Convergence Analysis of Classes of Asymmetric Networks of Cucker-Smale Type with Deterministic Perturbations

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Abstract

We introduce and discuss two nonlinear perturbed extensions of the Cucker-Smale model with asymmetric coupling weights. The first model assumes a finite collection of autonomous agents aiming to perform a consensus process in the presence of identical internal dynamics. The second model describes a similar population of agents that perform velocity alignment with the restriction of collision-free orbits. Although qualitatively different, we explain how these two non-trivial types of perturbations are analyzed under a unified framework. Rigorous analysis is conducted towards establishing new sufficient conditions for asymptotic flocking to a synchronized motion. Applications of our results are compared with simulations to illustrate the effectiveness of our theoretical estimates.

I. INTRODUCTION

For the past two decades, there has been a broad interest in the study of co-operative dynamic algorithms that run among a finite number of interconnected entities. Perhaps the most prominent family is this of consensus networks. The standard setting regards a finite number of agents n, where state of each agent $z_i \in \mathbb{R}$ for $i = 1, \ldots, n$ is governed by the following protocol

$$\dot{z}_i = \sum_{j=1}^{n} w_{ij}(t)(z_j - z_i). \tag{1}$$

The nonnegative numbers w_{ij} are coupling weights that characterize the effect of state of agent j onto the rate of change of the state of agent i. Certain criteria imposed on w_{ij} ensure the desired asymptotic consensus property

$$z_i(t) \to z^* \text{ as } t \to \infty \text{ for } i = 1, \ldots, n.$$

The common equilibrium point $z^* \in \mathbb{R}$ lies in the set defined by the convex hull of the initial values. The research on connectivity conditions for (1) is a saturated subject (see [11], [12], [18], [20], [21], [32] and references therein). On the contrary, there are relatively few works in non-linear consensus dynamics. In the next subsection we review a few such extensions of (1).

A. Types of Nonlinear Consensus Protocols

In [19] and [14] the authors study necessary and sufficient conditions of convergence to consensus in discrete and continuous time versions of non-linear networked co-operative systems represented by

$$\dot{x}_i(t+1) = f(t, x_1(t), \ldots, x_n(t))$$

and

$$\dot{x}_i = f(t, x_1, \ldots, x_n),$$

respectively. In [1], [23], the authors investigate nonlinear versions of (1)

$$\dot{z}_i = \sum_{j=1}^{n} w_{ij}(t, z_j - z_i) \tag{1.2}$$

and

$$\dot{z}_i = \sum_{j=1}^{n} w_{ij}(t, z_j) - \sum_{j=1}^{n} w_{ij}(t, z_i). \tag{1.3}$$

In network (1.2), the analysis is based on passivity properties of the coupling functions w_{ij}. For the second model given by (1.3), the investigation on the stability of the consensus network is derived through the standard Lyapunov methods combined

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with LMI techniques. In [6], [7], the authors introduce a dynamical network that models the emergence of bird flocking. In their setting, the state of each agent i consist of its vector position $x_i \in \mathbb{R}^r$ and velocity $v_i \in \mathbb{R}^r$, which is denoted by $(x_i, v_i) \in \mathbb{R}^r \times \mathbb{R}^r$. For an initial configuration (x_i^0, v_i^0) for $i = 1, \ldots, n$, the governing dynamics of the network are given by

$$\dot{x}_i = v_i$$
$$\dot{v}_i = \sum_{j=1}^n w_{ij}(x)(v_j - v_i). \tag{2}$$

The class of dynamical networks (2) are called as second-order flocking schemes. The simpler linear versions, where it is assumed that weights w_{ij} are independent of the state, have also appeared in [33] and [16]. In (2), the coupling weights are assumed to be decreasing functions of distance in the following form

$$w_{ij}(x) = \frac{K}{(\sigma^2 + \|x_i - x_j\|^2)^\beta}, \tag{3}$$

where $K, \sigma, \beta > 0$ are coupling parameters and $\|.\|$ is the Euclidean norm. The explicit choice of w_{ij} illustrates that the further the relative distance between two agents is, the less effect they have on each other. It may then occur that the agents will not be positioning themselves sufficiently close. This may in turn result the network to fail to preserve strong enough connectivity to achieve global speed alignment. The objective in (2) is to derive initial conditions that ensure a strong enough network to allow global flocking to a common consensus state. The authors exploit the explicit form of w_{ij}, in particular its symmetry, and use algebraic graph theory to establish a stability condition that involves the initial data and the parameters K, σ and β from (3). Since the seminal work of Cucker and Smale, the network model (2) has received considerable attention and it has been improved in several ways. In [9], a simple proof for general vanishing symmetric couplings is provided. In [17] the authors derive convergence results for general symmetric weights without the infininite distance connectivity condition. In [24] extend the results for asymmetric couplings. In [26] and [28], the authors provide extensions to asymmetric couplings with simple and switching connectivity in presence of time delay. Works [5] and [4] examine (2) under the effect of additive noise and repelling symmetric functions, respectively. A general convergence result that summarizes the contribution of most of the aforemenioned works, illustrates the kind of sufficient conditions for global asymptotic velocity alignment while the network does not get dissolved.

Theorem 1 ([24], [28]). *Suppose that the nonlinear dynamical network* (2) *with coupling weights* $w_{ij}(x) \geq \psi(\max_{ij} |x_i - x_j|)$, *where ψ is a non-negative integrable function. If the initial data satisfy*

$$\max_{i,j} |v_i^0 - v_j^0| < \int_{\max_{i,j} |x_i^0 - x_j^0|}^\infty \psi(s) \, ds,$$

then the solution (x, v) of (2) *satisfies*

  (i) $|v_i(t) - v_j(t)| \to 0$ *as* $t \to \infty$,
  (ii) $\sup_{t \geq 0} |x_i(t) - x_j(t)| < \infty$, *for* $i, j = 1, \ldots, n$.

The technical argument behind the inequality condition in Theorem 1 involves a rate of convergence estimate of the state of the network to its equilibrium that explicitly depends on the network parameters. The appropriate theoretical machinery is provided through algebraic graph theory for symmetric couplings [3] or through non-negative matrix theory [10], [25]. All the above references focus on (2) in exclusiveness. The stability conditions of Theorem 1 shed no light in the event that the agents are expected to collectively execute a more complex task different than convergence to a consensus state.

B. Our Contributions

In this paper, we introduce and discuss a class of extensions of the classic Cucker-Smale model represented by (2) that considers nonlinear deterministic perturbations. These models describe how a group of agents can benefit from asymmetric state-dependent graphs with spatially-decaying couplings to perform more elaborate collaborative tasks, rather than performing the simple task of converging to a consensus state.

The importance of considering perturbed versions of (2) is to understand the inherent interplay between being able to accomplish complex collective behaviors and the coupling structure of the underlying dynamical network. Our main goal is to investigate to what extent we can push the envelope to modify existing gold-standard and well-studied dynamical networks to allow them to exhibit more complex collective behaviors. Towards this end, we address two types of state-dependent perturbations, relevant to network models of type (2) that describe real-world paradigms.

The first extension takes into account the scenario of agents aiming to perform a consensus process in the presence of an internal generic dynamic rule. For instance, birds in a flock can fly individually or in coordination with other birds, possibly towards a synchronized complex motion different than a simple velocity co-ordination. This feature is modeled with an extra

non-linear term in (2) that makes the inequality flocking condition of Theorem 1 inapplicable. In Section III-A, it is shown how this class of networks can be analyzed.

The second extension is a perturbation of (2) that destabilizes the coordination process so long as the relative position of agents falls below a certain safety margin. Such a safety restriction can prevent a network with fast decaying coupling strength from achieving global convergence. Similar to the first case, an extended version of Theorem 1 is needed to ensure flocking conditions. In Section III-B, we show that under a new set of technical conditions, design of flocking without collision can be achieved.

Although qualitatively different, we demonstrate how these two non-trivial types of perturbations can be analyzed under a unified framework. We explore the resulting dynamics and establish sufficient stability relations between the nonlinear coupling forces and the inherent dynamics, towards collective convergence within a connected network. Furthermore, we discuss extensions of our results for networks with non-linear coupling terms and explain the potential difficulties that can occur in the event of looser connectivity. Finally, several carefully tailored examples are discussed to truthfully verify the strength of the imposed conditions.

C. Literature Review

The subject of synchronization is rather mature and well-documented in the context of linear networks, with static or varying communication topologies (see [8], [34] for control theoretic perspective or [22], [35] for a physics approach approach and references therein).

In [29] the authors analyze first order synchronization schemes with time-dependent topologies where the internal dynamics may destabilize the alignment process. The work establishes sufficient conditions between the internal rule and the coupling scheme, in order for asymptotic synchronization to occur. To the best of our knowledge, the literature lacks a study of synchronization networks where the state of the network is correlated to the strength of the coupling as such is the case in networks of Cucker-Smale type.

