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HIGHER ORDER RIESZ TRANSFORMS ON NONCOMPACT SYMMETRIC SPACES

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Abstract. In this note we prove various sharp boundedness results on suitable Hardy type spaces for Riesz transforms of arbitrary order on noncompact symmetric spaces of arbitrary rank.

1. Introduction

Suppose that $\mathbb{X}$ is a Riemannian symmetric spaces of the noncompact type, and denote by ∇ the covariant derivative and by $\mathcal{L}$ the Laplace–Beltrami operator on $\mathbb{X}$. The purpose of this paper is to prove estimates on the Hardy type spaces $H^1(\mathbb{X})$ and $X^k(\mathbb{X})$ for the (higher order) *Riesz transform* $\mathcal{R}^d$, defined, for any positive integer d, by

$$\mathcal{R}^d = \nabla^d \mathcal{L}^{-d/2}, \tag{1.1}$$

and the *shifted Riesz transform* $\mathcal{R}^d_c$, where $c > 0$, defined by

$$\mathcal{R}^d_c = \nabla^d (\mathcal{L} + c)^{-d/2}. \tag{1.2}$$

Here $H^1(\mathbb{X})$ is the space introduced by A. Carbonaro, Mauceri and Meda in [5], and $X^k(\mathbb{X})$ denotes the space introduced in [17], and further investigated in the series of papers [18, 19, 20].

Riesz transforms on $\mathbb{X}$ have been the object of a number of investigations in the last forty years, or so. Without any pretence of exhaustiveness, we recall the works of J.-Ph. Anker and his collaborators [1, 2] and of N. Lohoué [13, 14]. For more on the analysis of Riesz transforms on a wider class of Riemannian manifolds with spectral gap and bounded geometry, see [18, 19, 20] and the references therein. ALTRO?

To the best of our knowledge, there are very few endpoint results for $p = 1$ for (higher order) Riesz transforms on nondoubling Riemannian manifolds. Nonshifted Riesz transforms of order d are known to be bounded from $L^p(\mathbb{X})$ to the space $L^p(\mathbb{X}; T^d)$ of all p-integrable covariant tensors of order d on $\mathbb{X}$ for every p in $(1, \infty)$ [1, 2]. See also [13], where a similar result is proved for Riesz transforms of even order on certain Cartan–Hadamard manifolds. Anker [1] also proved that the first and second order Riesz transforms are of weak type 1, and observed that this is no longer true of Riesz transforms of order ≥ 3, at least in the rank one case. The same is presumably true

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in any noncompact symmetric space. Anker's proof relies on very fine estimates of the heat kernel and its derivatives. So does the proof of our main result concerning $\mathcal{R}^d$. In particular, the key point in our argument is to obtain good estimates of the kernel of the operator $\nabla^d\mathcal{L}^{1/2}$ (see Lemma 3.3). We are not able to prove similar estimates on a more general class of manifolds: this is the reason for which we restrict our analysis to the case of noncompact symmetric spaces. We prove the following:

- (i) suppose that d is a positive integer. Then
  - (a) $\mathcal{R}_c^d$ is bounded from $H^1(\mathbb{X})$ to $L^1(\mathbb{X};T^d)$;
  - (b) $\mathcal{R}^d$ is bounded from $X^k(\mathbb{X})$ to $L^1(\mathbb{X};T^d)$ for every $k \geq \lfloor (d+1)/2 \rfloor$;
- (ii) $\mathcal{R}^1$ is unbounded from $H^1(\mathbb{X})$ to $L^1(\mathbb{X};T^1)$.

Part (i) is proved in Theorem 3.1 and part (ii) in Theorem 4.1. The remarkable difference between the boundedness properties of $\mathcal{R}_c^d$ and $\mathcal{R}^d$ on $H^1(\mathbb{X})$ and $X^k(\mathbb{X})$ has a simple explanation. Fix a base point o in $\mathbb{X}$. On the one hand, the "convolution kernels" of $\mathcal{R}_c^d$ and $\mathcal{R}^d$ have a similar behaviour in a neighbourhood of o, where they are homologous to a kernel of a standard singular integral operator. On the other hand, the kernel of $\mathcal{R}_c^d$ is integrable at infinity, whereas that of $\mathcal{R}^d$ is not. Furthermore, the greater the order d is, the slower decay the kernel of $\mathcal{R}^d$ has at infinity.

We emphasize that the results in this paper aim at corroborating the fact that $X^k(\mathbb{X})$ does serve as an effective counterpart on $\mathbb{X}$ of the classical Hardy space $H^1(\mathbb{R}^n)$, whereas the effectiveness of the space $H^1(\mathbb{X})$ is somewhat limited to operators whose kernels are integrable at infinity. It is important to keep in mind that the following strict continuous containments hold

$$H^1(\mathbb{X}) \supset X^1(\mathbb{X}) \supset X^2(\mathbb{X}) \supset \cdots \supset X^k(\mathbb{X}) \supset \cdots$$

There is a huge literature concerning Hardy type spaces on Riemannian manifolds, and sometimes a bit of confusion about their effectiveness. As already mentioned, we prove that shifted Riesz transforms of order d are bounded from $H^1(\mathbb{X})$ to $L^1(\mathbb{X},T^d)$. We would like to make it clear that we could have proved similar results involving Taylor's version of Goldberg's Hardy space $\mathfrak{h}^1(\mathbb{X})$ [23] or the space of Carbonaro, A. McIntosh and A. Morris [6] instead of the space $H^1(\mathbb{X})$. All these spaces may be used to give endpoint results for $p=1$ for operators with kernels that are integrable at infinity and behave, roughly speaking, as Calderón–Zygmund operators near the origin. However, none of these is apt to serve as an endpoint result for nonshifted Riesz transforms (of whatsoever order) in the setting of noncompact symmetric spaces. In our paper we show that the nonshifted Riesz transform $\mathcal{R}^1$ does not map $H^1(\mathbb{X})$ into $L^1(\mathbb{X})$. *A fortiori*, it cannot map Taylor's space $\mathfrak{h}^1(\mathbb{X})$, or the Carbonaro–McIntosh–Morris space, into $L^1(\mathbb{X})$.

The paper is organised as follows. Section 2 contains the basic notions of analysis on $\mathbb{X}$ and the definitions of the Hardy spaces $H^1(\mathbb{X})$ and $X^k(\mathbb{X})$. In Section 3 we prove the positive results for the Riesz transforms (see Theorem 3.1). Finally, in the last section we prove that the Riesz potentials $\mathcal{L}^{-\tau}$, $\tau > 0$, and the first order Riesz transform $\mathcal{R}^1$ are unbounded from $H^1(\mathbb{X})$ to $L^1(\mathbb{X})$.

We will use the "variable constant convention", and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards.

2. Preliminaries

2.1. **Preliminaries on symmetric spaces.** In this subsection we recall the basic notions of analysis on noncompact symmetric spaces that we shall need in the sequel. Our main references are the books [11, 12] and the papers [**?**, 1, 2]. For the sake of the reader we recall also the notation, which is quite standard.

