Operads for algebraic quantum field theory

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Abstract

We construct a colored operad whose category of algebras is canonically isomorphic to the category of algebraic quantum field theories. This is achieved by a construction that depends on the choice of a category, whose objects provide the operad colors, equipped with an additional structure that we call an orthogonality relation. This allows us to describe different types of quantum field theories, including theories on a fixed Lorentzian manifold, locally covariant theories and also chiral conformal and Euclidean theories. Moreover, because the colored operad depends functorially on the orthogonal category, we obtain adjunctions between categories of different types of quantum field theories. These include novel and physically very interesting constructions, such as time-slicification and local-to-global extensions of quantum field theories. We compare the latter to Fredenhagen's universal algebra.

Keywords: algebraic quantum field theory, locally covariant quantum field theory, colored operads, change of color adjunctions, Fredenhagen's universal algebra

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Contents

1 Introduction and summary

Algebraic quantum field theory [HK64] is a conceptually clear axiomatic framework to define and investigate quantum field theories on Lorentzian spacetimes from a model-independent perspective. Combining the core principles of quantum theory and relativity, it describes a quantum field theory on a spacetime M in terms of a coherent assignment $M \supseteq U \mapsto \mathfrak{A}(U)$ of associative and unital algebras to suitable spacetime regions. $\mathfrak{A}(U)$ is interpreted as the algebra of quantum observables of the theory that can be measured in the region $U \subseteq M$. Given two spacetime regions U and V such that $U \subseteq V \subseteq M$, there is an algebra homomorphism $\mathfrak{A}(U) \to \mathfrak{A}(V)$ mapping observables in the smaller region U to the bigger region V. These maps are required to be coherent in the sense that $\mathfrak{A}$ is a pre-cosheaf on a suitable category of regions in the spacetime M. Moreover, given two causally disjoint regions U_1 and U_2 of some $V \subseteq M$, i.e. no causal curve in V links U_1 and U_2, the elements of $\mathfrak{A}(U_1)$ and $\mathfrak{A}(U_2)$ are required to commute within $\mathfrak{A}(V)$. This crucial property is called the *Einstein causality axiom* and it formalizes the physical principle that "nothing should propagate faster than light".

The traditional framework [HK64] of algebraic quantum field theory on a fixed Lorentzian spacetime M may be generalized and adapted in order to capture also other flavors of quantum field theory. For example, instead of focusing on regions in a fixed Lorentzian manifold M, one may also consider the category of *all* (globally hyperbolic) Lorentzian manifolds $\mathbf{Loc}$. This leads to quantum field theories defined coherently on *all* spacetimes, which are called locally covariant quantum field theories [BFV03, FV15]. Furthermore, there are also algebraic approaches to chiral conformal quantum field theory [Kaw15, Reh15, BDH15] and Euclidean quantum field theory [Sch99], where Lorentzian spacetimes are replaced respectively by intervals in the circle $\mathbb{S}^1$ or by Riemannian manifolds. The Einstein causality axiom, which is a concept intrinsic to Lorentzian geometry, is modified in such scenarios to the requirement that observables associated to disjoint regions $U_1 \cap U_2 = \emptyset$ commute.

From a more abstract point of view, one observes that all these flavors of algebraic quantum field theory have the following common features: There is a category $\mathbf{C}$ whose objects are the "spacetimes" of interest and whose morphisms describe the admissible "spacetime embeddings". In this category we single out a distinguished subset $\perp \subseteq \mathrm{Mor}\,\mathbf{C}\,{}_{\mathrm{t}}\!\times_{\mathrm{t}} \mathrm{Mor}\,\mathbf{C}$, which we call an *orthogonality relation* (cf. Definition 4.3), formed by certain pairs of morphisms

$$c_1 \xrightarrow{\;f_1\;} c \xleftarrow{\;f_2\;} c_2 \tag{1.1}$$

with the same target. Notice that for Lorentzian theories $\perp$ is characterized by causal disjointness and for chiral conformal or Euclidean theories by disjointness. We shall call elements of this subset *orthogonal pairs* of morphisms and write $f_1 \perp f_2$. We call the pair $\overline{\mathbf{C}} := (\mathbf{C}, \perp)$ consisting of a category $\mathbf{C}$ and an orthogonality relation $\perp$ an *orthogonal category*. The role of the orthogonal category $\overline{\mathbf{C}}$ is thus to specify the flavor or type of quantum field theory one would like to study. A quantum field theory on $\overline{\mathbf{C}}$ is then described by a functor $\mathfrak{A} : \mathbf{C} \to \mathbf{Alg}$ to the category of associative and unital algebras, which satisfies the $\perp$-*commutativity axiom*: For every orthogonal pair $f_1 \perp f_2$, the induced commutator

$$\big[-,-\big]_{\mathfrak{A}(c)} \circ \big(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)\big) \;:\; \mathfrak{A}(c_1) \otimes \mathfrak{A}(c_2) \;\longrightarrow\; \mathfrak{A}(c) \tag{1.2}$$

is equal to the zero map. Hence, the category of all quantum field theories on $\overline{\mathbf{C}}$ is the full subcategory of the functor category $\mathbf{Alg}^{\mathbf{C}}$ consisting of all functors satisfying the $\perp$-commutativity axiom.

The aim of this paper is to develop a more elegant and powerful description of the category of all quantum field theories on $\overline{\mathbf{C}}$ by using techniques from operad theory. Loosely speaking, operads are mathematical structures that encode n-ary operations and their composition properties on an abstract level. Let us illustrate what this means by a simple analogy: In ordinary algebra, one distinguishes between the abstract concept of an associative and unital algebra A and the concrete concept of the algebra of endomorphisms $\mathrm{End}(V)$ of some vector space V. These two concepts are linked by representations $\rho : A \to \mathrm{End}(V)$, which realize the abstract algebra elements $a \in A$ concretely as linear maps $\rho(a) : V \to V$ on V. Operads play a similar role, however at one level deeper. An abstract associative and unital algebra A is specified by its underlying vector space and its n-fold product maps $\mu^n : A^{\otimes n} \to A\,,~ a_1 \otimes \cdots \otimes a_n \mapsto a_1 \cdots a_n$ (note that $\mu^0 : \mathbb{K} \to A\,,~ k \mapsto k\,1$ is the unit), which satisfy the obvious composition properties. These n-ary operations and their composition properties can be encoded abstractly in an operad, called the associative operad. Our A is then a particular "representation", called an algebra in operad theory, of this operad.

We shall construct, for each orthogonal category $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$, a colored operad $\mathcal{O}_{\overline{\mathbf{C}}}$ whose category of algebras $\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ is canonically isomorphic to the category of quantum field theories on $\overline{\mathbf{C}}$, i.e. the category of $\perp$-commutative functors from $\mathbf{C}$ to $\mathbf{Alg}$. Therefore, in the spirit of the analogy explained above, we succeed in identifying and extracting the abstract algebraic structures underlying algebraic quantum field theory. It is worth emphasizing very clearly the key advantage of our novel operadic perspective in comparison to the traditional functor perspective: In the functor approach, $\perp$-commutativity is an additional *property* that a functor $\mathfrak{A} : \mathbf{C} \to \mathbf{Alg}$ may or may not satisfy. In contrast to that, in our operadic approach every algebra over the colored operad $\mathcal{O}_{\overline{\mathbf{C}}}$ satisfies the $\perp$-commutativity axiom because it is part of the *structure* that is encoded in the operad $\mathcal{O}_{\overline{\mathbf{C}}}$. Below we shall comment more on the crucial difference between property and structure and on the resulting advantages of our operadic approach to algebraic quantum field theory.

We will also prove that the assignment $\overline{\mathbf{C}} \mapsto \mathcal{O}_{\overline{\mathbf{C}}}$ of our colored operad to an orthogonal category is functorial. This means that for every orthogonal functor $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$, i.e. a functor preserving the orthogonality relations, there is an associated colored operad morphism $\mathcal{O}_F : \mathcal{O}_{\overline{\mathbf{C}}} \to \mathcal{O}_{\overline{\mathbf{D}}}$. This morphism induces an adjunction

$$\mathcal{O}_{F\,!} \,:\, \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{C}}}\big) \; \rightleftarrows \; \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{D}}}\big) \,:\, \mathcal{O}_F^{*} \tag{1.3}$$

between the corresponding categories of algebras, which allows us to relate the quantum field theories on $\overline{\mathbf{C}}$ to those on $\overline{\mathbf{D}}$. Hence, our operadic approach does not only provide us with powerful tools to describe the categories of quantum field theories of a fixed kind, but it also introduces novel techniques to connect and compare different types of theories. We will show that these include some novel and physically very interesting constructions. For example: (1) Using localizations of orthogonal categories, we obtain adjunctions that should be interpreted physically as *time-slicification*. This means that we can assign to theories that do not necessarily satisfy the time-slice axiom — a kind of dynamical law in Lorentzian quantum field theories — theories that do. (2) Using full orthogonal subcategory embeddings $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$, where $\overline{\mathbf{D}}$ are the "spacetimes" of interest and $\overline{\mathbf{C}}$ particularly "nice spacetimes" in $\overline{\mathbf{D}}$, we obtain an adjunction that should be interpreted physically as a *local-to-global extension* of quantum field theories from $\overline{\mathbf{C}}$ to $\overline{\mathbf{D}}$. In spirit, this is similar to Fredenhagen's universal algebra construction [Fre90, Fre93, FRS92], which is formalized as a left Kan extension of the functor underlying a quantum field theory [Lan14]. There is however one major difference: Left Kan extensions in general do not preserve the $\perp$-commutativity *property* of a functor, i.e. it is unclear whether the prescription of [Fre90, Fre93, FRS92, Lan14] succeeds in defining a quantum field theory on $\overline{\mathbf{D}}$. In stark contrast to that, our operadic version of the local-to-global extension does always define quantum field theories on $\overline{\mathbf{D}}$ because the $\perp$-commutativity axiom is encoded as a *structure* in our colored operad. We will study this particularly important example of an adjunction in detail in this paper and hope that it convinces the reader that our operadic framework is very useful for quantum field theoretic applications.

Our original motivation for developing an operadic approach to algebraic quantum field theory came from *homotopical algebraic quantum field theory* [BSS15, BS17, BSS17]. This is a longer-term research program of two of us (M.B. and A.S.), whose goal is to combine algebraic quantum field theory with techniques from homotopical algebra in order to capture the crucial higher categorical structures that are present in quantum gauge theories. While studying toy-models of such theories in [BS17], we observed that both the functorial structure and $\perp$-commutativity are in general only realized up to homotopy in homotopical algebraic quantum field theory. Combining the operadic approach we develop in the present paper with homotopical algebra will lead to a precise framework to describe algebraic quantum field theories up to coherent homotopies: These will be algebras over the colored operad $\mathcal{O}_{\overline{\mathbf{C}}}^{\infty}$ that is obtained by a cofibrant replacement of our operad $\mathcal{O}_{\overline{\mathbf{C}}}$, see e.g. [BM07, LV12]. We expect to address these points in detail in future works.

Finally, we would like to comment briefly on the relationship between our approach and the factorization algebra approach of Costello and Gwilliam [CG17]. From a superficial point of view, the two frameworks appear very similar as they both employ colored operads to encode the algebraic structures underlying quantum field theories. However, the factorization algebra operad is quite different from our family of colored operads $\mathcal{O}_{\overline{\mathbf{C}}}$ as it captures only the multiplication of those observables that are localized in disjoint spacetime regions. It is presently unclear to us if there is a way to relate factorization algebras and algebraic quantum field theory, for example by establishing maps between the relevant colored operads and analyzing the induced adjunctions between their associated categories of algebras. We hope to come back to this point in a future work.

The outline of the remainder of this paper is as follows: In Section 2 we fix our notation and review some definitions and constructions from category theory and categorical algebra that are crucial for our work. Section 3 provides a self-contained summary of the theory of colored operads and their algebras. Most of these results are well-known to experts in this field, however they often appear scattered throughout the literature or are proven under certain assumptions, e.g. for uncolored operads, that are too restrictive for our purpose. We therefore decided to include a concise summary of the relevant techniques from colored operad theory. Hopefully this will be very useful for readers without an operadic background to understand and follow the arguments and constructions in our paper. In Section 4 we construct and study our family of colored operads $\mathcal{O}_{\overline{\mathbf{C}}}$, where $\overline{\mathbf{C}}$ is an orthogonal category. In particular, we provide two different constructions of $\mathcal{O}_{\overline{\mathbf{C}}}$, a direct definition and a presentation by generators and relations. We then prove that the category $\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ of algebras over $\mathcal{O}_{\overline{\mathbf{C}}}$ is canonically isomorphic to the category of quantum field theories on $\overline{\mathbf{C}}$. Moreover, we provide concrete examples of orthogonal categories $\overline{\mathbf{C}}$ that are relevant for algebraic quantum field theory. In Section 5 we study in detail the properties of the adjunctions (1.3) that are induced by orthogonal functors $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$. We show that these include physically very interesting constructions such as time-slicification and local-to-global extensions of quantum field theories. In Section 6 we compare our operadic local-to-global extension to Fredenhagen's universal algebra construction, which is given by left Kan extension of the functor underlying a quantum field theory. The general result is that, whenever the left Kan extension yields a $\perp$-commutative functor, then it coincides with our operadic construction. We shall provide examples when this is the case, but also counterexamples for which Fredenhagen's construction yields a functor that is not $\perp$-commutative.

2 Categorical preliminaries

We briefly recall some standard tools from category theory. For a more thorough treatment we refer to [MacL98] and [Bor94a, Bor94b]. A detailed discussion of ends and coends is available in [Lor15], see also [FS16] for their applications to quantum field theory.

2.1 Closed symmetric monoidal categories

Recall that a *monoidal category* $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of a category $\mathbf{C}$, a functor $\otimes : \mathbf{C}\times\mathbf{C} \to \mathbf{C}$ (called *tensor product*), an object $I \in \mathbf{C}$ (called *unit object*) and three natural isomorphisms (called *associator* and *left/right unitor*)

$$\alpha \,:\, (c_1 \otimes c_2) \otimes c_3 \;\cong\; c_1 \otimes (c_2 \otimes c_3) \quad , \qquad \lambda \,:\, I \otimes c \;\cong\; c \quad , \qquad \rho \,:\, c \otimes I \;\cong\; c \quad , \tag{2.1}$$

which satisfy certain coherence conditions (*pentagon* and *triangle identities*). We follow the usual abuse of notation and denote a monoidal category by its underlying category $\mathbf{C}$.

A *symmetric monoidal category* is a monoidal category $\mathbf{C}$ together with a natural isomorphism (called *braiding*)

$$\tau \,:\, c_1 \otimes c_2 \;\cong\; c_2 \otimes c_1 \quad , \tag{2.2}$$

which satisfies certain coherence conditions (*hexagon identities*) and the *symmetry constraint* $\tau^2 = \mathrm{id}$. We shall also drop the braiding from our notation and simply denote a symmetric monoidal category by its underlying category $\mathbf{C}$.

A monoidal category $\mathbf{C}$ is called *right closed* if the functor $(-) \otimes c : \mathbf{C} \to \mathbf{C}$ admits a right adjoint, denoted by $[c, -] : \mathbf{C} \to \mathbf{C}$, for every $c \in \mathbf{C}$. Explicitly, this means that there exists a natural bijection

$$\mathbf{C}\big(c_1 \otimes c, c_2\big) \;\cong\; \mathbf{C}\big(c_1, [c, c_2]\big) \quad , \tag{2.3}$$

where $\mathbf{C}(-,-) : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$ denotes the Hom-functor. On the other hand, $\mathbf{C}$ is called *left closed* if the functor $c \otimes (-) : \mathbf{C} \to \mathbf{C}$ admits a right adjoint, for every $c \in \mathbf{C}$. Notice that for a symmetric monoidal category left and right closedness are equivalent. As a consequence, we may define a *closed symmetric monoidal category* to be a symmetric monoidal category $\mathbf{C}$ that is, say, right closed with *internal-hom functor* $[-,-] : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$.

The reader who feels unfamiliar with closed symmetric monoidal categories should always think of the following examples.

Example 2.1. The category $\mathbf{Set}$ of sets can be equipped with the structure of a closed symmetric monoidal category. Concretely, $\otimes = \times$ is the Cartesian product of sets, $I = \{*\}$ is any singleton set, $\tau : S \times T \to T \times S\,,~(s,t) \mapsto (t,s)$ is the flip map and $[S,T] = \mathbf{Set}(S,T)$ is the set of maps from S to T. $\triangledown$

Example 2.2. Let $\mathbb{K}$ be a field. The category $\mathbf{Vec}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$ can be equipped with the structure of a closed symmetric monoidal category. Concretely, $\otimes$ is the ordinary tensor product of vector spaces, $I = \mathbb{K}$ is the 1-dimensional vector space, $\tau : V \otimes W \to W \otimes V\,,~v \otimes w \mapsto w \otimes v$ is the flip map and $[V,W]$ is the vector space of linear maps from V to W. $\triangledown$

Let now $\mathbf{C}$ and $\mathbf{D}$ be two monoidal categories. A *lax monoidal functor* (F, F_2, F_0) from $\mathbf{C}$ to $\mathbf{D}$ consists of a functor $F : \mathbf{C} \to \mathbf{D}$, a natural transformation

$$F_2 \,:\, F(c) \otimes_{\mathbf{D}} F(c') \;\longrightarrow\; F(c \otimes_{\mathbf{C}} c') \tag{2.4a}$$

and a $\mathbf{D}$-morphism

$$F_0 \,:\, I_{\mathbf{D}} \longrightarrow F(I_{\mathbf{C}}) \quad , \tag{2.4b}$$

which satisfy certain coherence conditions involving the associators and unitors, see e.g. [MacL98, Chapter XI.2]. A *strong monoidal functor* is a lax monoidal functor (F, F_2, F_0) for which F_2 is a natural isomorphism and F_0 is an isomorphism. If both $\mathbf{C}$ and $\mathbf{D}$ are symmetric monoidal categories, a lax monoidal functor (F, F_2, F_0) is called a *symmetric lax monoidal functor* if F_2 is compatible with the braidings, i.e. the diagrams

$$\begin{tikzcd}
F(c) \otimes_{\mathbf{D}} F(c') \arrow[r, "\tau_{\mathbf{D}}"] \arrow[d, "F_2"'] & F(c') \otimes_{\mathbf{D}} F(c) \arrow[d, "F_2"] \\
F(c \otimes_{\mathbf{C}} c') \arrow[r, "F(\tau_{\mathbf{C}})"'] & F(c' \otimes_{\mathbf{C}} c)
\end{tikzcd} \tag{2.5}$$

in $\mathbf{D}$ commute, for all $c, c' \in \mathbf{C}$.

The dual concept of a lax monoidal functor is called an *oplax monoidal functor*. Concretely, this is a triple (F, F_2, F_0) consisting of a functor $F : \mathbf{C} \to \mathbf{D}$, a natural transformation

$$F_2 \,:\, F(c \otimes_{\mathbf{C}} c') \;\longrightarrow\; F(c) \otimes_{\mathbf{D}} F(c') \tag{2.6a}$$

and a $\mathbf{D}$-morphism

$$F_0 \,:\; F(I_{\mathbf{C}}) \longrightarrow I_{\mathbf{D}} \quad , \tag{2.6b}$$

going in the opposite direction as in (2.4), which satisfy similar coherence conditions as lax monoidal functors (obtained by reversing all arrows associated to F_2 and F_0). The notion of *symmetric oplax monoidal functors* is similar to that of symmetric lax monoidal functors by reversing the vertical arrows in (2.5).

Note that being a lax or oplax monoidal functor is a structure that a functor may or may not be endowed with. Symmetry, however, is a property that may or may not hold true with respect to specified braidings. The following is proven e.g. in [AM10, Propositions 3.84 and 3.85].

Proposition 2.3. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two monoidal categories and let*

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : G \tag{2.7}$$

be a pair of adjoint functors with F the left adjoint and G the right adjoint functor.

(i) Any lax monoidal structure (G_2, G_0) on $G : \mathbf{D} \to \mathbf{C}$ canonically induces an oplax monoidal structure (F_2, F_0) on $F : \mathbf{C} \to \mathbf{D}$. Additionally, if $\mathbf{C}$ and $\mathbf{D}$ are symmetric monoidal categories and (G, G_2, G_0) is symmetric lax monoidal, then (F, F_2, F_0) is symmetric oplax monoidal.

(ii) Vice versa, any oplax monoidal structure (F_2, F_0) on $F : \mathbf{C} \to \mathbf{D}$ canonically induces a lax monoidal structure (G_2, G_0) on $G : \mathbf{D} \to \mathbf{C}$. Additionally, if $\mathbf{C}$ and $\mathbf{D}$ are symmetric monoidal categories and (F, F_2, F_0) is symmetric oplax monoidal, then (G, G_2, G_0) is symmetric lax monoidal.

2.2 Ends and coends

Let $\mathbf{D}$ be a small category and $\mathbf{C}$ a complete and cocomplete category, i.e. all (small) limits and colimits exist in $\mathbf{C}$. Consider two functors $F, G : \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{C}$. A *dinatural transformation* $\alpha : F \overset{..}{\longrightarrow} G$ from F to G consists of a family of $\mathbf{C}$-morphisms $\alpha_d : F(d,d) \to G(d,d)$, for all $d \in \mathbf{D}$, such that for all $\mathbf{D}$-morphisms $f : d \to d'$ the diagram

$$\begin{tikzcd} F(d',d) \arrow[r, "F(f,d)"] \arrow[d, "F(d',f)"'] & F(d,d) \arrow[r, "\alpha_d"] & G(d,d) \arrow[d, "G(d,f)"] \\ F(d',d') \arrow[r, "\alpha_{d'}"'] & G(d',d') \arrow[r, "G(f,d')"'] & G(d,d') \end{tikzcd} \tag{2.8}$$

in $\mathbf{C}$ commutes.

For any object $c \in \mathbf{C}$, we may define a constant functor $\Delta(c) : \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{C}$. Explicitly, $\Delta(c)$ acts on objects as $(d,d') \mapsto c$ and on morphisms as $(f,f') \mapsto \mathrm{id}_c$. Given any functor $F : \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{C}$, a *wedge* for F is a pair (c, α) consisting of an object $c \in \mathbf{C}$ and a dinatural transformation $\alpha : \Delta(c) \overset{..}{\longrightarrow} F$. Dually, a *cowedge* for F is a pair (c, β) consisting of an object $c \in \mathbf{C}$ and a dinatural transformation $\beta : F \overset{..}{\longrightarrow} \Delta(c)$. Graphically, this means that for all $\mathbf{D}$-morphisms $f : d \to d'$ the diagrams

$$\begin{tikzcd} & F(d,d) \arrow[dr, "F(d,f)"] & \\ c \arrow[ur, "\alpha_d"] \arrow[dr, "\alpha_{d'}"'] \arrow[rr, phantom, "\text{(wedge)}"] & & F(d,d') \\ & F(d',d') \arrow[ur, "F(f,d')"'] & \end{tikzcd} \qquad \begin{tikzcd} & F(d,d) \arrow[dr, "\beta_d"] & \\ F(d',d) \arrow[ur, "F(f,d)"] \arrow[dr, "F(d',f)"'] \arrow[rr, phantom, "\text{(cowedge)}"] & & c \\ & F(d',d') \arrow[ur, "\beta_{d'}"'] & \end{tikzcd} \tag{2.9}$$

in $\mathbf{C}$ commute.

Definition 2.4. Let $F : \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{C}$ be a functor.

a) An *end* of F is a universal wedge $\pi : \Delta(\mathrm{end}(F)) \stackrel{..}{\longrightarrow} F$. The object $\mathrm{end}(F) \in \mathbf{C}$ is also denoted by the subscripted integral $\int_{d\in\mathbf{D}} F(d,d)$ or simply by $\int_d F(d,d)$.

b) A *coend* of F is a universal cowedge $\iota : F \stackrel{..}{\longrightarrow} \Delta(\mathrm{coend}(F))$. The object $\mathrm{coend}(F) \in \mathbf{C}$ is also denoted by the superscripted integral $\int^{d\in\mathbf{D}} F(d,d)$ or simply by $\int^d F(d,d)$.

Remark 2.5. Because we assume $\mathbf{C}$ to be complete and cocomplete, the end and coend of every functor $F : \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{C}$ exist. As a consequence of their definition by a universal property, different universal (co)wedges of a functor F are isomorphic via a unique isomorphism, hence it is justified to speak of *the* (co)end of a functor F. The universal property for an end of F concretely states that, given any other wedge $\alpha : \Delta(c) \stackrel{..}{\longrightarrow} F$, there exists a unique $\mathbf{C}$-morphism $c \to \int_d F(d,d)$ such that the diagram

$$\begin{tikzcd}
c \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\alpha_d"] \arrow[ddr, bend right, "\alpha_{d'}"'] & & \\
& \int_d F(d,d) \arrow[r, "\pi_d"] \arrow[d, "\pi_{d'}"'] & F(d,d) \arrow[d, "F(d,f)"] \\
& F(d',d') \arrow[r, "F(f,d')"'] & F(d,d')
\end{tikzcd} \tag{2.10}$$

commutes for all $\mathbf{D}$-morphisms $f : d \to d'$. Dually, the universal property for a coend of F concretely states that, given any other cowedge $\beta : F \stackrel{..}{\longrightarrow} \Delta(c)$, there exists a unique $\mathbf{C}$-morphism $\int^d F(d,d) \to c$ such that the diagram

$$\begin{tikzcd}
F(d',d) \arrow[r, "F(f,d)"] \arrow[d, "F(d',f)"'] & F(d,d) \arrow[d, "\iota_d"] \arrow[ddr, bend left, "\beta_d"] & \\
F(d',d') \arrow[r, "\iota_{d'}"'] \arrow[drr, bend right, "\beta_{d'}"'] & \int^d F(d,d) \arrow[dr, dashed, "\exists!"] & \\
& & c
\end{tikzcd} \tag{2.11}$$

commutes for all $\mathbf{D}$-morphisms $f : d \to d'$. $\triangle$

We collect some well-known properties of ends and coends which will be used in our paper. The proofs are relatively straightforward and can be found in the literature, see e.g. [MacL98, Chapters IX.5 and IX.6] and [Lor15, Sections 1 and 2].

Proposition 2.6. *Let $F : \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{C}$ be a functor and $c \in \mathbf{C}$ an object. There exist canonical bijections*

$$\mathbf{C}\Big(\int^d F(d,d), c\Big) ~\cong~ \int_d \mathbf{C}\big(F(d,d), c\big) \tag{2.12a}$$

and

$$\mathbf{C}\Big(c, \int_d F(d,d)\Big) ~\cong~ \int_d \mathbf{C}\big(c, F(d,d)\big) \quad . \tag{2.12b}$$

Theorem 2.7. *Let $\mathbf{E}$ be another small category and let $F : \mathbf{D}^{\mathrm{op}} \times \mathbf{E}^{\mathrm{op}} \times \mathbf{D} \times \mathbf{E} \to \mathbf{C}$ be a functor. There exist canonical isomorphisms*

$$\int_e \int_d F(d,e,d,e) ~\cong~ \int_{(d,e)} F(d,e,d,e) ~\cong~ \int_d \int_e F(d,e,d,e) \tag{2.13a}$$

and

$$\int^e \int^d F(d,e,d,e) ~\cong~ \int^{(d,e)} F(d,e,d,e) ~\cong~ \int^d \int^e F(d,e,d,e) \quad . \tag{2.13b}$$

Because $\mathbf{C}$ is by assumption complete and cocomplete, it is tensored and cotensored over $\mathbf{Set}$. Concretely, given $S \in \mathbf{Set}$ and $c \in \mathbf{C}$, we define

$$S \otimes c := \coprod_{s\in S} c \in \mathbf{C} \quad , \qquad c^S := \prod_{s\in S} c \in \mathbf{C} \quad . \tag{2.14}$$

There exist canonical bijections

$$\mathbf{C}(S \otimes c_1, c_2) \;\cong\; \mathbf{Set}\big(S, \mathbf{C}(c_1, c_2)\big) \;\cong\; \mathbf{C}(c_1, c_2^S) \quad , \tag{2.15}$$

for all $c_1, c_2 \in \mathbf{C}$ and $S \in \mathbf{Set}$. In particular, $S \otimes (-)$ is the left adjoint of $(-)^S$.

Theorem 2.8. *Let $X : \mathbf{D} \to \mathbf{C}$ and $Y : \mathbf{D}^{\mathrm{op}} \to \mathbf{C}$ be functors. There exist canonical isomorphisms*

$$X \;\cong\; \int^d \mathbf{D}(d,-) \otimes X(d) \quad , \tag{2.16a}$$

$$Y \;\cong\; \int^d \mathbf{D}(-,d) \otimes Y(d) \quad , \tag{2.16b}$$

$$X \;\cong\; \int_d X(d)^{\mathbf{D}(-,d)} \quad , \tag{2.16c}$$

$$Y \;\cong\; \int_d Y(d)^{\mathbf{D}(d,-)} \quad . \tag{2.16d}$$

We denote by $\mathbf{C}^{\mathbf{D}}$ the category of all functors $\mathbf{D} \to \mathbf{C}$. The morphisms in $\mathbf{C}^{\mathbf{D}}$ are given by natural transformations $\zeta : X \to Y$ between functors $X, Y : \mathbf{D} \to \mathbf{C}$.

Proposition 2.9. *Let $X, Y : \mathbf{D} \to \mathbf{C}$ be functors. There exists a canonical bijection*

$$\mathbf{C}^{\mathbf{D}}\big(X, Y\big) \;\cong\; \int_d \mathbf{C}\big(X(d), Y(d)\big) \quad . \tag{2.17}$$

2.3 Day convolution

In this subsection we assume that $\mathbf{D}$ is a small symmetric monoidal category and that $\mathbf{C}$ is a closed symmetric monoidal category which is complete and cocomplete. With an abuse of notation, we often denote the symmetric monoidal structures on $\mathbf{D}$ and $\mathbf{C}$ by the same symbols (i.e. $\otimes$, I and τ), as it will be clear from the context which one is meant. The internal-hom on $\mathbf{C}$ is denoted as $[-,-] : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$. Because $\mathbf{C}$ is by hypothesis closed, the tensor product $\otimes$ preserves colimits in both entries and hence in particular coproducts. This implies that there exist natural isomorphisms

$$S \otimes (c_1 \otimes c_2) \;\cong\; (S \otimes c_1) \otimes c_2 \;\cong\; c_1 \otimes (S \otimes c_2) \quad , \tag{2.18}$$

for all $S \in \mathbf{Set}$ and all $c_1, c_2 \in \mathbf{C}$, i.e. we can unambiguously write $S \otimes c_1 \otimes c_2$. We shall also suppress all associators and simply write $c_1 \otimes c_2 \otimes \cdots \otimes c_n$ for multiple tensor products. Notice further that there is a natural isomorphism

$$S_1 \otimes (S_2 \otimes c) \;\cong\; (S_1 \times S_2) \otimes c \quad , \tag{2.19}$$

for all $S_1, S_2 \in \mathbf{Set}$ and $c \in \mathbf{C}$, where $\times$ is the Cartesian product of sets.

Under our hypotheses, there exists a closed symmetric monoidal structure on the functor category $\mathbf{C}^{\mathbf{D}}$, where the tensor product is given by a kind of "convolution product" called the *Day convolution* [Day70]. As we will use this closed symmetric monoidal structure on $\mathbf{C}^{\mathbf{D}}$ in our work, we will briefly review the relevant parts of its construction.

Definition 2.10. The *Day convolution* on $\mathbf{C}^{\mathbf{D}}$ is the functor

$$\otimes_{\mathrm{Day}} \,:\, \mathbf{C}^{\mathbf{D}} \times \mathbf{C}^{\mathbf{D}} \;\longrightarrow\; \mathbf{C}^{\mathbf{D}} \tag{2.20a}$$

defined by

$$X \otimes_{\mathrm{Day}} Y \,:=\, \int^{(d_1,d_2)} \mathbf{D}(d_1 \otimes d_2, -) \otimes X(d_1) \otimes Y(d_2) \quad , \tag{2.20b}$$

for all $X, Y \in \mathbf{C}^{\mathbf{D}}$. The *Day unit* is the object $I_{\mathrm{Day}} \in \mathbf{C}^{\mathbf{D}}$ defined by

$$I_{\mathrm{Day}} \,:=\, \mathbf{D}(I, -) \otimes I \quad . \tag{2.21}$$

Proposition 2.11. $(\mathbf{C}^{\mathbf{D}}, \otimes_{\mathrm{Day}}, I_{\mathrm{Day}})$ *is a monoidal category.*

Proof. It is instructive to sketch some relevant parts of the proof to get familiar with the end/coend calculus. The main ingredients here are the properties of coends recalled in Theorems 2.7 and 2.8. The associator is given by the canonical isomorphisms

$$\begin{aligned}
(X \otimes_{\mathrm{Day}} Y) \otimes_{\mathrm{Day}} Z &= \int^{(d_1,d_2)} \mathbf{D}(d_1 \otimes d_2, -) \otimes (X \otimes_{\mathrm{Day}} Y)(d_1) \otimes Z(d_2) \\
&\cong \int^{(d_1,d_2,d_3,d_4)} \Big(\mathbf{D}(d_1 \otimes d_2, -) \times \mathbf{D}(d_3 \otimes d_4, d_1)\Big) \otimes X(d_3) \otimes Y(d_4) \otimes Z(d_2) \\
&\cong \int^{(d_2,d_3,d_4)} \mathbf{D}(d_3 \otimes d_4 \otimes d_2, -) \otimes X(d_3) \otimes Y(d_4) \otimes Z(d_2) \\
&\cong \int^{(d_1,d_2,d_3,d_4)} \Big(\mathbf{D}(d_3 \otimes d_1, -) \times \mathbf{D}(d_4 \otimes d_2, d_1)\Big) \otimes X(d_3) \otimes Y(d_4) \otimes Z(d_2) \\
&\cong \int^{(d_1,d_3)} \mathbf{D}(d_3 \otimes d_1, -) \otimes X(d_3) \otimes (Y \otimes_{\mathrm{Day}} Z)(d_1) \\
&= X \otimes_{\mathrm{Day}} (Y \otimes_{\mathrm{Day}} Z) \quad .
\end{aligned} \tag{2.22}$$

Similarly, the unitors are given by the canonical isomorphisms

$$\begin{aligned}
I_{\mathrm{Day}} \otimes_{\mathrm{Day}} X &\cong \int^{(d_1,d_2)} \Big(\mathbf{D}(d_1 \otimes d_2, -) \times \mathbf{D}(I, d_1)\Big) \otimes X(d_2) \\
&\cong \int^{d_2} \mathbf{D}(d_2, -) \otimes X(d_2) \;\cong X
\end{aligned} \tag{2.23a}$$

and

$$\begin{aligned}
X \otimes_{\mathrm{Day}} I_{\mathrm{Day}} &\cong \int^{(d_1,d_2)} \Big(\mathbf{D}(d_1 \otimes d_2, -) \times \mathbf{D}(I, d_2)\Big) \otimes X(d_1) \\
&\cong \int^{d_1} \mathbf{D}(d_1, -) \otimes X(d_1) \;\cong X \quad .
\end{aligned} \tag{2.23b}$$

The coherences may be checked by a straightforward, but lengthy calculation. $\square$

Using the braidings τ on $\mathbf{C}$ and $\mathbf{D}$, we can equip the monoidal category $(\mathbf{C}^{\mathbf{D}}, \otimes_{\mathrm{Day}}, I_{\mathrm{Day}})$ with a braiding too. From Definition 2.10, we observe that the Day convolution is the coend of the functor $F : (\mathbf{D} \times \mathbf{D})^{\mathrm{op}} \times \mathbf{D} \times \mathbf{D} \to \mathbf{C}^{\mathbf{D}}$ defined by

$$F(d_1, d_2, d_3, d_4) = \mathbf{D}(d_1 \otimes d_2, -) \otimes X(d_3) \otimes Y(d_4) \quad , \tag{2.24}$$

i.e. $X \otimes_{\mathrm{Day}} Y = \int^{(d_1,d_2)} F(d_1,d_2,d_1,d_2)$. Consider also the functor $\widetilde{F} : (\mathbf{D}\times\mathbf{D})^{\mathrm{op}} \times \mathbf{D}\times\mathbf{D} \to \mathbf{C}^{\mathbf{D}}$ defined by

$$\widetilde{F}(d_1,d_2,d_3,d_4) = \mathbf{D}(d_2\otimes d_1,-)\otimes Y(d_4)\otimes X(d_3) \tag{2.25}$$

and note that its coend is $Y\otimes_{\mathrm{Day}} X = \int^{(d_1,d_2)} \widetilde{F}(d_1,d_2,d_1,d_2)$. We define a natural transformation $\zeta : F \to \widetilde{F}$ by

$$\zeta := \mathbf{D}(\tau,-)\otimes\tau \ :\ F(d_1,d_2,d_3,d_4) \ \longrightarrow\ \widetilde{F}(d_1,d_2,d_3,d_4) \quad . \tag{2.26}$$

Then the braiding

$$\tau_{\mathrm{Day}} \ :\ X\otimes_{\mathrm{Day}} Y \ \cong\ Y\otimes_{\mathrm{Day}} X \tag{2.27}$$

on $(\mathbf{C}^{\mathbf{D}}, \otimes_{\mathrm{Day}}, I_{\mathrm{Day}})$ is induced by ζ and the functoriality of coends.

Proposition 2.12. $(\mathbf{C}^{\mathbf{D}}, \otimes_{\mathrm{Day}}, I_{\mathrm{Day}}, \tau_{\mathrm{Day}})$ *is a symmetric monoidal category. It is further a closed symmetric monoidal category with internal-hom* $[-,-]_{\mathrm{Day}} : \big(\mathbf{C}^{\mathbf{D}}\big)^{\mathrm{op}}\times\mathbf{C}^{\mathbf{D}}\to\mathbf{C}^{\mathbf{D}}$ *given by*

$$[Y,Z]_{\mathrm{Day}} \ =\ \int_d \big[Y(d), Z(-\otimes d)\big] \quad , \tag{2.28}$$

for all $Y,Z\in\mathbf{C}^{\mathbf{D}}$.

Proof. The braiding (2.27) is obviously symmetric, i.e. $\tau_{\mathrm{Day}}^2 = \mathrm{id}$. Proving that (2.28) is an internal-hom, i.e. that $[Y,-]_{\mathrm{Day}}$ is the right adjoint of $(-)\otimes_{\mathrm{Day}} Y$, is a simple calculation using the end/coend calculus, in particular Proposition 2.6, Theorem 2.8 and Proposition 2.9. $\square$

Let us now assume that $\mathbf{E}$ is another small symmetric monoidal category and that $F:\mathbf{D}\to\mathbf{E}$ is a symmetric lax monoidal functor.

