THE KORTEWEG-DE VRIES EQUATION ON A METRIC STAR GRAPH

MÁRCIO CAVALCANTE

Centro de Modelamiento Matemático, Universidad de Chile
Beauchef 851, Edificio Norte, Piso 7,
Santiago-Chile

ABSTRACT. We prove local well-posedness for the Cauchy problem associated to Korteweg-de Vries equation on a metric star graph with three semi-infinite bonds given by one negative half-line and two positives half-lines attached to a common vertex, for two classes of boundary conditions. The results are obtained in the low regularity setting by using the approach given by Colliander, Kenig (2002) and Holmer (2006).

Keywords: local well-posedness; Korteweg-de Vries equation; metric star graph; low regularity.

1. INTRODUCTION

Partial differential equations (PDE) on metric graphs, as known networks or quantum graphs, arise naturally in many topics of physics such as acoustics, optics, condensed matter and polymer physics, and, more recently, in connection with biological trees. We refer to [4] and [26] for further information and bibliography. Earlier, the linear Schrödinger equation on a metric graph was subject of extensive research due to its applications in quantum nanotechnologies, chemistry and mesoscopic physics (see [3], [4] and references therein). Studies of the nonlinear Schrödinger equation on graphs have started appearing recently. In particular, existence and stability of standing waves for nonlinear Schrödinger equation on a star graph with a power nonlinearity $|u|^{p-1}u$ have been studied extensively, see [1], [9] and [29] for a brief survey of the topic. Recently, Ardila [2] has obtained existence and stability of standing waves for the logarithmic Schrödinger equation on star graphs. Caudrelier [10] presented a method to solve the open problem of formulating the inverse scattering method for an integrable PDE on a star graph and the nonlinear Schrödinger equation was chosen to illustrate the method.

Another nonlinear dispersive equation, the Benjamin-Bona-Mahony (BBM) equation, is treated in [5, 28]. More precisely, Bona and Cascaval [5] obtained local well-posedness in Sobolev space H^1 and Mugnolo and Rault [28] obtained existence of travelling waves for the BBM equation on star graphs.

The well-known Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + u_x u = 0 \tag{1.1}$$

was first derived by Korteweg and de Vries [25] in 1895 as a model for long waves propagating on a shallow water surface. It is now commonly accepted as a mathematical model for the unidirectional propagation of small-amplitude long waves in nonlinear dispersive systems. In particular, the KdV equation is not only used to serve as a model to study surface water waves. In fact, recently the KdV equation has been used to serve as a model to study blood pressure waves in large arteries. In this way, for example, in [13] a new computer model for systolic pulse waves within the cardiovascular system based on the KdV equation is presented. In [15] some particular solutions of the KdV equation, the 2 and 3-soliton solutions, seem to be good candidates to match the observed pressure pulse waves.

In the mathematic context the Cauchy problem for the KdV posed on the real axis, torus, on the half-lines and on a finite interval have been well studied in the last years, we refer as a sample [6, 7, 14, 17, 18, 21, 20, 23, 24], but we point out that there are many further references in the works just cited.

1.1. **Previous works for the linearized KdV equation on star graphs.** Studies for the linearized Korteweg-de Vries equation on star graphs have started appearing recently. In [30],[31] and [32] was studied existence and uniqueness of solutions for the linearized KdV equation on metric star graphs by using potential theory, where the solutions were obtained in the class of Schwartz and in Sobolev classes with high order. Very recently in [27] was given a characterization of all boundary conditions under which the Airy-type evolution equation $u_t = \alpha u_{xxx} + \beta u_x$, for $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$ on star graphs given by $E = E_- \cup E_+$, where E_+ and E_- are finite or countable collections of semi-infinite edges parametrized by $(-\infty, 0)$ or $(0, +\infty)$ respectively and the half-lines are connected at a vertex v, generates a contraction semigroup.

As far as we know, the study of the nonlinear Korteweg-de Vries equation in star graphs is unknown. Here we extend the treatment of the KdV equation given in [21] from intervals and half-lines to a star graph.

1.2. **Formulation of the problem.** In this work, motivated by the work [31] we consider the KdV equation on a star graph $\mathcal{Y} = (-\infty, 0) \cup (0, +\infty) \cup (0, +\infty)$ with three semi-infinite bonds given by one negative half-line and two positives half-lines attached to a common vertex, also known as $\mathcal{Y}$-junction. More precisely,

$$\begin{cases} u_t + u_{xxx} + u_x u = 0, & (x,t) \in (-\infty, 0) \times (0,T), \\ v_t + v_{xxx} + v_x v = 0, & (x,t) \in (0, +\infty) \times (0,T), \\ w_t + w_{xxx} + w_x w = 0, & (x,t) \in (0, +\infty) \times (0,T). \end{cases} \tag{1.2}$$

We consider the initial conditions given by

$$u(x,0) = u_0(x), \ v(x,0) = v_0(x) \text{ and } w(x,0) = w_0(x), \tag{1.3}$$

where

$$(u_0, v_0, w_0) \in H^s(\mathbb{R}^-) \times H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+) := H^s(\mathcal{Y}). \tag{1.4}$$

Our goal in studying Cauchy problem (1.2)-(1.3) is to obtain low regularity results in Sobolev spaces.

1.3. **Choices of boundary conditions.** Determining the number of boundary conditions necessary for a well-posed problem is a nontrivial issue. As far we know, it's not at all clear which boundary conditions should be appropriates for physical applications, and therefore here we will consider two classes of boundary conditions that are coherent with uniqueness calculations for smooth decaying solutions of a linear version of the Cauchy problem. In this sense, suppose that $(u(x,t), v(x,t), w(x,t))$ is a smooth decaying solution of a linear version of (1.2), i.e.

$$\begin{cases} u_t + u_{xxx} = 0, & (x,t) \in (-\infty, 0) \times (0,T), \\ v_t + v_{xxx} = 0, & (x,t) \in (0, +\infty) \times (0,T), \\ w_t + w_{xxx} = 0, & (x,t) \in (0, +\infty) \times (0,T), \end{cases} \tag{1.5}$$

with homogeneous initial condition $(u_0, v_0, w_0) = (0,0,0)$. Multiplying the equations in (1.5) by u, v and w respectively, and integrating by parts we obtain

$$\int_{-\infty}^{0} u^2(x,T)dx + \int_{0}^{+\infty} v^2(x,T)dx + \int_{0}^{+\infty} w^2(x,T)dx$$
$$= \int_{0}^{T} (u_x^2(0,t) - v_x^2(0,t) - w_x^2(0,t))dt \tag{1.6}$$
$$- 2\int_{0}^{T} u(0,t)u_{xx}(0,t)dt + 2\int_{0}^{T} v(0,t)v_{xx}(0,t)dt + 2\int_{0}^{T} w(0,t)w_{xx}(0,t)dt.$$

By analyzing the expression (1.6), we are interesting in considering boundary conditions for Cauchy problem (1.2)-(1.3) that imply that the right hand side of (1.6) would have a negative sign.

In this sense, if we consider the particular following boundary conditions

$$u(0,t) = \alpha_2 v(0,t) = \alpha_3 w(0,t),\ t \in (0,T), \tag{1.7}$$

$$u_x(0,t) = \beta_2 v_x(0,t) + \beta_3 w_x(0,t),\ t \in (0,T) \tag{1.8}$$

and

$$u_{xx}(0,t) = \frac{1}{\alpha_2}v_{xx}(0,t) + \frac{1}{\alpha_3}w_{xx}(0,t),\ t \in (0,T), \tag{1.9}$$

where α_2, α_3, β_2 and β_3 are real constants satisfying $3\beta_i^2 \le 1$ for $i = 2,3$ we have that

$$\int_{-\infty}^{0} u^2(x,T)dx + \int_{0}^{+\infty} v^2(x,T)dx + \int_{0}^{+\infty} w^2(x,T)dx = 0.$$

It follows that $u(x,T) = v(x,T) = w(x,T) = 0$, that implies the uniqueness argument.

Remark 1.1. *The boundary conditions* (1.7)-(1.9) *also were considered by Sobirov, Akhmedov and Uecker* [31] *to treat a linearized version of KdV on a $\mathcal{Y}$-junction.*

Remark 1.2. *The calculation above is formal, but in the context of half-line this is very efficient to describe the boundary conditions for initial boundary value problems (IBVPs). In this sense, Holmer [21] describes boundary conditions for IBVPs associated to the KdV equation on the positive and negative half-lines that imply a result of well-posedness. For other nice discussion about the boundary condition for KdV on half-lines, based in the behavior of characteristic curves we refer the reader to [16].*

In the same way, if we consider the particular following boundary conditions

$$u(0,t) = \frac{1}{\alpha_2} v(0,t) + \frac{1}{\alpha_3} w(0,t), \ t \in (0,T), \tag{1.10}$$

$$u_x(0,t) = \beta_2 v_x(0,t) + \beta_3 w_x(0,t), \ t \in (0,T) \tag{1.11}$$

and

$$u_{xx}(0,t) = \alpha_2 v_{xx}(0,t) = \alpha_3 w_{xx}(0,t), \ t \in (0,T), \tag{1.12}$$

where α_2, α_3, β_2 and β_3 are real constants satisfying $3\beta_i^2 \leq 1$ for $i = 2,3$ also holds the uniqueness argument.

This motivate us to define the following two classes of boundary conditions which involves the particular conditions (1.7)-(1.9) and (1.10)-(1.12).

Definition 1.1. *Given a_2, a_3, b_2, b_3, c_2 and c_3 real constants, we call type 1 boundary conditions for the Cauchy problem (1.2)-(1.3) if these satisfies the following boundary conditions at the vertex:*

$$u(0,t) = a_2 v(0,t) = a_3 w(0,t), \ t \in (0,T), \tag{1.13}$$

$$u_x(0,t) = b_2 v_x(0,t) + b_3 w_x(0,t), \ t \in (0,T) \tag{1.14}$$

and

$$u_{xx}(0,t) = c_2 v_{xx}(0,t) + c_3 w_{xx}(0,t), \ t \in (0,T). \tag{1.15}$$

Definition 1.2. *Given a_2, a_3, b_2, b_3, c_2 and c_3 real constants, we call type 2 boundary conditions for the Cauchy problem (1.2)-(1.3) if these satisfies*

$$u(0,t) = a_2 v(0,t) + a_3 w(0,t), \ t \in (0,T), \tag{1.16}$$

$$u_x(0,t) = b_2 v_x(0,t) + b_3 w_x(0,t), \ t \in (0,T) \tag{1.17}$$

and

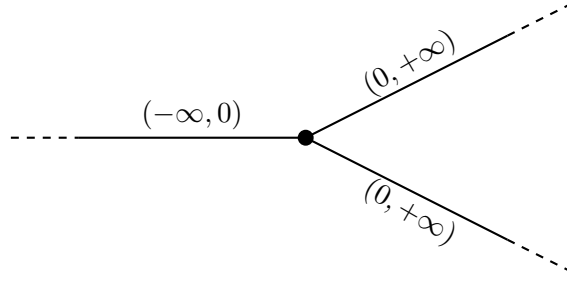$$u_{xx}(0,t) = c_2 v_{xx}(0,t) = c_3 w_{xx}(0,t), \ t \in (0,T). \tag{1.18}$$

It is well-known that the trace operator $\gamma_0 : u(x) \mapsto u(0)$ is well-defined on $H^s(\mathbb{R}^+)$ for $s > \frac{1}{2}$. Hence, on the case $s > \frac{1}{2}$ we will assume the following additional condition

$$u_0(0) = a_2 v_0(0) = a_3 w_0(0) \tag{1.19}$$

for initial data for the Cauchy problem (1.2)-(1.3) with type 1 boundary conditions and

$$u_0(0) = a_2 v_0(0) + a_3 w_0(0) \tag{1.20}$$

for type 2 boundary conditions.