We denote by G a noncompact connected real semisimple Lie group with finite centre, by K a maximal compact subgroup and by $\mathbb{X} = G/K$ the associated noncompact Riemannian symmetric space. The point $o = eK$, where e is the identity of G, is called the *origin* in $\mathbb{X}$. Let θ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan involution and Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We denote by Σ the restricted root system of $(\mathfrak{g}, \mathfrak{a})$ and by W the associated Weyl group. Once a positive Weyl chamber $\mathfrak{a}^+$ has been selected, Σ^+ denotes the corresponding set of positive roots, Σ_s the set of simple roots in Σ^+ and Σ_0^+ the set of positive indivisible roots. As usual, $\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ denotes the sum of the positive root spaces. Denote by m_α the dimension of $\mathfrak{g}_\alpha$ and set $\rho := (1/2)\sum_{\alpha\in\Sigma^+} m_\alpha\, \alpha$. We denote by $\mathbf{W}$ the interior of the convex hull of the points $\{w \cdot \rho : w \in W\}$. Clearly $\mathbf{W}$ is an open convex polyhedron in $\mathfrak{a}^*$. By $N = \exp \mathfrak{n}$ and $A = \exp \mathfrak{a}$ we denote the analytic subgroups of G corresponding to $\mathfrak{n}$ and $\mathfrak{a}$. The Killing form B induces the K–invariant inner product $\langle X, Y \rangle = -B\big(X, \theta(Y)\big)$ on $\mathfrak{p}$ and hence a G–invariant metric d on $\mathbb{X}$. The ball with centre $x \cdot o$ and radius r will be denoted by $B_r(o)$. The map $X \mapsto \exp X \cdot o$ is a diffeomorphism of $\mathfrak{p}$ onto $\mathbb{X}$. The distance of $\exp X \cdot o$ from the origin in $\mathbb{X}$ is equal to $|X|$, and will be denoted by $|\exp X \cdot o|$. We denote by n the dimension of $\mathbb{X}$ and by ℓ its rank, i.e. the dimension of $\mathfrak{a}$.

We identify functions on the symmetric space $\mathbb{X}$ with K–right-invariant functions on G, in the usual way. If $E(G)$ denotes a space of functions on G, we define $E(\mathbb{X})$ and $E(K\backslash\mathbb{X})$ to be the closed subspaces of $E(G)$ of the K–right-invariant and the K–bi-invariant functions, respectively. If $D = Z_1 Z_2 \cdots Z_d$, with $Z_i \in \mathfrak{g}$, then we denote by $Df(x)$ the right differentiation of f at the point x in G. Thus,

$$Df(x) = \frac{\partial^d}{\partial t_1 \cdots \partial t_d} f\big(x \exp(t_1 Z_1) \cdots \exp(t_d Z_d)\big)\Big|_{t_1=\ldots=t_d=0}.$$

We write $\mathrm{d}x$ for a Haar measure on G, and let $\mathrm{d}k$ be the Haar measure on K of total mass one. The Haar measure of G induces a G–invariant measure μ on $\mathbb{X}$ for which

$$\int_{\mathbb{X}} f(x \cdot o)\, \mathrm{d}\mu(x \cdot o) = \int_G f(x)\, \mathrm{d}x \qquad \forall f \in C_c(\mathbb{X}).$$

We shall often write $\big|E\big|$ instead of $\mu(E)$ for a measurable subset E of $\mathbb{X}$. We recall that

$$\int_G f(x)\,\mathrm{d}x = \int_K \int_{\mathfrak{a}^+} \int_K f\big(k_1 \exp H\, k_2\big)\, \delta(H)\, \mathrm{d}k_1\, \mathrm{d}H\, \mathrm{d}k_2\,, \tag{2.1}$$

where $\mathrm{d}H$ denotes a suitable nonzero multiple of the Lebesgue measure on $\mathfrak{a}$, and

$$\delta(H) = \prod_{\alpha\in\Sigma^+} \big(\sinh \alpha(H)\big)^{m_\alpha} \leq C\, \mathrm{e}^{2\rho(H)} \qquad \forall H \in \mathfrak{a}^+. \tag{2.2}$$

We recall the Iwasawa decomposition of G, which is $G = K\,A\,N$. For every x in G we denote by $H(x)$ the unique element of $\mathfrak{a}$ such that $x \in K \exp H(x) N$. For any linear form $\lambda:\mathfrak{a}\to\mathbb{C}$, the elementary spherical function φ_λ is defined by the rule

$$\varphi_\lambda(x)=\int_K \mathrm{e}^{-(i\lambda+\rho)H(x^{-1}k)}\,\mathrm{d}k \qquad \forall x\in G.$$

In the sequel we shall use the following estimate of the spherical function φ_0 [2, Proposition 2.2.12]:

$$\varphi_0(\exp H\cdot o)\leq (1+|H|)^{|\Sigma_0^+|}\,\mathrm{e}^{-\rho(H)} \qquad \forall H\in\mathfrak{a}^+. \tag{2.3}$$

The spherical transform $\mathcal{H}f$ of an $L^1(G)$ function f, also denoted by $\widetilde{f}$, is defined by the formula

$$\mathcal{H}f(\lambda)=\int_G f(x)\,\phi_{-\lambda}(x)\,\mathrm{d}x \qquad \forall\lambda\in\mathfrak{a}^*.$$

Harish-Chandra's inversion formula and Plancherel formula state that for "nice" K–bi-invariant functions f on G

$$f(x)=\int_{\mathfrak{a}^*}\widetilde{f}(\lambda)\,\phi_\lambda(x)\;\mathrm{d}\nu(\lambda) \qquad \forall x\in G \tag{2.4}$$

and

$$\|f\|_2=\left[\int_{\mathfrak{a}^*}|\widetilde{f}(\lambda)|^2\;\mathrm{d}\nu(\lambda)\right]^{1/2} \qquad \forall f\in L^2(K\backslash G/K),$$

where $\mathrm{d}\nu(\lambda)=c_{_G}|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda$, and $\mathbf{c}$ denotes the Harish-Chandra $\mathbf{c}$-function. We do not need the exact form of $\mathbf{c}$. It will be enough to know that there exists a constant C such that

$$|\mathbf{c}(\lambda)|^{-2}\leq C\,\big(1+|\lambda|\big)^{n-\ell}, \tag{2.5}$$

[11, IV.7].

Next, we recall the Cartan decomposition of G, which is $G=K\exp\overline{\mathfrak{a}^+}K$. In fact, for almost every x in G, there exists a unique element $A^+(x)$ in $\mathfrak{a}^+$ such that x belongs to $K\exp A^+(x)K$.

Lemma 2.1. *The map $A^+:G\to\mathfrak{a}$ is Lipschitz with respect to both left and right translations of G. More precisely*

$$|A^+(yx)-A^+(y)|\leq\; d(x\cdot o,o) \qquad \textit{and} \qquad |A^+(xy)-A^+(y)|\leq\; d(x\cdot o,o)\,,$$

for all x and y in G.

Proof. The first inequality follows from $|A^+(yx)-A^+(y)|\leq\; d(yx\cdot o,y\cdot o)$, see [2, Lemma 2.1.2], and the G–invariance of the metric d on $\mathbb{X}$.

The second inequality follows from the first, for $A^+(x^{-1})=-\sigma\,A^+(x)$, where σ is the element of the Weyl group that maps the negative Weyl chamber $-\mathfrak{a}^+$ to the positive Weyl chamber $\mathfrak{a}^+$. □

For every positive r we define

$$\mathfrak{b}_r=\{H\in\mathfrak{a}:|H|\leq r\} \qquad \text{and} \qquad B_r=K(\exp\mathfrak{b}_r)K. \tag{2.6}$$

The set B_r is the inverse image under the canonical projection $\pi:G\to\mathbb{X}$ of the ball $B_r(o)$ in the symmetric space $\mathbb{X}$. Thus, a function f on $\mathbb{X}$ is supported in $B_r(o)$ if and only if, as a K–right-invariant function on G, is supported in B_r.

2.2. **Hardy spaces on $\mathbb{X}$.** In this subsection we briefly recall the definitions and properties of $H^1(\mathbb{X})$ and $X^k(\mathbb{X})$. For more about $H^1(\mathbb{X})$ and $X^k(\mathbb{X})$ we refer the reader to [5] and [17, 18, 19], respectively.

Definition 2.2. An H^1-*atom* is a function a in $L^2(\mathbb{X})$, with support contained in a ball B of *radius at most* 1, and such that

(i) $\int_B a \,\mathrm{d}\mu = 0$;
(ii) $\|a\|_2 \leq \big|B\big|^{-1/2}$.

