LOOPING DIRECTIONS AND INTEGRALS OF EIGENFUNCTIONS OVER SUBMANIFOLDS

EMMETT L. WYMAN

ABSTRACT. Let (M, g) be a compact n-dimensional Riemannian manifold without boundary and e_λ be an L^2-normalized eigenfunction of the Laplace-Beltrami operator with respect to the metric g, i.e

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda \qquad \text{and} \qquad \|e_\lambda\|_{L^2(M)} = 1.$$

Let Σ be a d-dimensional submanifold and $d\mu$ a smooth, compactly supported measure on Σ. It is well-known (e.g. proved by Zelditch in [15] in far greater generality) that

$$\int_\Sigma e_\lambda \, d\mu = O(\lambda^{\frac{n-d-1}{2}}).$$

We show this bound improves to $o(\lambda^{\frac{n-d-1}{2}})$ provided the set of looping directions,

$$\mathcal{L}_\Sigma = \{(x, \xi) \in SN^*\Sigma : \Phi_t(x, \xi) \in SN^*\Sigma \text{ for some } t > 0\}$$

has measure zero as a subset of $SN^*\Sigma$, where here Φ_t is the geodesic flow on the cosphere bundle S^*M and $SN^*\Sigma$ is the unit conormal bundle over Σ.

1. Introduction.

In what follows, (M, g) will denote a compact, boundaryless, n-dimensional Riemannian manifold. Let Δ_g denote the Laplace-Beltrami operator and e_λ an L^2-normalized eigenfunction of Δ_g on M, i.e.

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda \qquad \text{and} \qquad \|e_\lambda\|_{L^2(M)} = 1.$$

In [13], Sogge and Zelditch investigate which manifolds have a sequence of eigenfunctions e_λ with $\lambda \to \infty$ which saturate the bound

$$\|e_\lambda\|_{L^\infty(M)} = O(\lambda^{\frac{n-1}{2}}).$$

They show that the bound above is necessarily $o(\lambda^{\frac{n-1}{2}})$ if at each each x, the set of looping directions through x,

$$\mathcal{L}_x = \{\xi \in S_x^*M : \Phi_t(x, \xi) \in S_x^*M \text{ for some } t > 0\}$$

has measure zero[1] as a subset of S_x^*M for each $x \in M$. Here, Φ_t denotes the geodesic flow on the unit cosphere bundle S^*M after time t. The hypotheses were later weakened by Sogge, Toth, and Zelditch in [10], where they showed

$$\|e_\lambda\|_{L^\infty(M)} = o(\lambda^{\frac{n-1}{2}})$$

provided the set of *recurrent* directions at x has measure zero for each $x \in M$.

[1]Let $\psi_j : U_j \subset \mathbb{R}^n \to M$ be coordinate charts of a general manifold M. We say a set $E \subset M$ has measure zero if the preimage $\psi_j^{-1}(E)$ has Lebesgue measure 0 in $\mathbb{R}^n$ for each chart ψ_j. Sets of Lebesgue measure zero are preserved under transition maps, ensuring this definition is intrinsic to the C^∞ structure of M.

We are interested in extending the result in [13] to integrals of eigenfunctions over submanifolds. Let Σ be a submanifold of dimension d with $d < n$ and a measure $d\mu(x) = h(x)d\sigma(x)$ where $d\sigma$ is the surface measure on Σ and h is a smooth function supported on a compact subset of Σ. In his 1992 paper [15], Zelditch proves, among other things, a Weyl law- type bound

$$\sum_{\lambda_j \leq \lambda} \left| \int_\Sigma e_j \, d\mu \right|^2 \sim \lambda^{n-d} + O(\lambda^{n-d-1}) \tag{1.1}$$

from which follows

$$\int_\Sigma e_\lambda \, d\mu = O(\lambda^{\frac{n-d-1}{2}}). \tag{1.2}$$

Though (1.2) is already well known, we will give a direct proof which will be illustrative for our main argument.

Theorem 1.1. *Let Σ be a d-dimensional submanifold with $0 \leq d < n$, and $d\mu(x) = h(x)d\sigma(x)$ where h is a smooth, real valued function supported on a compact neighborhood in Σ. Then,* (1.2) *holds.*

We let $SN^*\Sigma$ denote the unit conormal bundle over Σ. We define the set of looping directions through Σ by

$$\mathcal{L}_\Sigma = \{(x,\xi) \in SN^*\Sigma : \Phi_t(x,\xi) \in SN^*\Sigma \text{ for some } t > 0\}.$$

Our main result shows the bound (1.2) cannot be saturated whenever the set of looping directions through Σ has measure zero.

Theorem 1.2. *Assume the hypotheses of Theorem 1.1 and additionally that $\mathcal{L}_\Sigma$ has measure zero as a subset of $SN^*\Sigma$. Then,*

$$\int_\Sigma e_\lambda \, d\mu = o(\lambda^{\frac{n-d-1}{2}}).$$

The argument for Theorem 1.2 is modeled after Sogge and Zelditch's arguments in [13]. In fact if $d = 0$ we obtain the first part of [13, Theorem 1.2].

We expect the bound (1.2) to be saturated in the case $M = S^n$, since $\mathcal{L}_\Sigma = SN^*\Sigma$ always. The spectrum of $-\Delta_g$ on S^n consists of λ_j^2 where

$$\lambda_j = \sqrt{j(j+n-1)} \qquad \text{for } j = 0, 1, 2, \ldots$$

(see [9]). For each λ_j we select an eigenfunction e_j maximizing $\left|\int_\Sigma e_j \, d\mu\right|$. By Zelditch's Weyl law type bound (1.1), there exists an increasing sequence of λ with $\lambda \to \infty$ for which

$$\sum_{\lambda_j \in [\lambda, \lambda+1]} \left| \int_\Sigma e_j \, d\mu \right|^2 \gtrsim \lambda^{n-d-1}.$$

Since the gaps $\lambda_j - \lambda_{j-1}$ approach a constant width of 1 as $j \to \infty$, we may pick a subsequence of λ's so that only one λ_j falls in each band $[\lambda, \lambda+1]$. Hence,

$$\left| \int_\Sigma e_j \, d\mu \right| \gtrsim \lambda_j^{\frac{n-d-1}{2}}$$

for some subsequence of λ_j.

