On the convergence to a statistical equilibrium for the wave equations coupled to a particle

T.V. Dudnikova [1]

Elektrostal Polytechnical Institute, Elektrostal 144000, Russia

Abstract

We consider a linear Hamiltonian system consisting of a classical particle and a scalar field describing by the wave or Klein-Gordon equations with variable coefficients. The initial data of the system are supposed to be a random function which has some mixing properties. We study the distribution μ_t of the random solution at time moments $t \in \mathbb{R}$. The main result is the convergence of μ_t to a Gaussian probability measure as $t \to \infty$. The mixing properties of the limit measures are studied. The application to the case of Gibbs initial measures is given.

Key words and phrases: a wave field coupled to a particle; Cauchy problem; random initial data; mixing condition; Volterra integro-differential equation; compactness of measures; characteristic functional; convergence to statistical equilibrium; Gibbs measures

AMS Subject Classification: 35L15, 60Fxx, 60Gxx, 82Bxx

[1]Electronic mail: tdudnikov@mail.ru

1 Introduction

The paper concerns problems of long-time convergence to an equilibrium distribution for a coupled system consisting of a field and a particle. For one-dimensional chains of harmonic oscillators, the results have been established by Spohn and Lebowitz in [36], and by Boldrighini *et al.* in [2]. Ergodic properties of one-dimensional chains of anharmonic oscillators coupled to heat baths were studied by Jakšić, Pillet and others (see, e.g., [23, 14]). In [6, 7, 8, 10], we studied the convergence to equilibrium for the systems described by partial differential equations. Later on, similar results were obtained in [9] for d-dimensional harmonic crystals with $d \geq 1$, and in [11] for a scalar field coupled to a harmonic crystal.

Here we treat the linear Hamiltonian system consisting of the scalar wave or Klein–Gordon field $\varphi(x)$, $x \in \mathbb{R}^d$, coupled to a classical particle with position in $q \in \mathbb{R}^d$, $d \geq 3$. The Hamiltonian functional of the coupled system reads

$$H(\varphi, \pi, q, p) = H_A(q, p) + H_B(\varphi, \pi) + q \cdot \langle \nabla\varphi, \rho \rangle. \tag{1.1}$$

Here "$\cdot$" stands for the standard Euclidean scalar product in $\mathbb{R}^d$, $\langle \cdot, \cdot \rangle$ denotes the inner product in the real Hilbert space $L^2(\mathbb{R}^d)$ (or its extensions), H_A is the Hamiltonian of the particle,

$$H_A(q, p) = \frac{1}{2}\Big(|p|^2 + \omega^2 |q|^2\Big), \quad \text{with some} \ \ \omega > 0,$$

and H_B denotes the Hamiltonian for the wave or Klein-Gordon field. We suppose that

$$H_B(\varphi, \pi) = \frac{1}{2} \int\limits_{\mathbb{R}^d} \Big(\sum_{i,j=1}^{d} a_{ij}(x) \nabla_i \varphi(x) \nabla_j \varphi(x) + a_0(x)|\varphi(x)|^2 + |\pi(x)|^2 \Big)\, dx$$

in the case of the wave field (WF), and

$$H_B(\varphi, \pi) = \frac{1}{2} \int\limits_{\mathbb{R}^d} \Big(\sum_{j=1}^{d} |(\nabla_j - iA_j(x))\varphi(x)|^2 + m^2|\varphi(x)|^2 + |\pi(x)|^2 \Big)\, dx, \quad \text{with some} \ \ m > 0,$$

in the case of the Klein-Gordon field (KGF). We impose the conditions **A1**–**A5** below on the coefficients $a_{ij}(x)$, $a_0(x)$ and $A_j(x)$. In particular, the functions $a_{ij}(x) - \delta_{ij}$, $a_0(x)$ and $A_j(x)$ vanish outside a bounded domain.

We assume that the initial data $Y_0 := (\varphi_0, \pi_0, q_0, p_0)$ are a random element of a real functional space $\mathcal{E}$ consisting of states with finite local energy, see Definition 2.1 below. The distribution of Y_0 is a probability measure μ_0 of mean zero satisfying conditions **S1**–**S3** below. In particular, we assume that the initial measure μ_0 satisfies a mixing condition. Roughly speaking, it means that

$$Y_0(x) \quad \text{and} \quad Y_0(y) \quad \text{are asymptotically independent as} \ \ |x - y| \to \infty.$$

We study the distributions μ_t, $t \in \mathbb{R}$, of the random solution $Y_t := (\varphi_t, \pi_t, q_t, p_t)$ at time moments $t \in \mathbb{R}$. Our main objective is to prove the weak convergence of the measures μ_t to an equilibrium measure μ_∞,

$$\mu_t \rightharpoonup \mu_\infty \quad \text{as} \ \ t \to \infty, \tag{1.2}$$

where the limit measure μ_∞ is a Gaussian measure on $\mathcal{E}$. We derive the explicit formulas for the limiting correlation functions of μ_∞. The similar convergence holds for $t \to -\infty$ because our system is time-reversible. We prove that the dynamic group is mixing (and, in particular, ergodic) with respect to the limit measures μ_∞. Moreover, we extend results to the case of non translation-invariant initial measures μ_0 and give an application to the case of the Gibbs initial measures.

Let us outline the strategy of the proof. When the field variables (φ_t, π_t) are eliminated from the equations of the coupled system, the particle evolves according to a linear Volterra integro-differential equation of a form (see Eqn (3.2) below)

$$\ddot{q}_t = -\omega^2 q_t + \int_0^t D(t-s) q_s \, ds + F(t), \quad t \in \mathbb{R}, \tag{1.3}$$

where $D(t)$ is a matrix-valued function depending on the coupled function ρ, $F(t)$ is a vector-valued function depending on the initial field data (φ_0, π_0). Therefore, our first objective is to study the long-time behavior of the solutions to Eqn (1.3). We prove that for the solutions q_t of Eqn (1.3) with $F(t) \equiv 0$, the following bound holds

$$|q_t| + |\dot{q}_t| \le C\varepsilon_F(t), \tag{1.4}$$

where $\varepsilon_F(t) = e^{-\delta|t|}$ with some $\delta > 0$ for the WF, and $\varepsilon_F(t) = (1+|t|)^{-3/2}$ for the KGF (see Theorem 3.1 below).

The *deterministic* dynamics of the equations with delay has been extensively studied by many authors under some restrictions on the kernel $D(t)$: Myshkis [31], Grossman and Millet [17], Driver [4] and others. For details on the first results and problems in the theory of equations with delay, we refer to the survey paper by Corduneanu and Lakshmikantham [3]. For further development of the theory, see the monograph by Gripenberg, Londen and Staffans [16]. The stability properties for Volterra integro-differential equations can be found in the papers by Murakami [30], Hara [18], and Kordonis and Philos [26].

The linear *stochastic* Volterra equations of convolution type have been treated also by many authors, see, e.g., Appleby and Freeman [1], the survey article by Karczewska [24] and the references therein.

Note that in the literature frequently the asymptotic behavior of the solutions of Eqn (1.3) is studied assuming that $F(t)$ is a Gaussian with noise or (and) that the kernel $D(t)$ has the exponential decay or is of one sign. However, in our case, $F(t)$ is not Gaussian white-noise, in general. Moreover, in the case of the KGF, the decay of $D(t)$ is like $(1+|t|)^{-3/2}$.

In recent years the nonlinear *generalized Langevin equation*, i.e., the equation of a form (cf. Eqn (A.20) below)

$$\ddot{q}_t = -\nabla V(q_t) - \int_0^t \Gamma(t-s)\dot{q}_s \, ds + F(t), \quad t \in \mathbb{R}, \tag{1.5}$$

with a stationary Gaussian process $F(t)$ and with a smooth (confining or periodic) potential $V(q)$, has been investigated also extensively, see, e.g., [22, 32, 35, 41]. In particular, the ergodic properties of (1.5) were studied by Jakšić and Pillet in [22], the qualitative properties of solutions to Eqn (1.5) were established by Ottobre and Pavliotis in [32]. Rey-Bellet and Thomas [33] have investigated a model consisting of a chain of non-linear oscillators coupled to two heat reservoirs. The nonlinear stochastic integro-differential equations were studied also in Mao' works (see, e.g., [27, 28]).

In this paper, we study a linear "field-particle" model. However, we do not assume that the initial distribution of the system is a Gibbs measure or absolutely continuous with respect to a Gibbs measure. Therefore, in particular, the force $F(t)$ in Eqn (1.3) is non-Gaussian, in general.

The key step in our proof is the derivation of the asymptotic behavior for the solutions Y_t of the coupled field-particle system. Using bound (1.4), we prove the following asymptotics in mean (see Corollary 5.2 below):

$$\langle Y_t, Z\rangle \sim \langle W_t(\varphi_0, \pi_0), \Pi(Z)\rangle, \quad t \to \infty, \tag{1.6}$$

where W_t is a solving operator to the Cauchy problem for the wave or Klein-Gordon equations (2.13), (φ_0, π_0) is a initial state of the field, and the function $\Pi(Z)$ is defined in (2.24). This asymptotics allows us to apply the results from [8, 10], where the weak convergence of the statistical solutions has been proved for wave and Klein–Gordon equations with variable coefficients. We divide the proof of (1.2) into two steps: we first establish the weak compactness of the measures family $\{\mu_t,\, t \in \mathbb{R}\}$ (see Section 4), and then we prove the convergence of the characteristic functionals of the measures μ_t (Section 6).

In conclusion, note that convergence (1.2) remains true for a linear Hamiltonian system consisting of N wave fields coupled to a single particle. In this case, the Hamiltonian is

$$\sum_{k=1}^{N} H_B(\varphi_k, \pi_k) + H_A(q,p) + q \cdot \sum_{k=1}^{N} \langle \nabla\varphi_k, \rho_k\rangle.$$

The paper is organized as follows. In Section 2 we describe the model, impose the conditions on the coupled function ρ and on the initial measures μ_0 and state the main results. The limit behavior for solutions of Eqn (1.3) is studied in Section 3. In Section 4 we prove the compactness of the measures family $\{\mu_t,\, t \in \mathbb{R}\}$. The asymptotics (1.6) is proved in Section 5. In Section 6 we establish the convergence of characteristic functionals of μ_t to a limit and complete the proof of the main result. In Section 7 we study the mixing properties of the dynamics with respect to the limit measures μ_∞. In Section 8 we extend the results to the case of non translation–invariant initial measures. Appendix A concerns the case of Gibbs initial measures. The existence of the solutions of the coupled system is proved in Appendix B.

2 Main Results

2.1 Model

After taking formally variational derivatives in (1.1), the coupled dynamics becomes

$$\left.\begin{array}{ll} \dot\varphi_t(x)=\pi_t(x), & \dot\pi_t(x)=L_B\,\varphi_t(x)+q_t\cdot\nabla\rho(x), \quad x\in\mathbb{R}^d, \quad t\in\mathbb{R}, \\[2ex] \dot q_t=p_t, & \dot p_t=-\omega^2 q_t+\displaystyle\int\limits_{\mathbb{R}^d}\nabla\rho(x)\varphi_t(x)\,dx. \end{array}\right| \tag{2.1}$$

Here L_B is a differential operator of one of two types:

$$L_B=\left\{\begin{array}{l} L_W:=\sum\limits_{i,j=1}^{d}\nabla_i\left(a_{ij}(x)\nabla_j\right)-a_0(x), \\ L_{KG}:=\sum\limits_{j=1}^{d}(\nabla_j-iA_j(x))^2-m^2, \end{array}\right. \tag{2.2}$$

where $\nabla_i=\partial/\partial x_i$, $i=1,\dots,d$; $d\ge 3$, and d is odd in the case when $L_B=L_W$. For simplicity of exposition, we consider the case $d=3$ only.

We study the Cauchy problem for the system (2.1) with initial data

$$\varphi_t(x)|_{t=0}=\varphi_0(x), \quad \pi_t(x)|_{t=0}=\pi_0(x), \quad x\in\mathbb{R}^3, \quad q_t|_{t=0}=q_0, \quad p_t|_{t=0}=p_0. \tag{2.3}$$

Write $\phi_t=(\varphi_t(\cdot),\pi_t(\cdot))$, $\xi_t=(q_t,p_t)$, $Y_t=(\phi_t,\xi_t)$. Then the system (2.1)–(2.3) becomes

$$\dot Y_t=\mathcal{L}(Y_t), \quad t\in\mathbb{R}; \quad Y_t|_{t=0}=Y_0. \tag{2.4}$$

We assume that the coefficients of L_B satisfy the following conditions **A1**–**A5**.

A1. $a_{ij}(x), a_0(x), A_j(x)$ are real C^∞-functions.

A2. $a_{ij}(x)=\delta_{ij}$, $a_0(x)=0$, $A_j(x)=0$ for $|x|>R_a$, where $R_a<\infty$. Then

$$L_B\varphi_t(x)=(\Delta-m^2)\varphi_t(x) \quad \text{for} \quad |x|>R_a.$$

Here $m>0$ in the case of the Klein-Gordon field (KGF), i.e., $L_B=L_{KG}$, and $m=0$ in the case of the wave field (WF), i.e., $L_B=L_W$.

In the WF case, we impose the next conditions **A3** and **A4**.

A3. $a_0(x)\ge 0$, and the hyperbolicity condition holds: there exists a constant $\alpha>0$ such that

$$\sum_{i,j=1}^{3}a_{ij}(x)k_ik_j\ge\alpha\,|k|^2, \quad x,k\in\mathbb{R}^3. \tag{2.5}$$

A4. A non-trapping condition [39]: for $(x(0),k(0))\in\mathbb{R}^3\times\mathbb{R}^3$ with $k(0)\neq 0$,

$$|x(t)|\to\infty \quad \text{as} \quad t\to\infty, \tag{2.6}$$

where $(x(t),k(t))$ is a solution to the Hamiltonian system

$$\dot x(t)=\ \nabla_k h(x(t),k(t)), \quad \dot k(t)=-\nabla_x h(x(t),k(t)), \quad \text{with} \ \ h(x,k)=\frac{1}{2}\sum_{i,j=1}^{3}a_{ij}(x)k_ik_j.$$

Example. In the WF case, **A1**–**A4** hold for the acoustic equation with constant coefficients

$$\ddot{\varphi}_t(x) = \Delta\varphi_t(x), \quad x \in \mathbb{R}^3.$$

For instance, **A4** follows because $\dot{k}(t) \equiv 0 \Rightarrow x(t) \equiv k(0)t + x(0)$.

Write $M_a = \max\limits_{x\in\mathbb{R}^3} \max\limits_{i,j}\{|a_{ij}(x) - \delta_{ij}|, |a_0(x)|\}$, or $M_a = \max\limits_{x\in\mathbb{R}^3} \max\limits_{j} |A_j(x)|$.

A5. M_a is sufficiently small (we will specify this condition in the proof of Lemma 3.3).

Now we formulate the conditions **R1**–**R3** on $\rho(x)$ and $\omega > 0$.

R1. In the case of the WF, we assume that $\|\rho\|^2_{L^2} < \alpha\,\omega^2$ with α from condition (2.5). In the KGF case, $\|\nabla\rho\|^2_{L^2} < m^2\,\omega^2$.

R2. The function $\rho(x)$ is a real-valued smooth function, $\rho(-x) = \rho(x)$, $\rho(x) = 0$ for $|x| \geq R_\rho$.

R3. For any $k \in \mathbb{R}^3 \setminus \{0\}$, $\hat{\rho}(k) = \int e^{ik\cdot x}\rho(x)\,dx \neq 0$.

Remark. Condition **R1** implies that the Hamiltonian $H(\phi_t, \xi_t)$ is nonnegative for finite energy solutions (see Appendix B). In the case of the constant coefficients, i.e. $L_B = \Delta - m^2$, condition **R1** can be weakened as follows.

R1'. The matrix $\omega^2 I - K_m$ is positive definite, where $K_m = (K_{m,ij})^3_{i,j=1}$ stands for the 3×3 matrix with matrix elements $K_{m,ij}$,

$$K_{m,ij} := (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{k_i k_j |\hat{\rho}(k)|^2}{k^2 + m^2}\, dk, \quad m \geq 0. \tag{2.7}$$

However, to prove the main result in the case of the KGF, we need a stronger condition than **R1'**. Namely, the matrix $(\omega^2 - m^2)I - K_m$ is positive definite. This condition is fulfilled, in particular, if $\|\nabla\rho\|^2_{L^2} < m^2(\omega^2 - m^2)$.

2.2 Phase space for the coupled system

We introduce a phase space $\mathcal{E}$.

Definition 2.1 *(i) Choose a function $\zeta(x) \in C_0^\infty(\mathbb{R}^3)$ with $\zeta(0) \neq 0$. Denote by $H^s_{\rm loc}(\mathbb{R}^3)$, $s \in \mathbb{R}$, the local Sobolev spaces, i.e., the Fréchet spaces of distributions $\varphi \in D'(\mathbb{R}^3)$ with the finite seminorms $\|\varphi\|_{s,R} := \|\Lambda^s\left(\zeta(x/R)\varphi\right)\|_{L^2(\mathbb{R}^3)}$, where Λ^s stands for the pseudodifferential operator with the symbol $\langle k\rangle^s$, i.e.,*

$$\Lambda^s\psi := F^{-1}_{k\to x}(\langle k\rangle^s\hat{\psi}(k)), \quad \langle k\rangle := \sqrt{|k|^2+1},$$

and $\hat{\psi}$ is the Fourier transform of the tempered distribution ψ.

(ii) $\mathcal{H} \equiv H^1_{\rm loc}(\mathbb{R}^3) \oplus H^0_{\rm loc}(\mathbb{R}^3)$ is the Fréchet space of pairs $\phi \equiv (\varphi(x), \pi(x))$ with real valued functions $\varphi(x)$ and $\pi(x)$, which is endowed with the local energy seminorms

$$\|\phi\|^2_R = \int\limits_{|x|<R} (|\varphi(x)|^2 + |\nabla\varphi(x)|^2 + |\pi(x)|^2)dx < \infty, \quad R > 0.$$

In the case of the KGF, we assume that $\varphi(x)$ and $\pi(x)$ are complex valued functions.

(iii) $\mathcal{E} \equiv \mathcal{H} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ is the Fréchet space of vectors $Y \equiv (\phi(x), q, p)$ with the local energy seminorms

$$\|Y\|^2_{\mathcal{E},R} = \|\phi\|^2_R + |q|^2 + |p|^2, \quad R > 0. \tag{2.8}$$

(iv) For $s \in \mathbb{R}$, write $\mathcal{H}^s \equiv H^{1+s}_{\rm loc}(\mathbb{R}^3) \oplus H^s_{\rm loc}(\mathbb{R}^3)$ and $\mathcal{E}^s \equiv \mathcal{H}^s \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$. In particular, $\mathcal{H} \equiv \mathcal{H}^0$, $\mathcal{E} \equiv \mathcal{E}^0$.

Using the standard technique of pseudodifferential operators and Sobolev's Theorem (see, e.g., [19]), one can prove that $\mathcal{E}^0 \equiv \mathcal{E} \subset \mathcal{E}^{-\varepsilon}$ for every $\varepsilon > 0$, and the embedding is compact.

