Construction of constant mean curvature n-noids using the DPW method

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Abstract: we construct constant mean curvature surfaces in euclidean space with genus zero and n ends asymptotic to Delaunay surfaces using the DPW method.

1 Introduction

In [2], Dorfmeister, Pedit and Wu have shown that harmonic maps from a Riemann surface to a symmetric space admit a Weierstrass-type representation, which means that they can be represented in terms of holomorphic data. In particular, surfaces with constant mean curvature one (CMC-1 for short) in euclidean space admit such a representation, owing to the fact that the Gauss map of a CMC-1 surface is a harmonic map to the 2-sphere. This representation is now called the DPW method and has been widely used to construct examples of CMC-1 surfaces in $\mathbb{R}^3$ and also constant mean curvature surfaces in homogeneous spaces such as the sphere $\mathbb{S}^3$ or hyperbolic space $\mathbb{H}^3$.

- In the genus zero case, the full family of 3-noids (CMC-1 genus zero surfaces with 3 Delaunay ends) is constructed in [3, 17]. n-noids have been constructed only in the most symmetric cases in [13, 16].
- In the genus one case, closed CMC-1 tori in $\mathbb{R}^3$ are classified in [1] using the DPW method. More recently, integrable system methods akin to the DPW method have been used in [5] to classify closed minimal or CMC tori in the sphere $\mathbb{S}^3$ (the Lawson conjecture) and embedded minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$ in [6].
- In the higher genus case, CMC-1 surfaces with arbitrary genus and one end have been constructed in [12]. More recently, S. Heller was able to write down a DPW

potential for Lawson genus 2 minimal surface in $\mathbb{S}^3$ in [8], and for related CMC surfaces in $\mathbb{S}^3$ in subsequent papers [7, 9, 10]. All these higher genus examples are highly symmetric.

We have omitted from this list the construction of wildly immersed surfaces such as the Smyth surface, Mr Bubble and surfaces with bubbletons.

The main limitation to the construction of examples is the monodromy problem: only low dimensional monodromy problems have been solved. Symmetries are used to reduce the number of equations to be solved.

In constract, Kapouleas [11] has constructed embedded CMC-1 surfaces with no limitation on the genus or number of ends by gluing round spheres and pieces of Delaunay surfaces, using Partial Differential Equations techniques. An interesting question is wether similar results can be achieved with the DPW method. In this paper, we make a first step in this direction by constructing n-noids with no symmetry assumptions:

Theorem 1. *Given $n \geq 3$ distinct vectors $u_1, \cdots, u_n$ in $\mathbb{S}^2$ and n non-zero weights $\tau_1, \cdots, \tau_n$ satisfying the balancing condition*

$$\sum_{i=1}^{n} \tau_i u_i = 0$$

there exists a smooth 1-parameter family of CMC-1 surfaces $(M_t)_{0<t<\varepsilon}$ with genus zero, n Delaunay ends and the following properties:

1. *If we denote $w_{i,t}$ the weight of the i-th Delaunay end and $\Delta_{i,t}$ its axis, then*

$$\lim_{t\to 0} \frac{w_{i,t}}{t} = 8\pi\tau_i$$

   and $\Delta_{i,t}$ converges to the half-line through the origin directed by u_i.

2. *If all weights τ_i are positive and for all $j \neq i$, the angle between u_i and u_j is greater than $\frac{\pi}{3}$, then M_t is embedded.*

These examples can be described heuristically as the unit sphere with n half Delaunay surfaces attached at the points $u_1, \cdots, u_n$ (see Figure 1). They are a particular case of the construction of Kapouleas [11]. We construct them with the DPW method using a rather simple and natural DPW potential. As a bonus, we can locate explicitly the umbilics of M_t as $t \to 0$, because umbilics are the zeros of the Hopf differential which is central in the DPW theory (see Section 4.9). Also, our construction yields smooth dependence of

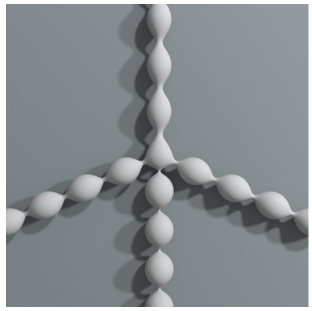

Figure 1: a symmetric 4-noid. Image by N. Schmitt

M_t on the parameters t, u_i and τ_i, because we use the implicit function theorem to solve equations, whereas Kapouleas used a topological fixed point argument.

Our main motivation is to make progress in the DPW method. An important theoretical issue that we address in this paper is that the underlying Riemann surface is allowed to depend on the spectral parameter (see Section 3).

2 Background

In this section, we recall standard notations and results used in the DPW method. For a comprehensive introduction to the DPW method, we suggest [4]. The DPW method has several avatars: we have chosen the "untwisted" setting.

2.1 Loop groups

A loop is a smooth map from the unit circle $\mathbb{S}^1$ to a matrix group. The circle variable is denoted λ and called the spectral parameter.

1. $\Lambda SL(2,\mathbb{C})$ is the set of smooth maps $\Phi : \mathbb{S}^1 \to SL(2,\mathbb{C})$.

2. $\Lambda SU(2) \subset \Lambda SL(2,C)$ is the set of smooth maps $F : \mathbb{S}^1 \to SU(2)$.

3. $\Lambda_+ SL(2,\mathbb{C}) \subset \Lambda SL(2,\mathbb{C})$ is the set of smooth maps $B : \mathbb{S}^1 \to SL(2,\mathbb{C})$ which extend holomorphically to the unit disk $\mathbb{D} = D(0,1)$.

4. $\Lambda_+^{\mathbb{R}} SL(2,\mathbb{C})$ is the set of $B \in \Lambda_+ SL(2,\mathbb{C})$ such that $B(0)$ is upper triangular with real elements on the diagonal.

Theorem 2 (Iwasawa decomposition). *The multiplication $\Lambda SU(2) \times \Lambda_+^{\mathbb{R}} SL(2,\mathbb{C}) \to \Lambda SL(2,\mathbb{C})$ is bijective. The unique splitting of an element $\Phi \in \Lambda SL(2,\mathbb{C})$ as $\Phi = FB$ with $F \in \Lambda SU(2)$ and $B \in \Lambda_+^{\mathbb{R}} SL(2,\mathbb{C})$ is called Iwasawa decomposition. F is called the unitary factor of Φ and B the positive factor.*

2.2 The matrix model of $\mathbb{R}^3$

In the DPW method, one identifies $\mathbb{R}^3$ with the Lie algebra $\mathfrak{su}(2)$ by

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3 \longleftrightarrow X = \frac{-i}{2} \begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \in \mathfrak{su}(2)$$

Under this identification, the euclidean scalar product and norm are given by

$$\langle x, y \rangle = -2\mathrm{tr}(XY)$$

$$||x||^2 = 4 \det(X)$$

The group $SU(2)$ acts as linear isometries on $\mathfrak{su}(2)$ by $H \cdot X = HXH^{-1}$. The kernel of this action is $\pm I_2$ so that $SO(3)$ is isomorphic to $SU(2)/\{\pm I_2\}$.

2.3 The DPW method

The input data for the DPW method is a quadruple $(\Sigma, \xi, z_0, \Phi_0)$ where:

1. Σ is a Riemann surface.

2. $\xi = \xi(z, \lambda)$ is a $\mathfrak{sl}(2,\mathbb{C})$-valued holomorphic 1-form on Σ called the DPW potential, depending on the spectral parameter $\lambda \in \mathbb{S}^1$. More precisely, ξ can be written in term of a local coordinate z on Σ as

$$\xi(z, \lambda) = \sum_{j=-1}^{\infty} \xi_j(z) \lambda^j dz$$

   where each matrix $\xi_j(z) \in \mathfrak{sl}(2,\mathbb{C})$ depends holomorphically on z and all entries of $\xi_{-1}(z)$ are zero except for the upper right.

3. $z_0 \in \Sigma$ is a base point.

4. $\Phi_0 \in \Lambda SL(2,\mathbb{C})$ is an initial condition.

Given this data, the DPW method is the following procedure:

1. Solve the Cauchy problem ($z \in \Sigma$, $\lambda \in \mathbb{S}^1$)

   $$d_z\Phi(z,\lambda) = \Phi(z,\lambda)\xi(z,\lambda)$$

   with initial condition

   $$\Phi(z_0,\lambda) = \Phi_0(\lambda)$$

   to obtain a solution $\Phi(z,\cdot) \in \Lambda SL(2,\mathbb{C})$. (The notation d_z means that we are considering the differential with respect to the z-variable.)

