On a lower bound for the energy functional on a family of Hamiltonian minimal Lagrangian tori in $\mathbb{C}P^2$

A. A. Kazhymurat

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Abstract

We study the energy functional on the set of Lagrangian tori in $\mathbb{C}P^2$. We prove that the value of the energy functional on a certain family of Hamiltonian minimal Lagrangian tori in $\mathbb{C}P^2$ is strictly larger than energy of the Clifford torus.

1 Introduction

As remarked in [1], one can naturally associate a 2D periodic Schrödinger operator with every Lagrangian torus in $\mathbb{C}P^2$. More precisely, any Lagrangian torus $\Sigma \subset \mathbb{C}P^2$ with induced metric

$$ds^2 = 2e^{v(x,y)}(dx^2 + dy^2)$$
 (1)

is the image of the composition of mappings

$$r: \mathbb{R}^2 \to S^5 \xrightarrow{\mathcal{H}} \mathbb{C}P^2$$

where r is a horizontal lift, \mathcal{H} is the Hopf projection. The vector-function r satisfies the Schrödinger equation

$$Lr = 0,$$
 $L = (\partial_x - \frac{i\beta_x}{2})^2 + (\partial_y - \frac{i\beta_y}{2})^2 + V(x, y),$ $V = 4e^v + \frac{1}{4}(\beta_x^2 + \beta_y^2) + \frac{i}{2}\Delta\beta,$

where β is the Lagrangian angle (see the definition below).

The existence of operator L allows us to introduce the energy functional E on the set of Lagrangian tori in $\mathbb{C}P^2$ (see [2])

$$E(\Sigma) = \frac{1}{2} \int_{\Sigma} V \, dx \wedge dy.$$

As shown in [2] the energy functional admits following geometric interpretation

$$E(\Sigma) = A(\Sigma) + \frac{1}{8}W(\Sigma), \qquad A(\Sigma) = \int_{\Sigma} d\sigma, \qquad W(\Sigma) = \int_{\Sigma} |H|^2 d\sigma,$$

where $d\sigma = 2e^v dx \wedge dy$ is the induced area element, H is the mean curvature vector field. For the Clifford torus Σ_{Cl} which is defined by the vector-function

$$r(x,y) = \left(\frac{1}{\sqrt{3}}e^{2\pi ix}, \frac{1}{\sqrt{3}}e^{2\pi i\left(-\frac{1}{2}x + \frac{\sqrt{3}y}{2}\right)}, \frac{1}{\sqrt{3}}e^{2\pi i\left(-\frac{1}{2}x - \frac{\sqrt{3}y}{2}\right)}\right),$$

energy equals

$$E(\Sigma_{Cl}) = \frac{4\pi^2}{3\sqrt{3}}.$$

Following conjecture was proposed in [2].

Conjecture 1. The minimum of the energy functional is attained on the Clifford torus.

Conjecture 1 has been verified for two families of Hamiltonian minimal Lagrangian tori: for homogeneous tori and for tori constructed in [3].

A homogeneous torus $\Sigma_{r_1,r_2,r_3} \subset \mathbb{C}P^2, r_1^2 + r_2^2 + r_3^2 = 1, r_i > 0$ is defined by the vector-function

$$r(x,y) = (r_1 e^{2\pi i x}, r_2 e^{2\pi i (a_1 x + b_1 y)}, r_3 e^{2\pi i (a_2 x + b_2 y)}),$$

with some restrictions on a_i, b_i . Following inequality holds

$$E(\Sigma_{r_1,r_2,r_3}) = \frac{\pi^2(1-r_1^2)(1-r_2^2)(1-r_3^2)}{2r_1r_2r_3} \geqslant \frac{4\pi^2}{3\sqrt{3}},$$

and equality is attained only for the Clifford torus.

