EXTENSIONS OF INTERPOLATION BETWEEN THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY FOR MATRICES

M. BAKHERAD[1], R. LASHKARIPOUR[2] AND M. HAJMOHAMADI[3]

ABSTRACT. In this paper, we present some extensions of interpolation between the arithmetic-geometric means inequality. Among other inequalities, it is shown that if A, B, X are $n \times n$ matrices, then

$$\|AXB^*\|^2 \leq \|f_1(A^*A)Xg_1(B^*B)\| \, \|f_2(A^*A)Xg_2(B^*B)\|,$$

where f_1, f_2, g_1, g_2 are non-negative continues functions such that $f_1(t)f_2(t) = t$ and $g_1(t)g_2(t) = t$ $(t \geq 0)$. We also obtain the inequality

$$|||AB^*|||^2 \leq \left\vert\left\vert\left\vert p(A^*A)^{\frac{m}{p}} + (1-p)(B^*B)^{\frac{s}{1-p}} \right\vert\right\vert\right\vert \, \left\vert\left\vert\left\vert (1-p)(A^*A)^{\frac{n}{1-p}} + p(B^*B)^{\frac{t}{p}} \right\vert\right\vert\right\vert,$$

in which m, n, s, t are real numbers such that $m + n = s + t = 1$, $||| \cdot |||$ is an arbitrary unitarily invariant norm and $p \in [0, 1]$.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{M}_n$ be the C^*-algebra of all $n \times n$ complex matrices and $\langle \cdot, \cdot \rangle$ be the standard scalar product in $\mathbb{C}^n$ with the identity I. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a Hermitian matrix A and the C^*-algebra generated by A and I.

A norm $||| \cdot |||$ on $\mathcal{M}_n$ is said to be unitarily invariant norm if $|||UAV||| = |||A|||$, for all unitary matrices U and V. For $A \in \mathcal{M}_n$, let $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ denote the singular values of A, i.e. the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$ arranged in a decreasing order with their multiplicities counted. Note that $s_j(A) = s_j(A^*) = s_j(|A|)$ $(1 \leq j \leq n)$ and $\|A\| = s_1(A)$. The Ky Fan norm of a matrix A is defined as $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A)$ $(1 \leq k \leq n)$. The Fan dominance theorem asserts that $\|A\|_{(k)} \leq \|B\|_{(k)}$ for $k = 1, 2, \cdots, n$ if and only if $|||A||| \leq |||B|||$ for every unitarily invariant norm (see [8, p.93]). The Hilbert-Schmidt norm is defined by $\|A\|_2 = \left(\sum_{j=1}^{n} s_j^2(A) \right)^{1/2}$, where is unitarily invariant.

The classical Cauchy-Schwarz inequality for $a_j \geq 0$, $b_j \geq 0$ $(1 \leq j \leq n)$ states that

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right)$$

with equality if and only if $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ are proportional [11]. Bhatia and Davis gave a matrix Cauchy-Schwarz inequality as follows

$$|||AXB^*|||^2 \leq |||A^*AX||| \, |||XB^*B||| \tag{1.1}$$

where $A, B, X \in \mathcal{M}_n$. (For further information about the Cauchy-Schwarz inequality, see [4, 6, 7] and references therein.) Recently, Kittaneh et al. [3] extended inequality (1.1) to the form

$$|||AXB^*|||^2 \leq |||(A^*A)^p X (B^*B)^{1-p}||| \, |||(A^*A)^{1-p} X (B^*B)^p|||, \tag{1.2}$$

where $A, B, X \in \mathcal{M}_n$ and $p \in [0, 1]$. Audenaert [2] proved that for all $A, B \in \mathcal{M}_n$ and all $p \in [0, 1]$, we have

$$|||AB^*|||^2 \leq |||pA^*A + (1-p)B^*B||| \, |||(1-p)A^*A + pB^*B|||. \tag{1.3}$$

In [14], the authors generalized inequality (1.3) for all $A, B, X \in \mathcal{M}_n$ and all $p \in [0, 1]$ to the form

$$|||AXB^*|||^2 \leq |||pA^*AX + (1-p)XB^*B||| \, |||(1-p)A^*AX + pXB^*B|||. \tag{1.4}$$