FIGURE 1. A star graph with three bonds ($\mathcal{Y}$-junction)

1.4. **Principal results.** Now we state our main result.

Theorem 1.1. *Let $-\frac{1}{2} < s < \frac{3}{2}$, with $s \neq \frac{1}{2}$. Assume that u_0, v_0 and w_0 satisfying (1.4).*
  (i) *For a fixed s suppose that there exists $\lambda_i(s)$ satisfying*

$$\max\{s - 1, 0\} < \lambda_i(s) < \min\left\{s + \frac{1}{2}, \frac{1}{2}\right\} \ \text{for } i = 1, 2, 3, 4, \tag{1.21}$$

such that the following matrix

$$\mathbf{M}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) := \begin{bmatrix} 2\sin\left(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6}\right) & -a_2 e^{i\pi\lambda_3} & 0 & 2\sin\left(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6}\right) \\ 2\sin\left(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6}\right) & 0 & -a_3 e^{i\pi\lambda_4} & 2\sin\left(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6}\right) \\ 2\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{6}\right) & -b_2 e^{i\pi(\lambda_3-1)} & -b_3 e^{i\pi(\lambda_4-1)} & 2\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{6}\right) \\ 2\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{2}\right) & -c_2 e^{i\pi(\lambda_3-2)} & -c_3 e^{i\pi(\lambda_4-2)} & 2\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{2}\right) \end{bmatrix} \tag{1.22}$$

is invertible. Then there exists a positive time $T > 0$ and a distributional solution (u, v, w) in the space $C([0, T] : H^s(\mathcal{Y}))$, for the Cauchy problem (1.2)-(1.3) with type 1 boundary condition, satisfying the additional compatibility condition (1.19) on the case $\frac{1}{2} < s < \frac{3}{2}$. Furthermore the data-to-solution map $(u_0, v_0, w_0) \mapsto (u, v, w)$ is locally Lipschitz continuous from $H^s(\mathcal{Y})$ to $C([0, T]; H^s(\mathcal{Y}))$.
  (ii) *For a fixed s suppose that there exists $\lambda_i(s)$ with*

$$\max\{s - 1, 0\} < \lambda_i(s) < \min\left\{s + \frac{1}{2}, \frac{1}{2}\right\} \ \text{for } i = 1, 2, 3, 4,$$

such that the following matrix

$$\mathbf{M}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) := \begin{bmatrix} 2\sin\left(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6}\right) & -a_2 e^{i\pi\lambda_3} & -a_3 e^{i\pi\lambda_4} & 2\sin\left(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6}\right) \\ 2\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{6}\right) & -b_2 e^{i\pi(\lambda_3-1)} & -b_3 e^{i\pi(\lambda_4-1)} & 2\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{6}\right) \\ 2\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{2}\right) & -c_2 e^{i\pi(\lambda_3-2)} & 0 & 2\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{2}\right) \\ 2\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{2}\right) & 0 & -c_3 e^{i\pi(\lambda_4-2)} & 2\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{2}\right) \end{bmatrix} \tag{1.23}$$

is invertible. Then there exists a positive time $T > 0$ and a distributional solution (u, v, w) in the space $C([0, T] : H^s(\mathcal{Y}))$, for the Cauchy problem (1.2)-(1.3) with type 2 boundary condition, satisfying the additional compatibility condition (1.20) on the

case $\frac{1}{2} < s < \frac{3}{2}$. *Furthermore the data-to-solution map* $(u_0, v_0, w_0) \mapsto (u, v, w)$ *is locally Lipschitz continuous from* $H^s(\mathcal{Y})$ *to* $C([0, T]; H^s(\mathcal{Y}))$.

As the consequence of Theorem 1.1, we can obtain the following result for the special boundary conditions (1.7)-(1.9), and (1.10)-(1.12), that is appropriate for our formal uniqueness calculations.

Corollary 1.1. *Let* $-\frac{1}{2} < s < \frac{3}{2}$ *with* $s \neq \frac{1}{2}$ *and* α_2, α_3, β_2, $\beta_3 \in \mathbb{R}$ *satisfying* $\frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2} \neq -1$. *Assume that* u_0, v_0 *and* w_0 *satisfying* (1.4). *Then there exists a positive time* $T > 0$ *and a distributional solution* $(u, v, w) \in C([0, T]; H^s(\mathcal{Y}))$ *for the Cauchy problem* (1.2)-(1.3) *with boundary condition* (1.7)-(1.9), *and the initial conditions satisfying additional conditions* (1.19) *for* $\frac{1}{2} < s < \frac{3}{2}$. *Furthermore the data-to-solution map* $(u_0, v_0, w_0) \mapsto (u, v, w)$ *is locally Lipschitz continuous from* $H^s(\mathcal{Y})$ *to* $C([0, T]; H^s(\mathcal{Y}))$.

Corollary 1.2. *The same result of the Corollary 1.1 is valid for the Cauchy problem* (1.2)-(1.3) *with boundary conditions* (1.10)-(1.12), *and the initial conditions satisfying additional condition* (1.20) *for* $\frac{1}{2} < s < \frac{3}{2}$.

The approach used to proof the main result is based on the arguments developed in [11], [12], [14], [19] and [21]. The main idea to prove Theorem 1.1 is the construction of an auxiliary forced Cauchy problem in all $\mathbb{R}$, analogous to the (1.2); more precisely:

$$\begin{cases} u_t + u_{xxx} + u_x u = \mathcal{T}_1(x) h_1(t) + \mathcal{T}_2(x) h_2(t), & (x, t) \in \mathbb{R} \times (0, T), \\ v_t + v_{xxx} + v_x v = \mathcal{T}_3(x) h_3(t), & (x, t) \in \mathbb{R} \times (0, T), \\ w_t + w_{xxx} + w_x w = \mathcal{T}_4(x) h_4(t), & (x, t) \in \mathbb{R} \times (0, T), \\ u(x, 0) = \widetilde{u}_0(x), \ v(x, 0) = \widetilde{v}_0(x) \ w(x, 0) = \widetilde{w}_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.24}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are distributions supported in a positive half-line $\mathbb{R}^+$, $\mathcal{T}_3$ and $\mathcal{T}_4$ are distributions supported in the negative half-line $\mathbb{R}^-$, $\widetilde{u}_0$, $\widetilde{v}_0$ and $\widetilde{w}_0$ are nice extensions of u_0, v_0 and w_0 in $\mathbb{R}$. The boundary forcing functions h_1, h_2, h_3 and h_4 are selected to ensure that the vertex conditions are satisfied.

The solution of forced Cauchy problem (1.24) satisfying the vertex conditions is constructed using the classical restricted norm method of Bourgain (see [8] and [23]) and the inversion of a Riemann-Liouville fractional integration operator.

Following [21] we consider the distributions $\mathcal{T}_1 = \frac{x_-^{\lambda_1 - 1}}{\Gamma(\lambda_1)}$, $\mathcal{T}_2 = \frac{x_-^{\lambda_2 - 1}}{\Gamma(\lambda_2)}$, $\mathcal{T}_3 = \frac{x_+^{\lambda_3 - 1}}{\Gamma(\lambda_3)}$ and $\mathcal{T}_4 = \frac{x_+^{\lambda_4 - 1}}{\Gamma(\lambda_4)}$, where

$$\left\langle \frac{x_+^{\lambda - 1}}{\Gamma(\lambda)}, \phi \right\rangle = \int_0^{+\infty} \frac{x^{\lambda - 1}}{\Gamma(\lambda)} \phi(x) dx, \text{ for Re } \lambda > 0. \tag{1.25}$$

For other values of λ we can define $\frac{x_+^{\lambda - 1}}{\Gamma(\lambda)} = \frac{d^k}{dx} \frac{x^{\lambda + k - 1}}{\Gamma(\lambda + k)}$, for any integer k satisfying $k + \text{Re}\lambda > 0$. Finally, we define $\frac{x_-^{\lambda - 1}}{\Gamma(\lambda)} = e^{i\pi\lambda} \frac{(-x)_+^{\lambda - 1}}{\Gamma(\lambda)}$.

The crucial point here are the appropriate choices of the parameters λ_i and the functions h_i, for $i = 1, 2, 3, 4$, that will depend on the regularity index s.

Remark 1.3. *We believe that the same approach used to prove Theorem 1.1 can provide similar results for the KdV equation in other star graphs and possibly for other nonlinear dispersive equations. For example, a treatment for the nonlinear Schröndiger equation on a star graphs can be done using the approach developed by Holmer [19] and Cavalcante [11].*

Actually we have uniqueness in the sense of Kato (see [22]) only for a reformulation of the Cauchy problem (1.2)-(1.3) for 1 and 2 types boundary conditions as an integral equation posed in $\mathbb{R}$, such that solves (1.2). As there are many ways to transform the Cauchy problem into an integral equation, we do not have uniqueness in the strong sense.

We note that our best regularity attained is $s = -\frac{1}{2}^+$ (see Remark 3.2) and well-posedness for less regular initial data exists in the literature for the KdV equation in other domains. For example; in full real line the best regularity obtained was $s = -\frac{3}{4}$ given independently by Guo [18] and Kishimoto [24], in context of half-lines the best regularity was $s = -\frac{3}{4}^+$ that was obtained independently by Holmer [21] and Bona, Sun and Zhang [6] in 2006.

1.5. **Organization of this paper.** This paper is organized as follows: in the next section, we discuss some notation, introduce function spaces and recall some needed properties of these function spaces, and review the definition and basic properties of the Riemann-Liouville fractional integral. Sections 3, are devoted to the needed estimates for the linear group and the Duhamel boundary forcing classes operators. In Section 4 we state the estimates for the classical Duhamel inhomogeneous solution operator. Section 5 is devoted to prove Theorem 1.1. Finally, Sections 6 and 7 are devoted to proof of Corollaries 1.1 and 1.2, respectively .

2. **Preliminaries**

Here we introduce some notations, function spaces and the operator Riemann-Liouville fractional integral.

2.1. **Notations.** For $\phi = \phi(x) \in S(\mathbb{R})$, $\hat{\phi}(\xi) = \int e^{-i\xi x}\phi(x)dx$ denotes the Fourier transform of ϕ. For $u = u(x,t) \in S(\mathbb{R}^2)$, $\hat{u} = \hat{u}(\xi,\tau) = \int e^{-i(\xi x + \tau t)}u(x,t)dxdt$ denotes its space-time Fourier transform, $\mathcal{F}_x u(\xi,t)$ its space Fourier transform and $\mathcal{F}_t u(x,\tau)$ its time Fourier transform.

