NONLINEAR FRACTIONAL ELLIPTIC PROBLEM WITH SINGULAR TERM AT THE BOUNDARY

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain, $0 < s < 1$ and $N > 2s$. We consider

$$(P)\begin{cases} (-\Delta)^s u &= \dfrac{u^q}{d^{2s}} \quad \text{in } \Omega,\\ u &> 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < q \le 2^* - 1$, $0 < s < 1$ and $d(x) = dist(x, \partial\Omega)$. The main goal of this paper is to analyze the existence of solution to problem (P) according to the value of s and q.

1. Introduction

In this paper we deal with the following problem

$$\begin{cases} (-\Delta)^s u = \dfrac{u^q}{d^{2s}} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.0}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, $0 < s < 1$, $q \ge 0$ and $d(x) = dist(x, \partial\Omega)$. For $0 < s < 1$, the fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) := a_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy,\ s \in (0,1), \tag{1.1}$$

where

$$a_{N,s} := 2^{2s-1}\pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}$$

is the normalization constant such that the identity

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u),\ \xi \in \mathbb{R}^N, s \in (0,1),\ u \in \mathcal{S}(\mathbb{R}^N),$$

holds, where $\mathcal{F}u$ denotes the Fourier transform of u and $\mathcal{S}(\mathbb{R}^N)$, the Schwartz class of tempered functions.

The problem (1.0) is related to the following Hardy inequality, proved in [15], see also [23] and the references therein. More precisely, it is well known that if there exists $x_0 \in \partial\Omega$ such that $\partial\Omega \cap B(x_0, r) \in \mathcal{C}^1$ and $s \in [\frac{1}{2}, 1)$, then there exists a positive constant $C \equiv C(\Omega, N, s)$ such that for all $\phi \in C_0^\infty(\Omega)$,

$$\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dxdy \ge C \int_\Omega \frac{\phi^2}{d^{2s}} dx. \tag{1.2}$$

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where

$$D_\Omega \equiv \mathbb{R}^N \times \mathbb{R}^N \setminus \big(\mathcal{C}\Omega \times \mathcal{C}\Omega\big).$$

In the case where Ω is a convex domain, then the constant C does not depend on Ω and it is given by

$$K_{N,s} = \frac{2^{1-2s}\pi^{\frac{N-2}{2}}\Gamma(1-s)\Gamma^2(s+\frac{2}{2})}{s\Gamma(\frac{N+2s}{2})}$$

We refer to [23] and the references therein for more details about the Hardy inequality. Notice that if $0 < C(\Omega, N, s) < K_{N,s}$, then $C(\Omega, N, s)$ is achieved, hence we get the existence of $\overline{u}$ that solves the eigenvalue problem

$$\left\{\begin{array}{rcll} (-\Delta)^s \overline{u} &=& C\dfrac{\overline{u}}{d^{2s}} & \text{in } \Omega, \\ \overline{u} &>& 0 & \text{in } \Omega, \\ \overline{u} &=& 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{array}\right. \tag{1.3}$$

In the local case, $s = 1$, problem (1.0) with general q was considered recently in [2]. The authors proved a strong non existence result if $q < 1$, however, for $q > 1$, they proved the existence of a positive solution using suitable blow-up technics and the concentration compactness argument. The main goal of this paper is to analyze the nonlocal case $s \in (0, 1)$.

In the case where $q < 1$, for all $s \in (0, 1)$, we are able to show the existence of solution u, in a suitable sense. This result make a significative difference in comparing with the local case, $s = 1$, where a strong non existence result is proved for all $q < 1$. This seems bo be surprising since the fractional Laplacian has less regularizing effect than the classical Laplacian.

In the case $1 < q \le 2^*$ and $s \in [\frac{1}{2}, 1)$, we will show the existence of an energy solution as in the local case, however, taking into consideration the nonlocal nature of the operator, the proofs are more complicated and fine computations are needed in order to get compactness results and apriori estimates. Notice that the hypothesis $s \in [\frac{1}{2}, 1)$ is needed since we will use systematically the Hardy inequality and some Liouville type results that hold for $s \ge \frac{1}{2}$.

The paper is organized as follows. In the first section we give some auxiliary results, the concepts of solutions that we will use and some functional tools that will be needed along of the paper.

The case $q < 1$ is considered in Section 2. We will prove that the situation is totally different comparing with the local case. Namely, for all $s \in (0, 1)$ and for all $q \in (0, 1)$, we show the existence of a distributional solution to problem (1.0). In the case where $s < \frac{1}{2}$ and under suitable condition on q, we are able to prove that the solution is in a suitable fractional Sobolev space. This makes the nonlocal case $s \in (0, 1)$ totally different with respect to the local case $s = 1$.

In Section 3, we treat the case $1 < q \le 2^*_s - 1$. The main idea is to combine extension of Caffarelli-Silvestre and the blowing-up arguments based on Liouville type theorems. Hence we ill assume that $s \in [\frac{1}{2}, 1)$. In the first subcritical we treat the case $q < 2^*_s - 1$, then, as in [2], using suitable variational arguments and Blow-up technics, we are able to prove the existence of a bounded positive solution. The critical case, $q = 2^*_s - 1$, is studied in subsection 4.2 under the hypothesis

$\Omega = B_R(0)$. Then taking advantage of the radial structure of the problem, we are able to show the existence of a nontrivial radial solution.

Finally, in the last section we deal with the case $q < 0$.

2. The functional setting and tools

Let $s \in (0,1)$ and $\Omega \subset \mathbb{R}^N$. We define the fractional Sobolev space $H^s(\Omega)$ as

$$H^s(\Omega) \equiv \Big\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy < +\infty \Big\}.$$

$H^s(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_{H^s(\Omega)} = ||u||_{L^2(\Omega)} + \Big(\frac{a_{d,s}}{2} \int_\Omega \int_\Omega |u(x)-u(y)|^2 ddxdy \Big)^{\frac{1}{2}}.$$

Since we are working in a bounded domain, then we will use the space $H^s_0(\Omega)$,

$$H^s_0(\Omega) = \{u \in H^s(\mathbb{R}^N) \text{ with } u = 0 \text{ a.e } \mathbb{R}^N \backslash \Omega\}$$

endowed with the norm

$$||u||^2_{H^s_0(\Omega)} = \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy,$$

where $D_\Omega = \mathbb{R}^{2N} \backslash (\Omega \times \Omega)$. It is clear that $(H^s_0(\Omega), ||.||_{H^s_0(\Omega)})$ is a Hilbert space. We refer to [16] and [4] for more properties of the previous spaces.

The the next Sobolev inequality is proved in [16], see also [21] for a simple proof.

Theorem 2.1. *Assume that $0 < s < 1$ with $2s < N$. There exists a positive constant $S \equiv S(N,s)$ such that for all $u \in C^\infty_0(\mathbb{R}^N)$, we have*

$$S\Big(\int_{\mathbb{R}^N} |u(x)|^{2^*_s} dx \Big)^{\frac{2}{2^*_s}} \le \frac{a_{d,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+ps}} dxdy$$

with $2^*_s = \dfrac{2N}{N-2s}$.

We introduce the notions of solution to be use later. Notice that, for problem (1.0), we are looking the second term of the equation as an $L^1(\Omega)$ function and then we need to precise the sense in which the solution is defined. More precisely, we have the next definition.

Definition 2.2. *Assume that $h \in L^1(\Omega)$. We say that $u \in L^1(\Omega)$ is a weak solution to problem*

$$\left\{ \begin{array}{rcll} (-\Delta)^s u &=& h & \text{in } \Omega, \\ u &=& 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{array} \right. \tag{2.4}$$

if $u = 0$ in $\mathbb{R}^N \backslash \Omega$ and for all $\psi \in \mathbb{X}_s$, we have

$$\int\limits_\Omega u((-\Delta)^s \psi) dx = \int\limits_\Omega h\psi dx.$$

where

$$\mathbb{X}_s \equiv \Big\{ \psi \in \mathcal{C}(\mathbb{R}^N) \,|\, supp(\psi) \subset \overline{\Omega},\ (-\Delta)^s \psi(x) \text{ pointwise defined and } |(-\Delta)^s \psi(x)| < C \text{ in } \Omega \Big\}.$$

In the same way we define the sense of distributional solution to (2.4).