Inequality (1.4) interpolates between the arithmetic-geometric mean inequality. In [3], the authors showed a refinement of inequality (1.4) for the Hilbert-Schmidt norm as follows

$$\|AXB^*\|_2^2 \leq \left(\|pA^*AX + (1-p)XB^*B\|_2^2 - r^2 \|A^*AX - XB^*B\|_2^2\right)$$
$$\times \left(\|(1-p)A^*AX + pXB^*B\|_2^2 - r^2 \|A^*AX - XB^*B\|_2^2\right), \tag{1.5}$$

in which $A, B, X \in \mathcal{M}_n$, $p \in [0, 1]$ and $r = \min\{p, 1-p\}$. The Young inequality for every unitarily invariant norm states that $|||A^p B^{1-p}||| \leq |||pA + (1-p)B|||$, where A, B are positive definite matrices and $p \in [0, 1]$ (see [1] and also [5]). Kosaki [10] extended the last inequality for the Hilbert-Schmidt norm as follows

$$\|A^p XB^{1-p}\|_2 \leq \|pAX + (1-p)XB\|_2, \tag{1.6}$$

where A, B are positive definite matrices, X is any matrix and $p \in [0, 1]$. In [9], the authors considered as a refined matrix Young inequality for the Hilbert-Schmidt norm

$$\|A^p XB^{1-p}\|_2^2 + r^2 \|AX - XB\|_2^2 \leq \|pAX + (1-p)XB\|_2^2, \tag{1.7}$$

in which A, B are positive semidefinite matrices, $X \in \mathcal{M}_n$, $p \in [0,1]$ and $r = \min\{p, 1-p\}$.

Based on the refined Young inequality (1.7), Zhao and Wu [13] proved that

$$\|A^p X B^{1-p}\|_2^2 + r_0 \|A^{\frac{1}{2}} X B^{\frac{1}{2}} - AX\|_2^2 + (1-p)^2 \|AX - XB\|_2^2 \le \|pAX + (1-p)XB\|_2^2,$$
(1.8)

for $0 < p \le \frac{1}{2}$ and

$$\|A^p X B^{1-p}\|_2^2 + r_0 \|A^{\frac{1}{2}} X B^{\frac{1}{2}} - XB\|_2^2 + p^2 \|AX - XB\|_2^2 \le \|pAX + (1-p)XB\|_2^2,$$

for $\frac{1}{2} < p < 1$ such that $r = \min\{p, 1-p\}$ and $r_0 = \min\{2r, 1-2r\}$.

In this paper, we obtain some operator and unitarily invariant norms inequalities. Among other results, we obtain a refinement of inequality (1.5) and we also extend inequalities (1.2), (1.3) and (1.5) for the function $f(t) = t^p$ ($p \in \mathbb{R}$).

2. Main results

In this section, by using some ideas of [3] we extend the Audenaert results for the operator norm.

Theorem 2.1. *Let* $A, B, X \in \mathcal{M}_n$ *and* f_1, f_2, g_1, g_2 *be non-negative continues functions such that* $f_1(t)f_2(t) = t$ *and* $g_1(t)g_2(t) = t$ ($t \ge 0$). *Then*

$$\|AXB^*\|^2 \le \|f_1(A^*A)Xg_1(B^*B)\| \, \|f_2(A^*A)Xg_2(B^*B)\|.$$
(2.1)

Proof. It follows from

$$
\begin{aligned}
\|AXB^*\|^2 &= \|BX^*A^*AXB^*\| \\
&= s_1(BX^*A^*AXB^*) \\
&= \lambda_{\max}(BX^*A^*AXB^*) \quad \text{(since } BX^*A^*AXB^* \text{ is positive semidefinite)} \\
&= \lambda_{\max}(X^*A^*AXB^*B) \\
&= \lambda_{\max}(X^*f_1(A^*A)f_2(A^*A)Xg_2(B^*B)g_1(B^*B)) \\
&= \lambda_{\max}(g_1(B^*B)X^*f_1(A^*A)f_2(A^*A)Xg_2(B^*B)) \\
&\le \|g_1(B^*B)X^*f_1(A^*A)f_2(A^*A)Xg_2(B^*B)\| \\
&\le \|g_1(B^*B)X^*f_1(A^*A)\| \, \|f_2(A^*A)Xg_2(B^*B)\| \\
&= \|f_1(A^*A)Xg_1(B^*B)\| \, \|f_2(A^*A)Xg_2(B^*B)\|
\end{aligned}
$$