For any real number ξ we put $\langle \xi \rangle := 1 + |\xi|$ and $f(\xi,\tau) \lesssim g(\xi,\tau)$ means that there is a constant C such that $f(\xi,\tau) \leq Cg(\xi,\tau)$ for all $(\xi,\tau) \in \mathbb{R}^2$. The characteristic function of an arbitrary set A is denoted by χ_A. Throughout the paper, we fix a cutoff function $\psi \in C_0^\infty(\mathbb{R})$ such that

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| \geq 2 \end{cases}$$

and $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

2.2. **Function Spaces.** For $s \geq 0$ we say that $\phi \in H^s(\mathbb{R}^+)$ if exists $\tilde{\phi} \in H^s(\mathbb{R})$ such that $\phi = \tilde{\phi}|_{\mathbb{R}+}$. In this case we set $\|\phi\|_{H^s(\mathbb{R}^+)} := \inf_{\tilde{\phi}} \|\tilde{\phi}\|_{H^s(\mathbb{R})}$. For $s \geq 0$ define

$$H_0^s(\mathbb{R}^+) = \left\{ \phi \in H^s(\mathbb{R}^+); \ \mathrm{supp}(\phi) \subset [0, +\infty) \right\}.$$

For $s < 0$, define $H^s(\mathbb{R}^+)$ and $H_0^s(\mathbb{R}^+)$ as the dual space of $H_0^{-s}(\mathbb{R}^+)$ and $H^{-s}(\mathbb{R}^+)$, respectively. We define the usual Sobolev spaces for functions defined on the junction $\mathcal{Y}$ as

$$H^s(\mathcal{Y}) = H^s(\mathbb{R}^-) \times H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+). \tag{2.1}$$

Also define

$$C_0^\infty(\mathbb{R}^+) = \left\{ \phi \in C^\infty(\mathbb{R}); \ \mathrm{supp}(\phi) \subset [0, +\infty) \right\}$$

and $C_{0,c}^\infty(\mathbb{R}^+)$ as those members of $C_0^\infty(\mathbb{R}^+)$ with compact support. We recall that $C_{0,c}^\infty(\mathbb{R}^+)$ is dense in $H_0^s(\mathbb{R}^+)$ for all $s \in \mathbb{R}$. A definition for $H^s(\mathbb{R}^-)$ and $H_0^s(\mathbb{R}^-)$ can be given analogous to that for $H^s(\mathbb{R}^+)$ and $H_0^s(\mathbb{R}^+)$.

The following results summarize useful properties of the Sobolev spaces on the half-line. For the proofs we refer the reader [14].

Lemma 2.1. *For all $f \in H^s(\mathbb{R})$ with $-\frac{1}{2} < s < \frac{1}{2}$ we have*

$$\|\chi_{(0,+\infty)} f\|_{H^s(\mathbb{R})} \leq c\|f\|_{H^s(\mathbb{R})}.$$

Lemma 2.2. *If $\frac{1}{2} < s < \frac{3}{2}$ the following statements are valid:*

(a) $H_0^s(\mathbb{R}^+) = \left\{ f \in H^s(\mathbb{R}^+); f(0) = 0 \right\}$,

(b) *If $f \in H^s(\mathbb{R}^+)$ with $f(0) = 0$, then $\|\chi_{(0,+\infty)} f\|_{H_0^s(\mathbb{R}^+)} \leq c\|f\|_{H^s(\mathbb{R}^+)}$.*

Lemma 2.3. *If $f \in H_0^s(\mathbb{R}^+)$ with $s \in \mathbb{R}$, we then have*

$$\|\psi f\|_{H_0^s(\mathbb{R}^+)} \leq c\|f\|_{H_0^s(\mathbb{R}^+)}.$$

Remark 2.1. *In Lemmas 2.1, 2.2 and 2.3 all the constants c only depend on s and ψ.*

We denote by $X^{s,b}$ the so called Bourgain spaces associated to linear KdV equation; more precisely, $X^{s,b}$ is the completion of $S'(\mathbb{R}^2)$ with respect to the norm

$$\|w\|_{X^{s,b}(\phi)} = \|\langle\xi\rangle^s \langle\tau - \xi^3\rangle^b \hat{w}(\xi, \tau)\|_{L_\tau^2 L_\xi^2}.$$

To obtain our results we also need define the following auxiliary modified Bougain spaces of [21]. Let $U^{s,b}$ and V^α the completion of $S'(\mathbb{R}^2)$ with respect to the norms:

$$\|w\|_{U^{s,b}} = \left(\int \int \langle\tau\rangle^{2s/3} \langle\tau - \xi^3\rangle^{2b} |\hat{w}(\xi, \tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}$$

and

$$\|w\|_{V^\alpha} = \left(\int \int \langle\tau\rangle^{2\alpha} |\hat{w}(\xi, \tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}.$$

Next nonlinear estimates, in the context of the KdV equation, for $b < \frac{1}{2}$, was derived by Holmer in [21].

Lemma 2.4. (a) *Given $s > -\frac{3}{4}$, there exists $b = b(s) < \frac{1}{2}$ such that for all $\alpha > \frac{1}{2}$ we have*

$$\left\|\partial_x(v_1 v_2)\right\|_{X^{s,-b}} \leq c\|v_1\|_{X^{s,b} \cap V^\alpha}\|v_2\|_{X^{s,b} \cap V^\alpha}.$$

(b) *Given $-\frac{3}{4} < s < 3$, there exists $b = b(s) < \frac{1}{2}$ such that for all $\alpha > \frac{1}{2}$ we have*

$$\left\|\partial_x(v_1 v_2)\right\|_{X^{s,-b}} \leq c\|v_1\|_{X^{s,b} \cap V^\alpha}\|v_2\|_{X^{s,b} \cap V^\alpha}.$$

2.3. **Riemman-Liouville fractional integral.** The tempered distribution $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ is defined as a locally integrable function for Re $\alpha > 0$, that is

$$\left\langle \frac{t_+^{\alpha-1}}{\Gamma(\alpha)},\ f \right\rangle := \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} f(t) dt.$$

For Re $\alpha > 0$, integration by parts implies that

$$\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} = \partial_t^k \left(\frac{t_+^{\alpha+k-1}}{\Gamma(\alpha+k)} \right)$$

for all $k \in \mathbb{N}$. This expression allows to extend the definition, in the sense of distributions, of $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ to all $\alpha \in \mathbb{C}$.

If $f \in C_0^\infty(\mathbb{R}^+)$, we define

$$\mathcal{I}_\alpha f = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} * f.$$

Thus, for Re $\alpha > 0$,

$$\mathcal{I}_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$$

and notice that

$$\mathcal{I}_0 f = f, \quad \mathcal{I}_1 f(t) = \int_0^t f(s) ds, \quad \mathcal{I}_{-1} f = f' \quad \text{and} \quad \mathcal{I}_\alpha \mathcal{I}_\beta = \mathcal{I}_{\alpha+\beta}.$$

The following results state important properties of the Riemman-Liouville fractional integral operator. The proof of them can be found in [21].

Lemma 2.5. *If $f \in C_0^\infty(\mathbb{R}^+)$, then $\mathcal{I}_\alpha f \in C_0^\infty(\mathbb{R}^+)$ for all $\alpha \in \mathbb{C}$.*

Lemma 2.6. *If $0 \leq \alpha < \infty$, $s \in \mathbb{R}$ and $\varphi \in C_0^\infty(\mathbb{R})$, then we have*

$$\|\mathcal{I}_{-\alpha} h\|_{H_0^s(\mathbb{R}^+)} \leq c\|h\|_{H_0^{s+\alpha}(\mathbb{R}^+)} \tag{2.2}$$

and

$$\|\varphi \mathcal{I}_\alpha h\|_{H_0^s(\mathbb{R}^+)} \leq c_\varphi \|h\|_{H_0^{s-\alpha}(\mathbb{R}^+)}. \tag{2.3}$$

3. The linear versions

3.1. Linear group associated to the KdV equation.

The linear unitary group $e^{-t\partial_x^3}$: $S'(\mathbb{R}) \to S'(\mathbb{R})$ associated to the linear KdV equation is defined by

$$e^{-t\partial_x^3}\phi(x) = \left(e^{it\xi^3}\widehat{\phi}(\xi)\right)^{\vee}(x),$$

that satisfies

$$\begin{cases} (\partial_t + \partial_x^3)e^{-t\partial_x^3}\phi(x,t) = 0 & \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \\ e^{-t\partial_x^3}(x,0) = \phi(x) & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{3.1}$$

The next estimates were proven in [21].

Lemma 3.1. *Let $s \in \mathbb{R}$ and $0 < b < 1$. If $\phi \in H^s(\mathbb{R})$, then we have*

(a) *(**space traces**)* $\|e^{-t\partial_x^3}\phi(x)\|_{C\left(\mathbb{R}_t; H^s(\mathbb{R}_x)\right)} \leq c\|\phi\|_{H^s(\mathbb{R})}$,

(b) *(**time traces**)* $\|\psi(t)e^{-t\partial_x^3}\phi(x)\|_{C\left(\mathbb{R}_x; H^{(s+1)/3}(\mathbb{R}_t)\right)} \leq c\|\phi\|_{H^s(\mathbb{R})}$,

(c) *(**derivative time traces**)* $\|\psi(t)\partial_x e^{-t\partial_x^3}\phi(x)\|_{C\left(\mathbb{R}_x; H^{s/3}(\mathbb{R}_t)\right)} \leq c\|\phi\|_{H^s(\mathbb{R})}$,

(d) *(**second derivative time traces**)* $\|\psi(t)\partial_x^2 e^{-t\partial_x^3}\phi(x)\|_{C\left(\mathbb{R}_x; H^{(s-1)/3}(\mathbb{R}_t)\right)} \leq c\|\phi\|_{H^s(\mathbb{R})}$,

(e) *(**Bourgain spaces**)* $\|\psi(t)e^{-t\partial_x^3}\phi(x)\|_{X^{s,b} \cap V^\alpha} \leq c\|\phi\|_{H^s(\mathbb{R})}$.

Remark 3.1. *The spaces V^α introduced in [21] give us useful auxiliary norms of the classical Bourgain spaces in order to validate the nonlinear estimates associated to the KdV equation for $b < \frac{1}{2}$.*

3.2. The Duhamel boundary forcing operator associated to the linear KdV equation.

Now we give the properties of the Duhamel boundary forcing operator introduced in [21], that is

$$\mathcal{V}g(x,t) = 3\int_0^t e^{-(t-t')\partial_x^3}\delta_0(x)\mathcal{I}_{-2/3}g(t')dt'$$
$$= 3\int_0^t A\left(\frac{x}{(t-t')^{1/3}}\right)\frac{\mathcal{I}_{-2/3}g(t')}{(t-t')^{1/3}}dt', \tag{3.2}$$

defined for all $g \in C_0^\infty(\mathbb{R}^+)$ and A denotes the Airy function

$$A(x) = \frac{1}{2\pi}\int_\xi e^{ix\xi}e^{i\xi^3}d\xi.$$

From definition of $\mathcal{V}$ it follows that

$$\begin{cases} (\partial_t + \partial_x^3)\mathcal{V}g(x,t) = 3\delta_0(x)\mathcal{I}_{-\frac{2}{3}}g(t) & \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \\ \mathcal{V}g(x,0) = 0 & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{3.3}$$

The proof of the results exhibited in this section was shown in [21].