Proposition 2.2 *Let conditions* **A1**–**A3**, **R1** *and* **R2** *hold. Then*
(i) for every $Y_0 \in \mathcal{E}$, the Cauchy problem (2.4) has a unique solution $Y_t \in C(\mathbb{R}, \mathcal{E})$.
(ii) For any $t \in \mathbb{R}$, the operator $S_t : Y_0 \mapsto Y_t$ is continuous on $\mathcal{E}$. Moreover, for any $T > 0$ and $R > \max\{R_\rho, R_a\}$,

$$\sup_{|t|\le T} \|S_t Y_0\|_{\mathcal{E},R} \le C(T) \|Y_0\|_{\mathcal{E},R+T}.$$

This proposition can be proved using a similar technique as in [25, Lemma 6.3] and [12, Proposition 2.3], and the proof is based on Lemma 2.3 below (cf. [12, Lemma 3.1]). Introduce a Hilbert space $H^1_F(\mathbb{R}^3)$ as follows. For the KGF, $H^1_F(\mathbb{R}^3)$ is the Sobolev space $H^1(\mathbb{R}^3)$. In the case of the WF, $H^1_F(\mathbb{R}^3)$ stands for the completion of real space $C_0^\infty(\mathbb{R}^3)$ with norm $\|\nabla\varphi\|_{L^2}$. Denote by E the Hilbert space $H^1_F(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with finite norm

$$\|Y\|_E^2 = \int_{\mathbb{R}^3} \left(|\nabla\varphi(x)|^2 + m^2|\varphi(x)|^2 + |\pi(x)|^2 \right) dx + |q|^2 + |p|^2 \quad \text{for } Y = (\varphi(x), \pi(x), q, p),$$

where $m > 0$ for the KGF case, and $m = 0$ for the WF case.

Lemma 2.3 *Let conditions* **A1**–**A3**, **R1** *and* **R2** *be valid. Then the following assertions hold.*
(i) For every $Y_0 \in E$, the Cauchy problem (2.4) has a unique solution $Y_t \in C(\mathbb{R}, E)$.
(ii) For $Y_0 \in E$, the energy is conserved, finite and nonnegative, $H(Y_t) = H(Y_0) \ge 0$, $t \in \mathbb{R}$.
(iii) For every $t \in \mathbb{R}$, the operator $S_t : Y_0 \mapsto Y_t$ is continuous on E. Moreover,

$$\|Y_t\|_E \le C \|Y_0\|_E \quad \textit{for} \;\; t \in \mathbb{R}. \tag{2.9}$$

We outline the proof of Lemma 2.3 and Proposition 2.2 in Appendix B.

2.3 Conditions on the initial measure

Let (Ω, Σ, P) be a probability space with expectation $\mathbb{E}$ and $\mathcal{B}(\mathcal{E})$ denote the Borel σ-algebra in $\mathcal{E}$. We assume that $Y_0 = Y_0(\omega, x)$ in (2.4) is a measurable random function with values in $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$. In other words, $(\omega, x) \mapsto Y_0(\omega, x)$ is a measurable map $\Omega \times \mathbb{R}^3 \to \mathbb{R}^8$ with respect to the (completed) σ-algebra $\Sigma \times \mathcal{B}(\mathbb{R}^3)$ and $\mathcal{B}(\mathbb{R}^8)$. Then $Y_t = S_t Y_0$ is also a measurable random function with values in $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$, by Proposition 2.2. Denote by $\mu_0(dY_0)$ the Borel probability measure in $\mathcal{E}$ giving the distribution of Y_0. Without loss of generality, we may assume that $(\Omega, \Sigma, P) = (\mathcal{E}, \mathcal{B}(\mathcal{E}), \mu_0)$ and $Y_0(\omega, x) = \omega(x)$ for $\mu_0(d\omega) \times dx$-almost all $(\omega, x) \in \mathcal{E} \times \mathbb{R}^3$.

Set $\mathcal{D} = \mathcal{D}_0 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$, $\mathcal{D}_0 := [C_0^\infty(\mathbb{R}^3)]^2$, and

$$\langle Y, Z \rangle := \langle \phi, f \rangle + q \cdot u + p \cdot v \quad \text{for} \;\; Y = (\phi, q, p) \in \mathcal{E} \quad \text{and} \;\; Z = (f, u, v) \in \mathcal{D}.$$

For a probability measure μ on $\mathcal{E}$, denote by $\hat\mu$ the characteristic functional (the Fourier transform)

$$\hat\mu(Z) \equiv \int \exp(i\langle Y, Z \rangle)\, \mu(dY), \quad Z \in \mathcal{D}.$$

A measure μ is called *Gaussian* (with zero expectation) if its characteristic functional is of the form $\hat\mu(Z) = \exp\{-(1/2)\mathcal{Q}(Z, Z)\}$, $Z \in \mathcal{D}$, where $\mathcal{Q}$ is a real nonnegative quadratic form on $\mathcal{D}$. A measure μ is called *translation-invariant* if $\mu(T_h B) = \mu(B)$ for any $B \in \mathcal{B}(\mathcal{E})$ and $h \in \mathbb{R}^3$, where $T_h Y(x) = Y(x - h)$.

We assume that the initial measure μ_0 has the following properties **S0**–**S3**.

S0 μ_0 has zero expectation value, $\mathbb{E}Y_0(x) \equiv \displaystyle\int Y_0(x)\,\mu_0(dY_0) = 0$ for $x \in \mathbb{R}^3$.

S1 μ_0 has finite mean energy density, i.e., $\mathbb{E}(|q_0|^2 + |p_0|^2) < \infty$, and

$$\mathbb{E}\Big(|\varphi_0(x)|^2 + |\nabla\varphi_0(x)|^2 + |\pi_0(x)|^2\Big) \le e_0 < \infty. \tag{2.10}$$

Write $\mu_0^B := P\mu_0$, where $P : (\phi_0, q_0, p_0) \in \mathcal{E} \to \phi_0 \in \mathcal{H}$. Now we impose conditions **S2** and **S3** on the measure μ_0^B. For simplicity of exposition, we assume that μ_0^B has translation-invariant correlation matrices (the case of non translation-invariant measures μ_0^B is considered in Section 8).

S2 The correlation functions of the measure μ_0^B,

$$Q_0^{ij}(x,y) := \int \phi_0^i(x)\phi_0^j(y)\,\mu_0^B(d\phi_0), \quad x,y \in \mathbb{R}^3, \quad \phi_0 = (\phi_0^0, \phi_0^1) \equiv (\varphi_0, \pi_0),$$

are *translation-invariant*, i.e., $Q_0^{ij}(x,y) = q_0^{ij}(x-y)$, $i,j = 0,1$.

Now we formulate the *mixing condition* for the measure μ_0^B.

Let $\mathcal{O}(r)$ be the set of all pairs of open convex subsets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^3$ at distance $d(\mathcal{A},\,\mathcal{B}) \ge r$, and let $\sigma(\mathcal{A})$ be the σ-algebra in $\mathcal{H}$ generated by the linear functionals $\phi \mapsto \langle \phi, f\rangle$, where $f \in [C_0^\infty(\mathbb{R}^3)]^2$ with $\operatorname{supp} f \subset \mathcal{A}$. Define the *Ibragimov mixing coefficient* of a probability measure μ_0^B on $\mathcal{H}$ by the rule (cf [20, Def. 17.2.2])

$$\varphi(r) \equiv \sup_{(\mathcal{A},\mathcal{B})\in\mathcal{O}(r)} \ \sup_{\substack{A\in\sigma(\mathcal{A}),\, B\in\sigma(\mathcal{B}) \\ \mu_0^B(B)>0}} \frac{|\mu_0^B(A\cap B) - \mu_0^B(A)\mu_0^B(B)|}{\mu_0^B(B)}. \tag{2.11}$$

Definition 2.4 *We say that the measure μ_0^B satisfies the strong uniform Ibragimov mixing condition if $\varphi(r) \to 0$ as $r \to \infty$.*

S3 The measure μ_0^B satisfies the strong uniform Ibragimov mixing condition, and

$$\int_0^{+\infty} r^{d_F}\, \varphi^{1/2}(r)dr < \infty, \tag{2.12}$$

where $d_F = d-1$ for the KGF, and $d_F = d-2$ for the WF, d is dimension of the space.

Remark 2.5 (i) The examples of the measures μ_0^B with zero mean satisfying conditions (2.10), **S2** and **S3** are given in [6, Section 2.6].

(ii) Instead of the *strong uniform* Ibragimov mixing condition, it suffices to assume the *uniform* Rosenblatt mixing condition [34] together with a higher degree (> 2) in the bound (2.10), i.e., to assume that there exists a δ, $\delta > 0$, such that

$$\mathbb{E}\left(|\varphi_0(x)|^{2+\delta} + |\nabla\varphi_0(x)|^{2+\delta} + |\pi_0(x)|^{2+\delta}\right) < \infty.$$

In this case, the condition (2.12) needs the following modification: $\displaystyle\int_0^{+\infty} r^{d_F}\,\alpha^p(r)dr < \infty$, where $p = \min(\delta/(2+\delta), 1/2)$, $\alpha(r)$ is the Rosenblatt mixing coefficient defined as in (2.11) but without $\mu_0^B(B)$ in the denominator.

2.4 Convergence to equilibrium for Klein-Gordon equations

We first consider the Cauchy problem for the wave (or Klein–Gordon) equation,

$$\left\{\begin{array}{l}\ddot{\varphi}_t(x)=L_B\varphi_t(x),\quad t\in\mathbb{R},\quad x\in\mathbb{R}^3,\\ \varphi_t(x)|_{t=0}=\varphi_0(x),\quad \dot{\varphi}_t(x)|_{t=0}=\pi_0(x).\end{array}\right. \tag{2.13}$$

Lemma 2.6 follows from [29, Thms V.3.1, V.3.2] as the speed of propagation for Eqn (2.13) is finite.

Lemma 2.6 *Let conditions* **A1**–**A4** *hold. Then (i) for any $\phi_0=(\varphi_0,\pi_0)\in\mathcal{H}$, there exists a unique solution $\phi_t=(\varphi_t(x),\dot{\varphi}_t(x))\in C(\mathbb{R},\mathcal{H})$ to the Cauchy problem (2.13).*
(ii) For any $t\in\mathbb{R}$, the operator $W_t:\phi_0\mapsto\phi_t$ is continuous on $\mathcal{H}$, and for any $T>0$, $R>R_a$,

$$\sup_{|t|\le T}\|W_t\phi_0\|_R\le C(T)\|\phi_0\|_{R+T}.$$

Let $\mathcal{E}_m(x)$ be the fundamental solution of the operator $-\Delta+m^2$, i.e., $(-\Delta+m^2)\mathcal{E}_m(x)=\delta(x)$. Since $d=3$, $\mathcal{E}_m(x)=e^{-m|x|}/(4\pi|x|)$. For almost all $x,y\in\mathbb{R}^3$, introduce the matrix-valued function $Q^B_\infty(x,y)=q^B_\infty(x-y)$, where

$$q^B_\infty=\frac{1}{2}\left(\begin{array}{cc} q_0^{00}+\mathcal{E}_m*q_0^{11} & q_0^{01}-q_0^{10}\\ q_0^{10}-q_0^{01} & q_0^{11}+(-\Delta+m^2)q_0^{00}\end{array}\right). \tag{2.14}$$

Here q_0^{ij}, $i,j=0,1$, are correlation functions of μ_0^B (see condition **S2**), $*$ stands for the convolution. We can rewrite q^B_∞ in the Fourier transform as

$$\hat{q}^B_\infty(k)=\frac{1}{2}\Big(\hat{q}_0(k)+\hat{C}(k)\hat{q}_0(k)\hat{C}^T(k)\Big), \tag{2.15}$$

where $(\,)^T$ denotes a matrix transposition, and

$$\hat{C}(k):=\left(\begin{array}{cc} 0 & \omega^{-1}(k)\\ -\omega(k) & 0\end{array}\right),\quad \omega(k)=\sqrt{|k|^2+m^2}. \tag{2.16}$$

Remark 2.7 Conditions **S0**, (2.10), **S2** and **S3** imply, by [20, Lemma 17.2.3], that the derivatives $D^\alpha q_0^{ij}$ are bounded by the mixing coefficient:

$$|D^\alpha q_0^{ij}(z)|\le Ce_0\,\varphi^{1/2}(|z|),\quad \text{for any }\; z\in\mathbb{R}^3,\quad |\alpha|\le 2-i-j,\quad i,j=0,1.$$

Therefore, $D^\alpha q_0^{ij}\in L^p(\mathbb{R}^3)$, $p\ge1$ (see [6, p.16]). Hence, $(q^B_\infty)^{ij}\in L^1(\mathbb{R}^3)$ if $m\ne0$ by (2.14). If $m=0$, then the bound (2.12) implies the existence of the convolution $\mathcal{E}_m*q_0^{11}$ in (2.14).

Denote by $\mathcal{Q}^{B,0}_\infty(f,f)$ the real quadratic form on $\mathcal{D}_0\equiv[C_0^\infty(\mathbb{R}^3)]^2$ defined by

$$\mathcal{Q}^{B,0}_\infty(f,f)=\langle Q^B_\infty(x,y),f(x)\otimes f(y)\rangle=\langle q^B_\infty(x-y),f(x)\otimes f(y)\rangle. \tag{2.17}$$

Definition 2.8 *μ_t^B is a Borel probability measure in $\mathcal{H}$ which gives the distribution of ϕ_t: $\mu_t^B(A)=\mu_0^B(W_t^{-1}A)$, for any $A\in\mathcal{B}(\mathcal{H})$ and $t\in\mathbb{R}$.*

For the measures μ_t^B, the following result was proved in [5]–[7].

Theorem 2.9 *Let conditions* **A1**–**A4** *hold and let the measure μ_0^B have zero mean and satisfy conditions (2.10),* **S2** *and* **S3**. *Then (i) the measures μ_t^B weakly converge as $t\to\infty$ on the space $\mathcal{H}^{-\varepsilon}$ for each $\varepsilon>0$. This means the convergence*

$$\int F(\phi)\mu_t^B(d\phi)\to\int F(\phi)\mu_\infty^B(d\phi)\quad as\quad t\to\infty \tag{2.18}$$

for any bounded continuous functional $F(\phi)$ on $\mathcal{H}^{-\varepsilon}$.
(ii) The limit measure μ_∞^B is a Gaussian measure on $\mathcal{H}$. The characteristic functional of μ_∞^B is of the form $\hat\mu_\infty^B(f)=\exp\{-(1/2)\mathcal{Q}_\infty^B(f,f)\}$. Here

$$\mathcal{Q}_\infty^B(f,f)=\mathcal{Q}_\infty^{B,0}(\Omega' f,\Omega' f),\quad f\in\mathcal{D}_0, \tag{2.19}$$

where Ω' is a linear continuous operator, and $\Omega'=I$ in the case of the constant coefficients (see Remark 2.10 below).
(iii) The correlation matrices of μ_t^B converge to a limit, i.e., for any $f_1,f_2\in\mathcal{D}_0$,

$$\int\langle\phi,f_1\rangle\langle\phi,f_2\rangle\,\mu_t^B(d\phi)\to\mathcal{Q}_\infty^B(f_1,f_2)\quad as\quad t\to\infty.$$

(iv) μ_∞^B is invariant, i.e., $W_t^*\mu_\infty^B=\mu_\infty^B$, $t\in\mathbb{R}$. Moreover, the flow W_t is mixing w.r.t. μ_∞^B, i.e., the convergence (7.1) holds.

Remark 2.10 Now we explain the sense of the operator Ω' in (2.19). To prove (2.18) in the case of variable coefficients, we constructed in [6, 7] a version of the scattering theory for solutions of infinite global energy. Namely, in the case of the WF, we introduce appropriate spaces $\mathcal{H}_\gamma$ of the initial data. By definition, $\mathcal{H}_\gamma$, $\gamma>0$, is the Hilbert space of the functions $\phi=(\varphi,\pi)\in\mathcal{H}$ with finite norm $\|\!|\phi|\!\|_\gamma^2=\int e^{-2\gamma|x|}\Big(|\pi(x)|^2+|\nabla\varphi(x)|^2+|\varphi(x)|^2\Big)\,dx<\infty$. It follows from (2.10) that μ_0^B is concentrated in $\mathcal{H}_\gamma$ for all $\gamma>0$, since

$$\int\|\!|\phi_0|\!\|_\gamma^2\,\mu_0^B(d\phi_0)\le e_0\int\exp(-2\gamma|x|)\,dx<\infty. \tag{2.20}$$

Denote by W_t the dynamical group of Eqn (2.13), while W_t^0 corresponds to the 'free' equation, with $L_B=\Delta-m^2$. In the WF case, the following long-time asymptotics holds (see [7])

$$W_t\phi_0=\Omega W_t^0\phi_0+r_t\phi_0,\quad t>0, \tag{2.21}$$

where Ω is a 'scattering operator'. $\Omega:\ \mathcal{H}_\gamma\to\mathcal{H}$ is a linear continuous operator for sufficiently small $\gamma>0$. The remainder r_t is small in local energy seminorms $\|\cdot\|_R$, $\forall R>0$:

$$\|r_t\phi_0\|_R\to 0,\quad t\to\infty.$$

The representation (2.21) is based on our version of the scattering theory for solutions of finite energy,

$$(W_t)'f=(W_t^0)'\Omega' f+r_t' f,\quad t>0, \tag{2.22}$$

where $(W_t)'$ and $(W_t^0)'$ are 'formal adjoint' to the groups W_t and W_t^0, respectively, see (2.23). $\Omega',r_t':\mathcal{H}'\to\mathcal{H}_\gamma'$, $\|\!|r_t'f|\!\|_\gamma'\to0$ as $t\to\infty$, where $\|\!|\cdot|\!\|_\gamma'$ denotes the norm in the Hilbert space $\mathcal{H}_\gamma'$ dual to $\mathcal{H}_\gamma$. In particular, for $f\in\mathcal{D}_0$, $\Omega' f\in\mathcal{H}_\gamma'$ and the quadratic form $\mathcal{Q}_\infty^{B,0}$ from (2.19) is continuous in $\mathcal{H}_\gamma'$ (for details, see Theorem 8.1 in [7, p.1245]).

In the case of the KGF, we derived in [6] the dual representation (2.22), where the remainder r_t' is small in mean: $\mathbb{E}|\langle\phi_0,r_t'f\rangle|^2\to0$, $t\to\infty$. Moreover, $\Omega' f\in H_m'\equiv L^2(\mathbb{R}^3)\oplus H^1(\mathbb{R}^3)$ for $f\in\mathcal{D}_0$, and the quadratic form $\mathcal{Q}_\infty^{B,0}$ is continuous in H_m'.

2.5 Convergence to equilibrium for the coupled system

To formulate the main result for the coupled system we introduce the following notations. Let W_t' denote the operator adjoint to W_t:

$$\langle \phi, W_t' f\rangle = \langle W_t\phi, f\rangle, \quad \text{for } f \in [S(\mathbb{R}^3)]^2, \quad \phi \in \mathcal{H}, \quad t \in \mathbb{R}. \tag{2.23}$$

Let $Z = (f, u, v) \in \mathcal{D}$, i.e., $f \in [C_0^\infty(\mathbb{R}^3)]^2$, $(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3$. Write

$$\Pi(Z) := f_*(x) + \alpha(x) \cdot u + \beta(x) \cdot v. \tag{2.24}$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$, where

$$\alpha_i(x) \ := \ \sum_{r=1}^{3} \int_0^{+\infty} \mathcal{N}_{ir}(s) W_{-s}' \nabla_r \rho_0 \, ds, \quad \text{with } \rho_0 := (\rho, 0), \tag{2.25}$$

$$\beta_i(x) \ := \ \sum_{r=1}^{3} \int_0^{+\infty} \mathcal{N}_{ir}(s) W_{-s}' \nabla_r \rho^0 \, ds, \quad \text{with } \rho^0 := (0, \rho), \quad i = 1, 2, 3, \tag{2.26}$$

the matrix-valued function $\mathcal{N}(s) = (\mathcal{N}_{ir}(s))_{i,r=1}^3$, $s > 0$, is introduced in Corollary 3.2, and

$$f_*(x) := f(x) + \sum_{i=1}^{3} \int_0^{+\infty} \left(W_{-s}'\alpha_i\right)(x) \left\langle W_s \nabla_i \rho^0, f\right\rangle \, ds. \tag{2.27}$$

Definition 2.11 *μ_t is a Borel probability measure in $\mathcal{E}$ which gives the distribution of Y_t: $\mu_t(B) = \mu_0(S_t^{-1}B)$, $\forall B \in \mathcal{B}(\mathcal{E})$, $t \in \mathbb{R}$.*

Our main result is as follows.

