On the metric entropy of the Banach-Mazur compactum

by
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Abstract

We study the metric entropy of the metric space $\mathcal{B}_n$ of all n-dimensional Banach spaces (the so-called Banach-Mazur compactum) equipped with the Banach-Mazur (multiplicative) "distance" d. We are interested either in estimates independent of the dimension or in asymptotic estimates when the dimension tends to ∞. For instance, we prove that, if $N(\mathcal{B}_n, d, 1+\varepsilon)$ is the smallest number of "balls" of "radius" $1 + \varepsilon$ that cover $\mathcal{B}_n$, then for any $\varepsilon > 0$ we have

$$0 < \liminf_{n \to \infty} n^{-1} \log \log N(\mathcal{B}_n, d, 1+\varepsilon) \le \limsup_{n \to \infty} n^{-1} \log \log N(\mathcal{B}_n, d, 1+\varepsilon) < \infty.$$

We also prove an analogous result for the metric entropy of the set of n-dimensional operator spaces equipped with the distance d_N naturally associated to $N \times N$-matrices with operator entries. In that case N is arbitrary but our estimates are valid independently of N. In the Banach space case (i.e. $N = 1$) the above upper bound is part of the folklore, and the lower bound is at least partially known (but apparently has not appeared in print). While we follow the same approach in both cases, the matricial case requires more delicate ingredients, namely estimates (from our previous work) on certain n-tuples of $N \times N$ unitary matrices known as "quantum expanders".

Let $\mathcal{B}_n$ denote the set of all n-dimensional Banach spaces. Actually we wish to identify two spaces if they are isometrically isomorphic. Thus, although it would be more proper (but heavier) to describe $\mathcal{B}_n$ as the set of equivalence classes for this equivalence relation, we adopt the common abuse to identify a space E with its equivalence class. We equip this space with the (multiplicative) metric defined for any pair $E, F \in \mathcal{B}_n$ by

$$d(E, F) = \inf\{\|u\|\|u^{-1}\|\}$$

where the inf runs over all the isomorphisms $u : E \to F$. For any $G \in \mathcal{B}_n$ we have $d(E, F) \le d(E, G)d(G, F)$, so that $(E, F) \mapsto \log d(E, F)$ is indeed a distance. It is easy to see (by the compactness of the unit ball of $B(E, F)$) that E, F are isometric iff $d(E, F) = 1$, and in that case we declare that $E = F$. Therefore $\mathcal{B}_n$ equipped with $\log d$ is a bona fide metric space. It is a

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classical fact (presumably due to Banach and Mazur) that this metric space is compact, and hence is often called the Banach-Mazur compactum. Since it is more convenient to work with d than with $\log d$, we will abusively define an open ball of radius r in $(\mathcal{B}_n, d)$ to be any set of the form $\{E \mid d(E, E_0) < r\}$, and we call it the ball of radius r centered at E_0. Note that this is non void only if $r > 1$.

In this note we wish to tackle the rather natural problem of estimating the metric entropy $H_n(\varepsilon) = \log_2 N(\mathcal{B}_n, d, 1+\varepsilon)$ of this compactum, or equivalently its covering number $N(\mathcal{B}_n, d, 1+\varepsilon)$. By $N(\mathcal{B}_n, d, 1 + \varepsilon)$ we mean the smallest number of open balls of radius $1 + \varepsilon$ that cover $\mathcal{B}_n$. By compactness we of course know that $N(\mathcal{B}_n, d, 1 + \varepsilon) < \infty$ for any $\varepsilon > 0$.
The metric entropy $H_n(\varepsilon)$ was already studied in [4] (see also the more recent survey [5]): Indeed, it is proved in [4] that $H_n(\varepsilon) \approx \varepsilon^{\frac{1-n}{2}}$. Note however, that this equivalence is for each fixed dimension n and for sufficiently small $\varepsilon > 0$ (the range of which may depend on n). In sharp contrast we are interested in estimates independent of the dimension, or with asymptotic estimates when $n \to \infty$ with a view to infinite dimensional applications. Our interest lies in estimates of $N(\mathcal{B}_n, d, 1+\varepsilon)$ (or possibly of $N(\mathcal{B}_n, d, 1+\varepsilon_n)$) when $n \to \infty$ and $\varepsilon > 0$ is either fixed or depends on n in a prescribed way.

Our proofs §1 are all rather simple and direct, and the results are partially known, but mostly unpublished (see however Remarks 0.1 and 1.12 below). One of our goals is to stimulate further research on this theme.

We prove in Theorem 1.1 that for any $\varepsilon > 0$ we have

$$(0.1) \qquad 0 < \liminf_{n \to \infty} n^{-1} \log \log N(\mathcal{B}_n, d, 1 + \varepsilon) \leq \limsup_{n \to \infty} n^{-1} \log \log N(\mathcal{B}_n, d, 1 + \varepsilon) < \infty.$$

Moreover, it turns out that for any $R > 1+\varepsilon$ and any choice of $E \in \mathcal{B}_n$, if we replace the whole space $\mathcal{B}_n$ by the single ball $B_{E,R} \subset \mathcal{B}_n$, with center E and radius R, then we still have

$$0 < \liminf_{n \to \infty} n^{-1} \log \log N(B_{E,R}, d, 1 + \varepsilon).$$

This refinement is treated in §2.

We should recall that by Fritz John's famous theorem (see e.g. [33, 20]), we have

$$d(E, \ell_2^n) \leq \sqrt{n}$$

and hence

$$\forall E, F \in \mathcal{B}_n \quad d(E, F) \leq n.$$

Thus the Euclidean space ℓ_2^n is a natural center for $\mathcal{B}_n$ and the d-diameter of $\mathcal{B}_n$ is at most n. Moreover, there is a number $\delta > 0$ such that for each n there are n-dimensional spaces E_n, F_n such that $d(E_n F_n) > \delta n$. This is due to Gluskin [9] (see also [25]). It would be interesting to estimate the growth of $N(\mathcal{B}_n, d, \varepsilon_n)$ when ε_n is of the same order as n. Similar questions can be raised for the "packing number" $M(\mathcal{B}_n, d, 1 + \varepsilon)$ which is defined as the maximal number of elements in $\mathcal{B}_n$ at mutual distance $\geq 1 + \varepsilon$. Note that $N(\mathcal{B}_n, d, 1 + \varepsilon) \leq M(\mathcal{B}_n, d, 1 + \varepsilon) \leq N(\mathcal{B}_n, d, (1 + \varepsilon)^{1/2})$ for any $\varepsilon > 0$. See Remarks 1.11 and 1.12 below for some results in that direction, based on Gluskin's "random Banach spaces".

We should mention that somewhat related results appear in [30, 31]. Also [10, 27, 26, 28, 2] is recommended reading to anyone interested in the subject of this note.

While §1 is devoted to the metric entropy of the space of n-dimensional Banach spaces, §3 treats the case of operator spaces, which can be viewed as "non-commutative" or "matricial" analogues of Banach spaces. The latter are just Banach spaces given together with a specific embedding in the

algebra $B(H)$ of all bounded operators on a Hilbert space H. There the Banach-Mazur distance d is replaced by the matricial analogue d_N defined in (3.1) below, depending on the size N of the matrices being used. When $N = 1$ we recover the distance d.