Lemma 3.2. *Let $g \in C_0^\infty(\mathbb{R}^+)$ and consider a fixed time $t \in [0,1]$. Then,*

(a) *the functions $\mathcal{V}g(\cdot, t)$ and $\partial_x \mathcal{V}g(\cdot, t)$ are continuous in x for all $x \in \mathbb{R}$. Moreover, they satisfy the spatial decay bounds*

$$|\mathcal{V}g(x,t)| + |\partial_x \mathcal{V}g(x,t)| \le c_k \|g\|_{H^{k+1}} \langle x \rangle^{-k} \quad \text{for all } k \ge 0;$$

(b) *the function $\partial_x^2 \mathcal{V}g(x,t)$ is continuous in x for all $x \ne 0$ and has a step discontinuity of size $3\mathcal{I}_{\frac{2}{3}}g(t)$ at $x = 0$. Also, $\partial_x^2 \mathcal{V}g(x,t)$ satisfies the spatial decay bounds*

$$|\partial_x^2 \mathcal{V}g(x,t)| \le c_k \|f\|_{H^{k+2}} \langle x \rangle^{-k} \quad \text{for all } k \ge 0.$$

Since $A(0) = \frac{1}{3\Gamma\left(\frac{2}{3}\right)}$ from (3.2) we have that $\mathcal{V}g(0,t) = g(t)$.

For the convenience to the reader, we present here an application of the operator $\mathcal{V}$ to solve a linear version of the IBVP associated to the KdV equation on the half-line, given in [14]. Set

$$v(x,t) = e^{-t\partial_x^3}\phi(x) + \mathcal{V}\big(g - e^{-\cdot\partial_x^3}\phi\big|_{x=0}\big)(x,t), \tag{3.4}$$

where $g \in C_0^\infty(\mathbb{R}^+)$ and $\phi \in S(\mathbb{R})$.

Then from (3.1) and (3.3) we see that v solves the linear problem

$$\begin{cases} (\partial_t + \partial_x^3)v(x,t) = 0 & \text{for} \quad (x,t) \in \mathbb{R}^* \times \mathbb{R}, \\ v(x,0) = \phi(x) & \text{for} \quad x \in \mathbb{R}, \\ v(0,t) = g(t) & \text{for} \quad t \in (0,+\infty), \end{cases} \tag{3.5}$$

in the sense of distributions, and then this would suffice to solve the IBVP on the right half-line associated to linear KdV equation.

Now, we consider the second boundary forcing operator associated to the linear KdV equation:

$$\mathcal{V}^{-1}g(x,t) = \partial_x \mathcal{V} \mathcal{I}_{\frac{1}{3}}g(x,t) = 3\int_0^t A'\left(\frac{x}{(t-t')^{1/3}}\right)\frac{\mathcal{I}_{-\frac{1}{3}}g(t')}{(t-t')^{2/3}}dt'. \tag{3.6}$$

From Lemma 3.2, for all $g \in C_0^\infty(\mathbb{R}^+)$ the function $\mathcal{V}^{-1}g(x,t)$ is continuous in x on $x \in \mathbb{R}$; moreover using that $A'(0) = -\frac{1}{3\Gamma(\frac{1}{3})}$ we get the relation $\mathcal{V}^{-1}g(0,t) = -g(t)$.

Also, the definition of $\mathcal{V}^{-1}g(x,t)$ allows us to ensure that

$$\begin{cases} (\partial_t + \partial_x^3)\mathcal{V}^{-1}g(x,t) = 3\delta_0'(x)\mathcal{I}_{-\frac{1}{3}}g(t) & \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \\ \mathcal{V}^{-1}g(x,0) = 0 & \text{for} \quad x \in \mathbb{R}, \end{cases} \tag{3.7}$$

in the sense of distributions.

Furthermore, Lemma 3.2 implies that the function $\partial_x \mathcal{V}f(x,t)$ is continuous in x for all $x \in \mathbb{R}$ and, since $A'(0) = -\frac{1}{3\Gamma(\frac{1}{3})}$,

$$\partial_x \mathcal{V}g(0,t) = -\mathcal{I}_{-\frac{1}{3}}g(t). \tag{3.8}$$

Also, $\partial_x \mathcal{V}^{-1} g(x,t) = \partial_x^2 \mathcal{V} \mathcal{I}_{\frac{1}{3}} g(x,t)$ is continuous in x for $x \neq 0$ and has a step discontinuity of size $3\mathcal{I}_{-\frac{1}{3}} g(t)$ at $x = 0$. Indeed,

$$
\begin{aligned}
\lim_{x \to 0^+} \partial_x^2 \mathcal{V} g(x,t) &= -\int_0^{+\infty} \partial_y^3 \mathcal{V} g(y,t) dy = \int_0^{+\infty} \partial_t \mathcal{V} g(y,t) dy \\
&= 3 \int_0^{+\infty} A(y) dy \int_0^t \partial_t \mathcal{I}_{-\frac{2}{3}} g(t') dt' = \mathcal{I}_{-\frac{2}{3}} g(t),
\end{aligned}
$$

then from Lemma 3.2 -(b) we have

$$
\lim_{x \to 0^-} \partial_x \mathcal{V}^{-1} g(x,t) = -2\mathcal{I}_{-\frac{1}{3}} g(t) \quad \text{and} \quad \lim_{x \to 0^+} \partial_x \mathcal{V}^{-1} g(x,t) = \mathcal{I}_{-\frac{1}{3}} g(t).
$$

Now, for convenience, we give an application of the operator $\mathcal{V}^{-1}$ to solve a IBVP linear associated to the KdV equation on the negative half-line with two boundary conditions given by Holmer [21]. Let $h_1(t)$ and $h_2(t)$ belonging to $C_0^\infty(\mathbb{R}^+)$ we have the relations:

$$
\begin{aligned}
\mathcal{V} h_1(0,t) + \mathcal{V}^{-1} h_2(0,t) &= h_1(t) - h_2(t), \\
\lim_{x \to 0^-} \mathcal{I}_{\frac{1}{3}} \partial_x (\mathcal{V} h_1(x,\cdot) + \partial_x \mathcal{V}^{-1} h_2(x,\cdot))(t) &= -h_1(t) - 2h_2(t), \\
\lim_{x \to 0^+} \mathcal{I}_{\frac{1}{3}} \partial_x (\mathcal{V} h_1(x,\cdot) + \partial_x \mathcal{V}^{-1} h_2(x,\cdot))(t) &= -h_1(t) + h_2(t).
\end{aligned}
$$

For given $v_0(x)$, $g(t)$ and $h(t)$ we assigned

$$
\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} := \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} g - e^{-\cdot \partial_x^3} v_0|_{x=0} \\ \mathcal{I}_{\frac{1}{3}} \left(h - \partial_x e^{-\cdot \partial_x^3} v_0|_{x=0} \right) \end{bmatrix}.
$$

Then, taking $v(x,t) = e^{-t\partial_x^3} v_0(x) + \mathcal{V} h_1(x,t) + \mathcal{V}^{-1} h_2(x,t)$ we get

$$
\begin{cases}
(\partial_t + \partial_x^3) v(x,t) = 0 & \text{for} \quad (x,t) \in \mathbb{R}^* \times \mathbb{R}, \\
v(x,0) = v_0(x) & \text{for} \quad x \in \mathbb{R}, \\
v(0,t) = g(t) & \text{for} \quad t \in \mathbb{R}, \\
\lim_{x \to 0^-} \partial_x v(x,t) = h(t) & \text{for} \quad t \in \mathbb{R},
\end{cases}
\tag{3.9}
$$

in the sense of distributions.

3.3. The Duhamel Boundary Forcing Operator Classes associated to linear KdV equation.
To get our results, we need to work with two classes of boundary forcing operators in order to obtain the required estimates for the second order derivative of traces. In this way, we define the generalization of operators $\mathcal{V}$ and $\mathcal{V}^{-1}$ given by Holmer [21].

Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > -3$ and $g \in C_0^\infty(\mathbb{R}^+)$. Define the operators

$$
\mathcal{V}_-^\lambda g(x,t) = \left[\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} * \mathcal{V}(\mathcal{I}_{-\frac{\lambda}{3}} g)(\cdot, t) \right](x)
$$

and

$$\mathcal{V}_+^\lambda g(x,t) = \left[\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} * \mathcal{V}\big(\mathcal{I}_{-\frac{\lambda}{3}}g\big)(\cdot,t) \right](x),$$

with $\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} = e^{i\pi\lambda}\frac{(-x)_+^{\lambda-1}}{\Gamma(\lambda)}$. Then, using (3.3) we have that

$$(\partial_t + \partial_x^3)\mathcal{V}_-^\lambda g(x,t) = 3\frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}}g(t)$$

and

$$(\partial_t + \partial_x^3)\mathcal{V}_+^\lambda g(x,t) = 3\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}}g(t).$$

The following lemmas state properties of the operators classes $\mathcal{V}_\pm^\lambda$. For the proofs we refer the reader [21].

Lemma 3.3 (**Spatial continuity and decay properties for $\mathcal{V}_\pm^\lambda g(x,t)$**)**.** *Let $g \in C_0^\infty(\mathbb{R}^+)$. Then, we have*

$$\mathcal{V}_\pm^{\lambda-2}g = \partial_x^2\mathcal{V}^\lambda\mathcal{I}_{-\frac{2}{3}}g, \quad \mathcal{V}_\pm^{\lambda-1}g = \partial_x\mathcal{V}^\lambda\mathcal{I}_{-\frac{1}{3}}g \quad and \quad \mathcal{V}_\pm^0 g = \mathcal{V}g.$$

Also, $\mathcal{V}_\pm^{-2}g(x,t)$ has a step discontinuity of size $3g(t)$ at $x = 0$, otherwise for $x \neq 0$, $\mathcal{V}_\pm^{-2}g(x,t)$ is continuous in x. For $\lambda > -2$, $\mathcal{V}_\pm^\lambda g(x,t)$ is continuous in x for all $x \in \mathbb{R}$. For $-2 \leq \lambda \leq 1$ and $0 \leq t \leq 1$, $\mathcal{V}_-^\lambda g(x,t)$ satisfies the following decay bounds:

$$|\mathcal{V}_-^\lambda g(x,t)| \leq c_{m,\lambda,g}\langle x\rangle^{-m}, \text{ for all } x \leq 0 \text{ and } m \geq 0,$$
$$|\mathcal{V}_-^\lambda g(x,t)| \leq c_{\lambda,g}\langle x\rangle^{\lambda-1} \text{ for all } x \geq 0.$$
$$|\mathcal{V}_+^\lambda g(x,t)| \leq c_{m,\lambda,g}\langle x\rangle^{-m}, \text{ for all } x \geq 0 \text{ and } m \geq 0,$$

and

$$|\mathcal{V}_+^\lambda g(x,t)| \leq c_{\lambda,g}\langle x\rangle^{\lambda-1} \text{ for all } x \leq 0.$$

Lemma 3.4 (**Values of $\mathcal{V}_\pm^\lambda g(x,t)$ at $x = 0$**)**.** *For $\operatorname{Re}\lambda > -2$ and $g \in C_0^\infty(\mathbb{R}^+)$ we have*

$$\mathcal{V}_-^\lambda g(0,t) = 2\sin\left(\frac{\pi}{3}\lambda + \frac{\pi}{6}\right)g(t)$$

and

$$\mathcal{V}_+^\lambda g(0,t) = e^{i\pi\lambda}g(t).$$

Lemma 3.5. *Let $s \in \mathbb{R}$. The following estimates are ensured:*

(a) *(**space traces**) $\|\mathcal{V}_\pm^\lambda g(x,t)\|_{C\left(\mathbb{R}_t; H^s(\mathbb{R}_x)\right)} \lesssim \|g\|_{H_0^{(s+1)/3}(\mathbb{R}^+)}$ for all $s - \frac{5}{2} < \lambda < s + \frac{1}{2}$, $\lambda < \frac{1}{2}$ and $\operatorname{supp}(g) \subset [0,1]$.*

(b) *(**time traces**) $\|\psi(t)\mathcal{V}_\pm^\lambda g(x,t)\|_{C\left(\mathbb{R}_x; H_0^{(s+1)/3}(\mathbb{R}_t^+)\right)} \lesssim c\|g\|_{H_0^{(s+1)/3}(\mathbb{R}^+)}$ for all $-2 < \lambda < 1$.*

(c) (**derivative time traces**) $\left\|\psi(t)\partial_x \mathcal{V}_\pm^\lambda g(x,t)\right\|_{C\left(\mathbb{R}_x;\, H_0^{s/3}(\mathbb{R}_t^+)\right)} \lesssim c\|g\|_{H_0^{(s+1)/3}(\mathbb{R}^+)}$ *for all*
$-1 < \lambda < 2$.

