STABILITY RESULTS FOR GABOR FRAMES AND THE p-ORDER HOLD MODELS.

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ABSTRACT. We prove stability results for a class of Gabor frames in $L^2(\mathbb{R})$. We consider window functions in the Sobolev spaces $H^1_0(\mathbb{R})$ and B-splines of order $p \geq 1$. Our results can be used to describe the effect of the timing jitters in the p-order hold models of signal reconstruction.

1. INTRODUCTION

In this paper we prove stability results for Gabor frames and bases of $L^2(\mathbb{R})$ that are relevant in electronics and communication theory. A *Gabor system* in $L^2(\mathbb{R})$ is a collection of functions $\mathcal{G} = \{e^{2\pi i b_n x} g(x - a_k)\}_{n,k\in\mathbb{Z}}$, where g (the *window function*) is a fixed function in $L^2(\mathbb{R})$ and $a_k,\ b_n \in \mathbb{R}$.

If $\mathcal{G}$ is *regular* , i.e., if $(a_k, b_n) = (ak, bn)$ for some $a, b > 0$, we let $\mathcal{G}(g, a, b) = \{e^{2\pi binx} g(x - ak)\}_{n,k\in\mathbb{Z}}$.

Gabor systems have had a fundamental impact on the development of modern time-frequency analysis and have been widely used in all branches of pure and applied sciences.

An important problem is to determine general and verifiable conditions on the window function g, the *time sampling* $\{a_k\}$ and the *frequency sampling* $\{b_n\}$ which imply that a Gabor system is a frame. In the regular case many necessary and sufficient conditions on g, a and b are known (see e.g. Christensen (2003) and the references cited there). An early article by Gröchenig Gröchenig (1993) provided some partial sufficient conditions for the existence of irregular Gabor frames. See also Feichtinger and Sun (2006) and Balan et al. (2006) and the articles cited in these papers.

Given a regular Gabor frame $\mathcal{G}(g, a, b)$, it is important to determine *stability bounds* $\delta_{n,k} > 0$ so that each set $\mathcal{F} = \{e^{2\pi i b\lambda_{n,k} x} g(x - a\mu_{n,k})\}_{n,k\in\mathbb{Z}}$ is a frame whenever $|\lambda_{n,k} - n| + |\mu_{n,k} - k| < \delta_{n,k}$. The main results of our paper concern the stability of Gabor frames $\mathcal{G}(\mathrm{rect}^{(p)}, a, b)$ where $\mathrm{rect}(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$ is the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$ and $\mathrm{rect}^{(p)}(x) = \mathrm{rect} * ... * \mathrm{rect}(x)$ is the $p-$times iterated convolution of $\mathrm{rect}(x)$. The function $\mathrm{rect}^{(p)}(x)$ is a piecewise polynomial function of degree $p - 1$ and a prime example of *B-spline* of order $p - 1$. See Schoenberg (1969), Prautzsch et al. (2002), Unser et al. (1993) and the references cited there.

Our investigation is motivated by the study of the timing jitter effect in $p-$order hold (pOH) devices. The pOH devices are used to transform a sequence of impulses $\{q_n\}$ originating from a continuous-time signal $f(t)$ into a piecewise polynomial function $f_p(t)$. The impulses are assumed to be evenly spaced, i.e., $q_n = f(Tn)$ for some $T > 0$, but in the presence of timing jitter we have instead $q_n = f(T(n \pm \epsilon_n))$ for some $\epsilon_n > 0$. It is natural to investigate whether $f(t)$ can be effectively reconstructed from the sequence $f(T(n \pm \epsilon_n))$.

It is proved Daubechies (1992), (but see also (Heil, 2011, chapt. 11)) that the condition $0 < ab \leq 1$ is necessary for a Gabor system $\mathcal{G}(g, a, b)$ to be a frame, so we will always assume (often without saying) that $ab \leq 1$. When $\mathcal{G}(g, a, b)$ is a *Riesz basis*, i.e., it is the image of an

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orthonormal basis of $L^2(\mathbb{R})$ through a linear, invertible and bounded transformation, we have $ab = 1$.

We consider sets of coefficients $\{\mu_{n,k}\}_{n,k\in\mathbb{Z}} \subset \mathbb{R}$, with

$$\boxed{L_n = \sup_{k\in\mathbb{Z}} |\mu_{n,k} - k| < 1; \quad L = \sum_{n\in\mathbb{Z}} L_n < \infty.} \tag{1}$$

The assumption $L_n < 1$ is made to simplify the statement of our result, but it is not necessary in the proofs.

We prove first a stability result for the frame $\mathcal{G}(\mathrm{rect}, a, b)$.

Theorem 1.1. *Assume* $0 < a \le 1$, *and* $4abL < 1$. *The set*

$$\mathcal{F} = \left\{e^{2\pi ibnt}\,\mathrm{rect}\,(a\mu_{n,k} - t)\right\}_{k,n\in\mathbb{Z}}$$

is a frame of $L^2(\mathbb{R})$ *with bounds* $A \ge (1 - 2(abL)^{\frac{1}{2}})^2$ *and* $B \le (1 + 2(abL)^{\frac{1}{2}})^2$. *In particular,* $\mathcal{F}$ *is a Riesz basis if* $ab = 1$ *and* $0 \le L < \frac{1}{4}$.

It is easy to verify that $\mathcal{G}(\mathrm{rect}, a, b)$ is not a frame when $a > 1$; indeed, all functions in $\mathcal{G}(\mathrm{rect}, a, b)$ vanish on the intervals $\{(ak + \frac{1}{2},\ a(k+1) - \frac{1}{2})\}_{k\in\mathbb{Z}}$, and so $\mathcal{G}(\mathrm{rect}, a, b)$ is not complete.

Observe also that if the coefficients $\mu_{n,k}$ in the definition of $\mathcal{F}$ are bounded below by a positive δ_k independent of n, then $\mathcal{F}$ is not a frame. For example, let $a = b = 1$ and let $\mu_{n,k} = k$ if $k \ne 0$ and $\mu_{n,0} = d$, where $d > 0$ is fixed. All functions in the set $\mathcal{F} = \left\{e^{2\pi int}\,\mathrm{rect}\,(k - t)\right\}_{\substack{k\ne 0\\ n\in\mathbb{Z}}} \cup \left\{e^{2\pi int}\,\mathrm{rect}\,(d - t)\right\}_{n\in\mathbb{Z}}$ vanish on the interval $(-\frac{1}{2}, -d)$ and so $\mathcal{F}$ is not complete.

These considerations are not new. See e.g. (Heil, 2011, Chapt 11) for similar observations.

Theorem 1.2 below generalizes Theorem 1.1.

