ON EXTENSIONS FOR GENTLE ALGEBRAS

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ABSTRACT. We develop an algorithmic method for determining the cohomology of complexes in the derived category of a gentle algebra. We then use this to give a complete description of a basis of the extensions between indecomposable modules in the module category of a gentle algebra thereby answering an open problem.

Introduction

The representation theory of finite-dimensional algebras plays an important role in many different areas of mathematics, such as for example, in many areas of Lie theory, in number theory in connection with the Langlands program and automorphic forms, in geometry ranging from invariant theory to non-commutative resolutions of singularities and as far afield as harmonic analysis where the representation theory of S^1 appears in the guise of Fourier analysis.

Most finite-dimensional algebras are of wild representation type, that is their representation theory is at least as complicated as that of the free associative algebra in two generators. For example, the only group algebras of finite groups that are not wild are those of finite groups with cyclic p-Sylow subgroups in characteristic p and those of finite groups with dihedral, semidihedral and (generalised) quarternion 2-Sylow subgroups in characteristic 2. All other group algebras are wild. An algebra that is not wild is either of tame or finite representation type.

One particular class of tame algebras, the so-called *gentle algebras* appear in a surprising number of different and apparently unrelated contexts. For example, in the context of Fukaya categories related to Kontsevich's homological mirror symmetry program [12], in the context of dimer models [5], in the context of the enveloping algebras of Lie algebras as zigzag algebras [14], and in cluster theory as (m-)cluster tilted and m-Calabi Yau tilted algebras and also as Jacobian algebras associated to unpunctured surfaces [2, 11, 18]. Furthermore, the class of derived-discrete algebras consists of gentle algebras [23].

But there are many other reasons why gentle algebras have been studied extensively. One of the main reasons being that they are string algebras and that their indecomposable representations are classified by string and band modules [24], see also [7]. The associated string combinatorics governs the representation theory of gentle algebras, examples of this are the classification of morphisms between string and band modules [10, 17] and a characterisation of almost split sequences in terms of string combinatorics [7]. Over last few years, interest in gentle algebras has intensified with many new results appearing, an

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example of this is the very recent work [20], where string combinatorics is used to classify support τ -tilting modules.

Another reason for the extensive investigation of gentle algebras is the fact that they are derived tame and the indecomposable objects in the derived category of a gentle algebra have been classified. They are given by the so-called homotopy strings and bands [4]. In [1] the morphisms between string and band complexes in the derived category of a gentle algebra were characterised in terms of homotopy string combinatorics and in [8] a graphical mapping cone calculus based on the morphisms described in [1] was developed.

Extensions between modules are one of the fundamental cohomological tools. Not only do they play an essential role in the definition of, for example, the Yoneda algebra or Hochschild cohomology, they are also essential in many of the newer developments in representation theory such as in cluster tilting in cluster theory and in higher representation theory.

The projective resolutions of indecomposable modules over gentle algebras are well understood, see, for example, [16]. So it is surprising that up to now, in general, no complete combinatorial description of the extensions between indecomposable modules over a gentle algebra is known. A description of certain combinatorially defined extensions between string modules was given in [21], and in [25] it was shown that the existence of such extensions is a necessary and sufficient condition for the non-vanishing of the Ext¹-space. However, it has remained an open problem for almost twenty years whether these extensions form a basis of the Ext¹-space between string modules. In fact, it has become apparent that string combinatorics in the module category of a gentle algebra might not be enough to answer this question. This has further been confirmed by the recent results in [9] where based on arguments using the associated cluster category, it was shown that in the context of surface gentle algebras, the extensions described in [21] do indeed give a basis.

In this paper, we show that this holds in general for any gentle algebra. Namely, building on [21], we give a complete description of the extension space between string and quasi-simple band modules over a gentle algebra by giving a combinatorial description of a basis of the Ext^1 space. We do this by working not in the module category of a gentle algebra, but we transfer the problem into the derived category, where we use the graphical mapping cone calculus developed in [8] as well as the results in [1] to obtain a combinatorial description of bases of the Ext^1 -spaces between string and quasi-simple band modules. Furthermore, our results use the algorithmic method for determining the cohomology $H^{\bullet}(Q^{\bullet})$ of a homotopy string or band Q^{\bullet} which we develop in Section 2.

We now state our main result, the combinatorial description of a basis of the extensions between indecomposable modules over a gentle algebra, both for string and quasi-simple band modules.

For this we recall the results on extensions from [21] in statements (1) and (2) of Theorem A. Statement (3) is new and its proof together with Theorem B is one of the main results in the second part of this paper. We refer to Figure 1 for a pictorial description of the extensions described in Theorem A. We will adopt the following notation: given a string w we denote the corresponding string module by M(w) and given a band b and a scalar $\mu \in K^*$, we denote the associated quasi-simple band module by $B(b, \mu)$, where use the convention that the twist by the scalar μ is placed on a direct arrow. For an arrow $a \in Q_1$ we denote its formal inverse by \bar{a} ; see Section 1.2 for details.

Theorem A. Let $\Lambda = KQ/I$ be a gentle algebra. Let v and w be strings and M(v) and M(w) the corresponding string modules.

(1) If there exists $a \in Q_1$ such that u = wav is a string then there is a non-split short exact sequence

$$0 \to M(w) \to M(u) \to M(v) \to 0.$$

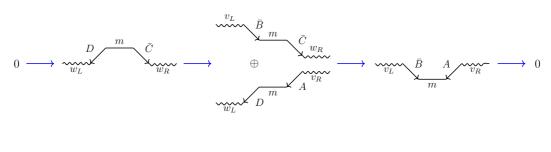
Such a short exact sequence will be called an **arrow extension** of M(v) by M(w).

(2) Suppose that $v = v_L \bar{B} m A v_R$ and $w = w_L D m \bar{C} w_R$ with $A, B, C, D \in Q_1$ and m, v_L, v_R, w_L, w_R (possibly trivial) strings such that v and w do not both start at the start of m or do not both end at the end of m. Then there is a non-split short exact sequence

$$0 \to M(w) \to M(u) \oplus M(u') \to M(v) \to 0$$

where $u = w_L DmAv_R$ and $u' = v_L \bar{B}m\bar{C}w_R$. Such a short exact sequence will be called an **overlap extension** of M(v) by M(w).

(3) The collection of arrow and overlap extensions of M(v) by M(w) form a basis of $\operatorname{Ext}^1_{\Lambda}(M(v), M(w))$.



$$0 \longrightarrow \overset{w}{\longrightarrow} \overset{a}{\longrightarrow} \overset{v}{\longrightarrow} 0$$

FIGURE 1. Presentation in terms of strings of an overlap extension (top picture) and an arrow extension (bottom picture).

When a band is involved, the situation becomes somewhat simpler: there are no arrow extensions and the overlap extensions have only one middle term.

Theorem B. Let $\Lambda = KQ/I$ be a gentle algebra. Let $\lambda, \mu \in K^*$.

(1) Suppose that $v = v_L \bar{B} m A v_R$ is a string and $w = w_L D m \bar{C} w_R$ a band, with $A, B, C, D \in Q_1$ and m, v_L, v_R, w_L, w_R (possibly trivial) strings such that v and w do not both start at the start of m or do not both end at the end of m. Then there is a non-split overlap extension

$$0 \to B(w,\mu) \to M(u) \to M(v) \to 0$$

where $u = v_L \bar{B} m \bar{C} w_R w_L D m A v_R$ is a string. The collection of such extensions forms a basis of $\operatorname{Ext}^1_{\Lambda}(M(v), B(w, \mu))$.

(2) Suppose that $v = v_L \bar{B} m A v_R$ is a band and $w = w_L D m \bar{C} w_R$ a string with $A, B, C, D \in Q_1$ and m, v_L, v_R, w_L, w_R (possibly trivial) strings such that v and w do not both start at the start of m or do not both end at the end of m. Then there is a non-split overlap extension

$$0 \to M(w) \to M(u) \to B(v,\lambda) \to 0$$

where $u = w_L Dm A v_R v_L \bar{B} m \bar{C} w_R$ is a string. The collection of such extensions forms a basis of $\operatorname{Ext}^1_{\Lambda}(B(v,\lambda),M(w))$.

(3) Suppose $v = v_L \bar{B} m A v_R$ and $w = w_L D m \bar{C} w_R$ are distinct bands with $A, B, C, D \in Q_1$ and m, v_L, v_R, w_L, w_R (possibly trivial) strings such that v and w do not both start at the start of m or do not both end at the end of m. Then there is a non-split overlap extension

$$0 \to B(w,\mu) \to B(u,\pm \lambda \mu^{-1}) \to B(v,\lambda) \to 0$$

where $u = v_L \bar{B} m \bar{C} w_R w_L D m A v_R$ is a band, where the sign \pm depends on the number of homotopy letters in the map corresponding to the extension; see [8, §4.3]. The collection of such extensions forms a basis of $\text{Ext}^1_{\Lambda}(B(v,\lambda), B(w,\mu))$.

Remark. In fact, the statement in Theorem B (3) above holds for almost all extensions when v = w: the only exception is when the overlap m = v. In this case, the middle term of the extension is the band module corresponding to the middle term of the almost split sequence starting and ending at B(v), in particular, the middle term no longer is a quasi-simple band module. However, in general the methods of this paper together with those in [8] can be extended to also cover the case of 'higher-dimensional' band modules and complexes.

We note that a basis for extensions between string modules over gentle algebras is also given, by different techniques, in [6] building on the work in [19].

We now briefly outline the content of the paper, including the general strategy of the proofs of Theorems A and B. Let Λ be a gentle algebra. We begin by recalling the basic notions of string and homotopy string combinatorics for gentle algebras in Section 1. In Section 2 we describe algorithmic methods for computing the homotopy string or band of the minimal projective resolution of a string or band module over Λ and the cohomology of a string or band complex in $\mathsf{K}^{b,-}(\operatorname{proj}(\Lambda))$. This provides the means to pass between homotopy string combinatorics and string combinatorics which will be used heavily in Sections 3 and 4.

In order to describe the content of Sections 3 and 4 more precisely, fix the following notation. Let v and w be strings or bands and M(v) and M(w) the corresponding string or quasi-simple band modules. We denote the homotopy strings or bands of their projective resolutions by $\pi(v)$ and $\pi(w)$ and the corresponding string or band complexes by $Q_{\pi(v)}^{\bullet}$ and $Q_{\pi(w)}^{\bullet}$. The standard basis of homomorphisms between string and/or band complexes is recalled from [1] in Section 1.4, enabling us to give an explicit description of a basis of $\operatorname{Hom}_{\mathsf{K}^{b,-}(\operatorname{proj}(\Lambda))}(Q_{\pi(v)}^{\bullet}, \Sigma Q_{\pi(w)}^{\bullet})$.

In the first step in the proof, we show in Section 3 that the image of every element of the standard basis under the canonical isomorphism

(1)
$$\Phi: \operatorname{Hom}_{\mathsf{K}^{b,-}(\operatorname{proj}(\Lambda))}(Q_{\pi(v)}^{\bullet}, \Sigma Q_{\pi(w)}^{\bullet}) \stackrel{\sim}{\to} \operatorname{Ext}_{\Lambda}^{1}(M(v), M(w))$$

is either an overlap or an arrow extension. In particular, this shows that the set of overlap and arrow extensions form a generating set for $\operatorname{Ext}^1_\Lambda(M(v), M(w))$.

The second step of the proof, comprising Section 4, shows that the set of overlap and arrow extensions forms a basis of $\operatorname{Ext}^1_\Lambda(M(v),M(w))$. To see this, we show that Φ restricts to a surjection from the standard basis of $\operatorname{Hom}_{\mathsf{K}^{b,-}(\operatorname{proj}(\Lambda))}(Q^{\bullet}_{\pi(v)},\Sigma Q^{\bullet}_{\pi(w)})$ to the set of arrow and overlap extensions in $\operatorname{Ext}^1_\Lambda(M(v),M(w))$.

We emphasise that, with the exception of the case highlighted in the remark above, the methods apply equally to (homotopy or classical) strings and bands. Furthermore, for ease of the already somewhat heavy notation, in the proofs in Section 3 and 4, whenever we have a map between two band complexes or an extension between two band modules,

implicitly and without loss of generality we assume that the parameters of the corresponding band complexes or band modules are equal to one. The reason that we can assume this is a direct consequence of the following: consider a map between two band complexes (resp. an extension between two band modules) with parameters $\lambda, \mu \in K^*$ each placed on a direct homotopy letter (resp. arrow). By [8] and since Φ is a linear map it follows that the parameter of the mapping cone (and hence the middle term of the extension) placed on a direct homotopy letter (resp. arrow) is $\pm \lambda \mu^{-1}$. We refer the reader to [8, §4.3] for a more detailed discussion on the placement of parameters with respect to taking mapping cones.

1. Background

In this section we briefly recall the definition of gentle algebras, background on string and band modules, string and band complexes and the standard basis of the morphism spaces between string and band complexes that will be needed in the article.

1.1. **Gentle algebras.** Throughout, K will be an algebraically closed field. We recall the following definition from [3].

Definition 1.1. A finite-dimensional K-algebra Λ is *gentle* if it is Morita equivalent to a bound path algebra KQ/I, where Q is a quiver and I an admissible ideal in KQ such that

- (1) for each vertex $i \in Q_0$ there are at most two arrows starting at i and at most two arrows ending at i;
- (2) for each arrow $a \in Q_1$ there is at most one arrow b with $ba \notin I$ and at most one arrow c with $ac \notin I$;
- (3) for each arrow $a \in Q_1$ there is at most one arrow b with $ba \in I$ and at most one arrow c with $ac \in I$;
- (4) the ideal I is generated by length-two monomial relations.

From now on $\Lambda = KQ/I$ will be a gentle algebra.

1.2. String and band modules. We now describe string and bands, which parametrise the indecomposable Λ -modules. The reference for this material is [7, 24]. Note that, in this paper all modules will be finitely generated left modules, and therefore paths in the quiver will be read from right to left.

For each arrow $a \in Q_1$ we introduce a formal inverse arrow $\overline{a} = a^{-1}$ with $s(\overline{a}) = e(a)$ and $e(\overline{a}) = s(a)$. We write \overline{Q}_1 for the set of formal inverse arrows. Similarly for a path $p = a_n \cdots a_1$ the inverse path is $\overline{p} = \overline{a}_1 \cdots \overline{a}_n$. Sometimes we shall assert the nonexistence of an arrow or inverse arrow a, and in this case we write $a = \emptyset$.

Definitions 1.2. We recall the following notions.

- (1) A walk of length l > 0 in (Q, I) is a sequence $w = w_l \cdots w_1$ satisfying $s(w_{i+1}) = e(w_i)$, where each w_i is either an arrow or an inverse arrow, and where the sequence does not contain any subsequence of the form $a\overline{a}$ or $\overline{a}a$ for an arrow $a \in Q_1$. We will call each arrow or inverse arrow w_i in w a letter of w.
- (2) A string is a walk that does not contain subwalks v such that $v \in I$ or $\overline{v} \in I$. In addition, there are trivial strings 1_x for each vertex $x \in Q_0$.
- (3) A band is a string $w = w_n \cdots w_1$ such that $e(w_n) = s(w_1)$, $w_1 \neq \overline{w}_n$ and $w \neq v^m$ for some substring v and m > 1.

Modulo the equivalence relation $w \sim \overline{w}$ the strings form an indexing set for the socalled *string modules*. Given a string w, we write M(w) for the corresponding string module. Note that if $w = 1_x$ is a trivial string M(w) = S(x) is the simple module at x. We refer to [7, 24] for more details on how to construct string modules from strings.

Modulo the equivalence relation given by inversion and cyclic permutation, the bands together with scalars $\lambda \in K^*$ form an indexing set for the so-called *band modules*, $B(w,\lambda)$. By abuse of notation, we will usually drop the scalar and write simply B(w) for the corresponding band module. Again we refer to [7] for the actual construction of the band modules.

By [24, Prop. 2.3], the string and band modules form a complete set of isomorphism classes of indecomposable Λ -modules.

The band modules given by representations in which each vertex is replaced by a 1-dimensional vector space all lie at the mouth of homogenous tubes and are referred to as quasi-simple (band) modules. They can be characterised as those band modules B such there exists an almost split sequence of the form $0 \to B \to E \to \tau^{-1}B \to 0$ where E is indecomposable, see for example [22]. In the following by abuse of notation, whenever we will use the term band module we will be referring to a quasi-simple band module.

1.3. String and band complexes. We now describe homotopy strings and bands, which parametrise the indecomposable complexes in the derived category $D^b(\Lambda)$. We will use the notation and terminology employed in [1, 8] and the references therein. However, for the sake of brevity we drop some of the formality of [1, 8] regarding the degrees.

