INTEGRALS OF LOGARITHMIC FORMS ON SEMI-ALGEBRAIC SETS AND A GENERALIZED CAUCHY FORMULA PART II: GENERALIZED CAUCHY FORMULA (DETAILED VERSION)

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Abstract. This paper is the continuation of the paper arXiv:1509.06950, which is Part I under the same title. In this paper, we prove a generalized Cauchy formula for the integrals of logarithmic forms on products of projective lines, and give an application to the construction of Hodge realization of mixed Tate motives.

[1]

Contents

1. Introduction

This paper is the continuation of the paper [10], which is Part I under the same title. In this paper, we prove a generalized Cauchy formula for the integrals of logarithmic forms on products $P^n = (\mathbb{P}^1_{\mathbb{C}})^n$ of projective lines $\mathbb{P}^1_{\mathbb{C}}$. As an application, we define a variant of the Hodge realization functor for the category of mixed Tate motives defined by Bloch and Kriz [4]. In the sequel to this paper, we prove that our construction coincides with the original one defined by Bloch and Kriz. The motivation of our series of papers is to understand the Hodge realization functor via integral of logarithmic differential forms.

Before going into the detail, we describe a simple example of the generalized Cauchy formula. Let $\omega_2 = (2\pi i)^{-2}\dfrac{dz_1}{z_1} \wedge \dfrac{dz_2}{z_2}$, $\omega_1 = (2\pi i)^{-1}\dfrac{dz_2}{z_2}$ be a holomorphic two resp. one form on $(\mathbb{C}-\{0\})^2$ resp. $\mathbb{C}-\{0\}$. Let $0 < a < b$ be real numbers and $\overline{D} = \{z_1 \in \mathbb{C} \mid |z_1| \leq 1\}$

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be the closed unit disk. Let $\sigma = \overline{D} \times [a,b]$ be a 3-chain in $\mathbb{C}^2$. Its topological boundary is denoted by $\delta\sigma$. Then we have the following identity, called the generalized Cauchy formula:

$$\begin{aligned}\int_{\delta\sigma} \omega_2 &= \int_{\delta\overline{D}\times[a,b]} \omega_2 + \int_{(\overline{D}\times\{b\})-(\overline{D}\times\{a\})} \omega_2 \\ &= \int_{[a,b]} \omega_1 = \int_{\sigma\cap(\{0\}\times\mathbb{C})} \omega_1.\end{aligned} \tag{1.0.1}$$

In the leftmost side of the above equality, although the differential form ω_2 is not defined on $\delta\sigma \cap (\{0\} \times \mathbb{C})$, the integral is defined as an improper integral. To show the second equality we use Fubini's theorem and the classical Cauchy formula. To generalize the above formula, we define

(1) suitable subspaces $AC_3(\mathbb{C}^2)$ and $AC_2(\mathbb{C}^2)$ of 3-chains and 2-chains in $\mathbb{C}^2$ and a sub space $AC_1(\mathbb{C})$ of 1-chains in $\mathbb{C}$, and
(2) a "face map" $\partial : AC_3(\mathbb{C}^2) \to AC_1(\mathbb{C})$ which generalizes the above operation $\sigma \mapsto \sigma \cap (\{0\} \times \mathbb{C})$ taking multiplicities into account.

satisfying the following properties.

(1) The topological boundary map δ induces the map $\delta : AC_3(\mathbb{C}^2) \to AC_2(\mathbb{C}^2)$.
(2) The improper integrals $\displaystyle\int_{\gamma_2} \omega_2$ and $\displaystyle\int_{\gamma_1} \omega_1$ converge for $\gamma_2 \in AC_2(\mathbb{C}^2)$ and $\gamma_1 \in AC_1(\mathbb{C})$.

Using the above setting, the generalized Cauchy formula (1.0.1) is the equality

$$I_1(\partial\gamma) + I_2(\delta\gamma) = 0$$

where the map I_2 (resp. I_1) is defined as the improper integral $\gamma_2 \mapsto -\displaystyle\int_{\gamma_2} \omega_2$ (resp. $\gamma_1 \mapsto \displaystyle\int_{\gamma_1} \omega_1$) for $\gamma_2 \in AC_2(\mathbb{C}^2)$ (resp. $\gamma_1 \in AC_1(\mathbb{C})$).

In this paper, we study the generalized Cauchy formula for $P^n = (\mathbb{P}^1_{\mathbb{C}})^n$. Let $(z_1, \dots, z_n)$ be the coordinates of P^n, and $\mathbf{D}^n$ be the divisor of P^n defined by $\prod_i^n (z_i - 1) = 0$. For a semi-algebraic triangulation K of P^n such that $\mathbf{D}^n$ is a subcomplex of K, the relative chain complex of $(K, \mathbf{D}^n)$ with the coefficients in $\mathbb{Q}$ is denoted by $C_\bullet(K, \mathbf{D}^n; \mathbb{Q})$. In §2 we define a subcomplex $AC_\bullet(K, \mathbf{D}^n; \mathbb{Q})$ of $C_\bullet(K, \mathbf{D}^n; \mathbb{Q})$ consisting of elements satisfying certain admissibility conditions (see Definition 2.5). By taking the inductive limit of $AC_\bullet(K, \mathbf{D}^n; \mathbb{Q})$ of all semi-algebraic triangulations under subdivisions, we get a complex $AC_\bullet(P^n, \mathbf{D}; \mathbb{Q})$. The moving lemma Proposition 2.8 asserts that imposing the admissibility condition does not change the homology groups.

In §3, we define a face map

$$\partial_{i,\alpha} : AC_i(P^n, \mathbf{D}^n; \mathbb{Q}) \to AC_{i-2}(P^{n-1}, \mathbf{D}^n; \mathbb{Q}) \quad (1 \le i \le n, \alpha = 0, \infty)$$

with respect to the hypersurface $H_{i,\alpha} = \{z_i = \alpha\}$. Roughly this map is taking the intersection with the face $H_{i,\alpha}$. More precisely, it is defined by the cap product with a Thom cocycle T of the face $H_{i,\alpha}$. The cubical differential ∂ is defined to be the alternating sum of the face maps $\partial_{i,\alpha}$.

In §4 we prove a generalized Cauchy formula. Let $\gamma = \sum a_\sigma \sigma$ be an element of $AC_n(P^n, \mathbf{D}; \mathbb{Q})$, where σ's are n-simplexes in a triangulation K of P^n and $a_\sigma \in \mathbb{Q}$. Let ω_n be the rational differential form on P^n defined by

$$\omega_n = \frac{1}{(2\pi i)^n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

By a result of Part I, the integral $\displaystyle\int_\sigma \omega_n$ converges absolutely if σ is admissible (Theorem 4.1) and we define a homomorphism I_n as the following integration:

$$I_n : AC_n(P^n, \mathbf{D}^n; \mathbb{Q}) \to \mathbb{C} : \sum a_\sigma \sigma \mapsto (-1)^{\frac{n(n-1)}{2}} \sum_\sigma \int_\sigma a_\sigma \omega_n.$$

Then the generalized Cauchy formula (Theorem 4.3) asserts that, for an element $\gamma \in AC_{n+1}(P^n, \mathbf{D}^n; \mathbb{Q})$, the equality

$$I_{n-1}(\partial\gamma) + (-1)^n I_n(\delta\gamma) = 0 \tag{1.0.2}$$

holds. If the chain γ does not meet any of the faces, then we have $\partial\gamma = 0$, and (1.0.2) holds by the Stokes formula. In general, the correction term for the Stokes formula arising from the logarithmic singularity is computed in terms of $\partial\gamma$. The main results of [10] are used in the proof of Theorem 4.3 which is fairly complex. If the reader is mainly interested in its applications and is willing to accept this fact, it is possible to read §5 separately.

We construct a variant of the Hodge realization functor for the category of mixed Tate motives in §5. We briefly recall the construction of the category of mixed Tate motives given in the paper of Bloch and Kriz ([4]). Let $\mathbf{k}$ be a subfield of $\mathbb{C}$. Bloch defines a graded DGA $N_{\mathbf{k}}$ of algebraic cycle complexes of $\mathbf{k}$. The 0-th cohomology $\mathcal{H} = H^0(B(N_{\mathbf{k}}))$ of the bar complex $B(N_{\mathbf{k}})$ of $N_{\mathbf{k}}$ becomes a commutative Hopf algebra with a grading $\mathcal{H} = \oplus_r \mathcal{H}_r$. They define the category of mixed Tate motives as that of graded comodules over the Hopf algebra $\mathcal{H}$. They also define the ℓ-adic and the Hodge realization functors from the category of mixed Tate motives over Spec($\mathbf{k}$) to that of ℓ-adic Galois representations of the field $\mathbf{k}$, and that of mixed Tate Hodge structures. The papers [9] is one of the works based on this construction.

In [4], they also present an alternative construction of the Hodge realization functor using integral of logarithmic differential forms ω_n on P^n. For this they assume the existence of a certain DGA $\mathcal{DP}$ of topological chains satisfying the following conditions.

(a) The DGA $\mathcal{DP}$ contains the DGA $N_{\mathbf{k}}$,
(b) The integral of the form $\dfrac{dz_1}{z_1} \wedge \cdots \wedge \dfrac{dz_n}{z_n}$ on elements in $\mathcal{DP}$ converges.
(c) The generalized Cauchy integral formula holds for the integral in (b).
(d) The natural map $\tau^* : H^*(B(N)) \to H^*(B(\mathcal{DP}, N, \mathbb{Q}))$ is 0, which implies the E_1-degeneracy of the spectral sequence obtained from a certain filtration on $B(\mathcal{DP}, N, \mathbb{Q})$ (for the precise statement see [4] (8.6)).

The cubical differential ∂ and the topological differential δ make the direct sum of $AC_i(P^n, \mathbf{D}^n; \mathbb{Q})^{\mathrm{alt}}$'s a double complex. Here $^{\mathrm{alt}}$ indicates that we take the subspace on which a certain group G_n acts by a certain character sign. The associated simple complex of this, which we denote by $AC^\bullet$, is a cohomological complex. Let I be the map from $AC^\bullet$ to $\mathbb{C}$ defined by $I = \sum_n I_n$. The equality (1.0.2) implies that the map I is a homomorphism of complexes. We use $AC^\bullet$ in place of $\mathcal{DP}$, which enjoys the following properties:

(1) There exists a natural injection $N_{\mathbf{k}} \to AC^\bullet$. Via this map, we define an $\mathcal{H}$ comodule $\mathcal{H}_B$ in Definition 5.4.
(2) There is a canonical map $\mathbb{Q} \to AC^\bullet$ which is a quasi-isomorphism.
(3) The above homomorphism I induces a quasi-isomorphism from $AC^\bullet \otimes \mathbb{C}$ to $\mathbb{C}$.

Using the properties (2) and (3), we show the E_1-degeneracy of the spectral sequence obtained from a similar filtration on $B(\mathbb{Q}, N, AC^\bullet)$ as in (d).

We give a rough sketch of our construction of the Hodge realization. We define an $\mathcal{H}$ comodule $\mathcal{H}_B$ resp. $\mathcal{H}_{dR}$ as $H^0(B(\mathbb{Q}, N, AC^\bullet))$ resp. $H^0(B(\mathbb{Q}, N, \mathbb{C}))$. The homomorphism I in (3) yields a comparison isomorphism $c : \mathcal{H}_B \otimes \mathbb{C} \to \mathcal{H}_{dR}$, and via this comparison map, we construct a "universal" mixed Tate Hodge structure $\mathcal{H}_{Hg} = (\mathcal{H}_B, \mathcal{H}_{dR}, c)$ with a left "coaction" Δ_{Hg} of $\mathcal{H}$ (see (5.5.5)). We define a functor Φ from the category of graded right $H^0(B(N_{\mathbf{k}}))$-comodules (V, Δ_V) to that of mixed Tate Hodge structures by the "twisted cotensor product"

$$\Phi(V) = \ker\left(\bigoplus_i V_i \otimes \mathcal{H}_{Hg}(-i) \xrightarrow{\Delta_V \otimes id - id \otimes \Delta_{Hg}} \bigoplus_{i,j} V_i \otimes \mathcal{H}_j \otimes \mathcal{H}_{Hg}(-i-j)\right).$$

More precisely we consider graded complexes $AC^\bullet \otimes \mathbb{Q}(*)$ resp. $\mathbb{C}(*)$ in stead of $AC^\bullet$ resp. $\mathbb{C}$. See (5.4.3) for details.

In the sequel to this paper, we will prove that the the above functor Φ is isomorphic to that defined by Bloch-Kriz.

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2. Admissible chain complexes

2.1. **Semi-algebraic triangulation.** Let k be a non-negative integer. A k-simplex in an Euclidean space $\mathbb{R}^n$ is the convex hull of affinely independent points $a_0, \cdots, a_k$ in $\mathbb{R}^n$. A *finite simplicial complex* of $\mathbb{R}^n$ is a finite set K consisting of simplexes such that (1) for all $s \in K$, all the faces of s belong to K and (2) for all $s, t \in K$, $s \cap t$ is either the empty set or a common face of s and t. We denote by K_p the set of p-simplexes of K. For a simplex σ in K, the relative interior of σ (=the complement of the union of the proper faces of σ) is denoted by σ°. For a finite simplicial complex K, the union of simplexes in K as a subset of $\mathbb{R}^n$ is denoted by $|K|$.

As for the definition of semi-algebraic set and their fundamental properties, see [5].

Theorem 2.1 ([5], Theorem 9.2.1). *Let P be a compact semi-algebraic subset of $\mathbb{R}^m$. The set P is triangulable, i.e. there exists a finite simplicial complex K and a semi-algebraic homeomorphism $\Phi_K : |K| \to P$. Moreover, for a given finite family $S = \{S_j\}_{j=1,\cdots,q}$ of semi-algebraic subsets of P, we can choose a finite simplicial complex K and a semi-algebraic homeomorphism $\Phi_K : |K| \to P$ such that every S_j is the union of a subset of $\{\Phi_K(\sigma^\circ)\}_{\sigma\in K}$.*

Remark 2.2. (1) *By [5] Remark 9.2.3 (a), the map Φ_K can be taken so that the map Φ_K is facewise regular embedding i.e. for each $\sigma \in K$, $\Phi_K(\sigma^\circ)$ is a regular submanifold of $\mathbb{R}^m$.*
(2) *The pair (K, Φ_K) as in Theorem 2.1 is called a semi-algebraic triangulation of P; we will then identify $|K|$ with P. A projective real or complex variety V is regarded*

as a compact semi-algebraic subset of an Euclidean space by [5] *Theorem 3.4.4, thus the above theorem applies to* V.

Notation 2.3. *For a subcomplex* L *of* K, *the space* $|L|$ *is a subspace of* $|K|$. *A subset of* $|K|$ *of the form* $|L|$ *is also called a subcomplex. If a subset* S *of* $|K|$ *is equal to* $|M|$ *for a subcomplex* M *of* K, *then* M *is often denoted by* $K \cap S$.

2.2. **The complex** $AC_\bullet(P^n, \mathbf{D}^n; \mathbb{Q})$**.** Let $(z_1, \cdots, z_n)$ be the coordinates of $P^n = (\mathbb{P}^1_{\mathbb{C}})^n$. For $i = 1, \cdots, n$ and $\alpha = 0, \infty$, we denote by $H_{i,\alpha}$ the subvariety of P^n defined by $z_i = \alpha$. The intersection of several $H_{i,\alpha}$'s is called a *cubical face.* We set

$$\mathbf{H}^n = \bigcup_{\substack{1\le i\le n\\ \alpha=0,\infty}} H_{i,\alpha}, \quad \mathbf{D}^n = \bigcup_{i=1}^{n} \{z_i = 1\}, \quad \square^n = P^n - \mathbf{D}^n. \tag{2.2.1}$$

In the following we suppose that each triangulation K of P^n is semi-algebraic , and that the divisor $\mathbf{D}^n$ is a subcomplex of K. Let K be a triangulation of P^n. We denote by $C_\bullet(K;\mathbb{Q})$ resp. $C_\bullet(K, \mathbf{D}^n;\mathbb{Q})$ the chain complex resp. the relative chain complex of K. An element of $C_p(K;\mathbb{Q})$ for $p \ge 0$ is written as $\sum a_\sigma \sigma$ where the sum is taken over p-simplexes of K. By doing so, it is agreed upon that an orientation has been chosen for each σ. By abuse of notation, an element of $C_\bullet(K, \mathbf{D}^n;\mathbb{Q})$ is often described similarly.

Definition 2.4. *For an element* $\gamma = \sum a_\sigma \sigma$ *of* $C_p(K;\mathbb{Q})$, *we define the support* $|\gamma|$ *of* γ *as the subset of* $|K|$ *given by*

$$|\gamma| = \bigcup_{\substack{\sigma\in K_p\\ a_\sigma\neq 0}} \sigma. \tag{2.2.2}$$

Under Notation 2.3, For an element $\gamma = \sum a_\sigma \sigma$ of $C_p(K;\mathbb{Q})$, $|\gamma|$ is sometimes regarded as a subcomplex of K.

Definition 2.5. *Let* $p \ge 0$ *be an integer.*

(1) *(Admissibility) A semi-algebraic subset* S *of* P *is said to be admissible if for each cubical face* H, *the inequality*

$$\dim(S \cap (H - \mathbf{D}^n)) \le \dim S - 2 \operatorname{codim} H$$

holds. Here note that $\dim(S \cap (H - \mathbf{D}^n))$ *and* $\dim S$ *are the dimensions as semi-algebraic sets, and* $\operatorname{codim} H$ *means the codimension of the subvariety* H *of* P^n.

(2) *Let* γ *be an element of* $C_p(K, \mathbf{D}^n;\mathbb{Q})$. *Then* γ *is said to be admissible if the support of a representative of* γ *in* $C_p(K;\mathbb{Q})$ *is admissible. This condition is independent of the choice of a representative.*

(3) *We set*

$$AC_p(K, \mathbf{D}^n;\mathbb{Q}) = \{\gamma \in C_p(K, \mathbf{D}^n;\mathbb{Q})) \mid \gamma \text{ and } \delta\gamma \text{ are admissible } \}$$

2.3. Subdivision and inductive limit.

Definition 2.6. *Let $(K, \Phi_K : |K| \to P)$ be a triangulation of a compact semi-algebraic set P. Another triangulation $(K', \Phi_{K'} : |K'| \to P)$ is* a subdivision of K *if :*

(1) *The image of each simplex of K' under the map $\Phi_{K'}$ is contained in the image of a simplex of K under the map Φ_K.*
(2) *The image of each simplex of K under the map Φ_K is the union of the images of simplexes of K' under $\Phi_{K'}$.*

If K' is a subdivision of a triangulation K, there is a natural homomorphism of complexes $\lambda : C_\bullet(K, \mathbf{D}^n; \mathbb{Q}) \to C_\bullet(K', \mathbf{D}^n; \mathbb{Q})$ called the subdivision operator. See for example [14] Theorem 17.3 for the definition. For a simplex σ of K, the chain $\lambda(\sigma)$ is carried by $K' \cap \sigma$, and so that the map λ sends $AC_\bullet(K, \mathbf{D}^n; \mathbb{Q})$ to $AC_\bullet(K', \mathbf{D}^n; \mathbb{Q})$. By Theorem 2.1 two semi-algebraic triangulations have a common subdivision. Since the map λ and the differential δ commute, the complexes $C_\bullet(K, \mathbf{D}^n; \mathbb{Q})$ and $AC_\bullet(K, \mathbf{D}^n; \mathbb{Q})$ form inductive systems indexed by triangulations K of P^n.

Definition 2.7. *We set*

$$C_\bullet(P^n, \mathbf{D}^n; \mathbb{Q}) = \varinjlim_K C_\bullet(K, \mathbf{D}^n; \mathbb{Q}), \quad AC_\bullet(P^n, \mathbf{D}^n; \mathbb{Q}) = \varinjlim_K AC_\bullet(K, \mathbf{D}^n; \mathbb{Q}).$$

Here the limit is taken on the directed set of triangulations.

A proof of the following Proposition will be given in Appendix A.

Proposition 2.8 (Moving lemma)**.** *The inclusion of complexes*

$$\iota : \ AC_\bullet(P^n, \mathbf{D}; \mathbb{Q}) \to C_\bullet(P^n, \mathbf{D}; \mathbb{Q}) \tag{2.3.3}$$

is a quasi-isomorphism.

3. Face map and cubical differential

Recall that a subcomplex L of a simplicial complex K is called a full subcomplex , if all the vertices of a simplex σ in K belong to L, then σ belongs to L.

Definition 3.1 (Good triangulation)**.** *We define a family $\mathcal{L}$ of subsets of P^n by*

$$\mathcal{L} = \{H_{J_1} \cup \cdots \cup H_{J_k}\}_{(J_1, \ldots, J_k)},$$

where H_{J_j} are cubical faces of P^n. In short, a member of $\mathcal{L}$ is the union of several cubical faces. A finite semi-algabraic triangulation K of P^n is called a good triangulation if K satisfies the following conditions.