Prior works on the subject of flocking with collision avoidance in Cucker-Smale type networks include [4], [27]. The authors in [4] study a collision-avoidance problem (model (12) of our work) on a framework based on the symmetry assumption of coupling weights. Their results, however, seize to hold when the couplings and the repelling functions are not symmetric. The asymmetry on the coupling terms has been highlighted in the literature as a very important feature in real-world biological and natural networks [2]. The present work is an outgrowth of [30] in various directions. The synchronization scheme is considered under milder assumptions and multi-dimensional internal dynamics. This gives rise to stability conditions that generalize the ones in [30]. For the collision free model we provide a detailed proof of the convergence result.

II. Preliminaries

Notations: For any $k \in \mathbb{N}$, $[k]$ is the set consisting of the first k natural numbers, i.e. $[k] := \{1, \ldots, k\}$. By $n < \infty$ we understand the number of autonomous agents. The communication scheme is represented by a weighted graph $G = ([n], E)$ where $[n]$ is the set of nodes and $E = \{w_{ij} : i, j \in [n]\}$ is the set of weighted edges. We denote the weighted degree of node i as $d_i = \sum_j w_{ij}$. The dynamics evolve in $\mathbb{R}^r$ for some $r \geq 1$ that is endowed with the inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\| \cdot \|$. Each agent $i \in [n]$ is characterized by the state $(x_i, v_i) \in \mathbb{R}^r \times \mathbb{R}^r$. Clearly $x_i = (x_i^{(1)}, \ldots, x_i^{(r)})$ and $v_i = (v_i^{(1)}, \ldots, v_i^{(r)})$ stand for the position and velocity of i, respectively. Compact representation of the overall network state include $x = (x_1, \ldots, x_n)$ and $v = (v_1, \ldots, v_n)$ for elements x and v in the augmented space $\mathbb{R}^{nr}$. For $l \in [r]$, the spread of $y \in \mathbb{R}^{nr}$ in the l-th dimension is

$$S_l(y) = \max_i y_i^{(l)} - \min_i y_i^{(l)} = \max_{i,j} |y_i^{(l)} - y_j^{(l)}|$$

and finally we denote $S(y) = \max_l S_l(y)$. Throughout this paper, we reserve the notation " $\cdot$ " for the classic derivative and $\frac{d}{dt}$ for the right-hand Dini derivative.

The Contraction Coefficient: The problem of stability of general consensus networks is associated with the asymptotic behavior of products of non-negative matrices [25]. The underlying mathematical concept used in the proofs of those results was first discussed in one of Markov's first papers [10]. This concept, known as the contraction coefficient (or coefficient of ergodicity), measures the worst case estimate on the averaging effect of non-negative matrices when they act on vectors. Markov demonstrated that for a $P = [p_{ij}]$ non-negative $n \times n$ matrix with constant row sums (i.e. $\sum_j p_{ij} \equiv m$ for some $m > 0$), it holds that

$$S(Pz) \leq \left(m - \min_{i,i'} \sum_k \{p_{ik}, p_{i'k}\}\right) S(z), \ \forall \ z \in \mathbb{R}^n. \tag{4}$$

A proof of this result can be found in [10]. The contraction coefficient is the quantity $\left(m - \min_{i,i'} \sum_k \{p_{ik}, p_{i'k}\}\right)$ and it provides an explicit estimate of the averaging property of P. While the framework is primarily compatible with discrete time dynamics, it can be adapted to a continuous time framework, [24], [28], [29]. The contributions of this paper critically rely on auxiliary results (Lemma 5 and Lemma 11, below) the proof of which is an elaborate adaptation of the proof of the contraction

estimate (4) [1]. These ideas provide us with with worst-case contraction estimates of the consensus term. Then combining ideas from [9] and [4] we are able to derive sufficient conditions for the desired asymptotic stability.

III. NETWORKS OF CUCKER-SMALE TYPE WITH DETERMINISTIC PERTURBATIONS

In this section we present the dynamic algorithms with the accompanying hypotheses. The discussion is concluded with the main convergence results.

Let n be a number of agents and $i \in [n]$ with the state $(x_i, v_i) \in \mathbb{R}^r \times \mathbb{R}^r$. The models we consider in this work can be cast as the following system of equations:

$$
\begin{aligned}
\dot{x}_i^{(l)} &= v_i^{(l)} \\
\dot{v}_i^{(l)} &= \sum_j \left(w_{ij}(t, x) + b_{ij}^{(l)}(t, x, v) \right) (v_j^{(l)} - v_i^{(l)}), t \geq t_0
\end{aligned}
\tag{5}
$$

for $l \in [r]$, w_{ij} are the coupling terms and $b_{ij} : \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}^l$ a non-trivial deterministic perturbation between agents j and i. This perturbation may, in general, depend on both time and state. We look at two meaningful scenarios of this general model, assuming corresponding special forms of b_{ij}. One form leads to flocking to a synchronization solution while the other form leads to collision-less velocity coordination.

A. Heterogeneous Synchronization Using Diffusion Processes

The first type of perturbation considers a situation where agents, in addition to the consensus averaging, attain an individual way of flying that is dictated by an internal dynamical behavior. For this case we take (5) with

$$
b_{ij}^{(l)}(t, x, v) = \frac{g^{(l)}(t, v_i)}{v_j^{(l)} - v_i^{(l)}}.
$$

Then for given $t_0 \in \mathbb{R}$ we are led to the initial value problem:

$$
\begin{aligned}
\dot{x}_i &= v_i \\
\dot{v}_i &= g(t, v_i) + \sum_j w_{ij}(t, x)(v_j - v_i), \quad t \geq t_0 \\
x_i(t_0) &= x_i^0, \ v_i(t_0) = v_i^0 \in \mathbb{R}^r, \quad \text{given.}
\end{aligned}
\tag{6}
$$

An agent's velocity is affected by both the state of the other nodes and by an inherent dynamical process. The state of the other nodes affects the agent at a rate that depends on time and on the position vector x It is, therefore, far from clear that the condition of Theorem 1 ensures convergence. Our objective is to reveal the interplay between the coupling forces of the consensus network, the initial configuration and the potential instability induced by the internal dynamics through a new stability condition. We proceed by stating assumptions on the acceptable behavior of the internal dynamic system and conclude with a condition on the coupling functions.

Let for $t \geq t_0$ the initial value problem

$$
\dot{z} = g(t, z), \qquad z(t_0) = z^0 \in \mathbb{R}^r
\tag{7}
$$

together with its solution $z = z(t, t_0, z^0)$ defined in a maximal interval $[t_0, T]$. The hypothesis below aims to establish the well-posedness of z.

Assumption 2. *The function $g(t, z)$ is defined in $\mathbb{R} \times V$, where V an open subset of $\mathbb{R}^r$. It is continuous in $t \in \mathbb{R}$ and attains continuous first derivative in $z \in V$.*

We impose the following smoothness and bound conditions on the coupling weights.

Assumption 3. *The following statements hold for the functions $w_{ij}(t, x)$:*
(1.) They are continuous functions of t and continuously differentiable functions of x.
(2.) They satisfy

$$
\bar{w} \geq \sup_{t \geq t_0} \sup_x \max_{i \neq j} w_{ij}(t, x) \geq \inf_t w_{ij}(t, x) \geq \psi(S(x))
$$

for $\bar{w} < \infty$ and $\psi \geq 0$ an integrable and non-increasing function.

The rates w_{ij} are uniformly bounded from above by $\bar{w}$ but not bounded from below as $\psi(\cdot)$ is allowed to vanish. This means that Assumption 3 allows the spatially decaying property. In addition, it includes symmetric couplings either as in (3) or the

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[1]The intereted reader can directly observe the similarities between the contraction estimates derived in those two Lemma with 4

ones proposed in [9]. Clearly, it includes non-symmetric coupling rates such as those proposed in [24]. For the statement of the first result we are in need of some additional notation. Let

$$K(t, l, y, w) = \int_0^1 \frac{\partial}{\partial z^{(l)}} g^{(l)}(t, qy + (1-q)w) \, dq + + \sum_{h \neq l} \left| \int_0^1 \frac{\partial}{\partial z^{(h)}} g^{(l)}(t, qy + (1-q)w) \, dq \right|$$

and for the maximal solution $\big(x(t), v(t)\big)$, $t \in [t_0, T)$ of (6):

$$K = \sup_{t \in [t_0, T)} \max_{i, i' \in [n], l \in [r]} K(t, l, v_i(t), v_{i'}(t)). \tag{8}$$

This quantity represents the effect of the internal dynamic rule g in the coupling process. Evaluated on the maximal solution, K, is the worst case estimate that we must take into account in order to derive initial conditions that compensate for the potential instability that g will induce in the system that will in turn weaken the coupling rate w_{ij}. Observe also in $K(t, l, y, w)$ that the cross terms $\frac{\partial}{\partial z^{(h)}} g^{(l)}$ for $h \neq l$, are added in absolute value, as opposed to $\frac{\partial}{\partial z^{(l)}} g^{(l)}$. This discrepancy is the result of the dimensionality problem. Seeking synchronization of v_i in all r dimensions, one must take into account the effect that the internal dynamics g have in all r dimensions. Due to lack of structure on g, we have no choice but to regard the rate at which g varies in different dimensions ($h \neq l$) as a purely negative perturbation against the synchronization along the lth dimensions.