that we get the desired result.   □

Corollary 2.2. *If $A, B, X \in \mathcal{M}_n$ and m, n, s, t are real numbers such that $m + n = s + t = 1$, then*

$$\|AXB^*\|^2 \le \|(A^*A)^m X(B^*B)^s\| \, \|(A^*A)^n X(B^*B)^t\|. \tag{2.2}$$

In the next results, we show some generalizations of inequality (1.3) for the operator norm.

Corollary 2.3. *Let $A, B \in \mathcal{M}_n$ and let f_1, f_2, g_1, g_2 be non-negative continues functions such that $f_1(t)f_2(t) = t$ and $g_1(t)g_2(t) = t$ $(t \ge 0)$. Then*

$$\|AB^*\|^2 \le \left\| pf_1(A^*A)^{\frac{1}{p}} + (1-p)g_1(B^*B)^{\frac{1}{1-p}} \right\| \left\| (1-p)f_2(A^*A)^{\frac{1}{1-p}} + pg_2(B^*B)^{\frac{1}{p}} \right\|,$$

where $p \in [0, 1]$.

Proof. Using Theorem 2.1 for $X = I$ we have

$$
\begin{aligned}
\|AB^*\|^2 &\le \|f_1(A^*A)g_1(B^*B)\| \, \|f_2(A^*A)g_2(B^*B)\| \\
&= \left\| \left(f_1(A^*A)^{\frac{1}{p}}\right)^p \left(g_1(B^*B)^{\frac{1}{1-p}}\right)^{1-p} \right\| \left\| \left(f_2(A^*A)^{\frac{1}{1-p}}\right)^{1-p} \left(g_2(B^*B)^{\frac{1}{p}}\right)^p \right\| \\
&\le \left\| pf_1(A^*A)^{\frac{1}{p}} + (1-p)g_1(B^*B)^{\frac{1}{1-p}} \right\| \left\| (1-p)f_2(A^*A)^{\frac{1}{1-p}} + pg_2(B^*B)^{\frac{1}{p}} \right\| \\
&\qquad \text{(by the Young inequality).}
\end{aligned}
$$

□

Corollary 2.4. *Let $A, B \in \mathcal{M}_n$ and let f, g be non-negative continues functions such that $f(t)g(t) = t^2$ $(t \ge 0)$. Then*

$$
\begin{aligned}
\|AB^*\|^2 &\le \|pf(A^*A) + (1-p)g(B^*B)\|^{\frac{1}{2}} \, \|(1-p)f(A^*A) + pg(B^*B)\|^{\frac{1}{2}} \\
&\quad \times \|pg(A^*A) + (1-p)f(B^*B)\|^{\frac{1}{2}} \, \|(1-p)g(A^*A) + pf(B^*B)\|^{\frac{1}{2}},
\end{aligned}
$$

where $p \in [0, 1]$.

Proof. Applying Theorem 2.1 and the Young inequality we get

$$
\begin{aligned}
\|AB^*\|^4 &\leq \left\|f(A^*A)^{\frac{1}{2}}g(B^*B)^{\frac{1}{2}}\right\|^2 \left\|g(A^*A)^{\frac{1}{2}}f(B^*B)^{\frac{1}{2}}\right\|^2 \\
&\qquad\qquad\qquad\qquad\text{(by Theorem 2.1 for } \sqrt{f} \text{ and } \sqrt{g}) \\
&\leq \left\|f(A^*A)^p g(B^*B)^{1-p}\right\| \left\|f(A^*A)^{1-p}g(B^*B)^p\right\| \\
&\quad \times \left\|g(A^*A)^p f(B^*B)^{1-p}\right\| \left\|g(A^*A)^{1-p}f(B^*B)^p\right\| \\
&\qquad\qquad\qquad\qquad\text{(by inequality (2.2))} \\
&\leq \|pf(A^*A) + (1-p)g(B^*B)\| \,\|(1-p)f(A^*A) + pg(B^*B)\| \\
&\quad \times \|pg(A^*A) + (1-p)f(B^*B)\| \,\|(1-p)g(A^*A) + pf(B^*B)\| \\
&\qquad\qquad\qquad\qquad\text{(by the Young inequality).}
\end{aligned}
$$

$\square$

3. Some interpolations for unitarily invariant norms

In this section, by applying some ideas of [3] we generalize some interpolations for an arbitrary unitarily invariant norm.