For the operator space case in §3, the general structure of our argument is modeled on that of §1. However, the ingredients are more involved. We make crucial use of our previous results from [22] on quantum expanders, and we carefully explain in §4 why they are needed in the matricial case.

Remark 0.1. I am grateful to S. Szarek for the following information. The upper bound in (0.1) is known and should be considered part of the folklore. The lower bound was known to Joram Lindenstrauss and Szarek for say $\varepsilon = 1$ at least in the 1990's. The referee of [30] insisted that it should be attributed to [4], which, in retrospect, was not appropriate since the entropy estimates in [4] were for a fixed dimension and for ε small enough. In any case, both upper and lower bounds in (0.1) did not seem to have appeared explicitly in print yet when we submitted this paper. See however Remark 1.12 for an update concerning [15].

1 Packing and covering the Banach-Mazur compactum

In this section, all the metric properties (such as nets and balls) are relative to the Banach-Mazur distance d. Thus to abbreviate, we will denote simply $N(\mathcal{B}_n, 1 + \varepsilon)$ instead of $N(\mathcal{B}_n, d, 1 + \varepsilon)$. More generally, for any subset $S \subset \mathcal{B}_n$ and $r > 1$ we denote by $N(S, r)$ the smallest number of open balls (centered in S) of radius r that cover S. Note that $N(S, r) \leq N(\mathcal{B}_n, r^{1/2})$.

Similarly we will denote by $M(S, r)$ ("packing number") the maximal cardinality of a finite subset $A \subset S$ such that $d(s, t) \geq r$ for all $s \neq t \in A$. It is well known (and easy to check) that for any $r \geq 1$ $N(S, r) \leq M(S, r) \leq N(S, r^{1/2})$.

Our main result is:

Theorem 1.1. *For any $r > 1$, there are positive constants b_r, c_r such that, for any integer n assumed large enough (more precisely for $n \geq n_0(r)$), we have*

$$\exp(\exp b_r n) \leq N(\mathcal{B}_n, r) \leq \exp(\exp c_r n).$$

While the upper bound is easy and already known to specialists, the lower bound is a bit more delicate. Curiously, for the latter we were inspired by the operator space analogues developed in [22] following the ideas in [14]. In the second part of this note, we expand on the first attempt of [22] and give the matricial analogue of the preceding theorem.

The first two Lemmas are classical facts.

Lemma 1.2. *Let X be an m-dimensional Banach space. Let $S(n, X) \subset \mathcal{B}_n$ be the subset formed by all the n-dimensional subspaces of X. Consider $0 < \xi < 1/n$ and let $R = (1 + \xi n)(1 - \xi n)^{-1}$. Then $S(n, X)$ admits an R-net with at most $(1 + 2/\xi)^{nm}$. In other words,*

$$N(S(n, X), R) \leq (1 + 2/\xi)^{nm}.$$

Proof. By Auerbach's classical Lemma, any $E \in \mathcal{B}_n$ admits a basis e_j such that for any n-tuple of scalars $x = (x_j)$ we have
$$\sup |x_j| \leq \|\sum x_j e_j\| \leq \sum |x_j|.$$

Let $T \subset B_X^m$ be a ξ-net in the unit ball of X with $|T| \leq (1 + 2/\xi)^m$ (see e.g. [20, p. 49]). Given an arbitrary $E \in S(n, X)$ with Auerbach basis (e_j) we may choose $f = (f_j) \in T^n$ such that

$\sup \|e_j - f_j\| \le \xi$. Then we have for any n-tuple of scalars $x = (x_j)$

$$\|\sum x_j e_j\| \le \|\sum x_j f_j\| + \xi \sum |x_j| \le \|\sum x_j f_j\| + \xi n \sup |x_j| \le \|\sum x_j f_j\| + \xi n \|\sum x_j e_j\|$$

$$\|\sum x_j e_j\| \ge \|\sum x_j f_j\| - \xi \sum |x_j| \ge \|\sum x_j f_j\| - \xi n \sup |x_j| \ge \|\sum x_j f_j\| - \xi n \|\sum x_j e_j\|$$

and hence

$$(1 - \xi n)\|\sum x_j e_j\| \le \|\sum x_j f_j\| \le (1 + \xi n)\|\sum x_j e_j\|$$

from which we deduce that if we set $E_f = \text{span}[f_j]$ we have

$$d(E, E_f) \le R.$$

Thus $\{E_f \mid f \in T^n\}$ is the desired R-net, and $|T^n| \le |T|^n \le (1 + 2/\xi)^{nm}$. $\qquad\square$

Lemma 1.3. *Fix $0 < \delta < 1$. Let m be the largest integer such that $m \le (1 + 2/\delta)^n$.*
For any n-dimensional space E there is a subspace $F \subset \ell_\infty^m$ such that

$$d(E, F) \le (1 - \delta)^{-1}.$$

Proof. Let T be a δ-net in the unit ball of E^* with $|T| \le (1 + 2/\delta)^m$ (see e.g. [20, p. 49]). We claim that for any $x \in E$ we have

$$(1 - \delta)\|x\| \le \sup_{t \in T} |t(x)| \le \|x\|.$$

Indeed, choose $s \in B_{E^*}$ such that $\|x\| = |s(x)|$ and then $t \in T$ such that $\|s - t\| \le \delta$. We have then

$$\|x\| = |s(x)| \le |t(x)| + |(s - t)(x)| \le \sup_{t \in T} |t(x)| + \delta\|x\|$$

from which the claim follows. The Lemma is then clear since this claim implies $d(E, F) \le (1 - \delta)^{-1}$ where $F = \{\hat{x} \mid x \in E\} \subset \ell_\infty(T)$ is the subspace defined by setting $\hat{x}(t) = t(x)$ for all $t \in T$. $\qquad\square$

Remark 1.4. See [1, Cor. 1.2] for a recent refinement of the estimate in Lemma 1.2.

Proof of Theorem 1.1 (Upper bound). Fix $0 < \varepsilon < 1$. Let $\xi = \varepsilon/n$. Then $R = (1 + \varepsilon)(1 - \varepsilon)^{-1}$ Let $m = [(1 + 2/\delta)^n]$. Combining the two Lemmas we find

$$N(\mathcal{B}_n, (1 - \delta)^{-1} R) \le (1 + 2/\xi)^{nm} \le (1 + 2n/\varepsilon)^{nm} \le (3n/\varepsilon)^{nm} = \exp(nm \log(3n/\varepsilon))$$

and a fortiori

$$N(\mathcal{B}_n, (1 - \delta)^{-1} R) \le \exp(n \log(3n/\varepsilon) \exp n \log(3/\delta)).$$

So if we take, say $\delta = \varepsilon$, there is clearly $c'_\varepsilon > 0$ such that $\exp(n \log(3n/\varepsilon) \exp n \log(3/\varepsilon)) \le \exp \exp c_\varepsilon n$ for all n, and $(1 - \delta)^{-1} R = (1 + \varepsilon)(1 - \varepsilon)^{-2}$, so we conclude

$$N(\mathcal{B}_n, (1 + \varepsilon)(1 - \varepsilon)^{-2}) \le \exp \exp c'_\varepsilon n,$$

which is clearly equivalent to an upper bound of the announced form, at least for all $r = 1 + \varepsilon$ with ε small enough. But, since the upper bound becomes easier for larger values of ε, the proof of the upper bound is complete. $\qquad\square$