Proposition 2.13. *The pullback functor* $F^* : \mathbf{C}^{\mathbf{E}}\to\mathbf{C}^{\mathbf{D}}$ *is canonically symmetric lax monoidal, which endows its left adjoint* $F_! : \mathbf{C}^{\mathbf{D}}\to\mathbf{C}^{\mathbf{E}}$ *with a symmetric oplax monoidal structure in the sense of Proposition 2.3.*

Proof. We first remark that the left adjoint exists by left Kan extension along F. Hence, we only have to show that $F^* : \mathbf{C}^{\mathbf{E}}\to\mathbf{C}^{\mathbf{D}}$ is symmetric lax monoidal, i.e. we have to construct the structure morphisms $F_2^* : F^*(X)\otimes^{\mathbf{D}}_{\mathrm{Day}} F^*(Y)\to F^*(X\otimes^{\mathbf{E}}_{\mathrm{Day}} Y)$, for all $X,Y\in\mathbf{C}^{\mathbf{E}}$, and $F_0^* : I^{\mathbf{D}}_{\mathrm{Day}}\to F^*(I^{\mathbf{E}}_{\mathrm{Day}})$. (To enhance readability, we label the Day convolutions, units and braidings on $\mathbf{C}^{\mathbf{E}}$ and $\mathbf{C}^{\mathbf{D}}$ by the appropriate source category $\mathbf{E}$ or $\mathbf{D}$.)

To define $F_0^* : I^{\mathbf{D}}_{\mathrm{Day}}\to F^*(I^{\mathbf{E}}_{\mathrm{Day}})$, we note that by the Yoneda lemma the set of natural transformations $\mathbf{D}(I_{\mathbf{D}},-)\to\mathbf{E}(I_{\mathbf{E}},F(-))$ is in natural bijection to $\mathbf{E}(I_{\mathbf{E}},F(I_{\mathbf{D}}))$. Hence, the given $\mathbf{E}$-morphism $F_0 : I_{\mathbf{E}}\to F(I_{\mathbf{D}})$ defines a canonical $\mathbf{C}^{\mathbf{D}}$-morphism

$$F_0^* \ :\ I^{\mathbf{D}}_{\mathrm{Day}} = \mathbf{D}(I_{\mathbf{D}},-)\otimes I \ \longrightarrow\ \mathbf{E}\big(I_{\mathbf{E}},F(-)\big)\otimes I = F^*(I^{\mathbf{E}}_{\mathrm{Day}}) \quad . \tag{2.29}$$

To define $F_2^* : F^*(X)\otimes^{\mathbf{D}}_{\mathrm{Day}} F^*(Y)\to F^*(X\otimes^{\mathbf{E}}_{\mathrm{Day}} Y)$, we recall (cf. Definition 2.10) that

$$\big(F^*(X)\otimes^{\mathbf{D}}_{\mathrm{Day}} F^*(Y)\big)(d) = \int^{(d_1,d_2)} \mathbf{D}(d_1\otimes_{\mathbf{D}} d_2, d)\otimes X(F(d_1))\otimes Y(F(d_2)) \quad , \tag{2.30a}$$

$$F^*(X\otimes^{\mathbf{E}}_{\mathrm{Day}} Y)(d) = \int^{(e_1,e_2)} \mathbf{E}(e_1\otimes_{\mathbf{E}} e_2, F(d))\otimes X(e_1)\otimes Y(e_2) \quad . \tag{2.30b}$$

Then F_2^* is induced by the universal property of coends and the morphisms

$$\begin{tikzcd}
\mathbf{D}(d_1 \otimes_{\mathbf{D}} d_2, d) \otimes X(F(d_1)) \otimes Y(F(d_2)) \arrow[d, "F\otimes \mathrm{id}\otimes \mathrm{id}"] \\
\mathbf{E}\big(F(d_1 \otimes_{\mathbf{D}} d_2), F(d)\big) \otimes X(F(d_1)) \otimes Y(F(d_2)) \arrow[d, "\mathbf{E}(F_2, F(d))\otimes \mathrm{id}\otimes \mathrm{id}"] \\
\mathbf{E}\big(F(d_1) \otimes_{\mathbf{E}} F(d_2), F(d)\big) \otimes X(F(d_1)) \otimes Y(F(d_2))
\end{tikzcd} \tag{2.31}$$

for all $d_1, d_2 \in \mathbf{D}$. (In the first step we apply F to the Hom-sets and in the second step we pull back along $F_2 : F(d_1) \otimes_{\mathbf{E}} F(d_2) \to F(d_1 \otimes_{\mathbf{D}} d_2)$.)

To prove symmetry of $F^* : \mathbf{C}^{\mathbf{E}} \to \mathbf{C}^{\mathbf{D}}$ we have to show that the square

$$\begin{tikzcd}
F^*(X) \otimes^{\mathbf{D}}_{\mathrm{Day}} F^*(Y) \arrow[r, "\tau^{\mathbf{D}}_{\mathrm{Day}}"] \arrow[d, "F_2^*"'] & F^*(Y) \otimes^{\mathbf{D}}_{\mathrm{Day}} F^*(X) \arrow[d, "F_2^*"] \\
F^*(X \otimes^{\mathbf{E}}_{\mathrm{Day}} Y) \arrow[r, "F^*(\tau^{\mathbf{E}}_{\mathrm{Day}})"'] & F^*(Y \otimes^{\mathbf{E}}_{\mathrm{Day}} X)
\end{tikzcd} \tag{2.32}$$

in $\mathbf{C}^{\mathbf{D}}$ commutes, for all $X, Y \in \mathbf{C}^{\mathbf{E}}$. (The Day braiding was defined in (2.27).) Both compositions in this diagram are induced by a family of morphisms

$$\begin{tikzcd}
\mathbf{D}(d_1 \otimes_{\mathbf{D}} d_2, d) \otimes X(F(d_1)) \otimes Y(F(d_2)) \arrow[d] \\
\mathbf{E}\big(F(d_2) \otimes_{\mathbf{E}} F(d_1), F(d)\big) \otimes Y(F(d_2)) \otimes X(F(d_1))
\end{tikzcd} \tag{2.33}$$

in $\mathbf{C}$. Moreover, both compositions agree on the second and third tensor factor, where they are simply given by applying the braiding $\tau_{\mathbf{C}}$ of $\mathbf{C}$. On the first tensor factor, we obtain that $F_2^*\, \tau^{\mathbf{D}}_{\mathrm{Day}}$ is the clockwise path and that $F^*(\tau^{\mathbf{E}}_{\mathrm{Day}})\, F_2^*$ is the counterclockwise path in the diagram

$$\begin{tikzcd}
\mathbf{D}(d_1 \otimes_{\mathbf{D}} d_2, d) \arrow[r, "\mathbf{D}(\tau_{\mathbf{D}}, d)"] \arrow[d, "F"'] & \mathbf{D}(d_2 \otimes_{\mathbf{D}} d_1, d) \arrow[d, "F"] \\
\mathbf{E}\big(F(d_1 \otimes_{\mathbf{D}} d_2), F(d)\big) \arrow[r, "\mathbf{E}(F(\tau_{\mathbf{D}}), F(d))"] \arrow[d, "\mathbf{E}(F_2, F(d))"'] & \mathbf{E}\big(F(d_2 \otimes_{\mathbf{D}} d_1), F(d)\big) \arrow[d, "\mathbf{E}(F_2, F(d))"] \\
\mathbf{E}\big(F(d_1) \otimes_{\mathbf{E}} F(d_2), F(d)\big) \arrow[r, "\mathbf{E}(\tau_{\mathbf{E}}, F(d))"] & \mathbf{E}\big(F(d_2) \otimes_{\mathbf{E}} F(d_1), F(d)\big)
\end{tikzcd} \tag{2.34}$$

in **Set**. Hence, it remains to show that this diagram commutes. Indeed, the upper square commutes by functoriality of F and the lower square by symmetry of F. □

2.4 Monoids, monads and algebras

We briefly recall some relevant concepts of categorical algebra which are needed in our work.

Definition 2.14. A *monoid* in a monoidal category $\mathbf{C}$ is an object $M \in \mathbf{C}$ together with two $\mathbf{C}$-morphisms $\mu : M \otimes M \to M$ (called *multiplication*) and $\eta : I \to M$ (called *unit*), such that the diagrams

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[r, "\cong"] \arrow[d, "\mu\otimes \mathrm{id}"'] & M \otimes (M \otimes M) \arrow[r, "\mathrm{id}\otimes\mu"] & M \otimes M \arrow[d, "\mu"] \\
M \otimes M \arrow[rr, "\mu"'] & & M
\end{tikzcd} \tag{2.35a}$$

$$\begin{tikzcd} I \otimes M \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\cong"'] & M \otimes M \arrow[d, "\mu"'] & M \otimes I \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl, "\cong"] \\ & M & \end{tikzcd} \tag{2.35b}$$

in $\mathbf{C}$ commute. A morphism of monoids is a $\mathbf{C}$-morphism preserving multiplications and units. The category of monoids in $\mathbf{C}$ is denoted by $\mathbf{Mon}_{\mathbf{C}}$. To simplify notation, we often denote monoids by their underlying object M.

Example 2.15. As a simple example, consider the monoidal category $\mathbf{Vec}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$, cf. Example 2.2. Monoids in $\mathbf{Vec}_{\mathbb{K}}$ are associative and unital algebras over $\mathbb{K}$. $\triangledown$

Example 2.16. Let $\mathbf{C}$ be a monoidal category that is right closed. For any object $c \in \mathbf{C}$, consider the internal-hom object $[c, c] \in \mathbf{C}$. This object may be equipped with a canonical monoid structure by using the adjunction $(-) \otimes c : \mathbf{C} \rightleftarrows \mathbf{C} : [c, -]$. Concretely, the unit $\eta : I \to [c, c]$ is given by the adjoint of left unitor $I \otimes c \to c$ and the multiplication $\mu : [c, c] \otimes [c, c] \to [c, c]$ is given by the adjoint of the composition of the $\mathbf{C}$-morphisms

$$\big([c, c] \otimes [c, c]\big) \otimes c \xrightarrow{\;\cong\;} [c, c] \otimes \big([c, c] \otimes c\big) \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} [c, c] \otimes c \xrightarrow{\;\mathrm{ev}\;} c \quad , \tag{2.36}$$

where $\mathrm{ev} : [c, c] \otimes c \to c$ is the evaluation map given by the adjoint of $\mathrm{id}_{[c,c]} : [c, c] \to [c, c]$. $\triangledown$

Definition 2.17. A *monad* in a category $\mathbf{C}$ is an endofunctor $T : \mathbf{C} \to \mathbf{C}$ together with two natural transformations $\mu : T^2 \to T$ and $\eta : \mathrm{id}_{\mathbf{C}} \to T$, such that the diagrams

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[d, "\eta T"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \tag{2.37}$$

of natural transformations commute. (See Remark 2.18 below for a more explicit component-wise description.) A morphism of monads is a natural transformation of the underlying endofunctors preserving the structure transformations. To simplify notation, we often denote monads by their underlying endofunctor T.

Remark 2.18. It is useful to provide a more explicit component description of monads in a category $\mathbf{C}$. Let us denote the components of the natural transformations μ and η by $\mu_c : TT(c) \to T(c)$ and $\eta_c : c \to T(c)$, for all objects $c \in \mathbf{C}$. The components of the diagrams of natural transformations in (2.37) then read as

$$\begin{tikzcd} TTT(c) \arrow[r, "T(\mu_c)"] \arrow[d, "\mu_{T(c)}"'] & TT(c) \arrow[d, "\mu_c"] \\ TT(c) \arrow[r, "\mu_c"'] & T(c) \end{tikzcd} \qquad \begin{tikzcd} T(c) \arrow[r, "T(\eta_c)"] \arrow[d, "\eta_{T(c)}"'] \arrow[dr, equal] & TT(c) \arrow[d, "\mu_c"] \\ TT(c) \arrow[r, "\mu_c"'] & T(c) \end{tikzcd} \tag{2.38}$$

for all objects $c \in \mathbf{C}$. $\triangle$

Definition 2.19. Let T be a monad in a category $\mathbf{C}$. An *algebra over* T, or shorter a T*-algebra*, is an object $A \in \mathbf{C}$ together with a $\mathbf{C}$-morphism $\alpha : T(A) \to A$ (called *structure map*), such that the diagrams

$$\begin{tikzcd} TT(A) \arrow[r, "T(\alpha)"] \arrow[d, "\mu_A"'] & T(A) \arrow[d, "\alpha"] \\ T(A) \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & T(A) \arrow[d, "\alpha"] \\ & A \end{tikzcd} \tag{2.39}$$

in $\mathbf{C}$ commute. A morphism $\varphi : (A, \alpha) \to (A', \alpha')$ of T-algebras is a $\mathbf{C}$-morphism $\varphi : A \to A'$ such that $\alpha' T(\varphi) = \varphi\,\alpha$. The category of T-algebras will be denoted by $\mathbf{Alg}(T)$. To simplify notation, we often denote T-algebras by their underlying object A.

Theorem 2.20. *Let $\mathbf{C}$ be a category and T a monad in $\mathbf{C}$. The forgetful functor $U : \mathbf{Alg}(T) \to \mathbf{C}$ has a left adjoint, i.e. there exists an adjunction*

$$F \,:\, \mathbf{C} \rightleftarrows \mathbf{Alg}(T) \,:\, U \quad . \tag{2.40}$$

The left adjoint $F : \mathbf{C} \to \mathbf{Alg}(T)$ is called the free T-algebra functor. Concretely, it assigns to an object $c \in \mathbf{C}$ the T-algebra $F(c) = \big(T(c), \mu_c : TT(c) \to T(c)\big)$.

We shall also need the following lemma, see e.g. [Bor94b, Lemma 4.3.3] for a proof.

Lemma 2.21. *Let T be a monad in a category $\mathbf{C}$ and $(A, \alpha) \in \mathbf{Alg}(T)$. Then*

$$\big(TT(A), \mu_{T(A)}\big) \overset{T(\alpha)}{\underset{\mu_A}{\rightrightarrows}} \big(T(A), \mu_A\big) \xrightarrow{\quad\alpha\quad} \big(A, \alpha\big) \tag{2.41}$$

is a coequalizer in $\mathbf{Alg}(T)$. This coequalizer is natural in $(A, \alpha) \in \mathbf{Alg}(T)$.

3 Colored symmetric sequences, operads and their algebras

In this section we provide a self-contained review of those aspects of the theory of colored operads and their algebras which are relevant for our work. We shall fix once and for all a closed symmetric monoidal category $(\mathbf{M}, \otimes, I, \tau)$ which is complete and cocomplete. For any non-empty set of colors $\mathfrak{C}$, we define the monoidal category $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ of $\mathfrak{C}$-colored symmetric sequences in $\mathbf{M}$. $\mathfrak{C}$-colored operads are then defined as monoid objects in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. This abstract point of view is useful to study categorical properties and constructions of colored operads and their algebras. Algebras over a $\mathfrak{C}$-colored operad $\mathcal{O}$ should be interpreted as concrete realizations of the abstract operations encoded by $\mathcal{O}$ on a $\mathfrak{C}$-colored object in $\mathbf{M}$. They can be described very elegantly as algebras over a certain monad that is associated to a $\mathfrak{C}$-colored operad. Further material and results about colored operads can be found in [WY15, Yau16, BM07, GJ17], see also [LV12, Fre17, Kel05, Rez96] for the uncolored case $\mathfrak{C} = \{*\}$.

3.1 Colored symmetric sequences

3.1.1 Definition and monoidal structure

We first introduce the concept of $\mathfrak{C}$-profiles and their groupoid structure.

Definition 3.1. Let $\mathfrak{C}$ be a non-empty set, whose elements are called *colors*.

a) A $\mathfrak{C}$*-profile* is a finite sequence $\underline{c} = (c_1, \dots, c_n)$ of elements in $\mathfrak{C}$. We denote by $|\underline{c}| := n$ its length. We also admit the empty sequence $\emptyset$, which by definition has length 0.

b) The *groupoid of* $\mathfrak{C}$*-profiles* $\Sigma_{\mathfrak{C}}$ is the groupoid whose objects are all $\mathfrak{C}$-profiles $\underline{c}$ and whose morphisms are right permutations

$$\sigma \,:\, \underline{c} = (c_1, \dots, c_n) \;\longrightarrow\; (c_{\sigma(1)}, \dots, c_{\sigma(n)}) =: \underline{c}\sigma \quad , \tag{3.1}$$

for all $\sigma \in \Sigma_n$, where Σ_n is the symmetric group on n letters.

Remark 3.2. In [WY15, Yau16] the groupoid of $\mathfrak{C}$-profiles is defined by using left permutations. Hence, our groupoid corresponds to the groupoid $\Sigma_{\mathfrak{C}}^{\mathrm{op}}$ in these references. We justify our different choice of convention by the fact that it minimizes the occurrences of opposite categories in the rest of our paper. $\triangle$

Concatenation of $\mathfrak{C}$-profiles endows $\Sigma_{\mathfrak{C}}$ with a symmetric monoidal structure. Explicitly, we define a functor

$$\otimes\,:\,\Sigma_{\mathfrak{C}}\times\Sigma_{\mathfrak{C}}\;\longrightarrow\;\Sigma_{\mathfrak{C}} \tag{3.2a}$$

by

$$\underline{c}\otimes\underline{d}\;:=\;(c_1,\dots,c_n,d_1,\dots,d_m)\quad, \tag{3.2b}$$

for all $\underline{c}=(c_1,\dots,c_n)\in\Sigma_{\mathfrak{C}}$ and $\underline{d}=(d_1,\dots,d_m)\in\Sigma_{\mathfrak{C}}$. The unit object is given by the empty $\mathfrak{C}$-profile $I:=\emptyset\in\Sigma_{\mathfrak{C}}$ and the braiding

$$\tau\,:\,\underline{c}\otimes\underline{d}\;\cong\;\underline{d}\otimes\underline{c} \tag{3.3}$$

is given by block transposition.

Proposition 3.3. *$(\Sigma_{\mathfrak{C}},\otimes,I,\tau)$ is a strict symmetric monoidal category.*

Definition 3.4. The category of $\mathfrak{C}$-*colored symmetric sequences* in $\mathbf{M}$ is the functor category

$$\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})\;:=\;\mathbf{M}^{\Sigma_{\mathfrak{C}}\times\mathfrak{C}}\quad, \tag{3.4}$$

where we regard the set $\mathfrak{C}$ as a discrete category (i.e. a category whose only morphisms are the identities). An object $X\in\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ is thus a functor $X:\Sigma_{\mathfrak{C}}\times\mathfrak{C}\to\mathbf{M}$, whose values on objects and morphisms will be denoted by

$$X\big(\substack{t\\ \underline{c}}\big)\in\mathbf{M}\quad, \tag{3.5a}$$

for all objects $(\underline{c},t)\in\Sigma_{\mathfrak{C}}\times\mathfrak{C}$, and

$$X(\sigma)\,:\,X\big(\substack{t\\ \underline{c}}\big)\;\longrightarrow\;X\big(\substack{t\\ \underline{c}\sigma}\big)\quad, \tag{3.5b}$$

for all $\Sigma_{\mathfrak{C}}\times\mathfrak{C}$-morphisms $\sigma:(\underline{c},t)\to(\underline{c}\sigma,t)$.

Given any object $Y\in\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ and any $\mathfrak{C}$-profile $\underline{a}=(a_1,\dots,a_m)\in\Sigma_{\mathfrak{C}}$, we define

$$Y^{\underline{a}}\;:=\;Y\big(\substack{a_1\\ -}\big)\otimes_{\mathrm{Day}}\cdots\otimes_{\mathrm{Day}}Y\big(\substack{a_m\\ -}\big)\in\mathbf{M}^{\Sigma_{\mathfrak{C}}}\quad, \tag{3.6}$$

where $\otimes_{\mathrm{Day}}$ is the Day convolution (cf. Definition 2.10) on the functor category $\mathbf{M}^{\Sigma_{\mathfrak{C}}}$. (Note that for any $a\in\mathfrak{C}$, $Y\big(\substack{a\\ -}\big):\Sigma_{\mathfrak{C}}\to\mathbf{M}$ is a functor and hence an object in $\mathbf{M}^{\Sigma_{\mathfrak{C}}}$.) By convention, we set $Y^{\emptyset}:=I_{\mathrm{Day}}\in\mathbf{M}^{\Sigma_{\mathfrak{C}}}$ for the empty $\mathfrak{C}$-profile $\emptyset\in\Sigma_{\mathfrak{C}}$. Using (3.6) and the definition of the Day convolution (cf. Definition 2.10), one finds

$$Y^{\underline{a}}(\underline{c})\;\cong\;\int^{(\underline{b}_1,\dots,\underline{b}_m)}\Sigma_{\mathfrak{C}}\big(\underline{b}_1\otimes\cdots\otimes\underline{b}_m,\underline{c}\big)\otimes Y\big(\substack{a_1\\ \underline{b}_1}\big)\otimes\cdots\otimes Y\big(\substack{a_m\\ \underline{b}_m}\big)\quad, \tag{3.7}$$

for all $\underline{c}\in\Sigma_{\mathfrak{C}}$. We notice that the assignment $\underline{a}\mapsto Y^{\underline{a}}$ defines a functor

$$Y^{(-)}\,:\,\Sigma_{\mathfrak{C}}^{\mathrm{op}}\;\longrightarrow\;\mathbf{M}^{\Sigma_{\mathfrak{C}}} \tag{3.8}$$

on the opposite of the groupoid of $\mathfrak{C}$-profiles. Indeed, any $\Sigma_{\mathfrak{C}}$-morphism $\sigma:\underline{a}=(a_1,\dots,a_m)\to\underline{a}\sigma=(a_{\sigma(1)},\dots,a_{\sigma(m)})$ induces a $\mathbf{M}^{\Sigma_{\mathfrak{C}}}$-morphism

$$Y^{\sigma}\,:\,Y^{\underline{a}\sigma}=Y\big(\substack{a_{\sigma(1)}\\ -}\big)\otimes_{\mathrm{Day}}\cdots\otimes_{\mathrm{Day}}Y\big(\substack{a_{\sigma(m)}\\ -}\big)\;\longrightarrow\;Y\big(\substack{a_1\\ -}\big)\otimes_{\mathrm{Day}}\cdots\otimes_{\mathrm{Day}}Y\big(\substack{a_m\\ -}\big)=Y^{\underline{a}}\quad, \tag{3.9}$$

which is defined by permuting via τ_{Day} the tensor factors from the order in the source to the one in the target, see e.g. [Yau16, Chapter 8.6]. (This construction uses that $(\mathbf{M}^{\Sigma_{\mathfrak{C}}},\otimes_{\mathrm{Day}},I_{\mathrm{Day}},\tau_{\mathrm{Day}})$ is a symmetric monoidal category, cf. Proposition 2.12.)

Definition 3.5. The $\mathfrak{C}$-*colored circle product* on $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ is the functor

$$\circ\,:\,\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})\times\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})~\longrightarrow~\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \tag{3.10a}$$

defined by

$$(X\circ Y)\big(\begin{smallmatrix}t\\ \underline{c}\end{smallmatrix}\big)~:=~\int^{\underline{a}}X\big(\begin{smallmatrix}t\\ \underline{a}\end{smallmatrix}\big)\otimes Y^{\underline{a}}(\underline{c})\quad, \tag{3.10b}$$

for all $(\underline{c},t)\in\Sigma_{\mathfrak{C}}\times\mathfrak{C}$ and all $X,Y\in\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. The $\mathfrak{C}$-*colored circle unit* is the object $I_\circ\in\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ defined by

$$I_\circ\big(\begin{smallmatrix}t\\ \underline{c}\end{smallmatrix}\big)~:=~\Sigma_{\mathfrak{C}}(t,\underline{c})\otimes I=\begin{cases}I\in\mathbf{M} & ,\quad\text{if }\underline{c}=t\ ,\\ \emptyset\in\mathbf{M} & ,\quad\text{else}\ ,\end{cases} \tag{3.11}$$

for all $(\underline{c},t)\in\Sigma_{\mathfrak{C}}\times\mathfrak{C}$, where we denote the initial object in $\mathbf{M}$ also by the symbol $\emptyset\in\mathbf{M}$.

Remark 3.6. Using (3.7), we may expand the $\mathfrak{C}$-colored circle product (3.10) as

$$(X\circ Y)\big(\begin{smallmatrix}t\\ \underline{c}\end{smallmatrix}\big)~\cong~\int^{\underline{a}}\int^{(\underline{b}_1,\dots,\underline{b}_m)}\Sigma_{\mathfrak{C}}\big(\underline{b}_1\otimes\cdots\otimes\underline{b}_m,\underline{c}\big)\otimes X\big(\begin{smallmatrix}t\\ \underline{a}\end{smallmatrix}\big)\otimes Y\big(\begin{smallmatrix}a_1\\ \underline{b}_1\end{smallmatrix}\big)\otimes\cdots\otimes Y\big(\begin{smallmatrix}a_m\\ \underline{b}_m\end{smallmatrix}\big)\quad, \tag{3.12}$$

where $m=|\underline{a}|$ denotes the length of $\underline{a}$. Because the Hom-sets are empty for profiles of different lengths, the $\mathfrak{C}$-colored circle product involves computing a coend over all possible factorizations of $\underline{c}$ into profiles $\underline{b}_j$ of length $|\underline{b}_j|\leq|\underline{c}|$ such that $|\underline{c}|=\sum_{j=1}^m|\underline{b}_j|$. $\triangle$

In order to prove that Definition 3.5 endows $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ with the structure of a monoidal category, we need the following technical lemma.

Lemma 3.7. *For every $Y,Z\in\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ and every $\underline{a}\in\Sigma_{\mathfrak{C}}$, there exists a canonical isomorphism*

$$(Y\circ Z)^{\underline{a}}~\cong~\int^{\underline{b}}Y^{\underline{a}}(\underline{b})\otimes Z^{\underline{b}}\quad. \tag{3.13}$$

Proof. For all $\underline{c}\in\Sigma_{\mathfrak{C}}$, we have the following chain of canonical isomorphisms

$$\begin{aligned}(Y\circ Z)^{\underline{a}}(\underline{c})~&\cong~\int^{(\underline{b}_1,\dots,\underline{b}_m)}\Sigma_{\mathfrak{C}}\big(\underline{b}_1\otimes\cdots\otimes\underline{b}_m,\underline{c}\big)\otimes(Y\circ Z)\big(\begin{smallmatrix}a_1\\ \underline{b}_1\end{smallmatrix}\big)\otimes\cdots\otimes(Y\circ Z)\big(\begin{smallmatrix}a_m\\ \underline{b}_m\end{smallmatrix}\big)\\
&\cong~\int^{(\underline{b}_1,\dots,\underline{b}_m,\underline{d}_1,\dots,\underline{d}_m)}\Sigma_{\mathfrak{C}}\big(\underline{b}_1\otimes\cdots\otimes\underline{b}_m,\underline{c}\big)\otimes Y\big(\begin{smallmatrix}a_1\\ \underline{d}_1\end{smallmatrix}\big)\otimes\cdots\otimes Y\big(\begin{smallmatrix}a_m\\ \underline{d}_m\end{smallmatrix}\big)\\
&\qquad\qquad\otimes Z^{\underline{d}_1}(\underline{b}_1)\otimes\cdots\otimes Z^{\underline{d}_m}(\underline{b}_m)\\
&\cong~\int^{(\underline{d}_1,\dots,\underline{d}_m)}Y\big(\begin{smallmatrix}a_1\\ \underline{d}_1\end{smallmatrix}\big)\otimes\cdots\otimes Y\big(\begin{smallmatrix}a_m\\ \underline{d}_m\end{smallmatrix}\big)\otimes\big(Z^{\underline{d}_1}\otimes_{\mathrm{Day}}\cdots\otimes_{\mathrm{Day}}Z^{\underline{d}_m}\big)(\underline{c})\\
&\cong~\int^{(\underline{d}_1,\dots,\underline{d}_m)}Y\big(\begin{smallmatrix}a_1\\ \underline{d}_1\end{smallmatrix}\big)\otimes\cdots\otimes Y\big(\begin{smallmatrix}a_m\\ \underline{d}_m\end{smallmatrix}\big)\otimes Z^{\underline{d}_1\otimes\cdots\otimes\underline{d}_m}(\underline{c})\\
&\cong~\int^{(\underline{d}_1,\dots,\underline{d}_m)}\int^{\underline{b}}\Sigma_{\mathfrak{C}}\big(\underline{d}_1\otimes\cdots\otimes\underline{d}_m,\underline{b}\big)\otimes Y\big(\begin{smallmatrix}a_1\\ \underline{d}_1\end{smallmatrix}\big)\otimes\cdots\otimes Y\big(\begin{smallmatrix}a_m\\ \underline{d}_m\end{smallmatrix}\big)\otimes Z^{\underline{b}}(\underline{c})\\
&\cong~\int^{\underline{b}}Y^{\underline{a}}(\underline{b})\otimes Z^{\underline{b}}(\underline{c})\quad.\end{aligned} \tag{3.14}$$

In the first step we used (3.7). In the second step we used the definition of the circle product (3.10) and symmetry of the monoidal category $\mathbf{M}$ to rearrange the tensor factors. Step three uses the definition of the Day convolution (cf. Definition 2.10) and step four uses (3.6). Step five is a consequence of applying Theorem 2.8 to the functor $Z^{(-)}(\underline{c}):\Sigma_{\mathfrak{C}}^{\mathrm{op}}\to\mathbf{M}$. Finally, the last step uses Theorem 2.7 and (3.7). $\square$

Proposition 3.8. $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), \circ, I_\circ)$ *is a monoidal category.*

Proof. The associator is given by the canonical isomorphisms

$$\begin{aligned}((X \circ Y) \circ Z)\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) &= \int^{\underline{a}} (X \circ Y)\big(\begin{smallmatrix} t \\ \underline{a}\end{smallmatrix}\big) \otimes Z^{\underline{a}}(\underline{c}) \\ &\cong \int^{(\underline{a},\underline{b})} X\big(\begin{smallmatrix} t \\ \underline{b}\end{smallmatrix}\big) \otimes Y^{\underline{b}}(\underline{a}) \otimes Z^{\underline{a}}(\underline{c}) \\ &\cong \int^{\underline{b}} X\big(\begin{smallmatrix} t \\ \underline{b}\end{smallmatrix}\big) \otimes (Y \circ Z)^{\underline{b}}(\underline{c}) = (X \circ (Y \circ Z))\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) \quad , \end{aligned} \tag{3.15}$$

where we used Lemma 3.7 in step three. The unitors are given by the canonical isomorphisms

$$(I_\circ \circ X)\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) \;\cong\; \int^{\underline{a}} \Sigma_{\mathfrak{C}}(t, \underline{a}) \otimes X^{\underline{a}}(\underline{c}) \;\cong\; X^t(\underline{c}) \;=\; X\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) \tag{3.16a}$$

and

$$\begin{aligned}(X \circ I_\circ)\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) \;&\cong\; \int^{\underline{a}} \int^{(\underline{b}_1,\dots,\underline{b}_m)} \Big(\Sigma_{\mathfrak{C}}\big(\underline{b}_1 \otimes \cdots \otimes \underline{b}_m, \underline{c}\big) \times \Sigma_{\mathfrak{C}}(a_1, \underline{b}_1) \times \cdots \times \Sigma_{\mathfrak{C}}(a_m, \underline{b}_m)\Big) \otimes X\big(\begin{smallmatrix} t \\ \underline{a}\end{smallmatrix}\big) \\ &\cong\; \int^{\underline{a}} \Sigma_{\mathfrak{C}}(\underline{a}, \underline{c}) \otimes X\big(\begin{smallmatrix} t \\ \underline{a}\end{smallmatrix}\big) \;\cong\; X\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) \quad , \end{aligned} \tag{3.16b}$$

where we used the definition of the circle unit (3.11). □

Remark 3.9. It is in general not possible to equip the monoidal category $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), \circ, I_\circ)$ with a braiding. Consider for example the two objects $X, Y \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ defined by

$$X\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) = \big(\Sigma_{\mathfrak{C}}(t, t_0) \times \Sigma_{\mathfrak{C}}(t_0, \underline{c})\big) \otimes I \quad , \qquad Y\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) = \Sigma_{\mathfrak{C}}(\emptyset, \underline{c}) \otimes y \quad , \tag{3.17}$$

for some fixed $t_0 \in \mathfrak{C}$ and $\emptyset \not\cong y \in \mathbf{M}$. Using (3.12), a calculation similar to the one in the proof of Proposition 3.8 shows that

$$(X \circ Y)\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) \;\cong\; \Sigma_{\mathfrak{C}}(t, t_0) \otimes Y\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) \quad , \qquad (Y \circ X)\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) \;\cong\; Y\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) \quad . \tag{3.18}$$

For $t \neq t_0$, we obtain $(X \circ Y)\big(\begin{smallmatrix} t \\ \emptyset\end{smallmatrix}\big) \;\cong\; \emptyset \;\not\cong\; y \;\cong\; (Y \circ X)\big(\begin{smallmatrix} t \\ \emptyset\end{smallmatrix}\big)$, hence there cannot exist a braiding in general. △

Proposition 3.10. *The monoidal category* $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), \circ, I_\circ)$ *is right closed. The right adjoint of the functor* $(-) \circ Y : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ *is the functor*

$$[Y, -]_\circ \;:\; \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \;\longrightarrow\; \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \tag{3.19a}$$

given by

$$[Y, Z]_\circ\big(\begin{smallmatrix} t \\ \underline{c}\end{smallmatrix}\big) = \int_{\underline{a}} \big[Y^{\underline{c}}(\underline{a}), Z\big(\begin{smallmatrix} t \\ \underline{a}\end{smallmatrix}\big)\big] \quad , \tag{3.19b}$$

for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$ *and all* $Y, Z \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. *(Recall that* $[-,-] : \mathbf{M}^{\mathrm{op}} \times \mathbf{M} \to \mathbf{M}$ *denotes the internal-hom functor on* $\mathbf{M}$.*)*

Proof. This is again a simple application of the end/coend calculus. □

Remark 3.11. Notice that the monoidal category $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), \circ, I_\circ)$ is in general *not* left closed. In fact, the occurrence of multiple tensor powers of Y in the colored circle product (3.12) implies that the functor $X \circ (-) : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ in general does not preserve coproducts. Hence, it cannot have a right adjoint. △

3.1.2 Free symmetrization

Let $\mathfrak{C}$ be a non-empty set of colors and denote by $\Omega_{\mathfrak{C}}$ the *set* of $\mathfrak{C}$-profiles, i.e. the set of objects underlying the groupoid $\Sigma_{\mathfrak{C}}$. There exists an obvious forgetful functor

$$U \,:\, \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \longrightarrow \mathbf{M}^{\Omega_{\mathfrak{C}} \times \mathfrak{C}} \tag{3.20}$$

given by forgetting the action of permutations. Recalling that $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) = \mathbf{M}^{\Sigma_{\mathfrak{C}} \times \mathfrak{C}}$, we notice that U is the pullback along the canonical functor $\Omega_{\mathfrak{C}} \times \mathfrak{C} \to \Sigma_{\mathfrak{C}} \times \mathfrak{C}$. The category $\mathbf{M}^{\Omega_{\mathfrak{C}} \times \mathfrak{C}}$ is called the category of $\mathfrak{C}$*-colored non-symmetric sequences* in $\mathbf{M}$. Every non-symmetric sequence may be freely symmetrized, which is a construction we frequently use in Section 4.

Proposition 3.12. *The functor* (3.20) *has a left adjoint, i.e. there exists an adjunction*

$$(-)^{\mathsf{sym}} \,:\, \mathbf{M}^{\Omega_{\mathfrak{C}} \times \mathfrak{C}} \rightleftarrows \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \,:\, U \quad . \tag{3.21}$$

For any $X \in \mathbf{M}^{\Omega_{\mathfrak{C}} \times \mathfrak{C}}$, we have the explicit formulae

$$X^{\mathsf{sym}}\binom{t}{\underline{c}} \;=\; \coprod_{\sigma \in \Sigma_{|\underline{c}|}} X\binom{t}{\underline{c}\sigma^{-1}} \quad , \tag{3.22}$$

for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$, and

$$\begin{array}{ccc} X^{\mathsf{sym}}\binom{t}{\underline{c}} & \xrightarrow{\;X^{\mathsf{sym}}(\sigma')\;} & X^{\mathsf{sym}}\binom{t}{\underline{c}\sigma'} \\ {\scriptstyle \iota_\sigma}\big\uparrow & & \big\uparrow{\scriptstyle \iota_{\sigma\sigma'}} \\ X\binom{t}{\underline{c}\sigma^{-1}} & =\!=\!=\!=\!= & X\binom{t}{(\underline{c}\sigma')(\sigma\sigma')^{-1}} \end{array} \tag{3.23}$$

for all $\Sigma_{\mathfrak{C}} \times \mathfrak{C}$-morphisms $\sigma' : (\underline{c}, t) \to (\underline{c}\sigma', t)$, where ι denotes the canonical inclusions into the coproducts.

Proof. Existence and the explicit formulae follow by left Kan extension, see e.g. [MacL98, Chapter X.4]. Concretely, we have that

$$\begin{aligned} X^{\mathsf{sym}}\binom{t}{\underline{c}} \;&=\; \int^{(\underline{c}', t') \in \Omega_{\mathfrak{C}} \times \mathfrak{C}} (\Sigma_{\mathfrak{C}} \times \mathfrak{C})\big((\underline{c}', t'), (\underline{c}, t)\big) \otimes X\binom{t'}{\underline{c}'} \\ &=\; \coprod_{(\underline{c}', t') \in \Omega_{\mathfrak{C}} \times \mathfrak{C}} (\Sigma_{\mathfrak{C}} \times \mathfrak{C})\big((\underline{c}', t'), (\underline{c}, t)\big) \otimes X\binom{t'}{\underline{c}'} \\ &\cong\; \coprod_{\underline{c}\sigma^{-1} \xrightarrow{\sigma} \underline{c}} X\binom{t}{\underline{c}\sigma^{-1}} \;\cong\; \coprod_{\sigma \in \Sigma_{|\underline{c}|}} X\binom{t}{\underline{c}\sigma^{-1}} \quad , \end{aligned} \tag{3.24}$$

where in the second step we used that $\Omega_{\mathfrak{C}} \times \mathfrak{C}$ is a discrete category (i.e. coends become coproducts) and in the third and fourth step the definition of $\Sigma_{\mathfrak{C}}$, see Definition 3.1. It is easy to deduce our explicit formula for $X^{\mathsf{sym}}(\sigma')$ from these isomorphisms. □

3.1.3 Change of color

So far we considered a fixed set of colors $\mathfrak{C}$. Given any map $f : \mathfrak{C} \to \mathfrak{D}$ of non-empty sets, a natural question is whether it induces mappings between $\mathfrak{C}$-colored symmetric sequences and $\mathfrak{D}$-colored symmetric sequences. Such constructions will be important in our applications to quantum field theory in Sections 5 and 6.

Any map $f : \mathfrak{C} \to \mathfrak{D}$ of non-empty sets canonically induces a symmetric strong monoidal functor (denoted by the same symbol)

$$f \,:\, \Sigma_{\mathfrak{C}} \longrightarrow \Sigma_{\mathfrak{D}} \tag{3.25}$$

between the groupoids of profiles (cf. Definition 3.1) and a functor (denoted again by the same symbol)

$$f\ :\ \Sigma_{\mathfrak{C}}\times\mathfrak{C}\ \longrightarrow\ \Sigma_{\mathfrak{D}}\times\mathfrak{D} \tag{3.26}$$

between the product categories. Recalling the definition of colored symmetric sequences as functor categories (cf. Definition 3.4), we obtain a functor

$$f^{*}\ :\ \mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M})\ \longrightarrow\ \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \tag{3.27}$$

by pulling back along (3.26). We will refer to this functor as pullback functor.

