FRÉCHET GLOBALISATIONS OF HARISH-CHANDRA SUPERMODULES

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ABSTRACT. For any Lie supergroup whose underlying Lie group is reductive, we prove an extension of the Casselman–Wallach globalisation theorem: There is an equivalence between the category of Harish-Chandra modules and the category of SF-representations (smooth Fréchet representations of moderate growth) whose module of finite vectors is Harish-Chandra. As an application, we extend to Lie supergroups a general general form of the Gel′fand–Kazhdan criterion due to Sun–Zhu.

INTRODUCTION

In the study of continuous representations of non-compact real-reductive Lie groups G_0, a fundamental obstacle is that almost all representations of interest are infinite-dimensional. A basic tool, which reduces many analytic questions to algebraic ones, is the passage to the module of K_0-finite vectors. The fundamental Casselman–Wallach theorem [16, 65] guarantees that every Harish-Chandra $(\mathfrak{g}_0, K_0)$-module occurs in this way. This is essential, in particular in applications to the classification problem for irreducible unitary representations.

Lie supergroups were introduced by Berezin, Kostant, and Leĭtes [8, 9, 40] in the 1970s as a mathematical framework for the study of the supersymmetries occurring in quantum field theory. Lie superalgebras had entered the stage three decades earlier, in the work of Whitehead, to gain more prominence the work of Frölicher–Nijenhuis on the deformation of complex structure, and subsequently in Gerstenhaber's work on the deformation of rings and algebras. Through the applications in physics, the subject of Lie superalgebras representations has come to the fore, and is at present well-established in both mathematics and physics, with a literature far too extensive to cite; compare the paper [23] for the development up to the 1970s, and the monographs [21, 49] for an up-to-date account of the theory. On the level of Lie supergroups, there is a sizeable literature in physics, but the subject has been hardly studied from a mathematical perspective.

Most mathematical works (*e.g.* Refs. [30, 35, 38, 52, 54]) consider the unitarisable Harish-Chandra modules, without exploring the issue whether they arise as the space of finite vectors of some 'global' representation. A 'global' perspective was taken by Dobrev–Petkova [27], who realise induced representations of the supergroup $SU(2, 2|N)$ on spaces of superfunctions. They classify unitary irreducible

represenations of positive energy [28, 29], thereby extending previous work of Flato–Fronsdal [33] for $N = 1$. On the basis of these seminal ideas, Carmeli–Cassinelli–Toigo–Varadarajan [19] introduce a notion of unitary Lie supergroup representations for arbitary Lie supergroups. This has spawned a flurry of further investigation [48, 50, 51, 57].

Meanwhile, beyond the obvious fact that non-unitary representations may occur as intermediates in the study of unitary ones, it has become clear that unitary representations alone are insufficient for the purpose of Fourier–Plancherel decomposition, even in simple cases [5]. This is confirmed by applications of supersymmetry to number theory and random matrices [22, 39], as well as in physics, for instance in the study of the Chalker–Coddington model with point contacts [12].

Finally, as has become increasingly clear in recent investigations of the Gel′fand and Gel′fand–Kazhdan properties for pairs of Lie groups beyond the setting of Riemannian symmetric pairs [1–3, 61] that Casselman–Wallach theory is eminently useful for the study of branching multiplicities. Here, we argue that similar statements hold true also for the setting of Lie supergroups.

Therefore, it seems paramount to study the globalisations of 'algebraic' representations, irrespective of unitarity, for Lie supergroups. In this paper, we generalise the Casselman–Wallach theorem to Lie supergroups, as follows.

Theorem A. *Let G be a Lie supergroup whose underlying Lie group G_0 is almost connected and real reductive, $\mathfrak{g}$ its Lie superalgebra, and let $K_0 \subseteq G_0$ be maximally compact. Then any Harish-Chandra $(\mathfrak{g}, K_0)$-module has a unique SF-globalisation.*

This defines an additive equivalence between the category $\mathbf{HC}(\mathfrak{g}, K_0)$ of Harish-Chandra $(\mathfrak{g}, K_0)$-modules and the category $\mathbf{CW}(G)$ of SF-representations of G whose module of K_0-finite vectors is Harish-Chandra.

Here, we follow Ref. [10] in using the term 'SF-representation' (resp. 'F-representation') instead of 'smooth Fréchet representation of moderate growth' (resp. 'Fréchet representation of moderate growth').

As an application of our results on globalisation, we study the Gel′fand–Kazhdan property for pairs of supergroups, to arrive by the following version of the Gel′fand–Kazhdan criterion, which generalises that given recently by Sun–Zhu [60].

Theorem B. *Let $H_1, H_2 \subseteq G$ be closed subsupergroups, χ_i, $i = 1, 2$, characters of H_i, $i = 1, 2$, and σ an antiautomorphism of G. Assume that any even relatively $(\chi_1^{-1} \otimes \chi_2^{-1})$-invariant tempered superfunction G that is a joint eigenvector of all even G-invariant $D \in \mathfrak{U}(\mathfrak{g})$ is fixed by σ.*

Then, for any contragredient pair (E, F) of F-representations of G such that E_∞ and F_∞ are irreducible G-representations whose modules of K_0-finite vectors are Harish-Chandra, we have

$$\dim \mathrm{Hom}_{H_1}(E_\infty, \chi_1) \dim \mathrm{Hom}_{H_2}(F_\infty, \chi_2) \leqslant 1.$$

Theorem A (Theorem 4.6) is derived in the framework of convolution algebras of Schwartz functions, as used by Bernstein–Krötz [10] for Lie groups in their proof of a Casselman–Wallach theorem for holomorphic families of Harish-Chandra modules.

As it turns out, the framework of convolution superalgebras of superdistributions and Berezinian densities is well-adapted to the study of the classes of continuous and weakly smooth representations, introduced here. In fact, a version of the Dixmier–Malliavin theorem holds (Proposition 2.15).

Moreover, the convolution algebra of Schwartz–Berezin densities is equally well suited for the study of F- and SF-representations (or moderate growth representations) of Lie supergroups. Indeed, we prove a Schwartzian Dixmier–Malliavin

theorem for F-representations (Proposition 3.8), generalising the corresponding result of Bernstein–Krötz [10].

What makes the proof of our main results tick is the fact that all of the convolution superalgebras in question can be presented as coinduced modules (Proposition 2.2, Corollary 2.9, Proposition 3.2, and Proposition 3.3), allowing for a passage from Lie supergroups to supergroup pairs. We can thus reduce many analytic questions to the underlying Lie group and use Hopf algebraic methods of computation to arrive by our conclusions.

The expression of the convolution superalgebras *via* coinduced modules, whilst preserving the convolution product, is, however, a non-trivial fact. It is based on an extension of Bruhat's regularity theorem for left-invariant distributions (Proposition 2.4), which, together with dualising module techniques, implies an expression of the invariant Berezin density in terms of the Haar density on the underlying Lie group (Proposition 2.8). Such an expression was previously only known in very special cases, where, in particular, the 'odd modular function' is trivial [24]. The present result is far more general and covers all kinds of Lie supergroups, including non-basic classical and even non-simple cases.

In the final Section 5, we apply our results to the generalisation of the Gel′fand–Kazhdan criterion in Theorem B (Theorem 5.7). The setting of Sun–Zhu [60] goes over more or less verbatim, due to our extension of the Casselman–Wallach theory.

We do not yet view these last results as the definitive statements on multiplicity freeness for supergroups. Indeed, whereas we have focused here on the extension of phenomena from the purely even setting, there are many issues special to the super case that yet need to be addressed, such as Q type modules and lack of semi-simplicity at the level of finite-dimensional modules. Moreover, non-trivial examples that verify the assumptions of Theorem B have yet to be supplied, and we intend to study this question in future work. However, the ease with which at least the purely even results transfer to the super case is to our mind a strong indication to the utility of the super Casselman–Wallach Theorem A.

1. Supergroup representations

In this section, we collect some preliminary material on supergroups and their representations.

1.1. **Preliminaries and notation.** Concerning supermanifolds, we will essentially work in the standard framework of Berezin and Leĭtes [9], and use standard facts concerning it, to be found in Refs. [17, 25, 44, 45]. We give some basic definitions to fix our terminology and to clarify in which points we deviate from this literature.

We consider sheaves of Abelian groups and will denote them by calligraphic Roman letters $\mathcal{E}, \mathcal{F}, \mathcal{O}$, *etc.* The set of sections of a sheaf $\mathcal{F}$ will be denoted by $\Gamma(\mathcal{F})$. The *support* of a section $f \in \Gamma(\mathcal{F})$ is $\{x \mid f_x \neq 0\}$ where f_x is the germ at x. We denote by $\Gamma_K(\mathcal{F})$ the set of sections with support contained in K, and by $\Gamma_c(\mathcal{F})$ the set of sections with compact support.

Let $\mathbb{K}$ be the field $\mathbb{R}$ of real or the field $\mathbb{C}$ of complex numbers. Consider the category of $\mathbb{K}$-superspaces: Its objects are pairs $X = (X_0, \mathcal{O}_X)$ comprised of a

topological space X_0 and a sheaf $\mathcal{O}_X$ on X_0 of supercommutative $\mathbb{K}$-superalgebras with local stalks; its morphisms $\varphi : X \longrightarrow Y$ are pairs $(\varphi_0, \varphi^\sharp)$ consisting of a continuous map $\varphi_0 : X_0 \longrightarrow Y_0$ and an even unital morphism of $\mathbb{K}$-superalgebra sheaves $\varphi^\sharp : \mathcal{O}_Y \longrightarrow (\varphi_0)_* \mathcal{O}_X$ where $(\varphi_0)_*$ denotes the direct image functor. By a standard adjunction [14, Chapter I.3], we may equivalently consider $\varphi^\sharp$ as a morphism of sheaves $\varphi_0^{-1} \mathcal{O}_Y \longrightarrow \mathcal{O}_X$, where φ_0^{-1} is the inverse image functor. Given some finite-dimensional super-vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over $\mathbb{R}$, together with a compatible $\mathbb{K}$-structure on the odd part $V_{\bar{1}}$, we define the *affine superspace* $\mathbb{A}(V)$ by

$$\mathbb{A}(V)_0 \coloneqq V_{\bar{0}}, \quad \mathcal{O}_{\mathbb{A}(V)} \coloneqq \mathcal{C}^\infty_{V_{\bar{0}}} \otimes_\mathbb{R} \textstyle\bigwedge_\mathbb{K} (V_{\bar{1}}^*).$$

Here, $\mathcal{C}^\infty_{V_{\bar{0}}}$ denotes the sheaf of smooth real-valued functions on $V_{\bar{0}}$, and $\bigwedge_\mathbb{K}(V_{\bar{1}}^*)$ denotes the exterior algebra of the $\mathbb{K}$-vector space $V_{\bar{1}}^*$. Here and in what follows, we denote the homogeneous parts of a given grading over $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ by the subscripts $\bar{0}$ (even) and $\bar{1}$ (odd).

Given a $\mathbb{K}$-superspace X, an *open subspace* is one of the form $X|_U \coloneqq (U, \mathcal{O}_X|_U)$ for some open subset $U \subseteq X_0$. A $\mathbb{K}$-superspace X is called a *supermanifold* if X_0 is Hausdorff and admits an open cover (U_i) such that for every index i, $X|_{U_i}$ is isomorphic to an open subspace of some affine superspace $\mathbb{A}(V)$. In this case, for $x \in U_i$, the tuple $\dim_\mathbb{R} V_{\bar{0}} | \dim_\mathbb{K} V_{\bar{1}}$ is denoted by $\dim_x X$ and called the *superdimension* of X at x.

For $\mathbb{K} = \mathbb{R}$, one customarily calls supermanifolds as defined above *real supermanifolds*; in the case $\mathbb{K} = \mathbb{C}$, they are called cs *manifolds* [25, §4.8]. Notice that due to our preference for representations on complex vector spaces, we have a natural bias towards working in the latter setting. This is the main point in which we do not follow the standard texts. Most aspects of real supermanifolds carry over to the *cs* case, with some notable exceptions related to real structures and the representability of vector bundles. We will take care to point these out to the reader.

1.2. **Supergroups and supergroup pairs.** It is known that the category of supermanifolds admits finite products [44, 3.1.6]. Thus, group objects and their morphisms in this category are well-defined [47, Chapter III.6]. A group object in the category of supermanifolds will be called a *Lie supergroup* or simply a *supergroup*. For $\mathbb{K} = \mathbb{R}$, these are real Lie supergroups, while for $\mathbb{K} = \mathbb{C}$, they are *cs* Lie supergroups.

For applications to linear representations, the following definition proves useful.

Definition 1.1 (Supergroup pairs)**.** Let G_0 be a real Lie group with Lie algebra $\mathfrak{g}_0$, $\mathfrak{g}$ be a Lie superalgebra over $\mathbb{K}$ such that $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0 \otimes_\mathbb{R} \mathbb{K}$, and $\mathrm{Ad} : G_0 \longrightarrow \mathrm{Aut}(\mathfrak{g})$ a smooth action of G_0 by Lie $\mathbb{K}$-superalgebra automorphisms. We say that $(\mathfrak{g}, G_0)$ (where the action is understood) is a *supergroup pair* if the differential $d\,\mathrm{Ad}$ of Ad is the restriction of the bracket $[\cdot, \cdot]$ of $\mathfrak{g}$ to $\mathfrak{g}_0 \times \mathfrak{g}$.

A *morphism of supergroup pairs* $(\mathfrak{g}, G_0) \longrightarrow (\mathfrak{h}, H_0)$ consists by definition of a morphism $\varphi_0 : G_0 \longrightarrow H_0$ of real Lie groups and a φ_0-equivariant Lie $\mathbb{K}$-superalgebra morphism $d\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ such that $d\varphi_0 = d\varphi|_{\mathfrak{g}_0}$.

In the literature, supergroup pairs are referred to as Harish-Chandra pairs. Since to our knowledge, Harish-Chandra never worked on supergroups, we prefer to use a less colourful nomenclature.

The following proposition is due to Kostant [40] and Koszul [42] in the case $\mathbb{K} = \mathbb{R}$; see [17, Chapter 7] for a detailed exposition. The extension to the case of $\mathbb{K} = \mathbb{C}$ presents no difficulty.

Proposition 1.2. *Consider the functor that assigns to a Lie supergroup G the supergroup pair $(\mathfrak{g}, G_0)$, where G_0 is the underlying Lie group of G, $\mathfrak{g}$ is its Lie superalgebra, and G_0 acts on $\mathfrak{g}$ by the natural adjoint action.*

This functor defines an equivalence of the category of Lie supergroups and their morphisms with the category of supergroup pairs and their morphisms.

Remark 1.3. In particular, we may associate with any real Lie supergroup G the *cs* Lie supergroup whose supergroup pair is $(G_0, \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$. On the level of superspaces, this sends G to the complex superspace $(G_0, \mathcal{O}_G \otimes_{\mathbb{R}} \mathbb{C})$.

We are mainly interested in complex representations, so we consider the case of *cs* Lie supergroups to be more relevant than the case of real Lie supergroups. Compare [25, § 4.9] for a list of five exemplary situations where it is more natural or even required to consider *cs* manifolds instead of real supermanifolds. In particular, Example 4.9.3 (*op. cit.*) describes a *cs* Lie supergroup which does not admit a real form. By contrast, any complex Lie supergroup whose underlying Lie group has a real form, has a *cs* form. (NB: It is known that there are nilpotent step-2 Lie algebras without a real form.)

1.3. **Smooth and continuous supergroup representations.** In what follows, let G_0 be a Lie group with Lie algebra $\mathfrak{g}_0$. To fix our terminology, we recall the following somewhat standard definitions.

Definition 1.4 (Continuous and smooth representations)**.** Let G_0 be a Lie group, E a topological vector space over $\mathbb{K}$ and $G_0 \times E \longrightarrow E$ a linear left action of G_0 on E. If the action is a continuous map, then we say that the induced map $\pi_0 : G \longrightarrow \mathrm{GL}(E)$ is a *continuous representation* of G_0 on E.

Let the topology on E be locally convex. A vector $v \in E$ is called *smooth* if the orbit map $\gamma_v : G_0 \longrightarrow E : g \longmapsto \pi_0(g)v$ is a smooth map. For $x \in \mathfrak{g}_0$, one defines

$$d\pi_0(x)v \coloneqq \frac{d}{dt}\Big|_{t=0} \pi_0(\exp(tx))v.$$

This defines an action of $\mathfrak{g}_0$ on the space E_∞ of all smooth vectors. One endows E_∞ with the coarsest locally convex topology such that for all $u \in \mathfrak{U}(\mathfrak{g}_0)$, the linear map

$$d\pi_0(u) : E_\infty \longrightarrow E$$

is continuous. Compare [10, 2.4.2; 15, §2.3; 62, Chapter 0; 66, Section 4.4.1] for alternative definitions of the topology. The representation π_0 is called *weakly smooth* if the canonical inclusion $E_\infty \longrightarrow E$ is an isomorphism of topological vector spaces. (For the reasons explained in [10, Remark 2.12], we reserve the term *smooth* for F-representations, to be defined below.)

