ON THE CLASSIFICATION OF NONCOMPACT STEADY QUASI-EINSTEIN MANIFOLD WITH VANISHING CONDITION ON THE WEYL TENSOR

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ABSTRACT. The aim of this paper is to study complete (noncompact) steady m-quasi-Einstein manifolds satisfying a fourth-order vanishing condition on the Weyl tensor. In this case, we are able to prove that a steady m-quasi-Einstein manifold ($m > 1$) on a simply connected n-dimensional manifold (M^n, g), ($n \geq 4$), with nonnegative Ricci curvature and zero radial Weyl curvature must be a warped product with $(n-1)-$dimensional Einstein fiber, provided that M has fourth order divergence-free Weyl tensor (i.e., $\mathrm{div}^4 W = 0$).

1. INTRODUCTION

Following the terminology used in [12, 24] we recall the definition of quasi-Einstein manifold. A complete Riemannian manifold (M^n, g), $n \geq 2$, will be called m-quasi-Einstein manifold or simply quasi-Einstein, if there exist a smooth potential function f on M, a constant m with $0 < m \leq +\infty$ and a constant λ satisfying

$$Ric + Hessf - \frac{1}{m} df \otimes df = \lambda g, \tag{1.1}$$

where $Hessf$ stands for a Hessian of f. We shall refer to this equation as the fundamental equation of a quasi-Einstein manifold (M^n, g, f, λ). We also recall the well known m-Bakry-Emery Ricci tensor that is given by

$$Ric_f^m = Ric + Hessf - \frac{1}{m} df \otimes df,$$

where f is a smooth function on M^n, for more details about this tensor see for instance [2, 23, 27]. Therefore, the fundamental equation of the quasi-Einstein (1.1) can be rewritten as

$$Ric_f^m = \lambda g. \tag{1.2}$$

A quasi-Einstein manifold is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$.

Date: July 28, 2017.
2010 *Mathematics Subject Classification.* Primary 53C25, 53C20, 53C21; Secondary 53C65.
Key words and phrases. Quasi-Einstein manifold; warped product; Weyl tensor.

Remark 1. *It is important highlight that, if* (M, g) *and* (F, g_F) *are Riemannian manifolds, the warped product* $(M \times F, \overline{g} = g + e^{-\frac{2}{m}f} g_F)$ *is an Einstein manifold with Einstein constant* λ *if and only if* (F, g_F) *is Einstein* ($Ric_{g_F} = \mu g_F$)*, satisfies (1.2) for metric* g, *and*

$$\Delta f = |\nabla f|^2 + m\lambda - m\mu e^{\frac{2}{m}f}. \tag{1.3}$$

In 2003, Kim and Kim showed that every quasi-Einstein manifold must satisfy (1.3) for some constant μ. *For more references on Einstein warped products and quasi-Einstein metrics, see* [3, 4, 5, 11, 19, 25].

Let us point out that if $m = \infty$ the equation (1.1) becomes the fundamental equation of the gradient Ricci solitons, in this case, we refer the reader to the survey papers [6, 7] and references therein for a overview on this subject. Moreover, recent results can be found in [8, 9, 10, 13, 15, 17] and [21].

In 2012, Chenxu He, Petersen and Wylie proved that, if a complete, simply connected manifold has harmonic Weyl tensor and satisfies $W(\cdot, \nabla f, \cdot, \nabla f) = 0$, then (M, g) is a m-quasi Einstein metric if and only if it is a warped product with Einstein fibers. More precisely, they proved the following result.

Theorem 1 (He-Petersen-Wylie, [18])**.** *Let* $m > 1$ *and suppose that* (M, g) *is complete, simply connected, and has harmonic Weyl tensor and* $W(\nabla f, \cdot, \cdot, \nabla f) = 0$, *then* (M, g, f) *is a nontrivial* m*-quasi-Einstein metric if and only if it is of the form*

$$g = dt^2 + \psi^2(t) g_L \quad \textit{and} \quad f = f(t),$$

where g_L *is an Einstein metric. Moreover, if* $\lambda \geq 0$ *then* (L, g_L) *has non-negative Ricci curvature, and if it is Ricci flat, then* ψ *is constant, i.e.,* (M^n, g) *is a Riemannian product.*

