

The Ricci flow on domains in cohomogeneity one manifolds

Artem Pulemotov*

a.pulemotov@uq.edu.au

January 9, 2018

Abstract

Suppose G is a compact Lie group, H is a closed subgroup of G , and the homogeneous space G/H is connected. The paper investigates the Ricci flow on a manifold M diffeomorphic to $[0, 1] \times G/H$. First, we prove a short-time existence and uniqueness theorem for a G -invariant solution $g(t)$ satisfying the boundary condition $\text{II}(g(t)) = F(t, g_{\partial M}(t))$ and the initial condition $g(0) = \hat{g}$. Here, $\text{II}(g(t))$ is the second fundamental form of ∂M , $g_{\partial M}$ is the metric induced on ∂M by $g(t)$, F is a smooth map and \hat{g} is a metric on M . Second, we study Perelman's \mathcal{F} -functional on M . Our results show, roughly speaking, that \mathcal{F} is non-decreasing on a G -invariant solution to the modified Ricci flow, provided that this solution satisfies boundary conditions inspired by the 2012 paper of Gianniotis.

1 Introduction

Developing the theory of the Ricci flow on manifolds with boundary is a long-standing open problem with numerous potential applications. The present paper addresses several aspects of this problem in the setting of spaces with symmetries. We focus on the short-time existence and the uniqueness of solutions, as well as the monotonicity of Perelman's \mathcal{F} -functional. Before we describe our results, let us review the history of the subject.

Suppose M is a d -dimensional manifold with $d \geq 3$. The Ricci flow on M is the partial differential equation

$$\frac{\partial}{\partial t} g(t) = -2 \text{Ric}(g(t)) \quad (1.1)$$

for a Riemannian metric $g(t)$ depending on the parameter $t \geq 0$. It is customary to interpret t as time. In the right-hand side, $\text{Ric}(g(t))$ stands for the Ricci curvature of $g(t)$. Given a Riemannian metric \hat{g} on M , we supplement (1.1) with the initial condition

$$g(0) = \hat{g}. \quad (1.2)$$

The Ricci flow on manifolds without boundary has been widely studied. The reader will find a wealth of information about it in the books [6, 27, 21].

Assume the manifold M is compact and ∂M is nonempty. When trying to develop the theory of the Ricci flow on M in this case, one faces a number of major roadblocks. For instance, it is necessary to find boundary conditions for equation (1.1) that would allow tractable analysis and admit a meaningful geometric interpretation. Doing so is difficult, as equation (1.1) is only weakly parabolic; see [27, Section 5.1], [13, Introduction] and also [2, Section 3]. Note that the Ricci flow on *surfaces* with boundary appears to be more approachable. For results in this area, consult the references in [13]. However, the higher-dimensional setting considered in the present paper encompasses a different set of difficulties and requires different techniques.

Initial progress regarding the Ricci flow on M under the above assumptions was made by Shen in [25, 26]. Let us briefly describe it. Suppose $\text{II}(g(t))$ is the second fundamental form of ∂M with respect to $g(t)$, $g_{\partial M}(t)$

*School of Mathematics and Physics, The University of Queensland, St Lucia, QLD 4072, Australia

is the metric induced on ∂M by $g(t)$, and λ is a fixed real number. Shen considered equation (1.1) subject to the boundary condition

$$\Pi(g(t)) = \lambda g_{\partial M}(t) \quad (1.3)$$

and the initial condition (1.2). He outlined the proof of a short-time existence theorem. He also obtained a formula for the t -derivative of $\Pi(g(t))$. Despite this success, further investigation of problem (1.1)–(1.3)–(1.2) turned out to be complicated. As of today, no uniqueness theorem for this problem is found in the literature. The long-time behaviour of solutions was investigated in [25, 26, 5, 8], but it is not yet deeply understood. In fact, the majority of available results concern the special case where $\lambda = 0$ (i.e., the boundary is totally geodesic). We invite the reader to see [3] for a discussion about letting the parameter λ depend on t . Aside from the discussion, that paper contains two gradient estimates for the heat equation under (1.1)–(1.3).

Further progress in the study of the Ricci flow on M was made by the author in [22]. More precisely, suppose $\mathcal{H}(g(t))$ is the mean curvature of ∂M with respect to $g(t)$. Fix a function $b : [0, \infty) \rightarrow \mathbb{R}$. The main result of [22] is a short-time existence theorem for solutions to (1.1) under the boundary condition

$$\mathcal{H}(g(t)) = b(t) \quad (1.4)$$

and the initial condition (1.2). This result was improved by Gianniotis in the paper [13]. More specifically, choose a t -dependent Riemannian metric $\eta(t)$ and a t -dependent real-valued function $\kappa(t)$ on ∂M . Gianniotis considered the Ricci flow subject to the boundary conditions

$$[g_{\partial M}(t)] = [\eta(t)], \quad \mathcal{H}(g(t)) = \kappa(t). \quad (1.5)$$

The square brackets here denote the conformal class. We emphasize that, in contrast with (1.4), formulas (1.5) allow the mean curvature $\mathcal{H}(g(t))$ to be nonconstant on ∂M . The reasoning in [13] yielded the short-time existence and the uniqueness of solutions to (1.1)–(1.5)–(1.2) under natural assumptions. At the same time, describing the behaviour of these solutions for large t remains a challenging open problem. In the recent work [14], Gianniotis made progress towards the resolution of this problem by producing several interesting estimates. However, a comprehensive long-time existence theorem is still out of reach. Note that Gianniotis's results were largely inspired by Anderson's work on the Einstein equation; see [2].

Another direction in the study of the Ricci flow on manifolds with boundary is the analysis of Perelman's \mathcal{F} -functional. The reader may consult, e.g., [27, Chapter 6] for the definition and the key properties of \mathcal{F} . Given a t -dependent Riemannian metric $g(t)$ and a t -dependent real-valued function $p(t)$ on M , it is well-known that the expression

$$\mathcal{F}(g(t), p(t)) = \int_M (R(g(t)) + |\nabla p(t)|^2) e^{-p(t)} d\mu$$

would be non-decreasing in t if ∂M were empty and the pair $(g(t), p(t))$ satisfied

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2(\text{Ric}(g(t)) + \text{Hess } p(t)), \\ \frac{\partial}{\partial t} p(t) &= -\Delta p(t) - R(g(t)). \end{aligned} \quad (1.6)$$

(In the equalities above, R and μ denote the scalar curvature and the volume measure.) In fact, one would be able to interpret system (1.6) as the gradient flow of \mathcal{F} . The metric $g(t)$ would be the pullback of a solution to (1.1) by a t -dependent diffeomorphism. Monotonicity properties of \mathcal{F} are substantially harder to discover when $\partial M \neq \emptyset$. Lott's paper [19] provides several formulas for $\frac{d}{dt} \mathcal{F}(g(t), p(t))$ assuming the pair $(g(t), p(t))$ satisfies (1.6) and, after appropriate diffeomorphisms are performed, ∂M evolves under the mean curvature flow. The works [7, 8, 9] contain related computations. However, none of the results in [7, 8, 9, 19] asserts that $\mathcal{F}(g(t), p(t))$ is non-decreasing.

The present paper focuses on boundary-value problems for the Ricci flow (1.1) under the assumption

$$M \simeq [0, 1] \times G/H, \quad (1.7)$$

where G is a compact Lie group, H is a closed subgroup of G , and G/H is connected. In a sense, equality (1.7) means M possesses axial symmetry. The boundary of M has two connected components. Spaces of the form (1.7) arise as (closures of) domains on cohomogeneity one manifolds. Recently, the author used them in the study of the prescribed Ricci curvature equation; see [24] and also [23]. It is worth mentioning that cohomogeneity one manifolds enjoy numerous applications in geometry and mathematical physics. In particular, they have been used to construct important examples of Einstein metrics; see, e.g., [10] and references therein. They were effectively employed in the paper [12] to investigate Ricci solitons. For more information on the basic properties and applications of cohomogeneity one manifolds, consult [16].

The literature devoted to the Ricci flow on spaces with symmetries is rather extensive. The papers [18, 4] are two examples of recent works on the subject. The introduction to [20] contains a survey of what is known in three dimensions. The vast majority of existing works, however, only consider manifolds without boundary.

Let us describe our results. In what follows, we assume M has the form (1.7) and the isotropy representation of G/H splits into pairwise inequivalent irreducible summands. The latter assumption is quite standard in the theory of cohomogeneity one manifolds; we will discuss it in detail before stating our first theorem. Section 2 considers the Ricci flow (1.1) on M subject to the boundary condition

$$\Pi(g(t)) = F(t, g_{\partial M}(t)). \quad (1.8)$$

The letter F here denotes a map with values in the space of symmetric G -invariant $(0,2)$ -tensor fields on ∂M . When $F(t, g_{\partial M}(t)) = \lambda g_{\partial M}(t)$, formula (1.8) becomes Shen's boundary condition (1.3). In Section 2, we establish the short-time existence and the uniqueness of G -invariant solutions to problem (1.1)–(1.8)–(1.2) assuming (1.8) holds at $t = 0$ and \hat{g} is G -invariant. The author intends to study the behaviour of these solutions for large t in subsequent papers. Note that, until now, (1.5) has been the only boundary condition known to guarantee both the short-time existence and the uniqueness for the Ricci flow with given initial data in dimensions three or higher.

Section 3 studies the Perelman \mathcal{F} -functional on M . We begin with an examination of system (1.6) subject to the boundary conditions (1.5) on $g(t)$ and the Neumann condition $\frac{\partial}{\partial \nu} p(t) = 0$ on $p(t)$. We first prove a short-time existence theorem for G -invariant solutions. Next, we show that $\mathcal{F}(g(t), p(t))$ is non-decreasing if $(g(t), p(t))$ is such a solution, $\eta(t)$ is independent of t , and $\kappa(t)$ is identically 0. In the process, we obtain a new formula for the t -derivative of \mathcal{F} under (1.6). The section ends with a discussion of how our results relate to those of [19].

2 Short-time existence and uniqueness

Consider a compact Lie group G and closed subgroup H of G . Suppose the homogeneous space G/H is connected and $(d-1)$ -dimensional with $d \geq 3$. The objective of this paper is to investigate the Ricci flow on a smooth manifold M diffeomorphic to $[0, 1] \times G/H$. It will be convenient for us to assume that

$$M = [0, 1] \times G/H.$$

Such an assumption does not lead to any loss of generality. Obviously, the manifold M has nonempty boundary ∂M consisting of two connected components, $\{0\} \times G/H$ and $\{1\} \times G/H$. We will use the notation M_0 for the interior $M \setminus \partial M$. The group G acts naturally on M .