In what follows, let G be a Lie supergroup with underlying Lie group G_0 and Lie superalgebra $\mathfrak{g}$. We continue to denote the Lie algebra of G_0 by $\mathfrak{g}_0$; in particular, $\mathfrak{g}_0$ is a real form of $\mathfrak{g}_{\bar{0}}$, that is $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{K}$. We intend to give a definition of what a representation of G is. For finite-dimensional representations, there are two possible ways to do this: A 'piecemeal' definition in terms of pairs of representations of G_0 and $\mathfrak{g}$ with a suitable compatibility, and a definition in functorial terms.

To state this precisely, let E be a super-vector space over $\mathbb{K}$ of dimension $p|q$. Choose any homogeneous $\mathbb{K}$-basis (z_j) of E, and let (z^j) be the dual basis. We let $E_{\mathbb{R}}$ be E, where we forget the $\mathbb{K}$-structure on $E_{\bar{0}}$ and retain only the $\mathbb{R}$-structure. If $|z^j| = \bar{0}$, then we decompose $z^j = x^j + iy^j$ where $x^j, y^j \in \mathrm{Hom}_{\mathbb{R}}(E_{\bar{0}}, \mathbb{R})$.

By [44, Theorem 2.1.7], there is for any supermanifold S a natural bijection

$$\left\{ v : S \longrightarrow \mathbb{A}(E_{\mathbb{R}}) \right\} \longrightarrow \Gamma(\mathcal{O}^{np}_{S,\bar{0},\mathbb{R}}) \times \Gamma(\mathcal{O}^q_{S,\bar{1}})$$

$$v \longmapsto \left(v^\sharp(x^1), v^\sharp(y^1), \dots, v^\sharp(x^p), v^\sharp(y^p), v^\sharp(z^{p+1}), \dots, v^\sharp(z^{p+q}) \right).$$

Here, $n = \dim_{\mathbb{R}} \mathbb{K}$, it is understood that $y^j = 0$ for $\mathbb{K} = \mathbb{R}$, and $\mathcal{O}_{S,\bar{0},\mathbb{R}}$ denotes the subsheaf of *real-valued* sections of $\mathcal{O}_{S,\bar{0}}$. That is, the canonical image of any germ f_s of a section in the residue field $\varkappa(s) = \mathcal{O}_{S,s}/\mathfrak{m}_{S,s} = \mathbb{C}$ is required to lie in the subfield $\mathbb{R}$ at every point s. The above bijection is natural in the choice of bases.

For any supermanifold S, we define $\mathrm{GL}(E)(S)$ to be set of invertible even matrices $g = (g_{k\ell})$ with entries in $\Gamma(\mathcal{O}_S)$. There is an obvious way to define $\mathrm{GL}(E)$ on morphisms $S' \longrightarrow S$, turning the assignment $S \longmapsto \mathrm{GL}(E)(S)$ into a set-valued cofunctor on $\mathbb{K}$-supermanifolds. Naturally, $\mathrm{GL}(E)$ is a functor in groups.

The following proposition is a minor variation on a well-known statement and can be easily derived from standard facts on supergroup actions given ample exposition in the literature, see *e.g.* [17, Chapter 8]. Compare also the recent work of Ostermayr [53, Section 2].

Proposition 1.5. *Then the following data are in one-to-one correspondence:*
  (i) *Pairs $(d\pi, \pi_0)$ of graded linear representations π_0 of G_0 on E and G_0-equivariant Lie superalgebra actions π of $\mathfrak{g}$ on E with $d\pi_0 = d\pi|_{\mathfrak{g}_0}$;*
  (ii) *left actions $a : G \times E \longrightarrow E$ that are $\mathbb{K}$-linear over G, in the sense that there is some $g = (g_{k\ell}) \in \mathrm{GL}(E)(G)$ such that*

$$\sum_k g_{kj} z^k = \begin{cases} a^\sharp(x^j) + i a^\sharp(y^j), & \text{if } j \leqslant p, \\ a^\sharp(z^j), & \text{if } j > p. \end{cases}$$

Remark 1.6. For $\mathbb{K} = \mathbb{R}$, $\mathrm{GL}(E)$ is represented by the real Lie supergroup $\mathrm{GL}(E, \mathbb{R})$ whose underlying supergroup pair is $(\mathrm{GL}(E_{\bar{0}}, \mathbb{R}) \times \mathrm{GL}(E_{\bar{1}}, \mathbb{R}), \mathfrak{gl}(E, \mathbb{R}))$. Thus, the data in Proposition 1.5 (ii) are just morphisms of Lie supergroups $G \longrightarrow \mathrm{GL}(E, \mathbb{R})$.

On the other hand, for $\mathbb{K} = \mathbb{C}$ (the case of *cs* Lie supergroups), the proposition does not admit a statement in terms of supergroup homomorphisms. Indeed, in this case, $\mathrm{GL}(E)$ is not representable in the category of supermanifolds (*i.e. cs* manifolds). Instead, if we extend $\mathrm{GL}(E)$ to a suitable subcategory of the category of $\mathbb{C}$-superspaces which contains complex supermanifolds as a (full) subcategory, then $\mathrm{GL}(E)$ can be seen to coincide on this subcategory with the point functor of the *complex* Lie supergroup $\mathrm{GL}(E, \mathbb{C})$, which is not a *cs* manifold for $E \neq 0$. See Ref. [53, Appendix 7.1].

Another way to see that the data in Proposition 1.5 (i) do not correspond to the G-points of a representable functor is to remark that the even part of the complex Lie superalgebra $\mathfrak{gl}(E, \mathbb{C})$ is not the complexification of the Lie algebra of $\mathrm{GL}(E_{\bar{0}}, \mathbb{C}) \times \mathrm{GL}(E_{\bar{1}}, \mathbb{C})$, considered as a real Lie group, and thus, these do not form a supergroup pair.

On grounds of the above equivalence, we adopt the following terminology.

Definition 1.7 (Continuous and smooth supergroup representations)**.** Let E be a locally convex super-vector space (*i.e. E* is a locally convex vector space with a grading that exhibits E as a locally convex direct sum). Assume given a continuous representation π_0 of G_0 on $E_{\bar{0}}$ and a Lie superalgebra representation $d\pi$ of $\mathfrak{g}$ on E_∞ such that the map $\mathfrak{g} \times E_\infty \longrightarrow E_\infty : (x, v) \longmapsto d\pi(x)v$ is continuous.

We say that $(d\pi, \pi_0)$ is a *continuous G-representation* if $d\pi$ is G_0-equivariant, *i.e.*

$$d\pi\big(\mathrm{Ad}(g)(x)\big) = \pi_0(g) d\pi(x) \pi_0(g^{-1})$$

for all $x \in \mathfrak{g}$ and $g \in G_0$, and $d\pi|_{\mathfrak{g}_0} = d\pi_0$. If in addition, E is weakly smooth as a G_0-representation, then we call E a *weakly smooth G-representation*.

The definition given above for continuous supergroup representations is compatible with the corresponding ones given in the literature for the case of unitary representations [19, 2.3; 48, Definition 4.1].

2. Convolution superalgebras and representations

In what follows, let G be a Lie supergroup, where G_0 is assumed to be σ-compact. Let $\mathfrak{g}$ be its Lie superalgebra. In this section, we introduce a convolution superalgebra of compactly supported Berezinian densities on G and show that there is a one-to-one correspondence between its (non-degenerate) representations and the smooth representations of G.

To that end, we will identify the sheaf of Berezinian densities of G within the sheaf of superdistributions as the $\mathfrak{g}$-module induced from the sheaf of densities on the underlying Lie group G_0. We begin by discussing superdistributions.

2.1. Superdistributions.

In this section, we introduce superdistributions on G, and show how to express them in terms of the underlying Lie group.

For any open $U \subseteq G_0$, we endow $\mathcal{O}_G(U)$ with the locally convex topology generated by the seminorms

$$p_{u,v,K}(f) := \sup_{x \in K} \left| (L_u R_v f)(x) \right|$$

where $K \subseteq U$ is compact and $u, v \in \mathfrak{U}(\mathfrak{g})$. Here, L and R, respectively, denote the left and right regular representation. It is known [17, Proposition 7.4.13; 42, Section 1] that there is an isomorphism

$$\mathcal{O}_G(U) \xrightarrow{\ \phi\ } \underline{\mathrm{Hom}}_{\mathfrak{g}_{\bar{0}}}(\mathfrak{U}(\mathfrak{g}), \mathcal{C}^\infty(U, \mathbb{K}))$$

given by

$$\phi(f)(u; x) := (-1)^{|f||u|}(R_u f)(x)$$

for all $f \in \mathcal{O}_G(U)$, $u \in \mathfrak{U}(\mathfrak{g})$, and $x \in U$. Here, the action of $\mathfrak{g}_{\bar{0}}$ on $\mathcal{C}^\infty(U, \mathbb{K})$ is by left-invariant differential operators (i.e. infinitesimal right translations), and the algebra product is expressed on the right-hand side by the rule

$$fh = m \circ (f \otimes h) \circ \Delta,$$

where m denotes multiplication in $\mathcal{C}^\infty$ and Δ denotes comultiplication in $\mathfrak{U}(\mathfrak{g})$. For future use, we note that the multiplication morphism m is given in terms of the isomorphism ϕ by

$$(2.1) \qquad \begin{aligned} \phi(m^\sharp(f))(u \otimes v; g, h) &= \phi(f)\big(\mathrm{Ad}(h^{-1})(u)v; gh\big) \\ &= (-1)^{(|u|+|v|)|f|}\big(L_{S(\mathrm{Ad}(g)(u))} R_v f\big)(gh). \end{aligned}$$

Since $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}_{\bar{0}}) \otimes \bigwedge \mathfrak{g}_{\bar{1}}$ as graded $\mathfrak{g}_{\bar{0}}$-modules [58, I.2.3], we have

$$\mathcal{O}_G(U) \cong \mathcal{C}^\infty(U, \mathbb{K}) \otimes_{\mathbb{K}} \bigwedge (\mathfrak{g}_{\bar{1}})^*.$$

Since the Grassmann factor is finite-dimensional, one readily checks that is an isomorphism of locally convex super-vector spaces, where $\mathcal{C}^\infty(U, \mathbb{K})$ is given the usual topology of uniform convergence with all derivatives on compact subsets. In particular, $\mathcal{O}_G(U)$ is an m-convex Fréchet algebra [46, 2.2]. Here, we recall that a locally convex algebra is called m-convex if its topology is generated by a set of submultiplicative seminorms.

Similarly, we give $\Gamma_c(\mathcal{O}_G)$ the locally convex inductive limit topology for the embeddings of the subspaces $\Gamma_K(\mathcal{O}_G)$ of sections f with support $\mathrm{supp}\, f \subseteq K$, where $K \subseteq G_0$ is compact. (See Subsection 1.1 for the notation.) The latter are given the relative topology induced by $\Gamma(\mathcal{O}_G)$. Then $\Gamma_c(\mathcal{O}_G)$ is an LF space, and the multiplication is jointly continuous.

Definition 2.1 (Superdistributions). For any open $U \subseteq G_0$, define

$$\mathcal{D}b_G(U) := \Gamma_c(\mathcal{O}_G|_U)',$$

the strong continuous dual space. Since $\mathcal{O}_G$ is a c-soft sheaf, we have by [14, Chapter V, §1, Proposition 1.6] that $U \longmapsto \Gamma_c(\mathcal{O}_G|_U)$ is a flabby cosheaf. The corestriction

maps are continuous by the definition of the topology. Thus, $\mathcal{D}b_G$ is a sheaf of locally convex super-vector spaces, called the *sheaf of superdistributions*. In particular, we let $\mathcal{D}'(G) \coloneqq \Gamma(\mathcal{D}b_G) = \Gamma_c(\mathcal{O}_G)'$.

The sheaf $\mathcal{D}b_G$ is naturally a right $\mathcal{O}_G$-module by

$$\langle \mu f, \varphi \rangle \coloneqq \langle \mu, f\varphi \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between $\mathcal{D}b_G(U)$ and $\Gamma_c(\mathcal{O}_G|_U)$.

The Lie supergroup G acts from the left on $\mathcal{D}b_G$, where the G_0- and $\mathfrak{g}$-action are given respectively by

$$\langle L_g \mu, \varphi \rangle \coloneqq \langle \mu, L_{g^{-1}}\varphi \rangle, \quad \langle L_x \mu, \varphi \rangle \coloneqq -\langle \mu, L_x \varphi \rangle.$$

Here, in terms of the isomorphism ϕ, we have

$$\phi(L_g\varphi)(u; h) = \phi(\varphi)(u; g^{-1}h), \quad \phi(L_x\varphi)(u; h) = -\phi(\varphi)(\mathrm{Ad}(h^{-1})(x)u; h).$$

In what follows, if H is a subsupergroup of G and $\mathcal{A}$ is a subalgebra of $\mathcal{O}_G$, we will call a sheaf on G_0 with a left H-action commuting with a right $\mathcal{A}$-action an $(H, \mathcal{A})$-module. Thus, $\mathcal{D}b_G$ is a $(G, \mathcal{O}_G)$-module.

In the following proposition, recall that any supermanifold X comes with a natural embedding of the underlying manifold X_0, denoted by $j_{X_0} : X_0 \longrightarrow X$. The underlying map of j_{X_0} is the identity; the sheaf map $j_{X_0}^\sharp$ assigns to any superfunction f its underlying function f_0.

Proposition 2.2. *Let $\mathcal{D}b_{G_0}$ be the sheaf of superdistributions on G_0. There is an isomorphism of $(G, \mathcal{O}_{G_0})$-modules*

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathcal{D}b_{G_0} \longrightarrow \mathcal{D}b_G,$$

given by

$$(2.2) \qquad \langle u \otimes \mu, \varphi \rangle = \langle \mu, j_{G_0}^\sharp(L_{S(u)}\varphi) \rangle = (-1)^{|u||\varphi|} \langle \mu_g, \varphi(\mathrm{Ad}(g^{-1})(u); g) \rangle$$

for all open $U \subseteq G_0$, $u \in \mathfrak{U}(\mathfrak{g})$, $\mu \in \mathcal{D}b_{G_0}(U)$, and $\varphi \in \Gamma_c(\mathcal{O}_G|_U)$.

Proof. First, we check that the map is well-defined. Indeed, we compute for $x \in \mathfrak{g}_{\bar{0}}$:

$$\langle ux \otimes \mu, \varphi \rangle = \langle \mu, j^\sharp(L_{S(ux)}\varphi) \rangle = -\langle \mu, j^\sharp(L_x L_{S(u)}\varphi) \rangle$$
$$= \langle L_x \mu, j^\sharp L_{S(u)}\varphi \rangle = \langle u \otimes L_x\mu, \varphi \rangle,$$

where we abbreviate $j = j_{G_0}$. Similarly, one verifies that the map is G-equivariant. Since it is manifestly right $\mathcal{O}_{G_0}$-linear, it is a morphism of $(G, \mathcal{O}_{G_0})$-modules.

To see that it is an isomorphism, we define an involutive anti-automorphism $(-)^\vee = i^\sharp : i_0^{-1}\mathcal{O}_G \longrightarrow \mathcal{O}_G$ (where $i_0(g) = g^{-1}$) by

$$f^\vee(u; g) \coloneqq f(\mathrm{Ad}(g)(S(u)); g^{-1}).$$

(This just the inversion morphism $i : G \longrightarrow G$.) Then we compute

$$(2.3) \qquad (-1)^{|u||\varphi|}\langle S(u) \otimes \mu, \check{\varphi} \rangle = \langle \mu_g, \check{\varphi}(\mathrm{Ad}(g^{-1})(u); g) \rangle = \langle \mu_{g^{-1}}, \varphi(u; g) \rangle.$$

We recall again that there is an isomorphism of right $\mathfrak{g}_{\bar{0}}$-modules $\mathfrak{U}(\mathfrak{g}) \cong \bigwedge \mathfrak{g}_{\bar{1}} \otimes \mathfrak{U}(\mathfrak{g}_{\bar{0}})$. One choice of such an isomorphism that we will repeatedly use is based on the supersymmetrisation map

$$\beta : S(\mathfrak{g}) \longrightarrow \mathfrak{U}(\mathfrak{g}).$$

Explicitly, it is given by

$$\bigwedge \mathfrak{g}_{\bar{1}} \otimes \mathfrak{U}(\mathfrak{g}_{\bar{0}}) \longrightarrow \mathfrak{U}(\mathfrak{g}) : \eta \otimes u \longmapsto \beta(\eta)u,$$

compare [42, Lemma 1]. Applying this decomposition in Equation (2.3) readily implies our claim. $\qquad \square$

The proof shows that

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathcal{D}b_{G_0} \cong \bigwedge \mathfrak{g}_{\bar{1}} \otimes_{\mathbb{K}} \mathcal{D}b_{G_0}.$$

If we consider on this sheaf the obvious tensor product locally convex topology (there is no choice which one to take, since $\bigwedge \mathfrak{g}_{\bar{1}}$ is finite-dimensional), then it is easy to check that the above isomorphism is in fact one of sheaves of locally convex super-vector spaces.

2.2. **Left-invariant superdistributions.** In this section, we show that left-invariant superdistributions are smooth and hence proportional to the invariant Berezinian density. To state this precisely, we recall the definition of Berezinian densities.

