CANONOID AND POISSONOID TRANSFORMATIONS, SYMMETRIES AND BIHAMILTONIAN STRUCTURES

GIOVANNI RASTELLI
MANUELE SANTOPRETE

Abstract. We give a characterization of linear canonoid transformations on symplectic manifolds and we use it to generate biHamiltonian structures for some mechanical systems. Utilizing this characterization we also study the behavior of the harmonic oscillator under canonoid transformations. We present a description of canonoid transformations due to E.T. Whittaker, and we show that it leads, in a natural way, to the modern, coordinate-independent definition of canonoid transformations. We also generalize canonoid transformations to Poisson manifolds by introducing Poissonoid transformations. We give examples of such transformations for Euler's equations of the rigid body (on $\mathfrak{so}^*(3)$ and $\mathfrak{so}^*(4)$) and for an integrable case of Kirchhoff's equations for the motion of a rigid body immersed in an ideal fluid. We study the relationship between biHamiltonian structures and Poissonoid transformations for these examples. We analyze the link between Poissonoid transformations, constants of motion, and symmetries.

Contents

1. Introduction

BiHamiltonian systems are, in a nutshell, dynamical systems described by a vector field that is Hamiltonian with respect to two distinct Poisson (or symplectic) structures and two associated (possibly distinct) Hamiltonian functions. Under certain additional hypothesis, possessing a biHamiltonian structure is enough to guarantee the integrability of the system (see for example [17]). During the last

2010 *Mathematics Subject Classification.* 53D05, 37K10, 53D17.
Key words and phrases. Canonoid transformations; biHamiltonian systems; symplectic geometry; Poisson geometry.

few decades it has been shown that many integrable systems are in fact biHamiltonian, consequently, biHamiltonian structures are now an important paradigm for understanding integrability. In some cases, a new Hamiltonian structure can be obtained with a transformation of coordinates. This may be possible when, on a symplectic manifold, the transformation changes the Hamiltonian characterization of a Hamiltonian vector field. Such transformations (in the case the symplectic manifold is $\mathbb{R}^{2n}$ and the symplectic form the standard one) were dubbed "canonoid" and popularized by Saletan and Cromer [25], and by Currie and Saletan [7], but they were know well before the 1970s, in fact, they were already present in the 1904 edition of the classical book of Whittaker [29]. This type of transformations include the well known canonical ones. The main difference between these transformations is that, while the canonoid ones are specific to the problem considered, the canonical ones preserve the Hamiltonian form of every Hamiltonian system on the manifold, and leave invariant the symplectic structure. Therefore, canonical transformations cannot be used to generate different symplectic structures. Strictly canonoid transformations (i.e., those canonoid transformations that are not canonical), in contrast, change the symplectic structure and only preserve the Hamiltonian form of some chosen Hamiltonian systems, and thus can be used to generate different symplectic structures. Canonoid transformations are the argument of about 20 papers, among them, we cite here the ones more closely related to the content of our article. A first set of papers concerns primarily the characterization of canonoid transformations and their relations with canonical transformations: [7, 22, 4, 28, 5, 3]. A second set deals primarily with applications of canonoid transformations to the analysis of Hamiltonian systems: [26, 14].

In this paper we use a modern geometrical definition of canonoid transformation based on locally Hamiltonian vector fields. This definition coincides to the so called quasi-canonical transformations of Marmo [18] and reduces to the definition of Saletan and Cromer [25] in the simplest case of a topologically trivial system, or at least when considering only local expressions for the system. By generalizing the approach of [10], we obtain simple explicit conditions for linear canonoid transformations on $\mathbb{R}^n$. We use this method to analyze some examples, including the harmonic oscillator in $\mathbb{R}^4$. We also recall the approach of Whittaker [29] and show that the modern definition of canonoid transformation we employ follows naturally from such approach. Moreover, we extend this type of transformations to the case of Poisson manifolds, by introducing a generalization of the canonoid transformations that we dub Poissonoid transformations. This type of transformations, as far as we know, have not been studied before, and they allow us to find biHamiltonian structures in the case of Poisson manifolds. The Casimirs of the new Poisson structures found this way provide first integrals of the Hamiltonian system. Furthermore, if the Poisson structures are compatible, the integrability of the systems follows from the theory of biHamiltonian systems. We also study the relationship between Linear Poissonoid transformations and biHamiltonian structures in some examples, namely Euler's equations for the rigid body (on $\mathfrak{so}^*(3)$ and $\mathfrak{so}^*(4)$) and an integrable case of Kirchhoff's equations for the motion of a rigid body immersed in an ideal fluid. We conclude with a study of the relations among infinitesimal Poissonoid transformations, Noether theorem and master symmetries, generalizing to the Poisson case some results obtained in [3] for canonoid transformations.

Our aim is to provide to the non-specialists an introduction to canonoid transformation and biHamiltonian systems through the analysis of several examples. For the specialists, we highlight the new definition of Poissonoid transformations, the role played by simple linear canonoid (and Poissonoid) transformations in the determination of several biHamiltonian structures, and the relationship between Poissonoid transformations, integrals of motion, and symmetries.

The paper is organized as follows. In section 2 we recall some essential facts concerning Poisson geometry, symplectic geometry and biHamiltonian structures, and we set the notations employed in the rest of the article. Section 2 can be skipped by readers already familiar with these topics. In section 3 we introduce canonoid transformations on symplectic manifolds and we study examples of linear canonoid transformations. In section 4 we analyze how the superintegrable structure of some simple systems behaves under linear canonoid transformations. In section 5 we translate into more modern language the characterization of canonoid transformations given in Whittaker [29]. In section 6, we extend the idea of canonoid transformations to Poisson manifolds by introducing Poissonoid transformations, and we give several examples of such transformations. In the last section we analyze the link between infinitesimal Poissonoid transformations, master symmetries and constants of motion.

2. Poisson, symplectic and biHamiltonian structures

2.1. **Poisson structures.** We now recall the fundamental definitions and some of the main results concerning Poisson structures, for a more detailed account we refer the reader to the following references: [19, 15, 6, 16, 24].

Definition 2.1. Let M be a smooth manifold, and let $C^\infty(M)$ be the set of smooth functions on M. A **Poisson bracket** or **Poisson structure** is a skew-symmetric bilinear operation $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which satisfies the Jacobi identity

$$\{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\} = 0$$

and the Leibnitz identity

$$\{f,gh\} = \{f,g\}h + g\{f,h\}.$$

Associated with the bracket there is a bivector field defined by

$$\{F,G\}(x) = \pi(x)(\mathbf{d}F(x), \mathbf{d}G(x)),$$

called **Poisson bivector** or **Poisson tensor** field. The pair (M,π) is called **Poisson manifold**.

Let α and β be differential forms, than we define the map $\pi^\sharp : T^*M \to TM$ as $\pi(\alpha,\beta) = \left\langle \alpha, \pi^\sharp \cdot \beta \right\rangle$. The **rank** of π at $x \in M$ is the rank of the linear map $\pi^\sharp_x : T^*_x M \to T_x M$. In general, the rank will vary from point to point.

Definition 2.2. A **regular point** of a Poisson manifold is a point where the rank of the Poisson bivector is locally constant, the remaining points are called **singular points**. A **regular Poisson bivector** π is a Poisson bivector whose rank is constant. Similar definitions apply to Poisson structures. A **regular Poisson manifold** is a Poisson manifold endowed with a regular Poisson structure.

Definition 2.3. Let (M, π) be a Poisson manifold. A smooth, real valued function $C : M \to \mathbb{R}$ is called a **Casimir function** if the Poisson bracket of C with any other real-valued function vanishes identically, i.e. $\{C, H\} = 0$ for all $H : M \to \mathbb{R}$.

An alternative way of introducing the Poisson bivector uses the so called Schouten-Nijenhuis bracket, namely an extension of the Lie bracket of vector fields to skew-symmetric multivector fields, see [19, 27]).

Proposition 2.4. *A bivector field π on M is the Poisson bivector of a Poisson structure on M if and only if $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket.*

Definition 2.5. Let (M, π) be a Poisson manifold. The **Hamiltonian vector field** of a smooth function $H : M \to M$ is the vector field such that

$$\mathcal{X}_H[F] = \{F, H\} = \left\langle \mathbf{d}F, \pi^\sharp \cdot \mathbf{d}H \right\rangle$$

for every smooth function F on M. The function H is called **Hamiltonian function**. The triplet (M, π, H) is called **Hamiltonian system**.

The definition of Hamiltonian system can be generalized to systems that are Hamiltonian in a neighborhood of each point of the manifold (M, π). Here we use a definition of locally Hamiltonian given in [15], note that this definition differs from the one used in [16] and [6].

Definition 2.6. A vector field $\mathcal{X}$ on a Poisson manifold (M, π) is called **locally Hamiltonian** if for every $x \in M$ there is a neighborhood U of x and a smooth function H_U defined on this neighborhood such that $\mathcal{X} = \pi^\sharp \cdot \mathbf{d}H_U$, that is $\mathcal{X}$ is Hamiltonian in U with the locally defined Hamiltonian H_U. A triplet $(M, \pi, \mathcal{X})$ as above is called a **locally Hamiltonian system**.

Let (M, π) be a Poisson manifold of dimension d. In a neighborhood U of a point $p \in M$, with local coordinates $\mathbf{x} = (x^1, \ldots, x^d)$, the bivector field π can be written as

$$\pi = \sum_{1 \le i < j \le d} \{x^i, x^j\} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

and the vector field

$$\mathcal{X}_H = \sum_{i,j=1}^{d} \{x^i, x^j\} \frac{\partial H}{\partial x^j} \frac{\partial}{\partial x^i}.$$

We recall that a function $F \in C^\infty(M)$ such that $\{F, H\} = 0$ is called a **constant of motion** or first integral of the Hamiltonian system.

Definition 2.7. Let (M, π) be a Poisson manifold, and let (M, π, H) be a Hamiltonian system on M with Hamiltonian vector field $\mathcal{X}_H$. A vector field ξ on M is called an **infinitesimal symmetry of the vector field** $\mathcal{X}_H$ if $\mathcal{L}_\xi \mathcal{X}_H = 0$. Moreover, ξ is called a **Poisson infinitesimal symmetry** of (M, π, H) if it is an infinitesimal symmetry of both π and H, that is if

$$\mathcal{L}_\xi \pi = 0 \quad \text{and} \quad \mathcal{L}_\xi H = \xi[H] = 0.$$

A Poisson infinitesimal symmetry ξ is called a **(locally) Hamiltonian infinitesimal symmetry** of (M, π, H) if, in addition, ξ is (locally) Hamiltonian. That is there is a (locally defined) function K such that $\xi = \pi^\sharp \cdot \mathbf{d}K$.

Note that, in the non-degenerate (symplectic) case, Poisson infinitesimal symmetries coincide with locally Hamiltonian infinitesimal symmetries.

It is important to keep in mind there is a distinction between Poisson infinitesimal symmetries of (M, π, H) and infinitesimal symmetries of the Hamiltonian vector field $\mathcal{X}_H = \pi^\sharp \cdot \mathbf{d}H$. The following proposition clarifies the relationship between the two types of symmetries

Proposition 2.8. *Let (M, π) be a Poisson manifold, and let (M, π, H) be a Hamiltonian system on M with Hamiltonian vector field $\mathcal{X}_H$. If ξ is a Poisson infinitesimal symmetry of (M, π, H) then it is also an infinitesimal symmetry of the vector field $\mathcal{X}_H$.*

Proof. Let α be an arbitrary 1-form. Since $\mathcal{X}_H = \pi^\sharp \cdot \mathbf{d}H$ we have

(2.1)
$$\begin{aligned}
0 &= \mathcal{L}_\xi \left\langle \alpha, \mathcal{X}_H - \pi^\sharp \cdot \mathbf{d}H \right\rangle = \mathcal{L}_\xi \left\langle \alpha, \mathcal{X}_H \right\rangle - \mathcal{L}_\xi \pi(\alpha, \mathbf{d}H) \\
&= \langle \mathcal{L}_\xi \alpha, \mathcal{X}_H \rangle + \langle \alpha, \mathcal{L}_\xi \mathcal{X}_H \rangle - (\mathcal{L}_\xi \pi)(\alpha, \mathbf{d}H) - \pi(\mathcal{L}_\xi \alpha, \mathbf{d}H) - \pi(\alpha, \mathcal{L}_\xi \mathbf{d}H) \\
&= \langle \mathcal{L}_\xi \alpha, \mathcal{X}_H \rangle + \langle \alpha, \mathcal{L}_\xi \mathcal{X}_H \rangle - (\mathcal{L}_\xi \pi)(\alpha, \mathbf{d}H) - \left\langle \mathcal{L}_\xi \alpha, \pi^\sharp \cdot (\mathbf{d}H) \right\rangle - \left\langle \alpha, \pi^\sharp \cdot (\mathcal{L}_\xi \mathbf{d}H) \right\rangle \\
&= \langle \alpha, \mathcal{L}_\xi \mathcal{X}_H \rangle - (\mathcal{L}_\xi \pi)(\alpha, \mathbf{d}H) - \left\langle \alpha, \pi^\sharp \cdot (\mathbf{d}(\mathcal{L}_\xi H)) \right\rangle .
\end{aligned}$$

If ξ is a Poisson infinitesimal symmetry of (M, π, H), then $\mathcal{L}_\xi \pi = 0$ and $\mathcal{L}_\xi H = 0$. Since α is arbitrary, by the equation above we have $\mathcal{L}_\xi \mathcal{X}_H = 0$. □

The converse of the proposition above is clearly not true in general, even when π is non-degenerate [3].

Note that Hamiltonian infinitesimal symmetries are very important because, through Noether's theorem (see below), they give rise to constants motion, which are very useful in the process of reduction the Hamiltonian system.

Theorem 2.9 (Noether's theorem)**.** *Let (M, π, H) an Hamiltonian system. If F is a constant of motion then its vector field is a Hamiltonian infinitesimal symmetry. Conversely, each Hamiltonian infinitesimal symmetry is the Hamiltonian vector field of a constant of motion, which is unique up to a (time dependent) Casimir function.*

Proof. Suppose F is a constant of motion. Then $\mathcal{X}_F = \pi^\sharp \cdot \mathbf{d}F$ is an Hamiltonian vector field, so that $\mathcal{L}_{\mathcal{X}_F} \pi = 0$. Moreover, since $\mathcal{X}_H[F] = 0$, we have that

$$0 = \mathcal{X}_H[F] = \{F, H\} = -\{H, F\} = \mathcal{X}_F[H] = 0.$$

Thus $\mathcal{X}_F$ is a Hamiltonian infinitesimal symmetry. Now suppose that V is a Hamiltonian infinitesimal symmetry of (M, π, H). Since V is Hamiltonian there is a function F such that $V = \mathcal{X}_F = \pi^\sharp \cdot \mathbf{d}F$. Since it is an infinitesimal symmetry of (M, π, H) we have that

$$0 = \mathcal{X}_F[H] = \{H, F\} = -\{F, H\} = \mathcal{X}_H[F]$$

so F is a constant of motion. If $\tilde{F}$ is another function that satisfies $\mathcal{X}_{\tilde{F}} = V = \mathcal{X}_F$, then

$$0 = \mathcal{X}_{\tilde{F}} - \mathcal{X}_F = \pi^\sharp \cdot (\mathbf{d}(\tilde{F} - F))$$

so $\tilde{F} - F$ must be a Casimir of π, since if we apply the above to an arbitrary function G we have that

$$0 = \left\langle \mathbf{d}G, \pi^\sharp \cdot \mathbf{d}(\tilde{F} - F) \right\rangle = \pi(\mathbf{d}G, \mathbf{d}(\tilde{F} - F)) = \{G, \tilde{F} - F\}.$$

□

Definition 2.10. A vector field $\mathcal{X}$ on a Poisson manifold (M, π) is called a **Poisson vector field** iff $\mathcal{L}_{\mathcal{X}}\pi = 0$.