Definitions 1.3. The original reference for the following definitions is [4].

- (1) A (finite) homotopy string is a walk of finite length in (Q, I). In addition there are trivial homotopy strings for each vertex $x \in Q_0$.
- (2) A subwalk $p = w_j \cdots w_i$ of a homotopy string $\sigma = w_l \cdots w_1$ is a homotopy letter if (a) p or \overline{p} is a path in (Q, I); and,
 - (b) $w_i \in Q_1$ and $w_{i-1} \in \overline{Q}_1$ or vice versa, or $w_i w_{i-1} \in I$, or $\overline{w_{i-1} w_i} \in I$; and,
 - (c) $w_j \in Q_1$ and $w_{j+1} \in \overline{Q}_1$ or vice versa, or $w_{j+1}w_j \in I$, or $\overline{w_jw_{j+1}} \in I$. We say that p is a direct homotopy letter if it is a path in (Q, I) and an inverse homotopy letter if \overline{p} is a path in (Q, I). In this way we partition a homotopy string σ into homotopy letters and write $\sigma = \sigma_n \cdots \sigma_1$ for this decomposition. A homotopy subletter of p is a subwalk of p.
- (3) A homotopy letter $p = w_l \cdots w_1$, with $w_i \in Q_1$ for $i = 1, \dots, l$ or $\bar{w}_i \in Q_1$ for $i = 1, \dots, l$, is said to have length l and we write length(p) = l. The length can be zero, in which case $p = 1_x$ for some $x \in Q_0$ and p is called a *trivial homotopy letter*. Sometimes we shall assert the nonexistence of homotopy letters, and in this case we write $p = \emptyset$.
- (4) Let $\sigma = \sigma_n \cdots \sigma_1$ be a homotopy string decomposed into its homotopy letters. A subwalk $\tau = \sigma_j \cdots \sigma_i$ with $1 \le i \le j \le n$ is called a homotopy substring of σ .
- (5) A homotopy band is a homotopy string $\sigma = \sigma_n \cdots \sigma_1$ with $s(\sigma) = e(\sigma)$, $\sigma_1 \neq \bar{\sigma}_n$, $\sigma \neq \tau^m$ for some homotopy substring τ and m > 1, and σ has equal numbers of direct and inverse homotopy letters.

Remark 1.4. Throughout the article, whenever we write a walk using Greek letters, such as $\sigma = \sigma_n \cdots \sigma_1$, we will always mean its decomposition into homotopy letters whereas, in general, we reserve Roman letters for (classical) strings and bands.

Modulo the equivalence relation $\sigma \sim \overline{\sigma}$ the homotopy strings form an indexing set for the so-called *string complexes*. Given a homotopy string σ , we write P_{σ}^{\bullet} for the

corresponding string complex. Note that if $\sigma = 1_x$ is a trivial homotopy string $P_{\sigma}^{\bullet} = P(x)$ is the stalk complex of the projective module at x. We refer to [4] for more details on how to construct string complexes from homotopy strings.

Modulo the equivalence relation given by inversion and cyclic permutation, the homotopy bands together with scalars $\lambda \in K^*$ form an indexing set for the so-called band complexes $B_{\sigma,\lambda}^{\bullet}$. Again we refer to [4] for the actual construction of the band complexes.

By [4, Thm. 3], the string and band complexes form a complete set of indecomposable perfect complexes in $\mathsf{D}^b(\Lambda)$. For the remaining objects of $\mathsf{D}^b(\Lambda)$ we need some further terminology.

Definitions 1.5. In the following, walks may now be infinite (on both sides).

- (1) A walk w is called a *direct antipath* if each letter is a direct homotopy letter; it is called an *inverse antipath* if each letter is an inverse homotopy letter.
- (2) A left infinite walk $w = \cdots w_n \cdots w_2 w_1$ is a left infinite homotopy string if there exists $m \ge 1$ such that $v = \cdots w_n \cdots w_{m+1} w_m$ is a direct antipath.
- (3) A right infinite walk $w = w_{-1}w_{-2}\cdots w_{-n}\cdots$ is a right infinite homotopy string if there exists $m \ge 1$ such that $v = w_{-m}w_{-m-1}\cdots w_{-n}\cdots$ is an inverse antipath.
- (4) A two sided infinite walk $w = \cdots w_2 w_1 w_0 w_{-1} \cdots$ is called a two-sided infinite homotopy string if there exist integers n > m such that $\cdots v_{n+1}v_n$ is a direct antipath and $v_m v_{m-1} \cdots$ is an inverse antipath.
- (5) By a one-sided infinite homotopy string we mean either a left infinite homotopy string or a right infinite homotopy string.

By [4, Thm. 3] the indecomposable non-perfect complexes in $\mathsf{D}^b(\Lambda)$ are parametrised by the one-sided and two-sided infinite homotopy strings; they are again called string complexes. In the following, we write

$$Q_{\sigma}^{\bullet} = \begin{cases} P_{\sigma}^{\bullet} & \text{if } \sigma \text{ is a (possibly infinite) homotopy string;} \\ B_{\sigma,\lambda}^{\bullet} & \text{if } \sigma \text{ is a homotopy band.} \end{cases}$$

From now on, by abuse of terminology, we say homotopy string for a (possibly infinite) homotopy string.

1.4. The standard basis. A basis for the morphism space between indecomposable complexes in $\mathsf{D}^b(\Lambda)$ was determined in [1]. Here we briefly recall this basis, which we shall refer to as the *standard basis*. Note that homotopy strings and bands correspond to an unfolding of the corresponding string and band complexes and we freely make use of the unfolded diagram notation for string and band complexes defined in more detail in [1, 8].

Theorem 1.6 ([1, Theorem 3.15]). Let σ and τ be homotopy strings or bands. Then there is a canonical basis of $\operatorname{Hom}_{\mathsf{D}^b(\Lambda)}(Q_{\sigma}^{\bullet},Q_{\tau}^{\bullet})$ given by:

- graph maps $f^{\bullet} \colon Q_{\sigma}^{\bullet} \to Q_{\tau}^{\bullet}$;
- singleton single maps f[•]: Q[•]_σ → Q[•]_τ;
 singleton double maps f[•]: Q[•]_σ → Q[•]_τ;
 quasi-graph maps φ: Q[•]_σ ↔ Σ⁻¹Q[•]_τ.

We note that a quasi-graph map is not a map, but in fact determines classes of homotopy equivalent single and double maps, which is why we denote it by \rightsquigarrow and not

Throughout the following description of the maps listed above, σ and τ will be homotopy strings or bands.

- 1.4.1. Graph maps. Suppose σ and τ are, up to inversion, of the form,
 - (1) $\sigma = \beta \sigma_L \rho \sigma_R \alpha$ and $\tau = \delta \tau_L \rho \tau_R \gamma$; or
 - (2) $\sigma = \rho \sigma_R \alpha$ and $\tau = \rho \tau_R$,

where α, β, γ and δ are homotopy substrings, $\sigma_L, \sigma_R, \tau_L$ and τ_R are (possibly trivial) homotopy letters, and ρ is a (possibly trivial) maximal common homotopy substring, and in the second case an infinite homotopy substring of σ and τ . We assume that ρ occurs in the same cohomological degrees in both homotopy strings. Then the corresponding graph maps can be represented by the following unfolded diagrams:

$$(2) \qquad Q_{\sigma}^{\bullet}: \qquad \stackrel{\beta}{\sim} \bullet \stackrel{\sigma_{L}}{\sim} \bullet \stackrel{\rho_{k}}{\sim} \bullet \stackrel{\rho_{k-1}}{\sim} \cdots \stackrel{\rho_{2}}{\sim} \bullet \stackrel{\rho_{1}}{\sim} \bullet \stackrel{\sigma_{R}}{\sim} \bullet \stackrel{\alpha}{\sim} \sim \\ Q_{\tau}^{\bullet}: \qquad \stackrel{\beta}{\sim} \bullet \stackrel{\sigma_{L}}{\sim} \bullet \stackrel{\rho_{k}}{\sim} \bullet \stackrel{\rho_{k-1}}{\sim} \cdots \stackrel{\rho_{2}}{\sim} \bullet \stackrel{\rho_{1}}{\sim} \bullet \stackrel{\sigma_{R}}{\sim} \bullet \stackrel{\alpha}{\sim} \sim \\ \stackrel{\gamma}{\sim} \bullet \stackrel{\sigma_{L}}{\sim} \bullet \stackrel{\rho_{k}}{\sim} \bullet \stackrel{\rho_{k-1}}{\sim} \cdots \stackrel{\rho_{2}}{\sim} \bullet \stackrel{\rho_{1}}{\sim} \bullet \stackrel{\sigma_{R}}{\sim} \bullet \stackrel{\alpha}{\sim} \sim \\ \stackrel{\gamma}{\sim} \bullet \stackrel{\sigma_{L}}{\sim} \bullet \stackrel{\sigma_{L}}{\sim} \bullet \stackrel{\rho_{k-1}}{\sim} \cdots \stackrel{\rho_{2}}{\sim} \bullet \stackrel{\rho_{1}}{\sim} \bullet \stackrel{\sigma_{R}}{\sim} \bullet \stackrel{\alpha}{\sim} \sim \\ \stackrel{\gamma}{\sim} \bullet \stackrel{\sigma_{L}}{\sim} \bullet \stackrel{$$

$$(3) \qquad P_{\sigma}^{\bullet}: \qquad \qquad \bullet \xrightarrow{\rho_{3}} \bullet \xrightarrow{\rho_{2}} \bullet \xrightarrow{\rho_{1}} \bullet \xrightarrow{\sigma_{R}} \bullet \xrightarrow{\alpha} \\ P_{\tau}^{\bullet}: \qquad \qquad \bullet \xrightarrow{\rho_{3}} \bullet \xrightarrow{\rho_{2}} \bullet \xrightarrow{\rho_{2}} \bullet \xrightarrow{\rho_{1}} \bullet \xrightarrow{\tau_{R}} \bullet \xrightarrow{\tau_{R}} \bullet \xrightarrow{\gamma}$$

where we require the squares marked (*) and (**) to commute; these are explicitly written down in [8, Def. 3.3]. The maximality of ρ as a common homotopy substring of σ and τ necessarily means that $\sigma_L \neq \tau_L$ and $\sigma_R \neq \tau_R$. Note that in the case of 1.4.1(2), ρ is an antipath and we say that the graph map f^{\bullet} is incident with ρ .

1.4.2. Single maps. The unfolded diagram of a single map $f^{\bullet}: Q_{\sigma}^{\bullet} \to Q_{\tau}^{\bullet}$ is given by

$$Q_{\sigma}^{\bullet}: \qquad \stackrel{\beta}{\sim} \bullet \stackrel{\sigma_{L}}{\longrightarrow} \bullet \stackrel{\sigma_{R}}{\longrightarrow} \bullet \stackrel{\alpha}{\sim} \\ f^{\bullet} \downarrow \qquad \qquad \downarrow f \qquad \qquad \downarrow f \qquad \qquad \downarrow \uparrow \\ Q_{\tau}^{\bullet}: \qquad \stackrel{\delta}{\sim} \bullet \stackrel{\tau_{L}}{\longrightarrow} \bullet \stackrel{\tau_{R}}{\longrightarrow} \bullet \stackrel{\alpha}{\longrightarrow}$$

where f is a nontrivial path in (Q, I), and satisfying the following conditions:

- (L1) σ_L is either inverse or is direct and $\sigma_L f$ has a subpath in I.
- (L2) τ_L is either direct or is inverse and $f\bar{\tau}_L$ has a subpath in I.
- (R1) σ_R is either direct or is inverse and $\bar{\sigma}_R f$ has a subpath in I.
- (R2) τ_R is either inverse or is direct and $f\tau_R$ has a subpath in I.

A single map $f^{\bullet}: Q_{\sigma}^{\bullet} \to Q_{\tau}^{\bullet}$ is called a *singleton single map* if its unfolded diagram, up to inversion of one of the homotopy strings/bands, is

where σ_L and τ_L never contain f as a subletter, nor does f contain σ_L or τ_L as a subletter, and any of σ_L , σ_R , τ_L and τ_R are permitted to be the empty homotopy letter \varnothing .

1.4.3. Double maps. The unfolded diagram of a double map $f^{\bullet}: Q_{\sigma}^{\bullet} \to Q_{\tau}^{\bullet}$ is

(6)
$$Q_{\sigma}^{\bullet}: \qquad \stackrel{\beta}{\sim} \bullet \xrightarrow{\sigma_{L}} \bullet \xrightarrow{\sigma_{C}} \bullet \xrightarrow{\sigma_{R}} \bullet \stackrel{\alpha}{\sim} \sim \\ Q_{\tau}^{\bullet}: \qquad \stackrel{\beta}{\sim} \bullet \xrightarrow{\tau_{L}} \bullet \xrightarrow{\tau_{C}} \bullet \xrightarrow{\tau_{R}} \bullet \stackrel{\alpha}{\sim} \sim \\ \stackrel{\beta}{\sim} \bullet \xrightarrow{\tau_{L}} \bullet \xrightarrow{\tau_{C}} \bullet \xrightarrow{\tau_{R}} \bullet \stackrel{\alpha}{\sim} \sim \\ \stackrel{\beta}{\sim} \bullet \xrightarrow{\tau_{L}} \bullet \xrightarrow{\tau_{C}} \bullet \xrightarrow{\tau_{C}} \bullet \xrightarrow{\tau_{R}} \bullet \stackrel{\alpha}{\sim} \sim \\ \stackrel{\beta}{\sim} \bullet \xrightarrow{\tau_{L}} \bullet \xrightarrow{\tau_{C}} \bullet \xrightarrow{\tau_{C}} \bullet \xrightarrow{\tau_{R}} \bullet \stackrel{\alpha}{\sim} \sim \\ \stackrel{\beta}{\sim} \bullet \xrightarrow{\tau_{L}} \bullet \xrightarrow{\tau_{C}} \bullet \xrightarrow{\tau_{C}} \bullet \xrightarrow{\tau_{R}} \bullet \stackrel{\alpha}{\sim} \sim \\ \stackrel{\beta}{\sim} \bullet \xrightarrow{\tau_{L}} \bullet \xrightarrow{\tau_{C}} \bullet \xrightarrow{\tau_{C}} \bullet \xrightarrow{\tau_{R}} \bullet \stackrel{\alpha}{\sim} \sim \\ \stackrel{\beta}{\sim} \bullet \xrightarrow{\tau_{C}} \bullet \xrightarrow{$$

where f_L and f_R are nontrivial paths in (Q, I) such that $f_L \tau_C = \sigma_C f_R$ has no subpath in I, conditions (L1) and (L2) hold for f_L and (R1) and (R2) hold for f_R .

A double map, as above, is called *singleton* if there is a nontrivial path f' in (Q, I) such that $\sigma_C = f_L f'$ and $\tau_C = f' f_R$.

1.4.4. Quasi-graph maps. If, in the situation of Section 1.4.1, the squares marked (*) and (**) of diagrams (2) and (3) do not commute, then such diagrams determine a quasi-graph map $\varphi \colon Q_{\sigma}^{\bullet} \leadsto Q_{\tau}^{\bullet}$. The non-commuting endpoint conditions are explicitly spelled out in [8, Def. 3.9]. Note that, while a quasi-graph map $Q_{\sigma}^{\bullet} \leadsto Q_{\tau}^{\bullet}$ does not define a map, a quasi-graph map $\varphi \colon Q_{\sigma}^{\bullet} \leadsto \Sigma^{-1}Q_{\tau}^{\bullet}$ determines a family of homotopy equivalent single and/or double maps. Indeed, all single and double maps that are not singleton arise in this way.

The following observation will be useful in the proofs in Section 4.