Definition 2.3 (Berezinian densities). Let $\mathcal{B}er_G$ denote the Berezian sheaf of G, compare [25, § 1.11, § 3.10; 44, 2.4.2; 45, 4.3.7, 4.6.1] for the definition. We let $|\Omega|_G := or_{G_0} \otimes_{\mathbb{Z}} \mathcal{B}er_G$, where or_{G_0} is the orientation sheaf of G_0. The set of global sections of $|\mathcal{B}er|_G$ is denoted by $|\Omega|(G)$; elements thereof are called *Berezinian densities*. The set of compactly supported sections of $|\mathcal{B}er|_G$ is denoted by $|\Omega|_c(G)$.

Then $|\mathcal{B}er|_G$ is naturally a $(G, \mathcal{O}_G)$-module. Moreover, if $U \subseteq G_0$ is open and $\omega \in \Gamma_c(|\Omega|_G|_U)$, then $\int_{G|_U} \omega \in \mathbb{K}$, the *Berezin integral* of ω, is well-defined [25, Proposition 3.10.5; 44, Theorem 2.4.5; 45, Theorem 4.6.3]. In particular, there is an embedding $|\Omega|_G \longrightarrow \mathcal{D}b_G$, given by

$$\langle \omega, \varphi \rangle := \int_{G|_U} \omega\varphi \quad \text{for all } \omega \in |\Omega|_G(U), \varphi \in \Gamma_c(\mathcal{O}_G|_U).$$

By [4, Theorem 4.13], $|\Omega|_G$ has a nowhere vanishing G-invariant section $|Dg|$, which is unique up to constant multiples. It furnishes a module basis of $|\Omega|_G$.

The following generalises a result due to Bruhat [15, Chapitre I, Proposition 3.1].

Proposition 2.4 (Super Bruhat regularity theorem). *Let $\mu \in \mathcal{D}'(G)$ be left-invariant under G. Then for some constant c, we have $\mu = c|Dg|$.*

The *proof* of the proposition uses the following definition and basic lemmas.

Definition 2.5 (Convolution of superdistributions). Let $\mu, \nu \in \Gamma(\mathcal{D}b_G)$. We say that (μ, ν) is a *proper pair* if $m_0 : \operatorname{supp}\mu \times \operatorname{supp}\nu \longrightarrow G_0$ is a proper map.

If $\varphi \in \Gamma_c(\mathcal{O}_G)$, then $K := (\operatorname{supp}\mu \times \operatorname{supp}\nu) \cap m_0^{-1}(\operatorname{supp}\varphi)$ is compact. Let $\chi \in \Gamma_c(\mathcal{O}_{G\times G})$ such that $\chi|_U = 1$ for some open neighbourhood of K. The quantity

$$\langle \mu * \nu, \varphi \rangle := \langle \mu \otimes \nu, \chi m^{\sharp}(\varphi) \rangle$$

is independent of χ. Moreover, it depends continuously on φ, thus defining an element $\mu * \nu \in \Gamma(\mathcal{O}_G)$, the *convolution* of μ and ν. Clearly, if either μ or ν is compactly supported, then (μ, ν) is a proper pair.

Lemma 2.6. *Let $\mu \in \Gamma(\mathcal{D}_G)$ and $\omega \in |\Omega|_c(G)$. Then $\mu * \omega \in |\Omega|(G)$.*

Proof. Let $\varphi \in \Gamma_c(\mathcal{O}_G)$ and set

$$\psi(g) := \int_G \omega(h) m^{\sharp}(\varphi)(g, h)$$

for any T and any $h \in_T G$. In the integral, h denotes the generic point $h = \operatorname{id}_G \in_G G$.

Then by Yoneda's lemma, we have $\psi \in \Gamma(\mathcal{O}_G)$, and this superfunction has compact support $\subseteq (\operatorname{supp}\varphi)(\operatorname{supp}\omega)^{-1}$. Hence, we find that

$$\langle \mu * \omega, \varphi \rangle = \langle \mu, \psi \rangle = \left\langle \mu_g, \int_G |Dh| f(h)\varphi(gh) \right\rangle.$$

Writing $\omega = |Dg| \, f$, we have

$$\varrho := (\mu \otimes \mathrm{id})(m \circ (i \times \mathrm{id}))^{\sharp}(f) \in \Gamma(\mathcal{O}_G),$$

since $\Gamma(\mathcal{O}_{G \times G}) = \Gamma(\mathcal{O}_G) \,\widehat{\otimes}_{\pi}\, \Gamma(\mathcal{O}_G)$, the completion of the projective tensor product [6, Corollary C.9], and $(\mu \otimes \mathrm{id})$ extends continuously this space. We thus compute

$$\langle \mu * \omega, \varphi \rangle = \left\langle \mu_g, \int_G |Dh| \, f(g^{-1}h)\varphi(h) \right\rangle = \langle |Dg| \, \varrho, \varphi \rangle,$$

so that $\mu * \omega = |Dg| \, \varrho$, proving the claim. $\square$

Lemma 2.7. *Let $\mathcal{U}$ be the filter of open neighbourhoods of $1 \in G_0$. There exist Berezinian densities $\chi_U = \check{\chi}_U \in |\Omega|_c(G)$, $\mathrm{supp}\,\chi_U \subseteq U \in \mathcal{U}$, such that*

$$\lim_{U \in \mathcal{U}} \chi_U * \mu = \lim_{U \in \mathcal{U}} \mu * \chi_U = \mu$$

in $\mathcal{D}'(G)$, for any $\mu \in D'(G)$. If $\mu \in |\Omega|_c(G)$, then the convergence is in $|\Omega|_c(G)$.

Proof. For U sufficiently small, we may choose local coordinates (u, ξ) and define

$$\chi_U := |D(u, \xi)| \, \xi_1 \cdots \xi_q \varrho_U,$$

where $\int_U |du_0| \varrho_U = 1$ and $\dim G = p|q$. Then for $\varphi \in \Gamma_c(\mathcal{O}_G)$, we have

$$\int_G \chi_U \varphi = \int_U |du_0| \, \varrho_U \varphi_0 \longrightarrow \varphi_0(1) = \varphi(1),$$

where the convergence is uniform for φ in compact subsets of $\Gamma_c(\mathcal{O}_G)$. Indeed, [34, Proposition 2.42] gives uniform convergence, and compactness is preserved when passing to a coarser topology.

Now, the computation in the proof of Lemma 2.6 shows that

$$\langle \mu * \chi_U, \varphi \rangle = \langle \mu, \chi_U * \varphi \rangle$$

where we set

$$(\chi_U * \varphi)(h) := \int_G \chi_U(g)\varphi(g^{-1}h)$$

for any T and any $h \in_T G$. Then for $h \in_T G$

$$(\chi_U * \varphi)(h) - \varphi(h) = \int_G \chi_U(g)\big(\varphi(g^{-1}h) - \varphi(h)\big) \longrightarrow 0,$$

the convergence being in $\Gamma_c(\mathcal{O}_T)$. Taking $T = G$ and $h = \mathrm{id}_G \in_G G$, the assertion follows for right convolutions, and the case of left convolutions is similar. $\square$

Proof of Proposition 2.4. The proof is the same as Bruhat's, based on the super-extensions of classical facts stated as the lemmas above. Let (χ_U) be as in the statement of Lemma 2.7. We have, for any $U \in \mathcal{U}$, $g \in G_0$, and $u \in \mathfrak{U}(\mathfrak{g})$:

$$L_g L_u(\mu * \chi_U) = (L_g L_u \mu) * \chi_U,$$

so the superdistribution $\mu * \chi_U$, which is a Berezin density by Lemma 2.6, is left-invariant under G and thus equals $c_U |Dg|$ for some constant c_U.

But by Lemma 2.7, we have $\mu = \lim_U \mu * \chi_U = \lim_U c_U |Dg|$, so that μ is contained in the closure of the line spanned by $|Dg|$. But this line is finite-dimensional, and hence a closed subspace of $\Gamma(\mathcal{D}b_G)$, since the unique Hausdorff vector space topology on $\mathbb{K}$ is complete. This shows the assertion. $\square$

2.3. **Berezinian densities *via* ordinary densities.** In this subsection, we show how Berezinian densities can be expressed in terms of ordinary densities on the underlying Lie groups.

To that end, let $|\Omega|_{G_0} := or_{G_0} \otimes_{\mathbb{Z}} \Omega_{G_0}^p$, where $p|q = \dim G$, denote the sheaf of $\mathbb{K}$-valued smooth densities on G_0. Its global sections will be denoted by $|\Omega|(G_0)$, and the subspace of compactly supported sections by $|\Omega|_c(G_0)$. As above, there is an embedding $|\Omega|_{G_0} \longrightarrow \mathcal{D}b_{G_0}$, given by

$$\langle \omega, \varphi \rangle := \int_U \omega\varphi \quad \text{for all } \omega \in |\Omega|_{G_0}(U), \varphi \in \Gamma_c(\mathcal{O}_{G_0}|_U) = \mathcal{C}_c^\infty(U).$$

The isomorphism in Proposition 2.2 suggests that we can indentify $|\Omega|_G$ and $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} |\Omega|_{G_0}$ within $\mathcal{D}b_G$. Although this is not completely straightforward, it turns out to be quite generally true, as we now proceed to explain.

Let $\delta_{\bar{1}}$ be the character by which $\mathfrak{g}_{\bar{0}}$ acts on $\mathrm{Ber}(\mathfrak{g}/\mathfrak{g}_{\bar{0}})$, *i.e.*

$$\delta_{\bar{1}}(x) = - \mathrm{tr}_{\mathfrak{g}_{\bar{1}}} \mathrm{ad}(x) \quad \text{for all } x \in \mathfrak{g}_{\bar{0}}.$$

This character extends naturally to $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$. It is the differential of the character $\Delta_{\bar{1}}$ of G_0, given by

$$\Delta_{\bar{1}}(g) := \mathrm{Ber}_{(\mathfrak{g}/\mathfrak{g}_{\bar{0}})^*}(\mathrm{Ad}(g)) = \left(\det_{\mathfrak{g}_{\bar{1}}} \mathrm{Ad}(g)\right)^{-1}.$$

For any $\mathfrak{g}_{\bar{0}}$-module N (say), there is a well-known [7, 18, 20, 31, 36] isomorphism of graded $\mathfrak{g}$-modules

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} N \xrightarrow{\Phi} \underline{\mathrm{Hom}}_{\mathfrak{g}_{\bar{0}}}\left(\mathfrak{U}(\mathfrak{g}), \mathrm{Ber}((\mathfrak{g}/\mathfrak{g}_{\bar{0}})^*) \otimes_{\mathbb{K}} N\right).$$

By the construction detailed in [36, 3.2.1], it is given explicitly by

$$(2.4) \qquad \Phi(u \otimes n)(v) = (-1)^{(|u|+|n|)|v|} \iota(vu)(\omega_{\bar{1}} \otimes n)$$

where $\omega_{\bar{1}} \in \mathrm{Ber}((\mathfrak{g}/\mathfrak{g}_{\bar{0}})^*)$ is an arbitrary non-zero element and $\iota : \mathfrak{U}(\mathfrak{g}) \longrightarrow \mathfrak{U}(\mathfrak{g}_{\bar{0}})$ is the left $\mathfrak{g}_{\bar{0}}$-linear map defined by

$$\iota(u\beta(\eta)) := u \int_{\mathfrak{g}_{\bar{1}}} \omega_{\bar{1}}\eta \quad \text{for all } u \in \mathfrak{U}(\mathfrak{g}_{\bar{0}}), \eta \in \bigwedge \mathfrak{g}_{\bar{1}}.$$

Here, β is supersymmetrisation, and the Berezin integral is normalised by $\int_{\mathfrak{g}_{\bar{1}}} \omega_{\bar{1}} = 1$.

A notable special case occurs when $N = \mathrm{Ber}(\mathfrak{g}/\mathfrak{g}_{\bar{0}})$. In this case, we may consider the action of G_0 on N, and $\mathrm{Ber}(\mathfrak{g}/\mathfrak{g}_{\bar{0}})^* \otimes_{\mathbb{K}} N \cong \mathbb{K}$ as G-modules. Moreover, if $g \in G_0$, then we have the equation

$$\iota\left(\mathrm{Ad}(g)(u\beta(\eta))\right) = \mathrm{Ad}(g)(u) \int_{\mathfrak{g}_{\bar{1}}} \omega_{\bar{1}} \mathrm{Ad}(g)(\eta) = \Delta_{\bar{1}}(g) \cdot \mathrm{Ad}(g)\left(\iota(u\beta(\eta))\right)$$

by the change of variables formula for the Berezin integral. Combining these facts with the definition of Φ, one arrives by the formula

$$(2.5) \qquad \Phi\left(\mathrm{Ad}(g)(u) \otimes n\right)(v) = \Delta_{\bar{1}}(g)\Phi(u \otimes n)\left(\mathrm{Ad}(g^{-1})(v)\right)$$

for $u, v \in \mathfrak{U}(\mathfrak{g})$ and $n \in \mathrm{Ber}(\mathfrak{g}/\mathfrak{g}_{\bar{0}})$.

Let $I_{\delta_{\bar{1}}}$ be the left ideal of $\mathfrak{U}(\mathfrak{g})$ generated by the set

$$\left\{x \in \mathfrak{g}_{\bar{0}} \mid x - \delta_{\bar{1}}(x)\right\}.$$

By [31, Proposition 3.5], the space of $\mathfrak{g}$-invariants in

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathrm{Ber}(\mathfrak{g}/\mathfrak{g}_{\bar{0}}) = \mathfrak{U}(\mathfrak{g})/I_{\delta_{\bar{1}}}$$

is one-dimensional. Let $\gamma \in \mathfrak{U}(\mathfrak{g})$ be a representative of a basis.

Proposition 2.8. *For a suitable normalisation of $|Dg|$ and $|dg|$, we have*

$$(2.6) \qquad\qquad |Dg| = L_\gamma(|dg|\,\Delta_{\bar{1}}).$$

Proof. Let us consider the isomorphism Φ for $N = \mathrm{Ber}(\mathfrak{g}/\mathfrak{g}_{\bar{0}})$. Since there is a canonical isomorphism

$$(2.7) \qquad\qquad \mathrm{Ber}((\mathfrak{g}/\mathfrak{g}_{\bar{0}})^*) \otimes_{\mathbb{K}} \mathrm{Ber}(\mathfrak{g}/\mathfrak{g}_{\bar{0}}) \longrightarrow \mathbb{K}$$

of $\mathfrak{g}_{\bar{0}}$-modules [31, Lemma 1.4], we may view Φ as an isomorphism

$$\Phi : \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathrm{Ber}(\mathfrak{g}/\mathfrak{g}_{\bar{0}}) \longrightarrow \underline{\mathrm{Hom}}_{\mathfrak{g}_{\bar{0}}}(\mathfrak{U}(\mathfrak{g}), \mathbb{K}).$$

Moreover, by [31, p. 150], the coset of γ corresponds under the canonical isomorphism Φ to the element $\varepsilon : \mathfrak{U}(\mathfrak{g}) \longrightarrow \mathbb{K}$, which is the extension of $0 : \mathfrak{g} \longrightarrow \mathbb{K}$ to a superalgebra morphism. Hence, by Equation (2.5), for any $g \in G_0$, we have

$$(2.8) \qquad\qquad \mathrm{Ad}(g)(\gamma) \equiv \Delta_{\bar{1}}(g)\gamma \quad (\mathrm{mod}\ I_{\delta_{\bar{1}}}).$$

Since $|dg|\,\Delta_{\bar{1}}$ is relatively $\mathfrak{g}_{\bar{0}}$-invariant for the character $\delta_{\bar{1}}$, this quantity is annihilated by $I_{\delta_{\bar{1}}}$. In particular, the superdistribution

$$\Omega := L_\gamma(|dg|\,\Delta_{\bar{1}}) \in \mathcal{D}'(G)$$

depends only on the coset of γ.

By Proposition 2.4, it will be sufficient to show that Ω is a $\mathfrak{g}$- and G_0-invariant functional. First, let $x \in \mathfrak{g}$ be homogeneous. Then we compute

$$\langle \Omega, L_x f \rangle = \langle |dg|\,\Delta_{\bar{1}}, L_{S(\gamma)x}(f) \rangle = (-1)^{|x||\gamma|} \langle L_{x\gamma}(|dg|\,\Delta_{\delta_{\bar{1}}}), f \rangle = 0$$

since by the choice of γ, we have $x\gamma \in I_{\delta_{\bar{1}}}$ for any $x \in \mathfrak{g}$.

Secondly, we compute

$$\langle \Omega, L_h f \rangle = \langle L_{\mathrm{Ad}(h^{-1})(\gamma)}(L_{h^{-1}}(|dg|\,\Delta_{\bar{1}})), f \rangle = \langle \Omega, f \rangle,$$

by the use of the relation $L_{h^{-1}}(|dg|\,\Delta_{\bar{1}}) = \Delta_{\bar{1}}(h)\,|dg|\,\Delta_{\bar{1}}$ and Equation (2.8). Thus, we reach our conclusion. $\qquad\square$

Corollary 2.9. *As $(G, \mathcal{O}_{G_0})$-submodules of $\mathcal{D}b_G$, we have*

$$|\Omega|_G = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} |\Omega|_{G_0}.$$

Writing $\Delta(\gamma) = \sum_i \gamma_i' \otimes \gamma_i''$, the Berezinian density $|Dg|\,f$ corresponds to

$$(2.9) \qquad\qquad \sum_i \gamma_i' \otimes |dg|\,\Delta_{\bar{1}}\, j_{G_0}^\sharp \big(L_{S(\gamma_i'')}(f)\big).$$

Conversely, the element $1 \otimes |dg|$ is mapped to $|Dg|\,\psi$, where $\psi \in \Gamma(\mathcal{O}_G)$ is defined by

$$(2.10) \qquad\qquad \psi(u; g) := (R_{\iota(u)}\Delta_{\bar{1}}^{-1})(g)$$

for all $u \in \mathfrak{U}(\mathfrak{g})$, $g \in G_0$.

