THE QUANTIZATION FOR MARKOV-TYPE MEASURES ON A CLASS OF RATIO-SPECIFIED GRAPH DIRECTED FRACTALS

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ABSTRACT. We study the asymptotic quantization error of order r for Markov-type measures μ on a class of ratio-specified graph directed fractals. We show that the quantization dimension of μ exists and determine its exact value s_r in terms of spectral radius of a related matrix. We prove that the s_r-dimensional lower quantization coefficient of μ is always positive. Moreover, inspired by Mauldin-Williams's work on the Hausdorff measure of graph directed fractals, we establish a necessary and sufficient condition for the s_r-dimensional upper quantization coefficient of μ to be finite.

1. Introduction

In this paper, we study the asymptotics of the quantization error for Markov-type measures on a class of ratio-specified graph directed fractals. One of the main mathematical aims of the quantization problem is to study the error in the approximation of a given probability measure with probability measures of finite support. We refer to [3, 4, 6, 7, 15] for more theoretical results and [13, 14] for promising applications of quantization theory. One may see [8, 16] for its deep background in information theory and engineering technology. In the following, let us recall some of the crucial definitions and known results.

1.1. **The upper (lower) quantization dimension and quantization coefficient.** We set $\mathcal{D}_n := \{\alpha \subset \mathbb{R}^q : 1 \leq \mathrm{card}(\alpha) \leq n\}$ for $n \in \mathbb{N}$. Let ν be a Borel probability measure on $\mathbb{R}^q$. For every $n \geq 1$, the nth quantization error for ν of order r is defined by (see [3] for a number of equivalent definitions):

$$e_{n,r}(\nu) := \inf_{\alpha \in \mathcal{D}_n} \left(\int d(x,\alpha)^r d\nu(x) \right)^{1/r}. \tag{1.1}$$

Here $d(x,\alpha) := \inf_{a\in\alpha} d(x,a)$ and $d(\cdot,\cdot)$ is the metric induced by an arbitrary norm on $\mathbb{R}^q$. For $r \geq 1$, $e_{n,r}(\nu)$ agrees with the error in the approximation of ν by discrete probability measures supported on at most n points, in the sense of the Wasserstein L_r-metric [3].

The convergence rate of $e_{n,r}(\nu)$ is characterized by the upper and lower quantization dimension of order r as defined below:

$$\overline{D}_r(\nu) := \limsup_{n\to\infty} \frac{\log n}{-\log e_{n,r}(\nu)}, \; \underline{D}_r(\nu) := \liminf_{n\to\infty} \frac{\log n}{-\log e_{n,r}(\nu)}.$$

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If $\overline{D}_r(\nu) = \underline{D}_r(\nu)$, the common value is denoted by $D_r(\nu)$ and called the quantization dimension for ν of order r. For $s > 0$, we define the s-dimensional upper and lower quantization coefficient for ν of order r by (cf. [3, 15])

$$\overline{Q}_r^s(\nu) := \limsup_{n\to\infty} n^{r/s} e_{n,r}(\nu)^r, \quad \underline{Q}_r^s(\nu) := \liminf_{n\to\infty} n^{r/s} e_{n,r}(\nu)^r.$$

According to [3, Proposition 11.3] (see also [15]), the upper (lower) quantization dimension is exactly the critical point at which the upper (lower) quantization coefficient jumps from zero to infinity. Compared with the dimensions, the coefficients provide us with more accurate information for the asymptotic properties of the quantization error whenever they are both positive and finite. Therefore, it is one of the standard topics in the quantization problem to examine the finiteness and positivity of the latter. So far, the upper (lower) quantization coefficient has been well studied for absolutely continuous probability measures [3, Theorem 6.2] and some classes of fractal measures, such as self-similar measures [4, 5] and diadic homogeneous Cantor measures [10].

Let $(f_i)_{i=1}^N$ be a family of contractive similitudes on $\mathbb{R}^q$. By [9], there exists a unique Borel probability measure ν satisfying $\nu = \sum_{i=1}^N q_i \nu \circ f_i^{-1}$ associated with $(f_i)_{i=1}^N$ and a probability vector $(q_i)_{i=1}^N$. Recall that $(f_i)_{i=1}^N$ is said to satisfy the open set condition (OSC), if there exists a non-empty bounded open set U such that $\bigcup_{i=1}^N f_i(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ for all $1 \le i \ne j \le N$. Graf and Luschgy proved the following result which often provides us with significant insight into the study for non-self-similar probability measures:

Theorem (Graf/Luschgy [4, 5])**.** Assume that $(f_i)_{i=1}^N$ satisfies the OSC. Let ν be the self-similar measure associated with $(f_i)_{i=1}^N$ and a probability vector $(q_i)_{i=1}^N$. Let k_r be the unique solution of the equation $\sum_{i=1}^N (q_i s_i^r)^{\frac{k_r}{k_r+r}} = 1$. Then

$$D_r(\nu) = k_r, \quad 0 < \underline{Q}_r^{k_r}(\nu) \le \overline{Q}_r^{k_r}(\nu) < \infty.$$

1.2. **A class of graph directed fractals and Markov-type measures.** In this subsection, we recall the definitions of a class of graph directed fractals and Markov-type measures on these sets. One may see [1, 2, 12] for more details.

Let $P := (p_{ij})_{N\times N}$ be a row-stochastic matrix, namely, $p_{ij} \ge 0, 1 \le i,j \le N$, and $\sum_{j=1}^N p_{ij} = 1, 1 \le i \le N$. We always assume

$$\text{card}(\{1 \le j \le N : p_{ij} > 0\}) \ge 2 \ \text{ for all } \ 1 \le i \le N. \tag{1.2}$$

We will need the following notations. Set

$$\begin{aligned}
&\theta := \text{empty word}, \ \ \Omega_0 := \{\theta\}, \ \Omega_1 := \{1, \dots N\};\\
&\Omega_k := \{\sigma \in \Omega_1^k : p_{\sigma_1\sigma_2} \cdots p_{\sigma_{k-1}\sigma_k} > 0\}, \ k \ge 2;\\
&\Omega^* := \bigcup_{k\ge 0} \Omega_k, \ \Omega_\infty := \{\sigma \in \Omega_1^{\mathbb{N}} : p_{\sigma_h\sigma_{h+1}} > 0 \ \text{ for all } \ h \ge 1\}.
\end{aligned}$$

We denote by $|\sigma|$ the length of σ, namely, $|\sigma| := k$ for $\sigma \in \Omega_k$ and $|\theta| := 0$. For every word $\sigma = (\sigma_1, \dots, \sigma_n)$ with $n \ge k$ or $\sigma \in \Omega_\infty$, we write $\sigma|_k := (\sigma_1, \dots, \sigma_k)$. If $\sigma, \omega \in \Omega^*$ and $(\sigma_{|\sigma|}, \omega_1) \in \Omega_2$, then we define

$$\sigma * \omega = (\sigma_1, \sigma_2, \dots, \sigma_{|\sigma|}, \omega_1, \dots, \omega_{|\omega|}) \in \Omega_{|\sigma|+|\omega|}.$$