Later on, assuming that the manifold is Bach-flat, Chen and He showed in [12] that any shrinking quasi-Einstein manifold is either Einstein or a finite quotient of a warped product with $(n-1)$-dimensional Einstein fiber. At this point, it is important to say that if m is positive then a quasi-Einstein manifold is compact if and only if $\lambda > 0$ (see, for example, [19, 23] and [26] for a complete description). More recently, Ranieri and Ribeiro Jr. [24] have studied steady quasi-Einstein metrics under Bach-flat assumption. In this case, the authors proved that a Bach-flat noncompact steady quasi-Einstein manifold with positive Ricci curvature must be a warped product with Einstein fiber. In this paper, motivated by the historical development on the study of the quasi-Einstein manifolds, we shall investigate such structure satisfying a fourth-order vanishing condition on the Weyl tensor.

Before presenting our first result, it is fundamental to remember that a Riemannian manifold (M^n, g) has *zero radial Weyl curvature* when

$$W(\cdot, \cdot, \cdot, \nabla f) = 0,$$

for a suitable smooth function f. This condition have been used to classify quasi-Einstein manifolds or more general generalized quasi-Einstein manifolds, see for instance, [14, 18, 20] and [22].

In the sequel, in the the same spirit of the recent work due Catino, Mastrolia and Monticelli [15], let us introduce the following definitions:

$$\mathrm{div}^4 W = \nabla_k \nabla_j \nabla_i \nabla_l W_{ijkl}$$

and

$$\mathrm{div}^3 C = \nabla_k \nabla_j \nabla_i C_{ijk},$$

where W and C are the Weyl and the Cotton tensors, respectively (see the definitions of this tensors in the Section 2). In [15], the authors showed that n-dimensional complete gradient shrinking Ricci solitons with fourth order divergence free Weyl tensor (i.e., $\mathrm{div}^4 W = 0$) are either Einstein or finite quotients of $N^{n-k} \times \mathbb{R}^k$, $(k > 0)$. That is, the product of a Einstein manifold N^{n-k} with the Gaussian shrinking soliton $\mathbb{R}^k$. Recently, other rigidity results have been obtained under vanishing condition on the Weyl tensor, see, for example, [29, 30].

With this notation in mind, Ranieri and Ribeiro Jr. showed in [24] that, under certain appropriate constraints, a Bach-flat noncompact steady quasi-Einstein manifold must have fourth order divergence-free Weyl tensor and satisfies the zero radial Weyl curvature condition (i.e., $W_{ijkl}\nabla_l f = 0$). It is natural to ask if the converse of this statement is also true. In this sense, inspired by ideas outlined in [10] and [24], we shall replace the assumption of Bach-flat in [24, Theorem 1] by the condition that M has fourth order divergence-free Weyl tensor. In fact, we have the following result.

Theorem 2. *Let $(M^n, g, f, m > 1)$, $n \geq 4$, be a noncompact steady quasi-Einstein manifold with positive Ricci curvature such that f has at least one critical point. If, in addition, (M, g) has zero Weyl radial curvature and satisfies $\mathrm{div}^4 W = 0$, then M^n has harmonic Weyl tensor.*

It is well known that 4-dimensional compact Riemannian manifolds have special behavior. In this case, we are able to conclude that when we restrict the Theorem 2 to the 4-dimensional case, then M^4 is actually a locally conformally flat manifold. Let us highlight that in [18] the authors provided some examples of quasi-Einstein which have $\mathrm{div}^4 W = 0$ and zero radial Weyl curvature but are not locally conformally flat, cf. [18, Section 3, Table 2]. However, these examples exist only for $n \geq 5$. Hence, after these considerations, we shall apply the Theorem 2 in order to get the following result.

Corollary 1. *Let $(M^4, g, f, m > 1)$ be a 4-dimensional noncompact steady quasi-Einstein manifold with positive Ricci curvature such that f has at least one critical point. If in addition (M, g) has zero Weyl radial curvature and satisfies $\mathrm{div}^4 W = 0$ then it is a locally conformally flat manifold.*

Finally, as an immediate consequence of Theorems 1 and 2 we get the following classification result for steady quasi-Einstein manifold.

