On the toric ideals of matroids of fixed rank

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Abstract. In 1980 White conjectured that the toric ideal associated to a matroid is generated by quadratic binomials corresponding to symmetric exchanges.

We prove White's conjecture for high degrees (w.r.t. rank of a matroid). That is, we prove that for all matroids of fixed rank r, homogeneous parts of degree at least $c(r)$ of the corresponding toric ideals are generated by quadratic binomials corresponding to symmetric exchanges. This extends our previous result [**13**] confirming the conjecture 'up to saturation'.

1. Introduction

Let M be a matroid on the ground set E with the set of bases $\mathfrak{B}$ and the rank function $r : \mathcal{P}(E) \to \mathbb{N}$. We denote the rank of M, that is $r(E)$, simply by r.

For a fixed field $\mathbb{K}$ consider a $\mathbb{K}$-homomorphism φ_M between polynomial rings:

$$\varphi_M : \mathbb{K}[y_B : B \in \mathfrak{B}] \ni y_B \to \prod_{e\in B} x_e \in \mathbb{K}[x_e : e \in E].$$

The *toric ideal of a matroid* M, denoted by I_M, is the kernel of the map φ_M. For a representable matroid M the toric variety associated with the toric ideal I_M has a very nice embedding as a subvariety of a Grassmannian [**10**]. It is the closure of the torus orbit of the point of the Grassmannian corresponding to the matroid M. Furthermore, any closure of a torus orbit in the Grassmannian is of this form for some representable matroid M.

When an ideal is defined only by combinatorial means, one expects to have a combinatorial description of its set of generators. An attempt to achieve this description often leads to surprisingly deep combinatorial questions. White's conjecture is an example. In 1980 Neil White stated in fact a bunch of conjectures that describe generators of the toric ideal of a matroid with increasing accuracy.

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Conjecture 1 (White, [**23**]). *The toric ideal I_M of a matroid M is generated in degree* 2*.*

The family $\mathfrak{B}$ of bases of M satisfies *symmetric exchange property* (the reader is referred to [**18**] for background of matroid theory, and to [**14**] for other exchange properties). That is, for every bases B_1, B_2 and $e \in B_1 \setminus B_2$ there exists $f \in B_2 \setminus B_1$, such that both sets $B_1' = (B_1 \setminus e) \cup f$ and $B_2' = (B_2 \setminus f) \cup e$ are bases. In this case we say that the quadratic binomial $y_{B_1}y_{B_2} - y_{B_1'}y_{B_2'}$ *corresponds to symmetric exchange*. It is clear that such binomials belong to the ideal I_M.

Conjecture 2 (White, [**23**]). *The toric ideal I_M of a matroid M is generated by quadratic binomials corresponding to symmetric exchanges.*

Conjecture 3 (White, [**23**]). *The toric ideal I_M of a matroid M considered in the noncommutative polynomial ring $\mathbb{K}\langle y_B : B \in \mathfrak{B}\rangle$ is generated by quadratic binomials corresponding to symmetric exchanges.*

Conjecture 3, the strongest among White's conjectures describing generators of the ideal I_M, turned out to be equivalent to Conjecture 2 when considered for all matroids (see the discussion in Section 4 of [**13**]).

Since every toric ideal is generated by binomials, it is not hard to rephrase the above conjectures in the combinatorial language. Conjecture 1 asserts that if two multisets of bases of a matroid have equal union (as a multiset), then one can pass between them by a sequence of steps, in each step exchanging two bases for another two bases with the same union (as a multiset). In Conjecture 2 additionally each step corresponds to a symmetric exchange. In Conjecture 3 we take sequences of bases instead of multisets, and similarly each step corresponds to a symmetric exchange between consecutive bases. Actually, this is the original formulation due to White. We immediately see that the conjectures do not depend on the field $\mathbb{K}$.

White's conjectures are known to be true for many special classes of matroids: graphic matroids [**1**], strongly base orderable matroids [**13**] (so also for transversal matroids), sparse paving matroids [**3**], and for matroids of rank at most 3 [**12**] (see also other related papers [**2, 5, 11, 19, 20**]).

The first general result, i.e. valid for arbitrary matroids, confirmed White's Conjecture 2 'up to saturation'. Let $\mathfrak{m}$ be the ideal generated by all variables in the polynomial ring $S_M = \mathbb{K}[y_B : B \in \mathfrak{B}]$ (so-called *irrelevant ideal*). Recall that the ideal $I : \mathfrak{m}^\infty = \{a \in S_M : a\mathfrak{m}^n \subset I \text{ for some } n \in \mathbb{N}\}$ is called the *saturation* of an ideal I with respect to the ideal $\mathfrak{m}$. Notice that the ideal I_M, as a prime ideal, is saturated. Let J_M be the ideal generated by quadratic binomials corresponding to symmetric exchanges. Clearly, $J_M \subset I_M$ and Conjecture 2 asserts that the ideals J_M and I_M are equal. In the language of algebraic geometry it means that both ideals define the same affine scheme.

In [**13**] we prove that the saturations of I_M and J_M with respect to $\mathfrak{m}$ are equal. That is, in the geometric language, that both ideals define the same projective scheme. In particular, they have the same affine set of zeros, so Conjecture 2 holds on set-theoretic level. Recall that two homogeneous ideals have equal saturations with respect to the ideal generated by all variables if and only if their homogeneous parts are equal starting from some degree. Thus we can rephrase the above in the following way.

THEOREM 4 (Lasoń, Michałek, [**13**]). *Let M be a matroid. Homogeneous parts of degree at least $c(M)$ of the toric ideal I_M are generated by quadratic binomials corresponding to symmetric exchanges.*

Here we study toric ideals of matroids of fixed rank. We obtain several finiteness results leading together to the following main result, which can be described as: 'White's conjecture for high degrees with respect to the rank of a matroid'.

THEOREM 5. *Let M be a matroid of rank r. Homogeneous parts of degree at least $c(r)$ of the toric ideal I_M are generated by quadratic binomials corresponding to symmetric exchanges.*

Namely, the degree bound from which ideals I_M and J_M agree, depends only on the rank of a matroid. That is, for an infinite class of matroids of fixed rank it is constant.

