Forced periodic solutions for nonresonant parabolic equations on $\mathbb{R}^N$

Aleksander Ćwiszewski (*), Renata Łukasiak (†)

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Abstract

Criteria for the existence of T-periodic solutions of nonautonomous parabolic equation $u_t = \Delta u + f(t, x, u)$, $x \in \mathbb{R}^N$, $t > 0$ with asymptotically linear f will be provided. It is expressed in terms of time average function $\widehat{f}$ of the nonlinear term f and the spectrum of the Laplace operator Δ on $\mathbb{R}^N$. One of them says that if the derivative $\widehat{f}_\infty$ of $\widehat{f}$ at infinity does not interact with the spectrum of Δ, i.e. $\mathrm{Ker}(\Delta + \widehat{f}_\infty) = \{0\}$, then the parabolic equation admits a T-periodic solution. Another theorem is derived in the situation, where the linearization at 0 and infinity differ topologically, i.e. the total multiplicities of positive eigenvalues of the averaged linearizations at 0 and ∞ are different mod 2.

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1 Introduction

We shall be concerned with time T-periodic solutions of the following parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + f(t, x, u(x,t)),\ x \in \mathbb{R}^N,\ t > 0, \\ u(\cdot, t) \in H^1(\mathbb{R}^N),\ t \geq 0, \end{cases} \tag{1}$$

where Δ is the Laplace operator (with respect to x) and a function $f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is T-periodic in time:

$$f(t, x, u) = f(t + T, x, u) \ \text{ for all } \ t \geq 0, u \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N. \tag{2}$$

Periodic problems for parabolic equations were widely studied by many authors by use of various methods. Some early results are due to Brezis and Nirenberg [5], Amman and Zehnder [2], Nkashama and Willem [18], Hirano [15, 16], Prüss [21], Hess [14], Shioji [24] and many others; see also [27] and the references therein. These

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†Nicolaus Copernicus University, Faculty of Mathematics and Computer Science, ul. Chopina 12/18, 87-100 Toruń, Poland

results treat the case where Ω is bounded and are based either on topological degree and coincidence index techniques in the spaces of functions depending both on x and time t or on the translation along trajectories operator to which fixed point theory is applied. In this paper we shall study the case $\Omega = \mathbb{R}^N$ by applying translation along trajectories approach together with fixed point index and Henry's averaging (see [13]) as in [7] (for a general reference see also [8]). In this case the semigroup compactness arguments are no longer valid (since the Rellich-Kondrachov theorem on $\mathbb{R}^N$ does not hold and the semigroup of bounded linear operators generated by the linear heat equation $u_t = \Delta u$ on $\mathbb{R}^N$ is not compact). Therefore adequate topological fixed point theory for noncompact maps and the adaptation of proper averaging techniques is required.

We shall assume that $f : [0, +\infty)\times\mathbb{R}^N\times \mathbb{R} \to \mathbb{R}$ is such that, for all $t, s \in [0, +\infty)$, $u, v \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^N$, one has

$$f(t, \cdot, u) \text{ is measurable and } \; |f(t, x, 0)| \le m_0(x); \tag{3}$$

$$|f(t, x, u) - f(s, x, v)| \le \left(\tilde{k}(x) + k(x)|u|\right) |t - s|^\theta + l(s, x)|u - v|; \tag{4}$$

$$\left(f(t, x, u) - f(t, x, v)\right)(u - v) \le -a|u - v|^2 + b(x)|u - v|^2 \tag{5}$$

where $m_0 \in L^2(\mathbb{R}^N)$, $\theta \in (0, 1)$, $\tilde{k} \in L^2(\mathbb{R}^N)$, $k = k_0 + k_\infty$, $l = l_0 + l_\infty$ with $k_0, l_0(t, \cdot) \in L^p(\mathbb{R}^N)$, $k_\infty, l_\infty(t, \cdot) \in L^\infty(\mathbb{R}^N)$, $\sup_{t\ge0}(\|l_0(t, \cdot)\|_{L^p} + \|l_\infty(t, \cdot)\|_{L^\infty}) < +\infty$, $a > 0$ and $b \in L^p(\mathbb{R}^N)$ where, if not stated otherwise,

$$2 < p < \infty \;\text{ for }\; N = 1, 2 \quad\text{and}\quad N \le p < \infty \;\text{ for }\; N \ge 3$$

(see Remark 1.3 for examples of functions satisfying these assumptions).

We shall consider the following linearization property of f at zero

$$\lim_{u\to0} \frac{f(t, x, u)}{u} = \alpha(t, x) := \alpha_0(t, x) - \alpha_\infty(t, x) \tag{6}$$

and at infinity

$$\lim_{|u|\to\infty} \frac{f(t, x, u)}{u} = \omega(t, x) := \omega_0(t, x) - \omega_\infty(t, x) \tag{7}$$

for all $x \in \mathbb{R}^N$ and $t \ge 0$, where $\alpha_0(t, \cdot), \omega_0(t, \cdot) \in L^p(\mathbb{R}^N)$, $\alpha_\infty, \omega_\infty(t, \cdot) \in L^\infty(\mathbb{R}^N)$, $\alpha_\infty(t, x) \ge \bar{\alpha}_\infty$, $\omega_\infty(t, x) \ge \bar{\omega}_\infty$, for all $t \ge 0$ and a.a. $x \in \mathbb{R}^N$, with some $\bar{\alpha}_\infty, \bar{\omega}_\infty > 0$ and

$$\sup_{t\ge0} \left(\|\alpha_0(t, \cdot)\|_{L^p} + \|\alpha_\infty(t, \cdot)\|_{L^\infty} + \|\omega_0(t, \cdot)\|_{L^p} + \|\omega_\infty(t, \cdot)\|_{L^\infty}\right) < +\infty.$$

We shall also assume that, for all $t, s \ge 0$ and a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned} &|\alpha_0(t, x) - \alpha_0(s, x)| \le k^0(x)|t - s|^\nu \;\text{ and }\; |\alpha_\infty(t, x) - \alpha_\infty(s, x)| \le k^\infty(x)|t - s|^\nu, \\ &|\omega_0(t, x) - \omega_0(s, x)| \le k^0(x)|t - s|^\nu \;\text{ and }\; |\omega_\infty(t, x) - \omega_\infty(s, x)| \le k^\infty(x)|t - s|^\nu \end{aligned}$$

with $k^0 \in L^p(\mathbb{R}^N)$, $k^\infty \in L^\infty(\mathbb{R}^N)$ and $\nu \in (0, 1)$.

Our main results are the following criteria for the existence of T-periodic solutions.

Theorem 1.1. *Suppose that f satisfies conditions* (2), (3), (4), (5) *and* (7)*. If*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \lambda \Delta u(x,t) + \lambda \omega(t,x) u(x,t), \ x \in \mathbb{R}^N, \ t > 0, \\ u(\cdot, t) \in H^1(\mathbb{R}^N), \ t \geq 0, \end{cases} \tag{8}$$

has no nonzero T-periodic solutions, for $\lambda \in (0,1]$ and $\mathrm{Ker}\,(\Delta + \widehat{\omega}) = \{0\}$*, where $\widehat{\omega} : \mathbb{R}^N \to \mathbb{R}$ is the time average function of ω, given by $\widehat{\omega}(x) := \frac{1}{T}\int_0^T \omega(t,x)\,\mathrm{d}t$, then the equation* (1) *admits a T-periodic solution*

$$u \in C([0,+\infty), H^2(\mathbb{R}^N)) \cap C^1([0,+\infty), L^2(\mathbb{R}^N)).$$

Our second result applies in the case where there exists a trivial periodic solution $u \equiv 0$ and the previous theorem does not imply the existence of a nontrivial periodic solution.

Theorem 1.2. *Suppose that all the assumptions of Theorem* 1.1 *are satisfied and additionally that (6) holds. If the equation*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \lambda \Delta u(x,t) + \lambda \alpha(t,x) u(x,t), \ x \in \mathbb{R}^N, \ t > 0, \\ u(\cdot, t) \in H^1(\mathbb{R}^N), \ t \geq 0, \end{cases} \tag{9}$$

has no nonzero T-periodic solutions for $\lambda \in (0,1]$ and $\mathrm{Ker}\,(\Delta + \widehat{\alpha}) = \mathrm{Ker}\,(\Delta + \widehat{\omega}) = \{0\}$ *and $m(0) \not\equiv m(\infty)$ mod 2, where $m(0)$ and $m(\infty)$ are the total multiplicities of the positive eigenvalues of $\Delta + \widehat{\alpha}$ and $\Delta + \widehat{\omega}$, respectively, then the equation* (1) *admits a nontrivial T-periodic solution $u \in C([0,+\infty), H^2(\mathbb{R}^N)) \cap C^1([0,+\infty), L^2(\mathbb{R}^N))$.*

Remark 1.3.

(a) Let us give an example of a class of functions satisfying (3), (4) and (5). Consider $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$f(t,x,u) := U(t,x) + V(t,x)u + g(W(t,x)u)$$

with $U, V, W : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that $U(t,\cdot) \in L^2(\mathbb{R}^N)$ for all $t \geq 0$, $\sup_{\tau \geq 0} \|U(\tau,\cdot)\|_{L^2} < +\infty$, $V = V_0 + V_\infty$, $W = W_0 + W_\infty$, $V_0(t,\cdot), W_0(t,\cdot) \in L^p(\mathbb{R}^N)$ for all $t \geq 0$ and $\sup_{\tau \geq 0} (\|V_0(\tau,\cdot)\|_{L^p} + \|W_0(\tau,\cdot)\|_{L^p}) < \infty$ and $V_\infty, W_\infty \in L^\infty([0,+\infty) \times \mathbb{R}^N)$. Moreover we assume that there are $L_U \in L^2(\mathbb{R}^N)$, $L_V, L_W \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ such that, for all $t, s \geq 0$ and a.e. $x \in \mathbb{R}^N$, $|U(t,x) - U(s,x)| \leq L_U(x)|t-s|^\theta$, $|V(t,x) - V(s,x)| \leq L_V(x)|t-s|^\theta$ and $|W(t,x) - W(s,x)| \leq L_W(x)|t-s|^\theta$. Furthermore, $g : \mathbb{R} \to \mathbb{R}$ is assumed to be a bounded Lipschitz function with a constant $L > 0$ such that $g(0) = 0$ and $g'(0)$ exists. Then the assumptions (3) and (4) are satisfied. If additionally there is $a > 0$ such that we have $V_\infty(t,x) + L|W_\infty(t,x)| \leq -a$ for all $t \geq 0$ and a.e. $x \in \mathbb{R}^N$, then (5) holds. As a concrete example one may give $f(t,x,u) := -2au + \sin(au + bu(1+|x|)^{-\rho}|\cos t|)$ where $a, b > 0$ and $\rho > 1$ if $N = 1, 2$ and $\rho > N/p$ if $N \geq 3$. Moreover, in this particular case $\omega_\infty \equiv 2a$, $\omega_0 \equiv 0$, $\alpha_\infty \equiv a$, $\alpha_0(t,x) = b(1+|x|)^{-\rho}|\cos t|$.

(b) The appearance of terms from the space $L^p(\mathbb{R}^N)$ is essential for our considerations. The function α_0 (or ω_0) assures that the positive part of the spectrum

$\sigma(\Delta+\widehat{\alpha})$ (or $\sigma(\Delta+\widehat{\omega})$) consists of a finite number of eigenvalues with finite dimensional eigenspaces (see Remark 6.2 for a more detailed discussion). That makes the numbers $m(0)$ and $m(\infty)$ well-defined, i.e. the formulation of the above result is correct. Note that in Theorem 1.2 the appearance of the nontrivial term either α_0 or ω_0 belonging to $L^p(\mathbb{R}^N)$ is necessary to satisfy the desired condition $m(0) \not\equiv m(\infty) \mod 2$.

(c) In this paper we focus on the case when there is no resonance both at 0 and at ∞, i.e. when the nonexistence assumptions for (8) and (9) hold, respectively. From the technical point of view they enable us to use continuation along the parameter λ up to $\lambda = 1$. For small parameter $\lambda > 0$ the lack of T-periodic assumptions of (8) and (9) is implied by the conditions $\mathrm{Ker}(\Delta+\widehat{\omega}) = \{0\}$ and $\mathrm{Ker}(\Delta+\widehat{\alpha}) = \{0\}$, respectively. The lack of nontrivial T-periodic solutions for the problem $\frac{\partial u}{\partial t} = \lambda(\Delta u - \alpha_\infty(t,x)u + \alpha_0(t,x)u)$, $u(\cdot,t) \in H^1(\mathbb{R}^N)$, $x \in \mathbb{R}^N$, $t > 0$, is obvious if α is independent of time (which is possible also when f depends on time). Moreover the nonexistence condition also holds in the general case if, for instance

$$\sup_{t\in[0,T]} \|\alpha_0(\cdot,t)\|_{L^p} < \begin{cases} \frac{p^{1/2p}\bar{\alpha}_\infty^{1-1/2p}}{2^{1/2p}}, & \text{if } N=1,\ p>2, \\ \frac{p^{1/p}\bar{\alpha}_\infty^{(1-1/p)}}{4^{1/p}}, & \text{if } N=2,\ p>2, \\ \frac{\bar{\alpha}_\infty^{1-N/2p}}{(N/2p)^{N/2p}C(N)^{N/p}}, & \text{if } N\geq 3, N \leq p < \infty, \end{cases}$$

where $C(N) > 0$ is the constant in the Sobolev inequality $\|u\|_{L^{\frac{2N}{N-2}}} \leq C(N)\|\nabla u\|_{L^2}$, $u \in H^1(\mathbb{R}^N)$ (for details see Remark 5.6).

(d) The resonant case was considered in [9]. □

Following the tail estimates techniques of Wang [28], who studied attractors, and Prizzi [20], who studied stationary states and connecting orbits by use of Conley index, we develop a fixed point index setting applicable to parabolic equations on $\mathbb{R}^N$. We shall show that the translation along trajectories operator $\mathbf{\Phi}_T : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ for (1) is ultimately compact, i.e. belongs to the class of maps for which the fixed point index $\mathrm{Ind}(\mathbf{\Phi}_T, U)$, with respect to open subsets of $H^1(\mathbb{R}^N)$, can be considered (see e.g. [1]). Clearly the nontriviality of that index will imply the existence of the fixed point of $\mathbf{\Phi}_T$ in U, which is the starting point of the corresponding periodic solution. In order to determine the index $\mathrm{Ind}(\mathbf{\Phi}_T, U)$, we use an averaging method, i.e. we embed the equation (1) into the family of problems

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + f\left(\dfrac{t}{\lambda}, x, u(x,t)\right), \ x \in \mathbb{R}^N,\ t > 0,\ \lambda > 0, \\ u(\cdot,t) \in H^1(\mathbb{R}^N),\ t \geq 0. \end{cases} \tag{10}$$

According to Henry's averaging principle solutions of (10) converge as $\lambda \to 0^+$ to a solution of the averaged equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + \widehat{f}(x, u(x,t)),\ x \in \mathbb{R}^N,\ t > 0, \\ u(\cdot,t) \in H^1(\mathbb{R}^N),\ t > 0, \end{cases} \tag{11}$$

where the time average function $\widehat{f} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ of f is given by

$$\widehat{f}(x,u) := \frac{1}{T}\int_0^T f(t,x,u)\,\mathrm{d}t.$$

Exploiting the tail estimate technique of Wang and Prizzi together with an extension of Henry's averaging principle we prove that asymptotic assumptions on f imply a sort of a priori bounds conditions, i.e. that there are no λT-periodic solution of (10), for $\lambda \in (0, 1]$, with initial states of large H^1 norm (in case of Theorem 1.1) and also of small H^1 norm (in case of Theorem 1.2), i.e. initial states of λT-periodic solutions are located outside some open bounded set $U \subset H^1(\mathbb{R}^N)$. This enables us to use a sort of the averaging index formula stating that

$$\mathrm{Ind}(\mathbf{\Phi}_T, U) = \lim_{t \to 0^+} \mathrm{Ind}(\widehat{\mathbf{\Phi}}_t, U) \tag{12}$$

where $\widehat{\mathbf{\Phi}}_t$ is the translation along trajectories operator for (11). In computation of $\mathrm{Ind}(\widehat{\mathbf{\Phi}}_t, U)$, for small $t > 0$, the spectral properties of the operators $\Delta + \widehat{\alpha}$ and $\Delta + \widehat{\omega}$ are crucial. We strongly use the fact that their essential spectrum is contained in $(-\infty, 0)$ and the rest consists of positive eigenvalues with finite dimensional eigenspaces and that numbers $m(0)$ and $m(\infty)$ are well-defined (i.e. finite).

