GEOMETRY AND INTERIOR NODAL SETS OF STEKLOV EIGENFUNCTIONS

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Abstract. We investigate the geometric properties of Steklov eigenfunctions in smooth manifolds. We derive the refined doubling estimates and Bernstein's inequalities. For the real analytic manifolds, we are able to obtain the sharp upper bound for the measure of interior nodal sets $H^{n-1}(\mathcal{N}_\lambda) \leq C\lambda$. Here the positive constant C depends only on the manifolds.

1. Introduction

In this paper, we address the geometric properties and interior nodal sets of Steklov eigenfunctions

$$\begin{cases} \triangle_g e_\lambda(x) = 0, \quad x \in \mathcal{M}, \\ \frac{\partial e_\lambda}{\partial \nu}(x) = \lambda e_\lambda(x), \quad x \in \partial\mathcal{M}, \end{cases} \tag{1.1}$$

where ν is a unit outward normal on $\partial\mathcal{M}$. Assume that $(\mathcal{M}, g)$ is a n-dimensional smooth, connected and compact manifold with smooth boundary $\partial\mathcal{M}$, where $n \geq 2$. The Steklov eigenfunctions were first studied by Steklov in 1902 for bounded domains in the plane. It is also regarded as eigenfunctions of the Dirichlet-to-Neumann map, which is a first order homogeneous, self-adjoint and elliptic pseudodifferential operator. The spectrum λ_j of Steklov eigenvalue problem consists of an infinite increasing sequence with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3, \cdots, \quad \text{and} \quad \lim_{j\to\infty} \lambda_j = \infty.$$

The eigenfunctions $\{e_{\lambda_j}\}$ form an orthonormal basis such that

$$e_{\lambda_j} \in C^\infty(\mathcal{M}), \quad \int_{\partial\mathcal{M}} e_{\lambda_j} e_{\lambda_k}\, dV_g = \delta_j^k.$$

Recently, the study of nodal geometry has been attracting much attentions. Estimating the Hausdorff measure of nodal sets has always been an important subject concerning the nodal geometry of eigenfunctions. The celebrated problem about nodal sets centers around the famous Yau's conjecture for smooth manifolds. Let e_λ be L^2 normalized eigenfunctions of

$$-\triangle_g e_\lambda = \lambda^2 e_\lambda \tag{1.2}$$

on compact manifolds $(\mathcal{M}, g)$ without boundary, Yau conjectured that the upper and lower bound of nodal sets of eigenfunctions in (1.2) are controlled by

$$c\lambda \leq H^{n-1}(\{x \in \mathcal{M} | e_\lambda(x) = 0\}) \leq C\lambda \tag{1.3}$$

where C, c depend only on the manifold $\mathcal{M}$. The conjecture is shown to be true for real analytic manifolds by Donnelly-Fefferman in [DF]. Lin [Lin] also proved the upper bound for the analytic manifolds using a different approach. For the smooth manifolds, there are some breakthrough by Lugonov on the polynomial upper bound [Lo1] and sharp lower bound of nodal sets [Lo2]. For

2010 *Mathematics Subject Classification.* 35P20, 35P15, 58C40, 28A78.
Key words and phrases. Nodal sets, Upper bound, Steklov eigenfunctions.
Research is partially supported by the NSF grant DMS 1656845.

detailed account about this subject, interested readers may refer to the book [HL] and survey [Z].

For the Steklov eigenfunctions, by the maximum principle, there exist nodal sets in $\mathcal{M}$ and those sets intersect the boundary $\partial\mathcal{M}$ traservasally. It is interesting to ask Yau's type questions about the Hausdorff measure of nodal sets of Steklov eigenfunctions on the boundary and interior of manifolds, respectively. The natural and corresponding conjecture for Steklov eigenfunctions should state exactly as

$$c\lambda \leq H^{n-2}(\{x \in \partial\mathcal{M}, e_\lambda(x) = 0\}) \leq C\lambda, \tag{1.4}$$

$$c\lambda \leq H^{n-1}(\{x \in \mathcal{M}, e_\lambda(x) = 0\}) \leq C\lambda. \tag{1.5}$$

See also the survey by Girouard and Polterovich in [GP] about these open questions.

Recently, much work has been devoted to the bounds of nodal sets of Steklov eigenfunctions on the boundary

$$Z_\lambda = \{x \in \partial\mathcal{M} | e_\lambda(x) = 0\}. \tag{1.6}$$

The study of (1.6) was initiated by Bellova and Lin [BL] who proved the $H^{n-2}(Z_\lambda) \leq C\lambda^6$ with C depending only on $\mathcal{M}$, if $\mathcal{M}$ is an analytic manifold. By microlocal analysis argument, Zelditch [Z1] was able to improve their results and gave the optimal upper bound $H^{n-2}(Z_\lambda) \leq C\lambda$ for analytic manifolds. For the smooth manifold $\mathcal{M}$, Wang and the author in [WZ] established a lower bound

$$H^{n-2}(Z_\lambda) \geq C\lambda^{\frac{4-n}{2}} \tag{1.7}$$

by considering the fact that the Steklov eigenfunctions are eigenfunctions of first order elliptic pseudodifferential operator. The polynomial lower bound (1.7) is the Steklov analogue of the lower bounds of nodal sets for classical eigenfunctions (1.2) obtained in [CM], [SZ], [SZ1] and [HSo].

Concerning about the bounds of interior nodal sets of eigenfunctions,

$$\mathcal{N}_\lambda = \{x \in \mathcal{M} | e_\lambda(x) = 0\},$$

Sogge, Wang and the author [SWZ] obtained a lower bound for interior nodal sets

$$H^{n-1}(\mathcal{N}_\lambda) \geq C\lambda^{\frac{2-n}{2}}$$

for a smooth manifold $\mathcal{M}$. The measure of nodal sets is more clear on surfaces. In [Zh1], the author was able to obtain an upper bound for the measure of interior nodal sets

$$H^1(\mathcal{N}_\lambda) \leq C\lambda^{\frac{3}{2}}.$$

The singular sets $\mathcal{S}_\lambda = \{x \in \mathcal{M} | e_\lambda = 0, \nabla e_\lambda = 0\}$ are finite points on the nodal surfaces. It was also shown that $H^0(\mathcal{S}_\lambda) \leq C\lambda^2$ in [Zh1]. Recently, Polterovich, Sher and Toth [PST] could verify Yau's type conjecture for upper and lower bounds in (1.5) for the real-analytic Riemannian surfaces $\mathcal{M}$. Georgiev and Roy-fortin [GR] obtained polynomial upper bounds for interior nodal sets on smooth manifolds. There are still many challenges for the study of Steklov eigenfunctions. For instance, it is well-known that the classical eigenfunctions in (1.2) are so dense that there are nodal sets in each geodesic ball with radius $C\lambda^{-1}$. This fundamental result is crucial to derive the lower bounds of nodal sets for classical eigenfunctions (1.2) in [DF] and [Br]. For the Steklov eigenfunctions, it is unknown whether such density results remain true on the boundary and interior of the manifold, which cause difficulties in studying the Steklov eigenfunctions, see e.g. [Z1].

An interesting topic in the study of eigenfunction is called as doubling inequality. Doubling inequality plays an important role in deriving strong unique continuation property, the vanishing order of eigenfunctions and obtaining the measure of nodal sets, see e.g. [DF], [DF1]. The doubling inequality for classical eigenfunctions (1.2)

$$\int_{\mathbb{B}(p,\,2r)} e_\lambda^2 \leq e^{C\lambda} \int_{\mathbb{B}(p,\,r)} e_\lambda^2 \tag{1.8}$$

is derived using Carleman estimates in [DF] for $0 < r < r_0$, where $\mathbb{B}(p, c)$ denotes as a ball in $\mathcal{M}$ centered at p with radius c and r_0 depends on $(\mathcal{M}, g)$. For the Steklov eigenfunctions on $\partial\mathcal{M}$, the author has obtained a similar type of doubling inequality on the boundary $\partial\mathcal{M}$ and derived that the sharp vanishing order is less than $C\lambda$ on the boundary $\partial\mathcal{M}$ in [Zh]. For Steklov eigenfunctions in $\mathcal{M}$, we were also able to get the doubling inequality as (1.8) in [Zh1]. For the classical eigenfunctions (1.2), a refined doubling inequality

$$\int_{\mathbb{B}(p,\,(1+\frac{1}{\lambda})r)} e_\lambda^2 \leq C \int_{\mathbb{B}(p,\,r)} e_\lambda^2 \tag{1.9}$$

was derived in [DF2] by stronger Carleman estimates. The refine doubling inequality also leads to Bernstein's gradient inequalities for classical eigenfunctions. The first goal in this note is to study a refined version doubling inequality for the Steklov eigenfunctions and its applications.

Theorem 1. *For the Steklov eigenfunctions in (1.1), there hold*
(A): a refined doubling inequality

$$\int_{\mathbb{B}(p,\,(1+\frac{1}{\lambda})r)} e_\lambda^2 \leq C \int_{\mathbb{B}(p,\,r)} e_\lambda^2,$$

(B): L^2-Bernstein's inequality

$$\int_{\mathbb{B}(p,\,r)} |\nabla e_\lambda|^2 \leq \frac{C\lambda^2}{r^2} \int_{\mathbb{B}(p,\,r)} e_\lambda^2,$$

(C) L^∞-Bernstein's inequality

$$\max_{\mathbb{B}(p,\,r)} |\nabla e_\lambda| \leq \frac{C\lambda^{\frac{n+2}{2}}}{r} \max_{\mathbb{B}(p,\,r)} |e_\lambda|$$

for $\mathbb{B}(p,\,(1+\frac{1}{\lambda})r) \subset \mathcal{M}$ and $0 < r < r_0$, where r_0 depends on $(\mathcal{M}, g)$.

Our second goal is to obtain the optimal upper bound of interior nodal sets for real analytic manifolds. Our work extends the optimal upper bound in [PST] to real analytic manifolds in any dimensions, which proves the upper bound of Yau's type conjecture for interior nodal sets in (1.5). We will transform the Steklov eigenvalue problem into a second order elliptic problem with a Neumann boundary condition. Adapting the ideas in [DF], [DF1] and doubling inequality, we obtain the following theorem.

