LATTICE WIDTH DIRECTIONS AND MINKOWSKI'S 3^d-THEOREM

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Abstract. We show that the number of lattice directions in which a d-dimensional convex body in $\mathbb{R}^d$ has minimum width is at most $3^d - 1$, with equality only for the regular cross-polytope. This is deduced from a sharpened version of the 3^d-theorem due to Hermann Minkowski (22 June 1864—12 January 1909), for which we provide two independent proofs.

1. Introduction

The *lattice width* of a non-empty subset S of $\mathbb{R}^d$ is a well-studied invariant in the geometry of numbers. It is defined to be the infimum of $\sup(u(S)) - \inf(u(S))$ as u ranges over the set of non-zero vectors in the lattice dual to $\mathbb{Z}^d \subset \mathbb{R}^d$ for which both $\sup(u(S))$ and $\inf(u(S))$ are finite. In the case that this set is empty, the lattice width is defined to be ∞. If the lattice width is finite, the vectors attaining this infimum are called *lattice width directions*.

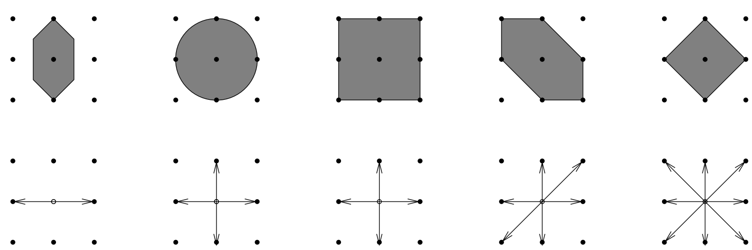

Figure 1. Convex bodies in $\mathbb{R}^2$ and their lattice width directions

The lattice width and the set of lattice width directions is invariant under the action of matrices in $\mathrm{GL}_d(\mathbb{Z})$ and under arbitrary translations of the convex body. The set of lattice width directions is also unchanged under scalings of the convex body. Note that in Figure 1 the polygon on the right has many lattice width directions. This is an instance of a *regular lattice cross-polytope*, which is defined as the convex hull of $x \pm \lambda e_1, \ldots, x \pm \lambda e_d$ for some $x \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$, and a lattice basis $e_1, \ldots, e_d$ of $\mathbb{Z}^d$. Our main result shows that this is indeed the only extreme case.

Theorem 1.1. *The number of lattice width directions of a non-empty subset S of $\mathbb{R}^d$ with $\dim(S) = d$ is at most $3^d - 1$. Equality holds if and only the closure of the convex hull of S is a regular lattice cross-polytope.*

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We prove this result in Section 2. The proof relies on the following strengthening of a theorem of Minkowski about centrally-symmetric convex sets with only one interior lattice point. Denote by $K_{\mathbb{Z}}$ the set of lattice points in the convex set K, and by $K_{\mathbb{Z}}^{\circ}$ the set of lattice points in the relative interior of K. While the most well-known lattice point theorem by Minkowski gives an upper bound on the volume of a centrally-symmetric convex set with only one interior lattice point, the result we are interested in yields an upper bound on the number of lattice points. We say that K is a *standard lattice cube* if there is a lattice basis $e_1, \ldots, e_d$ of $\mathbb{Z}^d$ such that K is the convex hull of $\pm e_1 \pm \cdots \pm e_d$.

Theorem 1.2. *Let $K \subseteq \mathbb{R}^d$ be a centrally-symmetric convex set. If $K_{\mathbb{Z}}^{\circ} = \{0\}$, then $|K_{\mathbb{Z}}| \leq 3^d$, with equality if and only if K is a standard lattice cube.*

We remark that there are centrally-symmetric compact convex sets K with $K_{\mathbb{Z}}^{\circ} = \{0\}$ that are *not* contained in a standard lattice cube, see Remark 4.9 in [Nill06b].

The upper bound in Theorem 1.2 was proved by Minkowski [Mink10, §31, p.79]; a reference in English is [Hanc64, Art. 45 p.149]. We give two proofs for the fact that only the standard lattice cube attains the upper bound. First, in Section 3 we use a geometric argument due to Groemer [Groe61]. Second, in Section 4, we give a self-contained proof. The latter proof is based on congruences modulo 3, in the line of Minkowski's original approach.