Lemma 1.5. *Let $a > 0$. Let $A \subset \{-1, 1\}^n$ be a (measurable) subset with $\mathbb{P}(A) > a$. Then, for any $0 < \theta < 1$, A contains a finite subset $T \subset A$ with*

$$|T| \geq (a/2) \exp \theta^2 n/2,$$

such that

$$\forall s \neq t \in T \quad |\sum s_j t_j| \leq \theta n.$$

Proof. Let $\Omega = \{-1, 1\}^n$ equipped with the uniform probability $\mathbb{P}$. Let T be a maximal subset of A such that $\forall s \neq t \in T \quad |\sum s_j t_j| \leq \theta n$. Then, by the maximality of T, for all $\omega \in A$ there is $t \in T$ such that $|\sum \omega_j t_j| > \theta n$. Therefore $\mathbb{P}(A) \leq \sum_{t \in T} \mathbb{P}\{|\sum \omega_j t_j| > \theta n\} \leq |T| \mathbb{P}\{|\sum \omega_j| > \theta n\}$. The last inequality holds because, by translation invariance on $\{-1, 1\}^n$, $\mathbb{P}\{\omega \mid |\sum \omega_j t_j| > \theta n\}$ is independent of $t = (t_j)$. But now it is a classical fact that $\mathbb{P}\{|\sum \omega_j| > \theta n\} \leq 2 \exp -\theta^2 n/2$ for any $\theta > 0$ and hence

$$|T| \geq (a/2) \exp \theta^2 n/2.$$

From this follows that for all n large enough (i.e. $n \geq n(a, \theta)$) we have, say,

$$|T| \geq \exp \theta^2 n/3.$$

$\square$

Lemma 1.6. *Let $\mathcal{T}$ be any finite set with even cardinality K. Let $P(\mathcal{T})$ denote the set of all the 2^K subsets of $\mathcal{T}$. Let $\mathcal{A} \subset P(\mathcal{T})$ be the subset formed of subsets with cardinality $K/2$. Then*

$$\forall x \neq y \in \mathcal{A} \quad x \not\subset y \quad \text{and} \quad y \not\subset x,$$

and we have $|\mathcal{A}| \approx \sqrt{2/\pi} K^{-1/2} 2^K$. A fortiori, we have for all K large enough

$$|\mathcal{A}| \geq \exp K/2.$$

Proof. Entirely elementary. $\square$

Proof of Theorem 1.1 (Lower bound). Let $\mathcal{T}$ be the set T appearing in Lemma 1.5 for $A = \{-1, 1\}^n$. We may clearly assume K even (indeed removing one point will not spoil the estimates appearing below). Let $\mathcal{A}$ be as in Lemma 1.6. Recall that for any $x \in \mathcal{A}$ we have $x \subset \mathcal{T} \subset \{-1, 1\}^n$. Let us denote by ε_j the coordinates on $\{-1, 1\}^n$, and by $\varepsilon_{j|x} \in \ell_\infty(x)$ the restriction of ε_j to the subset x. We denote by (e_j) the canonical basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$ depending on the context).
We use the same notation for the basis of ℓ_p^n for $1 \leq p \leq \infty$.
We then define for any $x \in \mathcal{A}$

$$f_j^x = e_j \oplus \varepsilon_{j|x} \in \ell_\infty^n \oplus_\infty \ell_\infty(x).$$

and

$$E_x = \text{span}[f_j^x, 1 \leq j \leq n] \subset \ell_\infty^n \oplus_\infty \ell_\infty(x).$$

Note $\dim(E_x) = n$. We will take the simplified viewpoint that each space E_x is equal to $\mathbb{R}^n$ (or $\mathbb{C}^n$) equipped with the norm

$$\forall a \in \mathbb{R}^n \quad \|\sum a_j e_j\| = \|\sum a_j f_j^x\|_{E_x} = \max\{\sup |a_j|, \sup_{t \in x} |\sum a_j t_j|\}.$$

Thus any linear map $u : E_y \to E_z$ can be identified with an $n \times n$-matrix. Since $\|f_j^x\| = 1$ we have for any any $x \in \mathcal{A}$

$$(1.1) \qquad \forall (a_j) \in \mathbb{C}^n \quad \sup |a_j| \leq \|\sum a_j f_j^x\| \leq \sum |a_j|.$$

Then, assuming n large enough (so that $\theta n \geq 1$), for any $y \neq z \in \mathcal{A}$ we can find $s \in y \setminus z$ such that

$$\|\sum s_j f_j^y\| = n \quad \text{but} \quad \|\sum s_j f_j^z\| \leq \theta n.$$

This proves a lower bound $\|u\|\|u^{-1}\| \geq 1/\theta$ (and actually $\geq 1/\theta^2$) when $u : E_y \to E_z$ is the map corresponding to the identity matrix. To obtain a similar lower bound for *any* u the rough idea is that there are "many more" x's in $\mathcal{A}$ than there are linear maps on an n-dimensional space. We will show that for any r with $1 < r < 1/\theta$ there is a subset $\mathcal{A}' \subset \mathcal{A}$ with $|\mathcal{A}'| \geq \exp\exp c_r n$ such that $d(E_x, E_y) > r$ for any $x \neq y \in \mathcal{A}'$. (Perhaps one can show this even for $1 < r < 1/\theta^2$ but we do not see how.)

Let $1 + \eta = 1/(r\theta)$. We will prove the following claim (assuming we work with real spaces, but the complex case is similar, requiring just $2n^2$ instead of n^2).

Claim: For any fixed $x \in \mathcal{A}$ we have

$$|\{y \in \mathcal{A} \mid d(E_x, E_y) < r\}| \leq (1 + 4n/\eta)^{n^2}.$$

To prove this claim we use the classical fact that the unit ball of the space $B(\ell_1^n, E_x)$ (being n^2 dimensional) contains a $\eta/2n$-net of cardinality at most $(1 + 4n/\eta)^{n^2}$.

Note that, by (1.1), for any y, we have for any $n \times n$-matrix u

$$(1.2) \qquad \|u : \ell_1^n \to E_x\| \leq \|u : E_y \to E_x\| \leq n\|u : \ell_1^n \to E_x\|.$$

For any y with $d(E_x, E_y) < r$ there is a matrix u^y such that

$$(1.3) \qquad \|u^y : E_y \to E_x\| \leq 1 \quad \text{and} \quad \|(u^y)^{-1} : E_x \to E_y\| < r.$$

Assume by contradiction that $|\{y \in \mathcal{A} \mid d(E_x, E_y) < r\}| > (1 + 4n/\eta)^{n^2}$. Then there must exist $y \neq z$ both in the set $|\{y \in \mathcal{A} \mid d(E_x, E_y) < r\}$ such that the associated u^y and u^z are $\eta/2n$-close to the same point of the $\eta/2n$-net, and hence such that $\|u^y - u^z : \ell_1^n \to E_x\| \leq \eta/n$. Therefore, by (1.2), we have $\|u^y - u^z : E_z \to E_x\| \leq \eta$, and hence by (1.3)

$$(1.4) \qquad \|u^y : E_z \to E_x\| \leq 1 + \eta.$$

But now recall that since $y \neq z$ with $y, z \in \mathcal{A}$, we already know that $1/\theta \leq \|Id : E_z \to E_y\|$. Thus we have

$$1/\theta \leq \|Id : E_z \to E_y\| = \|(u^y)^{-1} u^y : E_z \to E_y\| \leq \|(u^y)^{-1} : E_x \to E_y\|\|u^y : E_z \to E_x\|$$

and hence by (1.3) and (1.4)

$$1/\theta < r(1 + \eta).$$

Since this contradicts our initial choice of r, this proves the claim.

