NONNOETHERIAN HOMOTOPY DIMER ALGEBRAS AND NONCOMMUTATIVE CREPANT RESOLUTIONS

CHARLIE BEIL

Abstract. Noetherian dimer algebras form a prominent class of examples of noncommutative crepant resolutions (NCCRs). However, dimer algebras which are noetherian are quite rare, and we consider the question: how close are nonnoetherian homotopy dimer algebras to being NCCRs? To address this question, we introduce a generalization of NCCRs to nonnoetherian tiled matrix rings. We show that if a noetherian dimer algebra is obtained from a nonnoetherian homotopy dimer algebra A by contracting each arrow whose head has indegree 1, then A is a noncommutative desingularization of its nonnoetherian center. Furthermore, if any two arrows whose tails have indegree 1 are coprime, then A is a nonnoetherian NCCR.

1. Introduction

Let $(R, \mathfrak{m})$ be a local domain with an algebraically closed residue field k. In the mid 1950's, Auslander, Buchsbaum, and Serre established the famous homological characterization of regularity in the case R is noetherian [AB, AB2, S]: R is regular if and only if

$$\operatorname{gldim} R = \operatorname{pd}_R(k) = \dim R.$$

In 1984, Brown and Hajarnavis generalized this characterization to the setting of noncommutative noetherian rings which are module-finite over their centers [BH]: such a ring A with local center R is said to be homologically homogeneous if for each simple A-module V,

$$\operatorname{gldim} A = \operatorname{pd}_A(V) = \dim R.$$

In 2002, Van den Bergh placed this notion in the context of derived categories with the introduction of noncommutative crepant resolutions (henceforth NCCRs). Specifically, a homologically homogeneous ring A is a (local) NCCR if R is a normal Gorenstein domain and A is the endomorphism ring of a finitely generated reflexive R-module [V, Definition 4.1].[1]

2010 *Mathematics Subject Classification.* 13C15, 14A20.
Key words and phrases. Non-noetherian ring, noncommutative algebraic geometry, tiled matrix ring, dimer algebra, noncommutative crepant resolution.

[1] A proper birational map $f : Y \to X$ from a non-singular variety Y to a Gorenstein singularity X is a crepant resolution if $f^*\omega_X = \omega_Y$. Given an NCCR A of $R = k[X]$, Van den Bergh conjectured that the bounded derived category of A-modules is equivalent to the bounded derived category of coherent sheaves on Y [V, Conjecture 4.6].

A prominent class of NCCRs are noetherian dimer algebras on a torus (Definition 2.2) [Br, Bo, D, B4, B7]. In fact, every 3-dimensional affine toric Gorenstein singularity admits an NCCR given by such a dimer algebra [G, IU]. Although dimer quivers may be defined on any compact surface, in this article we consider the case where the surface is a torus.

A *homotopy algebra* is the quotient of a dimer algebra by homotopy-like relations on the paths in its quiver; a dimer algebra coincides with its homotopy algebra if and only if it is noetherian [B4, Theorem 1.1]. Homotopy algebras, just like noetherian dimer algebras, are tiled matrix rings over polynomial rings. The homotopy algebra of a nonnoetherian dimer algebra is also nonnoetherian and an infinitely generated module over its nonnoetherian center. Here we consider the question:

How close are nonnoetherian homotopy algebras to being NCCRs?

To address this question, we consider a relatively small but important class of nonnoetherian homotopy algebras: Let A be a homotopy algebra with quiver Q such that a noetherian dimer algebra is obtained by contracting each arrow of Q whose head has indegree 1, and no arrow of Q has head and tail of indegree both 1. Denote by R the center of A. The scheme $\operatorname{Spec} R$ has a unique closed point $\mathfrak{m}_0$ of positive geometric dimension [B6, Theorem 1.1]. Furthermore, $\mathfrak{m}_0$ is the unique closed point for which the localizations

$$R_{\mathfrak{m}_0} \quad \text{and} \quad A_{\mathfrak{m}_0} := A \otimes_R R_{\mathfrak{m}_0}$$

are nonnoetherian [B6, Section 3], [B3, Theorem 3.4]. An initial answer to our question appears to be negative:

- $A_{\mathfrak{m}_0}$ has infinite global dimension (Proposition 6.1).
- $A_{\mathfrak{m}_0}$ is typically not the endomorphism ring of a module over its center.

However, the underlying structure of $A_{\mathfrak{m}_0}$ is more subtle. To uncover this structure, we introduce a generalization of homological homogeneity and NCCRs for nonnoetherian tiled matrix rings. Let A be a nonnoetherian tiled matrix ring with local center $(R, \mathfrak{m})$. Firstly, we introduce

- the *cycle algebra* S of A, which is a commutative algebra that contains the center R as a subalgebra (but in general is not a subalgebra of A); and
- the *cyclic localization* $A_{\mathfrak{q}}$ of A at a prime ideal $\mathfrak{q}$ of S.

We then say A is *cycle regular* if for each $\mathfrak{q} \in \operatorname{Spec} S$ minimal over $\mathfrak{m}$ and each simple $A_{\mathfrak{q}}$-module V, we have

$$\operatorname{gldim} A_{\mathfrak{q}} = \operatorname{pd}_{A_{\mathfrak{q}}}(V) = \dim S_{\mathfrak{q}}.$$

Furthermore, we say A is a *nonnoetherian NCCR* if the cycle algebra S is a noetherian normal Gorenstein domain, A is cycle regular, and for each $\mathfrak{q} \in \operatorname{Spec} S$ minimal over $\mathfrak{m}$, $A_{\mathfrak{q}}$ is the endomorphism ring of a reflexive module over its center $Z(A_{\mathfrak{q}})$.

Our main result is the following.