The paper is organized as follows. In Section 2 we recall the concept of ultimately compact maps and fixed point index theory. In Section 3 we strengthen in a general setting of sectorial operators the initial condition continuity property and Henry's averaging principle. Section 4 is devoted the ultimate compactness property of the translation operator. In Section 5 we adapt the ideas of [7] to the case $\Omega = \mathbb{R}^N$, proving the averaging index formula (12) as well as verify a priori bounds conditions for λT-periodic solutions of (10) with $\lambda \in (0, 1]$. Finally, in Section 6 the main results are proved.

2 Preliminaries

Notation. If X is a normed space with the norm $\|\cdot\|$, then, for $x_0 \in X$ and $r > 0$, we put $B_X(x_0, r) := \{x \in X \mid \|x - x_0\| < r\}$. By ∂U and $\overline{U}$ we denote the boundary and the closure of $U \subset X$. $\mathrm{conv}\, V$ and $\overline{\mathrm{conv}}^X\, V$ stand for the convex hull and the closed (in X) convex hull of $V \subset X$, respectively. By $(\cdot, \cdot)_0$ is denoted the inner product in X.

Measure of noncompactness. If X is a Banach space and $V \subset X$ is bounded, then by $\beta_X(V)$ we denote the infimum over all $r > 0$ such that V can be covered with a finite number of open balls of radius r. Clearly $\beta_X(V)$ is finite and it is called the Hausdorff measure of noncompactness of the set V in the space X. It is not hard to show that $\beta_X(V) = 0$ implies that V is relatively compact in X. More properties of the measure of noncompactness can be found in [10] or [1].

Fixed point index. Below we recall basic definitions and facts from the fixed point index theory for ultimately compact maps. For details we refer to [1].

We say that a map $\Phi : D \to X$, defined on a subset D of a Banach space X is *ultimately compact* if $V \subset X$ is such that $\overline{\mathrm{conv}}\, \Phi(V \cap D) = V$, then V is compact. We shall say that an ultimately compact map $\Phi : \overline{U} \to X$, defined on the closure of an open bounded set $U \subset X$, is called *admissible* if $\Phi(u) \neq u$ for all $u \in \partial U$. By an *admissible homotopy* between two admissible maps $\Phi_0, \Phi_1 : \overline{U} \to X$ we mean a continuous map $\mathbf{\Psi} : \overline{U} \times [0,1] \to X$ such that $\mathbf{\Psi}(\cdot, 0) = \Phi_0$, $\mathbf{\Psi}(\cdot, 1) = \Phi_1$, $\mathbf{\Psi}(u, \mu) \neq u$ for all $u \in \partial U$ and $\mu \in [0,1]$, and, for any $V \subset X$, if $\overline{\operatorname{conv}}\, \mathbf{\Psi}((V \cap \overline{U}) \times [0,1]) = V$, then V is relatively compact. Φ_0, Φ_1 are called homotopic then. A fixed point index for ultimately compact maps was constructed in [1, 1.6.3 and 3.5.6]. Basic properties of the fixed point index are collected in the following

Proposition 2.1.
(i) (existence) *If* $\operatorname{Ind}(\Phi, U) \neq 0$, *then there exists* $u \in U$ *such that* $\Phi(u) = u$.
(ii) (additivity) *If* $U_1, U_2 \subset U$ *are open and* $\Phi(u) \neq u$ *for all* $u \in \overline{U \setminus (U_1 \cup U_2)}$, *then*

$$\operatorname{Ind}(\Phi, U) = \operatorname{Ind}(\Phi, U_1) + \operatorname{Ind}(\Phi, U_2).$$

(iii) (homotopy invariance) *If* $\Phi_0, \Phi_1 : \overline{U} \to X$ *are homotopic, then*

$$\operatorname{Ind}(\Phi_0, U) = \operatorname{Ind}(\Phi_1, U).$$

(iv) (normalization) *Let* $u_0 \in X$ *and* $\Phi_{u_0} : \overline{U} \to X$ *be defined by* $\Phi_{u_0}(u) = u_0$ *for all* $u \in \overline{U}$. *Then* $\operatorname{Ind}(\Phi_{u_0}, U)$ *is equal* 0 *if* $u_0 \notin U$ *and* 1 *if* $u_0 \in U$.

Remark 2.2. If $\Phi : \overline{U} \to X$ is a compact map then $\operatorname{Ind}(\Phi, U)$ is equal to the Leray-Schauder index $\operatorname{Ind}_{LS}(\Phi, U)$ (see e.g. [12]).

3 Remarks on abstract continuity and averaging principle

Let $A : D(A) \to X$ be a sectorial operator such that for some $a > 0$, $A + aI$ has its spectrum in the half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Let X^α, $0 \leq \alpha < 1$, be the fractional power space determined by $A + aI$. It is well-known that there exist $C_0, C_\alpha > 0$ such that for all $t > 0$

$$\|e^{-tA}u\|_\alpha \leq C_0 e^{at}\|u\|_\alpha \quad \text{for all} \quad u \in X^\alpha,$$

$$\|e^{-tA}u\|_\alpha \leq C_\alpha t^{-\alpha} e^{at}\|u\|_0 \quad \text{for all} \quad u \in X$$

where $\{e^{-tA}\}_{t \geq 0}$ is the semigroup generated by $-A$. Consider the equation

$$\begin{cases} \dot{u}(t) = -Au(t) + F(t, u(t)), \ t > 0, \\ u(0) = \bar{u}, \end{cases} \tag{13}$$

where $\bar{u} \in X^\alpha$ and $F : [0, +\infty) \times X^\alpha \to X$ is such that there exists $C \geq 0$ with

$$\|F(t, u)\| \leq C(1 + \|u\|_\alpha) \text{ for all } u \in X, \, t > 0, \tag{14}$$

and, for any bounded $V \subset X^\alpha \times [0, +\infty)$ there exist $D, L \geq 0$ and $\theta \in (0,1)$ with the property

$$\|F(t, u) - F(s, v)\| \leq D|t - s|^\theta + L\|u - v\|_\alpha \text{ for all } (u, t), (v, s) \in V. \tag{15}$$

We shall say that $u : [0, +\infty) \to X^\alpha$ is a solution of above initial value problem if

$$u \in C([0, +\infty), X^\alpha) \cap C((0, +\infty), D(A)) \cap C^1((0, +\infty), X)$$

and satisfies (13). By classical results (see [6] or [13]), the problem (13) admits a unique global solution $u \in C([0,+\infty), X^\alpha) \cap C((0,+\infty), D(A)) \cap C^1((0,+\infty), X)$. Moreover, it is known that u being solution of (13) satisfies the following Duhamel formula

$$u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-s)A}F(s,u(s))\,\mathrm{d}s, \qquad t > 0. \tag{16}$$

Remark 3.1. Assume that $u : [0,T] \to X^\alpha$ is a solution of (13) with $T > 0$. Then clearly, by (14) and (16), there is a constant $\tilde{C} = \tilde{C}(C, C_0, C_\alpha, a, T) > 0$ such that for all $t \in (0,T]$

$$\begin{aligned} \|u(t)\|_\alpha &\le C_0 e^{at}\|\bar{u}\|_\alpha + \int_0^t C_\alpha (t-s)^{-\alpha} e^{a(t-s)} \|F(s,u(s))\|\,\mathrm{d}s \\ &\le \tilde{C}(1 + \|\bar{u}\|_\alpha) + \tilde{C}\int_0^t (t-s)^{-\alpha}\|u(s)\|_\alpha\,\mathrm{d}s. \end{aligned}$$

This in view of [6, Lemma 1.2.9] implies that there exists $\bar{C} = \bar{C}(C, C_0, C_\alpha, a, T, \alpha) > 0$ such that

$$\|u(t)\|_\alpha \le \bar{C}(1 + \|\bar{u}\|_\alpha) \ \text{ for all } \ t \in [0,T]. \tag{17}$$

Theorem 3.2. *Assume that (α_n) is a sequence of positive numbers such that $\alpha_n \to \alpha_0$ as $n \to +\infty$ for some $\alpha_0 > 0$ and that $A_n := \alpha_n A$ for $n \ge 0$. Let $F_n : [0,T] \times X^\alpha \to X$, $T > 0$, $n \ge 0$, satisfy (14) and (15) with common constants C, L (independent of n) and let, for each $u \in X^\alpha$,*

$$\int_0^t F_n(s,u)\,\mathrm{d}s \to \int_0^t F_0(s,u)\,\mathrm{d}s \quad \text{in} \quad X \text{ as } n \to +\infty$$

uniformly with respect to $t \in [0,T]$. If $u_n : [0,T] \to X^\alpha$, $n \ge 0$, are solutions of

$$\dot{u}(t) = -A_n u(t) + F_n(t, u(t)), \ t \in [0,T],$$

and $u_n(0) \to u_0(0)$ in X, then $u_n(t) \to u_0(t)$ in X^α uniformly with respect to t from compact subsets of $(0,T]$.

Remark 3.3. Recall that Henry's result from [13] states that, under the above assumptions with $\alpha_n \equiv 1$, if $u_n(0) \to u_0(0)$ in X^α, as $n \to +\infty$, then $u_n(t) \to u_0(t)$ in X^α uniformly on compact subsets of $[0,T)$. Here, inspired by the proof of Proposition 2.3 of [20], we modify Henry's proof.

In the proof we shall use the following lemma.

Lemma 3.4. *Under the assumptions of Theorem 3.2, for any continuous $u : [0,T] \to X^\alpha$,*

$$\int_0^t e^{-(t-s)A_n}F_n(s,u(s))\,\mathrm{d}s \to \int_0^t e^{-(t-s)A_0}F_0(s,u(s))\,\mathrm{d}s \ \text{ in } X^\alpha \text{ as } \ n \to +\infty,$$

uniformly with respect to $t \in [0,T]$.

Proof: We shall adjust arguments from the proof of [13, Lemma 3.4.7]. First observe that due to the assumptions concerning the constant L for F_n's it is sufficient to show the assertion for $u \equiv \bar{u}$ where $\bar{u} \in X^\alpha$. Take any $\varepsilon > 0$. There exist $\delta > 0$, $\tilde{C} > 0$ and $\tilde{a} > 0$ such that, for any $n \geq 0$ and $t \in [0, \delta]$

$$\begin{aligned}\left\| \int_0^t e^{-(t-s)A_n} F_n(s, \bar{u})\, \mathrm{d}s \right\|_\alpha &\leq \int_0^t C_\alpha \alpha_n^{-\alpha} (t-s)^{-\alpha} e^{\alpha_n a (t-s)} C(1 + \|\bar{u}\|_\alpha)\, \mathrm{d}s \\ &\leq \tilde{C} \int_0^t \tau^{-\alpha} e^{\tilde{a}\tau}\, \mathrm{d}\tau \leq \tilde{C} e^{\tilde{a}T} (1-\alpha)^{-1} \delta^{1-\alpha} \leq \tilde{C} e^{\tilde{a}T} \delta^{1-\alpha} < \varepsilon/4 \end{aligned} \tag{18}$$

and, for any $n \geq 0$ and $t \in [\delta, T]$,

$$\left\| \int_{t-\delta}^t e^{-(t-s)A_n} F_n(s, \bar{u})\, \mathrm{d}s \right\|_\alpha \leq \tilde{C} \int_0^\delta \tau^{-\alpha} e^{\tilde{a}\tau}\, \mathrm{d}\tau \leq \tilde{C} e^{\tilde{a}T} (1-\alpha)^{-1} \delta^{1-\alpha} < \varepsilon/4. \tag{19}$$

Observe that, for any $n \geq 0$ and $t \in [\delta, T]$,

$$\begin{aligned}\int_0^{t-\delta} e^{-(t-s)A_n} F_n(s, \bar{u})\, \mathrm{d}s = e^{-tA_n} &\int_0^t F_n(\tau, \bar{u})\, \mathrm{d}\tau - e^{-\delta A_n} \int_{t-\delta}^t F_n(\tau, \bar{u})\, \mathrm{d}\tau \\ &+ \int_0^{t-\delta} A_n e^{-(t-s)A_n} \int_s^t F_n(\tau, \bar{u})\, \mathrm{d}\tau\ \mathrm{d}s.\end{aligned}$$

Clearly,

$$e^{-tA_n} \int_0^t F_n(\tau, \bar{u})\, \mathrm{d}\tau \to e^{-tA_0} \int_0^t F_0(\tau, \bar{u})\, \mathrm{d}\tau, \quad \text{in } X^\alpha,$$

uniformly with respect to $t \in [\delta, T]$. Note also that, for all $t \in [\delta, T]$ and all $n \geq 1$,

$$\left\| e^{-\delta A_n} \int_{t-\delta}^t F_n(\tau, \bar{u})\, \mathrm{d}\tau \right\|_\alpha \leq \tilde{C} e^{\tilde{a}\delta} \delta^{1-\alpha} \leq \varepsilon/4.$$

Finally, for large n and all $t \in [\delta, T]$ and $s \in [0, t-\delta]$, one has

$$\begin{aligned}&\left\| A_n e^{-(t-s)A_n} \int_s^t F_n(\tau, \bar{u})\, \mathrm{d}\tau - A_0 e^{-(t-s)A_0} \int_s^t F_0(\tau, \bar{u})\, \mathrm{d}\tau \right\|_\alpha \\ &\leq |\alpha_n - \alpha_0| \left\| A e^{-(t-s)A_n} \int_s^t F_n(\tau, \bar{u})\, \mathrm{d}\tau \right\|_\alpha \\ &\quad + \alpha_0 \left\| A e^{-(t-s)A_n} \int_s^t F_n(\tau, \bar{u})\, \mathrm{d}\tau - A e^{-(t-s)A_0} \int_s^t F_0(\tau, \bar{u})\, \mathrm{d}\tau \right\|_\alpha \\ &\leq \bar{C} |\alpha_n - \alpha_0| \left\| e^{-(t-s)A_n} \int_s^t F_n(\tau, \bar{u})\, \mathrm{d}\tau \right\|_{1+\alpha} \\ &\quad + \bar{C}\alpha_0 \left\| e^{-(t-s)A_n} \left(\int_s^t F_n(\tau, \bar{u})\, \mathrm{d}\tau - \int_s^t F_0(\tau, \bar{u})\, \mathrm{d}\tau \right) \right\|_{1+\alpha} \\ &\quad + \bar{C}\alpha_0 \left\| \left(e^{-(t-s)A_n} - e^{-(t-s)A_0} \right) \int_s^t F_0(\tau, \bar{u})\, \mathrm{d}\tau \right\|_{1+\alpha} \\ &\leq |\alpha_n - \alpha_0| \frac{\bar{C} C_{1+\alpha} e^{\tilde{a}T}}{\delta^{1+\alpha}} \left\| \int_s^t F_n(\tau, \bar{u})\, \mathrm{d}\tau \right\| + \alpha_0 \frac{\bar{C} C_{1+\alpha} e^{\tilde{a}T}}{\delta^{1+\alpha}} \left\| \int_s^t F_n(\tau, \bar{u})\, \mathrm{d}\tau - \int_s^t F_0(\tau, \bar{u})\, \mathrm{d}\tau \right\| \\ &\quad + \alpha_0 \frac{\bar{C} C_{1+\alpha} e^{\tilde{a}T}}{(\alpha_0\delta/2)^{1+\alpha}} \left\| \left(e^{-((t-s)\alpha_n/\alpha_0 - \delta/2)A_0} - e^{-(t-s-\delta/2)A_0} \right) \int_s^t F_0(\tau, \bar{u})\, \mathrm{d}\tau \right\|\end{aligned}$$

where $\bar{C} > 0$ is such that $\|Aw\|_\alpha \leq \bar{C}\|w\|_{1+\alpha}$ for all $w \in X^{1+\alpha}$. Therefore for large n and all $t \in [\delta, T]$

$$\left\| \int_0^{t-\delta} e^{-(t-s)A_n} F_n(s, \bar{u})\,\mathrm{d}s - \int_0^{t-\delta} e^{-(t-s)A_0} F_0(s, \bar{u})\,\mathrm{d}s \right\|_\alpha \leq \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = 3\varepsilon/4,$$

which together with (18) and (19) ends the proof. $\square$

Proof of Theorem 3.2: By the Duhamel formula, for $t \in (0, T]$ and $n \geq 1$,

$$\begin{aligned} u_n(t) - u_0(t) &= e^{-tA_n} u_n(0) - e^{-tA_0} u_0(0) + \\ &\quad + \int_0^t e^{-(t-s)A_n} F_n(s, u_0(s)) - e^{-(t-s)A_0} F_0(s, u_0(s)))\,\mathrm{d}s \\ &\quad + \int_0^t e^{-(t-s)A_n} (F_n(s, u_n(s)) - F_n(s, u_0(s)))\,\mathrm{d}s. \end{aligned}$$

