HIRSCH POLYTOPES WITH EXPONENTIALLY LONG COMBINATORIAL SEGMENTS

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Abstract. In their paper proving the Hirsch bound for flag normal simplicial complexes (Math. Oper. Res. 2014) Adiprasito and Benedetti define the notion of *combinatorial segment*. The study of the maximal length of these objects provides the upper bound $O(n2^d)$ for the diameter of any normal pure simplicial complex of dimension d with n vertices, and the Hirsch bound $n-d$ if the complexes are, moreover, flag. In the present article, we propose a formulation of combinatorial segments which is equivalent but more local, by introducing the notions of monotonicity and conservativeness of dual paths in pure simplicial complexes. We use this definition to investigate further properties of combinatorial segments. Besides recovering the two stated bounds, we show a refined bound for banner complexes, and study the behavior of the maximal length of combinatorial segments with respect to two usual operations, namely join and one-point suspension. Finally, we show the limitations of combinatorial segments by constructing pure normal simplicial complexes in which all combinatorial segments between two particular facets achieve the length $\Omega(n2^d)$. This includes vertex-decomposable—therefore Hirsch—polytopes.

1. Introduction

One of the main open questions in polyhedral combinatorics is the so-called polynomial Hirsch Conjecture:

> *Is there an upper bound for the (combinatorial) diameter of the graph of every d-polyhedron with n facets that is polynomial in n and d?*

Remember that it was conjectured by Hirsch in 1957 that $n-d$ is a valid upper bound. We say a polytope or polyhedron is *Hirsch* if it satisfies this bound. Although the Hirsch Conjecture was disproved by Klee and Walkup in the general case [KW67] and by Santos in the bounded case [San12], the known counterexamples exceed the bound only by a small fraction (25% in the unbounded case, 5% in the bounded case, see [San12, MSW15]). In contrast, the best upper bounds known are not polynomial. If we denote by $H(d,n)$ the maximum diameter of d-dimensional polytopes with n facets, these bounds are:

$$H(d,n) \leq \frac{2}{3}2^{d-3}n, \qquad H(d,n) \leq (n-d)^{\log_2 d}. \tag{1}$$

The first bound is *linear in fixed dimension*. Except for a constant factor, which was an improvement by Barnette [Bar74], this bound was first proved by Larman [Lar70]. The second, *quasi-polynomial* bound, was first proved by Kalai and Kleitman [KK92], although the version we state is an improvement by Todd [Tod14].

An approach to the question that has been attempted since long, starting with the introduction of *abstract polytopes* by Adler and Dantzig [AD74], is to generalize it to the setting of simplicial complexes. It is known that $H(d,n)$ is attained at a simple polytope, which is topologically dual to a simplicial sphere of dimension $d-1$. Thus, $H(d,n)$ is bounded by the maximum diameter of dual graphs of all simplicial $(d-1)$-spheres with n vertices, and generalizing the question to

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arbitrary pure simplicial complexes may help in finding the right proof strategy. Here and in the rest of the paper, a *pure simplicial complex* of dimension $d-1$ is a subset C of $\binom{[n]}{d}$. The elements of C are called *facets* and two facets are adjacent if they differ in a single element. This defines a *facet-adjacency* graph (or *dual graph*) of C. The (dual) diameter of C is the combinatorial diameter of this graph.

However, if one generalizes the question too much then exponential diameters arise:

Theorem A (Santos [San13]). *The maximum diameter of pure simplicial $(d-1)$-complexes with n vertices grows as $n^{\Theta(d)}$.*

Hence, some condition is needed if one wants to have (hopes of) a polynomial upper bound. The standard consensus is to look only at *normal* complexes. That is, complexes in which for every $\sigma \in [n]$ the following subcomplex (called the *star* of σ) has a connected dual graph:

$$\mathrm{st}_C(\sigma) := \{F \in C : \sigma \subset F\}.$$

Equivalently, C is normal if the *link* of every face of codimension at least two is connected [IJ03, Lemma 3.1.2], or if C can be obtained from a family of disjoint d-simplices by gluing only along codimension-one faces of them. The adjective "normal" for these complexes is quite established if they are pseudomanifolds, in which case the condition is also equivalent to the n-th local homology group $H_n(|C|, |C| \setminus \{p\})$ being isomorphic to $\mathbb{Z}$ for every point p (see e.g. [GM80, Sect. 4]). Without the pseudomanifold condition, the name has been used in [AB14, San13]. In the context of diameter bounds, pure simplicial complexes with this property have been called *locally connected* [AD74] and *ultraconnected* [Kal92]. Adler and Dantzig [AD74] call normal pseudomanifolds without boundary *abstract polytopes*. The property arises also in the context of unfoldings of simplicial complexes [Moh88, IJ03] under the name *locally strongly connected*.

Evidence that normality is the right condition to consider is that the two bounds stated in (1) for $H(d, n)$ are still valid, with minor adjustments, for the diameters of all normal complexes. This was already known to Kalai and Kleitman [KK92], and it was recently highlighted by the work of Eisenbrand et al. [EHRR10], who considered an even more abstract setting and proved the two bounds in this setting. Their work, in fact, suggests the following conjecture, much stronger than the polynomial Hirsch Conjecture:

Conjecture (Hähnle [Häh08, Conjecture 7.0.5]). *The diameter of every normal $(d-1)$-complex with n vertices is at most $(n-1)d$.*

Normality also plays a role in Adiprasito and Benedetti's proof of the (original) Hirsch bound for *flag polytopes* [AB14]; indeed, what they prove is the Hirsch bound for the diameter of every flag, normal simplicial complex. In this paper we revisit one of the two proofs of this result given by Adiprasito and Benedetti and study what it implies for normal complexes that are not flag.

Let us be more precise. Adiprasito–Benedetti introduce paths of certain types in dual graphs of pure complexes, that they call *combinatorial segments*. They then prove that:

- Between every pair of facets in a normal (not necessarily flag) complex there is always a combinatorial segment (see Corollary 2.10).
- Combinatorial segments in flag normal complexes are nonrevisiting, hence they satisfy the Hirsch bound (see Section 3.1).

Hereafter, we recast these proofs and the concept of combinatorial segment.

In Section 2 we introduce the notion of *monotone conservative* paths by extracting the key properties of Adiprasito and Benedetti's combinatorial segments. Although the two concepts are equivalent (see Theorem 2.13) our formulation is local: a path is *monotone and conservative* if each step from a facet F_i to the next facet F_{i+1} satisfies certain conditions. The original definition goes by posing global conditions on the path via a double recursion: on the vertex-distance between F_1 and F_2 and on the dimension of C. Here, we call *vertex-distance* between two subsets of vertices of C the distance between them in the 1-skeleton (the "usual" graph, whose vertices and edges are the 0-faces and 1-faces) of C. We achieve this local, yet equivalent, definition by considering the vertex set of each facet along the path to be ordered, so that the ordering encodes the recursions.

More precisely, we say that the order on $F_1 = \{x_1, \dots, x_d\}$ is *admissible* with respect to a target set S if the first vertex x_1 in this order realizes the vertex-distance between F_1 and S, and if the rest of F_1, which is itself a facet in the link of x_1 in the complex, is admissibly ordered with respect to a certain set S' of neighbors of x_1, derived from S. We are interested primarily in the case where S is the target facet F_2 of our path, but the lack of control on S' forces us to consider more general cases for S. We also define the *vector of distances* from F_1 to S, whose coordinates correspond to the vertices of F_1 in their given order. Its first coordinate is the vertex-distance of x_1 to S in C and the rest is the vector of distances from $F_1 \setminus \{x_1\}$ to S' in the link of x_1.

Now for a step $\Gamma = [F_1, F_2]$ in a dual path in C, consisting of two facets admissibly ordered with respect to some set S, we concentrate on the first vertices of their orders. One would expect that F_2 will be "closer" from S than F_1 if its first vertex is at smaller vertex-distance from S than the first vertex of F_1 and that this goes through via the recursion. This suggest to focus on paths whose steps are all *monotone*, i. e., such that the vector of distances lexicographically decreases at each step. We also require some additional "control" property that we call *conservativeness* and which can informally be expressed as: the step from F_1 to F_2 is conservative if F_2 tries to keep the order established in F_1 as much as admissibility allows.

In Section 3 we rephrase the proof of the Hirsch bound for flag normal complexes in the language of monotone conservative paths. We also include a proof of the Larman bound for normal complexes in this language, and we generalize the bound to the so-called banner complexes:

Theorem B (Adiprasito–Benedetti [AB14], see Theorem 3.5 and Corollary 3.2)**.** *The length of any monotone conservative path in a $(d-1)$-dimensional pure normal complex C on n vertices ($d \geq 2$) is at most $n2^{d-2}$. Moreover, if C is a pseudomanifold, the same statement holds with the bound $n2^{d-3}$; and if C is flag, the same statement holds with the Hirsch bound $n-d$.*

Theorem C (Novik 2013, unpublished, see Theorem 3.9)**.** *Let $k \geq 2$. If C is a k-banner pure normal complex on n vertices, then between every distinct facets F_1 and F_2 of C, there is a monotone conservative path of length bounded by $n2^{k-2}$.*

The property of being *k-banner* is a generalization of flagness (which is itself equivalent to 2-banner). See the precise definitions in Section 3.3. Theorem C interpolates (modulo a factor of 2 and an additive term of d, respectively) between two of the bounds of Theorem B: Since every $(d-1)$-complex is $(d+1)$-banner, we retrieve the bound $2^{d-1}n$ for arbitrary normal complexes. On the other extreme, substituting k by 2 into the theorem gives a bound of n for flag normal complexes.