Definition 2.3. The *Hardy space* $H^1(\mathbb{X})$ is the space of all functions g in $L^1(\mathbb{X})$ that admit a decomposition of the form

$$g = \sum_{j=1}^{\infty} c_j\, a_j, \tag{2.7}$$

where a_j is an H^1-atom, and $\sum_{j=1}^{\infty} |c_j| < \infty$. Then $\|g\|_{H^1}$ is defined as the infimum of $\sum_{j=1}^{\infty} |c_j|$ over all decompositions (2.7) of g.

Remark 2.4. A straightforward consequence of [17, Lemma 5.7] that we shall use repeatedly in the sequel is the following. If f is in $L^2(\mathbb{X})$, its support is contained in $B_R(o)$ for some $R > 1$, and its integral vanishes, then f is in $H^1(\mathbb{X})$, and

$$\big\|f\big\|_{H^1} \leq C\, R\, \big|B_R(o)\big|^{1/2}\, \big\|f\big\|_2\,.$$

The Hardy type spaces $X^k(\mathbb{X})$ were introduced in [17] as certain Banach spaces isometrically isomorphic to $H^1(\mathbb{X})$. An atomic characterisation of $X^k(\mathbb{X})$ was then established in [18], and refined in [19]. In this paper we adopt the latter as the definition of $X^k(\mathbb{X})$. We say that a (smooth) function Q on $\mathbb{X}$ is k–quasi-harmonic if $\mathcal{L}^k Q$ is constant on $\mathbb{X}$.

Definition 2.5. Suppose that k is a positive integer. An X^k-*atom* is a function A, with support contained in a ball B of *radius at most* 1, such that

(i) $\int_{\mathbb{X}} A\, Q \,\mathrm{d}\mu = 0$ for every k–quasi-harmonic function Q;
(ii) $\|A\|_2 \leq \big|B\big|^{-1/2}$.

Note that condition (i) implies that $\int_{\mathbb{X}} A \,\mathrm{d}\mu = 0$, because the constant function 1 is k–quasi-harmonic on $\mathbb{X}$.

Definition 2.6. The space $X^k(\mathbb{X})$ is the space of all functions F of the form $\sum_j c_j\, A_j$, where A_j are X^k-atoms and $\sum_j |c_j| < \infty$, endowed with the norm

$$\|F\|_{X^k} = \inf\big\{\sum_j |c_j| : F = \sum_j c_j\, A_j, \quad \text{where } A_j \text{ is an } X^k\text{-atom}\big\}.$$

2.3. **Estimate of operators.** We shall encounter various occurrences of the problem of estimating the $H^1(\mathbb{X})$ norm of functions of the form $a * \gamma$, where a is an $H^1(\mathbb{X})$-atom with support in $B_R(o)$ for some $R \leq 1$, and γ is a K–bi-invariant function with support contained in the ball $\overline{B_\beta(o)}$. The following lemma contains a version of such an estimate that we shall use frequently in the sequel. For the proof, see []. DECIDERE A COSA RIFERIRSI

Lemma 2.7. *Suppose that a and γ are as above. The following hold:*

(i) *there exists a constant C such that*

$$\|a*\gamma\|_{H^1} \leq \begin{cases} \big|B_{R+\beta}(o)\big|^{1/2} \min\big(\,\|\gamma\|_2\,, C\,R\,\,\|\nabla\gamma\|_2\,\big) & \text{if } R+\beta \leq 1 \\ C\,(R+\beta)\,\big|B_{R+\beta}(o)\big|^{1/2}\,\,\|\gamma\|_2 & \text{if } R+\beta > 1; \end{cases}$$

(ii) *suppose further that γ is of the form $\mathcal{A}^{-1}(\Phi\mathcal{A}\kappa)$, where Φ is a smooth function with compact support, and define $s := (n-\ell)/2$. Then there exists a constant C such that*

$$\big\|\mathcal{A}^{-1}(\Phi\mathcal{A}\kappa)\big\|_2 \leq C\,\,\|\Phi\mathcal{A}\kappa\|_{H^s(\mathfrak{a})}$$

and

$$\big\|\nabla\big[\mathcal{A}^{-1}(\Phi\mathcal{A}\kappa)\big]\big\|_2 \leq C\,\,\|\Phi\mathcal{A}\kappa\|_{H^{s+1}(\mathfrak{a})}\,,$$

where $H^s(\mathfrak{a})$ denotes the standard Sobolev space of order s on $\mathfrak{a}$.

3. Riesz transforms

Our analysis of Riesz transforms may be reduced to that of certain operators, called scalar Riesz transforms, which are convolution operators whose kernels are smooth functions on $\mathbb{X}\setminus\{o\}$. To describe these kernels we need more notation.

For y in G we denote by $L(y)$ left translation by y acting on G, by $\mathrm{d}L(y)$ and by $L(y)^*$ the differential and the pull-back of $L(y)$ acting on tangent vectors and covariant tensors on G, respectively. With a slight abuse of notation we shall also denote by $L(y)$, $\mathrm{d}L(y)$ and $L(y)^*$ the corresponding maps, acting on $\mathbb{X}$, on tangent vectors and on covariant tensors on $\mathbb{X}$. Thus $L(y)^*$ is an isometry between covariant tensors at the point $y\cdot o$ to covariant tensors at the point o. We recall that the tangent space of $\mathbb{X}$ at the point o is identified with $\mathfrak{p}$ and the space of covariant tensors of order d at the point o is identified with $(\mathfrak{p}^*)^{\otimes d}$.

For every $\mathcal{Z}=(Z_1,\ldots,Z_d)$ in $\mathfrak{p}^d$ the *scalar shifted Riesz transform* $\mathcal{R}^d_{c,\mathcal{Z}}$ of order d is the operator defined by

$$\mathcal{R}^d_{c,\mathcal{Z}}f(x\cdot o) = L(x)^*\big[\mathcal{R}^d_c f(x\cdot o)\big](Z_1,\ldots,Z_d) \qquad \forall x\in G. \tag{3.1}$$

It is well known that $\mathcal{R}^d$ is bounded from $L^2(\mathbb{X})$ to $L^2(\mathbb{X};T^d)$ [22, 3]. A straightforward argument shows that the same is true of $\mathcal{R}^d_c$, and, consequently, $\mathcal{R}^d_{c,\mathcal{Z}}$ is bounded on $L^2(\mathbb{X})$ for every $\mathcal{Z}$ in $\mathfrak{p}^d$. Our endpoint result for Riesz transforms is the following.

Theorem 3.1. *Suppose that d is a positive integer and $c>0$. The following hold:*

(i) *for every $\mathcal{Z}$ in $\mathfrak{p}^d$ the operator $\mathcal{R}^d_{c,\mathcal{Z}}$ extends to a bounded operator on $H^1(\mathbb{X})$;*
(ii) *$\mathcal{R}^d_c$ extends to bounded operator from $H^1(\mathbb{X})$ to $L^1(\mathbb{X};T^d)$;*
(iii) *$\mathcal{R}^d$ extends to a bounded operator from $X^{\lfloor (d+1)/2\rfloor}(\mathbb{X})$ to $L^1(\mathbb{X};T^d)$.*

For every $z\in\mathbb{C}$ and $c\geq 0$, the Bessel–Riesz potential $(\mathcal{L}+c)^{-z/2}$ maps the space of test functions $\mathcal{D}(\mathbb{X})$ into the space of distributions $\mathcal{D}'(\mathbb{X})$. If $z\neq 0,-2,-4,\ldots$, then its convolution kernel κ^z_c is a distribution, which, away from the origin o, coincides with the function

$$\kappa^z_c(x\cdot o) = \frac{1}{\Gamma(z/2)}\int_0^\infty t^{z/2-1}\,\mathrm{e}^{-ct}\,h_t(x\cdot o)\,\mathrm{d}t, \tag{3.2}$$

where h_t denotes the heat kernel on $\mathbb{X}$. In the sequel we shall use repeatedly the estimates for κ^z_c and their derivatives obtained by Anker and Ji [2, Thm 4.2.2]. They actually considered the case where $z \geq 0$, but their arguments extend almost *verbatim* to all complex $z \neq 0, -2, -4, \ldots$; in particular, their estimates apply to κ_0^{2iu}, the kernel of $\mathcal{L}^{-iu}$, for u real.