Proof. Consider the isomorphism

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathcal{D}b_{G_0} \longrightarrow \mathcal{D}b_G$$

from Proposition 2.2. For $f, \varphi \in \mathcal{O}_G(U)$, we compute

$$\big(L_{S(\gamma)}(f\varphi)\big)(g) = \sum_i (-1)^{|\gamma_i'|(|f|+|\gamma_i''|)} \big(L_{S(\gamma_i'')}(f)\big)(g)\big(L_{S(\gamma_i')}(\varphi)\big)(g).$$

For the non-zero summands, we have $|\gamma_i''| + |f| \equiv 0\,(2)$. Hence, under the isomorphism, the expression in Equation (2.9) is mapped to $|Dg|\,f$. Thus, $|\Omega|_G$ is contained in the image of the subsheaf

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} |\Omega|_{G_0} \subseteq \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathcal{D}b_{G_0}.$$

For the converse, *i.e.* that $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{g}_{\bar{0}}} |\Omega|_{G_0}$ is mapped to $|\Omega|_G$, we need only show that this is the case for the $\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{O}_{G_0}$-generator $1 \otimes |dg|$. To that end, consider the superfunction $\psi \in \Gamma(\mathcal{O}_G)$, defined by Equation (2.10). It is well-defined, because the map ι is by definition left $\mathfrak{g}_{\bar{0}}$-linear.

By [31, Theorem 3.1, Equation (65)], we have $\gamma \equiv \beta(x_1 \cdots x_q J) \ (I_{\delta_{\bar{1}}})$, where $J \in (\bigwedge \mathfrak{g}_{\bar{1}})_{\bar{0}}$ is the Jacobian of the exponential map (compare *loc. cit.*). Set $\tilde{\gamma} \coloneqq x_1 \cdots x_q J \in S(\mathfrak{g})$ and consider the grading with components

$$S^{\bullet,k} \coloneqq S^{\bullet,k}(\mathfrak{g}) \coloneqq S(\mathfrak{g}_{\bar{0}}) \otimes \bigwedge\nolimits^k \mathfrak{g}_{\bar{1}}.$$

Observe

$$\Delta(S^{\bullet,k}) \subseteq \bigoplus\nolimits_{a+b=k} S^{\bullet,a} \otimes S^{\bullet,b}.$$

In particular, we have

$$\Delta(\tilde{\gamma}) \equiv \tilde{\gamma} \otimes 1 \quad (\mathrm{mod}\ \bigoplus\nolimits_{a<q} S^{\bullet,q} \otimes S(\mathfrak{g})).$$

On the other hand, by the definition of ι, we have $\iota(u\beta(\eta)) = 0$ for $u \in \mathfrak{U}(\mathfrak{g}_{\bar{0}})$ and $\eta \in \bigwedge \mathfrak{g}_{\bar{1}}$, unless η has a non-zero component in top degree. Since $\beta : \mathfrak{U}(\mathfrak{g}) \longrightarrow S(\mathfrak{g})$ is an isomorphism of coalgebras [55, Theorem 8.1], we find

$$\Delta(\gamma) \equiv \gamma \otimes 1 \quad (\mathrm{mod}\ \ker \iota \otimes \mathfrak{U}(\mathfrak{g})).$$

As observed in the proof of Proposition 2.8, we have for all $v \in \mathfrak{U}(\mathfrak{g})$:

$$\varepsilon(v) = \Phi(\dot{\gamma})(v) = \iota(v\gamma),$$

where $\dot{\gamma} \in \mathfrak{U}(\mathfrak{g})/I_{\delta_{\bar{1}}}$ denotes the coset of γ and we have used Equation (2.4). No signs occur, since the left-hand side of the equation is independent of the odd part of v. In particular, $\iota(\gamma) = 1$.

Hence, we compute for any $\varphi \in \Gamma(\mathcal{O}_G)$ that

$$L_{S(\gamma)}(\psi\varphi)(1;g) = \Delta_{\bar{1}}(g)^{-1}\varphi(1;g).$$

For compactly supported φ, this implies that

$$\langle |Dg|\,\psi, \varphi \rangle = \langle |dg|, \Delta_{\bar{1}} j_{G_0}^{\sharp}\big(L_{S(\gamma)}(\psi\varphi)\big) \rangle = \langle |dg|, j_{G_0}^{\sharp}(\varphi) \rangle.$$

Thus, we find that $1 \otimes |dg|$ is mapped to $|\Omega|(G)$; this proves the claim. $\qquad\square$

2.4. Convolution of superdistributions and Berezinian densities.

Definition 2.10 (Compactly supported superdistributions)**.** We let $\mathcal{E}'(G)$ be the strong dual space of $\Gamma(\mathcal{O}_G) = \mathcal{O}_G(G_0)$ and call its elements *compactly supported superdistributions*. For $\mu, \nu \in \mathcal{E}'(G)$, the convolution $\mu * \nu \in \mathcal{E}'(G)$ from Definition 2.5 takes the form

$$\langle \mu * \nu, f \rangle \coloneqq \langle \mu \otimes \nu, m^{\sharp}(f) \rangle$$

for all $f \in \Gamma(\mathcal{O}_G)$. Here, $m : G \times G \longrightarrow G$ is the multiplication of G.

If A is a topological $\mathbb{K}$-vector space with an algebra structure, then we call A a *topological algebra* if multiplication is separately continuous. We allow non-unital algebras, but unless called 'non-unital' expressly, they are assumed to have a unit.

In the following, let $\mathcal{E}'(G_0)$ be the strong dual of $\Gamma(\mathcal{O}_{G_0})$. It carries a natural convolution, see [15, §1.4; 43, Chapter I.1; 62, Chapter 0.3]. Recall that $\mathfrak{U}(\mathfrak{g}_{\bar{0}}) \subseteq \mathcal{E}'(G_0)$ is a subalgebra *via* $u \longmapsto L_u \delta$, where δ denotes the Dirac delta distribution supported at the neutral element of G_0.

Proposition 2.11. *The convolution product on $\mathcal{E}'(G)$ is well-defined and turns it into an associative and unital topological superalgebra. We have $\mathcal{E}'(G) = \Gamma_c(\mathcal{D}b_G)$ and there is an isomorphism*

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathcal{E}'(G_0) \longrightarrow \mathcal{E}'(G)$$

of locally convex super-vector spaces. In terms of this isomorphism, the superalgebra structure is uniquely determined by the following facts:

(i) *The following are graded subalgebras:*

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathfrak{U}(\mathfrak{g}_{\bar{0}}), \quad \mathcal{E}'(G_0) = \mathfrak{U}(\mathfrak{g}_{\bar{0}}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathcal{E}'(G_0).$$

(ii) *For $\mu \in \mathcal{E}'(G_0)$ and $u \in \mathfrak{U}(\mathfrak{g})$, the products $u * \mu$ and $\mu * u$ are given by*

$$(2.11) \qquad \langle u * \mu, \varphi \rangle = \langle \mu, j_{G_0}^{\sharp}(L_{S(u)}\varphi) \rangle, \quad \langle \mu * u, \varphi \rangle = \langle \mu, j_{G_0}^{\sharp}(R_u \varphi) \rangle$$

for all superfunctions $\varphi \in \Gamma(\mathcal{O}_G)$.

Proof. Since $\Gamma_c(\mathcal{O}_G)$ is dense in $\Gamma(\mathcal{O}_G)$, $\mathcal{E}'(G)$ may be identified with a subspace of $\mathcal{D}'(G)$. On the other hand, one knows that $\mathcal{E}'(G_0) = \Gamma_c(\mathcal{D}b_{G_0})$. Therefore, Proposition 2.2 gives an isomorphism of super-vector spaces as stated and $\mathcal{E}'(G) = \Gamma_c(\mathcal{D}b_G)$. Moreover, it is straightforward to prove that it is indeed a homeomorphism for the topology on $\mathcal{E}'(G)$ and the natural topology on $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathcal{E}'(G_0) = \bigwedge \mathfrak{g}_{\bar{1}} \otimes_{\mathbb{K}} \mathcal{E}'(G_0)$.

It is clear that there is at most one algebra structure on $\mathcal{E}'(G)$ determined by the information stated in (i) and (ii). Conversely, we compute for $\mu, \nu \in \mathcal{E}'(G_0)$ and $u, v \in \mathfrak{U}(\mathfrak{g})$, by the use of Equations (2.1) and (2.2):

$$\langle (u \otimes \mu) * (v \otimes \nu), \varphi \rangle = (-1)^{|\varphi|(|u|+|v|)} \langle \mu_g \otimes \nu_h, (m^{\sharp}\varphi)(\mathrm{Ad}((g,h)^{-1})(u \otimes v); g, h) \rangle$$
$$= (-1)^{|\varphi|(|u|+|v|)} \langle \mu_g \otimes \nu_h, \varphi(\mathrm{Ad}(h^{-1})(\mathrm{Ad}(g^{-1})(u)v); gh) \rangle.$$

For $\mu = \delta$ and $v = 1$, we obtain

$$\langle u * \nu, \varphi \rangle = (-1)^{|\varphi||u|} \langle \nu_h, \varphi(\mathrm{Ad}(h^{-1})(u); h) \rangle = \langle \nu, L_{S(u)}\varphi \rangle,$$

and for $u = 1$ and $\nu = \delta$, we get

$$\langle \mu * v, \varphi \rangle = (-1)^{|\varphi||v|} \langle \mu_g, \varphi(v; g) \rangle = \langle \mu, R_v \varphi \rangle.$$

This shows Equation (2.11).

The convolution on $\mathcal{E}'(G)$ is an even bilinear map by definition. That it is an associative operation follows either from $m \circ (m \times \mathrm{id}) = m \circ (\mathrm{id} \times m)$, or also easily from Equation (2.11), together with the fact that $\mathfrak{U}(\mathfrak{g})$ and $\mathcal{E}'(G_0)$ are algebras and that the actions L and R commute. $\square$

The convolution algebra structure on $\mathcal{E}'(G)$ admits a natural $\mathbb{K}$-linear anti-involution, defined by

$$(2.12) \qquad \langle \check{\mu}, \varphi \rangle := \langle \mu, \check{\varphi} \rangle = \langle \mu, i^{\sharp}\varphi \rangle,$$

where $i : G \longrightarrow G$ is the inversion morphism, and $\check{\varphi} = i^{\sharp}\varphi$ was employed above in the proof of Proposition 2.2. Since Berezinian densities pull back under isomorphisms, the involution leaves $|\Omega|_c(G) \subseteq \mathcal{E}'(G)$ stable.

Corollary 2.12. *The dense subspace $|\Omega|_c(G) \subseteq \mathcal{E}'(G)$ is a graded ideal and a non-unital Fréchet algebra with the topology induced from $\Gamma_c(\mathcal{O}_G)$. In terms of the isomorphism $|\Omega|_c(G) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} |\Omega|_c(G_0)$, its $\mathcal{E}'(G)$-bimodule structure is determined uniquely by the following facts:*

(i) *The following is a non-unital graded subalgebra bi-invariant under $\mathcal{E}'(G_0)$:*

$$|\Omega|_c(G_0) = \mathfrak{U}(\mathfrak{g}_{\bar{0}}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} |\Omega|_c(G_0).$$

(ii) *For $u, v \in \mathfrak{U}(\mathfrak{g})$ and $\omega \in |\Omega|_c(G_0)$, we have*

$$u * (v \otimes \omega) = (u \otimes 1) * (v \otimes \omega) = uv \otimes \omega.$$

(iii) *For $\omega \in |\Omega|_c(G_0)$ and $u \in \mathfrak{U}(\mathfrak{g})$, the products $u * \omega$ and $\omega * u$ are given by*

(2.13)
$$\int_{G_0} (u * \omega)\varphi = \int_{G_0} \omega \, j_{G_0}^\sharp \big(L_{S(u)}\varphi \big) = \langle u \otimes \omega, \varphi \rangle,$$
$$\int_{G_0} (\omega * u)\varphi = \int_{G_0} \omega \, j_{G_0}^\sharp \big(R_u \varphi \big),$$

for all superfunctions $\varphi \in \Gamma(\mathcal{O}_G)$.

Proof. Let us verify that $|\Omega|_c(G)$ is indeed a convolution ideal in $\mathcal{E}'(G)$. Indeed, this follows from Lemma 2.6. Alternatively, one may proceed as follows.

Certainly, $|\Omega|_c(G_0)$ is an ideal of $\mathcal{E}'(G_0)$. Let ω and $u \in \mathfrak{U}(\mathfrak{g})$. Since $u * \omega$ corresponds to $u \otimes \omega$, it is obvious that $u * \omega \in \Omega_c(G)$. On the other hand, we have

$$\int_{G_0} (\omega * u)\varphi = (-1)^{|u||\varphi|} \int_{G_0} \omega(g)\varphi(u; g) = (-1)^{|u||\varphi|} \int_{G_0} \check{\omega}(g)\check{\varphi}(S(u); g) = \int_G \check{\Omega}\,\varphi,$$

where $\Omega \in |\Omega|_c(G)$ corresponds to $S(u) \otimes \check{\omega}$ and $\check{\Omega}$ was defined in Equation (2.12). This shows that $\omega * u \in |\Omega|_c(G)$.

Thus, $|\Omega|_c(G)$ is indeed a graded ideal of $\mathcal{E}'(G)$, and the remaining statements follow readily from Proposition 2.11. $\qquad\square$

2.5. **Convolution action on representations.** We now show how supergroup representations on Fréchet spaces can be characterised in terms of the action of convolution superalgebras. We will use the following terminology.

Definition 2.13. Left A be a topological algebra. A left A-module will be called a *continuous module* if the action map is separately continuous. An A-module E is called *non-degenerate* if

$$E = AE \coloneqq \big\langle av \,\big|\, a \in A, v \in E \big\rangle_{\mathbb{K}}.$$

Lemma 2.14. *Let (E, π) be a weakly smooth Fréchet G-representation. Then the $\mathcal{E}'(G_0)$-module structure inherited from $E|_{G_0}$ combines with the $\mathfrak{U}(\mathfrak{g})$-action on E to a unique continuous $\mathcal{E}'(G)$-module structure on E, denoted by Π. The action map $\mathcal{E}'(G) \times E \longrightarrow E$ is hypocontinuous.*

Proof. We begin with some preliminary considerations. Since $\Gamma(\mathcal{O}_{G_0})$ is a nuclear Fréchet space [63, Corollary to Theorem 51.4], we have $\mathcal{C}^\infty(G_0, E) = \Gamma(\mathcal{O}_{G_0}) \widehat{\otimes}_\pi E$ by [63, Theorems 44.1 and 50.1], where $\widehat{\otimes}_\pi$ denotes the completed projective tensor product. Moreover, $\Gamma(\mathcal{O}_{G_0})$ is barreled and reflexive, and $\mathcal{E}'(G_0)$ is nuclear and complete in the strong topology [63, Corollary 2 to Theorem 32.2, Corollary 1 to Proposition 33.2, Corollary to Proposition 36.9, Proposition 36.10, Proposition 50.6].

In particular, the abstract Kernels Theorem [63, Proposition 50.5] applies. Thus, if $\langle \cdot, \cdot \rangle$ denotes the pairing of $\mathcal{E}'(G_0)$ and $\Gamma(\mathcal{O}_{G_0})$, then the map

$$\mathcal{C}^\infty(G_0, E) = \Gamma(\mathcal{O}_{G_0}) \widehat{\otimes}_\pi E = \mathcal{E}'(G_0)'_\beta \widehat{\otimes}_\pi E \longrightarrow \underline{\mathrm{Hom}}(\mathcal{E}'(G_0), E)$$

induced by $\varphi \otimes e \longmapsto (\mu \longmapsto \langle \mu, \varphi \rangle e)$ is a continuous linear isomorphism. Here, $\underline{\mathrm{Hom}}$ denotes the space of continuous linear maps with the topology of uniform convergence on bounded subsets. This map is an element of

$$\underline{\mathrm{Hom}}\big(\mathcal{C}^\infty(G_0, E), \underline{\mathrm{Hom}}(\mathcal{E}'(G_0), E)\big),$$

so by [13, Chapitre III, § 5.3, Proposition 3], the corresponding bilinear map

$$\langle \cdot, \cdot \rangle : \mathcal{E}'(G_0) \times \mathcal{C}^\infty(G_0, E) \longrightarrow E,$$

is hypocontinuous with respect to the first argument. Since $\mathcal{E}'(G_0)$ is barreled, it is hypocontinuous [13, Chapitre III, § 5.3, Proposition 6].