(1) *The divisor $\mathbf{D}^n$ is a subcomplex of K.*
(2) *The map $\Phi_K : |K| \to P^n$ is facewise regular embedding. cf. Remark 2.2.*
(3) *Each element $L_i \in \mathcal{L}$ is a full subcomplex of K i.e. there exists a full subcomplex M_i of K such that $L_i = |M_i|$.*
(4) *For $i = 1, \cdots, n$, the subsets $\{|z_i| \le 1\}$ are subcomplexes of K.*

In particular, if K is a good triangulation, then for any simplex σ of K and $L_i \in \mathcal{L}$, the intersection $\sigma \cap L_i$ is a (simplicial) face of σ. This is the primary reason to consider the condition (3).

Remark 3.2. *For a simplicial complex K, we denote by $\mathrm{sd}\,K$ its barycentric subdivision. If L is a subcomplex of K, then $\mathrm{sd}\,L$ is a full subcomplex of $\mathrm{sd}\,K$ (See for example Exercise 3.2 of [15]). . It follows that if K is a semi-algebraic triangulation of P^n which is a facewise regular embedding, and such that $\mathbf{D}^n$ and each $L_i \in \mathcal{L}$ are subcomplexes of K, then $\mathrm{sd}\,K$ is a good triangulation.*

In the following, each triangluation K of $P^n = (\mathbb{P}^1_{\mathbb{C}})^n$ is assumed to be good in the sense of Definition 3.1.

3.1. **Thom cocycles.**

3.1.1. *Definition of Thom cocycles.* Let K be a good triangulation of $P^n = (\mathbb{P}^1_{\mathbb{C}})^n$.

Let Δ be the subset $\{|z_1| < 1\}$ of P^n. We set $L_1 = K \cap H_{1,0}$ and

$$N = \bigcup_{\sigma \in K,\, \sigma \cap H_{1,0} \neq \emptyset} \sigma, \quad W = \bigcup_{\sigma \in K,\, \sigma \cap H_{1,0} = \emptyset} \sigma.$$

For spaces X and Y such that $X \supset Y$, we denote by $H^i_{\mathrm{sing}}(X;\mathbb{Q})$ (resp. $H^i_{\mathrm{sing}}(X,Y;\mathbb{Q})$) the singular cohomology (resp. the relative singular cohomology) of X (resp. (X,Y)) with $\mathbb{Q}$-coefficients. Under the comparison isomorphism $H^1_{\mathrm{sing}}(\Delta - H_{1,0};\mathbb{C}) \simeq H^1_{dR}(\Delta - H_{1,0})$, the de Rham class $[\frac{dz_1}{2\pi i z_1}]$ of $\frac{dz_1}{2\pi i z_1}$ is contained in the subgroup $H^1_{\mathrm{sing}}(\Delta - H_{1,0};\mathbb{Q})$ of $H^1_{\mathrm{sing}}(\Delta - H_{1,0};\mathbb{C})$. Since L_1 is a full subcomplex of K, W is a deformation retract of $P^n - H_{1,0}$. (This can be seen as follows. Each simplex σ in N not contained in L_1 is the join $\tau * \eta$ where $\tau \in L_1$ and $\eta \in W$. The linear projection $\pi_{\sigma,\eta} : \sigma - \tau \to \eta$ in Definition 4.12 is glued to a deformation retract from $P^n - H_{1,0}$ to W.) Therefore we have isomorphisms

$$H^2(K,W;\mathbb{Q}) \xleftarrow{\simeq} H^2_{\mathrm{sing}}(P^n, P^n - H_{1,0};\mathbb{Q}) \xrightarrow{\simeq} H^2_{\mathrm{sing}}(\Delta, \Delta - H_{1,0};\mathbb{Q}). \tag{3.1.1}$$

Here the second isomorphism holds by excision.

Definition 3.3. *A simplicial cocycle T in $C^2(K,W;\mathbb{Q})$ is a Thom cocycle if its cohomology class is equal to $\delta[\frac{dz_1}{2\pi i z_1}]$ in $H^2_{\mathrm{sing}}(\Delta, \Delta - H_{1,0};\mathbb{Q})$ via the isomorphism (3.1.1). Here δ denotes the connecting homomorphism*

$$\delta : H^1_{\mathrm{sing}}(\Delta - H_{1,0};\mathbb{Q}) \to H^2_{\mathrm{sing}}(\Delta, \Delta - H_{1,0};\mathbb{Q}).$$

A $\mathbb{C}$-valued Thom cocycle in $C^2(K,W;\mathbb{C})$ is defined similarly.

We will give some examples of Thom cocycles.

3.1.2. *Thom form.* Let ϵ be a positive real number and Let ρ be a $[0,1]$-valued C^∞-function on Δ such that

(1) $\rho = 0$ on $\Delta \cap \{|z_1| < \frac{1}{2}\epsilon\}$.
(2) $\rho = 1$ on $\Delta \cap \{\epsilon < |z_1|\}$.

Then $c_\rho = \frac{1}{2\pi i}\rho\frac{dz_1}{z_1}$ defines an element of $C^1_{\mathrm{sing}}(P^n - H_{1,\infty};\mathbb{C})$. We set $T_\rho = dc_\rho = \frac{1}{2\pi i}d\rho \wedge \frac{dz}{z}$. For ϵ sufficiently small we have $c_\rho = \frac{1}{2\pi i}\frac{dz_1}{z_1}$ on $W - H_{1,\infty}$, and T_ρ defines the same class as $\delta[\dfrac{dz_1}{2\pi i z_1}]$ in $H^2_{\mathrm{sing}}(P^n - H_{1,\infty}, W - H_{1,\infty}; \mathbb{C}) \simeq H^2_{\mathrm{sing}}(P^n, W; \mathbb{C})$, so it is a rational class.

Definition 3.4. *The cocycle $T_\rho \in C^2(K, W; \mathbb{C})$ is a $\mathbb{C}$-valued Thom cocycle. We call the above cocycle T_ρ a Thom form.*

3.1.3. *Singular Thom cocycle $T^B_{H_{1,0}}$.* We first note that this cocycle is not used in the rest of the paper, and we omit the proof that this is a Thom cocycle. Suppose that Δ is a subcomplex of K and $\Delta \cap W \subset \Delta - \{0\}$ is a deformation retract. For a 1-simplex $\sigma \in C^1(\Delta)$, we set

$$L^B(\gamma) = \begin{cases} 0 & \text{if } \sigma \not\subset W \\ \left[\dfrac{1}{2\pi}\left(\mathrm{Im}\displaystyle\int_\sigma \frac{dz}{z} + \arg(\gamma(0))\right)\right] & \text{if } \sigma \subset W. \end{cases}$$

Here $[r]$ denotes the Gauss symbol of a real number r and $\arg(z)$ is the argument of a complex number z in $[0, 2\pi)$. Note that the cochain L^B counts the intersection number (with sign) of σ and the positive part of real axis. Then $T^B = dL^B \in C^2(\Delta, \Delta \cap W)$ becomes a Thom cocycle.

We remark that the cocycle T^B counts the winding number of the boundary of relative 2-cycle.

3.2. **The cap product with a Thom cocycle.**

3.2.1. *Simplicial cap product.*

Definition 3.5 (Ordering of complex, good ordering)**.** *Let K be a good triangulation of P^n.*

(1) *A partial ordering on the set of vertices in K is called an ordering of K, if the restriction of the ordering to each simplex is a total ordering.*
(2) *Let H_J be a cubical face of P^n. An ordering of K is said to be good with respect to H_J if it satisfies the following condition. If a vertex v is on H_J and $w \geq v$ for a vertex w, then $w \in H_J$.*

We denote by $[a_0, \cdots, a_k]$ the simplex spanned by $a_0, \cdots, a_k$. Let $\mathcal{O}$ be a good ordering of K with respect to $H_{1,0}$. We recall the definition of the cap product $\overset{\mathcal{O}}{\cap} : C^p(K) \otimes C_k(K) \to C_{k-p}(K)$. For a simplex $\alpha = [v_0, \cdots, v_k]$ such that $v_0 < \cdots < v_k$ and $u \in C^p(K)$, we define

$$u \overset{\mathcal{O}}{\cap} \alpha = u([v_0, \ldots, v_p])[v_p, \ldots, v_k]. \tag{3.2.2}$$

One has the boundary formula

$$\delta(u \overset{\mathcal{O}}{\cap} \alpha) = (-1)^p (u \overset{\mathcal{O}}{\cap} (\delta\alpha) - (du) \overset{\mathcal{O}}{\cap} \alpha) \tag{3.2.3}$$

where du denotes the coboundary of u, see [12], p.239 (note the difference in sign convention from [14]). Thus if u is a cocycle, $\delta(u \overset{\mathcal{O}}{\cap} \alpha) = (-1)^p u \overset{\mathcal{O}}{\cap} (\delta\alpha)$.

Proposition 3.6. *Let T be a Thom cocycle and $\mathcal{O}$ a good ordering of K with respect to $H_{1,0}$.*

(1) *The map $T\overset{\mathcal{O}}{\cap}$ and the differential δ commute.*

(2) *The image of the homomorphism* $T\overset{\mathcal{O}}{\cap}$ *is contained in* $C_{k-2}(L_1;\mathbb{Q})$*, where* $L_1 = K\cap H_{1,0}$*. As a consequence, we have a homomorphism of complexes*

$$T\overset{\mathcal{O}}{\cap}:\ C_k(K,\mathbf{D}^n;\mathbb{Q}) \to C_{k-2}(L_1,\mathbf{D}^{n-1};\mathbb{Q}). \tag{3.2.4}$$

Proof. (1). Since T is a cocycle of even degree, we have $\delta(T\overset{\mathcal{O}}{\cap}\sigma) = T\overset{\mathcal{O}}{\cap}(\delta\sigma)$ for a simplex σ.
(2). If $v_2 \notin H_{1,0}$, then $[v_0,v_1,v_2]\cap H_{1,0} = \emptyset$ and $T([v_0,v_1,v_2]) = 0$ since the cochain T vanishes on W. If $v_2\in H_{1,0}$, then the vertices $v_2,\cdots,v_k$ are on $H_{1,0}$, and we have $[v_2,\ldots,v_k]\subset H_{1,0}$ since $H_{1,0}$ is a full subcomplex of K. Thus the assertion holds. □

3.2.2. *Independence of* T *and ordering.* Let K be a good triangulation, and let L_1 be the subcomplex $K\cap H_{1,0}$.

Proposition 3.7. *Let* $T\in C^2(K,W;\mathbb{Q})$ *be a Thom cocycle of the face* $H_{1,0}$ *and let* γ *be an element of* $AC_k(K,\mathbf{D}^n;\mathbb{Q})$*. Then we have the following.*

(1) *The chain* $T\overset{\mathcal{O}}{\cap}\gamma$ *is an element in* $AC_{k-2}(L_1,\mathbf{D}^{n-1};\mathbb{Q})$*.*
(2) *The chain* $T\overset{\mathcal{O}}{\cap}\gamma$ *is independent of the choice of a Thom cocycle* T *and a good ordering* $\mathcal{O}$*. Thus the map*

$$T\overset{\mathcal{O}}{\cap}: AC_k(K,\mathbf{D}^n;\mathbb{Q})\to AC_{k-2}(L_1,\mathbf{D}^{n-1};\mathbb{Q}).$$

induced by (3.2.4) is denoted by $T\cap$*.*
(3) *Let* K' *be a good subdivision of* K*. We set*

$$W' = \bigcup_{\substack{\sigma\in K'\\ \sigma'\cap H_{1,0}=\emptyset}} \sigma'.$$

Let $T'\in C^2(K',W';\mathbb{Q})$ *be a Thom cocycle, and* $\mathcal{O}'$ *a good ordering of* K' *with respect to* $H_{1,0}$*. We set* $L_1' = K'\cap H_{1,0}$*. Then we have the following commutative diagram*

$$\begin{array}{ccc} AC_k(K,\mathbf{D}^n;\mathbb{Q}) & \xrightarrow{T\overset{\mathcal{O}}{\cap}} & AC_{k-2}(L_1,\mathbf{D}^{n-1};\mathbb{Q}) \\ \lambda\downarrow & & \downarrow\lambda \\ AC_k(K',\mathbf{D}^n;\mathbb{Q}) & \xrightarrow{T'\overset{\mathcal{O}'}{\cap}} & AC_{k-2}(L_1',\mathbf{D}^{n-1};\mathbb{Q}) \end{array} \tag{3.2.5}$$

where the vertical maps λ *are subdivision operators.*

Proof. (1). For an element $z\in C_k(K,\mathbf{D}^n;\mathbb{Q})$, we have $T\overset{\mathcal{O}}{\cap}z\in C_{k-2}(L_1,\mathbf{D}^n;\mathbb{Q})$ by Proposition 3.6 (2). By the definition of the cap product, we see that the set $|T\overset{\mathcal{O}}{\cap}z|\subset|z|\cap H_{1,0}$. It follows that if z is admissible i.e. $|z|-\mathbf{D}^n$ meets all the cubical faces properly, then $|T\overset{\mathcal{O}}{\cap}z|-\mathbf{D}^{n-1}$ meets all the cubical faces of $H_{1,0}$ properly. Similarly, if the chain δz is admissible, then $T\overset{\mathcal{O}}{\cap}(\delta z)$ is admissible in $H_{1,0}$. By Propisition 3.6 (1) we have the equality $\delta(T\overset{\mathcal{O}}{\cap}z) = T\overset{\mathcal{O}}{\cap}(\delta z)$.
(2) A proof will be given in Section B.2.
(3) A proof will be given in Section B.3.

□

By taking the inductive limit of the homomorphism

$$T\cap : AC_\bullet(K, \mathbf{D}^n; \mathbb{Q}) \to AC_{\bullet-2}(L_1, \mathbf{D}^{n-1}; \mathbb{Q}).$$

for subdivisions, we get a homomorphism

$$T\cap :\ AC_\bullet(P^n, \mathbf{D}^n; \mathbb{Q}) \to AC_{\bullet-2}(P^{n-1}, \mathbf{D}^n; \mathbb{Q}). \tag{3.2.6}$$

Definition 3.8. *The map $T\cap$ of (3.2.6) is denoted by $\partial_{1,0}$, and called the face map of the face $H_{1,0}$.*

By the symmetry, we have maps $\partial_{i,\alpha}$ for $i = 1, \cdots, n$ and $\alpha = 0, \infty$.

3.3. **Cubical differentials.**

Definition 3.9 (Cubical differential)**.** *The map ∂ is defined by the equality*

$$\partial = \sum_{i=1}^{n} (-1)^{i-1}(\partial_{i,0} - \partial_{i,\infty}) :\ AC_\bullet(P^n, \mathbf{D}^n; \mathbb{Q}) \to AC_{\bullet-2}(P^{n-1}, \mathbf{D}^n; \mathbb{Q}). \tag{3.3.7}$$

Proposition 3.10. *The composite*

$$\partial^2 : AC_\bullet(P^n, \mathbf{D}^n; \mathbb{Q}) \to AC_{\bullet-4}(P^{n-2}, \mathbf{D}^n; \mathbb{Q})$$

is the zero map.

A proof of this proposition will be given in Section B.4.

4. The Generalized Cauchy formula

4.1. **Statement of the generalized Cauchy formula.** Let K be a good triangulation of P^n. We define the rational differential form ω_n on P^n by

$$\omega_n = \frac{1}{(2\pi i)^n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

By applying Theorem 4.4 [10] to the case where $A = \sigma$, $m = \dim \sigma = n$ and $\omega = \omega_n$ we have the following theorem.

Theorem 4.1. *Let σ be an admissible n-simplex of K. The integral*

$$\int_\sigma \omega_n \tag{4.1.1}$$

converges absolutely.

Using Theorem 4.1, the following is well defined.

Definition 4.2. *Let γ be an element in $AC_n(K, \mathbf{D}^n; \mathbb{Q})$, and let $\sum_\sigma a_\sigma \sigma$ be a representative of γ in $C_n(K; \mathbb{Q})$. We define $I_n(\gamma) \in \mathbb{C}$ by*

$$I_n(\gamma) = (-1)^{\frac{n(n-1)}{2}} \sum_\sigma \int_\sigma a_\sigma \omega_n \tag{4.1.2}$$

where the sum is taken over the n-simplexes σ not contained in $\mathbf{D}^n$. The right hand side of (4.1.2) does not depend on the choice of a representative of γ. The map I_n is compatible with subdivisions of triangulations, and we obtain a map

$$I_n : AC_n(P^n, \mathbf{D}^n; \mathbb{Q}) \to \mathbb{C}.$$

In this section, we prove the following theorem.

Theorem 4.3 (Generalized Cauchy formula)**.** *Let γ be an element in $AC_{n+1}(P^n, \mathbf{D}^n; \mathbb{Q})$. Then we have the equality*

$$I_{n-1}(\partial\gamma) + (-1)^n I_n(\delta\gamma) = 0. \tag{4.1.3}$$

Let $\gamma = \sum a_\sigma \sigma$ be an element of $AC_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$ for a good triangulation K. By setting $\partial\gamma = \sum c_\tau \tau$ and $\delta\gamma = \sum b_\nu \nu$, the equality (4.1.3) can be written as

$$\sum_\nu \int_\nu b_\nu \omega_n = \sum_\tau \int_\tau c_\tau \omega_{n-1}. \tag{4.1.4}$$

Here it is understood that the integral of ω_n on a simplex contained in $\mathbf{D}^n$ is zero by definition.

Remark 4.4. *For a n-simplex ν contained in $\mathbf{D}^n$, the integral $\int_\nu \omega_n$ is defined to be zero. This will be justified as follows. Let σ be an admissible $(n+1)$-simplex of a good triangulation K. If $\sigma \subset \mathbf{D}^n$, then $\partial\sigma$ and $\delta\sigma$ are contained in $\mathbf{D}^n$, and σ is irrelevant to the problem. Suppose that $\sigma \not\subset \mathbf{D}^n$ and let ν be an n-simplex contained in $\sigma \cap \mathbf{D}^n$. If $\nu \not\subset \mathbf{H}^n$, then the integral $\int_\nu \omega_n$ should be zero since the restriction of ω_n to $\mathbf{D}^n - \mathbf{H}^n$ is zero. The critical case is that $\nu \subset \mathbf{H}^n$. In this case we have $\nu = \sigma \cap \mathbf{H}^n$ since K is a good triangulation, σ is admissible and $\sigma \not\subset \mathbf{D}^n$. This case will be considered in Proposition 4.5, and it will be clear that our definition is correct.*

4.2. **Outline of the proof of Theorem 4.3.** Let γ be an element in $AC_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$ and $\sum_\sigma a_\sigma \sigma$ be a representative of γ in $C_{n+1}(K; \mathbb{Q})$. We define elements $\gamma_{\mathbf{D}}$ and $\gamma_{\mathbf{D}^c}$ in $C_{n+1}(K, \mathbf{D}; \mathbb{Q})$ by

$$\begin{aligned} \gamma_{\mathbf{D}} &= \sum_{\sigma \cap \mathbf{H}^n \subset \mathbf{D}^n} a_\sigma \sigma, \\ \gamma_{\mathbf{D}^c} &= \gamma - \gamma_{\mathbf{D}}. \end{aligned} \tag{4.2.5}$$

Then $\gamma_{\mathbf{D}}$ is an element in $AC_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$, and as a consequence $\gamma_{\mathbf{D}^c}$ is also an element in $AC_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$. Theorem 4.3 is a consequence of the following Proposition 4.5 and Proposition 4.6.

Proposition 4.5. *Let γ be an element in $AC_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$. Then we have $I_n(\delta\gamma_{\mathbf{D}}) = 0$.*

Since $\partial\gamma_{\mathbf{D}} = 0$ by definition, Theorem 4.3 holds for $\gamma_{\mathbf{D}}$. The proof of Proposition 4.5 is given in §4.5.

Proposition 4.6. *Let γ be an element in $AC_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$. Suppose that γ has a representative $\sum_\sigma a_\sigma \sigma$ in $C_{n+1}(K; \mathbb{Q})$ such that $a_\sigma = 0$ if $\sigma \cap \mathbf{H}^n \subset \mathbf{D}^n$. Then Theorem 4.3 holds for γ.*

Note that if γ has a representative which satisfies the above condition, then it is uniquely determined. Therefore we denote $|\sum_\sigma a_\sigma \sigma|$ by $|\gamma|$.

Let $\{H_{i,\alpha}\}$ be the set of codimension one cubical faces defined in §2.2. We define $\mathbf{H}_h$ as the union of higher codimensional cubical faces, i.e.

$$\mathbf{H}_h = \bigcup_{\substack{1 \le i < i' \le n, \\ \alpha \in \{0,\infty\},\, \beta \in \{0,\infty\}}} (H_{i,\alpha} \cap H_{i',\beta}).$$

In §4.3, we prove the following theorem.

