

F -SINGULARITIES IN FAMILIES

ZSOLT PATAKFALVI, KARL SCHWEDE, AND WENLIANG ZHANG

ABSTRACT. We study the behavior of test ideals and F -singularities in families. In particular, we obtain generic (and non-generic) restriction theorems for test ideals and non- F -pure ideals, which imply for example openness of most of the F -singularity classes when the relative canonical sheaf is \mathbb{Q} -Cartier. Additionally, we study the global behavior of certain canonical linear systems (induced by Frobenius) associated to adjoint line bundles, in families. As a consequence, we obtain some positivity results for pushforwards of some adjoint line bundles and for certain subsheaves of these.

CONTENTS

1. Introduction	2
1.1. Organization	5
1.2. Acknowledgements	5
2. Notation and setup	5
2.13. Composing maps	9
2.16. Base change of φ	11
2.23. Passing to V^∞ and other perfect points	14
3. Relative non- F -pure ideals	14
3.11. Iterated non- F -pure ideals versus the absolute non- F -pure ideal	19
3.25. Application to sharply F -pure singularities and HSL numbers in families	27
4. Relative test ideals	28
4.10. Relative test ideals vs absolute test ideals	32
4.16. Applications to strong F -regularity and divisor pairs	33
5. Relative test submodules, F -rationality and F -injectivity	35
5.1. The definition and basic properties of $\sigma_n(X/V, \omega_{X/V})$	35
5.6. The definition and basic properties of $\tau_n(X/V, \omega_{X/V})$	37
5.12. Applications to families of F -injective and F -rational singularities	38
6. Global applications	39
6.1. Basic definitions	39
6.2. Auxiliary definition and stabilization	41
6.3. Base-change	45
6.4. Uniform stabilization	47
6.5. Global generation and semi-positivity	50
6.6. Relation to global canonical systems	52
Appendix A. Relative Serre's condition	56
References	58

2000 *Mathematics Subject Classification.* 14B05, 13A35, 14D05, 14F18, 13B40.

Key words and phrases. non- F -pure ideal, test ideal, F -pure, strongly F -regular, F -singularities, flat families.

The second author was partially supported by a Fellowship from the Sloan Foundation as well as NSF grant DMS #1064485, NSF FRG Grant DMS #1265261/1501115 and NSF CAREER Grant DMS #1252860/1501102.

The third author was partially supported by the NSF grants DMS #1068946/#1247354 and #1606414.

The material is based upon work supported by the National Science Foundation under Grant No. 0932078 000, while the second and third authors were in residence at the Mathematical Science Research Institute (MSRI) in Berkeley, California, during the spring semester 2013.

1. INTRODUCTION

In [Kun69] Kunz proved that a scheme over a field of characteristic $p > 0$ is regular if and only if the Frobenius (endo)morphism is flat, which initiated the study of singularities using the Frobenius morphism. Classes of singularities defined via the Frobenius morphism are referred to usually as F -singularities. Since Kunz's work, the study of F -singularities has become an active research area, cf. [HR76, GW77, Fed83, MR85, RR85, Sri91, Smi97, Har98, MS97, HY03, Tak04]. However, the methods of F -singularities have been widely applied to global geometry over a field of positive characteristic only recently [Sch11a, Mus11, Hac11, MS12, Zha12, CHMS12, Pat12, HX13, Tan13], at least outside of special classes of varieties [MR85, BK05]. A large field within global geometry is moduli theory, which requires understanding how varieties behave in flat families. In this direction, there have not been any positive results on the behavior of test ideals in families. We fill this gap.

Previously F -rational, Cohen-Macaulay F -injective and (Gorenstein) F -pure singularities have been studied in such a context [Has01, SZ09, Has10]. Additionally, [MY09, Example 4.7] showed that the test ideal τ does not satisfy the generic restriction theorem, at least as stated for multiplier ideals [Laz04b, Theorem 9.5.35]. We develop tools tackling this issue, obtaining generic (and non-generic) restriction theorems for test ideals, see Theorem A below.

In a flat family $f : X \rightarrow V$ the absolute Frobenius on X does not restrict to the Frobenius morphism on each fiber. Thus we systematically study the relative Frobenius morphism¹ $X' \rightarrow X \times_V V'$ where $X' \rightarrow X$ and $V' \rightarrow V$ are the Frobenius morphisms of X and V , respectively. Its fibers over V are morphisms between thickenings of the fibers of f . However, its fibers over V' are exactly Frobenius morphisms of fibers of f (at least over points with perfect residue field).

We begin by stating our main result in the local setting. We denote by X^n and V^n the domains of n -iterated Frobenius of X and V respectively. Let τ and σ denote the test ideal² [HH94] and non-sharply- F -pure ideal [FST11, BB11], respectively. These ideals play key roles in the theory of F -singularities, the test ideal τ being the unique smallest non-zero ideal fixed by p^{-e} -linear maps and σ being the unique largest. We define relative versions of these two ideals by using the relative instead of absolute Frobenius. The iterated relative Frobenius targets different spaces $X \times_V V^n$, and so we actually define a sequence ideals, one on each of these spaces. We explain the setup.

Suppose that $f : X \rightarrow V$ is a finite-type flat, equidimensional, reduced and S2 and G1 morphism of Noetherian F -finite schemes with V integral. Additionally suppose Δ is a \mathbb{Q} -divisor on X satisfying suitable conditions which allow it to be restricted to fibers and such that $K_{X/V} + \Delta$ is \mathbb{Q} -Cartier with index not divisible by $p > 0$ so that $(p^e - 1)(K_{X/V} + \Delta)$ is Cartier (see Remark 2.11 for a precise statement). Then for each integer $n > 0$, divisible by e , we define ideals $\sigma_n(X/V, \Delta)$ and $\tau_n(X/V, \Delta) \subseteq \mathcal{O}_{X \times_V V^n}$ called the *relative non F -pure* and *relative test ideals* (respectively).

Our main local theorem is that these ideals restrict to absolute non- F -pure and absolute test ideals on *all* of the geometric fibers, and can be used to prove *generic* restriction theorems for the usual absolute test ideals on $X \times_V V^n$ at least if V is regular.

Theorem A. (Corollary 3.10, Theorem 3.23, Corollary 4.8, Corollary 4.15) *With notation as above, there exists an $N > 0$ such that for every perfect point³ $s \in V$, and every $n \geq N$*

$$\sigma_{ne}(X/V, \Delta) \cdot \mathcal{O}_{X_{s^{ne}}} = \sigma(X_s, \Delta|_{X_s})$$

and

$$\tau_{ne}(X/V, \Delta) \cdot \mathcal{O}_{X_{s^{ne}}} = \tau(X_s, \Delta|_{X_s})$$

where $X_{s^{ne}}$ is the fiber of $X \times_V V^{ne} \rightarrow V^{ne}$ over $s^{ne} \in V^{ne}$, which is isomorphic to X_s since $k(s)$ is perfect. Additionally, both σ_n and τ_n map surjectively onto their arbitrary base changes.

¹Also called the Radu-André morphism, especially in commutative algebra.

²Technically, we are working with what has recently become known as the *big test ideal* [Hoc07].

³By definition, a morphism $\text{Spec } K \rightarrow V$ from a perfect field K , see Definition 2.2.

Furthermore, if V is regular and N is sufficiently large, then for all perfect points $s \in V$ we have that the absolute non- F -pure ideal restricts to all of the fibers for $n \geq N$

$$\sigma(X \times_V V^{ne}, \Delta \times_V V^{ne}) \cdot \mathcal{O}_{X_{s^{ne}}} = \sigma(X_s, \Delta|_{X_s})$$

and at least for an open dense set of the base $U \subseteq V$ that the same holds for the absolute test ideal

$$\tau(X \times_V V^{ne}, \Delta \times_V V^{ne}) \cdot \mathcal{O}_{X_{s^{ne}}} = \tau(X_s, \Delta|_{X_s})$$

for all perfect points $s \in U$.

Additionally, we show that over a dense open subset U of V with $W = f^{-1}(U)$, $\tau_{ne}(X/V, \Delta)|_W$ coincides with the absolute test ideal $\tau(X \times_V V^{ne}, \Delta \times_V V^{ne})|_W$ and likewise $\sigma_{ne}(X/V, \Delta)|_W = \sigma(X \times_V V^{ne}, \Delta \times_V V^{ne})|_W$ in [Theorem 4.13](#) and [Theorem 3.14](#) respectively.

The key method that allows to prove this result (at least stated for σ_n) is that if $m > n$, then $\text{Im}(\sigma_{ne}(X/V, \Delta) \otimes_{\mathcal{O}_{X \times_V V^{ne}}} \mathcal{O}_{X \times_V V^{me}} \rightarrow \mathcal{O}_{X \times_V V^{me}}) \supseteq \sigma_{me}(X/V, \Delta)$ and there is a dense open set of the base V over which equality holds for all sufficiently large n , this is [Proposition 3.3](#). This stabilization result should be viewed as a relative version of [\[HS77, Proposition 1.11\]](#), [\[Lyu97, Proposition 4.4\]](#), and [\[Gab04, Remark 13.6\]](#).

Remark 1.1. If one replaces the divisor Δ with the language of principal Cartier algebras, then the previous result still holds without the technical assumptions about divisors from [Remark 2.11](#).

Remark 1.2. There are a number of subtle issues in the statement above that we are suppressing. In particular, $\tau_n(X/V, \Delta)$ depends on the choice of some ideal I contained within the test ideal of every fiber, and shown to exist in [Proposition 4.7](#).

A particularly important related question is that of deformation of sharply F -pure singularities in flat families with \mathbb{Q} -Cartier relative canonical divisors. This would be important for a positive characteristic construction of the moduli of stable varieties, also known as the KSBA compactification. In characteristic zero, this is the moduli space given by the log-minimal model program. It classifies log-canonical models, hence birational equivalence classes of varieties of general type, and furthermore it contains some nodal varieties for the compactification. There is a conjectural framework of constructing this moduli space [\[Kol90\]](#). One of the main ingredients in this framework is to prove that log-canonical singularity deforms in flat families with \mathbb{Q} -Cartier relative log-canonical divisor. An important step in this direction in positive characteristic is the corresponding statement for sharply F -pure singularities. It is also an important ingredient in an upcoming paper of the first author where he is planning to address the question of the existence of an algebraic space structure on the space of sharply F -pure stable varieties. In this paper, we handle the deformation of F -pure and F -regular singularities. Indeed, openness of F -pure and F -regular singularities is a direct consequence of Theorem A above.

Theorem B. (Deformation of F -pure and F -regular singularities: [Corollary 3.31](#), [Corollary 4.21](#)) Suppose $f : X \rightarrow V$ and Δ is as in Theorem A and additionally assume that f is proper. If $s \in V$ is a perfect point and $(X_s, \Delta|_{X_s})$ is sharply F -pure (respectively, strongly F -regular⁴) then there exists an open set $U \subseteq V$ such that for all $u \in U$ that $(X_u, \Delta|_{X_u})$ is also sharply F -pure (respectively, strongly F -regular).

We also build relative test submodules and non- F -injective submodules of $\omega_{X \times_V V^{ne}/V^{ne}}$ and prove restriction theorems like Theorem A for them [Corollary 5.5](#) and [Corollary 5.10](#). As a consequence in [Theorem 5.16](#) we reprove a result of M. Hashimoto [\[Has01\]](#), deformation for Cohen-Macaulay F -injectivity and F -rationality.

Furthermore we apply our setup to global questions. One of the reasons for the recent global applications of F -singularity theorem is the lifting theorem shown by the second author in [\[Sch11a\]](#),

⁴In which case, you can even remove the index not divisible by p assumption from $K_{X/V} + \Delta$, see [Corollary 4.22](#).

Proposition 5.3]. This theorem can be used to replace some of the lifting arguments that use Kodaira vanishing in characteristic zero. One of the fundamental ideas in [Sch11a] is to try to lift only a big enough set of sections of adjoint bundles instead of all the sections. This canonical set of sections for a pair (X, Δ) with $(1 - p^e)(K_X + \Delta)$ Cartier and for a line bundle M is defined as [Sch11a, Definition 4.1]

$$S^0(X, \sigma(X, \Delta) \otimes M) := \bigcap_{m \geq 0} \operatorname{im} \left(H^0(X, F_*^{me} \mathcal{O}_X((1 - p^{me})(K_X + \Delta)) \otimes M) \rightarrow H^0(X, M) \right).$$

First, we investigate questions about how this canonical space of sections behaves in families: semicontinuity, stabilization of the intersection, etc.

Theorem C. *Let $f : X \rightarrow V$ and Δ be as in Theorem A, with f projective and V regular. Further, suppose that M is a line bundle on X such that $M - K_{X/V} - \Delta$ is f -ample (where by M we actually mean the Cartier divisor corresponding to M). Then the following statements hold.*

- (a) [Corollary 6.19, Example 6.22, Example 6.23, Example 6.25], cf. [Har98, Example 5.5], [Tan13, Theorem 8.3]. *The function*

$$(1.2.1) \quad s \mapsto \dim_{k(s)} S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$$

is not semicontinuous (here $s \in V$ is a perfect point) in either direction, however, there is a dense open subset $U \in V$, such that the function (1.2.1) is constant on U .

- (b) [Theorem 6.18]. *There exists $n > 0$ so that for all integers $m \geq n$ and perfect points $s \in V$,*

$$\begin{aligned} \operatorname{im} \left(H^0(X_s, F_*^{me} \mathcal{O}_{X_s}((1 - p^{me})(K_{X_s} + \Delta_s)) \otimes M_s) \rightarrow H^0(X_s, M_s) \right) \\ = S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s). \end{aligned}$$

- (c) [Theorem 6.20]. *If there is a perfect point $s_0 \in V$ such that*

$$H^0(X_{s_0}, M_{s_0}) = S^0(X_{s_0}, \sigma(X_{s_0}, \Delta_{s_0}) \otimes M_{s_0}),$$

then there is an open neighborhood U of s_0 , such that $f_ M|_U$ is locally free and compatible with base change and*

$$H^0(X_s, M_s) = S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$$

for every perfect point $s \in U$. In particular, the function (1.2.1) is constant for $s \in U$.

We would like to also mention the following natural question left open by Theorem C.

Question 1. Can one remove the f -ampleness assumption from the statements of Theorem C?

By Theorem C, it does make sense to talk about general value of $\dim_{k(s)} S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$ in the following theorem.

Theorem D. *With assumptions as in Theorem C such that V is projective over a perfect field and $(p^e - 1)(K_X + \Delta)$ is Cartier. Then for every $n \gg 0$ there is a subsheaf $S_{\Delta, ne}^0 f_*(M)$ of $(F_V^{ne})^*(f_* M)$, for which the following holds.*

- (a) [Corollary 6.19]. *The rank of $S_{\Delta, ne}^0 f_*(M)$ is the general value of $\dim_{k(s)} S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$.*
- (b) [Proposition 6.26]. *If $M - K_{X/V} - \Delta$ is ample, then $S_{\Delta, ne}^0 f_*(M)$ is globally generated for every $n \gg 0$.*
- (c) [Theorem 6.31]. *If $M - K_{X/V} - \Delta$ is nef, then $S_{\Delta, ne}^0 f_*(M)$ is weakly-positive for $n \gg 0$.*
- (d) [Corollary 6.14]. *If $M = Q^l \otimes P$ where Q is f -ample, then for all $l \gg 0$ and nef line bundle P , we have that $S_{\Delta, ne}^0 f_*(M)$ is contained in $(f_{V^{ne}})_*(\sigma_{ne}(X/V, \Delta) \otimes M_{V^{ne}})$ as subsheaves of $(f_{V^{ne}})_* M_{V^{ne}}$, and furthermore, these two subsheaves are generically equal.*

In fact points (a) and (d) are true without the projectivity assumption on V .

Further, note that if V in Theorem D is a curve, $M - K_{X/V} - \Delta$ is f -ample and nef and there is a $s \in V$, such that $H^0(X_s, M_s) = S^0(X_s, \sigma(X_s, \Delta_s))$, then points [Question 1](#) and [Question 1](#) of Theorem D with point (1) of Theorem C imply that $f_*\mathcal{O}_X(M)$ is a nef vector bundle. This strengthens [\[Pat12, Proposition 3.6\]](#), it removes the cohomology vanishing assumption made there. Of course our statements are considerably stronger. For example with the same assumptions except replacing $H^0(X_s, M_s) = S^0(X_s, \sigma(X_s, \Delta_s))$ with $\mathrm{rk} f_*(\sigma(X, \Delta) \otimes M)$ equals the general value of $S^0(X_s, \sigma(X_s, \Delta_s))$, we obtain that $f_*(\sigma(X, \Delta) \otimes M)$ being nef. In fact, the following even stronger statement can be made.

Theorem E. [\[Corollary 6.39\]](#) *With assumptions as in Theorem C such that V is projective over a perfect field, assume that $M - K_{X/V} - \Delta$ is nef and f -ample and that $\mathrm{rk} S^0 f_*(\sigma(X, \Delta) \otimes M)$ equals the general value of $H^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$. Then $S^0 f_*(\sigma(X, \Delta) \otimes M)$ is weakly positive. In particular, if V is a smooth curve then it is a nef vector bundle.*

Further, we show how $S^0_{\Delta, \mathrm{ne}} f_*(M)$ relates to the other similar notion $S^0 f_*(\sigma(X, \Delta) \otimes M)$ introduced in [\[HX13, Definition 2.14\]](#). In particular we obtain that $S^0 f_*(\sigma(X, \Delta) \otimes M)$ does not restrict to $S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$ in general for general $s \in S$. Intuitively, though $S^0 f_*(\sigma(X, \Delta) \otimes M)$ is a pushforward, it captures the global geometry of (X, Δ) rather than the geometry of the fibers. In positive characteristic these two can differ considerably, essentially because the function field of V is not perfect. Of course the relative and absolute $S^0 f_*$ are related. We study these similarities and differences in [Section 6.6](#).

1.1. Organization. In Section 2, we set up notation that we will follow throughout the paper, explain the interplay between p^{-e} -linear maps and \mathbb{Q} -Cartier divisors, and discuss the behaviors of such maps and divisors under base changes. In Sections 3, 4, and 5, we introduce the relative non- F -pure ideals, the relative test ideals and the relative test submodules, respectively. Applications to F -singularities in families are discussed in these 3 sections. Section 6 is devoted to the behaviors of S^0 (introduced in [\[Sch11a, Definition 4.1\]](#)) under base changes. Some semi-positivity results are also proved in this section. Finally, in the Appendix, we collect some statements that we can not find proper references for their generality but are needed in our paper.

1.2. Acknowledgements. The authors began working on this project while attending a workshop at the American Institute of Mathematics (AIM) titled “ACC for minimal log discrepancies and termination of flips”, May 14th – 18th 2012, organized by Tommaso de Fernex and Christopher Hacon. We would like to thank the organizers and the American Institute of Mathematics. Part of the work was done when the third author visited the Department of Mathematics at Pennsylvania State University; he would like to thank them for their hospitality. Additionally, the authors would like to thank Lawrence Ein, Christopher Hacon, János Kollár, Joseph Lipman, Mircea Mustața, and Kevin Tucker for useful discussions. Especially, we would like to thank Brian Conrad for pointing out compatibility of trace with base change. We would also like to thank Hiromu Tanaka for comments on a previous draft. Additionally, the authors of the paper would like to thank Leo Alonso, Damian Rössler, Graham Leuschke and Blup for providing answers and commentary to the following questions <http://mathoverflow.net/questions/120982/> and <http://mathoverflow.net/questions/120625/>.

2. NOTATION AND SETUP

Throughout this paper, all schemes are Noetherian and all maps of schemes are separated. We fix the following notation, which is in effect for the entire paper. In particular, for simplicity we do not state it in every statement even though it is assumed.

Notation 2.1. Suppose that $f : X \rightarrow V$ is a flat, equidimensional and geometrically reduced map⁵ of finite type from a scheme X to an excellent integral scheme V of equal characteristic $p > 0$ with a dualizing complex. We write $F^e = F_V^e : V = V^e \rightarrow V$ as the absolute e -iterated Frobenius on V , form the base change $f_{V^e} : X_{V^e} = X \times_V V^e \rightarrow V^e$ and define $F_{X^e/V^e}^e : X^e \rightarrow X_{V^e}$ to be the e -iterated relative Frobenius. Furthermore, we often assume that V is F -finite in which case we automatically assume that Frobenius $F_V : V \rightarrow V$ satisfies the following identity⁶, $(F_V)^! \omega_V^\bullet \simeq_{\text{qis}} \omega_V^\bullet$. If we say that V is a variety, it is always of finite type over a perfect field k .

Because we will be considering numerous different sheaves on the same topological space $X = X^e = X \times_V V^e = \text{etc.}$, but with respect to different schemes, we will adopt the following somewhat nonstandard notation. We use these because otherwise writing numerous $(F_{X^e/V^e}^e)_*$ and $(F_{X^e/V^e}^e)^{-1}$ operations, which do nothing to the underlying space, is confusing. We would need notations for projections of the form $X^e \times_{V^e} V^{e+d} \rightarrow V^{e+d}, X^e$ in as well as more general relative Frobenii $F_{(X^e)_{V^{e+d}/V^{e+d}}}^e : (X^e)_{V^{e+d}} := X^e \times_{V^e} V^{e+d} \rightarrow X_{V^{e+d}}$, and maps $X^e \times_{V^{e+d+c}} \rightarrow X^e \times_{V^{e+d}}$. By using simply modules, this becomes more transparent.

- (1) We will use R to denote \mathcal{O}_X .
- (2) We will use A to denote $f^{-1}\mathcal{O}_V$.
- (3) We will use A^{1/p^e} to denote $f^{-1}(F_V^e)_*\mathcal{O}_V$. We note that this is not an abuse of notation since V is integral.
- (4) We will use R^{1/p^e} to denote $(F_X^e)_*R$. This is a slight abuse of notation if X is not reduced.
- (5) We will use $R_{A^{1/p^e}}$ to denote $R \otimes_A A^{1/p^e}$.
- (6) We will use $(R_{A^{1/p^e}})^{1/p^d}$ to denote $R^{1/p^d} \otimes_{A^{1/p^d}} A^{1/p^{e+d}}$. This may be a slight abuse of notation if X_{V^e} is not reduced.
- (7) Given R -module M , we will use M^{1/p^e} to denote $F_*^e M$. This will generally not cause any confusion because typically M will be locally free or even a line bundle.
- (8) We use ω_R to denote ω_X and ω_A to denote $f^{-1}\omega_V$.
- (9) Etc.

Some of the main results of the paper concern restriction to fibers. These statements pertain only to a special set of fibers of f , the fibers over perfect points (see definition below). For example, if V is a curve over an algebraically closed field, then often we restrict to fibers over all closed points and the perfect closure of the generic point of V , but not the generic point itself.

Definition 2.2 (Perfect points). A *perfect point* of V , $s \in V$ is a morphism from the spectrum of a perfect field $k(s)$ to V , $s = \text{Spec } k(s) \rightarrow V$. It can also be viewed as a choice of a point $v \in V$ and a field extension $k(s) = K \supseteq k(v)$ such that K is perfect. Finally, a *neighborhood of a perfect point* $s \in V$ is simply a neighborhood of the image v of s .

The fact that $f : X \rightarrow V$ is of finite type implies that the relative Frobenius is a finite map. In some cases, this will allow us to avoid assuming that V is F -finite.

Lemma 2.3. *With notation as above, since $f : X \rightarrow V$ is of finite type, the relative Frobenius map $f : X^e \rightarrow X_{V^e}$ is a finite map.*

Proof. We work locally on V and X . It is sufficient to show that R^{1/p^e} is a finite $R_{A^{1/p^e}}$ -module. Write $S = A[x_1, \dots, x_n]$ and $R = S/I$. We first observe that $S_{A^{1/p^e}} \rightarrow S^{1/p^e}$ is a finite map. However, $x_1^{i_1} \cdots x_n^{i_n}$ for $0 \leq i_j < p^e$ clearly form a generating set for S^{1/p^e} over $S_{A^{1/p^e}}$.

⁵Meaning $X_T = X \times_V T$ is reduced for all $T \rightarrow V$ with T integral.

⁶For some discussion of this identity, which always holds for varieties or schemes of essentially finite type over a local ring with a dualizing complex, see [BSTZ10, Section 2].

Now, tensor the map $S_{A^{1/p^e}} \rightarrow S^{1/p^e}$ with $\otimes_S(S/I) = \otimes_S R$. We obtain

$$R_{A^{1/p^e}} = R \otimes_A A^{1/p^e} = (S/I) \otimes_S S_{A^{1/p^e}} \rightarrow (S/I) \otimes_S S^{1/p^e} = (S/I^{[p^e]})^{1/p^e} \rightarrow (S/I)^{1/p^e}$$

where the final map is the canonical surjection of rings. The map is finite since each part is. \square

We also recall the result of Radu and André.

Theorem 2.4. [Rad92, And93] *Suppose that $f : X \rightarrow V$ is a flat map of Noetherian schemes. Then f has geometrically regular fibers if and only if the relative Frobenius $R_{A^{1/p^e}} \cong A^{1/p^e} \otimes_A R \rightarrow R^{1/p^e}$ is flat.*

We immediately obtain the following corollary which will also be useful.

Corollary 2.5. *Suppose that $f : X \rightarrow V$ is as in [Notation 2.1](#) and additionally has geometrically regular fibers. Then for any $R_{A^{1/p^e}}$ -module M , the natural evaluation-at-1 map below surjects*

$$\mathcal{H}\text{om}_{R_{A^{1/p^e}}}(R^{1/p^e}, M) \rightarrow M$$

Proof. R^{1/p^e} is a locally free $R_{A^{1/p^e}}$ -module, and the result follows. \square

Now we state the object which we will study for the majority of the paper.

Definition 2.6 (φ). From here on, we will fix a line bundle L on $X \cong X^e$ and we will also fix a (possibly zero) $R_{A^{1/p^e}}$ -linear map $\varphi : L^{1/p^e} \rightarrow R_{A^{1/p^e}}$.

Remark 2.7 (Reflexive sheaves for G1 and S2 Morphisms). Suppose that $f : X \rightarrow V$ is G1 + S2. Note that there exists an open set $\iota : U \hookrightarrow X$ such that $X \setminus U$ has codimension ≥ 2 along each fiber and that $f|_U$ is a Gorenstein morphism.

Finally, suppose that M is any rank-1 reflexive R -module which is locally free on a set U as above. Then M^{1/p^e} is not only reflexive as an R^{1/p^e} -module, we claim it is also reflexive as an $R_{A^{1/p^e}}$ -module. Since $i_*M|_U = M$ already by [Proposition A.7](#) because M is reflexive, it is sufficient to replace X by U . Thus $\omega_{X/V}$ and M are both locally free as R -modules and ω_{X^e/V^e} is locally free as an $R_{A^{1/p^e}}$ -module. We work locally so as to trivialize all these modules. Then

$$\begin{aligned} M^{1/p^e} &\cong \omega_{X/V}^{1/p^e} \\ &= (F_{X^e/V^e}^e)_* \omega_{X/V} \\ &\cong \mathcal{H}\text{om}_{\mathcal{O}_{X^e/V^e}}((F_{X^e/V^e}^e)_* \mathcal{O}_X, \omega_{X^e/V^e}) \\ &\cong \mathcal{H}\text{om}_{R_{A^{1/p^e}}}(R^{1/p^e}, R_{A^{1/p^e}}) \end{aligned}$$

which is clearly reflexive (the second isomorphism follows from Grothendieck duality for the finite relative Frobenius map).

Conversely, if M is any R -module which is locally free on a set U and reflexive as an $R_{A^{1/p^e}}$ -module, then it is also reflexive as an R^{1/p^e} -module. To see this, note that $M|_U$ is reflexive as an R -module, and because M is reflexive as an $R_{A^{1/p^e}}$ -module it satisfies $i_*M|_U = M$.

Definition 2.8 (φ versus divisors). Observe the following identifications:

$$\begin{aligned}
 & \mathcal{H}\mathrm{om}_{R_{A^1/p^e}}(L^{1/p^e}, R_{A^1/p^e}) \\
 \cong & \underbrace{\left(\mathcal{H}\mathrm{om}_{R_{A^1/p^e}} \left(L^{1/p^e} \otimes_{R_{A^1/p^e}} (\omega_{R_{A^1/p^e}/A^{1/p^e}}), \omega_{R_{A^1/p^e}/A^{1/p^e}} \right) \right)^{**}}_{\text{This clearly holds on } U, \text{ note both sheaves are reflexive and use \textcolor{red}{Corollary A.8}.}} \\
 = & \underbrace{\left(\mathcal{H}\mathrm{om}_{\mathcal{O}_{X_{V^e}}} \left((F_{X^e/V^e}^e)_* \left(L \otimes ((F_{X^e/V^e}^e)^* \omega_{X_{V^e}/V^e}) \right), \omega_{X_{V^e}/V^e} \right) \right)^{**}}_{\text{This just rewrites the previous line using different notation.}} \\
 \cong & \underbrace{\left((F_{X^e/V^e}^e)_* \mathcal{H}\mathrm{om}_{\mathcal{O}_{X^e}} \left(L \otimes ((F_{X^e/V^e}^e)^* \omega_{X_{V^e}/V^e}), \omega_{X^e/V^e} \right) \right)^{**}}_{\text{Grothendieck duality for a finite map}} \\
 \cong & \underbrace{(F_{X^e/V^e}^e)_* \left(L^{-1} \otimes \omega_{X^e/V^e} \otimes (F_{X^e/V^e}^e)^* \omega_{X_{V^e}/V^e}^{-1} \right)^{**}}_{\text{By \textcolor{red}{Remark 2.7} and \textcolor{red}{Corollary A.8} we may take reflexive hull as } (F_{X^e/V^e}^e)_* \mathcal{O}_{X^e} \text{-modules.}}
 \end{aligned}
 \tag{2.8.1}$$

Now, observe that $\omega_{X/V}$ is compatible with base change up to reflexification. In particular, if our base change is the Frobenius $F_V^e : V^e \rightarrow V$, then writing $\pi_{V^e} : X_{V^e} = X \times_V V^e \rightarrow X$ as the projection, we have $\omega_{X_{V^e}/V^e} \cong (\pi_{V^e}^* \omega_{X/V})^{**}$ since both sheaves are reflexive and they certainly agree outside the non-relatively Cohen-Macaulay locus (which is of relative codimension at least 2) by [Corollary A.8](#). In particular

$$((F_{X^e/V^e}^e)^* \omega_{X_{V^e}/V^e})^{**} \cong ((F_{X^e/V^e}^e)^* \pi_{V^e}^* \omega_{X/V})^{**} = ((F_X^e)^* \omega_{X/V})^{**} = (\omega_{X/V}^{p^e})^{**}.$$

Plugging this into (2.8.1) we obtain

$$\begin{aligned}
 & \mathcal{H}\mathrm{om}_{R_{A^1/p^e}}(L^{1/p^e}, R_{A^1/p^e}) \\
 \cong & (F_{X^e/V^e}^e)_* \left(L^{-1} \otimes \omega_{X^e/V^e}^{1-p^e} \right)^{**}
 \end{aligned}
 \tag{2.8.2}$$

If additionally X is absolutely (instead of relatively) G1 and S2 (for example, if V is regular), then any choice of nondegenerate⁷ φ induces a non-zero, effective Weil divisorial sheaf D_φ such that $\mathcal{O}_X(D_\varphi) \cong \left(L^{-1} \otimes \omega_{X^e/V^e}^{1-p^e} \right)^{**}$ by [Har94]. We would like to generalize this to the case that $X \rightarrow V$ is *relatively* G1 and S2.

Definition 2.9. We say that φ is *relatively divisorial* if φ locally generates $\mathcal{H}\mathrm{om}_{R_{A^1/p^e}}(L^{1/p^e}, R_{A^1/p^e})$ as an R^{1/p^e} -module at

- (a) the generic points of each fiber and
- (b) at the generic point of every codimension-1 singular point of every geometric fiber.

In this case, by removing a set of relative codimension 2 so that f is relatively Gorenstein, we see that $\varphi \cdot R^{1/p^e} \subseteq \mathcal{H}\mathrm{om}_{R_{A^1/p^e}}(L^{1/p^e}, R_{A^1/p^e})$ is a rank-1 free submodule of an invertible R^{1/p^e} -module. To be able to associate a divisor to this submodule in a sensible way, we should show that it is the trivial (full) submodule at every singular codimension one point. Indeed, let ξ be a singular codimension 1 point. Then one of the following cases hold.

- $f(\xi)$ is a codimension 1 point. In this case, ξ is a general point of the fibers over $f(\xi)$, hence assumption (a) guarantees that φ generates a full submodule at ξ .
- $f(\xi)$ is the general point of V . In this case ξ is a codimension 1 point of the fiber over $f(\xi)$, and it is not in the smooth locus of f . Therefore, assumption (b) shows that again φ generates a full submodule at ξ .

Therefore, the submodule $\varphi \cdot R^{1/p^e}$ determines a Cartier divisor and also an honest Weil divisor on the original X . We denote this divisor D_φ as well.

⁷Nonzero on any irreducible component.

Definition 2.10 (φ as a divisor). If φ is relatively divisorial, we set Δ_φ to be the \mathbb{Q} -divisor $\frac{1}{p^e-1}D_\varphi$. This makes sense because D_φ is trivial along the codimension-1 components of the singular locus of X and so we avoid the pathological issues which occur for \mathbb{Q} -divisors on non-normal spaces.

We now explain how to recover φ from a \mathbb{Q} -divisor.

Remark 2.11 (Obtaining φ from divisors). We work under the conventions of [Definition 2.10](#). Untangling [Definition 2.8](#) yields a method to obtain φ from a divisor $\Delta \geq 0$ (which then coincides with Δ_φ) under the following assumptions:

- (a) $\Delta = \frac{1}{m}D$ for some Weil divisor D where $p \nmid m$.
- (b) D is a Weil divisor on X which is Cartier in relative codimension 1.
- (c) $\text{Supp } D$ does not contain any general point or any singular codimension one point of any geometric fiber.
- (d) $(p^e - 1)/m \in \mathbb{Z}$ and $(\omega_{X/V}^{1-p^e} \otimes \mathcal{O}_X((1-p^e)\Delta))^{**} = L$ is a line bundle.

In such a case, the integral divisor $(p^e - 1)\Delta$ (which is well defined since $\text{Supp } \Delta$ does not contain the codimension-1 components of the non-regular locus of X , cf. [\[Kc92, Pages 171–173\]](#) or [\[MS12, Section 2.2\]](#)) induces an inclusion

$$(F_{X^e/V^e}^e)_* L \hookrightarrow (F_{X^e/V^e}^e)_* ((\omega_{X/V}^{1-p^e})^{**}).$$

This composed with the natural Grothendieck trace map

$$(F_{X^e/V^e}^e)_* ((\omega_{X/V}^{1-p^e})^{**}) \rightarrow \mathcal{O}_{X_{V^e}}$$

yields a map φ . It is easy to see that conditions (a)–(d) above guarantee that φ is relatively divisorial. Furthermore, we also have that $\Delta = \Delta_\varphi$.

For future reference we make the following definition.

Definition 2.12. In the situation of [Notation 2.1](#), (X, Δ) is a pair, if f is G1 + S2 and Δ satisfies the assumptions of [Remark 2.11](#).

