CONGRUENCES FOR COEFFICIENTS OF LEVEL 2 MODULAR FUNCTIONS WITH POLES AT 0

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Abstract. We give congruences modulo powers of 2 for the Fourier coefficients of certain level 2 modular functions with poles only at 0, answering a question posed by Andersen and the first author. The congruences involve a modulus that depends on the binary expansion of the modular form's order of vanishing at ∞.

1. Introduction

A modular form $f(z)$ of level N and weight k is a function which is holomorphic on the upper half plane, satisfies the equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N),$$

and is holomorphic at the cusps of $\Gamma_0(N)$. Letting $q = e^{2\pi i z}$, these functions have Fourier series representations of the form $f(z) = \sum_{n=0}^{\infty} a(n)q^n$. A *weakly holomorphic* modular form is a modular form that is allowed to be meromorphic at the cusps. We define $M_k^\sharp(N)$ to be the space of weakly holomorphic modular forms of weight k and level N that are holomorphic away from the cusp at ∞, and define $M_k^\flat(N)$ similarly, but for forms holomorphic away from the cusp at 0.

The coefficients of many modular forms have interesting arithmetic properties; for instance, the coefficients $c(n)$ of the j-invariant $j(z) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n$ appear as linear combinations of dimensions of irreducible representations of the Monster group. Also, modular form coefficients often satisfy certain congruences. Lehner [10, 11] proved that the $c(n)$ satisfy the congruence

$$c(2^a 3^b 5^c 7^d n) \equiv 0 \pmod{2^{3a+8} 3^{2b+3} 5^{c+1} 7^d}.$$

Many others have obtained similar congruences, in particular for forms in the space $M_0^\sharp(N)$. Kolberg [8, 9], Aas [1], and Allatt and Slater [2] strengthened Lehner's congruences for $j(z)$, and Griffin [6] extended Kolberg's and Aas's results to all elements of a canonical basis for $M_0^\sharp(1)$. The first author, Andersen, and Thornton [3, 7] proved congruences for Fourier coefficients of canonical bases for $M_0^\sharp(p)$ with $p = 2$, 3, 5, and 7. A natural question is whether similar congruences hold for coefficients of canonical bases for spaces where we allow poles at some other cusp.

Recall that the cusps of $\Gamma_0(2)$ are 0 and ∞. Taking $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$ to be the Dedekind eta function, a Hauptmodul for $\Gamma_0(2)$ is

$$\phi(z) = \left(\frac{\eta(2z)}{\eta(z)}\right)^{24} = q + 24q^2 + \cdots,$$

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which vanishes at ∞ and has a pole only at 0. Note that the functions $\phi(z)^m$ for $m \geq 0$ are a basis for $M_0^\flat(2)$. Andersen and the first author used powers of $\phi(z)$ to prove congruences involving $\psi = \frac{1}{\phi} = q^{-1} - 24 + \cdots \in M_0^\sharp(2)$ in [3], and made the following remark: "Additionally, it appears that powers of the function $[\phi(z)]$ have Fourier coefficients with slightly weaker divisibility properties... It would be interesting to more fully understand these congruences." In this paper, we prove congruences for these Fourier coefficients.

Write $\phi(z)^m$ as $\sum_{n=m}^{\infty} a(m,n)q^n$. The main result of this paper is the following theorem.

Theorem 1. *Let $n = 2^\alpha n'$ where $2 \nmid n'$. Express the binary expansion of m as $a_r \dots a_2 a_1$, and consider the rightmost α digits $a_\alpha \dots a_2 a_1$, letting $a_i = 0$ for $i > r$ if $\alpha > r$. Let i' be the index of the rightmost 1, if it exists. Let*

$$\gamma(m,\alpha) = \begin{cases} \#\{i \mid a_i = 0, i > i'\} + 1 & \text{if } i' \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$a(m, 2^\alpha n') \equiv 0 \pmod{2^{3\gamma(m,\alpha)}}.$$

That the structure of the binary expansion of m appears in the modulus of this congruence is a surprising result. We note that this congruence is not sharp. For $m = 1$, Allatt and Slater in [2] proved a stronger result that provides an exact congruence for many n.

As an example, the binary expansion for $m = 40$ is $m = \cdots 000101000$. As we increment α, the γ function gives the values in Table 1.

