THE INDEX OF RUBIN-STARK UNITS

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Abstract. The aim of this paper is to compare the orders of the class groups and the quotients of the r-th exterior power of units modulo Rubin-Stark units.

1. Introduction and Preliminares

The class number associated with a number field is known to be related to L-functions, and this can provide valuable information about class groups using computations of special values of those functions. A direct way to link those two concepts is based on what is called class number formulas.

Class number formulas where the class number is compared to the index of special units within their group of units have been formulated in the abelian and imaginary cases for circular and elliptic units respectively. It seems, however, that such results that would use the Rubin-Stark units are absent from literature and it is in this perspective that this work has been conducted.

This paper has therefore for aim to formulate and prove a class number formula which involves the index of Rubin-Stark units within the group of S-units. We introduce first some of the notations that will be used for this purpose.

Let k be a totally real field of degree $r=[k:\mathbb{Q}]$ and let K/k be a finite abelian extension of totally real number fields with Galois group G. Fix a finite set S of places of k containing infinite places and all places ramified in K/k, and a second finite set T of places of k, disjoint from S. Let $\widehat{G}=\mathrm{Hom}(G,\mathbb{C}^\times)$. If $\chi\in\widehat{G}$, the modified Artin L-function attached to χ is defined for $s\in\mathbb{C}$, $\mathrm{Re}(s)>1$ by

$$L_{S,T}(s,\chi)=\prod_{\mathfrak{p}\notin S}(1-\chi(\sigma_\mathfrak{p})\mathbf{N}\mathfrak{p}^{-s})^{-1}\prod_{\mathfrak{p}\in T}(1-\chi(\sigma_\mathfrak{p})\mathbf{N}\mathfrak{p}^{1-s}),$$

where $\sigma_\mathfrak{p}\in G$ is the Frobenius of the (unramified) prime $\mathfrak{p}$. This function can be analytically continued to a meromorphic function on $\mathbb{C}$.

For each $\chi\in\widehat{G}$, there is an idempotent

$$e_\chi=\frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1}\in\mathbb{C}[G].$$

Following [6] we define the Stickelberger element

$$\Theta_{S,T}(s)=\Theta_{S,T,K/k}(s)=\sum_{\chi\in\widehat{G}}L_{S,T}(s,\chi^{-1})e_\chi$$

which is viewed as a $\mathbb{C}[G]$-valued meromorphic function on $\mathbb{C}$. Let $\chi\in\widehat{G}$ and let $r_S(\chi)$ be the order of vanishing of $L_{S,T}(s,\chi)$ at $s=0$. Recall that

$$r_S(\chi)=\mathrm{ord}_{s=0}L_{S,T}(s,\chi)=\begin{cases}|\{v\in S\ :\ \chi(D_v(K/k))=1\}|, & \chi\neq 1;\\ |S|-1, & \chi=1.\end{cases}$$

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(see *e.g.* [6, Proposition I.3.4]), where $D_v(K/k)$ is the decomposition group of v relative to K/k.

Before stating the Rubin-Stark conjecture we record some hypotheses $\mathbf{H}(K/k, S, T, r)$:

(1) S contains all the infinite primes of k and all the primes which ramify in K/k;
(2) S contains at least r places which split completely in K/k;
(3) $|S| \geq r+1$;
(4) $T \neq \emptyset$, $S \cap T = \emptyset$ and $U_{S,T}(K)$ is torsion-free.

Here $U_{S,T}(K)$ is the group of S-units of K which are congruent to 1 modulo all the primes in T.

Conditions (2) and (3) ensure that $s^{-r}\Theta_{S,T}(s)$ is holomorphic at $s=0$. Since K/k is an extension of totally real fields and S contains all infinite places the second condition is satisfied by default. The condition (4) is easily satisfied, for example if T contains primes of two different residue characteristics.

For any set V of places of k, we denote by V_K the set of places of K lying above places in V and by $\mathbb{Z}V_K$ the free abelian group on V_K. Let M be a $\mathbb{Z}$-module. If $\mathbf{R}$ is one of the fields $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, we denote by $\mathbf{R}M$ the tensor product $\mathbf{R} \otimes_{\mathbb{Z}} M$. We extend this notation to sets of primes of K, we denote by $\mathbf{R}V_K$ the tensor product $\mathbf{R} \otimes_{\mathbb{Z}} \mathbb{Z}V_K$. The exterior power over $\mathbb{Z}[G]$, and Hom of $\mathbb{Z}[G]$-modules are denoted by

$$\bigwedge_G, \ \mathrm{Hom}_G(-,-)$$

respectively.

Assume that V is finite and contains only finite primes. We denote by S_∞ the set of infinite places of k. Let $S = S_\infty \cup V$, so that

$$\mathbb{R}S_K = \mathbb{R}S_{\infty,K} \oplus \mathbb{R}V_K$$

(as $\mathbb{R}[G]$-modules) and let π_∞ denote the projection from $\mathbb{R}S_K$ to $\mathbb{R}S_{\infty,K}$. We define $\mathcal{L}_{S,\infty}$ as the composite $\pi_\infty \circ \mathcal{L}_S$:

$$\mathcal{L}_{S,\infty} : \mathbb{R}U_{S,T}(K) \xrightarrow{\mathcal{L}_S} \mathbb{R}S_K \xrightarrow{\pi_\infty} \mathbb{R}S_{\infty,K} \ , \tag{1}$$

where $\mathcal{L}_S$ is a logarithmic 'embedding' of $U_{S,T}(K)$:

$$\begin{array}{rccc} \mathcal{L}_S : & U_{S,T}(K) & \longrightarrow & \mathbb{R}S_K \\ & \varepsilon & \longmapsto & -\sum_{w\in S_K} \log(|\varepsilon|_w)w. \end{array}$$