- **Remark 1.7.** Suppose, in the unfolded diagram (2) above, ρ_1 is not the start of both σ and τ and ρ_k is not the end of both σ and τ . In this case, the diagram defines a graph map $f^{\bullet} \colon Q_{\sigma}^{\bullet} \to Q_{\tau}^{\bullet}$ if and only if the same diagram, when read upside down, i.e. from bottom to top, defines a quasi-graph map $\varphi \colon Q_{\tau}^{\bullet} \leadsto Q_{\sigma}^{\bullet}$.
- 1.5. Morphisms vs. extensions. For background on derived and homotopy categories we refer to [13]. One of the powerful features of the derived category is that it reformulates extensions in the module category in terms of morphisms. In particular, for any algebra Λ , and any Λ -modules M and N we have

$$\operatorname{Hom}_{\mathsf{K}}(P_{M}^{\bullet}, \Sigma P_{N}^{\bullet}) \simeq \operatorname{Ext}_{\mathsf{K}}^{1}(P_{M}, P_{N}^{\bullet}) \simeq \operatorname{Ext}_{\Lambda}^{1}(M, N),$$

$$P_{M}^{\bullet} \xrightarrow{f^{\bullet}} \Sigma P_{N}^{\bullet} \mapsto P_{N}^{\bullet} \longrightarrow C_{f^{\bullet}}^{\bullet} \longrightarrow P_{M}^{\bullet} \xrightarrow{f^{\bullet}} \Sigma P_{N}^{\bullet} \mapsto 0 \to N \to H^{0}(C_{f^{\bullet}}^{\bullet}) \to M \to 0$$

where $\mathsf{K} = \mathsf{K}^{b,-}(\mathrm{proj}(\Lambda))$, P_M^{\bullet} and P_N^{\bullet} are projective resolutions of M and N, respectively, and $C_{f^{\bullet}}^{\bullet}$ is the (negative shift of the) mapping cone of f^{\bullet} . In particular, computation of a basis of the Ext-space $\mathrm{Ext}_{\Lambda}^1(M,N)$ reduces to the computation of a basis of the Hom-space $\mathrm{Hom}_{\mathsf{K}^{b,-}(\mathrm{proj}(\Lambda))}(P_M^{\bullet},\Sigma P_N^{\bullet})$.

2. Cohomology of string and band complexes

Throughout σ will be a (possibly infinite) homotopy string or band, unless one is specified explicitly. When we wish to specify that σ is finite on the right we will write $\sigma = \cdots \sigma_2 \sigma_1$, finite on the left: $\sigma = \sigma_n \sigma_{n-1} \cdots$, and finite on both sides: $\sigma = \sigma_n \cdots \sigma_1$.

Given a homotopy string or band σ we will describe how to compute the cohomology of the string or band complex Q_{σ}^{\bullet} . The strategy is to divide σ up into various homotopy substrings each corresponding to appropriately chosen two-term complexes. We start with an important technical definition.

Definition 2.1. Let σ be a homotopy string or band. A homotopy substring $\tau = \sigma_j \cdots \sigma_i$ with i < j is a maximal alternating homotopy substring if

- (i) for each $i \leq k < j$, if σ_k is direct (resp., inverse) then σ_{k+1} is inverse (resp., direct);
- (ii) if σ_i is direct (resp., inverse) then σ_{i-1} is direct (resp., inverse) or \emptyset ; and,
- (iii) if σ_i is direct (resp., inverse) then σ_{i+1} is direct (resp., inverse) or \varnothing .

If only condition (i) holds, then τ is called an alternating homotopy substring.

Remark 2.2. Let σ be a homotopy string or band and $\tau = \sigma_j \cdots \sigma_i$ with i < j be a maximal alternating homotopy substring of σ .

(1) The homotopy string τ has at least two homotopy letters.

- (2) The string complex P_{τ}^{\bullet} is concentrated in precisely two cohomological degrees, namely deg $P(s(\sigma_i))$ and deg $P(e(\sigma_i))$, i.e. it is a 'two-term complex'.
- (3) A maximal homotopy substring of a homotopy string or band cannot be infinite: all infinite homotopy strings have antipaths to the left and/or to the right.
- (4) Since no two consecutive homotopy letters of τ 'pass through a relation', the underlying walk of τ also determines a string. In the case that $\sigma = \sigma_n \cdots \sigma_1$ is a homotopy band and $\tau = \sigma$, then the underlying walk of τ also determines a band.

Lemma 2.3 (Maximal alternating homotopy substring rule). Let σ be a homotopy string or band. Suppose $\tau = \sigma_j \cdots \sigma_i$ is a maximal alternating homotopy substring. Decompose the homotopy letters $\sigma_j = b_l \cdots b_1$ and $\sigma_i = a_k \cdots a_1$ into paths or inverse paths in (Q, I) and set

$$w := \begin{cases} b_{l-1} \cdots b_1 \sigma_{j-1} \cdots \sigma_{i+1} a_k \cdots a_2 & \text{if } \tau \neq \sigma \text{ or } \tau = \sigma \text{ and } \sigma \text{ is a homotopy string with} \\ \sigma_1 & \text{inverse and } \sigma_n \text{ direct;} \\ \text{if } \tau = \sigma \text{ and } \sigma \text{ is a homotopy band.} \end{cases}$$

Then the string module M(w) (resp., band module B(w)) is an indecomposable summand of the cohomology module $H^d(Q^{\bullet}_{\sigma})$, where $d = \max\{\deg P(s(\sigma_i)), \deg P(e(\sigma_i))\}$.

Proof. Suppose σ is a homotopy string and $(Q_{\sigma}^{\bullet}, \partial^{\bullet})$ is the corresponding string complex. We treat the case that the maximal alternating homotopy substring τ has unfolded diagram of the form below; the other cases, and the case that σ is a homotopy band, are similar.

$$\cdots \cdots \bullet \xrightarrow{\sigma_{j+1}} \bullet \xrightarrow{\sigma_j} \bullet \xrightarrow{\sigma_{j-1}} \bullet \cdots \cdots \bullet \xrightarrow{\sigma_{i+1}} \bullet \xrightarrow{\sigma_i} \bullet \xrightarrow{\sigma_{i-1}} \bullet \cdots \cdots \cdots \bullet$$

Note that, in this case $d = \deg P(s(\sigma_i))$ and the homotopy letters $\sigma_i, \ldots, \sigma_j$ are components of the differential ∂^{d-1} . In particular, we can wrap τ back up into a complex:

$$P(e(\sigma_{j+1})) \xrightarrow{\sigma_{j+1}} P(s(\sigma_{j+1})) \xrightarrow{\sigma_{j}} P(s(\sigma_{j}))$$

$$\oplus \qquad \qquad \oplus$$

$$P(s(\sigma_{j-1})) \xrightarrow{\sigma_{j-2}} P(s(\sigma_{j-2}))$$

$$\oplus \qquad \qquad \oplus$$

$$P(s(\sigma_{j-3})) \xrightarrow{\sigma_{j-4}} \vdots$$

$$\oplus \qquad \qquad \oplus$$

$$\vdots \xrightarrow{\sigma_{i+2}} P(s(\sigma_{i+2}))$$

$$\oplus \qquad \qquad \oplus$$

$$P(s(\sigma_{i+1})) \xrightarrow{\sigma_{i}} P(s(\sigma_{i})) \xrightarrow{\sigma_{i-2}} P(s(\sigma_{i-1}))$$

The other components of the differentials ∂^{d-2} , ∂^{d-1} and ∂^d are disconnected from the components of ∂^{d-1} indicated above. The components above therefore contribute a summand, M say, of the cohomology module $H^d(Q^{\bullet}_{\sigma})$; the other summands of $H^d(Q^{\bullet}_{\sigma})$ are contributed by other parts of σ . We claim that $M \cong M(w)$, where w is the string defined in the statement.

The projective modules $P(s(\sigma_{i+2})), P(s(\sigma_{i+4})), \ldots, P(s(\sigma_j)) \subset \ker(\partial^d)$. Consider the following components of the differential ∂^{d-1} ,

$$P(s(\sigma_{m+1})) \xrightarrow{\sigma_m} P(s(\sigma_{m+1}))$$

$$P(s(\sigma_m)) \xrightarrow{\sigma_{m-1}} P(s(\sigma_{m-1}))$$

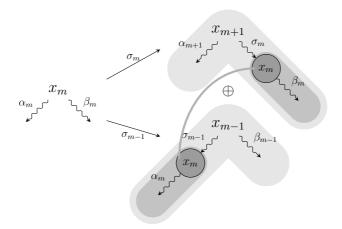
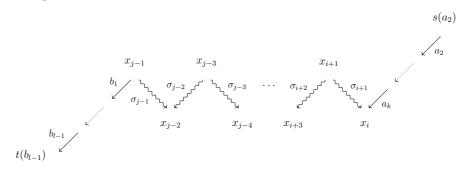


FIGURE 2. Schematic showing the computation of the cohomology, where $x_m = s(\sigma_m)$, i.e. is the start of the homotopy letter σ_m

which map diagonally into submodules of $P(s(\sigma_{m+1}))$ and $P(s(\sigma_{m-1}))$ with simple top $S(s(\sigma_m))$. Thus, in the quotient $\ker(\partial^d)/\operatorname{im}(\partial^{d-1})$ the action of σ_m on the basis vector at $s(\sigma_{m+1})$ is the same as the action of σ_{m-1} on the basis vector at $s(\sigma_{m-1})$, as indicated in Figure 2. Continuing in this way, we obtain that the summand M of $\ker(\partial^d)/\operatorname{im}(\partial^{d-1})$ has the following form.



that is, corresponds to the string $w = b_{l-1} \cdots b_1 \sigma_{j-1} \cdots \sigma_{i+1} a_k \cdots a_2$.

The following lemmas are computations analogous to that in Lemma 2.3 above. Thus we provide only their statements and leave the proofs to the reader.

Lemma 2.4 (Cokernel rule). Let $\sigma = \cdots \sigma_2 \sigma_1$ be a homotopy string in which $\sigma_1 = a_k \cdots a_1$ is a direct homotopy letter. If there exists c with $c \in \overline{Q}_1$ such that $\sigma_1 c$ is defined as a string, then take $u = c_m \cdots c_1$ to be the maximal inverse string ending with $c_m = c$. Set

$$w := \begin{cases} a_{k-1} \cdots a_1 u & \text{if there is such a } c; \\ a_{k-1} \cdots a_1 & \text{otherwise.} \end{cases}$$

Then the string module M(w) is an indecomposable summand of the cohomology module $H^d(P^{\bullet}_{\sigma})$, where $d = \deg P(s(\sigma_1))$.

Let $\sigma = \cdots \sigma_2 \sigma_1$ be a homotopy string. It is possible that there is a maximal alternating homotopy substring $\tau = \sigma_j \cdots \sigma_1$. If σ_1 is direct, we must combine the maximal alternating homotopy substring rule and the cokernel rule; dually for $\sigma = \sigma_n \sigma_{n-1} \cdots$ with $\tau = \sigma_n \cdots \sigma_i$ and σ_n inverse, we have a combined rule which we spell out below.

Lemma 2.5 (Combined rule). Let $\sigma = \cdots \sigma_2 \sigma_1$ be a homotopy string in which $\sigma_1 = a_k \cdots a_1$ is a direct homotopy letter and $\tau = \sigma_j \cdots \sigma_1$ is a maximal alternating homotopy

substring. Decompose the homotopy letter $\sigma_j = b_l \cdots b_1$ into a path or inverse path in (Q, I) and set

$$w := \begin{cases} b_{l-1} \cdots b_1 \sigma_{j-1} \cdots \sigma_1 u & \text{if there exist } c \text{ and } u \text{ as in Lemma 2.4;} \\ b_{l-1} \cdots b_1 \sigma_{j-1} \cdots \sigma_2 a_k \cdots a_1 & \text{otherwise.} \end{cases}$$

Then the string module M(w) is an indecomposable summand of the cohomology module $H^d(P^{\bullet}_{\sigma})$, where $d = \deg P(s(\sigma_1))$.

In the same vein, if $\tau = \sigma$ and σ is a homotopy string with σ_1 direct and σ_n inverse then Lemma 2.5 should be combined further with its dual statement.

Lemma 2.6 (Kernel rule). Let $\sigma = \sigma_n \sigma_{n-1} \cdots$ be a homotopy string in which $\sigma_n = b_l \cdots b_1$ is a direct homotopy letter. If there exists $c \in Q_1$ and $cb_l = 0$ then take $v = c_m \cdots c_1$ to be the maximal direct string starting with $c_1 = c$. Set

$$v := \begin{cases} c_m \cdots c_2 & \text{if there exists such a } c; \\ \varnothing & \text{otherwise.} \end{cases}$$

Then the string module M(v) is an indecomposable summand of the cohomology module $H^d(P^{\bullet}_{\sigma})$, where $d = \deg P(e(\sigma_n))$. If m = 1 then $v = 1_{e(c)}$ is the trivial string corresponding to the simple module S(e(c)).

Note that if $\sigma = \sigma_n \sigma_{n-1} \cdots$ is a homotopy string and $\tau = \sigma_n \cdots \sigma_i$ is a maximal alternating homotopy substring with σ_n direct, then Lemmas 2.3 and 2.6 do not need to be combined. In particular, the string module M(v) is an indecomposable summand of $H^d(P^{\bullet}_{\sigma})$ and the string module M(w) is an indecomposable summand of $H^{d+1}(P^{\bullet}_{\sigma})$, where v is defined as in Lemma 2.6, w is defined as in Lemma 2.3, and $d = \deg P(e(\sigma_n))$.

Lemma 2.7 (Nontrivial homotopy letter rule). Let σ be a homotopy string or band in which σ_i is a direct homotopy letter and $\sigma_{i+1}\sigma_i\sigma_{i-1}$ is not an alternating homotopy substring with σ_{i+1} possibly empty. Let $d = \deg P(s(\sigma_i))$.

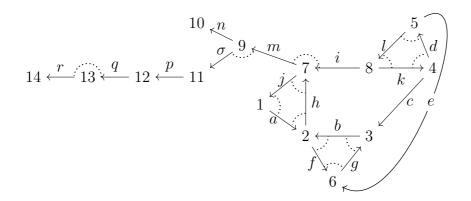
- (1) If $\sigma_i = a_k \cdots a_1$ with $a_j \in Q_1$ and k > 1 then set $w = a_{k-1} \cdots a_2$. The string module M(w) is an indecomposable summand of the cohomology module $H^d(Q^{\bullet}_{\sigma})$. If k = 2 then $w = 1_{e(a_1)} = 1_{s(a_2)}$ and $M(w) = S(e(a_1)) = S(s(a_2))$.
- (2) If $\sigma_i = a$ for some $a \in Q_1$ then the map $\sigma_i : P(e(\sigma_i)) \to P(s(\sigma_i))$ contributes the zero submodule to the cohomology module of $H^d(Q_{\bullet}^{\bullet})$.

Lemmas 2.4, 2.5, 2.6 and 2.7 admit obvious dual statements. When referring to these lemmas we shall freely include those dual statements. We summarise this section with the following theorem and illustrate with an example.

Theorem 2.8. Let σ be a homotopy string or band. Lemmas 2.3, 2.4, 2.5, 2.6 and 2.7 and their duals provide a complete description of the cohomology complex $H^{\bullet}(Q_{\sigma}^{\bullet})$.

Remark 2.9. Note that in computing the cohomology Lemmas 2.3, 2.4, 2.5, 2.6, 2.7 and their duals can be applied independently and therefore in any order. The only exception is that the combined rule Lemma 2.5 should always be applied instead of Lemma 2.3 whenever the homotopy string has the appropriate form.

Example 2.10. We consider the gentle algebra with the following quiver where the (length 2) relations are indicated by dotted lines.



Consider the following homotopy strings where the top line indicates the homological degree of the corresponding projective indecomposable:

$$0 \qquad 1 \qquad 0 \qquad -1 \qquad -2 \qquad -3 \qquad -2 \qquad -$$

$$\sigma \colon \qquad 7 \xrightarrow{i} 8 \xleftarrow{\bar{k}\bar{c}\bar{b}} 2 \xleftarrow{\bar{f}} 6 \xleftarrow{\bar{g}} 3 \xleftarrow{\bar{b}\bar{h}} 7 \xrightarrow{il} 5 \xrightarrow{d} 4$$
 and
$$-1 \qquad 0 \qquad 1 \qquad 2$$

$$\tau: 14 \xrightarrow{r} 13 \xrightarrow{qpo} 9 \xrightarrow{m} 7.$$

Examining the homotopy string σ , we see that there are four indecomposable summands of $H^{\bullet}(P_{\sigma}^{\bullet})$. We list them below in order of ascending cohomological degree.

- We have $H^{-2}(P_{\sigma}^{\bullet}) = M(w_1)$, where $w_1 = \bar{h}i$ coming from the maximal alternating homotopy substring rule (Lemma 2.3) applied to $3 \stackrel{\bar{b}\bar{h}}{\longleftarrow} 7 \stackrel{il}{\longrightarrow} 5$.
- We have $H^{-1}(P_{\sigma}^{\bullet}) = M(w_2)$, where $w_2 = \bar{c}\bar{b}\bar{h}\bar{m}\bar{n}$ coming from the cokernel rule (Lemma 2.4) applied to $5 \stackrel{d}{\longrightarrow} 4$.
- We have $H^0(P_{\sigma}^{\bullet}) = M(w_3)$, where $w_3 = n$ coming from the kernel rule (Lemma 2.6) applied to $7 \xrightarrow{i} 8$.
- We have $H^1(P^{\bullet}_{\sigma}) = M(w_4)$, where $w_4 = \bar{k}\bar{c}$ coming from the maximal alternating homotopy substring rule (Lemma 2.3) applied to $7 \xrightarrow{i} 8 \xleftarrow{\bar{k}\bar{c}\bar{b}} 2$.
- By Lemma 2.7, all remaining parts of the homotopy string σ contribute zero to the cohomology $H^{\bullet}(P_{\sigma}^{\bullet})$.