(d) (**second derivative time traces**) $\left\|\psi(t)\partial_x^2 \mathcal{V}_\pm^\lambda g(x,t)\right\|_{C\left(\mathbb{R}_x;\, H_0^{(s-1)/3}(\mathbb{R}_t^+)\right)} \lesssim c\|g\|_{H_0^{(s+1)/3}(\mathbb{R}^+)}$
for all $0 < \lambda < 3$.

(e) (**Bourgain spaces**) $\left\|\psi(t)\mathcal{V}_\pm^\lambda g(x,t)\right\|_{X^{s,b}\cap V^\alpha} \lesssim c\|g\|_{H_0^{(s+1)/3}(\mathbb{R}^+)}$ *for all* $s - 1 \le \lambda <$
$s + \frac{1}{2}$, $\lambda < \frac{1}{2}$, $\alpha \le \frac{s-\lambda+2}{3}$ *and* $0 \le b < \frac{1}{2}$.

Remark 3.2. *Note that for $\lambda = 0$ the second derivative time traces estimate is not obtained, for this reason we need to work with the family $\mathcal{V}_\pm^\lambda$. Also note that the set of regularity where the estimates are valid depends of the index λ.*

4. The Duhamel Inhomogeneous Solution Operator

The classical inhomogeneous solution operator $\mathcal{K}$ associated to the KdV equation is given by

$$\mathcal{K}w(x,t) = \int_0^t e^{-(t-t')\partial_x^3} w(x,t')dt',$$

that satisfies

$$\begin{cases} (\partial_t + \partial_x^3)\mathcal{K}w(x,t) = w(x,t) & \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \\ \mathcal{K}w(x,t) = 0 & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{4.1}$$

Now we summarize some useful estimates for the Duhamel inhomogeneous solution operators $\mathcal{K}$ that will be used later in the proof of the main results and its proof can be seen in [21].

Lemma 4.1. *For all $s \in \mathbb{R}$ we have the following estimates:*

(a) (**space traces**) *Let $-\frac{1}{2} < d < 0$, then*
$$\|\psi(t)\mathcal{K}w(x,t)\|_{C\left(\mathbb{R}_t;\, H^s(\mathbb{R}_x)\right)} \lesssim \|w\|_{X^{s,d}}.$$

(b) (**time traces**) *If $-\frac{1}{2} < d < 0$, then*
$$\|\psi(t)\mathcal{K}w(x,t)\|_{C\left(\mathbb{R}_x;\, H^{(s+1)/3}(\mathbb{R}_t)\right)} \lesssim \begin{cases} \|w\|_{X^{s,d}} & \text{if } -1 \le s \le \frac{1}{2}, \\ \|w\|_{X^{s,d}} + \|w\|_{U^{s,d}} & \text{for all } s \in \mathbb{R}. \end{cases}$$

(c) (**derivative time traces**) *If $-\frac{1}{2} < d < 0$, then*
$$\|\psi(t)\partial_x\mathcal{K}w(x,t)\|_{C\left(\mathbb{R}_x;\, H_t^{s/3}(\mathbb{R})\right)} \lesssim \begin{cases} \|w\|_{X^{s,d}} & \text{if } 0 \le s \le \frac{3}{2}, \\ \|w\|_{X^{s,d}} + \|w\|_{U^{s,d}} & \text{for all } s \in \mathbb{R}. \end{cases}$$

(d) (**second derivative time traces**) *If $-\frac{1}{2} < d < 0$, then*
$$\|\psi(t)\partial_x^2\mathcal{K}w(x,t)\|_{C\left(\mathbb{R}_x;\, H_s^{(s-1)/3}(\mathbb{R})\right)} \lesssim \begin{cases} \|w\|_{X^{s,d}} & \text{if } 1 \le s \le \frac{5}{2}, \\ \|w\|_{X^{s,d}} + \|w\|_{U^{s,d}} & \text{for all } s \in \mathbb{R}. \end{cases}$$

(e) (***Bourgain spaces estimates***) *Let* $0 < b < \frac{1}{2}$ *and* $\alpha > 1 - b$, *then*

$$\|\psi(t)\mathcal{K}w(x,t)\|_{X^{s,b}\cap V^\alpha} \lesssim \|w\|_{X^{s,-b}}.$$

Remark 4.1. *We note that the time-adapted Bourgain spaces* $U^{k,d}$ *used in Lemma 4.1-(c)-(d) are introduced in order to cover the full values of regularity* s.

5. Proof of Theorem 1.1

Here we show the proof of the main result announced of this work. We only proof the part (i) of the theorem, in effect the proof of part (ii) is very similar. We follow closely the arguments in [21] (see also [11] and [12]). The proof will be divided into five steps.

Step 1. We will first obtain a integral equation that solves the Cauchy problem (1.2)-(1.3)**, with type 1 boundary conditions, satisfying** (1.19) **for** $\frac{1}{2} < s < \frac{3}{2}$**.**

We start rewriting the vertex conditions (1.13), (1.14) and (1.15) in terms of matrices:

$$\begin{bmatrix} 1 & -a_2 & 0 \\ 1 & 0 & -a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(0,t) \\ v(0,t) \\ w(0,t) \end{bmatrix} = 0, \tag{5.1}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -b_2 & -b_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_x(0,t) \\ v_x(0,t) \\ w_x(0,t) \end{bmatrix} = 0 \tag{5.2}$$

and

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -c_2 & -c_3 \end{bmatrix} \begin{bmatrix} u_{xx}(0,t) \\ v_{xx}(0,t) \\ w_{xx}(0,t) \end{bmatrix} = 0. \tag{5.3}$$

Let $\widetilde{u}_0, \widetilde{v}_0$ and $\widetilde{w}_0$ nice extensions of u_0, v_0 and w_0, respectively satisfying

$$\|\widetilde{u}_0\|_{H^s(\mathbb{R})} \le c\|u_0\|_{H^s(\mathbb{R}^+)}, \ \|\widetilde{v}_0\|_{H^s(\mathbb{R})} \le c\|v_0\|_{H^s(\mathbb{R}^+)} \text{ and } \|\widetilde{w}_0\|_{H^s(\mathbb{R})} \le c\|w_0\|_{H^s(\mathbb{R}^+)}.$$

Initially, we look for solutions in the form

$$u(x,t) = \mathcal{V}_-^{\lambda_1}\gamma_1(x,t) + \mathcal{V}_-^{\lambda_2}\gamma_2(x,t) + F_1(x,t),$$
$$v(x,t) = \mathcal{V}_+^{\lambda_3}\gamma_3(x,t) + F_2(x,t),$$
$$w(x,t) = \mathcal{V}_+^{\lambda_4}\gamma_4(x,t) + F_3(x,t),$$

where γ_i $(i = 1, 2, 3, 4)$ are unknown functions and

$$F_1(x,t) = e^{it\partial_x^3}\widetilde{u}_0 + \mathcal{K}(uu_x)(x,t),$$
$$F_2(x,t) = e^{it\partial_x^3}\widetilde{v}_0 + \mathcal{K}(vv_x)(x,t),$$
$$F_3(x,t) = e^{it\partial_x^3}\widetilde{w}_0 + \mathcal{K}(ww_x)(x,t).$$

By using Lemma 3.4 we see that

$$u(0,t) = 2 \, sin\left(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6}\right)\gamma_1(t) + 2 \, sin\left(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6}\right)\gamma_2(t) + F_1(0,t), \tag{5.4}$$

$$v(0,t) = e^{i\pi\lambda_3}\gamma_3(t) + F_2(0,t), \tag{5.5}$$

$$w(0,t) = e^{i\pi\lambda_4}\gamma_4(t) + F_3(0,t). \tag{5.6}$$

Now we calculate the traces of first derivative functions. By Lemmas 3.3 and 3.4 we see that

$$u_x(0,t) = 2 \, sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{6}\right)\mathcal{I}_{-\frac{1}{3}}\gamma_1(t) + 2 \, sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{6}\right)\mathcal{I}_{-\frac{1}{3}}\gamma_2(t) + \partial_x F_1(0,t), \tag{5.7}$$

$$v_x(0,t) = e^{i\pi(\lambda_3-1)}\mathcal{I}_{-\frac{1}{3}}\gamma_3(t) + \partial_x F_2(0,t), \tag{5.8}$$

$$w_x(0,t) = e^{i\pi(\lambda_4-1)}\mathcal{I}_{-\frac{1}{3}}\gamma_4(t) + \partial_x F_3(0,t). \tag{5.9}$$

In the same way, we calculate the traces of second derivatives functions,

$$u_{xx}(0,t) = 2sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{2}\right)\mathcal{I}_{-\frac{2}{3}}\gamma_1(t) + 2 \, sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{2}\right)\gamma_2(t) + \partial_x^2 F_1(0,t), \tag{5.10}$$

$$v_{xx}(0,t) = e^{i\pi(\lambda_3-2)}\mathcal{I}_{-\frac{2}{3}}\gamma_3(t) + \partial_x^2 F_2(0,t), \tag{5.11}$$

$$w_{xx}(0,t) = e^{i\pi(\lambda_4-2)}\mathcal{I}_{-\frac{2}{3}}\gamma_4(t) + \partial_x^2 F_3(0,t). \tag{5.12}$$

Note that by Lemmas 3.3 and 3.4 these calculus are valid for Re $\lambda > 0$.