Theorem 4. *Consider the initial value problem* (6) *with Assumptions 2 and 3 to hold and its maximal solution* $(x(t), v(t))$, $t \in [t_0, T)$. *Assume also that*
(1.) there exists $d > 0$ *such that*

$$S(v^0) < \int_{S(x^0)}^d \big(n\psi(r) - K\big) \, dr,$$

(2.) There exist $\varepsilon > 0$ *such that*

$$n\psi(d^*) \geq K + \varepsilon$$

for $d^* > 0 : S(v^0) = \int_{S(x^0)}^{d^*} \big(n\psi(r) - K\big) \, dr$ *and* K *as in* (8). *Then,* (x, v) *satisfies*

$$S(v(t)) \leq e^{-\varepsilon t} S(v^0) \quad \& \quad \sup_{t \geq t_0} S(x(t)) < d^*, \ \forall \, t \in [t_0, T).$$

The proof of the Theorem relies on the following result, the proof of which is put in the Appendix:

Lemma 5. *Let* (x, v) *be the maximal solution of* (6)*, defined in* $[t_0, T)$. *Then*

$$\frac{d}{dt} S\big(v(t)\big) \leq \big[K - n\psi\big(S(x(t))\big)\big] S\big(v(t)\big), \tag{9}$$

for $t \in [t_0, T)$.

Proof of Theorem 4. Consider the functional

$$\mathcal{V}(x, v) = S(v) + \int_0^{S(x)} \big(n\psi(r) - K\big) \, dr \tag{10}$$

and evaluate it at the solution $\big(x(t), v(t)\big), t \in [t_0, T)$ with $\mathcal{V}(t) = \mathcal{V}\big(x(t), v(t)\big)$. From (1.) there exists $t_1 > t_0$ such that for $t \in [t_0, t_1)$

$$\frac{d}{dt} \mathcal{V}(t) \leq \frac{d}{dt} S(v(t)) + \big[n\psi\big(S(x(t)) - K\big] S\big(v(t)\big) \leq 0$$

in view of Lemma 5 and

$$\frac{d}{dt} S(x(t)) \leq S(v(t)).$$

The last claim is justified as follows: Note that $S(x(t)) = |x_i^{(l)}(t) - x_j^{(l)}(t)|$, for some $i, j \in [n]$ and $l \in [r]$ (possibly dependent on t). Then

$$\frac{d}{dt} S(x(t)) = \frac{d}{dt} |x_i^{(l)}(t) - x_j^{(l)}(t)| \leq \left| \frac{d}{dt} \big(x_i^{(l)}(t) - x_j^{(l)}(t)\big) \right|$$
$$= |v_i^{(l)}(t) - v_j^{(l)}(t)|$$
$$\leq S(v(t))).$$

Consequently, $\mathcal{V}(t) \leq \mathcal{V}(t_0)$ for $t < t_1$, equivalent to

$$S(v(t)) + \int_0^{S(x(t))} \big(n\psi(r) - K\big) \, dr \leq S(v^0) + \int_0^{S(x^0)} \big(n\psi(r) - K\big) \, dr$$

and obviously

$$\int_0^{S(x(t))} \left(n\psi(r) - K\right) dr \le S(v^0) + \int_0^{S(x^0)} \left(n\psi(r) - K\right) dr.$$

Condition (1.) also implies the existence of $d^* < d$ as in condition (2.) the inequality above yields

$$\int_0^{S(x(t))} \left(n\psi(r) - K\right) dr \le \int_0^{d^*} \left(n\psi(r) - K\right) dr$$

so

$$\int_{S(x(t))}^{d^*} \left(n\psi(r) - K\right) dr \ge 0$$

The last inequality implies that $S(x(t)) \le d^*$ for $t < t_1$ and $n\psi(d^*) - K \ge \varepsilon > 0$. Since no assumption was taken on t_1, the monotonicity of ψ yields that we can take $t_1 = T$ proving the second claim of the theorem. The differential inequality (9) then yields the first claim, concluding the proof. $\qquad\square$

This result establishes the connection between the internal and the position-dependent coupling dynamics. The power of Theorem 4 can be further extracted if we assume that the solution v is a-priori trapped within a region that possibly depends on the initial conditions. In such case, $T = \infty$ and the imposed conditions can be checked more easily. The following result asserts that for a particular type of compact subsets $\mathbb{R}^r$ into which z of (7) remains trapped, implies that v in (6) behaves likewise.

Theorem 6. *Assume that U be a compact, convex g-invariant subset of $\mathbb{R}^r$. Then $v_i^0 \in U$ for $i \in [n]$, guarantees that the solution (x, v) of (6) exists for all times. In addition, the results of Theorem 4 hold true with K as in (8) to be substituted by*

$$K = \sup_{t \ge t_0} \max_{y,w \in U, l \in [r]} K(t, l, y, w).$$

Proof. Consider the maximal solution $(x(t), v(t))$, $t \in [t_0, T)$ of (6). It suffices to show that $v_i(t_0) = v_i^0 \in U$ implies $v_i(t) \in U$ for all $t \in [t_0, T)$ and $i \in [n]$. Then T can be extended to ∞, establishing the solution (x, v) in the large. We can discretize the second part of (6) as follows

$$v_i(t + l) = v_i(t) + l\left[g(t, v_i(t)) + \sum_j w_{ij}(t)\left(v_j(t) - v_i(t)\right)\right] \tag{11}$$

where $w_{ij}(t) = w_{ij}(t, x(t))$, $t \in [t_0, T)$ and $i \in [n]$. The result is then proved if we can show that $v_i(t + l) \in U$ for l small and arbitrary $t \in [t_0, T)$. The collection of the initial values $\{v_i^0\}$ lies in U, and the convexity of U implies that the convex hull of $\{v_1^0, \ldots, v_n^0\}$ is also a subset of U. Pick $i \in [n]$ and let's elaborate on $v_i(t + l)$ in Eq. (11), at $t = t_0$. By the g-invariance property of U, one can discretize the uncoupled equation and conclude that there is $k' > 0$ such that

$$v_i^a(t_0 + k) := v_i^0 + kg(t_0, v_i^0) \in U$$

for any $k \in [0, k')$. In addition, if on ignores the internal dynamics (i.e. $g \equiv 0$), Assumption 3 on the weights, implies that there is k'' such that

$$v_i^b(t_0 + k) := v_i^0 + k\sum_j w_{ij}(t_0)(v_j^0 - v_i^0) \in U$$

for $k \in [0, k'')$. The latter claim is because the right hand side of the equation is a convex sum of $\{v_i^0\}$, thus places $v_i^b(t_0 + k)$ exactly in the convex hull of $\{v_1^0, \ldots, v_n^0\}$, which in turn is in U. We showed that both $v_i^a(t_0 + k)$ and $v_i^b(t_0 + k)$ are points in U for small enough $k \in [0, \min\{k', k''\})$. Then by convexity of U,

$$U \ni \frac{1}{2}v_i^a(t_0 + k) + \frac{1}{2}v_i^b(t_0 + k) =$$

$$= v_i^0 + \frac{k}{2}\left[g(t_0, v_i^0) + \sum_k w_{ij}(t_0)(v_j^0 - v_i^0)\right]$$

Observe that the right hand-side is precisely this of (11) with $l = k/2$ and $t = t_0$. We remark that in view of Assumptions 2 and 3, k' and k'' can be chosen independent of t_0 or i so that the argumentation remains valid for any other $t \in [t_0, T)$ and $i \in [n]$. $\qquad\square$

The applicability of this result includes, but not limited to, problems of synchronization of non-linear or chaotic oscillations where initial configuration near the synchronization manifold ensure boundedness of the solution of the network (see for example [8], [22], [35]).

Remark 7. In Theorems 4 and 6 we assumed that every agent has access to the states of the rest of the agents. This is an effectively all-to-all communication and it may be thought of as too demanding. The invoked argument allows for a small relaxation of the all-to-all sharing: We can ask for every two agents that do not communicate, the existence of a third agent with which both of them must communicate. In this case, the estimates of Theorem 4 are valid if $n\psi(r)$ in the initial setup conditions is substituted by $\psi(r)$.

B. Flocking with Guaranteed Collision Avoidance

The second dynamic model we will study rolls back to the classic consensus problem and convergence to a common constant value. Here the alignment should be achieved with agents positioning themselves in at least a minimum distance from each other. For this we will study a special case of (5) with

$$b_{ij}^{(l)}(t,x,v) = -\frac{1}{S(v)}f_{ij}(\|x_i - x_j\|^2)\langle x_i - x_j, v_i - v_j\rangle, \ l \in [r].$$

The protocol we propose is

$$\begin{aligned}
\dot{x}_i &= v_i \\
\dot{v}_i &= \sum_j \left(w_{ij}(t,x) - \frac{f_{ij}(\|x_i - x_j\|^2)\langle x_i - x_j, v_i - v_j\rangle}{S(v)}\right)(v_j - v_i) \\
x_i(t_0) &= x_i^0, \ v_i(t_0) = v_i^0 \in \mathbb{R}^r, \quad \text{given.}
\end{aligned} \tag{12}$$

where w_{ij} as in Assumption 3 and the additional terms that model the collision prevention mechanism. The functions f_{ij} are repelling forces that can be appropriately constructed so as to keep the agents at a prescribed relative distance.