In other words, the bilinear map

$$\mathcal{E}'(G_0) \times (\Gamma(\mathcal{O}_{G_0}) \otimes E) \longrightarrow E$$

that sends $(\mu, \varphi \otimes e)$ to $\langle \mu, \varphi \rangle e$ has a (unique) hypocontinuous bilinear extension.

Next, recall that $\Gamma(\mathcal{O}_G) \cong \underline{\mathrm{Hom}}_{\mathfrak{g}_{\bar{0}}}(\mathfrak{U}(\mathfrak{g}), \Gamma(\mathcal{O}_{G_0}))$, see the beginning of Subsection 2.1. Here, $\Gamma(\mathcal{O}_{G_0}) \otimes \bigwedge(\mathfrak{g}_{\bar{1}})^*$ induces the locally convex topology on the latter space. Thus, we define analogously:

$$\mathcal{C}^\infty(G, E) := \Gamma(\mathcal{O}_G) \,\widehat{\otimes}_\pi E.$$

The above arguments apply, and we get a natural hypocontinuous bilinear map

$$\langle \cdot, \cdot \rangle : \mathcal{E}'(G) \otimes \mathcal{C}^\infty(G, E) \longrightarrow E.$$

One sees that the maps thus constructed are compatible with the natural maps $\mathcal{E}'(G_0) \longrightarrow \mathcal{E}'(G)$ and $\mathcal{C}^\infty(G, E) \longrightarrow \mathcal{C}^\infty(G_0, E)$. We will therefore suppress these in the notation. Moreover, from the definition, it is clear that

$$(2.14) \qquad\qquad \langle \mu, T \circ \varphi \rangle = T(\langle \mu, \varphi \rangle)$$

for any continuous linear endomorphism T of E and any $\mu \in \mathcal{E}'(G)$, $\varphi \in \mathcal{C}^\infty(G, E)$. This completes our preliminaries.

Now, let $d\pi$ and π_0 denote the action of $\mathfrak{g}$ and G_0 on E, respectively. Take $v \in E$. Then $\pi_0(-)v : G \longrightarrow E : g \longmapsto \pi_0(g)v$ is a smooth map and there is an element $\pi_v = \pi(-)v \in \underline{\mathrm{Hom}}_{\mathfrak{g}_{\bar{0}}}(\mathfrak{U}(\mathfrak{g}), \mathcal{C}^\infty(G_0, E))$, defined by

$$\pi_v(u; g) := \pi(u; g)v := \big(\pi(-)v\big)(u)(g) := \pi_0(g)d\pi(u)v.$$

We may thus define for $u \in \mathfrak{U}(\mathfrak{g})$ and $\mu \in \mathcal{E}'(G_0)$:

$$(2.15) \qquad\qquad \Pi(u \otimes \mu)v := d\pi(u)\Pi_0(\mu)v = d\pi(u)\langle \mu_g, \pi_0(g)v \rangle,$$

where we let Π_0 denote the 'integrated' version (on distributions) of the G_0-representation π_0 on E [62, Chapter 0.3].

We compute with Equation (2.14) and Proposition 2.11 that

$$\Pi(u \otimes \mu)v = \big\langle \mu_g, d\pi(u)\pi_0(g)v \big\rangle = \big\langle \mu_g, \pi_0(g)d\pi(\mathrm{Ad}(g^{-1})(u))v \big\rangle$$

$$= (-1)^{|u||v|}\big\langle \mu, j^\sharp_{G_0}(L_{S(u)}(\pi_v)) \big\rangle = \langle u \otimes \mu, \pi_v \rangle,$$

i.e. $\Pi(\nu)v = \langle \nu, \pi_v \rangle$ for all $\nu \in \mathcal{E}'(G)$.

In particular, for $x \in \mathfrak{g}_{\bar{0}}$, we obtain

$$\Pi(ux \otimes \mu)v = -(-1)^{|u||v|}\big\langle \mu, L_x L_{S(u)}\pi_v \big\rangle$$

$$= (-1)^{|u||v|}\big\langle L_x\mu, L_{S(u)}\pi_v \big\rangle = \Pi(u \otimes L_x\mu)v.$$

This shows that the action Π is well-defined on $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathcal{E}'(G_0)$, and by our preliminary considerations, the action map $\mathcal{E}'(G) \times E \longrightarrow E$ is hypocontinuous.

To see that Π is an action, let $u \in \mathfrak{U}(\mathfrak{g})$, $\mu \in \mathcal{E}'(G_0)$, and $v \in E$. Then $\Pi(u \otimes 1)\Pi(1 \otimes \mu)v = \Pi(u \otimes \mu)v$ from the above. Moreover, we have

$$\pi_{d\pi(u)v}(u'; g) = \pi_0(g)d\pi(u')d\pi(u)v = \pi_0(g)d\pi(u'u)v = (-1)^{|u||u'|}(R_u\pi_v)(u'; g),$$

so

$$\Pi(1 \otimes \mu)\Pi(u \otimes 1)v = \Pi(1 \otimes \mu)d\pi(u)v = \big\langle \mu, \pi_{d\pi(u)v} \big\rangle$$

$$= \big\langle \mu, R_u\pi_v \big\rangle = \big\langle \mu * u, \pi_v \big\rangle = \Pi(\mu * u)v$$

by Equation (2.11). This proves the claim, since $(1 \otimes \mu)(u \otimes 1)$ and $\mu * u$ are identified within $\mathcal{E}'(G)$ by Proposition 2.11. $\qquad\qquad \square$

We call Π the *integrated action* of π. Restricting it to densities, we obtain the following proposition, which generalises a theorem of Dixmier–Malliavin [26].

Proposition 2.15 (Super Dixmier–Malliavin theorem). *Let E be a Fréchet super-vector space over $\mathbb{K}$. Then we have the following facts:*

(i) *If E carries the structure of a continuous G-representation, then the action of $|\Omega|_c(G)$ on E_∞ extends continuously to E. The induced action of $|\Omega|_c(G)$ on E_∞ is non-degenerate. More precisely, we have the equality*

$$(2.16) \qquad E_\infty = \Pi\big(|\Omega|_c(G)\big)E = \Pi\big(|\Omega|_c(G)\big)E_\infty.$$

(ii) *Conversely, let Π be a non-degenerate continuous action of $|\Omega|_c(G)$ on E. Then Π is integrated from a unique weakly smooth G-representation.*

In particular, the category of weakly smooth Fréchet G-representations and the category of non-degenerate continuous Fréchet $|\Omega|_c(G)$-modules are equivalent.

Proof. Assume that π is a continuous G-representation on E, so that we have by Lemma 2.14 the integrated representation Π of $\mathcal{E}'(G)$ on E_∞. Let Π_0 be the integrated version of the G_0-representation π_0 of G_0 on E. By a theorem of Gårding [62, (2.22)], we have $\Pi_0\big(|\Omega|_c(G_0)\big)E \subseteq E_\infty$.

Thus, it makes sense to define, for $u \in \mathfrak{U}(\mathfrak{g})$, $\omega \in |\Omega|_c(G_0)$, and $v \in E$:

$$\Pi(u \otimes \omega)v := d\pi(u)\Pi_0(\omega)v.$$

Indeed, this coincides with the definition of Π on E_∞ given in Equation (2.15). In addition, for $x \in \mathfrak{g}_{\bar{0}}$, we have

$$\Pi(ux \otimes \omega)v = d\pi(u)d\pi_0(x)\Pi_0(\omega)v = d\pi(u)\Pi_0(L_x\omega) = \Pi(u \otimes x\omega)v,$$

so that Π defines a continuous representation of $|\Omega|_c(G)$ by Corollary 2.9.

By the Dixmier–Malliavin theorem [26, Theorem 3.3], we have

$$E_\infty = \Pi_0\big(|\Omega|_c(G_0)\big)E_\infty.$$

Applying the definition of Π in Equation (2.15), we obtain

$$E_\infty = d\pi(\mathfrak{U}(\mathfrak{g}))E_\infty = d\pi(\mathfrak{U}(\mathfrak{g}))\Pi_0\big(|\Omega|_c(G_0)\big)E_\infty = \Pi\big(|\Omega|_c(G)\big)E_\infty.$$

But $\Pi(|\Omega|_c(G))E \subseteq E_\infty$, so we have proved part (i) of the proposition.

Conversely, assume that E is a non-degenerate continuous $|\Omega|_c(G)$-module. If $v \in E$ is a vector, then we may express it as $v = \sum_{j \in J} \Pi(\omega_j)v_j$ where J is finite and $\omega_j \in |\Omega|_c(G)$, $v_j \in V$. We wish to define π_0 and $d\pi$ for $g \in G_0$ and $u \in \mathfrak{U}(\mathfrak{g})$ by

$$(2.17) \qquad \pi_0(g)v := \sum_{j \in J} \Pi(L_g\omega_j)v_j, \quad d\pi(u)v := \sum_{j \in J} \Pi(L_u\omega_j)v_j.$$

The first task is to show that these quantities are independent of all choices.

To that end, let $\sum_{j \in J} \Pi(\omega_j)v_j = 0$ in E. Choose (χ_U) as in Lemma 2.7. Then $L_g\omega_j = \delta_g * \omega_j$, where δ_g is the Dirac distribution supported at g, and

$$\sum_{j \in J} \Pi(L_g\omega_j)v_j = \lim_{U \in \mathcal{U}} \sum_{j \in J} \Pi(\delta_g * \chi_U * \omega_j)v_j = \lim_{U \in \mathcal{U}} \Pi(L_g\chi_U)v = 0.$$

A similar argument applies for $u \in \mathfrak{U}(\mathfrak{g})$, and so Equation (2.17) indeed defines actions π_0 of G_0 and $d\pi$ of $\mathfrak{g}$.

Moreover, in case Π is already integrated from a weakly smooth G-representation π', then analogously

$$\pi_0'(g)v = \lim_{U \in \mathcal{U}} \sum_j \pi_0'(g)\Pi(\chi_U * \omega_j)v_j = \lim_{U \in \mathcal{U}} \Pi(L_g\chi_U)v = \pi_0(g)v.$$

Similarly, one shows that $d\pi' = d\pi$, so Π is integrated from at most one weakly smooth G-representation, and if it is, then the corresponding actions of G_0 and $\mathfrak{g}$ are given by π_0 and $d\pi$, respectively. It therefore remains to be shown that π_0 and $d\pi$ combine to a weakly smooth G-representation.

For this, we observe that the action map $\Pi : |\Omega|_c(G) \otimes_i E \longrightarrow E$ is continuous and surjective, $\otimes_i$ denoting injective tensor product. It extends to a continuous

surjective map $\hat{\Pi}$ on the completed tensor product $\widehat{\otimes}_i$. Since $|\Omega|_c(G)$ is nuclear [6, Proposition C.7], we have $\widehat{\otimes}_i = \widehat{\otimes}_\pi$ [63, Theorem 50.1], the latter denoting the completed projective tensor product. Since G acts weakly smoothly on $|\Omega|_c(G)$, it acts weakly smoothly on $|\Omega|_c(G) \widehat{\otimes}_\pi E$. Since E is as a Fréchet G_0- and $\mathfrak{g}$-representation a quotient of this space, it follows that it is a weakly smooth G-representation. □

Remark 2.16. The graded version of the Dixmier–Malliavin theorem offered above (part (i) of Proposition 2.15) admits an independent proof, which does not appeal to Corollary 2.9, but rather follows a similar path as Dixmier and Malliavin in their original proof, reducing the statement to low-dimensional cases. To simplify the exposition, we restrict ourselves to the case of a weakly smooth G-representation.

We need to show that $E \subseteq \Pi\big(|\Omega|_c(G)\big)E$. To that end, we introduce the following terminology: A closed Lie subsupergroup H of G is called *singly generated* if its Lie superalgebra $\mathfrak{h}$ is generated by a single homogeneous element.

Then the following is straightforward: Any singly generated Lie subsupergroup is locally isomorphic to one of the Abelian supergroups $\mathbb{A}^1$ and $\mathbb{A}^{0|1}$, or to $\mathbb{A}^{1|1}$, where the Lie superalgebra has the unique non-zero homogeneous relation $x = [y, y]$. Moreover, there exist singly generated closed Lie subsupergroups $H_1, \ldots, H_n$ such that the n-fold multiplication morphism $m : H_1 \times \cdots \times H_n \longrightarrow G$ is an isomorphism in a neighbourhood U of the identity.

Now, fix $v \in E$. We claim the following: For any singly generated sub-supergroup H and any neighbourhood $V \subseteq U$ of 1, there exist $\omega_0, \omega_1 \in |\Omega|_c(H) \subseteq \mathcal{E}'(H) \subseteq \mathcal{E}'(G)$ and $w \in E$ with $\operatorname{supp}\omega_j \subseteq H \cap V$, such that $v = \Pi(\omega_0)v + \Pi(\omega_1)w$. Since $\mathbb{A}^1 \subseteq \mathbb{A}^{1|1}$ as a closed Lie subsupergroup, this follows from Dixmier–Malliavin [26, Theorem 3.3] in case H_0 is locally isomorphic to $\mathbb{A}^1$.

In case H is isomorphic to $\mathbb{A}^{0|1}$, we have $\Gamma(\mathcal{O}_H) = \mathbb{K}[\tau]$ where τ is odd. It follows that $|D\tau|$, defined by $\int_H |D\tau| \, f = \frac{d}{d\tau} f$, is a smooth density, and $\int_H |D\tau| \, (\tau f) = f(0)$. Thus, the Dirac delta $\delta = |D\tau| \, \tau \in |\Omega|_c(H)$ is a smooth density and hence, the statement is obvious in this case.

Applying the statement inductively, we find $f_0^j, f_1^j \in |\Omega|_c(H_j) \subseteq \mathcal{E}'(G)$ and $w_{i_1,\ldots,i_n} \in E$, $i_j = 0, 1$, such that

$$v = \sum\nolimits_{i_1,\ldots,i_n=0,1} \Pi(f_{i_1}^1 * \cdots * f_{i_n}^n) w_{i_1,\ldots,i_n} \ .$$

Now, for $\omega_j \in |\Omega|_c(H_j)$ and $\varphi \in \Gamma(\mathcal{O}_G)$, we have

$$\langle \omega_1 * \cdots * \omega_n, \varphi \rangle = \langle \omega_1 \otimes \cdots \otimes \omega_n, m^\sharp \varphi \rangle.$$

Since $\omega_1 \otimes \cdots \otimes \omega_n$ is in $|\Omega|_c(H_1 \times \cdots \times H_n)$, we find $\omega_1 * \cdots * \omega_n \in |\Omega|_c(G)$, provided that the $\operatorname{supp}\omega_j$ are small enough. This finally proves the claim.

3. SF-REPRESENTATIONS

In this section, we extend the notion of smooth representations of moderate growth, or SF-*representations*, to the case of Lie supergroups. We construct a superalgebra of Schwartz–Berezin densities and show that its representations are in one-to-one correspondence with SF-representations of G.

3.1. **Schwartz–Berezin densities.** Following [10, 2.1], we will call a measurable function $s : G_0 \longrightarrow (0, \infty)$ a *scale* if s and $1/s$ are locally bounded and

$$s(gh) \leqslant s(g)s(h).$$

We write $s \preceq s'$ for scales s, s' if is a constant $C > 0$ and an integer $N \geqslant 0$ with

$$s(g) \leqslant Cs'(g)^N$$

for all $g \in G_0$. This defines a preorder. The equivalence classes for the largest equivalence relation contained in $\preceq$ are denoted by $[s]$ and called *scale structures*.

In what follows, we fix a scale s on G_0. We will always make the assumption that *s dominates the $\mathfrak{g}$-adjoint scale, i.e. $s \succeq s_\mathfrak{g}$ where*

$$s_\mathfrak{g}(g) := \max\left(\|\mathrm{Ad}(g)|_\mathfrak{g}\|, \|\mathrm{Ad}(g^{-1})|_\mathfrak{g}\|\right)$$

where we fix some norm on $\mathfrak{g}$. Observe that there is a constant $C > 0$ such that

$$\max\left(|\Delta_{\bar{1}}(g)|, |\Delta_{\bar{1}}(g)|^{-1}\right) \leqslant C s_\mathfrak{g}(g)^N,$$

where $N = \dim \mathfrak{g}_{\bar{1}}$.

Definition 3.1 (Schwartz–Berezin densities)**.** We define the space of *Schwartz–Berezin densities* to be

$$\mathscr{S}(G, [s]) := \left\{ |Dg|\, f \ \middle|\ \forall u, v \in \mathfrak{U}(\mathfrak{g}), N \geqslant 0 \ :\ \int_{G_0} |dg|\, s(g)^N \big|(L_u R_v f)(g)\big| < \infty \right\},$$

where $|Dg|$ and $|dg|$ are some choices of left invariant Berezin density on G resp. left invariant density on G_0. This space is endowed with the locally convex topology generated by the seminorms

$$p^s_{u,v,N}(|Dg|\, f) := \int_{G_0} |dg|\, s(g)^N \big|(L_u R_v f)(g)\big|.$$

Clearly, the locally convex super-vector space $\mathscr{S}(G, [s])$ is independent of the choice of $|Dg|$, $|dg|$, and the representative s of the scale structure $[s]$.