By substituting (5.4), (5.5) and (5.6) in (5.1); (5.6), (5.7) and (5.8) in (5.2), and (5.10), (5.11) and (5.12) in (5.3) we see that the functions γ_i and index λ_i, for $i = 1,2,3,4$, satisfies the expressions

$$\begin{bmatrix} 1 & -a_2 & 0 \\ 1 & 0 & -a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0. \end{bmatrix} \begin{bmatrix} 2 \, sin\left(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6}\right) & 0 & 0 & 2 \, sin\left(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6}\right) \\ 0 & e^{i\pi\lambda_3} & 0 & 0 \\ 0 & 0 & e^{i\pi\lambda_4} & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix}$$
$$= -\begin{bmatrix} 1 & -a_2 & 0 \\ 1 & 0 & -a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0. \end{bmatrix} \begin{bmatrix} F_1(0,t) \\ F_2(0,t) \\ F_3(0,t) \end{bmatrix}, \tag{5.13}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -b_2 & -b_3 \\ 0 & 0 & 0. \end{bmatrix} \begin{bmatrix} 2 \, sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{6}\right) & 0 & 0 & 2 \, sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{6}\right) \\ 0 & e^{i(\pi\lambda_3-1)} & 0 & 0 \\ 0 & 0 & e^{i(\pi\lambda_4-1)} & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix}$$
$$= -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -b_2 & -b_3 \\ 0 & 0 & 0. \end{bmatrix} \begin{bmatrix} \partial_x\mathcal{I}_{\frac{1}{3}}F_1(0,t) \\ \partial_x\mathcal{I}_{\frac{1}{3}}F_2(0,t) \\ \partial_x\mathcal{I}_{\frac{1}{3}}F_3(0,t) \end{bmatrix} \tag{5.14}$$

and

$$
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -c_2 & -c_3 \end{bmatrix}
\begin{bmatrix} 2\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{2}\right) & 0 & 0 & 2\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{2}\right) \\ 0 & e^{i(\pi\lambda_3 - 2)} & 0 & 0 \\ 0 & 0 & e^{i(\pi\lambda_4 - 2)} & 0 \end{bmatrix}
\begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix}
$$
$$
= -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -c_2 & -c_3. \end{bmatrix}
\begin{bmatrix} \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_1(0,t) \\ \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_2(0,t) \\ \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_3(0,t) \end{bmatrix}.
\tag{5.15}
$$

It follows that,

$$
\begin{bmatrix} 2\sin\left(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6}\right) & -a_2 e^{i\pi\lambda_3} & 0 & 2\sin\left(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6}\right) \\ 2\sin\left(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6}\right) & 0 & -a_3 e^{i\pi\lambda_4} & 2\sin\left(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6}\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix}
$$
$$
= -\begin{bmatrix} F_1(0,t) - a_2 F_2(0,t) \\ F_1(0,t) - a_3 F_3(0,t) \\ 0 \\ 0 \end{bmatrix}.
\tag{5.16}
$$

$$
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{6}\right) & -b_2 e^{i\pi(\lambda_3 - 1)} & -b_3 e^{i\pi(\lambda_4 - 1)} & 2\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{6}\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix}
$$
$$
= -\begin{bmatrix} 0 \\ 0 \\ \partial_x \mathcal{I}_{\frac{1}{3}} F_1(0,t) - b_2 \partial_x \mathcal{I}_{\frac{1}{3}} F_2(0,t) - b_3 \partial_x \mathcal{I}_{\frac{1}{3}} F_3(0,t) \\ 0 \end{bmatrix}
\tag{5.17}
$$

and

$$
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{2}\right) & -c_2 e^{i\pi(\lambda_3 - 2)} & -c_3 e^{i\pi(\lambda_4 - 2)} & 2\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{2}\right) \end{bmatrix}
\begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix}
$$
$$
= -\begin{bmatrix} 0 \\ 0 \\ 0 \\ \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_1(0,t) - c_2 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_2(0,t) - c_3 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_3(0,t) \end{bmatrix}.
\tag{5.18}
$$

From (5.16), (5.17) and (5.18) we need to get functions γ_i $(i = 1, 2, 3, 4)$ and parameters λ_i $(i = 1, 2, 3, 4)$ satisfying

$$
\begin{bmatrix}
2\sin\left(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6}\right) & -a_2 e^{i\pi\lambda_3} & 0 & 2\sin\left(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6}\right) \\
2\sin\left(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6}\right) & 0 & -a_3 e^{i\pi\lambda_4} & 2\sin\left(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6}\right) \\
2\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{6}\right) & -b_2 e^{i\pi(\lambda_3-1)} & -b_3 e^{i\pi(\lambda_4-1)} & 2\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{6}\right) \\
2\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{2}\right) & -c_2 e^{i\pi(\lambda_3-2)} & -c_3 e^{i\pi(\lambda_4-2)} & 2\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{2}\right)
\end{bmatrix}
\begin{bmatrix}
\gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2
\end{bmatrix}
$$

$$
= - \begin{bmatrix}
F_1(0,t) - a_2 F_2(0,t) \\
F_1(0,t) - a_3 F_3(0,t) \\
\partial_x \mathcal{I}_{\frac{1}{3}} F_1(0,t) - b_2 \partial_x \mathcal{I}_{\frac{1}{3}} F_2(0,t) - b_3 \partial_x \mathcal{I}_{\frac{1}{3}} F_3(0,t) \\
\partial_x^2 \mathcal{I}_{\frac{2}{3}} F_1(0,t) - c_2 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_2(0,t) - c_3 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_3(0,t)
\end{bmatrix}. \tag{5.19}
$$

We denote a simplified notation of (5.19) as

$$
\mathbf{M}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\boldsymbol{\gamma} = \mathbf{F}, \tag{5.20}
$$

where $\mathbf{M}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is the first matrix that appears in (5.19), $\boldsymbol{\gamma}$ is the matrix column given by vector $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and $\mathbf{F}$ is the last matrix in (5.19). By using the hypothesis of Theorem 1.1 we fix parameters λ_i, for $i = 1, 2, 3, 4$ such that

$$
\max\{s-1, 0\} < \lambda_i(s) < \min\left\{s + \frac{1}{2}, \frac{1}{2}\right\}. \tag{5.21}
$$

and the matrix $\mathbf{M}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is invertible.

Step 2. We will define the truncated integral operator and the appropriate functions space.

Given s as in the hypothesis of Theorem 1.1 we fix the parameters λ_i and the functions γ_i $(i = 1, 2, 3, 4)$ chosen as in the Step 1. Let $b = b(s) < \frac{1}{2}$ and $\alpha(b) > 1/2$ such that the estimates given in Lemma 2.4 are valid.

Define the operator

$$
\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) \tag{5.22}
$$

where

$$
\Lambda_1 u(x,t) = \psi(t)\mathcal{V}_-^{\lambda_1}\gamma_1(x,t) + \psi(t)\mathcal{V}_-^{\lambda_2}\gamma_2(x,t) + F_1(x,t),
$$
$$
\Lambda_2 v(x,t) = \psi(t)\mathcal{V}_+^{\lambda_3}\gamma_3(x,t) + F_2(x,t),
$$
$$
\Lambda_3 w(x,t) = \psi(t)\mathcal{V}_+^{\lambda_4}\gamma_4(x,t) + F_3(x,t),
$$

where

$$
F_1(x,t) = \psi(t)(e^{it\partial_x^3}\widetilde{u}_0 + \mathcal{K}(uu_x)(x,t)),
$$
$$
F_2(x,t) = \psi(t)(e^{it\partial_x^3}\widetilde{v}_0 + \mathcal{K}(vv_x)(x,t)),
$$
$$
F_3(x,t) = \psi(t)(e^{it\partial_x^3}\widetilde{w}_0 + \mathcal{K}(ww_x)(x,t)).
$$

We consider Λ on the Banach space $Z(s) = Z_1(s) \times Z_2(s) \times Z_3(s)$, where

$$Z_i(s) = \{w \in C(\mathbb{R}_t; H^s(\mathbb{R}_x)) \cap C(\mathbb{R}_x; H^{\frac{s+1}{3}}(\mathbb{R}_t)) \cap X^{s,b} \cap V^{\alpha};$$
$$w_x \in C(\mathbb{R}_x; H^{\frac{s}{3}}(\mathbb{R}_t)), w_{xx} \in C(\mathbb{R}_x; H^{\frac{s-1}{3}}(\mathbb{R}_t))\} \ (i = 1, 2, 3),$$

with norm

$$\|(u, v, w)\|_{Z(s)} = \|u\|_{Z_1(s)} + \|v\|_{Z_2(s)} + \|w\|_{Z_3(s)},$$

where

$$\begin{aligned}\|u\|_{Z_i(s)} =&\|u\|_{C(\mathbb{R}_t; H^s(\mathbb{R}_x))} + \|u\|_{C(\mathbb{R}_x; H^{\frac{s+1}{3}}(\mathbb{R}_t))} + \|u\|_{X^{s,b}} + \|u\|_{V^{\alpha}} \\ &+ \|u_x\|_{C(\mathbb{R}_x; H^{\frac{s}{3}}(\mathbb{R}_t))} + \|u_{xx}\|_{C(\mathbb{R}_x; H^{\frac{s-1}{3}}(\mathbb{R}_t))}.\end{aligned} \tag{5.23}$$

Step 3. We will prove that the functions $\mathcal{V}_-^{\lambda_1}\gamma_1(\mathbf{x}, \mathbf{t})$, $\mathcal{V}_-^{\lambda_2}\gamma_2(\mathbf{x}, \mathbf{t})$, $\mathcal{V}_+^{\lambda_3}\gamma_1(\mathbf{x}, \mathbf{t})$ and $\mathcal{V}_-^{\lambda_4}\gamma_4(\mathbf{x}, \mathbf{t})$ are well defined.

By Lemma (3.5) it suffices to show that these functions are in the closure of the spaces $C_0^\infty(\mathbb{R}^+)$. By using expression (5.19) we see that the functions γ_i ($i = 1, 2, 3, 4$) are linear combinations of the functions $F_1(0, t) - a_2 F_2(0, t), F_1(0, t) - a_3 F_3(0, t), \partial_x \mathcal{I}_{\frac{1}{3}} F_1(0, t) - b_2 \partial_x \mathcal{I}_{\frac{1}{3}} F_2(0, t) - b_3 \partial_x \mathcal{I}_{\frac{1}{3}} F_3(0, t)$ and $\partial_x^2 \mathcal{I}_{\frac{2}{3}} F_1(0, t) - c_2 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_2(0, t) - c_3 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_3(0, t)$. Thus, we need to show that the functions $F_i(0, t)$, $\partial_x \mathcal{I}_{\frac{1}{3}} F_i(0, t)$, $\partial_x^2 \mathcal{I}_{\frac{2}{3}} F_i(0, t)$ are in appropriate spaces. By using Lemmas 3.1, 3.5, 4.1 and 2.4 we obtain

$$\|F_1(0, t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)} \leq c(\|u_0\|_{H^s(\mathbb{R}^+)} + \|u\|_{X^{s,b}}^2 + \|u\|_{Y^\alpha}^2), \tag{5.24}$$

$$\|F_2(0, t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)} \leq c(\|v_0\|_{H^s(\mathbb{R}^+)} + \|v\|_{X^{s,b}}^2 + \|v\|_{Y^\alpha}^2), \tag{5.25}$$

$$\|F_3(0, t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)} \leq c(\|w_0\|_{H^s(\mathbb{R}^+)} + \|w\|_{X^{s,b}}^2 + \|w\|_{Y^\alpha}^2). \tag{5.26}$$

If $-\frac{1}{2} < s < \frac{1}{2}$ we have that $\frac{1}{6} < \frac{s+1}{3} < \frac{1}{2}$. Thus Lemma 2.1 implies that $H^{\frac{s+1}{3}}(\mathbb{R}^+) = H_0^{\frac{s+1}{3}}(\mathbb{R}^+)$. It follows that $F_i(0, t) \in H_0^{\frac{s+1}{3}}(\mathbb{R}^+)$ (for $i = 1, 2, 3$) for $-\frac{1}{2} < s < \frac{1}{2}$.