This gives, for all $t \in (0, T]$ and $n \geq 1$,

$$\|u_n(t) - u_0(t)\|_\alpha \leq \gamma_n(t) + C_\alpha L \int_0^t e^{a\alpha_n(t-s)} (\alpha_n(t-s))^{-\alpha} \|u_n(s) - u_0(s)\|_\alpha\,\mathrm{d}s$$

with

$$\begin{aligned} \gamma_n(t) := &\frac{C_\alpha e^{a\alpha_n t}}{(\alpha_n t)^\alpha} \|u_n(0) - u_0(0)\|_0 + \left\|(e^{-tA_n} - e^{-tA_0}) u_0(0)\right\|_\alpha \\ &+ \left\| \int_0^t \left(e^{-(t-s)A_n} F_n(s, u_0(s)) - e^{-(t-s)A_0} F_0(s, u_0(s))\right) \mathrm{d}s \right\|_\alpha. \end{aligned}$$

This means that there are $\tilde{a} > 0$ and $\tilde{C} > 0$ such that, for all $t \in (0, T]$ and $n \geq 1$,

$$\|u_n(t) - u_0(t)\|_\alpha \leq \gamma_n(t) + \tilde{C} \int_0^t e^{\tilde{a}(t-s)} (t-s)^{-\alpha} \|u_n(s) - u_0(s)\|_\alpha\,\mathrm{d}s.$$

By use of Lemma 7.1.1 of [13], we get

$$\|u_n(t) - u_0(t)\|_\alpha \leq \gamma_n(t) + K \int_0^t (t-s)^{-\alpha} \gamma_n(s)\,\mathrm{d}s$$

for some constant $K > 0$. Now let us fix $t \in [0, T]$ and take an arbitrary $\delta \in (0, t)$. Observe also that

$$\begin{aligned} \int_0^t (t-s)^{-\alpha} \gamma_n(s)\,\mathrm{d}s &\leq \frac{2^\alpha}{\delta^\alpha} \int_0^{t-\delta/2} \gamma_n(s)\,\mathrm{d}s + \int_{t-\delta/2}^t (t-s)^{-\alpha} \gamma_n(s)\,\mathrm{d}s \\ &\leq \frac{2^\alpha}{\delta^\alpha} \int_0^T \gamma_n(s)\,\mathrm{d}s + \frac{(\delta/2)^{1-\alpha}}{1-\alpha} \cdot \sup_{s \in [\delta/2, T]} \gamma_n(s). \end{aligned}$$

Since, in view of Lemma 3.4, $\gamma_n(t) \to 0$ uniformly with respect to t from compact subsets of $(0, T]$ and the functions γ_n, $n \geq 1$, are estimated from above by an integrable function we infer, by the dominated convergence theorem, that

$\|u_n(t) - u_0(t)\|_\alpha \to 0$ as $n \to +\infty$ uniformly with respect to $t \in [\delta, T]$. $\square$

The above theorem allows us to strengthen Henry's averaging principle. We assume that mappings $F_n : [0, +\infty) \times X^\alpha \to X$, $n \geq 1$, satisfy (14) and (15) with common constants C, L (independent of n) and that there exists $\widehat{F} : X^\alpha \to X$ such that, for all $\bar{u} \in X^\alpha$,

$$\lim_{\tau \to +\infty, n \to +\infty} \frac{1}{\tau} \int_0^\tau F_n(t, \bar{u})\,\mathrm{d}t = \widehat{F}(\bar{u}) \quad \text{in} \quad X. \tag{20}$$

Theorem 3.5. *Suppose F_n and $\widehat{F}$ are as above, $\bar{u}_n \to \bar{u}_0$ in X, $\lambda_n \to 0^+$ as $n \to +\infty$, and $u_n : [0, +\infty) \to X^\alpha$, $n \geq 1$, are solutions of*

$$\begin{cases} \dot{u}(t) = -Au(t) + F_n(t/\lambda_n, u(t)), \ t > 0, \\ u(0) = \bar{u}_n. \end{cases}$$

Then $u_n(t) \to \widehat{u}(t)$ in X^α uniformly with respect to t from compact subsets of $(0, +\infty)$ where $\widehat{u} : [0, +\infty) \to X^\alpha$ is the solution of

$$\begin{cases} \dot{u}(t) = -Au(t) + \widehat{F}(u(t)), \ t > 0, \\ u(0) = \bar{u}. \end{cases}$$

Proof: Let $\tilde{F}_n := F_n(\cdot/\lambda_n, \cdot)$ and $\tilde{F}_0 := \widehat{F}$. Observe that, using (20), we get, for any $\bar{u} \in X^\alpha$ and $t > 0$,

$$\int_0^t \tilde{F}_n(s, \bar{u})\,\mathrm{d}s = \lambda_n \int_0^{t/\lambda_n} F_n(\rho, \bar{u})\,\mathrm{d}\rho \to t\widehat{F}(\bar{u}) = \int_0^t \tilde{F}_0(\bar{u})\,\mathrm{d}s \quad \text{in } X, \text{ as } n \to +\infty.$$

Clearly, $\tilde{F}_n$, $n \geq 1$, and $\tilde{F}_0$ satisfy (14) and (15) with the common constants C, L. It can be easily verified that the convergence above is uniform with respect to t from bounded subintervals of $[0, +\infty)$. Now, an application of Theorem 3.2 yields the assertion. $\square$

Remark 3.6.

(a) The above result is an improvement of the continuation theorem and the Henry averaging principle [13, Th. 3.4.9] to the case when initial values from X^α converge in the topology of X (not X^α). This will appear crucial when establishing the ultimate compactness property and verifying a priori estimates in the proofs of main results. We shall need to consider solutions in the phase space $X^{1/2} = H^1(\mathbb{R}^N)$ (cf. Remark 1.3) while the compactness of sequences of initial values is possible with respect to the $L^2(\mathbb{R}^N)$ topology only.

(b) An averaging principle for parabolic equations on $\mathbb{R}^N$ was also proved in [3] where time dependent coefficients of the elliptic operator were considered. Here we have provided a general abstract approach.

4 Continuity, averaging and compactness for the parabolic equation

We transform (1) into an abstract evolution equation. To this end define an operator $\mathbf{A} : D(\mathbf{A}) \to X$ in the space $X := L^2(\mathbb{R}^N)$ by

$$\mathbf{A}u := -\sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u}{\partial x_j \partial x_i}, \text{ for } u \in D(\mathbf{A}) := H^2(\mathbb{R}^N),$$

where $a_{ij} \in \mathbb{R}$, $i,j = 1,\ldots,N$, are such that

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j > 0, \text{ for any } \xi \in \mathbb{R}^N,$$

and $a_{ij} = a_{ji}$ for $i,j = 1,\ldots,N$. It is well-known that $\mathbf{A}$ is a self-adjoint, positive and sectorial operator in $L^2(\mathbb{R}^N)$.

Suppose that f is as in Section 1 and define $\mathbf{F} : [0,+\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^2)$ by $[\mathbf{F}(t,u)](x) := f(t,x,u(x))$ for a.e. $x \in \mathbb{R}^N$.

Lemma 4.1. *Under the above assumptions there are constants $D > 0$, depending only on k, $\tilde{k}$, N and p, $L > 0$, depending only l, N and p, and $C > 0$, depending only on m_0, l, N and p, such that, for all $t_1, t_2 \geq 0$ and $u_1, u_2 \in H^1(\mathbb{R}^N)$,*

$$\|\mathbf{F}(t_1,u_1) - \mathbf{F}(t_2,u_2)\|_{L^2} \leq D(1 + \|u_1\|_{H^1})|t_1 - t_2|^{\theta} + L\|u_1 - u_2\|_{H^1} \quad \textit{and}$$

$$\|\mathbf{F}(t,u)\|_{L^2} \leq C(1 + \|u\|_{H^1}) \textit{ for any } t \geq 0 \textit{ and } u \in H^1(\mathbb{R}^N).$$

Before we pass to the proof of Lemma 4.1 we shall provide the following technical lemma.

Lemma 4.2. *There exist constants $C_1 = C_1(N,p) > 0$ and $C_2 = C_2(N,p) > 0$ such that for any $u \in H^1(\mathbb{R}^N)$*

$$\|u\|_{L^{2p/(p-1)}} \leq C_1\|u\|_{H^1}. \tag{21}$$

and

$$\|u\|_{L^{2p/(p-2)}} \leq C_2\|u\|_{H^1}. \tag{22}$$

Proof. Take any $u \in H^1(\mathbb{R}^N)$. If $N = 1$, then by use of the Hölder and interpolation inequalities together with the continuity of the embedding $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$, one gets

$$\|u\|_{L^{2p/(p-1)}} \leq \|u\|_{L^2}^{1-1/p}\|u\|_{L^\infty}^{1/p} \leq C\|u\|_{H^1}$$

and

$$\|u\|_{L^{2p/(p-2)}} \leq \|u\|_{L^2}^{1-2/p}\|u\|_{L^\infty}^{2/p} \leq C\|u\|_{H^1},$$

where $C > 0$ is such that $\|v\|_{L^\infty} \leq C\|v\|_{H^1}$ for all $v \in H^1(\mathbb{R})$. If $N = 2$, then

$$\|u\|_{L^{2p/(p-1)}} \leq \|u\|_{L^2}^{1-2/p}\|u\|_{L^4}^{2/p} \leq C\|u\|_{H^1},$$

where $C > 0$ is the constant from the inequality $\|v\|_{L^4} \le C\|v\|_{H^1}$ for all $v \in H^1(\mathbb{R}^2)$. Similarly,

$$\|u\|_{L^{2p/(p-2)}} \le \|u\|_{L^2}^{1-2q/(pq-2p)} \|u\|_{L^q}^{2q/(pq-2p)} \le C\|u\|_{H^1},$$

where q is an arbitrary fixed number from $(\frac{2p}{p-2}, +\infty)$ and $C > 0$ is the constant coming from the fact that $H^1(\mathbb{R}^2)$ embeds continuously into $L^s(\mathbb{R}^2)$ for any $s \in [2, +\infty)$. Finally, if $N \ge 3$, then, by the same techniques, we get

$$\|u\|_{L^{2p/(p-1)}} \le \|u\|_{L^2}^{1-N/2p} \|u\|_{L^{2N/(N-2)}}^{N/2p} \le C\|u\|_{H^1}$$

and

$$\|u\|_{L^{2p/(p-2)}} \le \|u\|_{L^2}^{1-N/p} \|u\|_{L^{2N/(N-2)}}^{N/p} \le C\|u\|_{H^1},$$

where $C > 0$ is the constant in the Sobolev inequality $\|v\|_{L^{2N/(N-2)}} \le C\|v\|_{H^1}$ for all $v \in H^1(\mathbb{R}^N)$. □

Proof of Lemma 4.1: By use of (4), the Hölder inequality and Lemma 4.2, one finds constants $D = D(k, \tilde{k}, N, p) > 0$ and $L = L(l, N, p) > 0$ such that, for any $t_1, t_2 \ge 0$ and $u_1, u_2 \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}\|\mathbf{F}(t_1, u_1) - \mathbf{F}(t_2, u_2)\|_{L^2} \le& (\|\tilde{k}\|_{L^2} + C_2\|k_0\|_{L^p}\|u_1\|_{H^1} + \|k_\infty\|_{L^\infty}\|u_1\|_{L^2})|t_1 - t_2|^\theta \\ &+ C_2\|l_0(t_2, \cdot)\|_{L^p}\|u_1 - u_2\|_{H^1} + \|l_\infty(t_2, \cdot)\|_{L^\infty}\|u_1 - u_2\|_{L^2} \\ \le& D(1 + \|u_1\|_{H^1})|t_1 - t_2|^\theta + L\|u_1 - u_2\|_{H^1}.\end{aligned}$$

Furthermore, by (4), one also has $|f(t, x, u)| \le |f(t, x, 0)| + l(t, x)|u|$ for $t \ge 0$, $x \in \mathbb{R}^N$, $u \in \mathbb{R}$. This gives the existence of $C = C(m_0, l, N, p) > 0$ such that

$$\|\mathbf{F}(t, u)\|_{L^2} \le \|m_0\|_{L^2} + C_2\|l_0(t, \cdot)\|_{L^p}\|u\|_{H^1} + \|l_\infty(t, \cdot)\|_{L^\infty}\|u\|_{L^2} \le C(1 + \|u\|_{H^1})$$

for any $t \ge 0$ and $u \in H^1(\mathbb{R}^N)$. □

Consider now the evolutionary problem

$$\dot{u}(t) = -\mathbf{A}u(t) + \mathbf{F}(t, u(t)),\ t \ge 0, \quad u(0) = \bar{u} \in H^1(\mathbb{R}^N). \tag{23}$$

Due to Lemma 4.1 and standard results in theory of abstract evolution equations (see [13] or [6]), the problem (23) admits a unique global solution $u \in C([0, +\infty), H^1(\mathbb{R}^N)) \cap C((0, +\infty), H^2(\mathbb{R}^N)) \cap C^1((0, +\infty), L^2(\mathbb{R}^N))$. We shall say that $u : [0, T_0) \to H^1(\mathbb{R}^N)$, $T_0 > 0$, is a *solution* (H^1-*solution*) of

$$\begin{cases} \dfrac{\partial u}{\partial t}(x, t) = \mathcal{A}u(x, t) + f(t, x, u(x, t)), & x \in \mathbb{R}^N,\ t \in (0, T_0), \\ \qquad\qquad u(x, 0) = \bar{u}(x), & x \in \mathbb{R}^N, \end{cases}$$

for some $\bar{u} \in H^1(\mathbb{R}^N)$, where $\mathcal{A} = \sum_{i,j=1}^N a_{ij} \frac{\partial^2}{\partial x_j \partial x_i}$, if

$$u \in C([0, +\infty), H^1(\mathbb{R}^N)) \cap C((0, +\infty), H^2(\mathbb{R}^N)) \cap C^1((0, +\infty), L^2(\mathbb{R}^N))$$

and (23) holds. In this sense we have global in time existence and uniqueness of solutions for the parabolic partial differential equation.

The continuity of solutions properties are collected below.