For each $c \geq 0$ and every positive integer d, we set

$$\mathbf{r}_c^d(x \cdot o) = L(x)^* \big[\nabla^d \kappa_c^d\big](x \cdot o) \qquad \forall x \in G \setminus K; \tag{3.3}$$

$\mathbf{r}_c^d$ is a $(\mathfrak{p}^*)^{\otimes d}$-valued smooth function on $\mathbb{X}\backslash\{o\}$. We recall that covariant differentiation on $\mathbb{X}$ has a simple expression in terms of left invariant derivatives on G [1, p. 264]. Thus,

$$\mathbf{r}_c^d(x \cdot o)(Z_1, \ldots, Z_d) = Z_1 \cdots Z_d \kappa_c^d(x) \qquad \forall x \in G \quad \forall Z_1, \ldots, Z_d \in \mathfrak{p}. \tag{3.4}$$

Lemma 3.2. *Suppose that $c \geq 0$ and that d is a nonnegative integer. Then*

$$L(x)^* \left[\mathcal{R}_c^d f\right](x \cdot o) = f * \mathbf{r}_c^d(x \cdot o) \tag{3.5}$$

whenever $x \cdot o$ does not belong to $\operatorname{supp} f$. Furthermore, for every $\mathcal{Z}$ in $\mathfrak{p}^d$

$$\mathcal{R}_{c,\mathcal{Z}}^d f(x \cdot o) = f * (Z_1 \cdots Z_d \kappa_c^d)(x \cdot o). \tag{3.6}$$

Proof. Suppose that f is in $C_c^\infty(\mathbb{X})$ and that $x \cdot o$ does not belong to $\operatorname{supp} f$. Then

$$\begin{aligned} \mathcal{R}_c^d f(x \cdot o) &= \nabla_x^d \int_G f(y \cdot o)\ \kappa_c^d(y^{-1}x \cdot o) \,\mathrm{d}y \\ &= \int_G f(y \cdot o)\ \nabla_x^d \big[\kappa_c^d(y^{-1}x \cdot o)\big] \,\mathrm{d}y. \end{aligned}$$

Since the map $L(y)$ is an isometry of $\mathbb{X}$,

$$\begin{aligned} \nabla_x^d \big[\kappa_c^d(y^{-1}x \cdot o)\big] &= \nabla^d \big[L(y^{-1})\kappa_c^d\big](x \cdot o) \\ &= L(y^{-1})^* \left[\nabla^d \kappa_c^d\right](y^{-1}x \cdot o). \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{R}_c^d f(x \cdot o) &= \int_G f(y \cdot o)\, L(y^{-1})^* \left[\nabla_x^d \kappa_c^d\right](y^{-1}x \cdot o) \,\mathrm{d}y \\ &= L(x^{-1})^* \int_G f(y \cdot o)\, \mathbf{r}_c^d(y^{-1}x \cdot o) \,\mathrm{d}y; \end{aligned}$$

we have used the fact that $L(y^{-1})^* = L(x^{-1})^* L(y^{-1}x)^*$ in the last inequality above. The proof of (3.5) is complete.

Formula (3.6) follows from (3.5), the definition of scalar Riesz transform (3.1), and (3.4). □

For technical reasons, which will be apparent shortly, we shall need to consider another tensor valued operator related to the Riesz transforms, namely $\nabla^d \mathcal{L}^{1/2}$. The following lemma will be useful in the proof of Theorem 3.1 (iii), as will be Lemma 3.4 below.

Lemma 3.3. *For every positive integer d there exists a constant C such that*

$$\big\|\nabla^d \mathcal{L}^{1/2} f\big\|_{L^1((4B)^c; T^d)} \leq C\, r_B^{-d-1}\ \big\|f\big\|_{L^1(B)} \qquad \forall f \in C_c^\infty(B)$$

for all balls B of radius $r_B \leq 1$.

Proof. Recall that

$$\big\|\nabla^d\mathcal{L}^{1/2}f\big\|_{L^1((4B)^c;T^d)} = \int_{(4B)^c}\big|\nabla^d\mathcal{L}^{1/2}f(x\cdot o)\big|_{x\cdot o}\,\mathrm{d}\mu(x\cdot o),$$

where $|\nabla^d\mathcal{L}^{1/2}f(x\cdot o)|_{x\cdot o}$ denotes the norm of the covariant tensor $\nabla^d\mathcal{L}^{1/2}f(x\cdot o)$. Since $L(x)^*$ is an isometry between covariant tensors at $x\cdot o$ and covariant tensors at o,

$$\begin{aligned}\big|\nabla^d\mathcal{L}^{1/2}f(x\cdot o)\big|_{x\cdot o} &= \big|L(x)^*[\nabla^d\mathcal{L}^{1/2}f(x\cdot o)]\big|_o \\ &= \sup_{|\mathcal{Z}|\leq 1}\big|L(x)^*\big[\nabla^d\,\mathcal{L}^{1/2}f(x\cdot o)\big](Z_1,\ldots,Z_d)\big|.\end{aligned}$$

For every d-tuple of vectors $\mathcal{Z}=(Z_1,\ldots,Z_d)$ in the unit ball of $\mathfrak{p}$, consider the scalar operator $\mathcal{S}^d_{\mathcal{Z}}$, defined by

$$\mathcal{S}^d_{\mathcal{Z}}\,f(x\cdot o) = L(x)^*\big[\nabla^d\,\mathcal{L}^{1/2}f(x\cdot o)\big](Z_1,\ldots,Z_d).$$

Thus, to prove the lemma it suffices to show that there exists a constant C such that

$$\Big\|\sup_{|\mathcal{Z}|\leq 1}|\mathcal{S}^d_{\mathcal{Z}}f|\Big\|_{L^1((4B)^c)} \leq C\,r_B^{-d-1}\ \|f\|_{L^1(B)}\,.$$

By arguing much as in in the proof of Lemma 3.2, it is straightforward to check that

$$\mathcal{S}^d_{\mathcal{Z}}\,f(x\cdot o) = f * Z_1\cdots Z_d\kappa_0^{-1}(x\cdot o),$$

where κ_0^{-1} is defined in (3.2). For the sake of brevity, for the duration of this proof, we write $s^d_{\mathcal{Z}}$ instead of $Z_1\cdots Z_d\kappa_0^{-1}$. Thus, it suffices to show that

$$\int_{(3B)^c}\sup_{|\mathcal{Z}|\leq 1}\big|s^d_{\mathcal{Z}}(x\cdot o)\big|\,\mathrm{d}\mu(x\cdot o)\leq C\,r_B^{-d-1}.$$

We write the integral as the sum of the integrals over the annulus $B_3(o)\setminus 3B$ and over $B_3(o)^c$, and estimate them separately.