2.13. Composing maps. Given φ as in [Definition 2.8](#), we can compose $\varphi : L^{1/p^e} \rightarrow R_{A^{1/p^e}}$ with itself (after twisting) similar to [\[BS13, Section 4\]](#) or [\[Sch09\]](#) and thus obtain new maps

$$\varphi^2 \in \text{Hom}_{R_{A^{1/p^{2e}}}} \left(\left(L^{(p^e+1)} \right)^{1/p^{2e}}, R_{A^{1/p^{2e}}} \right)$$

and more generally

$$\varphi^n \in \text{Hom}_{R_{A^{1/p^{ne}}}} \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{1/p^{ne}}, R_{A^{1/p^{ne}}} \right).$$

We explain this construction.

Begin by tensoring φ by L over R , and then taking $1/p^e$ th roots we obtain:

$$(2.13.1) \quad (L^{1+p^e})^{1/p^{2e}} = (L^{1/p^e} \otimes_R L)^{1/p^e} \rightarrow ((R \otimes_A A^{1/p^e}) \otimes_R L)^{1/p^e} = L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{2e}}$$

On the other hand, we can also tensor φ by $A^{1/p^{2e}}$ over A^{1/p^e} to obtain:

$$(2.13.2) \quad L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{2e}} \rightarrow R_{A^{1/p^{2e}}}$$

By composing (2.13.1) and (2.13.2) we obtain the desired map φ^2 . We now define φ^n inductively as follows. Given $\varphi^{n-1} : \left(L^{\frac{p^{(n-1)e}-1}{p^e-1}} \right)^{1/p^{(n-1)e}} \rightarrow R_{A^{1/p^{(n-1)e}}}$ tensor with L over R and then take

p^e th roots which yields

$$\begin{aligned}
& \left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{1/p^{ne}} \\
&= \left(\left(L^{\frac{p^{(n-1)e}-1}{p^e-1}} + p^{(n-1)e} \right)^{1/p^{(n-1)e}} \right)^{1/p^e} \\
&= \left(L \otimes_R \left(L^{\frac{p^{(n-1)e}-1}{p^e-1}} \right)^{1/p^{(n-1)e}} \right)^{1/p^e} \\
&\rightarrow \left(L \otimes_R R_{A^{1/p^{(n-1)e}}}} \right)^{1/p^e} \\
&= L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}.
\end{aligned}$$

We then apply φ to the first term in the final tensor product to obtain:

$$\left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{1/p^{ne}} \rightarrow L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}} \xrightarrow{\varphi \otimes \dots} \left(R \otimes_A A^{1/p^e} \right) \otimes_{A^{1/p^e}} A^{1/p^{ne}} \cong R_{A^{1/p^{ne}}}$$

which we take as the official definition of φ^n . On the other hand, for every $0 < m < n$ one can look at the following composition of φ^m and φ^{n-m}

(2.13.3)

$$\begin{aligned}
\gamma : \left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} &\cong \left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R L^{\frac{p^{ne-p^{me}}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \cong \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \right)^{\frac{1}{p^{me}}} \otimes_R L^{\frac{p^{(n-m)e}-1}{p^e-1}} \right)^{\frac{1}{p^{(n-m)e}}} \\
&\xrightarrow{(\varphi^m \otimes \text{id}_L \dots)} \left(\left(R \otimes_A A^{\frac{1}{p^{me}}} \right) \otimes_R L^{\frac{p^{(n-m)e}-1}{p^e-1}} \right)^{\frac{1}{p^{(n-m)e}}} \cong \left(L^{\frac{p^{(n-m)e}-1}{p^e-1}} \right)^{\frac{1}{p^{(n-m)e}}} \otimes_{A^{p^{\frac{1}{p^{(n-m)e}}}}} A^{\frac{1}{p^{ne}}} \\
&\xrightarrow{\varphi^{n-m} \otimes_A \dots A^{\dots}} \left(R \otimes_A A^{\frac{1}{p^{(n-m)e}}} \right) \otimes_{A^{p^{\frac{1}{p^{(n-m)e}}}}} A^{\frac{1}{p^{ne}}} \cong R \otimes_A A^{\frac{1}{p^{ne}}}
\end{aligned}$$

In particular, taking $m = 1$ gives an a priori different map which we could also define as φ^n . We now explain why this map is actually equal to the *official* φ^n .

Lemma 2.14. *With notation as above, $\gamma = \varphi^n$.*

Proof. The statement is local, and so we may suppose that $L = R$. With the (obscuring) powers of L removed, φ^n is described as the following composition:

$$\begin{aligned}
& R^{1/p^{ne}} \\
&\xrightarrow{\varphi^{1/p^{(n-1)e}}} R^{1/p^{(n-1)e}} \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}} \\
&\xrightarrow{\varphi^{1/p^{(n-2)e}} \otimes \dots} \left(R^{1/p^{(n-2)e}} \otimes_{A^{1/p^{(n-2)e}}} A^{1/p^{(n-1)e}} \right) \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}} \\
&\xrightarrow{\varphi^{1/p^{(n-3)e}} \otimes \dots} \dots \\
&\quad \dots \quad \dots \\
&\xrightarrow{\varphi^{1/p^e} \otimes \dots} R^{1/p^e} \otimes_{A^{1/p^e}} \dots \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}} \\
&\xrightarrow{\varphi \otimes \dots} R \otimes_A \dots \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}}.
\end{aligned}$$

The first m entries in the composition make up $(\varphi^m \otimes \dots)_{A^{1/p(n-m)e}}^{1/p^{(n-m)e}}$ in (2.13.3) and the last $n - m$ entries clearly yield $\varphi^{n-m} \otimes_{A^{1/p(n-m)e}} A^{1/p^n}$ as desired. The result is then obvious. \square

Lemma 2.15. *With notation as in Definition 2.8 and additionally that φ is relatively divisorial, then φ^n is relatively divisorial and $\Delta_\varphi = \Delta_{\varphi^n}$ for every integer $n \geq 1$.*

Proof. For showing any of the two statements we may remove the non relatively Gorenstein locus. That is, by possibly further restricting R we may assume that R is relatively Gorenstein over V and that L is trivial (since our map $f : X \rightarrow V$ is geometrically G1+S2). Hence by Remark 2.7,

$$\mathrm{Hom}_{R \otimes_A A^{1/p^{ie}}} (R^{1/p^{ie}}, R \otimes_A A^{1/p^{ie}}) \cong R^{1/p^{ie}} \text{ for all } i.$$

From [Kun86, Appendix F], [Sch09, Lemma 3.9], or composition of Grothendieck trace, we have

$$(2.15.1) \quad \left(\begin{array}{cc} \mathrm{Hom}_{R^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}} (R^{1/p^{ne}}, R^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}) & \\ \otimes_{R^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}} \mathrm{Hom}_{R_{A^{1/p^{ne}}}} (R^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}, R_{A^{1/p^{ne}}}) & \end{array} \right) \cong \mathrm{Hom}_{R_{A^{1/p^{ne}}}} (R^{1/p^{ne}}, R_{A^{1/p^{ne}}})$$

via the homomorphism induced by composition. Let ϑ_e be the R^{1/p^e} -module generator of

$$\mathrm{Hom}_{R \otimes_A A^{1/p^e}} (R^{1/p^e}, R \otimes_A A^{1/p^e})$$

for each e and let ϑ_e^n be defined as φ^n , but φ replaced by ϑ . Then by using (2.15.1) iteratively, $\vartheta_e^n = \vartheta_{ne}$ up to multiplication by a unit.

Further let $r \in R$ such that $\varphi(-) = \vartheta_e (r^{1/p^e} \cdot -)$. Then it is easy to verify that

$$(2.15.2) \quad \varphi^n = \vartheta_{ie} \left(\left(r^{\frac{p^{ne}-1}{p^e-1}} \right)^{1/p^{ne}} \cdot - \right)$$

Then we see that if φ was generating at a point $P \in X$, or equivalently r is a unit at P , then so is φ^n . This shows that φ^n is relatively divisorial. Furthermore (2.15.2) shows that $\Delta_{\varphi^n} = \Delta_\varphi$. \square

2.16. Base change of φ . Suppose that we are given $g : T \rightarrow V$ any morphism of schemes such that T is also excellent, integral and has a dualizing complex. For example, we could set T to be a closed point of V and let g be the inclusion. Alternately, we could let g be a regular alteration over some closed subscheme of V . We list the following maps:

$$(2.16.3) \quad \begin{array}{llll} p_1 : & X \times_V T & \rightarrow & X & \text{(the projection).} \\ (p_1)^{1/p^i} : & X^i \times_{V^i} T^i & \rightarrow & X^i & \text{(the projection for any } i). \\ q_i : & X \times_V T^i & \rightarrow & X \times_V V^i & \text{(base change)} \\ p_1 = q_0 : & X \times_V T & \rightarrow & X. \end{array}$$

These are pictured below.

$$\begin{array}{ccc} X^i \times_{V^i} T^i & \xrightarrow{(p_1)^{1/p^i}} & X^i \\ \downarrow F_{(X_T)^i/T^i}^i & & \downarrow F_{X^i/V^i}^i \\ X \times_V T^i & \xrightarrow{q_i} & X \times_V V^i \end{array}$$

Notice that given $\varphi : L^{1/p^e} \rightarrow R_{A^{1/p^e}} = \mathcal{O}_{X_{V^e}}$, we can form $(q_e)^* \varphi$ which we denote by

$$\varphi_T : L_T^{1/p^e} \cong q_e^* L^{1/p^e} \rightarrow q_e^* \mathcal{O}_{X_{V^e}} = \mathcal{O}_{X_{T^e}}.$$

We explain the isomorphism $L_T^{1/p^e} \cong q_e^* L^{1/p^e}$ briefly. Working locally, let $V = \operatorname{Spec} A$, $T = \operatorname{Spec} B$ and $X = \operatorname{Spec} R$. Then the map φ_T is identified with the map:

$$L^{1/p^e} \otimes_{R_{A^{1/p^e}}} R_{B^{1/p^e}} \cong L^{1/p^e} \otimes_{A^{1/p^e}} B^{1/p^e} \rightarrow ((R \otimes_A A^{1/p^e}) \otimes_{A^{1/p^e}} B^{1/p^e}) \cong R \otimes_A B^{1/p^e}.$$

The isomorphism in the definition of φ_T is now immediate.

The next lemma shows that base change of φ commutes with the self-composition defined in [Section 2.13](#).

Lemma 2.17. *Suppose that $g : T \rightarrow V$ is as above. Then $(\varphi^n)_T = (\varphi_T)^n$.*

Proof. It is sufficient to prove this in the affine case assuming our isomorphism is sufficiently canonical (which will be clear). We notice that $(\varphi_T)^2$ is the composition:

$$\begin{aligned} ((L^{p^e+1})^{1/p^e} \otimes_{A^{1/p^e}} B^{1/p^e})^{1/p^e} &\xrightarrow{(\varphi_T \otimes_{R_{B^{1/p^e}}} L_T)^{1/p^e}} L^{1/p^e} \otimes_{A^{1/p^e}} B^{1/p^e} \otimes_{B^{1/p^e}} B^{1/p^{2e}} \\ &\xrightarrow{\varphi_T \otimes_{B^{1/p^e}} B^{1/p^{2e}}} ((R \otimes_A A^{1/p^e}) \otimes_{A^{1/p^e}} B^{1/p^e}) \otimes_{B^{1/p^e}} B^{1/p^{2e}} \end{aligned}$$

Now unraveling the definitions we obtain

$$\begin{aligned} (\varphi_T)^2 &= \left((\varphi_T \otimes_{R_{B^{1/p^e}}} L_T)^{1/p^e} \right) \circ (\varphi_T \otimes_{B^{1/p^e}} B^{1/p^{2e}}) \\ &= \left(((\varphi \otimes_{A^{1/p^e}} B^{1/p^e}) \otimes_{R_{B^{1/p^e}}} (L \otimes_{A^{1/p^e}} B^{1/p^e}))^{1/p^e} \right) \circ (\varphi \otimes_{A^{1/p^e}} B^{1/p^e} \otimes_{B^{1/p^e}} B^{1/p^{2e}}) \\ &= \left((\varphi \otimes_{R_{A^{1/p^e}}} L \otimes_{A^{1/p^e}} B^{1/p^e})^{1/p^e} \right) \circ (\varphi \otimes_{A^{1/p^e}} B^{1/p^{2e}}) \\ &= \left((\varphi \otimes_{R_{A^{1/p^e}}} L)^{1/p^e} \otimes_{A^{1/p^{2e}}} B^{1/p^{2e}} \right) \circ ((\varphi \otimes_{A^{1/p^e}} A^{1/p^{2e}}) \otimes_{A^{1/p^{2e}}} B^{1/p^{2e}}) \\ &= \left((\varphi \otimes_{R_{A^{1/p^e}}} L)^{1/p^e} \circ (\varphi \otimes_{A^{1/p^e}} A^{1/p^{2e}}) \right) \otimes_{A^{1/p^{2e}}} B^{1/p^{2e}} \\ &= (\varphi^2)_T \end{aligned}$$

as desired. The general n th self composition is similar. \square

Our next goal is to describe how divisors, corresponding to maps φ , fare under base change. We thank Brian Conrad for pointing us in the right direction – [\[Con00, Theorem 3.61\]](#).

Lemma 2.18. *Suppose that $f : X \rightarrow V$ is a finite type Cohen-Macaulay morphism over V an excellent scheme of characteristic $p > 0$ and that $g : T \rightarrow V$ is as above. Then the Grothendieck-trace map*

$$(F_{X^e/V^e})_* \omega_{X^e/V^e} \cong \mathcal{H}om_{\mathcal{O}_{X_{V^e}}} ((F_{X^e/V^e})_* \mathcal{O}_{X^e}, \omega_{X_{V^e}/V^e}) \rightarrow \omega_{X_{V^e}/V^e}$$

is compatible with base change.

Proof. We clearly have a commutative diagram:

$$\begin{array}{ccc} q_e^* \mathcal{H}om_{\mathcal{O}_{X_{V^e}}} ((F_{X^e/V^e})_* \mathcal{O}_{X^e}, \omega_{X_{V^e}/V^e}) & \longrightarrow & q_e^* \omega_{X_{V^e}/V^e} \\ \alpha \downarrow & & \downarrow \beta \\ \mathcal{H}om_{\mathcal{O}_{X_{T^e}}} ((F_{(X_T)^e/T^e})_* \mathcal{O}_{(X_T)^e}, \omega_{X_{T^e}/T^e}) & \longrightarrow & \omega_{X_{T^e}/T^e} \end{array}$$

and by [\[Con00, Theorem 3.6.1\]](#) the map β is an isomorphism. It is sufficient to verify that α is an isomorphism as well. We work locally on some affine chart on X and hence assume that $X_V \subseteq \mathbb{A}_V^N =: P_V$ embeds as a closed subscheme since f is of finite type. The map α can then be identified with

$$Q_e^* \mathcal{E}xt_{\mathcal{O}_{P_{V^e}}}^j ((F_{X^e/V^e})_* \mathcal{O}_{X^e}, \omega_{P_{V^e}/V^e}) \rightarrow \mathcal{E}xt_{\mathcal{O}_{P_{T^e}}}^j ((F_{(X_T)^e/T^e})_* \mathcal{O}_{(X_T)^e}, \omega_{P_{T^e}/T^e})$$

where $Q_e : P_{T^e} \rightarrow P_{V^e}$ is the induced map and $j = N - \dim(X/V)$ (we leave off the pushforward for the inclusion $i : X_V \rightarrow P_V$ above). This in turn can be identified with

$$Q_e^* \mathcal{E}xt_{\mathcal{O}_{(P_V)^e}}^j(\mathcal{O}_{X^e}, \omega_{(P_V)^e/V^e}) \rightarrow \mathcal{E}xt_{\mathcal{O}_{(P_T)^e}}^j(\mathcal{O}_{(X_T)^e}, \omega_{(P_T)^e/T^e})$$

This last map is exactly the bottom row of [Con00, Diagram (3.6.1) in Theorem 3.6.1], which is an isomorphism, and hence the proof is complete. \square

Lemma 2.18 above allows us to show if the divisor D_φ or Δ_φ is trivial, then it stays trivial after base change.

Lemma 2.19 (Divisors which are zero stay zero). *Suppose that $f : X \rightarrow V$ is a G1 and S2 morphism and that $\varphi : L^{1/p^e} \rightarrow R_{A^{1/p^e}}$ is as above. Additionally suppose that $g : T \rightarrow V$ is any base change. Finally suppose that the natural map*

$$\varphi \cdot R^{1/p^e} \rightarrow \mathcal{H}om_{R_{A^{1/p^e}}} \left(L^{1/p^e}, R_{A^{1/p^e}} \right) \cong (F_{X^e/V^e}^e)_* \left(L^{-1} \otimes \omega_{X^e/V^e}^{1-p^e} \right)^{**}$$

is an isomorphism of R^{1/p^e} -modules (here we define $\varphi \cdot R^{1/p^e}$ to be the R^{1/p^e} -submodule of the set $\mathcal{H}om_{R_{A^{1/p^e}}} (L^{1/p^e}, R_{A^{1/p^e}})$ generated by φ). Then

$$\varphi_T \cdot (R_T)^{1/p^e} \rightarrow \mathcal{H}om_{R_{T^{1/p^e}}} \left(L_T^{1/p^e}, R_{T^{1/p^e}} \right) \cong (F_{(X_T)^e/T^e}^e)_* \left(L_T^{-1} \otimes \omega_{(X_T)^e/T^e}^{1-p^e} \right)^{**}$$

is also an isomorphism.

In particular, if φ is relatively divisorial (see Definition 2.9), then the following holds: If D_φ is zero, then so is D_{φ_T} .

Proof. The statement about divisors is trivial since it is easy to see that a divisor being zero corresponds to the map above being an isomorphism. Thus we merely need to prove the assertion. However, since f and all the sheaves involved are relatively S2, it suffices to prove the statement off a set of relative codimension 2 by Corollary A.8. Therefore, by removing a set of codimension 2, we can assume that f is a Gorenstein morphism. Thus, working locally if needed, φ can be identified (up to multiplication by a unit in R^{1/p^e}) with the Grothendieck trace $(F_{X^e/V^e}^e)_* \omega_{X^e/V^e} \rightarrow \omega_{X_{V^e}/V^e}$. Hence by Lemma 2.18, so can φ_T . The proof is complete. \square

We now move on to a discussion of base change with respect to divisors. First we observe that by Lemma 2.19 if φ is relatively divisorial, then so is φ_T for every base change $g : T \rightarrow V$.

Definition 2.20. Suppose that for some relatively divisorial φ we have Δ_φ is as above and that $g : T \rightarrow V$ is a base change. Then we write $\Delta_\varphi|_{X_T}$ (respectively $D_\varphi|_{X_T}$) to denote the divisor Δ_{φ_T} (respectively D_{φ_T}).

Lemma 2.21 (Pulling back Δ_φ). *Suppose that $f : X \rightarrow V$ is G1 and S2 and that φ is relatively divisorial. Then for any $g : T \rightarrow V$ we have $\Delta_\varphi|_{X_T} = (p_1^{1/p^e})^* \Delta_\varphi$ (recall $p_1^{1/p^e} : X^e \times_{V^e} T^e \rightarrow X^e$ from (2.16.3)).*

Here, even though Δ_φ is not necessarily \mathbb{Q} -Cartier, $(p_1^{1/p^e})^* \Delta_\varphi$ can be defined after removing a set of relative codimension 2 outside of which it is \mathbb{Q} -Cartier (since D_φ is Cartier on such a set).

Proof. The statement is local on X and can be checked after removing a relative codimension 2 set and so we may assume that D_φ is a Cartier divisor. Thus we assume that L is trivial on the affine scheme $X = \text{Spec } R$ and that f is a Gorenstein morphism. However then shrinking the neighborhood further if necessary, the map $\varphi : R^{1/p^e} \rightarrow R_{A^{1/p^e}}$ can be identified with $s^{1/p^e} \cdot \text{Tr}$ where $\text{Tr} : (F_{X^e/V^e}^e)_* \omega_{X^e/V^e} \rightarrow \omega_{X_{V^e}/V^e}$ is the Grothendieck trace. The divisor D_φ is then easily seen to be the divisor $\text{div}_X(s)$. On the other hand, it is clear by Lemma 2.18 that then φ_T is trace on X_T multiplied by s as well. In particular, it equals $(p_1^{1/p^e})^* \Delta_\varphi$ as desired. \square

Corollary 2.22. *Suppose that $g : z \rightarrow V$ is inclusion of a point. Suppose that φ is relatively divisorial and corresponds to Δ_φ . Then $\Delta_\varphi|_{X_z} = \Delta_{\varphi_z}$.*

Proof. The divisor Δ_{φ_z} is determined in codimension 1 where $\Delta_\varphi|_{X_z}$ and Δ_{φ_z} agree. \square

2.23. Passing to V^∞ and other perfect points. Suppose first that V is the spectrum of a perfect field A . Then the map $L^{1/p^e} \rightarrow R_{A^{1/p^e}}$ is also a map $L^{1/p^e} \rightarrow R$ since $R_{A^{1/p^e}} \cong R$. In such a case, we often also typically write the map as ψ to help distinguish how we are thinking about it. We note that maps such as $\psi : L^{1/p^e} \rightarrow R$ can be composed with themselves as in [BS13, Section 4.1] and furthermore this composition ψ^n coincides with φ^n . For the rest of the subsection, we discuss base change to perfect points. We first consider the perfection of the generic point of V .

Set V^∞ to denote the not-necessarily-Noetherian scheme $\text{Spec } \mathcal{O}_V^{1/p^\infty}$ and we set η_∞ to be the generic point of V^∞ with perfect fraction field $k(V^\infty)$ (note that this is a perfect point). We obtain a map $f_{k(V^\infty)} : X_{k(V^\infty)} = X \times_V \text{Spec } k(V^\infty) \rightarrow \text{Spec } k(V^\infty)$ now a scheme of finite type over a perfect field.

As in Section 2.16, the map φ then induces a map

$$\varphi_\infty : L^{1/p^e} \otimes_{k(V^e)} k(V^\infty) \rightarrow R_{k(V^\infty)} := R \otimes_A k(A^{1/p^\infty}).$$

which can be identified with

$$\psi_\infty : (L_{k(V^\infty)})^{1/p^e} \rightarrow R_{k(V^\infty)}$$

since $((F_{X^e/V^e}^e)_* \mathcal{O}_{X^e}) \otimes_{V^e} k(V^\infty) \cong F_*^e \mathcal{O}_{X_{k(V^\infty)}}$.

More generally, if $s \in V$ is any perfect point, we can induce

$$\varphi_s : L_s^{1/p^e} := L^{1/p^e} \otimes_{A^{1/p^e}} k(s)^{1/p^e} \rightarrow R \otimes_A k(s)^{1/p^e}.$$

Since $k(s) = k(s)^{1/p^e}$ is perfect, we can identify this with a p^{-e} -linear map:

$$\psi_s : L_s^{1/p^e} \rightarrow R_s.$$

As above, we see that the composition of ψ_s in the sense of [BS13, Section 4.1] coincides with the composition φ as in Section 2.13. Finally, we study this process with respect to divisors.

Lemma 2.24. *With notation as above, $\Delta_\varphi|_{X_s} = \Delta_{\psi_s}$ where Δ_φ is as in Definition 2.10 and Δ_{ψ_s} is as in [BS13, Section 4].*

Proof. By Lemma 2.21, we simply must show that Δ_{φ_s} coincides with Δ_{ψ_s} . Working locally, and removing a set of relative codimension 2, we may assume that $\Phi \in \text{Hom}_{R_{s^{1/p^e}}}((R_s)^{1/p^e}, R_{s^{1/p^e}})$ generates the Hom as an $(R_s)^{1/p^e}$ -module and that $\varphi(_) = \Phi(z^{1/p^e} \cdot _)$. Thus $\Delta_\varphi = \frac{1}{p^e-1} \text{div}_X(z)$.

We then identify $R_{s^{1/p^e}}$ with R_s and hence φ with ψ . Likewise, we can identify Φ with Ψ which now generates $\text{Hom}_{R_s}((R_s)^{1/p^e}, R_s)$ as an $(R_s)^{1/p^e}$ -module. Hence $\psi(_) = \Psi(z^{1/p^e} \cdot _)$ and so $\Delta_\psi = \frac{1}{p^e-1} \text{div}_X(z)$ as desired. \square

3. RELATIVE NON-F-PURE IDEALS

With notation as above, by the Hartshorne-Speiser-Lyubeznik-Gabber (HSLG-)Theorem [Gab04, Lemma 13.1] cf. [HS77, Lyu97, Bli13], we know that the chain

$$R_{k(V^\infty)} \supseteq \psi_\infty((L_{k(V^\infty)})^{1/p^e}) \supseteq \psi_\infty^2((L_{k(V^\infty)}^{p^e+1})^{1/p^{2e}}) \supseteq \cdots \supseteq \psi_\infty^n((L_{k(V^\infty)}^{\frac{p^{ne}-1}{p^e-1}})^{1/p^{ne}}) \supseteq \cdots$$

eventually stabilizes. Say that it stabilizes at

$$(3.0.1) \quad n \geq n_0.$$

For the rest of this section, we fix this integer n_0 , and make the following definition.

Definition 3.1. With notation as above, we define the integer n_0 to be the *uniform integer for σ over the generic point of V* , and in general, it will be denoted by $n_{\sigma(\varphi), k(V)}$. We notice that for any point $\eta \in V$, we can base change $\text{Spec } k(\eta) \rightarrow V$ and form a corresponding integer $n_{\sigma(\varphi_\eta), k(\eta)}$.

On the other hand, without the passing to $k(V^\infty)$. We have the images

$$\begin{aligned} \mathfrak{a}_1 &:= \varphi^1(L^{1/p^e}) && \subseteq R_{A^{1/p^e}}, \\ \mathfrak{a}_2 &:= \varphi^2((L^{(p^e+1)})^{1/p^{2e}}) && \subseteq R_{A^{1/p^{2e}}}, \\ &\dots && \\ \mathfrak{a}_n &:= \varphi^n((L^{\frac{p^{ne}-1}{p^e-1}})^{1/p^{ne}}) && \subseteq R_{A^{1/p^{ne}}}, \\ &\dots && \end{aligned}$$

These are ideals of different rings. However, we do have the following relation for any $i > j$

$$\text{Im}(\mathfrak{a}_j \otimes_{A^{1/p^j}} A^{1/p^i} \rightarrow R_{A^{1/p^i}}) \supseteq \mathfrak{a}_i.$$

This is straightforward so we leave it to the reader to check (note that the image of (2.13.2) contains the image of φ^2). Additionally, observe that if A was regular, A^{1/p^e} would be flat over A by [Kun69], and so we could identify the tensor product $\mathfrak{a}_j \otimes_{A^{1/p^j}} A^{1/p^i}$ with its image in $R_{A^{1/p^i}}$. Therefore, for any integer $n \geq i$, we set

$$\mathfrak{a}_{i,n} = \text{Im}(\mathfrak{a}_i \otimes_{A^{1/p^i}} A^{1/p^n} \rightarrow R_{A^{1/p^n}})$$

and consider the chain of ideals:

$$R_{A^{1/p^{ne}}} \supseteq \mathfrak{a}_{1,n} \supseteq \mathfrak{a}_{2,n} \supseteq \dots \supseteq \mathfrak{a}_{n-1,n} \supseteq \mathfrak{a}_{n,n}.$$

Definition 3.2 (*n th relative non- F -pure ideal*). For every integer $n \geq n_0 = n_{\sigma(\varphi), k(V)}$, we define the *n th limiting relative non- F -pure ideal* to be $\mathfrak{a}_{n,n} = \mathfrak{a}_n$. It is denoted by $\sigma_n(X/V, \varphi)$.

We now obtain a relative version of the Hartshorne-Speiser-Lyubeznik-Gabber theorem.

Proposition 3.3. *Fix notation as above. Then there exists a nonempty open subset $U \subseteq V$ of the base scheme V satisfying the following for every integer $m \geq n \geq n_0$*

$$(3.3.1) \quad \sigma_m(X/V, \varphi)|_{f^{-1}(U)} = \text{Im}\left(\sigma_n(X/V, \varphi) \otimes_{A^{1/p^{ne}}} A^{1/p^{me}} \rightarrow R_{A^{1/p^{me}}}\right)\Big|_{f^{-1}(U)}.$$

Proof. For any $m \geq n$, we consider the containment:

$$\mathfrak{a}_{n,m} \supseteq \mathfrak{a}_{m,m}.$$

Fix $k(V)$ to be the residue field of the generic point $\eta \in V$ and consider the induced containment:

$$\mathfrak{a}_{n,m} \otimes_A k(V) = \mathfrak{a}_{n,m} \otimes_{A^{1/p^m}} k(V)^{1/p^m} \supseteq \mathfrak{a}_{m,m} \otimes_{A^{1/p^m}} k(V)^{1/p^m} = \mathfrak{a}_{m,m} \otimes_A k(V).$$

since inverting an element is the same as inverting its p th power. We notice two identifications

$$\begin{aligned} \mathfrak{a}_{n,m} \otimes_{A^{1/p^m}} k(V)^{1/p^\infty} &= \mathfrak{a}_{n,m} \otimes_{A^{1/p^m}} k(V^\infty) = \psi_\infty^n \left(\left(L_{k(V^\infty)}^{\frac{p^{ne}-1}{p^e-1}} \right)^{1/p^{ne}} \right), \\ \mathfrak{a}_{m,m} \otimes_{A^{1/p^m}} k(V)^{1/p^\infty} &= \mathfrak{a}_{m,m} \otimes_{A^{1/p^m}} k(V^\infty) = \psi_\infty^m \left(\left(L_{k(V^\infty)}^{\frac{p^{me}-1}{p^e-1}} \right)^{1/p^{me}} \right). \end{aligned}$$

Thus

$$\mathfrak{a}_{n,m} \otimes_A k(V)^{1/p^\infty} = \mathfrak{a}_{m,m} \otimes_A k(V)^{1/p^\infty}$$

since $m > n \geq n_0$. It immediately follows that

$$\mathfrak{a}_{n,m} \otimes_{A^{1/p^m}} k(V)^{1/p^m} = \mathfrak{a}_{m,m} \otimes_{A^{1/p^m}} k(V)^{1/p^m}$$

since $k(V)^{1/p^m} \subseteq k(V)^{1/p^\infty}$ is a faithfully flat extension. But now by generic freeness [Eis95, Theorem 14.4], for a fixed m , (3.3.1) follows for some open set $U_{n,m}$ (in the case that V is affine, which we can certainly reduce to, we can invert a single element of A to form $U_{n,m}$).

We now vary m . Choose $U = U_{n,n+1}$ that works for $m = n + 1$ and consider the diagram:

$$\begin{array}{c}
 \left(L^{\frac{p^{(n+2)e}-1}{p^e-1}} \right)^{\frac{1}{p^{(n+2)e}}} \\
 \swarrow \quad \searrow \\
 \left(L^{\frac{p^{(n+1)e}-1}{p^e-1}} \right)^{\frac{1}{p^{(n+1)e}}} \otimes_{A^{1/p^{(n+1)e}}} A^{1/p^{(n+2)e}} \quad \quad (\varphi^{n+1})^{1/p^e} \otimes \dots \\
 \swarrow \quad \searrow \\
 \left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_{A^{1/p^{ne}}} A^{1/p^{(n+2)e}} \quad \quad (\varphi^n)^{1/p^e} \otimes \dots \\
 \swarrow \quad \searrow \\
 \dots \\
 L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{(n+2)e}} \\
 \downarrow \varphi \otimes \dots \\
 R_{A^{1/p^{(n+2)e}}}
 \end{array}$$

The diagram shows a sequence of maps from φ^{n+2} down to $R_{A^{1/p^{(n+2)e}}}$. The maps are labeled with tensor products of powers of L and A , and the maps $(\varphi^n)^{1/p^e} \otimes \dots$ and $(\varphi^{n+1})^{1/p^e} \otimes \dots$ are induced as in (2.13.1).

where the maps labeled $(\varphi^n)^{1/p^e} \otimes \dots$ and $(\varphi^{n+1})^{1/p^e} \otimes \dots$ are induced as in (2.13.1). We know that over U , φ^{n+1} and φ^n have the same image. Therefore, so do $(\varphi^n)^{1/p^e} \otimes \dots$ and $(\varphi^{n+1})^{1/p^e} \otimes \dots$ again over U (since tensor is right exact). But then, composing with $(\varphi \otimes \dots)$ one more time, we know $\varphi^{n+1} = (\varphi \otimes \dots) \circ ((\varphi^n)^{1/p^e} \otimes \dots)$ and $\varphi^{n+2} = (\varphi \otimes \dots) \circ ((\varphi^{n+1})^{1/p^e} \otimes \dots)$ also have the same image over U . Thus they also share the image with φ^n over U . Hence, if $\mathfrak{a}_{n,n+1}|_U = \mathfrak{a}_{n+1,n+1}|_U$, then $\mathfrak{a}_{n,n+2}|_U = \mathfrak{a}_{n+2,n+2}|_U$. We use this iteratedly to obtain that $\mathfrak{a}_{n,m}|_U = \mathfrak{a}_{m,m}|_U$, which is exactly the statement of the proposition. \square

We give three examples of these σ_n and U_n . In the first example, we show that it is possible that the images $\sigma_n(X/V, \varphi)$ never stabilize in the sense of Proposition 3.3 on all of V but only over an open set. We do the same in the second example, but with respect to a more interesting choice of X and φ . Finally, we give an example where stabilization occurs at $n = 2$ (instead of at $n = 1$).

Example 3.4. Fix k to be an algebraically closed field of characteristic $p > 2$, set $A = k[t]$ and set $R = k[x, t]$ with the obvious map $X \rightarrow V$. Let $\varphi : R^{1/p} = k[x^{1/p}, t^{1/p}] \rightarrow R_{A^{1/p}} = k[x, t^{1/p}]$ be the composition of the local generator $\beta \in \text{Hom}_{R_{A^{1/p}}}(R^{1/p}, R_{A^{1/p}})$ with pre-multiplication by $t^{\frac{1}{p}}$ (note since φ is $k[t^{1/p}]$ -linear, this is also post-multiplication by $t^{1/p}$, and it corresponds to the divisor $\Delta_\varphi = \frac{1}{p-1}\{t = 0\}$). It easily follows that the image of φ is $\langle t^{1/p} \rangle \subseteq k[x, t^{1/p}]$. By tensoring with $\otimes_{k[t^{1/p}]} k[t^{1/p^2}]$, we obtain that $\mathfrak{a}_{1,2} = \langle t^{1/p} \rangle = t^{1/p} \cdot k[x, t^{1/p^2}]$. More generally, we see that $\mathfrak{a}_{1,b} = t^{1/p} \cdot k[x, t^{1/p^n}]$. Now we compute $\mathfrak{a}_{2,2}$ as the image:

$$k[x^{1/p^2}, t^{1/p^2}] \xrightarrow{\varphi^{1/p}} k[x^{1/p}, t^{1/p^2}] \xrightarrow{\varphi \otimes \dots} k[x, t^{1/p^2}].$$

The image of $\varphi^{1/p}$ is t^{1/p^2} and since the map $\varphi \otimes \dots$ is $k[t^{1/p^2}]$ -linear, we see that the composition has image $\langle t^{1/p} \cdot t^{1/p^2} \rangle = \langle t^{(1+p)/p^2} \rangle = \mathfrak{a}_{2,2}$. In general, we see that

$$\mathfrak{a}_n = \mathfrak{a}_{n,n} = \langle t^{(1+p+\dots+p^{n-1})/p^n} \rangle$$

In particular, while we may take $U_i = \mathbb{A}^1 \setminus \{0\} = \text{Spec } A \setminus \langle t \rangle$, we see that $\mathfrak{a}_n = \sigma_n(X/V, \varphi)$ never stabilizes over all of V .