If $\frac{1}{2} < s < \frac{3}{2}$, then $\frac{1}{2} < \frac{s+1}{3} < \frac{5}{6}$. Using the compatibility condition (1.19) we have that

$$F_1(0, 0) - a_2 F_2(0, 0) = u(0, 0) - a_2 v(0, 0) = u_0(0) - a_2 v_0(0) = 0,$$

$$F_1(0, 0) - a_3 F_3(0, 0) = u(0, 0) - a_3 w(0, 0) = u_0(0) - a_3 w_0(0) = 0.$$

Then Lemma 2.2 implies

$$\begin{aligned}F_1(0, t) - a_2 2 F_2(0, t) &\in H_0^{\frac{s+1}{3}}(\mathbb{R}^+), \\ F_1(0, t) - a_3 F_3(0, t) &\in H_0^{\frac{s+1}{3}}(\mathbb{R}^+)\end{aligned} \tag{5.27}$$

Now using Lemmas 3.1, 3.5, 4.1 and 2.4 we see that

$$\|\partial_x F_1(0, t)\|_{H^{\frac{s}{3}}(\mathbb{R}^+)} \leq c(\|u_0\|_{H^s(\mathbb{R}^+)} + \|u\|_{X^{s,b}}^2 + \|u\|_{Y^\alpha}^2),$$

$$\|\partial_x F_2(0, t)\|_{H^{\frac{s}{3}}(\mathbb{R}^+)} \leq c(\|v_0\|_{H^s(\mathbb{R}^+)} + \|v\|_{X^{s,b}}^2 + \|v\|_{Y^\alpha}^2),$$

$$\|\partial_x F_3(0, t)\|_{H^{\frac{s}{3}}(\mathbb{R}^+)} \leq c(\|w_0\|_{H^s(\mathbb{R}^+)} + \|w\|_{X^{s,b}}^2 + \|w\|_{Y^\alpha}^2).$$

Since $-\frac{1}{2} < s < \frac{3}{2}$ we have $-\frac{1}{6} < \frac{s}{3} < \frac{1}{2}$, then Lemma 2.1 implies that the functions $\partial_x F_i(0,t) \in H_0^{\frac{s}{3}}(\mathbb{R}^+)$, for $i = 1, 2, 3, 4$. Then using Lemma 2.2 we have that

$$\|\partial_x \mathcal{I}_{\frac{1}{3}} F_1(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \le c(\|u_0\|_{H^s(\mathbb{R}^+)} + \|u\|_{X^{s,b}}^2 + \|u\|_{Y^\alpha}^2),$$

$$\|\partial_x \mathcal{I}_{\frac{1}{3}} F_2(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \le c(\|v_0\|_{H^s(\mathbb{R}^+)} + \|v\|_{X^{s,b}}^2 + \|v\|_{Y^\alpha}^2),$$

$$\|\partial_x \mathcal{I}_{\frac{1}{3}} F_3(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \le c(\|w_0\|_{H^s(\mathbb{R}^+)} + \|w\|_{X^{s,b}}^2 + \|w\|_{Y^\alpha}^2).$$

Thus we have

$$\partial_x \mathcal{I}_{\frac{1}{3}} F_1(0,t) - b_2 \partial_x \mathcal{I}_{\frac{1}{3}} F_2(0,t) - b_3 \partial_x \mathcal{I}_{\frac{1}{3}} F_3(0,t) \in H_0^{\frac{s+1}{3}}(\mathbb{R}^+). \tag{5.28}$$

In the same way we can obtain

$$\|\partial_x^2 \mathcal{I}_{\frac{2}{3}} F_1(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \le c(\|u_0\|_{H^s(\mathbb{R}^+)} + \|u\|_{X^{s,b}}^2 + \|u\|_{Y^\alpha}^2),$$

$$\|\partial_x^2 \mathcal{I}_{\frac{2}{3}} F_2(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \le c(\|v_0\|_{H^s(\mathbb{R}^+)} + \|v\|_{X^{s,b}}^2 + \|v\|_{Y^\alpha}^2),$$

$$\|\partial_x^2 \mathcal{I}_{\frac{2}{3}} F_3(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \le c(\|w_0\|_{H^s(\mathbb{R}^+)} + \|w\|_{X^{s,b}}^2 + \|w\|_{Y^\alpha}^2).$$

It follows that

$$\partial_x^2 \mathcal{I}_{\frac{2}{3}} F_1(0,t) - c_2 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_2(0,t) - c_3 \partial_x \mathcal{I}_{\frac{2}{3}} F_3(0,t) \in H_0^{\frac{s-1}{3}}(\mathbb{R}^+). \tag{5.29}$$

Thus (5.27), (5.28) and (5.29) imply that the functions $\mathcal{V}_-^{\lambda_1} \gamma_1(x,t)$, $\mathcal{V}_-^{\lambda_2} \gamma_2(x,t)$, $\mathcal{V}_+^{\lambda_3} \gamma_1(x,t)$ and $\mathcal{V}_-^{\lambda_4} \gamma_4(x,t)$ are well defined.

Step 4. We will obtain a fixed point for Λ in a ball of Z.

Using Lemmas 2.2, 3.1, 3.5, 4.1 and 2.4 we obtain

$$\|\Lambda(u_2, v_2, w_2) - \Lambda(u_1, v_1, w_1)\|_Z$$
$$\le c(\|(u_2, v_2, w_2)\|_Z + \|(u_1, v_1, w_1)\|_Z) \|(u_2, v_2, w_2) - (u_1, v_1, w_1)\|_Z \tag{5.30}$$

and

$$\|\Lambda(u, v, w)\|_Z \le c(\|u_0\|_{H^s(\mathbb{R}^+)} + \|v_0\|_{H^s(\mathbb{R}^+)} + \|w_0\|_{H^s(\mathbb{R}^+)}$$
$$+ \|u\|_{X^{s,b}}^2 + \|u\|_{Y^\alpha}^2 + \|v\|_{X^{s,b}}^2 + \|v\|_{Y^\alpha}^2 + \|w\|_{X^{s,b}}^2 + \|w\|_{Y^\alpha}^2). \tag{5.31}$$

By taking $\|u_0\|_{H^s(\mathbb{R}^+)} + \|v_0\|_{H^s(\mathbb{R}^+)} + \|w_0\|_{H^s(\mathbb{R}^+}} < \delta$ for $\delta > 0$ suitable small, we obtain a fixed point $\Lambda(\widetilde{u}, \widetilde{v}, \widetilde{w}) = (\widetilde{u}, \widetilde{v}, \widetilde{w})$ in a ball

$$B = \{(u, v, w) \in Z, \|(u, v, w)\|_Z \le 2c\delta\}.$$

It follows that the restriction

$$(u, v, w) = (\widetilde{u}\big|_{\mathbb{R}^- \times (0,1)}, \widetilde{v}\big|_{\mathbb{R}^+ \times (0,1)}, \widetilde{w}\big|_{\mathbb{R}^+ \times (0,1)}) \tag{5.32}$$

solves the Cauchy problem (1.2)-(1.3) with 1 boundary conditions in the sense of distributions.

Existence of solutions for any data in $H^s(\mathcal{Y})$ follows by the standard scaling argument. Suppose we are given data $\widetilde{u}_0, \widetilde{v}_0$ and $\widetilde{w}_0$ with arbitrary size for the Cauchy problem (1.2)-(1.3)

with 1 boundary conditions. For $\lambda << 1$ (to be selected after) define $u_0(x) = \lambda^2 \widetilde{u}_0(\lambda x)$, $v_0(x) = \lambda^2 \widetilde{v}_0(\lambda x)$ and $w_0(x) = \lambda^2 \widetilde{w}_0(\lambda x)$. Taking λ sufficiently small so that

$$\|u_0\|_{H^s} + \|v_0\|_{H^s} + \|w_0\|_{H^s} \leq \lambda^{\frac{3}{2}}(1+\lambda^s)(\|\widetilde{u}_0\|_{H^s(\mathbb{R}^+)} + \|\widetilde{v}_0\|_{H^s(\mathbb{R}^+)} + \|\widetilde{w}_0\|_{H^s(\mathbb{R}^+)}) < \delta. \quad (5.33)$$

Then using the previous argument, there exists a solution $u(x,t)$ for the problem (1.2)-(1.3), with type 1 boundary conditions, on $0 \leq t \leq 1$. Then $\widetilde{u}(x,t) = \lambda^{-2}u(\lambda^{-1}x, \lambda^{-3})$ solves the Cauchy problem for initial data $\widetilde{u}_0, \widetilde{v}_0$ and $\widetilde{w}_0$ on time interval $0 \leq t \leq \lambda^3$.

Step 5. Proof of locally Lipschitz continuity of map data-to-solution.

Let $\{(u_{0n}, v_{0n}, w_{0n})\}_{n \in \{1,2\}}$ two initial data in $H^s(\mathcal{Y})$ such that $\|u_{0n}\| + \|v_{0n}\| + \|w_{0n}\| < \delta$, $(i = 1,2)$ where δ is sufficiently small.

Let (u_n, v_n, w_n) $(n = 1,2)$ the solution of Cauchy problem (1.2)-(1.3) with 1 boundary condition on the space $C([0,1] : H^s(\mathcal{Y}))$ with initial data (u_{0n}, v_{0n}, w_{0n}). According to Step 4 the lifespans of these solutions is $[0,1]$.

By using the arguments used in Step 4 we have that

$$\begin{aligned}
\|(u_2, v_2, w_2) - (u_1, v_1, w_1)\|_{Z|_{[0,1]}} &\leq c\|(u_{02}, v_{02}, w_{02}) - (u_{01}, v_{01}, w_{01})\|_{H^s(\mathcal{Y})} \\
&+ c(\|(u_2, v_2, w_2) + (u_1, v_1, w_1)\|_{Z|_{[0,1]}})\|(u_2, v_2, w_2) - (u_1, v_1, w_1)\|_{Z|_{[0,1]}},
\end{aligned} \quad (5.34)$$

where $Z|_{[0,1]}$ denotes the restrictions of functions of Z in the interval $[0,1]$. In a ball of $Z|_{[0,1]}$ we have that

$$\|(u_2, v_2, w_2) - (u_1, v_1, w_1)\|_{Z|_{[0,1]}} \leq c\|(u_{01}, v_{02}, w_{02}) - (u_{01}, v_{01}, w_{01})\|_{H^s(\mathcal{Y})}. \quad (5.35)$$

which completes the proof for small data assumptions. The local Lipschitz continuity for any data can be showed by a scaling argument.

6. Proof of Corollary 1.1

By using Theorem 1.1 given a regularity index s it suffices to get scalars $\lambda_i(s)$ satisfying (1.21) such that the matrix $M(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ given by (1.22) is invertible. These choices of scalars is a crucial point to get Corollary 1.1. We will divide this analysis in 2 cases.

Case 1. Regularity: $-\frac{1}{2} < s < 1$, for $s \neq \frac{1}{2}$.

