On the validity of the local Fourier analysis

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Abstract

Local Fourier analysis (LFA) is a useful tool in predicting the convergence factors of geometric multigrid methods (GMG). As is well known, on rectangular domains with periodic boundary conditions this analysis gives the *exact* convergence factors of such methods. In this work, using the Fourier method, we extend these results by proving that such analysis yields the exact convergence factors for a wider class of problems.

Keywords: Local Fourier analysis, multigrid, Fourier method

1. Introduction

The local Fourier analysis (LFA) introduced by A. Brandt [1], is a tool which provides realistic quantitative estimates of the asymptotic convergence factors of the GMG algorithms. For discretizations of partial differential equations, the traditional LFA is based on a discrete Fourier transform and is accurate if the influence of the boundary conditions is negligible. In fact, it is well known (see [2, 3]), that for model problems on rectangular domains and with periodic boundary conditions this analysis gives the exact convergence rate of GMG.

In this work we focus on the question whether the LFA can be made rigorous for a wider class of problems with boundary conditions that are not necessarily periodic. We answer to this question positively. Our approach relies on the embedding of the model problem into a periodic problem. Similar ideas have also been explored in works on circulant preconditioners for elliptic problems [4, 5] and also for preconditioning the indefinite Helmholtz equation [6]. We introduce a class of operators called LFA-compatible operators here and prove that for such operators the LFA gives the exact multigrid convergence factors. Our studies include the Dirichlet, the Neumann and the mixed boundary condition problem for a constant coefficient, reaction-diffusion equation on a d-dimensional tensor product grid.

2. Preliminaries

2.1. The Dirichlet problem and its discretization

We consider a reaction-diffusion problem in d spatial dimensions on the domain $\Omega^D = (0,1)^d$,

$$-\Delta u(\boldsymbol{x}) + cu(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^D, \quad \text{and} \quad u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega^D, \tag{1}$$

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where $c > 0$ is a constant. First, let us consider the simplest case when $d = 1$ (one dimensional problem). The computational domain then is the interval $\Omega^D = (0, 1)$ and the corresponding two-point boundary value problem (1) is:

$$-u''(x) + cu(x) = f(x), \quad x \in \Omega^D, \quad u(0) = u(1) = 0. \tag{2}$$

For $d = 1$, we introduce a uniform grid $\Omega_h^D = \{x_k = kh\}_{k=0}^n$, with step size $h = 1/n$, $n \in \mathbb{N}$ and we discretize this problem by the standard central difference scheme. As a result, we obtain the linear system of algebraic equations with tri-diagonal matrix:

$$A_h^D \boldsymbol{u} = \boldsymbol{f} \text{ where } A_h^D = T_h^D + cI_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}, \tag{3}$$

where $\boldsymbol{u} = (u_1, \dots, u_{n-1})^T$, $\boldsymbol{f} = (f_1, \dots, f_{n-1})^T$, $I_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$ is the identity matrix, and

$$T_h^D = \frac{1}{h^2} \operatorname{diag}(-1, 2, -1) \in \mathbb{R}^{(n-1)\times(n-1)}. \tag{4}$$

This was the simple, but very important, one dimensional case. In the case of higher spatial dimensions and on a uniform grid with the same step size $h = 1/n$ in all the directions the linear systems are written in compact form by using the standard tensor product $\otimes$ for matrices. We recall the following properties of the tensor product

$$(X + Y) \otimes Z = (X \otimes Z) + (Y \otimes Z), \quad (X_1 \otimes X_2)(Y_1 \otimes Y_2) = (X_1 Y_1 \otimes X_2 Y_2). \tag{5}$$

We further denote the k-th tensor power of a matrix X by $X^{\otimes k} = \underbrace{X \otimes \ldots \otimes X}_{k}$. Finally, let us note that the generalization to different step sizes in different directions is straightforward.