Theorem 3. *Let $(M^n, g, f, \lambda, m > 1)$, $n \geq 4$, be a noncompact, simply connected steady quasi-Einstein manifold with positive Ricci curvature such that f has at least one critical point. If in addition (M, g) has zero Weyl radial curvature and satisfies* $\mathrm{div}^4 W = 0$, *then (M^n, g) is a warped product with*

$$g = dt^2 + \psi^2(t) g_L \quad \textit{and} \quad f = f(t),$$

where g_L is an Einstein metric. Moreover, if $\lambda \geq 0$ then (L, g_L) has non-negative Ricci curvature, and if it is Ricci flat, then ψ is constant, i.e., (M^n, g) is a Riemannian product.

2. Preliminaries

Throughout this section we recall some informations and basic results that will be useful in the proof of our main result. Firstly, by the trace of the fundamental equation (1.1), we verify the relation

$$R + \Delta f - \frac{1}{m}|\nabla f|^2 = \lambda n. \tag{2.1}$$

For sake of simplicity, we now rewrite equation (1.1) in the tensorial language as follows

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{m}\nabla_i f \nabla_j f = \lambda g_{ij}. \tag{2.2}$$

In order to proceed, we recall two special tensors in the study of curvature for a Riemannian manifold $(M^n,\, g)$, $n \geq 3$. The first one is the Weyl tensor W which is defined by the following decomposition formula

$$\begin{aligned} W_{ijkl} &= R_{ijkl} - \frac{1}{n-2}\big(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\big) \\ &\quad + \frac{R}{(n-1)(n-2)}\big(g_{jl}g_{ik} - g_{il}g_{jk}\big), \end{aligned} \tag{2.3}$$

where R_{ijkl} stands for the Riemann curvature operator Rm, whereas the second one is the Cotton tensor C given by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}\big(\nabla_i R g_{jk} - \nabla_j R g_{ik}\big). \tag{2.4}$$

Note that C_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices, i.e.,

$$C_{ijk} = -C_{jik} \quad \textit{and} \quad g^{ij}C_{ijk} = g^{ik}C_{ijk} = 0.$$

These two above tensors are related as follows

$$C_{ijk} = -\frac{(n-2)}{(n-3)}\nabla_l W_{ijkl}, \tag{2.5}$$

provided $n \geq 4$. Now, we recall a well-known tensor that was introduced by Bach [1] in the study of conformal relativity, namely, the Bach tensor. On a Riemannian manifold (M^n, g), $n \geq 4$, the Bach tensor is defined in term of the components of the Weyl tensor W_{ikjl} as follows

$$B_{ij} = \frac{1}{n-3}\nabla_k\nabla_l W_{ikjl} + \frac{1}{n-2}R_{kl}W_{ikjl}, \tag{2.6}$$

while for $n = 3$ it is given by

$$B_{ij} = \nabla_k C_{kij}. \tag{2.7}$$

We say that (M^n, g) is Bach-flat when $B_{ij} = 0$. It is easy to check that locally conformally flat metrics as well as Einstein metrics are Bach-flat. Moreover, for 4-dimensional case, we have that, on any compact manifold (M^4, g), Bach-flat metrics are precisely the critical point of the *conformally invariant* functional on the space of the metrics,

$$\mathcal{W}(g) = \int_M |W_g|^2 dV_g,$$

for more details see, for example [5] or [16]. Furthermore, it is worth reporting here the following interesting formula for the divergence of the Bach tensor

$$\nabla_p B_{pi} = \frac{n-4}{(n-2)^2}C_{ijk}R_{jk}. \tag{2.8}$$

We refer reader to [9, Lemma 5.1], for its proof.

For the purposes of this work, let us recall some well-known properties about quasi-Einstein manifolds. For more detail, see [11, 12, 18, 24, 28] and references therein. The next lemma, for instance, can be found in [11].

Lemma 1. *Let (M^n, g, f, λ) be a quasi-Einstein manifold. Then we have:*

$$\frac{1}{2}\nabla_i R + \frac{(n-1)}{m}\lambda\nabla_i f = \frac{m-1}{m}R_{is}\nabla_s f + \frac{1}{m}R\nabla_i f.$$

Now, we remembered the following 3-tensor defined in [12],

$$\begin{aligned} D_{ijk} \;\; = \;\; & \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) + \frac{1}{(n-1)(n-2)}(g_{jk}R_{is}\nabla_s f - g_{ik}R_{js}\nabla_s f) \\ & - \frac{R}{(n-1)(n-2)}(g_{jk}\nabla_i f - g_{ik}\nabla_j f). \end{aligned} \tag{2.9}$$

Note that D_{ijk} has the same symmetry properties as the Cotton tensor. Still in [12], the authors showed that D_{ijk} is related to the Cotton tensor C_{ijk} and the Weyl tensor W_{ijkl}. Since its proof is very short, we include here for the sake of completeness. More precisely, we have the following identity.