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2. Proof of Theorem 1.1

Let a non-empty set $S \subseteq \mathbb{R}^d$ be given. We define $\mathcal{D}(S)$ to be the set of vectors $v \in (\mathbb{R}^d)^*$ such that $\sup v(S) < \infty$ and $\inf v(S) > -\infty$. The set $\mathcal{D}(S)$ is easily seen to be a linear subspace of $\mathbb{R}^d$. For $v \in \mathcal{D}(S)$, we define the *width of S in direction v* to be

$$\mathrm{w}(S, v) := \sup v(S) - \inf v(S).$$

Note that the width does not change if we replace S by the closure of its convex hull. For S convex the map sending v to the first term is sometimes referred to as the *Stützfunktion* of S. Now let $\mathcal{L}(S)$ be the intersection of $\mathcal{D}(S)$ with the lattice $(\mathbb{Z}^d)^*$. The *lattice width* of S is given by

$$\mathrm{w}(S) = \inf\{\mathrm{w}(S, v) \,:\, v \in \mathcal{L}(S) \setminus \{0\}\}.$$

The set over which the infimum is taken may be empty, in which case we set the lattice width equal to ∞. The set of *lattice width directions* of S is defined as

$$S' := \{v \in \mathcal{L}(S) \setminus \{0\} \,:\, \mathrm{w}(S, v) = \mathrm{w}(S)\}.$$

We now show how $\mathrm{w}(S)$ and S' can be determined from a certain compact convex set related to S. Let e be the rank of $\mathcal{L}(S)$. Denote by $\mathcal{L}(S)_{\mathbb{R}}$ the vector subspace of $(\mathbb{R}^d)^*$ spanned by $\mathcal{L}(S)$; this is an e-dimensional, potentially strict, subspace of $\mathcal{D}(S)$. Let $V \subseteq \mathbb{R}^d$ be the subspace of $\mathbb{R}^d$ where all elements of $\mathcal{L}(S)_{\mathbb{R}}$ are zero, and let π denote the natural projection $\mathbb{R}^d \to \mathbb{R}^d/V$. This π maps $\mathbb{Z}^d$ to a lattice Λ of full rank e in the e-dimensional space $\mathbb{R}^d/V$, and the lattice dual to this lattice is canonically isomorphic to $\mathcal{L}(S) \subseteq \mathcal{L}(S)_{\mathbb{R}}$. The following lemma is straightforward, and reduces the study of lattice width and lattice width directions to the case where S has $\mathcal{D}(S) = \mathbb{R}^d$, i.e., to *bounded* sets S.

Lemma 2.1. *The lattice width of S relative to $\mathbb{Z}^d$ is equal to that of $\pi(S)$ relative to Λ. Similarly, S' equals $\pi(S)'$ under the identification $\Lambda^* = \mathcal{L}(S)$.*

Furthermore, if S is bounded, then we can make it compact and convex by passing to the closure of its convex hull.

Example 2.2. Let S be given as $\mathbb{R}_{\geq 0}\,(1, \sqrt{2}, 0) + [0,1]\,(0,0,1) \subseteq \mathbb{R}^3$. Then S is convex, unbounded, and contained in an affine hyperplane. Identify $\mathbb{Z}^3$ with $(\mathbb{Z}^3)^*$ via the usual scalar product. Then $\mathcal{D}(S) = \mathbb{R}(-\sqrt{2}, 1, 0) + \mathbb{R}(0,0,1) \supsetneq \mathcal{L}(S) = \mathbb{Z}(0,0,1)$. In particular $\mathrm{w}(S) = 1$ and $S' = \pm(0,0,1)$. In the notation of Lemma 2.1, we have $\pi(S) = [0,1] \subseteq \Lambda \cong \mathbb{Z}$.

Note that $\operatorname{Conv} S$ is not compact in the previous example.

Proposition 2.3. *Let $S \subseteq \mathbb{R}^d$ be a non-empty, compact and convex subset. Then $\mathrm{w}(S) < \infty$ if and only if $d > 0$. In this case:*

1. *If $\dim(S) < d$, then $\mathrm{w}(S) = 0$. Moreover, $S' \neq \emptyset$ if and only if S is contained in an affine hyperplane with a rational defining vector.*
2. *If $\dim(S) = d$, then $\mathrm{w}(S) > 0$ and $S' \neq \emptyset$.*

Proof. (1) After translating S, which does not effect $\mathrm{w}(S)$ or S', we may assume that S lies in the hyperplane through the origin defined by a non-zero element $w \in (\mathbb{R}^d)^*$. If w can be chosen in the lattice, then $\mathrm{w}(S) \leq \mathrm{w}(S, w) = 0$ and $w \in S'$ and we are done. If not, then the following argument shows that $\mathrm{w}(S) = 0$ still holds, while $S' = \emptyset$. Fix $\epsilon > 0$ and consider the set

$$Z := \{v \in (\mathbb{R}^d)^* \mid v(S) \subseteq (-\epsilon/2, +\epsilon/2)\}.$$

By compactness of S this set contains a d-dimensional ball B centered at the origin. Moreover, Z is stable under translation over multiples of w. These facts imply that Z has infinite volume. Moreover, Z is centrally-symmetric and convex since the interval $(-\epsilon/2, +\epsilon/2)$ is. By Minkowski's well-known lattice point theorem [Mink10, Hanc64] Z contains a non-zero lattice point v. But then $\mathrm{w}(S, v) < \epsilon$.