With this notation, the standard second order central difference scheme for discretization of the Dirichlet problem (1) results in the linear system

$$A_h^D \boldsymbol{u} = \boldsymbol{f}, \quad A_h^D = \sum_{j=1}^{d} \left(I_{n-1}^{\otimes(j-1)} \otimes T_h^D \otimes I_{n-1}^{\otimes(d-j)} \right) + cI_{n-1}^{\otimes d} \in \mathbb{R}^{(n-1)^d \times (n-1)^d}. \tag{6}$$

2.2. A periodic problem

We now consider a finite difference discretization on a grid with step size $h = 1/n$ of a periodic problem on $\Omega^P = (0, 2)$:

$$A_h^P \widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{f}}, \quad \text{where} \quad A_h^P = T_h^P + cI_N \in \mathbb{R}^{N\times N}, \tag{7}$$

with $N = 2n$ and $T_h^P = \frac{1}{h^2} \operatorname{diag}(-1, 2, -1) - \boldsymbol{e}_1^N (\boldsymbol{e}_N^N)^T - \boldsymbol{e}_N^N (\boldsymbol{e}_1^N)^T \in \mathbb{R}^{N\times N}$. Here, we have denoted $\widetilde{\boldsymbol{u}} = (\widetilde{u}_1, \dots, \widetilde{u}_N)^T$, $\widetilde{\boldsymbol{f}} = \left(\widetilde{f}_1, \dots, \widetilde{f}_N\right)^T$, and $\boldsymbol{e}_k^m$ is the k-th canonical Euclidean basis vector in $\mathbb{R}^m$. Finally, let us point out that by a periodic problem here we mean the problem (2) defined on Ω^P with boundary conditions $u(0) - u(2) = u'(0) - u'(2) = 0$.

The extension to higher dimension $d > 1$ is obvious and we have the linear system $A_h^P \widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{f}}$, with

$$A_h^P = \sum_{j=1}^{d} \left(I_N^{\otimes(j-1)} \otimes T_h^P \otimes I_N^{\otimes(d-j)} \right) + cI_N^{\otimes d} \in \mathbb{R}^{N^d \times N^d}. \tag{8}$$

2.3. Relation between the Dirichlet and the periodic problem

Our goal now is to describe how the discretized Dirichlet problem relates to the periodic problem defined in section 2.2. To begin, we consider the 1-dimensional case given in (3) and we define the *odd extension operator* as the linear operator $E_{o,h}: \mathbb{R}^{n-1} \to \mathbb{R}^N$, $N = 2n$ such that

$$E_{o,h}\boldsymbol{e}_i^{n-1} = \boldsymbol{e}_i^N - \boldsymbol{e}_{N-i}^N, \quad i = 1, \dots, n-1. \tag{9}$$

The *restriction operator* $R_{o,h}$ is defined as $R_{o,h} = \frac{1}{2}E_{o,h}^T$. It is easy to see that the following relations hold in the one dimensional case: $R_{o,h}E_{o,h} = I_{n-1}$, and $E_{o,h}R_{o,h}\boldsymbol{u} = \boldsymbol{u}$, for all $\boldsymbol{u} \in \text{range}(E_{o,h})$. Notice also that $\text{range}(E_{o,h}) = \{\boldsymbol{u} \in \mathbb{R}^N \mid u_n = u_N = 0, u_j = -u_{N-j}, j = 1, \dots, n-1\}$ and $\widetilde{\boldsymbol{f}} = E_{o,h}\boldsymbol{f}$. For $d > 1$ the restriction and extensions are $R_{o,h}^{\otimes d}$ and $E_{o,h}^{\otimes d}$ and we have:

$$R_{o,h}^{\otimes d}E_{o,h}^{\otimes d} = I_{n-1}^{\otimes d}, \quad \text{and} \quad E_{o,h}^{\otimes d}R_{o,h}^{\otimes d}\boldsymbol{u} = \boldsymbol{u}, \quad \text{for all} \quad \boldsymbol{u} \in \text{range}(E_{o,h}^{\otimes d}). \tag{10}$$

2.3.1. LFA-compatibility

We now clarify the relation between the Dirichlet and the periodic problem. We begin with a very general definition of LFA-compatibility.