Assumption 8. *For any $i \neq j$, $f_{ij}(\cdot)$ is a continuous non-negative function defined in (d_0, ∞), for some constant $d_0 > 0$ such that*

$$\int_{d_0}^{d_1} f_{ij}(r)\,dr = +\infty \ and \ \int_{d_1}^{+\infty} f_{ij}(r)\,dr < +\infty,$$

for all $d_1 > d_0$.

A simple example of symmetric repelling function (also to be used in §IV) is $f(r) = (r - d_0)^{-\varepsilon}$ for any fixed $\varepsilon > 1$. More examples can be found in [4]. The objective of this section is the derivation of sufficient conditions for flocking of (8). One should expect a formula that connects the coupling strength with the functions f_{ij}.

Theorem 9. *Consider the initial value problem* (12) *with Assumptions 3 and 8 to hold, and its maximal solution $(x(t), v(t))$ for $t \in [t_0, T)$. If $i \neq j$ implies $\|x_i^0 - x_j^0\| > d_0$ and*

$$\frac{S(v^0)}{n} < \frac{1}{2}\int_{S(x^0)}^\infty \psi(r)\,dr - \max_{i \neq j}\int_{\|x_i^0 - x_j^0\|^2}^\infty f_{ij}(r)\,dr. \tag{13}$$

Then:

1) $T = \infty$,
2) $\|x_i(t) - x_j(t)\| > d_0$, for all $t \geq t_0$,
3) *the solution satisfies*

$$S(v(t)) \to 0 \ as \ t \to \infty \ and \ \sup_{t \geq t_0} S(x(t)) < \infty.$$

The proof of Theorem 9 follows partly the steps of the proof of Theorem 4 and partly the arguments developed in [4]. We begin with a preliminary result that establishes a solution estimate of v similar to Lemma 5. Both the preliminary and the main results rely on the following observation.

Remark 10. Note that $b_{ij}(t,x,v)$ can be written as

$$b_{ij}(t,x,v) = \frac{1}{2S(v)}\frac{d}{dt}\int_{\|x_i(t) - x_j(t)\|^2}^\infty f_{ij}(r)\,dr. \tag{14}$$

Given a solution (x,v) of (12) defined in $t \in [t_0, T)$ we set $h = h_t$ and $h' = h_t' \in [n]$ the agents that lie closes to each other, i.e. the indices that minimize $\|x_i - x_{i'}\|$. While this mapping may not be unique, it is a piece-wise constant function of t.

Lemma 11. *The maximal solution (x,v) of* (12) *defined in $[t_0, T)$ satisfies for all $i, i' \in [n]$ and $l \in [r]$*

$$\frac{d}{dt}|v_i^{(l)} - v_{i'}^{(l)}| \leq -m|v_i^{(l)} - v_{i'}^{(l)}| + (m - \rho_{i,i'})S(v) - \Gamma_{i,i'}, \tag{15}$$

where $m = m(t)$ is an arbitrary but fixed, non-negative, integrable function, $\rho_{i,i'} = \sum_j \min\{w_{ij}(t, x(t)), w_{i'j}(t, x(t))\}$ and

$$\Gamma_{i,i'} = \frac{1}{2} \sum_{j \neq i} \min\left\{\frac{d}{dt} \int_{||x_i - x_j||^2}^{\infty} f_{ij}(r)\, dr, \frac{d}{dt} \int_{||x_{i'} - x_j||^2}^{\infty} f_{i'j}(r)\, dr\right\}$$

.

The proof of Lemma 11 is put in the Appendix. When i and i' are chosen so as to maximize $\max_l |v_i^{(l)} - v_{i'}^{(l)}| = S(v)$ then one obtains the estimate

$$\frac{d}{dt} S(v) \leq -\rho_{i,i'} S(v) - \Gamma_{i,i'}. \tag{16}$$

Proof of Theorem 9. For the reader's convenience we make the proof to consist of the following four steps: Collision avoidance conditions for the maximal solution, existence of the maximal solution for all times, persistent connectivity of the flock and velocity alignment.

a) Collision Avoidance: Define $\mathcal{E} : \mathbb{R}^{nr} \times \mathbb{R}^{nr} \to \mathbb{R}$

$$\mathcal{E}(x, v) = S(v) + |v_h^{(l)} - v_{h'}^{(l)}| + \sum_j \gamma_j^{i,i'} + \sum_j \gamma_j^{h,h'}$$

where $l \in [r]$ is arbitrary but fixed,

$$\gamma_j^{i,i'} = \gamma_j^{i,i'}(x, v) = \int_{||x_k - x_j||^2}^{\infty} f_{kj}(r)\, dr \text{ with } k = k_{i,i'} :$$

$$f_{kj}(||x_k - x_j||^2)\langle x_k - x_j, v_k - v_j\rangle = \min\left\{f_{ij}(||x_i - x_j||^2)\langle x_i - x_j, v_i - v_j\rangle, f_{i'j}(||x_{i'} - x_j||^2)\langle x_{i'} - x_j, v_{i'} - v_j\rangle\right\}$$

We remind that $h, h' \in [n]$ are as in Remark 10 and $i, i' \in [n]$ are chosen to maximize the velocity diameter and consequently satisfy Eq. (16). All of these indices are potentially time dependent but piecewise constant. Differentiating $\mathcal{E}$ along $(x(t), v(t)), t \geq t_0$, Lemma 11 for $m(t) = \rho_{i,i'}(t, x(t))$ and (16) will yield

$$\frac{d}{dt}\mathcal{E}(x(t), v(t)) = -\rho_{i,i'} S(v) - \Gamma_{i,i'} - \rho_{i,i'}|v_h^{(l)} - v_{h'}^{(l)}| + (\rho_{i,i'} - \rho_{h,h'})S(v) - \Gamma_{h,h'}$$

$$+ \frac{d}{dt}\sum_j \gamma_j^{i,i'}(x(t), v(t)) + \frac{d}{dt}\sum_j \gamma_j^{h,h'}(x(t), v(t))$$

$$= -\rho_{i,i'}|v_h^{(l)} - v_{h'}^{(l)}| - \rho_{h,h'} S(v) \leq 0$$

Note that

$$\text{either} \int_{||x_h(t) - x_{h'}(t)||^2}^{\infty} f_{hh'}(r)\, dr \text{ or } \int_{||x_h(t) - x_{h'}(t)||^2}^{\infty} f_{h'h}(r)\, dr$$

is a member of the sum $\sum_j \gamma_j^{h,h'}$. Then $\mathcal{E}(t) \leq \mathcal{E}(t_0)$ implies that for the two agents h and h' that are in closest distance from each other

$$\min\left\{\int_{||x_h(t) - x_{h'}(t)||^2}^{\infty} f_{hh'}(r)\, dr, \int_{||x_h(t) - x_{h'}(t)||^2}^{\infty} f_{hh'}(r)\, dr\right\} \leq \mathcal{E}(t) \leq \mathcal{E}(t_0) < \infty$$

Since both $f_{hh'}$ and $f_{h'h}$ satisfy Assumption 8, we proved that $||x_i(t) - x_j(t)|| \geq d^*$ for some $d^* > d_0$ for all $i \neq j \in [n]$ and $t \in [t_0, T)$.

b) Existence in the Large: From $\mathcal{E}(t) \leq \mathcal{E}(t_0)$ we also deduce that $S(v(t)) \leq \mathcal{E}(t_0) < \infty$ hence

$$S(x(t)) \leq S(x^0) + T\mathcal{E}(t_0) := \bar{X}$$

and the solution lies for $[t_0, T)$ in

$$\Omega = \{(x, v) : S(x) \leq \bar{X}, ||x_{i,j}|| \geq d^*, i \neq j, S(v) \leq \mathcal{E}(t_0)\}$$

where d^* defined in the first part of the proof. However Ω is a compact subset of

$$\{(x, v) : S(x) \leq \bar{X}, ||x_{i,j}|| > d_0, i \neq j\}.$$

The fundamentals in the theory of differential equations assure, however, that this cannot occur if $T < \infty$ (see for example Theorem 1.21 in [15]) hence it is ensured that the solution is eventually defined for all $t \geq t_0$.

c) Bounded Flock: We will show now that the initial settings we consider in the statement of the Theorem keeps all agents in finite relative distance. We recall from the first step that

$$\frac{d}{dt}\mathcal{E}(x(t), v(t)) \leq -\rho_{i,i'}|v_h^{(l)} - v_{h'}^{(l)}| - \rho_{h,h'} S(v) \leq -\rho_{h,h'} S(v)$$

If we integrate from t_0 to t and use the lower bound of $\rho_{h,h'}$ taken in view of Assumption 3 we obtain

$$\mathcal{E}(t) - \mathcal{E}(t_0) \leq -\int_{t_0}^{t} n\psi\big(S(x(r))\big)S(v(r))\,dr \Rightarrow$$

$$\mathcal{E}(t_0) \geq \int_{t_0}^{t} n\psi\big(S(x(r))\big)S(v(r))\,dr. \tag{17}$$