Theorem 1.2. *Let* $p \ge 1$ *be an integer; let* $0 < a \le 1$ *when* $p = 1$ *and* $0 < a < p$ *when* $p \ge 2$. *If* $4abL < \left(\mathrm{rect}^{(p)}\left(\frac{a}{2}\right)\right)^2$, *the set* $\mathcal{F} = \left\{e^{2\pi ibnt}\,\mathrm{rect}^{(p)}(t - a\mu_{k,n})\right\}_{n,k\in\mathbb{Z}}$ *is a frame with bounds* $A \ge (1 - \frac{2(abL)^{\frac{1}{2}}}{\mathrm{rect}^{(p)}\left(\frac{a}{2}\right)})^2$ *and* $B \le (1 + \frac{2(abL)^{\frac{1}{2}}}{\mathrm{rect}^{(p)}\left(\frac{a}{2}\right)})^2$.

We also consider window functions in the Sobolev space $H_0^1(I)$, where I is an interval of $\mathbb{R}$. We recall that $H^1(I)$ is the space of functions in $L^2(I)$ whose distributional derivative is also in $L^2(I)$ and that $H_0^1(I)$ is the closure of $C_0^\infty(I)$ in $H^1(I)$. We also recall that functions in $H^1(I)$ are continuous. See e.g. Brezis (2011) for definitions and results on Sobolev spaces.

We prove the following

Theorem 1.3. *Let* $\psi \in H_0^1(-\frac{p}{2}, \frac{p}{2})$, *with* $p > a > 0$. *Let* ψ' *be the distributional derivative of* ψ. *Assume that* $0 < m \le \psi(x) \le M$ *in* $[-\frac{a}{2}, \frac{a}{2}]$. *Then,*

- $\mathcal{G}(\psi, a, b)$ *is a frame with bounds* $A \ge \frac{m^2}{b}$, $B \le [\frac{p}{a}]\frac{M^2}{b}$.
- *Let* $\mathcal{F} = \left\{e^{2\pi ibnt}\psi\,(a\mu_{n,k} - t)\right\}_{k,n\in\mathbb{Z}}$ *with* $\{\mu_{n,k}\}_{n,k\in\mathbb{Z}} \subset \mathbb{R}$ *as in* (1). *If*

$$C^2 = ba^2||\psi'||_2^2 \sum_{s\in\mathbb{Z}} \left[\frac{p}{a} + L_s\right] L_s^2 < m^2,$$

$\mathcal{F}$ *is a frame of* $L^2(\mathbb{R})$ *with bounds* $A \ge \left(1 - \frac{C}{m}\right)^2$ *and* $B \le \left(1 + \frac{C}{m}\right)^2$

The following is a short (and most likely incomplete) survey of results related to our work. In Feichtinger and Kaiblinger (2004) the stability of Gabor frames is tested under perturbation of the lattice constants a and b. In Favier and Zalik (1995) the stability of Gabor frames and bases $\mathcal{G}(\phi, a, b)$ under some perturbation of the sampling sequence $\{ak\}_{k\in\mathbb{Z}}$ is discussed. See also Christensen (1996). The assumptions of Theorem 16 in Favier and Zalik (1995) do not apply to the frames that we have considered in this paper. Some results in Favier and Zalik (1995) have been improved by W. Sun and X. Zhou in Sun and Zhou (2001). A proof of the main theorem in Sun and Zhou (2001) is in (Christensen, 2003, Theorem 15.4.3). In this theorem the window function g is continuously differentiable while in our Theorem 1.3 we consider functions in $H^1(I)$.

The stability of Gabor frames with irregular sampling points is considered in Feichtinger and Sun (2006), Sun and Zhou (2003). It is also worth mentioning that in (Christensen, 2003, Theorem 15.4.1) and in (Christensen, 2003, Corollary 15.4.2), the stability of Gabor frame under perturbation of the window function g is discussed.

We prove Theorems 1.1 – 1.3 in Section 3. In Section 2 we recall definitions and preliminary results and we prove some useful Lemmas. In Section 4 we prove corollaries and a generalization of Theorem 1.1 in dimension $d > 1$. We have described the p-order hold models in the Appendix.

2. Preliminary

We refer the reader to the the excellent textbooks Heil (2011) and Young (2001) for the definitions of frame and Riesz bases and preliminary results. See also Christensen (2003), Gröchenig (2001).

We recall from (Heil, 2011, Chapt. 11) that a Gabor system $\mathcal{G}(g, a, b)$ is not a frame if $ab > 1$. If $\mathcal{G}(g, a, b)$ is a frame with bounds A and B, then

$$Ab \le \sum_{k\in\mathbb{Z}} |g(x + ak)|^2 \le Bb \tag{2}$$

holds almost everywhere (a.e.) in $\mathbb{R}$.

When g is supported in an interval of measure $\ge a$ and $0 < ab \le 1$, the set $\mathcal{G}(g, a, b)$ is a frame of $L^2(\mathbb{R})$ with constants A and B if and only if (2) is satisfied; if $\mathcal{G}(g, a, b)$ is a frame and $ab = 1$, then it is a Riesz basis. It follows from (2) that if g is continuous with compact support in $\mathbb{R}$ and $\mathcal{G}(g, a, b)$ is a frame, the translates of g must overlap. Indeed, it is easy to verify that if $g \in C(\mathbb{R})$ is supported in the interval $(x_0, x_0 + \frac{1}{b})$ for some $x_0 \in \mathbb{R}$, and if $a \le \frac{1}{b}$, the inequality in (2) can only hold with $A = 0$.

Example. Let $p \ge 1$ and $\mathcal{G}_p = \mathcal{G}(\mathrm{rect}^{(p)}, a, b)$; when $p \ge 2$, the functions $\mathrm{rect}^{(p)}$ are continuous and satisfy (2) with a constant > 0 only when $0 < a < p$ and $0 < b \le \frac{1}{a}$. The set $\mathcal{G}_p$ is a Riesz basis when $b = \frac{1}{a}$; because the functions $\mathrm{rect}^{(p)}(x - ka)$ overlap, $\mathcal{G}_p$ can never be orthogonal. On the other hand, the set $\mathcal{G}_1 = \mathcal{G}(\mathrm{rect}, a, b)$ is a frame if $0 \le a \le 1$ and $0 < b \le \frac{1}{a}$. When $a = b = 1$, $\mathcal{G}_1$ is orthonormal.

Let us recall some stability results that we need for our proofs. The following is Theorem 2.3 in Sun and Zhou (1999).

Theorem 2.1. *Let $\phi \in L^2(\mathbb{R})$ be supported in an interval of length $\frac{1}{b}$. Assume that $\mathcal{G}(\phi, a, b)$ is a frame in $L^2(\mathbb{R})$ with bounds A and B. Let $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ be such that $|\lambda_n - n| \le L < \dfrac{1}{4}$ for every $n \in \mathbb{Z}$. The set $\{e^{2\pi i b\lambda_n t}\phi(t - ak)\}_{n,k\in\mathbb{Z}}$ is a frame in $L^2(\mathbb{R})$ with bounds $A' \ge A(\cos(\pi L) - \sin(\pi L))$ and $B' \le B(2 - \cos(\pi L) + \sin(\pi L))$.*

The following theorem is a consequence of (Christensen, 1995, Theorem 1) (see also Sun and Zhou (2001)).