Lemma 2. *(Chen-He, [12]) Let (M^n, g, f) be a quasi-Einstein manifold. Then the following identity holds:*

$$C_{ijk} = \frac{m+n-2}{m}D_{ijk} - W_{ijkl}\nabla_l f.$$

Proof. First of all, substitute (2.2) into (2.4) to deduce

$$\begin{aligned} C_{ijk} &= (\nabla_j\nabla_i\nabla_k f - \nabla_i\nabla_j\nabla_k f) + \frac{1}{m}(\nabla_j f\nabla_i\nabla_k f - \nabla_i f\nabla_j\nabla_k f) \\ &\quad -\frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}) \\ &= -R_{ijkl}\nabla_l f + \frac{1}{m}(\nabla_j f\nabla_i\nabla_k f - \nabla_i f\nabla_j\nabla_k f) \\ &\quad -\frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}), \end{aligned}$$

where in the last step we use the Ricci identities.

Next, by Eq. (2.2) again and Lemma 1, we have that

$$\begin{aligned} C_{ijk} &= -R_{ijkl}\nabla_l f + \frac{1}{m}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) \\ &\quad -\frac{\lambda}{m}(\nabla_i f g_{jk} - \nabla_j f g_{ik}) - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}) \\ &= -R_{ijkl}\nabla_l f + \frac{1}{m}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) \\ &\quad -\frac{m-1}{m(n-1)}(R_{il}\nabla_l f g_{jk} - R_{jl}\nabla_l f g_{ik}) - \frac{R}{m(n-1)}(\nabla_i f g_{jk} - \nabla_j f g_{ik}). \end{aligned}$$

Finally, substituting (2.3) in the above expression, and after some computation, we get

$$\begin{aligned} C_{ijk} &= -W_{ijkl}\nabla_l f + D_{ijk} + \frac{1}{m}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) \\ &\quad +\frac{1}{m(n-1)}(R_{il}\nabla_l f g_{jk} - R_{jl}\nabla_l f g_{ik}) - \frac{R}{m(n-1)}(\nabla_i f g_{jk} - \nabla_j f g_{ik}) \\ &= -W_{ijkl}\nabla_l f + D_{ijk} + \frac{n-2}{m}D_{ijk} \\ &= -W_{ijkl}\nabla_l f + \frac{m+n-2}{m}D_{ijk}, \end{aligned}$$

as desired. □

Under these notations we get the following lemma.

Lemma 3. *Let (M^n, g, f, λ), $n \geq 4$, be a quasi-Einstein manifold. Then we have:*

$$B_{ij} = \frac{m+n-2}{n(n-2)}\nabla_k D_{kij} + \frac{n-3}{(n-2)^2}C_{kji}\nabla_k f + \frac{1}{m(n-2)}W_{ikjl}\nabla_k f\nabla_l f.$$

Proof. To beginning with, we use (2.2) and (2.5) to infer

$$\begin{aligned} \nabla_k(W_{ikjl}\nabla_l f) &= \nabla_k W_{ikjl}\nabla_l f + W_{ikjl}\left(-R_{kl} + \frac{1}{m}\nabla_k\nabla_l f + \lambda g_{kl}\right) \\ &= \frac{n-3}{n-2}C_{lji}\nabla_l f - W_{ikjl}R_{kl} + \frac{1}{m}W_{ikjl}\nabla_k f\nabla_l f. \end{aligned}$$

Then, from Lemma 2, we immediately obtain

$$\begin{aligned}\nabla_k C_{kij} + W_{ikjl}R_{kl} &= \frac{m+n-2}{m}\nabla_k D_{kij} + \frac{n-3}{n-2}C_{lji}\nabla_l f \\ &\quad + \frac{1}{m}W_{ikjl}\nabla_k f \nabla_l f.\end{aligned} \tag{2.10}$$