Theorem C was proven by Novik (unpublished, 2013) as an attempt to answer the following stronger question:

Question D. *Let $k \geq 2$ and C be a pure complex on n vertices in which any minimal nonface has dimension at most k, then the diameter of C is bounded by $n2^{k-2}$.*

Finally in Section 4 we show the limitations of combinatorial segments by constructing monotone conservative paths of exponential length in normal complexes. We do this in two versions. In the first one the complex is a topological ball, in the second it is the boundary complex of a vertex-decomposable simplicial polytope:

Theorem E (See Theorem 4.5)**.** *For every $d \geq 2$ and every $N \geq 4$ there is a simplicial d-polytope with $N + \Theta(d^2)$ vertices with vertex-decomposable boundary complex and two facets F_1 and F_2 in it such that every monotone conservative path between them has length at least $2^{d-3}N$.*

Vertex-decomposability is interesting in this context since vertex-decomposable simplicial polytopes are known to satisfy the Hirsch bound [PB80, Corollary 2.11]. Observe that the paths stated in this theorem are within a factor of $1+\varepsilon$ from the upper bound of Theorem B, with ε going to zero as N goes to infinity.

2. Combinatorial segments and monotone conservative paths

2.1. **Preliminaries on simplicial complexes.** A simplicial complex, or simply a complex, on a *vertex* set V is a collection C of subsets of V such that $f \subset g \in C$ implies $f \in C$. The elements of

C are called *faces*. The *dimension* of a face f is $|f|-1$. If all its *facets* (that is, inclusion maximal faces) have d elements, C is said to be *pure* of dimension $d-1$, and sometimes called $(d-1)$-complex. When describing a complex C we usually only list its maximal faces. For example, for $f \subset V$ and $v \in V$ we denote $\{f\}$ and $\{v\}$ the complexes $\{f' : f' \subset f\}$ and $\{\{v\}, \emptyset\}$, respectively. Observe that from this perspective a *pure simplicial $(d-1)$-complex* is nothing but a subset of $\binom{V}{d}$. A *minimal nonface* of a complex is a nonempty set of its vertices which is not a face but such that all its proper subset are faces. A complex is called *flag* if all its minimal nonfaces are 1-dimensional, i. e., edges.

The *star*, *deletion* (sometimes called the *antistar* or *face-deletion*), and *link* of a face f in C are the following subcomplexes of C, where $\sqcup$ denotes the disjoint union:

$$\begin{aligned}
\mathrm{st}_C(f) &:= \{f' \in C : f \cup f' \in C\}, \\
\mathrm{del}_C(f) &:= \{f' \in C : f \not\subseteq f'\}, \\
\mathrm{lk}_C(f) &:= \{f' \in C : f \sqcup f' \in C\}.
\end{aligned}$$

The *join* of two simplicial complexes C_1 and C_2 is the simplicial complex

$$C_1 * C_2 := \{f_1 \sqcup f_2 : f_1 \in C_1, f_2 \in C_2\}.$$

Observe that $\mathrm{st}_C(f) = \mathrm{lk}_C(f) * \{f\}$.

The *suspension* of a simplicial complex C is the join of C with a simplicial complex consisting in two singletons. Given a vertex v of a simplicial complex C, the *one-point-suspension* of C with respect to v is the simplicial complex

$$\mathrm{ops}_C(v) := (\mathrm{lk}_C(v) * \{\{v_1, v_2\}\}) \ \cup \ (\mathrm{del}_C(v) * \{\{v_1\}, \{v_2\}\}),$$

where v_1 and v_2 are new elements, called *suspension vertices*, that were not vertices of C. Equivalently, $\mathrm{ops}_C(v)$ is obtained contracting the edge vv_2 in the suspension $C * \{v_1, v_2\}$ of C. Observe that

$$\mathrm{lk}_{\mathrm{ops}_C(v)}(v_1) \cong \mathrm{lk}_{\mathrm{ops}_C(v)}(v_2) \cong C \qquad \text{and} \qquad \mathrm{lk}_{\mathrm{ops}_C(v)}(v_1v_2) = \mathrm{lk}_C(v).$$

One-point-suspensions are called *wedges* in [PB80].

Let f be a face of C. The *stellar subdivision* of f in C is the simplicial complex denoted by $\mathrm{stell}_C(f)$ and given by

$$\mathrm{stell}_C(f) := \mathrm{del}_{C\cup(\mathrm{st}_C(f)*\{a\})}(f) = \{f' \in C : f \not\subseteq f'\} \cup \{f' \cup \{a\} : f \not\subseteq f' \in \mathrm{st}_C(f)\},$$

where a is a new vertex that is not a vertex of C. Notice that in the particular case where $f = \{v_1, \ldots, v_d\}$ is a facet of C, then $\mathrm{stell}_C(f)$ consists of C in which the facet f has been replaced by the d facets obtained from f by substituting one of its vertices by a.

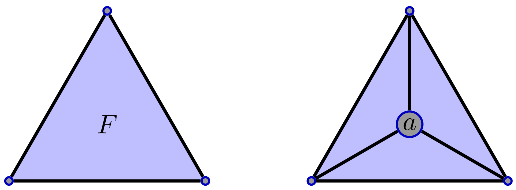


FIGURE 1. The stellar subdivision of a simplex F.

A pure $(d-1)$-complex C is *vertex-decomposable* [DLK12, PB80] if it is either a simplex or if there exists a vertex $x \in C$ such that $\mathrm{lk}_C(x)$ and $\mathrm{del}_C(x)$ are both vertex-decomposable and $\mathrm{del}_C(x)$ is still pure of dimension $d-1$.

Lemma 2.1 ([PB80, Proposition 2.5 and Theorem 2.7])**.** *Let C be a d-complex, x a vertex of C and f a face of C.*

- *The complex C is vertex-decomposable if and only if $\mathrm{ops}_C(x)$ is vertex-decomposable.*
- *If C is vertex-decomposable, then $\mathrm{stell}_C(f)$ is a vertex-decomposable d-complex.*

The *facet-adjacency graph* or *dual graph* of C has the facets of C as nodes with two facets F and F' being adjacent in the graph if they differ in a single vertex. A pure simplicial complex C is called *normal* if between every two facets F, F' there is a *dual path* (a path in the dual graph) consisting only of facets that contain $F \cap F'$. A pure simplicial complex C is called a *pseudomanifold* if every codimension-one simplex lies in either 1 or 2 facets. The codimension-one simplices lying in only one facet form the *boundary* of C.

Although our main question of interest is the diameter of the dual graph, we use also the distance in the "usual graph" (or 1-skeleton) of C, whose vertices and edges are those of C itself. We refer to it as the *vertex-distance* between two vertices u and v of C and denote it $\mathrm{vdist}_C(u, v)$. For two sets of vertices S and S' we denote

$$\mathrm{vdist}_C(S, S') := \min_{u \in S, v \in S'} \mathrm{vdist}_C(u, v),$$

with the standard convention that if one or both of S and S' are empty the minimum is ∞. If $\dim(C) = 0$, we also take the convention that for nonempty sets S and S', $\mathrm{vdist}_C(S, S') = 0$ if $S \cap S' \neq \emptyset$, and $\mathrm{vdist}_C(S, S') = 1$ otherwise.

2.2. **Monotone conservative paths.** The following definitions are crucial in order to define monotonicity and conservativeness of paths.

Definition 2.2. Let $F = (v_1, \ldots, v_d)$ be a facet of a pure $(d-1)$-dimensional simplicial complex C with its vertices given in a specific order. We call this an *ordered facet*. Let S be a subset of vertices of C, refered to as the *target set*. The *vector of distances* $\Lambda = (\lambda_1, \ldots, \lambda_d)$ from F to S is defined as follows:

- $\lambda_1 := \mathrm{vdist}(v_1, S)$.
- $(\lambda_2, \ldots, \lambda_d)$ is the vector of distances from $F \setminus v_1$ to S' in $\mathrm{lk}_C(v_1)$, where S' is the following subset of $V_1 := \mathrm{vertices}(\mathrm{lk}_C(v_1))$:

$$S' := \begin{cases} \{v \in V_1 : \mathrm{vdist}_C(v, S) = \mathrm{vdist}_C(v_1, S) - 1\} & \text{if } v_1 \notin S, \\ S \cap \mathrm{lk}_C(v_1) & \text{if } v_1 \in S. \end{cases}$$

We say the ordering is *admissible* with respect to S if $\lambda_1 = \mathrm{vdist}(v_1, S) = \mathrm{vdist}(F, S)$ and $F \setminus v_1$ (with the induced ordering) is admissible with respect to S' in $\mathrm{lk}_C(v_1)$.

Example 2.3. Consider the pure flag 2-dimensional complex given in Figure 2, with target set $S = \{s_1, s_2\}$. The ordered facet $F_1 = (a_1, a_2, a_3)$ is not admissible, whereas the ordered facet $F_2 = (b_1, b_2, b_3)$ and $F_3 = (c_1, c_2, c_3)$ are admissible. The facet F_1 would be admissible with respect to other orderings, e. g. (a_2, a_3, a_1). The vectors of distances for the facets F_1, F_2 and F_3 are $(5, 0, 0)$, $(2, 1, 1)$, and $(0, 0, \infty)$ respectively. In particular, it is possible to have infinite values in a vector of distances.

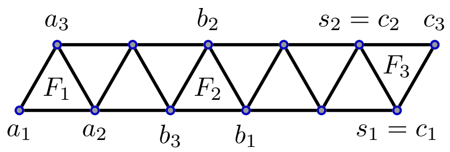


FIGURE 2. Examples of admissible and nonadmissible ordered facets in a 2-dimensional complex.