2.1 The existence and uniqueness theorem

The Ricci flow is the partial differential equation

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}(g(t)) \quad (2.1)$$

for a Riemannian metric $g(t)$ on M depending on the parameter $t \geq 0$. In the right-hand side, $\operatorname{Ric}(g(t))$ stands for the Ricci curvature of $g(t)$. As we explained in the introduction, one may learn about the history, the intuitive meaning, the technical peculiarities and the geometric applications of equation (2.1) from many books, such as [6, 27, 21].

Suppose $T^*\partial M \hat{\otimes} T^*\partial M$ is the bundle of symmetric $(0, 2)$ -tensors on ∂M . Consider a smooth map

$$F : [0, \infty) \times (T^*\partial M \hat{\otimes} T^*\partial M) \rightarrow T^*\partial M \hat{\otimes} T^*\partial M$$

such that $F(t, \cdot)$ is fiber-preserving for all $t \in [0, \infty)$. We will use F to supplement equation (2.1) with boundary conditions. Before we can do so, however, we need to make some preparations. Namely, suppose $\Gamma(T^*\partial M \hat{\otimes} T^*\partial M)$ is the space of continuous sections of $T^*\partial M \hat{\otimes} T^*\partial M$. Observe that F induces a map from $[0, \infty) \times \Gamma(T^*\partial M \hat{\otimes} T^*\partial M)$ to $\Gamma(T^*\partial M \hat{\otimes} T^*\partial M)$. It will be convenient for us to use the same letter F for this map. We assume the images of G -invariant sections of $T^*\partial M \hat{\otimes} T^*\partial M$ under $F(t, \cdot)$ are themselves G -invariant for all t .

Let $\text{II}(g(t))$ be the second fundamental form of ∂M computed in $g(t)$ with respect to the outward unit normal. Thus, $\text{II}(g(t))$ is a t -dependent $(0, 2)$ -tensor field on ∂M . Our sign convention is such that $\text{II}(g(t))$ is positive-definite when M is a closed ball in \mathbb{R}^3 and $g(t)$ is Euclidean. Suppose $g_{\partial M}(t)$ is the Riemannian metric induced on ∂M by $g(t)$. In this section, we study the Ricci flow equation (2.1) under the boundary condition

$$\text{II}(g(t)) = F(t, g_{\partial M}(t)). \quad (2.2)$$

Note that Y. Shen's works [25, 26] investigated the situation where $F(t, g_{\partial M}(t)) = \lambda g_{\partial M}(t)$ for some $\lambda \in \mathbb{R}$. The arguments from [25, 26] also apply when λ is allowed to depend on t .

Fix a smooth G -invariant Riemannian metric \hat{g} on M . We supplement the Ricci flow equation (2.1) with the initial condition

$$g(0) = \hat{g}. \quad (2.3)$$

Our objective in this section is to prove the short-time existence and the uniqueness of solutions to problem (2.1)–(2.2)–(2.3). Before we can state our result, however, we need to impose an assumption on the homogeneous space G/H .

Let \mathfrak{g} be the Lie algebra of the group G . Pick an $\text{Ad}(G)$ -invariant scalar product Q on \mathfrak{g} . Suppose \mathfrak{p} is the orthogonal complement of the Lie algebra of H in \mathfrak{g} with respect to Q . We standardly identify \mathfrak{p} with the tangent space of G/H at H . The isotropy representation of G/H then yields the structure of an H -module on \mathfrak{p} . We assume the following property of \mathfrak{p} throughout the paper.

Hypothesis 2.1. *The H -module \mathfrak{p} appears as an orthogonal sum*

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_n \quad (2.4)$$

of pairwise non-isomorphic irreducible H -modules $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.

Hypothesis 2.1 is rather standard. It has come up in a number of papers, such as [11, 12]. Roughly speaking, it ensures that G -invariant $(0, 2)$ -tensor fields on G/H are diagonal. Indeed, every such tensor field is determined by its restriction to \mathfrak{p} . Because the summands $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ in (2.4) are non-isomorphic and irreducible, this restriction must be diagonal with respect to (2.4). For a slightly more detailed discussion of Hypothesis 2.1, including a possible alternative to it, see the author's work [24].

Theorem 2.2. *Suppose*

$$\text{II}(\hat{g}) = F(0, \hat{g}_{\partial M}). \quad (2.5)$$

*For some number $T > 0$, there exists $g : M \times [0, T) \rightarrow T^*M \otimes T^*M$ such that the following statements hold:*

1. *The map g is smooth on $M_0 \times (0, T)$ and continuous on $M \times [0, T)$. Suppose X and Y are G -invariant vector fields on M tangent to $\{r\} \times G/H$ for each $r \in [0, 1]$. If X and Y are smooth, then the derivative of the map $M \times [0, T) \ni (x, t) \mapsto g(x, t)(X, Y) \in \mathbb{R}$ in the variable x exists and is continuous on $M \times [0, T)$.*
2. *For every $x \in M$ and $t \in [0, T)$, the tensor $g(x, t)$ is a symmetric positive-definite tensor at the point x . Thus, $g(\cdot, t)$ is a Riemannian metric on M . We will use the notation $g(t)$ for this metric.*

3. The equality $g(t) = \gamma^*g(t)$ holds for all $\gamma \in G$ and $t \in [0, T)$. In other words, $g(t)$ is G -invariant.
4. The t -dependent Riemannian metric $g(t)$ solves equation (2.1) on $M_0 \times (0, T)$. This metric satisfies the boundary condition (2.2) on $\partial M \times (0, T)$ and the initial condition (2.3) on M .

If, for some number $T > 0$, two smooth maps $g_1, g_2 : M \times [0, T) \rightarrow T^*M \otimes T^*M$ possess the above properties 2, 3 and 4, then $g_1 = g_2$.

Remark 2.3. One may be able to improve the differentiability of g near $\partial M \times [0, T)$ by imposing higher-order compatibility conditions on F and \hat{g} ; cf. [13, Section 5]. We will not discuss this further in the present paper.

Remark 2.4. It is convenient for us to assume that the maps g_1 and g_2 are smooth on $M \times [0, T)$. However, this assumption can be relaxed. It suffices to demand, for example, that the following two statements hold:

1. The maps g_1 and g_2 are smooth on $M_0 \times (0, T)$.
2. The derivatives $\frac{\partial^i}{\partial t^i} \hat{\nabla}^j g_1$ and $\frac{\partial^i}{\partial t^i} \hat{\nabla}^j g_2$ exist and are continuous on $M \times [0, T)$ when $2i + j \leq 3$.

Here, $\hat{\nabla}$ denotes covariant differentiation with respect to \hat{g} . The details are left to the reader; cf. [13, Theorem 1.3].

2.2 Three lemmas

Before we can prove Theorem 2.2, we need to make some preparations and state three lemmas. Note that the material laid out here will also be essential to the arguments in Section 3. Let us begin by fixing a geodesic $\alpha : [0, 1] \rightarrow M$ with respect to the metric \hat{g} . We choose α so that it intersects all the G -orbits on M orthogonally and $\alpha(r)$ lies in $\{r\} \times G/H$ for all $r \in [0, 1]$. The map $\Theta : M \rightarrow M$ given by the formula $\Theta(r, \gamma H) = \gamma \alpha(r)$ is a diffeomorphism. The equality

$$\Theta^* \hat{g} = \hat{h}^2(r) dr \otimes dr + \hat{g}^r, \quad r \in [0, 1],$$

holds true. In the right hand side, $\hat{h} : [0, 1] \rightarrow (0, \infty)$ is a smooth function, and \hat{g}^r is a G -invariant Riemannian metric on G/H for every $r \in [0, 1]$. Note that \hat{g}^r is fully determined by its restriction to \mathfrak{p} . Hypothesis 2.1 implies the existence of smooth functions $\hat{f}_1, \dots, \hat{f}_n : [0, 1] \rightarrow (0, \infty)$ such that

$$\hat{g}^r(X, Y) = \hat{f}_1^2(r) Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + \hat{f}_n^2(r) Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}.$$

The notation $\text{pr}_{\mathfrak{p}_k} X$ and $\text{pr}_{\mathfrak{p}_k} Y$ here stands for the projection of X and Y onto \mathfrak{p}_k for $k = 1, \dots, n$. In what follows, we assume that the diffeomorphism Θ is the identity map on M . This assumption does not lead to loss of generality. Thus, the equality

$$\hat{g} = \hat{h}^2(r) dr \otimes dr + \hat{g}^r, \quad r \in [0, 1], \tag{2.6}$$

holds true.

Our first lemma essentially shows that any G -invariant solution to (2.1), subject to the initial condition (2.3), must have the form (2.6). This fact is crucial to the proof of Theorem 2.2. It is also important to the arguments in Section 3.

Lemma 2.5. Assume $w : M_0 \times [0, T) \rightarrow T^*M \otimes T^*M$ is a smooth map satisfying the following requirements:

1. For every $x \in M$ and $t \in [0, T)$, the tensor $w(x, t)$ is a symmetric positive-definite tensor at the point x .
2. Given $\gamma \in G$ and $t \in [0, T)$, the Riemannian metric $w(t) = w(\cdot, t)$ satisfies the formula $w(t) = \gamma^* w(t)$.
3. The equality

$$\frac{\partial}{\partial t} w(t) = -2 \text{Ric}(w(t))$$

holds on $M_0 \times (0, T)$, and the equality

$$w(0) = \hat{g}$$

holds on M_0 .

Then

$$w(t) = z^2(r, t) dr \otimes dr + w^r(t), \quad r \in (0, 1), t \in [0, T).$$

In the right-hand side, z is a function on $(0, 1) \times [0, T)$ with positive values, and $w^r(t)$ is a G -invariant Riemannian metric on G/H for each $r \in (0, 1)$ and $t \in [0, T)$.

Remark 2.6. Let us emphasize that Lemma 2.5 does not require any boundary conditions on $w(t)$.

Proof. Fix $r_0 \in (0, 1)$. Given $t \in [0, T)$, suppose $v(t)$ is a unit normal to $\{r_0\} \times G/H$ at (r_0, H) with respect to the metric $w(t)$. We assume $v(t)$ points in the direction of $\{1\} \times G/H$. Our plan is to show that $v(t)$ is a scalar multiple of $v(0)$. The assertion of the lemma will follow immediately.