Theorem 2.2. *Let $\{x_n\}_{n\in\mathbb{Z}}$ be a frame for a Hilbert space H with bounds A and B. Let $\{y_n\}_{n\in\mathbb{Z}} \subset H$ be such that the inequality $\|\sum_n a_n(x_n - y_n)\|^2 \le C\sum_n |a_n|^2$ is valid with a constant $0 < C < A$. Then, $\{y_n\}_{n\in\mathbb{Z}}$ is a frame with bounds $A' \ge (1 - (\frac{C}{A})^{\frac{1}{2}})^2$ and $B' \le (1 + (\frac{C}{A})^{\frac{1}{2}})^2$. If $\{x_n\}_{n\in\mathbb{Z}}$ is a Riesz basis, then $\{y_n\}_{n\in\mathbb{Z}}$ is also a Riesz basis.*

We conclude this section with the following observation. Let α, β, $T \in \mathbb{R}$ with $T > 0$; let $\psi \in L^2(\mathbb{R})$ and let $\hat{\psi}(x) = \int_{\mathbb{R}} \psi(t)e^{-2\pi i x t}dt$ be the Fourier transform of ψ. The Fourier transform of $t \to e^{2\pi i\beta\frac{t}{T}}\psi(\frac{t}{T} - \alpha)$ is $x \to Te^{2\pi i\alpha(Tx-\beta)}\hat{\psi}(\beta - Tx)$. In particular, the Fourier transform of $t \to e^{\frac{2\pi i t\beta}{T}}\,\mathrm{rect}^{(p)}\left(\frac{t}{T} - \alpha\right)$ is $x \to Te^{2\pi i\alpha(\beta - Tx)}(\mathrm{sinc}(\beta - Tx))^p$, where $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$ when $t \ne 0$

and $\operatorname{sinc}(0) = 1$. If $\{f_j\}_{j\in\mathbb{Z}}$ is a frame in $L^2(\mathbb{R})$, by Plancherel's theorem the set $\{\hat{f}_j\}_{j\in\mathbb{Z}}$ is also a frame in $L^2(\mathbb{R})$ with the same frame bounds (see e.g. (Christensen, 1996, Prop. 11.2.5)). In particular, the Fourier transform maps $\mathcal{G}(\operatorname{rect}^{(p)}, a, b)$ into $\mathcal{G}(e^{2\pi iab}\operatorname{sinc}^p,\ b-a)$, and stability results for frames $\mathcal{G}(\operatorname{rect}^{(p)}, a, b)$ yield stability results for frames $\mathcal{G}(\operatorname{sinc}^p, b, -a)$ and vice versa. A sample result is Corollary 4.2 in Section 4.

2.1. **Two useful lemmas.** We recall that for every $p \ge 1$, $\operatorname{rect}^{(p)}(t)$ is the $p-$fold convolution of $\operatorname{rect}(t)$. Thus, for every $p \ge 1$,

$$\operatorname{rect}^{(p+1)}(x) = \operatorname{rect}(t) * \operatorname{rect}^{(p)}(t) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \operatorname{rect}^{(p)}(t)dt. \tag{3}$$

Using the identity (3) we can easily prove that $\operatorname{rect}^p(t)$ is supported in the interval $[-\frac{p}{2}, \frac{p}{2}]$, that $0 \le \operatorname{rect}^{(p)}(t) \le 1$ for every $t \in \mathbb{R}$ and that $\operatorname{rect}^{(p)}$ is continuous when $p \ge 2$ and is differentiable when $p \ge 3$. It is not too difficult to verify that $\operatorname{rect}^{(p)}(x)$ is increasing in $(-\infty,\ 0)$ and is decreasing in $(0, \infty)$.

In Schoenberg (1988) the following identity has been proved for every $p \ge 2$.

$$\operatorname{rect}^{(p)}(x) = \frac{1}{(p-1)!} \sum_{j=0}^{p} (-1)^j \binom{p}{j} \left(x + \frac{p}{2} - j\right)_+^{p-1} \tag{4}$$

where $x_+ = \begin{cases} x, & \text{if } x \ge 0; \\ 0, & \text{if } x < 0. \end{cases}$ See also Schoenberg (1969). This formula shows that $\operatorname{rect}^{(p)}(x)$ is a polynomial of degree $p-1$ in intervals $\Lambda_i^p(x) = [-\frac{p}{2} + i - 1, -\frac{p}{2} + i]$, with $1 \le i \le p$.

We use the notation $[x]$ to denote the integer part of a real number x. When $x \ge 0$, $[x]$ is the integer $n \ge 0$ that satisfies $n \le x < n+1$.

Lemma 2.3. *For every $p \ge 1$, the optimal frame bounds A_p and B_p of $\mathcal{G}(\operatorname{rect}^{(p)}, a, b)$ satisfy $B_p \le \frac{1}{b}[\frac{p}{a}]$ and $A_p \ge \frac{1}{b}(\operatorname{rect}^{(p)}(\frac{a}{2}))^2$.*

Remark. The exact frame constants of $\mathcal{G}(\operatorname{rect}^{(p)}, 1, 1)$ are evaluated in Mishchenko (2010) (see also Antony Selvan and Radha (2016)). They are $B_p = 1$ and $A_p = \frac{2^{2p+1}}{\pi^{2p}} \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^{2p}}$. The expression $K_m = \frac{2}{\pi} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(m+1)}}{(2\nu+1)^{m+1}}$ is known as the *Krein-Favard constant.*

Proof of Lemma 2.3. Let $S_p(a, x) = \sum_{n\in\mathbb{Z}} (\operatorname{rect}^{(p)}(x - an))^2$. By (2),

$$A = \frac{1}{b} \inf_{x\in\mathbb{R}} S_p(a, x) \quad \text{and} \quad B = \frac{1}{b} \sup_{x\in\mathbb{R}} S_p(a, x)$$

are frame bounds of $\mathcal{G}(\operatorname{rect}^{(p)}, a, b)$. Note that $S_p(a, x)$ is periodic with period a, i.e., $S_p(a, x) = S_p(a, a + x)$. Thus, bA and bB equal the minimum and maximum of $S_p(a, x)$ in the interval $[-\frac{a}{2}, \frac{a}{2}]$.

The maximum of $S_p(a, x)$ is easy to evaluate: the functions $\operatorname{rect}^{(p)}(t - na)$ are supported in the interval $J_n = [na - \frac{p}{2}, na + \frac{p}{2}]$ and each point in the interval $[-\frac{a}{2}, \frac{a}{2}]$ belongs to at most $[\frac{p}{a}]$ overlapping J_n's. Since $\operatorname{rect}^{(p)}(t) \le 1$ we have that $S_p(a, x) \le [\frac{p}{a}]$.