We are primarily interested in the case where S is a facet, but the recursive definition forces us to consider more general target sets. Observe that the definition may even make the new target set S' obtained from S be empty, in which case all entries in the vector of distances starting at that point will be infinite. The following statement shows that this never happens in a normal complex if the initial target set is a facet.

Lemma 2.4. *Let C be a normal $(d-1)$-complex, S be a nonempty set of vertices of C, and $F = (v_1, \dots, v_d)$ be an ordered facet of C. If F is admissible with respect to S and that one of the following condition is true*

- *$F \cap S = \emptyset$,*
- *S is a face of C not strictly contained in F,*

then the vector of distances $\Lambda = (\lambda_1, \dots, \lambda_d)$ from F to S has no infinite entry.

Proof. We use induction on the dimension $d-1$ of C. For $d = 1$, in both cases the vector of distances has a single entry which is either zero or one. Indeed the set F is a singleton, and S is a nonempty set. For $d > 1$, the entry λ_1 of the vector of distances is finite since S is nonempty and C is normal thus has connected 1-skeleton. Notice that the simplicial complex C being pure and normal ensures that $\mathrm{lk}_C(v_1)$ is also pure, normal and of dimension one less. Further, as the ordering on F is admissible with respect to S, so is the ordering on $F \smallsetminus v_1$ with respect to S'. Suppose first that $F \cap S = \emptyset$. Because the ordering on F is admissible with respect to S, the vertex v_1 is at smallest (and nonzero) vertex-distance of S in the facet F. Therefore the facet $F \smallsetminus v_1$ and the set S' in Definition 2.2 are disjoint. Suppose now that S is a face of C such that $v_1 \in S$. In this case $S' = S \cap \mathrm{lk}_C(v_1)$ is a face of $\mathrm{lk}_C(v_1)$ which is not strictly contained in $F \smallsetminus v_1$. In both cases, the induction hypothesis ensures that the vector $(\lambda_2, \dots, \lambda_d)$ has no infinite entry. ■

The previous proof also shows that if F is admissible with respect to S and the vector of distances contains some zero, then all zeroes are at the beginning of the vector. That is to say: no entry of its vector of distances may be zero if a previous entry is nonzero.

We now introduce the main definitions used in this article.

Definition 2.5. Let C be a pure and normal complex, S be a nonempty target set of vertices of C, and $\Gamma = [F_0, \dots, F_N]$ be a path of ordered facets in the dual graph of C.

(1) We say that Γ is *(combinatorially) monotone towards S* if the sequence $(\Lambda_0, \dots, \Lambda_N)$ of vectors of distances from its facets to S is lexicographically decreasing. That is, Λ_{i+1} is lexicographically smaller than Λ_i for every i.

(2) For each $i = 1, \dots, N$ we call *index* of the step from F_{i-1} to F_i in Γ the minimum k for which $v_k \neq v'_k$, where $F_{i-1} = (v_1, \dots, v_d)$ and $F_i = (v'_1, \dots, v'_d)$. Alternatively we call it sometimes the index of F_i when there is no ambiguity on the path Γ that we are considering. We say that the step from F_{i-1} to F_i, is *conservative towards S* if the following two things happen:
   - $F_{i-1} \smallsetminus F_i$ is the last vertex in the ordering of F_{i-1}. That is, F_i keeps the initial vertices of F_{i-1} as much as possible.
   - F_i is admissible with respect to S and it has the maximum index among all possible choices of admissible reorderings of F_i. That is, F_i tries to keep the order established in F_{i-1} as much as admissible.

   We say that Γ itself is *conservative towards S* if F_0 is admissible and every step is conservative.

Example 2.6. Consider the pure flag 2-dimensional complex given in Figure 3, with target set $S = \{a_8\}$. The ordered facet $F = (a_1, a_2, a_3)$ has distance vector $(2, 2, 1)$ and is admissible. The ordered facet $G_1 = (a_5, a_1, a_3)$ and $G_2 = (a_1, a_5, a_3)$ both have distance vector $(2, 1, 1)$ and are admissible. The ordered facet $H_1 = (a_4, a_1, a_2)$ and $H_2 = (a_1, a_4, a_2)$ both have distance vector $(2, 1, 1)$ and are admissible. The step $[F, G_1]$ is monotone, but does not satisfy either conditions of conservativeness: a_2 is not the last vertex of F and G_1 does not have maximal index. The step $[F, G_2]$ is monotone but not conservative: G_2 has maximal index, but a_2 is not the last vertex of F. The step $[F, H_1]$ is monotone but not conservative: a_3 is the last vertex of F, but H_1 is not of maximal index. The step $[F, H_2]$ is monotone and conservative: a_3 is the last vertex of F and H_2 has maximal index. Note that these four steps form initial segments of shortest paths in the dual graph from F to S, but only one of them is monotone and conservative.

To obtain a nonmonotone but conservative step, consider the complex $\{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$ with $F = (1, 2)$, $G = (1, 3)$, and $S = \{4\}$. The path $[F, G]$ is not monotone but is conservative.

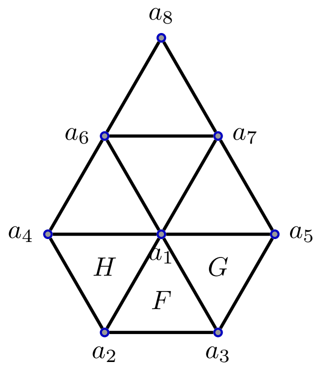


FIGURE 3. A 2-dimensional flag simplicial complex having monotone and non-conservative paths.

Remark 2.7. In a conservative step $[F_{i-1}, F_i]$ towards a target set S, the index somehow denotes the "depth" at which the step occurs in the recursive definition.

It is clear from this definition that for a path Γ of ordered facets in the dual graph and a target set S,

- if Γ is monotone (resp. conservative) towards S, then any subpath of Γ is also monotone (resp. conservative) towards S,
- if Γ_1 and Γ_2 are monotone (resp. conservative) towards S and the last ordered facet in Γ_1 equals the first one in Γ_2, then the concatenation of Γ_1 and Γ_2 is monotone (resp. conservative) towards S.

Define the *anchor* of an ordered facet to be its first vertex. The anchors along a monotone path towards S form a vertex-path in the 1-skeleton of C along which the distance to S decreases. In particular, they form a shortest path between the first and last anchor. Notice that in a monotone and conservative path Γ from F towards S, the simplices with the same anchor x_0 form a subpath Γ_0 of Γ. Moreover, the deletion of x_0 in the path Γ_0 is a monotone conservative path Γ'_0 in $\mathrm{lk}_C(x_0)$ towards the set $\{x \in V : \mathrm{vdist}_C(x, x_0) = 1, \mathrm{vdist}_C(x, S) = \mathrm{vdist}_C(x_0, S) - 1\}$ which is the set S' of Definition 2.2.

We now show the existence of monotone conservative paths in any pure normal complex, and bound their lengths. The following theorem deals with the existence part, up to the (immediate) existence of admissible orderings.

Theorem 2.8. *Let C be a pure and normal complex, S be a nonempty target set, and F_0 be an ordered facet of C. If $F_0 \cap S = \emptyset$ and F_0 is admissible with respect to S, then there exists an ordered facet F_1 adjacent to F_0 such that the step $[F_0, F_1]$ is monotone and conservative.*

Proof. Lemma 2.4 ensures that all entries in the vector of distances of F_0 are finite, so that we do not need to define any comparison convention on potential infinite values.

The proof works by induction on the dimension $d - 1$ of C, the case $d = 1$ being clear.

Let us also deal with the case $d = 2$ separately. In this case, C is a graph and F_0 is an ordered edge (u, v). The vector of distances is $(a, 1)$, where $a \geq 1$ is the distance from u to S in the graph. Simply take $F_1 = (w, u)$, where w is any neighbor of u at distance $a - 1$ from S.

For arbitrary d, let $F_0 = \{v_1, \dots, v_d\}$ (in this order) and let $F'_0 = \{v_2, \dots, v_d\} = F_0 \smallsetminus v_1$. Observe that (as in the proof of Lemma 2.4) the condition $F_0 \cap S = \emptyset$ is inherited into $F'_0 \cap S' = \emptyset$, where

$$S' = \{v \in \mathrm{vertices}(\mathrm{lk}_C(v_1)) : \mathrm{vdist}_C(v, S) = \mathrm{vdist}_C(v_1, S) - 1\}$$

is as in the definition of admissibility. In particular, by inductive hypothesis, there is an F'_1 containing $F'_0 \smallsetminus \{v_d\}$ such that (with a suitable ordering for F'_1) $[F'_0, F'_1]$ is a monotone conservative path from F_0 to S' in $\mathrm{lk}_{v_1}(C)$. Let $F_1 = \{v_1\} \cup F'_1$ considered, for now, with v_1 as the first vertex, followed by the rest in the order of F'_1. Then:

- The vector of distances of F_1 with respect to S has the same first entry as that of F_0, and the rest of entries are lexicographically smaller than those of F_0 by inductive hypothesis. Thus, $[F_0, F_1]$ is monotone.
- Again by inductive hypothesis, $F_0 \smallsetminus F_1$ is the last vertex in the ordering of F_0.

