|p_1p_2\|_{L^2(X)} + \|(r_1 + r_2)(\cdot, x_0)\|_{L^2(X)})$$
$$\leq C_1\|q_1 - q_2\|_{L^2(X)} \sup_{x_0\in X} (\tau\|p_1p_2\|_{L^2(X)} + \frac{1}{\sqrt{\tau}}).$$

By (25), (28) and Proposition 4.3

$$\begin{aligned}
\sup_{x_0\in X} \|p_1p_2\|_{L^2(X)} \quad \leq \quad & \sup_{x_0\in X} (\|r_1\|_{L^4(X)}\|r_2\|_{L^4(X)} \\
& + \frac{1}{4}\|\partial^{-1}(e^{i\tau(\overline{\Phi}+\Phi)}(\partial^{-1}q_2 - \partial^{-1}q_2(x_0)))\|_{L^\infty(X)}\|r_1\|_{L^2(X)} \\
& + \frac{1}{4}\|\overline{\partial}^{-1}(e^{i\tau(\overline{\Phi}+\Phi)}(\overline{\partial}^{-1}q_1 - \overline{\partial}^{-1}q_1(x_0)))\|_{L^\infty(X)}\|r_2\|_{L^2(X)} \\
& + \frac{1}{16}\|\overline{\partial}^{-1}(e^{i\tau(\overline{\Phi}+\Phi)}(\overline{\partial}^{-1}q_1 - \overline{\partial}^{-1}q_1(x_0)))\|_{L^2(X)} \\
& \|\partial^{-1}(e^{i\tau(\overline{\Phi}+\Phi)}(\partial^{-1}q_2 - \partial^{-1}q_2(x_0)))\|_{L^\infty(X)}) \\
\leq \quad & C\Big(\frac{1}{\tau^{\frac{3}{2}}} + \frac{1}{\tau^p}(\|r_1\|_{L^2(X)} + \|r_2\|_{L^2(X)}) \\
& + \frac{1}{\tau^p}\|\overline{\partial}^{-1}(e^{i\tau(\overline{\Phi}+\Phi)}(\overline{\partial}^{-1}q_1 - \overline{\partial}^{-1}q_1(x_0)))\|_{L^2(X)}\Big).
\end{aligned}$$

Applying (25), (28) and Proposition 4.2 with $\epsilon = \frac{p}{2}$, we obtain:

$$\sup_{x_0\in X} \|p_1p_2\|_{L^2(X)} \leq C(\frac{1}{\tau^{\frac{3}{2}}} + \frac{1}{\tau^p}(\frac{1}{\tau^{\frac{3}{2}}} + \frac{1}{\tau^{1-\frac{p}{2}}})). \tag{37}$$

Hence there exists τ_1 independent of z_0 such that

$$\mathcal{I} \leq \frac{1}{2}\|q_1 - q_2\|_{L^2(X)} \quad \forall\tau \geq \tau_1. \tag{38}$$

Combining estimates (33)-(38) and setting $R_0 = 8R^2 + 1$, we obtain

$$\|q_1 - q_2\|_{L^2(X)} \leq C(e^{\tau R_0} d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) + \tau^{-s/2}), \quad \forall \tau \geq \tau_1. \tag{39}$$

Replacing τ and C by $\tau + \tau_1$ and $Ce^{R_0 \tau_1}$ respectively, we have (39) for all $\tau > 0$. For obtaining the conditional stability, we should make the right-hand side of (39) as small as possible by choosing $\tau > 0$. For this we make the following choice of τ depending on the value of $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})$.

Case 1: $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) < 1$.

We choose

$$\tau = \frac{\alpha}{R_0} \left(1 + \ln \frac{1}{d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})} \right) > 0$$

with arbitrarily fixed $\alpha \in (0,1)$. Then $e^{\tau R_0} d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) = e^{\alpha} d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})^{1-\alpha}$ and

$$\tau^{-s/2} = \left(\frac{R_0}{\alpha} \right)^{s/2} \left(1 + \ln \frac{1}{d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})} \right)^{-s/2}.$$

Since for $0 < \alpha < 1$, there exists a constant $C > 0$ such that $\eta^{1-\alpha} \leq C \left(1 + \ln \frac{1}{\eta} \right)^{-s/2}$ for $0 \leq \eta < 1$, with this choice of τ, estimate (39) yields

$$\|q_1 - q_2\|_{L^2(X)} \leq C \left(1 + \ln \frac{1}{d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})} \right)^{-s/2}.$$

Case 2: $d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \geq 1$.

Since $\|q_1\|_{W_2^s(X)} \leq M$ and $\|q_2\|_{W_2^s(X)} \leq M$, we have $\|q_1 - q_2\|_{L^2(X)} \leq 2M \leq 2M d(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})$.

Therefore combining the two cases, we complete the proof of Theorem 2.1. ■

6 Proof of theorem 2.2.

For any point $x_0 \in X$ let $u_1, u_2 \in W_2^1(X)$ be the solutions to the Schrödinger equation given by (24) and (27) respectively.

Since the Dirichlet-to-Neumann maps are the same, we have $\int_X (q_1 - q_2) u_1 u_2 dx = 0$. Then plugging formulas (24) and (27) into it and adding $(q_1 - q_2)(x_0)$ to both sides, we have

$$
\begin{aligned}
(q_1-q_2)(x_0) = & \left((q_1-q_2)(x_0) - \int_X \frac{2\tau}{\pi} e^{i\tau(\Phi+\overline{\Phi})}(q_1-q_2)(x)dx\right) \\
& -\frac{2\tau}{\pi}\int_X \bar\partial^{-1}(q_1-q_2)(\partial^{-1}q_2 - \partial^{-1}q_2(x_0))e^{i\tau(\bar\Phi+\Phi)}dx \\
& -\frac{2\tau}{\pi}\int_X \partial^{-1}(q_1-q_2)(\bar\partial^{-1}q_1 - \bar\partial^{-1}q_1(x_0))e^{i\tau(\bar\Phi+\Phi)}dx \\
& -\frac{2\tau}{\pi}\int_X e^{i\tau(\Phi+\overline{\Phi})}(q_1-q_2)(x)(p_1p_2+r_1+r_2)(x,x_0)dx, \qquad (40)
\end{aligned}
$$

where the functions p_j are determined by (31) and (32).

Since the estimates (35), (38) hold true for all sufficiently large τ, we obtain from (40):

$$
\begin{aligned}
\|q_1-q_2\|_{L^2(X)} \le C & \left\| q_1 - q_2 - \int_X \frac{2\tau}{\pi} e^{i\tau(\Phi+\overline{\Phi})}(q_1-q_2)(x)dx \right\|_{L^2(X;dx_0)} \\
= C & \left\| E_0(q_1-q_2) - \int_{\mathbb{R}^2} \frac{2\tau}{\pi} e^{i\tau(\Phi+\overline{\Phi})} E_0(q_1-q_2)(x)dx \right\|_{L^2(\mathbb{R}^2;dx_0)}.
\end{aligned}
$$

In view of Lemma 3.3 we obtain

$$
\left\| q_1 - q_2 - \int_X \frac{2\tau}{\pi} e^{i\tau(\Phi+\overline{\Phi})}(q_1-q_2)(x)dx \right\|_{L^2(X;dx_0)} \to 0 \text{ as } \tau \to +\infty.
$$

The proof of the theorem is complete. ■

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