Theorem 1.1. *(Theorems 5.7, 6.15, 7.10.) Let A be a nonnoetherian homotopy algebra such that a noetherian dimer algebra is obtained by contracting each arrow whose head has indegree 1, and no arrow of A has head and tail of indegree both 1. Then*

(1) $A_{\mathfrak{m}_0}$ is cycle regular.
(2) For each prime $\mathfrak{q}$ of the cycle algebra S which is minimal over $\mathfrak{m}_0$, we have

$$\operatorname{gldim} A_{\mathfrak{q}} = \dim S_{\mathfrak{q}} = \operatorname{ght}_R(\mathfrak{m}_0) = 1 < 3 = \operatorname{ht}_R(\mathfrak{m}_0) = \dim R_{\mathfrak{m}_0},$$

where $\operatorname{ght}_R(\mathfrak{m}_0)$ and $\operatorname{ht}_R(\mathfrak{m}_0)$ denote the geometric height and height of $\mathfrak{m}_0$ in R respectively. Furthermore, for each prime $\mathfrak{q}$ of S minimal over $\mathfrak{q} \cap R$,

$$\operatorname{gldim} A_{\mathfrak{q}} = \operatorname{ght}_R(\mathfrak{q} \cap R).$$

(3) If the arrows whose tails have indegree 1 are pairwise coprime, then $A_{\mathfrak{m}_0}$ is a nonnoetherian NCCR.

The second claim suggests that geometric height, rather than height, is the 'right' notion of codimension for nonnoetherian commutative rings, noting that geometric height and height coincide for noetherian rings [B5, Theorem 3.8]. An example of a dimer algebra which is a nonnoetherian NCCR is given in Figure 1, and described in Example 7.12.

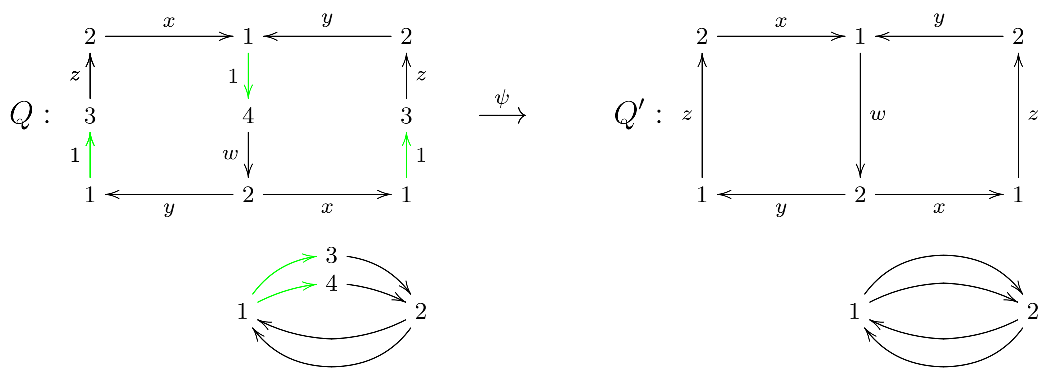
This work is a continuation of [B3], where the author considered localizations $A_{\mathfrak{p}} := A \otimes_R R_{\mathfrak{p}}$ of nonnoetherian dimer and homotopy algebras A at points $\mathfrak{p} \in \operatorname{Spec} R$ away from $\mathfrak{m}_0$. We focus exclusively on homotopy algebras here since the localization of a dimer algebra at $\mathfrak{m}_0$ is much less tractable than its homotopy counterpart; for example, any dimer algebra satisfying the assumptions of Theorem 1.1 has a free subalgebra, whereas its homotopy algebra does not [B4].

In future work we hope to explore the implications of the definitions we have introduced in terms of derived categories and tilting theory, and to study larger classes of nonnoetherian homotopy algebras, as well as other classes of tiled matrix rings.

2. Preliminary definitions

Throughout, let k be an algebraically closed field, let S be an integral domain and a k-algebra, and let R be a (possibly nonnoetherian) subalgebra of S. Denote by $\operatorname{Max} S$, $\operatorname{Spec} S$, and $\dim S$ the maximal spectrum (or variety), prime spectrum (or affine scheme), and Krull dimension of S respectively; similarly for R. For a subset $I \subset S$, set $\mathcal{Z}(I) := \{\mathfrak{n} \in \operatorname{Max} S \mid \mathfrak{n} \supseteq I\}$.

A quiver $Q = (Q_0, Q_1, \mathrm{t}, \mathrm{h})$ consists of a vertex set Q_0, an arrow set Q_1, and head and tail maps $\mathrm{h}, \mathrm{t} : Q_1 \to Q_0$. Denote by $\deg^+ i$ the indegree of a vertex $i \in Q_0$; by kQ the path algebra of Q; and by $e_i \in kQ$ the idempotent at vertex i. Path concatenation is read right to left. By module and global dimension we mean left module and left



$$A = \begin{bmatrix} S & I & zI & wI \\ J & S & zS & wS \\ S & I & k + zI & wI \\ S & I & zI & k + wI \end{bmatrix} \qquad A' = \begin{bmatrix} S & I \\ J & S \end{bmatrix}.$$

FIGURE 1. (Example 7.12.) The homotopy algebra A is a nonnoetherian NCCR. The quivers Q and Q' on the top line are each drawn on a torus, and the two contracted arrows of Q are drawn in green. Here, $S = k[xz, yz, xw, yw]$ is the coordinate ring for the quadric cone, considered as a subalgebra of the polynomial ring $k[x, y, z, w]$, and I and J are the respective S-modules $(x, y)S$ and $(z, w)S$.

global dimension, unless stated otherwise. In a fixed matrix ring, denote by e_{ij} the matrix with a 1 in the ij-th slot and zeros elsewhere, and set $e_i := e_{ii}$.

The following definitions were introduced in [B5] to formulate a theory of geometry for nonnoetherian rings with finite Krull dimension.

Definition 2.1. [B5, Definition 3.1]

- We say S is a *depiction* of R if S is a finitely generated k-algebra, the morphism
$$\iota_{S/R} : \operatorname{Spec} S \to \operatorname{Spec} R, \qquad \mathfrak{q} \mapsto \mathfrak{q} \cap R,$$
is surjective, and
$$\{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}\} = \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n}\cap R} \text{ is noetherian}\} \neq \emptyset.$$
- The *geometric height* of $\mathfrak{p} \in \operatorname{Spec} R$ is the minimum
$$\operatorname{ght}(\mathfrak{p}) := \min\left\{\operatorname{ht}_S(\mathfrak{q}) \mid \mathfrak{q} \in \iota^{-1}_{S/R}(\mathfrak{p}),\ S \text{ a depiction of } R\right\}.$$
The *geometric dimension* of $\mathfrak{p}$ is
$$\operatorname{gdim} \mathfrak{p} := \dim R - \operatorname{ght}(\mathfrak{p}).$$

The algebras that we will consider in this article are called homotopy (dimer) algebras. Dimer algebras are a type of quiver with potential, and were introduced in string theory [BFHMS] (see also [BD]). Homotopy algebras are special quotients of dimer algebras, and were introduced in [B2].