If $\operatorname{vdist}(F_1, S) = \operatorname{vdist}(v_1, S)$, the ordering of F_1 is admissible and, by inductive hypothesis, it has the maximum index among admissible orderings; hence $[F_0, F_1]$ is admissible and conservative. If $\operatorname{vdist}(F_1, S) < \operatorname{vdist}(v_1, S)$, the first vertex of any admissible ordering of F_1 cannot be v_1, therefore the index of this step has to be 0. Then choosing any reordering for F_1 that is admissible makes the step $[F_0, F_1]$ conservative. It is also still monotone since the first entry of the vector of distances of F_1 is then smaller than the first entry of the vector of distances of F_0. ■

Observe that the reordering of F_1 in the last part of the previous proof produces a change of anchor, from v_1 to v_1'. In fact, every change of anchor is produced by such a reordering. When an anchor is changed in the step from F_0 to F_1 (say from an anchor x to an anchor y), vertex x is still in the new facet F_1 because the vertex in $F_0 \smallsetminus F_1$ must be the last vertex in the order of F_0. So, the change of anchor is not due to the disappearance of x from the facet, but rather to the fact that x is no longer a closest vertex to S in F_1 and a reordering is needed.

Corollary 2.9. *Let C, S and F_0 be as in Theorem 2.8. There exists a monotone conservative path towards S starting at F_0 and ending in a facet that meets S.*

Proof. If F_0 meets S then F_0 alone is the desired path. If $F_0 \cap S = \emptyset$ then we apply Theorem 2.8, which gives us an F_1 to start the path. ■

Corollary 2.10. *Let C, S and F_0 be as in Theorem 2.8. If S is the vertex set of a facet, then there is a monotone conservative path towards S starting at F_0 and with S as its last facet.*

Proof. By the previous corollary, there is no loss of generality in assuming $F_0 \cap S \neq \emptyset$. That is, $v_1 \in S$. Now, $S' := S \smallsetminus \{v_1\}$ is a facet in $\operatorname{lk}_C(v_1)$. By induction on the dimension, there is a monotone path in $\operatorname{lk}_C(v_1)$ from $F_0' := F_0 \smallsetminus \{v_1\}$ which is conservative towards $S' := S \smallsetminus \{v_1\}$ and ending in S'. Adding v_1 as the first vertex in all facets of that path gives the desired path from F_0 to S. ■

Theorem 2.8 also has the following consequence:

Proposition 2.11. *In a pseudomanifold (with or without boundary), a conservative path is automatically monotone.*

Proof. Let $[F_0, F_1]$ be a conservative step in a pseudomanifold. If $F_0 \cap S \neq \emptyset$, then we induct on the dimension, by taking the link of the first vertex v_1 of F_0, which is still a pseudomanifold. So we can assume that $F_0 \cap S = \emptyset$.

By Corollary 2.10, from F_0 there is a monotone and conservative step to a certain facet F_1'. Now, since the definition of conservativeness determines which vertex of F_0 is to be removed, and since we are in a pseudomanifold, we have $F_1 = F_1'$ up to reordering. Being conservative implies that the first vertex in which the orderings of F_1 and F_1' differ is also the first vertex in which they differ from F_0, and that the corresponding entry in the vector of distances of F_1 and F_1' is smaller than the same entry in the vector of distances of F_0. Hence, the step $[F_0, F_1]$ is monotone. ■

2.3. **Combinatorial segments as monotone conservative paths.** We now relate these concepts with the notion of combinatorial segment by Adiprasito and Benedetti [AB14]. The following definition is taken from [San13].

Definition 2.12. A dual path $\Gamma = [F_0, \dots, F_N]$ with $x_0 \in F_0$ in a simplicial complex C is a *combinatorial segment from a facet F to a set S anchored at the vertex x_0* if it satisfies the following recursive definition:

1) if $F_0 \cap S \neq \emptyset$, then $N = 0$,
2) if $d = 1$ and $F_0 \cap S = \emptyset$, then $N = 1$ and $\Gamma = (\{x_0\}, \{v\})$ with $v \in S$,
3) if $d > 1$ and $F_0 \cap S = \emptyset$, then:

a) the facet F_N is the unique facet of Γ intersecting S,
b) let $\ell = \mathrm{vdist}_C(F_0, S)$, let $k = \min\{i \in [N] | \mathrm{vdist}_C(F_i, S) < \ell\}$ and let y be the unique vertex in F_k such that $\mathrm{vdist}_C(\{y\}, S) = \ell - 1$. Then $x_0 \in F_0 \cap \cdots \cap F_k$ and the link Γ_1' of x_0 in the path $\Gamma_1 = (F_0, F_1, \dots, F_k)$ is a combinatorial segment in $\mathrm{lk}_C(x_0)$ from the facet $F_0 \setminus x_0$ to the set of neighbors of x_0 being at vertex-distance $\ell - 1$ from S in C (that is the set S' of Definition 2.2 of admissibility),
c) the path $\Gamma_2 = [F_k, \dots, F_N]$ is a combinatorial segment from F_k to S anchored at y.

In [San13], the anchor of the path Γ is the vertex x_0. In our definition x_0 is only the "first anchor" of the path, after which come all the anchors in the path Γ_2. Anchors, in the sense of [San13], are called *pearls* in [AB14].

Observe that the facets of a combinatorial segment come with an implicit ordering of their vertices. For the facets in Γ_1, except the last one F_k, x_0 is the first vertex in the ordering and the rest of the ordering is obtained by induction on the dimension. For facets in Γ_2, the ordering is obtained by induction on N (or, alternatively, on the vertex-distance from F_0 to S). These orderings make combinatorial segments and monotone conservative paths essentially equivalent:

Theorem 2.13. *Let C be a simplicial complex, S be a nonempty set of vertices of C, $\Gamma = [F_0, \dots, F_N]$ be a dual path in C in which F_N is the only facet intersecting S. The path Γ is a combinatorial segment from F_0 to S if and only if it is monotone and conservative towards S.*

Proof. We make the proof follow the inductive definition of combinatorial segment. Observe that it has a double induction, first on the dimension and then on N (or, alternatively, on $\mathrm{vdist}(F_0, S)$).

In case (1) of the definition, the result trivially holds. In case (2) observe that for $d = 1$ the "vector of distances" between a vertex F and a set S is just a number, zero if $F \in S$ and one if not. Since monotonicity implies these numbers to be decreasing, a monotone path must have $N = 1$ and be of the form $[\{x_0\}, \{v\}]$ with $v \in S$.

So suppose that $F_0 \cap S = \emptyset$ and that $d > 1$. Assume for now that Γ is a combinatorial segment from F_0 to S. Then we already know that the path Γ_2 of the definition is also a combinatorial segment from F_k to S, implying by induction that any step of this path is both monotone and conservative towards S. By the induction hypothesis, since the path Γ_1' of the definition is a combinatorial segment, it is monotone and conservative towards the set S' of Definition 2.2. The vectors of distances in the path Γ_1 are all, except for the one in the last facet F_k, obtained from those in the path Γ_1' by adding the distance of the anchor x_0 as first coordinate, so that Γ_1 is still monotone. Moreover, the dual graph of $\mathrm{lk}_C(x_0)$ is the same as that of $\mathrm{st}_C(x_0)$, so that conservativeness is also preserved from Γ_1 to Γ_1' and conversely, except perhaps for the last step. Monotonicity in the step $[F_{k-1}, F_k]$ is clear, since $\mathrm{vdist}(y, S) < \mathrm{vdist}(x_0, S)$. Let us check the conditions for conservativeness:

- The vertex in $F_{k-1} \setminus F_k$ is the last vertex of F_{k-1} because it is the last vertex in $F_{k-1}' := F_{k-1} \setminus \{x_0\}$ (induction on the dimension).
- The ordering in F_k is admissible and has maximal index among admissible ones because Γ_2 is monotone and conservative, by induction on N.

Since Γ_2 is also monotone and conservative, the path Γ is monotone and conservative.

Conversely, assume now that Γ is monotone and conservative towards S. Let x_0 be the first anchor of Γ and $[F_0, \dots, F_{k-1}]$ be the part of Γ anchored at x_0 and let $\Gamma_1 = [F_0, \dots, F_k]$. Observe that, by conservativeness, F_k still contains x_0. Moreover, the link Γ_1' of x_0 in Γ_1 is monotone and conservative towards S' except for the last step. In this last step, the order on F_k in Γ cannot be restricted to an ordering of $F_k' := F_k \setminus \{x_0\}$ in $\mathrm{lk}_C(x_0)$, since x_0 is not the first vertex in F_k. Nevertheless the same arguments as above still apply and any admissible ordering on F_k with respect to S' with highest index guarantees monotonicity and conservativeness for the step $[F_{k-1}, F_k]$. So Γ_1' is a combinatorial segment to S' by the induction hypothesis. Moreover, the path $\Gamma_2 = [F_k, \dots, F_N]$ is a subpath of Γ and so is monotone and conservative towards S, and so is a combinatorial segment from F_k to S by induction on N. Thus Γ is a combinatorial segment from F_0 to S in C. ∎

Adiprasito and Benedetti also define a combinatorial segment "between two facets" as follows: Let F, F' be two facets of C and let Γ be a combinatorial segment from the facet F to the *set of vertices* F'. Then, only the last facet F'' in Γ intersects F', and F'' and F' intersect in a single vertex x. Since $F'' \smallsetminus x$ and $F' \smallsetminus x$ are both facets in $\mathrm{lk}_C(x)$ we can recursively define a combinatorial segment from the facet F to the facet F' to be Γ followed by $x * \Gamma'$, where Γ' is a combinatorial segment in $\mathrm{lk}_C(x)$ from the facet $F'' \smallsetminus x$ to the facet $F' \smallsetminus x$.

The following result follows easily:

Corollary 2.14. *Let F, F' be facets in a normal complex C. A combinatorial segment from F to F' is a monotone and conservative path that starts in F towards the target set F' and finishes in F'.*

Proof. By Theorem 2.13, the combinatorial segment starting at F and ending at the first facet intersecting F' at the vertex x is monotone and conservative towards the vertex set of F'. By induction on the dimension, we join x to the monotone and conservative path in the link of x and concatenate it to the previous path. The result is clearly monotone and conservative. ∎

2.4. **Joins and one-point-suspensions.** Let $\omega(C)$ denote the maximum length of monotone conservative paths in a pure normal complex C.