Let $J_i, 1 \le N$ be non-empty compact subsets of $\mathbb{R}^q$ with $J_i = \overline{\text{int}(J_i)}$ for all $1 \le i \le N$, where $\overline{B}$ and $\text{int}(B)$ respectively denote the closure and interior in $\mathbb{R}^q$ of a set $B \subset \mathbb{R}^q$. For convenience, we always assume that

$$\text{diam}(J_i) = 1 \ \text{ for all } \ 1 \le i \le N.$$

Let $(c_{ij})_{N\times N}$ be a non-negative matrix such that $c_{ij}>0$ if and only if $p_{ij}>0$. For each pair $1\le i,j\le N$ with $p_{ij}>0$, let T_{ij} be a contracting similitude on $\mathbb{R}^q$ of contraction ratio c_{ij}. Assume that, $T_{ij}(J_j)$, $(i,j)\in\Omega_2$, are non-overlapping subsets of J_i. By [1, Corollary 3.5] (see also [12, Theorem 3]), there exists a unique vector compact sets $(K_i)_{i=1}^N\subset\prod_{i=1}^N J_i$ such that

$$K_i=\bigcup_{j:(i,j)\in\Omega_2} T_{ij}(K_j),\ \ 1\le i\le N. \tag{1.3}$$

We call $K:=\bigcup_{i=1}^N K_i$ the recurrent self-similar set associated with the contracting similitudes $T_{ij}, 1\le i,j\le N$. One can see that K is also a *map-specified MW-fractal* which is defined in terms of a directed graph [12].

Assume that P is irreducible. Let $v=(q_i)_{i=1}^N$ be the unique normalized positive left eigenvector of P with respect to the Perron-Frobenius eigenvalue 1, or equivalently,

$$\sum_{i=1}^N v_i=1,\ v_i>0,\ 1\le i\le N;\ \sum_{i=1}^N v_ip_{ij}=v_j.$$

We accordingly have a unique vector $(\nu_i)_{i=1}^N$ of probability measures such that

$$\nu_i=\sum_{j:(i,j)\in\Omega_2} p_{ij}\nu_j\circ T_{ij}^{-1} \tag{1.4}$$

and ν_i is supported by K_i for each $1\le i\le N$. Hence, we get a Markov-type measure $\nu:=\sum_{i=1}^N p_i\nu_i$ supported on K.

Assuming the irreducibility of the corresponding transition matrices (or strong connectedness of the corresponding graphs), Lindsay has studied the quantization problem for Markov-type measures on map-specified graph directed fractals in [11]; in there he expressed the quantization dimension D_r in terms of temperature functions and showed that the D_r-dimensional upper quantization coefficient is finite. Let us note the following facts:

1. the arguments in [11] depend on the invariance properties (1.3) and (1.4); these arguments are not applicable to ratio-specified cases due to the absence of the invariance properties;
2. the interesting cases, where the transition matrices are reducible, have not been explored.

In the present paper, we consider the Markov-type measures μ on a class of *ratio-specified graph directed fractals* E. Mauldin and Williams [12, Theorem 4] have established a necessary and sufficient condition for the Hausdorff measure of a graph directed fractal to be positive and finite. Significantly inspired by this result, we will establish a necessary and sufficient condition for the upper and lower quantization coefficient of μ to be both positive and finite, allowing the corresponding transition matrices to be reducible.

Let $J_i, P=(p_{ij})_{N\times N}$, be given as above. We call J_i, $1\le i\le N$, cylinder sets of order one. For each $1\le i\le N$, let J_{ij}, $(i,j)\in\Omega_2$, be non-overlapping subsets of J_i such that J_{ij} is geometrically similar to J_j with $\text{diam}(J_{ij})/\text{diam}(J_j)=c_{ij}$. We call these sets cylinder sets of order two. Assume that cylinder sets of order k are determined, namely, for each $\sigma:=(i_1,\dots,i_k)\in\Omega_k$, we have a cylinder set J_σ. Let $J_{\sigma*i_{k+1}}$, $\sigma*i_{k+1}\in\Omega_{k+1}$, be non-overlapping subsets of J_σ such that $J_{\sigma*i_{k+1}}$ is geometrically similar to J_σ with $\text{diam}(J_{\sigma*i_{k+1}})/\text{diam}(J_\sigma)=c_{i_ki_{k+1}}$. Inductively, cylinder sets of order k are determined for all $k \geq 1$. The *ratio specified MW-fractal* is then given by

$$E := \bigcap_{k\geq 1} \bigcup_{\sigma\in\Omega_k} J_\sigma.$$

Note that we only fix the contraction ratios c_{ij}, $1 \leq i \leq j \leq N$, and we do not fix the similarity mappings, so a ratio-specified MW-fractal typically does not enjoy the invariance property (1.3) of K.

Let $(\chi_i)_{i=1}^N$ be an arbitrary probability vector with $\min_{1\leq i\leq N} \chi_i > 0$. By Kolmogorov consistency theorem, there exists a unique probability measure $\widetilde{\mu}$ on Ω_∞ such that $\widetilde{\mu}([\sigma]) = \chi_{\sigma_1} p_{\sigma_1\sigma_2} \cdots p_{\sigma_{k-1}\sigma_k}$ for every $k \geq 1$ and $\sigma = (\sigma_1, \ldots, \sigma_k) \in \Omega_k$, where $[\sigma] := \{\omega \in \Omega_\infty : \omega|_{|\sigma|} = \sigma\}$. Let π denote the projection from Ω_∞ to E: $\pi(\sigma) = x$, where

$$\{x\} := \bigcap_{k\geq 1} J_{\sigma|_k}, \text{ for } \sigma \in \Omega_\infty.$$

To overcome the difficulty caused by the absence of invariance properties, we assume the following separation property for E: there is some constant $0 < t < 1$ such that for every $\sigma \in \Omega^*$ and distinct $i_1, i_2 \in \Omega_1$ with $p_{\sigma_{|\sigma|} i_l} > 0, l = 1, 2$,

$$d(J_{\sigma*i_1}, J_{\sigma*i_2}) := \inf\{|x - y| : x \in J_{\sigma*i_1}, y \in J_{\sigma*i_2}\} \geq t \max\{|J_{\sigma*i_1}|, |J_{\sigma*i_2}|\}. \tag{1.5}$$

Here $|A|$ denotes the diameter of a set $A \subset \mathbb{R}^q$. Under this assumption, π is a bijection. We consider the image measure of $\widetilde{\mu}$ under the projection π: $\mu := \widetilde{\mu} \circ \pi^{-1}$. We call μ a Markov-type measure which satisfies

$$\mu(J_\sigma) = \chi_{\sigma_1} p_{\sigma_1\sigma_2} \cdots p_{\sigma_{k-1}\sigma_k} \text{ for } \sigma = (\sigma_1, \ldots, \sigma_k) \in \Omega_k. \tag{1.6}$$

As there are infinitely many similitudes corresponding to given contraction ratios c_{ij}, μ generally does not enjoy the invariance property (1.4).

1.3. **Statement of main results.** Before we state our main result, we need to recall some facts on spectral radius of matrices and some notations on directed graphs.