Theorem 2. *Let $\mathcal{M}$ be a real analytic compact and connected manifold with boundary. There exists a positive constant $C(\mathcal{M})$ such that,*

$$H^{n-1}(\mathcal{N}_\lambda) \leq C\lambda$$

for the Steklov eigenfunctions.

The outline of the paper is as follows. In section 2, we reduce the Steklov eigenvalue problem into an equivalent elliptic equation without boundary. Then we obtain the refined doubling inequality and show Theorem 1. Section 3 is devoted to the upper bound of interior nodal sets for real analytic manifolds. Section 4 is the appendix which provides the proof of some arguments for the Carleman estimates. The letter c, C, C_i denote generic positive constants and do not depend on λ. They may vary in different lines and sections.

Acknowledgement. It is my pleasure to thank Professor Christopher D. Sogge and Joel Spruck for support and helpful discussions about the topic of eigenfunctions. Especially, the author thanks Professor Steve Zelditch for critical comments and constructive suggestions on the nodal sets estimates in section 3.

2. Refined doubling inequality

In this section, we will establish a stronger Carleman estimate than that in [Zh]. We will transform the Steklov eigenvalue problem into a second order elliptic equation on a boundaryless manifold. The eigenvalue λ will be reflected in the coefficient functions of the elliptic equation.

To make the Steklov eigenvalue problem into an elliptic equation, adapting the ideas in [BL], we choose an auxiliary function involving the distance function. Let $d(x) = dist\{x, \partial\mathcal{M}\}$ be the distance function from $x \in \mathcal{M}$ to the boundary $\partial\mathcal{M}$. If $\mathcal{M}$ is smooth, $d(x)$ is smooth in the small neighborhood $\mathcal{M}_\rho$ of $\partial\mathcal{M}$ in $\mathcal{M}$. By the partition of unity, we extend $d(x)$ in a smooth manner by introducing

$$\varrho(x) = \begin{cases} d(x) & x \in \mathcal{M}_\rho, \\ l(x) & x \in \mathcal{M}\backslash\mathcal{M}_\rho. \end{cases}$$

Therefore, the extended function $\varrho(x)$ is a smooth function in $\mathcal{M}$. We consider an auxiliary function

$$u(x) = e_\lambda \exp\{\lambda\varrho(x)\}.$$

Then the new function $u(x)$ satisfies

$$\begin{cases} \triangle_g u + b(x)\cdot\nabla_g u + q(x)u = 0 & \text{in } \mathcal{M}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{M} \end{cases} \tag{2.1}$$

with

$$\begin{cases} b(x) = -2\lambda\nabla_g\varrho(x), \\ q(x) = \lambda^2|\nabla_g\varrho(x)|^2 - \lambda\triangle_g\varrho(x). \end{cases} \tag{2.2}$$

In order to construct a boundaryless model, we attach two copies of $\mathcal{M}$ along the boundary and consider a double manifold $\overline{\mathcal{M}} = \mathcal{M}\cup\mathcal{M}$. Then induced metric g' of g on the double manifold $\overline{\mathcal{M}}$ is Lipschitz. We consider a canonical involutive isometry $\mathcal{F} : \overline{\mathcal{M}} \to \overline{\mathcal{M}}$ which interchanges the two copies of $\mathcal{M}$. In this sense, the function $u(x)$ can be extended to the double manifold $\overline{\mathcal{M}}$ by $u\circ\mathcal{F} = u$. Thus, $u(x)$ satisfies

$$\triangle_{g'} u + \bar{b}(x)\cdot\nabla_{g'} u + \bar{q}(x)u = 0 \quad \text{in } \overline{\mathcal{M}}. \tag{2.3}$$

From the assumptions in (2.2), it follows that

$$\begin{cases} \|\bar{b}\|_{W^{1,\infty}(\overline{\mathcal{M}})} \leq C\lambda, \\ \|\bar{q}\|_{W^{1,\infty}(\overline{\mathcal{M}})} \leq C\lambda^2. \end{cases} \tag{2.4}$$

By a standard regularity argument for dealing with Lipschitz metrics in [DF1], we can established a similar Carleman inequality as that in [Zh] for the general second order elliptic equation (2.3). See also e.g. [BC] for similar estimates for smooth manifolds.

Lemma 1. *Let $u \in C_0^\infty\Big(\mathbb{B}(p,\epsilon_0)\backslash\mathbb{B}(p,\epsilon_1)\Big)$. If $\beta > C(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|^{1/2}_{W^{1,\infty}})$, then*

$$\int r^4 e^{2\beta\psi(r)}|\triangle_{g'} u + \bar{b}\cdot\nabla_{g'} u + \bar{q}u|^2\, dvol \geq C\beta^3 \int r^\epsilon e^{2\beta\psi(r)} u^2\, dvol, \tag{2.5}$$

where $\psi(r) = -\ln r(x) + r^\epsilon(x)$ and $r(x)$ is the geodesic distance from x to p, $0 < \epsilon_0, \epsilon_1, \epsilon < 1$ are some fixed constants.

We include the major argument of the proof of Lemma 1 in the appendix. By the above Carleman estimates, we can derive a Hadamard's three-ball theorem. Based on a propagation of smallness argument, we have obtained the following doubling inequality in $\overline{\mathcal{M}}$ in [Zh1].

Proposition 1. *There exist positive constants r_0 and C depending only on $\overline{\mathcal{M}}$ such that for any $0<r<r_0$ and $p\in\overline{\mathcal{M}}$, there holds*

$$\|u\|_{L^2(\mathbb{B}(p,\,2r))}\leq e^{C\lambda}\|u\|_{L^2(\mathbb{B}(p,\,r))} \tag{2.6}$$

for any solutions of (2.3).

From the proposition, it is easy to see that the doubling inequality for Steklov eigenfunctions as (1.8) holds in $\mathcal{M}$ if $\mathbb{B}(p,2r)\subset\mathcal{M}$, since $\varrho(x)$ is a bounded function. By standard elliptic estimates, the L^∞ norm of doubling inequality

$$\|u\|_{L^\infty(\mathbb{B}(p,\,2r))}\leq e^{C\lambda}\|u\|_{L^\infty(\mathbb{B}(p,\,r))}$$

holds, which also implies that

$$\|e_\lambda\|_{L^\infty(\mathbb{B}(p,\,2r))}\leq e^{C\lambda}\|e_\lambda\|_{L^\infty(\mathbb{B}(p,\,r))}.$$

Next we will establish a stronger Carleman inequality than that in Lemma 1 with weight function $\exp\{\beta\psi(x)\}$, where the function ψ satisfies some convexity properties. Choosing a fixed number ϵ such that $0<\epsilon<1$ and $T_0<0$, we define the function ϕ on $(-\infty,\ T_0]$ by $\phi(t)=t-e^{\epsilon t}$. If $|T_0|$ is sufficiently large, the function $\phi(t)$ satisfies the following properties

$$1-\epsilon e^{\epsilon T_0}\leq\phi'(t)\leq 1, \tag{2.7}$$

$$\lim_{t\to-\infty}\frac{-\phi''(t)}{e^t}=+\infty. \tag{2.8}$$

Let $\psi(x)=-\phi\big(\ln r(x)\big)$, where $r(x)=d(x,\,p)$ is geodesic distance between x and p. The stronger Carleman estimate is stated as follows.

Proposition 2. *Let $u\in C_0^\infty\Big(\mathbb{B}(p,h)\backslash\mathbb{B}(p,\delta)\Big)$. If $\beta>C(1+\|\bar b\|_{W^{1,\infty}}+\|\bar q\|^{1/2}_{W^{1,\infty}})$, then*

$$\begin{aligned}\int_{B(p,h)}r^4e^{2\beta\psi}|\triangle u+\bar b\cdot\nabla u+\bar q u|^2\,dvol\quad&\geq\quad C\beta^3\int_{\mathbb{B}(p,h)}r^\epsilon e^{2\beta\psi}u^2dvol\\ &+\quad C\beta^4\int_{\mathbb{B}(p,\delta(1+\frac{C}{\beta}))}e^{2\beta\psi}u^2dvol,\end{aligned} \tag{2.9}$$

where $\psi(r)=-\ln r(x)+r^\epsilon(x)$ and $r(x)$ is the geodesic distance, $0<\epsilon<1$ is some fixed constant.

Proof. By the standard argument in dealing with Lipschitz Riemannian manifold in [DF1] and [AKS], using a conformal change, we can still use polar geodesic coordinates (r,ω). The change only results in the change of C in the norm estimates of coefficient functions in (2.4). For simplicity, we still keep the notations in (2.3). We introduce the polar geodesic coordinates (r,ω) near p. Following the Einstein notation, we denote the Laplace-Beltrami operator as

$$r^2\triangle v=r^2\partial_r^2v+r^2\big(\partial_r\ln(\sqrt{\gamma})+\frac{n-1}{r}\big)\partial_rv+\frac{1}{\sqrt{\gamma}}\partial_i\big(\sqrt{\gamma}\gamma^{ij}\partial_jv\big),$$

where $\partial_i=\frac{\partial}{\partial\omega_i}$ and $\gamma_{ij}(r,\omega)$ is a metric on S^{n-1}, $\gamma=\det(\gamma_{ij})$. One can check that, for r small enough,

$$\begin{cases}\partial_r(\gamma_{ij})\leq C(\gamma_{ij})\ \text{ in term of tensors},\\ |\partial_r(\gamma)|\leq C,\\ C^{-1}\leq\gamma\leq C.\end{cases} \tag{2.10}$$

Set a new coordinate as $\ln r=t$. Using this new coordinate,

$$e^{2t}\triangle v=\partial_t^2v+(n-2+\partial_t\ln\sqrt{\gamma})\partial_tv+\frac{1}{\sqrt{\gamma}}\partial_i\big(\sqrt{\gamma}\gamma^{ij}\partial_jv\big) \tag{2.11}$$

and

$$e^{2t}\bar{b} = e^{2t}\bar{b}_t\partial_t + e^{2t}\bar{b}_i\partial_i.$$

Since u is supported in a small neighborhood, then u is supported in $(-\infty,\ T_0) \times S^{n-1}$ with $T_0 < 0$ and $|T_0|$ large enough. Under this new coordinate, the condition (2.10) becomes

$$\begin{cases} \partial_t(\gamma_{ij}) \leq Ce^t(\gamma_{ij}) \ \text{ in term of tensors,} \\ |\partial_t(\gamma)| \leq Ce^t, \\ C^{-1} \leq \gamma \leq C. \end{cases} \tag{2.12}$$