α	0	1	2	3	4	5	6	7	8	9	$\cdots$	α	$\cdots$
$\gamma(40,\alpha)$	0	0	0	0	1	2	2	3	4	5	$\cdots$	$\alpha - 4$	$\cdots$

TABLE 1. Values of $\gamma(m,\alpha)$ for $m = 40$

Notice that once α surpasses 6—and the leftmost 1 in the binary expansion of m occurs in the 6th place—γ always increases by 1 as α increases by 1. This illustrates that $\gamma(m,\alpha)$ is unbounded for a fixed m.

We also prove the following result on the parity of $a(1,n)$.

Theorem 2. *The nth coefficient $a(1,n)$ of $\phi(z)$ is odd if and only if n is an odd square.*

Section 2 contains the machinery and definitions we use in the proof of Theorem 1. The proof of Theorem 1 is in Section 3, and the proof of Theorem 2 is in Section 4.

2. Preliminary Lemmas

The operator U_p on a function $f(z)$ is given by

$$U_p f(z) = \frac{1}{p} \sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right).$$

We have $U_p : M_k^!(N) \to M_k^!(N)$ if p divides N. If $f(z)$ has the Fourier expansion $\sum_{n=n_0}^{\infty} a(n)q^n$, then the effect of U_p is given by $U_p f(z) = \sum_{n=n_0}^{\infty} a(pn)q^n$.

The following result describes how U_p applied to a modular function behaves under the Fricke involution. This will help us in Lemma 6 to write $U_2\phi^m$ as a polynomial in ϕ.

Lemma 3. [4, Theorem 4.6] *Let p be prime and let $f(z)$ be a level p modular function. Then*

$$p(U_pf)\left(\frac{-1}{pz}\right) = p(U_pf)(pz) + f\left(\frac{-1}{p^2z}\right) - f(z).$$

The Fricke involution $\left(\begin{smallmatrix} 0 & -1 \\ 2 & 0 \end{smallmatrix}\right)$ swaps the cusps of $\Gamma_0(2)$, which are 0 and ∞. We will use this fact in the proof of Lemma 6, and the following relations between $\phi(z)$ and $\psi(z)$ will help us compute this involution.

Lemma 4. [3, Lemma 3] *The functions $\phi(z)$ and $\psi(z)$ satisfy the relations*

$$\phi\left(\frac{-1}{2z}\right) = 2^{-12}\psi(z),$$
$$\psi\left(\frac{-1}{2z}\right) = 2^{12}\phi(z).$$

The following lemma is a special case of a result from one of Lehner's papers [11]. It provides a polynomial whose roots are modular forms used in the proof of Theorem 8.

Lemma 5. [11, Theorem 2] *There exist integers b_j such that*

$$U_2\phi(z) = 2(b_1\phi(z) + b_2\phi(z)^2).$$

Furthermore, let $h(z) = 2^{12}\phi(z/2)$, $g_1(z) = 2^{14}\left(b_1\phi(z) + b_2\phi(z)^2\right)$, and $g_2(z) = -2^{14}b_2\phi(z)$. Then

$$h(z)^2 - g_1(z)h(z) + g_2(z) = 0.$$

In the following lemma, we extend the result from the first part of Lemma 5, writing $U_2\phi^m$ as an integer polynomial in ϕ. In particular, we give the least and greatest powers of the polynomial's nonzero terms.

Lemma 6. *For all $m \geq 1$, $U_2\phi^m \in \mathbb{Z}[\phi]$. In particular,*

$$U_2\phi^m = \sum_{j=\lceil m/2\rceil}^{2m} d(m,j)\phi^j$$

where $d(m,j) \in \mathbb{Z}$, and $d(m, \lceil m/2\rceil)$ and $d(m, 2m)$ are not 0.