Proposition 3.13. *The pullback functor* (3.27) *has a left adjoint, i.e. there exists an adjunction*

$$f_{!}\ :\ \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})\ \rightleftarrows\ \mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M})\ :\ f^{*}\quad . \tag{3.28}$$

For any $X\in\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, *we have the explicit formula*

$$f_{!}(X)\big(\begin{smallmatrix}s\\ \underline{d}\end{smallmatrix}\big)\ =\ \coprod_{t\in f^{-1}(s)}\int^{\underline{c}\in\Sigma_{\mathfrak{C}}}\Sigma_{\mathfrak{D}}\big(f(\underline{c}),\underline{d}\big)\otimes X\big(\begin{smallmatrix}t\\ \underline{c}\end{smallmatrix}\big)\quad , \tag{3.29}$$

for all $(\underline{d},s)\in\Sigma_{\mathfrak{D}}\times\mathfrak{D}$.

Proof. Existence and the explicit formula follow by left Kan extension, see e.g. [MacL98, Chapter X.4]. (Use that $\mathfrak{C}$ is a discrete category to express coends in terms of coproducts.) □

Proposition 3.14. *The pullback functor* f^{*} (3.27) *is canonically lax monoidal, which endows its left adjoint* $f_{!}$ (3.28) *with an oplax monoidal structure in the sense of Proposition 2.3.*

Proof. We have to construct the structure morphisms $f^{*}_{2}:f^{*}(X)\circ^{\mathfrak{C}}f^{*}(Y)\to f^{*}(X\circ^{\mathfrak{D}}Y)$, for all $X,Y\in\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M})$, and $f^{*}_{0}:I^{\mathfrak{C}}_{\circ}\to f^{*}(I^{\mathfrak{D}}_{\circ})$. (To enhance readability, we label the circle products and units on $\mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M})$ and $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ by the appropriate set $\mathfrak{D}$ or $\mathfrak{C}$.) The latter is given by

$$f^{*}_{0}\ :\ I^{\mathfrak{C}}_{\circ}\big(\begin{smallmatrix}t\\ \underline{c}\end{smallmatrix}\big)=\Sigma_{\mathfrak{C}}(t,\underline{c})\otimes I\ \xrightarrow{\ \ f\otimes\mathrm{id}\ \ }\ \Sigma_{\mathfrak{D}}\big(f(t),f(\underline{c})\big)\otimes I=f^{*}(I^{\mathfrak{D}}_{\circ})\big(\begin{smallmatrix}t\\ \underline{c}\end{smallmatrix}\big)\quad , \tag{3.30}$$

for all $(\underline{c},t)\in\Sigma_{\mathfrak{C}}\times\mathfrak{C}$, i.e. by applying the functor f on the Hom-sets. In order to describe f^{*}_{2}, we recall (cf. (3.10)) that

$$\big(f^{*}(X)\circ^{\mathfrak{C}}f^{*}(Y)\big)\big(\begin{smallmatrix}t\\ \underline{c}\end{smallmatrix}\big)\ =\ \int^{\underline{c}'}X\big(\begin{smallmatrix}f(t)\\ f(\underline{c}')\end{smallmatrix}\big)\otimes\big(f^{*}(Y)\big)^{\underline{c}'}(\underline{c})\quad , \tag{3.31a}$$

$$f^{*}(X\circ^{\mathfrak{D}}Y)\big(\begin{smallmatrix}t\\ \underline{c}\end{smallmatrix}\big)\ =\ \int^{\underline{d}}X\big(\begin{smallmatrix}f(t)\\ \underline{d}\end{smallmatrix}\big)\otimes Y^{\underline{d}}(f(\underline{c}))\quad . \tag{3.31b}$$

We would like to define f^{*}_{2} by the universal property of coends and a family of morphisms

$$X\big(\begin{smallmatrix}f(t)\\ f(\underline{c}')\end{smallmatrix}\big)\otimes\big(f^{*}(Y)\big)^{\underline{c}'}(\underline{c})\ \longrightarrow\ X\big(\begin{smallmatrix}f(t)\\ f(\underline{c}')\end{smallmatrix}\big)\otimes Y^{f(\underline{c}')}(f(\underline{c}))\quad , \tag{3.32}$$

for all $t\in\mathfrak{C}$ and $\underline{c},\underline{c}'\in\Sigma_{\mathfrak{C}}$. Notice that by definition

$$\begin{aligned}\big(f^{*}(Y)\big)^{\underline{c}'}(\underline{c})\ &=\ \Big(Y\big(\begin{smallmatrix}f(c'_1)\\ f(-)\end{smallmatrix}\big)\otimes^{\Sigma_{\mathfrak{C}}}_{\mathrm{Day}}\cdots\otimes^{\Sigma_{\mathfrak{C}}}_{\mathrm{Day}}Y\big(\begin{smallmatrix}f(c'_m)\\ f(-)\end{smallmatrix}\big)\Big)(\underline{c})\\ &=\ \Big(f^{*}\big(Y\big(\begin{smallmatrix}f(c'_1)\\ -\end{smallmatrix}\big)\big)\otimes^{\Sigma_{\mathfrak{C}}}_{\mathrm{Day}}\cdots\otimes^{\Sigma_{\mathfrak{C}}}_{\mathrm{Day}}f^{*}\big(Y\big(\begin{smallmatrix}f(c'_m)\\ -\end{smallmatrix}\big)\big)\Big)(\underline{c})\quad ,\end{aligned} \tag{3.33}$$

where after the second equality f^* denotes the pullback functor $f^* : \mathbf{M}^{\Sigma_{\mathfrak{D}}} \to \mathbf{M}^{\Sigma_{\mathfrak{C}}}$ corresponding to the symmetric strong monoidal functor (3.25). Because the latter pullback functor is canonically lax monoidal (cf. Proposition 2.13), we obtain canonical morphisms

$$f^*\Big(Y\big(\tbinom{f(c_1')}{-}\big)\Big) \otimes^{\Sigma_{\mathfrak{C}}}_{\mathrm{Day}} \cdots \otimes^{\Sigma_{\mathfrak{C}}}_{\mathrm{Day}} f^*\Big(Y\big(\tbinom{f(c_m')}{-}\big)\Big) \;\longrightarrow\; f^*\Big(Y\big(\tbinom{f(c_1')}{-}\big) \otimes^{\Sigma_{\mathfrak{D}}}_{\mathrm{Day}} \cdots \otimes^{\Sigma_{\mathfrak{D}}}_{\mathrm{Day}} Y\big(\tbinom{f(c_m')}{-}\big)\Big) \quad , \tag{3.34}$$

which allow us to define the family of morphisms in (3.32). (Concretely, we apply these morphisms on the second tensor factor and the identities on the first tensor factor.) Using now the crucial property that $f^* : \mathbf{M}^{\Sigma_{\mathfrak{D}}} \to \mathbf{M}^{\Sigma_{\mathfrak{C}}}$ is symmetric (cf. Proposition 2.13), one easily shows that the family of morphisms (3.32) descends to the coends. □

Remark 3.15. Notice that the pullback functor f^* (3.27) is in general *not* a strong monoidal functor but only a lax monoidal functor. For example, if the map of colors $f : \mathfrak{C} \to \mathfrak{D}$ is not injective, then f_0^* given in (3.30) is clearly not an isomorphism. △

3.2 Colored operads

3.2.1 Definition

Recall the monoidal category of $\mathfrak{C}$-colored symmetric sequences (cf. Definitions 3.4 and 3.5) and the concept of monoids in monoidal categories (cf. Definition 2.14).

Definition 3.16. A $\mathfrak{C}$-*colored operad* with values in $\mathbf{M}$ is a monoid $\mathcal{O}$ in the monoidal category $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), \circ, I_\circ)$. Following [WY15, Yau16], we denote the multiplication by $\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ and the unit by $\mathbb{1} : I_\circ \to \mathcal{O}$. The category of $\mathfrak{C}$-colored operads is denoted by $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$. To simplify notation, we often denote colored operads by their underlying symmetric sequence $\mathcal{O}$.

Remark 3.17. One may unpack this definition to recover the usual component-wise definition of $\mathfrak{C}$-colored operads, see e.g. [Yau16] and [BM07]. The underlying object $\mathcal{O} \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ assigns to each $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$ an object

$$\mathcal{O}\big(\tbinom{t}{\underline{c}}\big) \in \mathbf{M} \tag{3.35a}$$

and to each $\Sigma_{\mathfrak{C}} \times \mathfrak{C}$-morphism $\sigma : (\underline{c}, t) \to (\underline{c}\sigma, t)$ an $\mathbf{M}$-morphism

$$\mathcal{O}(\sigma) \,:\, \mathcal{O}\big(\tbinom{t}{\underline{c}}\big) \;\longrightarrow\; \mathcal{O}\big(\tbinom{t}{\underline{c}\sigma}\big) \quad . \tag{3.35b}$$

Using (3.12), the multiplication $\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ is specified by the components

$$\gamma \,:\, \int^{\underline{a}} \int^{(\underline{b}_1,\dots,\underline{b}_m)} \Sigma_{\mathfrak{C}}\big(\underline{b}_1 \otimes \cdots \otimes \underline{b}_m, \underline{c}\big) \otimes \mathcal{O}\big(\tbinom{t}{\underline{a}}\big) \otimes \mathcal{O}\big(\tbinom{a_1}{\underline{b}_1}\big) \otimes \cdots \otimes \mathcal{O}\big(\tbinom{a_m}{\underline{b}_m}\big) \;\longrightarrow\; \mathcal{O}\big(\tbinom{t}{\underline{c}}\big) \quad , \tag{3.36}$$

for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$. Due to naturality of these components and the universal property of coends, γ is determined by a family of $\mathbf{M}$-morphisms

$$\gamma \,:\, \mathcal{O}\big(\tbinom{t}{\underline{a}}\big) \otimes \bigotimes_{i=1}^{m} \mathcal{O}\big(\tbinom{a_i}{\underline{b}_i}\big) \;\longrightarrow\; \mathcal{O}\big(\tbinom{t}{\underline{b}_1 \otimes \cdots \otimes \underline{b}_m}\big) \quad , \tag{3.37}$$

for all $t \in \mathfrak{C}$, $\underline{a} \in \Sigma_{\mathfrak{C}}$ with length $|\underline{a}| = m \geq 1$ and $\underline{b}_i \in \Sigma_{\mathfrak{C}}$ with length $|\underline{b}_i| = k_i \geq 0$, for $i = 1, \dots, m$. This family of morphisms satisfies the equivariance axioms of [Yau16, Definition 11.2.1]. Concretely, the first equivariance axiom is that, for each permutation $\sigma \in \Sigma_m$, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}\big(\tbinom{t}{\underline{a}}\big) \otimes \bigotimes\limits_{i=1}^{m} \mathcal{O}\big(\tbinom{a_i}{\underline{b}_i}\big) \arrow[rr, "\mathcal{O}(\sigma) \otimes \tau_\sigma"] \arrow[d, "\gamma"'] && \mathcal{O}\big(\tbinom{t}{\underline{a}\sigma}\big) \otimes \bigotimes\limits_{i=1}^{m} \mathcal{O}\big(\tbinom{a_{\sigma(i)}}{\underline{b}_{\sigma(i)}}\big) \arrow[d, "\gamma"] \\
\mathcal{O}\big(\tbinom{t}{\underline{b}_1 \otimes \cdots \otimes \underline{b}_m}\big) \arrow[rr, "\mathcal{O}(\sigma\langle k_1,\dots,k_m\rangle)"'] && \mathcal{O}\big(\tbinom{t}{\underline{b}_{\sigma(1)} \otimes \cdots \otimes \underline{b}_{\sigma(m)}}\big)
\end{tikzcd} \tag{3.38a}$$

where τ_σ denotes the permutation of tensor factors induced by the symmetric braiding τ of $\mathbf{M}$ and $\sigma\langle k_1,\dots,k_m\rangle \in \Sigma_{\sum_{i=1}^m k_i}$ is the block permutation induced by σ. The second equivariance axiom is that, for each family of permutations $\sigma_i \in \Sigma_{k_i}$, for $i=1,\dots,m$, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_{i=1}^m \mathcal{O}\binom{a_i}{\underline{b}_i} \arrow[r, "\mathrm{id}\otimes\bigotimes_{i=1}^m \mathcal{O}(\sigma_i)"] \arrow[d, "\gamma"'] & \mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_{i=1}^m \mathcal{O}\binom{a_i}{\underline{b}_i\sigma_i} \arrow[d, "\gamma"] \\
\mathcal{O}\binom{t}{\underline{b}_1\otimes\cdots\otimes\underline{b}_m} \arrow[r, "\mathcal{O}(\sigma_1\oplus\cdots\oplus\sigma_m)"'] & \mathcal{O}\binom{t}{\underline{b}_1\sigma_1\otimes\cdots\otimes\underline{b}_m\sigma_m}
\end{tikzcd} \tag{3.38b}$$

where $\sigma_1\oplus\cdots\oplus\sigma_m \in \Sigma_{\sum_{i=1}^m k_i}$ is the block sum permutation induced by the σ_i. Using (3.11), the unit $\mathbb{1} : I_\circ \to \mathcal{O}$ is specified by the components

$$\mathbb{1} \,:\, \Sigma_{\mathfrak{C}}(t,\underline{c})\otimes I \,\longrightarrow\, \mathcal{O}\binom{t}{\underline{c}} \quad , \tag{3.39}$$

for all $(\underline{c},t)\in\Sigma_{\mathfrak{C}}\times\mathfrak{C}$. Because $\Sigma_{\mathfrak{C}}(t,\underline{c})=\emptyset$, for $\underline{c}\neq t$, the unit is determined by a family of $\mathbf{M}$-morphisms

$$\mathbb{1} \,:\, I \,\longrightarrow\, \mathcal{O}\binom{t}{t} \quad , \tag{3.40}$$

for all $t\in\mathfrak{C}$. The associativity and unitality axioms for the monoid $\mathcal{O}$ imply the associativity and unitality axioms of [Yau16, Definition 11.2.1] for the components (3.37) and (3.40). Concretely, the associativity axiom says that all diagrams of the form

$$\begin{tikzcd}
\mathcal{O}\binom{t}{\underline{a}} \otimes \Big(\bigotimes_{i=1}^m \mathcal{O}\binom{a_i}{\underline{b}_i}\Big) \otimes \Big(\bigotimes_{i=1}^m\bigotimes_{j=1}^{k_i} \mathcal{O}\binom{b_{ij}}{\underline{c}_{ij}}\Big) \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[d, "\mathrm{permute}"'] & \mathcal{O}\binom{t}{\underline{b}_1\otimes\cdots\otimes\underline{b}_m} \otimes \Big(\bigotimes_{i=1}^m\bigotimes_{j=1}^{k_i} \mathcal{O}\binom{b_{ij}}{\underline{c}_{ij}}\Big) \arrow[dd, "\gamma"] \\
\mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_{i=1}^m \Big(\mathcal{O}\binom{a_i}{\underline{b}_i} \otimes \bigotimes_{j=1}^{k_i} \mathcal{O}\binom{b_{ij}}{\underline{c}_{ij}}\Big) \arrow[d, "\mathrm{id}\otimes\bigotimes_{i=1}^m\gamma"'] & \\
\mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_{i=1}^m \mathcal{O}\binom{a_i}{\underline{c}_{i1}\otimes\cdots\otimes\underline{c}_{ik_i}} \arrow[r, "\gamma"'] & \mathcal{O}\binom{t}{\underline{c}_{11}\otimes\cdots\otimes\underline{c}_{mk_m}}
\end{tikzcd} \tag{3.41}$$

commute, where we used the notation $\underline{b}_i=(b_{i1},\dots,b_{ik_i})$ and $|\underline{b}_i|=k_i\geq 1$. The unitality axiom is given by commutativity of the diagrams

$$\begin{tikzcd}
\mathcal{O}\binom{t}{\underline{a}}\otimes I^m \arrow[r, "\cong"] \arrow[d, "\mathrm{id}\otimes\bigotimes_{i=1}^m\mathbb{1}"'] & \mathcal{O}\binom{t}{\underline{a}} & I\otimes\mathcal{O}\binom{t}{\underline{a}} \arrow[l, "\cong"'] \arrow[d, "\mathbb{1}\otimes\mathrm{id}"] \\
\mathcal{O}\binom{t}{\underline{a}}\otimes\bigotimes_{i=1}^m\mathcal{O}\binom{a_i}{a_i} \arrow[ur, "\gamma"'] & & \mathcal{O}\binom{t}{t}\otimes\mathcal{O}\binom{t}{\underline{a}} \arrow[ul, "\gamma"]
\end{tikzcd} \tag{3.42}$$

for all $(\underline{a},t)\in\Sigma_{\mathfrak{C}}\times\mathfrak{C}$. △

Example 3.18. Recall from Proposition 3.10 that $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ is right closed. Given any object $X\in\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, we thus may form the internal-hom object $[X,X]_\circ\in\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, which by Example 2.16 carries a canonical monoid structure. Hence, this construction defines a $\mathfrak{C}$-colored operad $[X,X]_\circ\in\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$. These operads are particularly important when $X\in\mathbf{M}^{\mathfrak{C}}\subseteq\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ lies in the full subcategory of colored objects (cf. Definition 3.28 below). In this case one also writes $\mathrm{End}(X):=[X,X]_\circ$ and calls it the *endomorphism operad* of X. ▽

3.2.2 Categorical properties and constructions

There is an obvious forgetful functor

$$U \,:\, \mathbf{Op}_{\mathfrak{C}}(\mathbf{M}) \;\longrightarrow\; \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \tag{3.43}$$

from the category of $\mathfrak{C}$-colored operads to the category of $\mathfrak{C}$-colored symmetric sequences.

Theorem 3.19. *The forgetful functor* (3.43) *has a left adjoint, i.e. there exists an adjunction*

$$F \,:\, \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \;\rightleftarrows\; \mathbf{Op}_{\mathfrak{C}}(\mathbf{M}) \,:\, U \quad . \tag{3.44}$$

The left adjoint $F : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$ is called the free $\mathfrak{C}$-colored operad functor.

Remark 3.20. There exist various (equivalent) models for the free $\mathfrak{C}$-colored operad functor. For example, one may use the free monoid functor of [Rez96, Appendix A] or argue as in [WY15, Example 4.1.13] that Theorem 3.19 is a special instance of Corollary 3.36 below by describing $\mathfrak{C}$-colored operads as algebras over a suitable $\mathrm{Ob}(\Sigma_{\mathfrak{C}}) \times \mathfrak{C}$-colored operad. There exists also a more explicit description in terms of planar $\mathfrak{C}$-colored rooted trees, see [BM07, Section 3] and also [Yau16]. We do not have to explain any of these models in detail because existence of the free $\mathfrak{C}$-colored operad functor is sufficient for our work. △

Recall that the category $\mathbf{M}$ is by assumption complete and cocomplete. This implies that $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ is complete and cocomplete as well, because it is a category of functors with values in $\mathbf{M}$ (cf. Definition 3.4). (Recall that (co)limits in the functor category $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ may be computed object-wise as (co)limits in $\mathbf{M}$.)

Proposition 3.21. *The category $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$ is complete and cocomplete. Moreover, the forgetful functor $U : \mathbf{Op}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ creates all small limits, the filtered colimits and the coequalizers which are reflexive in the category $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$.*

Proof. The relevant arguments are given in [PS14, Theorem 3.8]. See also [Fre17, Proposition 1.2.4] for a more explicit proof in the uncolored case $\mathfrak{C} = \{*\}$. The latter generalizes to colored operads in a straightforward way. □

Remark 3.22. Recall that a pair of parallel morphisms $f, g : c \rightrightarrows c'$ in a category $\mathbf{C}$ is called *reflexive* if there exists a $\mathbf{C}$-morphism $s : c' \to c$ (called *reflector*) such that $f\,s = \mathrm{id}_{c'} = g\,s$. The colimit $\mathrm{colim}(f, g : c \rightrightarrows c')$ of a reflexive pair of parallel morphisms is called a *reflexive coequalizer.* △

The existence of free $\mathfrak{C}$-colored operads and coequalizers in $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$ allows us to construct examples of $\mathfrak{C}$-colored operads in terms of generators and relations. Let $X \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ and consider the corresponding free $\mathfrak{C}$-colored operad $F(X) \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$. In order to implement relations in $F(X)$, consider a pair of parallel $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$-morphisms $r_1, r_2 : R \rightrightarrows F(X)$, where here and in the following we suppress the forgetful functor $U : \mathbf{Op}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. The adjunction in Theorem 3.19 defines a pair of parallel morphisms (denoted by the same symbols) $r_1, r_2 : F(R) \rightrightarrows F(X)$ in $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$.

Definition 3.23. The $\mathfrak{C}$-colored operad presented by the generators $X \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ and relations $r_1, r_2 : R \rightrightarrows F(X)$ is defined as the coequalizer

$$F(X\!:\!r_1, r_2) \;:=\; \mathrm{colim}\big(r_1, r_2 : F(R) \rightrightarrows F(X)\big) \tag{3.45}$$

in $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$.

3.2.3 Change of color

Let $f : \mathfrak{C} \to \mathfrak{D}$ be a map of non-empty sets. Recall that there exists an induced pullback functor $f^* : \mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, which by Proposition 3.14 carries a canonical lax monoidal structure. This immediately implies

Corollary 3.24. *Any map $f : \mathfrak{C} \to \mathfrak{D}$ of non-empty sets induces a pullback functor*

$$f^* \,:\, \mathbf{Op}_{\mathfrak{D}}(\mathbf{M}) \longrightarrow \mathbf{Op}_{\mathfrak{C}}(\mathbf{M}) \tag{3.46}$$

on the categories of colored operads.

Proof. On the underlying symmetric sequences, the pullback functor (3.46) is given by $f^* : \mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. Given $\mathcal{P} \in \mathbf{Op}_{\mathfrak{D}}(\mathbf{M})$ with multiplication γ and unit $\mathbb{1}$, the multiplication $\widetilde{\gamma}$ and unit $\widetilde{\mathbb{1}}$ on $f^*(\mathcal{P})$ are defined by the lax monoidal structure (f_2^*, f_0^*) of Proposition 3.14. Concretely, they are defined by the commutative diagrams

$$\begin{tikzcd}
f^*(\mathcal{P}) \circ f^*(\mathcal{P}) \arrow[rr, "\widetilde{\gamma}"] \arrow[dr, "f_2^*"'] & & f^*(\mathcal{P}) \\
& f^*(\mathcal{P} \circ \mathcal{P}) \arrow[ur, "f^*(\gamma)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I_\circ \arrow[rr, "\widetilde{\mathbb{1}}"] \arrow[dr, "f_0^*"'] & & f^*(\mathcal{P}) \\
& f^*(I_\circ) \arrow[ur, "f^*(\mathbb{1})"'] &
\end{tikzcd} \tag{3.47}$$

in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. $\square$

Remark 3.25. Using arguments as in [BM07, Section 1.6], this corollary can be strengthened to a change of color adjunction $f_! : \mathbf{Op}_{\mathfrak{C}}(\mathbf{M}) \rightleftarrows \mathbf{Op}_{\mathfrak{D}}(\mathbf{M}) : f^*$ for the categories of colored operads. We do not have to explain this adjunction as we only need the pullback functor (3.46) in our work. $\triangle$

Using pullbacks of colored operads along change of color maps, one may define a category of operads with varying colors, see e.g. [MW07].

Definition 3.26. We denote by $\mathbf{Op}(\mathbf{M})$ the *category of colored operads* with values in $\mathbf{M}$. Concretely, the objects are pairs $(\mathfrak{C}, \mathcal{O})$, where $\mathfrak{C}$ is a non-empty set and $\mathcal{O}$ is a $\mathfrak{C}$-colored operad. A morphism $(\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$ in $\mathbf{Op}(\mathbf{M})$ is a pair (f, ϕ) consisting of a map $f : \mathfrak{C} \to \mathfrak{D}$ of sets and a morphism $\phi : \mathcal{O} \to f^*(\mathcal{P})$ of $\mathfrak{C}$-colored operads. Given two morphisms $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$ and $(g, \psi) : (\mathfrak{D}, \mathcal{P}) \to (\mathfrak{F}, \mathcal{Q})$ in $\mathbf{Op}(\mathbf{M})$, their composition is

$$(g, \psi)\ (f, \phi) = \big(g\, f, f^*(\psi)\, \phi\big) \,:\, (\mathfrak{C}, \mathcal{O}) \longrightarrow (\mathfrak{F}, \mathcal{Q}) \quad . \tag{3.48}$$

Remark 3.27. Projecting to the underlying set of colors defines a functor $\pi : \mathbf{Op}(\mathbf{M}) \to \mathbf{Set}$ whose fiber $\pi^{-1}(\mathfrak{C})$ over $\mathfrak{C} \in \mathbf{Set}$ is the category $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$ of $\mathfrak{C}$-colored operads. $\triangle$

3.3 Algebras over colored operads

3.3.1 Definition

Recall the category of $\mathfrak{C}$-colored symmetric sequences $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ (cf. Definition 3.4).

Definition 3.28. An object $X \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ is called a $\mathfrak{C}$-*colored object* in $\mathbf{M}$ if

$$X\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) = \emptyset \in \mathbf{M} \tag{3.49}$$

is the initial object, for all $(\underline{c}, t) \in \Sigma_{\mathfrak{C}} \times \mathfrak{C}$ with length $|\underline{c}| \geq 1$. The full subcategory of $\mathfrak{C}$-colored objects is isomorphic to the functor category $\mathbf{M}^{\mathfrak{C}}$. We shall use the simplified notation $X_t := X\big(\begin{smallmatrix} t \\ \emptyset \end{smallmatrix}\big) \in \mathbf{M}$, for all $t \in \mathfrak{C}$.

Lemma 3.29. *For any object $X \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, the functor $X \circ (-) : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ descends to a functor*

$$X \circ (-) \,:\, \mathbf{M}^{\mathfrak{C}} \longrightarrow \mathbf{M}^{\mathfrak{C}} \tag{3.50}$$

between the full subcategories of $\mathfrak{C}$-colored objects.

Proof. Let $Y \in \mathbf{M}^{\mathfrak{C}} \subseteq \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ be any $\mathfrak{C}$-colored object. The explicit expression for the colored circle product (3.12) implies that $(X \circ Y)\binom{t}{\underline{c}} = \emptyset \in \mathbf{M}$, for $|\underline{c}| \geq 1$, and

$$(X \circ Y)_t \,:=\, (X \circ Y)\tbinom{t}{\emptyset} \,\cong\, \int^{\underline{a}} X\tbinom{t}{\underline{a}} \otimes Y_{a_1} \otimes \cdots \otimes Y_{a_m} \quad , \tag{3.51}$$

for all $t \in \mathfrak{C}$. In particular, $X \circ Y \in \mathbf{M}^{\mathfrak{C}} \subseteq \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. □

This observation allows us to associate to a $\mathfrak{C}$-colored operad a monad in $\mathbf{M}^{\mathfrak{C}}$.

Proposition 3.30. *Let $\mathcal{O}$ be a $\mathfrak{C}$-colored operad. The endofunctor $\mathcal{O} \circ (-) : \mathbf{M}^{\mathfrak{C}} \to \mathbf{M}^{\mathfrak{C}}$ of Lemma 3.29 naturally carries the structure of a monad in $\mathbf{M}^{\mathfrak{C}}$.*

Proof. The structure transformations $\gamma : \mathcal{O} \circ (\mathcal{O} \circ (-)) \to \mathcal{O} \circ (-)$ and $\mathbb{1} : \mathrm{id}_{\mathbf{M}^{\mathfrak{C}}} \to \mathcal{O} \circ (-)$ of the monad $\mathcal{O} \circ (-)$ are defined by the components

$$\begin{tikzcd} \mathcal{O} \circ \big(\mathcal{O} \circ X\big) \arrow[r, "\gamma_X"] & \mathcal{O} \circ X \\ \big(\mathcal{O} \circ \mathcal{O}\big) \circ X \arrow[u, "\cong"] \arrow[ur, "\gamma \circ \mathrm{id}"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mathbb{1}_X"] & \mathcal{O} \circ X \\ I_\circ \circ X \arrow[u, "\cong"] \arrow[ur, "\mathbb{1} \circ \mathrm{id}"'] & \end{tikzcd} \tag{3.52}$$

for all objects $X \in \mathbf{M}^{\mathfrak{C}}$. The fact that $\mathcal{O}$ is a monoid in $(\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}), \circ, I_\circ)$ immediately implies that these natural transformations satisfy the axioms of a monad (cf. Definition 2.17). □

Definition 3.31. The category $\mathbf{Alg}(\mathcal{O})$ of *algebras over a $\mathfrak{C}$-colored operad* $\mathcal{O}$, or shorter $\mathcal{O}$-*algebras*, is the category of algebras over the monad $\mathcal{O} \circ (-) : \mathbf{M}^{\mathfrak{C}} \to \mathbf{M}^{\mathfrak{C}}$, cf. Definition 2.19. Concretely, an $\mathcal{O}$-algebra is an object $A \in \mathbf{M}^{\mathfrak{C}}$ together with an $\mathbf{M}^{\mathfrak{C}}$-morphism $\alpha : \mathcal{O} \circ A \to A$ (called *structure map*) that satisfies the conditions of Definition 2.19. To simplify notation, we often denote $\mathcal{O}$-algebras by their underlying object A.

Remark 3.32. Similarly to Remark 3.17, we may unpack this definition to recover the usual component-wise definition of algebras over a colored operad $\mathcal{O}$, see e.g. [Yau16]. The underlying object $A \in \mathbf{M}^{\mathfrak{C}}$ is simply a collection of objects $A_t \in \mathbf{M}$, for all colors $t \in \mathfrak{C}$. Using (3.51), the structure map $\alpha : \mathcal{O} \circ A \to A$ is specified by a collection of $\mathbf{M}$-morphisms

$$\alpha \,:\, \int^{\underline{a}} \mathcal{O}\tbinom{t}{\underline{a}} \otimes A_{a_1} \otimes \cdots \otimes A_{a_m} \longrightarrow A_t \quad , \tag{3.53}$$

for all $t \in \mathfrak{C}$. Due to the universal property of coends, α is determined by a family of $\mathbf{M}$-morphisms

$$\alpha \,:\, \mathcal{O}\tbinom{t}{\underline{a}} \otimes \bigotimes_{i=1}^{m} A_{a_i} \longrightarrow A_t \quad , \tag{3.54}$$

for all $t \in \mathfrak{C}$ and $\underline{a} \in \Sigma_{\mathfrak{C}}$, which satisfy the equivariance axiom of [Yau16, Definition 13.2.3]. The axioms of algebras over a monad in Definition 2.19 imply the associativity and unitality axiom of [Yau16, Definition 13.2.3] for these components. △

There is an equivalent point of view on algebras over a $\mathfrak{C}$-colored operad $\mathcal{O}$ which is particularly useful when $\mathcal{O}$ is presented by generators and relations (cf. Definition 3.23). Let $A \in \mathbf{Alg}(\mathcal{O})$ be an $\mathcal{O}$-algebra with structure map $\alpha : \mathcal{O} \circ A \to A$. Using that $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ is right closed (cf. Proposition 3.10), we can consider the adjoint morphism $\phi_\alpha : \mathcal{O} \to \mathrm{End}(A) := [A, A]_\circ$. Let us recall from Example 3.18 that $\mathrm{End}(A)$ carries a canonical $\mathfrak{C}$-colored operad structure and is called the endomorphism operad of $A \in \mathbf{M}^{\mathfrak{C}}$. The axioms for (A, α) to be an $\mathcal{O}$-algebra (cf. Definition 3.31) are equivalent to $\phi_\alpha : \mathcal{O} \to \mathrm{End}(A)$ being an $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$-morphism, i.e. ϕ_α preserves multiplications and units. Summing up, we obtain

Proposition 3.33. *Let $\mathcal{O}$ be a $\mathfrak{C}$-colored operad and A a $\mathfrak{C}$-colored object. The adjunction of Proposition 3.10 induces a bijection $\alpha \mapsto \phi_\alpha$ between $\mathcal{O}$-algebra structures $\alpha : \mathcal{O} \circ A \to A$ on A and $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$-morphisms $\phi_\alpha : \mathcal{O} \to \mathrm{End}(A)$ to the endomorphism operad of A. Furthermore, given two $\mathcal{O}$-algebras (A, α) and (A', α'), an $\mathbf{M}^{\mathfrak{C}}$-morphism $\psi : A \to A'$ defines an $\mathbf{Alg}(\mathcal{O})$-morphism $\psi : (A, \alpha) \to (A', \alpha')$ if and only if the square*

$$\begin{tikzcd} \mathcal{O} \arrow[r, "\phi_\alpha"] \arrow[d, "\phi_{\alpha'}"'] & \mathrm{End}(A) \arrow[d, "{[A,\psi]_\circ}"] \\ \mathrm{End}(A') \arrow[r, "{[\psi,A']_\circ}"'] & {[A, A']_\circ} \end{tikzcd} \tag{3.55}$$

in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ commutes.

Remark 3.34. Using the explicit formula for the internal-hom in Proposition 3.10, we obtain

$$\mathrm{End}(A)\tbinom{t}{\underline{c}} \;\cong\; \Big[A_{c_1} \otimes \cdots \otimes A_{c_n}, A_t\Big] \quad , \tag{3.56}$$

for all $\underline{c} = (c_1, \dots, c_n) \in \Sigma_{\mathfrak{C}}$ and $t \in \mathfrak{C}$. Writing out explicitly the (components of the) multiplication and unit of $\mathrm{End}(A)$, one recovers the usual description of the endomorphism operad as e.g. in [Yau16, Definition 13.8.1]. The interpretation of the endomorphism operad of $A \in \mathbf{M}^{\mathfrak{C}}$ is thus as the collection of all operations $A_{c_1} \otimes \cdots \otimes A_{c_n} \to A_t$ with n colored inputs and one colored output on A, together with their natural composition law, colored identities and equivariance properties under permutations. Therefore, regarding $\mathcal{O}$-algebras as $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$-morphisms $\phi : \mathcal{O} \to \mathrm{End}(A)$ amounts to realizing the abstract operations encoded in $\mathcal{O}\tbinom{t}{\underline{c}}$ in terms of concrete operations $A_{c_1} \otimes \cdots \otimes A_{c_n} \to A_t$ on A. $\triangle$

Let us now consider the $\mathfrak{C}$-colored operad $F(X : r_1, r_2)$ which is presented by the generators $X \in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ and relations $r_1, r_2 : R \to F(X)$ (cf. Definition 3.23). Being defined as the coequalizer of $r_1, r_2 : F(R) \rightrightarrows F(X)$, with F denoting the free $\mathfrak{C}$-colored operad functor, we obtain from Proposition 3.33 the following explicit characterization of $F(X : r_1, r_2)$-algebras.

Corollary 3.35. *The adjunctions of Proposition 3.10 and Theorem 3.19 induce a bijection $\alpha \mapsto \phi_\alpha$ between $F(X : r_1, r_2)$-algebra structures $\alpha : F(X : r_1, r_2) \circ A \to A$ on $A \in \mathbf{M}^{\mathfrak{C}}$ and $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$-morphisms $\phi_\alpha : X \to \mathrm{End}(A)$ such that $\phi_\alpha : F(X) \to \mathrm{End}(A)$ coequalizes $r_1, r_2 : R \rightrightarrows F(X)$ in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$. Furthermore, given two $F(X : r_1, r_2)$-algebras (A, α) and (A', α'), an $\mathbf{M}^{\mathfrak{C}}$-morphism $\psi : A \to A'$ defines an $\mathbf{Alg}(F(X : r_1, r_2))$-morphism $\psi : (A, \alpha) \to (A', \alpha')$ if and only if the square*

$$\begin{tikzcd} X \arrow[r, "\phi_\alpha"] \arrow[d, "\phi_{\alpha'}"'] & \mathrm{End}(A) \arrow[d, "{[A,\psi]_\circ}"] \\ \mathrm{End}(A') \arrow[r, "{[\psi,A']_\circ}"'] & {[A, A']_\circ} \end{tikzcd} \tag{3.57}$$

in $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ commutes.

3.3.2 Categorical properties and constructions

As a consequence of Theorem 2.20, there exists a free $\mathcal{O}$-algebra functor.

Corollary 3.36. *Let $\mathcal{O}$ be a $\mathfrak{C}$-colored operad. The forgetful functor $U_{\mathcal{O}} : \mathbf{Alg}(\mathcal{O}) \to \mathbf{M}^{\mathfrak{C}}$ has a left adjoint, i.e. there exists an adjunction*

$$\mathcal{O} \circ (-) \,:\, \mathbf{M}^{\mathfrak{C}} \rightleftarrows \mathbf{Alg}(\mathcal{O}) \,:\, U_{\mathcal{O}} \quad . \tag{3.58}$$

Explicitly, for $X \in \mathbf{M}^{\mathfrak{C}}$, the structure map $\gamma_X : \mathcal{O} \circ (\mathcal{O} \circ X) \to \mathcal{O} \circ X$ of the free $\mathcal{O}$-algebra $\mathcal{O} \circ X$ is the X-component of the structure transformation γ of the monad $\mathcal{O} \circ (-)$ and

$$(\mathcal{O} \circ X)_t \;\cong\; \int^{\underline{a}} \mathcal{O}\big(\begin{smallmatrix} t \\ \underline{a} \end{smallmatrix}\big) \otimes X_{a_1} \otimes \cdots \otimes X_{a_m} \quad , \tag{3.59}$$

for all $t \in \mathfrak{C}$. Moreover, the unit and counit of this adjunction explicitly read as

$$\mathbb{1}_X \,:\, X \longrightarrow \mathcal{O} \circ X \quad , \qquad \alpha \,:\, \mathcal{O} \circ A \longrightarrow A \quad , \tag{3.60}$$

for all $X \in \mathbf{M}^{\mathfrak{C}}$ and $(A, \alpha) \in \mathbf{Alg}(\mathcal{O})$, where we suppressed as usual the forgetful functor.

The following statement is proven in [WY15, Proposition 4.2.1]. See also [Fre17, Proposition 1.3.6] for a more concrete proof for the case of uncolored operads, which generalizes to colored operads in a straightforward way.

Proposition 3.37. *Let $\mathcal{O}$ be a $\mathfrak{C}$-colored operad. The category $\mathbf{Alg}(\mathcal{O})$ is complete and cocomplete. Moreover, the forgetful functor $U_{\mathcal{O}} : \mathbf{Alg}(\mathcal{O}) \to \mathbf{M}^{\mathfrak{C}}$ creates all small limits, the filtered colimits and the coequalizers which are reflexive in the category $\mathbf{M}^{\mathfrak{C}}$.*

The existence of free $\mathcal{O}$-algebras and coequalizers in $\mathbf{Alg}(\mathcal{O})$ allows us to construct examples of $\mathcal{O}$-algebras in terms of generators and relations. Let $X \in \mathbf{M}^{\mathfrak{C}}$ and consider a pair of parallel $\mathbf{M}^{\mathfrak{C}}$-morphisms $r_1, r_2 : R \rightrightarrows \mathcal{O} \circ X$.

Definition 3.38. The $\mathcal{O}$-algebra presented by the generators $X \in \mathbf{M}^{\mathfrak{C}}$ and relations $r_1, r_2 : R \rightrightarrows \mathcal{O} \circ X$ is defined as the coequalizer

$$F_{\mathcal{O}}(X \,{:}\, r_1, r_2) \;:=\; \mathrm{colim}\big(r_1, r_2 : \mathcal{O} \circ R \rightrightarrows \mathcal{O} \circ X\big) \tag{3.61}$$

in $\mathbf{Alg}(\mathcal{O})$.