To estimate the first integral, we observe that, by [2, Remark 4.2.3 (iii)], there exists a constant C, independent of $Z_1,\ldots,Z_d$ in the unit ball of $\mathfrak{p}$, such that

$$\big|s^d_{\mathcal{Z}}(x\cdot o)\big|\leq C\,|x\cdot o|^{-1-n-d}\qquad\forall x\cdot o\in B_3(o).$$

Therefore

$$\int_{B_3(o)\setminus 3B}\sup_{|\mathcal{Z}|\leq 1}\big|s^d_{\mathcal{Z}}(x\cdot o)\big|\,\mathrm{d}\mu(x\cdot o)\leq C\,r_B^{-d-1}.$$

To estimate the second integral, we observe that, by [2, Thm 4.2.2], there exists a constant C, independent of $Z_1,\ldots,Z_d$ in the unit ball of $\mathfrak{p}$, such that

$$\big|s^d_{\mathcal{Z}}(x\cdot o)\big|\leq C\,\big(1+|x\cdot o|\big)^{-|\Sigma_0^+|-1-\ell/2}\,\varphi_0(x\cdot o)\ \mathrm{e}^{-|\rho|\,|x\cdot o|}.$$

Then we integrate in polar co-ordinates (2.1); by combining this estimate with estimates (2.2) for the density δ and (2.3) for φ_0, we get

$$\begin{aligned}\int_{B_3(o)^c}\sup_{|\mathcal{Z}|\leq 1}\big|s^d_{\mathcal{Z}}(x\cdot o)\big|\,\mathrm{d}\mu(x\cdot o) &= C\int_{\mathfrak{b}_3^c}\sup_{|\mathcal{Z}|\leq 1}\big|s^d_{\mathcal{Z}}(\exp H\cdot o)\big|\,\delta(H)\,\mathrm{d}H \\ &\leq C\int_{\mathfrak{b}_3^c}|H|^{-1-\ell/2}\,\mathrm{e}^{\rho(H)-|\rho||H|}\,\mathrm{d}H.\end{aligned}$$

Denote by $(H_1, \ldots, H_\ell)$ the coordinates of H with respect to an orthonormal basis of $\mathfrak{a}$ such that $H_1 = \rho(H)/|\rho|$. Then the latter integral is dominated by

$$\int_{\mathfrak{b}_3^c} |H|^{-1-\ell/2}\, \mathrm{e}^{|\rho|(H_1-|H|)}\, \mathrm{d}H_1 \cdots \mathrm{d}H_\ell,$$

which is easily seen to converge. This concludes the proof of the lemma. $\square$

Lemma 3.4. *Suppose that k is a positive integer. For every X^k-atom A with support contained in B, the support of $\mathcal{L}^{-k}A$ is contained in $\overline{B}$. Furthermore, there exists a positive constant C, independent of A, such that*

$$\big\|\mathcal{L}^{-k}A\big\|_2 \le C\, r_B^{2k}\, |B|^{-1/2}.$$

Proof. The support of $\mathcal{L}^{-k}A$ is contained in $\overline{B}$ by [18, Remark 3.5]. Denote by $\lambda_1(B)$ the smallest eigenvalue of the Dirichlet Laplacian on B. By Faber-Krahn's inequality [10] there exists a positive constant C, independent of the ball B, such that $\lambda_1(B) \ge C r_B^{-2}$. Hence the desired conclusion for $k=1$ follows from [20, Corollary 3.3]. The general case follows from this by a straightforward induction argument. $\square$

We are now in position to prove Theorem 3.1.

Proof. For the sake of simplicity, for the duration of this proof we shall denote the kernel $Z_1 \cdots Z_d \kappa_c^d$ of $\mathcal{R}^d_{c,\mathcal{Z}}$ (see Lemma 3.2 above) simply by κ, and set

$$A_j := \{x \cdot o \in \mathbb{X} : j \le |x \cdot o| < j+2\}.$$

In view of [16, Theorem 4.1] and the translation invariance of $\mathcal{R}^d_{c,\mathcal{Z}}$, to prove (i) it suffices to show that

$$\sup_a \big\|\mathcal{R}^d_{c,\mathcal{Z}}\, a\big\|_{H^1} < \infty,$$

where the supremum is taken over all H^1-atoms a with support contained in $B_R(o)$ for some $R \le 1$. We analyse the cases where $R \ge 10^{-1}$ and $R < 10^{-1}$ separately.

Suppose first that $R \ge 10^{-1}$. We consider a partition of unity on $\mathbb{X}$ of the form

$$1 = \varphi + \sum_{j=1}^{\infty} \psi_j, \tag{3.7}$$

where φ and ψ_j are smooth K–invariant functions on $\mathbb{X}$, the support of φ is contained in $B_2(o)$, and the support of ψ_j is contained in the annulus A_j. Then we write

$$\mathcal{R}^d_{c,\mathcal{Z}} a = a * (\varphi\kappa) + \sum_{j=1}^{\infty} a * (\psi_j \kappa).$$

We denote by $\big\|\varphi\kappa\big\|_{Cv_2}$ the norm of the convolution operator $f \mapsto f * (\varphi\kappa)$, acting on $L^2(\mathbb{X})$. Observe that $\big\|\kappa\big\|_{Cv_2} < \infty$, because $\mathcal{R}^d_{c,\mathcal{Z}}$ is bounded on $L^2(\mathbb{X})$. Since $Cv_2(\mathbb{X})$ is a $C_c^\infty(K\backslash G/K)$-module, $\big\|\varphi\kappa\big\|_{Cv_2} < \infty$. Thus,

$$\begin{aligned} \big\|a * (\varphi\kappa)\big\|_{H^1} &\le \big|B_{R+2}(o)\big|^{1/2}\ \big\|a\big\|_2\ \big\|\varphi\kappa\big\|_{Cv_2} \\ &\le \sqrt{\frac{\big|B_{R+2}(o)\big|}{\big|B_R(o)\big|}}\ \big\|\varphi\kappa\big\|_{Cv_2} \\ &\le C\ \big\|\varphi\kappa\big\|_{Cv_2}\ ; \end{aligned} \tag{3.8}$$

in the last inequality we have used the assumption $R \geq 10^{-1}$ and the local doubling condition. Furthermore, by Lemma 2.7 (in the case where $R+\beta > 1$),

$$\big\|a * (\psi_j \kappa)\big\|_{H^1} \leq (R+j+2)\,\big|B_{R+j+2}(o)\big|^{1/2}\ \big\|\psi_j\kappa\big\|_2\,.$$

To estimate the L^2 norm of $\psi_j\kappa$ observe that [2, Thm 4.2.2] and estimate (2.3) imply that there exists a constant C such that

$$|\kappa(x\cdot o)| \leq C\,\big(1+|x\cdot o|\big)^{(d-\ell-1)/2}\ \mathrm{e}^{-\rho(A^+(x))-|x\cdot o|\sqrt{c^2+|\rho|^2}}.$$

By integrating in Cartan co-ordinates, and using the estimate above and the estimate (2.2) of the density function δ, we see that

$$\begin{aligned}\big\|\psi_j\kappa\big\|_2^2 &\leq C\int_{A_j} |H|^{d-\ell-1}\,\mathrm{e}^{-2|H|\sqrt{c^2+|\rho|^2}}\ \mathrm{d}H \\ &\leq C\,j^{d-2}\,\mathrm{e}^{-2j\sqrt{c^2+|\rho|^2}}.\end{aligned} \tag{3.9}$$

By combining the estimates above, we obtain that

$$\begin{aligned}\big\|\mathcal{R}^d_{c,\mathcal{Z}}a\big\|_{H^1} &\leq \big\|a*(\varphi\kappa)\big\|_{H^1} + \sum_{j=1}^\infty\ \big\|a*(\psi_j\kappa)\big\|_{H^1} \\ &\leq C\ \big\|\varphi\kappa\big\|_{Cv_2} + C\sum_{j=1}^\infty j\,\big|B_{j+3}(o)\big|^{1/2}\,j^{d/2-1}\,\mathrm{e}^{-j\sqrt{c^2+|\rho|^2}}.\end{aligned}$$

We use the estimate $|B_{j+3}(o)| \leq C\,j^{\ell-1}\,\mathrm{e}^{2|\rho|j}$, and conclude that

$$\big\|\mathcal{R}^d_{c,\mathcal{Z}}a\big\|_{H^1} \leq C\ \big\|\varphi\kappa\big\|_{Cv_2} + C\sum_{j=1}^\infty j^{(d+\ell-1)/2}\,\mathrm{e}^{j(|\rho|-\sqrt{c^2+|\rho|^2})},$$

which is easily seen to be finite (and independent of a).