It is worth remarking that there are some cases where the hypotheses of Theorem 1.2 are naturally fulfilled and we obtain an improvement over (1.2). Chen and

Sogge [2] proved that if M is 2-dimensional and has negative sectional curvature, and Σ is a geodesic in M,

$$\int_\Sigma e_\lambda \, d\mu = o(1).$$

They consider a lift $\tilde{\Sigma}$ of Σ to the universal cover of M. Using the Gauss-Bonnet theorem, they show for each non-identity deck transformation α, there is at most one geodesic which intersects both $\tilde{\Sigma}$ and $\alpha(\tilde{\Sigma})$ perpendicularly. Since there are only countably many deck transformations, $\mathcal{L}_\Sigma$ is at most a countable subset of $SN^*\Sigma$ and so satisfies the hypotheses of Theorem 1.2. This result was extended to a larger class of curves in [14] which similarly have countable $\mathcal{L}_\Sigma$.

Acknowledgements. The author would like to thank Yakun Xi for pointing out an error at the end of the proof of Proposition 2.1.

2. Proof of Theorem 1.1.

Theorem 1.1 is a consequence of this stronger result.

Proposition 2.1. *Given the hypotheses of Theorem 1.1, we have*

$$\sum_{\lambda_j \in [\lambda, \lambda+1]} \left| \int_\Sigma e_\lambda \, d\mu \right|^2 \leq C\lambda^{n-d-1}.$$

We lay out some local coordinates which we will use repeatedly. Fix $p \in \Sigma$, and consider local coordinates $x = (x_1, \ldots, x_n) = (x', \bar{x})$ centered about p, where x' denotes the first d coordinates and $\bar{x}$ the remaining $n - d$ coordinates. We let $(x', 0)$ parametrize Σ on a neighborhood of p in such a way that dx' agrees with the surface measure on Σ. Let g denote the metric tensor with respect to our local coordinates. We require

$$g = \begin{bmatrix} * & 0 \\ 0 & I \end{bmatrix} \qquad \text{wherever } \bar{x} = 0,$$

where I here is the $(n-d)\times(n-d)$ identity matrix. This is ensured after inductively picking smooth sections $v_j(x')$ of $SN\Sigma$ for $j = d+1, \ldots, n$ with $\langle v_i, v_j \rangle = \delta_{ij}$, and then using

$$(x_1, \ldots, x_n) \mapsto \exp(x_{d+1} v_{d+1}(x') + \cdots + x_n v_n(x')) \tag{2.1}$$

as our coordinate map.

Now we prove Proposition 2.1. For simplicity, we assume without loss of generality that $d\mu$ is a real measure. We set[2] $\chi \in C^\infty(\mathbb{R})$ with $\chi \geq 0$ and $\hat{\chi}$ supported on a small neighborhood of 0. It suffices to show

$$\sum_j \chi(\lambda_j - \lambda) \left| \int_\Sigma e_j \, d\mu \right|^2 \leq C\lambda^{n-d-1}.$$

[2] This reduction is standard and appears in [13], [2], proofs of the sharp Weyl law as presented in [9] and [8], and in many other similar problems.

By Fourier inversion, we write the left hand side as

$$
\begin{aligned}
&\sum_j \int_\Sigma \int_\Sigma \chi(\lambda_j - \lambda) e_j(x) \overline{e_j(y)}\, d\mu(x)\, d\mu(y) \\
&\qquad = \frac{1}{2\pi} \sum_j \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \hat\chi(t) e^{-it\lambda} e^{it\lambda_j} e_j(x) \overline{e_j(y)}\, d\mu(x)\, d\mu(y)\, dt \\
&\qquad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \hat\chi(t) e^{-it\lambda} e^{it\sqrt{-\Delta_g}}(x,y)\, d\mu(x)\, d\mu(y)\, dt
\end{aligned} \tag{2.2}
$$

where $e^{it\sqrt{-\Delta_g}}$ is the half wave operator with kernel

$$
e^{it\sqrt{-\Delta_g}}(x,y) = \sum_j e^{it\lambda_j} e_j(x) \overline{e_j(y)}.
$$

Using the coordinates $x = (x', \bar x)$ as in (2.1), the last line of (2.2) is written

$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat\chi(t) e^{-it\lambda} e^{it\sqrt{-\Delta_g}}(x',y') h(x') h(y')\, dx'\, dy'\, dt \tag{2.3}
$$

where h is a smooth function on $\mathbb{R}^d$ such that $d\mu(x) = h(x')dx'$, and where by abuse of notation x' is taken to mean $(x', 0)$ where appropriate. We now use Hörmander's parametrix as presented in [8], i.e

$$
e^{it\sqrt{-\Delta_g}}(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\varphi(x,y,\xi) + tp(y,\xi))} q(t,x,y,\xi)\, d\xi
$$

modulo a smooth kernel, where

$$
p(y,\xi) = \sqrt{\sum_{j,k} g^{jk}(y) \xi_j \xi_k}
$$

is the principal symbol of $\sqrt{-\Delta_g}$ and φ is smooth for $|\xi| > 0$, homogeneous of degree 1 in ξ, and satisfies

$$
|\partial_\xi^\alpha (\varphi(x,y,\xi) - \langle x - y, \xi \rangle)| \le C_\alpha |x-y|^2 |\xi|^{1-|\alpha|} \tag{2.4}
$$

for multiindices $\alpha \ge 0$ and for x and y sufficiently close. Moreover, q satisfies bounds

$$
|\partial_\xi^\alpha \partial_{t,x,y}^\beta q(t,x,y,\xi)| \le C_{\alpha,\beta} (1 + |\xi|)^{-|\alpha|}, \tag{2.5}
$$

and where for $t \in \operatorname{supp} \hat\chi$, q is supported on a small neighborhood of $x = y$. Hence, we write (2.3) as

$$
= \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} e^{i(\varphi(x',y',\xi) + tp(y',\xi) - t\lambda)} \hat\chi(t) q(t,x',y',\xi) h(x') h(y') \\ d\xi\, dx'\, dy'\, dt,
$$

and after making a change of coordinates $\xi \mapsto \lambda\xi$ is

$$
= \frac{\lambda^n}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} e^{i\lambda(\varphi(x',y',\xi) + t(p(y',\xi) - 1))} \hat\chi(t) q(t,x',y',\lambda\xi) h(x') h(y') \\ d\xi\, dx'\, dy'\, dt.
$$