Taking $\lambda_i = 0$ for $i = 1, 3, 4$ and $0 < \lambda_2 = \frac{3}{\pi}\epsilon << 1$, a simple computations gives that the determinant of $\mathbf{M}$ is given by

$$\det \mathbf{M}\left(0, \frac{3}{\pi}\epsilon, 0, 0\right) = 2\sqrt{3}\, \alpha_2\, \alpha_3\, \sin(\epsilon)\left(1 + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2}\right) \neq 0, \quad (6.1)$$

where we have used the hypothesis of the theorem about the parameters α_i and β_i $(i \in \{1,2\})$, and the fact $0 < \epsilon << 1$. Note that the condition (1.21) given in Theorem 1.1 is not valid for $\lambda = 0$. Then, by a continuity argument we will take the parameters λ_i $(i = 1, 3, 4)$ close to zero. In fact, for fixed $\alpha_2, \alpha_3, \beta_3$ and β_4 satisfying the hypothesis, we have that the function $\lambda \mapsto \det \mathbf{M}(\lambda, \frac{3}{\pi}\epsilon, \lambda, \lambda)$ is continuous from $\mathbb{R}$ to $\mathbb{C}$. It follows that there exists a positive number $\delta(\epsilon) << 1$, depending of ϵ, such that $\det \mathbf{M}(\lambda, \frac{3\epsilon}{\pi}, \lambda, \lambda) \neq 0$, for $0 < \lambda < \delta$.

Thus, given $-\frac{1}{2} < s < 1$ we can choice $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda, \frac{3}{\pi}\epsilon, \lambda, \lambda)$ satisfying

$$
\begin{aligned}
0 < \frac{3\epsilon}{2\pi} &< \min\{s + \frac{1}{2}, \frac{1}{2}\} \\
0 < \lambda &< \min\left\{\delta(\epsilon), s + \frac{1}{2}\right\}.
\end{aligned}
\tag{6.2}
$$

Note that with this choice the all hypothesis of Theorem 1.1 part (i) are valid and the proof of Corollary 1.1 on the **Case 1** is complete.

Case 2. Regularity: $1 \leq s < \frac{3}{2}$.

Taking $\lambda_i = \frac{1}{2}$ for $i = 1, 3, 4$ and $0 < \lambda_2 = \frac{1}{2} - \frac{3\epsilon}{\pi}$, for $0 < \epsilon << 1$. A simple calculation shows that the determinant is given by

$$
\det \mathbf{M}\left(\frac{1}{2}, \frac{1}{2} - \frac{3\epsilon}{\pi}, \frac{1}{2}, \frac{1}{2}\right) = 2\sqrt{3}\, \alpha_2\, \alpha_3\, \sin(\epsilon)\left(1 + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2}\right) \neq 0, \tag{6.3}
$$

where we have used the hypothesis $\frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2} \neq -1$ and the fact $0 < \epsilon << 1$.

As the estimate condition (1.21) in part (i) of Theorem (1.1) is not valid for $\lambda = \frac{1}{2}$, then we shall make a few perturbation in λ.

For fixed α_2, α_3, β_3 and β_4 satisfying the hypothesis, we have that the function $\lambda \mapsto \det \mathbf{M}(\lambda, 1 - \frac{3}{\pi}\epsilon, \lambda, \lambda)$ is continuous from $\mathbb{R}$ to $\mathbb{C}$. It follows that there exists a positive number $\delta(\epsilon) << 1$, depending of ϵ, such that $\det \mathbf{M}(\lambda, \frac{1}{2} - \frac{3}{2}\epsilon, \lambda, \lambda) \neq 0$, for $0 < \frac{1}{2} - \lambda < \delta$.

Thus, given $1 < s < \frac{3}{2}$ we can choice $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda, \frac{1}{2} - \frac{3}{\pi}\epsilon, \lambda, \lambda)$ satisfying

$$
\begin{aligned}
s - 1 < \frac{1}{2} - \frac{3\epsilon}{\pi} &< s + \frac{1}{2}, \\
\max\left\{s - 1, \frac{1}{2} - \delta\right\} &< \lambda < s + \frac{1}{2}.
\end{aligned}
\tag{6.4}
$$

This finish the proof of Corollary 1.1.

7. Proof of Corollary 1.2

For the regularity $-\frac{1}{2} < s < 1$ the result follows a similar idea of the proof of Corollary 1.1, by using the fact

$$
\det \mathbf{M}\left(0, \frac{3}{\pi}\epsilon, 0, 0\right) = 2\sqrt{3}\alpha_2\alpha_3\sin(\epsilon)\left(1 + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2}\right) \neq 0, \tag{7.1}
$$

Similarly, the case $1 \leq s < \frac{3}{2}$ with $s \neq \frac{1}{2}$ follows from the fact

$$
\det \mathbf{M}\left(\frac{1}{2}, \frac{1}{2} - \frac{3\epsilon}{\pi}, \frac{1}{2}, \frac{1}{2}\right) = 2\sqrt{3}\alpha_2\alpha_3\sin(\epsilon)\left(1 + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2}\right) \neq 0. \tag{7.2}
$$

Acknowledgments

The author wishes to thank the Centro de Modelamiento Matemático (CMM) and Universidad de Chile and Núcleo Milenio CAPDE, for the financial support and nice scientific infrastructure that allowed to conclude the paper during his postdoctoral stay. The author gratefully acknowledges the many suggestions of the referee, that has improved considerably this work.

References

[1] R. Adami, C. Cacciapuoti , D. Finco , D. Noja, *Variational properties and orbital stability of standing waves for NLS equation on a star graph.* J. Differential Equations 257 (2014) 3738–3777.

[2] A. H. Ardila,*Logarithmic NLS equation on star graphs: existence and stability of standing waves.* Differential and Integral Equations, 30 (2017), no. 9/10, 735–762 .

[3] B. Bellazzini, M. Mintchev. *Quantum field theory on star graphs.* Phys. A: Math. Theor. 39 (2006) 1101–1117.

[4] G. Berkolaiko, C. Carlson, S. Fulling, P. Kuchment. *Quantum Graphs and Their Applications,* volume 415 of Contemporary Math. American Math. Society, Providence, RI, 2006.

[5] J.L. Bona, R.Cascaval, Nonlinear dispersive waves on trees. Can. J. App. Math. 16 (2008) 1–18.

[6] J. L. Bona, S. M. Sun, B.- Y. Zhang, *Boundary smoothing properties of the Korteweg-de Vries equation in a quarter plane and applications,* Dyn. Partial Differ. Equ. 3 (2006), 1–70.

[7] J. Bona, S. M. Sun, B.-Y Zhang, *Non-homogeneous boundary value problems for the Korteweg-de Vries and the Korteweg-de Vries-Burgers equations in a quarter plane.* Ann. Inst. H. Poincar Anal. Non Linéaire 25 (2008), no. 6, 1145–1185.

[8] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations,* Geometric and Funct. Anal., Vol.3 (1993), 107–156.

[9] Cacciapuoti C, Finco D, Noja D, *Ground state and orbital stability for the NLS equation on a general starlike graph with potentials,* Nonlinearity, 30, 8, 3271–3303 (2017)

[10] V. Caudrelier, *On the Inverse Scattering Method for Integrable PDEs on a Star Graph.* Communications in Mathematical Physics, 338 (2015), no. 2, 893–917.

[11] M. Cavalcante, *The initial-boundary-value problem for some quadratic nonlinear Schrödinger equations on the half-line,* Differential and Integral Equations, Vol. 30 (2017), no. 7/8, 521–554.

[12] M. Cavalcante, A. J. Corcho, *The initial boundary value problem for the Schrödinger-Korteweg-de Vries system on the half-line*, preprint, arxiv: 1608.06308.

[13] G. P. Chuiko, O. V. Dvornik, S. I. Shyian, y. A. Baganov. *A new age-related model for blood stroke volume.* Computers in Biology and Medicine, 79 (2016) 144–148.

[14] J. E. Colliander, C. E. Kenig, *The generalized Korteweg-de Vries equation on the half line*, Comm. Partial Differential Equations, 27 (2002), no. 11/12, 2187–2266.

[15] E. Crépeau, M. Sorine *A reduced model of pulsatile flow in an arterial compartment.* Chaos Solitons Fractals, 34 (2007), no. 2, 594–605.

[16] B. Deconinck, N. E. Sheils, D. A. Smith,*The Linear KdV Equation with an Interface.* Commun. Math. Phys. 347 (2016), 489509.

[17] A. V. Faminskii, *An initial boundary-value problem in a half-strip for the Korteweg-de Vries equation in fractional-order Sobolev spaces,* Comm. Partial Differential Eq. 29 (2004), no. 11/12, 1653–1695.

[18] Z. Guo, *well-posedness of Korteweg-de Vries equation in $H^{-\frac{3}{4}}(\mathbb{R})$.* J. Math. Pures Appl. (9) 91 , no. 6, (2009) 583–597.

[19] J. Holmer, *The initial-boundary value problem for the 1D nonlinear Schrödinger equation on the half-line*, Differential and Integral equations, 18 (2005) 647–668.

[20] C. Jia, I. Rivas, B. Y. Zhang, *Lower regularity solutions of a class of non-homogeneous boundary value problems of the Korteweg-de Vries equation on a finite domain.* Adv. Differential Equations 19, no. 5-6, (2014) 559–584.

[21] J. Holmer, *The initial-boundary value problem for the Korteweg-de Vries equation*, Comm. Partial Differential Equations, 31 (2006) 1151–1190.

[22] T. Kato, *On nonlinear Scrödinger equations. II. H^s- solutions and unconditional well-posedness,* J. d'Analyse Math., 67 (1995), 281–306.

[23] C. E. Kenig, G. Ponce, L. Vega, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices.* Duke Math. J. 71(1) (1993) 121.

[24] N. Kishimoto.*Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity.* Differential Integral Equations, 22(5/6) (2009) 447–464.

[25] D. J. Korteweg, G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves.* Philos. Mag. 39 (1895) 422–443.

[26] F. A. Mehmeti, J. von Below, S. Nicaise, editors.*Partial Differential equations on multistrucutres,* number 219 in Lecture Notes in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, (2001).

[27] D. Mugnolo, D. Noja, C. Seifert. *Airy-type evolution equations on star graphs.* arXiv preprint arXiv:1608.01461 (2016).

[28] D. Mugnolo, J.F. Rault,*Construction of exact travelling waves for the Benjamin-Bona-Mahony equation on networks,* Bull. Belg. Math. Soc. Simon Stevin. 21 (2014) 415–436.

[29] D. Noja, *Nonlinear Schrödinger equation on graphs: recent results and open problems,* Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 372 no. 2007, 20130002 (2014), 20 pp.

[30] Z.A. Sobirov, M.I.Akhmedov, H.Uecker, *Cauchy problem for the linearized KdV equation on general metric star graph,* Nanosystems, 6 (2015) 198–204.

[31] Z.A. Sobirov, M. I. Akhmedov, H. Uecker, *Exact solution of the Cauchy problem for the linearized KdV equation on metric star graph,* Uzbek Math. J. 3 (2015) 143–154.

[32] Z.A. Sobirov, M.I.Akhmedov, O.V.Karpova, B. Jabbarova, *Linearized KdV equation on a metric graph,* Nanosystems, 6 (2015) 757–761.