Strong solutions of the thin film equation in spherical geometry.

Roman M. Taranets
Institute of Applied Mathematics and Mechanics of the NASU,
Sloviansk, Ukraine, taranets_r@yahoo.com

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Abstract

We study existence and long-time behaviour of strong solutions for the thin film equation using a priori estimates in a weighted Sobolev space. This equation can be classified as a doubly degenerate fourth-order parabolic and it models coating flow on the outer surface of a sphere. It is shown that the strong solution asymptotically decays to the flat profile.

1 Introduction

In this paper, we study the following doubly degenerate fourth-order parabolic equation

$$u_t + \big((1-x^2)|u|^n((1-x^2)u_x)_{xx}\big)_x = 0 \text{ in } Q_T, \tag{1.1}$$

where $Q_T = \Omega \times (0,T)$, $n > 0$, $T > 0$, and $\Omega = (-1,1)$. This equation describes the dynamics of a thin viscous liquid film on the outer surface of a solid sphere. More general dynamics of the liquid film for the cases when the draining of the film due to gravity were balanced by centrifugal forces arising from the rotation of the sphere about a vertical axis and by capillary forces due to surface tension was considered in [9]. In addition, Marangoni effects due to temperature gradients were taken into account in [10]. The spherical model without the surface tension and Marangoni effects was studied in [12, 13].

In [9], the authors derived the following equation for no-slip regime in dimensionless form

$$h_t + \tfrac{1}{\sin\theta}(h^3 \sin\theta\, J)_\theta = 0,$$

$$J := a\sin\theta + b\sin\theta\cos\theta + c[2h + \tfrac{1}{\sin\theta}(\sin\theta\, h_\theta)_\theta]_\theta,$$

where $h(\theta, t)$ represent the thickness of the thin film, $\theta \in (0, \pi)$ is the polar angle in spherical coordinates, with t denoting time; the dimensionless parameters a, b and c describe the effects of gravity, rotation and surface tension, respectively. After the change of variable $x = -\cos\theta$, this equation can be written in the form:

$$u_t + [u^3(1 - x^2)(a - bx + c(2h + ((1 - x^2)u_x)_x)_x)]_x = 0, \qquad (1.2)$$

where $x \in (-1, 1)$. As a result, equation (1.1) for $n = 3$ is a particular case of (1.2) for no-slip regime. On the other hand, (1.1) for $n < 3$ generalises (1.2) with $a = b = 0$ for different slip regimes, for example, like weak or partial wetting.

In contrast to the classical thin film equation:

$$u_t + \left(|u|^n u_{xxx}\right)_x = 0, \qquad (1.3)$$

which describes the behavior of a thin viscous film on a flat surface under the effect of surface tension, the equation (1.1) is not yet well analysed. To the best of our knowledge, there is only one analytical result [11] where the authors proved existence of weak solutions in a weighted Sobolev space. In 1990, Bernis and Friedman [2] constructed non-negative weak solutions of the equation (1.3) when $n \geqslant 1$, and it was also shown that for $n \geqslant 4$, with a positive initial condition, there exists a unique positive classical solution. In 1994, Bertozzi et al. [3] generalised this positivity property for the case $n \geqslant \frac{7}{2}$. In 1995, Beretta et al. [1] proved the existence of non-negative weak solutions for the equation (1.3) if $n > 0$, and the existence of strong ones for $0 < n < 3$. Also, they could show that this positivity-preserving property holds for almost every time t in the case $n \geqslant 2$. A similar result on a cylindrical surface was obtained in [7]. Regarding the long-time behaviour, Carrillo and Toscani [6] proved the convergence to a self-similar solution for equation (1.3) with $n = 1$ and Carlen and Ulusoy [5] gave an upper bound on the distance from the self-similar solution. A similar result on a cylindrical surface was obtained in [4].

