# Dispersion of the Fibonacci and the Frolov point sets

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**Abstract**

It is proved that the Fibonacci and the Frolov point sets, which are known to be very good for numerical integration, have optimal rate of decay of dispersion with respect to the cardinality of sets. This implies that the Fibonacci and the Frolov point sets provide universal discretization of the uniform norm for natural collections of subspaces of the multivariate trigonometric polynomials. It is shown how the optimal upper bounds for dispersion can be derived from the upper bounds for a new characteristic – the smooth fixed volume discrepancy. It is proved that the Fibonacci point sets provide the universal discretization of all integral norms.

## 1 Introduction

The concept of *dispersion* of a point set is an important geometric characteristic of a point set. It was established in a recent paper [21] that the property of a point set to have the minimal in the sense of order dispersion is equivalent, in a certain sense, to the property of the set to provide universal discretization in the $L_\infty$ norm for natural collections of subspaces of the multivariate trigonometric polynomials. In this paper we study decay of dispersion of the Fibonacci and the Frolov point sets with respect to the cardinality of sets. We remind the definition of dispersion. Let $d \geq 2$ and $[0,1)^d$ be the $d$-dimensional unit cube. For $\mathbf{x}, \mathbf{y} \in [0,1)^d$ with $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$ we write $\mathbf{x} < \mathbf{y}$ if this inequality holds coordinate-wise. For

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$\mathbf{x} < \mathbf{y}$ we write $[\mathbf{x}, \mathbf{y})$ for the axis-parallel box $[x_1, y_1) \times \cdots \times [x_d, y_d)$ and define

$$\mathcal{B} := \{[\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in [0,1)^d, \mathbf{x} < \mathbf{y}\}.$$

For $n \geq 1$ let $T$ be a set of points in $[0,1)^d$ of cardinality $|T| = n$. The volume of the largest empty (from points of $T$) axis-parallel box, which can be inscribed in $[0,1)^d$, is called the dispersion of $T$:

$$\operatorname{disp}(T) := \sup_{B \in \mathcal{B}: B \cap T = \emptyset} vol(B).$$

An interesting extremal problem is to find (estimate) the minimal dispersion of point sets of fixed cardinality:

$$\operatorname{disp}^*(n, d) := \inf_{T \subset [0,1)^d, |T| = n} \operatorname{disp}(T).$$

It is known that

$$\operatorname{disp}^*(n, d) \leq C^*(d)/n. \tag{1.1}$$

Inequality (1.1) with $C^*(d) = 2^{d-1} \prod_{i=1}^{d-1} p_i$, where $p_i$ denotes the $i$th prime number, was proved in [6] (see also [10]). The authors of [6] used the Halton-Hammersly set of $n$ points (see [8]). Inequality (1.1) with $C^*(d) = 2^{7d+1}$ was proved in [1]. The authors of [1], following G. Larcher, used the $(t, r, d)$-nets (see [9] and [8] for results on $(t, r, d)$-nets).

In this paper we are interested in optimal behavior of dispersion with respect to the cardinality of sets. A trivial lower bound $\operatorname{disp}^*(n, d) \geq (n+1)^{-1}$ combined with (1.1) shows that the optimal rate of decay of dispersion with respect to cardinality $n$ of sets is $1/n$. In this paper we prove that the Fibonacci and the Frolov point sets have optimal in the sense of order rate of decay of dispersion. We present results on the Fibonacci point sets in Section 2 and results on the Frolov point sets in Section 4. In Section 5 we introduce a new concept of discrepancy – the smooth fixed volume discrepancy – and show how good upper bounds of it can be used for proving optimal (in the sense of order) upper bounds for dispersion. These are the main results of the paper. At the end of the paper, in Section 7 we give a comment on the universal discretization of the uniform norm. In Section 8 we prove that the Fibonacci point sets provide the universal discretization of all integral norms. The main technical result of the paper is Lemma 3.1. This lemma is used in the direct proof of the optimal rate of convergence of dispersion of the Frolov point sets (see Theorem 4.1). Moreover, Lemma 3.1 is used in the proof of the upper bounds for a more delicate quantity – the smooth fixed volume discrepancy (see Theorem 5.1). Theorem 5.1 implies Theorem 4.1. We have the same phenomenon for the Fibonacci point sets: Theorem 5.3 on the behavior of the smooth fixed volume discrepancy implies Theorem 2.1 on the behavior of dispersion. For further recent results on dispersion we refer the reader to papers [23], [12], [14] and references therein.

# 2 The Fibonacci point sets

Let $\{b_n\}_{n=0}^{\infty}$, $b_0 = b_1 = 1$, $b_n = b_{n-1} + b_{n-2}$, $n \geq 2$, – be the Fibonacci numbers. Denote the $n$th *Fibonacci point set* by

$$\mathcal{F}_n := \{(\mu/b_n, \{\mu b_{n-1}/b_n\}),\ \mu = 1, \dots, b_n\}.$$

In this definition $\{a\}$ is the fractional part of the number $a$. In this section we prove the following upper bound on the dispersion of the $\mathcal{F}_n$.

**Theorem 2.1.** *There is an absolute constant $C$ such that for all $n$ we have*

$$disp(\mathcal{F}_n) \leq C/b_n. \tag{2.1}$$

*Proof.* We prove bound (2.1) for the set $\mathcal{F}_{n,\pi} := \{2\pi\mathbf{x} : \mathbf{x} \in \mathcal{F}_n\}$. For the continuous functions of two variables, which are $2\pi$-periodic in each variable, we define cubature formulas

$$\Phi_n(f) := b_n^{-1} \sum_{\mu=1}^{b_n} f\big(2\pi\mu/b_n, 2\pi\{\mu b_{n-1}/b_n\}\big),$$

called the *Fibonacci cubature formulas.* Denote

$$\mathbf{y}^\mu := \big(2\pi\mu/b_n, 2\pi\{\mu b_{n-1}/b_n\}\big), \quad \mu = 1, \dots, b_n,$$

$$\Phi(\mathbf{k}) := b_n^{-1} \sum_{\mu=1}^{b_n} e^{i(\mathbf{k},\mathbf{y}^\mu)}.$$

Note that

$$\Phi_n(f) = \sum_{\mathbf{k}} \hat{f}(\mathbf{k})\Phi(\mathbf{k}), \quad \hat{f}(\mathbf{k}) := (2\pi)^{-2} \int_{\mathbb{T}^2} f(\mathbf{x}) e^{-i(\mathbf{k},\mathbf{x})} d\mathbf{x}, \tag{2.2}$$

where for the sake of simplicity we may assume that $f$ is a trigonometric polynomial. It is clear that (2.2) holds for $f$ with absolutely convergent Fourier series.

It is easy to see that the following relation holds

$$\Phi(\mathbf{k}) = \begin{cases} 1 & \text{for} \quad \mathbf{k} \in L(n) \\ 0 & \text{for} \quad \mathbf{k} \notin L(n), \end{cases} \tag{2.3}$$

where

$$L(n) := \left\{ \mathbf{k} = (k_1, k_2) : k_1 + b_{n-1} k_2 \equiv 0 \pmod{b_n} \right\}.$$

Denote $L(n)' := L(n) \setminus \{\mathbf{0}\}$. For $N \in \mathbb{N}$ define the *hyperbolic cross* in dimension $d$ as follows:

$$\Gamma(N) := \left\{ \mathbf{k} \in \mathbb{Z}^d : \prod_{j=1}^{d} \max(|k_j|, 1) \leq N \right\}.$$

The following lemma is well known (see, for instance, [16]).

**Lemma 2.1.** *There exists an absolute constant $\gamma > 0$ such that for any $n > 2$ for the 2-dimensional hyperbolic cross we have*

$$\Gamma(\gamma b_n) \cap \left( L(n) \setminus \{\mathbf{0}\} \right) = \emptyset.$$

For $u \in (0, 1]$ define the even $2\pi$-periodic hat function $h_u(t)$, $t \in [-\pi, \pi]$, as follows: $h_u(0) = 1$, $h_u(t) = 0$ for $t \in [u, \pi]$, linear on $[0, u]$. Then $h_u(t) = |1 - t/u|$ on $[-u, u]$ and equal to 0 on $[-\pi, \pi] \setminus (-u, u)$. Therefore,

$$\hat{h}_u(k) := (2\pi)^{-1} \int_{-\pi}^{\pi} h_u(t) e^{-ikt} dt = \pi^{-1} \int_0^u (1 - t/u) \cos(|k|t) dt.$$

From here, using the formula

$$\int_0^u (1 - t/u) \cos(|k|t) dt = \frac{1 - \cos(|k|u)}{k^2 u}, \quad k \neq 0,$$

we easily obtain the following bound for $k \neq 0$

$$|\hat{h}_u(k)| \leq \frac{C}{|k|} \min\left( |k|u, \frac{1}{|k|u} \right). \tag{2.4}$$