Let (y_1, \dots, y_d) be a local coordinate system on M centred at (r_0, H) . Assume that, at (r_0, H) , the vectors $\frac{\partial}{\partial y_i}$ are tangent to $\{r_0\} \times G/H$ for $i = 1, \dots, d-1$, and $\frac{\partial}{\partial y_d}$ coincides with $v(t)$. The formula

$$v(\tau) = \sum_{i=1}^d \frac{w^{id}(\tau)}{(w^{dd}(\tau))^{\frac{1}{2}}} \frac{\partial}{\partial y_i}, \quad \tau \in [0, T),$$

holds true. In the right-hand side, $w^{id}(\tau)$ are the components of the inverse of $w(\tau)$ at (r_0, H) in the coordinates (y_1, \dots, y_d) . Taking advantage of assumption 3, we find

$$\begin{aligned} \frac{d}{d\tau} v(\tau)|_{\tau=t} &= \sum_{i,j,l=1}^d \left(\frac{2W_{jl}(t)w^{ji}(t)w^{dl}(t)}{(w^{dd}(t))^{\frac{1}{2}}} - \frac{W_{jl}(t)w^{jd}(t)w^{dl}(t)w^{id}(t)}{(w^{dd}(t))^{\frac{3}{2}}} \right) \frac{\partial}{\partial y_i} \\ &= - \sum_{i=1}^d \left(W_{dd}(t)w^{id}(t)(w^{dd}(t))^{\frac{1}{2}} - 2 \sum_{j=1}^d W_{jd}(t)w^{ji}(t)(w^{dd}(t))^{\frac{1}{2}} \right) \frac{\partial}{\partial y_i} \\ &= W_{dd}(t)(w^{dd}(t))^{\frac{3}{2}} \frac{\partial}{\partial y_d} = W_{dd}(t)w^{dd}(t)v(t) \\ &= \text{Ric}(w(t))(v(t), v(t))v(t). \end{aligned}$$

Here, we write $W_{jl}(t)$ for the components of $\text{Ric}(w(t))$ at (r_0, H) . To pass from the second line to the third, we used that fact that $W_{id}(t) = 0$ for all $i = 1, \dots, d-1$. This follows from Hypothesis 2.1 (see [15, Proposition 1.14]).

The above equalities imply

$$v(t) = v(0) \exp \left(\int_0^t \text{Ric}(w(\tau))(v(\tau), v(\tau)) d\tau \right).$$

Consequently, $v(t)$ is a scalar multiple of $v(0)$, and the assertion of the lemma becomes evident. \square

Given $T > 0$, suppose h, f_1, \dots, f_n are functions acting from $[0, 1] \times [0, T)$ to $(0, \infty)$. Assume these functions are smooth on $(0, 1) \times [0, T)$, h is continuous on $[0, 1] \times [0, T)$, and f_1, \dots, f_n have first derivatives in r that are continuous on $[0, 1] \times [0, T)$. For every $t \in [0, T)$, define a Riemannian metric $g(t)$ on M by setting

$$g(t) = h^2(r, t) dr \otimes dr + g^r(t), \quad r \in [0, 1]. \quad (2.7)$$

This formula is analogous to (2.6). The notation $g^r(t)$ stands for the G -invariant Riemannian metric on G/H such that

$$g^r(X, Y) = f_1^2(r, t)Q(\text{pr}_{\mathfrak{p}_1}X, \text{pr}_{\mathfrak{p}_1}Y) + \dots + f_n^2(r, t)Q(\text{pr}_{\mathfrak{p}_n}X, \text{pr}_{\mathfrak{p}_n}Y), \quad X, Y \in \mathfrak{p}. \quad (2.8)$$

We will demonstrate that it is possible to choose T and h, f_1, \dots, f_n in such a way that $g(t)$ solves the initial-boundary-value problem (2.1)–(2.2)–(2.3). This will prove the existence portion of Theorem 2.2.

Our second lemma provides an expression of the Ricci curvature of the metric $g(t)$ in terms of the functions h, f_1, \dots, f_n . To formulate it, we need more notation. Let $[\cdot, \cdot]$ and P be the Lie bracket and the

Killing form of the Lie algebra \mathfrak{g} . The irreducibility of the summands in the decomposition (2.4) implies the existence of nonnegative numbers β_1, \dots, β_n such that

$$P(X, Y) = -\beta_k Q(X, Y), \quad k = 1, \dots, n, \quad X, Y \in \mathfrak{p}_k.$$

Because the group G is compact and Hypothesis 2.1 holds, at least one of these numbers must be strictly positive. Let d_k be the dimension of \mathfrak{p}_k . We choose a Q -orthonormal basis $(e_j)_{j=1}^{d-1}$ of the space \mathfrak{p} adapted to (2.4). In addition to β_1, \dots, β_n , we define

$$\gamma_{i,k}^l = \frac{1}{d_i} \sum Q([e_{\iota_i}, e_{\iota_k}], e_{\iota_l})^2$$

for $i, k, l = 1, \dots, n$. The sum here is taken over all $\iota_i \in \mathfrak{p}_i$, $\iota_k \in \mathfrak{p}_k$ and $\iota_l \in \mathfrak{p}_l$. Note that $\gamma_{i,k}^l$ is independent of the choice of $(e_j)_{j=1}^{d-1}$. One easily checks that $d_i \gamma_{i,k}^l = d_k \gamma_{k,i}^l = d_l \gamma_{l,i}^k$. For more identities satisfied by $(\gamma_{i,k}^l)_{i,k,l=1}^n$, consult [15] and references therein.

Lemma 2.7. *The Ricci curvature of the Riemannian metric $g(t)$ given by (2.7)–(2.8) obeys the equality*

$$\text{Ric}(g(t)) = - \sum_{k=1}^n d_k \left(\frac{f_{krr}}{f_k} - \frac{h_r f_{kr}}{h f_k} \right) dr \otimes dr + \text{Ric}^r(g(t)), \quad r \in (0, 1),$$

where $\text{Ric}^r(g(t))$ is the G -invariant $(0, 2)$ -tensor field on G/H such that

$$\begin{aligned} & \text{Ric}^r(g(t))(X, Y) \\ &= \sum_{i=1}^n \left(\frac{\beta_i}{2} + \sum_{k,l=1}^n \gamma_{i,k}^l \frac{f_i^4 - 2f_k^4}{4f_k^2 f_l^2} - \frac{f_i f_{ir}}{h} \sum_{k=1}^n d_k \frac{f_{kr}}{h f_k} + \frac{f_{ir}^2}{h^2} - \frac{f_i f_{irr}}{h^2} + \frac{f_i h_r f_{ir}}{h^3} \right) Q(\text{pr}_{\mathfrak{p}_i} X, \text{pr}_{\mathfrak{p}_i} Y) \end{aligned}$$

for $X, Y \in \mathfrak{p}$. The subscript r here means differentiation in $r \in (0, 1)$.

The reader will find the proof of Lemma 2.7 in [24]. The computations were essentially made in [15].

Lastly, we need an existence and uniqueness result for parabolic systems of partial differential equations on the interval $[0, 1]$ under non-homogeneous Neumann boundary conditions. Given $T > 0$, we will deal with the Sobolev-type space $W_5^{2,1}((0, 1) \times (0, T))$. The reader may see [17, Section I.1] for its rigorous definition. It will be convenient for us to abbreviate $W_5^{2,1}((0, 1) \times (0, T))$ to $W_{5,T}^{2,1}$. Throughout the paper, we will also encounter Hölder-type spaces $H^{\delta, \frac{\delta}{2}}([0, 1] \times [0, T])$ with $\delta > 0$. We refer to [17, Sections I.1] for their precise definition. To keep our notation short, we will abbreviate $H^{\delta, \frac{\delta}{2}}([0, 1] \times [0, T])$ to $H_T^{\delta, \frac{\delta}{2}}$. Note that, according to [17, Lemma 3.3 in Chapter II], every function in $W_{5,T}^{2,1}$ must lie in $H_T^{\frac{6}{5}, \frac{3}{5}}$. In particular, the derivative of such a function with respect to the first variable is continuous on $[0, 1] \times [0, T]$. This fact is crucial to our further arguments.

Lemma 2.8. *Fix $m \in \mathbb{N}$ and consider smooth functions*

$$\begin{aligned} a &: [0, 1] \times [0, \infty) \times \mathbb{R}^m \rightarrow (0, \infty), \\ A_i &: [0, 1] \times [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}, \\ B_i &: \{0, 1\} \times [0, \infty) \times \mathbb{R}^m \rightarrow \mathbb{R}, \\ \hat{v}_i &: [0, 1] \rightarrow \mathbb{R}, \quad i = 1, \dots, m. \end{aligned}$$

Assume that

$$\hat{v}_{ir}(j) = B_i(j, 0, \hat{v}(j)), \quad j = 0, 1, \quad i = 1, \dots, m,$$

where $\hat{v} = (\hat{v}_1, \dots, \hat{v}_m)$. For some $T > 0$, there exist

$$v_i : [0, 1] \times [0, T] \rightarrow \mathbb{R}, \quad i = 1, \dots, m,$$

satisfying the following statements:

1. Each v_i is smooth on $(0, 1) \times [0, T]$ and lies in the space $W_{5,T}^{2,1}$.

2. For every $i = 1, \dots, m$, the function v_i solves the equation

$$v_{it}(r, t) = a(r, t, v(r, t))v_{irr}(r, t) + A_i(r, t, v(r, t), v_r(r, t)), \quad r \in (0, 1), \quad t \in (0, T), \quad (2.9)$$

subject to the boundary conditions

$$v_{ir}(j, t) = B_i(j, t, v(j, t)), \quad j = 0, 1, \quad t \in (0, T),$$

and the initial condition

$$v_i(r, 0) = \hat{v}_i(r), \quad r \in [0, 1].$$

Here, v stands for (v_1, \dots, v_m) , subscript t denotes the derivative in t , and v_r denotes the component-wise derivative in r . If, for some $T > 0$, the arrays

$$\begin{aligned} v_{1,i} &: [0, 1] \times [0, T] \rightarrow \mathbb{R}, \\ v_{2,i} &: [0, 1] \times [0, T] \rightarrow \mathbb{R}, \quad i = 1, \dots, m, \end{aligned}$$

possess the above properties 1 and 2, then

$$v_{1,i} = v_{2,i}, \quad i = 1, \dots, m.$$

One may establish Lemma 2.8 by repeating the reasoning from the proof of [22, Theorem 2.1] and from [28, Remark 2.2 (i)] with minor adjustments. We will not discuss these adjustments here, as they are fairly straightforward. The need for them arises primarily because the boundary of $[0, 1]$ is 0-dimensional and, as a consequence, the definitions of Sobolev-type spaces on $\partial[0, 1] \times [0, T]$ require clarification.

The result in [22] is essentially the existence portion of Lemma 2.8 with the interval $[0, 1]$ replaced by a Riemannian manifold of dimension two or higher. The method of proof employed in [22] relies on a fixed-point argument, as executed by Weidemaier in the proof of [28, Theorem 2.1]. A solution to the initial-boundary-value problem is first constructed in a Sobolev-type space. Its regularity is then established via a bootstrapping argument. Classical results from [17] are used in the process. The uniqueness portion of Lemma 2.8 follows from the arguments in [28, Remark 2.2 (i)]. Note that the reader may find results closely related to the lemma in Amann's paper [1]; specifically, see the theorem in the introduction.