Taking r-th exterior powers over the commutative ring $\mathbb{R}[G]$ gives an $\mathbb{R}[G]$-linear map

$$\textstyle\bigwedge_{\mathbb{R}[G]}^r \mathcal{L}_{S,\infty} : \ \bigwedge_{\mathbb{R}[G]}^r U_{S,T}(K) \longrightarrow \bigwedge_{\mathbb{R}[G]}^r \mathbb{R}S_{\infty,K} = \mathbb{R}[G](w_1 \wedge ... \wedge w_r) \ ,$$

where $w_1, ..., w_r$ is a choice of r-places of K above the infinite places $\{v_1, ..., v_r\}$ of k. Since $w_1 \wedge ... \wedge w_r$ is a free generator we can define a unique $\mathbb{R}[G]$-linear 'regulator' R_w, called Rubin-Stark regulator:

$$\mathbb{R}\bigwedge_G^r U_{S,T}(K) \longrightarrow \mathbb{R}[G] \text{ by } \bigwedge_{\mathbb{R}[G]}^r \mathcal{L}_{S,\infty}(x) = \mathrm{R}_w(x)(w_1 \wedge ... \wedge w_r).$$

Explicitly, every element of $\mathbb{R}\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)$ is a finite sum of terms of the form $\varepsilon_1 \wedge \cdots \wedge \varepsilon_r$ with $\varepsilon_i \in \mathbb{R}U_{S,T}(K)$ and

$$\begin{array}{rccc} \mathrm{R}_w & : \mathbb{R}\bigwedge_G^r U_{S,T}(K) & \longrightarrow & \mathbb{R}[G] \\ & \varepsilon = \varepsilon_1 \wedge \cdots \wedge \varepsilon_r & \longmapsto & \det(-\sum_{\sigma\in G} \log|\varepsilon_i^\sigma|_{w_j}\sigma^{-1})_{i,j=1}^r. \end{array}$$

Definition 1.1. *For a finitely generated G-module M and $r \in \mathbb{Z}_{\geq 0}$, we define Rubin's lattice by*

$$\bigcap_G^r M = \{m \in \mathbb{Q}\bigwedge_G^r M \mid \Phi(m) \in \mathbb{Z}[G] \text{ for all } \Phi \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])\}.$$

Remark 1.2. *Let M' be a finitely generated G-module. If $M \longrightarrow M'$ is a G-homomorphism, then it induces a natural G-homomorphism*

$$\bigcap_G^r M \longrightarrow \bigcap_G^r M' .$$

Besides, if $M \longrightarrow M'$ is injective and its cokernel is torsion-free, then the induced map

$$\bigcap_G^r M \longrightarrow \bigcap_G^r M' .$$

*is injective (*e.g. [5, Lemma 2.11]*).*
Note that the Sinnott index $(\bigcap_G^s M : \widetilde{\bigwedge_G^s} M)$ is finite (e.g. [3, Proposition 1.2]*), where $\widetilde{\bigwedge_G^s} M$ denotes the image of $\bigwedge_G^s M$ via the canonical morphism*

$$\bigwedge_G^s M \longrightarrow \mathbb{Q}\bigwedge_G^s M .$$

Let $\Theta_{S,T}^{(r)}(0)$ be the coefficient of s^r in the Taylor series of $\Theta_{S,T}$;

$$\Theta_{S,T}^{(r)}(0) := \lim_{x \to 0} s^{-r} \Theta_{S,T}^{(r)}(s).$$

Conjecture B′ (Rubin-Stark conjecture) of [3] predicts the existence of certain elements

$$\eta_{K,S,T} \in \bigcap_G^r U_{S,T}(K) \quad \text{such that} \quad \mathrm{R}_w(\eta_{K,S,T}) = \Theta_{S,T}^{(r)}(0).$$

Let $\mathfrak{f}$ denote the finite part of the conductor of K/k (we assume that $\mathfrak{f} \neq (1)$). For any ideal $\mathfrak{a}$ we denote the product of all distinct prime ideals dividing $\mathfrak{a}$ by $\widehat{\mathfrak{a}}$ and $\mathrm{T}_{\mathfrak{a}}(K)$ the subgroup of G generated by the inertia groups $I_{\mathfrak{q}}(K/k)$ with $\mathfrak{q} \mid \mathfrak{a}$. If $\mathfrak{a} = (1)$ we set $\mathrm{T}_{(1)} = \{1\}$. For any cycle $\mathfrak{g} \mid \widehat{\mathfrak{f}}$, we denote the maximal subextension of K whose conductor is prime to $\mathfrak{f}\mathfrak{g}^{-1}$ by $K_{\mathfrak{g}} = K^{I_{\mathfrak{f}\mathfrak{g}^{-1}}}$. In the sequel, we will fix a finite set S' of finite places of k which contains at least one finite place, and will denote by $S_{\mathfrak{g}}$ the set

$$S_{\mathfrak{g}} = S_\infty \cup \{\mathfrak{q} : \mathfrak{q} \mid \mathfrak{g}\} \cup S'.$$

Let us also denote by S the set $S = S_{\widehat{\mathfrak{f}}}$.
Since $K_{\mathfrak{g}}$ is totally real then the hypothesis $\mathbf{H}(K_{\mathfrak{g}}/k, S_{\mathfrak{g}}, T, r)$ is satisfied.

In the rest of this paper we assume the validity of Rubin-Stark conjecture.

Definition 1.3. *We denote by $\mathrm{St}_{K,T}$ the $\mathbb{Z}[G]$-module generated by $\eta_{K_{\mathfrak{g}},S_{\mathfrak{g}},T}$ for all $\mathfrak{g} \mid \widehat{\mathfrak{f}}$.*

We will see that

$$\bigcap_{\mathrm{Gal}(K_{\mathfrak{g}}/k)}^r U_{S_{\mathfrak{g}},T}(K_{\mathfrak{g}}) \hookrightarrow \bigcap_G^r U_{S,T}(K)$$

(see remark 1.2), which justifies our definition.

Recall that a $\mathbb{Z}[G]$-lattice is a finitely generated $\mathbb{Z}[G]$-module which is a torsion-free $\mathbb{Z}$-module.

Let $e_{S,r} := \Sigma_{\chi\in\hat{G},r_S(\chi)=r} e_\chi$. Note that $e_{S,r} \in \mathbb{Q}[G]$ and for any $\mathbb{Z}[G]$-lattice M, the $\mathbb{Z}[G]$-module

$$e_{S,r}M = \{e_{S,r}m, m \in M\}$$

is a lattice of the $\mathbb{Q}$-vector space $e_{S,r}(\mathbb{Q}M)$.