In the present article, we obtain the existence of weak solutions in a wider weighted classes of functions than it was done in [11]. Moreover, we show the existence of non-negative strong solutions and we also prove that this solution decays asymptotically to the flat profile. Note that (1.1) loses its parabolicity not only at $u = 0$ (as in (1.3)) but also at $x = \pm 1$. For this reason, it is natural to seek solution in a Soblev space with weight $1 - x^2$. For example, it is the well-known that the non-negative steady state of equation (1.3) for $x \in (-1, 1)$ has the form

$$u_s(x) = c_1(1 - x^2) + c_2, \text{ where } c_i \geqslant 0.$$

On the other hand, the equation (1.1) has the following non-negative steady state

$$u_s(x) = (c_1 + c_2) \ln(1 + x) + (c_1 - c_2) \ln(1 - x) + c_3,$$

where $0 \leqslant |c_2| \leqslant -c_1$, $c_3 \geqslant -(c_1 + c_2) \ln(1 + \frac{c_2}{c_1}) + (c_1 - c_2) \ln(1 - \frac{c_2}{c_1})$, hence $u_s(x) \to +\infty$ as $x \to \pm 1$.

2 Existence of Strong Solutions

We study the following thin film equation

$$u_t + \left((1 - x^2)|u|^n \left((1 - x^2)u_x\right)_{xx}\right)_x = 0 \text{ in } Q_T \tag{2.1}$$

with the no-flux boundary conditions

$$(1 - x^2)u_x = (1 - x^2)\left((1 - x^2)u_x\right)_{xx} = 0 \text{ at } x = \pm 1,\ t > 0, \tag{2.2}$$

and the initial condition

$$u(x,0)=u_0(x). \tag{2.3}$$

Here $n>0$, $Q_T=\Omega\times(0,T)$, $\Omega:=(-1,1)$, and $T>0$. Integrating the equation (2.1) by using boundary conditions (2.2), we obtain the mass conservation property

$$\int\limits_{\Omega} u(x,t)dx=\int\limits_{\Omega} u_0(x)dx=:M. \tag{2.4}$$

Consider initial data $u_0(x)\geqslant 0$ for all $x\in\bar{\Omega}$ satisfying

$$\int\limits_{\Omega}\{(1-x^2)^{\beta}u_0^2(x)+(1-x^2)u_{0,x}^2(x)\}dx<\infty,\ \beta\in(0,\tfrac{2}{n}]. \tag{2.5}$$

Definition 2.1. *[weak solution] Let $n>0$. A function u is a weak solution of the problem (2.1)–(2.3) with initial data u_0 satisfying (2.5) if $u(x,t)$ has the following properties*

$$(1-x^2)^{\frac{\beta}{2}}u\in C_{x,t}^{\frac{\alpha}{2},\frac{\alpha}{8}}(\bar{Q}_T),\ 0<\alpha<\beta\leqslant\tfrac{2}{n},$$

$$u_t\in L^2(0,T;(H^1(\Omega))^*),\ (1-x^2)^{\frac{1}{2}}u_x\in L^\infty(0,T;L^2(\Omega)),$$

$$(1-x^2)^{\frac{1}{2}}|u|^{\frac{n}{2}}((1-x^2)u_x)_{xx}\in L^2(P),$$

u satisfies (2.1) in the following sense:

$$\int\limits_0^T\langle u_t,\phi\rangle\,dt-\iint\limits_P(1-x^2)|u|^n((1-x^2)u_x)_{xx}\phi_x\,dxdt=0$$

for all $\phi\in L^2(0,T;H^1(\Omega))$, where $P:=\bar{Q}_T\setminus\{\{u=0\}\cup\{t=0\}\}$,

$$(1-x^2)^{\frac{1}{2}}u_x(.,t)\to(1-x^2)^{\frac{1}{2}}u_{0,x}(.)\ \textit{strongly in}\ L^2(\Omega)\ \textit{as}\ t\to 0,$$

and boundary conditions (2.2) hold at all points of the lateral boundary, where $\{u>0\}$.

Let us denote by

$$0 \leqslant G_0(z) := \begin{cases} \frac{z^{2-n}-A^{2-n}}{(n-1)(n-2)} - \frac{A^{1-n}}{1-n}(z-A) \text{ if } n \neq 1,2, \\ z\ln z - (z-A)(\ln A + 1) \text{ if } n = 1, \\ \ln(\frac{A}{z}) + \frac{z}{A} - 1 \text{ if } n = 2, \end{cases} \tag{2.6}$$

where $A = 0$ if $n \in (1,2)$ and $A > 0$ if else. Next, we establish existence of a more regular solution u of (2.1) than a weak solution in the sense of Definition 2.1. Besides, we show that this strong solution u with some weight exponentially decays to zero.