2.3 The argument for existence and uniqueness

According to Lemma 2.7, the Riemannian metric $g(t)$ satisfies the Ricci flow equation (2.1) if

$$\begin{aligned} h_t &= \sum_{k=1}^n d_k \left(\frac{f_{kr}}{h f_k} - \frac{h_r f_{kr}}{h^2 f_k} \right), \\ f_{it} &= -\frac{\beta_i}{2f_i} - \sum_{k,l=1}^n \gamma_{i,k}^l \frac{f_i^4 - 2f_k^4}{4f_i f_k^2 f_l^2} + \frac{f_{ir}}{h} \sum_{k=1}^n d_k \frac{f_{kr}}{h f_k} - \frac{f_{ir}^2}{h^2 f_i} + \frac{f_{irr}}{h^2} - \frac{h_r f_{ir}}{h^3}, \\ r &\in (0, 1), \quad t \in (0, T), \quad i = 1, \dots, n. \end{aligned} \quad (2.10)$$

Let us now write the boundary condition (2.2) and the initial condition (2.3) in terms of h, f_1, \dots, f_n . Given a G -invariant section $u \in \Gamma(T^* \partial M \hat{\otimes} T^* \partial M)$ and $j = 0, 1$, the restriction of u to $\{j\} \times G/H$ is fully determined by how u acts on the tangent space $T_{\{j\} \times G/H}(\{j\} \times G/H)$. Identifying this space with \mathfrak{p} in the natural way, we define the numbers $u_{j,1}, \dots, u_{j,n} \in \mathbb{R}$ by the equality

$$u|_{\{j\} \times G/H}(X, Y) = u_{j,1} Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) + \dots + u_{j,n} Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}.$$

There exist smooth functions $F_{j,1}, \dots, F_{j,n}$ from $[0, \infty) \times \mathbb{R}^n$ to \mathbb{R} such that, for all $t \in [0, \infty)$ and all G -invariant $u \in \Gamma(T^* \partial M \hat{\otimes} T^* \partial M)$, we have

$$\begin{aligned} F(t, u)|_{\{j\} \times G/H}(X, Y) &= F_{j,1}(t, u_{j,1}, \dots, u_{j,n}) Q(\text{pr}_{\mathfrak{p}_1} X, \text{pr}_{\mathfrak{p}_1} Y) \\ &\quad + \dots + F_{j,n}(t, u_{j,1}, \dots, u_{j,n}) Q(\text{pr}_{\mathfrak{p}_n} X, \text{pr}_{\mathfrak{p}_n} Y), \quad X, Y \in \mathfrak{p}. \end{aligned}$$

A computation (cf. [23, Lemma 2]) shows that $g(t)$ obeys (2.2) when

$$f_{ir}(j, t) = (-1)^{j+1} \frac{h(j, t) F_{j,i}(t, f_1^2(j, t), \dots, f_n^2(j, t))}{f_i(j, t)}, \quad j = 0, 1, \quad t \in (0, T), \quad i = 1, \dots, n. \quad (2.11)$$

It is also easy to see that (2.3) holds when

$$h(r, 0) = \hat{h}(r), \quad f_i(r, 0) = \hat{f}_i(r), \quad r \in [0, 1], \quad i = 1, \dots, n. \quad (2.12)$$

Proof of Theorem 2.2. We first prove the existence of a map g possessing the listed properties. The method we use is an adaptation of the DeTurck trick (to be specific, the version of the DeTurck trick described in [6, Section 2.6]). The main idea is to replace (2.10) with the more tractable system (2.13). Lemma 2.8 will guarantee the short-time existence of a solution to (2.13) under appropriate boundary and initial conditions. Modifying this solution, we will produce functions h, f_1, \dots, f_n which satisfy (2.10)–(2.11)–(2.12). The G -invariant t -dependent metric given by (2.7) and (2.8) will define a mapping with the properties asserted in the theorem. In particular, this metric will solve the Ricci flow.

Lemma 2.8 yields the existence, for some $T > 0$, of functions $\bar{h}, \bar{f}_1, \dots, \bar{f}_n : [0, 1] \times [0, T] \rightarrow (0, \infty)$ satisfying the equations

$$\begin{aligned} \bar{h}_t &= \frac{\bar{h}_{rr}}{\bar{h}^2} - \frac{2\bar{h}_r^2}{\bar{h}^3} + \sum_{k=1}^n d_k \frac{\bar{f}_{kr}^2}{\bar{h}\bar{f}_k^2} - \left(\frac{\hat{h}_r}{\hat{h}^2} + \sum_{k=1}^n d_k \frac{\hat{f}_{kr}}{\hat{h}\hat{f}_k} \right)_r, \\ \bar{f}_{it} &= \frac{\bar{f}_{irr}}{\bar{h}^2} - \frac{\bar{f}_{ir}^2}{\bar{h}^2\bar{f}_i} - \sum_{k,l=1}^n \gamma_{i,k}^l \frac{\bar{f}_i^4 - 2\bar{f}_k^4}{4\bar{f}_i\bar{f}_k^2\bar{f}_l^2} - \frac{\beta_i}{2\bar{f}_i} - \frac{\hat{h}_r\bar{f}_{ir}}{\bar{h}\hat{h}^2} + \sum_{k=1}^n d_k \frac{\bar{f}_{ir}\hat{f}_{kr}}{\bar{h}\hat{h}\hat{f}_k}, \\ r &\in (0, 1), \quad t \in (0, T), \quad i = 1, \dots, n, \end{aligned} \quad (2.13)$$

the boundary conditions

$$\begin{aligned} \bar{h}_r(j, t) &= (-1)^{j+1} \sum_{k=1}^n d_k \frac{\bar{h}^2(j, t) F_{j,k}(t, \bar{f}_1^2(j, t), \dots, \bar{f}_n^2(j, t))}{\bar{f}_k^2(j, t)} + \frac{\bar{h}^2(j, t)}{\hat{h}(j)} \left(\frac{\hat{h}_r(j)}{\hat{h}(j)} - \sum_{k=1}^n d_k \frac{\hat{f}_{kr}(j)}{\hat{h}\hat{f}_k(j)} \right), \\ \bar{f}_{ir}(j, t) &= (-1)^{j+1} \frac{\bar{h}(j, t) F_{j,i}(t, \bar{f}_1^2(j, t), \dots, \bar{f}_n^2(j, t))}{\bar{f}_i(j, t)}, \quad j = 0, 1, \quad t \in (0, T), \quad i = 1, \dots, n, \end{aligned} \quad (2.14)$$

and the initial conditions

$$\bar{h}(r, 0) = \hat{h}(r), \quad \bar{f}_i(r, 0) = \hat{f}_i(r), \quad r \in [0, 1], \quad i = 1, \dots, n. \quad (2.15)$$

These functions are smooth on $(0, 1) \times [0, T]$. They lie in the space $W_{5,T}^{2,1}$ and, consequently, the space $H_T^{\frac{6}{5}, \frac{3}{5}}$. Equality (2.5) and classical regularity results (see, e.g., [17, Theorem 5.3 in Chapter IV]) imply that the derivatives $\bar{h}_{rr}, \bar{f}_{irr}, \dots, \bar{f}_{nrr}$ exist and are continuous on $[0, 1] \times [0, T]$. This enables us to define a function $\phi : [0, 1] \times [0, T] \rightarrow \mathbb{R}$ by the formula

$$\begin{aligned} \phi_t(r, t) &= - \left(\frac{\bar{h}_\rho(\rho, t)}{\bar{h}^3(\rho, t)} - \sum_{k=1}^n d_k \frac{\bar{f}_{k\rho}(\rho, t)}{\bar{h}^2(\rho, t)\bar{f}_k(\rho, t)} - \frac{\hat{h}_\rho(\rho)}{\bar{h}(\rho, t)\hat{h}^2(\rho)} + \sum_{k=1}^n d_k \frac{\hat{f}_{k\rho}(\rho)}{\bar{h}(\rho, t)\hat{h}(\rho)\hat{f}_k(\rho)} \right) \Big|_{\rho=\phi(r, t)}, \\ r &\in [0, 1], \quad t \in (0, T), \end{aligned} \quad (2.16)$$

and the requirement that $\phi(r, 0) = r$ when $r \in [0, 1]$. Here, the subscript ρ denotes differentiation in ρ . We will use ϕ to modify $\bar{h}, \bar{f}_1, \dots, \bar{f}_n$ and obtain a solution to (2.10)–(2.11)–(2.12). Remark 2.9 will explain the thought process that lead us to considering system (2.13) and to (2.16).

The function ϕ is smooth on $(0, 1) \times [0, T]$. Its derivative with respect to r exists and is continuous on $[0, 1] \times [0, T]$. Moreover, this derivative is strictly positive on $[0, 1] \times [0, T]$. The boundary conditions (2.14) imply that the right-hand side of (2.16) is 0 when $\phi(r, t) \in \{0, 1\}$ and $t \in (0, T)$. Consequently, the equality

$$\phi(j, t) = j, \quad j = 0, 1, \quad t \in [0, T], \quad (2.17)$$

holds true. Consider the real-valued functions h, f_1, \dots, f_n on $[0, 1] \times [0, T]$ defined as

$$h(r, t) = \phi_r(r, t)\bar{h}(\phi(r, t), t), \quad f_i(r, t) = \bar{f}_i(\phi(r, t), t), \quad r \in [0, 1], t \in [0, T], i = 1, \dots, n. \quad (2.18)$$

Remark 2.9 discusses the geometric meaning of (2.18). Keeping in mind that $\bar{h}, \bar{f}_1, \dots, \bar{f}_n$ satisfy (2.13), we can verify by direct computation that h, f_1, \dots, f_n satisfy (2.10). One performs a computation of similar nature when one carries out the DeTurck trick on closed manifolds; cf. [6, Section 2.6] and [27, Section 5.2]. The boundary conditions (2.14) and formulas (2.17) imply that

$$\begin{aligned} f_{ir}(j, t) &= (-1)^{j+1} \phi_r(j, t) \frac{\bar{h}(j, t) F_{j,i}(t, \bar{f}_1^2(j, t), \dots, \bar{f}_n^2(j, t))}{\bar{f}_i(j, t)} \\ &= (-1)^{j+1} \frac{h(j, t) F_{j,i}(t, f_1^2(j, t), \dots, f_n^2(j, t))}{f_i(j, t)}, \quad j = 0, 1, t \in (0, T), i = 1, \dots, n. \end{aligned}$$

Therefore, h, f_1, \dots, f_n satisfy (2.11). Finally, equality (2.12) holds for h, f_1, \dots, f_n because (2.15) holds for $\bar{h}, \bar{f}_1, \dots, \bar{f}_n$ and $\phi(\cdot, 0)$ is the identity map on $[0, 1]$.