Proposition 2.15. *The function ω has the following properties.*

(i) $\omega(C_1 * C_2) = \omega(C_1) + \omega(C_2)$

(ii) $\omega(\mathrm{ops}_C(v)) \in \{\omega(C), \omega(C) + 1\}$

Proof. (i) It is clear that every dual path Γ in $C_1 * C_2$ restricts as a path Γ_1 in C_1 and a path Γ_2 in C_2 so that $\mathrm{length}(\Gamma_1) + \mathrm{length}(\Gamma_2) = \mathrm{length}(\Gamma)$. To prove $\omega(C_1 * C_2) \leq \omega(C_1) + \omega(C_2)$ we only need to check that if Γ is monotone and conservative, then Γ_1 and Γ_2 are also monotone and conservative.

Let Γ be a monotone and conservative path from the facet $F = F_1 * F_2$ to the facet $G = G_1 * G_2$, where F_1, G_1 (resp. F_2, G_2) are facets of C_1 (resp. C_2). Observe that flipping a vertex in C_1 does not influence the vertex-distances from vertices of a facet in C_1 to the facet G_2. Therefore the restriction of Γ to C_1 always flips according to vertex-distances within C_1. Further, the flip does not depend on the position of the anchor, i.e., it occurs either in C_1 or in $\mathrm{lk}_{C_1}(v)$, which respects the definition of vector of distances in C_1. Therefore, the restriction of Γ to C_1 (and symmetrically to C_2) forms a monotone and conservative path.

Conversely, taking monotone and conservative paths in C_1 and C_2 determines a monotone and conservative path in $C_1 * C_2$ by shuffling the consecutive subsegments depending on the minimal vertex-distances of the anchors. Therefore $\omega(C_1) + \omega(C_2) \leq \omega(C_1 * C_2)$, which concludes.

(ii) Since C is isomorphic to the link of v_1 in $\mathrm{ops}_C(v)$, we have $\omega(\mathrm{ops}_C(v)) \geq \omega(C)$. So, we only need to check that $\omega(\mathrm{ops}_C(v)) \leq \omega(C) + 1$. Let Γ be a longest monotone conservative path in the one-point-suspension $\mathrm{ops}_C(v)$ and let Γ' be the path restricted to C in the following sense: facets of $\mathrm{ops}_C(v)$ of the type $F * v_1$ or $F * v_2$ are sent to F, facets $F * \{v_1, v_2\}$ are sent to $F * v$. If one of v_1 or v_2 (say v_1) belongs to all the facets in Γ, then Γ' is a monotone and conservative path in $\mathrm{lk}_{\mathrm{ops}_C(v)}(v_1)$, which is isomorphic to C. If none of v_1 or v_2 belongs to all the facets in Γ, then assume that the starting facet contains v_1 but not v_2 and the final facet contains v_2 but not v_1. The path Γ' does the same flips as Γ except for the flips that change v_1 to v_2, or vice-versa. We need to show that there is only one of those. Since the stars of v_1 and v_2 contain all vertices, a step $[F, G]$ flipping v_1 to v_2 (or vice-versa) occurs with a vector of distances $(1, \dots, 1)$ for F. Therefore, G has to contain a vertex in the original target set S. Since v_2 is the only vertex not in F, we have $v_2 \in S$. Then, by the recursive definition of monotone conservative paths, the remaining part of the monotone conservative path in $\mathrm{ops}_C(v)$ is formed in $\mathrm{lk}_{\mathrm{ops}_C(v)}(v_2)$ which is isomorphic to C and v_2 is contained in the rest of the monotone conservative path. It remains to show that Γ' is monotone and conservative. The path Γ' consists of two parts, the one containing v_1 and the one containing v_2. By the argument above they are both monotone and conservative with respect to a same target set so concatenating them gives a monotone and conservative path up to the repetition of a unique facet of C. ∎

3. Upper bounds for the length of monotone conservative paths

The motivation for the definition of combinatorial segment was the proof of the Hirsch upper bound for flag normal complexes, and it gave as a byproduct the linear bound in fixed dimension. In this section we rework these two bounds in the language of monotone conservative paths. We also include a bound for banner complexes that interpolates between the two.

3.1. Hirsch bound for flag normal complexes.

A path $\Gamma = [F_0, \dots, F_N]$ in a complex C is said to be *nonrevisiting* if for every vertex v, the facets of Γ containing v form a subpath of Γ. That is to say, if $v \in F_i \cap F_j$ then $v \in F_k$ for every $k \in [i, j]$. It is easy to show (see below) that the length of nonrevisiting paths in a pure complex cannot exceed the Hirsch bound $n - d$.

Lemma 3.1 ([AB14, Section 3], see also [San13, Corollary 4.19])**.** *Let C be a flag normal complex, x and y be two consecutive anchors, with x coming first, along a monotone conservative path $\Gamma = [F_0, \dots, F_N]$. If z is a neighbor of y that belongs to a facet F_i anchored at x, then z belongs to all facets between F_i and the first one anchored at y.*

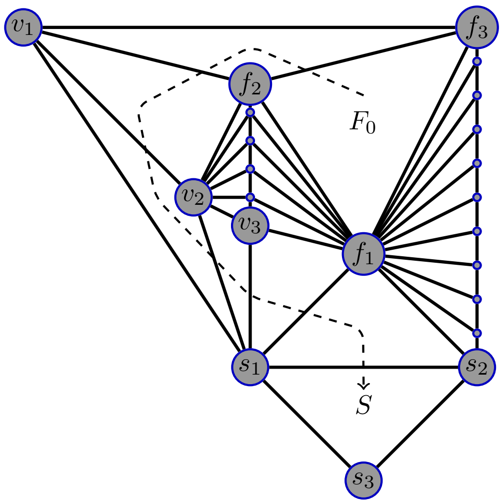
Proof. The proof goes by induction on the dimension of C and is obvious in dimension 1. Now in higher dimension, we can assume without loss of generality that x and y are the only two anchors in the path, that $i = 0$, that y first appears in F_N, and that $x \neq z$. Consider the monotone and conservative path Γ' obtained from Γ in $\mathrm{lk}_C(x)$. Since y is an anchor of Γ, it is in the target set of Γ' and Γ' finishes at a facet containing y. Since $\{x, y\}, \{x, z\}$ and $\{y, z\}$ are edges of C, which is flag, the set $\{x, y, z\}$ is a face of C so that $\{x, z\}$ and $\{y, z\}$ are in $\mathrm{lk}_C(x)$. The vertex z is in the first facet of Γ' and at distance 1 of the target set. So either it is the anchor of Γ' and we are done, or the first anchor of Γ' then has to be at distance 1 of y to give an admissible order. Call this anchor x'. Then, Γ' satisfies all the conditions in the statement, with respect to the vertices x', y and z. Since flagness is preserved under taking links, we can apply the induction hypothesis in $\mathrm{lk}_C(x)$. Thus, z is in all facets of the path Γ', and so of the path Γ. ■

Corollary 3.2 ([AB14, Section 3], see also [San13, Corollary 4.20])**.** *Every monotone and conservative path in a flag normal complex is a nonrevisiting path. In particular, its length is bounded by the Hirsch bound.*

Proof. Let $\Gamma = [F_0, \dots, F_N]$ be a monotone conservative path towards a target set S in a flag normal complex C. The proof is by a double induction on the dimension of C and the length N of the path Γ. For dimension $d = 1$ or for $N = 1$, the result is clear.

Now, assume $d > 1$ and $N > 1$. Let x be the first anchor of Γ and y be the second anchor, if any. We call Γ' the path $[F_0 \smallsetminus \{x\}, \dots, F_k \smallsetminus \{x\}]$ where $[F_0, \dots, F_{k-1}]$ is the part of Γ anchored at x. By induction on the dimension, the monotone and conservative path Γ' in $\mathrm{lk}_C(x)$ is nonrevisiting. By induction on N, the tail $\Gamma_2 := [F_k, \dots, F_N]$ of Γ once x is not an anchor is nonrevisiting. So it only remains to show that, if there is a vertex z used in Γ' and Γ_2, then it has to be contained in the last facet of Γ', where a change of anchor occurs. Since z is a vertex in $\mathrm{lk}_C(x)$, it is also a neighbor of x in C. Moreover x and y are consecutive anchors of Γ and so are neighbors in C. Set $\ell = \mathrm{vdist}_C(y, S)$ and suppose, for the sake of contradiction, that z is not a neighbor of y. Since y is the anchor of Γ following x, we have $\mathrm{vdist}_C(x, S) = \ell + 1$, and so $\mathrm{vdist}_C(z, S) \geq \ell + 1$ because of admissibility. Since z appears in a facet of Γ_2, the anchor of this facet has to be at vertex-distance at most $\ell - 1$ from S. Indeed it cannot be y and the sequence of anchors forms a shortest vertex path to S. Since z is a neighbor of this anchor in C, we have $\mathrm{vdist}_C(z, S) \leq \ell$; a contradiction. Hence y and z are neighbors in C; and Lemma 3.1 applies to C and x, y and z. Thus z belongs to the last facet of Γ'; which concludes. ■

Remark 3.3. Both the monotonicity and conservativeness assumptions are necessary in Corollary 3.2. On the one hand, consider the nonmonotone but conservative path obtained in Example 2.6 in the complex $C = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. The complex C is normal and flag. Let $F = (1, 2)$, $G = (1, 3)$, and $S = \{4\}$, the path $[F, G, F]$ is nonmonotone, conservative and revisiting. On the other hand, consider the complex presented in Figure 4 and the shown monotone but not conservative path which is revisiting. Again, the complex is flag and normal.