Theorem 1 (strong solution)**.** *Assume that $n \geqslant 1$ and initial data u_0 satisfies $\int\limits_{\Omega} G_0(u_0)\,dx < +\infty$ then the problem (2.1)–(2.3) has a non-negative weak solution, u, in the sense of Definition 2.1, such that*

$$(1-x^2)u_x \in L^2(0,T;H^1(\Omega)),\ (1-x^2)^{\frac{\gamma}{2}}u_x \in L^2(Q_T),\ \gamma \in (0,1).$$

$$(1-x^2)^{\frac{\mu}{2}}u \in L^2(Q_T),\ \mu \in (-1,\beta).$$

Moreover, there exist positive constants A, B depending on initial data such that

$$\tfrac{1}{2}\int\limits_{\Omega}(1-x^2)u_x^2(x,t)\,dx \leqslant A\,e^{-B\,t}\ \forall\, t \geqslant 0,$$

hence

$$(1-x^2)^{\frac{\beta}{2}}|u - \tfrac{M}{|\Omega|}| \to 0 \ \text{as } t \to +\infty.$$

3 Proof of Theorem 1

3.1 Approximating problems

Equation (2.1) is doubly degenerate when $u = 0$ and $x = \pm 1$. For this reason, for any $\epsilon > 0$ and $\delta > 0$ we consider two-parametric regularised equations

$$u_{\epsilon\delta,t} + \left[(1-x^2+\delta)(|u_{\epsilon\delta}|^n + \epsilon)\left((1-x^2+\delta)u_{\epsilon\delta,x}\right)_{xx}\right]_x = 0 \text{ in } Q_T \tag{3.1}$$

with boundary conditions

$$u_{\epsilon\delta,x} = \big((1 - x^2 + \delta)u_{\epsilon\delta,x}\big)_{xx} = 0 \text{ at } x = \pm 1, \tag{3.2}$$

and initial data

$$u_{\epsilon\delta}(x, 0) = u_{0,\epsilon\delta}(x) \in C^{4+\gamma}(\bar{\Omega}),\ \gamma > 0, \tag{3.3}$$

where

$$u_{0,\epsilon\delta}(x) \geqslant u_{0\delta}(x) + \epsilon^{\theta}, \quad \theta \in (0, \tfrac{1}{2(n-1)}), \tag{3.4}$$

$$u_{0,\epsilon\delta} \to u_{0\delta} \text{ strongly in } H^1(\Omega) \text{ as } \epsilon \to 0, \tag{3.5}$$

$$(1 - x^2 + \delta)^{\frac{1}{2}} u_{0x,\delta} \to (1 - x^2) u_{0,x} \text{ strongly in } L^2(\Omega) \text{ as } \delta \to 0. \tag{3.6}$$

The parameters $\epsilon > 0$ and $\delta > 0$ in (3.1) make the problem regular up to the boundary (i.e. uniformly parabolic). The existence of a solution of (3.1) in a small time interval is guaranteed by the Schauder estimates in [8]. Now suppose that $u_{\epsilon\delta}$ is a solution of equation (3.1) and that it is continuously differentiable with respect to the time variable and fourth order continuously differentiable with respect to the spatial variable.

3.2 Existence of weak solutions

In order to get an *a priori* estimation of $u_{\epsilon\delta}$, we multiply both sides of equation (3.1) by $-[(1 - x^2 + \delta)u_{\epsilon\delta,x}]_x$ and integrate over Ω by (3.2). This gives us

$$\tfrac{1}{2}\tfrac{d}{dt}\int_{\Omega} (1 - x^2 + \delta) u_{\epsilon\delta,x}^2 \, dx + \int_{\Omega} (1 - x^2 + \delta)(|u_{\epsilon\delta}|^n + \epsilon)[(1 - x^2 + \delta)u_{\epsilon\delta,x}]_{xx}^2 \, dx = 0. \tag{3.7}$$

Integrating (3.7) in time, we get

$$\tfrac{1}{2}\int\limits_{\Omega}(1-x^2+\delta)u^2_{\epsilon\delta,x}\,dx+ \iint\limits_{Q_T}(1-x^2+\delta)(|u_{\epsilon\delta}|^n+\epsilon)[(1-x^2+\delta)u_{\epsilon\delta,x}]^2_{xx}\,dxdt = \tfrac{1}{2}\int\limits_{\Omega}(1-x^2+\delta)u^2_{0x,\epsilon\delta}\,dx. \quad (3.8)$$