The goal of this paper is the following theorem

Theorem 1.4. *The Sinnott index* $(e_{S,r}\bigcap_G^r U_{S,T}(K) : e_{S,r}\mathrm{St}_{K,T})$ *is finite, and we have*

$$[e_{S,r}\bigcap_G^r U_{S,T}(K) : e_{S,r}\mathrm{St}_{K,T}] = h_K.(e_{S,r}\mathbb{Z}[G] : e_{S,r}U_K^{(r)}).(e_{S,r}\bigcap_G^r U_{S,T}(K) : e_{S,r}\widetilde{\bigwedge_G^r} U_{S,T}(K)).\beta_K.$$

where $U_K^{(r)}$ *is the Sinnott module (see Definition 3.1) and* β_K *is well determined, see* (4).

2. Image by the Rubin-Stark regulator

Throughout this section, let $F = K_{\mathfrak{g}}$, $\eta_F = \eta_{K_{\mathfrak{g}},S_{\mathfrak{g}},T}$ the Rubin-Stark element in $K_{\mathfrak{g}}$. Let H (resp. Δ) denote the Galois group $\mathrm{Gal}(K/F)$ (resp. $\mathrm{Gal}(F/k)$). Let

$$\pi_F : \mathbb{C}[G] \longrightarrow \mathbb{C}[\Delta]$$

denote the homomorphism induced by the natural surjection $G \twoheadrightarrow \Delta$, and let us fix $\gamma_1, \cdots, \gamma_d \in G$, such that

(1) $\gamma_1 = 1$
(2) $\{\pi_F(\gamma_1), \cdots, \pi_F(\gamma_d)\} = \Delta$.

Proposition 2.1. *Let* $R_{w'}$ *be the restriction of the regulator map* R_w *to the subfield* F *defined by using the infinites places* $w_1', .., w_r'$ *of* F *below the places* $w_1, .., w_r$ *of* K*. Then for any element* $u_F \in \mathbb{R}\bigwedge_\Delta^r U_{S,T}(F)$ *we have*

$$\pi_F(R_w(u_F)) = |H|^r R_{w'}(u_F).$$

Proof. By definition

$$R_w(u_F) = \det(a_{i,j})_{i,j},$$

where

$$a_{i,j} = -\Sigma_{\sigma\in G}\log \mid (u_F)_i^{\sigma^{-1}} \mid_{w_j} \sigma,$$

here we denote by $u_F = (u_F)_1 \wedge .. \wedge (u_F)_i \wedge .. \wedge (u_F)_r$. Let us first calculate the coefficient $a_{i,j}$ for some given (i, j). To simplify notations we refer to $(u_F)_i$ simply as u. Then

$$\begin{aligned}\pi_F(\Sigma_{\sigma\in G}\log \mid u^{\sigma^{-1}} \mid_{w_j} \sigma) &= \Sigma_{i=1}^d \Sigma_{h\in H}\log \mid u^{\gamma_i^{-1}h^{-1}} \mid_{w_j} \pi_F(\gamma_i h)\\ &= \Sigma_{i=1}^d \Sigma_{h\in H}\log \mid u^{\pi_F(\gamma_i)^{-1}} \mid_{w_j} \pi_F(\gamma_i)\ , \quad (u \in \mathbb{R}U_{S,T}(F))\\ &= |H|(\Sigma_{i=1}^d \log \mid u^{\pi_F(\gamma_i)^{-1}} \mid_{w_j} \pi_F(\gamma_i)).\end{aligned}$$

Since $w_j' = w_{j|F}$ is completely decomposed in K/F, we obtain $\mid u^{\gamma_i^{-1}} \mid_{w_j} = \mid u^{\gamma_i^{-1}} \mid_{w_j'}$. Finally we have

$$\pi_F(\mathrm{R}_w(u_F)) = |H|^r \mathrm{R}_{w'}(u_F)$$

where $R_{w'}$ is the same as R_w but defined over F instead of K using the infinite places $w_1', .., w_r'$ of F below the places $w_1, .., w_r$ of K. ∎

For any character $\psi \in \hat{\Delta}$, let $\mathfrak{f}_\psi$ denote the conductor of ψ. Let $\hat{\psi}$ denote the associated primitive character obtained by restricting ψ to $\Delta/\ker(\psi)$ (so that we obtain a faithful character). Let us denote by $L(s,\widehat{\psi})$ the primitive Hecke L-function defined for $\mathrm{Re}(s) > 1$ by the Euler product

$$L(s,\widehat{\psi}) = \prod_{\mathfrak{p}\nmid\mathfrak{f}_\psi}(1-\widehat{\psi}(\sigma_\mathfrak{p})N\mathfrak{p}^{-s})^{-1}.$$

The function $L(s,\widehat{\psi})$ can be analytically continued to an analytic function on $\mathbb{C}$ (meromorphic when $\psi = 1$). For any $s \in \mathbb{C}$ and any non trivial character ψ we have

$$L_S(s,\psi) = \prod_{\substack{\mathfrak{p}\mid\mathfrak{f}_F \\ \mathfrak{p}\nmid\mathfrak{f}_\psi}}(1-\widehat{\psi}(\sigma_\mathfrak{p})N\mathfrak{p}^{-s})L(s,\widehat{\psi})$$

where $\mathfrak{f}_F$ is the conductor of F/k. Since F/k is an extension of totally real fields, we have

$$\mathrm{ord}_{s=0}(L_{S,T}(s,\psi)) = \mathrm{ord}_{s=0}(L_S(s,\psi)) = \mathrm{ord}_{s=0}(L_S(s,\widehat{\psi})).$$

Then

$$\begin{aligned} L^{(r)}_{S,T}(0,\psi) &= L^{(r)}_S(0,\psi).\prod_{\mathfrak{q}\in T}(1-\psi(\sigma_\mathfrak{q})\mathrm{N}\mathfrak{q}) \\ &= \prod_{\mathfrak{q}\in T}(1-\psi(\sigma_\mathfrak{q})\mathrm{N}\mathfrak{q}).\prod_{\substack{\mathfrak{p}\mid\mathfrak{f}_F \\ \mathfrak{p}\nmid\mathfrak{f}_\psi}}(1-\widehat{\psi}(\sigma_\mathfrak{p}))L^{(r)}(0,\widehat{\psi}). \end{aligned}$$