$F_0 = (f_1, f_2, f_3)$ $\Lambda_{F_0} = (1, 6, 1)$
$F_1 = (v_1, f_2, f_3)$ $\Lambda_{F_1} = (1, 2, 1)$
$F_2 = (v_1, v_2, f_2)$ $\Lambda_{F_2} = (1, 1, 1)$
$F_3 = (s_1, v_2, v_1)$ $\Lambda_{F_3} = (0, 3, 1)$
$F_4 = (s_1, v_3, v_2)$ $\Lambda_{F_4} = (0, 2, 1)$
$F_5 = (s_1, f_1, v_3)$ $\Lambda_{F_5} = (0, 1, 1)$
$F_6 = (s_1, s_2, f_1)$ $\Lambda_{F_5} = (0, 0, 1)$

FIGURE 4. A flag normal complex with a monotone nonconservative path which is revisiting. The step $[F_0, F_1]$ is not conservative.

Even in flag normal complexes, monotone conservative paths do not yield, or even approximate, shortest paths:

Lemma 3.4. *There are flag 2-balls (and flag 3-polytopes) with the following behaviors:*

(a) the difference in length in monotone conservative paths between two given facets can be arbitrarily large.
(b) no monotone conservative path is a shortest path between two given facets.

Proof. In Figure 5, we see two examples of flag 2-balls with the needed properties. In the one on the left, the first anchor can be f_1 or f_2. If we choose f_1, the path to the facet S is going to be short. Otherwise, choosing f_2 may lead to an arbitrary large path depending on the number of vertices involved in the "comb" region. In the complex on the right, the first anchor has to be f_2 and the path can be very long for the same reason. The shortest path from F to S can thus be arbitrarily shorter than any monotone conservative path.

It is not difficult to make these examples as flag 3-polytopes by adding a vertex v joined to the boundary and making edge-subdivisions of the edge $\{v, f_3\}$ sufficiently many times. Adding v and the edges to the boundary gives a flag 2-sphere since there are no empty triangles. Finally, flagness and polytopality is preserved by doing edge-subdivisions.

■

3.2. **A linear bound in fixed dimension.** We show here that monotone conservative paths lead to the classical bound of Larman [Lar70], Barnette [Bar74], and Eisenbrand et al. [EHRR10].

Theorem 3.5 ([Lar70, Bar74, EHRR10])**.** *The length of a monotone conservative path in a $(d-1)$-dimensional pure complex C on n vertices ($d \geq 2$) is at most $n2^{d-2}$. In particular if C is normal, the same inequality holds for its diameter. Moreover, if C is a pseudomanifold without boundary, the same statement holds with the bound $n2^{d-3}$.*

The statement is meant with respect to any target set. In particular we never make the target set explicit in the proof for brevity reasons.

As mentioned in the introduction the bound $n2^{d-2}$ is, modulo a (small) constant, one of the best that are proved for the diameter of normal simplicial complexes, or even for the more restricted case of polytopal simplicial complexes.

Proof. The proof is by induction on d. In the case $d = 1$, either $n = 1$, and then $N = 0 \leq n2^{d-2} = 1/2$, or $n \geq 2$, and then $N \leq 1 = 2 \cdot \frac{1}{2} \leq n2^{d-2}$.

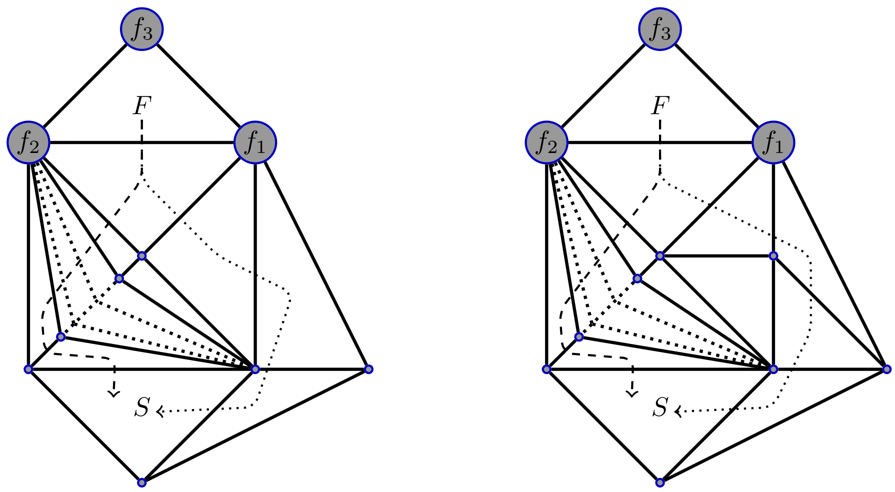


FIGURE 5. The flag 2-ball on the left has two monotone conservative paths from F to S whose difference in length is large. The flag 2-ball on the right has a monotone conservative path from F to S much longer than the shortest path.

For $d \geq 2$ let $x_1, \ldots, x_k$ be the sequence of anchors along Γ. Let $\Gamma_i = [F_0^i, \ldots, F_{N_i}^i]$ be the subpath of Γ anchored at x_i, including the facet at which the change of anchor takes places (that is, $F_{N_i}^{i-1} = F_0^i$ is the first facet anchored at x_i, obtained in a step that was still anchored at x_{i-1}).

The path Γ_i is monotone and conservative in the star of x_i. Deleting x_i from it gives a monotone conservative path in $\mathrm{lk}_C(x_i)$ which, by inductive hypothesis, has length at most

$$n_i 2^{d-3},$$

where n_i is the number of vertices other than x_i used in Γ_i. Now, since the distances $\mathrm{vdist}(x_i, S)$ are decreasing, no vertex of C can be in more than two links of the form $\mathrm{lk}_C(x_i)$. This shows that $\sum_{i=1}^k n_i \leq 2n$. Using this fact and the induction hypothesis, we have

$$N = \sum_{i=1}^{k} N_i \leq \sum_{i=1}^{k} n_i 2^{d-3} \leq 2^{d-2} n.$$

For pseudomanifolds the proof follows the same ideas, but the induction starts with $d = 2$. A 1-dimensional pseudomanifold without boundary is a cycle on n vertices, and the length of monotone conservative paths in it is indeed bounded by $n/2$.

Now the normality hypothesis ensures the existence of monotone conservative paths between any two facets, which concludes for the bound on the diameter in this case. ■

3.3. **Refined bound for banner complexes.** We now look at monotone conservative paths in banner complexes. We begin by recalling the definition by Klee and Novik [KN13]. Then, we prove upper bounds for the length of monotone conservative paths for these complexes, and therefore on their diameter.

Definition 3.6. Let C be a pure simplicial complex of dimension $(d-1)$. A *critical clique* is a set T of vertices of C forming a clique in the 1-skeleton of C and such that there exists a vertex $v \in T$ such that $T \setminus \{v\}$ is a face of C. For $k \in \{2, \ldots, d-1\}$, the complex C is said to be *k-banner* if every critical clique of size at least $(k+1)$ is a face of C.

Notice that any $(d-1)$-dimensional complex has no face with $(d+1)$ vertices, and thus cannot contain a critical clique of size $(d+2)$. So any $(d-1)$-dimensional complex is $(d+1)$-banner. Other properties on banner complexes are to be found in the article [KN13]. Among them is the fact that any k-banner complex is also $(k+1)$-banner, and that flag and 2-banner complexes coincide. Thus, banner complexes form a nested sequence of families of complexes starting with flag complexes for $k = 2$ to finish with all complexes for $k = d+1$. Let us also mention that the original definition allows for $k = 1$, but since 1-banner and 2-banner complexes are exactly the same, we prefer not to consider 1-banner complexes.

Remark 3.7. Every minimal nonface of size at least 3 in a complex is also a critical clique. In particular, if C is k-banner then every minimal nonface of it has size at most k. In other words, the Stanley–Reisner ring of a k-banner complex is generated in degree at most k. The converse may not hold.

We also recall the following result on links in banner complexes.

Lemma 3.8 ([KN13, Lemma 3.4])**.** *Let $k \geq 2$, C be a simplicial complex and x be a vertex of C. If C is a k-banner, then the link $\mathrm{lk}_C(x)$ of x in C is $(k-1)$-banner.*

Proof. Any critical clique in $\mathrm{lk}_C(x)$ is, after adding x, a critical clique of C of size one more. ∎

This lemma allows us to provide an upper bound for the length of monotone conservative paths in banner complexes, combining the proofs of Theorem 3.5 and Corollary 3.2, which are special cases of the following:

Theorem 3.9. *Let $d \geq 2$, $k \geq 2$, and C be a normal k-banner $(d-1)$-complex on n vertices. If $\Gamma = [F_0, \dots, F_\ell]$ is a monotone conservative path in C towards a set S, then the length of Γ satisfies $\ell \leq n2^{k-2}$.*

Proof. We show the result by induction on the dimension $d-1$. For $d = 2$, monotone conservative paths are the shortest paths in C so that $\ell \leq n$. The inequality thus holds in this case. Suppose now that $d > 2$. If $k = 2$, then we know that C is flag, implying the Hirsch bound [AB14], and the inequality holds.

So suppose now that $k \geq 3$ and consider the dual paths $\Gamma_1, \dots, \Gamma_r$ obtained from splitting Γ according to its sequence of anchors, that is Γ_i is the greatest subpath of Γ in which the anchor of any facet is the i^{th} anchor x_i of the sequence of anchors of Γ, for $i \in [r]$. In particular all Γ_i's are monotone conservative paths from their starting facet towards the set S. We fix $i \in [r]$ and consider the link Γ_i' of x_i in Γ_i. It is a monotone conservative path towards the set S' of Definition 2.2. Moreover it is a monotone conservative path in the link $\mathrm{lk}_C(x_i)$ of x_i in C.