Definition 2.2.
• Let Q be a finite quiver whose underlying graph $\overline{Q}$ embeds into a two-dimensional real torus T^2, such that each connected component of $T^2 \setminus \overline{Q}$ is simply connected and bounded by an oriented cycle, called a *unit cycle*.[2,3,4] The *dimer algebra* of Q is the quiver algebra kQ/I with relations

$$I := \langle p - q \mid \exists\, a \in Q_1 \text{ such that } pa \text{ and } qa \text{ are unit cycles} \rangle \subset kQ,$$

where p and q are paths.

Since I is generated by certain differences of paths, we may refer to a path modulo I as a *path* in the dimer algebra kQ/I.

• Two paths $p, q \in kQ/I$ form a *non-cancellative pair* if $p \neq q$, and there is a path $r \in kQ/I$ such that

$$rp = rq \neq 0 \quad \text{or} \quad pr = qr \neq 0.$$

kQ/I and Q are called *non-cancellative* if there is a non-cancellative pair; otherwise they are called *cancellative*. By [B4, Theorem 1.1], kQ/I is noetherian if and only if it is cancellative.

• We call the quotient algebra

$$A := (kQ/I)/\langle p - q \mid p, q \text{ is a non-cancellative pair} \rangle$$

the *homotopy (dimer) algebra* of Q.[5] (For the definition of a homotopy algebra on a general surface, see [B2].)

• Let A be a (homotopy) dimer algebra with quiver Q.
  – A *perfect matching* $D \subset Q_1$ is a set of arrows such that each unit cycle contains precisely one arrow in D.
  – A *simple matching* $D \subset Q_1$ is a perfect matching such that $Q \setminus D$ supports a simple A-module of dimension 1^{Q_0} (that is, $Q \setminus D$ contains a cycle that passes through each vertex of Q). Denote by $\mathcal{S}$ the set of simple matchings of A.

---

[2]In contexts such as cluster algebras, $\overline{Q}$ may be embedded into any compact surface; see for example [BKM].

[3]Note that for any vertex $i \in Q_0$, the indegree and outdegree of i are equal.

[4]In [B1], it useful to allow length 1 unit cycles. Consequently, it is possible for a length 1 path $a \in Q_1$ to equal a vertex modulo I; in this case, a is called a 'pseudo-arrow' rather than an 'arrow', in order to avoid modifying standard definitions such as perfect matchings.

[5]A dimer algebra coincides with its homotopy algebra if and only if its quiver is cancellative.

3. Cycle algebra and nonnoetherian NCCRs

In this section we introduce the cycle algebra, cyclic localization, and nonnoetherian NCCRs. Let B be an integral domain and a k-algebra. Let

$$A = \left[A^{ij}\right] \subset M_d(B)$$

be a tiled matrix algebra; that is, each diagonal entry $A^i := A^{ii}$ is a unital subalgebra of B. Denote by $Z = Z(A)$ the center of A.

Definition 3.1. Set

$$R := k\left[\cap_{i=1}^d A^i\right] \quad \text{and} \quad S := k\left[\cup_{i=1}^d A^i\right].$$

We call S the *cycle algebra* of A. Furthermore, for $\mathfrak{q} \in \operatorname{Spec} S$, set

$$A_{\mathfrak{q}} := \left\langle \begin{bmatrix} A^1_{\mathfrak{q}\cap A^1} & A^{12} & \cdots & A^{1d} \\ A^{21} & A^2_{\mathfrak{q}\cap A^2} & & \\ \vdots & & \ddots & \\ A^{d1} & & & A^d_{\mathfrak{q}\cap A^d} \end{bmatrix} \right\rangle \subset M_d(\operatorname{Frac} B).$$

We call $A_{\mathfrak{q}}$ the *cyclic localization* of A at $\mathfrak{q}$.

Note that R and S are integral domains since they are subalgebras of B. The following definitions aim to generalize homological homogeneity and NCCRs to the nonnoetherian setting.

Definition 3.2. Suppose R is a local domain with unique maximal ideal $\mathfrak{m}$.

- We say A is *cycle regular* if for each $\mathfrak{q} \in \operatorname{Spec} S$ minimal over $\mathfrak{m}$ and each simple $A_{\mathfrak{q}}$-module V,
$$\operatorname{gldim} A_{\mathfrak{q}} = \operatorname{pd}_{A_{\mathfrak{q}}}(V) = \dim S_{\mathfrak{q}}.$$
- We say A is a *noncommutative desingularization* if A is cycle regular, and $A \otimes_R \operatorname{Frac} R$ and $\operatorname{Frac} R$ are Morita equivalent.
- We say A is a *nonnoetherian noncommutative crepant resolution* if S is a normal Gorenstein domain, A is cycle regular, and for each $\mathfrak{q} \in \operatorname{Spec} S$ minimal over $\mathfrak{m}$, $A_{\mathfrak{q}}$ is the endomorphism ring of a reflexive $Z(A_{\mathfrak{q}})$-module.

Remark 3.3. Suppose B is a finitely generated k-algebra, and k is uncountable. Further suppose the embedding $\tau : A \hookrightarrow M_d(B)$ has the properties that

(i) for generic $\mathfrak{b} \in \operatorname{Max} B$, the composition
$$A \xrightarrow{\tau} M_d(B) \xrightarrow{1} M_d\left(B/\mathfrak{b}\right)$$
is surjective;

(ii) the morphism

$$\operatorname{Max} B \to \operatorname{Max} \tau(Z), \quad \mathfrak{b} \mapsto \mathfrak{b}\mathbf{1}_d \cap \tau(Z),$$

is surjective; and

(iii) for each $\mathfrak{n} \in \operatorname{Max} S$, $R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}$ iff $R_{\mathfrak{n}\cap R}$ is noetherian.

(τ, B) is then said to be an *impression* of A [B7, Definition 2.1].