2. Iwasawa decompose $\Phi(z,\lambda) = F(z,\lambda)B(z,\lambda)$. This is done for fixed $z \in \Sigma$, but it is known that $F(z,\lambda)$ and $B(z,\lambda)$ depend real-analytically on z.

3. Define $f : \Sigma \to \mathfrak{su}(2) \sim \mathbb{R}^3$ by the Sym-Bobenko formula:

   $$f(z) = \mathrm{i}\frac{\partial F}{\partial \lambda}(z,1)F(z,1)^{-1}$$

   Then f is a CMC-1 branched conformal immersion. Its Gauss map is given by

   $$N(z) = \frac{-\mathrm{i}}{2}F(z,1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}F(z,1)^{-1}$$

   Moreover, f is regular at z (meaning unbranched) if and only if the upper right entry of $\xi_{-1}(z)$ is non-zero.

In principle, any conformal CMC-1 immersion can be obtained this way. Here are two basic examples:

1. Round spheres are obtained with the data

   $$\Sigma = \mathbb{C} \cup \{\infty\} \qquad \xi(z,\lambda) = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} dz \qquad z_0 = 0 \qquad \Phi_0 = I_2$$

2. Delaunay surfaces are obtained with the data

   $$\Sigma = \mathbb{C} \setminus \{0\} \qquad \xi(z,\lambda) = \begin{pmatrix} 0 & r\lambda^{-1} + s \\ r\lambda + s & 0 \end{pmatrix} \frac{dz}{z} \qquad z_0 = 1 \qquad \Phi_0 = I_2$$

   where r, s are non-zero real numbers such that $r + s = \frac{1}{2}$.

2.4 The monodromy problem

When Σ is not simply connected, Φ is only well defined on the universal cover $\widetilde{\Sigma}$ of Σ. We identify the fundamental group $\pi_1(\Sigma, z_0)$ with the group of deck transforms of the universal cover. The action of an element $\gamma \in \pi_1(\Sigma, z_0)$ on $z \in \widetilde{\Sigma}$ will be denoted $\gamma \cdot z$. The monodromy of Φ along γ is

$$\mathcal{M}_\gamma(\Phi)(\lambda) = \Phi(\gamma \cdot z, \lambda)\Phi(z, \lambda)^{-1}$$

which does not depend on z. The standard condition which ensures that the immersion f is well defined on Σ is the following set of equations, called the monodromy problem:

$$\forall \gamma \in \pi_1(\Sigma, z_0) \quad \begin{cases} \mathcal{M}_\gamma(\Phi) \in \Lambda SU(2) & (i) \\ \mathcal{M}_\gamma(\Phi)(1) = \pm I_2 & (ii) \\ \frac{\partial \mathcal{M}_\gamma(\Phi)}{\partial \lambda}(1) = 0 & (iii) \end{cases} \tag{1}$$

2.5 Dressing and isometries

Let $H \in \Lambda SL(2, \mathbb{C})$. Let $\Phi(z, \lambda)$ be a solution of $d_z\Phi = \Phi\xi$. Then $\widetilde{\Phi} = H\Phi$ is a solution of $d_z\widetilde{\Phi} = \widetilde{\Phi}\xi$. This is called “dressing” and amounts to change the initial value Φ_0. In general, the effect of dressing on the immersion f is quite mysterious. However if $H \in \Lambda SU(2)$ then by uniqueness in the Iwasawa decomposition we have

$$\widetilde{F}(z, \lambda) = H(\lambda)F(z, \lambda)$$

and by the Sym-Bobenko formula :

$$\widetilde{f}(z) = H(1)f(z)H(1)^{-1} + i\frac{\partial H}{\partial \lambda}(1)H(1)^{-1}$$

so $\widetilde{f}$ is obtained from f by an affine isometry (the first term is an element of $SO(3)$, the second is a translation).

2.6 Gauging

Definition 1. *A gauge on Σ is a map $G : \Sigma \to \Lambda_+ SL(2, \mathbb{C})$ such that $G(z, \lambda)$ depends holomorphically on z and $G(z, 0)$ is upper triangular (with no restriction on its diagonal elements).*

Let Φ be a solution of $d_z\Phi = \Phi\xi$ and G be a gauge. Let $\widehat{\Phi} = \Phi \times G$. It turns out that $\widehat{\Phi}$ and Φ define the same immersion f. This is called "gauging". The gauged potential is

$$\widehat{\xi} = \widehat{\Phi}^{-1} d_z \widehat{\Phi} = G^{-1}\xi G + G^{-1} d_z G$$

and will be denoted $\xi \cdot G$, the dot denoting the action of the gauge group on the potential. Because of gauging, there is a lot of freedom in choosing the potential.

3 General remarks about the construction of examples

To construct an example of CMC-1 surface, we must specify the data $(\Sigma, \xi, z_0, \Phi_0)$, depending on some parameters, and adjust the parameters so that the monodromy problem is solved. For many of the examples constructed so far, the Riemann surface Σ and the potential ξ were fixed, and the only parameter that was adjusted was the initial condition $\Phi_0 \in \Lambda SL(2,\mathbb{C})$. Now for fixed $\lambda \in \mathbb{S}^1$, $\Phi_0(\lambda) \in SL(2,\mathbb{C})$ is 3 complex parameters and the monodromy problem $\mathcal{M}_\gamma(\Phi)(\lambda) \in SU(2)$ is 3 real equations. So essentially, if only Φ_0 is used, we can hope to solve the monodromy problem for two generators of the fundamental group. Hence only examples of low genus and small number of ends can be obtained this way, or symmetries must be assumed to reduce the number of equations to be solved.

In this paper we go the opposite way: we fix the initial condition $\Phi_0 = I_2$ and all parameters are in the potential ξ and the Riemann surface Σ. One thing we learnt from constructing minimal surfaces using Weierstrass Representation is that when solving period problems, the conformal type Σ can in general not be fixed and is part of the parameters that must be adjusted.

Next, because the monodromy $\mathcal{M}_\gamma(\Phi)$ is a function of λ, the parameters used to solve the monodromy problem must also be functions of λ. As a consequence, once the monodromy problem is solved, the Riemann surface Σ will depend on λ. This raises an interesting question:

If Σ_λ depends on λ, can we define an immersion f by the DPW method, and where is f defined ?

At first thought, one might think that f is defined on Σ_1 because the Sym-Bobenko formula is taken at $\lambda = 1$. But this is not what happens: as we shall see in Section 3.2, f is defined on Σ_0.

Another issue is regularity.

Definition 2. 1. *We say that the potential ξ is regular at p if ξ is holomorphic at p and the coefficient of λ^{-1} in its upper-right entry does not vanish at p.*

2. *Assume that the potential ξ has a pole at p. We say that p is a removable singularity if ξ is locally gauge-equivalent to a DPW potential which is regular at p.*

The first point ensures that f is a regular immersion at p. The second point ensures that f extends analytically at p. For example, the standard data for the round sphere (see Section 2.3) has a removable singularity at ∞. Indeed, if we consider the gauge

$$G(z,\lambda) = \begin{pmatrix} z & 0 \\ -\lambda & z^{-1} \end{pmatrix}$$

the gauged potential is

$$\xi \cdot G = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} \frac{dz}{z^2}$$

which is regular at ∞.

3.1 Change of variable depending on λ

In this section and the next one, it will be convenient to denote the dependence on λ with a subscript, so we write $\Phi_\lambda(z) = \Phi(z,\lambda)$ and $\xi_\lambda(z) = \xi(z,\lambda)$. The following theorem allows us to use λ-dependent changes of variables in the DPW method. Let U and V be fixed domains in the complex plane.