Let $Q_{k,n}$ denote the set of all strictly increasing k-tuples chosen from $1, 2, \cdots, n$, i.e. $I \in Q_{k,n}$ if $I = (i_1, i_2, \cdots, i_k)$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. The following lemma gives some properties of the kth antisymmetric tensor powers of matrices in $\mathcal{M}_n$; see [8, p.18].

Lemma 3.1. *Let $A, B \in \mathcal{M}_n$. Then*
(a) $(\wedge^k A)(\wedge^k B) = \wedge^k(AB)$ for $k = 1, \cdots, n$.
(b) $(\wedge^k A)^* = \wedge^k A^*$ for $k = 1, \cdots, n$.
(c) $(\wedge^k A)^{-1} = \wedge^k A^{-1}$ for $k = 1, \cdots, n$.
(d) If $s_1, s_2, \cdots, s_n$ are the singular values of A, then the singular values of $\wedge^k A$ are $s_{i_1}, s_{i_2}, \cdots, s_{i_k}$, where $(i_1, i_2, \cdots, i_k) \in Q_{k,n}$.

Now, we show inequality (2.2) for an arbitrary unitarily invariant norm.

Theorem 3.2. *Let $A, B, X \in \mathcal{M}_n$ and $||| \cdot |||$ be an arbitrary unitarily invariant norm. Then*

$$
|||AXB^*|||^2 \leq |||(A^*A)^m X(B^*B)^s||| \, |||(A^*A)^n X(B^*B)^t|||, \tag{3.1}
$$

where m, n, s, t are real numbers such that $m + n = s + t = 1$. In particular, if A, B are positive definite

$$|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||^2 \leq |||A^pXB^{1-p}||| \, |||A^{1-p}XB^p|||, \qquad (3.2)$$

where $p \in [0, 1]$.

Proof. If we replace A, B and X by $\wedge^k A$, $\wedge^k B$ and $\wedge^k X$, their kth antisymmetric tensor powers in inequality (2.1) and apply Lemma 3.1 , then we have

$$\left\|\wedge^k AXB^*\right\|^2 \leq \left\|\wedge^k (A^*A)^m X(B^*B)^s\right\| \, \left\|\wedge^k (A^*A)^n X(B^*B)^t\right\|$$

that is equivalent to

$$s_1^2 \left(\wedge^k AXB^*\right) \leq s_1 \left(\wedge^k (A^*A)^m X(B^*B)^s\right) \, s_1 \left(\wedge^k (A^*A)^n X(B^*B)^t\right).$$

Applying Lemma 3.1(d), we have

$$\prod_{j=1}^k s_j (AXB^*) \leq \prod_{j=1}^k s_j^{\frac{1}{2}} ((A^*A)^m X(B^*B)^s) \prod_{j=1}^k s_j^{\frac{1}{2}} ((A^*A)^n X(B^*B)^t)$$

$$\leq \prod_{j=1}^k s_j^{\frac{1}{2}} ((A^*A)^m X(B^*B)^s) \, s_j^{\frac{1}{2}} ((A^*A)^n X(B^*B)^t), \qquad (3.3)$$

where $k = 1, \cdots, n$. Inequality (3.3) implies that

$$\sum_{j=1}^k s_j (AXB^*) \leq \sum_{j=1}^k s_j^{\frac{1}{2}} ((A^*A)^m X(B^*B)^s) \, s_j^{\frac{1}{2}} ((A^*A)^n X(B^*B)^t)$$

$$\leq \left(\sum_{j=1}^k s_j ((A^*A)^m X(B^*B)^s)\right)^{\frac{1}{2}} \left(\sum_{j=1}^k s_j ((A^*A)^n X(B^*B)^t)\right)^{\frac{1}{2}}$$