Since C is k-banner, Lemma 3.8 ensures that this link is $(k-1)$-banner. Since $k \geq 3$, we can thus apply the induction hypothesis to Γ_i'. Let n_i be the number of vertices of C that appear in at least one facet of Γ_i', the induction hypothesis says that Γ_i', and thus Γ_i, has length at most $n_i 2^{k-3}$. So we have

$$\ell \leq \sum_{i=1}^{r} n_i 2^{k-3} = 2^{k-3} \sum_{i=1}^{r} n_i.$$

Now recall that each facet of the path Γ_i contains x_i and vertices that are either at distance $\mathrm{vdist}(x_i, S)$ or $\mathrm{vdist}(x_i, S) + 1$ of the set S. Since $\mathrm{vdist}(x_i, S) = \mathrm{vdist}(x_{i+1}, S) + 1$, a vertex of C cannot appear in more than two consecutive Γ_i's. So the sum of the n_i's is smaller than $2n$, which concludes. ∎

Observe that Theorem 3.9 somehow interpolates between Theorem 3.5 and Corollary 3.2:

- Since every $(d-1)$-complex is $(d+1)$-banner, substituting k by $d+1$, we recover (except for a factor of two) the bounds in Theorem 3.5.
- Since flag complexes are (the same as) 2-banner complexes, substituting k by 2, we recover (almost) the Hirsch bound of Corollary 3.2.

4. Monotone conservative paths can be exponentially long

In this final section, we present two constructions of normal complexes that contain exponentially long monotone conservative paths. The first one is a vertex-decomposable ball and the second, a simplicial polytope whose boundary complex is vertex-decomposable. Both construction rely on lemmas (Lemma 4.1 and 4.4) that provide the inductive procedure of the construction.

4.1. **A simplicial ball with exponential monotone conservative paths.**

Lemma 4.1. *Let $\mathcal{B}$ be a $(d-1)$-dimensional simplicial ball with n vertices. Suppose there are facets F_1 and F_2 and vertices x_1 and x_2 satisfying the following properties:*

(1) F_i is the only facet containing x_i, for $i = 1, 2$.
(2) Every monotone conservative path from F_1 to F_2 or from F_2 to F_1 has length at least L.

Then, for every $k \geq 2d$ there is a d-dimensional simplicial ball $\mathcal{B}'$ with $n + k + 1$ vertices having facets F'_1 and F'_2 and vertices x'_1 and x'_2 satisfying:

(1) F'_i is the only facet containing x'_i, for $i = 1, 2$.
(2) Every monotone conservative path from F'_1 to F'_2 or from F'_2 to F'_1 has length at least $2L + k$.

Observe that the statement does not specify the target sets for the monotone conservative paths. The expression "for every monotone conservative path" is then meant whatever the target set S may be; as long as that target set S allows for the existence of the claimed monotone conservative paths; in the case of the lemma this is equivalent to $x_2 \subset S \subset F_2$.

Proof. The condition on the vertex x_i implies that the facet F_i has a codimension-one face G_i contained in the boundary of $\mathcal{B}$. Consider the one-point-suspension $\mathrm{ops}_{\mathcal{B}}(x_1)$ of $\mathcal{B}$ on the vertex x_1, and call u_1 and u_2 the suspension vertices. Observe that $F_2 \cup u_i$ is a facet containing the codimension-one face $G_2 \cup u_i$, for $i = 1, 2$.

Let C_ℓ be the following d-complex on $\ell + d$ vertices $\{u, v_1, v_2, \ldots, v_{\ell+d-1}\}$ and ℓ facets:

$$C_\ell := \{\{u, v_i, \ldots, v_{i+d-1}\} : i = 1, \ldots \ell\}.$$

Observe that for $\ell \geq d$ the codimension-one boundary face $G := \{u, v_1, \ldots, v_{d-1}\}$ and the facet $F := \{u, v_\ell, \ldots, v_{\ell+d-1}\}$ only have u in common.

Let now $k_1, k_2 \in \mathbb{N}$ be such that $k_1 + k_2 = k$ and $k_i \geq d$, $i = 1, 2$. Our complex $\mathcal{B}'$ consists of the one-point-suspension $\mathrm{ops}_{\mathcal{B}}(x_1)$ of $\mathcal{B}$ with a copy of C_{k_1} glued by identifying $G_2 \cup u_1$ with $G \cup u$ and a copy of C_{k_2} glued by identifying $G_2 \cup u_2$ with $G \cup u$. The facets F'_1 and F'_2 are the F's in C_{k_1} and C_{k_2}, and the vertices x'_i are the last vertices v_{k_1+d-1} and v_{k_2+d-1} in them. Let us see that the stated properties are satisfied. We concentrate on paths from F'_1 to F'_2 but all the assertions are valid for the reverse paths, by symmetry.

- The number of vertices in $\mathcal{B}'$ equals the $n+1$ in $\mathrm{ops}_{\mathcal{B}}(x_1)$ plus the $k + 2d$ vertices in C_{k_1} plus C_{k_2}, minus the $2d$ vertices along which we glue.
- The facets F'_i and vertices x'_i clearly satisfy property 1 in the statement.
- Since $\mathcal{B}'$ is a ball, it is normal and, by Lemma 2.10, there is at least one monotone conservative path from F'_1 to F'_2.
- Every facet path (monotone and conservative or not) from F'_1 to F'_2 starts by going through the k_1 facets of C_{k_1} and finishes by going through the k_2 facets C_{k_2}. Apart of these k steps, the path consists of a path between $F_2 \cup u_1$ and $F_2 \cup u_2$ inside $\mathrm{ops}_{\mathcal{B}}(x_1)$.
- The unique closest vertices between F'_1 and F'_2 are u_1 and u_2, which form an edge. In particular, every monotone conservative path from F'_1 to F'_2 (or vice versa) has two anchors, u_1 and u_2.
- The part of the path anchored at u_1 goes from $F_2 \cup u_1$ to $\mathrm{ops}_{F_1}(x_1) = F_1 \setminus \{x_1\} \cup \{u_1, u_2\}$. Its link at u_1 is hence a monotone conservative path from F_2 to F_1 in $\mathcal{B}$, so it has length at least L.
- The part of the path anchored at u_2 goes from $\mathrm{ops}_{F_1}(x_1) = F_1 \setminus \{x_1\} \cup \{u_1, u_2\}$ to $F_2 \cup u_2$. Its link at u_2 is hence a monotone conservative path from F_1 to F_2 in $\mathcal{B}$, so it has length at least L. ■

Theorem 4.2. *For every $d \geq 2$ and every $N \geq 4$ there is a $(d-1)$-ball $\mathcal{B}_d$ with $N + d^2$ vertices and two facets F_1 and F_2 in it such that every monotone conservative path between them has length at least $2^{d-2}(N + 3)$.*

Proof. Let $\mathcal{B}_2$ be a path with $N + 4$ vertices and iterate the process of Lemma 4.1 with $k = 2d$. ■

Remark 4.3. It is easy to check that the construction of Lemma 4.1 preserves shellability and vertex-decomposability. Moreover, if every monotone conservative path in $\mathcal{B}$ is a Hamiltonian path in the dual graph, then the same happens in $\mathcal{B}'$. In particular, in the statement of Theorem 4.2 we can add that $\mathcal{B}_d$ is vertex-decomposable and that every monotone conservative path from F_1 to F_2 is Hamiltonian.

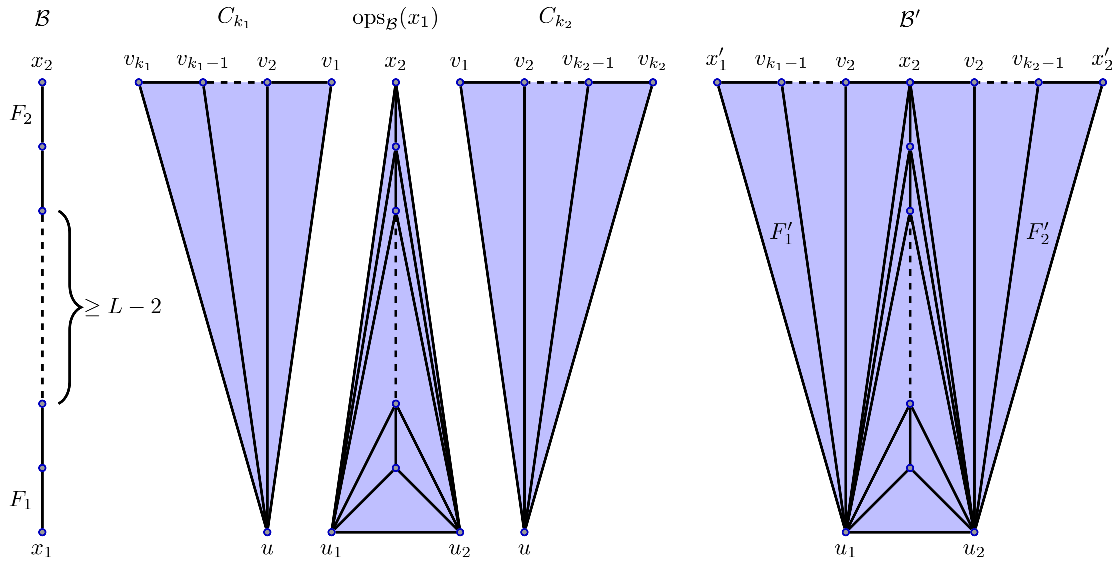


FIGURE 6. A 2-dimensional illustration of the example constructed in Theorem 4.2

Figure 6 illustrates the kind of objects that are produced by the previous proof.

