arXiv:1709.06798v2 [math.DG] 2 Oct 2017

Scalar conformal invariants of weight zero

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(Dated: 3 October 2017)

In the class of metrics of a generic conformal structure there exists a distinguishing metric. This was noticed by Albert Einstein in a lesser-known paper[1] of 1921. We explore this finding from a geometrical point of view. Then, we obtain a family of scalar conformal invariants of weight 0 for generic pseudo-Riemannian conformal structures $[g]$ in more than three dimensions. In particular, we define the conformal scalar curvature of $[g]$ and calculate it for some well-known conformal spacetimes, comparing the results with the Ricci scalar and the Kretschmann scalar. In the cited paper, Einstein also announced that it is possible to add an scalar equation to the field equations of General Relativity.

PACS numbers: 04.20.Cv, 04.50.Kd, 02.40.Hw
Keywords: general relativity; conformal differential geometry; tensor methods; differential invariants; Einstein Maxwell equations.

I. INTRODUCTION

We are accustomed to think that in a conformal pseudo-Riemannian class of metrics there is no preferred metric. *This is not true* for reasonably generic conformal structures in four or more dimensions (neither is it true for three dimensions[2]).

In a lesser-known paper in March 1921 Einstein wrote[1] (my notes in squared brackets):

> If we put $g'_{\mu\nu} = J g_{\mu\nu}$ [J is given next] then $d\sigma^2 = g'_{\mu\nu} dx_\mu dx_\nu$ is an invariant that depends only upon the ratios of the $g_{\mu\nu}$ [the conformal structure $[g]$]. All Riemann tensors formed as fundamental invariants from $d\sigma$ in the customary manner are—when seen as functions of the $g_{\mu\nu}$—Weyl tensors of weight 0. [. . .] Therefore, to every law of nature $T(g) = 0$ of the general theory of relativity, there corresponds a law $T(g') = 0$, which contains only the ratios of the $g_{\mu\nu}$.

We are going to develop the idea of Einstein and we will expose a collection of scalar conformal invariants, which, to my knowledge, has been no discussed up to now. We take the following steps:

- Describe the concept of conformal invariant that we are going to consider.
- Emphasize the existence of a distinguishing metric g' in the class of a generic conformal structure $[g]$.
- Establish that scalar invariants of g' of r-order are scalar conformal invariants of $[g]$ of $(r+2)$-order
- In particular, I show that the Ricci scalar of g' of $[g]$ is a fourth order scalar conformal invariant of $[g]$, which we called *conformal scalar curvature*. Then, for some well-known spacetimes, we compare the results with some usual scalar metric invariants.

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II. BUNDLE OF G-STRUCTURES

Given a closed subgroup G of $Gl_n \equiv GL(n,\mathbb{R})$, we get the *bundle M_G of G-structures* on a manifold M that is an associated bundle to the linear frame bundle $LM \equiv F^1M$, which typical fiber is ${}^{Gl_n}\!/_{G}$. Local sections of M_G correspond to local G-structures on M.

The *bundle of metrics* on M is obtained by taking $G = O_n$, the orthogonal group of a given signature (p,q). There is a bijective correspondence between ${}^{Gl_n}\!/_{O_n}$ and the set of symmetric invertible matrices of signature (p,q); therefore, when we give a chart, we can recognize a pseudo-Riemannian metric g of M as a section of M_{O_n} given by a matrix (g_{ij}) over the domain of the chart. We write $\sigma_g\colon M \to M_{O_n}$ for the section corresponding to the metric g.

The *bundle of conformal structures* arises when $G = C_n := \mathbb{R}^+ \cdot O_n$, the conformal orthogonal group of signature (p,q). There is a bijective correspondence between ${}^{Gl_n}\!/_{C_n}$ and the set of symmetric invertible matrices of signature (p,q) and absolute value of its determinant one. Now, we can recognize a conformal structure $[g]$ on M as a section of M_{C_n}, which is given by the matrix $(c_{ij}) = |\det(g_{ij})|^{-1/n}(g_{ij})$ over the domain of a chart. We write $\sigma_{[g]}\colon M \to M_{C_n}$ for the section corresponding to the conformal structure $[g]$.

Let F^rM be the frame bundle of r-th order—an r-frame is an r-jet at 0 of inverses of charts of M—that is a principal bundle with structural group the *jet group* G^r_n—group of r-jets at 0 of diffeomorphisms of $\mathbb{R}^n$ fixing 0. The bundle M^r_G of r-jets of local sections of M_G (i.e., r-jets of local G-structures on M) is an associated bundle to the frame bundle $F^{r+1}M$, which typical fiber is the space $(\mathbb{R}^n_G)^r_0 \equiv J^r_0(\mathbb{R}^n, {}^{Gl_n}\!/_{G})$ of r-jets at 0 of maps of $\mathbb{R}^n$ to ${}^{Gl_n}\!/_{G}$ (i.e., r-jets of G-structures on a neighborhood of 0).

