Sequential Gibbs Measures and Factor Maps

Giovane Ferreira and Krerley Oliveira

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Abstract

We define the notion of *sequential Gibbs measures*, inspired by on the classical notion of Gibbs measures and recent examples from the study of non-uniform hyperbolic dynamics. Extending previous results of Kempton-Pollicott [7] and Ugalde-Chazottes [3], we show that the images of one block factor maps of a sequential Gibbs measure are also a sequential Gibbs measure, with the same sequence of Gibbs times. We obtain some estimates on the regularity of the potential of the image measure at almost every point.

1 Introduction

Introduced in the early seventies in the realm of Dynamical Systems, Gibbs measures plays an important role in the understanding of the Ergodic Theory of hyperbolic or expanding maps. These measures are equilibrium states of expanding maps for regular potentials. However, even the existence of such measures requires strong forms of regularity and hyperbolicity. This makes it difficult to make use such measures beyond the uniformly hyperbolic dynamics and creates the need of adapt and extend this concept in the non-uniform hyperbolic setting.

Trying to understand the dynamics of intermittent maps and study its equilibrium states, Yuri generalized Gibbs measures defining the notion of weak Gibbs measure, where uniform control of the measure of dynamic balls at every point by a constant is replaced by a subexponential sequence of constants. Compare with Definition 3.1 and see more in [18, 19, 20].

More recently, with the rapid growth of the study and understanding of non-uniformly hyperbolic maps, several works were carried out in the context of non-uniformly expanding dynamics dealing with more general measures inspired by Gibbs measures, such as non-lacunary Gibbs measures. See [10, 15, 13], just to refer some of them. These measures are equilibrium states for some non-uniformly expanding maps and potentials and their Gibbs-like property is even weaker than analogous property of weak Gibbs measures in the sense of Yuri. The subexponential sequence of constants is replaced by a subexponential sequence of functions defined almost everywhere. It means that the non-uniform control at every point is replaced by a non-uniform control at *almost* every point, as was present in ([10], Proposition 3.17).

Here, we study the behavior of Gibbs-like properties under factor maps. To describe this problem precisely, let us consider two full shifts spaces $\Sigma_i = \{1, \dots, k_i\}^{\mathbb{N}}$, for $i = 1, 2$, and a surjective map $\pi : \{1, \dots, k_1\} \to \{1, \dots, k_2\}$ and extend π to a surjective map $\Pi : \Sigma_1 \to \Sigma_2$, defining

$$\Pi(x_1 x_2 ...) = \pi(x_1)\pi(x_2)....$$

[1] [2].

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This map is called an *one block* factor map.

Given a continuous potential ϕ on a full shift, one can not expect that Gibbs measures exists or they are unique, as was shown by Hofbauer[[5], page 230]. The regularity of the potential plays a important role in the existence and uniqueness of Gibbs and equilibrium measures. To analyze it in detail, consider the *n-variation* of ϕ defined by

$$var_n(\phi) = \sup\{|\phi(\underline{z}) - \phi(\underline{w})| : w_i = z_i, \text{ for } i = 0, \ldots, n-1\}.$$

The uniform continuity of the function ϕ corresponds to $var_n(\phi) \to 0$ as $n \to \infty$ and the Hölder continuity of ϕ corresponds to the existence of constants $C > 0$ and $\theta \in (0,1)$ such that $var_n(\phi) < C\theta^n$, for $n \geq 1$.

Given a Gibbs measure μ on Σ_1 for a continuous function ψ_1, we consider its image $\nu := \Pi_*\mu$ under Π. One interesting question that arises in the Theory of Hidden Markov Chains is to show that ν is also a Gibbs measure for some continuous function ψ_2. In the case where μ is a Markov measure, sufficient conditions for ν to be a Gibbs measure were given in [2, 17]. The case where μ is a Gibbs measure and Σ_1 is a full shift, we list some important recent contributions:

- ([7], Theorem 1.1 and [3], Theorem 3.1) If ψ_1 continuous, then ν is a Gibbs measure for some continuous potential ψ_2.
- ([14],Theorem 2) If ψ_1 Hölder continuous, then ν is a Gibbs measure for some Hölder continuous potential ψ_2.

  We say that ψ is *stretched* Hölder, if there are constants $t, C > 0$ and $\theta \in (0,1)$ such that $var_n(\psi) < C\theta^{n^t}$, for $n \geq 1$.
- ([7], Theorem 5.3 and [3], Theorem 4.1) If ψ_1 is stretched Hölder, then ψ_2 can be chosen stretched Hölder.
- ([7], Theorem 5.1) If $\sum_{n\geq 1} n^{d+1} var_n(\psi_1) < +\infty$ for some $d \geq 0$, then ψ_2 is such that $\sum_{n\geq 1} n^d var_n(\psi_2) < +\infty$.
- ([6],Theorem 1.8 and [12], Theorem 2) Similar results for the case of subshift of finite type under fiber-wise mixing assumption.
- ([16],Theorem 3.1) A non-additive version of the same problem: if μ is a Gibbs measure for a sequence of almost additives potentials $\Psi = \{\psi_n\}_n$ on Σ_1 with bounded variation, then $\Pi_*\mu$ is a Gibbs measure for a sequence of continuous potentials $\Phi = \{\phi_n\}_n$.

The main object of this paper is the notion of *sequential* Gibbs measures. We say that μ is a sequential Gibbs measure with respect to $\phi : \Sigma \to \mathbb{R}$ if there are constants K, P such that for μ-a.e. $\underline{x} \in \Sigma$, there exists an increasing sequence of natural numbers $n_i(\underline{x}) \in \mathbb{N}$ such that for every $0 \leq j \leq n_i - 1$

$$K^{-1} \leq \frac{\mu([x_j \ldots x_{n_i-1}])}{e^{\phi^{n_i-j}(\underline{x})-(n_i-j)P}} \leq K, \tag{1}$$

where $\phi^n(\underline{x}) = \sum_{i=0}^{n-1} \phi(f^i(\underline{x}))$ and $[x_0x_1\cdots x_{n-1}] = \{\underline{y} = y_0y_1\cdots \in \Sigma;$ such that $y_i = x_i$, for $i = 0, \ldots, n-1\}$. If Equation (1) holds for the sequence $n_i(\underline{x}) = i$ and every $\underline{x} \in \Sigma$, the sequential Gibbs measures is just a standard Gibbs measure.

The maximal subsequence $n_i(\underline{x})$ satisfying the Equation (1) is called the sequence of *Gibbs times* of $\underline{x}$. Note that if $n_i(\underline{x})$ is a Gibbs time of $\underline{x}$ then for $n \leq n_i(\underline{x})$, $n_i(\underline{x}) - n$ is Gibbs time of $\sigma^n(\underline{x})$.

We give natural examples of sequential Gibbs measures that are not standard Gibbs measures in Section 4, where we discuss equilibrium states on shifts constructed in [5] and image of non-lacunary Gibbs measures of local diffeomorphisms for Hölder potentials studied in [10], [15] and [13] under coding by some partition.

The results that we obtain here are a kind of non-uniform counterparts of those in [7, 3] adapted for sequential Gibbs measures, for much less regular potentials and more suitable for the study of non-uniformly hyperbolic dynamical systems. We prove that given a sequential Gibbs measure μ for a continuous potential ψ_1 on a shift Σ_1 and $\Pi : \Sigma_1 \to \Sigma_2$ a one block factor map that is regular with respect to μ, then the measure $\nu := \Pi_* \mu$ on Σ_2 is a sequential Gibbs measure for some almost everywhere continuous potential $\psi_2 : \Sigma_2 \to \mathbb{R}$. We also obtain local estimates almost everywhere for the regularity of ψ_2 based on the regularity almost everywhere of ψ_1.

The results obtained in this paper can be extended to the case of subshift of type finite with the property of topologically mixing in the fibers, as in [6]. We also expect that the theorems obtained here would be useful to obtain a Central Limit Theorem for pointwise dimension of non-lacunary Gibbs measures, using the approach of [8].

2 Results

The definition of sequential Gibbs measures depends on the constants K and P. However, if we denote by $G = \{\underline{x} \in \Sigma; \underline{x} \text{ has infinitely many Gibbs times}\}$, we prove that P is uniquely determined by the *pressure* $P_G(\psi)$ of ψ with respect to G.

To define $P_G(\psi)$, denote by $\mathcal{C}_n$ the set of all cylinders of length n. We consider the family $m_\alpha(\cdot, \psi, N)$ of exterior measures defined by

$$m_\alpha(G, \psi, N) = \inf_{\mathcal{U}} \left\{ \sum_{C \in \mathcal{U}} e^{-\alpha n(C) + \sup_{x \in C} \psi^{n(C)}(x)} \right\},$$

where the infimum is take over all open covers $\mathcal{U} \subset \cup_{n \geq N} \mathcal{C}_n$ of G and $n(C)$ is the length of C. Then, we set

$$m_\alpha(G, \psi) = \lim_{N \to \infty} m_\alpha(G, \psi, N)$$

and define

$$P_G(\psi) = \inf\{\alpha \in \mathbb{R}; m_\alpha(G, \psi) = 0\}.$$

For more details and properties about $P_G(\psi)$, we suggest [11, Section 11, Chapter 4]. Now, we prove that

Proposition 2.1. *If ψ admits a sequential Gibbs measure μ, then $P = P_G(\psi)$ is the unique number that satisfy Equation* (1)*, where G is the set of points with infinitely many Gibbs times. If μ is an ergodic invariant measure, $P_G(\psi) = h_\mu(\sigma) + \int \psi \, d\mu$.*

Proof. In fact, assume that μ is a sequential Gibbs measure with constants K and P satisfying Equation (1). For the first part, denote by $\mathcal{G}_n$ the collection of all cylinders $C = [x_0 \dots x_{n-1}]$ such that n is a Gibbs time of some $\underline{x} \in C$. Fixed k, by definition of G we have that $\mathcal{U}_k = \cup_{n > k} \mathcal{G}_n$ is an open cover of G and

$$\mathcal{V}_k = \bigcup_{n>k} \{[x_0 \dots x_{n-1}] \in \mathcal{G}_n; [x_0 \dots x_{l-1}] \notin \mathcal{G}_l, \text{ for } k \leq l < n\}$$

is an open partition of G. For any $\gamma > P$ we have that

$$m_\gamma(G,\psi,k) \le \sum_{C\in\mathcal{V}_k} e^{-\gamma n(C)+\sup_{\underline{x}\in C}\psi^{n(C)}(\underline{x})} =$$

$$= \sum_{C\in\mathcal{V}_k} e^{-(\gamma-P)n(C)}e^{-Pn(C)+\sup_{\underline{x}\in C}\psi^{n(C)}(\underline{x})} \le \quad Ke^{-(\gamma-P)k}\sum_{C\in\mathcal{V}_k}\mu(C).$$

Since $\sum_{C\in\mathcal{V}_k}\mu(C) \le 1$, taking $k\to\infty$, we have that $m_\gamma(G,\psi,k) = 0$ and $P \ge P_G(\psi)$. The opposite inequality follows in a similar fashion from the fact that for every cylinder $C = [x_0 \dots x_{n-1}]$ such that n is a Gibbs time of some point $\underline{x}$ of C we have that $e^{-Pn+\psi^n(\underline{x})} \ge K^{-1}\mu(C)$.