Definition 2.3. *Assume that $h \in L^1_{loc}(\Omega)$, we say that $u \in L^1(\Omega)$ is a distributional solution to problem (2.4) if for all $\psi \in \mathcal{C}_0^\infty(\Omega)$, we have*

$$\int\limits_{\Omega} u((-\Delta)^s \psi) dx = \int\limits_{\Omega} h\psi dx.$$

The next existence result is proved in [18], [13] and [3].

Theorem 2.4. *Assume that $h \in L^1(\Omega)$, then problem (2.4) has a unique weak solution u that is obtained as the limit of $\{u_n\}_{n\in\mathbb{N}}$, the sequence of the unique solutions to the approximating problems*

$$\left\{ \begin{array}{rcll} (-\Delta)^s u_n & = & h_n(x) & \text{in } \Omega, \\ u_n & = & 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{array} \right. \tag{2.5}$$

with $h_n = T_n(h)$ and $T_n(\sigma) = T_n(\sigma) = \max(-n, \min(n, \sigma))$. Moreover,

$$T_k(u_n) \to T_k(u) \text{ strongly in } H_0^s(\Omega), \quad \forall k > 0, \tag{2.6}$$

$$u \in L^\theta(\Omega)\,, \qquad \forall\ \theta \in \big(1, \frac{N}{N-2s}\big) \tag{2.7}$$

and

$$\big|(-\Delta)^{\frac{s}{2}} u\big| \in L^r(\Omega)\,, \qquad \forall\ r \in \big(1, \frac{N}{N-s}\big)\,. \tag{2.8}$$

Moreover,

$$u_n \to u \text{ strongly in } W_0^{s,q_1}(\Omega) \text{ for all } q_1 < \frac{N}{N-2s+1}. \tag{2.9}$$

Moreover, u is an entropy solution to problem (2.4) in the sense that

$$\iint_{R_\rho} \frac{|u(x)-u(y)|}{|x-y|^{N+2s}} dxdy \to 0 \text{ as } \rho \to \infty, \tag{2.10}$$

where

$$R_\rho = \Big\{(x,y) \in \mathbb{R}^{2N} : \rho{+}1 \le \max\{|u(x)|, |u(y)|\} \text{ with } \min\{|u(x)|, |u(y)|\} \le \rho \text{ or } u(x)u(y) < 0\Big\},$$

and for all $k > 0$ and $\varphi \in H_0^s(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{array}{l} \dfrac{1}{2}\displaystyle\iint_{D_\Omega} \frac{(u(x)-u(y))[T_k(u(x)-\varphi(x)) - T_k(u(y)-\varphi(y))]}{|x-y|^{N+2s}} dxdy \le \\ \displaystyle\int\limits_{\Omega} h(x) T_k(u(x)-\varphi(x))\, dx. \end{array} \tag{2.11}$$

As a consequence of Picone inequality to fractional operator, see [18], we have the next comparison principle proved in [18] that extends the one obtained by Brezis-Kamin in [9] for the local case.

Lemma 2.5. *Assume that $0 < s < 1$ and let $f(x,t)$ be a Caratheodory function such that $\dfrac{f(x,t)}{t}$ is decreasing for $t > 0$. Suppose $u, v \in H_0^s(\Omega)$ are such that*

$$\left\{ \begin{array}{ll} (-\Delta)^s u \ge f(x,u) & \text{in } \Omega, \\ (-\Delta)^s v \le f(x,v) & \text{in } \Omega. \end{array} \right. \tag{2.12}$$

Then $u \ge v$ a.e Ω.

Following the approach of Caffarelli-Silvestre in [11], we can formulate our problem (1.0) in a local setting by means of an auxiliary variable.

More precisely, setting $C_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$, the point in C_Ω is denoted by (x,y). The lateral boundary of cylinder will be denoted by $\partial_L\Omega = \partial\Omega \times (0,\infty)$. Now for $u \in H^s_0(\Omega)$, we define the $s-$hamonic extension $w = E_s(u)$ to the cylinder C_Ω as the solution to the problem

$$\left\{\begin{array}{rl} -\mathtt{div}(y^{1-2s}w) = 0 & \text{in } C_\Omega, \\ w = 0 & \text{in } \partial_L\Omega, \\ w = u & \text{on } \Omega \times \{0\}. \end{array}\right. \tag{2.13}$$

The extension function belongs to the space

$$X^s_0(C_\Omega) = \overline{\mathcal{C}^\infty_0(C_\Omega)}^{||.||_{X^s_0(C_\Omega)}}, \text{ with } ||w||_{X^s_0(C_\Omega)} = \left(\int_{C_\Omega} y^{1-2s}|\nabla w|^2 dxdy\right)^{\frac{1}{2}},$$

where k_s is a normalization constant, that gives an isometry between $H^s_0(\Omega)$ and $X^s_0(C_\Omega)$, as consequence, for all $\phi \in H^s_0(\Omega)$, we have that,

$$||E_s(\phi)||_{X^s_0(C_\Omega)} = ||\phi||_{H^s_0(\Omega)}. \tag{2.14}$$

Moreover, for any function $\psi \in X^s_0(C_\Omega)$, we have the following trace inequality

$$||\psi(.,0)||_{X^s_0(C_\Omega)} \le ||\psi||_{H^s_0(\Omega)}. \tag{2.15}$$

Now, going back to problem (2.13), we have

$$-\frac{1}{k_s}\lim_{y\to 0^+}\frac{\partial w(x,y)}{\partial y} = (-\Delta)^s u(x). \tag{2.16}$$

Now using the above approach we can formulate the nonlocal problem (1.0) in a local way,

$$\left\{\begin{array}{rl} -\mathtt{div}(y^{1-2s}w) = 0 & \text{in } C_\Omega, \\ w = 0 & \text{in } \partial_L\Omega, \\ -\dfrac{1}{k_s}\lim_{y\to 0^+}\dfrac{\partial w(x,y)}{\partial y} = \dfrac{u^q}{d^{2s}} & \text{on } \Omega \times \{0\}. \end{array}\right. \tag{2.17}$$

When dealing with blow-up arguments, we will use the next Liouville type results proved in [8].

Theorem 2.6. *Assume that $\frac{1}{2} \le s < 1$ and $N > 2s$. Then the problem*

$$\left\{\begin{array}{ll} -div(y^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \dfrac{\partial v}{\partial \nu^s} = Cv^q(x,0) & \text{on } \partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N, \end{array}\right.$$

has non bounded positive solution provided that $q < 2^*_s - 1$.

Theorem 2.7. *Let $\frac{1}{2} \le s < 1$ and $2s < N$. Suppose that $q < 2^*_s - 1$ and consider the first quarter*

$$\mathbb{R}^{N+1}_{++} = \{z = (x', x_N, y) | x' \in \mathbb{R}^{N-1}, x_N > 0, y > 0\}.$$

Then the problem

$$\left\{\begin{array}{ll} -div(y^{1-2s}\nabla v) = 0 & \text{in } x_N > 0, y > 0, \\ v(x',0,y) = 0 & , \\ \dfrac{\partial v}{\partial \nu^s} = Cv^q(x', x_N), & \end{array}\right.$$

has no positive bounded solution.

Finally in order to prove apriori estimates for approximating problem, we will use the next existence result obtained in [1].

