ON AMPLENESS OF CANONICAL BUNDLE

Robert Treger

In the present note we extend the results of [Tr1]. We assume the reader is familiar with [Tr1]. The main new observation of the note is the lemma below.

Let $\phi : X \hookrightarrow \mathbf{P}^r$ be a projective manifold of dimension $n \geq 1$. Let U_X denote the universal covering of X. We assume U_X is equipped with an arbitrary real analytic $\pi_1(X)$-invariant Kahler metric. Until Theorem 2, we assume $\pi_1(X)$ is nonamenable.

Let R be a Riemannian manifold which is a Galois covering of a compact manifold N with nonamenable Galois group. In the fundamental paper [LS, Theorems 3, 3′], Lyons and Sullivan proved R admits a non-constant bounded harmonic function. Employing their theorem, Toledo proved that the space of bounded harmonic functions on R is infinite dimensional [To1] (see a brief review in [Tr1, (2.6)]).

Assuming $\pi_1(X)$ is nonamenable, let V^b denote the vector space generated by all bounded positive pluriharmonic functions on U_X with the sup norm. The fundamental group $\pi_1(X)$ acts on V^b by isometries. We get a representation

$$\rho : \pi_1(X) \longrightarrow Isom(V^b).$$

Let Ξ denote the kernel of ρ. Let $U := U_X/\Xi$ denote a Galois covering of X with the Galois group denoted by Γ. Our Kahler metric on U_X induces the Γ-invariant real analytic Kahler metric on U. Let V^b_U denote the vector space generated by all *bounded* positive pluriharmonic functions on U with the sup norm. As before, the spaces V^b_U and V^b are naturally isomorphic *infinite-dimensional* vector spaces.

Let $PHar(U_X)$ be the vector space generated by *all* positive pluriharmonic functions on U_X. We will integrate pluriharmonic functions with respect to the measure

$$dv := p_{U_X}(s, x, \mathbf{Q})d\mu = p_{U_X}(x)d\mu,$$

where $\mathbf{Q} \in U_X$ is a fixed point, $d\mu$ is the corresponding Riemannian measure, and $p_{U_X}(x) := p_{U_X}(s, x, \mathbf{Q})$ is the corresponding heat kernel. We obtain a pre-Hilbert space generated by the *bounded* positive pluriharmonic square integrable functions on U_X (compare [Tr1, Sect. 2.4 and Sect. 4]).

The latter pre-Hilbert space has a completion in the Hilbert space H generated by all positive pluriharmonic $L_2(dv)$ functions:

$$H := \left\{ h \in PHar(U_X) \;\middle|\; \parallel h \parallel_H^2 := \int_{U_X} |h(x)|^2 dv = \int_{U_X} |h(x)|^2 p_{U_X}(x) d\mu < \infty \right\}.$$

Let $H^b \subseteq H$ be the Hilbert subspace generated by V^b. These Hilbert spaces are separable infinite-dimensional Hilbert spaces with reproducing kernels.

Similarly, we consider the vector space $PHar(U)$, the corresponding heat kernel $p_U(s.x, \mathbf{Q})$, the measure dv_U in place of dv, the Hilbert space H_U in place of H, and the Hilbert subspace $H^b_U \subseteq H_U$ generated by V^b_U.

Given $u \in V^b_U$, there exists a holomorphic $L_2(dv_U)$ function $f = u + \sqrt{-1}\tilde{u}$ on U. Namely, we set $\tilde{u}(x) = \sqrt{-1}\int_{x_0}^{x}(\bar{\partial}u - \partial u)$, where $x_0 \in U$ is a fixed point and $x \in U$ is a variable point (see, e.g., [FG, Chap. 6.1, p. 318] and [Tr1, Sections 2.5, 4.1, 4.2]).

Similarly to [Kl1, Prop. 2.12], X is said to be Γ-large if U contains no compact positive dimensional analytic subsets.

Theorem 1 (Uniformization II). *We keep the above notation. If $\pi_1(X)$ is non-amenable, Γ-large and non-residually finite then the canonical bundle $\mathcal{K}_X$ is ample and U is a bounded Stein domain in $\mathbf{C}^n$. It follows U_X is Stein as well.*

Proof. Let $\mathcal{L}$ by a very ample line bundle defining ϕ. As in [Tr1], we can construct the real analytic Kahler metrics $\Lambda_{U,\mathcal{L}}$, $\Sigma_{U,\mathcal{L}}$ and β_U on U in place of $\Lambda_{\mathcal{L}}$, $\Sigma_{\mathcal{L}}$ and β on U_X. Then we show show that U is a bounded domain in $\mathbf{C}^n$, provided we can establish that Γ is *residually finite* (see Lemma below).