For $1 \leq i, j \leq N$, we define $a_{ij}(s) := (p_{ij} c_{ij}^r)^s$. Then we get an $N \times N$ matrix $A(s) = (a_{ij}(s))_{N\times N}$. Let $\psi(s)$ denote the spectral radius of $A(s)$. By [12, Theorem 2], $\psi(s)$ is continuous and strictly decreasing. Note that, by the assumption (1.2), $\psi(0) \geq 2$; by Perron-Frobenius theorem, we have,

$$\psi(1) \leq \max_{1\leq i\leq N} \sum_{j=1}^N a_{ij}(1) < \max_{1\leq i\leq N} \sum_{j=1}^N p_{ij} = 1.$$

Intermediate-value theorem implies that there exists a unique number $\xi \in (0, 1)$ such that $\psi(\xi) = 1$. Thus, for every $r > 0$, there exists a unique positive number s_r such that $\psi(s_r/(s_r + r)) = 1$.

As in [12], we consider the directed graph G associated with the transition matrix $(p_{ij})_{N\times N}$. Namely, G has vertices $1, 2, \ldots, N$; there is an edge from i to j if and only if $p_{ij} > 0$. In the following, we will simply denote by $G = \{1, \ldots, N\}$ both the directed graph and its vertex sets. We also write

$$b_{ij}(s) := (p_{ij} c_{ij}^r)^{s/(s+r)}, \ A_{G,s} := (b_{ij}(s))_{N\times N}; \ \Psi_G(s) := \psi(s/(s + r)).$$

We also refer to an element $(i_1, \ldots, i_k) \in \Omega_k$ as a path in G. We call $H \subset G$, with edges inherited from G, a subgraph of G. A subgraph H of G is called strongly connected if for very pair $i_1, i_2 \in H$, there exists a path γ in H which begins at i_1 and ends at i_2. A strongly connected component of G refers to a maximal strongly connected subgraph. Let $\mathrm{SC}(G)$ denote the set of all strongly connected components of G. For $H_1, H_2 \in \mathrm{SC}(G)$, we write $H_1 \prec H_2$, if there is a path $\gamma = (i_1, \ldots, i_k)$ in G such that $i_1 \in H_1$ and $i_k \in H_2$. If we have neither $H_1 \prec H_2$ nor $H_2 \prec H_1$, then we say H_1, H_2 are incomparable.

For every $H \in \mathrm{SC}(G)$, we denote by $A_{H,s}$ the sub-matrix $(b_{ij}(s))_{i,j\in H}$ of $A_G(s)$. Let $\Psi_H(s)$ be the spectral radius of $A_{H,s}$ and $s_r(H)$ be the unique positive number satisfying $\Psi_H(s_r(H)) = 1$. With assumption (1.2), one can see by pigeon-hole principle that G has at least one strongly connected component H with $\mathrm{card}(H) \geq 2$. As our main result, we will prove

Theorem 1.1. *Assume that (1.2) and (1.5) are satisfied and let μ be the Markov-type measure defined in (1.6), and let s_r be the unique positive number satisfying $\Psi_G(s_r) = 1$. Then we have,*

$$D_r(\mu) = s_r \ \text{and} \ \underline{Q}_r^{s_r}(\mu) > 0.$$

Furthermore, $\overline{Q}_r^{s_r}(\mu) < \infty$ if and only if $\mathcal{M} := \{H \in \mathrm{SC}(G) : s_r(H) = s_r\}$ consists only of incomparable elements; otherwise, we have $\underline{Q}_r^{s_r}(\mu) = \infty$.

At this point, we remark that, although the quantization problem for probability measures and the Hausdorff measure of sets are two substantially different objects, we benefit significantly from some methods previously developed in [12].

2. Notations and preliminary facts

For every $k \geq 2$ and $\sigma = (\sigma_1, \ldots, \sigma_k) \in \Omega_k$, we write

$$\sigma^- := (\sigma_1, \ldots, \sigma_{k-1}); \ \ p_\sigma := p_{\sigma_1\sigma_2} \cdots p_{\sigma_{k-1}\sigma_k}, \ c_\sigma := c_{\sigma_1\sigma_2} \cdots c_{\sigma_{k-1}\sigma_k}.$$

If $|\sigma| = 1$, we set $\sigma^- = \theta$, where θ denotes the empty word; we also define $p_\sigma := 1, c_\sigma := 1$ for $\sigma \in \Omega_1 \cup \{\theta\}$. If $\sigma, \omega \in \Omega^*$ satisfy $|\sigma| \leq |\omega|$ and $\sigma = \omega|_{|\sigma|}$, then we call ω a descendant of σ and write $\sigma \prec \omega$. Two words $\sigma, \omega \in \Omega^*$ are said to be incomparable if neither $\sigma \prec \omega$, nor $\omega \prec \sigma$. A finite subset Γ of Ω^* is called a finite antichain if any two words in Γ are incomparable; a finite antichain Γ is said to be maximal, if every $\tau \in \Omega_\infty$ is the descendant of some word $\sigma \in \Gamma$. Set

$$\underline{p} := \min_{(i,j)\in\Omega_2} p_{ij}, \ \underline{c} := \min_{(i,j)\in\Omega_2} c_{ij}, \ \overline{p} := \max_{(i,j)\in\Omega_2} p_{ij}, \ \overline{c} := \max_{(i,j)\in\Omega_2} c_{ij}.$$

For $r > 0$, let $\eta := \underline{p}\underline{c}^r$. To study the quantization error $e_{n,r}(\mu)$, we define

$$\Lambda_{j,r} := \{\sigma \in \Omega^* : p_{\sigma^-} c_{\sigma^-}^r \geq \eta^j > p_\sigma c_\sigma^r\}. \tag{2.1}$$

Then $(\Lambda_{j,r})_{j=1}^\infty$ is a sequence of finite maximal antichains. This type of sets were constructed by Graf and Luschgy in their work on the quantization for self-similar measures (cf. [3]). The spirit of these constructions is to seek some kind of uniformity while general measures are not uniform. We define

$$\phi_{j,r} := \mathrm{card}(\Lambda_{j,r}), \ l_{1j} := \min_{\sigma\in\Lambda_{j,r}} |\sigma|, \ l_{2j} := \max_{\sigma\in\Lambda_{j,r}} |\sigma|;$$

$$\underline{P}_r^s(\mu) := \liminf_{j\to\infty} \phi_{j,r}^{\frac{r}{s}} e^r_{\phi_{j,r},r}(\mu), \ \ \overline{P}_r^s(\mu) := \limsup_{j\to\infty} \phi_{j,r}^{\frac{r}{s}} e^r_{\phi_{j,r},r}(\mu).$$

Lemma 2.1. *We have*

$$\underline{Q}_r^s(\mu) > 0 \iff \underline{P}_r^s(\mu) > 0 \ \text{and} \ \overline{Q}_r^s(\mu) < \infty \iff \overline{P}_r^s(\mu) < \infty.$$