If we set $p(s) = S(x(s))$ we observe that $\frac{dp}{ds} \leq S(v(s))$ and deduce

$$\int_{S(x^0)}^{S(x(t))} n\psi(s)\,ds \leq \mathcal{E}(t_0).$$

Had the flock been dissolved, there should be a sequence $\{t_i\}_{i \geq 0}$ with $t_i \to \infty$ as $i \to \infty$, so that $S(x(t_i)) \to \infty$ as $t_i \to \infty$. This would mean that

$$\int_{S(x^0)}^{\infty} n\psi(r)\,dr \leq \mathcal{E}(t_0) \leq 2S(v^0) + 2n \max_{i \neq j} \int_{||x_i^0 - x_j^0||^2}^{\infty} f_{ij}(r)\,dr$$

that is not possible in view of (13). Thus

$$\sup_{t \geq t_0} S(x(t)) < \infty, \tag{18}$$

i.e. the flock remains connected and bounded. *d) Convergence to flocking:* At first we combine (17) and (18) to conclude

$$\int^{\infty} S(v(r))\,dr < \infty.$$

Moreover, we used $\frac{d}{dt}\mathcal{E}(x(t), v(t)) < 0$ to show that $\sup_{t \geq t_0} S(v(t)) < \mathcal{E}(t_0)$ and this implies that $|S(v(t))|$ is uniformly bounded. It only remains to show that $S(v(t))$ is uniformly continuous so that Barbalat's Lemma (Lemma 8.2 in [13]) applies to $k(t) := \int_{t_0}^{t} S(v(s))\,ds$ to conclude

$$\dot{k}(t) = S(v(t)) \to 0 \text{ as } t \to \infty.$$

The last condition is shown by direct application of the definition of uniform continuity: For any $t_1, t_2 \geq t_0$ close to each other, there exist $i, i' \in [n]$ and $l \in [r]$ such that

$$|S(v(t_1)) - S(v(t_2))| = |(v_i^{(l)}(t_1) - v_{i'}^{(l)}(t_1)) - (v_i^{(l)}(t_2) - v_{i'}^{(l)}(t_2))| \leq 2 \max_{h \in \{i, i'\}} |v_h^{(l)}(t_1) - v_h^{(l)}(t_2)|$$

$$\leq 2 \max_{h \in \{i, i'\}} |\dot{v}_h^{(l)}(t^*)| \cdot |t_1 - t_2|$$

Since $\dot{v}_h^{(l)}(t)$ satisfied (12), its absolute value is bounded above by finite number of terms $|w_{ij}| \leq \bar{w}$,

$$|b_{ij}(x, v)| \leq \max_{ij} f_{ij}(\underline{d}^2)\big(\sup_t S(x(t))\big)^2 \mathcal{E}(t_0)$$

where $\underline{d} = \inf_t \min_{i \neq j} |x_{ij}(t)| > d_0$ and $|S(v(t))| \leq \mathcal{E}(t_0)$, each of which is independent of time and the uniform continuity property holds true. Barbalat's lemma can then be applied concluding the proof. $\square$

IV. EXAMPLES AND SIMULATIONS

In this section we will briefly present applications of our rigorous results. We will use a group of $n = 5$ and examine networks of different dimensions and types of solutions. The initial time is taken $t_0 = 0$. The coupling rates are set $w_{ij}(t, x) = w \frac{1.5 + 0.5 \sin(t)}{(|x| + \beta_{ij}^2)^\delta}$ for $\delta \geq 0$, $\beta_{ij} \in (0, \sqrt{2})$ and $w \geq 0$ a uniform control parameter. All simulations are carried with the `ode23` routine in `MATLAB`.

A. Scalar Networks

We consider (7) with $g(t, z) = \cos(t)(z - 1)(z - 2)$. Its solution $z(t) = z(t, 0, z^0)$, $t \geq 0$ is

$$z(t) = \frac{2 - \frac{z_0 - 2}{z_0 - 1}e^{\sin(t)}}{1 - \frac{z_0 - 2}{z_0 - 1}e^{\sin(t)}}.$$

It can be easily checked that for $z^0 \in (1, 2)$ the solution exists for all times and is periodic with period 1. We consider now the network (6) and its maximal solution (x, v) with initial setup $v^0 = (1.2, 1.4, 1.1, 1.5, 1.3)$. We select $w = 1$ throughout this example. The solution (x, v) exists for all times remaining trapped in $U = [1, 2]$ but we can in fact say much more. Due

to the monotonicity of solution z, using the same arguments as in the proof of Theorem 6 we can calculate the estimate K by evaluating it over $z(t, 0, 1.5)$, since no solution v_i will essentially exceed $z(t, 0, 1.5)$. Thus we estimate K as:

$$K \leq 2\frac{2 + e^{\sin(t)}}{1 + e^{\sin(t)}} - 3 \leq \frac{1 - e^{-1}}{1 + e^{-1}} \approx 0.462$$

In addition, $w_{ij}(t, x) \geq \frac{1}{(|x|+2)^\delta}$. From the conditions of Theorem 6, exponential convergence to flocking with rate $\varepsilon > 0$ can occur if we can find number d such that

$$0.4 < 5\frac{(d + 2)^{(1-\delta)} - (S(x^0) + 2)^{(1-\delta)}}{1 - \delta} - 0.462(d - S(x_0))$$

and

$$\frac{5}{(d^* + 2)^\delta} - 0.462 > 0$$

for d^* that after substituting it to d, it can achieve equality in the first condition. At $\delta = 0$ one can easily verify that the first condition holds true for every d^* and that the first inequality is always satisfied for d large enough. As a brief numerical exploration we took $S(x^0) = 1, 3, 5$ and examined the two conditions one after the other. The results are plotted in Figure 1 and suggest that all the conditions can be satisfied for $\delta < 1.1$. Figure, 2 presents realizations of (x, v) with $S(x^0) = 5$. The first simulation depicts a network with the weak coupling $\delta = 10$, where no synchronization can happen. In the second attempt we adopt a stronger coupling with $\delta = 4$ that however still does not yield a feasible d to satisfy our conditions. Finally for $\delta = 0.9$ we can see synchronization of solutions that occur exponentially fast.
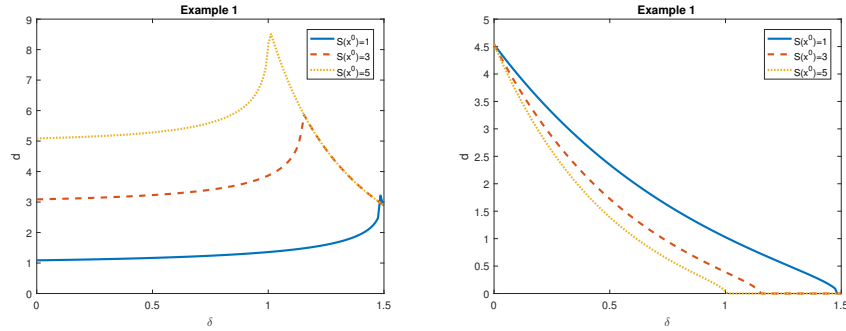

Fig. 1: The stability condition curves for different values of $S(x^0)$. The larger the initial spread of the relative positions, the weaker the couplings initially are. This makes the existence of a solution d harder.

B. Example 2. Chaotic Flocking.

The next example is on the problem of synchronizing chaotic oscillators. The nominal equation is chosen to be the Lorenz system

$$g(z^{(1)}, z^{(2)}, z^{(3)}) = \begin{bmatrix} 10(z^{(2)} - z^{(1)}) \\ -z^{(2)} + z^{(1)}(28 - z^{(3)}) \\ -\frac{8}{3}z^{(3)} + z^{(1)}z^{(2)} \end{bmatrix}.$$

The solutions of (7) converge for almost all initial values to a strange attractor [31]. This means that there is a g-invariant, convex subset $U \subset \mathbb{R}^3$ that includes the limit set. It can be verified that $U \subset \tilde{U} = [-17, 17.5] \times [-22, 24.5] \times [7, 45]$. Furthermore, $\tilde{U}$ can be numerically verified to be g-invariant while it is clearly compact and convex. We can apply Theorem 6 to (6) with g as in this example and initial conditions in $v_i^0 \in \tilde{U}$, $i \in [5]$. Then we can calculate an estimate on K based on U

$$K \leq \max_{z \in \tilde{U}}\{0, |28 - z^{(3)}| - 1 + |z^{(1)}|, |z^{(2)}| + |z^{(1)}| - \frac{8}{3}\} \approx 39.4^2$$

We set $S(v^0) = S(x^0) = 9$ and Theorem 6 applies if one can find d such that

$$9 < \int_9^d \left(\frac{5w}{(r + 2)^\delta} - 39.4\right) dr \quad \text{and} \quad \frac{5w}{(d^* + 2)^\delta} - 39.4 > 0$$