Theorem 2.12 *Let conditions* **A1**–**A5**, **R1**–**R3** *and* **S0**–**S3** *hold. Then*
(i) the measures μ_t weakly converge in the Fréchet spaces $\mathcal{E}^{-\varepsilon}$ for each $\varepsilon > 0$,

$$\mu_t \ \overset{\mathcal{E}^{-\varepsilon}}{\longrightarrow} \ \mu_\infty \quad as \quad t \to \infty, \tag{2.28}$$

where μ_∞ is a limit measure on $\mathcal{E}$. This means the convergence

$$\int F(Y)\mu_t(dY) \to \int F(Y)\mu_\infty(dY) \quad as \quad t \to \infty$$

for any bounded continuous functional $F(Y)$ on $\mathcal{E}^{-\varepsilon}$.
(ii) The limit measure μ_∞ is a Gaussian equilibrium measure on $\mathcal{E}$. The limit characteristic functional is of the form $\hat{\mu}_\infty(Z) = \exp\{-(1/2)\mathcal{Q}_\infty(Z, Z)\}$, $Z \in \mathcal{D}$. $\mathcal{Q}_\infty(Z, Z)$ denotes the real quadratic form on $\mathcal{D}$,

$$\mathcal{Q}_\infty(Z, Z) = \mathcal{Q}_\infty^B(\Pi(Z), \Pi(Z)) = \mathcal{Q}_\infty^{B,0}(\Omega'\Pi(Z), \Omega'\Pi(Z)), \tag{2.29}$$

where $\mathcal{Q}_\infty^{B,0}$ is defined in (2.17), and $\Pi(Z)$ is defined in (2.24).
(iii) The correlation functions of μ_t converge to a limit, i.e., for any $Z_1, Z_2 \in \mathcal{D}$,

$$\int \langle Y, Z_1\rangle \langle Y, Z_2\rangle \, \mu_t(dY) \to \mathcal{Q}_\infty(Z_1, Z_2) \quad as \ t \to \infty. \tag{2.30}$$

(iv) The measure μ_∞ is invariant, i.e., $S_t^*\mu_\infty = \mu_\infty$, $t \in \mathbb{R}$.
(v) The flow S_t is mixing w.r.t. μ_∞, i.e., $\forall F, G \in L^2(\mathcal{E}, \mu_\infty)$ the following convergence holds,

$$\lim_{t\to\infty} \int F(S_tY)G(Y)\mu_\infty(dY) = \int F(Y)\mu_\infty(dY) \int G(Y)\mu_\infty(dY).$$

The assertions (i) and (ii) of Theorem 2.12 follow from Propositions 2.13 and 2.14 below.

Proposition 2.13 *The family of the measures $\{\mu_t,\, t \ge 0\}$ is weakly compact in $\mathcal{E}^{-\varepsilon}$ with any $\varepsilon > 0$.*

Proposition 2.14 *For any $Z \in \mathcal{D}$,*

$$\hat\mu_t(Z) \equiv \int \exp(i\langle Y, Z\rangle)\, \mu_t(dY) \to \exp\{-\frac{1}{2}\mathcal{Q}_\infty(Z, Z)\}, \quad t \to \infty.$$

Proposition 2.13 (Proposition 2.14) provides the existence (the uniqueness, resp.) of the limit measure μ_∞. Proposition 2.13 is proved in Section 4, Proposition 2.14 and the assertion (iii) of Theorem 2.12 are proved in Section 6. Theorem 2.12 (iv) follows from (2.28) since the group S_t is continuous in $\mathcal{E}$ by Proposition 2.2 (ii). The assertion (v) is proved in Section 7.

3 Long-time behavior of the solutions

Using the operator W_t, we rewrite the system (2.1) in the form

$$\phi_t = W_t\phi_0 + \int_0^t q_s \cdot W_{t-s}\nabla\rho^0\, ds, \tag{3.1}$$

$$\ddot q_t = -\omega^2 q_t + \langle \nabla\rho_0, \phi_t\rangle = -\omega^2 q_t + \int_0^t D(t-s)q_s\, ds + F(t), \tag{3.2}$$

where $\phi_t = (\varphi_t(\cdot), \pi_t(\cdot))$, $\rho^0 = (0, \rho)$, $\rho_0 = (\rho, 0)$, $F(t)$ denotes the vector-valued function, $F(t) = \langle \nabla\rho_0, W_t\phi_0\rangle$, $D(t)$ stands for the matrix-valued function with entries

$$D_{ij}(t) := \langle \nabla_i\rho_0, W_t\nabla_j\rho^0\rangle, \quad i, j = 1, 2, 3. \tag{3.3}$$

Note that in the case of the constant coefficients, i.e., $a_{ij}(x) \equiv \delta_{ij}$ and $a_0(x) \equiv 0$ or $A_j(x) \equiv 0$,

$$D_{ij}(t) = (2\pi)^{-3} \int_{\mathbb{R}^3} k_i k_j \frac{\sin\omega(k)t}{\omega(k)} |\hat\rho(k)|^2\, dk, \quad \omega(k) = \sqrt{|k|^2 + m^2}, \quad m \ge 0. \tag{3.4}$$

In sections 3 and 5, we study the long-time behavior of the solutions $Y_t = (\phi_t, \xi_t)$ of problem (2.4) by the following way. In Section 3.1, we prove the time decay for the solutions q_t of (3.2) with $F(t) \equiv 0$. Then we establish the time decay for the solutions Y_t of (2.4) in the case when the initial data of the field vanish for $|x| \ge R_0$ (Section 3.2). Finally, for any initial data $Y_0 \in \mathcal{E}$, we derive the long-time asymptotics of the solution Y_t in the mean (Section 5).

At first, consider the Cauchy problem for Eqn (3.2) with $F(t) \equiv 0$, i.e.,

$$\ddot q_t = -\omega^2 q_t + \int_0^t D(t-s)q_s\, ds, \quad t > 0, \tag{3.5}$$

$$q_t|_{t=0} = q_0, \quad \dot q_t|_{t=0} = p_0. \tag{3.6}$$

For the solutions of problem (3.5)–(3.6), the following assertion holds.

Theorem 3.1 *Let conditions* **A1**–**A5** *and* **R1**–**R3** *be satisfied. Then* $|q_t|+|\dot{q}_t| \le C\varepsilon_F(t)(|q_0|+|p_0|)$ *for any* $t \ge 0$. *Here*

$$\varepsilon_F(t) = \left\{ \begin{array}{ll} e^{-\delta t} \quad \textit{with a } \delta > 0, & for\ the\ WF, \\ (1+t)^{-3/2}, & for\ the\ KGF. \end{array} \right. \tag{3.7}$$

Corollary 3.2 *Denote by* $V(t)$ *a solving operator of the Cauchy problem (3.5), (3.6). Then the variation constants formula gives the following representation for the solution of problem (3.2), (3.6):*

$$\left(\begin{array}{c} q_t \\ \dot{q}_t \end{array} \right) = V(t) \left(\begin{array}{c} q_0 \\ p_0 \end{array} \right) + \int\limits_0^t V(s) \left(\begin{array}{c} 0 \\ F(t-s) \end{array} \right) ds, \quad t > 0.$$

Evidently, $V(0) = I$. *The matrix* $V(t)$, $t > 0$, *is called the resolvent or principal matrix solution for Eqn (3.2). Theorem 3.1 implies that* $|V(t)| \le C\varepsilon_F(t)$ *with* $\varepsilon_F(t)$ *from (3.7), and for the solutions of (3.2) the following bound holds:*

$$|q_t| + |\dot{q}_t| \le C_1\varepsilon_F(t)(|q_0| + |p_0|) + C_2 \int\limits_0^t \varepsilon_F(s)|F(t-s)|\, ds, \quad \textit{for } t \ge 0. \tag{3.8}$$

Moreover, the matrix $V(t)$ *has a form* $\left(\begin{array}{cc} \dot{\mathcal{N}}(t) & \mathcal{N}(t) \\ \ddot{\mathcal{N}}(t) & \dot{\mathcal{N}}(t) \end{array} \right)$, *with matrix-valued entries satisfying the bound:*

$$|\mathcal{N}^{(j)}(t)| \le C\varepsilon_F(t), \quad t > 0, \quad j = 0, 1, 2. \tag{3.9}$$

In next subsection, we prove Theorem 3.1 for the WF case. In the case of the KGF, Theorem 3.1 can be proved combining the technique of [21] and [12, Appendix], where Theorem 3.1 was proved for the Klein-Gordon equation with constant coefficients, the methods of Section 3.1, where the result is established in the case of the wave equations with variable coefficients, and Vainberg' results [38] for Klein-Gordon equations with variable coefficients.

3.1 Exponential stability of the zero solution in the WF case

To prove Theorem 3.1, we solve the Cauchy problem (3.5), (3.6) by using the Laplace transform,

$$\tilde{q}(\lambda) = \int\limits_0^{+\infty} e^{-\lambda t} q_t \, dt, \quad \Re\lambda > 0.$$

Then Eqn (3.5) becomes

$$\lambda^2 \tilde{q}(\lambda) = -\omega^2 \tilde{q}(\lambda) + \tilde{D}(\lambda)\tilde{q}(\lambda) + p_0 + \lambda q_0. \tag{3.10}$$

Let $H^s \equiv H^s(\mathbb{R}^3)$ denote the Sobolev space with norm $\|\cdot\|_s$. Denote by $R_\lambda : H^0 \to H^2$, $\Re\lambda > 0$, an operator such that $R_\lambda f = \varphi_\lambda(x)$ is a solution to the following equation

$$(\lambda^2 - L_B)\varphi_\lambda(x) = f(x).$$

Then the entries of $\tilde{D}(\lambda)$ are

$$\tilde{D}_{ij}(\lambda) = \langle \nabla_i \rho, R_\lambda(\nabla_j \rho) \rangle, \quad i, j = 1, 2, 3. \tag{3.11}$$

Denote by R^0_λ, $\Re\lambda>0$, the operator R_λ in the case when $L_B=\Delta$. As shown in [37, Lemma 3], the operator $R^0_\lambda(R_\lambda)$, $\Re\lambda>0$, is analytic (finite-meromorphic, resp.) depends on λ. By conditions **A1**–**A3**, the operator R_λ, with $\Re\lambda>0$, has not the poles and equals $R_\lambda f=\int\limits_0^{+\infty} e^{-\lambda t}\varphi_t(x)\,dt$, where $\varphi_t(x)$ is the solution to the Cauchy problem (2.13) with initial data $\varphi_0\equiv 0$, $\pi_0=f\in H^0(\mathbb{R}^3)$. Moreover, by energy estimates, the following bound holds (see [37, Theorem 2]),

$$\|R_\lambda f\|_1+|\lambda|\|R_\lambda f\|_0\le C\|f\|_0. \tag{3.12}$$

We rewrite Eqn (3.10) as

$$\tilde q(\lambda)=\Big[(\lambda^2+\omega^2)I-\tilde D(\lambda)\Big]^{-1}(p_0+\lambda q_0)\equiv\tilde{\mathcal{N}}(\lambda)(p_0+\lambda q_0),$$

where $\tilde{\mathcal{N}}(\lambda)$ stands for the 3×3 matrix of the form

$$\tilde{\mathcal{N}}(\lambda)=A^{-1}(\lambda),\ \text{with}\ \ A(\lambda):=(\lambda^2+\omega^2)I-\tilde D(\lambda)\quad\text{for}\ \ \Re\lambda>0. \tag{3.13}$$

We first study properties of $A(\lambda)$. Write $\mathbb{C}_\beta:=\{\lambda\in\mathbb{C}:\ \Re\lambda>\beta\}$ for $\beta\in\mathbb{R}$.

Lemma 3.3 *Let conditions* **A1**–**A5** *and* **R1**–**R3** *hold. Then*
(i) $A(\lambda)$ admits an finite-meromorphic continuation to $\mathbb{C}$; and there exists a $\delta>0$ such that $A(\lambda)$ has not poles in $\mathbb{C}_{-\delta}$;
(ii) for every $\beta\in(0,\delta)$, $\exists N_\beta>0$ such that $v\cdot A(\lambda)v\ge C|\lambda|^2|v|^2$ for $\lambda\in\mathbb{C}_{-\beta}$ with $|\lambda|\ge N_\beta$ and for every $v\in\mathbb{R}^3$.
(iii) There exists a $\delta_*>0$ such that $v\cdot A(\lambda)v\ne 0$ for $\lambda\in\overline{\mathbb{C}}_{-\delta_*}$ and for every $v\ne 0$.

In the case when $\rho(x)=\rho_r(|x|)$ and $L_B=\Delta$, Lemma 3.3 was proved in [25, Lemma 7.2] (see also [12, Lemma 4.3] in the case of the constant coefficients).

Proof Let ψ be a smooth positive function which is like $e^{-|x|^2}$ as $|x|\to\infty$. By $\hat R_\lambda$ ($\hat R^0_\lambda$) we denote the operator R_λ (R^0_λ, resp.) which is considered as an operator from H^0_b to H^1_ψ, where $H^s_b=\{f\in H^s:\ f(x)=0$ for $|x|\ge b\}$ with a norm $\|\cdot\|_{s,b}$, H^s_ψ is the space with a norm $\|\varphi\|_{s,\psi}=\|\psi\varphi\|_s$. We choose a b such that $b\ge\max\{R_\rho,R_a\}$ (see conditions **A2** and **R2**).

Now we state properties (**V1**)–(**V4**) of the operator $\hat R_\lambda$ which follow from Vainberg's results [37, 39].
(**V1**) (see [37, Theorem 3]) The operator $\hat R^0_\lambda$ admits an analytic continuation on $\mathbb{C}$, and for any $\gamma>0$,

$$\|\hat R^0_\lambda f\|_{1,\psi}+|\lambda|\|\hat R^0_\lambda f\|_{0,\psi}\le C(\gamma)\|f\|_{0,b},\quad |\Re\lambda|<\gamma.$$

The operator $\hat R_\lambda$ admits a finite-meromorphic continuation on $\mathbb{C}$, and for any $\gamma>0$ there exists $N=N(\gamma)$ such that in the region $M_{\gamma,N}:=\{\lambda\in\mathbb{C}:\ |\Re\lambda|\le\gamma,\ |\Im\lambda|\ge N\}$ the following estimate holds: $\|\hat R_\lambda f\|_{j,\psi}\le 2\|\hat R^0_\lambda f\|_{j,\psi}$, $j=0,1$, for $f\in H^0_b$ (see [37, Theorem 4]).
(**V2**) For any $\gamma>0$, $\hat R_\lambda$ has at most a finite number of poles in the domain $\mathbb{C}_{-\gamma}$.
(**V3**) $\hat R_\lambda$ has not poles for $\Re\lambda\ge 0$, by conditions **A1**–**A3**.
(**V4**) There exist constants $C,T,\alpha,\beta>0$ such that for any $f\in H^0_b$,

$$\|\hat R_\lambda f\|_{0,\psi}\le C|\lambda|^{-1}e^{T|\Re\lambda|}\|f\|_{0,b},\quad \text{for}\ \ \lambda\in U_{\alpha,\beta}=\{\lambda\in\mathbb{C}:\ |\Re\lambda|<\alpha\ln|\Im\lambda|-\beta\}.$$

We return to the proof of Lemma 3.3.

(i) In the case of the constant coefficients, i.e., when $L_B = \Delta$, $\tilde{D}_{ij}(\lambda) = \langle \nabla_i \rho, R^0_\lambda(\nabla_j \rho)\rangle$ admits an analytic continuation to $\mathbb{C}$. Therefore, in this case, $A(\lambda)$ admits an analytic continuation to $\mathbb{C}$. In the general case, item (i) of Lemma 3.3 follows from (**V1**)–(**V3**).

(ii) By (3.11) and (3.12), $\tilde{D}_{ij}(\lambda) \to 0$ as $|\lambda| \to \infty$ with $\Re\lambda > 0$. On the other hand, property (**V1**) implies that for any $\gamma > 0$ there exists $N = N(\gamma) > 0$ such that

$$|\tilde{D}_{ij}(\lambda)| \le C(\gamma)|\lambda|^{-1} \quad \text{for} \;\; \lambda \in M_{\gamma,N}. \tag{3.14}$$

Hence, there exists a $\beta > 0$ such that $|\tilde{D}_{ij}(\lambda)| \le C|\lambda|^{-1} \to 0$ as $|\lambda| \to \infty$ with $\lambda \in \mathbb{C}_{-\beta}$. This implies the assertion (ii) of Lemma 3.3.