Under these conditions, the center Z of A is equal to R,

$$Z = R\mathbf{1}_d,$$

and is depicted by S [B5, Theorem 4.1.1]. Furthermore, by [B5, Theorem 4.1.2],

$$\begin{aligned} R = S \quad &\Leftrightarrow \quad A \text{ is a finitely generated } R\text{-module} \\ &\Leftrightarrow \quad R \text{ is noetherian} \\ &\Rightarrow \quad A \text{ is noetherian} \end{aligned}$$

In particular, if R is noetherian, then the cyclic and central localizations of A at $\mathfrak{q} \in \operatorname{Spec} S$ are isomorphic algebras,

$$A_{\mathfrak{q}} \cong A \otimes_R R_{\mathfrak{q}\cap R}.$$

If $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathfrak{q} \in \operatorname{Spec} S$, then we denote by $A_{\mathfrak{p}}$ and $A_{\mathfrak{q}}$ the central and cyclic localizations of A respectively; no ambiguity arises since the two localizations coincide whenever $R = S$.

4. A class of nonnoetherian homotopy algebras

For the remainder of this article, we will consider a class of homotopy algebras whose quivers contain vertices with indegree 1. Such quivers are necessarily non-cancellative. Unless stated otherwise, let A be a nonnoetherian homotopy algebra with quiver $Q = (Q_0, Q_1, \mathrm{t}, \mathrm{h})$ such that

(A) a cancellative dimer algebra $A' = kQ'/I'$ is obtained by contracting each arrow of Q whose head has indegree 1; and

(B) for each $a \in Q_1$, the indegrees $\deg^+ \mathrm{t}(a)$ and $\deg^+ \mathrm{h}(a)$ are not both 1.

Set

$$Q_1^* = \left\{ a \in Q_1 \mid \deg^+ \mathrm{h}(a) = 1 \right\} \quad \text{and} \quad Q_1^{\mathrm{t}} := \left\{ a \in Q_1 \mid \deg^+ \mathrm{t}(a) = 1 \right\}.$$

The quiver $Q' = (Q'_0, Q'_1, \mathrm{t}', \mathrm{h}')$ is then defined by

$$Q'_0 = Q_0 / \left\{ \mathrm{h}(a) \sim \mathrm{t}(a) \mid a \in Q_1^* \right\}, \quad Q'_1 = Q_1 \setminus Q_1^*,$$

and for each arrow $a \in Q'_1$,

$$\mathrm{h}'(a) = \mathrm{h}(a) \quad \text{and} \quad \mathrm{t}'(a) = \mathrm{t}(a).$$

The homotopy algebras A and A' are isomorphic to tiled matrix rings. Indeed, consider the k-linear map

$$\psi : A \to A'$$

defined by

$$\psi(a) = \begin{cases} a & \text{if } \ a \in Q_0 \cup Q_1 \setminus Q_1^* \\ e_{\mathrm{t}(a)} & \text{if } \ a \ \in Q_1^* \end{cases}$$

and extended multiplicatively to (nonzero) paths and k-linearly to A. Furthermore, consider the polynomial ring generated by the simple matchings $\mathcal{S}'$ of A',

$$B = k\left[x_D \mid D \in \mathcal{S}'\right].$$

By [B2, Theorem 1.1], there are injective algebra homomorphisms

$$\tau : A' \hookrightarrow M_{|Q_0'|}(B) \quad \text{and} \quad \tau_\psi : A \hookrightarrow M_{|Q_0|}(B)$$

defined by

$$\tau(a) = \begin{cases} e_{ii} & \text{if } a = e_i \in Q_0' \\ \left(\prod_{D \in \mathcal{S}' : D \ni a} x_D\right) e_{\mathrm{h}(a),\mathrm{t}(a)} & \text{if } a \in Q_1' \end{cases}$$

$$\tau_\psi(a) = \begin{cases} e_{ii} & \text{if } a = e_i \in Q_0 \\ \left(\prod_{D \in \mathcal{S}' : D \ni \psi(a)} x_D\right) e_{\mathrm{h}(a),\mathrm{t}(a)} & \text{if } a \in Q_1 \end{cases}$$

and extended multiplicatively and k-linearly to A' and A.

For $p \in e_j A e_i$ and $p' \in e_j A' e_i$, denote by

$$\bar{\tau}_\psi(p) = \bar{p} \in B \quad \text{and} \quad \bar{\tau}(p') = \bar{p}' \in B$$

the single nonzero matrix entry of $\tau_\psi(p)$ and $\tau(p')$, respectively. Note that

$$\bar{\tau}_\psi(p) = \bar{\tau}(\psi(p)).$$

Furthermore, for each $a \in Q_1$ and $D \in \mathcal{S}'$,

$$x_D | \bar{a} \quad \Longleftrightarrow \quad \psi(a) \in D.$$

Since A' is cancellative, each $a' \in Q_1'$ is contained in a simple matching by [B4, Theorem 1.1]; in particular, $\bar{a}' \neq 1$. Therefore, for each $a \in Q_1$,

$$\bar{a} = 1 \quad \Longleftrightarrow \quad \deg^+ \mathrm{h}(a) = 1.$$

Lemma 4.1.

(1) The cycle algebras of A and A' are equal,[6]

$$k\left[\cup_{i \in Q_0} \bar{\tau}_\psi\left(e_i A e_i\right)\right] = k\left[\cup_{i \in Q_0'} \bar{\tau}\left(e_i A' e_i\right)\right] = S.$$

(2) The center Z' of A' is isomorphic to S, and the center Z of A is isomorphic to the intersection

$$Z \cong k\left[\cap_{i \in Q_0} \bar{\tau}_\psi(e_i A e_i)\right] = R.$$

(3) S is a depiction of R.

[6]The map ψ is therefore called a 'cyclic contraction' [B2, Section 3].

(4) If the indegree of a vertex $i \in Q_0$ is at least 2, then

$$\bar{\tau}_\psi(e_i A e_i) = S.$$

In particular, for each arrow $a \in Q_1$,

$$\bar{\tau}_\psi(e_{\mathrm{t}(a)} A e_{\mathrm{t}(a)}) = S \quad \text{or} \quad \bar{\tau}_\psi(e_{\mathrm{h}(a)} A e_{\mathrm{h}(a)}) = S.$$

Proof. (1) By assumption (A), for each cycle p' in Q', there is a cycle p in Q such that $\psi(p) = p'$. Therefore the cycle algebras of A and A' are equal.