We derive that U is Stein by Siegel's theorem; see a discussion of Siegel's theorem and references in [Tr1, (A.0)]. Because U_X is a covering of U, U_X is Stein as well.

The Prolongation Lemma (see [Tr1, Lemma A in Appendix]) is valid on U. Indeed, the diastasic potential on U is induced by the diastasic potential in the target Fubini space. Furthermore, the Bochner canonical coordinates in U are holomorphic functions on the whole U.

In the proof that U is a bounded domain in $\mathbf{C}^n$, we proceed as in [Tr1, (5.3)] which rely on the fundamental paper by Calabi [C, Theorem 7 on p. 15, Proposition 7 on p. 14, Theorem 6 on p. 13, and Theorem 12 and its proof on pp. 20-21]. We observe that the natural image of U in the Fubini space $\mathbf{F_C}(\infty, 1)$ does not intersect the corresponding antipolar hyperplane because, in our case, the diastasic potential of $U \hookrightarrow \mathbf{F_C}(\infty, 1)$ is a *function* on U.

This completes the prove of the theorem provided Γ is residually finite.

Remark 1. Recall that Toledo constructed an example of projective manifold with nonamenable and non-residually finite fundamental group [To2].

If M is an arbitrary compact real analytic Kahler manifold with generically large and nonamenable fundamental group then M is projective by [Tr3] (there we have proved that M is Moishezon hence projective). Recall that if the corresponding fundamental group is residually finite then the well-known H. Wu conjecture about Kahler manifolds with negative sectional curvature is valid [Tr2].

Corollary. *With notation and assumptions of the theorem, U_X is not a bounded domain in $\mathbf{C}^n$.*

Proof. Suppose U_X is a bounded domain in $\mathbf{C}^n$. Then the functions of V^b separate points on U_X. Since V^b and V^b_U are naturally isomorphic and $U_X \to U$ is a nontrivial covering, we get a contradiction.

To complete the proof of the theorem and its corollary, we need the following lemma which is a generalization of a classical theorem of Maltsev about finitely generated subgroups of $GL(m)$ for $m < \infty$ (see, e.g., [Z, Chap. 1.2]).

Lemma. *Let V be a vector space over $\mathbf{C}$ with a countable base. Let $\mathcal{G}(\mathbf{C})$ denote the group of all $\mathbf{C}$-linear automorphisms of V. Let $H \subset \mathcal{G}(\mathbf{C})$ be a finitely generated subgroup. Then H is residually finite.*

Proof. We will assume $\dim V = \infty$. The case $\dim V < \infty$ was treated by Maltsev. We fix a base in V. Each $g \in \mathcal{G}(\mathbf{C})$ is given by an $(\infty \times \infty)$ matrix with entries in $\mathbf{C}$. Given an $(\infty \times \infty)$ matrix M, let M_i $(1 \le i < \infty)$ denote the matrix whose all entries outside the upper left $(i \times i)$ corner block of M are replaced by zeros.

Roughly speaking, all $(\infty \times \infty)$ matrices (x_{jk}) $(1 \le j, k < \infty)$ with indeterminate entries have a structure of an affine ind-algebraic variety $\mathfrak{M}$ (an affine infinite-dimensional variety in the sense of Shafarevich [S]) defined by finite-dimensional subvarieties $\mathfrak{M}_i$ in the obvious way, where $\mathfrak{M}_i = \{M_i\}$.