Theorem 3. *Let $\xi_\lambda(z)$ be a DPW potential, $\Phi_\lambda(z)$ a solution of $d_z\Phi_\lambda = \Phi_\lambda\xi_\lambda$ and $f(z)$ the immersion obtained by the DPW method, all defined for $z \in V$. Let $\psi_\lambda : U \xrightarrow{\sim} \psi_\lambda(U) \subset V$ be a family of diffeomorphisms depending holomorphically on $z \in U$ and $\lambda \in \mathbb{D}$. On the domain U, define $\widetilde{\xi}_\lambda = \psi_\lambda^*\xi_\lambda$ and $\widetilde{\Phi}_\lambda = \Phi_\lambda \circ \psi_\lambda$, so that $\widetilde{\xi}_\lambda$ is a DPW potential and $\widetilde{\Phi}_\lambda$ solves $d_z\widetilde{\Phi}_\lambda = \widetilde{\Phi}_\lambda\widetilde{\xi}_\lambda$. Let $\widetilde{f} : U \to \mathbb{R}^3$ be the immersion obtained by the DPW method from $\widetilde{\Phi}_\lambda$. Then $\widetilde{f} = f \circ \psi_0$ in U.*

Proof: define for $z \in U$ and $\lambda \in \mathbb{D} \setminus \{0\}$:

$$\widehat{\Phi}_\lambda(z) = \Phi_\lambda(\psi_0(z))$$

Let $\widehat{f}$ be the immersion obtained from $\widehat{\Phi}_\lambda$ by the DPW method. Since $\psi_0(z)$ does not depend on λ, we have by uniqueness of the Iwasawa decomposition (for fixed z)

$$\widehat{F}(z,\lambda) = F(\psi_0(z),\lambda)$$

This implies by Sym-Bobenko formula (again, z is fixed)

$$\widehat{f}(z) = f(\psi_0(z))$$

Define for $z \in U$ and $\lambda \in \mathbb{D} \setminus \{0\}$:

$$G(z,\lambda) = \widehat{\Phi}_\lambda(z)^{-1} \times \widetilde{\Phi}_\lambda(z) = \Phi_\lambda(\psi_0(z))^{-1} \times \Phi_\lambda(\psi_\lambda(z))$$

The following claim implies that $\widehat{\Phi}_\lambda$ and $\widetilde{\Phi}_\lambda$ are gauge-equivalent, so $\widehat{f} = \widetilde{f}$, which proves Theorem 3. □

Claim 1. *G is a gauge.*

Proof: by Gromwall inequality:

$$\begin{aligned}
||G(z,\lambda)|| &\leq \exp \int_{\psi_0(z)}^{\psi_\lambda(z)} ||\xi_\lambda|| \\
&\leq \exp\left(\frac{c_1}{\lambda}|\psi_\lambda(z) - \psi_0(z)|\right) \\
&\leq c_2
\end{aligned}$$

for some constants c_1 and c_2. (In the second line, we have used that ξ has a simple pole at $\lambda = 0$). By Riemann extension theorem, $G(z,\lambda)$ extends holomorphically at $\lambda = 0$. It remains to prove that $G(z,0)$ is upper triangular. Define

$$K(\lambda) = \begin{pmatrix} \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$$

$$\check{\Phi}_\lambda = \Phi_\lambda K(\lambda)$$

Then $\check{\Phi}_\lambda$ solves $d_z\check{\Phi}_\lambda = \check{\Phi}_\lambda\check{\xi}_\lambda$ where

$$\check{\xi}_\lambda = \xi_\lambda \cdot K(\lambda) = K(\lambda)^{-1}\xi_\lambda K(\lambda)$$

Observe that K is not a gauge and $\check{\xi}$ is not a DPW potential. But if we write

$$\xi_\lambda = \begin{pmatrix} \alpha & \lambda^{-1}\beta \\ \gamma & -\alpha \end{pmatrix}$$

with α, β, γ holomorphic at $\lambda = 0$, we have

$$\check{\xi}_\lambda = \begin{pmatrix} \alpha & \beta \\ \lambda^{-1}\gamma & -\alpha \end{pmatrix}$$

so $\check{\xi}_\lambda$ has (at most) a simple pole at $\lambda = 0$. Define

$$\check{G}(z,\lambda) = \check{\Phi}_\lambda(\psi_0(z))^{-1} \times \check{\Phi}_\lambda(\psi_\lambda(z))$$

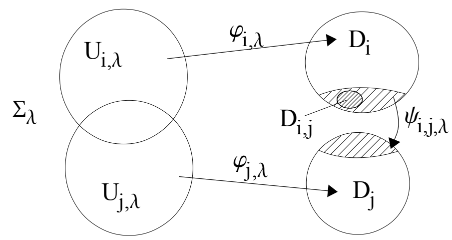

Figure 2: illustration of Definition 3

By the above Gromwall argument, $\check{G}(z,\lambda)$ extends holomorphically at $\lambda = 0$. But G and $\check{G}$ are related by

$$G(z,\lambda) = K(\lambda)\check{G}(z,\lambda)K(\lambda)^{-1}$$

This gives $G_{21}(z,\lambda) = \lambda\check{G}_{21}(z,\lambda)$, so $G_{21}(z,0) = 0$. This proves that $G(z,\lambda)$ is a gauge. $\square$

3.2 Riemann surface depending on λ

In the next definition, we allow the Riemann surface Σ to depend, in a limited way, on the spectral parameter λ. To understand the purpose of the definition, keep in mind that Iwasawa decomposition is done for fixed z, which is why we require certain domains to be independent of λ.

Definition 3. *We say a family of Riemann surfaces $(\Sigma_\lambda)_{\lambda\in\mathbb{D}}$ is admissible if each member admits an atlas $(U_{i,\lambda}, \varphi_{i,\lambda})_{i\in I}$ such that:*

1. *For all $i \in I$, $\varphi_{i,\lambda}(U_{i,\lambda}) = D_i \subset \mathbb{C}$ is independent of λ.*

2. *The change of chart $\psi_{i,j,\lambda} = \varphi_{j,\lambda} \circ \varphi_{i,\lambda}^{-1}$, when defined, is holomorphic with respect to (z,λ).*

3. *If $U_{i,0} \cap U_{j,0} \neq \emptyset$, then for all $\lambda \in \mathbb{D}$, $U_{i,\lambda} \cap U_{j,\lambda} \neq \emptyset$. Moreover, $\varphi_{i,\lambda}(U_{i,\lambda} \cap U_{j,\lambda})$ contains an open set $D_{i,j}$ independent of λ. In case $U_{i,0} \cap U_{j,0}$ is not connected, we require that $D_{i,j}$ intersects all components of $\varphi_{i,0}(U_{i,0} \cap U_{j,0})$.*

Definition 4. *For $\lambda \in \mathbb{D}$, let ξ_λ be a $\mathfrak{sl}(2,\mathbb{C})$-valued 1-forms on Σ_λ. We say that $\xi = (\xi_\lambda)_{\lambda\in\mathbb{D}}$ is an admissible potential if $\xi_{i,\lambda} = (\varphi_{i,\lambda}^{-1})^*\xi_\lambda$ is a DPW potential in D_i for all $i \in I$ (which makes sense because D_i is a fixed domain).*

In other words, in each coordinate $\varphi_{i,\lambda}$, ξ is a DPW potential in the usual sense. The following theorem shows that when the Riemann surface depends on λ, the immersion f resulting from the DPW method is defined on Σ_0. We use this result in Section 4.8.

Theorem 4. *Let $(\Sigma_\lambda)_{\lambda\in\mathbb{D}}$ be an admissible family of Riemann surfaces and $(\xi_\lambda)_{\lambda\in\mathbb{D}}$ be an admissible potential. Let Φ_λ be a solution of $d_z\Phi_\lambda = \Phi_\lambda\xi_\lambda$ on Σ_λ. Let $\Phi_{i,\lambda} = \Phi_\lambda \circ \varphi_{i,\lambda}^{-1}$ and $f_i : D_i \to \mathbb{R}^3$ be the CMC-1 immersion obtained from $\Phi_{i,\lambda}$ by the DPW method. Then whenever $U_{i,0} \cap U_{j,0} \neq \emptyset$, we have*

$$f_i \circ \varphi_{i,0} = f_j \circ \varphi_{j,0} \quad \text{in } U_{i,0} \cap U_{j,0}$$

Consequently, we may define a CMC-1 immersion $f : \Sigma_0 \to \mathbb{R}^3$ by $f = f_i \circ \varphi_{i,0}$ in each $U_{i,0}$.