III. INVARIANTS OF G-STRUCTURES

A paradigm of scalar invariant for a differential geometric structure is the scalar curvature, or Ricci scalar, $R_g\colon M \to \mathbb{R}$ of a Riemannian manifold (M,g). For $m \in M$, $R_g(m)$ is made from the partial derivatives up to second order of the metric g at m and does not depend on which chart has been used to perform the derivatives, in other words, $R_g(m)$ is a function of the 2-jet of g at m. Therefore, the Ricci scalar is defined by the function $R\colon M^2_{O_n} \to \mathbb{R}$, $R(j^2_m\sigma_g) = R_g(m)$, with σ_g being the section of M_{O_n} corresponding to the metric g.

The Ricci scalar of any metric g is said *invariant by diffeomorphisms* because, for any local diffeomorphism φ of M, it is verified that $R_{\varphi^* g} = R_g \circ \varphi$. This property is equivalent to say[3] that $R \circ \widehat{\varphi}^2 = R$, $\forall \varphi$, with $\widehat{\varphi}^2$ being the case $r = 2$ of the typical action of a diffeomorphism φ on M^r_G, which is defined by:

$$\widehat{\varphi}^r\colon M^r_G \to M^r_G, \quad j^r_p\sigma \mapsto j^r_{\varphi(p)}(\bar{\varphi} \circ \sigma \circ \varphi^{-1}), \tag{1}$$

with $\bar{\varphi}$ being the standard lifting of φ to M_G.

Following the model of the Ricci scalar function $R\colon M^2_{O_n} \to \mathbb{R}$, we define:

Definition III.1. An *scalar invariant of r-order of G-structures on M* is a function $f\colon M^r_G \to \mathbb{R}$ verifying $f \circ \widehat{\varphi}^r = f$ over the domain of φ, for all local diffeomorphism φ of M.

IV. MAIN RESULTS

In the setting of conformal differential geometry there is another concept of invariance, Let M be a manifold with a conformal structure $[g]$. A tensor T on M, obtained from the r-jet of g by an specific formula, is said *conformally invariant*—or invariant by conformal transformations—if, $\forall \bar{g} \in [g]$, T is equal to the tensor $\bar{T}$ obtained from the r-jet of $\bar{g}$ by the same formula.

It is well known that the conformal curvature or Weyl tensor $C^i_{\ jkl}$ is conformally invariant. We define the *square of the Weyl tensor* to be the scalar $H = C_{ijkl}C^{ijkl}$, which is not conformally invariant because it depends of the metric g used to raising and lowering indices. In this context, a point $p \in M$ is called *generic* if $H(p) \neq 0$ for some (and then for all) $g \in [g]$; and a conformal structure $[g]$ is called *generic* if all the points of M are generic.

The following result is obtained by Einstein[1] with the help, as he himself says, of the mathematician Wilhelm Wirtinger.

Theorem IV.1. *Let $[g]$ be a generic conformal structure on M. The metric $g' := Jg \in [g]$, where $J = |H|^{1/2}$, do not depends of the metric g in the class of $[g]$.*

In other words, in the class of metrics of a generic conformal structure *there exists a preferred metric*. The theorem can be formulated in a neighborhood of a generic point.

We need to calculate the dependence on g of the factor $J = |H|^{1/2}$. We can write:

$$H = C_{ijkl}C^{ijkl} = g_{ia}C^{a}_{\ jkl}C^{i}_{\ bcd}g^{bj}g^{ck}g^{dl}.$$

Now, if we change the metric g for $\bar{g} = \alpha g$ then $\bar{g}_{ij} = \alpha g_{ij}$ and $\bar{g}^{ij} = \alpha^{-1}g^{ij}$, and we obtain:

$$\bar{H} = \bar{C}_{ijkl}\bar{C}^{ijkl} = \bar{g}_{ia}C^{a}_{\ jkl}C^{i}_{\ bcd}\bar{g}^{bj}\bar{g}^{ck}\bar{g}^{dl} = \alpha^{-2}H, \tag{2}$$

because of the Weyl tensor is conformally invariant: $\bar{C}^{i}_{\ jkl} = C^{i}_{\ jkl}$. Then, taking absolute values and square roots in Eq. (2), we obtain $\bar{J} = \alpha^{-1}J$. Finally, we get the result:

$$g' = Jg = \alpha^{-1}J\alpha g = \bar{J}\bar{g}. \tag{3}$$

Furthermore, the scalar J' corresponding to g' is constant one, hence we can say that *the preferred metric of a generic conformal structure is the only metric in the class that normalizes to ± 1 the square of the Weyl tensor.*