(iii) Note first that $\det A(\lambda) \neq 0$ for $\Re\lambda > 0$, by (2.9). Further, the matrix $A(\lambda)$ is positive definite for $\Im\lambda = 0$. Indeed, let $\lambda = \mu \in \mathbb{R} \setminus 0$, and put $f = \nabla\rho \cdot v$ with $v \in \mathbb{R}^3$. Then $f \in H^0_b$ and $\hat{f}|k|^{-1} \in H^0$. Denoting $\varphi_\mu = R_\mu f \in H^2$, we obtain

$$\langle f, R_\mu f\rangle = \langle \varphi_\mu, (\mu^2 - L_B)\varphi_\mu\rangle \ge \alpha\|\nabla\varphi_\mu\|_0^2 = \alpha\|\nabla(R_\mu f)\|_0^2,$$

by condition **A3**. On the other hand, $\langle f, R_\mu f\rangle \le \|\nabla(R_\mu f)\|_0 \cdot \|F^{-1}(|k|^{-1}\hat{f})\|_0$. Hence,

$$\|\nabla(R_\mu f)\|_0 \le \frac{1}{\alpha}\|F^{-1}(|k|^{-1}\hat{f})\|_0.$$

Therefore, for any $\mu > 0$ and $v \in \mathbb{R}^3 \setminus \{0\}$, we obtain

$$\begin{aligned} v \cdot \tilde{D}(\mu)v &= \langle f, R_\mu f\rangle \le \frac{1}{\alpha}\left\|F^{-1}\left(|k|^{-1}\hat{f}\right)\right\|_0^2 = \frac{1}{\alpha(2\pi)^3}\left\||k|^{-1}\hat{f}\right\|_0^2 \\ &= \frac{1}{\alpha(2\pi)^3}\int \frac{(k\cdot v)^2}{|k|^2}|\hat{\rho}(k)|^2\,dk < \omega^2|v|^2, \end{aligned} \tag{3.15}$$

by condition **R1**. In the case $\mu = 0$, put $\hat{R}_0 f := \lim_{\varepsilon\to+0}\hat{R}_\varepsilon f$, where the limit is understood in the space H^1_ψ. Then $|\langle f, \hat{R}_0 f\rangle| < \omega^2|v|^2$ by (3.15). Hence, for any $v \in \mathbb{R}^3 \setminus 0$ and $\mu \in \mathbb{R}$,

$$v \cdot A(\mu)v = (\mu^2 + \omega^2)|v|^2 - v \cdot \tilde{D}(\mu)v > 0.$$

Moreover, there exists a δ_0, $\delta_0 > 0$, such that

$$v \cdot A(\lambda)v \neq 0 \quad \text{for} \;\; |\lambda| < \delta_0 \quad \text{and for any} \;\; v \in \mathbb{R}^3 \setminus \{0\}.$$

Now let $\lambda = iy + 0$ with $y \in \mathbb{R}$, and put again $f = \nabla\rho \cdot v \in H^0_b$. By property (**V1**), there exists $N_0 > 0$ such that $v \cdot A(iy)v \sim (\omega^2 - y^2)|v|^2 + C|v|^2/|y| \neq 0$ for $|y| \ge N_0$ and $v \neq 0$. Hence, to prove the assertion (iii) of Lemma 3.3, it suffices to show that

$$\det A(iy+0) = \det\left[(\omega^2 - y^2)I - \tilde{D}(iy+0)\right] \neq 0 \quad \text{for} \;\; \delta_0 \le |y| \le N_0.$$

In [12], we have proved that in the case when $L_B = \Delta$, condition **R3** and the Plemelj formula [15] yield

$$\Im\langle f, \hat{R}^0_{iy+0} f\rangle = -\frac{\pi}{2}y^3(2\pi)^{-3}\int\limits_{|\theta|=1}(v\cdot\theta)^2|\hat{\rho}(|y|\theta)|^2\,dS_\theta \neq 0 \quad \text{for any} \;\; v, y \in \mathbb{R}^3 \setminus \{0\}, \tag{3.16}$$

where $\hat{R}^0_{iy+0} f := \lim_{\varepsilon\to+0}\hat{R}^0_{iy+\varepsilon} f$. In the case when $L_B = L_W$, we can choose M_a so small that

$$v \cdot \Im\tilde{D}(iy+0)v \equiv \Im\langle f, \hat{R}_{iy+0} f\rangle \neq 0 \quad \text{for all} \;\; v \in \mathbb{R}^3 \setminus \{0\} \quad \text{and} \;\; |y| \in (\delta_0, N_0) \tag{3.17}$$

(see condition **A5**). In fact, we split $\langle f, \hat{R}_{iy+0} f\rangle$ into two terms

$$\langle f, \hat{R}_{iy+0} f\rangle = \langle f, \hat{R}^0_{iy+0} f\rangle + \langle f, (\hat{R}_{iy+0} - \hat{R}^0_{iy+0}) f\rangle. \tag{3.18}$$

Since $f = \nabla\rho \cdot v$, then there exists a constant $C_0 > 0$ such that for $|y| \in (\delta_0, N_0)$ we have

$$|\langle f, (\hat{R}_{iy+0} - \hat{R}^0_{iy+0}) f\rangle| = |\langle f, \hat{R}_{iy+0}(L_B - \Delta)\hat{R}^0_{iy+0} f\rangle| \le C_0 \|\rho\|_1^2 \, |v|^2 M_a, \tag{3.19}$$

where $M_a = \max_{x\in\mathbb{R}^3}\{|a_{ij}(x) - \delta_{ij}|, |a_0(x)|\}$. Hence, (3.16) and (3.19) imply that if M_a is enough small, then (3.17) holds. For example, assume that

$$M_a \le \frac{M}{2C_0\|\rho\|_1^2}, \quad \text{with} \ \ M = \min_{\delta_0 \le |y| \le N_0} \min_{v \ne 0} \frac{|v \cdot S(y) v|}{|v|^2} > 0,$$

where $S(y)$, $y \in \mathbb{R}^3$, stands for the 3×3 matrix with the entries $S_{ij}(y)$,

$$S_{ij}(y) = \frac{\pi}{2} y^3 (2\pi)^{-3} \int\limits_{|\theta|=1} \theta_i \theta_j |\hat{\rho}(|y|\theta)|^2 \, dS_\theta, \quad i,j = 1,2,3.$$

Hence, for $|y| \in (\delta_0, N_0)$, $|\Im\langle f, \hat{R}^0_{iy+0} f\rangle| = |v \cdot S(y) v| \ge 2C_0\|\rho\|_1^2 \, |v|^2 M_a$. Therefore, (3.18) and (3.19) imply bound (3.17). Finally, $v \cdot \Im A(iy+0) v = -v \cdot \Im\tilde{D}(iy+0) v \ne 0$. Therefore, there exists $\delta_* > 0$ such that $v \cdot \Im A(iy+x) v \ne 0$ for $|x| \le \delta_*$. Lemma 3.3 is proved. ■

For any $\delta < \delta_*$, denote by $\mathcal{N}(t)$ the inverse Laplace transformation of $\tilde{\mathcal{N}}(\lambda)$,

$$\mathcal{N}(t) = \frac{1}{2\pi i} \int\limits_{-i\infty-\delta}^{i\infty-\delta} e^{\lambda t} \tilde{\mathcal{N}}(\lambda) \, d\lambda, \quad t > 0.$$

Lemma 3.4 *Let $L_B = L_W$ and conditions* **A1**–**A5**, **R1**–**R3** *hold. Then, for $j = 0, 1, \dots$ and any $\delta < \delta_*$,*

$$|\mathcal{N}^{(j)}(t)| \le C e^{-\delta t}, \quad t > 1. \tag{3.20}$$

Proof By Lemma 3.3, the bound on $\mathcal{N}(t)$ follows. To prove the bound for $\dot{\mathcal{N}}(t)$, we consider $\lambda\tilde{\mathcal{N}}(\lambda)$ and prove the bound

$$\Big| v \cdot (\lambda \tilde{\mathcal{N}}(\lambda))' v \Big| \le \frac{C|v|^2}{1 + |\lambda|^2} \quad \text{for} \ \ \lambda \in \overline{\mathbb{C}}_{-\delta}. \tag{3.21}$$

Therefore,

$$\Big| t\dot{\mathcal{N}}(t) \Big| = C \left| \int\limits_{\Re\lambda = -\delta} e^{\lambda t} (\lambda \tilde{\mathcal{N}}(\lambda))' \, d\lambda \right| \le C_1 e^{-\delta t},$$

and bound (3.20) for $\dot{\mathcal{N}}(t)$ follows. By Lemma 3.3 (ii), to prove bound (3.21), it suffices to show that $|\tilde{\mathcal{N}}'_{ij}(\lambda)| \le C(1 + |\lambda|)^{-3}$. Since $R'_\lambda f = -2\lambda R^2_\lambda f$, then by formulas (3.11), (3.12) and property (**V1**), we have

$$|\tilde{D}'_{ij}(\lambda)| \le 2|\lambda| |\langle \nabla_i \rho, R^2_\lambda \nabla_j \rho\rangle| \le C < \infty \quad \text{as} \ \ |\lambda| \to \infty \quad \text{with} \quad \lambda \in \mathbb{C}_{-\delta}.$$

Therefore, (3.13) and Lemma 3.3 imply that, for $i, j = 1, 2, 3$,

$$|\tilde{\mathcal{N}}'_{ij}(\lambda)| \le \frac{C_1}{1 + |\lambda|^3} \quad \text{as} \ \ |\lambda| \to \infty.$$

This yields (3.21). Bound (3.20) with $j \ge 2$ can be proved in a similar way. ■

Corollary 3.5 *The solution of the Cauchy problem (3.5)–(3.6) is $q_t = \dot{\mathcal{N}}(t) q_0 + \mathcal{N}(t) p_0$. Therefore, in the case of the WF, Lemma 3.4 implies Theorem 3.1 with any $\delta < \delta_*$.*

3.2 Time decay for Y_t when $\phi_0(x) = 0$ for $|x| \geq R_0$

For the solution Y_t of (2.4), the following bound holds.

Lemma 3.6 *Let conditions* **A1**–**A5** *and* **R1**–**R3** *hold and let* $Y_0 \in \mathcal{E}$ *be such that*

$$\varphi_0(x) = \pi_0(x) = 0 \quad \textit{for } |x| > R_0, \tag{3.22}$$

with some $R_0 > 0$. *Then for every* $R > 0$ *there exists a constant* $C = C(R, R_0) > 0$ *such that*

$$\|Y_t\|_{\mathcal{E},R} \leq C\varepsilon_F(t)\|Y_0\|_{\mathcal{E},R_0}, \quad t \geq 0. \tag{3.23}$$

Here $\varepsilon_F(t) = (1+t)^{-3/2}$ *for the KGF. In the case of the WF,* $\varepsilon_F(t) = e^{-\delta t}$ *with a* $\delta \in (0, \min(\delta_*, \gamma))$, *where constants* δ_* *and* γ *are introduced in Lemma 3.3 (iii) and in bound (3.24), respectively.*

Proof Step (i): At first, we prove bound (3.23) for $\xi_t = (q_t, p_t)$. In the case of the WF, condition (3.22) and the Vainberg bounds (see [39] or [7, Proposition 10.1]) imply that, for any $R > 0$, there exist constants $\gamma = \gamma(R, R_0) > 0$ and $C = C(R, R_0) > 0$ such that

$$\|W_t\phi_0\|_R \leq Ce^{-\gamma t}\|\phi_0\|_{R_0}, \quad t \geq 0. \tag{3.24}$$

Therefore, bound (3.8) with $F(t) \equiv \langle \nabla\rho_0, W_t\phi_0\rangle = -\langle \rho_0, \nabla W_t\phi_0\rangle$ and condition **R2** yield

$$|\xi_t| \leq C_1 e^{-\delta t}|\xi_0| + C(\rho)\int_0^t e^{-\delta s}\|\nabla(W_{t-s}\phi_0)^0\|_{L^2(B_{R_\rho})}\,ds \leq Ce^{-\delta t}\|Y_0\|_{\mathcal{E},R_0}, \tag{3.25}$$

with any $\delta < \min(\delta_*, \gamma)$. If $L_B = L_{KG}$, then we apply the Vainberg bound [38]:

$$\|W_t\phi_0\|_R \leq C(1+t)^{-3/2}\|\phi_0\|_{R_0}, \quad t \geq 0. \tag{3.26}$$

Hence, $|F(t)| \leq C(1+t)^{-3/2}\|\phi_0\|_{R_0}$, and bound (3.23) for ξ_t follows from (3.8).

Step (ii): Now we prove bound (3.23) for ϕ_t. In the case of the WF, Eqn (3.1), condition (3.22), bounds (3.24) and (3.25) yield

$$\|\phi_t\|_R \leq C_1 e^{-\gamma t}\|\phi_0\|_{R_0} + C_2\int_0^t e^{-\delta s}\|Y_0\|_{\mathcal{E},R_0}e^{-\gamma(t-s)}\,ds \leq Ce^{-\delta t}\|Y_0\|_{\mathcal{E},R_0}, \quad t \geq 0,$$

with any $\delta < \min(\delta_*, \gamma)$. For the KGF, the bound $\|\phi_t\|_R \leq C(1+t)^{-3/2}\|Y_0\|_{\mathcal{E},R_0}$ follows from Eqn (3.1), bound (3.26), and estimate (3.23) for q_t. This proves Lemma 3.6. ■

4 Compactness of the measures μ_t

Proposition 2.13 can be deduced from bound (4.1) below by the Prokhorov Theorem [40, Lemma II.3.1] using the method of [40, Theorem XII.5.2], since the embedding $\mathcal{E} \equiv \mathcal{E}^0 \subset \mathcal{E}^{-\varepsilon}$ is compact for every $\varepsilon > 0$.

Lemma 4.1 *Let conditions* **A1**–**A5**, **R1**–**R3** *and* **S0**–**S2** *hold. Then*

$$\sup_{t\geq 0}\mathbb{E}\|S_tY_0\|^2_{\mathcal{E},R} \leq C(R) < \infty, \quad \forall R > 0. \tag{4.1}$$

Proof Let $\rho \equiv 0$. In this case, we denote by S_t^0 the solving operator S_t. Note first that

$$\sup_{t\ge 0} \mathbb{E}\|S_t^0 Y_0\|^2_{\mathcal{E},R} \le C(R), \quad \forall R>0. \tag{4.2}$$

Indeed, by the notation (2.8), $\|S_t^0 Y_0\|^2_{\mathcal{E},R} = \|W_t\phi_0\|^2_R + |q_t^0|^2 + |\dot q_t^0|^2$, where q_t^0 is a solution to the Cauchy problem

$$\ddot q_t^0 + \omega^2 q_t^0 = 0, \quad t\in\mathbb{R}, \quad (q_t^0, \dot q_t^0)|_{t=0} = (q_0, p_0).$$

Hence, $|q_t^0| + |\dot q_t^0| \le C(|q_0| + |p_0|)$. By [6, bound (11.2)] and [7, bound (9.2)], we have

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|W_t\phi_0\|^2_R \le C(R), \quad \forall R>0. \tag{4.3}$$

This implies (4.2). Further, we represent the solution to problem (2.4) as

$$S_tY_0 = S_t^0 Y_0 + \int\limits_0^t S_{t-\tau} B S_\tau^0 Y_0 \, d\tau,$$

where, by definition, $BY = (0, 0, q\cdot\nabla\rho, \langle\varphi, \nabla\rho\rangle)$ for $Y = (\varphi, q, \pi, p)$. Hence, condition **A2**, (3.23), and (4.2) yield

$$\begin{array}{rcl} \mathbb{E}\|S_tY_0\|^2_{\mathcal{E},R} & \le & \mathbb{E}\|S_t^0Y_0\|^2_{\mathcal{E},R} + \mathbb{E}\int\limits_0^t \|S_{t-\tau}BS_\tau^0 Y_0\|^2_{\mathcal{E},R}\, d\tau \\ & \le & C(R) + \int\limits_0^t \varepsilon_F^2(t-\tau)\,\mathbb{E}\|S_\tau^0 Y_0\|^2_{\mathcal{E},R_\rho}\, d\tau \le C_1(R) < \infty. \quad \blacksquare \end{array}$$

5 Asymptotic behavior for $Y_t = (\phi_t, q_t, p_t)$ in mean

Proposition 5.1 *Let conditions* **A1**–**A5**, **R1**–**R3** *and* **S0**–**S2** *be satisfied.*

(i) The following bounds hold,

$$\begin{array}{rcl} \mathbb{E}|q_t - \langle W_t\phi_0, \alpha\rangle|^2 & \le & C\tilde\varepsilon_F(t), \\ \mathbb{E}|p_t - \langle W_t\phi_0, \beta\rangle|^2 & \le & C\tilde\varepsilon_F(t), \quad t>0, \end{array} \tag{5.1, 5.2}$$

where the functions α and β are defined in (2.25) and (2.26), $\tilde\varepsilon_F(t) = (1+t)^{-1}$ for the KGF, and $\tilde\varepsilon_F(t) = \varepsilon_F^2(t) = e^{-2\delta t}$ with a $\delta>0$ for the WF.

(ii) Let $f\in[C_0^\infty(\mathbb{R}^3)]^2$ with $\operatorname{supp} f\subset B_R$. Then, for $t\ge 1$,

$$\mathbb{E}\Big|\langle\phi_t, f\rangle - \langle W_t\phi_0, f_*\rangle\Big|^2 \le C\tilde\varepsilon_F(t), \tag{5.3}$$

where the function f_* is defined in (2.27).

Proof (i) At first, Theorem 3.1 and Corollary 3.2 yield

$$\mathbb{E}\Big|q_t - \int\limits_0^t \mathcal{N}(s)\,\langle W_{t-s}\phi_0, \nabla\rho_0\rangle\, ds\Big|^2 \le C\varepsilon_F^2(t) \tag{5.4}$$

with $\varepsilon_F(t)$ from (3.7). Further,

$$
\begin{aligned}
&\mathbb{E}\left|\int\limits_t^{+\infty}\mathcal{N}_{ir}(s)\langle W_{t-s}\phi_0,\nabla_r\rho_0\rangle\,ds\right|^2\\
=&\int\limits_t^{+\infty}\mathcal{N}_{ir}(s_1)\,ds_1\int\limits_t^{+\infty}\mathcal{N}_{ir}(s_2)\mathbb{E}\Big(\langle W_{t-s_1}\phi_0,\nabla_r\rho_0\rangle\langle W_{t-s_2}\phi_0,\nabla_r\rho_0\rangle\Big)\,ds_2.
\end{aligned}
$$

For any $t,s_1,s_2\in\mathbb{R}$,

$$
\begin{aligned}
\left|\mathbb{E}\Big(\langle W_{t-s_1}\phi_0,\nabla_r\rho_0\rangle\langle W_{t-s_2}\phi_0,\nabla_r\rho_0\rangle\Big)\right| &\le C\sup_{\tau\in\mathbb{R}}\mathbb{E}|\langle W_\tau\phi_0,\nabla_r\rho_0\rangle|^2\\
&\le C_1\sup_{\tau\in\mathbb{R}}\mathbb{E}\|W_\tau\phi_0\|^2_{R_\rho}\le C_2<\infty
\end{aligned}
$$

by bound (4.3). Hence, using (3.9), we obtain

$$
\mathbb{E}\left|\int\limits_t^{+\infty}\mathcal{N}(s)\langle W_{t-s}\phi_0,\nabla\rho_0\rangle\,ds\right|^2\le\left(\int\limits_t^{+\infty}\varepsilon_F(s)\,ds\right)^2=C_1\,\tilde\varepsilon_F(t). \tag{5.5}
$$

Therefore, (5.1) follows from (5.4), (5.5) and (2.25) because

$$
\langle W_{t-s}\phi_0,\nabla\rho_0\rangle=\left\langle W_t\phi_0,W'_{-s}\nabla\rho_0\right\rangle.
$$

The bound (5.2) can be proved in a similar way.

(ii) Let $f\in[C_0^\infty(\mathbb{R}^3)]^2$ with $\operatorname{supp}f\subset B_R$. By Eqn (3.1), we have

$$
\langle\phi_t,f\rangle=\langle W_t\phi_0,f\rangle+\int\limits_0^t q_{t-s}\cdot\left\langle W_s\nabla\rho^0,f\right\rangle\,ds. \tag{5.6}
$$

Using Vainberg's bounds [38, 39], we obtain

$$
\langle W_s\nabla\rho^0,f\rangle=\left\{\begin{array}{ll}\mathcal{O}(e^{-\gamma|s|})\quad\text{with a }\gamma>0 & \text{if } L_B=L_W,\\ \mathcal{O}((1+|s|)^{-3/2}) & \text{if } L_B=L_{KG}.\end{array}\right. \tag{5.7}
$$

If $L_B=L_W$ we put $\tilde\varepsilon_F(t)=\varepsilon_F^2(t)=e^{-2\delta t}$ with any $\delta<\min(\delta_*,\gamma)$, see Lemma 3.6. Applying the Parseval inequality and bounds (5.1) and (5.7), we get

$$
\begin{aligned}
\mathbb{E}\Big|\int\limits_0^t\Big(q_{t-s}-\langle W_{t-s}\phi_0,\alpha\rangle\Big)\cdot\left\langle W_s\nabla\rho^0,f\right\rangle\,ds\Big|^2 &\le \left(\int_0^t(\tilde\varepsilon_F(t-s))^{1/2}|\left\langle W_s\nabla\rho^0,f\right\rangle|\,ds\right)^2\\
&\le C\tilde\varepsilon_F(t).
\end{aligned} \tag{5.8}
$$

Write $I(t):=\mathbb{E}\Big|\int\limits_t^\infty\langle W_{t-s}\phi_0,\alpha\rangle\cdot\langle W_s\nabla\rho^0,f\rangle\,ds\Big|^2$. Then

$$
|I(t)|\le C\tilde\varepsilon_F(t). \tag{5.9}
$$

This follows from (5.7) and from the following estimate:

$$\mathbb{E}|\langle W_\tau\phi_0,\alpha\rangle|^2=\sum_{i=1}^3\mathbb{E}\Big|\sum_{r=1}^3\int\limits_0^{+\infty}\mathcal{N}_{ir}(s)\,\langle W_{\tau-s}\phi_0,\nabla_r\rho_0\rangle\,ds\Big|^2\le C<\infty,\quad\text{for}\quad\tau\in\mathbb{R},$$

by (4.3) and (3.9). Relation (5.6) and bounds (5.8) and (5.9) imply (5.3). ■

Corollary 5.2 *Let $Z=(f,u,v)\in\mathcal{D}=[C_0^\infty(\mathbb{R}^3)]^2\times\mathbb{R}^3\times\mathbb{R}^3$. Then*

$$\langle Y_t,Z\rangle=\langle W_t\phi_0,\Pi(Z)\rangle+r(t),$$

where $\Pi(Z)$ is defined in (2.24), $\langle Y_t,Z\rangle=\langle\phi_t,f\rangle+q_t\cdot u+p_t\cdot v$, $Y_t=(\phi_t,q_t,p_t)$ is a solution to the Cauchy problem (2.4), and $\mathbb{E}\,(|r(t)|^2)\le C\tilde\varepsilon_F(t)$.