Proposition 4.7 (Generalized Cauchy formula for codimension one face)**.** *Let γ be an element in $AC_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$. Suppose that γ has the representative $\gamma = \sum_\sigma a_\sigma \sigma$ in $C_{n+1}(K;\mathbb{Q})$ such that (1) $a_\sigma = 0$ if $\sigma \cap \mathbf{H}^n \subset \mathbf{D}^n$ and (2) $|\gamma| \cap \mathbf{H}_h = \emptyset$. Then Theorem 4.3 holds for γ.*

Proposition 4.7 will be proved in §4.3. Proposition 4.6 is a consequence of Proposition 4.7 and a limit argument which will be given in §4.4.

4.3. **Proof of Proposition 4.7.** First we observe the following fact.

Lemma 4.8. *Let K be a good triangulation of P^n, and let $\sigma \in K$ be a simplex such that $\sigma \cap \mathbf{H}^n \neq \emptyset$. If $\sigma \cap \mathbf{H}_h = \emptyset$, then there is a unique codimension one cubical face $H_{i,\alpha}$ such that $\sigma \cap \mathbf{H}^n \subset H_{i,\alpha}$. Moreover we have $\sigma \cap H' = \emptyset$ for any other cubical face H'.*

Proof. This follows from the facts that $\mathbf{H}^n = \underset{i,\alpha}{\cup} H_{i,\alpha}$, that each $H_{i,\alpha}$ is a subcomplex of K and that $\sigma \cap \mathbf{H}^n$ is a simplicial face of σ. $\square$

Let γ be an element in $AC_{n+1}(K, \mathbf{D}^n, \mathbb{Q})$, and let $\sum_\sigma a_\sigma \sigma$ be the representative of γ satisfying the condition of Proposition 4.7. We set $\gamma_{i,\alpha} = \sum_{\substack{\sigma\cap\mathbf{H}^n \neq\emptyset \\ \sigma\cap\mathbf{H}^n\subset H_{i,\alpha}}} a_\sigma \sigma$. Then by Lemma 4.8 we have the equality

$$\gamma = \sum_{(i,\alpha)} \gamma_{i,\alpha} \tag{4.3.6}$$

and $|\gamma_{i,\alpha}| \cap H' = \emptyset$ for any cubical face H' other than $H_{i,\alpha}$. Therefore, each $\gamma_{i,\alpha}$ is an element in $AC_{n+1}(K, \mathbf{D}^n, \mathbb{Q})$. It suffices to prove the assertion for each $\gamma_{i,\alpha}$. Therefore to prove Proposition 4.7, we may and do assume that $|\gamma| \cap \mathbf{H}^n \subset H = H_{1,0}$.

Notation 4.9. *We fix a good ordering of K with respect to H. For an $(n+1)$-simplex $\sigma = [v_0, v_1, \cdots, v_{n+1}]$ such that that $v_0 < v_1 < \cdots < v_{n+1}$, we denote $\sigma_f = [v_0, v_1, v_2]$ and $\sigma_b = [v_2, \cdots, v_{n+1}]$.*

To compute the image $\partial\gamma$ of the face map, we choose a Thom form as follows. Let $\rho : \mathbf{R}_+ \to [0,1]$ be a C^∞ function such that

$$\rho(r) = \begin{cases} 0 & (r \leq \frac{1}{2}), \\ 1 & (r \geq 1). \end{cases}$$

Let ϵ be a positive number, and let ρ_ϵ be the function on $\mathbb{P}^1$ defined by $\rho_\epsilon(z_1) = \rho(\frac{|z_1|}{\epsilon})$. The function on P^n given by $(z_1, \ldots, z_n) \mapsto \rho_\epsilon(z_1)$ is also denoted by ρ_ϵ. We take ϵ small enough so that the set $W = \underset{\sigma\in K,\, \sigma\cap H=\emptyset}{\cup} \sigma$ is contained in the set $\{|z_1| \geq \epsilon\}$. Then $T_\epsilon = d\rho_\epsilon \wedge \omega_1$ is a Thom form i.e. $T_\epsilon(\sigma) = \int_\sigma T_\epsilon = 0$ for each 2-simplex σ in W.
We set $\delta\gamma = \sum_\nu b_\nu \nu$. Using the above Thom form, the image of γ under the face map is computed as

$$\partial\gamma = \sum_\sigma \left(\int_{\sigma_f} d\rho_\epsilon \wedge \omega_1 \right) a_\sigma \sigma_b$$

Note that if $\sigma_f \subset W$, then the above integral is zero. If $\sigma_f \not\subset W$, then we have $\sigma_b \subset H$ since $K \cap H$ is a full subcomplex and we take a good ordering. Therefore the assertion (4.1.4) is written as follows:

$$\sum_\nu \int_\nu b_\nu \omega_n = \sum_{\sigma_b \subset H} a_\sigma \int_{\sigma_f} d\rho_\epsilon \wedge \omega_1 \cdot \int_{\sigma_b} \omega_{n-1}. \tag{4.3.7}$$

Let σ be an $(n+1)$-simplex of K such that $a_\sigma \neq 0$. Since the form $a_\sigma \rho_\epsilon \omega_n$ is smooth on a neighborhood of σ, we have the equality

$$\int_\sigma a_\sigma d\rho_\epsilon \wedge \omega_n = \int_{\delta\sigma} a_\sigma \rho_\epsilon \omega_n \tag{4.3.8}$$

by the Stokes formula Theorem 2.9 [10]. The summation of the right hand side of (4.3.8) for σ is equal to

$$\sum_\sigma \int_{\delta\sigma} a_\sigma \rho_\epsilon \omega_n = \sum_\nu \int_\nu b_\nu \rho_\epsilon \omega_n.$$

By Theorem 4.1 and Lebesgue's convergence theorem, we have

$$\lim_{\epsilon\to 0} \int_\nu \rho_\epsilon b_\nu \omega_n = \int_\nu b_\nu \omega_n$$

for each ν with $b_\nu \neq 0$. By summing up (4.3.8) for all σ and taking the limit for $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0} \sum_\sigma \int_\sigma a_\sigma d\rho_\epsilon \wedge \omega_n = \sum_\nu \int_\nu b_\nu \omega_n.$$

Comparing with (4.3.7), to prove Proposition 4.7, it is enough to show the equality:

$$\lim_{\epsilon\to 0} \sum_\sigma \int_\sigma a_\sigma d\rho_\epsilon \wedge \omega_n = \sum_{\sigma_b \subset H} a_\sigma \int_{\sigma_f} d\rho_\epsilon \wedge \omega_1 \cdot \int_{\sigma_b} \omega_{n-1} \tag{4.3.9}$$

We reduce the proof of (4.3.9) to the case where $|\gamma| \cap H$ is a simplex. For this purpose, we prepare the following definition.

Definition 4.10. *Let $\gamma = \sum a_\sigma \sigma$ be an element in $C_{n+1}(K;\mathbb{Q})$, and let τ be a simplex such that $\tau \subset H = H_{1,0}$. We define an element $\gamma^{(\tau)}$ in $C_{n+1}(K, \mathbf{D}^n;\mathbb{Q})$ by*

$$\gamma^{(\tau)} = \sum_{\sigma \cap H = \tau} a_\sigma \sigma. \tag{4.3.10}$$

Suppose that $\gamma \in AC_{n+1}(K, \mathbf{D}^n;\mathbb{Q})$ has the representative $\gamma = \sum_\sigma a_\sigma \sigma$ such that (1) $a_\sigma = 0$ if $\sigma \cap \mathbf{H}^n \subset \mathbf{D}^n$ and (2) $|\gamma| \cap \mathbf{H}^n \subset H = H_{1,0}$. Then we have an equality

$$\gamma = \sum_{\substack{\tau \subset |\gamma| \cap H \\ \tau \not\subset \mathbf{D}^n}} \gamma^{(\tau)}. \tag{4.3.11}$$

Since we have assumed that $\gamma = \gamma_{1,0}$, we have the equality (4.3.11).

Proposition 4.11. *Let γ be an element of $AC_{n+1}(K, \mathbf{D}^n;\mathbb{Q})$ and let $\gamma = \sum_\sigma a_\sigma \sigma$ be its representative in $C_{n+1}(K;\mathbb{Q})$ such that*

(1) *$a_\sigma = 0$ if $\sigma \cap \mathbf{H}^n \subset \mathbf{D}^n$,*
(2) *$|\gamma| \cap \mathbf{H}^n \subset H = H_{1,0}$ and*

(3) $|\gamma| \cap \mathbf{H}_h = \emptyset$.

Let τ be a simplex in $|\gamma| \cap H$ not contained in $\mathbf{D}^n$. Then $\gamma^{(\tau)}$ is an element in $AC_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$.

Proof. We prove that $|\delta\gamma^{(\tau)}|$ is admissible. Since $|\gamma| \cap \mathbf{H}_h = \emptyset$, we have $|\gamma| \cap H' = \emptyset$ for any other cubical face H' than H by Lemma 4.8. So it suffices to show that $|\delta\gamma^{(\tau)}| - \mathbf{D}^n$ meets H properly. Since $|\gamma| - \mathbf{D}^n$ meets H properly, we have $\dim\tau \le n-1$. If $\dim\tau < n-1$, then $|\delta(\gamma^{(\tau)})| - \mathbf{D}^n$ meets H properly since $|\delta(\gamma^{(\tau)})| \cap H \subset \tau$. We consider the case where $\dim\tau = n-1$. We have

$$\delta(\gamma^{(\tau)}) = \sum_{\nu \subset |\gamma^{(\tau)}|} b_\nu \nu,\ b_\nu = \sum_{\substack{\nu \prec \sigma \\ \sigma \cap H = \tau}} [\sigma : \nu] a_\sigma$$

and

$$\delta\gamma = \sum_{\nu \subset |\gamma|} c_\nu \nu,\ c_\nu = \sum_{\nu \prec \sigma} [\sigma : \nu] a_\sigma$$

Here $\nu \prec \sigma$ means that ν is a codimension one face of σ, and $[\sigma : \nu] \in \{\pm 1\}$ is the index of σ with respect to ν. To prove the admissibility of $\delta\gamma^{(\tau)}$, it is sufficient to show the following claim.

Claim. Let ν be an n-simplex in K such that (1) $\nu \subset |\gamma^{(\tau)}|$, and (2) $\nu - \mathbf{D}^n$ does not meet H properly. Then we have $b_\nu = 0$.

Proof of the claim. Let ν be a simplex with the conditions in the claim. Then $\nu \cap H$ is a face of ν, since K is a good triangulation. By condition (1), we have $\nu \cap H \subset |\gamma^{(\tau)}| \cap H = \tau$ and $\dim(\nu \cap H) \le \dim(\tau)$. By condition (2), we have $n-1 \le \dim \nu \cap H$. Since $\dim\tau = n-1$, the above inequalities are equalities, and we have $\nu \cap H = \tau$. We consider each term appeared on the right hand side of the equality defining c_ν. Let σ be a $n+1$-simplex such that $\nu \subset \sigma$ and $a_\sigma \ne 0$. The set $\sigma \cap H$ is a face of σ. By the admissibility condition, we have $\dim \sigma \cap H \le n-1$. Since $\dim\tau = n-1$ and $\tau \subset \nu \subset \sigma$, we have $\sigma \cap H = \tau$. Thus this term appears in the right hand side of the equation defining b_ν. So we have $b_\nu = c_\nu$. Since $c_\nu = 0$ by the admissibility of $\delta\gamma$, we have $b_\nu = 0$.

$\square$

By Proposition 4.11 the chain $\gamma^{(\tau)} \in AC_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$ for each τ. It suffices to prove (4.3.9) for each $\gamma^{(\tau)}$ which we do. So until the end of §4.3, we assume that $\gamma = \gamma^{(\tau)}$ for a simplex $\tau \subset H$.

Let σ be a $(n+1)$-simplex in γ and assume given a smooth $(n-1)$-form φ on a neighborhood U of σ. The inclusion $i : H \to P^n$ restricts to an inclusion $i_U : U \cap H \to U$ (when there is no fear of confusion, we abbreviate $i_U^*\varphi$ to $i^*\varphi$.) Since $\tau^\circ \subset H$ is a smooth submanifold, $i^*\varphi$ restricts to a smooth form on τ°, denoted by the same $i^*\varphi$ (this is where the facewise regularity is used); it is zero if $\dim\tau < n-1$.

Definition 4.12 (Barycentric coordinate, linear projection)**.** *Let $\sigma = [a_0, \cdots, a_p]$ be a p-simplex. A point x in σ is expressed uniquely as $x = \sum_{i=0}^k \lambda_i a_i$ with $\sum_{i=0}^k \lambda_i = 1, \lambda_i \ge 0$. The vector $(\lambda_0, \ldots, \lambda_p)$ is called the barycentric coordinate of x.*

Let $\sigma = [v_0, \ldots, v_p]$ be a p-simplex and $\tau = [v_k, \ldots, v_p]$ be a proper $(p-k)$-face of σ $(0 < k \le p)$. We set $\tau' = [v_0, \cdots, v_{k-1}]$. We define a linear projection $\pi_{\sigma,\tau} : \sigma - \tau' \to \tau$ by

$$\pi_{\sigma,\tau}(x) = \frac{1}{\sum_{i=k}^p \lambda_i}(\lambda_k, \ldots, \lambda_p),$$

where $(\lambda_0, \dots, \lambda_p)$ *is the barycentric coordinate of* x*.*

Let $\pi_\sigma = \pi_{\sigma,\tau}$ be the linear projection $\sigma^\circ \to \tau$ defined in Definition 4.12. The map π_σ restricts to a smooth map between submanifolds $\sigma^\circ \to \tau^\circ$, thus the pull-back $\pi_\sigma^* i^* \varphi$ defines a smooth form on σ°.

The following proposition will be proved in §4.3.1.

Proposition 4.13. *Let* σ *be a* $(n+1)$*-simplex in* γ*.*

(1) *We have*

$$\lim_{\epsilon \to 0} \int_\sigma d\rho_\epsilon \wedge \omega_1 \wedge \left(a_\sigma \omega_{n-1} - \pi_\sigma^* i^*(a_\sigma \omega_{n-1})\right) = 0 \tag{4.3.12}$$

(2) *If the dimension of* $\tau < n-1$*, then the equality*

$$\lim_{\epsilon \to 0} \int_\sigma a_\sigma d\rho_\epsilon \wedge \omega_1 \wedge \omega_{n-1} = 0$$

holds.

We assume Proposition 4.13 for the moment and prove the equality (4.3.9). For an $(n+1)$-simplex σ in γ, if $\dim \tau = \dim \sigma \cap H < n-1$, then τ is a proper face of σ_b, and it follows that the right hand side of (4.3.9) is zero. Therefore by Proposition 4.13 (2), it is sufficient to prove (4.3.9) for the case where $\gamma = \gamma^{(\tau)}$ and $\dim \tau = n-1$. Under this assumption, we have $\sigma_b = \tau$ for each $(n+1)$-simplex σ such that $\sigma \subset |\gamma|$.

Proposition 4.14. *For a positive* ϵ*, we have the equality*

$$\sum_\sigma \int_\sigma a_\sigma d\rho_\epsilon \wedge \omega_1 \wedge \pi_\sigma^* i^*(\omega_{n-1}) = \sum_\sigma a_\sigma \left(\int_{\sigma_f} d\rho_\epsilon \wedge \omega_1\right) \cdot \left(\int_\tau i^*(\omega_{n-1})\right) \tag{4.3.13}$$

Proof. We recall the formulation of projection formula for integrals of differential forms. Let M, N be oriented smooth manifolds of dimension m, n, respectively. Then $M \times N$ is equipped with the product orientation. Let $\pi : M \times N \to N$ be the projection to N. For φ an m-form on $M \times N$ and ψ an n-form on N, we have *projection formula*

$$\int_{M \times N} \varphi \wedge \pi^* \psi = \int_N (\pi_* \varphi) \psi\,.$$

Here $\pi_* \varphi$ is the function

$$(\pi_* \varphi)(y) = \int_M \varphi|_{M \times \{y\}}\,.$$

(The precise meaning of the equality is that, if the left hand side is absolutely convergent, then the function $\pi_* \varphi$ is measurable, the right hand side is also absolutely convergent, and the equality holds.) This formula follows from Fubini's theorem for Lebesgue integrals.

For a $(n+1)$-simplex $\sigma = [v_0, \cdots, v_{n+1}]$ in γ with $v_0 < \cdots < v_{n+1}$, we have $\tau = [v_2, \cdots, v_{n+1}]$ and for a point $t \in \tau$, we have $\pi_\sigma^{-1}(t) = [v_0.v_1, t] - [v_0, v_1]$ where $[v_0, v_1, t]$ is the simplex spanned by the points v_0, v_1, t. By the projection formula, we have the equality

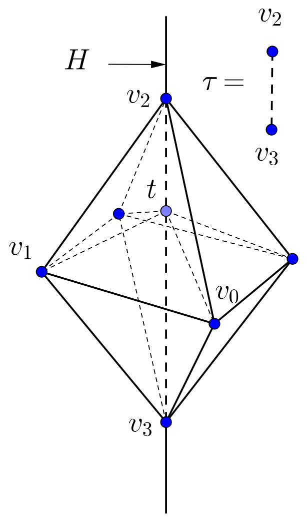
$$\begin{aligned}
(4.3.14) \qquad & \sum_\sigma a_\sigma \int_\sigma d\rho_\epsilon \wedge \omega_1 \wedge \pi_\sigma^* i^*(\omega_{n-1}) \\
&= \sum_\sigma \int_\tau \left(\int_{[v_0,v_1,t]} a_\sigma d\rho_\epsilon \wedge \omega_1 \right) i^* \Big(\omega_{n-1} \Big) \\
&= \int_\tau \sum_\sigma \left(\int_{[v_0,v_1,t]} a_\sigma d\rho_\epsilon \wedge \omega_1 \right) i^* \Big(\omega_{n-1} \Big).
\end{aligned}$$

Lemma 4.15. *We have the equality*

$$\sum_\sigma \int_{[v_0,v_1,t]} a_\sigma d\rho_\epsilon \wedge \omega_1 = \sum_\sigma \int_{[v_0,v_1,v_2]} a_\sigma d\rho_\epsilon \wedge \omega_1 \tag{4.3.15}$$

for all $t \in \tau$.

Note that we have $\sigma_f = [v_0, v_1, v_2]$ cf. Notation 4.9.



Proof. Let σ and ν be an $(n+1)$-simplex and an n-simplex respectively, such that $\sigma \succ \nu \succ \tau$. We set $\sigma = [v_0, v_1, v_2, \cdots, v_{n+1}]$, $\tau = [v_2, \cdots, v_{n+1}]$ and $\nu = [v, v_2, \cdots, v_{n+1}]$. Here the vertex v is v_0 or v_1. For a point $t \in \tau$ we set $\sigma_t = [v_0, v_1, v_2, t]$ and $\nu_t = [v, v_2, t]$. Then we have $[\sigma : \nu] = [\sigma_t : \nu_t]$. Since $\dim \nu \cap H = \dim \tau = n-1$, ν is not admissible. Therefore the coefficient of ν in $\delta\gamma$ is zero by the admissibility of $\delta\gamma$. It follows that we have

$$0 = \sum_{\{\sigma | \sigma \succ \nu\}} [\sigma : \nu] a_\sigma = \sum_{\{\sigma | \sigma \succ \nu\}} [\sigma_t : \nu_t] a_\sigma.$$

and obtain the equality

$$\begin{aligned}
(4.3.16) \qquad \sum_\sigma a_\sigma \delta\sigma_t &= \sum_\sigma a_\sigma \Big([v_0, v_1, t] - [v_0, v_1, v_2] \Big) + \sum_{\{\nu | \nu \succ \tau\}} \left(\sum_{\{\sigma | \sigma \succ \nu\}} [\sigma_t, \nu_t] a_\sigma \right) \nu_t \\
&= \sum_\sigma a_\sigma \Big([v_0, v_1, t] - [v_0, v_1, v_2] \Big).
\end{aligned}$$

The equality (4.3.16) implies the equality

$$\sum_\sigma a_\sigma\Big(\int_{[v_0,v_1,t]-[v_0,v_1,v_2]} d\rho_\epsilon\wedge\omega_1\Big)=\sum_\sigma a_\sigma\int_{\delta\sigma_t} d\rho_\epsilon\wedge\omega_1=0$$

by the Stokes formula and we finish the proof. □

Therefore the last line of (4.3.14) is equal to

$$\begin{aligned}&\int_\tau\sum_\sigma\Big(\int_{[v_0,v_1,t]} a_\sigma d\rho_\epsilon\wedge\omega_1\Big)i^*\Big(\omega_{n-1}\Big)\\ =&\int_\tau\sum_\sigma\Big(\int_{\sigma_f} a_\sigma d\rho_\epsilon\wedge\omega_1\Big)i^*\Big(\omega_{n-1}\Big)\quad\text{(Lemma 4.15)}\\ =&\Big(\sum_\sigma\int_{\sigma_f} a_\sigma d\rho_\epsilon\wedge\omega_1\Big)\cdot\int_\tau i^*(\omega_{n-1})\end{aligned}$$

Thus we have proved the assertion. □

Since $\gamma=\sum a_\sigma\sigma\in AC_{n+1}(K,\mathbf{D}^n;\mathbb{Q})$, the chain $T_\epsilon\cap\gamma=\Big(\sum_\sigma\int_{\sigma_f}a_\sigma d\rho_\epsilon\wedge\omega_1\Big)\tau$ is independent of the choice of a Thom cocycle i.e. the choice of a sufficiently small ϵ by Proposition 3.7 (2). By Proposition 4.14 and Proposition 4.13 (1), we have

Proposition 4.16. *If* $\dim\tau=n-1$, *then for a sufficiently small real number* $\epsilon_0>0$, *we have the equality*

$$\begin{aligned}&\lim_{\epsilon\to 0}\sum_\sigma a_\sigma\int_\sigma d\rho_\epsilon\wedge\omega_1\wedge\omega_{n-1}\\ =&\sum_\sigma\Big(\int_{\sigma_f}a_\sigma d\rho_{\epsilon_0}\wedge\omega_1\Big)\Big(\int_\tau i^*(\omega_{n-1})\Big).\end{aligned}$$

Here the sum is taken over the $(n+1)$*-simplexes of* γ.