Let $\mathcal{X} \subset \mathcal{A}$ be a maximal subset such that $d(E_x, E_y) \geq r$ for any $x \neq y \in \mathcal{X}$. Then $\mathcal{A}$ is covered by the balls of radius r centered in the points of $\mathcal{X}$, and hence by the claim

$$|\mathcal{A}| \leq |\mathcal{X}|(1 + 4n/\eta)^{n^2} \leq |\mathcal{X}| \exp 4n^3/\eta.$$

Thus we conclude
$$|\mathcal{X}| \geq |\mathcal{A}| \exp -4n^3/\eta.$$

We now recall that we take $|A| = 2^n$, $a = 1$, $|T| = K \geq (1/2) \exp \theta^2 n/2$ and assuming n large enough we have $|\mathcal{A}| \geq \exp K/2 = \exp((1/4) \exp \theta^2 n/2)$. Therefore, elementary verifications lead to

(1.5) $$|\mathcal{X}| \geq \exp[\exp (\theta^2 n/4)].$$

for all n large enough (say $n \geq n_0(\theta, r)$). At this point the lower bound follows immediately since $N(\mathcal{X}, r^{1/2}) \geq M(\mathcal{X}, r) = |\mathcal{X}|$. $\qquad\square$

Remark 1.7. One can replace e_j by a suitable system of unimodular functions w_j (such as the Walsh system, assuming n dyadic) for which it is known that $\|\sum \varepsilon_j w_j\|_\infty \in o(n)$ on a set of large probability of choices of signs (ε_j). The proof then leads to a similar family of spaces (E_x) but spanned by unimodular functions in ℓ_∞^m for $m = n + K/2$.

Remark 1.8. The preceding argument actually shows more precisely:
For all $r > 1$
$$\liminf_{n \to \infty} \frac{\log \log N(\mathcal{B}_n, r)}{n} \geq \liminf_{n \to \infty} \frac{\log \log M(\mathcal{B}_n, r^2)}{n} \geq \frac{1}{2r^2}.$$

Remark 1.9. Note that the Elton-Pajor theorem from [8, 18] applies here to any of the spaces E_x. Indeed, it is easy to check that the average over all $\pm$ signs of $\|\sum \pm f_j^x\|$ is $\geq c_\theta n$ for some $c_\theta > 0$. This shows that all these spaces (uniformly) contain ℓ_1^k's with dimension k proportional to n.

Remark 1.10. The preceding construction of the spaces E_x can be done using a subset T of the unit sphere of ℓ_2^n, denoted by $S(\ell_2^n)$. Indeed, assume that we have

$$\forall s \neq t \in T \quad |\langle s.t \rangle| \leq \theta.$$

We set $K = |T|$ and (assuming K even) consider again the subset $\mathcal{A} \subset P(T)$ of subsets $x \subset T \subset S(\ell_2^n)$ with $|x| = K/2$. Then for any $x \in \mathcal{A}$ we set

$$e_j^x = \theta e_j \oplus \varepsilon_j^x \in \ell_2^n \oplus_\infty \ell_\infty(x),$$

and $\mathcal{E}_x = \mathrm{span}[e_j^x] \subset \ell_2^n \oplus_\infty \ell_\infty(x)$. A similar reasoning leads to a separated family in $\mathcal{B}_n$. More precisely, if we denote by K_θ the smallest even K for which there exists such a set T, and if we take, say, $\delta = \theta^2$ then we find for any $0 < \theta < 1$

$$M(\mathcal{B}_n, \theta^{-1}(1 + \theta)^{-1}) \geq \gamma 2^{K_\theta} K_\theta^{-1/2} \exp -4n^2/\theta^2,$$

which is significant if K_θ is significantly larger than n^2/θ^2. See [6, Chapter 9] for an account of the works of Kabatiansky and Levenshtein estimating K_θ.

Remark 1.11. Given a classical metric space (T, d), suppose given a probability measure ν on T such that any open ball of radius r has ν-measure $\leq f(r)$ (resp. $\geq g(r)$). Then obviously we have $M(T, d, r) \geq N(T, d, r) \geq 1/f(r)$ (resp. $N(T, d, r) \leq M(T, d, r) \leq 1/g(r/2)$). Gluskin's method in [9, 10] uses elements chosen at random in $\mathcal{B}_n$ according to a certain probability measure ν_n for which he proves that (here d denotes the Banach-Mazur "multiplicative" distance) for some numerical constant $0 < c < 1$, for any E in the support of ν_n, we have

$$\nu_n\{F \in \mathcal{B}_n \mid d(E, F) < cn\} \leq 2^{-n^2}.$$

See [10, Prop. 1]. Thus it follows that, if we denote by $\mathcal{S}_n$ the support of ν_n, we have

$$M(\mathcal{B}_n, cn) \geq M(\mathcal{S}_n, cn) \geq N(\mathcal{S}_n, cn) \geq 2^{n^2},$$

and a fortiori $N(\mathcal{B}_n, \sqrt{cn}) \geq 2^{n^2}$.

Remark 1.12. After a first version of this paper had been submitted, the paper [15] by Litvak, Rudelson and Tomczak-Jaegermann was brought to our attention. It turns out that their [15, Corollary 2.4] implies our Theorem 1.1 (lower bound). Their method is a variation of the Gluskin method. For any M such that $2n \leq M \leq e^n$ they produce a probability $\nu_{n,M}$ on $\mathcal{B}_n$ such that

$$(1.6) \qquad \nu_{n,M} \times \nu_{n,M} \left(\{(E,F) \in \mathcal{B}_n \times \mathcal{B}_n \mid d(E,F) < Cn/\log(M/n)\} \right) \leq 2e^{-nM}$$

and consequently for any $E_0 \in \mathcal{B}_n$

$$\nu_{n,M}(\{E \mid d(E,E_0) < \sqrt{Cn/\log(M/n)}\}) \leq \sqrt{2}e^{-nM/2}$$

where $C > 0$ is an absolute constant. This implies obviously

$$1 = \nu_{n,M}(\mathcal{B}_n) \leq N(\mathcal{B}_n, d, \sqrt{Cn/\log(M/n)})\sqrt{2}e^{-nM/2},$$

so that choosing $M \approx ne^{nC/r^2}$ $(r > 1)$ one obtains the lower bound of our Theorem 1.1 in the form

$$\exp \exp C'n/r^2 \leq N(\mathcal{B}_n, d, r)$$

where $C' > 0$ is an absolute constant and we assume n large enough (i.e. $n \geq n_0(r)$ for some $n_0(r)$).