To estimate the minimum of $S_p(a, x)$ we observe that

$$\min_{x\in\mathbb{R}} S_p(a, x) = \min_{|x|\le\frac{a}{2}} S_p(a, x) \ge \min_{|x|\le\frac{a}{2}} (\operatorname{rect}^{(p)}(x))^2.$$

Recalling that $(\operatorname{rect}^{(p)}(x))^2$ is even, and is increasing when $x < 0$ and decreasing when $x > 0$, we can see at once that $\min_{|x|\le\frac{a}{2}} (\operatorname{rect}^{(p)}(x))^2 = (\operatorname{rect}^{(p)}(\frac{a}{2}))^2$ as required. $\square$

The following lemma will be used to prove Theorems 1.1 and 1.2.

Lemma 2.4. *Let $\{\mu_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be such that $\ell = \sup_{k\in\mathbb{Z}} |\mu_k - k| < \infty$. Then, for every finite sequence $\{\alpha_k\} \subset \ell^2$ we have that*

$$\left\|\sum_k \alpha_k(\mathrm{rect}(t-ak) - \mathrm{rect}(t-a\mu_k)\right\|_2^2 < 2a(z+1)\ell\sum_k |\alpha_k|^2 \tag{5}$$

where $z = 0$ when $0 < \ell \le \frac{a-1}{2a}$ and $z = [\ell] + 1$ in all other cases.

Proof. We can assume $\sum_k |\alpha_k|^2 = 1$. Let

$$f(t) = \sum_k \alpha_k \left(\mathrm{rect}(t-ak) - \mathrm{rect}(t-a\mu_k)\right) = \sum_k \alpha_k\chi_{I_k}(t), \tag{6}$$

where I_j denotes the support of $\mathrm{rect}(t-aj) - \mathrm{rect}(t-a\mu_j)$. When $\mu_j \neq j$, I_j is union of two intervals that we denote with I_j^+ and I_j^-. When $\mu_j > j$ we let

$$I_j^- = (aj - \frac{1}{2},\ a\mu_j - \frac{1}{2}), \quad I_j^+ = (aj + \frac{1}{2},\ a\mu_j + \frac{1}{2}).$$

We use (improperly) the same notation also when $\mu_j < j$. We can write (6) as:

$$f(t) = \sum_k \alpha_k\chi_{I_k^+}(t) + \sum_k \alpha_k\chi_{I_k^-}(t).$$

If $a > 1$ and if $a-1$, the measure of the "gap" between the supports of the $\mathrm{rect}(t-ak)$, is larger than $2a\ell$, the I_j's do not intersect (see Figure 1). From (6) follows that $||f||_2^2 = \sum_{k\in\mathbb{Z}} |\alpha_k|^2|I_k| \le 2a\ell$ whenever $a > 1$ and $\ell < \frac{a-1}{2a}$.

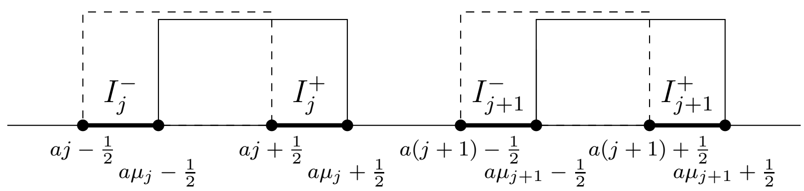

FIGURE 1

In all other cases, the intervals $I_j^\pm$ may intersect. Let I_h be fixed, and let $z = z(h)$ be the maximum number of intervals I_j, with $j \neq h$, that intersect I_h. The sum in (6) has at most $z+1$ terms because there are at most $z+1$ functions χ_{I_k} that overlap at each point. The elementary inequality

$$(x_1 + \cdots + x_m)^2 \le m\left(x_1^2 + \cdots + x_m^2\right) \tag{7}$$

which is valid for every $x_1, \ldots, x_m \in \mathbb{R}$, yields $|f(x)|^2 \le (z+1)\sum_k |\alpha_k|^2\chi_{I_k}(x)$, and $||f||_2^2 \le (z+1)\sup_k |I_k| = 2\ell(z+1)$.

It remains to determine z. Clearly z is maximum when $\mu_j = j \pm \ell$, so we assume that this is the case. For simplicity we let $I_h = I_0$, with $I_0^+ = (\frac{1}{2},\ a\ell + \frac{1}{2})$ and $I_0^- = (-\frac{1}{2},\ a\ell - \frac{1}{2})$, and $\mu_j = j + \ell$ (the case $\mu_j = j - \ell$ is similar).

The interval $I_j^+ = (aj + \frac{1}{2},\ a(j+\ell) + \frac{1}{2})$ intersects I_0^+ if either: (a) $\frac{1}{2} < aj + \frac{1}{2} < a\ell + \frac{1}{2}$, or (b) $\frac{1}{2} < a(j+\ell) + \frac{1}{2} < a\ell + \frac{1}{2}$. The inequality a) is equivalent to $0 < j < \ell$; the inequality b) is equivalent to $0 < j + \ell < \ell$, which is satisfied when $-\ell < j < 0$; since a) and b) cannot be verified simultaneously, there are at most $[\ell]$ integers j for which I_j^+ intersects I_0^+.

The interval $I_j^- = (aj - \frac{1}{2},\ a(j+\ell) - \frac{1}{2})$ intersects I_0^+ if either: (a) $\frac{1}{2} < aj - \frac{1}{2} < a\ell + \frac{1}{2}$, or (b) $\frac{1}{2} < a(j+\ell) - \frac{1}{2} < a\ell + \frac{1}{2}$. Equivalently, a) $\frac{1}{a} < j < \ell + \frac{1}{a}$, or *b*) $\frac{1}{a} - \ell < j < \frac{1}{a}$ and so there are at most $[\ell] + 1$ integers j for which I_j^- intersects I_0^+.

To summarize: we have $z = 0$ when $a > 1$ and $\ell < \frac{a-1}{2a}$, and $z \le 2([\ell]+1)$ in all other cases, and the proof of Lemma 2.4 is concluded. □

3. Proofs of Theorems 1.1, 1.2 and 1.3

Proof of Theorem 1.1. By Lemma 2.3, the lower frame bound of $\mathcal{G}(\text{rect}, a, b)$ is $A \geq \frac{1}{b}$. Fix a finite set of coefficients $\{\alpha_{n,k}\} \subset \mathbb{C}$ with $\sum_{n,k} |\alpha_{n,k}|^2 = 1$; let

$$f(t) = \sum_{n,k\in\mathbb{Z}} \alpha_{n,k} e^{2\pi ibnt} \left(\text{rect}\,(t - ak) - \text{rect}\,(t - a\mu_{k,n})\right).$$

If we show that $||f||_2 < \frac{1}{b}$ for every set of coefficients $\{\alpha_{n,k}\}$, by Theorem 2.2 the set $\mathcal{F}$ in Theorem 1.1 is a frame in $L^2(\mathbb{R})$ and a Riesz basis when $ab = 1$.