Remark that for any prime $\mathfrak{p}$ we have

$$\sigma_\mathfrak{p}^{-1}e_{I_\mathfrak{p}}e_{\psi^{-1}} = \hat{\psi}(\sigma_\mathfrak{p})e_{\psi^{-1}}$$

where $e_{I_\mathfrak{p}} = \frac{1}{|I_\mathfrak{p}|}\sum_{\sigma\in I_\mathfrak{p}}\sigma$. Hence we have the following proposition

Proposition 2.2. *There exists an element $\omega_K \in \mathbb{C}[G]$ independent of the choice of the field F which verifies*

$$\pi_F(e_{S,r})R_{w'}(\eta_F) = \pi_F\Big(e_{S,r}\omega_K(\delta_T\prod_{\mathfrak{p}\mid\mathfrak{f}_F}(1-\sigma_\mathfrak{p}^{-1}e_{I_\mathfrak{p}}))\Big),$$

where

$$\omega_K := \sum_{\chi\in\widehat{G},\, r_S(\chi)=r} L^{(r)}(0,\widehat{\chi})e_{\chi^{-1}} \quad \textit{and} \quad \delta_T := \prod_{\mathfrak{q}\in T}(1-\sigma_\mathfrak{q}^{-1}\mathbf{N}\mathfrak{q})$$

Proof. As we previously stated

$$R_{w'}(\eta_F) = \Theta^{(r)}_{S,T,F/k}(0) = \Sigma_{\psi\in\hat{\Delta}}L^{(r)}_{S,T}(0,\psi)e_{\psi^{-1}}.$$

Since $L^{(r)}_{S,T}(0,\psi) = \prod_{\mathfrak{q}\in T}(1-\psi(\sigma_\mathfrak{q})\mathrm{N}\mathfrak{q}).\prod_{\mathfrak{p}\mid\mathfrak{f}_F,\,\mathfrak{p}\nmid\mathfrak{f}_\psi}(1-\widehat{\psi}(\sigma_\mathfrak{p}))L^{(r)}(0,\widehat{\psi})$ holds, we obtain

$$\begin{aligned} \mathrm{R}_{w'}(\eta_F) &= \Sigma_{\psi\in\hat{\Delta},r_S(\psi)=r}\Big((\prod_{\mathfrak{q}\in T}(1-\psi(\sigma_\mathfrak{q})\mathbf{N}\mathfrak{q}))(\textstyle\prod_{\mathfrak{p}\mid\mathfrak{f}_F,\mathfrak{p}\nmid\mathfrak{f}_\psi}(1-\hat{\psi}(\sigma_\mathfrak{p}))\Big)L^{(r)}(0,\hat{\psi})e_{\psi^{-1}} \\ &= \Sigma_{\psi\in\hat{\Delta},r_S(\psi)=r}\Big((\prod_{\mathfrak{q}\in T}(1-\sigma_\mathfrak{q}^{-1}\mathbf{N}\mathfrak{q}))(\textstyle\prod_{\mathfrak{p}\mid\mathfrak{f}_F,\mathfrak{p}\nmid\mathfrak{f}_\psi}(1-\sigma_\mathfrak{p}^{-1}e_{I_\mathfrak{p}}))e_{\psi^{-1}}\Big)(L^{(r)}(0,\hat{\psi})e_{\psi^{-1}}) \\ &= \Big(\Sigma_{\psi\in\hat{\Delta},r_S(\psi)=r}(\prod_{\mathfrak{q}\in T}(1-\sigma_\mathfrak{q}^{-1}\mathbf{N}\mathfrak{q}).\textstyle\prod_{\mathfrak{p}\mid\mathfrak{f}_F,\mathfrak{p}\nmid\mathfrak{f}_\psi}(1-\sigma_\mathfrak{p}^{-1}e_{I_\mathfrak{p}}))e_{\psi^{-1}}\Big)\Big(\Sigma_{\psi\in\hat{\Delta},r_S(\psi)=r}L^{(r)}(0,\hat{\psi})e_{\psi^{-1}}\Big) \end{aligned}$$

where $I_\mathfrak{p}$ is the inertia group of $\mathfrak{p}$ in F/k. Using the fact that each character of $\Delta = \mathrm{Gal}(F/k)$ can be seen as a character of $G = \mathrm{Gal}(K/k)$ trivial on $H = \mathrm{Gal}(K/F)$, we get

$$\pi_F(e_{\psi^{-1}\circ\pi_F}) = e_{\psi^{-1}} \quad \text{and} \quad \sigma_\mathfrak{p}^{-1}e_{I_\mathfrak{p}}e_{\psi^{-1}\circ\pi_F} = \hat{\psi}(\sigma_\mathfrak{p})e_{\psi^{-1}\circ\pi_F}$$

where $I_{\mathfrak{p}}$ denotes also the inertia group of $\mathfrak{p}$ in K/k. Therefore

$$\pi_F(e_{S,r})R_{w'}(\eta_F) = \pi_F\left(e_{S,r}\left(\Sigma_{\substack{\chi\in\hat{G},\, r_S(\chi)=r\\ \chi(H)=1}}(\prod_{\mathfrak{q}\in T}(1-\sigma_{\mathfrak{q}}^{-1}\mathbf{N}\mathfrak{q}))(\prod_{\mathfrak{p}|\mathfrak{f}_F,\mathfrak{p}\nmid\mathfrak{f}_\chi}(1-\sigma_{\mathfrak{p}}^{-1}e_{I_{\mathfrak{p}}}))e_{\chi^{-1}}\right)\omega_K\right)$$

where

$$\omega_K := \Sigma_{\chi\in\hat{G},\, r_S(\chi)=r}L^{(r)}(0,\hat{\chi})e_{\chi^{-1}}.$$

Since

$$\pi_F(e_\chi) = \begin{cases} 0, & \text{if } \chi(H)\neq 1; \\ \frac{1}{|\Delta|}\sum_{\sigma\in\Delta}\chi(\sigma)\sigma^{-1}, & \text{if } \chi(H)=1. \end{cases}$$

holds, we get

$$\pi_F(e_{S,r})\mathrm{R}_{w'}(\eta_F) = \pi_F\left(e_{S,r}\omega_K\left(\delta_T.\prod_{\mathfrak{p}|\mathfrak{f}_F}(1-\sigma_{\mathfrak{p}}^{-1}e_{I_{\mathfrak{p}}})\right)\right)$$

where $\delta_T = \prod_{\mathfrak{q}\in T}(1-\sigma_{\mathfrak{q}}^{-1}\mathbf{N}\mathfrak{q})$. This finishes the proof of the proposition. □