In particular it follows that any locally Hamiltonian vector field is Poisson:

Proposition 2.11. *If $\mathcal{X}$ is a locally Hamiltonian vector field on a Poisson manifold (M, π), then it is a Poisson vector field.*

Proof. See [15]. □

The converse is not true in general. For example, if the Poisson structure is trivial, then any vector field is Poisson, while the only Hamiltonian vector field is the trivial one. In the special case of a symplectic manifold, a vector field is Poisson if and only if it is locally Hamiltonian (see Proposition 2.18).

Theorem 2.12 (Weinstein's splitting theorem)**.** *Let (M, π) be a Poisson manifold, let $x \in M$ be an arbitrary point and denote the rank of π at x by $2r$. There exists a coordinate neighborhood U of x with coordinates $(q^1, \dots q^r, p_1, \dots, p_r, z^1, \dots z^s)$ centered at x, such that, on U,*

$$\pi = \sum_{i=1}^{r} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \sum_{1 \le k < l \le s} \phi^{kl}(z) \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial z^l}, \tag{2.2}$$

where the functions ϕ^{kl} are smooth functions which depend on $z = (z^1, \dots, z^s)$ only, and which vanish when $z = 0$. Such local coordinates are called **splitting coordinates***, centered at x.*

In particular, if there is a neighborhood V of x such that the rank is constant and equal to $2r$, then there exists a coordinate neighborhood U of x with coordinates $(q^1, \dots q^r, p_1, \dots, p_r, z^1, \dots z^s)$ centered at x, such that, on U,

$$\pi = \sum_{i=1}^{r} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \tag{2.3}$$

Moreover, π is locally of the form above, in terms of arbitrary splitting coordinates on M. Such coordinates are called **Darboux coordinates***.*

Proof. See [15]. □

Remark 1. For a given point x on a Poisson manifold M, splitting coordinates are not unique. The Poisson structure, which is defined in a neighborhood of $z = 0$ by the second term of (2.2), however, is unique up to a Poisson diffeomorphism (see for example [15]).

Proposition 2.13. *Let (M, π) be a Poisson manifold and let $x \in M$. Suppose $\mathcal{X}$ is a locally Hamiltonian vector field. Let U be a neighborhood of x with splitting coordinates $(\mathbf{q}, \mathbf{p}, \mathbf{z}) = (q^1, \dots, q^r, p_1, \dots, p_r, z^1, \dots, z^s)$, and H_U such that $\mathcal{X} = \pi^\sharp \cdot \mathbf{d}H_U = \pi(\cdot, \mathbf{d}H_U)$. In these coordinates*

$$\mathcal{X} = \begin{bmatrix} \frac{\partial H_U}{\partial \mathbf{p}} \\ -\frac{\partial H_U}{\partial \mathbf{q}} \\ \Phi^T \frac{\partial H_U}{\partial \mathbf{z}} \end{bmatrix}$$

where Φ *is the matrix of entries* $[\Phi]_{kl} = \tilde{\phi}^{kl}(z)$ *(where* $\tilde{\phi}^{kl}(z) = \phi^{kl}(z)$ *for* $k < l$, $\tilde{\phi}^{kl}(z) = -\phi^{lk}(z)$ *for* $k > l$ *and* $\tilde{\phi}^{kl}(z) = 0$ *for* $k = l$*).*

Thus $(\mathbf{q}(t), \mathbf{p}(t), z(t))$ *is an integral curve of* $\mathcal{X}$ *if an only if* **Hamilton's equations** *hold:*

$$\dot{q}^i = \frac{\partial H_U}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_U}{\partial q^i}, \quad \dot{z}^k = \sum_l \tilde{\phi}^{kl}(z)\frac{\partial H_U}{\partial z^l}$$

for $i = 1, \ldots, r$, *and* $k = 1, \ldots, s$, *where the functions* $\tilde{\phi}^{ij}(z)$ *depend on the choice of splitting coordinates. Moreover, if the rank is locally constant at the point* x, *then the vector field, written in Darboux coordinates, is*

$$\mathcal{X} = \begin{bmatrix} \frac{\partial H_U}{\partial \mathbf{p}} \\ -\frac{\partial H_U}{\partial \mathbf{q}} \\ 0 \end{bmatrix}.$$

and Hamilton's equations take the simpler form

$$\dot{q}^i = \frac{\partial H_U}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_U}{\partial q^i}, \quad \dot{z}^k = 0$$

for $i = 1, \ldots, r$, *and* $k = 1, \ldots, s$.

Proof. See [15]. □

2.2. **Symplectic structures.** We now give a brief account of symplectic structures, for more details see [1, 19].

Definition 2.14. A **symplectic form** (or **symplectic structure**) on a manifold M is a nondegenerate, closed two-form ω on M. A **symplectic manifold** (M, ω) is a manifold M together with a symplectic form ω on M.

Definition 2.15. Let (M, ω) be a symplectic manifold, and $\mathcal{X}$ a vector field on M. If there is a smooth function $H : M \to \mathbb{R}$ such that

$$\mathbf{i}_{\mathcal{X}}\omega = \mathbf{d}H$$

we say that $\mathcal{X}$ is a **Hamiltonian vector field** .

Definition 2.16. Let (M, ω) be a symplectic manifold. The **Poisson bracket associated with** ω is defined by $\{F, G\} = \omega(\mathcal{X}_F, \mathcal{X}_G)$.

From the above definition it follows that, associated to ω there is a Poisson bivector π, so that a symplectic structure is a regular Poisson structure of maximal rank. The basic link between the Poisson bivector π and the symplectic form ω is that they are associated to the same Poisson bracket

$$\{F, H\} = \pi(\mathbf{d}F, \mathbf{d}H) = \omega(\mathcal{X}_F, \mathcal{X}_H),$$

that is $\langle \mathbf{d}F, \pi^\sharp \cdot \mathbf{d}H\rangle = \langle \mathbf{d}F, \mathcal{X}_H\rangle$. On the other hand, by definition, $\omega(\mathcal{X}_H, v) = \mathbf{d}H \cdot v$, and so $\langle \omega^\flat \cdot \mathcal{X}_H, v\rangle = \langle \mathbf{d}H, v\rangle$, whence

$$\mathcal{X}_H = \omega^\sharp \cdot \mathbf{d}H$$

since $\omega^\sharp = (\omega^\flat)^{-1}$ (see [1]). Thus $\pi^\sharp \cdot \mathbf{d}H = \omega^\sharp \cdot \mathbf{d}H$, for all H, and thus

$$\pi^\sharp = \omega^\sharp.$$

Definition 2.17. A vector field on a symplectic manifold (M,ω) is called **locally Hamiltonian** if for every $x \in M$ there is a neighborhood U of x and a smooth function H_U defined on this neighborhood such that $\mathbf{i}_{\mathcal{X}}\omega = \mathbf{d}H_U$, that is $\mathcal{X}$ is Hamiltonian in U with the locally defined Hamiltonian H_U.

If the manifold M has zero first group of real homology $H^1(M,\mathbb{R})$, then all local Hamiltonian vector fields are globally Hamiltonian [11].

Proposition 2.18. *The following statements are equivalent:*

(i) $\mathcal{X}$ *is locally Hamiltonian.*
(ii) $\mathbf{d}(\mathbf{i}_{\mathcal{X}}\omega) = 0$*, that is* $\mathbf{i}_{\mathcal{X}}\omega$ *is closed.*
(iii) $\mathcal{L}_{\mathcal{X}}\omega = 0$.

Proof. See [1]. □

Remark 2. Another equivalent way to define locally Hamiltonian vector fields is the following. A vector field is locally Hamiltonian if there exists a closed 1-form α such that $\mathcal{X} = \omega^\sharp \cdot \alpha$. In fact, if $\mathcal{X} = \omega^\sharp \cdot \alpha$, then $\mathbf{i}_{\mathcal{X}}\omega = \alpha$. So saying that α is closed is equivalent to saying that $\mathbf{i}_{\mathcal{X}}\omega$ is closed.

Theorem 2.19 (Darboux' Theorem for the symplectic case)**.** *Let* (M,ω) *be a symplectic manifold of dimension* $2n$*, then for each point* $x \in M$ *there exists a neighborhood* U *of* x *with coordinates* $(q^1,\dots,q^n,p_1,\dots,p_n)$ *(called* **canonical coordinates** *) such that*

$$\omega|_U = \sum_{i=1}^{n} \mathbf{d}q^i \wedge \mathbf{d}p_i.$$

Proof. See [1]. □

Proposition 2.20. *Let* (M,ω) *be a symplectic manifold and let* $x \in M$*, and let* $\mathcal{X}$ *be a locally Hamiltonian vector field. Let* U *be a neighborhood of* x *with canonical coordinates* $(\mathbf{q},\mathbf{p}) = (q^1,\dots,q^n,p_1,\dots,p_n)$ *, and* H_U *such that* $\mathbf{i}_{\mathcal{X}}\omega = \mathbf{d}H_U$*. In these coordinates*

$$\mathcal{X} = \begin{bmatrix} \frac{\partial H_U}{\partial \mathbf{p}} \\ -\frac{\partial H_U}{\partial \mathbf{q}} \end{bmatrix} = \mathbb{J}\nabla H_U$$

with

$$\mathbb{J} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} \text{ and } \nabla H_U = \begin{bmatrix} \frac{\partial H_U}{\partial \mathbf{q}} \\ \frac{\partial H_U}{\partial \mathbf{p}} \end{bmatrix}$$

where $\mathbf{1}$ *and* $\mathbf{0}$ *define the* $n \times n$ *identity and zero matrix, respectively.*

Thus $(\mathbf{q}(t),\mathbf{p}(t))$ *is an integral curve of* $\mathcal{X}$ *if an only if* **Hamilton's equations** *hold:*

$$\dot{q}^i = \frac{\partial H_U}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_U}{\partial q^i}$$

for $i = 1,\dots,n$.

Proof. See [1].

□

2.3. **BiHamiltonian structures.** Here we briefly recall some of the most important facts about biHamiltonian structures.

Definition 2.21. Let π_1 and π_2 be two Poisson bivector fields defined on a manifold M. We say that π_1 and π_2 are **compatible** if their Schouten-Nijenhuis bracket is zero, that is if $[\pi_1, \pi_2] = 0$. The triple (M, π_1, π_2) is called a **biHamiltonian manifold**.

We now recall the local coordinate representations of the pull-back of two forms and bivectors, and of the push-forward of vector fields. These expressions are useful in some of the computations done in the following sections. Let M and N be manifolds, $f : M \to N$ be a smooth map, and let ρ be a two-form on N. Recall that $f^*\rho$ denotes the pull-back of ρ by f, that in local coordinates takes the form

$$(f^*\rho)_{jk} = \sum_{rs} (\rho_{rs} \circ f) \left(\frac{\partial f^r}{\partial x^j} \right) \left(\frac{\partial f^s}{\partial x^k} \right).$$

Now suppose f is a diffeomorphism, and let π be a bivector field on N, then the pull-back of π by f in local coordinates is

$$(f^*\pi)^{jk} = \sum_{rs} \left(\frac{\partial (f^{-1})^j}{\partial X^r} \circ f \right) \left(\frac{\partial (f^{-1})^k}{\partial X^s} \circ f \right) \pi^{rs} \circ f. \tag{2.4}$$

Suppose $\mathcal{X}$ is a vector field on M. Then the pushforward $f_*\mathcal{X}$ of $\mathcal{X}$ by f, in local coordinates, takes the form

$$(f_*\mathcal{X})^j = \sum_k \left(\frac{\partial f^j}{\partial x^k} \circ f^{-1} \right) (\mathcal{X}^k \circ f^{-1}).$$

Proposition 2.22. *Let M be a manifold and let (N, π) be a Poisson manifold. Suppose $f : M \to N$ is a diffeomorphism. Then $f^*\pi$ is a Poisson tensor on M.*

Proof. Since the bracket has the following property (see [27]):

$$f^*[\pi, \pi] = [f^*\pi, f^*\pi].$$

it follows that $[\pi, \pi] = 0$ implies $[f^*\pi, f^*\pi] = 0$. □

Corollary 2.23. *Let M be a manifold and let (N, π_1, π_2) be a biHamiltonian manifold, that is π_1 and π_2 are compatible. Suppose $f : M \to N$ is a diffeomorphism. Then $f^*\pi_1$ and $f^*\pi_2$ are compatible and thus $(M, f^*\pi_1, f^*\pi_2)$ is a biHamiltonian manifold.*

Proof. Since

$$[\pi_1 + \pi_2, \pi_1 + \pi_2] = [\pi_1, \pi_1] + [\pi_2, \pi_2] + 2[\pi_1, \pi_2]$$

we have

$$f^*[\pi_1 + \pi_2, \pi_1 + \pi_2] = f^*[\pi_1, \pi_1] + f^*[\pi_2, \pi_2] + 2f^*[\pi_1, \pi_2]$$

and,

$$[f^*(\pi_1 + \pi_2), f^*(\pi_1 + \pi_2)] = [f^*\pi_1, f^*\pi_1] + [f^*\pi_2, f^*\pi_2] + 2[f^*\pi_1, f^*\pi_2].$$

Comparing the two equations above, by Proposition 2.22 we obtain that $f^*[\pi_1, \pi_2] = [f^*\pi_1, f^*\pi_2]$. □

Corresponding to the bivector fields π_1 and π_2 we can define the Poisson brackets $\{F, G\}_1 = \pi_1(\mathbf{d}F, \mathbf{d}G)$ and $\{F, G\}_2 = \pi_2(\mathbf{d}F, \mathbf{d}G)$. With these notations we give the following

Definition 2.24. Let (M, π_1, π_2) be a biHamiltonian manifold and suppose there exist functions H_1 and H_2 on M for which

$$\mathcal{X}[F] = \{F, H_1\}_1 = \{F, H_2\}_2$$

for every function F on M. Then $\mathcal{X}$ is called a **biHamiltonian vector field**.

The importance of biHamiltonian structures lies in the fact that, in certain situations, they can be used to show complete integrability. We do not give a complete account, but the main idea is that one can use them to construct a set of first integrals in involution by constructing a biHamiltonian hierarchy [17, 15, 2]

Definition 2.25. Let (M, π_1, π_2) be a biHamiltonian manifold. A **biHamiltonian hierarchy** on M is a sequence of functions $\{F_i\}_{i\in\mathbb{Z}}$ such that

$$\{\cdot, F_{i+i}\}_1 = \{\cdot, F_i\}_2$$

for every $i \in \mathbb{Z}$.

The following lemma explains why a biHamiltonian hierarchy yields functions in involution.

Proposition 2.26. *Suppose $\{F_i\}_{i\in\mathbb{Z}}$ is a biHamiltonian hierarchy, then $\{F_i, F_j\}_1 = \{F_i, F_j\}_2 = 0$ for all $i < j \in \mathbb{Z}$.*

Proof.

$$\begin{aligned}\{F_i, F_j\}_1 &= \{F_i, F_{j-1}\}_2\\ &= \{F_{i+1}, F_{j-1}\}_1\\ &= \dots\\ &= \{F_j, F_i\}_1,\end{aligned}$$

so that $\{F_i, F_j\}_1 = 0$ by skew-symmetry. Hence, the F_i's are in involution with respect to the Poisson bracket $\{\cdot,\cdot\}_1$, and also with respect to $\{\cdot,\cdot\}_2$, since $\{F_i, F_j\}_2 = \{F_i, F_{j+1}\}_1$.