We let

$$f_n(t) = \sum_{k\in\mathbb{Z}} \alpha_{n,k} \left(\text{rect}\,(t - ak) - \text{rect}\,(t - a\mu_{k,n})\right) \quad \text{and} \quad f(t) = \sum_{n\in\mathbb{Z}} e^{2\pi ibnt} f_n(t).$$

By the triangle inequality, $||f||_2 \leq \sum_{n\in\mathbb{Z}} ||f_n||_2$; we can apply Lemma 2.4 with $z = 1$ because we have assumed in (1) that $L_n = \sup_k |\mu_{n,k} - k| < 1$. We gather $||f_n||_2^2 \leq 4aL_n \sum_k |\alpha_{n,k}|^2$ and

$$||f||_2 \leq \sum_{n\in\mathbb{Z}} \sqrt{4aL_n} \left(\sum_k |\alpha_{n,k}|^2\right)^{\frac{1}{2}}.$$

By Hölder's inequality, $||f||_2^2 \leq 4a \sum_{n\in\mathbb{Z}} L_n = 4aL$. By Theorem 2.2, the set $\mathcal{F}$ in Theorem 1.1 is a frame with constants $A' \geq (1 - 2(abL)^{\frac{1}{2}})^2$ and $B' \leq \frac{1}{b}(1 + 2(abL)^{\frac{1}{2}})^2$ whenever $4aL < \frac{1}{b}$, as required. □

Proof of Theorem 1.2. Let $\{\alpha_{n,k}\} \subset \mathbb{C}$ be a finite sequence such that $\sum_{n,k} |\alpha_{n,k}|^2 = 1$. Let

$$f_p(t) = \sum_{n,k\in\mathbb{Z}} \alpha_{n,k} e^{2\pi ibnt} \left(\text{rect}^{(p)}(t - ak) - \text{rect}^{(p)}(t - a\mu_{k,n})\right). \tag{8}$$

We prove that, for every $p \geq 1$,

$$||f_p||_2^2 \leq 4aL \tag{9}$$

where $L = \sum_{n\in\mathbb{Z}} \sup_{h\in\mathbb{Z}} |\mu_{n,h} - h|$ is as in the proof of Theorem 1.1. By Lemma 2.3, the frame constants of $\mathcal{G}(\text{rect}^{(p)}, a, b)$ are $A_p \geq \frac{1}{b}(\text{rect}^{(p)}(\frac{a}{2}))^2$ and $B_p \leq \frac{1}{b}[p/a]$ and by assumption $4aL < \frac{1}{b}(\text{rect}^{(p)}(\frac{a}{2}))^2 \leq A_p$. We can use Theorem 2.2 to conclude the proof of the theorem.

We prove (9) by induction on p. The $L^2(\mathbb{R})$ norm of the function $f_1(t)$ has been estimated in Theorem 1.1 and we have proved (9). Assume that (9) is satisfied by f_{p-1}, with $p \geq 2$. Recalling that $\text{rect}^{(k)}(x) = \text{rect} * \text{rect}^{(k-1)}(x) = \int_{\mathbb{R}} \text{rect}(x - y)\,\text{rect}^{(k-1)}(y)dy$, (8) yields

$$\begin{aligned} f_p(t) &= \int_{-\infty}^{\infty} \text{rect}(t - y) \sum_{n,k\in\mathbb{Z}} \alpha_{n,k} e^{2\pi ibnt} \left(\text{rect}^{(p-1)}(y - ak) - \text{rect}^{(p-1)}(y - a\mu_{k,n})\right) dy \\ &= \int_{-\infty}^{\infty} \text{rect}(t - y) e^{2\pi ibn(t-y)} f_{p-1}(y)dy = \left(e^{2\pi ibnt}\,\text{rect}\,t\right) * f_{p-1}(t). \end{aligned}$$

By Young's inequality for convolution and (9),

$$||f_p||_2 \leq ||\,\text{rect}\,||_1 ||f_{p-1}||_2 \leq (4aL)^{\frac{1}{2}} \leq \frac{1}{b^{\frac{1}{2}}}(\text{rect}^{(p)}(\tfrac{a}{2}))$$

as required. □

Proof of Theorem 1.3 . Let $S(x) = \sum_{k\in\mathbb{Z}} |\psi(x+ak)|^2$. We can argue as in the proof of Lemma 2.3 to show that the frame constants of $\mathcal{G}(\psi, a, b)$ are $A = \frac{1}{b}\min_{x\in[-\frac{a}{2},\frac{a}{2}]} S(x)$ and $B = \frac{1}{b}\max_{x\in[-\frac{a}{2},\frac{a}{2}]} S(x)$. Clearly,

$$A \geq \frac{1}{b} \min_{x\in[-\frac{a}{2},\frac{a}{2}]} |\psi(x)|^2 \geq \frac{m^2}{b}.$$

Since each point in $[-\frac{a}{2}, \frac{a}{2}]$ belongs to the support of at most $[\frac{p}{a}]$ functions in the sum above, we have that $B \leq \frac{1}{b}[\frac{p}{a}] \max_{x\in[-\frac{a}{2},\frac{a}{2}]} |\psi(x)|^2 \leq \frac{1}{b}[\frac{p}{a}]M^2$, as required.

Fix a finite set of coefficients $\{\alpha_{n,k}\} \subset \mathbb{C}$ with $\sum_{n,k} |\alpha_{n,k}|^2 = 1$; let

$$f(t) = \sum_{n,k\in\mathbb{Z}} \alpha_{n,k} e^{2\pi i bnt} \left(\psi\left(t - ak\right) - \psi\left(t - a\mu_{k,n}\right)\right).$$

If we show that $||f||_2^2 < \frac{C^2}{b} < A$, by Theorem 2.1 we can conclude that $\mathcal{F}$ is a frame in $L^2(\mathbb{R})$.

We argue as in the proof of Theorem 1.1. We let $f_n(t) = \sum_{k\in\mathbb{Z}} \alpha_{n,k}(\psi(t-ak) - \psi(t-a\mu_{k,n})$ and $f(t) = \sum_{n\in\mathbb{Z}} e^{2\pi i bnt} f_n(t)$. By the triangle inequality, $||f||_2 \leq \sum_{n\in\mathbb{Z}} ||f_n||_2$;

To estimate the norm of f_n we argue as in Lemma 2.4. Fix $n \in \mathbb{Z}$; let $\{\mu_{n,k}\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, with $L_n = \sup_{k\in\mathbb{Z}} |\mu_{n,k} - k|$. Let $\{\alpha_{n,k}\} \subset \mathbb{C}$ be a finite sequence. We show that

$$\left\| \sum_k \alpha_{m,k}(\psi(t-ak) - \psi(t-a\mu_k) \right\|_2^2 \leq \left[\frac{p}{a} + L_n\right] (aL_n)^2 ||\psi'||_2^2. \tag{10}$$

