Synthetic Homology in Homotopy Type Theory

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Abstract

This paper defines homology in homotopy type theory, in the process stable homotopy groups are also defined. Previous research in synthetic homotopy theory is relied on, in particular the definition of cohomology. This work lays the foundation for a computer checked construction of homology.

Contents

1 Introduction

Homotopy type theory (HoTT) is a new variant of Martin-Löf's intuitionistic type theory (ITT) and a potential foundation for all of mathematics. Key to understanding HoTT is the fact that in ITT the identity endows each type with an ∞-groupoid structure [6, 9, 13]. This realization led to the use of homotopy theory techniques to understand ITT [14, 15, 1]. Formally HoTT extends ITT with two new concepts: the univalence axiom, and higher inductive types. The univalence axiom states that any two isomorphic types are actually equal. Higher inductive types are types in which the user can specify identities not just terms.

The univalence axiom and higher inductive types have many different applications but in this paper we will focus on how they relate to space. Since an ∞-groupoid can be viewed as a space, it follows that types can be viewed as spaces. In this context the univalence axiom says that any two homotopic spaces are in fact equal, whereas higher inductive types allow us to define freely generated ∞-groupoids. We can thus do homotopy theory inside of homotopy type theory. This viewpoint is often called synthetic homotopy theory as concepts like paths are primitive instead of being defined in terms of a set of points (what one could think of as the analytic approach).

This paper aims to define homology in homotopy type theory, that is, for each type X we want a group $H_n(X)$ that satisfies the Eilenberg–Steenrod axioms of homology (Defintion 2). To this end we would like a definition of H_n that does not depend on the cell structure of a space. Luckily there is such a construction in the literature, details of which can be found in Hatcher [5]. We will broadly follow the approach of Hatcher, however, since our definitions apply to types, our proofs will often be unique, combining ideas from type theory with ideas from classical homotopy theory. As in the case of cohomology, which has already been defined in [3], we drop the additivity axiom from our definition of homology. However, unlike the case of cohomology, it appears that this should not be necessary. Other researchers have communicated to me progress towards including additivity, but as of now this is not complete to my knowledge. Working in HoTT can often involve burdensome computations involving higher paths. As done elsewhere we do not formally prove certain properties of the smash product that should be true but seem too technical to formalize nicely (see Lemma 18). Other researchers are currently working on this (see [16]). Moreover we often do not explicitly work out details concerning the pointed structure or group structure on a type.

The plan of the paper is as follows. To begin with we must first define stable homotopy groups and prove that they form an homology theory. To our knowledge stable homotopy theory has not previously been defined in the literature, but the main tools for working with them have already been fleshed out (in particular the Freudenthal suspension theorem (Proposition 7) and the Blakers-Massey theorem (Theorem 17)). Our main result is proving the stable homotopy groups $\pi_n^s(-)$ form a homology theory (Corollary 25). After which we extend this to showing $\pi_n^s(- \wedge K)$ is a homology theory for a fixed K (Corollary 28) and finally to showing $\texttt{colim}_i\, \pi_{n+i}^s(X \wedge K_i)$ is a homology theory for a fixed prespectrum K_i (Corollary 36). Regular homology $H_n(X, G)$ is then defined using the spectrum of Eilenberg-Maclane spaces $K(G, i)$.

Our motivation is to ultimately reproduce the work of Serre [10] in HoTT. In particular we would like to define Serre spectral sequences and import the proof that all homotopy groups are finitely generated (amoung other things). Michael Shulman has already sketched the construction for cohomology Serre spectral sequences [11].

1.1 Review of higher inductive types

As mentioned above, higher inductive types (HIT) allow us to define various spaces in homotopy type theory by specifying the generators of a freely generated ∞-groupoid. Unlike regular types, HITs come with the ability to define equalities between terms. As an introductory example lets consider defining the circle, S^1, in homotopy type theory. In this case the circle is generated by

- a point $\texttt{base} : S^1$,
- a path $\texttt{loop} : \texttt{base} =_{S^1} \texttt{base}$.

In other words the circle is a ∞-groupoid generated by a single point base and a single path loop. Recall that in this context equalities are thought of as paths. The question now becomes what is the induction principle of this type? The recursion principle is not too hard to guess. For a type C, to define a function $F : S^1 \to C$ we must have the following:

- $\texttt{base}' : C$,
- $\texttt{loop}' : \texttt{base}' =_C \texttt{base}'$.

So, we must specify where the generators go to define a function from S^1 to another ∞-groupoid. As we would expect $F(\texttt{base}) \equiv \texttt{base}'$ and $F(\texttt{loop}) = \texttt{loop}'$. Note the last equality is propositional, this theory (as described in the HoTT book [12]) does not compute. The induction principle is slightly more complicated. Consider $C : S^1 \to \text{Type}$, this can be thought of as a fibration over S^1 (for each point in $s : S^1$ we have a space $C(s)$), so a function $F : \Pi(t : S^1).C(t)$ is a section of the fibration. Thus to define F we require a point

- $\texttt{base}' : C(\texttt{base})$,

and we'll need a *dependent* path over loop

- $\texttt{loop}' : \texttt{transport}(\texttt{loop})(\texttt{base}')=\texttt{base}$.

Perhaps this is best understood by considering the total space of the fibration which here is just the product type $\Sigma(t : S^1).C(t)$. A section $\Pi(t : S^1).C(t)$ is then really just a function $S^1 \to \Sigma(t : S^1).C(t)$ *where the first coordinate is given by the identity*. But we can see how to define such a function using the recursion principle and the definition of the product type Σ. We would need some $\texttt{base}' : C(\texttt{base})$ so that we can get $(\texttt{base}, \texttt{base}') : \Sigma(t : S^1).C(t)$, and we need $\texttt{loop}' : \texttt{transport}(\texttt{loop})(\texttt{base}') = \texttt{base}$ so that we can get $(\texttt{loop}, \texttt{loop}') : (\texttt{base}, \texttt{base}') = (\texttt{base}, \texttt{base}')$.

We will mainly be dealing with pushouts (not surprising, since most spaces are presented as gluing constructions which are in essence iterated pushouts). A pushout can be thought of as two spaces glued together in some way, in HoTT this gluing is accomplished by identifying points with an equality. The classical version of what we are defining is known as the double mapping cylinder. For spaces X, Y, Z and continuous maps $f \in Z \to X$ and $g \in Z \to Y$, we define the double mapping cylinder as $Z \times I \sqcup X \sqcup Y$ quotiented by the relation $(z, 0) \simeq f(z)$ and $(z, 1) \simeq g(z)$, so for each point in z there is a path between $f(z)$ and $g(z)$. Fix types X, Y, Z. Given functions $f : Z \to X$ and $g : Z \to Y$, we define the pushout $X +_Z Y$ by the generators

- $\texttt{inl} : X \to X +_Z Y$,
- $\texttt{inr} : Y \to X +_Z Y$ and,

- $\texttt{glue} : \Pi(z : Z).\, \texttt{inl}(f(z)) = \texttt{inr}(g(z))$.

Intuitively this says we have a copy of X and a copy of Y and for every $z : Z$ we set $f(z)$ equal to $g(z)$. The recursion principle is as follows. For a type C, to define a function $F : X +_Z Y \to C$ we must of course have functions

- $\texttt{inl}' : X \to C$
- and $\texttt{inr}' : Y \to C$,

but since $\texttt{inl}(f(z)) = \texttt{inr}(g(z))$ we also would expect to need an equality

- $\texttt{glue}' : \Pi(z : Z)\, \texttt{inl}'(f(z)) = \texttt{inr}'(g(z))$.

Of course $F(\texttt{inl}(x)) \equiv \texttt{inl}'(x)$, $F(\texttt{inr}(y)) \equiv \texttt{inr}'(x)$ and $F(\texttt{glue}(z)) = \texttt{glue}'(z)$. The induction principle is defined similarly. Given $C : X +_Z Y \to \text{Type}$, to define $F : \Pi(t : X +_Z Y).C(t)$ we need functions

- $\texttt{inl}' : \Pi(x : X).C(\texttt{inl}(x))$,
- $\texttt{inr}' : \Pi(y : Y).C(\texttt{inr}(y))$,

together with a dependent path over $\texttt{glue}$, namely

- $\texttt{glue}' : \Pi(z : Z).\, \texttt{transport}(\texttt{glue})(\texttt{inl}'(f(z)) = \texttt{inr}'(g(y))$.

Note that in homotopy type theory we do not need to worry about defining topology or any such thing. Indeed our definition looks like the definition of set pushout and yet we are really dealing with ∞-groupoids!