For $\mathbf{u} = (u_1, u_2)$, $\mathbf{x} = (x_1, x_2)$, consider

$$h_{\mathbf{u}}(\mathbf{x}) := h_{u_1}(x_1) h_{u_2}(x_2).$$

We now prove that for some large enough absolute constant $c > 0$ any rectangle $R$ of the form $R = (a_1, a_2) \times (v_1, v_2) \subset \mathbb{T}^2 := [0, 2\pi]^2$ with area $|R| = c/b_n$ contains at least one point from the set $\mathcal{F}_{n,\pi}$. Our proof goes by contradiction. Let $u_1$, $u_2$ be such that $u_1 u_2 = c_0/b_n$. We choose $c_0 > 0$ later. Take an $R \subset [0, 2\pi]^2$ and write it in the form $R = (x_1^0 - u_1, x_1^0 + u_1) \times (x_2^0 - u_2, x_2^0 + u_2)$. Assuming that $R$ does not contain any points from $\mathcal{F}_{n,\pi}$ we get $h_{\mathbf{u}}(y^\mu - \mathbf{x}^0) = 0$ for all $\mu = 1, \dots, b_n$. Then, clearly

$$E := (2\pi)^{-2} \int_{\mathbb{T}^2} h_{\mathbf{u}}(\mathbf{x} - \mathbf{x}^0) d\mathbf{x} - \Phi_n(h_{\mathbf{u}}(\mathbf{x} - \mathbf{x}^0))$$

$$= (2\pi)^{-2} \int_{\mathbb{T}^2} h_{\mathbf{u}}(\mathbf{x} - \mathbf{x}^0) d\mathbf{x} = (2\pi)^{-2} u_1 u_2 = \frac{c_0}{(2\pi)^2 b_n}. \tag{2.5}$$

To obtain a contradiction we estimate the above error $E$ of the Fibonacci cubature formula from above.

For $\mathbf{s} \in \mathbb{N}_0^d$ – the set of vectors with nonnegative integer coordinates, define

$$\rho(\mathbf{s}) := \{\mathbf{k} \in \mathbb{Z}^d : [2^{s_j - 1}] \le |k_j| < 2^{s_j}, \quad j = 1, \dots, d\}$$

where $[a]$ denotes the integer part of $a$.

By formulas (2.2) and (2.3) we obtain

$$E \le \sum_{\mathbf{k} \ne 0} |\hat{h}_{\mathbf{u}}(\mathbf{k})| \Phi(\mathbf{k}) = \sum_{v=1}^{\infty} \sum_{\|\mathbf{s}\|_1 = v} \sum_{\mathbf{k} \in \rho(\mathbf{s})} |\hat{h}_{\mathbf{u}}(\mathbf{k})| \Phi(\mathbf{k}). \tag{2.6}$$

Lemma 2.1 implies that if $v \ne 0$ is such that $2^v \le \gamma b_n$ then for $\mathbf{s}$ with $\|\mathbf{s}\|_1 = v$ we have $\rho(\mathbf{s}) \subset \Gamma(\gamma b_n)$ and $\Phi(\mathbf{k}) = 0$, $\mathbf{k} \in \rho(\mathbf{s})$. Let $v_0 \in \mathbb{N}$ be the smallest number satisfying $2^{v_0} > \gamma b_n$. Then we have

$$E \le \sum_{v=v_0}^{\infty} \sum_{\|\mathbf{s}\|_1 = v} \sum_{\mathbf{k} \in \rho(\mathbf{s})} |\hat{h}_{\mathbf{u}}(\mathbf{k})| \Phi(\mathbf{k}). \tag{2.7}$$

Lemma 2.1 implies that for $v \ge v_0$ we have

$$|\rho(\mathbf{s}) \cap L(n)| \le C_1 2^{v - v_0}, \quad \|\mathbf{s}\|_1 = v. \tag{2.8}$$

Relations (2.7), (2.8), and (2.4) imply

$$E \le C_2 2^{-v_0} \sum_{v=v_0}^{\infty} \sum_{\|\mathbf{s}\|_1=v} \min\left(2^{s_1}u_1, \frac{1}{2^{s_1}u_1}\right) \min\left(2^{s_2}u_2, \frac{1}{2^{s_2}u_2}\right). \tag{2.9}$$

We now need the following technical lemma.

**Lemma 2.2.** *Let $u_1$, $u_2$ be such that $u_1u_2 \ge 2^{-v}$, $v \in \mathbb{N}$. Then we have*

$$\sigma(v) := \sum_{\|\mathbf{s}\|_1=v} \min\left(2^{s_1}u_1, \frac{1}{2^{s_1}u_1}\right) \min\left(2^{s_2}u_2, \frac{1}{2^{s_2}u_2}\right) \le C_3 \frac{\log(2^{v+1}u_1u_2)}{2^v u_1u_2}.$$

*Proof.* Our condition $u_1u_2 \ge 2^{-v}$ guarantees that we have $2^v u_1u_2 \ge 1$. We split $\sigma(v) = \sigma_1(v) + \sigma_2(v) + \sigma_3(v)$ into three sums with summation over $s_1$ from the following three sets

$$S_1 := \{s_1 \ge 0 : 2^{s_1} \le 1/u_1\},$$

$$S_2 := \{s_1 : 1/u_1 < 2^{s_1} \le 2^v u_2\},$$

$$S_3 := \{s_1 \le v : 2^{s_1} > 2^v u_2\}.$$

We now estimate separately the $\sigma_i(v)$, $i = 1, 2, 3$. Using the inequality $2^v u_1u_2 \ge 1$ mentioned above, we see that for $s_1 \in S_1$ we have $2^{s_2}u_2 \ge 1$ and therefore

$$\sigma_1(v) = \sum_{s_1 \in S_1} 2^{s_1}u_1 \frac{1}{2^{v-s_1}u_2} = 2^{-v}\frac{u_1}{u_2} \sum_{s_1 \in S_1} 2^{2s_1} \le \frac{4}{3}(2^v u_1u_2)^{-1}. \tag{2.10}$$

In the same way, replacing the role of $s_1$, $u_1$ by $s_2$ and $u_2$ we obtain

$$\sigma_3(v) \le \frac{4}{3}(2^v u_1u_2)^{-1}. \tag{2.11}$$

Finally, for $\sigma_2(v)$ we have

$$\sigma_2(v) = \sum_{s_1 \in S_2} \frac{1}{2^{s_1}u_1} \frac{1}{2^{v-s_1}u_2} = (2^v u_1u_2)^{-1}|S_2|$$

$$\le (2^v u_1u_2)^{-1}(1 + \log(2^v u_1u_2)). \tag{2.12}$$

Combining inequalities (2.10)–(2.12) we complete the proof of Lemma 2.2. $\square$

We now complete the proof of Theorem 2.1. Assume that $u_1u_2 = c_0/b_n$, $c_0 \geq 2$. Then the relation $2^{v_0} \asymp b_n$, relation (2.9) and Lemma 2.2 imply

$$E \leq C_4 \frac{\log c_0}{c_0 b_n}.$$

Obviously, this contradicts (2.5) for large enough $c_0$.

Theorem 2.1 is now proved.

$\square$

# 3 Technical lemmas

In Section 2 we discussed the two-dimensional case. In the next Sections 4 and 5 we discuss the general $d$-dimensional case. There we need a generalization of the two-dimensional Lemma 2.2. This section deals with such a generalization. It is somewhat technically involved. We begin with some notations, which are used here. For $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{R}^d_+$ we denote $\mathbf{u}^d := (u_1, \dots, u_{d-1}) \in \mathbb{R}^{d-1}_+$. It is convenient for us to use the following notation for products

$$pr(\mathbf{u}, \nu) := u_1 \cdots u_\nu, \quad \nu \leq d.$$

Thus, for instance, $pr(\mathbf{u}, d) = pr(\mathbf{u}, d-1)u_d$.

We are interested in the behavior of special sums

$$\sigma(v, \mathbf{u}) := \sum_{\|\mathbf{s}\|_1 = v} \prod_{j=1}^{d} \min\left(2^{s_j}u_j, \frac{1}{2^{s_j}u_j}\right), \quad v \in \mathbb{N}_0.$$

Clearly, for $d \geq 3$ we have

$$\sigma(v, \mathbf{u}) = \sum_{s_d=0}^{v} \min\left(2^{s_d}u_d, \frac{1}{2^{s_d}u_d}\right) \sigma(v - s_d, \mathbf{u}^d). \tag{3.1}$$

The main result of this section is the following lemma.