(2) Since A' is cancellative, for each $i, j \in Q'_0$,

$$\bar{\tau}(e_i A' e_i) = \bar{\tau}(e_j A' e_j),$$

by [B4, Theorem 1.1]. Whence for each $i \in Q'_0$,

$$\bar{\tau}(e_i A' e_i) = S. \tag{1}$$

Furthermore, the centers Z and Z' are isomorphic to the intersections

$$Z \cong k\left[\cap_{i \in Q_0} \bar{\tau}_\psi(e_i A e_i)\right] = R \quad \text{and} \quad Z' \cong k\left[\cap_{i \in Q'_0} \bar{\tau}(e_i A' e_i)\right],$$

by [B2, Theorem 1.1]. Therefore Z' is isomorphic to S by (1).

(3) Since A and A' have equal cycle algebras, $Z \cong R$ is depicted by $Z' \cong S$, by [B6, Theorem 1.1].

(4) By assumption (A), if a vertex $i \in Q_0$ has indegree at least 2, then

$$\bar{\tau}_\psi(e_i A e_i) = \bar{\tau}(e_{\psi(i)} A' e_{\psi(i)}) \overset{(\mathrm{I})}{=} S,$$

where (I) holds by (1). Furthermore, by assumption (B), the head or tail of each arrow $a \in Q_1$ has indegree at least 2. □

5. Prime decomposition of the origin

Recall that A is a nonnoetherian homotopy algebra with center R, satisfying assumptions (A) and (B) given in Section 4. Consider the origin of $\operatorname{Max} R$,

$$\mathfrak{m}_0 := (x_D \mid D \in \mathcal{S}')\, B \cap R.$$

For a monomial $g \in B$, denote by $\mathfrak{q}_g$ the ideal in S generated by all monomials in S that are divisible by g in B. If $g = x_D$ for some simple matching $D \in \mathcal{S}'$, then set

$$\mathfrak{q}_D := \mathfrak{q}_{x_D}.$$

We will write $h \mid g$ if h divides g in B, unless stated otherwise.

Lemma 5.1. *Let $g \in B$ be a monomial. Then the ideal $\mathfrak{q}_g \subset S$ is prime if and only if $g = x_D$ for some $D \in \mathcal{S}'$.*

Proof. Let $n := |\mathcal{S}'|$, and enumerate the simple matchings of A', $\mathcal{S}' = \{D_1, \ldots, D_n\}$. Set $x_i := x_{D_i}$.

(i) We first claim that for each pair of distinct simple matchings $D_i, D_j \in \mathcal{S}'$, there is a cycle $s \in A$ satisfying

$$x_i \mid \bar{s} \quad \text{and} \quad x_j \nmid \bar{s}. \tag{2}$$

Indeed, fix $i \neq j$. Since $D_i \neq D_j$, there is an arrow $a \in Q_1'$ for which $a \in D_i \setminus D_j$. Furthermore, since D_j is simple, there is a path $p \in e_{\mathrm{t}(a)} A' e_{\mathrm{h}(a)}$ supported on $Q' \setminus D_j$. Whence $s := pa$ is a cycle satisfying (2). But A and A' have equal cycle algebras by Lemma 4.1.1. Therefore $\bar{s}$ is the $\bar{\tau}_\psi$-image of a cycle in A, proving our claim.

(ii) We now claim that if $g \in B$ is a monomial and $\mathfrak{q}_g$ is a prime ideal of S, then $g = x_D$ for some $D \in \mathcal{S}'$. It suffices to consider a monomial $g = \prod_{i=1}^{n'} x_i^{m_i}$, where $2 \leq n' \leq n$, and for each i, $m_i \geq 1$. By Claim (i), there are cycles $s_1, \ldots, s_{n'} \in A$ such that

$$x_1 \mid \bar{s}_1, \quad x_2 \nmid \bar{s}_1,$$

and for each $2 \leq i \leq n'$,

$$x_1 \nmid \bar{s}_i, \quad x_i \mid \bar{s}_i.$$

Set

$$h_1 := \bar{s}_1^{m_1} \quad \text{and} \quad h_2 := \prod_{i=2}^{n'} \bar{s}_i^{m_i}.$$

Then $h_1 h_2 \in \mathfrak{q}_g$. But $h_1 \notin \mathfrak{q}_g$ and $h_2 \notin \mathfrak{q}_g$ since $x_2 \nmid h_1$ and $x_1 \nmid h_2$. Therefore $\mathfrak{q}_g$ is not prime.

(iii) Finally, consider a simple matching $D \in \mathcal{S}'$. If $s, t \in e_i A e_i$ are cycles for which $x_D \mid \overline{st}$, then $x_D \mid \bar{s}$ or $x_D \mid \bar{t}$, since B is the polynomial ring generated by $\mathcal{S}'$. Therefore the ideal $\mathfrak{q}_{x_D}$ is prime. $\square$

Lemma 5.2. *Let $i, j \in Q_0$ and $D \in \mathcal{S}'$. If $\deg^+ i \geq 2$, or i is not the tail of an arrow $a \in Q_1^{\mathrm{t}}$ for which $x_D \mid \bar{a}$, then there is a path $p \in e_j A e_i$ such that $x_D \nmid \bar{p}$.*

Proof. (i) First suppose $\deg^+ i \geq 2$. Since D is simple, there is a path $q \in e_{\psi(j)} A' e_{\psi(i)}$ supported on $Q' \setminus D$; whence $x_D \nmid \bar{q}$. Furthermore, since $\deg^+ i \geq 2$, there is a path $p \in e_j A e_i$ such that $\psi(p) = q$, by assumption (A). In particular, $x_D \nmid \bar{q} = \bar{p}$.