4.2. **A Hirsch polytope with exponential monotone conservative paths.** We now show a second construction which achieves essentially the same long monotone and conservative paths but in a simplicial complex that is a *polytopal sphere*. The construction is similar in spirit to the previous one, based on the use of one-point-suspensions, but a bit more complicated since in a sphere we cannot have the vertices contained in a unique facet that were instrumental in the previous proof.

In the inductive step, instead of gluing *stacks* to the one-point-suspension of the previously constructed sphere, we do stellar subdivisions at edges. Let us first highlight some properties of these stellar subdivisions that will be used in the construction:

Let ab be an edge in a simplicial complex C, and let C' be the complex obtained by a stellar subdivision of the edge ab. Let v be the new vertex.

- Let c be a vertex of C other than a and b and assume that abc is not a triangle. Then, $\mathrm{lk}_{C'}(v) = \mathrm{lk}_C(v)$.
- The same is true for the links at a (resp. b) except the role of b (resp. a) in it is now played by c.
- The degree of v in the 1-skeleton of C' is 2 plus the number of triangles containing ab in C, and is bounded above by $\min\{\deg_C(a), \deg_C(b)\}$.

In particular, the proof makes use of the following idea. Let a be a vertex in a simplicial complex C and let C' be obtained from C by stellar subdivision of all the edges ab containing a (in any order). Then, for every vertex $v \neq a$ of C we have that

$$\mathrm{vdist}_{C'}(a, v) = \mathrm{vdist}_C(a, v) + 1.$$

We are now ready to state and prove the lemma that gives the induction step:

Lemma 4.4. *Let C be a $(d-1)$-complex on N vertices. If C has two facets F_1 and F_2 and vertices $x_i \in F_i$ $(i = 1, 2)$ such that:*

(1) Every monotone conservative path from F_i towards $\{x_j\}$ (and ending in a facet containing $\{x_j\}$) ends precisely in F_j and has length at least L, for $(i, j) \in \{(1, 2), (2, 1)\}$.
(2) Every monotone conservative path from F_i towards F_j (and ending in F_j) has length at least L, for $(i, j) \in \{(1, 2), (2, 1)\}$.
(3) $\mathrm{vdist}_C(x_1, x_2) \geq 3$.
(4) The degrees of x_1 and x_2 in (the 1-skeleton of) C are $2d - 2$.

Then, there exists a d-complex C' on $N' := N + 4d + 1$ vertices having the same properties (1) to (4) with $L' := 2L$ and $d' = d + 1$, and with respect to facets F'_i and vertices $x'_i \in F'_i$ $(i = 1, 2)$.

Proof. Consider first the complex $\mathrm{ops}_C(x_1)$ and let u and v be the suspension vertices. In it do the following stellar subdivisions:

- Let $\{y_1, \dots, y_{2d-2}\}$ be the neighbor vertices of x_2 in C. For each of them, subdivide the edges uy_i and vy_i. Call u_i and v_i the new vertices introduced.
- Then subdivide the edges ux_2 and vx_2. Call u' and v' the new vertices.
- Finally subdivide the edges uu' and vv' and call the new vertices x'_1 and x'_2.

Let C' be the final complex so obtained. Observe that the number of vertices has increased by one via the one-point-suspension, then by $4d - 4$ (vertices u_i and v_i, $i = 1, \dots, 2d - 2$), then, finally by another four (u', v', x'_1 and x'_2). In particular, the new complex indeed has $N' := N + 4d + 1$ vertices.

Observe that the links of u and v do not change by the subdivisions, except some of the new vertices introduced play the role of vertices of C. For example, in $\mathrm{lk}_{C'}(u)$ the role of x_1 and x_2 is played by x'_1 and v, respectively, and the role of each neighbor y of x_1 is played by the new vertex produced by subdividing uy. Here we are using the assumption $\mathrm{vdist}(x_1, x_2) \geq 3$. If this didn't hold, some of the y_i's would form a triangle with u and v, in which case the subdivision of uy_i would change the link at v, and vice-versa.

We let F'_1 consist of u together with the vertices of $\mathrm{lk}_{C'}(u)$ corresponding to F_2 and let F'_2 consist of v together with the vertices of $\mathrm{lk}_{C'}(v)$ corresponding to F_2. Put differently:

$$F'_1 := \{u, x'_1\} \cup \{y'_i : y_i \in F_2\}, \qquad F'_2 := \{v, x'_2\} \cup \{y'_i : y_i \in F_2\}.$$

Let us show some properties of this construction, among which are the claimed properties (1) to (4).

- Let us look at the degree of x'_1 (and x_2 symmetrically). Observe that the number $2d - 2$ of neighbors of x_2 in C equals the number of triangles containing ux_2 in $\mathrm{ops}_C(x_1)$. This number does not change by the subdivision of the edges uy_i, and it becomes the number of triangles containing uu' when we subdivide ux_2. So, when we create the vertex x'_1 by subdividing uu', the degree of the new vertex x'_1 equals 2 plus that number, which shows property (4).
- The path (x'_1, u, v, x'_2) is the unique shortest path from x'_1 to x'_2. Indeed, observe that any other path from x'_1 to x'_2 (or from a vertex on the "u side" to a vertex on the "v side" of the construction, for that matter) needs to pass through a vertex of C. The claim then follows by noticing that no vertex of C is a neighbor of neither x'_1 nor x'_2, so all other paths between them have length at least four. This implies part (3), but it also shows that:
  - (u, v) is the unique shortest vertex-path between F'_1 and F'_2.
  - (u, v, x'_2) is the unique shortest vertex-path between F'_1 and x'_2.
  - (x'_1, u, v) is the unique shortest vertex-path between x'_1 and F'_2.
- Then every monotone conservative path from F'_1 to x'_2 has (u, v, x'_2) as its sequence of anchors. While we are in the first anchor u the path is a monotone conservative path along $\mathrm{lk}_{C'}(u)$ towards the vertex set $S' := \{v\}$, which is the same as a monotone conservative path from F_2 to x_1 in C. By hypothesis (1), this path has length at least L and finishes in the facet $\mathrm{ops}_C(F_1)$. The part anchored at v is, by the same arguments, a monotone

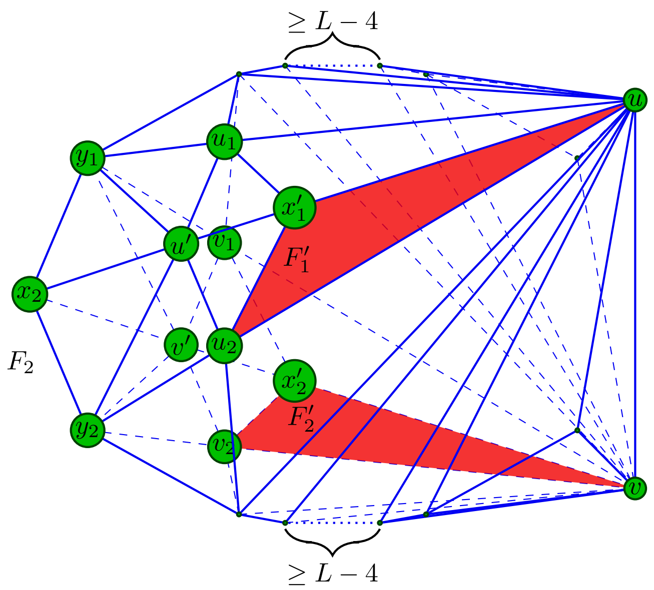


Figure 7. A 2-sphere illustration of the example constructed in Theorem 4.5

conservative path from $\mathrm{ops}_C(F_1)$ to x'_2, which finishes in F'_2 and has length also at least L by the same hypothesis. This proves (1) for C'.

- Similarly, every monotone conservative path from F'_1 towards F'_2 has (u, v) as its first two anchors. While we are anchored at u the path is a monotone conservative path along $\mathrm{lk}_{C'}(u)$ towards the vertex set $S' := \{v\}$, which is the same as a monotone conservative path from F_2 to x_1 in C. By hypothesis (1), this path has length at least L and finishes in the facet $\mathrm{ops}_C(F_1)$. The rest of the path (the part anchored at v, and what comes later, since v belongs to our target set S) is a monotone conservative path from $\mathrm{ops}_C(F_1)$ to F'_2 and has length also at least L by hypothesis (2). This proves (2) for C'. ∎

Theorem 4.5. *For every $d \geq 2$ and every $N \geq 4$, there is a vertex-decomposable polytopal $(d-1)$-sphere $\mathcal{S}_d$ with $N + \Theta(d^2)$ vertices and two facets F_1 and F_2 in it such that every monotone conservative path between them has length at least $2^{d-3}N$.*

See Figure 7 for a 2-sphere example.

Proof. Let $\mathcal{S}_2$ be a cycle with N vertices and iterate on it the process of Lemma 4.4. Observe that the construction in the lemma preserves polytopality and vertex-decomposability, since both one-point-suspension and stellar subdivision preserve them. For vertex-decomposability this follows from Lemma 2.1. For polytopality, it suffices to add a vertex outside the polytope close enough to the barycenter of the face which is stellar-subdivided to realize the stellar subdivision. ∎

Remark 4.6.
- Observe that, for fixed d, the bounds of Theorems 4.2 and 4.5 are both within a $1 + \varepsilon$ ratio of the upper bounds of Theorem 3.5, with ε going to zero as N goes to infinity.
- The fact that the polytopes constructed in Theorem 4.5 are vertex-decomposable implies that they satisfy the Hirsch bound [PB80], and it also dramatically illustrates the need of flagness for the proof of Adiprasito and Benedetti [AB14] to work.

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