Theorem 2.8. *Assume that $s \in (0,1)$ and $\beta \in (0, s+1)$, then the problem*

$$\left\{\begin{array}{rcll} (-\Delta)^s \phi & = & \dfrac{1}{d^\beta(x)} & \text{in } \Omega, \\ \phi & = & 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{array}\right. \tag{2.18}$$

has a distributional solution such that

A) *if $\beta < s$, then $\phi \backsimeq d^s$,*
B) *if $\beta = s$, then $\phi \backsimeq d^s \log(\frac{D}{d(x)})$,*
C) *if $\beta \in (s, s+1)$, then $\phi \backsimeq d^{2s-\beta}$.*

3. The sublinear case: $0 < q < 1$

In this section we are interested to find a positive solution to problem (1.0) for $0 < q < 1$, more precisely we have the following result.

Theorem 3.1. *Assume that $0 < q < 1$ and $0 < s < 1$, then problem (1.0) has a distributional positive solution in the sense of definition (2.3) such that $u(x) \geq Cd^s(x)$ in Ω.*

Proof. We divide the proof into two main cases according to the value of s.

The first case: $0 < s < \frac{1}{2}$

We proceed by approximation. Let u_n be the unique positive solution to

$$\left\{\begin{array}{rl} (-\Delta)^s u_n = \dfrac{u_n^q}{(d(x) + \frac{1}{n})^{2s}} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{array}\right. \tag{3.19}$$

It is clear that the existence of u_n follows using classical variational argument. However, the uniqueness holds using the comparison principle in Proposition 2.5. In the same way we reach that $u_n \leq u_{n+1}$ for all n.

Let ρ be the solution to

$$\left\{\begin{array}{rcll} (-\Delta)^s \rho & = & 1 & \text{in } \Omega, \\ \rho & = & 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{array}\right. \tag{3.20}$$

From [12], we know that $\rho \in C^\alpha(\overline{\Omega})$ where $\alpha \in (0, \min(2s, 1))$. In particular, since $0 < s < \frac{1}{2}$, then $\rho \leq Cd^{2s}$.

Using ρ as a test function in (3.20) and taking into consideration the previous estimate on ρ, it holds

$$\int\limits_\Omega u_n dx = \int\limits_\Omega \frac{u_n^q \rho}{(d(x) + \frac{1}{n})^{2s}} dx \leq C \int\limits_\Omega u_n^q dx.$$

Since $q < 1$, by Hölder inequality, we obtain

$$\int\limits_\Omega u_n dx \leq C, \text{ for all } n.$$

Thus we get the existence of a measurable function u such that $u_n \uparrow u$ strongly in $L^1(\Omega)$ as $n \to \infty$. It is clear that

$$\frac{u_n^q}{(d(x)+\frac{1}{n})^{2s}} \uparrow \frac{u^q}{d^{2s}} \quad \text{strongly in} \quad L^1_{loc}(\Omega).$$

Hence u is, at least, a distributional solution to problem (1.0).

Moreover, if $q < 1-2s$, then using Hölder inequality we can prove that $\frac{u^q}{d^{2s}} \in L^1(\Omega)$. Thus u is an entropy solution to (1.0) and then $u \in W_0^{s,\sigma}(\Omega)$ for all $\sigma < \frac{N}{N-2s-1}$, see [18] and [3].

Let prove now that $u \geq Cd^s$. Denote by ϕ_1 the first positive eigenfunction of the fractional operator, then we know that $\phi_1 \backsimeq d^s$. It is not difficult to see that $C\phi_1$ is a subsolution to problem (3.19) where $C > 0$ can be chosen independently of n. Hence by the comparison principle in Lemma 2.5, we conclude that $u_n \geq C\phi_1$. Passing to the limit as $n \to \infty$, we get the desired estimate.

The second case: $\frac{1}{2} \leq s < 1$. Let ϕ be the unique solution to problem (2.18) with $\frac{1}{2} \leq s < \beta < 1$. Taking ϕ as a test function in (3.19) and taking into consideration that $\phi \backsimeq d^{2s-\beta}$, we reach that

$$\int\limits_{\Omega} \frac{u_n}{d^\beta(x)} dx = \int\limits_{\Omega} \frac{u_n^q \phi}{(d(x)+\frac{1}{n})^{2s}} dx \leq C \int\limits_{\Omega} \frac{u_n^q}{d^\beta} dx.$$

Since $\beta < 1$, by Hölder inequality, we obtain

$$\int\limits_{\Omega} \frac{u_n}{d^\beta(x)} dx \leq \left(\int\limits_{\Omega} \frac{u_n}{d^\beta(x)} dx \right)^q \left(\int\limits_{\Omega} \frac{1}{d^\beta} dx \right)^{1-q},$$

Hence

$$\int\limits_{\Omega} \frac{u_n}{d^\beta} dx \leq C.$$

As consequence, we get the existence of a measurable function u such that $u_n \uparrow u$ in $L^1(\Omega)$ and $\frac{u_n}{(d(x)+\frac{1}{n})^\beta} \uparrow \frac{u}{d^\beta}$ strongly in $L^1(\Omega)$. It is clear that u solves problem (1.0), at least, in the distributional sense.

Notice that, since $u_n = 0$ in the set $\mathbb{R}^N \backslash \Omega$, then, in any case, it holds that $u = 0$ a.e. in $\mathbb{R}^N \backslash \Omega$. $\square$

Remarks 3.2. *In the local case, $s = 1$, the authors in* [2] *proved a strong non existence result to problem* (1.0) *for all $q < 1$ and as a consequence they get a complete blow-up for the approximating problems. Hence our existence result in Theorem 3.1 show a significative difference between the local and the non local case.*

4. The superlinear case: $1 < q \leq 2_s^* - 1$

In this section we are interested to find a positive solution to problem (1.0) in the superlinear case $1 < q \leq 2_s^* - 1$. According to the value of q, we will consider two main cases: the subcritical case $q < 2_s^* - 1$ and the critical case $q = 2_s^* - 1$.

4.1. **Subcritical case:** $q < 2_s^* - 1$**.** The main idea in this case is to use the Caffarelli-Silvestre extension developed in [11]. More precisely, in order to prove the existence of solution to problem (1.0), we will prove that the problem

$$\left\{\begin{array}{ll} -\mathtt{div}(y^{1-2s}\nabla w) = 0 & \text{in } C_\Omega, \\ w = 0 & \text{in } \partial_L\Omega, \\ \dfrac{\partial w(x,y)}{\partial \nu^s} = \dfrac{w^q(x,0)}{d^{2s}} & \text{on } \Omega\times\{0\}, \end{array}\right. \tag{4.21}$$

where

$$\frac{\partial w}{\partial \nu^s}(x,y) = -\frac{1}{k_s}\lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}(x,y),$$

has a non negative solution. More precisely we have the next result.

Theorem 4.1. *Assume that* $\frac12 \le s < 1$ *and* $1 < q < 2_s^* - 1$*, then problem* (4.21) *has a bounded positive solution* $w \in X_0^s(C_\Omega)$*.*

Proof. We follow closely the arguments used in [2]. We argue by approximation. Let $w_n \in X_0^s(C_\Omega)) \cap L^\infty(\Omega)$ be the "mountain pass solution " to the approximated problem

$$\left\{\begin{array}{ll} -\mathtt{div}(y^{1-2s}\nabla w_n) = 0 & \text{in } C_\Omega, \\ w = 0 & \text{in } \partial_L\Omega, \\ \dfrac{\partial w_n}{\partial \nu^s}(x,y) = \dfrac{w_n^q}{(d+\frac1n)^{2s}}(x,0) & \text{on } \Omega\times\{0\}, \end{array}\right. \tag{4.22}$$

thus w_n is a critical point of the functional

$$J_n(v) = \frac{k_s}{2}\int_{C_\Omega} y^{1-2s}|\nabla v|^2 dxdy - \frac{1}{q+1}\int\limits_\Omega \frac{|v(x,0)|^{q+1}}{(d(x)+\frac1n)^2}dx.$$

It is clear that $J_n(w_n) = c_n$ is the mountain pass energy level defined by

$$c_n = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$$

and

$$\Gamma = \{\gamma \in \mathcal{C}([0,1],\mathbb{R} \text{ with } \gamma(0)=0 \text{ and } \gamma(1) = v_1 \in X_0^s(C_\Omega), J_n(v_1) < 0\},$$

where $v_1 \in X_0^s(C_\Omega)$ is chosen such that

$$J_n(v_1) << 0 \quad \text{uniformly in} \quad n.$$

Since

$$c_n = \frac{k_s(q-1)}{2(q+1)}\int\limits_\Omega y^{1-2s}|\nabla w_n|^2 dxdy$$

and

$$0 \le c_n \le \max_{t\in[0,\infty)} J_n(tv_1) \le C \text{ for all } n \in I\!N,$$

as a consequence we have that the sequence $\{w_n\}_n$ is bounded in $X_0^s(C_\Omega)$.