Examining the homotopy string τ , in a similar fashion we obtain the following for $H^{\bullet}(P_{\tau}^{\bullet})$.

- We have $H^1(P_\tau) = M(w_1)$ where $w_1 = p$ coming from the nontrivial homotopy letter rule (Lemma 2.7) applied to $13 \xrightarrow{qpo} 9 \xrightarrow{m} 7$.
- We have $H^2(P_\tau) = M(w_2)$ where $w_2 = j$ coming from the cokernel rule (Lemma 2.4) applied to $9 \xrightarrow{m} 7$.

• There is no non-zero contribution to the cohomology coming from the nontrivial homotopy letter rule (Lemma 2.7) applied to $14 \xrightarrow{r} 13 \xrightarrow{qpo} 9$ or from the kernel rule (Lemma 2.6) applied to $14 \xrightarrow{r} 13$.

We end this section by giving the homotopy string or band of the minimal projective resolution of a string or quasi-simple band module, which will be heavily used in the next sections. We note that gentle algebras are string algebras and that there is a large body of work on string algebras. In particular, projective resolutions and syzygies, have been considered before, see for example [15, 16]. In [24], minimal projective presentations of string and band modules were given in terms of string combinatorics, which in the case of gentle algebras can be formulated in terms of homotopy string combinatorics. These projective presentations correspond to maximal alternating homotopy substrings sitting between degrees -1 and 0. Before stating the result, we set up some notation.

Definition 2.11. Let a and b be such that $\bar{a}, b \in Q_1$. Define

- $\operatorname{inv}(a) := \sigma_{-1}\sigma_{-2}\cdots$ to be maximal inverse antipath ending with $\sigma_{-1} = a$;
- $\operatorname{dir}(b) := \cdots \sigma_2 \sigma_1$ to be the maximal direct antipath starting with $\sigma_1 = b$.

Corollary 2.12. Let $w = w_n \cdots w_1$ be a string. Define a homotopy string $\pi(w)$ as follows:

- (1) $\pi(w) = \operatorname{dir}(b) w' \operatorname{inv}(a)$ if there are a and b such that $\bar{a}, b \in Q_1$ and bwa is defined as a string and where w' = w.
- (2) $\pi(w) = w' \operatorname{inv}(a)$ if there is an a with $\bar{a} \in Q_1$ such that wa is defined as a string but no $b \in Q_1$ with bw defined as a string, where $w' = w_j \cdots w_1$ after removing a maximal direct substring $w_n \cdots w_{j+1}$ of w.
- (3) $\pi(w) = \operatorname{dir}(b) w'$ if there is $b \in Q_1$ with bw defined as a string but no a with $\bar{a} \in Q_1$ such that wa is defined as a string, where $w' = w_n \cdots w_i$ after removing a maximal inverse substring $w_{i-1} \cdots w_1$ of w.
- (4) $\pi(w) = w'$ if there are no a and b such that $\bar{a}, b \in Q_1$ and bwa is defined as a string, where $w' = w_j \cdots w_i$ after removing a maximal direct substring $w_n \cdots w_{j+1}$ and a maximal inverse substring $w_{i-1} \cdots w_1$.
- (5) $\pi(w) = w$ if w is a band.

Then $P_{\pi(w)}^{\bullet}$ (resp., $B_{\pi(w)}^{\bullet}$ when w is a band) is a projective resolution of M(w) (resp., B(w)).

Proof. The computation of the cohomology of $P_{\pi(w)}^{\bullet}$ (resp., $B_{\pi(w)}^{\bullet}$) in Theorem 2.8 gives M(w) (resp., B(w)) in cohomological degree zero and zero in all other degrees.

Corollary 2.13. Let A be a gentle algebra. Then any band module has projective dimension one.

The maximal direct substring $w_n \cdots w_{j+1}$ removed from w in Corollary 2.12(2) will be called a maximal direct suffix. Likewise, the maximal inverse substring $w_{i-1} \cdots w_1$ removed from w in Corollary 2.12(3) will be called a maximal inverse prefix.

Definition 2.14. For the homotopy string $\sigma = \pi(w)$ defined in Corollary 2.12 above we call the homotopy substrings inv(a) and dir(b) the antipath part of $\pi(w)$. By abuse of notation we write inv(w) = inv(a) and dir(w) = dir(b). In the notation of Corollary 2.12, we will call w' the module part of $\pi(w)$, this is the (possibly truncated) maximal alternating homotopy substring part of $\pi(w)$.

An inverse homotopy letter $\sigma_i = \bar{a}_1 \cdots \bar{a}_k$ of σ is incident with inv(a) if $\bar{a}_k = \bar{a}$. Likewise, a direct homotopy letter $\sigma_j = b_l \cdots b_1$ of σ is incident with dir(b) if $b_1 = b$.

In the following, as usual, we write $Q_{\pi(w)}^{\bullet}$ when we do not wish to specify whether w is a string or a band.

Remark 2.15. We make the following straightforward observations regarding the forms of the homotopy strings occurring in Corollary 2.12.

- (1) If there is no a such that wa is defined as a string then the homotopy string $\pi(w)$ starts with a direct homotopy letter whose target lies in degree 0.
- (2) If there is no b such that bw is defined as a string then the homotopy string $\pi(w)$ ends with an inverse homotopy letter whose target lies in degree 0.
- (3) If $\sigma_{i+1}\sigma_i$ are consecutive homotopy letters with the same orientation then at least one of them lies in the antipath part and the other either lies in the antipath part or else is incident with $\operatorname{dir}(w)$ or $\operatorname{inv}(w)$.
- (4) Owing to being a projective resolution of a module, the string/band complex $Q_{\pi(w)}^{\bullet}$ attains its maximal cohomological degree in degree 0. Moreover, homotopy letters occurring in the module part of $\pi(w)$ provide components of the differential in $Q_{\pi(w)}^{\bullet}$ from degree -1 to degree 0. Indeed, together with those homotopy letters incident with $\operatorname{dir}(w)$ and $\operatorname{inv}(w)$ these provide all components of the differential in $Q_{\pi(w)}^{\bullet}$ from degree -1 to degree 0.
- (5) Suppose σ_k is a homotopy letter of $\pi(w)$. If length $(\sigma_k) > 1$ then $\deg(P(e(\sigma_k))) \in \{0, -1\}$ and $\deg(P(s(\sigma_k))) \in \{-1, 0\}$, where $\deg(P(x))$ denotes the cohomological degree in which P(x) occurs.

3. Determining extensions in the module category

Recall that in [21] extensions for string modules are given in terms of string combinatorics. Namely, for v, w two strings, we have

1) (Arrow extension) If there exists $a \in Q_1$ such that u = wav is a string then there is a non-split short exact sequence

$$0 \to M(w) \to M(u) \to M(v) \to 0.$$

2) (Overlap extension) Suppose that $v = v_L \bar{B} m A v_R$ and $w = w_L D m \bar{C} w_R$ with $A, B, C, D \in Q_1$ and m, v_L, v_R, w_L, w_R (possibly trivial) strings such that v and w do not both start at the start of m and do not both end at the end of m. Then there is a non-split short exact sequence

$$0 \to M(w) \to M(u) \oplus M(u') \to M(v) \to 0$$

where $u = w_L D m A v_R$ and $u' = v_L \bar{B} m \bar{C} w_R$.

Recall the canonical isomorphism in (1)

$$\Phi: \operatorname{Hom}_{\mathsf{K}^{b,-}(\operatorname{proj}(\Lambda))}(Q_{\pi(v)}^{\bullet}, \Sigma Q_{\pi(w)}^{\bullet}) \xrightarrow{\sim} \operatorname{Ext}_{\Lambda}^{1}(M(v), M(w)).$$

Theorem 3.1. With the notation above, let M(v) and M(w) be indecomposable Λ -modules with strings or bands w and v respectively and let Q_{σ}^{\bullet} and Q_{τ}^{\bullet} with $\sigma = \pi(v)$ and $\tau = \pi(w)$ be their projective resolutions. Then for any standard basis element f^{\bullet} in $\operatorname{Hom}_{\mathsf{K}^{b,-}(\operatorname{proj}(\Lambda))}(Q_{\sigma}^{\bullet}, \Sigma Q_{\tau}^{\bullet})$ the corresponding extension $\Phi(f^{\bullet})$ in $\operatorname{Ext}_{\Lambda}^{1}(M(v), M(w))$ is given by an arrow or an overlap extension. In particular, the set of overlap and arrow extensions form a generating set for $\operatorname{Ext}_{\Lambda}^{1}(M(v), M(w))$.

In the rest of this section we prove Theorem 3.1 by considering each of type of map of the standard basis of $\operatorname{Hom}_{\mathsf{K}^{b,-}(\operatorname{proj}(\Lambda))}(Q_{\sigma}^{\bullet}, \Sigma Q_{\tau}^{\bullet})$ as defined in [1]. We start by showing that the Theorem 3.1 holds for graph maps.

3.1. **Graph maps.** Throughout this subsection we fix the following setup.

Setup 3.2. Let v and w be strings or bands and M(v) and M(w) be the corresponding string or band modules. Let $\sigma = \pi(v)$ and $\tau = \pi(w)$ be the homotopy strings or bands corresponding to the projective resolutions Q_{σ}^{\bullet} and Q_{τ}^{\bullet} of M(v) and M(w) as given in Corollary 2.12, respectively.

Lemma 3.3. Let $f^{\bullet}: Q^{\bullet}_{\sigma} \to \Sigma Q^{\bullet}_{\tau}$ be a graph map incident with an antipath in Q^{\bullet}_{σ} and an antipath in $\Sigma Q^{\bullet}_{\tau}$. Then $\Phi(f^{\bullet})$ gives rise to an arrow extension in $\operatorname{Ext}^{1}_{\Lambda}(M(w), M(v))$.

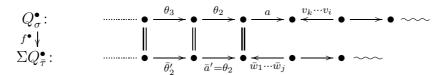
Proof. Recall that $\pi(v) = \sigma$ and $\pi(w) = \tau$. We assume that $\pi(v) = \operatorname{dir}(a)v\varphi$ and $\pi(w) = \varphi'w \operatorname{inv}(a')$ where $a, \bar{a}' \in Q_1$; the other cases for the other ends of v and w can be checked similarly.

To simplify the notation, set $\operatorname{dir}(a) = \theta = \cdots \theta_2 \theta_1$ and $\operatorname{inv}(a') = \theta' = \theta'_1 \theta'_2 \cdots$ with $\theta_i, \bar{\theta}'_i \in Q_1$ and $\theta_1 = a$ and $\theta'_1 = a'$. Suppose that f^{\bullet} induces an isomorphism of projective modules lying in θ and θ' and suppose this isomorphism is in degree n. Then as homotopy letters in antipaths are of length 1 and since Λ is gentle, there exists an isomorphism $\theta_n \simeq \bar{\theta}'_{n-1}$ and we obtain an isomorphism in degree n-1. We now continue inductively to the left and right.

Let $v = v_k \cdots v_1$ and $w = w_l \cdots w_1$. We now analyse in turn the different cases when v_k and w_1 correspond to direct or inverse arrows.

Case 1: v_k is inverse and w_1 is direct.

We have the following unfolded diagram



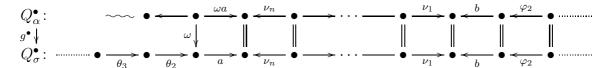
where $1 \leq i \leq k$ and $1 \leq j \leq l$. Then by [8, Thm. 4.3] the homotopy string of the (shift of the) mapping cone of f^{\bullet} is given by $\alpha = \varphi' w a v \varphi$. By the form of $\pi(w)$ and $\pi(v)$, it follows from Corollary 2.12 that there exist $\bar{b}, b' \in Q_1$ such that $\varphi = b\rho$ and $\varphi' = \rho'b'$ and b'wavb is a string. Then by Lemma 2.3, M(wav) is the cohomology (in cohomological degree zero) of Q_{α}^{\bullet} .

The shift of the mapping cone, Q_{α}^{\bullet} , by definition sits in a distinguished triangle

(7)
$$Q_{\tau}^{\bullet} \xrightarrow{h^{\bullet}} Q_{\alpha}^{\bullet} \xrightarrow{g^{\bullet}} Q_{\sigma}^{\bullet} \xrightarrow{f^{\bullet}} \Sigma Q_{\tau}^{\bullet}.$$

We now observe that $H^0(g^{\bullet}) := g : M(wav) \to M(v)$ is the canonical map in the arrow extension, showing that the corresponding graph map does indeed induce the claimed arrow extension.

Decompose $v = \nu_n \cdots \nu_1$ into homotopy letters so that $\sigma = \theta \nu_n \cdots \nu_1 b \varphi$. We assume that ν_1 is direct so that b is a homotopy letter; the case ν_1 is inverse is similar. Set $\omega = w_j \cdots w_1$. The map $g^{\bullet} : Q^{\bullet}_{\alpha} \to Q^{\bullet}_{\sigma}$ is given by the following unfolded diagram



which is supported in cohomological degree -1 at the left endpoint. Wrapping α and σ back up into complexes as in the proof of Lemma 2.3, where we have taken a 'mirror

image' of σ in order to more easily match up the cohomological degree 0 parts, we get the following diagram.

$$P(e(\omega a)) \xrightarrow{\omega a} P(s(\nu_n)) = P(s(\nu_n)) \xrightarrow{a} P(e(a))$$

$$P(s(\nu_{n-1})) \xrightarrow{\nu_{n-1}} P(s(\nu_{n-2})) = P(s(\nu_{n-2})) \xrightarrow{\nu_{n-1}} P(s(\nu_{n-1}))$$

$$P(s(\nu_{n-2})) \xrightarrow{\nu_{n-3}} \vdots \qquad \vdots \qquad \vdots \qquad P(s(\nu_{n-2}))$$

$$P(s(\nu_{n-2})) \xrightarrow{\bar{\nu}_{n-3}} P(s(\nu_{n-2})) = P(s(\nu_{n-2}))$$

$$P(s(\bar{\varphi}_2)) \xrightarrow{\bar{\varphi}_2} P(s(\bar{b}))$$

$$P(s(\bar{\varphi}_2)) \xrightarrow{\bar{\varphi}_2} P(s(\bar{b}))$$

Re-writing this diagram in terms of the strings defining the indecomposable projective modules occurring as in Figure 2, it is straightforward to see that $H^0(g^{\bullet})$ is the canonical map $M(wav) \to M(v)$ given by the obvious substring/factor string decomposition. Taking the long exact cohomology sequence associated to the triangle (7) gives a short exact sequence

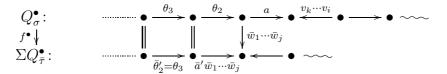
$$0 \longrightarrow M(w) \xrightarrow{H^0(h^{\bullet})} M(wav) \xrightarrow{H^0(g^{\bullet})} M(v) \longrightarrow 0,$$

in which $H^0(g^{\bullet})$ is the canonical map, whence it follows immediately that $H^0(h^{\bullet})$ is also the canonical map associated to the obvious substring/factor string decomposition.

It now follows that f^{\bullet} induces an arrow extension corresponding to the arrow induced by a, where the middle term of the extension is given by the string module M(wav).

Case 2: Both v_k and w_1 are inverse.

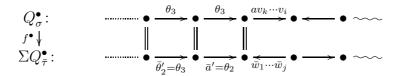
We have the following unfolded diagram



where $1 \leq i \leq k$ and $1 \leq j \leq l$. Then $a' = \theta_1$ and by [8, Thm. 4.3] the homotopy string of the mapping cone of f^{\bullet} is given by $\alpha = \rho' b' w a v b \rho$ where $\varphi = b \rho$ and $\varphi' = \rho' b'$ with $\bar{b}, b' \in Q_1$ such that b' w a v b is a string. As in Case 1 above, one can check that the map $H^0(g^{\bullet}): M(wav) \to M(v)$ is the canonical map given by the obvious substring/factor string decomposition. It then follows that, taking cohomology, f^{\bullet} induces an arrow extension, corresponding to the arrow a, whose middle term is M(wav).

Case 3: Both v_k and w_1 are direct.