Proposition 4.3. (compare [20, Prop. 2.3]) *Assume that functions* $f_n : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $n \ge 0$, *satisfy the assumptions* (3) *with common* m_0 *and* (4) *with common* l *and that* $f_n(t,x,u) \to f_0(t,x,u)$, *for all* $t \ge 0$, $u \in \mathbb{R}$, *a.e.* $x \in \mathbb{R}^N$, *and* $f_n(t,\cdot,0) \to f_0(t,\cdot,0)$ *in* $L^2(\mathbb{R}^N)$ *for all* $t \ge 0$. *Suppose that* (α_n) *is a sequence of positive numbers such that* $\alpha_n \to \alpha_0$, *as* $n \to +\infty$, *for some* $\alpha_0 > 0$. *Let* $u_n : [0,T] \to H^1(\mathbb{R}^N)$, $n \ge 0$, *be a solution of*

$$\frac{\partial u}{\partial t}(x,t) = \alpha_n \mathcal{A}u(x,t) + f_n(t,x,u(x,t)),\ x \in \mathbb{R}^N,\ t \in (0,T],$$

such that, for some $R > 0$, $\|u_n(t)\|_{H^1} \le R$, *for all* $t \in [0,T]$ *and* $n \ge 0$. *Then* $f_n(t,\cdot,u(\cdot)) \to f_0(t,\cdot,u(\cdot))$ *in* $L^2(\mathbb{R}^N)$ *for any* $u \in H^1(\mathbb{R}^N)$ *and* $t \ge 0$ *and*
(i) *if* $u_n(0) \to u_0(0)$ *in* $L^2(\mathbb{R}^N)$ *as* $n \to \infty$, *then* $u_n(t) \to u(t)$ *in* $H^1(\mathbb{R}^N)$ *for* t *from compact subsets of* $(0,T]$.
(ii) *if* $u_n(0) \to u_0(0)$ *in* $H^1(\mathbb{R}^N)$ *as* $n \to \infty$, *then* $u_n(t) \to u_0(t)$ *in* $H^1(\mathbb{R}^N)$ *uniformly for* $t \in [0,T]$.

Proof: Define $\mathbf{F}_n : [0,+\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, $n \ge 0$, by $[\mathbf{F}_n(t,u)](x) := f_n(t,x,u(x))$. Note that, in view of (4), for any $t \ge 0$ and $u \in H^1(\mathbb{R}^N)$ and a.e. $x \in \mathbb{R}^N$

$$|f_n(t,x,u(x)) - f_0(t,x,u(x))|^2 \le 2|f_n(t,x,0) - f_0(t,x,0)|^2 + 4|l(x,t)u|^2.$$

Since, for any $t \ge 0$, $f_n(t,\cdot,0) \to f_0(t,\cdot,0)$ in $L^2(\mathbb{R}^N)$ as $n \to +\infty$, the right hand side can be estimated by an integrated function, which due to the Lebesgue dominated convergence theorem implies $\mathbf{F}_n(t,u) \to \mathbf{F}_0(t,u)$ in $L^2(\mathbb{R}^N)$ as $n \to +\infty$. Moreover, by use of Lemma 4.1, we may pass to the limit under the integral to get $\int_0^t \mathbf{F}_n(s,u)\,\mathrm{d}s \to \int_0^t \mathbf{F}_0(s,u)\,\mathrm{d}s$ in $L^2(\mathbb{R}^N)$ for any $u \in H^1(\mathbb{R}^N)$ and $t \ge 0$. This in view of Theorem 3.2 implies the assertion (i). The assertion (ii) comes from the standard continuity theorem from [13]. □

Let us also state an averaging principle.

Proposition 4.4. *Assume that functions* $f_n : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $n \ge 0$, *satisfy the assumptions of Proposition* 4.3 *and additionally* (2). *Suppose that* $\bar{u}_n \to \bar{u}_0$ *in* $L^2(\mathbb{R}^N)$, $\lambda_n \to 0^+$ *as* $n \to +\infty$ *and that* $u_n : [0,+\infty) \to H^1(\mathbb{R}^N)$, $n \ge 1$, *are solutions of*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{A}u + f_n(t/\lambda_n, x, u), & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = \bar{u}_n(x), & x \in \mathbb{R}^N. \end{cases}$$

Then $u_n(t) \to \widehat{u}(t)$ *in* $H^1(\mathbb{R}^N)$ *uniformly on compact subsets of* $(0,+\infty)$, *where* $\widehat{u} : [0,+\infty) \to H^1(\mathbb{R}^N)$ *is the solution of*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{A}u + \widehat{f}_0(x,u), & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = \bar{u}_0(x), & x \in \mathbb{R}^N, \end{cases}$$

with $\widehat{f}_0 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *given by* $\widehat{f}_0(x,u) := \frac{1}{T}\int_0^T f_0(t,x,u)\,\mathrm{d}t$ *for all* $u \in \mathbb{R}$ *and a.e.* $x \in \mathbb{R}^N$.

Proof: Define $\mathbf{F}_n : [0,+\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, $n \geq 0$, by $[\mathbf{F}_n(t,u)](x) := f_n(t,x,u(x))$ and $\widehat{\mathbf{F}}_0 : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by $\widehat{\mathbf{F}}_0(u) := \frac{1}{T}\int_0^T \mathbf{F}_0(t,u)\,\mathrm{d}t$. Clearly, for all $u \in H^1(\mathbb{R}^N)$, $\widehat{\mathbf{F}}(u)(x) = \widehat{f}(x,u(x))$ for a.e. $x \in \mathbb{R}^N$. Fix any $\bar{u} \in H^1(\mathbb{R}^N)$ and (τ_n) in $(0,+\infty)$ such that $\tau_n \to +\infty$. Clearly, $\mathbf{F}_n$, $n \geq 1$, are T-periodic in time. Consequently, one has

$$I_n := \frac{1}{\tau_n}\int_0^{\tau_n} \mathbf{F}_n(t,\bar{u})\,\mathrm{d}t = \frac{[\tau_n/T]}{\tau_n/T}\cdot\frac{1}{T}\int_0^T \mathbf{F}_n(t,\bar{u})\,\mathrm{d}t + \frac{1}{\tau_n}\int_0^{\tau_n-[\tau_n/T]T} \mathbf{F}_n(t,\bar{u})\,\mathrm{d}t.$$

Hence, to see that $I_n \to \widehat{\mathbf{F}}_0(\bar{u})$ it is is sufficient to prove that

$$I_n^{(T)} := \frac{1}{T}\int_0^T \mathbf{F}_n(t,\bar{u})\,\mathrm{d}t \to \widehat{\mathbf{F}}_0(\bar{u}) \ \text{ in } L^2(\mathbb{R}^N), \quad \text{as } n \to +\infty.$$

To this end observe that, for a.e. $x \in \mathbb{R}^N$,

$$I_n^{(T)}(x) = \frac{1}{T}\int_0^T f_n(t,x,\bar{u}(x))\,\mathrm{d}t \to \frac{1}{T}\int_0^T f_0(t,x,\bar{u}(x))\,\mathrm{d}t = \widehat{f}_0(x,u(x)) = [\widehat{\mathbf{F}}_0(\bar{u})](x).$$

Moreover, by use of the assumptions on f_n's, one has

$$|I_n^{(T)}(x)| \;=\; \left|\frac{1}{T}\int_0^T f_n(t,x,\bar{u}(x))\,\mathrm{d}t\right| \leq m_0(x) + g(x)$$

where $g(x) := \frac{1}{T}\int_0^T |l(t,x)||\bar{u}(x)|\,\mathrm{d}t$. and, by use of Jensen's inequality,

$$\int_{\mathbb{R}^N} |g(x)|^2\,\mathrm{d}x \leq \frac{1}{T}\int_{\mathbb{R}^N}\int_0^T |l(t,x)|^2|\bar{u}(x)|^2\,\mathrm{d}t\,\mathrm{d}x < +\infty$$

(see the proof of Lemma 4.1). Hence, by the dominated convergence theorem we infer that $I_n^{(T)} \to \widehat{\mathbf{F}}_0(\bar{u})$ in $L^2(\mathbb{R}^N)$. Since (τ_n) was arbitrary it follows that

$$\lim_{\tau\to+\infty,\ n\to+\infty} \frac{1}{\tau}\int_0^\tau \mathbf{F}_n(t,\bar{u})\,\mathrm{d}t \to \widehat{\mathbf{F}}_0(\bar{u}).$$

Finally, we get the assertion by use of Theorem 3.5. □

Now we pass to compactness issues that we treat with use of tail estimates technique.

Lemma 4.5. *Assume that $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies* (3), (4) *and* (5). *Suppose that $u : [0,T] \to H^1(\mathbb{R}^N)$ is a solution of* (1) *such that $\|u(t)\|_{H^1} \leq R$ for all $t \in [0,T]$. Then there exists a sequence (α_n) with $\alpha_n \to 0$ as $n \to \infty$ such that*

$$\int_{\mathbb{R}^N\setminus B(0,n)} |u(t)|^2\,\mathrm{d}x \leq R^2 e^{-2at} + \alpha_n \quad \textit{for all } t \in [0,T],\ n \geq 1,$$

where α_n's depend only on N, p, R, m_0, a, b and a_{ij}'s.

Proof: it goes along the lines of [20, Prop. 2.2]. The only difference is that here we have the modified dissipativity condition (5), i.e., (5) implies

$$f(t,x,u)u \leq -a|u|^2 + b(x)|u|^2 + f(t,x,0)u$$

for $t \geq 0$, $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, and one needs to modify the proof in a rather obvious way. $\square$

Now suppose that $a_{ij} \in C([0,1],\mathbb{R})$, $i,j = 1,\ldots,N$, are such that $\sum_{i,j=1}^{N} a_{ij}(\mu)\xi_i\xi_j > 0$ for any $\xi \in \mathbb{R}^N$ and $\mu \in [0,1]$. Let $\mathbf{A}^{(\mu)} : D(\mathbf{A}^{(\mu)}) \to L^2(\mathbb{R}^N)$, $\mu \in [0,1]$, be given by

$$\mathbf{A}^{(\mu)}u := -\sum_{i,j=1}^{N} a_{ij}(\mu)\frac{\partial^2 u}{\partial x_j \partial x_i},\ u \in D(\mathbf{A}^{(\mu)}) := H^2(\mathbb{R}^N).$$

Let $h : [0,+\infty)\times\mathbb{R}^N\times\mathbb{R}\times[0,1] \to \mathbb{R}$ be such that, for all $t,s \geq 0$, $u,v \in \mathbb{R}$, $\mu,\nu \in [0,1]$ and a.e. $x \in \mathbb{R}^N$,

$$h(t,\cdot,u,\mu) \text{ is measurable and } |h(t,x,0,\mu)| \leq m_0(x), \tag{24}$$

$$|h(t,x,u,\mu) - h(s,x,v,\mu)| \leq (\tilde{k}(x) + k(x)|u|)|t-s|^\theta + l(s,x)|u-v|, \tag{25}$$

$$|h(t,x,u,\mu) - h(t,x,u,\nu)| \leq l(t,x)\,|u|\,|\rho(\mu) - \rho(\nu)|, \tag{26}$$

$$(h(t,x,u,\mu) - h(t,x,v,\mu))(u-v) \leq -a|u-v|^2 + b(x)|u-v|^2 \tag{27}$$

where $m_0 \in L^2(\mathbb{R}^N)$, $\theta \in (0,1)$, $\tilde{k} \in L^2(\mathbb{R}^N)$, $k = k_0 + k_\infty$ with $k_0 \in L^p(\mathbb{R}^N)$, $k_\infty \in L^\infty(\mathbb{R}^N)$, $l = l_0 + l_\infty$ with $l_0(t,\cdot) \in L^p(\mathbb{R}^N)$, $l_\infty(t,\cdot) \in L^\infty(\mathbb{R}^N)$ for all $t \geq 0$, and $\sup_{t\geq 0}(\|l_0(t,\cdot)\|_{L^p} + \|l_\infty(t,\cdot)\|_{L^\infty}) < +\infty$, $\rho \in C([0,1],\mathbb{R})$, $a > 0$ and $b \in L^p(\mathbb{R}^N)$.

Under these assumptions consider

$$\dot{u}(t) = -\mathbf{A}^{(\mu)}u(t) + \mathbf{H}(t,u(t),\mu),\ t > 0, \tag{28}$$

where $\mathbf{H} : [0,+\infty) \times H^1(\mathbb{R}^N) \times [0,1] \to L^2(\mathbb{R}^N)$ is defined by

$$[\mathbf{H}(t,u,\mu)](x) := h(t,x,u(x),\mu) \text{ for } t \geq 0,\ u \in H^1(\mathbb{R}^N),\ \mu \in [0,1],\ \text{a.e. } x \in \mathbb{R}^N.$$

Clearly, due to Lemma 4.1, we get the existence and uniqueness of solutions on $[0,+\infty)$. Denote by $u(\cdot;\bar{u},\mu)$ the solution of (28) satisfying the initial value condition $u(0) = \bar{u}$.

The following tail estimates will be crucial in studying the compactness properties of the translation along trajectories operator of (28).

Lemma 4.6. *Take any $\bar{u}_1, \bar{u}_2 \in H^1(\mathbb{R}^N)$ and $\mu_1, \mu_2 \in [0,1]$ and suppose that there are solutions $u(\cdot;\bar{u}_i,\mu_i) : [0,T] \to H^1(\mathbb{R}^N)$, $i = 1,2$ of (28), for some fixed $T > 0$. If $\|u(t;\bar{u}_1,\mu_1)\|_{H^1} \leq R$ and $\|u(t;\bar{u}_2,\mu_2)\|_{H^1} \leq R$ for all $t \in [0,T]$ and some fixed $R > 0$, then there exists a sequence (α_n) with $\alpha_n \to 0$ as $n \to \infty$ such that*

$$\int_{\mathbb{R}^N\setminus B(0,n)} |u(t;\bar{u}_1,\mu_1) - u(t;\bar{u}_2,\mu_2)|^2\,\mathrm{d}x \leq e^{-2at}\|\bar{u}_1 - \bar{u}_2\|_{L^2}^2 + Q\eta(\mu_1,\mu_2) + \alpha_n,$$

for all $t \in [0,T]$ *and* $n \geq 1$*, where* $\alpha_n \geq 0$ *and* $Q > 0$ *depend only on* N, p, R, l, a, b *and* $a'_{ij}s$*,*

$$\eta(\mu_1, \mu_2) := \max\left\{ |\rho(\mu_1) - \rho(\mu_2)|, \max_{i,j=1,\dots,N} |a_{ij}(\mu_1) - a_{ij}(\mu_2)| \right\}.$$

Proof: Let $\phi : [0,+\infty) \to \mathbb{R}$ be a smooth function such that $\phi(s) \in [0,1]$ for $s \in [0,+\infty)$, $\phi_{|[0,\frac{1}{2}]} \equiv 0$ and $\phi_{|[1,+\infty)} \equiv 1$ and let $\phi_n : \mathbb{R}^N \to \mathbb{R}$ be defined by $\phi_n(x) := \phi(|x|^2/n^2)$, $x \in \mathbb{R}^N$. Put $u_1 := u(\cdot; \bar{u}_1, \mu_1)$, $u_2 := u(\cdot; \bar{u}_2, \mu_2)$ and $v := u_1 - u_2$. Observe that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(v(t), \phi_n v(t))_0 &= \frac{1}{2}\left((v(t), \phi_n \dot{v}(t))_0 + (\dot{v}(t), \phi_n v(t))_0\right) = (\phi_n v(t), \dot{v}(t)))_0 \\
&= I_1(t) + I_2(t) + I_3(t)
\end{aligned}$$

where

$$\begin{aligned}
I_1(t) &:= (\phi_n v(t), -\mathbf{A}^{(\mu_1)} u_1(t) + \mathbf{A}^{(\mu_1)} u_2(t))_0, \\
I_2(t) &:= (\phi_n v(t), -\mathbf{A}^{(\mu_1)} u_2(t) + \mathbf{A}^{(\mu_2)} u_2(t))_0, \\
I_3(t) &:= (\phi_n v(t), \mathbf{H}(t, u_1(t), \mu_1) - \mathbf{H}(t, u_2(t), \mu_2))_0.
\end{aligned}$$