Let

$$u = e^{-\beta\psi(x)}v.$$

Define the conjugate operator,

$$\begin{aligned} \mathcal{L}_\beta(v) &= r^2 e^{\beta\psi(x)}\triangle(e^{-\beta\psi(x)}v) + r^2 e^{\beta\psi(x)}\bar{b}\cdot\nabla(e^{-\beta\psi(x)}v) + r^2\bar{q}v \\ &= e^{2t}e^{-\beta\phi(t)}\triangle(e^{\beta\phi(t)}v) + e^{2t}e^{-\beta\phi(t)}\bar{b}\cdot\nabla(e^{\beta\phi(t)}v) + e^{2t}\bar{q}v. \end{aligned} \tag{2.13}$$

From (2.11), straightforward calculations show that

$$\begin{aligned} \mathcal{L}_\beta(v) &= \partial_t^2 v + \big(2\beta\phi' + e^{2t}\bar{b}_t + (n-2) + \partial_t \ln\sqrt{\gamma}\big)\partial_t v + e^{2t}\bar{b}_i\partial_i v \\ &+ \big(\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + \beta\phi'' + (n-2)\beta\phi' + \beta\partial_t\ln\sqrt{\gamma}\phi'\big)v + \triangle_\omega v + e^{2t}\bar{q}v, \end{aligned} \tag{2.14}$$

where

$$\triangle_\omega v = \frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}\gamma^{ij}\partial_j v).$$

We will work in the following L^2 norm

$$\|v\|_\phi^2 = \int_{(-\infty,\ T_0]\times S^{n-1}} |v|^2\sqrt{\gamma}\phi'^{-3}\, dtd\omega,$$

where $d\omega$ is measure on S^{n-1}. By the triangle inequality,

$$\|\mathcal{L}_\beta(v)\|_\phi^2 \geq \frac{1}{2}\mathcal{A} - \mathcal{B},$$

where

$$\begin{aligned} \mathcal{A} = \|\partial_t^2 v &+ \triangle_\omega v + \big(2\beta\phi' + e^{2t}\bar{b}_t\big)\partial_t v + e^{2t}\bar{b}_i\partial_i v \\ &+ \big(\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q}\big)v\|_\phi^2 \end{aligned} \tag{2.15}$$

and

$$\mathcal{B} = \|\beta^2\phi'' + \beta\partial_t\ln\sqrt{\gamma}\phi' v + (n-2)\partial_t v + \partial_t\ln\sqrt{\gamma}\partial_t v\|_\phi^2. \tag{2.16}$$

By integration by parts argument, we can absorb $\mathcal{B}$ into $\mathcal{A}$. It holds that

$$\|\mathcal{L}_\beta(v)\|_\phi^2 \geq \frac{1}{4}\mathcal{A}. \tag{2.17}$$

We can also obtain a lower bound for $\mathcal{A}$,

$$\begin{aligned} C\mathcal{A} \ &\geq \ \beta^3\int |\phi''||v|^2\phi'^{-3}\sqrt{\gamma}\, dtd\omega + \beta\int |\phi''||D_\omega v|^2\phi'^{-3}\sqrt{\gamma}\, dtd\omega \\ &+\beta\int |\partial_t v|^2\phi'^{-3}\sqrt{\gamma}\, dtd\omega. \end{aligned} \tag{2.18}$$

For the completeness of the presentation, we include the proof of (2.17) and (2.18) in the Appendix.

We also want to find another refined lower bound for $\mathcal{A}$. We write $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4,$$

where

$$\mathcal{A}_1=\|\partial_t^2 v+\big(\beta^2\phi'^2+\beta\phi'\bar b_t e^{2t}+(n-2)\beta\phi'+e^{2t}\bar q\big)v+\triangle_\omega v\|^2_\phi$$

and

$$\mathcal{A}_2=\|\big(2\beta\phi'+e^{2t}\bar b_t\big)\partial_t v+e^{2t}\bar b_i\partial_i v+\beta g v\|^2_\phi$$

and

$$\begin{aligned}\mathcal{A}_3=2<\partial_t^2 v+\big(\beta^2\phi'^2+\beta\phi'\bar b_t e^{2t}+(n-2)\beta\phi'+e^{2t}\bar q\big)v+\triangle_\omega v-\beta g v,\\ \big(2\beta\phi'+e^{2t}\bar b_t\big)\partial_t v+e^{2t}\bar b_i\partial_i v>_\phi\end{aligned}$$

and

$$\mathcal{A}_4=-\beta^2\|gv\|^2_\phi,$$

and $g(t)$ is a function to be determined. We continue to break $\mathcal{A}_3$ down as

$$\mathcal{A}_3=\mathcal{I}_1+\mathcal{I}_2, \tag{2.19}$$

where

$$\begin{aligned}\mathcal{I}_1=2<\partial_t^2 v+\big(\beta^2\phi'^2+\beta\phi'\bar b_t e^{2t}+(n-2)\beta\phi'+e^{2t}\bar q\big)v+\triangle_\omega v,\\ \big(2\beta\phi'+e^{2t}\bar b_t\big)\partial_t v+e^{2t}\bar b_i\partial_i v>_\phi\end{aligned}$$

and

$$\mathcal{I}_2=-2<\beta g v,\ \big(2\beta\phi'+e^{2t}\bar b_t\big)\partial_t v+e^{2t}\bar b_i\partial_i v>_\phi .$$

Integration by part arguments as [BC] shows that

$$\begin{aligned}\mathcal{I}_1\quad\geq\quad & 3\beta\int|\phi''||D_\omega v|^2\phi'^{-3}\sqrt{\gamma}\,dtd\omega-c\beta^3\int e^t|v|^2\phi'^{-3}\sqrt{\gamma}\,dtd\omega\\ & -c\beta\int|\phi''||\partial_t v|^2\phi'^{-3}\sqrt{\gamma}\,dtd\omega-c\beta^2\int|\phi''||v|^2\phi'^{-3}\sqrt{\gamma}\,dtd\omega,\end{aligned} \tag{2.20}$$

where $|D_\omega v|^2$ stands for

$$|D_\omega v|^2=\gamma^{ij}\partial_i v\partial_j v.$$

From (2.18) and (2.20), it follows that

$$\mathcal{I}_1+C'\mathcal{A}\geq 0$$

for some positive constant C'. That is,

$$\mathcal{I}_1\geq -C'\mathcal{A}. \tag{2.21}$$

We compute $\mathcal{I}_2$. Integrating by parts with respect to t gives

$$\begin{aligned}\mathcal{I}_2=&\int\beta g'(2\beta\phi'+e^{2t}\bar b_t)v^2\phi'^{-3}\sqrt{\gamma}\,dtd\omega\\ &+\int\beta g(2\beta\phi''+2e^{2t}\bar b_t+e^{2t}\partial_t\bar b_t)v^2\phi'^{-3}\sqrt{\gamma}\,dtd\omega\\ &-\int 3\beta g(2\beta\phi'+e^{2t}\bar b_t)\phi'^{-1}\phi''v^2\phi'^{-3}\sqrt{\gamma}\,dtd\omega\\ &+\int\beta g(2\beta\phi'+e^{2t}\bar b_t)\partial_t\ln\sqrt{\gamma}v^2\phi'^{-3}\sqrt{\gamma}\,dtd\omega\\ &+\int\beta g e^{2t}(\partial_i\bar b_i+\bar b_i\partial_i\ln\sqrt{\gamma})v^2\phi'^{-3}\sqrt{\gamma}\,dtd\omega.\end{aligned}$$

Combining terms in the later identity yields that

$$
\begin{aligned}
\mathcal{I}_2 = \beta^2 \int \Big\{ & g'(2\phi' + \frac{e^{2t}\bar{b}_t}{\beta}) + g\Big(2\phi'' + \frac{2e^{2t}\bar{b}_t + e^{2t}\partial_t \bar{b}_t}{\beta} - 6\phi'' - 3\frac{e^{2t}\bar{b}_t\phi'^{-1}\phi''}{\beta} \\
& + (2\phi' + \frac{e^{2t}\bar{b}_t}{\beta})\partial_t \ln\sqrt{\gamma} + \frac{e^{2t}\partial_i\bar{b}_i + e^{2t}\bar{b}_i\partial_i \ln\sqrt{\gamma}}{\beta}\Big)\Big\} v^2\phi'^{-3}\sqrt{\gamma}\, dtd\omega.
\end{aligned} \tag{2.22}
$$

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are nonnegative, from (2.19) and (2.21), we have

$$
\begin{aligned}
\mathcal{A} &\geq \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{A}_4 \\
&\geq -C'\mathcal{A} + \mathcal{I}_2 + \mathcal{A}_4.
\end{aligned}
$$

By (2.22), we have a lower bound of $\mathcal{A}$ as

$$
\begin{aligned}
C\mathcal{A} \ \geq \ \beta^2 \int \Big\{ \Big[& g'(2\phi' + \frac{e^{2t}\bar{b}_t}{\beta}) + g\Big(\frac{2e^{2t}\bar{b}_t + e^{2t}\partial_t \bar{b}_t}{\beta} - 4\phi'' - 3\frac{e^{2t}\bar{b}_t\phi'^{-1}\phi''}{\beta} \\
& + (2\phi' + \frac{e^{2t}\bar{b}_t}{\beta})\partial_t \ln\sqrt{\gamma} + \frac{e^{2t}\partial_i\bar{b}_i + e^{2t}\bar{b}_i\partial_i \ln\sqrt{\gamma}}{\beta}\Big)\Big] - g^2 \Big\} v^2\phi'^{-3}\sqrt{\gamma}\, dtd\omega.
\end{aligned} \tag{2.23}
$$