The equality (4.3.9) follows from Proposition 4.16.

4.3.1. *Proof of Proposition 4.13.* We prove the following proposition from which Proposition 4.13 follows.

Proposition 4.17. *Let* σ *be an* $(n+1)$*-simplex in* γ, *and let* φ *be a smooth* $(n-1)$*-form on a neighborhood of* σ.

(1) *When* ϵ *is sufficiently small, the integral* $\int_\sigma d\rho_\epsilon\wedge\omega_1\wedge\big(\varphi-\pi_\sigma^*i^*\varphi\big)$ *converges absolutely.*
(2) *We have the equality*

$$\lim_{\epsilon\to 0}\int_\sigma d\rho_\epsilon\wedge\omega_1\wedge\big(\varphi-\pi_\sigma^*i^*\varphi\big)=0. \tag{4.3.17}$$

Proof of Proposition 4.17. The form φ is a sum of the forms $f\,du_1\wedge\cdots\wedge du_{n-1}$, where u_i are from the set $\{x_1,y_1,\cdots,x_n,y_n\}$ (we denote $z_j=x_j+iy_j$), and f is a smooth function. One may thus assume $\varphi=f\,du_1\wedge\cdots\wedge du_{n-1}$.

(1) We wish to apply [10] Theorem 2.6, which reads as follows: Let S be a compact semi-algebraic set of dimension m, $h : S \to \mathbb{R}^\ell$ be a continuous semi-algebraic map, and ψ be a smooth m-form defined on an open set of $\mathbb{R}^\ell$ containing $h(S)$. Then the integral $\int_S |h^*\psi|$ is convergent.

It is useful note that differential forms on S of the form $h^*\psi$, with $h : S \to \mathbb{R}^\ell$ continuous semi-algebraic, and ψ a smooth p-form ($0 \le p \le m = \dim S$) are closed under wedge product. Indeed, if $h' : S \to \mathbb{R}^{\ell'}$ is another continuous semi-algebraic map, and ψ' a smooth p'-form on an neighborhood of $h'(S')$, then $(h^*\psi) \wedge (h'^*\psi')$ equals the pull-back by the product map $(h, h') : S \to \mathbb{R}^\ell \times \mathbb{R}^{\ell'}$ of the smooth form $(p_1^*\psi) \wedge (p_2^*\psi')$ defined on a neighborhood of $(h, h')(S)$ in $\mathbb{R}^{\ell+\ell'}$.

In order to show the absolute convergence of $\int_\sigma d\rho_\epsilon \wedge \omega_1 \wedge \pi_\sigma^* i^*\varphi$, let S be the compact semi-algebraic set obtained from σ by removing a small neighborhood of τ', and note that the integral in question equals $\int_S d\rho_\epsilon \wedge \omega_1 \wedge \pi_\sigma^* i^*\varphi$ for ϵ sufficiently small. We consider the projection $\pi_\sigma : \sigma - \tau' \to \tau \subset H$ restricted to S,

$$\pi_\sigma : S \to \tau \subset H\,,$$

and the smooth form $i_U^*\varphi$ defined on a neighborhood of τ; then $\pi_\sigma^* i_U^*\varphi$ is a form of the above-mentioned shape $h^*\psi$. Also, pull-back by the inclusion $S \hookrightarrow P^n$ of the smooth form $d\rho_\epsilon \wedge \omega_1$ gives us another form of the shape $h^*\psi$. Thus the wedge product of them, $d\rho_\epsilon \wedge \omega_1 \wedge \pi_\sigma^* i^*\varphi$, is also a form of the same kind, and we conclude absolute convergence of $\int_S d\rho_\epsilon \wedge \omega_1 \wedge \pi_\sigma^* i^*\varphi$ by the theorem we recalled.

Similarly (and more easily) the absolute convergence of $\int_\sigma d\rho_\epsilon \wedge \omega_1 \wedge \varphi$ is obtained by applying the same theorem to the inclusion $\sigma \hookrightarrow P^n$ and the smooth form $d\rho_\epsilon \wedge \omega_1 \wedge \varphi$.

(2). We need the following lemma.

Lemma 4.18. *For a complex number ζ_1, we set $\sigma(\zeta_1) = \sigma \cap \{z_1 = \zeta_1\}$. There exists a closed semi-algebraic set C of $\mathbb{C}$ of dimension ≤ 1 for which the equality*

$$\lim_{|\zeta_1| \to 0,\, \zeta_1 \notin C} \int_{\sigma(\zeta_1)} |\varphi - \pi_\sigma^* i^*\varphi| = 0. \tag{4.3.18}$$

holds.

Proof of Lemma 4.18. By Semi-algebraic triviality of semi-algebraic maps as stated in Theorem 9.3.2, [5], there exists a closed semi-algebraic set C of $\mathbb{C}$ of dimension ≤ 1 such that if $\zeta_1 \notin C$, then the inequality $\dim \sigma(\zeta_1) \le \dim \sigma - 2$ holds. We have an equality

(4.3.19)

$$\begin{aligned} & \varphi - \pi_\sigma^* i^*\varphi \\ = \ & f du_1 \wedge \cdots \wedge du_{n-1} - \pi_\sigma^* i^*(f du_1 \wedge \cdots \wedge du_{n-1}) \\ = \ & \big(f - \pi_\sigma^* i^* f\big) du_1 \wedge \cdots \wedge du_{n-1} \\ & + \textstyle\sum_{k=1}^{n-1} \pi_\sigma^* i^* f du_1 \wedge \cdots \wedge du_{k-1} \wedge \big(du_k - \pi_\sigma^* i^* du_k\big) \wedge \pi_\sigma^* i^*(du_{k+1} \wedge \cdots \wedge du_{n-1}) \end{aligned}$$

We estimate the integral of the first term on the right hand side of (4.3.19). Let g be the map defined by

$$\sigma \to \mathbb{C} \times \mathbb{R}^{n-1} : z \to (z_1, u_i).$$

By Proposition 2.7 of [10], we have the inequality

$$\left| \int_{\sigma(\zeta_1)} (f - \pi_\sigma^* f|_\tau) \wedge du_1 \wedge \cdots \wedge du_{n-1} \right| \le \operatorname*{Max}_{\sigma(\zeta_1)} |f - \pi_\sigma^* i^* f| \delta(g) \int_{g(\sigma(\zeta_1))} |du_1 \wedge \cdots \wedge du_{n-1}|$$

Here $\delta(g)$ denotes the maximal of the cardinalities of finite fibers of g. For the precise definition, see Definition 2.2. of [10]. Note that $\delta(g|_{\sigma(\zeta_1)}) \leq \delta(g)$.

Claim. $\mathrm{Max}_{\sigma(\zeta_1)}|f - \pi_\sigma^* i^* f| \to 0$ as ζ_1 tends to 0 (outside C).

Proof of the claim. The function $f - \pi_\sigma^* i^* f$ is continuous semi-algebraic on $\sigma(\zeta_1)$, and vanishes on τ. If the claim were false, there exists an $\epsilon > 0$ and a sequence $P_j \in \sigma$ with $|z_1(P_j)| \to 0$ and $|(f - \pi_\sigma^* i^* f)(P_j)| \geq \epsilon$. Taking a subsequence we may assume that the sequence converges to a point $P \in \sigma$. Then $z_1(P) = 0$, thus $P \in \tau$, while $|(f - \pi_\sigma^* i^* f)(P)| \geq \epsilon$, contradicting the function $f - \pi_\sigma^* i^* f$ being zero on τ. □

The integral $\displaystyle\int_{g(\sigma(\zeta_1))} | \wedge_i du_i|$ is bounded by the volume of $g(\sigma)$ which is independent of ζ_1. We conclude that the integral of the first term on the right hand side of (4.3.19) converges to zero as $|\zeta_1| \to 0$.

We estimate the integral of the second term of the right hand side of (4.3.19). Let h be the map defined by

$$\sigma \to \mathbb{C} \times \mathbb{R}^{n-1} : z \mapsto (z_1, v_i) = (z_1, u_1, \ldots, u_{k-1}, u_k - \pi_\sigma^* i^* u_k, \pi_\sigma^* i^* u_{k+1}, \ldots \pi_\sigma^* i^* u_{n-1}).$$

By Proposition 2.7 of [10], we have the inequality

$$\begin{aligned}&\int_{\sigma(\zeta_1)} |\pi_\sigma^* i^* f(z) \wedge du_1 \wedge \cdots \wedge du_{k-1} \wedge \big(du_k - \pi_\sigma^* i^* du_k\big) \wedge \pi_\sigma^* i^* (du_{k+1} \wedge \cdots \wedge du_{n-1})| \\ &\leq \mathop{\mathrm{Max}}_{\sigma(\zeta_1)} |\pi_\sigma^* i^* f| \delta(h|_{\sigma(\zeta_1)}) \int_{h(\sigma(\zeta_1))} |dv_1 \wedge \cdots \wedge dv_{n-1}|\end{aligned}$$

where $v_1, \cdots, v_{n-1}$ are the coordinates of $\mathbb{R}^{n-1}$. Note that $\delta(h|_{\sigma(\zeta_1)})$ is bounded by $\delta(h)$ which is independent of ζ_1. The function $|\pi_\sigma^* i^* f|$ is bounded by $\max\{|f(z)| \mid z \in \tau\}$. By the same proof as for the Claim, $M_{\zeta_1} := \mathrm{Max}_{\sigma(\zeta_1)} |u_k - \pi_\sigma^* i^* u_k|$ tends to zero as $|\zeta_1| \to 0$. There exist numbers $a < b$ such that

$$\begin{aligned} u_i(\sigma) \subset [a, b] \quad & \text{for } 1 \leq i \leq k-1, \text{ and} \\ \pi_\sigma^* i^* u_i(\sigma) \subset [a, b] \quad & \text{for } k+1 \leq i \leq n-1\,, \end{aligned}$$

thus

$$h(\sigma(\zeta_1)) \subset [a, b]^{k-1} \times [-M_{\zeta_1}, M_{\zeta_1}] \times [a, b]^{n-1-k}\,,$$

hence $\displaystyle\int_{h(\sigma(\zeta_1))} dv_1 \wedge \cdots \wedge dv_{n-1} \to 0.$ □

We go back to the proof of Proposition 4.17 (2). One has

$$\int_{\mathbb{C}} d\rho \wedge \omega_1 = 1\,,$$

as follows from the identity

$$d\rho \wedge \frac{dz_1}{z_1} = i\rho'(r_1) dr_1 \wedge d\theta_1\,.$$

For the form $d\rho_\epsilon(z_1) \wedge \omega_1$, the change of variables $z_1' = z_1/\epsilon$ yields

$$d\rho_\epsilon(z_1) \wedge \omega_1 = d\rho(z_1') \wedge \frac{1}{2\pi i} \frac{dz_1'}{z_1'}\,.$$

So we have

$$\int_{\mathbb{C}} d\rho_\epsilon \wedge \omega_1 = 1\,.$$

Also, $d\rho_\epsilon \wedge \omega_1$ has support in $|z_1| \leq \epsilon$. Therefore

$$\left|\int_\sigma d\rho_\epsilon \wedge \omega_1 \wedge (\varphi - \pi_\sigma^* i^* \varphi)\right| \leq \operatorname*{Max}_{|\zeta_1|\leq\epsilon,\, \zeta_1 \notin C} (\int_{\sigma(\zeta_1)} |\varphi - \pi_\sigma^* i^* \varphi|) \int_{\mathbb{C}} d\rho_\epsilon \wedge \omega_1$$

$$= \operatorname*{Max}_{|\zeta_1|\leq\epsilon,\, \zeta_1 \notin C} (\int_{\sigma(\zeta_1)} |\varphi - \pi_\sigma^* i^* \varphi|)$$

and the assertion follows from Lemma 4.18 . □

4.4. **Proof of Proposition 4.6.** We prove Proposition 4.6 assuming Proposition 4.7 by a limit argument. Let ϵ be a positive real number and for $i = 1, \cdots, n$, set $z_i^{(0)} = z_i$ and $z_i^{(\infty)} = z_i^{-1}$. We define a neighborhood N_ϵ of $\mathbf{H}_h$ by

$$N_\epsilon = \bigcup_{\substack{1\leq i<i'\leq n,\\ \alpha\in\{0,\infty\},\beta\in\{0,\infty\}}} \{z \in P^n \mid |z_i^{(\alpha)}| \leq \epsilon, |z_{i'}^{(\beta)}| \leq \epsilon\}.$$

We set $N_\epsilon^* = \overline{P^n - N_\epsilon}$.

By the semi-algebraic triviality of the semi-algebraic maps ([5], Theorem 9.3.2), the following **condition (P)** holds when ϵ is sufficiently small.

(P) For each simplex σ of K and each pair (i, α), we have $\dim \sigma \cap \{|z_i^{(\alpha)}| = \epsilon\} \leq \dim \sigma - 1$ if the set $\sigma \cap \{|z_i^{(\alpha)}| = \epsilon\}$ is not empty. Here if the dimension of a set is negative, then the set must be empty.

In the rest of this section and §4.5 we assume that ϵ satisfies the condition (P). Let K_ϵ be a good subdivision of K such that N_ϵ and N_ϵ^* are subcomplexes of K_ϵ. We denote by λ the subdivision operator from $C_\bullet(K)$ to $C_\bullet(K_\epsilon)$. Note that by the condition (P) we have the following. If σ is a simplex of K and $\lambda(\sigma) = \sum \sigma'$, then each σ' is contained in one of $\{N_\epsilon, N_\epsilon^*\}$ exclusively.

Definition 4.19. (1) *Let σ be a m-simplex in K and we set $\lambda(\sigma) = \sum \sigma'$. We put $\sigma_{\geq\epsilon} = \sum_{\sigma' \subset N_\epsilon^*} \sigma'$. For an element $\gamma = \sum_\sigma a_\sigma \sigma$ in $C_m(K, \mathbf{D}^n; \mathbb{Q})$, we set $\gamma_{\geq\epsilon} = \sum_\sigma a_\sigma \sigma_{\geq\epsilon}$.*
(2) *We put $\sigma_{=\epsilon} = \delta(\sigma_{\geq\epsilon}) - (\delta\sigma)_{\geq\epsilon}$.*
(3) *For an element $\gamma = \sum_\sigma a_\sigma \sigma$ in $C_m(K, \mathbf{D}^n; \mathbb{Q})$, we set $\gamma_{=\epsilon} = \sum_\sigma a_\sigma \sigma_{=\epsilon}$.*

Lemma 4.20. *Let ϵ be a positive number which satisfies the condition (P).*
(1) *Let σ be an m-simplex in K. We have $|\sigma_{=\epsilon}| \subset N_\epsilon^* \cap N_\epsilon$.*
(2) *Let σ be an admissible m-simplex in K. Then the chain $\sigma_{=\epsilon}$ is admissible.*
(3) *Let $\gamma = \sum a_\sigma \sigma$ be an element of $AC_m(K, \mathbf{D}^n; \mathbb{Q})$. We have $(\partial\gamma)_{\geq\epsilon} = \partial(\gamma_{\geq\epsilon})$.*

Proof. (1). Let $\lambda(\sigma) = \sum \sigma'$ and we put $\sigma_{\leq\epsilon} = \sum_{\sigma' \subset N_\epsilon} \sigma'$. By condition (P), we have $\lambda(\sigma) = \sigma_{\geq\epsilon} + \sigma_{\leq\epsilon}$ and $\delta(\lambda(\sigma)) = \lambda(\delta\sigma) = (\delta\sigma)_{\geq\epsilon} + (\delta\sigma)_{\leq\epsilon}$. Let $\sigma_\epsilon^- = \delta(\sigma_{\leq\epsilon}) - (\delta\sigma)_{\leq\epsilon}$. Then we have $\sigma_{=\epsilon} + \sigma_{=\epsilon}^- = 0$. Since $|\sigma_{=\epsilon}| \subset N_\epsilon^*$ and $|\sigma_{=\epsilon}^-| \subset N_\epsilon$, we have the assertion.

(2). We set $\tau = \sigma \cap \mathbf{H}^n$. If $\tau \subset \mathbf{D}^n$, then $\sigma_{=\epsilon}$ is admissible by definition. We assume that $\tau \not\subset \mathbf{D}^n$. If $\tau \subset \mathbf{H}_h$, then $|\sigma_{=\epsilon}| \cap \mathbf{H}^n = \emptyset$ by the definition of N^*_ϵ. If $\tau \not\subset \mathbf{H}_h$, then there is a unique codimension one face $H_{i,\alpha}$ such that $\tau \subset H_{i,\alpha}$. Since $|\sigma_{=\epsilon}| \cap \mathbf{H}_h = \emptyset$, $\sigma_{=\epsilon}$ does not meet other cubical face than $H_{i,\alpha}$. Hence it suffices to show that $\sigma_{=\epsilon}$ meets $H_{i,\alpha}$ properly. By the admissibility of σ we have $\dim \sigma \cap H_{i,\alpha} \leq m-2$. Since the set $\tau = \sigma \cap H_{i,\alpha}$ is a face of σ, we have the inequality $\dim \tau \cap \{|z^{(\alpha')}_{i'}| = \epsilon\} \leq m-3$ for any (i', α') by the condition (P). By the assertion (1) we have $|\sigma_{=\epsilon}| \cap H_{i,\alpha} \subset \sigma \cap H_{i,\alpha} \cap \big(\bigcup_{(i',\alpha')} \{|z^{(\alpha')}_{i'}| = \epsilon\}\big)$. Therefore we have $\dim(|\sigma_{=\epsilon}| \cap H_{i,\alpha}) \leq m-3$.

(3). We denote the cubical face $\{z_1 = 0\}$ by H. Let σ be a m-simplex in γ. We show that $\partial(\sigma_{\geq\epsilon}) = (\partial\sigma)_{\geq\epsilon}$. We denote the simplex $\sigma \cap H$ by τ. We set $\lambda(\sigma) = \sum \sigma'$ and $\lambda(\tau) = \sum \tau'$. Then we have $\sigma_{\geq\epsilon} = \displaystyle\sum_{\sigma' \subset N^*_\epsilon} \sigma'$. Let T be a Thom cocycle for $K_\epsilon \cap H$ and $\mathcal{O}$ be a good ordering of K_ϵ with respect to H. We set $T \overset{\mathcal{O}}{\cap} \lambda(\sigma) = \sum b_{\tau'}\tau'$ and $T \overset{\mathcal{O}}{\cap} (\sigma_{\geq\epsilon}) = \sum c_{\tau'}\tau'$. We have $b_{\tau'}\tau' = \displaystyle\sum_{\sigma' \cap H = \tau'} T \overset{\mathcal{O}}{\cap} \sigma'$ and $c_{\tau'}\tau' = \displaystyle\sum_{\sigma' \cap H = \tau',\, \sigma' \subset N^*_\epsilon} T \overset{\mathcal{O}}{\cap} \sigma'$.

Suppose first that τ is a $(m-2)$-simplex which is not contained in $\mathbf{D}^n$. We will show that for each $(m-2)$-simplex τ' contained in N^*_ϵ, we have $b_{\tau'} = c_{\tau'}$. Such a τ' is not contained in N_ϵ by the condition (P). Each m-simplex σ' in $\lambda(\sigma)$ is contained either N_ϵ or N^*_ϵ. If $\sigma' \cap H = \tau'$ then we have $\sigma' \subset N^*_\epsilon$. It follows that $b_{\tau'} = c_{\tau'}$.