---

[2]Clearly, better estimates on K may be achieved when $\tilde{U}$ is substituted with a sharper estimate of U.
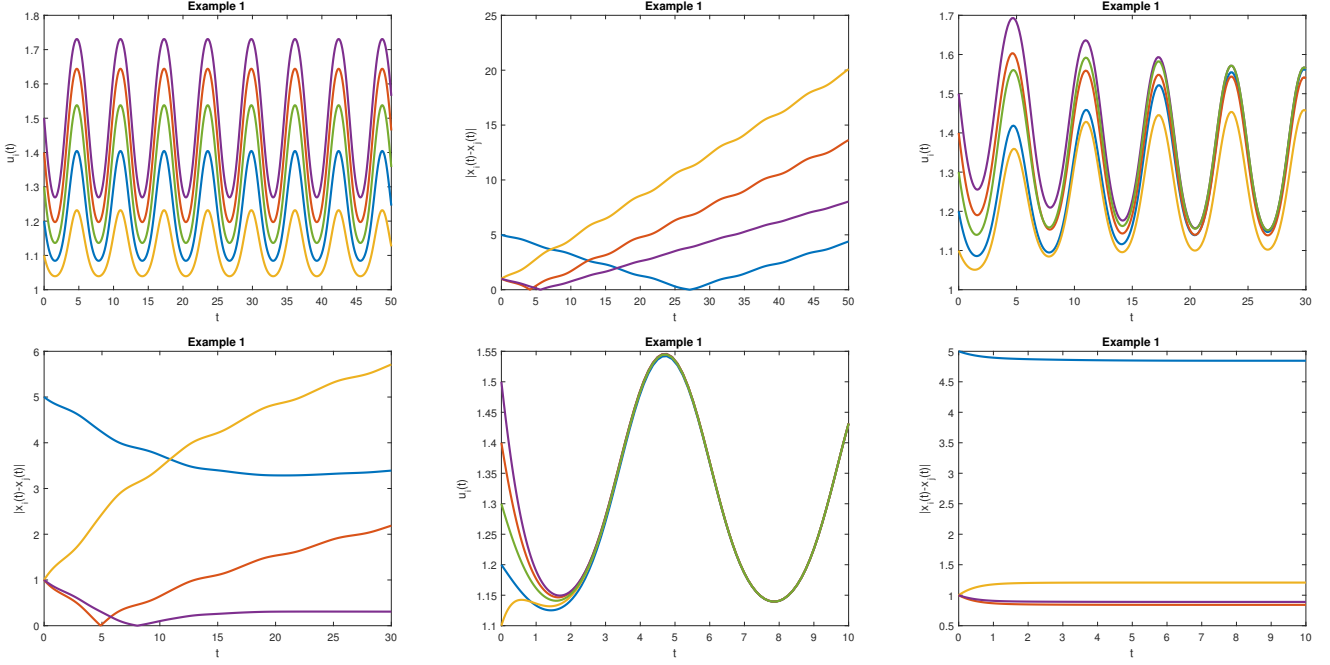
Fig. 2: Realization of (x, v) of Example 1 with $\delta = 10$, $\delta = 4$ and $\delta = 0.9$. The first column presents v_i, $i \in [5]$ and the second column presents the differences $|x_i - x_j|$, $i < j$.

where d^* when substitutes d in the first condition achieves equality. Note that for any δ small enough one can always find d^* such that

$$9 + 39.4(d^* - 9) > 39.4(d^* + 2)^\delta \int_9^{d^*} \frac{dr}{(r+2)^\delta}$$

This is a relation after solving for w in the first condition (that is an equality for $d = d^*$) and substituting in the second condition. If for example we take $\delta = 0.5$ then the last condition is satisfied with $d^* = 11.67$ and $w \approx 150$. Figure 3 depicts our results for strong $\delta = 0.5$ and loose $\delta = 7$ coupling, respectively. The rest of the parameters w and initial configurations were kept identical. These two illustrative choices represent the case that conditions of Theorem 6 are met and when the conditions are violated, respectively.

C. Example 3. Collision Avoidance.

We conclude this section with an illustration of Theorem 9. We consider the network (12) with $r = 2$. The repelling functions functions are taken

$$f_{ij}(r) = \frac{C_{ij}}{(r - r_0)^\varphi}$$

for numbers C_{ij} arbitrarily chosen from $(1, 2)$, $\varphi = 1.5$ and $r_0 = 0.25$. Our initial data are $S(v^0) = 6$ and $S(x^0) = 4$. The first simulation runs with $\delta = 1$ so that the condition of Theorem 9 is clearly satisfied and asymptotic flocking with collision avoidance is achieved. See Fig. 4 for the simulation results. The repelling functions destabilize the agents' velocities in an uncontrollable manner. It then takes the strength of the coupling networks to determine the stability of flocking solutions. In the first case $\delta = 1$ ensures the stability because it makes the right hand side of condition of Theorem 9 unbounded.

V. DISCUSSION

We addressed two variations of the classic non-linear flocking algorithm of Cucker-Smale type with asymmetric coupling rates. The striking similarity in the analysis of these two evidently different algorithms is clear: Both networks include the consensus-based stabilizing term and a potentially destabilizing term. The perturbations affect the stability of the network because it leads to fluctuations in the relative velocities $v_i - v_j$. This in turn results in weaker coupling strength the level of which can dissolve the network to fail to converge to consensus. This interplay is reflected in the initial configuration sufficient conditions that ensure convergence of the overall network to flocking behavior.

As in Theorem 1, the flocking to synchronization results in Theorem 4 require convergence conditions to be parametrized by the initial setup states, the coupling strength in addition to the growth of the first derivative of the internal dynamics function.
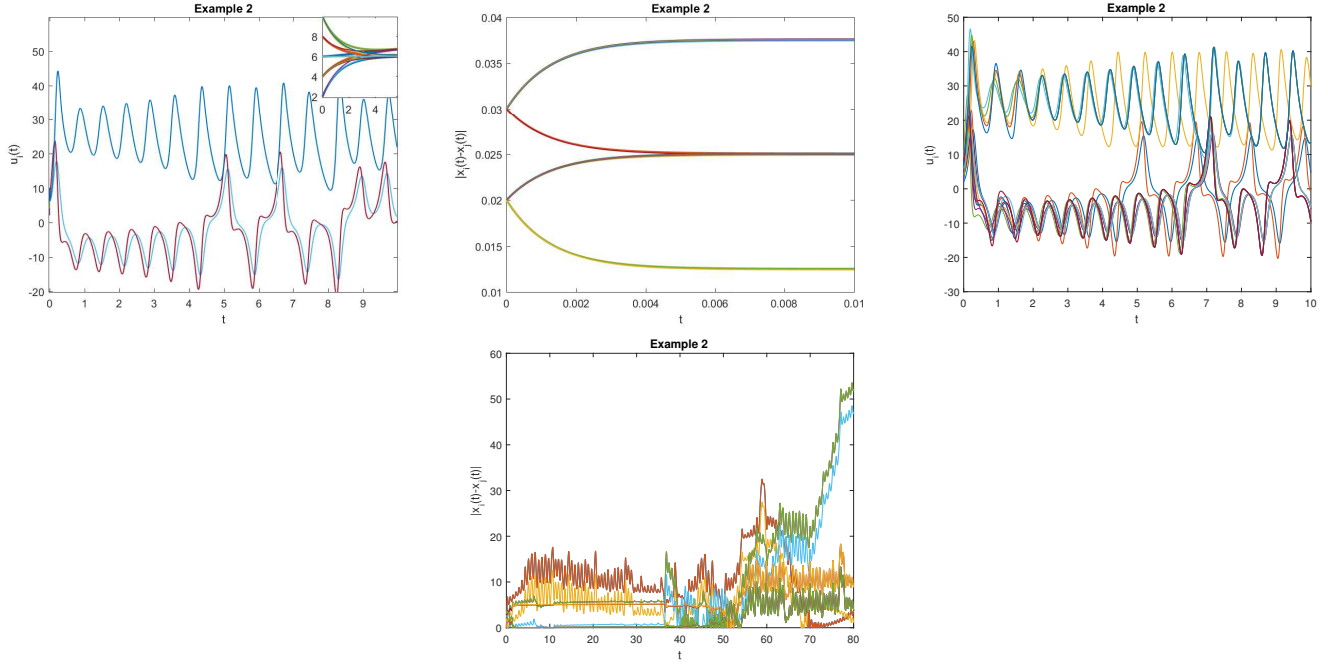
Fig. 3: Simulation Example 2. Synchronization of chaotic oscillators with $r = 3$. In the first two figures we observe extremely strong synchronization under the theoretical sufficient conditions, $w = 150$ and $\delta = 0.5$. In the two figures that follow we present loose coupling with the same w and $\delta = 7$. We observe the of our network to syhnchronize.

The argumentation is very similar for the case of the collision-free extension. We remark the striking similarity of Theorem 9 to the results of [4]. In fact Theorem 9 may be considered as the the asymmetric alternative of [4].

Non-linearity imposes investigation of solutions that may not necessarily exist for all times. Theorem 4 can accept such solutions and may be applied to networks where the solutions escape in finite time. Since this event is of little interest we readily turned to a result concerning solutions of the nominal network that exist for all times and derived claims for the coupled network.

It is yet to be investigated on the dynamics that can emerge with abstract nonlinear perturbations. Our strategy was to focus on reasonable forms of b_{ij} that correspond to real-world mathematical models. In any case one could reasonably assume that positive valued b_{ij}'s in (5) tend to stabilize the network towards a common constant equilibrium, while negative valued perturbation destabilize the network possibly towards more complex behaviors. We conclude this paper on a brief discussion on extensions to our protocols that can be analyzed using similar tools, together with the corresponding limitations that emerge every time.