□

3. Canonoid Transformations

Definition 3.1. Let (M, ω) be a symplectic manifold, and let $\mathcal{X}$ be a locally Hamiltonian vector field on M, that is, for each $x \in M$, there is a neighborhood U of x and locally defined function H_U such that $\mathbf{i}_{\mathcal{X}}\omega = \mathbf{d}H_U$. A diffeomorphism $f : M \to M$ is said to be **canonoid** with respect to the vector field $\mathcal{X}$ if the transformed vector field $f_*\mathcal{X}$ is also locally Hamiltonian, that is, for each $x \in M$, there is a neighborhood V of $f(x)$ and a locally defined function K_V such that $\mathbf{i}_{f_*\mathcal{X}}\omega = \mathbf{d}K_V$.

This is equivalent to saying that, for each $y \in M$, there is a neighborhood V of y and a locally defined function K_V such that $\mathbf{i}_{\mathcal{X}}(f^*\omega) = f^*\mathbf{d}K_V$. This is also equivalent to $\mathcal{L}_{\mathcal{X}}(f^*\omega) = 0$.

Remark 3. By 2.20 the previous definition means that, for each point $x \in M$ the system of equations associated with $\mathbf{i}_{\mathcal{X}}\omega = \mathbf{d}H_U$ can be written, in Darboux coordinates on the neighborhood U of x, as:

$$\dot{q}^i = \frac{\partial H_U}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_U}{\partial q^i} \tag{3.1}$$

and the system associated with $\mathbf{i}_{f_*\mathcal{X}}\omega = \mathbf{d}K_V$ can be written, in Darboux coordinates on the neighborhood V of $f(x)$, as:

$$\dot{Q}^i = \frac{\partial K_V}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K_V}{\partial Q^i}. \tag{3.2}$$

so that the transformation f carries the system of Hamilton's equations 3.1 again into a system of Hamilton's equations 3.2.

We modify an example found in [10] to give a general framework to construct linear canonoid transformations. Let us consider Hamiltonian systems on the symplectic manifold $(M, \omega) = (\mathbb{R}^{2n}, \omega)$, let $x = (\mathbf{q}, \mathbf{p})$ be Darboux coordinates on $\mathbb{R}^{2n}$, then the symplectic form can be written as $\omega = \sum \mathbf{d}q^i \wedge \mathbf{d}p_i$. In this case a diffeomorphism $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called a **canonical transformation** if f preserves the 2-form $\omega = \sum \mathbf{d}q^i \wedge \mathbf{d}p_i$, that is if $f^*\omega = \omega$. A diffeomorphism is canonical if and only if the matrix representation of $\mathbf{d}f$ in the canonical basis of $\mathbb{R}^{2n}$, namely $[\mathbf{d}f]$, is a symplectic matrix, that is $[\mathbf{d}f]^t \mathbb{J}[\mathbf{d}f] = \mathbb{J}$.

Any quadratic Hamiltonian can be written as

$$H(x) = \frac{1}{2}x^t S x,$$

where S is a real symmetric constant $2n \times 2n$ matrix. With these notations, the Hamiltonian vector field corresponding to H can be written as $\mathbb{J}Sx$, and Hamilton's equations take the form

$$\dot{x} = \mathbb{J}Sx.$$

Hamilton's equations above define a linear Hamiltonian system with constant coefficients. Consider the transformation $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, defined by $X = f(x) = Ax$, with A an invertible matrix. Then the vector field $\mathcal{X}$ is transformed to $f_*\mathcal{X} = A\mathbb{J}SA^{-1}X$ and the system of Hamilton's equations is transformed into a new system of $2n$ differential equations

$$\dot{X} = A\mathbb{J}SA^{-1}X$$

expressed in terms of the variables $X = (\mathbf{Q}, \mathbf{P})$. In general, the new system does not have the canonical structure, that is, it is not necessarily true that there exists a Hamiltonian $K(X)$ such that

$$\dot{X} = A\mathbb{J}SA^{-1}X = \mathbb{J}\nabla_X K$$

and thus not every transformation of this type is canonoid. However, it is easy to see that, in order to preserve the canonical structure, we must have

$$A\mathbb{J}SA^{-1} = \mathbb{J}C$$

for some symmetric matrix C. We can rewrite this condition as $A^t\mathbb{J}A\mathbb{J}S = -A^tCA$. It follows that the existence of a symmetric matrix C is equivalent to the symmetry condition

$$A^t\mathbb{J}A\mathbb{J}S = S\mathbb{J}A^t\mathbb{J}A$$

or

$$\Gamma^t\mathbb{J}S + S\mathbb{J}\Gamma = 0 \tag{3.3}$$

with $\Gamma = A^t\mathbb{J}A = -\Gamma^t$. Thus, in this case, the condition for having a canonoid transformation reduces to equation (3.3).

Remark 4. If the transformation is canonical the matrix A is symplectic ($A^T\mathbb{J}A = \mathbb{J}$), and thus, $\Gamma = \mathbb{J}$ and the condition is satisfied. The same is true if $\Gamma = a\mathbb{J}$ (with $a \neq 0$).

Remark 5. If A represents a rescaling of the given coordinates, namely

$$\begin{aligned} A_{ii} &= a_i,\, 1 \leq i \leq n, \\ A_{ii} &= b_i,\, n < i \leq 2n, \\ A_{ij} &= 0,\, i \neq j, \end{aligned}$$

then, $\Gamma\mathbb{J} = \mathbb{J}\Gamma = B$, a diagonal matrix determined by

$$\begin{aligned} B_{ii} &= -a_i b_i,\, 1 \leq i \leq n, \\ B_{ii} &= -a_i b_i,\, n < i \leq 2n, \\ B_{ij} &= 0,\, i \neq j. \end{aligned}$$

The transformation is canonoid if and only if (3.3) holds, i.e.

$$BS = SB.$$

When the rescaling is a point-transformation, then $a_i b_i = 1$ and it is always canonoid.

We now find more explicit conditions to have canonoid transformations. Write the matrices Γ and S in terms of $n \times n$ blocks as follows:

$$\Gamma = \begin{bmatrix} \lambda & \mu \\ -\mu^t & \nu \end{bmatrix}, \qquad S = \begin{bmatrix} \alpha & \beta \\ \beta^t & \gamma \end{bmatrix}$$

where $\lambda^t = -\lambda$, $\nu^t = -\nu$, $\alpha^t = \alpha$, and $\gamma^t = \gamma$. The equations $\Gamma\mathbb{J}S = S\mathbb{J}\Gamma$ leads to the system

$$\begin{aligned} -\lambda\beta^t + \mu\alpha &= \alpha\mu^t + \beta\lambda \\ -\lambda\gamma + \mu\beta &= -\alpha\nu + \beta\mu \\ \mu^t\beta^t + \nu\alpha &= \beta^t\mu^t + \gamma\lambda \\ \mu^t\gamma + \nu\beta &= -\beta^t\nu + \gamma\mu. \end{aligned}$$

If we let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$\begin{aligned} \Gamma &= \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} -c^t a + a^t c & -c^t b + a^t d \\ -d^t a + b^t c & -d^t b + b^t d \end{bmatrix} = \begin{bmatrix} \lambda & \mu \\ -\mu^t & \nu \end{bmatrix} \end{aligned} \tag{3.4}$$

Proposition 3.2. *Given the linear Hamiltonian system of Hamiltonian* $H(x) = \frac{1}{2}x^t Sx$*, the invertible transformation* $X = Ax$ *preserves the canonical structure of Hamilton's equations for all Hamiltonians if and only if* $\Gamma = a\mathbb{J}$ *for some constant* $a \neq 0$.

Proof. Showing that $\Gamma = a\mathbb{J}$ for some constant $a \neq 0$ satisfies the conditions is trivial Conversely, consider the particular case $\alpha = \gamma = \mathbf{0}$. We find that μ must commute with every $n \times n$ matrix, and therefore $\mu = a\mathbf{1}$. Choosing $\alpha = \beta = \mathbf{0}$ we find $\lambda = \mathbf{0}$. From $\beta = \gamma = \mathbf{0}$ it follows that $\nu = \mathbf{0}$. Hence $\Gamma = a\mathbb{J}$, and in addition, from $A^t\mathbb{J}A = a\mathbb{J}$ it follows that $\mathbb{J}A\mathbb{J} = -a(A^{-1})^t$. We finally find that $C = a(A^{-1})^t SA^{-1}$ and the new Hamiltonian is $K(X) = \frac{1}{2}X^tCX$. If A is symplectic it holds that $K(X) = H(x)$, and if $a \neq 1$ we find $K(X) = aH(x)$. $\square$

Example 3.3. Let S be such that $\beta = \alpha = \mathbf{0}$, and $\gamma = \mathbf{1}$. Then the equations reduce to

$$\begin{aligned}\mathbf{0} &= \mathbf{0}\\ -\lambda\gamma &= -\lambda\mathbf{1} = \mathbf{0}\\ \gamma\lambda &= \mathbf{1}\lambda = \mathbf{0}\\ \mu^t\gamma &= \gamma\mu\end{aligned}$$

Hence, from the first (or second) equation $\lambda = \mathbf{0}$, from the last equation $\mu^t = \mu$. Hence, by equation (3.4) the only requirements are that $c^ta = a^tc$ (i.e. c^ta is symmetric), and that $-c^tb + a^td = d^ta - b^tc$ (i.e. $-c^tb + a^td$ is symmetric).

Example 3.4. We specialize the previous example. Let $H = \frac{1}{2}(p_1^2 + p_2^2)$ then S is a 4×4 matrix with $\beta = \alpha = \mathbf{0}$ and $\gamma = \mathbf{1}$. Suppose that $a = \mathbf{1}$, $b = c = \mathbf{0}$, and that

$$d^{-1} = \begin{bmatrix} m & l \\ l & n \end{bmatrix}$$

Clearly $-c^tb + a^td = a^td = d$ is symmetric, since d is symmetric. Moreover, $c^ta = \mathbf{0}$ and so it is symmetric. Therefore, this transformation satisfies the conditions obtained in the previous example. Then we can compute C as $C = -\mathbb{J}A\mathbb{J}SA^{-1}$. We obtain

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m & l \\ 0 & 0 & l & n \end{bmatrix}$$

and hence $K = \frac{1}{2}(mP_1^2 + nP_2^2 + 2lP_1P_2)$.

We now show that given a canonoid transformation it is possible to find an additional symplectic structure and an additional first integral, and thus canonoid transformations can be used to find bihamiltonian structures and to study the integrability of Hamiltonian systems.

Let $\omega_1 = \omega$ be the symplectic form defined by

$$\omega(x, y) = x^t\mathbb{J}y.$$

Then, the Hamiltonian vector field with Hamiltonian H satisfy the equation

$$\omega_1(\mathcal{X}_H, v) = \mathbf{d}H(x) \cdot v$$

for all $v \in \mathbb{R}^{2n}$. Similarly, let Ω be the symplectic form in the "transformed space". Ω is defined as follows:

$$\Omega(X, Y) = X^t\mathbb{J}Y.$$

Then, the Hamiltonian vector field with Hamiltonian K is satisfies the equation

$$\Omega(\mathcal{X}_K, v) = \mathbf{d}K \cdot v$$

for all $v \in \mathbb{R}^{2n}$, where $\mathcal{X}_K = f_*(\mathcal{X}_H)$. Let f be the linear transformation defined as $X = f(x) = Ax$, where A is the $2n \times 2n$ invertible matrix introduced above.

We can use f to define a new canonical form in the "x" space by pulling back the canonical form Ω:

$$\begin{aligned}\omega_2(x,y) &= (f^*\Omega)(x,y) = \Omega(f(x),f(y))\\ &= \Omega(Ax,Ay) = (Ax)^t\mathbb{J}(Ay)\\ &= x^t(A^t\mathbb{J}A)y\end{aligned}$$

which gives an explicit expression of the symplectic form ω_2 in terms of the matrix A.

Then we can write, by pulling back the equation $\Omega(\mathcal{X}_K,v) = \mathbf{d}K\cdot v$

$$\omega_2(\mathcal{X}_H,v) = (f^*\Omega)(\mathcal{X}_H,v) = f^*(\mathbf{d}K)(v) = \mathbf{d}(f^*K)(v)$$

where $(f^*K)(x) = K(f(x)) = K(Ax)$. If we introduce $H_2(x) = (f^*K)(x)$ we can write

$$\omega_2(\mathcal{X}_H,v) = \mathbf{d}H_2\cdot v$$

Hence, the vector field $\mathcal{X}_H$ is Hamiltonian with respect to the symplectic form ω_1 and also Hamiltonian with respect to the symplectic form ω_2.

Example 3.5. We now continue example 3.4. We compute ω_2 and H_2 for this example. The transformation is given by the matrix

$$A = \begin{bmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & d\end{bmatrix}$$

where $\mathbf{1}$ and $\mathbf{0}$ are the 2×2 identity matrix and zero matrix, respectively. The matrix d is given by

$$d = \frac{1}{mn-l^2}\begin{bmatrix}n & -l\\ -l & m\end{bmatrix}$$

Then the matrix representative of ω_2 is given by

$$[\omega_2] = A^t\mathbb{J}A = \begin{bmatrix}\mathbf{0} & d\\ -d & \mathbf{0}\end{bmatrix},$$

note that the matrix A is symplectic if and only if $l=0$ and $m=n=1$, so that this transformation is symplectic if and only if it is the identity. The new Hamiltonian, obtained after some computations, is

$$H_2(x) = K(Ax) = \frac{1}{2(mn-l^2)}[np_1^2 + mp_2^2 - 2lp_1p_2].$$

Clearly H_2 is a first integral of the system with Hamiltonian H, since $\{H,H_2\} = 0$.

Example 3.6. We now consider a more interesting example, namely the harmonic oscillator. In this case $\beta = \mathbf{0}$, and $\alpha = \gamma = \mathbf{1}$. The conditions for having a canonoid transformation reduce to $\nu = \lambda$, and $\mu = \mu^t$ (i.e. μ is symmetric). Now suppose S is a 2×2 matrix.