Definition 2.1. Let $R_{o,h}$ and $E_{o,h}$ be operators satisfying (10). We say that the pair of operators (M_h^D, M_h^P) is an *LFA-compatible* pair if and only if $M_h^D = R_{o,h}M_h^PE_{o,h}$ and $M_h^P\boldsymbol{v} \in \text{range}(E_{o,h})$ for all $\boldsymbol{v} \in \text{range}(E_{o,h})$.

The LFA-compatibility is, in some sense, the minimal requirement which allows for building relations between solutions to a periodic and the corresponding Dirichlet problems, or the iterates constructed in an iterative method for these problems. In a more abstract setting, the operators M_h^P and M_h^D do not have to be a periodic or a Dirichlet problem, they only need to be connected via a compatibility relation based on operators $E_{o,h}$ and $R_{o,h}$ satisfying the relations in (10). In the following, however, we only use $E_{o,h}$ and $R_{o,h}$ as defined above.

Now we prove several results, which follow directly from the definition of LFA-compatibility.

Lemma 2.2. *Let A_h^D and A_h^P be the coefficient matrices related to the Dirichlet and periodic problems. Then, (A_h^D, A_h^P) is an LFA-compatible pair.*

Proof. The standard properties of the tensor product imply that

$$\begin{aligned}
R_{o,h}^{\otimes d}A_h^PE_{o,h}^{\otimes d} &= R_{o,h}^{\otimes d}\left(\sum_{j=1}^{d}\left(I_N^{\otimes(j-1)} \otimes T_h^P \otimes I_N^{\otimes(d-j)}\right) + cI_N^{\otimes d}\right)E_{o,h}^{\otimes d} \\
&= R_{o,h}^{\otimes d}\left(\sum_{j=1}^{d}\left(E_{o,h}^{\otimes(j-1)} \otimes T_h^PE_{o,h} \otimes E_{o,h}^{\otimes(d-j)}\right) + cE_{o,h}^{\otimes d}\right) \\
&= \sum_{j=1}^{d}\left(I_{n-1}^{\otimes(j-1)} \otimes R_{o,h}T_h^PE_{o,h} \otimes I_{n-1}^{\otimes(d-j)}\right) + cI_{n-1}^{\otimes d}.
\end{aligned}$$

Further, taking into account that $R_{o,h}T_h^PE_{o,h} = T_h^D$, we also have $A_h^D = R_{o,h}^{\otimes d}A_h^PE_{o,h}^{\otimes d}$. If $\boldsymbol{u} \in \text{range}(E_{o,h}^{\otimes d})$, then there exists $\boldsymbol{v} \in \mathbb{R}^{(n-1)^d}$ such that $\boldsymbol{u} = E_{o,h}^{\otimes d}\boldsymbol{v}$ and we have

$$\begin{aligned}
A_h^P\boldsymbol{u} &= \left(\sum_{j=1}^{d} I_N^{\otimes(j-1)} \otimes T_h^P \otimes I_N^{\otimes(d-j)}\right)E_{o,h}^{\otimes d}\boldsymbol{v} + cI_N^{\otimes d}E_{o,h}^{\otimes d}\boldsymbol{v} \\
&= \left(\sum_{j=1}^{d} E_{o,h}^{\otimes(j-1)} \otimes T_h^PE_{o,h} \otimes E_{o,h}^{\otimes(d-j)}\right)\boldsymbol{v} + cE_{o,h}^{\otimes d}\boldsymbol{v}.
\end{aligned}$$

A straightforward computation shows that $T_h^P \boldsymbol{u} \in \text{range}(E_{o,h})$ for any $\boldsymbol{u} \in \text{range}(E_{o,h})$, and this completes the proof. $\square$

Lemma 2.3. *If $\boldsymbol{u}$ satisfies $A_h^D \boldsymbol{u} = \boldsymbol{f}$, then $A_h^P(E_{o,h}^{\otimes d}\boldsymbol{u}) = E_{o,h}^{\otimes d}\boldsymbol{f}$.*