We claim that

$$||w_n||_\infty \le C, \qquad \text{for all } n. \tag{4.23}$$

Assume by contradiction that there exists a sequence $\{w_n\} \subset X_0^s(C_\Omega)$ of solutions to (4.22) such that $||w_n||_\infty = M_n \to \infty$ as $n \to \infty$. By the Maximum Principle, it hods that the maximum of w_n is attained at a point $(x_n, 0)$ where $x_n \in \Omega$. Since

$\{x_n\}_n \subset \Omega$ is a bounded sequence, we get the existence of $\overline{x} \in \overline{\Omega}$ such that, up to a subsequence, $x_n \to \overline{x}$.

(1) First case : $\overline{x} \in \Omega$. We consider the scaled function

$$v_n(x,y) = \frac{w_n(\mu_n x + x_n, \mu_n y)}{M_n} \text{ for } x_n \in \Omega, y > 0,$$

where $\mu_n = M_n^{\frac{1-q}{2s}}$. Then v_n solves

$$\left\{\begin{array}{rcl} -div(y^{1-2s}\nabla v_n) = 0 & & \text{in } C_{\Omega_n}, \\ v_n = 0 & & \text{in } \partial_L \Omega_n, \\ \dfrac{\partial v_n}{\partial \nu^s} = \dfrac{v_n^q}{(d(\mu_n x + x_n) + \frac{1}{n})^{2s}} & & \text{on } \Omega_n \times \{0\}, \end{array}\right. \tag{4.24}$$

and $\Omega_n = \dfrac{1}{\mu_n}(\Omega - x_n)$. It is clear that for (x,y) fixed, $d(\mu_n x + x_n) + \frac{1}{n} \to d(\overline{x}) = C$ as $n \to \infty$.

By proposition 5.4 in [8], we have that $v_n \in C^\gamma$ and $||v_n||_{C^\gamma} \le C$, with $0 < \gamma < 1$. Passing to the limit as $n \to \infty$, we get the existence of $v \in \mathcal{C}^\gamma(\mathbb{R}_+^{N+1}) \cap L^\infty(\mathbb{R}_+^{N+1})$ such that $v(x,y)) \le v(0,0) = 1$ and $v > 0$ solves

$$\left\{\begin{array}{rcl} -div(y^{1-2s}\nabla v) = 0 & & \text{in } \mathbb{R}_+^{N+1}, \\ \dfrac{\partial v}{\partial \nu^s} = Cv^q & & \text{on } \partial\mathbb{R}_+^{N+1}. \end{array}\right.$$

Since $q < 2^* - 1$, we get a contradiction with the non existence result of Theorem 2.6.

(2) Second case : $\overline{x} \in \partial\Omega$. In this case we set $\mu_n = M_n^{\frac{1-q}{2s}}(d(x_n) + \frac{1}{n})$, then v_n solves

$$\left\{\begin{array}{rcl} -div(y^{1-2s}\nabla v_n) = 0 & & \text{in } C_{\Omega_n}, \\ v_n = 0 & & \text{in } \partial_L \Omega_n, \\ \dfrac{\partial v_n}{\partial \nu^s} = \Big(\dfrac{d(x_n) + \frac{1}{n}}{d(\mu_n x + x_n) + \frac{1}{n}}\Big)^{2s} v_n^q & & \text{on } \Omega_n \times \{0\}. \end{array}\right. \tag{4.25}$$

Fixed $z \in \mathbb{R}^N$, then $\dfrac{d(x_n) + \frac{1}{n}}{d(\mu_n x + x_n) + \frac{1}{n}} \to 1$, as $n \to \infty$. Thus passing to the limit as $n \to \infty$, we get the existence of v such that either, $v \in \mathcal{C}^\gamma(\mathbb{R}_+^{N+1}) \cap L^\infty(\mathbb{R}_+^{N+1})$ where $\gamma \in (0,1)$, $v(x,y) \le v(0,0) = 1$ and $v > 0$ solves

$$\left\{\begin{array}{rcl} -div(y^{1-2s}\nabla v) = 0 & & \text{in } \mathbb{R}_+^{N+1}, \\ \dfrac{\partial v}{\partial \nu^s} = Cv^q & & \text{on } \partial\mathbb{R}_+^{N+1}, \end{array}\right.$$

or , if the domains Ω_n converge (up to a rotation) to some half space

$$D_s = \{x \in \mathbb{R}^N \text{ with } x_n > 0\},$$

we obtain here that v is solution to

$$\left\{\begin{array}{rcl} -div(y^{1-2s}\nabla v) = 0 & & \text{in } D_s \times (0,\infty), \\ \dfrac{\partial v}{\partial \nu^s} = Cv^q & & \text{on } D_s \times \{0\}, \end{array}\right.$$

with $||v||_\infty = 1$ and $v(0,0) = 1$. Since $q < 2^* - 1$, we again get a contradiction with the non existence result of Theorem 2.7. Hence the claim follows.

Let us prove now that the sequence $\{w_n\}$ is bounded from below, namely that

$$||w_n||_\infty \geq \overline{C} > 0 \text{ for all } n. \tag{4.26}$$

If the conclusion (4.26) is false, then we get the existence of a subsequence of $\{w_n\}$ denoted also by $\{w_n\}_n$ such that $||w_n||_\infty \to 0$ as $n \to \infty$. Hence

$$\int_{C_\Omega} y^{1-2s}|\nabla w_n|^2 dxdy = \frac{1}{k_s}\int\limits_\Omega \frac{w_n^q(x,0)}{(d(x)+\frac{1}{n})^2}dx \leq \frac{||w_n||_\infty^{q-1}}{k_s}\int\limits_\Omega \frac{w_n^2(x,0)}{(d(x)+\frac{1}{n})^2}dx.$$

Taking n large, we obtain that $||w_n||_{L^\infty}^{q-1} << d_s$. This leads to a contradiction with the trace Hardy inequality . Hence we conclude that $||w_n||_{L^\infty} \geq \overline{C}$ for all n.

Recall that $w_n(x_n, 0) = ||w_n||_{L^\infty}$. We claim that

$$d(x_n) \equiv \text{dist}(x_n, \partial\Omega) > C_1 > 0, \quad \text{for all} \quad n.$$

We proceed by contradiction. Assume that, for some subsequence, $x_n \to \overline{x} \in \partial\Omega$ and $||w_n||_{L^\infty} = w_n(x_n) \to C_2 \geq \overline{C}$ as $n \to \infty$. Then as above, we set

$$v_n(x,y) = \frac{w_n(\mu_n x + x_n, \mu_n y)}{M_n}$$

where

$$\mu_n = M_n^{\frac{1-q}{2s}}\Big(d^2(x_n) + \frac{1}{n}\Big)^{\frac{1}{2s}}.$$

Thus, we obtain that $\mu_n \to 0$ as $n \to \infty$. Following the same blow-up analysis As above, we reach that $v_n \to v$ strongly in $\mathcal{C}^\gamma(\mathbb{R}_+^{N+1}) \cap L^\infty(\mathbb{R}_+^{N+1})$ where v solves

$$\begin{cases} -div(y^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \dfrac{\partial v}{\partial \nu^s} = Cv^q & \text{on } \partial\mathbb{R}_+^{N+1}. \end{cases}$$

This is a contradiction with Theorem 2.6. Hence the claim follows.

