Preprint, arXiv:1707.06223

SOME UNIVERSAL QUADRATIC SUMS OVER THE INTEGERS

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ABSTRACT. Let $a,b,c,d,e,f \in \mathbb{N}$ with $a \geqslant c \geqslant e > 0$, $b \leqslant a$ and $b \equiv a \pmod 2$, $d \leqslant c$ and $d \equiv c \pmod 2$, $f \leqslant e$ and $f \equiv e \pmod 2$. If any nonnegative integer can be written as $x(ax+b)/2+y(cy+d)/2+z(ez+f)/2$ with $x,y,z \in \mathbb{Z}$, then the tuple (a,b,c,d,e,f) is said to be universal over $\mathbb{Z}$. Recently, Z.-W. Sun found all candidates of such universal tuples over $\mathbb{Z}$. In this paper, we use the theory of ternary quadratic forms to show that 47 concrete tuples (a,b,c,d,e,f) in Sun's list of candidates are indeed universal over $\mathbb{Z}$. For example, we prove the universality of $(16,4,2,0,1,1)$ over $\mathbb{Z}$ which is related to the sophisticated form $x^2+y^2+32z^2$.

arXiv:1707.06223v4 [math.NT] 9 Oct 2017

1. INTRODUCTION

Those $T_x = x(x+1)/2$ with $x \in \mathbb{Z}$ are called triangular numbers. In 1796 Gauss proved Fermat's assertion that each $n \in \mathbb{N} = \{0,1,2,\ldots\}$ can be expressed as the sum of three triangular numbers.

For polynomials $f_1(x), f_2(x), f_3(x)$ with $f_i(\mathbb{Z}) = \{f_i(x) : \ x \in \mathbb{Z}\} \subseteq \mathbb{N}$ for $i = 1,2,3$, if any $n \in \mathbb{N}$ can be written as $f_1(x)+f_2(y)+f_3(z)$ with $x,y,z \in \mathbb{Z}$ then we call the sum $f_1(x)+f_2(y)+f_3(z)$ *universal over* $\mathbb{Z}$. For example, $T_x+T_y+T_z$ is universal over $\mathbb{Z}$ by Gauss' result.

In 1862 Liouville (cf. [2, p. 82]) determined all universal sums $aT_x+bT_y+cT_z$ over $\mathbb{Z}$ with $a,b,c \in \mathbb{Z}^+$. Z.-W. Sun [23, 24] studied universal sums of the form $ap_i(x)+bp_j(y)+cp_k(z)$ with $a,b,c \in \mathbb{N}$ and $i,j,k \in \{3,4,\ldots\}$, where $p_m(x)$ denotes the generalized polygonal number $(m-2)\binom{x}{2}+x$; see also [11, 19, 10, 18, 16] for subsequent work on some of Sun's conjectures posed in [23, 24]. In 2017 Sun [26] investigated universal sums $x(ax+1)+y(by+1)+z(cz+1)$ over $\mathbb{Z}$ with $a,b,c \in \mathbb{Z}^+ = \{1,2,3,\ldots\}$ and also universal sums $x(ax+b)+y(ay+c)+z(az+d)$ with $a,b,c,d \in \mathbb{N}$ and $a \geqslant b \geqslant c \geqslant d$. Quite recently, Sun [27] investigated for what tuples (a,b,c,d,e,f) with $a \geqslant c \geqslant e \geqslant 1$, $b \equiv a \pmod 2$ and $|b| \leqslant a$, $d \equiv c \pmod 2$ and $|d| \leqslant c$,

2010 *Mathematics Subject Classification*. Primary 11E25; Secondary 11D85, 11E20.

Keywords. Universal sums, quadratic polynomials, ternary quadratic forms.

Supported by the National Natural Science Foundation of China (Grant No. 11571162).

$f \equiv e \pmod 2$ and $|f| \leqslant e$, the sum
$$\frac{x(ax+b)}{2}+\frac{y(cy+d)}{2}+\frac{z(ez+f)}{2}$$
is universal over $\mathbb{Z}$. Such tuples (a,b,c,d,e,f) are said to be universal over $\mathbb{Z}$. He showed such tuples with $|b|<a$, $|d|<c$, $|f|<e$, and $b \geqslant d$ if $a=c$, and $d \geqslant f$ if $c=e$, must be in his list of 12082 candidates (cf. [28, A286944]), and conjectured that all such candidates are indeed universal over $\mathbb{Z}$. Note that
$$\left\{\frac{x(x-1)}{2}:\ x\in\mathbb{Z}\right\}=\{T_x:\ x\in\mathbb{Z}\}=\{x(2x+1):\ x\in\mathbb{Z}\}.$$
Sun [27] proved that some candidates (a,b,c,d,e,f) are universal over $\mathbb{Z}$, e.g. $(5,1,3,1,1,1)$ (equivalent to $(5,1,4,2,3,1)$) is universal over $\mathbb{Z}$. Sun even conjectured that any $n\in\mathbb{N}$ can be written as $x(x+1)/2+y(3y+1)/2+z(5z+1)/2$ with $x,y,z\in\mathbb{N}$.

In this paper, via the theory of ternary quadratic forms, we establish the universality (over $\mathbb{Z}$) of 47 concrete tuples (a,b,c,d,e,f) in Sun's list of candidates.

Theorem 1.1. *The tuples*
$$(5,1,2,2,1,1),\ (6,0,3,3,3,1),\ (6,2,5,5,1,1),\ (6,6,3,3,3,1),$$
$$(8,2,3,1,1,1),\ (8,6,3,1,1,1),\ (8,8,3,1,1,1)$$
are universal over $\mathbb{Z}$.

Remark 1.1. Our proof of Theorem 1.1 uses some special techniques. Sun [24] conjectured that any $n\in\mathbb{N}$ can be written as $T_x+2T_y+p_7(z)$ with $x,y,z\in\mathbb{N}$, and J. Ju, B.-K. Oh and B. Seo [16] proved that $T_x+2T_y+p_7(z)$ (or the tuple $(5,3,2,2,1,1)$) is universal over $\mathbb{Z}$.

Theorem 1.2. *The tuples*
$$(6,0,5,1,3,1),\ (6,0,5,3,3,1),\ (7,1,1,1,1,1),\ (7,1,2,0,1,1),$$
$$(7,1,2,2,1,1),\ (7,1,3,1,1,1),\ (7,1,3,3,1,1),\ (7,3,1,1,1,1),$$
$$(7,3,2,0,1,1),\ (7,3,2,2,1,1),\ (7,3,3,1,1,1),\ (7,3,3,3,1,1),$$
$$(7,5,1,1,1,1),\ (7,5,3,1,1,1),\ (7,5,3,3,1,1),\ (15,3,3,1,1,1),$$
$$(15,5,1,1,1,1),\ (15,5,3,1,2,0),\ (15,5,3,1,2,2),\ (15,9,3,1,1,1),$$
$$(21,7,3,1,2,2)$$
are universal over $\mathbb{Z}$.

Remark 1.2. Our proof of Theorem 1.2 involves the theory of genera of ternary quadratic forms. Sun [24] conjectured that any $n \in \mathbb{N}$ can be written as $T_x + y^2 + p_9(z)$ (or $T_x + 2T_y + p_9(z)$) with $x, y, z \in \mathbb{N}$, and Ju, Oh and Seo [16] proved that $T_x + y^2 + p_9(z)$ and $T_x + 2T_y + p_9(z)$ are universal over $\mathbb{Z}$, i.e., the tuples $(7, 5, 2, 0, 1, 1)$ and $(7, 5, 2, 2, 1, 1)$ are universal over $\mathbb{Z}$.

Theorem 1.3. (i) *The tuples* $(5, 5, 3, 1, 3, 1)$, $(5, 5, 3, 3, 3, 1)$, $(6, 4, 5, 5, 1, 1)$ *and* $(7, 7, 3, 1, 1, 1)$ *are universal over* $\mathbb{Z}$.

(ii) *All the five tuples*

$$(6, 2, 5, 1, 1, 1), (6, 2, 5, 5, 1, 1), (6, 4, 5, 1, 1, 1), (15, 5, 6, 2, 1, 1), (15, 5, 6, 4, 1, 1)$$

are universal over $\mathbb{Z}$.

Remark 1.3. Our proof of Theorem 1.3(i) employs the Minkowski-Siegel formula (cf. [17, pp. 173–174]). Sun [24] conjectured that any $n \in \mathbb{N}$ can be written as $T_x + p_7(y) + 2p_5(z)$ (or $T_x + p_7(y) + p_8(z)$) with $x, y, z \in \mathbb{N}$, and Ju, Oh and Seo [16] proved that $T_x + p_7(y) + 2p_5(z)$ and $T_x + p_7(y) + p_8(z)$ are universal over $\mathbb{Z}$, i.e., the tuples $(6, 2, 5, 3, 1, 1)$ and $(6, 4, 5, 3, 1, 1)$ are universal over $\mathbb{Z}$.

Similarly to [27, Theorem 1.4], we observe that

$$\{T_x + p_5(y) : \ x, y \in \mathbb{Z}\} = \{p_5(x) + 3p_5(y) : \ x, y \in \mathbb{Z}\}. \tag{1.1}$$

In fact,

$$\begin{aligned} & n \in \{T_x + p_5(y) : \ x, y \in \mathbb{Z}\} \\ \iff & 24n + 4 \in \{3(2x+1)^2 + (6y-1)^2 : \ x, y \in \mathbb{Z}\} \\ \iff & 24n + 4 \in \{3u^2 + v^2 : \ u, v \in \mathbb{Z} \ \& \ 2 \nmid uv\} \end{aligned}$$

and

$$\begin{aligned} & n \in \{3p_5(x) + p_5(y) : \ x, y \in \mathbb{Z}\} \\ \iff & 24n + 4 \in \{3(6x-1)^2 + (6y-1)^2 : \ x, y \in \mathbb{Z}\} \\ \iff & 24n + 4 \in \{3u^2 + v^2 : \ u, v \in \mathbb{Z}, \ 2 \nmid uv \ \& \ 3 \nmid u\}. \end{aligned}$$

If u and v are odd integers with $3 \mid u$ and $3 \nmid v$, then

$$3u^2 + v^2 = 3\left(\frac{u \pm v}{2}\right)^2 + \left(\frac{3u \mp v}{2}\right)^2$$

with $(u \pm v)/2$ not divisible by 3. Therefore (1.1) holds. In view of (1.1) and Theorems 1.1-1.3, we have the following consequence.