We introduce a function $\beta \in C_0^\infty(\mathbb{R})$ with $\beta \equiv 1$ near 0 and support contained in a small neighborhood of 0, and cut the integral into $\beta(\log p(y',\xi))$ and $1 -$

$\beta(\log p(y',\xi))$ parts. $|p(y',\xi)-1|$ is bounded away from 0 on the support of $1-\beta(\log p(y',\xi))$, so integrating by parts in t yields

$$\frac{\lambda^n}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^n} e^{i\lambda(\varphi(x',y',\xi)+t(p(y',\xi)-1))} \\ (1-\beta(\log p(y',\xi)))\hat{\chi}(t)q(t,x',y',\lambda\xi)h(x')h(y')\,d\xi\,dx'\,dy'\,dt. \\ = O(\lambda^{-N})$$

for each $N=1,2,\ldots$. What is left to bound is the $\beta(\log p(y',\xi))$ part, i.e.

$$\lambda^n\int_{-\infty}^{\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^n} e^{i\lambda\Phi(t,x',y',\xi)}a(\lambda;t,x',y',\xi)\,d\xi\,dx'\,dy'\,dt = O(\lambda^{n-d-1}) \tag{2.6}$$

where we have set the amplitude

$$a(\lambda;t,x',y',\xi)=\frac{1}{(2\pi)^{n+1}}\beta(\log|\xi|)\hat{\chi}(t)q(t,x',y',\lambda\xi)h(x')h(y')$$

and the phase

$$\Phi(t,x',y',\xi)=\varphi(x',y',\xi)+t(p(y',\xi)-1).$$

By (2.5) and since a has compact support in t, x', y', and ξ, a and all of its derivatives are uniformly bounded in λ.

We are now in a position to apply stationary phase. Write $\xi=(\xi',\bar{\xi})$ and write $\bar{\xi}=r\omega$ in polar coordinates with $r\geq 0$ and $\omega\in S^{n-d-1}$. The integral in (2.6) is then written

$$\lambda^n\int_{-\infty}^{\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{S^{n-d-1}}\int_0^{\infty} e^{i\lambda\Phi(t,x',y',\xi)}a(\lambda;t,x',y',\xi) \\ r^{n-d-1}\,dr\,d\omega\,d\xi'\,dx'\,dy'\,dt$$

We will fix y' and ω and use the method of stationary phase in the remaining variables t, x', ξ', and r (a total of $2d+2$ dimensions). We assert that, for fixed y' and ω, there is a nondegenerate stationary point at $(t,x',\xi',r)=(0,y',0,1)$. $\Phi=0$ at such a stationary point, and after perhaps shrinking the support of a we apply [8, Corollary 1.1.8] to write the left hand side of (2.6) as

$$\lambda^{n-d-1}\int_{\mathbb{R}^d}\int_{S^{n-d-1}} a(\lambda;y',\omega)\,dy'\,d\omega$$

for some amplitude $a(\lambda;y',\omega)$ uniformly bounded with respect to λ. (2.6) follows.

We have

$$\begin{aligned}\partial_t\Phi &= p(y',\xi)-1\\ \nabla_{x'}\Phi &= \nabla_{x'}\varphi(x',y',\xi)\\ \nabla_{\xi'}\Phi &= \nabla_{\xi'}\varphi(x',y',\xi)+t\nabla_{\xi'}p(y',\xi)\\ \partial_r\Phi &= \partial_r\varphi(x',y',\xi)+t\partial_r p(y',\xi).\end{aligned}$$

Note for fixed y' and ω, $(t,x',\xi',r)=(0,y',0,1)$ is a critical point of Φ. Now we compute the second derivatives at this point. We immediately see that $\partial_t^2\Phi$, $\partial_t\nabla_{x'}\Phi$, $\nabla_{\xi'}^2\Phi$, $\partial_r\nabla_{\xi'}\Phi$, and $\partial_r^2\Phi$ all vanish. Moreover, $\partial_r\partial_r\Phi=1$ since $p(y',\xi)=r$

where $\xi' = 0$. By our coordinates (2.1) and the fact that $[g^{ij}]_{i,j\le d}$ is necessarily positive definite,

$$p(y',\xi) = \sqrt{\sum_{j,k} g^{jk}\xi_j\xi_k} = \sqrt{r^2 + \sum_{j,k\le d} g^{jk}\xi_j'\xi_k'} \ge r = p(y', r\omega).$$

Hence, $\partial_t \nabla_{\xi'}\Phi = \nabla_{\xi'} p(y',\xi) = 0$. Since φ is homogeneous of degree 1 in ξ, at $\xi' = 0$ and $t = 0$,

$$\nabla_{x'}\partial_r \Phi = \nabla_{x'}\partial_r \varphi(x',y',\xi) = \nabla_{x'}\varphi(x',y',\omega) = 0$$

since $\varphi(x',y',\omega) = O(|x'-y'|^2)$ by (2.4) and the fact that $\langle x'-y',\omega\rangle = 0$. Finally by (2.4),

$$\nabla_{\xi'}\varphi(x',y',\xi'+\omega) = x' + O(|x'-y'|^2)$$

whence at the critical point

$$\nabla_{x'}\nabla_{\xi'}\Phi = I,$$

the $d\times d$ identity matrix. In summary, the Hessian matrix of Φ at the critical point $(t,x',\xi',r) = (0,y',0,1)$ is

$$\nabla^2_{t,x',\xi',r}\Phi = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & * & I & 0 \\ 0 & I & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which has full rank.

3. **Microlocal tools.**

The hypotheses on the looping directions in Theorem 1.2 ensure that the wavefront sets of μ and $e^{it\sqrt{-\Delta_g}}\mu$ have minimal intersection for any given t. We can then use pseudodifferential operators to break μ into two parts, the first whose wavefront set is disjoint from that of $e^{it\sqrt{-\Delta_g}}\mu$ and the second which contributes a small, controllable term to the bound. The following propositions will allow us to handle these cases, respectively.

Proposition 3.1. *Let u and v be distributions on M for which*

$$\mathrm{WF}(u)\cap \mathrm{WF}(v) = \emptyset.$$

Then

$$t \mapsto \int_M e^{it\sqrt{-\Delta_g}} u(x)\overline{v(x)}\,dx$$

is a smooth function of t on some neighborhood of 0.