(2) Since S is compact, $\mathcal{D}(S) = \mathbb{R}^d$ and $\mathrm{w}(S) < \infty$. Since S contains a ball B of dimension d, it is clear that $\mathrm{w}(S, v) \geq \mathrm{w}(B, v) \geq \mathrm{w}(S) + 1$ for v outside some large ball in $(\mathbb{R}^d)^*$. This large ball has only finitely many lattice points, hence, $\mathrm{w}(S)$ is attained by one of these lattice points. In particular, $S' \neq \emptyset$. Since S is not contained in an affine hyperplane, we have $\mathrm{w}(S) > 0$. $\square$

Example 2.4. Let us illustrate the previous proposition for $S = \{(0,0), (1, \sqrt{2})\} \subset \mathbb{R}^2$. Then there exist $a, b \in \mathbb{Z} \setminus \{0\}$ such that $\sqrt{2} \approx \frac{a}{b}$. Therefore, for $v := (a, -b)$ we see $\mathrm{w}(S, v) \approx 0$. Hence, $\mathrm{w}(S) = 0$. However, $S' = \emptyset$, since $\mathcal{L}(S) \cap S^\perp = \{0\}$. Moreover, note that for $S = \mathbb{R}_{\geq 0}(1, \sqrt{2})$ we have $\mathrm{w}(S) = \infty$.

Combining Lemma 2.1 and Proposition 2.3 yields the following observation.

Corollary 2.5. *Let $\emptyset \neq S \subseteq \mathbb{R}^d$ with $\dim(S) = d$. Then $\mathrm{w}(S) > 0$.*

When S is a full-dimensional and compact convex set, observe that lattice width directions are necessarily primitive lattice vectors—that is, they are not properly divisible by an integer. The following result shows that even more is true.

Theorem 2.6. *Let S be a subset of $\mathbb{R}^d$ such that $0 < \mathrm{w}(S) < \infty$. Then $\operatorname{Conv} S'$ is a non-empty, convex, centrally-symmetric set that contains no lattice point other* *than the origin in its relative interior. Moreover, the lattice points on the boundary of* $\operatorname{Conv} S'$ *are precisely the elements of* S'.

Proof. Convexity and central-symmetry are immediate from the definition of S'. Non-emptiness follows from Lemma 2.1 and Proposition 2.3.

It is easy to verify that $\mathrm{w}(S, -): \mathcal{D}(S) \to \mathbb{R}$ is a convex homogeneous function of degree 1. Suppose that $v \in \operatorname{Conv} S'$. By Carathéodory's theorem, there exist $v_1, \ldots, v_n \in S'$ and coefficients $0 \le \lambda_1, \ldots, \lambda_n$ with $\lambda_1 + \cdots + \lambda_n = 1$ such that $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$. Hence,

$$\mathrm{w}(S, v) \le \lambda_1 \mathrm{w}(S, v_1) + \cdots + \lambda_n \mathrm{w}(S, v_n) = \mathrm{w}(S).$$

In particular, if v is a nonzero lattice point, so that $\mathrm{w}(S, v) \ge \mathrm{w}(S)$, then we have $\mathrm{w}(S, v) = \mathrm{w}(S)$, so that $v \in S'$. Therefore, the non-zero lattice points in $\operatorname{Conv} S'$ are precisely the elements of S'.

Moreover, we cannot have v in the relative interior of $\operatorname{Conv} S'$, since, as shown above, that would imply that, for some $\epsilon > 0$, $\mathrm{w}(S, (1+\epsilon)v) \le \mathrm{w}(S)$, contradicting the fact that $\mathrm{w}(S, (1+\epsilon)v) = (1+\epsilon)\,\mathrm{w}(S, v) > \mathrm{w}(S)$ by minimality. □

Remark 2.7. As we have seen, the notion of lattice width directions is quite subtle. Here is an important case where everything works out nice. Let S be a *rational polyhedron*, namely, a convex set given by finitely many linear inequalities

$$S = \{x \in \mathbb{R}^d \ : \ f_i(x) \ge c_i \ \forall\, i = 1, \ldots, m\},$$

where $f_i \in (\mathbb{Z}^d)^*$ and $c_i \in \mathbb{Z}$. By standard arguments in convex geometry it follows that $\mathcal{D}(S) = \operatorname{Rec}(S)^{\perp}$, where

$$\operatorname{Rec}(S) = \{y \in \mathbb{R}^d \ : \ \exists\, x \in S \text{ with } x + \mathbb{R}_{>0} y \subseteq S\}$$

is the *recession cone* of S. Since

$$\operatorname{Rec}(S) = \{y \in \mathbb{R}^d \ : \ f_i(y) \ge 0 \ \forall\, i = 1, \ldots, m\},$$

we see that $\mathcal{D}(S) = \mathcal{L}(S)_{\mathbb{R}}$ is the largest subspace contained in the rational polyhedral cone spanned by $f_1, \ldots, f_m$. The criterion in Theorem 2.6, $0 < \mathrm{w}(S) < \infty$, holds if and only if $\dim(S) = d$ and $\dim(\operatorname{Rec}(S)) < d$.