Proof. Using Lemmas 3 and 4, we have that

$$\begin{aligned} U_2\phi(-1/2z)^m &= U_2\phi(2z)^m + 2^{-1}\phi(-1/4z)^m - 2^{-1}\phi(z)^m \\ &= U_2\phi(2z)^m + 2^{-1-12m}\psi(2z)^m - 2^{-1}\phi(z)^m \\ &= 2^{-1-12m}q^{-2m} + O(q^{-2m+2}) \\ 2^{1+12m}U_2\phi(-1/2z)^m &= q^{-2m} + O(q^{-2m+2}). \end{aligned}$$

Because $\phi(z)^m$ is holomorphic at ∞, $U_2\phi(z)^m$ is holomorphic at ∞. So $U_2\phi(-1/2z)^m$ is holomorphic at 0 and, since it starts with q^{-2m}, must be a polynomial of degree

$2m$ in $\psi(z)$. Let $b(m,j) \in \mathbb{Z}$ such that

$$2^{1+12m} U_2 \phi(-1/2z)^m = \sum_{j=0}^{2m} b(m,j)\psi(z)^j,$$

and we note that $b(m,2m)$ is not 0. Now replace z with $-1/2z$ and use Lemma 4 to get

$$2^{1+12m} U_2 \phi(z)^m = \sum_{j=0}^{2m} b(m,j) 2^{12j} \phi(z)^j,$$

which gives

$$U_2 \phi(z)^m = \sum_{j=0}^{2m} b(m,j) 2^{12(j-m)-1} \phi(z)^j.$$

If m is even, the leading term of the above sum is $q^{m/2}$, and if m is odd, the leading term is $q^{(m+1)/2}$, so the sum starts with $j = \lceil m/2 \rceil$ as desired. Notice that $b(m,j)2^{12(j-m)-1}$ is an integer because the coefficients of $\phi(z)^m$ are integers. $\square$

We may repeatedly use Lemma 6 to write $U_2^\alpha \phi^m$ as a polynomial in ϕ. Let

$$f(\ell) = \lceil \ell/2 \rceil, \ f^0(\ell) = \ell, \text{ and } f^k(\ell) = f(f^{k-1}(\ell)). \tag{1}$$

Using Lemma 6, the smallest power of q appearing in $U_2^\alpha \phi^m$ is $f^\alpha(m)$. Lemma 7 provides a connection between $\gamma(m,\alpha)$ and the integers $f^\alpha(m)$.

Lemma 7. *The function $\gamma(m,\alpha)$ as defined in Theorem 1 is equal to the number of odd integers in the list*

$$m, f(m), f^2(m), \ldots, f^{\alpha-1}(m).$$

Proof. Write the binary expansion of m as $a_r \ldots a_2 a_1$, and consider its first α digits, $a_\alpha \ldots a_2 a_1$, where $a_i = 0$ for $i > r$ if $\alpha > r$. If all $a_i = 0$, then all of the integers in the list are even. Otherwise, suppose that $a_i = 0$ for $1 \le i < i'$ and $a_{i'} = 1$. Apply f repeatedly to m, which deletes the beginning 0s from the expansion, until $a_{i'}$ is the rightmost remaining digit; that is, $f^{i'-1}(m) = a_\alpha \ldots a_{i'-1} a_{i'}$. In particular, this integer is odd. Having reduced to the odd case, we now treat only the case where m is odd.

If m in the list is odd, then $a_1 = 1$, which corresponds to the $+1$ in the definition of $\gamma(m,\alpha)$. Also, $f(m) = \lceil m/2 \rceil = (m+1)/2$. Applied to the binary expansion of m, this deletes a_1 and propagates a 1 leftward through the binary expansion, flipping 1s to 0s, and then terminating upon encountering the first 0 (if it exists), which changes to a 1. As in the even case, we apply f repeatedly to delete the new leading 0s, producing one more odd output in the list once all the 0s have been deleted. Thus, each 0 to the left of $a_{i'}$ corresponds to one odd number in the list. $\square$

3. Proof of the Main Theorem

Theorem 1 will follow from the following theorem.

Theorem 8. *Let $f(\ell)$ be as in (1). Let $\gamma(m,\alpha)$ be as in Theorem 1, and let $\alpha \ge 1$. Define*

$$c(m,j,\alpha) = \begin{cases} -1 & \textit{if } f^{\alpha-1}(m) \textit{ is even and is not } 2j, \\ 0 & \textit{otherwise.} \end{cases}$$

Write $U_2^\alpha \phi^m = \sum_{j=f^\alpha(m)}^{2^\alpha m} d(m,j,\alpha)\phi^j$. *Then*

$$\nu_2(d(m,j,\alpha)) \geq 8(j - f^\alpha(m)) + 3\gamma(m,\alpha) + c(m,j,\alpha). \tag{2}$$

The α of Theorem 8 corresponds to the α in $n = 2^\alpha n'$ of Theorem 1, and because our methods use the U_2 operator, they do not give meaningful congruences for the case when $\alpha = 0$. Theorem 8 is an improvement on the following result by Lehner [11].