Theorem IV.2. *To each scalar metric invariant of r-order on M corresponds an scalar conformal invariant of $(r+2)$-order on M.*

In fact, an scalar invariant of r-order of the preferred metric $g' \in [g]$ is a conformal invariant of $[g]$ of $(r+2)$-order. In particular, the Ricci scalar of the metric g' is an scalar conformal invariant of fourth order that we call the *conformal scalar curvature* of $[g]$.

We will return to this theorem at the end of the section, but first let us go back to the notion of invariant of Def. III.1. The action of the group G_n^{r+1} on $(\mathbb{R}^n_G)^r_0$, which makes M^r_G into an associated bundle to $F^{r+1}M$, is defined[3] by

$$j_0^{r+1}\xi \cdot j_0^r\mu := j_0^r\left((D\xi \cdot \mu) \circ \xi^{-1}\right) \tag{4}$$

being ξ a local diffeomorphism of $\mathbb{R}^n$ with $\xi(0) = 0$, μ a smooth map from a neighborhood of 0 to ${}^{Gl_n}/_{G}$ (for G closed), and the last dot for the action of Gl_n on ${}^{Gl_n}/_{G}$. Its orbit space is ${}^{(\mathbb{R}^n_G)^r_0}/_{G_n^{r+1}}$, consisting of classes of r-jets at 0 of G-structures over $\mathbb{R}^n$ "modulo diffeomorphisms".

It is proved[3] that r-order scalar invariants of G-structures on M are in a natural bijective correspondence with functions $\psi\colon {}^{(\mathbb{R}^n_G)^r_0}/_{G_n^{r+1}} \to \mathbb{R}$ such that $\psi \circ \pi$ is smooth, with π being the projection of the quotient space. We will call a such ψ an *scalar G-invariant of r-order* (it does not depend on M!). We are stating that the nature of an invariant of r-order can be appreciated when it works on r-jets at 0 of G-structures on $\mathbb{R}^n$ in a neighborhood of 0.

With this simplified concept of scalar invariant it is easier to prove the following properties of invariants (in the rest of the section all the invariants are scalar invariants):

- *An invariant of r-order is invariant of $(r+s)$-order*, $\forall s$. Because there exists the projection

  $$\Pi\colon (\mathbb{R}^n_G)^{r+s}_0 \to (\mathbb{R}^n_G)^r_0, \quad j_0^{r+s}\mu \mapsto j_0^r\mu,$$

  which passes to a map Π' between quotient spaces, we obtain the result on invariants from:

  $${}^{(\mathbb{R}^n_G)^{r+s}_0}/_{G_n^{r+s+1}} \xrightarrow{\Pi'} {}^{(\mathbb{R}^n_G)^r_0}/_{G_n^{r+1}} \xrightarrow{\psi} \mathbb{R}.$$

- *If $U \subset G$ is a subgroup, a G-invariant of r-order is a U-invariant of r-order*. In effect: the map $\beta\colon {}^{Gl_n}/_U \to {}^{Gl_n}/_G,\ aU \mapsto aG$, verifies $\beta \circ L_b = L_b \circ \beta$, $\forall b \in Gl_n$; hence the map

$$\gamma\colon (\mathbb{R}^n_U)^r_0 \to (\mathbb{R}^n_G)^r_0, \quad j^r_0\mu \mapsto j^r_0(\beta \circ \mu)$$

passes to a map γ' between quotient spaces, and we obtain this result on invariants from:

$$(\mathbb{R}^n_U)^r_0/G^{r+1}_n \xrightarrow{\gamma'} (\mathbb{R}^n_G)^r_0/G^{r+1}_n \xrightarrow{\psi} \mathbb{R}.$$

Therefore, *conformal invariants of a given order are metric invariants of the same order*. Moreover, the conformal invariants are the metric invariants which are conformally invariant.