Now, we prove that $P_G(\psi) = h_\mu(\sigma) + \int \psi\, d\mu$ for an ergodic invariant sequential Gibbs measure. To finish the proof, just observe that by Brin-Katok's local entropy formula we have that for almost every $\underline{x} \in G$, if $n_i(\underline{x})$ is the sequence of Gibbs times of $\underline{x}$, then

$$h_\mu(\sigma) = -\lim \frac{1}{n_i}\log\mu([x_0\dots x_{n_i-1}]) = P_G(\psi) - \int \psi\, d\mu.$$

□

By the previous Lemma, since P is uniquely defined, we call it the *pressure* of the sequential Gibbs measure μ. Through this paper, we assume that the constant K in Equation (1) is fixed. Without loss of generality, we assume that $P = 0$ in Equation (1), since μ is a sequential Gibbs measure for ψ with pressure P if, and only if, μ is a sequential Gibbs measure for $\psi - P$ with pressure zero.

Definition 2.1. *We say that an one block factor map $\Pi : \Sigma_1 \to \Sigma_2$ is regular with respect to a sequential Gibbs measure μ on Σ_1, if there exists a μ-full measure set $D \subset G \subset \Sigma_1$, such that given $\underline{x} \in D$ then $\Pi^{-1}(\Pi(\underline{x})) \subset G$ and $n_1(\underline{x}) = n_1(\underline{y})$, for every $\underline{y} \in \Pi^{-1}(\Pi(\underline{x}))$.*

From now on, we consider only measures μ such that $\mu(\sigma^{-1}(A)) = 0$ for every set $A \subset \Sigma_1$ with $\mu(A) = 0$. Since $n_k(\underline{x}) = n_1(\sigma^{n_{k-1}(\underline{x})}(\underline{x}))$, if Π is regular with respect μ then we may define $E = \Pi(\cap_{k\ge 0}\sigma^{-k}(D))$ and observe that $E \subset \Sigma_2$ is a ν-full measure set such that given $\underline{x}, \underline{y} \in \Pi^{-1}(E)$, with $\Pi(\underline{x}) = \Pi(\underline{y})$ then $n_k(\underline{x}) = n_k(\underline{y})$, for every $k \ge 1$.

We define the n-th variation of ϕ at $\underline{x} = x_0x_1...x_n...$ as

$$var_n(\phi,\underline{x}) = \sup\{|\phi(\underline{x}) - \phi(\underline{w})| : \underline{w} \in [x_0...x_{n-1}]\}.$$

where $[x_0...x_{n-1}] = \{\underline{w} \in \Sigma : w_0...w_{n-1} = x_0...x_{n-1}\}$ is the cylinder of length n at $\underline{x}$.

We define the variation of a potential $\phi : \Sigma \to \mathbb{R}$ on the set $K \subset \Sigma$ with respect to X by

$$var_n(\phi, K) := \sup_{\underline{x}\in K} var_n(\phi,\underline{x}).$$

Now, we state the first result of this paper:

Theorem 1. *Let μ be a sequential Gibbs measure for a continuous potential $\psi_1 : \Sigma_1 \to \mathbb{R}$. If Π is regular with respect to μ, then the measure $\nu := \Pi_*\mu$ on Σ_2 is a sequential Gibbs measure for some potential $\psi_2 : \Sigma_2 \to \mathbb{R}$, continuous at ν almost every point.*

Now we study the modulus of continuity of ψ_2 at some point with respect to the modulus of ψ_1 at its preimages.

Theorem 2. *Let μ be a sequential Gibbs measure for a continuous potential $\psi_1 : \Sigma_1 \to \mathbb{R}$. If for ν-a.e. $\underline{z} \in \Sigma_2$, we have that $\limsup n_k(z)/k < +\infty$ and there exist a decreasing positive function $f_{\underline{z}} : \mathbb{N} \to \mathbb{R}$ such that $\limsup f_{\underline{z}}(k)k < +\infty$ and for every $1 \le j \le n_k$ we have*

$$var_j(\psi_1, \Pi^{-1}(\sigma^{n_k - j}(\underline{z}))) < f_{\underline{z}}(j).$$

Then, given any $\gamma < 1$ there are constants $0 < \alpha < 1$ and $C > 0$ such that for ν-almost every point $\underline{z} \in \Sigma_2$, there exists $k_0(\underline{z})$ such that for each $k > k_0(\underline{z})$ given $\underline{z}' \in [z_0, \dots, z_k]$, then

$$|\psi_2(\underline{z}) - \psi_2(\underline{z}')| < C \max\{\alpha^{k^{1-\gamma}}, f_{\underline{z}}([k^\gamma])k\}.$$

It follows directly from the Theorem 2 that:

Corollary 1. *(Local stretched Hölder decay): Suppose that there are constants $\Gamma_1 > 0$ and $\beta_1, \theta_1 \in (0,1)$ such that for almost every $\underline{w} \in E$, if $n_k(\underline{w})$ is the sequence of Gibbs times of $\underline{w}$ and $1 \le j \le n_k$, we have that for k big enough*

$$var_j(\psi_1, \Pi^{-1}(\sigma^{n_k - j}(\underline{w})) < \Gamma_1 {\theta_1}^{j^{\beta_1}}.$$

Then, we may choose ψ_2 in such way that ν is a sequential Gibbs measure for ψ_2 and there are constants $\Gamma_2 > 0$ and $\beta_2, \theta_2 \in (0,1)$ such that for almost every $\underline{w} \in \Sigma_2$, there exists $k_0(\underline{w})$ such that given $\underline{w}' \in [w_0, \dots, w_k]$ then

$$|\psi_2(\underline{w}) - \psi_2(\underline{w}')| < \Gamma_2 {\theta_2}^{k^{\beta_2}},$$

where $\theta_2 = \max \alpha, \theta_1 \in (0,1)$ and $\Gamma_2 > 0$.

Proof. In Theorem 2 we put $f_{\underline{w}}(j) = \Gamma_1 \cdot {\theta_1}^{j^{\beta_1}}$. Then, for $\gamma < 1$, we have that for $\underline{w}' \in [w_0, \dots, w_k]$ then

$$|\psi_2(\underline{w}) - \psi_2(\underline{w}')| < C \max\{\alpha^{k^{1-\gamma}}, \Gamma_1 {\theta_1}^{[k^\gamma]^{\beta_1}} k\}$$

Let k_0 such that $\theta := \theta_1 k_0^{\frac{1}{[k_0^\gamma]^{\beta_1}}} < 1$ and $\beta < 1$ such that $[k^\gamma]^{\beta_1} \ge k^\beta$. Then, for $k \ge k_0$, we have

$$|\psi_2(\underline{w}) - \psi_2(\underline{w}')| < C \max\{\alpha^{k^{1-\gamma}}, \Gamma_1 \theta^{\beta}\} < \Gamma_2 \theta_2^{k^{\beta_2}}$$

where $\theta_2 = \max\{\alpha, \theta\}$ and $\beta_2 = \max\{1-\gamma, \beta\}$. □

Corollary 2. *(Local polynomial decay): Suppose that there constants $\Gamma_1 > 0$ and $r > 2$ such that for every $\underline{w} \in E$, if $n_k(\underline{w})$ is the sequence of Gibbs times of $\underline{w}$, we have that for any $1 \le j \le n_k$*

$$var_j(\psi_1, \Pi^{-1}(\sigma^{n_k - j}(\underline{w}))) < \Gamma_1 j^{-r}.$$

Then, we may choose ψ_2 in such way that ν is a sequential Gibbs measure for ψ_2 and there are constants $\Gamma_2 > 0$ and for every $s < r - 1$ such that for ν-a.e. there exists $k_0(\underline{w})$ such that given $\underline{w}' \in [w_0, \dots, w_k]$ then

$$|\psi_2(\underline{w}) - \psi_2(\underline{w}')| < \Gamma_2 k^{-s}.$$

Proof. In Theorem 2 we put $f_{\underline{w}}(k) = \Gamma_1 k^{-r}$. For ν-a.e. there exists $k_0(\underline{w})$ such that given $\underline{w}' \in [w_0, \dots, w_k]$ then

$$|\psi_2(\underline{w}) - \psi_2(\underline{w}')| < C \max\{\alpha^{k^{1-\gamma}}, \Gamma_1 [k^\gamma]^{-r} k\} \le C \, \Gamma_1 [k^\gamma]^{-r} k$$

We can choose $\lambda < 1$ such that $[k^\gamma] > k^\lambda$ for every k. Indeed,

$$k^{\gamma\cdot\lambda} - 1 \leq \frac{k^\gamma - 1}{k^\lambda} \leq \frac{[k^\gamma]}{k^\lambda}$$

Then

$$|\psi_2(\underline{w}) - \psi_2(\underline{w}')| < C\,\Gamma_1 k^{1-\lambda r} < C\,\Gamma_1 k^{-s}$$

To finish the proof, just put $\Gamma_2 = C\,\Gamma_1$.