Suppose that $\dim \sigma \cap H \leq m-3$. We have $\dim \sigma' \cap H \leq m-3$ for each σ'. If $\sigma' = \pm[v_0, \cdots, v_m]$ with $v_0 < \cdots < v_m$, then then we have $T \overset{\mathcal{O}}{\cap} \sigma' = \pm T([v_0, v_1, v_2])[v_2, \cdots, v_m] = 0$ since $[v_0, v_1, v_2] \cap H = \emptyset$, and $T \in C^2(K_\epsilon, W)$ where $W = \displaystyle\bigcup_{\xi \in K_\epsilon, \xi \cap H = \emptyset} \xi$. Therefore we have $b_{\tau'} = c_{\tau'} = 0$ for each τ'. The same argument applies to each codimension one cubical face. $\square$

Proof of Proposition 4.6 assuming Proposition 4.7. Let $\gamma = \sum_\sigma a_\sigma \sigma$ be the representative of γ such that $a_\sigma = 0$ if $\sigma \cap \mathbf{H}^n \subset \mathbf{D}^n$. We show that $\gamma_{\geq\epsilon}$ is an element in $AC_{n+1}(K_\epsilon, \mathbf{D}^n; \mathbb{Q})$ as follows. The set $|\gamma_{\geq\epsilon}|$ is contained in $|\gamma|$ which is admissible by the assumption. Hence $|\gamma_{\geq\epsilon}|$ is admissible. We have $|\delta(\gamma_{\geq\epsilon})| = |(\delta\gamma)_{\geq\epsilon}| \cup |\gamma_{=\epsilon}|$. The set $|(\delta\gamma)_{\geq\epsilon}|$ is admissible by the same reason as above. For an $\epsilon > 0$ which satisfies (P), $|\gamma_{=\epsilon}|$ is admissible by Lemma 4.20 (2). Therefore $\gamma_{\geq\epsilon}$ is an element in $AC_{n+1}(K_\epsilon, \mathbf{D}^n; \mathbb{Q})$.

Since $N^*_\epsilon \cap \mathbf{H}_h = \emptyset$, the element $\gamma_{\geq\epsilon}$ satisfies the condition of Proposition 4.7 and we have $I_{n-1}(\partial(\gamma_{\geq\epsilon})) + (-1)^n I_n(\delta(\gamma_{\geq\epsilon})) = 0$. We prove the equality

$$\lim_{\epsilon\to 0}\big(I_{n-1}(\partial(\gamma_{\geq\epsilon})) + (-1)^n I_n(\delta(\gamma_{\geq\epsilon}))\big) = I_{n-1}(\partial\gamma) + (-1)^n I_n(\delta\gamma). \tag{4.4.20}$$

We set $\delta\gamma = \sum b_\nu \nu$ and $\partial\gamma = \sum \tau \cdot c_\tau$. By the admissibility of $\delta\gamma$ and $\partial\gamma$, the integrals $\displaystyle\int_\nu b_\nu \omega_n$ and $\displaystyle\int_\tau c_\tau \omega_{n-1}$ converge absolutely by Theorem 4.1. By Lebesgue's convergence theorem, we have

$$\lim_{\epsilon\to 0} \int_{\nu_{\geq\epsilon}} b_\nu \omega_n = \int_\nu b_\nu \omega_n \text{ and } \lim_{\epsilon\to 0} \int_{\tau_{\geq\epsilon}} c_\tau \omega_{n-1} = \int_\tau c_\tau \omega_{n-1}.$$

Therefore we have

$$\lim_{\epsilon\to 0} I_n((\delta\gamma)_{\geq\epsilon}) = I_n(\delta\gamma)$$
$$\lim_{\epsilon\to 0} I_{n-1}(\partial(\gamma_{\geq\epsilon})) = \lim_{\epsilon\to 0} I_{n-1}((\partial\gamma)_{\geq\epsilon}) = I_{n-1}(\partial\gamma).$$

By the equality $\delta(\gamma_{\geq\epsilon}) = (\delta\gamma)_{\geq\epsilon} + \gamma_{=\epsilon}$, to show the equality (4.4.20), it is enough to prove the equality

$$\lim_{\epsilon\to 0} I_n(\gamma_{=\epsilon}) = 0. \tag{4.4.21}$$

For a positive real number t and $1 \leq i \neq j \leq n$, $\alpha \in \{0,\infty\}$, $\beta \in \{0,\infty\}$, we set

$$A_t^{(i,\alpha),(j,\beta)} = \{z \in P^n |\ |z_i^{(\alpha)}| \leq |z_j^{(\beta)}| = t\}.$$

Let σ be an $(n+1)$-simplex of K. By Lemma 4.20 (1), the set $|\sigma_{=\epsilon}|$ is contained in the topological boundary of N_ϵ. Therefore we have the relation

$$|\sigma_{=\epsilon}| \subset \bigcup_{\substack{1\leq i\neq j\leq n,\\ \alpha\in\{0,\infty\},\beta\in\{0,\infty\}}} \sigma \cap A_\epsilon^{(i,\alpha),(j,\beta)}.$$

By applying Theorem 4.7 [10] to the case where $A = \sigma$, $m = n+1$ and $\omega = \omega_n$, we obtain the following proposition.

Proposition 4.21. *Let σ be an $(n+1)$-simplex. Assume that σ is admissible. Then for a sufficiently small $t > 0$, the dimension of $\sigma \cap A_t^{(i,\alpha),(j,\beta)}$ is equal to or less than n, and we have*

$$\lim_{t\to 0} \int_{\sigma\cap A_t^{(i,\alpha),(j,\beta)}} |\omega_n| = 0.$$

By Proposition 4.21, we have

$$\lim_{\epsilon\to 0} \int_{\sigma_{=\epsilon}} |\omega_n| \leq \sum_{\substack{i\neq j\\ \alpha,\beta}} \lim_{\epsilon\to 0} \int_{\sigma\cap A_\epsilon^{(i,\alpha),(j,\beta)}} |\omega_n| = 0.$$

and as a consequence, we have the equality (4.4.21). □

4.5. **Proof of Proposition 4.5.** Supose that an element $\gamma \in AC_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$ satisfies the condition $|\gamma| \cap \mathbf{H}^n \subset \mathbf{D}^n$. If $\sum a_\sigma \sigma$ is a representative of γ in $C_{n+1}(K;\mathbb{Q})$, then for each σ with $a_\sigma \neq 0$ we have $\sigma \cap \mathbf{H}^n \subset \mathbf{D}^n$, and $\sigma \in AC_{n+1}(K, \mathbf{D}^n; \mathbb{Q})$. So it is necessary and sufficient to prove the assertion for each such σ. Since K is a good triangulation, for an $(n+1)$-simplex σ, $\sigma \cap \mathbf{H}^n$ is a face of σ. Therefore $\sigma \cap \mathbf{H}^n \subset H_{i,\alpha}$ for some (i,α) because $\mathbf{H}^n$ is the union of the codimension one faces $H_{i,\alpha}$. We may assume that $H = H_{1,0}$.

For a positive number ϵ which satisfies the condition (P), we set

$$N_\epsilon^* = \{|z_1| \geq \epsilon\} \subset P^n \text{ and } N_\epsilon = \{|z_1| \leq \epsilon\} \subset P^n.$$

Let K_ϵ be a subdivision of K such that N_ϵ^* and N_ϵ are subcomplexes of K_ϵ, and let $\lambda : C_\bullet(K) \to C_\bullet(K_\epsilon)$ be the subdivision operator. Let $\lambda(\sigma) = \sum \sigma'$, and put $\sigma_{\geq\epsilon} = \sum_{\sigma',\sigma'\subset N_\epsilon^*} \sigma.$ We set $\sigma_{=\epsilon} = \delta(\sigma_{\geq\epsilon}) - (\delta\sigma)_{\geq\epsilon}$. By the same argument as the proof of Lemma 4.20 (1), we have $|\sigma_{=\epsilon}| \subset \{|z_1| = \epsilon\}$.

Since $\sigma_{\geq\epsilon}$ does not meet $\mathbf{H}^n$, we have the equality

$$0=\int_{\delta(\sigma_{\geq\epsilon})} a_\sigma\omega_n=\int_{\delta(\sigma)_{\geq\epsilon}} a_\sigma\omega_n+\int_{\sigma_{=\epsilon}} a_\sigma\omega_n \tag{4.5.22}$$

by the Stokes formula. We consider the limit of this as $\epsilon\to 0$. As the chain $\delta\sigma$ is admissible, the integral $\int_{\delta\sigma} a_\sigma\omega_n$ converges absolutely. By Lebesgue's convergence theorem, we have

$$\lim_{\epsilon\to 0}\int_{(\delta\sigma)_{\geq\epsilon}} a_\sigma\omega_n=\int_{\delta\sigma} a_\sigma\omega_n$$

By applying Theorem 4.8 [10] to the case where $A=\sigma$, $\omega=\omega_n$, $m=n+1$ and $\epsilon=1$, we have the following proposition.

Proposition 4.22. *Let σ be an $(n+1)$-simplex. Assume that $\sigma\cap\mathbf{H}^n\subset\mathbf{D}^n$. Then we have*

$$\lim_{t\to 0}\int_{\sigma\cap\{|z_1|=t\}}|\omega_n|=0.$$

By Proposition 4.22, we have the equality

$$\lim_{\epsilon\to 0}\int_{\sigma_{=\epsilon}} a_\sigma\omega_n=0.$$

By taking the limit of (4.5.22) for $\epsilon\to 0$, we have

$$\int_{\delta\sigma} a_\sigma\omega_n=0$$

and we finish the proof of Proposition 4.5.

5. Construction of the Hodge realization functor.

In this section, we give a construction of Hodge realization functor for the category of mixed Tate motives.

5.1. **Cycle complexes and graded DGA N.** Let $\mathbf{k}$ be a field. Following [4], we recall that the cycle complex of $\operatorname{Spec}\mathbf{k}$ may be viewed as a DGA over $\mathbb{Q}$.

Bloch defined the cycle complex for any quasi-projective variety, but we will restrict to the case of $\operatorname{Spec}\mathbf{k}$. Let $\square^n=\square^n_{\mathbf{k}}=(\mathbf{P}^1_{\mathbf{k}}-\{1\})^n$, which is isomorphic to affine n-space as a variety (and which coincides with $\square^n$ of §2 if $\mathbf{k}=\mathbb{C}$). As in §2, $(z_1,\cdots,z_n)$ are the coordinates of $\square^n$.

For $n\geq 0$ and $r\geq 0$, let $Z^r(\operatorname{Spec}\mathbf{k},n)$ be the $\mathbb{Q}$-vector space with basis irreducible closed subvarieties of $\square^n$ of codimension r which meet the faces properly. The cubical differential $\partial: Z^r(\operatorname{Spec}\mathbf{k},n)\to Z^r(\operatorname{Spec}\mathbf{k},n-1)$ is defined by the same formula as Definition 3.9.

The group $G_n=\{\pm 1\}^n\rtimes S_n$ acts naturally on $\square^n$ as follows. The subgroup $\{\pm1\}^n$ acts by the inversion of the coordinates z_i, and the symmetric group S_n acts by permutation of z_i's. This action induces an action of G_n on $Z^r(\operatorname{Spec}\mathbf{k},n)$. Let sign : $G_n\to\{\pm1\}$ be the character which sends $(\epsilon_1,\cdots,\epsilon_n;\sigma)$ to $\epsilon_1\cdot\cdots\cdot\epsilon_n\cdot\operatorname{sign}(\sigma)$. The idempotent $\mathrm{Alt}=\mathrm{Alt}_n:=(1/|G_n|)\sum_{g\in G_n}\operatorname{sign}(g)g$ in the group ring $\mathbb{Q}[G_n]$ is called the alternating projector. For a $\mathbb{Q}[G_n]$-module M, the submodule

$$M^{\mathrm{alt}}=\{\alpha\in M\mid \mathrm{Alt}\,\alpha=\alpha\}=\mathrm{Alt}(M)$$

is called the alternating part of M. By Lemma 4.3 [4], one knows that the cubical differential ∂ maps $Z^r(\operatorname{Spec}\mathbf{k},n)^{\mathrm{alt}}$ to $Z^r(\operatorname{Spec}\mathbf{k},n-1)^{\mathrm{alt}}$. For convenience let $Z^r(\operatorname{Spec}\mathbf{k},n)=0$ if $n<0$. Product of cycles induces a map of complexes $\times : Z^r(\operatorname{Spec}\mathbf{k},n)\otimes Z^s(\operatorname{Spec}\mathbf{k},m)\to Z^{r+s}(\operatorname{Spec}\mathbf{k},n+m)$, $z\otimes w\mapsto z\times w$. This induces a map of complexes on alternating parts

$$Z^r(\operatorname{Spec}\mathbf{k},n)^{\mathrm{alt}}\otimes Z^s(\operatorname{Spec}\mathbf{k},m)^{\mathrm{alt}}\to Z^{r+s}(\operatorname{Spec}\mathbf{k},n+m)^{\mathrm{alt}}$$

given by $z\otimes w\mapsto z\cdot w=\mathrm{Alt}(z\times w)$. We set $N_r^i=Z^r(\operatorname{Spec}\mathbf{k},2r-i)^{\mathrm{alt}}$ for $r\geq 0$ and $i\in\mathbb{Z}$.

One thus has an associative product map

$$N_r^i\otimes N_s^j\to N_{r+s}^{i+j}\,,\qquad z\otimes w\mapsto z\cdot w\,,$$

One verifies that the product is graded-commutative: $w\cdot z=(-1)^{ij}z\cdot w$ for $z\in N_r^i$ and $w\in N_s^j$.

Let $N=\underset{r\geq 0,\, i\in\mathbb{Z}}{\oplus} N_r^i$, and $N_r=\underset{i\in\mathbb{Z}}{\oplus} N_r^i$. By Lemma 4.3 [4], N with the above product and the differential ∂ is a differential graded algebra (DGA) over $\mathbb{Q}$.

By definition, one has $N_0=\mathbb{Q}$ and $1\in N_0$ is the unit for the product. Thus the projection $\epsilon : N\to N_0=\mathbb{Q}$ is an augmentation, namely it is a map of DGA's and the composition with the unit map $\mathbb{Q}\to N$ is the identity.

5.2. **The complex $AC^\bullet$.** Assume now that $\mathbf{k}$ is a subfield of $\mathbb{C}$.

Definition 5.1 (The complex $AC^\bullet$)**.** *For an integer k, let*

$$AC^k=\bigoplus_{n-i=k} AC_i(P^n,\mathbf{D}^n;\mathbb{Q})^{\mathrm{alt}}$$

Here we set

$$AC_i(P^0,\mathbf{D}^0;\mathbb{Q})^{\mathrm{alt}}=\begin{cases}\mathbb{Q} & i=0\\ 0 & i\neq 0\end{cases}$$

We define the differential d of $AC^\bullet$ by the equality

(5.2.1) $$d(\alpha)=\partial\alpha+(-1)^n\delta\alpha \text{ for } \alpha\in AC_i(P^n,\mathbf{D}^n;\mathbb{Q})^{\mathrm{alt}}.$$

Here we use the fact that the maps ∂ and δ commute with the projector Alt. *cf. Lemma 4.3* [4].

We have a natural inclusion $Z^r(\operatorname{Spec}\mathbf{k},2r-i)\to AC_{2r-2i}(P^{2r-i},\mathbf{D}^{2r-i};\mathbb{Q})$ which induces an inclusion $\iota : N\to AC^\bullet$.

Proposition 5.2. (1) *Let $u:\mathbb{Q}\to AC^\bullet$ be the inclusion defined by identifying $\mathbb{Q}$ with $AC_0(P^0,\mathbf{D}^0;\mathbb{Q})^{\mathrm{alt}}$. Then the map u is a quasi-isomorphism.*

(2) *The inclusion ι and the product defined by*

$$N_r^i\otimes AC_j(P^n,\mathbf{D}^n;\mathbb{Q})^{\mathrm{alt}}\to AC_{2r-2i+j}(P^{n+2r-i},\mathbf{D}^{n+2r-i};\mathbb{Q})^{\mathrm{alt}}$$
$$z\times\gamma\mapsto z\cdot\gamma:=\mathrm{Alt}(\iota(z)\times\gamma)$$

makes $AC^\bullet$ a differential graded N-module i.e. the product sends $N_r^i\otimes AC^j$ to AC^{i+j}, and one has the derivation formula; For $z\in N_r^i$ and $\gamma\in AC^\bullet$, one has

$$d(z\cdot\gamma)=\partial z\cdot\gamma+(-1)^i z\cdot d\gamma$$

(3) *The map I defined by $I = \sum_{n\geq 0} I_n : AC^\bullet \to \mathbb{C}$ is a map of complexes. Here the map I on $AC_0(P^0, D^0; \mathbb{Q})^{\mathrm{alt}} = \mathbb{Q}$ is defined to be the natural inclusion of $\mathbb{Q}$ to $\mathbb{C}$. The map I is a homomorphism of N-modules; For $z \in N$ and $\gamma \in AC^\bullet$, we have an equality*

$$I(z \cdot \gamma) = \epsilon(z) I(\gamma)$$

(4) *Let $I_{\mathbb{C}} : AC^\bullet \otimes \mathbb{C} \to \mathbb{C}$ be the map defined by the composition $AC^\bullet \otimes \mathbb{C} \overset{I\otimes \mathrm{id}}{\to} \mathbb{C} \otimes \mathbb{C} \overset{m}{\to} \mathbb{C}$. Here the map m is the multiplication. Then this map $I_{\mathbb{C}}$ is a quasi-isomorphism.*

Proof. (1) Since $(P^n, \mathbf{D}^n) = (\mathbb{P}^1, \{1\})^n$ (the n-fold self product), the Künneth formula tells us

$$H_*(C_\bullet(P^n, \mathbf{D}^n; \mathbb{Q})) = H_*(P^n, \mathbf{D}^n; \mathbb{Q}) = H_*(\mathbb{P}^1, \{1\}; \mathbb{Q})^{\otimes n}.$$

It follows that $H_i(P^n, \mathbf{D}^n; \mathbb{Q}) = 0$ for $i \neq 2n$, and $H_{2n}(P^n, \mathbf{D}^n; \mathbb{Q}) = \mathbb{Q}[P^n]$, where $[P^n]$ denotes the image of the orientation class $[P^n] \in H_{2n}(P^n; \mathbb{Q})$. Since $[P^n]$ is fixed by all $g \in G_n$, the alternating part $H_*(P^n, \mathbf{D}^n; \mathbb{Q})^{\mathrm{alt}}$ is zero for $n > 0$. By Proposition 2.8 , the complex $AC_\bullet(P^n, \mathbf{D}^n; \mathbb{Q})$ is quasi-isomorphic to $C_\bullet(P^n, \mathbf{D}^n; \mathbb{Q})$. It follows that $AC_\bullet(P^n, \mathbf{D}^n; \mathbb{Q})$ is an acyclic complex for $n > 0$. Therefore the map $u : \mathbb{Q} \to AC^\bullet(\mathbb{Q})$ is a quasi-isomorphism.

(2) The proof is similar to that of Lemma 4.3 [4]. The points are that the cubical differential ∂ and the topological differential δ commute with the projector Alt, and that we have the equality

$$\delta(z \times \gamma) = z \times \delta(\gamma)$$

since z is a topological cycle of even dimension.

(3) By Theorem 4.3 the map I is a homomorphism of complexes. The equality holds because the integral of the logarithmic differential form ω_n on an algebraic cycle is zero for the reason of type.

(4) This follows from the assertion (1) and the fact that the map $I_{\mathbb{C}} \circ (u \otimes \mathrm{id}) : \mathbb{C} = \mathbb{Q} \otimes \mathbb{C} \to \mathbb{C}$ is the identity map. □

5.3. **The bar complex.** Let M (resp. L) be a complex which is a differential left N-module (resp. right N-module). We recall the definition of the bar complex $B(L, N, M)$ (see [8], [11] for details.)

Let $N_+ = \oplus_{r>0} N_r$. As a module, $B(L, N, M)$ is equal to $L \otimes \left(\bigoplus_{s\geq 0}(\otimes^s N_+)\right) \otimes M$, with the convention $(\otimes^s N_+) = \mathbb{Q}$ for $s = 0$. An element $l \otimes (a_1 \otimes \cdots \otimes a_s) \otimes m$ of $L \otimes (\otimes^s N_+) \otimes M$ is written as $l[a_1|\cdots|a_s]m$ (for $s = 0$, we write $l[\ ]m$ for $l \otimes 1 \otimes m$ in $L \otimes \mathbb{Q} \otimes M$).