As a first step, in Section 3, we bound in terms of the rank of a matroid the degree in which the corresponding toric ideal is generated.

THEOREM 6. *The toric ideal I_M of a matroid M of rank r is generated in degree at most $(r+3)!$.*

Further, White's conjectures for matroids of fixed rank become finite problems.

COROLLARY 7. *Checking if Conjecture 1, 2 or 3 is true for matroids of fixed rank is decidable (it is enough to check connectivity of a finite number of graphs).*

The main part in the proof of Theorem 4 is [**13**, Claim 4]. It asserts that if $b \in I_M$ is a binomial of degree n, then for every variable y_B we have $y_B^{rn} b \in J_M$. Suppose the ideal I_M is generated in degree d. If $b \in I_M$ is a binomial of degree at least $d + rd|\mathfrak{B}|$, then $b = a_1b_1 + \cdots + a_kb_k$ where b_i are generators of I_M of degree d, and a_i are monomials of degree at least $rd|\mathfrak{B}|$. From the pigeon hole principle, every monomial a_i contains some variable y_B in degree at least rd, hence by [**13**, Claim 4] $a_ib_i \in J_M$, and finally $b \in J_M$. Therefore, the constant $c(M)$ from Theorem 4 is at most $d + rd|\mathfrak{B}|$.

By Theorem 6 we have a bound $d \leq (r+3)!$. But, the size of the set of bases $|\mathfrak{B}|$ can not be bounded for matroids of rank r. Also, we have to be able to generate by quadratic binomials corresponding to symmetric exchanges binomials $y_{B_1} \cdots y_{B_n} - y_{B'_1} \cdots y_{B'_n} \in I_M$ of high degree with respect to the rank ($n \gg r$) for which bases $B_1, \ldots, B_n$ are pairwise disjoint. For them there is no hope for a single variable in high degree, as every variable can appear in degree at most one.

To overcome this difficulty, in Section 4, we introduce a Ramsey-type result for blow-ups of bases. It asserts that if a matroid contains sufficiently many disjoint bases, then it contains an arbitrarily large k-th blow-up of a basis – a matroid obtained by replacing every element of a basis by k parallel elements. Moreover, if we modify this bases by only symmetric exchanges, then we can guarantee that this k-th blow-up agrees with some k bases. This allows us to 'reveal' a single variable in high degree in any monomial $y_{B_1} \cdots y_{B_n}$ of sufficiently large degree.

Having these three ingredients – [**13**, Claim 4], Theorem 6, and a Ramsey-type result for blow-ups of bases, we finally prove Theorem 5 in the last Section 5. Notice that by the discussion after Remark 15 from [**13**] we can deduce the following.

REMARK 8. *Theorems 5, 6 and Corollary 7 are true for discrete polymatroids.*

2. Graphs on bases of a matroid

This section contains preliminaries, in particular notions used throughout the paper. We discuss here how White's conjectures translate into problems on graphs on bases of a matroid.

We say that two bases of a matroid are *neighboring* if one is obtained from the other by a symmetric exchange. That is, if their symmetric difference has exactly two elements. A graph on bases of a matroid M with edges between neighboring bases is called the *basis graph* of M, and denoted by $\mathfrak{B}(M)$. Basis graphs have been studied in 1960s and 1970s, and they are well understood. In particular, basis graphs are Hamiltonian (with only two trivial exceptions), even a characterization is known (see [**15, 16, 4**] and references within).

For $k \geq 1$, a *k-matroid* is a matroid whose ground set can be partitioned into k pairwise disjoint bases. We call a basis of a k-matroid *complementary* if its complement can be partitioned into $k-1$ pairwise disjoint bases. That is, when it is an element of some partition of the ground set into bases. When $\mathfrak{B}$ is the set of bases of a k-matroid, then we denote the set of complementary bases by $\mathfrak{B}^c$.

We recall one of the versions of the matroid union theorem, which will be used several times in this paper. It characterizes k-matroids in terms of rank function.

Theorem 9 (Nash-Williams [**17**], Edmonds [**7**]). *A matroid M is a k-matroid, if and only if for every $A \subset E$ the inequality $kr(A) \geq |A|$ holds, and $kr(E) = |E|$.*

Blasiak [**1**] proposed a very nice and simple translation of the problem of generating the toric ideal of a matroid to the problem of connectivity of some graphs naturally associated to k-matroids. We are going to use this approach for the proof of Theorem 6 and Corollary 7. Following Blasiak, for $k \geq 3$ the *k-base graph* of a k-matroid M, denoted by $\mathfrak{B}_k(M)$, is a graph on sets of k pairwise disjoint bases of M (partitions of the ground set into bases), where edges join vertices with nonempty intersection. That is, sets of bases $\{B_1, \ldots, B_k\}$ and $\{B'_1, \ldots, B'_k\}$ are connected in $\mathfrak{B}_k(M)$ if for some i, j the equality $B_i = B'_j$ holds. Recall that if $\{e, f\}$ is a circuit in a matroid M, then elements e and f are said to be *parallel*. In this case B is a basis of M containing e if and only if $(B \cup f) \setminus e$ is a basis of M containing f. Via this property one can add to a matroid elements parallel to a fixed element (enlarging its ground set), or remove them. Notice that the reflexive closure of being parallel is an equivalence relation. A simple corollary of the proof of Proposition 2.1 from [**1**] gives the following.

Proposition 10 (Blasiak, [**1**]). *Let $\mathfrak{C}$ be a class of matroids that is closed under deletions that do not lower the rank of a matroid, and adding parallel elements. Then the following conditions are equivalent:*

(1) *for every $k > d$ and for every k-matroid M in $\mathfrak{C}$ the k-base graph $\mathfrak{B}_k(M)$ is connected,*
(2) *for every matroid M in $\mathfrak{C}$ the ideal I_M is generated in degree at most d.*

In particular, in order to prove Conjecture 1 it is enough to show that for every $k > 2$ and for every k-matroid M the k-base graph $\mathfrak{B}_k(M)$ is connected.