From the assumption (2.7), we know ϕ' is close to 1 as $|T_0|$ is sufficiently large. By the assumption of $\bar{b}$ and the condition $\beta > C(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|^{1/2}_{W^{1,\infty}})$, it is clear that $|\frac{e^{2t}\bar{b}_t}{\beta}|$ is small. Thus, the condition

$$
2\phi' + \frac{e^{2t}\bar{b}_t}{\beta} > 0
$$

holds. Let

$$
\begin{aligned}
& g'(2\phi' + \frac{e^{2t}\bar{b}_t}{\beta}) + g\Big(\frac{2e^{2t}\bar{b}_t + e^{2t}\partial_t \bar{b}_t}{\beta} - 4\phi'' - 3\frac{e^{2t}\bar{b}_t\phi'^{-1}\phi''}{\beta} + (2\phi' + \frac{e^{2t}\bar{b}_t}{\beta})\partial_t \ln\sqrt{\gamma} \\
& \qquad + \frac{e^{2t}\partial_i\bar{b}_i + e^{2t}\bar{b}_i\partial_t \ln\sqrt{\gamma}}{\beta}\Big) - g^2 = \beta^2(2\phi' + \frac{e^{2t}\bar{b}_t}{\beta})\varphi(\beta(t - t_*)),
\end{aligned} \tag{2.24}
$$

where $\varphi(t) = 0$ for $t \geq 0$, $\varphi(t) > 0$ for $t < 0$, and $|t_*|$ is an arbitrary large number with $t_* < 0$. We attempt to solve (2.24) with $g = 0$ for $t \geq t_*$. Making the change of rescale,

$$
g = \beta G, \qquad z = \beta(t - t_*).
$$

Then (2.24) is transformed into an equation of the form

$$
\begin{cases}
\frac{\partial G}{\partial z} = H_1(z) + H_2(z)G + H_3(z)G^2, \\
G(0) = 0,
\end{cases}
$$

with H_1, H_2 and H_3 are uniformly in C^2. Standard existence theorem from ordinary differential equations shows a solution to (2.24) for $-C_1 \leq \beta(t - t_*) \leq 0$ with a fixed small positive constant C_1. Then (2.24) can be solved for $\frac{-C_1}{\beta} + t_* \leq t \leq t_*$. If we assume that $\operatorname{supp} v \subset \{\frac{-C_1}{\beta} + t_* \leq t \leq T_0\}$ with $T_0 < 0$, then (2.23) implies that

$$
C\mathcal{A} \geq \beta^4 \int (2\phi' + \frac{e^{2t}\bar{b}_t}{\beta})\varphi(\beta(t - t_*)) v^2\phi'^{-3}\sqrt{\gamma}\, dtd\omega. \tag{2.25}
$$

There exist $0 < -T_0 < C_2 < C_3 < C_1$ such that

$$
\varphi(z) > C_4 \qquad \text{for } -C_3 < z < -C_2
$$

and C_4 depends on C_2, C_3. It follows from the last inequality that

$$
C\mathcal{A} \geq C_4\beta^4 \int_{t_* - \frac{C_3}{\beta} < t < t_* - \frac{C_2}{\beta}} v^2\phi'^{-3}\sqrt{\gamma}\, dtd\omega. \tag{2.26}
$$

Since $r=e^t$ and recall that $u=e^{-\beta\psi(x)}v$, the previous estimates yield that

$$\mathcal{A} \geq C_5\beta^4 \int_{t_*-\frac{C_3}{\beta}<\ln r<t_*-\frac{C_2}{\beta}} e^{2\beta\psi(x)}u^2 r^{-1}\phi'^{-3}\sqrt{\gamma}\, dr d\omega. \tag{2.27}$$

Set $e^{t_*}=r_*$. If $r_*\exp\{-\frac{C_3}{\beta}\}<r<r_*\exp\{-\frac{C_2}{\beta}\}$, there exist positive constants C_6 and C_7 such that $r_*(1-\frac{C_6}{\beta})<r<r_*(1-\frac{C_7}{\beta})$. Recall the estimates (2.17), it follows that

$$\|\mathcal{L}_\beta(v)\|^2_\phi \geq C_5\beta^4 \int_{r_*(1-\frac{C_6}{\beta})<r<r_*(1-\frac{C_7}{\beta})} e^{2\beta\psi(x)}u^2 r^{-1}\phi'^{-3}\sqrt{\gamma} dr d\omega. \tag{2.28}$$

Note that ϕ' is close to 1, we have

$$\|\mathcal{L}_\beta(v)\|^2 \geq C_5\beta^4 \int_{r_*(1-\frac{C_6}{\beta})<r<r_*(1-\frac{C_7}{\beta})} e^{2\beta\psi(x)}u^2\, dvol \tag{2.29}$$

by a constant change of the value of β. Since $u\in C_0^\infty\big(\mathbb{B}(p,h)\backslash\mathbb{B}(p,\delta)\big)$, choosing $r_*=\frac{\delta}{1-\frac{C_6}{\beta}}$, we have

$$\|\mathcal{L}_\beta(u)\|^2 \geq C_5\beta^4 \int_{\delta<r<\delta(1+\frac{C_8}{\beta})} e^{2\beta\psi(x)}u^2\, dvol. \tag{2.30}$$

From Lemma 1, we have established that

$$\|\mathcal{L}_\beta(u)\| \geq C_9\beta^{\frac{3}{2}}\|r^{\frac{\epsilon}{2}}e^{\beta\psi}u\|. \tag{2.31}$$

Combining those two Carleman inequalities (2.30) and (2.31) yields that

$$\begin{aligned}\int_{B(p,h)} r^4 e^{2\beta\psi}|\triangle u+\bar{b}\cdot\nabla u+\bar{q}u|^2\, dvol \quad &\geq \quad C\beta^3\int_{\mathbb{B}(p,h)} r^\epsilon e^{2\beta\psi}u^2 dvol \\ &+ \quad C\beta^4\int_{\mathbb{B}(p,\delta(1+\frac{C_8}{\beta}))} e^{2\beta\psi}u^2 dvol\end{aligned} \tag{2.32}$$

for $u\in C_0^\infty\big(\mathbb{B}(p,h)\backslash\mathbb{B}(p,\delta)\big)$ and $\beta>C(1+\|\bar{b}\|_{W^{1,\infty}}+\|\bar{q}\|_{W^{1,\infty}})$. □

With the aid of the Carleman estimates (2.32), we are in the position to give the proof of Theorem 1. The refined doubling inequality and Bernstein's inequalities have been obtained for classical eigenfunctions in [DF2].

Proof of Theorem 1. We introduce a cut-off function $\theta(x)\in C_0^\infty\big(\mathbb{B}(p,h)\backslash B(p,\delta)\big)$ satisfying the following properties:
(i): $\theta=1$ in $\mathbb{B}(p,\frac{h}{2})\backslash\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})$,
(ii): $|\nabla\theta|\leq\frac{C\beta}{\delta}$, $|\triangle\theta|\leq\frac{C\beta^2}{\delta^2}$ in $\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})$,
(iii): $|\nabla\theta|\leq C$ in $\mathbb{B}(p,h)\backslash\mathbb{B}(p,\frac{h}{2})$.
Let $w(x)=\theta(x)u(x)$. Since u satisfies

$$\triangle u+\bar{b}\cdot\nabla u+\bar{q}u=0,$$

then w satisfies

$$\triangle w+\bar{b}\cdot\nabla w+\bar{q}w=\triangle\theta u+2\nabla\theta\cdot\nabla u+\bar{b}\cdot\nabla\theta u.$$

Substituting w into the left hand side of the stronger inequality (2.32) and calculating its integrals gives that

$$\begin{aligned}&\int_{\big(\mathbb{B}(p,h)\backslash\mathbb{B}(p,\frac{h}{2})\big)\bigcup\big(\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})\backslash\mathbb{B}(p,\delta)\big)} r^4e^{2\beta\psi}|\triangle\theta u+2\nabla\theta\cdot\nabla u+\bar{b}\cdot\nabla\theta u|^2 \\ &\leq C\beta^2\int_{\mathbb{B}(p,h)\backslash\mathbb{B}(p,\frac{h}{2})} r^4e^{2\beta\psi}(u^2+|\nabla u|^2)+C\int_{\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})\backslash\mathbb{B}(p,\delta)} r^4e^{2\beta\psi}(\frac{\beta^4}{\delta^4}u^2+\frac{\beta^2}{\delta^2}|\nabla u|^2+\frac{\beta^4}{\delta^2}u^2),\end{aligned}$$

where we have used the assumption for $\bar{b}$ and $\bar{q}$ in (2.4) and the assumption $\beta > C(1+\|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|^{1/2}_{W^{1,\infty}})$.

Substituting w into the right hand side of (2.32) and taking the later inequality into consideration yields that

$$
\begin{aligned}
&C\int_{\mathbb{B}(p,h)\setminus\mathbb{B}(p,\frac{h}{2})} r^4 e^{2\beta\psi}(u^2+|\nabla u|^2) + C\int_{\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})\setminus\mathbb{B}(p,\delta)} r^4 e^{2\beta\psi}(\frac{\beta^2}{\delta^4}u^2+\frac{1}{\delta^2}|\nabla u|^2) \\
&\geq \beta\int_{\mathbb{B}(p,\frac{h}{2})\setminus\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} r^\epsilon e^{2\beta\psi}u^2 + \beta^2\int_{\mathbb{B}(p,\delta+\frac{C\delta}{\beta})\setminus\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} e^{2\beta\psi}u^2. \qquad (2.33)
\end{aligned}
$$

Using the fact that ψ is a decreasing function and the standard elliptic estimates, we have

$$
\begin{aligned}
\int_{\mathbb{B}(p,h)\setminus\mathbb{B}(p,\frac{h}{2})} r^4 e^{2\beta\psi}|\nabla u|^2 &\leq Ch^4 e^{2\beta\psi(\frac{h}{2})}\int_{\mathbb{B}(p,h)\setminus\mathbb{B}(p,\frac{h}{2})}|\nabla u|^2 \\
&\leq C\lambda^2 h^2 e^{2\beta\psi(\frac{h}{2})}\int_{\mathcal{M}} u^2. \qquad (2.34)
\end{aligned}
$$

From the doubling inequality in [Zh1], for a suitable small $\mathbb{B}'(\tilde{p},\delta)\subset\mathbb{B}(p,\frac{h}{10})\setminus\mathbb{B}(p,2\delta)$, we have

$$
\int_{\mathbb{B}'} u^2 \geq C(\delta)e^{-C\lambda}\int_{\mathcal{M}} u^2. \qquad (2.35)
$$