We define a G -invariant t -dependent metric $g(t)$ on M through formulas (2.7) and (2.8). One may interpret $g(t)$ as a map from $M \times [0, T]$ to $T^*M \otimes T^*M$. This map obviously possesses the desired properties.

Let us prove the uniqueness portion of the theorem. To do so, we first rewrite the Ricci flow equations for g_1 and g_2 in the form (2.10). We then replace the obtained systems with more tractable systems analogous to (2.13). The approach we take is rather classical; cf. [6, Section 2.6]. To make it work in our setting, however, we need an auxiliary result (specifically, Lemma 2.5 above). The purpose of this result is to help us demonstrate that g_1 and g_2 can be simultaneously diagonalized. Roughly speaking, it states that the normals to G -orbits on M with respect to g_1 and g_2 point in the same direction.

We argue by contradiction. Suppose g_1 and g_2 do not coincide. Without loss of generality, assume that

$$\sup\{\tau \in [0, T] \mid g_1 = g_2 \text{ on } M \times [0, \tau]\} = 0.$$

Lemma 2.5 and Hypothesis 2.1 imply the existence of smooth positive functions $h_1, f_{1,1}, \dots, f_{1,n}$ and $h_2, f_{2,1}, \dots, f_{2,n}$ on $[0, 1] \times [0, T]$ satisfying the formula

$$g_m(t) = h_m^2(r, t) dr \otimes dr + g_m^r(t), \quad m = 1, 2, r \in [0, 1], t \in [0, T].$$

Here, $g_m(t)$ is the t -dependent Riemannian metric given by the map g_m , and $g_m^r(t)$ is the G -invariant Riemannian metric on G/H such that

$$g_m^r(t)(X, Y) = f_{m,1}^2(r, t)Q(\text{pr}_{\mathfrak{p}_1}X, \text{pr}_{\mathfrak{p}_1}Y) + \dots + f_{m,n}^2(r, t)Q(\text{pr}_{\mathfrak{p}_n}X, \text{pr}_{\mathfrak{p}_n}Y), \quad X, Y \in \mathfrak{p}.$$

It is clear that formulas (2.10)–(2.11)–(2.12) will still hold if we replace h, f_1, \dots, f_n in them by $h_m, f_{m,1}, \dots, f_{m,n}$ for either $m = 1$ or $m = 2$.

According to classical existence results for parabolic problems (see, e.g., [17, Theorem 6.1 in Chapter V]), for some $S \in (0, \frac{T}{2})$, we can find $\phi_1, \phi_2 : [0, 1] \times [0, 2S] \rightarrow \mathbb{R}$ obeying the equation

$$\begin{aligned} \phi_{mt}(r, t) &= \frac{\phi_{mrr}(r, t)}{h_m^2(r, t)} - \frac{\phi_{mr}(r, t)h_{mr}(r, t)}{h_m^3(r, t)} + \sum_{k=1}^n d_k \frac{\phi_{mr}(r, t)(f_{m,k})_r(r, t)}{h_m^2(r, t)f_{m,k}(r, t)} \\ &\quad + \frac{\phi_{mr}(r, t)}{h_m(r, t)} \left(\frac{\hat{h}_{m\rho}(\rho)}{\hat{h}_m^2(\rho)} - \sum_{k=1}^n d_k \frac{(\hat{f}_{m,k})_\rho(\rho)}{\hat{h}_m(\rho)\hat{f}_{m,k}(\rho)} \right) \Big|_{\rho=\phi_m(r, t)}, \quad m = 1, 2, r \in (0, 1), t \in (0, 2S), \end{aligned}$$

the boundary conditions

$$\phi_m(j, t) = j, \quad m = 1, 2, j = 0, 1, t \in (0, 2S),$$

and the initial condition

$$\phi_m(r, 0) = r, \quad m = 1, 2, r \in [0, 1].$$

Given $\epsilon \in (0, 1)$, these ϕ_1 and ϕ_2 lie in $H_S^{2+\epsilon, 1+\frac{\epsilon}{2}}$. In fact, they have third derivatives in r that are continuous on $[0, 1] \times [0, S]$, and they are smooth on $(0, 1) \times [0, S]$. By choosing S sufficiently small, we ensure that ϕ_{1r} and ϕ_{2r} are strictly positive on $[0, 1] \times [0, S]$. The formula

$$\phi_m([0, 1] \times \{t\}) = [0, 1], \quad m = 1, 2, \quad t \in [0, S],$$

holds true. The thought process behind introducing ϕ_1 and ϕ_2 is explained in Remark 2.10.

Let $\phi_m^{-1}(\cdot, t)$ denote the inverse of the map $\phi_m(\cdot, t) : [0, 1] \rightarrow [0, 1]$ for each $t \in [0, S]$ and $m = 1, 2$. Consider the real-valued functions $\bar{h}_1, \bar{f}_{1,1}, \dots, \bar{f}_{1,n}$ and $\bar{h}_2, \bar{f}_{2,1}, \dots, \bar{f}_{2,n}$ on $[0, 1] \times [0, S]$ defined as

$$\begin{aligned} \bar{h}_m(r, t) &= (\phi_m^{-1})_r(r, t) h_m(\phi_m^{-1}(r, t), t), \\ \bar{f}_{m,i}(r, t) &= f_{m,i}(\phi_m^{-1}(r, t), t), \quad m = 1, 2, \quad r \in [0, 1], \quad t \in [0, S], \quad i = 1, \dots, n. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} \phi_{mt}(r, t) &= - \left(\frac{\bar{h}_{m\rho}(\rho, t)}{\bar{h}_m^3(\rho, t)} - \sum_{k=1}^n d_k \frac{(\bar{f}_{m,k})_\rho(\rho, t)}{\bar{h}_m^2(\rho, t) \bar{f}_{m,k}(\rho, t)} \right) \Big|_{\rho=\phi_m(r, t)} \\ &\quad + \left(\frac{\hat{h}_{m\rho}(\rho)}{\bar{h}_m(\rho, t) \hat{h}_m^2(\rho)} - \sum_{k=1}^n d_k \frac{(\hat{f}_{m,k})_\rho(\rho)}{\bar{h}_m(\rho, t) \hat{h}_m(\rho) \hat{f}_{m,k}(\rho)} \right) \Big|_{\rho=\phi_m(r, t)}, \\ m &= 1, 2, \quad r \in [0, 1], \quad t \in [0, S]. \end{aligned} \tag{2.19}$$

Formulas (2.13)–(2.14)–(2.15) will still hold if we replace T in them by S and $\bar{h}, \bar{f}_1, \dots, \bar{f}_n$ by $\bar{h}_m, \bar{f}_{m,1}, \dots, \bar{f}_{m,n}$ for either $m = 1$ or $m = 2$. Also, $\bar{h}_1, \bar{f}_{1,1}, \dots, \bar{f}_{1,n}$ and $\bar{h}_2, \bar{f}_{2,1}, \dots, \bar{f}_{2,n}$ have two derivatives in r and one in t that are continuous on $[0, 1] \times [0, S]$. Lemma 2.8 tells us that $\bar{h}_1 = \bar{h}_2$ and $\bar{f}_{1,i} = \bar{f}_{2,i}$ when $i = 1, \dots, n$. Furthermore, according to (2.19) and the standard uniqueness theorems for ordinary differential equations, ϕ_1 must coincide with ϕ_2 on $[0, 1] \times [0, S]$. Because

$$\begin{aligned} h_m(r, t) &= \phi_{mr}(r, t) \bar{h}_m(\phi_m(r, t), t), \\ f_{m,i}(r, t) &= \bar{f}_{m,i}(\phi_m(r, t), t), \quad m = 1, 2, \quad r \in [0, 1], \quad t \in [0, S], \quad i = 1, \dots, n, \end{aligned}$$

it becomes clear that $h_1 = h_2$ and $f_{1,i} = f_{2,i}$ on $[0, 1] \times [0, S]$ when $i = 1, \dots, n$. This is a contradiction. \square

Remark 2.9. The following principle underlies the DeTurck trick: if the metric $g(t)$ solves the Ricci flow equation on $M_0 \times (0, T)$, then for a properly chosen t -dependent diffeomorphism $\Phi(\cdot, t)$ of M_0 , the pullback $(\Phi^{-1})^*(\cdot, t)g(t)$ must solve a more tractable equation. The proof of the existence part of Theorem 2.2 is based on this principle. To clarify, we need to make two observations.

1. The function ϕ defines a mapping $\Phi : M \times [0, T) \rightarrow M$ via the formula

$$\Phi((r, \gamma H), t) = (\phi(r, t), \gamma H), \quad r \in [0, 1], \quad \gamma \in G, \quad t \in [0, T).$$

Because the derivative of ϕ in the first variable is positive on $[0, 1] \times [0, T)$, and because (2.17) holds, $\Phi(\cdot, t)$ must be a diffeomorphism of M for every $t \in [0, T)$. Assuming $g(t)$ is given by (2.7) and (2.8), we can easily check that

$$(\Phi^{-1})^*(\cdot, t)g(t) = \bar{h}^2(r, t) dr \otimes dr + \bar{g}^r(t), \quad r \in [0, 1], \quad t \in [0, T),$$

where $\bar{g}^r(t)$ is the G -invariant metric on G/H with

$$\bar{g}^r(t)(X, Y) = \bar{f}_1^2(r, t)Q(\text{pr}_{\mathfrak{p}_1}X, \text{pr}_{\mathfrak{p}_1}Y) + \dots + \bar{f}_n^2(r, t)Q(\text{pr}_{\mathfrak{p}_n}X, \text{pr}_{\mathfrak{p}_n}Y), \quad X, Y \in \mathfrak{p}.$$

Here, $\bar{h}, \bar{f}_1, \dots, \bar{f}_n$ obey (2.18).