The second family of tori $\Sigma_{m,n,k} \subset \mathbb{C}P^2, m,n,k \in \mathbb{Z}, m \geqslant n > 0, k < 0$ is of form $\mathcal{H}(\tilde{\Sigma}_{m,n,k})$ where

$$\tilde{\Sigma}_{m,n,k} = \left\{ (u_1 e^{2\pi i m y}, u_2 e^{2\pi i n y}, u_3 e^{2\pi i k y}) \right\} \subset S^5,$$

the numbers u_1, u_2, u_3 satisfy the equation

$$u_1^2 + u_2^2 + u_3^2 = 1,$$
 $mu_1^2 + nu_2^2 + ku_3^2 = 0.$

The parameters m, n, k should be chosen so that the involution

$$(u_1, u_2, u_3) \longrightarrow (u_1 \cos(m\pi), u_2 \cos(n\pi), u_3 \cos(k\pi))$$

on the surface $mu_1^2 + nu_2^2 + ku_3^2 = 0$ preserves its orientation (otherwise $\mathcal{H}(\tilde{\Sigma}_{m,n,k})$ is homeomorphic to Klein bottle, see [3]). Following inequality is proved in [2]

$$E(\Sigma_{m,n,k}) > E(\Sigma_{Cl}).$$

In the case of minimal Lagrangian tori the function v(x,y) satisfies the Tzizeica equation (see [5]). Smooth periodic solutions of this equation are finite-gap, i.e. can be expressed in terms of the theta-function on the Jacobian variety of the spectral curve. The results of [6] imply the conjecture for minimal Lagrangian tori corresponding to spectral curve of sufficiently high genus.

One should note that for embedded Lagrangian tori with non-trivial Floer cohomology one can derive lower bounds for the area functional. For instance, following inequality holds for any Lagrangian torus Σ Hamiltonian isotopic to the Clifford torus [4]

$$A(\Sigma) \geqslant \frac{3}{\pi} E(\Sigma_{Cl}).$$

It is unclear at present whether one can derive symplectic-topological bounds for the Willmore functional. This seems to be related to the question whether every monotone Lagrangian torus in $\mathbb{C}P^2$ is Hamiltonian isotopic to a minimal torus.

The aim of the present work is to verify the conjecture 1 for the family of Hamiltonian minimal Lagrangian tori constructed in [5] (also see [7]).

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$, $b = -\alpha_1 - \alpha_2 - \alpha_3$, $c = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$, $c_1 = -\alpha_1\alpha_2\alpha_3$, $a_1 > a_2 > 0$ be some real numbers satisfying the inequalities (4), (5) (see below). Following theorem has been proved in [5].

Theorem 1. The mapping $\psi: \mathbb{R}^2 \to \mathbb{C}P^2$ defined by the formula

$$\psi(x,y) = \left(F_1(x)e^{i(G_1(x) + \alpha_1 y)} : F_2(x)e^{i(G_2(x) + \alpha_2 y)} : F_3(x)e^{i(G_3(x) + \alpha_3 y)}\right),$$

is a conformal Hamiltonian minimal Lagrangian immersion, where

$$F_{i} = \sqrt{\frac{2e^{v} + \alpha_{i+1}\alpha_{i+2}}{(\alpha_{i} - \alpha_{i+1})(\alpha_{i} - \alpha_{i+2})}}, \qquad G_{i} = \alpha_{i} \int_{0}^{x} \frac{c_{2} - ae^{v}}{2\alpha_{i}e^{v} - c_{1}} dz,$$

$$2e^{v(x)} = a_{1} \left(1 - \frac{a_{1} - a_{2}}{a_{1}} \operatorname{sn}^{2} \left(x\sqrt{a_{1} + a_{3}}, \frac{a_{1} - a_{2}}{a_{1} + a_{3}}\right)\right)$$
(2)

(index i runs modulo 3), $\operatorname{sn}(x)$ is the Jacobi's elliptic function, c_2 is a real root of (3), $a_3 = \frac{c_1^2 + c_2^2}{a_1 a_2}$.

Moreover, if the rationality constraints (8) are met, ψ is a doubly periodic mapping and the image of the plane is a Hamiltonian minimal Lagrangian torus $\Sigma_M \subset \mathbb{C}P^2$.

The principal result of the present work is following theorem.

Theorem 2. The inequality

$$E(\Sigma_M) > E(\Sigma_{Cl})$$

holds if $\alpha_1 - \alpha_3, \alpha_2 - \alpha_3$ are relatively prime.

The theorem 2 thus confirms the conjecture 1.