Fix $k \in \mathbb{Z}$; if $\mu_{n,k} > k$ (resp. $\mu_{n,k} < k$) the support of the function $g_k(t) = \psi(t) - \psi(t - a(\mu_{n,k} - k))$ is in the interval $[-\frac{p}{2}, aL_n + \frac{p}{2}]$ (resp. $[-\frac{p}{2} - aL_n, \frac{p}{2}]$) and so there are at most $[\frac{p}{a} + L_n]$ supports of functions $g_k(t-ak)$ that overlap at each point. In view of the elementary inequality (7) we obtain

$$\begin{aligned}
\left\| \sum_k \alpha_{m,k}(\psi(t-ak) - \psi(t-a\mu_{n,k})) \right\|_2^2 &= \left\| \sum_k \alpha_{n,k} g_k(t-ak) \right\|_2^2 \\
&\leq \left[\frac{p}{a} + L_n\right] \sum_k \alpha_{n,k}^2 \|\psi(.-ak) - \psi(.-a\mu_{n,k}))\|_2^2 \\
&= \left[\frac{p}{a} + L_n\right] \sum_k \alpha_{n,k}^2 \left\| \int_{t-ak}^{t-a\mu_{n,k}} \psi'(\xi) d\xi \right\|_2^2 \\
&\leq \left[\frac{p}{a} + L_n\right] \sum_k \alpha_{n,k}^2 \left(\int_{ak}^{a\mu_{n,k}} \|\psi'(.-s)\|_2 \, ds \right)^2 \\
&\leq \left[\frac{p}{a} + L_n\right] \sum_k \alpha_{n,k}^2 (aL_n)^2 ||\psi'||_2^2 \\
&= \left[\frac{p}{a} + L_n\right] (aL_n)^2 ||\psi'||_2^2 \sum_j |\alpha_{n,j}|^2.
\end{aligned}$$

We have used the change of variables $\xi = t - s$ in the integral on the third line and Minkoswky's inequality. The proof of Theorem 1.3 is concluded.

□

4. Corollaries and generalizations

In this section we prove corollaries and generalizations of Theorems 1.1 and 1.2. We start with two corollaries of Theorem 1.2.

Corollary 4.1. *Let $p \geq 1$ be an integer. Let $0 < a < p$ when $p > 1$ and $0 \leq a \leq 1$ when $p = 1$. Let $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ that satisfies $|\lambda_n - n| \leq \ell < \frac{1}{4}$; let $\{\mu_{n,k}\}_{n,k\in\mathbb{Z}} \subset \mathbb{R}$ be as in (1). If*

$$4abL < \left[\mathrm{rect}^{(p)}(a/2)\right]^2 (\cos(\pi\ell) - \sin(\pi\ell)),$$

the set $\{e^{2\pi i\lambda_n bt}\,\mathrm{rect}^{(p)}(t - a\mu_{n,k})\}_{n,k\in\mathbb{Z}}$ is a frame with constants

$$A''_p = \left(1 - \frac{2}{\mathrm{rect}^{(p)}(a/2)}\left(\frac{abL}{\cos(\pi\ell) - \sin(\pi\ell)}\right)^{\frac{1}{2}}\right)^2$$

and

$$B''_p = \left(1 + \frac{2}{\mathrm{rect}^{(p)}(a/2)}\left(\frac{abL}{\cos(\pi\ell) - \sin(\pi\ell)}\right)^{\frac{1}{2}}\right)^2 .$$

Proof. Let $c = 1 - \cos(\pi\ell) + \sin(\pi\ell)$. Recalling that the frame constants of $\mathcal{G}(\mathrm{rect}^{(p)}, a, b)$ are $A_p \geq \frac{1}{b}(\mathrm{rect}^{(p)}(a/2))^2$ and $B_p \leq \frac{1}{b}[\frac{p}{a}]$, by Theorem 2.1 the set $\{e^{2\pi i\lambda_n bt}\,\mathrm{rect}^{(p)}(t - ak)\}_{n,k\in\mathbb{Z}}$ is a frame with constants $A'_p \geq \frac{1}{b}(\mathrm{rect}^{(p)}(a/2))^2(1 - c)$ and $B'_p \leq \frac{1}{b}[\frac{p}{a}](1 + c)$.

The proof of Theorem 1.2 can be repeated line by line for the set $\mathcal{F} = \{e^{2\pi i\lambda_n bt}\,\mathrm{rect}(t - a\mu_{n,k})\}_{n,k\in\mathbb{Z}}$, with λ_n in place of n. We conclude that if $4aL < A'_p$, the set $\{e^{2\pi i\lambda_n bt}\,\mathrm{rect}(t - a\mu_{n,k})\}_{n,k\in\mathbb{Z}}$ is a frame with constants $A''_p = (1 - \frac{2}{\mathrm{rect}^{(p)}(a/2)}(\frac{abL}{1-c})^{\frac{1}{2}})^2$ and $B'' = (1 + \frac{2}{\mathrm{rect}^{(p)}(a/2)}(\frac{abL}{1-c})^{\frac{1}{2}})^2$. □

Plancherel's theorem and Corollary 4.1 yield the following

Corollary 4.2. *With the notations and assumptions of Corollary 4.1, the set*

$$\left\{e^{-2\pi i a\mu_{n,k}t}\,\mathrm{sinc}^p\,(t - b\lambda_n)\right\}_{k,n\in\mathbb{Z}}$$

is frame of $L^2(\mathbb{R})$ with constants A''_p and B''_p.

We prove a multi-dimensional version of Theorem 1.1. We will use the following notation: for any two vectors $\mathbf{v} = (v_1, \dots, v_d)$ and $\mathbf{w} = (w_1, \dots, w_d) \in \mathbb{R}^d$, we let $\langle \mathbf{v},\ \mathbf{w}\rangle = \sum_{j=1}^d v_j w_j$ and $|\mathbf{w}| = \sqrt{\langle \mathbf{w},\ \mathbf{w}\rangle}$ be the usual scalar product and norm in $\mathbb{R}^d$. We denote with $\mathbf{v}\cdot\mathbf{w} = (v_1\, w_1, \dots, c_d\, w_d)$ the Hadamard (componentwise) product of the vectors $\mathbf{v}$ and $\mathbf{w}$. We let $\mathrm{rect}\,(\mathbf{x}) = \mathrm{rect}(x_1, ..., x_d)$ be the product of the functions $\mathrm{rect}(x_1), \dots,\ \mathrm{rect}(x_d)$.

Theorem 4.3. *Let $\boldsymbol{a} = (a_1, \dots, a_d)$ and $\boldsymbol{b} = (b_1, \dots, b_d)$, where $0 < a_j \leq 1$ and $0 < a_j b_j \leq 1$ for $j = 1, \dots, d$. If $\{\boldsymbol{\mu}_{\boldsymbol{k},\boldsymbol{n}}\}_{\boldsymbol{k},\boldsymbol{n}\in\mathbb{Z}^d} \subset \mathbb{R}^d$, with $\boldsymbol{\mu}_{\boldsymbol{k},\boldsymbol{n}} = (\mu_{k_1,n_1}, \dots, \mu_{k_d,n_d})$, is a set of vectorial coefficients that satisfy*

$$\mathcal{L}_j = \sum_{n_j\in\mathbb{Z}} \sup_{h_j\in\mathbb{Z}} \left|\mu_{n_j,h_j} - h_j\right| < \frac{1}{4a_j b_j}, \quad j = 1, \dots, d, \tag{11}$$

the set $\mathcal{B} = \left\{e^{2\pi i\langle \boldsymbol{b}\cdot\boldsymbol{n},\boldsymbol{t}\rangle}\,\mathrm{rect}\,(\boldsymbol{a}\cdot\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}} - \boldsymbol{t})\right\}_{\boldsymbol{n},\boldsymbol{k}\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ with constants $A \geq \prod_{j=1}^d (1 - 2(a_j b_j \mathcal{L}_j)^{\frac{1}{2}})^2$ and $B \leq \prod_{j=1}^d (1 + 2(a_j b_j \mathcal{L}_j)^{\frac{1}{2}})^2$.