- *A metric invariant of r-order is a conformal invariant of $(r+2)$-order* (hence Theorem IV.2 holds). Because J is made from the partial derivatives up to second order of the metric g of M, the well-defined map between conformal structures and metrics, given by $[g] \mapsto g' = Jg$, depends of the 2-jet of the components g_{ij} at each point of M or, equivalently, of the 2-jet of the components c_{ij} that characterize $[g]$. The same is true for conformal structures $[g]$ on $\mathbb{R}^n$ in a neighborhood of 0; then the map between conformal structures and metrics induces, for successive orders, the maps

$$\Gamma\colon (\mathbb{R}^n_{C_n})^{r+2}_0 \to (\mathbb{R}^n_{O_n})^r_0, \quad j^{r+2}_0\sigma_{[g]} \mapsto j^r_0\sigma_{g'},$$

each one of which passes to a map Γ' between quotient spaces, and we obtain the result from:

$$(\mathbb{R}^n_{C_n})^{r+2}_0/G^{r+3}_n \xrightarrow{\Gamma'} (\mathbb{R}^n_{O_n})^r_0/G^{r+1}_n \xrightarrow{\psi} \mathbb{R}.$$

V. CONFORMAL SCALAR CURVATURE

I have calculated the Ricci scalar of g', which we have called *conformal scalar curvature of* $[g]$ and denoted by $S_{[g]}$, for some well known spacetime metrics g. To do this for a metric g, we calculate J and the Ricci scalar of $g' = Jg$. I used the package *xAct`xCoba`* for Mathematica. In the Table I below, I give some results comparing the Ricci scalar and Kretschmann scalar of g with the conformal scalar curvature of $[g]$; this table shows the calculations for the following metrics:

- Schwarzschild: $-(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
- Reissner-Nordstrøm: $-(1-\frac{2M}{r}+\frac{q^2}{r^2})dt^2 + (1-\frac{2M}{r}+\frac{q^2}{r^2})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
- Gödel: $-dt^2 + (1+\frac{r^2}{4a^2})^{-1}dr^2 + r^2(1-\frac{r^2}{4a^2})d\phi^2 - \frac{\sqrt{2}r^2}{a}\,dt\,d\phi + dz^2$
- Barriola-Vilenkin: $-dt^2 + dr^2 + k^2r^2(d\theta^2 + \sin^2\theta d\phi^2)$

TABLE I. Ricci scalar R_g, Kretschmann scalar K_g and conformal scalar curvature $S_{[g]}$ for some solutions g of the Einstein equations.

Metrics	R_g	K_g	$S_{[g]}$
Schwarzschild	0	$\frac{48M^2}{r^6}$	$\frac{9\sqrt{3}}{4}(1-\frac{r}{6M})$
Reissner-Nordstrøm	0	$\frac{8(7q^4-12Mq^2r+6M^2r^2)}{r^8}$	$\frac{9\sqrt{3}}{8}(1-\frac{r}{3M}+\frac{q^2}{3M^2})$
Gödel	$-\frac{1}{a^2}$	$\frac{3}{a^4}$	$-\frac{\sqrt{3}}{2}$
Barriola-Vilenkin	$\frac{2(1-k^2)}{k^2r^2}$	$\frac{4(1-k^2)^2}{k^4r^4}$	$-\sqrt{3}$

Let me highlight that replacing the metric g by a conformally related metric αg, Ricci and Kretschmann scalars get complicated with terms including up to second order derivatives of α. In contrast, the conformal scalar curvature keeps unalterable, meaning that its weight is zero in the Weyl's sense[4].

VI. CONCLUSION

At the end of his article[1], Einstein proposed the obvious addition of the differential equation $J = J_0$ (constant), free of genericity conditions, to the field equations of General Relativity because it is "*a logical possibility that is worthy of publication, which may be useful for physics or not.*"

In dimension four, the existence of a distinguishing metric in the class of a conformal structure with $H \neq 0$ arises also from the theory of Weyl gravity (or conformal gravity). This theory is governed by the Lagrangian $H\,\Omega_g$, with Ω_g being the metric volume form. This Lagrangian, in dimension 4, is invariant by conformal transformations (in dimension greater than 4 what is conformally invariant is $|H|^{n/4}\,\Omega_g$, which is the former Lagrangian, except sign, when $n = 4$). Then $H\Omega_g$ is a distinctive volume form provided only by the conformal structure $[g]$. Therefore, it can be concluded that there exists a unique metric $g' \in [g]$ such that $\Omega_{g'} = H\Omega_g$.

These previous facts suggest new perspectives of research using that preferred metric of the conformal class, which is surprisingly almost unknown in the research bibliography (I only found one reference—not on subject of history—to this paper of Einstein pointing at this distinguishing metric[5]).

ACKNOWLEDGMENTS

This work was presented for the first time at the Spanish-Portuguese Relativity Meeting – EREP 2017 held in Málaga (Spain), 12-15 September 2017. I am very grateful to all the colleagues at the meeting who, through their comments, have contributed to improve this paper. This research was partially supported by Junta de Andalucía (Spain), under grant P.A.I. FQM-324.

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