2 The proof of the theorem 2

Lagrangianity of Σ , horizontality of the mapping $r: \mathbb{R}^2 \to S^5$ and the form of the induced metric (1) imply

$$R = \begin{pmatrix} r \\ \frac{r_x}{|r_x|} \\ \frac{r_y}{|r_y|} \end{pmatrix} \in U(3).$$

The Lagrangian angle $\beta(x,y)$ is defined by the equation $e^{i\beta}=\det R$. The mean curvature vector field can be expressed in terms of the Lagrangian angle $H=J\nabla\beta$ where J is the complex structure on $\mathbb{C}P^2$. For minimal tori $\beta=\mathrm{const}$. As demonstrated in [8] in the case of Hamiltonian minimal tori β is a linear function in the conformal coordinates x,y.

Let us consider the Hamiltonian minimal immersion ψ [5] defined in the theorem 2. The equation

$$(a_1-a_2)^2x^4+2(a_1^3a_2^2+a_1^2a_2^3+(a_1^2a_2+a_1a_2^2)bc_1+(a_1^2+a_2^2)c_1^2+2a_1^2a_2^2c)x^2+$$

$$+((a_1+a_2)c_1^2-a_1^2a_2^2+a_1a_2bc_1)^2=0. (3)$$

has a real root $x = c_2$ iff following inequalities are satisfied

$$P = a_1^3 a_2^2 + a_1^2 a_2^3 + (a_1^2 a_2 + a_1 a_2^2) b c_1 + (a_1^2 + a_2^2) c_1^2 + 2a_1^2 a_2^2 c \le 0,$$
(4)

$$P^{2} - (a_{1} - a_{2})^{2} ((a_{1} + a_{2})c_{1}^{2} - a_{1}^{2}a_{2}^{2} + a_{1}a_{2}bc_{1})^{2} \geqslant 0.$$
 (5)

Recall that $\operatorname{sn}(u,k) = \sin \theta$ where

$$u(\theta) = \int_0^\theta \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$
 (6)

The function $\operatorname{sn}^2(u)$ is periodic with period $2u(\frac{\pi}{2})$ (see, for instance, [9]). Therefore v(x) has period

$$T = \frac{2u\left(\frac{\pi}{2}\right)}{\sqrt{a_1 + a_3}}. (7)$$

Further we assume that $(\alpha_1 - \alpha_3, \alpha_2 - \alpha_3) = 1$. The immersion $\psi : \mathbb{R}^2 \to \mathbb{C}P^2$ is doubly periodic if there exists $\tau \in \mathbb{R}$ such that

$$\lambda_1 = \frac{G_1(T) - G_3(T) + (\alpha_1 - \alpha_3)\tau}{2\pi}, \qquad \lambda_2 = \frac{G_2(T) - G_3(T) + (\alpha_2 - \alpha_3)\tau}{2\pi} \in \mathbb{Q}.$$
 (8)

Then the vectors of period can be expressed as follows

$$e_1 = (0, 2\pi), \qquad e_2 = N(T, \tau),$$

where N is some natural number. If the condition (8) is met, $\Sigma_M \subset \mathbb{C}P^2$ is an immersed torus with Lagrangian angle $\beta = ax + by$ where

$$a = \frac{bc_1 + a_1a_3 + a_2a_3 - a_1a_2}{c_2}.$$

Following equality holds

$$|H|^2 = \frac{1}{2}e^{-v}(a^2 + b^2).$$

Let us find lower bounds for $W(\Sigma_M)$ and $A(\Sigma_M)$.

Using (7) and $a_3 > 0$ we arrive at the inequalities

$$u(\frac{\pi}{2}) > \frac{\pi}{2}, \qquad T > \frac{\pi}{\sqrt{a_1 + a_3}}.$$

Thus

$$W(\Sigma_M) = \int_{\Sigma_M} |H|^2 d\sigma = \int_{\Lambda} \frac{1}{2} e^{-v} (a^2 + b^2) 2e^v dx \wedge dy = 2\pi N T(a^2 + b^2).$$

Therefore, following lower bound for $W(\Sigma_M)$ holds

$$W(\Sigma_M) > 2\pi^2 \frac{a^2 + b^2}{\sqrt{a_1 + a_3}}. (9)$$

Following lemma provides a lower bound for $A(\Sigma_M)$.