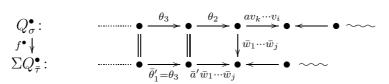
This case is similar to case 1. We have the following unfolded diagram



where $1 \le i \le k$ and $1 \le j \le l$. Then as above the cohomology of the mapping cone induces an arrow extension corresponding to the arrow a.

Case 4: v_k is direct and w_1 is inverse.

This case is similar to case 2. We have the following unfolded diagram



where $1 \le i \le k$ and $1 \le j \le l$. Then as above the cohomology of the mapping cone induces an arrow extension corresponding to the arrow a.

Case 5: v or w or both are trivial.

If v is trivial but w is not, this is a degenerate case of Case 1 or 2. If v is not trivial but w is, this is a degenerate case of Case 1 or 3. If both v and w are trivial, this is a degenerate case of Case 1.

Lemma 3.4. Let $f^{\bullet}: Q^{\bullet}_{\pi(v)} \to \Sigma Q^{\bullet}_{\pi(w)}$ be a graph map and let ν in $\pi(v)$ and ω in $\pi(w)$ be the maximal alternating homotopy substrings corresponding to the module parts of $\pi(v)$ and $\pi(w)$ respectively. Suppose that f^{\bullet} is supported in projective modules lying in ν and ω . Then f^{\bullet} is supported in a single indecomposable projective Λ -module P in degree -1 unless it is incident with antipaths in both Q^{\bullet}_{σ} and $\Sigma Q^{\bullet}_{\tau}$.

Furthermore, $\Phi(f^{\bullet})$ gives rise to either an arrow extension or an overlap extension where the overlap is given by the simple Λ -module P/rad(P).

Proof. There are two cases to be considered. For the first case suppose that f^{\bullet} is supported in ν , and f^{\bullet} is not incident with any antipath of $\pi(v)$. Then we must have at least one isomorphism between projective modules in degree -1 as follows

(8)
$$Q_{\pi(v)}^{\bullet}: \qquad \sim \bullet \stackrel{\nu_{i}}{\longleftarrow} x \stackrel{\nu_{i-1}}{\longrightarrow} \bullet \sim \sim$$

$$\Sigma Q_{\pi(w)}^{\bullet}: \qquad \sim \bullet \stackrel{}{\longrightarrow} x \stackrel{\nu_{i-1}}{\longleftarrow} \bullet \sim \sim$$

$$-2 \qquad -1 \qquad -2$$

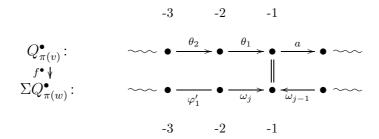
where $x \in Q_0$. Since the projectives in ν as a substring of $\pi(v)$ are in homological degrees 0 and -1 and the projectives in ω as a substring of the homotopy string corresponding to $\Sigma Q_{\pi(w)}^{\bullet}$ are in degrees -1 and -2, the graph map f^{\bullet} can only be supported in a single degree, as shown.

Now, the homotopy letters ν_{i-1} , ν_i , ω_{j-1} and ω_j have the form $\nu_{i-1} = A\nu'_{i-1}$, $\nu_i = \nu'_i \bar{B}$, $\omega_{j-1} = \bar{C}\omega'_{j-1}$ and $\omega_j = \omega'_j D$, where $A, B, C, D \in Q_1$ and the primed symbols are homotopy subletters. Then $v = v_L \bar{B} A v_R$ and $w = w_L D \bar{C} w_R$ where v_L, v_R are (possibly trivial) subwords of v and v_L, v_R are (possibly trivial) subwords of v. Set $v_L = e(A) (e(\bar{B}) = s(\bar{C}) = s(D))$ and let $v_L = v_L \bar{B} v_L = v_L \bar{B} v_$

We now show that given the hypotheses on f^{\bullet} , the following situation cannot occur, namely that $v = v_L \bar{B}$ and $w = w_L D$. Suppose for contradiction that $v = v_L \bar{B}$ and $w = w_L D$. Suppose furthermore, that there exists $a \in Q_1$ such that $w\bar{a}$ is a string. Then either $v\bar{a}$ is defined as a string or not. Suppose first that $v\bar{a}$ is not defined as a string, that is $aB \in I$. Then since Λ is gentle, if there exists an arrow $b \in Q_1$, such that $b \neq B$ and

 $ab \notin I$ then this contradicts that f^{\bullet} is supported in S(x) because the homotopy letter ν_i would not have the given form. If such an arrow b does not exist, then by Corollary 2.12 this again contradicts our assumption on f^{\bullet} . So $v\bar{a}$ must be defined as a string and $aB \notin I$. But then we get a contradiction because the homotopy letter ν_i is again not of the required form. Therefore, there is no such $a \in Q_1$ so that $w\bar{a}$ is defined as a string. By the same argument we must have $DB \in I$. Then by Corollary 2.12, in $\pi(v)$ we must remove a maximal inverse prefix, so in particular we remove \bar{B} . This contradicts the setup of f^{\bullet} . So we cannot have $v = v_L \bar{B}$ and $v = w_L D$. Similarly, there cannot be a graph map as in (8) such that $v = Av_R$ and $v = \bar{C}w_R$.

Finally for the second case suppose that f^{\bullet} is incident with an antipath in $Q_{\pi(v)}^{\bullet}$ and the module part in $\Sigma Q_{\pi(w)}^{\bullet}$. In this case we obtain the following diagram for f^{\bullet} .



Since θ_1 is a homotopy letter of length 1, we must have $\theta_1 = \omega_j$. If φ'_1 is inverse or zero then we reach a non-commuting endpoint condition. This contradicts the fact that f^{\bullet} is a graph map. Thus φ'_1 must be direct and $\varphi'_1 = \theta_2$. We are therefore in the setup of Lemma 3.3 and the corresponding extension in the module category is an arrow extension.

The case that f^{\bullet} is incident with an antipath in $Q^{\bullet}_{\pi(v)}$ and the module part in $\Sigma Q^{\bullet}_{\pi(w)}$ cannot happen for degree reasons.

3.2. Quasi-graph maps. In this section we consider a quasi-graph map $\varphi \colon Q_{\sigma}^{\bullet} \leadsto Q_{\tau}^{\bullet}$, corresponding to a homotopic family of single and double maps in the basis of $\operatorname{Hom}_{\mathsf{K}^{b,-}(\operatorname{proj}(\Lambda))}(Q_{\sigma}^{\bullet}, \Sigma Q_{\tau}^{\bullet})$; see [1, Def. 3.12].

We start by placing a restriction on the cohomological degrees in which a quasi-graph map $\varphi \colon Q_{\sigma}^{\bullet} \leadsto Q_{\tau}^{\bullet}$ can be supported.

Lemma 3.5. Under the hypotheses of Setup 3.2, a quasi-graph map $\varphi: Q_{\sigma}^{\bullet} \leadsto Q_{\tau}^{\bullet}$ is supported in cohomological degrees -1 and 0 only.

Proof. If one of Q_{σ}^{\bullet} or Q_{τ}^{\bullet} is a band complex then, by Corollary 2.13, it is supported in cohomological degrees -1 and 0 only, and therefore any quasi-graph map $\varphi \colon Q_{\sigma}^{\bullet} \leadsto Q_{\tau}^{\bullet}$ is trivially supported in only those cohomological degrees. Therefore we assume that $Q_{\sigma}^{\bullet} = P_{\sigma}^{\bullet}$ and $Q_{\tau}^{\bullet} = P_{\tau}^{\bullet}$ are string complexes.

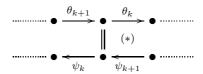
Suppose, for a contradiction, that $\varphi \colon P_{\sigma}^{\bullet} \leadsto P_{\tau}^{\bullet}$ is supported in cohomological degree $-k \leq -2$. By Remark 2.15(4), any component of φ supported in degrees $-k \leq -2$ occurs in antipath parts of P_{σ}^{\bullet} and P_{τ}^{\bullet} . Without loss of generality, we may assume, up to inversion if necessary, that $\sigma = \operatorname{dir}(b)w'\sigma_R$ and $\tau = \operatorname{dir}(d)v'\tau_R$, where σ_R is either an inverse antipath or empty; likewise for τ_R . Thus, the antipath parts have the form

$$\operatorname{dir}(b) = \cdots \theta_n \cdots \theta_2 \theta_1$$
 and $\operatorname{dir}(c) = \cdots \psi_n \cdots \psi_2 \psi_1$,

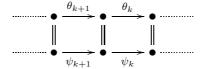
where $\theta_1 = b$ and $\psi_1 = d$ and $b, d \in Q_1$ are such that bw and dv are defined as strings.

Since $\varphi \colon P_{\sigma}^{\bullet} \leadsto P_{\tau}^{\bullet}$ is supported in cohomological degree $-k \leq -2$, we have the following subdiagram of the unfolded diagram for φ .

We first show that φ is supported in degrees -k-1 and -k+1. Suppose that φ was not supported in cohomological degree -k+1, then inverting τ the unfolded diagram of φ would have the form,



where (*) corresponds to the graph map right endpoint condition (RG3) in [8, Def. 3.3], whence by [1, Rem. 4.9] corresponds to a family of null-homotopic maps. Similarly, one can show that φ is supported in cohomological degree -k-1. This means that we can extend the subdiagram of the unfolded diagram of φ to the following,



showing that $\theta_k = \psi_k$ for each $k \geq 2$. But this mean that the unfolded diagram of φ satisfies (LG3) or (LG ∞) (cf. [8, Def. 3.3]) and, therefore, invoking [1, Rem. 4.9] again, we see that φ corresponds to a null-homotopic family of single and double maps. This contradicts our assumption that φ is a quasi-graph map, therefore φ cannot be supported in cohomological degrees smaller than -2, as claimed.

We now consider the endpoints of a quasi-graph map $\varphi: Q_{\sigma}^{\bullet} \leadsto Q_{\tau}^{\bullet}$. Lemma 3.5 says that they must occur in degrees -1 or 0. Recall the definition of homotopy strings or bands σ and τ being compatibly oriented for a quasi-graph map φ from [8, Def. 7.1]; note that if a quasi-graph map is supported in more than one degree it is automatically compatibly oriented in its unfolded form.

Lemma 3.6. Suppose the quasi-graph map $\varphi \colon Q_{\sigma}^{\bullet} \leadsto Q_{\tau}^{\bullet}$ has right endpoint in degree 0.

(1) The compatibly oriented unfolded diagram for φ has the following form at the right endpoint of φ :

such that $\sigma_s, \sigma_R \neq \emptyset$, $\tau_t = \emptyset$ or $\tau_t = \sigma'_s \sigma_s$ for some (possibly nontrivial) σ'_s , and

 $\tau_{R} = \varnothing \text{ or } \tau_{R} = \sigma_{R}\sigma'_{R} \text{ for some nontrivial } \sigma'_{R}.$ (2) Write $\sigma_{R} = \bar{a}_{k} \cdots \bar{a}_{1} \text{ and } \sigma_{s} = b_{l} \cdots b_{1} \text{ for } k, l \geq 1 \text{ and } a_{i}, b_{j} \in Q_{1}.$ Then (i) v has a substring of the form

$$\widetilde{v} = \begin{cases} b_{l-1} \cdots b_1 \overline{a}_k \cdots \overline{a}_2 & \text{if } \sigma_R \text{ is incident with inv}(v), \\ b_{l-1} \cdots b_1 \overline{a}_k \cdots \overline{a}_1 a & \text{for some } a \in Q_1 \text{ otherwise;} \end{cases}$$

(ii) w has a substring of the form

$$\widetilde{w} = \begin{cases} b_{l-1} \cdots b_1 \overline{a}_k \cdots \overline{a}_1 \overline{a}' & \text{for some } a' \in Q_1 \text{ if } \tau_R \neq \emptyset, \\ b_{l-1} \cdots b_1 \overline{a}_k \cdots \overline{a}_1 & \text{otherwise.} \end{cases}$$

- Proof. (1) Since P(x) sits in degree zero it must be a sink for any differential incident with it because Q_{σ}^{\bullet} and Q_{τ}^{\bullet} are projective resolutions. If $\sigma_s = \varnothing$ or $\sigma_R = \varnothing$, then the diagram indicates a graph map endpoint and $\varphi \colon Q_{\sigma}^{\bullet} \to Q_{\tau}^{\bullet}$ is not a quasi-graph map. Therefore, $\sigma_s, \sigma_R \neq \varnothing$. If $\tau_R \neq \varnothing$, the orientation of the differentials means that φ must satisfy the quasi-graph map right endpoint condition (RQ2), whence $\tau_R = \sigma_R \sigma_R'$ for some nontrivial σ_R' . The statement regarding τ_t just lists the possible cases that may occur with the given orientation.
- (2)(i) First note that $b_{l-1} \cdots b_1$ is a substring of v by Corollary 2.12. If σ_R is incident with $\operatorname{inv}(v)$ then the first statement is clear. By Remark 2.15(1), σ cannot start with the inverse homotopy letter σ_R unless it is incident with $\operatorname{inv}(v)$. Thus, if σ_R is not incident with $\operatorname{inv}(v)$ then α must end with a direct homotopy letter, whose last arrow we denote by $a \in Q_1$, say, giving the required form for \widetilde{v} .
- (2)(ii) We treat this in cases. Firstly, if $\tau_t, \tau_R = \emptyset$, then Q_{τ}^{\bullet} is the stalk complex P(x) concentrated in degree zero. Using the form of σ_s and σ_R we see that $P(x) \cong M(u)$ for some string $u = qb_l \cdots b_1 \bar{a}_k \cdots \bar{a}_1 \bar{p}$, where q is a maximal direct string and \bar{p} a maximal inverse string composable with b_l and \bar{a}_k , respectively, as strings. The claim is now clear in this case.

Now assume that $\tau_t \neq \emptyset$ and $\tau_R = \emptyset$. By (1), $\tau_t = \sigma_s' \sigma_s$, where σ_s' is possibly trivial. Since $\tau_R = \emptyset$, w either starts with b_1 (a direct arrow) or else w has had a maximal inverse prefix removed. The former case cannot occur because $b_1 \bar{a}_k$ is defined as a string, which by Corollary 2.12 would make $\tau_R \neq \emptyset$. Thus, by gentleness, $w = u\bar{a}_1w_R$ for some (possibly trivial) inverse string w_R . If $\bar{a}_k \cdots \bar{a}_1$ is a (possibly equal) substring of $\bar{a}_k w_R$ then w contains the substring \tilde{w} as claimed. So suppose $\bar{a}_k w_R = \bar{a}_k \cdots \bar{a}_i$ for some $1 < i \le k$. Then, $\bar{a}_k w_R \bar{a}_{i-1}$ is defined as a string, again rendering $\tau_R \neq \emptyset$ by Corollary 2.12; a contradiction.

Suppose now that $\tau_t = \emptyset$ and $\tau_R \neq \emptyset$. Since $\tau_t = \emptyset$, w ends with a direct substring which has been removed by Remark 2.15(2). By gentleness, the maximal direct suffix that has been removed is $pb_l \cdots b_1$, where again p is the maximal direct path composable with b_l as a string. Now since $\tau_R = \sigma_R \sigma_R'$ is a strictly longer inverse homotopy letter than σ_R , it follows that \widetilde{w} is a substring of w, where $\sigma_R' = \overline{a}' \sigma_R''$ for some $a' \in Q_1$ and σ_R'' is possibly trivial.

Finally, if $\tau_t, \tau_R \neq \emptyset$, then arguing as above shows that \widetilde{w} is a substring of w.

Lemma 3.7. Suppose the quasi-graph map $\varphi \colon Q_{\sigma}^{\bullet} \leadsto Q_{\tau}^{\bullet}$ has right endpoint in degree -1.

(1) The compatibly oriented unfolded diagram for φ has the following form at the right endpoint of φ :

$$(a) \quad \stackrel{\sigma_s}{\longleftarrow} x \xrightarrow{\sigma_R} \bullet \qquad (b) \quad \stackrel{\sigma_s}{\longleftarrow} x \xrightarrow{\sigma_R} \bullet \sim \qquad ,$$

$$\stackrel{\tau_t}{\longrightarrow} x \xrightarrow{\tau_R} \bullet \sim \qquad \stackrel{\tau_t}{\longrightarrow} x \xrightarrow{\tau_R} \bullet \sim \qquad ,$$

where $\tau_t \neq \varnothing$. In case (a), $\sigma_s = \varnothing$ or $\sigma_s = \tau_t' \tau_t$ for some τ_t' and we require $\tau_R \neq \varnothing$ and $\sigma_R = \varnothing$ or else $\sigma_R = \tau_R \tau_R'$ for some nontrivial τ_R' . In case (b) $\sigma_s = \tau_t' \tau_t$ for some τ_t' and we require one of $\tau_R \neq \varnothing$ or $\sigma_R \neq \varnothing$ and if both are not empty letters then $\sigma_R \tau_R \neq \varnothing$. In both cases τ_t' may be trivial.