Proof. Using a partition of unity, we write

$$I = \sum_j A_j$$

modulo a smoothing operator where $A_j \in \Psi^0_{\mathrm{cl}}(M)$ with essential supports in small conic neighborhoods. We then write, formally,

$$\int e^{it\sqrt{-\Delta_g}} u(x)\overline{v(x)}\,dx = \sum_{j,k}\int A_j e^{it\sqrt{-\Delta_g}} u(x)\overline{A_k v(x)}\,dx.$$

We are done if for each i and j,

$$\int_M A_j e^{it\sqrt{-\Delta_g}} u(x)\overline{A_k v(x)}\,dx \qquad \text{is smooth for } |t| \ll 1. \tag{3.1}$$

If the essential supports of A_j and A_k are disjoint, then $A_j^* A_k$ is a smoothing operator, and so $A_j^* A_k v$ is a smooth function and the contributing term

$$\int u(x)\overline{e^{it\sqrt{-\Delta_g}} A_j^* A_k v(x)}\,dx$$

is smooth is t. Assume the essential support of A_j are small enough so that for each j there exists a small conic neighborhood Γ_j which fully contains the essential support of A_k if it intersects the essential support of A_j. We in turn take Γ_j small enough so that for each j, $\overline{\Gamma_j}$ either does not intersect $\mathrm{WF}(u)$ or does not intersect $\mathrm{WF}(v)$. In the latter case, $A_k v$ is smooth and we have (3.1) as before. In the former case,

$$\overline{\Gamma_j} \cap \mathrm{WF}(e^{it\sqrt{-\Delta_g}} u) = \emptyset \qquad \text{for } |t| \ll 1$$

since both sets above are closed and the geodesic flow is continuous. Then $A_j e^{it\sqrt{-\Delta_g}} u(x)$ is smooth as a function of t and x, and we have (3.1). □

The second piece of our argument requires the following generalization of Proposition 2.1, modeled after [9, Lemma 5.2.2]. In the proof we will come to a point where it seems like we may have to perform a stationary phase argument involving an eight-by-eight Hessian matrix. Instead, we appeal to Proposition 5.1 in the appendix to break the argument into two steps involving two four-by-four Hessian matrices.

Proposition 3.2. *Let $b(x,\xi)$ be smooth for $\xi \neq 0$ and homogeneous of degree 0 in the ξ variable. We define $b \in \Psi^0_{cl}(M)$ by*

$$b(x,D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} b(x,\xi)\,dy\,d\xi$$

for x, y, and ξ expressed locally according to our coordinates (2.1). Then,

$$\sum_{\lambda_j \in [\lambda,\lambda+1]} \left| \int_\Sigma b e_j(x)\,d\mu(x) \right|^2 \leq C \left(\int_{\mathbb{R}^d} \int_{S^{n-d-1}} |b(x',\omega)|^2 h(x')^2\,d\omega\,dx' \right) \lambda^{n-d-1} + C_b \lambda^{n-d-2}$$

where C is a constant independent of b and λ and C_b is a constant independent of λ but which depends on b.

Proof. We may by a partition of unity assume that $b(x,D)$ has small x-support. Let χ be as in the proof of Proposition 2.1. It suffices to show

$$\sum_j \int_\Sigma \int_\Sigma \chi(\lambda_j - \lambda) b(x,D)e_j(x)\overline{b(y,D)e_j(y)}\,d\mu(x)\,d\mu(y) \sim \left(\int_{\mathbb{R}^d} \int_{S^{n-d-1}} |b(y',\omega)|^2 h(y')^2\,d\omega\,dy' \right) \lambda^{n-d-1} + O_b(\lambda^{n-d-2}).$$

Using the same reduction as in Proposition 2.1, the left hand side is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_\Sigma\int_\Sigma \hat\chi(t)e^{-it\lambda}be^{it\sqrt{-\Delta_g}}b^*(x,y)\,d\mu(x)\,d\mu(y)\,dt. \tag{3.2}$$

Set $\beta \in C_0^\infty(\mathbb{R})$ with small support and where $\beta \equiv 1$ near 0. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^d} b(x',D)f(x')h(x')\,dx' \\
&\qquad = \frac{\lambda^n}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d} e^{i\lambda\langle x'-w,\eta\rangle}b(x',\eta)f(w)h(x')\,dx'\,dw\,d\eta \\
&\qquad = \frac{\lambda^n}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d} e^{i\lambda\langle x'-w,\eta\rangle}\beta(\log|\eta|)b(x',\eta)f(w)h(x')\,dx'\,dw\,d\eta \\
&\qquad\qquad + O(\lambda^{-N}),
\end{aligned} \tag{3.3}$$

where the second line is obtained by a change of variables $\eta \mapsto \lambda\eta$, and the third line is obtained after multiplying in the cutoff $\beta(\log|\eta|)$ and bounding the discrepancy by $O(\lambda^{-N})$ by integrating by parts in x'. Additionally,

$$\begin{aligned}
b^*(z,D)d\mu(z) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d} e^{i\langle z-y',\zeta\rangle}b(y',\zeta)h(y')\,dy'\,d\zeta \\
&= \frac{\lambda^n}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d} e^{i\lambda\langle z-y',\zeta\rangle}b(y',\zeta)h(y')\,dy'\,d\zeta \\
&= \frac{\lambda^n}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d} e^{i\lambda\langle z-y',\zeta\rangle}\beta(\log|\zeta|)b(y',\zeta)h(y')\,dy'\,d\zeta + O(\lambda^{-N}) \\
&= \frac{\lambda^n}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d} e^{i\lambda\langle z-y',\zeta\rangle}\beta(\log|\zeta|)\beta(|z-y'|)b(y',\zeta)h(y')\,dy'\,d\zeta \\
&\qquad + O(\lambda^{-N}),
\end{aligned} \tag{3.4}$$

where the second and third lines are obtained similarly as before and the fourth line is obtained after multiplying by $\beta(\log|z-y'|)$ and integrating the remainder by parts in ζ. Using Hörmander's parametrix,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat\chi(t)e^{-it\lambda}e^{it\sqrt{-\Delta_g}}(w,z)\,dt \\
&\quad = \frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e^{i(\varphi(w,z,\xi)+tp(z,\xi)-t\lambda)}\hat\chi(t)q(t,w,z,\xi)\,d\xi\,dt \\
&\quad = \frac{\lambda^n}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e^{i\lambda(\varphi(w,z,\xi)+t(p(z,\xi)-1))}\hat\chi(t)q(t,w,z,\lambda\xi)\,d\xi\,dt \\
&\quad = \frac{\lambda^n}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e^{i\lambda(\varphi(w,z,\xi)+t(p(z,\xi)-1))}\beta(\log p(z,\xi))\hat\chi(t)q(t,w,z,\lambda\xi)\,d\xi \\
&\qquad + O(\lambda^{-N}).
\end{aligned} \tag{3.5}$$