Proof. Using that $A_h^D = R_{o,h}^{\otimes d} A_h^P E_{o,h}^{\otimes d}$, we have that $R_{o,h}^{\otimes d} A_h^P E_{o,h}^{\otimes d}\boldsymbol{u} = \boldsymbol{f}$. Applying $E_{o,h}^{\otimes d}$ on the left and taking into account that $A_h^P E_{o,h}^{\otimes d}\boldsymbol{u} \in \text{range}(E_{o,h}^{\otimes d})$, completes the proof. $\square$

Theorem 2.4. *The pair $((A_h^D)^{-1}, (A_h^P)^{-1})$ is LFA-compatible.*

Proof. We consider $\boldsymbol{f} \in \text{range}(E_{o,h}^{\otimes d})$. Then, there exists $\boldsymbol{g} \in \mathbb{R}^{(n-1)^d}$ such that $E_{o,h}^{\otimes d}\boldsymbol{g} = \boldsymbol{f}$. If $\boldsymbol{u} = (A_h^D)^{-1}\boldsymbol{g}$, by using Lemma 2.3 we have that $E_{o,h}^{\otimes d}\boldsymbol{u} = (A_h^P)^{-1}\boldsymbol{f}$, which implies that $(A_h^P)^{-1}\boldsymbol{f} \in \text{range}(E_{o,h}^{\otimes d})$. Next, again from Lemma 2.3, it follows that if $\boldsymbol{u} = (A_h^D)^{-1}\boldsymbol{f}$, then $(A_h^P)^{-1}E_{o,h}^{\otimes d}\boldsymbol{f} = E_{o,h}^{\otimes d}\boldsymbol{u}$. Hence, $R_{o,h}^{\otimes d}(A_h^P)^{-1}E_{o,h}^{\otimes d}\boldsymbol{f} = R_{o,h}^{\otimes d}E_{o,h}^{\otimes d}\boldsymbol{u} = \boldsymbol{u}$ and the proof is complete. $\square$

3. Linear iterative methods and multigrid

Let us consider a general stationary iterative method for the Dirichlet and the periodic problems:

$$\boldsymbol{u}^{k+1} = \boldsymbol{u}^k + B_h^D(\boldsymbol{f} - A_h^D \boldsymbol{u}^k), \quad \widetilde{\boldsymbol{u}}^{k+1} = \widetilde{\boldsymbol{u}}^k + B_h^P(\widetilde{\boldsymbol{f}} - A_h^P \widetilde{\boldsymbol{u}}^k), \tag{11}$$

where $B_h^{D,P}$ are linear operators (called iterators). We have the following theorem which shows that the LFA-compatibility of the iterators provides a relation between the iterates.

Theorem 3.1. *Let (B_h^D, B_h^P) be an LFA-compatible pair and $\widetilde{\boldsymbol{f}} = E_{o,h}^{\otimes d}\boldsymbol{f}$. If $\widetilde{\boldsymbol{u}}^0 = E_{o,h}^{\otimes d}\boldsymbol{u}^0$, then $\widetilde{\boldsymbol{u}}^k = E_{o,h}^{\otimes d}\boldsymbol{u}^k$, $k = 1, 2, \ldots$*

Proof. We prove the result by showing that if $\widetilde{\boldsymbol{u}}^k = E_{o,h}^{\otimes d}\boldsymbol{u}^k$ then $\widetilde{\boldsymbol{u}}^{k+1} = E_{o,h}^{\otimes d}\boldsymbol{u}^{k+1}$. Clearly, from (11), and the fact that $\widetilde{\boldsymbol{u}}^k = E_{o,h}^{\otimes d}\boldsymbol{u}^k$, we have $\widetilde{\boldsymbol{u}}^{k+1} = E_{o,h}^{\otimes d}\boldsymbol{u}^k + B_h^P(E_{o,h}^{\otimes d}\boldsymbol{f} - A_h^P E_{o,h}^{\otimes d}\boldsymbol{u}^k)$. Next, we use Lemmata 2.2–2.4 to obtain that,