(a) We can specialize the previous transformation by taking $a = d = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, $b = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$, and $c = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$. Then

$$\Gamma = \begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix},$$

where $E_3 = \Gamma$ is the matrix of the new symplectic form. Moreover,

$$C = \begin{bmatrix} -2 & 3 & 0 & 3 \\ 3 & 4 & -6 & 0 \\ 0 & -6 & 4 & -3 \\ 3 & 0 & -3 & -2 \end{bmatrix}$$

so that $K = \frac{1}{2}X^tCX$ is the Hamiltonian of the transformed system. Transforming K to the old coordinates yields the Hamiltonian

$$W_1 = (q_2p_1 - q_1p_2).$$

(b) We can also specialize the previous transformation by taking $b = c = 0$ and taking a and d to be symmetric matrices, then

$$\Gamma = \begin{bmatrix} 0 & a^td \\ -d^ta & 0 \end{bmatrix}, \quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \quad d = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix}$$

and $\mu = a^td$ is a symmetric matrix. Then

$$C = \begin{bmatrix} C_1 & \mathbf{0} \\ \mathbf{0} & C_2 \end{bmatrix}$$

where

$$C_1 = \frac{1}{\det a}\left[\begin{smallmatrix} a_{22}d_{11}-a_{12}d_{12} & a_{11}d_{12}-a_{12}d_{11} \\ a_{22}d_{12}-a_{12}d_{22} & a_{11}d_{22}-a_{12}d_{12} \end{smallmatrix}\right], \quad C_2 = \frac{1}{\det d}\left[\begin{smallmatrix} a_{11}d_{22}-a_{12}d_{12} & a_{12}d_{11}-a_{11}d_{12} \\ a_{12}d_{22}-a_{22}d_{12} & a_{22}d_{11}-a_{12}d_{12} \end{smallmatrix}\right]$$

then $K = \frac{1}{2}X^tCX$, and $H_2 = \frac{1}{2}x^t(A^tCA)x$, where

$$A^tCA = \left[\begin{smallmatrix} a_{11}d_{11}+a_{12}d_{12} & a_{11}d_{12}+a_{12}d_{22} & 0 & 0 \\ a_{12}d_{11}+a_{22}d_{12} & a_{12}d_{12}+a_{22}d_{22} & 0 & 0 \\ 0 & 0 & a_{11}d_{11}+a_{12}d_{12} & a_{12}d_{11}+a_{22}d_{12} \\ 0 & 0 & a_{11}d_{12}+a_{12}d_{22} & a_{12}d_{12}+a_{22}d_{22} \end{smallmatrix}\right]$$

If, in particular, we set $a_{11} = d_{12} = a_{22} = 0$, $a_{12} = 1$, $d_{11} = 1$, and $d_{22} = 1$, then we obtain $W_2 = (q_1q_2 + p_1p_2)$, which is a first integral, and the corresponding symplectic form has the following matrix representation

$$E_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

If, instead, we set $a_{12} = d_{12} = 0$, $a_{11} = a_{22} = d_{11} = 1$, and $d_{22} = -1$ then we obtain $W_3 = \frac{1}{2}(q_1^2 + p_1^2 - q_2^2 - p_2^2)$, which is a first integral, and the corresponding symplectic has the following matrix representation

$$E_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now, let $W_4 = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)$ and $E_4 = \mathbb{J}$. It is easy to see that

$$W_4^2 = W_1^2 + W_2^2 + W_3^2.$$

Suppose $\mathfrak{u}(2)$ is the Lie algebra of the Lie group $U(2)$ of 2×2 unitary matrices. If we consider $\mathfrak{u}(2)$ as a subspace of $sp(4, \mathbb{R})$, then $\{E_1, E_2, E_3, E_4\}$ is a basis for $\mathfrak{u}(2)$. Moreover, the functions W_1, W_2, W_3, and W_4 are a basis for the vector spaces of all quadratic integrals of the harmonic oscillator vector field. The map

$$\mathcal{H} : \mathbb{R}^4 \to \mathbb{R}^4 : (q, p) \to (w_1(q, p), w_2(q, p), w_3(q, p), w_4(q, p))$$

where $w_1 = 2W_1,\ w_2 = 2W_2,\ w_3 = 2W_3$, and $w_4 = 2W_4$ is called the **Hopf map**.

4. Linear Canonoid Transformations and the harmonic oscillator

When we perform a canonoid transformation of some Hamiltonian system, the integrability or superintegrability of the system are preserved. Indeed, a canonoid transformation is essentially a change of coordinates and, consequently, the existence of intrinsic structures like foliations made of invariant tori (Liouville or complete integrability) or the closure of the finite orbits (maximal superintegrability) are left unchanged. A canonoid transformation may only make these structures more or less evident and easy to handle by allowing the determination, together with the new coordinates, of a new Hamiltonian function and a new symplectic structure for the same dynamical system. From Prop. 6.3, it is clear that canonoid and Poissonoid transformations of a Hamiltonian system preserve the functionally independent constants of the motion of the system. Therefore, the transformed of a superintegrable system is again superintegrable with the transformed constants of motion. If the transformation is linear, then the degree of the polynomial constants of the motion is also preserved by the transformation. We see below how the two-dimensional harmonic oscillator, that admits three quadratic in the momenta first integrals, W_2, W_3 and W_4 seen above, behaves under linear canonoid transformations.

For our purpose, it is useful that the canonoid transformation of the two dimensional harmonic oscillator leads to a system with Hamiltonian in either one of the forms

$$K_1 = \frac{1}{2}(P_1^2 + P_2^2 + V(Q^1, Q^2)), \quad K_2 = P_1 P_2 + V(Q^1, Q^2).$$

We remark that the manifold where K_1 is defined as the Euclidean plane, while K_2 is defined in the Minkowski plane. Moreover, in the Euclidean case the form of the Hamiltonian is non-restrictive, since any non-degenerate Hamiltonian can be put in this form by a real canonical point-transformation. A linear canonical point-transformation changes K_2 into $P_1^2 - P_2^2 + V$. The Hamiltonians W_2 and W_3 are then recovered by K_2. Under the same constraints, we apply linear canonoid transformations also to the system of Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2),$$

that we can consider as an embedding of the one-dimensional harmonic oscillator in $\mathbb{E}^2$. This system possesses two evident quadratic first integrals, plus a functionally independent third-one

$$q_2 - p_2 \arctan\left(\frac{q_1}{p_1}\right),$$

not globally defined.

We recall that the linear transformation A is canonoid if and only if (3.3) holds. In order to obtain Hamiltonians of the prescribed form, we must constrain the 2×2 submatrix in the lower-right corner of $C = -\mathbb{J}A\mathbb{J}SA^{-1}$ to be

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad or \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

respectively, where the matrix S is determined by the original Hamiltonian. Moreover, the 2×2 submatrices in the upper-right and lower-left corners of C must be equal to zero. We can check if the transformations are canonical thanks to Proposition 3.2.

With these constraints, we search first for canonoid transformations of the isotropic harmonic oscillator $H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)$. In this case

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In order to perform computations with a computer-algebra software, we put $c_{12} = b_{21} = 0$. With this simplification we find that only one matrix C determines a Hamiltonian of the form K_1 and can be obtained by several different matrices A. The Hamiltonian K_1 is

$$K_1 = \frac{1}{2}(P_1^2 + P_2^2 + Q_1^2 + Q_2^2).$$

Therefore, the isotropic harmonic oscillator corresponds only to itself under a canonoid transformation of the prescribed type. We remark in particular that it is impossible to obtain by this way anisotropic harmonic oscillators, even if they are superintegrable when the ratio of the parameters is a rational number. This because one of the constants of the motion must be of degree higher than two in momenta or coordinates [12, 13].

If we start from the system of Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2), \tag{4.1}$$

then

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We find that, under the same assumptions regarding A and C, the canonoid transformations maps (4.1) into either

$$K_1 = \frac{1}{2}(P_1^2 + P_2^2 + Q_1^2),$$

a Hamiltonian identical to (4.1), or

$$K_1 = \frac{1}{2}(P_1^2 + P_2^2) + k(\alpha_1 Q_1 - \alpha_2 Q_2)^2,$$

where k, α_1 and α_2 are constants. In the last case, the point-transformation

$$Q_1 = (\alpha_2 Y - \alpha_1 X)\sqrt{(\alpha_1^2 + \alpha_2^2)}, \quad Q_2 = (\alpha_1 Y + \alpha_2 X)\sqrt{(\alpha_1^2 + \alpha_2^2)},$$

makes K_1 in the pristine form

$$\frac{1}{2}(P_X^2 + P_Y^2 + k' X^2),$$

for some suitable constant k'.

As an example of the second type of canonoid transformations, when $c_{21} = c_{22} = 0$, $b_{12}d_{22} = d_{12}b_{22}$, $d_{21}a_{22} = a_{21}d_{22}$, $b_{11} = -(2a_{21}^2 a_{22} d_{12}^2 d_{22}^2 + a_{22} d_{12}^3 a_{21} d_{11} d_{22} -$

$d_{12}^2 a_{11} a_{22} d_{11} d_{22}^2 + d_{12}^3 a_{11} a_{21} d_{22}^2 + a_{22} d_{12}^4 a_{21}^2 - d_{12}^4 a_{21}^2 d_{22} + a_{21}^2 a_{22} d_{22}^4)/(a_{22} d_{22}^2 d_{12}^2 c_{11})$ we have

$$K_1 = \frac{1}{2}(P_1^2 + P_2^2) + (d_{22}^2 + d_{12}^2)(d_{22}Q_1 - d_{12}Q_2)^2.$$

It can be checked that none of the transformations leading to the last form of K_1 is canonical.

By imposing the constraint corresponding to K_2, we are mapping the Euclidean harmonic oscillator into the Minkowski plane. Let us apply first the canonoid transformation to the isotropic oscillator. We obtain, with the same assumptions on A, that, for the admissible solutions, K_2 is always in the form

$$K_2 = P_1 P_2 + kQ_1 Q_2,$$

for some constant k. This is essentially the form of W_2 of the previous section. For example, if $a_{21} = c_{11} = b_{22} = d_{11} = 0$, $a_{12} d_{12} = a_{22} d_{22}$, $b_{12} d_{12} = -a_{22} c_{22}$, $a_{11} = (a_{22}^2 d_{21}^2 + c_{21}^2 a_{22}^2 + c_{21} b_{11} d_{12}^2)/(d_{12}^2 d_{21})$, then $k = d_{12}^2/a_{12}^2$, where we assume that A is invertible, i.e. $\det A = -a_{22}(d_{21}^2 + c_{21}^2)/d_{12} \neq 0$. None of the corresponding transformations is canonical.

This type of Hamiltonians corresponds to a well known class of quadratically superintegrable systems of the Minkowski plane, classified as Class II in [8] (in this reference, manifolds and Hamiltonians are considered in general complex while we limit ourselves to the real case).

We search now for canonoid transformations of the system (4.1) leading to Hamiltonians of the form K_2. It is possible in this case to consider the matrix A in full generality. We find that the Hamiltonian of the transformed system, when the transformation is canonoid, is always in the form

$$K_2 = P_1 P_2 + k(\alpha_1 Q_1 - \alpha_2 Q_2)^2, \tag{4.2}$$

where k, α_1 and α_2 are constants. After the point-transformation determined by

$$X = \alpha_1 Q_1 - \alpha_2 Q_2, \quad Y = \alpha_1 Q_1 + \alpha_2 Q_2,$$

we have

$$K_2 = \alpha_1 \alpha_2 (-P_X^2 + P_Y^2) + kX^2.$$

We can divide the Hamiltonian by the constant $\alpha_1 \alpha_2 \neq 0$ and see that, similarly to the original one, admits two evident quadratic first integrals and the functionally independent local third integral

$$Y + P_Y \ln(X + P_X).$$

In this case the local superintegrable structure remains unchanged. After the computation, we observe that no canonoid transformation such that $\alpha_1 \alpha_2 = 0$ do exist. Nevertheless, we can analyze the superintegrability of the system of Hamiltonian (4.2) in this case. If, say, $\alpha_2 = 0$, then the system admits two evident quadratic first integrals plus the third-one

$$P_2(Q_2 P_2 - Q_1 P_1) - \frac{2}{3} k Q_1^3,$$

and the system is quadratically superintegrable.

As an example, for $c_{22} = c_{12} = 0$, $c_{21} a_{22} = b_{21} d_{22}$, $d_{21} a_{22} = -a_{21} d_{22}$, $b_{12} d_{12} = d_{22} b_{22}$, $a_{12} d_{12} = d_{22} a_{22}$, $a_{11} = (-4 d_{22} a_{22} b_{21}^2 - d_{12}^2 b_{11} b_{21} - 4 d_{22} a_{21}^2 a_{22} + b_{21}^2 d_{22} d_{12} +$

$a_{21}^2 d_{22} d_{12} + a_{22} d_{12} c_{11} b_{11} + a_{22} d_{11} d_{22} a_{21} - c_{11} d_{22} b_{21} a_{22})/(d_{12}(a_{22} d_{11} + a_{21} d_{12}))$, with $\det A = -4 a_{22} d_{22}^2 (a_{21}^2 + b_{21}^2)/d_{12} \neq 0$, we have

$$K_2 = P_1 P_2 + \frac{d_{12}}{2 a_{22} d_{22}} (d_{12} Q_1 - d_{22} Q_2)^2.$$

A computation shows that none of the transformations leading to the last form of K_2 is canonical.

5. Whittaker's characterization

Since the first edition (1904) of his celebrated Treatise on Analytical Mechanics [29], E. T. Whittaker characterizes what we call here canonoid transformations. Given a system of ODEs

$$\frac{d}{dt} x^r = \mathcal{X}^r (x^1, \ldots, x^n, t), \quad r = 1, \ldots, n, \tag{5.1}$$

and a one-form $M(x^r, t)$, the absolute and relative integral invariants of the differential equations are defined following Poincaré [23]. We do not need here to recall the definitions of integral invariants (for this, see [29], §§112-116), but only their characterization in modern notation. We have that M determines an absolute invariant integral if and only if

$$\frac{\partial}{\partial t} M + \mathcal{L}_{\mathcal{X}} M = 0,$$

where $\mathcal{L}_{\mathcal{X}} M$ is the Lie derivative of M along the vector field $\mathcal{X}$. M determines a relative invariant integral if and only if $\mathbf{d}M$ is an absolute integral invariant. If the coordinates (x^i) can be divided into two sets (q^i, p_i), such that $n = 2N$, then, as stated in §116 of [29],

Proposition 5.1. *The ODEs (5.1) in coordinates (q^i, p_i) are in Hamiltonian form if and only if*

$$\Sigma_{i=1}^N p_i \delta q^i,$$

determines a relative invariant integral of (5.1).

Indeed, if we consider time-independent systems and if we identify the variational quantities δq^i with the differentials $\mathbf{d}q^i$ then $\Sigma_{i=1}^N p_i \delta q^i$ becomes the Liouville one-form $\theta = \sum_i p_i \mathbf{d}q^i$, and $-\mathbf{d}\theta = \omega = \sum \mathbf{d}q^i \wedge \mathbf{d}p_i$ becomes the symplectic form. Hence, by Cartan's magic formula, we have that

$$\mathcal{L}_{\mathcal{X}} (\mathbf{d}\theta) = -\mathcal{L}_{\mathcal{X}}(\omega) = -\mathbf{d}(\mathbf{i}_{\mathcal{X}}(\omega)) - \mathbf{i}_{\mathcal{X}} \mathbf{d}\omega = -\mathbf{d}(\mathbf{i}_{\mathcal{X}}(\omega)) = 0, \tag{5.2}$$

and thus the vector field $\mathcal{X}$ is locally Hamiltonian. Moreover, if the manifold is contractible, thanks to the Poincaré lemma we have

$$\mathbf{i}_{\mathcal{X}}(\omega) = \mathbf{d}H,$$

for some function H, that means that the system is Hamiltonian, and the previous statement follows in the case of the relative integral invariance condition. For the absolute integral invariance we have

$$0 = \mathcal{L}_{\mathcal{X}}(\theta) = \mathbf{i}_{\mathcal{X}} \mathbf{d}(\theta) + \mathbf{d}\mathbf{i}_{\mathcal{X}}(\theta) = -\mathbf{i}_{\mathcal{X}}\omega + \mathbf{d}\mathbf{i}_{\mathcal{X}}(\theta), \tag{5.3}$$

and the system is clearly Hamiltonian with Hamilton function $\mathbf{i}_{\mathcal{X}}(\theta)$.

Finally, in §136 of [29], the transformations of coordinates $(P_j(q^i, p_i), Q^j(q^i, p_i))$ that maintain the Hamiltonian form of (5.1), our canonoid transformations, are naturally characterized as those for which the form $\mathbf{PdQ}$ determines an invariant integral (relative or absolute) of the ODEs. Canonical transformations are defined in the same section of [29].