It follows from (2.34) and (2.35) that

$$
\begin{aligned}
\beta\int_{\mathbb{B}(p,\frac{h}{2})\setminus\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} r^\epsilon e^{2\beta\psi}u^2 &\geq \beta\min_{\mathbb{B}'}(r^\epsilon e^{2\beta\psi})\int_{\mathbb{B}'}u^2 \\
&\geq \beta C(\delta)e^{-C\lambda}\min_{\mathbb{B}'}(r^\epsilon e^{2\beta\psi})\int_{\mathcal{M}}u^2 \\
&\geq C(\delta)\min_{\mathbb{B}'}(e^{2\beta\psi})e^{-C\lambda}e^{-2\beta\psi(\frac{h}{2})}\int_{\mathbb{B}(p,h)\setminus\mathbb{B}(p,\frac{h}{2})} r^4 e^{2\beta\psi}|\nabla u|^2. \qquad (2.36)
\end{aligned}
$$

If δ is appropriately small, it follows that

$$
\begin{aligned}
\beta\int_{\mathbb{B}(p,\frac{h}{2})\setminus\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} r^\epsilon e^{2\beta\psi}u^2 &\geq Ce^{2\beta\psi(\frac{h}{10})}e^{-C\lambda}e^{-2\beta\psi(\frac{h}{2})}\int_{\mathbb{B}(p,h)\setminus\mathbb{B}(p,\frac{h}{2})} r^4 e^{2\beta\psi}|\nabla u|^2 \\
&\geq C(h)\int_{\mathbb{B}(p,h)\setminus\mathbb{B}(p,\frac{h}{2})} r^4 e^{2\beta\psi}|\nabla u|^2. \qquad (2.37)
\end{aligned}
$$

It is also true that

$$
\begin{aligned}
\beta\int_{\mathbb{B}(p,\frac{h}{2})\setminus\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} r^\epsilon e^{2\beta\psi}u^2 &\geq \beta\min_{\mathbb{B}'}(r^\epsilon e^{2\beta\psi})\int_{\mathbb{B}'}u^2 \\
&\geq \beta C(\delta)e^{-C\lambda}\min_{\mathbb{B}'}(r^\epsilon e^{2\beta\psi})\int_{\mathcal{M}}u^2 \\
&\geq C(\delta)\min_{\mathbb{B}'}(e^{2\beta\psi})e^{-C\lambda}e^{-2\beta\psi(\frac{h}{2})}\int_{\mathbb{B}(p,h)\setminus\mathbb{B}(p,\frac{h}{2})} r^4 e^{2\beta\psi}u^2 \\
&\geq Ce^{2\beta\psi(\frac{h}{10})}e^{-C\lambda}e^{-2\beta\psi(\frac{h}{2})}\int_{\mathbb{B}(p,h)\setminus\mathbb{B}(p,\frac{h}{2})} r^4 e^{2\beta\psi}u^2 \\
&\geq C(h)\int_{\mathbb{B}(p,h)\setminus\mathbb{B}(p,\frac{h}{2})} r^4 e^{2\beta\psi}u^2. \qquad (2.38)
\end{aligned}
$$

Thus, together with (2.37), we have

$$\beta\int_{\mathbb{B}(p,\frac{h}{2})\backslash\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} r^{\epsilon}e^{2\beta\psi}u^2 \geq C(h)\int_{\mathbb{B}(p,h)\backslash\mathbb{B}(p,\frac{h}{2})} r^4 e^{2\beta\psi}(|\nabla u|^2+u^2). \tag{2.39}$$

The combination of (2.33) and (2.39) yields that

$$\int_{\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})\backslash\mathbb{B}(p,\delta)} r^4 e^{2\beta\psi}(\frac{\beta^2}{\delta^4}u^2+\frac{1}{\delta^2}|\nabla u|^2) \geq \beta^2\int_{\mathbb{B}(p,\delta+\frac{C\delta}{\beta})\backslash\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} e^{2\beta\psi}u^2. \tag{2.40}$$

We continue to simplify the last inequality,

$$\begin{aligned}(\delta+\frac{C\delta}{10\beta})^4 e^{2\beta\psi(\delta)}\frac{\beta^2}{\delta^4}\int_{\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})\backslash\mathbb{B}(p,\delta)} u^2 \quad &+ \quad (\delta+\frac{C\delta}{10\beta})^4 e^{2\beta\psi(\delta)}\frac{1}{\delta^2}\int_{\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})\backslash\mathbb{B}(p,\delta)} |\nabla u|^2 \\ &\geq \quad \beta^2 e^{2\beta\psi(\delta+\frac{C\delta}{\beta})}\int_{\mathbb{B}(p,\delta+\frac{C\delta}{\beta})\backslash\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} u^2. \end{aligned}\tag{2.41}$$

From the explicit form of $\psi(x)$, there exists some small positive constant c such that

$$\exp\{2\beta\psi(\delta+\frac{C\delta}{\beta})-2\beta\psi(\delta)\}>c$$

for β large enough. Thus,

$$\frac{\beta^2}{\delta^2}\int_{\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})\backslash\mathbb{B}(p,\delta)} u^2+\int_{\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})\backslash\mathbb{B}(p,\delta)} |\nabla u|^2 \geq c\frac{\beta^2}{\delta^2}\int_{\mathbb{B}(p,\delta+\frac{C\delta}{\beta})\backslash\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} u^2. \tag{2.42}$$

Let

$$\frac{C\delta}{10\beta}\leq\lambda^{-1}.$$

Since u satisfies (2.3), standard elliptic theory yields that

$$|\nabla u(x)|^2\leq C(\frac{\beta}{\delta})^{n+2}\int_{y\in\mathbb{B}(x,\frac{C\delta}{10\beta})} u^2(y)\,dy. \tag{2.43}$$

We integrate last inequality for $x\in\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})\backslash\mathbb{B}(p,\delta)$. It follows that

$$\begin{aligned}\int_{\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})\backslash\mathbb{B}(p,\delta)} |\nabla u|^2 \quad &\leq \quad C(\frac{\beta}{\delta})^{n+2}\int_{\{x\in\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})\backslash\mathbb{B}(p,\delta),\ y\in\mathbb{B}(x,\frac{C\delta}{10\beta})\}} u^2(y)\,dydx \\ &\leq \quad C\frac{\beta^2}{\delta^2}\int_{y\in\mathbb{B}(p,\delta+\frac{C\delta}{5\beta})\backslash\mathbb{B}(p,\delta-\frac{C\delta}{10\beta})} u^2(y)\,dy,\end{aligned}\tag{2.44}$$

where we have changed the order of integration in the last inequlity. Substituting last inequality into (2.42) gives that

$$\int_{\mathbb{B}(p,\delta+\frac{C\delta}{5\beta})\backslash\mathbb{B}(p,\delta-\frac{C\delta}{10\beta})} u^2\geq\int_{\mathbb{B}(p,\delta+\frac{C\delta}{\beta})\backslash\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} u^2. \tag{2.45}$$

Recall that $u(x)=e_\lambda(x)\exp\{\lambda\varrho(x)\}$. Let

$$\varrho(x_0)=\max_{\mathbb{B}(p,\delta+\frac{C\delta}{5\beta})\backslash\mathbb{B}(p,\delta-\frac{C\delta}{10\beta})}\varrho(x), \quad \varrho(x_1)=\min_{\mathbb{B}(p,\delta+\frac{C\delta}{\beta})\backslash\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})}\varrho(x).$$

Then

$$\lambda|\varrho(x_0)-\varrho(x_1)|\leq C\max_{\overline{\mathcal{M}}}|\nabla\varrho(x)|\delta,$$

since $\beta\geq C\lambda$. Furthermore, thanks to the fact that $\nabla\varrho(x)$ is a bounded function in $\overline{\mathcal{M}}$, we have

$$C\int_{\mathbb{B}(p,\delta+\frac{C\delta}{5\beta})\backslash\mathbb{B}(p,\delta-\frac{C\delta}{10\beta})} e_\lambda^2\geq\int_{\mathbb{B}(p,\delta+\frac{C\delta}{\beta})\backslash\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} e_\lambda^2. \tag{2.46}$$

Adding $\int_{\mathbb{B}(p,\delta+\frac{C\delta}{10\beta})} e_\lambda^2$ to both sides of (2.46) yields that

$$C\int_{\mathbb{B}(p,\delta+\frac{C\delta}{5\beta})} e_\lambda^2 \geq \int_{\mathbb{B}(p,\delta+\frac{C\delta}{\beta})} e_\lambda^2. \tag{2.47}$$

If we replace $\delta = \frac{\delta'}{1+\frac{C}{5\beta}}$, we get

$$C\int_{\mathbb{B}(p,\delta')} e_\lambda^2 \geq \int_{\mathbb{B}(p,\delta'+\frac{C\delta'}{\beta})} e_\lambda^2. \tag{2.48}$$

Since we can choose $\beta = C\lambda$, by finite iteration, we arrive at

$$\int_{\mathbb{B}(p,\delta)} e_\lambda^2 \geq C\int_{\mathbb{B}(p,\delta(1+\frac{1}{\lambda}))} e_\lambda^2. \tag{2.49}$$

This completes conclusion (A) in Theorem 1. Next we show the L^2-Bernstein's inequality. By the standard elliptic estimates,

$$|\nabla e_\lambda(x)|^2 \leq \frac{C}{r^{2+n}}\int_{\mathbb{B}(x,r)} e_\lambda^2(y)\,dy \tag{2.50}$$

if $\lambda r\leq 1$ and $\mathbb{B}(x,r)\subset\mathcal{M}$. Choosing $r=\frac{\delta}{\lambda}$ and integrating over $x\in\mathbb{B}(p,\delta)$,

$$\begin{aligned}\int_{\mathbb{B}(p,\delta)} |\nabla e_\lambda(x)|^2\,dx &\leq \frac{C}{r^{2+n}}\int_{\{y\in\mathbb{B}(x,r),\ x\in\mathbb{B}(p,\delta)\}} e_\lambda^2(y)dydx\\ &\leq \frac{C}{r^2}\int_{\mathbb{B}(p,\delta+r)} e_\lambda^2(x)\,dx,\end{aligned} \tag{2.51}$$

where we have changed the order of integration in last inequality. Application of (2.49) yields that

$$\int_{\mathbb{B}(p,\delta)} |\nabla e_\lambda(x)|^2\,dx \leq \frac{C\lambda^2}{\delta^2}\int_{\mathbb{B}(p,\delta)} e_\lambda^2(x)\,dx. \tag{2.52}$$

Thus, we arrive at the conclusion (B).