We combine the results of the two previous sections and get

Corollary 2.3. *Recall that* $H := \mathrm{Gal}(K/F)$. *Then*

$$\pi_F(e_{S,r}\mathrm{R}_w(\eta_F)) = \pi_F\left(\omega_K(|H|^r(\delta_T.\prod_{\mathfrak{p}|\mathfrak{f}_F}(1-\sigma_p^{-1}e_{I_{\mathfrak{p}}}))e_{S,r}\right).$$

3. Index of the "Stark" module

3.1. **The generalised Sinnott index.** We recall some data about the generalised Sinnott index. For a more complete exhibit of the properties of this index the reader is invited to refer to [4]. Let p be a prime rational and v_p its normalised valuation ($v_p(p)=1$). Let $\mathbb{F}$ be one of the fields $\mathbb{Q}$, $\mathbb{Q}_p$ or $\mathbb{R}$, and let

$$\mathcal{O} := \begin{cases} \mathbb{Z}, & \mathbb{F}=\mathbb{Q} \text{ or } \mathbb{R}; \\ \mathbb{Z}_p, & \mathbb{F}=\mathbb{Q}_p. \end{cases}$$

Let E be an $\mathbb{F}$-vector space of finite dimension d. An $\mathcal{O}$-lattice Λ is a free $\mathcal{O}$-submodule of E of rank d such that the $\mathbb{F}$-vector space generated by Λ is E. If M and N are two lattices of E, we define the generalised Sinnott index as follows

$$(M:N) = \begin{cases} |\det(\gamma)| & \text{if } \mathbb{F}=\mathbb{Q} \text{ or } \mathbb{R} \\ p^{v_p(\det(\gamma))} & \text{if } \mathbb{F}=\mathbb{Q}_p \end{cases}$$

where γ is an automorphism of the $\mathbb{F}$-vector space E such that $\gamma(M)=N$.

Recall that $\mathrm{T}_{\mathfrak{r}}(K)$ denotes the subgroup of G generated by the inertia groups $I_{\mathfrak{q}}(K/k)$ with $\mathfrak{q}\mid\mathfrak{r}$.

Definition 3.1. *Let* $\mathfrak{f}$ *be the conductor of* K/k. *Let* $\mathfrak{s}$ *be a divisor of* $\widehat{\mathfrak{f}}$. *If* $\mathfrak{s}\neq(1)$, *then we denote by* $U_{\mathfrak{s}}^{(r)}$ *or* $U_{\mathfrak{s},K}^{(r)}$ *the* $\mathbb{Z}[\mathrm{Gal}(K/k)]$*-submodule of* $\mathbb{Q}[\mathrm{Gal}(K/k)]$ *generated by all the elements*

$$\alpha(\mathfrak{r},\mathfrak{s}) = s(\mathrm{T}_{\mathfrak{r}}(K))^r\prod_{\mathfrak{p}|\mathfrak{s}/\mathfrak{r}}(1-\sigma_{\mathfrak{p}}^{-1}e_{I_{\mathfrak{p}}});\ \mathfrak{r}\mid\mathfrak{s},\ \textit{where}\ s(\mathrm{T}_{\mathfrak{r}}(K)) = \sum_{\sigma\in\mathrm{T}_{\mathfrak{r}}(K)}\sigma.$$

Moreover we set $U_{(1)}^{(r)} = \mathbb{Z}[\mathrm{Gal}(K/k)]$, $U_K^{(r)} = U_{\widehat{\mathfrak{f}}}^{(r)}$ *and* $U_{\mathfrak{s},K}^{(1)} = U_{\mathfrak{s}}$.

Remark 3.2. *The modules* $U_{\mathfrak{s}}$ *were introduced in* [4] *when* k *is equal to the field of rational numbers* $\mathbb{Q}$. *Sinnott used these modules to study the index of cyclotomic units in the cyclotomic* $\mathbb{Z}_p$*-extension. This technique has been followed in the case of circular units or in* [2] *for the elliptic units case.*

Lemma 3.3. *The following generalized Sinnott indices are well defined*

(1) $(e_{S,r}U_K^{(r)} : e_{S,r}\omega_K U_K^{(r)})$
(2) $(e_{S,r}\mathbb{Z}[G] : e_{S,r}U_K^{(r)})$
(3) $(e_{S,r}\mathbb{Z}[G] : \mathrm{R}_w(e_{S,r}\bigcap_G^r U_{S,T}(K))$

Proof. The assertions (1) and (2) are a direct consequence of the fact that $U_K^{(r)}$ is a lattice of $\mathbb{Q}[G]$ and the definition of the generalized Sinnott index. The image of $e_{S,r}\bigcap_G^r U_{S,T}(K)$ by the Rubin-Stark regulator is a lattice of $e_{S,r}\mathbb{Q}[G]$ and hence, the index in (3) is well defined. □

Corollary 3.4. *The generalized Sinnott index* $(e_{S,r}\bigcap_G^r U_{S,T}(K) : e_{S,r}\mathrm{Stark}_{K,T})$ *is well defined and we have the equality*

$$(e_{S,r}\bigcap_G^r U_{S,T}(K) : e_{S,r}\mathrm{Stark}_{K,T}) = \frac{(e_{S,r}\mathbb{Z}[G] : e_{S,r}\delta_T U_K^{(r)})}{(e_{S,r}\mathbb{Z}[G] : \mathrm{R}_w(e_{S,r}\bigcap_G^r U_{S,T}(K)))}.(e_{S,r}U_K^{(r)} : \omega_K e_{S,r}U_K^{(r)}).$$