Here the third line comes from a change of coordinates $\xi \mapsto \lambda\xi$. The fourth line follows after applying the cutoff $\beta(\log p(z,\zeta))$ and integrating the discrepancy by parts in t. Combining (3.3), (3.4), and (3.5), we write (3.2) as

$$\begin{aligned}
&\lambda^{3n}\int\cdots\int e^{i\Phi(t,x',y',w,z,\eta,\zeta,\xi)}a(\lambda;t,x',y',w,z,\eta,\zeta,\xi) \\
&\qquad\qquad dx'\,dy'\,dw\,dz\,d\eta\,d\zeta\,d\xi + O(\lambda^{-N})
\end{aligned} \tag{3.6}$$

with amplitude

$$a(\lambda; t, x', y', w, z, \eta, \zeta, \xi) = \frac{1}{(2\pi)^{3n+1}} \hat{\chi}(t) q(t, w, z, \lambda\xi) \beta(\log p(z, \xi)) \beta(\log |\eta|) \\ \beta(\log |\zeta|) \beta(|z - y'|) b(x', \eta) b(y', \zeta) h(x') h(y')$$

and phase

$$\Phi(t, x', y', w, z, \eta, \zeta, \xi) = \langle x' - w, \eta \rangle + \varphi(w, z, \xi) + t(p(z, \xi) - 1) + \langle z - y', \zeta \rangle.$$

We pause here to make a couple observations. First, a has compact support in all variables, support which we may adjust to be smaller by controlling the supports of $\hat{\chi}$, β, b, and the support of q near the diagonal. Second, the derivatives of a are bounded independently of $\lambda \geq 1$. We are now in a position to use the method of stationary phase – not in all variables at once, though. First, we fix t, x', y' and ξ, and use stationary phase in w, z, η, and ζ. We have

$$\begin{aligned} \nabla_w \Phi &= -\eta + \nabla_w \varphi(w, z, \xi) \\ \nabla_z \Phi &= \nabla_z \varphi(w, z, \xi) + t \nabla_z p(z, \xi) + \zeta \\ \nabla_\eta \Phi &= x' - w \\ \nabla_\zeta \Phi &= z - y' \end{aligned}$$

which all simultaneously vanishes if and only if

$$(w, z, \eta, \zeta) = (x', y', \nabla_x \varphi(x', y', \xi), -\nabla_y \varphi(x', y', \xi) - t \nabla_y p(y', \xi)). \tag{3.7}$$

At such a critical point we have the Hessian matrix

$$\nabla^2_{w,z,\eta,\zeta} \Phi = \begin{bmatrix} * & * & -I & 0 \\ * & * & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix},$$

which has determinant -1. By Proposition 5.1 in the appendix, (3.6) is equal to complex constant times

$$\lambda^n \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\lambda \Phi(t, x', y', \xi)} a(\lambda; t, x', y', \xi) \, dx' \, dy' \, d\xi' \, dt \\ + \lambda^{n-1} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\lambda \Phi(t, x', y', \xi)} R(\lambda; t, x', y', \xi) \, dx' \, dy' \, d\xi' \, dt + O(\lambda^{-N})$$

where we have phase

$$\Phi(t, x', y', \xi) = \varphi(x', y', \xi) + t(p(y', \xi) - 1),$$

amplitude

$$a(\lambda; t, x', y', \xi) = a(\lambda; t, x', y', w, z, \eta, \zeta, \xi)$$

with w, z, η, and ζ subject to the constraints (3.7), and where R is a compactly supported smooth function in t, x', y', and ξ whose derivatives are bounded uniformly with respect to λ. Our phase function matches that in the proof of Proposition 2.1, and so we repeat that argument – we write $\bar{\xi} = r\omega$ and fix y' and ω. We obtain unique nondegenerate stationary points

$$(t, x', \xi', r) = (0, y', 0, 1).$$

Now,

$$a(\lambda; 0, y', y', \omega) \sim |b(y', \omega)|^2 h(y')^2.$$

Hence, we have

$$\lambda^n \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\lambda\Phi(t,x',y',\xi)} a(\lambda; t, x', y', \xi)\, dx'\, dy'\, d\xi'\, dt$$
$$\sim \lambda^{n-d-1} \left(\int_{\mathbb{R}^d} \int_{S^{n-d-1}} b(y',\omega)^2 h(y')^2\, d\omega\, dy' \right) + O(\lambda^{n-d-2})$$

by Proposition 5.1 as desired. The same argument applied to the remainder term gives

$$\lambda^{n-1} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\lambda\Phi(t,x',y',\xi)} R(\lambda; t, x', y', \xi)\, dx'\, dy'\, d\xi'\, dt = O(\lambda^{n-d-2})$$

as desired. □

4. **Proof of Theorem 1.2.**

Theorem 1.2 follows from the following stronger statement.

Proposition 4.1. *Given the hypotheses of Theorem 1.2, we have*

$$\sum_{\lambda_j \in [\lambda, \lambda+\varepsilon]} \left| \int_\Sigma e_\lambda\, d\mu \right|^2 \le C\varepsilon\lambda^{n-d-1} + C_\varepsilon \lambda^{n-d-2},$$

where C is a constant independent of ε and λ, and C_ε is a constant depending on ε but not λ.

We make a few convenient assumptions. First, we take the injectivity radius of M to be at least 1 by scaling the metric g. Second, we assume the support of $d\mu$ has diameter less than $1/2$ by a partition of unity. We reserve the right to further scale the metric g and restrict the support of $d\mu$ as needed, finitely many times.