We continue to obtain L^∞ version of Bernstein's inequality. For $x\in\mathbb{B}(p,\delta)$, choosing $r=\frac{\delta}{\lambda}$, the refined doubling inequality (2.49) and (2.50) yield that

$$\begin{aligned}|\nabla e_\lambda(x)|^2 \leq \frac{C}{r^{2+n}}\int_{\mathbb{B}(x,r)} e_\lambda^2 &\leq \frac{C}{r^{2+n}}\int_{\mathbb{B}(p,\delta+r)} e_\lambda^2\\ &\leq \frac{C}{r^{2+n}}\int_{\mathbb{B}(p,\delta)} e_\lambda^2\\ &\leq \frac{C}{r^{2+n}}\delta^n \max_{\mathbb{B}(p,\delta)} e_\lambda^2.\end{aligned} \tag{2.53}$$

Therefore,

$$|\nabla e_\lambda(x)| \leq \frac{C\lambda^{\frac{n+2}{2}}}{\delta}\max_{\mathbb{B}(p,\delta)} |e_\lambda| \tag{2.54}$$

for any $x\in\mathbb{B}(p,\delta)$. The conclusion (C) in Theorem 1 is arrived.

□

3. Upper bound of Steklov eigenfunctions

In this section, we will prove the optimal upper bound for interior Steklov eigenfunctions. Assume that $\mathcal{M}$ is a real analytic Riemannian manifold with boundary. We first show the measure of nodal sets in the neighborhood close to boundary, then show the upper bound of nodal sets away from the boundary $\partial\mathcal{M}$. Since $\mathcal{M}$ is a real analytic Riemannian manifold with boundary, we may embed $\mathcal{M} \subset \mathcal{M}_1$ as a relatively compact subset, where $\mathcal{M}_1$ is an open real analytic Riemannian manifold. The real analytic Riemannian manifold $\mathcal{M}$ and $\mathcal{M}_1$ are of the same dimension. We denote $d(x) = dist\{x,\ \partial\mathcal{M}\}$ as the geodesic distance function from $x \in \mathcal{M}$ to the boundary $\partial\mathcal{M}$. Then $d(x)$ is a real analytic function in a small neighborhood of $\partial\mathcal{M}$ in $\mathcal{M}$. Let the small neighborhood be $\mathcal{M}_\rho = \{x \in \mathcal{M} | d(x) \leq \rho\}$. As the arguments in [BL], we construct

$$\hat{u}(x) = e_\lambda(x)\exp\{\lambda d(x)\}.$$

Simple calculations show that the new function $\hat{u}(x)$ satisfies

$$\begin{cases} \triangle_g \hat{u} + b(x)\cdot\nabla_g\hat{u} + q(x)\hat{u} = 0 & \text{in } \mathcal{M}_{\frac{\varrho}{2}}, \\ \frac{\partial\hat{u}}{\partial\nu} = 0 & \text{on } \partial\mathcal{M}, \end{cases} \tag{3.1}$$

where

$$\begin{cases} b(x) = -2\lambda\nabla_g d(x), \\ q(x) = \lambda^2|\nabla_g d(x)|^2 - \lambda\triangle_g d(x). \end{cases} \tag{3.2}$$

Note that $b(x)$, $q(x)$ are analytic functions in $\mathcal{M}_{\frac{\varrho}{2}}$. Our strategy is inspired by the study of nodal sets of classical eigenfunctions with Dirichlet or Neumann boundary conditions in [DF1].

Proposition 3. *The function $\hat{u}(x)$ can be analytically extended to some neighborhood $\mathcal{M}_2$ of $\mathcal{M}_{\frac{\varrho}{2}}$ across $\partial\mathcal{M}$ in $\mathcal{M}_1$, where $\mathcal{M}_{\frac{\varrho}{2}} \subset \mathcal{M}_2 \subset \mathcal{M}_1$. For each $p \in \mathcal{M}_{\frac{\varrho}{4}}$, there exists a ball $\mathbb{B}(p,h) \subset \mathcal{M}_2$ so that for $h_1 < h$,*

$$\sup_{x\in\mathbb{B}(p,h)} |\hat{u}(x)| \leq e^{C_1\lambda} \sup_{x\in\mathbb{B}(p,h_1)\cap\mathcal{M}} |\hat{u}(x)|, \tag{3.3}$$

where the positive constant C_1 depends on h_1.

Proof. For a fixed point $p \in \partial\mathcal{M}$, we consider $\hat{u}_{p,\lambda}(x) = \hat{u}(p + \frac{x}{\lambda})$, then $\hat{u}_{p,\lambda}$ satisfies

$$\triangle_g \hat{u}_{p,\lambda} + \tilde{b}(x)\cdot\nabla_g\hat{u}_{p,\lambda} + \tilde{q}(x)\hat{u}_{p,\lambda} = 0 \quad \text{in } \mathcal{M}_{p,\lambda} \tag{3.4}$$

with the norm of $\tilde{b}(x)$ and $\tilde{q}(x)$ bounded independent of λ, where $\mathcal{M}_{p,\lambda} := \{x | p + \frac{x}{\lambda} \in \mathcal{M}_{\frac{\varrho}{2}}\}$. We can extend $d(x)$ analytically as signed distance functions across the boundary $\partial\mathcal{M}$. Then $\tilde{b}(x)$ and $\tilde{q}(x)$ can be extended analytically across the boundary $\partial\mathcal{M}$. Since (3.4) is an elliptic equation, by the Cauchy-Kowaleski theorem [T], $\hat{u}_{p,\lambda}$ can be analytically extended to $\mathbb{B}(0, r_0)$, where r_0 depends only on $\mathcal{M}$. By the compactness of $\partial\mathcal{M}$, $u(x)$ is analytically extended to a $\frac{C_2}{\lambda}$ neighborhood of $\partial\mathcal{M}$, say $\tilde{\mathcal{M}_1}$. We also know that

$$\|\hat{u}\|_{L^\infty(\tilde{\mathcal{M}_1})} \leq C_3\|\hat{u}\|_{L^\infty(\mathcal{M}_{\frac{\varrho}{2}})}.$$

Iterating this process λ times, $\hat{u}(x)$ is extended to an analytic function in some neighborhood of $\partial\mathcal{M}$, i.e. $\mathcal{M}_2$. Thus,

$$\|\hat{u}\|_{L^\infty(\mathcal{M}_2)} \leq e^{C_4\lambda}\|\hat{u}\|_{L^\infty(\mathcal{M}_{\frac{\varrho}{2}})}. \tag{3.5}$$

The double inequality is establish for u in smooth manifolds. Recall that $u = e_\lambda(x)\exp\{\lambda\varrho(x)\}$. Note that the distance function in the definition $\varrho(x)$ in (2.1) may be different from the $d(x)$ in

Section 3 because of the different metrics in smooth manifolds and analytic manifolds. However, $\hat{u}(x)$ and $u(x)$ are comparable in $\mathcal{M}_{\frac{\varrho}{2}}$. We have

$$\hat{u}(x) = u(x)\exp\{-\lambda\varrho(x) + \lambda d(x)\}$$

in $\mathcal{M}_{\frac{\varrho}{2}}$. Since there exists some constant C such that $-C \le \varrho(x) - d(x) \le C$ in $\mathcal{M}_{\frac{\varrho}{2}}$, then

$$e^{-C\lambda}|u(x)| \le |\hat{u}(x)| \le |u(x)|e^{C\lambda}.$$

We have obtained the doubling inequality for $u(x)$ in (2.6). By the doubling inequality (2.6) and (3.5),

$$\begin{aligned} \sup_{x\in\mathbb{B}(p,h)} |\hat{u}(x)| &\le \|\hat{u}\|_{L^\infty(\mathcal{M}_2)} \le e^{C\lambda}\|\hat{u}\|_{L^\infty(\mathcal{M}_{\frac{\varrho}{2}})} \le e^{C\lambda}\|u\|_{L^\infty(\mathcal{M}_{\frac{\varrho}{2}})} \\ &\le e^{C\lambda}\|u\|_{L^\infty(\mathbb{B}(p,\ h_1)\cap\mathcal{M})} \le e^{C_5\lambda}\|\hat{u}\|_{L^\infty(\mathbb{B}(p,\ h_1)\cap\mathcal{M})}, \end{aligned} \tag{3.6}$$

where C_5 depends on h_1. If p is the interior point in $\mathcal{M}_{\frac{\varrho}{4}}$, we can carry out the same argument as (3.6). Therefore, (3.3) is reached. □

Next we extend $u(x)$ locally as a holomorphic function in $\mathbb{C}^n$. Applying elliptic estimates in a ball $\mathbb{B}(p,\ C_6\lambda^{-1})$, we have

$$|\frac{D^\alpha \hat{u}(p)}{\alpha!}| \le C^{|\alpha|}\lambda^{|\alpha|}\|\hat{u}\|_{L^\infty}. \tag{3.7}$$

Without loss of generality, we may consider the point p as origin. Summing a geometric series, we can extend $\hat{u}(x)$ to be a holomorphic function $\hat{u}(z)$ with $z \in \mathbb{C}^n$. Moreover, we have

$$\sup_{|z|\le C_7\lambda^{-1}} |\hat{u}(z)| \le C \sup_{|x|\le C_8\lambda^{-1}} |\hat{u}(x)|. \tag{3.8}$$

with $C_7 < C_8$. Iterating λ times gives that

$$\sup_{|z|\le\rho_1} |\hat{u}(z)| \le e^{C\lambda} \sup_{|x|\le\rho_2} |\hat{u}(x)|,$$

where $\rho_1 < \rho_2$. By considering $\hat{u}(x)$ in manifold $\mathcal{M}_2$ and Proposition 3, we obtain that

$$\sup_{|z|\le\rho_1} |\hat{u}(z)| \le e^{C_9\lambda} \sup_{|x|\le\rho_3} |\hat{u}(x)| \tag{3.9}$$

with $\rho_3 < \rho_1$, where C_9 depends on ρ_3. Especially,

$$\sup_{|z|\le 2r} |\hat{u}(z)| \le e^{C\lambda} \sup_{|x|\le r} |\hat{u}(x)| \tag{3.10}$$

for $0 < r < r_0$ with r_0 depending on $\mathcal{M}$.