Lemma 1. The inequality

$$A(\Sigma_M) > \pi^2 \frac{a_1 + a_2}{\sqrt{a_1 + a_3}}$$

 $is\ true.$

Proof of the lemma 1. We have

$$A(\Sigma_M) = \int_{\Sigma_M} d\sigma = \int_{\Lambda} 2e^{v(x)} dx \wedge dy = 2\pi \int_0^{NT} 2e^{v(x)} dx \geqslant 2\pi \int_0^T 2e^{v(x)} dx =$$

$$= 2\pi \int_0^T a_1 \left(1 - \frac{a_1 - a_2}{a_1} \operatorname{sn}^2 \left(x\sqrt{a_1 + a_3}, \frac{a_1 - a_2}{a_1 + a_3} \right) \right) dx =$$

$$= \frac{2\pi a_1}{\sqrt{a_1 + a_3}} \int_0^{2u(\frac{\pi}{2})} \left(1 - \frac{a_1 - a_2}{a_1} \operatorname{sn}^2 \left(u, \frac{a_1 - a_2}{a_1 + a_3} \right) \right) du.$$

Using (6) we arrive at

$$\int_0^T 2e^{v(x)} dx = \frac{a_1}{\sqrt{a_1 + a_3}} \int_0^\pi \frac{1 - \frac{a_1 - a_2}{a_1} \sin^2 \theta}{\sqrt{1 - \left(\frac{a_1 - a_2}{a_1 + a_3}\right)^2 \sin^2 \theta}} d\theta.$$

As $0 < \frac{a_1 - a_2}{a_1 + a_3} < 1$, following estimate is true

$$\int_0^T 2e^{v(x)}dx > \frac{a_1}{\sqrt{a_1 + a_3}} \int_0^\pi \left(1 - \frac{a_1 - a_2}{a_1} \sin^2 \theta \right) d\theta = \frac{\pi(a_1 + a_2)}{2\sqrt{a_1 + a_3}}.$$

Lemma 1 is proved.

The inequalities (4), (5) are invariant under simultaneous change of sign $\alpha_1, \alpha_2, \alpha_3$ and their permutations. If $\alpha_1, \alpha_2, \alpha_3$ are all of the same sign, the inequality (4) has no positive solutions. Therefore we assume without loss of generality that $\alpha_1 \geqslant \alpha_2 \geqslant 0 \geqslant \alpha_3$.

Lemma 2. If $\alpha_1 \geqslant \alpha_2 \geqslant 0 \geqslant \alpha_3$ $a_1 > a_2 > 0$, the inequalities (4) and (5) are satisfied simultaneously iff

$$-\alpha_2 \alpha_3 \leqslant a_2 < a_1 \leqslant -\alpha_1 \alpha_3. \tag{10}$$

Proof of the lemma 2. Denote

$$Q(x) = -(x + \alpha_1 \alpha_2)(x + \alpha_1 \alpha_3)(x + \alpha_2 \alpha_3).$$

Then (3) assumes the form

$$(a_1 - a_2)^2 \left(x^2 - \left(\frac{a_1 \sqrt{Q(a_2)} - a_2 \sqrt{Q(a_1)}}{a_1 - a_2} \right)^2 \right) \left(x^2 - \left(\frac{a_1 \sqrt{Q(a_2)} + a_2 \sqrt{Q(a_1)}}{a_1 - a_2} \right)^2 \right) = 0.$$

This equation has a positive root iff $Q(a_1) \ge 0$, $Q(a_2) \ge 0$. This is equivalent to $-\alpha_2 \alpha_3 \le a_2 < a_1 \le -\alpha_1 \alpha_3$. Lemma 2 is proved.

It follows from the proof of the lemma 2 that if $\alpha_3 = 0$ or $\alpha_1 = \alpha_2$ inequalities (4), (5) are not satisfied for $a_1 > a_2$. Therefore we assume without loss of generality

$$\alpha_1 > \alpha_2 \geqslant 0 > \alpha_3. \tag{11}$$

The inequality (9) and lemma 1 imply

$$E(\Sigma_M) > \pi^2 \frac{a_1 + a_2 + \frac{a^2 + b^2}{4}}{\sqrt{a_1 + a_3}}.$$

Let us prove $E(\Sigma_M) > E(\Sigma_{Cl})$. We will consider two cases: $\alpha_2 > 0$ and $\alpha_2 = 0$. Assume $\alpha_2 > 0$.