Corollary 1.1. *The tuples*

$$(9,3,7,1,3,1),\ (9,3,7,3,3,1),\ (9,3,7,5,3,1),$$
$$(9,3,7,7,3,1),\ (9,3,8,2,3,1),\ (9,3,8,6,3,1),$$
$$(9,3,8,8,3,1),\ (15,3,9,3,3,1),\ (15,9,9,3,3,1)$$

are universal over $\mathbb{Z}$.

Theorem 1.4. *The tuple* $(16,4,2,0,1,1)$ *is universal over* $\mathbb{Z}$. *In other words, any* $n \in \mathbb{N}$ *can be written as* $T_x + y^2 + 2z(4z+1)$ *with* $x,y,z \in \mathbb{Z}$.

Remark 1.4. This result is closely related to the sophisticated form $x^2 + y^2 + 32z^2$. Sun [27] even conjectured that any $n \in \mathbb{N}$ can be written as $T_x + y^2 + 2z(4z-1)$ with $x,y,z \in \mathbb{N}$.

We will show Theorems 1.1-1.4 in Sections 2-5 respectively.

In view of Theorems 1.1-1.3, [27, Theorem 1.4], and some basic facts on regular quadratic forms, among those conjectural universal tuples (a,b,c,d,e,f) with $a = 6 \geqslant c \geqslant e \geqslant 2$, $b \in (-a,a)$, $d \in (-c,c)$, $f \in (-e,e)$ and $a-b, c-d, e-f$ all even listed in [28, A286944], only the universality of the tuples

$$(6,0,5,1,4,2),\ (6,0,5,3,4,2),\ (6,2,5,3,4,0),\ (6,2,5,3,5,3),$$
$$(6,2,6,0,5,3),\ (6,2,6,2,5,3),\ (6,4,5,1,4,0),\ (6,4,5,1,5,1),$$
$$(6,4,5,3,2,0),\ (6,4,5,3,4,0),\ (6,4,5,3,5,3),\ (6,4,6,0,5,1),$$
$$(6,4,6,0,5,3)$$

remains open.

2. Proof of Theorem 1.1

Lemma 2.1. *Let* V *be a quadratic space. For any isometry* $T \in O(V)$ *of infinite order,*

$$V_T = \{x \in V : \text{there is a positive integer } k \text{ such that } T^k(x) = x\}.$$

is a subspace of V *with dimension one, and* $T(x) = \det(T)x$ *for any* $x \in V_T$.

Remark 2.1. Any unexplained notation in the theory of quadratic forms can be found in [4, 17, 20]. Lemma 2.1 is a known result, see, e.g., [18].

Lemma 2.2. (i) *For any* $n \in \mathbb{N}$, *we can write* $12n+5$ *as* $x^2 + y^2 + (6z)^2$ *with* $x,y,z \in \mathbb{Z}$.

(ii) *Let* $n \in \mathbb{Z}^+$ *and* $\delta \in \{0,1\}$. *Then we can write* $6n+1$ *as* $x^2+3y^2+6z^2$ *with* $x,y,z \in \mathbb{Z}$ *and* $x \equiv \delta \pmod 2$.

Remark 2.2. Lemma 2.2 is a known result due to the second author, see [24, Theorem 1.7(iii) and Lemma 3.3] and [26, Remark 3.1].

John S. Hsia, in a letter to Irving Kaplansky in 1993, proved that $x^2 + y^2 + 10z^2$ represents all eligible numbers of the form $3m+2$ (cf. [14, pp. 12–14]). As all positive odd numbers are eligible, we have the following lemma.

Lemma 2.3. *For each* $n \in \mathbb{N}$, *we can write* $6n+5$ *as* $x^2+y^2+10z^2$ *with* $x,y,z \in \mathbb{Z}$.

For $a,b,c \in \mathbb{Z}^+$, we define

$$E(a,b,c) = \{n \in \mathbb{N}:\ n \neq ax^2+by^2+cz^2 \text{ for all } x,y,z \in \mathbb{Z}\}.$$

L.E. Dickson [7, pp. 112-113] listed all the 102 primitive regular diagonal quadratic forms $ax^2+by^2+cz^2$ for which the structure of $E(a,b,c)$ is known explicitly. For example, the Gauss-Legendre theorem asserts that $E(1,1,1) = \{4^k(8l+7):\ k,l \in \mathbb{N}\}$.

In 1996 W. Jagy [12] investigated so-called *nearly regular* quadratic forms, and showed the following result (cf. [14, pp. 25–26]).

Lemma 2.4. *We have*

$$E(1,4,9) = \{2\} \cup \bigcup_{k,l \in \mathbb{N}} \{4^k(8l+7),\ 8l+3,\ 9l+3\}.$$

Proof of Theorem 1.1. (i) Let $n \in \mathbb{N}$ and $r \in \{1,3\}$. Apparently,

$$\begin{aligned} & n = T_x + y(y+1) + \frac{z(5z+r)}{2} \\ \iff & 40n + r^2 + 15 = 5(2x+1)^2 + 10(2y+1)^2 + (10z+r)^2. \end{aligned}$$

Since

$$E(1,5,10) = \{25^k m:\ k,m \in \mathbb{N} \text{ and } m \equiv 2,3 \pmod 5\},$$

we have $40 + r^2 + 15 \in \{x^2+5y^2+10z^2:\ x,y,z \in \mathbb{N}\}$. Thus we can write

$$40n + r^2 + 15 = (2^k x_0)^2 + 5(2^k y_0)^2 + 10(2^k z_0)^2 = 4^k(x_0^2 + 5y_0^2 + 10z_0)^2$$

with $k \in \mathbb{N}$, $x_0, y_0, z_0 \in \mathbb{Z}$, and x_0, y_0, z_0 not all even. In the case $k=0$, if $2 \mid z_0$ then $x_0^2 + 5y_0^2 \equiv r^2 + 15 \equiv 0 \pmod 8$ and hence $x_0 \equiv y_0 \equiv 0 \pmod 2$

which contradicts that x_0, y_0, z_0 are not all even, thus $2 \nmid z_0$ and also $2 \nmid x_0y_0$ since $x_0^2 + 5y_0^2 \equiv r^2 + 15 - 10z_0^2 \equiv 6 \pmod 8$.

It is easy to verify the following new identity:

$$4^2(x^2+5y^2+10z^2) = (x-5y-10z)^2+5(x+3y-2z)^2+10(x-y+2z)^2. \quad (2.1)$$

If x, y, z are odd integers, then by (2.1) we have

$$4(x^2 + 5y^2 + z^2) = \bar{x}^2 + 5\bar{y}^2 + 10\bar{z}^2$$

with

$$\tilde{x} = \frac{x-y}{2} - 2y - 5z,\ \tilde{y} = \frac{x-y}{2} + 2y - z,\ \tilde{z} = \frac{x-y}{2} + z$$

all odd. Thus, if $2 \nmid x_0y_0z_0$ then

$$40n+r^2+15 = 4^k(x_0^2+5y_0^2+10z_0^2) \in \{x^2+5y^2+10z^2 :\ x,y,z \text{ are odd}\}. \quad (2.2)$$

If $x_0 \not\equiv y_0 \pmod 2$, then $x_0^2 + 5y_0^2 + 10z_0^2 \equiv 1 \pmod 2$ and $k \geqslant 2$ since $40n + r^2 + 15 \equiv 0 \pmod 8$, hence by (2.1) we have

$$4^2(x_0^2 + 5y_0^2 + 10z_0^2) = \bar{x}_0^2 + 5\bar{y}_0^2 + 10\bar{z}_0^2$$

with $\bar{x}_0 = x_0 - 5y_0 - 10z_0$, $\bar{y}_0 = x_0 + 3y_0 - 2z_0$ and $\bar{z}_0 = x_0 - y_0 + 2z_0$ all odd, and therefore (2.2) holds.

Now we suppose that $k > 0$, $2 \mid x_0y_0z_0$ and $x_0 \equiv y_0 \pmod 2$. By (2.1),

$$4(x_0^2 + 5y_0^2 + 10z_0^2) = x_1^2 + 5y_1^2 + 10z_1^2$$

with

$$x_1 = \frac{x_0-y_0}{2} - 2y_0 - 5z_0,\ y_1 = \frac{x_0-y_0}{2} + 2y_0 - z_0,\ z_1 = \frac{x_0-y_0}{2} + z_0$$

If x_0 and y_0 are odd, then we may assume $x_0 \not\equiv y_0 - 2z_0 \pmod 4$ without loss of generality (otherwise we replace x_0 by $-x_0$), and hence x_1, y_1, z_1 are all odd. If $x_0, y_0, (x_0-y_0)/2$ are all even, then z_0 is odd and so are x_1, y_1, z_1. If x_0 and y_0 are even with $x_0 \not\equiv y_0 \pmod 4$, then z_0 is odd and we may assume $z_0 \equiv (y_0 - x_0)/2 \pmod 4$ without loss of generality (otherwise we replace z_0 by $-z_0$), hence $z_1 \equiv 0 \pmod 4$, $y_1 = z_1 + 2(y_0 - z_0) \equiv 0 \pmod 2$ and $(x_1 - y_1)/4 \equiv -y_0 - z_0 \equiv 1 \pmod 2$, therefore by (2.1) we have

$$x_1^2 + 5y_1^2 + 10z_1^2 = x_2^2 + 5y_2^2 + 10z_2^2$$

with

$$x_2 = \frac{x_1 - 5y_1 - 10z_1}{4},\ y_2 = \frac{x_1 + 3y_1 - 2z_1}{4},\ z_2 = \frac{x_1 - y_1 + 2z_1}{4}$$

all odd. So we still have (2.2).

By the above, there always exist odd integers x,y,z such that $40n+r^2+15=x^2+5y^2+10z^2$. Write $y=2u+1$ and $z=2v+1$ with $u,v\in\mathbb{Z}$. As $x^2\equiv r^2 \pmod 5$, either x or $-x$ has the form $10w+r$ with $w\in\mathbb{Z}$. Therefore

$$40n+r^2+15=(10w+r)^2+5(2u+1)^2+10(2v+1)^2$$

and hence $n=T_u+v(v+1)+w(5w+r)/2$. This proves the universality of $(5,r,2,2,1,1)$ over $\mathbb{Z}$.

There is an alternative way using (2.1) and Lemma 2.1 with

$$T=\begin{pmatrix}1/4 & -5/4 & -5/2\\ 1/4 & 3/4 & -1/2\\ 1/4 & -1/4 & 1/2\end{pmatrix}$$

to explain that $40n+r^2+15=x^2+5y^2+10z^2$ for some odd integers x,y,z.