(by the Cauchy-Schwarz inequality),

where $k = 1, \cdots, n$. Hence

$$\|AXB^*\|_{(k)}^2 \leq \|(A^*A)^m X(B^*B)^s\|_{(k)} \|(A^*A)^n X(B^*B)^t\|_{(k)}.$$

Now, using the Fan dominace theorem [8, p.98], we get the desired result. □

Now, by using inequality (3.2), Theorem 3.2 and a same argument in the proof of Corollaries 2.3 and 2.4 , we get the following results that these inequalities are some generalizations of the Audenaert inequality (1.3).

Corollary 3.3. *Let $A, B \in \mathcal{M}_n$, m, n, s, t be real numbers such that $m + n = s + t = 1$ and $||| \cdot |||$ be an arbitrary unitarily invariant norm. Then*

$$|||AB^*|||^2 \leq \left|\left|\left| p(A^*A)^{\frac{m}{p}} + (1-p)(B^*B)^{\frac{s}{1-p}} \right|\right|\right| \ \left|\left|\left| (1-p)(A^*A)^{\frac{n}{1-p}} + p(B^*B)^{\frac{t}{p}} \right|\right|\right|,$$

where $p \in [0, 1]$.

Corollary 3.4. *Let $A, B \in \mathcal{M}_n$, m, n, s, t be real numbers such that $m + n = s + t = 2$ and $||| \cdot |||$ be an arbitrary unitarily invariant norm. Then*

$$|||AB^*|||^2 \leq |||p(A^*A)^m + (1-p)(B^*B)^s|||^{\frac{1}{2}} |||(1-p)(A^*A)^m + p(B^*B)^s|||^{\frac{1}{2}}$$

$$\times |||p(A^*A)^n + (1-p)(B^*B)^t|||^{\frac{1}{2}} |||(1-p)(A^*A)^n + p(B^*B)^t|||^{\frac{1}{2}}, \quad (3.4)$$

in which $p \in [0, 1]$.

Remark 3.5. If we put $n = m = s = t = 1$ in inequality (3.4), then we obtain the Audenaert inequality (1.3). Also, if we use inequality (1.6), Corollaries 3.3 and 3.4, then similar to Corollaries 2.3 and 2.4 we get the following inequalities

$$\|AXB^*\|_2^2 \leq \|p(A^*A)^{\frac{m}{p}}X + (1-p)X(B^*B)^{\frac{s}{1-p}}\|_2\|(1-p)(A^*A)^{\frac{n}{1-p}}X + pX(B^*B)^{\frac{t}{p}}\|_2,$$
$$(3.5)$$

where $A, B \in \mathcal{M}_n$, m, n, s, t are real numbers such that $m + n = s + t = 1$, $p \in [0, 1]$ and

$$\|AXB^*\|_2^2 \leq \|p(A^*A)^mX + (1-p)X(B^*B)^s\|_2^{\frac{1}{2}}\|(1-p)(A^*A)^mX + pX(B^*B)^s\|_2^{\frac{1}{2}}$$

$$\times \|p(A^*A)^nX + (1-p)X(B^*B)^t\|_2^{\frac{1}{2}}\|(1-p)(A^*A)^nX + pX(B^*B)^t\|_2^{\frac{1}{2}}$$

for $A, B \in \mathcal{M}_n$, real numbers m, n, s, t such that $m + n = s + t = 2$ and $p \in [0, 1]$. These inequalities are generalizations of (1.4) for the Hilbert-Schmidt norms.

In the following theorem, we show a refinement of inequality (3.5) for the Hilbert-Schmidt norm.