As for the first term we notice that

$$\begin{aligned}
I_1(t) &= (\phi_n v(t), -\mathbf{A}^{(\mu_1)} v(t))_0 \\
&= -\int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(\mu_1) \frac{\partial}{\partial x_j}(\phi_n(x) v(t)) \frac{\partial}{\partial x_i}(v(t))\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^N} \phi_n(x) \sum_{i,j=1}^N a_{ij}(\mu_1) \frac{\partial}{\partial x_j}(v(t)) \frac{\partial}{\partial x_i}(v(t))\,\mathrm{d}x \\
&\quad -\frac{2}{n^2}\int_{\mathbb{R}^N} \sum_{i,j=1}^N \phi'(|x|^2/n^2) v(t) x_j a_{ij}(\mu_1) \frac{\partial}{\partial x_i}(v(t))\,\mathrm{d}x \\
&\leq \frac{2L_\phi}{n^2} \int_{\left\{\frac{\sqrt{2}}{2}n \leq |x| \leq n\right\}} \sum_{i,j=1}^N a_{ij}(\mu_1) |x| |v(t)| |\nabla_x v(t)|\,\mathrm{d}x \\
&\leq \frac{2L_\phi M N^2}{n} \|v(t)\|_{L^2} \|v(t)\|_{H^1}
\end{aligned}$$

where $L_\phi := \sup_{s \in [0,+\infty)} |\phi'(s)| < \infty$ (as ϕ' is smooth and nonzero on a bounded interval) and $M := \max_{1 \leq i,j \leq N,\, \mu \in [0,1]} |a_{ij}(\mu)|$. Further, in a similar manner

$$\begin{aligned}
I_2(t) &= -\int_{\mathbb{R}^N} \sum_{i,j=1}^N \frac{\partial}{\partial x_j}(\phi_n v(t))(a_{ij}(\mu_1) - a_{ij}(\mu_2)) \frac{\partial}{\partial x_i}(u_2(t))\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^N} \phi_n(x) \sum_{i,j=1}^N (a_{ij}(\mu_1) - a_{ij}(\mu_2)) \frac{\partial}{\partial x_j}(v(t)) \frac{\partial}{\partial x_i}(u_2(t))\,\mathrm{d}x \\
&\quad -\frac{2}{n^2}\int_{\mathbb{R}^N} \sum_{i,j=1}^N \phi'(|x|^2/n^2) v(t) x_j (a_{ij}(\mu_1) - a_{ij}(\mu_2)) \frac{\partial}{\partial x_i}(u_2(t))\,\mathrm{d}x \\
&\leq \eta(\mu_1,\mu_2) \|v(t)\|_{H^1} \|u_2(t)\|_{H^1} + \frac{4L_\phi \eta(\mu_1,\mu_2) N^2}{n} \|v(t)\|_{L^2} \|u_2(t)\|_{H^1}.
\end{aligned}$$

To estimate $I_3(t)$ we see that (27) implies

$$\begin{aligned} I_3(t) &= \int_{\mathbb{R}^N} \phi_n(x) \left(\mathbf{H}(t,u_1(t),\mu_1) - \mathbf{H}(t,u_2(t),\mu_2)\right) v(t)\,\mathrm{d}x \\ &\le \int_{\mathbb{R}^N} \phi_n(x) \left(\mathbf{H}(t,u_1(t),\mu_1) - \mathbf{H}(t,u_2(t),\mu_1)\right) v(t)\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \phi_n(x) \left(\mathbf{H}(t,u_2(t),\mu_1) - \mathbf{H}(t,u_2(t),\mu_2)\right) v(t)\,\mathrm{d}x \\ &\le -a \int_{\mathbb{R}^N} \phi_n(x)|v(t)|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} \phi_n(x) b(x)|v(t)|^2\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} l(t,x)|\rho(\mu_1)-\rho(\mu_2)||u_2(t)||v(t)|\,\mathrm{d}x. \end{aligned}$$

By use of the Hölder inequality together with Lemma 4.2 one can get

$$\begin{aligned} I_3(t) \quad &\le -a \int_{\mathbb{R}^N} \phi_n(x)|v(t)|^2\,\mathrm{d}x + C_1\|v(t)\|_{H^1}^2 \left(\int_{\left\{|x|\ge \frac{\sqrt{2}}{2}n\right\}} b(x)^p\,\mathrm{d}x \right)^{1/p} \\ &\quad + \eta(\mu_1,\mu_2)(C_2\|l_0(t,\cdot)\|_{L^p}\|u_2(t)\|_{H^1} + \|l_\infty(t,\cdot)\|_{L^\infty}\|u_2(t)\|_{L^2})\|v(t)\|_{L^2}. \end{aligned} \tag{29}$$

Hence we get, for any $n \ge 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(v(t),\phi_n v(t))_0 \le -2a(v(t),\phi_n v(t))_0 + \tilde{C}\eta(\mu_1,\mu_2) + \alpha_n$$

for some constant $\tilde{C} = \tilde{C}(l,p,N,R) > 0$, where $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence such that $\alpha_n \to 0$ as $n \to +\infty$. Multiplying by e^{2at} and integrating over $[0,\tau]$ one obtains

$$e^{2a\tau}(v(\tau),\phi_n v(\tau))_0 - (v(0),\phi_n v(0))_0 \le (2a)^{-1}(e^{2a\tau}-1)\,(\tilde{C}\eta(\mu_1,\mu_2)+\alpha_n),$$

which gives

$$(v(\tau),\phi_n v(\tau))_0 \le e^{-2a\tau}\|v(0)\|_{L^2}^2 + (2a)^{-1}\left(\tilde{C}\eta(\mu_1,\mu_2)+\alpha_n\right).$$

And this finally implies the assertion as $\|\phi_n v(\tau)\|_{L^2}^2 \le (v(\tau),\phi_n v(\tau))_0$. □

Let $\mathbf{\Psi}_t : H^1(\mathbb{R}^N)\times[0,1] \to H^1(\mathbb{R}^N)$, $t>0$, be the translation operator for (28), i.e. $\mathbf{\Psi}_t(\bar{u},\mu) = u(t;\bar{u},\mu)$ for $\bar{u}\in H^1(\mathbb{R}^N)$ and $\mu\in[0,1]$.

Proposition 4.7. *Suppose that* (24), (25), (26) *and* (27) *are satisfied.*
(i) *For any bounded* $V \subset H^1(\mathbb{R}^N)$ *and* $t>0$, $\beta_{L^2}(\mathbf{\Psi}_t(V\times[0,1])) \le e^{-at}\beta_{L^2}(V)$*;*
(ii) *If a bounded* $V \subset H^1(\mathbb{R}^N)$ *is relatively compact as a subset of* $L^2(\mathbb{R}^N)$*, then* $\mathbf{\Psi}_t(V\times[0,1])$ *is relatively compact in* $H^1(\mathbb{R}^N)$*;*
(iii) *If* $V \subset \overline{\mathrm{conv}}^{H^1}\mathbf{\Psi}_t(V\times[0,1])$ *for some bounded* $V\subset H^1(\mathbb{R}^N)$ *and* $t>0$*, then* V *is relatively compact in* $H^1(\mathbb{R}^N)$*.*

Proof: (i) Observe that, for each $n\ge 1$,

$$\mathbf{\Psi}_t(V\times[0,1]) \subset \{u(t;\bar{u},\mu) \mid \bar{u}\in V,\ \mu\in[0,1]\} \subset W_n + R_n$$

where $W_n := \{\chi_n u(t;\bar{u},\mu) \mid \bar{u}\in V,\, \mu\in[0,1]\}$ and $R_n := \{(1-\chi_n)u(t;\bar{u},\mu) \mid \bar{u}\in V,\, \mu\in[0,1]\}$ where χ_n is the characteristic function of the ball $B(0,n)$. Note that W_n may be viewed as a subset of $H^1(B(0,n))$. Therefore, due to the Rellich-Kondrachov theorem, W_n is relatively compact in $L^2(\mathbb{R}^N)$. Hence

$$\beta_{L^2}(\mathbf{\Psi}_t(V\times[0,1])) \le \beta_{L^2}(R_n), \quad \text{for all} \quad n\ge 1. \tag{30}$$

Now we need to estimate the measure of noncompactness of R_n in $L^2(\mathbb{R}^N)$. To this end fix an arbitrary $\varepsilon>0$. Choose a finite covering of V consisting of balls $B_{L^2}(\bar{u}_k, r_\varepsilon)$, $k=1,\ldots,m_\varepsilon$, with $r_\varepsilon := \beta_{L^2}(V)+\varepsilon$ and such that $\bar{u}_k\in V$ for each $k=1,\ldots,m_\varepsilon$ and cover $[0,1]$ with intervals $(\mu_l-\delta,\mu_l+\delta)$, $l=1,\ldots,n_\delta$ where $\delta>0$ is such that $\eta(\mu_1,\mu_2)<\varepsilon$ whenever $|\mu_1-\mu_2|<\delta$. Put $\bar{u}_{k,l} := (1-\chi_n)u(t;\bar{u}_k,\mu_l)$, $k=1,\ldots,m_\varepsilon$, $l=1,\ldots,n_\delta$.

Now take any $\bar{v}\in R_n$. There are $\bar{u}\in V$ and $\mu\in[0,1]$ such that $\bar{v}=(1-\chi_n)u(t;\bar{u},\mu)$. Clearly there exist $k_0\in\{1,\ldots,m_\varepsilon\}$ and $l_0\in\{1,\ldots,n_\delta\}$ such that $\|\bar{u}-\bar{u}_{k_0}\|<r_\varepsilon$ and $|\mu-\mu_{l_0}|<\delta$. In view of Lemma 4.6

$$\begin{aligned}
\|\bar{v}-\bar{u}_{k_0,l_0}\|_{L^2}^2 &= \int_{\mathbb{R}^N\setminus B(0,n)} |u(t;\bar{u},\mu)-u(t;\bar{u}_{k_0},\mu_{l_0})|^2\,\mathrm{d}x \\
&\le e^{-2at}\|\bar{u}-\bar{u}_{k_0}\|_{L^2}^2 + Q\,\eta(\mu,\mu_{l_0}) + \alpha_n \\
&\le r_{\varepsilon,n} := e^{-2at}r_\varepsilon^2 + Q\,\varepsilon + \alpha_n,
\end{aligned}$$

which means that R_n is covered by the balls $B_{L^2}(\bar{u}_{k,l},\sqrt{r_{\varepsilon,n}})$, $k=1,\ldots,m_\varepsilon$, $l=1,\ldots,n_\delta$. This means that $\beta_{L^2}(R_n)\le\sqrt{r_{\varepsilon,n}}$ for any $\varepsilon>0$, and, in consequence, $\beta_{L^2}(R_n)\le(e^{-2at}(\beta_{L^2}(V))^2+\alpha_n)^{1/2}$. Using (30) we get

$$\beta_{L^2}(\mathbf{\Psi}_t(V\times[0,1])) \le (e^{-2at}(\beta_{L^2}(V))^2+\alpha_n)^{1/2}, \text{ for } n\ge 1.$$

Finally, by a passage to the limit with $n\to\infty$ we obtain the required inequality as $\alpha_n\to 0^+$.

(ii) Take any $(\bar{u}_n)$ in V and (μ_n) in $[0,1]$. We may assume that $\mu_n\to\mu_0$ for some $\mu_0\in[0,1]$, as $n\to+\infty$. Since $(\bar{u}_n)$ is bounded, by the Banach-Alaoglu theorem, we may suppose that $(\bar{u}_n)$ converges weakly in $H^1(\mathbb{R}^N)$ to some $\bar{u}\in H^1(\mathbb{R}^N)$. By the relative compactness of V in $L^2(\mathbb{R}^N)$ we may assume that $\bar{u}_n\to\bar{u}$ in $L^2(\mathbb{R}^N)$. Therefore, by use of Proposition 4.3, one has $\mathbf{\Psi}_t(\bar{u}_n,\mu_n)\to\mathbf{\Psi}_t(\bar{u},\mu_0)$ in $H^1(\mathbb{R}^N)$, which ends the proof.

(iii) Observe that here, by use of (i), one gets

$$\beta_{L^2}(V)\le\beta_{L^2}(\mathbf{\Psi}_t(V\times[0,1]))\le e^{-at}\beta_{L^2}(V).$$

This implies $\beta_{L^2}(V)=0$, i.e. that V is relatively compact in $L^2(\mathbb{R}^N)$. To see that V is relatively compact in $H^1(\mathbb{R}^N)$ observe that, by (ii), $\mathbf{\Psi}_t(V\times[0,1])$ is relatively compact in $H^1(\mathbb{R}^N)$. □

5 Averaging index formula

Consider the following parameterized equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + h(t/\lambda, x, u(x,t),\mu),\ t>0,\ x\in\mathbb{R}^N, \\ u(\cdot,t)\in H^1(\mathbb{R}^N),\ t>0, \end{cases} \tag{31}$$

where h is as in the previous section and $\lambda > 0$. Combining the compactness result with averaging principle we get the following result.

Lemma 5.1. *Suppose h satisfies conditions* (24), (25), (26) *and* (27) *and is T-periodic in the time variable ($T > 0$). If $(\bar{u}_n)$ is a bounded sequence in $H^1(\mathbb{R}^N)$, (μ_n) in $[0, 1]$, (λ_n) in $(0, +\infty)$ with $\lambda_n \to 0^+$ as $n \to +\infty$ and $u_n : [0, +\infty) \to H^1(\mathbb{R}^N)$ are solutions of* (31) *with $\lambda = \lambda_n$, $\mu = \mu_n$ such that $u_n(0) = u_n(\lambda_n T) = \bar{u}_n$, then there are a subsequence $(\bar{u}_{n_k})$ of $(\bar{u}_n)$ converging in $H^1(\mathbb{R}^N)$ to some $\bar{u}_0 \in H^2(\mathbb{R}^N)$ and a subsequence (μ_{n_k}) of (μ_n) converging to some $\mu_0 \in [0, 1]$, as $k \to +\infty$, such that $\bar{u}_0$ is a solution of*

$$\Delta u(x) + \widehat{h}(x, u(x), \mu_0) = 0, \ x \in \mathbb{R}^N,$$

where $\widehat{h} : \mathbb{R}^N \times \mathbb{R} \times [0, 1] \to \mathbb{R}$, $\widehat{h}(x, u, \mu) := \dfrac{1}{T}\displaystyle\int_0^T h(t, x, u, \mu)\, \mathrm{d}t$, $(x, u, \mu) \in \mathbb{R}^N \times \mathbb{R} \times [0, 1]$. *Moreover, $u_{n_k}(t) \to \bar{u}_0$ in $H^1(\mathbb{R}^N)$, as $k \to +\infty$, uniformly with respect to t from compact subsets of $(0, +\infty)$.*

Proof: Recall that u_n are solutions of $\dot{u} = -\mathbf{A}u + \mathbf{H}(t/\lambda_n, u, \mu_n)$ with $u_n(0) = u_n(\lambda_n T) = \bar{u}_n$, $n \geq 1$, where $\mathbf{A}$ and $\mathbf{H}$ are as in the previous section (with $a_{ij} = 0$ if $i \neq j$ and $a_{ij} = 1$ if $i = j$). Clearly, by the sublinear growth, there exists $R > 0$ such that $\|u_n(t)\|_{H^1} \leq R$ for all $t > 0$ and $n \geq 1$. For an arbitrary $M > 0$ and $n \geq 1$ take $k_n \in \mathbb{N}$ such that $k_n \lambda_n T > M$. In view of Lemma 4.5, for all $m \geq 1$ and $n \geq 1$,

$$\|(1 - \chi_m)\bar{u}_n\|_{L^2}^2 = \|(1 - \chi_m)u_n(k_n \lambda_n T)\|_{L^2}^2 \leq R^2 e^{-2ak_n\lambda_n T} + \alpha_m \leq R^2 e^{-2aM} + \alpha_m,$$

where χ_m is the characteristic function of $B(0, m)$. Since $M > 0$ is arbitrary we see that $\|(1 - \chi_m)\bar{u}_n\|_{L^2} \leq \sqrt{\alpha_m}$. Since, due to the Rellich-Kondrachov for any $m \geq 1$, the set $\{\chi_m \bar{u}_n\}_{n \geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$, we infer that $\{\bar{u}_n\}_{n\geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$. And since it is bounded in $H^1(\mathbb{R}^N)$ we get a subsequence $(\bar{u}_{n_k})$, denoted in the sequel again by $(\bar{u}_n)$, such that $\bar{u}_{n_k} \to \bar{u}_0$ in $L^2(\mathbb{R}^N)$ for some $\bar{u}_0 \in H^1(\mathbb{R}^N)$. We may also assume that $\mu_{n_k} \to \mu_0$ for some $\mu_0 \in [0, 1]$. Hence, in view of Theorem 3.5, $u_n(t) \to \widehat{u}(t)$ uniformly for t from compact subsets of $(0, +\infty)$ where $\widehat{u} : [0, +\infty) \to H^1(\mathbb{R}^N)$ is a solution to

$$\dot{u} = -\mathbf{A}u + \widehat{\mathbf{H}}(u, \mu_0), \ t > 0,$$

with $\widehat{\mathbf{H}}(u, \mu) := \frac{1}{T}\int_0^T \mathbf{H}(t, u, \mu)\, \mathrm{d}t$ for $u \in H^1(\mathbb{R}^N)$, $\mu \in [0, 1]$. Here note that, for each $u \in H^1(\mathbb{R}^N)$ and $\mu \in [0, 1]$,

$$[\widehat{\mathbf{H}}(u, \mu)](x) = \widehat{h}(x, u(x), \mu) \ \text{ for all a.a. } x \in \mathbb{R}^N.$$