1.2 Preliminaries

Unless otherwise stated, all our definitions will be for pointed types, pointed functions, pointed isomorphisms, and so on. We will often avoid explicitly showing that our definitions are pointed but in each case it should be straightforward to do so. For a type pointed X we will denote the point by x_0, similarly y_0 for Y, etc.

We will also often not explicitly work out the group structure. For example we may show two types are isomorphic and avoid explicitly showing the maps to be homomorphism. Again, it should be straightforward for the reader to work out the details.

We use the name of isomorphisms, say $f : X \simeq Y$, to refer to the function $X \to Y$ that belongs to the definition of $X \simeq Y$.

We will make use of many basic higher inductive types, all details can be found in the HoTT book [12]. We will use four different particular types of pushouts: suspension, wedge, smash product, and cofibers. In the classical setting this would be like defining suspension, wedge, smash product and the mapping cylinder using double mapping cylinders.

We define the suspension ΣX by the constructors $N : \Sigma X$, $S : \Sigma X$ and $\texttt{merid} : \Pi(x : X).N = S$. As we have seen this says we have two elements

N and S ('north and south pole') and for every element of X we have a path between N and S. For example the circle is the suspension of the two element type S^0, i.e. we have two points N, S and two paths between them. Classically this corresponds to the double mapping cylinder $X \times I \sqcup \{N\} \sqcup \{S\}$ quotiented by the relation $(x, 0) \simeq N$ and $(x, 1) \simeq S$. We leave it to the reader to understand the remaining definitions in terms of their classical analogues.

We define the wedge $X \vee Y$ by the constructors $\texttt{left} : X \to X \vee Y$, $\texttt{right} : Y \to X \vee Y$, and $\texttt{wglue} : \texttt{left}(x_0) = \texttt{right}(y_0)$.

We define the smash product $X \wedge Y$ by the constructors $\texttt{smbase} : X \wedge Y$, $\texttt{smin} : X \times Y \to X \wedge Y$ and $\texttt{smglue} : \Pi(p : X \vee Y).\, \texttt{smbase} = \texttt{smin}(f(p))$ where f is the canonical map $X \vee Y \to X \times Y$ defined by mapping $\texttt{left}(x)$ to (x, y_0), $\texttt{right}(y)$ to (x_0, y) and $\texttt{wglue}$ to $\texttt{refl}_{(x_0, y_0)}$. We often write $\texttt{smglue}(x)$ for $\texttt{smglue}(\texttt{left}(x))$ and $\texttt{smglue}(y)$ for $\texttt{smglue}(\texttt{right}(y))$ when we feel it is clear from context.

Finally for a function $f : X \to Y$ we define the cofiber C_f by constructors $\texttt{cfbase} : C_f$, $\texttt{cfcod}(f) : Y \to C_f$ and $\texttt{cfglue}(f) : \Pi(x : X).\, \texttt{cfbase} = \texttt{cfcod}(f)(f(x))$. in the later two cases we will sometimes omit the f when it is clear from context.

We also make use of the circle S^1 defined by constructors $\texttt{base} : S^1$ and $\texttt{loop} : \texttt{base} = \texttt{base}$ and set truncation $\|X\|_0$ defined by constructors $| - |_0 : X \to \|X\|_0$ and $\texttt{isset} : \Pi(y : \|X\|_0)$.

Recall the path space ΩX is defined by $x_0 = x_0$. We define $\Sigma^k X$ inductively as $\Sigma(\Sigma^{k-1} X)$, whereas we define $\Omega^k X$ as $\Omega^{k-1}(\Omega X)$. This is worth knowing as occasionally we make use of the definitional equality.

Lemma 1. *(Functor Lemma) Let K be some fixed type. The following can be given actions on functions and become functors from pointed types to pointed types: $\Sigma-$, $\Omega-$, $\| - \|_0$, $- \wedge K$, $K \to -$. Moreover $\pi_k(-) :\equiv \|\Omega^k - \|_0$ is a functor from pointed types to pointed sets, to groups for $k \geq 1$ and to abelian groups for $k \geq 2$.*

Proof. This is fairly straightforward and much of it as been proven elsewhere, see [3, 12]. ☐

Definition 2. A collection of functors H_n from pointed types to groups is a homology theory when the following axioms are satisfied.

- (Suspension) There exists an isomorphism $\texttt{susp} : H_n(X) \simeq H_{n+1}(\Sigma X)$ and moreover this is natural, namely given $f : X \to Y$ we have $\texttt{susp} \circ (H_{n+1}(\Sigma f)) \simeq H_n(f) \circ \texttt{susp}$.
- (Exactness) For any $f : X \to Y$ we have

$$H_n(X) \xrightarrow{H_n(f)} H_n(Y) \xrightarrow{H_n(\text{cfcod(f)})} H_n(C_f)$$

is an exact sequence.

2 Stable Homotopy Theory

In this section we will define the stable homotopy groups in HoTT and show that they satisfy the axioms of homology.

Lemma 3. *For every X and n there is a function $\pi_n(X) \to \pi_{n+1}(\Sigma X)$.*

Proof. This argument comes from chapter 8 of the HoTT book. We can define $\mathtt{merloop}(X) : X \to \Omega\Sigma X$ by sending x to $\mathtt{merid}(x) \circ \mathtt{merid}(x_0)^{-1}$, then the required map is simply $\pi_n(\mathtt{merloop}(X))$. □

Definition 4. Given a sequence of types X_i and maps $f_i : X_i \to X_{i+1}$ we define the colimit $\mathtt{colim}_{i\to\infty} X_i$ as a higher inductive type with point constructors $\mathtt{in}_i : X_i \to \mathtt{colim}_{i\to\infty} X_i$ and path constructors $\mathtt{coglue}_i : \Pi(x : X_i)\mathtt{in}_i(x) = f_i \circ \mathtt{in}_{i+1}(x)$.

Definition 5. For a type X the stable homotopy groups are defined as

$$\pi_n^s(X) \coloneqq \mathtt{colim}_{i\to\infty} \pi_{n+i}(\Sigma^i X),$$

where the map from $\pi_{n+i}(\Sigma^i X) \to \pi_{n+i+1}(\Sigma^{i+1} X)$ is given by Lemma 3, namely $\pi_{n+i}(\mathtt{merloop}(\Sigma^i X))$.

We denote $\pi_{n+i}(\mathtt{merloop}(\Sigma^i X))$ by $\phi_i(n, X)$, where we may drop the n and X if it is clear from context. For $j > i$ the composition $\phi_i \circ \phi_{i+1} ... \circ \phi_{j-1} : \pi_{n+i}(\Sigma^i X) \to \pi_{n+j}(\Sigma^j X)$ will be denoted by $\Phi_i^j(n, X)$.

As the name suggests this sequence is eventually constant. To prove this we will now recall some concepts and results from the homotopy type theory book.

Definition 6. We say a function $f : X \to Y$ is $n-$connected provided

$$\| \mathtt{fib}_f(y) \|_n$$

is contractible for all $y : Y$. Recall $\mathtt{fib}_f(y) = \Sigma(x : X).f(x) = y$. A type X is $n-$connected provided $\|A\|_n$ is contractible.

Proposition 7. *(Freudenthal Suspension Theorem) Suppose X is $n-$connected then the map $\mathtt{merloop}(X)$ is $2n-$connected.*

Proposition 8. *(8.8.5 from HoTT book) If $f : X \to Y$ is $n-$ connected and $k \leq n$ then $\pi_k(f)$ is an isomorphism.*

Proposition 9. *(8.2.1 from HoTT book) If X is $n-$connected then ΣX is $n+1$-connected.*

Lemma 10. *Suppose we have a sequence of types X_i and functions $f_i : X_i \to X_{i+1}$ and suppose there exists an i_0 such that for all $i \geq i_0$, f_i is an isomorphism, then*

$$\mathtt{colim}_i X_i \simeq X_{i_0}.$$

Proof. We'll start by constructing a map $\mathtt{colim}_i X_i \to X_{i_0}$. For each $x : X_i$ we want an element of X_{i_0}. If $i < i_0$ we use $f_{i_0-1}(f_{i_0-2}(...f_i(x))...))$. If $i > i_0$ we do a similar trick with f_i^{-1}'s. Finally if $i = i_0$ we merely use x. It then becomes trivial to show that our maps satisfy the required equalities and hence we get a map $\mathtt{colim}_i X_i \to X_{i_0}$ as desired. On the other hand we use the canonical map $X_{i_0} \to \mathtt{colim}_i X_i$. It is not difficult to see that these two maps form an inverse. □

Theorem 11. *(Stability) Given a type X and $n : \mathbb{N}$ there exists an $i_{X,n}$ such that each $\phi_i(n, X)$ is an isomorphism for $i \geq i_{X,n}$. Moreover this will imply that there is an isomorpshim*

$$\mathtt{stab}(n, X) : \pi_n^s(X) \simeq \pi_{n+i_{X,n}}(\Sigma^{i_{X,n}} X).$$

Proof. Proposition 9 tells us that if X is $n-$connected then ΣX is $n+1$-connected. Therefore as soon as i is large enough so that $2i \geq n + i$ then $\mathtt{merloop}(\Sigma^i X)$ will be $2i$ connected by Prop 7 and hence $\pi_{n+i}(\mathtt{merloop}(\Sigma^i X))$ will be an isomorphism by Prop 8. The second result follows from the above Lemma. □

The stability theorem will let us avoid working explicitly with colimits when proving facts about the stable homotopy groups. Indeed one could take the above theorem as the definition of the stable homotopy groups.