(2) Write $\tau_t = \bar{d}_q \cdots \bar{d}_1$ and $\tau_R = c_p \cdots c_1$ for $k, l \geq 1$ and $c_i, d_j \in Q_1$. Then

(i) v has a substring of the form

$$\widetilde{v} = \begin{cases} \overline{d}_q \cdots \overline{d}_2 & \text{if } \sigma_s \text{ is incident with inv}(v), \\ c_{p-1} \cdots c_1 c & \text{for some } c \in Q_1 \text{ if } \sigma_s = \varnothing \text{ and } \sigma_R \text{ is incident with } \operatorname{dir}(v), \\ \overline{d}_q \cdots \overline{d}_1 c_p \cdots c_1 c & \text{for some } c \in Q_1 \text{ if } \sigma_s \neq \varnothing \text{ and } \sigma_R \neq \varnothing \text{ is direct}; \end{cases}$$

(ii) w has a substring of the form

$$\widetilde{w} = \begin{cases} \overline{d}_q \cdots \overline{d}_1 c_p \cdots c_1 & \tau_R \neq \emptyset, \\ \overline{d}_q \cdots \overline{d}_2 & otherwise. \end{cases}$$

Proof. (1) There are three possible orientations for the homotopy strings σ and τ with right endpoint in degree -1, where in the following diagrams x sits in degree -1:

$$(I) \quad \bullet \longleftarrow x \longrightarrow \bullet \quad (II) \quad \bullet \longleftarrow x \longleftarrow \bullet \quad (III) \quad \bullet \longrightarrow x \longrightarrow \bullet .$$

Note that the fourth possible orientation does not occur because then the corresponding string of band complex would have maximal cohomological degree -1, contradicting Remark 2.15(4). One can check that if σ has orientation (I) then so does τ : the other orientations produce graph map endpoint conditions (and hence null-homotopies; see [1, Rem. 4.9), this gives case (a) above. Observe that in case (a), $\tau_t \neq \emptyset$ and $\tau_R \neq \emptyset$, for otherwise we would have a graph map endpoint condition.

If σ has orientation (II) then τ cannot have orientation (III) because this again gives a graph map endpoint condition. If τ has orientation (II) then we may assume $\tau_R \neq \emptyset$ (the case $\tau_R = \emptyset$ is trivial can be considered as a subcase of τ having orientation (I)), in which case length(τ_R) ≥ 1 . However, for degree reasons, it must be incident with inv(w) and hence length(τ_R) = 1. Therefore τ cannot have orientation (II). This gives us case (b). Note in this case that since x sits in degree -1, $\sigma_s \neq \emptyset$ by Remark 2.15(2); as above, $\tau_s \neq \emptyset$ otherwise we have a graph map endpoint condition.

When σ has orientation (III), the unfolded diagrams are those for the dual left endpoint conditions, and can be properly stated in the dual of this lemma.

(2)(i) First observe that, in both cases, either $\sigma_s \neq \emptyset$ or $\sigma_R \neq \emptyset$ (or both) for degree reasons: if both were empty homotopy letters, Q_{σ}^{\bullet} would be a stalk complex concentrated in degree -1, contradicting Remark 2.15(4).

Suppose we are in case (a) of part (1). Suppose $\sigma_s = \emptyset$ but $\sigma_R \neq \emptyset$. Then $\sigma_R = \tau_R \tau_R'$ for some nontrivial τ_R' by the (RQ1) endpoint condition. By Remark 2.15(2), σ cannot end with a direct homotopy letter unless it is incident with dir(v). Let $c \in Q_1$ be the final arrow of the homotopy (sub)letter τ'_R . Then since σ_R is incident with $\operatorname{dir}(v)$, we have that $\widetilde{v} = c_{p-1} \cdots c_1 c$ is a substring of v.

If $\sigma_s \neq \varnothing$ but $\sigma_R = \varnothing$, then Remark 2.15(1) shows that σ_s is incident with inv(v), giving $\widetilde{v} = d_q \cdots d_2$ as a substring of v.

If $\sigma_1, \sigma_R \neq \emptyset$, then neither is incident with $\operatorname{dir}(v)$ or $\operatorname{inv}(v)$, in which case $\widetilde{v} =$ $\bar{d}_q \cdots \bar{d}_1 c_p \cdots c_1 c$, where $c \in Q_1$ is as above, is a substring of v.

Now suppose we are in case (b) of part (1). If $\sigma_R = \emptyset$ then using Remark 2.15(1) again we have σ_s is incident with $\operatorname{inv}(v)$ and $\widetilde{v} = \overline{d}_q \cdots \overline{d}_2$ is a substring of v. If $\sigma_R \neq \varnothing$, then by Remark 2.15(5), length(σ_R) = 1 and σ_R is incident with inv(v), in which case $\tilde{v} = \bar{d}_q \cdots \bar{d}_2$ is again a substring of v.

(2)(ii) Suppose we are in case (a) of part (1). Since $\tau_t, \tau_R \neq \emptyset$, the homotopy substring $\tau_1 \tau_R$ cannot be incident with dir(w) nor inv(w) for degree reasons. Thus, $\widetilde{w} = d_q \cdots d_1 c_p \cdots c_1$ is a substring of w.

Finally, suppose we are in case (b) of (1). If $\tau_R = \emptyset$, then Remark 2.15(1) shows that τ_1 is incident with inv(w), giving $\widetilde{w} = \overline{d}_q \cdots \overline{d}_2$ as a substring of w. If $\tau_R \neq \emptyset$, then as above the homotopy substring $\tau_1 \tau_R$ cannot be incident with $\operatorname{dir}(w)$ nor $\operatorname{inv}(w)$. Thus, $\widetilde{w} = \overline{d}_q \cdots \overline{d}_1 c_p \cdots c_1$ is a substring of w.

Lemmas 3.6 and 3.7 admit obvious duals for the left endpoints of quasi-graph maps.

Now applying the graphical calculus for the mapping cones of the homotopy set determined by a quasi-graph map [8, Prop. 7.2] determines the middle term of the extension $Q_{\tau}^{\bullet} \to E^{\bullet} \to Q_{\sigma}^{\bullet} \to \Sigma Q_{\tau}^{\bullet}$ in $\mathsf{K}^{b,-}(\mathrm{proj}(\Lambda))$. Lemmas 3.6 and 3.7 and their duals, Theorem 2.8, together with a calculation as in the proof of Lemma 3.3 allows us to take cohomology to determine the extension $0 \to M(w) \to H^0(E^{\bullet}) \to M(v) \to 0$. We summarise this computation in the next proposition.

Proposition 3.8. Suppose $\varphi \colon Q_{\sigma}^{\bullet} \leadsto Q_{\tau}^{\bullet}$ is a quasi-graph map with the following unfolded diagram, with $t \geq 0$ and, when t = 0 we mean a quasi-graph map supported in precisely one degree and we replace ρ_1 by σ_L and τ_L as appropriate.

$$deg: \qquad h' \qquad h$$

$$Q_{\sigma}^{\bullet}: \qquad \stackrel{\beta}{\sim} \bullet \stackrel{\sigma_L}{\longrightarrow} \bullet \stackrel{\rho_t}{\longrightarrow} \bullet \stackrel{\rho_{t-1}}{\longrightarrow} \cdots \stackrel{\rho_2}{\longrightarrow} \bullet \stackrel{\rho_1}{\longrightarrow} \bullet \stackrel{\sigma_R}{\longrightarrow} \bullet \stackrel{\alpha}{\sim} \sim \stackrel{\alpha}{\sim} \qquad $

Let $f^{\bullet}: Q_{\sigma}^{\bullet} \to \Sigma Q_{\tau}^{\bullet}$ be any representative of φ , then $\Phi(f^{\bullet})$ is an overlap extension with overlap $m = m_L \rho_{t-1} \cdots \rho_2 m_R$, where

$$m_{R} = \begin{cases} \widetilde{\rho_{1}} \overline{a}_{k} \cdots \overline{a_{2}} & \text{if } h = 0 \text{ and } \sigma_{R} = \overline{a}_{k} \cdots \overline{a_{2}} \text{ is incident with inv}(v); \\ \widetilde{\rho_{1}} \overline{a}_{k} \cdots \overline{a_{1}} & \text{if } h = 0 \text{ and } \sigma_{R} = \overline{a}_{k} \cdots \overline{a_{1}} \text{ is not incident with inv}(v); \\ \overline{d}_{q} \cdots \overline{d_{2}} & \text{if } h = -1 \text{ and } \rho_{1} \neq \varnothing \text{ is incident with inv}(v); \\ \widetilde{\rho_{1}} c_{p} \cdots c_{2} & \text{if } h = -1, \ \rho_{1} \neq \varnothing \text{ and } \sigma_{R} = c_{p} \cdots c_{1} \text{ with } p > 0; \end{cases}$$

$$m = c_{p-1} \cdots c_{1} \qquad \text{if } \rho_{1} = \varnothing \text{ and } \sigma_{R} = c_{p} \cdots c_{1} \text{ is incident with } \operatorname{dir}(v),$$

where $a_i, d_i, c_i \in Q_1$, m_L is defined dually, and

$$\widetilde{\rho_1} = \begin{cases} \rho_1 & \text{if } t \geq 1, \\ \text{the last homotopy letter of } m_L & \text{if } t \geq 1. \end{cases}$$

3.3. Singleton maps. As before, throughout this subsection $\sigma = \pi(v)$ and $\tau = \pi(w)$ for some strings or bands v and w. We now examine the kinds of extensions that arise from singleton (single and double) maps $f^{\bullet} \colon Q_{\sigma}^{\bullet} \to \Sigma Q_{\tau}^{\bullet}$. We first note that singleton double maps never occur as morphisms between projective resolutions of modules.

Lemma 3.9. There are no singleton double maps $f^{\bullet}: Q_{\sigma}^{\bullet} \to \Sigma Q_{\tau}^{\bullet}$.

Proof. By definition, the unfolded diagram of a singleton double map has the form

$$Q_{\sigma}^{\bullet}: \qquad \stackrel{\beta}{\sim} \bullet \xrightarrow{\sigma_{L}} \bullet \xrightarrow{\sigma_{C} = f_{L} f'} \bullet \xrightarrow{\sigma_{R}} \bullet \stackrel{\alpha}{\sim} \\ \Sigma Q_{\tau}^{\bullet}: \qquad \stackrel{\delta}{\sim} \bullet \xrightarrow{\tau_{L}} \bullet \xrightarrow{\tau_{C} = f' f_{R}} \bullet \xrightarrow{\tau_{R}} \bullet \stackrel{\alpha}{\sim} \\ \stackrel{\gamma}{\sim} \sim \stackrel{\gamma}{\sim} \bullet \xrightarrow{\tau_{L}} \bullet \xrightarrow{\tau_{C} = f' f_{R}} \bullet \xrightarrow{\tau_{R}} \bullet \stackrel{\alpha}{\sim} \stackrel{\alpha}{\sim}$$

where f_L , f' and f_R are nontrivial. By Remark 2.15(5), length(σ_C) > 1 and length(τ_C) > 1. In particular, since σ is a homotopy string or band corresponding to a projective resolution, σ_C is a homotopy letter occurring between degrees -1 and 0. On the other hand, τ is also a homotopy string or band corresponding to a projective resolution, but $\Sigma Q_{\tau}^{\bullet}$ has been shifted, whence τ_C must be a homotopy letter occurring between degrees -2 and -1. Hence there are no such maps.

Recall the notation and unfolded diagram for a singleton single map $f^{\bullet}: Q_{\sigma}^{\bullet} \to \Sigma Q_{\tau}^{\bullet}$ from Section 1.4.2(5). Throughout this section, whenever $\sigma_R \neq \emptyset$ or $\tau_R \neq \emptyset$ in (5) we assume, without loss of generality, that $f_R \in Q_1$ and $f_L \in Q_1$, respectively.

Lemma 3.10. Suppose $f^{\bullet}: Q_{\sigma}^{\bullet} \to \Sigma Q_{\tau}^{\bullet}$ as a singleton single map with single component $f = f_n \cdots f_1$. Then the component f occurs in cohomological degree -1.

Proof. Suppose f^{\bullet} is supported in cohomological degree d. Since Q^{\bullet}_{τ} is a projective resolution, $\Sigma Q^{\bullet}_{\tau}$ attains its maximal degree in degree -1, thus $d \leq -1$. By Remark 2.15(5), if in (5) either $\sigma_R \neq \emptyset$ or $\tau_R \neq \emptyset$ then d = -1. So assume $\sigma_R, \tau_R = \emptyset$ and d < -1. By Corollary 2.12, since τ_L is the endpoint of a homotopy string occurring in degree d it must be inverse (otherwise there would be nontrivial cohomology in degree d, contradicting the fact that $\Sigma Q^{\bullet}_{\tau}$ is a (shifted) projective resolution). Moreover, for degree reasons, τ_L must be the first homotopy letter of inv(w). Writing $\tau_L = \bar{b}_l \cdots \bar{b}_1$ for some $b_i \in Q_1$, $i = 1, \ldots, l$, the definition of single maps gives us that $\bar{b}_1 \bar{f}_1 = 0$. This contradicts the fact that inv(w) is the longest inverse antipath incident with w' (see Corollary 2.12). Therefore, d = -1, as claimed.

Corollary 3.11. Suppose $f^{\bullet}: Q_{\sigma}^{\bullet} \to \Sigma Q_{\tau}^{\bullet}$ as a singleton single map. In the unfolded diagram (5), τ_L must be a direct homotopy letter or $\tau_L = \emptyset$.

Proof. Since $\Sigma Q_{\tau}^{\bullet}$ attains its maximal cohomological degree in degree -1 and f^{\bullet} is supported in degree -1 by Lemma 3.10, τ_L cannot be inverse.

Corollary 3.11 allows us to further specialise the setup in Section 1.4.2(5) in the statement of the next proposition.

Proposition 3.12. Suppose $f^{\bullet}: Q_{\sigma}^{\bullet} \to \Sigma Q_{\tau}^{\bullet}$ is a singleton single map with single component $f = f_n \cdots f_1$. Write $\tau_L = b_l \cdots b_1$ with $b_i \in Q_1$ for $i = 1, \dots, l$. Whenever σ_L is an inverse homotopy letter we shall write $\sigma_L = \bar{a}_k \cdots \bar{a}_1$, where $a_i \in Q_1$ for $i = 1, \dots, k$ with $k \geq 1$.

(1) If $\sigma_R = \emptyset$ then $\tau_R = \emptyset$ and σ_L is inverse. The corresponding extension $\Phi(f^{\bullet}) \in \operatorname{Ext}^1_{\Lambda}(M(\bar{v}), M(w))$ is an arrow extension given by a_1 , i.e. $\Phi(f^{\bullet})$ gives rise to an extension of M(w) by M(v) with middle term M(u) where $u = wa_1\bar{v}$.

Suppose $\sigma_R \neq \emptyset$. If σ_R is not incident with dir(v) then σ_L is inverse and we have:

(2) If $\tau_R = \emptyset$ then the corresponding extension $\Phi(f^{\bullet}) \in \operatorname{Ext}_{\Lambda}^1(M(\bar{v}), M(w))$ is an overlap extension whose middle term is given by

$$m = \bar{f}_1 \cdots \bar{f}_{n-1}, \ A = \varnothing, \ B = f_R, \ C = f_n \ and \ D = b_1$$
 if σ_R is incident with $\operatorname{dir}(v)$; $m = \bar{f}_1 \cdots \bar{f}_n, \ A = a_1, \ B = f_R, \ C = \varnothing \ and \ D = b_1$ otherwise.

(3) If $\tau_R = \emptyset$ then the corresponding extension $\Phi(f^{\bullet}) \in \operatorname{Ext}^1_{\Lambda}(M(\bar{v}), M(w))$ is an overlap extension which when σ_R is incident with $\operatorname{dir}(v)$ has its middle term given by,

$$m = \bar{f}_1 \cdots \bar{f}_{n-1}, \quad A = \emptyset, \quad B = f_R, \quad C = f_n \quad and \quad D = c_1 \qquad if \ \tau_L = \emptyset;$$

 $m = \bar{f}_1 \cdots \bar{f}_{n-1}, \quad A = \emptyset, \quad B = f_R, \quad C = f_n \quad and \quad D = b_1 \qquad if \ \tau_L \neq \emptyset,$

and is an overlap extension which when σ_R is not incident with dir(v) has its middle term given by,

$$m = \bar{f}_1 \cdots \bar{f}_n$$
, $A = a_1$, $B = f_R$, $C = \varnothing$ and $D = c_1$ if $\tau_L = \varnothing$; $m = \bar{f}_1 \cdots \bar{f}_n$, $A = a_1$, $B = f_R$, $C = \varnothing$ and $D = b_1$ if $\tau_L \neq \varnothing$.