We need a lemma concerning the growth of a complex analytic function with the number of zeros. See e.g. [BL] and [HL].

Lemma 2. *Suppose $f :\ \boldsymbol{B}(0,1) \subset \mathbb{C} \to \mathbb{C}$ is an analytic function satisfying*

$$f(0) = 1 \quad and \quad \sup_{\boldsymbol{B}(0,1)} |f| \le 2^N$$

for some positive constant N. Then for any $r \in (0,1)$, there holds

$$\sharp\{z \in \boldsymbol{B}(0,r) : f(z) = 0\} \le cN$$

where c depends on r. Especially, for $r = \frac{1}{2}$, there holds

$$\sharp\{z \in \boldsymbol{B}(0,1/2) : f(z) = 0\} \le N.$$

Proof of Theorem 2. We first prove the nodal sets in a neighborhood $\mathcal{M}_{\frac{\varrho}{4}}$. By rescaling and translation, we can argue on scales of order one. Let $p \in \mathbb{B}_{1/2}$ be the point where the maximum of $|\hat{u}|$ in $\mathbb{B}_{1/2}$ is attained. For each direction $\omega \in S^{n-1}$, set $\hat{u}_\omega(z) = \hat{u}(p+z\omega)$ in $z \in \mathbf{B}(0,1) \subset \mathbb{C}$. By the doubling property (3.10) and the lemma above,

$$\begin{aligned} \sharp\{x \in \mathbb{B}(p,1/4) \quad &| \quad x-p \text{ is parallel to } \omega \text{ and } \hat{u}(x) = 0\} \\ &\leq \quad \sharp\{z \in \mathbf{B}(0,1/2) \subset \mathcal{C} | \hat{u}_\omega(z) = 0\} \\ &= \quad N(\omega) \leq C\lambda. \end{aligned} \tag{3.11}$$

With the aid of integral geometry estimates, it implies that

$$\begin{aligned} H^{n-1}\{x \in \mathbb{B}(p,1/4) | \hat{u}(x) = 0\} \quad &\leq \quad c(n) \int_{S^{n-1}} N(\omega)\, d\omega \\ &\leq \quad \int_{S^{n-1}} C\lambda\, d\omega = C\lambda. \end{aligned} \tag{3.12}$$

By covering the compact manifold $\mathcal{M}_{\frac{\varrho}{4}} \subset \mathcal{M}_2$ by finite number of coordinate charts, we arrive at

$$H^{n-1}\{x \in \mathcal{M}_{\frac{\varrho}{4}} | \hat{u}(x) = 0\} \leq C\lambda. \tag{3.13}$$

It implies that

$$H^{n-1}\{x \in \mathcal{M}_{\frac{\varrho}{4}} | e_\lambda(x) = 0\} \leq C\lambda. \tag{3.14}$$

Next we deal with the measure of nodal sets in $\mathcal{M}\backslash\mathcal{M}_{\frac{\varrho}{4}}$. We have obtained the doubling inequality in the interior of the manifold, i.e.

$$\|u\|_{L^\infty(\mathbb{B}(p,\,2r))} \leq e^{C\lambda} \|u\|_{L^\infty(\mathbb{B}(p,\,r))}.$$

Since $u(x) = e_\lambda(x) \exp\{\lambda \varrho(x)\}$ and $-C_0 < \varrho(x) \leq C_0$ for some constant C_0 depending on $\mathcal{M}$, it is true that

$$\|e_\lambda\|_{L^\infty(\mathbb{B}(p,\,2r))} \leq e^{C\lambda} \|e_\lambda\|_{L^\infty(\mathbb{B}(p,\,r))} \tag{3.15}$$

holds for $p \in \mathcal{M}\backslash\mathcal{M}_{\frac{\varrho}{4}}$ and $0 < r \leq r_0 \leq \frac{\varrho}{4}$. We will similarly extend $e_\lambda(x)$ locally as a holomorphic function in $\mathbb{C}^n$. Since $e_\lambda(x)$ is harmonic in $\mathcal{M}\backslash\mathcal{M}_{\frac{\varrho}{4}}$, applying elliptic estimates in a small ball $\mathbb{B}(p,\ C_{10})$, we have

$$|\frac{D^\alpha e_\lambda(p)}{\alpha!}| \leq C^{|\alpha|} \|e_\lambda\|_{L^\infty}. \tag{3.16}$$

We also consider the point p as the origin. Summing a geometric series, we can extend $e_\lambda(x)$ to be a holomorphic function $e_\lambda(z)$ with $z \in \mathbb{C}^n$. Moreover, we have

$$\sup_{|z| \leq C_{11}} |e_\lambda(z)| \leq C \sup_{|x| \leq C_{12}} |e_\lambda(x)| \tag{3.17}$$

with $C_{11} < C_{12}$.

Thanks to the doubling inequality (3.15), we obtain that

$$\sup_{|z| \leq \rho_1} |e_\lambda(z)| \leq e^{C_{13}\lambda} \sup_{|x| \leq \rho_3} |e_\lambda(x)| \tag{3.18}$$

with $\rho_3 < \rho_1$, where C_{13} depends on ρ_3. In particular,

$$\sup_{|z| \leq 2r} |e_\lambda(z)| \leq e^{C\lambda} \sup_{|x| \leq r} |e_\lambda(x)| \tag{3.19}$$

holds for $0 < r < r_0$ with r_0 depending on $\mathcal{M}$. Using the same argument as obtaining the nodal sets in the neighborhood of the boundary, we take advantage of lemma 2 and the inequality (3.19). By rescaling and translation, we can argue on scales of order one. Let $p \in \mathbb{B}_{1/2}$ be the point where the maximum of $|e_\lambda|$ in $\mathbb{B}_{1/2}$ is achieved. For each direction $\omega \in S^{n-1}$, set

$e_\lambda^\omega(z) = e_\lambda(p+z\omega)$ in $z \in \mathbf{B}(0,1) \subset \mathbb{C}$. From the doubling property (3.19) and the lemma 2 above,

$$\begin{aligned} \sharp\{x \in \mathbb{B}(p,1/4) \quad &| \quad x-p \text{ is parallel to } \omega \text{ and } e_\lambda(x)=0\} \\ &\leq \sharp\{z \in \mathbf{B}(0,1/2) \subset \mathcal{C} | e_\lambda^\omega(z) = 0\} \\ &= N(\omega) \leq C\lambda. \end{aligned} \tag{3.20}$$

Thanks to the integral geometry estimates, we get

$$\begin{aligned} H^{n-1}\{x \in \mathbb{B}(p,1/4) | e_\lambda(x) = 0\} \quad &\leq \quad c(n) \int_{S^{n-1}} N(\omega)\, d\omega \\ &\leq \int_{S^{n-1}} C\lambda \, d\omega = C\lambda. \end{aligned} \tag{3.21}$$

Using the finite number of coordinate charts to cover the compact manifold $\mathcal{M}\backslash\mathcal{M}_{\frac{\varrho}{4}}$, we obtain

$$H^{n-1}\{x \in \mathcal{M}\backslash\mathcal{M}_{\frac{\varrho}{4}} | e_\lambda(x) = 0\} \leq C\lambda. \tag{3.22}$$

Together with (3.14) and (3.22), we arrive at the conclusion in Theorem 2.

□

4. Appendix

In this section, we provide the proof of Lemma 1 and some arguments stated in the proof Proposition 2. Recall that

$$\|\mathcal{L}_\beta(v)\|_\phi^2 \geq \frac{1}{2}\mathcal{A} - \mathcal{B},$$

where

$$\begin{aligned} \mathcal{L}_\beta(v) \quad &= \quad \partial_t^2 v + \big(2\beta\phi' + e^{2t}\bar{b}_t + (n-2) + \partial_t \ln\sqrt{\gamma}\big)\partial_t v + e^{2t}\bar{b}_i\partial_i v \\ &+ \quad \big(\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + \beta\phi'' + (n-2)\beta\phi' + \beta\partial_t \ln\sqrt{\gamma}\phi'\big)v + \triangle_\omega v + e^{2t}\bar{q}v \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \mathcal{A} = \|\partial_t^2 v \quad &+ \quad \triangle_\omega v + \big(2\beta\phi' + e^{2t}\bar{b}_t\big)\partial_t v + e^{2t}\bar{b}_i\partial_i v \\ &+ \quad \big(\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q}\big)v\|_\phi^2 \end{aligned} \tag{4.2}$$

and

$$\mathcal{B} = \|\beta^2\phi'' + \beta\partial_t \ln\sqrt{\gamma}\phi' v + (n-2)\partial_t v + \partial_t \ln\sqrt{\gamma}\partial_t v\|_\phi^2. \tag{4.3}$$

Modifying the arguments in [BC] and [Zh], we can obtain the following lemma, which verifies the proof of (2.17) and (2.18) in Proposition 2.