6 Convergence of characteristic functionals and correlation functions

Proof of Proposition 2.14 By the triangle inequality,

$$\Big|\mathbb{E}e^{i\langle Y_t,Z\rangle}-e^{-\frac12\mathcal{Q}_\infty(Z,Z)}\Big|\le\Big|\mathbb{E}\Big(e^{i\langle Y_t,Z\rangle}-e^{i\langle W_t\phi_0,\Pi(Z)\rangle}\Big)\Big|+\Big|\mathbb{E}e^{i\langle W_t\phi_0,\Pi(Z)\rangle}-e^{-\frac12\mathcal{Q}_\infty(Z,Z)}\Big|.\quad(6.1)$$

Applying Corollary 5.2, we estimate the first term in the r.h.s. of (6.1) by

$$\mathbb{E}\Big|\langle Y_t,Z\rangle-\langle W_t\phi_0,\Pi(Z)\rangle\Big|\le\mathbb{E}|r(t)|\le\Big(\mathbb{E}|r(t)|^2\Big)^{1/2}\le C\tilde\varepsilon_m^{1/2}(t)\to0\quad\text{as }\;t\to\infty,$$

It remains to prove the convergence $\mathbb{E}\,(\exp\{i\langle W_t\phi_0,\Pi(Z)\rangle\})\equiv\hat\mu_t^B(\Pi(Z))$ to a limit as $t\to\infty$.

In [6, 7], we have proved the convergence of $\hat\mu_t^B(f)$ to a limit for $f\in\mathcal{D}_0\equiv[C_0^\infty(\mathbb{R}^3)]^2$. However, $\Pi(Z)\notin\mathcal{D}_0$ in general. Consider the cases of the WF and KGF separately.

In the WF case, $\Pi(Z)\in\mathcal{H}'_\gamma$ if $Z\in\mathcal{D}$, for sufficiently small $\gamma>0$, where $\mathcal{H}'_\gamma$ is introduced in Remark 2.10. This follows from formulas (2.24)–(2.27), from the bound (3.9), and from the estimate

$$|||W'_tf|||'_\gamma\le Ce^{\gamma|t|}|||f|||'_\gamma,\quad t\in\mathbb{R},\quad\text{for any}\quad f\in\mathcal{H}'_\gamma.$$

The last estimate can be proved in a similar way as the same estimate for $(W_t^0)'$ in [7, lemma 8.2] using the energy estimates.

Lemma 6.1 *Let $L_B=L_W$. Then the quadratic forms $\mathcal{Q}_t^B(f,f):=\displaystyle\int|\langle\phi_0,f\rangle|^2\mu_t^B(d\phi_0)$, $t\in\mathbb{R}$, and the characteristic functionals $\hat\mu_t^B(f)$, $t\in\mathbb{R}$, are equicontinuous on $\mathcal{H}'_\gamma$.*

Proof In the case when $L_B=\Delta$, Lemma 6.1 was proved in [7, Corollary 4.3]. In the general case, i.e., when $L_B=L_W$, this lemma can be proved by a similar way and the proof is based on the bound $\mathbb{E}|||W_t\phi_0|||_\gamma^2\le C<\infty$ for any $\gamma>0$. Now we prove this bound. By (4.3), we have

$$e_t:=\mathbb{E}(|\varphi_t(x)|^2+|\nabla\varphi_t(x)|^2+|\pi_t(x)|^2)\le C<\infty,\qquad(6.2)$$

since $\mathbb{E}\|W_t\phi_0\|_R^2=e_t|B_R|$, where $|B_R|$ denotes the volume of the ball $B_R=\{x\in\mathbb{R}^3:|x|\le R\}$. Hence, the bound (6.2) implies, similarly to (2.20), that for any $\gamma>0$ there is a constant $C=C(\gamma)>0$ such that

$$\mathbb{E}|||W_t\phi_0|||_\gamma^2=e_t\int\exp(-2\gamma|x|)\,dx\le C<\infty.\quad■$$

In the case of KGF, we write $H'_m = L^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3)$. Then $\Pi(Z) \in H'_m$ if $Z \in \mathcal{D}$. This follows from formulas (2.24)–(2.27) and the bound (3.9).

Lemma 6.2 *Let $L_B = L_{KG}$. Then (i) the quadratic forms $\mathcal{Q}^B_t(f,f)$, $t \in \mathbb{R}$, are equicontinuous on H'_m, (ii) the characteristic functionals $\hat{\mu}^B_t(f)$, $t \in \mathbb{R}$, are equicontinuous on H'_m.*

Proof (i) It suffices to prove the uniform bound

$$\sup_{t\in\mathbb{R}} |\mathcal{Q}^B_t(f,f)| \le C\|f\|^2_{H'_m} \quad \text{for any} \quad f \in H'_m. \tag{6.3}$$

At first, note that $\mathcal{Q}^B_t(f,f) = \langle Q_0(x,y), W'_t f(x) \otimes W'_t f(y)\rangle$. On the other hand, by conditions **S0**, **S2** and **S3**, the correlation functions $Q^{ij}_0(x,y)$ of the measure μ^B_0 satisfy the following bound: for $\alpha, \beta \in \mathbb{Z}^3$, $|\alpha| \le 1-i$, $|\beta| \le 1-j$, $i,j = 0,1$,

$$|D^{\alpha,\beta}_{x,y} Q^{ij}_0(x,y)| \le C e_0 \varphi^{1/2}(|x-y|), \quad x,y \in \mathbb{R}^3, \tag{6.4}$$

according to [20, Lemma 17.2.3]. Therefore, by (2.12),

$$\int\limits_{\mathbb{R}^3} |D^{\alpha,\beta}_{x,y} Q^{ij}_0(x,y)|^p \, dy \le C e_0^p \int\limits_{\mathbb{R}^3} \varphi^{p/2}(|x-y|)\, dy \le C_1 e_0^p \int\limits_0^\infty r^2 \varphi^{1/2}(r) dr < \infty, \quad p \ge 1.$$

Hence, by the Shur lemma, the quadratic form $\langle Q_0(x,y), f(x)\otimes f(y)\rangle$ is continuous in $[L^2(\mathbb{R}^3)]^2$. Therefore,

$$\sup_{t\in\mathbb{R}} |\mathcal{Q}^B_t(f,f)| = \sup_{t\in\mathbb{R}} |\langle Q_0(x,y), W'_t f(x) \otimes W'_t f(y)\rangle| \le C \sup_{t\in\mathbb{R}} \|W'_t f\|^2_{L^2} \le C\|f\|^2_{H'_m}.$$

The last inequality follows from the energy conservation for the Klein–Gordon equation.
(ii) By the Cauchy-Schwartz inequality and (6.3), we obtain

$$\begin{aligned}
\left|\hat{\mu}^B_t(f_1) - \hat{\mu}^B_t(f_2)\right| &= \left|\int \left(e^{i\langle\phi_0,f_1\rangle} - e^{i\langle\phi_0,f_2\rangle}\right)\mu^B_t(d\phi_0)\right| \le \int \left|e^{i\langle\phi_0,f_1-f_2\rangle} - 1\right| \mu^B_t(d\phi_0) \\
&\le \int |\langle\phi_0, f_1 - f_2\rangle| \mu^B_t(d\phi_0) \le \sqrt{\int |\langle\phi_0, f_1-f_2\rangle|^2 \mu^B_t(d\phi_0)} \\
&= \sqrt{\mathcal{Q}^B_t(f_1-f_2, f_1-f_2)} \le C\|f_1-f_2\|_{H'_m}. \qquad \blacksquare
\end{aligned}$$

We return to the proof of Proposition 2.14. By [8, Proposition 2.3] (or [10, Proposition 3.3]), and by Lemmas 6.1 and 6.2, the characteristic functionals $\hat{\mu}^B_t(\Pi(Z))$ converge to a limit as $t \to \infty$. This completes the proof of Proposition 2.14 and Theorem 2.12, (i)–(ii). $\blacksquare$

Lemma 6.3 *Let all assumptions of Theorem 2.12 be satisfied. Then convergence (2.30) holds.*

Proof It suffices to prove the convergence of $\int |\langle Y, Z\rangle|^2 \mu_t(dY) = \mathbb{E}|\langle Y_t, Z\rangle|^2$ to a limit as $t \to \infty$. It follows from Corollary 5.2 that for $Z \in \mathcal{D}$,

$$\mathbb{E}|\langle Y_t, Z\rangle|^2 = \mathbb{E}|\langle W_t\phi_0, \Pi(Z)\rangle|^2 + o(1) = \mathcal{Q}^B_t(\Pi(Z), \Pi(Z)) + o(1), \quad t \to \infty,$$

where $\Pi(Z)$ is defined in (2.24). Therefore, by the results from [8, 10] and by Lemmas 6.1 and 6.2, the quadratic forms $\mathcal{Q}^B_t(\Pi(Z), \Pi(Z))$ converge to a limit as $t \to \infty$. Formula (2.29) implies (2.30). $\blacksquare$

7 Ergodicity and mixing for the limit measures

Denote by $\mathbb{E}_\infty$ ($\mathbb{E}_\infty^B$) the integral w.r.t. μ_∞ (μ_∞^B, respectively). In [5], we have proved that W_t is mixing w.r.t. μ_∞^B, i.e., for any $f, g \in L_2(\mathcal{H}, \mu_\infty^B)$, the following convergence holds,

$$\mathbb{E}_\infty^B \left(f(W_t\phi)g(\phi)\right) \to \mathbb{E}_\infty^B(f(\phi))\,\mathbb{E}_\infty^B(g(\phi)) \quad \text{as} \quad t \to \infty. \tag{7.1}$$

Recall that the limit measure μ_∞ is invariant by Theorem 2.12 (iv). Now we prove that the flow S_t is mixing w.r.t. μ_∞. This mixing property means that the convergence (2.28) holds for the initial measures μ_0 that are absolutely continuous w.r.t. μ_∞, and the limit measure coincides with μ_∞.

Theorem 7.1 *The phase flow S_t is mixing w.r.t. μ_∞, i.e., for any $F, G \in L_2(\mathcal{E}, \mu_\infty)$ we have*

$$\mathbb{E}_\infty \left(F(S_tY)G(Y)\right) \to \mathbb{E}_\infty(F(Y))\,\mathbb{E}_\infty(G(Y)) \quad \text{as} \quad t \to \infty.$$

In particular, the flow S_t is ergodic w.r.t. μ_∞, i.e., for any $F \in L_2(\mathcal{E}, \mu_\infty)$,

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T F(S_tY)\,dt = \mathbb{E}_\infty(F(Y)) \pmod{\mu_\infty}.$$

To prove Theorem 7.1, we introduce new notations. Represent $Y \in \mathcal{E}$ as $Y = (Y^0, Y^1)$ with $Y^0 = (\varphi(\cdot), q) \in H^1_{\rm loc}(\mathbb{R}^d) \times \mathbb{R}^d$, $Y^1 = (\pi(\cdot), p) \in L^2_{\rm loc}(\mathbb{R}^d) \times \mathbb{R}^d$, and $Z \in \mathcal{D}$ as $Z = (Z^0, Z^1)$ with $Z^0 = (f^0(\cdot), u^0)$, $Z^1 = (f^1(\cdot), u^1) \in C_0^\infty(\mathbb{R}^d) \times \mathbb{R}^d$. For $t \in \mathbb{R}$, introduce a "formal adjoint" operator S'_t on the space $\mathcal{D}$ by the rule

$$\langle S_tY, Z\rangle = \langle Y, S'_tZ\rangle, \quad Y \in \mathcal{E}, \quad Z \in \mathcal{D}. \tag{7.2}$$

Lemma 7.2 *For $Z \in \mathcal{D}$,*

$$S'_tZ = (\dot f_t(\cdot), \dot u_t, f_t(\cdot), u_t), \tag{7.3}$$

where $(f_t(x), u_t)$ is the solution of system (2.1) with the initial data (see (2.3))
$(\varphi_0, q_0, \pi_0, p_0) = (f^1, u^1, f^0, u^0)$.

Proof Differentiating (7.2) in t with $Y, Z \in \mathcal{D}$, we obtain $\langle \dot S_tY, Z\rangle = \langle Y, \dot S'_tZ\rangle$. The group S_t has the generator

$$\mathcal{L} = \begin{pmatrix} 0 & 1 \\ \mathcal{A} & 0 \end{pmatrix}, \quad \text{with} \quad \mathcal{A}\begin{pmatrix} \varphi \\ q \end{pmatrix} = \begin{pmatrix} L_B\varphi + q\cdot\nabla\rho \\ -\omega^2 q + \langle\nabla\rho, \varphi\rangle \end{pmatrix}. \tag{7.4}$$

The generator of S'_t is the conjugate operator $\mathcal{L}' = \begin{pmatrix} 0 & \mathcal{A} \\ 1 & 0 \end{pmatrix}$. Hence, (7.3) holds with $\begin{pmatrix} \ddot f_t(x) \\ \ddot u_t \end{pmatrix} = \mathcal{A}\begin{pmatrix} f_t(x) \\ u_t \end{pmatrix}$. ■

Since the limit measure μ_∞ is Gaussian with zero mean, the proof of Theorem 7.1 reduces to that of the following convergence.

Lemma 7.3 *For any $Z_1, Z_2 \in \mathcal{D}$,*

$$\mathbb{E}_\infty\Big(\langle S_t Y, Z_1\rangle\langle Y, Z_2\rangle\Big) \to 0, \quad t \to \infty. \tag{7.5}$$

Proof First we note that, by relation (2.29),

$$\mathbb{E}_\infty\Big(\langle Y, Z_1\rangle\langle Y, Z_2\rangle\Big) = \mathbb{E}^B_\infty\Big(\langle \phi, \Pi(Z_1)\rangle\langle \phi, \Pi(Z_2)\rangle\Big),$$

where $\Pi(Z)$ is defined in (2.24). Secondly, for fixed t, we have $S'_t Z \in \mathcal{D}$. Further,

$$\begin{aligned}
\mathbb{E}_\infty\Big(\langle S_t Y, Z_1\rangle\langle Y, Z_2\rangle\Big) &= \mathbb{E}_\infty\Big(\langle Y, S'_t Z_1\rangle\langle Y, Z_2\rangle\Big) = \mathbb{E}^B_\infty\Big(\langle \phi, \Pi(S'_t Z_1)\rangle\langle \phi, \Pi(Z_2)\rangle\Big)\\
&= \mathbb{E}^B_\infty\Big(\langle \phi, (\Pi S'_t - W'_t\Pi) Z_1\rangle\langle \phi, \Pi(Z_2)\rangle\Big) + \mathbb{E}^B_\infty\Big(\langle \phi, W'_t\Pi(Z_1)\rangle\langle \phi, \Pi(Z_2)\rangle\Big)\\
&= I_1 + I_2.
\end{aligned} \tag{7.6}$$

Note that $\langle \phi, \Pi(Z)\rangle \in L_2(\mathcal{H}, \mu^B_\infty)$ for all $Z \in \mathcal{D}$. Indeed, by (2.29),

$$\mathbb{E}^B_\infty|\langle \phi, \Pi(Z)\rangle|^2 = \mathcal{Q}^B_\infty(\Pi(Z), \Pi(Z)) = \mathcal{Q}_\infty(Z, Z) < \infty.$$

Therefore, the convergence (7.1) implies

$$\begin{aligned}
I_2 &\equiv \mathbb{E}^B_\infty\Big(\langle \phi, W'_t\Pi(Z_1)\rangle\langle \phi, \Pi(Z_2)\rangle\Big)\\
&= \mathbb{E}^B_\infty\Big(\langle W_t\phi, \Pi(Z_1)\rangle\langle \phi, \Pi(Z_2)\rangle\Big) \to \mathbb{E}^B_\infty\Big(\langle \phi, \Pi(Z_1)\rangle\Big)\mathbb{E}^B_\infty\Big(\langle \phi, \Pi(Z_2)\rangle\Big), \quad t \to \infty.
\end{aligned}$$

On the other hand, $\mathbb{E}^B_\infty\langle \phi, \Pi(Z_i)\rangle = \mathbb{E}_\infty\langle Y, Z_i\rangle = 0$, for $Z_i \in \mathcal{D}$, because μ_∞ has zero mean. Therefore,

$$I_2 \to 0, \quad t \to \infty. \tag{7.7}$$

Now we prove that $\mathbb{E}^B_\infty\,|\langle \phi, \Pi(S'_t Z) - W'_t\Pi(Z)\rangle|^2 = 0$ for all $t > 0$. This yields

$$I_1 \equiv \mathbb{E}^B_\infty\Big(\langle \phi, \Pi(S'_t Z_1) - W'_t\Pi(Z_1)\rangle\langle \phi, \Pi(Z_2)\rangle\Big) = 0. \tag{7.8}$$

Indeed, by Corollary 5.2,

$$\mathbb{E}|\langle S_{\tau+t} Y, Z\rangle - \langle W_{\tau+t}\phi, \Pi(Z)\rangle|^2 \to 0, \quad \tau \to \infty.$$

On the other hand, since $\langle S_{\tau+t} Y, Z\rangle = \langle S_\tau Y, S'_t Z\rangle$, we have, for all $t > 0$,

$$\mathbb{E}|\langle S_\tau Y, S'_t Z\rangle - \langle W_\tau\phi, \Pi(S'_t Z)\rangle|^2 \to 0, \quad \tau \to \infty.$$

Therefore, by the triangle inequality,

$$A := \mathbb{E}|\langle W_\tau\phi, \Pi(S'_t Z)\rangle - \langle W_{\tau+t}\,\phi, \Pi(Z)\rangle|^2 \to 0, \quad \tau \to \infty.$$

Since $\langle W_{\tau+t}\,\phi, \Pi(Z)\rangle = \langle W_\tau\,\phi, W'_t\Pi(Z)\rangle$, we obtain

$$A = \mathbb{E}|\langle W_\tau\phi, \Pi(S'_t Z) - W'_t\Pi(Z)\rangle|^2 \to 0, \quad \tau \to \infty.$$

Hence, by Theorem 2.9 (iii) and Lemmas 6.1 and 6.2,

$$\mathbb{E}^B_\infty|\langle \phi, \Pi(S'_t Z) - W'_t\Pi(Z)\rangle|^2 = \lim_{\tau\to\infty}\mathbb{E}|\langle W_\tau\phi, \Pi(S'_t Z) - W'_t\Pi(Z)\rangle|^2 = 0 \quad \text{for all} \quad t > 0.$$

Finally, (7.6)–(7.8) imply the convergence (7.5). Theorem 7.1 is proved. ■

8 Non translation invariant initial measures

In this section we extend the results of Theorem 2.12 to the case of non translation-invariant initial measures. Note that the proof of Theorem 2.12 is based on two assertions. We first derive the asymptotic behavior of solutions Y_t in mean: $\langle Y_t, Z\rangle \sim \langle W_t\phi_0, \Pi(Z)\rangle$ as $t\to\infty$ (see Corollary 5.2). This asymptotics allows us to reduce the convergence analysis for the coupled system to the same problem for the wave (or Klein-Gordon) equation. The second assertion is the weak convergence of the measures $\mu_t^B = W_t^*\mu_0^B$ to a limit as $t\to\infty$ (see Theorem 2.9). However, the weak convergence of μ_t^B holds under weaker conditions on μ_0^B than **S2** and **S3**. Now we formulate these conditions (see [8] for $L_B = L_W$ and [10] for $L_B = L_{KG}$).