Proof. (1) First note that σ_L is inverse, since if it were direct or empty Q_{σ}^{\bullet} would have nontrivial cohomology in degree -1, contradicting the fact that it is a projective resolution. Therefore, $\sigma_L = \bar{a}_k \cdots \bar{a}_1$ with $a_i \in Q_1$ for $i = 1, \ldots, k$ for some $k \geq 1$. Moreover, σ_L is the start of inv(v), for otherwise σ would start in degree 0 after the removal of a maximal inverse prefix. It follows that v starts with the inverse substring $\bar{a}_k \cdots \bar{a}_2$, whence \bar{v} ends with the direct substring $a_2 \cdots a_k$.

Consider the local subquiver of Q, where, without loss of generality, we assume $f_L \in Q_1$,



If $\tau_R \neq \emptyset$ then $\bar{a}_1 \bar{f}_L = 0$, contradicting the fact that σ_L is the start of inv(v). Thus, $\tau_R = \emptyset$.

Since f is not a subletter of τ_L or vice versa we must have $f_1 \neq b_1$ and $b_1\bar{f}_1$ is defined as a string. This means that a maximal inverse prefix, whose last (inverse) arrow is \bar{f}_1 , has been removed from w in the computation to $\tau = \pi(w)$ for otherwise $\tau_R \neq \varnothing$. We claim that \bar{f} is precisely the maximal inverse prefix that has been removed. Clearly, the maximal inverse prefix cannot be a proper substring of \bar{f} for the computation of $\tau = \pi(w)$ in Corollary 2.12 would require us to compose this with w giving $\tau_R \neq \varnothing$. However, if \bar{f} were a proper substring of the maximal inverse prefix then there would be an arrow $f_{n+1} \in Q_1$ such that $\bar{a}_1\bar{f}_{n+1} = 0$ giving us a contradiction as above. Therefore, w starts with the substring $\tau_L\bar{f}$. Applying [8, Thm. 5.2], Theorem 2.8 and a computation as in Lemma 3.3 shows that that $\Phi(f^{\bullet})$ gives an arrow extension corresponding to the arrow a_1 with middle term M(u), where $u = wa_1\bar{v}$.

Suppose that $\sigma_R \neq \emptyset$. If σ_R is not incident with $\operatorname{dir}(v)$ then by Remark 2.15(1), $\sigma_L \neq \emptyset$ and is inverse and we write $\sigma_L = \bar{a}_k \cdots \bar{a}_1$, where $a_i \in Q_1$ for $i = 1, \ldots, k$ with $k \geq 1$.

(2) Suppose that $\tau_R = \varnothing$. First observe that, by Corollary 2.12, w has a substring of the form $b_{l-1} \cdots b_1 \bar{f}_1 \cdots \bar{f}_n$. If σ_R is incident with $\operatorname{dir}(v)$ (in which case so is σ_L regardless of whether it is empty), then v ends with a substring $f_{n-1} \cdots f_1 f_R$, i.e. \bar{v} starts with a substring $\bar{f}_R \bar{f}_1 \cdots \bar{f}_{n-1}$, by Corollary 2.12. Applying [8, Thm. 5.2], taking cohomology using Theorem 2.8 and a calculation as in Lemma 3.3 then gives us an overlap extension between M(w) and $M(\bar{v})$:

$$m = \bar{f}_1 \cdots \bar{f}_{n-1}$$
, $A = \emptyset$, $B = f_R$, $C = f_n$ and $D = b_1$.

Now suppose σ_R is not incident with $\operatorname{dir}(v)$. By Corollary 2.12 that v has a substring $\sigma_L f_n \cdots f_1 f_R$, i.e. \bar{v} has a substring $\bar{f}_R \bar{f}_1 \cdots \bar{f}_n \bar{\sigma}_L$. Again applying [8, Thm. 5.2], Theorem 2.8 and a calculation as in Lemma 3.3 gives us the following overlap extension between M(w) and $M(\bar{v})$:

$$m = \bar{f}_1 \cdots \bar{f}_n$$
, $A = a_1$, $B = f_R$, $C = \emptyset$ and $D = b_1$.

(3) Suppose that $\tau_R \neq \emptyset$. First we assume σ_R is incident with $\operatorname{dir}(v)$, whence \bar{v} starts with the substring $\bar{f}_R \bar{f}_1 \cdots \bar{f}_{n-1}$ as above. If $\tau_L = \emptyset$, then, by Corollary 2.12, w ends with a substring $c_t \cdots c_1 \bar{f}_1 \cdots \bar{f}_n \bar{f}_L$, where $c_i \in Q_1$ for $i = 1, \dots, t$ and $t \geq 0$. In this case the application of [8, Thm. 5.2], Theorem 2.8 and a calculation as in Lemma 3.3 gives us the following overlap extension between M(w) and $M(\bar{v})$:

$$m = \bar{f}_1 \cdots \bar{f}_{n-1}$$
, $A = \emptyset$, $B = f_R$, $C = f_n$ and $D = c_1$.

If $\tau_L \neq \emptyset$, then w has a substring $b_{l-1} \cdots b_1 \bar{f}_1 \cdots \bar{f}_n \bar{f}_L$. Applying [8, Thm. 5.2] and Theorem 2.8 gives us the following overlap extension between M(w) and $M(\bar{v})$:

$$m = \bar{f}_1 \cdots \bar{f}_{n-1}$$
, $A = \emptyset$, $B = f_R$, $C = f_n$ and $D = b_1$.

Now assume that σ_R is not incident with $\operatorname{dir}(v)$, whence \bar{v} has a substring $\bar{f}_R \bar{f}_1 \cdots \bar{f}_n \bar{\sigma}_L$ as above. Using the calculations of substrings of w for $\tau_L = \varnothing$ and $\tau_L \neq \varnothing$ above respectively, and the application of [8, Thm. 5.2], Theorem 2.8 and a calculation as in Lemma 3.3 gives us the following overlap extensions between M(w) and $M(\bar{v})$:

$$m = \bar{f}_1 \cdots \bar{f}_n$$
, $A = a_1$, $B = f_R$, $C = \varnothing$ and $D = c_1$ if $\tau_L = \varnothing$; $m = \bar{f}_1 \cdots \bar{f}_n$, $A = a_1$, $B = f_R$, $C = \varnothing$ and $D = b_1$ if $\tau_L \neq \varnothing$. \square

4. Surjectivity of Φ onto overlap and arrow extensions

In this section, we use the combinatorics of an overlap or arrow extension to show that the isomorphism $\Phi \colon \operatorname{Hom}_{\mathsf{K}^{b,-}(\operatorname{proj}(\Lambda))}(Q^{\bullet}_{\pi(v)}, \Sigma Q^{\bullet}_{\pi(w)}) \to \operatorname{Ext}^{1}_{\Lambda}(M(v), M(w))$ restricts to a surjection,

$$\Phi \colon \left\{ \begin{array}{l} \text{standard basis elements of} \\ \operatorname{Hom}_{\mathsf{K}^{b,-}(\operatorname{proj}(\Lambda))}(Q_{\pi(v)}^{\bullet}, \Sigma Q_{\pi(w)}^{\bullet}) \end{array} \right\} \twoheadrightarrow \left\{ \begin{array}{l} \text{overlap and arrow extensions} \\ \eta \in \operatorname{Ext}_{\Lambda}^{1}(M(v), M(w)) \end{array} \right\}.$$

4.1. Overlap extensions. Throughout this section we shall have the following setup.

Setup 4.1. Let v and w be strings or bands and $\pi(v)$ and $\pi(w)$ be the corresponding homotopy strings or bands of their projective resolutions.

Suppose $0 \neq \eta \in \operatorname{Ext}_{\Lambda}^{1}(M(v), M(w))$ is an overlap extension corresponding to the decompositions $v = v_{L} \bar{B} m A v_{R}$ and $w = w_{L} D m \bar{C} w_{R}$. We consider m and decompose it into its homotopy letters $m = \mu_{l} \cdots \mu_{1}$ with $l \geq 0$. When l = 0, m is a trivial string, i.e. m = x for some $x \in Q_{0}$ and we call it a trivial overlap. If l = 1, we say m is a direct or inverse overlap. If l > 1, we say that m is a zigzag overlap.

4.1.1. Zigzag overlaps. We start with the zigzag overlap case.

Lemma 4.2. Suppose in Setup 4.1, the string m is a zigzag overlap. Then the map $f: M(w) \to M(v)$ associated with this decomposition induces a graph map $f^{\bullet}: Q_{\pi(w)}^{\bullet} \to Q_{\pi(v)}^{\bullet}$ of homotopy string or band complexes such that $H^{0}(f^{\bullet}) = f$.

Proof. We first show that the given decomposition induces a graph map $Q_{\pi(w)}^{\bullet} \to Q_{\pi(v)}^{\bullet}$. It is sufficient only to consider the endpoints of the map, as determined by the decomposition. We consider only the right endpoints; the analysis for left endpoints is analogous.

Before breaking the argument up into a case analysis, first note that one of A and \bar{C} must exist (i.e. be nonempty) since η is a non-split extension. By gentleness, if both A and \bar{C} exist we must have CA = 0.

Case: μ_1 is a direct homotopy letter.

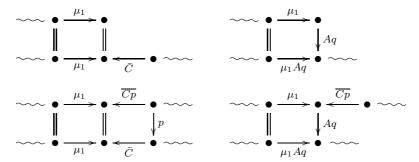
By Corollary 2.12, the homotopy string or band $\pi(w)$ has the following form:

$$\pi(w) = \begin{cases} & \sim & \stackrel{\mu_1}{\longrightarrow} \bullet & \text{if } \bar{C}w_R = \varnothing \text{ or is removed;} \\ & \sim & \stackrel{\mu_1}{\longrightarrow} \bullet & \stackrel{\overline{C}p}{\longleftarrow} \bullet \sim & \text{for some path } p \text{ in } (Q, I) \text{ otherwise.} \end{cases}$$

Similarly, the homotopy string or band $\pi(v)$ has the following form:

$$\pi(v) = \begin{cases} & & \xrightarrow{\mu_1} \bullet \xrightarrow{\bar{C}} \bullet & & \text{if } A = \emptyset; \\ & & & & \xrightarrow{\mu_1 Aq} \bullet & & & \text{for some path } q \text{ in } (Q, I) \text{ otherwise.} \end{cases}$$

Combining these, we get the following unfolded diagrams of graph map right endpoint conditions, showing the claim in this case.



Case: μ_1 is an inverse homotopy letter.

$$\pi(w) = \begin{cases} & \sim & \text{if } \bar{C} = \varnothing \text{ or } \bar{C}w_R \text{ is inverse, and there is no } a \in Q_1 \\ & \text{with } w\bar{a} \text{ defined as a string;} \end{cases}$$

$$\sim & \bullet \xrightarrow{\mu_1\bar{a}} \bullet \sim & \text{if } \bar{C} = \varnothing \text{ and } \exists a \in Q_1 \text{ with } \mu_1\bar{a} \text{ a string;}$$

$$\sim & \bullet \xrightarrow{\mu_1\bar{C}\bar{p}} \bullet \sim & \text{for some (possibly trivial) path } p \text{ in } (Q, I), \text{ otherwise,} \end{cases}$$

where the homotopy string in the first case starts with μ_2 if it exists, or a single projective or the start of an antipath otherwise. Similarly, the homotopy string or band $\pi(v)$ has the following form:

$$\pi(v) = \begin{cases} & \sim \sim \stackrel{\mu_1}{\longleftarrow} \bullet \xrightarrow{Aq} \bullet \sim \sim & \text{for some path } q \text{ in } (Q, I) \text{ if } A \neq \emptyset; \\ & \sim \sim \bullet \stackrel{\mu_1\bar{C}}{\longleftarrow} \bullet \sim \sim & \text{if } A = \emptyset. \end{cases}$$

We leave it to the reader to match up the various forms of the projective resolutions and confirm that they give rise to graph map right endpoint conditions as above.

Now examining the components of $f^{\bullet}: Q_{\pi(w)}^{\bullet} \to Q_{\pi(v)}^{\bullet}$ consisting of identity maps between indecomposable projective modules and following these maps through a calculation of the kind in Lemma 3.3 shows that the $H^0(f^{\bullet}) = f: M(w) \to M(v)$, i.e. f^{\bullet} is indeed induced from f.

Applying Remark 1.7 we get the following corollary.

Corollary 4.3. Keep the setup as in Lemma 4.2. The map $f: M(w) \to M(v)$ induces a quasi-graph map $\varphi: Q_{\pi(v)}^{\bullet} \leadsto Q_{\pi(w)}^{\bullet}$ of homotopy string or band complexes, and hence a homotopy family of maps $Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$.

Let $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$ be a representative of the homotopy family of single or double maps defined by the quasi-graph map $\varphi: Q_{\pi(v)}^{\bullet} \leadsto Q_{\pi(w)}^{\bullet}$ obtained in Corollary 4.3 above. Then, by Proposition 3.8 one obtains $\Phi(g^{\bullet}) = \eta$.

4.1.2. Direct or inverse overlaps. Here we consider the case of Setup 4.1 in which m a direct overlap; the case that m is an inverse overlap is analogous. As in previous sections $\sigma = \pi(v)$ and $\tau = \pi(w)$. Again, we use the combinatorics of the overlap to define a map $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$ such that $\Phi(g^{\bullet}) = \eta$. In this case, g^{\bullet} is either a singleton single map or a representative of a homotopy family of maps defined by a quasi-graph map $\varphi: Q_{\pi(v)}^{\bullet} \leadsto Q_{\pi(w)}^{\bullet}$.

In the following we do a detailed analysis of the different type of standard basis maps which are induced by the different possible forms the strings v and w can take. We present

the results by grouping the different cases giving rise to the same type of standard basis element in $\operatorname{Hom}_{\mathsf{K}^{b,-}(\operatorname{proj}(\Lambda))}(Q^{\bullet}_{\pi(v)}, \Sigma Q^{\bullet}_{\pi(w)})$.

Case: $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$ is a singleton single map.

The unfolded diagram of the singleton single map is one of the diagrams below; we explain in which cases they arise. In each case the precise description of τ_R is irrelevant, we note only that in each case it is necessarily empty or an inverse homotopy letter not containing m as a substring, or vice versa.

$$(I) \quad \sigma : \quad \sim \bullet \stackrel{\bar{q}\bar{B}}{\longleftrightarrow} \bullet \stackrel{mAp}{\longleftrightarrow} \bullet \quad \sim \quad (II) \quad \sigma : \quad \sim \bullet \stackrel{DmAp}{\longleftrightarrow} \bullet \quad \sim \quad \Sigma \tau : \quad \sim \bullet \stackrel{\tau_L}{\longleftrightarrow} \bullet \stackrel{\tau_R}{\longleftrightarrow} \bullet \quad \sim \quad (II) \quad \sigma : \quad \sim \bullet \stackrel{DmAp}{\longleftrightarrow} \bullet \quad \sim \quad (II) \quad \sigma : \quad \sim \bullet \stackrel{DmAp}{\longleftrightarrow} \bullet \quad \sim \quad (II) \quad \sigma : \quad \sim \bullet \stackrel{DmAp}{\longleftrightarrow} \bullet \quad \sim \quad (II) \quad \sigma : \quad \sim \bullet \stackrel{DmAp}{\longleftrightarrow} \bullet \quad \sim \quad (II) \quad \sigma : \quad \sim \bullet \stackrel{DmAp}{\longleftrightarrow} \bullet \quad \sim \quad (II) \quad \sigma : \quad \sim \bullet \stackrel{DmAp}{\longleftrightarrow} \bullet \quad \sim 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where p and q are (possibly trivial) paths in (Q, I). Diagram (I) occurs precisely when both $A \neq \emptyset$ and $\bar{B} \neq \emptyset$: the pertinent part of the projective resolution of M(v) has this form by Corollary 2.12. Now, applying Corollary 2.12 to w we see that,

$$\tau_L = \begin{cases} dm & \text{if } D = \varnothing \text{ but there exists } d \in Q_1 \text{ with } dm \text{ defined as a string;} \\ q'Dm & \text{for some (possibly trivial) path } q' \text{ in } (Q,I) \text{ if } D \neq \varnothing \text{ and } w_L \text{ is not direct or } w_LD \text{ is direct and there exists } d \in Q_1 \text{ with } dw \text{ defined as a string;} \\ \varnothing & \text{otherwise.} \end{cases}$$

Diagram (II) occurs in the case that $A \neq \emptyset$ but $\bar{B} = \emptyset$; in this case to avoid η being a split extension we must have $D \neq \emptyset$. In this case we have

$$\tau_L = \begin{cases} \varnothing & \text{if } w_L D \text{ is direct and } \nexists \, d \in Q_1 \text{ with } dw \text{ defined as a string;} \\ q' D m & \text{for some nontrivial path } q' \text{ in } (Q, I) \text{ if the first letter of } w_L \text{ is not inverse} \\ & \text{and we are not in the case above.} \end{cases}$$

Note that the case above when the first letter of w_L is inverse, we do not get a singleton single map, hence this case is included in this argument but is treated in the next case below. In each case it is straightforward to verify that the diagram defines a singleton single map. One now applies Proposition 3.12 to see that $\Phi(g^{\bullet}) = \eta$.