Therefore, substituting (2.5) in (2.6) we get

$$(n-2)B_{ij} = \nabla_k C_{kij} + W_{ikjl}R_{kl}, \tag{2.11}$$

which combined with (2.10) gives the requested result. □

Now, restricting to the steady quasi-Einstein case, we can take $u = e^{-\frac{f}{m}}$ to deduce, after a straightforward computation, the following identities

$$\nabla_i u = -\frac{u}{m}\nabla_i f$$

and

$$\nabla_i \nabla_j u = -\frac{u}{m}\left(\nabla_i\nabla_j f - \frac{1}{m}\nabla_i f \nabla_j f\right) = \frac{u}{m}R_{ij}.$$

Hence, taking the trace in the last equality, it is easy to verify that

$$\Delta u = \frac{1}{u}|\nabla u|^2 - \frac{u}{m}\Delta f = \frac{u}{m}R. \tag{2.12}$$

Moreover, substituting (1.3) into (2.12), we get

$$m(m-1)|\nabla u|^2 + Ru^2 = m\mu. \tag{2.13}$$

We finalize this section with a lemma that will be very useful for our purposes, whose its proof can be check in [24].

Lemma 4. *Let $(M^n, g, f, m > 1)$ be a complete (noncompact) steady quasi Einstein manifold with positive Ricci curvature and such that f has at last one critical point. Then there exist positive constant c_1 and c_2 such that the function $u = e^{-\frac{f}{m}}$ satisfies the estimates*

$$c_1 r(x) - c_2 \le u(x) \le \sqrt{\frac{\mu}{m-1}}\,r(x) + |u(p)|,$$

where $r(x) = d(p, x)$ is the distance function from some fixed critical point $p \in M$, c_1 and c_2 are positive constants depending only on n and the geometry of g_{ij} on the unit ball $B_p(1)$.

3. Quasi-Einstein manifold with $\mathrm{div}^4 W = 0$

In this section we shall prove Theorems 2 and 3 announced in Section 1. To do this, under our assumption, we shall first derive a useful integral formula for the norm square of the Cotton tensor for steady quasi-Einstein metric with zero radial Weyl tensor. This formula plays an important role in our conclusion of the desired theorems.

3.1. **Proof of the Theorem 2.**

Proof. Firstly, by direct computation using (2.11), we get

$$\begin{aligned}(n-2)\mathrm{div}^2 B &= \nabla_j(\nabla_i\nabla_k C_{kij} + \nabla_i W_{ikjl}R_{kl} + W_{ijkl}\nabla_i R_{kl})\\ &= \nabla_j(\nabla_i\nabla_k C_{kij} + \nabla_i W_{ikjl}R_{kl} + \frac{1}{2}W_{ijkl}(\nabla_i R_{kl} - \nabla_k R_{il}).\end{aligned}$$

In the sequel, using the expressions (2.4) and (2.5) and the fact that the Weyl tensor W has null trace, we arrive at

$$(n-2)\mathrm{div}^2 B = \mathrm{div}^3 C + \nabla_j\left(-\frac{n-3}{n-2}C_{ljk}R_{kl} + \frac{1}{2}W_{ikjl}C_{ikl}\right). \tag{3.1}$$

Next, consider a critical point $p \in M$ and take the ball $B_p(s)$ of radius s centered at p. Also, let $\phi : M \to \mathbb{R}$ be a smooth test function defined on M. Thus, integrating (3.1) by parts and changing the indices conveniently, we obtain

$$\begin{aligned}(n-2)\int_{B_p(s)}\phi\mathrm{div}^2 B dV_g &= -\frac{n-3}{n-2}\int_{B_p(s)} C_{ijk}\nabla_i\phi R_{jk}dV_g + \frac{n-3}{n-2}\int_{\partial B_p(s)}\phi C_{ijk}R_{jk}\nu_i d\sigma\\ &+\frac{1}{2}\int_{B_p(s)} W_{ijkl}\nabla_l\phi C_{ijk}dV_g - \frac{1}{2}\int_{\partial B_p(s)}\phi W_{ijkl}C_{ijk}\nu_l d\sigma\\ &+\int_{B_p(s)}\phi\mathrm{div}^3 C dV_g,\end{aligned}$$

where ν is the outward unit normal on $\partial B_p(s)$ and $d\sigma$ is its volume form.