Here we propose another approach to White's conjecture. Consider other graphs that can be naturally associated to k-matroids. The *complementary basis graph* of a k-matroid, denoted by $\mathfrak{B}^c(M)$, is a graph on complementary bases of M with edges between neighboring bases. That is, the complementary basis graph of a k-matroid is the restriction of its basis graph to complementary bases $\mathfrak{B}^c(M) = \mathfrak{B}(M)|_{\mathfrak{B}^c}$.

Graphs $\mathfrak{B}^c(M)$ have been already studied for 2-matroids M. In 1985 Farber, Richter and Shank [**9**] proved that for a graphic 2-matroid M the graph $\mathfrak{B}^c(M)$ is connected, they also conjectured connectivity for arbitrary 2-matroids. In [**1**] after the proof of Proposition 2.1 Blasiak observes the following easy equivalence.

PROPOSITION 11 (Blasiak, [**1**]). *Let $\mathfrak{C}$ be a class of matroids that is closed under deletions that do not lower the rank of a matroid, and adding parallel elements. Then the following conditions are equivalent:*

(1) *for every 2-matroid M in $\mathfrak{C}$ the complementary basis graph $\mathfrak{B}^c(M)$ is connected,*
(2) *for every matroid M in $\mathfrak{C}$, elements of degree 2 in I_M considered in the noncommutative polynomial ring $\mathbb{K}\langle y_B : B \in \mathfrak{B}\rangle$ are generated by quadratic binomials corresponding to symmetric exchanges.*

We state the following two conjectures strongly related to White's conjectures.

CONJECTURE 12. *Complementary basis graph of a k-matroid is connected.*

CONJECTURE 13. *Let $k \geq 2$, and let M be a matroid of rank r on the ground set E of size $kr+1$. Suppose $x, y \in E$ are two elements such that both sets $E \setminus x$ and $E \setminus y$ can be partitioned into k pairwise disjoint bases. Then there exist partitions of $E \setminus x$ and $E \setminus y$ into k pairwise disjoint bases which share a common basis.*

We learned from Joseph Bonin that Conjecture 13 for $k = 2$ was studied in 1980s by Paul Seymour and Neil White, but it was not resolved.

PROPOSITION 14. *Let $\mathfrak{C}$ be a class of matroids that is closed under deletions that do not lower the rank of a matroid, and adding parallel elements. Then, considered for all matroids in $\mathfrak{C}$, the following implications between conjectures hold:*

(1) *the strongest White's Conjecture 3 implies complementary basis graph Conjecture 12,*
(2) *conjunction of complementary basis graph Conjecture 12 and Conjecture 13 implies the strongest White's Conjecture 3,*
in general, for every $d \geq 2$ conjunction of Conjecture 12 for k-matroids for $k > d$ and Conjecture 13 for $k \geq d$ implies that the toric ideal of a matroid, considered in the noncommutative polynomial ring, is generated in degree at most d.

PROOF. We begin with implication (1). Let M be a k-matroid in $\mathfrak{C}$, and let B_1, B_1' be complementary bases in M. So, there exist bases $B_2, \dots, B_k$, $B_2', \dots, B_k'$ such that entries of the sequences $\mathcal{A} = (B_1, \dots, B_k)$ and $\mathcal{A}' = (B_1', \dots, B_k')$ form partitions of the ground set E. Then $b = y_{B_1} \cdots y_{B_k} - y_{B_1'} \cdots y_{B_k'} \in I_M$, or equivalently sequences of bases $\mathcal{A}$ and $\mathcal{A}'$ have equal union (as a multiset). By the assumption, we can generate b using quadratic binomials corresponding to symmetric exchanges, or equivalently we can pass between $\mathcal{A}$ and $\mathcal{A}'$ by a sequence of steps, in each step making a symmetric exchange. Notice that all bases appearing during this process are complementary bases of M. Observe that the first bases of two sequences joined by a single step are either the same or neighboring. Thus we get a path in $\mathfrak{B}^c(M)$ between B_1 and B_1'.

For the implication (2), by Propositions 10 and 11 it is enough to show that for every $k \geq 3$ and for every k-matroid M in $\mathfrak{C}$ the k-base graph $\mathfrak{B}_k(M)$ is connected, and for every 2-matroid M in $\mathfrak{C}$ the complementary basis graph $\mathfrak{B}^{\mathfrak{c}}(M)$ is connected. The second part we get directly from complementary basis graph Conjecture 12. For the first part, let M be a k-matroid in $\mathfrak{C}$ (for $k \geq 3$) and let $\{B_1, \dots, B_k\}$, $\{B_1', \dots, B_k'\}$ be two vertices in $\mathfrak{B}_k(M)$. Since $\mathfrak{B}^{\mathfrak{c}}(M)$ is connected, it is enough to show that vertices of $\mathfrak{B}_k(M)$ containing neighboring bases are connected in $\mathfrak{B}_k(M)$. If the symmetric difference of B_1 and B_1' is $\{x, y\}$, then consider the restriction of M to the set $E \setminus (B_1 \cap B_1')$. This matroid satisfies assumptions of Conjecture 13 with points x, y. Thus in $\mathfrak{B}_k(M)$ there are vertices $\{B_1, B_2'', \dots, B_k''\}, \{B_1', B_2''', \dots, B_k'''\}$ connected by an edge. The first one is connected by an edge with $\{B_1, \dots, B_k\}$, while the second with $\{B_1', \dots, B_k'\}$. We get that any two vertices $\{B_1, \dots, B_k\}$, $\{B_1', \dots, B_k'\}$ in $\mathfrak{B}_k(M)$ containing neighboring bases are connected by a path. □