Proof. We prove the theorem only for $d = 2$ (the proof for $d > 2$ is similar).

To prove that $\mathcal{B}$ is a frame it is enough to show that for every $f \in L^2(\mathbb{R}^2)$ with $||f||_{L^2(\mathbb{R}^2)} = 1$, we have that

$$A \leq \sum_{(n_j,k_j)\in\mathbb{Z}^2} \left(\int_{\mathbb{R}^2} f(t_1, t_2) e^{-2\pi i(b_1 n_1 t_1 + b_2 n_2 t_2)} \prod_{j=1,2} \mathrm{rect}(t_j - a_j\mu_{n_j,k_j}) dt_1 dt_2\right)^2 \leq B$$

where A and B are as in the statement of the theorem.

Fix t_2, $\mu_{n_2,k_2} \in \mathbb{R}$; let $g(t_1) = \int_{\mathbb{R}} f(t_1,t_2)e^{-2\pi i b_2 n_2 t_2}\operatorname{rect}(t_2 - a_2\mu_{n_2,k_2}))dt_2$. With this notation, the inequality above can be written as

$$A \leq \sum_{(n_j,k_j)\in\mathbb{Z}^2} \left(\int_{\mathbb{R}} g(t_1)e^{-2\pi i b_1 n_1 t_1}\operatorname{rect}(t_1 - a_1\mu_{n_1,k_1})dt_1\right)^2 \leq B. \tag{12}$$

By Theorem 1.1, the sets $\mathcal{B}_j = \left\{e^{2\pi i b_j n_j t_j}\operatorname{rect}\left(t_j - a_j\mu_{n_j,k_j}\right)\right\}_{k_j,n_j\in\mathbb{Z}}$, with $j = 1, 2$, are frames for $L^2(\mathbb{R})$ with bounds $A_j \geq (1 - 2(a_j b_j \mathcal{L}_j)^{\frac{1}{2}})^2$, $B_j \leq (1 + 2(a_j b_j \mathcal{L}_j)^{\frac{1}{2}})^2$. Thus,

$$\begin{aligned} A_1||g||^2_{L^2(\mathbb{R})} &\leq \sum_{(n_1,k_1)\in\mathbb{Z}^2} \left(\int_{\mathbb{R}} g(t_1)e^{-2\pi i b_1 n_1 t_1}\operatorname{rect}(a_1\mu_{n_1,k_1} - t_1)dt_1\right)^2 \\ &\leq B_1||g||^2_{L^2(\mathbb{R})}. \end{aligned} \tag{13}$$

But $||g||^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(t_1,t_2)e^{-2\pi i b_2 n_2 t_2}\operatorname{rect}(t_2 - a_2\mu_{n_2,k_2}))dt_2\right)^2 dt_1$ and

$$\begin{aligned} \sum_{(n_2,k_2)\in\mathbb{Z}^2} ||g||^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \sum_{(n_2,k_2)\in\mathbb{Z}^2}\left(\int_{\mathbb{R}} f(t_1,t_2)e^{-2\pi i b_2 n_2 t_2}\operatorname{rect}(t_2 - a_2\mu_{n_2,k_2}))dt_2\right)^2 dt_1 \\ &\leq B_2 \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(t_1,t_2)|^2 dt_2\right)dt_1 = B_2. \end{aligned}$$

Similarly, $\sum_{(n_2,k_2)\in\mathbb{Z}^2} ||g||^2_{L^2(\mathbb{R})} \geq A_2$. These inequalities and (13) yield (12). $\square$

A version of Theorem 4.3 can be proved for functions $\operatorname{rect}^{(p)}(\mathbf{x}) = \operatorname{rect}^{(p)}(x_1)\operatorname{rect}^{(p)}(x_2)\ldots\operatorname{rect}^{(p)}(x_d)$, with $p \geq 2$. We leave the details to the interested reader.

5. Remarks and open problems

When $a = b = 1$, the stability bound in Theorem 1.1 is $L = \frac{1}{4}$. We do not know whether $\frac{1}{4}$ can be replaced by any larger constant or not. A famous example by Ingham Ingham (1936) shows that the constant $\frac{1}{4}$ in Kadec's theorem is optimal, but we could not generalize Ingham's example.

From Theorem 2.1 and Plancherel's theorem, stability results for Riesz bases on spaces of band-limited functions easily follow. Recall that a function $f \in L^2(\mathbb{R}^d)$ is *band-limited* to a bounded measurable set D (or: *f is in the Paley-Wiener space PW_D*) if its Fourier transform vanishes outside D. By Plancherel's theorem, PW_D is a closed subspace of $L^2(\mathbb{R}^d)$ which is isometrically isomorphic to $L^2(D)$. The importance of exponential bases in the reconstruction of bandlimited functions is emphasized by the classical sampling theorem, attributed to Shannon, Whittaker, Kotel'nikov and others (Shannon (1949), Unser (2000)). By Theorem 2.1, the set $\{e^{2\pi i\lambda_n t}\operatorname{rect}(t-k)\}_{n\in\mathbb{Z}}$ is a Riesz basis of the subspace of $L^2(\mathbb{R})$ spanned by the functions $\{e^{2\pi i n t}\operatorname{rect}(t-k)\}_{n\in\mathbb{Z}}$ whenever $|\lambda_n - n| < \ell < \frac{1}{4}$; thus, for every $k \in \mathbb{Z}$, the set $\{e^{2\pi i k t}\operatorname{sinc}(t - \lambda_n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of $PW_{(k-\frac{1}{2},\ k+\frac{1}{2})}$. Our Theorems 1.1 and 1.2 deal with frames of functions with compact support, but do not yield stability theorems in Paley-Wiener spaces. For example, consider the set $\mathcal{B} = \{e^{2\pi i n t}\operatorname{rect}(t-\mu_n)\}_{n\in\mathbb{Z}} \subset \mathcal{F}$, with $\sup_{n\in\mathbb{Z}}|\mu_n| = L$; the set $\widehat{\mathcal{B}} = \{-e^{2\pi i\mu_n t}\operatorname{sinc}(t-n)\}_{n\in\mathbb{Z}}$ is a subset of $PW_{(-\frac{1}{2}-L,\frac{1}{2}+L)}$, but it is not a Riesz basis because $\mathcal{B}$ is not a Riesz basis of $L^2(-\frac{1}{2}-L, \frac{1}{2}+L)$.