Proceeding, by Lemma 2 together with fact which our manifold satisfies zero radial Weyl curvature (i.e., $W_{ijkl}\nabla_l f = 0$), we get

$$\begin{aligned}(n-2)\int_{B_p(s)}\phi\mathrm{div}^2 B dV_g &= -\frac{n-3}{n-2}\int_{B_p(s)} C_{ijk}\nabla_i\phi R_{jk}dV_g + \frac{n-3}{n-2}\int_{\partial B_p(s)}\phi C_{ijk}R_{jk}\nu_i d\sigma\\ &+\frac{m+n-2}{2m}\Big\{\int_{B_p(s)} W_{ijkl}\nabla_l\phi D_{ijk}dV_g\\ &-\int_{\partial B_p(s)}\phi W_{ijkl}D_{ijk}\nu_l d\sigma\Big\} + \int_{B_p(s)}\phi\mathrm{div}^3 C dV_g.\end{aligned}$$

So, as the tensor D has the same skew-symmetric of the Cotton tensor, we arrive at

$$\begin{aligned}(n-2)\int_{B_p(s)}\phi\mathrm{div}^2 B dV_g &= -\frac{n-3}{n-2}\int_{B_p(s)} C_{ijk}\nabla_i\phi R_{jk}dV_g + \frac{n-3}{n-2}\int_{\partial B_p(s)}\phi C_{ijk}R_{jk}\nu_i d\sigma\\ &+\frac{m+n-2}{m(n-2)}\Big\{\int_{B_p(s)} W_{ijkl}\nabla_l\phi R_{jk}\nabla_i f dV_g\\ &-\int_{\partial B_p(s)}\phi W_{ijkl}R_{jk}\nabla_i f\nu_l d\sigma\Big\} + \int_{B_p(s)}\phi\mathrm{div}^3 C dV_g\\ &= -\frac{n-3}{n-2}\int_{B_p(s)} C_{ijk}\nabla_i\phi R_{jk}dV_g + \frac{n-3}{n-2}\int_{\partial B_p(s)}\phi C_{ijk}R_{jk}\nu_i d\sigma\\ &+\int_{B_p(s)}\phi\mathrm{div}^3 C dV_g,\end{aligned} \tag{3.2}$$

where in the last step we use the condition $W_{ijkl}\nabla_l f = 0$ again.

On the other hand, multiplying the equation (2.8) by ϕ and integrating by parts, we deduce

$$\begin{aligned}(n-2)\int_{B_p(s)}\phi\mathrm{div}^2 B dV_g &= \frac{n-4}{n-2}\int_{B_p(s)}\phi\nabla_i(C_{ijk}R_{jk})dV_g\\ &= \frac{n-4}{n-2}\left\{\int_{B_p(s)}\nabla_i(\phi C_{ijk}R_{jk})dV_g - \int_{B_p(s)}\nabla_i\phi C_{ijk}R_{jk}dV_g\right\}\\ &= \frac{n-4}{n-2}\int_{\partial B_p(s)}\phi C_{ijk}R_{jk}\nu_i d\sigma - \frac{n-4}{n-2}\int_{B_p(s)}C_{ijk}\nabla_i\phi R_{jk}dV_g. \qquad (3.3)\end{aligned}$$

Thus, comparing (3.2) with (3.3) and using the Lemma 2, we have

$$\begin{aligned}\int_{B_p(s)}\nabla_i\phi C_{ijk}R_{jk}dV_g &= \int_{\partial B_p(s)}\phi C_{ijk}R_{jk}\nu_i d\sigma + (n-2)\int_{B_p(s)}\phi\mathrm{div}^3 C dV_g\\ &= \frac{m+n-2}{m}\int_{\partial B_p(s)}\phi D_{ijk}R_{jk}\nu_i d\sigma + (n-2)\int_{B_p(s)}\phi\mathrm{div}^3 C dV_g. \qquad (3.4)\end{aligned}$$