The algebra $\mathbf{C}[\mathfrak{M}]$ of regular functions on $\mathfrak{M}$ has a structure of topological algebra. Let $\mathbf{C}[\mathfrak{M} \times \mathfrak{M}] = \mathbf{C}[\mathfrak{M}] \hat{\otimes}_{\mathbf{C}} \mathbf{C}[\mathfrak{M}]$ be the algebra of regular functions on $\mathfrak{M} \times \mathfrak{M}$. The regular functions on $\mathfrak{M}_i$ form an algebra and a coalgebra with the standard comultiplication

$$\Delta_i : \mathbf{C}[\mathfrak{M}_i] \to \mathbf{C}[\mathfrak{M}_i \times \mathfrak{M}_i],\ (\Delta_i f_i)(g_i', g_i'') = f_i(g_i' g_i'') \quad (f_i \in \mathbf{C}[\mathfrak{M}_i];\ g_i', g_i'' \in \mathfrak{M}_i).$$

We consider each $\mathfrak{M}_i$ with its Zariski topology, and $\mathfrak{M}$ is equipped with the topology of inductive limit [S, Sect 1]. In fact, $\mathfrak{M}_i$ is a finite-dimensional algebraic semigroup variety. Since $\mathfrak{M} = \varinjlim \mathfrak{M}_i$, we get a natural multiplication map $\mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ associated with a natural map of topological algebras $\Delta_{\mathfrak{M}} : \mathbf{C}[\mathfrak{M}] \to \mathbf{C}[\mathfrak{M} \times \mathfrak{M}]$, which is the key point of the lemma. Let $\mathfrak{M}(\mathbf{C})$ be the $\mathbf{C}$-points of $\mathfrak{M}$.

A set G which is a group and an ind-algebraic variety is said to be an ind-algebraic group if the inversion map $G \to G$ and the multiplication map $G \times G \to G$ are morphisms of ind-algebraic varieties ([S, Sect. 1], [AT, Sect. 3.2]).

Let $\boldsymbol{e}$ be the $(\infty \times \infty)$ unit matrix. Let $\mathfrak{M}'$ and $\mathfrak{M}''$ denote two copies of $\mathfrak{M}$. The structure of ind-affine variety on $\mathfrak{M}' \times \mathfrak{M}''$ is defined by finite-dimensional subvarieties $\mathfrak{M}_i' \times \mathfrak{M}_i'' \subset \mathfrak{M}' \times \mathfrak{M}''$, where $\mathfrak{M}_i'$ and $\mathfrak{M}_i''$ are two copies of $\mathfrak{M}_i$.

The equation $\mathfrak{p} \circ \mathfrak{q} = \boldsymbol{e}$ $(\mathfrak{p} \in \mathfrak{M}', \mathfrak{q} \in \mathfrak{M}'')$ defines an ind-algebraic group as well as an ind-algebraic subvariety $\mathfrak{G} \subset \mathfrak{M}' \times \mathfrak{M}''$ (compare [S, Sect. 2]; the equation produces the projective system of algebras of the form $R_i = \mathbf{C}[x_{jk}'; x_{jk}'']/\{\text{relations}\}$ defining $\mathfrak{G}$). From the projection $\mathfrak{M}' \times \mathfrak{M}'' \to \mathfrak{M}'$, we get the embedding $\mathfrak{G} \subset \mathfrak{M}$.

Let $\mathfrak{G}(\mathbf{C})$ denote the $\mathbf{C}$-points of $\mathfrak{G}$. We get a map of groups $\eta_H : H \to \mathfrak{G}(\mathbf{C})$; η_H arises from the map $H \to (H, H)$, $h \mapsto (h, h^{-1})$. The closure of $\eta_H(H)$ in $\mathfrak{G}(\mathbf{C})$, denoted by $\overline{\eta_H(H)}$, is a group as in the finite-dimensional case (compare [M, Lemma 1.2.6]). Therefore $\overline{\eta_H(H)} \subset \mathfrak{G}(\mathbf{C}) \subset \mathfrak{M}(\mathbf{C})$ is an ind-algebraic subgroup as in the finite-dimensional case. In fact, the algebra of regular functions on $\overline{\eta_H(H)}$ is a topological Hopf algebra with the standard comultiplication and antipode and the structure maps satisfy the well-known identities.

Let $\mathcal{M}(\mathbf{C})$ be the affine space of $(\infty \times \infty)$ matrices with entries in $\mathbf{C}$ with its Zariski topology. Let $I : \mathfrak{M}(\mathbf{C}) \to \mathcal{M}(\mathbf{C})$ be the natural continuous map.

Let $\mathcal{H}(\mathbf{C})$ be the closure of H in $I(\mathfrak{G}(\mathbf{C}))$. Clearly, I is an isomorphism of the group $H \subset \mathfrak{M}(\mathbf{C})$ onto the group $H \subset I(\mathfrak{G}(\mathbf{C}))$. Also, I maps $\overline{\eta_H(H)}$ onto $\mathcal{H}(\mathbf{C})$.