□

Corollary 3. *(Local summable variations) Suppose that for any $\underline{w} \in E$, if n_k is a Gibbs time of $\underline{w}$, we have that*

$$\sum_{k\geq 1} k var_k(\psi_1, \Pi^{-1}(\underline{w})) < \infty$$

Then, we may choose ψ_2 in such way that ν is a sequential Gibbs measure for ψ_2 and such that for ν-a.e. $\underline{w} \in \Sigma_2$ there exists $k_0(\underline{w})$ such that $\underline{w}' \in [w_0...w_k]$ then

$$\sum_{k>k_0(\underline{w})} k|\psi_2(\underline{w}) - \psi_2(\underline{w}')| < \infty.$$

Proof. Let $\beta > 0$ and $\gamma = \frac{1}{\beta+1}$, as in proof of the Theorem 2. Define $f_{\underline{w}}(k) = var_{[k^{\frac{1}{\gamma}}]+1}(\psi_1, \Pi^{-1}(\underline{w}))$. For ν-a.e. there exists $k_0(\underline{w})$ such that given $\underline{w}' \in [w_0, \dots, w_k]$

$$\begin{aligned} |\psi_2(\underline{w}) - \psi_2(\underline{w}')| &< C\max\{\alpha^{k^{1-\gamma}}, k\, var_{[[k^\gamma]^{\frac{1}{\gamma}}]+1}(\psi_1, \Pi^{-1}(\underline{w}))\} \\ &\leq C\max\{\alpha^{k^{1-\gamma}}, k\, var_k(\psi_1, \Pi^{-1}(\underline{w}))\} \end{aligned}$$

Obviously, $\sum_{k\geq k_0(\underline{w})} k\alpha^{k^{1-\gamma}} < \infty$. Then, jointly with the hypothesis, we have

$$\sum_{k>k_0(\underline{w})} k|\psi_2(\underline{w}) - \psi_2(\underline{w}')| < \infty.$$

□

Before start the proofs of Theorem 1 and Theorem 2, we prove some properties of sequential Gibbs measures. First, we observe that the first Gibbs time function give us some information about the growth of the function n_k. In fact, define the function $n_1 : G \to \mathbb{R}$ the *first Gibbs time* of $\underline{x}$. Then,

Proposition 2.2. *If μ is an ergodic sequential Gibbs measure such that $\int n_1\, d\mu < +\infty$, then for μ-a.e. $\underline{x} \in \Sigma_1$, there exists $b(\underline{x})$ such that for every $k \geq 0$ we have that $n_k(\underline{x}) \leq bk$.*

Proof. Let G be the set of points with infinitely many Gibbs times. We may define $g : G \to G$ by

$$g(\underline{x}) = \sigma^{n_1(\underline{x})}(\underline{x}),$$

Since $\int n_1\, d\mu < +\infty$, using Theorem 1.1 of [21] we have that there is an ergodic g-invariant measure μ_g absolutely continuous with respect to μ. Moreover, if G_k is the subset of points $\underline{x} \in G$ such that $n_1(\underline{x}) = k$ then, we may characterize this measure defining

$$\mu(E) = \sum_{n=0}^{\infty} \sum_{k>n} \mu_g(\sigma^{-n}(E) \cap G_k), \tag{2}$$

for every measurable set $E \subset \Sigma_1$. Thus, by Birkhoff Ergodic Theorem applied to the system (g, μ_g), we have that for μ_g almost everywhere $\underline{x}$

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} n_1(g^j(\underline{x})) = \int n_1 \, d\mu_g. \tag{3}$$

Observe that $n_k(\underline{x}) = \sum_{j=0}^{k-1} n_1(g^j(\underline{x}))$. Consequently, we have that

$$\lim_{k\to\infty} \frac{n_k(\underline{x})}{k} = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} n_1(g^j(\underline{x})) = \int n_1 \, d\mu_g, \tag{4}$$

and this finish the proof.

□

3 Conformal Measures and Weak Gibbs Measures

We observe that eigen-measures for the Ruelle-Perron-Frobenius operator are natural candidates for sequential Gibbs measures. To recall, denote by $\mathcal{C}(\Sigma)$ the set of real-valued continuous functions on Σ. The Ruelle-Perron-Frobenius operator $\mathcal{L}_\psi : \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma)$ associated to $\psi \in \mathcal{C}(\Sigma)$ is defined by

$$\mathcal{L}_\psi \phi(\underline{x}) := \sum_{\underline{y}\in\sigma^{-1}(\underline{x})} e^{\psi(\underline{y})} \phi(\underline{y}).$$

Observe, that for $n \in \mathbb{N}$

$$\mathcal{L}^n_\psi \phi(\underline{x}) = \sum_{\underline{y}\in\sigma^{-n}(\underline{x})} e^{\psi^n(\underline{y})} \phi(\underline{y})$$

This operator is *positive*, i.e., preserves the cone of positive functions $C(\Sigma)^+$. Therefore, we may restrict the dual operator $\mathcal{L}^*_\psi$ to the dual cone $(C(\Sigma)^+)^*$. If we identify the cone $(C(\Sigma)^+)^*$ with the space of positive finite measures $\mathcal{M}(\Sigma)$ by Riesz Theorem, the operator $\mathcal{L}^*_\psi$ is defined by:

$$\begin{aligned} \mathcal{L}^*_\psi : \mathcal{M}(\Sigma) &\to \mathcal{M}(\Sigma) \\ \mu &\mapsto \mathcal{L}^*_\psi(\mu) : \mathcal{C}(\Sigma) \to \mathbb{R} \\ &\qquad\qquad \phi \mapsto \mathcal{L}^*_\psi(\mu)(\phi) = \int_\Sigma \mathcal{L}_\psi(\phi) d\mu \end{aligned}$$

If $r(T)$ denotes the spectral radius of T, we have that $r(\mathcal{L}_\psi) = r(\mathcal{L}^*_\psi)$. Since $C(\Sigma)^+$ is a normal cone with non-empty interior, the spectral theory of positive operators on cones (see [4] or [1], for instance) give us that that $r(\mathcal{L}_\psi)$ is an eigenvalue of $\mathcal{L}^*_\psi$ with some eigenvector $\mu \in \mathcal{M}(\Sigma)$, i.e,

$$\mathcal{L}_\psi^* \mu = r(\mathcal{L}_\psi)\mu.$$

We say these measures are *conformal measures* with respect to ψ. A main feature of a conformal measure is the relation between the variation of ψ and distortion properties:

Proposition 3.1. *Let $\mu \in \mathcal{M}(\Sigma)$ be a conformal measure for $\psi \in \mathcal{C}(\Sigma)$. Then for every $n \in \mathbb{N}$ and $\underline{y} \in \sigma^{-j}([x_j...x_{n-1}])$, with $0 \le j < n$, we have that*

$$e^{-var_{n-j}(\psi^{n-j},\sigma^j(\underline{x}))} \le \frac{\mu([x_j...x_{n-1}])}{e^{\psi^{n-j}(\sigma^j(\underline{y}))-(n-j)P}} \le e^{var_{n-j}(\psi^{n-j},\sigma^j(\underline{x}))}, \tag{5}$$

where $\log r(\mathcal{L}_\psi) = P$.

Proof. For all $[x_j...x_{n-1}]$ and $\underline{y} \in \sigma^{-j}([x_j...x_{n-1}])$, with $0 \le j < n$, we have

$$\begin{aligned}
\mu([x_j...x_{n-1}]) &= \int 1_{[x_j...x_{n-1}]} d\mu \\
&= \lambda^{-(n-j)} \int \mathcal{L}_\psi^{n-j} 1_{[x_j...x_{n-1}]} d\mu \\
&= \lambda^{-(n-j)} \int \sum_{\sigma^j(\underline{y}) \in \sigma^{-(n-j)}(\underline{z})} 1_{[x_j...x_{n-1}]}(\sigma^j(\underline{y})) e^{\psi^{n-j}(\sigma^j(\underline{y}))} d\mu(\underline{z}) \\
&\le \lambda^{-(n-j)} e^{var_{n-j}(\psi^{n-j},\sigma^j(\underline{x}))} e^{\psi^{n-j}(\sigma^j(\underline{x}))}
\end{aligned}$$

similarly

$$\lambda^{-(n-j)} e^{-var_{n-j}(\psi^{n-j},\sigma^j(\underline{x}))} e^{\psi^{n-j}\sigma^j(\underline{x}))} \le \mu([x_j...x_{n-1}])$$

Then,

$$e^{-var_{n-j}(\psi^{n-j},\sigma^j(\underline{x}))} \le \frac{\mu([x_j...x_{n-1}])}{e^{\psi^{n-j}(\sigma^j(\underline{x}))-(n-j)P}} \le e^{var_{n-j}(\psi^{n-j},\sigma^j(\underline{x}))}.$$

□

To analyze sufficient conditions such that a conformal measure is sequential Gibbs measure, we define the sequence of functions $\xi_n(\underline{x})$ at $\underline{x} = x_0x_1...$ by

$$\xi_n(\underline{x}) = \sup_{\underline{y} \in [x_0...x_{n-1}]} \left\{ \sum_{i=0}^{n-1} |\psi(\sigma^j(\underline{x}) - \psi(\sigma^j(\underline{y})| \right\}.$$

We have that

Proposition 3.2. *Given a conformal measure μ such that $\liminf_{n\to\infty} \xi_n(\underline{x}) \le C$ at μ-a.e $\underline{x} \in \Sigma$, for some constant $C > 0$, then μ is a sequential Gibbs measure.*

Proof. Observe that $var_n(\psi^n, \underline{x}) \le \xi_n(\underline{x})$ and $\xi_{n-j}(\sigma^j(\underline{x})) \le \xi_n(\underline{x})$, for each $0 \le j \le n$. Then,

$$var_{n-j}(\psi^{n-j}, \sigma^j(\underline{x})) \le \xi_{n-j}(\sigma^j(\underline{x})) \le \xi_n(\underline{x})$$

By hypothesis, for almost every $\underline{x} \in \Sigma$ there is a sequence $n_i(\underline{x})$ such that $\xi_{n_i}(\underline{x}) \le C$. Thus, by Equation (5), we have for any $n_i(\underline{x})$

$$e^{-C} \le e^{-var_{n_i-j}(\psi^{n-j},\sigma^j(\underline{x}))} \le \frac{\mu([x_j...x_{n_i-1}])}{e^{\psi^{n_i-j}(\sigma^j(\underline{y})-(n_i-j)P}} \le e^{var_{n_i-j}(\psi^{n_i-j},\sigma^j(\underline{x}))} \le e^C. \tag{6}$$

And this finish the proof.