By (3.6) we have

$$\int\limits_{\Omega}(1-x^2+\delta)u^2_{\epsilon\delta,x}\,dx \leqslant C_0, \quad (3.9)$$

where $C_0>0$ is independent of ϵ and δ. From (3.9) and (3.8) it follows that

$$\{u_{\epsilon\delta}\}_{\epsilon>0} \text{ is uniformly bounded in } L^{\infty}(0,T;H^1(\Omega)). \quad (3.10)$$

$$\{(1-x^2+\delta)^{\frac{1}{2}}(|u_{\epsilon\delta}|^n+\epsilon)^{\frac{1}{2}}[(1-x^2+\delta)u_{\epsilon\delta,x}]_{xx}\}_{\epsilon,\,\delta>0} \text{ is u.b. in } L^2(Q_T). \quad (3.11)$$

By (3.10) and (3.11), using the same method as [2], we can prove that solutions $u_{\epsilon\delta}$ have uniformly (in ϵ) bounded $C^{1/2,1/8}_{x,t}$-norms. By the Arzelà-Ascoli theorem, this equicontinuous property, together with the uniformly boundedness shows that every sequence $\{u_{\epsilon\delta}\}_{\epsilon>0}$ has a subsequence such that

$$u_{\epsilon\delta}\to u_{\delta} \text{ uniformly in } Q_T \text{ as } \epsilon\to 0. \quad (3.12)$$

As a result, we obtain a solution u_{δ} of the problem (3.1)–(3.3) with $\epsilon=0$ in the sense of [2, Theorem 3.1, pp. 185–186].

Next, we show that the family of solutions $\{u_{\delta}\}_{\delta>0}$ is uniformly bounded in some weighted space. Using the mass conservation property

$$\int\limits_{\Omega}u_{\delta}(x,t)dx = M>0,$$

we arrive at

$$|u_\delta - \tfrac{M}{|\Omega|}| = \Big|\int\limits_{x_0}^{x} u_x\,dx\Big| \leqslant \Big(\int\limits_{\Omega} (1-x^2)u_x^2\,dx\Big)^{\frac12}\Big(\int\limits_{x_0}^{x} \tfrac{dx}{1-x^2}\Big)^{\frac12}. \tag{3.13}$$

Multiplying (3.13) by $(1-x^2)^{\frac{\beta}{2}}$ for any $\beta>0$, by (3.9) we deduce that

$$(1-x^2)^{\frac{\beta}{2}}|u_\delta - \tfrac{M}{|\Omega|}| \leqslant (\tfrac{C_0}{2})^{\frac12}\Big((1-x^2)^{\beta}\ln(\tfrac{(1+x)(1-x_0)}{(1-x)(1+x_0)})\Big)^{\frac12} \leqslant C_1 \tag{3.14}$$

for all $x\in\bar\Omega$, where $C_1>0$ is independent of $\delta>0$. From (3.14) we find that

$$\{(1-x^2)^{\frac{\beta}{2}}u_\delta\}_{\delta>0} \text{ is u.b. in } Q_T \text{ for any } \beta>0. \tag{3.15}$$

In particular, by (3.9) we get

$$(1-x^2)^{\frac{\beta}{2}}|u_\delta(x_1,t)-u_\delta(x_2,t)| \leqslant C_2|x_1-x_2|^{\frac{\alpha}{2}} \ \ \forall\, x_1,x_2\in\Omega,\ \alpha\in(0,\beta). \tag{3.16}$$

By (3.11), (3.15) and (3.16) with $\beta\in(0,\frac{2}{n}]$, using the same method as [2, Lemma 2.1, p.183], we can prove similarly that

$$(1-x^2)^{\frac{\beta}{2}}|u_\delta(x,t_1)-u_\delta(x,t_2)| \leqslant C_3|t_1-t_2|^{\frac{\alpha}{8}} \ \ \forall\, t_1,t_2\in(0,T). \tag{3.17}$$