8.1 Conditions on μ_0^B

In the case of the KGF, we assume that μ_0^B has zero mean, satisfies a mixing condition **S3** and has a finite mean energy density (see (2.10)), i.e.,

$$\begin{aligned}\int \Big(|\varphi_0(x)|^2 + |\nabla\varphi_0(x)|^2 + |\pi_0(x)|^2\Big)\mu_0^B(d\phi_0) \;&=\; Q_0^{00}(x,x) + [\nabla_x\nabla_y Q_0^{00}(x,y)]\Big|_{x=y} + Q_0^{11}(x,x)\\ &\le\; e_0 < \infty. \qquad (8.1)\end{aligned}$$

However, condition **S2** of translation invariance for μ_0^B can be weakened as follows.
S2' The correlation functions of the measure μ_0^B have the form

$$Q_0^{ij}(x,y) = q_-^{ij}(x-y)\zeta_-(x_1)\zeta_-(y_1) + q_+^{ij}(x-y)\zeta_+(x_1)\zeta_+(y_1), \quad i,j=0,1. \qquad (8.2)$$

Here $q_\pm^{ij}(x-y)$ are the correlation functions of some translation-invariant measures $\mu_\pm^B$ with zero mean value in $\mathcal{H}$, $x=(x_1,\dots,x_d)$, $y=(y_1,\dots,y_d)\in\mathbb{R}^d$, the functions $\zeta_\pm\in C^\infty(\mathbb{R})$ such that

$$\zeta_\pm(s) = \begin{cases} 1, & \text{for } \pm s > a,\\ 0, & \text{for } \pm s < -a,\end{cases} \qquad (8.3)$$

and $a>0$. The measure μ_0^B is not translation-invariant if $q_-^{ij}\neq q_+^{ij}$.

In the case of WF, instead of **S2** and **S3** we impose the following conditions **S2'** and **S3'**.
S2' The correlation functions of μ_0^B have the form

$$Q_0^{ij}(x,y) = \begin{cases} q_-^{ij}(x-y), & x_1,y_1 < -a,\\ q_+^{ij}(x-y), & x_1,y_1 > a,\end{cases} \qquad (8.4)$$

with some $a>0$ and $q_\pm^{ij}$ as in (8.2). However, in the WF case, instead of (8.1) we impose a stronger condition (8.5). Namely, the following derivatives are continuous and the bounds hold,

$$|D_{x,y}^{\alpha,\beta}Q_0^{ij}(x,y)| \le \begin{cases} C\nu_\kappa(|x-y|) & \text{if } \kappa=0,1,\dots,d-2\\ C\nu_{d-1}(|x-y|) & \text{if } \kappa=d-1,d,d+1\end{cases}\Bigg|\; \kappa=i+j+|\alpha|+|\beta|, \qquad (8.5)$$

with $|\alpha|\le(d-3)/2+i$, $|\beta|\le(d-3)/2+j$, $i,j=0,1$. Here $\nu_\kappa\in C[0,\infty)$ ($\kappa=0,\dots,d-1$) denote some continuous nonnegative nonincreasing functions in $[0,\infty)$ with the finite integrals $\int\limits_0^\infty(1+r)^{\kappa-1}\nu_\kappa(r)dr<\infty$. Moreover, for $d\ge5$, $\int\limits_0^\infty(1+r)^{d-4+\kappa}\nu_\kappa(r)dr<\infty$ with $\kappa=0,2$.

S3' Let $\mathcal{O}(r)$ be the set of all pairs of open convex subsets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^d$ at distance $d(\mathcal{A}, \mathcal{B}) \geq r$, and let $\alpha = (\alpha_1, \dots, \alpha_d)$ with integers $\alpha_i \geq 0$. Denote by $\sigma_{i\alpha}(\mathcal{A})$ the σ-algebra of the subsets in $\mathcal{H}$ generated by all linear functionals

$$\phi_0 = (\phi_0^0, \phi_0^1) \mapsto \langle D^\alpha \phi_0^i, f\rangle, \quad \text{with } |\alpha| \leq 1 - i, \quad i = 0, 1,$$

where $f \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} f \subset \mathcal{A}$. For $\kappa = 0, 1$, let $\sigma_\kappa(\mathcal{A})$ be the σ-algebra generated by $\sigma_{i\alpha}(\mathcal{A})$ with $i + |\alpha| \geq \kappa$, i.e., $\sigma_\kappa(\mathcal{A}) \equiv \bigvee\limits_{i+|\alpha|\geq\kappa} \sigma_{i\alpha}(\mathcal{A})$. We define the (Ibragimov) mixing coefficient of μ_0^B on $\mathcal{H}$ as (cf. (2.11))

$$\varphi_{\kappa_1,\kappa_2}(r) \equiv \sup_{(\mathcal{A},\mathcal{B})\in \mathcal{O}(r)} \sup_{\substack{A \in \sigma_{\kappa_1}(\mathcal{A}), B \in \sigma_{\kappa_2}(\mathcal{B}) \\ \mu_0^B(B) > 0}} \frac{|\mu_0^B(A \cap B) - \mu_0^B(A)\mu_0^B(B)|}{\mu_0^B(B)}, \quad \kappa_1, \kappa_2 = 0, 1.$$

We assume that the measure μ_0^B satisfies the strong uniform Ibragimov mixing condition, i.e., for any $\kappa_1, \kappa_2 = 0, 1$, $\varphi_{\kappa_1,\kappa_2}(r) \to 0$, $r \to \infty$. Moreover,

$$\varphi_{\kappa_1,\kappa_2}(r) \leq C\nu_\kappa^2(r), \quad \text{where } \kappa = \kappa_1 + \kappa_2, \quad \kappa_1, \kappa_2 = 0, 1.$$

Remark 8.1 (i) In [8, 10], we have constructed the generic examples of the initial measures μ_0^B satisfying all assumptions imposed.
(ii) Condition **S3** and the bound (8.1) imply the bound (6.4).
(iii) Condition (8.5) implies (8.1). Condition **S3'** implies estimates (8.5) with $i + |\alpha| \leq 1$, $j + |\beta| \leq 1$. The mixing condition **S3'** is weaker than condition **S3**. On the other hand, the estimates (8.5) with $\kappa > 2$ are not required for translation-invariant initial measures μ_0^B or in the KGF case.
(iv) The conditions **S2** and **S3** admit various modifications. We choose the variant which allows an application to the case of the Gibbs measures $\mu_\pm^B$ (see Section A.3 below).

8.2 Convergence to equilibrium

Theorem 8.2 *(see [8, 10]) Let conditions* **A1**–**A4** *and all conditions imposed on μ_0^B in Section 8.1 be satisfied. Then the assertions of Theorem 2.9 remains true with the matrix $Q_\infty^B(x, y) = q_\infty^B(x - y)$ of the following form. In the Fourier transform, $\hat{q}_\infty^B(k) = \hat{q}_\infty^+(k) + \hat{q}_\infty^-(k)$, where (cf. (2.15))*

$$\begin{aligned} \hat{q}_\infty^+(k) &= \tfrac{1}{2}\Big(\hat{\mathbf{q}}^+(k) + \hat{C}(k)\hat{\mathbf{q}}^+(k)\hat{C}^T(k)\Big), \\ \hat{q}_\infty^-(k) &= i\,\mathrm{sgn}(k_1)\tfrac{1}{2}\Big(\hat{C}(k)\hat{\mathbf{q}}^-(k) - \hat{\mathbf{q}}^-(k)\hat{C}^T(k)\Big), \end{aligned}$$

with $\mathbf{q}^+ = (q_+ + q_-)/2$, $\mathbf{q}^- = (q_+ - q_-)/2$, and $\hat{C}(k)$ from (2.16).

Theorem 8.3 *Let conditions* **A1**–**A5**, **R1**–**R3**, **S0**, **S1**, *and all assumptions imposed on μ_0^B be satisfied. Then the assertions of Theorem 2.12 hold.*

This theorem can be proved in a similar way as Theorem 2.12 (see Sections 4–6).

In Appendix A we will give an application of Theorems 8.2 and 8.3 to the case when the measures $\mu_\pm^B$ from condition **S2'** are Gibbs measures with different temperatures $T_+ \neq T_-$.

ACKNOWLEDGMENTS

This work was supported partly by the research grant of RFBR (Grant No. 12-01-00203). The author is grateful to Alexander Komech for useful discussions concerning several aspects of this paper.

Appendix A: Gibbs measures

Here we study the case $L_B = \Delta - m^2$ only. Consider first the 'free' wave (or Klein–Gordon) equation,

$$\left\{ \begin{array}{l} \ddot{\varphi}_t(x) = (\Delta - m^2)\varphi_t(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d, \\ \varphi_t(x)|_{t=0} = \varphi_0(x), \quad \dot{\varphi}_t(x)|_{t=0} = \pi_0(x), \end{array} \right. \tag{A.1}$$

where $m \geq 0$, $d \geq 3$, and d is odd if $m = 0$. Denoting $\phi_t = (\varphi_t, \pi_t)$, $t \in \mathbb{R}$, we rewrite (A.1) in the form

$$\dot{\phi}_t = \mathcal{L}_B(\phi_t), \quad t \in \mathbb{R}, \quad \phi_t|_{t=0} = \phi_0, \tag{A.2}$$

with $\mathcal{L}_B = \begin{pmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{pmatrix}$. In the Fourier transform representation, system (A.1) becomes $\dot{\hat{\phi}}_t(k) = \hat{\mathcal{L}}_B(k)\hat{\phi}_t(k)$, hence $\hat{\phi}_t(k) = \hat{\mathcal{G}}_t(k)\hat{\phi}_0(k)$, where $\hat{\mathcal{G}}_t(k) = \exp(\hat{\mathcal{L}}_B(k)t)$. Here we denote

$$\hat{\mathcal{L}}_B(k) = \begin{pmatrix} 0 & 1 \\ -\omega^2(k) & 0 \end{pmatrix}, \qquad \hat{\mathcal{G}}_t(k) = \begin{pmatrix} \cos \omega t & \dfrac{\sin \omega t}{\omega} \\ -\omega \sin \omega t & \cos \omega t \end{pmatrix},$$

with $\omega \equiv \omega(k) = \sqrt{|k|^2 + m^2}$. Hence, the solution of (A.2) is $\phi_t = W_t^0 \phi_0 = \mathcal{G}_t(\cdot) * \phi_0$, where $\mathcal{G}_t(x) = F^{-1}_{k\to x}[\hat{\mathcal{G}}_t(k)]$. For simplicity of exposition, we omit below the index 0 in the notation of the group W_t^0.

A.1 Phase space

We define the weighted Sobolev spaces with any $s, \alpha \in \mathbb{R}$.

Definition A.1 *(i) $H^s_\alpha(\mathbb{R}^d)$ is the Hilbert space of the distributions $\varphi \in S'(\mathbb{R}^d)$ with finite norm*

$$\|\varphi\|_{s,\alpha} \equiv \|\langle x\rangle^\alpha \Lambda^s \varphi\|_{L^2(\mathbb{R}^d)} < \infty, \quad \Lambda^s\varphi \equiv F^{-1}\left[\langle k\rangle^s \hat{\varphi}(k)\right], \quad s, \alpha \in \mathbb{R}. \tag{A.3}$$

(ii) $\mathcal{H}^s_\alpha \equiv H^{s+1}_\alpha(\mathbb{R}^d) \oplus H^s_\alpha(\mathbb{R}^d)$ is the Hilbert space of pairs $\phi \equiv (\varphi(x), \pi(x))$ with finite norm

$$\|\phi\|_{s,\alpha} = \|\varphi\|_{s+1,\alpha} + \|\pi\|_{s,\alpha}, \quad s, \alpha \in \mathbb{R}. \tag{A.4}$$

(iii) $\mathcal{E}^s_\alpha \equiv \mathcal{H}^s_\alpha \oplus \mathbb{R}^d \oplus \mathbb{R}^d$ is the Hilbert space of vectors $Y \equiv (\phi(x), q, p)$ with finite norm

$$\|Y\|_{s,\alpha} = \|\phi\|_{s,\alpha} + |q| + |p|, \quad s, \alpha \in \mathbb{R}.$$

Note that $\mathcal{H}^{\bar{s}}_{\bar{\alpha}} \subset \mathcal{H}^s_\alpha$ (and also $\mathcal{E}^{\bar{s}}_{\bar{\alpha}} \subset \mathcal{E}^s_\alpha$) if $\bar{s} > s$ and $\bar{\alpha} > \alpha$, and this embedding is compact. Moreover, for any α, $\mathcal{H}^0_\alpha \subset \mathcal{H}$, $\mathcal{E}^0_\alpha \subset \mathcal{E}$ (see Definition 2.1).

Lemma A.2 *Let $L_B = \Delta - m^2$, $s, \alpha \in \mathbb{R}$, and conditions* **A1'** *and* **A2** *hold. Then (i) for every $Y_0 \in \mathcal{E}^s_\alpha$, the Cauchy problem (2.4) has a unique solution $Y_t \in C(\mathbb{R}, \mathcal{E}^s_\alpha)$.*
(ii) For every $t \in \mathbb{R}$, the operator $S_t : Y_0 \mapsto Y_t$ is continuous on $\mathcal{E}^s_\alpha$. Moreover, there exist positive constants $C_1, C_2 > 0$ such that $\|S_t Y_0\|_{s,\alpha} \leq C_1 \langle t\rangle^{C_2} \|Y_0\|_{s,\alpha}$.

This lemma can be proved by the similar technique from [23], where the nonlinear "wave field–particle" system was studied.

A.2 Gibbs measures for the Klein-Gordon equation

Write $\phi = (\varphi, \pi)$. We introduce the (normalized) Gibbs measures g_β^B on the space $\mathcal{H}_\alpha^s$. Formally,

$$g_\beta^B(d\phi) = \frac{1}{Z_B} e^{-\beta H_B(\phi)} \prod_{x\in\mathbb{R}^d} d\phi(x), \quad H_B(\phi) = \frac{1}{2}\int \left(|\nabla\varphi(x)|^2 + m^2|\varphi(x)|^2 + |\pi(x)|^2\right) dx.$$

Now we adjust the definition of the Gibbs measures g_β^B. Write $\phi = (\phi^0, \phi^1) \equiv (\varphi, \pi)$, and denote by $Q^{ij}(x,y)$, $i,j = 0,1$, the correlation functions of g_β^B,

$$Q^{ij}(x,y) = \int \phi^i(x)\phi^j(y)\, g_\beta^B(d\phi) = q^{ij}(x-y), \quad x,y \in \mathbb{R}^d.$$

We will define the Gibbs measures g_β^B as the Gaussian measures with the correlation functions

$$q^{00}(x-y) = T\mathcal{E}_m(x-y), \quad q^{11}(x-y) = T\delta(x-y), \quad q^{01}(x-y) = q^{10}(x-y) = 0, \qquad \text{(A.5)}$$

where $T = 1/\beta$, $\mathcal{E}_m(x)$ is the fundamental solution of the operator $-\Delta + m^2$. The correlation functions q^{ii} do not satisfy condition (8.1) because of singularity at $x = y$. The singularity means that the measures g_β^B are not concentrated in the space $\mathcal{H}$.

Definition A.3 *For $\beta > 0$, define the Gibbs measures $g_\beta^B(d\phi)$ as the Borel probability measures $g_\beta^B(d\phi) = g_\beta^0(d\varphi) \times g_\beta^1(d\pi)$ in $\mathcal{H}_\alpha^s = H_\alpha^{s+1}(\mathbb{R}^d) \otimes H_\alpha^s(\mathbb{R}^d)$, $s, \alpha < -d/2$, where $g_\beta^0(d\varphi)$ and $g_\beta^1(d\pi)$ are Gaussian Borel probability measures in spaces $H_\alpha^{s+1}(\mathbb{R}^d)$ and $H_\alpha^s(\mathbb{R}^d)$, respectively, with characteristic functionals*

$$\left.\begin{array}{rcl} \hat{g}_\beta^0(f) &=& \displaystyle\int \exp\{i\langle\varphi, f\rangle\}\, g_\beta^0(d\varphi) = \exp\left\{-\frac{1}{2\beta}\langle(-\Delta+m^2)^{-1}f, f\rangle\right\} \\ \hat{g}_\beta^1(f) &=& \displaystyle\int \exp\{i\langle\pi, f\rangle\}\, g_\beta^1(d\pi) = \exp\left\{-\frac{1}{2\beta}\langle f, f\rangle\right\} \end{array}\right| \quad f \in C_0^\infty(\mathbb{R}^d). \quad \text{(A.6)}$$

By the Minlos theorem, the Borel probability measures g_β^0 and g_β^1 exist in the spaces $H_\alpha^{s+1}(\mathbb{R}^d)$ and $H_\alpha^s(\mathbb{R}^d)$, respectively, because *formally*

$$\int \|\varphi\|_{s+1,\alpha}^2\, g_\beta^0(d\varphi) < \infty, \quad \int \|\pi\|_{s,\alpha}^2\, g_\beta^1(d\pi) < \infty, \quad s, \alpha < -d/2. \qquad \text{(A.7)}$$

We verify (A.7). Definition (A.3) implies, for $\varphi \in H_\alpha^s(\mathbb{R}^d)$,

$$\|\varphi\|_{s,\alpha}^2 = (2\pi)^{-2d} \int_{\mathbb{R}^d} \langle x\rangle^{2\alpha} \Big(\int_{\mathbb{R}^{2d}} e^{-ix(k-k')} \langle k\rangle^s \langle k'\rangle^s \hat{\varphi}(k)\overline{\hat{\varphi}}(k')\, dk\, dk' \Big) dx. \qquad \text{(A.8)}$$

Let $g(d\varphi)$ be a translation invariant measure in $H_\alpha^s(\mathbb{R}^d)$ with a correlation function $Q(x,y) = q(x-y)$. Let us introduce the following correlation function

$$C(k,k') \equiv \int \hat{\varphi}(k)\overline{\hat{\varphi}}(k')\, g(d\varphi)$$

in the sense of distributions. Since $\varphi(x)$ is real-valued, we have

$$C(k,k') = F_{x\to k} F_{x'\to -k'} Q(x,x') = (2\pi)^d \delta(k-k')\hat{q}(k).$$

Then, integrating (A.8) with respect to the measure $g(d\varphi)$, we obtain the formula

$$\int \|\varphi\|_{s,\alpha}^2 \, g(d\varphi) = (2\pi)^{-d} \int_{\mathbb{R}^d} \langle x \rangle^{2\alpha} dx \int_{\mathbb{R}^d} \langle k \rangle^{2s} \hat{q}(k) \, dk.$$

Substituting $\hat{q}(k) = T$ (see (A.5)) we obtain the second bound in (A.7). To obtain the first bound in (A.7) we replace s into $s+1$ and put $\hat{q}(k) = T\hat{\mathcal{E}}_m(k) = T(|k|^2 + m^2)^{-1}$.

Below for spaces $\mathcal{E}_\alpha^s$ and $\mathcal{H}_\alpha^s$, we put $s, \alpha < -d/2$. Definition A.3 implies the following lemma (cf the convergence (7.1)).

Lemma A.4 *The Gibbs measures g_β^B are invariant w.r.t. W_t. Moreover, the flow W_t is mixing w.r.t. g_β^B.*

In Section A.4, we will define the Gibbs measures g_β for the coupled system and check the mixing property for the dynamics S_t w.r.t. g_β.