Lemma 3. *There holds that*

$$\begin{aligned} \|\mathcal{L}_\beta(v)\|_\phi^2 &\geq \frac{1}{4}\mathcal{A} \\ &\geq C\beta^3 \int |\phi''||v|^2\phi'^{-3}\sqrt{\gamma}\, dt d\omega + C\beta \int |\phi''||D_\omega v|^2\phi'^{-3}\sqrt{\gamma}\, dt d\omega \\ &+ C\beta \int |\partial_t v|^2 \phi'^{-3}\sqrt{\gamma}\, dt d\omega. \end{aligned} \tag{4.4}$$

Proof. We decompose $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_1' + \mathcal{A}_2' + \mathcal{A}_3',$$

where

$$\mathcal{A}_1' = \|\partial_t^2 v + \big(\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q}\big)v + \triangle_\omega v\|_\phi^2$$

and

$$\mathcal{A}_2' = \|\big(2\beta\phi' + e^{2t}\bar{b}_t\big)\partial_t v + e^{2t}\bar{b}_i\partial_i v\|_\phi^2$$

and

$$\begin{aligned}\mathcal{A}_3' = 2 < \partial_t^2 v + \big(\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q}\big)v + \triangle_\omega v,\\ \big(2\beta\phi' + e^{2t}\bar{b}_t\big)\partial_t v + e^{2t}\bar{b}_i\partial_i v >_\phi .\end{aligned}$$

We first compute $\mathcal{A}_1'$. Let α be some small positive constant. Since $|\phi^{''}| \leq 1$ and β is large enough, it is true that

$$\mathcal{A}_1' \geq \frac{\alpha}{\beta}\mathcal{A}_1^{''}, \tag{4.5}$$

where $\mathcal{A}_1^{''}$ is given by

$$\mathcal{A}_1' = \left\|\sqrt{|\phi''|}[\partial_t^2 v + \big(\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q}\big)v + \triangle_\omega v]\right\|_\phi^2.$$

We split $\mathcal{A}_1'$ into three parts:

$$\mathcal{A}_1^{''} = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3, \tag{4.6}$$

where

$$\mathcal{K}_1 = \left\|\sqrt{|\phi''|}\big(\partial_t^2 v + \triangle_\omega v\big)\right\|_\phi^2$$

and

$$\mathcal{K}_2 = \left\|\sqrt{|\phi''|}\big(\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q}\big)v\right\|_\phi^2$$

and

$$\mathcal{K}_3 = 2\Big\langle|\phi^{''}|(\partial_t^2 v + \triangle_\omega v),\ \big(\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q}\big)v\Big\rangle_\phi.$$

The expression $\mathcal{K}_1$ is considered to be a nonnegative term. We estimate $\mathcal{K}_2$. By the triangle inequality,

$$\mathcal{K}_2 \geq \beta^4\left\|\sqrt{|\phi''|}\phi' v\right\|_\phi^2 - \left\|\sqrt{|\phi''|}\big(\beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q}\big)v\right\|_\phi^2. \tag{4.7}$$

Using the fact that $\beta > C(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}}^{1/2})$, we have

$$\begin{aligned}\left\|\sqrt{|\phi''|}\big(\beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + (n-2)\beta\phi' + e^{2t}\bar{q}\big)v\right\|_\phi^2 \leq C\beta^4\left\|\sqrt{|\phi''|}e^t v\right\|_\phi^2\\ + C\beta^2\left\|\sqrt{|\phi''|}v\right\|_\phi^2.\end{aligned} \tag{4.8}$$

Since t is close to negative infinity and then ϕ' is close to 1, from (4.7) and (4.8), we obtain

$$\mathcal{K}_2 \geq C\beta^4\left\|\sqrt{|\phi''|}v\right\|_\phi^2, \tag{4.9}$$

where we also used the fact that ϕ' is close to 1. We derive a lower bound for $\mathcal{K}_3$. Integration by parts shows that

$$\begin{aligned}
\mathcal{K}_3 = &-2\int |\phi''||\partial_t v|^2\big(\beta^2\phi'^2+\beta\phi'\bar{b}_t e^{2t}+(n-2)\beta\phi'+(n-2)\beta\phi'+e^{2t}\bar{q}\big)\phi'^{-3}\sqrt{\gamma}\ dtd\omega \\
&-2\int \partial_t v v\partial_t\big[|\phi''|\big(\beta^2\phi'^2+\beta\phi'\bar{b}_t e^{2t}+(n-2)\beta\phi'+(n-2)\beta\phi'+e^{2t}\bar{q}\big)\phi'^{-3}\sqrt{\gamma}\big]\ dtd\omega \\
&-2\int |\phi''||D_\omega v|^2\big(\beta^2\phi'^2+\beta\phi'\bar{b}_t e^{2t}+(n-2)\beta\phi'+(n-2)\beta\phi'+e^{2t}\bar{q}\big)\phi'^{-3}\sqrt{\gamma}dtd\omega \\
&-2\int \beta|\phi''|\phi'\gamma^{ij}\partial_i v\partial_j\bar{b}_t e^{2t}\phi'^{-3}\sqrt{\gamma}dtd\omega \\
&-2\int |\phi''|\gamma^{ij}\partial_i v\partial_j\bar{q}e^{2t}v\phi'^{-3}\sqrt{\gamma}dtd\omega.
\end{aligned}\tag{4.10}$$

By the Cauchy-Schwartz inequality and the condition that $\beta > C(1+\|\bar{b}\|_{W^{1,\infty}}+\|\bar{q}\|^{1/2}_{W^{1,\infty}})$, we arrive at

$$\mathcal{K}_3 \geq -C\beta^2\int |\phi''|(|\partial_t v|^2+|D_\omega v|^2+v^2)\phi'^{-3}\sqrt{\gamma}dtd\omega.\tag{4.11}$$

Since $\mathcal{K}_1$ is nonnegative, the combination of (4.6), (4.9) and (4.11) yields that

$$\begin{aligned}
\mathcal{A}_1'' \geq\ & C\beta^4\left\|\sqrt{|\phi''|}v\right\|^2_\phi - C\beta^2\left\|\sqrt{|\phi''|}\partial_t v\right\|^2_\phi \\
&- C\beta^2\left\|\sqrt{|\phi''|}|D_\omega v|\right\|^2_\phi.
\end{aligned}\tag{4.12}$$

From (4.5), it follows that

$$\begin{aligned}
\mathcal{A}_1' \geq\ & C\alpha\beta^3\left\|\sqrt{|\phi''|}v\right\|^2_\phi - C\alpha\beta\left\|\sqrt{|\phi''|}\partial_t v\right\|^2_\phi \\
&- C\alpha\beta\left\|\sqrt{|\phi''|}|D_\omega v|\right\|^2_\phi.
\end{aligned}\tag{4.13}$$

Recall that

$$\mathcal{A}_2' = \|\big(2\beta\phi'+e^{2t}\bar{b}_t\big)\partial_t v+e^{2t}\bar{b}_i\partial_i v\|^2_\phi.$$

By the triangle inequality, one has

$$\mathcal{A}_2' \geq 2\beta^2\|\phi'\partial_t v\|^2_\phi - \|e^{2t}\bar{b}_t\partial_t v+e^{2t}\bar{b}_i\partial_i v\|^2_\phi.$$

It is obvious that

$$\mathcal{A}_2' \geq \frac{1}{\beta}\mathcal{A}_2'.$$

From the assumption that $\beta > C(1+\|\bar{b}\|_{W^{1,\infty}}+\|\bar{q}\|^{1/2}_{W^{1,\infty}})$, we obtain that

$$\begin{aligned}
\mathcal{A}_2' &\geq C\beta\|\phi'\partial_t v\|^2_\phi - C\beta\|e^t\partial_t v\|^2_\phi - C\beta\|e^t|D_\omega v|\|^2_\phi \\
&\geq C\beta\|\phi'\partial_t v\|^2_\phi - C\beta\|e^t|D_\omega v|\|^2_\phi.
\end{aligned}\tag{4.14}$$

For the inner product $\mathcal{A}_3'$, using the arguments of integration by parts, we can show a lower bound of $\mathcal{A}_3'$,

$$\begin{aligned}\mathcal{A}_3' \geq C\beta \left\| \sqrt{|\phi''|}|D_\omega v| \right\|_\phi^2 - C\beta^3 \left\| e^t v \right\|_\phi^2 - C\beta \left\| \sqrt{|\phi''|}\partial_t v \right\|_\phi^2 \\ - C\beta^2 \left\| \sqrt{|\phi''|} v \right\|_\phi^2 .\end{aligned} \tag{4.15}$$

Recall that $\mathcal{A} = \mathcal{A}_1' + \mathcal{A}_2' + \mathcal{A}_3'$. From (4.13), (4.14) and (4.15), it follows that

$$\begin{aligned}\mathcal{A} \geq & \, C\alpha\beta^3 \int |\phi''| v^2 \phi'^{-3}\sqrt{\gamma} dt d\omega + C\beta \int |\partial_t v|^2 \phi'^{-3} \sqrt{\gamma} \\ & + C\beta \int |\phi''||D_\omega v|^2 \phi'^{-3}\sqrt{\gamma} dt d\omega - C\beta^2 \int |\phi''| v^2 \phi'^{-3}\sqrt{\gamma} dt d\omega \\ & - C\beta^3 \int e^{2t} v^2 \phi'^{-3}\sqrt{\gamma} dt d\omega - C\beta \int |\phi''||\partial_t v|^2 \phi'^{-3}\sqrt{\gamma} dt d\omega \\ & - C\alpha\beta \int |\phi''||D_\omega v|^2 \phi'^{-3}\sqrt{\gamma} dt d\omega - C\beta \int e^{2t} |D_\omega v|^2 \phi'^{-3}\sqrt{\gamma} dt d\omega.\end{aligned} \tag{4.16}$$

If we choose α to be appropriately small and take the fact $|\phi''| > e^t$ into account, we obtain that

$$\begin{aligned} C\mathcal{A} \ \geq \ & \beta^3 \int |\phi''||v|^2 \phi'^{-3}\sqrt{\gamma}\, dt d\omega + \beta \int |\phi''||D_\omega v|^2 \phi'^{-3} \sqrt{\gamma}\, dt d\omega \\ & + \beta \int |\partial_t v|^2 \phi'^{-3}\sqrt{\gamma}\, dt d\omega.\end{aligned} \tag{4.17}$$

Now we show $\mathcal{B}$ can be absorbed in to $\mathcal{A}$ for large $|T_0|$ and large β. Since $|\partial_t \ln \sqrt{\gamma}| \leq Ce^t \leq |\phi''|$,

$$\begin{aligned}\mathcal{B} &= \|\beta^2 \phi'' + \beta \partial_t \ln \sqrt{\gamma} \phi' v + (n-2)\partial_t v + \partial_t \ln \sqrt{\gamma} \partial_t v\|_\phi^2 \\ &\leq \beta^2 \int |\phi''| v^2 \phi'^{-3} \sqrt{\gamma} dt d\omega + C \int |\partial_t v|^2 e^{2t} \phi'^{-3}\sqrt{\gamma} dt d\omega.\end{aligned} \tag{4.18}$$

Thus, the right hand side of (4.18) can be incorporated by the right hand side of (4.17). Hence the proof of the lemma is arrived. □

If we recall that $u = e^{-\beta\varphi(x)} v$, the proof of Lemma 3 also implies Lemma 1 stated in section 2.

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