This characterization provides a simple direct way to characterize the possible canonoid transformations, or, equivalently, the possible alternative Hamiltonian representations for the field $\mathcal{X}$. Given a system of Hamiltonian H on a symplectic manifold with symplectic form ω, such that $\mathbf{i}_{\mathcal{X}}\omega = \mathbf{d}H$, we can determine another local Hamiltonian structure for the field $\mathcal{X}$ whenever we know some non-closed one-form Θ, such that $\mathbf{d}\Theta$ is non degenerate, satisfying

$$\mathcal{L}_{\mathcal{X}}\mathbf{d}\Theta = 0. \tag{5.4}$$

In this case, by (5.2), we know that, at least locally, $i_{\mathcal{X}}\mathbf{d}\Theta = \mathbf{d}K$ for some Hamiltonian function K. A stronger, global, condition is provided if Θ is an absolute invariant integral with $\mathbf{d}\Theta$ non degenerate. In this case, by (5.3) $\Omega = -\mathbf{d}\Theta$ is the new symplectic form and the new Hamiltonian K of the system is

$$K = \mathbf{i}_{\mathcal{X}}\Theta.$$

In both cases, when we can write $\Theta = P_i\mathbf{d}Q^i$ for some coordinate system (P_i, Q^i), the transformation $(p_i, q^i) \leftrightarrow (P_i, Q^i)$ is canonoid. We remark that, if $\Theta - \sum p_i\mathbf{d}q^i = \mathbf{d}f$ for some function f, then the transformation is the identity.

By putting $\mathcal{X} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$, the condition $\mathcal{L}_{\mathcal{X}}\Theta = 0$ becomes a system of first-order PDEs in $\Theta_i(q^j, p_j)$ involving the Hamiltonian H. These last two conditions are not very different from those given for generating functions of canonoid transformations in Carinena and Ranada [4].

We can call the one-forms Θ such that $\mathcal{L}_{\mathcal{X}}\Theta = 0$ **absolute generators** (or global generators) of a canonoid transformation. We call Θ a **relative generator** (or local generator) of a canonoid transformation when

$$\mathcal{L}_{\mathcal{X}}\mathbf{d}\Theta = 0.$$

If (p_i, q^i) are canonical coordinates, then the Liouville one-form $\Theta = \sum p_i\mathbf{d}q^i$ is a relative generator of the identity transformation.

For example, if H is the harmonic oscillator with coordinates $(x^1, x^2, x^3, x^4) = (q_1, q_2, p_1, p_2)$

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2),$$

then

$$\mathcal{X} = p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2} - q_1\frac{\partial}{\partial p_1} - q_2\frac{\partial}{\partial p_2}.$$

Therefore, the absolute generators of canonoid transformations of the harmonic oscillator are characterized by

$$\mathcal{L}_{\mathcal{X}}\Theta = 0 \Leftrightarrow \begin{cases} \mathcal{X}(\Theta_1) = \Theta_3, \\ \mathcal{X}(\Theta_2) = \Theta_4, \\ \mathcal{X}(\Theta_3) = -\Theta_1, \\ \mathcal{X}(\Theta_4) = -\Theta_2, \end{cases}$$

with the evident integrability conditions $\mathcal{X}^2(\Theta_i) = -\Theta_i$, $i = 1, \ldots, 4$.

A solution is $\Theta_1 = -p_1 + q_2$, $\Theta_2 = q_2$, $\Theta_3 = q_1 + p_2$, $\Theta_4 = p_2$. We have in this case

$$\mathbf{d}\Theta = \begin{bmatrix} 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

a non degenerate form with $\mathbf{d}(\mathbb{J} - \Theta) \neq 0$, and

$$K = \mathbf{i}_{\mathcal{X}}\Theta = -(p_1^2 + q_1^2 + q_1 p_2 - p_1 q_2),$$

that is a first integral of H. Then, the form Θ and the function K provide an alternative Hamiltonian structure for the harmonic oscillator. In the case when $H^1(M, \mathbb{R})$ is zero, relative generators also determine global Hamiltonian structures.

6. Poissonoid Transformations

The following definition is a natural extension of the definition of canonoid transformations to the case of regular Poisson manifolds.

Definition 6.1. Let (M, π) be a Poisson manifold, and let $\mathcal{X}$ be a locally Hamiltonian vector field on M, that is, for each $x \in M$, there is a neighborhood U of x and locally defined function H_U such that $\mathcal{X} = \pi^\sharp \mathbf{d} H_U$. A diffeomorphism $f : M \to M$ is said to be **Poissonoid** with respect to the vector field $\mathcal{X}$ if the transformed vector field $f_*\mathcal{X}$ is also locally Hamiltonian, that is, for each $x \in M$, there is a neighborhood V of $f(x)$ and a locally defined function K_V such that $f_*\mathcal{X} = \pi^\sharp \cdot \mathbf{d} K_V$.

This is equivalent to saying that, for each $y \in M$, there is a neighborhood V of y and a locally defined function K_V such that $\mathcal{X} = (f^*\pi^\sharp) \cdot \mathbf{d}(f^* K_V)$.

Remark 6. By Proposition 2.13 the previous definition means that, for each point $x \in M$ the system of equations associated with $\mathcal{X} = \pi^\sharp \cdot \mathbf{d} H_U$ can be written, in splitting coordinates in the neighborhood U of x, as:

$$\dot{q}^i = \frac{\partial H_U}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_U}{\partial q^i}, \quad \dot{z}^k = \sum_l \tilde{\phi}^{kl}(z) \frac{\partial H_U}{\partial z^l} \tag{6.1}$$

and the system associated with $f_*\mathcal{X} = \pi^\sharp \cdot \mathbf{d} K_U$ can be written, choosing appropriate splitting coordinates in a neighborhood V of $f(x)$, as:

$$\dot{Q}^i = \frac{\partial K_U}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K_U}{\partial Q^i}, \quad \dot{Z}^k = \sum_l \tilde{\phi}^{kl}(Z) \frac{\partial K_U}{\partial Z^l} \tag{6.2}$$

so that the transformation f carries the system of Hamilton's equations 6.1 again into a system of Hamilton's equations 6.2.

Note that, in the case of a Poisson manifold, not all Poisson vector fields are locally Hamiltonian. This means that if f is a Poissonoid map then $\mathcal{L}_{\mathcal{X}}(f^*\pi) = 0$, but the converse is in general not true. For instance, if π is the trivial Poisson bivector, then any diffeomorphism pushes π to the trivial bivector and $\mathcal{L}_{\mathcal{X}}(f^*\pi) = 0$, for any $\mathcal{X}$. On the other hand, the only locally Hamiltonian vector field is the trivial one. If $\mathcal{L}_{\mathcal{X}}(f^*\pi) = 0$, we say that the map is **weakly Poissonoid** with respect to the vector field $\mathcal{X}$. Weakly Poissonoid maps, in general, do not lead to the nice structure described in Remark 6.

Remark 7. Since Poissonoid transformations are diffeomorphisms, by Corollary 2.23 they send compatible Poisson bivectors into compatible Poisson bivectors. This may be of use in finding hierarchies of (compatible) Poisson structures.

6.1. **Poissonoid transformations and integrals of motion.** The definition of Poissonoid transformations can be specialized to Hamiltonian systems instead of locally-Hamiltonian ones: If $\mathcal{X}$ is a Hamiltonian vector field on a Poisson manifold (M, π), a diffeomorphism $f : M \to M$ is a Poissonoid transformation with respect to $\mathcal{X}$ if the transformed field $f_*\mathcal{X}$ is also Hamiltonian with respect to π, that is if there is a smooth function K on M such that $f_*\mathcal{X} = \pi^\sharp \cdot \mathbf{d}K$.

Since f is a diffeomorphism, we have that the vector field $f_*\mathcal{X}$ is Hamiltonian with respect to π if and only if $\mathcal{X}$ is Hamiltonian with respect the transformed bivector $f^*\pi$, i.e. there exists a smooth function K' on M such that

$$\mathcal{X} = (f^*\pi^\sharp) \cdot \mathbf{d}(K').$$

This means that, if f is a Poissonoid transformation for $\mathcal{X}$, then $\mathcal{X}$ admits a new and possibly different Hamiltonian structure. If, in addition π and $f^*\pi$ are compatible, then the vector field $\mathcal{X}$ will be biHamiltonian. Under some additional conditions these facts are enough to show that the system has a complete set of integrals in involution.

More explicitly, the existence of the additional Poisson bivector $f^*\pi$, provides a concrete way of obtaining additional constants of motion from the Casimirs of $f^*\pi$

Proposition 6.2. *Let (M, π) a Poisson manifold, and let $\{\ ,\ \}$ the Poisson bracket associated with π. Suppose $\mathcal{X}$ is an Hamiltonian vector field, so that there exist a function H such that $\mathcal{X} = \pi^\sharp \cdot \mathbf{d}H$, and suppose that f is a Poissonoid transfomation for $\mathcal{X}$, so that there exist a function K' such that $\mathcal{X} = (f^*\pi^\sharp) \cdot \mathbf{d}(K')$. Then K' and any Casimir of the Poisson bivector π are constants of motion of the Hamiltonian system (M, π, H).*

Proof. Since f is a Poissonoid transformation for $\mathcal{X}$ we have that

$$\mathcal{X} = \pi^\sharp \cdot \mathbf{d}H = (f^*\pi^\sharp) \cdot \mathbf{d}K'.$$

Suppose C is a Casimir of $(f^*\pi)$, and that $\{\ ,\ \}'$ is the Poisson bracket associated to $(f^*\pi)$, then

$$\begin{aligned}
0 =& \{C, K'\}' = (f^*\pi)(\mathbf{d}C, \mathbf{d}K') \\
=& \left\langle \mathbf{d}C, (f^*\pi^\sharp) \cdot \mathbf{d}K' \right\rangle = \langle \mathbf{d}C, \mathcal{X} \rangle \\
=& \left\langle \mathbf{d}C, \pi^\sharp \cdot \mathbf{d}H \right\rangle = \pi(\mathbf{d}C, \pi^\sharp \cdot \mathbf{d}H\rangle \\
=& \{C, H\}
\end{aligned}$$

Hence, $\{C, H\} = 0$, and C is a constant of motion of the Hamiltonian system (M, π, H). The proof that K' is a costant of motion is similar. □

Another important property is that Poissonoid transformations preserve constants of motion:

Proposition 6.3. *Let (M, π, H) be a Hamiltonian system, and let f be a Poissonoid transformation such that $f_*\mathcal{X}_H = \pi^\sharp \cdot \mathbf{d}K$, then F is a constant of motion of the transformed system if and only if f^*F is a constant of motion for (M, π, H).*

Proof.
$$\frac{dF}{dt} = \{F, K\} = \left\langle \mathbf{d}F, \pi^\sharp \cdot \mathbf{d}K \right\rangle = \langle \mathbf{d}F, f_*\mathcal{X}_H \rangle = \left\langle \mathbf{d}F, f_*\pi^\sharp \cdot \mathbf{d}H \right\rangle$$
since $\mathcal{X}_K = f_*\mathcal{X}_H$. Taking the pull-back of the above yields
$$f^*\left(\frac{dF}{dt}\right) = \frac{df^*F}{dt} = f^*\left\langle \mathbf{d}F, f_*(\pi^\sharp \cdot \mathbf{d}H) \right\rangle = \left\langle \mathbf{d}(f^*F), \pi^\sharp \cdot \mathbf{d}H \right\rangle = \{f^*F, H\}.$$
The proof follows. □

Remark 8. Since the transformation f is a diffeomorphism, functionally independent constants of motion are sent into functionally independent constants of motion. In the case when the Poisson structure coincides with a symplectic structure, Proposition 6.3 applies to canonoid transformations.

Example 6.4 (Euler's Equations for the rigid body)**.** Let $\mathfrak{so}(3)$ be the Lie algebra of $SO(3)$, the group of rotations in $\mathbb{R}^3$, and let π be the Poisson tensor associated to the Lie-Poisson bracket. Then (M, π), is a Poisson manifold. On the manifold $M = \mathfrak{so}^*(3)$ we introduce the coordinates $\mathbf{m} = (m_1, m_2, m_3) \in \mathfrak{so}^*(3) \cong \mathbb{R}^3$. In these notations the Poisson tensor has the form
$$\pi_{\mathbf{m}} = \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix}.$$
Euler's rigid body equations, are the Hamiltonian equations on $(\mathfrak{so}^*(3), \pi)$ with Hamiltonian function
$$H = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_2^2}{I_3}\right),$$
where I_1, I_2, I_3 are the principal moments of inertia of the rigid body. The corresponding Hamiltonian vector field, given by $\mathcal{X} = \pi^\sharp \cdot \mathbf{d}H$ is:
$$\mathcal{X} = \begin{bmatrix} \frac{I_2 - I_3}{I_2 I_3} m_2 m_3 \\ \frac{I_3 - I_1}{I_3 I_1} m_3 m_1 \\ \frac{I_1 - I_2}{I_1 I_2} m_1 m_2 \end{bmatrix}$$
Let $N = \mathfrak{so}^*(3)$ we introduce the coordinates $\mathbf{n} = (n_1, n_2, n_3) \in \mathfrak{so}^*(3) \cong \mathbb{R}^3$, then $(N, \pi_{\mathbf{n}})$ is a Poisson manifold. Consider the diffeomorphism $f : M \to N$ such that $(n_1, n_2, n_3) = f(m_1, m_2, m_3)$ defined by the equations $n_1 = am_1, n_2 = bm_2, n_3 = cm_3$, where a, b and c are non-zero constants. The pushforward of $\mathcal{X}$ can be expressed as
$$f_*\mathcal{X} = \begin{bmatrix} \frac{(I_2 - I_3)a}{I_2 I_3 bc} n_2 n_3 \\ \\ \frac{(I_3 - I_1)b}{I_3 I_1 ac} n_3 n_1 \\ \frac{(I_1 - I_2)c}{I_1 I_2 ab} n_1 n_2 \end{bmatrix}$$
If $a = \sqrt{I_2 I_3}, b = \sqrt{I_1 I_3}$ and $c = \sqrt{I_1 I_2}$

$$f_*\mathcal{X} = \begin{bmatrix} \dfrac{(I_2 - I_3)}{I_1 I_2 I_3} n_2 n_3 \\ \dfrac{(I_3 - I_1)}{I_1 I_2 I_3} n_3 n_1 \\ \dfrac{(I_1 - I_2)}{I_1 I_2 I_3} n_1 n_2 \end{bmatrix}$$

and one possible corresponding Hamiltonian, obtained from $f_*\mathcal{X} = \pi_{\mathbf{n}} \mathbf{d}K$, is

$$K = -\frac{1}{2}\left(\frac{n_1^2}{I_2 I_3} + \frac{n_2^2}{I_1 I_3} + \frac{n_3^2}{I_1 I_2}\right).$$

Pulling back K we get $f^*K = -\frac{1}{2}(m_1^2 + m_2^2 + m_2^2)$. Pulling back the form $\pi_{\mathbf{n}}$ yields

$$f^*\pi_{\mathbf{n}} = \begin{bmatrix} 0 & -\dfrac{m_3}{I_3} & \dfrac{m_2}{I_2} \\ \dfrac{m_3}{I_3} & 0 & -\dfrac{m_1}{I_1} \\ -\dfrac{m_2}{I_2} & \dfrac{m_1}{I_1} & 0 \end{bmatrix}$$

and thus $\mathcal{X} = \pi^\sharp_{\mathbf{m}} \cdot \mathbf{d}H = (f^*\pi_{\mathbf{n}})^\sharp \cdot \mathbf{d}(f^*K)$. This shows that the rigid body equations are biHamiltonian.