$$\widetilde{\boldsymbol{u}}^{k+1} = E_{o,h}^{\otimes d}\boldsymbol{u}^k + B_h^P(E_{o,h}^{\otimes d}\boldsymbol{f} - E_{o,h}^{\otimes d}R_{o,h}^{\otimes d}A_h^P E_{o,h}^{\otimes d}\boldsymbol{u}^k) = E_{o,h}^{\otimes d}\boldsymbol{u}^k + B_h^P E_{o,h}^{\otimes d}(\boldsymbol{f} - A_h^D\boldsymbol{u}^k).$$

Since, $E_{o,h}^{\otimes d}(\boldsymbol{f} - A_h^D\boldsymbol{u}^k) \in \text{range}(E_{o,h}^{\otimes d})$, and $B_h^P E_{o,h}^{\otimes d}(\boldsymbol{f} - A_h^D\boldsymbol{u}^k) \in \text{range}(E_{o,h}^{\otimes d})$ we have that

$$\widetilde{\boldsymbol{u}}^{k+1} = E_{o,h}^{\otimes d}\boldsymbol{u}^k + E_{o,h}^{\otimes d}R_{o,h}^{\otimes d}B_h^P E_{o,h}^{\otimes d}(\boldsymbol{f} - A_h^D\boldsymbol{u}^k).$$

Finally, we use that (B_h^D, B_h^P) is an LFA-compatible pair to obtain that $\widetilde{\boldsymbol{u}}^{k+1} = E_{o,h}^{\otimes d}(\boldsymbol{u}^k + B_h^D(\boldsymbol{f} - A_h^D\boldsymbol{u}^k)) = E_{o,h}^{\otimes d}\boldsymbol{u}^{k+1}$ which is what we wanted to show. $\square$

3.1. Two grid methods

We now consider the two-grid and multigrid methods. We begin by defining the coarse grids for the Dirichlet and periodic problems in one spatial dimension ($d = 1$). In a standard fashion, we define

$$\Omega_{2h}^D = \{x_i = 2ih \mid i = 0, \ldots, n/2\}, \quad \text{and} \quad \Omega_{2h}^P = \{x_i = 2ih \mid i = 0, \ldots, n\}.$$

We denote by $\mathcal{G}(\Omega_h^{D,P})$, $\mathcal{G}(\Omega_{2h}^{D,P})$ the subspaces of grid-functions defined on $\Omega_h^{D,P}$ and $\Omega_{2h}^{D,P}$, respectively. On such coarse grid, we also define $A_{2h}^{D,P}$ by (6) but with $2h$ instead of h. The extension to higher spatial dimensions is done using standard tensor products of grids and operators.

We now consider the two-grid algorithms, which are linear iterative methods already defined in (11) with special iterators $B_{TG} = B_{TG}^{D,P}$ as follows:

$$B_{TG} = \left(I - (I - I_{2h,h}(A_{2h})^{-1} I_{h,2h} A_h)(I - S_h A_h)\right)(A_h)^{-1}, \tag{12}$$

In (12) all operators change depending on whether we consider Dirichlet or periodic problem, namely, we have A_h^D, A_h^P, $I_{2h,h}^D$, $I_{2h,h}^P$, etc. Here, $S_h^{D,P}$ are relaxation (smoothing) operators, $I_{h,2h}^{D,P} : \mathcal{G}(\Omega_h^{D,P}) \to \mathcal{G}(\Omega_{2h}^{D,P})$ are the restriction operators and $I_{2h,h}^{D,P} : \mathcal{G}(\Omega_{2h}^{D,P}) \to \mathcal{G}(\Omega_h^{D,P})$ are the prolongation operators. To prove the main result, we need to introduce LFA-compatible restriction and prolongation operators. We say that the pairs $(I_{2h,h}^D, I_{2h,h}^P)$ and $(I_{h,2h}^D, I_{h,2h}^P)$ are *LFA-compatible* if and only if