Next suppose that $R < 10^{-1}$. We denote by D_{2^hR} the dyadic annulus

$$\{x\cdot o\in\mathbb{X} : 2^{h-1}R \leq |x\cdot o| < 2^{h+1}R\},$$

and consider a partition of unity on $\mathbb{X}$ of the form

$$1 = \phi + \sum_{h=1}^N \eta_h + \psi_0 + \sum_{j=1}^\infty \psi_j, \tag{3.10}$$

where ψ_j, $j = 1,2,\ldots$, is as in (3.7), ϕ is a K–invariant function on $\mathbb{X}$ with support contained in $B_{2R}(o)$, η_h are smooth K–invariant functions on $\mathbb{X}$, with support contained in D_{2^hR}, N is the least integer for which $2^{N+1}R > 10^{-1}$, and the support of ψ_0 is contained in the annulus $\{x\cdot o\in\mathbb{X} : 10^{-1} \leq |x\cdot o| \leq 2\}$. We also require that there exists a constant C such that

$$\big|\nabla\eta_h(x\cdot o)\big| \leq C\,(2^hR)^{-1} \qquad \forall h\in\{1,\ldots,N\}.$$

It is important to keep in mind that ϕ and η_j depend on R.

By arguing *verbatim* as above, we may prove that the $H^1(\mathbb{X})$ norm of $\sum_{j=1}^\infty a*(\psi_j\kappa)$ is uniformly bounded with respect to R in $(0,10^{-1}]$, and that the same is true of

$a * (\psi_0 \kappa)$. Also, much as for the estimate of the $H^1(\mathbb{X})$ norm of $a * (\varphi\kappa)$ above,

$$\begin{aligned} \big\|a * (\phi\kappa)\big\|_{H^1} &\leq \big|B_{3R}(o)\big|^{1/2}\ \big\|a\big\|_2\ \big\|\phi\kappa\big\|_{Cv_2} \\ &\leq \sqrt{\frac{\big|B_{3R}(o)\big|}{\big|B_R(o)\big|}}\ \big\|\phi\kappa\big\|_{Cv_2} \\ &\leq C\ \big\|\kappa\big\|_{Cv_2}\ ; \end{aligned} \tag{3.11}$$

in the last inequality we have used the local doubling condition, and the fact that multiplication by ϕ is a bounded operator on $Cv_2(\mathbb{X})$, with norm independent of R, as long as R stays bounded.

Thus, to conclude the proof of (i) it suffices to show that the $H^1(\mathbb{X})$ norm of $\sum_{h=1}^N a * (\eta_h\kappa)$ is uniformly bounded with respect to R in $(0, 10^{-1}]$. Since the support of $a*(\eta_h\kappa)$ is contained in $B_1(o)$, we may apply the first estimate in Lemma 2.7 (i), and conclude that

$$\big\|a * (\eta_h\kappa)\big\|_{H^1} \leq \big|B_{R+2^{h+1}R}(o)\big|^{1/2} \min\big(\ \big\|\eta_h\kappa\big\|_2\ , C\,R\ \big\|\nabla(\eta_h\kappa)\big\|_2\ \big). \tag{3.12}$$

By [2, Remark 4.2.3 (iii)], there exists a constant C such that

$$\big|\kappa(x\cdot o)\big| \leq C\,|x|^{-n} \qquad \big|\nabla\kappa(x\cdot o)\big| \leq C\,|x|^{-n-1} \qquad \forall x \in G : |x\cdot o| \leq 1.$$

This, and the fact that the support of $\eta_h\kappa$ is contained in D_{2^hR} imply that

$$\begin{aligned} \big|\nabla(\eta_h\kappa)(x\cdot o)\big| &\leq C\,\big[(2^hR)^{-1}\,|\kappa(x)| + \big|\nabla\kappa(x)\big|\big] \\ &\leq C\,\big[(2^hR)^{-1}\,|x\cdot o|^{-n} + |x\cdot o|^{-n-1}\big] \qquad \forall x \in D_{2^hR}. \end{aligned}$$

Therefore

$$\begin{aligned} \big\|\nabla(\eta_h\kappa)\big\|_2 &\leq C\,(2^hR)^{-1}\,\Big[\int_{D_{2^hR}} |x|^{-2n}\,\mathrm{d}x\Big]^{1/2} + \Big[\int_{D_{2^hR}} |x|^{-2n-2}\,\mathrm{d}x\Big]^{1/2} \\ &\leq C\,(2^hR)^{-n-1}\,(2^hR)^{n/2} \\ &= C\,(2^hR)^{-n/2-1}. \end{aligned}$$

This estimate, (3.12), and the fact that $|B_{(2^{h+1}+1)R}| \leq C\,(2^hR)^n$ imply that there exists a constant C, independent of a, such that

$$\big\|a * (\eta_h\kappa)\big\|_{H^1} \leq C\,2^{-h},$$

so that $\sup_{R\leq 10^{-1}}\ \big\|\sum_{h=1}^N a * (\eta_h\kappa)\big\|_{H^1} < \infty$, and the proof of (i) is complete.

Next we prove (ii), i.e. that $\mathcal{R}_c^d$ is bounded from $H^1(\mathbb{X})$ to $L^1(\mathbb{X}; T^d)$. A careful examination of the proof of (i) reveals that there exists a constant C, independent of $Z_1, \ldots, Z_d$ in the unit ball of $\mathfrak{p}$, such that

$$\Big\|\sup_{|\mathcal{Z}|\leq 1} \big|\mathcal{R}_{c,\mathcal{Z}}^d\,a\big|\Big\|_{L^1} \leq C. \tag{3.13}$$

As in the proof of Lemma 3.3 we use the fact that $L(x)^*$ is an isometry between covariant tensors at the point $x\cdot o$ and covariant tensors at o, and conclude that

$$\big|\mathcal{R}_c^d a(x\cdot o)\big|_{x\cdot o} = \sup_{|\mathcal{Z}|\leq 1} \big|\mathcal{R}_{c,\mathcal{Z}}^d a(x\cdot o)\big|.$$

The required estimate follows directly from this and (3.13).

Finally, we prove (iii). If d is even, then $\lfloor (d+1)/2\rfloor = d/2$ and the result is already known [18, Theorem 5.2]. Thus, we only need to consider the case when d is odd, for which $\lfloor (d+1)/2 \rfloor = (d+1)/2$. By [19, Corollary 6.2 and Proposition 6.3] and the translation invariance of $\mathcal{R}^d$, it suffices to prove that

$$\sup_A \left\| \mathcal{R}^d A \right\|_{L^1(\mathbb{X};T^d)} < \infty, \tag{3.14}$$

where the supremum is taken over all $X^{(d+1)/2}$-atoms A supported in balls centred at o. Given such an atom A, denote by $B_R(o)$ the ball associated to it. Observe that

$$\left\| \mathcal{R}^d A \right\|_{L^1(\mathbb{X};T^d)} = \left\| |\mathcal{R}^d A| \right\|_{L^1(4B)} + \left\| |\mathcal{R}^d A| \right\|_{L^1((4B)^c)} . \tag{3.15}$$

We shall estimate the two summands on the right hand side separately. Clearly

$$\begin{aligned} \left\| |\mathcal{R}^d A| \right\|_{L^1(4B)} &\leq |4B|^{1/2} \, \left\| |\mathcal{R}^d A| \right\|_{L^2(4B)} \\ &\leq C \sqrt{|4B|/|B|} \\ &\leq C; \end{aligned}$$

here we have applied the L^2-boundedness of $\mathcal{R}^d$, the size property of A and the local doubling property of μ.