2. If the functions h, f_1, \dots, f_n are to satisfy (2.10), then $\bar{h}, \bar{f}_1, \dots, \bar{f}_n$ must satisfy

$$\begin{aligned}\bar{h}_t(r, t) &= \sum_{k=1}^n d_k \left(\frac{\bar{f}_{kr}(r, t)}{\bar{h}(r, t)\bar{f}_k(r, t)} - \frac{\bar{h}_r(r, t)\bar{f}_{kr}(r, t)}{\bar{h}^2(r, t)\bar{f}_k(r, t)} \right) \\ &\quad - \bar{h}_r(r, t)\phi_t(\phi^{-1}(r, t), t) - \bar{h}(r, t)(\phi_t(\phi^{-1}(r, t), t))_r, \\ \bar{f}_{it}(r, t) &= \frac{\bar{f}_{irr}(r, t)}{\bar{h}^2(r, t)} - \frac{\bar{f}_{ir}^2(r, t)}{\bar{h}^2(r, t)\bar{f}_i(r, t)} - \sum_{k,l=1}^n \gamma_{i,k}^l \frac{\bar{f}_i^4(r, t) - 2\bar{f}_k^4(r, t)}{4\bar{f}_i(r, t)\bar{f}_k^2(r, t)\bar{f}_l^2(r, t)} \\ &\quad - \frac{\beta_i}{2\bar{f}_i(r, t)} - \frac{\bar{h}_r(r, t)\bar{f}_{ir}(r, t)}{\bar{h}^3(r, t)} + \sum_{k=1}^n d_k \frac{\bar{f}_{ir}(r, t)\bar{f}_{kr}(r, t)}{\bar{h}^2(r, t)\bar{f}_k(r, t)} - \bar{f}_{ir}(r, t)\phi_t(\phi^{-1}(r, t), t), \\ r &\in (0, 1), t \in (0, T), i = 1, \dots, n.\end{aligned}$$

We define ϕ by (2.16) to ensure that $\frac{\bar{h}_r}{\bar{h}^2}$ is the only second-order term in the first equation. The above system for $\bar{h}, \bar{f}_1, \dots, \bar{f}_n$ then takes the form (2.9) (in fact, it coincides with (2.13)), and Lemma 2.8 guarantees the existence of a solution. Formulas (2.18) enable us to obtain the functions h, f_1, \dots, f_n from this solution.

Remark 2.10. A simple computation based on (2.16) and (2.18) yields

$$\begin{aligned}\phi_t(r, t) &= \frac{\phi_{rr}(r, t)}{h^2(r, t)} - \frac{\phi_r(r, t)h_r(r, t)}{h^3(r, t)} + \sum_{k=1}^n d_k \frac{\phi_r(r, t)f_{kr}(r, t)}{h^2(r, t)f_k(r, t)} \\ &\quad + \frac{\phi_r(r, t)}{h(r, t)} \left(\frac{\hat{h}_\rho(\rho)}{\hat{h}^2(\rho)} - \sum_{k=1}^n d_k \frac{\hat{f}_{k\rho}(\rho)}{\hat{h}(\rho)\hat{f}_k(\rho)} \right) \Big|_{\rho=\phi(r, t)}, \quad r \in (0, 1), t \in (0, T).\end{aligned}$$

This equation motivates the definition of the maps ϕ_1 and ϕ_2 in the proof of the theorem.

3 Perelman's \mathcal{F} -functional

Suppose w is a smooth Riemannian metric on M and q is a smooth real-valued function on M . By definition, the Perelman \mathcal{F} -functional takes the pair (w, q) to the number

$$\mathcal{F}(w, q) = \int_M (R(w) + |\nabla q|^2) e^{-q} d\mu.$$

Here, $R(w)$ is the scalar curvature of w . The absolute value and the gradient are taken with respect to w . The letter μ denotes the w -volume measure on M . The purpose of this section is to relate the Ricci flow on M to the functional \mathcal{F} and its monotonicity properties. The main challenge, of course, lies in the nonemptiness of ∂M .

3.1 The modified Ricci flow

Fix a smooth Riemannian metric $\eta(t)$ and a smooth real-valued function $\kappa(t)$ on ∂M depending on a parameter $t \in [0, \infty)$. For some $T > 0$ and $\delta > 0$, suppose $\tilde{g}(t)$ is a G -invariant solution to the equation

$$\frac{\partial}{\partial t} \tilde{g}(t) = -2 \text{Ric}(\tilde{g}(t)) \quad (3.1)$$

on $M_0 \times (0, T + \delta)$ subject to the boundary conditions

$$[\tilde{g}_{\partial M}(t)] = [\eta(t)], \quad \mathcal{H}(\tilde{g}(t)) = \kappa(t), \quad t \in (0, T + \delta). \quad (3.2)$$

The square brackets denote the conformal class. Thus, for example, $[\eta(t)]$ is the set of smooth metrics of the form $\theta\eta(t)$, where θ is a positive function on ∂M . The notation $\mathcal{H}(\tilde{g}(t))$ stands for the mean curvature of ∂M with respect to $\tilde{g}(t)$. By definition, $\mathcal{H}(\tilde{g}(t))$ is the trace of $\Pi(\tilde{g}(t))$. We impose the initial condition

$$\tilde{g}(0) = \hat{g}. \quad (3.3)$$

Here, \hat{g} is the G -invariant metric on M fixed in Section 2.1. Equality (2.6) holds true. It will be convenient for us to assume that $\tilde{g}(t)$ is smooth on $M \times [0, T + \delta)$. Remark 3.2 below explains how this assumption can be relaxed. Note that the paper [13] offers a comprehensive existence theorem for solutions to (3.1)–(3.2)–(3.3). Corollary 5.1 in that paper provides a simple sufficient condition for the G -invariance of such solutions.

Let us consider the system of equations

$$\begin{aligned}\frac{\partial}{\partial t}g(t) &= -2(\text{Ric}(g(t)) + \text{Hess } p(t)), \\ \frac{\partial}{\partial t}p(t) &= -\Delta p(t) - R(g(t)).\end{aligned}\tag{3.4}$$

The unknowns here are the Riemannian metric $g(t)$ and the real-valued function $p(t)$ on M depending on t . The notation Hess and Δ refers to the Hessian and the Laplacian with respect to $g(t)$. The relationship between system (3.4) and the Ricci flow is well-understood on closed manifolds. It is explained in detail in, e.g., [27, Chapter 6]. Essentially, solutions to the first equation of (3.4) are pullbacks of solutions to the Ricci flow by t -dependent diffeomorphisms. If the pair $(g(t), p(t))$ satisfies system (3.4) on a closed manifold, then the expression $\mathcal{F}(g(t), p(t))$ is non-decreasing in t .

We supplement (3.4) with the boundary conditions

$$[g_{\partial M}(t)] = [\eta(t)], \quad \mathcal{H}(g(t)) = \kappa(t), \quad \frac{\partial}{\partial \nu}p(t) = 0.\tag{3.5}$$

In the third equality, $\frac{\partial}{\partial \nu}$ denotes differentiation along the outward unit normal vector field on ∂M with respect to $g(t)$. Proposition 3.1 will explain how the boundary-value problem (3.4)–(3.5) relates to the Ricci flow on M . For an analogous result on closed manifolds, see [27, Theorem 6.4.1]. In Section 3.2, we will demonstrate that \mathcal{F} is monotone on solutions to (3.4)–(3.5).

Proposition 3.1. *There exist a smooth map $g : M \times [0, T) \rightarrow T^*M \otimes T^*M$, a smooth map $\Psi : M \times [0, T) \rightarrow M$ and a smooth function $p : M \times [0, T) \rightarrow \mathbb{R}$ such that the following statements hold:*

1. *For every $x \in M$ and $t \in (0, T)$, the tensor $g(x, t)$ is a symmetric positive-definite tensor at the point x . Thus, $g(t) = g(\cdot, t)$ is a Riemannian metric on M .*
2. *Given $\gamma \in G$, $x \in M$ and $t \in (0, T)$, the equalities $g(t) = \gamma^*g(t)$ and $p(\gamma x, t) = p(x, t)$ hold true.*
3. *For some $h, f_1, \dots, f_n : [0, 1] \times (0, T) \rightarrow (0, \infty)$, the metric $g(t)$ satisfies (2.7)–(2.8) on $M \times (0, T)$.*
4. *The pair $(g(t), p(t))$, where $p(t)$ denotes the function $p(\cdot, t)$, solves system (3.4) on $M_0 \times (0, T)$.*
5. *The boundary conditions (3.5) are satisfied on $\partial M \times (0, T)$, and $g(0) = \hat{g}$ on M .*
6. *The equality $\Psi(x, t) = x$ holds when $x \in \partial M$ and $t \in (0, T)$. The map $\Psi(\cdot, t)$ is a diffeomorphism of M for each $t \in (0, T)$. The metric $g(t)$ coincides with the pullback $\Psi^*(\cdot, t)\tilde{g}(t)$ of the solution $\tilde{g}(t)$ to problem (3.1)–(3.2)–(3.3).*

Remark 3.2. We assumed above that $\tilde{g}(t)$ was smooth on $M \times [0, T + \delta)$. One may obtain results analogous to Proposition 3.1 under weaker hypotheses. For example, take a natural number k greater than 3. Instead of demanding that \tilde{g} be smooth on $M \times [0, T + \delta)$, assume it is only smooth on $M_0 \times [0, T + \delta)$. In addition, let $\frac{\partial^i}{\partial t^i}\hat{\nabla}^j\tilde{g}$ exist and be continuous on $M \times [0, T + \delta)$ whenever $2i + j \leq k$. The notation $\hat{\nabla}$ here stands for the covariant derivative with respect to \hat{g} . Proposition 3.1 will hold under these hypotheses if one modifies the differentiability properties of g , Ψ and p in its formulation. Specifically, one may assert that g is smooth on $M_0 \times (0, T)$, while $\frac{\partial^i}{\partial t^i}\hat{\nabla}^jg$ exists and is continuous on $M \times [0, T)$ when $2i + j \leq k - 3$. The adjustments required for Ψ and p are of the same nature. We will not discuss them in the present paper.

Proof. Lemma 2.5 implies the equality

$$\tilde{g}(t) = \tilde{h}^2(r, t) dr \otimes dr + \tilde{g}^r(t), \quad r \in [0, 1], \quad t \in [0, T + \delta).$$

Here, \tilde{h} is a positive function, and $\tilde{g}^r(t)$ is a t -dependent G -invariant metric on G/H . Because Hypothesis 2.1 holds, there exist $\tilde{f}_1, \dots, \tilde{f}_n : [0, 1] \times [0, T + \delta] \rightarrow (0, \infty)$ such that

$$\tilde{g}^r(t)(X, Y) = \tilde{f}_1^2(r, t)Q(\text{pr}_{\mathfrak{p}_1}X, \text{pr}_{\mathfrak{p}_1}Y) + \dots + \tilde{f}_n^2(r, t)Q(\text{pr}_{\mathfrak{p}_n}X, \text{pr}_{\mathfrak{p}_n}Y), \quad X, Y \in \mathfrak{p}.$$

In order to construct g , Ψ and p , we need to introduce auxiliary functions $\tilde{p} : [0, 1] \times [0, T] \rightarrow (0, \infty)$ and $\psi : [0, 1] \times [0, T] \rightarrow [0, 1]$. The definitions of \tilde{p} and ψ will involve $\tilde{h}, \tilde{f}_1, \dots, \tilde{f}_n$.