Moreover by (1.6), Fubini's equality and Tchebyshev's inequality, if we let

$$A = \{E \mid \nu_{n,M}(\{F \mid d(E,F) < Cn/\log(M/n)\}) \leq 4e^{-nM}\}$$

we have

$$\nu_{n,M}(\mathcal{B}_n \setminus A) \leq 1/2,$$

and hence

$$1/2 \leq \nu_{n,M}(A) \leq N(A, d, Cn/\log(M/n))4e^{-nM},$$

so that choosing $M = n \exp n^\tau$ $(0 < \tau < 1)$ we find (assuming $n \geq n_0(\tau)$)

$$N(A, d, Cn^{1-\tau}) \geq (1/8) \exp(n^2 \exp n^\tau) \geq \exp \exp n^\tau.$$

A fortiori we have

$$M(\mathcal{B}_n, d, Cn^{1-\tau}) \geq M(A, d, Cn^{1-\tau}) \geq N(A, d, Cn^{1-\tau}) \geq \exp \exp n^\tau.$$

Remark 1.13. A natural question arises on the behaviour of the best possible constant c_r in Theorem 1.1 when $r \to \infty$. In [29] Szarek proves that this is $O(1/r)$. More precisely, given $r > 2$ and $n \in \mathbb{N}$, his result is that every n-dimensional normed space embeds r-isomorphically in ℓ_∞^m for some $m \leq n \exp c'n/r$. Consequently (see Lemma 1.2 and note that we may assume $r \leq \sqrt{n}$), the space $\mathcal{B}_n$ admits an r-net of cardinality at most $\exp \exp c''n/r$ where c', c'' are positive absolute constants. See also [1, Cor. 1.3] for a related result.

Remark 1.14. We return to the set $S_n(X)$ of n-dimensional subspaces of X defined in Lemma 1.2. We first observe that, if a Banach space X contains ℓ_∞^n's uniformly, then $S_n(X) = \mathcal{B}_n$, so the metric entropy of $S_n(X)$ is maximal. In sharp contrast, if $X = \ell_2$, $S_n(X)$ is reduced to ℓ_2^n so the metric entropy of $S_n(X)$ is minimal.

Assume that a Banach space X has the following property: any n-dimensional subspace $E \subset X$ embeds c-isomorphically into a fixed Banach space X_n with $\dim(X_n) = m$. Consider $0 < \xi < 1/n$ and let $R = (1 + \xi n)(1 - \xi n)^{-1}$. Then Lemma 1.2 implies

$$N(S(n,X), cR) \leq (1 + 2/\xi)^{nm}.$$

If this holds for $m = m(n)$ with $m(n) = 2^{o(n)}$ then (choosing $\xi = \varepsilon/n$) we have

$$\log \log N(S_n(X), d, 1 + \varepsilon) \in o(n)$$

for any $0 < \varepsilon < 1$.

When $X = \ell_1$ or L_1, it is proved in [3] (see also [23] and [32]) that for any $\varepsilon > 0$ any n-dimensional subspace $E \subset X$ embeds $1 + \varepsilon$-isomorphically into $\ell_1^{c_\varepsilon n (\log n)^3}$. This implies that $N(S_n(X), d, 1 + \varepsilon)$ is much smaller than the metric entropy of $\mathcal{B}_n$. In particular we have

$$\log \log N(S_n(X), d, 1 + \varepsilon) \in o(n).$$

By our initial observation, if, for some $\varepsilon > 0$, we have

$$\log \log N(S_n(X), d, 1 + \varepsilon) \in o(n)$$

then X does not contains ℓ_∞^n's uniformly, and hence has finite cotype.

It is tempting to wonder whether the converse holds (recall that $X = \ell_1$ or L_1 are examples of cotype 2 spaces). This question is reminiscent of the conjecture formulated in [19].

Problem. Characterize the infinite dimensional Banach spaces X such that

$$\log \log N(S_n(X), d, 1 + \varepsilon) \in o(n) \quad (n \to \infty)$$

for some (or all) $\varepsilon > 0$.

Note that this includes all L_p spaces ($1 \le p < \infty$) by [3, Th. 7.3 and Th. 7.4].

Same question for the class $SQ_n(X)$ of all n-dimensional subspaces of quotients of X. In that case, the $o(n)$-condition implies that X does not contains ℓ_1^n's uniformly, and hence has finite type.

Remark 1.15. Our main result suggests a number of other questions. Among them in the style of [33] it is natural to wonder how behaves the metric entropy of the class of spaces in $\mathcal{B}_n$ with either 1-symmetric or 1-unconditional bases.

2 Large local metric entropy

In this section we formulate a refinement of our main result in §1: even the balls around a single fixed space E in $\mathcal{B}_n$ have extremely large metric entropy. While for *some* E this can be deduced a priori from §1 by an elementary argument involving the maximal metric entropy of a ball of a fixed radius, it seems more surprising that it holds for *all* E. However, it turns out that, using [17], we can essentially reduce to the case $E = \ell_2^n$.

More precisely we have:

Theorem 2.1. *For all r, R such that $1 < r < R$ there is a constant $c > 0$ such that for any n assumed n large enough (i.e. $n \ge n_0(r, R)$), for any n-dimensional Banach space E there is a subset $T_n \subset \mathcal{B}_n$ with*

$$|T_n| \ge \exp \exp cn$$

such that

$$\forall s, t \in T_n \quad d(t, s) > r.$$

and

$$\forall t \in T_n \quad d(t, E) < R.$$

Remark 2.2. Note that it would be optimal if R could be any number such that $r < R^2$, with $c = c_{r,R}$, but we could not reach this degree of precision.

The proof is a simple modification of the proof of Theorem 1.1 (lower bound).

Lemma 2.3. *Let E be any n-dimensional space and $0 < \theta < 1$. Fix $c > 0, C \geq 1$. Assume there a finite subset $(x_t, x_t^*)_{t \in T} \subset E \times E^*$ with cardinality $|T| \geq \exp cn$ such that $\forall s \neq t \in T$ we have*

$$\|x_t\|_E \leq C \quad \|x_t\|_{E^*} \leq C$$

and

$$x_t^*(x_t) = 1 \quad |x_s^*(x_t)| \leq \theta.$$

Then for any $r < 1/\theta$, assuming n large enough (i.e. $n \geq n_0(r, \theta, c)$) there is a family of n-dimensional spaces $\{E_x \mid x \in \mathcal{X}\}$ with $|\mathcal{X}| \geq \exp\exp(cn/2)$ such that

$$\forall x \in \mathcal{X} \quad d(E, E_x) \leq C^2/\theta$$

and

$$\forall y \neq x \in \mathcal{X} \quad d(E_y, E_x) \geq r.$$

Therefore, if $R = C^2/\theta$ and if $B_{E,R}$ denotes the d-ball centered in E with radius R in $\mathcal{B}_n$ we have

$$\exp\exp(cn/2) \leq M(B_{E,R}, d, r) \leq N(B_{E,R}, d, \sqrt{r}).$$

Proof. We assume $|T|$ even and we let $\mathcal{A}$ denote the set of all subsets $x \subset \mathcal{A}$ with $|x| = |T|/2$. We then define E_x as equal to E equipped with the norm

$$\forall a \in E \quad \|a\|_{E_x} = \max\{C^{-1}\theta\|a\|_E \ , \ \sup_{t \in x} |\xi_t(a)|\}.$$