Proof: we have in $D_{i,j}$

$$\xi_{i,\lambda} = \psi_{i,j,\lambda}^* \xi_{j,\lambda}$$

$$\phi_{i,\lambda} = \phi_{j,\lambda} \circ \psi_{i,j,\lambda}$$

By Theorem 3, we have in $D_{i,j}$

$$f_i = f_j \circ \psi_{i,j,0}$$

Since f_i and $f_j \circ \psi_{i,j,0}$ are real-analytic, the last equality extends to all of $\varphi_{i,0}(U_{i,0} \cap U_{j,0})$, thanks to the last hypothesis of Definition 3. Composing by $\varphi_{i,0}$, this gives

$$f_i \circ \varphi_{i,0} = f_j \circ \varphi_{j,0} \quad \text{in } U_{i,0} \cap U_{j,0}$$

$\square$

4 Construction of n-noids

4.1 The DPW potential

Consider the n-punctured Riemann sphere

$$\Sigma = \mathbb{C} \cup \{\infty\} \setminus \{p_1, \cdots, p_n\}$$

Consider a meromorphic 1-form ω with double poles at $p_1, \cdots, p_n$

$$\omega(z) = \sum_{i=1}^n \left(\frac{a_i}{(z-p_i)^2} + \frac{b_i}{z-p_i} \right) dz$$

We take the following DPW potential

$$\xi(z,\lambda) = \begin{pmatrix} 0 & \lambda^{-1}dz \\ t(\lambda-1)^2\omega(z) & 0 \end{pmatrix}$$

with the initial data

$$z_0 = 0 \qquad \Phi_0 = I_2$$

Here t is a real parameter close to 0. We assume all a_i are non-zero real numbers and all p_i are distinct complex numbers. We denote $\gamma_i \in \pi_1(\Sigma, z_0)$ the homotopy class of a loop encircling p_i.

Potentials of this form have been introduced for 3-noids in [17]. Before working out the equations that must be solved, let us mention some nice properties of this potential:

1. If $t = 0$, we get the standard data for the round sphere (see Section 2.3). Therefore, away from the poles, we are constructing a perturbation of the round sphere.

2. If $\lambda = 1$, the potential is holomorphic at p_i so $\mathcal{M}_{\gamma_i}(\Phi)(1) = I_2$. Therefore, Equations (ii) of the monodromy problem (1) is automatically solved, and also Equation (iii) because of the factor $(\lambda-1)^2$ in the potential.

3. At each p_i, the potential is locally gauge equivalent to a potential with a simple pole and the same residue as the standard Delaunay potential. Indeed, let $(r,s) \in \mathbb{R}^2$ be the solution of the system

$$\begin{cases} rs = ta_i \\ r + s = \frac{1}{2} \\ r > s \end{cases} \tag{2}$$

Let $w = z - p_i$ and consider the gauge

$$G_i(w,\lambda) = \begin{pmatrix} \sqrt{\frac{w}{r+s\lambda}} & 0 \\ \frac{-\lambda}{2\sqrt{(r+s\lambda)w}} & \sqrt{\frac{r+s\lambda}{w}} \end{pmatrix} \tag{3}$$

The gauged potential is

$$\widehat{\xi} = \xi \cdot G_i = \begin{pmatrix} 0 & \frac{(r\lambda^{-1}+s)dw}{w} \\ \frac{t(\lambda-1)^2w^2\omega + \frac{\lambda}{4}dw}{(r+s\lambda)w} & 0 \end{pmatrix} \tag{4}$$

It has a simple pole at p_i with residue

$$\mathrm{Res}_{p_i}\widehat{\xi} = \begin{pmatrix} 0 & r\lambda^{-1} + s \\ \frac{ta_i(\lambda-1)^2 + \frac{\lambda}{4}}{r+s\lambda} & \end{pmatrix}$$

Now using $r+s=\frac{1}{2}$, we have

$$(r+s\lambda)(r\lambda+s) = rs(\lambda-1)^2 + \frac{\lambda}{4} = ta_i(\lambda-1)^2 + \frac{\lambda}{4} \tag{5}$$

Hence the residue of $\widehat{\xi}$ is equal to the matrix

$$A_{r,s}(\lambda) = \begin{pmatrix} 0 & r\lambda^{-1}+s \\ r\lambda+s & 0 \end{pmatrix} \tag{6}$$

which is the residue of the standard Delaunay potential (see Section 2.3). Therefore, provided the monodromy problem is solved, the surface will have a Delaunay end at p_i of weight $8\pi rs = 8\pi ta_i$ by the work of Kilian, Rossman and Schmitt in [14].

4.2 Regularity at ∞

We want ∞ to be a removable singularity of ξ. Consider the gauge

$$G_\infty(z,\lambda) = \begin{pmatrix} z & 0 \\ -\lambda & z^{-1} \end{pmatrix}$$

The gauged potential is

$$\xi \cdot G_\infty = \begin{pmatrix} 0 & \lambda^{-1}\frac{dz}{z^2} \\ t(\lambda-1)^2 z^2\omega & 0 \end{pmatrix}$$

which is regular at ∞ provided ω has a double zero at ∞. Using the coordinate $w=1/z$ in a neighborhood of ∞, we obtain

$$\omega = -\sum_{i=1}^{n} \left(a_i(1+2p_iw) + b_i(w^{-1}+p_i+p_i^2w) + O(w^2)\right) dw$$

So we have to solve the following three equations:

$$\begin{cases} \sum_{i=1}^n b_i = 0 \\ \sum_{i=1}^n (a_i + b_ip_i) = 0 \\ \sum_{i=1}^n (2a_ip_i + b_ip_i^2) = 0 \end{cases} \tag{7}$$

4.3 The monodromy problem

Let $\Phi(z,\lambda)$ be the solution of the Cauchy Problem $d_z\Phi = \Phi\xi$ with initial condition $\Phi(0,\lambda) = I_2$. Let $x = (a_i, b_i, p_i)_{1\leq i\leq n}$ be the collection of all parameters (except t) which are involved in the definition of the potential ξ. Of course $\Phi(z,\lambda)$ also depends on t and x. We write $\Phi_{t,x}$ when we need to emphasize the dependence of Φ on these parameters. Let

$$M_i(t,x,\lambda) := \mathcal{M}_{\gamma_i}(\Phi_{t,x})(\lambda)$$

be the monodromy of $\Phi_{t,x}$ along γ_i. We need to solve Equation (i) of the monodromy problem (1), namely

$$M_i(t,x,\cdot) \in \Lambda SU(2) \quad \text{for } 1 \leq i \leq n-1 \tag{8}$$

Indeed, the fundamental group of the n-punctured sphere is generated by $\gamma_1, \cdots, \gamma_{n-1}$. Recall that the matrix exponential is a local diffeomorphism from a neighborhood of 0 in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ (respectively $\mathfrak{su}(2)$) to a neighborhood of I_2 in $SL(2,\mathbb{C})$ (respectively $SU(2)$). The inverse diffeomorphism is denoted log. Let

$$\widetilde{M}_i(t,x,\lambda) := \frac{\lambda}{t(\lambda-1)^2}\log M_i(t,x,\lambda)$$

Observe that $\frac{(\lambda-1)^2}{\lambda} = \lambda + \lambda^{-1} - 2$ is real on the unit circle. Assuming that t is real, (8) is equivalent to

$$\widetilde{M}_i(t,x,\cdot) \in \Lambda\mathfrak{su}(2) \quad \text{for } 1 \leq i \leq n-1 \tag{9}$$

Proposition 1. *$\widetilde{M}_i(t,x,\lambda)$ extends holomorphically at $t=0$ and $\lambda=1$. Moreover, we have at $t=0$*

$$\widetilde{M}_i(0,x,\lambda) = 2\pi\mathrm{i}\begin{pmatrix} a_i + b_ip_i & -\lambda^{-1}(2a_ip_i + b_ip_i^2) \\ \lambda b_i & -a_i - b_ip_i \end{pmatrix}$$

Proof: let $\mu = t(\lambda-1)^2$ and let us temporarily see μ as a complex parameter independent of (t,λ) in the definition of ξ. If $\mu = 0$, then ξ is holomorphic in a neighborhood of p_i so $M_i = I_2$. Consequently, $\frac{1}{\mu}\log M_i$ extends holomorphically at $\mu = 0$. Moreover,

$$\widetilde{M}_i(0,x,\lambda) = \lambda\frac{\partial M_i}{\partial\mu}(0,x,\lambda)$$

By Proposition 9 in [19], the derivative of the monodromy is given by the following formula (the order of the products is reversed because the differential equation is $d_z\Phi = \Phi\xi$ instead of $d_z\Phi = \xi\Phi$)

$$\frac{\partial M_i}{\partial\mu}(0,x,\lambda) = \int_{z\in\gamma_i}\Phi_{0,x}(z,\lambda)\frac{\partial\xi}{\partial\mu}\Phi_{0,x}(z,\lambda)^{-1}$$