The scalar curvature of the metric $\tilde{g}(t)$ at the point $(r, \gamma H) \in M$ does not depend on $\gamma \in G$. Indeed, the Ricci flow equation (2.1) implies

$$R(\tilde{g}(t))((r, \gamma H)) = -\frac{\tilde{h}_t(r, t)}{\tilde{h}(r, t)} - \sum_{k=1}^n d_k \frac{\tilde{f}_{kt}(r, t)}{\tilde{f}_k(r, t)}.$$

In what follows, we will abbreviate $R(\tilde{g}(t))((r, \gamma H))$ to $\tilde{R}(r, t)$. We thus obtain a function $\tilde{R} : [0, 1] \times [0, T + \delta] \rightarrow \mathbb{R}$. The classical theory of linear parabolic problems (see, e.g., [17, Theorem 5.3 in Chapter IV and Theorem 12.2 in Chapter III]) yields the existence of a continuous $\tilde{p} : [0, 1] \times [0, T] \rightarrow \mathbb{R}$, which is smooth on $[0, 1] \times [0, T)$, satisfying the equation

$$\begin{aligned} \tilde{p}_t(r, t) = -\frac{1}{\tilde{h}^2(r, t)}\tilde{p}_{rr}(r, t) + \frac{\tilde{h}_r(r, t)}{\tilde{h}^3(r, t)}\tilde{p}_r(r, t) - \sum_{k=1}^n d_k \frac{\tilde{f}_{kr}(r, t)}{\tilde{h}^2(r, t)\tilde{f}_k(r, t)}\tilde{p}_r(r, t) + \tilde{R}(r, t)\tilde{p}(r, t), \\ r \in (0, 1), \quad t \in (0, T), \end{aligned}$$

the boundary conditions

$$\tilde{p}_r(j, t) = 0, \quad j = 0, 1, \quad t \in [0, T),$$

and the terminal condition

$$\tilde{p}(r, T) = 1, \quad r \in [0, 1].$$

According to the Hopf Lemma, \tilde{p} must be positive. We define $\psi : [0, 1] \times [0, T] \rightarrow \mathbb{R}$ by the formula

$$\psi(r, t) = \frac{\tilde{p}_\rho(\rho, t)}{\tilde{h}^2(\rho, t)\tilde{p}(\rho, t)} \Big|_{\rho=\psi(r, t)}, \quad r \in [0, 1], \quad t \in (0, T),$$

and the requirement that $\psi(r, 0) = r$ for $r \in [0, 1]$. Note that, because $\tilde{p}_r(j, t) = 0$ when $j = 0, 1$ and $t \in [0, T)$, the range of ψ is actually the interval $[0, 1]$. We will now use the functions \tilde{p} and ψ to produce g , Ψ and p . The properties listed in the theorem will be easy to verify.

Let us set

$$h(r, t) = \psi_r(r, t)\tilde{h}(\psi(r, t), t), \quad f_i(r, t) = \tilde{f}_i(\psi(r, t), t), \quad r \in [0, 1], \quad t \in [0, T], \quad i = 1, \dots, n. \quad (3.6)$$

Consider the t -dependent metric $g(t)$ on M given by the formulas (2.7)–(2.8). This metric defines a map $g : M \times [0, T] \rightarrow T^*M \otimes T^*M$ in a natural way. Further, we introduce $\Psi : M \times [0, T] \rightarrow M$ and $p : M \times [0, T] \rightarrow \mathbb{R}$ through the equalities

$$\begin{aligned} \Psi((r, \gamma H), t) &= (\psi(r, t), \gamma H), \\ p((r, \gamma H), t) &= -\log \tilde{p}(\psi(r, t), t), \quad \gamma \in G, \quad r \in [0, 1], \quad t \in [0, T]. \end{aligned}$$

It is evident that g , Ψ and p possess the properties 1, 2, 3 and 6 listed in the proposition. A direct verification demonstrates that the pair $(g(t), p(t))$ satisfies (3.4) on $M_0 \times (0, T)$. One performs an analogous verification when analysing (3.4) on closed manifolds; cf. [27, Section 6.4]. With the aid of (3.6), we find

$$\begin{aligned} [g_{\partial M}(t)] &= [\tilde{g}_{\partial M}(t)] = [\eta(t)], \\ \mathcal{H}(g(t))|_{\{j\} \times G/H} &= \sum_{k=1}^n (-1)^{j+1} d_k \frac{f_{kr}(j, t)}{h(j, t)f_k(j, t)} \\ &= \sum_{k=1}^n (-1)^{j+1} d_k \frac{\tilde{f}_{kr}(j, t)}{\tilde{h}(j, t)\tilde{f}_k(j, t)} \\ &= \mathcal{H}(\tilde{g}(t))|_{\{j\} \times G/H} = \kappa(t)|_{\{j\} \times G/H}, \quad j = 0, 1, \quad t \in (0, T). \end{aligned}$$

Finally,

$$\frac{\partial}{\partial \nu} p(t)|_{\{j\} \times G/H} = \frac{(-1)^j \psi_r(j, t) \tilde{p}_r(j, t)}{h(j, t) \tilde{p}(j, t)} = 0, \quad j = 0, 1, \quad t \in (0, T),$$

and $\psi(\cdot, 0)$ is the identity map on $[0, 1]$. Thus, g and p satisfy statements 4 and 5 in the formulation of the proposition. \square

3.2 Monotonicity of \mathcal{F}

The following result demonstrates the connection between (3.4)–(3.5) and the monotonicity of the functional \mathcal{F} on M .

Theorem 3.3. *Consider a smooth map $g : M \times (0, T) \rightarrow T^*M \otimes T^*M$ and a smooth function $p : M \times (0, T) \rightarrow \mathbb{R}$ satisfying statements 1 through 4 of Proposition 3.1. Suppose the boundary conditions (3.5) hold on $\partial M \times (0, T)$ with $\eta(t)$ independent of t and $\kappa(t)$ identically equal to 0. Then*

$$\frac{d}{dt} \mathcal{F}(g(t), p(t)) = 2 \int_M |\text{Ric}(g(t)) + \text{Hess } p(t)|^2 e^{-p(t)} d\mu, \quad t \in (0, T),$$

where the absolute value, the Hessian and the volume measure μ are computed with respect to $g(t)$. Consequently, the quantity $\mathcal{F}(g(t), p(t))$ is non-decreasing in $t \in (0, T)$.

Remark 3.4. While it is convenient for us to assume in Theorem 3.3 that g and p are smooth on $M \times (0, T)$, we can establish the result under weaker hypotheses. It suffices to demand that $\frac{\partial^i}{\partial t^i} \hat{\nabla}^j g$ and $\frac{\partial^i}{\partial t^i} \hat{\nabla}^j p$ exist and be continuous on $M \times (0, T)$ when $2i + j \leq 4$ and $2i + j \leq 3$, respectively. We remind the reader that $\hat{\nabla}$ denotes covariant differentiation with respect to \hat{g} .

Proof. A computation shows that

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(g(t), p(t)) &= 2 \int_M |\text{Ric}(g(t)) + \text{Hess } p(t)|^2 e^{-p(t)} d\mu \\ &\quad + 2 \int_{\partial M} (\text{div Ric}(g(t)))(\nu) e^{-p(t)} d\sigma \\ &\quad - 2 \int_{\partial M} (\text{Ric}(g(t)) + \text{Hess } p(t))(\nu, \nabla p(t)) e^{-p(t)} d\sigma \\ &\quad + 2 \int_{\partial M} \frac{\partial}{\partial \nu} R(g(t)) e^{-p(t)} d\sigma. \end{aligned}$$

Here, div is the divergence with respect to $g(t)$, and ν is the outward unit normal vector field on ∂M with respect to $g(t)$. The letter σ denotes the volume measure of the metric induced by $g(t)$ on ∂M . The above formula for $\frac{d}{dt} \mathcal{F}(g(t), p(t))$ is well-known (see, for example, [9] and the related computations in [27, Section 6.2]). The author first learned of it from Xiaodong Cao's unpublished notes in 2007.

Statement 2 of Proposition 3.1 implies that, given $t \in (0, T)$, the function $p(t)$ is constant on $\{r\} \times G/H$ for each $r \in [0, 1]$. Also, $p(t)$ satisfies the boundary condition $\frac{\partial}{\partial \nu} p(t) = 0$. Consequently, the gradient $\nabla p(t)$ vanishes on ∂M . This means we can rewrite the above formula for $\frac{d}{dt} \mathcal{F}(g(t), p(t))$ as

$$\frac{d}{dt} \mathcal{F}(g(t), p(t)) = 2 \int_M |\text{Ric}(g(t)) + \text{Hess } p(t)|^2 e^{-p(t)} d\mu + 2 \int_{\partial M} \mathfrak{F}(g(t)) e^{-p(t)} d\sigma,$$

where

$$\mathfrak{F}(g(t)) = (\text{div Ric}(g(t)))(\nu) + \frac{\partial}{\partial \nu} R(g(t)). \quad (3.7)$$

To prove the theorem, it suffices to show that $\mathfrak{F}(g(t)) = 0$ for all $t \in (0, T)$. Note that we could further simplify the expression in the right-hand side of (3.7) by utilizing the contracted second Bianchi identity. Such a simplification, however, would only hinder our proof.

The metric $g(t)$ is given by (2.7)–(2.8). Equalities (3.4) yield

$$\text{Ric}(g(t)) = \zeta(r, t) dr \otimes dr + \text{Ric}^r(g(t)), \quad r \in [0, 1], \quad t \in (0, T),$$

with $\zeta : [0, 1] \times (0, T) \rightarrow \mathbb{R}$ a smooth function and $\text{Ric}^r(g(t))$ the G -invariant $(0, 2)$ -tensor field on G/H satisfying

$$\text{Ric}^r(g(t))(X, Y) = - \sum_{k=1}^n d_k \left(f_k(r, t) f_{kt}(r, t) + \frac{f_k(r, t) f_{kr}(r, t) \bar{p}_r(r, t)}{h^2(r, t)} \right) Q(\text{pr}_{\mathfrak{p}_k} X, \text{pr}_{\mathfrak{p}_k} Y), \quad X, Y \in \mathfrak{p}.$$

Here, $\bar{p} : [0, 1] \times (0, T) \rightarrow \mathbb{R}$ is such that

$$\bar{p}(r, t) = p((r, \gamma H), t), \quad \gamma \in G, \quad r \in [0, 1], \quad t \in (0, T).$$

We compute $(\text{div Ric}(g(t)))(\nu)$ (cf. [24, Lemma 4.2]) and find

$$\begin{aligned} (\text{div Ric}(g(t)))(\nu)|_{\{j\} \times G/H} &= (-1)^j \left(\frac{\zeta_r(j, t)}{h^3(j, t)} - \frac{2\zeta(j, t)h_r(j, t)}{h^4(j, t)} \right) \\ &\quad + (-1)^j \sum_{k=1}^n d_k \left(\frac{\zeta(j, t)f_{kr}(j, t)}{h^3(j, t)f_k(j, t)} + \frac{f_{kt}(j, t)f_{kr}(j, t)}{h(j, t)f_k^2(j, t)} \right), \quad j = 0, 1, \quad t \in (0, T). \end{aligned}$$