Example 3.5. Let f be the morphism $X := \text{Spec} \left(\frac{k[x, y, t]}{(y^2 + x^3 + t)} \right) \rightarrow V := \text{Spec}(k[t])$. Let the standard trace map $\varphi : (F_{X^1/V^1})_* \omega_{X^1/V^1} \rightarrow \mathcal{O}_{X_{V^1}}$, for which $\Delta_\varphi = 0$. This map can be identified with the descent of the following map to the quotient

$$k[x, y, t] \xrightarrow{\cdot(y^2 + x^3 + t)^{p-1}} k[x, y, t] \xrightarrow{\text{Tr}} k[x, y, t]$$

where Tr is a k -linear map, such that

$$\text{Tr}(x^i y^j t^l) = \begin{cases} x^{\frac{i+1-p}{p}} y^{\frac{j+1-p}{p}} t^l & \text{if } p|i+1 \text{ and } p|j+1 \\ 0 & \text{otherwise} \end{cases}.$$

Then φ^n is given by

$$k[x, y, t] \xrightarrow{\cdot(y^2 + x^3 + t)^{p^n-1}} k[x, y, t] \xrightarrow{\text{Tr}_n} k[x, y, t]$$

where Tr_n is a k -linear map, such that

$$(3.5.1) \quad \text{Tr}_n(x^i y^j t^l) = \begin{cases} x^{\frac{i+1-p^n}{p^n}} y^{\frac{j+1-p^n}{p^n}} t^l & \text{if } p^n|i+1 \text{ and } p^n|j+1 \\ 0 & \text{otherwise} \end{cases}.$$

Assume that p is a prime such that $p \equiv 1 \pmod{6}$ so that $2, 3|p^n - 1$ for all $n > 0$. Let us try to compute now the following number.

$$d := \min\{c | t^c \in \text{im } \varphi^n\}$$

To have $t^c \in \text{im } \varphi^n$ it is necessary to have a polynomial in the ideal generated by $(y^2 + x^3 + t)^{p^n-1}$ one of the non-zero monomials of which is $x^{p^n-1} y^{p^n-1} t^c$. Therefore, $(y^2 + x^3 + t)^{p^n-1}$ itself has to have a non-zero monomial which divides $x^{p^n-1} y^{p^n-1} t^c$. That is, there have to be integers $0 \leq a \leq \frac{p^n-1}{2}$ and $0 \leq b \leq \frac{p^n-1}{3}$, such that $c = p^n - 1 - a - b$. So, we see that

$$d \geq p^n - 1 - \frac{p^n - 1}{2} - \frac{p^n - 1}{3} = \frac{p^n - 1}{6}$$

On the other hand we claim that $\text{Tr}_n(y^2 + x^3 + t)^{p^n-1} = t^{\frac{p^n-1}{6}}$, which will show that $d = \frac{p^n-1}{6}$ and also that $\sigma_n(X/V, \varphi)$ does not stabilize. First, note that $x^{\frac{p^n-1}{2}} y^{\frac{p^n-1}{3}} t^{\frac{p^n-1}{6}}$ is the only monomial in the expansion of $(y^2 + x^3 + t)^{p^n-1}$ the image of which via Tr_n is not zero. Indeed, the expansion can contain only monomials of the form $y^{2a} x^{3b} t^{p^n-1-a-b}$, where $a, b \geq 0$ are integers. Hence, higher powers $(2p^n - 1, 3p^n - 1, \text{etc})$ of x and y that do not go to zero by Tr_n cannot be obtained. Hence, as we stated $x^{\frac{p^n-1}{2}} y^{\frac{p^n-1}{3}} t^{\frac{p^n-1}{6}}$ is the only interesting monomial that can show up in the expansion. Lastly we have to verify that the monomial it has non-zero coefficient in the expansion. That is,

$$\frac{(p^n - 1)!}{\frac{p^n-1}{2}! \frac{p^n-1}{3}! \frac{p^n-1}{6}!} \neq 0$$

Equivalently, we have to show that the power of p in the prime factorization of the numerator is the same as in the denominator. For an arbitrary number m this number for $m!$ is

$$\sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor.$$

Now assume that $m = \frac{p^n-1}{r}$, where $r|p-1$. Then the times p divides $m!$ is

$$\begin{aligned} \sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor &= \sum_{i=1}^{n-1} \left\lfloor \frac{p^n-1}{rp^i} \right\rfloor = \sum_{i=1}^{n-1} \left\lfloor \sum_{j=0}^{n-1} p^j \frac{p-1}{rp^i} \right\rfloor = \sum_{i=1}^{n-1} \left\lfloor \sum_{j=0}^{n-1} p^{j-i} \frac{p-1}{r} \right\rfloor \\ &= \sum_{i=1}^{n-1} \sum_{j=i}^{n-1} p^{j-i} \frac{p-1}{r} = \sum_{i=1}^{n-1} \frac{p^{n-i}-1}{p-1} \frac{p-1}{r} = \sum_{i=1}^{n-1} \frac{p^{n-i}-1}{r} = \sum_{i=1}^{n-1} \frac{p^i-1}{r} \\ &= \sum_{i=0}^{n-1} \frac{p^i-1}{r} = \frac{p^n-1}{r} \end{aligned}$$

Therefore, we see that p divides p^n-1 times both $(p^n-1)!$ and $\frac{p^n-1}{2}! \frac{p^n-1}{3}! \frac{p^n-1}{6}!$, as claimed earlier. Summarizing, $t^{\frac{p^n-1}{6}}$ is the smallest power of t that is in $\sigma_n(X/V, \varphi)$. Hence there is no stabilization over all of V .

Based on this example, one might ask:

Question 3.6. Is there a relation between the asymptotics of the σ_n over the non-stable locus, and the singularities of those fibers (for example, the F -pure threshold)?

We now give an example that stabilizes at the second step (over the entire base).

Example 3.7. (cf. [MY09, Example 4.7]) Fix k to be an algebraically closed field of characteristic $p > 2$, set $A = k[t]$ and set $R = k[x, t]$ with the obvious map $X \rightarrow V$. Let $\varphi : R^{1/p} = k[x^{1/p}, t^{1/p}] \rightarrow R_{A^{1/p}} = k[x, t^{1/p}]$ be the composition of the local generator $\beta \in \text{Hom}_{R_{A^{1/p}}}(R^{1/p}, R_{A^{1/p}})$ with pre-multiplication by $(x^{p^2} + t)^{\frac{1}{p}} = f^{\frac{1}{p}}$ (which corresponds to $\Delta_\varphi = \frac{1}{p-1}\{x^{p^2} = t\}$ restricting to $\frac{p^2}{p-1}\{x = \lambda^{1/p^2}\}$ on the fiber over $t = \lambda$). Note that in $R_{A^{1/p}}$ or $R^{1/p}$, $f^{\frac{1}{p}}$ can be written as $x^p + t^{1/p}$. In particular, $f^{\frac{1}{p}}$ is already an element of $R_{A^{1/p}}$. Therefore, since φ is $R_{A^{1/p}}$ -linear and β is clearly surjective, the image of φ is just $\langle f^{1/p} \rangle = \sigma_1(X/V, \varphi) = \langle x^p + t^{1/p} \rangle$.

On the other hand, we now compose φ with itself as described above. In this case, $\varphi^2 : R^{1/p^2} \rightarrow R_{A^{1/p^2}}$ is induced by taking the generator $\beta^2 : R^{1/p^2} \rightarrow R_{A^{1/p^2}}$ and pre-multiplying by $f^{\frac{p+1}{p^2}}$. Now, $f^{\frac{p+1}{p^2}} = (x^1 + t^{1/p^2})^{p+1}$. This again is already an element of $R_{A^{1/p^2}}$, and so by the same argument as above, we see that

$$\sigma_2(X/V, \varphi) = \langle x + t^{1/p^2} \rangle^{p+1} = \sigma_1(X/V, \varphi) \cdot \langle x + t^{1/p^2} \rangle = \left\langle f^{\frac{1+p}{p^2}} \right\rangle.$$

Next we form φ^3 . In this case, it is obtained by taking a generator $R^{1/p^3} \rightarrow R_{A^{1/p^3}}$ and pre-multiplying by $f^{\frac{1+p+p^2}{p^3}}$. However, this time $f^{\frac{1+p+p^2}{p^3}}$ is *not* contained in $R_{A^{1/p^3}}$, and so we cannot argue as above. However, f still has a $1/p^2$ -root in $R_{A^{1/p^3}}$, and so

$$\varphi^3 \left(\left\langle f^{\frac{1+p+p^2}{p^3}} \right\rangle \right) = \varphi^3 \left(f^{p(1+p)} \left\langle f^{1/p^3} \right\rangle \right) = f^{\frac{1+p}{p^2}} \varphi^3 \left(\left\langle f^{1/p^3} \right\rangle \right).$$

So we must only compute $\varphi^3(\langle f^{1/p^3} \rangle)$. Now, $\varphi^3 : k[x^{1/p^3}, t^{1/p^3}] \rightarrow k[x, t^{1/p^3}]$ we may take as the map which sends $x^{\frac{p^3-1}{p^3}}$ to 1 and other monomials in x to 0 (those monomials form a basis for R^{1/p^3} over $R_{A^{1/p^3}}$). It is thus obvious that $\varphi^3(\langle f^{1/p^3} \rangle) = k[x, t^{1/p^3}] = R_{A^{1/p^3}}$. In particular,

$$\sigma_3(X/V, \varphi) = \left\langle f^{\frac{1+p}{p^2}} \right\rangle = \sigma_2(X/V, \varphi) \otimes_{A^{1/p^2}} A^{1/p^3}.$$

We now obtain a surprising base change statement.

Proposition 3.8. *Suppose that $T \rightarrow V$ is a map from an excellent integral scheme with a dualizing complex, then using the notation of [Section 2.16](#)*

$$\mathrm{Im}((q_{ne})^* \sigma_n(X/V, \varphi) \rightarrow \mathcal{O}_{X_{T^{ne}}}) = \sigma_n(X/V, \varphi) \cdot \mathcal{O}_{X_{T^{ne}}} = \sigma_n(X_T/T, \varphi_T).$$

Furthermore, if U satisfies condition (3.3.1) from [Proposition 3.3](#), then

$$W = g^{-1}(U) \subseteq T$$

satisfies the same condition for $\sigma_n(X_T/T, \varphi_T)$.

Proof. Indeed, images of maps are compatible with arbitrary base change by the right exactness of tensor. Thus the first statement follows immediately. The second statement follows from the first since if the two images $\mathfrak{a}_{n,n}$ and $\mathfrak{a}_{n-1,n}$ are equal, they are also equal after base change. \square

Theorem 3.9 (Base change for σ_n). *There exists an integer $N \geq 0$, such that for all points $s \in V$, $N \geq n_{\sigma(\varphi_s), k(s)}$. In other words, we have both that $\sigma_n(X/V, \varphi) \cdot \mathcal{O}_{X_{s^{ne}}} = \sigma_n(X_s/s, \varphi_s)$ (which always holds) and also that for all $m \geq n \geq N$ that*

$$\sigma_n(X/V, \varphi) \otimes_{V^n} k(s)^{1/p^m} = \sigma_m(X_s/s, \varphi_s).$$

Proof. Of course, the statement already holds on U with respect to some integer N_0 . Set $V'_1 := V \setminus U$ to be the complement with reduced scheme structure and let

$$i_1 : V_1 = (V'_1)_{\mathrm{reg}} \hookrightarrow V'_1$$

be the regular locus. We notice that V_1 has dimension strictly smaller than $\dim V$. We consider the base change $X_{V_1} \rightarrow V_1$. Each fiber of $X_{V_1} \rightarrow V_1$ is isomorphic to a fiber of $X \rightarrow V$. Then choose an open set $U_1 \subseteq V_1$ for which the statement holds for some integer N_1 .

Now fix $V'_2 = V'_1 \setminus U_1$ and $i_2 : V_2 = (V'_2)_{\mathrm{reg}} \hookrightarrow V'_2$ to be the regular locus and repeat. This process terminates by Noetherian induction.

Setting $N = \max\{N_0, N_1, N_2, \dots\}$ completes the proof. \square

Suppose now that $s \in V$ is a perfect point, see [Definition 2.2](#). It follows that X_s/s is a variety over a perfect field and so since $s^e \cong s$, we have $X_{s^e} \cong X_s$. Thus we can identify $\varphi_s : L_s^{1/p^e} \rightarrow R_{s^e}$ with a p^{-e} -linear map ψ_s as in [Section 2.23](#). Thus under these identifications $\bigcap_{n \geq 0} \sigma_n(X_s/s, \varphi) = \sigma(X_s, \psi_s)$. However, if we have the stabilization $\sigma_n(X_s/s, \varphi) \cong \sigma_n(X_s/s, \varphi) \otimes_{k(s)^{1/p^{ne}}} k(s)^{1/p^{me}} \cong \sigma_m(X_s/s, \varphi)$ for every $m \geq n$, the equality $\sigma_n(X_s/s, \varphi) = \sigma(X_s, \psi)$ holds. Combining this with our previous work we obtain the following corollary.

Corollary 3.10. *With notation as above, there exists an integer $N \geq 0$ such that*

$$\sigma_n(X/V, \varphi) \cdot \mathcal{O}_{X_{s^{ne}}} = \sigma(X_s, \psi_s)$$

for all perfect points $s \in V$ and $n \geq N$.

3.11. Iterated non- F -pure ideals versus the absolute non- F -pure ideal. We now compare the iterated non- F -pure ideal with the absolute non- F -pure ideal. First however, we make the following assumption.

Convention 3.12. In this subsection, [Section 3.11](#), we always assume that our base V is an F -pure quasi-Gorenstein (i.e., ω_V is a line bundle) F -finite integral scheme.

Let $\Psi_V^e : F_*^e \mathcal{O}_V((1 - p^e)K_V) \rightarrow \mathcal{O}_V$ be a map generating $\mathcal{H}\mathrm{om}_V(F_*^e \mathcal{O}_V((1 - p^e)K_V), \mathcal{O}_V)$ as an $F_*^e \mathcal{O}_V$ -module. By taking p^{ne} th roots, applying f^{-1} , and then writing the A -module $M := f^{-1}(\mathcal{O}_V((1 - p^e)K_V))$, we obtain:

$$(\Phi_A^e)^{1/p^{ne}} : M^{1/p^{(n+1)e}} = f^{-1}F_*^{(n+1)e}(\mathcal{O}_V((1 - p^e)K_V)) \rightarrow f^{-1}F_*^{ne} \mathcal{O}_V = A^{1/p^{ne}}.$$

We tensor with $L^{1/p^e} \otimes_{A^{1/p^e}} _$ and so obtain:

$$(3.12.1) \quad \Phi' : L^{1/p^e} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \rightarrow L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}.$$

Set $N = L \otimes_A M^{1/p^{ne}}$. We observe that N is a line bundle on $X_{V^{ne}}$. In fact, if $q_1 : X_{V^{ne}} = X \times_V V^{ne} \rightarrow X$ is the first projection and $q_2 : X_{V^{ne}} \rightarrow V^{ne}$ is the second projection, then

$$N = q_1^* L \otimes_{\mathcal{O}_{X_{V^{ne}}}} q_2^* \mathcal{O}_{V^{ne}}((1-p^e)K_{V^{ne}})$$

and so it follows for any integer $l > 0$ that

$$(3.12.2) \quad N^l \cong q_1^* L^l \otimes_{\mathcal{O}_{X_{V^{ne}}}} q_2^* \mathcal{O}_{V^{ne}}(l(1-p^e)K_{V^{ne}}) = L^l \otimes_A \left(M^l\right)^{1/p^{ne}}$$

Finally, we compose (3.12.1) with $\varphi' := \varphi \otimes_{A^{1/p^e}} A^{1/p^{ne}}$ to construct:

$$(3.12.3) \quad \begin{aligned} \gamma : \quad & N^{1/p^e} \\ &= L^{1/p^e} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \\ &\xrightarrow{\Phi'} L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}} \\ &\xrightarrow{\varphi'} R_{A^{1/p^e}} \otimes_{A^{1/p^e}} A^{1/p^{ne}} \\ &= R \otimes_A A^{1/p^{ne}} \\ &= R_{A^{1/p^{ne}}}. \end{aligned}$$

This is a map from an invertible sheaf on $X_{V^{(n+1)e}}^e = (X_{V^{ne}})^e$ to the structure sheaf on $X_{V^{ne}}$. In particular, it is a map such as one studied in [BS13, Section 4].

Let us point out an alternate way to construct γ . Indeed, take

$$\varphi'' := \varphi \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} : L^{1/p^e} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \rightarrow R_{A^{1/p^e}} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \cong R \otimes_A M^{1/p^{(n+1)e}}.$$

We can then compose this with $R_{A^{1/p^e}} \otimes_{A^{1/p^e}} (\Phi_A^e)^{1/p^{ne}} = (\Phi_A^e)'$ to obtain

$$N^{1/p^e} = L^{1/p^e} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \xrightarrow{\varphi''} R_{A^{1/p^e}} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \xrightarrow{(\Phi_A^e)'} R_{A^{1/p^e}} \otimes_{A^{1/p^e}} A^{1/p^{ne}} = R_{A^{1/p^{ne}}}.$$

This composition is easily seen to coincide with γ .

We can then compose γ with itself m -times as in [BS13, Section 4] or [Sch09]. We recall this construction for the benefit of the reader. To construct γ^2 , we tensor the map

$$\gamma : N^{1/p^e} \rightarrow R \otimes_A A^{1/p^{ne}} = R_{A^{1/p^{ne}}}$$

with the line bundle $\otimes_{R_{A^{1/p^{ne}}}} N$, take p^e th roots and then compose it with γ to obtain:

$$\gamma^2 : (N^{1+p^e})^{1/p^{2e}} \rightarrow N^{1/p^e} \xrightarrow{\gamma} R \otimes_A A^{1/p^{ne}}$$

Recursively, we can construct:

$$\begin{aligned} \gamma^m : \quad & \left(N^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \\ &= \left(L^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \otimes_{A^{1/p^{me}}} \left(M^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{(n+m)e}} \\ &\rightarrow R_{A^{1/p^{ne}}} \end{aligned}$$

We are now in a position to relate the absolute $\sigma(X_{V^{1/p^{ne}}}, \gamma)$ with the relative $\sigma_n(X/V, \varphi)$.

First we observe that for each $m \leq n$, Φ_A^m induces maps

$$\begin{aligned} \mu_m : \quad & \left(N^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \\ &= \left(L^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \otimes_{A^{1/p^{me}}} \left(M^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{(n+m)e}} \\ &\xrightarrow{\dots \otimes (\Phi_A^m)^{1/p^{ne}}} \left(L^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \otimes_{A^{1/p^{me}}} A^{1/p^{ne}}. \end{aligned}$$

Furthermore these maps are surjective since Φ_A^m is.

Lemma 3.13. *Assuming [Convention 3.12](#), we have the following factorization of γ^n and γ^{n-1} as indicated by the diagram below*

$$\begin{array}{ccc}
 \left(N^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} & & \left(N^{\frac{p^{(n-1)e}-1}{p^e-1}}\right)^{1/p^{(n-1)e}} \\
 \parallel & & \parallel \\
 \left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} \left(M^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{(n+n)e}} & \xrightarrow{\gamma'} & \left(L^{\frac{p^{(n-1)e}-1}{p^e-1}}\right)^{1/p^{(n-1)e}} \otimes_{A^{1/p^{(n-1)e}}} \left(M^{\frac{p^{(n-1)e}-1}{p^e-1}}\right)^{1/p^{(n+n-1)e}} \\
 \downarrow \mu_n & & \downarrow \mu_{n-1} \\
 \left(L^{\frac{p^{ne}-1}{p^e-1}}\right)^{1/p^{ne}} & \xrightarrow{\varphi^n} & \left(L^{\frac{p^{(n-1)e}-1}{p^e-1}}\right)^{1/p^{(n-1)e}} \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}} \quad \gamma^{n-1} \\
 & \searrow \varphi^n & \downarrow \varphi^{n-1} \otimes \dots \\
 & & R_{A^{1/p^{ne}}} \leftarrow
 \end{array}$$

where the arrow γ' is induced by γ and $\varphi^{n-1} \otimes \dots$ is simply $\varphi^{n-1} \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}}$. Furthermore, since μ_{n-1} is surjective we have

$$\text{Im}(\gamma^{n-1}) = \mathfrak{a}_{n-1,n} = \text{Im}(\varphi^{n-1} \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}}).$$

Likewise since μ_n is surjective, we have

$$\text{Im}(\gamma^n) = \mathfrak{a}_{n,n} = \mathfrak{a}_n = \text{Im}(\varphi^n).$$

Proof. The two equalities are immediate from the surjectivities of μ_{n-1} and μ_n and the commutativity of our diagram. The surjectivities of μ_{n-1} and μ_n are clear since $M^{1/p^{(n+1)e}} \rightarrow A^{1/p^{ne}}$ is surjective and tensor is right-exact. It remains to prove the commutativity of our diagram. It suffices to prove the commutativity of the square:

$$\begin{array}{ccc}
 \left(L^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \otimes_{A^{1/p^{me}}} \left(M^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{(n+m)e}} & \xrightarrow{\gamma'} & \left(L^{\frac{p^{(m-1)e}-1}{p^e-1}}\right)^{1/p^{(m-1)e}} \otimes_{A^{1/p^{(m-1)e}}} \left(M^{\frac{p^{(m-1)e}-1}{p^e-1}}\right)^{1/p^{(n+m-1)e}} \\
 \downarrow \mu_m & & \downarrow \mu_{m-1} \\
 \left(L^{\frac{p^{me}-1}{p^e-1}}\right)^{1/p^{me}} \otimes_{A^{1/p^{me}}} A^{1/p^{ne}} & \longrightarrow & \left(L^{\frac{p^{(m-1)e}-1}{p^e-1}}\right)^{1/p^{(m-1)e}} \otimes_{A^{1/p^{(m-1)e}}} A^{1/p^{ne}}
 \end{array}$$

and we will prove it by induction on m . It is clear that, if the above square is commutative for m , then, by tensoring it with $L \otimes_R (-) \otimes_{A^{1/p^{ne}}} M^{1/p^{(n+1)e}}$ and taking p^e th roots, we will have the commutativity for $m+1$. Hence it remains to prove the commutativity when $m=2$. To this end, we will denote the map $M^{1/p^{(n+1)e}} \rightarrow A^{1/p^{ne}}$ by α . Then α induces a map

$$\begin{aligned}
 \alpha' &= (id \otimes \alpha)^{1/p^e} : \left(M^{\frac{p^{2e}-1}{p^e-1}}\right)^{1/p^{(n+2)e}} = \left(M^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} M^{1/p^{(n+1)e}}\right)^{1/p^e} \\
 &\rightarrow M^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} A^{1/p^{ne}} = M^{1/p^{(n+1)e}}
 \end{aligned}$$

It is clear that $\alpha \circ \alpha' = \alpha^2$. Likewise, the map $\varphi : L^{1/p^e} \rightarrow R \otimes_A A^{1/p^e}$ induces

$$\varphi' = (id \otimes \varphi)^{1/p^e} : \left(L^{\frac{p^{2e}-1}{p^e-1}}\right)^{1/p^{2e}} = \left(L^{1/p^e} \otimes_{R^{1/p^e}} L^{1/p^{2e}}\right)^{1/p^e} \rightarrow L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{2e}}$$

We will analyze the 4 maps involved in the square. The left vertical map

$$\mu_2 : \left(L^{\frac{p^{2e}-1}{p^e-1}}\right)^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} \left(M^{\frac{p^{2e}-1}{p^e-1}}\right)^{1/p^{(n+2)e}} \rightarrow \left(L^{\frac{p^{2e}-1}{p^e-1}}\right)^{1/p^{2e}}$$

is given by $id \otimes (\alpha \circ \alpha')$; similarly the right vertical map μ_1 is $id \otimes \alpha$. The top horizontal map is given by

$$\begin{aligned} \left(L^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} \left(M^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{(n+2)e}} &\xrightarrow{\varphi' \otimes \alpha'} L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} M^{1/p^{(n+1)e}} \\ &\xrightarrow{\sim} L^{1/p^e} \otimes_{A^{1/p^e}} M^{1/p^{(n+1)e}} \end{aligned}$$

And the bottom horizontal map is give by

$$\begin{aligned} \left(L^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} A^{1/p^{ne}} &\xrightarrow{\varphi' \otimes id} L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} A^{1/p^{ne}} \\ &\xrightarrow{\sim} L^{1/p^e} \otimes_{A^{1/p^e}} A^{1/p^{ne}}. \end{aligned}$$

Now given an arbitrary $x \otimes z \in \left(L^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{2e}} \otimes_{A^{1/p^{2e}}} \left(M^{\frac{p^{2e}-1}{p^e-1}} \right)^{1/p^{(n+2)e}}$, write $\varphi'(x) = \sum_i x_i \otimes y_i$ with $x_i \in L^{1/p^e}$ and $y_i \in A^{1/p^{2e}}$. Then, on one hand, if we follow the top horizontal map and then the right vertical map, we will have

$$x \otimes z \mapsto \left(\sum_i x_i \otimes y_i \right) \otimes \alpha'(z) \mapsto \sum_i x_i \otimes y_i \alpha'(z) \mapsto \sum_i x_i \alpha(y_i \alpha'(z)) = \sum_i x_i \otimes y_i \alpha(\alpha'(z)),$$

where the last equality holds since α is $A^{1/p^{ne}}$ -linear and hence $A^{1/p^{2e}}$ -linear.

On the other hand, if we follow the other path (*i.e.* the left vertical map first and then the bottom horizontal map), we will have

$$x \otimes z \mapsto x \otimes \alpha'(z) \mapsto x \otimes \alpha(\alpha'(z)) \mapsto \left(\sum_i x_i \otimes y_i \right) \otimes \alpha(\alpha'(z)) \mapsto \sum_i x_i \otimes y_i \alpha(\alpha'(z)).$$

This proves that the square is indeed commutative when $m = 2$ and concludes the proof of our Lemma. \square

Theorem 3.14. *With notation as in [Convention 3.12](#) and below, choose $n > n_0 = n_{\sigma(\varphi), k(V)}$. Then there exists a dense open set $U \subseteq V \cong V^e$ with $W = f^{-1}(U) \subseteq X$ such that*

$$\sigma(X_{V^{1/p^{ne}}}, \gamma)|_W = \sigma_n(X/V, \varphi)|_W.$$

Furthermore, shrinking U further if necessary we can require for all perfect points $u \in U$, that

$$\sigma(X_{V^{1/p^{ne}}}, \gamma) \cdot \mathcal{O}_{X_u} = \sigma_n(X_u/u, \varphi_u) = \sigma(X_u, \psi_u)$$

where ψ_u is φ_u viewed as an absolute p^{-e} -linear map.

Proof. The second statement is immediate from the first by [Proposition 3.8](#), so we need only prove the first statement. By the diagram in [Lemma 3.13](#) and applying [Proposition 3.3](#), we see that

$$\mathrm{Im}(\gamma^n)|_W = \mathfrak{a}_{n,n}|_W = \mathfrak{a}_{n-1,n}|_W = \mathrm{Im}(\gamma^{n-1})|_W$$

for all $n > n_0$. Hence $\mathrm{Im}(\gamma^n)|_W = \sigma(X_{V^{1/p^{ne}}}, \gamma)|_W$ and the result follows. \square

Convention 3.15. Assuming everything in [Convention 3.12](#), we additionally assume that φ is relatively divisorial.

Before we proceed, we prove a lemma about interpreting maps as divisors in the relative vs absolute setting.

Lemma 3.16. *With notation as in [Convention 3.15](#), if Δ_φ is the divisor corresponding to φ as in [Definition 2.8](#). Then Δ_γ , the divisor on $X_{V^{ne}}$ corresponding to γ as in [\[BS13, Section 4\]](#), is equal to $h^* \Delta$ where $h : X_{V^{ne}} = X \times_V V^{ne} \rightarrow X$ is the projection.*

Proof. Working off a set of relative codimension 2, and then working locally on X and V , we can assume that L and M are trivial, and further assume that $\varphi(_) = \Phi(z^{1/p^e} \cdot _)$ where Φ generates $\mathcal{H}\text{om}_{R_{A^{1/p^e}}}(R^{1/p^e}, R_{A^{1/p^e}})$ as an R^{1/p^e} -module. Then Δ_φ is the divisor corresponding to $\frac{1}{p^e-1} \text{div}_X(z)$. On the other hand, since the map Γ corresponding to Φ is a composition of generating maps, Γ generates $\mathcal{H}\text{om}_{R_{A^{1/p^{ne}}}}(R_{A^{1/p^{(n+1)e}}}, R_{A^{1/p^{ne}}})$, see for example [Kun86, Appendix F], and so corresponds to the zero divisor. Furthermore, we see that $\gamma(_) = \Gamma(\bar{z}^{1/p^e} \cdot _)$ where \bar{z} is the element corresponding to $z \otimes 1 \in R_{A^{1/p^{ne}}}$. Hence $\Delta_\gamma = \frac{1}{p^e-1} \text{div}_{X_{V^{ne}}}(\bar{z})$ and the claim follows. \square

We now need a Lemma which says roughly that in an F -pure F -finite ring A with maximal ideal \mathfrak{m} , then at least locally, one can always choose a $\alpha : A^{1/p^e} \rightarrow A$ such that the closed point $V(\mathfrak{m})$ is an F -pure center of Δ_α . To this end, we introduce the following.

$$M_{e,A} := \{\varphi \in \text{Hom}_A(A^{1/p^e}, A) \mid \varphi(\mathfrak{m}^{1/p^e}) \subseteq \mathfrak{m}\}$$

and

$$N_{e,A} := \{\varphi \in \text{Hom}_A(A^{1/p^e}, A) \mid \varphi(A^{1/p^e}) \subseteq \mathfrak{m}\}.$$

It is straightforward to check that both $M_{e,A}$ and $N_{e,A}$ are A^{1/p^e} -submodules of $\text{Hom}_A(A^{1/p^e}, A)$ and that $N_{e,A} \subseteq M_{e,A}$. We now observe that:

Lemma 3.17. *For any F -finite reduced G1 and S2 ring A with maximal ideal \mathfrak{m} , the formation of $M_{e,A}$ and $N_{e,A}$ commute with localization and completion completion, i.e. if W is a multiplicative system and $\hat{_}$ denotes completion along \mathfrak{m} then*

$$W^{-1}M_{e,A} \cong (W^{-1}M)_{e,W^{-1}A} \text{ and } W^{-1}N_{e,A} \cong (W^{-1}N)_{e,W^{-1}A} \text{ and} \\ M_{e,A} \otimes_A \hat{A} \cong M_{e,\hat{A}} \text{ and } N_{e,A} \otimes_A \hat{A} \cong N_{e,\hat{A}}$$

The following proof was suggested to us by Manuel Blickle and Kevin Tucker. We believe there are more general proofs that work without the G1 and S2 hypothesis (but this proof is short).

Proof. We have a natural injective map $\text{Hom}_A(\mathfrak{m}^{1/p^e}, \mathfrak{m}) \hookrightarrow \text{Hom}_A(\mathfrak{m}^{1/p^e}, A)$, but $\text{Hom}_A(\mathfrak{m}^{1/p^e}, A) \cong \text{Hom}_A(A^{1/p^e}, A)$ since both modules are S2 (since a reflexive module is S2 in a G1 and S2 ring, [Har94]). Thus we have $\text{Hom}_A(\mathfrak{m}^{1/p^e}, \mathfrak{m}) \hookrightarrow \text{Hom}_A(A^{1/p^e}, A)$, the image of which is $M_{e,A}$. Therefore $M_{e,A}$ clearly commutes with localization and completion since the formation of both Hom sets commutes with localization and completion. Similarly, the formation of $N_{e,A}$ commutes with localization and completion. \square

Lemma 3.18. *Suppose that A is an F -pure F -finite G1 and S2 ring with maximal ideal \mathfrak{m} . Then there exists an A -linear map $\alpha_A : A^{1/p^e} \rightarrow A$ such that α_A is surjective and $\alpha_A(\mathfrak{m}^{1/p^e}) \subseteq \mathfrak{m}$.*

Proof. We first assume that A is a local ring. In this case, it suffices to show that $N_{e,A} \neq M_{e,A}$. By Lemma 3.17, we may assume that A is complete since \hat{A} is faithfully flat over A . By the Cohen Structure Theorem, there is a ring of formal power series $S = k[[x_1, \dots, x_n]]$ over a coefficient field k of A with surjection $S \twoheadrightarrow A$. Write $A = S/I$. Recall that $\text{Hom}_S(S^{1/p^e}, S)$ is generated (as an S^{1/p^e} -module) by $g : S^{1/p^e} \rightarrow S$ that sends $(x_1 \cdots x_n)^{\frac{p^e-1}{p^e}}$ to 1 and other basis monomials (including 1) to zero. By Fedder's lemma [Fed83, Lemma 1.6], each element of $\text{Hom}_A(A^{1/p^e}, A)$ has the form $f(u^{1/p^e} -)$ with $u \in (I^{[p^e]} : I)$. Since A is F -pure, $(I^{[p^e]} : I) \not\subseteq \mathfrak{n}^{[p^e]}$ where $\mathfrak{n} = (x_1, \dots, x_n)$. Let u be an element of $(I^{[p^e]} : I) \setminus \mathfrak{n}^{[p^e]}$. Then u must have a monomial term $cx_1^{a_1} \cdots x_n^{a_n}$ with $c \in k$ and $a_i < p^e$ for each i . Choose such a monomial appearing in u with the least degree and still denote it by $cx_1^{a_1} \cdots x_n^{a_n}$. Then it is clear that $(x_1^{p^e-1-a_1} \cdots x_n^{p^e-1-a_n})u - c(x_1 \cdots x_n)^{p^e-1} \in \mathfrak{n}^{[p^e]}$. Now define $\alpha : A^{1/p^e} \rightarrow A$ by $\alpha(-) = g((x_1^{p^e-1-a_1} \cdots x_n^{p^e-1-a_n}u)^{1/p^e} -)$. Then α is in $M_{e,A} \setminus N_{e,A}$. This completes the local case by Lemma 3.17.

For the non-local case, we have the map $M_{e,A} \rightarrow A$ which is evaluation at 1. Since the formation of $M_{e,A}$ commutes with localization, this map is surjective if and only if it is surjective locally. The above work proves the result after localizing at \mathfrak{m} . However, clearly $(M_{e,A})_{\mathfrak{n}} = (\text{Hom}_A(A^{1/p^e}, A))_{\mathfrak{n}}$ for prime ideals not equal to \mathfrak{m} . The result follows. \square

For the moment, we work sufficiently locally so that $X = \text{Spec } R$ and $V = \text{Spec } A$ are affine and that L and M are isomorphic to R and A respectively. Continue to assume that A is F -pure and quasi-Gorenstein, and fix a point $s \in \text{Spec } A$. Shrinking V if necessary, choose a map $\alpha_s \in \text{Hom}_A(A^{1/p^e}, A)$ which is surjective and which satisfies $\alpha(\mathfrak{m}_s^{1/p^e}) \subseteq \mathfrak{m}_s$ (whose existence is guaranteed by [Lemma 3.18](#)). We write $\alpha_s(_) = \Phi_A^e(z^{1/p^e} \cdot _)$ for some $z \in A$. Note we have an induced map $\overline{\alpha}_s : (A/\mathfrak{m}_s)^{1/p^e} \rightarrow A/\mathfrak{m}_s$.