Note that

$$\forall a \in E \quad C^{-1}\theta\|a\|_E \leq \|a\|_{E_x} \leq C\|a\|_E,$$

and hence

$$d(E, E_x) \leq C^2/\theta.$$

Moreover, for any $y \neq z \in \mathcal{A}$ we can find $s \in y \setminus z$ so that $\|x_s\|_{E_y} = 1$ but $\|x_s\|_{E_z} \leq \theta$, and hence $\|E_z \to E_y\| \geq \theta^{-1}$. Thus the proof can be completed just as above for Theorem 1.1 (lower bound). $\square$

Proof of Theorem 2.1. We first observe that, e.g. by Lemma 1.5, the assumption of Lemma 2.3 is satisfied when $E = \ell_2^n$ for any $0 < \theta < 1$ (with $x_t = \xi_t$ and $C = 1$). Moreover, by adjusting c the same still holds if E admits either a subspace (resp. a quotient) or a subspace of a quotient of proportional dimension (with a fixed proportion) which is isometric to a Hilbert space. Indeed, the required extension (resp. lifting) is provided by the classical Hahn-Banach Theorem. Then the general case follows from Milman's proportional QS-theorem from [17] (see also [20, p. 108]). Indeed, given $r < R$, we may choose $0 < \theta < 1$ and $C > 1$ so that $r < \theta^{-1}$ and $C^2/\theta < R$. Then by [17] (see also [20, p. 108]) there is a proportion $0 < \delta < 1$ (depending only on C) such that any $E \in \mathcal{B}_n$ admits a quotient of a subspace of dimension $> \delta n$ that is C-isomorphic to a Hilbert space. By the same (extension/lifting) argument as for the isometric case, the assumption (and hence the conclusion) of Lemma 1.5 holds for some adjusted value of c. $\square$

3 The matricial case

We now turn to operator spaces, i.e. closed subspaces of the algebra $B(H)$ of all bounded operators on Hilbert space. The analogue of the norm becomes a sequence of norms: for each N we consider the space $M_N(E)$ of $N \times N$ matrices with entries in E equipped with the norm induced by $B(H \oplus \cdots \oplus H)$. It is then customary to think of the sequence $\{\|.\|_{M_N(E)} \mid N \geq 1\}$ as the operator space analogue of the usual norm (corresponding to $N = 1$). We refer to the books [7, 21] for all unexplained terminology and for more background.

We wish to study the metric space OS_n equipped with the completely bounded analogue d_{cb} of the distance d. However, although it is complete, this space is not compact, and by [14] it is not even separable. Nevertheless, one may study the distance associated to a fixed size $N \geq 1$ for the matrix coefficients, as follows.

Let $u : E \to F$ be a linear map between operator spaces. We denote

$$u_N = Id \otimes u : M_N(E) \to M_N(F).$$

If E, F are two operator spaces that are isomorphic as Banach spaces, we set

(3.1) $$d_N(E, F) = \inf\{\|u_N\|\|(u^{-1})_N\|\}$$

where the inf runs over all the isomorphisms $u : E \to F$.
We set $d_N(E, F) = \infty$ if E, F are not isomorphic.
Recall that

$$\|u\|_{cb} = \sup_{N \geq 1} \|u_N\|.$$

Recall also that, if E, F are completely isomorphic, we set

$$d_{cb}(E, F) = \inf\{\|u\|_{cb}\|u^{-1}\|_{cb}\}$$

where the inf runs over all the complete isomorphisms $u : E \to F$.
When E, F are both n-dimensional, using a compactness argument, it is easy to show that $d_N(E, F) = 1$ iff there is an isomorphism $u : E \to F$ such that u_N is isometric. Moreover, again by a compactness argument, one can check easily that

$$d_{cb}(E, F) = \sup_{N \geq 1} d_N(E, F).$$

Our main result is :

Theorem 3.1. *For any $r > 1$, there are positive constants b_r, c_r such that, for any n assumed large enough (more precisely $n \geq n_0(r)$), we have for all $N \geq 1$*

$$\exp(\exp b_r n N^2) \leq N(\mathcal{O}S_n, d_N, r) \leq \exp(\exp c_r n N^2).$$

As observed in [22, Lemma 2.11], for any $E \in \mathcal{O}S_n$ and $\varepsilon > 0$ there is $F \subset \ell_\infty^k \otimes M_N$ such that $d_N(E, F) \leq (1 - \delta)^{-1}$ provided $k \leq (1 + 2/\delta)^{nN^2}$. The argument for this is entirely analogous to the one above for Lemma 1.2. Using this, the upper bound in Theorem 3.1 can be proved just like the one for Theorem 1.1. For the lower bound, as in [22] we crucially use quantum expanders following on [12] (see [22] for more information and references).

Let τ_N denote the normalized trace on M_N. We will denote by $S_\varepsilon = S_\varepsilon(n, N) \subset U(N)^n$ the set of all n-tuples $u = (u_j) \in U(N)^n$ such that $\forall x \in M_N$, we have

$$\|\sum u_j(x - \tau_N(x)I)u_j^*\|_{L_2(\tau_N)} \leq \varepsilon n \|x\|_{L_2(\tau_N)}.$$

We will use the following result from [22].

Theorem 3.2. *For any $0 < \delta < 1$ there is a constant $\beta_\delta > 0$ such that for each $0 < \varepsilon < 1$ and for all sufficiently large integer n (i.e. $n \geq n_0$ with n_0 depending on ε and δ) and for all $N \geq 1$, there is a subset $T \subset S_\varepsilon(n, N)$ with $|T| \geq \exp \beta_\delta n N^2$ that is δ-separated in the following sense:*

$$\forall s = (s_j) \neq t = (t_j) \in T \quad \|\sum s_j \otimes \overline{t_j}\| \leq (1-\delta)n,$$

where the norm is in the space $M_N(M_N)$ or equivalently M_{N^2}.