Finally, for any $t > 0$, we put $k_n := [t/\lambda_n T]$, $n \geq 1$, and see that

$$\bar{u}_n = u_n(0) = u_n(k_n \lambda_n T) \to \widehat{u}(t) \ \text{ in } H^1(\mathbb{R}^N), \ \text{as } n \to +\infty.$$

Hence $\widehat{u}(t) = \widehat{u}(0) = \bar{u}_0$ and $\bar{u}_n \to \bar{u}_0$ in $H^1(\mathbb{R}^N)$. $\square$

Remark 5.2. Clearly that it follows from the proof of Lemma 5.1 that if $f_n : [0,+\infty)\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ are as in Proposition 4.3 and satisfy (5) with common a and b, then for any bounded sequence $(\bar{u}_n)$ in $H^1(\mathbb{R}^N)$, (λ_n) in $(0,+\infty)$ with $\lambda_n\to 0^+$ as $n\to+\infty$ and $u_n:[0,+\infty)\to H^1(\mathbb{R}^N)$ being $\lambda_n T$-periodic solutions of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f_n(t/\lambda_n, x, u),\ x\in\mathbb{R}^N,\ t>0,\\ u(x,0)=u(x,\lambda_n T)=\bar{u}_n(x),\ x\in\mathbb{R}^N, \end{cases}$$

there is a subsequence $(\bar{u}_{n_k})$ of $(\bar{u}_n)$ converging in $H^1(\mathbb{R}^N)$ to some $\bar{u}_0\in H^2(\mathbb{R}^N)$ being a solution of

$$\Delta u(x) + \widehat{f}_0(x,u(x)) = 0 \text{ on } \mathbb{R}^N.$$

Moreover, $u_{n_k}(t)\to\bar{u}_0$ in $H^1(\mathbb{R}^N)$, as $k\to+\infty$, uniformly with respect to t from compact subsets of $(0,+\infty)$. $\square$

Now consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + f\left(\dfrac{t}{\lambda}, x, u(x,t)\right),\ t>0,\ x\in\mathbb{R}^N,\\ u(\cdot,t)\in H^1(\mathbb{R}^N),\ t>0, \end{cases} \tag{32}$$

where f satisfies conditions (2), (3), (4) and (5). We intend to prove an averaging index formula that allows to express the fixed point index of translation along trajectories operator for (32) in terms of the averaged equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + \widehat{f}(x,u(x,t)),\ t>0,\ x\in\mathbb{R}^N,\\ u(\cdot,t)\in H^1(\mathbb{R}^N),\ t\geq 0, \end{cases} \tag{33}$$

where $\widehat{f}:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is defined by

$$\widehat{f}(x,u) := \frac{1}{T}\int_0^T f(t,x,u)\,\mathrm{d}t,\ x\in\mathbb{R}^N,\ u\in\mathbb{R}.$$

Theorem 5.3. *Let $U\subset H^1(\mathbb{R}^N)$ be an open bounded set and by $\mathbf{\Phi}_t^{(\lambda)}$ and $\widehat{\mathbf{\Phi}}_t$, $t>0$, denote the translation along trajectories operators (by time t) for the equations* (32) *and* (33)*, respectively. If the problem*

$$\begin{cases} -\Delta u(x) = \widehat{f}(x,u(x)),\ x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N), \end{cases} \tag{34}$$

has no solution in ∂U, then there exists $\lambda_0>0$ such that, for all $\lambda\in(0,\lambda_0]$, $\mathbf{\Phi}_{\lambda T}^{(\lambda)}(\bar{u})\neq\bar{u}$, $\widehat{\mathbf{\Phi}}_{\lambda T}(\bar{u})\neq\bar{u}$ for all $\bar{u}\in\partial U$, and

$$\mathrm{Ind}(\mathbf{\Phi}_{\lambda T}^{(\lambda)}, U) = \mathrm{Ind}(\widehat{\mathbf{\Phi}}_{\lambda T}, U).$$

Proof: Define $\mathbf{H}:[0,+\infty)\times H^1(\mathbb{R}^N)\times[0,1]\to L^2(\mathbb{R}^N)$ by

$$[\mathbf{H}(t,u,\mu)](x) := (1-\mu)f(t,x,u(x)) + \mu\widehat{f}(x,u(x)),\ \text{ for a.e. } x\in\mathbb{R}^N,$$

and all $t > 0$, $u \in H^1(\mathbb{R}^N)$. For a parameter $\lambda > 0$ consider

$$\dot{u}(t) = -\mathbf{A}u(t) + \mathbf{H}(t/\lambda, u(t), \mu),\ t \in [0,T], \tag{35}$$

and the parameterized translation operator $\mathbf{\Psi}_t^{(\lambda)} : H^1(\mathbb{R}^N)\times[0,1] \to H^1(\mathbb{R}^N)$ defined by

$$\mathbf{\Psi}_t^{(\lambda)}(\bar{u},\mu) := u(t)$$

where $u : [0,T] \to H^1(\mathbb{R}^N)$ is the solution of (35) with $u(0) = \bar{u}$. Observe that for $\mu = 0$, (35) becomes

$$\dot{u}(t) = -\mathbf{A}u(t) + \mathbf{F}(t/\lambda, u(t)),\ t \in [0,T],$$

and we have $\mathbf{\Phi}_t^{(\lambda)} = \mathbf{\Psi}_t^{(\lambda)}(\cdot, 0)$. In the same way for $\mu = 1$ the equation (35) becomes

$$\dot{u}(t) = -\mathbf{A}u(t) + \widehat{\mathbf{F}}(u(t)),\ t \in [0,T]$$

and one has $\widehat{\mathbf{\Phi}}_t = \mathbf{\Psi}_t^{(\lambda)}(\cdot, 1)$ (it does not depend on λ).

We claim that there exists $\lambda_0 > 0$ such that, for all $\lambda \in (0, \lambda_0]$,

$$\mathbf{\Psi}_{\lambda T}^{(\lambda)}(\bar{u},\mu) \neq \bar{u} \quad \text{for all} \quad \bar{u} \in \partial U,\ \mu \in [0,1]. \tag{36}$$

Suppose the claim does not hold. Then there exist $(\bar{u}_n)$ in ∂U, (μ_n) in $[0,1]$ and (λ_n) with $\lambda_n \to 0^+$ as $n \to \infty$ such that

$$\mathbf{\Psi}_{\lambda_n T}^{(\lambda_n)}(\bar{u}_n,\mu_n) = \bar{u}_n \quad \text{for all} \quad n \geq 1.$$

This means that for each $n \geq 1$ there is a $\lambda_n T$-periodic solution $u_n : [0,+\infty) \to H^1(\mathbb{R}^N)$ of (35) with $\lambda = \lambda_n$, $\mu = \mu_n$ and $u_n(0) = \bar{u}_n$. By Lemma 5.1 we may assume that $\bar{u}_n \to \bar{u}_0$ in $H^1(\mathbb{R}^N)$. Therefore $\bar{u}_0 \in \partial U \cap D(\mathbf{A})$ and $0 = -\mathbf{A}\bar{u}_0 + \widehat{\mathbf{F}}(\bar{u}_0)$, a contradiction with the assumption. This proves the existence of $\lambda_0 > 0$ such that, for all $\lambda \in (0, \lambda_0]$, (36) holds.

Now, due to Proposition 4.7 (iii), for each $\lambda \in (0, \lambda_0]$, $\mathbf{\Psi}_{\lambda T}^{(\lambda)}$ is an admissible homotopy in the sense of fixed point index theory for ultimately compact maps. Finally, by Proposition 2.1(iii), we get the desired equality of the indices. $\square$

As a consequence we get the following *continuation principle*.

Corollary 5.4. *Suppose that an open bounded $U \subset H^1(\mathbb{R}^N)$ is such that* (34) *has no solution in ∂U, and for any $\lambda \in (0,1)$ the problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \lambda\Delta u + \lambda f(t,x,u), x \in \mathbb{R}^N,\ t > 0, \\ u(\cdot,t) \in H^1(\mathbb{R}^N),\ t \geq 0 \\ u(x,0) = u(x,T),\ x \in \mathbb{R}^N, \end{cases} \tag{37}$$

has no solution $u : [0,+\infty) \to H^1(\mathbb{R}^N)$ with $u(\cdot,0) \in \partial U$. Then

$$\mathrm{Ind}(\mathbf{\Phi}_T, U) = \lim_{t\to 0^+} \mathrm{Ind}(\widehat{\mathbf{\Phi}}_t, U)$$

where $\mathbf{\Phi}_T$ is the translation along trajectories operator for (1).

Proof: Let $\lambda_0 > 0$ be as in Theorem 5.3. Since there are no solutions to (37), we infer that

$$\mathbf{\Phi}_{\lambda T}^{(\lambda)}(\bar{u}) \neq \bar{u} \text{ for any } \bar{u} \in \partial U,\ \lambda \in (0,1).$$

Now by Proposition 4.7 (iii) and the homotopy invariance of the index, for any $\lambda \in (0,1]$, we get $\mathrm{Ind}(\mathbf{\Phi}_T, U) = \mathrm{Ind}(\widetilde{\mathbf{\Phi}}_T^{(1)}, U) = \mathrm{Ind}(\widetilde{\mathbf{\Phi}}_T^{(\lambda)}, U) = \mathrm{Ind}(\mathbf{\Phi}_{\lambda T}^{(\lambda)}, U)$, where $\widetilde{\mathbf{\Phi}}_T^{(\lambda)}$ is the translation along trajectories operator for the parabolic equation in (37) with the parameter λ and the last equality comes from a time rescaling argument saying that $\widetilde{\mathbf{\Phi}}_T^{(\lambda)} = \mathbf{\Phi}_{\lambda T}^{(\lambda)}$. Now an application of Theorem 5.3 completes the proof. $\square$

The rest of the section is devoted to methods of verification the *a priori* bounds conditions occurring in the above corollary and computation of fixed point index. We shall use a linearization approach.

Proposition 5.5. *Suppose that f satisfies conditions* (2), (3), (4), (5) *and* $f(t,x,0) = 0$ *for all* $x \in \mathbb{R}^N$ *and* $t \geq 0$.
(i) *If* (7) *holds,* $\mathrm{Ker}\,(\Delta + \widehat{\omega}) = \{0\}$ *and the linear equation*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \lambda \Delta u(x,t) + \lambda \omega(t,x) u(x,t),\ x \in \mathbb{R}^N,\ t > 0, \\ u(\cdot,t) \in H^1(\mathbb{R}^N),\ t \geq 0, \end{cases} \tag{38}$$

has no nonzero T-periodic solutions for $\lambda \in (0,1]$, then there exists $R > 0$ such that, for any $\lambda \in (0,1]$ the problem (37) *has no T-periodic solutions $u : [0,+\infty) \to H^1(\mathbb{R}^N)$ with $\|u(0)\|_{H^1} \geq R$.*

(ii) *If* (6) *holds,* $\mathrm{Ker}\,(\Delta + \widehat{\alpha}) = \{0\}$ *and the linear equation*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \lambda \Delta u(x,t) + \lambda \alpha(t,x) u(x,t),\ x \in \mathbb{R}^N,\ t > 0, \\ u(\cdot,t) \in H^1(\mathbb{R}^N),\ t \geq 0, \end{cases} \tag{39}$$

has no nonzero T-periodic solutions, then there exists $r > 0$ such that, for any $\lambda \in (0,1]$ the problem (37) *has no T-periodic solutions $u : [0,+\infty) \to H^1(\mathbb{R}^N)$ with $0 < \|u(0)\|_{H^1} \leq r$.*

Proof: (i) Suppose to the contrary, i.e. that for any $n \geq 1$ there exist $\lambda_n \in (0,1)$ and a time T-periodic solution $u_n : [0,+\infty) \to H^1(\mathbb{R}^N)$ of

$$\frac{\partial u}{\partial t} = \lambda_n \Delta u + \lambda_n f(t,x,u),\ x \in \mathbb{R}^N,\ t > 0$$

with $\|u_n(0)\|_{H^1} \to +\infty$. This means that z_n given by $z_n(t) := \frac{u_n}{\rho_n}$, $\rho_n := 1 + \|u_n(0)\|_{H^1}$, is a T-periodic solution of

$$\frac{\partial z}{\partial t} = \lambda_n \Delta z + \lambda_n \rho_n^{-1} f(t,x,\rho_n z),\ x \in \mathbb{R}^N,\ t > 0. \tag{40}$$

It is also clear that $v_n : [0,+\infty) \to H^1(\mathbb{R}^N)$ given by $v_n(t) := z_n(t/\lambda_n)$ satisfies

$$\frac{\partial v}{\partial t} = \Delta v + \rho_n^{-1} f(t/\lambda_n, x, \rho_n v),\ x \in \mathbb{R}^N,\ t > 0, \tag{41}$$

and that $\rho_n \to +\infty$. Define $g_n : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $g_n(t, x, v) := \rho_n^{-1} f(t/\lambda_n, x, \rho_n v)$, $n \geq 1$, $t \geq 0$, $x \in \mathbb{R}^N$, $v \in \mathbb{R}$. Since the functions g_n, $n \geq 1$, satisfy (3) with a common m_0 and (4) with common l and $\{v_n(0)\}_{n\geq 1}$ is bounded, by use of Lemma 4.1 and Remark 3.1, we obtain a constant $R_0 > 0$ such that $\|v_n(t)\|_{H^1} \leq R_0$ for all $n \geq 1$ and $t \geq 0$. For a moment fix an arbitrary $M > 0$ and for any $n \geq 1$ take an integer $k_n \geq 1$ such that $k_n \lambda_n T > M$. Observe that Lemma 4.5 gives, for all $m \geq 1$ and $n \geq 1$,

$$\|(1 - \chi_m)v_n(0)\|_{L^2}^2 = \|(1 - \chi_m)v_n(k_n\lambda_n T)\|_{L^2}^2 \leq R_0^2 e^{-2ak_n\lambda_n T} + \alpha_m \leq R_0^2 e^{-2aM} + \alpha_m$$

with $\alpha_m \to 0^+$ as $m \to +\infty$. Since $M > 0$ is arbitrary we see that $\|(1 - \chi_m)v_n(0)\|_{L^2} \leq \sqrt{\alpha_m}$ for $m, n \geq 1$. Due to the Rellich-Kondrachov for any $m \geq 1$, the set $\{\chi_m v_n(0)\}_{n\geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$. Therefore $\{v_n(0)\}_{n\geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$, since $\alpha_m \to 0^+$ as $m \to +\infty$. As a bounded sequence in $H^1(\mathbb{R}^N)$, $(v_n(0))$ contains a subsequence convergent in $L^2(\mathbb{R}^N)$ to some $\bar{v}_0 \in H^1(\mathbb{R}^N)$. Therefore we may assume that $v_n(0) \to \bar{v}_0$ in $L^2(\mathbb{R}^N)$. Moreover, we may suppose that $\lambda_n \to \lambda_0$, as $n \to +\infty$ for some $\lambda_0 \in [0, 1]$.