Corollary 12. *Consider X, n, F, $i_{X,n}$ defined as in the previous result. For any $i \geq i_{X,n}$ we can show $\pi_n^s(X) \simeq \pi_{n+i}(\Sigma^i X)$.*

Proof. This is straightforward, first invoke the previous Theorem to get $\mathtt{stab}(n, X) : \pi_n^s(X) \simeq \pi_{n+i_X}(\Sigma^{i_X} X)$ we can then get $\pi_{n+i_X}(\Sigma^{i_X} X) \simeq \pi_{n+i}(\Sigma^i X)$ by composing with $\Phi_{i_X}^i$. □

We wish to show that $\pi_n^s(-)$ is a homology theory but first we must show it is a functor. To that end we start by defining its action on functions.

Definition 13. Given an $f : X \to Y$ we let $k = \mathtt{max}(i_X, i_Y)$, Note that from Lemma 1 we already have $\pi_{n+k}(\Sigma^k -)$ is a functor. We define $\pi_n^s(f)$ as

$$(\mathtt{stab}(n, X) \circ \Phi_{i_x}^k) \circ \pi_{n+k}(\Sigma^k f) \circ (\mathtt{stab}(n, X) \circ \Phi_{i_y}^k)^{-1}.$$

Note that it is trivial to show $\pi_n^s(id) = id$ but showing that π_n^s preserves composition is not quite so easy. We require the following lemma which allows us to replace k in the definition of $\pi_n^s(f)$.

Lemma 14. *Let k be as above then for any $k' \geq k$ we have*

$$\pi_n^s(f) = (\mathtt{stab} \circ \Phi_{i_x}^{k'}) \circ \pi_{n+k'}(\Sigma^{k'} f) \circ (\mathtt{stab} \circ \Phi_{i_y}^{k'})^{-1}.$$

Proof. Clearly by definition and simple properties of repeated composition it suffices to show

$$\pi_{n+k}(\Sigma^k f) = \Phi_k^{k'} \circ \pi_{n+k'}(\Sigma^{k'} f) \circ (\Phi_k^{k'})^{-1}.$$

Its also easy to see that it suffices to prove this for $k' = k+1$. So we need the following square to commute:

$$\begin{tikzcd}
\pi_{n+k}(\Sigma^k X) \arrow[r, "\pi_{n+k}(\Sigma^k f)"] \arrow[d, "\phi_k(X)"] & \pi_{n+k}(\Sigma^k Y) \arrow[d, "\phi_k(Y)"] \\
\pi_{n+k+1}(\Sigma^{k+1} X) \arrow[r, "\pi_{n+k+1}(\Sigma^{k+1} f)"] & \pi_{n+k+1}(\Sigma^{k+1} Y)
\end{tikzcd}$$

But by the functoriality Lemma above we can reduce this to showing the following commutes:

$$\begin{tikzcd}
\Sigma^k X \arrow[r, "\Sigma^k f"] \arrow[d] & \Sigma^k Y \arrow[d] \\
\Omega\Sigma^{k+1} X \arrow[r, "\Omega\Sigma^{k+1} f"] & \Omega\Sigma^{k+1} Y
\end{tikzcd}$$

where the left and right arrows are $\texttt{merloop}(\Sigma^k X)$ and $\texttt{merloop}(\Sigma^k Y)$ respectively. In fact it will be clearer to prove a more general statement namely for any $f : X \to Y$ the following commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
\Omega\Sigma X \arrow[r, "\Omega\Sigma f"] & \Omega\Sigma Y
\end{tikzcd}$$

This is easy to show. For any $x : X$ we wish to show

$$\texttt{merid}(f(x)) \circ \texttt{merid}(y_0)^{-1} = \texttt{ap}_{\Sigma f}(\texttt{merid}(x) \circ \texttt{merid}(x_0)^{-1})$$

but the right hand side is simply

$$\texttt{merid}(f(x)) \circ \texttt{merid}(f(x_0))^{-1},$$

and of course $f(x_0) = y_0$, completing the proof. $\square$

Theorem 15. *$\pi_n^s(-)$ is a functor. In particular it remains to show given $f : X \to Y$ and $g : Y \to Z$ we have $\pi_n^s(g \circ f) = \pi_n^s(g) \circ \pi_n^s(f)$.*

Proof. Let $k_1 = \texttt{max}(i_x, i_y)$ $k_2 = \texttt{max}(i_y, i_z)$ and $k_3 = \texttt{max}(i_x, i_z)$. Written out in full our goal is to show that

$$\texttt{stab}(X) \circ \Phi_{i_x}^{k_1}(X) \circ \pi_{n+k_1}(\Sigma^{k_1} f) \circ (\texttt{stab}(Y) \circ \Phi_{i_Y}^{k_1}(Y))^{-1}$$

composed with

$$\texttt{stab}(Y) \circ \Phi_{i_x}^{k_2}(Y) \circ \pi_{n+k_2}(\Sigma^{k_2} g) \circ (\texttt{stab}(Z) \circ \Phi_{i_Z}^{k_2}(Z))^{-1}$$

is equal to

$$\texttt{stab}(X) \circ \Phi_{i_x}^{k_3}(X) \circ \pi_{n+k_3}(\Sigma^{k_3}(f \circ g)) \circ (\texttt{stab}(Z) \circ \Phi_{i_Z}^{k_3}(Z))^{-1}.$$

But applying the above Lemma to each of these components we can replace k_1, k_2, k_3 with $k := \mathtt{max}(k_1, k_2, k_3)$. Once this is done the equation can be simplified until it suffices to show

$$\pi_{n+k}(\Sigma^k f) \circ \pi_{n+k}(\Sigma^k g) = \pi_{n+k}(\Sigma^k f \circ g),$$

which follows from functoriality of π_{n+k} and Σ^k □

Theorem 16. *The functors π_n^s satisfy the suspension axiom of homology theories.*

Proof. We first construct $\mathtt{susp}(n, X) : \pi_n^s(X) \simeq \pi_{n+1}^s(\Sigma X)$. Let $k = \mathtt{max}(i_{X,n}, i_{\Sigma X,n+1})$ then we have

$$\mathtt{stab}(n, X) \circ \Phi_{i_x}^{k+1} : \pi_n^s(X) \simeq \pi_{n+(k+1)}(\Sigma^{k+1} X).$$

Similarly we have

$$\mathtt{stab}(\Sigma X, n+1) \circ \Phi_{i_{\Sigma X}}^{k} : \pi_{n+1}^s(\Sigma X) \simeq \pi_{(n+1)+k}(\Sigma^k \Sigma X).$$

Thus it suffices to find

$$\mathtt{susp}_0(X, n, k) : \pi_{n+(k+1)}(\Sigma^{k+1} X) \simeq \pi_{(n+1)+k}(\Sigma^k \Sigma X).$$

We define $\mathtt{susp}_0$ by first transporting along the equality $n+(k+1) = (n+1)+k$ and then transporting along the equality$\Sigma^{k+1} X = \Sigma^k \Sigma X$. Therefore (omitting many dependencies) we have

$$\mathtt{susp}(n, X) := \mathtt{stab}(X) \circ \Phi_{i_x}^{k+1} \circ \mathtt{susp}_0 \circ (\Phi_{i_{\Sigma X}}^{k})^{-1} \circ \mathtt{stab}(\Sigma X)^{-1}.$$