$$I_{h,2h}^D = R_{o,2h} I_{h,2h}^P E_{o,h}, \quad I_{h,2h}^P v \in \operatorname{range}(E_{o,2h}), \quad \text{for all} \quad v \in \operatorname{range}(E_{o,h}), \tag{13}$$

$$I_{2h,h}^D = R_{o,h} I_{2h,h}^P E_{o,2h}, \quad I_{2h,h}^P v \in \operatorname{range}(E_{o,h}) \quad \text{for all} \quad v \in \operatorname{range}(E_{o,2h}). \tag{14}$$

The multigrid iterator is obtained from the two grid by recursion, namely,

$$B_h = (I - (I - I_{2h,h} B_{2h} I_{h,2h} A_h)(I - S_h A_h))(A_h)^{-1}, \tag{15}$$

where $B_{nh} = A_{nh}^{-1}$ for both the Dirichlet and the periodic problem.

We have the following theorem, showing that the iterations via two grid are related.

Theorem 3.2. *If (A_h^D, A_h^P), $((A_{2h}^D)^{-1}, (A_{2h}^P)^{-1})$, (S_h^D, S_h^P), $(I_{2h,h}^D, I_{2h,h}^P)$, $(I_{h,2h}^D, I_{h,2h}^P)$ are LFA compatible, then (B_h^D, B_h^P) is LFA-compatible.*

Proof. We prove this theorem for the case $d = 1$ only and $B_h = B_{TG}$ as the general case follows from recursive application of this argument and the properties of tensor product listed earlier.

$$\begin{aligned} R_{o,h} B_{TG}^P E_{o,h} &= R_{o,h}\left(I - (I - I_{2h,h}^P (A_{2h}^P)^{-1} I_{h,2h}^P A_h^P) E_{o,h} R_{o,h} (I - S_h^P A_h^P)\right) E_{o,h} R_{o,h} (A_h^P)^{-1} E_{o,h} \\ &= \left(I - (I - R_{o,h} I_{2h,h}^P (A_{2h}^P)^{-1} I_{h,2h}^P A_h^P E_{o,h})(I - S_h^D A_h^D)\right)(A_h^D)^{-1}. \end{aligned}$$

Moreover, because of the invariant properties it follows that

$$R_{o,h} I_{2h,h}^P (A_{2h}^P)^{-1} I_{h,2h}^P A_h^P E_{o,h} = (R_{o,h} I_{2h,h}^P E_{2h})(R_{2h} (A_{2h}^P)^{-1} E_{2h})(R_{2h} I_{h,2h}^P E_{o,h})(R_{o,h} A_h^P E_{o,h}).$$

By using the properties in the assumptions in the theorem we have that $B_{TG}^D = R_{o,h} B_{TG}^P E_{o,h}$. The invariant property of B_{TG}^P follows from the invariant properties of all the operators involved in the two-grid method. $\square$

4. Examples and extensions

The compatibility result in Theorem 3.2 shows that the LFA, which is strictly justified for periodic problems, provides rigorous results also for the Dirichlet problems. Of course, this is for particular choices of S_h, $I_{h,2h}$ and the rest of the involved operators. LFA-compatible smoothers include the weighted Jacobi method, the Red-Black Gauss-Seidel, line relaxation methods and polynomial smoothers. The frequently used inter-grid transfer operators full-weighting and bilinear interpolation are LFA-compatible restriction and prolongation operators, respectively. Therefore, multigrid methods based on these components applied to problems with Dirichlet boundary conditions can be analyzed rigorously by LFA.

Moreover, problems with other boundary conditions can also be put into this framework. For example, all the results presented in this work are easily reproduced for the pure Neumann problem by using an even extension operator instead the odd extension operator. Problems with mixed boundary conditions can also be included in this framework by using an even extension operator followed by an odd extension operator.

We conclude that for a wide range of multigrid components and for problems with other boundary conditions than the periodic ones, the LFA provides rigorous asymptotic multigrid convergence factors.

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