We thus induce the following map $\beta : (R_{A^{1/p^{ne}}})^{1/p^e} \rightarrow R_{A^{1/p^{ne}}}$ defined by the rule $\beta(_) = \gamma(z^{1/p^{(n+1)e}} \cdot _)$. Certainly using the notation of [\(3.12.3\)](#),

$$\begin{aligned} & \beta(\mathfrak{m}_s^{1/p^{(n+1)e}} \cdot R_{A^{1/p^{(n+1)e}}}^{1/p^e}) \\ &= \gamma(z^{1/p^{(n+1)e}} \mathfrak{m}_s^{1/p^{(n+1)e}} \cdot R_{A^{1/p^{(n+1)e}}}^{1/p^e}) \\ &= \left(\varphi' \circ ((\Phi_A^e)^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} R^{1/p^e}) \right) \left(z^{1/p^{(n+1)e}} \mathfrak{m}_s^{1/p^{(n+1)e}} \cdot R_{A^{1/p^{(n+1)e}}}^{1/p^e} \right) \\ &= \left(\varphi' \circ ((\alpha_s)^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} R^{1/p^e}) \right) \left(\mathfrak{m}_s^{1/p^{(n+1)e}} \cdot R_{A^{1/p^{(n+1)e}}}^{1/p^e} \right) \\ &\subseteq \varphi'(\mathfrak{m}_s^{1/p^{ne}} \cdot R_{A^{1/p^{ne}}}^{1/p^e}) \\ &\subseteq \mathfrak{m}_s^{1/p^{ne}} R_{A^{1/p^{ne}}}. \end{aligned}$$

And hence we induce a map

$$\overline{\beta} : R_{(A/\mathfrak{m}_s)^{1/p^{(n+1)e}}}^{1/p^e} \rightarrow R_{(A/\mathfrak{m}_s)^{1/p^{ne}}}.$$

Lemma 3.19. *With notation as above, $\Delta_{\overline{\beta}} = \Delta_{\gamma}|_{X_{s^{ne}}}$ where the divisors are induced as in [\[BS13, Section 4\]](#).*

Proof. Since φ is relatively divisorial, by working in relative codimension 1, it suffices to show the result in the case that Δ_{γ} (and so Δ_{φ}) is trivial (since locally φ or ψ is a generator pre-multiplied by the element which determines Δ_{φ} or Δ_{ψ}). But now consider the composition constructing β .

$$R_{A^{1/p^{(n+1)e}}}^{1/p^e} \xrightarrow{\cdot z^{1/p^{(n+1)e}}} R_{A^{1/p^{(n+1)e}}}^{1/p^e} \xrightarrow{\Phi'} R_{A^{1/p^{ne}}}^{1/p^e} \xrightarrow{\varphi'} R_{A^{1/p^{ne}}}$$

The composition of the first two maps sends $\mathfrak{m}^{1/p^{(n+1)e}} \cdot R_{A^{1/p^{(n+1)e}}}^{1/p^e}$ to $\mathfrak{m}^{1/p^{ne}} \cdot R_{A^{1/p^{ne}}}^{1/p^e}$ and modding out by these ideals induces a generating map for $\text{Hom}_{R_{(A/\mathfrak{m})^{1/p^{ne}}}}^{1/p^e}(R_{(A/\mathfrak{m})^{1/p^{(n+1)e}}}^{1/p^e}, R_{(A/\mathfrak{m})^{1/p^{ne}}}^{1/p^e})$ via the fact that α_s is surjective and so the induced map $(A/\mathfrak{m})^{1/p^{(n+1)e}} \rightarrow (A/\mathfrak{m})^{1/p^{ne}}$ is a generating map as well. On the other hand, if φ is a generating map, so is $\varphi' \otimes_{A^{1/p^{ne}}} (A/\mathfrak{m})^{1/p^{ne}}$ by [Lemma 2.19](#). Since the composition of two generating maps is a generating map, we are done. \square

We now explain how the absolute σ behaves when restricting to fibers. First we need two lemmas.

Lemma 3.20. *With notation as above, and $v \in V$ a closed point.*

$$\sigma(X_{V^{ne}}, \Delta_{\gamma}) \cdot \mathcal{O}_{X_{v^{ne}}} = \sigma(X_{V^{ne}}, \gamma) \cdot \mathcal{O}_{X_{v^{ne}}} \supseteq \sigma(X_{v^{ne}}, \overline{\beta}) = \sigma(X_{v^{ne}}, \Delta_{\overline{\beta}}) = \sigma(X_{v^{ne}}, \Delta_{\gamma}|_{X_{v^{ne}}}).$$

Proof. Obviously $\sigma(X_{V^{ne}}, \gamma) \supseteq \sigma(X_{V^{ne}}, \beta)$. On the other hand, it follows easily that $\sigma(X_{V^{ne}}, \beta)$ restricts to $\sigma(X_{v^{ne}}, \overline{\beta})$ by F -adjunction, see for instance [\[FST11\]](#). \square

The next lemma is a generalization of the fact that for an ideal in a regular ring, $I \subseteq (I^{[1/p]})^{[p]}$ using the $\bullet^{[1/p]}$ notation from [\[BMS08\]](#).

Lemma 3.21. *Let B be an F -finite regular ring such that $B^{1/p}$ is a free B -module and that $\mathrm{Hom}_B(B^{1/p}, B) \cong B^{1/p}$ (for example this happens if B is local). Further suppose that Q is a B^{1/p^a} submodule of $B^{1/p^a} \otimes_B P$ for some B -module P ,*

$$Q \subseteq B^{1/p^a} \otimes_B P.$$

Then $Q \subseteq B^{1/p^a} \otimes_B ((\vartheta \otimes \mathrm{id}_P)(Q))$ where $\vartheta \in \mathrm{Hom}_B(B^{1/p^a}, B)$ is the generator.

Proof. The strategy is similar to [BMS08, Proposition 2.5]. By [Kun69, Theorem 2.1], any element of Q can be written as a finite sum $\sum \lambda_i \otimes m_i$, where $\{\lambda_i\}$ form a basis for B^{1/p^a} over B and $m_i \in P$. Now since ϑ is a local generating map of $\mathrm{Hom}_B(B^{1/p^a}, B)$, the projection maps onto the λ_i are multiples of ϑ . In other words, for each λ_i there exists a $u_i \in B^{1/p^a}$ such that both $\vartheta(u_i \lambda_i) = 1 \in R$ and $\vartheta(u_i \lambda_j) = 0 \in R$ for $j \neq i$.

We observe that each m_j is then in $(\vartheta \otimes \mathrm{id}_P)(Q)$ since $\vartheta(u_j \sum \lambda_i \otimes m_i) = m_j$. It follows then that $\sum \lambda_i \otimes m_i \in B^{1/p^a} \otimes_B ((\vartheta \otimes \mathrm{id}_P)(Q))$. \square

Lemma 3.22. *Suppose that $A = k$ is an F -finite field and $K \supseteq k$ is a field extension of k such that K is perfect. Choose $X = \mathrm{Spec} R \rightarrow V = \mathrm{Spec} A = \mathrm{Spec} k$ a flat map of finite type with $X \rightarrow V$ possessing geometrically reduced, $G1$ and $S2$ fibers. Additionally suppose that $\gamma : R^{1/p^e} \rightarrow R$ is any R -linear map which is a local generator of $\mathrm{Hom}_R(R^{1/p^e}, R)$ at the generic points of the codimension-1 components of the non-smooth locus of X . Then, setting Δ_γ as in [BS13, Section 4].*

$$\sigma(X, \Delta_\gamma) \otimes_k K \supseteq \sigma(X_K, (\Delta_\gamma) \times_k K)$$

Proof. First choose $n > 0$ such that $\sigma(X, \Delta_\gamma) = \gamma^n(R^{1/p^{ne}})$. Since $\mathrm{Hom}_k(k^{1/p^{ne}}, k)$ is a free $k^{1/p^{ne}}$ -module of rank 1, by [Kun86, Appendix F] we can factor γ^n as

$$\begin{array}{ccccc} R^{1/p^{ne}} & \xrightarrow{\beta} & R_{k^{1/p^{ne}}} & \xrightarrow{\vartheta} & R_k \\ & \searrow \gamma^n & & \nearrow & \\ & & & & \end{array}$$

with ϑ a local generator of $\mathcal{H}\mathrm{om}_{R_k}(R_{k^{1/p^{ne}}}, R_k)$ and so that Δ_β coincides with Δ_γ . To see this last point, notice that if γ is a local generator of $\mathrm{Hom}_R(R^{1/p^e}, R)$, then so is γ^n of $\mathrm{Hom}_R(R^{1/p^{ne}}, R)$ and hence β is also a local generator of $\mathrm{Hom}_{R_{A^{1/p^{ne}}}}(R^{1/p^{ne}}, R_{A^{1/p^{ne}}})$. More generally then, if γ^n is a local generator pre-multiplied by some $g^{1/p^{ne}} \in R^{1/p^{ne}}$ (working in codimension 1) then β is also a generator times $g^{1/p^{ne}}$ (possibly times a unit depending on our choice of ϑ). This proves that $\Delta_\beta = \Delta_\gamma$ as asserted.

We now observe that $\gamma^n(R^{1/p^{ne}}) \otimes_k k^{1/p^{ne}} = \vartheta(\beta(R^{1/p^{ne}})) \otimes_k k^{1/p^{ne}} \supseteq \beta(R^{1/p^{ne}})$ by Lemma 3.21 (here $B = k$ and $P = \beta(R^{1/p^{ne}})$). After embedding $k^{1/p^{ne}} \subseteq K$ we tensor β by K to obtain a map

$$\beta_K : R^{1/p^{ne}} \otimes_{k^{1/p^{ne}}} K \rightarrow R_{k^{1/p^{ne}}} \otimes_{k^{1/p^{ne}}} K.$$

We see immediately that $\Delta_{\beta_K} = (\Delta_\gamma) \times_k K$ (again by an argument about local generators). Then, by Lemma 3.21,

$$\sigma(X, \gamma) \otimes_k K \cong \gamma^n(R^{1/p^{ne}}) \otimes_k k^{1/p^{ne}} \otimes_{k^{1/p^{ne}}} K \supseteq \beta(R^{1/p^{ne}}) \otimes_{k^{1/p^{ne}}} K \supseteq \sigma(X, \beta_K).$$

This completes the proof. \square

Theorem 3.23. *With notation as in Convention 3.15, there exists an $N > 0$ such that for all $n > N$ we have $\sigma(X_{V^{1/p^{ne}}}, \Delta_\gamma) \cdot \mathcal{O}_{X_{s^{ne}}} = \sigma(X_{s^{ne}}, (\Delta_\gamma)|_{X_{s^{ne}}})$ for all perfect points $s \in V$.*

Proof. The statement is local, so we may assume X and V are affine and that L and M are trivial just as above in Lemma 3.19. We know

$$\sigma(X_{V^{1/p^{ne}}}, \gamma^n) \subseteq \mathrm{Im}(\gamma^n) = \sigma_n(X/V, \varphi)$$

by [Lemma 3.13](#). On the other hand, by [Theorem 3.9](#), we have $\sigma_n(X/V, \varphi) \cdot \mathcal{O}_{X_s} = \sigma(X_s, \varphi_s)$. Therefore, it is sufficient to show that $\sigma(X_{V^{1/p^{ne}}}, \gamma) \cdot \mathcal{O}_{X_{s^{ne}}} \supseteq \sigma(X_{s^{ne}}, \psi_{s^{ne}})$.

For this we may assume that V is the spectrum of an F -finite F -pure local ring and that $\text{Spec } K = s \mapsto V$ has image the closed point $\text{Spec } k = v \in V$. We first see that $\sigma(X_{V^{ne}}, \Delta_\gamma) \cdot \mathcal{O}_{X_{v^{ne}}} \supseteq \sigma(X_{v^{ne}}, \Delta_{\bar{\beta}}) = \sigma(X_{v^{ne}}, \Delta_\gamma|_{X_{v^{ne}}})$ by and using the notation of [Lemma 3.20](#). On the other hand, by [Lemma 3.22](#)

$$\sigma(X_{v^{ne}}, \Delta_\gamma|_{X_{v^{ne}}}) \cdot \mathcal{O}_{X_{s^{ne}}} = \sigma(X_{v^{ne}}, \Delta_{\bar{\beta}}) \cdot \mathcal{O}_{X_{s^{ne}}} \supseteq \sigma(X_{s^{ne}}, \Delta_{\bar{\beta}} \times_{v^{ne}} s^{ne}) = \sigma(X_{s^{ne}}, \Delta_\gamma|_{X_{s^{ne}}}).$$

This completes the proof. \square

Using the same method as [Lemma 3.20](#), we also obtain the following result which may also be of independent interest.

Proposition 3.24. *Assume that $f : X \rightarrow V$ is a flat finite type reduced $G1$ and $S2$ morphism and that V is regular. Additionally assume that $\gamma : R^{1/p^e} \rightarrow R$ is an R -linear map which is a local generator at the generic points of the codimension-1 components of the non-smooth locus of f for each fiber of f . Then for any $a > 0$ we have*

$$\sigma(X, \Delta_\gamma) \otimes_A A^{1/p^a} \supseteq \sigma(X_{V^a}, \Delta_\gamma \times_V V^a).$$

Proof. The statement is local over the base and so we may assume that $K_V \sim 0$ and that A^{1/p^e} is a free A -module since V is regular [[Kun69](#)]. Note then that $K_{X/V} \cong K_X$. Choose $ne \geq a > 0$ such that $\sigma(X, \Delta_\gamma) = \gamma^n(R^{1/p^{ne}})$. Since $\text{Hom}_A(A^{1/p^a}, A)$ is a free $A^{1/p^{ne}}$ -module of rank 1, by [[Kun86](#), Appendix F] we can factor γ^n as

$$\begin{array}{ccccc} R^{1/p^{ne}} & \xrightarrow{\beta} & R_{A^{1/p^a}} & \xrightarrow{\vartheta} & R \\ & \searrow & & \nearrow & \\ & & \gamma^n & & \end{array}$$

where ϑ is a local generator of $\mathcal{H}\text{om}_R(R_{A^{1/p^a}}, R)$. Note $\beta \in \text{Hom}_{R_{A^{1/p^a}}}(R^{1/p^{ne}}, R_{A^{1/p^a}})$ which is identified with

$$\begin{aligned} & \mathcal{H}\text{om}_{R_{A^{1/p^a}}}(R^{1/p^{ne}} \otimes_{R_{A^{1/p^a}}} (A^{1/p^a} \otimes_A \omega_{R/A}), (A^{1/p^a} \otimes_A \omega_{R/A})) \\ \cong & \mathcal{H}\text{om}_{R_{A^{1/p^a}}}((\mathcal{O}_X(p^{ne}K_{X/V}))^{1/p^{ne}}, \omega_{R_{A^{1/p^a}}/A^{1/p^a}}) \\ \cong & \mathcal{H}\text{om}_{R_{A^{1/p^a}}}((\mathcal{O}_X(p^{ne}K_X))^{1/p^{ne}}, \omega_{R_{A^{1/p^a}}}) \\ \cong & \mathcal{H}\text{om}_{R^{1/p^{ne}}}((\mathcal{O}_X(p^{ne}K_X))^{1/p^{ne}}, \omega_{R^{1/p^{ne}}}) \\ \cong & (\mathcal{O}_X((1-p^{ne})K_X))^{1/p^{ne}}. \end{aligned}$$

Note that since the base is regular, all these sheaves are automatically reflexive and so there is no need to double dualize as we did before in [Definition 2.8](#). By dividing by $(p^{ne} - 1)$ we see that β coincides with a divisor Δ_β such that $(1 - p^{ne})(K_X + \Delta_\beta) \sim 0$. Furthermore, it is easy to see that Δ_β coincides with Δ_γ just as in [Lemma 3.20](#) since ϑ is a local generator.

By composing β with a local generating (and surjective) map $R^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} A^{1/p^{ne+a}} \xrightarrow{\dots \otimes (\Phi_A^a)^{1/p^{ne}}} R^{1/p^{ne}}$ we then obtain a map $\gamma' : (R_{A^{1/p^a}})^{1/p^{ne}} \cong R^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} A^{1/p^{ne+a}} \rightarrow R_{A^{1/p^a}}$ satisfying the condition $\Delta_{\gamma'} = \Delta_\gamma \times_V V^a$. We also notice that γ' has the same image as β .

Now, we observe that $\gamma^n(R^{1/p^{ne}}) \otimes_A A^{1/p^a} = \vartheta(\beta(R^{1/p^{ne}})) \otimes_A A^{1/p^a} \supseteq \beta(R^{1/p^{ne}})$ by Lemma 3.21 setting $B = A$ and $P = \beta(R^{1/p^{ne}})$. The remainder of the proof is easy since

$$\begin{aligned} & \sigma(X, \Delta_\gamma) \otimes_A A^{1/p^a} \\ &= \gamma^n(R^{1/p^{ne}}) \otimes_A A^{1/p^a} \\ &\supseteq \beta(R^{1/p^{ne}}) \\ &= \gamma'((R_{A^{1/p^a}})^{1/p^{ne}}) \\ &\supseteq \sigma(X_{V^a}, \Delta_{\gamma'}) \\ &= \sigma(X_{V^a}, \Delta_\gamma \times_V V^a). \end{aligned}$$

□

3.25. Application to sharply F -pure singularities and HSL numbers in families. We observe that Corollary 3.10 has an immediate application. Recall that if $\gamma : L^{1/p^e} \rightarrow R$ is an R -linear map, then the *HSL number* is the first integer n such that $\text{Im}(\gamma^n) = \text{Im}(\gamma^{n+1}) = \sigma(R, \gamma)$, cf. [Sha07].

Corollary 3.26 (Uniform behavior of HSL numbers). *Given a flat family $f : X \rightarrow V$ over a excellent integral scheme V with a dualizing complex of characteristic $p > 0$ and $\varphi : L^{1/p^e} \rightarrow \mathcal{O}_{X_{V^e}}$ as before, there exists an integer $N > 0$ such that gives an upper bound on the HSL number of $(X_s, \varphi_s : L_s^{1/p^e} \rightarrow \mathcal{O}_{X_s})$ for every perfect point $s \in V$.*

Proof. The statement immediately follows from Theorem 3.9 since clearly $\sigma_m(X_s/s, \varphi_s)$ agrees with $\sigma(X_s, \gamma_s = \varphi_s)$ for $m \gg n$. □

Now we study the deformation of sharp F -purity, a question which has been studied before in [SZ09, Has10]. We believe that the hypothesis that f is proper is necessary in Theorem 3.30.

Definition 3.27. Recall that a pair $(X, \psi : L^{1/p^e} \rightarrow \mathcal{O}_X = R)$ is called *sharply F -pure* if $\sigma(X, \psi) = \mathcal{O}_X$. Given $(X/S, \varphi : L^{1/p^e} \rightarrow R_{A^{1/p^e}})$, we define $(X/S, \varphi)$ to be *relatively sharply F -pure* if $\sigma_n(X/S, \varphi) = R_{A^{1/p^{ne}}}$ for some $n > 0$, cf. [Has10].

Lemma 3.28. *If $\sigma_n(X/S, \varphi) = R_{A^{1/p^{ne}}}$ for some $n > 0$, then the same holds for all $n > 0$.*

Proof. Suppose $\sigma_n(X/S, \varphi) = R_{A^{1/p^{ne}}}$ for some $n > 0$. Choose $m < n$ to start. Then we know $\mathfrak{a}_{m,n} \supseteq \mathfrak{a}_{n,n} = \sigma_n(X/S, \varphi)$ and so $\mathfrak{a}_{m,n} = R_{A^{1/p^{ne}}}$. But $\mathfrak{a}_{m,n} = \text{Im}(\mathfrak{a}_m \otimes_{A^{1/p^m}} A^{1/p^n} \rightarrow R_{A^{1/p^n}})$ which is just the extension of $\sigma_m(X/S, \varphi) \subseteq R_{A^{1/p^{me}}}$ to $R_{A^{1/p^{ne}}}$. Additionally, $R_{A^{1/p^{me}}} \subseteq R_{A^{1/p^{ne}}}$ is an integral extension and hence we must have $\sigma_m(X/S, \varphi) = R_{A^{1/p^{me}}}$ as well.

We finish the proof by showing $\sigma_{2n}(X/S, \varphi) = R_{A^{1/p^{2ne}}}$. This follows quickly since $\sigma_{2n}(X/S, \varphi)$ factors as a composition of two surjective maps (as \otimes is right exact) as in (2.13.1) and (2.13.1). □

Remark 3.29. If S is a point, then $(X/S, \varphi)$ being relatively sharply F -pure is equivalent to $(X/S, \varphi)$ being geometrically sharply F -pure (i.e., that (X_t, γ_t) is sharply F -pure for every geometric point $t \rightarrow S$ where γ_t is induced from φ_t as in Section 3.11). To see this, certainly observe that if $(X/S, \varphi)$ is relatively sharply F -pure then so is any base change $(X_t/t, \varphi_t)$. But then (X_t, γ) is sharply F -pure by Theorem 3.14. Conversely, if (X_t, γ_t) is sharply F -pure, then certainly $(X_t/t, \varphi_t)$ is relatively sharply F -pure by Lemma 3.13. But then so is $(X/S, \varphi)$ near that t by Nakayama.

Theorem 3.30 (Openness of sharp F -purity). *With notation as before, assume that $f : X \rightarrow V$ is proper. Assume that $s \in V$ is a point and that $(X_s/s, \varphi_s)$ is relatively sharply F -pure (in other words, geometrically F -pure). Then there exists a dense open set $U \subseteq V$ containing s such that $(X_u/u, \varphi_u)$ is relatively sharply F -pure for all $u \in U$ (in particular, (X_u, φ_u) is sharply F -pure for all perfect points $u \in U$).*

Proof. Choose $n \gg 0$ such that $\sigma_n(X_s/s, \varphi_s) = R_{\mathcal{O}_s^{1/p^{ne}}}$ since $(X_s/s, \varphi_s)$ is sharply F -pure. By [Theorem 3.9](#), we know that $\sigma_n(X/V, \varphi) = R_{A^{1/p^{ne}}}$ in a neighborhood $W \subseteq X$ of X_s . Let $Z = X \setminus W \subseteq X$ be the complement of that neighborhood. Since f is proper, $f(Z)$ is closed, and also does not contain s . Set $U = V \setminus f(Z)$. Then $\sigma_n(X_U/U, \varphi_U) = R_{\mathcal{O}_U^{1/p^{ne}}}$. It follows from [Theorem 3.9](#) that all the fibers $(X_u/u, \varphi_u)$ are relatively sharply F -pure for perfect $u \in U$ as desired. \square

We now state the above theorem in the context of divisors.

Corollary 3.31 (Openness of sharp F -purity for divisor pairs). *Suppose that $f : X \rightarrow V$ is a proper $G1$ and $S2$ morphism as before and now additionally suppose that Δ is a \mathbb{Q} -divisor satisfying conditions (a)–(d) from [Remark 2.11](#). Additionally suppose that $s \in V$ is a perfect point and that $(X_s, \Delta|_{X_s})$ is sharply F -pure (or more generally, if s is not perfect, that $(X_{s^e}, \Delta|_{X_{s^e}})$ is sharply F -pure for some/all $e > 0$). Then there exists an open set $U \subseteq V$ containing s such that for all $u \in U$ we have $(X_u, \Delta|_{X_u})$ is sharply F -pure (respectively, we have $(X_{u^e}, \Delta|_{X_{u^e}})$ is sharply F -pure for some/all $e > 0$ and all $u \in U$).*

Proof. Associate to Δ a relatively divisorial $\varphi : L^{1/p^e} \rightarrow R_{A^{1/p^e}}$ as in [Remark 2.11](#). The result then follows immediately from [Theorem 3.30](#) and [Corollary 2.22](#). \square

4. RELATIVE TEST IDEALS

In this section we define a notion of a relative test ideal. We construct it using the following method. We first find some ideal I which when restricted to every geometric fiber is contained in the test ideal of that fiber. We then sum up the image of $I^{1/p^{ne}}$ under φ^n (the same method is used to construct the test ideal in the absolute case). This sounds simple enough, but it actually leads to a somewhat disappointing construction. The problem is that the ideal we pick is in no sense canonical and we do not see a way to make it both canonical and stable under base change. In the case that φ corresponds to a trivial divisor, we can make our choices canonical by setting I to be the Jacobian ideal (this is done later when we discuss relative F -rationality in [Section 5](#)), but for more general φ we don't know of an analogous choice.

We fix the notation of the previous sections (in particular, we fix $\varphi : L^{1/p^e} \rightarrow R_{A^{1/p^e}}$). We begin by choosing an *arbitrary ideal* $I \subseteq \mathcal{O}_X$. After base change to $k(V^\infty)$ as before, setting $I_\infty = I \cdot R_\infty$, we can form the following sum:

$$\tau(R_{k(V^\infty)}, \psi_\infty I_\infty) := \sum_{i=0}^{\infty} \psi_\infty^i ((I_\infty \cdot L_{k(V^\infty)}^{\frac{p^{ie}-1}{p^e-1}})^{1/p^{ie}}).$$

If it happens that I_∞ is contained in the test ideal of $(R_{k(V^\infty)}, \varphi_\infty)$ and non-zero on any component of X_∞ , then we see that the absolute test ideal $\tau(R_{k(V^\infty)}, \psi_\infty(I_\infty))$ is equal to $\tau(R_{k(V^\infty)}, \psi_\infty)$ for example by [\[ST12, Section 7\]](#).

We can also describe this as follows, first essentially observed in [\[Kat08\]](#). Fix ideals

$$\begin{aligned} \mathfrak{b}_0^\infty &:= I_\infty = \psi_\infty^0(I_\infty) \\ \mathfrak{b}_1^\infty &:= \mathfrak{b}_0 + \psi_\infty((\mathfrak{b}_0^\infty \cdot L_{k(V^\infty)})^{1/p^e}) = I_\infty + \psi_\infty((I_\infty \cdot L_{k(V^\infty)})^{1/p^e}) \\ \mathfrak{b}_2^\infty &:= \mathfrak{b}_1^\infty + \psi_\infty((\mathfrak{b}_1^\infty \cdot L_{k(V^\infty)})^{1/p^e}) \\ &= \mathfrak{b}_1^\infty + \psi_\infty((I_\infty + \psi_\infty((I_\infty \cdot L_{k(V^\infty)})^{1/p^e}) \cdot L_{k(V^\infty)})^{1/p^e}) \\ &= \sum_{i=0}^2 \psi_\infty^i((I_\infty \cdot L_{k(V^\infty)}^{\frac{p^{ie}-1}{p^e-1}})^{1/p^{ie}}) \\ &\dots \dots \\ \mathfrak{b}_n^\infty &:= \mathfrak{b}_{n-1}^\infty + \psi_\infty((\mathfrak{b}_{n-1}^\infty \cdot L_{k(V^\infty)})^{1/p^e}) = \sum_{i=0}^n \psi_\infty^i((I_\infty \cdot L_{k(V^\infty)}^{\frac{p^{ie}-1}{p^e-1}})^{1/p^{ie}}) \end{aligned} \tag{4.0.1}$$

And we notice that this ascending chain stabilizes say at t . Also note that the first time $\mathfrak{b}_t^\infty = \mathfrak{b}_{t+1}^\infty$, we then have $\mathfrak{b}_t^\infty = \mathfrak{b}_{t+1}^\infty = \mathfrak{b}_{t+2}^\infty = \dots$. We fix this integer t .

Definition 4.1. With notation as above, we define the integer t to be the *uniform integer for τ and I over the generic point of V* , and in general, it will be denoted by $n_{\tau(\varphi I), k(V)}$. We notice that for any point $\eta \in V$, by base change we can replace V by $\text{Spec } k(\eta)$ and form a corresponding integer $n_{\tau(\varphi_\eta I_\eta), k(\eta)}$.

On the other hand, without the passing to $k(V^\infty)$, we have the images

$$\begin{aligned} \mathfrak{b}_1 &:= \text{Im} \left(I \otimes_A A^{1/p^e} \rightarrow R_{A^{1/p^e}} \right) + \varphi((I \cdot L)^{1/p^e}) \subseteq R_{A^{1/p^e}}, \\ \mathfrak{b}_2 &:= \sum_{i=0}^2 \text{Im} \left(\varphi^i((I \cdot L^{\frac{p^{ie}-1}{p^e-1}})^{1/p^{ie}}) \otimes_{A^{1/p^{ie}}} A^{1/p^{2e}} \rightarrow R_{A^{1/p^{2e}}} \right) \subseteq R_{A^{1/p^{2e}}}, \\ &\dots \\ \mathfrak{b}_n &:= \sum_{i=0}^n \text{Im} \left(\varphi^i((I \cdot L^{\frac{p^{ie}-1}{p^e-1}})^{1/p^{ie}}) \otimes_{A^{1/p^{ie}}} A^{1/p^{ne}} \rightarrow R_{A^{1/p^{ne}}} \right) \subseteq R_{A^{1/p^{ne}}}, \\ &\dots \end{aligned}$$

Notice that $\text{Im} \left(\mathfrak{b}_1 \otimes_{A^{1/p^e}} A^{1/p^{2e}} \rightarrow R_{A^{1/p^{2e}}} \right) \subseteq \mathfrak{b}_2$ and more generally for $j > i$ that $\text{Im} \left(\mathfrak{b}_i \otimes_{A^{1/p^{ie}}} A^{1/p^{je}} \rightarrow R_{A^{1/p^{je}}} \right) \subseteq \mathfrak{b}_j$. Also observe that this is the *opposite* containment compared to what we had in [Section 3](#).

By the same argument as in [Section 3](#), we know that there exists an open set $U \subseteq V$ with $W = f^{-1}(U)$ such that

$$(4.1.1) \quad \text{Im}(\mathfrak{b}_t \otimes_{A^{1/p^{te}}} A^{1/p^{(t+1)e}} \rightarrow R_{A^{1/p^{(t+1)e}}})|_W = \mathfrak{b}_{t+1}|_W.$$

As before in [Proposition 3.3](#), we claim that:

Lemma 4.2. *With notation as above, $\text{Im}(\mathfrak{b}_n \otimes_{A^{1/p^{ne}}} A^{1/p^{(n+1)e}} \rightarrow R_{A^{1/p^{(n+1)e}}})|_W = \mathfrak{b}_{n+1}|_W$ for all $n \geq t$.*

Proof. Replacing V by U and X by W , we may assume that

$$\mathfrak{b}_{t+1} = \text{Im} \left(\mathfrak{b}_t \otimes_{A^{1/p^{te}}} A^{1/p^{(t+1)e}} \rightarrow R_{A^{1/p^{(t+1)e}}} \right)$$

By induction on n , it suffices to show that, if $\mathfrak{b}_{n+1} = \text{Im}(\mathfrak{b}_n \otimes_{A^{1/p^{ne}}} A^{1/p^{(n+1)e}} \rightarrow R_{A^{1/p^{(n+1)e}}})$, then $\mathfrak{b}_{n+2} = \text{Im}(\mathfrak{b}_{n+1} \otimes_{A^{1/p^{(n+1)e}}} A^{1/p^{(n+2)e}} \rightarrow R_{A^{1/p^{(n+2)e}}})$. Note that $\mathfrak{b}_{n+1} = \text{Im}(\mathfrak{b}_n \otimes_{A^{1/p^{ne}}} A^{1/p^{(n+1)e}} \rightarrow R_{A^{1/p^{(n+1)e}}})$ if and only if

$$\begin{aligned} &\text{Im}(\varphi^{n+1}((I \cdot L^{\frac{p^{(n+1)e}-1}{p^e-1}})^{1/p^{(n+1)e}}) \rightarrow R_{A^{1/p^{(n+1)e}}}) \\ &\subseteq \sum_{i=0}^n \text{Im} \left(\varphi^i((I \cdot L^{\frac{p^{ie}-1}{p^e-1}})^{1/p^{ie}}) \otimes_{A^{1/p^{ie}}} A^{1/p^{(n+1)e}} \rightarrow R_{A^{1/p^{(n+1)e}}} \right) \end{aligned}$$

Tensoring with $\otimes_R L$, taking $1/p^e$ th roots, and applying φ yields

$$\begin{aligned} &\text{Im}(\varphi^{n+2}((I \cdot L^{\frac{p^{(n+2)e}-1}{p^e-1}})^{1/p^{(n+2)e}}) \rightarrow R_{A^{1/p^{(n+2)e}}}) \\ &\subseteq \sum_{i=1}^{n+1} \text{Im} \left(\varphi^i((I \cdot L^{\frac{p^{ie}-1}{p^e-1}})^{1/p^{ie}}) \otimes_{A^{1/p^{ie}}} A^{1/p^{(n+2)e}} \rightarrow R_{A^{1/p^{(n+2)e}}} \right) \end{aligned}$$

Hence,

$$\mathfrak{b}_{n+2} = \text{Im}(\mathfrak{b}_{n+1} \otimes_{A^{1/p^{(n+1)e}}} A^{1/p^{(n+2)e}} \rightarrow R_{A^{1/p^{(n+2)e}}}).$$

This finishes the proof. \square

As before, we define relative test ideals as follows.

Definition 4.3. The n th limiting relative test ideal with respect to I (of the pair $(X/V, \varphi)$) is defined to be $\mathfrak{b}_n \subseteq R_{A^{1/p^{ne}}}$ and is denoted by $\tau_n(X/V, \varphi I)$.

Before proceeding to various properties of relative test ideals, we give one example of relative test ideals.

Example 4.4. (Example 3.7 revisited) Fix k to be an algebraically closed field of characteristic $p > 2$, set $A = k[t]$ and set $R = k[x, t]$ with the obvious map $X = \text{Spec}(R) \rightarrow V = \text{Spec}(A)$. Let $\varphi : R^{1/p} = k[x^{1/p}, t^{1/p}] \rightarrow R_{A^{1/p}} = k[x, t^{1/p}]$ be the composition of the local generator $\beta \in \text{Hom}_{R_{A^{1/p}}}(R^{1/p}, R_{A^{1/p}})$ with pre-multiplication by $(x^{p^2} + t)^{\frac{1}{p}} = f^{\frac{1}{p}}$. Set $I = \langle x^p + t \rangle$. Then

$$\mathfrak{b}_1 = \text{Im}(I \otimes_A A^{1/p} \rightarrow R_{A^{1/p}}) + \varphi(I^{1/p}) = \langle x^p + t \rangle + \langle (x^p + t^{1/p})(x + t^{1/p}) \rangle = \langle x^p + t, (x^p + t^{1/p})(x + t^{1/p}) \rangle.$$

To calculate \mathfrak{b}_2 , it suffices to calculate $\text{Im}(\varphi^2(I^{1/p^2})) = \text{Im}(\beta^2((x^{p^2} + t)^{\frac{1+p}{p^2}} I^{1/p^2}))$. Since $(x^{p^2} + t)^{\frac{1+p}{p^2}} \in R_{A^{1/p^2}}$, we can see that $\text{Im}(\varphi^2(I^{1/p^2})) \subseteq \langle (x^{p^2} + t)^{\frac{1+p}{p^2}} \rangle$. On the other hand, it is straightforward to check that $\beta^2((x^{p^2} + t)^{\frac{1+p}{p^2}} x^{\frac{p^2-p-1}{p^2}} (x^p + t)^{1/p^2}) = (x^{p^2} + t)^{\frac{1+p}{p^2}} = (x + t^{1/p^2})(x^p + t^{1/p})$. Therefore,

$$\mathfrak{b}_2 = \langle x^p + t, (x^p + t^{1/p})(x + t^{1/p}), (x + t^{1/p^2})(x^p + t^{1/p}) \rangle.$$

For each $i \geq 2$, to calculate \mathfrak{b}_{i+1} , it suffices to calculate $\text{Im}(\beta^{i+1}((x^{p^2} + t)^{\frac{1+p+\dots+p^i}{p^{i+1}}} I^{1/p^{i+1}}))$. It is straightforward to check that it is contained in $\langle (x^p + t^{1/p})(x + t^{1/p^2}) \rangle$ and hence is contained in \mathfrak{b}_2 . Therefore, $\mathfrak{b}_{i+1} = \mathfrak{b}_2$ as well. So we have

$$\mathfrak{b}_n = \begin{cases} \langle x^p + t, (x^p + t^{1/p})(x + t^{1/p}) \rangle & \text{when } n = 1 \\ \mathfrak{b}_2 = \langle x^p + t, (x^p + t^{1/p})(x + t^{1/p}), (x + t^{1/p^2})(x^p + t^{1/p}) \rangle & n \geq 2. \end{cases}$$

We now prove a base change statement for relative test ideals. Compare with Proposition 3.8. As before, we fix $q_i : X \times_V T^i \rightarrow X \times_V V^i$ to be the natural map.