Proof. The index $(R_w(e_{S,r}\bigcap_G^r U_{S,T}(K)) : R_w(e_{S,r}\mathrm{Stark}_{K,T}))$ is well defined and the map R_w is injective, thus

$$(e_{S,r}\bigcap_G^r U_{S,T}(K) : e_{S,r}\mathrm{Stark}_{K,T}) = (\mathrm{R}_w(e_{S,r}\bigcap_G^r U_{S,T}(K)) : \mathrm{R}_w(e_{S,r}\mathrm{Stark}_{K,T})).$$

Since $\mathrm{R}_w(e_{S,r}\mathrm{Stark}_K) = \omega_K e_{S,r}\delta_T U_K^{(r)}$ (see Corollary 2.3) holds, we obtain

$$(R_w(e_{S,r}\bigcap_G^r U_{S,T}(K)) : \mathrm{R}_w(e_{S,r}\mathrm{Stark}_{K,T}))$$
$$=$$
$$\frac{(e_{S,r}\mathbb{Z}[G]:e_{S,r}\delta_T U_K^{(r)})}{(e_{S,r}\mathbb{Z}[G]:\mathrm{R}_w(e_{S,r}\bigcap_G^r U_{S,T}(K)))}(e_{S,r}\delta_T U_K^{(r)} : \omega_K e_{S,r}\delta_T U_K^{(r)}).$$

Using the fact that $\delta_T = \prod_{\mathfrak{q}\in T}(1-\sigma_{\mathfrak{q}}^{-1}\mathbf{N}\mathfrak{q})$ is a non-zero-divisor, we get

$$(e_{S,r}\delta_T U_K^{(r)} : \omega_K e_{S,r}\delta_T U_K^{(r)}) = (e_{S,r}U_K^{(r)} : \omega_K e_{S,r}U_K^{(r)}).$$

Hence the corollary follows. □

3.2. **The class number Formula.** Next, we use the previous result to prove the class number formula shown in Theorem 1.4.

Let F/k be an intermediate extension in K/k, we denote by $\mathrm{Ram}(F/k)$ the set of primes that ramify in the extension F/k. We make some further notations

(1) $X(F) := \{\Sigma a_w w \in \mathbb{Z}S_{\infty,F}, \Sigma a_w = 0\}$.
(2) $\lambda_F : U_{S_\infty}(F) \longrightarrow X(F)\otimes\mathbb{R}$ is the map defined by
$$\lambda_F(\alpha) = -\Sigma_{w\in S_{\infty,F}}\log(\mid \alpha \mid_w)w.$$
(3) $\mathrm{Reg}_F =\mid \det(\lambda_F)\mid$ the regulator associated to λ_F.
(4) We assume that $\mathrm{Ram}(K/k) = \{\mathfrak{p}_1, .., \mathfrak{p}_{|Ram(K/k)|}\}$. For $I \subset \{1, .., \mid \mathrm{Ram}(K/k)\mid\}$ we define the field
$$K_I := K^{D_I}$$
where D_I is the subgroup of G generated by the decomposition groups D_i of $\mathfrak{p}_i$ in K/k, $i \in I$.

Lemma 3.5. *One has* $e_{S,r}U_{S,T}(K) = e_{S,r}U_{S_\infty}(K)$.

Proof. Let S_1 be a finite set of places of K, and let $S_2 = S_1 \cup \{\mathfrak{q}_v\}$. Let $\{u_1, \cdots, u_t\}$ be fundamental units of $\mathcal{O}_{S_1}^*$. We claim that if $\mathfrak{q}_v^m = a\mathcal{O}_{S_1}$ then $\{u_1, \cdots, u_t, a\}$ are fundamental units for $\mathcal{O}_{S_2}^*$, and $a^{1-e_{D_v}} \in \mathcal{O}_{S_1}^*$, where m is the order of $\mathfrak{q}_v$ in the ideal class group of $\mathcal{O}_{S_1}$,D_v is the decomposition group of $\mathfrak{q}_v$ in K/k and $e_{D_v} = \frac{1}{|D_v|}N_{D_v}$. First we prove that this claim will give the desired result. Since $|S| > r+1$, we obtain

$$e_{S,r} = \prod_{v \in S - S_\infty} (1 - e_{D_v}).$$

Iterating our claim gives

$$e_{S,r}U_{S,T}(K) \subset e_{S,r}U_{S_\infty}(K)$$

as desired.

It remains to prove our claim that $\{u_1, \cdots, u_t, a\}$ are fundamental units for $\mathcal{O}_{S_2}^*$. Let u be a unit of $\mathcal{O}_{S_2}$. By scaling by an appropriate power of a, we may assume that $0 \leq i = v_{\mathfrak{q}_v}(u) \leq m-1$. Then $\mathfrak{q}_v^m = u\mathcal{O}_{S_1}$. Since the order of $\mathfrak{q}_v$ in the ideal class group of $\mathcal{O}_{S_1}$ is m, we must have $i = 0$, so that $u \in \mathcal{O}_{S_1}^*$. Then we have $\mathfrak{q}_v^{1-e_{D_v}} = \mathcal{O}_{S_1}$, and hence $a^{1-e_{D_v}} \in \mathcal{O}_{S_1}^*$. $\square$

Recall that for a G-module, $\widetilde{\bigwedge_G^s} M$ denotes the image of $\bigwedge_G^s M$ via the canonical morphism

$$\bigwedge_G^s M \longrightarrow \mathbb{Q}\bigwedge_G^s M \ .$$

Using the properties of det and the fact that the category of $\mathbb{Q}[G]$-modules is semi-simple, we obtain the following lemma

Lemma 3.6. *Let M and N be $\mathbb{Z}[G]$-lattices, such that the Sinnott index $(M:N)$ is defined. Then, we have*

$$(M:N) = (\widetilde{\bigwedge_G^s} M : \widetilde{\bigwedge_G^s} N),$$

where s is maximal.

Proof. Exercise . $\square$

Definition 3.7. *Let M be a $\mathbb{Z}[G]$-lattice. We denote by $S(M)$ the semi-simplified of M. It is the smallest module completely decomposable containing M, and definite by*

$$S(M) := \bigoplus_{\chi \in \mathcal{X}} e_\chi M \subset \mathbb{Q}M$$

where $\mathcal{X}$ is the set of all irreducible characters of G over $\mathbb{Q}$.