We will need to show that we can replace k with any $k' > k$ as we did for $\pi_n^s(f)$. In this case, as before, begin by performing the obvious cancellations and reduce to the case $k' = k+1$ then it will suffice to show the following commutes

$$\begin{tikzcd}
\pi_{n+(k+1)}(\Sigma^{k+1} X) \arrow[r, "\mathtt{susp}_0(X\ n\ k)"] \arrow[d, "\phi_{k+1}(n\ X)"] & \pi_{(n+1)+k}(\Sigma^k \Sigma X) \arrow[d, "\phi_k(n+1\ \Sigma X)"] \\
\pi_{(n+(k+1))+1} \Sigma^{(k+1)+1} X \arrow[r, "\mathtt{susp}_0(X\ n+1\ k+1)"] & \pi_{((n+1)+k)+1}(\Sigma^{k+1} \Sigma X)
\end{tikzcd}$$

Note, since $n+(k+1) \equiv (n+k)+1$ and $\Sigma^{k+1} X \equiv \Sigma \Sigma^k X$, we see that both the top and bottom are merely transporting along the equalities $(n+(k+1)) = (n+1)+k$ and $\Sigma^{k+1} X = \Sigma^k \Sigma X$. Moreover $\phi_{k+1}(n, X)$ is $\pi_{n+(k+1)}(\mathtt{merloop}(\Sigma^{k+1} X))$ and $\phi_k(n+1, \Sigma X)$ is $\pi_{(n+1)+k}(\mathtt{merloop}(\Sigma^k \Sigma X))$, notice that they depend on the same terms as the types in the corners. Therefore filling this square becomes a simple exercise in using path induction.

Now that we know we can increase k in the definition above, we can show the $\mathtt{susp}$ is natural. We must show

$$\mathtt{susp}(n, X) \circ \pi_{n+1}^s(\Sigma f) = \pi_n^s(f) \circ \mathtt{susp}(n, Y).$$

By expanding we see that we must show

$$\begin{aligned}\texttt{stab}(n,X)\circ\Phi_{i_x}^{k_1+1}\circ\texttt{susp}_0(X,n,k_1)\circ(\Phi_{i_{\Sigma X}}^{k_1})^{-1}\circ\texttt{stab}(n+1,\Sigma X)^{-1}\\ \circ(\texttt{stab}(n+1,\Sigma X)\circ\Phi_{i_{\Sigma X}}^{k_2})\circ\pi_{(n+1)+k_2}(\Sigma^{k_2}\Sigma f)\circ(\texttt{stab}(n+1,\Sigma Y)\circ\Phi_{i_{\Sigma Y}}^{k_2})^{-1}\end{aligned}$$

is equal to

$$\begin{aligned}(\texttt{stab}(n,X)\circ\Phi_{i_x}^{k_3})\circ\pi_{n+k_3}(\Sigma^{k_3}f)\circ(\texttt{stab}(n,Y)\circ\Phi_{i_Y}^{k_3})^{-1}\circ\\ \texttt{stab}(n,Y)\circ\Phi_{i_Y}^{k_4+1}\circ\texttt{susp}_0(Y,n,k_4)\circ(\Phi_{i_{\Sigma Y}}^{k_4})^{-1}\circ\texttt{stab}(n+1,\Sigma Y)^{-1}.\end{aligned}$$

Replace everything so that $k_3=k_4+1=k_2+1=k_1+1$, we are then left with showing

$$\texttt{susp}_0(X,n,k)\circ\pi_{(n+1)+k}(\Sigma^k\Sigma f)=\pi_{n+k+1}(\Sigma^{k+1}f)\circ\texttt{susp}_0(Y,n,k).$$

Split this into two squares. The first we transport along the equality $(n+1)+k=n+k+1$ and so easily get that it commutes. The second we transport along the equality $\Sigma^{k+1}X=\Sigma^k\Sigma X$. After applying the functoriality of π_{n+k+1} it suffices to show the following commutes:

$$\begin{tikzcd}
\Sigma^{k+1}X \arrow[r] \arrow[d, "\Sigma^{k+1}f"] & \Sigma^k\Sigma X \arrow[d, "\Sigma^k\Sigma f"] \\
\Sigma^{k+1}Y \arrow[r] & \Sigma^k\Sigma Y
\end{tikzcd}$$

We do this by induction. For $k=1$ this is trivial, otherwise if it is true for $k-1$ recall that we get the equality $\Sigma^{k+1}X=\Sigma^k\Sigma X$ by rewriting $\Sigma^{k+1}X=\Sigma\Sigma^kX$ and $\Sigma^k\Sigma X=\Sigma\Sigma^{k-1}\Sigma X$ and then by using the fact that $k=(k-1)+1$ and applying Σ to the equality for $k-1$. Thus the entire square can be again be decomposed. First we transport along $k=(k-1)+1$ then we have simply Σ applied to the $k-1$ square. Both these new squares are easy to fill. □

It remains to show that $\pi_n^s(-)$ satisfies the exactness axiom. To this end we need the following facts.

Theorem 17. *(Blakers-Massey Theorem) if A is $n-$connected and $f:A\to B$ is m connected then $\pi_n(A)\to\pi_n(B)\to\pi_n(C_f)$ is exact for $n\leq n+m$.*

Proof. This is a straightforward consequence of the version of Blakers-Massey Theorem proven in [4]. □

We will need to know that suspension preserves cofibers. In fact later we will need to know that smash product preserves cofibers. So we first prove that suspension is a special case of smash and then we show the more general result. Classically this follows since smash is left adjoint to pointed arrow and hence preserves colimits. However this general statement is not easy to prove in HoTT and so instead we prove the result directly.

Lemma 18. *there exists isomorphisms $\Sigma X \simeq S^1 \wedge X$, $\texttt{suspsm}(X,K) : \Sigma(X \wedge K) \simeq (\Sigma X) \wedge K$, $\texttt{suspsm2}(X,K) : \Sigma(X \wedge K) \simeq X \wedge (\Sigma K)$, moreover this is natural, in particular we will use the following facts for $f : X \to Y,\ g : K \to L$*

$$\texttt{suspsm}(X,K) \circ (\Sigma f) \wedge K = \Sigma(f \wedge K) \circ \texttt{suspsm}(Y,K),$$

$$\texttt{suspsm2}(X,K) \circ (f \wedge \Sigma K) = \Sigma(f \wedge K) \circ \texttt{suspsm}\,2(Y,K),$$
$$X \wedge g \circ f \wedge L = f \wedge K \circ Y \wedge g,$$

$$\begin{aligned} &\texttt{suspsm2}(\Sigma X, K) \circ \Sigma X \wedge g \circ \texttt{suspsm}(X,L)^{-1} \\ &= \Sigma\big(\texttt{suspsm}(X,K)^{-1} \circ \texttt{suspsm2}(X,K) \circ (X \wedge g)\big). \end{aligned}$$

Proof. In [2] we see that $\Sigma X \simeq S^1 \wedge X$. Moreover a proof is sketched that $\wedge$ is a symmetric monoidal product (in particular associative and communative in a natural way). We will accept this proof sketch for this paper. Since this paper was first typed up progress has been made on this problem by Floris Van Doorn [16]. Assuming the result we can prove the first isomorphism $\texttt{suspsm}$ as follows

$$\Sigma(X \wedge K) \simeq S^1 \wedge (X \wedge K) \simeq (S^1 \wedge X) \wedge K \simeq \Sigma X \wedge K.$$

To prove the first equation we can use the symmetric monoidal structure of $\wedge$, the only thing missing is naturality of $\Sigma X \simeq S^1 \wedge X$ but this can be easily deduced as follows. Let $\Phi(X) : \Sigma X \to S^1 \wedge X$ be the isomorphism given in [2]. Let $f : X \to Y$, we wish to show the following commutes

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma f"] \arrow[d, "\Phi(X)"] & \Sigma Y \arrow[d, "\Phi(Y)"] \\ S^1 \wedge X \arrow[r, "S^1 \wedge f"] & S^1 \wedge Y \end{tikzcd}$$

That is to say we want to show for $q : \Sigma X$ that

$$\Sigma f \circ \Phi(Y)(q) = \Phi(X) \circ S^1 \wedge f(q).$$

We proceed by induction on $q : \Sigma X$ for $q \equiv N$ we trace through the definitions and see we just need to show $\texttt{cfbase} = \texttt{cfbase}$ which we can do by $\texttt{refl}_{\texttt{cfbase}}$. Similarly for $q \equiv S$. Finally for $q = \texttt{merid}(x)$ it will suffices to show

$$\Sigma f \circ \Phi(Y)(\texttt{merid}(x)) = \Phi(X) \circ S^1 \wedge f(\texttt{merid}(x)).$$