Using the definition of ξ and the residue theorem, we obtain

$$\begin{aligned}\widetilde{M}_i(0,x,\lambda) &= 2\pi\mathrm{i}\,\lambda\,\mathrm{Res}_{p_i}\begin{pmatrix}1 & \lambda^{-1}z\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ \omega & 0\end{pmatrix}\begin{pmatrix}1 & -\lambda^{-1}z\\ 0 & 1\end{pmatrix}\\ &= 2\pi\mathrm{i}\ \mathrm{Res}_{p_i}\begin{pmatrix}z & -\lambda^{-1}z^2\\ \lambda & -z\end{pmatrix}\left(\frac{a_i}{(z-p_i)^2}+\frac{b_i}{z-p_i}\right)\end{aligned}$$

Proposition 1 follows from the following elementary residue computation for $k\in\mathbb{N}$

$$\mathrm{Res}_p\frac{z^k}{(z-p)^2}=kp^{k-1}$$

$\square$

4.4 Functional spaces

Our goal is to solve Equations (7) and (9) using the implicit function theorem at $t=0$. This will determine the parameters a_i, b_i and p_i as functions of t. Since Equation (9) depends on λ, the parameters must be functions of λ. In this section, we introduce suitable functional spaces for this problem. We decompose a function $f:\mathbb{S}^1\to\mathbb{C}$ in Fourier series

$$f(\lambda)=\sum_{i\in\mathbb{Z}}f_i\lambda^i$$

Fix some $\rho>1$ and define

$$||f||=\sum_{i\in\mathbb{Z}}|f_i|\rho^{|i|}$$

Let $\mathcal{W}$ be the space of functions f with finite norm. This is a Banach algebra (classically called the Wiener algebra when $\rho=1$, whence the letter $\mathcal{W}$). Functions in $\mathcal{W}$ are holomorphic in the annulus $\frac{1}{\rho}<|\lambda|<\rho$, and in particular are smooth on $\mathbb{S}^1$ as required for the DPW method.

We define $\mathcal{W}^{\geq 0}$, $\mathcal{W}^{+}$, $\mathcal{W}^{\leq 0}$ and $\mathcal{W}^{-}$ as the subspaces of functions f such that $f_i=0$ for $i<0$, $i\leq 0$, $i>0$ and $i\geq 0$, respectively. Functions in $\mathcal{W}^{\geq 0}$ extend holomorphically to the disk $D(0,\rho)$. We also write $\mathcal{W}^0$ for the subspace of constant functions, so we have a direct sum $\mathcal{W}=\mathcal{W}^-\oplus\mathcal{W}^0\oplus\mathcal{W}^+$. A function f will be decomposed as $f=f^-+f^0+f^+$. We define the star operator by

$$f^*(\lambda)=\overline{f\left(\frac{1}{\overline{\lambda}}\right)}=\sum_{i\in\mathbb{Z}}\overline{f_{-i}}\lambda^i$$

The involution $f \mapsto f^*$ exchanges $\mathcal{W}^{\geq 0}$ and $\mathcal{W}^{\leq 0}$. We have $\lambda^* = \lambda^{-1}$ and $c^* = \overline{c}$ if c is a constant.

The parameters a_i, b_i and p_i must be holomorphic in $\mathbb{D}$ for ξ to be an admissible DPW potential, so we take them in the space $\mathcal{W}^{\geq 0}$. Recall that $x = (a_i, b_i, p_i)_{1 \leq i \leq n} \in (\mathcal{W}^{\geq 0})^{3n}$ denotes the collection of all parameters. We define the following functions:

$$\mathcal{F}_i(t, x, \lambda) = \widetilde{M}_{i,11}(t, x, \lambda) + \widetilde{M}^*_{i,11}(t, x, \lambda)$$

$$\mathcal{G}_i(t, x, \lambda) = \lambda \left(\widetilde{M}_{i,12}(t, x, \lambda) + \widetilde{M}^*_{i,21}(t, x, \lambda) \right)$$

where we write $\widetilde{M}_i = (\widetilde{M}_{i,jk})_{1 \leq j,k \leq 2}$ and

$$\mathcal{H}_1(x) = \sum_{i=1}^{n} b_i$$

$$\mathcal{H}_2(x) = \sum_{i=1}^{n} a_i + b_i p_i$$

$$\mathcal{H}_3(x) = \sum_{i=1}^{n} 2a_i p_i + b_i p_i^2$$

Equations (7) and (9) are equivalent to

$$\begin{cases} \mathcal{F}_i(t, x, \lambda) = 0 & \text{for } 1 \leq i \leq n-1 \\ \mathcal{G}_i(t, x, \lambda) = 0 & \text{for } 1 \leq i \leq n-1 \\ \mathcal{H}_i(x) = 0 & \text{for } 1 \leq i \leq 3 \end{cases} \tag{10}$$

Note that the coefficients of ξ are in the space $\mathcal{W}$ so by the theorem on smooth dependence on parameters for the solution of an ODE, the coefficients of $\Phi_{t,x}(z, \lambda)$ are in $\mathcal{W}$ and depend holomorphically on $x \in (\mathcal{W}^{\geq 0})^{3n}$. Hence $\mathcal{F}_i$ and $\mathcal{G}_i$ are smooth maps between Banach spaces.

4.5 Solving the equations at $t = 0$

Proposition 2. *When $t = 0$, the system (10) is equivalent to the following conditions, for $1 \leq i \leq n$:*

(i) a_i is a real constant.

(ii) p_i is constant.

(iii) $b_i = \dfrac{-2a_i\overline{p_i}}{1+|p_i|^2}$

(iv) $\sum_{i=1}^{n} a_i\pi^{-1}(p_i) = 0$ *where π is the stereographic projection from the south pole:*

$$\pi^{-1}(z) = \left(\frac{2\,\mathrm{Re}z}{1+|z|^2}, \frac{2\,\mathrm{Im}z}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\right) \tag{11}$$

Proof: Using Proposition 1, we have at $t = 0$:

$$\mathcal{F}_i(0, x, \lambda) = 2\pi\mathrm{i}\left((a_i + b_i p_i) - (a_i + b_i p_i)^*\right) \tag{12}$$

$$\mathcal{G}_i(0, x, \lambda) = -2\pi\mathrm{i}(2a_i p_i + b_i p_i^2 + b_i^*) \tag{13}$$

Hence at $t = 0$:

$$\sum_{i=1}^{n} \mathcal{F}_i = 2\pi\mathrm{i}(\mathcal{H}_2 - \mathcal{H}_2^*) \tag{14}$$

$$\sum_{i=1}^{n} \mathcal{G}_i = -2\pi\mathrm{i}(\mathcal{H}_3 + \mathcal{H}_1^*) \tag{15}$$

Assume that $t = 0$ and let $x = (a_i, b_i, p_i)_{1\leq i\leq n}$ be a solution of System (10). From Equations (14) and (15) we infer that $\mathcal{F}_n = \mathcal{G}_n = 0$. From Equation (13), we see that $b_i \in \mathcal{W}^{\leq 0} \cap \mathcal{W}^{\geq 0} = \mathcal{W}^0$ hence b_i is constant. From Equation (12), we obtain by the same argument

$$a_i + b_i p_i = c_i \text{ (a real constant)} \tag{16}$$

Eliminating a_i from Equations (13) and (16), we obtain

$$-b_i p_i^2 + 2c_i p_i + \overline{b_i} = 0$$

Hence p_i can take only two values, so must be constant, and so a_i is constant. (If $b_i = 0$, it is straightforward that a_i and p_i are constant.) Multipliying (13) by $\overline{p_i}$ we obtain

$$(2a_i + b_i p_i)|p_i|^2 + \overline{b_i p_i} = 0 \tag{17}$$

Taking the imaginary part and using $\mathrm{Im}(b_i p_i) = -\mathrm{Im}(a_i)$, we obtain

$$\mathrm{Im}(a_i)(|p_i|^2 + 1) = 0$$

Hence $a_i \in \mathbb{R}$ and so $b_i p_i \in \mathbb{R}$. Equation (17) gives

$$b_i = \frac{-2a_i\overline{p_i}}{1+|p_i|^2}$$

$$\mathcal{H}_1 = -2\sum_{i=1}^{n} a_i \frac{\overline{p_i}}{1+|p_i|^2} = 0$$

$$\mathcal{H}_2 = \sum_{i=1}^{n} a_i \frac{1-|p_i|^2}{1+|p_i|^2} = 0$$

which gives Point (iv). The proof of the reciprocal statement is a straightforward computation and is omitted. □

4.6 Solving the equations using the implicit function theorem

Fix a value $x^\circ = (a_i, b_i, p_i)_{1\le i\le n}$ of the parameters satisfying Conditions (i)-(iv) of Proposition 2 and such that $a_i \neq 0$ for $1 \le i \le n$.