**Lemma 3.1.** *Let $v \in \mathbb{N}_0$ and $\mathbf{u} \in \mathbb{R}^d_+$. Then we have the following inequalities.*

*(I) Under condition $2^v pr(\mathbf{u},d) \geq 1$ we have*

$$\sigma(v,\mathbf{u}) \leq C(d)\frac{(\log(2^{v+1}pr(\mathbf{u},d)))^{d-1}}{2^v pr(\mathbf{u},d)}. \tag{3.2}$$

*(II) Under condition $2^v pr(\mathbf{u},d) \leq 1$ we have*

$$\sigma(v,\mathbf{u}) \leq C(d)2^v pr(\mathbf{u},d)\left(\log\frac{2}{2^v pr(\mathbf{u},d)}\right)^{d-1}. \tag{3.3}$$

*Proof.* Our proof goes by induction on $d$. First, we establish Lemma 3.1 for $d=2$. Inequality (3.2) follows directly from Lemma 2.2. We now prove inequality (3.3). As in the proof of Lemma 2.2 we split the sum $\sigma(v,\mathbf{u})$ into three sums respectively over the index sets

$$S_1' := \{s_1 \geq 0 : 2^{s_1} \leq 2^v u_2\},$$

$$S_2' := \{s_1 : 2^v u_2 < 2^{s_1} < 1/u_1\}.$$

Note that our condition $2^v pr(\mathbf{u},2) = 2^v u_1 u_2 \leq 1$ implies $2^v u_2 \leq 1/u_1$.

$$S_3' := \{s_1 \leq v : 2^{s_1} \geq 1/u_1\}.$$

We now estimate the corresponding $\sigma_i'$, $i=1,2,3$ separately. For the first sum we have

$$\sigma_1' = \sum_{s_1\in S_1'} 2^{s_1}u_1\frac{1}{2^{v-s_1}u_2} = 2^{-v}\frac{u_1}{u_2}\sum_{s_1\in S_1'}2^{2s_1}$$
$$\leq \frac{4}{3}2^{-v}\frac{u_1}{u_2}(2^v u_2)^2 = \frac{4}{3}2^v u_1 u_2.$$

For the second sum we have

$$\sigma_2' = \sum_{s_1\in S_2'} 2^{s_1}u_1 2^{v-s_1}u_2 = 2^v u_1 u_2 |S_2'| \leq 2^v u_1 u_2\left(1+\log\frac{1}{2^v u_1 u_2}\right).$$

The third sum $S_3'$ is similar to the sum $S_1'$. In $S_1'$ we have a condition, that can be rewritten as $2^{s_2}u_2 \geq 1$ and in $S_3'$ we have the condition $2^{s_1}u_1 \geq 1$. Thus, for the third sum we have

$$\sigma_3' \leq \frac{4}{3}2^v u_1 u_2.$$

Summing up the above bounds for $\sigma_i'$, $i = 1, 2, 3$, we obtain (3.3) in case $d = 2$, which completes the proof of Lemma 3.1 in case $d = 2$.

We now proceed to the induction step from $d-1$ to $d$. Suppose Lemma 3.1 holds for $d-1 \geq 2$. We derive from here Lemma 3.1 for $d$. We begin with the case (I), i.e. assume that inequality $2^v pr(\mathbf{u}, d) \geq 1$ holds. We use identity (3.1) and Lemma 3.1 for $d-1$. We split the sum $\sigma(v, \mathbf{u})$ into three sums over the following index sets

$$U_1 := \{s_d \geq 0 : 2^{s_d} \leq 1/u_d\},$$

$$U_2 := \{s_d : 1/u_d < 2^{s_d} < 2^v pr(\mathbf{u}, d-1)\}.$$

Note that our assumption $2^v pr(\mathbf{u}, d) \geq 1$ guarantees $1/u_d \leq 2^v pr(\mathbf{u}, d-1)$.

$$U_3 := \{s_d \leq v : 2^{s_d} \geq 2^v pr(\mathbf{u}, d-1)\}.$$

Then, for the first sum we have

$$\sigma_1(v, \mathbf{u}) = \sum_{s_d \in U_1} 2^{s_d} u_d \sigma(v - s_d, \mathbf{u}^d).$$

For $s_d \in U_1$ we have $2^{s_d} u_d \leq 1$, which combined with the condition (I) inequality

$$2^{v-s_d} pr(\mathbf{u}, d-1) 2^{s_d} u_d = 2^v pr(\mathbf{u}, d) \geq 1$$

implies $2^{v-s_d} pr(\mathbf{u}, d-1) \geq 1$. Thus, applying inequality (3.2) of Lemma 3.1 for $d-1$ we get (for convenience, here and later we write $\alpha \ll \beta$ instead of $\alpha \leq C(d)\beta$)

$$\sigma_1(v, \mathbf{u}) \ll \sum_{s_d \in U_1} 2^{s_d} u_d \frac{(\log(2^{v+1-s_d} pr(\mathbf{u}, d-1)))^{d-2}}{2^{v-s_d} pr(\mathbf{u}, d-1)}$$

$$= \frac{u_d}{2^v pr(\mathbf{u}, d-1)} \sum_{s_d \in U_1} 2^{2s_d} (\log(2^{v+1-s_d} pr(\mathbf{u}, d-1)))^{d-2}. \tag{3.4}$$

Here we need the following simple technical lemma, which we formulate without proof.

**Lemma 3.2.** *Let two numbers $A \geq 1$ and $B \geq A$ be given. Then for $\nu \in \mathbb{N}$ we have*

$$\sum_{k: 2^k \leq A} 2^{2k} \left(\log \frac{2B}{2^k}\right)^\nu \leq C(\nu) A^2 \left(\log \frac{2B}{A}\right)^\nu.$$

Using Lemma 3.2 we continue relation (3.4)

$$\ll \frac{u_d}{2^v pr(\mathbf{u}, d-1)} \left(\frac{1}{u_d}\right)^2 \log(2^{v+1} pr(\mathbf{u}, d))^{d-2}$$

$$= \frac{1}{2^v pr(\mathbf{u}, d)} \log(2^{v+1} pr(\mathbf{u}, d))^{d-2}.$$

Next, for the second sum we have

$$\sigma_2(v, \mathbf{u}) = \sum_{s_d \in U_2} \frac{1}{2^{s_d} u_d} \sigma(v - s_d, \mathbf{u}^d).$$

For $s_d \in U_2$ we have $2^{s_d} < 2^v pr(\mathbf{u}, d-1)$, which implies $2^{v-s_d} pr(\mathbf{u}, d-1) > 1$. Therefore, we continue, using the first inequality of Lemma 3.1 for $d-1$.

$$\sigma_2(v, \mathbf{u}) \ll \sum_{s_d \in U_2} \frac{1}{2^{s_d} u_d} \frac{(\log(2^{v+1-s_d} pr(\mathbf{u}, d-1))^{d-2}}{2^{v-s_d} pr(\mathbf{u}, d-1)}$$

$$= \frac{1}{2^v pr(\mathbf{u}, d)} \sum_{s_d \in U_2} (\log(2^{v+1-s_d} pr(\mathbf{u}, d-1))^{d-2}$$

$$\leq \frac{1}{2^v pr(\mathbf{u}, d)} (\log(2^{v+1} pr(\mathbf{u}, d))^{d-2} |U_2| \leq \frac{\log(2^{v+1} pr(\mathbf{u}, d))^{d-1}}{2^v pr(\mathbf{u}, d)}.$$

Finally, for the third sum we have

$$\sigma_3(v, \mathbf{u}) = \sum_{s_d \in U_3} \frac{1}{2^{s_d} u_d} \sigma(v - s_d, \mathbf{u}^d).$$

For $s_d \in U_3$ we have $2^{s_d} \geq 2^v pr(\mathbf{u}, d-1)$, which is the same as $2^{v-s_d} pr(\mathbf{u}, d-1) \leq 1$. Applying inequality (3.3) of Lemma 3.1 for $d-1$ we obtain

$$\sigma_3(v, \mathbf{u}) \ll \sum_{s_d \in U_3} \frac{1}{2^{s_d} u_d} 2^{v-s_d} pr(\mathbf{u}, d-1) \left(\log \frac{2}{2^{v-s_d} pr(\mathbf{u}, d-1)}\right)^{d-2}$$

$$= \frac{2^v pr(\mathbf{u}, d-1)}{u_d} \sum_{s_d \in U_3} 2^{-2s_d} \left(\log \frac{2^{s_d+1}}{2^v pr(\mathbf{u}, d-1)}\right)^{d-2}. \tag{3.5}$$

Here we need the following simple technical lemma, which we formulate without proof.

**Lemma 3.3.** *Let two numbers $A \ge 1$ and $0 < B \le A$ be given. Then for $\nu \in \mathbb{N}$ we have*

$$\sum_{k:2^k \ge A} 2^{-2k} \left(\log \frac{2^{k+1}}{B}\right)^{\nu} \le C(\nu) A^{-2} \left(\log \frac{2A}{B}\right)^{\nu}.$$

Using Lemma 3.3 we continue relation (3.5)

$$\ll \frac{2^v pr(\mathbf{u}, d-1)}{u_d} \frac{1}{(2^v pr(\mathbf{u}, d-1))^2} = \frac{1}{2^v pr(\mathbf{u}, d)}.$$

Combining the above inequalities for all three sums $\sigma_i(v, \mathbf{u})$ we complete the proof of Lemma 3.1 in the first case (I).