Note that the index of M in $S(M)$ is finite. Indeed, let $g = |G|$. Since $gS(M) \subset M$ and M is a finitely generated module, we get

$$(S(M):M) \mid g^{\operatorname{rank}_{\mathbb{Z}}(M)}.$$

To go further, we need some notations. For any subextension F of K/k, we put

$$c_F = \frac{(S(\lambda_K(U_{S_\infty}(K)^{N_H})) : \lambda_K(U_{S_\infty}(K)^{N_H}))}{(S(X(K)^{N_H}) : X(K)^{N_H})}.|\widehat{H}^0(H, U_{S_\infty}(K))|^{-1} \tag{2}$$

and

$$c_{K,r} = \frac{(S(e_{S,r}\lambda_K(U_S(K)) : e_{S,r}\lambda_K(U_S(K))}{(S(e_{S,r}X(K)) : e_{S,r}X(K))} \tag{3}$$

where $H = \operatorname{Gal}(K/F)$ and $N_H = \sum_{\sigma \in H} \sigma$.

The following proposition is crucial for our purpose.

Proposition 3.8.

$$(e_{S,r}\mathbb{Z}[G] : R_w(e_{S,r}\widetilde{\bigwedge_G^r} U_{S,T}(K))) = \mathrm{Reg}_K c_{K,r}.c_K^{-1} \prod_{I\subset\{1,\cdots,|\mathrm{Ram}(K/k)|\}} c_{K_I}^{(-1)^{|I|+1}} \mathrm{Reg}_{K_I}^{(-1)^{|I|}}.$$

Proof. Let $S = S_\infty \cup V$ and let $\mathcal{L}_{S,\infty}$ the map defined in (1). The facts that $e_{S,r}\mathbb{R}V_K = 0$ ($|S| > r+1$) and that the map

$$e_{S,r}\mathcal{L}_{S,\infty} : e_{S,r}\mathbb{R}U_{S,T}(K) \xrightarrow{e_{S,r}\mathcal{L}_S} e_{S,r}\mathbb{R}S_K \xrightarrow{\mathrm{id}} e_{S,r}\mathbb{R}S_{\infty,K} := e_{s,r}X(K)$$

is an isomoprhism, show that

$$(e_{S,r}X(K) : e_{S,r}\mathcal{L}_{S,\infty}(e_{S,r}U_{S,T}(K))) = \det(e_{S,r}\mathcal{L}_S).$$

Then, using the facts

$$\begin{aligned}
\Big(e_{S,r}X(K) : e_{S,r}\mathcal{L}_{S,\infty}(e_{S,r}U_{S,T}(K))\Big) &= \Big(e_{S,r}\mathbb{Z}S_{\infty,K} : e_{S,r}\mathcal{L}_{S,\infty}(e_{S,r}U_{S,T}(K))\Big)\\
&= \Big(e_{S,r}\widetilde{\bigwedge_G^r}\mathbb{Z}S_{\infty,K} : e_{S,r}\bigwedge^r \mathcal{L}_{S,\infty}(e_{S,r}\widetilde{\bigwedge_G^r}U_{S,T}(K))\Big)\\
&= \Big(e_{S,r}\mathbb{Z}[G](w_1\wedge\ldots\wedge w_r) : \mathrm{R}_w(e_{S,r}\widetilde{\bigwedge_G^r}U_{S,T}(K))(w_1\wedge\ldots\wedge w_r)\Big)\\
&= \Big(e_{S,r}\mathbb{Z}[G] : \mathrm{R}_w(e_{S,r}\widetilde{\bigwedge_G^r}U_{S,T}(K)\Big)\\
&= \Big(e_{S,r}\mathbb{Z}[G] : R_w(e_{S,r}\widetilde{\bigwedge_G^r}U_{S,T}(K))\Big),
\end{aligned}$$

we obtain $\Big(e_{S,r}\mathbb{Z}[G] : R_w(e_{S,r}\widetilde{\bigwedge_G^r}U_{S,T}(K))\Big) = \det(e_{S,r}\mathcal{L}_S)$. Therefore, using lemma 3.5, we get

$$\begin{aligned}
(e_{S,r}X(K) : e_{S,r}\mathcal{L}_{S,\infty}(e_{S,r}U_{S,T}(K))) &= (e_{S,r}X(K) : e_{S,r}\lambda_K(U_{S_\infty}(K))\\
&= c_{K,r}.(S(e_{S,r}X(K)) : S(e_{S,r}\lambda_K(U_{S_\infty}(K))\\
&= c_{K,r}.\prod_{\substack{\chi\in\widehat{G}\\ r_S(\chi)=r}} (e_\chi X(K) : e_\chi\lambda_K(U_{S_\infty}(K))
\end{aligned}$$

Let F be a subextension of K/k. On the one hand, the commutative diagram

$$\begin{tikzcd}
\mathbb{C}U_{S_\infty}(F) \arrow[r, "\lambda_F"] \arrow[d, "i"', "\wr"] & \mathbb{C}X(F) \arrow[d, "j", "\wr"'] \\
\mathbb{C}U_{S_\infty}(K)^H \arrow[r, "\lambda_K"] & \mathbb{C}X(K)^H
\end{tikzcd}$$

shows that

$$\mathrm{Reg}_F = (X(K)^H : \lambda_K(U_{S_\infty}(K)^H)).(X(K)^H : j(X(F)))^{-1}.(U_{S_\infty}(K)^H : i(U_{S_\infty}(F)))$$

where

- $j(w_F) := \sum_{w|w_F}[K_w : F_{w_F}]w = N_H w_K$, where $w_K \mid w_F$ is a place of K laying above w_F
- $i(x) = x$.