(ii) Let $n\in\mathbb{N}$ and $r\in\{1,3\}$. Apparently,

$$\begin{aligned} & n=T_x+\frac{y(3y+1)}{2}+z(4z+r)\\ \iff & 48n+3r^2+8=6(2x+1)^2+2(6y+1)^2+3(8z+r)^2. \end{aligned}$$

Since

$$E(2,3,6)=\{3q+1:\ q\in\mathbb{N}\}\cup\{4^k(8l+7):\ k,l\in\mathbb{N}\}$$

by Dickson [7, pp. 112-113], we see that $48n+3r^2+8=2x^2+3y^2+6z^2$ for some $x,y,z\in\mathbb{Z}$. Clearly, $y^2+2z^2\neq 0$, and hence by [24, Lemma 2.1] we have $y^2+2z^2=y_0^2+2z_0^2$ for some $y_0,z_0\in\mathbb{Z}$ not all divisible by 3. Thus, without any loss of generality, we simply assume that $3\nmid y$ or $3\nmid z$. Note that $3\nmid x$, $2\nmid y$, and $x\equiv z \pmod 2$ since $2(x^2+z^2)\equiv 2x^2+6z^2\equiv 3r^2+8-3y^2\equiv 0 \pmod 4$. If $3\mid y$ and $3\nmid z$, then z or $-z$ is congruent to $x+y$ modulo 3. If $3\nmid y$ and $3\mid z$, then y or $-y$ is congruent to $x+z$ modulo 3. If $3\nmid yz$, then $\varepsilon_1 y\equiv\varepsilon_2 z\equiv x \pmod 3$ for some $\varepsilon_1,\varepsilon_2\in\{\pm1\}$. So, without loss of generality, we may assume that $x+y+z\equiv 0 \pmod 3$ (otherwise we may change signs of x,y,z suitably). Note that

$$48n+3r^2+8=2x^2+3y^2+6z^2=2a^2+3b^2+6c^2,$$

where $a=y+z$, $b=(2x-y+2z)/3$ and $c=(x+y-2z)/3$ are integers. If $x\equiv z\equiv 1 \pmod 2$, then x,y,z are all odd. If $x\equiv z\equiv 0 \pmod 2$, then a,b,c are all odd.

By the above, $48n+3r^2+8=2a^2+3b^2+6c^2$ for some odd integers a,b,c. Since $3b^2\equiv 3r^2+8-2a^2-6c^2\equiv 3r^2 \pmod{16}$, we can write b or $-b$ as

$8w+r$ with $w\in\mathbb{Z}$. Clearly, a or $-a$ has the form $6u+1$ with $u\in\mathbb{Z}$, and $c=2v+1$ for some $v\in\mathbb{Z}$. Therefore

$$48n+3r^2+8=2(6u+1)^2+3(8w+r)^2+6(2v+1)^2$$

and hence $n=u(3u+1)/2+T_v+w(4w+r)$. This proves the universality of $(8,2r,3,1,1,1)$ over $\mathbb{Z}$.

(iii) Let $n\in\mathbb{N}$. By Lemma 2.2(ii), we can write $6n+7$ in the form $x^2+3y^2+6z^2$ with $x,y,z\in\mathbb{Z}$ and $x\equiv n+1 \pmod 2$. Clearly, $y\equiv n \pmod 2$. Since $6z^2\equiv 6n+7-(n+1)^2-3n^2\equiv 6 \pmod 4$, we have $2\nmid z$. Hence

$$24n+28=4(6n+7)=4(x^2+3y^2+6z^2)=(x-3y)^2+3(x+y)^2+24z^2$$

with $x-2y$, $x+2y$ and z all odd. Note that $x-3y$ or $3y-x$ has the form $6w+1$ with $w\in\mathbb{Z}$. Write $x+y=2u+1$ and $z=2v+1$ with $u,v\in\mathbb{Z}$. Then

$$24n+28=(6w+1)^2+3(2u+1)^2+24(2v+1)^2$$

and hence $n=w(3w+1)/2+T_u+8T_v$. This proves the universality of $(8,8,3,1,1,1)$.

(iv) Let $n\in\mathbb{N}$. By Lemma 2.3, we can write $6n+5$ as $x^2+y^2+10z^2$ with $x,y,z\in\mathbb{Z}$. Clearly, $x\not\equiv y \pmod 2$. Since $x^2+y^2+z^2\equiv 2 \pmod 3$, exactly one of x,y,z is divisible by 3. Without loss of generality, we may assume that $x+y+z\equiv 0 \pmod 3$ (other we adjust signs of x,y,z suitably to meet our purpose). Observe that

$$4(x^2+y^2+10z^2)=2(x-y)^2+3\left(\frac{x+y+10z}{3}\right)^2+15\left(\frac{x+y-2z}{3}\right)^2.$$

So, $4(6n+5)=2a^2+3b^2+15c^2$ for some odd integers a,b,c. As $3\nmid a$, we may write a or $-a$ as $6w+1$ with $w\in\mathbb{Z}$. Write $b=2u+1$ and $c=2v+1$ with $u,v\in\mathbb{Z}$. Then

$$24n+20=2(6w+1)^2+3(2u+1)^2+15(2v+1)^2$$

and hence $n=T_u+5T_v+w(3w+1)$. This proves the universality of $(6,2,5,5,1,1)$ over $\mathbb{Z}$.

(v) Let $n\in\mathbb{N}$. By Lemma 2.2(i), we can write $12n+5$ in the form $x^2+y^2+(6z)^2$ with $x,y,z\in\mathbb{Z}$. It follows that $24n+10=(x+y)^2+(x-y)^2+72z^2$. As $(x+y)^2+(x-y)^2\equiv 10\equiv 2 \pmod 4$, both $x+y$ and $x-y$ are odd. Since $(x+y)^2+(x-y)^2\equiv 10\equiv 1 \pmod 3$, exactly one of $x+y$ and $x-y$ is divisible by 3. So $(x+y)^2+(x-y)^2=(6u+1)^2+(6v+3)^2$

for some $u, v \in \mathbb{Z}$. Therefore

$$24n + 10 = (6u + 1)^2 + (6v + 3)^2 + 72z^2$$

and hence $n = u(3u + 1)/2 + 3T_v + 3z^2$. This proves the universality of $(6, 0, 3, 3, 3, 1)$ over $\mathbb{Z}$.

By Lemma 2.4, we can write $12n + 14$ in the form $x^2 + 4y^2 + 9z^2$ with $x, y, z \in \mathbb{Z}$. Since $x^2 + z^2 \equiv 14 \pmod 4$, we have $2 \nmid xz$. Observe that

$$24n + 28 = 2(x^2 + 4y^2 + 9z^2) = (x - 2y)^2 + (x + 2y)^2 + 18z^2$$

with $x \pm 2y$ and z all odd. Clearly, exactly one of $x - 2y$ and $x + 2y$ is divisible by 3. So, for some $u, v, w \in \mathbb{Z}$ we have

$$24n + 28 = (6x + 1)^2 + 9(2y + 1)^2 + 18(2z + 1)^2$$

and hence $n = x(3x + 1)/2 + 3T_y + 6T_z$. This proves the universality of $(6, 6, 3, 3, 3, 1)$ over $\mathbb{Z}$.

The proof of Theorem 1.1 is now complete. $\square$

3. Proof of Theorem 1.2

The following lemma is one of the most important theorems about integral representations of quadratic forms (cf. [4, pp.129]).

Lemma 3.1. *Let f be a nonsingular integral quadratic form and let m be a nonzero integer which is represented by f over the real field $\mathbb{R}$ and the ring $\mathbb{Z}_p$ of p-adic integers for each prime p. Then m is represented by some form f^* over $\mathbb{Z}$ where f^* is in the same genus of f.*

Lemma 3.2. (i) [24, Lemma 3.2] *If $x^2 + 3y^2 \equiv 4 \pmod 8$ with $x, y \in \mathbb{Z}$, then $x^2 + 3y^2 = u^2 + 3v^2$ for some odd integers u and v.*

(ii) [24, Lemma 3.6] *If $w = x^2 + 7y^2 > 0$ with $x, y \in \mathbb{Z}$ and $8 \mid w$, then $w = u^2 + 7v^2$ for some odd integers u and v.*

(iii) [27, Lemma 5.1] *If $w = 3x^2 + 5y^2 > 0$ with $x, y \in \mathbb{Z}$ and $8 \mid w$, then $w = 3u^2 + 5v^2$ for some odd integers u and v.*

Proof of Theorem 1.2. (i) Let $n \in \mathbb{N}$. Clearly,

$$n = T_x + T_y + 5z(3z+1)/2 \iff 24n + 11 = 3(2x+1)^2 + 3(2y+1)^2 + 5(6z+1)^2.$$

There are two classes in the genus of $3x^2+3y^2+5z^2$, and the one not containing $3x^2+3y^2+5z^2$ has the representative

$$\begin{aligned}3x^2+2y^2+8z^2-2yz =&3x^2+3\left(\frac{y}{2}+z\right)^2+5\left(\frac{y}{2}-z\right)^2\\ =&3x^2+3\left(\frac{y-3z}{2}\right)^2+5\left(\frac{y+z}{2}\right)^2\end{aligned}$$

If $24n+11=3x^2+2y^2+8z^2-2yz$ with y odd and z even, then $3x^2\equiv 11-2y^2\equiv 9 \pmod 4$ which is impossible. Thus, if $24n+11\in\{3x^2+2y^2+8z^2-2yz:\ x,y,z\in\mathbb{Z}\}$ then $24n+11\in\{3x^2+3y^2+5z^2:\ x,y,z\in\mathbb{Z}\}$. With the help of Lemma 3.1, there are $x,y,z\in\mathbb{Z}$ such that $24n+11=3x^2+3y^2+5z^2$. As $5z^2\not\equiv 11 \pmod 4$, x and y cannot be both even. Without loss of generality, we assume that $2\nmid x$. Then $3y^2+5z^2\equiv 11-3x^2\equiv 0 \pmod 8$ and $3y^2+5z^2\neq 0$. By Lemma 3.2(iii), $3y^2+5z^2=3y_0^2+5z_0^2$ for some odd integers y_0 and z_0. Write $x=2u+1$ and $y_0=2v+1$ with $u,v\in\mathbb{Z}$. As $2\nmid z_0$ and $3\nmid z_0$, z_0 or $-z_0$ has the form $6w+1$ with $w\in\mathbb{Z}$. Thus $24n+11=3(2u+1)^2+3(2v+1)^2+5(6w+1)^2$ and hence $n=T_u+T_v+5w(3w+1)/2$. This proves the universality of $(15,5,1,1,1,1)$ over $\mathbb{Z}$.