We now proceed to the second case (II). In this case we split the summation over $s_d$ into three index sets:

$$U_1' := \{s_d \ge 0 : 2^{s_d} \le 2^v pr(\mathbf{u}, d-1)\},$$
$$U_2' := \{s_d : 2^v pr(\mathbf{u}, d-1) < 2^{s_d} < 1/u_d\},$$
$$U_3' := \{s_d \le v : 2^{s_d} \ge 1/u_d\}.$$

We now estimate the corresponding sums separately. For $s_d \in U_1'$ we have $2^{s_d} u_d \le 2^v pr(\mathbf{u}, d) \le 1$ by the condition for case (II). Also, for $s_d \in U_1'$ we have $2^{v-s_d} pr(\mathbf{u}, d-1) \ge 1$. Therefore, by (3.2) of Lemma 3.1 for $d-1$ we get

$$\sigma_1'(v, \mathbf{u}) \ll \sum_{s_d \in U_1'} 2^{s_d} u_d \frac{\left(\log(2^{v+1-s_d} pr(\mathbf{u}, d-1))\right)^{d-2}}{2^{v-s_d} pr(\mathbf{u}, d-1)}$$
$$= \frac{u_d}{2^v pr(\mathbf{u}, d-1)} \sum_{s_d \in U_1'} 2^{2s_d} \left(\log(2^{v+1-s_d} pr(\mathbf{u}, d-1))\right)^{d-2}. \tag{3.6}$$

Using Lemma 3.2 we continue relation (3.6)

$$\ll \frac{u_d}{2^v pr(\mathbf{u}, d-1)} (2^v pr(\mathbf{u}, d-1))^2 = 2^v pr(\mathbf{u}, d).$$

For $s_d \in U_2'$ we have $2^{v-s_d} pr(\mathbf{u}, d-1) < 1$. Therefore, by (3.3) of Lemma 3.1 for $d-1$ we obtain

$$\sigma_2'(v, \mathbf{u}) \ll \sum_{s_d \in U_2'} 2^{s_d} u_d 2^{v-s_d} pr(\mathbf{u}, d-1) \left(\log \frac{2}{2^{v-s_d} pr(\mathbf{u}, d-1)}\right)^{d-2}$$

$$= 2^v pr(\mathbf{u},d) \sum_{s_d \in U_2'} \left(\log \frac{2}{2^{v-s_d} pr(\mathbf{u},d-1)}\right)^{d-2}$$

$$\le 2^v pr(\mathbf{u},d) \left(\log \frac{2}{2^v pr(\mathbf{u},d)}\right)^{d-2} |U_2'| \le 2^v pr(\mathbf{u},d) \left(\log \frac{2}{2^v pr(\mathbf{u},d)}\right)^{d-1}.$$

Finally, for $s_d \in U_3'$ the case (II) assumption $2^v pr(\mathbf{u},d) \le 1$ implies $2^{v-s_d} pr(\mathbf{u},d-1) \le 1$. Thus, using inequality (3.3) from Lemma 3.1 for $d-1$ we get

$$\sigma_3'(v,\mathbf{u}) \ll \sum_{s_d \in U_3'} \frac{1}{2^{s_d} u_d} 2^{v-s_d} pr(\mathbf{u},d-1) \left(\log \frac{2}{2^{v-s_d} pr(\mathbf{u},d-1)}\right)^{d-2}$$

$$= \frac{2^v pr(\mathbf{u},d-1)}{u_d} \sum_{s_d \in U_3'} 2^{-2s_d} \left(\log \frac{2}{2^{v-s_d} pr(\mathbf{u},d-1)}\right)^{d-2}.$$

Using Lemma 3.3 we continue

$$\ll \frac{2^v pr(\mathbf{u},d-1)}{u_d} \left(\frac{1}{u_d}\right)^{-2} \left(\log \frac{2}{2^v pr(\mathbf{u},d)}\right)^{d-2}$$

$$= 2^v pr(\mathbf{u},d) \left(\log \frac{2}{2^v pr(\mathbf{u},d)}\right)^{d-2}.$$

Combining the above inequalities for all three sums $\sigma_i'(v,\mathbf{u})$ we complete the proof of Lemma 3.1 in the second case (II) and complete the proof of Lemma 3.1.

$\square$

**Remark 3.1.** *It is easy to check that the above proof of Lemma 3.1 allows us to obtain the following bound on the constant $C(d) \le C^d$ with an absolute constant $C$.*

# 4 The Frolov point sets

In this section we study dispersion of point sets, which are known to be very good for numerical integration, – the *Frolov point sets*. We refer the reader for detailed presentation of the theory of the Frolov cubature formulas to [16], [17], [22], and [5]. We begin with a description of the Frolov point sets. The following lemma plays a fundamental role in the construction of such point sets (see [16] for its proof).

**Lemma 4.1.** *There exists a matrix $A$ such that the lattice $L(\mathbf{m}) = A\mathbf{m}$*

$$L(\mathbf{m}) = \begin{bmatrix} L_1(\mathbf{m}) \\ \vdots \\ L_d(\mathbf{m}) \end{bmatrix},$$

*where $\mathbf{m}$ is a (column) vector with integer coordinates, has the following properties*

$1^0$. $\left|\prod_{j=1}^d L_j(\mathbf{m})\right| \geq 1$ *for all* $\mathbf{m} \neq \mathbf{0}$*;*

$2^0$ *each parallelepiped $P$ with volume $|P|$ whose edges are parallel to the coordinate axes contains no more than $|P|+1$ lattice points.*

Let $a > 1$ and $A$ be the matrix from Lemma 4.1. We consider the cubature formula

$$\Phi(a, A)(f) := \left(a^d |\det A|\right)^{-1} \sum_{\mathbf{m} \in \mathbb{Z}^d} f\left(\frac{(A^{-1})^T \mathbf{m}}{a}\right)$$

for $f$ with compact support.

We call the *Frolov point set* the following set associated with the matrix $A$ and parameter $a$

$$\mathcal{F}(a, A) := \left\{\left(\frac{(A^{-1})^T \mathbf{m}}{a}\right)\right\}_{\mathbf{m} \in \mathbb{Z}^d} \cap [0,1]^d =: \{z^\mu\}_{\mu=1}^N.$$

Clearly, the number $N = |\mathcal{F}(a, A)|$ of points of this set does not exceed $C(A)a^d$.

The main result of this section is the following theorem.

**Theorem 4.1.** *Let $A$ be a matrix from Lemma 4.1. There is a constant $C(d, A)$, which may only depend on $A$ and $d$, such that for all $a$ we have*

$$disp(\mathcal{F}(a, A)) \leq C(A, d)a^{-d}. \tag{4.1}$$

*Proof.* The idea of the proof of this theorem is the same as of the proof of Theorem 2.1. For $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{R}_+^d$, $\mathbf{x} = (x_1, \dots, x_d)$, consider

$$h_{\mathbf{u}}(\mathbf{x}) := \prod_{j=1}^d h_{u_j}(x_j), \tag{4.2}$$

where $h_u(t) = |1 - t/u|$ on $[-u, u]$ and equal to 0 for $|t| \geq u$. We now prove that for some large enough constant $C(A, d) > 0$ any $d$-dimensional box $B$ of the form $B = \prod_{j=1}^d (a_1^j, a_2^j) \subset [0, 1]^d$ with area $|B| = C(A, d)a^{-d}$ contains at least one point from the set $\mathcal{F}(a, A)$. Our proof goes by contradiction. Let $\mathbf{u}$ be such that $pr(\mathbf{u}, d) = c_0 a^{-d}$. We choose $c_0 > 0$ later. Take a $B \subset [0, 1]^d$ and write it in the form $B = \prod_{j=1}^d (x_j^0 - u_j, x_j^0 + u_j)$. Assuming that $B$ does not contain any points from $\mathcal{F}(a, A)$ we get $h_{\mathbf{u}}(z^\mu - \mathbf{x}^0) = 0$ for all $\mu = 1, \dots, N$. Then, clearly

$$e := \int_{[0,1]^d} h_{\mathbf{u}}(\mathbf{x} - \mathbf{x}^0) d\mathbf{x} - \Phi(a, A)(h_{\mathbf{u}}(\mathbf{x} - \mathbf{x}^0))$$

$$= \int_{[0,1]^d} h_{\mathbf{u}}(\mathbf{x} - \mathbf{x}^0) d\mathbf{x} = pr(\mathbf{u}, d) = c_0 a^{-d}. \tag{4.3}$$

To obtain a contradiction we estimate the above error $e$ of the Frolov cubature formula from above. Denote for $f \in L_1(\mathbb{R}^d)$

$$\hat{f}(\mathbf{y}) := \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-2\pi i (\mathbf{y}, \mathbf{x})} d\mathbf{x}.$$

For a function $f$ with finite support and absolutely convergent series $\sum_{\mathbf{m} \in \mathbb{Z}^d} \hat{f}(aA\mathbf{m})$ we have for the error of the Frolov cubature formula (see [16])

$$\Phi(a, A)(f) - \hat{f}(\mathbf{0}) = \sum_{\mathbf{m} \neq \mathbf{0}} \hat{f}(aA\mathbf{m}). \tag{4.4}$$

The proof of this formula is based on the Poisson formula, which we formulate in the form convenient for us (see [16] for the proof).