Similarly, there is a space of Schwartz densities $\mathscr{S}(G_0, [s]) \subseteq |\Omega|(G_0)$. According to [10, 2.5], it is defined as the space of smooth vectors for the bi-regular representation $L_0 \times R_0$ of $G_0 \times G_0$ on the space $\mathcal{R}(G_0, [s])$, the set of *continuous* densities ω that are *rapidly decreasing* in the sense that

$$\forall N \in \mathbb{N} \ :\ \int_{G_0} |\omega| s^N < \infty.$$

We have the following description of $\mathscr{S}(G, [s])$ in terms of $\mathscr{S}(G_0, [s])$.

Proposition 3.2. *The isomorphism from Corollary 2.9 induces an isomorphism*

$$(3.1) \qquad \mathscr{S}(G, [s]) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathscr{S}(G_0, [s]).$$

In particular, $\mathscr{S}(G, [s])$ is nuclear space and G-invariant for the left regular representation L, as a subspace of $|\Omega|(G)$.

Proof. By the above definitions, we have $|Dg|\, f \in \mathscr{S}(G, [s])$ if and only if for any $u, v \in \mathfrak{U}(\mathfrak{g})$, we have

$$\omega := |dg|\, j^\sharp(L_u R_v f) \in \mathcal{R}(G_0, [s]),$$

where we abbreviate $j := j_{G_0}$. Such a density ω is smooth, and for $x \in \mathfrak{g}_{\bar{0}}$, we have

$$L_x \omega = |dg|\, j^\sharp(L_{xu} R_v f) \in \mathcal{R}(G_0, [s]).$$

One argues similarly for R_x, so that $\omega \in \mathscr{S}(G_0, [s])$.

Now, let $|Dg|\, f \in |\Omega|(G)$. We may assume w.l.o.g. that $|Dg|$ and $|dg|$ are related by Equation (2.6). Then Equation (2.9) implies that $|Dg|\, f$ corresponds to

$$\sum_i \gamma'_i \otimes |dg|\, \Delta_{\bar{1}}\, j^\sharp\big(L_{S(\gamma''_i)}(f)\big)$$

where $\Delta(\gamma) = \sum_i \gamma'_i \otimes \gamma''_i$. For any $u \in \mathfrak{U}(\mathfrak{g})$, we therefore have

$$|dg|\, \Delta_{\bar{1}}\, j^\sharp\big(L_u(f)\big) \in \mathscr{S}(G_0, [s]),$$

since s dominates the $\mathfrak{g}$-adjoint scale by assumption.

Conversely, let the Berezinian density $|Dg|\,f$ correspond to $u \otimes \omega$, where we assume $\omega = |dg|\,h \in \mathscr{S}(G_0, [s])$ and $u \in \mathfrak{U}(\mathfrak{g})$. By Corollary 2.9, we have $f = L_u(\psi h)$, with ψ defined in Equation (2.10).

For $v, w \in \mathfrak{U}(\mathfrak{g})$, $g \in G_0$, we expand

$$(-1)^{|u||w|} \Delta(vu) = \sum_i v_i' \otimes v_i'', \quad \Delta(w) = \sum_j w_j' \otimes w_j''.$$

Then we compute for $\varkappa_{ij} := (-1)^{|\psi|(|v_i''|+|w_j''|)+|v_i'||w_j'|}$ that

$$(3.2) \qquad j^\sharp(L_v R_w(\psi f))(g) = \sum_{i,j} \varkappa_{ij} \big(L_{v_i'} R_{w_j'}(\psi)\big)(g) \big(L_{v_i''} R_{w_j''}(h)\big)(g)$$

with

$$\begin{aligned}
\big(L_{v'} R_{w'}(\psi)\big)(g) &= \big(R_{\iota(\mathrm{Ad}(g^{-1})(S(v'))w')} \Delta_{\bar{1}}^{-1}\big)(g) \\
&= \delta_{\bar{1}}\big(S(\iota(\mathrm{Ad}(g^{-1})(S(v'))w'))\big) \Delta_{\bar{1}}(g)^{-1}.
\end{aligned}$$

We have

$$\iota(xay) = x\iota(a)(y - \delta_{\bar{1}}(y))$$

for all $x, y \in \mathfrak{g}_{\bar{0}}$ and $a \in \mathfrak{U}(\mathfrak{g})$ [36, Equation (3)]. Moreover, $\delta_{\bar{1}}$ is a character of $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$ and in particular $\mathrm{Ad}(G_0)$-invariant. Finally, there is a constant $C > 0$ such that

$$\big|\iota(\beta(\mathrm{Ad}(g)(\xi)\eta)\big| = \left|\int_{\mathfrak{g}_{\bar{1}}} \mathrm{Ad}(g)(\xi)\eta\right| \leqslant C\|\mathrm{Ad}(g)|_{\mathfrak{g}_{\bar{1}}}\|^k \|\xi\| \|\eta\|$$

for all $\xi \in \bigwedge^k \mathfrak{g}_{\bar{1}}$, $\eta \in \bigwedge \mathfrak{g}_{\bar{1}}$. (Here, $\|\cdot\|$ denotes some submultiplicative norm on $\bigwedge \mathfrak{g}_{\bar{1}}$.) It follows that there exist a constant $C > 0$ and an integer $N \geqslant 0$ such that

$$\big|\delta_{\bar{1}}\big(S(\iota(\mathrm{Ad}(g^{-1})(S(v'))w'))\big)\big| \leqslant C s_{\mathfrak{g}}(g)^N.$$

for all $g \in G_0$. The sum in Equation (3.2) is finite, so we may conclude that $|dg|\,j^\sharp(L_v R_w(f))$ is a Schwartz density on $(G_0, [s])$ once so is $|dg|\,\big(L_{u''} R_{v''}(h)\big)$.

To that end, similarly as above, we note that

$$j^\sharp\big(L_{xa} R_{yb}(h)\big) = L_x R_y j^\sharp\big(L_a R_b h\big)$$

for all $x, y \in \mathfrak{g}_{\bar{0}}$, $a, b \in \mathfrak{U}(\mathfrak{g})$, that

$$\big(R_{\beta(\mathrm{Ad}(g)(\xi)\eta)} h\big)(g) = \varepsilon(\beta(\mathrm{Ad}(g)(\xi)\eta)),$$

and that there is a constant $C > 0$ such that

$$\big|\varepsilon(\beta(\mathrm{Ad}(g)(\xi)\eta))\big| \leqslant C\|\mathrm{Ad}(g)|_{\mathfrak{g}_{\bar{1}}}\|^k \|\xi\| \|\eta\|.$$

Thus, there is an integer $N \geqslant 0$ such that for all $g \in G_0$, we have

$$\big|\big(L_{v''} R_{w''}(h)\big)(g)\big| \leqslant s_{\mathfrak{g}}(g)^N |H(g)|$$

where $H = \sum_\ell (L_{a_\ell} R_{b_\ell} h)$ for some $a_\ell, b_\ell \in \mathfrak{U}(\mathfrak{g}_{\bar{0}})$.

In summary, we have shown the isomorphism in Equation (3.1), in particular, $\mathscr{S}(G, [s])$ is a G-invariant subspace of $|\Omega|(G)$. Inspecting the above formulæ, it is evident that it is an isomorphism of topological vector spaces, if $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathscr{S}(G_0, [s])$ is endowed with the natural topology on $\bigwedge \mathfrak{g}_{\bar{1}} \otimes \mathscr{S}(G_0, [s])$. The nuclearity now follows from that of $\mathscr{S}(G_0, [s])$ [10, Corollary 5.6]. $\qquad\square$

Proposition 3.3. *The subspace $\mathscr{S}(G, [s]) \subseteq \mathcal{D}'(G)$ is bi-invariant under the regular representation of G. Via the isomorphism in Equation (3.1), it inherits a non-unital Fréchet superalgebra structure with continuous multiplication, determined uniquely by the following facts:*

(i) *The following is a non-unital graded subalgebra bi-invariant under G_0:*

$$\mathscr{S}(G_0, [s]) = \mathfrak{U}(\mathfrak{g}_{\bar{0}}) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathscr{S}(G_0, [s]).$$

(ii) *For $u, v \in \mathfrak{U}(\mathfrak{g})$ and $\omega \in \mathscr{S}(G_0, [s])$, we have*

$$u * (v \otimes \omega) = (u \otimes 1) * (v \otimes \omega) = uv \otimes \omega.$$

(iii) *For $\omega \in \mathscr{S}(G_0, [s])$ and $u \in \mathfrak{U}(\mathfrak{g})$, the product $u * \omega$ is given by*

$$(3.3) \qquad \int_{G_0} (u * \omega)\varphi = \int_{G_0} \omega\, j_{G_0}^\sharp \big(L_{S(u)}\varphi\big) = \langle u \otimes \omega, \varphi \rangle,$$

for all compactly supported superfunctions $\varphi \in \Gamma_c(\mathcal{O}_G)$.

Proof. We already know that $\mathscr{S}(G, [s])$ is invariant under L_u and L_g for any $u \in \mathfrak{U}(\mathfrak{g})$, $g \in G_0$. To see that $\mathscr{S}(G, [s])$ is also invariant under the right regular action R, it will be sufficient to show that $\mathscr{S}(G, [s])$ is stable under $(-)^\vee$, defined in Equation (2.12).

Choose a basis $x_1, \ldots, x_q$ of $\mathfrak{g}_{\bar{1}}$, and let $x^1, \ldots, x^q$ be the dual basis of $\mathfrak{g}_{\bar{1}}^*$. We write $x_I := x_{i_1} \cdots x_{i_k} \in \bigwedge \mathfrak{g}_{\bar{1}}$ and $x^I := x^{i_1} \cdots x^{i_k} \in \bigwedge \mathfrak{g}_{\bar{1}}^*$ for $I = (1 \leqslant i_1 < \cdots < i_k \leqslant q)$. Then we compute by Equation (2.2)

$$\langle (\beta(x_I) \otimes \omega)^\vee, \varphi \rangle = (-1)^{|I||\varphi|} \int_{G_0} \omega(g)\varphi(S(\beta(x_I)); g^{-1})$$

$$= \sum_J \int_{G_0} (-1)^{|J||\varphi|} \breve{\omega}(g) \langle x^J, \mathrm{Ad}(g)(x_I) \rangle \varphi(\mathrm{Ad}(g^{-1})(S(x_J)); g)$$

$$= \left\langle \sum_J \beta(x_J) \otimes \breve{\omega}\langle x^J, \mathrm{Ad}(\cdot)(x_I) \rangle, \varphi \right\rangle$$

for all $\varphi \in \Gamma_c(\mathcal{O}_G)$. Here, observe that $|J| = |I|$, because the adjoint action by G_0 on $\bigwedge \mathfrak{g}_{\bar{1}}$ respects the $\mathbb{Z}$-grading.

By the assumption on the scale s, we have $\breve{\omega} \in \mathscr{S}(G_0, [s])$ and thus

$$|dg| \sum_J \breve{\omega}\langle x^J, \mathrm{Ad}(\cdot)(x_I) \rangle \in \mathscr{S}(G_0, [s]).$$

In view of Proposition 3.2, this shows that $(\beta(x_I) \otimes \omega)^\vee \in \mathscr{S}(G, [s])$. Therefore, $\mathscr{S}(G, [s])$ is invariant under $(-)^\vee$ and bi-invariant under G.

In follows that there is a well-defined operation $*$ on $\mathscr{S}(G, [s])$, defined by

$$(3.4) \qquad (u \otimes \omega) * (v \otimes \varpi) := \sum_j uv_j \otimes (\omega_j * \varpi),$$

for arbitrary $u, v \in \mathfrak{U}(\mathfrak{g})$ and ω, ϖ, where we decompose $R_{S(v)}\omega = \sum_j v_j \otimes \omega_j$.

If ω, ϖ are compactly supported, then by Equation (2.2), we have

$$\langle (u \otimes \omega) * (v \otimes \varpi), \varphi \rangle = \sum_j (-1)^{|uv_j||\varphi|} \int_{G_0} (\omega_j * \varpi)(g)\varphi\big(\mathrm{Ad}(g^{-1})(uv_j); g\big)$$

$$= \sum_j (-1)^{|uv_j||\varphi|} \int_{G_0 \times G_0} \omega_j(g)\varpi(h)\varphi\big(\mathrm{Ad}((gh)^{-1})(uv_j); gh\big)$$

$$= \sum_j (-1)^{|uv_j||\varphi|} \int_{G_0 \times G_0} \omega_j(g)\varpi(h)m^\sharp(\varphi)\big(\mathrm{Ad}(g^{-1})(uv_j) \otimes 1; g, h\big)$$

$$= \left\langle \sum_j (uv_j \otimes \omega_j) \otimes \varpi, m^\sharp(\varphi) \right\rangle$$

$$= \langle L_u R_{S(v)}\omega \otimes \varpi, m^\sharp(\varphi) \rangle$$

$$= (-1)^{|uv||\varphi|} \int_{G_0 \times G_0} \omega(g)\varpi(h)m^\sharp(\varphi)(\mathrm{Ad}(g^{-1})(u)v \otimes 1; g, h)$$

$$= (-1)^{|uv||\varphi|} \int_{G_0 \times G_0} \omega(g)\varpi(h)m^\sharp(\varphi)(\mathrm{Ad}(g^{-1})(u) \otimes \mathrm{Ad}(h^{-1})(v); g, h)$$

$$= \langle (u \otimes \omega) \otimes (v \otimes \varpi), m^\sharp(\varphi) \rangle,$$

so that $*$ on $\mathscr{S}(G, [s])$ extends the convolution on $|\Omega|_c(G)$. By Proposition 3.2, $|\Omega|_c(G) \subseteq \mathscr{S}(G, [s])$ is a dense subspace and $\mathscr{S}(G, [s])$ is nuclear. To finish the proof

of our assertion, it is by [63, Theorem 50.1] sufficient to show that the convolution
$*$ on $|\Omega|_c(G)$ is separately continuous in the topology induced by $\mathscr{S}(G, [s])$.

Since $(-)^\vee$ is continuous on $\mathscr{S}(G, [s])$, it will be sufficient to show continuity in
the second argument. In view of Corollary 2.12 (i)–(iii), we have the identity

$$L_u R_v(\omega * \varpi) = (-1)^{|v||\omega|}(L_u\omega) * (R_v\varpi)$$

for all $u, v \in \mathfrak{U}(\mathfrak{g})$ and $\omega, \varpi \in |\Omega|_c(G)$. Together with the fact that for any $v \in \mathfrak{U}(\mathfrak{g})$,
R_v is continuous on $\mathscr{S}(G, [s])$, it follows that it is sufficient to show that

$$\varpi \longmapsto p^s_{1,1,N}\big((L_u R_{S(v)}\omega) * \varpi\big) : \mathscr{S}(G_0, [s]) \longrightarrow \mathbb{R}$$

is a continuous seminorm for any integer $N \geqslant 0$. But this follows from the continuity
of the convolution on $\mathscr{S}(G_0, [s])$ [64, Theorem 7.1]. $\qquad\square$

3.2. SF-representations of supergroups.
Fix a scale s on G_0 dominating the
$\mathfrak{g}$-adjoint scale. Recall [10, Definition 2.6, Lemma 2.10] that a continuous Fréchet
G-representation π_0 on E is called an F-representation or a Fréchet representation
of moderate growth of $(G_0, [s])$ if the topology of E is generated by a countable
collection (p_j) of seminorms such that for any j, there exist an index k, a constant
$C > 0$, and an integer $N \geqslant 0$ with

$$(3.5) \qquad p_j\big(\pi_0(g)v\big) \leqslant Cs(g)^N p_k(v)$$

for all $v \in E$ and $g \in G_0$. It is called an SF-representation or smooth if it is in
addition weakly smooth.

In view of this terminology, we make the following definition.

Definition 3.4 (SF-representations)**.** Let π be a continuous representation G on
a Fréchet super-vector space E. Then π is called an F-representation of $(G, [s])$
if the topology of E is generated by a countable collection (p_j) of seminorms such
that for some norm $\|\cdot\|$ on $\bigwedge \mathfrak{g}_{\bar{1}}$ and for any index j, there is an index k, a constant
$C > 0$, and an integer $N \geqslant 0$ with

$$(3.6) \qquad p_j\big(d\pi(\beta(\eta))\pi_0(g)v\big) \leqslant C\|\eta\|s(g)^N p_k(v)$$

for all $v \in E_\infty$, $g \in G_0$, and $\eta \in \bigwedge \mathfrak{g}_{\bar{1}}$. If in addition, π is a weakly smooth
G-representation, then it is called an SF-representation of $(G, [s])$.

Remark 3.5. If π is an F-representation (resp. an SF-representation) of $(G, [s])$,
then π_0 is an F-representation (resp. an SF-representation) of $(G_0, [s])$. Indeed, E_∞
is dense in E, and taking $\eta = 1$ in Equation (3.6), we obtain Equation (3.5). Also
by definition, if π is an F-representation of $(G, [s])$ on E, then the subrepresentation
on the space E_∞ of smooth vectors is an SF-representation [10, Corollary 2.16].

In particular, using [10, (2.2)], Proposition 3.2, and Proposition 3.3, we find
that the left and right regular representations L and R on $\mathscr{S}(G, [s])$ are SF-
representations of $(G, [s])$.