Indeed, given $h \in \mathcal{H}(\mathbf{C}) \backslash H$, we take a general curvilinear section C $(\subset \mathcal{M}(\mathbf{C}))$ though h. Then $I^{-1}(C \cap \mathcal{H}(\mathbf{C}))$ will be closed in $\overline{\eta_H(H)}$ and $I^{-1}(h)$ will be a unique point in $\overline{\eta_H(H)}$, i.e., I maps $\overline{\eta_H(H)}$ one-to-one onto $\mathcal{H}(\mathbf{C})$. Further, $\mathcal{H}(\mathbf{C})$ is an open subset of its closure in $\mathcal{M}(\mathbf{C})$ (Generalized Chevalley theorem).

Let $\mathcal{A}$ and $\mathcal{B}$ denote the commutative (reduced) $\mathbf{C}$-algebras of regular functions on $\mathfrak{H}(\mathbf{C})$ and $\mathfrak{H}(\mathbf{C}) \times \mathfrak{H}(\mathbf{C})$, respectfully. Our aim is to show that $\mathfrak{H}(\mathbf{C})$ has a structure of an *affine group* [AT, Chap. 3.2]. We will show $\mathcal{A}$ has a natural structure of a discrete commutative Hopf algebra.

Let $A(H) := \mathcal{A}|_H$ and $B(H \times H) := \mathcal{B}|_{H\times H}$ denote $\mathbf{C}$-algebras of the corresponding $\mathbf{C}$-maps $H \to \mathbf{C}$ and $B(H \times H) \to \mathbf{C}$, respectfully ($H \times H$ is a subset of $\mathfrak{H}(\mathbf{C}) \times \mathfrak{H}(\mathbf{C})$). We observe that $\mathcal{B} = \mathcal{A} \otimes_{\mathbf{C}} \mathcal{A}$ hence $B(H \times H) = A(H) \otimes_{\mathbf{C}} A(H)$. It follows the existence of a discrete Hopf algebra structure on $A(H)$ with the coidentity ϵ_H, the comultiplication Δ_H and the antipod S_H (see, e.g., [M, Chap. 3]),

$$\epsilon_H f_H := f_H(1), \quad \Delta_H : A(H) \longrightarrow A(H \times H),\ (\Delta_H f_H)(h', h'') := f_H(h'h''),$$

$$S_H : A(H) \longrightarrow A(H),\ (S_H f_H)(h) := f_H(h^{-1}) \quad (f_H \in A(H);\ h, h', h'' \in H).$$

A regular function $f_{\mathcal{A}}$ on $\mathfrak{H}(\mathbf{C})$ in the infinite number of variables $\{x_{jk}\}$ defines a projective system $\{f_{\mathcal{A},i}\}$, where each $f_{\mathcal{A},i}$ is obtained from $f_{\mathcal{A}}$ by letting the variables $\{x_{jk}\}$ equal to zero when $j > i$ or $k > i$.

Now, we consider arbitrary $f_{\mathcal{A}} \in \mathcal{A}$ and $h', h'' \in \mathfrak{H}(\mathbf{C})$. Let $h' = \lim h'_t$ and $h'' = \lim h''_t$ ($h'_t, h''_t \in H;\ t \in \mathbf{N}$). As before, we assume $\{h'_t, h'\}$ as well as $\{h''_t, h''\}$ are contained in the corresponding curvilinear sections. Clearly, $I^{-1}(h'_t)I^{-1}(h''_t) = I^{-1}(h'_t h''_t)$. It follows $\lim I^{-1}(h'_t)$, $\lim I^{-1}(h''_t)$ and $\lim I^{-1}(h'_t h''_t)$ exist.

Furthermore, $(\lim I^{-1}(h_t))^{-1} = \lim((I^{-1}(h_t))^{-1}) = \lim I^{-1}(h_t^{-1})$ and the limits exist where $h_t := h'_t$. The $\lim I^{-1}(h'_t) \lim I^{-1}(h''_t)$ makes sense because $\overline{\eta_H(H)}$ is a group.