The inequalities (3.16) and (3.17) show the uniform (in δ) boundedness of a sequence $\{(1-x^2)^{\frac{\beta}{2}}u_\delta\}_{\delta>0}$ in the $C_{x,t}^{\frac{\alpha}{2},\frac{\alpha}{8}}$-norm. By the Arzelà-Ascoli theorem, this a priori bound together with (3.15) shows that as $\delta\to0$, every sequence $\{(1-x^2)^{\frac{\beta}{2}}u_\delta\}_{\delta>0}$ has a subsequence $\{(1-x^2)^{\frac{\beta}{2}}u_{\delta_k}\}_{\delta_k>0}$ such that

$$(1-x^2)^{\frac{\beta}{2}}u_{\delta_k} \to (1-x^2)^{\frac{\beta}{2}}u \text{ uniformly in } \bar Q_T \text{ as } \delta_k\to0. \tag{3.18}$$

Following the idea of proof [2, Theorem 3.1], we obtain a solution u of the problem (3.1)–(3.3) in the sense of Definition 2.1.

3.3 Existence of strong solutions

Let us denote by $G_\epsilon(z)$ the following function

$$G_\epsilon(z) \geqslant 0 \; \forall\, z \in \mathbb{R},\;\; G''_\epsilon(z) = \tfrac{1}{|s|^n+\epsilon}.$$

Now we multiply equation (3.1) by $G'_\epsilon(u_{\epsilon\delta})$ and integrate over Ω to get

$$\tfrac{d}{dt} \int\limits_\Omega G_\epsilon(u_{\epsilon\delta}(x,t))\,dx + \int\limits_\Omega [(1-x^2+\delta)u_{\epsilon\delta,x}]_x^2\,dx = 0. \tag{3.19}$$

After integration in time, equation (3.19) becomes

$$\int\limits_\Omega G_\epsilon(u_{\epsilon\delta}(x,T))\,dx + \iint\limits_{Q_T} [(1-x^2+\delta)u_{\epsilon\delta,x}]_x^2\,dxdt = \int\limits_\Omega G_\epsilon(u_{0,\epsilon\delta}(x))\,dx. \tag{3.20}$$

We compute

$$G''_0(z) - G''_\epsilon(z) = \tfrac{\epsilon}{|z|^n(|z|^n+\epsilon)},$$

and consequently

$$G_0(z) - G_\epsilon(z) = \epsilon \int\limits_A^z \int\limits_A^v \tfrac{dsdv}{|s|^n(|s|^n+\epsilon)},$$

where A is some positive constant. As $u_{0,\epsilon\delta}(x)$ is bounded then by (3.4) it follows that

$$|G_0(u_{0,\epsilon\delta}(x)) - G_\epsilon(u_{0,\epsilon\delta}(x))| \leqslant C\,\epsilon^{1-2\theta(n-1)} \to 0 \text{ as } \epsilon \to 0,$$

and therefore, due to (3.5), we have

$$\int\limits_\Omega G_\epsilon(u_{0,\epsilon}(x))\,dx \to \int\limits_\Omega G_0(u_{0\delta}(x))\,dx \text{ as } \epsilon \to 0. \tag{3.21}$$

As a result, by (3.20), (3.21) we deduce that

$$\int\limits_\Omega G_\epsilon(u_{\epsilon\delta}(x,T))\,dx \leqslant C_4, \tag{3.22}$$

$$\{(1-x^2+\delta)u_{\epsilon\delta,x}\}_{\epsilon,\,\delta>0} \text{ is u. b. in } L^2(0,T;H^1(\Omega)), \tag{3.23}$$

where $C_1>0$ is independent of ϵ and δ. Similar to [2, Theorem 4.1, p. 190], using (3.10) and (3.22), we can show that the limit solution u_δ is non-negative if $n\in[1,4)$ and positive if $n\geqslant 4$. Next, letting $\delta\to 0$, we get a non-negative strong solution.