Then we conclude that $\{w_n\}_n$ is bounded in $X_0^s(C_\Omega)\cap L^\infty(C_\Omega)$ and hence there exists $w \in X_0^s(C_\Omega) \cap L^\infty(C_\Omega)$ such that

$$w_n \rightharpoonup w \text{ weakly in } X_0^s(C_\Omega) \qquad \text{and} \quad w_n \to w \text{ strongly in } L^p(C_\Omega),$$

for all $p \geq 1$.

To finish, we have just to prove that $w \not\equiv 0$. Assume by contradiction that $w \equiv 0$, then $w_n \to 0$ strongly in $L^p(\Omega)$ for all $p \geq 1$. We claim that

$$\int\limits_\Omega y^{1-2s}|\nabla w_n|^2\phi_1 \to 0 \qquad \text{as} \quad n \to \infty,$$

where ϕ_1 is the first eigenfunction to

$$\begin{cases} -div(y^{1-2s}\nabla\phi_1) = 0 & \text{in } \Omega, \\ \phi_1 = 0 & \text{in } \partial_L\Omega, \\ \dfrac{\partial\phi_1}{\partial\nu^s} = \lambda_1\phi_1(x,0) & \text{on } \Omega\times\{0\}. \end{cases}$$

To prove the claim, we take $w_n(\phi_1 + \frac{c}{n})$ as a test function in (4.22) with $c \geq \sup\limits_{\bar{\Omega}} \dfrac{\phi_1(x,0)}{d^s(x)}$. Then, it holds that

$$\int_{C_\Omega} y^{1-2s}|\nabla w_n|^2(\phi_1 + \frac{c}{n})dxdy + \int_{C_\Omega} y^{1-2s} w_n \nabla w_n \nabla \phi_1 dxdy \leq C \int\limits_\Omega \frac{w_n^{q+1}}{d^s + \frac{1}{n}} dx.$$

Thus

$$\begin{aligned}
\int_{C_\Omega} y^{1-2s}|\nabla w_n|^2(\phi_1 + \frac{c}{n})dxdy + \frac{\lambda_1}{2}\int\limits_\Omega w_n^2\phi_1 dx &\leq C\int\limits_\Omega \frac{w_n^{q+1}}{d^s + \frac{1}{n}}dx \\
&\leq C\Big(\int\limits_\Omega \frac{w_n^{q+1}}{(d^s + \frac{1}{n})^2}dx\Big)^{\frac{1}{2}}\Big(\int\limits_\Omega w_n^{q+1}dx\Big)^{\frac{1}{2}} \\
&\leq C\Big(\int\limits_\Omega w_n^{q+1}dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

Taking into consideration that

$$\int\limits_\Omega w_n^{q+1}dx \to 0 \text{ as } n \to \infty,$$

we reach that $\int_{C_\Omega} y^{1-2s}|\nabla w_n|^2\phi_1 dxdy \to 0$ and the claim follows.

By the elliptic regularity, we conclude that $w_n \to 0$ strongly in $\mathcal{C}^\gamma_{loc}(\overline{C_\Omega})$. Since $d(x_n) \geq C > 0$ for all n, then up to a subsequence, $w_n(x_n, 0) \to 0$ as $n \to \infty$, a contradiction with (4.26). Hence $w \gneqq 0$ and then the existence result follows. $\square$

4.2. **The critical case $q = 2^*_s - 1$.** In this section we will consider (1.0) with $q = 2^*_s - 1$ and $\Omega = B_R(0)$ is the ball of radius R centered at the origin. We define the space

$$H^s_{0,ra}(B_R(0)) \equiv \{u \in H^s_0(B_R(0)) : \ u \text{ radial } \}.$$

Our main existence result is the following.

Theorem 4.2. *Assume that $q = 2^* - 1$ and that $\Omega = B_R(0) \subset \mathbb{R}^N$ with $N > 2s$. Then problem (1.0) has a bounded positive radial solution $u \in H^s_{0,ra}(B_R(0))$.*

Let us define $S(R)$ as follows,

$$S(R) \equiv \inf_{\phi \in H^s_{0,ra}(B_R(0))} \frac{\dfrac{a_{N,s}}{2}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\phi(x)-\phi(y))^2}{|x-y|^{N+2s}}dxdy}{\Big(\displaystyle\int_{B_R(0)} \frac{|\phi|^{2^*_s}}{d^{2s}(x)}dx\Big)^{\frac{2}{2^*_s}}}. \tag{4.27}$$

In order to prove Theorem 4.2, it suffices to show that $S(R)$ is archived.

We begin by the next Proposition.

Proposition 4.3. *We have*

(1) $S(R) > 0$ *for all* $R > 0$,
(2) $S(R) = R^{\frac{4s}{2^*_s}} S(1)$.

Proof. The proof follows the same arguments as in [2]. For the reader convenience, we include here some details. Since $\phi \in H^s_{0,ra}(B_R(0))$, then from [19], it holds that

$$|\phi(x)| \leq C|x|^{-\frac{N-2s}{2}} ||\phi||_{H^s_{0,ra}(B_R(0))} \tag{4.28}$$

with $C \equiv C(N,s)$. Let $0 < R_1 < R$, then

$$\begin{aligned}\int_0^R \frac{|\phi|^{2^*_s}}{(R-r)^{2s}} r^{N-1} dr &= \int_0^{R_1} \frac{|\phi|^{2^*_s}}{(R-r)^{2s}} r^{N-1} dr + \int_{R_1}^R \frac{|\phi|^{2^*_s}}{(R-r)^{2s}} r^{N-1} dr \\ &= I_1(R_1) + I_2(R_1).\end{aligned}$$

Using Sobolev inequality, we obtain that

$$I_1(R_1) \leq \frac{1}{(R-R_1)^{2s}} \int_0^R |\phi|^{2^*_s} r^{N-1} dr \leq C(R, R_1, N)\, ||\phi||^{2^*_s}_{H^s_{0,ra}(B_R(0))}. \tag{4.29}$$

Respect to I_2, since $\phi \in H^s_{0,ra}(B_R(0))$, using (4.28), we reach that

$$\begin{aligned}|\phi(x)|^{2^*_s} &= |\phi(x)|^{2^*_s - 2} |\phi(x)|^2 \\ &\leq C|x|^{-2s} ||\phi||^{2^*_s - 2}_{H^s_{0,ra}(B_R(0))} |\phi(x)|^2 \\ &\leq CR_1^{-2s} ||\phi||^{2^*_s - 2}_{H^s_{0,ra}(B_R(0))} |\phi(x)|^2.\end{aligned}$$

Thus, using Hardy inequality,

$$I_2(R) \leq CR_1^{-2s} ||\phi||^{2^*_s - 2}_{H^s_{0,ra}(B_R(0))} \int_0^R \frac{|\phi|^2}{(R-r)^{2s}} r^{N-1} dr \leq CR_1^{-2s} ||\phi||^{2^*_s}_{H^s_{0,ra}(B_R(0))}. \tag{4.30}$$

Therefore, by (4.29) and (4.30) we reach that $S(R) \geq \dfrac{1}{C(N, R, R_1)} > 0$.

The second point follows using a rescaling argument. □

We are now able to proof Theorem 4.2.

Proof of Theorem 4.2.