The internal differential d_I is defined by

$$\begin{aligned}
&d_I(l[a_1|\cdots|a_s]m) \\
=&dl[a_1|\cdots|a_s]m \\
+&\sum_{i=1}^{s}(-1)^i Jl[Ja_1|\cdots|Ja_{i-1}|da_i|\cdots|a_s]m + (-1)^s Jl[Ja_1|\cdots|Ja_s]dm
\end{aligned}$$

where $Ja = (-1)^{\deg a} a$. The external differential d_E is defined by

$$\begin{aligned} & d_E(l[a_1|\cdots|a_s]m) \\ & = -\,(Jl)a_1[a_2|\cdots|a_s]m \\ & + \sum_{i=1}^{s-1}(-1)^{i+1}Jl[Ja_1|\cdots|(Ja_i)a_{i+1}|\cdots|a_s]m \\ & +(-1)^{s-1}Jl[Ja_1|\cdots|Ja_{s-1}]a_s m. \end{aligned}$$

Then we have $d_I d_E + d_E d_I = 0$ and the map $d_E + d_I$ defines a differential on $B(L, N, M)$. The degree of an element $l[a_1|\cdots|a_s]m$ is defined by $\sum_{i=1}^{s} \deg a_i + \deg l + \deg m - s$.

If $L = \mathbb{Q}$ and the right N-module structure is given by the augmentation ϵ, the complex $B(L, N, M)$ is denoted by $B(N, M)$ and omit the first factor "$1\otimes$". If $L = M = \mathbb{Q}$ with the N-module structure given by the augmentation ϵ, we set

$$B(N) := B(\mathbb{Q}, N, \mathbb{Q}).$$

we omit the first and the last tensor factor "$1\otimes$" and "$\otimes 1$" for an element in $B(N)$.

The complex $B(N)$ is graded by non-negative integers as a complex: $B(N) = \oplus_{r\geq 0} B(N)_r$ where $B(N)_0 = \mathbb{Q}$ and, for $r > 0$,

$$B(N)_r = \oplus_{r_1+\cdots+r_s=r,\, r_i>0} N_{r_1} \otimes \cdots \otimes N_{r_s}.$$

Let $\Delta : B(N) \to B(N) \otimes B(N)$ be the map given by

$$\Delta([a_1|\cdots|a_s]) = \sum_{i=0}^{s} \big([a_1|\cdots|a_i]\big) \otimes \big([a_{i+1}|\cdots|a_s]\big).$$

and $e : B(N) \to \mathbb{Q}$ be the projection to $B(N)_0$. These are maps of complexes, and they satisfy coassociativity $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ and counitarity $(1 \otimes e)\Delta = (e \otimes 1)\Delta = id$, in other words Δ is a coproduct on $B(N)$ with counit e. In addition, the shuffle product (see e.g. [8]) makes $B(N)$ a DG algebra with unit $\mathbb{Q} = B(N)_0 \subset B(N)$. The shuffle product is graded-commutative. Further, the maps Δ and e are compatible with product and unit. We summarize:

(1) $B(N) = \oplus_{r\geq 0} B(N)_r$ is a DG bi-algebra over $\mathbb{Q}$. (It follows that $B(N)$ is a DG Hopf algebra, since it is a fact that antipode exists for a graded bi-algebra.)
(2) $B(N) = \oplus_{r\geq 0} B(N)_r$ is a direct sum decomposition to subcomplexes, and product, unit, co- product and counit are compatible with this decomposition.
(3) The product is graded-commutative with respect to the cohomological degree which we denote by i.

With due caution one may say that $B(N)$ is a "graded" DG Hopf algebra over $\mathbb{Q}$, with graded- commutative product; the first "grading" refers to r, and the second grading refers to i, while graded-commutativity of product is with respect to the grading i (the product is neither graded- commutative or commutative with respect to r). We recall that graded Hopf algebra in the literature means a graded Hopf algebra with graded-commutative product, so our $B(N)$ is a graded Hopf algebra in this sense with respect to the grading i, but is not one with respect to the "grading" r.

Let $\mathcal{H} := H^0(B(N))$. The product, unit, coproduct, counit on $B(N)$ induce the corresponding maps on $\mathcal{H}$, hence $\mathcal{H}$ is a "graded" Hopf algebra over $\mathbb{Q}$ in the following sense:

(1) $\mathcal{H}$ is a Hopf algebra over $\mathbb{Q}$.
(2) With $\mathcal{H}_r := H^0(B(N)_r)$, one has $\mathcal{H} = \oplus_{r\geq 0}\mathcal{H}_r$ a direct sum decomposition to subspaces; the product, unit, coproduct and counit are compatible with this decomposition.

We also have the coproduct map $\Delta : H^0(B(\mathbb{Q}, N, M)) \to \mathcal{H} \otimes H^0(B(\mathbb{Q}, N, M))$ obtained from the homomorphism of complexes $\Delta : B(\mathbb{Q}, N, M) \to B(N) \otimes B(\mathbb{Q}, N, M)$ given by

$$\Delta([a_1|\cdots|a_s]m) = \sum_{i=0}^{s} \big([a_1|\cdots|a_i]\big) \otimes \big([a_{i+1}|\cdots|a_s]m\big).$$

We define the category of mixed Tate motives after Bloch-Kriz [4].

Definition 5.3 (Graded $\mathcal{H}$-comodules, mixed Tate motives, [4]). (1) *Let $V = \oplus_i V_i$ be a graded vector space (to be precise, a finite dimensional $\mathbb{Q}$-vector space equipped with a grading by integers i). A linear map*

$$\Delta_V : V \to V \otimes \mathcal{H}$$

is called a graded coaction of $\mathcal{H}$ if the following conditions hold.
(a) $\Delta_V(V_i) \subset \oplus_{p+q=i} V_p \otimes \mathcal{H}_q$.
(b) *(Coassociativity) The following diagram commutes.*

$$\begin{array}{ccc} V & \xrightarrow{\Delta_V} & V \otimes \mathcal{H} \\ \Delta_V \downarrow & & \downarrow \mathrm{id}_V \otimes \Delta_{\mathcal{H}} \\ V \otimes \mathcal{H} & \xrightarrow{\Delta_V \otimes \mathrm{id}_{\mathcal{H}}} & V \otimes \mathcal{H} \otimes \mathcal{H} \end{array}$$

(c) *(Counitarity) The composite*

$$V \to V \otimes \mathcal{H} \xrightarrow{\mathrm{id}_V \otimes e} V,$$

is the identity map, where e is the counit of $\mathcal{H}$.

A graded vector space V with a graded coaction Δ_V of $\mathcal{H}$ is called a graded right comodule over $\mathcal{H}$. For graded right comodules V, W over $\mathcal{H}$, a linear map $V \to W$ is called a homomorphism of graded right comodules over $\mathcal{H}$ if it preserves the gradings and the coactions of $\mathcal{H}$. The category of graded right comodules over $\mathcal{H}$ is denoted by $(\mathrm{Com}^{gr}_{\mathcal{H}})$.

(2) *The category of mixed Tate motives $(MTM) = (MTM_{\mathbf{k}})$ over $\mathrm{Spec}(\mathbf{k})$ is defined as the category $(\mathrm{Com}^{\mathrm{gr}}_{\mathcal{H}})$ of graded right comodules over $\mathcal{H}$.*

5.4. **Mixed Hodge structure $\mathcal{H}_{Hg}$.** We recall the definition of the Tate Hodge structure. For an integer r, let $\mathbb{Q}(r) = \mathbb{Q}(2\pi i)^r$ with the weight filtration W defined by $\mathbb{Q}(r) = W_{-2r} \supset W_{-2r-1} = 0$, and let $\mathbb{C}(r) = \mathbb{C}$ with the Hodge filtration F defined by $\mathbb{C}(r) = F^{-r} \supset F^{-r+1} = 0$. We define the mixed Tate Hodge structure $\mathbb{Q}_{Hg}(r)$ of weight $-2r$ by the $\mathbb{Q}$-mixed Hodge structure $(\mathbb{Q}(r), \mathbb{C}(r), F, W)$ where the comparison map $c : \mathbb{Q}(r) \to \mathbb{C}(r)$ is defined by the identity map. This is a Hodge structure of type $(-r, -r)$. For a mixed Hodge structure H_{Hg}, $H_{Hg} \otimes \mathbb{Q}_{Hg}(r)$ is denoted by $H_{Hg}(r)$. A (finite dimensional) mixed Hodge structure is called a mixed Tate Hodge structure if the weight graded quotients are isomorphic to direct sums of Tate Hodge structures. An inductive limit of mixed Tate Hodge structure is called a ind-mixed Tate Hodge structure.

In this section, we define a mixed Hodge structure $\mathcal{H}_{Hg}$ with a left coaction of $\mathcal{H}$. We define the following two homomorphisms of complexes

$$\begin{aligned} &N_r \otimes \big(AC^\bullet \otimes \mathbb{Q}(s)\big) \to AC^\bullet \otimes \mathbb{Q}(r+s) : z \otimes \big(\gamma \otimes (2\pi i)^s\big) \mapsto z \cdot \gamma \otimes (2\pi i)^{r+s}, \\ &N_r \otimes \mathbb{C}(s) \to \mathbb{C}(r+s) : z \otimes a \mapsto \epsilon(z) a (2\pi i)^r. \end{aligned} \tag{5.4.2}$$

Here $z \cdot \gamma$ in the first line is defined in Proposition 5.2. We define the graded complexes $AC^\bullet \otimes \mathbb{Q}(*)$ and $\mathbb{C}(*)$ by

$$AC^\bullet \otimes \mathbb{Q}(*) = \oplus_r AC^\bullet \otimes \mathbb{Q}(r), \quad \mathbb{C}(*) = \oplus_r \mathbb{C}(r), \tag{5.4.3}$$

where the grading is given by r. Then by the product map (5.4.2), $AC^\bullet \otimes \mathbb{Q}(*)$ and $\mathbb{C}(*)$ become graded left N modules. We define the bar complexes $\mathcal{B}_B$ and $\mathcal{B}_{dR}$ by

$$\mathcal{B}_B = B(N, AC^\bullet \otimes \mathbb{Q}(*)) \text{ and } \mathcal{B}_{dR} = B(N, \mathbb{C}(*)).$$

They are equipped with the gradings

$$\mathcal{B}_B = \oplus_r \mathcal{B}_{B,r}, \mathcal{B}_{dR} = \oplus_r \mathcal{B}_{dR,r}$$

defined by (**??**). By Proposition 5.2 (3), the map I induces a morphism of complexes

$$I_r = I \otimes c : AC^\bullet \otimes \mathbb{Q}(r) \to \mathbb{C}(r),$$

and the direct sum $\oplus_r I_r : AC^\bullet \otimes \mathbb{Q}(*) \to \mathbb{C}(*)$ of I_r becomes a homomorphism of graded left N modules. Therefore the homomorphism $\oplus_r I_r$ induces a homomorphism of graded complexes $c : \mathcal{B}_B \to \mathcal{B}_{dR}$.

Definition 5.4. (1) *The homomorphism of complexes $c : \mathcal{B}_B \to \mathcal{B}_{dR}$ is called the comparison homomorphism.*

(2) *We define the Betti part $\mathcal{H}_B$ and the de Rham part $\mathcal{H}_{dR}$ of $\mathcal{H}_{Hg}$ by $\mathcal{H}_B = H^0(\mathcal{B}_B)$ and $\mathcal{H}_{dR} = H^0(\mathcal{B}_{dR})$. The gradings on $\mathcal{B}_B$ and $\mathcal{B}_{dR}$ induces the gradings $\mathcal{H}_B = \oplus_r \mathcal{H}_{B,r}$ and $\mathcal{H}_{dR} = \oplus_r \mathcal{H}_{dR,r}$.*

The comparison map c induces a homomorphism of graded vector spaces

$$c = \oplus_r c_r : \oplus_r (\mathcal{H}_{B,r} \otimes \mathbb{C}) \to \oplus_r \mathcal{H}_{dR,r}.$$

We introduce the weight filtration $W_\bullet$ on $\mathcal{B}_B$ and $\mathcal{B}_{dR}$ as follows.

$$W_n \mathcal{B}_B = \bigoplus_{2r \le n, s \ge 0} (\otimes^s N_+ \otimes AC^\bullet \otimes \mathbb{Q}(-r)).$$

$$W_n \mathcal{B}_{dR} = \bigoplus_{2r \le n, s \ge 0} (\otimes^s N_+ \otimes \mathbb{C}(-r)).$$

The Hodge filtration $F^\bullet$ on $\mathcal{B}_{dR}$ is defined by

$$F^p \mathcal{B}_{dR} = \bigoplus_{r \ge p, s \ge 0} (\otimes^s N_+ \otimes \mathbb{C}(-r)).$$

By Proposition 5.2 (3), the map c induces a homomorphism

$$c : W_n \mathcal{B}_B \otimes \mathbb{C} \to W_n \mathcal{B}_{dR} \tag{5.4.4}$$

The weight and Hodge filtrations are direct sum of the the induced filtrations on each gradings. The weight and Hodge filtrations on $\mathcal{B}_B$ and $\mathcal{B}_{dR}$ induce those of $\mathcal{H}_B$ and $\mathcal{H}_{dR}$, respectively.

Proposition-Definition 5.5. (1) *The map c induces a filtered quasi-isomorphism $\mathcal{B}_B \otimes \mathbb{C} \to \mathcal{B}_{dR}$ with respect to W.*

(2) *We have a canonical isomorphism of vector spaces*

$$Gr^W_{2r}\mathcal{H}_B \to \mathcal{H} \otimes \mathbb{Q}(-r)$$

(3) *Via the isomorphism $c : \mathcal{H}_B \otimes \mathbb{C} \to \mathcal{H}_{dR}$, the pair of filtered vector spaces $\mathcal{H}_{Hg} = (\mathcal{H}_B, \mathcal{H}_{dR}, W, F)$ becomes a ind-mixed Tate Hodge structure with a grading*

$$\mathcal{H}_{Hg} = \oplus_r \mathcal{H}_{Hg,r}.$$

Proof. (1) One sees that the quotient $Gr^W_{2r}\mathcal{B}_B$ is the tensor product

$$B(N) \otimes AC^\bullet \otimes \mathbb{Q}(-r)$$

as a complex. By Proposition 5.2 (4) the map c induces a quasi-isomorphism $Gr^W_{2r}\mathcal{B}_B \otimes \mathbb{C} \to Gr^W_{2r}\mathcal{B}_{dR}$.

(2) We consider the spectral sequences for the filtration W:

$$E_1^{p,q} = H^{p+q}(Gr^W_{-p}\mathcal{B}_B) \Rightarrow E^{p+q} = H^{p+q}(\mathcal{B}_B)$$
$$'E_1^{p,q} = H^{p+q}(Gr^W_{-p}\mathcal{B}_{dR}) \Rightarrow \ 'E^{p+q} = H^{p+q}(\mathcal{B}_{dR}).$$

Since the morphism of complexes $c : \mathcal{B}_B \otimes \mathbb{C} \to \mathcal{B}_{dR}$ is a filtered quasi-isomorphism, the morphism of spectral sequences

$$E_*^{*,*} \otimes \mathbf{C} \to \ 'E_*^{*,*}$$

is an isomorphism. Since the complex $\mathcal{B}_{dR}$ is the direct sum $\underset{r}{\oplus} B(N) \otimes \mathbb{C}(r)$, the spectral sequence $'E_*^{*,*}$ degenerates at E_1-term, and as a consequence $E_*^{*,*}$ also degenerates at E_1-term. Therefore the vector space $Gr^W_{2r}H^0(\mathcal{B}_B)$ is canonically isomorphic to $H^0(Gr^W_{2r}\mathcal{B}_B)$. By Proposition 5.2 (1) we have

$$H^0(Gr^W_{2r}\mathcal{B}_B) = H^0(B(N)) \otimes H^0(AC^\bullet \otimes \mathbb{Q}(-r)) = \mathcal{H} \otimes \mathbb{Q}(-r).$$

(3) We need to show the following.

(1) The filtrations F and $\overline{F}$ on $Gr^W_{2r}\mathcal{H}_{dR}$ are $2r$-opposite ([6], (1.2.3)).

(2) $(F^p \cap \overline{F^{2r-p}})Gr^W_{2r}\mathcal{H}_{dR} = 0$ for $p \neq r$.

We denote $Gr^W_{2r}\mathcal{H}_{dR} = A$. We have

$$F^p(A) = \begin{cases} A & p \leq r \\ 0 & p > r \end{cases}$$

By taking complex conjugate, a similar fact holds for $\overline{F^p}(A)$. The assertions (1) and (2) follow from this.

□

5.5. **Coaction of $\mathcal{H}$ on $\mathcal{H}_{Hg}$ and a Hodge realization functor.** We have a homomorphism of complexes

$$\Delta_B : \mathcal{B}_B \to B(N) \otimes \mathcal{B}_B, \quad \Delta_{dR} : \mathcal{B}_{dR} \to B(N) \otimes \mathcal{B}_{dR}$$

which are compatible with the gradings and homomorphism c. Here the grading of the target of Δ_B (resp. Δ_{dR}) is given by tensor product of those of $B(N)$ and $\mathcal{B}_B$ (resp. $\mathcal{B}_{dR}$). By taking the cohomology of degree zero, we have a homomorphism

$$\Delta_{Hg} : \mathcal{H}_{Hg} \to \mathcal{H} \otimes \mathcal{H}_{Hg} \tag{5.5.5}$$

of ind-mixed Tate Hodge structures with gradings, i.e. $\Delta_{Hg}(\mathcal{H}_{Hg,r}) \subset \oplus_{s+t=r}\mathcal{H}_s \otimes \mathcal{H}_{Hg,t}$.

We define the Hodge realization functor Φ from the category $(\mathrm{Com}^{\mathrm{gr}}_{\mathcal{H}})$ of graded $\mathcal{H}$-comodules to the category MTH of mixed Tate Hodge structure as follows. Let $V = \oplus_i V_i$ be a graded comodule over $\mathcal{H}$ given by the comodule structure

$$\Delta_V : V \to V \otimes \mathcal{H}, \quad \Delta_V(V_i) \subset \oplus_{j+k=i} V_j \otimes \mathcal{H}_k.$$

Using the comodule structure Δ_V and the coproduct homomorphism Δ_{Hg} of (5.5.5), we define the following graded homomorphisms of graded ind-mixed Hodge structures.

$$\Delta_V \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_{Hg} : V \otimes \mathcal{H}_{Hg} \to V \otimes \mathcal{H} \otimes \mathcal{H}_{Hg}. \tag{5.5.6}$$

The degree zero part of the kernel of a graded homomorphism φ is denoted by $\ker_0(\varphi)$.

Definition 5.6 (Realization functor, functor ω_{gr}). (1) *We define the functor Φ from the category $(\mathrm{Com}^{gr}_{\mathcal{H}})$ of graded $\mathcal{H}$-comodules to that of mixed Tate Hodge structures (MTH) by*

$$\Phi(V) = \ker_0(\Delta_V \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_{Hg})$$

It is called the realization functor.

(2) *Let $\mathbf{V}$ be a mixed Tate Hodge structure. We define a graded module $\omega_{\mathrm{gr}}(\mathbf{V})$ by $\oplus_i \omega_{\mathrm{gr}}(\mathbf{V})_i$ where*

$$\omega_{\mathrm{gr}}(\mathbf{V})_i = Hom_{MTH}(\mathbb{Q}_{Hg}(-i), Gr^W_{2i}(\mathbf{V})).$$

Then ω becomes an exact functor from the category (MTH) of mixed Tate Hodge structures to the category $(\mathrm{Vect}^{\mathrm{gr}})$ of finite dimensional graded vector spaces. The category of ind-mixed Tate Hodge structure is denoted by $(\mathrm{ind}-MTH)$ and that of ind-finite dimensional graded vector space is denoted by $(\mathrm{ind}-\mathrm{Vect}^{gr}_{\mathbb{Q}})$. The functor ω_{gr} is extended to the functor from $(\mathrm{ind}-MTH)$ to $(\mathrm{ind}-\mathrm{Vect}^{gr}_{\mathbb{Q}})$.

Proposition 5.7. (1) *For objects M, N in (MTH), we have*

$$\omega_{gr}(M \otimes N) = \omega_{gr}(M) \otimes \omega_{gr}(N).$$

(2) *We have an equality of graded vector spaces*

$$\omega_{gr}(\mathcal{H}_{Hg}) = \mathcal{H} \otimes \omega_{gr}(\mathbb{Q}_{Hg}(*)),$$

where $\mathbb{Q}_{Hg}(*) = \oplus_r \mathbb{Q}_{Hg}(r)$. This equality is compatible with the left coaction of $\mathcal{H}$.

(3) *Let $\varphi :\ (\mathrm{Com}^{\mathrm{gr}}_{\mathcal{H}}) \to (\mathrm{Vect}^{\mathrm{gr}}_{\mathbb{Q}})$ be the functor of forgetting the $\mathcal{H}$-comodule structure. Then we have the following equality of functors.*

$$\omega_{\mathrm{gr}} \circ \Phi = \varphi :\ (\mathrm{Com}^{\mathrm{gr}}_{\mathcal{H}}) \to (\mathrm{Vect}^{\mathrm{gr}}_{\mathbb{Q}}).$$

As a consequence, the functor Φ is an exact functor.