Proposition 15. *If $k \geq 2^{r-1} + 1$, then Conjecture 13 holds.*

Proof. Proof by contradiction. Let $B_1, \dots, B_k$ be a partition of the set $E \setminus y$ into k pairwise disjoint bases. Without loss of generality $x \in B_1$. If the assertion is not true, then for each $i = 2, \dots, k$ the basis B_i can not be completed to a partition of $E \setminus x$ into k bases. Thus from the matroid union Theorem 9 it follows that for each $i = 2, \dots, k$ there is a set $A_i \subset E \setminus (x \cup B_i)$ such that $(k-1)r(A_i) < |A_i|$. On the other hand, since $E \setminus (y \cup B_i)$ has a partition into $k-1$ pairwise disjoint bases (namely $B_1, \dots, \hat{B}_i, \dots, B_k$), for every set $A \subset E \setminus (y \cup B_i)$ the inequality $(k-1)r(A) \geq |A|$ holds. Thus for each i we have $y \in A_i$, $r(A_i) = r(A_i \setminus y)$, and $(k-1)r(A_i \setminus y) = |A_i \setminus y|$. The last equality implies that for every basis B_j (for $j \neq i$) $|B_j \cap A_i| = r(A_i)$. Moreover, since there is equality in the inequality $(k-1)r(A_i \setminus y) \geq |A_i \setminus y|$, each A_i is closed in $E \setminus B_i$, and it is equal to the closure of $B_j \cap A_i$ (for $j \neq i$) in $E \setminus B_i$. Consider sets $B_1 \cap A_i \subset B_1 \setminus x$. None of them is empty, since otherwise $r(A_i) = 0$ and y would be a loop. Thus, since there are $k - 1 \geq 2^{r-1}$ of them, for some $i \neq j$ the equality $B_1 \cap A_i = B_1 \cap A_j$ holds. But since A_i is the closure of $B_1 \cap A_i$ in $E \setminus B_i$ and A_j is the closure of $B_1 \cap A_j$ in $E \setminus B_j$, we get that the set $A := A_i \cup A_j$ is the closure of $B_1 \cap A_i = B_1 \cap A_j$ in E, so it is closed, $y \in A$, and $|B_l \cap A| = r(A)$ for every $l = 1, \dots, k$. Therefore, $|A| = kr(A) + 1$ and $x \notin A$, which by the matroid union Theorem 9 contradicts the assumption that $E \setminus x$ can be partitioned into k pairwise disjoint bases. □

3. Degree bounds for generating the toric ideal of a matroid

By Hilbert's basis theorem the ideal I_M is finitely generated. However, it is not easy to give any explicit bound on degree in which it is generated. A bound follows from a more general theorem about toric ideals. Theorem 13.14 from [**21**] asserts that if a graded set $\mathcal{A} \subset \mathbb{Z}^d$ generates a normal semigroup, then the corresponding toric ideal $I_{\mathcal{A}}$ is generated in degree at most d. For a matroid M we consider a set $\mathcal{A}_M = \{\chi_B : B \in \mathfrak{B}\} \subset \mathbb{Z}^{|E|}$, where χ_B is a characteristic function of B in E. By [**22**, Theorem 1] the semigroup generated by $\mathcal{A}_M$ is normal (it is also an easy consequence of the matroid union theorem – Theorem 9). The toric ideal corresponding to $\mathcal{A}_M$ is the ideal I_M. Hence, the toric ideal of a matroid is generated in degree at most the size of its ground set.

If we fix the size of the ground set, then there are only finitely many matroids on it. So a common bound is not surprising. But, when we fix only the rank, then the number of matroids of that rank is infinite. Theorem 6 asserts that in this case there is also a common bound on the degree. In order to prove it we need the following structural statement.

THEOREM 16. *Let M be a k-matroid of rank r. Then among every $k-s$ pairwise disjoint bases in M there are at most $r(r+2)!+s(r+1)!$ not complementary bases.*

PROOF. We prove by induction on r, that among every $k-s$ pairwise disjoint bases there are at most $r(r+2)!+s((r+1)!-2)$ not complementary bases. When $r=1$, then the statement becomes trivial. Suppose $r\geq 2$, and fix k,s. Let $B_1,\dots,B_k$ be disjoint bases of the k-matroid M. Their union is the whole ground set E. Let $D_1,\dots,D_{k-s}$ be arbitrary pairwise disjoint bases in M. If every D_j is complementary, then the assertion clearly holds. So, we can assume that some basis D_j is not complementary.

Due to the matroid union theorem – Theorem 9, there exists a set $A\subset E\setminus D_j$, such that

$$(k-1)r(A)<|A|.$$

Of course $0<r(A)<r$. Indeed, otherwise either A would have to be empty (in a k-matroid M there are no loops) and we would have $0<0$, or we would have $(k-1)r<|A|\leq|E\setminus D_j|=(k-1)r$.

Let $A_i=A\cap B_i$ for every $i=1,\dots,k$. Since every B_i is a basis, inequalities $|A_i|\leq r(A)$ hold. And, all together

$$(k-1)r(A)<|A|=|A_1|+\cdots+|A_k|\leq kr(A).$$

Therefore for every i, except at most $r(A)-1\leq r$, we have $|A_i|=r(A)$. Without loss of generality the equality holds for $i=1,\dots,k-r$.

Let $E'=B_1\cup\cdots\cup B_{k-r}$ and let $A'=A_1\cup\cdots\cup A_{k-r}$. We are going to reduce the problem to the $(k-r)$-matroid $M'=M|_{E'}$ (restriction of M to the set E'), and then use the set A' to split it into smaller instances – for $M'|_{A'}$ (restriction of M' to A') and for M'/A' (contraction of A' in M').

Notice that there are at most r^2 bases among bases D_i which have non-empty intersection with $B_{k-r+1}\cup\cdots\cup B_k$. Thus, without loss of generality, bases $D_1,\dots,D_{k-s-r^2}$ are contained in E'.

Let $C_i=A'\cap D_i$ for $i=1,\dots,k-s-r^2$. Since every D_i is a basis, inequalities $|C_i|\leq r(A')=r(A)$ hold. In order to split the problem for $M'|_{A'}$ and M'/A' we need bases D_i satisfying $|C_i|=|A'\cap D_i|=r(A)$.

Since $D_1,\dots,D_{k-s-r^2}$ cover all except $(s+r^2)r$ elements of E we get

$$(k-s-r^2)r(A)-(s+r^2)r\leq(k-r)r(A)-(s+r^2)r=|A'|-(s+r^2)r\leq$$

$$\leq|A'\cap(D_1\cup\cdots\cup D_{k-s-r^2})|=|C_1|+\cdots+|C_{k-s-r^2}|\leq(k-s-r^2)r(A).$$

Therefore for every $i=1,\dots,k-s-r^2$, except at most $(s+r^2)r$, the equality $|C_i|=r(A)$ holds. Without loss of generality it holds for $i=1,\dots,k-(s+r^2)(r+1)$. Denote $s'=(s+r^2)(r+1)-r$. Now we can pass to the matroids $M'|_{A'}$ and M'/A'.