Taking into consideration the second point in Proposition 4.3, then, we have just to show that $S(R)$ is achieved for some $R > 0$.

From the second point in Proposition 4.3, we get the existence of $R < 1$ such that $S(R) < S$, the Sobolev constant defined in Theorem 2.1. Fix a such R and let $\{u_n\}_n \subset H^s_{0,ra}(B_R(0))$, be a minimizing sequence of $S(R)$ with

$$\int_0^R \frac{|u_n|^{2^*_s}}{(R-r)^{2s}} r^{N-1} dr = 1.$$

Without loss of generality we can assume that $u_n \geq 0$. Thus

$$||u_n||_{H^s_{0,ra}(B_R(0))} \leq C.$$

Hence we get the existence of $u \in H^s_{0,ra}(B_R(0))$ such that $u_n \rightharpoonup u$ weakly in $H^s_{0,ra}(B_R(0))$, $u_n \to u$ strongly in $L^\sigma(B_R(0))\, \forall\, \sigma < 2^*_s$ and $u_n \to u$ strongly in $L^\sigma(B_R(0) \backslash B_\varepsilon(0))$ for all $\sigma > 1$ and for all $\varepsilon > 0$.

We claim that $u \neq 0$ and then u solves (1.0) with $q = 2^*_s - 1$.

We argue by contradiction. Assume that $u \equiv 0$, then $u_n \to 0$ strongly in $L^\sigma(B_R(0)\backslash B_\varepsilon(0))$ for all $\sigma > 1$ and for all $\varepsilon > 0$. Fix $0 < R_1 < R$, then

$$\frac{u_n^{2_s^*}}{(R-r)^{2s}} r^{N-1} \leq C R_1^{-2s} ||u_n||^{2_s^*-2}_{H^s_{0,ra}(B_R(0))} \frac{u_n^2}{(R-r)^{2s}} r^{N-1}.$$

Since by Hardy inequality $\displaystyle\int_0^R \frac{u_n^2}{(R-r)^{2s}} r^{N-1} < \infty$, then by the dominated convergence theorem, it follows that

$$\int_{R_1}^R \frac{|u_n|^{2_s^*}}{(R-r)^{2s}} r^{N-1} dr \to 0 \text{ as } n \to \infty.$$

Thus, for all $1 < R_1 < R$, we have

$$\int_{B_{R_1}(0)} \frac{|u_n|^{2_s^*}}{(R-|x|)^{2s}} dx \to 1 \text{ as } n \to \infty.$$

Hence, in order to show the compactness of the sequence $\{u_n\}_n$ we have to ovoid any concentration in zero.

Using Ekeland variational principle, we obtain that, up to a subsequence,

$$(-\Delta)^s u_n = S(R) \frac{u_n^{2_s^*-1}}{(R-|x|)^{2s}} + o(1). \tag{4.31}$$

Now, by the concentration compactness principle, see [20], and using the fact that u_n is a radial function, it follows that

$$|u_n|^{2_s^*} \rightharpoonup \nu = \nu_0 \delta_0, \quad |(\Delta)^{\frac{s}{2}} u_n|^2 \rightharpoonup \mu \geq \widetilde{\mu} + \mu_0 \delta_0, \tag{4.32}$$

with

$$S \nu_0^{\frac{2}{2_s^*}} \leq \mu_0, \tag{4.33}$$

where $\widetilde{\mu}$ is a positive measure with $supp(\widetilde{\mu}) \subset \overline{B_R(0)}$.
For $\varepsilon > 0$, we consider $\phi_\varepsilon \in \mathcal{C}_0^\infty(B_R(0)) \cap H^s_{0,ra}(B_R(0))$ such that $0 \leq \phi_\varepsilon \leq 1$,

$$\phi_\varepsilon \equiv 1 \quad \text{in } B_\varepsilon(0), \quad \phi_\varepsilon \equiv 0 \quad \text{in } B_R(0)\backslash B_{2\varepsilon}(0) \text{ and } \frac{|\phi_\varepsilon(x) - \phi_\varepsilon(y)|}{|x-y|} \leq \frac{C}{\varepsilon}.$$

Using $u_n \phi_\varepsilon$ as a test function in (4.31), it holds that

$$\int_{B_R(0)} \phi_\varepsilon u_n (-\Delta)^s u_n dx = S(R) \int_{B_R(0)} \frac{u_n^{2_s^*} \phi_\varepsilon}{(R-|x|)^{2s}} dx + o(1). \tag{4.34}$$

It is clear that

$$S(R) \int_{B_R(0)} \frac{u_n^{2_s^*} \phi_\varepsilon}{(R-|x|)^{2s}} dx \to \nu_0 S(R) \text{ as } n \to \infty \text{ and } \varepsilon \to 0.$$

On the other hand, taking into consideration the properties of operator $(-\Delta)^s$, we obtain that

$$\begin{aligned}
\int_{B_R(0)} \phi_\varepsilon u_n (-\Delta)^s u_n dx &= \int_{B_R(0)} \phi_\varepsilon |(\Delta)^{\frac{s}{2}} u_n|^2 + \int_{B_R(0)} u_n (-\Delta)^{\frac{s}{2}} \phi_\varepsilon (-\Delta)^{\frac{s}{2}} u_n dx \\
&- \frac{a_{N,s}}{2} \int_{B_R(0)} u_n(x) \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(\phi_\varepsilon(x) - \phi_\varepsilon(y))}{|x-y|^{N+s}} dx dy \\
&= A_1(\varepsilon, n) + A_2(\varepsilon, n) + A_3(\varepsilon, n).
\end{aligned}$$

We will estimate each term in the last identity.

Since $supp(\widetilde{\mu}) \subset \overline{B_R(0)}$, then using (4.32), letting $n \to \infty$ and $\varepsilon \to 0$, it holds that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} A_1(\varepsilon,n) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{B_R(0)} \phi_\varepsilon |(\Delta)^{\frac{s}{2}} u_n|^2 dx \to \mu \geq \mu_0.$$

We estimate now the term $A_2(\varepsilon,n)$. Recall that $\dfrac{|\phi_\varepsilon(x)-\phi_\varepsilon(y)|}{|x-y|} \leq \dfrac{C}{\varepsilon}$, then it follows that $|(-\Delta)^{\frac{s}{2}}\phi_\varepsilon| \leq \dfrac{C}{\varepsilon}$.

We have

$$\begin{aligned}\int_{B_R(0)} \left|u_n(-\Delta)^{\frac{s}{2}}\phi_\varepsilon(-\Delta)^{\frac{s}{2}}u_n\right| dx &= \int_{B_{2\varepsilon}(0)} \left|u_n(-\Delta)^{\frac{s}{2}}\phi_\varepsilon(-\Delta)^{\frac{s}{2}}u_n\right| dx \\ &+ \int_{B_R(0)\setminus B_{2\varepsilon}(0)} \left|u_n(-\Delta)^{\frac{s}{2}}\phi_\varepsilon(-\Delta)^{\frac{s}{2}}u_n\right| dx = J_1(\varepsilon,n)+J_2(\varepsilon,n).\end{aligned}$$

Since

$$\int_{B_{2\varepsilon}(0)} \left|u_n(-\Delta)^{\frac{s}{2}}\phi_\varepsilon(-\Delta)^{\frac{s}{2}}u_n dx\right| \leq \frac{C}{\varepsilon}\int_{B_{2\varepsilon}(0)} \left|u_n(-\Delta)^{\frac{s}{2}}u_n\right| dx,$$

taking into consideration that

$$\int_{B_{2\varepsilon}(0)} \left|u_n(-\Delta)^{\frac{s}{2}}u_n\right| dx \to \int_{B_{2\varepsilon}(0)} \left|u(-\Delta)^{\frac{s}{2}}u\right| dx \text{ as } n\to\infty,$$

$$\int_{B_{2\varepsilon}(0)} \left|u(-\Delta)^{\frac{s}{2}}u\right| dx \leq C\varepsilon ||u||_{L^{2^*_s}(B_{2\varepsilon}(0))},$$

and since we have assumed that $u=0$, we conclude that $\lim\limits_{\varepsilon\to 0}\lim\limits_{n\to\infty} J_1(\varepsilon,n)=0$. In the same way and using a duality argument we get $\lim\limits_{\varepsilon\to 0}\lim\limits_{n\to\infty} J_2(\varepsilon,n)=0$.