If $(a_1 + a_2)a_3 \ge \frac{7}{4}(a_1a_2 - bc_1)$ then

$$a^2 = \frac{((a_1 + a_2)a_3 - (a_1a_2 - bc_1))^2}{c_2^2} \geqslant \frac{9}{49}(a_1 + a_2)^2 \frac{a_3^2}{c_2^2} = \frac{9}{49}(a_1 + a_2)^2 \frac{a_3}{a_1a_2} \frac{c_1^2 + c_2^2}{c_2^2} \geqslant \frac{9}{49}(a_1 + a_2)^2 \frac{a_3}{a_1a_2}.$$

As $a_1 > a_2 \ge 1$ $(a_1 + a_2)^2 > 4a_1a_2$ we have

$$E(\Sigma_M) > \pi^2 \frac{a_1 + a_2 + \frac{9(a_1 + a_2)^2 a_3}{196a_1 a_2}}{\sqrt{a_1 + a_3}} > \pi^2 \frac{a_1 + \frac{9a_3}{49}}{\sqrt{a_1 + a_3}} = \pi^2 \sqrt{a_1} \frac{1 + \frac{9a_3}{49a_1}}{\sqrt{1 + \frac{a_3}{a_1}}} > \pi^2 \frac{1 + \frac{9a_3}{49a_1}}{\sqrt{1 + \frac{a_3}{a_1}}}.$$

Note that for positive x we have $\frac{1+\frac{9x}{49}}{\sqrt{1+x}} > \frac{4}{3\sqrt{3}}$ holds. Consequently, $E(\Sigma_M) > E(\Sigma_{Cl})$. Now consider the case

$$(a_1 + a_2)a_3 < \frac{7}{4}(a_1a_2 - bc_1).$$

We analyse two cases: $\alpha_1 > -\frac{3}{2}\alpha_2\alpha_3$ and $\alpha_1 \leqslant -\frac{3}{2}\alpha_2\alpha_3$.

If $\alpha_1 > -\frac{3}{2}\alpha_2\alpha_3$ then

$$\alpha_1 < -3b = 3(\alpha_1 + \alpha_2 + \alpha_3),$$

as $\alpha_1 > -\frac{3}{2}(\alpha_2 + \alpha_3)$. From (10)

$$-\frac{bc_1}{a_1 + a_2} = \frac{b\alpha_1\alpha_2\alpha_3}{a_1 + a_2} < \frac{b(3b)\alpha_2\alpha_3}{2\alpha_2\alpha_3} = \frac{3}{2}b^2.$$

Hence

$$E(\Sigma_M) > \pi^2 \frac{a_1 + a_2 + \frac{b^2}{4}}{\sqrt{a_1 + a_3}} > \pi^2 \frac{a_1 + a_2 + \frac{b^2}{4}}{\sqrt{a_1 + \frac{7}{4} \frac{a_1 a_2}{a_1 + a_2} - \frac{7}{4} \frac{bc_1}{a_1 + a_2}}} > \pi^2 \frac{a_1 + a_2 + \frac{b^2}{4}}{\sqrt{a_1 + \frac{7}{4} a_2 + \frac{21}{8} b^2}} >$$

$$> \pi^2 \frac{a_1 + a_2 + \frac{b^2}{4}}{\sqrt{\frac{7}{4}a_1 + \frac{7}{4}a_2 + \frac{21}{8}b^2}} = \pi^2 \sqrt{\frac{4(a_1 + a_2)}{7}} \frac{1 + \frac{b^2}{4(a_1 + a_2)}}{\sqrt{1 + \frac{3}{2}\frac{b^2}{a_1 + a_2}}} > \pi^2 \sqrt{\frac{8}{7}} \frac{1 + \frac{b^2}{4(a_1 + a_2)}}{\sqrt{1 + \frac{3}{2}\frac{b^2}{a_1 + a_2}}} > E(\Sigma_{Cl}).$$

The last inequality can be seen by considering the function $f(x) = \sqrt{\frac{8}{7}} \frac{1 + \frac{x}{4}}{\sqrt{1 + \frac{3}{2}x}}$ for x > 0.