Theorem 9. [11, Equation 3.4] *Write* $U_2^\alpha \phi^m$ *as* $\sum d(m,j,\alpha)\phi^j \in \mathbb{Z}[\phi]$. *Then* $\nu_2(d(m,j,\alpha)) \geq 8(j-1) + 3(\alpha - m + 1) + (1-m)$.

In particular, Lehner's bound sometimes only gives the trivial result that the 2-adic valuation of $d(m,j,\alpha)$ is greater than some negative integer.

We prove Theorem 8 by induction on α. The base case is similar to Lemma 6 from [3], which gives a subring of $\mathbb{Z}[\phi]$ which is closed under the U_2 operator. The polynomials are useful because their coefficients are highly divisible by 2. Here, we employ a similar technique to prove divisibility properties of the polynomial coefficients in Lemma 6. We then induct to extend the divisibility results to the polynomials that arise from repeated application of U_2.

Proof of Theorem 8. For the base case, we let $\alpha = 1$, and seek to prove the statement

$$U_2\phi^m = \sum_{j=\lceil m/2\rceil}^{2m} d(m,j,1)\phi^j$$

with

$$\nu_2(d(m,j,1)) \geq 8(j - \lceil m/2\rceil) + c(m,j) \tag{3}$$

where

$$c(m,j) = \begin{cases} 3 & m \text{ is odd,} \\ 0 & m = 2j, \\ -1 & \text{otherwise.} \end{cases}$$

The term $c(m,j)$ combines $c(m,j,\alpha)$ and $3\gamma(m,\alpha)$ for notational convenience. We prove (3) by induction on m.

We follow the proof techniques used in Lemmas 5 and 6 of [3]. From the definition of U_2, we have

$$U_2\phi^m = 2^{-1}\left(\phi\left(\frac{z}{2}\right)^m + \phi\left(\frac{z+1}{2}\right)^m\right) = 2^{-1-12m}\left(h_0(z)^m + h_1(z)^m\right)$$

where $h_\ell(z) = 2^{12}\phi\left(\frac{z+\ell}{2}\right)$. To understand this form, we construct a polynomial whose roots are $h_0(z)$ and $h_1(z)$. Let $g_1(z) = 2^{16}\cdot 3\phi(z) + 2^{24}\phi(z)^2$ and $g_2(z) = -2^{24}\phi(z)$. Then by Lemma 5, the polynomial $F(x) = x^2 - g_1(z)x + g_2(z)$ has $h_0(z)$ as a root. It also has $h_1(z)$ as a root because under $z \mapsto z+1$, $h_0(z) \mapsto h_1(z)$ and the g_ℓ are fixed.

Recall Newton's identities for the sum of powers of roots of a polynomial. For a polynomial $\prod_{i=1}^n (x - x_i)$, let $S_\ell = x_1^\ell + \cdots + x_n^\ell$ and let g_ℓ be the ℓth symmetric polynomial in the $x_1, \ldots, x_n$. Then

$$S_\ell = g_1 S_{\ell-1} - g_2 S_{\ell-2} + \cdots + (-1)^{\ell+1}\ell g_\ell.$$

We apply this to the polynomial $F(x)$, which has only two roots, to find that

$$h_0(z)^m + h_1(z)^m = S_m = g_1 S_{m-1} - g_2 S_{m-2}.$$

Furthermore,

$$U_2\phi^m = 2^{-1-12m} S_m. \tag{4}$$

Lastly, let R be the set of polynomials of the form $d(1)\phi(z) + \sum_{n=2}^{N} d(n)\phi(z)^n$ where for $n \geq 2$, $\nu_2(d(n)) \geq 8(n-1)$. Now we rephrase the theorem statement in terms of S_m and elements of R. When m is odd, we wish to show that for some $r \in R$, $U_2\phi^m = 2^{-8(\lceil m/2\rceil -1)+3} r$. Performing straightforward manipulations using (4), this is equivalent to $S_m = 2^{8(m+1)} r$ for some $r \in R$. Similarly, when m is even and is not $2j$, we wish to show that $U_2\phi^m = 2^{-8(\lceil m/2\rceil -1)-1} r$ for some $r \in R$. This again reduces to showing that $S_m = 2^{8(m+1)} r$ for some $r \in R$. If $m = 2j$, then (3) gives $8(j - \lceil 2j/2\rceil) + 0 = 0$, which means the polynomial has integer coefficients, which is true by Lemma 6.