Combining the above estimates, there results that $\lim\limits_{\varepsilon\to 0}\lim\limits_{n\to\infty} A_2(\varepsilon,n)=0$.

We deal now with the last term $A_3(\varepsilon,n)$. We have

$$\begin{aligned}&\int_{B_R(0)} u_n(x)\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+s}}dxdy = \\ &\int_{B_R(0)} u_n\int_{B_R(0)} \frac{(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+s}}dxdy + \\ &\int_{B_R(0)} u_n\int_{\mathcal{C}B_R(0)} \frac{(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+s}}dxdy \\ &\equiv B_1(\varepsilon,n)+B_2(\varepsilon,n).\end{aligned}$$

Respect to $B_1(\varepsilon,n)$, we have

$$\begin{aligned}&\left|\int_{B_R(0)} u_n(x)\int_{B_R(0)} \frac{(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+s}}dxdy\right| \leq \\ &\Big(\int_{B_R(0)}\int_{B_R(0)} \frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+s}}dxdy\Big)^{\frac12} \times \Big(\int_{B_R(0)} u_n^2(x)\int_{B_R(0)} \frac{(\phi_\varepsilon(x)-\phi_\varepsilon(y))^2dy}{|x-y|^{N+s}}dx\Big)^{\frac12} \leq \\ &C\Big(\int_{B_R(0)} u_n^2(x)\int_{B_R(0)} \frac{(\phi_\varepsilon(x)-\phi_\varepsilon(y))^2dy}{|x-y|^{N+s}}dx\Big)^{\frac12},\end{aligned}$$

where we have used the fact that the sequence $\{u_n\}_n$ is bounded in $H^s_{0,ra}(B_R(0))$. Since $B_R(0)\times B_R(0)$ is a bounded domain, then as in the estimate of the term $J_1(\varepsilon,n)$, we can show that

$$\int_{B_R(0)} u_n^2(x)\int_{B_R(0)}\frac{(\phi_\varepsilon(x)-\phi_\varepsilon(y))^2dy}{|x-y|^{N+s}}dx\to 0 \text{ as } n\to 0 \text{ and } n\to\infty.$$

Respect to $B_2(\varepsilon,n)$, since $supp(\phi_\varepsilon), supp(u_n)\subset B_R(0)$, using Hölder inequality and taking into consideration that $B_R(0)\times CB_R(0)\subset D_{B_R(0)}$, it holds that

$$\Big|\int_{B_R(0)} u_n\int_{\mathcal{C}B_R(0)}\frac{(u_n(x)-u_n(y))(\phi_\varepsilon(x)-\phi_\varepsilon(y))}{|x-y|^{N+s}}dxdy\Big|\le$$

$$\Big(\iint_{D_{B_R(0)}}\frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+s}}dxdy\Big)^{\frac12}\times\Big(\int_{B_R(0)}u_n^2(x)\phi_\varepsilon^2(x)\int_{\mathcal{C}B_R(0)}\frac{dy}{|x-y|^{N+s}}dx\Big)^{\frac12}\le$$

$$C\Big(\int_{B_R(0)}u_n^2(x)\phi_\varepsilon^2(x)dx\Big)^{\frac12},$$

where in the last estimate, we have used the fact that $\int_{\mathcal{C}B_R(0)}\frac{dy}{|x-y|^{N+s}}dx\le C(R)$. Hence $\lim\limits_{\varepsilon\to0}\lim\limits_{n\to\infty}B_2(\varepsilon,n)=0$.

Therefore, combining the above estimates and passing to the limit in n and ε in (4.34), we conclude that

$$\mu_0\le S(R)\nu_0.$$

Since $S\nu_0^{\frac{2}{2^*_s}}\le\mu_0$, then $S\nu_0^{\frac{2}{2^*_s}}\le S(R)\nu_0$. If $\nu_0=0$, then $\mu_0=0$. Hence

$$\int_{B_R(0)}\frac{|u_n|^{2^*_s}}{(R-|x|)^{2s}}dx\to\int_{B_R(0)}\frac{|u|^{2^*_s}}{(R-|x|)^{2s}}dx=1$$

a contradiction with the fact that $u\equiv 0$.

Assume that $\nu_0>0$, then $S\le S(R)\nu_0^{1-\frac{2}{2^*_s}}$. Recall that we have chosen $R<1$ such that $S(R)\equiv R^{\frac{4s}{2^*_s}}S(1)<S$. In this way we get easily that $\nu_0<1$. Hence $S\le R^{\frac{4s}{2^*_s}}S(1)$. Taking into consideration that the Sobolev constant S is independent of the domain, and in particular it is independent of R, then letting $R\to0$, we reach a contradiction.

Thus $u\ne0$ and it solves (1.0) with $q=2^*_s-1$. The strong maximum principle allows us to get that $u>0$ in $B_R(0)$.

Notice that, from the above computation, we can conclude that

$$\int_{B_R(0)}\frac{|u_n|^{2^*_s}}{(R-|x|)^{2}}dx\to\int_{B_R(0)}\frac{|u|^{2^*_s}}{(R-|x|)^{2}}dx=1$$

and then u realize $S(R)$. ∎

5. The case : $q<0$

In this section, we consider the following problem

$$\left\{\begin{array}{rcll}(-\Delta)^s u&=&\dfrac{f}{u^\sigma d^\alpha(x)} & \text{in } \Omega,\\ u&>&0 & \text{in } \Omega,\\ u&=&0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{array}\right. \tag{5.35}$$

where $\sigma = -q > 0$, $\alpha > 0$, $0 < s < 1$, $d(x) = d(x, \partial\Omega)$ and f is a nonnegative function under suitable summability conditions that will be specified later.

Notice that problem (5.35) has been treated in the local case ($s = 1$) (see [14] and [7]). In the case where $0 < s < 1$ and $\alpha = 0$, see [6] and the references therein.

In order to study the solvability of problem (5.35), we will analyze the associated approximating problem. Indeed for every $n \in I\!N^*$, we consider the following problem

$$
\left\{
\begin{array}{rcll}
(-\Delta)^s u_n & = & \dfrac{f_n}{(u_n + \frac{1}{n})^\sigma (d(x) + \frac{1}{n})^\alpha} & \text{in } \Omega, \\
u_n & > & 0 & \text{in } \Omega, \\
u_n & = & 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{array}
\right. \tag{5.36}
$$

where $f_n := \min(n, f)$.

Notice that the existence of solution $u_n \in H_0^s(\Omega)$ to (5.36) follows using the Schauder fixed point theorem. It is clear that $u_n \in L^\infty(\Omega)$.

We start proving the next result.