A. Further Nonlinearities

Models (6) and (12) can be easily extended to non-linearities in the spirit of (1.2) or (1.3). In this case one would require additional assumptions on the type of these non-linearities. For instance note that (1.3) can be re-written as

$$\dot{z}_i = \sum_j w_{ij}(t, z)(z_i - z_j)$$

where $w_{ij}(t, z) = \int_0^1 \frac{\partial}{\partial z} w_{ij}(t, qz_i - (1 - q)z_j) \, dq$. Then typical smoothness or uniformity assumptions on the non-linear w_{ij} can make a second order version possible to be analyzed with our framework, under appropriate modifications. Similar arguments can be adopted for couplings like (1.2).

B. The problem of dimensionality and different metrics

For dynamics in one dimension, all metrics boil down to the absolute value making the analysis simple and elegant. In higher dimensional systems one must carefully chose the most appropriate metric that suits the geometry of the generated solutions. Our analysis dictates the option of $S(y)$ and any discrepancy with the solutions of the nominal dynamics may result in more conservative convergence conditions (i.e. larger K). Attempts towards sharper sufficient conditions should probably allow some structure on the nominal dynamics g. In this case, the researcher can leverage on the geometry of the nominal solutions and their qualitative properties so as to come up with an appropriate contraction metric for the synchronization analysis.
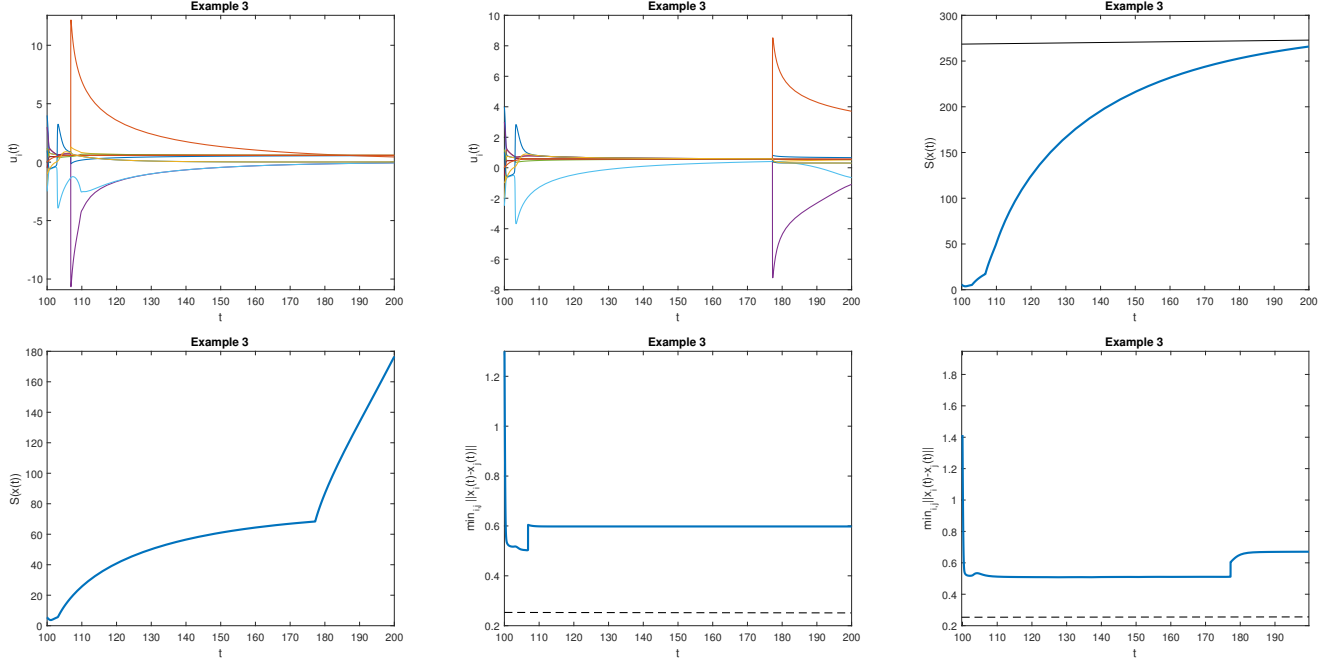
Fig. 4: Simulation Example 3. Flocking with collision avoidance. The first column is a strong coupling simulation of the velocity curves, the maximum distance and the minimum distance. The second column is a weak coupling simulation with the same quantities. While both simulations achieve collision avoidance, only the first one achieves flocking.

C. Connection Topologies

A careful inspection of (4) reveals that the contraction coefficient that provides non-trivial estimates of the stochastic matrix P, must be associated with a graph that is rich in edges. In particular, it is asked that for every couple of two agents there is at least one agent that affects both of the aforementioned two. The connectivity we assumed in this work corresponds to a complete graph and clearly covers this case. The interest in networks of Cucker-Smale type lies primarily on the strength of the coupling weights and not on the topological structure. That is why they are usually assumed under all-to-all communication schemes. Looser connectivities are feasible for the synchronization network (6) but it involves more elaborate arguments, perhaps along the lines of [28]. A collision free model without complete connectivity is, however, rather unreasonable. If two agents do not communicate somehow their positions are clearly prone to approach each other arbitrarily close as they will simply not notice each other. In either case one should expect even more conservative stability conditions. The weaker the connectivity over the network, the smaller the rate estimates become. In either case, the problem of nonlinear synchronization under weaker connectivity regimes remains of interest and will be considered in the future.

APPENDIX

Proof of Lemma 5. Let $(x(t), v(t))$, $t \in [t_0, T)$ the maximal solution of (6). Fix $i, i' \in [n]$, $l \in [r]$ and take $m = \bar{n}\bar{w} + K$. where $\bar{n} > n + 1$, $\bar{w}$ as in Assumption 2. From the second line of (6) we have

$$\dot{v}_j^{(l)} = -mv_j^{(l)} + (m - d_j)v_j^{(l)} + \sum_k w_{jk}v_j^{(l)} + g^{(l)}(t, v_j)$$

for $j = i, i'$, $w_{jk} = w_{jk}(t, x(t))$, $d_j = \sum_{k \neq j} w_{jk}$. Set $\Delta_{i,i'}(t) = e^{-m(t-t_0)} \frac{d}{dt}(e^{m(t-t_0)}[v_i^{(l)}(t) - v_{i'}^{(l)}(t)])$ and substituting the equations on i and i' we obtain

$$\Delta_{i,i'}(t) = \sum_k (\tilde{w}_{ik} - \tilde{w}_{i'k})v_k^{(l)} + \sum_{h \neq l} \left[\int_0^1 \frac{\partial}{\partial z^{(h)}} g^{(l)}(t, qv_i + (1-q)v_{i'}) \, dq \right] (v_i^{(h)} - v_{i'}^{(h)})$$

where $\tilde{w}_{ij} = \tilde{w}_{ij}(t, l)$ defined as

$$\tilde{w}_{ij} = \begin{cases} m - d_i + c_{i,i'}, & j = i \\ w_{ij}, & j \neq i. \end{cases}$$

for $c_{i,i'} = \int_0^1 \frac{\partial}{\partial z^{(l)}} g^{(l)}(t, qv_i(t) + (1-q)v_{i'}(t)) \, dq$. Similarly for $\tilde{w}_{i'j}$. Take $a_j = (\tilde{w}_{ij} - \tilde{w}_{i'j})$ and observe that

$$\sum_j a_j = \sum_{j \neq i} w_{ij} + w_{ii} - \sum_{j \neq i'} w_{i'j} - w_{i'i'} = d_i + m - d_i + c_{i,i'} - d_{i'} - m + d_{i'} - c_{i,i'} = 0.$$

The index j for which $a_j > 0$ is denoted by j^+ and the index for which $a_j \leq 0$ is denoted by j^-. Set $\theta = \theta(t)$

$$\theta = \sum_{j^+} a_{j^+} = \sum_{j^+} |a_{j^+}| = -\sum_{j^-} a_{j^-} = \sum_{j^-} |a_{j^-}| = \frac{1}{2} \sum_j |a_j| = \frac{1}{2} \sum_j |\tilde{w}_{ij} - \tilde{w}_{i'j}|$$

Then for $t \in [t_0, T)$

$$\Delta_{i,i'}(t) = \theta \left(\frac{\sum_{j^+} |a_{j^+}| v_{j^+}}{\theta} - \frac{\sum_{j^-} |a_{j^-}| v_{j^-}}{\theta} \right) + \sum_{h \neq l} \left[\int_0^1 \frac{\partial}{\partial z^{(h)}} g^{(l)}(t, qv_i + (1-q)v_{i'}) \, dq \right] \left(v_i^{(h)} - v_{i'}^{(h)} \right)$$

$$\leq \left(\theta + \sum_{h \neq l} \left| \int_0^1 \frac{\partial}{\partial z^{(h)}} g^{(l)}(t, qv_i + (1-q)v_{i'}) \, dq \right| \right) S(v)$$