We now show how Theorems 1.2 and 2.6 imply Theorem 1.1. Note that in general a full-dimensional compact convex set S is not uniquely determined by S', as exemplified in Figure 1.

Proof of Theorem 1.1. We may assume $\mathrm{w}(S) < \infty$. By Corollary 2.5 and Theorem 2.6, we can apply Theorem 1.2 to $\operatorname{Conv} S'$. This yields the desired upper bound $|S'| \le 3^d - 1$ on the set S' of lattice width directions. Note that the bound is actually at most $3^{d-1} - 1$ if $\operatorname{Conv} S'$ does not have full dimension. Hence, if $|S'|$ equals $3^d - 1$, then $\operatorname{Conv} S'$ is d-dimensional and by Theorem 1.2 there exists a lattice basis $e_1^*, \ldots, e_d^*$ of $(\mathbb{Z}^d)^*$ such that $\operatorname{Conv} S'$ is the standard lattice cube with vertices $\pm e_1 \pm \ldots \pm e_d$. After replacing S by the closure of its convex hull we may assume that S is closed and convex. Since all coordinates e_i^* are bounded on S, the latter set is bounded, hence compact.

We now show that S is then a regular cross-polytope. After translating S, we may assume that all coordinates take the same maximum λ and the same minimum $-\lambda$ on S. For $i = 1, \ldots, d$, let $p_i = \sum_{j=1}^d p_{ij} e_j \in S$ be a point with i-th coordinate

$p_{ii} = \lambda$, and let $q_i = \sum_{j=1}^d q_{ij} e_j \in S$ be a point with i-th coordinate $q_{ii} = -\lambda$. By assumption, for every direction $v \in \{-1, 0, 1\}^d$, there exists a $t_v \in \mathbb{R}$ such that

$$(*) \qquad -\lambda + t_v \leq v(p) \leq \lambda + t_v$$

for all $p \in S$. In particular, for distinct i, j we have

$$\begin{aligned} p_{ij} &\leq t_{e_i^*+e_j^*}, & t_{e_i^*+e_j^*} &\leq q_{ij} \\ -p_{ij} &\leq t_{e_i^*-e_j^*}, \text{ and} & t_{e_i^*-e_j^*} &\leq -q_{ij}, \end{aligned}$$

so that $p_{ij} = t_{e_i^*+e_j^*} = -t_{e_i^*-e_j^*} = q_{ij}$. Similarly,

$$\begin{aligned} p_{ji} &\leq t_{e_i^*+e_j^*}, & t_{e_i^*+e_j^*} &\leq q_{ji} \\ t_{e_i^*-e_j^*} &\leq p_{ji}, \text{ and} & q_{ji} &\leq t_{e_i^*-e_j^*}, \end{aligned}$$

so that $p_{ji} = t_{e_i^*+e_j^*} = t_{e_i^*-e_j^*} = q_{ji}$. Combining these, we find that $p_{ij} = q_{ij} = 0$ for all distinct i, j, so that $p_i = \lambda e_i = -q_i$. But then the inequalities $(*)$ for v, by filling in p_i, q_i for some i for which $v_i \neq 0$, give $t_v = 0$ for all v. The inequalities thus reduce to inequalities cutting out the cross-polytope spanned by p_i and q_i. Hence, S contains this cross-polytope and is contained in it. □

3. A geometric proof of Theorem 1.2

In this section we give a geometric proof of Theorem 1.2, inspired by Minkowski's proof of his lattice point theorem. It is based on Groemer's article [Groe61]. We start with the following observation, a folklore result for which we could not find an explicit reference in the literature.

Theorem 3.1. *Let $K \subseteq \mathbb{R}^d$ be a centrally-symmetric convex set with $K_{\mathbb{Z}}^\circ = \{0\}$. Then the union of the elements of*

$$\mathscr{K} = \{K + 2\alpha : \alpha \in K_{\mathbb{Z}}\}$$

is contained in $3K$ and the relative interiors of these elements are pairwise disjoint.

Proof. For $x \in K$ and $\alpha \in K_{\mathbb{Z}}$ we have $x + 2\alpha = 3(\frac{1}{3}x + \frac{2}{3}\alpha) \in 3K$ by convexity of K. This shows that $\bigcup \mathscr{K} \subseteq 3K$. To see that the relative interiors of the elements of $\mathscr{K}$ are disjoint, suppose otherwise. Then there exist x, y in the relative interior of K and distinct $\alpha, \beta \in K_{\mathbb{Z}}$ such that $x + 2\alpha = y + 2\beta$. By central symmetry $(x - y)/2$ is in the relative interior of K, while it equals $\beta - \alpha$, which is a non-zero lattice point. This contradicts the assumption that $K_{\mathbb{Z}}^\circ = \{0\}$. □

As a straightforward consequence of this result we can prove the 3^d-bound.