$\square$

We discuss some conditions on the sequence $(\xi_n)_{n\geq 1}$ that give us more information about conformal sequential Gibbs measures. The results here will not be used elsewhere in this paper and are included to help to clarify the relation between our notion of sequential Gibbs measures and weak forms of Gibbs measures studied before. We begin recalling the notion of weak Gibbs measure for continuous potentials, studied by Yuri in [18]:

Definition 3.1. *A measure μ is a weak Gibbs measure for the potential $\psi : \Sigma \to \mathbb{R}$ if there is a constant P and a sequence of positive numbers K_n satisfying*

$$\lim_{n\to\infty} \frac{\log K_n}{n} = 0,$$

such that for each $n \in \mathbb{N}$, $\underline{x} = x_0x_1...$ and $\underline{y} \in [x_0...x_{n-1}]$ we have

$$\frac{1}{K_n} \leq \frac{\mu([x_0...x_{n-1}])}{e^{\psi^n(\underline{y})-nP}} \leq K_n \tag{7}$$

In [18], the author discussed some examples of nonuniformly hyperbolic maps with potentials with an unique equilibrium measure that fails to be a Gibbs measure, but has the weak Gibbs property. Now, we establish the relation between sequential and weak Gibbs conformal measures, using the sequence ξ_n:

Proposition 3.3. *If* $\lim_{n\to\infty}(1/n)\|\xi_n\|_\infty = 0$*, then any conformal measure is a weak Gibbs measure.*

Proof. Following the steps of the proof of the Proposition 3.2 and observing the Equation (6), we have that

$$\begin{aligned} e^{-\xi_n(\underline{x})} &\leq e^{-var_n(\psi^n,\underline{y})} \leq \frac{\nu([x_0,...,x_{n-1}])}{e^{\psi^n(\underline{y})-nP}} \\ &\leq e^{var_n(\psi^n,\underline{y})} \leq e^{\xi_n(\underline{x})} \end{aligned}$$

for every $\underline{y} \in [x_0...x_{n-1}]$. Just take $K_n := e^{\|\xi_n\|_\infty}$. $\square$

Now, we discuss the notion of *non-lacunary Gibbs measure*, studied in [10]. We say that a sequence of natural numbers $a_1 < a_2 < ...$ is *non-lacunary*, if $\lim_{i\to\infty} a_{i+1}/a_i = 0$.

A non-lacunary Gibbs measure is a sequential Gibbs measure such that the sequence $n_i(\underline{x})$ is *non-lacunary* at almost every point $\underline{x} \in \Sigma$. The proof of next lemma follows, mutatis mutandis, from the proof of Proposition 3.8 of [10].

Lemma 3.1. *Let μ be a invariant sequential Gibbs measure such that the function n_1 is integrable. Then, for almost every $\underline{x} \in \Sigma$, the sequence $n_i(\underline{x})$ is non-lacunary.*

In view of Lemma 3.1 and following the proof of Proposition 3.17 of [10], we are able to show that:

Proposition 3.4. *If μ is sequential Gibbs measure and the function n_1 is integrable, then exist a sequence of positive functions $K_n > 1$ such that μ-a.e. $\underline{x}$ and for all $n \in \mathbb{N}$, we have*

$$K_n^{-1}(\underline{x}) \leq \frac{\mu([x_0...x_{n-1}])}{e^{\phi^n(\underline{y})-nP}} \leq K_n(\underline{x}) \tag{8}$$

and $\limsup_{n\to\infty} \frac{\log K_n(\underline{x})}{n} = 0$.

Remark 3.1. *We observe that given a conformal sequential Gibbs measure, it is always possible to choose $K_n(x)$ as in Proposition 3.4 in such way that $(1/n)\log K_n(\underline{x})$ converges almost everywhere.*

Indeed, since ξ_n is a subadditive sequence of non-negative functions, we use the Ergodic Subadditive Theorem of Kingman (see [9], Theorem 3.3.3) to have that $(1/n)\xi_n$ converge almost everywhere. Then, take $K_n(x) := e^{\xi_n(\underline{x})}$ and observe that the Equation (8) *is satisfied.*

4 Examples

In this section we discuss some examples of sequential Gibbs measures. The first one was introduced in [5], where Hofbauer gave an interesting example of a family of continuous potentials with phase transitions and equilibrium states that are not a Gibbs measures. We reproduce this example and show that, despite the fact that they not satisfy the Gibbs property, these measure are sequential Gibbs measure with integrable first Gibbs time function.

Example 4.1. *For simplicity, let $\Sigma_2^+ = \{1,0\}^{\mathbb{N}}$ be an one-sided shift space with two symbols. Consider the partition of Σ_2^+ by sets $(M_k)_{k\geq 0}$ plus the point $\underline{1} = 1111..$ where M_k is defined by $M_0 = [0]$ and for $k = 1, 2, ...$*

$$M_k = [\underbrace{11..1}_{k\ times}0] = \{\underline{x} \in \Sigma_2^+ : x_i = 1 \text{ for } 0 \leq i < k-1,\ x_k = 0\}.$$

Let (a_k) be a sequence of real numbers with $\lim a_k = 0$. Set $s_k = a_0 + ... + a_k$. Define a continuous potential $g \in C(\Sigma_2^+)$ by

$$g(\underline{x}) = a_k \text{ for } \underline{x} \in M_k \text{ and } g(11...) = 0.$$

As was pointed out at Section 2, there exists some conformal measure ν with respect to g. By results in ([5], page 230), g admits a Gibbs measure if, and only if, $\sum_{k\geq 0} a_k$ is convergent. Assume that g has no Gibbs measures, i.e., $\sum_{k\geq 0} a_k$ diverges.

If $\sum_{k\geq 0} e^{s_k} > 1$, by ([5], page 226) we have that there exists some positive continuous function h such that $\mu = h\nu$ is the unique equilibrium state of g. We prove that in this case, despite the fact that it do not satisfy the Gibbs property, μ is a sequential Gibbs measure.

Indeed, since $\mu = h\nu$ and h is bounded from above and below, it follows from the fact that ν is a sequential Gibbs measure. As we observed before in Proposition 3.2, it is enough to show that $\liminf \xi_n(\underline{x})$ is bounded at ν almost everywhere. In fact, we prove that $\liminf \xi_n(\underline{x}) = 0$ at ν almost every point.

Firstly, observe that from the definition, $\xi_{k+1}(\underline{x}) = 0$, if $\sigma^k(\underline{x}) \in M_0$. On the other hand, since $\mu \neq \delta_{11...}$, we have that for almost every point $\underline{x}$ there exists a sequence $k_1(\underline{x}) < k_2(\underline{x}) < \dots$ such that $\sigma^{k_i}(\underline{x}) \in M_0$. In particular, $\xi_{k_i+1}(\underline{x}) = 0$.

From this, we have that the first Gibbs time is integrable. Indeed, using Proposition 3.2, we have that the first Gibbs time function n_1 of μ is smaller than the first return time to M_0. Then, by Kac's Lemma, we have that n_1 is integrable with respect to μ.

In the next example we discuss the non-lacunary Gibbs measures studied in [10, 15, 13]. These measures are equilibrium states for Hölder *hyperbolic* potentials of some C^1 local diffeomorphisms on compact Riemannian manifolds and they have only positive Lyapunov exponents.

Example 4.2. *Let $f : M \to M$ be an C^1 local diffeomorphism of a compact connected manifold M such that there exists sets $R_1, ..., R_q$ of M that are domains of injectivity of f such that $\overline{R_i} \cap \overline{R_j} = \emptyset$, for $i \neq j$, and $f(R_i) = M$, for $1 \leq i \leq q$. Consider $\mathcal{R} = R_1 \cup \cdots \cup R_q$ and the invariant set $\Lambda = \cap_{n\geq 1} f^{-n}(\mathcal{R})$. We may define a semiconjugacy*

$$\pi : \Lambda \to \Sigma_q^+,$$

between $f|_\Lambda$ and $\sigma : \Sigma_q^+ \to \Sigma_q^+$, where $\Sigma_q^+ = \{1,\dots,q\}^{\mathbb{N}}$, just considering $\pi(x)$ as the itinerary of x with respect to the partition $\mathcal{P} = \{P_1,\dots,P_q\}$ of Λ, defined by $P_i = R_i \cap \Lambda$. Denote by $P(x)$ the element of $\mathcal{P}$ that contains x and assume that there exists $\sigma_1, \sigma_2 > 1$ such that:

- $\|Df(x)^{-1}\| < \sigma_2$, *for every $x \in M$ and*
- $\|Df(x)^{-1}\| < \sigma_1^{-1} < 1$, *for x in the complement of an open set containing R_1.*

In [10] and [15], the authors proved that if ϕ is a Hölder continuous potential such that $\max\phi - \min\phi$ *is small enough, then there exists an unique equilibrium state η for ϕ and this measure has only positive Lyapunov exponents and it is a non-lacunary Gibbs measure, in the sense that if we define*

$$P^n(x) = P(x) \cap f^{-1}(P(f(x))) \cap \cdots \cap f^{-(n-1)}(P(f^{n-1}(x))),$$

then there exist a constant K, such that for η almost every $x \in \Lambda$ there exists a sequence $n_i(x)$ such that $\lim_{i\to\infty} n_{i+1}(x)/n_i(x) = 1$ *and*

$$K^{-1} \le \frac{\eta(P^{n_k}(x))}{e^{\phi^{n_i}(x) - n_i P(\phi)}} \le K.$$

If we consider the push-forward measure $\mu = \pi_*\eta$ on Σ_q^+, then the map π is invertible in a set of μ-full measure and the measure μ is a sequential Gibbs measure with respect to the potential $\psi = \phi \circ \pi^{-1}$.