But from the identity $|x - y| = x + y - 2\min\{x, y\}$ and the particular m we chose, it can be deduced that

$$\theta = m + c_{i,i'} - \sum_{k \neq i, i'} \min\{w_{ik}(t), w_{i'k}(t)\}$$

So that

$$\Delta_{i,i'}(t) \leq \left(m + K - \sum_{k \neq i, i'} \min\{w_{ik}(t), w_{i'k}(t)\} \right) S(v).$$

Finally, for i, i' and l that maximize $|v_i^{(l)}(t) - v_{i'}^{(l)}(t)|$, we have

$$\frac{d}{dt} S(v(t)) = \frac{d}{dt} \left[e^{-m(t-t_0)} S(e^{m(t-t_0)} v(t)) \right] = -mS(v(t)) + \Delta_{i,i'}(t) \leq \left[K - n\psi(S(x(t))) \right] S(v(t)).$$

that is true for $t \in [t_0, T)$. $\qquad \square$

Proof of Lemma 11 . Consider the maximal solution $(x(t), v(t))$ on $[t_0, T)$. In any $l \in [r]$ dimension, $v_i^{(l)}$ satisfies

$$\dot{v}_i = -mv_i + (m - d_i)v_i + \sum_j (w_{ij} + b_{ij})v_j,$$

where $w_{ij} = w_{ij}(t, x(t))$, $b_{ij} = b_{ij}(t, x(t), v(t))$ as in (14), $d_i = d_i(t) = \sum_j (w_{ij} + b_{ij})$ and $m = m(t)$ to be determined below. Note that we removed the l dependency because the dynamics are identical in all $[r]$ dimensions. We set $\phi(t, t_0) = \exp\{\int_{t_0}^t m(s) \, ds\}$. Following the proof of Theorem 4, $\Delta_{i,i'}(t) := \phi^{-1}(t, t_0) \frac{d}{dt} [\phi(t, t_0) |v_i(t) - v_{i'}(t)|]$, yields

$$\Delta_{i,i'}(t) = \sum_j \alpha_j v_j$$

for

$$\alpha_j = \begin{cases} (w_{ij} - w_{i'j}) + (b_{ij} - b_{i'j}), & j \neq i, i' \\ m - d_i - w_{ii'} - b_{i'i}, & j = i, \\ -m + d_{i'} + w_{i'i} + b_{ii'}, & j = i' \end{cases}$$

Let j' denote the indices j for which $\alpha_j > 0$ and j'' the indices for which $\alpha_j \leq 0$. Then

$$\Delta_{i,i'}(t) = \sum_{j'} |\alpha_{j'}| v_{j'} - \sum_{j''} |\alpha_{j''}| v_{j''}.$$

Again, it holds that $\sum_j \alpha_j \equiv 0$ so we take $\theta_{ii'} = \theta_{ii'}(t)$ to be the sum of the positive α_j.

$$\theta_{ii'} = \sum_{j'} |\alpha_{j'}| = -\sum_{j''} \alpha_{j''} = \sum_{j''} |\alpha_{j''}| = \frac{1}{2} \sum_j |\alpha_j|$$

$$\leq \frac{1}{2} \sum_j |w_{ij} - w_{i'j}| + \frac{1}{2} \sum_j |b_{ij} - b_{i'j}|$$

$$= m - \sum_{j \neq i} \min\{w_{ij}, w_{i'j}\} - \sum_{j \neq i} \min\{b_{ij}, b_{i'j}\}$$

in view of the identity $2\min\{x,y\} = x + y - |x-y|$.

$$\Delta_{ii'}(t) = \theta\left(\frac{\sum_{j'}|w_{j'}|v_{j'}}{\sum_{j'}|w_{j'}|} - \frac{\sum_{j''}|w_{j''}|v_{j''}}{\sum_{j''}|w_{j''}|}\right)$$

Invoking the estimate on θ

$$\Delta_{ii'}(t) \leq \left(m - \sum_{j\neq i}\min\{w_{ij}, w_{i'j}\} - \sum_{j\neq i}\min\{b_{ij}, b_{i'j}\}\right)S(v) = \left(m - \rho_{i,i'}\right)S(v) - \Gamma_{i,i'}$$

where $\rho_{i,i'}$, $\Gamma_{i,i'}$ as in the statement of the Lemma. Observe that $\frac{d}{dt}|v_i(t) - v_{i'}(t)| = -m|v_i - v_{i'}(t)| + \Delta_{ii'}(t)$ and substitute the estimate of $\Delta_{i,i'}(t)$ to conclude. □

REFERENCES

[1] M. Arcak. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 52(8):1380–1390, 2007.
[2] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic. Interaction ruling animal collective behavior depends on topological rather than metric distance. *Proc. Natl. Acad. Sci. USA*, 105:12321237, 2008.
[3] F. R K Chung. *Spectral Graph Theory*, volume 92. American Mathematical Society, 1997.
[4] F. Cucker and J. G. Dong. A general collision-avoiding flocking framework. *IEEE Transactions on Automatic Control*, 56(5):1124–1129, 2011.
[5] F. Cucker and E. Mordecki. Flocking in noisy environments. *J. Math. Pures Appl.*, 89:278–296, 2008.
[6] F. Cucker and S. Smale. Emergent behavior in flocks. *IEEE Transactions on Automatic Control*, 52(5):852–862, 2007.
[7] F. Cucker and S. Smale. On the mathematics of emergence. *Japan Journal of Mathematics*, 2:197–227, 2007.
[8] J.M. González-Miranda. *Synchronization and Control of Chaos: An Introduction for Scientists and Engineers*. Imperial College Press, 2004.
[9] S-Y. Ha and J. G. Liu. A simple proof of the cucker-smale flocking dyanics and mean-field limit. *Commun. Math. Sci.*, 7(2):297–325, 2009.
[10] D. J. Hartfiel. *Markov Set-Chains*, volume 1695. Springer, 1998.
[11] J. M. Hendrickx and J. N. Tsitsiklis. Convergence of type-symmetric and cut-balanced consensus seeking systems. *IEEE Transactions on Automatic Control*, 58(1):214–218, Jan 2013.
[12] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.
[13] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, 2000.
[14] Sabato Manfredi and David Angeli. Necessary and sufficient conditions for consensus in nonlinear monotone networks with unilateral interactions. *Automatica*, 77:51 – 60, 2017.
[15] N. G. Markley. *Principles of Differential Equations*. Wiley-Interscience, 2004.
[16] S. Martin. Multi-agent flocking under topological interactions. *Systems & Control Letters*, 69:53 – 61, 2014.
[17] S. Martin, A. Girard, A. Fazeli, and A. Jadbabaie. Multiagent flocking under general communication rule. *IEEE Transactions on Control of Network Systems*, 1(2):155–166, June 2014.
[18] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.
[19] L. Moreau. Stability of continuous-time distributed consensus algorithms. In *43rd IEEE Conference on Decision and Control*. IEEE, 2004.
[20] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(02):169–182, 2005.
[21] R. Olfati-Saber and R. M. Murray. Consensus problems in networks with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(09), 2004.
[22] Ed. Ott. *Chaos in Dynamical Systems*. Cambridge University Press, 2002.
[23] H. Qing and W. M. Haddad. Distributed nonlinear control algorithms for network consensus. *Automatica*, 44(9):2375 – 2381, 2008.
[24] Motsch S. and Tadmor E. Heterophilious dynamics enhances consensus. *SIAM: Review*, 56(4):577–621, 2014.
[25] E. Seneta. *Non-negative Matrices and Markov Chains*. Springer, 2006.
[26] C. Somarakis and J. S. Baras. Delay-independent convergence for linear consensus networks with applications to non-linear flocking systems. In *Proceedings of the 12th IFAC Workshop on Time Delay Systems*, pages 159–164, Ann Arbor, MI, USA, June 2015.
[27] C. Somarakis and J. S. Baras. Non-linear flocking networks with collision avoidance. In *Proceedings of the 34th Chinese Control Conference*, July 2015.
[28] C. Somarakis and J. S. Baras. A simple proof of the continuous time linear consensus problem with applications in non-linear flocking networks. In *14th European Control Conference*, Linz, Austria, July 2015.
[29] C. Somarakis, R. Hadadi, and J. S. Baras. Dynamics of collectives with opinionated agents: The case of scrambling connectivity. In *ECC'16 (to appear)*, 2016.
[30] C. Somarakis, E. Paraskevas, J. S. Baras, and N. Motee. Synchronization and collision avoidance in non-linear flocking networks of autonomous agents. In *24th Mediterranean Conference on Control & Automation*, pages 1089–1094, June 2016.
[31] C. Sparrow. *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. Applied Mathematical Sciences. Springer New York, 1982.
[32] J. Tsitsiklis, D. Bertsekas, and M. Athans. Distributed asynchronous deterministic and stochastic gradient optimization algorithms. *IEEE Transactions on Automatic Control*, 31(9):803–812, 1986.
[33] Y. Wenwu, C. Guanrong, and C. Ming. Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems. *Automatica*, 46:1089–1095, 2010.
[34] Peter Wieland, Rodolphe Sepulchre, and Frank Allgwer. An internal model principle is necessary and sufficient for linear output synchronization. *Automatica*, 47(5):1068 – 1074, 2011.
[35] Chai Wah Wu. *Synchronization in coupled chaotic circuits and systems*. World Scientific, 2002.