We have

(1) $(k-r)$-matroid $M'|_{A'}$ of rank $r(A)<r$ with $(k-r)-s'$ pairwise disjoint bases $D_1\cap A',\dots,D_{k-r-s'}\cap A'$, and
(2) $(k-r)$-matroid M'/A' of rank $r-r(A)<r$ with $(k-r)-s'$ pairwise disjoint bases $D_1\setminus A',\dots,D_{k-r-s'}\setminus A'$.

For both cases we use the inductive assumption. In the case (1) there are at most $r(A)(r(A)+2)! + s'(r(A)+1)! - 2s'$ bases not complementary in $M'|_{A'}$. In the case (2) there are at most $(r-r(A))(r-r(A)+2)! + s'(r-r(A)+1)! - 2s'$ bases not complementary in M'/A'.

Notice that if $D_i \cap A'$ is a basis complementary in $M'|_{A'}$ and $D_i \setminus A'$ is a basis complementary in M'/A', then D_i is a basis complementary in M. Therefore all together there are at most

$$t = (s'+r) + r(A)(r(A)+2)! + s'(r(A)+1)! - 2s'$$
$$+(r-r(A))(r-r(A)+2)! + s'(r-r(A)+1)! - 2s'$$

bases not complementary in M. Denote $r' = r(A)$ and recall that $0 < r' < r$. Now,

$$t = r + r'(r'+2)! + (r-r')(r-r'+2)! + (r^2(r+1)-r)(1+(r'+1)!-2+(r-r'+1)!-2) +$$
$$+ s(r+1)(1+(r'+1)!-2+(r-r'+1)!-2) \leq r(r+2)! + s((r+1)!-2).$$

It is easy to verify the last inequality, because for $s \geq 0$ and $r \geq 2$ we have both

$$s(r+1)(1+r!-2+2!-2) \leq s((r+1)!-2),$$
$$r + (r-1)(r+1)! + 3! + (r^2(r+1)-r)(1+r!-2+2!-2) \leq r(r+2)!. \qquad \square$$

Proof of Theorem 6. We apply Proposition 10 to the class of matroids of rank r, and $d = (r+3)!$. Clearly, this class is closed under deletions that do not lower the rank of a matroid, and adding parallel elements. Let $k > (r+3)!$, and let M be a k-matroid of rank r. Suppose that $\{B_1, \dots, B_k\}$, $\{D_1, \dots, D_k\}$ are two vertices in $\mathfrak{B}_k(M)$. At most r bases D_i intersect basis B_1, so without loss of generality bases $D_{r+1}, \dots, D_k$ do not intersect it. Hence, $D_{r+1}, \dots, D_k$ are $(k-1)-(r-1)$ disjoint bases in a $(k-1)$-matroid $M' = M|_{B_2 \cup \dots \cup B_k}$. By Theorem 16 applied to r and $s = r-1$ we get that at most $r(r+2)! + (r-1)(r+1)! \leq (r+3)! - r < k - r$ bases among $D_{r+1}, \dots, D_k$ are not complementary in M'. Hence, some basis D_i is complementary in M'. Let $G_1, \dots, G_{k-2}$ be a partition of its complement into bases. Now, vertices $\{B_1, \dots, B_k\}$ and $\{B_1, D_i, G_1, \dots, G_{k-2}\}$ are connected by an edge in $\mathfrak{B}_k(M)$. Also, vertices $\{B_1, D_i, G_1, \dots, G_{k-2}\}$ and $\{D_1, \dots, D_k\}$ are connected by an edge in $\mathfrak{B}_k(M)$. Therefore, the graph $\mathfrak{B}_k(M)$ is connected. We get even that its diameter is 2. By Proposition 10 the ideal I_M of a matroid M of rank r is generated in degree at most $(r+3)!$. $\square$

Proof of Corollary 7. Again using Proposition 10, in order to check if Conjecture 1 is true for matroids of rank r it is enough to check if for every k-matroid M of rank r (for every $k > 2$), the k-base graph $\mathfrak{B}_k(M)$ is connected. By Theorem 6 it is enough to consider k from the range $(r+3)! \geq k > 2$, since for $k > (r+3)!$ the statement is true. That is, the problem reduces to checking connectivity of a finite number of graphs.

To check if Conjectures 1 and 3 are equivalent for matroids of rank r, by Proposition 11 it suffices to check connectivity of a finite number of graphs.

Analogously, to check if Conjectures 1 and 2 are equivalent for matroids of rank r it suffices to check connectivity of graphs from Proposition 11 modified by adding an edge between every complementary basis B and its complement B^c (these are bases of a 2-matroid). This completes the proof of Corollary 7. $\square$

We also get a new class of discrete polymatroids for which White's Conjecture 1 is true (for an extension of White's conjectures to discrete polymatroids see [**11**]).

COROLLARY 17. *Let P be a discrete polymatroid which is a join of $c \geq \frac{1}{2}(r+3)!$ copies of a matroid M of rank r (a basis of P is a union, as a multiset, of c bases of M). Then the toric ideal I_P is generated in degree 2.*

PROOF. We will prove the following claim. Let P be a discrete polymatroid which is a join of c copies of a matroid M. Suppose that the toric ideal I_M is generated in degree at most $2c$. Then the toric ideal I_P is generated in degree 2.

Let $D_1,\dots,D_k,\tilde{D}_1,\dots,\tilde{D}_k$ be bases of P with $y_{D_1}\cdots y_{D_k} - y_{\tilde{D}_1}\cdots y_{\tilde{D}_k} \in I_P$. For $i=1,\dots,k$ and $j=1,\dots,c$ let B_i^j and $\tilde{B}_i^j$ be bases of M such that $D_i = \bigcup_j B_i^j$ and $\tilde{D}_i = \bigcup_j \tilde{B}_i^j$. Then $\prod_{i,j} y_{B_i^j} - \prod_{i,j} y_{\tilde{B}_i^j} \in I_M$.