3.3.3 Change of operad and color

Let $f : \mathfrak{C} \to \mathfrak{D}$ be a map of non-empty sets. Recall that there exists an induced pullback functor $f^* : \mathbf{SymSeq}_{\mathfrak{D}}(\mathbf{M}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$, which by Proposition 3.14 carries a canonical lax monoidal structure (f_2^*, f_0^*). Notice that this pullback functor restricts to the full subcategories of colored objects

$$f^* \,:\, \mathbf{M}^{\mathfrak{D}} \longrightarrow \mathbf{M}^{\mathfrak{C}} \quad . \tag{3.62}$$

The following statement is the analog of Proposition 3.13 for colored objects.

Proposition 3.39. *The pullback functor* (3.62) *has a left adjoint, i.e. there exists an adjunction*

$$f_! \,:\, \mathbf{M}^{\mathfrak{C}} \rightleftarrows \mathbf{M}^{\mathfrak{D}} \,:\, f^* \quad . \tag{3.63}$$

For any $X \in \mathbf{M}^{\mathfrak{C}}$, we have the explicit formula

$$f_!(X)_s \;=\; \coprod_{t \in f^{-1}(s)} X_t \quad , \tag{3.64}$$

for all $s \in \mathfrak{D}$. Moreover, the unit and counit of this adjunction explicitly read as

$$\iota_t \,:\, X_t \longrightarrow \coprod_{t' \in f^{-1}(f(t))} X_{t'} \quad , \qquad \coprod_{t' \in f^{-1}(s)} \mathrm{id}_{Y_s} \,:\, \coprod_{t' \in f^{-1}(s)} Y_s \longrightarrow Y_s \quad , \tag{3.65}$$

for all $X \in \mathbf{M}^{\mathfrak{C}}$, $Y \in \mathbf{M}^{\mathfrak{D}}$, $t \in \mathfrak{C}$ and $s \in \mathfrak{D}$, where ι denotes the inclusion morphisms into the coproducts.

Let $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$ be a morphism in the category $\mathbf{Op}(\mathbf{M})$ of operads with varying colors. Recall from Definition 3.26 that $f : \mathfrak{C} \to \mathfrak{D}$ is a map of sets and $\phi : \mathcal{O} \to f^*(\mathcal{P})$ is an $\mathbf{Op}_{\mathfrak{C}}(\mathbf{M})$-morphism to the pullback along f of the $\mathfrak{D}$-colored operad $\mathcal{P}$, see (3.46). We define a pullback functor

$$(f, \phi)^* \,:\, \mathbf{Alg}(\mathcal{P}) \longrightarrow \mathbf{Alg}(\mathcal{O}) \tag{3.66}$$

on the associated categories of algebras. Explicitly, to any $(B, \beta) \in \mathbf{Alg}(\mathcal{P})$ this functor assigns the $\mathcal{O}$-algebra $(f, \phi)^*(B, \beta)$ whose underlying $\mathfrak{C}$-colored object is $f^*(B)$ and whose structure map $\widetilde{\beta}$ is defined by the commutative diagram

$$\begin{tikzcd}
\mathcal{O} \circ f^*(B) \arrow[r, "\widetilde{\beta}"] \arrow[d, "\phi \circ \mathrm{id}"'] & f^*(B) \\
f^*(\mathcal{P}) \circ f^*(B) \arrow[r, "f_2^*"'] & f^*(\mathcal{P} \circ B) \arrow[u, "f^*(\beta)"']
\end{tikzcd} \tag{3.67}$$

in $\mathbf{M}^{\mathfrak{C}}$.

Theorem 3.40. *For any morphism $(f, \phi) : (\mathfrak{C}, \mathcal{O}) \to (\mathfrak{D}, \mathcal{P})$ in $\mathbf{Op}(\mathbf{M})$, the pullback functor* (3.66) *has a left adjoint, i.e. there exists an adjunction*

$$(f, \phi)_! \,:\, \mathbf{Alg}(\mathcal{O}) \rightleftarrows \mathbf{Alg}(\mathcal{P}) \,:\, (f, \phi)^* \quad . \tag{3.68}$$

Proof. This is an application of the adjoint lifting theorem, see e.g. [Bor94b, Chapter 4.5]. Concretely, we have the following diagram of categories and functors

$$\begin{tikzcd}
\mathbf{Alg}(\mathcal{O}) \arrow[d, shift left, "U_{\mathcal{O}}"] & \mathbf{Alg}(\mathcal{P}) \arrow[l, "(f,\phi)^*"'] \arrow[d, shift left, "U_{\mathcal{P}}"] \\
\mathbf{M}^{\mathfrak{C}} \arrow[u, shift left, "\mathcal{O} \circ (-)"] \arrow[r, shift left, "f_!"] & \mathbf{M}^{\mathfrak{D}} \arrow[u, shift left, "\mathcal{P} \circ (-)"] \arrow[l, shift left, "f^*"]
\end{tikzcd} \tag{3.69}$$

where the vertical adjunctions have been established in Corollary 3.36 and the bottom horizontal adjunction in Proposition 3.39. By definition of $(f, \phi)^*$, one observes that $U_{\mathcal{O}}\, (f, \phi)^* = f^*\, U_{\mathcal{P}}$. Existence of the left adjoint functor $(f, \phi)_!$ is then a consequence of [Bor94b, Theorem 4.5.6]. $\square$

Analyzing the proof of the adjoint lifting theorem [Bor94b, Chapter 4.5] one obtains, for any $(A, \alpha) \in \mathbf{Alg}(\mathcal{O})$, an explicit construction of $(f, \phi)_!(A, \alpha) \in \mathbf{Alg}(\mathcal{P})$ in terms of a reflexive coequalizer in $\mathbf{Alg}(\mathcal{P})$. In the following we will again suppress all forgetful functors to simplify our notation. The relevant pair of parallel morphisms in $\mathbf{Alg}(\mathcal{P})$ is then of the form

$$\begin{tikzcd}
\mathcal{P} \circ f_!\big(\mathcal{O} \circ A\big) \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & \mathcal{P} \circ f_!(A)
\end{tikzcd} \quad , \tag{3.70}$$

where ∂_0 and ∂_1 are described as follows: The $\mathbf{Alg}(\mathcal{P})$-morphism ∂_0 is defined by the structure map $\alpha : \mathcal{O} \circ A \to A$ via

$$\begin{tikzcd}
\mathcal{P} \circ f_!\big(\mathcal{O} \circ A\big) \arrow[r, "\partial_0 := \mathrm{id} \circ f_!(\alpha)"] & \mathcal{P} \circ f_!(A)
\end{tikzcd} \quad . \tag{3.71}$$

The $\mathbf{Alg}(\mathcal{P})$-morphism ∂_1 is defined by the units and counits of the adjunctions in (3.69) via

$$\begin{tikzcd}
\mathcal{P}\circ f_!\big(\mathcal{O}\circ A\big) \arrow[rr, "\partial_1"] \arrow[d, "\text{unit of } f_! \dashv f^*"'] & & \mathcal{P}\circ f_!(A) \\
\mathcal{P}\circ f_!\big(\mathcal{O}\circ f^* f_!(A)\big) \arrow[d, "\text{unit of } \mathcal{P}\circ(-)\dashv U_{\mathcal{P}}"'] & & \mathcal{P}\circ\big(\mathcal{P}\circ f_!(A)\big) \arrow[u, "\text{counit of } \mathcal{P}\circ(-)\dashv U_{\mathcal{P}}"'] \\
\mathcal{P}\circ f_!\big(\mathcal{O}\circ f^*\big(\mathcal{P}\circ f_!(A)\big)\big) \arrow[rr, "\text{counit of } \mathcal{O}\circ(-)\dashv U_{\mathcal{O}}"'] & & \mathcal{P}\circ f_! f^*\big(\mathcal{P}\circ f_!(A)\big) \arrow[u, "\text{counit of } f_!\dashv f^*"']
\end{tikzcd} \tag{3.72}$$

Using that the (co)units of the vertical adjunctions in (3.69) are given by structure maps (cf. Corollary 3.36), we can simplify the diagram (3.72) defining ∂_1 as

$$\begin{tikzcd}
\mathcal{P}\circ f_!\big(\mathcal{O}\circ A\big) \arrow[rr, "\partial_1"] \arrow[d, "\mathrm{id}\circ f_!(\phi\circ\mathrm{id})"'] & & \mathcal{P}\circ f_!(A) \\
\mathcal{P}\circ f_!\big(f^*(\mathcal{P})\circ A\big) \arrow[d, "\text{unit of } f_!\dashv f^*"'] & & \mathcal{P}\circ\big(\mathcal{P}\circ f_!(A)\big) \arrow[u, "\gamma_{f_!(A)}"'] \\
\mathcal{P}\circ f_!\big(f^*(\mathcal{P})\circ f^* f_!(A)\big) \arrow[rr, "\mathrm{id}\circ f_!(f_2^*)"'] & & \mathcal{P}\circ f_! f^*\big(\mathcal{P}\circ f_!(A)\big) \arrow[u, "\text{counit of } f_!\dashv f^*"']
\end{tikzcd} \tag{3.73}$$

Here we also used (3.67) to express the bottom horizontal arrow in (3.72). Notice that (3.70) is reflexive with reflector given by the unit of $\mathcal{O}\circ(-)\dashv U_{\mathcal{O}}$. Explicitly,

$$\begin{tikzcd}
\mathcal{P}\circ f_!(A) \arrow[r, "\mathrm{id}\circ f_!(\mathbb{1}_A)"] & \mathcal{P}\circ f_!\big(\mathcal{O}\circ A\big)
\end{tikzcd} \quad . \tag{3.74}$$

Proposition 3.37 implies that the colimit of the reflexive pair of parallel morphisms (3.70) is created by the forgetful functor to colored objects. Summing up, we obtain

Proposition 3.41. *The left adjoint functor $(f,\phi)_!$ of Theorem 3.40 maps an $\mathcal{O}$-algebra $(A,\alpha)\in\mathbf{Alg}(\mathcal{O})$ to the $\mathcal{P}$-algebra given by the reflexive coequalizer*

$$(f,\phi)_!(A,\alpha) \;=\; \mathrm{colim}\Big(\begin{tikzcd} \mathcal{P}\circ f_!\big(\mathcal{O}\circ A\big) \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & \mathcal{P}\circ f_!(A) \end{tikzcd}\Big) \tag{3.75}$$

in $\mathbf{Alg}(\mathcal{P})$, where ∂_0 and ∂_1 are given in (3.71) *and* (3.73)*, and the reflector is* (3.74)*.*

We shall also need the following lemmas for our work.

Lemma 3.42. *Forming the pullback functor along an $\mathbf{Op}(\mathbf{M})$-morphism according to* (3.66) *preserves identities and compositions, i.e.*

$$(\mathrm{id}_{\mathfrak{C}},\mathrm{id}_{\mathcal{O}})^* \;=\; \mathrm{id}_{\mathbf{Alg}(\mathcal{O})} \quad,\qquad \big((g,\psi)\,(f,\phi)\big)^* \;=\; (f,\phi)^*\,(g,\psi)^* \quad, \tag{3.76}$$

for all objects $(\mathfrak{C},\mathcal{O})\in\mathbf{Op}(\mathbf{M})$ and all composable $\mathbf{Op}(\mathbf{M})$-morphisms $(f,\phi):(\mathfrak{C},\mathcal{O})\to(\mathfrak{D},\mathcal{P})$ and $(g,\psi):(\mathfrak{D},\mathcal{P})\to(\mathfrak{F},\mathcal{Q})$. Consequently, forming its left adjoint functor according to (3.68) *preserves identities and compositions up to natural isomorphisms, i.e.*

$$(\mathrm{id}_{\mathfrak{C}},\mathrm{id}_{\mathcal{O}})_! \;\cong\; \mathrm{id}_{\mathbf{Alg}(\mathcal{O})} \quad,\qquad \big((g,\psi)\,(f,\phi)\big)_! \;\cong\; (g,\psi)_!\,(f,\phi)_! \quad, \tag{3.77}$$

for all objects $(\mathfrak{C},\mathcal{O})\in\mathbf{Op}(\mathbf{M})$ and all composable $\mathbf{Op}(\mathbf{M})$-morphisms $(f,\phi):(\mathfrak{C},\mathcal{O})\to(\mathfrak{D},\mathcal{P})$ and $(g,\psi):(\mathfrak{D},\mathcal{P})\to(\mathfrak{F},\mathcal{Q})$.

Proof. The first equality in (3.76) is obvious. Recalling that the composition in $\mathbf{Op}(\mathbf{M})$ of (f,ϕ) and (g,ψ) is given by $(g,\psi)\,(f,\phi) = \big(g\,f, f^*(\psi)\,\phi\big)$ (cf. Definition 3.26), the second equality in (3.76) can be proven easily by a short calculation using (3.67). The natural isomorphisms in (3.77) follow from (3.76) and uniqueness (up to natural isomorphism) of adjoint functors. $\square$

Lemma 3.43. *For any* $\mathbf{Op}(\mathbf{M})$*-morphism* $(f,\phi) : (\mathfrak{C},\mathcal{O}) \to (\mathfrak{D},\mathcal{P})$*, the right adjoint functor* $(f,\phi)^* : \mathbf{Alg}(\mathcal{P}) \to \mathbf{Alg}(\mathcal{O})$ *preserves reflexive coequalizers.*

Proof. Given any reflexive coequalizer

$$\begin{tikzcd} A \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\varphi_2"'] & B \arrow[r, "\chi"] & C \end{tikzcd} \tag{3.78}$$

in $\mathbf{Alg}(\mathcal{P})$, we have to show that

$$\begin{tikzcd} (f,\phi)^*A \arrow[r, shift left, "(f,\phi)^*\varphi_1"] \arrow[r, shift right, "(f,\phi)^*\varphi_2"'] & (f,\phi)^*B \arrow[r, "(f,\phi)^*\chi"] & (f,\phi)^*C \end{tikzcd} \tag{3.79}$$

is a coequalizer in $\mathbf{Alg}(\mathcal{O})$. By functoriality of $(f,\phi)^*$, it is obvious that $(f,\phi)^*\chi$ coequalizes the pair of parallel morphisms in (3.79) and that it is reflexive too. Hence, it remains to prove the universal property of a coequalizer. Using also Proposition 3.37, we observe that (3.79) is a coequalizer in $\mathbf{Alg}(\mathcal{O})$ if and only if

$$\begin{tikzcd} U_{\mathcal{O}}\,(f,\phi)^*A \arrow[r, shift left, "U_{\mathcal{O}}\,(f,\phi)^*\varphi_1"] \arrow[r, shift right, "U_{\mathcal{O}}\,(f,\phi)^*\varphi_2"'] & U_{\mathcal{O}}\,(f,\phi)^*B \arrow[r, "U_{\mathcal{O}}\,(f,\phi)^*\chi"] & U_{\mathcal{O}}\,(f,\phi)^*C \end{tikzcd} \tag{3.80}$$

is a coequalizer in $\mathbf{M}^{\mathfrak{C}}$. Because of $U_{\mathcal{O}}\,(f,\phi)^* = f^*\,U_{\mathcal{P}}$, the latter is equivalent to

$$\begin{tikzcd} f^*\,U_{\mathcal{P}}A \arrow[r, shift left, "f^*\,U_{\mathcal{P}}\varphi_1"] \arrow[r, shift right, "f^*\,U_{\mathcal{P}}\varphi_2"'] & f^*\,U_{\mathcal{P}}B \arrow[r, "f^*\,U_{\mathcal{P}}\chi"] & f^*\,U_{\mathcal{P}}C \end{tikzcd} \tag{3.81}$$

being a coequalizer in $\mathbf{M}^{\mathfrak{C}}$. This holds true because $U_{\mathcal{P}}$ preserves reflexive coequalizers (cf. Proposition 3.37) and $f^* : \mathbf{M}^{\mathfrak{D}} \to \mathbf{M}^{\mathfrak{C}}$ preserves all colimits because it has a right adjoint (given by the right Kan extension). $\square$

4 Quantum field theory operads

The aim of this section is to construct colored operads whose algebras describe quantum field theories. Abstractly, we assign to any small category $\mathbf{C}$ with orthogonality relation $\perp$ (cf. Definition 4.3 below) a colored operad that is inspired by the structures underlying algebraic quantum field theory. The category $\mathbf{C}$ should be interpreted as the category of "spacetimes" of interest, and the orthogonality relation $\perp$ encodes the commutative behavior of certain observables. Our construction is very flexible and in particular it reveals an operadic structure underlying various kinds of quantum field theories, including Haag-Kastler theories on the Minkowski spacetime [HK64], locally covariant theories on all Lorentzian spacetimes [BFV03, FV15], chiral conformal theories [Kaw15, Reh15, BDH15] and also Euclidean theories [Sch99]. Each of these models is obtained by a different choice of category $\mathbf{C}$ and orthogonality relation $\perp$, however the operadic structure is formally the same.

4.1 Motivation

Let $\mathbf{M}$ be a complete and cocomplete closed symmetric monoidal category and $\mathbf{C}$ a small category which we interpret as the category of spacetimes of interest. The basic idea of algebraic quantum field theory is to assign coherently to each spacetime $c \in \mathbf{C}$ a monoid $\mathfrak{A}(c)$ in $\mathbf{M}$, i.e. we consider a functor

$$\mathfrak{A} : \mathbf{C} \longrightarrow \mathbf{Mon}_{\mathbf{M}} \quad . \tag{4.1}$$

(In practice, one often takes $\mathbf{M} = \mathbf{Vec}_{\mathbb{K}}$, whose monoids are associative and unital algebras over the field $\mathbb{K}$, see Example 2.15.) The monoid $\mathfrak{A}(c)$ should be interpreted as the "algebra of quantum observables" that can be measured in the spacetime $c \in \mathbf{C}$. The $\mathbf{Mon}_{\mathbf{M}}$-morphism $\mathfrak{A}(f) : \mathfrak{A}(c) \to \mathfrak{A}(c')$ is the pushforward of observables along the $\mathbf{C}$-morphism $f : c \to c'$ (spacetime embedding). It is crucial to demand that the functor $\mathfrak{A}$ satisfies a collection of physically motivated axioms. Inspired by the time-slice axiom in Lorentzian theories [HK64, BFV03, FV15] we propose the following formalization and generalization.

Definition 4.1. Let $W \subseteq \operatorname{Mor} \mathbf{C}$ be a subset of the set of morphisms in $\mathbf{C}$. A functor $\mathfrak{A} : \mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$ is called *W-constant* if it sends every morphism $f : c \to c'$ in W to a $\mathbf{Mon}_{\mathbf{M}}$-isomorphism $\mathfrak{A}(f) : \mathfrak{A}(c) \xrightarrow{\cong} \mathfrak{A}(c')$.

Remark 4.2. Our notion of W-constancy is similar to and generalizes the concept of *local constancy* of factorization algebras [CG17]. This motivates our choice of terminology. △

Because $\mathbf{C}$ is a small category, the localization $\mathbf{C}[W^{-1}]$ of $\mathbf{C}$ at W exists as a small category, see e.g. [Bor94a, Proposition 5.2.2]. Hence, we may equivalently describe W-constant functors as functors $\mathfrak{A} : \mathbf{C}[W^{-1}] \to \mathbf{Mon}_{\mathbf{M}}$ on the localized category. This means that the W-constancy axiom may be implemented formally by choosing $\mathbf{C}[W^{-1}]$ instead of $\mathbf{C}$ as the underlying category. In the following we shall always denote the underlying category by $\mathbf{C}$ in order to simplify notation. More generally, if one is interested in theories satisfying a certain constancy axiom, one takes $\mathbf{C}$ to be the localization of some other category.

Another important property of quantum field theories is that certain pairs of observables behave commutatively. For example, in Lorentzian theories [HK64, BFV03, FV15] any two observables which are associated to spacelike separated regions commute with each other. Similarly, in Euclidean [Sch99] and chiral conformal [Kaw15, Reh15, BDH15] theories any two observables associated to disjoint regions commute. We propose the following formalization and generalization of this concept: Let us denote by $\operatorname{Mor} \mathbf{C} \,{}_{\mathrm{t}}{\times}_{\mathrm{t}} \operatorname{Mor} \mathbf{C}$ the set of pairs of $\mathbf{C}$-morphisms whose targets coincide. Its elements (f_1, f_2) may be visualized as $\mathbf{C}$-diagrams of the form $c_1 \xrightarrow{f_1} c \xleftarrow{f_2} c_2$, where $c_1, c_2, c \in \mathbf{C}$.

Definition 4.3. Let $\mathbf{C}$ be a small category. An *orthogonality relation* $\perp$ on $\mathbf{C}$ is a subset $\perp \subseteq \operatorname{Mor} \mathbf{C} \,{}_{\mathrm{t}}{\times}_{\mathrm{t}} \operatorname{Mor} \mathbf{C}$ satisfying the following properties:

(1) *Symmetry:* $(f_1, f_2) \in \perp \implies (f_2, f_1) \in \perp$.

(2) *Stability under post-composition:* $(f_1, f_2) \in \perp \implies (g\, f_1, g\, f_2) \in \perp$, for all composable $\mathbf{C}$-morphisms g.

(3) *Stability under pre-composition:* $(f_1, f_2) \in \perp \implies (f_1\, h_1, f_2\, h_2) \in \perp$, for all composable $\mathbf{C}$-morphisms h_1 and h_2.

We call elements $(f_1, f_2) \in \perp$ *orthogonal pairs* and write also $f_1 \perp f_2$. To simplify notation, we often denote the pair consisting of a small category and an orthogonality relation by $\overline{\mathbf{C}} := (\mathbf{C}, \perp)$ and call it an *orthogonal category*.

Definition 4.4. Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ be an orthogonal category.

a) A functor $\mathfrak{A} : \mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$ is called $\perp$-*commutative over the object* $c \in \mathbf{C}$ if for every $f_1 \perp f_2$ with target c the diagram

$$\begin{tikzcd}
\mathfrak{A}(c_1) \otimes \mathfrak{A}(c_2) \arrow[r, "\mathfrak{A}(f_1)\otimes\mathfrak{A}(f_2)"] \arrow[d, "\mathfrak{A}(f_1)\otimes\mathfrak{A}(f_2)"'] & \mathfrak{A}(c) \otimes \mathfrak{A}(c) \arrow[d, "\mu_c^{\mathrm{op}}"] \\
\mathfrak{A}(c) \otimes \mathfrak{A}(c) \arrow[r, "\mu_c"'] & \mathfrak{A}(c)
\end{tikzcd} \tag{4.2}$$

in $\mathbf{M}$ commutes. Here μ_c denotes the multiplication on $\mathfrak{A}(c)$ and $\mu_c^{\mathrm{op}} := \mu_c\,\tau$ the opposite multiplication on $\mathfrak{A}(c)$, with τ the symmetric braiding on $\mathbf{M}$.

b) A functor $\mathfrak{A} : \mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$ is called $\perp$-*commutative* if it is $\perp$-commutative over every object $c \in \mathbf{C}$.

c) The full subcategory of $\mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}$ whose objects are all $\perp$-commutative functors is denoted by $\mathbf{Mon}_{\mathbf{M}}^{\overline{\mathbf{C}}}$. There exists a fully faithful forgetful functor

$$U \,:\, \mathbf{Mon}_{\mathbf{M}}^{\overline{\mathbf{C}}} \longrightarrow \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}} \quad , \tag{4.3}$$

which forgets $\perp$-commutativity.

Notice that, in contrast to W-constancy, $\perp$-commutativity can not be implemented easily by adjusting the categories $\mathbf{C}$ and $\mathbf{Mon}_{\mathbf{M}}$: It imposes a non-trivial condition on functors $\mathfrak{A} : \mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$ that relates orthogonal pairs of morphisms $f_1 \perp f_2$ in $\mathbf{C}$ to a certain commutative behavior of the algebraic structures on the objects of $\mathbf{M}$ underlying $\mathfrak{A}$. The colored operads we develop in this section solve this problem in the sense that they allow us to encode $\perp$-commutativity as part of the structure instead of enforcing it as a property. This is very useful for the study of universal constructions on quantum field theories, see Sections 5 and 6.

4.2 Definition in Set and properties

Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ be an orthogonal category (cf. Definition 4.3) and denote by $\mathbf{C}_0$ the set of objects of $\mathbf{C}$. The aim of this section is to define a $\mathbf{C}_0$-colored operad $\mathcal{O}_{\overline{\mathbf{C}}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$ with values in $\mathbf{Set}$ that is inspired by the algebraic structures underlying quantum field theories. For this we first construct an auxiliary $\mathbf{C}_0$-colored operad $\mathcal{O}_{\mathbf{C}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$, neglecting the orthogonality relation, and then define $\mathcal{O}_{\overline{\mathbf{C}}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$ in terms of an equivalence relation determined by $\perp$. Our construction can be promoted from $\mathbf{Set}$ to any complete and cocomplete closed symmetric monoidal category $\mathbf{M}$ by a change of base category construction. This will be addressed in Section 4.4. The reason why we focus in the present section on the special case of $\mathbf{Set}$-valued operads is that this simplifies the explicit calculations below.

Let us introduce the following useful notation: Given any $\mathbf{C}_0$-profile $\underline{c} = (c_1, \dots, c_n) \in \Sigma_{\mathbf{C}_0}$ and $t \in \mathbf{C}_0$, we write

$$\mathbf{C}(\underline{c}, t) \,:=\, \prod_{i=1}^{n} \mathbf{C}(c_i, t) \tag{4.4a}$$

for the product of Hom-sets and denote its elements by

$$\underline{f} = (f_1, \dots, f_n) \in \mathbf{C}(\underline{c}, t) \quad , \tag{4.4b}$$

where f_i is a $\mathbf{C}$-morphism $f_i : c_i \to t$. Note that for the empty profile $\underline{c} = \emptyset$, we have $\mathbf{C}(\emptyset, t) = \{*\}$. For $\underline{f} = (f_1, \dots, f_m) \in \mathbf{C}(\underline{a}, t)$ with $|\underline{a}| = m \geq 1$ and $\underline{g}_i = (g_{i1}, \dots, g_{ik_i}) \in \mathbf{C}(\underline{b}_i, a_i)$ with $|\underline{b}_i| = k_i \geq 0$, for $i = 1, \dots, m$, we define

$$\underline{f}(\underline{g}_1, \dots, \underline{g}_m) := \big(f_1\, g_{11}, \dots, f_1\, g_{1k_1}, \dots, f_m\, g_{m1}, \dots, f_m\, g_{mk_m}\big) \in \mathbf{C}(\underline{b}_1 \otimes \cdots \otimes \underline{b}_m, t) \tag{4.5}$$

by using the composition of $\mathbf{C}$-morphisms.

Definition 4.5. Let $\mathbf{C}$ be a small category.

a) To any $(\underline{c}, t) \in \Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$, we assign the set

$$\mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{c}}\big) := \Sigma_{|\underline{c}|} \times \mathbf{C}(\underline{c}, t) \quad . \tag{4.6}$$

b) To any $\Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$-morphism $\sigma' : (\underline{c}, t) \to (\underline{c}\sigma', t)$, we assign the map of sets

$$\mathcal{O}_{\mathbf{C}}(\sigma') \,:\, \mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{c}}\big) \longrightarrow \mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{c}\sigma'}\big) \quad , \qquad (\sigma, \underline{f}) \longmapsto (\sigma\sigma', \underline{f}\sigma') \quad , \tag{4.7}$$

where $\underline{f}\sigma' := (f_{\sigma'(1)}, \dots, f_{\sigma'(n)}) \in \mathbf{C}(\underline{c}\sigma', t)$.

c) To any $t \in \mathbf{C}_0$, $\underline{a} \in \Sigma_{\mathbf{C}_0}$ with length $|\underline{a}| = m \geq 1$ and $\underline{b}_i \in \Sigma_{\mathbf{C}_0}$ with length $|\underline{b}_i| = k_i \geq 0$, for $i = 1, \dots, m$, we assign the map of sets

$$\begin{aligned} \gamma \,:\, \mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{a}}\big) \otimes \bigotimes_{i=1}^{m} \mathcal{O}_{\mathbf{C}}\big(\substack{a_i\\ \underline{b}_i}\big) &\longrightarrow \mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{b}_1 \otimes \cdots \otimes \underline{b}_m}\big) \quad , \\ (\sigma, \underline{f}) \otimes \bigotimes_{i=1}^{m} (\sigma_i, \underline{g}_i) &\longmapsto \Big(\sigma(\sigma_1, \dots, \sigma_m), \underline{f}(\underline{g}_1, \dots, \underline{g}_m)\Big) \quad , \end{aligned} \tag{4.8}$$

where

$$\sigma(\sigma_1, \dots, \sigma_m) := \sigma\langle k_{\sigma^{-1}(1)}, \dots, k_{\sigma^{-1}(m)}\rangle \; (\sigma_1 \oplus \cdots \oplus \sigma_m) \tag{4.9}$$

is the group multiplication in $\Sigma_{\sum_{i=1}^m k_i}$ of the block permutation $\sigma\langle k_{\sigma^{-1}(1)}, \dots, k_{\sigma^{-1}(m)}\rangle$ induced by σ and the block sum permutation $\sigma_1 \oplus \cdots \oplus \sigma_m$ induced by the σ_i, and $\underline{f}(\underline{g}_1, \dots, \underline{g}_m)$ is defined in (4.5).

d) To any $t \in \mathbf{C}_0$, we assign the map of sets

$$\mathbb{1} \,:\, I \longrightarrow \mathcal{O}_{\mathbf{C}}\big(\substack{t\\ t}\big) \quad , \qquad * \longmapsto (e, \mathrm{id}_t) \quad , \tag{4.10}$$

where $e \in \Sigma_1$ is the group identity.

We now show that $(\mathcal{O}_{\mathbf{C}}, \gamma, \mathbb{1})$ as per Definition 4.5 is a colored operad by verifying the component-wise axioms stated in Remark 3.17. For this we shall need some standard properties of permutations which are easy to verify.

Lemma 4.6. *(i) For every $\sigma, \sigma' \in \Sigma_m$ with $m \geq 1$ and $k_i \geq 0$, for $i = 1, \dots, m$, we have that*

$$(\sigma\sigma')\langle k_1, \dots, k_m\rangle = \sigma\langle k_1, \dots, k_m\rangle \; \sigma'\langle k_{\sigma(1)}, \dots, k_{\sigma(m)}\rangle \quad . \tag{4.11}$$

(ii) For every $\sigma \in \Sigma_m$ with $m \geq 1$ and $\sigma_i \in \Sigma_{k_i}$ with $k_i \geq 0$, for $i = 1, \dots, m$, we have that

$$(\sigma_1 \oplus \cdots \oplus \sigma_m) \; \sigma\langle k_1, \dots, k_m\rangle = \sigma\langle k_1, \dots, k_m\rangle \; (\sigma_{\sigma(1)} \oplus \cdots \oplus \sigma_{\sigma(m)}) \quad . \tag{4.12}$$

(iii) For every $m \geq 1$ and $\sigma_i, \sigma_i' \in \Sigma_{k_i}$ with $k_i \geq 0$, for $i = 1, \dots, m$, we have that

$$(\sigma_1\, \sigma_1') \oplus \cdots \oplus (\sigma_m\, \sigma_m') = (\sigma_1 \oplus \cdots \oplus \sigma_m) \; (\sigma_1' \oplus \cdots \oplus \sigma_m') \quad . \tag{4.13}$$

(iv) For every $\sigma \in \Sigma_m$ with $m \geq 1$, $k_i \geq 1$, for $i = 1, \dots, m$, and $\ell_{ij} \geq 0$, for $i = 1, \dots, m$ and $j = 1, \dots, k_i$, we have that

$$\big(\sigma\langle k_1, \dots, k_m\rangle\big)\big\langle \ell_{11}, \dots, \ell_{1k_1}, \dots, \ell_{m1}, \dots, \ell_{mk_m}\big\rangle = \sigma\Big\langle \sum_{j=1}^{k_1} \ell_{1j}, \dots, \sum_{j=1}^{k_m} \ell_{mj}\Big\rangle \quad . \tag{4.14}$$

(v) For every $m \geq 1$, $\sigma_i \in \Sigma_{k_i}$ with $k_i \geq 1$, for $i = 1, \dots, m$, and $\ell_{ij} \geq 0$, for $i = 1, \dots, m$ and $j = 1, \dots, k_i$, we have that

$$(\sigma_1 \oplus \cdots \oplus \sigma_m)\big\langle \ell_{11}, \dots, \ell_{1k_1}, \dots, \ell_{m1}, \dots, \ell_{mk_m}\big\rangle \\ = \sigma_1\langle \ell_{11}, \dots, \ell_{1k_1}\rangle \oplus \cdots \oplus \sigma_m\langle \ell_{m1}, \dots, \ell_{mk_m}\rangle \quad . \tag{4.15}$$

(vi) For every $m \geq 1$ and $\sigma \in \Sigma_m$, we have that

$$\sigma\langle\underbrace{1, \dots, 1}_{m \text{ times}}\rangle = \sigma \quad , \qquad \underbrace{e \oplus \cdots \oplus e}_{m \text{ times}} = e \quad , \tag{4.16}$$

where in the second equality $e \in \Sigma_1$ on the left-hand side, while $e \in \Sigma_m$ on the right-hand side.

Proposition 4.7. *$\mathcal{O}_{\mathbf{C}}$ given in Definition 4.5 is a $\mathbf{C}_0$-colored operad with values in* **Set***.*

Proof. It is easy to confirm that $\mathcal{O}_{\mathbf{C}}$ is an object in $\mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{Set})$, i.e. a functor $\mathcal{O}_{\mathbf{C}} : \Sigma_{\mathbf{C}_0} \times \mathbf{C}_0 \to \mathbf{Set}$.

First equivariance axiom (3.38a)*:* For $\sigma' \in \Sigma_m$, we obtain

$$\gamma\left(\big(\mathcal{O}_{\mathbf{C}}(\sigma') \otimes \tau_{\sigma'}\big)\left((\sigma, \underline{f}) \otimes \bigotimes_{i=1}^{m} (\sigma_i, \underline{g}_i)\right)\right) \\ = \Big((\sigma\sigma')\big(\sigma_{\sigma'(1)}, \dots, \sigma_{\sigma'(m)}\big), (\underline{f}\sigma')\big(\underline{g}_{\sigma'(1)}, \dots, \underline{g}_{\sigma'(m)}\big)\Big) \tag{4.17a}$$

and

$$\mathcal{O}_{\mathbf{C}}(\sigma'\langle k_1, \dots, k_m\rangle)\left(\gamma\left((\sigma, \underline{f}) \otimes \bigotimes_{i=1}^{m} (\sigma_i, \underline{g}_i)\right)\right) \\ = \Big(\sigma(\sigma_1, \dots, \sigma_m)\ \sigma'\langle k_1, \dots, k_m\rangle, \underline{f}(\underline{g}_1, \dots, \underline{g}_m)\, \sigma'\langle k_1, \dots, k_m\rangle\Big) \quad , \tag{4.17b}$$

for all $(\sigma, \underline{f}) \otimes \bigotimes_{i=1}^m (\sigma_i, \underline{g}_i) \in \mathcal{O}_{\mathbf{C}}\big(\substack{t \\ \underline{a}}\big) \otimes \bigotimes_{i=1}^m \mathcal{O}_{\mathbf{C}}\big(\substack{a_i \\ \underline{b}_i}\big)$. Using Lemma 4.6 (i) and (ii), we observe that both expressions coincide, i.e. the first equivariance axiom holds true.

Second equivariance axiom (3.38b)*:* For $\sigma_i' \in \Sigma_{k_i}$, for $i = 1, \dots, m$, we obtain

$$\gamma\left((\sigma, \underline{f}) \otimes \bigotimes_{i=1}^{m} \mathcal{O}_{\mathbf{C}}(\sigma_i')(\sigma_i, \underline{g}_i)\right) = \Big(\sigma\big(\sigma_1\sigma_1', \dots, \sigma_m\sigma_m'\big), \underline{f}\big(\underline{g}_1\sigma_1', \dots, \underline{g}_m\sigma_m'\big)\Big) \tag{4.18a}$$

and

$$\mathcal{O}_{\mathbf{C}}(\sigma_1' \oplus \cdots \oplus \sigma_m')\left(\gamma\left((\sigma, \underline{f}) \otimes \bigotimes_{i=1}^{m} (\sigma_i, \underline{g}_i)\right)\right) \\ = \Big(\sigma(\sigma_1, \dots, \sigma_m)\ (\sigma_1' \oplus \cdots \oplus \sigma_m'), \underline{f}\big(\underline{g}_1, \dots, \underline{g}_m\big)\ (\sigma_1' \oplus \cdots \oplus \sigma_m')\Big) \quad , \tag{4.18b}$$

for all $(\sigma, \underline{f}) \otimes \bigotimes_{i=1}^m (\sigma_i, \underline{g}_i) \in \mathcal{O}_{\mathbf{C}}\big(\substack{t \\ \underline{a}}\big) \otimes \bigotimes_{i=1}^m \mathcal{O}_{\mathbf{C}}\big(\substack{a_i \\ \underline{b}_i}\big)$. Using Lemma 4.6 (iii), we observe that both expressions coincide, i.e. the second equivariance axiom holds true.

Associativity axiom (3.41)*:* For all $(\sigma, \underline{f}) \in \mathcal{O}_{\mathbf{C}}\big(\substack{t \\ \underline{a}}\big)$, $(\sigma_i, \underline{g}_i) \in \mathcal{O}_{\mathbf{C}}\big(\substack{a_i \\ \underline{b}_i}\big)$, for $i = 1, \dots, m$, and $(\sigma_{ij}, \underline{h}_{ij}) \in \mathcal{O}_{\mathbf{C}}\big(\substack{b_{ij} \\ \underline{c}_{ij}}\big)$, for $i = 1, \dots, m$ and $j = 1, \dots, k_i$, we obtain

$$\gamma\left(\gamma\left((\sigma, \underline{f}) \otimes \bigotimes_{i=1}^{m} (\sigma_i, \underline{g}_i)\right) \otimes \bigotimes_{i=1}^{m} \bigotimes_{j=1}^{k_i} (\sigma_{ij}, \underline{h}_{ij})\right) \\ = \begin{pmatrix} \big(\sigma(\sigma_1, \dots, \sigma_m)\big)(\sigma_{11}, \dots, \sigma_{1k_1}, \dots, \sigma_{m1}, \dots, \sigma_{mk_m}) \\ \big(\underline{f}(\underline{g}_1, \dots, \underline{g}_m)\big)(\underline{h}_{11}, \dots, \underline{h}_{1k_1}, \dots, \underline{h}_{m1}, \dots, \underline{h}_{mk_m}) \end{pmatrix} \tag{4.19a}$$

and

$$
\gamma\left((\sigma,\underline{f})\otimes\bigotimes_{i=1}^{m}\gamma\left((\sigma_i,\underline{g}_i)\otimes\bigotimes_{j=1}^{k_i}(\sigma_{ij},\underline{h}_{ij})\right)\right)
$$
$$
=\begin{pmatrix}\sigma\big(\sigma_1(\sigma_{11},\dots,\sigma_{1k_1}),\dots,\sigma_m(\sigma_{m1},\dots,\sigma_{mk_m})\big)\\ \underline{f}\big(\underline{g}_1(\underline{h}_{11},\dots,\underline{h}_{1k_1}),\dots,\underline{g}_m(\underline{h}_{m1},\dots,\underline{h}_{mk_m})\big)\end{pmatrix}\quad . \tag{4.19b}
$$

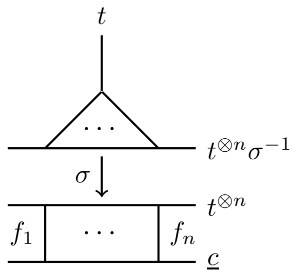
Using Lemma 4.6 (i), (iii), (iv) and (v), we observe that both expressions coincide, i.e. the associativity axiom holds true.