**Lemma 4.2.** *Let $f(\mathbf{x})$ be continuous and have compact support and the series $\sum_{\mathbf{k} \in \mathbb{Z}^d} \hat{f}(\mathbf{k})$ converges. Then*

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \hat{f}(\mathbf{k}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}).$$

By (4.4) we obtain

$$e \leq \sum_{\mathbf{m} \neq 0} |\hat{h}_{\mathbf{u}}(aA\mathbf{m})| = \sum_{v=1}^{\infty} \sum_{\|\mathbf{s}\|_1 = v} \sum_{\mathbf{m}: aA\mathbf{m} \in \rho(\mathbf{s})} |\hat{h}_{\mathbf{u}}(aA\mathbf{m})|. \tag{4.5}$$

Lemma 4.1 implies that if $v \neq 0$ is such that $2^v < a^d$ then for $\mathbf{s}$ with $\|\mathbf{s}\|_1 = v$ there is no $\mathbf{m}$ such that $aA\mathbf{m} \in \rho(\mathbf{s})$. Let $v_0 \in \mathbb{N}$ be the smallest number satisfying $2^{v_0} \geq a^d$. Then we have

$$e \leq \sum_{v=v_0}^{\infty} \sum_{\|\mathbf{s}\|_1=v} \sum_{\mathbf{m}:aA\mathbf{m}\in\rho(\mathbf{s})} |\hat{h}_{\mathbf{u}}(aA\mathbf{m})|. \tag{4.6}$$

Lemma 4.1 implies that for $v \geq v_0$ we have

$$|\rho(\mathbf{s}) \cap \{aA\mathbf{m}\}_{\mathbf{m}\in\mathbb{Z}^d}| \leq C_1 2^{v-v_0}, \quad \|\mathbf{s}\|_1 = v. \tag{4.7}$$

Relations (4.6), (4.7), and (2.4) imply

$$e \leq C_2 2^{-v_0} \sum_{v=v_0}^{\infty} \sum_{\|\mathbf{s}\|_1=v} \prod_{j=1}^{d} \min\left(2^{s_j}u_j, \frac{1}{2^{s_j}u_j}\right). \tag{4.8}$$

We now assume that $c_0 \geq 2$ and $c_0 a^{-d} \geq 2^{-v_0}$. Using inequality (3.2) of Lemma 3.1 and an analog of Lemma 3.3 we obtain from here

$$e \leq C(d,A) 2^{-v_0} \frac{(\log c_0)^{d-1}}{c_0},$$

which is in contradiction with (4.3) for large enough $c_0$.

$\square$

**Remark 4.1.** *The above proof of Theorem 4.1 and Remark 3.1 allow us to obtain the following bound on the constant $C(A,d) \leq d^{c(A)d}$ with $c(A)$ depending only on $A$.*

**Remark 4.2.** *Right after the first version of this paper, which contained Theorem 4.1, has been published in arXiv Mario Ullrich informed me that he has an unpublished note, where he obtained a bound similar to (4.1). His argument is based on different ideas. It certainly does not apply to the study of the smooth fixed volume discrepancy (see Section 5 below).*

# 5 A remark on smooth discrepancy

We begin with a classical definition of discrepancy ("star discrepancy", $L_\infty$-discrepancy) of a point set $T := \Xi_m := \{\xi^\mu\}_{\mu=1}^m \subset [0,1)^d$. Introduce a class of special $d$-variate characteristic functions

$$\chi^d := \{\chi_{[\mathbf{0},\mathbf{b})}(\mathbf{x}) := \prod_{j=1}^{d} \chi_{[0,b_j)}(x_j), \quad b_j \in [0,1), \quad j = 1, \dots, d\}$$

where $\chi_{[a,b)}(x)$ is a univariate characteristic function of the interval $[a,b)$. The classical definition of discrepancy of a set $T$ of points $\{\xi^1, \dots, \xi^m\} \subset [0,1)^d$ is as follows

$$D(T,m,d)_\infty := \max_{\mathbf{b} \in [0,1)^d} \left| \prod_{j=1}^{d} b_j - \frac{1}{m} \sum_{\mu=1}^{m} \chi_{[\mathbf{0},\mathbf{b})}(\xi^\mu) \right|.$$

It is equivalent within multiplicative constants, which may only depend on $d$, to the following definition

$$D^1(T) := \sup_{B \in \mathcal{B}} \left| vol(B) - \frac{1}{m} \sum_{\mu=1}^{m} \chi_B(\xi^\mu) \right|, \tag{5.1}$$

where for $B = [\mathbf{a},\mathbf{b}) \in \mathcal{B}$ we denote $\chi_B(\mathbf{x}) := \prod_{j=1}^{d} \chi_{[a_j,b_j)}(x_j)$. We use here definition (5.1) because it is more in a spirit of the definition of dispersion. Moreover, we consider the following optimized version of $D^1(T)$

$$D^{1,o}(T) := \inf_{w_1,\dots,w_m} \sup_{B \in \mathcal{B}} \left| vol(B) - \sum_{\mu=1}^{m} w_\mu \chi_B(\xi^\mu) \right|. \tag{5.2}$$

We now modify definitions (5.1) and (5.2), replacing the characteristic function $\chi_B$ by a smoother hat function $h_B$. Let $B \in \mathcal{B}$ be written in the form

$$B = \prod_{j=1}^{d} [x_j^0 - u_j, x_j^0 + u_j).$$

Then we define

$$h_B(\mathbf{x}) := pr(\mathbf{u},d) h_{\mathbf{u}}(\mathbf{x} - \mathbf{x}^0),$$

where $h_{\mathbf{u}}(\mathbf{x})$ is defined in (4.2).

The 2-smooth discrepancy is now defined as

$$D^2(T) := \sup_{B \in \mathcal{B}} \left| \int h_B(\mathbf{x}) d\mathbf{x} - \frac{1}{m} \sum_{\mu=1}^{m} h_B(\xi^\mu) \right| \tag{5.3}$$

and its optimized version as

$$D^{2,o}(T) := \inf_{w_1,\dots,w_m} \sup_{B\in\mathcal{B}} \left| \int h_B(\mathbf{x})d\mathbf{x} - \sum_{\mu=1}^{m} w_\mu h_B(\xi^\mu) \right|. \tag{5.4}$$

Note that the known concept of $r$-discrepancy with $r = 2$ (see, for instance, [16] and [17]) is close to the above concepts of 2-smooth discrepancy.

Along with $D^2(T)$ and $D^{2,o}(T)$ we consider a more refined quantity – 2-*smooth fixed volume discrepancy* – defined as follows

$$D^2(T, V) := \sup_{B\in\mathcal{B}:vol(B)=V} \left| \int h_B(\mathbf{x})d\mathbf{x} - \frac{1}{m}\sum_{\mu=1}^{m} h_B(\xi^\mu) \right|; \tag{5.5}$$

$$D^{2,o}(T, V) := \inf_{w_1,\dots,w_m} \sup_{B\in\mathcal{B}:vol(B)=V} \left| \int h_B(\mathbf{x})d\mathbf{x} - \sum_{\mu=1}^{m} w_\mu h_B(\xi^\mu) \right|. \tag{5.6}$$

Clearly,

$$D^2(T) = \sup_{V\in(0,1]} D^2(T, V).$$

**The Frolov point sets.** The main result of this section is the following Theorem 5.1 on the Frolov point set $T = \mathcal{F}(a, A)$.

**Theorem 5.1.** *There exists a constant $c(d, A) > 0$ such that for any $V \geq V_0 := c(d, A)a^{-d}$ we have*

$$D^{2,o}(\mathcal{F}(a, A), V) \leq C(d, A)a^{-2d}(\log(2V/V_0))^{d-1}. \tag{5.7}$$

In the definition (5.6) of the quantity $D^{2,o}(T, V)$ we optimize over weights $w_1, \dots, w_m$, when $V$ is fixed. Therefore, the optimal weights may depend on parameter $V$. We prove a somewhat stronger version of Theorem 5.1 where the weights, which provide the bound (5.7), do not depend on $V$. We formulate it as a theorem.