Proof. Let $N_1 := \min\{h \in \mathbb{N} : (\overline{p}\overline{c}^r)^h < \eta\}$. For every $\sigma \in \Lambda_{j,r}$, we have, $p_\sigma c_\sigma^r < \eta^j$. Hence, for every $\omega \in \Omega_{N_1}$ with $(\sigma_{|\sigma|}, \omega_1) \in \Omega_2$, we have

$$p_{\sigma*\omega} c^r_{\sigma*\omega} = (p_\sigma c_\sigma^r)(p_{\sigma_{|\sigma|}\omega_1}) p_\omega c_\omega^r \leq (p_\sigma c_\sigma^r)(\overline{p}\overline{c}^r)^{N_1} < \eta^{j+1}.$$

Hence, $\Lambda_{j+1,r} \subset \bigcup_{h=1}^{N_1} \bigcup_{\sigma\in\Lambda_{j,r}} \Gamma(\sigma,h)$, where

$$\Gamma(\sigma,h) := \left\{\omega \in \Omega_{|\sigma|+h} : \sigma \prec \omega\right\}. \tag{2.2}$$

It follows that $\phi_{j,r} \leq \phi_{j+1,r} \leq N^{N_1}\phi_{j,r}$. This and [18, Lemma 2.4] completes the proof of the lemma. □

If the infimum in (1.1) is attained at some α with $\mathrm{card}(\alpha) \leq n$, we call α an n-optimal set for ν of order r. The collection of all n-optimal sets for ν of order r is denoted by $C_{n,r}(\nu)$. For two sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ of positive real numbers, we write $a_n \ll b_n$ if there is some constant B independent of n such that $a_n \leq B \cdot b_n$. If $a_n \ll b_n$ and $b_n \ll a_n$ we write $a_n \asymp b_n$.

For every $k \geq 1$ and a vector $w = (w_i)_{i=1}^k \in \mathbb{R}^k$, we define

$$\overline{w} := \max_{1\leq i\leq k} w_i, \; \underline{w} := \min_{1\leq i\leq k} w_i.$$

Lemma 2.2. *For all large $j \geq 1$, we have*

$$e^r_{\phi_{j,r},r}(\mu) \asymp \sum_{\sigma\in\Lambda_{j,r}} p_\sigma c^r_\sigma. \tag{2.3}$$

Proof. For every $\sigma \in \Lambda_{j,r}$, let a_σ be an arbitrary point in J_σ. We have

$$\begin{aligned} e^r_{\phi_{j,r},r}(\mu) &\leq \sum_{\sigma\in\Lambda_{j,r}} \int_{J_\sigma} d(x,a_\sigma)^r d\mu(x) \\ &\leq \sum_{\sigma\in\Lambda_{j,r}} \mu(J_\sigma)|J_\sigma|^r = \sum_{\sigma\in\Lambda_{j,r}} \chi_{\sigma_1} p_\sigma c^r_\sigma \leq \overline{\chi} \sum_{\sigma\in\Lambda_{j,r}} p_\sigma c^r_\sigma. \end{aligned}$$

Using (1.5) and the method in [17, Lemma 3], one can find a constant $L \geq 1$, which is independent of j, and a set $\beta(\sigma)$ with $\mathrm{card}(\beta(\sigma)) \leq L$ such that

$$e^r_{\phi_{j,r},r}(\mu) \geq \sum_{\sigma\in\Lambda_{j,r}} \int_{J_\sigma} d(x,\beta(\sigma))^r d\mu(x). \tag{2.4}$$

Then by (1.2) and the arguments in [17, Lemma 4], one may find a constant $D > 0$ which is independent of $\sigma \in \Omega^*$, such that

$$\int_{J_\sigma} d(x,\beta(\sigma))^r d\mu(x) \geq D\mu(J_\sigma)|J_\sigma|^r \geq D\underline{\chi} p_\sigma c^r_\sigma. \tag{2.5}$$

By (2.4) and (2.5), we conclude that $e^r_{\phi_{j,r},r}(\mu) \geq D\underline{\chi}\sum_{\sigma\in\Lambda_{j,r}} p_\sigma c^r_\sigma$. □

3. Proof of Theorem 1.1

We will treat the irreducible and non-irreducible case separately.

3.1. **Markov measures with irreducible transition matrix.**

Lemma 3.1. *Assume that $P = (p_{ij})_{N\times N}$ is irreducible. Then there exist constants $\delta_i > 0, i = 1,2$ such that, for every finite maximal antichain $\Gamma \subset \Omega^*$,*

$$\delta_1 \leq \sum_{\sigma\in\Gamma} (p_\sigma c^r_\sigma)^{s_r/(s_r+r)} \leq \delta_2.$$

Proof. As $\Psi_G(s_r) = 1$ and A_{G,s_r} is a non-negative irreducible matrix, by Perron-Frobenius theorem, 1 is an eigenvalue of A_{G,s_r} and there is a positive vector $\xi = (\xi_i)_{i=1}^N$ with $\sum_{i=1}^N \xi_i = 1$ such that $A_{G,s_r}\xi = \xi$. As before, for $k \geq 2$ and $\sigma \in \Omega_k$, we set $[\sigma] := \{\omega \in \Omega_\infty : \omega|_{|\sigma|} = \sigma\}$. We define

$$\nu_1([\sigma]) := (p_\sigma c_\sigma^r)^{s_r/(s_r+r)}\, \xi_{\sigma_{|\sigma|}}.$$

Then one can easily see

$$\begin{aligned} \sum_{i=1}^N \nu_1([\sigma * i]) &= (p_\sigma c_\sigma^r)^{s_r/(s_r+r)} \sum_{i=1}^N (p_{\sigma_{|\sigma|} i} c_{\sigma_{|\sigma|} i}^r)^{s_r/(s_r+r)} \xi_i = \nu_1([\sigma]), \\ \sum_{\sigma \in \Omega_2} (p_\sigma c_\sigma^r)^{s_r/(s_r+r)} \xi_{\sigma_{|\sigma|}} &= \sum_{i=1}^N \sum_{j=1}^N (p_{ij} c_{ij}^r)^{s_r/(s_r+r)} \xi_j = \sum_{i=1}^N \xi_i = 1. \end{aligned}$$

Thus, by Kolmogorov consistency theorem, ν_1 extends a probability measure on Ω_∞. Since Γ is a finite maximal antichain, we have

$$\sum_{\sigma \in \Gamma} (p_\sigma c_\sigma^r)^{s_r/(s_r+r)}\, \xi_{\sigma_{|\sigma|}} = \nu_1(\Omega_\infty) = 1.$$

As an immediate consequence, we have

$$\overline{\xi}^{-1} \leq \sum_{\sigma \in \Gamma} (p_\sigma c_\sigma^r)^{s_r/(s_r+r)} \leq \underline{\xi}^{-1}.$$

The lemma follows by setting $\delta_1 := \overline{\xi}^{-1}$ and $\delta_2 := \underline{\xi}^{-1}$. $\square$

Proposition 3.2. *Assume that $P = (p_{ij})_{N\times N}$ is irreducible. Then we have*

$$0 < \underline{Q}_r^{s_r}(\mu) \leq \overline{Q}_r^{s_r}(\mu) < \infty. \tag{3.1}$$

Proof. By Lemma 3.1, for $j \geq 1$, we see

$$\phi_{j,r}\eta^{js_r/(s_r+r)} \asymp \sum_{\sigma \in \Lambda_{j,r}} (p_\sigma c_\sigma^r)^{s_r/(s_r+r)} \asymp 1, \text{ implying } \eta^j \asymp \phi_{j,r}^{-(s_r+r)/s_r}.$$