Unit axiom (3.42)*:* For all $(\sigma,\underline{f})\in\mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{a}}\big)$, we obtain

$$
\gamma\Big((\sigma,\underline{f})\otimes\bigotimes_{i=1}^{m}(e,\mathrm{id}_{a_i})\Big)=(\sigma,\underline{f})=\gamma\Big((e,\mathrm{id}_t)\otimes(\sigma,\underline{f})\Big)\quad , \tag{4.20}
$$

where Lemma 4.6 (vi) has been used. This shows that the unit axiom holds true. □

Remark 4.8. The operad $\mathcal{O}_{\mathbf{C}}$ has a natural physical interpretation coming from algebraic quantum field theory. We may graphically visualize an element $(\sigma,\underline{f})\in\mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{c}}\big)$ by



(4.21)

This picture should be read from bottom to top and it should be understood as the following 3-step operation in algebraic quantum field theory. [The presentation below is intentionally slightly heuristic. It can be made precise by considering algebras over $\mathcal{O}_{\mathbf{C}}$. This is however not the aim of this motivational remark.] (1) Apply the morphisms $\underline{f}$ to observables on $\underline{c}=(c_1,\dots,c_n)$; (2) Permute the resulting observables on $t^{\otimes n}$ by acting with σ^{-1} from the right; (3) Multiply the resulting observables on $t^{\otimes n}\sigma^{-1}$ according to the order in which they appear. Concretely, given observables Φ_i on c_i, for $i=1,\dots,n$, the 3-step operation formally looks like

$$
\begin{aligned}
\Phi_1\otimes\cdots\otimes\Phi_n \;&\overset{(1)}{\longmapsto}\; f_1(\Phi_1)\otimes\cdots\otimes f_n(\Phi_n)\\
&\overset{(2)}{\longmapsto}\; f_{\sigma^{-1}(1)}\big(\Phi_{\sigma^{-1}(1)}\big)\otimes\cdots\otimes f_{\sigma^{-1}(n)}\big(\Phi_{\sigma^{-1}(n)}\big)\\
&\overset{(3)}{\longmapsto}\; f_{\sigma^{-1}(1)}\big(\Phi_{\sigma^{-1}(1)}\big)\,\cdots\, f_{\sigma^{-1}(n)}\big(\Phi_{\sigma^{-1}(n)}\big)\quad .
\end{aligned}\tag{4.22}
$$

The operad structure on $\mathcal{O}_{\mathbf{C}}$ we introduced in Definition 4.5 is motivated by this physical interpretation and it precisely encodes how such 3-step operations in algebraic quantum field theory compose. △

The $\mathbf{C}_0$-colored operad $\mathcal{O}_{\mathbf{C}}$ established in Definition 4.5 and Proposition 4.7 does not yet encode the orthogonality relation $\perp$ on the category $\mathbf{C}$. We shall now equip the sets $\mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{c}}\big)$ with an equivalence relation that is inspired by the $\perp$-commutativity axiom (cf. Definition 4.4) and our quantum field theoretic interpretation of the elements of $\mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{c}}\big)$ given in (4.22).

Definition 4.9. Let $\overline{\mathbf{C}}=(\mathbf{C},\perp)$ be an orthogonal category. For each $(\underline{c},t)\in\Sigma_{\mathbf{C}_0}\times\mathbf{C}_0$, we equip the set $\mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{c}}\big)$ given in (4.6) with the following equivalence relation: $(\sigma,\underline{f})\sim_\perp(\sigma',\underline{f'})$ if and only if

(1) $\underline{f} = \underline{f}'$ as elements in $\mathbf{C}(\underline{c}, t)$;

(2) the right permutation $\sigma\sigma'^{-1} : \underline{f}\sigma^{-1} \to \underline{f}\sigma'^{-1}$ is generated by transpositions of *adjacent orthogonal pairs*. Precisely, this means that $\sigma\sigma'^{-1} = \tau_1 \cdots \tau_N$ may be factored into a product of transpositions $\tau_1, \dots, \tau_N \in \Sigma_{|\underline{c}|}$, for some $N \in \mathbb{N}$, such that the right permutation

$$\tau_k \,:\, \underline{f}\sigma^{-1}\tau_1 \cdots \tau_{k-1} \longrightarrow \underline{f}\sigma^{-1}\tau_1 \cdots \tau_k \tag{4.23}$$

is a transposition of a pair of $\mathbf{C}$-morphisms that are orthogonal and adjacent in the sequence $\underline{f}\sigma^{-1}\tau_1 \cdots \tau_{k-1}$, for all $k = 1, \dots, N$.

We denote the corresponding quotient set by

$$\mathcal{O}_{\overline{\mathbf{C}}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \,:=\, \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big)/\!\sim_\perp \quad , \tag{4.24}$$

for all $(\underline{c}, t) \in \Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$, and its elements by $[\sigma, \underline{f}] \in \mathcal{O}_{\overline{\mathbf{C}}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big)$.

Example 4.10. Let us illustrate the equivalence relation $\sim_\perp$ with a simple example: Let $\underline{c} = (c_1, c_2) \in \Sigma_{\mathbf{C}_0}$ and $t \in \mathbf{C}_0$. Elements in $\mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big)$ are either of the form $(e, (f_1, f_2))$ or of the form $(\tau, (f_1, f_2))$, where $e \in \Sigma_2$ denotes the identity permutation and $\tau \in \Sigma_2$ the transposition. It follows that $(e, (f_1, f_2)) \sim_\perp (\tau, (f_1, f_2))$ if and only if $f_1 \perp f_2$. Indeed, using $\tau^{-1} = \tau \in \Sigma_2$, the right permutation

$$e\tau^{-1} = \tau \;:\; (f_1, f_2)e^{-1} = (f_1, f_2) \quad\longrightarrow\quad (f_1, f_2)\tau^{-1} = (f_2, f_1) \tag{4.25}$$

is a transposition of an adjacent orthogonal pair if and only $f_1 \perp f_2$. $\triangledown$

Proposition 4.11. *Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ be an orthogonal category. The operad structure on $\mathcal{O}_{\mathbf{C}}$ given in Definition 4.5 descends to the quotients described in Definition 4.9. Consequently, $\mathcal{O}_{\overline{\mathbf{C}}}$ is a $\mathbf{C}_0$-colored operad with values in* $\mathbf{Set}$.

Proof. We have to prove that the equivalence relation $\sim_\perp$ from Definition 4.9 is compatible with the maps in Definition 4.5 b), c) and d). It then follows from Proposition 4.7 that $(\mathcal{O}_{\overline{\mathbf{C}}}, \gamma, \mathbb{1})$ is a $\mathbf{C}_0$-colored operad.

Compatibility with right permutations (4.7)*:* Let $\sigma' : (\underline{c}, t) \to (\underline{c}\sigma', t)$ be any $\Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$-morphism and consider any two elements $(\sigma, \underline{f}), (\widetilde{\sigma}, \underline{f}) \in \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big)$ such that $(\sigma, \underline{f}) \sim_\perp (\widetilde{\sigma}, \underline{f})$. We obtain that

$$\mathcal{O}_{\mathbf{C}}(\sigma')(\sigma, \underline{f}) = (\sigma\sigma', \underline{f}\sigma') \;\sim_\perp\; (\widetilde{\sigma}\sigma', \underline{f}\sigma') = \mathcal{O}_{\mathbf{C}}(\sigma')(\widetilde{\sigma}, \underline{f}) \quad , \tag{4.26}$$

because the right permutation

$$(\sigma\sigma')(\widetilde{\sigma}\sigma')^{-1} = \sigma\widetilde{\sigma}^{-1} \;:\; \underline{f}\sigma'(\sigma\sigma')^{-1} = \underline{f}\sigma^{-1} \quad\longrightarrow\quad \underline{f}\sigma'(\widetilde{\sigma}\sigma')^{-1} = \underline{f}\widetilde{\sigma}^{-1} \tag{4.27}$$

is precisely the one corresponding to $(\sigma, \underline{f}) \sim_\perp (\widetilde{\sigma}, \underline{f})$ and hence it is generated by transpositions of adjacent orthogonal pairs.

Compatibility with compositions (4.8)*:* Let $t \in \mathbf{C}_0$, $\underline{a} \in \Sigma_{\mathbf{C}_0}$ with length $|\underline{a}| = m \geq 1$ and $\underline{b}_i \in \Sigma_{\mathbf{C}_0}$ with length $|\underline{b}_i| = k_i \geq 0$, for $i = 1, \dots, m$.

Firstly, let us consider any two elements $(\sigma, \underline{f}), (\widetilde{\sigma}, \underline{f}) \in \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{a} \end{smallmatrix}\big)$ such that $(\sigma, \underline{f}) \sim_\perp (\widetilde{\sigma}, \underline{f})$ and any family of elements $(\sigma_i, \underline{g}_i) \in \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} a_i \\ \underline{b}_i \end{smallmatrix}\big)$, for $i = 1, \dots, m$. We have to show that the two elements

$$\gamma\left((\sigma, \underline{f}) \otimes \bigotimes_{i=1}^m (\sigma_i, \underline{g}_i)\right) = \Big(\sigma(\sigma_1, \dots, \sigma_m), \underline{f}(\underline{g}_1, \dots, \underline{g}_m)\Big) \quad , \tag{4.28a}$$

$$\gamma\left((\widetilde{\sigma}, \underline{f}) \otimes \bigotimes_{i=1}^m (\sigma_i, \underline{g}_i)\right) = \Big(\widetilde{\sigma}(\sigma_1, \dots, \sigma_m), \underline{f}(\underline{g}_1, \dots, \underline{g}_m)\Big) \tag{4.28b}$$

in $\mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{b}_1 \otimes \cdots \otimes \underline{b}_m \end{smallmatrix}\big)$ are equivalent under $\sim_\perp$. Using Lemma 4.6, we can express the relevant right permutation

$$
\begin{array}{c}
\underline{f}(\underline{g}_1, \dots, \underline{g}_m) \ \big(\sigma(\sigma_1, \dots, \sigma_m)\big)^{-1} \\
\Big\downarrow {\scriptstyle \sigma(\sigma_1,\dots,\sigma_m)\ (\widetilde{\sigma}(\sigma_1,\dots,\sigma_m))^{-1}} \\
\underline{f}(\underline{g}_1, \dots, \underline{g}_m) \ \big(\widetilde{\sigma}(\sigma_1, \dots, \sigma_m)\big)^{-1}
\end{array}
\tag{4.29a}
$$

by

$$
\begin{array}{c}
(\underline{f}\sigma^{-1})\big(\underline{g}_{\sigma^{-1}(1)}\sigma^{-1}_{\sigma^{-1}(1)}, \dots, \underline{g}_{\sigma^{-1}(m)}\sigma^{-1}_{\sigma^{-1}(m)}\big) \\
\Big\downarrow {\scriptstyle (\sigma\widetilde{\sigma}^{-1})\langle k_{\sigma^{-1}(1)},\dots,k_{\sigma^{-1}(m)}\rangle} \\
(\underline{f}\widetilde{\sigma}^{-1})\big(\underline{g}_{\widetilde{\sigma}^{-1}(1)}\sigma^{-1}_{\widetilde{\sigma}^{-1}(1)}, \dots, \underline{g}_{\widetilde{\sigma}^{-1}(m)}\sigma^{-1}_{\widetilde{\sigma}^{-1}(m)}\big)
\end{array}
\tag{4.29b}
$$

From the latter expression we deduce that this right permutation is generated by transpositions of adjacent orthogonal pairs because (1) $\sigma\widetilde{\sigma}^{-1} : \underline{f}\sigma^{-1} \to \underline{f}\widetilde{\sigma}^{-1}$ is by hypothesis generated by transpositions of adjacent orthogonal pairs, (2) $\perp$ is by definition stable under pre-composition in $\mathbf{C}$, and (3) transpositions of adjacent blocks are generated by adjacent transpositions of block elements belonging to different blocks. Hence, the elements in (4.28) are equivalent under $\sim_\perp$.

Secondly, let us consider any element $(\sigma, \underline{f}) \in \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{a} \end{smallmatrix}\big)$ and any two families of elements $(\sigma_i, \underline{g}_i), (\widetilde{\sigma}_i, \underline{g}_i) \in \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} a_i \\ \underline{b}_i \end{smallmatrix}\big)$ such that $(\sigma_i, \underline{g}_i) \sim_\perp (\widetilde{\sigma}_i, \underline{g}_i)$, for $i = 1, \dots, m$. We have to show that the two elements

$$
\gamma\left((\sigma, \underline{f}) \otimes \bigotimes_{i=1}^{m} (\sigma_i, \underline{g}_i)\right) = \Big(\sigma(\sigma_1, \dots, \sigma_m), \underline{f}(\underline{g}_1, \dots, \underline{g}_m)\Big) \quad , \tag{4.30a}
$$

$$
\gamma\left((\sigma, \underline{f}) \otimes \bigotimes_{i=1}^{m} (\widetilde{\sigma}_i, \underline{g}_i)\right) = \Big(\sigma(\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_m), \underline{f}(\underline{g}_1, \dots, \underline{g}_m)\Big) \tag{4.30b}
$$

in $\mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{b}_1 \otimes \cdots \otimes \underline{b}_m \end{smallmatrix}\big)$ are equivalent under $\sim_\perp$. Using Lemma 4.6, we can express the relevant right permutation by

$$
\begin{array}{c}
(\underline{f}\sigma^{-1})\big(\underline{g}_{\sigma^{-1}(1)}\sigma^{-1}_{\sigma^{-1}(1)}, \dots, \underline{g}_{\sigma^{-1}(m)}\sigma^{-1}_{\sigma^{-1}(m)}\big) \\
\Big\downarrow {\scriptstyle \sigma_{\sigma^{-1}(1)}\widetilde{\sigma}^{-1}_{\sigma^{-1}(1)} \oplus \cdots \oplus \sigma_{\sigma^{-1}(m)}\widetilde{\sigma}^{-1}_{\sigma^{-1}(m)}} \\
(\underline{f}\sigma^{-1})\big(\underline{g}_{\sigma^{-1}(1)}\widetilde{\sigma}^{-1}_{\sigma^{-1}(1)}, \dots, \underline{g}_{\sigma^{-1}(m)}\widetilde{\sigma}^{-1}_{\sigma^{-1}(m)}\big)
\end{array}
\tag{4.31}
$$

This right permutation is generated by transpositions of adjacent orthogonal pairs because (1) $\sigma_i\widetilde{\sigma}_i^{-1} : \underline{g}_i\sigma_i^{-1} \to \underline{g}_i\widetilde{\sigma}_i^{-1}$ is by hypothesis generated by transpositions of adjacent orthogonal pairs, for all $i = 1, \dots, m$, (2) $\perp$ is by definition stable under post-composition in $\mathbf{C}$, and (3) being generated by transpositions of adjacent orthogonal pairs is a property that is inherited by the block sum permutation. Hence, the elements in (4.30) are equivalent under $\sim_\perp$.

Compatibility with units (4.10)*:* There is nothing to prove in this case. □

The colored operad $\mathcal{O}_{\overline{\mathbf{C}}}$ may also be described more intrinsically as the coequalizer of a suitable pair of parallel $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphisms. In the following we frequently make use of $\mathbf{C}_0$-colored non-symmetric sequences $\mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$ and their free symmetrization $(-)^{\mathsf{sym}} : \mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0} \to \mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{Set})$, see Section 3.1.2 and in particular Proposition 3.12. (As usual, we will suppress all forgetful functors to simplify notation.)

Definition 4.12. Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ be an orthogonal category. We define a non-symmetric sequence $R_\perp \in \mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$ by setting

$$R_\perp\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \;:=\; \begin{cases} \perp \cap\, \mathbf{C}(\underline{c}, t) & , \;\text{if } |\underline{c}| = 2 \quad , \\ \emptyset & , \;\text{else} \quad , \end{cases} \tag{4.32}$$

for all $(\underline{c}, t) \in \Omega_{\mathbf{C}_0} \times \mathbf{C}_0$. We further define a pair of parallel $\mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$-morphisms $s_{\perp,1}, s_{\perp,2} : R_\perp \rightrightarrows \mathcal{O}_{\mathbf{C}}$ by setting

$$s_{\perp,1} \,:\, R_\perp\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \;\longrightarrow\; \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \quad , \qquad (f_1, f_2) \;\longmapsto\; \big(e, (f_1, f_2)\big) \quad , \tag{4.33a}$$

$$s_{\perp,2} \,:\, R_\perp\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \;\longrightarrow\; \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \quad , \qquad (f_1, f_2) \;\longmapsto\; \big(\tau, (f_1, f_2)\big) \quad , \tag{4.33b}$$

for all $(\underline{c}, t) \in \Omega_{\mathbf{C}_0} \times \mathbf{C}_0$ with length $|\underline{c}| = 2$, where $e, \tau \in \Sigma_2$ are the group identity and the transposition. (For $|\underline{c}| \neq 2$, $s_{\perp,1}$ and $s_{\perp,2}$ are the unique maps from the initial object $\emptyset \in \mathbf{Set}$.)

Using the adjunction of Proposition 3.12, we obtain $R_\perp^{\mathsf{sym}} \in \mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{Set})$ together with a pair of parallel $\mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{Set})$-morphisms (denoted by the same symbols) $s_{\perp,1}, s_{\perp,2} : R_\perp^{\mathsf{sym}} \rightrightarrows \mathcal{O}_{\mathbf{C}}$. Using further the free-forgetful adjunction for colored operads of Theorem 3.19, we obtain a pair of parallel $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphisms (denoted again by the same symbols) $s_{\perp,1}, s_{\perp,2} : F(R_\perp^{\mathsf{sym}}) \rightrightarrows \mathcal{O}_{\mathbf{C}}$. The definition of the colored operad $\mathcal{O}_{\overline{\mathbf{C}}}$ as an object-wise quotient of $\mathcal{O}_{\mathbf{C}}$ (cf. Definition 4.9 and Proposition 4.11) allows us to define an $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism $p_{\overline{\mathbf{C}}} : \mathcal{O}_{\mathbf{C}} \to \mathcal{O}_{\overline{\mathbf{C}}}$ by setting

$$p_{\overline{\mathbf{C}}} \,:\, \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \;\longrightarrow\; \mathcal{O}_{\overline{\mathbf{C}}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \quad , \qquad (\sigma, \underline{f}) \;\longmapsto\; [\sigma, \underline{f}] \quad , \tag{4.34}$$

for all $(\underline{c}, t) \in \Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$.

Proposition 4.13. *Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ be an orthogonal category. Then*

$$\begin{tikzcd} F(R_\perp^{\mathsf{sym}}) \arrow[r, shift left, "s_{\perp,1}"] \arrow[r, shift right, "s_{\perp,2}"'] & \mathcal{O}_{\mathbf{C}} \arrow[r, "p_{\overline{\mathbf{C}}}"] & \mathcal{O}_{\overline{\mathbf{C}}} \end{tikzcd} \tag{4.35}$$

is a coequalizer in $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$.

Proof. We first show that $p_{\overline{\mathbf{C}}}$ coequalizes the pair of parallel morphisms $s_{\perp,1}, s_{\perp,2}$. In the second step we prove that it possesses the universal property of a coequalizer.

Coequalizing property: Using the adjunctions of Proposition 3.12 and Theorem 3.19, it is enough to prove that $p_{\overline{\mathbf{C}}}\, s_{\perp,1} = p_{\overline{\mathbf{C}}}\, s_{\perp,2} : R_\perp \to \mathcal{O}_{\overline{\mathbf{C}}}$ as $\mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$-morphisms. Recalling (4.32), this is equivalent to showing that $p_{\overline{\mathbf{C}}}\, s_{\perp,1} = p_{\overline{\mathbf{C}}}\, s_{\perp,2} : R_\perp\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \to \mathcal{O}_{\overline{\mathbf{C}}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big)$ as maps of sets, for all $(\underline{c}, t) \in \Omega_{\mathbf{C}_0} \times \mathbf{C}_0$ with $|\underline{c}| = 2$. Using the explicit definitions in (4.33) and (4.34), we obtain

$$p_{\overline{\mathbf{C}}}\, s_{\perp,1}(f_1, f_2) = [e, (f_1, f_2)] = [\tau, (f_1, f_2)] = p_{\overline{\mathbf{C}}}\, s_{\perp,2}(f_1, f_2) \quad , \tag{4.36}$$

where in the second step we used that every $(f_1, f_2) \in R_\perp\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big)$ is an orthogonal pair (cf. (4.32)) and the definition of the equivalence relation $\sim_\perp$ (cf. Definition 4.9).

Universal property: We have to prove that any $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism $\phi : \mathcal{O}_{\mathbf{C}} \to \mathcal{P}$ coequalizing the parallel pair $s_{\perp,1}, s_{\perp,2} : F(R_\perp^{\mathsf{sym}}) \rightrightarrows \mathcal{O}_{\mathbf{C}}$ factors uniquely through $p_{\overline{\mathbf{C}}} : \mathcal{O}_{\mathbf{C}} \to \mathcal{O}_{\overline{\mathbf{C}}}$. If such a factorization exists, it is unique because all components of $p_{\overline{\mathbf{C}}}$ are surjective maps of sets (cf. (4.34)). To prove existence, we will show that the component maps $\phi : \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \to \mathcal{P}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big)$ descend to the quotients $\mathcal{O}_{\overline{\mathbf{C}}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) = \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big)/{\sim_\perp}$ by the equivalence relation of Definition 4.9, for all $(\underline{c}, t) \in \Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$. These induced maps then define an $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism $\mathcal{O}_{\overline{\mathbf{C}}} \to \mathcal{P}$ because the operad structure on $\mathcal{O}_{\overline{\mathbf{C}}}$ is canonically induced by the one on $\mathcal{O}_{\mathbf{C}}$, see Proposition 4.11.

Hence, it remains to show that for any two $(\sigma, \underline{f}), (\sigma', \underline{f}) \in \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big)$ such that $(\sigma, \underline{f}) \sim_\perp (\sigma', \underline{f})$, we have that $\phi(\sigma, \underline{f}) = \phi(\sigma', \underline{f})$ in $\mathcal{P}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big)$. Applying $\mathcal{P}(\sigma'^{-1})$ and using that ϕ is an $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism (in particular, it preserves the permutation actions), we obtain the equivalent condition

$\phi(\sigma\sigma'^{-1}, \underline{f}\sigma'^{-1}) = \phi(e, \underline{f}\sigma'^{-1})$. By definition of the equivalence relation $\sim_\perp$ (cf. Definition 4.9), there exists a factorization $\sigma\sigma'^{-1} = \tau_1 \cdots \tau_N : \underline{f}\sigma^{-1} \to \underline{f}\sigma'^{-1}$ into transpositions of adjacent orthogonal pairs. Let us focus first on the inverse of the rightmost transposition, i.e. $\tau_N^{-1} : \underline{f}\sigma'^{-1} \to \underline{f}\sigma'^{-1}\tau_N^{-1}$. It transposes a pair of adjacent orthogonal morphisms, say $(f_{\sigma'^{-1}(k)}, f_{\sigma'^{-1}(k+1)}) \in \perp$, in the sequence $\underline{f}\sigma'^{-1} = (f_{\sigma'^{-1}(1)}, \dots, f_{\sigma'^{-1}(n)})$. Using the composition in $\mathcal{O}_{\mathbf{C}}$ (cf. (4.8)), we may factorize $(e, \underline{f}\sigma'^{-1})$ for example as

$$\gamma\Big(\big(e, \underbrace{(\mathrm{id}_t, \dots, \mathrm{id}_t)}_{n-1 \text{ times}}\big) \otimes \bigotimes_{i=1}^{k-1} \big(e, f_{\sigma'^{-1}(i)}\big) \otimes \big(e, (f_{\sigma'^{-1}(k)}, f_{\sigma'^{-1}(k+1)})\big) \otimes \bigotimes_{i=k+2}^{n} \big(e, f_{\sigma'^{-1}(i)}\big)\Big) \quad . \tag{4.37}$$

Now apply ϕ to this expression and use that it preserves compositions because it is by hypothesis an $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism. We observe that the factor $\phi\big(e, (f_{\sigma'^{-1}(k)}, f_{\sigma'^{-1}(k+1)})\big)$ can be replaced by $\phi\big(\tau, (f_{\sigma'^{-1}(k)}, f_{\sigma'^{-1}(k+1)})\big)$ because ϕ coequalizes the pair of morphisms in (4.33). Evaluating the resulting expression proves the identity

$$\phi\big(e, \underline{f}\sigma'^{-1}\big) = \phi\big(\tau_N, \underline{f}\sigma'^{-1}\big) = \mathcal{P}(\tau_N)\phi\big(e, \underline{f}\sigma'^{-1}\tau_N^{-1}\big) \quad . \tag{4.38}$$

Iterating this construction then proves our desired identity

$$\begin{aligned} \phi\big(e, \underline{f}\sigma'^{-1}\big) &= \mathcal{P}(\tau_N) \cdots \mathcal{P}(\tau_1)\phi\big(e, \underline{f}\sigma'^{-1}\tau_N^{-1} \cdots \tau_1^{-1}\big) \\ &= \phi\big(\tau_1 \cdots \tau_N, \underline{f}\sigma'^{-1}\big) = \phi\big(\sigma\sigma'^{-1}, \underline{f}\sigma'^{-1}\big) \quad . \end{aligned} \tag{4.39}$$

where in the second step we used that ϕ preserves the permutation actions. $\square$

We conclude this subsection by showing that our constructions are functorial. Let us first address functoriality of the assignment $\mathbf{C} \mapsto \mathcal{O}_{\mathbf{C}}$ of the colored operad described in Definition 4.5 and Proposition 4.7. For this recall from Definition 3.26 the category $\mathbf{Op}(\mathbf{Set})$ of $\mathbf{Set}$-valued operads with varying colors.

Proposition 4.14. *The assignment $\mathbf{C} \mapsto \mathcal{O}_{\mathbf{C}}$ naturally extends to a functor $\mathcal{O}_{(-)} : \mathbf{Cat} \to \mathbf{Op}(\mathbf{Set})$ on the category $\mathbf{Cat}$ of small categories.*

Proof. Let $F : \mathbf{C} \to \mathbf{D}$ be any $\mathbf{Cat}$-morphism, i.e. a functor between small categories, and denote its underlying map of object sets by $F : \mathbf{C}_0 \to \mathbf{D}_0$. We have to define an $\mathbf{Op}(\mathbf{Set})$-morphism $\mathcal{O}_F := (F, \phi_F) : (\mathbf{C}_0, \mathcal{O}_{\mathbf{C}}) \to (\mathbf{D}_0, \mathcal{O}_{\mathbf{D}})$, i.e. a morphism $\phi_F : \mathcal{O}_{\mathbf{C}} \to F^*(\mathcal{O}_{\mathbf{D}})$ of $\mathbf{C}_0$-colored operads. We set

$$\phi_F \,:\, \mathcal{O}_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \;\longrightarrow\; F^*(\mathcal{O}_{\mathbf{D}})\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) = \mathcal{O}_{\mathbf{D}}\big(\begin{smallmatrix} F(t) \\ F(\underline{c}) \end{smallmatrix}\big) \quad , \qquad \big(\sigma, \underline{f}\big) \;\longmapsto\; \big(\sigma, F(\underline{f})\big) \quad , \tag{4.40}$$

for all $(\underline{c}, t) \in \Sigma_{\mathbf{C}_0} \times \mathbf{C}_0$, where $F(\underline{f}) := (F(f_1), \dots, F(f_n)) \in \mathbf{D}(F(\underline{c}), F(t))$. It is straightforward to show that ϕ_F is an $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism, i.e. that its components (4.40) are compatible with the actions of the permutation group and preserve compositions and units. $\square$

The assignment $\overline{\mathbf{C}} = (\mathbf{C}, \perp) \mapsto \mathcal{O}_{\overline{\mathbf{C}}}$ of the colored operad described in Definition 4.9 and Proposition 4.11 is functorial as well for the following choice of source category.

Definition 4.15. We denote by **OrthCat** the *category of orthogonal categories*. The objects are orthogonal categories $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$. A morphism $F : \overline{\mathbf{C}} = (\mathbf{C}, \perp_{\mathbf{C}}) \to \overline{\mathbf{D}} = (\mathbf{D}, \perp_{\mathbf{D}})$ in **OrthCat** is a functor $F : \mathbf{C} \to \mathbf{D}$ that preserves the orthogonality relations, i.e. $f_1 \perp_{\mathbf{C}} f_2$ implies $F(f_1) \perp_{\mathbf{D}} F(f_2)$. We shall call such morphisms *orthogonal functors.*

Proposition 4.16. *The assignment $\overline{\mathbf{C}} \mapsto \mathcal{O}_{\overline{\mathbf{C}}}$ naturally extends to a functor $\mathcal{O}_{(-)} : \mathbf{OrthCat} \to \mathbf{Op}(\mathbf{Set})$.*

Proof. The definition of morphisms in **OrthCat** immediately implies that the components (4.40) descend to the quotients in Definition 4.9. It then follows from Proposition 4.14 and Proposition 4.11 that the induced map $\phi_F : \mathcal{O}_{\overline{\mathbf{C}}} \to F^*(\mathcal{O}_{\overline{\mathbf{D}}})$ is a morphism of $\mathbf{C}_0$-colored operads. $\square$

Corollary 4.17. *Let* Π : **OrthCat** $\to$ **Cat** *be the canonical projection functor given by* $\overline{\mathbf{C}} = (\mathbf{C}, \perp) \mapsto \mathbf{C}$*. The* $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$*-morphisms* $p_{\overline{\mathbf{C}}} : \mathcal{O}_{\mathbf{C}} \to \mathcal{O}_{\overline{\mathbf{C}}}$ *given in* (4.34) *are the components of a natural transformation*

$$p : \mathcal{O}_{\Pi(-)} \longrightarrow \mathcal{O}_{(-)} \tag{4.41}$$

between functors from **OrthCat** *to* **Op**(**Set**)*.*

Remark 4.18. Notice that for the empty orthogonality relation $\emptyset \subseteq \mathrm{Mor}\,\mathbf{C}\,{}_{\mathrm{t}}{\times}_{\mathrm{t}}\,\mathrm{Mor}\,\mathbf{C}$ the colored operad $\mathcal{O}_{(\mathbf{C},\emptyset)}$ coincides with $\mathcal{O}_{\mathbf{C}}$. Given any $\overline{\mathbf{C}} = (\mathbf{C}, \perp) \in$ **OrthCat**, the identity functor $\mathrm{id}_{\mathbf{C}} : (\mathbf{C}, \emptyset) \to \overline{\mathbf{C}}$ is clearly orthogonal, hence we obtain by Proposition 4.16 an $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphisms $\mathcal{O}_{\mathrm{id}_{\mathbf{C}}} : \mathcal{O}_{(\mathbf{C},\emptyset)} \to \mathcal{O}_{\overline{\mathbf{C}}}$. By a direct comparison, we observe that this morphism is equal to $p_{\overline{\mathbf{C}}}$ given in (4.34), i.e.

$$\mathcal{O}_{(\mathbf{C},\emptyset)} = \mathcal{O}_{\mathbf{C}} \xrightarrow{\ \mathcal{O}_{\mathrm{id}_{\mathbf{C}}} = p_{\overline{\mathbf{C}}}\ } \mathcal{O}_{\overline{\mathbf{C}}} \quad . \tag{4.42}$$

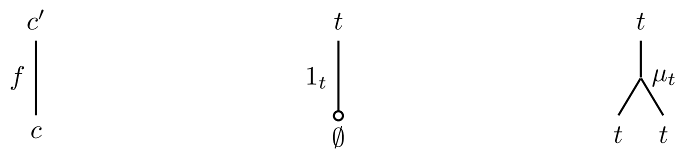
This simple observation will be useful later on. $\triangle$

4.3 Presentation by generators and relations

The aim of this subsection is to show that the colored operads $\mathcal{O}_{\mathbf{C}}$ and $\mathcal{O}_{\overline{\mathbf{C}}}$ introduced in Section 4.2 admit a convenient presentation in terms of generators and relations. This is particularly useful for characterizing their algebras, see Section 4.5.

Let us start with the colored operad $\mathcal{O}_{\mathbf{C}}$ described in Definition 4.5 and Proposition 4.7, where $\mathbf{C}$ is a small category. We shall prove that this operad is presented by generators and relations which all admit a natural quantum field theoretic interpretation. We find it instructive to provide first a more informal presentation of our construction by using a convenient graphical notation. After this we shall give formal definitions of the relevant non-symmetric sequences and pair of parallel morphisms which presents our colored operad $\mathcal{O}_{\mathbf{C}}$.

- *Generators* $G_{\mathbf{C}}$*:* We introduce three types of generators



(4.43)

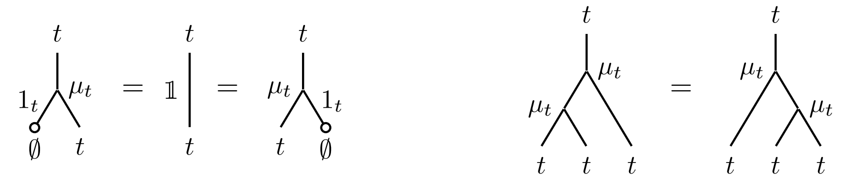
for every $t \in \mathbf{C}_0$ and $(f : c \to c') \in \mathrm{Mor}\,\mathbf{C}$. The quantum field theoretic interpretation is as follows: For every spacetime embedding $f : c \to c'$, we introduce a 1-ary operation f to push forward observables. Moreover, for every spacetime t, we introduce a 0-ary operation 1_t and a 2-ary operation μ_t to obtain a unit and a product for the observables on t.

- *Functoriality relations* R_{Fun}*:* We impose the relations

(4.44)

for every $t \in \mathbf{C}_0$ and every pair of composable morphisms $(f : c' \to c'', g : c \to c') \in \mathrm{Mor}\, \mathbf{C} \,{}_{\mathrm{s}}\!\times_{\mathrm{t}} \mathrm{Mor}\, \mathbf{C}$. These relations capture the functoriality of pushing forward observables by identifying the operadic unit $\mathbb{1}$ with the identity morphisms id_t and operadic compositions of spacetime embeddings with their categorical compositions in $\mathbf{C}$.

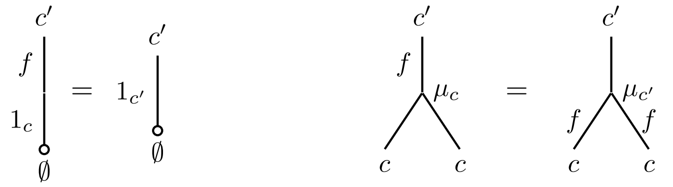
- *Monoid relations* R_{Mon}*:* We impose the relations



(4.45)

  for every $t \in \mathbf{C}_0$. These relations capture the unitality and associativity property of the unit 1_t and product μ_t of observables.

- *Compatibility relations* R_{FM}*:* We impose the relations



(4.46)

  for every $(f : c \to c') \in \mathrm{Mor}\, \mathbf{C}$. These relations capture the compatibility between pushing forward observables along spacetime embeddings and the unit and product.

Similarly to Definition 4.12, one can obtain a more formal description of these generators and relations by using $\mathbf{C}_0$-colored non-symmetric sequences $\mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$, their free symmetrization (cf. Proposition 3.12) and free colored operads (cf. Theorem 3.19). Concretely, the generators $G_{\mathbf{C}} \in \mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$ are given by setting

$$G_{\mathbf{C}}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) := \begin{cases} \{1_t\} & , \quad \text{if } \underline{c} = \emptyset \quad , \\ \mathbf{C}(\underline{c}, t) & , \quad \text{if } |\underline{c}| = 1 \quad , \\ \{\mu_t\} & , \quad \text{if } \underline{c} = (t,t) \quad , \\ \emptyset & , \quad \text{else} \quad , \end{cases} \tag{4.47}$$

for all $(\underline{c}, t) \in \Omega_{\mathbf{C}_0} \times \mathbf{C}_0$. Let $F(G_{\mathbf{C}}^{\mathsf{sym}}) \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$ be the free colored operad of the free symmetrization of $G_{\mathbf{C}}$ and notice that the adjunctions of Proposition 3.12 and Theorem 3.19 define a canonical $\mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$-morphism $G_{\mathbf{C}} \to F(G_{\mathbf{C}}^{\mathsf{sym}})$. For notational convenience, we shall always suppress this morphism and, for example, simply write $\mu_t \in F(G_{\mathbf{C}}^{\mathsf{sym}})\big(\begin{smallmatrix} t \\ (t,t) \end{smallmatrix}\big)$ for the image of $\mu_t \in G_{\mathbf{C}}\big(\begin{smallmatrix} t \\ (t,t) \end{smallmatrix}\big)$ under this morphism.