When one exchanges bases B_i^j and $B_{i'}^{j'}$ between bases D_i and $D_{i'}$ of P, then the corresponding elements of I_P differ by an element generated in degree 2. Thus we can rearrange bases B_i^j (and $\tilde{B}_i^j$) into an arbitrary k multisets of c bases. Since I_M is generated in degree $2c$, one can pass between the multisets of bases $\{B_i^j : i,j\}$ and $\{\tilde{B}_i^j : i,j\}$ by a sequence of steps, in each step exchanging $2c$ bases for another $2c$ bases of the same union (as a multiset). We partition these $2c$ bases into an arbitrary 2 parts of c bases. Each part corresponds to a basis of P. This way we are able to pass between the multisets of bases $\{D_i\}_i$ and $\{\tilde{D}_i\}_i$ of P by a sequence of steps, in each step exchanging only 2 bases and preserving multiset union. $\square$

4. Ramsey-type results for blow-ups of bases

By *k-th blow-up of a matroid M* we mean a matroid obtained from M by replacing every element of its ground set E by k parallel elements. By *k-th blow-up of a set $A \subset E$ in M* we mean a matroid obtained from M by replacing every element of A by k parallel elements.

Let N, M be two matroids of the same rank r on the same ground set E. We say that N is a *submatroid* of M, if the complex of independent sets of N is a subcomplex of the complex of independent sets of M, or equivalently, if the set of bases of N is contained in the set of bases of M.

We define a convenient notion of morphisms between matroids. Let M and M' be two matroids of the same rank r on the corresponding ground sets E, E'. A *morphism* ψ from M to M' is a function $\psi : E \to E'$, such that if B' is a basis in M', then any choice of representatives of sets $\{\psi^{-1}(b')\}_{b' \in B'}$ forms a basis in M. That is, a function ψ is a morphism if M contains a compatible submatroid which is obtained from M' by replacing every element $e \in E'$ by $|\psi^{-1}(e)|$ parallel copies of e. In particular, there is a natural morphism from the k-th blow up of a matroid to the original matroid. Let us formulate the key observation.

OBSERVATION 18. *Suppose $\psi : M \to M'$ is a morphism between matroids that sends variables of $b \in I_M$ to variables (i.e. images of the corresponding bases are bases), so $\psi(b)$ makes sense. Then $\psi(b) \in J_{M'}$ implies $b \in J_M$.*

PROOF. Suppose $b = y_{B_1}\cdots y_{B_k} - y_{D_1}\cdots y_{D_k}$. The condition $\psi(b) \in J_{M'}$ means that one can modify the monomial $y_{\psi(B_1)}\cdots y_{\psi(B_k)}$ using quadratic binomials corresponding to symmetric exchanges in M' to get $y_{\psi(D_1)}\cdots y_{\psi(D_k)}$. But, every symmetric exchange in M' between bases $\psi(B_1), \psi(B_2)$ lifts to a symmetric exchange in M between B_1, B_2. Therefore, we can modify $y_{B_1}\cdots y_{B_k}$ using symmetric exchanges in M to get $y_{B'_1}\cdots y_{B'_k}$ such that $\psi(B'_i) = \psi(D_i)$. Since any choice of representatives of bases from M' forms bases in M, and $b \in I_M$, we can modify $y_{B'_1} \cdots y_{B'_k}$, step by step using symmetric exchanges, to get $y_{D_1} \cdots y_{D_k}$. $\square$

We say that k disjoint *bases* $B_1, \ldots, B_k$ of a matroid M *contain the k-th blow-up of the basis* B_1 if there exists a morphism $\psi : M|_{B_1 \cup \cdots \cup B_k} \to M|_{B_1}$, such that $\psi(B_1) = \cdots = \psi(B_k) = B_1$ and ψ is an identity on B_1. That is, if one can label the elements of $B_1 \cup \cdots \cup B_k$ with labels $l_1, \ldots, l_r$, elements of every basis B_i with distinct labels, such that every set of r elements of distinct labels is a basis in M.

Our Ramsey-type result asserts that if a matroid contains sufficiently many disjoint bases, then it contains an arbitrarily large k-th blow-up of a basis. Moreover, if we modify these bases by only symmetric exchanges, we can guarantee that this k-th blow-up agrees with some k bases.

LEMMA 19. *For every positive integers r, k there exists an integer $n = n(r, k)$, such that if M is an n-matroid of rank r with disjoint bases $B_1, \ldots, B_n$, then there exists a modification of these bases by symmetric exchanges to $B'_1, \ldots, B'_n$ from which one can pick k bases $B_{i_1}, \ldots, B_{i_k}$ that contain the k-th blow-up of basis B_{i_1}.*

PROOF. The proof goes by induction on the rank r. If $r = 1$, then M itself is the n-th blow-up of a basis. In particular, we do not need to make a modification and any k bases contain the k-th blow-up of a basis. Thus $n(1, k) = k$.

Suppose $r \geq 2$, and fix also a positive integer k. We will show that for

$$n = n(r, k) = rn(r-1, k) + r^{rn(r-1,k)} 2^{2^{rn(r-1,k)}} n(r-1, k)$$

the desired property holds. Denote $s = rn(r-1, k)$, and $t = n - s$.

Let M be an n-matroid of rank r with disjoint bases $B_1, \ldots, B_n$. Choose an element b_i in each basis B_i among $B_1, \ldots, B_s$. Consider symmetric exchanges between bases $B_1, \ldots, B_s$ and bases $B_{s+1}, \ldots, B_{s+t}$. For $j = s+1, \ldots, s+t$ and $i = 1, \ldots, s$ let $b_{j,i}$ be an element of B_j that exchanges symmetrically with $b_i \in B_i$.