If $\alpha_1 \leqslant -\frac{3}{2}\alpha_2\alpha_3$, the inequalities (10) and (11) imply

$$-bc_1 \leqslant -2\alpha_1^2 \alpha_2 \alpha_3 < \frac{9}{2}a_1 a_2^2.$$

Therefore

$$E(\Sigma_M) > \pi^2 \frac{a_1 + a_2}{\sqrt{a_1 + \frac{7}{4} \frac{a_1 a_2 - b c_1}{a_1 + a_2}}} = \pi^2 \frac{(a_1 + a_2)\sqrt{a_1 + a_2}}{\sqrt{a_1(a_1 + a_2) + \frac{7}{4}a_1a_2 - \frac{7}{4}bc_1}} >$$

$$> \pi^2 \frac{(a_1 + a_2)\sqrt{a_1 + a_2}}{\sqrt{a_1^2 + \frac{11}{4}a_1a_2 + \frac{63}{8}a_1a_2^2}} > \pi^2 \frac{(a_1 + a_2)\sqrt{a_1 + a_2}}{\sqrt{a_1^3 + \frac{11}{4}a_1^2a_2 + \frac{63}{8}a_1a_2^2}} = \pi^2 \frac{(1 + \frac{a_2}{a_1})\sqrt{1 + \frac{a_2}{a_1}}}{\sqrt{1 + \frac{11}{4}\frac{a_2}{a_1} + \frac{63}{8}\frac{a_2^2}{a_1^2}}} > E(\Sigma_{Cl}).$$

Let us consider the case $\alpha_2=0$. Introduce $p=-\alpha_1\alpha_3$, $x=\frac{a_1}{p},y=\frac{a_2}{p}$. Note that $0 < y < x \leqslant 1$ due to (11). Then inequalities (4), (5) assume following form

$$p^5 x^2 y^2 (x+y-2) \le 0,$$
 $4p^{10} x^4 y^4 (1-x)(1-y) \ge 0.$

The equation (3) implies

$$c_2^2 = p^3 x^2 y^2 \frac{2 - x - y \pm \sqrt{(2 - x - y)^2 - (x - y)^2}}{(x - y)^2}.$$
 (12)

As 2 - x - y > 0 we have $\sqrt{(2 - x - y)^2 - (x - y)^2} = (2 - x - y)\sqrt{1 - \frac{(x - y)^2}{(2 - x - y)^2}}$. Note that by Bernoulli inequality

$$1 - \frac{(x-y)^2}{(2-x-y)^2} \leqslant \sqrt{1 - \frac{(x-y)^2}{(2-x-y)^2}} \leqslant 1 - \frac{(x-y)^2}{2(2-x-y)^2}.$$

Consequently,

$$2 - x - y - \frac{(x - y)^2}{2 - x - y} \le \sqrt{(2 - x - y)^2 - (x - y)^2} \le 2 - x - y - \frac{(x - y)^2}{2(2 - x - y)}.$$
 (13)

Consider two cases: sign '+' and '-' in (12). For the '-' sign (12) and (13) imply the inequalities

$$p^{3} \frac{x^{2} y^{2}}{2(2-x-y)} \leqslant c_{2}^{2} \leqslant p^{3} \frac{x^{2} y^{2}}{2-x-y}.$$

As $c_1 = 0$ we have following bound for a_3

$$a_3 = \frac{c_2^2}{a_1 a_2}, \qquad p \frac{xy}{2(2-x-y)} \leqslant a_3 \leqslant p \frac{xy}{2-x-y}.$$

These estimates and lemma 1 imply

$$A(\Sigma_M) \geqslant \pi^2 \sqrt{p} \frac{x+y}{\sqrt{x+\frac{xy}{2-x-y}}}.$$

Following inequality holds

$$a = \frac{(a_1 + a_2)a_3 - a_1a_2}{c_2} \geqslant \frac{(xp + yp)p\frac{xy}{2(2 - x - y)} - xyp^2}{c_2} \geqslant \sqrt{p} \left(\frac{x + y}{2(2 - x - y)} - 1\right) \sqrt{2 - x - y}.$$