Also, we have

$$\begin{aligned} \frac{\partial}{\partial \nu} R(g(t))|_{\{j\} \times G/H} &= \frac{(-1)^{j+1}}{h(j, t)} \left(\frac{\zeta(r, t)}{h^2(r, t)} - \sum_{k=1}^n d_k \left(\frac{f_{kt}(r, t)}{f_k(r, t)} + \frac{f_{kr}(r, t)\bar{p}_r(r, t)}{h^2(r, t)f_k(r, t)} \right) \right) \Big|_{r=j} \\ &= \frac{(-1)^j}{h(j, t)} \left(\sum_{k=1}^n d_k \frac{f_{kr}(j, t)}{f_k(j, t)} \right)_t + (-1)^j \sum_{k=1}^n d_k \frac{f_{kr}(j, t)\bar{p}_{rr}(j, t)}{h^3(j, t)f_k(j, t)} \\ &\quad + (-1)^{j+1} \left(\frac{\zeta_r(j, t)}{h^3(j, t)} - \frac{2\zeta(j, t)h_r(j, t)}{h^4(j, t)} \right), \quad j = 0, 1, \quad t \in (0, T). \end{aligned}$$

Therefore, the formula

$$\begin{aligned} \mathfrak{F}(g(t))|_{\{j\} \times G/H} &= (-1)^j \sum_{k=1}^n d_k \left(\frac{\zeta(j, t)f_{kr}(j, t)}{h^3(j, t)f_k(j, t)} + \frac{f_{kt}(j, t)f_{kr}(j, t)}{h(j, t)f_k^2(j, t)} \right) \\ &\quad + \frac{(-1)^j}{h(j, t)} \left(\sum_{k=1}^n d_k \frac{f_{kr}(j, t)}{f_k(j, t)} \right)_t + (-1)^j \sum_{k=1}^n d_k \frac{f_{kr}(j, t)\bar{p}_{rr}(j, t)}{h^3(j, t)f_k(j, t)} \\ &= - \frac{\zeta(j, t)}{h^2(j, t)} \mathcal{H}(g(t))|_{\{j\} \times G/H} + (-1)^j \sum_{k=1}^n d_k \frac{f_{kt}(j, t)}{f_k(j, t)} \frac{f_{kr}(j, t)}{h(j, t)f_k(j, t)} \\ &\quad - \frac{1}{h(j, t)} (h(j, t)\mathcal{H}(g(t))|_{\{j\} \times G/H})_t - \frac{\bar{p}_{rr}(j, t)}{h^2(j, t)} \mathcal{H}(g(t))|_{\{j\} \times G/H}, \quad j = 0, 1, \quad t \in (0, T), \end{aligned} \tag{3.8}$$

must hold.

Because $[g_{\partial M}(t)]$ is independent of $t \in (0, T)$, there exist positive functions ξ_0 and ξ_1 on $(0, T)$ such that

$$f_k(j, t) = \xi_j(t) f_k \left(j, \frac{T}{2} \right), \quad j = 0, 1, \quad t \in (0, T), \quad k = 1, \dots, n.$$

This implies

$$\frac{f_{kt}(j, t)}{f_k(j, t)} = \frac{\xi_{jt}(t)}{\xi_j(t)}, \quad j = 0, 1, \quad t \in (0, T), \quad k = 1, \dots, n.$$

Consequently,

$$\begin{aligned}\mathfrak{F}(g(t))|_{\{j\} \times G/H} &= -\frac{\zeta(j, t)}{h^2(j, t)} \mathcal{H}(g(t))|_{\{j\} \times G/H} - \frac{\xi_{jt}(t)}{\xi_j(t)} \mathcal{H}(g(t))|_{\{j\} \times G/H} \\ &\quad - \frac{1}{h(j, t)} (h(j, t) \mathcal{H}(g(t))|_{\{j\} \times G/H})_t - \frac{\bar{p}_{rr}(j, t)}{h^2(j, t)} \mathcal{H}(g(t))|_{\{j\} \times G/H}, \quad j = 0, 1, \quad t \in (0, T).\end{aligned}$$

The assumption that $\mathcal{H}(g(t)) = 0$ now yields $\mathfrak{F}(g(t)) = 0$ for all $t \in (0, T)$. \square

The paper [19] conducts a detailed study of the weighted Gibbons-Hawking-York functional I_∞ on manifolds with boundary. Its Proposition 2 computes the variation of I_∞ . One can derive Theorem 3.3 from that result instead of arguing as above. However, the calculations in the present paper are somewhat simpler than those in [19] because they exploit the symmetries of M . Besides, they provide a new formula for the derivative $\frac{d}{dt} \mathcal{F}(g(t), p(t))$ under the assumptions that g and p are smooth and satisfy statements 1 through 4 of Proposition 3.1: equality (3.8) implies

$$\begin{aligned}\frac{d}{dt} \mathcal{F}(g(t), p(t)) &= 2 \int_M |\text{Ric}(g(t)) + \text{Hess } p(t)|^2 e^{-p(t)} d\mu \\ &\quad + 2 \int_{\partial M} (-\text{Ric}(g(t))(\nu, \nu) \mathcal{H}(g(t)) + \langle (\text{Ric}(g(t)))_{\partial M}, \Pi(g(t)) \rangle) e^{-p(t)} d\sigma \\ &\quad - 2 \int_{\partial M} \left(\frac{1}{|\hat{\nu}|} (|\hat{\nu}| \mathcal{H}(g(t)))_t + \Delta p(t) \mathcal{H}(g(t)) \right) d\sigma \\ &= 2 \int_M |\text{Ric}(g(t)) + \text{Hess } p(t)|^2 e^{-p(t)} d\mu \\ &\quad + 2 \int_{\partial M} (\langle (\text{Ric}(g(t)))_{\partial M}, \Pi(g(t)) \rangle - (\mathcal{H}(g(t)))_t) e^{-p(t)} d\sigma, \quad t \in (0, T).\end{aligned}$$

The angular brackets here mean the scalar product in the tensor bundle over ∂M induced by $g_{\partial M}(t)$. Interpreting $\text{Ric}(g(t))$ as a map from $TM \otimes TM$ to \mathbb{R} , we write $(\text{Ric}(g(t)))_{\partial M}$ for the restriction of this map to $T\partial M \otimes T\partial M$. The notation $\hat{\nu}$ stands for the outward unit normal vector field on ∂M with respect to the metric \hat{g} .

Acknowledgements

I am grateful to Xiaodong Cao and Jeff Streets for the stimulating discussions on monotone functionals and the Ricci flow.

References

- [1] H. Amann, Dynamic theory of quasilinear parabolic equations II. Reaction-diffusion systems, *Diff. Int. Equ.* 3 (1990) 13–75.
- [2] M.T. Anderson, On boundary value problems for Einstein metrics, *Geom. Topol.* 12 (2008) 2009–2045.
- [3] M. Bailesteanu, X. Cao, A. Pulemotov, Gradient estimates for the heat equation under the Ricci flow, *J. Funct. Anal.* 258 (2010) 3517–3542.
- [4] M. Buzano, Ricci flow on homogeneous spaces with two isotropy summands, *Ann. Global Anal. Geom.* 45 (2014) 25–45.
- [5] X.-Z. Chen, T. Dong, Ricci deformation of a metric on a Riemannian manifold with boundary, *J. Zhejiang Univ. Sci. Ed.* 33 (2006) 496–499.
- [6] B. Chow, P. Lu, L. Ni, *Hamilton’s Ricci flow*, Amer. Math. Soc., Providence, RI, 2006.

- [7] J.C. Cortisoz, The Ricci flow on the two ball with a rotationally symmetric metric, *Russian Math. (Iz. VUZ)* 51 (2007), no. 12, 30–51.
- [8] J.C. Cortisoz, Three-manifolds of positive curvature and convex weakly umbilic boundary, *Geom. Dedicata* 138 (2009) 83–98.
- [9] J. Cortisoz, A. Murcia, The Ricci flow on surfaces with boundary, [arXiv:1209.2386 \[math.DG\]](#).
- [10] A.S. Dancer, M.Y. Wang, Integrable cases of the Einstein equations, *Comm. Math. Phys.* 208 (1999) 225–243.
- [11] A.S. Dancer, M.Y. Wang, The cohomogeneity one Einstein equations from the Hamiltonian viewpoint, *J. reine angew. Math.* 524 (2000) 97–128.
- [12] A.S. Dancer, M.Y. Wang, On Ricci solitons of cohomogeneity one, *Ann. Global Anal. Geom.* 39 (2011) 259–292.
- [13] P. Gianniotis, The Ricci flow on manifolds with boundary, [arXiv:1210.0813 \[math.DG\]](#).
- [14] P. Gianniotis, Boundary estimates for the Ricci flow, [arXiv:1311.3162 \[math.DG\]](#).
- [15] K. Grove, W. Ziller, Cohomogeneity one manifolds with positive Ricci curvature, *Invent. Math.* 149 (2002) 619–646.
- [16] C.A. Hoelscher, Classification of cohomogeneity one manifolds in low dimensions, *Pacific J. Math.* 246 (2010) 129–185.
- [17] O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural’ceva, *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc., Providence, RI, 1968.
- [18] J. Lauret, Ricci flow of homogeneous manifolds, *Math. Z.* 274 (2013) 373–403.
- [19] J. Lott, Mean curvature flow in a Ricci flow background, *Comm. Math. Phys.* 313 (2012) 517–533.
- [20] J. Lott, N. Sesum, Ricci flow on three-dimensional manifolds with symmetry, *Comment. Math. Helv.* 89 (2014) 1–32.
- [21] J. Morgan, G. Tian, *Ricci flow and the Poincaré Conjecture*, Amer. Math. Soc., Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.
- [22] A. Pulemotov, Quasilinear parabolic equations and the Ricci flow on manifolds with boundary, *J. reine angew. Math.* 683 (2013) 97–118.
- [23] A. Pulemotov, Metrics with prescribed Ricci curvature near the boundary of a manifold, *Math. Ann.* 357 (2013) 969–986.
- [24] A. Pulemotov, The Dirichlet problem for the prescribed Ricci curvature equation on cohomogeneity one manifolds, submitted, [arXiv:1303.2419 \[math.AP\]](#).
- [25] Y. Shen, New results on some dynamical and stationary problems in geometry, Ph.D. Dissertation, Stanford University, 1992.
- [26] Y. Shen, On Ricci deformation of a Riemannian metric on manifold with boundary, *Pacific J. Math.* 173 (1996) 203–221.
- [27] P. Topping, *Lectures on the Ricci flow*, Cambridge University Press, Cambridge, 2006.
- [28] P. Weidemaier, Local existence for parabolic problems with fully nonlinear boundary condition; an L_p -approach, *Ann. Mat. Pura Appl.* 160 (1991) 207–222.