Again we trace through the definitions and see it suffices to show

$$\begin{aligned} &S^1 \wedge f\big(\texttt{smglue}(\texttt{left}(x)) \circ \texttt{ap}_{\lambda s.\, \texttt{smin}(s,x)}(\texttt{loop}) \circ \texttt{smglue}(\texttt{left}(x))^{-1}\big) \\ &= \texttt{smglue}(\texttt{left}(f(x))) \circ \texttt{ap}_{\lambda s.\, \texttt{smin}(s,fx)}(\texttt{loop}) \circ \texttt{smglue}(\texttt{left}(f(x))^{-1}, \end{aligned}$$

which is clear. The second isomorphism $\texttt{suspsm2}$ and the second equality can be shown in a similar manner. The third equality follows directly from the symmetric monoid structure, and finally the last equality looks more formidable but can also be easily deduced from the above square and the properties in [2]. ∎

Lemma 19. *To construct a term $F : \Pi(x : X \wedge Y).P(x)$ it suffices to give the following ingredients:*

1. $\mathtt{smbase}' : P(\mathtt{smbase})$.
2. $\mathtt{smin}' : \Pi(p : X \times Y).P(\mathtt{smin}(p))$.
3. $\mathtt{smglueleft}' : \Pi(x : X)\,\mathtt{smbase}' =_{\mathtt{smglue}(\mathtt{left}\,x))} \mathtt{smin}'(x, y_0)$.
4. $\mathtt{smglueright}' : \Pi(y : Y)\,\mathtt{smbase}' =_{\mathtt{smglue}(\mathtt{right}(y))} \mathtt{smin}'(x_0, y)$.

Moreover F will satisfy $F(\mathtt{smbase}) \equiv \mathtt{smbase}'$, $F(\mathtt{smin}(p)) \equiv \mathtt{smin}'(p)$, $F(\mathtt{smglue}(\mathtt{left}(x)) = \mathtt{smglueleft}'$ and $F(\mathtt{smglue}(\mathtt{right}(y))) = \mathtt{smglueright}'$.

Now let $f : X \to Y$ be fixed. To construct a term $F : \Pi(x : C_{f\wedge K}).P(x)$ it suffices to give the following ingredients:

1. $\mathtt{cfbase}' : P(\mathtt{cfbase})$.
2. $\mathtt{cfcod}' : \Pi(p : Y \wedge K).P(\mathtt{cfcod}(p))$.
3. $\mathtt{cfgluesmbase}' : \mathtt{cfbase}' =_{\mathtt{cfglue}(\mathtt{smbase})} \mathtt{cfcod}'(f \wedge K(\mathtt{smbase}))$.
4. $\mathtt{cfgluesmin}' : \Pi(q : X \times K)\,\mathtt{smbase}' =_{\mathtt{cfglue}(\mathtt{smin}(q))} \mathtt{cfcod}'(f \wedge K(\mathtt{smin}(q))$.

Moreover F will satisfy $F(\mathtt{cfbase}) \equiv \mathtt{cfbase}'$, $F(\mathtt{cfcod}(p)) \equiv \mathtt{cfcod}'(p)$, $F(\mathtt{cfglue}(\mathtt{smbase})) = \mathtt{cfgluesmbase}'$ and $F(\mathtt{cfglue}(\mathtt{smin}(q))) = \mathtt{cfgluesmin}'$.

Proof. The first part follows from the fact that the smash product $X \wedge Y$ can be defined as the pushout of $2 \leftarrow X + Y \to X \times Y$. This is shown in the PhD prospectus of Floris Van Doorn [16]. Using this definition it is clear that the given ingredients suffice to construct an element of $\Pi(x : X \wedge Y).P(x)$. Similarly the second part will follow from the following claim: $C_{f\wedge K}$ can be defined as the pushout of $2 \leftarrow 1 + X \times K \to Y \wedge K$ where the map $1 + X \times K \to Y \wedge K$ is defined by sending the point to the basepoint of $Y \wedge K$ and sending (x, k) to $f(\mathtt{smin}(x, k))$. To prove this claim consider the following diagram.

$$\begin{tikzcd}
1+X+K \arrow[r] \arrow[d] & 1+X\times K \arrow[r] \arrow[d] & Y\wedge K \arrow[d] \\
1+2 \arrow[r] & 1+X\wedge K \arrow[r] \arrow[d] & Y\wedge K \arrow[d] \\
 & 2 \arrow[r] & C_{f\wedge K}
\end{tikzcd}$$

We want to show te right rectangle is a pushout. It is not hard to show that the top left square, the bottom right square and the top rectangle are all pushouts. It hence follows from the pushout lemma that the top right square and hence the right rectangle are also pushouts. □

Theorem 20. *There exists an isomorphism*

$$\mathtt{smcf} : C_{f\wedge K} \simeq C_f \wedge K,$$

and $\mathtt{cfcod}(f \wedge K) \circ \mathtt{smcf} = \mathtt{cfcod}(f) \wedge K$.

Proof. We will make use of 19 whenever we can.

Define $\phi : C_{f\wedge K} \to C_f \wedge K$ by sending $\texttt{cfbase}_{f\wedge K}$ to $\texttt{smbase}$, $\texttt{cfcod}_{f\wedge K}(q)$ to $\texttt{cfcod}\wedge K(q)$ then for $p : A \wedge K$ we need to show

$$\texttt{smbase} = \texttt{cfcod}_f \wedge K(f \wedge K(p)).$$

For $p \equiv \texttt{smbase}$ we can just use $\texttt{refl}_{\texttt{smbase}}$. For $p \equiv \texttt{smin}(a, k)$ we need to show

$$\texttt{smbase} = \texttt{smin}(\texttt{cfcod}(f(a)), k),$$

which can be done by $\texttt{smglue}(k) \circ \texttt{smin}(\texttt{cfglue}(a), k)$.

Next we construct a map $\psi : C_f \wedge K \to C_{f\wedge K}$. Map $\texttt{smbase}$ to $\texttt{cfbase}$. For $\texttt{smin}(t, k)$ with $t : C_f$ we induct on t. For $t \equiv \texttt{cfbase}_f$ use $\texttt{cfbase}_{f\wedge K}$, for $t \equiv \texttt{cfcod}_f(b)$ we use $\texttt{cfcod}_{f\wedge K}(\texttt{smin}(b, k))$ and finally for $a : A$ we must show

$$\texttt{cfbase}_{f\wedge K} = \texttt{cfcod}(\texttt{smin}(f(a), k)).$$

But since $\texttt{smin}(f(a), k) \equiv f \wedge K(\texttt{smin}(a, k))$ we can simply apply $\texttt{cfglue}_{f\wedge K}(a, k)$. For $k : K$ we need that $\texttt{cfbase}_{f\wedge K} = \texttt{cfbase}_{f\wedge K}$ which we of course do with $\texttt{refl}$, and for $t : \texttt{cfbase}_f$ we must show

$$\texttt{cfbase} = \psi(\texttt{smin}(t, k_0)),$$

(ψ standing in for the partially defined function above), which we prove by induction on t. For $t \equiv \texttt{cfbase}$ we need $\texttt{cfbase} = \texttt{cfbase}$ which we do by $\texttt{refl}$. For $t \equiv \texttt{cfcod}_f(b)$ we need to show

$$\texttt{cfbase} = \texttt{cfcod}(\texttt{smin}(b, k_0)).$$

Now we have $\texttt{smglue}(\texttt{left}(b)) : \texttt{smin}(b, k_0) = \texttt{smbase}$ and $\texttt{cfglue}(\texttt{smbase}) : \texttt{cfbase} = \texttt{cfcod}(\texttt{smbase})$ so we simply use $\texttt{cfglue}(\texttt{smbase}) \circ \texttt{ap}_{\texttt{cfcod}}(\texttt{smglue}(\texttt{left}(b)))$. For the case of $t = \texttt{cfglue}(a)$ this reduces to showing

$$\texttt{cfglue}(\texttt{smbase}) \circ \texttt{ap}_{\texttt{cfcod}}(\texttt{smglue}(\texttt{left}(f(a))) \circ \psi(\texttt{glue}(a))^{-1} = \texttt{refl}_{\texttt{cfbase}},$$

which is

$$\texttt{cfglue}(\texttt{smbase}) \circ \texttt{ap}_{\texttt{cfcod}}(\texttt{smglue}(\texttt{left}(f(a)) \circ \texttt{cfglue}(\texttt{smin}(f(a), k_0))^{-1} = \texttt{refl}_{\texttt{cfbase}}.$$

But $\texttt{smglue}(\texttt{left}(f(a)) = \texttt{ap}_{f\wedge K}(\texttt{smglue}(\texttt{left}(a))$, so we replace the middle with $\texttt{ap}_{f\wedge K\circ\texttt{cfcod}}(\texttt{smglue}(\texttt{left}(a))$. Note $f \wedge K \circ \texttt{cfcod}$ is constant and the proof of this is simply $\texttt{cfglue}$, therefore by a simple path induction argument the above equality is satisfied.