Example 6.5 (Euler's equations on $\mathfrak{so}^*(4)$)**.** Here we use the same notations as in [9]. The manifold $M = \mathfrak{so}^*(4)$ is six dimensional. Since $\mathfrak{so}(4)$ is isomorphic to the space of 4×4 skew-symmetric matrices, identifying $\mathfrak{so}^*(4)$ with $\mathfrak{so}(4)$ we can write any element of $\mathfrak{so}^*(4)$ as

$$\mathcal{M} = \sum_{i<j=1}^{4} m_{ij}\left(E_{ij} - E_{ji}\right).$$

where E_{ij} denotes the elementary matrix whose (i, j) entry is 1. The manifold $M = \mathfrak{so}^*(4)$ is endowed with a Lie-Poisson structure that, in the variables $\mathbf{m} = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34})$, can be written as

$$\pi_{\mathbf{m}} = \begin{bmatrix} 0 & -m_{23} & -m_{24} & m_{13} & m_{14} & 0 \\ m_{23} & 0 & -m_{34} & -m_{12} & 0 & m_{14} \\ m_{24} & m_{34} & 0 & 0 & -m_{12} & -m_{13} \\ -m_{13} & m_{12} & 0 & 0 & -m_{34} & m_{24} \\ -m_{14} & 0 & m_{12} & m_{34} & 0 & -m_{23} \\ 0 & -m_{14} & m_{13} & -m_{24} & m_{23} & 0 \end{bmatrix}.$$

The Euler-Manakov equations for the rigid body on $(\mathfrak{so}^*(4), \pi_{\mathbf{m}})$ are the Hamiltonian equations with the following quadratic Hamiltonian

$$H = \frac{1}{2}\sum_{i<j} a_{ij} m_{ij}^2$$

where the coefficients a_{ij} can be written as

$$a_{ij} = J_l^2 + J_k^2$$

with $\{i,j,l,k\}$ a permutation of $\{1,2,3,4\}$. The rank of the $\mathfrak{so}^*(4)$ Lie-Poisson structure is 4 almost everywhere, the Casimirs are

$$C_1 = m_{12}^2 + m_{13}^2 + m_{14}^2 + m_{23}^2 + m_{24}^2 + m_{34}^2, \quad C_2 = m_{12}m_{34} + m_{14}m_{23} - m_{13}m_{24}$$

where $C_1 = -\frac{1}{2}\operatorname{Tr}(\mathcal{M})$, and $C_2 = \operatorname{Pf}(\mathcal{M})$ (with $\operatorname{Pf}(\mathcal{M})$ the Pfaffian of $\mathcal{M}$). The corresponding Hamiltonian vector field, given by $\mathcal{X} = \pi^\sharp \cdot \mathbf{d}H$ is

$$\mathcal{X} = \begin{bmatrix} J_1^2(m_{13}m_{23} + m_{14}m_{24}) - J_2^2(m_{13}m_{23} + m_{14}m_{24}) \\ J_3^2(m_{12}m_{23} - m_{14}m_{34}) + J_1^2(m_{14}m_{34} - m_{12}m_{23}) \\ J_4^2(m_{12}m_{24} + m_{13}m_{34}) - J_1^2(m_{12}m_{24} + m_{13}m_{34}) \\ J_2^2(m_{12}m_{13} + m_{24}m_{34}) - J_3^2(m_{12}m_{13} + m_{24}m_{34}) \\ J_4^2(m_{23}m_{34} - m_{12}m_{14}) + J_2^2(m_{12}m_{14} - m_{23}m_{34}) \\ J_3^2(m_{13}m_{14} + m_{23}m_{24}) - J_4^2(m_{13}m_{14} + m_{23}m_{24}) \end{bmatrix}$$

Let $N = \mathfrak{so}(4)$ we introduce the coordinates $\mathbf{n} = (n_{12}, n_{13}, n_{14}, n_{23}, n_{24}, n_{34})$ then $(N, \pi_{\mathbf{n}})$ is a Poisson manifold. Consider the diffeomorphism $f : M \to N$ defined by the equations

$$n_{12} = \frac{m_{12}}{J_1J_2},\ n_{13} = \frac{m_{13}}{J_1J_3},\ n_{14} = \frac{m_{14}}{J_1J_4}$$
$$n_{23} = \frac{m_{23}}{J_2J_3},\ n_{24} = \frac{m_{24}}{J_2J_4},\ n_{34} = \frac{m_{34}}{J_3J_4}$$

The pushforward of $\mathcal{X}$ can be expressed as

$$f_*\mathcal{X} = \begin{bmatrix} J_1^2(J_3^2n_{13}n_{23} + J_4^2n_{14}n_{24}) - J_2^2(J_3^2n_{13}n_{23} + J_4^2n_{14}n_{24}) \\ J_3^2(J_2^2n_{12}n_{23} - J_4^2n_{14}n_{34}) + J_1^2(J_4^2n_{14}n_{34} - J_2^2n_{12}n_{23}) \\ J_4^2(J_2^2n_{12}n_{24} + J_3^2n_{13}n_{34}) - J_1^2(J_2^2n_{12}n_{24} + J_3^2n_{13}n_{34}) \\ J_2^2(J_1^2n_{12}n_{13} + J_4^2n_{24}n_{34}) - J_3^2(J_1^2n_{12}n_{13} + J_4^2n_{24}n_{34}) \\ J_4^2(J_3^2n_{23}n_{34} - J_1^2n_{12}n_{14}) + J_2^2(J_1^2n_{12}n_{14} - J_3^2n_{23}n_{34}) \\ J_3^2(J_1^2n_{13}n_{14} + J_2^2n_{23}n_{24}) - J_4^2(J_1^2n_{13}n_{14} + J_2^2n_{23}n_{24}) \end{bmatrix}$$

and one possible Hamiltonian, corresponding to $f_*\mathcal{X}$, and obtained from $f_*\mathcal{X} = \pi_{\mathbf{n}}\mathbf{d}K$, is

$$K = -\frac{1}{2}\left(J_1^2J_2^2n_{12}^2 + J_1^2J_3^2n_{13}^2 + J_1^2J_4^2n_{14}^2 + J_2^2J_3^2n_{23}^2 + J_2^2J_4^2n_{24}^2 + J_3^2J_4^2n_{34}^2\right).$$

Pulling back K we get

$$f^*K = -\frac{1}{2}(m_{12}^2 + m_{13}^2 + m_{14}^2 + m_{23}^2 + m_{24}^2 + m_{34}^2),$$

a Casimir of $\pi_{\mathbf{m}}$. Pulling back the form $\pi_{\mathbf{n}}$ yields

$$f^*\pi_{\mathbf{n}} = \begin{bmatrix} 0 & -J_1^2m_{23} & -J_1^2m_{24} & J_2^2m_{13} & J_2^2m_{14} & 0 \\ J_1^2m_{23} & 0 & -J_1^2m_{34} & -J_3^2m_{12} & 0 & J_3^2m_{14} \\ J_1^2m_{24} & J_1^2m_{34} & 0 & 0 & -J_4^2m_{12} & -J_4^2m_{13} \\ -J_2^2m_{13} & J_3^2m_{12} & 0 & 0 & -J_2^2m_{34} & J_3^2m_{24} \\ -J_2^2m_{14} & 0 & J_4^2m_{12} & J_2^2m_{34} & 0 & -J_4^2m_{23} \\ 0 & -J_3^2m_{14} & J_4^2m_{13} & -J_3^2m_{24} & J_4^2m_{23} & 0 \end{bmatrix}.$$

Therefore, $\mathcal{X} = \pi_{\mathbf{m}}^{\sharp} \cdot \mathbf{d}H = (f^*\pi_{\mathbf{n}})^{\sharp} \cdot \mathbf{d}(f^*K)$. This computation recovers the Poisson structure obtained in [2, 21, 9]. It can be shown that this Poisson structure is compatible with the original one. Thus, we rediscovered that the Euler-Manakov equations of motion admit a bihamiltonian formulation (compare our expressions with [9]). Using this additional Poisson structure it is possible to prove the integrability of the Euler-Manakov equations (see [2, 21, 9]), in fact they admit one additional integral of motion

$$I_1 = J_3^2 J_4^2 m_{12}^2 + J_2^2 J_4^2 m_{13}^2 + J_2^2 J_3^2 m_{14}^2 + J_1^2 J_4^2 m_{23}^2 + J_1^2 J_3^2 m_{24}^2 + J_1^2 J_2^2 m_{34}^2.$$

Example 6.6 (Kirchhoff's equations)**.** In the previous examples it was possible to find a biHamiltonian structure by finding a Poissonoid transformation. In those examples, the transformation was a rescaling, and thus it was quite simple. We now we give an example where the Poissonoid transformation is more complex.

The motion of a rigid body in an ideal fluid can be described by the so called **Kirchhoff's equations**. These equations can be written as Hamiltonian equations on $\mathfrak{e}^*(3)$ (the Lie algebra of the group $E(3)$ of motions of the three-dimensional Euclidean space) with the Lie-Poisson bracket. The Lie algebra $\mathfrak{e}(3)$ is a semidirect sum of $\mathfrak{so}(3)$ and the group of translations in $\mathbb{R}^3$, that is $\mathfrak{e}(3) = \mathfrak{so}(3) \oplus \mathbb{R}^3$. On the manifold $\mathcal{M} \cong \mathfrak{e}^*(3)$, we introduce the coordinates $\mathbf{z} = (\mathbf{p}, \mathbf{m})$, where

$$\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3, \quad \mathbf{m} = (m_1, m_2, m_3) \in \mathbb{R}^3 \cong \mathfrak{so}(3)$$

are two three dimensional vectors. The hat map $\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$ defined as

$$\mathbf{v} = (v_1, v_2, v_3) \to \hat{\mathbf{v}} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

defines an isomorphism of the Lie algebras $(\mathbb{R}^3, \times)$ and $(\mathfrak{so}(3), [,])$. If we identify the Lie algebra $\mathfrak{e}(3)$ with its dual, in these notations, the Lie-Poisson tensor has the form

$$\pi = \begin{bmatrix} 0 & 0 & 0 & 0 & -p_3 & p_2 \\ 0 & 0 & 0 & p_3 & 0 & -p_1 \\ 0 & 0 & 0 & -p_2 & p_1 & 0 \\ 0 & -p_3 & p_2 & 0 & -m_3 & m_2 \\ p_3 & 0 & -p_1 & m_3 & 0 & -m_1 \\ -p_2 & p_1 & 0 & -m_2 & m_1 & 0. \end{bmatrix}$$

This Poisson tensor has the following quadratic Casimirs:

$$C_1 = p_1^2 + p_2^2 + p_3^2, \quad C_2 = m_1 p_1 + m_2 p_2 + m_3 p_3.$$

Hamilton's equations corresponding to a quadratic Hamiltonian are called Kirchhoff equations.

A famous integrable case of the Kirchhoff equations was discovered by Clebsch and it is characterized by the Hamiltonian

$$H_1 = \frac{1}{2}\left(m_1^2 + m_2^2 + m_3^2 + \omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2\right)$$

The Hamiltonian vector field in this case is

$$\mathcal{X} = \pi^\sharp \cdot \mathbf{d}H_1 = \begin{bmatrix} m_3p_2 - m_2p_3 \\ m_1p_3 - m_3p_1 \\ m_2p_1 - m_1p_2 \\ (\omega_3-\omega_2)p_2p_3 \\ (\omega_1-\omega_3)p_1p_3 \\ (\omega_2-\omega_1)p_1p_2 \end{bmatrix}$$

and an additional integral of motion is

$$I = \frac{1}{2}\left(\omega_1 m_1^2 + \omega_2 m_2^2 + \omega_3 m_3^2 - \omega_2\omega_3 p_1^2 - \omega_3\omega_1 p_2^2 - \omega_1\omega_2 p_3^2\right).$$

A Poisson bivector compatible with π is

$$\widetilde{\eta} = \begin{bmatrix} 0 & -m_3 & m_2 & 0 & 0 & 0 \\ m_3 & 0 & -m_1 & (\omega_1-\omega_2)p_3 & 0 & (\omega_2-\omega_1)p_1 \\ -m_2 & m_1 & 0 & (\omega_3-\omega_1)p_2 & (\omega_1-\omega_3)p_1 & 0 \\ 0 & (\omega_2-\omega_1)p_3 & (\omega_1-\omega_3)p_2 & 0 & (\omega_3-\omega_1)m_3 & (\omega_1-\omega_2)m_2 \\ 0 & 0 & (\omega_3-\omega_1)p_1 & (\omega_1-\omega_3)m_3 & 0 & 0 \\ 0 & (\omega_1-\omega_2)p_1 & 0 & (\omega_2-\omega_1)m_2 & 0 & 0 \end{bmatrix}$$

Note that a linear change of variables transforms the Euler equations on $\mathfrak{so}^*(4)$ to equations $\mathfrak{e}^*(3)$, which in the case of a positive definite quadratic Hamiltonian are the Kirchhoff's equations describing the motion of a rigid body in an ideal fluid [2]. Thus, in principle, the Poisson bivector above and the Poissonoid transformation we find below could be obtained from the previous example. However, we prefer to find them with a direct computation.

In this case, it is easy to verify that $\mathcal{X} = \pi^\sharp \cdot \mathbf{d}H_1 = (\widetilde{\eta})^\sharp \cdot \mathbf{d}(-\frac{1}{2}C_1)$. It can also be shown that this Poisson structure is compatible with the original one. Hence, the Clebsch system is biHamiltonian.