First consider the case when $\lambda_0 \in (0, 1]$. Let $f_n : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $n \geq 1$, be given by

$$f_n(t, x, z) := \rho_n^{-1} f(t, x, \rho_n z), \quad \text{for all } t \geq 0,\ z \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}.$$

Note that (7) and (3) yield

$$\lim_{n\to+\infty} f_n(t, x, z) = \omega(t, x)z, \text{ for all } t \geq 0,\ z \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R},$$

and $\|f_n(t, \cdot, 0)\|_{L^2} = \rho_n^{-1}\|f(t, \cdot, 0)\|_{L^2} \leq \rho_n^{-1}\|m_0\|_{L^2} \to 0$, as $n \to +\infty$. It allows us to apply Proposition 4.3 to (41). As a result we infer that $z_n(t) \to z_0(t)$ in $H^1(\mathbb{R}^N)$ uniformly with respect to t from compact subsets of $(0, +\infty)$, where $z_0 : [0, +\infty) \to H^1(\mathbb{R}^N)$ is a T-periodic solution of

$$\frac{\partial z}{\partial t} = \lambda_0 \Delta z + \lambda_0 \omega(t, x)z.$$

Since $\|z(0)\|_{H^1} = \|v(0)\|_{H^1} \neq 0$, we get a nontrivial T-periodic solution of (38) with $\lambda = \lambda_0$, a contradiction proving the desired assertion.

In the situation when $\lambda_0 = 0$, we apply Proposition 4.4 to (41) to see that $v_n(t) \to \widehat{v}(t)$ uniformly with respect to t from compact subsets of $[0, +\infty)$, where $\widehat{v} : [0, +\infty) \to H^1(\mathbb{R}^N)$ is a nontrivial solution of

$$\frac{\partial v}{\partial t} = \Delta v + \widehat{\omega}(x)v,\ x \in \mathbb{R}^N,\ t > 0.$$

Now observe that, for any $t > 0$ and $k_n := [t/\lambda_n T]$, $n \geq 1$, one has

$$v_n(0) = v_n(k_n\lambda_n T) \to \widehat{v}(t) \text{ in } H^1(\mathbb{R}^N), \text{ as } n \to +\infty.$$

Hence $\widehat{v} \equiv \bar{v}_0$ and, as a consequence,

$$0 = \Delta\bar{v}_0(x) + \widehat{\omega}(x)\bar{v}_0(x),\ x \in \mathbb{R}^N,$$

which contradicts the assumption and completes the proof of (i).

To see (ii), suppose that assertion does not hold. Then there exist $\lambda_n \in (0,1)$ and a T-periodic solutions $u_n : [0,+\infty) \to H^1(\mathbb{R}^N)$ of

$$\frac{\partial u}{\partial t} = \lambda_n \Delta u + \lambda_n f(t,x,u),\ x \in \mathbb{R}^N,\ t > 0$$

with $\|u_n(0)\|_{H^1} > 0$, $n \geq 1$, and $\|u_n(0)\|_{H^1} \to 0^+$ as $n \to +\infty$. Put $z_n := \frac{u_n}{\rho_n}$ and let $v_n(t) := z_n(t/\lambda_n)$ with $\rho_n := \|u_n(0)\|_{H^1}$. Then, for each $n \geq 1$, z_n is a solution of

$$\frac{\partial z}{\partial t} = \lambda_n \Delta z + \lambda_n \rho_n^{-1} f(t,x,\rho_n z),\ x \in \mathbb{R}^N,\ t > 0.$$

and v_n is a solution of

$$\frac{\partial v}{\partial t} = \Delta v + \rho_n^{-1} f(t/\lambda_n, x, \rho_n v),\ x \in \mathbb{R}^N,\ t > 0.$$

The rest of the proof goes along the lines of the proof for (i). □

Remark 5.6.
Let us remark that the nonexistence of solutions for (38) or (39) may be also verified if α or ω are time dependent. Assume that

$$\sup_{t\in[0,T]} \|\omega_0(\cdot,t)\|_{L^p} < \begin{cases} \frac{p^{1/2p}\bar{\omega}_\infty^{1-1/2p}}{2^{1/2p}}, & \text{if } N=1,\ p>2, \\ \frac{p^{1/p}\bar{\omega}_\infty^{(1-1/p)}}{4^{1/p}}, & \text{if } N=2,\ p>2, \\ \frac{\bar{\omega}_\infty^{1-N/2p}}{(N/2p)^{N/2p}C(N)^{N/p}}, & \text{if } N\geq 3, N \leq p < \infty, \end{cases} \tag{42}$$

where $C(N) > 0$ is the constant in the Sobolev inequality $\|u\|_{L^{\frac{2N}{N-2}}} \leq C(N)\|\nabla u\|_{L^2}$, $u \in H^1(\mathbb{R}^N)$. Suppose that u is a nonzero T-periodic solution of (38). Then , for all $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2\lambda}\|u(t)\|_{L^2}^2 = -\int_{\mathbb{R}^N} |\nabla u(t)|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} \omega_\infty(t,x)|u(t)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} \omega_0(t,x)|u(t)|^2 \,\mathrm{d}x. \tag{43}$$

Assume first that $N = 1$. Then, by use of the Hölder inequality,

$$\begin{aligned} \int_0^T \left(\|\nabla u(t)\|_{L^2}^2 + \bar{\omega}_\infty \|u(t)\|_{L^2}^2\right) \mathrm{d}t &\leq \int_0^T \|\omega_0(t,\cdot)\|_{L^p} \|u(t)\|_{L^2}^{2-2/p} \|u(t)\|_{L^\infty}^{2/p} \,\mathrm{d}t, \\ &\leq 2^{1/p} \int_0^T \|\omega_0(t,\cdot)\|_{L^p} \|\nabla u(t)\|_{L^2}^{1/p} \|u(t)\|_{L^2}^{2-1/p} \,\mathrm{d}t, \end{aligned}$$

where the latter inequality follows by the fact that $\|u\|_{L^\infty}^2 \leq 2\|\nabla u\|_{L^2}\|u\|_{L^2}$ for $u \in H^1(\mathbb{R})$. In the Young inequality $ab \leq \frac{a^r}{\epsilon^r r} + \frac{b^s \epsilon^s}{s}$ where $a, b \geq 0$, $\epsilon > 0$ and $r \in (1,+\infty)$ such that $\frac{1}{r} + \frac{1}{s} = 1$, put $a := \|\omega_0(\cdot,t)\|_{L^p}\|\nabla u(t)\|_{L^2}^{1/p}$, $b := \|u(t)\|_{L^2}^{2-1/p}$ and $r := 2p$ to obtain

$$\|\omega_0(t,\cdot)\|_{L^p}\|\nabla u(t)\|_{L^2}^{1/p}\|u(t)\|_{L^2}^{2-1/p} \leq \frac{\|\omega_0(t,\cdot)\|_{L^p}^{2p}\|\nabla u(t)\|_{L^2}^2}{2p\epsilon^{2p}} + \frac{\epsilon^{2p/(2p-1)}\|u(t)\|_{L^2}^2}{2p/(2p-1)}$$

for any $\epsilon > 0$ and fixed $t \in [0,T]$. If we take $\epsilon = \epsilon(t)$ so that $2^{1/p}\frac{\|\omega_0(t,\cdot)\|_{L^p}^{2p}}{2p\epsilon^{2p}} = 1$, i.e. $\epsilon(t) := \left(\frac{2^{(1-p)/p}}{p}\right)^{1/2p} \|\omega_0(t,\cdot)\|_{L^p}$ and apply (42), then

$$\begin{aligned}
\bar{\omega}_\infty \int_0^T \|u(t)\|_{L^2}^2 \,\mathrm{d}t &\le 2^{1/p}\frac{2p-1}{2p}\int_0^T \epsilon(t)^{2p/(2p-1)}\|u(t)\|_{L^2}^2 \,\mathrm{d}t \\
&\le 2^{1/p}\left(\frac{2^{(1-p)/p}}{p}\right)^{1/(2p-1)} \sup_{t\in[0,T]} \|\omega_0(t,\cdot)\|_{L^p}^{2p/(2p-1)} \int_0^T \|u(t)\|_{L^2}^2 \,\mathrm{d}t \\
&= \left(\frac{2}{p}\right)^{1/(2p-1)} \sup_{t\in[0,T]} \|\omega_0(t,\cdot)\|_{L^p}^{2p/(2p-1)} \int_0^T \|u(t)\|_{L^2}^2 \,\mathrm{d}t \\
&< \bar{\omega}_\infty \int_0^T \|u(t)\|_{L^2}^2 \,\mathrm{d}t,
\end{aligned}$$

a contradiction proving that (38) has no nontrivial T-periodic solutions. Assume now that $N = 2$. Then by (43) and the Hölder inequality it follows that

$$\int_0^T \left(\|\nabla u(t)\|_{L^2}^2 + \bar{\omega}_\infty\|u(t)\|_{L^2}^2\right) \,\mathrm{d}t \le \int_0^T \|\omega_0(t,\cdot)\|_{L^p}\|u(t)\|_{L^4}^{4/p}\|u(t)\|_{L^2}^{2-4/p} \,\mathrm{d}t,$$

which in view of the Sobolev inequality $\|u\|_{L^4}^2 \le 2\|u\|_{L^2}\|\nabla u\|_{L^2}$, $u \in H^1(\mathbb{R}^2)$ implies

$$\int_0^T \left(\|\nabla u(t)\|_{L^2}^2 + \bar{\omega}_\infty\|u(t)\|_{L^2}^2\right) \mathrm{d}t \le 2^{2/p}\int_0^T \|\omega_0(t,\cdot)\|_{L^p}\|\nabla u(t)\|_{L^2}^{2/p}\|u(t)\|_{L^2}^{2-2/p} \,\mathrm{d}t. \tag{44}$$

By use of the Young inequality, we obtain for any $\epsilon > 0$ and fixed $t \in [0,T]$,

$$\|\omega_0(t,\cdot)\|_{L^p}\|\nabla u(t)\|_{L^2}^{2/p}\|u(t)\|_{L^2}^{2-2/p} \le \frac{\|\omega_0(t,\cdot)\|_{L^p}^p\|\nabla u(t)\|_{L^2}^2}{p\epsilon^p} + \frac{\epsilon^{p/(p-1)}\|u(t)\|_{L^2}^2}{p/(p-1)}. \tag{45}$$

Choose $\epsilon = \epsilon(t) > 0$ such that $\frac{2^{2/p}}{p\epsilon^p}\|\omega_0(\cdot,t)\|_{L^p}^p = 1$, i.e. $\epsilon(t) := \frac{2^{2/p^2}}{p^{1/p}}\|\omega_0(\cdot,t)\|_{L^p}$. Then, by applying (42), we have

$$\begin{aligned}
\bar{\omega}_\infty \int_0^T \|u(t)\|_{L^2}^2 &\le 2^{2/p}\frac{(p-1)}{p}\int_0^T \epsilon(t)^{p/(p-1)}\|u(t)\|_{L^2}^2 \,\mathrm{d}t \\
&\le \left(\frac{4}{p}\right)^{1/(p-1)} \sup_{t\in[0,T]} \|\omega_0(t,\cdot)\|_{L^p}^{p/(p-1)} \int_0^T \|u(t)\|_{L^2}^2 \,\mathrm{d}t \\
&< \bar{\omega}_\infty \int_0^T \|u(t)\|_{L^2}^2 \,\mathrm{d}t,
\end{aligned}$$

which contradicts the existence of nonzero T-periodic solution for (38). Finally, for $N \ge 3$, by use of the Hölder and the Sobolev inequalities we get

$$\begin{aligned}
\int_0^T \left(\|\nabla u(t)\|_{L^2}^2 + \bar{\omega}_\infty\|u(t)\|_{L^2}^2\right) \,\mathrm{d}t &\le \int_0^T \|\omega_0(t,\cdot)\|_{L^p}\|u(t)\|_{L^{2N/(N-2)}}^{N/p}\|u(t)\|_{L^2}^{2-N/p} \,\mathrm{d}t \\
&\le C(N)^{N/p}\int_0^T \|\omega_0(\cdot,t)\|_{L^p}\|\nabla u(t)\|_{L^2}^{N/p}\|u(t)\|_{L^2}^{2-N/p} \,\mathrm{d}t.
\end{aligned}$$

In view of the Young inequality, for any $\epsilon > 0$ and fixed $t \in [0, T]$,

$$\|\omega_0(\cdot,t)\|_{L^p}\|\nabla u(t)\|_{L^2}^{\frac{N}{p}}\|u(t)\|_{L^2}^{2-\frac{N}{p}} \leq \frac{N/2p}{\epsilon^{\frac{2p}{N}}}\|\omega_0(\cdot,t)\|_{L^p}^{\frac{2p}{N}}\|\nabla u(t)\|_{L^2}^2 + \left(1-\frac{N}{2p}\right)\epsilon^{\frac{2p}{2p-N}}\|u(t)\|_{L^2}^2$$

Take $\epsilon = \epsilon(t)$ so that $\frac{N}{2p}\cdot\frac{C(N)^{N/p}}{\epsilon(t)^{\frac{2p}{N}}}\|\omega_0(\cdot,t)\|_{L^p}^{\frac{2p}{N}} = 1$, i.e. $\epsilon(t) = (N/2p)^{N/2p}C(N)^{N^2/2p^2}\|\omega_0(\cdot,t)\|_{L^p}$ and apply (42), then

$$\begin{aligned}
\bar{\omega}_\infty \int_0^T \|u(t)\|_{L^2}^2 \, \mathrm{d}t &\leq C(N)^{N/p}\left(1-\frac{N}{2p}\right)\int_0^T \epsilon(t)^{\frac{2p}{2p-N}}\|u(t)\|_{L^2}^2 \, \mathrm{d}t \\
&\leq (N/2p)^{N/(2p-N)}C(N)^{2N/(2p-N)}\int_0^T \|\omega_0(\cdot,t)\|_{L^p}^{\frac{2p}{2p-N}}\|u(t)\|_{L^2}^2 \, \mathrm{d}t \\
&\leq (N/2p)^{N/(2p-N)}C(N)^{2N/(2p-N)}\sup_{t\in[0,T]}\|\omega_0(\cdot,t)\|_{L^p}^{\frac{2p}{2p-N}}\int_0^T \|u(t)\|_{L^2}^2 \, \mathrm{d}t \\
&< \bar{\omega}_\infty \int_0^T \|u(t)\|_{L^2}^2 \, \mathrm{d}t,
\end{aligned}$$

a contradiction proving that (38) has no nontrivial T-periodic solutions.

6 Proofs of Theorems 1.1 and 1.2

We start with a linearization method for computing the fixed point index of the translation operator in the autonomous case.

Proposition 6.1. *Assume that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies conditions* (3), (4) *and* (5) (*in their time-independent versions*) *and let $\mathbf{\Phi}_t$ be the translation along trajectories for the autonomous equation*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + f(x,u(x,t)), \ x \in \mathbb{R}^N, \ t > 0, \\ u(\cdot,t) \in H^1(\mathbb{R}^N), \ t \geq 0. \end{cases}$$

(i) *If* (7) *holds and* $\mathrm{Ker}(\Delta+\omega) = \{0\}$, *then there exists $R_0 > 0$ such that $-\Delta u(x) = f(x,u(x))$, $x \in \mathbb{R}^N$, has no solutions $u \in H^1(\mathbb{R}^N)$ with $\|u\|_{H^1} \geq R_0$ and there exists $\bar{t} > 0$ such that, for all $t \in (0,\bar{t}]$, $\mathbf{\Phi}_t(\bar{u}) \neq \bar{u}$ for all $\bar{u} \in H^1(\mathbb{R}^N) \setminus B_{H^1}(0,R_0)$ and, for all and all $t \in (0,\bar{t}]$ and $R \geq R_0$,*

$$\mathrm{Ind}(\mathbf{\Phi}_t, B_{H^1}(0,R)) = (-1)^{m(\infty)}$$

where $m(\infty)$ is the total multiplicity of the positive eigenvalues of $\Delta+\omega$.
(ii) *If* (6) *holds and* $\mathrm{Ker}(\Delta+\alpha) = \{0\}$, *then there exists $r_0 > 0$ such that $-\Delta u(x) = f(x,u(x))$, $x \in \mathbb{R}^N$, has no solutions with $0 < \|u\|_{H^1} \leq r_0$ and there exists $\bar{t} > 0$ such that, for all $t \in (0,\bar{t}]$, $\mathbf{\Phi}_t(\bar{u}) \neq \bar{u}$ for all $\bar{u} \in B_{H^1}(0,r_0) \setminus \{0\}$ and, for each $t \in (0,\bar{t}]$,*

$$\mathrm{Ind}(\mathbf{\Phi}_t, B_{H^1}(0,r_0)) = (-1)^{m(0)}$$

where $m(0)$ is the total multiplicity of the positive eigenvalues of $\Delta+\alpha$.