(ii) Let $n\in\mathbb{N}$ and $r\in\{1,3\}$. Obviously,

$$\begin{aligned}&n=T_x+\frac{y(3y+1)}{2}+3\frac{z(5z+r)}{2}\\ \iff& 120n+9r^2+20=15(2x+1)^2+5(6y+1)^2+9(10z+r)^2.\end{aligned}$$

There are two classes in the genus of $x^2+15y^2+5z^2$, and the one not containing $x^2+15y^2+5z^2$ has the representative

$$\begin{aligned}4x^2+4y^2+5z^2+2xy=&\left(\frac{x}{2}+2y\right)^2+15\left(\frac{x}{2}\right)^2+5z^2\\ =&\left(2x+\frac{y}{2}\right)^2+15\left(\frac{y}{2}\right)^2+5z^2.\end{aligned}$$

If $120n+9r^2+20=4x^2+4y^2+5z^2+2xy$ with $x,y,z\in\mathbb{Z}$, then $2xy\equiv 9r^2-5z^2\equiv 0 \pmod 4$ and hence x or y is even. Thus, with the help of Lemma 3.1, we can always write $120n+9r^2+20=x^2+15y^2+5z^2$ with $x,y,z\in\mathbb{Z}$. Since $x^2+5z^2\equiv 20\equiv 2 \pmod 3$, $x=3x_0$ for some $x_0\in\mathbb{Z}$. As $15y^2\not\equiv 9r^2 \pmod 4$, x and z cannot be both even. If $2\nmid x$, then $5(3y^2+z^2)\equiv 9r^2+20-x^2\equiv 4 \pmod 8$ and hence by Lemma 3.2(i) we can write $3y^2+z^2$ as $3y_0^2+z_0^2$ with y_0 and z_0 both odd. If $2\nmid z$, then $x^2+15y^2\neq 0$ and $x^2+15y^2=3(3x_0^2+5y^2)\equiv 9r^2+20-5z^2\equiv 0 \pmod 8$, hence by Lemma 3.2(iii) we can write $3x_0^2+5y^2$ as $3x_1^2+5y_1^2$ with x_1 and y_1 both odd.

By the above, there are odd integers x, y, z such that $120n + 9r^2 + 20 = 9x^2 + 15y^2 + 5z^2$. Write $y = 2u + 1$ with $u \in \mathbb{Z}$. As $3 \nmid z$, we can write z or $-z$ as $6v + 1$ with $v \in \mathbb{Z}$. Since $x^2 \equiv r^2 \pmod 5$, we can write x or $-x$ as $10w + r$ with $w \in \mathbb{Z}$. Thus

$$120n + 9r^2 + 20 = 15(2u + 1)^2 + 5(6v + 1)^2 + 9(10z + r)^2$$

and hence $n = T_x + y(3y+1)/2 + 3z(5z+r)/2$ with $x, y, z \in \mathbb{Z}$. This proves the universality of $(15, 3r, 3, 1, 1, 1)$ over $\mathbb{Z}$.

(iii) Let $n \in \mathbb{N}$ and $r \in \{1, 3\}$. Apparently,

$$\begin{aligned} & n = 3x^2 + \frac{y(3y+1)}{2} + \frac{z(5z+r)}{2} \\ \iff & 120n + 3r^2 + 5 = 360x^2 + 5(6y + s)^2 + 3(10z + r)^2. \end{aligned}$$

If $60n + (3r^2 + 5)/2 = 4x^2 + 4y^2 + 5z^2 + 2xy$ with $x, y, z \in \mathbb{Z}$, then x or y must be even. Thus, as in part (ii), $60n + (3r^2 + 5)/2 = x^2 + 5y^2 + 15z^2$ for some $x, y, z \in \mathbb{Z}$. Note that $x^2 + y^2 \equiv z^2 \pmod 4$. If y is odd, then $2 \mid x$, $2 \nmid z$ and we may assume $y \not\equiv z \pmod 4$ (otherwise it suffices to change the sign of z), hence

$$y^2 + 3z^2 = \left(\frac{y - 3z}{2}\right)^2 + 3\left(\frac{y + z}{2}\right)^2$$

with $y_1 = (y - 3z)/2$ and $z_1 = (y + z)/2$ both even. So, without loss of generality, we may simply assume that $2 \mid y$ and $x \equiv z \pmod 2$. Observe that

$$120n + 3r^2 + 5 = 2(x^2 + 5y^2 + 15z^2) = 3a^2 + 5b^2 + 10y^2.$$

with $a = (x + 5z)/2$ and $b = (x - 3z)/2$ both integral. Since $3a^2 + 5b^2 \equiv 5s^2 + 3t^2 - 10y^2 \equiv 0 \pmod 8$ and $3a^2 + 5b^2 > 0$, by Lemma 3.2(iii) we can write $3a^2 + 5b^2 = 3c^2 + 5d^2$ with c and d both odd. Thus

$$120n + 3r^2 + 5 = 3c^2 + 5d^2 + 40\left(\frac{y}{2}\right)^2.$$

As $(y/2)^2 \equiv 5(1 - d^2) \equiv d^2 - 1 \pmod 3$, we must have $3 \nmid d$ and $3 \mid y$. Write $y = 6u$ with $u \in \mathbb{Z}$. Clearly, d or $-d$ has the form $6v + 1$ with $v \in \mathbb{Z}$. Since $c^2 \equiv r^2 \pmod 5$, we may write c or $-c$ as $10w + r$ with $w \in \mathbb{Z}$. Therefore

$$120n + 3r^2 + 5 = 3(10w + r)^2 + 5(6v + 1)^2 + 40(3u)^2$$

and hence $n = 3u^2 + v(3v+1)/2 + w(5w+r)/2$. This proves the universality of $(6, 0, 5, r, 3, 1)$ over $\mathbb{Z}$.

(iv) Let $n\in\mathbb{N}$ and $\delta\in\{0,1\}$. Clearly,

$$\begin{aligned} &n=x(x+\delta)+\frac{y(3y+1)}{2}+5\frac{z(3z+1)}{2}\\ \iff &24n+6(\delta+1)=6(2x+\delta)^2+(6y+1)^2+5(6z+1)^2. \end{aligned}$$

There are two classes in the genus of $x^2+5y^2+6z^2$, and the one not containing $x^2+5y^2+6z^2$ has the representative $3x^2+3y^2+4z^2-2yz+2zx$. If $24n+6(\delta+1)=3x^2+3y^2+4z^2-2yz+2zx$, then $u=(x+y)/2$ and $v=(x-y)/2$ are integers, and

$$24n+6(\delta+1)=6u^2+6v^2+4z^2+4vz=6u^2+5v^2+(v+2z)^2.$$

Thus, by Lemma 3.1, $24n+6(\delta+1)=x^2+5y^2+6z^2$ for some $x,y,z\in\mathbb{Z}$. Since $x^2\equiv-5y^2\equiv y^2\pmod 3$, we may assume that $x\equiv y\pmod 3$ without loss of generality. If $z\not\equiv\delta\pmod 2$, then $x^2+5y^2\equiv 6(\delta+1)-6z^2\equiv 6(\delta+1)-6(1-\delta)\equiv 4\delta\pmod 8$, hence both x and y are even and $(x-y)/2\equiv\delta\pmod 2$, and thus

$$x^2+5y^2+6z^2=\left(z-\frac{5(x-y)}{6}\right)^2+5\left(\frac{x-y}{6}+z\right)^2+6\left(\frac{x-y}{6}+y\right)^2$$

with $(x-y)/6+y\equiv(x-y)/2\equiv\delta\pmod 2$.

By the above, $24n+6(\delta+1)=x^2+5y^2+6z^2$ for some $x,y,z\in\mathbb{Z}$ with $x,y,z\in\mathbb{Z}$ with $z\equiv\delta\pmod 2$. Since x^2+5y^2 is a positive multiple of 3, by [24, Lemma 2.1] we can write $x^2+5y^2=x_0^2+5y_0^2$ with $x_0y_0\in\mathbb{Z}$ and $3\nmid x_0y_0$. So, there are $x,y,z\in\mathbb{Z}$ with $x\equiv y\not\equiv 0\pmod 3$ and $z\equiv\delta\pmod 2$ such that $24n+6(\delta+1)=x^2+5y^2+6z^2$. Write $z=2w+\delta$ with $w\in\mathbb{Z}$. Since $x^2+5y^2\equiv 6\pmod 8$, both x and y are odd. Thus x or $-x$ has the form $6u+1$ with $u\in\mathbb{Z}$, and y or $-y$ has the form $6v+1$ with $v\in\mathbb{Z}$. Therefore

$$24n+6(\delta+1)=(6u+1)^2+5(6v+1)^2+6(2w+\delta)^2$$

and hence $n=w(w+\delta)+u(3u+1)/2+5v(3v+1)/2$. This proves the universality of $(15,5,3,1,2,2\delta)$ over $\mathbb{Z}$.

(v) Let $n\in\mathbb{N}$. Apparently,

$$\begin{aligned} &n=x(x+1)+\frac{y(3y+1)}{2}+7\frac{z(3z+1)}{2}\\ \iff &24n+14=6(2x+1)^2+(6y+1)^2+7(6z+1)^2. \end{aligned}$$

There are two classes in the genus of $x^2+6y^2+7z^2$, and the one not containing $x^2+6y^2+7z^2$ has the representative

$$2x^2+5y^2+5z^2-4yz=2x^2+10u^2+10v^2-4(u+v)(u-v)=2x^2+6u^2+14v^2$$

with $u = (y+z)/2$ and $v = (y-z)/2$. If $24n+14 = 2x^2+6u^2+14v^2$ for some $x,u,v \in \mathbb{Z}$ with $x \not\equiv v \pmod 2$, then $14 \equiv 2+6u^2 \pmod 8$ which is impossible. If $24n+14 = 2x^2+6u^2+14v^2$ with $x,u,v \in \mathbb{Z}$ and $x \equiv v \pmod 2$, then

$$24n+14 = 6u^2 + \left(\frac{x-7v}{2}\right)^2 + 7\left(\frac{x+v}{2}\right)^2.$$

By the above and Lemma 3.1, there are $x,y,z \in \mathbb{Z}$ such that $24n+14 = 6x^2+y^2+7z^2$. If $2 \mid x$, then $y^2+7z^2 \equiv 6-6x^2 \equiv 6 \pmod 8$ which is impossible. So $x = 2u+1$ for some $u \in \mathbb{Z}$. Note that $y^2+7z^2 \equiv 6-6x^2 \equiv 0 \pmod 8$ and $y^2+7z^2 \neq 0$. Applying Lemma 3.2(ii) we can write y^2+7z^2 as $y_0^2+7z_0^2$ with y_0 and z_0 both odd. Note that $y_0^2+z_0^2 \equiv y_0^2+7z_0^2 \equiv 14 \equiv 2 \pmod 3$. So y_0 or $-y_0$ can be written as $6v+1$ with $v \in \mathbb{Z}$, and z_0 or $-z_0$ has the form $6w+1$ with $w \in \mathbb{Z}$. Thus

$$24n+14 = 6x^2+y_0^2+7z_0^2 = 6(2u+1)^2+(6v+1)^2+7(6w+1)^2$$

and hence $n = u(u+1)+v(3v+1)/2+7z(3z+1)/2$. This proves the universality of $(21,7,3,1,2,2)$.