A.3 Application of Theorems 8.2 and 8.3 to Gibbs measures $\mu_\pm^B$

Let $\mu_\pm^B$ (see condition **S2'**) be the Gibbs measures $g_\pm^B \equiv g_{\beta_\pm}^B$ (with $\beta_\pm = 1/T_\pm$) corresponding to different positive temperatures $T_- \neq T_+$. We define the Gibbs measures $g_\pm^B$ in the space $\mathcal{H}_\alpha^s$ (see Definition A.3) as the Gaussian measures with the correlation functions (cf. (A.5))

$$q_\pm^{00}(x-y) = T_\pm \, \mathcal{E}_m(x-y), \quad q_\pm^{11}(x-y) = T_\pm \delta(x-y), \quad q_\pm^{01}(x-y) = q_\pm^{10}(x-y) = 0, \quad \text{(A.9)}$$

where $x, y \in \mathbb{R}^d$.

Let us introduce (ϕ_-, ϕ_+) as a unit random function in the probability space $(\mathcal{H}_\alpha^s \times \mathcal{H}_\alpha^s, g_-^B \times g_+^B)$. Then $\phi_\pm$ are Gaussian independent vectors in $\mathcal{H}_\alpha^s$. Define a Borel probability measure $\mu_0^B \equiv g_0^B$ on $\mathcal{H}_\alpha^s$ as the distribution of the random function

$$\phi_0(x) = \zeta_-(x_1)\phi_-(x) + \zeta_+(x_1)\phi_+(x),$$

where functions $\zeta_\pm$ are introduced in (8.3). Then correlation functions of g_0^B are of the form (8.2) with $q_\pm^{ij}$ from (A.9). Hence, the measure g_0^B has zero mean and satisfies condition (8.2) or (8.4). However, g_0^B does not satisfy (8.1) because of singularity at $x = y$. Therefore, Theorem 8.2 cannot be applied directly to $\mu_0^B \equiv g_0^B$. The embedding $\mathcal{H}_\alpha^s \subset \mathcal{H}^s$ is continuous by the standard arguments of pseudodifferential equations, [19]. The next lemma follows by Fourier transform and the finite speed of propagation for the wave and Klein-Gordon equation.

Lemma A.5 *The operators $W_t : \phi_0 \mapsto \phi_t$ allow a continuous extension $\mathcal{H}^s \mapsto \mathcal{H}^s$.*

Let ϕ_0 be the random function with the distribution g_0^B. Hence $\phi_0 \in \mathcal{H}_\alpha^s$ a.s. Denote by g_t^B the distribution of $W_t\phi_0$. For the measures g_t^B, the following result was proved in [8, Theorem 3.1] and [10, Section 4].

Lemma A.6 *Let $s < -d + 1/2$. Then there exists a Gaussian Borel probability measure g_∞^B on the space $\mathcal{H}^s$ such that*

$$g_t^B \xrightarrow{\mathcal{H}^s} g_\infty^B, \quad t \to \infty. \quad \text{(A.10)}$$

The correlation matrix $Q^B_\infty(x,y)=(q^{B,ij}_\infty(x-y))_{i,j=0,1}$ *of the limit measure* g^B_∞ *has a form*

$$\left.\begin{array}{rcl} q^{B,00}_\infty(x-y) &=& \frac{1}{2}(T_+ + T_-)\mathcal{E}_m(x-y), \\ q^{B,10}_\infty(x-y) &=& -q^{B,01}_\infty(x-y)=\frac{1}{2}(T_+ - T_-)\mathcal{P}(x-y), \\ q^{B,11}_\infty(x-y) &=& \frac{1}{2}(T_+ + T_-)\delta(x-y), \end{array}\right| \tag{A.11}$$

where $\mathcal{P}(x)=-iF^{-1}_{k\to x}[\mathrm{sgn}(k_1)/\omega(k)]$. *In particular, the limiting mean energy current density is formally*

$$\nabla q^{B,10}_\infty(0)=\frac{T_+ - T_-}{2}\,\nabla\mathcal{P}(0)=-\frac{T_+ - T_-}{2(2\pi)^d}\int_{\mathbb{R}^d}\frac{k\,\mathrm{sgn}(k_1)}{\sqrt{|k|^2+m^2}}\,dk=-\infty\cdot(T_+ - T_-,0,\ldots,0).$$

The infinity means the 'ultraviolet divergence'.

Denote by g_0 a Borel probability measure on $\mathcal{E}^s_\alpha$ such that $Pg_0=g^B_0$, where $P:(\phi_0,q_0,p_0)\in\mathcal{E}^s_\alpha\to\phi_0\in\mathcal{H}^s_\alpha$, and g^B_0 is the probability measure on $\mathcal{H}^s_\alpha$ constructed above. Then, by the asymptotic behavior of Y_t (see Section 5) and by Lemma A.6, the following result holds (cf. Theorems 2.12 and 8.3).

Lemma A.7 *Let* $s<-d+1/2$. *Then the measures* $g_t=S^*_t g_0$ *weakly converge to a limit measure* g_∞ *as* $t\to\infty$ *on the space* $\mathcal{E}^s$. *The limit measure* g_∞ *is Gaussian and its characteristic functional is* $\hat{g}_\infty(Z)=\exp\{-(1/2)\mathcal{Q}_\infty(Z,Z)\}$, *where* $\mathcal{Q}_\infty(Z,Z)=\langle q^B_\infty(x-y),\Pi(Z)\otimes\Pi(Z)\rangle$ *with* q^B_∞ *from (A.11).*

A.4 Gibbs measure for the coupled system

For $\beta>0$, we introduce the (normalized) Gibbs measures g_β on the space $\mathcal{E}^s_\alpha$. Formally,

$$g_\beta(d\phi\,d\xi)=\frac{1}{Z}e^{-\beta H(\phi,\xi)}\prod_{x\in\mathbb{R}^d}d\phi(x)\,d\xi.$$

Definition A.8 *For* $\beta>0$, *define the Gibbs measures* $g_\beta(d\phi\,d\xi)$ *in* $\mathcal{E}^s_\alpha$, $s,\alpha<-d/2$, *as*

$$g_\beta(d\phi\,d\xi)=\frac{1}{Z}e^{-\beta q\cdot\langle\rho,\nabla\varphi\rangle}\,g^B_\beta(d\phi)\times g^A_\beta(d\xi). \tag{A.12}$$

Here $\beta=1/T$ *is an inverse temperature,* $g^B_\beta(d\phi)$ *is defined in Definition A.3, and* g^A_β *is the Gibbs measure on* $\mathbb{R}^d\times\mathbb{R}^d$,

$$g^A_\beta(d\xi)=\frac{1}{Z_A}e^{-\beta H_A(\xi)}\,d\xi,\quad H_A(\xi)=\frac{1}{2}(|p|^2+\omega^2|q|^2). \tag{A.13}$$

In Section A.6 we will prove the invariance of the Gibbs measures g_β w.r.t. the group S_t.

Lemma A.9 *The flow* S_t *is mixing w.r.t.* g_β, *i.e., for any functions* $F_1,F_2\in L_2(\mathcal{E}^s_\alpha,g_\beta)$, *we have*

$$\int F_1(S_tY)F_2(Y)g_\beta(dY)\to\int F_1(Y)g_\beta(dY)\int F_2(Y)g_\beta(dY)\quad\text{as }\ t\to\infty.$$

Proof It suffices to check that for any $Z_1, Z_2 \in \mathcal{D}$,

$$\int \langle S_t Y_0, Z_1\rangle \langle Y_0, Z_2\rangle \, g_\beta(dY_0) \to 0, \quad t \to \infty. \tag{A.14}$$

Let $Z_1 = (f, u, v) \in \mathcal{D}$. By Corollary 3.2 and formulas (2.25)–(2.27), we obtain

$$|q_t - \langle W_t \phi_0, \alpha\rangle| + |p_t - \langle W_t \phi_0, \beta\rangle| \le C_1 \varepsilon_m(t)|\xi_0| + C_2 \sqrt{\tilde{\varepsilon}_m(t)} \sup_{\tau\in\mathbb{R}} |\langle W_\tau \phi_0, \nabla\rho_0\rangle|,$$

and

$$|\langle \phi_t, f\rangle - \langle W_t\phi_0, f_*\rangle| \le C_1\varepsilon_m(t)|\xi_0| + C_2\sqrt{\tilde{\varepsilon}_m(t)} \left(\sup_{\tau\in\mathbb{R}} |\langle W_\tau\phi_0, \nabla\rho_0\rangle| + \sup_{\tau\in\mathbb{R}} |\langle W_\tau\phi_0, \alpha\rangle| \right).$$

These bounds can be proved similarly to Proposition 5.1. Hence, to prove (A.14) it suffices to verify that

$$\int \langle \phi_0, W_t'\chi\rangle \langle Y_0, Z_2\rangle \, g_\beta(dY_0) \to 0 \quad \text{as} \quad t \to \infty, \tag{A.15}$$

with $\chi = \alpha, \beta, f_*$. Since

$$F_{x\to k}[W_t' f] = \begin{pmatrix} \cos\omega(k)t & -\omega(k)\sin\omega(k)t \\ \omega^{-1}(k)\sin\omega(k)t & \cos\omega(k)t \end{pmatrix} \begin{pmatrix} \hat{f}^0(k) \\ \hat{f}^1(k) \end{pmatrix}, \tag{A.16}$$

then Definition A.8, equalities (A.5), and the Lebesgue–Riemann theorem imply (A.15). ■

A.5 Effective Hamiltonian

To prove the invariance of the Gibbs measures g_β we use notations introduced by Jakšić and Pillet in [23]. At first, we rewrite the system (3.1)–(3.2) in new variables. Introduce an *effective potential* by

$$V_{eff}(q) = \frac{1}{2}(\omega^2|q|^2 - q\cdot K_m q), \tag{A.17}$$

where K_m is the 'coupling constant matrix' defined in (2.7). By condition **R1'**, $V_{eff}(q) \ge 0$. Define $\mathbb{R}^d$-valued function $h(x)$,

$$h(x) = (\Delta - m^2)^{-1}\nabla\rho(x), \quad x \in \mathbb{R}^d, \tag{A.18}$$

where ρ is the coupled function, and put $h_0 = (h, 0) \in \mathbb{R}^d \times \mathbb{R}^d$. Then the fist equations in (2.1) become

$$\dot{\phi}_t = \mathcal{L}_B\phi_t + q_t\cdot\nabla\rho^0 = \mathcal{L}_B(\phi_t + q_t\cdot h_0), \quad \text{with} \;\; \mathcal{L}_B = \begin{pmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{pmatrix}, \tag{A.19}$$

because $\mathcal{L}_B h_0 = (0, \nabla\rho)$. Define a $\mathbb{R}^2$-valued function $\psi \equiv \psi(x)$, $x \in \mathbb{R}^d$, where

$$\psi = (\psi^0, \psi^1): \quad \psi^0 = \varphi + q\cdot h, \quad \psi^1 = \pi.$$

Then (A.19) becomes $\dot{\psi}_t = \mathcal{L}_B\psi_t + \dot{q}_t\cdot h_0$. Recall that $\mathcal{L}_B$ is the generator of the group W_t. Hence, in new variables (ψ_t, ξ_t) the system (3.1)–(3.2) becomes

$$\begin{aligned} \psi_t &= W_t\psi_0 + \int_0^t W_{t-s}h_0\cdot\dot{q}_s\,ds, \quad x\in\mathbb{R}^d, \quad t\in\mathbb{R}, \\ \ddot{q}_t &= -\nabla V_{eff}(q_t) - \int_0^t \Gamma(t-s)\dot{q}_s\,ds + \mathcal{F}(t), \end{aligned} \tag{A.20}$$

where $\mathcal{F}(t) := \langle \nabla \rho_0, W_t \psi_0 \rangle$, $\nabla V_{eff}(q_t) = (\omega^2 I - K_m) q_t$, the matrix K_m is defined in (2.7), its entries are

$$K_{m,ij} = -\langle \nabla_i \rho_0, h_0^j \rangle = (2\pi)^{-d} \int \frac{k_i k_j |\hat{\rho}(k)|^2}{k^2 + m^2}\, dk,$$

and $\Gamma(t)$ stands for the $\mathbb{R}^d \times \mathbb{R}^d$ matrix with entries $\Gamma_{ij}(t)$,

$$\Gamma_{ij}(t) := -\langle \nabla_i \rho_0, W_t h_0^j \rangle = (2\pi)^{-d} \int k_i k_j \frac{\cos \omega(k) t}{\omega^2(k)} |\hat{\rho}(k)|^2\, dk, \quad i, j = 1, \ldots, d. \tag{A.21}$$

The equation (A.20) is called the *generalized or retarded Langevin equation* with the *random force* $\mathcal{F}(t)$ and with the *memory kernel* $\Gamma(t)$.

Remark A.10 (i) By (A.21), we have $\Gamma(0) = K_m$. Moreover, $\dot{\Gamma}_{ij}(t) = -D_{ij}(t)$, where $D_{ij}(t)$ are the entries of the matrix $D(t)$ defined in (3.4).
(ii) $\rho_0(x) = h_0(x) = 0$ for $|x| \geq R_\rho$, by condition **R2**. Hence, $\Gamma(t) = 0$ for $|t| > 2R_\rho$ if $m = 0$ due to a *strong* Huyghen's principle, and $|\Gamma(t)| \leq C(1 + |t|)^{-d/2}$ if $m \neq 0$.
(iii) It follows from (A.5), (A.16) and (A.21) that $\int (\mathcal{F}(t) \otimes \mathcal{F}(s))\, g_\beta^B(d\psi) = (1/\beta)\, \Gamma(t - s)$ (*fluctuation–dissipation* relation).
(iv) The force $\mathcal{F}(t)$ equals $F(t) - \Gamma(t) q_0$ with $F(t)$ from (3.2).

Introduce an *effective Hamiltonian* $H_A^{eff}(\xi) = |p|^2/2 + V_{eff}(q)$. Hence, by (1.1),

$$H(\phi, \xi) = H_B(\psi) + H_A^{eff}(\xi).$$

Definition A.11 *(i) Define a map* $\mathbf{T}$ *on* $\mathcal{E}_\alpha^s$ *by the rule*

$$\mathbf{T} : (\phi, \xi) \to (\psi, \xi), \quad \psi = \phi + q \cdot h_0.$$

(ii) Denote $g_\beta^{\mathbf{T}}(d\psi d\xi) := g_\beta(\mathbf{T}^{-1}(d\psi d\xi))$. *Then,* $g_\beta^{\mathbf{T}}(d\psi d\xi) = g_\beta^B(d\psi) \times g_\beta^{eff}(d\xi)$, *where* g_β^B *is defined in Definition A.3,* g_β^{eff} *is a Gaussian measure defined by* $g_\beta^{eff}(d\xi) = (1/Z) e^{-\beta H_A^{eff}(\xi)} d\xi$.

A.6 Invariance of Gibbs measures g_β

Proposition A.12 *Let conditions* **R1** *and* **R2** *hold. Then the Gibbs measures* g_β, $\beta > 0$, *are invariant with respect to the dynamics, i.e.*

$$S_t^* g_\beta(\omega) := g_\beta(S_t^{-1} \omega) = g_\beta(\omega), \quad \textit{for } \omega \in \mathcal{B}(\mathcal{E}_\alpha^s) \quad \textit{and } t \in \mathbb{R}. \tag{A.22}$$

Here $\mathcal{B}(\mathcal{E}_\alpha^s)$ *is the Borelian* σ*-algebra of subsets in* $\mathcal{E}_\alpha^s$.

Proof For simplicity, we omit indices α, s in notations $\mathcal{E}_\alpha^s$ and $\mathcal{H}_\alpha^s$. The invariance (A.22) is equivalent to the identity:

$$\frac{d}{dt} \int_{\mathcal{E}} F(S_t Y) g_\beta(dY) = 0, \quad t \in \mathbb{R}, \tag{A.23}$$

for any bounded continuous functional $F(Y)$ on $\mathcal{E}$, i.e., $F(Y) \in C_b(\mathcal{E})$. It suffices to prove (A.23) with $t = 0$ only. Indeed, since $S_{t+\tau} = S_t\, S_\tau$, we have

$$\frac{d}{d\tau} \int_{\mathcal{E}} F(S_\tau S_t Y) g_\beta(dY) = \frac{d}{d\tau} \int_{\mathcal{E}} F(S_{\tau+t} Y) g_\beta(dY) = \frac{d}{dt} \int_{\mathcal{E}} F(S_t S_\tau Y) g_\beta(dY). \tag{A.24}$$

Let $\frac{d}{d\tau}\int_{\mathcal{E}} F(S_\tau Y)g_\beta(dY)\Big|_{\tau=0} = 0$. Since $F(S_tY)\in C_b(\mathcal{E})$, then $\frac{d}{d\tau}\int_{\mathcal{E}} F(S_\tau S_t Y)g_\beta(dY)\Big|_{\tau=0} = 0$ for any fixed $t\in\mathbb{R}$. Hence, (A.24) implies

$$0 = \frac{d}{d\tau}\int_{\mathcal{E}} F(S_\tau S_t Y)g_\beta(dY)\Big|_{\tau=0} = \frac{d}{dt}\int_{\mathcal{E}} F(S_t Y)g_\beta(dY),$$

and (A.23) follows. Moreover, it suffices to verify (A.23) with $t=0$ and $F(Y)=\exp(i\langle Y,Z\rangle)$ for every $Z=(f_0(x),f_1(x),u,v)\in\mathcal{D}$. Then, by (2.4), identity (A.23) with $t=0$ becomes

$$\frac{d}{dt}\int_{\mathcal{E}} e^{i\langle S_tY,Z\rangle}g_\beta(dY)\Big|_{t=0} = \int_{\mathcal{E}} e^{i\langle Y,Z\rangle} i\langle \mathcal{L}(Y),Z\rangle g_\beta(dY) = 0, \tag{A.25}$$

where

$$\mathcal{L}(\varphi,\pi,q,p) = (\pi, (\Delta-m^2)\varphi + q\cdot\nabla\rho, p, -\omega^2 q + \langle\nabla\rho,\varphi\rangle). \tag{A.26}$$

Now we prove (A.25). Denote by I the integral

$$I := \int_{\mathcal{E}} e^{i\langle Y,Z\rangle} i\langle\mathcal{L}(Y),Z\rangle g_\beta(dY),$$

and check that $I=0$. Definition A.11 implies $g_\beta(dY) = g_\beta^{\mathbf{T}}(\mathbf{T}dY)$. Hence,

$$\int\limits_{\mathcal{E}} F(Y)\, g_\beta(dY) = \int\limits_{\mathbb{R}^{2d}} g_\beta^{eff}(d\xi)\int\limits_{\mathcal{H}} F(\psi - q\cdot h_0,\xi)\, g_\beta^B(d\psi). \tag{A.27}$$

Using (A.26), (A.27), and (A.18), we rewrite I in the form

$$\begin{aligned} I \;=\; & \int\limits_{\mathbb{R}^{2d}} e^{i(u\cdot q+v\cdot p)} g_\beta^{eff}(d\xi)\int\limits_{\mathcal{H}} e^{i\langle\psi^0 - q\cdot h, f_0\rangle + i\langle\psi^1,f_1\rangle}\Big(i\langle\psi^1,f_0\rangle + i\langle(\Delta-m^2)\psi^0,f_1\rangle \\ & + iu\cdot p + iv\cdot[-\omega^2 q + \langle\nabla\rho,\psi^0 - q\cdot h\rangle]\Big) g_\beta^0(d\psi^0)g_\beta^1(d\psi^1). \end{aligned}$$