Proposition 3. *For t in a neighborhood of 0, there exists a smooth map $x(t) = (a_{i,t}, b_{i,t}, p_{i,t})_{1\le i\le n}$ with value in $(\mathcal{W}^{\geq 0})^{3n}$ such that $x(0) = x^\circ$ and $\mathcal{F}_i(t, x(t), \lambda) = \mathcal{G}_i(t, x(t), \lambda) = 0$ for $1 \le i \le n-1$ and $\mathcal{H}_i(x(t)) = 0$ for $1 \le i \le 3$. Moreover, for $1 \le i \le n-1$ we have $\mathrm{Re}(a_{i,t}(0)) = a_i$ and $p_{i,t}(0) = p_i$.*

Proof: We compute the partial differentials of Equations (12) and (13) with respect to x and get

$$d_x\mathcal{F}_i(0, x^\circ, \lambda) = 2\pi \mathrm{i}\,(da_i + p_i db_i + b_i dp_i) - 2\pi\mathrm{i}\,(da_i + p_i db_i + b_i dp_i)^*$$

$$d_x\mathcal{G}_i(0, x^\circ, \lambda)\cdot X = -2\pi\mathrm{i}\left(2p_i da_i + p_i^2 db_i + 2(a_i + b_i p_i)dp_i + db_i^*\right)$$

By definition, $\mathcal{F}_i^- = (\mathcal{F}_i^+)^*$ so we do not need to solve the equation $\mathcal{F}_i^- = 0$. Keep in mind that a_i, b_i, p_i are constant, whereas da_i, db_i, dp_i are in $\mathcal{W}^{\geq 0}$. Projecting on $\mathcal{W}^+$, $\mathcal{W}^0$ and $\mathcal{W}^-$ we obtain

$$d_x\mathcal{F}_i^+ = 2\pi\mathrm{i}\left(da_i^+ + p_i db_i^+ + b_i dp_i^+\right)$$

$$d_x\mathcal{F}_i^0 = -4\pi\,\mathrm{Im}(da_i^0 + p_i db_i^0 + b_i dp_i^0)$$

$$d_x\mathcal{G}_i^+ = -2\pi\mathrm{i}\left(2p_i da_i^+ + p_i^2 db_i^+ + 2(a_i + b_i p_i)dp_i^+\right)$$

$$d_x\mathcal{G}_i^0 = -2\pi\mathrm{i}\left(2p_i da_i^0 + p_i^2 db_i^0 + 2(a_i + b_i p_i)dp_i^0 + \overline{db_i^0}\right)$$

$$d_x\mathcal{G}_i^- = -2\pi\mathrm{i}(db_i^+)^*$$
$$(d_x\mathcal{G}_i^-)^* = 2\pi\mathrm{i}db_i^+$$

We restrict the parameters to the subspaces defined by $\mathrm{Re}(da_i^0) = 0$ and $dp_i^0 = 0$ for $1 \leq i \leq n-1$.

Claim 2. *1. For $1 \leq i \leq n-1$, the partial differential of $(\mathcal{F}_i^+, \mathcal{G}_i^+, (\mathcal{G}_i^-)^*, \mathcal{F}_i^0, \mathcal{G}_i^0)$ only depends on the variables $(a_i^+, b_i^+, p_i^+, \mathrm{Im}(a_i^0), b_i^0)$ and is an $\mathbb{R}$-linear automorphism of $(\mathcal{W}^+)^3 \times \mathbb{R} \times \mathbb{C}$.*

2. The partial differential of $(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ with respect to (a_n, b_n, p_n) is an automorphism of $(\mathcal{W}^{\geq 0})^3$.

Proof:

1. We can write in matrix form
$$\begin{pmatrix} d_x\mathcal{F}_i^+ \\ d_x\mathcal{G}_i^+ \\ (d_x\mathcal{G}_i^-)^* \end{pmatrix} = 2\pi\mathrm{i} \begin{pmatrix} 1 & p_i & b_i \\ -2p_i & -p_i^2 & -2(a_i + b_i p_i) \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} da_i^+ \\ db_i^+ \\ dp_i^+ \end{pmatrix}$$
This constant matrix has determinant $2a_i$ so is invertible. If $p_i \neq 0$, we multiply $d_x\mathcal{G}_i^0$ by $\overline{p_i}$ and write in matrix form
$$\begin{pmatrix} d_x\mathcal{F}_i^0 \\ \mathrm{Re}(\overline{p_i}d_x\mathcal{G}_i^0) \\ \mathrm{Im}(\overline{p_i}d_x\mathcal{G}_i^0) \end{pmatrix} = 2\pi \begin{pmatrix} -2 & 0 & -2 \\ 2|p_i|^2 & 0 & |p_i|^2 - 1 \\ 0 & -|p_i|^2 - 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Im}(da_i^0) \\ \mathrm{Re}(p_i db_i^0) \\ \mathrm{Im}(p_i db_i^0) \end{pmatrix}$$
This matrix has determinant $16\pi^3(|p_i|^2 + 1)^2$. This proves Point 1 of Claim 2 if $p_i \neq 0$. If $p_i = 0$ then $d_x\mathcal{F}_i^0 = -4\pi\,\mathrm{Im}(da_i^0)$ and $d_x\mathcal{G}_i^0 = -2\pi\mathrm{i}\,\overline{db_i^0}$ so the conclusion is straightforward.

2. The partial differential of $(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ with respect to the remaining variable $x_n = (a_n, b_n, p_n)$ can be written in matrix form as
$$\begin{pmatrix} d_{x_n}\mathcal{H}_1 \\ d_{x_n}\mathcal{H}_2 \\ d_{x_n}\mathcal{H}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & p_n & b_n \\ 2p_n & p_n^2 & 2(a_n + b_n p_n) \end{pmatrix} \begin{pmatrix} da_n \\ db_n \\ dp_n \end{pmatrix}$$
This matrix has determinant $-2a_n$. This proves Point 2 of Claim 2. □

Proposition 3 follows from the implicit function theorem. (The partial differential of all equations with respect to all parameters has lower triangular block form.) □

4.7 Each parameter $a_{i,t}$ is a real constant

Let $x(t) = (a_{i,t}, b_{i,t}, p_{i,t})_{1\leq i\leq n}$ be determined by Proposition 3. A priori, $a_{i,t} \in \mathcal{W}^{\geq 0}$ is a function of λ. This is a problem to ensure that the ends are asymptotic to Delaunay surfaces since the solution (r,s) of System (2) then depends on λ. Fortunately:

Proposition 4. *Each coefficient $a_{i,t}$ is a real constant (with respect to λ).*

Proof: we use the standard theory of fuchsian systems. Consider again the gauge G_i defined in Equation (3). The gauged potential $\widehat{\xi}$ has a simple pole at p_i so $d_z\widehat{\Phi} = \widehat{\Phi}\widehat{\xi}$ is a fuchsian system. Using Equation (5), the eigenvalues of the residue matrix $A_{r,s}$ are $\pm\Lambda$ with

$$\Lambda = \sqrt{\frac{1}{4} + t a_{i,t}\lambda^{-1}(\lambda-1)^2}$$

The system is resonant when $2\Lambda \in \mathbb{Z}$. Hence provided t is small enough, the system is resonant if and only if $\lambda = 1$. Fix some $\lambda \neq 1$ on the unit circle. Since the system is non-resonant, its solution has the standard form (see Proposition 11.2 in [18])

$$\widehat{\Phi}(z,\lambda) = V(\lambda)\exp(A_{r,s}(\lambda)\log(z-p_i))U(z,\lambda)$$

where $U(z,\lambda)$ is a well defined holomorphic function of z in a neighborhood of p_i. Hence

$$\mathcal{M}_{\gamma_i}(\widehat{\Phi})(\lambda) = V(\lambda)\exp(2\pi \mathrm{i} A_{r,s}(\lambda))V(\lambda)^{-1}$$

with eigenvalues $\exp(\pm 2\pi\mathrm{i}\Lambda)$. Since gauging does not change monodromy, $\mathcal{M}_{\gamma_i}(\widehat{\Phi})(\lambda) \in SU(2)$. Hence its eigenvalues have modulus 1 so $\Lambda \in \mathbb{R}$. This implies that $a_{i,t}$ is real on $\mathbb{S}^1 \setminus \{1\}$. Since $a_{i,t} \in \mathcal{W}^{\geq 0}$, it is constant. $\square$

Remark 1. *Another strategy is to impose a priori that a_i is a real constant for $1 \leq i \leq n-1$ and use the information that the eigenvalues of $\mathcal{M}_{\gamma_i}(\Phi)$ have modulus one to solve the monodromy problem with the parameters (b_i, p_i). This works but is not really any simpler.*

4.8 Proof of Theorem 1

Consider n unit vectors $u_1, \cdots, u_n$ and n non-zero weights $\tau_1, \cdots, \tau_n$ satisfying the balancing condition of Theorem 1. Without loss of generality, we may assume that no vector u_i is equal to $(0,0,-1)$. In view of Proposition 2, we take

$$a_i = \tau_i \qquad p_i = \pi(u_i) \qquad b_i = \frac{-2\tau_i\overline{p_i}}{1+|p_i|^2} \tag{18}$$

where $\pi : \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$ is the stereographic projection from the south pole. Equation (iv) of Proposition 2 is equivalent to the balancing condition of Theorem 1. Proposition 3 gives us functions $a_{i,t}$, $b_{i,t}$ and $p_{i,t}$ in $\mathcal{W}^{\geq 0}$ such that the regularity problem (7) and the monodromy problem (8) are solved. Moreover, by Proposition 4, each $a_{i,t}$ is a real constant.