Now, from the definition of the tensor D_{ijk}, it is easy to check that

$$\begin{aligned}D_{ijk}R_{jk}\nu_i &= \frac{1}{n-2}|Ric|^2\langle\nabla f,\nu\rangle - \frac{n}{(n-1)(n-2)}R_{ik}R_{jk}\nabla_i f\nu_j\\ &\quad + \frac{2R}{(n-1)(n-2)}R_{ij}\nabla_i f\nu_j - \frac{R^2}{(n-1)(n-2)}\langle\nabla f,\nu\rangle,\end{aligned}$$

which joint with fact that $|R_{ij}| \leq R$ (this follows directly from our hypothesis which M has positive Ricci curvature), allows us to conclude the following estimate

$$|D_{ijk}R_{jk}\nu_i| \leq CR^2|\nabla f|, \qquad (3.5)$$

for some constant $C > 0$. Furthermore, since we are working in the steady case, the scalar curvature is nonnegative (cf. [28] for more details) and consequently, from (2.13) we get

$$|\nabla u|^2 \leq \frac{\mu}{m-1} \quad \textit{and} \quad u^2 R \leq m\mu.$$

Hence, (3.5) becomes

$$|D_{ijk}R_{jk}\nu_i| \leq C\frac{1}{u^5},$$

where we denote the same constant for simplicity.

Next, consider s sufficiently large such that $c_1 s - c_2$ is a positive number, where c_1 and c_2 are constant provided in the Lemma 4. Thus, by the inequality obtained in Lemma 4 and taking our test function as $\phi = u^{-n+\frac{11}{2}}$, we deduce

$$\begin{aligned}\left|\int_{\partial B_p(s)}\phi D_{ijk}R_{jk}\nu_i d\sigma\right| &\leq \int_{\partial B_p(s)}u^{-n+\frac{11}{2}}|D_{ijk}R_{jk}\nu_i|d\sigma\\ &\leq C\int_{\partial B_p(s)}\frac{1}{u^{n-\frac{1}{2}}}d\sigma\\ &\leq \frac{C}{(c_1 s - c_2)^{n-\frac{1}{2}}}Area(\partial B_p(s)). \qquad (3.6)\end{aligned}$$

Therefore, since we are assuming positive Ricci curvature, it follows from the well-known Bishop-Gromov's theorem, that

$$Area(\partial B_p(s)) \leq \widetilde{C} s^{n-1},$$

where $\widetilde{C}$ is a positive constant. Thus, returning to inequality (3.6), we deduce

$$\left| \int_{\partial B_p(s)} \phi D_{ijk} R_{jk} \nu_i d\sigma \right| \quad \leq \quad C \left(\frac{s}{c_1 s - c_2} \right)^{n-\frac{1}{2}} \frac{1}{\sqrt{s}},$$

again we consider the same constant. In particular, if we take $s \to \infty$ in (3.4), then it is easy to verify that

$$\begin{aligned} \int_M u^{-n+\frac{11}{2}} \operatorname{div}^3 C dV_g &= \frac{2n-11}{2m(n-2)} \int_M u^{-n+\frac{11}{2}} C_{ijk} \nabla_i f R_{jk} dV_g \\ &= \frac{2n-11}{4m(n-2)} \int_M u^{-n+\frac{11}{2}} C_{ijk} (\nabla_i f R_{jk} - \nabla_j f R_{ik}) dV_g \\ &= \frac{2n-11}{4m} \int_M u^{-n+\frac{11}{2}} C_{ijk} D_{ijk} dV_g, \end{aligned}$$

where we use the skew-symmetry of Cotton tensor jointly with the definition of the auxiliary tensor D. So, we may use Lemma 2 in order to get the following integral formula for steady quasi-Einstein metric with zero radial Weyl tensor,

$$\int_M u^{-n+\frac{11}{2}} \operatorname{div}^3 C dV_g = \frac{(2n-11)}{4(m+n-2)} \int_M u^{-n+\frac{11}{2}} |C_{ijk}|^2 dV_g. \tag{3.7}$$

Finally, since we are suppose that M has fourth order divergence-free Weyl tensor (i.e., $\operatorname{div}^4 W = 0$), it follows from (3.7) that M has null Cotton tensor, which is equivalent to say, using Eq. (2.5), that M has harmonic Weyl tensor. This is what we wanted to prove. □

3.2. **Conclusion of the proof of Corollary 1 and Theorem 3.**

Proof. Under the conditions of Theorem 2, since we already know that M has null Cotton tensor, then by Lemma 3 we get immediately that M is a Bach-flat manifold and consequently, the Corollary 1 and Theorem 3 follows from Theorem 2 and Corollary 1 in [24], respectively. □

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