**Theorem 5.2.** *There exists a constant $c(d, A) > 0$ such that for any $V \geq V_0 := c(d, A)a^{-d}$ we have for all $B \in \mathcal{B}$, $vol(B) = V$,*

$$|\Phi(a, A)(h_B) - \hat{h}_B(\mathbf{0})| \leq C(d, A)a^{-2d}(\log(2V/V_0))^{d-1}. \tag{5.8}$$

*Proof.* By (4.6) we have for the error (with $M := a^d|\det A|$)

$$\delta_B := \left|\int h_B(\mathbf{x})d\mathbf{x} - \frac{1}{M}\sum_{\mu=1}^{N} h_B(z^\mu)\right| \le \sum_{v=v_0}^{\infty}\sum_{\|\mathbf{s}\|_1=v}\sum_{\mathbf{m}:aA\mathbf{m}\in\rho(\mathbf{s})} |\hat{h}_B(aA\mathbf{m})|.$$

Using (4.7) we obtain by (2.4)

$$\delta_B \ll \sum_{v=v_0}^{\infty}\sum_{\|\mathbf{s}\|_1=v} 2^{v-v_0} pr(\mathbf{u},d)2^{-v}\prod_{j=1}^{d}\min\left(2^{s_j}u_j, \frac{1}{2^{s_j}u_j}\right).$$

We now assume that the constant $c(d,A)$ is such that $V_0 = 2^{d-v_0}$. Then for $B \in \mathcal{B}$ such that $vol(B) \ge V_0$ we have $pr(\mathbf{u},d) \ge 2^{-v_0}$. Using inequality (3.2) of Lemma 3.1 and an analog of Lemma 3.3 we obtain from here

$$\delta_B \ll 2^{-v_0}\sum_{v=v_0}^{\infty} 2^{-v}\left(\log\left(2^{v+1}pr(\mathbf{u},d)\right)\right)^{d-1}$$

$$\ll 2^{-2v_0}\left(\log\left(2V/V_0\right)\right)^{d-1} \le a^{-2d}\left(\log\left(2V/V_0\right)\right)^{d-1}.$$

□

We now make comments on the relation between discrepancy and dispersion. It is obvious from (5.1) that

$$\mathrm{disp}(T) \le D^1(T). \tag{5.9}$$

The best known upper bounds for discrepancy $D^1(T)$ for sets of cardinality $m$ are of the form $D^1(T) \ll m^{-1}(\log m)^{d-1}$. Also, the classical result of Roth [11] gives the lower bound $D^1(T) \gg m^{-1}(\log m)^{(d-1)/2}$. There are very interesting improvements of the above lower bound (see [13], [2], [3]), which we do not discuss here. Therefore, inequality (5.9) can give us the bound $\mathrm{disp}(T) \ll m^{-1}(\log m)^{d-1}$ and, for sure, we cannot get the bound $\mathrm{disp}(T) \ll m^{-1}$ on this way.

Relations

$$vol(B) = 2^d pr(\mathbf{u},d) \quad \text{and} \quad \int h_B(\mathbf{x})d\mathbf{x} = pr(\mathbf{u},d)^2 \tag{5.10}$$

imply that

$$\mathrm{disp}(T) \le 2^d(D^2(T))^{1/2}. \tag{5.11}$$

This inequality is better than (5.9) but still cannot give us the desired bound $\mathrm{disp}(T) \ll m^{-1}$. Thus, the step from discrepancy to smooth discrepancy does not solve the problem. It turns out that the critical step here is to the smooth fixed volume discrepancy. It is clear that

$$\mathrm{disp}(T) =: V \leq 2^d (D^{2,o}(T,V))^{1/2}. \tag{5.12}$$

Inequality (5.12) applied to $T = \mathcal{F}(a,A)$ combined with Theorem 5.1 gives for $V := \mathrm{disp}(T)$

$$\text{either} \quad V \leq V_0 \quad \text{or} \quad V \ll (\log(2V/V_0))^{(d-1)/2} V_0,$$

which implies

$$\mathrm{disp}(\mathcal{F}(a,A)) \ll V_0 \ll a^{-d} \asymp |\mathcal{F}(a,A)|^{-1}.$$

**The Fibonacci point sets.** We have discussed above new concepts of discrepancy and their applications for the upper bounds for dispersion, in particular, for the Frolov point sets. The crucial role in the proof of Theorem 5.2 is played by Lemma 3.1. In the same way, using Lemma 2.2 instead of Lemma 3.1 we can prove the following version of Theorem 5.2 for the Fibonacci point sets.

**Theorem 5.3.** *Let $d = 2$. There exists an absolute constant $c > 0$ such that for any $V \geq V_0 := c/b_n$ we have for all $B \in \mathcal{B}$, $vol(B) = V$*

$$\left| b_n^{-1} \sum_{\mu=1}^{b_n} h_B(\mu/b_n, \{\mu b_{n-1}/b_n\}) - \hat{h}_B(\mathbf{0}) \right| \leq C \log(2V/V_0)/b_n^2. \tag{5.13}$$

Theorem 5.3 implies Theorem 2.1. Also, Theorem 5.3 provides the following inequalities for the Fibonacci point sets $\mathcal{F}_n$

$$D^{2,o}(\mathcal{F}_n, V) \leq D^2(\mathcal{F}_n, V) \leq C(\log(2V/V_0))/b_n^2, \qquad V \geq V_0.$$

# 6 Generalization for higher smoothness

In the definition of $D^1(T)$ and $D^{1,o}(T)$ – the 1-smooth discrepancy – we used as a building block the univariate characteristic function $\chi_{[-u/2,u/2)}(x)$ identified by the parameter $u \in \mathbb{R}_+$. In numerical integration $L_1$-smoothness of a function plays an important role. A characteristic function of an interval has smoothness 1 in the $L_1$ norm. This is why we call the corresponding discrepancy characteristics the 1-smooth discrepancy. In the definition of $D^2(T)$, $D^{2,o}(T)$, $D^2(T,V)$, and $D^{2,o}(T,V)$ we use the hat function $h_{[-u,u)}(x) = |u-x|$ for $|x| \le u$ and $h_{[-u,u)}(x) = 0$ for $|x| \ge u$ instead of the characteristic function $\chi_{[-u/2,u/2)}(x)$. Function $h_{[-u,u)}(x)$ has smoothness 2 in $L_1$. This fact gives the corresponding name. Note that

$$h_{[-u,u)}(x) = \chi_{[-u/2,u/2)}(x) * \chi_{[-u/2,u/2)}(x),$$

where

$$f(x) * g(x) := \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Now, for $r = 1, 2, 3, \dots$ we inductively define

$$h^1(x,u) := \chi_{[-u/2,u/2)}(x), \qquad h^2(x,u) := h_{[-u,u)}(x),$$

$$h^r(x,u) := h^{r-1}(x,u) * h^1(x,u), \qquad r = 3, 4, \dots.$$

Then $h^r(x,u)$ has smoothness $r$ in $L_1$ and has support $(-ru/2, ru/2)$. Represent a box $B \in \mathcal{B}$ in the form

$$B = \prod_{j=1}^{d} [x_j^0 - ru_j/2, x_j^0 + ru/2)$$

and define

$$h_B^r(\mathbf{x}) := \prod_{j=1}^{d} h^r(x_j - x_j^0, u_j).$$

We define the quantities $D^r(T)$, $D^{r,o}(T)$, $D^r(T,V)$, and $D^{r,o}(T,V)$ replacing $h_B(\mathbf{x})$ by $h_B^r(\mathbf{x})$ in the definitions (5.3) – (5.6). By the properties of convolution we obtain

$$\hat{h}^r(y,u) = \hat{h}^{r-1}(y,u)\hat{h}^1(y,u),$$

which implies for $y \neq 0$

$$\hat{h}^r(y,u) = \left(\frac{\sin(\pi y u)}{\pi y}\right)^r.$$

Therefore,

$$|\hat{h}^r(y,u)| \le \min\left(|u|^r, \frac{1}{|y|^r}\right) = \left(\frac{|u|}{|y|}\right)^{r/2} \min\left(|yu|^{r/2}, \frac{1}{|yu|^{r/2}}\right).$$

Consider

$$\sigma^r(v,\mathbf{u}) := \sum_{\|\mathbf{s}\|_1=v} \prod_{j=1}^{d} \min\left((2^{s_j}u_j)^{r/2}, \frac{1}{(2^{s_j}u_j)^{r/2}}\right), \quad v \in \mathbb{N}_0.$$

In the case $r=2$ we have $\sigma^2(v,\mathbf{u}) = \sigma(v,\mathbf{u})$ with $\sigma(v,\mathbf{u})$ defined and estimated in Section 3. In the same way as Lemma 3.1 has been proved we can prove the following its generalization for all $r \in \mathbb{N}$.

**Lemma 6.1.** *Let $v \in \mathbb{N}_0$ and $\mathbf{u} \in \mathbb{R}^d_+$. Then we have the following inequalities.*

*(I) Under condition $2^v pr(\mathbf{u},d) \ge 1$ we have*

$$\sigma^r(v,\mathbf{u}) \le C(d)\frac{(\log(2^{v+1}pr(\mathbf{u},d)))^{d-1}}{(2^v pr(\mathbf{u},d))^{r/2}}. \tag{6.1}$$

*(II) Under condition $2^v pr(\mathbf{u},d) \le 1$ we have*

$$\sigma^r(v,\mathbf{u}) \le C(d)(2^v pr(\mathbf{u},d))^{r/2}\left(\log\frac{2}{2^v pr(\mathbf{u},d)}\right)^{d-1}. \tag{6.2}$$

Using Lemma 6.1 instead of Lemma 3.1 we prove the following generalization of Theorem 5.2.