5 Proof of Theorem 1

In this subsection we construct the potential ψ_2 as in Theorem 1, obtained as the limit of a converging sequence of functions. Given $z \in E \subset \Sigma_2$, as in the Definition 2.1, denote by $(n_k(\underline{z}))_{i\ge 1}$ the sequence of Gibbs times of any $\underline{x} \in \Pi^{-1}(\underline{z}) \cap \Sigma_1$. By Hypothesis 2.1, $n_k(\underline{z})$ is well defined for a set of full ν measure.

Definition 5.1. *Given $k \in \mathbb{N}$ and $\underline{w} \in \Sigma_1$, define $u_{\underline{w},k} : E \subset \Sigma_2 \to \mathbb{R}$ by*

$$u_{\underline{w},k}(\underline{z}) = \frac{\sum_{\underline{x}=x_0\dots x_{n_k}} e^{\psi_1^{n_k+1}(\underline{xw})}}{\sum_{\underline{x'}=x_1\dots x_{n_k}} e^{\psi_1^{n_k}(\underline{x'w})}},$$

where $\sum_{\underline{x}=x_0\dots x_{n_k}}$ represents the sum over finite words $\underline{x} = x_0x_1\dots x_{n_k}$ such that $\pi(x_i) = z_i$, for $i = 0,\dots,n_k$ and $\underline{xw} = x_0\dots x_n w_0 w_1\dots$.

We show that

Proposition 5.1. *The limit $u(\underline{z}) := \lim_{k\to\infty} u_{\underline{w},k}(\underline{z})$ is well defined and independent of $\underline{w}$.*

The proof of Proposition 5.1 is the central point of this article. We postpone this proof to the next section. However, assume it to be true for a moment and let us prove that ν is a sequential Gibbs measure for $\psi_2 = \log u$. First, we prove that:

Lemma 5.1. *There is a constant $C > 0$ depending only ψ_1, such that for every $\underline{w}, \underline{w}'$ and a sequence of Gibbs times $(n_i(\underline{x}))_{i\geq 1}$ and $0 \leq l \leq n_i$, we have*

$$\frac{e^{\psi_1^{n_i-l+1}(\sigma^l(\underline{xw}))}}{e^{\psi_1^{n_i-l+1}(\sigma^l(\underline{xw}'))}} \leq C$$

Proof. Note that the definition of sequential Gibbs measures we have for every choice $\underline{w}$, $\underline{w}'$ and $\underline{x} = x_0...x_{n_i}$ (with $0 \leq l \leq n_i$),

$$K^{-1}e^{\psi_1^{n_i-l+1}(\sigma^l(\underline{xw}))} \leq \mu[x_l...x_{n_i}] \leq Ke^{\psi_1^{n_i-l+1}(\sigma^l(\underline{xw}'))}$$

Thus,

$$\frac{e^{\psi_1^{n_i-l+1}(\sigma^l(\underline{xw}))}}{e^{\psi_1^{n_i-l+1}(\sigma^l(\underline{xw}'))}} \leq K^2 = C$$

□

Corollary 5.1. *With the same hypothesis of the previous lemma, we have that there is a constant $C > 0$, depending only ψ_1, such that for every n_i and $0 \leq l \leq n_i$, we have*

$$\frac{\sum_{\underline{x}=x_l...x_{n_i}} e^{\psi_1^{n_i-l+1}(\underline{xw})}}{\sum_{\underline{x}=x_l...x_{n_i}} e^{\psi_1^{n_i-l+1}(\underline{xw}')}} \leq C$$

We can now define the potential for ν.

Definition 5.2. *We define the potential $\psi_2 : \Sigma_2 \to \mathbb{R}$ by $\psi_2(\underline{z}) := \log u(\underline{z})$.*

The main problem is precisely to show that the potential ψ_2 is well defined. We follow the lines of [7] and also prove the assertions of Theorem 1. Suppose, for a moment, that the Proposition 5.1 is true. We have the following lemma.

Lemma 5.2. *The measure $\nu = \Pi_*\mu$ is sequential Gibbs measure for the potential $\psi_2(\underline{z}) = \log u(\underline{z})$.*

Proof. Let us fix $n \geq 1$. We can write

$$\psi_2^{n+1}(\underline{z}) = \sum_{i=0}^{n} \log u(\sigma^i(\underline{z})) = \lim_{k\to\infty} \log u_{\underline{w},k}(\underline{z}) + ... + \lim_{k\to\infty} \log u_{\underline{w},k}(\sigma^n(\underline{z}))$$

Since $n_k(\underline{z}) - l$ is a Gibbs time of $\sigma^l(z)$, given $1 \leq l \leq n_k$, we may choose a subsequence $(i_k^l)_{k\geq 1}$, such that $n_{i_k^l}(\sigma^l(\underline{z})) = n_k(\underline{z}) - l$. Consequently, given n

$$\begin{aligned}\psi_2^{n+1}(\underline{z}) &= \lim_{k\to\infty} \log \left(u_{\underline{w},i_k^0(\underline{z})}(\underline{z}) \cdot ... \cdot u_{\underline{w},i_k^n(\sigma^n(\underline{z}))}(\sigma^n(\underline{z})\right) = \\ &= \lim_{k\to\infty} \log \left(\frac{\sum_{\underline{x}=x_0...x_{n_k}} e^{\psi_1^{n_k+1}(\underline{xw})}}{\sum_{\underline{x}'=x_1...x_{n_k}} e^{\psi_1^{n_k}(\underline{x}'\underline{w})}} \cdot \frac{\sum_{\underline{x}=x_1...x_{n_k}} e^{\psi_1^{n_k}(\underline{xw})}}{\sum_{\underline{x}'=x_2...x_{n_k}} e^{\psi_1^{n_k}(\underline{x}'\underline{w})}} \cdot ... \right. \\ &\left. \cdot \frac{\sum_{\underline{x}=x_n...x_{n_k}} e^{\psi_1^{n_k-n+1}(\underline{xw})}}{\sum_{\underline{x}'=x_{n+1}...x_{n_k}} e^{\psi_1^{n_k-n}(\underline{x}'\underline{w})}} \right) = \lim_{k\to\infty} \log \left(\frac{\sum_{\underline{x}=x_0...x_{n_k}} e^{\psi_1^{n_k+1}(\underline{xw})}}{\sum_{\overline{x}=x_{n+1}...x_{n_k}} e^{\psi_1^{n_k-n}(\overline{x}\underline{w})}} \right)\end{aligned}$$

Note that, in the same way for $1 \leq l \leq n$, we have

$$\psi_2^{n-l+1}(\sigma^l(\underline{z})) = \lim_{k\to\infty} \log \left(\frac{\sum_{\underline{x}=x_l...x_{n_k}} e^{\psi_1^{n_k-l+1}(\underline{x}\underline{w})}}{\sum_{\overline{x}=x_{n+1}...x_{n_k}} e^{\psi_1^{n_k-n}(\overline{x}\underline{w})}} \right) \tag{9}$$

Moreover, for $n_k > n_i$ and for $0 \leq l \leq n_i$ we can write

$$\sum_{\underline{x}=x_l...x_{n_k}} e^{\psi_1^{n_k-l+1}(\underline{x}\underline{w})} = \sum_{\underline{x}=x_l...x_{n_i}} \sum_{\overline{x}=x_{n_i+1}...x_{n_k}} e^{\psi_1^{n_i-l+1}(\underline{x}\overline{x}\underline{w})} e^{\psi_1^{n_k-n_i}(\overline{x}\underline{w})}$$

By Corollary 5.1, for $0 \leq l \leq n_i$ we have

$$\sum_{\underline{x}=x_l...x_{n_k}} e^{\psi_1^{n_k-l+1}(\underline{x}\underline{w})} \leq C \sum_{\underline{x}'=x_l...x_{n_i}} e^{\psi_1^{n_i-l+1}(\underline{x}'\underline{w})} \sum_{\overline{x}=x_{n_i+1}...x_{n_k}} e^{\psi_1^{n_k-n_i}(\overline{x}\underline{w})},$$

and also follows from Corollary 5.1

$$C^{-1} \cdot \sum_{\underline{x}'=x_l...x_{n_i}} e^{\psi_1^{n_i-l+1}(\underline{x}'\underline{w})} \leq \frac{\sum_{\underline{x}=x_l...x_{n_k}} e^{\psi_1^{n_k-l+1}(\underline{x}\underline{w})}}{\sum_{\overline{x}=x_{n_i+1}...x_{n_k}} e^{\psi_1^{n_k-n_i}(\overline{x}\underline{w})}} \leq C \cdot \sum_{\underline{x}'=x_l...x_{n_i}} e^{\psi_1^{n_i-l+1}(\underline{x}'\underline{w})}$$

Taking $k \to \infty$, we have

$$C^{-1} \cdot \sum_{\underline{x}'=x_l...x_{n_i}} e^{\psi_1^{n_i-l+1}(\underline{x}'\underline{w})} \leq e^{\psi_2^{n_i-l+1}(\sigma^l(\underline{z}))} \leq C \cdot \sum_{\underline{x}'=x_l...x_{n_i}} e^{\psi_1^{n_i-l+1}(\underline{x}'\underline{w})}$$

And so

$$e^{\psi_2^{n_i-l+1}(\sigma^l(\underline{z}))} \approx \sum_{\underline{x}'=x_l...x_{n_i}} e^{\psi_1^{n_i-l+1}(\underline{x}'\underline{w})}$$

Since μ is an sequential Gibbs measure for ψ_1, there are constant K, such that for each $\underline{x} \in \Pi^{-1}(\underline{z}) \cap \Sigma_1$ and an sequence $(n_i(\underline{x}))_{i\geq 1}$,

$$K^{-1} e^{\psi_1^{n_i+1-l}(\sigma^l(\underline{x}))} \leq \mu_1[x_l...x_{n_i}] \leq K e^{\psi_1^{n_i+1-l}(\sigma^l(\underline{x}))}.$$

for $0 \leq l \leq n_i$. Adding over all words $\underline{x}$ that are projected in $\underline{z}$, we have

$$\begin{aligned} K^{-1} \sum_{\underline{x}'=x_l...x_{n_i}} e^{\psi_1^{n_i-l+1}(\underline{x}'\underline{w})} &\leq \sum_{x_l...x_{n_i}} \mu[x_l...x_{n_i}] \\ &\leq K \sum_{\underline{x}'=x_l...x_{n_i}} e^{\psi_1^{n_i-l+1}(\underline{x}'\underline{w})} \end{aligned}$$

So,

$$\sum_{x_l...x_{n_i}} \mu[x_l...x_{n_i}] \approx \sum_{\underline{x}'=x_l...x_{n_i}} e^{\psi_1^{n_i-l+1}(\underline{x}'\underline{w})}$$

Therefore,

$$\nu[z_l...z_{n_i}] = \sum_{x_l...x_{n_i}} \mu[x_l...x_{n_i}] \approx e^{\psi_2^{n_i-l+1}(\sigma^l(\underline{z}))}$$

for each $0 \leq l \leq n_i$, proving that ν is sequential Gibbs measure for $\sigma : \Sigma_2 \to \Sigma_2$ and ψ_2. $\square$

In this section, we prove that ψ_2 is well defined. We give definitions that help us in this purpose.