Proof of upper bound in Theorem 1.2. Let $d' \leq d$ be the dimension of K. It follows from the theorem just proved that

$$(1) \qquad |K_{\mathbb{Z}}| \operatorname{Vol}(K) \leq \operatorname{Vol}(3K) = 3^{d'} \operatorname{Vol}(K),$$

so that $|K_{\mathbb{Z}}| \leq 3^{d'} \leq 3^d$, as claimed. □

For the equality case we use Hilfssatz 2 of [Groe61]. For this recall that a *parallelepiped* is any $\mathbb{R}^d$-translate of the convex hull of the points $\pm e_1 \pm \cdots \pm e_d$ for an $\mathbb{R}$-basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$. By a *homothetic copy* of a subset K of $\mathbb{R}^d$ we mean any set of the form $\alpha + \lambda K$ for some $\alpha \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}_+$.

Theorem 3.2 (Groemer 1961). *Let $K \subseteq \mathbb{R}^d$ be a d-dimensional compact convex subset of $\mathbb{R}^d$ that can be covered with finitely many homothetic copies of K, whose interiors are mutually disjoint. Then K is a parallelepiped.*

Using this geometric result we can finish the proof of Theorem 1.2.

Proof of equality case in Theorem 1.2. By Equation (1) we may assume that K is d-dimensional. Let us first argue that it suffices to consider the case where K is compact. We already know by Minkowski's fundamental lattice point theorem that the volume of K is bounded. Now, let $\overline{K}$ be the closure of K. Then $\overline{K}$ is also a d-dimensional centrally-symmetric convex set such that $\overline{K}^{\circ}_{\mathbb{Z}} = \{0\}$. Therefore, the 3^d-bound yields $\left|\overline{K}_{\mathbb{Z}}\right| = 3^d$. Assume we already showed that $\overline{K}$ is a standard lattice cube. Since K has the same number of lattice points as $\overline{K}$, all of the 2^d vertices of the standard lattice cube $\overline{K}$ also have to be contained in K. This shows $K = \overline{K}$.

Hence, we may assume that K is compact. Now, by Theorem 3.1 we see that that the translates of K by its 3^d lattice points together cover $3K$ and that their interiors do not intersect. Applying Theorem 3.2 to $3K$ yields that $3K$ is a parallelepiped, hence so is K. By central symmetry, K equals the convex hull of the 2^d vertices $\pm e_1 \pm \ldots \pm e_d$ for some $\mathbb{R}$-basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$. It remains to show that $e_1, \ldots, e_d$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^d$. We do this by arguing that there is only one way to cover the parallelepiped $3K$ with 3^d translates of K, namely with the translates over the vectors $2\sum_{i=1}^d \epsilon_i e_i$ with each $\epsilon_i \in \{-1, 0, 1\}$. Indeed, this follows from a simple induction on d: consider any covering of $3K$ with 3^k translates of K. Then their interiors do not intersect for volume reasons. Now consider the facet F of $3K$ where the e_1-coordinate equals 3. This facet is a $(d-1)$-dimensional parallelepiped which is covered by facets of translates K_i of K. Since the interiors of the K_i do not intersect, the relative interiors of their facets F_i on F do not intersect either. Hence, for volume reasons there are exactly 3^{d-1} facets of translates of K covering F. By induction, the K_i are the translates of K over the vectors $2e_1 + 2\sum_{i=2}^d \epsilon_i e_i$ with $\epsilon_i \in \{-1, 0, 1\}$. The same argument applies to the remaining two layers of $3K$ in the e_1-direction, so that the covering of $3K$ equals the standard covering above. Now, since the translates of K over the vectors in $2K_{\mathbb{Z}}$ also cover $3K$, the vectors $\sum_{i=1}^d \epsilon_i e_i$ with each $\epsilon_i \in \{-1, 0, 1\}$ are precisely the lattice points in K. In particular, all e_i are in $\mathbb{Z}^d$. Finally, they must generate $\mathbb{Z}^d$, or else K would contain more than 3^d lattice points. This proves that K is a standard cube. □

4. A Minkowski-style proof of Theorem 1.2

Minkowski's original proof of the 3^d bound relies on considering congruences of lattice points. By the same method he also provided a sharpening of this bound in an important subcase. Let us recall his elegant proof of these results. For this let us denote for a subset S of $\mathbb{R}^d$ by $\mathrm{Conv}^{\circ}(S)$ the relative interior of $\mathrm{Conv}(S)$. Moreover, by $\partial K_{\mathbb{Z}}$ we denote the set of lattice points on the boundary of a convex set K.