As before, we set $\chi \in C^\infty(\mathbb{R})$ with $\chi(0) = 1$, $\chi \ge 0$, and $\operatorname{supp} \hat{\chi} \subset [-1, 1]$. It suffices to show

$$\sum_j \chi(T(\lambda_j - \lambda)) \left| \int_\Sigma e_\lambda\, d\mu \right|^2 \le CT^{-1}\lambda^{n-d-1} + C_T \lambda^{n-d-2}$$

for $T > 1$. Similar to the reduction in the proof of Proposition 2.1, we have

$$\sum_j \int_\Sigma \int_\Sigma \chi(T(\lambda_j - \lambda)) e_j(x)\overline{e_j(y)}\, d\mu(x)\, d\mu(y)$$
$$= \frac{1}{2\pi} \sum_j \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \hat{\chi}(t) e^{itT(\lambda_j - \lambda)} e_j(x)\overline{e_j(y)}\, d\mu(x)\, d\mu(y)\, dt$$
$$= \frac{1}{2\pi T} \sum_j \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \hat{\chi}(t/T) e^{-it\lambda} e^{it\lambda_j} e_j(x)\overline{e_j(y)}\, d\mu(x)\, d\mu(y)\, dt$$
$$= \frac{1}{2\pi T} \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \hat{\chi}(t/T) e^{-it\lambda} e^{it\sqrt{-\Delta_g}}(x,y)\, d\mu(x)\, d\mu(y)\, dt.$$

Hence, it suffices to show

$$\left| \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \hat{\chi}(t/T) e^{-it\lambda} e^{it\sqrt{-\Delta_g}}(x,y)\, d\mu(x)\, d\mu(y)\, dt \right| \le C\lambda^{n-d-1} + C_T\lambda^{n-d-2}. \tag{4.1}$$

Set $\beta \in C_0^\infty(\mathbb{R})$ with $\beta(t) \equiv 1$ near 0 and β. We cut the integral in (4.1) into $\beta(t)$ and $1-\beta(t)$ parts. Since $\beta(t)\hat\chi(t/T)$ and its derivatives are all bounded independently of $T \geq 1$,

$$\left| \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma \beta(t)\hat\chi(t/T) e^{-it\lambda} e^{it\sqrt{-\Delta_g}}(x,y)\, d\mu(x)\, d\mu(y)\, dt \right| \leq C\lambda^{n-d-1}$$

by the proof of Proposition 2.1. Hence, it suffices to show

$$\left| \int_{-\infty}^{\infty} \int_\Sigma \int_\Sigma (1-\beta(t))\hat\chi(t/T) e^{-it\lambda} e^{it\sqrt{-\Delta_g}}(x,y)\, d\mu(x)\, d\mu(y)\, dt \right| \leq C\lambda^{n-d-1} + C_T \lambda^{n-d-2}. \tag{4.2}$$

Here we shrink the support of μ so that $\beta(d_g(x,y)) = 1$ for $x, y \in \operatorname{supp}\mu$. We now state and prove a useful decomposition based off of those in [13], [10], and Chapter 5 of [9]. We let $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$ denote the subset of $\mathcal{L}_\Sigma$ relevant to the support of μ and the timespan $[1, T]$, specifically

$$\mathcal{L}_\Sigma(\operatorname{supp}\mu, T) = \{(x,\xi) \in SN^*\Sigma : \Phi_t(x,\xi) = (y,\eta) \in SN^*\Sigma \text{ for some } t \in [1,T] \text{ and where } x, y \in \operatorname{supp}\mu\}.$$

Lemma 4.2. *Fix $T > 1$ and $\varepsilon > 0$. There exist $b, B \in \Psi^0_{cl}(M)$ supported on a neighborhood of* $\operatorname{supp}\mu$ *with the following properties.*

1. *$b(x,D) + B(x,D) = I$ modulo a smoothing operator on* $\operatorname{supp}\mu$.
2. *Using coordinates* (2.1),
$$\int_{\mathbb{R}^d} \int_{S^{n-d-1}} |b(x',\omega)|^2 \, d\omega\, dx' < \varepsilon,$$
where $b(x,\xi)$ is the principal symbol of $b(x,D)$.
3. *The essential support of $B(x,D)$ contains no elements of $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$.*

Proof. As shorthand, we write

$$SN^*_{\operatorname{supp}\mu}\Sigma = \{(x,\xi) \in SN^*\Sigma : x \in \operatorname{supp}\mu\}.$$

We first argue that $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$ is closed for each $T > 1$. However, $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$ is the projection of the set

$$\{(t,x,\xi) \in [1,T] \times SN^*_{\operatorname{supp}\mu}\Sigma : \Phi_t(x,\xi) \in SN^*_{\operatorname{supp}\mu}\Sigma\} \tag{4.3}$$

onto $SN^*_{\operatorname{supp}\mu}\Sigma$, and since $[1,T]$ is compact it suffices to show that (4.3) is closed. However, (4.3) is the intersection of $[1,T] \times SN^*_{\operatorname{supp}\mu}\Sigma$ with the preimage of $SN^*_{\operatorname{supp}\mu}\Sigma$ under the continuous map

$$(t,x,\xi) \mapsto \Phi_t(x,\xi).$$

Since $SN^*_{\operatorname{supp}\mu}\Sigma$ is closed, (4.3) is closed.

Since $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$ is closed and has measure zero, there is $\tilde b \in C^\infty(SN^*\Sigma)$ supported on a neighborhood of $SN^*_{\operatorname{supp}\mu}\Sigma$ with $0 \leq \tilde b(x,\xi) \leq 1$, $\tilde b(x,\xi) \equiv 1$ on an open neighborhood of $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$, and

$$\int_{\mathbb{R}^d} \int_{S^{n-d-1}} |b(x',\omega)|^2 \, d\omega\, dx' < \varepsilon.$$

We use the coordinates in (2.1) and define

$$b(x,D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \tilde b(x, \xi/|\xi|) f(y)\, dy\, d\xi,$$

hence (2). We set $\psi \in C_0^\infty(\Sigma)$ to be a cutoff function supported on a neighborhood of $\operatorname{supp}\mu$ with $\psi \equiv 1$ on $\operatorname{supp}\mu$. Defining

$$B(x,D)f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}\psi(x)(1-\tilde{b}(x,\xi/|\xi|))f(y)\,dy\,d\xi$$

yields (1). We have (3) since the support of $1-\tilde{b}(x,\xi)$ contains no elements of $\mathcal{L}_\Sigma(\operatorname{supp}\mu, T)$. □

Returning to the proof of Proposition 4.1, let X_T denote the function with

$$\hat{X}_T(t) = (1-\beta(t))\hat{\chi}(t/T),$$

and let $X_{T,\lambda}$ denote the operator with kernel

$$X_{T,\lambda}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{X}_T(t)e^{-it\lambda}e^{it\sqrt{-\Delta_g}}(x,y)\,dt.$$