In analogy with the previous examples, we search now for a transformation $\phi = f^{-1}$ such that

$$-\frac{1}{2}\tilde{\eta} = f^*\pi_F, \tag{6.3}$$

(we consider here $-\frac{1}{2}\tilde{\eta}$ instead of $\tilde{\eta}$ for computational convenience) and, consequently, $\mathcal{X} = (f^*\pi_F)d(C_1)$, where $F^1, \ldots, F^6$ are new coordinates and

$$\pi_F = \begin{bmatrix} 0 & 0 & 0 & 0 & -F^3 & F^2 \\ 0 & 0 & 0 & F^3 & 0 & -F^1 \\ 0 & 0 & 0 & -F^2 & F^1 & 0 \\ 0 & -F^3 & F^2 & 0 & -F^6 & F^5 \\ F^3 & 0 & -F^1 & F^6 & 0 & -F^4 \\ -F^2 & F^1 & 0 & -F^5 & F^4 & 0 \end{bmatrix}.$$

While in all the previous examples f was a simple rescaling of the old coordinates, here we must assume that f, and thus ϕ, are linear and homogeneous in the new coordinates F^i, so that

$$\phi : \phi^i = p_i = a^i_j F^j, \qquad \phi^{i+3} = m_i = a^{i+3}_j F^j, \quad i = 1, \ldots, 3;\ j = 1, \ldots, 6; \tag{6.4}$$

with a^k_j constants. The previous relations allow to write the components of $\tilde{\eta}$ with respect to the (ϕ^j) as functions of the (F^j), therefore, we can solve the equation (6.3) without inverting the transformation ϕ. Substituting $\phi = f^{-1}$ and $X = F$ in equation (2.4), and using (6.4), yields:

$$\begin{aligned}(f^*\pi)^{jk} &= \left(\frac{\partial \phi^j}{\partial F^r}\circ f\right)\left(\frac{\partial \phi^k}{\partial F^s}\circ f\right)\pi^{rs}\circ f\\ &= \left(a^j_r\circ f\right)\left(a^k_s\circ f\right)\pi^{rs}\circ f\\ &= a^j_r a^k_s \pi^{rs}\circ(\phi^{-1})\end{aligned}$$

Therefore, condition (6.3) written explicitly in local coordinates is

$$-\frac{1}{2}\tilde{\eta}^{jk}(x) = a^j_r a^k_s \pi^{rs}(f(x)) = a^j_r a^k_s \pi^{rs}(\phi^{-1}(x))$$

or, if we take $x = \phi(F)$

$$-\frac{1}{2}\tilde{\eta}^{jk}(\phi(F)) = a^j_r a^k_s \pi^{rs}(\phi^{-1}(\phi(F))) = a^j_r a^k_s \pi^{rs}(F)$$

that can also be written as

$$-\frac{1}{2}\tilde{\eta}^{jk} = a^j_r a^k_s \pi^{rs}_F$$

This last equation can be solved for the a^j_k giving for example the following linear transformation

$$\begin{bmatrix} p_1\\ p_2\\ p_3\\ m_1\\ m_2\\ m_3\end{bmatrix} = \begin{bmatrix} a & 0 & -\frac{a}{2\epsilon}B & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2}A & 0\\ 0 & 0 & 0 & \frac{1}{2}B & 0 & \epsilon\\ 0 & 0 & 0 & -\epsilon A & 0 & \frac{1}{2}AB\\ 0 & -\frac{a}{2\epsilon}C & 0 & 0 & 0 & 0\\ -\frac{a}{2\epsilon}AB & 0 & -aA & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} F_1\\ F_2\\ F_3\\ F_4\\ F_5\\ F_6\end{bmatrix},$$

where a, ϵ are real parameters, $A = \sqrt{\omega_1-\omega_2}$, $B = \sqrt{\omega_1-\omega_3-4\epsilon^2}$, $C = \omega_1-\omega_3$, and whose determinant is

$$-\left(\frac{a}{4\epsilon}AC\right)^3.$$

If $\omega_1 > \omega_2 > \omega_3 \geq 0$ then a positive value of ϵ can always be found such that the transformation is real. In the new coordinates we have

$$\begin{aligned}C_1 =& a^2(F^1)^2 + \left(\frac{a}{2\epsilon}B\right)^2(F^3)^2 + \frac{1}{4}B^2(F^4)^2 + \frac{1}{4}A^2(F^5)^2\\ &+ \epsilon^2(F^6)^2 - \frac{a^2}{\epsilon}BF^1F^3 + \epsilon BF^4F^6.\end{aligned}$$

7. Infinitesimal Poissonoid transformations and symmetries

Let $(M, \pi, \mathcal{X})$ be a locally Hamiltonian vector field with a locally defined Hamiltonian function H, and f_t a one-parameter group of Poissonoid diffeomorphisms, then $(f_t)_*\mathcal{X} = \pi^\sharp \cdot \mathbf{d}K_t$. This expression can be rewritten as

$$\mathcal{X} = f_t^*(\pi)^\sharp \cdot (f_t^*\mathbf{d}K_t).$$

Let ξ be the vector field on M that is the infinitesimal generator of f_t. Differentiating the previous equation with respect to t yields

$$
\begin{aligned}
0 &= \frac{d}{dt}\bigg|_{t=0}\left[f_t^*(\pi)^\sharp\cdot(f_t^*\mathbf{d}K_t)\right] \\
&= (\mathcal{L}_\xi\pi^\sharp)\cdot(\mathbf{d}H)+\pi^\sharp\cdot(\mathcal{L}_\xi\mathbf{d}H)+\pi^\sharp\cdot(\mathbf{d}\dot{K}) \\
&= \mathcal{L}_\xi(\pi^\sharp\cdot\mathbf{d}H)+\pi^\sharp\cdot(\mathbf{d}\dot{K})
\end{aligned} \tag{7.1}
$$

where we used the definition of Lie derivative and $\dot{K}$ is the derivative of K with respect to t computed at $t = 0$. Hence

$$\mathcal{L}_\xi(\pi^\sharp\cdot\mathbf{d}H) = [\xi,\mathcal{X}] = -\pi^\sharp\cdot(\mathbf{d}\dot{K}) = \pi^\sharp\cdot(\mathbf{d}F)$$

where $F = -\dot{K}$. A vector field ξ satisfying the equation above is called a **infinitesimal Poissonoid transformation**.

Proposition 7.1. *A vector field ξ is a infinitesimal Poissonoid transformation for a locally Hamiltonian vector field $\mathcal{X}$ if and only if $[\xi,\mathcal{X}]$ is a locally Hamiltonian vector field. Moreover, if ξ is an infinitesimal symmetry of $\mathcal{X}$ then it is also an infinitesimal Poissonoid transformation.*

Proof. By definition of infinitesimal Poissonoid transformation we have $[\xi,\mathcal{X}] = \pi^\sharp\cdot(\mathbf{d}F)$, and thus ξ is locally Hamiltonian if and only if ξ is an infinitesimal Poissonoid transformation. If ξ is an infinitesimal symmetry of $\mathcal{X}$ then $[\xi,\mathcal{X}] = 0$. Consequently, $[\xi,\mathcal{X}]$ is locally Hamiltonian with F the constant function, and thus it is an infinitesimal Poissonoid transformation. □

In [3] canonoid transformations are studied using cohomology techniques. If $(M,\pi,\mathcal{X})$ is a locally Hamiltonian system, then, in analogy with [3] one can introduce twisted boundary and coboundary operators defined as follows:

$$\partial_\mathcal{X} = \mathbf{i}_\mathcal{X}\circ\mathbf{d}\circ\mathbf{i}_\mathcal{X}, \qquad \mathbf{d}_\mathcal{X} = \mathbf{d}\circ\mathbf{i}_\mathcal{X}\circ\mathbf{d}$$

where $\mathbf{d}$ is the usual de Rham differential and $\mathbf{i}_\mathcal{X}$ denotes contraction.

Proposition 7.2. *Suppose $(M,\pi,\mathcal{X})$ is a locally Hamiltonian system with (locally defined) Hamiltonian H, then $\mathcal{X}_F$ is a Hamiltonian infinitesimal symmetry of $(M,\pi,\mathcal{X})$ if and only if $\mathbf{d}_\mathcal{X}F\in\ker(\pi^\sharp)$.*

Proof. Let $F\in C^\infty(M)$, and let $\mathcal{X}_F = \pi^\sharp\cdot\mathbf{d}F$ be its Hamiltonian vector field. Let α be an arbitrary one form, then the preceding equation can be written as

$$\langle\alpha,\mathcal{X}_F\rangle = \left\langle\alpha,\pi^\sharp\cdot\mathbf{d}F\right\rangle.$$

Taking the Lie derivative of the left hand side yields

$$\mathcal{L}_\mathcal{X}\langle\alpha,\mathcal{X}_F\rangle = \langle\mathcal{L}_\mathcal{X}\alpha,\mathcal{X}_F\rangle + \langle\alpha,\mathcal{L}_\mathcal{X}(\mathcal{X}_F)\rangle. \tag{7.2}$$

Taking the derivative of the right hand side yields

$$
\begin{aligned}
\mathcal{L}_\mathcal{X}(\left\langle\alpha,\pi^\sharp\cdot\mathbf{d}F\right\rangle) &= \mathcal{L}_\mathcal{X}(\pi(\alpha,\mathbf{d}F)) \\
&= (\mathcal{L}_\mathcal{X}\pi)(\alpha,\mathbf{d}F)+\pi(\mathcal{L}_\mathcal{X}(\alpha),\mathbf{d}F)+\pi(\alpha,\mathcal{L}_\mathcal{X}(\mathbf{d}F)) \\
&= \pi(\mathcal{L}_\mathcal{X}(\alpha),\mathbf{d}F)+\pi(\alpha,\mathcal{L}_\mathcal{X}(\mathbf{d}F)) \\
&= \left\langle\mathcal{L}_\mathcal{X}(\alpha),\pi^\sharp\cdot\mathbf{d}F\right\rangle+\left\langle\alpha,\pi^\sharp\cdot(\mathcal{L}_\mathcal{X}(\mathbf{d}F))\right\rangle
\end{aligned} \tag{7.3}
$$

since $\mathcal{X}$ is locally Hamiltonian. Since α is arbitrary, comparing 7.2 and 7.3 gives

$$\mathcal{L}_\mathcal{X}(\mathcal{X}_F) = [\mathcal{X},\mathcal{X}_F] = \pi^\sharp\cdot(\mathcal{L}_\mathcal{X}(\mathbf{d}F)). \tag{7.4}$$

Moreover, by Cartan's magic formula

$$\mathcal{L}_{\mathcal{X}}(\mathbf{d}F) = \mathbf{d}(\mathbf{i}_{\mathcal{X}}(\mathbf{d}F)) + \mathbf{i}_{\mathcal{X}}(\mathbf{d}^2 F) = \mathbf{d}(\mathbf{i}_{\mathcal{X}}(\mathbf{d}F)) = \mathbf{d}_{\mathcal{X}} F$$

and hence

$$\mathcal{L}_{\mathcal{X}}(\mathcal{X}_F) = [\mathcal{X}, \mathcal{X}_F] = \pi^\sharp \cdot (\mathbf{d}_{\mathcal{X}} F). \tag{7.5}$$

Thus $[\mathcal{X}, \mathcal{X}_F] = 0$ if and only if $\mathbf{d}_{\mathcal{X}} F$ is in $\ker(\pi^\sharp)$. Since $\mathcal{X}$ is locally Hamiltonian we also have that $\mathcal{L}_{\mathcal{X}} H = 0$ □

If π is non-degenerate and $H^0_{\mathcal{X}}(M) = \{F \in C^\infty(M) | \mathbf{d}_{\mathcal{X}} F = 0\}$ is the zero cohomology group of $\mathbf{d}_{\mathcal{X}}$, then $H^0_{\mathcal{X}}(M)$ coincides with the set of Hamiltonian infinitesimal symmetries of $(M, \pi, \mathcal{X})$, reproducing the result given in [3] for the symplectic case. This result suggests that the cohomology approach introduced in [3], is not well adapted to the Poisson case. It may be possible to give a nice cohomological interpretation of symmetries in this case by using certain cohomology groups associated with a foliated space, called tangential cohomology groups (see [20]) and references therein).

Let α be a differential one-form on (M, π). For the vector field associated to α we use the following notation, $\alpha^\sharp := \pi^\sharp(\alpha)$.

Proposition 7.3. *Let $(M, \pi, \mathcal{X})$ be a locally Hamiltonian system. The vector field associated to the $\mathbf{d}_{\mathcal{X}}$ exact one-form $\beta = \mathbf{d}_{\mathcal{X}} F$ is $\beta^\sharp = [\mathcal{X}, \mathcal{X}_F]$, where $\mathcal{X}_F$ is the Hamiltonian vector field of F.*

Proof. Equation 7.4 yields

$$\beta^\sharp = [\mathcal{X}, \mathcal{X}_F] = \pi^\sharp \cdot (\mathcal{L}_{\mathcal{X}}(\mathbf{d}F)) = \pi^\sharp \cdot \mathbf{d}_{\mathcal{X}} F = \pi^\sharp \cdot \beta.$$

□

Proposition 7.4. *Let $(M, \pi, \mathcal{X})$ be a locally-Hamiltonian system and let β be a one-form on (M, π). Then the vector field $\beta^\sharp = \pi^\sharp \cdot \beta$ is an infinitesimal Poissonoid transformation, if and only if $\mathbf{d}_{\mathcal{X}} \beta = \mathbf{d}\alpha_1$, where $\alpha_1 \in \ker \pi^\sharp$. In particular, if $\mathbf{d}_{\mathcal{X}} \beta = 0$, then $\beta^\sharp = \pi^\sharp \cdot \beta$ is an infinitesimal Poissonoid transformation. If π is non-degenerate then the vector field $\beta^\sharp = \pi^\sharp \cdot \beta$ is an infinitesimal Poissonoid transformation, if and only if $\mathbf{d}_{\mathcal{X}} \beta = 0$.*

Proof. $\beta^\sharp$ is infinitesimally Poissonoid if and only if $[\beta^\sharp, \mathcal{X}] = \pi^\sharp \cdot \mathbf{d}F$ for some (locally defined) function F. The left hand side can be written as

$$\begin{aligned}
[\beta^\sharp, \mathcal{X}] &= [\pi^\sharp \cdot \beta, \mathcal{X}] \\
&= -\mathcal{L}_{\mathcal{X}}(\pi^\sharp \cdot \beta) \\
&= -\pi^\sharp \cdot (\mathcal{L}_{\mathcal{X}} \beta) \\
&= -\pi^\sharp \cdot (\mathbf{d}(\mathbf{i}_{\mathcal{X}} \beta) + \mathbf{i}_{\mathcal{X}}(\mathbf{d}\beta))
\end{aligned}$$

If $(\mathbf{d}(\mathbf{i}_{\mathcal{X}} \beta) + \mathbf{i}_{\mathcal{X}}(\mathbf{d}\beta))$ is closed, that is if, locally, we have

$$0 = \mathbf{d}((\mathbf{d}(\mathbf{i}_{\mathcal{X}} \beta) + \mathbf{i}_{\mathcal{X}}(\mathbf{d}\beta))) = \mathbf{d}^2(\mathbf{i}_{\mathcal{X}} \beta) + \mathbf{d}(\mathbf{i}_{\mathcal{X}}(\mathbf{d}\beta)) = \mathbf{d}(\mathbf{i}_{\mathcal{X}}(\mathbf{d}\beta)) = \mathbf{d}_{\mathcal{X}} \beta$$

then $[\beta^\sharp, \mathcal{X}]$ can be written, locally, as $\pi^\sharp \cdot \mathbf{d}F$. The converse is not true unless $\pi^\sharp$ is non-degenerate. In order to do the general case let $\mathbf{i}_{\mathcal{X}}(\mathbf{d}\beta) = \alpha_0 + \alpha_1$ where α_0 is closed and $\alpha_1 \in \ker \pi^\sharp$. This is equivalent to writing $\mathbf{d}_{\mathcal{X}} \beta = \mathbf{d}\alpha_1$ with $\alpha_1 \in \ker \pi^\sharp$.

Then there is a locally defined function G such that we can write $\alpha_0 = \mathbf{d}G$, and thus

$$\begin{aligned}[\beta^\sharp, \mathcal{X}] &= -\pi^\sharp \cdot (\mathbf{d}(\mathbf{i}_{\mathcal{X}}\beta) + \mathbf{i}_{\mathcal{X}}(\mathbf{d}\beta)) \\ &= -\pi^\sharp \cdot (\mathbf{d}(\mathbf{i}_{\mathcal{X}}\beta + G) + \alpha_1) \\ &= -\pi^\sharp \cdot (\mathbf{d}(\mathbf{i}_{\mathcal{X}}\beta + G)) \\ &= \pi^\sharp \cdot \mathbf{d}F,\end{aligned}$$

with $F = -\mathbf{d}(\mathbf{i}_{\mathcal{X}}\beta + G)$. □

This result generalizes the well known fact that infinitesimal canonical transformations are generated by closed forms in the de Rahm cohomology (that is the basis of the theory of generating functions) and the fact that infinitesimal canonoid transformations are generated by closed forms in the twisted cohomology introduced in [3] to the case of Poissonoid transformations. In this case, however, not all the infinitesimal Poissonoid transformations are generated by $\mathbf{d}_{\mathcal{X}}$-closed differential forms.

It seems possible to state the proposition above more elegantly by introducing differential forms that are tangential to the foliation, see [20] and references therein for possible definitions of these forms.