Theorem 4.1 (Minkowski 1910). *Let $K \subseteq \mathbb{R}^d$ be a d-dimensional centrally-symmetric convex set.*

1. *If $K^{\circ}_{\mathbb{Z}} = \{0\}$, then $|K_{\mathbb{Z}}| \leq 3^d$.*
2. *If $K^{\circ}_{\mathbb{Z}} = \{0\}$ and no boundary lattice point of K is in the convex hull of some others, then $|K_{\mathbb{Z}}| \leq 2^{d+1} - 1$.*

Proof. (1) We regard the canonical map $\gamma\colon \mathbb{Z}^d \to (\mathbb{Z}/3\mathbb{Z})^d$. We claim that γ is injective on $K_\mathbb{Z}$. For let $x, y \in K_\mathbb{Z}$ such that $\gamma(x) = \gamma(y)$ be given. Then $\gamma(x - y) = 0$, so $z := (x - y)/3 \in \mathbb{Z}^d$. Since K is centrally-symmetric, we see that $z \in \mathrm{Conv}^\circ(0, x, -y) \subset K^\circ$. This implies $z \in K^\circ_\mathbb{Z} = \{0\}$, so $z = 0$. We deduce $x = y$, so γ is injective, as claimed.

(2) In this case, we look at the canonical map $\delta\colon \mathbb{Z}^d \to (\mathbb{Z}/2\mathbb{Z})^d$. Assume there is a boundary lattice point $v \in \partial K_\mathbb{Z}$ such that $\delta(v) = 0$. Then $v/2 \in \mathbb{Z}^d$, in particular, $0 \neq v/2 \in K^\circ_\mathbb{Z}$, a contradiction. Hence, $\delta^{-1}(0) \cap \partial K_\mathbb{Z} = \emptyset$. Let $0 \neq f \in (\mathbb{Z}/2\mathbb{Z})^d$ be fixed. We claim that $\left|\delta^{-1}(f) \cap \partial K_\mathbb{Z}\right| \leq 2$. From this we immediately get the upper bound. So, assume that there are $x, y \in \partial K_\mathbb{Z}$, $x \neq y \neq -x$, such that $\delta(x) = \delta(y)$. Then $\delta(x - y) = 0$ and therefore $z := (x - y)/2$ lies in $\mathbb{Z}^d$. Since K is centrally-symmetric and $x \neq -y$, we see that $z \in \mathrm{Conv}^\circ(x, -y)$. Since $x \neq y$, we have $z \neq 0$. Therefore $z \in \partial K_\mathbb{Z}$, a contradiction to the assumption. □

In the remainder of this section we prove Theorem 1.2 following Minkowski's approach. As it will turn out, it is enough to consider the case of lattice polytopes. For this let us recall that a *lattice polytope* is the convex hull of finitely many lattice points in $\mathbb{Z}^d$. Now, the main idea is to use the modulo map to inductively construct lattice points until we find a lattice point in the interior of a facet. This goal is inspired by the proof of Theorem 1.1 in a special case, see [Nill06a, Theorem 6.1]. Then we show that P has to be a prism over this facet by applying a lattice point addition method analogous to [Nill06a, Lemma 5.9]. This allows us to proceed by induction on the dimension.

From now on let $d \geq 2$, and $P \subseteq \mathbb{R}^d$ be a d-dimensional centrally-symmetric lattice polytope with $P^\circ_\mathbb{Z} = \{0\}$ and $|P_\mathbb{Z}| = 3^d$.

The following result is the key-lemma for our proof.

Lemma 4.2. *For $x, y \in P_\mathbb{Z}$ there exists a unique $z \in P_\mathbb{Z}$ such that*

$$w := \frac{x + y + z}{3} \in \mathbb{Z}^d.$$

The lattice point lies in $P_\mathbb{Z}$, and if $x \neq y$ then also $x \neq z \neq y$.

Proof. Consider the canonical map $\gamma\colon \mathbb{Z}^d \to (\mathbb{Z}/3\mathbb{Z})^d$. As was shown in the proof of Theorem 4.1(1) the map γ is injective on $P_\mathbb{Z}$. Since $|Pz| = 3^d$ it is actually a bijection. Therefore, there exists a unique $z \in P_\mathbb{Z}$ such that $\gamma(x) + \gamma(y) + \gamma(z) = 0$. This latter equality is equivalent to $w \in \mathbb{Z}^d$. The point w is a convex combination of x, y, z and hence lies in $P_\mathbb{Z}$. Finally, if $x \neq y$ then $\gamma(x) \neq \gamma(y)$ and hence $\gamma(z) = -\gamma(x) - \gamma(y)$ equals neither $\gamma(x)$ nor $\gamma(y)$. Hence $x \neq z \neq y$, as desired. □

We are going to use this observation in an inductive way. For this, let us write $F \leq P$ if F is a face of P, and let us denote by $\mathcal{V}(P)$ the set of vertices, *i.e.*, 0-dimensional faces, of P, and by $\mathcal{F}(P)$ the set of facets, i.e., $(d-1)$-dimensional faces. If F is a facet of P, we denote by $u_F \in (\mathbb{Q}^d)^*$ the unique outer normal of F determined by $u_F(F) = 1$ and $u_F(P) \leq 1$.