This, together with (2.3), leads to

$$e_{\phi_{j,r}}^r(\mu) \asymp \sum_{\sigma \in \Lambda_{j,r}} p_\sigma c_\sigma^r \asymp \phi_{j,r} \cdot \eta^j \asymp \phi_{j,r} \cdot \phi_{j,r}^{-(s_r+r)/s_r} = \phi_{j,r}^{-r/s_r}.$$

It follows that $\phi_{j,r}^{r/s_r} e_{\phi_{j,r}}^r(\mu) \asymp 1$. Hence, (3.1) follows by Lemma 2.1. $\square$

3.2. **Markov measures with reducible transition matrix.** For every $H \in \mathrm{SC}(G)$, we write

$$\begin{aligned} H_\infty &:= \{\sigma \in \Omega_\infty : \sigma_i \in H \text{ for } i \geq 1\}, \;\; H^* := \bigcup_{k=1}^\infty H^k; \\ H_k(i) &:= \left\{\sigma \in H^k : \sigma_1 = i\right\}, \;\; H^*(i) := \bigcup_{k=1}^\infty H_k(i); \\ H_\infty(i) &:= \{\sigma \in H_\infty : \sigma_1 = i\}; \;\; i \in H. \end{aligned}$$

We define finite maximal antichains in H^* or $H^*(i)$ in the same way as we did for those in Ω^*.

Lemma 3.3 (cf. [12]). *We have $s_r = \max_{H \in \mathrm{SC}(G)} s_r(H)$.*

Proof. By the factor theorem [12, pp. 813], we have

$$0=\det(I-A_{G,s_r})=\prod_{H\in\mathrm{SC}(G)}\det(I-A_{H,s_r}). \tag{3.2}$$

Since A_{G,s_r} is non-negative, by the definition of s_r, 1 is an eigenvalue of A_{G,s_r}. For every $H\in\mathrm{SC}(G)$, we know that $A_{H,s_r(H)}$ is irreducible since H is strongly connected. Thus, there exists an eigenvalue $\lambda_H>0$ which equals the spectral radius $\Psi_H(s_r)$ of A_{H,s_r}, namely, λ_H is the Perron root of A_{H,s_r}. It follows that $1=\max_{H\in\mathrm{SC}(G)}\lambda_H$ and there exists an $H\in\mathrm{SC}(G)$ such that $\lambda_H=1$. If $\lambda_H=1$, then $s_r=s_r(H)$; otherwise, we have $\lambda_H=\Psi_H(s_r)<1$. By [12, Theorem 2], $\Psi_H(s)$ is strictly decreasing with respect to s. Thus, $s_r>s_r(H)$. Combining the above analysis, the lemma follows. □

Remark. The factor theorem is an easy consequence of the following facts. For the non-negative reducible matrix $A_{G,s}$, there exists some permutation matrix T (which is necessarily orthogonal) such that $TA_{G,s}T^{-1}$ is a block upper triangular matrix, where the blocks on the diagonal are either irreducible matrices or 1×1 null matrices; thus the set of the eigenvalues of $A_{G,s}$ are exactly the union of those of all the blocks on the diagonal.

Further, the non-zero blocks on the diagonal are the images of the maximal irreducible sub-matrices of $A_{G,s}$ corresponding to the strongly connected components under symmetric permutations which are orthogonal and preserve eigenvalues. Therefore, if we denote by m (possibly zero) the number of 1×1 null matrices on the diagonal of $TA_{G,s}T^{-1}$, then

$$\det(\lambda I-A_{G,s})=\lambda^m\prod_{H\in\mathrm{SC}(G)}\det(\lambda I-A_{H,s}).$$

Lemma 3.4. *There exists constant $M_1,M_2>0$ such that*

$$M_1\le\sum_{\sigma\in\Gamma}(p_\sigma c_\sigma^r)^{s_r(H)/(s_r(H)+r)}\le M_2. \tag{3.3}$$

for every finite maximal antichain in H^* or $H^*(i), i\in H$, with $H\in\mathrm{SC}(G)$.

Proof. By Lemma 3.1, for each H, one can choose a constant M_H such that the second inequality in (3.3) holds with M_H in place of M_2. Set $M_2:=\max\{M_H : H\in\mathrm{SC}(G)\}$. Then for every $H\in\mathrm{SC}(G)$ and every finite maximal antichain Γ in H^*, we have

$$\sum_{\sigma\in\Gamma}(p_\sigma c_\sigma^r)^{s_r(H)/(s_r(H)+r)}\le M_2 \tag{3.4}$$

Since every finite maximal antichain $\Gamma(i)$ in $H^*(i)$ is contained in a finite maximal antichain in H^*, we conclude that the second inequality of (3.3) also holds for such a $\Gamma(i)$. Next we need to choose M_1 such that the first inequality also holds for such a $\Gamma(i)$.

Let $H\in\mathrm{SC}(G)$ and $i\in H$. We denote by ζ the unique normalized positive right eigenvector of $A_{H,s_r(H)}$ with respect to the Perron-Frobenius eigenvalue 1. We consider the measure $\nu_{1,i}$ on $H_\infty(i)$ satisfying

$$\begin{aligned}\nu_{1,i}([\sigma]) &:= (p_\sigma c_\sigma^r)^{s_r(H)/(s_r(H)+r)}\,\zeta_{\sigma_{|\sigma|}},\ \sigma\in H^*(i);\\ \nu_{1,i}(H_\infty(i) &= \sum_{j=1}^N\left(p_{ij}c_{ij}^r\right)^{s_r(H)/(s_r(H)+r)}\zeta_j=\zeta_i,\end{aligned}$$

where $[\sigma] := \{\omega \in H_\infty(i) : \omega|_{|\sigma|} = \sigma\}$. Then for every finite maximal antichain $\Gamma(i) \subset H^*(i)$, we have

$$\overline{\zeta}^{-1}\zeta_i \leq \sum_{\sigma\in\Gamma(i)} (p_\sigma c_\sigma^r)^{s_r(H)/(s_r(H)+r)} \leq \underline{\zeta}^{-1}\zeta_i.$$

Let $m_H := \min\{\overline{\zeta}^{-1}\zeta_i : i \in H\}$ and $M_1 := \min\{m_H : H \in \mathrm{SC}(G)\}$. Then (3.3) holds for every every finite maximal antichain $\Gamma(i)$ in $H^*(i)$ and finite maximal antichain in H^*. This completes the proof of the lemma. □

Lemma 3.5. *Let $H \in \mathrm{SC}(G)$ and $i \in H$. Then $\underline{Q}_r^{s_r(H)}(\mu(\cdot|J_i)) > 0$.*

Proof. For each $k \geq 1$, we write

$$\begin{aligned} \Lambda_{k,r}(i) &:= \left\{\sigma \in \Omega^* : \sigma_1 = i, p_{\sigma^-} c_{\sigma^-}^r \geq \eta^k > p_\sigma c_\sigma^r\right\}, \\ \Lambda_{k,r}(i, H^c) &:= \{\sigma \in \Lambda_{k,r}(i) : \sigma_h \notin H \text{ for some } \ h\}. \end{aligned} \tag{3.5}$$