Case: $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$ is a representative of a homotopy family determined by a quasi-graph map $\varphi: Q_{\pi(v)}^{\bullet} \leadsto Q_{\pi(w)}^{\bullet}$.

We actually check that we get a graph map $f^{\bullet}: Q_{\pi(w)}^{\bullet} \to Q_{\pi(v)}^{\bullet}$ in the opposite direction and apply Remark 1.7.

In the case that $A \neq \emptyset$ but $\bar{B} = \emptyset$, and the first letter of w_L is inverse, i.e. the one case excluded in treating diagram (II) above, then we get the following graph map, in which p is some (possibly trivial) path in (Q, I).

$$\pi(w): \sim \sim \bullet \leftarrow \xrightarrow{Dm} \bullet \leftarrow \bullet \sim \sim$$

$$\pi(v): \sim \sim \bullet \xrightarrow{DmAp} \bullet \sim \sim$$

Now suppose $A = \emptyset$, whence $\bar{C} \neq \emptyset$. The overlap data gives rise a graph map with the following unfolded diagram,

$$\pi(w): \sim \sim \bullet \xrightarrow{f_L} \bullet \xrightarrow{\tau_L} \bullet \xrightarrow{\tau_R} \bullet \xrightarrow{\tau_0} \bullet \sim \sim$$

$$\pi(v): \sim \sim \bullet \xrightarrow{\sigma_L} \bullet \xrightarrow{\overline{C}} \bullet \sim \sim$$

where $\sigma_L = m$ in each case but the one specified and τ_L and τ_R are given by the following.

- $\tau_L = \emptyset$ and $f_L = \emptyset$ whenever $w_L D$ is direct (or empty) and there is no $d \in Q_1$ such that dw is defined as a string. In this case $\sigma_L = Dm$.
- $\tau_L = Dm$ and $f_L = D$ whenever the first letter of w_L is inverse.
- $\tau_L = qDm$ and $f_L = qD$ for some nontrivial path q in (Q, I) if the first letter of w_L is not inverse and, in the case that w_LD is direct, there is an arrow $d \in Q_1$ such that dw is defined as a string.
- $\tau_R = \emptyset$ and $f_R = \emptyset$ whenever $w_R = \emptyset$ or is inverse with no $c \in Q_1$ such that $\bar{C}w_R\bar{c}$ is defined as a string.
- $\tau_R = \bar{C}\bar{p}$ and $f_R = p$ for some (possibly trivial) path p in (Q, I) otherwise.

Note in the final case, if p is trivial then τ_0 necessarily exists and is direct, for otherwise \bar{C} would have been removed in the computation of $\pi(w)$. One now uses Proposition 3.8 to see that $\Phi(g^{\bullet}) = \eta$.

4.1.3. Trivial overlaps. We finally turn our attention to trivial overlaps. Suppose $m=1_x$ for some $x\in Q_0$. In this case, we fix the orientation of our strings and bands by requiring, whenever the relevant arrows exist, that $CB\neq 0$ and $DA\neq 0$. We again describe in each case how the combinatorics of the overlap can be used to construct a standard basis map $g^{\bullet}\colon Q_{\pi(v)}^{\bullet}\to Q_{\pi(w)}^{\bullet}$ such that $\Phi(g^{\bullet})=\eta$.

Case: $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$ is a graph map supported in one degree.

This is simply a degeneration of diagram (I) in the singleton single map case of Section 4.1.2, where instead $m=1_x$ for some vertex $x \in Q_0$, i.e. providing a graph map concentrated in one degree. Applying Lemma 3.4 we get $\Phi(g^{\bullet}) = \eta$.

Case: $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$ is a singleton single map.

If $A = \emptyset$ and $B \neq \emptyset$, in which case $\bar{C} \neq \emptyset$, then by Corollary 2.12, the homotopy string $\pi(v)$ has the form

$$\pi(v)$$
: $\sim \bullet \stackrel{\bar{q}\bar{B}\bar{C}}{\longleftarrow} \bullet \stackrel{\sigma_R}{\longleftarrow} \bullet \sim \sim$

where σ_R may be an empty homotopy letter. Similarly, the homotopy string $\pi(w)$ has the form

$$\pi(w): \sim \bullet \xrightarrow{\tau_L} \bullet \xleftarrow{\bar{C}\bar{p}} \bullet \xrightarrow{\tau_R} \bullet \sim ,$$

where p is a (possibly trivial) path in (Q, I), and τ_L and τ_R are possibly empty homotopy letters. The form of τ_L depends on the form of the substring $w_L D$, but is not relevant for the description of the map. In the case that p is nontrivial, we get a singleton single map, given by the following unfolded diagram.

$$\pi(v): \sim \sim \bullet \xrightarrow{\bar{q}\bar{B}\bar{C}} \bullet \xrightarrow{\sigma_R} \bullet \sim \sim \\ \Sigma \pi(w): \sim \sim \bullet \xrightarrow{\tau_L} \bullet \xrightarrow{\bar{C}\bar{p}} \bullet \xrightarrow{\tau_R} \bullet \sim \sim$$

The case that p is trivial gives rise to a quasi-graph map, which is dealt with below. There are obvious dual considerations when $A \neq \emptyset$ and $B = \emptyset$. Now apply Proposition 3.12.

Case: $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$ is a representative of a homotopy family determined by a quasi-graph map $\varphi: Q_{\pi(v)}^{\bullet} \leadsto Q_{\pi(w)}^{\bullet}$.

In the case that $A=\varnothing$ but $B\neq\varnothing$ above, in which the path p occurring in the homotopy string $\pi(w)$ is trivial, we must have that $\tau_R\neq\varnothing$ and is direct by Corollary 2.12. This gives rise to a graph map $f^{\bullet}Q_{\pi(w)}^{\bullet}\to Q_{\pi(v)}^{\bullet}$ given by the following unfolded diagram.

$$\pi(w): \sim \sim \bullet \xrightarrow{\tau_L} \bullet \xleftarrow{\bar{C}} \bullet \xrightarrow{\tau_R} \bullet \sim \sim$$

$$\pi(v): \sim \sim \bullet \xleftarrow{\bar{B}\bar{C}} \bullet \xleftarrow{\sigma_R} \bullet \sim \sim$$

By Remark 1.7, this gives rise to the quasi-graph map $\varphi \colon Q_{\pi(v)}^{\bullet} \leadsto Q_{\pi(w)}^{\bullet}$, as claimed. Indeed, one can see that the map given in the unfolded diagram above is one member of the homotopy family determined by φ . Dual considerations apply for the case $A \neq \emptyset$ and $B^{\bullet} = \emptyset$.

Finally, the case $A=\varnothing$ and $B=\varnothing$ gives rise to a graph map $f^{\bullet}\colon Q_{\pi(w)}^{\bullet}\to Q_{\pi(v)}^{\bullet}$, whence a quasi-graph map $\varphi\colon Q_{\pi(v)}^{\bullet}\leadsto Q_{\pi(w)}^{\bullet}$ by Remark 1.7. Note that, necessarily, $C\neq\varnothing$ and $D\neq\varnothing$. In this case, by Corollary 2.12, $\pi(v)$ has the form,

$$\pi(v)$$
: $\sim \bullet \xrightarrow{\sigma_L} \bullet \xrightarrow{D} x \xleftarrow{\bar{C}} \bullet \xleftarrow{\sigma_R} \bullet \sim \sim$,

in which the homotopy letters σ_L and σ_R may be empty. The homotopy string $\pi(w)$ has one of the following four forms

$$\sim \sim \bullet \xrightarrow{\tau_L} \bullet \xrightarrow{qD} x \xleftarrow{\bar{C}\bar{p}} \bullet \xrightarrow{\tau_R} \bullet \sim \sim \qquad x$$

$$\sim \sim \bullet \xrightarrow{\tau_L} \bullet \xrightarrow{qD} x \qquad \qquad x \xleftarrow{\bar{C}\bar{p}} \bullet \xrightarrow{\tau_R} \bullet \sim \sim \sim$$

where p and q are (possibly trivial) paths in (Q, I). Whenever p is trivial $\tau_R \neq \emptyset$ and is direct; whenever q is trivial $\tau_L \neq \emptyset$ and is inverse. The graph map $f^{\bullet}Q_{\pi(w)}^{\bullet} \to Q_{\pi(v)}^{\bullet}$ can be read off from the following unfolded diagrams, interpreting p and q as trivial paths (whence isomorphisms) and deleting homotopy letters as appropriate to fit the cases.

$$\pi(w): \sim \sim \bullet \xrightarrow{\tau_L} \bullet \xrightarrow{qD} x \xleftarrow{\bar{C}\bar{p}} \bullet \xrightarrow{\tau_R} \bullet \sim \sim$$

$$\pi(v): \sim \sim \bullet \xrightarrow{\sigma_L} \bullet \xrightarrow{D} x \xleftarrow{\bar{C}} \bullet \xrightarrow{\bar{C}\bar{p}} \bullet \sim \sim$$

As above, we apply Proposition 3.8 to get $\Phi(q^{\bullet}) = \eta$.

4.2. **Arrow Extensions.** Let $v = v_m \cdots v_1$ and $w = w_n \cdots w_1$ where $v_i, w_i \in Q_1 \cup \bar{Q}_1$. Suppose that $\eta \in \operatorname{Ext}^1_\Lambda(M(v), M(w))$ is an arrow extension corresponding to an arrow $a \in Q_1$, i.e. η corresponds to an extension with M(u) as the middle term where u = wav.

Since we know av is defined as a string, then we are in case (1) or (3) in Corollary 2.12 so that $\pi(v) = \operatorname{dir}(a)\widetilde{v}$, where $\widetilde{v} = v'\operatorname{inv}(b)$ for some $\overline{b} \in Q_1$ or $\widetilde{v} = v'$ depending on whether we fall into case (1) or (3), respectively. We set $\operatorname{dir}(a) = \cdots \theta_2 \theta_1 a$. Likewise,

$$\pi(w) = \begin{cases} \widetilde{w} \operatorname{inv}(c) & \text{if there exists } c \in Q_1 \text{ such that } w_1 \bar{c} \text{ is defined as a string;} \\ \widetilde{w} & \text{otherwise,} \end{cases}$$

where \widetilde{w} is defined in a manner analogous to \widetilde{v} , depending on considerations at its end. We write $\operatorname{inv}(c) = \overline{c}\overline{\varphi}_1 \cdots \overline{\varphi}_2 \cdots$.

The form of the map $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to Q_{\pi(w)}^{\bullet}$ such that $\Phi(g^{\bullet}) = \eta$ depends on whether v ends with an inverse or direct letter and w starts with an inverse or direct letter. We deal with the cases in turn.

Case: $w_1 \in Q_1 \text{ and } v_m \in \bar{Q}_1.$

If \widetilde{w} inv(c) is defined, then we get the unfolded diagram of a (one-sided) graph map, $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$, below, where we have used $\overline{\pi(w)}$ in the diagram.

$$Q_{\pi(v)}^{\bullet}: \qquad \qquad \stackrel{\theta_2}{\longrightarrow} \bullet \xrightarrow{\theta_1} \bullet \xrightarrow{a} \bullet \xrightarrow{v_m \cdots v_i} \sim \\ g^{\bullet} \downarrow \qquad \qquad \qquad \parallel \qquad \parallel \qquad \parallel \\ \Sigma Q_{\pi(w)}^{\bullet}: \qquad \qquad \stackrel{\bullet}{\longrightarrow} \bullet \xrightarrow{\varphi_1} \bullet \xrightarrow{c=\theta_1} \bullet \xrightarrow{\bar{w}_1 \cdots \bar{w}_i} \bullet \sim \sim$$

Since w_1a is defined as a string, we have $w_1a \notin I$, whence $c = \theta_1$ by gentleness. Continuing, we see that $\varphi_i = \theta_{i+1}$ for each i > 1. Applying Lemma 3.3 one verifies that $\Phi(g^{\bullet}) = \eta$.

If \widetilde{w} inv(c) is not defined, then we get the following unfolded diagram of a (one-sided) graph map supported in one degree, $g^{\bullet} \colon Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$; applying Lemma 3.3 shows $\Phi(g^{\bullet}) = \eta$.

Note that since $w_1 a \notin I$ then $\theta_1 = \emptyset$ (i.e. dir(a) = a) because otherwise θ_1 would provide such a c by gentleness of Λ .

Case: $w_1 \in Q_1 \text{ and } v_m \in Q_1.$

By the same argument as above, we have one of the following unfolded diagram of a (one-sided) graph map, $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$, depending on whether \widetilde{w} inv(c) is defined. In both cases, one then applies Lemma 3.3.

$$Q_{\pi(v)}^{\bullet}: \qquad \sim \bullet \xrightarrow{\theta_{2}} \bullet \xrightarrow{\theta_{1}} \bullet \xrightarrow{av_{m} \cdots v_{i}} \bullet \sim \sim \quad \text{or} \quad Q_{\pi(v)}^{\bullet}: \qquad \bullet \xrightarrow{av_{m} \cdots v_{i}} \bullet \sim \sim \\ g^{\bullet} \downarrow \qquad \qquad \parallel \qquad \parallel \qquad \parallel \qquad \qquad \qquad \qquad g^{\bullet} \downarrow \qquad \qquad \parallel \\ \Sigma Q_{\pi(w)}^{\bullet}: \qquad \sim \bullet \xrightarrow{\varphi_{1}} \bullet \xrightarrow{c=\theta_{1}} \bullet \xrightarrow{\bar{w}_{1} \cdots \bar{w}_{i}} \bullet \sim \sim \qquad \qquad \Sigma Q_{\pi(w)}^{\bullet}: \qquad \bullet \xrightarrow{\bar{w}_{1} \cdots \bar{w}_{i}} \bullet \sim \sim$$

Case: $w_1 \in \bar{Q}_1$ and $v_m \in \bar{Q}_1$.

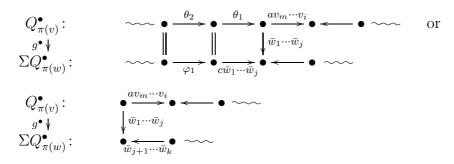
Suppose \widetilde{w} inv(c) is defined. Since $\theta_1 a \in I$ we have that $\theta_1 \overline{w}_1$ is a string and $c = \theta_1$ is the unique arrow such that $c\overline{w}_1 \notin I$. Continuing we have $\varphi_i = \theta_{i+1}$ for $i \geq 1$. This gives the following unfolded diagram of a (one-sided) graph map, $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$; now apply Lemma 3.3 again.

If \widetilde{w} inv(c) is not defined, then suppose $w_j \cdots w_1$ is the maximal inverse substring starting w, in particular, \widetilde{w} starts with w_{j+1} is either direct or empty. Furthermore, $\theta_1 = \emptyset$ for otherwise $w_1\overline{\theta}_1$ would be defined as a string and we could take $c = \theta_1$. Hence we get the following unfolded diagram of a singleton single map $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$ and we apply Proposition 3.12.

$$Q_{\pi(v)}^{\bullet}: \qquad \xrightarrow{a} \bullet \stackrel{v_m \cdots v_i}{\longleftarrow} \bullet \sim \sim \\ g^{\bullet} \lor \qquad \qquad \downarrow \bar{w}_1 \cdots \bar{w}_j \\ \Sigma Q_{\pi(w)}^{\bullet}: \qquad \xrightarrow{\bar{w}_{i+1} \cdots \bar{w}_k} \bullet \sim \sim$$

Case: $w_1 \in \bar{Q}_1 \text{ and } v_m \in Q_1.$

Arguing as above, we get the following unfolded diagram of a (one-sided) graph map or a singleton single map, $g^{\bullet}: Q_{\pi(v)}^{\bullet} \to \Sigma Q_{\pi(w)}^{\bullet}$, when \widetilde{w} inv(c) is defined and when it is not, respectively. One then applies Lemma 3.3 or Proposition 3.12, respectively.



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