To estimate the second summand in (3.15) we write

$$\mathcal{R}^d A = \nabla^d \mathcal{L}^{1/2} (\mathcal{L}^{-(d+1)/2} A).$$

By Schwarz's inequality and Lemma 3.4 there exists a constant C, independent of A, such that

$$\begin{aligned} \left\| \mathcal{L}^{-(d+1)/2} A \right\|_{L^1(B)} &\leq |B|^{1/2} \, \left\| \mathcal{L}^{-(d+1)/2} A \right\|_{L^2(B)} \\ &\leq |B|^{1/2} \, C \, R^{d+1} \, |B|^{-1/2} \\ &\leq C \, R^{d+1}. \end{aligned}$$

Now, Lemma 3.3 and this estimate imply that

$$\begin{aligned} \left\| \mathcal{R}^d A \right\|_{L^1((4B)^c;T^d)} &= \left\| |\nabla^d \mathcal{L}^{1/2} \, (\mathcal{L}^{-(d+1)/2} A)| \right\|_{L^1((4B)^c)} \\ &\leq C \, R^{-d-1} \, \left\| \mathcal{L}^{-(d+1)/2} A \right\|_{L^1(B)} \\ &\leq C. \end{aligned}$$

This concludes the proof for odd m. □

We notice that, by interpolation, Theorem 3.1 implies the L^p boundedness of $\mathcal{R}^d_c$ and $\mathcal{R}^d$ for every $c > 0$ and $p \in (1, 2]$ (see [5, 17] for interpolation properties of $H^1(\mathbb{X})$ and $X^k(\mathbb{X})$).

4. Unboundedness on $H^1(\mathbb{X})$ of Riesz potentials and Riesz transform

In this section we prove that the Riesz potentials $\mathcal{L}^{-\sigma/2}$, $\sigma > 0$, and the Riesz transform $\mathcal{R}^1$ are unbounded from $H^1(\mathbb{X})$ to $L^1(\mathbb{X})$. Thus the endpoint result in Theorem 3.1 (iii) is sharp.

Theorem 4.1. *The operators $\mathcal{L}^{-\sigma/2}$, $\sigma > 0$, do not map $H^1(\mathbb{X})$ to $L^1(\mathbb{X})$ and the Riesz transform $\mathcal{R}^1$ does not map $H^1(\mathbb{X})$ to $L^1(\mathbb{X}; T^1)$.*

The proof of this theorem requires some Harnack type estimates, which will be established in the next lemma. For each positive real number R, denote by H_R the element in the positive Weyl chamber $\mathfrak{a}^+$ such that $|H_R| = R$ and

$$\langle H_R, H\rangle = R\,\frac{\rho(H)}{|\rho|} \qquad \forall H \in \mathfrak{a}.$$

Set $a_R = \exp H_R$. Recall that κ_0^σ is the convolution kernel of $\mathcal{L}^{-\sigma/2}$ (see formula (3.2)).

Lemma 4.2. *For each $\varepsilon > 0$ the following hold:*

(i) *there exists a positive number η_0 such that*

$$\sup_{B_\eta(y\cdot o)} \kappa_0^\sigma \leq (1+\varepsilon) \inf_{B_\eta(y\cdot o)} \kappa_0^\sigma \qquad \forall \eta \leq \eta_0 \quad \forall y\cdot o \in B_2(o)^c;$$

(ii) *for each $R > 0$ there exists a neighbourhood U of the identity in K such that*

$$\kappa_0^\sigma(a_R u a\cdot o) \leq (1+\varepsilon)\,\kappa_0^\sigma(a_R a\cdot o) \qquad \forall u \in U \quad \forall a \in \exp \mathfrak{b}_2^c.$$

Proof. First we prove (i). Suppose that $x_1 \cdot o, x_2\cdot o$ are points in $B_\eta(y\cdot o)$. By the mean value theorem

$$\kappa_0^\sigma(x_1\cdot o) - \kappa_0^\sigma(x_2\cdot o) \leq 2\eta \sup_{B_\eta(y\cdot o)} \big|\nabla \kappa_0^\sigma\big|.$$

By [2, Thm 4.2.2], there exists a constant C such that

$$\big|\nabla\kappa_0^\sigma(x\cdot o)\big| \leq C\,\kappa_0^\sigma(x\cdot o) \qquad \forall x\cdot o \notin B_1(o).$$

Therefore for all $y\cdot o$ in $B_2(o)^c$

$$\begin{aligned}\kappa_0^\sigma(x_2\cdot o) &\geq \kappa_0^\sigma(x_1\cdot o) - 2\eta \sup_{B_\eta(y\cdot o)}\big|\nabla\kappa_0^\sigma\big| \\ &\geq \kappa_0^\sigma(x_1\cdot o) - 2\,\eta\, C \sup_{B_\eta(y\cdot o)} \kappa_0^\sigma.\end{aligned}$$

By taking the supremum over all x_1 in $B_\eta(y\cdot o)$, we obtain that

$$\kappa_0^\sigma(x_2\cdot o) \geq (1 - 2\,\eta\, C) \sup_{B_\eta(y\cdot o)} \kappa_0^\sigma.$$

Now, if $\eta < 1/(2C)$, then we may take the infimum of both sides over all x_2 in $B_\eta(y\cdot o)$, and obtain the required conclusion with $\eta_0 = \varepsilon/2C(1+\varepsilon)$.

To prove (ii), write $a_R u a\cdot o = u^{a_R} a_R a\cdot o$, where u^{a_R} is short for $a_R u a_R^{-1}$. By Lemma 2.1,

$$\begin{aligned}\big|A^+(u^{a_R}a_R a) - A^+(a_R a)\big| &\leq \mathrm{d}(u^{a_R}\cdot o, o) \\ &= \mathrm{d}\big(\exp(\mathrm{Ad}(a_R)X)\cdot o, o\big) \\ &\leq \big|\mathrm{Ad}(a_R)X\big|.\end{aligned}$$

Here X is in $\mathfrak{k}$, $\exp X = u$, and $|X| \leq s$ with s small, for we assume that u belongs to a small neighbourhood of the origin in K. Notice that $\mathrm{Ad}(a_R)X = \mathrm{e}^{\mathrm{ad} H_R} X$ so that

$$\big|\mathrm{Ad}(a_R)X\big| \leq \mathrm{e}^{\|\mathrm{ad} H_R\|}\,\big|X\big| \leq \mathrm{e}^{\|\mathrm{ad} H_R\|}\, s.$$

Therefore $\exp\big(A^+(u^{a_R}a_R a)\big)\cdot o$ lies in the ball with centre $a_R a\cdot o$ and radius $\mathrm{e}^{\|\mathrm{ad} H_R\|}\,s$. Assume that the latter quantity is smaller than η_0, i.e. that $s < \varepsilon\,\mathrm{e}^{-\|H_R\|}/2C(1+\varepsilon)$. Then (i) implies that

$$\begin{aligned}\kappa_0^\sigma(a_R u a\cdot o) &= \kappa_0^\sigma(\exp(A^+(u^{a_R}a_R a)\cdot o) \\ &\leq (1+\varepsilon)\,\kappa_0^\sigma(a_R a\cdot o) \qquad \forall u \in U \quad \forall a \in \exp\mathfrak{b}_2^c,\end{aligned}$$

as required to conclude the proof of (ii), and of the lemma. □

We now prove Theorem 4.1.

Proof. Fix $\varepsilon > 0$. Consider the function $f = b\,\mathbf{1}_{B_{\eta_0}(o)} - b\,\mathbf{1}_{B_{\eta_0}(a_R^{-1}\cdot o)}$, where η_0 is as in Lemma 4.2 (i), $b = \mu\big(B_{\eta_0}(o)\big)^{-1}$, and $\mathbf{1}_E$ denotes the characteristic function of E. Clearly f is in $L^2(\mathbb{X})$, its integral vanishes and its support is contained in $\overline{B_{R+1}(o)}$. Then f belongs to $H^1(\mathbb{X})$, by Remark 2.4.

We shall prove that if R is large enough, then $\mathcal{L}^{-\sigma/2}f$ is not in $L^1(\mathbb{X})$. We observe that this implies that the Riesz transform $\mathcal{R}^1$ does not map $H^1(\mathbb{X})$ into $L^1(\mathbb{X};T^1)$. Indeed, by Cheeger's inequality, there exists a positive constant c such that

$$\big\||\mathcal{R}^1 f|\big\|_1 \geq c\, \big\|\mathcal{L}^{-1/2}f\big\|_1\,,$$

and the right hand side is infinite.