Theorem 4.5 (Relative test ideals and base change). *Fix $g : T \rightarrow V$ to be any morphism with T excellent, integral and admitting a dualizing complex. Then*

$$\text{Im}((q_{ne})^* \tau_n(X/V, \varphi I) \hookrightarrow \mathcal{O}_{X_{T^{ne}}}) := \tau_n(X/V, \varphi I) \cdot \mathcal{O}_{X_{T^{ne}}} = \tau_n(X_T/T, \varphi_T I_T).$$

Furthermore, if $U = U_n$ satisfies the condition of Lemma 4.2, then $W = g^{-1}(U) \subseteq T$ satisfies the same condition for $\tau_n(X_T/T, \varphi_T I_T)$.

Proof. We write $B = \mathcal{O}_T$ and work locally. By construction and right exactness of tensor:

$$\tau_n(X/V, \varphi) \cdot \mathcal{O}_{X_{T^{ne}}} = \mathfrak{b}_n \otimes_{A^{1/p^{ne}}} B^{1/p^{ne}} = \sum_{i=0}^n (\varphi^i((I \cdot L^{\frac{p^{ie}-1}{p^e-1}})^{1/p^{ie}}) \otimes_{A^{1/p^{ie}}} B^{1/p^{ie}}) \subseteq R_{B^{1/p^{ne}}}.$$

But by Lemma 2.17, we have

$$\begin{aligned} & \sum_{i=0}^n (\varphi^i((I \cdot L^{\frac{p^{ie}-1}{p^e-1}})^{1/p^{ie}}) \otimes_{A^{1/p^{ie}}} B^{1/p^{ie}}) \\ &= \sum_{i=0}^n (\varphi_T^i((I_T \cdot L_T^{\frac{p^{ie}-1}{p^e-1}})^{1/p^{ie}}) \otimes_{B^{1/p^{ie}}} B^{1/p^{ie}}) \\ &= \tau_n(X_T/T, \varphi_T I_T) \end{aligned}$$

which completes the proof. \square

Theorem 4.6 (Restriction of τ_n to fibers). *With notation as above, there exists an integer $N \geq 0$, such that for all points $s \in V$, $N \geq n_{\tau(\varphi_s I), k(s)}$. In other words, we have both that $\tau_n(X/V, \varphi) \cdot \mathcal{O}_{X_{s^{ne}}} = \tau_n(X_s/s, \varphi_s)$ (which always holds) and also that for all $m \geq n \geq N$ that*

$$\tau_n(X/V, \varphi I) \otimes_{V^n} k(s)^{1/p^m} = \tau_m(X_s/s, \varphi_s I_s).$$

Proof. The idea is the same as for [Theorem 3.9](#) and we only sketch it here. We stratify V as follows. The result holds on a dense open subset of $U_0 \subseteq V$. Let V'_1 be the complement and let V_1 denote the regular locus of V'_1 . Base change to V_1 , and then repeat. This procedure stops after finitely many iterations by Noetherian induction and we choose an N that works for them all. \square

We now construct $I \subseteq R$ whose restriction is contained in the test ideal of every geometric fiber.

Proposition 4.7 (The existence of relative test elements). *With notation as above suppose additionally that $f : X \rightarrow V$ is relatively G1 and S2. Then there exists an ideal $I \subset \mathcal{O}_X$ such that for every point $s \in V$ and every perfect extension $K \supseteq k(s)$, we have that $I_K \subseteq \tau(X_K, \psi_K)$ where again ψ_K is simply φ_K interpreted as in [\[BS13\]](#). Additionally, we can assume that $I = \mathcal{O}_X$ at all points of X such that both X/V is smooth and that φ locally generates $\mathcal{H}\text{om}_{\mathcal{O}_X}(L^{1/p^e}, R_{A^{1/p^e}})$. Furthermore, if V is strongly F -regular and quasi-Gorenstein, then I can additionally be chosen so that I_{V^e} is within the absolute test ideal of (X_{V^e}, γ) (where γ is as in [Section 3.11](#)).*

We caution the reader that it is possible that ψ_s (and thus ψ_K) could be the zero-map, and hence I_K the zero ideal.

Proof. First let $Z_1 \subseteq X$ denote the locus where $\varphi R^{1/p^e} \subseteq \mathcal{H}\text{om}_{R_{A^{1/p^e}}}(L^{1/p^e}, R_{A^{1/p^e}})$ is not an isomorphism. Additionally let Z_2 denote the locus where X is not smooth over V . Set $Z = Z_1 \cup Z_2$. Set $W = X \setminus Z$ (and note that we can also view this as a subset of X_{V^e} for any e) and notice that on W we know that φ can be identified with the trace map up to multiplication by a unit.

Since $f|_W$ is smooth, we observe that φ (which is identified with trace) is surjective when restricted to W by [Corollary 2.5](#). Now choose an exponent $m > 0$ such that

$$I_Z^m \cdot R_{A^{1/p^e}} \subseteq \mathfrak{a}_{1,1} = \sigma_1(X/S, \varphi)$$

Now choose an integer $l > 0$ such that I_Z^{ml} is locally generated by cubes of elements of I_Z^m (this only depends on the number of local generators of I_Z). Then the formation of I_Z^{ml} is obviously compatible with base change (in that the extension of I_Z^{ml} to the base change will also satisfy the same containment condition.). Thus set $I = I_Z^{ml}$. The first result follows immediately from [\[Hoc07, Theorem on Page 90\]](#) or [\[Sch11b, Proof of Proposition 3.21\]](#).

For the second result, we notice that γ locally generates $\mathcal{H}\text{om}_{R_{A^{1/p^e}}}(N^{1/p^e}, R_{A^{1/p^e}})$ and that I_Z^m is also in the image of γ . Therefore, I^{ml} works by the above references. \square

Therefore we obtain:

Corollary 4.8. *Using the notation of [Theorem 4.6](#) assume further that I satisfies the condition of [Proposition 4.7](#). If s is a perfect point then for all $n \geq N$ as in [Theorem 3.9](#)*

$$\tau_n(X/V, \varphi I) \otimes_{V^n} k(s)^{1/p^m} = \tau(X_s, \psi_s)$$

where ψ_s is φ_s interpreted as a p^{-e} -linear map.

Proof. Simply observe that in general the ascending $\tau_j(X_s/s, \varphi_s I_s) \subseteq \tau_{j+1}(X_s/s, \varphi_s I_s) \subseteq \dots$ stabilize to $\tau(X_s, \psi_s)$. Furthermore, by [Theorem 4.6](#) this ascending chain stabilizes. \square

Corollary 4.9. *Using the notation of [Theorem 4.6](#) and assume further that I satisfies the condition of [Proposition 4.7](#). If the perfect closure of the generic fiber of $f : X \rightarrow V$ is strongly F -regular, i.e. $\tau(R_{k(V^\infty)}, \psi_\infty I_\infty) = R_{k(V^\infty)}$, then there exists an open subset $U \subseteq V$ such that (X_s, ψ_s) is also strongly F -regular for each perfect point $s \in U$.*

Proof. It follows immediately from the proof of [Proposition 3.3](#) and [Corollary 4.8](#). \square

4.10. Relative test ideals vs absolute test ideals. We make the following assumption.

Convention 4.11. For the rest of [Section 4.10](#), we assume that V is regular and F -finite.

It is natural to try to relate the relative test ideal $\tau_n(X/V, \varphi I)$ with the absolute test ideal $\tau(X_{V^{ne}}, \gamma^n)$ as in [Section 3.11](#). Indeed, if we construct I as we did in the proof of [Proposition 4.7](#) then it follows easily that $I \cdot R_{A^{1/p^{ne}}} \subseteq \tau(X_{V^{ne}}, \gamma^n)$. Thus fix such an I .

We now recall the following diagram from [Lemma 3.13](#)

$$\begin{array}{ccc}
 \left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{1/p^{ne}} \otimes_{A^{1/p^{ne}}} \left(M^{\frac{p^{ne}-1}{p^e-1}} \right)^{1/p^{(n+n)e}} & \xrightarrow{\gamma'} & \left(L^{\frac{p^{(n-1)e}-1}{p^e-1}} \right)^{1/p^{(n-1)e}} \otimes_{A^{1/p^{(n-1)e}}} \left(M^{\frac{p^{(n-1)e}-1}{p^e-1}} \right)^{1/p^{(n+n-1)e}} \\
 \downarrow \mu_n & & \downarrow \mu_{n-1} \\
 \left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{1/p^{ne}} & \xrightarrow{\quad \quad \quad} & \left(L^{\frac{p^{(n-1)e}-1}{p^e-1}} \right)^{1/p^{(n-1)e}} \otimes_{A^{1/p^{(n-1)e}}} A^{1/p^{ne}} \quad \gamma^{n-1} \\
 & \searrow \varphi^n & \downarrow \varphi^{n-1} \otimes \dots \\
 & & R_{A^{1/p^{ne}}} \leftarrow \dots
 \end{array}$$

γ^n

We let $I_{A^{1/p^{ke}}}^{1/p^{je}}$ denote the extension of $I^{1/p^{je}}$ to $R_{A^{1/p^{ke}}}^{1/p^{je}}$. Then notice that μ_j sends $I_{A^{1/p^{(j+n)e}}}^{1/p^{je}}$ (times μ_j 's domain) back onto $I^{1/p^{je}}$ since each μ_j is surjective.

We then observe that:

Lemma 4.12. *With notation as above and assuming [Convention 4.11](#), then $\tau_n(X/V, \varphi I) \subseteq \tau(X_{V^{ne}}, \gamma)$.*

Proof. We know $\tau_n(X/V, \varphi I)$ is the sum

$$(4.12.1) \quad \sum_{j=0}^n \left(\varphi^j \otimes_{A^{1/p^{je}}} A^{1/p^{ne}} \right) \left(I^{1/p^{je}} \cdot \left(L^{\frac{p^{je}-1}{p^e-1}} \right)^{1/p^{je}} \otimes_{A^{1/p^{je}}} A^{1/p^{ne}} \right)$$

On the other hand, $\tau(X, \gamma)$ is the sum

$$(4.12.2) \quad \sum_{j=0}^{\infty} \gamma^j \left(I_{A^{1/p^{(j+n)e}}}^{1/p^{je}} \cdot \left(L^{\frac{p^{je}-1}{p^e-1}} \right)^{1/p^{je}} \otimes_{A^{1/p^{je}}} \left(M^{\frac{p^{je}-1}{p^e-1}} \right)^{1/p^{(j+n)e}} \right)$$

By our above observations about the surjectivities of the μ_j above, the sum of the 0th through j th terms of [\(4.12.2\)](#) is equal to the sum [\(4.12.1\)](#). The result follows. \square

Additionally, at least over an open subset of the base, we actually have that the relative test ideal and absolute test ideal agree, cf. [Theorem 3.14](#).

Theorem 4.13. *With notation as above and assuming [Convention 4.11](#), choose $n > t = n_{\tau(\varphi I), k(V)}$. Then there exists a dense open set $U \subseteq V \cong V^e$ with $W = f^{-1}(U) \subseteq X$ such that $\tau(X_{V^{1/p^{ne}}}, \gamma)|_W = \tau_n(X/V, \varphi I)|_W$. Furthermore, shrinking U further if necessary and possibly increasing n , we can require for all perfect points $u \in U$, that*

$$\tau(X_{V^{1/p^{ne}}}, \gamma) \cdot \mathcal{O}_{X_u} = \tau_n(X_u/u, \varphi_u I_u) = \tau(X_u, \psi_u).$$

where again ψ_u is φ_u viewed as a p^{-e} -linear map.

Proof. First we observe that by taking U as in [\(4.1.1\)](#), we can assume that $\mathfrak{b}_{n-1, n} = \mathfrak{b}_{n, n}$. But by the diagram above, we see that the $n-1$ st partial sum defining $\tau(X_{V^{1/p^{ne}}}, \gamma)$ in [\(4.12.2\)](#) is equal to $\mathfrak{b}_{n-1, n}$. Likewise the n th partial sum is equal to $\mathfrak{b}_{n, n}$. However, once two adjacent partial sums defining $\tau(X_{V^{1/p^{ne}}}, \gamma)$ coincide, the sum stabilizes for further powers by the computation we made when defining the \mathfrak{b}_n^∞ in [\(4.0.1\)](#). The second statement follows from the first statement and from [Theorem 4.5](#), cf. the argument of [Corollary 4.8](#). \square

When dealing with relatively non- F -pure ideals σ , we actually obtained the above restriction theorem without shrinking X to U in [Theorem 3.23](#). The difference is that for σ we have the easy

containment $\sigma(X_{V^{1/p^{ne}}}, \gamma^n) \subseteq \sigma_n(X/V, \varphi)$ by [Lemma 3.13](#). For τ however, the easy containment is reversed. This leads us to the following question:

Question 4.14. Is it true that $\tau(X_{V^{1/p^{ne}}}, \gamma) \cdot \mathcal{O}_{X_{s^{ne}}} = \tau(X_s, \psi_s)$ for all perfect points $s \in V$ at least when $n \gg 0$?

Rephrasing [Theorem 4.13](#) for divisors we obtain:

Corollary 4.15. *Suppose that $f : X \rightarrow V$ is a flat finite map to an excellent regular scheme V . Additionally suppose that f is relatively G1 and S2 and I is chosen as in [Proposition 4.7](#). Choose Δ satisfying conditions (a)–(d) of [Remark 2.11](#). Then there exists an open dense set $U \subseteq V$ such that*

$$\tau(X_{V^{1/p^{ne}}}, \Delta) \cdot \mathcal{O}_{X_{u^{1/p^{ne}}}} = \tau(X_u, \Delta|_{X_u})$$

for all perfect points $u \in U$.

Proof. Using [Lemma 2.24](#) and [Lemma 3.16](#), we see that Δ corresponds to some $\varphi : L^{1/p^e} \rightarrow R_{A^{1/p^e}}$. Thus we simply apply [Corollary 2.22](#) and [Theorem 4.13](#). \square

4.16. Applications to strong F -regularity and divisor pairs.

Definition 4.17. Recall that a pair $(X, \psi : L^{1/p^e} \rightarrow \mathcal{O}_X = R)$ is called *strongly F -regular* if $\tau(X, \psi) = \mathcal{O}_X$. Given $(X/S, \varphi : L^{1/p^e} \rightarrow R_{A^{1/p^e}})$ and some $I \subseteq R$ satisfying the condition of [Proposition 4.7](#), we say that $(X/S, \varphi I)$ is *relatively strongly F -regular* if we have $\tau_n(X/S, \varphi I) = R_{A^{1/p^{ne}}}$ for some $n > 0$ (equivalently, all $n \gg 0$ since the τ_n ascend), cf. [\[Has10\]](#).

Remark 4.18. Suppose that S is a point and I is chosen as in [Proposition 4.7](#), then $(X/S, \varphi I)$ is relatively strongly F -regular if and only if it is geometrically strongly F -regular (i.e., (X_t, γ_t) is strongly F -regular for every point $t \rightarrow S$). The argument is the same as in [Remark 3.29](#).

The same ideas imply the strongly F -regular locus of a proper map is open.

Corollary 4.19 (Openness of the strongly F -regular locus). *With notation as before, assume additionally that $f : X \rightarrow V$ is proper and that I satisfies the condition of [Proposition 4.7](#). Assume that $s \in V$ is a point and that $(X_s/s, \varphi_s I_s)$ is relatively strongly F -regular (for example, if s is a perfect point, this just means it is strongly F -regular and is independent of I). Then there exists a dense open set $U \subseteq V$ containing s such that $(X_u/u, \varphi_u I_u)$ is relatively strongly F -regular for all $u \in U$ (in particular, (X_u, φ_u) is strongly F -regular for all perfect points $u \in U$).*

Proof. Choose $n \gg 0$ such that $\tau_n(X_s/s, \varphi_s I_s) = R_{\mathcal{O}_s^{1/p^{ne}}}$ since $(X_s/s, \varphi_s I_s)$ is strongly F -regular. By [Theorem 4.6](#), we know that $\tau_n(X/V, \varphi I) = R_{A^{1/p^{ne}}}$ in a neighborhood $W \subseteq X$ of X_s . Let $Z = X \setminus W \subseteq X$ be the complement of that neighborhood. Since f is proper, $f(Z)$ is closed, and also doesn't contain s . Set $U = V \setminus f(Z)$. Then $\tau_n(X_U/U, \varphi_U I|_U) = R_{\mathcal{O}_U^{1/p^{ne}}}$. It follows from [Theorem 4.6](#) that all the fibers $(X_u/u, \varphi_u I_u)$ are relatively strongly F -regular as desired. \square

At least for proper maps, we also obtain that the definition of relative strongly F -regularity [Definition 4.17](#) is independent of the choice of I .

Lemma 4.20. *Suppose that $f : X \rightarrow V$ is proper and that $(X/V, \varphi I)$ is relatively strongly F -regular for some I satisfying the condition of [Proposition 4.7](#). Then for all $J \subseteq R$ such that J_s is non-zero on every component of every fiber X_s , we have that $\tau_n(X/S, \varphi J) = R_{A^{1/p^{ne}}}$ for some $n > 0$ (equivalently, all $n \gg 0$).*

In particular in [Definition 4.17](#), it would be equivalent to require $\tau_n(X/S, \varphi J) = R_{A^{1/p^{ne}}}$ for all such J and some n .

Proof. Choose a closed point $s \in V$ and a perfect extension $K \supseteq k(s)$. Then $\tau_m(X_K/K, \varphi_K I_K) = R_{K^{1/p^m}}$ for some $m > 0$ by [Theorem 4.6](#). But since K is a perfect field extension, this implies that $\tau_m(X_K, \psi_K I_K) = R_{K^{1/p^m}}$ as well. On the other hand, since (X_K, ψ_K) is strongly F -regular, we see that $\tau_n(X_K/K, \varphi_K J_K) = R_{K^{1/p^n}}$ for some $n > 0$. Hence since $k(s) \subseteq K$ is faithfully flat, we see that $\tau_n(X_s/s, \varphi_s J_s) = R_{k(s)^{1/p^n}}$. But then since $\tau_n(X/V, \varphi J)$ restricts to $\tau_n(X_s/s, \varphi_s J_s)$ we observe that $\tau_n(X/V, \varphi J) = R_{A^{1/p^n}}$ at least in a neighborhood of s using that f is proper and the same argument we made in [Corollary 4.19](#). But we can find such an n for every $s \in V$, and so the lemma holds by the quasi-compactness of V . \square

We now state our result in the divisorial case.

Corollary 4.21. *With notation as above, suppose that $f : X \rightarrow V$ is a proper map and that Δ is a \mathbb{Q} -divisor satisfying conditions (a)–(d)⁸ of [Remark 2.11](#). Additionally suppose that for some perfect point $s \in V$, the fiber $(X_s, \Delta|_{X_s})$ is strongly F -regular. Then there exists a dense open set $U \subseteq V$ containing s such that $(X_u, \Delta|_{X_u})$ is strongly F -regular for all perfect $u \in U$.*

Proof. Using [Remark 2.11](#) we construct a relatively divisorial $\varphi : L^{1/p^e} \rightarrow R_{A^{1/p^e}}$ corresponding to Δ . Choose now I satisfying the condition of [Proposition 4.7](#). It follows that (X_s, ψ_s) is strongly F -regular and hence that $(X_s/s, \varphi_s I_s)$ is relatively strongly F -regular since s is a perfect point. Then [Corollary 4.19](#) and [Corollary 2.22](#) complete the proof. \square

By a perturbation trick we can also handle the case that the index of $K_X + \Delta$ is divisible by $p > 0$, at least over curves.

Corollary 4.22. *With notation as above, suppose that $f : X \rightarrow V$ is a projective map to a regular 1-dimensional base, and that Δ is a \mathbb{Q} -divisor satisfying the following 4 conditions:*

- (a') $\Delta' = \frac{1}{m}D$ for some Weil divisor D .
- (b) D is a Weil divisor on X which is Cartier in codimension 1 and Cartier at every codimension 1 point of every fiber.
- (c) D is trivial along the codimension-1 components of the non-smooth locus of $X \rightarrow V$ and the codimension-1 components of the non-smooth locus of every fiber.
- (d') $l/m \in \mathbb{Z}$ and $(\omega_{X/V}^l \otimes \mathcal{O}_X(l\Delta'))^{**}$ is a line bundle.

Additionally suppose that for some perfect point $s \in V$, the fiber $(X_s, \Delta|_{X_s})$ is strongly F -regular. Then there exists a dense open set $U \subseteq V$ containing s such that $(X_u, \Delta|_{X_u})$ is strongly F -regular for all perfect $u \in U$.

Proof. We may certainly suppose that V is affine. Since the fiber X_s is normal, and hence geometrically normal, we may also assume that the nearby fibers satisfy the same condition (using that f is proper). Thus we may assume that all the fibers are geometrically normal and in fact that the map $f : X \rightarrow V$ is geometrically integral. Without loss of generality, by base change we can assume that V is normal and hence X is normal itself. Thus we may assume that X is normal.

Now, since the base is 1-dimensional, we claim that can assume that $K_{X/V}$ doesn't contain any fiber. The only fiber we must worry about is X_s (as the others can be handled by shrinking V). We argue as follows: note first that $K_{X/V}$ can be viewed as an honest Weil divisor since we already assumed that X is normal. On the other hand, each fiber is a Cartier divisor. Hence, if $K_{X/V}$ is non-trivial along the generic point of X_s , by twisting by the pullback of s , we can assume that $K_{X/V}$ does not contain X_s .

Write $l\Delta' = cp^{e_0}\Delta'$ where p does not divide c . Let E be an effective Cartier divisor on X not containing any fiber such that $K_{X/V} + E$ is effective. We also observe that $K_{X/V} + E$ satisfies

⁸For example, these conditions hold if V is regular, $f : X \rightarrow V$ is geometrically normal, $K_X + \Delta$ is \mathbb{Q} -Cartier with index not divisible by p and Δ does not contain any fiber of f in its support.

conditions (b) and (c) above. Let $\Gamma = \frac{E+K_{X/V}+\Delta'}{p^e-1}$ for some $e \gg e_0$. Then consider

$$\Delta := \Delta' + \Gamma.$$

Note that

$$\begin{aligned} & c(p^e - 1)(K_{X/V} + \Delta) \\ &= c(p^e - 1)(K_{X/V} + \Delta' + \Gamma) \\ &= c(p^e - 1)(K_{X/V} + \Delta') + c(E + K_{X/V} + \Delta') \\ &= cp^e(K_{X/V} + \Delta') + cE \end{aligned}$$

which is certainly Cartier. Hence $K_{X/V} + \Delta$ satisfies conditions (a)–(d) from [Remark 2.11](#). On the other hand, since $e \gg 0$, we know that $(X_s, \Delta|_{X_s}) = (X_s, \Delta'|_{X_s} + \Gamma|_{X_s})$ is still strongly F -regular and so by [Corollary 4.21](#) there exists an open set U such that $(X_u, \Delta|_{X_u})$ is strongly F -regular for all $u \in U$. But $\Delta \geq \Delta'$ and so the result follows for Δ' as well. \square

Remark 4.23. In the case of a normal X and in the non-relative case, we know that $\tau(X; \Delta) = \tau(\omega_X, K_X + \Delta)$. Furthermore, we then know that

$$\mathrm{Tr}^e(F_*^e \tau(\omega_X, K_X + \Delta)) = \tau(\omega_X, \frac{1}{p^e}(K_X + \Delta)) = \tau(X, \frac{1}{p^e}(K_X + \Delta) - K_X).$$

Reversing this process gives us a nice means to compute $\tau(X, \Delta)$ when the index of $K_X + \Delta$ is divisible by p . It would be natural try to prove a relative version of this, which may yield a suitable definition of relative test ideals for $K_X + \Delta$ of any index. We won't attempt this here.

5. RELATIVE TEST SUBMODULES, F -RATIONALITY AND F -INJECTIVITY

Our goal in this (somewhat shorter) section is to explore relative test submodules and non- F -injective modules. Throughout this section, we assume that $f : X \rightarrow V$ is a Cohen-Macaulay morphism. This provides us with base change for relative canonical sheaves $\omega_{X/V}$ [[Con00](#)].

We first define relative non- F -injective modules $\sigma_n(X/V, \omega_{X/V})$ and relative test submodules $\tau_n(X/V, \omega_{X/V})$. Recall the trace map, [Lemma 2.18](#)

$$\Phi_{X/V,n} : \omega_{X/V}^{1/p^n} = \omega_{X^n/V^n} \rightarrow \omega_{X_{V^n}/V^n} \cong \omega_{X/V} \otimes_A A^{1/p^{ne}}.$$

Here the final isomorphism follows from [[Con00](#), Theorem 3.6.1] since f is a Cohen-Macaulay morphism. We will form σ and τ relative to these maps, instead of relative to the map φ discussed previously. One key point to remember is that by [Lemma 2.18](#), the map $\Phi_{X/V,n}$ is compatible with arbitrary base change (since in this section, f is a Cohen-Macaulay morphism). We also observe that the composition of trace maps

$$(5.0.1) \quad \omega_{X^m/V^m} \xrightarrow{\Phi_{X/V,m-n}^{1/p^{m-n}}} \omega_{X_{V^m}^n/V^m} \xrightarrow{\Phi_{X/V,n} \otimes A^{1/p^n} A^{1/p^m}} \omega_{X/V} \otimes_A A^{1/p^m} \cong \omega_{X_{V^m}/V^m}.$$

can be identified with $\Phi_{X/V,m}$.

5.1. The definition and basic properties of $\sigma_n(X/V, \omega_{X/V})$. For each integer $n > 0$, define $\mathfrak{c}_n := \mathrm{Im}(\Phi_{X/V,n}) \subseteq \omega_{X_{V^n}/V^n} \cong \omega_{X/V} \otimes_A A^{1/p^n}$. Furthermore, for each $m \geq n$, using the factorization in (5.0.1) it is easy to see that

$$(5.1.2) \quad \mathfrak{c}_m \subseteq \mathrm{Im}(\mathfrak{c}_n \otimes_{A^{1/p^n}} A^{1/p^m} \rightarrow \omega_{R_{A^{1/p^m}}})$$

just as we observed in [Section 3](#).

Definition 5.2. With notation as above, we define the n th relative non- F -injective submodule, denoted $\sigma_n(X/V, \omega_{X/V})$ to be $\mathfrak{c}_n \subseteq \omega_{X_{V^n}/V^n}$.

As before, we will prove a stabilization statement for \mathfrak{c}_n and \mathfrak{c}_m , in particular that the containment (5.1.2) is an equality for all $m > n \gg 0$. By base changing with $k(V^\infty)$, the perfection of the residue field of the generic point of V , we can again find an integer $n_0 > 0$ and an open set $U \subseteq V$ such that

$$\mathfrak{c}_{n_0+1}|_{f^{-1}U} = \text{Im} \left(\mathfrak{c}_{n_0} \otimes_{A^{1/p^{n_0}}} A^{1/p^{n_0+1}} \rightarrow \omega_{R_{A^{1/p^{n_0+1}}}} \right)|_{f^{-1}U}.$$

Again observe that for U inside the regular locus of V , we can identify the image above with $(\mathfrak{c}_{n_0} \otimes_{A^{1/p^{n_0}}} A^{1/p^{n_0+1}})|_U$ by the flatness of A^{1/p^m} over A^{1/p^n} . We then obtain:

Proposition 5.3. *Fix notation as above. For every integer $n \geq n_0$, there exists a nonempty open subset $U_n \subseteq V$ of the base scheme V satisfying the following condition. If one sets $X_n = f^{-1}(U_n)$, then we have that for every $m \geq n$*

$$(5.3.1) \quad \sigma_m(X/V, \omega_{X/V})|_{X_n} = \text{Im} \left(\sigma_n(X/V, \omega_{R/A}) \cdot \mathcal{O}_{X_n} \otimes_{A^{1/p^n}} A^{1/p^m} \rightarrow \omega_{R_{A^{1/p^m}}} \right).$$

Furthermore we may assume that $U_{n_0} \subseteq U_{n_0+1} \subseteq \dots \subseteq U_n \subseteq U_{n+1} \subseteq \dots$ form an ascending chain of open sets.

Proof. The proof is identical to that of Proposition 3.3 and so we omit it. \square

We now point out that relative non- F -injective modules behave well with respect to base change. Recall that if $g : T \rightarrow V$ is a map, then $q_n : X_{T^n} \rightarrow X_{V^n}$ is the induced map.

Proposition 5.4 (Base change for $\sigma_n(X/V, \omega_{X/V})$). *Suppose that $g : T \rightarrow V$ is a map from an excellent scheme with a dualizing complex, then using the notation of Section 2.16*

$$\text{Im} \left((q_n)^* \sigma_n(X/V, \omega_{X/V}) \rightarrow \omega_{X_{T^n}/T^n} \right) = \sigma_n(X_T/T, \omega_{X_T/T}).$$

Furthermore, if $U = U_n$ satisfies condition (5.3.1) from Proposition 5.3, then $W = g^{-1}(U) \subseteq T$ satisfies the same condition for $\sigma_n(X_T/T, \omega_{X_T/T})$.

Proof. The first statement is an immediate consequence of base change relative canonical sheaves and trace Lemma 2.18. For the second, if $\Phi_{X/V,n} \otimes_{A^{1/p^n}} A^{1/p^{n+1}}$ and $\Phi_{X/V,n+1}$ have the same image in $\omega_{X_{V^n}/V^n}$, then its easy to see that the base changed maps also have the same images. \square

Recall for any F -finite scheme Y with canonical module ω_Y , then $\sigma(\omega_Y)$ is equal to $\Im(F_*^e \omega_Y \xrightarrow{\text{Tr}^e} \omega_Y)$ for any $e \gg 0$.

Corollary 5.5 (Restriction theorem for $\sigma_n(X/V, \omega_{X/V})$). *With notation as above, there exists an integer $N > 0$ such that for every perfect point $s \in V$, we have*

$$\sigma(\omega_{X_s}) = \text{Im} \left(\sigma_n(X/V, \omega_{X/V}) \otimes_{A^{1/p^{ne}}} k(s)^{1/p^{ne}} \rightarrow \omega_{X_s/s} \right).$$

for all $n > N$.

Proof. Taking $g : s \rightarrow V$ in the previous theorem and so obtain that the image is equal to $\sigma_n(X_s/s, \omega_{X_s/s})$. Furthermore, since $k(s)$ is perfect, we have containments $\dots \supseteq \sigma_n(X_s/s, \omega_{X_s/s}) \supseteq \sigma_{n+1}(X_s/s, \omega_{X_s/s}) \supseteq \dots$ with a descending intersection that coincides with $\sigma(\omega_{X_s})$. Furthermore, by Proposition 5.4, over a dense open set $U \subseteq V$ and some $N > 0$ we have $\sigma_n(X_s/s, \omega_{X_s/s}) = \sigma_{n+1}(X_s/s, \omega_{X_s/s})$ and hence $\sigma_n(X_s/s, \omega_{X_s/s}) = \sigma(\omega_{X_s})$ by the construction of $\sigma(\omega_{X_s})$ for all $n \geq N_0$. Let $V_1 = V \setminus U$, base change with V_1 and obtain the result over a dense open subset U_1 of V_1 (for some N_1). By Noetherian induction, this process terminates. \square

5.6. The definition and basic properties of $\tau_n(X/V, \omega_{X/V})$. The goal of this section is to develop the basics of a relative theory of test submodules. We fix the notation of the previous sections and additionally assume that X is geometrically normal over V which we now also assume is regular. We let $J = J_{X/V} \subseteq R$ be the Jacobian ideal sheaf of X over V . We observe that the formation of $J_{X/V}$ commutes with base change in the following sense: for any $T \rightarrow V$, we have $J_{X/V} \cdot \mathcal{O}_{X_T} = J_{X_T/T}$. To see this, just note that $J_{X/V}$ can be defined as a Fitting ideal of $\Omega_{X/V}$ ([HS06, Discussion 4.4.7]) and that the formation of $\Omega_{X/V}$ ([Eis95, Proposition 16.4]) and Fitting ideals ([HS06, Discussion 4.4.7]) commutes with arbitrary base change. Note that J is nonzero at any generic point of X since X is geometrically reduced. Furthermore, on every perfect fiber, the Jacobian ideal is contained in the (big) test ideal [Hoc07, Theorem on Page 213].

We have the images

$$\begin{aligned} \mathfrak{d}_1 &:= \text{Im} \left(J \cdot \omega_{R_{A^{1/p}/A^{1/p}}} \rightarrow \omega_{R_{A^{1/p}/A^{1/p}}} + \Phi_{X/V,1}((J \cdot \omega_R)^{1/p}) \subseteq \omega_{R_{A^{1/p}}}, \right. \\ \mathfrak{d}_2 &:= \sum_{i=0}^2 \text{Im} \left((\Phi_{X/V,i}(J \cdot \omega_{R/A})^{1/p^i}) \otimes_{A^{1/p^i}} A^{1/p^2} \rightarrow \omega_{R_{A^{1/p^2}}} \right) \subseteq \omega_{R_{A^{1/p^2}}}, \\ &\dots \\ \mathfrak{d}_n &:= \sum_{i=0}^n \text{Im} \left((\Phi_{X/V,i}(J \cdot \omega_{R/A})^{1/p^i}) \otimes_{A^{1/p^i}} A^{1/p^n} \rightarrow \omega_{R_{A^{1/p^n}}} \right) \subseteq \omega_{R_{A^{1/p^n}}}, \\ &\dots \end{aligned}$$

Notice that $\mathfrak{d}_{1,2} := \text{Im} \left(\mathfrak{d}_1 \otimes_{A^{1/p}} A^{1/p^2} \rightarrow \omega_{R_{A^{1/p^2}}} \right) \subseteq \mathfrak{d}_2$ and more generally for $j > i$ that $\mathfrak{d}_{i,j} := \text{Im} \left(\mathfrak{d}_i \otimes_{A^{1/p^i}} A^{1/p^j} \rightarrow \omega_{R_{A^{1/p^j}}} \right) \subseteq \mathfrak{d}_j$.

By the same argument as in Section 3, we know that there exists an open set $U \subseteq V$ with $W = f^{-1}(U)$ such that $\text{Im} \left(\mathfrak{d}_t \otimes_{A^{1/p^{te}}} A^{1/p^{(t+1)e}} \rightarrow \omega_{R_{A^{1/p^{t+1}}}} \right)|_W = \mathfrak{d}_{t+1}|_W$.