Let's now show $\Pi(z : C_{f\wedge K}).\psi(\phi(z)) = z$. For $z \equiv \texttt{cfbase}$ we use $\texttt{refl}_{\texttt{cfbase}}$ for $z \equiv \texttt{cfcod}(y)$ for $y : B \wedge K$ we must induct on y. For $y \equiv \texttt{smbase}$ we use the proof that

$$\texttt{cfbase} = \texttt{cfcod}(\texttt{smbase}),$$

namely $\texttt{cfglue}(\texttt{smbase})$. For $y \equiv \texttt{smin}(b, k)$ we can just use $\texttt{refl}$. Now for $b : B$ we need to check

$$\psi(\texttt{cfcod}\wedge K(\texttt{smglue}(b))^{-1} \circ \texttt{cfglue}(\texttt{smbase}) \circ \texttt{cfcod}(\texttt{smglue}(b)) = \texttt{refl},$$

but the first term is by definition equal to

$$(\texttt{cfglue}(\texttt{smbase}) \circ \texttt{cfcod}(\texttt{smglue}(\texttt{left}(b)))^{-1}.$$

so this is clear. Next for $k : K$ we must show

$$\psi(\texttt{cfcod} \wedge K(\texttt{smglue}(k))^{-1} \circ \texttt{cfglue}(\texttt{smbase}) \circ \texttt{cfcod}(\texttt{smglue}(k)) = \texttt{refl}\,.$$

but here the first term computes to

$$\psi(\texttt{smglue}_{C_f \wedge K}(k) \circ \texttt{ap}_{\texttt{smin}(-,k)}((\texttt{cfglue}(a_0) \circ \texttt{ap}_{\texttt{cfcod}}(p)))^{-1}).$$

where p is the proof that $f(a_0) = b_0$. which is

$$(\texttt{cfglue}(\texttt{smin}(a_0, k)) \circ \texttt{ap}_{\lambda x.\, \texttt{cfcod}(\texttt{smin}(x,k))}(p))^{-1}.$$

The proof can then be completed by noting that since $\texttt{cfcod}(f \wedge K(x))$ is constant by $\texttt{cfglue}$ we have

$$\texttt{cfcod}(f \wedge K(\texttt{smglue}(k)) = \texttt{cfglue}(\texttt{smbase})^{-1} \circ \texttt{cfglue}(\texttt{smin}(a_0, k)),$$

while on the other hand, by definition of $f \wedge K$,

$$\texttt{cfcod}(f \wedge K(\texttt{smglue}(k)) = \texttt{cfcod}(\texttt{smglue}(k)) \circ \texttt{cfcod}(\texttt{smin}(p, k)),$$

so we solve for $\texttt{cfcod}(\texttt{smglue}(k))$, plug it into our original equation and everything easily cancels.

We have two more cases to check. First corresponding to $\texttt{cfglue}(\texttt{smbase}_{A \wedge K})$, we would like the following

$$\psi(\phi(\texttt{cfglue}(\texttt{smbase})) = \texttt{refl},$$

however this is immediate from the definition. Next corresponding to $\texttt{cfglue}(\texttt{smin}(a, k))$ we would like to show

$$\psi(\phi(\texttt{cfglue}(\texttt{smin}(a, k)) = \texttt{cfglue}(\texttt{smin}(a, k)),$$

but again this follows from the definition since the left hand side is

$$\psi(\texttt{smglue}(k) \circ \texttt{smin}(\texttt{cfglue}(a)), k),$$

which is

$$\texttt{refl} \circ \texttt{cfglue}(\texttt{smin}(a, k)).$$

Finally we need to prove $\Pi(z : C_f \wedge K).\phi(\psi(z)) = z$. We first consider $z \equiv \texttt{smbase}$ then we can just use $\texttt{refl}$. If $z \equiv \texttt{smin}(t, k)$ we induct on t. For $t \equiv \texttt{cfbase}$ we must show

$$\texttt{smbase} = \texttt{smin}(\texttt{cfbase}, k),$$

which we prove by $\mathtt{smglue}_{C_f \wedge K}(k)$. Then for $t \equiv \mathtt{cfcod}(b)$ we must show

$$\mathtt{smin}(\mathtt{cfcod}(b), k) \equiv \mathtt{smin}(\mathtt{cfcod}(b), k),$$

which is done by $\mathtt{refl}$. Finally we need to consider the case for $t = \mathtt{glue}(a)$. In other words we must show

$$\phi(\psi(\mathtt{smin}(\mathtt{glue}(a), k))) \circ \mathtt{smin}(\mathtt{cfglue}(a), k)^{-1} \circ \mathtt{smglue}(k)^{-1} = \mathtt{refl}.$$

The first term computes to

$$\phi(\mathtt{cfglue}(\mathtt{smin}(a, k))) = \mathtt{smglue}(k) \circ \mathtt{smin}(\mathtt{cfglue}(a), k),$$

so the required equality is trivial.

Next consider the case $z = \mathtt{smglue}(k)$ where $k : K$. We must show

$$\phi(\psi(\mathtt{smglue}(k))) \circ \mathtt{smglue}(k) \circ \mathtt{smglue}(k)^{-1} = \mathtt{refl},$$

so it suffices to show

$$\phi(\psi(\mathtt{smglue}(k)) = \mathtt{refl},$$

which is by definition of ψ.

Next consider the case $z = \mathtt{smglue}(t)$, where $t : C_f$. We induct on t. For $t \equiv \mathtt{cfbase}$ show

$$\phi(\psi(\mathtt{smglue}(\mathtt{cfbase})) \circ \mathtt{smglue}(k_0) \circ \mathtt{smglue}(\mathtt{cfbase})^{-1} = \mathtt{refl}.$$

However $\mathtt{smglue}(k_0) = \mathtt{smglue}(\mathtt{cfbase})$, so we just need

$$\phi(\psi(\mathtt{smglue}(\mathtt{cfbase})) = \mathtt{refl},$$

which follows by definition. For $t \equiv \mathtt{cfcod}(b)$ we must show

$$\phi(\psi(\mathtt{smglue}(\mathtt{cfcod}(b)) = \mathtt{smglue}(\mathtt{cfcod}(b)),$$

this follows from the definition since the left hand side is

$$\phi(\mathtt{cfglue}(\mathtt{smbase}) \circ \mathtt{ap}_{\mathtt{cfcod}}(\mathtt{smglue}(\mathtt{left}(b)))),$$

which is

$$\mathtt{refl} \circ \mathtt{cfcod} \wedge K(\mathtt{smglue}(b)) = \mathtt{smglue}(\mathtt{cfcod}(b)).$$

Finally for $t = \mathtt{glue}(a)$ we must show some 3 dimensional condition. Using terminology from [7] we have already filled the following squares

$$\begin{tikzcd}
\mathtt{smglue} \arrow[r, "\mathtt{smglue}(\mathtt{cfbase})"] \arrow[d, "\mathtt{refl}"] & \mathtt{smin}(\mathtt{cfbase}, k_0) \arrow[d, "\mathtt{smglue}(k_0)"] \\
\phi(\psi(\mathtt{smbase})) \arrow[r, "\phi(\psi(\mathtt{smglue}(\mathtt{cfbase})))"] & \phi(\psi(\mathtt{smin}(\mathtt{cfbase}, k_0))
\end{tikzcd}$$

$$\begin{array}{ccc}
\texttt{smglue} & \xrightarrow{\texttt{smglue}(\texttt{cfcod}(b))} & \texttt{smin}(\texttt{cfcod}(b), k_0) \\
\big\downarrow \texttt{refl} & & \big\downarrow \texttt{refl} \\
\phi(\psi(\texttt{smbase})) & \xrightarrow{\phi(\psi(\texttt{smglue}(\texttt{cfcod}(b))))} & \phi(\psi(\texttt{smin}(\texttt{cfcod}(b), k_0))
\end{array}$$

and we want a cube between them. We have $\texttt{cfglue}(a) : \texttt{cfbase} = \texttt{cfcod}(a)$. With some contemplation (and since square fillers are unique) it can be seen that we are essentially in the following case: Suppose we have types X, A, $f : X \to X$, $x : X$, $x' : A \to X$, $p : \Pi(a : A).x = x'(a)$, $\ell : f(x) = x$, $a_1 : A$, $a_2 : A$, and finally $q : a_1 = a_2$.

$$\begin{array}{ccc}
x & \xrightarrow{p(a_1)} & x'(a_1) \\
\big\downarrow \ell & & \big\downarrow \text{filler} \\
f(x) & \xrightarrow{f(p(a_1))} & f(x'(a_1)
\end{array}$$

$$\begin{array}{ccc}
x & \xrightarrow{p(a_2)} & x'(a_2) \\
\big\downarrow \ell & & \big\downarrow \text{filler} \\
f(x) & \xrightarrow{f(p(a_2))} & f(x'(a_2))
\end{array}$$

These two squares have a cube between them. This is clear since they are the same square if path induction is applied to q.