(ii) Now suppose $\deg^+ i = 1$. Let $a \in Q_1^{\mathrm{t}}$ be such that $\mathrm{t}(a) = i$. Then $\deg^+ \mathrm{h}(a) \geq 2$ by assumption (B). Thus there is a path $t \in e_j A e_{\mathrm{h}(a)}$ for which $x_D \nmid \bar{t}$, by Claim (i). Therefore if $x_D \nmid \bar{a}$, then the path $p := ta \in e_j A e_i$ satisfies $x_D \nmid \bar{p}$. $\square$

Notation 5.3. Denote by σ_i the unit cycle at vertex $i \in Q_0$, and by

$$\sigma := \bar{\tau}_\psi(\sigma_i) = \prod_{D \in \mathcal{S}'} x_D$$

the common $\bar{\tau}_\psi$-image of each unit cycle in Q. (σ is also the $\bar{\tau}$-image of each unit cycle in Q'.) Furthermore, consider a covering map of the torus, $\pi : \mathbb{R}^2 \to T^2$, such that for some $i \in Q_0$,

$$\pi(\mathbb{Z}^2) = i.$$

Denote by

$$Q^+ := \pi^{-1}(Q) \subset \mathbb{R}^2$$

the covering quiver of Q. For each path p in Q, denote by p^+ a path in Q^+ with tail in $[0,1) \times [0,1) \subset \mathbb{R}^2$ satisfying $\pi(p^+) = p$.

Lemma 5.4. *Let $a \in A'$ be an arrow and let $s \in e_{\mathrm{t}(a)} A' e_{\mathrm{t}(a)}$ be a cycle satisfying $\bar{a} \mid \bar{s}$. Then there is a path $p \in e_{\mathrm{t}(a)} A' e_{\mathrm{h}(a)}$ such that*

$$s = pa.$$

Proof. We use the notation in [B3, Notation 2.1]. Suppose the hypotheses hold.[7] It suffices to assume $\sigma \nmid \bar{s}$ by [B2, Lemma 2.1]. Whence $s \in \hat{\mathcal{C}}$ by [B2, Lemma 4.8.3]. Let $u \in \mathbb{Z}^2$ be such that $s \in \hat{\mathcal{C}}^u$. Since A' is cancellative, for each $i \in Q'_0$ we have

$$\hat{\mathcal{C}}^u_i \neq \emptyset, \tag{3}$$

by [B2, Proposition 4.10]. Consider $t \in \hat{\mathcal{C}}^u_{\mathrm{h}(a)}$. Then $\bar{s} = \bar{t}$ by [B2, Proposition 4.20.2].

Now the paths $(as)^+$ and $(ta)^+$ bound a compact region

$$\mathcal{R}_{as,ta} \subset \mathbb{R}^2.$$

Furthermore, since A' is cancellative, if a cycle p is formed from subpaths of cycles in $\hat{\mathcal{C}}^u$, then p is in $\hat{\mathcal{C}}^u$, by [B2, Proposition 4.20.3]. Therefore we may suppose that the interior of $\mathcal{R}_{as,ta}$ does not contain any vertices of Q'^+, by (3).

Assume to the contrary that s^+ and t^+ do not intersect (modulo I). Then a is contained in a simple matching D of A' such that $x_D \nmid \bar{s}$, by [B2, Lemma 4.15]; see Figure 2.i. In particular, $x_D \mid \bar{a}$. But by assumption, $\bar{a} \mid \bar{s}$. Thus $x_D \mid \bar{s}$, a contradiction.

Therefore s^+ and t^+ intersect at a vertex i^+; see Figure 2.ii. By assumption, $\sigma \nmid \bar{s} = \bar{t}$. Whence $\sigma \nmid \overline{as}$ and $\sigma \nmid \overline{ta}$ since $\bar{a} \mid \bar{s} = \bar{t}$. Thus

$$\bar{s}_1 = \bar{t}_1 \bar{a} \quad \text{and} \quad \bar{a}\bar{s}_2 = \bar{t}_2,$$

by [B2, Lemma 4.3]. Consequently,

$$\overline{s_2 t_1 a} = \bar{s}_2 \bar{s}_1 = \bar{s}.$$

Therefore, since $\tau : A' \to M_{|Q'_0|}(B)$ is injective, we have

$$s_2 t_1 a = s.$$

In particular, we may take $p = s_2 t_1$. $\square$

[7] This proof is similar to [B3, Claim (i) in proof of Lemma 2.4].

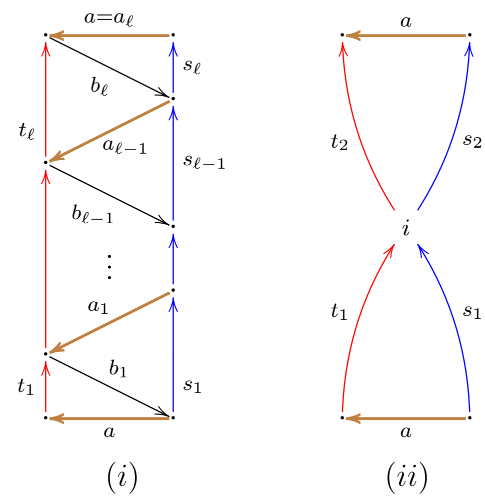


FIGURE 2. Cases for Lemma 5.4. In case (i), s and t factor into paths $s = s_\ell \cdots s_2 s_1$ and $t = t_\ell \cdots t_2 t_1$, where $a_1, \ldots, a_\ell, b_1, \ldots, b_\ell$ are arrows, and the cycles $b_j a_j s_j$ and $a_{j-1} b_j t_j$ are unit cycles. The a_j arrows, drawn in thick brown, belong to a simple matching D of A'. In case (ii), s and t factor into paths $s = s_2 e_i s_1$ and $t = t_2 e_i t_1$.

Proposition 5.5. *For each arrow $a \in Q_1 \setminus Q_1^*$, $\bar{\tau}_\psi(e_{\mathrm{t}(a)} A a)$ is an ideal of S with prime decomposition*

$$\bar{\tau}_\psi(e_{\mathrm{t}(a)} A a) = \bigcap_{D \in \mathcal{S}' \,:\, x_D \mid \bar{a}} \mathfrak{q}_D. \tag{4}$$

Consequently, the prime decomposition of $\mathfrak{m}_0 \in \operatorname{Max} R$, as an ideal of S, is

$$\mathfrak{m}_0 = \bigcap_{a \in Q_1^{\mathrm{t}}} \bar{\tau}_\psi(e_{\mathrm{t}(a)} A a) = \bigcap_{\substack{D \in \mathcal{S}' : \\ x_D \mid \bar{a} \text{ where } a \in Q_1^{\mathrm{t}}}} \mathfrak{q}_D.$$

Proof. $\bar{\tau}_\psi(e_{\mathrm{t}(a)} A a)$ is an ideal of S by Lemma 4.1.4. Set $\mathfrak{q}_a := \bigcap_{D \in \mathcal{S}' : x_D \mid \bar{a}} \mathfrak{q}_D$. The inclusion $\bar{\tau}_\psi(e_{\mathrm{t}(a)} A a) \subseteq \mathfrak{q}_a$ is clear. So suppose $t \in e_j A e_j$ is a cycle such that $\bar{t} \in \mathfrak{q}_a$, that is, $\bar{a} \mid \bar{t}$. We want to show that $\bar{t} \in \bar{\tau}_\psi(e_{\mathrm{t}(a)} A a)$.