Remark 6.2. Recall the known arguments on the spectrum of $-\Delta - \omega_0 + \omega_\infty$. To this end, define $\mathbf{B}_0 : D(\mathbf{B}_0) \to L^2(\mathbb{R}^N)$ with $D(\mathbf{B}_0) := H^1(\mathbb{R}^N)$ by $\mathbf{B}_0 u := \omega_0 u$ and $\mathbf{B}_\infty : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by $\mathbf{B}_\infty u := \omega_\infty u$. By [19], $\mathbf{A}_\infty := \mathbf{A} - \mathbf{B}_0 + \mathbf{B}_\infty$ is a C_0 semigroup generator and its spectrum $\sigma(\mathbf{A}_\infty)$ is contained in an interval $(-c, +\infty)$ with some $c > 0$. It is clear that $\sigma(\mathbf{A} + \mathbf{B}_\infty) \subset [\bar{\omega}_\infty, +\infty)$. Moreover, it is known, that $\mathbf{B}_0(\mathbf{A}+\mathbf{B}_\infty)^{-1} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a compact linear operator – for the proof we refer to [20, Lem. 3.1], where the result is obtained under assumption $N \geq 3$. However a proper restatement, i.e. exploiting Sobolev embeddings $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ for $N = 1$ and $H^1(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$ - in case $N = 2$ together with the Rellich-Kondrachov Theorem, leads to the same conclusion. Therefore, by use of the Weyl theorem on essential spectra, we obtain $\sigma_{ess}(\mathbf{A}_\infty) = \sigma_{ess}(\mathbf{A} + \mathbf{B}_\infty) \subset \sigma(\mathbf{A}+\mathbf{B}_\infty) \subset [\bar{\omega}_\infty, +\infty)$ (see e.g. [22]). Hence, by general characterizations of essential spectrum, we see that $\sigma(\mathbf{A}_\infty) \cap (-\infty, 0)$ consists of isolated eigenvalues with finite dimensional eigenspaces (see [22]).

Proof of Proposition 6.1: (i) We start with an observation that there exists $R_0 > 0$ such that the problem

$$0 = \Delta u + (1-\mu) f(x,u) + \mu\omega(x)u,\ x \in \mathbb{R}^N, \tag{46}$$

has no weak solutions in $H^1(\mathbb{R}^N) \setminus B_{H^1}(0, R_0)$. To see this, suppose to the contrary that there exist a sequence (μ_n) in $[0,1]$ and solutions $\bar{u}_n$, $n \geq 1$, of (46) with $\mu = \mu_n$ such that $\|\bar{u}_n\|_{H^1} \to +\infty$ as $n \to +\infty$. Put $\rho_n := 1 + \|\bar{u}_n\|_{H^1}$ and observe that $\bar{v}_n := \frac{\bar{u}_n}{\rho_n}$ are solutions of

$$0 = \Delta v + (1-\mu_n)\rho_n^{-1} f(x, \rho_n v) + \mu_n \omega(x) v,\ x \in \mathbb{R}^N.$$

Clearly

$$\rho_n^{-1} f(x, \rho_n v) \to \omega(x) v \text{ as } n \to +\infty \text{ for all } t \geq 0 \text{ and a.a. } x \in \mathbb{R}^N.$$

Hence, by use of Remark 5.2 we see that $(\bar{u}_n)$ contains a sequence convergent to some $\bar{u}_0 \in H^1(\mathbb{R}^N)$ being a weak nonzero solution of $0 = \Delta u + \omega(x) u$, $x \in \mathbb{R}^N$, a contradiction proving that (46) has no solutions outside some ball $B_{H^1}(0, R_0)$.

Now consider the equation

$$\frac{\partial u}{\partial t} = \Delta u + (1-\mu) f(x,u) + \mu\omega(x)u,\ x \in \mathbb{R}^N,\ t > 0, \tag{47}$$

where $\mu \in [0,1]$ is a parameter. Let $\mathbf{\Psi}_t : H^1(\mathbb{R}^N) \times [0,1] \to H^1(\mathbb{R}^N)$, $t > 0$, be the parameterized translation along trajectories operator for the above equation. In view of Theorem 5.3, there exists $\bar{t} > 0$ such that

$$\mathbf{\Psi}_t(\bar{u}, \mu) \neq \bar{u} \quad \text{for all} \quad t \in (0, \bar{t}], \bar{u} \in \partial B_{H^1}(0, R_0).$$

By Proposition 4.7 (iii), the homotopy $\mathbf{\Psi}_t$ is admissible in the sense of the fixed point theory for ultimately compact maps (see Section 2). Therefore using the homotopy invariance one has, for $t \in (0, \bar{t}]$,

$$\mathrm{Ind}(\mathbf{\Phi}_t, B_{H^1}(0, R_0)) = \mathrm{Ind}(e^{-t\mathbf{A}_\infty}, B_{H^1}(0, R_0)) \tag{48}$$

where $\mathbf{A}_\infty := \mathbf{A} - \mathbf{B}_0 + \mathbf{B}_\infty$.

It is left to determine the fixed point index of $e^{-t\mathbf{A}_\infty}$. We note that the set $\sigma(\mathbf{A}_\infty)\cap(-\infty,0)$ is bounded and closed. Hence, in view of the spectral theorem (see [26]) there are closed subspaces X_- and X^+ of $L^2(\mathbb{R}^N)$ such that $X_-\oplus X^+ = L^2(\mathbb{R}^N)$, $\dim X_- < +\infty$, $\mathbf{A}_\infty(X_-) \subset X_-$, $\mathbf{A}_\infty(D(\mathbf{A}_\infty)\cap X^+) \subset X^+$, $\sigma(\mathbf{A}_\infty|_{X_-}) = \sigma(\mathbf{A}_\infty)\cap(-\infty,0)$, $\sigma(\mathbf{A}_\infty|_{X^+}) = \sigma(\mathbf{A}_\infty)\cap(0,+\infty)$. Define $\mathbf{\Theta}_t : H^1(\mathbb{R}^N)\times[0,1] \to H^1(\mathbb{R}^N)$ by

$$\mathbf{\Theta}_t(\bar{u},\mu) := (1-\mu)e^{-t\mathbf{A}_\infty}\bar{u} + \mu e^{-t\mathbf{A}_\infty}\mathbf{P}_-\bar{u},$$

where $\mathbf{P}_- : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ is the restriction of the projection onto $X_-\cap H^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$. Since $\dim X_- < +\infty$ we infer that $\mathbf{P}_-$ is continuous. W also claim that $\mathbf{\Theta}_t$ is ultimately compact. To see this take a bounded set $V \subset H^1(\mathbb{R}^N)$ such that $V = \overline{\mathrm{conv}}^{H^1}\mathbf{\Theta}_t(V\times[0,1])$. This means that $V \subset \overline{\mathrm{conv}}^{H^1}e^{-t\mathbf{A}_\infty}(V\cup\mathbf{P}_-V)$. Since $V\cup\mathbf{P}_-V$ is bounded, Proposition 4.7 (ii) implies that V is relatively compact in $H^1(\mathbb{R}^N)$, which proves the ultimate compactness of $\mathbf{\Theta}_t$. Since $\mathrm{Ker}(I-\mathbf{\Theta}_t(\cdot,\mu)) = \{0\}$ for $\mu\in[0,1]$, by the homotopy invariance and the restriction property of the Leray-Schauder fixed point index, one gets

$$\begin{aligned}\mathrm{Ind}(e^{-t\mathbf{A}_\infty}, B_{H^1}(0,R_0)) &= \mathrm{Ind}_{LS}(e^{-t\mathbf{A}_\infty}\mathbf{P}_-, B_{H^1}(0,R_0))\\ &= \mathrm{Ind}_{LS}(e^{-t(\mathbf{A}_\infty|_{X_-})}, B_{H^1}(0,R_0)\cap X_-) = (-1)^{m(\infty)}.\end{aligned}$$

The latter equality comes from the fact that $\sigma(\mathbf{A}_\infty|_{X_-}) \subset (-\infty,0)$ consists of isolated eigenvalues of finite dimensional eigenspaces. This ends the proof of (i) together with (48).

(ii) First we shall prove the existence of $r_0 > 0$ such that the problem

$$0 = \Delta u + (1-\mu)f(x,u) + \mu\alpha(x)u,\ x\in\mathbb{R}^N, \tag{49}$$

has no solutions in $B_{H^1}(0,r_0)\setminus\{0\}$. Suppose to the contrary that there exist a sequence (μ_n) in $[0,1]$ and solutions $\bar{u}_n : [0,+\infty) \to H^1(\mathbb{R}^N)$, $n\geq 1$, of (49) with $\mu = \mu_n$ such that $\|\bar{u}_n\|_{H^1} \to 0^+$ as $n\to+\infty$ and $\|\bar{u}_n\|_{H^1} \neq 0$, $n\geq 1$. Put $\rho_n := \|\bar{u}_n\|_{H^1}$. Then $\bar{v}_n := \frac{\bar{u}_n}{\rho_n}$ are solutions of

$$0 = \Delta v + (1-\mu_n)\rho_n^{-1}f(x,\rho_n v) + \mu_n\alpha(x)v,\ x\in\mathbb{R}^N.$$

Observe that

$$\rho_n^{-1}f(x,\rho_n v) \to \alpha(x)v \text{ as } n\to\infty \text{ for a.a. } x\in\mathbb{R}^N.$$

Using again Remark 5.2 one can see that $(\bar{u}_n)$ (up to a subsequence) converges to some nonzero solution of $0 = \Delta u + \alpha(x)u$, $x\in\mathbb{R}^N$, a contradiction. Summing up, there is $r_0 > 0$ such that (49) has no solutions $u\in H^1(\mathbb{R}^N)$ with $0 < \|u\|_{H^1} \leq r_0$. The rest of the proof runs as before: by $\mathbf{\Psi}_t : H^1(\mathbb{R}^N)\times[0,1] \to H^1(\mathbb{R}^N)$, $t>0$ we denote the translation along trajectories operator for the equation

$$\frac{\partial u}{\partial t} = \Delta u + (1-\mu)f(x,u) + \mu\alpha(x)u,\ x\in\mathbb{R}^N,\ t>0,\ \mu\in[0,1], \tag{50}$$

and, by applying Theorem 5.3 we obtain the existence of $\bar{t} > 0$ such that

$$\mathbf{\Psi}_t(\bar{u},\mu) \neq \bar{u} \quad \text{for all} \quad t\in(0,\bar{t}], \bar{u}\in\partial B_{H^1}(0,r_0).$$

Next Proposition 4.7 (iii) ensures the admissibility of $\mathbf{\Psi}_t$ and by homotopy invariance, for $t \in (0, \bar{t}]$, we have

$$\mathrm{Ind}(\mathbf{\Phi}_t, B_{H^1}(0, r_0)) = \mathrm{Ind}(e^{-t\mathbf{A}_0}, B_{H^1}(0, r_0)) \tag{51}$$

where $\mathbf{A}_0 := \mathbf{A} - \mathbf{C}_0 + \mathbf{C}_\infty$ and operators $\mathbf{C}_i : D(\mathbf{C}_i) \to L^2(\mathbb{R}^N)$ with $D(\mathbf{C}_i) = H^1(\mathbb{R}^N)$ are given by $\mathbf{C}_i u := \alpha_i u$, $i \in \{0, \infty\}$. Now one can easily determine fixed point index of $e^{-t\mathbf{A}_0}$ by arguing as in part (i) (with $\mathbf{A}_\infty$ replaced by $\mathbf{A}_0$ and $B_{H^1}(0, R_0)$ replaced by $B_{H^1}(0, r_0)$) and, as a consequence, obtain that

$$\begin{aligned}\mathrm{Ind}(e^{-t\mathbf{A}_0}, B_{H^1}(0, r_0)) &= \mathrm{Ind}_{LS}(e^{-t\mathbf{A}_0}\mathbf{P}_-, B_{H^1}(0, r_0)) \\ &= \mathrm{Ind}_{LS}(e^{-t(\mathbf{A}_0|_{X_-})}, B_{H^1}(0, r_0) \cap X_-) = (-1)^{m(0)}.\end{aligned}$$

This completes the proof. □

Now we are ready to conclude and provide proofs of our main results.

Proof of Theorem 1.1: Let $\mathbf{\Phi}_t$, $t > 0$, be the translation operator for (1). It is clear that

$$\lim_{|u| \to +\infty} \frac{\widehat{f}(x, u)}{u} = \widehat{\omega}(x), \text{ for any } x \in \mathbb{R}^N.$$

Hence, by applying Proposition 6.1 (i) we obtain $R_0 > 0$ such that

$$\Delta u(x) + \widehat{f}(x, u(x)) = 0, \ x \in \mathbb{R}^N,$$

has no solutions in the set $H^1(\mathbb{R}^N) \setminus B_{H^1}(0, R_0)$ and there exists $t_0 > 0$ such that, for $t \in (0, t_0]$,

$$\mathrm{Ind}(\widehat{\mathbf{\Phi}}_t, B_{H^1}(0, R_0)) = (-1)^{m(\infty)}. \tag{52}$$

Due to Proposition 5.5 and the assumption, increasing R_0 if necessary, we can assume that (38) has no T-periodic solutions starting from $H^1(\mathbb{R}^N) \setminus B_{H^1}(0, R_0)$. Taking $U := B_{H^1}(0, R_0)$ and applying Corollary 5.4 we get

$$\mathrm{Ind}(\mathbf{\Phi}_T, B_{H^1}(0, R_0)) = \lim_{t \to 0^+} \mathrm{Ind}(\widehat{\mathbf{\Phi}}_t, B_{H^1}(0, R_0)),$$

which along with (52) yields $\mathrm{Ind}(\mathbf{\Phi}_T, B_{H^1}(0, R_0)) = (-1)^{m(\infty)}$. This and the existence property of the fixed point index imply that there exists $\bar{u} \in B_{H^1}(0, R_0)$ such that $\mathbf{\Phi}_T(\bar{u}) = \bar{u}$, i.e. there exists a T-periodic solution of (1). □

Proof of Theorem 1.2: First use Proposition 6.1 to get $R_0, r_0 > 0$ such that

$$\lim_{t \to 0^+} \mathrm{Ind}(\widehat{\mathbf{\Phi}}_t, B_{H^1}(0, R)) = (-1)^{m_-(\infty)} \text{ if } R \geq R_0 \tag{53}$$

and

$$\lim_{t \to 0^+} \mathrm{Ind}(\widehat{\mathbf{\Phi}}_t, B_{H^1}(0, r)) = (-1)^{m(0)} \text{ if } 0 < r \leq r_0. \tag{54}$$

Now, due to Proposition 5.5 there exist $R \geq R_0$ and $r \in (0, r_0]$ such that, for any $\lambda \in (0,1]$, (37) has no solutions with $u(0) \in B_{H^1}(0,r) \cup \big(H^1(\mathbb{R}^N) \setminus B_{H^1}(0,R)\big)$. Next we put $U := B_{H^1}(0,R) \setminus \overline{B_{H^1}(0,r)}$ and apply Corollary 5.4 to get

$$\mathrm{Ind}(\mathbf{\Phi}_T, U) = \lim_{t \to 0^+} \mathrm{Ind}(\widehat{\mathbf{\Phi}}_t, U).$$

This together with (53) and (54), by use of the additivity property of the fixed point index, yields

$$\begin{aligned} \mathrm{Ind}(\mathbf{\Phi}_T, U) &= \lim_{t \to 0^+} \mathrm{Ind}(\widehat{\mathbf{\Phi}}_t, B_{H^1}(0,R)) - \lim_{t \to 0^+} \mathrm{Ind}(\widehat{\mathbf{\Phi}}_t, B_{H^1}(0,r)) \\ &= (-1)^{m(\infty)} - (-1)^{m(0)} \neq 0, \end{aligned}$$

which gives the existence of the fixed point of $\mathbf{\Phi}_T$ in U. □

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