(vi) Let $r \in \{1,3,5\}$ and $n \in \mathbb{N}$. Clearly,

$$n = T_x+T_y+\frac{z(7z+r)}{2} \iff 56n+14+r^2 = 7(2x+1)^2+7(2y+1)^2+(14z+r)^2.$$

There are two classes in the genus of $x^2+7y^2+7z^2$, and the one not containing $x^2+7y^2+7z^2$ has the representative

$$\begin{aligned} 2x^2+4y^2+7z^2+2xy &= \left(\frac{x}{2}+2y\right)^2+7\left(\frac{x}{2}\right)^2+7z^2. \\ &= \left(\frac{x-3y}{2}\right)^2+7\left(\frac{x+y}{2}\right)^2+7z^2 \end{aligned}$$

If $56n+14+r^2 = 2x^2+4y^2+7z^2+2xy$ with x odd and y even, then $15 \equiv 14+r^2 \equiv 2x^2+7z^2 \equiv 9 \pmod 4$ which is impossible. Thus, if $56n+14+r^2 \in \{2x^2+4y^2+7z^2+2xy:\ x,y,z \in \mathbb{Z}\}$ then $56n+14+r^2 \in \{x^2+7y^2+7z^2:\ x,y,z \in \mathbb{Z}\}$. With the help of Lemma 3.1, there are $x,y,z \in \mathbb{Z}$ such that $56n+14+r^2 = x^2+7y^2+7z^2$. As $x^2 \not\equiv 14+r^2 \equiv 15 \pmod 4$, y and z cannot be both even. Without loss of generality, we assume that $2 \nmid z$. Then $x^2+7y^2 \equiv 14+r^2-7z^2 \equiv 0 \pmod 8$ and $x^2+7y^2 \neq 0$. By Lemma 3.2(ii), $x^2+7y^2 = x_0^2+7y_0^2$ for some odd integers x_0 and y_0. Now $56n+14+r^2 = x_0^2+7y_0^2+7z^2$. Clearly, x_0 or $-x_0$ has the form $14w+r$ with $w \in \mathbb{Z}$. Write $y_0 = 2u+1$ and $z = 2v+1$ with $u,v \in \mathbb{Z}$. Then

$$56n+14+r^2 = (14w+r)^2+7(2u+1)^2+7(2v+1)^2$$

and hence $n = T_u + T_v + w(7w + r)/2$. This proves the universality of $(7, r, 1, 1, 1, 1)$ over $\mathbb{Z}$.

(vii) Let $n \in \mathbb{N}$, $s \in \{1, 3\}$ and $t \in \{1, 3, 5\}$. Clearly,

$$\begin{aligned} & n = T_x + \frac{y(3y+s)}{2} + \frac{z(7z+t)}{2} \\ \iff & 168n + 21 + 7s^2 + 3t^2 = 21(2x+1)^2 + 7(6y+s)^2 + 3(14z+t)^2. \end{aligned}$$

There are two classes in the genus of $3x^2 + 21y^2 + 7z^2$, and the one not containing $3x^2 + 21y^2 + 7z^2$ has the representative

$$\begin{aligned} 6x^2 + 12y^2 + 7z^2 + 6xy =& 3\left(\frac{x}{2} + 2y\right)^2 + 21\left(\frac{x}{2}\right)^2 + 7z^2. \\ =& 3\left(\frac{x-3y}{2}\right)^2 + 21\left(\frac{x+y}{2}\right)^2 + 7z^2 \end{aligned}$$

If $168n + 21 + 7s^2 + 3t^2 = 6x^2 + 12y^2 + 7z^2 + 6xy$ with x odd and y even, then $31 \equiv 21 + 7s^2 + 3t^2 \equiv 6x^2 + 7z^2 \equiv 13 \pmod 4$ which is impossible. Thus, if $168n + 21 + 7s^2 + 3t^2 \in \{6x^2 + 12y^2 + 7z^2 + 6xy : \ x, y, z \in \mathbb{Z}\}$ then $168n + 21 + 7s^2 + 3t^2 \in \{3x^2 + 21y^2 + 7z^2 : \ x, y, z \in \mathbb{Z}\}$. With the help of Lemma 3.1, there are $x, y, z \in \mathbb{Z}$ such that $168n + 21 + 7s^2 + 3t^2 = 3x^2 + 21y^2 + 7z^2$. As $21y^2 \not\equiv 21 + 7s^2 + 3t^2 \equiv 31 \pmod 4$, x and z cannot be both even. If $2 \nmid x$, then $21y^2 + 7z^2 \equiv 21 + 7s^2 + 3t^2 - 3x^2 \equiv 4 \pmod 8$ and hence by Lemma 3.2(i) we can write $3y^2 + z^2$ as $3y_0^2 + z_0^2$ with y_0, z_0 odd integers. Note that $x^2 + 7y^2 \neq 0$ since $7 \nmid t$. If $2 \nmid z$, then $3(x^2 + 7y^2) \equiv 21 + 7s^2 + 3t^2 - 7z^2 \equiv 0 \pmod 8$ and hence by Lemma 3.2(ii) $x^2 + 7y^2 = x_0^2 + 7y_0^2$ for some odd integers x_0 and y_0.

By the above, there are odd integers x, y, z such that $168n+21+7s^2+3t^2 = 3x^2 + 7y^2 + 21z^2$. Write $z = 2u + 1$ with $u \in \mathbb{Z}$. As $y^2 \equiv s^2 \pmod 3$, y or $-y$ has the form $6v + s$ with $v \in \mathbb{Z}$. Since $x^2 \equiv t^2 \pmod 7$, x or $-x$ has the form $14w + t$ with $w \in \mathbb{Z}$. Thus

$$168n + 21 + 7s^2 + 3t^2 = 3(14w + t)^2 + 7(6v + s)^2 + 21(2u + 1)^2$$

and hence $n = T_u + v(3v + s)/2 + w(7w + t)/2$. This proves the universality of $(7, t, 3, s, 1, 1)$ over $\mathbb{Z}$.

(viii) Let $\delta \in \{0, 1\}$ and $r \in \{1, 3, 5\}$. Clearly,

$$\begin{aligned} & n = T_x + y(y + \delta) + \frac{z(7z+r)}{2} \\ \iff & 56n + 14\delta + r^2 + 7 = 7(2x+1)^2 + 14(2y+\delta)^2 + (14z+r)^2. \end{aligned}$$

There are two classes in the genus of $x^2+7y^2+14z^2$, the one not containing $x^2+7y^2+14z^2$ has the representative

$$2x^2+7y^2+7z^2=2x^2+14\left(\frac{y+z}{2}\right)^2+14\left(\frac{y-z}{2}\right)^2.$$

If $56n+14\delta+r^2+7=2x^2+14y^2+14z^2$ with $x,y,z\in\mathbb{Z}$ and $y\equiv z\pmod 2$, then $2x^2\equiv 14\delta+r^2+7\equiv 2\delta\pmod 4$, hence $x^2\equiv\delta\pmod 4$ and $y\equiv z\equiv\delta\pmod 2$ since

$$-2(y^2+z^2)\equiv 14(y^2+z^2)\equiv 14\delta+r^2+7-2\delta\equiv -4\delta\pmod 8.$$

If $56n+14\delta+r^2+7=2x^2+14y^2+14z^2$ with $x,y,z\in\mathbb{Z}$ and $x\equiv y\pmod 2$, then

$$56n+14\delta+r^2+7=\left(\frac{x-7y}{2}\right)^2+7\left(\frac{x+y}{2}\right)^2+14z^2.$$

In view of Lemma 3.1 and the above, there are $x,y,z\in\mathbb{Z}$ such that $56n+14\delta+r^2+7=x^2+7y^2+14z^2$. If $z\not\equiv\delta\pmod 2$, then

$$x^2+7y^2\equiv 14\delta+r^2+7-14z^2\equiv 14\delta-14(1-\delta)\equiv 2\pmod 4$$

which is impossible. Thus $z\equiv\delta\pmod 2$ and $x^2+7y^2\equiv r^2+7\equiv 0\pmod 8$. Note that $x^2+7y^2\neq 0$ since $7\nmid r$. Applying Lemma 3.2(ii) we can write x^2+7y^2 as $x_0^2+7y_0^2$ with x_0 and y_0 both odd. Since $x_0^2\equiv r^2\pmod 7$, either x_0 or $-x_0$ has the form $14w+r$ with $w\in\mathbb{Z}$. Write $y_0=2u+1$ and $z=2v+\delta$ with $u,v\in\mathbb{Z}$. Then

$$56n+14\delta+r^2+7\equiv(14w+r)^2+7(2u+1)^2+14(2v+\delta)^2$$

and hence $n=T_u+v(v+\delta)+w(7w+r)/2$. This proves the universality of $(7,r,2,2\delta,1,1)$ over $\mathbb{Z}$.

The proof of Theorem 1.2 is now complete. □

4. Proof of Theorem 1.3

For a positive definite integral ternary quadratic form $f(x,y,z)$ and an integer n, as usual we define

$$r(n,f):=|\{(x,y,z)\in\mathbb{Z}^3:\ f(x,y,z)=n\}|$$

and adopt the standard notation $r(n,\mathrm{gen}(f))$ introduced in [17, pp. 173–174].