Proposition 4.3. *For $k = 1, \ldots, d-1$ there exists a face $F \lneq P$ such that* $\dim(F) \geq$ *k and $F^\circ_\mathbb{Z} \neq \emptyset$.*

Proof. Let $k = 1$ and assume the statement were false. In this case, $\mathcal{V}(P) = \partial P_\mathbb{Z}$, hence Theorem 4.1(2) yields $3^d = |P_\mathbb{Z}| \leq 2^{d+1} - 1$, in contradiction to $d \geq 2$.

We proceed by induction. Let $2 \leq k \leq d-1$. Then, by the induction hypothesis, there exists a face $F \lneq P$ such that $\dim(F) \geq k-1$ and $F_{\mathbb{Z}}^{\circ} \neq \emptyset$. We may assume that $\dim(F) = k-1$ and $x \in F_{\mathbb{Z}}^{\circ}$. Let us choose a face G of P of dimension k such that $F \subset G$. Since $k < d$, we have $G \neq P$. Because F is a facet of G, there exists a vertex $y \in \mathcal{V}(G)$, $y \notin F$, such that $\mathrm{Conv}^{\circ}(x, y) \subseteq G^{\circ}$. Let z, w be chosen as in Lemma 4.2. We distinguish two cases.

(1) $\dim(x, y, z) = 1$. We have three subcases to consider:
   (a) $x \in \mathrm{Conv}^{\circ}(y, z)$. Since $x \in F$, we get $y \in F$, a contradiction.
   (b) $y \in \mathrm{Conv}^{\circ}(x, z)$. This is a contradiction to $y \in \mathcal{V}(P)$.
   (c) $z \in \mathrm{Conv}^{\circ}(x, y)$. Hence, $z \in G_{\mathbb{Z}}^{\circ}$, so G satisfies the conditions of the Proposition, as desired.

(2) $\dim(x, y, z) = 2$. Therefore, $w = (x + y + z)/3 \in \mathrm{Conv}^{\circ}(x, y, z)$. Let H be a face of P such that $w \in H_{\mathbb{Z}}^{\circ}$. Then, $x, y, z \in H$. In particular, $\mathrm{Conv}^{\circ}(x, y) \subseteq H$, so also $G \subseteq H$. Hence, $\dim(H) \geq \dim(G) = k$. We claim that H satisfies the conditions of the Proposition. It remains to show that $H \neq P$. So, assume $H = P$. In this case, $w \in P_{\mathbb{Z}}^{\circ}$, so $w = 0$, in particular, $x + y + z = 0$. Now, let u be the unique outer normal of a facet of P containing G. By central-symmetry, $-u$ is also an outer normal of a facet of P. However, $-u(z) = -u(-x-y) = 2$, in contradiction to $z \in P$.

□

Applying the Proposition for $k = d - 1$ yields:

Corollary 4.4. *There exists a facet $F \in \mathcal{F}(P)$ such that $F_{\mathbb{Z}}^{\circ} \neq \emptyset$.*

From now on, we will intensively use this corollary.

Proposition 4.5. *Let $x \in F_{\mathbb{Z}}^{\circ}$ for $F \in \mathcal{F}(P)$. Then*

$$x + (P_{\mathbb{Z}} \setminus F_{\mathbb{Z}}) \subseteq P_{\mathbb{Z}}.$$

Proof. Let $y \in P_{\mathbb{Z}} \setminus F_{\mathbb{Z}}$. Therefore, $\mathrm{Conv}^{\circ}(x, y) \subseteq P^{\circ}$. We may assume $y \notin \{0, -x\}$. Let z, w be chosen as in Lemma 4.2. Again, we distinguish two cases.

(1) $\dim(x, y, z) = 1$.
   (a) $x \in \mathrm{Conv}^{\circ}(y, z)$. Since $x \in F$, we get $y \in F$, a contradiction.
   (b) $y \in \mathrm{Conv}^{\circ}(x, z)$. Since $y \notin F$, we get $z \notin F$. Therefore, $y \in P_{\mathbb{Z}}^{\circ} = \{0\}$, a contradiction.
   (c) $z \in \mathrm{Conv}^{\circ}(x, y)$. Hence, $z \in P_{\mathbb{Z}}^{\circ} = \{0\}$, so necessarily $y = -x$, a contradiction.