Label elements of each B_j among $B_{s+1}, \ldots, B_{s+t}$ with distinct labels $l_1, \ldots, l_r$. Then each B_j gets a label from $\{l_1, \ldots, l_r\}^s$ which is a sequence of labels of $b_{j,1}, \ldots, b_{j,s}$. From the pigeon hole principle at least $2^{2^{rn(r-1,k)}} n(r-1, k)$ bases B_j have the same label. Without loss of generality (statement of the lemma is independent on the order of bases) they are $B_{s+1}, \ldots, B_{s+t'}$, for $t' = 2^{2^{rn(r-1,k)}} n(r-1, k)$. Now for $j = s+1, \ldots, s+t'$ and $i = 1, \ldots, s$ the label of $b_{j,i}$ is the same for all j's. So, we label B_i with it. Again, from the pigeon hole principle at least $n(r-1, k)$ bases B_i have the same label, without of loss of generality they are $B_1, \ldots, B_{n(r-1,k)}$. The label of $b_{j,i}$ is the same for all these i's. Thus, we can define $b_j := b_{j,i}$. Now, for every $i = 1, \ldots, n(r-1, k)$ and every $j = s+1, \ldots, s+t'$, $b_i \in B_i$ exchanges symmetrically with $b_j \in B_j$.

Consider matroids $M_j := (M/b_j)|_{B_1 \cup \cdots \cup B_{n(r-1,k)}}$ for $j = s+1, \ldots, s+t'$. Since there are at most $2^{2^{rn(r-1,k)}}$ matroids on the ground set $B_1 \cup \cdots \cup B_{n(r-1,k)}$ (of size $rn(r-1, k)$), there are at least $n(r-1, k)$ indices j for which M_j is the same, without loss of generality for $j = s+1, \ldots, s+n(r-1, k)$.

Now, we make a first modification of bases $B_1, \ldots, B_n$. We exchange $b_i \in B_i$ symmetrically with $b_{s+i} \in B_{s+i}$ for every $i = 1, \ldots, n(r-1, k)$, obtaining bases $D_1, \ldots, D_n$. Each D_i for $i = 1, \ldots, n(r-1, k)$ has a distinguished element d_i (former b_{s+i}), such that $d_i \in D_i$ exchanges symmetrically with $d_{i'} \in D_{i'}$ and matroids $N_i := (M/d_i)|_{(D_1 \setminus d_1) \cup \cdots \cup (D_{n(r-1,k)} \setminus d_{n(r-1,k)})}$ are the same (as restrictions of matroids M_j).

Consider $n(r-1,k)$-matroid N_i of rank $r-1$ with $n(r-1,k)$ disjoint bases $F_1 = D_1 \setminus d_1, \ldots, F_{n(r-1,k)} = D_{n(r-1,k)} \setminus d_{n(r-1,k)}$ (here we use the fact that $d_i \in D_i$ exchanges symmetrically with $d_{i'} \in D_{i'}$). From the inductive assumption it follows that there are disjoint bases $F'_1, \ldots, F'_{n(r-1,k)}$ obtained from $F_1, \ldots, F_{n(r-1,k)}$ by symmetric exchanges, among which there are k bases (without loss of generality) $F'_1, \ldots, F'_k$ that contain the k-th blow-up of a basis of rank $r-1$. That is, one can label the elements of $F'_1 \cup \cdots \cup F'_k$ with labels $l_1, \ldots, l_{r-1}$, each F'_j with distinct labels, such that every set of $r-1$ elements of distinct labels is a basis in N_i.

Notice that bases $B'_1 = F'_1 \cup d_1, \ldots, B'_{n(r-1,k)} = F'_{n(r-1,k)} \cup d_{n(r-1,k)}$ are obtained from bases $D_1, \ldots, D_{n(r-1,k)}$, hence also from bases $B_1, \ldots, B_{n(r-1,k)}$, by symmetric exchanges in M. Moreover, bases $B'_1, \ldots, B'_k$ contain the k-th blow-up of a basis of rank r. Namely, one can label the elements of $B'_1 \cup \cdots \cup B'_k$ with labels $l_1, \ldots, l_r$ (we use the former labeling of $F'_1 \cup \cdots \cup F'_k$, additionally elements d_i get label l_r), each basis with distinct labels, such that every set of r elements of distinct labels forms a basis in M. This proves the inductive assertion. □

We are going to prove a generalization of Lemma 19, which asserts that additionally the desired k-th blow-up can be compatible with a fixed subset of a matroid. For this purpose we will use Ramsey theory for hypergraphs. A result of Erdős [**8**] implies the following lemma (see [**6**] for possible generalizations).

Lemma 20. *For every integers r, k and c there exists an integer $R = R(r, k, c)$, such that if H is a c-colored (edges receive one of c colors) complete r-uniform r-partite hypergraph of size R (size of each part is R), then one can find in it a monochromatic complete subhypergraph H' of size k.*

Let $B_1, \ldots, B_k$ be disjoint bases of a matroid M on the ground set E. Denote $F = E \setminus (B_1 \cup \cdots \cup B_k)$. Furthermore, we say that *bases $B_1, \ldots, B_k$ contain the k-th blow-up of a basis B_1 in $B_1 \cup F$* if there exists a morphism $\psi : M|_{B_1 \cup \cdots \cup B_k \cup F} \to M|_{B_1 \cup F}$, such that $\psi(B_1) = \cdots = \psi(B_k) = B_1$ and ψ is an identity on $B_1 \cup F$.

Lemma 21. *For every positive integers r, k and nonnegative integer l there exists an integer $m = m(r, k, l)$, such that if M is a matroid of rank r on the ground set E, containing m disjoint bases $B_1, \ldots, B_m$ whose complement $F = E \setminus (B_1 \cup \cdots \cup B_m)$ is of size l, then there exists a modification of bases $B_1, \ldots, B_m$ by symmetric exchanges to bases $B'_1, \ldots, B'_m$ from which one can pick k bases $B'_{i_1}, \ldots, B'_{i_k}$ that contain the k-th blow-up of basis B_1 in $B_1 \cup F$.*

Proof. We will show that for $m = m(r, k, l) = n(r, R(r, k, 2^{r+l}))$ the desired property holds, where n, R are the functions from Lemmas 19 and 20.

Let M be a matroid of rank r on the ground set E, containing m disjoint bases $B_1, \ldots, B_m$ whose complement $F = E \setminus (B_1 \cup \cdots \cup B_m)$ has l elements. Due to Lemma 19 there exists a modification of bases $B_1, \ldots, B_m$ by symmetric exchanges to bases $B'_1, \ldots, B'_m$ from which one can pick $R := R(r, k, 2^{r+l})$ bases (without loss of generality) $B'_1, \ldots, B'_R$ that contain the R-th blow-up of basis B'_1. So, there exists a morphism $\psi : M|_{B'_1 \cup \cdots \cup B'_R} \to M|_{B'_1}$, such that $\psi(B'_1) = \cdots = \psi(B'_R) = B'_1$ and ψ is an identity on B'_1.