Lemma 4.1. *Let f be a positive ternary quadratic form with determinant $d(f)$. Suppose that $m \in \mathbb{Z}^+$ is represented by the genus of f. Then, for each prime $p \nmid 2md(f)$, we have*

$$\frac{r(mp^2, \mathrm{gen}(f))}{r(m, \mathrm{gen}(f))} = p + 1 - \left(\frac{-md(f)}{p}\right). \tag{4.1}$$

Proof. By the Minkowski-Siegel formula [17, pp. 173–174],

$$r(mp^2, \mathrm{gen}(f)) = 2\pi\sqrt{\frac{mp^2}{d(f)}} \prod_q \alpha_q(mp^2, f),$$

where q runs over all primes and α_q is the local density. As $p \nmid 2md(f)$, by [29] we have

$$\alpha_p(mp^2, f) = 1 + \frac{1}{p} - \frac{1}{p^2} + \left(\frac{-md(f)}{p}\right)\frac{1}{p^2},$$
$$\alpha_p(m, f) = 1 + \left(\frac{-md(f)}{p}\right)\frac{1}{p}.$$

Thus

$$\frac{r(mp^2, \mathrm{gen}(f))}{r(m, \mathrm{gen}(f))} = p\frac{\alpha_p(mp^2, f)}{\alpha_p(m, f)} = p + 1 - \left(\frac{-md(f)}{p}\right).$$

This concludes the proof. □

Lemma 4.2. *Let $w = u^2 + 15v^2 > 0$ with $u, v \in \mathbb{Z}$ and $8 \mid w$. Then $w = x^2 + 15y^2$ for some odd integers x and y.*

Proof. Let k be the 2-adic order of $\gcd(u, v)$, and write $u = 2^k u_0$ and $v = 2^k v_0$ with $u_0, v_0 \in \mathbb{Z}$ not all even. If $k = 0$, then both u_0 and v_0 are odd since w is even. Below we assume that $k > 0$.

We observe the identity

$$4^2(x^2 + 15y^2) = (x - 15y)^2 + 15(x + y)^2.$$

If $u_0 \not\equiv v_0 \pmod 2$, then $k \geqslant 2$ (since $8 \mid w$) and $4^2(u_0^2+15v_0^2) = s^2+15t^2$ with $s = u_0-15v_0$ and $t = u_0+v_0$ both odd. For $j \in \mathbb{N}$, if $4^j(u_0^2+15v_0^2) = u_j^2+15v_j^2$ for some odd integers u_j and v_j, then we may assume $u_j \equiv v_j \pmod 4$ without loss of generality (otherwise we may replace v_j by $-v_j$), and hence

$$4^{j+1}(u_0^2 + 15v_0^2) = 4(u_j^2 + 15v_j^2) = u_{j+1}^2 + 15v_{j+1}^2$$

with $u_{j+1} = (u_j - 15v_j)/2$ and $v_{j+1} = (u_j + v_j)/2$ both odd. Thus, for some odd integers u_k and v_k, we have

$$w = 4^k(u_0^2 + 15v_0^2) = u_k^2 + 15v_k^2.$$

This concludes the proof. □

Proof of Theorem 1.3(i). (a) We first prove that $(7,7,3,1,1,1)$ is universal over $\mathbb{Z}$. Let $n\in\mathbb{N}$. Clearly,

$$\begin{aligned} &n=T_x+7T_y+\frac{z(3z+1)}{2}\\ \iff &24n+25=3(2x+1)^2+21(2y+1)^2+(2z+1)^2. \end{aligned}$$

There are two classes in the genus of $x^2+3y^2+21z^2$ and the one not containing $x^2+3y^2+21z^2$ has the representative

$$\begin{aligned} x^2+6y^2+12z^2-6yz =&x^2+3\left(\frac{y}{2}-2z\right)^2+21\left(\frac{y}{2}\right)^2\\ =&x^2+3\left(\frac{y+3z}{2}\right)^2+21\left(\frac{y-z}{2}\right)^2. \end{aligned} \tag{4.2}$$

If $24n+25=x^2+6y^2+12z^2-6yz$ with $x,y,z\in\mathbb{Z}$, then the equality modulo 4 yields $y(y-z)\equiv 0 \pmod 2$. Thus, by (4.2) and Lemma 3.1, we have

$$24n+25\in\{x^2+3y^2+21z^2:\ x,y,z\in\mathbb{Z}\}. \tag{4.3}$$

Now we claim that $24n+25=x^2+3y^2+21z^2$ for some $x,y,z\in\mathbb{Z}$ with $y^2+7z^2>0$. This holds by (4.3) if $24n+25$ is not a square. Suppose that $24n+25=m^2$ with $m\in\mathbb{Z}^+$. Let p be any prime divisor of m. Clearly, $p\geqslant 5$. Note that $r(7^2,x^2+3y^2+21z^2)>2$ since $7^2=(\pm5)^2+3\times(\pm1)^2+21\times(\pm1)^2$. If $p\neq 7$ and $r(p^2,x^2+6y^2+12z^2-6yz)>2$, then $p^2=x^2+6y^2+12z^2-6yz$ for some $x,y,z\in\mathbb{Z}$ with $2\mid y(y-z)$ and $y^2+z^2>0$, hence by (4.2) we have $p^2=x^2+3u^2+21v^2$ for some $x,u,v\in\mathbb{Z}$ with $u^2+7v^2>0$, and thus $r(p^2,x^2+3y^2+21z^2)>2$. By Lemma 4.1, if $p\neq 7$ then

$$\frac{r(p^2,\mathrm{gen}(x^2+3y^2+21z^2))}{r(1,\mathrm{gen}(x^2+3y^2+21z^2))}=p+1-\left(\frac{-7}{p}\right)$$

and hence

$$r(p^2,x^2+3y^2+21z^2)+r(p^2,x^2+6y^2+12z^2-6yz)=4\left(p+1-\left(\frac{-7}{p}\right)\right)>4.$$

So we still have $r(p^2,x^2+3y^2+21z^2)>2$ if $r(p^2,x^2+6y^2+12z^2-6yz)\leqslant 2$. As $r(m^2,x^2+3y^2+21z^2)\geqslant r(p^2,x^2+3y^2+21z^2)>2$, we can write $24n+25=m^2$ as $x^2+3y^2+21z^2$ with $x,y,z\in\mathbb{Z}$ and $y^2+7z^2>0$. This proves the claim.

By the claim, there are $x,y,z\in\mathbb{Z}$ such that $24n+25=x^2+3y^2+21z^2$ and $y^2+7z^2>0$. As $3y^2\not\equiv 25\equiv 1 \pmod 4$, either x or z is odd. If $2\nmid x$, then $3(y^2+7z^2)\equiv 25-x^2\equiv 0 \pmod 8$ and hence by Lemma 3.2(ii) we can write y^2+7z^2 as $y_0^2+7z_0^2$ with y_0 and z_0 both odd. If $2\nmid z$, then $x^2+3y^2\equiv 25-21z^2\equiv 4 \pmod 8$ and hence by Lemma 3.2(i) we can write

x^2+3y^2 as $x_1^2+3y_1^2$ with x_1 and y_1 both odd. Thus $24n+25=a^2+3b^2+21c^2$ for some odd integers a,b,c. As $3\nmid a$, either a or $-a$ has the form $6w+1$ with $w\in\mathbb{Z}$. Write $b=2u+1$ and $c=2v+1$ with $u,v\in\mathbb{Z}$. Then

$$24n+25=(6w+1)^2+3(2u+1)^2+21(2v+1)^2$$

and hence $n=T_u+7T_v+w(3w+1)/2$. This proves the universality of $(7,7,3,1,1,1)$ over $\mathbb{Z}$.

(b) Let $n\in\mathbb{N}$ and $r\in\{1,3\}$. Clearly,

$$\begin{aligned} &n=5T_x+\frac{y(3y+1)}{2}+\frac{z(3z+r)}{2}\\ \iff&24n+r^2+16=15(2x+1)^2+(6y+1)^2+(6z+r)^2. \end{aligned}$$

There are two classes in the genus of $x^2+y^2+15z^2$, and the one not containing $x^2+y^2+15z^2$ has the representative

$$\begin{aligned} x^2+4y^2+4z^2-2yz=&x^2+\left(\frac{y}{2}-2z\right)^2+15\left(\frac{y}{2}\right)^2\\ =&x^2+\left(2y-\frac{z}{2}\right)^2+15\left(\frac{z}{2}\right)^2. \end{aligned}\tag{4.4}$$

If $24n+r^2+16=x^2+4y^2+4z^2-2yz$ with $x,y,z\in\mathbb{Z}$, then $2\nmid x$ and $2\mid yz$. Thus, in view of (4.4) and Lemma 3.1, we have

$$24n+r^2+16\in\{x^2+y^2+15z^2:\ x,y,z\in\mathbb{Z}\}.\tag{4.5}$$

We claim that $24n+r^2+16=x^2+y^2+15z^2$ for some $x,y,z\in\mathbb{Z}$ with $(x^2+15z^2)(y^2+15z^2)>0$. This holds by (4.5) if $24n+r^2+16$ is not a square. Now suppose that $24n+r^2+16=m^2$ with $m\in\mathbb{Z}^+$. Let p be any prime divisor of m. Clearly, $p\geqslant 5$. Note that $r(5^2,x^2+y^2+15z^2)>4$ since

$$5^2=(\pm5)^2+0^2+15\times0^2=0^2+(\pm5)^2+15\times0^2=(\pm3)^2+(\pm4)^2+15\times0^2.$$

If $r(p^2,x^2+4y^2+4z^2-2yz)>2$, then $p^2=x^2+4y^2+4z^2-2yz$ for some $x,y,z\in\mathbb{Z}$ with $2\mid yz$ and $y^2+z^2>0$, hence by (4.4) $p^2=x^2+u^2+15v^2$ for some $x,u,v\in\mathbb{Z}$ with $(x^2+15v^2)(u^2+15v^2)>0$, and thus $r(p^2,x^2+y^2+15z^2)>4$. When $p>5$, by Lemma 4.1 we have

$$\frac{r(p^2,\mathrm{gen}(x^2+y^2+15z^2))}{r(1,\mathrm{gen}(x^2+y^2+15z^2))}=p+1-\left(\frac{-15}{p}\right)$$

and hence

$$r(p^2,x^2+y^2+15z^2)+2r(p^2,x^2+4y^2+4z^2-2yz)=8\left(p+1-\left(\frac{-15}{p}\right)\right)>50.$$

Thus we still have $r(p^2, x^2+y^2+15z^2) > 4$ if $r(p^2, x^2+4y^2+4z^2-2yz) \leqslant 2$. As $r(m^2, x^2+y^2+15z^2) \geqslant r(p^2, x^2+y^2+15z^2) > 4$, we can write $24n+r^2+16$ as $x^2+y^2+15z^2$ with $(x^2+15z^2)(y^2+15z^2) > 0$. This proves the claim.