(2) $\dim(x, y, z) = 2$. Therefore, $w = (x + y + z)/3 \in \mathrm{Conv}^{\circ}(x, y, z)$. Hence, $w \in P_{\mathbb{Z}}^{\circ} = \{0\}$. This implies $x + y = -z \in P_{\mathbb{Z}}$ by central-symmetry, as desired.

□

Here is a direct consequence. For this let us define $u_F^{\perp} := \{v \in \mathbb{R}^d : u_F(v) = 0\}$ for a facet $F \in \mathcal{F}(P)$.

Corollary 4.6. *Let $x \in F_{\mathbb{Z}}^{\circ}$ for $F \in \mathcal{F}(P)$. Then*

$$P_{\mathbb{Z}} = F_{\mathbb{Z}} \sqcup (P_{\mathbb{Z}} \cap u_F^{\perp}) \sqcup (-F_{\mathbb{Z}}).$$

Moreover, the map $\mathbb{Z}^d \to \mathbb{Z}^d$, $y \mapsto x + y$, induces bijections

$$(-F_{\mathbb{Z}}) \to (P_{\mathbb{Z}} \cap u_F^{\perp}) \to F_{\mathbb{Z}}.$$

In particular, $\mathcal{V}(P) \subseteq F_{\mathbb{Z}} \sqcup (-F_{\mathbb{Z}})$.

Proof. Assume the first statement is wrong. Then, by central-symmetry, there exists $y \in P_{\mathbb{Z}}$, $y \notin F_{\mathbb{Z}}$, such that $u_F(y) > 0$. Proposition 4.5 yields $x + y \in P_{\mathbb{Z}}$. However, $u_F(x + y) > 1$, a contradiction.

The second statement follows by central-symmetry from Proposition 4.5. For the last statement, note that, since P is a lattice polytope, we have $\mathcal{V}(P) \subseteq P_{\mathbb{Z}}$. So assume $y \in \mathcal{V}(P)$ with $y \in u_F^{\perp}$. Then $y \in \mathrm{Conv}^{\circ}(-x + y, x + y)$ with $-x + y \in -F_{\mathbb{Z}}$ and $x + y \in F_{\mathbb{Z}}$, a contradiction. $\square$

Now, we can easily finish the proof of Theorem 1.2.

Proof of Theorem 1.2. By Theorem 4.1(1) it remains to prove the equality case. For this, let us first deal with the case of a lattice polytope P as before. The proof is by induction on the dimension d. We may assume $d \geq 2$. By Corollary 4.4 there exists a facet $F \in \mathcal{F}(P)$ such that F has an interior lattice point x. Now, Corollary 4.6 actually shows that $-F = F - 2x$ and $P = \mathrm{Conv}(F, F - 2x)$, i.e., P is a prism over F. Moreover, we see that $F - x = P \cap u_F^{\perp}$ is a $(d-1)$-dimensional centrally-symmetric lattice polytope (with respect to the lattice $\mathbb{Z}^d \cap u_F^{\perp}$) such that $(F - x)^{\circ} = \{0\}$ and $|(F - x)_{\mathbb{Z}}| = 3^{d-1}$. Hence, the induction hypothesis yields that $F - x$ is a standard lattice cube (with respect to a lattice basis $e_1, \ldots, e_{d-1}$). It remains to show that $e_1, \ldots, e_{d-1}, x$ is a lattice basis of $\mathbb{Z}^d$. This follows, since any lattice point in $\mathbb{Z}^d$ can be successively translated via $e_1, \ldots, e_{d-1}, x$ into P, and $P_{\mathbb{Z}} \subseteq \{\pm e_1 + \cdots + \pm e_{d-1} \pm x\}$ by Corollary 4.6.

In the general case, let $K \subseteq \mathbb{R}^d$ be a d-dimensional centrally-symmetric convex set with $K_{\mathbb{Z}}^{\circ} = \{0\}$ and $|K_{\mathbb{Z}}| = 3^d$. We define $P \coloneqq \mathrm{Conv}(K_{\mathbb{Z}})$. This is a centrally-symmetric lattice polytope with $P_{\mathbb{Z}}^{\circ} = \{0\}$ and $|P_{\mathbb{Z}}| = 3^d$, in particular $\dim(P) = d$ by Theorem 4.1(1). Therefore, P is a standard lattice cube (with respect to a lattice basis $e_1, \ldots, e_d$). Assume $P \subsetneq K$. Then there exists $x \in K$, $x \notin P$. Hence, there is a facet $F \in \mathcal{F}(P)$ such that $u_F(x) > 1$. We may assume $F = e_1 + [-1, 1]e_2 + \cdots + [-1, 1]e_d$. However, this implies that $e_1 \in \mathrm{Conv}^{\circ}(0, \mathcal{V}(F), x)$, a contradiction to $K_{\mathbb{Z}}^{\circ} = \{0\}$. $\square$

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