3.4 Asymptotic behaviour

Let us denote by

$$\mathcal{E}_\delta(u(t)) := \tfrac{1}{2}\int\limits_\Omega (1-x^2+\delta)u_x^2\,dx, \quad \tfrac{\delta\mathcal{E}_\delta(u)}{\delta u} := -((1-x^2+\delta)u_x)_x.$$

By using the notations, we rewrite (3.7) and (3.19) with $\epsilon=0$ in the form

$$\tfrac{d}{dt}\mathcal{E}_\delta(u_\delta(t)) + \int\limits_\Omega (1-x^2+\delta)u_\delta^n[\tfrac{\delta\mathcal{E}_\delta(u_\delta)}{\delta u}]_x^2\,dx = 0, \tag{3.24}$$

$$\tfrac{d}{dt}\int\limits_\Omega G_0(u_\delta)\,dx + \int\limits_\Omega [\tfrac{\delta\mathcal{E}_\delta(u_\delta)}{\delta u}]^2\,dx = 0. \tag{3.25}$$

Next, we will use the following Hardy's inequality

$$\int\limits_{-1}^{1} (1-x^2)^{-2+\gamma}v^2(x)\,dx \leqslant C\int\limits_{-1}^{1}(1-x^2)^{\gamma}v_x^2(x)\,dx \tag{3.26}$$

for any $\gamma>0$ and for all $v\in H^1(-1,1)$ such that $v(\pm1)=0$. Really, using integration by parts and Cauchy inequality, we have

$$\int\limits_{-1}^{1}(1-x^2)^{-2+\gamma}v^2(x)\,dx = v^2(x)g(x)\Big|_{-1}^{1} - 2\int\limits_{-1}^{1}v(x)v_x(x)g(x)\,dx \leqslant$$

$$2\Big(\int\limits_{-1}^{1}(1-x^2)^{-2+\gamma}v^2(x)\,dx\Big)^{\frac12}\Big(\int\limits_{-1}^{1}(1-x^2)^{2-\gamma}g^2(x)v_x^2(x)\,dx\Big)^{\frac12},$$

where

$$|(1-x^2)^{1-\frac{\gamma}{2}}g(x)| = \left|(1-x^2)^{1-\frac{\gamma}{2}} \int^x (1-x^2)^{-2+\gamma}dx\right| \leqslant$$
$$C\,(1-x^2)^{\frac{\gamma}{2}}\ \forall\, x \in [-1,1],\ \gamma \geqslant 0,$$

and as $v(x) \in C^{\frac{1}{2}}[-1,1]$ and $g(x) \sim (1-x^2)^{-1+\gamma}$ at $x = \pm 1$ then $v^2(x)g(x) = 0$ at $x = \pm 1$. From here we find that

$$\int_{-1}^{1} (1-x^2)^{-2+\gamma} v^2(x)\,dx \leqslant$$
$$C\Big(\int_{-1}^{1} (1-x^2)^{-2+\gamma} v^2(x)\,dx\Big)^{\frac{1}{2}} \Big(\int_{-1}^{1} (1-x^2)^{\gamma} v_x^2(x)\,dx\Big)^{\frac{1}{2}},$$

whence it follows (3.26).

Applying (3.26) to $v = (1-x^2)u_x$ with $\gamma = 1$, we obtain that

$$\int_\Omega (1-x^2)u_x^2\,dx \leqslant C_5 \int_\Omega (1-x^2)[(1-x^2)u_x]_x^2\,dx \leqslant C_5 \int_\Omega [(1-x^2)u_x]_x^2\,dx,$$

i. e.

$$2\mathcal{E}_0(u(t)) \leqslant C_5 \int_\Omega [\tfrac{\delta \mathcal{E}_0(u)}{\delta u}]^2\,dx. \tag{3.27}$$

Summing (3.24) and (3.25), after integrating in time, taking $\delta \to 0$, and using (3.27), we arrive at

$$\mathcal{E}_0(u(t)) + B\int_0^t \mathcal{E}_0(u(s))\,ds \leqslant A := \mathcal{E}(u_0) + \int_\Omega G_0(u_0)\,dx, \tag{3.28}$$

where $B := \frac{2}{C_5}$. From (3.28) by comparing to the solution $y(t)$ of the problem for ODE

$$y'(t) + By(t) = 0, \quad y(0) = A,$$

we get

$$0 \leqslant \mathcal{E}_0(u(t)) \leqslant A\, e^{-B\,t} \to 0 \text{ as } t \to +\infty. \tag{3.29}$$

By (3.14) and (3.29) we deduce that

$$(1-x^2)^{\frac{\beta}{2}}|u - \tfrac{M}{|\Omega|}| \leqslant \tilde{A}\, e^{-\tilde{B}\,t} \to 0 \text{ as } t \to +\infty.$$

This proves Theorem 1 completely. □

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