Theorem 3.6. *Let $A, B, X \in \mathcal{M}_n$. Then*

$$\|AXB^*\|_2^2 \leq \left(\left\| p(A^*A)^{\frac{m}{p}}X + (1-p)X(B^*B)^{\frac{s}{1-p}} \right\|_2^2 - r^2 \left\| (A^*A)^{\frac{m}{p}}X - X(B^*B)^{\frac{s}{1-p}} \right\|_2^2 \right)$$

$$\times \left(\left\| (1-p)(A^*A)^{\frac{n}{1-p}}X + pX(B^*B)^{\frac{t}{p}} \right\|_2^2 - r^2 \left\| (A^*A)^{\frac{n}{1-p}}X - X(B^*B)^{\frac{t}{p}} \right\|_2^2 \right),$$

in which m, n, s, t are real numbers such that $m + n = s + t = 1$, $p \in [0, 1]$ and $r = \min\{p, 1 - p\}$.

Proof. Using inequality (3.1), we obtain

$$
\begin{aligned}
\|AXB^*\|_2^2 &\leq \left\|(A^*A)^m X(B^*B)^s\right\|_2 \left\|(A^*A)^n X(B^*B)^t\right\|_2 \\
&= \left\|\left((A^*A)^{\frac{m}{p}}\right)^p X\left((B^*B)^{\frac{s}{1-p}}\right)^{1-p}\right\|_2 \left\|\left((A^*A)^{\frac{n}{1-p}}\right)^{1-p} X\left((B^*B)^{\frac{t}{p}}\right)^p\right\|_2 \\
&\leq \left(\left\|p(A^*A)^{\frac{m}{p}}X + (1-p)X(B^*B)^{\frac{s}{1-p}}\right\|_2^2 - r^2\left\|(A^*A)^{\frac{m}{p}}X - X(B^*B)^{\frac{s}{1-p}}\right\|_2^2\right) \\
&\quad \times \left(\left\|(1-p)(A^*A)^{\frac{n}{1-p}}X + pX(B^*B)^{\frac{t}{p}}\right\|_2^2 - r^2\left\|(A^*A)^{\frac{n}{1-p}}X - X(B^*B)^{\frac{t}{p}}\right\|_2^2\right),
\end{aligned}
$$

where $p \in [0,1]$ and $r = \min\{p, 1-p\}$, and the proof is complete. $\qquad\square$

Theorem 3.6 includes a special case as follows.

Corollary 3.7. [3, Theorem 2.5] *Let $A, B, X \in \mathcal{M}_n$. Then*

$$
\begin{aligned}
\|AXB^*\|_2^2 &\leq \left(\|pA^*AX + (1-p)XB^*B\|_2^2 - r^2\|A^*AX - XB^*B\|_2^2\right) \\
&\quad \times \left(\|(1-p)(A^*A)X + pXB^*B\|_2^2 - r^2\|A^*AX - XB^*B\|_2^2\right),
\end{aligned}
$$

where $p \in [0,1]$ and $r = \min\{p, 1-p\}$.

Proof. For $p \in [0,1]$, if we put $m = t = p$ and $n = s = 1 - p$ in Theorem 3.6, then we get the desired result. $\qquad\square$

The next result is a refinement of inequality (1.5).

Theorem 3.8. *Let $A, B, X \in \mathcal{M}_n(\mathbb{C})$ and let $p \in (0,1)$. Then*

(i) *for $0 < p \leq \frac{1}{2}$,*

$$\|AXB^*\|_2^2 \leq$$

$$
\left(\|pA^*AX + (1-p)XB^*B\|_2^2 - r_0\|(A^*A)^{\frac{1}{2}}X(B^*B)^{\frac{1}{2}} - A^*AX\|_2^2 - (1-p)^2\|A^*AX - XB^*B\|_2^2\right)^{\frac{1}{2}}
$$

$$
\times \left(\|(1-p)A^*AX + pXB^*B\|_2^2 - r_0\|(A^*A)^{\frac{1}{2}}X(B^*B)^{\frac{1}{2}} - A^*AX\|_2^2 - p^2\|A^*AX - XB^*B\|_2^2\right)^{\frac{1}{2}}.
$$

$$(3.6)$$

(ii) *for $\frac{1}{2} < p < 1$,*

$$\|AXB^*\|_2^2 \leq$$

$$
\left(\|pA^*AX + (1-p)XB^*B\|_2^2 - r_0\|(A^*A)^{\frac{1}{2}}X(B^*B)^{\frac{1}{2}} - X(B^*B)\|_2^2 - (1-p)^2\|A^*AX - XB^*B\|_2^2\right)^{\frac{1}{2}}
$$

$$
\times \left(\|(1-p)A^*AX + pXB^*B\|_2^2 - r_0\|(A^*A)^{\frac{1}{2}}X(B^*B)^{\frac{1}{2}} - X(B^*B)\|_2^2 - p^2\|A^*AX - XB^*B\|_2^2\right)^{\frac{1}{2}},
$$

$$(3.7)$$

where $r = \min\{p, 1-p\}$ and $r_0 = \min\{2r, 1-2r\}$.