Let $\hat{\mathcal{A}}$ be the topological Hopf algebra of regular functions on $\overline{\eta_H(H)}$; we have $\hat{\mathcal{A}} = \varprojlim \mathbf{C}[\overline{\eta_H(H)}_i]$. Let $f_{\hat{\mathcal{A}}} := \varprojlim f_{\mathcal{A},i}$ be the image of $f_{\mathcal{A}}$ in $\hat{\mathcal{A}}$. We get

$$(\Delta_{\hat{\mathcal{A}}} f_{\hat{\mathcal{A}}})(\lim I^{-1}(h'_t), \lim I^{-1}(h''_t)) = f_{\hat{\mathcal{A}}}(\lim I^{-1}(h'_t) \lim I^{-1}(h''_t)) = \lim f_{\hat{\mathcal{A}}}(I^{-1}(h'_t h''_t)).$$

Hence, we can define a comultiplication in $\mathcal{A}$ as follows:

$$(\Delta_{\mathcal{A}} f_{\mathcal{A}})(h', h'') = (\Delta_{\mathcal{A}} f_{\mathcal{A}})(\lim h'_t, \lim h''_t) := \lim f_{\mathcal{A}}(h'_t h''_t).$$

The definition is independent of choices of h'_t and h''_t. Similarly, we define the antipode $S_{\mathcal{A}}$. Set $\epsilon f_{\mathcal{A}} := f_H(1)$. The structure maps satisfy the well-known identities.

Thus we obtain a discrete Hopf $\mathbf{C}$-algebra structure on $\mathcal{A}$ extending the Hopf algebra $A(H)$. Hence $\mathfrak{H}(\mathbf{C})$ has a structure of an affine group associated with $\mathcal{A}$ (see [AT, Sect. 3.2]). We get $\mathcal{A} = \varinjlim \mathcal{A}_\alpha$ is the inductive limit of finitely generated sub-Hopf algebras $\mathcal{A}_\alpha \subset \mathcal{A}$ (see [A, Lemma 3.4.5]). Hence $\mathfrak{H}(\mathbf{C}) = \varprojlim \mathfrak{H}_\alpha(\mathbf{C})$ is the projective limit of (finite-dimensional) affine algebraic groups $\mathfrak{H}_\alpha(\mathbf{C})$.

So $\mathfrak{H}(\mathbf{C})$ is a pro-affine algebraic group. The projection of H in each $\mathfrak{H}_\alpha(\mathbf{C})$ is residually finite by Maltsev's theorem. Hence H is residually finite.

The next theorem is a higher-dimensional generalization of the Poincaré ampleness theorem ($\dim X = 1$). In the sequel, $\pi_1(X)$ is not necessary nonamenable.

Theorem 2. *We keep the above notation. Let X be a projective manifold with residually finite and large fundamental group. If the genus $g(C)$ of a general curvilinea section $C \subset X$ is at least 2 then the canonical bundle $\mathcal{K}_X$ is ample.*

Proof. Let $\mathcal{L}$ by a very ample line bundle defining ϕ. We can construct the real analytic Kahler metrics $\Lambda_{\mathcal{L}}$ (a generalization of Poincaré metric) [Tr1, Sect. 3.3]. Let $D_{U_X}(\mathbf{Q}, p)$ denote the diastasic potential at $\mathbf{Q} \in U_X$ of our Kahler metric.

We will follow [Tr1] with some corrections. Because the diastasis is inductive on complex submanifolds, various questions about higher-dimensional manifolds are reduced to the one-dimensional case (a generalization of the Poincaré metric, the proof of Prolongation lemma, the Shafarevich conjecture, etc. [Tr1]).

We consider the real-valued function $\tilde{\Phi}_{\mathbf{Q}}(z(p), \bar{z}(p)) := D_{U_X}(\mathbf{Q}, p)$ in a small neighborhood $\mathcal{V}_{\mathbf{Q}} \subset U_X$.

Let C be a general curvilinear section of X and R is its inverse image on U_X. With the notation from [Tr1, Proposition-Definition 2], we can construct a real analytic $Gal(R/C)$-invariant Kahler metric $\Lambda_R := \lim_{t\to\infty} \frac{1}{t} g_{R,t}$. It follows from the prolongation over U_X [Tr1, Appendix] that the diastasic potential of Λ_R has the prolongation over R.

We consider the complexification of $\tilde{\Phi}_{\mathbf{Q}}(z(p), \bar{z}(p))$ and obtain a complex holomorphic function $F_{(p,\bar{p})}$ on a small neighborhood of $(p, \bar{p})$ in $\Delta \subset U_X \times \bar{U}_X$, where $\Delta := \{(z, \bar{z})\}$.