By the claim, there are $x, y, z \in \mathbb{Z}$ such that $24n+r^2+16 = x^2+y^2+15z^2$ and $(x^2+15z^2)(y^2+15z^2) > 0$. Since $15z^2 \not\equiv r^2 \equiv 1 \pmod 4$, either x or y is odd. Without any loss of generality, we assume that $2 \nmid x$. Since $y^2+15z^2 > 0$ and $y^2+15z^2 \equiv r^2 - x^2 \equiv 0 \pmod 8$, by Lemma 4.2 we can write $y^2+15z^2 = y_0^2+15z_0^2$ with y_0 and z_0 both odd. Now, $24n+r^2+16 = x^2+y_0^2+15z_0^2$. Since $x^2+y_0^2 \equiv r^2+1 \pmod 3$, one of x^2 and y_0^2 is congruent to r^2 modulo 3 and the other one is congruent to 1 modulo 3. Thus $x^2+y_0^2 = (6u+r)^2+(6v+1)^2$ for some $u, v \in \mathbb{Z}$. Write $z_0 = 2w+1$ with $v \in \mathbb{Z}$. Then

$$24n+r^2+16 = (6u+r)^2+(6v+1)^2+15(2w+1)^2$$

and hence $n = u(3u+r)/2+v(3v+1)/2+5T_w$. This proves the universality of $(5,5,3,r,3,1)$ over $\mathbb{Z}$.

(c) Let $n \in \mathbb{N}$. Apparently,

$$\begin{aligned} & n = T_x + 5T_y + z(3z+2) \\ \Longleftrightarrow\ & 24n+26 = 3(2x+1)^2+15(2y+1)^2+2(6z+2)^2. \end{aligned}$$

There are two classes in the genus of $2x^2+3y^2+15z^2$, and the one not containing $2x^2+3y^2+15z^2$ has the representative

$$g(x,y,z) = 2x^2+5y^2+11z^2+2yz+2x(y-z) = 2\,(x+v)^2+3(u-2v)^2+15u^2 \tag{4.6}$$

with $u = (y+z)/2$ and $v = (y-z)/2$. If $24n+26 = g(x,y,z)$ with $x, y, z \in \mathbb{Z}$, then $y \equiv z \pmod 2$, and hence by (4.6) we have $24n+26 = 2a^2+3b^2+15c^2$ for some $a, b, c \in \mathbb{Z}$. So, in view of Lemma 3.1, we always have

$$24n+26 \in \{2x^2+3y^2+15z^2 :\ x, y, z \in \mathbb{Z}\}. \tag{4.7}$$

We claim that $24n+26 = 2x^2+3y^2+15z^2$ for some $x, y, z \in \mathbb{Z}$ with $y^2+5z^2 > 0$. This holds by (4.7) if $12n+13$ is not a square. Now suppose that $12n+13 = m^2$ with $m \in \mathbb{Z}^+$. Let p be any prime divisor of m. Clearly, $p \geqslant 5$. Note that $r(2\times 5^2, 2x^2+3y^2+15z^2) > 2$ since

$$2\times 5^2 = 2\times(\pm 5)^2+3\times 0^2+15\times 0^2 = 2(\pm 1)^2+3(\pm 4)^2+30\times 0^2.$$

If $r(2p^2, g(x,y,z)) > 2$, then $2p^2 = g(x,y,z)$ for some $x, y, z \in \mathbb{Z}$ with $y^2+z^2 > 0$, hence by (4.6) $2p^2 = 2x^2+3b^2+15c^2$ for some $x, b, c \in \mathbb{Z}$ with

$b^2+c^2>0$, and thus $r(2p^2,2x^2+3y^2+15z^2)>2$. When $p>5$, by Lemma 4.1 we have

$$\frac{r(2p^2,\mathrm{gen}(2x^2+3y^2+15z^2))}{r(2,\mathrm{gen}(2x^2+3y^2+15z^2))}=p+1-\left(\frac{-5}{p}\right)$$

and hence

$$r(2p^2,2x^2+3y^2+15z^2)+2r(2p^2,g(x,y,z))=6\left(p+1-\left(\frac{-5}{p}\right)\right)>40.$$

Thus we still have $r(2p^2,2x^2+3y^2+15z^2)>2$ if $r(2p^2,g(x,y,z))\leqslant 2$. As $r(2m^2,2x^2+3y^2+15z^2)\geqslant r(2p^2,2x^2+3y^2+15z^2)>2$, we can write $24n+26$ as $2x^2+3y^2+15z^2$ with $y^2+5z^2>0$. This proves the claim.

By the claim, there are $x,y,z\in\mathbb{Z}$ such that $24n+26=2x^2+3(y^2+5z^2)$ and $y^2+5z^2>0$. By [24, Lemma 2.1], $y^2+5z^2=y_0^2+5z_0^2$ for some integers y_0 and z_0 not all divisible by 3. Without any loss of generality, we simply assume that $3\nmid y$ or $3\nmid z$. Note that $3\nmid x$ and $y\equiv z\pmod 2$. If $3\nmid yz$, then $\varepsilon_1 y\equiv\varepsilon_2 z\equiv x\pmod 3$ for some $\varepsilon_1,\varepsilon_2\in\{\pm1\}$. If $3\mid y$ and $3\nmid z$ then $x+y+\varepsilon z\equiv 0\pmod 3$ for some $\varepsilon\in\{\pm1\}$; similarly, if $3\nmid y$ and $3\mid z$ then $x+\varepsilon y+z\equiv 0\pmod 3$. So, without loss of generality we may suppose that $x+y+z\equiv 0\pmod 3$ (otherwise we adjust signs of x,y,z suitably to meet our purpose). If $y\equiv z\equiv 0\pmod 2$, then $2x^2\equiv 26\pmod 4$, hence $2\nmid x$ and $y\equiv z\pmod 4$ since $y^2+5z^2\equiv 0\pmod 8$, therefore

$$2x^2+3y^2+15z^2=2\left(\frac{y-5z}{2}\right)^2+3\left(\frac{2x+5y+5z}{6}\right)^2+15\left(\frac{2x-y-z}{6}\right)^2 \tag{4.8}$$

with $(2x+5y+5z)/6$ and $(2x-y-z)/6$ both odd.

By the above, $24n+26=2a^2+3b^2+15c^2$ for some $a,b,c\in\mathbb{Z}$ with $2\nmid bc$. As $3\nmid a$ and $2a^2\equiv 26-3-15\equiv 0\pmod 8$, a or $-a$ has the form $2(3w+1)$ with $w\in\mathbb{Z}$. Write $b=2u+1$ and $c=2v+1$ with $u,v\in\mathbb{Z}$. Then

$$24n+26=2(2(3w+1))^2+3(2u+1)^2+15(2v+1)^2$$

and hence $n=T_u+5T_v+w(3w+2)$. This proves the universality of $(6,4,5,5,1,1)$ over $\mathbb{Z}$. □

Proof of Theorem 1.3(ii). (a) Let $n\in\mathbb{N}$ and $r\in\{1,2\}$. Apparently,

$$\begin{aligned} &n=T_x+5\frac{y(3y+1)}{2}+z(3z+r)\\ \iff &24n+2r^2+8=3(2x+1)^2+5(6y+1)^2+2(6z+r)^2.\end{aligned}$$

As mentioned Part (b) in the proof of Theorem 1.3(i), there are two classes in the genus of $x^2+y^2+15z^2$, and the one not containing $x^2+y^2+15z^2$ has the representative $x^2+4y^2+4z^2-2yz$. If $12n+r^2+4=x^2+4y^2+4z^2-2yz$ with $x,y,z\in\mathbb{Z}$, then $2\mid yz$ since $r^2\not\equiv x^2-2\pmod 4$. Thus, in view of (4.4) and Lemma 3.1, $12n+r^2+4=x^2+y^2+15z^2$ for some $x,y,z\in\mathbb{Z}$. If $x\equiv y\pmod 2$, then $z\equiv r\pmod 2$, $x^2+y^2\equiv r^2-15z^2\equiv 2r^2\pmod 4$ and hence $x\equiv y\equiv r\equiv z\pmod 2$. So, x or y has the same parity with z. Without loss of generality we may assume that $y\equiv z\pmod 2$. Since $y^2+15z^2\equiv 0\pmod 4$, we have $x\equiv r\pmod 2$. If $r=2$ and $y^2+15z^2=0$, then $12n+r^2+4=0^2+x^2+15\times 0^2$ with $x\equiv 0\equiv r\pmod 2$ and $x^2+15\times 0^2>0$. If $r=1$, then $12n^2+r^2+4=12n+5$ is congruent to 2 modulo 3 and hence not a square. Thus, without loss of generality we may assume that $y^2+15z^2>0$.

Observe that

$$24n+2r^2+8=2(x^2+y^2+15z^2)=2x^2+3u^2+5v^2$$

with $u=(y+5z)/2$ and $v=(y-3z)/2$ both odd. Since $3u^2+5v^2\equiv 2r^2-2x^2\equiv 0\pmod 8$ and $2(3u^2+5v^2)=y^2+15z^2>0$, by Lemma 3.2(iii) we can write $3u^2+5v^2$ as $3y_0^2+5z_0^2$ with y_0 and z_0 both odd. As $2(x^2+z_0^2)\equiv 2x^2+5z_0^2\equiv 2r^2+8\pmod 3$, we have $x^2+z_0^2\equiv r^2+1\equiv 2\pmod 3$ and hence we may write x or $-x$ as $6u+r$, z_0 or $-z_0$ as $6v+1$, and $y_0=2w+1$, where u,v,w are integers. Therefore

$$24n+2r^2+8=2x^2+3y_0^2+5z_0^2=2(6u+r)^2+3(2w+1)^2+5(6v+1)^2$$

and hence $n=u(3u+r)/2+5v(3v+1)/2+T_w$. This proves the universality of $(15,5,6,2r,1,1)$ over $\mathbb{Z}$.