Proof. The proof of inequality (3.7) is similar to that of inequality (3.6). Thus, we only need to prove the inequality (3.6).

If $0 < p \leq \frac{1}{2}$, then we replace A and B by A^*A and B^*B in inequality (1.8), respectively, we have

$$\|(A^*A)^p X (B^*B)^{1-p}\|_2 \leq$$

$$\left(\|pA^*AX + (1-p)XB^*B\|_2^2 - r_0\|(A^*A)^{\frac{1}{2}}X(B^*B)^{\frac{1}{2}} - A^*AX\|_2^2 - (1-p)^2\|A^*AX - XB^*B\|_2^2 \right)^{\frac{1}{2}}. \tag{3.8}$$

Interchanging the roles of p and $1-p$ in the inequality (3.8), we get

$$\|(A^*A)^{1-p} X (B^*B)^{p}\|_2 \leq$$

$$\left(\|(1-p)A^*AX + pXB^*B\|_2^2 - r_0\|(A^*A)^{\frac{1}{2}}X(B^*B)^{\frac{1}{2}} - A^*AX\|_2^2 - p^2\|A^*AX - XB^*B\|_2^2 \right)^{\frac{1}{2}}. \tag{3.9}$$

Applying inequalities (3.1), (3.8) and (3.9), we get the desired result. $\qquad\square$

Corollary 3.9. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$ and $p \in (0, 1)$. Then*

(i) *for $0 < p \leq \frac{1}{2}$,*

$$\|AB^*\|_2^2 \leq$$

$$\left(\|pA^*A + (1-p)B^*B\|_2^2 - r_0\|(A^*A)^{\frac{1}{2}}(B^*B)^{\frac{1}{2}} - A^*A\|_2^2 - (1-p)^2\|A^*A - B^*B\|_2^2 \right)^{\frac{1}{2}}$$

$$\times \left(\|(1-p)A^*A + pB^*B\|_2^2 - r_0\|(A^*A)^{\frac{1}{2}}(B^*B)^{\frac{1}{2}} - A^*A\|_2^2 - p^2\|A^*A - B^*B\|_2^2 \right)^{\frac{1}{2}}.$$

(ii) *for $\frac{1}{2} < p < 1$,*

$$\|AB^*\|_2^2 \leq$$

$$\left(\|pA^*A + (1-p)B^*B\|_2^2 - r_0\|(A^*A)^{\frac{1}{2}}(B^*B)^{\frac{1}{2}} - (B^*B)\|_2^2 - (1-p)^2\|A^*A - B^*B\|_2^2 \right)^{\frac{1}{2}}$$

$$\times \left(\|(1-p)A^*A + pB^*B\|_2^2 - r_0\|(A^*A)^{\frac{1}{2}}(B^*B)^{\frac{1}{2}} - (B^*B)\|_2^2 - p^2\|A^*A - B^*B\|_2^2 \right)^{\frac{1}{2}},$$

where $r = \min\{p, 1-p\}$ and $r_0 = \min\{2r, 1-2r\}$.

We would like obtain upper bound for $|||AXB^*|||$, for every unitary invariant norm. The following lemma has been shown in [12], and considered as a refined matrix Youngs inequality for every unitary invariant norm.

Lemma 3.10. *Let $A, B, X \in \mathcal{M}_n$ such that A, B are positive semidefinite. Then for $0 \leq p \leq 1$, we have*

$$|||A^p X B^{1-p}|||^2 + r_0(|||AX||| - |||XB|||)^2 \leq (p|||AX||| + (1-p)|||XB|||)^2, \tag{3.10}$$

where $r_0 = \min\{p, 1-p\}$.