Definition 5.3. *Let be $k \in \mathbb{N}$ and $\underline{z} \in E \subset \Sigma_2$. We define the closed interval*

$$\Lambda_k(\underline{z}) := \left[\min_{\underline{w}} u_{\underline{w},k}(\underline{z}), \max_{\underline{w}',} u_{\underline{w}',k}(\underline{z})\right].$$

Definition 5.4. *Given $k \in \mathbb{N}$ e $\underline{z} \in E \subset \Sigma_2$. We define*

$$\lambda_k(\underline{z}) := \sup \left\{ \frac{u_{\underline{w},k}(\underline{z})}{u_{\underline{w}',k}(\underline{z})} : \underline{w}, \underline{w}' \in \Sigma_1 \right\}.$$

Definition 5.5. *We say that a sequence of intervals I_n is monotonically nested if we have*

$$I_0 \supseteq I_1 \supseteq ... \supseteq I_n \supseteq ...$$

In the next lemma, we show that the sequence $(\Lambda_k(\underline{z}))_{k\geq 1}$ is monotonically nested. Then, the existence of ψ_2 at $\nu-$a.e. $\underline{z} \in \Sigma_2$ corresponds to the convergence to 1 of the sequence $\lambda_k(\underline{z})$.

Lemma 5.3. *The sequence of intervals $(\Lambda_k(\underline{z}))_{k\geq 1}$ is monotonically nested.*

Proof. Given $z \in E$, observe that

$$\begin{aligned}
u_{\underline{w},k+1}(\underline{z}) &= \frac{\sum_{\underline{x}=x_0...x_{n_{k+1}}} e^{\psi_1^{n_{k+1}+1}(\underline{x}\underline{w})}}{\sum_{\underline{x}'=x_1...x_{n_{k+1}}} e^{\psi_1^{n_{k+1}}(\underline{x}'\underline{w})}} \\
&= \frac{\sum_{\underline{x}=x_0...x_{n_k}} \sum_{\overline{x}=x_{n_k}+1...x_{n_{k+1}}} e^{\psi_1^{n_k+1}(\underline{x}\overline{x}\underline{w})} e^{\psi_1^{n_{k+1}-n_k}(\overline{x}\underline{w})}}{\sum_{\underline{x}'=x_1...x_{n_k}} \sum_{\overline{x}=x_{n_k}+1...x_{n_{k+1}}} e^{\psi_1^{n_k}(\underline{x}\overline{x}\underline{w})} e^{\psi_1^{n_{k+1}-n_k}(\overline{x}\underline{w})}} \\
&\leq \max_{\overline{x}} \frac{\sum_{\underline{x}=x_0...x_{n_k}} e^{\psi_1^{n_k+1}(\underline{x}\overline{x}\underline{w})}}{\sum_{\underline{x}'=x_1...x_{n_k}} e^{\psi_1^{n_k}(\underline{x}'\overline{x}\underline{w})}} \\
&\leq \max_{\overline{x}} u_{\overline{x}\underline{w},k}(\underline{z}) \\
&\leq \max_{\underline{w}'} u_{\underline{w}',k}(\underline{z})
\end{aligned}$$

On the other hand, for given $\overline{x}$

$$\begin{aligned}
\min_{\underline{w}'} u_{\underline{w}',k}(\underline{z}) &\leq \min_{\overline{x}\underline{w}} u_{\overline{x}\underline{w},k}(\underline{z}) \\
&= \min_{\overline{x}\underline{w}} \frac{\sum_{\underline{x}=x_0\ldots x_{n_k}} e^{\psi_1^{n_k+1}(\underline{x}\overline{x}\underline{w})}}{\sum_{\underline{x}'=x_1\ldots x_{n_k}} e^{\psi_1^{n_k}(\underline{x}'\overline{x}\underline{w})}} \\
&\leq \frac{\sum_{\underline{x}=x_0\ldots x_{n_k}} \sum_{\overline{x}=x_{n_k}+1\ldots x_{n_{k+1}}} e^{\psi_1^{n_k+1}(\underline{x}\overline{x}\underline{w})} e^{\psi_1^{n_{k+1}-n_k}(\overline{x}\underline{w})}}{\sum_{\underline{x}'=x_1\ldots x_{n_k}} \sum_{\overline{x}=x_{n_k}+1\ldots x_{n_{k+1}}} e^{\psi_1^{n_k}(\underline{x}\overline{x}\underline{w})} e^{\psi_1^{n_{k+1}-n_k}(\overline{x}\underline{w})}} \\
&= \frac{\sum_{\underline{x}=x_0\ldots x_{n_{k+1}}} e^{\psi_1^{n_{k+1}+1}(\underline{x}\underline{w})}}{\sum_{\underline{x}'=x_1\ldots x_{n_{k+1}}} e^{\psi_1^{n_{k+1}+1}(\underline{x}'\underline{w})}} \\
&= u_{\underline{w},k+1}(\underline{z})
\end{aligned}$$

$\square$

Lemma 5.4. *For ν-a.e. $\underline{z} \in \Sigma_2$, the sequence $\lambda_0(\underline{z}) \geq \lambda_1(\underline{z}) \geq \ldots \geq \lambda_n(\underline{z}) \geq \lambda_{n+1}(\underline{z}) \geq \ldots \geq 1$ is a decreasing sequence.*

Proof. From inequalities of Lemma 5.3, to $\underline{w}, \underline{w}' \in \Sigma_1$, we have

$$\frac{u_{\underline{w},k+1}(\underline{z})}{u_{\underline{w}',k+1}(\underline{z})} \leq \sup_{\underline{v},\underline{v}' \in \Sigma_1} \left\{ \frac{u_{\underline{v},k}(\underline{z})}{u_{\underline{v}',k}(\underline{z})} \right\} = \lambda_k(\underline{z})$$

Taking the supremum over $\underline{w}, \underline{w}' \in \Sigma_1$, we have $\lambda_{k+1}(\underline{z}) \leq \lambda_k(\underline{z})$.

$\square$

Now, we show that $\lambda_k(\underline{z}) \to 1$ for ν-a.e. $\underline{z} \in \Sigma_2$. Let $n_i < n_k$ be Gibbs times of $\underline{z}$ and words $x_0, \ldots, x_{n_i}$ that projects on $\underline{z}$. We define

$$P^{k,i}(\overline{x}, \underline{w}) = \frac{\sum_{\underline{x}=x_1\ldots x_{n_i}} e^{\psi_1^{n_k}(\underline{x}\overline{x}\underline{w})}}{\sum_{\underline{x}'=x_1\ldots x_{n_k}} e^{\psi_1^{n_k}(\underline{x}'\underline{w})}}$$

The probability vector $P^{k,i}(\overline{x}, \underline{w})$ allow us to express the function $u_{\underline{w},k}$ in terms of the function $u_{\underline{w},i}$, for $i < k$. Indeed, we have the

Lemma 5.5. *Let $n_i < n_k$ be Gibbs times of $\underline{z} \in B$. Then, we have that*

$$u_{\underline{w},k}(\underline{z}) = \sum_{\overline{x}=x_{n_i+1}\ldots x_{n_k}} u_{\overline{x}\underline{w},i}(\underline{z}) P^{k,i}(\overline{x}, \underline{w}).$$

where the sum above is over words $x_0 \ldots x_{n_k}$ that project onto $z_0 \ldots z_{n_k}$.

Proof. By definition, the numerator of $u_{\underline{w},k}(\underline{z})$ is

$$\sum_{\underline{x}=x_0\ldots x_{n_k}} e^{\psi_1^{n_k+1}(\underline{x}\underline{w})} = \sum_{\underline{x}=x_0\ldots x_{n_i}} \sum_{\overline{x}=x_{n_i+1}\ldots x_{n_k}} e^{\psi_1^{n_i+1}(\underline{x}\overline{x}\underline{w})} e^{\psi_1^{n_k-n_i}(\overline{x}\underline{w})} \tag{10}$$

Further, we can rewrite the right hand side of Equation (10) as

$$= \sum_{\overline{x}=x_{n_i+1}\dots x_{n_k}} \underbrace{\left(\frac{\sum_{\underline{x}=x_0\dots x_{n_i}} e^{\psi_1^{n_i+1}(\underline{x}\overline{x}\underline{w})}}{\sum_{\underline{x}'=x_1\dots x_{n_i}} e^{\psi_1^{n_i}(\underline{x}'\overline{x}\underline{w})}} \right)}_{u_{\overline{x}\underline{w},i}(\underline{z})} \underbrace{\left(\sum_{\underline{x}'=x_1\dots x_{n_i}} e^{\psi_1^{n_i}(\underline{x}'\overline{x}\underline{w})} \right) e^{\psi_1^{n_k-n_i}(\overline{x}\underline{w})}}_{\sum_{\underline{x}'=x_1\dots x_{n_i}} e^{\psi_1^{n_k}(\underline{x}'\overline{x}\underline{w})}} \tag{11}$$

Then, dividing both members of Equation (11) by $\sum_{\underline{x}=x_1\dots x_{n_k}} e^{\psi_1^{n_k}}(\underline{x}\,\underline{w})$, we have

$$u_{\underline{w},k}(\underline{z}) = \sum_{\overline{x}=x_{n_i+1}\dots x_{n_k}} u_{\overline{x}\underline{w},i}(\underline{z}) \cdot P^{k,i}(\overline{x}, \underline{w})$$

□

Corollary 5.2.

$$\frac{u_{\underline{w},k}(\underline{z})}{u_{\underline{w}',k}(\underline{z})} = \frac{\sum_{\overline{x}=x_{n_i+1}\dots x_{n_k}} u_{\overline{x}\underline{w},i}(\underline{z}) P^{k,i}(\overline{x}, \underline{w})}{\sum_{\overline{x}=x_{n_i+1}\dots x_{n_k}} u_{\overline{x}\underline{w}',i}(\underline{z}) P^{k,i}(\overline{x}, \underline{w}')}$$

Proof. Follows directly from Lemma 5.5.