When $m = 1$ or 2, we have that $S_m = 2^{8(m+1)} r$ for some $r \in R$, as

$$S_1 = g_1 = 2^{8(2)}(3\phi + 2^8\phi^2),$$
$$S_2 = g_1 S_1 - 2g_2 = 2^{8(3)}(2\phi + 2^8 3^2 \phi^2 + 2^{17}\phi^3 + 2^{24}\phi^4).$$

Now assume the equality is true for positive integers less than m with m at least 3. Then for some $r_1, r_2 \in R$,

$$\begin{aligned} S_m &= g_1 S_{m-1} - g_2 S_{m-2} \\ &= (2^{16}(3\phi + 2^8\phi^2))(2^{8m} r_1) + (2^{24}\phi)(2^{8(m-1)} r_2) \\ &= 2^{8(m+1)}[(3\cdot 2^8\phi + 2^{16}\phi^2) r_1 + 2^8\phi r_2], \end{aligned}$$

completing the proof where $\alpha = 1$.

Assume the theorem is true for $U_2^\alpha \phi^m = \sum\limits_{j=s}^{2^\alpha m} d(j)\phi^j$, meaning

$$\nu_2(d(j)) \geq 8(j - f^\alpha(m)) + 3\gamma(m,\alpha) + c(m,j,\alpha). \tag{5}$$

Note that $s = f^\alpha(m)$. Letting $s' = f(s)$ and $U_2\phi^j = \sum_{i=\lceil j/2\rceil}^{2j} b(j,i)\phi^i$, we define $d'(j)$ as the integers satisfying the following equation:

$$\begin{aligned} U_2^{\alpha+1}\phi^m &= U_2\left(\sum_{j=s}^{2^\alpha m} d(j)\phi^j\right) \\ &= \sum_{j=s}^{2^\alpha m} d(j) U_2\phi^j \\ &= \sum_{j=s}^{2^\alpha m} \sum_{i=\lceil j/2\rceil}^{2j} d(j) b(j,i)\phi^i \\ &= \sum_{j=s'}^{2^{\alpha+1} m} d'(j)\phi^j. \end{aligned} \tag{6}$$

We wish to prove that

$$\nu_2(d'(j)) \geq 8(j - f^{\alpha+1}(m)) + 3\gamma(m,\alpha+1) + c(m,j,\alpha+1). \tag{7}$$

We will prove inequalities that imply (7). Observe that

$$c(m,j,\alpha+1)=\begin{cases}-1 & \text{if } s \text{ is even and not } 2j,\\ 0 & \text{if } s \text{ is odd or } s=2j,\end{cases}$$

and

$$\gamma(m,\alpha+1)=\begin{cases}\gamma(m,\alpha) & \text{if } s \text{ is even,}\\ \gamma(m,\alpha)+1 & \text{if } s \text{ is odd.}\end{cases}$$

Also, $c(m,s,\alpha)=0$ because if $f^{\alpha-1}(m)$ is even, then $s=f^{\alpha-1}(m)/2$ so $f^{\alpha-1}(m)=2s$. Therefore, $\nu_2(d(s))\geq 3\gamma(m,\alpha)$ by (5).

If s is even, we will show that

$$\nu_2(d'(j))\geq \max\left\{8\left(j-s'\right)-1+\nu_2(d(s)),\nu_2(d(s))\right\}, \tag{8}$$

because then if $j=s'$, we have

$$\begin{aligned}\nu_2(d'(s'))&\geq \nu_2(d(s))\\ &\geq 8(s'-s')+3\gamma(m,\alpha)+c(m,s',\alpha+1),\end{aligned}$$

and for all j,

$$\begin{aligned}\nu_2(d'(j))&\geq 8(j-s')+3\gamma(m,\alpha)+c(m,j,\alpha+1)\\ &=8(j-f^{\alpha+1}(m))+3\gamma(m,\alpha+1)+c(m,j,\alpha+1),\end{aligned}$$

so that (8) implies (7). If s is odd we will show that

$$\nu_2(d'(j))\geq 8\left(j-s'\right)+3+\nu_2(d(s)), \tag{9}$$

because then

$$\begin{aligned}\nu_2(d'(j))&\geq 8(j-s')+3\gamma(m,\alpha)+3\\ &=8(j-s')+3(\gamma(m,\alpha)+1)\\ &=8(j-f^{\alpha+1}(m))+3\gamma(m,\alpha+1)+c(m,j,\alpha+1),\end{aligned}$$

which is (7).