Since i is injective and $j(X(F)) = N_H(X(K))$, we obtain

$$\mathrm{Reg}_F = |\widehat{H}^0(H, X(K))|^{-1}.(X(K)^H : \lambda_K(U_{S_\infty}(K))^H).$$

Using the fact that $U_{S_\infty}(K) \xrightarrow{\lambda_K} \mathbb{R}X(K)$ is injective as G-module, we get

$$(X(K)^H : \lambda_K(U_{S_\infty}(K))^H).|\widehat{H}^0(H, U_{S_\infty}(K))| = (X(K)^{N_H} : \lambda_K(U_{S_\infty}(K))^{N_H}).|\widehat{H}^0(H, X(K))|.$$

It follows that

$$\begin{aligned}\mathrm{Reg}_F &= |\widehat{H}^0(H, U_{S_\infty}(K))|^{-1}.(X(K)^{N_H} : \lambda_K(U_{S_\infty}(K))^{N_H}) \\ &= c_F.(S(X(K)^{N_H}) : S(\lambda_K(U_{S_\infty}(K)^{N_H}))).\end{aligned}$$

where c_F is defined in (2). On the other hand, for any $\widetilde{\chi} \in \widehat{\mathrm{Gal}(F/k)}$, we have

$$\begin{aligned}(e_{\widetilde{\chi}}X(K)^{N_H} : e_{\widetilde{\chi}}\lambda_K(U_{S_\infty}(K))^{N_H}) &= (|H|e_{\widetilde{\chi}o\pi_F}X(K) : |H|e_{\widetilde{\chi}o\pi_F}\lambda_K(U_{S_\infty}(K)) \\ &= (e_{\widetilde{\chi}o\pi_F}X(K) : e_{\widetilde{\chi}o\pi_F}\lambda_K(U_{S_\infty}(K)).\end{aligned}$$

Then

$$\mathrm{Reg}_F = c_F. \prod_{\substack{\chi \in \widehat{G} \\ \chi(\mathrm{Gal}(K/F))=1}} (e_\chi X(K) : e_\chi \lambda_K(U_{S_\infty}(K)).$$

Therefore, a simple inclusion-exclusion argument gives

$$\prod_{\substack{\chi \in \widehat{G} \\ r_S(\chi)=r}} (e_\chi X(K) : e_\chi \lambda_K(U_{S_\infty}(K))) = c_K^{-1}\mathrm{Reg}_K \prod_{I \subset \{1,\cdots,|\mathrm{Ram}(K/k)|\}} c_{K_I}^{(-1)^{|I|+1}} \mathrm{Reg}_{K_I}^{(-1)^{|I|}}$$

Finally

$$(e_{S,r}\mathbb{Z}[G] : R_w(e_{S,r}\widetilde{\bigwedge_G^r} U_{S,T}(K))) = c_{K,r}.c_K^{-1}\mathrm{Reg}_K \prod_{I \subset \{1,\cdots,|\mathrm{Ram}(K/k)|\}} c_{K_I}^{(-1)^{|I|+1}} \mathrm{Reg}_{K_I}^{(-1)^{|I|}}.$$

□

We prove now Theorem 1.4

Theorem 1.4. The Sinnott index $(e_{S,r}\bigcap_G^r U_{S,T}(K) : e_{S,r}\mathrm{St}_{K,T})$ is finite, and we have

$$[e_{S,r}\bigcap_G^r U_{S,T}(K) : e_{S,r}\mathrm{St}_{K,T}] = h_K.(e_{S,r}\mathbb{Z}[G] : e_{S,r}U_K^{(r)}).(e_{S,r}\bigcap_G^r U_{S,T}(K) : e_{S,r}\widetilde{\bigwedge_G^r} U_{S,T}(K)).\beta_K.$$

where

$$\beta_K = c_K c_{K,r}^{-1} \prod_{I \subset \{1,..,|\mathrm{Ram}(K/k)|\}} c_{K_I}^{(-1)^{|I|}} h_{K_I}^{(-1)^{|I|}}.$$

Proof. We begin by the expression obtained in Corollary 3.4 and analyse each term. We have

$$(e_{S,r}U_K^{(r)} : \omega_K e_{S,r}U_K^{(r)}) =| \det(m_{\omega_K}) |$$

where $\omega_K := \Sigma_{\chi \in \hat{G}, r_S(\chi)=r} L^{(r)}(0, \hat{\chi})e_{\chi^{-1}}$ and m_{ω_K} is the multiplication by w_K. Since the set $\{e_\chi, r_S(\chi) = r\}$ is an $\mathbb{R}$-base of the vector space $e_{S,r}\mathbb{R}[G]$, and $e_{S,r}U_K^{(r)}$ is a lattice of it,

$$\det(m_{\omega_K}) = \prod_{\chi \in \hat{G}, r_S(\chi)=r} L^{(r)}(0, \hat{\chi})$$

A simple inclusion-exclusion argument gives

$$\prod_{\chi \in \hat{G}, r_S(\chi)=r} L^{(r)}(0, \hat{\chi}) = \zeta_K^*(0) \prod_{I \subset \{1,..,|\mathrm{Ram}(K/k)|\}} \zeta_{K_I}^*(0)^{(-1)^{|I|}}$$

where $\zeta^*_{K_I}(0)$ is the first non trivial term in the Taylor expansion of the function $\zeta_{K_I}(s)$ at 0 given by

$$\zeta^*_{K_I}(0) := \lim_{s\to 0} s^{-\mathrm{ord}_{s=0}(\zeta_{K_I}(s))}\zeta_{K_I}(s)$$

Recall the following well known class number formula (see e.g. [6, Corollaire I.1.2])

$$\zeta^*_{K_I}(0) = -\frac{h_{K_I}.\mathrm{Reg}_{K_I}}{|\mu(K_I)|}$$

This formula combined with the previous work gives

$$(e_{S,r}U_K^{(r)} : \omega_K e_{S,r}U_K^{(r)}) = h_K \mathrm{Reg}_K \prod_{I\subset\{1,..,|\mathrm{Ram}(K/k)|\}} h_{K_I}^{(-1)^{|I|}} \mathrm{Reg}_{K_I}^{(-1)^{|I|}}.$$

Using Proposition 3.8 and Corollary 3.4, we get

$$(e_{S,r}\bigcap_G^r U_{S,T}(K) : e_{S,r}\mathrm{St}_{K,T}) = h_K.(e_{S,r}\mathbb{Z}[G] : e_{S,r}U_K^{(r)}).(e_{S,r}\bigcap_G^r U_{S,T}(K) : e_{S,r}\widetilde{\bigwedge_G^r} U_{S,T}(K)).\beta_K.$$

where

$$\beta_K = c_K c_{K,r}^{-1} \prod_{I\subset\{1,..,|\mathrm{Ram}(K/k)|\}} c_{K_I}^{(-1)^{|I|}} h_{K_I}^{(-1)^{|I|}}. \tag{4}$$

□

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