We would like to obtain a complex-valued Hermitian positive definite function $\mathcal{F}$ on $U_X \times U_X$ [Tr1, (2.3.2.1)] holomorphic in the first variable. If $\mathcal{F}(z, z)$ is, in addition, positive for every $z \in U_X$ then we can define $\log \mathcal{F}(z, z)$. Then we will get a positive semidefinite Hermitian form, called the Bergman pseudo-metric

$$ds_U^2 = 2 \sum g_{jk} dz_j d\overline{z}_k, \qquad g_{jk} := \frac{\partial^2 \log \mathcal{F}(z, z)}{\partial z_j \partial \overline{z}_k}.$$

Recall that the positive definite $\mathcal{F}$ is a positive matrix in the sense of E. H. Moore, i.e., $\mathcal{F}$ satisfies the property (ii) of [Tr1, (2.3.2.1)]:

$$\forall\, \tilde{q}_1, \ldots, \tilde{q}_N \in U_X,\ \forall\, a_1, \ldots, a_N \in \mathbf{C} \implies \sum_{j,k}^{N} \mathcal{F}(\tilde{q}_k, \tilde{q}_j) a_j \bar{a}_k \geq 0.$$

The required $\mathcal{F}$ will arise from the function $F_{(p,\bar{p})}$ which, in turn, arises from the diastasis of the metric $\Lambda_{\mathcal{L}}$. First, we will establish the above property (ii) in the one-dimensional case which is the key point of the proof. Then we will derive (ii) in general. In short, first we get the Hilbert space H_R. Second, we consider the diastasis in the one-dimensional case. Third, we consider the diastasis in the general case. Finally, we obtain the separable Hilbert space H_{U_X}.

In the one-dimensional case, we get the diastasis on R, denoted by D_R, as well as $\frac{1}{t} D_{R,t}$ and $\lim_{t\to\infty} \frac{1}{t} D_{R,t}$ corresponding $\mathcal{L}_R^t := \mathcal{L}^t|_R$. We get the Hilbert space H_R of square-integrable holomorphic functions on R and, then, the corresponding Hermitian positive definite complex-valued function on $R \times R$ (see [Tr1, (2.3.2.1) and (3.2.2)]).

The general case follows from the one-dimensional case because $\pi_1(X)$ is large (compare [Tr1, Appendix, (A.1.2)]). Suppose we are given arbitrary N points $\tilde{q}_1, \ldots, \tilde{q}_N$ in U_X and a vector $\mathbf{v}_{\tilde{p}} \in \mathbf{T}_{\tilde{p}} U_X$, where $\tilde{p}$ is a point of U_X. The diastasis on U_X will produce the Hermitian positive definite function on $U_X \times U_X$ as follows.

We consider an appropriate finite Galois covering of X, denoted by X_i ($i \gg 0$), containing the finite set of points $q_1, \ldots, q_N \in X_i$ in a general one-dimensional nonsingular curve (compact Riemann surface) $C_i \subset X_i$ tangent to $\mathbf{v}_p \in \mathbf{T}_p X_i$, where $p \in C_i$ is a given point and X_i *depends* on N. Furthermore, we assume $\tilde{q}_k \mapsto q_k$ $(\forall k, 1 \le k \le N)$, $\tilde{p} \mapsto p$ and $\mathbf{v}_{\tilde{p}} \mapsto \mathbf{v}_p$.

We conclude the proof as in [Tr1, (4.6)]. From the one-dimensional case, we get the desired Hermitian positive definite function on $U_X \times U_X$. In fact, the complex-valued *holomorphic* function on the "diagonal" $\Delta \subset U_X \times \bar{U}_X$ $(\Delta := \{(z, \bar{z})\})$ obtained from the diastasic potential on U_X determines a unique complex-valued function on $U_X \times U_X$ (compare [Kl2, Prop. 7.6]).