It still remains to show the equality $\texttt{cfcod}(f \wedge K) \circ \texttt{smcf} = \texttt{cfcod}(f \wedge K)$ but this follows quickly from the definition. □

Corollary 21. *Given the above two results it follows that*

$$\texttt{suspcf} : C_{\Sigma f} \simeq \Sigma C_f,$$

and $\texttt{cfcod}(\Sigma f) \circ \texttt{suspcf} = \Sigma\, \texttt{cfcod}(f)$.

Corollary 22. *It follows that Σ^k preserves cofibers in the same manner as Σ.*

Lemma 23. *For any group homomorphism $f : X \to Y$ and $g : Y \to Z$, if we have isomorphisms $\ell_1 : X_0 \to X$, $\ell_2 : Y \to Y_0$ and $\ell_3 : Z \to Z_0$ and then $\ell_1 \circ f \circ \ell_2$, $\ell_2^{-1} \circ g \circ \ell_3$ is exact if f, g is exact.*

Proof. Straightforward. □

Theorem 24. *$\pi_n^s(-)$ satisfies the exactness axiom.*

Proof. For any $f : X \to Y$ we wish to show that

$$\pi_n^s(X) \xrightarrow{\pi_n^s f} \pi_n^s(Y) \xrightarrow{\pi_n^s\ \texttt{cfcod}(f)} \pi_n^s(C_f)$$

is exact. We start by expanding the definitions of $\pi_n^s(f)$ and $\pi_n^s(\texttt{cfcod}(f))$ and using our replacement lemma as we've done above. It then becomes immediately clear that we are in the same position as the lemma above and so it suffices to show

$$\pi_{n+k}(\Sigma^k X) \xrightarrow{\pi_{n+k}(\Sigma^k f)} \pi_{n+k}\Sigma^k(Y) \xrightarrow{\pi_{n+k}\Sigma^k\ \texttt{cfcod}(f)} \pi_{n+k}\Sigma^k(C_f)$$

is exact where $k \geq i_X, i_Y, i_{C_f}$. But from Corollary 22 we can replace this with

$$\pi_{n+k}(\Sigma^k X) \xrightarrow{\pi_{n+k}(\Sigma^k f)} \pi_{n+k}\Sigma^k(Y) \xrightarrow{\pi_{n+k}\,\texttt{cfcod}(\Sigma^k f)} \pi_{n+k}(C_{\Sigma^k f}).$$

Now $\Sigma^k X$ is k connected as long as $k \geq n$ (by 9). Moreover $\Sigma^k Y$ is k connected so the domain and codomain of $\Sigma^k f$ is k connected and hence $\Sigma^k f$ must be $k-1$ connected (for a proof combine 7.5.6 and 7.5.11 from the HoTT book). Therefore by the Blakers Massey Theorem (17) this sequence is exact. ☐

Corollary 25. *$\pi_n^s(-)$ is a homology theory.*

3 Extending the result to homology

We first wish to extend the previous result by showing for a fixed type K, $\pi_n^s(- \wedge K)$ is a homology theory.

It is clear that $\pi_n^s(- \wedge K)$ is a functor since it is the composition of π_n^s and $- \wedge K$ both of which we've already shown are functors.

Theorem 26. *$\pi_n^s(- \wedge K)$ satisfies the suspension axiom.*

We start by constructing an isomorphism $\texttt{susp2}(n, X) : \pi_n^s(X \wedge K) \simeq \pi_{n+1}^s((\Sigma X) \wedge K)$. Of course we already have $\pi_n^s(X \wedge K) \simeq \pi_{n+1}^s(\Sigma(X \wedge K))$ by $\texttt{susp}(n, X \wedge K)$. It therefore suffices to compose this with the isomorphism $\pi_{n+1}^s(\texttt{suspsm}) : \pi_{n+1}^s(\Sigma(X \wedge K)) \simeq \pi_{n+1}^s((\Sigma X) \wedge K))$. Next we must show this is natural. We wish to show

$$\texttt{susp2}(n, X) \circ \pi_{n+1}^s((\Sigma f) \wedge K) = \pi_n^s(f \wedge K) \circ \texttt{susp2}(n, Y).$$

Therefore, we must show

$$\begin{aligned} &\texttt{susp}(n, X \wedge K) \circ \pi_{n+1}^s(\texttt{suspsm}(X)) \circ \pi_{n+1}^s((\Sigma f) \wedge K) = \\ &\qquad \pi_n^s(f \wedge K) \circ \texttt{susp}(n, Y \wedge K) \circ \pi_{n+1}^s(\texttt{suspsm}(Y). \end{aligned}$$

Using naturality of $\texttt{susp}$ we can simplify this to showing

$$\pi_{n+1}^s(\texttt{suspsm}(X)) \circ \pi_{n+1}^s((\Sigma f) \wedge K) = \pi_{n+1}^s(\Sigma(f \wedge K) \circ \pi_{n+1}^s(\texttt{suspsm}(Y)),$$

which would follow provided we show

$$\texttt{suspsm}(X) \circ (\Sigma f) \wedge K = \Sigma(f \wedge K) \circ \texttt{suspsm}(Y).$$

This is true by Lemma 18 above.

Theorem 27. *$\pi_n^s(- \wedge K)$ satisfies the exactness axiom.*

Proof. We wish to show

$$\pi_n^s(X \wedge K) \xrightarrow{\pi_n^s f \wedge K} \pi_n^s(Y \wedge K) \xrightarrow{\pi_n^s\,\texttt{cfcod}_f \wedge K} \pi_n^s(C_f \wedge K)$$

is exact. Just like in the proof of exactness of π_n^s we can replace this with

$$\pi_n^s(X \wedge K) \xrightarrow{\pi_n^s f \wedge K} \pi_n^s(Y \wedge K) \xrightarrow{\pi_n^s\,\texttt{cfcod}_{f \wedge K}} \pi_n^s(C_{f \wedge K}).$$

We are then done by exactness of π_n^s. ☐

Corollary 28. *$\pi_n^s(- \wedge K)$ is a homology theory.*

3.1 Homology

We are now ready to construct ordinary homology.

Definition 29. A prespectrum is a sequence of types K_i and maps $\kappa_i : \Sigma K_i \to K_{i+1}$.

Lemma 30. *Given a prespectrum (K_i, κ_i) and a type X we have a prespectrum with types $X \wedge K_i$.*

Proof. From Lemma 18 we have $\texttt{suspsm2}(X, K_i) : \Sigma(X \wedge K_i) \to X \wedge \Sigma K_i$. Compose this with $X \wedge \kappa_i$. □

Lemma 31. *Given a prespectrum (K_i, κ_i) we can construct a map $\pi^s_{n+i}(K_i) \to \pi^s_{n+i+1}(K_{i+1})$.*

Proof. We first use $\texttt{susp}(K_i)$ from above to get $\pi^s_n(K_i) \to \pi^s_{n+i+1}(\Sigma K_i)$ then we compose with $\pi^s_{n+i+1}(\kappa_i)$. □

Theorem 32. *Using the functions we just computed above we define $H_n(X) = \| \texttt{colim}_i \, \pi^s_{n+i}(X \wedge K_i)\|_0$. This is a homology theory.*

Remark 33. Truncated colimits of sets behave much like we expect form the classical setting. In particular note the following. To define a function from $\| \texttt{colim}_i(X_i, \phi_i)\|_0 \to \| \texttt{colim}_i(Y, \psi_i)\|_0$ where each X_i, Y_i is a set it suffices to find maps $f_i : X_i \to Y_i$ such that $f_i \circ \psi_i = \phi_i \circ f_{i+1}$. Moreover because of the truncation the exact proof used is irrelevant. For this reason we conclude that if we have a map $\| \texttt{colim}_i(Y_i, \psi_i)\|_0 \to \| \texttt{colim}_i(Z_i, \epsilon_i)\|_0$ given by some $g_i : Y_i \to Z_i$ satisfying the appropriate relations then the map $\| \texttt{colim}_i(X_i, \phi_i)\|_0 \to \| \texttt{colim}_i(Z_i, \epsilon_i)\|_0$ given by $f_i \circ g_i$ is the same as we would get from composing the two maps $\| \texttt{colim}_i(X_i, \phi_i)\|_0 \to \| \texttt{colim}_i(Y, \psi_i)\|_0 \to \| \texttt{colim}_i(Z_i, \epsilon_i)\|_0$. Finally if the f_i are isomorphisms then the maps we get are isomorphisms.