Proof of Theorem 3.1 (Lower bound). Let T be as in the preceding Theorem. Let $K = |T| \geq \exp \beta_\delta n N^2$ (assumed even) and let $\mathcal{A}$ denote again the class of subsets $x \subset T$ such that $|x| = K/2$, so that again for all $x \neq y \in \mathcal{A}$ we have $x \not\subset y$, and $|\mathcal{A}| \geq \gamma 2^K / \sqrt{K}$. We denote by e_{jj} the canonical basis of ℓ_∞^n. We then define for any subset $x \in \mathcal{A}$, (note $x \subset T \subset U(N)^n$)

$$u_j^x = e_{jj} \oplus [\oplus_{t \in x} t_j],$$

and we define

$$F_x = \text{span}[u_j^x, 1 \leq j \leq n] \subset \ell_\infty^n \oplus_\infty [\ell_\infty(x) \otimes M_N].$$

Here the space $\ell_\infty(x) \otimes M_N$ is equipped with its (unique) C^*-norm, namely the norm defined for a function $f : x \to M_N$ by $\|f\| = \sup_{t \in x} \|f(t)\|_{M_N}$. With this notation, $[\oplus_{t \in x} t_j] \in \oplus_{t \in x} M_N$ is identified with the function $f_j : x \to M_N$ defined by $f_j(t) = t_j \; \forall t \in x$.
We equip $M_N \otimes M_N$ (resp. $M_N \otimes [\ell_\infty(x) \otimes M_N]$) with the classical minimal (or spatial) tensor norm, or equivalently the norm of M_{N^2} (resp. the sup-norm on x of M_{N^2}-valued functions). Let $\theta = 1 - \delta$. Then, whenever $x \neq y \in \mathcal{A}$, since $x \not\subset y$ there is $s \in x \setminus y$ and hence (assuming as we may $(1-\delta)n \geq 1$)

$$\|\sum \bar{s}_j \otimes u_j^x\| = n \quad \|\sum \bar{s}_j \otimes u_j^y\| \leq (1-\delta)n = \theta n.$$

We will now use the metric entropy estimate of the unit ball of $B(\ell_1^n, E_x)$, just like before. Fix r such that $1 < r < \theta^{-1}$ and let η be as before so that $1 + \eta = 1/(r\theta)$. We find that for a fixed $x \in \mathcal{A}$ we have

$$|\{y \in \mathcal{A} \mid d_N(F_x, F_y) < r\}| \leq (1 + 4n/\eta)^{2n^2}.$$

Let $\mathcal{X} \subset \mathcal{A}$ be a maximal subset such that $d_N(F_x, F_y) \geq r$ for any $x \neq y \in \mathcal{X}$. Then $\mathcal{A}$ is covered by the d_N-balls of radius r centered in the points of $\mathcal{X}$, and hence by the claim

$$|\mathcal{A}| \leq |\mathcal{X}|(1 + 4n/\eta)^{2n^2} \leq |\mathcal{X}| \exp 8n^3/\eta.$$

Thus we conclude as before

$$|\mathcal{X}| \geq |\mathcal{A}| \exp -8n^3/\eta,$$

and for n large enough, the announced lower bound follows, just like for Theorem 1.1. $\square$

Remark 3.3. Using the stronger separation property considered in [22, Lemma 2.14], it is easy to check the same lower bound (but possibly for a smaller value of b_r) as in Theorem 3.1 for the class denoted by $\mathcal{HH}_n$ of n-dimensional Hilbertian homogeneous operator spaces. However, this argument requires $N^2/n \geq N_0$. Note that if E, F are two n-dimensional Hilbertian homogeneous operator spaces, then $d_{cb}(E, F) = \|Id : E \to F\|\|Id : F \to E\|$. This is due to C. Zhang (see [21, p. 217]). With this fact the completion of the proof is much simpler for this case.

Remark 3.4. At the time of this writing, we do not see how to prove the analogue for OS_n of Szarek's result described in Remark 1.13. However we do have a very simple argument to show that for any $r > 1$ $\forall n \geq (r \log r)/4$ $\forall N \geq 1$ we have

$$\log \log N(\mathcal{HH}_n, d_N, r) \leq (c/r)nN^2$$

for some absolute constant $c > 0$. The idea is that any $E \in \mathcal{HH}_{2n}$ is completely 2-isomorphic to $F \oplus F$ where F is any n dimensional subspace of E (note that, by homogeneity, these are all mutually completely isometric). Thus if we let $m(n, N, r)$ be the smallest k such that for any $E \in \mathcal{HH}_n$ there is $\hat{E} \subset \ell_\infty^k(M_N)$ with $d_N(E, \hat{E}) \leq r$, then we have

(3.2) $$m(2n, N, 2r) \leq 2m(n, N, r).$$

But as we already mentioned we also know that $m(n, N, r) \leq \exp\left(2nN^2 r(r-1)^{-1}\right)$ (see [22, Lemma 2.11]), and in particular, say, $m(n, N, 2) \leq \exp\left(4nN^2\right)$. Iterating (3.2) (assuming for simplicity that n is a power of 2) we obtain

$$m(n, N, 2^{k+1}) \leq 2^k m(n/2^k, N, 2) \leq 2^k \exp\left(4nN^2 2^{-k}\right).$$

Thus if $r = 2^k \leq \exp\left(4n/r\right)$ (i.e. if $n \geq (r \log r)/4$) then $m(n, N, 2r) \leq \exp\left(8nN^2/r\right)$ for any $N \geq 1$. From this the announced result follows, using the operator space version of Lemma 1.2.

4   Why quantum expanders ?

It seems worthwhile to clarify why quantum expanders are needed in the proof of Theorem 3.2 and why the latter is crucial to extend §1.

   A direct generalization of Lemma 1.5 to matrices would require to produce a subset $T \subset U(N)^n$ with $|T| \geq \exp \beta nN^2$ such that

(4.1) $$\forall u = (u_j) \neq v = (v_j) \in T \quad \|\sum u_j \otimes \overline{v_j}\| \leq (1 - \delta)n,$$

with β, δ as in Theorem 3.2. One could argue similarly and consider a maximal subset $T \subset U(N)^n$ satisfying (4.1). Then denoting by m the normalized Haar measure on $U(N)^n$, we note that, by maximality, $U(N)^n$ is covered by the union of the sets

$$C_u(\delta) = \{v \in U(N)^n \mid \|\sum u_j \otimes \overline{v_j}\| > (1 - \delta)n\}$$

for $u \in T$. Thus if we set

$$F(\delta) = \sup_{u \in T} m(C_u(\delta))$$

we find

$$1 \leq |T| F(\delta).$$

In §1 we use the easy fact that when $N = 1$ there is $c > 0$ so that we have $m(C_u(\delta)) \leq \exp -cn$ for all $u \in \mathbb{T}^n$ or actually for all $u \in \{-1, 1\}^n$. For arbitrary N, one can show by concentration of measure arguments an upper bound of the form $m(C_u(\delta)) \leq \exp -cnN$. This follows e.g. from a Gaussian matrix result due to Haagerup and Thorbjoernsen [11, Th. 3.3] together with a comparison principle between Gaussian and unitary random matrices (see [16, p. 82]).

However, in our present setting the corresponding result we would need to prove Theorem 3.1 by direct analogy with §1 should be

$$(4.2) \qquad \sup_{u \in U(N)^n} m(C_u(\delta)) \le \exp -cnN^2.$$

This is simply not true: Take $u_j = I$ for all j, and note $\|\sum u_j \otimes \overline{v_j}\| = \|\sum v_j\| \ge \Re(\sum \langle v_j e, e \rangle)$ for any fixed e in the unit sphere of ℓ_2^N. Then $\cap_{1 \le j \le n} \{v \mid \Re(\langle v_j e, e \rangle) > (1 - \delta)\} \subset C_u(\delta)$, and hence (here m_1 denotes Haar measure on $U(N)$)

$$m(C_u(\delta)) \ge (m_1\{v \in U(N) \mid \Re(\langle ve, e \rangle) > (1 - \delta)\})^n.$$

But it is well known and easy to check that for any fixed $\delta > 0$, there is $c_1 > 0$ such that $m_1\{v \in U(N) \mid \Re(\langle ve, e \rangle) > (1 - \delta)\} \ge \exp -c_1 N$ for all N large enough. Thus we must have $m(C_u(\delta)) \ge \exp -c_1 nN$ when $N \to \infty$, contradicting the desired bound $m(C_u(\delta)) \le \exp -cnN^2$ at least if we want it to hold for any u.