First suppose $\deg^+ \mathrm{t}(a) \geq 2$. Then $e_{\mathrm{t}(a)} A e_{\mathrm{t}(a)} = S e_{\mathrm{t}(a)}$ by Lemma 4.1.4. In particular, there is a cycle $s \in e_{\mathrm{t}(a)} A e_{\mathrm{t}(a)}$ for which $\bar{s} = \bar{t}$. Furthermore, there is a path $p \in e_{\mathrm{t}(a)} A e_{\mathrm{h}(a)}$ such that $s = pa$, by Lemma 5.4 and assumption (A).

Now suppose $\deg^+ \mathrm{t}(a) = 1$. Then $\deg^+ \mathrm{h}(a) \geq 2$ by assumption (B). Whence $e_{\mathrm{h}(a)} A e_{\mathrm{h}(a)} = S e_{\mathrm{h}(a)}$. In particular, there is a cycle $s \in e_{\mathrm{h}(a)} A e_{\mathrm{h}(a)}$ for which $\bar{s} = \bar{t}$. Furthermore, there is a path $p \in e_{\mathrm{t}(a)} A e_{\mathrm{h}(a)}$ such that $s = ap$, again by Lemma 5.4 and assumption (A).

Thus, in either case,

$$\bar{t} = \bar{s} \in \bar{\tau}_\psi(e_{\mathrm{t}(a)} A a).$$

Therefore (4) holds. Finally, each $\mathfrak{q}_D$ is prime by Lemma 5.1. $\square$

In the following, we show that although the ideal $\mathfrak{q}_D$ may not be principal in S, it becomes principal over the localization $S_{\mathfrak{q}_D}$.

Proposition 5.6. *Let $D \in \mathcal{S}'$ and set $\mathfrak{q} := \mathfrak{q}_D$. Then the maximal ideal $\mathfrak{q}S_{\mathfrak{q}}$ of $S_{\mathfrak{q}}$ is generated by σ,*

$$\mathfrak{q}S_{\mathfrak{q}} = \sigma S_{\mathfrak{q}}.$$

Proof. Let $g \in \mathfrak{q}$ be a nonzero monomial. Then there is a cycle $s \in A$ with $\bar{s} = g$. By possibly cyclically permuting the arrow subpaths of s, we may assume s factors into paths $s = pa$, where $x_D \mid \bar{a}$ and either

- $a \in Q_1 \setminus (Q_1^* \cup Q_1^{\mathrm{t}})$, or
- $a = a'\delta$ where $\delta \in Q_1^*$ and $a' \in Q_1^{\mathrm{t}}$.

In either case, $\deg^+ \mathrm{t}(a) \geq 2$.

Let b be a path such that ba is a unit cycle. Then $x_D \nmid \bar{b}$ since $x_D \mid \bar{a}$ and $\overline{ba} = \sigma$. Furthermore, since $\deg^+ \mathrm{h}(b) = \deg^+ \mathrm{t}(a) \geq 2$, there is a path $t \in e_{\mathrm{t}(b)} A e_{\mathrm{h}(b)}$ for which $x_D \nmid \bar{t}$, by Lemma 5.2. In particular, tp and tb are cycles, and $x_D \nmid \overline{tb}$. Whence

$$\overline{tp} \in S \quad \text{and} \quad \overline{tb} \in S \setminus \mathfrak{q}.$$

Therefore

$$g = \bar{a}\bar{p}\,\frac{\overline{tb}}{\overline{tb}} = \bar{a}\bar{b}\,\frac{\overline{tp}}{\overline{tb}} = \sigma\,\frac{\overline{tp}}{\overline{tb}} \in \sigma S_{\mathfrak{q}}.$$

$\square$

Recall that an ideal I is unmixed if for each minimal prime $\mathfrak{q}$ over I, $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(I)$.

Theorem 5.7.

(1) For each $D \in \mathcal{S}'$, the height of $\mathfrak{q}_D$ in S is 1.
(2) The set of minimal primes of S over $\mathfrak{m}_0$ are the ideals $\mathfrak{q}_D \in \operatorname{Spec} S$ for which D contains the ψ-image of some $a \in Q_1^{\mathrm{t}}$.
(3) $\mathfrak{m}_0$ is an unmixed ideal of S. Furthermore, $\mathfrak{m}_0$ has height 1 as an ideal of S and height 3 as an ideal of R,

$$\mathrm{ht}_S(\mathfrak{m}_0) = 1 \quad \textit{and} \quad \mathrm{ht}_R(\mathfrak{m}_0) = 3.$$

Proof. (1) Set $\mathfrak{q} := \mathfrak{q}_D$. Then

$$1 \overset{(\mathrm{I})}{\leq} \mathrm{ht}_S(\mathfrak{q}) = \mathrm{ht}_{S_{\mathfrak{q}}}(\mathfrak{q}S_{\mathfrak{q}}) \overset{(\mathrm{II})}{=} \mathrm{ht}_{S_{\mathfrak{q}}}(\sigma S_{\mathfrak{q}}) \overset{(\mathrm{III})}{\leq} 1.$$

Indeed, (I) holds since S is an integral domain and $\mathfrak{q}$ is nonzero; (II) holds by Proposition 5.6; and (III) holds by Krull's principal ideal theorem.

(2) Follows from Claim (1) and Proposition 5.5.