□

Lemma 5.6. *There exist $c > 0$ such that for any $n_i < n_k$ Gibbs times of $\underline{z}$ and for $\overline{x} = x_{n_i+1}\dots x_{n_k}$ that projects onto $\overline{z} = z_{n_i+1}\dots z_{n_k}$ and $\underline{w}, \underline{w}'$ we have*

$$\frac{P^{k,i}(\overline{x}, \underline{w})}{P^{k,i}(\overline{x}, \underline{w}')} \geq c$$

Proof. Since $\underline{x}\overline{x}\underline{w}$ and $\underline{x}\overline{x}\underline{w}'$ agree in n_k places by the Gibbs Property (1) we have

$$\frac{e^{\psi_1^{n_k}(\underline{x}\overline{x}\underline{w})}}{e^{\psi_1^{n_k}(\underline{x}\overline{x}\underline{w}')}} \leq K^2$$

We can write

$$\begin{aligned} \frac{P^{k,i}(\overline{x}, \underline{w})}{P^{k,i}(\overline{x}, \underline{w}')} &= \frac{\sum_{\underline{x}=x_1\dots x_{n_i}} e^{\psi_1^{n_k}(\underline{x}\overline{x}\underline{w})}}{\sum_{\underline{x}=x_1\dots x_{n_i}} e^{\psi_1^{n_k}(\underline{x}\overline{x}\underline{w}')}} \cdot \frac{\sum_{\underline{x}'=x_1\dots x_{n_k}} e^{\psi_1^{n_k}(\underline{x}'\underline{w}')}}{\sum_{\underline{x}'=x_1\dots x_{n_k}} e^{\psi_1^{n_k}(\underline{x}'\underline{w})}} \\ &\leq K^4 \end{aligned}$$

To finish the proof of Lemma 5.6, just take

$$c = \frac{1}{K^4}.$$

□

Lemma 5.7. *With the same notations of Lemma 5.6 we have*

$$\frac{u_{\overline{x}\underline{w},i}(\underline{z})}{u_{\overline{x}\underline{w}',i}(\underline{z})} \leq e^{2\sum_{n=n_k-n_i}^{n_k} var_n(\psi_1, \Pi^{-1}(\sigma^{n_k-n}(\underline{z})))}$$

Proof. Considering first the numerators, we have

$$\frac{\text{numerator}(u_{\overline{x}\underline{w},i}(\underline{z}))}{\text{numerator}(u_{\overline{x}\underline{w}',i}(\underline{z}))} = \frac{\sum_{\underline{x}=x_0\ldots x_{n_i}} e^{\psi_1^{n_i+1}(\underline{x}\overline{x}w)}}{\sum_{\underline{x}=x_0\ldots x_{n_i}} e^{\psi_1^{n_i+1}(\underline{x}\overline{x}w')}}$$

comparing termwise we see that $\sigma^j(\underline{x}\overline{x}w)$ and $\sigma^j(\underline{x}\overline{x}w')$ agree to $n_k - n_i + (n_i - j)$ places, and thus for any choice of $\overline{x}$,

$$\frac{e^{\psi_1^{n_i+1}(\underline{x}\overline{x}w)}}{e^{\psi_1^{n_i+1}(\underline{x}\overline{x}w')}} \leq e^{\sum_{n=n_k-n_i}^{n_k} var_n(\psi_1,\sigma^{n_k-n}(\underline{x}))} \leq e^{\sum_{n=n_k-n_i}^{n_k} var_n(\psi_1,\Pi^{-1}(\sigma^{n_k-n}(\underline{z})))}$$

Summing over all choices of $\overline{x}$ and making the identical calculations for the denominator the lemma is proved. □

Corollary 5.3.

$$\frac{u_{\overline{x}\underline{w}^{max},i}(\underline{z})}{u_{\overline{x}\underline{w}^{min},i}(\underline{z})} \leq e^{2\sum_{n=n_k-n_i}^{n_k} var_n(\psi_1,\Pi^{-1}(\sigma^{n_k-n}(\underline{z})))}$$

where $\underline{w}^{max}$ and $x\underline{w}^{min}$ are concatenation $\overline{x}\underline{w}$ which maximizes and minimizes $u_{\overline{x}\underline{w},i}(\underline{z})$, respectively.

Lemma 5.8. *Let be $(n_i)_{i\geq 1}$ the sequence of Gibbs times of $\underline{z}$. For $k \geq i$, we have*

$$\lambda_k(\underline{z}) \leq c \cdot e^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1,\Pi^{-1}(\sigma^n(\underline{z})))} + (1-c)\lambda_i(\underline{z}) \tag{12}$$

Proof. Suppose that for $k \geq i$

$$\lambda_i(\underline{z}) = \max_{\overline{x}} \left(\frac{u_{\overline{x}\underline{w}^{max},i}(\underline{z})}{u_{\overline{x}\underline{w}^{min},i}(\underline{z})} \right) \leq e^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1,\Pi^{-1}(\sigma^n(\underline{z})))}.$$

Then, by Lemma 5.4, $(\lambda_k)_k$ is decreasing and we have that

$$\begin{aligned} \lambda_k(\underline{z}) &\leq \lambda_i(\underline{z}) = c\lambda_i(\underline{z}) + (1-c)\lambda_i(\underline{z}) \\ &\leq c \cdot e^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1,\Pi^{-1}(\sigma^n(\underline{z})))} + (1-c)\lambda_i(\underline{z}). \end{aligned}$$

Proving the claim in the Lemma. Now assume that

$$\lambda_i(\underline{z}) = \max_{\overline{x}} \left(\frac{u_{\overline{x}\underline{w}^{max},i}(\underline{z})}{u_{\overline{x}\underline{w}^{min},i}(\underline{z})} \right) > e^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1,\Pi^{-1}(\sigma^n(\underline{z})))}.$$

By Corollary 5.2 we have

$$\frac{u_{\underline{w}^{max},k}(\underline{z})}{u_{\underline{w}^{min},k}(\underline{z})} = \frac{\sum_{\overline{x}=x_{n_i+1}\ldots x_{n_k}} u_{\overline{x}\underline{w}^{max},i}(\underline{z}) P^{k,i}(\overline{x},\underline{w}^{max})}{\sum_{\overline{x}=x_{n_i+1}\ldots x_{n_k}} u_{\overline{x}\underline{w}^{min},i}(\underline{z}) P^{k,i}(\overline{x},\underline{w}^{min})}$$

To simplify notation, we will fix $\underline{z}$ and enumerate the set $\overline{X}$ of all possible choices $\overline{x} \in \Pi^{-1}(\underline{z})$. So, the sum $\sum_{\overline{x}=x_{n_i+1}\ldots x_{n_k}}$ can be denoted by a sum $\sum_{l\in\overline{X}}$.

Let P_1 be the probability vector $P^{k,i}(\overline{x}, \underline{w}^{max})$ and P_2 the vector $P^{k,i}(\overline{x}, \underline{w}^{min})$. Also denote by A the vector $(u_{\overline{x}\underline{w}^{\max},i}(\underline{z}))$ and B the vector $(u_{\overline{x}\underline{w}^{\min},i}(\underline{z}))$, where $\overline{x}$ run over all choices in $\overline{X}$. If a_l and b_l represent the l-th term of A and B, respectively, and I is vector of length $|\overline{X}|$ with 1 in all of its coordinates, we can summarize the above equality as

$$\frac{u_{\underline{w}^{max},k}(\underline{z})}{u_{\underline{w}^{min},k}(\underline{z})} = \frac{P_1 \cdot A}{P_2 \cdot B} = \frac{cP_1 \cdot A + (1-c)P_1 \cdot A}{cP_1 \cdot B + (P_2 - cP_1) \cdot B} \tag{13}$$

where the signal $\cdot$ is represent the inner product in $\mathbb{R}^{|\overline{X}|}$.

By Lemma 5.7, $a_l \leq b_l \, e^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1, \Pi^{-1}(\sigma^n(\underline{z})))}$ for each l. Thus,

$$P_1 \cdot A \leq e^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1, \Pi^{-1}(\sigma^n(\underline{z})))} P_1 \cdot B.$$

Therefore, in the Equation 13 we have

$$\begin{aligned}
\frac{u_{\underline{w}^{max},k}(\underline{z})}{u_{\underline{w}^{min},k}(\underline{z})} &\leq \frac{ce^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1, \Pi^{-1}(\sigma^n(\underline{z})))} P_1 \cdot B + (1-c)P_1 \cdot A}{cP_1 \cdot B + (P_2 - cP_1) \cdot B} \\
&\leq \frac{ce^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1, \Pi^{-1}(\sigma^n(\underline{z})))} P_1 \cdot B + (1-c)P_1 \cdot I \max_l a_l}{cP_1 \cdot B + (P_2 - cP_1) \cdot I \min_l b_l}
\end{aligned}$$

We prove the following lemma

Lemma 5.9. *Putting* $\alpha_1 = ce^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1, \Pi^{-1}(\sigma^n(\underline{z})))} P_1 \cdot B$, $\beta_1 = cP_1 \cdot B$, $\alpha_2 = (1-c)P_1 \cdot I \max_l a_l$ *and* $\beta_2 = (P_2 - cP_1) \cdot I \min_l b_l$. *Then* $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$.