Thus we get the separable Hilbert space H_{U_X}, the Bergman-diastasic form $ds^2_{U_X} = 2\sum g_{jk} dz_j d\bar{z}_k$ and a natural *immersion* into a projective space. Note that H_{U_X} is separable because the reproducing kernel determines a countable total subset of H_{U_X}. The corresponding metric $ds^2_{U_X}$ arises via the immersion

$$\Upsilon : U_X \longrightarrow \mathbf{P}(H^*_{U_X}), \quad \Upsilon^* ds^2_{\mathbf{P}(H^*_{U_X})} = ds^2_{U_X}$$

[Kb, Chap. 4.10, p. 228]. If $\{\varphi_j\}$ is an orthonormal basis of H_{U_X} then Υ is given by $u \mapsto [\varphi_0(u) : \varphi_1(u) : \ldots]$. The fundamental group $\pi_1(X)$ acts on H_{U_X} as follows

$$T_\gamma : \varphi \mapsto (\varphi \circ \gamma) \cdot Jac_\gamma \qquad (\gamma \in \pi_1(X))$$

where Jac_γ is the (complex) Jacobian determinant of T_γ. We get a $\pi_1(X)$-invariant volume form on U_X defined in Bochner canonical coordinates as follows:

$$\Big(\sum_{j=0}^{\infty} |\varphi_j(z)|^2\Big)\prod_{\alpha=1}^{n}\Big(\frac{\sqrt{-1}}{2} dz_\alpha \wedge d\bar{z}_\alpha\Big) = \Big(\sum_{j=0}^{\infty} |\varphi_j(z)|^2\Big)\prod_{\alpha=1}^{n}\big(dx_\alpha \wedge dy_\alpha\big)$$

where $z_\alpha = x_\alpha + \sqrt{-1} y_\alpha$. The Ricci form of the volume form is negative definite [Kb, Chap. 2.4.4]. It follows the canonical bundle of X is ample by Kodaira.

In Theorem 3, we consider the following conjecture by Kobayashi [Kb]: if X is a Kobayashi hyperbolic projective manifold then $\mathcal{K}_X$ is ample. The conjecture was suggested to the author by J.-P. Demailly.

Theorem 3. *If X is a Kobayashi hyperbolic projective manifold then $\mathcal{K}_X$ is ample.*

Proof. We proceed by induction on $\dim X$. Let g_{FS} denote the Fubini-Study metric in projective space. Let $Y \subset X$ be a projective submanifold of dimension $\nu < \dim X$ with very ample $\mathcal{K}_Y^{t_Y}$ $(t_Y \gg 0)$. It is equipped with the real analytic Kahler metric

$$G_Y := \psi^*_{\mathcal{K}_Y^{t_Y}} g_{FS} \quad \text{where} \quad \psi_{\mathcal{K}_Y^{t_Y}} : Y \hookrightarrow \mathbf{P}^{N_Y(t_Y)}.$$

A section $\sigma \in H^0(\mathcal{K}_Y^{t_Y})$ can be expressed locally as $\sigma = f(dy_1 \wedge \cdots \wedge dy_\nu)^{t_Y}$, where $y_1, \ldots, y_\nu$ is a Bochner canonical coordinate system on Y with center $\mathbf{Q}$ and f is a function holomorphic in the coordinate neighborhood. Let $\sigma_0, \ldots, \sigma_{N(t_Y)}$ be a basis of $H^0(\mathcal{K}_Y^{t_Y})$, and let

$$\sigma_{i_Y} = f_{i_Y}(y)(dy_1 \wedge \cdots \wedge dy_\nu)^{t_Y} \quad (0 \le i_Y \le N(t_Y)).$$

Define a volume form of G_Y by setting

$$v_{G_Y} := \Big(\sum_{i_Y=0}^{N_Y(t_Y)} |f_{i_Y}(y)|^2\Big)^{1/t_Y} (\sqrt{-1})^{\nu^2} dy_1 \wedge \cdots \wedge dy_\nu \wedge d\bar{y}_1 \wedge \cdots \wedge d\bar{y}_\nu.$$

Let $D_Y(\mathbf{Q}, p)$ be the diastasic potential at $\mathbf{Q} \in Y$ (see [C, Chap. 4, especially Theorem 12], [Tr1, (2.2)]). It is defined (extended) everywhere except perhaps the inverse image on Y of the antipolar hyperplane of $\mathbf{Q} \in \mathbf{P}^{N_Y(t_Y)}$ (so-called infinity). According to Calabi [C, Chap. 1, (5)], we have the real analytic function in p in a neighborhood of $\mathbf{Q}$:

$$D_Y(\mathbf{Q}, p) = F(y(\mathbf{Q}), \overline{y(\mathbf{Q})}) \;+\; F(y(p), \overline{y(p)}) \;-\; F(y(\mathbf{Q}), \overline{y(p)}) \;-\; F(y(p), \overline{y(\mathbf{Q})}),$$