Lemma 5.7. *With notation as above, $\text{Im} \left(\mathfrak{d}_n \otimes_{A^{1/p^{ne}}} A^{1/p^{(n+1)e}} \rightarrow \omega_{R_{A^{1/p^{(n+1)e}}}} \right)|_W = \mathfrak{d}_{n+1}|_W$ for all $n > t$.*

Proof. The proof is the same as the proof of Lemma 4.2 and so we omit it. \square

Definition 5.8 (Relative test submodules). With notation as above (in particular, $\omega_{X/V}$ is still compatible with base change), we define the n th iterated relative test submodule to be \mathfrak{d}_n and denote it by $\tau_n(X/V, \omega_{X/V})$.

We now discuss base change for relative test ideals.

Proposition 5.9 (Base change for $\tau_n(X/V, \omega_{X/V})$). *Suppose that $g : T \rightarrow V$ is a map from a excellent scheme with a dualizing complex, then using the notation of Section 2.16*

$$\text{Im} \left((q_n)^* \tau_n(X/V, \omega_{X/V}) \rightarrow \omega_{X_{T^n}/T^n} \right) = \tau_n(X_T/T, \omega_{X_T/T}).$$

Furthermore, if $U = U_n$ satisfies condition from Lemma 5.7, then $Y = g^{-1}(U) \subseteq T$ satisfies the same condition for $\tau_n(X_T/T, \omega_{X_T/T})$.

Proof. It is just as before since $\omega_{X/V}$ and J are compatible with arbitrary base change. \square

Corollary 5.10 (Restriction theorem for $\tau_n(X/V, \omega_{X/V})$). *With notation as above, there exists an integer $N > 0$ such that for every perfect point $s \in V$, we have*

$$\text{Im} \left(\tau_n(X/V, \omega_{X/V}) \otimes_{A^{1/p^{ne}}} k(s)^{1/p^{ne}} \rightarrow \omega_{X_s/s} \right) = \tau(\omega_{X_s}).$$

for all $n > N$.

Proof. Taking $g : s \rightarrow V$ in [Proposition 5.9](#) and so obtain that the image is equal to $\tau_n(X_s/s, \omega_{X_s/s})$. Furthermore, since $k(s)$ is perfect, we can make identifications so as to have containments $\dots \subseteq \tau_n(X_s/s, \omega_{X_s/s}) \subseteq \tau_{n+1}(X_s/s, \omega_{X_s/s}) \subseteq \dots$ with an ascending union that coincides with $\tau(\omega_{X_s})$. Furthermore, by [Proposition 5.4](#), over a dense open set $U \subseteq V$ and some $N > 0$ we have $\tau_n(X_s/s, \omega_{X_s/s}) = \tau_{n+1}(X_s/s, \omega_{X_s/s})$ for all $n \geq N_0$ and hence $\tau_n(X_s/s, \omega_{X_s/s}) = \tau(\omega_{X_s})$ by the computation of [\(4.0.1\)](#). Let $V_1 = V \setminus U$, base change with V_1 and obtain the result over a dense open subset U_1 of V_1 (for some N_1). By Noetherian induction, this process must terminate. \square

Remark 5.11 (Relative versus absolute $\tau(\omega)$). It would be natural to relate the relative $\sigma(\omega)$ and $\tau(\omega)$ with the absolute $\sigma(\omega)$ and $\tau(\omega)$. While the authors believe that this is possible along the lines of [Section 3.11](#) or [Section 4.10](#), we won't work out the details here.

5.12. Applications to families of F -injective and F -rational singularities. In this section, we obtain new proofs of results of M. Hashimoto [[Has01](#)], deformation of F -injectivity and F -rationality in proper flat families. Note Hashimoto proved the local version of these results (and in the F -rationality case, over a variety). However, the local results generalize to the non-local case via straightforward computations. We begin with a definition essentially first made by Hashimoto.

Definition 5.13 (cf. [[Has01](#), [Has10](#)]). We say that X/V (which is still assumed to be relatively Cohen-Macaulay) is *relatively F -injective* if for some $n > 0$ we have $\sigma_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$. Likewise, X/V is *relatively F -rational* if for some $n > 0$ we have $\tau_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$.

Remark 5.14. If V is the spectrum of a perfect field k , it is easy to see that $f : X \rightarrow V$ is relatively F -injective (respectively relatively F -rational) if and only if X is F -injective (respectively F -rational) in the usual sense [[ST12](#), Section 8] via an identification of $k \cong k^{1/p}$. Note we are implicitly assuming that X is Cohen-Macaulay in this case.

Lemma 5.15. (cf. [[Has01](#), Proposition 5.5]) *With notation as above, if $\sigma_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$ for some $n > 0$, then $\sigma_m(X/V, \omega_{X/V}) = \omega_{X_{V^m}/V^m}$ for all $m > 0$ divisible by n . Furthermore, if V is regular, then the result holds for all $m \geq n$. Additionally, if $\tau_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$ for some $n > 0$, then $\tau_m(X/V, \omega_{X/V}) = \omega_{X_{V^m}/V^m}$ for all $m \gg 0$.*

Proof. We begin with σ . We notice that

$$\sigma_{2n}(X/V, \omega_{X/V}) = \text{Im} \left(\text{Im} (\omega_{X^n/V^n} \rightarrow \omega_{X_V^n/V^n}) \otimes_{A^{1/p^n}} A^{1/p^{2n}} \rightarrow \omega_{X_{V^{2n}}/V^{2n}} \right)$$

and our hypothesis $\sigma_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$ implies that the inner map is surjective. But then the outer map is surjective too by right exactness of tensor and so $\sigma_{2n}(X/V, \omega_{X/V}) = \omega_{X_{V^{2n}}/V^{2n}}$. The general case repeats this process and so follows similarly.

Now we assume that V is regular for the second statement about σ_n . We fix an $n > 0$ such that $\sigma_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n}$. Then for all $m \leq n$, we have

$$\sigma_m(X/V, \omega_{X/V}) \otimes_{A^{1/p^m}} A^{1/p^n} \supseteq \sigma_n(X/V, \omega_{X/V}) = \omega_{X_{V^n}/V^n} \cong \omega_{X_{V^m}/V^m} \otimes_{A^{1/p^m}} A^{1/p^n}$$

and so $\sigma_m(X/V, \omega_{X/V}) \otimes_{A^{1/p^m}} A^{1/p^n} = \omega_{X_{V^m}/V^m} \otimes_{A^{1/p^m}} A^{1/p^n}$. On the other hand, if we know $\sigma_m(X/V, \omega_{X/V}) \subsetneq \omega_{X_{V^m}/V^m}$ then the previous equality is impossible since A^{1/p^n} faithfully flat over A [[Kun69](#)] (since V is regular).

Handling τ is easy. Again use that $\omega_{X/V}$ is compatible with base change, since

$$\begin{aligned} \omega_{X_V^m/V^m} &= \text{Im} (\omega_{X/V} \otimes_{A^{1/p^n}} A^{1/p^m} \rightarrow \omega_{X_V^m/V^m}) \\ &= \text{Im} (\tau_n(X/V, \omega_{X/V}) \otimes_{A^{1/p^n}} A^{1/p^m} \rightarrow \omega_{X_V^m/V^m}) \\ &= \mathfrak{d}_{n,m} \\ &\subseteq \mathfrak{d}_m \\ &= \tau_m(X/V, \omega_{X/V}) \\ &\subseteq \omega_{X_V^m/V^m} \end{aligned}$$

□

Theorem 5.16. (Deformation of F -rationality and F -injectivity, cf. [Has01, Theorem 5.8, Remark 6.7]) Suppose that $f : X \rightarrow V$ is a proper flat finite type equidimensional reduced Cohen-Macaulay morphism to an excellent integral scheme V with a dualizing complex. Suppose that for some point $s \in V$, the fiber X_s/s is relatively F -injective (respectively, F -rational). Then there exists an open neighborhood $U \subseteq V$ containing s such that $X_u \rightarrow u$ is relatively F -injective (respectively, F -rational) for all $u \in U$.

Proof. We first show that $\sigma_n(\omega_{X/V}) = \omega_{X_{V^{ne}}/V^{ne}}$ at each point of the fiber X_s . Indeed, let $z \in X$ be a point on X_s and let I denote the ideal sheaf of X_s . We observe that for some n , the natural map $\sigma_n(\omega_{X/V})/(I \cdot \sigma_n(\omega_{X/V})) \rightarrow \omega_{X/V}/(I \cdot \omega_{X/V}) = \omega_{X_s/V_s}$ is surjective. This is preserved after localizing at z , and so since $I \subseteq \mathfrak{m}_z$, we see that the generators of the stalk $(\sigma_n(\omega_{X/V}))_z$ generate $(\omega_{X/V})_z$ by Nakayama's lemma. Hence $(\sigma_n(\omega_{X/V}))_z = (\omega_{X/V})_z$. Since this holds at every point $z \in X_s \subseteq X$, it holds in a neighborhood of X_s .

As before, let Z denote the locus where $\sigma_n(\omega_{X/V}) \neq \omega_{X_{V^{ne}}/V^{ne}}$. This is closed and since f is proper, its image $f(Z)$ is closed too. But $f(Z)$ is then a closed set not containing s . The result follows for F -injectivity. The proof for F -rationality is the same. □

6. GLOBAL APPLICATIONS

The purpose of this section is to develop a global theory of the previous sections for a projective family $f : X \rightarrow V$. In the last few years, there has been a new push to use Frobenius and the trace map to replace the Kodaira vanishing theorem. In this section we extend some of these ideas to families. We study how the canonical linear subsystems $S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s) \subseteq H^0(X_s, M_s)$, introduced in [Sch11a], behave as we vary $s \in V$. Furthermore, as mentioned in the introduction we also obtain some global generation and semi-positivity statements.

6.1. Basic definitions. We use the following setup throughout Section 6.

Notation 6.1. In the situation of Notation 2.1, assume also that $f : X \rightarrow V$ is projective and additionally that V is regular (which implies that $F_V^e : V^e \rightarrow V$ is flat). Furthermore, fix a line bundle M on X . Sometimes we also assume the following (in which case we write Notation 6.1^{*}):

(^{*}): there is an integer $N \geq 0$, such that for every integer $m \geq N$,

$$\sigma_m(X/V, \varphi) = \sigma_N(X/V, \varphi) \otimes_{A^{1/p^{Ne}}} A^{1/p^{me}}.$$

In this situation we denote $\sigma_N(X/V, \varphi)$ by σ_N . We notice that this condition (^{*}) always holds over a dense open set of the base by Proposition 3.3.

Remark 6.2. Note that since X (resp. V) is topologically isomorphic to $X_{V^{ne}}^{ne}$ (resp. V^{me}) for every $m \geq n$, f_* can be identified with g_* , where g is any of the induced morphisms $X_{V^{ne}}^{ne} \rightarrow V^{me}$. Hence, we use only f_* for all purposes, even when g_* for one of the above maps $g : X_{V^{ne}}^{ne} \rightarrow V^{me}$ would be more natural. The downside of this notation is that it does not show if a sheaf has a $A^{1/p^{me}}$ structure, and consequently its pushforward by f a $\mathcal{O}_{V^{me}}$ structure. We decided to still use it, because it simplifies greatly the notations.

Summarizing: when reading the following arguments it is important to trace through the space $X_{V^{ne}}^{ne}$ on which the adequate sheaves live. Then f_* of these sheaves will live on V^{me} .

Notation 6.3. When dealing with sheaves G on V^{ne} , we will frequently pull them back to V^{me} for $m \geq n$. When doing this, instead of writing $\mathcal{O}_{V^{me}} \otimes_{\mathcal{O}_{V^{ne}}} G$ or $\mathcal{O}_V^{1/p^{me}} \otimes_{\mathcal{O}_V^{1/p^{ne}}} G$, we will write simply $V^{me} \times_{V^{ne}} G$ or $G_{V^{me}}$. We trust this abuse of notation will cause no confusion, as it helps compactify the notation substantially.

Our approach to understanding how the canonical linear systems of [Sch11a] behave in families is to define a relative version of them, and then show certain base-change properties. These objects will be relative versions of σ for global sections, and they play the same role for σ_n that T^0 and S^0 plays for τ and σ in [BST11] and [Sch11a] respectively.

Definition 6.4. In the situation of [Notation 6.1](#), define

$$S_{\varphi^n}^0 f_*(M) := \operatorname{im} \left(f_* \left(\left(L^{\frac{p^{ne}-1}{p^{ne}}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) \xrightarrow{f_*(\varphi^n \otimes_R \operatorname{id}_M)} f_* \left(A^{\frac{1}{p^{ne}}} \otimes_A M \right) \right).$$

Note that $S_{\varphi^n}^0 f_*(M)$ is a sheaf on V^{ne} and it is a subsheaf of $(f_*(M))_{V^{ne}}$ by flat base-change. In case (X, Δ) is a pair we define $S_{\Delta, ne}^0 f_*(M) := S_{\varphi^n}^0 f_*(M)$ (assuming $(p^{ne} - 1)(K_X + \Delta)$ is Cartier and φ is the corresponding map). If $\Delta = 0$, then we write $S_{ne}^0 f_*(M)$ for $S_{\Delta, ne}^0 f_*(M)$.

Since the image of $\varphi^n \otimes_R \operatorname{id}_M$ in the above definition is $\sigma_n(X/V, \varphi) \otimes_R M$ the following proposition is immediate.

Proposition 6.5. *In the situation of [Notation 6.1](#),*

$$S_{\varphi^n}^0 f_*(M) \subseteq f_*(\sigma_n(X/V, \varphi) \otimes_R M).$$

Our first goal is to show that the images in [Definition 6.4](#) descend (up to appropriate base change by Frobenius). Compare with the containments $\mathfrak{a}_{i,n} \supseteq \mathfrak{a}_{i+1,n} \supseteq \cdots \supseteq \mathfrak{a}_{n,n}$ of [Section 3](#).

Proposition 6.6. *For all integers $m \geq n \geq 0$,*

$$(6.6.1) \quad V^{me} \times_{V^{ne}} S_{\varphi^n}^0 f_*(M) \supseteq S_{\varphi^m}^0 f_*(M)$$

as subsheaves of $V^{me} \times_V f_(M)$.*

Remark 6.7. To be precise the left and right hand side of (6.6.1) are subsheaves of

$$V^{me} \times_{V^{ne}} f_*(A^{\frac{1}{p^{ne}}} \otimes_A M) \quad \text{and of} \quad f_*(A^{\frac{1}{p^{me}}} \otimes_A M),$$

respectively. However, both $V^{me} \times_{V^{ne}} f_*(A^{\frac{1}{p^{ne}}} \otimes_A M)$ and $f_*(A^{\frac{1}{p^{me}}} \otimes_A M)$ are canonically isomorphic to $V^{me} \times_V f_*(M)$ via flat base-change (since V is regular).

Proof of Proposition 6.6. The following commutative diagram shows that $f_*(\varphi^m \otimes_R \text{id}_M)$ factors through $V^{me} \times_{V^{ne}} f_*(\varphi^n \otimes_R \text{id}_M)$.

$$\begin{array}{ccc}
V^{me} \times_{V^{ne}} f_* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) & \xrightarrow{V^{me} \times_{V^{ne}} f_*(\varphi^n \otimes_R \text{id}_M)} & V^{me} \times_{V^{ne}} f_* \left(A^{\frac{1}{p^{ne}}} \otimes_A M \right) \\
\parallel \begin{array}{c} \text{flat base-change [Har77,} \\ \text{Proposition III.9.3] of } f_* \\ \text{by } V^{me} \rightarrow V^{ne} \end{array} & & \parallel \begin{array}{c} \text{flat base-} \\ \text{change} \end{array} \\
f_* \left(A^{\frac{1}{p^{me}}} \otimes_{A^{\frac{1}{p^{ne}}}} \left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) & \xrightarrow{f_* \left(A^{\frac{1}{p^{me}}} \otimes_{A^{\frac{1}{p^{ne}}}} (\varphi^n \otimes_R \text{id}_M) \right)} & f_* \left(A^{\frac{1}{p^{me}}} \otimes_A M \right) \\
\parallel & & \parallel \\
f_* \left(\left(R_{A^{\frac{1}{p^{(m-n)e}}}} \otimes_R L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) & \xrightarrow{\hspace{10em}} & f_* \left(A^{\frac{1}{p^{me}}} \otimes_A M \right) \\
\uparrow f_* \left(\left(\varphi^{m-n} \otimes_R \text{id}_L \right)^{\frac{1}{p^{ne}}} \otimes_R \text{id}_M \right) & & \uparrow f_*(\varphi^m \otimes_R \text{id}_M) \\
f_* \left(\left(\left(L^{\frac{p^{(m-n)e}-1}}{p^{e-1}} \right)^{\frac{1}{p^{(m-n)e}}} \otimes_R L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) & & \\
\parallel \text{projection formula} & & \\
f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \right)^{\frac{1}{p^{me}}} \otimes_R M \right) & \xrightarrow{\hspace{10em}} & f_* \left(A^{\frac{1}{p^{me}}} \otimes_A M \right)
\end{array}$$

Hence, the statement of the proposition holds by the following computation.

$$\begin{aligned}
S_{\varphi^m}^0 f_*(M) &= \text{im} (f_*(\varphi^m \otimes_R \text{id}_M)) \\
&\subseteq \text{im} (V^{me} \times_{V^{ne}} f_*(\varphi^n \otimes_R \text{id}_M)) \\
&= \underbrace{V^{me} \times_{V^{ne}} \text{im} (f_*(\varphi^n \otimes_R \text{id}_M))}_{V^{ne} \rightarrow V^{me} \text{ is flat}} \\
&= V^{me} \times_{V^{ne}} S_{\varphi^n}^0 f_*(M)
\end{aligned}$$

□

6.2. Auxiliary definition and stabilization. We would now like to obtain a global result similar to Proposition 3.3. In particular, we'd like the containments of Proposition 6.6 to be equality over a dense open subset of the base V . There is a complicating factor however, while we can still find an open set U of V such that

$$\left(V^{(n+1)e} \times_{V^{ne}} S_{\varphi^n}^0 f_*(M) \right) \Big|_U = \left(S_{\varphi^{n+1}}^0 f_*(M) \right) \Big|_U$$

we do not see how to use this to show that we have the $n+1$ to $n+2$ equality without additional assumptions. The issue is that in the proof of Proposition 3.3, twisting by line bundles is exact. For $S_{\varphi^n}^0$ however we also push forward. Therefore, in order to obtain our stabilization over a dense open set of the base we need additional positivity assumptions on M and L .

Furthermore, we need an auxiliary version of S_φ^0 which involves the $\sigma_N := \sigma_N(X/V, \varphi)$ from [Notation 6.1*](#). To do this, first observe that since f is flat, the tensor product $L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \sigma_N$ is naturally identified with a subsheaf of $L^{\frac{p^{me}-1}{p^e-1}} \otimes_R R_{A^{1/p^{Ne}}} \cong L^{\frac{p^{me}-1}{p^e-1}} \otimes_A A^{1/p^{Ne}}$. In order to motivate this auxiliary definition, we make the following observation:

Lemma 6.8. *In the situation of [Notation 6.1*](#), we have that the image of the natural map*

$$\alpha_{N+m} : \left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \sigma_N \right)^{\frac{1}{p^{me}}} \hookrightarrow \left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R R_{A^{1/p^{Ne}}} \right)^{\frac{1}{p^{me}}} \xrightarrow{\varphi^m \otimes_{A^{1/p^{me}}} A^{1/p^{(N+m)e}}} R_{A^{1/p^{(N+m)e}}}$$

is equal to $\sigma_N \otimes_{A^{1/p^{Ne}}} A^{1/p^{(N+m)e}} \cong \sigma_{m+N}(X/V, \varphi)$.

Proof. The first term is the image of the p^{me} th root of $L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \varphi^N$. Therefore the image of the composition also equals $\sigma_{m+N}(X/V, \varphi)$ by our construction of φ^j in [Section 2.13](#). \square

This stabilization suggests it is reasonable to make the following definition.

Definition 6.9. In the situation of [Notation 6.1*](#),

$$S_{\varphi^n, \sigma_N}^0 f_*(M) := \text{im} \left(f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \sigma_N \right)^{\frac{1}{p^{me}}} \otimes_R M \right) \xrightarrow{f_*(\alpha_{N+m} \otimes_R M)} f_* \left(A^{\frac{1}{p^{(n+N)e}}} \otimes_A M \right) \right)$$

where α_{N+m} is as in [Lemma 6.8](#).

Consider the following proposition relating the various $S^0 f_*$ objects so far described.

Proposition 6.10. *In the situation of [Notation 6.1*](#), for every integer $m \geq 0$,*

$$V^{(N+m)e} \times_{V^{me}} S_{\varphi^m}^0 f_*(M) \supseteq S_{\varphi^m, \sigma_N}^0 f_*(M) \supseteq S_{\varphi^{m+N}}^0 f_*(M)$$

(Here all sheaves are regarded as subsheaves of $V^{(N+m)e} \times_V f_*(M)$ via flat base-change.)

Proof. This follows directly from the definitions. The first containment is trivial, and the second follows from a factorization similar to the one from [Proposition 6.6](#). \square

Proposition 6.11. *In the situation of [Notation 6.1*](#),*

(a) *for all integers $m \geq n \geq 0$,*

$$V^{(N+m)e} \times_{V^{(N+n)e}} S_{\varphi^n, \sigma_N}^0 f_*(M) \supseteq S_{\varphi^m, \sigma_N}^0 f_*(M),$$

as subsheaves of $V^{(N+m)e} \times_V f_(M)$, and*

(b) *if furthermore $L \otimes M^{p^e-1}$ is f -ample, then there is an integer $n > 0$, such that for every integer $m \geq n$, the above inclusion is equality.*

(c) *if furthermore $M = Q^l \otimes P$, where Q is f -ample, then there is an integer $l_0 > 0$, such that for every integer $l \geq l_0$, $m \geq n \geq 0$ and nef line bundle P the above inclusion is equality.*

Proof. Define the coherent sheaf B_φ on $X_{V^{(N+1)e}}$ as the kernel of the top horizontal map in the following commutative diagram.

$$(6.11.1) \quad \begin{array}{ccc} B_\varphi \hookrightarrow & (L \otimes_R \sigma_N)^{\frac{1}{p^e}} & \xrightarrow{\varphi \otimes} A^{\frac{1}{p^{(N+1)e}}} \otimes_{A^{\frac{1}{p^{Ne}}}} \sigma_N \\ & \uparrow (\text{id}_L \otimes_R \varphi^N)^{\frac{1}{p^e}} & \uparrow \varphi^{N+1} \\ & \left(L \otimes \left(L^{\frac{p^{Ne}-1}{p^e-1}} \right)^{\frac{1}{p^{Ne}}} \right)^{\frac{1}{p^e}} & \cong \left(L^{\frac{p^{(N+1)e}-1}{p^e-1}} \right)^{\frac{1}{p^{(N+1)e}}} \end{array}$$

Note here we used that the image of φ^{N+1} is $A^{\frac{1}{p^{(N+1)e}}} \otimes_{A^{\frac{1}{p^{Ne}}}} \sigma_N$ by the assumptions made in [Notation 6.1](#)^{*}. Also, the horizontal arrow is surjective by either [Lemma 6.8](#) or from the diagram (which is how we proved [Lemma 6.8](#)). Then one can apply $f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R _ \right)^{\frac{1}{p^{me}}} \otimes_R M \right)$ to the top row of [\(6.11.1\)](#), which is shown in the following commutative diagram. We have also included important isomorphisms to the different terms of the exact sequence.

$$\begin{array}{ccc}
f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\varphi \right)^{\frac{1}{p^{me}}} \otimes_R M \right) & & \\
\downarrow & & \\
f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R (L \otimes \sigma_N)^{\frac{1}{p^e}} \right)^{\frac{1}{p^{me}}} \otimes_R M \right) & \cong & f_* \left(\left(L^{\frac{p^{(m+1)e}-1}{p^e-1}} \otimes_R \sigma_N \right)^{\frac{1}{p^{(m+1)e}}} \otimes_R M \right) \\
\downarrow \nu & & \downarrow \\
f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \left(\sigma_N \otimes_{A^{\frac{1}{p^{Ne}}}} A^{\frac{1}{p^{(N+1)e}}} \right) \right)^{\frac{1}{p^{me}}} \otimes_R M \right) & & \\
\downarrow & \swarrow \cong & \\
R^1 f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\varphi \right)^{\frac{1}{p^{me}}} \otimes_R M \right) & & \\
\downarrow & \swarrow & \\
f_* \left(\left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \sigma_N \right)^{\frac{1}{p^{me}}} \otimes_{A^{\frac{1}{p^{(N+m)e}}}} A^{\frac{1}{p^{(N+m+1)e}}} \right) \otimes_R M \right) & & \\
\parallel \text{ by flat base change} & & \\
V^{(N+m+1)e} \times_{V^{(N+m)e}} f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R \sigma_N \right)^{\frac{1}{p^{me}}} \otimes_R M \right) & & \\
\downarrow & & \downarrow \\
V^{(N+m+1)e} \times_{V^{(N+m)e}} S_{\varphi^m, \sigma_N}^0 f_*(M) & \longleftarrow & S_{\varphi^{m+1}, \sigma_N}^0 f_*(M)
\end{array}$$

Both $V^{(N+m+1)e} \times_{V^{(N+m)e}} S_{\varphi^m, \sigma_N}^0 f_*(M)$ and $S_{\varphi^{m+1}, \sigma_N}^0 f_*(M)$ can be regarded as subsheaves of $V^{(N+m+1)e} \times_V f_*(M)$ via flat base change. Furthermore, the bottom horizontal arrow becomes a map of subsheaves, i.e., an injection, this way. This shows point [\(a\)](#).

To prove point [\(b\)](#), we are supposed to prove that whenever $L \otimes M^{p^e-1}$ is f -ample, this arrow is also surjective for $m \gg 0$. This would follow if the third vertical arrow (from above), labeled ν , was surjective. Therefore, it is sufficient to show that for $m \gg 0$,

$$\begin{aligned}
0 = R^1 f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\varphi \right)^{\frac{1}{p^{me}}} \otimes_R M \right) &= R^1 f_* \left(\left(L^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\varphi \otimes_R M^{p^{me}} \right)^{\frac{1}{p^{me}}} \right) \\
&= R^1 f_* \left(\left((L \otimes M^{p^e-1})^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\varphi \otimes_R M \right)^{\frac{1}{p^{me}}} \right).
\end{aligned}$$

Note that since applying $(_)^{\frac{1}{p^{me}}}$ does not change the sheaf of abelian groups structure, this is equivalent to showing

$$\begin{aligned} 0 &= R^1 f_* \left((L \otimes M^{p^e-1})^{\frac{p^{me}-1}{p^e-1}} \otimes_R B_\varphi \otimes_R M \right) \\ &= R^1 f_* \left(\left(L \otimes M^{p^e-1} \otimes_A A^{\frac{1}{p^{N+1}}} \right)^{\frac{p^{me}-1}{p^e-1}} \otimes_{R_{A^{1/p^{(N+1)e}}}} (B_\varphi \otimes_R M) \right). \end{aligned}$$

Furthermore, $L \otimes M^{p^e-1} \otimes_A A^{\frac{1}{p^{N+1}}}$ is a relatively ample line bundle by the assumption of point (b) and $B_\varphi \otimes_R M$ is a coherent sheaf on $X_{V^{(N+1)e}}$. Therefore, relative Serre vanishing concludes our proof.

Point (c) follows immediately from the above argument. Indeed, if $M = Q^l \otimes P$, then by relative Fujita vanishing [Kee03, Theorem 1.5] there is an integer $l_0 > 0$, such that the above vanishing holds for every $m \geq 0$, $l \geq l_0$ and nef line bundle P . \square

Corollary 6.12. *In the situation of Notation 6.1*, if $L \otimes M^{p^e-1}$ is f -ample, then there is an integer $n > 0$, such that for every integer $m \geq n$*

$$V^{me} \times_{V^{ne}} S_{\varphi^n}^0 f_*(M) = S_{\varphi^m}^0 f_*(M) \quad \left(= S_{\varphi^{m-N}, \sigma_N}^0 f_*(M) \right)$$

as subsheaves of $V^{me} \times_V f_*(M)$. Furthermore in the situation of point (c) of Proposition 6.11 where $M = Q^l \otimes P = (\text{ample})^l \otimes (\text{nef})$ with $l \gg 0$, we can pick $n = N$.

Proof. By Proposition 6.10,

$$(6.12.1) \quad V^{(m+2N)e} \times_{V^{(m+N)e}} S_{\varphi^m, \sigma_N}^0 f_*(M) \supseteq V^{(m+2N)e} \times_{V^{(m+N)e}} S_{\varphi^{m+N}}^0 f_*(M) \supseteq S_{\varphi^{m+N}, \sigma_N}^0 f_*(M).$$

Furthermore, by Proposition 6.11.b, we know the statement holds when $S_{\varphi^m}^0 f_*(M)$ is replaced by $S_{\varphi^m, \sigma_N}^0 f_*(M)$. Hence the inclusion of the right end of (6.12.1) to the left end is an equality for $m \gg 0$. However, then all inclusions in (6.12.1) are equalities. In particular from the first equality of (6.12.1), using that $V^{(m+2N)e} \rightarrow V^{(m+N)e}$ is faithfully flat, we obtain that $S_{\varphi^m, \sigma_N}^0 f_*(M) = S_{\varphi^{m+N}}^0 f_*(M)$ for $m \gg 0$. Using Proposition 6.11.b once more concludes our proof of the main statement. The final statement is similar. \square

We can now obtain our promised analog of Proposition 3.3.

Theorem 6.13. *In the situation of Notation 6.1, if $L \otimes M^{p^e-1}$ is ample, there exists an integer n_0 and a dense open set $U \subseteq V$ such that for all $m \geq n \geq n_0$ we have that*

$$\left(V^{me} \times_{V^{ne}} S_{\varphi^n}^0 f_*(M) \right) \Big|_U = \left(S_{\varphi^m}^0 f_*(M) \right) \Big|_U.$$

Proof. By shrinking V and applying Proposition 3.3 we can reduce to the case of Notation 6.1*. The result follows directly from Corollary 6.12. \square

Corollary 6.14. *In the situation of Notation 6.1, there is a natural inclusion*

$$S_{\varphi^n}^0 f_*(M) \subseteq f_*(\sigma_n(X/V, \varphi) \otimes_R M).$$

as subsheaves of $V^{ne} \times_V f_*(M)$.

Further, in the situation of Notation 6.1*, if $M = Q^l \otimes P$ where Q is f -ample, then there is an integer $l_0 > 0$, such that for every integer $l \geq l_0$, $n \geq N$ and f -nef line bundle P , the above inclusion is equality.

Proof. Consider the surjection.

$$\xi : \left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \xrightarrow{\varphi^n} \sigma_n(X/V, \varphi)$$

Then $S_{\varphi^n}^0 f_*(M) = \text{im } f_*(\xi \otimes_R \text{id}_M)$, which is a subsheaf of $f_*(\sigma_n(X/V, \varphi) \otimes_R M)$.

We prove the addendum by induction on n . If $n = N$,

$$S_{\varphi^N}^0 f_*(M) = \underbrace{S_{\varphi^0, \sigma_N}^0 f_*(M)}_{\text{by Corollary 6.12}} = \underbrace{f_*(\sigma_N \otimes_R M)}_{\text{by Definition 6.9}}.$$

Let us assume then that $n > N$. By flat base change and the assumption of [Notation 6.1](#)*

$$f_*(\sigma_n(X/V) \otimes_R M) = f_*(\sigma_N(X/V) \otimes_R M) \times_{V^{Ne}} V^{ne}.$$

Hence, using the inductual hypothesis, it is enough to see that for every $n \geq N$,

$$S_{\varphi^n}^0 f_*(M) = S_{\varphi^{n-1}}^0 f_*(M).$$

However, this follows from point (c) of [Proposition 6.11](#) and [Corollary 6.12](#) □

6.3. Base-change. We now prove base change for $S_{\varphi^n}^0 f_*$.

Notation 6.15. In the situation of [Notation 6.1](#), choose

- T a regular integral excellent scheme with a dualizing complex.
- $T \rightarrow V$ a morphism.

We indicate base-change via this morphism by T in subscript. Define the following sheaves on X_T .

- (a) $B := (f_T)^{-1} \mathcal{O}_T$
- (b) $Q := \mathcal{O}_{X_T}$

Denote by $(p_1)^{\frac{1}{p^i}}$ the natural projection morphism $X^i \times_{V^i} T^i \rightarrow X^i$.

Proposition 6.16. *In the situation of [Notation 6.15](#), if $L \otimes M^{p^e-1}$ is f -ample then for $n \gg 0$ the natural base change morphism*

$$f_* \left(A^{\frac{1}{p^{ne}}} \otimes_A M \right) \times_{V^{ne}} T^{ne} \rightarrow (f_T)_* \left(B^{\frac{1}{p^{ne}}} \otimes_B M_T \right)$$

induces a surjective morphism on subsheaves

$$(6.16.1) \quad S_{\varphi^n}^0 f_*(M) \times_{V^{ne}} T^{ne} \rightarrow S_{\varphi_T^n}^0 (f_T)_*(M_T).$$

Furthermore,

- (a) *the lower bound on the n for which the above statement holds depends only on f and M . In particular, it is independent of T .*
- (b) *If $M = Q^l \otimes P$ for some f -ample line bundle Q , nef line bundle P and integer $l > 0$, then there is a uniform lower bound on n independent of l and P .*
- (c) *Even if $L \otimes M^{p^e-1}$ is not assumed to be f -ample, (6.16.1) is an isomorphism if $T \rightarrow V$ is flat.*
- (d) *There is a dense open set $U \subseteq V$, such that if the image of $T \rightarrow V$ is contained in U , then (6.16.1) is an isomorphism.*

Proof of Proposition 6.16. First, note that

$$\left((p_1)^{\frac{1}{p^{ne}}} \right)^* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \otimes_R M^{p^{ne}} \right)^{\frac{1}{p^{ne}}} \right) \cong \left((p_1)^{\frac{1}{p^{ne}}} \right)^* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) \cong \left(L_T^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_Q M_T.$$

Hence, there is a natural base-change morphism below in (6.16.2). We will show that it is an isomorphism for $n \gg 0$. Furthermore, this n can be chosen independently of T .

$$(6.16.2) \quad \left(f_* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) \right) \times_{V^{ne}} T^{ne} \rightarrow (f_T)_* \left(\left(L_T^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_Q M_T \right)$$

Indeed by cohomology and base-change [Sta, Lemma 25.20.1] it is enough to show that for every integer $i > 0$,

$$R^i f_* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) = 0.$$

Since $V^{ne} \rightarrow V$ is affine, this is equivalent to showing that for $i > 0$,

$$0 = R^i (f_{V^{ne}})_* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) = R^i (f_{V^{ne}})_* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \otimes_R M^{p^{ne}} \right)^{\frac{1}{p^{ne}}} \right),$$

which is further equivalent to showing that for $i > 0$,

$$(6.16.3) \quad 0 = R^i f_* \left(L^{\frac{p^{ne}-1}{p^e-1}} \otimes_R M^{p^{ne}} \right).$$

However, the last vanishing holds for $n \gg 0$ by relative Serre vanishing, independently of T . Hence the base change homomorphism of (6.16.2) is isomorphism indeed for $n \gg 0$, which can be chosen independently of T . In particular, for $n \gg 0$ there is a commutative base change diagram as follows, which implies the statement of the proposition together with addendum (a).

$$(6.16.4) \quad \begin{array}{ccc} f_* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) \times_{V^{ne}} T^{ne} & \longrightarrow & f_* \left(A^{\frac{1}{p^{ne}}} \otimes_A M \right) \times_{V^{ne}} T^{ne} \\ \parallel & & \downarrow \\ (f_T)_* \left(\left(L_T^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_Q M_T \right) & \longrightarrow & (f_T)_* \left(B^{\frac{1}{p^{ne}}} \otimes_B M_T \right) \end{array}$$

For addendum (c) note that if $T \rightarrow V$ is flat, then both vertical morphisms in the above diagram are isomorphisms even if $L \otimes M^{p^e-1}$ is not assumed to be f -ample. Furthermore, images via flat pullbacks are pullbacks of images, which concludes the proof of (c).