For the sake of brevity, we treat here only the case where s is odd. The case where s is even has a similar proof. This case breaks into subcases. We will only show the proof where $j\leq 2s$, but the other cases are $2s<j\leq 2^{\alpha-1}m$ and $2^{\alpha-1}m<j\leq 2^{\alpha+1}m$, using the same subcases for when s is even. These subcases are natural to consider because in the first range of j-values, the $d(s)$ term is included for computing $d'(j)$, in the second range, there are no $d(s)$ or $d(2^{\alpha}m)$ terms, and in the third range, there is a $d(2^{\alpha}m)$ term.

Let $j\leq 2s$. Using (6), we know that $d'(j)=\sum_{i=s}^{2j}d(i)b(i,j)$ by collecting the coefficients of ϕ^j. Let $\delta(i)$ be given by

$$\delta(i)=\nu_2(d(i))+\nu_2(b(i,j)).$$

Let $D=\{\delta(i)\mid s\leq i\leq 2j\}$. Therefore we have

$$\begin{aligned}\nu_2(d'(j))&\geq \min\left\{\nu_2(d(i))+\nu_2(b(i,j))\mid s\leq i\leq 2j\right\}\\ &=\min D.\end{aligned}$$

We claim that $\delta(i)$ achieves its minimum with $\delta(s)$, which proves (9). For that element of D, we know by inequality (3) that

$$\delta(s)\geq \nu_2(d(s))+8(j-s')+3.$$

Now suppose $i > s$. Then every element of D satisfies the following inequality:

$$\begin{aligned}\delta(i) &= \nu_2(d(i)) + 8\,(j - \lceil i/2 \rceil) + c(i,j)\\ &\geq 8\,(i-s) - 1 + \nu_2(d(s)) + 8\,(j - \lceil i/2 \rceil) + c(i,j)\\ &\geq 8\,(s+1-s+j-\lceil (s+1)/2 \rceil) - 2 + \nu_2(d(s))\\ &= 8\,(j-s') + 6 + \nu_2(d(s)),\end{aligned}$$

but this is clearly greater than $\delta(s)$. Therefore, if $j \leq 2s$ and s is odd, then $\nu_2(d'(j)) \geq 8\,(j-s') + 3 + \nu_2(d(s))$. The other cases are similar. □

Now Theorem 1 follows easily from Theorem 8.

Theorem 1. *Let $n = 2^\alpha n'$ where $2 \nmid n'$. Express the binary expansion of m as $a_r \dots a_2 a_1$, and consider the rightmost α digits $a_\alpha \dots a_2 a_1$, letting $a_i = 0$ for $i > r$ if $\alpha > r$. Let i' be the index of the rightmost 1, if it exists. Let*

$$\gamma(m,\alpha) = \begin{cases} \#\,\{i \mid a_i = 0, i > i'\} + 1 & \text{if } i' \text{ exists,}\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$a(m, 2^\alpha n') \equiv 0 \pmod{2^{3\gamma(m,\alpha)}}.$$

Proof. Letting $j = f^\alpha(m)$ in (2), the right hand side reduces to

$$3\gamma(m,\alpha) + c(m, f^\alpha(m), \alpha).$$

Notice that $c(m, f^\alpha(m), \alpha) = 0$, because if $f^{\alpha-1}(m)$ is even, then $f^\alpha(m) = f^{\alpha-1}(m)/2$ so $f^{\alpha-1}(m) = 2f^\alpha(m)$. The right hand side of (2) is minimized when $j = f^\alpha(m)$, so we conclude that $\nu_2(a(m, 2^\alpha n')) \geq 3\gamma(m,\alpha)$. □

4. The Parity of $a(1,n)$

Table 2 contains all odd coefficients of $\phi(z) = \sum_{n=1}^\infty a(1,n)q^n$ up to $n = 225$. The table shows that, up to $n = 225$, the coefficient $a(1,n)$ is odd if and only if n is an odd square. This holds in general.