where $F(w, \overline{z})$ denote the complex holomorphic function in a neighborhood of $\mathbf{Q} \times \bar{\mathbf{Q}}$ in $Y \times \bar{Y}$ (compare [Tr1, (2.2.1)]) arising from a potential on Y. Furthermore,

$$v_{G_Y} = e^{D_Y(\mathbf{Q},p)/t_Y} (\sqrt{-1})^{\nu^2} dy_1 \wedge \cdots \wedge dy_\nu \wedge d\bar{y}_1 \wedge \cdots \wedge d\bar{y}_\nu.$$

The associated Ricci form $\operatorname{Ric} v_{G_Y}$ is negative.

Let Δ denote the unit disk. We consider the family

$$H_Y = \{(g, h)\} := \mathfrak{Hol}_{\mathbf{Q}\times\bar{\mathbf{Q}}}(\Delta \times \bar{\Delta}, Y \times \bar{Y}), \quad g : \Delta \to Y,\, h : \bar{\Delta} \to \bar{Y}, 0\times 0 \mapsto \mathbf{Q}\times\bar{\mathbf{Q}},$$

where g and h are holomorphic maps. We define H_X by replacing Y in H_Y by X. The families H_Y and H_X are equicontinuous [Kb, Theorem 2.2.23]. With a help of classical theorems of Ascoli, Arzelá and Montel, we will obtain a real analytic Kahler metric with the diastasic potential $D_X(\mathbf{Q}, p)$ at $\mathbf{Q} \in X$ as follows.

We consider an element $\mathfrak{h} \in H_Y$ where $\operatorname{im}(g)$ and $\operatorname{im}(h)$ do not intersect the infinity. Let $D_Y(\mathbf{Q}, p)$ be the diastasic potential at $\mathbf{Q} \in Y$. We get the real analytic function $D_{Y,\mathfrak{h}}(0, p)$ on Δ (by abuse of notation, now $p \in \Delta$).

Let $\{Y^\alpha\}$ be a collection of all Y's as above. For the various Y^α's and the various elements $\mathfrak{h}_\alpha \in H_{Y^\alpha}$ as above $\mathfrak{h}$, we take the limit of functions $D_{Y^\alpha,\mathfrak{h}_\alpha}(0, p)$'s as well as $D_{Y^\alpha}(\mathbf{Q}, p)$'s. The limit exists.

We, then, obtain the real analytic function denoted by $D_X(\mathbf{Q}, p)$. Further, if $\mathbf{Q}' \in X$ is another point then we get $D_X(\mathbf{Q}', p)$. If $\mathbf{Q}'$ is close to $\mathbf{Q}$ (in the Kobayashi hyperbolic topology) then $D_X(\mathbf{Q}', p)$ is close to $D_X(\mathbf{Q}, p)$. So, we get the real analytic Kahler metric on X with the potential $D_X(\mathbf{Q}, p)$ at $\mathbf{Q}$.

Finally, we can apply Wirtinger's theorem. Let ω be the fundamental form of the Kahler metric on X. The induced metric on each Y^α coincides with the corresponding G_{Y^α}. Thus $\omega^\nu/\nu!$ restricted to Y^α coincides with volume form on Y^α. One can compare the volume forms on X and Y^α's as well as the Ricci forms of the corresponding volume forms. It follows the Ricci form of a volume form on X will be negative. Hence $\mathcal{K}_X$ is ample by Kodaira.

Remark 2. In Theorem 2, U_X is not necessary a bounded domain in $\mathbf{C}^n$. Recently D. Wu and S.-T. Yau [WY] have established that a projective manifold X which admits a Kahler metric with negative holomorphic sectional curvature has the ample canonical bundle. We observe that negative holomorphic sectional curvature of a Kahler manifold does not imply its fundamental group is nonamenable while the negative sectional curvature yields the fundamental group is nonamenable.

Acknowledgments. The author would like to thank Domingo Toledo for his email. The author would like to thank Fedor Bogomolov and Frédéric Campana for pointing an error in an earlier statement of Theorem 1. The author had also benefited from remarks by Campana. The last but not list, the author would like to thank Jean-Pierre Demailly for his encouragement.

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Princeton, NJ 08540

E-mail address: `roberttreger117@gmail.com`