In fact, the F-representations are characterised among the continuous represen-
tations of G by the growth of the underlying G_0-representation.

Lemma 3.6. *Let π be a continuous (resp. weakly smooth) representation of G
on a Fréchet super vector-space E. Then π is an F-representation (resp. an SF-
representation) of $(G, [s])$ if and only if π_0 is an F-representation (resp. an SF-
representation) of $(G_0, [s])$.*

Proof. It is sufficient to consider the case of F-representations. As noted above,
if π is an F-representation of $(G, [s])$, then π_0 is an F-representation of $(G_0, [s])$.
Conversely, assume that π_0 is an F-representation of $(G_0, [s])$. Since s dominates
the $\mathfrak{g}$-adjoint scale, the adjoint representation of G_0 on $\bigwedge \mathfrak{g}_{\bar{1}}$ is an F-representation.
Hence, so is $\bigwedge \mathfrak{g}_{\bar{1}} \otimes E$. Manifestly, this gives the condition in Equation (3.6). $\quad\square$

Remark 3.7. From Lemma 3.6, we obtain the following: Let π be a continuous G-representation on a Banach super vector-space E. Then π is an F-representation of $(G, [s])$ if and only if π_0 is s-bounded in the sense that $s \succeq s_{\pi_0}$ where

$$s_{\pi_0}(g) := \max\big(\|\pi_0(g)\|, \|\pi_0(g^{-1})\|\big).$$

In particular, in this case, the G-representation on E_∞ is an SF-representation [10, Corollary 2.16].

For F-representations of G, we obtain the following variant of the Dixmier–Malliavin theorem, generalising [10, Proposition 2.20]. Compare [32, Exemple 2.3.3].

Proposition 3.8. *Let E be a Fréchet super-vector space over $\mathbb{K}$. Then we have the following facts:*

(i) If E carries the structure of an F-representation π of $(G, [s])$, then the integrated action Π extends continuously to an action of $\mathscr{S}(G, [s])$, also called the integrated action of π. We have the equality

$$(3.7) \qquad E_\infty = \Pi\big(\mathscr{S}(G, [s])\big)E = \Pi\big(\mathscr{S}(G, [s])\big)E_\infty.$$

(ii) Conversely, let $\mathscr{S}(G, [s])$ act continuously and non-degenerately via Π on E. Then Π is integrated from a unique SF-representation of $(G, [s])$.

In particular, we obtain an equivalence of the category of SF-representations of $(G, [s])$ with the category of non-degenerate continuous Fréchet $\mathscr{S}(G, [s])$-modules.

Proof. If E is an F-representation of $(G_0, [s])$, then $\mathscr{S}(G_0, [s])$ acts continuously on E, and $E_\infty = \mathscr{S}(G_0, [s])E = \mathscr{S}(G_0, [s])E_\infty$ [10, Proposition 2.20]. Conversely, if E carries a continuous non-degenerate action of $\mathscr{S}(G_0, [s])$, then this action is integrated from a unique SF-representation of $(G_0, [s])$ (*loc. cit.*). Using these facts, together with Proposition 3.2 and Proposition 3.3, the proof of the claim is the same as that of Proposition 2.15. We therefore leave the details to the reader. $\square$

4. Harish-Chandra supermodules

In this section, we come to our main result, a generalisation of the Casselman–Wallach theorem to supergroups.

4.1. **Basic facts and definitions.** In what follows, we assume that the underlying Lie group G_0 of G is almost connected and real reductive [64, 2.1] and let $K_0 \subseteq G_0$ be a maximal compact subgroup. We fix on G_0 the maximal scale structure [10, 2.1.1] and omit the mention of $[s]$ in our notation. In particular, any Banach representation of G is an F-representation.

Definition 4.1 (Harish-Chandra supermodules)**.** A $(\mathfrak{g}, K_0)$-*module* is by definition a complex, $\mathbb{Z}/2\mathbb{Z}$ graded, locally finite K_0-representation V, endowed with a K_0-equivariant $\mathfrak{g}$-module structure, which extends the derived $\mathfrak{k}_0$-action on V. A *morphism of $(\mathfrak{g}, K_0)$-modules* $\phi : U \longrightarrow V$ is an even $\mathbb{C}$-linear map that is equivariant for the actions of $\mathfrak{g}$ and K_0.

A $(\mathfrak{g}, K_0)$-module is called *Harish-Chandra* or a *Harish-Chandra supermodule* if it is K_0-multiplicity finite and finitely generated over $\mathfrak{U}(\mathfrak{g})$. The full subcategory of the category of $(\mathfrak{g}, K_0)$-modules whose objects are the Harish-Chandra supermodules is denoted by $\mathbf{HC}(\mathfrak{g}, K_0)$.

The following observation is elementary, but effective.

Lemma 4.2. *Let V be a $(\mathfrak{g}, K_0)$-module. Then $V \in \mathbf{HC}(\mathfrak{g}, K_0)$ if and only if its restriction $V|_{(\mathfrak{g}_{\bar{0}}, K_0)}$ to a $(\mathfrak{g}_{\bar{0}}, K_0)$-module lies in $\mathbf{HC}(\mathfrak{g}_{\bar{0}}, K_0)$.*

Proof. We need only observe that $\mathfrak{U}(\mathfrak{g})$ is finitely generated as a $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module. $\square$

Lemma 4.3. *Let E be an F-representation of G (for instance, a Banach representation). Then the space $E_\infty^{(K_0)}$ of K_0-finite and smooth vectors is a $(\mathfrak{g}, K_0)$-module.*

Proof. Since the action of K_0 on $\mathfrak{U}(\mathfrak{g})$ is locally finite, we see that the $\mathfrak{g}$-action on E_∞ leaves $E_\infty^{(K_0)}$ invariant. $\square$

Remark 4.4. Let π_0 be a continuous G-representation on a complex Banach supervector space E. Denoting by C the Casimir element of $\mathfrak{g}_{\bar{0}}$, assume that either

    (i) $d\pi_0(C) \in \mathrm{End}(E_\infty)$ extends continuously to E, or
    (ii) $P(d\pi_0(C)) = 0$ on E_∞ for some polynomial P.

Then it is known that the space $E^{(K_0)}$ of K_0-finite vectors is contained in E_∞ [10, Corollary 3.10]. Hence, if π_0 is the G_0 part of a continuous G-representation, then $E^{(K_0)}$ is a $(\mathfrak{g}, K_0)$-module, by Lemma 4.3.

4.2. Globalisation of Harish-Chandra supermodules.

Definition 4.5 (Casselman–Wallach representations)**.** An SF-representation (E, π) of G is called *Casselman–Wallach* or a *CW representation* if the space $E^{(K_0)}$ of K_0-finite and smooth vectors is in $\mathbf{HC}(\mathfrak{g}, K_0)$.

If $V \in \mathbf{HC}(\mathfrak{g}, K_0)$, then an isomorphism $\phi : V \longrightarrow E^{(K_0)}$ of $(\mathfrak{g}, K_0)$-modules, where (E, π) is an SF-representation, is called an SF-*globalisation* of V. Any SF-globalisation of a Harish-Chandra supermodule is a CW representation of G.

A CW globalisation $\phi : V \longrightarrow E$ is called *minimal* if for any CW globalisation $\psi : V \longrightarrow H$, there exists an even continuous G-equivariant map $\tilde{\psi} : E \longrightarrow H$ such that $\tilde{\psi} \circ \varphi = \varphi$. Since the K_0-finite vectors are dense in E, such a map $\tilde{\psi}$ is unique. Thus, minimal globalisations (if they exist) are unique up to canonical isomorphism.

Dually, a CW globalisation $\phi : V \longrightarrow E$ is called *maximal* if for any CW globalisation $\psi : V \longrightarrow H$, there exists an even continuous G-equivariant map $\tilde{\psi} : H \longrightarrow E$ such that $\tilde{\psi} \circ \psi = \varphi$. Again, maximal globalisations (if they exist) are unique up to canonical isomorphism.

We are now ready to state our main theorem.

Theorem 4.6 (Super Casselman–Wallach theorem)**.** *Let $V \in \mathbf{HC}(\mathfrak{g}, K_0)$. Up to isomorphism, there is a unique CW globalisation of V.*

We postpone the proof to Subsection 4.3 and give a number of corollaries. The derivation of these follows the same procedures as in the Lie group case [65, 11.6.8].

Corollary 4.7. *The functor mapping (E, π) to $E_\infty^{(K_0)}$ sets up an additive equivalence between the category $\mathbf{CW}(G)$ of CW representations of G and the category $\mathbf{HC}(\mathfrak{g}, K_0)$ of Harish-Chandra supermodules. In particular, the category $\mathbf{CW}(G)$ is Abelian.*

Corollary 4.8. *Let $f : E \longrightarrow F$ be a morphism of CW G-representations. Then f is a topological morphism with closed image.*

Here, $f : E \longrightarrow F$ is called a topological morphism *if the induced map*

$$E/\ker f \longrightarrow \mathrm{im}\, f$$

is an isomorphism of topological vector spaces.

As a corollary to the proof of Theorem 4.6, we obtain the following.

Corollary 4.9. *Any $E \in \mathbf{CW}(G)$ is the space of smooth vectors of a continuous Hilbert G-representation.*

4.3. **Proof of Theorem 4.6.** Having stated our main result, together with some immediate corollaries, let us come to its proof.

Proof of Theorem 4.6. First, we show that V has a minimal SF-globalisation $V_+ \supseteq V$. We mimic the construction detailed in [10, § 6].

By Lemma 4.2, we have $U := V|_{(\mathfrak{g}_{\bar{0}}, K_0)} \in \mathbf{HC}(\mathfrak{g}_{\bar{0}}, K_0)$. Thus, there is a finite set $v_1, \ldots, v_n$ of homogeneous vectors generating the $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module V and a continuous Hilbert representation (E, π_0) of G_0 such that $E_\infty^{(K_0)} = U$ [10, § 5.1].

Since $\mathscr{S}(G)$ is invariant under $(-)^\vee$, Proposition 3.2 shows that the map

$$(4.1) \qquad \mathscr{S}(G_0) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathfrak{U}(\mathfrak{g}) \longrightarrow \mathscr{S}(G) : \omega \otimes u \longmapsto R_{S(u)}(\omega)$$

is an isomorphism of right $\mathfrak{U}(\mathfrak{g})$-modules. Here, $\mathfrak{U}(\mathfrak{g})$ acts from the right on $\mathscr{S}(G)$ by $\omega u := (-1)^{|\omega||u|} R_{S(u)}(\omega)$. We define, for $\omega \in \mathscr{S}(G_0)$ and $v \in V$

$$(4.2) \qquad \Pi(\omega)v := \sum_i \Pi_0(\omega_i)u_i v,$$

where

$$\omega = \sum_I R_{S(u_i)}(\omega_i)$$

is any decomposition with $\omega_i \in \mathscr{S}(G_0)$ and $u_i \in \mathfrak{U}(\mathfrak{g})$. To see that this is well-defined, we need only remark that

$$\Pi_0(R_{-x}\omega) = \Pi_0(\omega)d\pi_0(x)$$

for all $\omega \in \mathscr{S}(G_0)$ and $x \in \mathfrak{g}_{\bar{0}}$.

Now, consider the graded subspace $\mathscr{N} \subseteq \mathscr{S}(G)^n$, defined by

$$\mathscr{N} := \left\{ (\omega_1, \ldots, \omega_n) \in \mathscr{S}(G)^n \ \middle| \ \sum_j \Pi(\omega_j)v_j = 0 \right\}.$$

We claim that it is closed and invariant under the action of $\mathscr{S}(G)$ by left convolution. To prepare the proof of this claim, we briefly suspend our argument and establish some ancillary lemmas. $\qquad \square$

Let $\tilde{V} \in \mathbf{HC}(\mathfrak{g}, K_0)$ be the dual Harish-Chandra module of V, defined as the set of K_0-finite vectors in the algebraic dual V^*. Then $\tilde{V}$ is also the dual of $V|_{(\mathfrak{g}_{\bar{0}}, K_0)}$ [10, § 4], and in particular $\tilde{V}$ is contained in the space $\tilde{E}$ of continuous vectors of the topological dual E' of E [10, Lemma 5.3].

Lemma 4.10. *Let $v \in V$, $u \in \mathfrak{U}(\mathfrak{g})$ and $g \in G_0$. We have the identity*

$$\langle \xi, \pi_0(g^{-1}) \operatorname{Ad}(g)(u)v \rangle = (-1)^{|\xi||u|} \langle S(u)\xi, \pi_0(g^{-1})v \rangle.$$

Proof. The equality is obvious for $g \in K_0$. Since $G_0' K_0 = G_0$, where G_0' is the connected component of the identity of G_0, we may assume that G_0 is connected.

To prove the assertion in that case, assume first that $u \in \beta(\bigwedge \mathfrak{g}_{\bar{1}})$. The image F of $\beta(\bigwedge \mathfrak{g}_{\bar{1}})$ in $\operatorname{End}(V)$ is finite-dimensional, so the linear map

$$F \longrightarrow V \subseteq E : u \longmapsto uv$$

is continuous. For $x \in \mathfrak{g}_0$, we may hence exchange limits and compute

$$\frac{d}{dt}\Big|_{t=0} \operatorname{Ad}(\exp(tx))(u)v = [x, u]v = d\pi_0(x)uv - uxv.$$

Thus, we have

$$\frac{d}{dt}\Big|_{t=0} \pi_0(\exp(-tx)) \operatorname{Ad}(\exp(tx))(u)v = -d\pi_0(x)uv + [x, u]v = -uxv,$$

by the smoothness of the G_0-representation E_∞. Hence

$$\frac{d}{dt}\Big|_{t=0}\langle\xi,\pi_0(e^{-tx})\operatorname{Ad}(e^{tx})(u)v\rangle = -(-1)^{|\xi||u|}\langle S(u)\xi, d\pi_0(x)v\rangle$$

$$= (-1)^{|\xi||u|}\frac{d}{dt}\Big|_{t=0}\langle S(u)\xi, \pi_0(e^{-tx})v\rangle.$$

By the uniqueness of initial value problems, the equality follows for $g = e^x$. Since exp is a local diffeomorphism and G_0, being connected, is generated by a neighbourhood of the identity, the equality holds for arbitrary $g \in G_0$.

To remove the restriction on u, recall that $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}_{\bar{0}})\beta(\bigwedge \mathfrak{g}_{\bar{1}})$. By linearity in u, it is sufficient to consider $u = u'u''$ for $u' \in \mathfrak{U}(\mathfrak{g}_{\bar{0}})$ and $u'' \in \beta(\bigwedge \mathfrak{g}_{\bar{1}})$. Then

$$\langle\xi,\pi_0(g^{-1})\operatorname{Ad}(g)(u)v\rangle = \langle\xi,\pi_0(g^{-1})\operatorname{Ad}(g)(u')\operatorname{Ad}(g)(u'')v\rangle$$

$$= \langle S(u')\xi, \pi_0(g^{-1})\operatorname{Ad}(g)(u'')v\rangle$$

$$= (-1)^{|\xi||u|}\langle S(u'')S(u')\xi, \pi_0(g^{-1})v\rangle$$

$$= (-1)^{|\xi||u|}\langle S(u)\xi, \pi_0(g^{-1})v\rangle.$$

This proves the claim in general. $\qquad\square$

Lemma 4.11. *For $u \in \mathfrak{U}(\mathfrak{g})$, $\omega \in \mathscr{S}(G)$, $v \in V$, and $\xi \in \tilde{V}$, we have*

$$\langle\xi,\Pi(L_u(\omega))v\rangle = (-1)^{|\xi||u|}\langle S(u)\xi, \Pi(\omega)v\rangle.$$

Proof. For $v \in V$ and $\xi \in \tilde{V}$, we define $M_{\xi,v} \in \Gamma(\mathcal{O}_G)$ by

$$M_{\xi,v}(u;g) := (-1)^{|u||v|}\langle\xi, \pi_0(g)v\rangle.$$

Clearly, this is well-defined.

For $u \in \mathfrak{U}(\mathfrak{g})$ and $\omega \in \mathscr{S}(G_0)$, we compute

$$\langle\xi,\Pi(R_{S(u)}(\omega))v\rangle = (-1)^{|u||v|}\langle\xi,\Pi_0(\omega)uv\rangle = (-1)^{|u||v|}\int_{G_0}\omega(g)\langle\xi,\pi_0(g)uv\rangle$$

$$= (-1)^{|\xi||u|}\langle R_{S(u)}(\omega), M_{\xi,v}\rangle.$$

By Equation (4.1), it follows that

$$\langle\xi,\Pi(\omega)v\rangle = (-1)^{|\xi||\omega|}\langle\omega, M_{\xi,v}\rangle$$

for any $\omega \in \mathscr{S}(G)$. In particular, if $u \in \mathfrak{U}(\mathfrak{g})$, we have

$$\langle\xi,\Pi(L_u(\omega))v\rangle = (-1)^{|\xi|(|u|+|\omega|)+|u||\omega|}\langle\omega, L_{S(u)}(M_{\xi,v})\rangle$$

$$= (-1)^{|\xi|(|u|+|\omega|)+|u||\omega|}\langle\omega, M_{S(u)\xi,v}\rangle = (-1)^{|\xi||u|}\langle S(u)\xi, \Pi(\omega)v\rangle,$$

since

$$L_u(M_{\xi,v})(u';g) = (-1)^{|u||\xi|+|u'||v|}\langle\xi, \pi_0(g)\operatorname{Ad}(g^{-1})(S(u))u'v\rangle$$

$$= (-1)^{|u'||v|}\langle uv, \pi_0(g)u'v\rangle = M_{u\xi,v}(u';g),$$

by Lemma 4.10. This proves the assertion. $\qquad\square$

We now again take up the proof of our main theorem.