For addendum (b), note that if $M = Q^l \otimes P$ for an f -ample line bundle Q and a nef line bundle P , then (6.16.3) holds for every $l > 0$ uniformly, by relative Fujita vanishing [Kee03].

For addendum (d) note that by [Har77, Theorem 12.8 and Corollary 12.9] there is a dense open set $U \subseteq V$ (i.e., the open set where $h^0(X_s, M_s)$ is constant), such that if T maps into U then the right vertical arrow in (6.16.4) is an isomorphism. In particular then the homomorphism (6.16.1) is injective, because it is a homomorphism between subsheaves of the two sheaves involved in the above vertical map. Therefore, by the already proven surjectivity (6.16.1) is an isomorphism. This concludes addendum (d). \square

Remark 6.17. In particular if in Notation 6.15, $T = s$ is a perfect point of V (that is, $T = \text{Spec}(K)$ for some perfect field K), then $S_{\varphi_T^n}^0(f_T)_*(M_T)$ can be interpreted as follows. Since, the natural map $K \rightarrow (K)^{1/p^i}$ is an isomorphism, the natural projection $X_{T^i} \rightarrow X_T$ is an isomorphism. Furthermore $(f_T)_*(M_{T^i})$ can be identified with $(f_T)_*(M_T) = H^0(X_T, M_T)$ via $K \rightarrow K^{1/p^i}$. Thus

$S_{\varphi_T^n}^0(f_T)_*(M_T)$ is identified with

$$(6.17.1) \quad S_{s,m}^0 := \operatorname{im} \left(H^0 \left(X_s, \left(L_s^{\frac{p^{me}-1}{p^{me}}} \right)^{\frac{1}{p^{me}}} \otimes M_s \right) \rightarrow H^0(X_s, M_s) \right),$$

In particular, for $m \gg 0$, (6.17.1) can be identified with $S^0(X_s, \sigma(X_s, \varphi_s) \otimes M_s)$.

6.4. Uniform stabilization.

Theorem 6.18. *In the situation of Notation 6.1, if $L \otimes M^{p^e-1}$ is f -ample, then there is an integer $n > 0$, such that for all integers $m \geq n$ and perfect points $s \in V$,*

$$\operatorname{im} \left(S_{\varphi^m}^0 f_*(M) \otimes_{\mathcal{O}_{V^{me}}} k(s)^{\frac{1}{p^{me}}} \rightarrow H^0(X_s, M_s) \right) = S_{s,m}^0 = S^0(X_s, \sigma(X_s, \psi_s) \otimes M_s).$$

(Note ψ_s is defined in Section 2.23 and $S_{s,m}^0$ in Remark 6.17.)

Proof. We show the statement by induction on the dimension of V . There is nothing to prove if $\dim V = 0$. Hence, we may proceed to the inductive step and assume that $\dim V > 0$. By Proposition 6.16, there is an $n > 0$ such that for every perfect point $s \in V$ and every $m \geq n$

$$\operatorname{im} \left(S_{\varphi^m}^0 f_*(M) \otimes_{\mathcal{O}_{V^{me}}} k(s)^{\frac{1}{p^{me}}} \rightarrow H^0(X_s, M_s) \right) = S_{s,m}^0.$$

According to Lemma 2.17, by possibly increasing n , we may find a non-empty, dense open set U such that for all $m \geq n$,

$$\sigma_m(X/V, \varphi)|_U = \sigma_n(X/V, \varphi) \times_{V^{ne}} V^{me}|_U.$$

Therefore, applying Corollary 6.12, yields (also by possibly increasing n) that for all $m \geq n$,

$$V^{me} \times_{V^{ne}} S_{\varphi^m}^0 f_*(M)|_U = S_{\varphi^m}^0 f_*(M)|_U.$$

It follows then that for every perfect point $s \in U$,

$$\operatorname{im} \left(S_{\varphi^m}^0 f_*(M) \otimes_{\mathcal{O}_{V^{me}}} k(s)^{\frac{1}{p^{me}}} \rightarrow H^0(X_s, M) \right)$$

is the same for all $m \geq n$. Therefore, $S_{s,m}^0$ is stabilized for all perfect $s \in U$ for values of $m \geq n$. Hence it is equal to $S^0(X_s, \sigma(X_s, \psi_s) \otimes M_s)$. This shows the statement of the proposition for all perfect $s \in U$. We fix the $n_0 = n$ used above for future use.

On the other hand, consider the reduced scheme $V_1 = V \setminus U$. Write $V_1 = \coprod V_{1,i}$ as a disjoint union of regular locally closed integral subschemes. Each $V_{1,i}$ has dimension smaller than V . In particular, there exists an n_i such that the statement holds for $f_i : X_{V_i} \rightarrow V_i$. Letting n be the maximum of n_0 and the n_i we obtain our result since every perfect point of X factors through a point of U or of one of the V_i . \square

Corollary 6.19. *In the situation of Notation 6.1, if $L \otimes M^{p^e-1}$ is f -ample, then there is a dense open set $U \subseteq V$, such that for every perfect point $s \in U$,*

$$S_{\varphi^m}^0 f_*(M) \otimes_{\mathcal{O}_{V^{me}}} k(s)^{\frac{1}{p^{me}}} = S^0(X_s, \sigma(X_s, \psi_s) \otimes M_s).$$

In particular, $\dim_{k(s)} S^0(X_s, \sigma(X_s, \psi_s) \otimes M_s)$ is constant on an open set, and the rank of $S_{\varphi^m}^0 f_(m)$ equals this dimension.*

Proof. The main statement follows immediately from Theorem 6.18 and point (d) of Proposition 6.16. The corollary follows from basic properties of coherent sheaves [Har77, Exercise II.5.8] \square

If there is a point $s_0 \in V$, such that in X_{s_0} we have $H^0 = S^0$, then we can say more than the above corollary: it turns out that it can be assumed that $s_0 \in U$, and further that U has other useful properties. This is proved in the following theorem.

Theorem 6.20. *In the situation of [Notation 6.1](#), if $L \otimes M^{p^e-1}$ is f -ample and there is a perfect point $s_0 \in V$ such that $H^0(X_{s_0}, M_{s_0}) = S^0(X_{s_0}, \sigma(X_{s_0}, \psi_{s_0}) \otimes M_{s_0})$, then there is an open neighborhood U of s_0 , such that*

- (a) $f_*M|_U$ is locally free and compatible with base change and
- (b) $H^0(X_v, M_v) = S^0(X_v, \sigma(X_v, \psi_v) \otimes M_v)$ for every perfect point $v \in U$.

In particular, $\dim S^0(X_v, \sigma(X_v, \psi_v) \otimes M_v)$ is constant for $v \in U$.

Proof. By [Theorem 6.18](#), there is an n , such that for every $v \in V$ and integer $m \geq n$, the natural base-change homomorphism induces a surjection as follows

$$(6.20.1) \quad \begin{array}{ccc} S^0_{\varphi^m} f_*(M) \otimes_{\mathcal{O}_{V^{me}}} k(v)^{\frac{1}{p^{me}}} & \longrightarrow & S^0(X_v, \sigma(X_v, \psi_v) \otimes M_v) \\ \downarrow & & \downarrow \\ f_* \left(M \otimes_A A^{\frac{1}{p^{me}}} \right) \otimes_{\mathcal{O}_{V^{me}}} k(v)^{\frac{1}{p^{me}}} & \longrightarrow & H^0(X_v, M_v). \end{array}$$

Furthermore, the right vertical arrow is surjective for $v = s_0$. It follows that so is the bottom horizontal arrow then. Using [\[Har77, Corollary III.12.11\]](#) concludes [\(a\)](#).

To prove point [\(b\)](#), let us consider [\(6.20.1\)](#) for $v = s_0$ again. Now we know that the bottom horizontal arrow is isomorphism. However, then the left vertical arrow is also surjective. By Nakayama's-lemma, there is an equality $(S^0_{\varphi^n} f_*(M))_v = (f_*(M))_v$ of stalks, which means that by possibly restricting U , for every $v \in U$, the left vertical arrow of [\(6.20.1\)](#) is surjective. However then, since the bottom horizontal arrow is isomorphism for every $v \in U$, point [\(b\)](#) follows. \square

The surjection in [Proposition 6.16](#) is not an isomorphism in general by the following two examples. In the first example the global geometry of the smooth fibers causes the anomaly, while in the second, the degeneration of F -pure singularities to non F -pure singularities is the main culprit. However, we first show a lemma.

Lemma 6.21. *Let Y be a smooth curve over k and G an effective divisor on Y . Define \mathcal{B}_Y^1 as the cokernel of $\mathcal{O}_Y \rightarrow F_*\mathcal{O}_Y$. Then*

$$S^0(Y, \sigma(X, 0) \otimes \omega_Y(G)) = H^0(Y, \omega_Y(G)) \Leftrightarrow H^0(Y, \mathcal{B}_Y^1(-p^n G)) = 0 \quad (\forall n > 0).$$

Proof. Consider the following exact sequence.

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow F_*\mathcal{O}_Y \longrightarrow \mathcal{B}_Y^1 \longrightarrow 0$$

Twist this sequence by $-G$, apply cohomology and use that since G is effective, either

- (a) $G > 0$ and hence $H^0(Y, (F_*\mathcal{O}_Y)(-G)) \cong H^0(\mathcal{O}_Y(-F^*G)) = 0$, or
- (b) $G = 0$ and then $H^0(Y, \mathcal{O}_Y) \rightarrow H^0(Y, F_*\mathcal{O}_Y)$ is an isomorphism.

In either case, we have another exact sequence:

$$0 \longrightarrow H^0(Y, \mathcal{B}_Y^1(-G)) \longrightarrow H^1(Y, \mathcal{O}_Y(-G)) \longrightarrow H^1(Y, F_*\mathcal{O}_Y(-F^*G)),$$

where the last map is the Serre dual to $H^0(Y, (F_*\omega_Y)(G)) \rightarrow H^0(Y, \omega_Y(G))$. Furthermore, observe that the map $H^0(Y, (F_*^{n+1}\omega_Y)(G)) \rightarrow H^0(Y, (F_*^n\omega_Y)(G))$ is harmlessly identified with the map $H^0(Y, (F_*\omega_Y)(p^n G)) \rightarrow H^0(Y, (\omega_Y)(p^n G))$. Therefore, combining the previous statements, we see that $H^0(Y, (F_*^{n+1}\omega_Y)(G)) \rightarrow H^0(Y, (F_*^n\omega_Y)(G))$ surjects if and only if $H^0(Y, \mathcal{B}_Y^1(-p^n G)) = 0$. \square

Example 6.22. We choose two smooth projective curves C and D of genus 3 over an algebraically closed field k of prime characteristic with two ample line bundles N_C and N_D (of degree 1), such that $S^0(C, \omega_C \otimes N_C) \neq H^0(C, \omega_C \otimes N_C)$, but $S^0(D, \omega_D \otimes N_D) = H^0(D, \omega_D \otimes N_D)$. Since the relative Picard scheme over the moduli stack of smooth curves of genus g is smooth and irreducible, there is a (possibly reducible) curve connecting (C, N_C) and (D, N_D) in the above moduli space. However,

then by possibly replacing (C, N_C) and (D, N_D) we may find also an irreducible curve connecting them. Therefore, by passing to the normalization of this curve, we may assume that there is a family $f : X \rightarrow V$, a line bundle N on X and two points $c, d \in C$, such that if we set $(C, N_C) := (X_c, N|_{X_c})$ and $(D, N_D) := (X_d, N|_{X_d})$, then

- (a) V is a smooth curve,
- (b) X is a family of smooth curves of genus 3,
- (c) $\deg_{X/V} N = 1$,
- (d) $S^0(C, \omega_C \otimes N_C) \neq H^0(C, \omega_C \otimes N_C)$ and
- (e) $S^0(D, \omega_D \otimes N_D) = H^0(D, \omega_D \otimes N_D)$.

Furthermore, fix $e = 1$, $L := \omega_{X/V}^{1-p}$ and $\varphi : \omega_{X/V}^{\frac{1}{p}} \rightarrow \mathcal{O}_{X_{V^1}}$ the map with $D_\varphi = 0$ (see [Definition 2.8](#) for the definition of D_φ). Note that by [Lemma 2.24](#), $D_{\varphi_s} = 0$ as well. Set $M := \omega_{X/V} \otimes N$.

By assumption (c), for every $v \in V$ and $i > 0$, $H^i(X_v, N_v) = 0$. Therefore, f_*N is a vector bundle of rank 3, and its formation is compatible with arbitrary base-change. In particular, by [Theorem 6.20](#) and assumption (e), $S_\varphi^n f_*(\omega_{X/V} \otimes M) \subseteq f_*(\omega_{X/V} \otimes M)$ is a subsheaf that is isomorphic generically to $f_*(\omega_{X/V} \otimes M)$. Furthermore, by being a subsheaf of $f_*(\omega_{X/V} \otimes M)$ it is torsion free, and by V being a curve, it is locally free of rank 3. Therefore,

$$\dim_k S_\varphi^n f_*(\omega_{X/V} \otimes M) \otimes k(c) = 3$$

However,

$$\dim_k S^0(C, \omega_C \otimes M_C) < \dim_k H^0(C, \omega_C \otimes M_C) = 3$$

This shows that the base change morphism

$$S_{\varphi^n}^0 f_*(\omega_{X/V} \otimes M) \otimes k(c) \rightarrow S^0(C, \omega_C \otimes M_C)$$

cannot be isomorphism.

We are left to give the polarized curves (C, M_C) and (D, M_D) . For that consider the situation of [Lemma 6.21](#). By [\[Tan72, Lemma 12\]](#), $H^0(Y, \mathcal{B}^1(-p^n G)) = 0$ if $p^n G \geq n(Y)$ where $n(Y)$ is a numerical invariant of the curve, which is at most 1 for genus 3 curves [\[Tan72, Lemma 10\]](#). Hence $H^0(Y, \mathcal{B}^1(-p^n G)) = 0$ if $\deg G \geq 1$ and $n > 1$. In particular, in our special case (i.e., if Y is of genus 3 and $\deg G \geq 1$) [Lemma 6.21](#) states that

$$S^0(Y, \sigma(Y, 0) \otimes \omega_Y(G)) = H^0(Y, \omega_Y(G)) \Leftrightarrow H^0(Y, \mathcal{B}_Y^1(-G)) = 0.$$

Finding a curve where $H^0(Y, \mathcal{B}_Y^1(-G)) = 0$ is quite easy, because again using [\[Tan72, Lemma 12\]](#) yields that if $n(Y) = 1$, then there is a degree one divisor G for which $H^0(Y, \mathcal{B}_Y^1(-G)) \neq 0$, and then by passing to a linearly equivalent divisor we may also assume that G is effective. Further, [\[Tan72, Example 1\]](#) gives a curve $(x^3y + y^3z + z^3x = 0)$ for which $n(Y) = 1$. Therefore we have found C and N_C .

To find D and N_D assume further that $k = \overline{\mathbb{F}}_3$. Then by [\[Mil72, Kob75\]](#) a general genus three curve is ordinary, or equivalently $S^0(Y, \sigma(Y, 0) \otimes \omega_Y) = H^0(Y, \omega_Y)$. Therefore then $H^0(Y, \mathcal{B}_Y^1) = 0$ for such a Y . Take now an arbitrary effective degree 1 divisor G . Then $\mathcal{B}_Y^1(-G)$ embeds into \mathcal{B}_Y^1 and hence $H^0(Y, \mathcal{B}_Y^1(-G)) = 0$ as well. In particular, we can choose D to be a generic curve and N_D to be an arbitrary degree one line bundle on D .

Example 6.23. In the following example for an f -ample line bundle Q the surjection

$$(6.23.1) \quad S_{\varphi^n}^0 f_*(Q^l) \otimes_{\mathcal{O}_{V^{ne}}} \left(k(s)^{\frac{1}{p^{ne}}} \right) \rightarrow S^0(X_s, \sigma(X_s, \varphi_s) \otimes Q_s^l).$$

is not an isomorphism for any integer $n \geq n_0$ and $l > 0$. Therefore, the isomorphism in [Proposition 6.16](#) cannot be obtained by stronger positivity assumptions.

Let C be the projective cone over a supersingular elliptic curve, and let D be a non-singular cubic surface. Then these can be put into a family $f : X \rightarrow V$ as above. More precisely, we may find a family $f : X \rightarrow V$, and a point $c \in V$, such that $C := X_c$, and hence the following hold

- (a) V is a smooth curve,
- (b) X is a flat family of normal surfaces,
- (c) X_c is not sharply F -pure at on point $P \in X_c$ and
- (d) X_s is sharply F -pure for every $s \in V \setminus \{c\}$.

Let $L := \omega_{X/V}^{1-p}$ and $\varphi := \varphi_0$ as in the previous example. Then by [Corollary 3.10](#) and Nakayama lemma, for every $n \gg 0$, $\sigma_n(X/V, \varphi)|_{V \setminus \{c\}} \cong \mathcal{O}_{f^{-1}(V \setminus \{c\})}$. Choose now an arbitrary sufficiently f -ample line bundle M . By [Corollary 6.14](#), for every $n \gg 0$, $S_{\varphi^n}^0 f_*(M^l)|_{V \setminus \{c\}} \cong (f_*(M))_{V^{ne} \setminus \{c\}}$ for every integer $l > 0$. In particular the $\text{rk } S_{\varphi^n}^0 f_*(M^l) = \text{rk } f_*(M^l)$ for every $n \gg 0$ and $l > 0$ (where n does not depend on l). Consequently,

$$(6.23.2) \quad \dim S_{\varphi^n}^0 f_*(M^l) \otimes k(c) \geq \text{rk } f_*(M^l) = \underbrace{\dim_k H^0(X_c, M_c^l)}_{M \text{ is relatively ample enough}}$$

On the other hand for all $n \gg 0$ (independent of l)

$$(6.23.3) \quad S_{\varphi^n}^0 f_*(M^l) \otimes k(c) \rightarrow \underbrace{S_{\varphi_c^n}^0 (f_c)_*(M_c^l)}_{\text{by point (b) of Proposition 6.16}} = \underbrace{H^0(X_c, \sigma(X_c, 0) \otimes M_c^l)}_{\text{since } M_c \text{ is ample enough ([Pat12]) and Lemma 2.24}}$$

However, by the relative ampleness assumption, M_c is globally generated, hence $\dim_k H^0(X_c, \sigma(X_c, 0) \otimes M_c^l) < \dim_k H^0(X_c, M_c^l)$ (because $\sigma(X_c, 0) \subsetneq \mathcal{O}_{X_c}$). Therefore, (6.23.2) and (6.23.3) implies that indeed the surjection (6.23.1) is not an isomorphism for any $l > 0$ and any $n \gg 0$ (independently bounded of l).

Remark 6.24. The fundamental reason for the above example is that $\sigma_n(X/V, \varphi)$ is NOT flat in general.

The above two examples show that $\dim_k(S^0(X_v, \sigma(X_v, \varphi_v) \otimes M_v))$ is not upper semicontinuous. One might guess then it is lower semicontinuous. The next example shows that that is also not the case (this can also be deduced from [Har98, Example 5.5] and [Tan13, Theorem 8.3] as pointed out to us by Tanaka). So, $\dim_k S^0(X_v, \sigma(X_v, \varphi_v) \otimes M_v)$ is not semicontinuous in either direction.

Example 6.25. Take a flat family $f : X \rightarrow V$ of ordinary elliptic curves, and a line bundle M on X of relative degree 0, such that $M_{X_{v_0}} \cong \mathcal{O}_{X_{v_0}}$ for a special $v_0 \in V$, and $M_{X_v} \not\cong \mathcal{O}_{X_v}$ for generic $v \in V$. Then, by the ordinarity of the fibers, for all $v \in V$,

$$H^0(X_v, M_v) = S^0(X_v, \sigma(X_v, 0) \otimes M_v).$$

However, $\dim H^0(X_v, M_v) = 1$ for the special fiber and 0 for the generic one. So, $\dim S^0(X_v, \sigma(X_v, 0) \otimes M_v)$ is not lower semicontinuous in this example. One can also easily modify this example by taking an M with higher relative degree to obtain higher values of dimension for the generic fiber.

6.5. Global generation and semi-positivity. Suppose now V is a projective variety over a perfect field k . In this section, we explore global generation results if $L \otimes M^{p^e-1}$ is ample (instead of just relatively ample). In particular, $S_{\varphi^n}^0 f_*(M)$ is globally generated for all large n . This should not be surprising since $S_{\varphi^n}^0 f_*(M)$ lives on V^{ne} where ampleness is amplified.

Proposition 6.26. *In the situation of Notation 6.1, if V is projective over a perfect field k and $L \otimes M^{p^e-1}$ is ample, then $S_{\varphi^n}^0 f_*(M)$ is globally generated for every $n \gg 0$.*

Proof. Choose a globally generated ample divisor H on V and let $d := \dim V$. Since $S_{\varphi^n}^0 f_*(M)$ is defined as the image of

$$f_* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right),$$

it is enough to show that this sheaf is globally generated as an $\mathcal{O}_{V^{ne}}$ -module. By Mumford's criterion [Laz04a, Theorem 1.8.5], it is enough to show that for every $i > 0$ and for the divisor H_{ne} on V^{ne} identified with H via the isomorphism $V^{ne} \cong V$,

$$(6.26.1) \quad H^i \left(V^{ne}, \mathcal{O}_{V^{ne}}(-iH_{ne}) \otimes_{\mathcal{O}_{V^{ne}}} f_* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) \right) = 0.$$

By relative Serre vanishing for $i > 0$,

$$R^i f_* \left(\left(L^{\frac{p^{ne}-1}{p^e-1}} \right)^{\frac{1}{p^{ne}}} \otimes_R M \right) = 0.$$

In particular, then to prove (6.26.1) it is enough to show

$$H^i \left(X, L^{\frac{p^{ne}-1}{p^e-1}} \otimes M^{p^{ne}}(-if^*H) \right) = 0.$$

However this is equivalent to showing that,

$$H^i \left(X, \left(L \otimes M^{(p^e-1)} \right)^{\frac{p^{ne}-1}{p^e-1}} \otimes M(-if^*H) \right) = 0,$$

which holds by Serre-vanishing for $n \gg 0$ and $i > 0$. \square

We recall the following definition.

Definition 6.27 (Definition 2.11 of [Vie95]). Let \mathcal{F} be a sheaf on a normal, quasi-projective (over a perfect field k) variety V and $U \subseteq V$ a dense open set. Let U_{lf} be the open locus of V where $\mathcal{F}' := \mathcal{F}/(\text{torsion})$ is locally free. Then \mathcal{F} is *weakly positive over U* , if for a fixed (or equivalently every: [Vie95, Lemma 2.14.a]) ample line bundle \mathcal{H} for every $a > 0$ there is a $b > 0$ such that $S^{(ab)}(\mathcal{F}) \otimes \mathcal{H}^b$ is globally generated over $U \cap U_{\text{lf}}$ (here $S^{(ab)}(\mathcal{F})$ denotes the ab th symmetric reflexive power of \mathcal{F}). We say that \mathcal{F} is *weakly-positive* if it is weakly-positive over some dense open set.

Lemma 6.28. *If $g : Y \rightarrow Z$ is a finite morphism of normal varieties, quasi-projective over k , $U \subseteq Z$ a dense open set and \mathcal{F} a sheaf on Z , then \mathcal{F} is weakly positive over U if and only if $g^*\mathcal{F}$ is weakly positive over $g^{-1}(U)$.*

Proof. The only if direction is shown in [Vie95, Lemma 2.15.1]. For the other direction, according to Definition 6.27 by throwing out codimension two subset we may assume that \mathcal{F} is locally free and Y is flat over Z . In particular then $g^*\mathcal{F}$ is also locally free. Choose a very ample divisor \mathcal{H} on Z , such that

- (a) $\mathcal{H} \otimes (g_*\mathcal{O}_Y)$ is globally generated and
- (b) $g_*\mathcal{O}_Y \otimes \mathcal{H}$ is globally generated.

Fix then an $a > 0$. By the weak positivity of $g^*\mathcal{F}$, there is a $b > 0$, such that there is a homomorphism $\alpha : \mathcal{O}_Y^{\oplus N} \rightarrow S^{(ab)}(g^*\mathcal{F}) \otimes g^*\mathcal{H}^b$ surjective over g^*U . Consider then for every choice of $s \in \text{Hom}(g_*\mathcal{O}_Y, \mathcal{H})$ and $s' \in H^0(\mathcal{H}^{b-1})$ the following composition.

$$\begin{array}{ccc} (g_*\mathcal{O}_Y)^{\oplus N} & \xrightarrow{\gamma := g_*\alpha} & g_*(S^{(ab)}(g^*\mathcal{F}) \otimes g^*\mathcal{H}^b) \cong S^{(ab)}(\mathcal{F}) \otimes \mathcal{H}^b \otimes g_*\mathcal{O}_Y \\ & \searrow \beta_s & \downarrow \delta_{s,s'} := \text{id}_{S^{(ab)}(\mathcal{F}) \otimes \mathcal{H}^b} \otimes s \otimes s' \\ & & S^{(ab)}(\mathcal{F}) \otimes \mathcal{H}^{2b} \end{array}$$

Choose a point $P \in U$ and an element f of the fiber $S^{(ab)}(\mathcal{F}) \otimes k(P)$ over P . Then by assumption (b) for any preimage Q of P there is a section $t' \in \mathcal{O}_Y^{\oplus N}$, such that $\alpha(t')_Q = g^*f \times \text{"generator"}$. The section t' descends to a section $t := g_*(t') \in (g_*\mathcal{O}_Y)^{\oplus N}$, such that $\gamma(t)_P = f \times \text{"generator"} \times h$

for some $h \in (g_* \mathcal{O}_Y)_P$. However then by (a) and the very ampleness of \mathcal{H} for a suitable choice of s and s' , $\delta_{s,s'}(\gamma(t))_P = f \times \text{“generator”}$ is not zero at P . Therefore, for every point P in U and every element f in the fiber of $S^{(ab)}(\mathcal{F}) \otimes \mathcal{H}^{2b}$ at that point we find a section of $S^{(ab)}(\mathcal{F}) \otimes \mathcal{H}^{2b}$ whose image in the fiber is f up to multiplication by unit. This finishes our proof. \square

Corollary 6.29. *In the situation of Notation 6.1, let V be projective over a perfect field k , $L \otimes M^{p^e-1}$ ample and $n > 0$ be an integer such that there is an open set $U \subseteq V$ for which*

$$S_{\varphi^n}^0 f_*(M) \times_{U^{ne}} U^{me} = S_{\varphi^m}^0 f_*(M)|_{U^{me}},$$

for every integer $m \geq n$. Then $S_{\varphi^n}^0 f_(M)$ is weakly-positive. Note that an integer as above always exists by Corollary 6.12 and Proposition 5.3.*

Proof. By the assumption, there is an embedding

$$S_{\varphi^n}^0 f_*(M) \hookrightarrow S_{\varphi^n}^0 f_*(M) \times_{V^{ne}} V^{me},$$

which is generically isomorphism. Since the left sheaf is globally generated and hence weakly-positive, so is the right one. However then by Lemma 6.28, so is $S_{\varphi^n}^0 f_*(M)$. \square

Lemma 6.30. *If \mathcal{F} is a coherent sheaf on a normal variety V over a perfect field k , then \mathcal{F} is weakly-positive if and only if for every ample line bundle \mathcal{H} and every integer $p \nmid r > 0$ there is a finite morphism $\tau : T \rightarrow V$, such that $\tau^* \mathcal{H} \cong (\mathcal{H}')^r$ for some line bundle \mathcal{H}' and $\tau^* \mathcal{F} \otimes \mathcal{H}'$ is weakly-positive.*

Proof. The proof is identical to the (b) \Rightarrow (a) part of [Vie95, Lemma 2.15.1]. \square

Theorem 6.31. *In the situation of Notation 6.1, if V is projective and $L \otimes M^{p^e-1}$ is a nef and f -ample, then $S_{\varphi^n}^0 f_*(M)$ is weakly positive for $n \gg 0$.*

Proof. Choose an integer $n > 0$, as in Corollary 6.29. Fix an ample line bundle \mathcal{H} on V , an integer $p \nmid r > 0$ and a finite morphism $\tau : T \rightarrow V$, such that $\tau^* \mathcal{H} \cong (\mathcal{H}')^r$ for some line bundle \mathcal{H}' . Such a morphism exists by [Vie95, Lemma 2.1]. By Lemma 6.30, we are supposed to prove that $\left(\tau^{\frac{1}{p^{ne}}}\right)^* S_{\varphi^n}^0 f_*(M) \otimes_{\mathcal{O}_{V^{ne}}} (\mathcal{H}')^{\frac{1}{p^{ne}}}$ is weakly positive. By disregarding codimension two closed sets, we may assume that T is regular as well. Then by point (c) of Proposition 6.16, we see that n satisfies also the assumptions of Corollary 6.29 but for f and M replaced by f_T and $M_T \otimes f_T^* \mathcal{H}'_T$, respectively. In particular, since $M_T \otimes f_T^* \mathcal{H}'_T$ is ample,

$$S_{\varphi_T^n}^0 (f_T)_*(M_T \otimes f_T^* \mathcal{H}'_T) \cong \mathcal{H}'_T \otimes S_{\varphi_T^n}^0 (f_T)_*(M_T)$$

is weakly-positive over T^{ne} . \square

6.6. Relation to global canonical systems. There has been another subsheaf $S^0 f_*(\sigma(X, \Delta) \otimes M)$ of $f_*(M)$ introduced in [HX13, Definition 2.14] with a definition similar to that of $S_{\varphi^n}^0 f_*(M)$. In this section we show some of the similarities and differences between the two sheaves. The advantage of $S^0 f_*(\sigma(X, \Delta) \otimes M)$ over $S_{\varphi^n}^0 f_*(M)$ is that it lives on one V , there is no involvement of V^{ne} at all. On the other hand we show that contrary to $S_{\varphi^n}^0 f_*(M)$ it does not restrict even generically to $S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$.

Throughout the section we use the divisorial language since the presentation seems to be more straightforward this way, hence Δ satisfies the conditions of Remark 2.11. Recall that we defined the notion of pair in our setting in Definition 2.12. Whenever we say pair, we mean everything assumed there. In particular, all the assumptions on $f : X \rightarrow V$ from Notation 2.1: flatness, equidimensionality, etc.

First, we recall the definition of $S^0 f_*(M)$.

Definition 6.32. [HX13, Definition 2.14] Let Δ be an effective \mathbb{Q} -divisor, such that (X, Δ) is a pair and assume that $f : X \rightarrow V$ is projective. Let e be the smallest⁹ positive integer such that $(p^e - 1)(K_X + \Delta)$ is Cartier. Then given a line bundle M on X , we define

$$S^0 f_*(\sigma(X, \Delta) \otimes M) := \bigcap_{n>0} \text{im}(f_*(F_*^{ne} \mathcal{O}_X((1 - p^{ne})(K_X + \Delta)) \otimes M) \rightarrow f_*(M)).$$

We say that $S^0 f_*(\sigma(X, \Delta) \otimes M)$ stabilizes if the above intersection stabilizes. Note that in that case $S^0 f_*(\sigma(X, \Delta) \otimes M)$ is a coherent sheaf.

Recall that $S^0 f_*(\sigma(X, \Delta) \otimes M)$ stabilizes if $M - K_X - \Delta$ is relatively ample, see [HX13, Proposition 2.15] for a proof.

Proposition 6.33. *Let Δ be an effective \mathbb{Q} -divisor, such that (X, Δ) is a pair and assume that $f : X \rightarrow V$ is projective. Let M be a line bundle on X such that $S^0 f_*(\sigma(X, \Delta) \otimes M)$ stabilizes. Further assume that V is regular. Then for every $n \gg 0$*

$$S_{\Delta, ne}^0 f_*(M) \subseteq (S^0 f_*(\sigma(X, \Delta) \otimes M))_{V^{ne}}$$

as subsheaves of $(f_*(M))_{V^{ne}}$, where $S_{\Delta, ne}^0 f_*(M)$ is defined in Definition 6.4.

Proof. The statement is local over the base hence we can assume that V is affine, $K_V \sim 0$, and that $\mathcal{H}om_{\mathcal{O}_V}(F_*^i \mathcal{O}_V, \mathcal{O}_V) \cong F_*^i \mathcal{O}_V$ is a free \mathcal{O}_V -module for all $i \geq 0$. Note then also we can identify $K_{X/V}$ with K_X . We introduce the notation

$$L := \mathcal{O}_X((1 - p^{ne})(K_X + \Delta)).$$

We have the evaluation-at-1 map $F_*^{ne} \mathcal{O}_V \cong \mathcal{H}om_{\mathcal{O}_V}(F_*^{ne} \mathcal{O}_V, \mathcal{O}_V) \rightarrow \mathcal{O}_V$ which we identify with the trace of V based on our previous assumption $\omega_V \cong \mathcal{O}_V$. This map pulls back to X to provide us with a map which we also denote as $\text{Tr}_{F_V}^{ne} : \mathcal{O}_{X_{V^{ne}}} \cong \mathcal{O}_X \otimes f^* F_*^{ne} \mathcal{O}_V \rightarrow \mathcal{O}_X \otimes_{\mathcal{O}_X} f^* F_*^{ne} \mathcal{O}_V \cong \mathcal{O}_X$. Then there is a diagram as follows, which is commutative up to multiplication by a unit cf. [Sch09, Lemma 3.9].

$$\begin{array}{ccccc} F_*^{ne} L & \xrightarrow{\quad} & F_*^{ne} \mathcal{O}_X((1 - p^{ne})K_X) & \xrightarrow{\quad \text{Tr}_{F_X} \quad} & \mathcal{O}_X \\ \parallel & & \parallel & & \uparrow \text{Tr}_{F_V}^{ne} \\ F_*^{ne} \mathcal{O}_X((1 - p^{ne})(K_{X/V} + \Delta)) & \xrightarrow{\quad} & F_*^{ne} \mathcal{O}_X((1 - p^{ne})K_{X/V}) & \xrightarrow{\quad \text{Tr}_{F_{X/V}}^{ne} \quad} & \mathcal{O}_X \otimes_{\mathcal{O}_X} f^* F_*^{ne} \mathcal{O}_V \end{array}$$

Applying the functor $f_*(- \otimes_{\mathcal{O}_X} M)$ to the above diagram, we obtain the following.

$$\begin{array}{ccc} f_*((F_*^{ne} L) \otimes_{\mathcal{O}_X} M) & \xrightarrow{\quad} & f_*(M) \\ \parallel & & \uparrow \kappa \\ f_*((F_*^{ne} \mathcal{O}_X((1 - p^{ne})(K_{X/V} + \Delta))) \otimes_{\mathcal{O}_X} M) & \xrightarrow{\quad} & f_*(M \otimes_{\mathcal{O}_X} f^* F_*^{ne} \mathcal{O}_V) \end{array}$$

Note that for every $n \gg 0$ the image of the top horizontal row is $S^0 f_*(\sigma(X, \Delta) \otimes M)$ and for every $n > 0$ the image of the bottom horizontal row is $S_{\Delta, n}^0 f_*(M)$. Hence for every $n \gg 0$,

$$S^0 f_*(\sigma(X, \Delta) \otimes M) = \text{im}(S_{\Delta, n}^0 f_*(M) \rightarrow f_*(M)),$$

where the above map is induced by Tr_{F_V} .