For $\sigma \in \Lambda_{k,r}(i, H^c)$, we write $h(\sigma) := \min\{h : \sigma_h \notin H\}$. Then, since H is a strongly connected component, we deduce that $\sigma_l \notin H$ for all $l \geq h(\sigma)$. Note that $A_{H,s_r(H)}$ is irreducible and that $\Lambda_{k,r}(i) \setminus \Lambda_{k,r}(i, H^c)$ is a maximal finite antichain in $H^*(i)$. Hence, by Lemma 3.4, we have

$$\sum_{\sigma\in\Lambda_{k,r}(i)\setminus\Lambda_{k,r}(i,H^c)} (p_\sigma c_\sigma^r)^{s_r(H)/(s_r(H)+r)} \geq M_1 > 0.$$

Let $\phi_{k,r}(i)$ denote the cardinality of $\Lambda_{k,r}(i)$. As we did for (2.3), one can show

$$e^r_{\phi_{k,r}(i),r}\big(\mu(\cdot|J_i)\big) \gg \sum_{\sigma\in\Lambda_{k,r}(i)} p_\sigma c_\sigma^r.$$

By Hölder's inequality for exponent less than one, we have

$$\begin{aligned} e^r_{\phi_{k,r}(i),r}\big(\mu(\cdot|J_i)\big) &\gg \left(\sum_{\sigma\in\Lambda_{k,r}(i)} (p_\sigma c_\sigma^r)^{\frac{s_r(H)}{s_r(H)+r}}\right)^{\frac{s_r(H)+r}{s_r(H)}} \cdot \phi_{k,r}(i)^{-r/s_r(H)} \\ &\geq \left(\sum_{\sigma\in\Lambda_{k,r}(i)\setminus\Lambda_{k,r}(i,H^c)} (p_\sigma c_\sigma^r)^{\frac{s_r(H)}{s_r(H)+r}}\right)^{\frac{s_r(H)+r}{s_r(H)}} \cdot \phi_{k,r}(i)^{-r/s_r(H)} \\ &\gg \phi_{k,r}(i)^{-r/s_r(H)}. \end{aligned}$$

This and Lemma 2.1 yields that $\underline{Q}_r^{s_r(H)}(\mu(\cdot|J_i)) > 0$. The lemma follows. □

Proposition 3.6. *For any Markov-type measure μ as defined in (1.6) we have $D_r(\mu) = s_r$ and $\underline{Q}_r^{s_r}(\mu) > 0$.*

Proof. Let $s > s_r$. By [12, Theorem 2], we have, $\Psi_G(s) < \Psi_G(s_r) = 1$. Let $u = (u_i)_{i=1}^N$ be the column vector with $u_i = 1$ for all $1 \leq i \leq N$. We choose t such that $\Psi_G(s) < t < 1$. By Gelfand's formula, we have

$$\lim_{k\to\infty} \|A_{G,s}^k u\|_1^{1/k} = \Psi_G(s) < t < 1.$$

Thus, for large k, we have that $\|A_{G,s}^k u\|_1 < t^k$. It follows that

$$\sum_{\sigma\in\Omega_k} (p_\sigma c_\sigma^r)^{s/(s+r)} = \|A_{G,s}^{k-1} u\|_1 < t^{k-1}. \tag{3.6}$$

Let $\Lambda_{j,r}$ be as defined in (2.1). It is immediate to see that, there exist two constants $A_1, A_2 > 0$ such that

$$A_1 j \leq l_{1j} \leq l_{2j} < A_2 j.$$

Applying (3.6) to every $l_{1j} \leq k \leq l_{2j}$, we deduce

$$\begin{aligned}\sum_{\sigma\in\Lambda_{j,r}}(p_\sigma c_\sigma^r)^{s/(s+r)} &\leq \sum_{k=l_{1j}}^{l_{2j}}\sum_{\sigma\in\Omega_k}(p_\sigma c_\sigma^r)^{s/(s+r)}\\ &\leq \sum_{k=l_{1j}}^{l_{2j}} t^{k-1} \leq \frac{t^{l_{1j}-1}}{1-t} < 1 \text{ for large j.}\end{aligned}$$

Thus, by (2.1), for all large j, we have $\phi_{j,r} \leq \eta^{-s(j+1)/(s+r)}$. Also, by (2.3),

$$e_{\phi_{j,r},r}(\mu) \leq \overline{\chi}\sum_{\sigma\in\Lambda_{j,r}}(p_\sigma c_\sigma^r)^{s/(s+r)}(p_\sigma c_\sigma^r)^{r/(s+r)} \leq \overline{\chi}\eta^{rj/(s+r)} \leq \overline{\chi}\phi_{j,r}^{-r/s}.$$

Thus, by Lemma 2.1, we have, $\overline{D}_r(\mu) \leq s$. Since $s > s_r$ was chosen arbitrarily, we obtain that $\overline{D}_r(\mu) \leq s_r$.

Let $H \in \mathcal{M}$. Then we have $s_r(H) = s_r$. We take an arbitrary vertex $i_0 \in H$ and consider the conditional probability measure $\mu_{i_0} := \mu(\cdot|J_{i_0})$. By Lemma 3.5, we have, $\underline{Q}_r^{s_r}(\mu_{i_0}) > 0$. Hence,

$$\underline{Q}_r^{s_r}(\mu) \geq \mu(J_{i_0})\underline{Q}_r^{s_r}(\mu_{i_0}) \geq \underline{\chi}\,\underline{Q}_r^{s_r}(\mu_{i_0}) > 0.$$

In particular, by [3, Proposition 11.3], we have, $\underline{D}_r(\mu) \geq s_r$. Combing this and the first part of the proof, we conclude that $D_r(\mu)$ exists and equals s_r. This completes the proof of the theorem. $\square$

Define the set $F := G \setminus \bigcup_{H\in\mathcal{M}} H$ which is possibly empty. Whenever $F \neq \emptyset$, there corresponds a sub-matrix A_{F,s_r} of A_{G,s_r}. We write

$$F_0 := \{\theta\},\ \ F_k := \{\sigma \in \Omega_k : \sigma_h \in F, 1 \leq h \leq k\},\ k \geq 1;\ \ F^* := \bigcup_{k=0}^{\infty} F_k.$$

Lemma 3.7. *There exists a constant $t \in (0,1)$ such that for $n \in \mathbb{N}$ large*

$$\sum_{\sigma\in F_n}(p_\sigma c_\sigma^r)^{s_r/(s_r+r)} \ll t^n.$$

Proof. Let $s_r(F)$ denote the unique number with $\Psi_F(s_r(F))$=1. Then by Lemma 3.3 and the definition of F, we deduce

$$s_r(F) = \max\{s_r(H) : \mathrm{SC}(G) \ni H \subset F\} < s_r.$$

According to [12, Theorem 3], $\Psi_F(s)$ is strictly decreasing with respect to s. Thus, $\Psi_F(s_r) < 1$ and we may choose some $t > 0$ such that $\Psi_F(s_r) < t < 1$. Following the proof of Proposition 3.6, one can see that, there exists a constant $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have

$$\sum_{\sigma\in F_k}(p_\sigma c_\sigma^r)^{s_r/(s_r+r)} \leq t^{k-1},\ \ k \geq k_1.$$