The relations are described by an object $R_{\mathbf{C}} \in \mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$ together with a pair of parallel $\mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$-morphisms $r_{\mathbf{C},1}, r_{\mathbf{C},2} : R_{\mathbf{C}} \rightrightarrows F(G_{\mathbf{C}}^{\mathsf{sym}})$. Because our relations are naturally divided into three different types (functoriality, monoid and compatibility relations), there exists a decomposition into coproducts

$$r_{\mathbf{C},i} := r_{\mathrm{Fun},i} \sqcup r_{\mathrm{Mon},i} \sqcup r_{\mathrm{FM},i} \; : \; R_{\mathbf{C}} := R_{\mathrm{Fun}} \sqcup R_{\mathrm{Mon}} \sqcup R_{\mathrm{FM}} \longrightarrow F(G_{\mathbf{C}}^{\mathsf{sym}}) \quad , \tag{4.48}$$

for $i = 1, 2$. It is easy, however slightly cumbersome, to read off the formal definition of these three individual components from our graphical presentation above: We define $R_{\mathrm{Fun}} \in \mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$ by setting

$$R_{\mathrm{Fun}}\tbinom{t}{\underline{c}} := \begin{cases} \Big(\coprod\limits_{c'\in\mathbf{C}_0} \mathbf{C}(c',t)\times\mathbf{C}(t,c')\Big) \sqcup \{*_t\} & , \quad \text{if } \underline{c}=t \quad , \\ \Big(\coprod\limits_{c'\in\mathbf{C}_0} \mathbf{C}(c',t)\times\mathbf{C}(\underline{c},c')\Big) & , \quad \text{if } |\underline{c}|=1 \text{ and } \underline{c}\neq t \quad , \\ \emptyset & , \quad \text{else} \quad , \end{cases} \tag{4.49a}$$

for all $(\underline{c},t)\in\Omega_{\mathbf{C}_0}\times\mathbf{C}_0$. We define the components $r_{\mathrm{Fun},1}, r_{\mathrm{Fun},2} : R_{\mathrm{Fun}}\tbinom{t}{\underline{c}} \rightrightarrows F(G_{\mathbf{C}}^{\mathsf{sym}})\tbinom{t}{\underline{c}}$ by setting

$$r_{\mathrm{Fun},1} \,:\, (f,g) \longmapsto \gamma\big(f\otimes g\big) \quad , \qquad r_{\mathrm{Fun},2} \,:\, (f,g) \longmapsto f\,g \quad , \tag{4.49b}$$

for $|c|=1$, and additionally

$$r_{\mathrm{Fun},1} \,:\, *_t \longmapsto \mathbb{1} \quad , \qquad r_{\mathrm{Fun},2} \,:\, *_t \longmapsto \mathrm{id}_t \quad , \tag{4.49c}$$

in the case of $\underline{c}=t$, where γ and $\mathbb{1}$ denote the operadic composition and unit in $F(G_{\mathbf{C}}^{\mathsf{sym}})$. Next, we define $R_{\mathrm{Mon}}\in\mathbf{Set}^{\Omega_{\mathbf{C}_0}\times\mathbf{C}_0}$ by setting

$$R_{\mathrm{Mon}}\tbinom{t}{\underline{c}} := \begin{cases} \{l_t,r_t\} & , \quad \text{if } \underline{c}=t \quad , \\ \{a_t\} & , \quad \text{if } \underline{c}=(t,t,t) \quad , \\ \emptyset & , \quad \text{else} \quad , \end{cases} \tag{4.50a}$$

for all $(\underline{c},t)\in\Omega_{\mathbf{C}_0}\times\mathbf{C}_0$. We define the components $r_{\mathrm{Mon},1}, r_{\mathrm{Mon},2} : R_{\mathrm{Mon}}\tbinom{t}{\underline{c}} \rightrightarrows F(G_{\mathbf{C}}^{\mathsf{sym}})\tbinom{t}{\underline{c}}$ by setting

$$r_{\mathrm{Mon},1} \,:\, \begin{pmatrix} l_t \longmapsto \gamma\big(\mu_t\otimes 1_t\otimes\mathbb{1}\big) \\ r_t \longmapsto \gamma\big(\mu_t\otimes\mathbb{1}\otimes 1_t\big) \end{pmatrix} \quad , \qquad r_{\mathrm{Mon},2} \,:\, l_t,\, r_t \longmapsto \mathbb{1} \quad , \tag{4.50b}$$

for $\underline{c}=t$, and

$$r_{\mathrm{Mon},1} \,:\, a_t \longmapsto \gamma\big(\mu_t\otimes\mu_t\otimes\mathbb{1}\big) \quad , \qquad r_{\mathrm{Mon},2} \,:\, a_t \longmapsto \gamma\big(\mu_t\otimes\mathbb{1}\otimes\mu_t\big) \quad , \tag{4.50c}$$

for $\underline{c}=(t,t,t)$. Finally, we define $R_{\mathrm{FM}}\in\mathbf{Set}^{\Omega_{\mathbf{C}_0}\times\mathbf{C}_0}$ by setting

$$R_{\mathrm{FM}}\tbinom{t}{\underline{c}} := \begin{cases} \coprod\limits_{c'\in\mathbf{C}_0} \mathbf{C}(c',t) & , \quad \text{if } \underline{c}=\emptyset \quad , \\ \mathbf{C}(c_1,t) & , \quad \text{if } |\underline{c}|=2 \text{ and } c_1=c_2 \quad , \\ \emptyset & , \quad \text{else} \quad , \end{cases} \tag{4.51a}$$

for all $(\underline{c},t)\in\Omega_{\mathbf{C}_0}\times\mathbf{C}_0$. We define the components $r_{\mathrm{FM},1}, r_{\mathrm{FM},2} : R_{\mathrm{FM}}\tbinom{t}{\underline{c}} \rightrightarrows F(G_{\mathbf{C}}^{\mathsf{sym}})\tbinom{t}{\underline{c}}$ by setting

$$r_{\mathrm{FM},1} \,:\, f \longmapsto \gamma\big(f\otimes 1_{c'}\big) \quad , \qquad r_{\mathrm{FM},2} \,:\, f \longmapsto 1_t \quad , \tag{4.51b}$$

for $\underline{c}=\emptyset$, and

$$r_{\mathrm{FM},1} \,:\, f \longmapsto \gamma\big(f\otimes\mu_c\big) \quad , \qquad r_{\mathrm{FM},2} \,:\, f \longmapsto \gamma\big(\mu_t\otimes f\otimes f\big) \quad , \tag{4.51c}$$

for $\underline{c}=(c,c)$.

In order to relate the description in terms of generators and relations to our colored operad $\mathcal{O}_{\mathbf{C}}$ described in Definition 4.5, we define a $\mathbf{Set}^{\Omega_{\mathbf{C}_0}\times\mathbf{C}_0}$-morphism

$$q_{\mathbf{C}} \,:\, G_{\mathbf{C}} \longrightarrow \mathcal{O}_{\mathbf{C}} \tag{4.52a}$$

by setting

$$q_{\mathbf{C}} : \begin{pmatrix} 1_t & \longmapsto & (e,*) \\ f & \longmapsto & (e,f) \\ \mu_t & \longmapsto & \big(e,(\mathrm{id}_t,\mathrm{id}_t)\big) \end{pmatrix} \quad , \tag{4.52b}$$

for the non-trivial components $(\emptyset,t)$, (c,t) and $((t,t),t)$, where e denotes the identity permutations. Our first main result of this section is

Theorem 4.19. *Let* $\mathbf{C}$ *be a small category. Then*

$$F(R^{\mathsf{sym}}_{\mathbf{C}}) \overset{r_{\mathbf{C},1}}{\underset{r_{\mathbf{C},2}}{\rightrightarrows}} F(G^{\mathsf{sym}}_{\mathbf{C}}) \xrightarrow{\quad q_{\mathbf{C}}\quad} \mathcal{O}_{\mathbf{C}} \tag{4.53}$$

is a coequalizer in $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$.

Proof. Verifying that $q_{\mathbf{C}}$ coequalizes the pair of parallel morphisms $r_{\mathbf{C},1}$ and $r_{\mathbf{C},2}$ is a simple calculation using the explicit expressions for the relations (cf. (4.49), (4.50) and (4.51)) and for $q_{\mathbf{C}}$ (cf. (4.52)), the operad structure on $\mathcal{O}_{\mathbf{C}}$ (cf. Definition 4.5) and that $q_{\mathbf{C}}$ in (4.53) is an $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism. As an illustration, let us show this for the first relation in (4.49):

$$q_{\mathbf{C}}\big(r_{\mathrm{Fun},1}(f,g)\big) = q_{\mathbf{C}}\big(\gamma(f\otimes g)\big) = \gamma\big((e,f)\otimes(e,g)\big) = (e,f\,g) = q_{\mathbf{C}}\big(r_{\mathrm{Fun},2}(f,g)\big) \quad . \tag{4.54}$$

It thus remains to prove that the $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism $q_{\mathbf{C}}$ in (4.53) has the universal property of a coequalizer.

As a crucial preparation, we show that every $(\sigma,\underline{f}) \in \mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{c}}\big)$, for any $(\underline{c},t)\in\Sigma_{\mathbf{C}_0}\times\mathbf{C}_0$, admits a factorization in terms of images (under $q_{\mathbf{C}}$) of the generators (cf. (4.52)). Let us define recursively the elements $\mu_t^n \in F(G^{\mathsf{sym}}_{\mathbf{C}})\big(\substack{t\\ t^{\otimes n}}\big)$ by

$$\mu_t^0 := 1_t \quad , \qquad \mu_t^n := \gamma\big(\mu_t\otimes \mathbb{1}\otimes\mu_t^{n-1}\big) \quad , \tag{4.55}$$

and observe that

$$q_{\mathbf{C}}(\mu_t^n) \;=\; \big(e,\underbrace{(\mathrm{id}_t,\dots,\mathrm{id}_t)}_{n \text{ times}}\big) \,\in\, \mathcal{O}_{\mathbf{C}}\big(\substack{t\\ t^{\otimes n}}\big) \quad . \tag{4.56}$$

Using the operad structure on $\mathcal{O}_{\mathbf{C}}$ (cf. Definition 4.5), we obtain a factorization

$$(\sigma,\underline{f}) = \mathcal{O}_{\mathbf{C}}(\sigma)\,\gamma\Big(q_{\mathbf{C}}(\mu_t^n)\otimes\bigotimes_{i=1}^n q_{\mathbf{C}}\big(f_{\sigma^{-1}(i)}\big)\Big) \quad , \tag{4.57}$$

where $n=|\underline{c}|$. In particular, this implies that each component $q_{\mathbf{C}} : F(G^{\mathsf{sym}}_{\mathbf{C}})\big(\substack{t\\ \underline{c}}\big) \to \mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{c}}\big)$ is a surjective map of sets.

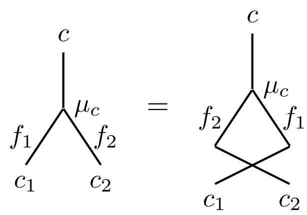
We now prove the universal property. Given any $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism $\phi : F(G^{\mathsf{sym}}_{\mathbf{C}}) \to \mathcal{P}$ which coequalizes the pair of parallel morphisms $r_{\mathbf{C},1}, r_{\mathbf{C},2} : F(R^{\mathsf{sym}}_{\mathbf{C}}) \rightrightarrows F(G^{\mathsf{sym}}_{\mathbf{C}})$, we have to construct a unique factorization $\phi = \phi'\,q_{\mathbf{C}}$ of ϕ through $q_{\mathbf{C}} : F(G^{\mathsf{sym}}_{\mathbf{C}}) \to \mathcal{O}_{\mathbf{C}}$. Uniqueness is immediate because $q_{\mathbf{C}}$ is component-wise surjective. To prove existence, we use (4.57) and define

$$\phi' : \mathcal{O}_{\mathbf{C}}\big(\substack{t\\ \underline{c}}\big) \longrightarrow \mathcal{P}\big(\substack{t\\ \underline{c}}\big) \quad , \qquad (\sigma,\underline{f}) \longmapsto \mathcal{P}(\sigma)\,\gamma\Big(\phi(\mu_t^n)\otimes\bigotimes_{i=1}^n \phi\big(f_{\sigma^{-1}(i)}\big)\Big) \quad , \tag{4.58}$$

for all $(\underline{c},t)\in\Sigma_{\mathbf{C}_0}\times\mathbf{C}_0$, where $n=|\underline{c}|$. One easily observes that this defines a $\mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{Set})$-morphism and that $\phi=\phi'\,q_{\mathbf{C}}$. By an elementary but slightly lengthy calculation one shows that ϕ' is an $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism, i.e. that it preserves compositions and units. (Hint: Due to the equivariance axioms of colored operads (cf. Remark 3.17) it is sufficient to consider elements $(e,\underline{f})$ with trivial permutations in these calculations, which leads to drastic simplifications.) $\square$

Let us now consider the colored operad $\mathcal{O}_{\overline{\mathbf{C}}}$ described in Definition 4.9 and Proposition 4.11, where $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ is an orthogonal category. We already observed in Proposition 4.13 that $\mathcal{O}_{\overline{\mathbf{C}}}$ is obtained from the colored operad $\mathcal{O}_{\mathbf{C}}$ by enforcing relations to capture the $\perp$-commutativity axiom, see also Definition 4.12. For our presentation by generators and relations, this amounts to adding one more type of relations, which we again first describe more informally using a graphical notation. (See below for a formal description.)

- $\perp$*-commutativity relations* $R_\perp$*:* We impose the relations



(4.59)

for every orthogonal pair of $\mathbf{C}$-morphisms $(f_1 : c_1 \to c, f_2 : c_2 \to c) \in \perp$. These relations capture the commutative behavior of pairs of observables on c which arise from pushforwards along an orthogonal pair of spacetime embeddings.

More formally, these relations are described by a pair of parallel $\mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$-morphisms $r_{\perp,1}, r_{\perp,2} : R_\perp \rightrightarrows F(G^{\mathsf{sym}}_{\mathbf{C}})$. We define $R_\perp \in \mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$ as in Definition 4.12 by setting

$$R_\perp\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \,:=\, \begin{cases} \perp \cap\, \mathbf{C}(\underline{c}, t) & , \ \text{if } |\underline{c}| = 2 \quad , \\ \emptyset & , \ \text{else} \quad , \end{cases} \tag{4.60a}$$

for all $(\underline{c}, t) \in \Omega_{\mathbf{C}_0} \times \mathbf{C}_0$. We define the components $r_{\perp,1}, r_{\perp,2} : R_\perp\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \rightrightarrows F(G^{\mathsf{sym}}_{\mathbf{C}})\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big)$ by setting

$$r_{\perp,1} \,:\, (f_1, f_2) \longmapsto \gamma\big(\mu_t \otimes f_1 \otimes f_2\big) \quad , \tag{4.60b}$$
$$r_{\perp,2} \,:\, (f_1, f_2) \longmapsto F(G^{\mathsf{sym}}_{\mathbf{C}})(\tau)\,\big(\gamma\big(\mu_t \otimes f_2 \otimes f_1\big)\big) \quad , \tag{4.60c}$$

where $\tau \in \Sigma_2$ denotes the transposition. To implement simultaneously all four types of relations (functoriality, monoid, compatibility and $\perp$-commutativity relations), we consider

$$R_{\overline{\mathbf{C}}} \,:=\, R_{\mathbf{C}} \sqcup R_\perp \,=\, R_{\mathrm{Fun}} \sqcup R_{\mathrm{Mon}} \sqcup R_{\mathrm{FM}} \sqcup R_\perp \,\in\, \mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0} \quad , \tag{4.61a}$$

together with

$$r_{\overline{\mathbf{C}},i} \,:=\, r_{\mathbf{C},i} \sqcup r_{\perp,i} \,=\, r_{\mathrm{Fun},i} \sqcup r_{\mathrm{Mon},i} \sqcup r_{\mathrm{FM},i} \sqcup r_{\perp,i} \,:\, R_{\overline{\mathbf{C}}} \longrightarrow F(G^{\mathsf{sym}}_{\mathbf{C}}) \quad , \tag{4.61b}$$

for $i = 1, 2$. Composing $q_{\mathbf{C}}$ given in (4.52) with the $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism $p_{\overline{\mathbf{C}}}$ given in (4.34), we obtain a $\mathbf{Set}^{\Omega_{\mathbf{C}_0} \times \mathbf{C}_0}$-morphism

$$p_{\overline{\mathbf{C}}}\, q_{\mathbf{C}} \,:\, G_{\mathbf{C}} \longrightarrow \mathcal{O}_{\overline{\mathbf{C}}} \quad . \tag{4.62}$$

Our second main result of this section is

Corollary 4.20. *Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ be an orthogonal category. Then*

$$\begin{tikzcd} F(R^{\mathsf{sym}}_{\overline{\mathbf{C}}}) \arrow[r, shift left, "r_{\overline{\mathbf{C}},1}"] \arrow[r, shift right, "r_{\overline{\mathbf{C}},2}"'] & F(G^{\mathsf{sym}}_{\mathbf{C}}) \arrow[r, "p_{\overline{\mathbf{C}}}\, q_{\mathbf{C}}"] & \mathcal{O}_{\overline{\mathbf{C}}} \end{tikzcd} \tag{4.63}$$

is a coequalizer in $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$.

Proof. This is a simple consequence of Proposition 4.13, Theorem 4.19 and the diagram

$$\begin{tikzcd}
F(R^{\mathsf{sym}}_{\mathbf{C}}) \arrow[d, "\iota_1"'] \arrow[dr, shift left, "r_{\mathbf{C},1}"] \arrow[dr, shift right, "r_{\mathbf{C},2}"'] & & & \mathcal{P} \\
F(R^{\mathsf{sym}}_{\mathbf{C}}) \sqcup F(R^{\mathsf{sym}}_{\perp}) \arrow[r, shift left, "r_{\overline{\mathbf{C}},1}"] \arrow[r, shift right, "r_{\overline{\mathbf{C}},2}"'] & F(G^{\mathsf{sym}}_{\mathbf{C}}) \arrow[urr, "\phi"] \arrow[r, "q_{\mathbf{C}}"] & \mathcal{O}_{\mathbf{C}} \arrow[ur, dashed, "\exists!\,\phi'"'] \arrow[r, "p_{\overline{\mathbf{C}}}"] & \mathcal{O}_{\overline{\mathbf{C}}} \arrow[u, dashed, "\exists!\,\phi''"'] \\
F(R^{\mathsf{sym}}_{\perp}) \arrow[u, "\iota_2"] \arrow[urr, shift left, "s_{\perp,1}"] \arrow[urr, shift right, "s_{\perp,2}"'] & & &
\end{tikzcd} \tag{4.64}$$

in $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$, where $\phi : F(G^{\mathsf{sym}}_{\mathbf{C}}) \to \mathcal{P}$ is any $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set})$-morphism which coequalizes $r_{\overline{\mathbf{C}},1}$ and $r_{\overline{\mathbf{C}},2}$. Let us explain this in more detail: (1) We have that $F(R^{\mathsf{sym}}_{\overline{\mathbf{C}}}) \cong F(R^{\mathsf{sym}}_{\mathbf{C}}) \sqcup F(R^{\mathsf{sym}}_{\perp})$ because F and $(-)^{\mathsf{sym}}$ are left adjoint functors, hence they preserve coproducts. (2) By definition of $r_{\overline{\mathbf{C}},i}$ the upper left triangles commute, i.e. $r_{\mathbf{C},i} = r_{\overline{\mathbf{C}},i}\,\iota_1$ for $i = 1, 2$. (3) By a short calculation using (4.60), (4.52) and (4.33) one shows that the lower left triangles commute, i.e. $s_{\perp,i} = q_{\mathbf{C}}\, r_{\overline{\mathbf{C}},i}\,\iota_2$ for $i = 1, 2$. (4) Because of (2), $\phi : F(G^{\mathsf{sym}}_{\mathbf{C}}) \to \mathcal{P}$ also coequalizes $r_{\mathbf{C},1}$ and $r_{\mathbf{C},2}$. Existence and uniqueness of the morphism $\phi' : \mathcal{O}_{\mathbf{C}} \to \mathcal{P}$ in (4.64) is then a consequence of Theorem 4.19. (5) Because of (3), $\phi' : \mathcal{O}_{\mathbf{C}} \to \mathcal{P}$ coequalizes $s_{\perp,1}$ and $s_{\perp,2}$. Existence and uniqueness of the morphism $\phi'' : \mathcal{O}_{\overline{\mathbf{C}}} \to \mathcal{P}$ in (4.64) is then a consequence of Proposition 4.13. $\square$

We conclude this subsection by noting that our constructions are functorial. The following statements are easily proven, hence we may omit the proofs.

Proposition 4.21. *The coequalizer* (4.53) *is natural in* $\mathbf{C} \in \mathbf{Cat}$ *and the coequalizer* (4.63) *is natural in* $\overline{\mathbf{C}} = (\mathbf{C}, \perp) \in \mathbf{OrthCat}$.

4.4 Change of base category $\mathbf{Set} \to \mathbf{M}$

We now promote our $\mathbf{Set}$-valued colored operads $\mathcal{O}_{\mathbf{C}}$ and $\mathcal{O}_{\overline{\mathbf{C}}}$ to colored operads with values in a general complete and cocomplete closed symmetric monoidal category $\mathbf{M}$ via a change of base category construction, see e.g. [Yau16, Theorem 11.5.1] and [Fre17, Chapter 3.1].

Recalling that any such $\mathbf{M}$ is tensored over $\mathbf{Set}$, there exists an adjunction

$$(-) \otimes I \,:\, \mathbf{Set} \rightleftarrows \mathbf{M} \,:\, \mathbf{M}(I, -) \quad , \tag{4.65}$$

where $I \in \mathbf{M}$ denotes the unit object. The left adjoint functor may be equipped with a canonical *strong* monoidal structure given by the isomorphisms

$$(S_1 \otimes I) \otimes (S_2 \otimes I) \;\cong\; \coprod_{s_1 \in S_1} \coprod_{s_2 \in S_2} I \otimes I \;\cong\; \coprod_{(s_1, s_2) \in S_1 \times S_2} I \;\cong\; (S_1 \times S_2) \otimes I \quad , \tag{4.66a}$$

and

$$I \;\cong\; \coprod_{* \in \{*\}} I \;\cong\; \{*\} \otimes I \quad . \tag{4.66b}$$

Note that this strong monoidal structure is clearly compatible with the braidings on $\mathbf{Set}$ and $\mathbf{M}$, hence $(-) \otimes I$ is a symmetric strong monoidal functor.

Let $\mathfrak{C}$ be any non-empty set of colors. Because adjunctions lift to functor categories, (4.65) induces adjunctions (denoted with abuse of notation by the same symbols)

$$(-) \otimes I \,:\, \mathbf{Set}^{\Omega_{\mathfrak{C}} \times \mathfrak{C}} \rightleftarrows \mathbf{M}^{\Omega_{\mathfrak{C}} \times \mathfrak{C}} \,:\, \mathbf{M}(I, -) \quad , \tag{4.67}$$

and

$$(-)\otimes I \,:\, \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{Set}) \rightleftarrows \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M}) \,:\, \mathbf{M}\big(I,-\big) \quad , \tag{4.68}$$

between the categories of $\mathfrak{C}$-colored non-symmetric sequences and the categories of $\mathfrak{C}$-colored symmetric sequences.

Lemma 4.22. *The left adjoint functor* $(-)\otimes I : \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{Set}) \to \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{M})$ *in* (4.68) *inherits from* (4.66) *a strong monoidal structure. (Here the categories of symmetric sequences are endowed with the monoidal structure from Definition 3.5.)*

Proof. In this proof we decorate the circle products and units with a superscript to indicate the underlying base category **Set** or **M**. The structure natural isomorphism $(X\otimes I)\circ^{\mathbf{M}}(Y\otimes I) \xrightarrow{\cong} (X\circ^{\mathbf{Set}} Y)\otimes I$, for all $X,Y\in \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{Set})$, is given by

$$\begin{aligned}\big((X\otimes I)\circ^{\mathbf{M}}(Y\otimes I)\big)\big(\begin{smallmatrix} t\\ \underline{c}\end{smallmatrix}\big) \;&\cong\; \int^{\underline{a}}\int^{(\underline{b}_1,\dots,\underline{b}_m)} \Big(\Sigma_{\mathfrak{C}}\big(\underline{b}_1\otimes\cdots\otimes\underline{b}_m,\underline{c}\big)\times X\big(\begin{smallmatrix} t\\ \underline{a}\end{smallmatrix}\big)\times\prod_{i=1}^m Y\big(\begin{smallmatrix} a_i\\ \underline{b}_i\end{smallmatrix}\big)\Big)\otimes I\\ &\cong\; \big((X\circ^{\mathbf{Set}} Y)\otimes I\big)\big(\begin{smallmatrix} t\\ \underline{c}\end{smallmatrix}\big) \quad .\end{aligned} \tag{4.69}$$

In the first step we used the definition of the circle product (3.12) and (4.66a). In the second step we used that $(-)\otimes I : \mathbf{Set}\to\mathbf{M}$ is a left adjoint (cf. (4.65)), hence it preserves coends. The structure isomorphism $I_{\circ}^{\mathbf{M}} \xrightarrow{\cong} I_{\circ}^{\mathbf{Set}}\otimes I$ is given by

$$I_{\circ}^{\mathbf{M}}\big(\begin{smallmatrix} t\\ \underline{c}\end{smallmatrix}\big) \;\cong\; \Big(\Sigma_{\mathfrak{C}}(t,\underline{c})\times\{*\}\Big)\otimes I \;=\; \big(I_{\circ}^{\mathbf{Set}}\otimes I\big)\big(\begin{smallmatrix} t\\ \underline{c}\end{smallmatrix}\big) \quad , \tag{4.70}$$

where we used the definition of the circle unit (3.11) and (4.66b). $\square$

Using Proposition 2.3, we observe that *both* functors $(-)\otimes I$ and $\mathbf{M}\big(I,-\big)$ in (4.68) are canonically lax monoidal and hence they preserve monoids, which by Definition 3.16 are $\mathfrak{C}$-colored operads. Even more, because $(-)\otimes I$ is *strongly* monoidal, we obtain from [AM10, Propositions 3.91 and 3.94] the following result.

Proposition 4.23. *The adjunction* (4.68) *induces an adjunction (denoted by the same symbol)*

$$(-)\otimes I \,:\, \mathbf{Op}_{\mathfrak{C}}(\mathbf{Set}) \rightleftarrows \mathbf{Op}_{\mathfrak{C}}(\mathbf{M}) \,:\, \mathbf{M}\big(I,-\big) \quad , \tag{4.71}$$

between the categories of $\mathfrak{C}$*-colored operads.*

These results allow us promote all constructions in Sections 4.2 and 4.3 from **Set** to **M**.

Corollary 4.24. *Let* **M** *be any complete and cocomplete closed symmetric monoidal category.*

(i) For any $\mathbf{C}\in\mathbf{Cat}$*, there exists a* $\mathbf{C}_0$*-colored operad*

$$\mathcal{O}_{\mathbf{C}}\otimes I\in\mathbf{Op}_{\mathbf{C}_0}(\mathbf{M}) \quad , \tag{4.72}$$

which is obtained by applying (4.71) *to the* **Set***-valued colored operad* $\mathcal{O}_{\mathbf{C}}$ *described in Definition 4.5 and Proposition 4.7. This assignment gives a functor* $\mathcal{O}_{(-)}\otimes I : \mathbf{Cat}\to\mathbf{Op}(\mathbf{M})$.

(ii) For any $\overline{\mathbf{C}} = (\mathbf{C},\perp)\in\mathbf{OrthCat}$*, there exists a* $\mathbf{C}_0$*-colored operad*

$$\mathcal{O}_{\overline{\mathbf{C}}}\otimes I\in\mathbf{Op}_{\mathbf{C}_0}(\mathbf{M}) \quad , \tag{4.73}$$

which is obtained by applying (4.71) *to the* **Set***-valued colored operad* $\mathcal{O}_{\overline{\mathbf{C}}}$ *described in Definition 4.9 and Proposition 4.11. This assignment gives a functor* $\mathcal{O}_{(-)}\otimes I : \mathbf{OrthCat}\to \mathbf{Op}(\mathbf{M})$ *and Corollary 4.17 induces a natural transformation*

$$p\otimes I \,:\, \mathcal{O}_{\Pi(-)}\otimes I \longrightarrow \mathcal{O}_{(-)}\otimes I \quad . \tag{4.74}$$

(iii) For any $\mathbf{C} \in \mathbf{Cat}$, there exists an $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$-coequalizer

$$F\big((R_{\mathbf{C}} \otimes I)^{\mathsf{sym}}\big) \underset{r_{\mathbf{C},2}\otimes I}{\overset{r_{\mathbf{C},1}\otimes I}{\rightrightarrows}} F\big((G_{\mathbf{C}} \otimes I)^{\mathsf{sym}}\big) \xrightarrow{\;q_{\mathbf{C}}\otimes I\;} \mathcal{O}_{\mathbf{C}} \otimes I \quad , \tag{4.75}$$

which is obtained by applying (4.71) *to* (4.53)*. This coequalizer is natural in* $\mathbf{C} \in \mathbf{Cat}$*.*

(iv) For any $\overline{\mathbf{C}} = (\mathbf{C}, \perp) \in \mathbf{OrthCat}$, there exists an $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$-coequalizer

$$F\big((R_{\overline{\mathbf{C}}} \otimes I)^{\mathsf{sym}}\big) \underset{r_{\overline{\mathbf{C}},2}\otimes I}{\overset{r_{\overline{\mathbf{C}},1}\otimes I}{\rightrightarrows}} F\big((G_{\mathbf{C}} \otimes I)^{\mathsf{sym}}\big) \xrightarrow{\;(p_{\overline{\mathbf{C}}}\, q_{\mathbf{C}})\otimes I\;} \mathcal{O}_{\overline{\mathbf{C}}} \otimes I \quad , \tag{4.76}$$

which is obtained by applying (4.71) *to* (4.63)*. This coequalizer is natural in* $\overline{\mathbf{C}} \in \mathbf{OrthCat}$*.*

Proof. Item (i) and (ii) are obvious. To prove item (iii), let us first note that applying (4.71) to (4.53) produces a coequalizer

$$F(R_{\mathbf{C}}^{\mathsf{sym}}) \otimes I \underset{r_{\mathbf{C},2}\otimes I}{\overset{r_{\mathbf{C},1}\otimes I}{\rightrightarrows}} F(G_{\mathbf{C}}^{\mathsf{sym}}) \otimes I \xrightarrow{\;q_{\mathbf{C}}\otimes I\;} \mathcal{O}_{\mathbf{C}} \otimes I \tag{4.77}$$

in $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$, because $(-) \otimes I$ is left adjoint (cf. Proposition 4.23) and hence it preserves coequalizers. It remains to show that the functor $(-) \otimes I$ commutes (up to natural isomorphism) with the free colored operad functor F and the free symmetrization functor $(-)^{\mathsf{sym}}$. Consider the diagram of adjunctions

$$\begin{tikzcd}
\mathbf{Set}^{\Omega_{\mathbf{C}_0}\times\mathbf{C}_0} \arrow[r, shift left, "(-)^{\mathsf{sym}}"] \arrow[d, shift right, "(-)\otimes I"'] & \mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{Set}) \arrow[l, shift left, "U"] \arrow[r, shift left, "F"] \arrow[d, shift right, "(-)\otimes I"'] & \mathbf{Op}_{\mathbf{C}_0}(\mathbf{Set}) \arrow[l, shift left, "U"] \arrow[d, shift right, "(-)\otimes I"'] \\
\mathbf{M}^{\Omega_{\mathbf{C}_0}\times\mathbf{C}_0} \arrow[r, shift left, "(-)^{\mathsf{sym}}"] \arrow[u, shift right, "\mathbf{M}(I,-)"'] & \mathbf{SymSeq}_{\mathbf{C}_0}(\mathbf{M}) \arrow[l, shift left, "U"] \arrow[r, shift left, "F"] \arrow[u, shift right, "\mathbf{M}(I,-)"'] & \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M}) \arrow[l, shift left, "U"] \arrow[u, shift right, "\mathbf{M}(I,-)"']
\end{tikzcd} \tag{4.78}$$

where the horizontal adjunctions are given in Proposition 3.12 and Theorem 3.19, and the vertical adjunctions are (4.67), (4.68) and (4.71). Notice that all squares in (4.78) which are formed by right adjoint functors commute. Due to uniqueness (up to natural isomorphism) of left adjoint functors, this implies that all squares in (4.78) which are formed by left adjoint functors commute up to a natural isomorphism. This in particular proves item (iii), and item (iv) is shown by the same argument. □

Remark 4.25. In the following we shall drop the change of base category functor $(-) \otimes I$ from our notation. For example, the $\mathbf{M}$-valued colored operad given in Corollary 4.24 (ii) is from now on simply denoted by $\mathcal{O}_{\overline{\mathbf{C}}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$. △

4.5 Algebras

We shall now characterize the categories of algebras over the colored operads $\mathcal{O}_{\mathbf{C}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ and $\mathcal{O}_{\overline{\mathbf{C}}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ defined in Corollary 4.24. (Recall our simplified notation explained in Remark 4.25.) This will establish a relationship between these colored operads and algebraic quantum field theory (cf. Section 4.1).

Theorem 4.26. *Let $\mathbf{C} \in \mathbf{Cat}$ be a small category. There exists an isomorphism*

$$\mathbf{Alg}(\mathcal{O}_{\mathbf{C}}) \;\cong\; \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}} \tag{4.79}$$

between the category of algebras over the colored operad $\mathcal{O}_{\mathbf{C}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ of Corollary 4.24 (i) and the category of functors from $\mathbf{C}$ to monoids in $\mathbf{M}$. This isomorphism is natural in $\mathbf{C} \in \mathbf{Cat}$.

Proof. By Corollary 4.24 (iii), the colored operad $\mathcal{O}_{\mathbf{C}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ is naturally presented by the generators $G_{\mathbf{C}} \in \mathbf{M}^{\Omega_{\mathbf{C}_0}\times\mathbf{C}_0}$ and the relations defined by the pair of parallel $\mathbf{M}^{\Omega_{\mathbf{C}_0}\times\mathbf{C}_0}$-morphisms $r_{\mathbf{C},1}, r_{\mathbf{C},2} : R_{\mathbf{C}} \rightrightarrows F(G_{\mathbf{C}}^{\mathsf{sym}})$. Corollary 3.35 and the adjunction in Proposition 3.12 imply that an algebra over $\mathcal{O}_{\mathbf{C}}$ is a $\mathbf{C}_0$-colored object $A \in \mathbf{M}^{\mathbf{C}_0}$ together with an $\mathbf{M}^{\Omega_{\mathbf{C}_0}\times\mathbf{C}_0}$-morphism $\phi : G_{\mathbf{C}} \to \mathrm{End}(A)$ such that $\phi : F(G_{\mathbf{C}}^{\mathsf{sym}}) \to \mathrm{End}(A)$ coequalizes $r_{\mathbf{C},1}, r_{\mathbf{C},2} : R_{\mathbf{C}} \rightrightarrows F(G_{\mathbf{C}}^{\mathsf{sym}})$ in $\mathbf{M}^{\Omega_{\mathbf{C}_0}\times\mathbf{C}_0}$. We now show that this data is equivalent to a functor $\mathfrak{A} : \mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$. Let us set

$$\mathfrak{A}(c) \,:=\, A_c \in \mathbf{M} \quad , \tag{4.80}$$

for all $c \in \mathbf{C}_0$. Using the expression for $G_{\mathbf{C}}$ in (4.47) and for the endomorphism operad in (3.56), we observe that an $\mathbf{M}^{\Omega_{\mathbf{C}_0}\times\mathbf{C}_0}$-morphism $\phi : G_{\mathbf{C}} \to \mathrm{End}(A)$ is equivalent to the following data:

- For every $c \in \mathbf{C}_0$, an $\mathbf{M}$-morphism $1_c : I \to \mathfrak{A}(c)$;
- For every $f \in \mathbf{C}(c, c')$, an $\mathbf{M}$-morphism $\mathfrak{A}(f) : \mathfrak{A}(c) \to \mathfrak{A}(c')$;
- For every $c \in \mathbf{C}_0$, an $\mathbf{M}$-morphism $\mu_c : \mathfrak{A}(c) \otimes \mathfrak{A}(c) \to \mathfrak{A}(c)$.

Recall that the relations decompose into three different types (functoriality, monoid and compatibility relations (4.48)). Their individual effects are as follows:

(1) $\phi : F(G_{\mathbf{C}}^{\mathsf{sym}}) \to \mathrm{End}(A)$ coequalizes $r_{\mathrm{Fun},1}, r_{\mathrm{Fun},2} : R_{\mathrm{Fun}} \rightrightarrows F(G_{\mathbf{C}}^{\mathsf{sym}})$ if and only if

$$\mathrm{id}_{\mathfrak{A}(c)} \,=\, \mathfrak{A}(\mathrm{id}_c) \quad , \qquad \mathfrak{A}(f)\,\mathfrak{A}(g) \,=\, \mathfrak{A}(f\,g) \quad , \tag{4.81}$$

for all $c \in \mathbf{C}_0$ and all composable $\mathbf{C}$-morphisms $(f : c' \to c'', g : c \to c')$;

(2) $\phi : F(G_{\mathbf{C}}^{\mathsf{sym}}) \to \mathrm{End}(A)$ coequalizes $r_{\mathrm{Mon},1}, r_{\mathrm{Mon},2} : R_{\mathrm{Mon}} \rightrightarrows F(G_{\mathbf{C}}^{\mathsf{sym}})$ if and only if

$$\begin{aligned} \mu_c\big(1_c \otimes \mathrm{id}_{\mathfrak{A}(c)}\big) \,&=\, \mathrm{id}_{\mathfrak{A}(c)} \,=\, \mu_c\big(\mathrm{id}_{\mathfrak{A}(c)} \otimes 1_c\big) \quad , \\ \mu_c\big(\mu_c \otimes \mathrm{id}_{\mathfrak{A}(c)}\big) \,&=\, \mu_c\big(\mathrm{id}_{\mathfrak{A}(c)} \otimes \mu_c\big) \quad , \end{aligned} \tag{4.82}$$

for all $c \in \mathbf{C}_0$;

(3) $\phi : F(G_{\mathbf{C}}^{\mathsf{sym}}) \to \mathrm{End}(A)$ coequalizes $r_{\mathrm{FM},1}, r_{\mathrm{FM},2} : R_{\mathrm{FM}} \rightrightarrows F(G_{\mathbf{C}}^{\mathsf{sym}})$ if and only if

$$\mathfrak{A}(f)\,1_c \,=\, 1_{c'} \quad , \qquad \mathfrak{A}(f)\,\mu_c \,=\, \mu_{c'}\big(\mathfrak{A}(f) \otimes \mathfrak{A}(f)\big) \quad , \tag{4.83}$$

for all $(f : c \to c') \in \mathrm{Mor}\,\mathbf{C}$.

Hence, we obtain a canonical identification between $\mathcal{O}_{\mathbf{C}}$-algebras and functors $\mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$.

Next, we show that there is a canonical identification of morphisms of $\mathcal{O}_{\mathbf{C}}$-algebras and natural transformations between their associated functors $\mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$. Let $(A, \phi), (A', \phi') \in \mathbf{Alg}(\mathcal{O}_{\mathbf{C}})$ and denote their associated functors by $\mathfrak{A}, \mathfrak{A}' : \mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$. Corollary 3.35 and the adjunction in Proposition 3.12 imply that a morphism of $\mathcal{O}_{\mathbf{C}}$-algebras is an $\mathbf{M}^{\mathbf{C}_0}$-morphism $\psi : A \to A'$ such that the square

$$\begin{tikzcd} G_{\mathbf{C}} \arrow[r, "\phi"] \arrow[d, "\phi'"'] & \mathrm{End}(A) \arrow[d, "{[A,\psi]_\circ}"] \\ \mathrm{End}(A') \arrow[r, "{[\psi,A']_\circ}"'] & {[A, A']_\circ} \end{tikzcd} \tag{4.84}$$

in $\mathbf{M}^{\Omega_{\mathbf{C}_0}\times\mathbf{C}_0}$ commutes. Using the expression for $G_{\mathbf{C}}$ in (4.47) and for the endomorphism operad in (3.56), this is equivalent to the following conditions which say that $\psi : \mathfrak{A} \to \mathfrak{A}'$ defines a natural transformation of functors $\mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$:

- For every $c \in \mathbf{C}_0$, $\psi_c \, 1_c = 1'_c$;
- For every $f \in \mathbf{C}(c, c')$, $\psi_{c'} \, \mathfrak{A}(f) = \mathfrak{A}'(f) \, \psi_c$;
- For every $c \in \mathbf{C}_0$, $\psi_c \, \mu_c = \mu'_c \, (\psi_c \otimes \psi_c)$.

This establishes the isomorphism of categories in (4.79), which is natural due to naturality of the presentation by generators and relations, cf. Corollary 4.24 (iii). □

Theorem 4.27. *Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp) \in \mathbf{OrthCat}$ be an orthogonal category. There exists an isomorphism*

$$\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}}) \;\cong\; \mathbf{Mon}_{\mathbf{M}}^{\overline{\mathbf{C}}} \tag{4.85}$$

between the category of algebras over the colored operad $\mathcal{O}_{\overline{\mathbf{C}}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ of Corollary 4.24 (ii) and the category of $\perp$-commutative functors from $\mathbf{C}$ to monoids in $\mathbf{M}$ (cf. Definition 4.4). This isomorphism is natural in $\overline{\mathbf{C}} \in \mathbf{OrthCat}$.