Theorem 2. *The nth coefficient $a(1,n)$ of $\phi(z)$ is odd if and only if n is an odd square.*

Proof. Substitute $\eta(z)$ into the definition of $\phi(z)$:

$$\phi(z) = \left(\frac{\eta(2z)}{\eta(z)}\right)^{24} = \left(\frac{q^{2/24}\prod_{n=1}^\infty (1-q^{2n})}{q^{1/24}\prod_{n=1}^\infty (1-q^n)}\right)^{24}.$$

By recognizing that $(1-q^{2n}) = (1-q^n)(1+q^n)$ and simplifying, it is easy to see that

$$\phi(z) = q\prod_{n=1}^\infty (1+q^n)^{24}.$$

Reducing this mod 2, the odd coefficients will be the only nonzero terms. But $\binom{24}{i}$ is odd if and only if $i = 0, 8, 16, 24$. It follows that

$$\phi(z) \equiv q\prod_{n=1}^\infty (1+q^{8n}+q^{16n}+q^{24n}) \pmod 2.$$

Immediately, it is clear that the coefficient of q^n in the Fourier expansion of $\phi(z)$ is even if $n \not\equiv 1 \pmod 8$.

Note that the coefficient of q^n in the product $\prod_{n=1}^{\infty}(1+q^n+q^{2n}+q^{3n})$ is odd if and only if the coefficient of q^{8n+1} is odd in the Fourier expansion of $\phi(z)$. Furthermore, this product can be interpreted as the generating function for the number of partitions of n where each part is repeated at most 3 times. The nth coefficient of the generating function is equivalent mod 2 to T_n of [5]. Theorem 2.1 of [5] shows that n is a triangular number if and only if T_n is odd. Therefore, the coefficient of q^n is odd in the Fourier expansion of $\phi(z)$ if and only if

$$n = 8\frac{k(k+1)}{2} + 1 = 4k^2 + 4k + 1 = (2k+1)^2,$$

meaning that n is an odd square. □

n	$a(1,n)$
1	1
9	10400997
25	254038914924791
49	8032568516459357451913
81	28827450451683687172361829572‌1
121	1115664686143980561311817219902403825‌3
169	45398829054388718939196306308933722268484668‌7
225	19146547947132951990683661128349583597266368489785587

Table 2. All odd coefficients of $\phi(z)$ up to $n = 225$.

References

1. Hans-Fredrik Aas, *Congruences for the coefficients of the modular invariant $j(\tau)$*, Math. Scand. **15** (1964), 64–68. MR 0179138
2. P. Allatt and J. B. Slater, *Congruences on some special modular forms*, J. London Math. Soc. (2) **17** (1978), no. 3, 380–392. MR 0491502
3. Nickolas Andersen and Paul Jenkins, *Divisibility properties of coefficients of level p modular functions for genus zero primes*, Proc. Amer. Math. Soc. **141** (2013), no. 1, 41–53. MR 2988709
4. Tom Apostol, *Modular functions and dirichlet series in number theory*, 2 ed., vol. 41, Springer-Verlag, 1990.
5. Alex Fink, Richard Guy, and Mark Krusemeyer, *Partitions with parts occurring at most thrice*, Contrib. Discrete Math. **3** (2008), no. 2, 76–114. MR 2458135
6. Michael Griffin, *Divisibility properties of coefficients of weight 0 weakly holomorphic modular forms*, Int. J. Number Theory **7** (2011), no. 4, 933–941. MR 2812644
7. Paul Jenkins and D. J. Thornton, *Congruences for coefficients of modular functions*, Ramanujan J. **38** (2015), no. 3, 619–628. MR 3423017
8. O. Kolberg, *The coefficients of $j(\tau)$ modulo powers of* 3, Arbok Univ. Bergen Mat.-Natur. Ser. **1962** (1962), no. 16, 7. MR 0158061
9. ———, *Congruences for the coefficients of the modular invariant $j(\tau)$*, Math. Scand. **10** (1962), 173–181. MR 0143735
10. Joseph Lehner, *Divisibility properties of the Fourier coefficients of the modular invariant $j(\tau)$*, Amer. J. Math. **71** (1949), 136–148. MR 0027801
11. ———, *Further congruence properties of the Fourier coefficients of the modular invariant $j(\tau)$*, Amer. J. Math. **71** (1949), 373–386. MR 0027802

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