From this the lemma follows. $\square$

Proposition 3.8. *If $\mathcal{M}$ consists of pairwise incomparable elements then $\overline{Q}_r^{s_r}(\mu) < \infty$.*

Proof. First note that in this situation we have that any $\omega \in \Omega^*$ has the form $\omega = \nu' * \tau * \nu''$ where $\nu', \nu'' \in F^*$ and $\tau \in H^*$ for some $H \in \mathcal{M}$. Further, for

$M_3 := \min\left\{k \in \mathbb{N} : \overline{p}\,\overline{c}^r/\underline{p}\underline{c}^r < k\right\}$, $H \in \mathcal{M}$ and any choice $\nu', \nu'' \in F^*$, we have that

$$H^*_{j,r}\left(\nu', \nu''\right) := \{\tau \in H^* : \nu' * \tau * \nu'' \in \Lambda_{j,r}\} \subset \bigcup_{k=1}^{M_3} \Gamma_k^{H,j}\left(\nu', \nu''\right),$$

where $\Gamma_k^{H,j}\left(\nu', \nu''\right)$ is some antichain for each $k = 1, \ldots, M_3$. With this notation and using Lemmata 3.4 and 3.7 and the definition of t and M_2 therein we estimate

$$\begin{aligned}
&\sum_{\omega \in \Lambda_{j,r}} \left(p_\omega c_\omega^r\right)^{s_r/(s_r+r)} \\
&= \sum_{H \in \mathcal{M}} \sum_{\nu', \nu'' \in F^*} \sum_{\tau \in H^*_{j,r}(\nu', \nu'')} \left(p_{\nu' * \tau_1} c^r_{\nu' * \tau_1} p_\tau c_\tau^r p_{\tau_{|\tau|} * \nu''} c^r_{\tau_{|\tau|} * \nu''}\right)^{s_r/(s_r+r)} \\
&\leq \sum_{H \in \mathcal{M}} \sum_{\nu', \nu'' \in F^*} \left(p_{\nu'} c^r_{\nu'} p_{\nu''} c^r_{\nu''}\right)^{\frac{s_r}{s_r+r}} \left(\sum_{\rho \in \Omega_2 \cup \{\theta\}} \left(p_\rho c_\rho^r\right)^{\frac{s_r}{s_r+r}}\right)^2 \sum_{k=1}^{M_3} \sum_{\Gamma_k^{H,j}(\nu', \nu'')} \left(p_\tau c_\tau^r\right)^{\frac{s_r}{s_r+r}} \\
&\leq \mathrm{card}\,(\mathcal{M}) \left(\sum_{n=0}^{\infty} \sum_{\nu \in F_n} \left(p_\nu c_\nu^r\right)^{s_r/(s_r+r)}\right)^2 \left(N^2+1\right)^2 M_3 M_2 \ \ll \left(\sum_{n=0}^{\infty} t^n\right)^2 \ll 1.
\end{aligned}$$

Combining this with Lemma 3.1, for $j \geq 1$, we get

$$\phi_{j,r} \eta^{j s_r/(s_r+r)} \asymp \sum_{\sigma \in \Lambda_{j,r}} \left(p_\sigma c_\sigma^r\right)^{s_r/(s_r+r)} \ll 1$$

and hence $\eta^j \ll \phi_{j,r}^{-(s_r+r)/s_r}$. This, together with (2.3), leads to

$$e^r_{\phi_{j,r}}(\mu) \asymp \sum_{\sigma \in \Lambda_{j,r}} p_\sigma c_\sigma^r \asymp \phi_{j,r} \cdot \eta^j \ll \phi_{j,r} \cdot \phi_{j,r}^{-(s_r+r)/s_r} = \phi_{j,r}^{-r/s_r}.$$

It follows that $\phi_{j,r}^{r/s_r} e^r_{\phi_{j,r}}(\mu) \ll 1$. Hence, the assertion follows by Lemma 2.1.

□

In order to estimate the quantization error from below, we need an auxiliary measure of Mauldin-Williams-type. One may see [12, p. 823] for more details.

Assume that, there are two elements $H_1, H_2 \in \mathcal{M}$ such that $H_1 \prec H_2$, i.e., there exists a path $\gamma = (i_1, \ldots, i_h)$ satisfying

$$i_1 \in H_1, \ \ i_2, \ldots, i_{h-1} \notin H_1 \cup H_2, \ \ i_h \in H_2. \tag{3.7}$$

Let $v = (v_i)_{i=m_1+1}^{m_2}$ be the positive normalized right eigenvector of A_{H_2, s_r} with respect to the Perron-Frobenius eigenvector 1. Set

$$\begin{aligned}
E_q &:= \{\tau \in H_1^q : \tau_q = 1\}, \ \widetilde{\gamma} := \{i_2, \ldots, i_{h-1}\}, \\
F_q &:= \left\{\tau * \widetilde{\gamma} * \rho : \tau \in E_q, \rho \in H_2^{\mathbb{N}}, \rho_1 = i_h\right\}.
\end{aligned}$$

For every $\tau \in E_q$ and $\rho \in H_2^*(i_h)$, we define

$$\nu_q([\tau * \widetilde{\gamma} * \rho]) = \left(p_\tau c_\tau^r p_{\widetilde{\gamma}} c^r_{\widetilde{\gamma}} p_\rho c_\rho^r\right)^{s_r/(s_r+r)} v_{\rho_{|\rho|}}. \tag{3.8}$$

where $[\tau * \widetilde{\gamma} * \rho] := \{\tau * \widetilde{\gamma} * \omega : \omega \in (H_2)_\infty, \omega|_{|\rho|} = \rho\}$. By (3.8), we have

$$\begin{aligned}
\sum_{i \in H_2} \nu_q([\tau * \widetilde{\gamma} * \rho * i]) &= \sum_{i \in H_2} \left(p_\tau c_\tau^r p_{\widetilde{\gamma}} c^r_{\widetilde{\gamma}}\right)^{s_r/(s_r+r)} \left(p_{\rho * i} c^r_{\rho * i}\right)^{s_r/(s_r+r)} v_i \\
&= \left(p_\tau c_\tau^r p_{\widetilde{\gamma}} c^r_{\widetilde{\gamma}} p_\rho c_\rho^r\right)^{s_r/(s_r+r)} \sum_{i \in H_2} \left(p_{\rho_{|\rho|} i} c^r_{\rho_{|\rho|} i}\right)^{s_r/(s_r+r)} v_i \\
&= \left(p_\tau c_\tau^r p_{\widetilde{\gamma}} c^r_{\widetilde{\gamma}} p_\rho c_\rho^r\right)^{s_r/(s_r+r)} v_{\rho_{|\rho|}}.
\end{aligned}$$

Thus, by Kolmogorov consistency theorem, we get a unique measure ν_q on F_q.