The estimate (9) implies

$$W(\Sigma_M) \geqslant 2\pi^2 \frac{a^2}{\sqrt{a_1 + a_3}} \geqslant 2\pi^2 \sqrt{p} \left(\frac{x + y}{2(2 - x - y)} - 1 \right)^2 \frac{2 - x - y}{\sqrt{x + \frac{xy}{2 - x - y}}}.$$

Henceforth

$$E(\Sigma_M) \geqslant \pi^2 \sqrt{p} \left(\frac{x+y}{\sqrt{x + \frac{xy}{2-x-y}}} + \frac{1}{4} \left(\frac{x+y}{2(2-x-y)} - 1 \right)^2 \frac{2-x-y}{\sqrt{x + \frac{xy}{2-x-y}}} \right).$$

As $p \ge 1$ we have

$$E(\Sigma_M) \geqslant \pi^2 B_1(x,y), \qquad B_1(x,y) = \frac{16 - 7x^2 + 8x - 14yx + 8y - 7y^2}{16\sqrt{(2-x)(2-x-y)x}}.$$

Lemma 3. If $0 < y < x \le 1$, then $B_1(x, y) > 1$.

Proof of the lemma 3. One can check by direct computation that there are no critical points $\partial_x B_1 = \partial_y B_1 = 0$ inside the triangle $0 < y < x \le 1$ while on the boundary of the triangle $B_1(x,y) > 1$ holds. Lemma 3 is proved.

Therefore, $E(\Sigma_M) > E(\Sigma_{Cl})$ holds for the '-' sign in (12).

For the '+' sign in (12) (13) implies the inequalities

$$p^3 f(x,y) \leqslant c_2^2 \leqslant p^3 g(x,y),$$

where

$$f(x,y) = x^{2}y^{2} \frac{2(2-x-y) - \frac{(x-y)^{2}}{2-x-y}}{(x-y)^{2}}, \qquad g(x,y) = x^{2}y^{2} \frac{2(2-x-y) - \frac{(x-y)^{2}}{2(2-x-y)}}{(x-y)^{2}}.$$

Analogously one establishes the inequalities

$$p\frac{f(x,y)}{xy} \leqslant a_3 \leqslant p\frac{g(x,y)}{xy},$$

$$a \geqslant \sqrt{p} \frac{(x+y)\frac{f(x,y)}{xy} - xy}{\sqrt{g(x,y)}}.$$

The inequality (9) and lemma 1 imply

$$A(\Sigma_{M}) \geqslant \pi^{2} \sqrt{p} \frac{x+y}{\sqrt{x+\frac{g(x,y)}{xy}}},$$

$$W(\Sigma_{M}) \geqslant 2\pi^{2} \frac{a^{2}}{\sqrt{a_{1}+a_{3}}} \geqslant 2\pi^{2} \sqrt{p} \frac{((x+y)\frac{f(x,y)}{xy}-xy)^{2}}{g(x,y)\sqrt{x+\frac{g(x,y)}{xy}}},$$

$$E(\Sigma_{M}) \geqslant \pi^{2} \sqrt{p} \frac{x+y+\frac{1}{4}\frac{((x+y)\frac{f(x,y)}{xy}-xy)^{2}}{g(x,y)}}{\sqrt{x+\frac{g(x,y)}{xy}}} \geqslant \pi^{2}B_{2}(x,y),$$

$$B_{2}(x,y) = \frac{x+y+\frac{1}{4}\frac{((x+y)\frac{f(x,y)}{xy}-xy)^{2}}{g(x,y)}}{\sqrt{x+\frac{g(x,y)}{xy}}}.$$

The following lemma is established similarly to the lemma 3.

Lemma 4. If $0 < y < x \le 1$, then $B_1(x, y) > 0.9$.

This finishes the proof of the theorem 2.

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NIS of Physics and Mathematics Zhamakaev St, Almaty 55000, Kazakhstan akkazhymurat@gmail.com