Proof. (1) The proof will be left to the reader.

(2) By the definition of ω_{gr}, it is a consequence of Proposition 5.5 (1).

(3) Let V be a graded right $\mathcal{H}$-comodule. Since the functor ω_{gr} is exact ([6], Theorem(1.2.10) (iv)), we have

$$\begin{aligned}
&\omega_{gr}(\ker(\Delta_V \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_{Hg})) \\
&\xrightarrow{\simeq} \ker(\omega_{gr}(\Delta_V \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_{Hg})) \\
&\simeq \ker(V \otimes \mathcal{H} \otimes \omega_{gr}(\mathbb{Q}_{Hg}(*)) \xrightarrow{\Delta_V \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_{\mathcal{H}}} V \otimes \mathcal{H} \otimes \mathcal{H} \otimes \omega_{gr}(\mathbb{Q}_{Hg}(*))) \\
&\simeq V \otimes \omega_{gr}(\mathbb{Q}_{Hg}(*)) = \bigoplus_{r,s} (V_r \otimes \omega_{gr}(\mathbb{Q}_{Hg}(s))).
\end{aligned}$$

By taking the direct summand of the above isomorphisms corresponding to $\Phi(V)$, we have

$$\omega_{gr}(\Phi(V)) = \bigoplus_{r} (V_r \otimes \omega_{gr}(\mathbb{Q}_{Hg}(-r))) = \varphi(V),$$

Thus we have the statement (3). $\square$

Remark 5.8. *The functor Φ is isomorphic to the functor obtained by the map $Real_{TMH}$ defined in* [4]. *The proof with slightly different formulation will be given in the sequel paper.*

5.6. **In the case of dilogarithm.** Using the above notations, we describe the Hodge realization of the comodule over $\mathcal{H}$ associated to dilogarithm functions after Section 9 of [4].

We assume $\mathbf{k} \subset \mathbb{C}$ and $a \in \mathbf{k}^\times - \{1\}$. We define elements $\rho_1(a) \in N_1^1$ and $\rho_2(a) \in N_2^1$ by

$$\begin{aligned}
\rho_1(a) &= \{(1-a) \in \mathbb{P}^1\}^{\mathrm{alt}} \\
\rho_2(a) &= \{(x_1, 1-x_1, 1-\frac{a}{x_1}) \in (\mathbb{P}^1)^3\}^{\mathrm{alt}}
\end{aligned}$$

Then we have the following relations:

$$\partial \rho_2(a) = -\rho_1(1-a) \cdot \rho_1(a), \quad \partial \rho_1(a) = \partial \rho_1(1-a) = 0.$$

Therefore the elements $\mathbf{Li}_1(a), \mathbf{Li}_2(a)$ defined as follows are closed elements in $B(N)$, and thus they define elements in $\mathcal{H}$.

$$\begin{aligned}
\mathbf{Li}_2(a) &= [\rho_2] - [\rho_1(1-a)|\rho_1(a)], \\
\mathbf{Li}_1(a) &= [\rho_1(a)].
\end{aligned}$$

Let $V = V_2 \oplus V_1 \oplus V_0$ be the sub $\mathcal{H}$-comodule of $\mathcal{H}$ generated by $e_2 := \mathbf{Li}_2(a)$, $e_1 := \mathbf{Li}_1(1-a)$ and $e_0 := 1$. We have

$$\begin{aligned}
\Delta_V(e_2) &= e_2 \otimes 1 - e_1 \otimes \mathbf{Li}_1(a) + e_0 \otimes \mathbf{Li}_2(a), \\
\Delta_V(e_1) &= e_1 \otimes 1 + e_0 \otimes \mathbf{Li}_1(1-a), \\
\Delta_V(e_0) &= e_0 \otimes 1.
\end{aligned}$$

Assume that a is contained in $\mathbb{R}$ and assume that $0 < a < 1$. We consider elements $\eta_1(0), \eta_2(1), \eta_2(0)$ in AC^0 defined by

$$\begin{aligned}
\eta_1(0) &= \{(1-t_0) \in \mathbb{P}^1 \mid 0 < t_0 < a\}^{\mathrm{alt}}, \\
\eta_2(1) &= \{(x_1, 1-x_1, 1-\frac{t_1}{x_1}) | \, x_1 \in \mathbb{P}^1_{\mathbb{C}} - \{1\}, \, 0 < t_1 < a\}^{\mathrm{alt}}, \\
\eta_2(0) &= \{(t_1, 1-t_0) | \, 0 < t_0 < t_1 < a\}^{\mathrm{alt}},
\end{aligned}$$

where the orientation of $\eta_2(0)$ is defined so that (t_1, t_0) are positive coordinates. We have the relations

$$\begin{aligned}
\delta\eta_1(0) &= \rho_1(a), \\
\partial\eta_1(0) &= 0, \\
\delta\eta_2(1) &= \rho_2(a), \\
\delta\eta_2(0) &= -\partial\eta_2(1) + \rho(1-a)\cdot\eta_1(0), \\
\partial\eta_2(0) &= 0.
\end{aligned}$$

Let $\xi_1(a) = \eta_1(0)$, and $\xi_2(a) = \eta_2(1) + \eta_2(0)$ be chains of AC^0. We have the equalities

$$\begin{aligned}
d(\xi_1(a)) &= -\rho_1(a), \\
d(\xi_2(a)) &= -\rho_2(a) + \rho_1(1-a)\cdot\xi_1(a).
\end{aligned}$$

Then the elements $Z_0, Z_1(a), Z_2$ of $\mathcal{B}_B$ defined by

$$\begin{aligned}
&Z_2 = \mathbf{Li}_2(a) \otimes 1 \otimes (2\pi i)^{-2} - \mathbf{Li}_1(1-a) \otimes \xi_1(a) \otimes (2\pi i)^{-1} + 1 \otimes \xi_2(a) \otimes (2\pi i)^0 \in \mathcal{B}_{B,0}, \\
&Z_1(a) = \mathbf{Li}_1(a) \otimes 1 \otimes (2\pi i)^{-2} + 1 \otimes \xi_1(a) \otimes (2\pi i)^{-1} \in \mathcal{B}_{B,-1}, \\
&Z_i = 1 \otimes (2\pi i)^i \in \mathcal{B}_{B,i} \ (i = 0, -1, -2)
\end{aligned}$$

are cocycles. The elements

$$\begin{aligned}
v_2 &= e_2 \otimes Z_{-2} - e_1 \otimes Z_1(a) + e_0 \otimes Z_2, \\
v_1 &= e_1 \otimes Z_{-1} + e_0 \otimes Z_1(1-a)\cdot(2\pi i), \\
v_0 &= e_0 \otimes Z_0
\end{aligned}$$

forms a basis of the kernel of the map

$$\Delta_V \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_B : \bigoplus_i V_i \otimes \mathcal{H}_{B,-i} \to \bigoplus_{i,j} V_i \otimes \mathcal{H}_j \otimes \mathcal{H}_{B,-i-j}.$$

Similarly, the elements

$$\begin{aligned}
w_2 &= e_2 \otimes 1 \otimes 1 - e_1 \otimes \mathbf{Li}_1(a) \otimes 1 + e_0 \otimes \mathbf{Li}_2(a) \otimes 1 \\
&\quad \in V_2 \otimes \mathcal{H}_{dR,-2} \oplus V_1 \otimes \mathcal{H}_{dR,-1} \oplus V_0 \otimes \mathcal{H}_{dR,0}, \\
w_1 &= e_1 \otimes 1 \otimes 1 + e_0 \otimes \mathbf{Li}_1(1-a) \otimes 1 \\
&\quad \in V_1 \otimes \mathcal{H}_{dR,-1} \oplus V_0 \otimes \mathcal{H}_{dR,0}, \\
w_0 &= e_0 \otimes 1 \otimes 1 \in V_0 \otimes \mathcal{H}_{dR,0}
\end{aligned}$$

form a basis of the kernel of the map

$$\Delta_V \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_{dR} : \bigoplus_i V_i \otimes \mathcal{H}_{dR,-i} \to \bigoplus_{i,j} V_i \otimes \mathcal{H}_j \otimes \mathcal{H}_{dR,-i-j}.$$

The images of Z_i under the comparizon map $\mathcal{B}_B \to \mathcal{B}_{dR}$ are equal to

$$\begin{aligned} c(Z_2) &= (\mathbf{Li}_2(a) \otimes 1 + \mathbf{Li}_1(1-a) \otimes Li_1(a) + 1 \otimes Li_2(a))(2\pi i)^{-2}, \\ c(Z_1(a)) &= (\mathbf{Li}_1(a) \otimes 1 + 1 \otimes \log(1-a))(2\pi i)^{-2}, \\ c(Z_0) &= 1 \otimes 1. \end{aligned} \tag{5.6.7}$$

Therefore we have the relation

$$\begin{aligned} c(v_2) &= (2\pi i)^{-2}(w_2 + w_1 Li_1(a) + w_0 Li_2(a)), \\ c(v_1) &= (2\pi i)^{-1}(w_1 + w_0 \log a), \\ c(v_0) &= w_0. \end{aligned}$$

So the period matrix of the Hodge realization $\Phi(V)$ of V with respect to the de Rham basis $w_0 \in W_0 \cap F^0$, $w_1 \in F^1 \cap W_1$ and $w_2 \in F^2 \cap W_2$ is equal to

$$\begin{pmatrix} (2\pi i)^{-2} & 0 & 0 \\ Li_1(a)(2\pi i)^{-2} & (2\pi i)^{-1} & 0 \\ Li_2(a)(2\pi i)^{-2} & \log a (2\pi i)^{-1} & 1 \end{pmatrix}$$

which is compatible with the one given in [1].

Appendix A. Proof of the moving lemma

Before the proof, we recall the following three theorems.

Theorem A.1. *([16] Ch.6, Theorem 15) Let M be a compact PL-manifold and let K be a PL-triangulation of M. Let X, X_0 and Y be subpolyhedra of M such that $X_0 \subset X$. Then there exists an ambient PL isotopy $h : M \times [0,1] \to M$ such that $h(x,t) = x$ for any point $x \in X_0$ and $t \in [0,1]$ (we refer to this fact by saying that h fixes X_0), and such that $h_1(|X| - X_0)$ is in general position with respect to Y i.e. the inequality*

$$\dim(h_1(|X| - X_0) \cap Y) \leq \dim(|X| - X_0) + \dim Y - \dim M$$

holds. Here $h_t(m) = h(m,t)$ for $m \in M$ and $t \in [0,1]$.

We need the following variant of this theorem.

Theorem A.2. *Let M be a compact PL-manifold and let K be a triangulation of M. Let X_0, X and $Y_1, \cdots, Y_n$ be subpolyhedra of M. Then there exists an ambient PL isotopy $h : M \times [0,1] \to M$ which fixes X_0, and such that $h_1(|X| - X_0)$ are in general position with respect to Y_i for $1 \leq i \leq n$.*

Theorem A.3 (Lemma 1.10, [13])**.** *Let $f : |K| \to |L|$ be a PL map of the realizations of simplicial complexes K and L. Then there exist subdivisions K' and L' of K and L respectively, such that f is induced by a simplicial map $K' \to L'$.*

Proof of Proposition 2.8. (1) We prove the surjectivity of the map ι on homology groups. Let γ be a closed element in $C_p(K, \mathbf{D}^n; \mathbb{Q})$ for a triangulation K. Let $\gamma = \sum a_\sigma \sigma$ be a representative of γ in $C_p(K; \mathbb{Q})$. By applying Theorem A.2 to the case where $M = P^n$, $X = |\gamma|$, $X_0 = \mathbf{D}^n$ and Y_i the set of cubical faces, we have a PL isotopy $h : P^n \times [0,1] \to P^n$ such that

(1) h fixes $\mathbf{D}^n$, and
(2) $h_1(|\gamma| - \mathbf{D}^n)$ meets each cubical face properly.

By Theorem A.3 there exists a triangulation $\mathcal{K}$ of $P^n \times [0,1]$ and a subdivision K' of K such that

(1) The map h is induced by a simplicial map from $\mathcal{K}$ to K', and
(2) $\sigma \times [0,1]$, $\sigma \times \{0\}$ and $\sigma \times \{1\}$ are subcomplexes of $\mathcal{K}$ for each simplex σ of K.

Let $\lambda : C_p(K) \to C_p(K')$ be the subdivision operator. Let

$$h_* : C_\bullet(\mathcal{K}, \mathbf{D}^n \times [0,1]; \mathbb{Q}) \to C_\bullet(K', \mathbf{D}^n; \mathbb{Q})$$

be the map of complexes induced by h. For a simplex $\sigma \in K$, the product $\sigma \times [0,1]$, $\sigma \times \{0\}$ and $\sigma \times \{1\}$ are regarded elements of $C_\bullet(\mathcal{K})$. For an element $\sigma \in K_p$, we set

$$\begin{aligned} h_\sigma &= h_*(\sigma \times [0,1]) \in C_{p+1}(K', \mathbf{D}^n), \\ h_i(\sigma) &= h_*(\sigma \times \{i\}) \in C_p(K', \mathbf{D}^n) \quad (i = 0,1). \end{aligned} \tag{1.0.1}$$

Then we have $h_0(\sigma) = \lambda(\sigma)$. Let θ be the chain $\sum_\sigma a_\sigma h_\sigma$. Then we have

$$\begin{aligned} \delta\theta &= \sum_\sigma a_\sigma \delta h_\sigma \\ &= \sum_\sigma a_\sigma((-1)^p(h_1(\sigma) - h_0(\sigma)) + h_{\delta\sigma}) \\ &= \sum_\sigma a_\sigma h_{\delta\sigma} + (-1)^p \left(\sum_\sigma a_\sigma h_1(\sigma) - \sum_\sigma a_\sigma \lambda(\sigma) \right) \\ &= \sum_\rho \left(\sum_{\rho \prec \sigma} [\sigma : \rho] a_\sigma \right) h_\rho + (-1)^p \left(\sum_\sigma a_\sigma h_1(\sigma) - \sum_\sigma a_\sigma \lambda(\sigma) \right) \end{aligned}$$

in $C_p(K'; \mathbb{Q})$. Here in the last line, the sum in the first term is taken over $(p-1)$-simplexes of K, and $\rho \prec \sigma$ means that ρ is a face of σ. The index of σ with respect to ρ is denoted by $[\sigma : \rho]$. It is in the set $\{\pm 1\}$. Since γ is closed, we have $\sum_{\rho \prec \sigma} [\sigma : \rho] a_\sigma = 0$ if $\rho \not\subset \mathbf{D}^n$. Since $\mathbf{D}^n$ is fixed by h, if $\rho \subset \mathbf{D}^n$, then we have $h_\rho = 0$. Thus we have the equality

$$\delta\theta = (-1)^p \left(\sum_\sigma a_\sigma h_1(\sigma) - \lambda(\gamma) \right).$$

Since $|\sum_\sigma a_\sigma h_1(\sigma)| = h_1(|\gamma|)$ by the construction of h, it follows that $\sum_\sigma a_\sigma h_1(\sigma) \in AC_p(K', \mathbf{D}^n; \mathbb{Q})$.

(2) We prove the injectivity of the map ι on homology. Let γ be an element in $AC_p(P^n, \mathbf{D}^n; \mathbb{Q})$ and suppose that γ is the boundary of an element ξ in $C_{p+1}(P^n, \mathbf{D}; \mathbb{Q})$. Representatives of γ and ξ in $AC_p(K; \mathbb{Q})$ and $C_{p+1}(K; \mathbb{Q})$ are also denoted by γ and ξ. By applying Theorem A.2 to the case where $M = P^n$, $X = |\xi|$, $X_0 = \mathbf{D}^n \cup |\gamma|$ and Y_i the set of cubical faces, we obtain a PL isotopy $h : P^n \times [0,1] \to P^n$ of P^n such that

(1) h fixes $\mathbf{D}^n \cup |\gamma|$, and
(2) $h_1(|\xi| - (\mathbf{D}^n \cup |\gamma|))$ intersects the cubical faces properly.

By a similar argument as the surjectivity, we see that the chain $h_1(\xi)$ is an element of $AC_{p+1}(K', \mathbf{D}^n; \mathbb{Q})$ for a subdivision K' of K, and that $\delta h_1(\xi) = \lambda(\gamma)$ where $\lambda : C_\bullet(K, \mathbf{D}^n; \mathbb{Q}) \to C_\bullet(K', \mathbf{D}^n, \mathbb{Q})$ is the subdivision operator. □

Appendix B. Properties of cap products

B.1. **Some facts on homotopy.** The following proposition is known as acyclic carrier theorem (See [14], Theorem 13.4, p76).

Proposition B.1. *Let K be a simplicial complex and $D_\bullet$ be a (homological) complex such that $D_i = 0$ for $i < 0$. Let p be a non-negative integer, and let $\varphi_\bullet : C_\bullet(K) \to D_{\bullet-p}$ be a homomorphism of complexes. We suppose that there exists a family of subcomplexes $\{L^\sigma_\bullet\}_{\sigma\in K}$ of $D_\bullet$ indexed by $\sigma \in K$ which satisfies the following conditions.*

(1) *$L^\tau \subset L^\sigma$ for $\tau, \sigma \in K$ such that $\tau \subset \sigma$.*
(2) *$\varphi_\bullet(\sigma) \in L^\sigma_{\bullet-p}$ for all $\sigma \in K$.*
(3) *The homology groups $H_k(L^\sigma_\bullet) = 0$ for $k > 0$.*
(4) *The homology class of the cycle $\varphi_p(\sigma)$ in $H_0(L^\sigma)$ is zero for each p-simplex σ.*

Under the above assumptions, there exist homomorphisms $\theta_{p+q} : C_{p+q}(K) \to D_{q+1}$ $(q \geq 0)$, satisfying the following conditions:

(a) *$\delta\theta_{p+q} + \theta_{p+q-1}\delta = \varphi_{p+q}$ for $q \geq 0$. Here we set $\theta_{p-1} = 0$.*
(b) *$\theta_i(\sigma) \in L^\sigma_\bullet$.*

Proof. We construct maps θ_{p+q} inductively on q. We consider the case where $q = 0$. Let σ be a simplex of K_p. Since the homology class of $\varphi_p(\sigma)$ in $H_0(L^\sigma_\bullet)$ is zero, there exists an element $t_\sigma \in L^\sigma_1$ such that $\delta t_\sigma = \varphi(\sigma)$. By setting $\theta_p(\sigma) = t_\sigma$, we have a map $\theta_p : C_p(K) \to D_1$.

We assume that θ_{p+q} is constructed and construct θ_{p+q+1}. Let σ be a $(p+q+1)$-simplex of K regarded as an element of $C_{p+q+1}(K)$. Using the inductive assumption of the equality (a), we have

$$\begin{aligned}\delta(\varphi(\sigma) - \theta_{p+q}(\delta\sigma)) =& \varphi(\delta\sigma) - \delta\theta_{p+q}(\delta\sigma)\\ =& \varphi(\delta\sigma) + \theta_{p+q-1}(\delta\delta\sigma) - \varphi(\delta\sigma) = 0.\end{aligned}$$

By the inductive assumption of (b) and the assumption (1), we have $\theta_{p+q}(\delta\sigma) \in L^\sigma_{q+1}$. Since we have $\varphi(\sigma) \in L^\sigma$ by the assumption (2), $\varphi(\sigma) - \theta_{p+q}(\delta\sigma)$ is a closed element in L^σ_{q+1}. By the assumption (3), there exists an element $t_\sigma \in L^\sigma_{p+q+2}$ such that $\delta t_\sigma = \varphi(\sigma) - \theta_{p+q}(\delta\sigma)$. We define a morphism θ_{p+q+1} to be $\theta_{p+q+1}(\sigma) = t_\sigma$ and the map θ_{p+q+1} satisfies conditions (a) and (b) for $q + 1$. □

B.2. **Independence of ordering.** Let K be a finite simplicial complex, L a full subcomplexes of K, and $\mathcal{O}$ be a good ordering with respect to L. We set

$$W = \bigcup_{\sigma\cap L=\emptyset} \sigma.$$

Let p be a positive even integer, and let T be a p-cocycle in $C^p(K, W)$ and $\varphi : C_r(K; \mathbb{Q}) \to C_{r-p}(K; \mathbb{Q})$ be the map defined by $\varphi(\alpha) = T \overset{\mathcal{O}}{\cap} \alpha$ (3.2.2). Then the map φ is a homomorphism of complexes, and its image is contained in $C_{\bullet-p}(L;\mathbb{Q})$. Thus we have a homomorphism of complexes:

$$\varphi : C_\bullet(K;\mathbb{Q}) \to C_{\bullet-p}(L;\mathbb{Q}).$$

Let T' be a p-cocycle in $C^p(K,W)$ and set $\varphi'(\alpha) = T' \overset{\mathcal{O}}{\cap} \alpha$. If $T - T'$ is the coboundary of $w \in C^\bullet(K,W)$, i.e. $dw = T - T'$, then we have

$$\delta(w \overset{\mathcal{O}}{\cap} \alpha) - w \overset{\mathcal{O}}{\cap} \delta\alpha = -(\varphi - \varphi')(\alpha)$$

for each $\alpha \in C_\bullet(K)$. Therefore the homomorphism of homologies $[\varphi] : H_{p+q}(K;\mathbb{Q}) \to H_q(L;\mathbb{Q})$ induced by φ depends only on the cohomology class $[T]$ of T.