Proof of Theorem 4.6 (continued). For $v' \in E$, we have

$$v' = 0 \iff \forall\xi \in \tilde{V} : \langle\xi, v'\rangle = 0.$$

Hence, by Lemma 4.11, the subspace $\mathcal{N}$ is invariant under L^n, where L is the regular G-representation. That it is invariant under left convolution by $\mathscr{S}(G)$ now follows from the identity

$$R_{S(u)}(\omega) * \varpi = \omega * (L_u(\varpi))$$

valid for $u \in \mathfrak{U}(\mathfrak{g})$, $\omega \in \mathscr{S}(G_0)$, and $\varpi \in \mathscr{S}(G)$, together with Equation (4.1).

Since $\mathscr{S}(G) \cong \mathscr{S}(G_0) \otimes \bigwedge \mathfrak{g}_{\bar{1}}$ is the locally convex direct sum of finitely many copies of $\mathscr{S}(G_0)$, it follows directly from the definition in Equation (4.2) that

$$\phi : \mathscr{S}(G)^n \longrightarrow E : (\omega_1, \ldots, \omega_n) \longmapsto \sum_j \Pi(\omega_j) v_j$$

is continuous, so that $\mathscr{N}$ is also closed, as claimed.

Hence, if we define

$$V_+ \coloneqq \mathscr{S}(G)^n / \mathscr{N},$$

then this is a continuous non-degenerate Fréchet $\mathscr{S}(G)$-module. By Proposition 3.8, the $\mathscr{S}(G)$-action is integrated from a unique SF-representation π of G.

The map induced by ϕ identifies V_+ (as a super-vector space) with the subspace

$$U_+ \coloneqq \Pi(\mathscr{S}(G))V = \Pi(\mathscr{S}(G_0))V$$

of E. By construction [10, § 6], U_+ is, with the quotient topology defined by the natural map $\mathscr{S}(G_0)^n \longrightarrow U_+$ induced by ϕ, the minimal globalisation of the module $U \in \mathbf{HC}(\mathfrak{g}_{\bar{0}}, K_0)$. But by the Casselman–Wallach theorem [10, Theorem 10.6], it holds that $U_+ = E_\infty$ as locally convex spaces.

Since $\mathfrak{U}(\mathfrak{g})$ is $\mathrm{Ad}(K_0)$-locally finite, the space of $K_0 \times K_0$-finite vectors is

$$\mathscr{S}(G)^{(K_0 \times K_0)} = \mathscr{S}(G_0)^{(K_0 \times K_0)} \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathfrak{U}(\mathfrak{g}).$$

From this, it is easy to deduce that V_+ is an SF-globalisation of V. In particular, $(V_+)|_{G_0}$ is an SF-globalisation of U. From the Casselman–Wallach theorem [10, Theorem 10.6] again, it follows that the map $V_+ \longrightarrow U_+$ induced by ϕ is an isomorphism of locally convex vector spaces. In particular, V_+ is the space of smooth vectors of a continuous Hilbert G-representation.

Now, let F be any SF-globalisation of V, so that we are given an isomorphism $\psi : V \longrightarrow F^{(K_0)}$ of $(\mathfrak{g}, K_0)$-modules. Invoking the Casselman–Wallach theorem (*loc. cit.*), there is a unique isomorphism $\tilde{\psi} : V_+ \longrightarrow F$ of SF-representations of G_0 extending ψ. For any $u \in \mathfrak{U}(\mathfrak{g})$, the action by u on V_+ and F is continuous. Hence, by the density of V in V_+, it follows that $\tilde{\psi}$ is $\mathfrak{g}$-equivariant. This shows that V_+ is a minimal SF-globalisation. The same argument shows that it is maximal, and hence follows the claim. $\qquad\square$

5. Application: Gel'fand–Kazhdan representations

In this section, we show, by way of application of our results in Section 4, that the Gel'fand–Kazhdan criterion for multiplicity freeness carries over to the case of Lie supergroups. Therein, we build on the work of Sun–Zhu [60] who have shown how to present this within the framework of Lie group Casselman–Wallach theory. Antecedents are the classical results of Gel'fand–Kazhdan [37] and Shalika [59], as well as theorems of Kostant [41], Yamashita [67], and Prasad [56].

We retain our assumptions on the Lie supergroup G from Section 4.

Definition 5.1 (Contragredient pairs). A pair (E, F) of continuous G-representations is called *contragredient* if there exists a G_0-invariant continuous bilinear map

$$\langle \cdot, \cdot \rangle : E \times F \longrightarrow \mathbb{K}$$

that is a perfect pairing whose restriction to $E_\infty \times F_\infty$ is G-invariant.

Here, by a *perfect pairing* we mean that the canonical maps

$$E \longrightarrow F', \quad F \longrightarrow E'$$

are isomorphisms of topological vector spaces.

Remark 5.2. Assume (E_∞, F_∞) is a pair of SF-representations of G and

$$\langle \cdot, \cdot \rangle : E_\infty \times F_\infty \longrightarrow \mathbb{K}$$

is a non-degenerate continuous bilinear form that is G-invariant. If U is a Hilbert globalisation of E_∞ (which exists if E_∞ is CW), then the space of G_0-smooth vectors in $F \coloneqq E'$ coincides with F_∞. Thus, (E, F) a contragredient pair with underlying SF-representations E_∞ and F_∞.

As we shall presently see, contragredient pairs of representations allow for an abstract matrix coefficient map. To state this precisely, we introduce the following definition.

Definition 5.3 (Tempered superfunctions). A superfunction $f \in \Gamma(\mathcal{O}_G)$ is called *tempered* if for all $u, v \in \mathfrak{U}(\mathfrak{g})$

$$t_{u,v,N}(f) \coloneqq \sup_{g \in G_0} s(g)^{-N} \big| (L_u R_v f)(g) \big| < \infty$$

for some $N \geqslant 0$. Here, s denotes the maximal scale, see [10, 2.1.1].

The space of tempered superfunctions is denoted by $\mathcal{T}(G)$. It is topologised as the locally convex inductive limit of the spaces $\mathcal{T}_N(G) \coloneqq \bigcap_{u,v} \{t_{u,v,N} < \infty\}$, endowed with the locally convex topology generated by the seminorms $t_{u,v,N}$, $u, v \in \mathfrak{U}(\mathfrak{g})$.

For any $\omega \in \mathscr{S}(G)$, the Berezin integral

$$\varphi \longmapsto \langle \omega, \varphi \rangle \coloneqq \int_G \omega \varphi$$

extends uniquely to a continuous functional on $\mathcal{T}(G)$. This is easy to deduce from Proposition 2.8 and the corresponding classical facts.

Define $\mathscr{S}'(G)$, the space of *tempered generalised functions*, to be the strong dual of $\mathscr{S}(G)$. There is a continuous linear injection

$$\mathcal{T}(G) \longrightarrow \mathscr{S}'(G).$$

The following proposition generalises [60, Theorem 2.1].

Proposition 5.4. *Let (E, F) be a contragredient pair of continuous F-representations of G. Then the map*

$$M : E_\infty \times F_\infty \longrightarrow \mathcal{T}(G), \quad M_{v \otimes v'}(u; g) \coloneqq (-1)^{|u|(|v| + |v|')} \langle \pi_{E,0}(g) d\pi_E(u)v, v' \rangle$$

extends continuously to a $G \times G$-equivariant separately continuous bilinear map

$$M^{-\infty} : E_{-\infty} \times F_{-\infty} \longrightarrow \mathscr{S}'(G),$$

where $E_{-\infty} \coloneqq (F_\infty)'$, $F_{-\infty} \coloneqq (E_\infty)'$, and $(d\pi_E, \pi_{E,0})$ is the G-action on E.

If, moreover, E_∞ and F_∞ are CW G-representations, then $M^{-\infty}$ is continuous and the induced $(G \times G)$-equivariant continuous linear map

$$E_{-\infty} \widehat{\otimes}_\pi F_{-\infty} \longrightarrow \mathscr{S}'(G)$$

is a topological morphism with closed image.

The structure of the *proof* is manifestly the same as the one given by Sun–Zhu [60], so we shall be brief. We begin with the following lemma.

Lemma 5.5. *Let E be an F-representation of G. Then the bilinear map*

$$\Phi_E : \mathscr{S}(G) \times E \longrightarrow E_\infty : (\omega, v) \longmapsto \Pi_E(\omega)v$$

is well-defined and continuous.

Proof. That the map is well-defined follows from Proposition 3.8 (i). The continuity is an immediate consequence of Proposition 3.2 and [60, Lemma 3.3]. $\qquad\square$

Proof of Proposition 5.4. For $\omega \in \mathscr{S}(G)$, we may define $\Pi_E^{-\infty}(\omega) : E^{-\infty} \longrightarrow E$ by

$$\langle \Pi_E^{-\infty}(\omega)v, v' \rangle := (-1)^{|\omega||v|}\langle v, \Phi_F(\check{\omega}, v') \rangle, \quad v \in E^{-\infty}, v' \in F.$$

Then $\Pi_E^{-\infty}(\omega)$ is continuous, and the bilinear map

$$\Phi_E^{-\infty} : \mathscr{S}(G) \times E^{-\infty} \longrightarrow E : (\omega, v) \longmapsto \Pi_E^{-\infty}(\omega)v$$

is separately continuous, both by Lemma 5.5. Applying Proposition 3.2 and [60, Lemma 3.5], we see that it takes values in E_∞ and is separately continuous with respect to the natural topology on this space.

We compute for $v \in E$ and $v' \in F$:

$$\langle \Phi_E^{-\infty}(\omega, v), v' \rangle = (-1)^{|\omega||v|}\langle v, \Pi_F(\check{\omega})v' \rangle = \langle \Pi_E(\omega)v, v' \rangle = \langle \Phi_E(\omega, v), v' \rangle,$$

since for $\omega = u \otimes \varpi$, $u \in \mathfrak{U}(\mathfrak{g})$, $\varpi \in \mathscr{S}(G_0)$, we have

$$\langle v, \Pi_F(\check{\omega})v' \rangle = \langle v, \Pi_{F,0}(\check{\varpi})d\pi_F(S(u))v' \rangle$$
$$= (-1)^{|u||v|}\langle d\pi_E(u)\Pi_{E,0}(\varpi)v, v' \rangle = (-1)^{|\omega||v|}\langle \Pi_E(\omega)v, v' \rangle,$$

in view of Proposition 3.8 (i). Thus, $\Phi_E^{-\infty}$ extends Φ_E.

Altogether, the map $M^{-\infty} : E_{-\infty} \times F_{-\infty} \longrightarrow \mathscr{S}'(G)$,

$$(5.1) \qquad \langle \omega, M^{-\infty}(v, v') \rangle := \langle \Phi_E^{-\infty}(\omega, v), v' \rangle = (-1)^{|\omega||v|}\langle v, \Phi_F^{-\infty}(\check{\omega}, v') \rangle,$$

is well-defined, separately continuous, and extends M.

Now, assume that E_∞ and F_∞ are *CW* G-representations. As such, they are nuclear Fréchet spaces [10, Corollary 5.6] and hence reflexive [63, Corollary 3 to Proposition 50.2, Corollary to Proposition 36.9]. The same holds for $\mathscr{S}(G)$, by Proposition 3.2. Thus, $E^{-\infty}$, $F^{-\infty}$, and $\mathscr{S}'(G)$ are strong duals of reflexive Fréchet spaces, and $M^{-\infty}$ is automatically continuous (*op. cit.*, Theorem 41.1). The final statement now follows from Corollary 4.8. $\qquad\square$

We now generalise Sun–Zhu's version of the Gel′fand–Kazhdan criterion [60, Theorem 2.3 (i)] to Lie supergroups.

Definition 5.6 (Irreducible representations). Let U be an SF-representation of G. We say that U is *irreducible* if there is no non-zero proper closed subspace of U that is G-invariant.

Theorem 5.7 (Super Gel′fand–Kazhdan criterion). *Let H_1, H_2 be closed subsupergroups of G, $\chi_i : H_i \longrightarrow \mathbb{K}^\times$ characters of H_i, and $\sigma : G \longrightarrow G$ an anti-automorphism. Assume that any $T \in \mathscr{S}'(G)_{\bar{0}}$, which is at once $(H_1 \times H_2)$-relatively invariant for the character $\chi_1^{-1} \otimes \chi_2^{-1}$ and a joint eigenvector of all $D \in \mathfrak{U}(\mathfrak{g})_{\bar{0}}^G$, is fixed by σ.*

Then, for any contragredient pair (E, F) of F-representations of G such that E_∞, F_∞ are irreducible CW G-representations, we have

$$\dim \mathrm{Hom}_{H_1}(E_\infty, \chi_1) \dim \mathrm{Hom}_{H_2}(F_\infty, \chi_2) \leqslant 1.$$

Here, Hom_H denotes continuous even linear maps that are equivariant with respect to the supergroup H.

Proof. Again, our argument is largely that of Sun–Zhu [60], with appropriate modifications and references to our results. Let

$$0 \neq v \in \underline{\mathrm{Hom}}_{H_1}(E_\infty, \chi_1) \subseteq F_{-\infty}, \quad 0 \neq u \in \underline{\mathrm{Hom}}_{H_1}(F_\infty, \chi_2) \subseteq E_{-\infty},$$

and set $T := M_{u \otimes v}^{-\infty} \in \mathscr{S}'(G)$, appealing to Proposition 5.4. (Here, $\underline{\mathrm{Hom}}_{H_1}$ denotes the space of H_1-equivariant continuous linear maps.)

For $D \in \mathfrak{U}(\mathfrak{g})$ and $\omega \in \mathscr{S}(G)$, we compute

$$\langle \omega, DT \rangle = (-1)^{|D||\omega|} \langle R_D \omega, M_{u \otimes v}^{-\infty} \rangle$$
$$= (-1)^{|D||\omega|} \langle \Pi_E^{-\infty}(R_D \omega)u, v \rangle = \langle \Pi_E^{-\infty}(\omega) d\pi_E^{-\infty}(D)u, v \rangle,$$

by the use of Equations (5.1) and (3.4). If now D is even and G-invariant, then D commutes with the G-action on $E^{-\infty}$.

The Harish-Chandra $(\mathfrak{g}, K_0)$-module $E_\infty^{(K_0)}$ is countable-dimensional, and $\mathfrak{U}(\mathfrak{g})$ acts irreducibly, hence Dixmier's Lemma [64, Lemma 0.5.2] applies, and $S(D)$ acts by a scalar. Since $E_\infty^{(K_0)} \subseteq E_\infty$ is dense, it follows that D acts by a scalar on $E_{-\infty}$. Thus, by the computation above, T is an eigenvector of D.

On the other hand, as a similar computation shows, T is also relatively $(\chi_1^{-1} \otimes \chi_2^{-1})$-invariant under $(H_1 \times H_2)$. By assumption, T is fixed by σ.

Let $\omega \in \mathscr{S}(G)$ and $g \in G_0$. We compute

$$\langle \Pi_E^{-\infty}(\omega)u, \pi_F(\sigma(g))^{-\infty}v \rangle = \langle \omega, R_{\sigma(g)}T \rangle = \langle \omega, R_{\sigma(T)}\sigma(T) \rangle$$
$$= \langle \omega, \sigma(L_g T) \rangle = (-1)^{|u||\omega|} \langle \pi_E^{-\infty}(g)u, \Pi_F^{-\infty}(\check{\omega})v \rangle.$$

By the irreducibility of E_∞ and F_∞, we conclude that

$$\Pi_E^{-\infty}(\omega)u = 0 \iff \Pi_F^{-\infty}(\check{\omega})v = 0.$$

Hence, for any other $0 \neq u' \in \underline{\mathrm{Hom}}_{H_1}(F_\infty, \chi_2)$, the continuous linear maps

$$\mathscr{S}(G) \longrightarrow E_\infty : \omega \longmapsto \Pi_E^{-\infty}(\omega)u, \quad \mathscr{S}(G) \longrightarrow E_\infty : \omega \longmapsto \Pi_E^{-\infty}(\omega)u',$$

have the same kernel W (say), and induce continuous linear maps

$$\varphi, \varphi' : \mathscr{S}(G)/W \longrightarrow E_\infty.$$

These are G-equivariant by their definition, so they are isomorphisms with closed image, by the token of Corollary 4.8. They are non-zero, and therefore surjective, by the assumption of irreducibility.

Hence, $\psi := \varphi'^{-1} \circ \varphi$ is a well-defined continuous even linear and G-equivariant automorphism of E_∞. Restricted to $E_\infty^{(K_0)}$, it is a constant, by Dixmier's Lemma (loc. cit.) again. This shows that $u' \in \mathbb{K}u$, by applying Lemma 2.7. A similar argument applies to v, proving the assertion. $\square$

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