Integrals over Gaussian measures $g_\beta^0(d\psi^0)$ and $g_\beta^1(d\psi^1)$ can be represented as variational derivatives of their characteristic functionals $\hat g_\beta^0(f_0)$ and $\hat g_\beta^1(f_1)$:

$$\int e^{i\langle\psi,f\rangle} i\langle\psi,\cdot\rangle g_\beta^i(d\psi) = \langle\frac{\delta}{\delta f}\,\hat g_\beta^i(f),\cdot\rangle,\quad i=0,1,\quad f\in C_0^\infty(\mathbb{R}^d).$$

Then

$$\begin{aligned} I \;=\; & \int\limits_{\mathbb{R}^{2d}} e^{i(u\cdot q+v\cdot p)} e^{-iq\cdot\langle h,f_0\rangle}\Big(\langle\frac{\delta}{\delta f_1},f_0\rangle + \langle(\Delta-m^2)\frac{\delta}{\delta f_0},f_1\rangle \\ & + iu\cdot p + v\cdot\Big[-i\omega^2 q + \langle\frac{\delta}{\delta f_0},\nabla\rho\rangle - i\langle\nabla\rho,q\cdot h\rangle\Big]\Big)\hat g_\beta^0(f_0)\hat g_\beta^1(f_1)\, g_\beta^{eff}(d\xi). \end{aligned} \tag{A.28}$$

Using (A.6), we calculate

$$\left.\begin{aligned} \langle\frac{\delta}{\delta f}\,\hat g_\beta^0(f),\cdot\rangle &= -\frac{1}{\beta}\,e^{-\frac{1}{2\beta}\langle(-\Delta+m^2)^{-1}f,f\rangle}\langle(-\Delta+m^2)^{-1}f,\cdot\rangle \\ \langle\frac{\delta}{\delta f}\,\hat g_\beta^1(f),\cdot\rangle &= -\frac{1}{\beta}\,e^{-\frac{1}{2\beta}\langle f,f\rangle}\langle f,\cdot\rangle \end{aligned}\right|\quad f\in C_0^\infty(\mathbb{R}^d).$$

Therefore, we reduce (A.28) to the following integral

$$\begin{aligned} I &= C \int\limits_{\mathbb{R}^{2d}} e^{i(u\cdot q+v\cdot p)} e^{-iq\cdot\langle h,f_0\rangle} \Big(iu\cdot p - iv\cdot\nabla V_{eff}(q) + \frac{1}{\beta}\, v\cdot\langle f_0,h\rangle\Big) e^{-\beta H_A^{eff}(\xi)}\, d\xi \\ &= C_1 \int\limits_{\mathbb{R}^{2d}} e^{i(u\cdot q+v\cdot p)} (u\cdot\nabla_p - v\cdot\nabla_q)\Big[e^{-iq\cdot\langle h,f_0\rangle - \beta H_A^{eff}(q,p)}\Big]\, dq\, dp, \end{aligned}$$

by (A.17) and (A.18). Partial integration in q and in p leads to

$$I = C_2 \int\limits_{\mathbb{R}^{2d}} e^{i(u\cdot q+v\cdot p)} (-u\cdot(iv) + v\cdot(iu)) e^{-iq\cdot\langle h,f_0\rangle - \beta H_A^{eff}(q,p)}\, dqdp = 0. \quad \blacksquare$$

Appendix B: Existence of solutions

Proposition 2.2 can be proved by using the methods of [25, Lemma 6.3]. In this section, we outline the proof of this proposition.

Proof of Lemma 2.3. *Step (i)* If $\rho = 0$, then the existence and uniqueness of the solution $Y_t \in C(\mathbb{R}, E)$ to problem (2.4) is well-known (see, for example, [29]). Represent the solution Y_t as the pair of the functions (Y_t^0, Y_t^1), where $Y_t^0 = (\varphi_t, q_t)$, $Y_t^1 = (\pi_t, p_t)$. Therefore, problem (2.4) for $Y_t \in C(\mathbb{R}, E)$ is equivalent to

$$Y_t = e^{\mathcal{L}_0 t} Y_0 + \int\limits_0^t e^{\mathcal{L}_0(t-s)} B Y_s\, ds, \tag{B.1}$$

where $Y_0 = (\varphi_0, q_0, \pi_0, p_0) \in E = H_F^1(\mathbb{R}^3) \otimes \mathbb{R}^3 \otimes L^2(\mathbb{R}^3) \otimes \mathbb{R}^3$,

$$\left.\begin{array}{l} \mathcal{L}_0 = \begin{pmatrix} 0 & I \\ \mathcal{A}_0 & 0 \end{pmatrix}, \qquad \mathcal{A}_0 \begin{pmatrix} \varphi \\ q \end{pmatrix} = \begin{pmatrix} L_B\varphi \\ -\omega^2 q \end{pmatrix} \\ B(Y^0, Y^1) = (0, RY^0), \quad RY^0 := \Big(q\cdot\nabla\rho, \langle\varphi, \nabla\rho\rangle\Big) \end{array}\right| \quad \text{for} \ \ Y^0 = (\varphi, q), \tag{B.2}$$

(cf (7.4)). Note that $\|e^{\mathcal{L}_0 t} Y_0\|_E \le C\|Y_0\|_E$; and the second term in (B.1) is estimated by

$$\sup_{|t|\le T} \| \int\limits_0^t e^{\mathcal{L}_0(t-s)} B Y_s\, ds\|_E \le C\, T \sup_{|s|\le T} \|Y_s\|_E.$$

This bound and the contraction mapping principle imply the existence and uniqueness of the local solution $Y_t \in C([-\varepsilon, \varepsilon], E)$ for some $\varepsilon > 0$.

Step (ii) To prove the energy conservation

$$H(Y_t) = H(Y_0) \ \text{ for } \ t \in \mathbb{R}, \tag{B.3}$$

we first assume that $\phi_0 = (\varphi_0, \pi_0) \in C^3(\mathbb{R}^3) \times C^2(\mathbb{R}^3)$ and $\phi_0(x) = 0$ for $|x| \ge R_0$. Then $\varphi_t(x) \in C^2(\mathbb{R}_x^3 \times \mathbb{R}_t)$ and

$$\varphi_t(x) = 0 \quad \text{for} \ \ |x| \ge |t| + \max\{R_0, R_a, R_\rho\}$$

by the integral representation (B.1) and conditions **A2** and **R2**. Therefore, for such initial data, relation (B.3) can be proved by integrating by parts. Hence, for $Y_0 \in E$, (B.3) follows from the continuity of S_t and from the fact that $C_0^3(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus C_0^2(\mathbb{R}^3) \oplus \mathbb{R}^3$ is dense in E.

Step (iii) In the case of WF, we apply condition **A3** and obtain

$$\frac{1}{2}\int\Big(\sum_{ij}\nabla_i\varphi(x)a_{ij}(x)\nabla_j\varphi(x)+2q\cdot\nabla\varphi(x)\rho(x)\Big)\,dx$$
$$\geq \frac{1}{2}\int\Big(\alpha|\nabla\varphi(x)|^2+2q\cdot\nabla\varphi(x)\rho(x)\Big)\,dx=\frac{\alpha}{2}\|\nabla\varphi+\frac{q\rho}{\alpha}\|^2-\frac{1}{2\alpha}|q|^2\|\rho\|^2,$$

where $\|\cdot\|$ stands for the norm in L^2. In the case of KGF,

$$\frac{1}{2}\int\Big(m^2|\varphi(x)|^2-2\varphi(x)q\cdot\nabla\rho(x)\Big)\,dx\geq\frac{1}{2}\,m^2\|\varphi-\frac{q\cdot\nabla\rho}{m^2}\|^2-\frac{1}{2m^2}|q|^2\|\nabla\rho\|^2.$$

Hence, the Hamiltonian functional $H(Y)$ is nonnegative. Indeed, in the case of WF,

$$\begin{aligned}H(Y) \;&\geq\; \frac{1}{2}\int\Big(|\pi(x)|^2+\alpha\Big|\nabla\varphi(x)+\frac{q\rho(x)}{\alpha}\Big|^2+a_0(x)|\varphi(x)|^2\Big)\,dx\\&\quad+\frac{1}{2}\Big(\omega^2-\frac{1}{\alpha}\|\rho\|^2\Big)|q|^2+\frac{1}{2}|p|^2\geq 0\end{aligned}\tag{B.4}$$

by condition **R1**. In the case of KGF,

$$\begin{aligned}H(Y) \;&\geq\; \frac{1}{2}\int\Big(|\pi(x)|^2+\sum_j|(\nabla_j-iA_j(x))\varphi(x)|^2+m^2\Big|\varphi(x)-\frac{q\cdot\nabla\rho(x)}{m^2}\Big|^2\Big)\,dx\\&\quad+\frac{1}{2}\Big(\omega^2-\frac{1}{m^2}\|\nabla\rho\|^2\Big)|q|^2+\frac{1}{2}|p|^2\geq 0\end{aligned}\tag{B.5}$$

by condition **R1**. Moreover, by (B.3), (B.4) and (B.5), we obtain

$$\|Y_t\|_E^2\leq C\,H(Y_t)=C\,H(Y_0).\tag{B.6}$$

On the other hand, in the case of KGF, we have

$$\begin{aligned}H(Y) \;&\leq\; \frac{1}{2}\Big\{\sum_j\|(\nabla_j-iA_j(x))\varphi\|^2+\|\nabla\varphi\|^2+\|\pi\|^2+m^2\|\varphi\|^2+(\omega^2+\|\rho\|^2)|q|^2+|p|^2\Big\}\\&\leq\; C\|Y\|_E^2,\end{aligned}\tag{B.7}$$

since $|q\cdot\langle\nabla\varphi,\rho\rangle|\leq(\|\nabla\varphi\|^2+|q|^2\|\rho\|^2)/2$. In the WF case,

$$H(Y)\leq C\left(\|\nabla\varphi\|^2+\|\pi\|^2+(\omega^2+\|\rho\|^2)|q|^2+|p|^2+\int a_0(x)|\varphi(x)|^2\,dx\right).$$

Since $Y\in E$, $\varphi\in H_F^1$. For the WF case, H_F^1 is the completion of real space $C_0^\infty(\mathbb{R}^3)$ with the norm $\|\nabla\varphi\|$. Therefore, $H_F^1=\{\varphi\in L^6(\mathbb{R}^3):\,|\nabla\varphi|\in L^2\}$ by Sobolev's embedding theorem. Hence,

$$\int a_0(x)|\varphi(x)|^2\,dx\leq C\|\varphi\|_{L^6}^2\leq C_1\|\nabla\varphi\|^2.$$

Using (B.6) and (B.7), we obtain the *a priori* estimate

$$\|Y_t\|_E \le C_1\|Y_0\|_E \quad \text{for} \;\; t\in\mathbb{R}. \tag{B.8}$$

Therefore, properties (i)–(iii) of Lemma 2.3 for arbitrary $t\in\mathbb{R}$ follow from bound (B.8). ■

We return to the proof of Proposition 2.2. Let us choose $R>\max\{R_a,R_\rho\}$ with R_a and R_ρ from conditions **A2** and **R2**. Then, by the integral representation (B.1), the solution Y_t for $|x|<R$ depends only on the initial data $Y_0(x)$ with $|x|<R+|t|$. Thus, the continuity of S_t in $\mathcal{E}$ follows from the continuity in E.

For every $R>0$, define the local energy seminorms by

$$\|Y\|^2_{E(R)} := \int\limits_{|x|<R} \Big(|\nabla\varphi(x)|^2+m^2|\varphi(x)|^2+|\pi(x)|^2\Big)\,dx+|q|^2+|p|^2, \quad Y=(\varphi,\pi,q,p),$$

where $m>0$ for the KGF case, and $m=0$ for the WF case. By estimate (B.8), we obtain the following local energy estimates:

$$\|S_tY_0\|^2_{E(R)} \le C\|Y_0\|^2_{E(R+|t|)} \quad \text{for} \;\; R>\max\{R_\rho,R_a\} \quad \text{and} \;\; t\in\mathbb{R}.$$

Hence, for any $T>0$ and $R>\max\{R_\rho,R_a\}$,

$$\sup_{|t|\le T}\|S_tY_0\|_{\mathcal{E},R} \le C(T)\|Y_0\|_{\mathcal{E},R+T}. \quad ■$$

References

[1] Appleby, J.A. and Freeman, A., "Exponential asymptotic stability of linear Itô–Volterra equations with damped stochastic perturbations," Electronic J. of Probability **8**, 1–22 (2003).

[2] Boldrighini, C., Pellegrinotti, A., and Triolo, L., "Convergence to stationary states for infinite harmonic systems," J. Stat. Phys. **30**, 123-155 (1983).

[3] Corduneanu, C. and Lakshmikantham, V., "Equations with unbounded delay: a survey," Nonlinear Analysis, TMA **4** (5), 831-877 (1980).

[4] Driver, R.D., *Ordinary and Delay Differential Equations* (Springer–Verlag, New York, 1977).

[5] Dudnikova, T.V. and Komech, A.I., "Ergodic properties of hyperbolic equations with mixing," Theory Probab. Appl. **41** (3), 436-448 (1996).

[6] Dudnikova, T.V., Komech, A.I., Kopylova, E.A., and Suhov, Yu.M., "On convergence to equilibrium distribution, I. The Klein-Gordon equation with mixing," Commun. Math. Phys. **225** (1), 1-32 (2002). e-print: ArXiv:math-ph/0508042.

[7] Dudnikova, T.V., Komech, A.I., Ratanov, N.E. and Suhov, Yu.M., "On convergence to equilibrium distribution, II. The wave equation in odd dimensions, with mixing," J. Stat. Phys. **108** (4), 1219-1253 (2002). e-print: ArXiv:math-ph/0508039.

[8] Dudnikova, T.V., Komech, A.I., and Spohn, H., "On a two-temperature problem for wave equation," Markov Processes and Related Fields **8**, 43-80 (2002). e-print: ArXiv:math-ph/0508044.

[9] Dudnikova, T.V., Komech, A.I., and Spohn, H., "On the convergence to statistical equilibrium for harmonic crystals," J. Math. Phys. **44** (6), 2596-2620 (2003). ArXiv: math-ph/0210039.

[10] Dudnikova, T.V. and Komech, A.I., "On a two-temperature problem for the Klein-Gordon equation," Teor. Veroyatn. Ee Primen. **50**, 675-710 (2005) [Russian] (English translation: Theory Prob. Appl. **50** (4), 582-611 (2006)).

[11] Dudnikova, T.V. and Komech, A.I., "On the convergence to a statistical equilibrium in the crystal coupled to a scalar field," Russian J. Math. Phys. **12** (3), 301-325 (2005). e-print: ArXiv:math-ph/0508053.

[12] Dudnikova, T.V., "Convergence to equilibrium distribution. The Klein-Gordon equation coupled to a particle," Russian J. Math. Phys. **17** (1), 77-95 (2010). e-print: arXiv:0711.1091.

[13] Egorov, Yu.V., Komech, A.I., and Shubin, M.A., *Elements of the Modern Theory of Partial Differential Equations* (Springer, Berlin, 1999).

[14] Eckmann, J.-P., Pillet, C.-A., and Rey-Bellet, L., "Non-equilibrium statistical mechanics of anharmonic chains coupled to two heat baths at different temperatures," Commun. Math. Phys. **201**, 657-697 (1999).

[15] Gel'fand, I.M. and Shilov, G.E., *Generalized Functions. Vol.I: Properties and Operations* (Academic Press, New York, 1964).

[16] Gripenberg, G., Londen, S.-O., and Staffans, O., *Volterra Integral and Functional Equations*, vol. 34, in: *Encyclopedia of Mathematics and its Applications* (Cambridge University Press, Cambridge, 1990).

[17] Grossman, S.I. and Miller, R.K., "Nonlinear Volterra integrodifferential systems with L^1-kernels," J. Differential Equations **13**, 551–566 (1973).

[18] Hara, T., "Exponential asymptotic stability for Volterra integrodifferential equations of nonconvolution type," Funkcialaj Ekvacioj **37**, 373-382 (1994).

[19] Hörmander, L., *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators* (Springer-Verlag, 1985).

[20] Ibragimov, I.A. and Linnik, Yu.V., *Independent and Stationary Sequences of Random Variables* (Wolters-Noordhoff, Groningen, 1971).

[21] Imaikin, V., Komech, A., and Vainberg, B., "On scattering of solitons for the Klein–Gordon equation coupled to a particle," Comm. Math. Phys. **268** (3), 321-367 (2006).

[22] Jakšić, V. and Pillet, C.-A., "Ergodic properties of the non-Markovian Langevin equation," Lett. Math. Phys. **41**(1), 49-57 (1997).

[23] Jakšić, V. and Pillet, C.-A., "Ergodic properties of classical dissipative systems. I," Acta Math. **181** (2), 245-282 (1998).

[24] Karczewska, A., "Convolution type stochastic Volterra equations," - Torun: Juliusz Schauder Center for Nonlinear Studies; Nicolaus Copernicus Univ. (2007).– (Lecture Notes in Nonlinear Analysis **10**). e-print: arXiv:0712.4357 (2007).

[25] Komech, A., Spohn, H., and Kunze, M., "Long-time asymptotics for a classical particle interacting with a scalar wave field," Comm. Partial Diff. Equations **22**, 307-335 (1997).

[26] Kordonis, I.-G.E. and Philos, Ch.G., "The behavior of solutions of linear integro-differential equations with unbounded delay," Computers and Mathematics with Appl. **38**, 45–50 (1999).

[27] Mao, X., "Stability of stochastic integro-differential equations," Stochastic Analysis and Appl. **18** (6), 1005–1017 (2000).

[28] Mao, X. and Riedle, M., "Mean square stability of stochastic Volterra integro-differential equations," Systems & Control Letters **55**, 459–465 (2006).

[29] Mikhailov, V.P., *Partial Differential Equations* (Mir, Moscow, 1978).

[30] Murakami, S., "Exponential asymptotic stability for scalar linear Volterra equations," Differential and Integral Equations **4**, 519–525 (1991).

[31] Myshkis, A.D., *Linear Differential Equations with Retarded Argument* (Russian, 2-nd edition, Nauka, Moscow, 1972).

[32] Ottobre, M. and Pavliotis, G.A., "Asymptotic analysis for the generalized Langevin equation," Nonlineartity, **24**(5), 1629-1653 (2011).

[33] Rey-Bellet, L. and Thomas, L.E., "Exponential convergence to non-equilibrium stationary states in classical statistical mechanics," Commun. Math. Phys. **225**, 305-329 (2002).

[34] Rosenblatt, M.A., "A central limit theorem and a strong mixing condition," Proc. Nat. Acad. Sci. U.S.A. **42** (1), 43-47 (1956).

[35] Snook, I., *The Langevin and Generalized Langevin Approach to the Dynamics of Atomic, Polymeric and Colloidal Systems* (Elsevier, 2006).

[36] Spohn, H. and Lebowitz, J., "Stationary non-equilibrium states of infinite harmonic systems," Comm. Math. Phys. **54** (2), 97-120 (1977).

[37] Vainberg, B.R., "Behavior of the solution of the Cauchy problem for a hyperbolic equation as $t \to \infty$," Math. of the USSR-Sbornik **7** (4), 533-568 (1969); trans. Mat. Sb. **78** (4), 542-578 (1969).

[38] Vainberg, B.R., "Behaviour for large time of solutions of the Klein-Gordon equation," Trans. Moscow Math. Soc. **30**, 139-158 (1974).

[39] Vainberg, B.R., *Asymptotic Methods in Equations of Mathematical Physics* (Gordon and Breach, New York, 1989).

[40] Vishik, M.I. and Fursikov, A.V., *Mathematical Problems of Statistical Hydromechanics* (Kluwer Academic Publishers, 1988).

[41] Zwanzig, R., "Nonlinear generalized Langevin equations," J. Stat. Phys. **9**, 215-220 (1973).