We continue the proof of the fact that $\mathcal{L}^{-\sigma/2}f$ is not in $L^1(\mathbb{X})$. Observe that

$$\begin{aligned}
f*\kappa_0^\sigma(x\cdot o) &= b\int_{B_{\eta_0}(o)} \kappa_0^\sigma(y^{-1}x\cdot o)\,\mathrm{d}\mu(y\cdot o) - b\int_{B_{\eta_0}(a_R^{-1}\cdot o)} \kappa_0^\sigma(y^{-1}x\cdot o)\,\mathrm{d}\mu(y\cdot o)\\
&= b\int_{B_{\eta_0}(o)} \big[\kappa_0^\sigma(y^{-1}x\cdot o) - \kappa_0^\sigma(y^{-1}a_R x\cdot o)\big]\,\mathrm{d}\mu(y\cdot o)\\
&= b\int_{B_{\eta_0}(o)} \big[\kappa_0^\sigma(x^{-1}y\cdot o) - \kappa_0^\sigma(x^{-1}a_R^{-1}y\cdot o)\big]\,\mathrm{d}\mu(y\cdot o).
\end{aligned}$$

We have used the fact that $\kappa_0^\sigma(v^{-1}\cdot o) = \kappa_0^\sigma(v\cdot o)$ in the last equality. By Lemma 4.2 (i), the last integrand above is bounded from below by

$$\begin{aligned}
&\frac{1}{1+\varepsilon}\,\kappa_0^\sigma(x^{-1}\cdot o) - (1+\varepsilon)\,\kappa_0^\sigma(x^{-1}a_R^{-1}\cdot o)\\
&\qquad\qquad = \frac{1}{1+\varepsilon}\,\kappa_0^\sigma(x\cdot o)\,\Big[1\,-(1+\varepsilon)^2\,\frac{\kappa_0^\sigma(a_R x\cdot o)}{\kappa_0^\sigma(x\cdot o)}\Big].
\end{aligned}$$

We have used again the fact that $\kappa_0^\sigma(v^{-1}\cdot o) = \kappa_0^\sigma(v\cdot o)$ in the last equality.

We now restrict x to $U\cdot\exp\big(\mathfrak{c}_\delta\cap\mathfrak{b}_{2R}^c\big)$, where U is a small neighbourhood of the identity in K (as in Lemma 4.2 (ii)), and, for δ in $(0,1)$, $\mathfrak{c}_\delta$ denotes the proper subcone of the positive Weyl chamber $\mathfrak{a}^+$, defined by

$$\mathfrak{c}_\delta = \big\{H\in\mathfrak{a}^+ : \rho(H)\geq\delta\,|\rho|\,|H|\big\}.$$

Then Lemma 4.2 (ii) implies that $\kappa_0^\sigma(a_R x\cdot o)\leq(1+\varepsilon)\,\kappa_0^\sigma(a_R a\cdot o)$ for all such x, and we are left with the problem of estimating

$$\frac{1}{1+\varepsilon}\,\kappa_0^\sigma(a\cdot o)\,\Big[1\,-(1+\varepsilon)^3\,\frac{\kappa_0^\sigma(a_R a\cdot o)}{\kappa_0^\sigma(a\cdot o)}\Big]\qquad\forall a\in\exp\big[\mathfrak{c}_\delta\cap\mathfrak{b}_{2R}^c\big]$$

from below.

A straightforward consequence of [2, Theor. 4.2.2], and of the sharp estimate of the spherical function φ_0, is that for every δ close to 1, there exist positive constants c and C such that

$$c\,|H|^{(\sigma-\ell-1)/2}\,\mathrm{e}^{-\rho(H)-|\rho||H|}\leq\kappa_0^\sigma\big(u\exp H\cdot o\big)\leq C\,|H|^{(\sigma-\ell-1)/2}\,\mathrm{e}^{-\rho(H)-|\rho||H|}\tag{4.1}$$

for all u in K and for all H in $\mathfrak{c}_\delta \cap \mathfrak{b}_1^c$. Therefore

$$
\begin{aligned}
\frac{\kappa_0^\sigma(a_R a\cdot o)}{\kappa_0^\sigma(a\cdot o)} &\leq C\left(\frac{|H_R+H|}{|H|}\right)^{(\sigma-\ell-1)/2} \mathrm{e}^{-\rho(H_R+H)-|\rho|\,|H_R+H|+\rho(H)+|\rho|\,|H|} \\
&\leq C\left(\frac{|H_R+H|}{|H|}\right)^{(\sigma-\ell-1)/2} \mathrm{e}^{-|\rho|(R+|H_R+H|-|H|)}.
\end{aligned}
\tag{4.2}
$$

Observe that

$$
\frac{1}{2} \leq \frac{|H_R+H|}{|H|} \leq \frac{3}{2} \qquad \forall H \in \mathfrak{c}_\delta \cap \mathfrak{b}_{2R}^c,
$$

and that the exponential on the right hand side of (4.2) is dominated by $\mathrm{e}^{-|\rho|R}$. By choosing R large enough, we may conclude that

$$
\frac{\kappa_0^\sigma(a_R a\cdot o)}{\kappa_0^\sigma(a\cdot o)} < \varepsilon \qquad \forall a \in \exp\left[\mathfrak{c}_\delta \cap \mathfrak{b}_{2R}^c\right].
$$

Altogether, we have proved that for R large enough

$$
\int_{U\cdot\exp(\mathfrak{c}_\delta\cap\mathfrak{b}_{2R}^c)} |f*\kappa_0^\sigma|\,\mathrm{d}\mu \geq b\,\frac{1-(1+\varepsilon)^3\varepsilon}{1+\varepsilon}\int_{U\cdot\exp(\mathfrak{c}_\delta\cap\mathfrak{b}_{2R}^c)} \kappa_0^\sigma\,\mathrm{d}\mu.
$$

It is not hard to prove that the last integral is equal to infinity. Indeed, integrate in Cartan co-ordinates, use the fact that κ_0^σ is K–bi-invariant, and obtain that the last integral is equal to

$$
\mu_K(U)\int_{\mathfrak{c}_\delta\cap\mathfrak{b}_{2R}^c} \kappa_0^\sigma(\exp H\cdot o)\,\delta(H)\,\mathrm{d}H \geq c\,\mu_K(U)\int_{\mathfrak{c}_\delta\cap\mathfrak{b}_{2R}^c} |H|^{(\sigma-\ell-1)/2}\,\mathrm{e}^{\rho(H)-|\rho||H|}\,\mathrm{d}H.
$$

In the last inequality we have used the fact that $\delta(H) \geq c\,\mathrm{e}^{2\rho(H)}$ for some positive constant c when H is in $\mathfrak{c}_\delta$ (see (2.2)). If the rank ℓ is equal to one, then $\rho(H) = |\rho|\,|H|$, $\mathfrak{c}_\delta \cap \mathfrak{b}_r^c$ reduces to the half line $[r,\infty)$, and the integrand becomes $|H|^{-1+\sigma/2}$, which is nonintegrable on $[r,\infty)$. If $\ell \geq 2$, then we pass to polar co-ordinates in $\mathfrak{a}$ and see that the last integral is equal to

$$
c\int_0^{\arccos\delta} \mathrm{d}\theta\,(\sin\theta)^{\ell-2}\int_r^\infty s^{(\sigma+\ell-3)/2}\,\mathrm{e}^{|\rho|\,(\cos\theta-1)s}\,\mathrm{d}s,
$$

which is easily seen to diverge for all $\sigma \geq 0$.

This concludes the proof that $\mathcal{L}^{-\sigma/2}f$ is not in $L^1(\mathbb{X})$. □

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