**Theorem 6.1.** *Let $r \ge 2$. There exists a constant $c(d,A,r) > 0$ such that for any $V \ge V_0 := c(d,A,r)a^{-d}$ we have for all $B \in \mathcal{B}$, $vol(B) = V$,*

$$|\Phi(a,A)(h^r_B) - \hat{h}^r_B(\mathbf{0})| \le C(d,A,r)a^{-rd}(\log(2V/V_0))^{d-1}. \tag{6.3}$$

**Corollary 6.1.** *For $r \ge 2$ there exists a constant $c(d,A,r) > 0$ such that for any $V \ge V_0 := c(d,A,r)a^{-d}$ we have*

$$D^{r,o}(\mathcal{F}(a,A),V) \le C(d,A,r)a^{-rd}(\log(2V/V_0))^{d-1}. \tag{6.4}$$

Similar generalizations can be obtained for the Fibonacci point sets. Using Lemma 6.1 instead of Lemma 2.2 we obtain the following results.

**Theorem 6.2.** *Let $d = 2$, $r \geq 2$. There exists a constant $c(r) > 0$ such that for any $V \geq V_0 := c(r)/b_n$ we have for all $B \in \mathcal{B}$, $vol(B) = V$*

$$\left| b_n^{-1} \sum_{\mu=1}^{b_n} h_B^r(\mu/b_n, \{\mu b_{n-1}/b_n\}) - \hat{h}_B^r(\mathbf{0}) \right| \leq C(r) \log(2V/V_0)/b_n^r. \tag{6.5}$$

Theorem 6.2 provides the following inequalities for the Fibonacci point sets $\mathcal{F}_n$ in case $r \geq 2$

$$D^{r,o}(\mathcal{F}_n, V) \leq D^r(\mathcal{F}_n, V) \leq C(r)(\log(2V/V_0))/b_n^r, \qquad V \geq V_0.$$

# 7 Universal discretization of the uniform norm

In this section we demonstrate an application of results from Sections 2 and 4 to the problem of universal discretization. For a more detailed discussion of universality in approximation and learning theory we refer the reader to [15], [16], [17], [5], [21], [7], [4], [18]. We remind the discretization problem setting, which we plan to discuss (see [19] and [20]).

**Marcinkiewicz problem.** Let $\Omega$ be a compact subset of $\mathbb{R}^d$ with the probability measure $\mu$. We say that a linear subspace $X_N$ of the $L_q(\Omega)$, $1 \leq q < \infty$, admits the Marcinkiewicz-type discretization theorem with parameters $m$ and $q$ if there exist a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ and two positive constants $C_j(d, q)$, $j = 1, 2$, such that for any $f \in X_N$ we have

$$C_1(d,q)\|f\|_q^q \leq \frac{1}{m} \sum_{\nu=1}^{m} |f(\xi^\nu)|^q \leq C_2(d,q)\|f\|_q^q. \tag{7.1}$$

In the case $q = \infty$ we define $L_\infty$ as the space of continuous on $\Omega$ functions and ask for

$$C_1(d)\|f\|_\infty \leq \max_{1 \leq \nu \leq m} |f(\xi^\nu)| \leq \|f\|_\infty. \tag{7.2}$$

We will also use a brief way to express the above property: the $\mathcal{M}(m, q)$ theorem holds for a subspace $X_N$ or $X_N \in \mathcal{M}(m, q)$.

**Universal discretization problem.** This problem is about finding (proving existence) of a set of points, which is good in the sense of the above Marcinkiewicz-type discretization for a collection of linear subspaces (see [21]). We formulate it in an explicit form. Let $\mathcal{X}_N := \{X_N^j\}_{j=1}^k$ be a collection of linear subspaces $X_N^j$ of the $L_q(\Omega)$, $1 \leq q \leq \infty$. We say that a set

$\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ provides *universal discretization* for the collection $\mathcal{X}_N$ if, in the case $1 \le q < \infty$, there are two positive constants $C_i(d,q)$, $i = 1,2$, such that for each $j \in [1,k]$ and any $f \in X_N^j$ we have

$$C_1(d,q)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^m |f(\xi^\nu)|^q \le C_2(d,q)\|f\|_q^q. \tag{7.3}$$

In the case $q = \infty$ for each $j \in [1,k]$ and any $f \in X_N^j$ we have

$$C_1(d)\|f\|_\infty \le \max_{1\le\nu\le m} |f(\xi^\nu)| \le \|f\|_\infty. \tag{7.4}$$

In [21] we studied the universal discretization for the collection of subspaces of trigonometric polynomials with frequencies from parallelepipeds (rectangles). For $\mathbf{s} \in \mathbb{N}_0^d$ define

$$R(\mathbf{s}) := \{\mathbf{k} \in \mathbb{Z}^d : |k_j| < 2^{s_j}, \quad j = 1, \dots, d\}.$$

Let $Q$ be a finite subset of $\mathbb{Z}^d$. We denote

$$\mathcal{T}(Q) := \{f : f = \sum_{\mathbf{k}\in Q} c_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}\}.$$

Consider the collection $\mathcal{C}(n,d) := \{\mathcal{T}(R(\mathbf{s})), \|\mathbf{s}\|_1 = n\}$.

The following theorem was proved in [21].

**Theorem 7.1.** *Let a set $T$ with cardinality $|T| = 2^r =: m$ have dispersion satisfying the bound disp$(T) < C(d)2^{-r}$ with some constant $C(d)$. Then there exists a constant $c(d) \in \mathbb{N}$ such that the set $2\pi T := \{2\pi\mathbf{x} : \mathbf{x} \in T\}$ provides the universal discretization in $L_\infty$ for the collection $\mathcal{C}(n,d)$ with $n = r - c(d)$.*

Theorem 7.1 in a combination with Theorems 2.1 and 4.1 guarantees that the appropriately chosen Fibonacci ($d = 2$) and Frolov (any $d \ge 2$) point sets provide universal discretization in $L_\infty$ for the collection $\mathcal{C}(n,d)$.

# 8 Universal discretization of the $L_q$ norm

We begin this section with proving a general conditional result. Then we derive from it universality of the Fibonacci point sets for discretization of the $L_q$ norm for all $1 \le q \le \infty$. We formulate the universality problem with weights.

**Marcinkiewicz problem with weights.** We say that a linear subspace $X_N$ of the $L_q(\Omega)$, $1 \le q < \infty$, admits the weighted Marcinkiewicz-type discretization theorem with parameters $m$ and $q$ if there exist a set of knots $\{\xi^\nu \in \Omega\}$, a set of weights $\{w_\nu\}$, $\nu = 1, \dots, m$, and two positive constants $C_j(d,q)$, $j = 1,2$, such that for any $f \in X_N$ we have

$$C_1(d,q)\|f\|_q^q \le \sum_{\nu=1}^{m} w_\nu |f(\xi^\nu)|^q \le C_2(d,q)\|f\|_q^q. \tag{8.1}$$

Then we also say that the $\mathcal{M}^w(m,q)$ theorem holds for a subspace $X_N$ or $X_N \in \mathcal{M}^w(m,q)$. Obviously, $X_N \in \mathcal{M}(m,q)$ implies that $X_N \in \mathcal{M}^w(m,q)$.

**Universal discretization problem with weights.** This problem is about finding (proving existence) of a set of points and a set of weights which are good in the sense of the above Marcinkiewicz-type discretization with weights for a collection of linear subspaces. We formulate it in an explicit form. Let $\mathcal{X}_N := \{X_N^j\}_{j=1}^k$ be a collection of linear subspaces $X_N^j$ of the $L_q(\Omega)$, $1 \le q < \infty$. We say that a set of knots $\{\xi^\nu \in \Omega\}$ and a set of weights $\{w_\nu\}$, $\nu = 1, \dots, m$, provide *universal discretization with weights* for the collection $\mathcal{X}_N$ if there are two positive constants $C_i(d,q)$, $i = 1,2$, such that for each $j \in [1,k]$ and any $f \in X_N^j$ we have

$$C_1(d,q)\|f\|_q^q \le \sum_{\nu=1}^{m} w_\nu |f(\xi^\nu)|^q \le C_2(d,q)\|f\|_q^q. \tag{8.2}$$

For a set of knots $\Xi_m := \{\xi^\nu\}_{\nu=1}^m \subset \Omega$ and a set of weights $W_m := \{w_\nu\}_{\nu=1}^m$ consider the cubature formula

$$I_m := I_m(\Xi_m, W_m)(f) := \sum_{\nu=1}^{m} w_\nu f(\xi^\nu). \tag{8.3}$$

For $\mathbf{N} \in \mathbb{N}_0^d$ define a subspace of trigonometric polynomials

$$\mathcal{T}(\mathbf{N}) := \left\{ f(\mathbf{x}) : f(\mathbf{x}) = \sum_{\mathbf{k}: |k_j| \le N_j, j=1,\dots,d} c_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})} \right\}.$$

The following Lemma 8.1 is the conditional result that we mentioned above.