Proof. The proof proceeds analogously to the one of Theorem 4.26 by using Corollary 4.24 (iv) instead of (iii). The additional $\perp$-commutativity relations (4.61) have the following effect:

(4) $\phi : F(G_{\mathbf{C}}^{\mathsf{sym}}) \to \mathrm{End}(A)$ coequalizes $r_{\perp,1}, r_{\perp,2} : R_\perp \rightrightarrows F(G_{\mathbf{C}}^{\mathsf{sym}})$ if and only if

$$\mu_c \big(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)\big) \;=\; \mu_c \, \tau \, \big(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)\big) \quad , \tag{4.86}$$

for all orthogonal pairs $f_1 \perp f_2$, where τ denotes the symmetric braiding on $\mathbf{M}$.

This is precisely the $\perp$-commutativity axiom of Definition 4.4, which completes the proof. □

Remark 4.28. The results of this section have the following quantum field theoretic interpretation. As in Section 4.1, we interpret $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ as a category of spacetimes $\mathbf{C}$, together with a specification $\perp$ of pairs of observables that are supposed to behave commutatively. The category $\mathbf{Mon}_{\mathbf{M}}^{\overline{\mathbf{C}}}$ of $\perp$-commutative functors describes all possible quantum field theories for this scenario. Theorem 4.27 deepens our understanding of the abstract algebraic structures underlying such quantum field theories by proving that they are precisely the algebras over our colored operad $\mathcal{O}_{\overline{\mathbf{C}}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$. It is worth noticing the following similarity: One of the key ideas of algebraic quantum field theory is to shift the focus from (Hilbert space) representations of algebras to the underlying abstract algebras in order to analyze structural properties of quantum field theories. Our operadic framework goes one level deeper by shifting the focus from particular realizations of the algebraic structures of quantum field theories to the underlying abstract operads. △

4.6 Examples

We present examples of orthogonal categories $\overline{\mathbf{C}} = (\mathbf{C}, \perp) \in \mathbf{OrthCat}$ that are motivated by algebraic quantum field theory. We will obtain that the algebras over their colored operads $\mathcal{O}_{\overline{\mathbf{C}}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ canonically correspond to quantum field theories.

To present our examples, we need the following constructions and properties.

Lemma 4.29. *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between small categories.*

(i) Given any orthogonality relation $\perp_{\mathbf{D}} \subseteq \mathrm{Mor}\,\mathbf{D} \,{}_{\mathrm{t}}\!\times_{\mathrm{t}} \mathrm{Mor}\,\mathbf{D}$ on $\mathbf{D}$, then

$$F^*(\perp_{\mathbf{D}}) \,:=\, \Big\{ (f_1, f_2) \in \mathrm{Mor}\,\mathbf{C} \,{}_{\mathrm{t}}\!\times_{\mathrm{t}} \mathrm{Mor}\,\mathbf{C} \;:\; F(f_1) \perp_{\mathbf{D}} F(f_2) \Big\} \tag{4.87}$$

is an orthogonality relation on $\mathbf{C}$. We call $F^*(\perp_{\mathbf{D}})$ the pullback of $\perp_{\mathbf{D}}$ along F and note that $F : (\mathbf{C}, F^*(\perp_{\mathbf{D}})) \to (\mathbf{D}, \perp_{\mathbf{D}})$ is an orthogonal functor.

(ii) Given any orthogonality relation $\perp_{\mathbf{C}} \subseteq \operatorname{Mor}\mathbf{C}\,{}_{\mathrm{t}}\!\times_{\mathrm{t}} \operatorname{Mor}\mathbf{C}$ *on* $\mathbf{C}$*, then*

$$F_*(\perp_{\mathbf{C}}) := \Big\{ \big(g\,F(f_1)\,h_1, g\,F(f_2)\,h_2\big) \in \operatorname{Mor}\mathbf{D}\,{}_{\mathrm{t}}\!\times_{\mathrm{t}} \operatorname{Mor}\mathbf{D} \ : \ f_1 \perp_{\mathbf{C}} f_2 \text{ and } g, h_1, h_2 \in \operatorname{Mor}\mathbf{D} \Big\} \tag{4.88}$$

is an orthogonality relation on $\mathbf{D}$*. We call* $F_*(\perp_{\mathbf{C}})$ *the pushforward of* $\perp_{\mathbf{C}}$ *along* F *and note that* $F : (\mathbf{C}, \perp_{\mathbf{C}}) \to (\mathbf{D}, F_*(\perp_{\mathbf{C}}))$ *is an orthogonal functor.*

Proof. Obvious by checking the conditions in Definitions 4.3 and 4.15. □

Lemma 4.30. *Let* $\overline{\mathbf{C}} = (\mathbf{C}, \perp) \in \mathbf{OrthCat}$ *and* $W \subseteq \operatorname{Mor}\mathbf{C}$ *any subset. Define* $\overline{\mathbf{C}[W^{-1}]} := (\mathbf{C}[W^{-1}], L_*(\perp)) \in \mathbf{OrthCat}$ *by pushing forward the orthogonality relation* $\perp$ *on* $\mathbf{C}$ *along the localization functor* $L : \mathbf{C} \to \mathbf{C}[W^{-1}]$*. There exists a canonical identification between*

(1) $\perp$*-commutative and* W*-constant functors* $\mathfrak{B} : \mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$*;*

(2) $L_*(\perp)$*-commutative functors* $\mathfrak{A} : \mathbf{C}[W^{-1}] \to \mathbf{Mon}_{\mathbf{M}}$*.*

Proof. Given any $L_*(\perp)$-commutative functor $\mathfrak{A} : \mathbf{C}[W^{-1}] \to \mathbf{Mon}_{\mathbf{M}}$, define $\mathfrak{B} := \mathfrak{A}\,L : \mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$. The functor $\mathfrak{B}$ is obviously W-constant and $\perp$-commutative.

Conversely, let $\mathfrak{B} : \mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$ be any $\perp$-commutative and W-constant functor. Define $\mathfrak{A} : \mathbf{C}[W^{-1}] \to \mathbf{Mon}_{\mathbf{M}}$ by using the universal property of localizations, i.e. $\mathfrak{A}$ is the unique functor such that $\mathfrak{A}\,L = \mathfrak{B}$. We have to show that $\mathfrak{A}$ is $L_*(\perp)$-commutative. By definition of the pushforward orthogonality relation (cf. Lemma 4.29), any element in $L_*(\perp)$ may be written as $\big(g\,L(f_1)\,h_1, g\,L(f_2)\,h_2\big)$ with $f_1 \perp f_2$ and $g, h_1, h_2 \in \operatorname{Mor}\mathbf{C}[W^{-1}]$. Using that $\mathfrak{A} : \mathbf{C}[W^{-1}] \to \mathbf{Mon}_{\mathbf{M}}$ is a functor, the diagram in Definition 4.4 corresponding to $\big(g\,L(f_1)\,h_1, g\,L(f_2)\,h_2\big)$ may be decomposed into five smaller squares. One observes that it suffices to prove that the square

$$\begin{tikzcd} \mathfrak{A}(c_1) \otimes \mathfrak{A}(c_2) \arrow[r, "\mathfrak{A}\,L(f_1)\otimes\mathfrak{A}\,L(f_2)"] \arrow[d, "\mathfrak{A}\,L(f_1)\otimes\mathfrak{A}\,L(f_2)"'] & \mathfrak{A}(c) \otimes \mathfrak{A}(c) \arrow[d, "\mu_c^{\mathrm{op}}"] \\ \mathfrak{A}(c) \otimes \mathfrak{A}(c) \arrow[r, "\mu_c"'] & \mathfrak{A}(c) \end{tikzcd} \tag{4.89}$$

commutes, which is true because $\mathfrak{A}\,L = \mathfrak{B}$ is by hypothesis $\perp$-commutative and $f_1 \perp f_2$. □

Example 4.31 (Locally covariant quantum field theory without time-slice axiom)**.** In locally covariant quantum field theory [BFV03, FV15] one considers the category **Loc** of oriented, time-oriented and globally hyperbolic Lorentzian manifolds of a fixed dimension $m \geq 2$. Concretely, the objects in **Loc** are tuples $\mathbb{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ where M is an m-dimensional manifold (Hausdorff and second-countable), g is a globally hyperbolic Lorentzian metric on M, $\mathfrak{o}$ is an orientation of M and $\mathfrak{t}$ is a time-orientation of (M, g). A **Loc**-morphism $f : \mathbb{M} \to \mathbb{M}'$ is an orientation and time-orientation preserving isometric embedding, such that the image $f(M) \subseteq M'$ is causally compatible and open. (For an introduction to Lorentzian geometry we refer the reader to e.g. [BGP07].) Note that the category **Loc** is *not* small, however it is equivalent to a small category, i.e. it is essentially small. As usual, this follows by using Whitney's embedding theorem to realize (up to diffeomorphism) all m-dimensional manifolds M as submanifolds of $\mathbb{R}^{2m+1}$. In the following we always choose a small category equivalent to **Loc** and denote it with abuse of notation also by **Loc**. Our results in Section 5.4 imply that it does not matter which small subcategory equivalent to **Loc** we choose. More precisely, different choices define equivalent categories of algebras over their associated colored operads.

We equip the category **Loc** with the following orthogonality relation: Two **Loc**-morphisms $f_1 : \mathbb{M}_1 \to \mathbb{M}$ and $f_2 : \mathbb{M}_2 \to \mathbb{M}$ are orthogonal, $f_1 \perp f_2$, if and only if their images $f_1(M_1)$

and $f_2(M_2)$ are causally disjoint subsets in $\mathbb{M}$, i.e. $f_1(M_1) \cap J_{\mathbb{M}}(f_2(M_2)) = \emptyset$, where $J_{\mathbb{M}}(S) := J^+_{\mathbb{M}}(S) \cup J^-_{\mathbb{M}}(S) \subseteq M$ denotes the union of the causal future and past of a subset $S \subseteq M$. It is easy to verify the symmetry and composition stability properties of Definition 4.3. Hence, $\overline{\mathbf{Loc}} := (\mathbf{Loc}, \perp) \in \mathbf{OrthCat}$ is an orthogonal category.

By Theorem 4.27, we obtain that algebras over the colored operad $\mathcal{O}_{\overline{\mathbf{Loc}}} \in \mathbf{Op}_{\mathbf{Loc}_0}(\mathbf{M})$ are canonically identified with functors $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Mon}_{\mathbf{M}}$ from $\mathbf{Loc}$ to the category of monoids in $\mathbf{M}$ that satisfy the $\perp$-commutativity axiom (cf. Definition 4.4). For $\mathbf{M} = \mathbf{Vec}_{\mathbb{K}}$, these are locally covariant quantum field theories [BFV03, FV15] satisfying the Einstein causality axiom, but not necessarily the time-slice axiom. $\triangledown$

Example 4.32 (Locally covariant quantum field theory with time-slice axiom)**.** In the setting of Example 4.31, recall that a morphism $f : \mathbb{M} \to \mathbb{M}'$ is called a *Cauchy morphism* if its image $f(M) \subseteq M'$ contains a Cauchy surface of $\mathbb{M}'$. We denote the subset of all Cauchy morphisms by $W \subseteq \mathrm{Mor}\,\mathbf{Loc}$ and consider the localized category $\mathbf{Loc}[W^{-1}]$, together with the localization functor $L : \mathbf{Loc} \to \mathbf{Loc}[W^{-1}]$. Using Lemma 4.29, we may equip $\mathbf{Loc}[W^{-1}]$ with the pushforward orthogonality relation $L_*(\perp)$ and obtain an orthogonal functor $L : \overline{\mathbf{Loc}} \to \big(\mathbf{Loc}[W^{-1}], L_*(\perp)\big) =: \overline{\mathbf{Loc}[W^{-1}]}$.

By Theorem 4.27, we obtain that algebras over $\mathcal{O}_{\overline{\mathbf{Loc}[W^{-1}]}} \in \mathbf{Op}_{\mathbf{Loc}[W^{-1}]_0}(\mathbf{M})$ are canonically identified with $L_*(\perp)$-commutative functors $\mathfrak{A} : \mathbf{Loc}[W^{-1}] \to \mathbf{Mon}_{\mathbf{M}}$. By Lemma 4.30, such functors are canonically identified with $\perp$-commutative and W-constant functors $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Mon}_{\mathbf{M}}$ on $\mathbf{Loc}$ (cf. Definitions 4.4 and 4.1). For $\mathbf{M} = \mathbf{Vec}_{\mathbb{K}}$, these are locally covariant quantum field theories [BFV03, FV15] satisfying the Einstein causality axiom and the time-slice axiom.

Using Corollary 4.24 (ii) and Theorem 3.40, the orthogonal functor $L : \overline{\mathbf{Loc}} \to \overline{\mathbf{Loc}[W^{-1}]}$ defines an adjunction

$$\mathcal{O}_{L!} \,:\, \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{Loc}}}\big) \xrightleftharpoons{} \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{Loc}[W^{-1}]}}\big) \,:\, \mathcal{O}_L^* \quad . \tag{4.90}$$

We call the left adjoint $\mathcal{O}_{L!}$ the *W-constantification functor* or, more specifically, the *time-slicification functor*. To a quantum field theory which may not satisfy the time-slice axiom it assigns one which does. We shall analyze such adjunctions in more detail in Section 5.2. $\triangledown$

Example 4.33 (Algebraic quantum field theory on a fixed spacetime)**.** Let us fix any object $\mathbb{M} \in \mathbf{Loc}$, e.g. the Minkowski spacetime. Let $\mathbf{gh}(\mathbb{M})$ be the category of all globally hyperbolic open subsets $U \subseteq M$ of $\mathbb{M}$ with morphisms given by subset inclusions $U \subseteq V \subseteq M$. There is a functor $K : \mathbf{gh}(\mathbb{M}) \to \mathbf{Loc}$ that assigns to $U \subseteq M$ the $\mathbf{Loc}$-object $K(U) := (U, g|_U, \mathfrak{o}|_U, \mathfrak{t}|_U)$ obtained by restricting the metric, orientation and time-orientation of $\mathbb{M}$ to $U \subseteq M$. To a morphism $U \subseteq V \subseteq M$ it assigns the $\mathbf{Loc}$-morphism $K(U) \to K(V)$ that is induced by the subset inclusion $U \hookrightarrow V$. Using Lemma 4.29, we equip $\mathbf{gh}(\mathbb{M})$ with the pullback orthogonality relation $K^*(\perp)$ and obtain an orthogonal functor $K : \overline{\mathbf{gh}(\mathbb{M})} := (\mathbf{gh}(\mathbb{M}), K^*(\perp)) \to \overline{\mathbf{Loc}}$. Explicitly, $U_1 \subseteq V \subseteq M$ and $U_2 \subseteq V \subseteq M$ are $K^*(\perp)$-orthogonal if and only if U_1 and U_2 are causally disjoint subsets in $\mathbb{M}$. By Theorem 4.27, we obtain that algebras over $\mathcal{O}_{\overline{\mathbf{gh}(\mathbb{M})}} \in \mathbf{Op}_{\mathbf{gh}(\mathbb{M})_0}(\mathbf{M})$ are canonically identified with $K^*(\perp)$-commutative functors $\mathfrak{A} : \mathbf{gh}(\mathbb{M}) \to \mathbf{Mon}_{\mathbf{M}}$. For $\mathbf{M} = \mathbf{Vec}_{\mathbb{K}}$, these are causal nets of $\mathbb{K}$-algebras on $\mathbb{M}$ that do not necessarily satisfy the time-slice axiom.

The time-slice axiom may be implemented again by a localization: Let $W_{\mathbb{M}} \subseteq \mathrm{Mor}\,\mathbf{gh}(\mathbb{M})$ be the subset of all morphisms $U \subseteq V \subseteq M$ such that $(K(U) \to K(V)) \in W$ is a Cauchy morphism in $\mathbf{Loc}$ (cf. Example 4.32), i.e. $U \subseteq V$ contains a Cauchy surface of $K(V) \in \mathbf{Loc}$. We use the localization functor $L_{\mathbb{M}} : \mathbf{gh}(\mathbb{M}) \to \mathbf{gh}(\mathbb{M})[W_{\mathbb{M}}^{-1}]$ to equip $\mathbf{gh}(\mathbb{M})[W_{\mathbb{M}}^{-1}]$ with the pushforward orthogonality relation $L_{\mathbb{M}*}K^*(\perp)$. We obtain an orthogonal functor $L_{\mathbb{M}} : \overline{\mathbf{gh}(\mathbb{M})} \to \big(\mathbf{gh}(\mathbb{M})[W_{\mathbb{M}}^{-1}], L_{\mathbb{M}*}K^*(\perp)\big) =: \overline{\mathbf{gh}(\mathbb{M})[W_{\mathbb{M}}^{-1}]}$. Analogously to Example 4.32, we obtain that algebras over $\mathcal{O}_{\overline{\mathbf{gh}(\mathbb{M})[W_{\mathbb{M}}^{-1}]}} \in \mathbf{Op}_{\mathbf{gh}(\mathbb{M})[W_{\mathbb{M}}^{-1}]_0}(\mathbf{M})$ are canonically identified with $K^*(\perp)$-commutative and $W_{\mathbb{M}}$-constant functors $\mathfrak{A} : \mathbf{gh}(\mathbb{M}) \to \mathbf{Mon}_{\mathbf{M}}$. For $\mathbf{M} = \mathbf{Vec}_{\mathbb{K}}$, these are causal nets of $\mathbb{K}$-algebras on $\mathbb{M}$ that satisfy the time-slice axiom, cf. [HK64]. $\triangledown$

Example 4.34 (Chiral conformal quantum field theory)**.** Inspired by the algebraic structures underlying chiral conformal quantum field theory [Kaw15, Reh15, BDH15], let us consider the category $\mathbf{Man}$ of oriented manifolds of a fixed dimension $m \geq 1$ with morphisms given by orientation preserving open embeddings. (In practice, one considers $m = 1$, however the following statements hold for any dimension m.) Similarly to Example 4.31, the category $\mathbf{Man}$ is only essentially small, hence we have to choose a small category equivalent to $\mathbf{Man}$, which we denote with abuse of notation also by $\mathbf{Man}$.

We equip the category $\mathbf{Man}$ with the following orthogonality relation: $f_1 : M_1 \to M$ and $f_2 : M_2 \to M$ are orthogonal, $f_1 \perp f_2$, if and only if their images are disjoint, i.e. $f_1(M_1) \cap f_2(M_2) = \emptyset$ as subsets of M. Let us denote by $\overline{\mathbf{Man}} := (\mathbf{Man}, \perp)$ the corresponding orthogonal category. By Theorem 4.27, we obtain that algebras over $\mathcal{O}_{\overline{\mathbf{Man}}} \in \mathbf{Op}_{\mathbf{Man}_0}(\mathbf{M})$ are canonically identified with functors $\mathfrak{A} : \mathbf{Man} \to \mathbf{Mon}_{\mathbf{M}}$ that satisfy the $\perp$-commutativity axiom. For $\mathbf{M} = \mathbf{Vec}_{\mathbb{K}}$ and dimension $m = 1$, these are coordinate-free chiral conformal nets of $\mathbb{K}$-algebras that satisfy the commutativity axiom for observables localized in disjoint regions [BDH15].

Similarly to Example 4.33, we may fix any $M \in \mathbf{Man}$, e.g. the circle $\mathbb{S}^1$ for dimension $m = 1$, and introduce the category $\mathbf{op}(M)$ of all open subsets of $U \subseteq M$ with morphisms given by subset inclusions. There is an obvious functor $K : \mathbf{op}(M) \to \mathbf{Man}$ which we can use to pull back the orthogonality relation $\perp$ on $\mathbf{Man}$ to an orthogonality relation $K^*(\perp)$ on $\mathbf{op}(M)$. We denote the resulting orthogonal functor by $K : \overline{\mathbf{op}(M)} := (\mathbf{op}(M)\, K^*(\perp)) \to \overline{\mathbf{Man}}$. Algebras over $\mathcal{O}_{\overline{\mathbf{op}(M)}} \in \mathbf{Op}_{\mathbf{op}(M)_0}(\mathbf{M})$ are canonically identified with $K^*(\perp)$-commutative functors $\mathfrak{A} : \mathbf{op}(M) \to \mathbf{Mon}_{\mathbf{M}}$. For $\mathbf{M} = \mathbf{Vec}_{\mathbb{K}}$, dimension $m = 1$ and $M = \mathbb{S}^1$, these are chiral conformal nets of $\mathbb{K}$-algebras on the circle, cf. [Kaw15, Reh15].

One is sometimes also interested in (coordinate-free) chiral conformal nets which are defined only on intervals. Let us formalize such theories in our framework. Let $\mathbf{Disc} \subseteq \mathbf{Man}$ denote the full subcategory whose objects M are diffeomorphic to $\mathbb{R}^m$ and $j : \mathbf{Disc} \to \mathbf{Man}$ the inclusion functor. We equip $\mathbf{Disc}$ with the pullback orthogonality relation $j^*(\perp)$ and obtain an orthogonal functor $j : \overline{\mathbf{Disc}} := (\mathbf{Disc}, j^*(\perp)) \to \overline{\mathbf{Man}}$. Algebras over $\mathcal{O}_{\overline{\mathbf{Disc}}} \in \mathbf{Op}_{\mathbf{Disc}_0}(\mathbf{M})$ are $j^*(\perp)$-commutative functors $\mathfrak{A} : \mathbf{Disc} \to \mathbf{Mon}_{\mathbf{M}}$. For $\mathbf{M} = \mathbf{Vec}_{\mathbb{K}}$ and dimension $m = 1$, these are coordinate-free chiral conformal nets of $\mathbb{K}$-algebras defined only on intervals, which satisfy the commutativity axiom for observables localized in disjoint intervals [BDH15]. By Corollary 4.24 (ii) and Theorem 3.40, the orthogonal functor $j : \overline{\mathbf{Disc}} \to \overline{\mathbf{Man}}$ induces an adjunction

$$\mathcal{O}_{j_!} : \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{Disc}}}) \rightleftarrows \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{Man}}}) : \mathcal{O}_j^* \quad . \tag{4.91}$$

The right adjoint $\mathcal{O}_j^*$ is the restriction functor from quantum field theories defined on all of $\mathbf{Man}$ to theories on $\mathbf{Disc}$. More interestingly, the left adjoint $\mathcal{O}_{j_!}$ is an extension functor that extends quantum field theories defined on $\mathbf{Disc}$ to all of $\mathbf{Man}$. Such ideas of extending quantum field theories from a subcategory of "nice" spacetimes to the whole category of spacetimes are of course also interesting in other scenarios, e.g. in locally covariant quantum field theory (cf. Example 4.31). We shall analyze these adjunctions in more detail in Section 5.3. In Section 6, we compare our constructions to Fredenhagen's universal algebra [Fre90, Fre93, FRS92], which is obtained by left Kan extension of monoid-valued functors [Lan14]. $\triangledown$

Example 4.35 (Euclidean quantum field theories)**.** Similarly to the examples above, we may also describe Euclidean algebraic quantum field theories [Sch99] in our framework. The relevant category is (any small category equivalent to) $\mathbf{Riem}$, the category of m-dimensional oriented Riemannian manifolds $\mathbb{M} = (M, g, \mathfrak{o})$ with morphisms given by orientation preserving isometric open embeddings. The relevant orthogonality relation is given by $(f_1 : \mathbb{M}_1 \to \mathbb{M}) \perp (f_2 : \mathbb{M}_2 \to \mathbb{M})$ if and only if $f_1(M_1) \cap f_2(M_2) = \emptyset$ as subsets of M. Algebras over the $\mathbf{Vec}_{\mathbb{K}}$-valued colored operad $\mathcal{O}_{\overline{\mathbf{Riem}}}$ are canonically identified with (locally covariant versions of) Euclidean quantum field theories that satisfy the commutativity axiom for observables localized in disjoint regions. As in the other examples above, one may again restrict to Euclidean theories on a fixed Riemannian manifold $\mathbb{M} \in \mathbf{Riem}$, e.g. the Euclidean plane [Sch99]. $\triangledown$

5 Algebra adjunctions

Given an orthogonal functor $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$, we obtain by Corollary 4.24 (ii) an $\mathbf{Op}(\mathbf{M})$-morphism $\mathcal{O}_F : \mathcal{O}_{\overline{\mathbf{C}}} \to \mathcal{O}_{\overline{\mathbf{D}}}$ and thus by Theorem 3.40 an adjunction

$$\mathcal{O}_{F\,!} \,:\, \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{C}}}\big) \xrightarrow{\quad} \xleftarrow{\quad} \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{D}}}\big) \,:\, \mathcal{O}_F^* \tag{5.1}$$

between the categories of algebras. The examples in Section 4.6 showed that such adjunctions lead to interesting constructions in quantum field theory, for example W-constantification/time-slicification (cf. Example 4.32) and local-to-global extensions (cf. Example 4.34). The aim of this section is to study these adjunctions for particularly interesting classes of orthogonal functors in more detail. We will also explain the physical significance of our results for quantum field theory.

5.1 General orthogonal functors

In this subsection we establish a relation between the adjunction (5.1) for a general orthogonal functor F and the adjunction

$$\mathrm{Lan}_F \,:\, \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}} \xrightarrow{\quad} \xleftarrow{\quad} \mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}} \,:\, F^* \tag{5.2}$$

obtained by left Kan extension of monoid-valued functors along the functor $F : \mathbf{C} \to \mathbf{D}$. Notice that the latter neglects the orthogonality relations on $\mathbf{C}$ and $\mathbf{D}$.

In order to compare these two adjunctions, let us recall from Theorem 4.26 that there exists a natural isomorphism $\mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}} \cong \mathbf{Alg}(\mathcal{O}_{\mathbf{C}})$, where $\mathcal{O}_{\mathbf{C}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$ is our auxiliary colored operad that does not encode the $\perp$-commutativity relations. By Corollary 4.24 (ii), there exists a natural $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{M})$-morphism $p_{\overline{\mathbf{C}}} : \mathcal{O}_{\mathbf{C}} \to \mathcal{O}_{\overline{\mathbf{C}}}$, hence we obtain from Theorem 3.40 a natural adjunction

$$p_{\overline{\mathbf{C}}\,!} \,:\, \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}} \xrightarrow{\quad} \xleftarrow{\quad} \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}}) \,:\, p_{\overline{\mathbf{C}}}^* \quad . \tag{5.3}$$

Notice that (5.3) involves a slight abuse of notation. According to our notation in Theorem 3.40, the right adjoint functor would be denoted as $(\mathrm{id}_{\mathbf{C}_0}, p_{\overline{\mathbf{C}}})^*$ and the left adjoint functor as $(\mathrm{id}_{\mathbf{C}_0}, p_{\overline{\mathbf{C}}})_!$. We decided to drop the identity maps $\mathrm{id}_{\mathbf{C}_0}$ in order to simplify notation. Theorem 4.27 implies that $\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}}) \cong \mathbf{Mon}_{\mathbf{M}}^{\overline{\mathbf{C}}}$ and it is easy to verify that under this isomorphism the right adjoint functor $p_{\overline{\mathbf{C}}}^*$ in (5.3) is given by the functor $U : \mathbf{Mon}_{\mathbf{M}}^{\overline{\mathbf{C}}} \to \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}$ that forgets $\perp$-commutativity. Because the latter is a full subcategory embedding, we observe that $\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ is a *full reflective subcategory* of $\mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}$, see e.g. [MacL98, Chapter IV.3]. Summing up, we obtain

Lemma 5.1. *The natural adjunction* (5.3) *exhibits* $\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ *as a full reflective subcategory of* $\mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}$*, for every* $\overline{\mathbf{C}} = (\mathbf{C}, \perp) \in \mathbf{OrthCat}$*. In particular, the counit* $\epsilon : p_{\overline{\mathbf{C}}\,!}\, p_{\overline{\mathbf{C}}}^* \to \mathrm{id}_{\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})}$ *of this adjunction is a natural isomorphism.*

Remark 5.2. The adjunction (5.3) admits a quantum field theoretic interpretation. The category $\mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}$ contains also functors that do not satisfy the $\perp$-commutativity axiom and hence should not be regarded as quantum field theories. The left adjoint functor $p_{\overline{\mathbf{C}}\,!}$ in (5.3) allows us to assign to any functor $\mathfrak{B} : \mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$ a bona fide quantum field theory $p_{\overline{\mathbf{C}}\,!}(\mathfrak{B}) \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$. This construction may be called $\perp$-*abelianization* due to its structural similarity with the abelianization of algebraic structures such as groups or monoids. In locally covariant quantum field theory (cf.

Examples 4.31 and 4.32), this construction amounts to a *causalization* of theories that do not necessarily obey the Einstein causality axiom. By Lemma 5.1, we know that the counit of the adjunction (5.3) is a natural isomorphism. Concretely, this means that the $\perp$-abelianization of the functor $\mathfrak{B} = p_{\overline{\mathbf{C}}}^*(A)$ underlying a bona fide quantum field theory $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ is isomorphic to A itself via $\epsilon : p_{\overline{\mathbf{C}}!}\, p_{\overline{\mathbf{C}}}^*(A) \to A$. This is definitely a physically reasonable property. $\triangle$

Using Remark 4.18 and Lemma 3.42, we observe that there exists a diagram of adjunctions

$$
\begin{tikzcd}
\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}}) \arrow[r, shift left, "\mathcal{O}_{F!}"] \arrow[d, shift left, "p_{\overline{\mathbf{C}}}^*"] & \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}}) \arrow[l, shift left, "\mathcal{O}_F^*"] \arrow[d, shift left, "p_{\overline{\mathbf{D}}}^*"] \\
\mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}} \arrow[r, shift left, "\mathrm{Lan}_F"] \arrow[u, shift left, "p_{\overline{\mathbf{C}}!}"] & \mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}} \arrow[l, shift left, "F^*"] \arrow[u, shift left, "p_{\overline{\mathbf{D}}!}"]
\end{tikzcd}
\tag{5.4}
$$

in which the square formed by the right adjoint functors commutes, i.e. $p_{\overline{\mathbf{C}}}^*\, \mathcal{O}_F^* = F^*\, p_{\overline{\mathbf{D}}}^*$. This allows us to prove the main result of this subsection.

Proposition 5.3. *Let $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ be an orthogonal functor. There exists a natural isomorphism*

$$
\mathcal{O}_{F!} \;\cong\; p_{\overline{\mathbf{D}}!}\ \mathrm{Lan}_F\ p_{\overline{\mathbf{C}}}^* \tag{5.5}
$$

of functors $\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}}) \to \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$.

Proof. Let $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ and $B \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$. Using (5.4), we obtain the following chain of natural bijections of Hom-sets

$$
\begin{aligned}
\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})\big(p_{\overline{\mathbf{D}}!}\ \mathrm{Lan}_F\ p_{\overline{\mathbf{C}}}^*(A), B\big) \;&\cong\; \mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}\big(\mathrm{Lan}_F\ p_{\overline{\mathbf{C}}}^*(A), p_{\overline{\mathbf{D}}}^*(B)\big)\\
&\cong\; \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}\big(p_{\overline{\mathbf{C}}}^*(A), F^*\, p_{\overline{\mathbf{D}}}^*(B)\big)\\
&=\; \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}\big(p_{\overline{\mathbf{C}}}^*(A), p_{\overline{\mathbf{C}}}^*\, \mathcal{O}_F^*(B)\big) \quad ,
\end{aligned}
\tag{5.6}
$$

where in the last step we used that the square formed by the right adjoint functors commutes. Using also that the functor $p_{\overline{\mathbf{C}}}^*$ is fully faithful (cf. Lemma 5.1), we obtain

$$
\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})\big(p_{\overline{\mathbf{D}}!}\ \mathrm{Lan}_F\, p_{\overline{\mathbf{C}}}^*(A), B\big) \;\cong\; \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})\big(A, \mathcal{O}_F^*(B)\big) \quad , \tag{5.7}
$$

which implies that $p_{\overline{\mathbf{D}}!}\ \mathrm{Lan}_F\ p_{\overline{\mathbf{C}}}^*$ is a left adjoint of $\mathcal{O}_F^*$. The uniqueness (up to natural isomorphism) of adjoint functors implies the assertion. $\square$

5.2 Orthogonal localizations

Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp) \in \mathbf{OrthCat}$ be an orthogonal category and $W \subseteq \mathrm{Mor}\,\mathbf{C}$ a subset of the set of morphisms. Consider the localized category $\mathbf{C}[W^{-1}]$ together with the localization functor $L : \mathbf{C} \to \mathbf{C}[W^{-1}]$. We define an orthogonality relation $L_*(\perp)$ on $\mathbf{C}[W^{-1}]$ by using the pushforward construction from Lemma 4.29. We obtain an orthogonal functor $L : \overline{\mathbf{C}} \to (\mathbf{C}[W^{-1}], L_*(\perp)) =: \overline{\mathbf{C}[W^{-1}]}$ and hence by Corollary 4.24 (ii) and Theorem 3.40 an adjunction

$$
\mathcal{O}_{L!} \;:\; \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{C}}}\big) \rightleftarrows \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{C}[W^{-1}]}}\big) \;:\; \mathcal{O}_L^* \quad . \tag{5.8}
$$

Recall from Example 4.32 that the left adjoints of such adjunctions should be interpreted physically in terms of W-constantification/time-slicification. This particular class of adjunctions obtained from localization functors enjoys the following properties.

Proposition 5.4. *The adjunction* (5.8) *exhibits $\mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{C}[W^{-1}]}}\big)$ as a full reflective subcategory of $\mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{C}}}\big)$. In particular, the counit $\epsilon : \mathcal{O}_{L!}\, \mathcal{O}_L^* \to \mathrm{id}_{\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}[W^{-1}]}})}$ of this adjunction is a natural isomorphism.*

Proof. Up to the isomorphisms established Theorem 4.27, the right adjoint functor $\mathcal{O}_L^*$ in (5.8) is given by restricting the pullback functor

$$L^* \,:\, \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}[W^{-1}]} \longrightarrow \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}} \tag{5.9}$$

to the full subcategories $\mathbf{Mon}_{\mathbf{M}}^{\overline{\mathbf{C}[W^{-1}]}}$ and $\mathbf{Mon}_{\mathbf{M}}^{\overline{\mathbf{C}}}$ of $\perp$-commutative functors. Due to the universal property of localizations (see also [GZ67, Chapter 1]), the functor (5.9) is a fully faithful embedding and hence so is $\mathcal{O}_L^*$ by restriction to full subcategories. The statement about the counit is a consequence of [MacL98, Chapter IV.3]. □

Remark 5.5. In the context of Example 4.32, we interpret the left adjoint $\mathcal{O}_{L!}$ as the W-constantification/time-slicification functor and the right adjoint $\mathcal{O}_L^*$ as the functor forgetting the W-constancy/time-slice axiom. Furthermore, Proposition 5.4 has the following pleasant physical interpretation: Take any quantum field theory $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}[W^{-1}]}})$ that does satisfy the W-constancy axiom and forget this property by considering $\mathcal{O}_L^*(A) \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$. Applying the W-constantification functor then determines a quantum field theory that is isomorphic to our original theory via $\epsilon : \mathcal{O}_{L!}\, \mathcal{O}_L^*(A) \to A$. This is definitely a physically reasonable property. △

5.3 Full orthogonal subcategories

Let $\overline{\mathbf{D}} = (\mathbf{D}, \perp) \in \mathbf{OrthCat}$ be an orthogonal category. Let further $\mathbf{C} \subseteq \mathbf{D}$ be a full subcategory with embedding functor denoted by $j : \mathbf{C} \to \mathbf{D}$. We may equip $\mathbf{C}$ with the pullback orthogonality relation $j^*(\perp)$ of Lemma 4.29. We call $\overline{\mathbf{C}} = (\mathbf{C}, j^*(\perp)) \in \mathbf{OrthCat}$ a *full orthogonal subcategory* of $\overline{\mathbf{D}}$ and note that $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ is an orthogonal functor. By Corollary 4.24 (ii) and Theorem 3.40 we obtain an adjunction

$$\mathcal{O}_{j!} \,:\, \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{C}}}\big) \rightleftarrows \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{D}}}\big) \,:\, \mathcal{O}_j^* \quad . \tag{5.10}$$

Recall from Example 4.34 that in this case the left adjoints should be interpreted as extension functors from quantum field theories defined on $\overline{\mathbf{C}}$ to theories on $\overline{\mathbf{D}}$. This particular class of adjunctions obtained from full orthogonal subcategory embeddings enjoys the following properties.

Proposition 5.6. *The unit $\eta : \mathrm{id}_{\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})} \to \mathcal{O}_j^*\, \mathcal{O}_{j!}$ of the adjunction* (5.10) *is a natural isomorphism.*

Proof. Given any $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$, we use Proposition 3.41 to present $\mathcal{O}_{j!}(A) \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$ as the reflexive coequalizer

$$\mathcal{O}_{j!}(A) \;=\; \mathrm{colim}\Big(\mathcal{O}_{\overline{\mathbf{D}}} \circ j_!\big(\mathcal{O}_{\overline{\mathbf{C}}} \circ A\big) \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \mathcal{O}_{\overline{\mathbf{D}}} \circ j_!(A) \Big) \quad . \tag{5.11}$$

Applying the right adjoint functor $\mathcal{O}_j^*$ and recalling that it preserves reflexive coequalizers (cf. Lemma 3.43), we obtain a natural isomorphism

$$\mathcal{O}_j^*\, \mathcal{O}_{j!}(A) \;\cong\; \mathrm{colim}\Big(\mathcal{O}_j^*\big(\mathcal{O}_{\overline{\mathbf{D}}} \circ j_!\big(\mathcal{O}_{\overline{\mathbf{C}}} \circ A\big)\big) \overset{\mathcal{O}_j^*(\partial_0)}{\underset{\mathcal{O}_j^*(\partial_1)}{\rightrightarrows}} \mathcal{O}_j^*\big(\mathcal{O}_{\overline{\mathbf{D}}} \circ j_!(A)\big) \Big) \quad . \tag{5.12}$$

The fact that $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ is a full orthogonal subcategory embedding implies $j^*(\mathcal{O}_{\overline{\mathbf{D}}}) = \mathcal{O}_{\overline{\mathbf{C}}}$ (use Corollary 3.24) and

$$j_!(X)_s \;\cong\; \begin{cases} X_s & , \quad \text{if } s \in \mathbf{C}_0 \subseteq \mathbf{D}_0 \quad , \\ \emptyset & , \quad \text{else} \quad , \end{cases} \tag{5.13}$$

for all $s \in \mathbf{D}_0$ and $X \in \mathbf{M}^{\mathbf{C}_0}$ (use Proposition 3.39). By a straightforward calculation using Corollary 3.36 we then observe that the functor $\mathcal{O}_j^* \, (\mathcal{O}_{\overline{\mathbf{D}}} \circ (-)) \, j_! : \mathbf{M}^{\mathbf{C}_0} \to \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ is naturally isomorphic to the free $\mathcal{O}_{\overline{\mathbf{C}}}$-algebra functor $\mathcal{O}_{\overline{\mathbf{C}}} \circ (-)$. Applying this natural isomorphism to the right-hand side of (5.12), we obtain a natural isomorphism

$$\mathcal{O}_j^* \, \mathcal{O}_{j!}(A) \;\cong\; \mathrm{colim}\Big(\mathcal{O}_{\overline{\mathbf{C}}} \circ \big(\mathcal{O}_{\overline{\mathbf{C}}} \circ A\big) \overset{\mathcal{O}_{\overline{\mathbf{C}}}\circ\alpha}{\underset{\gamma_A}{\rightrightarrows}} \mathcal{O}_{\overline{\mathbf{C}}} \circ A \Big) \quad . \tag{5.14}$$

Applying now Lemma 2.21 to the right-hand side shows that the functor $\mathcal{O}_j^* \, \mathcal{O}_{j!} : \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}}) \to \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ is naturally isomorphic to the identity functor. This is sufficient to conclude that the unit of the adjunction is a natural isomorphism, see e.g. [JM89, Lemma 1.3]. □

Inspired by our quantum field theoretical framework explained in Example 4.34, we introduce the following concept.