Since we want containment for subsheaves of $(f_*(M))_{V^{ne}}$, we localize at a point of V . By setting $B := \Gamma(V, \mathcal{O}_V)$, $P := f_*(M)$ and $Q := F_{V,*}^{ne}(S_{\Delta, n}^0 f_*(M))$, we are in the following situation: if Q is a $B^{1/p^{ne}}$ submodule of $B^{1/p^{ne}} \otimes_B P$ for some B module P , we would like to show that $Q \subseteq \vartheta(Q) \otimes_B B^{1/p^{ne}}$ where $\vartheta \in \text{Hom}_B(B^{1/p^{ne}}, B)$ is the generator. This is simply Lemma 3.21. \square

⁹The intersection below is easily seen to be descending, hence if one chooses another e , the same object results.

Example 6.34. We provide an example where the two sheaves of [Proposition 6.33](#) have different ranks for every $n \gg 0$. Fix an algebraically closed field k of characteristic $p > 0$. Let $X := \mathbb{P}^1 \times \mathbb{A}^1$, $U = \mathbb{A}^1 \times \mathbb{A}^1 = \operatorname{Spec} k[y, x] \subseteq X$ and $V := \mathbb{A}^1 := \operatorname{Spec} k[x]$. Define the divisor $\Delta := \frac{1}{p-1}V(y^p - x)$ which is a priori a divisor in U , but it happens to have support which is a closed subvariety of X as well. Note that $V(y^p - x)$ is a subvariety of X isomorphic to \mathbb{A}^1 . In particular, (X, Δ) is sharply F -pure. Choose now a line bundle N on X which is relatively ample over V and let $M := N^l$ for some $l \gg 0$. Then by [\[HX13, Lemma 2.19\]](#) $S^0 f_*(\sigma(X, \Delta) \otimes M) = f_*(M)$. Therefore,

$$(6.34.1) \quad (S^0 f_*(\sigma(X, \Delta) \otimes M))_{V^{ne}} = (f_*(M))_{V^{ne}}$$

On the other hand by [Proposition 6.5](#), $S_{\Delta, n}^0 f_*(M) \subseteq f_*(\sigma_n(X/V, \Delta) \otimes M)$. By [Corollary 3.10](#) and [Lemma 2.24](#), for every $n \gg 0$ and $s \in V$, $\sigma_n(X/V, \Delta)|_{X_s} = \sigma(X_s, \Delta)$. However, $\Delta|_{X_s}$ is one point with multiplicity $\frac{p}{p-1}$, and hence not sharply F -pure. In particular, $\sigma(X_s, \Delta) \neq \mathcal{O}_{X_s}$ for every $s \in S$. Hence for every $n \gg 0$, $\sigma_n(X/V, \Delta)|_{X_s} \subsetneq \mathcal{O}_{X_s}$. It follows by the relative ampleness of M that $f_*(\sigma_n(X/V, \Delta) \otimes M)$ has smaller rank than $f_*(M)$. Then $S_{\Delta, n}^0 f_*(M)$ has also smaller rank than $(S^0 f_*(\sigma(X, \Delta) \otimes M))_{V^{ne}}$ by (6.34.1).

The above example has an important corollary, which follows immediately from (d) of [Proposition 6.16](#) and [Theorem 6.18](#).

Corollary 6.35. *In general $S^0 f_*(\sigma(X, \Delta) \otimes M) \otimes k(s)$ is not isomorphic to $S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$. In fact there are examples when the former has strictly bigger dimension than the latter for every closed point $s \in V$.*

Compare the following proposition with [Proposition 3.24](#).

Proposition 6.36. *Let Δ be an effective \mathbb{Q} -divisor, such that (X, Δ) is a pair and assume that $f : X \rightarrow V$ is projective. Let M be a line bundle on X and assume that V is regular. Then for every $n \geq m \geq 0$,*

$$S^0 f_{V^{ne},*}(\sigma(X_{V^{ne}}, \Delta_{V^{ne}}) \otimes M_{V^{ne}}) \subseteq (S^0 f_{V^{me},*}(\sigma(X_{V^{me}}, \Delta_{V^{me}}) \otimes M_{V^{me}}))_{V^{ne}}$$

as subsheaves of $(f_*(M))_{V^{ne}}$. Furthermore, if the above two sheaves stabilize then

$$\begin{aligned} & \kappa \left(F_{V,*}^{(n-m)e} S^0 f_{V^{ne},*}(\sigma(X_{V^{ne}}, \Delta_{V^{ne}}) \otimes f^* \mathcal{O}_V((1-p^{(n-m)e})K_{V^{ne}})) \otimes M_{V^{ne}} \right) \\ &= S^0 f_{V^{me},*}(\sigma(X_{V^{me}}, \Delta_{V^{me}}) \otimes M_{V^{me}}) \end{aligned}$$

where κ is induced by $\operatorname{Tr}_V^{(n-m)e}$.

Proof. Note that by replacing $f : X \rightarrow V$ by $f_{V^{me}} : X_{V^{me}} \rightarrow V^{me}$, we may assume that $m = 0$, and then by replacing e by $(n-m)e$ that $n = 1$. As in the previous proof, let us assume that V is affine, $K_V \sim 0$, and that $\mathcal{H}\operatorname{om}_{\mathcal{O}_V}(F_*^e \mathcal{O}_V, \mathcal{O}_V) \cong F_*^e \mathcal{O}_V$ is a free \mathcal{O}_V -module. Consider the following commutative diagram.

$$\begin{array}{ccccc} (X_{V^e})^e & \xrightarrow{\nu := (F_V^e)_{X^e}} & X^e & & \\ & \searrow F_{X_{V^e}}^e & \searrow F_X^e & & \\ & & X_{V^e} & \xrightarrow{\eta := (F_V^e)_X} & X \\ & & \downarrow f_{V^e} & & \downarrow f \\ & & V^e & \xrightarrow{F_V^e} & V \end{array}$$

We fix the notations

$$\mathcal{L}'_{\Delta} := \mathcal{O}_{X_{V^e}}((1-p^e)(K_{X_{V^e}} + \eta^* \Delta)), \quad \mathcal{L}_{\Delta} := \mathcal{O}_X((1-p^e)(K_X + \Delta)).$$

Then we have

$$(1 - p^e)K_{X_{V^e}} = (1 - p^e)(\eta^*K_{X/V} + f_{V^e}^*K_{V^e}), \text{ and } (1 - p^e)K_X = (1 - p^e)(K_{X/V} + f^*K_V).$$

Therefore,

- applying pullback of $\text{Tr}_{F_V^e}$ yields a homomorphism $\nu_*\mathcal{L}'_\Delta \rightarrow \mathcal{L}_\Delta$, which then induces a homomorphism $\eta_*F_{X_{V^e},*}^e\mathcal{L}'_\Delta \cong F_{X,*}^e\nu_*\mathcal{L}'_\Delta \rightarrow F_{X,*}^e\mathcal{L}_\Delta$,
- applying $\text{Tr}_{F_{X_{V^e}}^e}$ yields a homomorphism $F_{X_{V^e},*}^e\mathcal{L}'_\Delta \rightarrow \mathcal{O}_{X_{V^e}}$,
- applying $\text{Tr}_{F_X^e}$ yields a homomorphism $F_{X,*}^e\mathcal{L}_\Delta \rightarrow \mathcal{O}_X$,
- by the assumption $K_V \sim 0$, $\text{Tr}_{F_V^e}$ corresponds to a homomorphism $F_{V,*}^e\mathcal{O}_V \rightarrow \mathcal{O}_V$ generating $\text{Hom}_{\mathcal{O}_V}(F_{V,*}^e\mathcal{O}_V, \mathcal{O}_V)$ and
- the previous homomorphism also induces a pullback homomorphism $\eta_*\mathcal{O}_{X_{V^e}} \rightarrow \mathcal{O}_X$.

Furthermore, the above homomorphisms fit into the following diagram

$$\begin{array}{ccc} \eta_*F_{X_{V^e},*}^e\mathcal{L}'_\Delta & \longrightarrow & F_{X,*}^e\mathcal{L}_\Delta \\ & \searrow & \searrow \\ & \eta_*\mathcal{O}_{X_{V^e}} & \longrightarrow \mathcal{O}_X \end{array}$$

The diagram is a composition of trace maps (restricted to smaller domains determined by Δ), hence it is commutative. Applying now $f_*(- \otimes M)$ to the above diagram and using the projection formula we obtain the following commutative diagram

$$\begin{array}{ccc} f_*\eta_*(F_{X_{V^e},*}^e\mathcal{L}'_\Delta \otimes M_{V^e}) & \longrightarrow & f_*(F_{X,*}^e\mathcal{L}_\Delta \otimes M) \\ & \searrow & \searrow \\ & f_*\eta_*M_{V^e} & \longrightarrow f_*(M). \end{array}$$

Observing that $f_*\eta_* = F_{V,*}^ef_{V^e,*}$ yields another commutative diagram

$$\begin{array}{ccc} F_{V,*}^ef_{V^e,*}(F_{X_{V^e},*}^e\mathcal{L}'_\Delta \otimes M_{V^e}) & \longrightarrow & f_*(F_{X,*}^e\mathcal{L}_\Delta \otimes M) \\ & \searrow & \searrow \\ & F_{V,*}^ef_{V^e,*}M_{V^e} & \longrightarrow f_*(M). \end{array}$$

Note now that the top horizontal arrow is split, because it is induced from $\nu_*\mathcal{L}'_\Delta \rightarrow \mathcal{L}_\Delta$ which is split as well. Therefore, if we define $P := f_*(M)$ and

$$S := \text{im}(f_*(F_{X,*}^e\mathcal{L}_\Delta \otimes M) \rightarrow f_*(M)) \quad Q := \text{im}(f_{V^e,*}(F_{X_{V^e},*}^e\mathcal{L}'_\Delta \otimes M_{V^e}) \rightarrow f_{V^e,*}M_{V^e}),$$

then we have $Q \subseteq P \otimes_A A^{1/p^e}$, $S \subseteq P$ and $(\text{id}_P \otimes \text{Tr}_{F_V^e})(Q) = S$. Therefore, by Lemma 3.21, $Q \subseteq S \otimes_A A^{1/p^e}$. Since this holds for any n , the statement of the proposition follows. \square

Proposition 6.37. *Let Δ be an effective \mathbb{Q} -divisor, such that (X, Δ) is a pair and assume that $f : X \rightarrow V$ is projective and V is regular. Let M be a line bundle on X . Then for every $n \gg 0$,*

$$S^0 f_{V^{ne},*}(\sigma(X_{V^{ne}}, \Delta_{V^{ne}}) \otimes M_{V^{ne}}) \subseteq S_{\Delta, ne}^0 f_*(M)$$

as subsheaves of $(f_*(M))_{V^{ne}}$.

Proof. As before, let us assume that V is affine, $K_V \sim 0$, and that $\mathcal{H}\text{om}_{\mathcal{O}_V}(F_*^i\mathcal{O}_V, \mathcal{O}_V) \cong F_*^i\mathcal{O}_V$ is a free \mathcal{O}_V -module for all $i \geq 0$. By Lemma 3.13 and Lemma 3.16 there is a commutative diagram

$$\begin{array}{ccc} F_{X_{V^{ne}},*}^{ne}\mathcal{O}_{X_{V^{ne}}}((1 - p^{ne})(K_{X_{V^{ne}}} + \Delta_{V^{ne}})) & \longrightarrow & F_{X_{V^{ne}}/V^{ne},*}^{ne}\mathcal{O}_X((1 - p^{ne})(K_{X/V} + \Delta)) \\ & \searrow & \downarrow \\ & & \mathcal{O}_{X_{V^{ne}}}. \end{array}$$

Furthermore, the top horizontal arrow in the above diagram is split surjective. Therefore, after applying $f_{V^{ne},*}(- \otimes M_{V^{ne}})$ the image of the vertical map still agrees with the image of the diagonal map. The former is exactly $S^0_{\Delta,n} f_*(M)$, while the latter contains $S^0 f_{V^{ne},*}(\sigma(X, (\Delta)_{V^{ne}}) \otimes M)$, since it is one of the terms in the intersection defining $S^0 f_{V^{ne},*}(\sigma(X, (\Delta)_{V^{ne}}) \otimes M)$. \square

Remark 6.38. Assuming that $f : X \rightarrow V$ is projective and $M - K_X - \Delta$ is ample, it would be natural to ask whether $\text{Im}(S^0 f_{V^{ne},*}(\sigma(X_{V^{ne}}, \Delta_{V^{ne}}) \otimes M_{V^{ne}}) \rightarrow f_{V^{ne},*} M_s)$ equals $S^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$ for all perfect points $s \in V$, in analogy with [Theorem 3.23](#). We suspect this is true but will not try to prove it here.

Corollary 6.39. *Let $f : (X, \Delta) \rightarrow V$ be a projective morphism from a pair with V regular and projective over a perfect field k . Further, suppose that M is a line bundle on X such that $M - K_{X/V} - \Delta$ is nef and f -ample (here M denotes a Cartier divisor corresponding to M) and that $\text{rk } S^0 f_*(\sigma(X, \Delta) \otimes M)$ equals the general value of $H^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$. Then $S^0 f_*(\sigma(X, \Delta) \otimes M)$ is weakly positive. In particular, if V is a smooth curve then it is a nef vector bundle.*

Proof. From the assumption $\text{rk } S^0 f_*(\sigma(X, \Delta) \otimes M)$ equals the general value of $H^0(X_s, \sigma(X_s, \Delta_s) \otimes M_s)$ follows that the inclusion of [Proposition 6.33](#) is generically an isomorphism for every $n \gg 0$. However, since $S^0_{\varphi^n} f_*(M)$ is weakly-positive for every $n \gg 0$ by [Theorem 6.31](#), we obtain by the above generically isomorphic inclusion that $(S^0 f_*(\sigma(X, \Delta) \otimes M))_{V^{ne}}$ is also weakly positive. Then weak-positivity of $S^0 f_*(\sigma(X, \Delta) \otimes M)$ follows from [Lemma 6.28](#). \square

APPENDIX A. RELATIVE SERRE'S CONDITION

Here we collect the statements of [\[HK04\]](#) and other sources, that are important for the current paper for ease of reference. In some cases, we also state them in the greater generality that we need. All schemes are Noetherian, excellent and possess dualizing complexes and all maps are separable.

Definition A.1. Let $r > 0$ be an integer. A coherent sheaf \mathcal{E} on a Noetherian scheme X is S_r , if for every $x \in X$,

$$\text{depth}_{\mathcal{O}_{X,x}} \mathcal{E}_x \geq \min\{r, \dim_{\mathcal{O}_{X,x}} \mathcal{E}_x\}.$$

The sheaf \mathcal{E} is said to have *full support*, if $\text{Supp } \mathcal{E} = X$. It is *reflexive* if the natural map $\mathcal{E} \rightarrow \mathcal{E}^{**} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X), \mathcal{O}_X)$ is an isomorphism.

Definition A.2. If $f : X \rightarrow V$ is a morphism of Noetherian schemes and \mathcal{E} is a coherent sheaf on X flat over V , then \mathcal{E} is S_r over V if $\mathcal{E}|_{X_v}$ is S_r for every $v \in V$. That is, for every $x \in X$,

$$\text{depth}_{\mathcal{O}_{X_{f(x)}}} (\mathcal{E}|_{X_{f(x)}})_x \geq \min\left\{r, \dim_{\mathcal{O}_{X_{f(x)}}} (\mathcal{E}|_{X_{f(x)}})_x\right\}$$

We now recall two results from EGA about how depth and dimension behave in families.

Proposition A.3. [\[Gro67, Proposition 6.3.1\]](#) *Let $\varphi : A \rightarrow B$ be a local homomorphism of Noetherian local rings, k the residue field of A and M and N are finite A and B modules respectively. If N is a flat non-zero A -module, then*

$$\text{depth}_B(M \otimes_A N) = \text{depth}_A(M) + \text{depth}_{B \otimes_A k}(N \otimes_A k).$$

Proposition A.4. [\[Gro67, Corollaire 6.1.2\]](#) *Let $\varphi : A \rightarrow B$ be a local homomorphism of Noetherian local rings, k the residue field of A and M and N are non-zero finite A and B modules respectively. If N is a flat A -module, then*

$$\dim_B(M \otimes_A N) = \dim_A(M) + \dim_{B \otimes_A k}(N \otimes_A k).$$

From these we obtain the following corollary.

Corollary A.5. *Let $f : X \rightarrow V$ be a morphism of Noetherian schemes and \mathcal{E} a coherent sheaf on X flat over V , then \mathcal{E} is S_r over V if and only if for every $x \in X$,*

$$\min\{i \mid H_x^i(\mathcal{E}) \neq 0\} \geq \text{depth}_{\mathcal{O}_{V,v}} \mathcal{O}_{V,v} + \min\{r, \dim \mathcal{E}_x - \dim \mathcal{O}_{V,v}\} \\ (= \text{depth}_{\mathcal{O}_{V,v}} \mathcal{O}_{V,v} + \min\{r, \dim (\mathcal{E}|_{X_v})_x\})$$

where $v := f(x)$.

Proof. To say that $\mathcal{E}|_{X_v}$ has depth equal to t at x is equivalent to asserting that $\text{depth}_x \mathcal{E} = (\text{depth}_{\mathcal{O}_{V,v}} \mathcal{O}_{V,v}) + t$ by [Proposition A.3](#). The result follows. \square

The following vanishing allows us to extend sections over sets of relative codimension 2.

Proposition A.6. [[HK04](#), Proposition 3.3] *Let $f : X \rightarrow V$ be a morphism of Noetherian schemes and \mathcal{E} a coherent sheaf on X flat and S_r over V . Let $Z \subseteq X$ be a closed subscheme of X such that we have $\text{codim}_{\text{Supp } \mathcal{E}_v}(\text{Supp } \mathcal{E}_v \cap Z_v) \geq r$ for every $v \in V$. Then for each $0 \leq i < r$, $\mathcal{H}_Z^i(\mathcal{E}) = 0$.*

Proof. We follow the proof in [[HK04](#), Proposition 3.3]. First, we certainly have a spectral sequence $E_2^{p,q} := H_x^p \circ \mathcal{H}_Z^q \Rightarrow H_x^{p+q}$. We then induct on r , the base case of $r = 0$ being trivial. Now assume that $\mathcal{H}_Z^0(\mathcal{E}) = \dots = \mathcal{H}_Z^{r-2}(\mathcal{E}) = 0$ but $\mathcal{H}_Z^{r-1}(\mathcal{E}) \neq 0$. Then for any $x \in Z$ which is a generic point¹⁰ of the support of $\mathcal{H}_Z^{r-1}(\mathcal{E})$, we have that $H_x^0(\mathcal{H}_Z^{r-1}(\mathcal{E})) \neq 0$ (note here that x is probably not a closed point). Additionally, since x is a point of $Z \cap \text{Supp}(\mathcal{E})$, we have that $\dim(\mathcal{E}|_{X_v})_x \geq r$.

On the other hand, for all $i > 0$ we have $H_x^i(\mathcal{H}_Z^{r-1-i}(\mathcal{E})) = 0$, and so by the spectral sequence $H_x^{r-1}(\mathcal{E}) \cong H_x^0(\mathcal{H}_Z^{r-1}(\mathcal{E})) \neq 0$. This contradicts [Corollary A.5](#). \square

We now obtain a relative version of Hartog's phenomena, just as in [[HK04](#)].

Proposition A.7. [[HK04](#), Proposition 3.5, 3.6.1] *Let $f : X \rightarrow V$ be a morphism of Noetherian schemes and that \mathcal{E} a coherent sheaf on X which satisfies one of the following two conditions:*

- (i) \mathcal{E} is reflexive and f is flat and relatively S_2 , or
- (ii) \mathcal{E} is of full support, flat and S_2 over V .

Let $j : U \hookrightarrow X$ be an open set such that for $Z := X \setminus U$, $\text{codim}_{X_v} Z_v \geq 2$ for every $v \in V$. Then the following natural map is an isomorphism:

$$\mathcal{E} \rightarrow j_* \mathcal{E}|_U.$$

Proof. First, assume that \mathcal{E} is reflexive. Consider a presentation of \mathcal{E}^* by locally free sheaves

$$\mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}^* \longrightarrow 0 \quad \text{and the dual} \quad 0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_1^* \longrightarrow \mathcal{E}_2^*$$

Then by applying [Proposition A.6](#) to \mathcal{O}_X and the applying local cohomology to the above exact sequence yields that

$$(A.7.1) \quad \mathcal{H}_Z^1(\mathcal{E}) = \mathcal{H}_Z^0(\mathcal{E}) = 0.$$

If instead, \mathcal{E} is flat, S_2 and has full support, then we still have the vanishing (A.7.1) by [Proposition A.6](#). Consider then the exact sequence

$$0 \longrightarrow \mathcal{H}_Z^0(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow j_* \mathcal{E} \longrightarrow \mathcal{H}_Z^1(\mathcal{E}).$$

Applying (A.7.1) concludes our proof. \square

Corollary A.8. [[HK04](#), Proposition 3.6.2] *Let $f : X \rightarrow V$ be a flat, S_2 morphism of Noetherian schemes, \mathcal{E} a coherent X satisfying either [Proposition A.7\(i\)](#) or (ii). Let \mathcal{F} be another coherent sheaf satisfying [Proposition A.7\(i\)](#) or (ii). Let $j : U \hookrightarrow X$ be an open set, such that for $Z := X \setminus U$, $\text{codim}_{X_v} Z_v \geq 2$ for every $v \in V$. Assume also that $\mathcal{E}|_U \cong \mathcal{F}|_U$. Then $\mathcal{E} \cong \mathcal{F}$.*

¹⁰a minimal prime in the language of commutative algebra

Proof. In any case, $j_*\mathcal{E}|_U = \mathcal{E}$ and $j_*\mathcal{F}|_U = \mathcal{F}$ and so the result follows. \square

Proposition A.9. [HK04, Corollary 3.7] *Let $f : X \rightarrow V$ be a flat, S_2 morphism of Noetherian schemes and $j : U \hookrightarrow X$ an open set, such that for $Z := X \setminus U$, $\text{codim}_{X_v} Z_v \geq 2$ for every $v \in V$. Assume also that \mathcal{E} is a reflexive, coherent sheaf on U . Then $j_*\mathcal{E}$ is reflexive.*

Proof. Choose a coherent subsheaf $\mathcal{F} \subseteq j_*\mathcal{E}$ such that $\mathcal{F}|_U = \mathcal{E}$ [Har77, Ex II.5.15]. Then $j_*\mathcal{E}$ is reflexive by the following computation.

$$\mathcal{F}^{**} \cong \underbrace{j_*(\mathcal{F}^{**}|_U)}_{\text{by Proposition A.7}} \cong \underbrace{j_*\mathcal{E}^{**}}_{\mathcal{E} \cong \mathcal{F}|_U} \cong \underbrace{j_*\mathcal{E}}_{\mathcal{E} \text{ is reflexive}}$$

This completes the proof. \square

Proposition A.10. *If $f : X \rightarrow Y$ is a projective, flat, relatively S_2 and G_1 , equidimensional morphism, then $\omega_{X/Y}$ is reflexive.*

Proof. The proof given in [PS12, Lemma 4.8] works. \square

REFERENCES

- [And93] M. ANDRÉ: *Homomorphismes réguliers en caractéristique p* , C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), no. 7, 643–646. 1214408 (94e:13026)
- [Bli13] M. BLICKLE: *Test ideals via algebras of p^{-e} -linear maps*, J. Algebraic Geom. **22** (2013), no. 1, 49–83.
- [BB11] M. BLICKLE AND G. BÖCKLE: *Cartier modules: finiteness results*, J. Reine Angew. Math. **661** (2011), 85–123. 2863904
- [BMS08] M. BLICKLE, M. MUSTĂŢĂ, AND K. E. SMITH: *Discreteness and rationality of F -thresholds*, Michigan Math. J. **57** (2008), 43–61, Special volume in honor of Melvin Hochster. 2492440 (2010c:13003)
- [BS13] M. BLICKLE AND K. SCHWEDE: *p^{-1} -linear maps in algebra and geometry*, Commutative Algebra: Expository Papers Dedicated to David Eisenbud on the Occasion of His 65th Birthday (I. Peeva, ed.), Springer, Berlin, 2013, pp. 123–205.
- [BSTZ10] M. BLICKLE, K. SCHWEDE, S. TAKAGI, AND W. ZHANG: *Discreteness and rationality of F -jumping numbers on singular varieties*, Math. Ann. **347** (2010), no. 4, 917–949. 2658149
- [BST11] M. BLICKLE, K. SCHWEDE, AND K. TUCKER: *F -singularities via alterations*, arXiv:1107.3807.
- [BK05] M. BRION AND S. KUMAR: *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, vol. 231, Birkhäuser Boston Inc., Boston, MA, 2005. MR2107324 (2005k:14104)
- [CHMS12] P. CASCINI, C. HACON, M. MUSTĂŢĂ, AND K. SCHWEDE: *On the numerical dimension of pseudo-effective divisors in positive characteristic*, arXiv:1206.6521, to appear in the American Journal of Mathematics.
- [Con00] B. CONRAD: *Grothendieck duality and base change*, Lecture Notes in Mathematics, vol. 1750, Springer-Verlag, Berlin, 2000. MR1804902 (2002d:14025)
- [Eis95] D. EISENBUD: *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. 1322960 (97a:13001)
- [Fed83] R. FEDDER: *F -purity and rational singularity*, Trans. Amer. Math. Soc. **278** (1983), no. 2, 461–480. MR701505 (84h:13031)
- [FST11] O. FUJINO, K. SCHWEDE, AND S. TAKAGI: *Supplements to non- lc ideal sheaves*, Higher Dimensional Algebraic Geometry, RIMS Kôkyûroku Bessatsu, B24, Res. Inst. Math. Sci. (RIMS), Kyoto, 2011, pp. 1–47.
- [Gab04] O. GABBER: *Notes on some t -structures*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 711–734.
- [GW77] S. GOTO AND K. WATANABE: *The structure of one-dimensional F -pure rings*, J. Algebra **49** (1977), no. 2, 415–421. MR0453729 (56 #11989)
- [Gro67] A. GROTHENDIECK: *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361. MR0238860 (39 #220)
- [Hac11] C. D. HACON: *Singularities of pluri-theta divisors in char $p > 0$* , arXiv:1112.2219.
- [HX13] C. D. HACON AND C. XU: *On the three dimensional minimal model program in positive characteristic*, arXiv:1302.0298.
- [Har98] N. HARA: *A characterization of rational singularities in terms of injectivity of Frobenius maps*, Amer. J. Math. **120** (1998), no. 5, 981–996. MR1646049 (99h:13005)
- [HY03] N. HARA AND K.-I. YOSHIDA: *A generalization of tight closure and multiplier ideals*, Trans. Amer. Math. Soc. **355** (2003), no. 8, 3143–3174 (electronic). MR1974679 (2004i:13003)

- [Har77] R. HARTSHORNE: *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)
- [Har94] R. HARTSHORNE: *Generalized divisors on Gorenstein schemes*, Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992), vol. 8, 1994, pp. 287–339. MR1291023 (95k:14008)
- [HS77] R. HARTSHORNE AND R. SPEISER: *Local cohomological dimension in characteristic p* , Ann. of Math. (2) **105** (1977), no. 1, 45–79. MR0441962 (56 #353)
- [Has01] M. HASHIMOTO: *Cohen-Macaulay F -injective homomorphisms*, Geometric and combinatorial aspects of commutative algebra (Messina, 1999), Lecture Notes in Pure and Appl. Math., vol. 217, Dekker, New York, 2001, pp. 231–244. 1824233 (2002d:13007)
- [Has10] M. HASHIMOTO: *F -pure homomorphisms, strong F -regularity, and F -injectivity*, Comm. Algebra **38** (2010), no. 12, 4569–4596. 2764840 (2012a:13007)
- [HK04] B. HASSETT AND S. J. KOVÁCS: *Reflexive pull-backs and base extension*, J. Algebraic Geom. **13** (2004), no. 2, 233–247. MR2047697 (2005b:14028)
- [Hoc07] M. HOCHSTER: *Foundations of tight closure theory*, lecture notes from a course taught on the University of Michigan Fall 2007 (2007).
- [HH94] M. HOCHSTER AND C. HUNEKE: *F -regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), no. 1, 1–62. MR1273534 (95d:13007)
- [HR76] M. HOCHSTER AND J. L. ROBERTS: *The purity of the Frobenius and local cohomology*, Advances in Math. **21** (1976), no. 2, 117–172. MR0417172 (54 #5230)
- [HS06] C. HUNEKE AND I. SWANSON: *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006. MR2266432 (2008m:13013)
- [Kat08] M. KATZMAN: *Parameter-test-ideals of Cohen-Macaulay rings*, Compos. Math. **144** (2008), no. 4, 933–948. MR2441251 (2009d:13030)
- [Kee03] D. S. KEELER: *Ample filters of invertible sheaves*, J. Algebra **259** (2003), no. 1, 243–283. 1953719 (2003m:14026)
- [Kob75] N. KOBLITZ: *p -adic variation of the zeta-function over families of varieties defined over finite fields*, Compositio Math. **31** (1975), no. 2, 119–218. 0414557 (54 #2658)
- [Kol90] J. KOLLÁR: *Projectivity of complete moduli*, J. Differential Geom. **32** (1990), no. 1, 235–268. 1064874 (92e:14008)
- [Kc92] J. KOLLÁR AND 14 COAUTHORS: *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992). MR1225842 (94f:14013)
- [Kun69] E. KUNZ: *Characterizations of regular local rings for characteristic p* , Amer. J. Math. **91** (1969), 772–784. MR0252389 (40 #5609)
- [Kun86] E. KUNZ: *Kähler differentials*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1986. MR864975 (88e:14025)
- [Laz04a] R. LAZARSFELD: *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series. MR2095471 (2005k:14001a)
- [Laz04b] R. LAZARSFELD: *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004, Positivity for vector bundles, and multiplier ideals. MR2095472 (2005k:14001b)
- [Lyu97] G. LYUBEZNIK: *F -modules: applications to local cohomology and D -modules in characteristic $p > 0$* , J. Reine Angew. Math. **491** (1997), 65–130. MR1476089 (99c:13005)
- [MR85] V. B. MEHTA AND A. RAMANATHAN: *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. of Math. (2) **122** (1985), no. 1, 27–40. MR799251 (86k:14038)
- [MS97] V. B. MEHTA AND V. SRINIVAS: *A characterization of rational singularities*, Asian J. Math. **1** (1997), no. 2, 249–271. MR1491985 (99e:13009)
- [MS12] L. E. MILLER AND K. SCHWEDE: *Semi-log canonical vs F -pure singularities*, J. Algebra **349** (2012), 150–164. 2853631
- [Mil72] L. MILLER: *Curves with invertible Hasse-Witt-matrix*, Math. Ann. **197** (1972), 123–127. 0314849 (47 #3399)
- [Mus11] M. MUSTĂȚĂ: *The non-nef locus in positive characteristic*, arXiv:1109.3825, to appear in the volume dedicated to Joe Harris on the occasion of his 60th birthday.
- [MS12] M. MUSTĂȚĂ AND K. SCHWEDE: *A Frobenius variant of Seshadri constants*, arXiv:1203.1081.
- [MY09] M. MUSTĂȚĂ AND K.-I. YOSHIDA: *Test ideals vs. multiplier ideals*, Nagoya Math. J. **193** (2009), 111–128. MR2502910

- [Pat12] ZS. PATAKFALVI: *Semi-positivity in positive characteristics*, arXiv:1208.5391.
- [PS12] ZS. PATAKFALVI AND K. SCHWEDE: *Depth of F -singularities and base change of relative canonical sheaves*, arXiv:1207.1910, to appear in the Journal of the Institute of Mathematics of Jussieu.
- [Rad92] N. RADU: *Une classe d'anneaux noethériens*, Rev. Roumaine Math. Pures Appl. **37** (1992), no. 1, 79–82. 1172271 (93g:13014)
- [RR85] S. RAMANAN AND A. RAMANATHAN: *Projective normality of flag varieties and Schubert varieties*, Invent. Math. **79** (1985), no. 2, 217–224. MR778124 (86j:14051)
- [Sch09] K. SCHWEDE: *F -adjunction*, Algebra Number Theory **3** (2009), no. 8, 907–950. 2587408 (2011b:14006)
- [Sch11a] K. SCHWEDE: *A canonical linear system associated to adjoint divisors in characteristic $p > 0$* , arXiv:1107.3833, to appear in the Journal für die reine und angewandte Mathematik.
- [Sch11b] K. SCHWEDE: *Test ideals in non- \mathbb{Q} -Gorenstein rings*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 5925–5941.
- [ST12] K. SCHWEDE AND K. TUCKER: *A survey of test ideals*, Progress in Commutative Algebra 2. Closures, Finiteness and Factorization (C. Francisco, L. C. Klinger, S. M. Sather-Wagstaff, and J. C. Vassilev, eds.), Walter de Gruyter GmbH & Co. KG, Berlin, 2012, pp. 39–99.
- [Sha07] R. Y. SHARP: *On the Hartshorne-Speiser-Lyubeznik theorem about Artinian modules with a Frobenius action*, Proc. Amer. Math. Soc. **135** (2007), no. 3, 665–670 (electronic). 2262861 (2007f:13008)
- [SZ09] K. SHIMOMOTO AND W. ZHANG: *On the localization theorem for F -pure rings*, J. Pure Appl. Algebra **213** (2009), no. 6, 1133–1139. 2498803 (2010d:13006)
- [Smi97] K. E. SMITH: *F -rational rings have rational singularities*, Amer. J. Math. **119** (1997), no. 1, 159–180. MR1428062 (97k:13004)
- [Sri91] V. SRINIVAS: *Normal surface singularities of F -pure type*, J. Algebra **142** (1991), no. 2, 348–359. MR1127067 (93b:14060)
- [Sta] T. STACKS PROJECT AUTHORS: *Stacks Project*.
- [Tak04] S. TAKAGI: *An interpretation of multiplier ideals via tight closure*, J. Algebraic Geom. **13** (2004), no. 2, 393–415. MR2047704 (2005c:13002)
- [Tan13] H. TANAKA: *The trace map of Frobenius and extending sections for threefolds*, arXiv:1302.3134.
- [Tan72] H. TANGO: *On the behavior of extensions of vector bundles under the Frobenius map*, Nagoya Math. J. **48** (1972), 73–89. 0314851 (47 #3401)
- [Vie95] E. VIEHWEG: *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 30, Springer-Verlag, Berlin, 1995. MR1368632 (97j:14001)
- [Zha12] Y. ZHANG: *Pluri-canonical map of varieties of maximal Albanese dimension in positive characteristic*, arXiv:1203.1360.

EPFL, SB MATHGEOM CAG, MA B3 635 (BÂTIMENT MA), STATION 8, CH-1015 LAUSANNE
E-mail address: `zsolt.patakfalvi@epfl.ch`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UT 84112
E-mail address: `schwede@math.utah.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, IL 60607
E-mail address: `wlzhang@uic.edu`