Let K^* be a subcomplex of K and set $L^* = K^* \cap L$. By restricting the homomorphism φ to $C_\bullet(K^*)$, we have a homomorphism of subcomplexes $C_{p+q}(K^*;\mathbb{Q}) \to C_q(L^*;\mathbb{Q})$ and a homomorphism of relative homologies

$$[\varphi] : H_{p+q}(K,K^*;\mathbb{Q}) \to H_q(L,L^*;\mathbb{Q}). \tag{2.2.1}$$

This homomorphism also depends only on the cohomology class $[T]$ of T in $H^p(K,W)$.

Proposition B.2. *The homomorphism (2.2.1) is independent of the ordering $\mathcal{O}$.*

Proof. The assertion follows from the following lemma.

Lemma B.3. *Let T be a cocycle in $C^p(K,W)$ and $\mathcal{O}$ and $\mathcal{O}'$ be good orderings of K with respect to L. Then there exists a map $\theta_{p+q} : C_{p+q}(K) \to C_{q+1}(L)$ $(q \geq 0)$ satisfying the following conditions:*

1. $\delta\theta_{p+q}(x) + \theta_{p+q-1}(\delta x) = T \overset{\mathcal{O}}{\cap} x - T \overset{\mathcal{O}'}{\cap} x$. *for $q \geq 0$. Here we set $\theta_{p-1} = 0$.*
2. $\theta_{p+q}(\sigma) \in C_{q+1}(L \cap \sigma)$ *for each simplex $\sigma \in K_{p+q}$.*

Proof. We apply Proposition B.1 to $\varphi(x) = T \overset{\mathcal{O}}{\cap} x - T \overset{\mathcal{O}'}{\cap} x$, $D = C_\bullet(L)$ and $L^\sigma_\bullet = C_\bullet(L\cap\sigma)$. Conditions (1), (2) are easily verified. Since the complex L is a full subcomplex of K, the intersection $L\cap\sigma$ is a face of σ, and condition (3) is satisfied. We check condition (4). Let $\sigma = [v_0,\dots,v_p] = \pm[v'_0,\dots,v'_p]$ be a p-simplex. Here $v_0 < \cdots < v_p$ for the ordering $\mathcal{O}$ and $v'_0 < \cdots < v'_p$ for the ordering $\mathcal{O}'$. Then we have $T \overset{\mathcal{O}}{\cap} \sigma = T(\sigma)[v_p]$ and $T \overset{\mathcal{O}'}{\cap} \sigma = T(\sigma)[v'_p]$ Since $[v_p]$ and $[v'_p]$ are in the same homology class in $H_0(L\cap\sigma)$, and (4) is proved. Thus we have a map satisfying conditions (1) and (2) in the lemma. □

□

Since the homomorphism (2.2.1) depends only on the choice of cohomology class $[T]$ of T, it is written as $[T]\cap$.

Proof of Proposition 3.7 (2). Let γ be an element in $AC_k(K,\mathbf{D}^n;\mathbb{Q})$. By the admissibility condition for $\delta\gamma$, we have $L_1 \cap |\delta\gamma| \subset (L_1 \cap |\gamma|)^{(k-3)} \cup \mathbf{D}^n$, where $(L_1\cap|\delta\gamma|)^{(k-3)}$ is the $(k-3)$-skeleton of $L_1 \cap |\gamma|$. We denote the set $\mathbf{D}^n \cap |\gamma|$ by $\mathbf{D}^n_\gamma$ and the set $H_{1,0} \cap \mathbf{D}^n \cap |\gamma|$

by $\mathbf{D}_\gamma^{n-1}$. We have a homomorphism

$$
\begin{aligned}
(2.2.2) \qquad T\overset{\mathcal{O}}{\cap}: H_k(|\gamma|, |\delta\gamma| \cup \mathbf{D}_\gamma^n; \mathbb{Q}) &\to H_{k-2}(L_1 \cap |\gamma|, (L_1 \cap |\delta\gamma|) \cup \mathbf{D}_\gamma^{n-1}; \mathbb{Q}) \\
&\to H_{k-2}(L_1 \cap |\gamma|, (L_1 \cap |\gamma|)^{(k-3)} \cup \mathbf{D}_\gamma^{n-1}; \mathbb{Q}) \\
&\simeq \bigoplus_{\substack{\tau \in L_1 \cap |\gamma|,\ \tau \not\subset \mathbf{D}^{n-1} \\ \dim \tau = k-2}} \mathbb{Q}\tau.
\end{aligned}
$$

The chain $T\overset{\mathcal{O}}{\cap}\gamma$ is equal to the image of the homology class $[\gamma]$ of γ under the homomorphism (2.2.2). The map (2.2.2) is independent of the choice of Thom cocycle T and the ordering $\mathcal{O}$ by Proposition B.2. $\square$

B.3. **Compatibility with the subdivision map.** Let K be a simplicial complex, L a full subcomplex of K, and K' a subdivision of K. The subdivision of L induced by K' is denoted by L'. We assume that L' is a full subcomplex of K'. Then we have the following subdivision operators:

$$
\begin{aligned}
\lambda &: C_\bullet(K; \mathbb{Q}) \to C_\bullet(K'; \mathbb{Q}), \\
\lambda &: C_\bullet(L; \mathbb{Q}) \to C_\bullet(L'; \mathbb{Q}).
\end{aligned}
$$

Let W' be the subcomplex of K' defined as $\bigcup_{\sigma' \in K',\, \sigma \cap L' = \emptyset} \sigma'$ and let T be a closed element in $C^p(K', W')$, i.e. $T(\sigma') = 0$ if $\sigma' \cap L' = \emptyset$. Then the pull back $\lambda^* T$ is contained in $C^p(K, W)$ where $W = \bigcup_{\sigma \in K,\, \sigma \cap L = \emptyset} \sigma$. We choose a good ordering $\mathcal{O}$ resp. $\mathcal{O}'$ of K resp. K' with respect to L resp. L'. Then we have the following (generally non-commutative) diagram.

$$
(2.3.3) \qquad
\begin{array}{ccc}
C_\bullet(K; \mathbb{Q}) & \xrightarrow{\lambda^* T \overset{\mathcal{O}}{\cap}} & C_{\bullet - p}(L; \mathbb{Q}) \\
\lambda \downarrow & & \downarrow \lambda \\
C_\bullet(K'; \mathbb{Q}) & \xrightarrow{T \overset{\mathcal{O}'}{\cap}} & C_{\bullet - p}(L'; \mathbb{Q}).
\end{array}
$$

For a subcomplex K^* of K, we set $L^* = K^* \cap L$. The subdivisions of K^* and L^* induced by K' are denoted by $K^{*\prime}$ and $L^{*\prime}$, respectively. The diagram (2.3.3) induces the following diagram for relative homologies

$$
(2.3.4) \qquad
\begin{array}{ccc}
H_{p+q}(K, K^*; \mathbb{Q}) & \xrightarrow{[\lambda^* T]\cap} & H_q(L, L^*; \mathbb{Q}) \\
\lambda \downarrow & & \downarrow \lambda \\
H_{p+q}(K', K^{*\prime}; \mathbb{Q}) & \xrightarrow{[T]\cap} & H_q(L', L^{*\prime}; \mathbb{Q}).
\end{array}
$$

Proposition B.4. *The diagram (2.3.4) is commutative.*

Proof. The assertion follows from the following lemma.

Lemma B.5. *Consider the following two homomorphism of complexes*

$$
\lambda \circ (\lambda^* T \overset{\mathcal{O}}{\cap}\ ) \quad \textit{and} \quad (T \overset{\mathcal{O}'}{\cap}\ ) \circ \lambda : C_\bullet(K) \to C_{\bullet - p}(L')
$$

Then there exist maps $\theta_{p+q} : C_{p+q}(K) \to C_{q+1}(L')$ *such that*

(1) $\delta\theta_{p+q}(x) + \theta_{p+q-1}(\delta x) = \lambda(\lambda^* T \overset{\mathcal{O}}{\cap}\ x) - T \overset{\mathcal{O}'}{\cap} \lambda(x)$, $(q \geq 0,\ \theta_{p-1} = 0)$ *and*

(2) $\theta_{p+q}(\sigma) \in C_{q+1}(L' \cap \sigma)$ *for each simplex* $\sigma \in K_{p+q}$.

Proof. We apply Proposition B.1 to the case where $\varphi(x) = \lambda(\lambda^* T \overset{\mathcal{O}}{\cap} x) - T \overset{\mathcal{O}'}{\cap} \lambda(x)$, $D_\bullet = C_\bullet(L')$ and $L^\sigma_\bullet = C_\bullet(L' \cap \sigma)$. Conditions (1) and (2) are easily verified. Since L is a full subcomplex of K, for each $\sigma \in K$, the intersection $L \cap \sigma$ is a face of σ. The complex $L' \cap \sigma$ is a subdivision of $L \cap \sigma$, and so the condition (3) is satisfied. We claim that the condition (4) of Proposition B.1 holds for φ. Let $\sigma = [v_0, \dots, v_p] \in K_p$ and set $\lambda\sigma = \sum_j \sigma_j = \sum_j \pm[w^j_0, \dots, w^j_p]$. Here we assume that $v_0 < \cdots < v_p$ resp. $w^j_0 < \cdots < w^j_p$ under $\mathcal{O}$ resp. $\mathcal{O}'$ for each j. Then we have

$$\lambda(\lambda^* T \overset{\mathcal{O}}{\cap} \sigma) = \sum_j T(\sigma_j)[v_p]$$

and

$$T \overset{\mathcal{O}'}{\cap} (\lambda\sigma) = \sum_j T(\sigma_j)[w^j_p].$$

Since $L \cap \sigma$ is a simplex of σ, and $[v_p]$ and $[w^j_p]$ define the same homology class in $H_0(L^\sigma_\bullet)$. Thus condition (4) is satisfied. □

□

Proof of Proposition 3.7 (3). Let γ be an element in $AC_k(K, \mathbf{D}^n; \mathbb{Q})$. The homology class of γ in $H_k(|\gamma|, |\delta\gamma|; \mathbb{Q})$. is denoted by $[\gamma]$. We set $|\gamma|' = K' \cap |\gamma|$ and $|\delta\gamma|' = K' \cap |\delta\gamma|$. The element $\lambda(\gamma)$ defines a class $[\lambda(\gamma)]$ in $H_k(|\gamma|', |\delta\gamma|'; \mathbb{Q})$. Let $T' \in C^2(K', W')$ be a Thom cocycle of $H_{1,0}$. Then $\lambda^* T' \in C^2(K, W)$ is also a Thom cocycle of $H_{1,0}$. As in the proof of Proposition3.7 (2), for a chain $\gamma \in AC_k(K, \mathbf{D}^n; \mathbb{Q})$, the chain $\lambda^* T \overset{\mathcal{O}}{\cap} \gamma$ resp. $T \overset{\mathcal{O}'}{\cap} \lambda(\gamma)$ is determined by its class

$$\lambda^* T \cap [\gamma] \in H_{k-2}(L_1 \cap |\gamma|, (L_1 \cap |\gamma|)^{(k-3)} \cup \mathbf{D}^{n-1}_\gamma; \mathbb{Q})$$

resp.

$$T \cap [\lambda(\gamma)] \in H_{k-2}(L'_1 \cap |\gamma|', (L'_1 \cap |\gamma|')^{(k-3)} \cup \mathbf{D}^{n-1}_\gamma; \mathbb{Q})$$

Since these classes are independent of the orderings by Proposition 3.7 (2), we forget them. By Proposition B.4 applied to $K = |\gamma|$, $K^* = |\delta\gamma| \cup \mathbf{D}^n_\gamma$ and $L = L_1$, we have the following commutative diagram.

$$\begin{array}{ccc} H_k(|\gamma|, |\delta\gamma| \cup \mathbf{D}^n_\gamma; \mathbb{Q}) & \xrightarrow{\lambda^* T' \cap} & H_{k-2}(L_1 \cap |\gamma|, (L_1 \cap |\gamma|)^{(k-3)} \cup \mathbf{D}^{n-1}_\gamma; \mathbb{Q}) \\ \lambda \downarrow & & \downarrow \lambda \\ H_k(|\gamma|', |\delta\gamma|' \cup \mathbf{D}^n_\gamma; \mathbb{Q}) & \xrightarrow{T' \cap} & H_{k-2}(L'_1 \cap |\gamma|', (L'_1 \cap |\gamma|')^{(k-3)} \cup \mathbf{D}^{n-1}_\gamma; \mathbb{Q}) \end{array} \tag{2.3.5}$$

The assertion follows from this. □

B.4. **Relations between the cap product and the cup product.** Let K be a finite simplicial complex and L_1, L_2 be subcomplexes in K. Assume that L_1, L_2 and $L_1 \cup L_2$ are full subcomplexes of K. Let $\mathcal{O}$ be a good ordering with respect to L_1 and $L_{12} = L_1 \cap L_2$. We set

$$W_i = \bigcup_{\sigma \cap L_i = \emptyset} \sigma.$$

Lemma B.6. *Under the above assumtions, we have* $W_1 \cup W_2 = \bigcup_{\sigma \cap L_{12}=\emptyset} \sigma$.

Proof. We will show that $W_1 \cup W_2 \supset \bigcup_{\sigma \cap L_{12}=\emptyset} \sigma$. Suppose that $\sigma \cap L_{12} = \emptyset$. Since $L_1 \cup L_2$ is a full subcomplex of K, $\sigma \cap (L_1 \cup L_2)$ is a face of σ which we denote by τ. If $\tau \in L_1$, then $\tau \cap L_2 = \emptyset$. □

Definition B.7 (Cup product)**.** *For* $T_1 \in C^p(K, W_1)$ *and* $T_2 \in C^q(K, W_2)$*, we define the cup product* $T_1 \overset{\mathcal{O}}{\cup} T_2 \in C^{p+q}(K)$ *by*

$$(T_1 \overset{\mathcal{O}}{\cup} T_2)(\sigma) = T_1(v_0, \dots, v_p) T_2(v_p, \dots, v_{p+q})$$

where $\sigma = [v_0, \dots, v_{p+q}]$ *with* $v_0 < \cdots < v_{p+q}$*. The cup product induces a homomorphism of complexes:*

$$\overset{\mathcal{O}}{\cup} : C^\bullet(K, W_1) \otimes C^\bullet(K, W_2) \to C^\bullet(K).$$

Let K^* be a subcomplex of K and set $L_i^* = K^* \cap L_i$ and $L_{12}^* = K^* \cap L_{12}$. The proof of the following proposition is obvious and is omitted.

Proposition B.8. (1) *The restriction of the cup product* $T_1 \cup T_2$ *to* $W_1 \cup W_2$ *vanishes.*

(2) *Let* T_1 *and* T_2 *be closed elements in* $C^p(K, W_1)$ *and* $C^q(K, W_2)$ *and set* $T_{12} = T_1 \overset{\mathcal{O}}{\cup} T_2$*. Then the composite of the homomorphisms*

$$\begin{aligned} T_2 \overset{\mathcal{O}}{\cap} (T_1 \overset{\mathcal{O}}{\cap} *) : C_{p+q+r}(K, K^*; \mathbb{Q}) &\xrightarrow{T_1 \overset{\mathcal{O}}{\cap}} C_{q+r}(L_1, L_1^*; \mathbb{Q}) \\ &\xrightarrow{T_2 \overset{\mathcal{O}}{\cap}} C_r(L_{12}, L_{12}^*; \mathbb{Q}) \end{aligned}$$

is equal to the homomorphism $T_{12} \overset{\mathcal{O}}{\cap}$.

As a consequence the composite of the following morphisms coincides with the cap product with $[T_{12}]$.

$$\begin{aligned} H_{p+q+r}(K, K^*; \mathbb{Q}) &\xrightarrow{[T_1]\cap} H_{q+r}(L_1, L_1^*; \mathbb{Q}) \\ &\xrightarrow{[T_2]\cap} H_r(L_{12}, L_{12}^*; \mathbb{Q}) \end{aligned}$$

Proof of Proposition 3.10. Let K be a good triangulation of P^n, and let γ be an element of $AC_p(K, \mathbf{D}^n; \mathbb{Q})$. We set $H_1 = H_{1,0}$, $H_2 = H_{2,0}$, $H_{12} = H_1 \cap H_2$, $L_1 = K \cap H_{1,0}$, $L_2 = K \cap H_2$A $L_{12} = K \cap H_{12}$, $W_i = \bigcup_{\sigma \in K, \sigma \cap H_i = \emptyset} \sigma$ $(i = 1, 2)$ and $W_{12} = \bigcup_{\sigma \in K, \sigma \cap H_{12} = \emptyset} \sigma$. By Lemma B.6 we have $W_{12} = W_1 \cup W_2$. The face map $\partial_{i,0}$ $(i = 1, 2)$ is denoted by ∂_i. Considering the symmetry on $H_{i,\alpha}$ $(1 \le i \le n, \alpha = 0, \infty)$, it is enough to prove the commutativity of the following diagram

$$(2.4.6) \qquad \begin{array}{ccc} AC_p(K, \mathbf{D}^n; \mathbb{Q}) & \xrightarrow{\partial_1} & AC_{p-2}(L_1, \mathbf{D}^{n-1}; \mathbb{Q}) \\ \partial_2 \downarrow & & \downarrow \partial_2 \\ AC_{p-2}(L_2, \mathbf{D}^{n-1}; \mathbb{Q}) & \xrightarrow{\partial_1} & AC_{p-4}(L_{12}, \mathbf{D}^{n-2}; \mathbb{Q}). \end{array}$$

We denote the complex $\mathbf{D}^n \cap |\gamma|$ by $\mathbf{D}^n_\gamma$. Let $T_1 \in C^2(K, W_1; \mathbb{Q})$ resp. $T_2 \in C^2(K, W_2; \mathbb{Q})$ be a Thom cocycle of the face H_1 resp. H_2. We have the equality

$$[T_1] \cup [T_2] = [T_2] \cup [T_1]$$

in $H^4(K, W_{12}; \mathbb{Q})$. By Proposition B.8 applied to $K = |\gamma|$ and $K^* = |\delta\gamma| \cup \mathbf{D}^n_\gamma$, we have a commutative diagram

(2.4.7)

$$\begin{array}{ccc} H_p(|\gamma|, |\delta\gamma| \cup \mathbf{D}^n_\gamma; \mathbb{Q}) & \xrightarrow{[T_1]\cap} & H_{p-2}(L_1 \cap |\gamma|, L_1 \cap (|\delta\gamma| \cup \mathbf{D}^n_\gamma); \mathbb{Q}) \\ [T_2]\cap \downarrow & & \downarrow [T_2]\cap \\ H_{p-2}(L_2 \cap |\gamma|, L_2 \cap (|\delta\gamma| \cup \mathbf{D}^n_\gamma); \mathbb{Q}) & \xrightarrow{[T_1]\cap} & H_{p-4}(L_{12} \cap |\gamma|, L_{12} \cap (|\delta\gamma| \cup \mathbf{D}^n_\gamma); \mathbb{Q}) \end{array}$$

The chain γ defines a class $[\gamma]$ in $H_p(|\gamma|, |\delta\gamma| \cup \mathbf{D}^n_\gamma; \mathbb{Q})$. By the admissibility condition, the complex $L_{12} \cap (|\delta\gamma| \cup \mathbf{D}^n_\gamma)$ is contained in $(L_{12} \cap |\delta\gamma|)^{(p-5)} \cup (L_{12} \cap \mathbf{D}^n_\gamma)$ where $(L_{12} \cap |\delta\gamma|)^{(p-5)}$ denotes the $(p-5)$-skeleton of $L_{12} \cap |\delta\gamma|$. The chain $T_2 \cap (T_1 \cap \gamma)$ resp. $T_1 \cap (T_2 \cap \gamma)$ is determined by its class $[T_2] \cap ([T_1] \cap [\gamma])$ resp. $[T_1] \cap ([T_2] \cap [\gamma])$ in $H_{p-4}(L_{12} \cap |\gamma|, (L_{12} \cap |\delta\gamma|)^{(p-5)} \cup (L_{12} \cap \mathbf{D}^n_\gamma); \mathbb{Q})$. Hence we have the equality $\partial_2\partial_1(\gamma) = \partial_1\partial_2(\gamma)$. $\square$

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