Theorem 34. *$H_n(-)$ is a functor.*

Proof. First let's be clear about the functorial action. Since $\pi^s_{n+i}(- \wedge K_i)$ is a functor for each i it is easy to define an action on functions. We just need to check that the following commutes:

$$\begin{aligned}
&\texttt{susp}(X \wedge K_i) \circ \pi^s_{n+i+1}(\texttt{suspsm2}(X, K_i) \circ X \wedge \kappa_i) \circ \pi^s_{n+i+1}(f \wedge K_{i+1}) \\
&\quad = \pi^s_{n+i}(f \wedge K_i) \circ \texttt{susp}(Y \wedge K_i) \circ \pi^s_{n+i+1}(\texttt{suspsm2}(Y, K_i) \circ Y \wedge \kappa_i).
\end{aligned}$$

We use that $\texttt{susp}$ is natural and then cancel after which it suffices to show

$$\begin{aligned}
&\pi^s_{n+i+1}(\texttt{suspsm2}(X, K_i) \circ X \wedge \kappa_i) \circ \pi^s_{n+i+1}(f \wedge K_{i+1}) \\
&= \pi^s_{n+i+1}(\Sigma(f \wedge K_i)) \circ \pi^s_{n+i+1}(\texttt{suspsm2}(Y, K_I) \circ Y \wedge \kappa_i).
\end{aligned}$$

By functoriality it suffices to show

$$\mathtt{suspsm2}(X, K_i) \circ X \wedge \kappa_i \circ f \wedge K_{i+1} = (\Sigma(f \wedge K_i)) \circ \mathtt{suspsm2}(Y, K_i) \circ Y \wedge \kappa_i.$$

By Lemma 18

$$X \wedge \kappa_i \circ f \wedge K_{i+1} = f \wedge \Sigma K_i \circ Y \wedge \kappa_i,$$

and so it remains to show

$$\mathtt{suspsm2}(X, K_i) \circ f \wedge \Sigma K_i = (\Sigma(f \wedge K_i)) \circ \mathtt{suspsm2}(Y, K_i),$$

which again follows by the same techniques mentioned in Lemma 18. The fact that this preserves composition is immediate from the previous remark and the functoriality of $\pi^s_{n+i}(- \wedge K_i)$. □

Theorem 35. *$H_n(-)$ satisfies the suspension axiom.*

Proof. We use `susp2` on each component. By the above remark it suffices to show the following commutes:

$$\begin{aligned}
&\mathtt{susp}(n+i, X \wedge K_i) \circ \pi^s_{n+i+1}(\mathtt{suspsm2}(X, K_i) \circ X \wedge \kappa_i) \\
&\qquad\qquad \circ \mathtt{susp2}(n+1+i, X, K_{i+1}) \\
&= \mathtt{susp2}(n+i, X, K_i) \circ \mathtt{susp}(n+i+1, (\Sigma X) \wedge K_i) \circ \\
&\qquad\qquad \pi^s_{n+i+2}(\mathtt{suspsm2}(\Sigma X, K_i) \circ \Sigma X \wedge \kappa_i).
\end{aligned}$$

Expand the definition of `susp2` and cancel the first term. We get

$$\begin{aligned}
&\pi^s_{n+i+1}(\mathtt{suspsm2}(X, K_i) \circ X \wedge \kappa_i) \circ \mathtt{susp}(n+1+i, X \wedge K_{i+1}) \circ \\
&\qquad\qquad \pi_{n+i+2}(\mathtt{suspsm}(X, K_{i+1}) \\
&= \pi^s_{n+i+1}(\mathtt{suspsm}(X, K_i)) \circ \mathtt{susp}((n+i+1, (\Sigma X) \wedge K_i) \circ \\
&\qquad\qquad \pi^s_{n+i+2}(\mathtt{suspsm2}(\Sigma X, K_i) \circ \Sigma X \wedge \kappa_i)
\end{aligned}$$

We can then move the last term of the left hand side to the right and use functoriality of π_{n+i+2} to get $\pi^s_{n+i+2}(\mathtt{suspsm2}(\Sigma X, , K_i) \circ \Sigma X \wedge \kappa_i \circ \mathtt{suspsm}(X, K_{i+1})^{-1})$ on the right hand side. Then we use Lemma 18 to show

$$\begin{aligned}
&\mathtt{suspsm2}(\Sigma X, , K_i) \circ \Sigma X \wedge \kappa_i \circ \mathtt{suspsm}(X, K_{i+1})^{-1} \\
&= \Sigma\big(\mathtt{suspsm}(X, K_i)^{-1} \circ \mathtt{suspsm2}(X, K_i) \circ (X \wedge \kappa_i)\big).
\end{aligned}$$

So we can apply functoriality of Σ and naturality of `susp` to get

$$\pi^s_{n+i+1}(\mathtt{suspsm2}(X, K_i) \circ X \wedge \kappa_i) \circ \mathtt{susp}(n+1+i, X \wedge K_{i+1})$$

is equal to

$$\begin{aligned}
&\pi^s_{n+i+1}(\mathtt{suspsm}(X, K_i)) \circ \pi^s_{n+i+1}(\mathtt{suspsm}(X, K_i)^{-1} \\
&\circ \mathtt{suspsm2}(X, K_i) \circ (X \wedge \kappa_i)) \circ \mathtt{susp}(n+i+1, X \wedge K_{i+1}).
\end{aligned}$$

Finally we are done by simple canceling and using functoriality of π^s_{n+i+1}.

Naturality follows from the above remark and the fact that `susp2` is natural.

We conclude by noting exactness follows easily from exactness of $\pi^s_n(- \wedge K)$ and basic arguments about truncated colimits. So in conclusion we have: ☐

Corollary 36. *$H_n(-)$ is a homology theory.*

Remark 37. We get regular homology with coefficients in G by using the spectrum $K_i = K(G, i)$, the Eilenberg-Maclane spaces. These have been already defined in Homotopy type theory [8].

References

[1] Steve Awodey and Michael A Warren. Homotopy theoretic models of identity types. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 146, pages 45–55. Cambridge Univ Press, 2009.

[2] Guillaume Brunerie. *On the homotopy groups of spheres in homotopy type theory*. PhD thesis, Université de Nice Sophia Antipolis, 2016.

[3] Evan Cavallo. Synthetic cohomology in homotopy type theory. Master's thesis, Carnegie Mellon University, 2015.

[4] Kuen-Bang Hou Favonia, Eric Finster, Daniel R Licata, and Peter LeFanu Lumsdaine. A mechanization of the blakers-massey connectivity theorem in homotopy type theory. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 565–574. ACM, 2016.

[5] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.

[6] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. *Twenty-five years of constructive type theory (Venice, 1995)*, 36:83–111, 1998.

[7] Daniel R Licata and Guillaume Brunerie. A cubical approach to synthetic homotopy theory. In *Proceedings of the 2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 92–103. IEEE Computer Society, 2015.

[8] Daniel R Licata and Eric Finster. Eilenberg-maclane spaces in homotopy type theory. In *Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, page 66. ACM, 2014.

[9] Peter LeFanu Lumsdaine. Weak ω-categories from intensional type theory. In *International Conference on Typed Lambda Calculi and Applications*, pages 172–187. Springer, 2009.

[10] Jean-Pierre Serre. Homologie singuliere des espaces fibres. *Annals of Mathematics*, pages 425–505, 1951.

[11] Michael Shulman. Spectral sequences. `https://homotopytypetheory.org/2013/08/08/spectral-sequences/`, 2013.

[12] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

[13] Benno Van Den Berg and Richard Garner. Types are weak ω-groupoids. *Proceedings of the London Mathematical Society*, 102(2):370–394, 2011.

[14] Vladimir Voevodsky. Univalent foundations project (a modified version of an nsf grant application). 2010.

[15] Michael A Warren. *Homotopy theoretic aspects of constructive type theory*. PhD thesis, PhD thesis, Carnegie Mellon University, 2008.

[16] Floris Van Doorn. *On the Formalization of Higher Inductive Types and Synthetic Homotopy Theory*. Prospectus defense, Carnegie Mellon University, 2017.