We use part (1) of Lemma 4.2 to write the integral in (4.2) as

$$\begin{aligned}\int_\Sigma\int_\Sigma X_{T,\lambda}(x,y)\,d\mu(y)\,d\mu(x) &= \int_\Sigma\int_\Sigma BX_{T,\lambda}B^*(x,y)\,d\mu(y)\,d\mu(x)\\ &\quad+\int_\Sigma\int_\Sigma BX_{T,\lambda}b^*(x,y)\,d\mu(y)\,d\mu(x)\\ &\quad+\int_\Sigma\int_\Sigma bX_{T,\lambda}B^*(x,y)\,d\mu(y)\,d\mu(x)\\ &\quad+\int_\Sigma\int_\Sigma bX_{T,\lambda}b^*(x,y)\,d\mu(y)\,d\mu(x).\end{aligned}$$

We claim the first three terms on the right are $O_T(\lambda^{-N})$ for $N = 1,2,\ldots$. We will only prove this for the first term – the argument is the same for the second term and the bound for the third term follows since $X_{T,\lambda}$ is self-adjoint. Interpreting μ as a distribution on M, we write formally

$$\begin{aligned}&\int_\Sigma\int_\Sigma BX_{T,\lambda}B^*(x,y)\,d\mu(y)\,d\mu(x)\\ &\qquad= \int_M\int_M X_{T,\lambda}(x,y)B^*\mu(y)\overline{B^*\mu(x)}\,dx\,dy\\ &\qquad= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{X}_T(t)e^{-it\lambda}\int_M e^{it\sqrt{-\Delta_g}}(B^*\mu)(x)\overline{B^*\mu(x)}\,dx\,dt \qquad (4.4)\end{aligned}$$

Once we show

$$\mathrm{WF}(e^{it\sqrt{-\Delta_g}}B^*\mu)\cap B^*\mu = \emptyset \qquad \text{for all } t\in\operatorname{supp}\hat{X}_T, \qquad (4.5)$$

the integral over M will be smooth in t by Proposition 3.1. Integration by parts in t then gives the desired bound of $O_T(\lambda^{-N})$. To prove (4.5), suppose $(x,\xi)\in \mathrm{WF}(B^*\mu)$. By part (3) of Lemma 4.2, $\Phi_t(x,\xi)$ is not in $SN^*_{\operatorname{supp}\mu}\Sigma$ for any $1\le|t|\le T$. By propagation or singularities,

$$\mathrm{WF}(e^{it\sqrt{-\Delta_g}}B^*\mu) = \Phi_t\,\mathrm{WF}(B^*\mu),$$

hence

$$\mathrm{WF}(e^{it\sqrt{-\Delta_g}}B^*\mu)\cap\mathrm{WF}(B^*\mu) = \emptyset \qquad \text{for } 1\le|t|\le T. \qquad (4.6)$$

Since the support of μ has been made small, if there is $(x,\xi)\in SN^*_{\operatorname{supp}\mu}\Sigma$ and some $t>0$ in the support of $(1-\beta(t))\hat{\chi}(t/T)$ for which $\Phi_t(x,\xi)\in SN^*_{\operatorname{supp}\mu}\Sigma$, then $t\ge 1$

since the diameter of $\operatorname{supp}\mu$ is small and the injectivity radius of M is at least 1. We now have (4.5), from which follows (4.4) as promised.

What remains is to bound

$$\left| \int_\Sigma \int_\Sigma bX_{T,\lambda}b^*(x,y)\, d\mu(x)\, d\mu(y) \right| \le \lambda^{n-d-1} + C_{T,b}\lambda^{n-d-2}. \tag{4.7}$$

We have

$$bX_{T,\lambda}b^*(x,y) = \sum_j X_T(\lambda_j - \lambda) b e_j(x) \overline{b e_j(y)},$$

and so we write the integral in (4.7) as

$$\sum_j X_T(\lambda_j - \lambda) \left| \int_\Sigma b(x,D) e_j(x)\, d\mu(x) \right|^2. \tag{4.8}$$

By the bounds

$$|X_T(\tau)| \le C_{T,N}(1+|\tau|)^{-N} \qquad \text{for } N = 1,2,\dots$$

and Proposition 3.2,

$$\left| \sum_j X_T(\lambda_j - \lambda) \left| \int_\Sigma b(x,D)e_j(x)\, d\mu(x) \right|^2 \right|$$
$$\le C_T \lambda^{n-d-1} \int_{\mathbb{R}^d} \int_{S^{n-d-1}} |b(x',\omega)|^2 h(x')^2\, d\omega\, dx' + C_{T,b}\lambda^{n-d-2}.$$

Taking ε in part (2) of Lemma 4.2 small enough so that $\varepsilon C_T \le 1$ yields (4.7). This concludes the proof of Proposition 4.1.

5. **Appendix: Stationary phase tool.**

The following tool is a combination of Corollary 1.1.8 with the discussion at the end of Section 1.1 in [8]. Let $\phi(x,y)$ be a smooth phase function on $\mathbb{R}^m \times \mathbb{R}^n$ with

$$\nabla_y \phi(0,0) = 0 \qquad \text{and} \qquad \det \nabla_y^2 \phi(0,0) \ne 0,$$

and let $a(\lambda;x,y)$ be a smooth amplitude with small, adjustable support satisfying

$$|\partial_\lambda^j \partial_x^\alpha \partial_y^\beta a(\lambda;x,y)| \le C_{j,\alpha,\beta}\lambda^{-j} \qquad \text{for } \lambda \ge 1$$

for $j = 0,1,2,\dots$ and multiindices α and β. $\nabla_y^2\phi \ne 0$ on a neighborhood of 0 by continuity. There exists locally a smooth map $x \mapsto y(x)$ whose graph in $\mathbb{R}^m \times \mathbb{R}^n$ contains all points in a neighborhood of 0 such that $\nabla_y \phi = 0$, by the implicit function theorem. Let p and q be integers denoting the number of positive and negative eigenvalues of $\nabla_y^2\phi$, respectively, counting multiplicity. By continuity, p and q are constant on a neighborhood of 0. We adjust the support of a to lie in the intersection of these neighborhoods.