Definition 5.7. We say that an object $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$ is *j-local* if the corresponding component $\epsilon : \mathcal{O}_{j!} \, \mathcal{O}_j^*(A) \to A$ of the counit of the adjunction (5.10) is an isomorphism in $\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$. We denote the full subcategory of j-local objects by $\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})^{j\text{-loc}}$.

Corollary 5.8. *(i) For every $B \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$, the object $\mathcal{O}_{j!}(B) \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$ is j-local.*

(ii) The adjunction (5.10) *restricts to an adjoint equivalence*

$$\mathcal{O}_{j!} \,:\, \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{C}}}\big) \;\underset{\sim}{\rightleftarrows}\; \mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{D}}}\big)^{j\text{-loc}} \,:\, \mathcal{O}_j^* \quad . \tag{5.15}$$

Proof. This is a direct consequence of Proposition 5.6. Let us also prove these claims more explicitly. For any $B \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$, we have to show that $\epsilon : \mathcal{O}_{j!} \, \mathcal{O}_j^* \, \mathcal{O}_{j!}(B) \to \mathcal{O}_{j!}(B)$ is an isomorphism in $\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$. Pre-composing ϵ with $\mathcal{O}_{j!}(\eta) : \mathcal{O}_{j!}(B) \to \mathcal{O}_{j!} \, \mathcal{O}_j^* \, \mathcal{O}_{j!}(B)$ yields the identity, i.e. $\epsilon \, \mathcal{O}_{j!}(\eta) = \mathrm{id}_{\mathcal{O}_{j!}(B)}$, because η and ϵ are the unit and the counit of an adjunction. Since η is an isomorphism by Proposition 5.6, it follows that $\epsilon : \mathcal{O}_{j!} \, \mathcal{O}_j^* \, \mathcal{O}_{j!}(B) \to \mathcal{O}_{j!}(B)$ is an isomorphism too. This proves item (i) and implies that (5.15) is a well-defined adjunction (the image of the left adjoint lies in $\mathbf{Alg}\big(\mathcal{O}_{\overline{\mathbf{D}}}\big)^{j\text{-loc}}$). The unit of this adjunction is a natural isomorphism by Proposition 5.6 and the counit is a natural isomorphism by Definition 5.7. Hence, (5.15) is an adjoint equivalence. □

Remark 5.9. In the context of quantum field theory, the full orthogonal subcategory $\overline{\mathbf{C}} \subseteq \overline{\mathbf{D}}$ should be interpreted as a subcategory of particularly "nice" spacetimes, e.g. discs in Example 4.34 or diamonds in Lorentzian quantum field theory. A j-local object $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$ is then a quantum field theory on the bigger spacetime category $\overline{\mathbf{D}}$ which is already determined by its restriction $\mathcal{O}_j^*(A) \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ to the subcategory of "nice" spacetimes $\overline{\mathbf{C}}$. In this sense, j-local objects should be interpreted as quantum field theories that satisfy a local-to-global property. Corollary 5.8 states that the category of such quantum field theories that satisfy the local-to-global property is equivalent to the category $\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ of quantum field theories which are only defined on the category $\overline{\mathbf{C}}$ of "nice" spacetimes. △

5.4 Orthogonal equivalences

In this subsection we introduce a suitable notion of equivalence $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ between orthogonal categories. We then show that the adjunction (5.1) induced by an orthogonal equivalence F is an adjoint equivalence between the associated categories of algebras. In the terminology of [KM01], this means that the $\mathbf{Op}(\mathbf{M})$-morphism $\mathcal{O}_F : \mathcal{O}_{\overline{\mathbf{C}}} \to \mathcal{O}_{\overline{\mathbf{D}}}$ is a Morita equivalence.

Definition 5.10. An orthogonal functor $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ is called an *orthogonal equivalence* if the following two properties hold true: (1) $F : \mathbf{C} \to \mathbf{D}$ is an equivalence of small categories, i.e. a fully faithful and essentially surjective functor, and (2) $F^*(\perp_{\mathbf{D}}) = \perp_{\mathbf{C}}$.

Theorem 5.11. *Let $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ be an orthogonal equivalence. Then the induced adjunction* (5.1) *is an adjoint equivalence*

$$\mathcal{O}_{F!} \,:\, \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}}) \xrightarrow[\longleftarrow]{\;\sim\;} \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}}) \,:\, \mathcal{O}_F^* \quad . \tag{5.16}$$

Proof. Let us first consider the special case where the functor $F : \mathbf{C} \to \mathbf{D}$ is also injective on objects. Then the unit of the adjunction (5.1) is a natural isomorphism because of Proposition 5.6. We now show that the counit is a natural isomorphism too. Notice that there is the following chain of natural isomorphisms

$$\mathcal{O}_{F!}\,\mathcal{O}_F^* \;\cong\; p_{\overline{\mathbf{D}}!}\,\mathrm{Lan}_F\, p_{\overline{\mathbf{C}}}^*\,\mathcal{O}_F^* \;=\; p_{\overline{\mathbf{D}}!}\,\mathrm{Lan}_F\, F^*\, p_{\overline{\mathbf{D}}}^* \;\cong\; p_{\overline{\mathbf{D}}!}\,p_{\overline{\mathbf{D}}}^* \;\cong\; \mathrm{id}_{\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})} \quad . \tag{5.17}$$

In the first step we used Proposition 5.3 and in the second step we used that the square of right adjoints in (5.4) commutes. In step three we used that $F : \mathbf{C} \to \mathbf{D}$ is an equivalence of small categories, which implies that the pullback functor F^* is fully faithful and hence that the counit $\mathrm{Lan}_F\, F^* \to \mathrm{id}_{\mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}}$ of the adjunction (5.2) is a natural isomorphism. The last step follows from Lemma 5.1. The dual of [JM89, Lemma 1.3] allows us to conclude from (5.17) that the counit ϵ is a natural isomorphism.

The generic case can be reduced to the special case above by the following argument: Let us choose a skeleton $\mathbf{C}' \subseteq \mathbf{C}$, with embedding functor denoted by $j : \mathbf{C}' \to \mathbf{C}$, and define $\overline{\mathbf{C}}' := (\mathbf{C}', j^*(\perp_{\mathbf{C}}))$. Note that $\overline{\mathbf{C}}'$ is a full orthogonal subcategory of $\overline{\mathbf{C}}$. One easily confirms that both $j : \overline{\mathbf{C}}' \to \overline{\mathbf{C}}$ and $F\,j : \overline{\mathbf{C}}' \to \overline{\mathbf{D}}$ are orthogonal equivalences that are injective on objects. Our results above then imply that both $F\,j$ and j induce adjoint equivalences between the associated categories of algebras. To complete the proof, we notice that the 2-out-of-3 property of equivalences of categories and Lemma 3.42 implies that also F induces an adjoint equivalence. $\square$

Remark 5.12. The practical relevance of this result is the following: Recall from the examples in Section 4.6 that one is often interested in studying quantum field theories which are defined on an orthogonal category that is only *essentially* small. To avoid set theoretic issues, one has to replace such orthogonal categories by equivalent small orthogonal categories, whose choice is typically not unique. Different choices in general define non-isomorphic colored operads which, however, are Morita-equivalent because of Theorem 5.11, i.e. the associated categories of algebras are naturally equivalent. The physical implication is that the category of quantum field theories does not depend on the choice of a small model for the orthogonal category of interest. $\triangle$

5.5 Right adjoints and orbifoldization

Given an orthogonal functor $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$, our focus so far was on the induced adjunction (5.1) where the pullback $\mathcal{O}_F^*$ is a right adjoint functor and $\mathcal{O}_{F!}$ is its left adjoint. Forgetting for the moment the orthogonality relations on our categories, this reduces to the adjunction (5.2) obtained by left Kan extension. Because the underlying base category $\mathbf{M}$ is by hypothesis also complete, there exists another adjunction (obtained by right Kan extension)

$$F^* \,:\, \mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}} \xrightarrow[\longleftarrow]{} \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}} \,:\, \mathrm{Ran}_F \quad , \tag{5.18}$$

where F^* is the left adjoint. It is natural to ask whether also the functor $\mathcal{O}_F^* : \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}}) \to \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ admits a right adjoint. In general, this is not the case due to the following

Example 5.13. Consider the category $\{*\}$ consisting of one object $*$ and its identity morphism id_*. On this category there exist two different orthogonality relations $\perp_{\mathrm{min}} = \emptyset$ and $\perp_{\mathrm{max}} = \{(\mathrm{id}_*, \mathrm{id}_*)\}$. By Theorem 4.27, we obtain isomorphisms

$$\mathbf{Alg}(\mathcal{O}_{(\{*\},\perp_{\mathrm{min}})}) \;\cong\; \mathbf{Mon}_{\mathbf{M}} \quad , \qquad \mathbf{Alg}(\mathcal{O}_{(\{*\},\perp_{\mathrm{max}})}) \;\cong\; \mathbf{CMon}_{\mathbf{M}} \quad , \tag{5.19}$$

where $\mathbf{CMon_M}$ is the category of commutative monoids in $\mathbf{M}$. The identity functor defines an orthogonal functor $(\{*\},\perp_{\min}) \to (\{*\},\perp_{\max})$ which (under the isomorphisms above) induces the adjunction

$$\mathrm{Ab} \,:\, \mathbf{Mon_M} \rightleftarrows \mathbf{CMon_M} \,:\, U \quad . \tag{5.20}$$

The right adjoint U is the functor forgetting commutativity and the left adjoint Ab is the abelianization of monoids in $\mathbf{M}$. Since U fails to preserve coproducts, it can not be a left adjoint functor. This implies that $\mathcal{O}_F^* : \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}}) \to \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ in general does not admit a right adjoint. $\triangledown$

We now consider a special situation where it turns out that the functor $\mathcal{O}_F^*$ does admit a right adjoint. The motivation for this scenario comes from *orbifoldization*, which is the procedure of assigning to quantum field theories with group (or groupoid) actions their corresponding invariants [DVVV89]. Such constructions were studied by [BS17] in the context of locally covariant quantum field theory and by [SW17] in the context of topological quantum field theory. It is important to stress that the procedure of taking invariants is formalized by categorical limits and hence is related to right adjoints of the functor $\mathcal{O}_F^*$.

Our scenario is as follows: Let $\overline{\mathbf{D}} = (\mathbf{D},\perp)$ be an orthogonal category and $F : \mathbf{C} \to \mathbf{D}$ a category fibered in groupoids over $\mathbf{D}$, see e.g. [BS17] for a definition. We equip $\mathbf{C}$ with the pullback orthogonality relation $F^*(\perp)$ and call the resulting orthogonal functor $F : \overline{\mathbf{C}} := (\mathbf{C}, F^*(\perp)) \to \overline{\mathbf{D}}$ an *orthogonal category fibered in groupoids.*

Proposition 5.14. *Let $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ be an orthogonal category fibered in groupoids. Then the pullback functor $\mathcal{O}_F^* : \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}}) \to \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ has a right adjoint, i.e. there is an adjunction*

$$\mathcal{O}_F^* \,:\, \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}}) \rightleftarrows \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}}) \,:\, \mathcal{O}_{F*} \quad . \tag{5.21}$$

We call the right adjoint $\mathcal{O}_{F*}$ the orbifoldization functor.

Proof. In [BS17, Theorem 4.3] it was shown that under our hypotheses the right Kan extension $\mathrm{Ran}_F : \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}} \to \mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}$ preserves $\perp$-commutativity. Using Lemma 5.1, this implies the existence of a unique functor $\mathcal{O}_{F*} : \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}}) \to \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$ such that $\mathrm{Ran}_F\, p_{\overline{\mathbf{C}}}^* = p_{\overline{\mathbf{D}}}^* \mathcal{O}_{F*}$. It remains to show that $\mathcal{O}_{F*}$ is the right adjoint of $\mathcal{O}_F^*$. Given any $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ and $B \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$, there is the following chain of natural bijections of Hom-sets

$$\begin{aligned}
\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})\big(B, \mathcal{O}_{F*}(A)\big) &\cong \mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}\big(p_{\overline{\mathbf{D}}}^*(B), p_{\overline{\mathbf{D}}}^*\mathcal{O}_{F*}(A)\big)\\
&\cong \mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}\big(p_{\overline{\mathbf{D}}}^*(B), \mathrm{Ran}_F\, p_{\overline{\mathbf{C}}}^*(A)\big)\\
&\cong \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}\big(F^*\, p_{\overline{\mathbf{D}}}^*(B), p_{\overline{\mathbf{C}}}^*(A)\big)\\
&\cong \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}\big(p_{\overline{\mathbf{C}}}^*\,\mathcal{O}_F^*(B), p_{\overline{\mathbf{C}}}^*(A)\big)\\
&\cong \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})\big(\mathcal{O}_F^*(B), A\big) \quad .
\end{aligned} \tag{5.22}$$

In the first and last step we used Lemma 5.1 and in step four we used that the square formed by the right adjoints in (5.4) commutes. This proves that $\mathcal{O}_{F*}$ is the right adjoint of $\mathcal{O}_F^*$. $\square$

6 Comparison to Fredenhagen's universal algebra

In [Fre90, Fre93, FRS92], Fredenhagen studied extensions of quantum field theories that are defined only on certain open subsets $U \subseteq M$ of a spacetime manifold M to the whole of M. It was later recognized by Lang in his PhD thesis [Lan14] that this extension may be formalized as a left Kan extension of the functor underlying the quantum field theory. In our notation and language, Fredenhagen's universal algebra construction can be formalized as follows: Let $\overline{\mathbf{D}} = (\mathbf{D},\perp) \in \mathbf{OrthCat}$ be an orthogonal category. Let $\overline{\mathbf{C}} = (\mathbf{C}, j^*(\perp))$ be a full orthogonal subcategory (cf. Section 5.3) with orthogonal embedding functor denoted by $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$. The universal algebra construction [Fre90, Fre93, FRS92, Lan14] assigns to a quantum field theory $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ on $\overline{\mathbf{C}}$ the monoid-valued functor

$$\mathrm{Lan}_j \; p_{\overline{\mathbf{C}}}^*(A) \in \mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}} \tag{6.1}$$

on the category $\mathbf{D}$. Notice that the construction (6.1) consists of two steps: First, one applies the functor $p_{\overline{\mathbf{C}}}^* : \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}}) \to \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}$ that forgets $\perp$-commutativity, assigning to the quantum field theory $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ its underlying monoid-valued functor $p_{\overline{\mathbf{C}}}^*(A) \in \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}$ on $\mathbf{C}$. In the second step this underlying functor is extended from $\mathbf{C}$ to $\mathbf{D}$ via left Kan extension along the embedding functor $j : \mathbf{C} \to \mathbf{D}$.

A potential weakness of this construction is that it is unclear whether the extended functor (6.1) satisfies the $\perp$-commutativity axiom on $\overline{\mathbf{D}}$, i.e. whether it is a bona fide quantum field theory. This weakness is solved by our operadic construction explained in Section 5.3. Concretely, instead of using (6.1) to extend the quantum field theory $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ from $\overline{\mathbf{C}}$ to $\overline{\mathbf{D}}$, we use the left adjoint in (5.10) to assign

$$\mathcal{O}_{j!}(A) \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}}) \quad . \tag{6.2}$$

By construction, our extended theory satisfies the $\perp$-commutativity axiom on $\overline{\mathbf{D}}$, i.e. it is a bona fide quantum field theory. The aim of this section is to compare our construction (6.2) to the construction (6.1) of Fredenhagen and Lang. Our first result is that whenever (6.1) satisfies the $\perp$-commutativity axiom on $\overline{\mathbf{D}}$, then it agrees with our construction (6.2).

Proposition 6.1. *Let $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ be such that $\mathrm{Lan}_j \; p_{\overline{\mathbf{C}}}^*(A) \in \mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}$ is $\perp$-commutative on $\overline{\mathbf{D}}$. Then there exists an isomorphism*

$$\mathrm{Lan}_j \; p_{\overline{\mathbf{C}}}^*(A) \;\cong\; p_{\overline{\mathbf{D}}}^* \mathcal{O}_{j!}(A) \tag{6.3}$$

in $\mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}$.

Proof. By Lemma 5.1, we know that $\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$ is a full reflective subcategory of $\mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}$. Because $\mathrm{Lan}_j \; p_{\overline{\mathbf{C}}}^*(A) \in \mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}$ satisfies by hypothesis the $\perp$-commutativity axiom, it then follows that there exists $\widehat{A} \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{D}}})$ such that $p_{\overline{\mathbf{D}}}^*(\widehat{A}) \cong \mathrm{Lan}_j \; p_{\overline{\mathbf{C}}}^*(A)$. Applying $p_{\overline{\mathbf{D}}!}$ we obtain

$$\widehat{A} \;\cong\; p_{\overline{\mathbf{D}}!} \, p_{\overline{\mathbf{D}}}^*(\widehat{A}) \;\cong\; p_{\overline{\mathbf{D}}!} \, \mathrm{Lan}_j \; p_{\overline{\mathbf{C}}}^*(A) \;\cong\; \mathcal{O}_{j!}(A) \quad , \tag{6.4}$$

where in the first step we used that the counit of the adjunction $p_{\overline{\mathbf{D}}!} \dashv p_{\overline{\mathbf{D}}}^*$ is a natural isomorphism (cf. Lemma 5.1) and in the last step we used Proposition 5.3. $\square$

It thus remains to understand whether (6.1) does satisfy the $\perp$-commutativity axiom. Our strategy to address this question is to compute explicitly the functor (6.1) by using the operadic techniques from Section 3.3.3. To simplify the presentation, we assume that the underlying base category $\mathbf{M}$ is concrete and that the monoidal unit $I \ncong \emptyset$ is not isomorphic to the initial object. For example, we could take the typical example $\mathbf{M} = \mathbf{Vec}_{\mathbb{K}}$ (cf. Example 2.2). This allows us to think of the objects in $\mathbf{M}$ as sets with additional structures and of the morphisms as structure preserving functions. In particular, we can make element-wise computations. (Using the concept of generalized elements, there is no need to assume that $\mathbf{M}$ is concrete. However, we decided to add this reasonable assumption to avoid introducing generalized elements.)

The problem of computing left Kan extensions can be addressed within our operadic formalism. Recalling from Theorem 4.26 that $\mathbf{Mon}_{\mathbf{M}}^{\mathbf{E}} \cong \mathbf{Alg}(\mathcal{O}_{\mathbf{E}})$ for any small category $\mathbf{E}$, we may describe monoid-valued functors in terms of algebras over our auxiliary operad $\mathcal{O}_{\mathbf{E}}$, see Definition 4.5

and Proposition 4.7. Under these isomorphisms, the left Kan extension $\mathrm{Lan}_j : \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}} \to \mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}$ is identified with the left adjoint of the adjunction

$$\mathcal{O}_{j\,!} \,:\, \mathbf{Alg}\big(\mathcal{O}_{\mathbf{C}}\big) \;\xrightleftharpoons{}\; \mathbf{Alg}\big(\mathcal{O}_{\mathbf{D}}\big) \,:\, \mathcal{O}_j^* \quad , \tag{6.5}$$

which is induced by applying Theorem 3.40 to the $\mathbf{Op}(\mathbf{M})$-morphism $\mathcal{O}_j : \mathcal{O}_{\mathbf{C}} \to \mathcal{O}_{\mathbf{D}}$ between our auxiliary operads (cf. Proposition 4.14). Using Proposition 3.41, we obtain that the left Kan extension of a monoid-valued functor $B \in \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}$ can be computed by the reflexive coequalizer

$$\mathrm{Lan}_j(B) \;=\; \mathrm{colim}\Big(\, \mathcal{O}_{\mathbf{D}} \circ j_!\big(\mathcal{O}_{\mathbf{C}} \circ B\big) \;\overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}}\; \mathcal{O}_{\mathbf{D}} \circ j_!(B)\, \Big) \tag{6.6}$$

in $\mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}} \cong \mathbf{Alg}(\mathcal{O}_{\mathbf{D}})$. Using also Proposition 3.37, we obtain that the monoid $\mathrm{Lan}_j(B)_d \in \mathbf{Mon}_{\mathbf{M}}$ associated by the functor to $d \in \mathbf{D}$ can be computed by the point-wise reflexive coequalizer

$$\mathrm{Lan}_j(B)_d \;=\; \mathrm{colim}\Big(\, \big(\mathcal{O}_{\mathbf{D}} \circ j_!\big(\mathcal{O}_{\mathbf{C}} \circ B\big)\big)_d \;\overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}}\; \big(\mathcal{O}_{\mathbf{D}} \circ j_!(B)\big)_d\, \Big) \tag{6.7}$$

in the category $\mathbf{M}$. Explicitly, using (3.51) and that $j : \mathbf{C} \to \mathbf{D}$ is an inclusion on the sets of objects, we obtain

$$\begin{aligned} \big(\mathcal{O}_{\mathbf{D}} \circ j_!(B)\big)_d \;&=\; \int^{\underline{d}} \mathcal{O}_{\mathbf{D}}\big(\begin{smallmatrix} d \\ \underline{d} \end{smallmatrix}\big) \otimes j_!(B)_{d_1} \otimes \cdots \otimes j_!(B)_{d_n} \\ &\cong\; \int^{\underline{c}} \mathcal{O}_{\mathbf{D}}\big(\begin{smallmatrix} d \\ \underline{c} \end{smallmatrix}\big) \otimes B_{c_1} \otimes \cdots \otimes B_{c_n} \;=:\; \int^{\underline{c}} \mathcal{O}_{\mathbf{D}}\big(\begin{smallmatrix} d \\ \underline{c} \end{smallmatrix}\big) \otimes B_{\underline{c}} \quad , \end{aligned} \tag{6.8}$$

where we denote the objects of the subcategory $\mathbf{C} \subseteq \mathbf{D}$ by c's and generic objects of $\mathbf{D}$ by d's. Using Definition 4.5, we find that the elements of $\big(\mathcal{O}_{\mathbf{D}} \circ j_!(B)\big)_d$ are of the form

$$(\sigma, \underline{g}) \otimes \underline{b} \;:=\; (\sigma, \underline{g}) \otimes b_1 \otimes \cdots \otimes b_n \in \mathcal{O}_{\mathbf{D}}\big(\begin{smallmatrix} d \\ \underline{c} \end{smallmatrix}\big) \otimes B_{\underline{c}} \quad , \tag{6.9a}$$

modulo the equivalence relation (coming from the coend)

$$(\sigma\sigma', \underline{g}\sigma') \otimes \underline{b}\sigma' \;\sim\; (\sigma, \underline{g}) \otimes \underline{b} \quad , \tag{6.9b}$$

for all $\sigma' \in \Sigma_n$. Notice that every equivalence class has a *unique* representative whose permutation part $\sigma = e$ is the identity permutation. Hence, there is an $\mathbf{M}$-isomorphism

$$\big(\mathcal{O}_{\mathbf{D}} \circ j_!(B)\big)_d \;\cong\; \coprod_{\underline{c} \in \Sigma_{\mathbf{C}_0}} \mathbf{D}(\underline{c}, d) \otimes B_{\underline{c}} \tag{6.10}$$

and we may denote elements simply by

$$\underline{g} \otimes \underline{b} \;:=\; \big[(e, \underline{g}) \otimes \underline{b}\big] \in \big(\mathcal{O}_{\mathbf{D}} \circ j_!(B)\big)_d \quad . \tag{6.11}$$

The monoid structure on $\big(\mathcal{O}_{\mathbf{D}} \circ j_!(B)\big)_d$ is given by

$$\mu_d\big((\underline{g} \otimes \underline{b}) \otimes (\underline{g}' \otimes \underline{b}')\big) \;=\; (\underline{g}, \underline{g}') \otimes \underline{b} \otimes \underline{b}' \quad , \qquad 1_d \;=\; * \in \mathbf{D}(\emptyset, d) \quad , \tag{6.12}$$

where $(\underline{g}, \underline{g}') := (g_1, \dots, g_n, g_1', \dots, g_m')$ is the concatenation of $\underline{g}$ and $\underline{g}'$. The other object $\big(\mathcal{O}_{\mathbf{D}} \circ j_!\big(\mathcal{O}_{\mathbf{C}} \circ B\big)\big)_d$ and the morphisms ∂_0, ∂_1 in (6.7) can be computed similarly. The result is

Lemma 6.2. *The functor* $\mathrm{Lan}_j(B) : \mathbf{D} \to \mathbf{Mon}_{\mathbf{M}}$ *has the following explicit description: To an object* $d \in \mathbf{D}$*, it assigns the monoid*

$$\mathrm{Lan}_j(B)_d \;=\; \big(\mathcal{O}_{\mathbf{D}} \circ j_!(B)\big)_d / \sim \tag{6.13a}$$

given by implementing the equivalence relation

$$\underline{g}(\underline{f}_1,\dots,\underline{f}_n)\otimes\underline{b}_1\otimes\cdots\otimes\underline{b}_n ~\sim~ \underline{g}\otimes B(\underline{f}_1)(\underline{b}_1)\otimes\cdots\otimes B(\underline{f}_n)(\underline{b}_n) \quad , \tag{6.13b}$$

for all $\underline{g}\in\mathbf{D}(\underline{c},d)$ *with* $|\underline{c}|=n\geq 1$, $\underline{f}_i\in\mathbf{C}(\underline{c}_i,c_i)$, *for* $i=1,\dots,n$, *and* $\underline{b}_i\in B_{\underline{c}_i}$, *for* $i=1,\dots,n$. *Here* $\underline{g}(\underline{f}_1,\dots,\underline{f}_n)$ *is defined in* (4.5) *and the* n*-fold product* $B(\underline{f})(\underline{b}):=B(f_1)(b_1)\,\cdots\,B(f_n)(b_n)$ *is defined by the monoid-valued functor* $B:\mathbf{C}\to\mathbf{Mon}_{\mathbf{M}}$*. The monoid structure in* (6.12) *descends to* (6.13)*. To a* $\mathbf{D}$*-morphism* $h:d\to d'$*, the functor* $\operatorname{Lan}_j(B):\mathbf{D}\to\mathbf{Mon}_{\mathbf{M}}$ *assigns the* $\mathbf{Mon}_{\mathbf{M}}$*-morphism*

$$\operatorname{Lan}_j(B)(h)\,:\,\operatorname{Lan}_j(B)_d~\longrightarrow~\operatorname{Lan}_j(B)_{d'}\quad,\qquad \big[\underline{g}\otimes\underline{b}\big]~\longmapsto~\big[h(\underline{g})\otimes\underline{b}\big]\quad . \tag{6.14}$$

We can now answer the question when (6.1) satisfies the $\perp$-commutativity axiom. For this it will be useful to introduce the following terminology.

Definition 6.3. We say that an object $d\in\mathbf{D}$ is *j-closed* if for every pair of orthogonal morphisms $(g_1:c_1\to d)\perp(g_2:c_2\to d)$ with target d and sources $c_1,c_2\in\mathbf{C}$ there exists a commutative diagram

$$\begin{tikzcd}
 & d & \\
 & c \arrow[u, dashed, "g"] & \\
c_1 \arrow[ruu, bend left, "g_1"] \arrow[ru, dashed, "f_1"'] & & c_2 \arrow[luu, bend right, "g_2"'] \arrow[lu, dashed, "f_2"]
\end{tikzcd} \tag{6.15}$$

with $c\in\mathbf{C}$ and $(f_1,f_2)\in j^\ast(\perp)$.

Theorem 6.4. $\operatorname{Lan}_j\,p^\ast_{\overline{\mathbf{C}}}(A)\in\mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}$ *is* $\perp$*-commutative over* $d\in\mathbf{D}$*, for all* $A\in\mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$*, if and only if* d *is* j*-closed.*

Proof. Let us first prove the direction "$\Leftarrow$": We have to show that $\operatorname{Lan}_j\,p^\ast_{\overline{\mathbf{C}}}(A)\in\mathbf{Mon}_{\mathbf{M}}^{\mathbf{D}}$ is $\perp$-commutative over d, i.e. that, for all $(h_1:d_1\to d)\perp(h_2:d_2\to d)$ with target d,

$$\begin{aligned}\mu_d\Big(&\operatorname{Lan}_j\,p^\ast_{\overline{\mathbf{C}}}(A)(h_1)\big(\tilde{a}_1\big)\otimes\operatorname{Lan}_j\,p^\ast_{\overline{\mathbf{C}}}(A)(h_2)\big(\tilde{a}_2\big)\Big)\\ &=~\mu_d^{\mathrm{op}}\Big(\operatorname{Lan}_j\,p^\ast_{\overline{\mathbf{C}}}(A)(h_1)\big(\tilde{a}_1\big)\otimes\operatorname{Lan}_j\,p^\ast_{\overline{\mathbf{C}}}(A)(h_2)\big(\tilde{a}_2\big)\Big)\quad ,\end{aligned} \tag{6.16}$$

for all $\tilde{a}_i\in\operatorname{Lan}_j\,p^\ast_{\overline{\mathbf{C}}}(A)_{d_i}$, $i=1,2$. Using Lemma 6.2, this condition explicitly reads as

$$\Big[\big(h_1(\underline{g}_1),h_2(\underline{g}_2)\big)\otimes\underline{a}_1\otimes\underline{a}_2\Big]~=~\Big[\big(h_2(\underline{g}_2),h_1(\underline{g}_1)\big)\otimes\tau(\underline{a}_1\otimes\underline{a}_2)\Big]\quad , \tag{6.17}$$

for all $\underline{g}_i\otimes\underline{a}_i\in\coprod_{\underline{c}\in\Sigma_{\mathbf{C}_0}}\mathbf{D}(\underline{c},d_i)\otimes A_{\underline{c}}$, $i=1,2$, where τ is the symmetric braiding on $\mathbf{M}$. It is sufficient to prove (6.17) for elements $g_i\otimes a_i\in\mathbf{D}(c_i,d_i)\otimes A_{c_i}$, $i=1,2$, of length 1; the general case follows from this by iteration. Because by assumption $h_1\perp h_2$, it follows from composition stability of $\perp$ that $(h_1\,g_1:c_1\to d)\perp(h_2\,g_2:c_2\to d)$. Using further that d is by hypothesis j-closed, we find $g:c\to d$ in $\mathbf{D}$ and $(f_1:c_1\to c)\perp(f_2:c_2\to c)$ in $\mathbf{C}$, such that $(h_1\,g_1,h_2\,g_2)=(g\,f_1,g\,f_2)$. Using the relations in (6.13), we obtain

$$\begin{aligned}\Big[\big(h_1\,g_1,h_2\,g_2\big)\otimes a_1\otimes a_2\Big]~&=~\Big[g(f_1,f_2)\otimes a_1\otimes a_2\Big]~=~\Big[g\otimes\mu_c\big(A(f_1)(a_1)\otimes A(f_2)(a_2)\big)\Big]\\ &=~\Big[g\otimes\mu_c^{\mathrm{op}}\big(A(f_1)(a_1)\otimes A(f_2)(a_2)\big)\Big]~=~\Big[g(f_2,f_1)\otimes\tau(a_1\otimes a_2)\Big]\\ &=~\Big[\big(h_2\,g_2,h_1\,g_1\big)\otimes\tau(a_1\otimes a_2)\Big]\quad ,\end{aligned} \tag{6.18}$$

where in the third step we used that A is $\perp$-commutative on $\overline{\mathbf{C}}$.

We now prove "$\Rightarrow$": For each orthogonal pair of morphisms $(g_1 : c_1 \to d) \perp (g_2 : c_2 \to d)$, our strategy is to construct an object $A \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ such that $\perp$-commutativity of $\mathrm{Lan}_j\, p_{\overline{\mathbf{C}}}^*(A)$ implies the existence of a factorization as in Definition 6.3. Given any $(g_1 : c_1 \to d) \perp (g_2 : c_2 \to d)$, we define the colored object $X \in \mathbf{M}^{\mathbf{C}_0}$ by setting

$$X_c \,:=\, \big(\Sigma_{\mathbf{C}_0}(c, c_1) \otimes I\big) \sqcup \big(\Sigma_{\mathbf{C}_0}(c, c_2) \otimes I\big) \quad , \tag{6.19}$$

for all $c \in \mathbf{C}_0$. Because we assume that $I \ncong \emptyset$, there exists an element $x_1 \in X_{c_1}$ and an element $x_2 \in X_{c_2}$, such that x_1 and x_2 are different in the case of $c_1 = c_2$. Let us consider the free $\mathcal{O}_{\overline{\mathbf{C}}}$-algebra $A := \mathcal{O}_{\overline{\mathbf{C}}} \circ X \in \mathbf{Alg}(\mathcal{O}_{\overline{\mathbf{C}}})$ and note that $x_1 \in A_{c_1}$ and $x_2 \in A_{c_2}$ are generators. Because $\mathrm{Lan}_j\, p_{\overline{\mathbf{C}}}^*(A)$ is by hypothesis $\perp$-commutative over $d \in \mathbf{D}$, it follows that

$$\Big[(g_1, g_2) \otimes x_1 \otimes x_2\Big] \;=\; \Big[(g_2, g_1) \otimes \tau(x_1 \otimes x_2)\Big] \quad , \tag{6.20}$$

where $(g_1 : c_1 \to d) \perp (g_2 : c_2 \to d)$ is the given orthogonal pair of morphisms. Using the equivalence relation (6.13) and that x_1, x_2 are two distinct generators of the free $\mathcal{O}_{\overline{\mathbf{C}}}$-algebra A, one observes that this equality of equivalence classes can only hold true if both sides admit a representative of length 1, i.e. with only one tensor factor in A. Hence, there must exist a factorization $(g_1, g_2) = (g\, f_1, g\, f_2)$ with $f_i : c_i \to c$ in $\mathbf{C}$, for $i = 1, 2$, and $g : c \to d$ in $\mathbf{D}$. This yields the equality

$$\Big[g \otimes \mu_c\big(A(f_1)(x_1) \otimes A(f_2)(x_2)\big)\Big] \;=\; \Big[g \otimes \mu_c^{\mathrm{op}}\big(A(f_1)(x_1) \otimes A(f_2)(x_2)\big)\Big] \quad , \tag{6.21}$$

which ensures the existence of a further factorization $g = g'\, f'$ with $f' : c \to c'$ in $\mathbf{C}$ and $g' : c' \to d$ in $\mathbf{D}$ such that $\big(f'\, f_1, f'\, f_2\big) \in j^*(\perp)$. $\square$

Let us denote by $\mathbf{D}' \subseteq \mathbf{D}$ the full subcategory of j-closed objects. Notice that every object $c \in \mathbf{C}$ is j-closed, hence $\mathbf{C} \subseteq \mathbf{D}'$. Equipping $\mathbf{D}'$ with the pullback orthogonality relation, we obtain a factorization of $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ into two full orthogonal subcategory embeddings $j' : \overline{\mathbf{C}} \to \overline{\mathbf{D}}'$ and $j'' : \overline{\mathbf{D}}' \to \overline{\mathbf{D}}$. Combining Theorem 6.4 and Proposition 6.1, we obtain

Corollary 6.5. *Let $\overline{\mathbf{D}}'$ be the full orthogonal subcategory of j-closed objects in $\overline{\mathbf{D}}$ and consider the full orthogonal subcategory embedding $j' : \overline{\mathbf{C}} \to \overline{\mathbf{D}}'$. Then there exists a natural isomorphism*

$$\mathrm{Lan}_{j'}\, p_{\overline{\mathbf{C}}}^* \;\cong\; p_{\overline{\mathbf{D}}'}^*\, \mathcal{O}_{j'!} \quad . \tag{6.22}$$

We conclude this section by providing examples and counterexamples of j-closed objects in the context of the examples discussed in Section 4.6.

Example 6.6. Consider the full orthogonal subcategory embedding $j : \overline{\mathbf{Disc}} \to \overline{\mathbf{Man}}$ described in Example 4.34. We first notice that every disconnected manifold $M \in \mathbf{Man}$ is *not* j-closed: Two embeddings $g_1 : U_1 \to M$ and $g_2 : U_2 \to M$ of discs into different connected components of M are clearly orthogonal, $g_1 \perp g_2$, however they do not factorize through a common disc $g : U \to M$. Hence, Fredenhagen's universal algebra (6.1) in general fails to produce functors that are $\perp$-commutative over disconnected manifolds and our construction (6.2) solves this issue. On the other hand, the sphere $\mathbb{S}^m \in \mathbf{Man}$ is j-closed: Given two disjoint embeddings $g_1 : U_1 \to \mathbb{S}^m$ and $g_2 : U_2 \to \mathbb{S}^m$ of discs, $g_1 \perp g_2$, they factorize through the disc inclusion $g : U \to \mathbb{S}^m$, where $U = \mathbb{S}^m \setminus \{x\} \cong \mathbb{R}^m$ for some $x \in \mathbb{S}^m \setminus (g_1(U_1) \cup g_2(U_2))$. Hence, Fredenhagen's universal algebra (6.1) produces functors that are $\perp$-commutative over spheres and, when evaluated there, it coincides with our construction (6.2). This case includes Fredenhagen's original applications to chiral conformal quantum field theories on the circle [Fre90, Fre93, FRS92]. A complete characterization of the j-closed objects in $\overline{\mathbf{Man}}$ seems to be rather complicated and is beyond the scope of this article. $\triangledown$

Example 6.7. In the context of Example 4.31, consider the full orthogonal subcategory $j : \overline{\mathbf{Loc}_{©}} \to \overline{\mathbf{Loc}}$ characterized by all globally hyperbolic Lorentzian spacetimes $\mathbb{U}$ whose underlying manifold is diffeomorphic to $\mathbb{R}^m$. We call such objects *diamonds*. This scenario has been studied in [Lan14]. Similarly to the example above, every disconnected spacetime $\mathbb{M} \in \mathbf{Loc}$ is *not* j-closed. Hence, Fredenhagen's universal algebra (6.1) in general fails to produce functors that are $\perp$-commutative over disconnected spacetimes and our construction (6.2) solves this issue. On the other hand, objects $\mathbb{M} \in \mathbf{Loc}$ whose underlying manifold is diffeomorphic to $\mathbb{R}\times\mathbb{S}^{m-1}$ are j-closed: Given two causally disjoint embeddings $g_1 : \mathbb{U}_1 \to \mathbb{M}$ and $g_2 : \mathbb{U}_2 \to \mathbb{M}$ of diamonds, $g_1 \perp g_2$, they factorize through the diamond inclusion $g : \mathbb{U} \to \mathbb{M}$, where $\mathbb{U}$ is defined by restricting $\mathbb{M}$ to the globally hyperbolic open subset $U = M \setminus J_{\mathbb{M}}(\{x\}) \cong \mathbb{R}^m$ for some $x \in M \setminus J_{\mathbb{M}}(g_1(U_1) \cup g_2(U_2))$. As in Example 6.6, a complete characterization of the j-closed objects in $\overline{\mathbf{Loc}}$ seems to be rather complicated and is beyond the scope of this article. $\triangledown$

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