However, let us assume instead that

$$(4.3) \qquad m\{u \in U(N)^n \mid m(C_u(\delta)) \le \exp -cnN^2\} \ge a > 0,$$

and let $A = \{u \in U(N)^n \mid m(C_u(\delta)) \le \exp -cnN^2\}$. Then let T be a maximal subset $T \subset A$ satisfying (4.1). Again by maximality we find $A \subset \cup_{u \in T} C_u(\delta)$ and hence, since $m(C_u(\delta)) \le \exp -cnN^2$ for $u \in A$, we now have $a \le m(A) \le |T| \exp -cnN^2$ and we conclude

$$|T| \ge a \exp cnN^2,$$

which implies Theorem 3.2.

In [22] we show that this can be applied when A is the set $S_\varepsilon(n, N)$ for $0 < \varepsilon < 1$ small enough with respect to δ. For the convenience of the reader we will outline the argument of [22], with special emphasis on the key concentration of measure issues.
Assume $0 < \delta < 1$. We define $0 < \varepsilon, \varepsilon' < 1$ by the equalities

$$3\varepsilon'^{1/3} = (1 - \delta)/2 \quad 2\varepsilon = (1 - \delta)/2,$$

so that

$$1 - \delta = 3\varepsilon'^{1/3} + 2\varepsilon.$$

With these values (depending on δ) of $\varepsilon, \varepsilon'$, it is shown in [22, Lemma 1.11] that if $u, v \in S_\varepsilon(n, N)$ then

$$\inf_{U, V \in U(N)} N^{-1} \sum \mathrm{tr} |Uu_j V - v_j|^2 \ge 2n(1 - \varepsilon') \Rightarrow \|\sum u_j \otimes \bar{v}_j\| \le n(1 - \delta).$$

Equivalently we have

$$(4.4) \qquad C_u(\delta) \subset \{v \in U(N)^n \mid \sup_{U, V \in U(N)} \Re(N^{-1} \mathrm{tr}(\sum_1^n u_j U v_j^* V^*)) > \varepsilon' n\}.$$

Let $f(U, V) = \Re(N^{-1} \mathrm{tr}(\sum_1^n u_j U v_j^* V^*))$. Using the fact that for any U, U', V, V' in $U(N)$ we have

$$|f(U, V) - f(U', V')| \le n(\|U - U'\| + \|V - V'\|),$$

together with the well known fact (see e.g. [26, appendix]) that for any $\xi > 0$ there is an ξ-net $\mathcal{N}_\xi \subset U(N)$ with respect to the operator norm with $|\mathcal{N}_\xi| \le (\gamma/\xi)^{2N^2}$ for some numerical constant γ, we find that

$$\sup_{U, V \in U(N)} f(U, V) \le \sup_{U, V \in \mathcal{N}_\xi} f(U, V) + 2n\xi,$$

14

and hence by (4.4)

$$m(C_u(\delta)) \le \sum_{U,V \in \mathcal{N}_\xi} m\{v \mid \Re(N^{-1}\text{tr}(\sum_1^n u_j U v_j^* V^*)) > \varepsilon'n - 2n\xi\}.$$

We choose $\xi = \varepsilon'/4$ so that $\varepsilon'n - 2n\xi = n\varepsilon'/2$. By the translation invariance of m, $m\{v \mid \Re(N^{-1}\text{tr}(\sum_1^n u_j U v_j^* V^*)) > \varepsilon'n - 2n\xi\}$ is actually independent of U, V and u and equal simply to $m\{v \mid \Re(N^{-1}\text{tr}(\sum_1^n v_j)) > \varepsilon'n - 2n\xi\}$. Thus we find

$$m(C_u(\delta)) \le |\mathcal{N}_\xi|^2 m\{v \mid \Re(N^{-1}\text{tr}(\sum_1^n v_j)) > n\varepsilon'/2\}.$$

But now the last measure can be efficiently estimated by a simple subGaussian argument since it is known (see e.g. [13, §36.3]) that there is a numerical constant $c'' > 0$ such that for any $s > 0$

$$m\{\Re(N^{-1}\text{tr}(\sum_1^n v_j)) > s\} \le \exp-\frac{c''s^2N^2}{n}.$$

Recalling the bound for $|\mathcal{N}_\xi|$ for $\xi = \varepsilon'/4$ we obtain

$$m(C_u(\delta)) \le (4\gamma/\varepsilon')^{4N^2} \exp-\{c''(\varepsilon'/2)^2 N^2 n\}.$$

Note that for any $0 < \delta < 1$, when n is large enough (more precisely $n \ge n(\delta)$), there is $c = c_\delta > 0$ so that the last term is $\le \exp-c_\delta nN^2$ for all $N \ge 1$. Indeed, recalling that $\varepsilon' = \varepsilon'_\delta$, one can set e.g. $c_\delta = (1/2)c''(\varepsilon'/2)^2$ and then determine $n(\delta)$ by the requirement that $(4\gamma/\varepsilon')^4 \le \exp c_\delta n$ for all $n \ge n(\delta)$. Then for all $n \ge n(\delta)$ we have for any $N \ge 1$ (recall here that $\varepsilon = \varepsilon_\delta$)

(4.5) $$S_\varepsilon(n, N) \subset \{u \in U(N)^n \mid m(C_u(\delta)) \le \exp-c_\delta nN^2\}.$$

To complete this approach, we now need to check that for any $\varepsilon > 0$ $m(S_\varepsilon(n, N))$ is bounded below. This is proved in the appendix of [22], as an immediate consequence of the following inequality where C is an absolute constant and P is the orthogonal projection onto multiples of the $N \times N$-identity matrix

$$\forall n, N \ge 1 \quad \int_{U(N)^n} \|\sum_1^n u_j \otimes \bar{u}_j(1 - P)\| dm(u) \le C\sqrt{n}.$$

Indeed, by Tchebyshev's inequality, this implies

$$\forall n, N \ge 1 \quad m((S_\varepsilon(n, N)) \ge 1 - (C/\varepsilon)n^{-1/2}.$$

Thus we obtain finally by (4.5) for any $0 < \delta < 1$

$$m\{u \in U(N)^n \mid m(C_u(\delta)) \le \exp-c_\delta nN^2\} \ge 1 - (C/\varepsilon_\delta)n^{-1/2},$$

and if $n(\delta)$ is chosen large enough this last term is $> 1/2$ for all $n \ge n(\delta)$.

Thus the answer to the question raised in the title of this section is : Because quantum expanders are the key to prove (4.3) which can be substituted to the (false) (4.2) !

15

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