Proposition 5.1. *Let*

$$I(\lambda;x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)} a(\lambda;x,y)\, dy$$

with ϕ and a as above. Then,

$$I(\lambda;x) = (\lambda/2\pi)^{-n/2} |\det \nabla_y^2 \phi(x,y(x))|^{-1/2} e^{\pi i(p-q)/4} e^{i\lambda\phi(x,y(x))} a(\lambda;x,y(x))$$
$$+ \lambda^{-n/2-1} e^{i\lambda\phi(x,y(x))} R(\lambda;x) + O(\lambda^{-N})$$

for $N = 1, 2, \dots$, where R has compact support,

$$|\partial_\lambda^j \partial_x^\alpha R(\lambda; x)| \le C_{j,\alpha} \lambda^{-j} \qquad \text{for } \lambda \ge 1,$$

and the $O(\lambda^{-N})$ term is constant in x.

Proof. We have

$$e^{-i\lambda\phi(x,y(x))} I(\lambda; x) = \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} a(\lambda; x, y)\, dy \tag{5.1}$$

where we have set

$$\Phi(x, y) = \phi(x, y) - \phi(x, y(x)).$$

The proof of the Morse-Bott lemma in [1] lets us construct a smooth map $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ such that $y \mapsto F(x, y)$ is a diffeomorphism between neighborhoods of 0 in $\mathbb{R}^n$ for each x, and for which

$$F(x, 0) = y(x)$$

and

$$\Phi(x, F(x, y)) = \frac{1}{2}(y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2) = Q(y).$$

Applying a change of variables in y to (5.1) yields

$$\begin{aligned} &= \int_{\mathbb{R}^n} e^{i\lambda Q(y)} a(\lambda; x, F(x, y)) |\det D_y F(x, y)|\, dy \\ &= \int_{\mathbb{R}^n} e^{i\lambda Q(y)} a(\lambda; x, F(x, 0)) |\det D_y F(x, 0)|\, dy + \int_{\mathbb{R}^n} e^{i\lambda Q(y)} r(\lambda; x, y)\, dy \end{aligned}$$

where

$$r(\lambda; x, y) = a(\lambda; x, F(x, y)) |\det D_y F(x, y)| - a(\lambda; x, F(x, 0)) |\det D_y F(x, 0)|.$$

The first term evaluates to

$$(\lambda/2\pi)^{-n/2} |\det \nabla_y^2 \phi(x, y(x))|^{-1/2} e^{\pi i(p-q)/4} a(\lambda; x, y(x))$$

since

$$\int_{-\infty}^{\infty} e^{i\lambda t^2/2}\, dt = (\lambda/2\pi)^{-1/2} e^{\pi i/4}$$

and

$$|\det \nabla_y^2 \phi(x, y(x))| = |\det D_y F(x, 0)|^{-2}.$$

To estimate the second term, we let χ be a smooth compactly supported cutoff function with $\chi(|y|) = 1$ for all $y \in \operatorname{supp}_y a$. Then,

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{i\lambda Q(y)} r(\lambda; x, y)\, dy \\ &\qquad = \int_{\mathbb{R}^n} e^{i\lambda Q(y)} \chi(|y|) r(\lambda; x, y)\, dy + \int_{\mathbb{R}^n} e^{i\lambda Q(y)} (1 - \chi(|y|)) r(\lambda; x, y)\, dy \\ &\qquad = R_1(\lambda; x) + R_2(\lambda; x), \end{aligned}$$

respectively. Since $r(\lambda; x, y)$ vanishes for $y = 0$,

$$|\partial_\lambda^j \partial_x^\alpha R_1(\lambda; x)| \le C_{j,\alpha} \lambda^{-n/2-1-j}$$

by [8, Lemma 1.1.6] applied to the x-derivatives of R_1. Finally,

$$R_2(\lambda; x) = c \int_{\mathbb{R}^n} e^{i\lambda Q(y)} (1 - \chi(|y|))\, dy = O(\lambda^{-N})$$

by integration by parts. □

References

[1] A. Banyaga and D. E. Hurtubise, *A proof of the Morse-Bott Lemma*, Expo. Math., Volume 22, Issue 4, 2004, 365-373

[2] X. Chen and C. D. Sogge, *On integrals of eigenfunctions over geodesics*, Proceedings of the American Mathematical Society 143.1 (2015), 151-161.

[3] M. do Carmo, *Riemannian geometry*, Birkhäuser, Basel, Boston, Berlin, 1992.

[4] A. Good, *Local analysis of Selberg's trace formula*, Lecture Notes in Mathematics, vol. 1040, Springer-Verlag, Berlin, 1983.

[5] D. A. Hejhal, *Sur certaines séries de Dirichlet associées aux géodésiques fermées d'une surface de Riemann compacte*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 8, 273-276.

[6] A. Reznikov, *A uniform bound for geodesic periods of eigenfunctions on hyperbolic surfaces*, Forum Math. **27** (2015), no. 3, 1569–1590.

[7] R. Schoen and S. Yau, *Lectures on differential geometry*, International Press, 1994.

[8] C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics **105**, Cambridge University Press, Cambridge, 1993.

[9] C. D. Sogge, *Hangzhou lectures on eigenfunctions of the Laplacian*, to appear in Annals of Math Studies, Princeton Univ. Press.

[10] C. D. Sogge, J. A. Toth, S. Zelditch, *About the blowup of quasimodes on Riemannian manifolds*, J. Geom. Anal. 21 (2011), no. 1, 150-173.

[11] C. D. Sogge, Y. Xi, C. Zhang, *Geodesic period integrals of eigenfunctions on Riemann surfaces and the Gauss Bonnet theorem*, arXiv:1604.03189.

[12] C. D. Sogge and S. Zelditch, *On eigenfunction restriction estimates and L^4-bounds for compact surfaces with nonpositive curvature*, Advances in analysis: the legacy of Elias M. Stein, 447-461, Princeton Math. Ser., 50, Princeton Univ. Press, Princeton, NJ, 2014.

[13] C. D. Sogge and S. Zelditch, *Riemannian manifolds with maximal eigenfuntion growth*, Duke Math. J. 114 (2002), no. 3, 387-437.

[14] E. L. Wyman, *Integrals of eigenfunctions over curves in surfaces of nonpositive curvature*, arXiv:1702.03552.

[15] S. Zelditch, *Kuznecov sum formulae and Szegő limit formulae on manifolds*, Comm. Partial Differential Equations **17** (1992), no. 1-2, 221–260.

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218