Proposition 3.9. *Assume that there are two comparable elements in $\mathcal{M}$. Then we have, $\underline{Q}_r^{s_r}(\mu)=\infty$.*

Proof. Assume that, there are two elements $H_1, H_2 \in \mathcal{M}$ such that $H_1 \prec H_2$. Without loss of generality, as in [12], we assume that $H_1=\{1,\ldots,m_1\}, i_1=1$ and $H_2=\{m_2+1,\ldots,m_1+m_2\}, i_h=m_2+1$; and (3.7) holds . As above, let $\widetilde{\gamma}:=(i_2,\ldots,i_{h-1})$. For all large k, there exist some words $\sigma\in\Lambda_{k,r}$ taking the form $\sigma=\tau*\widetilde{\gamma}*\rho$ with $\tau\in E_q$ and $\rho\in H_2^*, \rho_1=m_2+1$.

For every $q\le l_{1k}-1-h$ and $\tau\in E_q$, we have, $p_{\tau*\widetilde{\gamma}}c^r_{\tau*\widetilde{\gamma}}\ge\eta^{-k}$, otherwise, $\min_{\sigma\in\Lambda_{k,r}}|\sigma|$ would be strictly less than l_{1k}, contradicting the definition of l_{1k}. This implies that $\Lambda_{k,r}$ includes some subset $F_q^\flat$ of F_q^* such that $\{\rho:\tau*\widetilde{\gamma}*\rho\in F_q^\flat\}$ forms a finite maximal antichain in $H_2^*(m_2+1):=\{\sigma\in H_2^*:\sigma_1=m_2+1\}$. For each $\sigma=\tau*\widetilde{\gamma}*\rho\in F_q^\flat$, we have

$$\begin{aligned}\left(p_{\tau*\widetilde{\gamma}*\rho}c^r_{\tau*\widetilde{\gamma}*\rho}\right)^{s_r/(s_r+r)} &= \left(p_\tau c^r_\tau p_{1i_2}c^r_{1i_2}p_{\widetilde{\gamma}}c^r_{\widetilde{\gamma}}p_{i_{h-1}\rho_1}c^r_{i_{h-1}\rho_1}p_\rho c^r_\rho\right)^{s_r/(s_r+r)}\\ &\ge \left(p_\tau c^r_\tau p_{\widetilde{\gamma}}c^r_{\widetilde{\gamma}}p_\rho c^r_\rho\right)^{s_r/(s_r+r)}\eta^{\frac{2s_r}{s_r+r}}\ge\eta^{\frac{2s_r}{s_r+r}}\nu([\sigma])\overline{v}^{-1}.\end{aligned}$$

Using this facts, we deduce

$$\begin{aligned}&\sum_{\sigma\in\Lambda_{k,r}}(p_\sigma c^r_\sigma)^{s_r/(s_r+r)}\\ \ge\ &\sum_{q=1}^{l_{1k}-1-h}\sum_{\sigma\in F_q^\flat}(p_\sigma c^r_\sigma)^{s_r/(s_r+r)}\ge\overline{v}^{-1}\eta^{2s_r/(s_r+r)}\sum_{q=1}^{l_{1k}-1-h}\nu_q(F_q)\\ =\ &\overline{v}^{-1}\eta^{2s_r/(s_r+r)}\left(p_{\widetilde{\gamma}}c_{\widetilde{\gamma}}\right)^{s_r/(s_r+r)}v_{m_1+1}\sum_{q=1}^{l_{1k}-1-h}\sum_{\tau\in E_q}(p_\tau c^r_\tau)^{s_r/(s_r+r)}=:Q_k,\end{aligned}$$

where $v=(v_i)_{i=1}^{m_2}$ is the positive eigenvector in the definition of the measures ν_q and $\overline{v}:=\max_{1\le i\le m_2}v_i$. Note that $s_r(H_1)=s_r$. Let $w=(w_i)$ be a positive left eigenvector A_{H_1,s_r} with respect to the Perron-Frobenius eigenvector 1. Then we have $wA^h_{H_1,s_r}=w$ for all $h\ge1$. Let $A^{q-1}_{H_1,s_r}=(c_{ij})_{m_1\times m_1}$. We have

$$\sum_{\tau\in E_q}(p_\tau c^r_\tau)^{s_r/(s_r+r)}=\sum_{i=1}^{m_1}c_{i1}\ge\overline{w}^{-1}w_1.$$

This implies that $Q_k\to\infty$. Thus, by (2.3) and Hölder's inequality, we have

$$e^r_{\phi_{k,r},r}(\mu)\ge D\bigg(\sum_{\sigma\in\Lambda_{k,r}}(p_\sigma c^r_\sigma)^{s_r/(s_r+r)}\bigg)^{(s_r+r)/s_r}\phi_{k,r}^{-r/s_r}\ge Q_k^{(s_r+r)/s_r}\phi_{k,r}^{-r/s_r}.$$

Hence, by Lemma 2.1, it follows that $\underline{Q}_r^{s_r}(\mu)=\infty$. The proposition follows. □

Proof of Theorem 1.1. For the proof of Theorem 1.1 we just have to combine Proposition 3.8 and 3.9. □

Next, we construct two examples illustrating Theorem 1.1.

Example 3.10. Let $Q=(q_{ij})_{2\times2}, T=(t_{ij})_{3\times3}$ be two positive matrices, i.e., $q_{ij}>0, 1\le i,j\le2$ and $t_{ij}>0, 1\le i,j\le3$. We define

$$P=\begin{pmatrix}Q_{2\times2}&0\\0&T_{3\times3}\end{pmatrix}.$$

Then P is a reducible matrix. Let μ be the Markov-type measure associated with P. Let $H_1 := \{1,2\}$ and $H_2 := \{3,4\}$. Clearly, $\mathcal{M} = \{H_1, H_2\}$ and H_1, H_2 are incomparable. Thus, by Theorem 3.8, $0 < \underline{Q}_r^{s_r}(\mu) \leq \overline{Q}_r^{s_r}(\mu) < \infty$.

Example 3.11. Let the transition matrix be given by

$$P = (p_{ij})_{4\times 4} = \begin{pmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Fix $r > 0$ we set $s := r/(2r+1)$ implying $2\left(4^{-1}4^{-r}\right)^{s/(s+r)} = 1$. Let $c_{i,j} = 1/8$ for $1 \leq i \leq 2$, $1 \leq j \leq 3$, and $c_{33} = c_{34} = c_{43} = c_{44} = 2^{-1/s}$. With $H_1 := \{1,2\}$ and $H_2 := \{3,4\}$ we have

$$A_{H_1,s} = A_{H_2,s} = \begin{pmatrix} \left(2^{-(2+2r)}\right)^{s/(s+r)} & \left(2^{-(2+2r)}\right)^{s/(s+r)} \\ \left(2^{-(2+2r)}\right)^{s/(s+r)} & \left(2^{-(2+2r)}\right)^{s/(s+r)} \end{pmatrix}.$$

Clearly $A_{H_1,s}$, $A_{H_2,s}$ are irreducible row-stochastic matrices. Hence, $s_r = s = s_r(H_i)$, $i = 1,2$. Since $H_1 \prec H_2$, by Theorem 1.1, we conclude that $\underline{Q}_r^{s_r}(\mu) = \infty$.

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