**Lemma 8.1.** *Let* $\mathbf{N} \in \mathbb{N}_0^d$ *and let a set of knots* $\Xi_m := \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$ *and a set of nonnegative weights* $W_m := \{w_\nu\}_{\nu=1}^m$ *be such that for any* $f \in \mathcal{T}(3\mathbf{N})$ *we have*

$$I_m(\Xi_m, W_m)(f) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathbf{x}) d\mathbf{x}. \tag{8.4}$$

*Then for any* $1 \le q \le \infty$ *we have for all* $f \in \mathcal{T}(\mathbf{N})$

$$C_1(d)\|f\|_q \le \left( \sum_{\nu=1}^m w_\nu |f(\xi^\nu)|^q \right)^{1/q} \le C_2(d)\|f\|_q \tag{8.5}$$

*with constants* $C_1(d)$ *and* $C_2(d)$*, which may only depend on* $d$.

*Proof.* We need some classical trigonometric polynomials. We begin with the univariate case. The Dirichlet kernel of order $n$:

$$\mathcal{D}_n(x) := \sum_{|k| \le n} e^{ikx} = e^{-inx}(e^{i(2n+1)x} - 1)(e^{ix} - 1)^{-1}$$

$$= \big(\sin(n + 1/2)x\big) \,/\, \sin(x/2) \tag{8.6}$$

is an even trigonometric polynomial. The Fejér kernel of order $n - 1$:

$$\mathcal{K}_n(x) := n^{-1} \sum_{k=0}^{n-1} \mathcal{D}_k(x) = \sum_{|k| \le n} \big(1 - |k|/n\big) e^{ikx}$$

$$= \big(\sin(nx/2)\big)^2 \,/\, \big(n(\sin(x/2))^2\big).$$

The Fejér kernel is an even nonnegative trigonometric polynomial in $\mathcal{T}(n-1)$. It satisfies the obvious relations

$$\|\mathcal{K}_n\|_1 = 1, \qquad \|\mathcal{K}_n\|_\infty = n. \tag{8.7}$$

The de la Vallée Poussin kernel

$$\mathcal{V}_n(x) := n^{-1} \sum_{l=n}^{2n-1} \mathcal{D}_l(x) = 2\mathcal{K}_{2n}(x) - \mathcal{K}_n(x)$$

is an even trigonometric polynomial of order $2n - 1$.

In the multivariate case define the Fejér and de la Vallée Poussin kernels as follows:

$$\mathcal{K}_{\mathbf{N}}(\mathbf{x}) := \prod_{j=1}^{d} \mathcal{K}_{N_j}(x_j), \qquad \mathcal{V}_{\mathbf{N}}(\mathbf{x}) := \prod_{j=1}^{d} \mathcal{V}_{N_j}(x_j), \qquad \mathbf{N} = (N_1, \dots, N_d).$$

For $f \in \mathcal{T}(\mathbf{N})$ we have for each $\mathbf{x} \in \mathbb{T}^d$ that $f(\mathbf{y})\mathcal{V}_{\mathbf{N}}(\mathbf{x}-\mathbf{y}) \in \mathcal{T}(3\mathbf{N})$ and by our condition (8.4) we obtain

$$f(\mathbf{x}) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathbf{y})\mathcal{V}_{\mathbf{N}}(\mathbf{x}-\mathbf{y})d\mathbf{y} = \sum_{\nu=1}^{m} w_\nu f(\xi^\nu)\mathcal{V}_{\mathbf{N}}(\mathbf{x}-\xi^\nu). \tag{8.8}$$

Define a space

$$\ell_q(W_m) := \{\mathbf{a} \in \mathbb{C}^m\} \quad \text{with norm} \quad \|\mathbf{a}\|_{q,w} := \left(\sum_{\nu=1}^{m} w_\nu |a_\nu|^q\right)^{1/q}.$$

Let $V_{\mathbf{N}}$ be the operator on $\ell_q(W_m)$ defined as follows:

$$V_{\mathbf{N}}(\mathbf{a}) := \sum_{\nu=1}^{m} w_\nu a_\nu \mathcal{V}_{\mathbf{N}}(\mathbf{x}-\xi^\nu).$$

Property (8.7) implies that $\|\mathcal{V}_{\mathbf{N}}\|_1 \le 3^d$. Therefore,

$$\|V_{\mathbf{N}}\|_{\ell_1(W_m)\to L_1} \le 3^d. \tag{8.9}$$

We now bound the norm $\|V_{\mathbf{N}}\|_{\ell_\infty(W_m)\to L_\infty}$. Clearly,

$$\|V_{\mathbf{N}}\|_{\ell_\infty(W_m)\to L_\infty} \le \left\|\sum_{\nu=1}^{m} w_\nu |\mathcal{V}_{\mathbf{N}}(\mathbf{x}-\xi^\nu)|\right\|_\infty. \tag{8.10}$$

We need the following simple technical lemma.

**Lemma 8.2.** *Under conditions of Lemma 8.1 we have*

$$\left\|\sum_{\nu=1}^{m} w_\nu |\mathcal{V}_{\mathbf{N}}(\mathbf{x}-\xi^\nu)|\right\|_\infty \le 3^d.$$

*Proof.* Represent

$$\mathcal{V}_{N_j}(t) = 2\mathcal{K}_{2N_j}(t) - \mathcal{K}_{N_j}(t).$$

Using the fact that the Fejér kernel is a nonnegative polynomial we obtain

$$\sum_{\nu=1}^{m} w_\nu |\mathcal{V}_{\mathbf{N}}(\mathbf{x}-\xi^\nu)| \le \sum_{\nu=1}^{m} w_\nu \prod_{j=1}^{d} \left(2\mathcal{K}_{2N_j}(x_j-\xi^\nu_j) + \mathcal{K}_{N_j}(x_j-\xi^\nu_j)\right)$$

and by (8.4) and (8.7) we continue

$$\le \sum_{k=0}^{d} \binom{d}{k} 2^k 1^{d-k} = 3^d.$$

$\square$

Lemma 8.2 and inequality (8.9) imply by the Riesz-Thorin interpolation theorem that

$$\|V_{\mathbf{N}}\|_{\ell_q(W_m)\to L_q} \le 3^d, \qquad 1 \le q \le \infty. \tag{8.11}$$

By representation (8.8) and inequality (8.11) we obtain for $f \in \mathcal{T}(\mathbf{N})$

$$\|f(\mathbf{x})\|_q = \left\|\sum_{\nu=1}^{m} w_\nu f(\xi^\nu)\mathcal{V}_{\mathbf{N}}(\mathbf{x}-\xi^\nu)\right\|_q \le 3^d \left(\sum_{\nu=1}^{m} w_\nu |f(\xi^\nu)|^q\right)^{1/q},$$

which proves the first inequality in (8.5) of Lemma 8.1.

We now prove the second inequality in (8.5) of Lemma 8.1 for $1 \le q < \infty$. In the case $q=\infty$ it is trivial. We have ($q' := \frac{q}{q-1}$)

$$\sum_{\nu=1}^{m} w_\nu |f(\xi^\nu)|^q = \sum_{\nu=1}^{m} w_\nu f(\xi^\nu)\varepsilon_\nu \left|f(\xi^\nu)\right|^{q-1} =$$

$$= (2\pi)^{-d}\int_{\mathbb{T}^d} f(\mathbf{x}) \sum_{\nu=1}^{m} w_\nu \varepsilon_\nu \left|f(\xi^\nu)\right|^{q-1} \mathcal{V}_{\mathbf{N}}(\mathbf{x}-\xi^\nu)d\mathbf{x} \le$$

$$\le \|f\|_q \left\|\sum_{\nu=1}^{m} w_\nu \varepsilon_\nu \left|f(\xi^\nu)\right|^{q-1}\mathcal{V}_{\mathbf{N}}(\mathbf{x}-\xi^\nu)\right\|_{q'}.$$

Using (8.11) we see that the last term is

$$\le 3^d \|f\|_q \left(\sum_{\nu=1}^{m} w_\nu \left|f(\xi^\nu)\right|^q\right)^{(q-1)/q},$$

which implies the required inequality.

Lemma 8.1 is proved.

$\square$

**Universality of the Fibonacci point sets.** We use Lemmas 2.1 and 8.1. Lemma 2.1 and identity (2.2) imply that for any

$$f \in \mathcal{T}(\Gamma(\gamma b_n)) := \left\{ f : f(\mathbf{x}) = \sum_{\mathbf{k}\in\Gamma(\gamma b_n)} c_{\mathbf{k}} e^{(\mathbf{k},\mathbf{x})} \right\}$$

we have

$$\Phi_n(f) = \hat{f}(\mathbf{0}).$$

Therefore, condition (8.4) of Lemma 8.1 is satisfied for $m = b_n$, $I_m = \Phi_n$, $w_\nu = 1/m$, $\nu = 1,\dots,m$, with $\mathbf{N} = (2^{s_1},\dots,2^{s_d})$ under condition $\mathbf{s} \in \mathbb{N}_0^d$ is such that $3\cdot 2^{\|\mathbf{s}\|_1} \le \gamma b_n$. Lemma 8.1 implies the following result.

**Theorem 8.1.** *The Fibonacci point set $\mathcal{F}_n$ provides the universal discretization in $L_q$, $1 \le q \le \infty$, for the collection $\mathcal{C}(r,2)$ with $r$ satisfying the condition $3\cdot 2^r \le \gamma b_n$.*

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