We denote $\xi_t(z, \lambda)$ the potential that we have constructed, $\Phi_t(z, \lambda)$ the solution of $d_z\Phi_t = \Phi_t\xi_t$ with initial condition $\Phi_t(0) = I_2$ and $f_t(z)$ the immersion obtained from the DPW method, translated by the vertical vector $(0, 0, 1)$. The Riemann surface is

$$\Sigma_{t,\lambda} = \mathbb{C} \cup \{\infty\} \setminus \{p_{1,t}(\lambda), \cdots, p_{n,t}(\lambda)\}$$

For fixed t, the family $(\Sigma_{t,\lambda})_{\lambda\in\mathbb{D}}$ is admissible in the sense of Definition 3 by considering the following atlas:

$$\varphi_{i,\lambda}(z) = z - p_{i,t}(\lambda) \qquad U_{i,\lambda} = D^*(p_{i,t}(\lambda), \epsilon) \qquad D_i = D^*(0, \epsilon) \qquad \text{for } 1 \leq i \leq n$$

$$\varphi_{0,\lambda}(z) = z \qquad U_{0,\lambda} = D_0 = D(0, \tfrac{1}{\epsilon}) \setminus \bigcup_{i=1}^{n} \overline{D}(p_i, \tfrac{\epsilon}{2})$$

$$\varphi_{\infty,\lambda}(z) = \frac{1}{z} \qquad U_{\infty,\lambda} = \mathbb{C} \cup \{\infty\} \setminus \overline{D}(0, \tfrac{1}{2\epsilon}) \qquad D_\infty = D(0, 2\epsilon)$$

Here ϵ is a small enough positive number. By Theorem 4, f_t is defined on $\Sigma_{t,0}$. As seen in Section 4.1, f_t has a Delaunay end of weight $w_{i,t} = 8\pi t a_{i,t}$ at $p_{i,t}(0)$ for $1 \leq i \leq n$. It remains to compute the axes of the ends and to prove the embeddedness statement of Theorem 1. For that purpose, the result obtained in [14] is not enough: it gives us for each t a positive ε such that f_t is close to a Delaunay immersion in the punctured disk $D^*(p_{i,t}(0), \varepsilon)$, but that ε depends on t and goes to 0 as $t \to 0$. Thomas Raujouan has improved this result in [15] and was able to obtain a uniform ε under additional assumptions:

Theorem 5. *[15] Let $\xi_t(z, \lambda)$ be a family of DPW potentials depending on the parameter $t \geq 0$ and defined for z in a punctured neighborhoof of 0. Let $\Phi_t(z, \lambda)$ a solution of $d_z\Phi_t = \Phi_t\xi_t$ and $f_t(z)$ be the immersion obtained by the DPW method, translated by $(0, 0, 1)$. Assume the following hypotheses:*

1. *$\xi_t = A_{r,s}\frac{dz}{z} + O(t, z^0)$ where (r, s) is the solution of $r + s = \frac{1}{2}$, $rs = t\tau$, $r > s$ and $A_{r,s}$ is the standard Delaunay residue given by Equation (6).*

2. *The monodromy of Φ_t around the origin is in $\Lambda SU(2)$.*

3. $\Phi_0(1,\lambda) = \left(\begin{smallmatrix} a & \lambda^{-1}b \\ \lambda c & d \end{smallmatrix}\right)$ *where* a, b, c, d *are complex numbers.*

Then there exists uniform positive numbers ε, c, α *and a family of Delaunay immersions* $f_t^D : \mathbb{C}^* \to \mathbb{R}^3$ *such that for* $t \neq 0$ *small enough and* $z \in D^*(0,\varepsilon)$

$$||f_t(z) - f_t^D(z)|| \leq ct|z|^\alpha$$

The end of f_t^D *at* $z = 0$ *has weight* $8\pi\tau t$ *and its axis converges when* $t \to 0$ *to the half-line through the origin spanned by the vector* $H(1) \cdot e_3$ *where* (e_1, e_2, e_3) *denotes the canonical basis of* $\mathbb{R}^3$*, the dot denotes the isometric action of* $SU(2)$ *on* $\mathbb{R}^3$ *and* $H(\lambda)$ *is the unitary factor in the Iwasawa decomposition of*

$$M(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} b+a & \lambda^{-1}(b-a) \\ \lambda(c+d) & d-c \end{pmatrix}$$

Moreover, if $\tau > 0$ *then the restriction of* f_t *to* $D^*(0,\varepsilon)$ *is an embedding.*

Fix some index i and consider the change of coordinate $w = \varphi_{i,\lambda}(z) = z - p_{i,t}(\lambda)$. We apply Theorem 5 to the gauged potential $\widehat{\xi_t} = \xi_t \cdot G_i$ where G_i is the gauge defined by Equation (3). By Equation (4), $\widehat{\xi_t}$ satisfies Hypothesis 1 of Theorem 5. Hypothesis 2 is satisfied because gauging does not change monodromy. To check Hypothesis 3, we compute $\widehat{\Phi}_0$, observing that when $t = 0$, $p_{i,0}(\lambda) = p_i$, $r = \frac{1}{2}$ and $s = 0$ so

$$\widehat{\Phi}_0(w,\lambda) = \begin{pmatrix} 1 & \lambda^{-1}(p_i + w) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2w} & 0 \\ \frac{-\lambda}{\sqrt{2w}} & \frac{1}{\sqrt{2w}} \end{pmatrix}$$

This gives

$$\widehat{\Phi}_0(1,\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - p_i & \lambda^{-1}(1 + p_i) \\ -\lambda & 1 \end{pmatrix}$$

Theorem 5 gives us a uniform $\varepsilon > 0$ such that on each punctured disk $D^*(p_{i,t}(0), \varepsilon)$, f_t is t-close to a Delaunay immersion $f_{i,t}^D$. To find the limit axis of $f_{i,t}^D$ as $t \to 0$ we compute

$$M(\lambda) = \begin{pmatrix} 1 & \lambda^{-1}p_i \\ 0 & 1 \end{pmatrix}$$

$$H(\lambda) = \frac{1}{\sqrt{1 + |p_i|^2}} \begin{pmatrix} 1 & \lambda^{-1}p_i \\ -\lambda\overline{p_i} & 1 \end{pmatrix}$$

$$
\begin{aligned}
H(1)\cdot e_3 &= \frac{-\mathrm{i}}{2} H(1) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} H(1)^{-1} \\
&= \frac{-\mathrm{i}}{2} \frac{1}{1+|p_i|^2} \begin{pmatrix} |p_i|^2-1 & 2p_i \\ 2\overline{p_i} & 1-|p_i|^2 \end{pmatrix} \in \mathfrak{su}(2) \\
&= \frac{1}{1+|p_i|^2}\left(2\operatorname{Re}(p_i), 2\operatorname{Im}(p_i), 1-|p_i|^2\right) \in \mathbb{R}^3 \\
&= \pi^{-1}(p_i) \quad \text{by Equation (11)} \\
&= u_i
\end{aligned}
$$

This proves Point 1 of Theorem 1. Let Ω_r be the compact domain $\mathbb{C}\cup\{\infty\}$ minus the disks $D(p_i,r)$ for $1\leq i\leq n$. As $t\to 0$, ξ_t converges on Ω_r to the spherical potential $\xi_0 = \left(\begin{smallmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{smallmatrix}\right) dz$ and Φ_t converges to $\Phi_0 = \left(\begin{smallmatrix} 1 & \lambda^{-1}z \\ 0 & 1 \end{smallmatrix}\right)$. The corresponding immersion is the standard conformal immersion of the sphere $f_0(z)=\pi^{-1}(z)$. So $f_t(\Omega_r)$ converges smoothly as $t\to 0$ to the round sphere minus n spherical caps (almost) centered at $u_1,\cdots,u_n$.