Proposition 3.11. *Let $A, B, X \in \mathcal{M}_n$. Then*

$$|||AXB^*||||^2 \leq \left((p|||A^*AX||| + (1-p)|||XB^*B|||)^2 - r_0^2(|||A^*AX||| - |||XB^*B|||)^2\right)^{\frac{1}{2}}$$

$$\times \left(((1-p)|||A^*AX||| + p|||XB^*B|||)^2 - r_0^2(|||A^*AX||| - |||XB^*B|||)^2\right)^{\frac{1}{2}},$$

where $p \in [0,1]$ and $r_0 = \min\{p, 1-p\}$.

Proof. In inequality (3.10), we put $A = A^*A$ and $B = B^*B$, we get

$$|||(A^*A)^p X (B^*B)^{1-p}||| \leq \left((p|||A^*AX||| + (1-p)|||XB^*B|||)^2 - r_0^2(|||A^*AX||| - |||XB^*B|||)^2\right)^{\frac{1}{2}}.$$
$$(3.11)$$

Interchanging p with $1-p$ in inequality (3.11), we get

$$|||(A^*A)^{1-p} X (B^*B)^p||| \leq \left(((1-p)|||A^*AX||| + p|||XB^*B|||)^2 - r_0^2(|||A^*AX||| - |||XB^*B|||)^2\right)^{\frac{1}{2}}.$$
$$(3.12)$$

Now by inequalities (3.1), (3.11) and (3.12) we get the desired inequality. $\square$

REFERENCES

1. T. Ando, *Matrix Young inequality*, J. Oper. Theor. Adv. Appl. **75** (1995), 33–38.

2. K.M.R. Audenaert, *Interpolating between the arithmetic-geometric mean and Cauchy-Schwarz matrix norm inequalities*, Oper. Matrices. **9** (2015) 475–479.

3. M. Al-khlyleh and F. Kittaneh, *Interpolating inequalities related to a recent result of Audenaert*, Linear Multilinear Algebra **65** (2017), no. 5, 922–929.

4. M. Bakherad, *Some Reversed and Refined Callebaut Inequalities Via Kontorovich Constant*, Bull. Malay. Math. Sci. Soc. (to appear), doi: 10.1007/s40840-016-0364-9.

5. M. Bakherad, M. Krnic and M.S. Moslehian, *Reverse Young-type inequalities for matrices and operators*, Rocky Mountain J. Math. **46** (2016), no. 4, 1089–1105.

6. M. Bakherad and M.S. Moslehian, *Reverses and variations of Heinz inequality*, Linear Multilinear Algebra 63 (2015), no. 10, 1972–1980.

7. M. Bakherad and M.S. Moslehian, *Complementary and refined inequalities of Callebaut inequality for operators*, Linear Multilinear Algebra 63 (2015), no. 8, 1678–1692.

8. R. Bhatia, *Matrix analysis*, New York (NY): Springer-Verlag; 1997.

9. O. Hirzallah and F. Kittaneh, *Matrix Young inequalities for the Hilbert-Schmidt norm*, Linear Algebra Appl. **308** (2000) 77–84.

10. H. Kosaki, *Arithmetic-geometric mean and related inequalities for operators*, J. Funct. Anal. **156** (1998), 429–451.

11. D.S. Mitrinović, J.E. Pečarić and A.M Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic, 1993.

12. M. Sababheh, *Interpolated inequalities for unitarily invariant norms*, Linear Algebra Appl. **475** (2015), 240-250.

13. J. Zho and J. Wu, *Operator inequalities involving improved Young and its reverse inequalities*, J. Math. Anal. Appl. **421** (2015), 1779–1789.

14. L. Zou and Y. Jiang, *A note on interpolation between the arithmetic–geometric mean and Cauchy-Schwarz matrix norm inequalities*, J. Math. Inequal. **10** (2016), no. 4, 1119–1122.

[123]Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, I.R.Iran.

E-mail address: [1]`mojtaba.bakherad@yahoo.com; bakherad@member.ams.org`

E-mail address: [2]`lashkari@hamoon.usb.ac.ir`

E-mail address: [2]`monire.hajmohamadi@yahoo.com`