Proof.

$$\begin{aligned}
\alpha_2\beta_1 = cP_1 \cdot B(1-c)P_1 \cdot I \max_l a_l &\geq ce^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1, \Pi^{-1}(\sigma^n(\underline{z})))} P_1 \cdot B(1-c)P_1 \cdot I \min_l b_l \\
&\geq ce^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1, \Pi^{-1}(\sigma^n(\underline{z})))} P_1 \cdot B(P_2 - cP_1) \cdot I \min_l b_l = \alpha_1\beta_2
\end{aligned}$$

□

It is an elementary fact that $\dfrac{x_1}{y_1} < \dfrac{x_2}{y_2}$ implies $\dfrac{cx_1 + x_2}{cy_1 + y_2} > \dfrac{x_1 + x_2}{y_1 + y_2}$, $\forall \, c \in (0,1)$ and positive real numbers x_1, x_2, y_1 and y_2. Using Lemma 5.9 we have that

$$\frac{u_{\underline{w}^{max},k}(\underline{z})}{u_{\underline{w}^{min},k}(\underline{z})} \leq \frac{ce^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1, \Pi^{-1}(\sigma^n(\underline{z})))} P_1 \cdot I \min_l b_l + (1-c)P_1 \cdot I \max_l a_l}{cP_1 \cdot I \min_l b_l + (P_2 - cP_1) \cdot I \min_l b_l}$$

As P_1 and P_2 are probability vectors, then $1 = P_1 \cdot I = P_2 \cdot I$. Dividing by $\min_l b_l$ we have

$$\begin{aligned}
\frac{u_{\underline{w}^{max},k}(\underline{z})}{u_{\underline{w}^{min},k}(\underline{z})} &\leq \frac{ce^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1, \Pi^{-1}(\sigma^n(\underline{z})))} + (1-c)\frac{\max_i(a_i)}{\min_i(b_i)}}{c + (1-c)} \\
&= ce^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1, \Pi^{-1}(\sigma^n(\underline{z})))} + (1-c)\frac{\max_i(a_i)}{\min_i(b_i)}
\end{aligned}$$

resulting, $\lambda_k(\underline{z}) \leq ce^{2\sum_{n=0}^{n_i} var_{n_k-n}(\psi_1,\Pi^{-1}(\sigma^n(\underline{z})))} + (1-c)\lambda_i(\underline{z})$ as we desire. $\square$

Corollary 5.4. *In $\nu-$a.e. $\underline{z} \in \Sigma_2$, the sequence $(\lambda_k(\underline{z}))_k$ converges to* 1.

Proof. For $k > i$, we have $n_k - n_i = (n_k - n_{k-1}) + (n_{k-1} - n_{k-2}) + ... + (n_{i+1} - n_i) \geq k - i$. In particular, $n_{2k} - n_k \geq k$. By Lemma 5.8, we have

$$
\begin{aligned}
\lambda_{2k}(\underline{z}) &\leq c\, e^{2\sum_{n=0}^{n_k} var_{n_{2k}-n}(\psi_1,\Pi^{-1}(\sigma^n(\underline{z})))} + (1-c)\,\lambda_k(\underline{z}) \\
\lambda_{2k}(\underline{z}) - \lambda_k(\underline{z}) &\leq c\, e^{2\sum_{n=0}^{n_k} var_{n_{2k}-n}(\psi_1,\Pi^{-1}(\sigma^n(\underline{z})))} - c\lambda_k(\underline{z})
\end{aligned}
$$

Since ψ_1 is continuous at z, it follows that $e^{\sum_{n=0}^{n_k} var_{n_{2k}-n}(\psi_1,\Pi^{-1}(\sigma^n(\underline{z})))} \leq e^{\sum_{n=n_{2k}-n_k}^{n_{2k}} var_n(\psi_1)} \to 1$, when $k \to \infty$. As the sequence $(\lambda_k(\underline{z}))_k$ is decreasing,

$$\lim_{k\to\infty} \lambda_k(\underline{z}) \leq 1$$

By Lemma 5.4 we have that $\lim_{k\to\infty} \lambda_k(\underline{z}) \geq 1$ and, therefore, $\lambda_k(\underline{z}) \to 1$. $\square$

To finish the proof of the statement of main result of this section, we prove the following lemma.

Lemma 5.10. *For $\nu-$a.e. $\underline{z} \in \Sigma_2$ the limit $u(\underline{z}) = \lim_{k\to\infty} u_{\underline{w},k}(\underline{z})$ exist and $\psi_2(\underline{z}) := \log(u(\underline{z}))$ is continuous a.e.*

Proof. For the existence of the limit it sufficient to use the fact that $\lambda_k(\underline{z}) \to 1$ in $\nu-$a.e. $\underline{z} \in \Sigma_2$.

Now, to prove the continuity almost everywhere. Let $(n_i)_{i\geq 1}$ be the sequence of Gibbs times of $\underline{z}$ and n such that $n \geq n_k$ (note that $k \leq n$). Let $\underline{z}' \in [z_0...z_n]$. By definition of sequential Gibbs measure we have that $\Lambda_k(\underline{z}) = \Lambda_k(\underline{z}')$. And the fact the sequence $(\Lambda_n(\underline{z}))_n$ be monotonically nested, both $u(\underline{z})$ and $u(\underline{z}')$ are in interval $\Lambda_k(\underline{z})$. Therefore,

$$\frac{u(\underline{z})}{u(\underline{z}')} \leq \sup_{\underline{w},\underline{w}'\in\Sigma_1} \left\{ \frac{u_{\underline{w},k}(\underline{z})}{u_{\underline{w}',k}(\underline{z}')} \right\} = \sup_{\underline{w},\underline{w}'\in\Sigma_1} \left\{ \frac{u_{\underline{w},k}(\underline{z})}{u_{\underline{w}',k}(\underline{z})} \right\} = \lambda_k(\underline{z})$$

Then

$$|\log(u(\underline{z})) - \log(u(\underline{z}'))| \leq \log \lambda_k(\underline{z})$$

that implies $\psi_2 = \log u$ is continuous at $\underline{z} \in \Sigma_2$. Therefore, is continuous at ν- a.e. $\underline{z} \in \Sigma_2$. $\square$

6 Proof of Theorem 2: Modulus of continuity of ψ_2

Take $k_0(\underline{x})$ such that $k \geq k_0(\underline{x})$ implies $n_k(\underline{x}) \leq bk$. Thus, we have that for a.e. $\underline{z} \in \Sigma_2$, if $\underline{z}' \in [z_0, \ldots, z_{bk}]$

$$|\psi_2(\underline{z}) - \psi_2(\underline{z}')| \leq \log \lambda_k, \tag{14}$$

where $[x]$ denotes the greatest integer smaller or equal to x. It follows directly that the speed of convergence of $\log \lambda_k$ to zero give us the modulus of continuity of ϕ_2 at $\underline{z}$.

To estimate $\log \lambda_k$ we observe that given $k > k_0$ and $l \geq 2$, for every $2 \leq i \leq l$ we have that $n_{ik} - n_{(i-1)k} \geq k$ and that $n_{ik} \leq bik \leq blk$. Thus, by the integral test for series we have that:

$$\sum_{j=n_{ik}-n_{(i-1)k}}^{n_{ik}} var_j(\psi_1, \Pi^{-1}(\sigma^{n_{ik}-j}(\underline{z}))) \leq \sum_{j=k}^{n_{ik}} var_j(\psi_1, \Pi^{-1}(\sigma^{n_{ik}-j}(\underline{z}))) \leq f(k)n_{ik} \leq f(k)blk. \quad (15)$$

By Lemma 5.8, for ν-a.e $\underline{z} \in \Sigma_2$, for every $k > k_0(z)$ and $i = 2, \ldots, l$:

$$\lambda_{ik}(\underline{z}) \leq ce^{2\sum_{j=n_{ik}-n_{(i-1)k}}^{n_{ik}} var_j(\psi_1, \Pi^{-1}(\sigma^{n_{ik}-j}(\underline{z})))} + \alpha\lambda_{(i-1)k}(\underline{z}).$$

Thus, using Equation (15):

$$\lambda_{ik}(\underline{z}) \leq ce^{2f(k)blk} + \alpha\lambda_{(i-1)k}(\underline{z}).$$

Multiplying by α^{l-i} both sides:

$$\alpha^{l-i}\lambda_{ik}(\underline{z}) \leq c\alpha^{l-i}e^{2f(k)blk} + \alpha^{l-i+1}\lambda_{(i-1)k}(\underline{z})$$

Adding all equations as above and canceling the respective terms, we have that:

$$\lambda_{lk}(\underline{z}) \leq c\sum_{i=2}^{l} \alpha^{l-i}e^{2f(k)blk} + \alpha^{l-1}\lambda_k(\underline{z}) \leq e^{2f(k)bkl} + \alpha^{l-1}\lambda_k(\underline{z}). \quad (16)$$

Take $l = \omega_k$ above any sequence for $k \geq k_0$. Dividing by $e^{2f(k)bk\omega_k}$ and using that $\lambda_k \to 1$, $\log(1+x) \approx x$ for x small enough we have that for k big enough that

$$\log \lambda_{\omega_k k} \leq \frac{1}{\alpha}\left(\frac{\alpha}{e^{2bf_{\underline{z}}(k)k}}\right)^{\omega_k}\lambda_k + 2bf_{\underline{z}}(k)\omega_k k \leq \frac{2}{\alpha}\alpha^{\omega_k} + 2bf_{\underline{z}}(k)\omega_k k. \quad (17)$$

Giving $n \in \mathbb{N}$ big enough and $\gamma > 0$, define $\beta = 1 - \gamma$ and consider $k_n = [n^\gamma]$ and $\omega_n = [n^\beta]$. Thus, for every n big than some n_0, we have that $w_n k_n \leq n$ and $\log \lambda_{\omega_n k_n} \geq \log \lambda_n$. By Equation (17) and (14) we have that for almost every z there exist $n_0(z)$ such that for every $n > n_0(z)$:

$$\log \lambda_n \leq \log \lambda_{\omega_n k_n} \leq 2\alpha^{n^{1-\gamma}} + 4bf_{\underline{z}}([n^\gamma])n, \quad (18)$$

as we wish to show.

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Giovane Ferreira
Universidade Federal do Maranhão
Campus do Bacanga, São Luís, Maranhão Brazil
giovane.ferreira@ufma.com

Krerley Oliveira
Universidade Federal de Alagoas
Campus A.C. Simões, Maceió, Alagoas Brazil
krerley@gmail.com