7.1. **Master Symmetries.** Let $(M, \pi, \mathcal{X})$ be a locally-Hamiltonian system. Recall that an infinitesimal symmetry of $\mathcal{X}$ is a vector field ξ such that $[\xi, \mathcal{X}] = 0$. A **master symmetry** for $\mathcal{X}$ is a vector field $\mathcal{X}$ such that

$$[\xi, \mathcal{X}] \neq 0, \qquad [[\xi, \mathcal{X}], \mathcal{X}] = 0.$$

More generally, a **master symmetry of degree** m for $\mathcal{X}$ is a vector field $\mathcal{X}$ such that

$$[\xi, \mathcal{X}] \neq 0, \ldots, [\cdots[[\xi, \underbrace{\mathcal{X}], \mathcal{X}], \cdots, \mathcal{X}]}_{m} \neq 0,\; [\cdots[[\xi, \underbrace{\mathcal{X}], \mathcal{X}], \cdots, \mathcal{X}]}_{m+1} = 0.$$

The last condition in the equation above can also be written as $\mathcal{L}_{\mathcal{X}}^{m+1}(\xi) = 0$. Neither master symmetries nor Poissonoid infinitesimal transformation do, in general, generate constants of motion. However, the next proposition shows that some special infinitesimal Poissonoid transformations generate constants of motion.

Proposition 7.5. *Suppose ξ is an infinitesimal Poissonoid transformation of the Hamiltonian system (M, π, H), such that the relationship $[\xi, \mathcal{X}_H] = \pi^\sharp \cdot \mathbf{d}F$ is satisfied globally and $\mathcal{L}_{[\xi, \mathcal{X}_H]}H = 0$. Then F is a constant of motion of (M, π, H) and ξ is a master symmetry of degree 1.*

Proof. Since $[\xi, \mathcal{X}_H] = \pi^\sharp \cdot \mathbf{d}F$ holds globally and $\mathcal{L}_{[\xi, \mathcal{X}_H]}H = 0$ then $\mathcal{X}_F = [\xi, \mathcal{X}_H]$ is a Hamiltonian symmetry of the system (M, π, H). Thus, by Noether's theorem, F is a constant of motion, that is $\{F, H\} = 0$. It follows that,

$$0 = \{F, H\} = [\mathcal{X}_F, \mathcal{X}_H] = [[\xi, \mathcal{X}_H], \mathcal{X}_H],$$

and thus ξ is a master symmetry of degree 1. □

Let $(M, \pi, \mathcal{X})$ be a locally-Hamiltonian system. A function T is called a **generator of constants of motion of degree** m for $\mathcal{X}$ if

$$\mathcal{L}_{\mathcal{X}}T \neq 0, \ldots,\; \mathcal{L}_{\mathcal{X}}^m T \neq 0,\; \mathcal{L}_{\mathcal{X}}^{m+1}T = 0.$$

Suppose ξ is a Hamiltonian vector field with Hamiltonian function T, with $\xi = \pi^\sharp \cdot \mathbf{d}T$. Taking the Lie derivative of both sides yields

$$\mathcal{L}_\mathcal{X}(\xi) = \mathcal{L}_\mathcal{X}(\pi^\sharp \cdot \mathbf{d}T) = \mathcal{L}_\mathcal{X}(\pi^\sharp) \cdot \mathbf{d}T + \pi^\sharp \cdot (\mathcal{L}_\mathcal{X}(\mathbf{d}T)) = \pi^\sharp \cdot \mathbf{d}(\mathcal{L}_\mathcal{X}(T)).$$

This relation can be generalized to

$$\mathcal{L}^{m+1}_\mathcal{X}(\xi) = \pi^\sharp \cdot \mathbf{d}(\mathcal{L}^{m+1}_\mathcal{X}(T)).$$

We call T **the generator of a Hamiltonian master symmetry of degree** m if and only if $\mathbf{d}(\mathcal{L}^{m+1}_\mathcal{X}(T)) \in \ker \pi^\sharp$. Note that, in particular, If T is the generator of constants of motion of degree m, then $\mathcal{L}^{m+1}_\mathcal{X} T = 0$, and thus $\mathbf{d}(\mathcal{L}^{m+1}_\mathcal{X} T) = 0$. It follows that $\mathcal{L}^{m+1}_\mathcal{X}(\xi) = 0$, and thus ξ is a master symmetry of degree m. The converse is clearly not true if $\pi^\sharp$ is degenerate. The following proposition shows that Poissonoid transformations are very general: every master symmetry, and also every infinitesimal symmetry of the vector field can be generated using Poissonoid transformations.

Proposition 7.6. *Let $(M, \pi, \mathcal{X})$ be a locally-Hamiltonian vector field. Suppose that ξ is a master symmetry of degree $m+1$. Then the vector field $\mathcal{L}^m_\mathcal{X}(\xi)$ is an infinitesimal Poissonoid transformation.*

Proof. Since ξ is a master symmetry of degree $m+1$, we have

$$\mathcal{L}^{m+1}_\mathcal{X}(\xi) = [\mathcal{L}^m_\mathcal{X}(\xi), \mathcal{X}] = 0.$$

Then $\mathcal{L}^m_\mathcal{X}(\xi)$ is an infinitesimal Poissonoid transformation by Proposition 7.1. □

Proposition 7.7. *Let $(M, \pi, \mathcal{X})$ be a locally-Hamiltonian system and let β be a one-form on (M, π). Suppose the vector field $\beta^\sharp = \pi^\sharp \cdot \beta$ is an infinitesimal Poissonoid transformation such that $\mathbf{d}_\mathcal{X}\beta = 0$. Then $\beta^\sharp$ is master symmetry of degree m ($m \geq 1$) if and only if $\mathbf{i}_\mathcal{X}\beta$ is the generator of a Hamiltonian master symmetry of degree $m-1$.*

Proof. $\beta^\sharp$ is a master symmetry of degree m if and only if

$$\mathcal{L}^{m+1}_\mathcal{X}\beta^\sharp = 0, \quad \text{and} \quad \mathcal{L}^m_\mathcal{X}\beta^\sharp \neq 0. \tag{7.6}$$

Since $\beta^\sharp = \pi^\sharp \cdot \beta$, we have

$$\begin{aligned}
\mathcal{L}^{m+1}_\mathcal{X}(\beta^\sharp) &= \mathcal{L}^{m+1}_\mathcal{X}(\pi^\sharp \cdot \beta) = \mathcal{L}^m_\mathcal{X}(\mathcal{L}_\mathcal{X}(\pi^\sharp \cdot \beta)) \\
&= \mathcal{L}^m_\mathcal{X}(\mathcal{L}_\mathcal{X}(\pi^\sharp) \cdot \beta + \pi^\sharp \cdot (\mathcal{L}_\mathcal{X}(\beta))) \\
&= \mathcal{L}^m_\mathcal{X}(\pi^\sharp \cdot (\mathcal{L}_\mathcal{X}(\beta))) \\
&= \cdots \\
&= \pi^\sharp \cdot (\mathcal{L}^{m+1}_\mathcal{X}\beta),
\end{aligned}$$

where we used that $\mathcal{L}_\mathcal{X}\pi^\sharp = 0$ since $\mathcal{X}$ is locally Hamiltonian. Hence, equation 7.6 can be equivalently written as

$$\mathcal{L}^{m+1}_\mathcal{X}\beta^\sharp = \pi^\sharp \cdot (\mathcal{L}^{m+1}_\mathcal{X}\beta) = 0, \quad \text{and} \quad \mathcal{L}^m_\mathcal{X}\beta^\sharp = \pi^\sharp \cdot (\mathcal{L}^m_\mathcal{X}\beta) \neq 0,$$

or

$$(\mathcal{L}^{m+1}_\mathcal{X}\beta) \in \ker \pi^\sharp, \quad \text{and} \quad (\mathcal{L}^m_\mathcal{X}\beta) \notin \ker \pi^\sharp.$$

Using Cartan's magic formula yields

$$\begin{aligned}\mathcal{L}_{\mathcal{X}}^{m+1}\beta &= \mathcal{L}_{\mathcal{X}}^{m}(\mathcal{L}_{\mathcal{X}}\beta) = \mathcal{L}_{\mathcal{X}}^{m}(\mathbf{d}(\mathbf{i}_{\mathcal{X}}\beta) + \mathbf{i}_{\mathcal{X}}\mathbf{d}\beta)\\ &= \mathbf{d}(\mathcal{L}_{\mathcal{X}}^{m}(\mathbf{i}_{\mathcal{X}}\beta)) + \mathcal{L}_{\mathcal{X}}^{m}(\mathbf{i}_{\mathcal{X}}\mathbf{d}\beta)\\ &= \mathbf{d}(\mathcal{L}_{\mathcal{X}}^{m}(\mathbf{i}_{\mathcal{X}}\beta)) + \mathcal{L}_{\mathcal{X}}^{m-1}(\mathbf{i}_{\mathcal{X}}\mathbf{d}(\mathbf{i}_{\mathcal{X}}\mathbf{d}\beta) + \mathbf{d}(\mathbf{i}_{\mathcal{X}}\mathbf{i}_{\mathcal{X}}(\mathbf{d}\beta)))\\ &= \mathbf{d}(\mathcal{L}_{\mathcal{X}}^{m}(\mathbf{i}_{\mathcal{X}}\beta)) + \mathcal{L}_{\mathcal{X}}^{m-1}(\mathbf{i}_{\mathcal{X}}(\mathbf{d}_{\mathcal{X}}\beta))\\ &= \mathbf{d}(\mathcal{L}_{\mathcal{X}}^{m}(\mathbf{i}_{\mathcal{X}}\beta))\end{aligned}$$

where we used the fact that $\mathbf{i}_{\mathcal{X}}\mathbf{i}_{\mathcal{X}}\alpha = 0$ for any k-form α, and the fact that $\mathcal{X}$ is an infinitesimal Poissonoid transformations such that $\mathbf{d}_{\mathcal{X}}\beta = 0$. It follows that $\mathcal{L}_{\mathcal{X}}^{m+1}\beta^\sharp = 0$ if and only if $\mathbf{d}(\mathcal{L}_{\mathcal{X}}^{m}(\mathbf{i}_{\mathcal{X}}\beta)) \in \ker\pi^\sharp$, that is if and only if $\mathbf{i}_{\mathcal{X}}\beta$ is the generator of a Hamiltonian master symmetry of degree $m-1$.

□

Acknowledgments

The authors wish to thank Luca Degiovanni, Francesco Fassò, and Anton Izosimov for comments and suggestions, and Shengda Hu for several discussions concerning this work. We would also like to thank both reviewers for their insightful comments on the paper, as these comments led us to an improvement of our work. The second author was supported by an NSERC Discovery Grant.

References

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, American Mathematical Soc., 1978.

[2] A. V. Bolsinov, Compatible {P}oisson Brackets on {L}ie Algebras and Completeness of Families of Functions in Involution, *Mathematics of the USSR-Izvestiya*, **38** (1992), 69.

[3] J. F. Cariñena, F. Falceto and M. F. Rañada, Canonoid transformations and master symmetries, *Journal of Geometric Mechanics*, **5** (2013), 151–166.

[4] J. F. Cariñena and M. F. Rañada, Canonoid transformations from a geometric perspective, *Journal of Mathematical Physics*, **29** (1988), 2181–2186.

[5] J. F. Cariñena and M. F. Rañada, Generating functions, bi-Hamiltonian systems, and the quadratic-Hamiltonian theorem, *Journal of Mathematical Physics*, **31** (1990), 801–807.

[6] Y. Choquet-Bruhat, *Analysis, Manifolds and Physics, Part II - Revised and Enlarged Edition*, Elsevier, 2000.

[7] D. G. Currie and E. J. Saletan, Canonical transformations and quadratic hamiltonians, *Il Nuovo Cimento B Series 11*, **9** (1972), 143–153.

[8] C. Daskaloyannis and K. Ypsilantis, Unified treatment and classification of superintegrable systems with integrals quadratic in momenta on a two-dimensional manifold, *Journal of Mathematical Physics*, **47** (2006), 042904.

[9] G. Falqui, A Note on the Rotationally Symmetric $SO(4)$ {E}uler Rigid Body, *Symmetry, Integrability and Geometry: Methods and Applications.*

[10] A. Fasano and S. Marmi, *Analytical Mechanics : An Introduction*, Oxford University Press, 2006.

[11] A. T. Fomenko, *Integrability and Nonintegrability in Geometry and Mechanics*, Kluwer Academic Publishers, Dordrecht, 1988.

[12] J. M. Jauch and E. L. Hill, On the Problem of Degeneracy in Quantum Mechanics, *Physical Review*, **57** (1940), 641–645.

[13] E. G. Kalnins, J. M. Kress, G. S. Pogosyan and W. M. Jr, Completeness of superintegrability in two-dimensional constant-curvature spaces, *Journal of Physics A: Mathematical and General*, **34** (2001), 4705.

[14] G. Landolfi and G. Soliani, On certain canonoid transformations and invariants for the parametric oscillator, *Journal of Physics A: Mathematical and Theoretical*, **40** (2007), 3413.

[15] C. Laurent-Gengoux, A. Pichereau and P. Vanhaecke, *Poisson Structures*, Springer, 2012.

[16] P. Libermann and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics*, Springer, 1987.
[17] F. Magri, P. Casati, G. Falqui and M. Pedroni, Eight lectures on integrable systems, in *Integrability of Nonlinear Systems* (eds. Y. Kosmann-Schwarzbach, K. M. Tamizhmani and B. Grammaticos), no. 638 in Lecture Notes in Physics, Springer Berlin Heidelberg, 2004, 209–250.
[18] G. Marmo, Equivalent Lagrangians and quasicanonical transformations, in *Group Theoretical Methods in Physics* (eds. A. Janner, T. Janssen and M. Boon), no. 50 in Lecture Notes in Physics, Springer Berlin Heidelberg, 1976, 568–572.
[19] J. E. Marsden and T. S. Ratiu, *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*, Springer, 1999.
[20] C. C. Moore and C. Schochet, Tangential Cohomology, in *Global Analysis on Foliated Spaces*, no. 9 in Mathematical Sciences Research Institute Publications, Springer New York, 1988, 68–91.
[21] C. Morosi and L. Pizzocchero, On the {E}uler equation: Bi-{H}amiltonian structure and integrals in involution, *Letters in Mathematical Physics*, **37** (1996), 117–135.
[22] L. J. Negri, L. C. Oliveira and J. M. Teixeira, Canonoid transformations and constants of motion, *Journal of Mathematical Physics*, **28** (1987), 2369–2372.
[23] H. Poincaré, Sur le probléme des trois corps et les équations de la dynamique, *Acta Mathematica*, **13** (1890), A3–A270.
[24] G. Rudolph and M. Schmidt, *Differential Geometry and Mathematical Physics: Part I. Manifolds, Lie Groups and Hamiltonian Systems*, Springer, 2012.
[25] E. J. Saletan and A. H. Cromer, *Theoretical mechanics*, Wiley, 1971.
[26] P. Tempesta, E. Alfinito, R. A. Leo and G. Soliani, Quantum models related to fouled Hamiltonians of the harmonic oscillator, *Journal of Mathematical Physics*, **43** (2002), 3538–3553.
[27] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Springer, 1994.
[28] F. J. D. Vegas, On the canonical transformation theorem of Currie and Saletan, *Journal of Physics A: Mathematical and General*, **22** (1989), 1927.
[29] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, 1988.

Dipartimento di Matematica, Universitá di Torino, Torino, via Carlo Alberto 10, Italia,
E-mail address: `giovanni.rastelli@unito.it`

Department of Mathematics, Wilfrid Laurier University, 75 University Avenue West, Waterloo, ON,Canada,
E-mail address: `msantoprete@wlu.ca`