It would be interesting to study the stability properties of Riesz bases or frames of $L^2(\mathbb{R})$ in the form of $\{e^{2\pi i n t}\operatorname{rect}(T_k t - k)\}_{n,k\in\mathbb{Z}}$, where $\{T_k\}_{k\in\mathbb{Z}}$ is a sequence of positive real numbers. We hope to address this problem in a subsequent paper.

Appendix A. The $p-$order holds

Let $p \geq 0$ be an integer and let $T > 0$. A *$p-$order hold* (pOH) is a device which models a sequence of impulses $\{q_n\}$ into a piecewise polynomial function of degree p. If the sequence $\{q_n\}$ originates from a continuous-time signal, i.e. $q_n = f(nT)$ for some $f \in L^2(\mathbb{R})$, the function $f_p(t)$

obtained through the $p-$order hold can be viewed as an approximation of the original function $f(t)$. The zero-order hold (ZOH) is the simplest and most widely used model: for a given function $f \in C(\mathbb{R})$, we let $f_{ZOH}(t) = \sum_{n=-\infty}^{\infty} f(nT)\ \mathrm{rect}\left(\frac{t}{T} - n\right)$. The zero order hold model is unambiguously defined in the literature (Eshbach et al. (1990) Hinrichsen and Pritchard (2005) Oppenheim et al. (2014)) but the definition of p-order hold varies. The extrapolation formulas that are most used in pOH are discrete versions of the Taylor expansion for differentiable function (Bonivento et al. (1995)) but other interpolation polynomials can be considered. When $p \geq 1$ we can let

$$f_p(t) = \sum_{n=-\infty}^{\infty} f(nT)\, \chi_{(Nt-\frac{1}{2}, NT+\frac{1}{2})}\, \mathrm{rect}^{(p+1)}\left(\frac{t-nT}{T}\right).$$

This model is adopted, for example, in (Easton Jr, 2010, pg. 495) for $p = 1$. The function $f_{ZOH}(t) = \sum_{n=-\infty}^{\infty} f(\mu_n)\ \mathrm{rect}\left(\frac{t-\mu_n T}{T}\right)$, where $\mu_n = n+\epsilon_n$, is considered in Angrisani and D'Arco (2009). The term ϵ_n models the so-called *timing jitter*, an unwelcome phenomenon of electronic systems. It is natural to investigate whether signals in $L^2(\mathbb{R})$ can effectively be reconstructed from ZOH devices with jitter. Similarly, we can model the effect of the timing jitter also in pOH devices as $f_{pOH}(t) = \sum_{n=-\infty}^{\infty} f(\mu_n)\ \mathrm{rect}^{(p)}\left(\frac{t-\mu_n T}{T}\right)$. Our Theorems 1.1 and 1.2 can be used to estimate how much jitter can be allowed in ZOH and pOH models. For example, suppose that in a ZOH model a given signal $f \in L^2(\mathbb{R})$ is approximated with linear combinations of $\mathrm{rect}(t-k)$, with $k \in \mathbb{Z}$. By Theorem 1.1 (with $a = b = 1$), the set $\mathcal{B} = \{e^{2\pi int}\,\mathrm{rect}(k-t)\}_{k\in\mathbb{Z}, n\neq 0} \cup \{\mathrm{rect}(\mu_{0,k}-t)\}_{k\in\mathbb{Z}}$ is a Riesz basis in $L^2(\mathbb{R})$ if $|\epsilon_k| = |\mu_{0,k}-k| < L < \frac{1}{4}$. If the timing jitter ϵ_k satisfies this inequality, signals in $L^2(\mathbb{R})$ can be effectively reconstructed from the functions in $\mathcal{B}$.

In Lim (1990) an extension of the ZOH model in dimension $d = 2$ is considered. Let $f(t_1, t_2)$ be an analog signal, and let $\mathrm{rect}(t_1, t_2) = \mathrm{rect}(t_1)\ \mathrm{rect}(t_2)$. We consider a sequence of equally spaced points $(T_1 n_1,\ T_2 n_2)$, where $T_1,\ T_2 > 0$ and $n_1, n_2 \in \mathbb{Z}$. The zero-hold reconstruction of the signal f is

$$f_{ZOH}(t_1, t_2) = \sum_{(n_1, n_2)\in\mathbb{Z}^2} f(n_1, n_2)\ \mathrm{rect}\left(\frac{t_1 - n_1 T_1}{T_1},\ \frac{t_2 - n_2 T_2}{T_2}\right).$$

In the presence of jitter, the sampling points $\mathbf{n} = (n_1, n_2)$ are replaced by $\mu_{\mathbf{n}} = (n_1 + \epsilon_{n_1},\ n_2 + \epsilon_{n_2})$; we can use Theorem 4.3 to conclude that if the timing jitter $(\epsilon_{n_1}, \epsilon_{n_2})$ satisfies the inequality: $\max\{|\epsilon_{n_1}|,\ |\epsilon_{n_1}|\} < \ell < \frac{1}{4}$, signals in $L^2(\mathbb{R}^2)$ can be effectively reconstructed from functions in $\mathcal{B} = \{e^{2\pi i\langle \mathbf{n},\ \mathbf{t}\rangle}\,\mathrm{rect}(\mathbf{k}-\mathbf{t})\}_{\mathbf{k}\in\mathbb{Z}^2, \mathbf{n}\neq 0} \cup \{\mathrm{rect}(\mu_{\mathbf{0},\mathbf{k}} - \mathbf{t})\}_{\mathbf{k}\in\mathbb{Z}^2}$.

Jitter can appear also in the orthogonal frequency-division multiplexing (OFDM), a method of encoding digital data on multiple carrier frequencies. OFDM has developed into a popular scheme for wideband digital communication, used in applications such as digital television and audio broadcasting. According to the basic OFDM realization Hrasnica et al. (2005),Rohling (2011), the transmitted signal $f(t)$ can be often expressed by

$$f(t) = \sum_{n=0}^{N-1}\sum_{k\in\mathbb{Z}} \alpha_{n,k}\ \mathrm{rect}(t-k)\, e^{2\pi i\lambda_n t}$$

where $\lambda_n = n/T$. The *frequency jitter* is modeled by $\lambda_n = \frac{n+\epsilon_n}{T}$, for some $\epsilon_n > 0$. As in the previous example, it is important to understand how much jitter can be tolerated in order to obtain a good signal reconstruction.

Let $\Gamma = \{0, 1, \ldots, N-1\}$. By Theorem 2.1 (with $a = b = 1$), the set $\mathcal{C} = \{e^{2\pi int}\,\mathrm{rect}(k-t)\}_{n,k\in\mathbb{Z}, n\notin\Gamma} \cup \{e^{2\pi i\lambda_n t}\,\mathrm{rect}(k-t)\}_{k\in\mathbb{Z}, n\in\Gamma}$ is a Riesz basis in $L^2(\mathbb{R})$ if $|\epsilon_n| = |\lambda_n - n| < L < \frac{1}{4}$, and so signals in $L^2(\mathbb{R})$ can be effectively reconstructed also in terms of the functions in $\mathcal{C}$.

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