There is only one possible extension of the morphism ψ to $\psi' : M|_{B'_1 \cup \cdots \cup B'_R \cup F} \to M|_{B'_1 \cup F}$, such that ψ' is an identity on $B'_1 \cup F$. It is a morphism if for every $i = 1, \ldots, r-1$, for every i-element proper subset $S \subset B'_1$, and for every choice of representatives of sets $\{\psi^{-1}(b)\}_{b \in S}$, their union with a $(r-i)$-element subset

$T \subset F$ is a basis of M if and only if $S \cup T$ is a basis in M. We will show that, but for a smaller blow-up.

Consider a complete r-uniform r-partite hypergraph H with parts $\psi^{-1}(e)$ for $e \in B_1'$, each part of size R. Edges of H are bases in M, since ψ is a morphism from M. Define a 2^{r+l}-coloring c of edges of H. For each i-element proper subset $S \subset B_1'$, for each $(r-i)$-element subset $T \subset F$ and for an edge D of H, let the bit $c_{S,T}(D)$ of $c(D)$ be 1 if $(\psi^{-1}(S) \cap D) \cup T$ is a basis in M, and 0 otherwise.

Using Lemma 20 for H with coloring c we get a monochromatic subhypergraph H' of size k. Let $D_1, \dots, D_k$ be k disjoint edges of H' (they are also bases of M), and let $D_{k+1}, \dots, D_R$ be disjoint edges of H completing a partition of the ground set of H. Then, since H is a complete hypergraph, bases $D_1, \dots, D_R$ are obtained from $B_1', \dots, B_R'$ by symmetric exchanges. Hence also bases $D_1, \dots, D_R, B_{R+1}', \dots, B_m'$ are obtained from bases $B_1, \dots, B_m$ by symmetric exchanges. Directly from the construction, there exists a desired morphism $\psi : M|_{D_1 \cup \dots \cup D_k \cup F} \to M|_{D_1 \cup F}$. $\square$

5. Main result

Proof of Theorem 5. Let $c(r) = (r+3)! + m(r, r(r+3)!, r(r+3)!)$, where m is the function from Lemma 21.

Let M be a matroid of rank r. We have to show that $J_M^d = I_M^d$ for $d \geq c(r)$. The inclusion $J_M \subset I_M$ implies that $J_M^d \subset I_M^d$ for every d. To prove the opposite inclusion, let $b \in I_M$ be a binomial of degree $d \geq c(r)$. By Theorem 6 we have that $b = \sum_{i=1}^n a_i b_i$, where $b_i \in I_M$ is a binomial of degree $(r+3)!$ and a_i is a monomial of degree greater or equal to $m := m(r, r(r+3)!, r(r+3)!)$. We show that $ab \in J_M$, for every binomial $b \in I_M$ of degree $(r+3)!$ and every monomial a of degree m.

Suppose that $b = y_{D_1} \cdots y_{D_{(r+3)!}} - y_{G_1} \cdots y_{G_{(r+3)!}}$ and $a = y_{B_1} \cdots y_{B_m}$. Denote the union $D_1 \cup \dots \cup D_{(r+3)!} \cup B_1 \cup \dots \cup B_m$ by S as a set, and by (S, μ) as a multiset in which μ is the multiplicity function. Let M' be a matroid obtained from $M|_S$ by replacing every element e by $\mu(e)$ parallel elements. That is, there is a morphism $\psi' : M' \to M|_S$. Let $D_1', \dots, D_{(r+3)!}', B_1', \dots, B_m'$ be disjoint bases of M' such that $\psi'(D_i') = D_i$ and $\psi'(B_i') = B_i$. Let $G_1', \dots, G_{(r+3)!}'$ be disjoint bases of M' such that $D_1' \cup \dots \cup D_{(r+3)!}' = G_1' \cup \dots \cup G_{(r+3)!}'$ and $\psi'(G_i') = G_i$. Let b', a' be the corresponding polynomials. Clearly, $b' \in I_{M'}$. Observe that $ab \in J_M$ if and only if $a'b' \in J_{M'}$. Therefore, we need to show that $a'b' \in J_{M'}$.

Due to Lemma 21 applied for m disjoint bases $B_1', \dots, B_m'$ in M' whose complement $F = D_1' \cup \dots \cup D_{(r+3)!}'$ is of size $l = r(r+3)!$, there exists a modification of bases $B_1', \dots, B_m'$ by symmetric exchanges to bases $B_1'', \dots, B_m''$ among which $k = r(r+3)!$ bases (without loss of generality) $B_1'', \dots, B_k''$ contain the k-th blow-up of basis B_1'' in $B_1'' \cup F$. Let $\psi : M'|_{B_1'' \cup \dots \cup B_k'' \cup F} \to M'|_{B_1'' \cup F}$ be a morphism, such that $\psi(B_1'') = \dots = \psi(B_k'') = B_1''$ and ψ is an identity on $B_1'' \cup F$. Denote $a'' = y_{B_1''} \cdots y_{B_m''}$. Then, $a' - a'' \in J_{M'}$. Hence, it is enough to show that $a''b' \in J_{M'}$, or even that $y_{B_1''} \cdots y_{B_k''} b' \in J_{M'}$.

We use Observation 18 for morphism ψ and $f = y_{B_1''} \cdots y_{B_k''} b' \in I_M$. We have $\psi(y_{B_1''} \cdots y_{B_k''} b') = y_{B_1''}^{r(r+3)!} b'$, so it is enough to show that $y_{B_1''}^{r(r+3)!} b' \in J_{M'}$.

Finally, we use [**13**, Claim 4]. It asserts that for every basis B of M', if $b' \in I_{M'}$ is a binomial, then $y_B^{\deg_B(b) - \deg(b)} b' \in J_{M'}$ (where $\deg_B(y_{B'}) = |B' \setminus B|$). In our case $y_{B_1''}^{r(r+3)!} b' \in J_{M'}$, since the degree of b' is $(r+3)!$. This finishes the proof. $\square$

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