For ease of notation, we forget from now on the index $\lambda=0$ so we write $\Sigma_t=\Sigma_{t,0}$ and $p_{i,t}=p_{i,t}(0)$. Assume that all weights τ_i are positive and that for $j\neq i$, the angle between u_i and u_j is greater than $\frac{\pi}{3}$. Let $M_t=f_t(\Sigma_t)$. By Theorem 5, $f_t(D^*(p_{i,t},\varepsilon))$ is embedded and close to a piece of a Delaunay surface. By the angle hypothesis, these pieces do not intersect. It should be rather clear at this point that M_t is embedded. In the rest of this section, we give a formal proof of embeddedness.

Define $h_i(x)=\langle x,u_i\rangle$ for $x\in\mathbb{R}^3$. Consider the following domains for $1\leq i\leq n$ and non-negative numbers θ,d (see Figure 3):

- $C_{i,\theta}\subset\mathbb{R}^3$ is the (solid) cone with vertex at the origin, axis u_i and angle $\frac{\pi}{6}+\theta$ defined by the inequality $h_i(x)>\cos(\frac{\pi}{6}+\theta)||x||$.
- $C_{i,\theta,d}=\{x\in\mathbb{C}_{i,\theta}\mid h_i(x)>d\}$ is a truncated cone.
- $K_d\subset\mathbb{R}^3$ is the convex set defined by $h_i(x)<d$ for $1\leq i\leq n$.

Thanks to the angle hypothesis, we can fix a small $\theta>0$ such that the cones $C_{i,\theta}$ for $1\leq i\leq n$ are disjoint.

Claim 3. *1. For all $r>0$, there exists $\delta_1(r)<1$ such that for t small enough, $f_t(\Omega_r)\subset K_{\delta_1(r)}$.*

2. For all $r\in(0,\varepsilon]$, there exists $\delta_2(r)<1$ such that for t small enough, $f_t(D^*(p_{i,t},r))\subset C_{i,\theta,\delta_2(r)}$. Moreover, $\lim_{r\to 0}\delta_2(r)=1$.

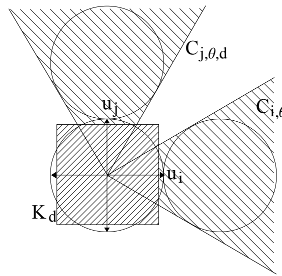

Figure 3: the domains $C_{i,\theta,d}$ and K_d.

Proof: for $a \in \mathbb{R}^3$, let $S(a)$ denote the unit sphere centered at a.

1. Let $r > 0$. As we have seen, $\lim_{t\to 0} f_t(\Omega_r)$ is equal to $S(0)$ minus n spherical caps centered at $u_1, \cdots, u_n$. There exists $d < 1$ such that this set is included in the convex set K_d. Let $\delta_1(r) = \frac{1+d}{2} > d$. Then for t small enough, $f_t(\Omega_r) \subset K_{\delta_1(r)}$.

2. Let $r \in (0, \varepsilon]$. As $t \to 0$, the image of the Delaunay immersion f_t^D converges to the union of the spheres $S(2ku_i)$ for $k \in \mathbb{Z}$ (a chain of spheres). Hence $f_t^D(D^*(p_{i,t}, r))$ converges to the union of the spheres $S(2ku_i)$ for $k \geq 1$ and a small spherical cap centered at u_i in $S(0)$. Take the largest value of d such that this set is included in the closure of $C_{i,0,d}$. (Observe that the sphere $S(2u_i)$ is tangent to the boundary of the cone $C_{i,0}$, see Figure 3.) Clearly $d < 1$. Let $\delta_2(r) = 2d - 1 < d$. Then for t small enough $f_t(D^*(p_{i,t}, r)) \subset C_{i,\theta,\delta_2(r)}$. As $r \to 0$, the radius of the spherical cap goes to 0 so $d \to 1$ and $\delta_2(r) \to 1$. □

Using Claim 3, we now prove that $M_t = f_t(\Sigma_t)$ is embedded for t small enough.

- Fix some index $i \in [1, n]$. Let $d_1 = \delta_1(\frac{\varepsilon}{2})$. By Point 1 of Claim 3, $h_i < d_1$ on $f_t(\Omega_{\frac{\varepsilon}{2}})$ so $f_t^{-1}(C_{i,\theta,d_1}) \cap \Omega_{\frac{\varepsilon}{2}} = \emptyset$. Also for $j \neq i$, $C_{i,\theta} \cap C_{j,\theta} = \emptyset$ so $f_t^{-1}(C_{i,\theta,d_1}) \cap D^*(p_{j,t}, \varepsilon) = \emptyset$. Hence $f_t^{-1}(C_{i,\theta,d_1}) \subset D^*(p_{i,t}, \varepsilon)$. By the last sentence of Theorem 5, $M_t \cap C_{i,\theta,d_1}$ is a submanifold of $\mathbb{R}^3$.

- Let $d_2 = \frac{1+d_1}{2}$. There exists $r > 0$ such that $\delta_2(r) > d_2$. By Point 2 of Claim 3, $h_i > d_2$ on $f_t(D^*(p_{i,t}, r))$ for all i so $f_t^{-1}(K_{d_2}) \cap D^*(p_{i,t}, r) = \emptyset$. Hence for t small enough, $f_t^{-1}(K_{d_2}) \subset \Omega_{\frac{r}{2}}$. Since f_t converges smoothly to the spherical immersion f_0 on $\Omega_{\frac{r}{2}}$, $M_t \cap K_{d_2}$ is a submanifold of $\mathbb{R}^3$ for t small enough.

- It remains to see that the domains C_{i,θ,d_1} for $1 \leq i \leq n$ and K_{d_2} cover M_t. Let $z \in \Sigma_t$. If $z \in D^*(p_{i,t}, \varepsilon)$ then either $h_i(f_t(z)) > d_1$ so $f_t(z) \in C_{i,\theta,d_1}$, or $h_i(f_t(z)) \leq d_1 < d_2$

so $f_t(z) \in K_{d_2}$. (It is clear that $h_j(f_t(z)) < d_2$ for $j \neq i$.) If $z \in \Omega_{\frac{\varepsilon}{2}}$, then $f_t(z) \in K_{d_2}$ because $d_2 > \delta_1(\frac{\varepsilon}{2})$.

This implies that M_t is a submanifold of $\mathbb{R}^3$ and proves Point 2 of Theorem 1.

Remark 2. *If all weights τ_i are positive but we make no angle assumption, then M_t is Alexandrov embedded.*

4.9 Umbilics

Umbilics are points where the two principal curvatures are equal. On a CMC-1 surface, they are the zeros of the Hopf quadratic differential. In term of the DPW potential $\xi = \begin{pmatrix} \alpha & \lambda^{-1}\beta \\ \gamma & -\alpha \end{pmatrix}$, the Hopf differential is equal to $\beta^0\gamma^0$ (where the exponent 0 denotes the coefficient of λ^0). Hence in our case, the Hopf differential of M_t is equal to $t\omega_t^0\,dz$. Since ω_t has a double zero at ∞, the Hopf differential is holomorphic at ∞ and has n double poles at the punctures, so being a quadratic differential on the Riemann sphere, it has $2n-4$ zeros, so there are $2n-4$ umbilics (counting multiplicity). The umbilics of M_t converge as $t \to 0$ to the zeros of $\omega_0\,dz$, where by Equation (18),

$$\omega_0\,dz = \sum_{i=1}^n \left(\frac{\tau_i}{(z-p_i)^2} - \frac{2\tau_i\overline{p_i}}{(1+|p_i|^2)(z-p_i)} \right) dz^2$$

For example, in the case of the equilateral n-noid, where the points p_i are the n-th roots of unity and all weights τ_i are equal to 1, a computation gives

$$\omega_0\,dz = \frac{n^2 z^{n-2}}{(z^n-1)^2} dz^2$$

so there are two umbilics of multiplicity $n-2$ at 0 and ∞.

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