Lemma 5.1. *The sequence $\{u_n\}_n$ of the solutions to problem (5.36) is increasing in n and for every $\widetilde{\Omega} \subset\subset \Omega$, there exists a positive constant $C(\Omega)$ independent of n, such that*

$$u_n \geq C(\widetilde{\Omega}) > 0.$$

Proof. Fixed $m \in I\!N$, then by subtracting, it holds that

$$
\begin{array}{rcl}
(-\Delta)^s(u_n - u_{n+1}) & = & \dfrac{f_n}{(u_n + \frac{1}{n})^\sigma (d(x) + \frac{1}{n})^\alpha} - \dfrac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^\sigma (d(x) + \frac{1}{n+1})^\alpha} \\
& \leq & \dfrac{f_{n+1}}{(u_n + \frac{1}{n})^\sigma (d(x) + \frac{1}{n+1})^\alpha} - \dfrac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^\sigma (d(x) + \frac{1}{n+1})^\alpha} \\
& = & \dfrac{f_{n+1}}{(d(x) + \frac{1}{n+1})^\alpha} \left(\dfrac{(u_{n+1} + \frac{1}{n+1})^\sigma - (u_n + \frac{1}{n+1})^\sigma}{(u_{n+1} + \frac{1}{n+1})^\sigma (u_n + \frac{1}{n+1})^\sigma} \right).
\end{array}
$$

Using $(u_n - u_{n+1})_+$ as a test function, we obtain that

$$\int_{\mathbb{R}^N} \frac{f_{n+1}}{(d(x) + \frac{1}{n+1})^\alpha} \left(\frac{(u_{n+1} + \frac{1}{n+1})^\sigma - (u_n + \frac{1}{n+1})^\sigma}{(u_{n+1} + \frac{1}{n+1})^\sigma (u_n + \frac{1}{n+1})^\sigma} \right) (u_n - u_{n+1})_+ dx \leq 0.$$

Since

$$\int_{\mathbb{R}^N} (-\Delta)^s(u_n - u_{n+1})(u_n - u_{n+1})_+ \geq \iint_{D_\Omega} \frac{(u_n - u_{n+1})_+)^2}{|x - y|^{N+2s}} dxdy,$$

we conclude that $(u_n - u_{n+1})_+ = 0$, and then $u_n \leq u_{n+1}$, for all n. On other hand, we know that $u_1 \in L^\infty(\Omega)$ and

$$(-\Delta)^s u_1 = \frac{f_1}{(u_1 + 1)^\sigma (d(x) + 1)^\alpha} \geq \frac{f_1}{(||u_1||_\infty + 1)^\sigma} \geq \frac{f_1}{(||u_1||_\infty + 1)^\sigma}.$$

Thus by the strong Maximum principle $u_1 > 0$ in Ω. Hence for every $\widetilde{\Omega} \subset\subset \Omega$, there exists a positive constant $C(\Omega)$ independent of n such that

$$u_n \geq u_1 \geq C(\widetilde{\Omega}) > 0.$$

$\square$

Remarks 5.2. *As a conclusion of the above computation, we obtain that u_n is the unique positive solution to problem* (5.36).

Let us begin by the case $\alpha < 1$, then we have the next existence result.

Theorem 5.3. *Assume that $\alpha < 1$ and $f \in L^p(\Omega)$ with $p'\alpha < 1$, then the problem* (5.35) *has a distributional solution u such that $u^{\frac{\sigma+1}{2}} \in H_0^s(\Omega)$.*

Proof. Recall that u_n is the unique solution to problem (5.36). Using u_n^σ as a test function in (5.36) it follows that

$$\iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(u_n^\sigma(x) - u_n^\sigma(y))}{|x-y|^{N+2s}} dxdy \leq \int\limits_\Omega \frac{f_n}{(d(x) + \frac{1}{n})^\alpha} dx.$$

Notice that, for all for all $(a,b) \in (\mathbb{R}^+)^2$ and for all $\sigma > 0$, we have

$$(a-b)(a^\sigma - b^\sigma) \geq c_3 |a^{\frac{\sigma+1}{2}} - b^{\frac{\sigma+1}{2}}|^2, \tag{5.37}$$

hence using Hölder inequality, it follows that

$$\iint_{\mathbb{R}^{2N}} \frac{(u_n^{\frac{\sigma+1}{2}}(x) - u_n^{\frac{\sigma+1}{2}}(y))^2}{|x-y|^{N+2s}} dxdy \leq ||f||_{L^p(\Omega)} \Big(\int\limits_\Omega \frac{1}{d^{p'\alpha}(x)} dx \Big)^{\frac{1}{p'}}.$$

Since $p'\alpha < 1$, then $||u_n^{\frac{\sigma+1}{2}}||_{H_0^s(\Omega)} \leq C$. Hence we get the existence of a measurable function u such that $u_n \uparrow u$ a.e in Ω, $u^{\frac{\sigma+1}{2}} \in H_0^s(\Omega)$ and $u_n^{\frac{\sigma+1}{2}} \rightharpoonup u^{\frac{\sigma+1}{2}}$ weakly in $H_0^s(\Omega)$. It is not difficult to show that u is a distributional solution to problem (5.35). □

In the case of general datum f, we have the next existence result.

Theorem 5.4. *Assume $\alpha, \sigma > 0$, then we have:*

(1) *If $f \in L^1(\Omega, d^{s-\alpha})$, then for all $\sigma > 0$, there exists positive solution u to problem* (5.35) *in the sense of the definition* (2.8) *such that $\frac{u^{\sigma+1}}{d^\beta} \in L^1(\Omega)$ for all $0 < \beta < s$.*
(2) *If $f \in L^1(\Omega, d^{s-\alpha} \log(\frac{D}{d}))$, where $D > dist(x, \partial\Omega)$, then for all $\sigma > 0$, there exists a positive solution u to problem* (5.35) *in the sense of the definition 2.8 such that $\frac{u^{\sigma+1}}{d^s} \in L^1(\Omega)$.*
(3) *If $f \in L^1(\Omega, d^{2s-\beta-\alpha})$ for some $\beta \in (s, 2s)$, then for all $\sigma > 0$, there exists positive solution u to problem* (5.35) *in the sense of the definition 2.8 such that $\frac{u^{\sigma+1}}{d^\beta} \in L^1(\Omega)$.*

Proof. We proceed by iterations. Consider u_n to be unique positive solution to the problem (5.36), then using the Kato inequality it holds that

$$(-\Delta)^s u_n^{\sigma+1} \leq (\sigma+1) u_n^\sigma (-\Delta)^s u_n.$$

Define ϕ as the unique positive solution to (2.18) with $\beta \in (0, s+1)$. Using ϕ as a test function in the previous inequality, we obtain that,

$$\begin{aligned}\int\limits_{\Omega} \phi(-\Delta)^s u_n^{\sigma+1} dx &\le (\sigma+1)\int\limits_{\Omega} \phi u_n^{\sigma}(-\Delta u_n)^s dx \\ &= \int\limits_{\Omega} \frac{u_n^{\sigma}\phi f_n}{(u_n+\frac{1}{n})^{\sigma}(d(x)+\frac{1}{n})^{\alpha}} dx \\ &\le \int\limits_{\Omega} \frac{\phi f}{d^{\alpha}(x)} dx.\end{aligned}$$

Suppose that f satisfies the first condition in the Theorem 5.4, then choosing $\beta \in (0, s)$ in Theorem 2.8, we know that $\phi \backsimeq d^s$. Hence taking into consideration that $f \in L^1(\Omega, d^{s-\alpha})$, it follows that

$$\int\limits_{\Omega} \frac{u_n^{\sigma+1}}{d^{\beta}} dx \le C \int\limits_{\Omega} f(x) d^{s-\alpha}(x)\, dx < \infty.$$

Since $\{u_n\}_n$ is a monotone sequence in n, we get the existence of a measurable function u such that,

$$\frac{u_n^{\sigma+1}}{d^{\beta}} \to \frac{u^{\sigma+1}}{d^{\beta}}, \quad n \to \infty.$$

It is clear that $u = 0$ in $\mathbb{R}^N \setminus \Omega$, hence u is a positive solution to problem (5.35) in the sense of definition 2.3.

For the second (resp. the third cases), it suffices to take $\beta = s$ (resp. *beta* $\in (s, 2s)$) and to use the fact that $v \backsimeq d^s(x)\log(\frac{D}{d})$(resp. $v \backsimeq d^{2s-\beta}(x)$). Hence following the closely the same calculation as in first case and passing to the limit, we reach the desired result. $\square$

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