(b) Let $n\in\mathbb{N}$, $s\in\{1,3,5\}$ and $t\in\{1,2\}$ with $(s,t)\neq(5,2)$. Apparently,

$$\begin{aligned}&n=T_x+\frac{y(5y+s)}{2}+z(3z+t)\\ \iff &120n+3s^2+10t^2+15=15(2x+1)^2+3(10y+s)^2+10(6z+t)^2.\end{aligned}$$

There are two classes in the genus of $3x^2+10y^2+15z^2$, and the one not containing $3x^2+10y^2+15z^2$ has the representative

$$\begin{aligned}g(x,y,z)=&7x^2+7y^2+12z^2+6(x+y)z+4xy\\ =&3\left(\frac{x+y}{2}+2z\right)^2+10\left(\frac{x-y}{2}\right)^2+15\left(\frac{x+y}{2}\right)^2.\end{aligned}\tag{4.9}$$

If $120n+3s^2+10t^2+15=g(x,y,z)$ with $x,y,z\in\mathbb{Z}$, then we obviously have $x\equiv y\pmod 2$. Thus, in view of (4.9) and Lemma 3.1, $120n+3s^2+10t^2+15=3x^2+10y^2+15z^2$ for some $x,y,z\in\mathbb{Z}$. If $x=z=0$, then $120n+3s^2+10t^2+15=10y^2$, hence $(s,t)=(5,1)$ and $y^2=12n+10\equiv$

2 (mod 4) which is impossible. So $x^2+5z^2>0$, and hence by [24, Lemma 2.1] we can rewrite x^2+5z^2 as $x_0^2+5z_0^2$ with $x_0,z_0\in\mathbb{Z}$ not all divisible by 3. Without loss of generality, we simply assume that $3\nmid x$ or $3\nmid z$. Note that $3\nmid y$ since $3\nmid t$. If $3\nmid xz$, then $\varepsilon_1 x\equiv y\equiv \varepsilon_2 z$ for some $\varepsilon_1,\varepsilon_2\in\{\pm1\}$. If $3\mid x$ and $3\nmid z$, then $x+y+\varepsilon z\equiv 0 \pmod 3$ for some $\varepsilon\in\{\pm1\}$. If $3\nmid x$ and $3\mid z$, then $\varepsilon x+y+z\equiv 0\pmod 3$ for some $\varepsilon\in\{\pm1\}$. Without loss of generality, we just assume that $x+y+z\equiv 0\pmod 3$ (otherwise we may adjust signs of x,y,z suitably). Note that $x\equiv z\pmod 2$ and we have the identity

$$3\left(\frac{x+10y-5z}{6}\right)^2+10\left(\frac{x+z}{2}\right)^2+15\left(\frac{x-2y-5z}{6}\right)^2=3x^2+10y^2+15z^2 \tag{4.10}$$

with $x_1=(x+10y-5z)/6$, $y_1=(x+z)/2$ and $z_1=(x-2y-5z)/6$ all integral.

If $x\equiv z\equiv 1\pmod 2$, then $10y^2=120n+3s^2+10t^2+15-3x^2-15z^2\equiv 10t^2\pmod 4$ and hence $y\equiv t\pmod 2$.

Now suppose that $x\equiv z\equiv 0\pmod 2$. Then $2y^2\equiv 10y^2\equiv 3s^2+10t^2+15\equiv 2(t^2+1)\pmod 4$ and hence $y\not\equiv t\pmod 2$. Observe that

$$2t^2+2\equiv 120n+3s^2+10t^2+15=3x^2+10y^2+15z^2\equiv x^2+z^2+2(t+1)^2 \pmod 8$$

and hence

$$y_1=\frac{x+z}{2}\equiv\left(\frac{x}{2}\right)^2+\left(\frac{z}{2}\right)^2=\frac{x^2+z^2}{4}\equiv t\pmod 2.$$

Thus

$$z_1=x_1-2y\equiv x_1\equiv\frac{x+z}{2}-3z+5y\equiv t+y\equiv 1\pmod 2.$$

In view of the above, there are integers $x,y,z\in\mathbb{Z}$ with $x\equiv z\equiv 1\pmod 2$ and $y\equiv t\pmod 2$ such that $120n+3s^2+10t^2+15=3x^2+10y^2+15z^2$. Clearly, y or $-y$ has the form $6v+t$ with $v\in\mathbb{Z}$. Write $z=2w+1$ with $w\in\mathbb{Z}$. Since $x^2\equiv s^2\pmod 5$, we can write x or $-x$ as $10u+s$ with $w\in\mathbb{Z}$. Therefore

$$120n+3s^2+10t^2+15=3(10u+s)^2+10(6v+t)^2+15(2w+1)^2$$

and hence $n=T_w+u(5u+s)/2+v(3v+t)$. This proves the universality of $(6,2t,5,s,1,1)$ over $\mathbb{Z}$. $\square$

5. Proof of Theorem 1.4

B.W. Jones and G. Pall [15] proved the following celebrated result.

Lemma 5.1. *Let $n \in \mathbb{N}$ with $8n+1$ not a square. Then*

$$\begin{aligned} &|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+z^2=8n+1\ \&\ 4\mid x\}| \\ =&|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+z^2=8n+1\ \&\ x\equiv 2 \pmod 4\}|>0. \end{aligned}$$

A. G. Earnest [8, 9] showed the following useful result.

Lemma 5.2. *Let c be a primitive spinor exceptional integer for the genus of a positive ternary quadratic form $f(x,y,z)$, and let S be a spinor genus containing f. Let s be a fixed positive integer relatively prime to $2d(f)$ for which cs^2 can be primitively represented by S. If $t \in \mathbb{Z}^+$ is relatively prime to $2d(f)$, then ct^2 can be primitively represented by S if and only if*

$$\left(\frac{-cd(f)}{s}\right)=\left(\frac{-cd(f)}{t}\right).$$

Proof of Theorem 1.4. Fix $n \in \mathbb{N}$. Clearly,

$$n=T_x+y^2+2z(4z+1) \iff 8n+2=(2x+1)^2+8y^2+(8z+1)^2.$$

So, it suffices to show that $8n+2=x^2+y^2+8z^2$ for some $x,y,z\in\mathbb{Z}$ with $x\equiv\pm1 \pmod 8$.

Case 1. n is not twice a triangular number.

In this case, $4n+1$ is not a square. If $2\mid n$, then by Lemma 5.1 we can write $4n+1$ as $x^2+y^2+z^2$ with $2\nmid x$, $2\mid y$ and $z\equiv 2 \pmod 4$. If $2\nmid n$, then there are $x,y,z\in\mathbb{Z}$ with $2\nmid x$ and $y\equiv z\equiv 0 \pmod 2$ such that $4n+1=x^2+y^2+z^2$ and hence $y\not\equiv z \pmod 4$ since $y^2+z^2\equiv 5-x^2\equiv 4 \pmod 8$. So we can always write $4n+1=x^2+y^2+z^2$ with $2\nmid x$, $2\mid y$ and $z\equiv 2n-2 \pmod 4$, hence

$$8n+2=2(x^2+y^2+z^2)=(x+y)^2+(x-y)^2+8\left(\frac{z}{2}\right)^2$$

with $z/2\equiv n-1 \pmod 2$, thus

$$(x+y)^2+(x-y)^2\equiv 8n+2-8(n-1)=10\not\equiv 3^2+3^2 \pmod{16}$$

and hence $x+\varepsilon y\equiv\pm1 \pmod 8$ for some $\varepsilon\in\{\pm1\}$.

Case 2. $n=2T_m$ with $m\in\mathbb{N}$, and $2m+1$ has no prime factor of the form $4k+3$.

In this case, $2m+1$ can be expressed as the sum of two squares. If $4\mid m$, then

$$8n+2=2(2m+1)^2=(2m+1)^2+(2m+1)^2+8\times 0^2$$

with $2m+1\equiv 1 \pmod 8$. If $4\nmid m$, then $2m+1=u^2+(2v)^2$ for some odd integers u and v, and hence

$$\begin{aligned}8n+2=&2(u^2+4v^2)^2=2((u^2-4v^2)^2+(4uv)^2)\\ =&(u^2-4v^2+4uv)^2+(u^2-4v^2-4uv)^2+8\times 0^2\end{aligned}$$

with $u^2-4v^2\pm 4uv\equiv 1 \pmod 8$.

Case 3. $n=2T_m$ with $m\in\mathbb{N}$, and $2m+1$ has a prime factor $p\equiv 3 \pmod 4$.

By Lagrange's four-square theorem, we can write $p=a^2+b^2+c^2+d^2$, where a is an even number and b,c,d are odd numbers. Thus

$$\begin{aligned}p^2=&(a^2+b^2-c^2-d^2)^2+4(a^2+b^2)(c^2+d^2)\\ =&(a^2+b^2-c^2-d^2)^2+(2ac+2bd)^2+(2ad-2bc)^2\end{aligned}$$

and hence $(2m+1)^2=x^2+(2y)^2+(2z)^2$ for some odd integers x,y,z. Observe that

$$8n+2=2(2m+1)^2=(x+2y)^2+(x-2y)^2+8z^2$$

and $(x+2y)^2+(x-2y)^2\equiv 2-8z^2\equiv 10\not\equiv 3^2+3^2 \pmod{16}$. So one of $x+2y$ and $x-2y$ is congruent to 1 or -1 modulo 8.

Now we give an alternative approach to Case 3. There are three classes in the genus of $x^2+y^2+32z^2$ with the three representatives

$$\begin{aligned}f_1(x,y,z)=&x^2+y^2+32z^2,\\ f_2(x,y,z)=&2x^2+2y^2+9z^2+2yz-2zx,\\ f_3(x,y,z)=&x^2+4y^2+9z^2-4yz.\end{aligned}$$

The class of f_1 and the class of f_2 constitute a spinor genus while another spinor genus in the genus only contains the class of f_3. Since 2 is a primitive spinor exceptional integer for this genus, by Lemma 5.2 we can write $2p^2$ as

$$f_3(u,v,w)=u^2+4v^2+9w^2-4vw=u^2+(2v-w)^2+8w^2$$

with $u,v,w\in\mathbb{Z}$. Since $2\nmid uw$, we see that $8n+2=2(2m+1)^2=a^2+b^2+8c^2$ for some odd integers a,b,c. As $a^2+b^2\equiv 2-8c^2\equiv 10\not\equiv 3^2+3^2 \pmod{16}$, a or b is congruent to 1 or -1 modulo 8. This concludes our discussion of Case 3.

In view of the above, we have completed the proof of Theorem 1.4. $\square$

Remark 5.1. $f_3(x,y,z)$ in the proof of Theorem 1.4 is one of the very few spinor regular forms that are not regular. For more details, see [1].

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