(3) $\mathfrak{m}_0$ is a height 1 unmixed ideal of S by Claims (1) and (2), and Proposition 5.5. Furthermore, R admits a depiction by Lemma 4.1.3. Thus the height of each maximal ideal of R equals the Krull dimension of R by [B5, Lemma 3.7.2]. But the Krull dimension of R is 3 by [B6, Theorem 1.1]. Therefore $\operatorname{ht}_R(\mathfrak{m}_0) = 3$. □

Question 5.8. Let K be the function field of an algebraic variety. As shown in Theorem 5.7.3, a subset $\mathfrak{p}$ of K may be an ideal in different subalgebras of K, and the height of $\mathfrak{p}$ depends on the choice of such subalgebra. Is the geometric height of $\mathfrak{p}$ independent of the choice of subalgebra for which $\mathfrak{p}$ is an ideal? If this is the case, then the geometric height would be an intrinsic property of an ideal, whereas its height would not be.

The center and cycle algebra of $A_{\mathfrak{m}_0} := A \otimes_R R_{\mathfrak{m}_0}$ are respectively

$$Z(A_{\mathfrak{m}_0}) \cong R \otimes_R R_{\mathfrak{m}_0} \cong R_{\mathfrak{m}_0} \quad \text{and} \quad S \otimes_R R_{\mathfrak{m}_0} \cong SR_{\mathfrak{m}_0}.$$

Proposition 5.9. *The cycle algebra $SR_{\mathfrak{m}_0}$ of $A_{\mathfrak{m}_0}$ is a normal Gorenstein domain.*

Proof. Let $\mathfrak{t} \in \operatorname{Spec}(SR_{\mathfrak{m}_0})$ and set $\mathfrak{q} := \mathfrak{t} \cap S$.

(i) We claim that

$$(SR_{\mathfrak{m}_0})_{\mathfrak{t}} = S_{\mathfrak{q}}.$$

Clearly $(SR_{\mathfrak{m}_0})_{\mathfrak{t}} = S_{\mathfrak{q}} R_{\mathfrak{m}_0}$.[8] It thus suffices to show that

$$S_{\mathfrak{q}} R_{\mathfrak{m}_0} = S_{\mathfrak{q}}. \tag{5}$$

Indeed, we have

$$\mathfrak{t} \cap R \subseteq \mathfrak{m}_0. \tag{6}$$

Thus if $\mathfrak{m}_0 \subseteq \mathfrak{q}$, then $\mathfrak{q} \cap R = \mathfrak{m}_0$. Whence $R_{\mathfrak{m}_0} \subseteq S_{\mathfrak{q}}$. In particular, $S_{\mathfrak{q}} R_{\mathfrak{m}_0} = S_{\mathfrak{q}}$. Otherwise $\mathfrak{q} = 0 \subset \mathfrak{m}_0$ by Theorem 5.7.3; whence

$$S_{\mathfrak{q}} R_{\mathfrak{m}_0} = (\operatorname{Frac} S) R_{\mathfrak{m}_0} = \operatorname{Frac} S = S_{\mathfrak{q}}.$$

Therefore in either case (5) holds, proving our claim.

(ii) S is isomorphic to the center of A' by Lemma 4.1.2. Thus S is a normal Gorenstein domain since A' is an NCCR. Whence $S_{\mathfrak{q}}$ is a normal Gorenstein domain. But $(SR_{\mathfrak{m}_0})_{\mathfrak{t}} = S_{\mathfrak{q}}$ by Claim (i). Therefore $(SR_{\mathfrak{m}_0})_{\mathfrak{t}}$ is a normal Gorenstein domain. Since this holds for all $\mathfrak{t} \in \operatorname{Spec}(SR_{\mathfrak{m}_0})$, $SR_{\mathfrak{m}_0}$ is also a normal Gorenstein domain. □

[8] To show this, note that the elements of $SR_{\mathfrak{m}_0}$ are of the form s/r, with $s \in S$ and $r \in R \setminus \mathfrak{m}_0$. Thus an element of $(SR_{\mathfrak{m}_0})_{\mathfrak{t}}$ is of the form $\frac{s_1}{r_1}(\frac{s_2}{r_2})^{-1}$, with $s_1, s_2 \in S$, $r_1, r_2 \in R \setminus \mathfrak{m}_0$, and $\frac{s_2}{r_2} \not\in \mathfrak{t}$. Furthermore, $\frac{s_2}{r_2} \not\in \mathfrak{t}$ and (6) together imply $s_2 \not\in \mathfrak{t}$. Whence

$$s_2 \in S \setminus (\mathfrak{t} \cap S) = S \setminus \mathfrak{q}.$$

Therefore

$$\frac{s_1}{r_1}\left(\frac{s_2}{r_2}\right)^{-1} = \frac{s_1 r_2}{s_2} \cdot \frac{1}{r_1} \in S_{\mathfrak{q}} R_{\mathfrak{m}_0}.$$

6. Cycle regularity

Recall that A is a nonnoetherian homotopy algebra satisfying assumptions (A) and (B) given in Section 4, unless stated otherwise. Let $\mathfrak{q} \in \operatorname{Spec} S$ be a minimal prime over the origin $\mathfrak{m}_0$ of $\operatorname{Max} R$; then there is a simple matching $D \in \mathcal{S}'$ such that $\mathfrak{q} = \mathfrak{q}_D$, by Proposition 5.5. In this section, we will consider the cyclic localization $A_{\mathfrak{q}}$ of A at $\mathfrak{q}$.

The algebra homomorphism $\tau_\psi : A \hookrightarrow M_{|Q_0|}(B)$ extends to the cyclic localization, $\tau_\psi : A_{\mathfrak{q}} \hookrightarrow M_{|Q_0|}(\operatorname{Frac} B)$. For $p \in e_j A_{\mathfrak{q}} e_i$, we will denote by $\bar{\tau}_\psi(p) = \bar{p} \in \operatorname{Frac} B$ the single nonzero matrix entry of $\tau_\psi(p)$.

We begin by showing that a notion of homological regularity cannot be obtained by considering the central localization $A_{\mathfrak{m}_0} := A \otimes_R R_{\mathfrak{m}_0}$ alone.

Proposition 6.1. *The $A_{\mathfrak{m}_0}$-module $A_{\mathfrak{m}_0}/\mathfrak{m}_0 = A \otimes_R (R_{\mathfrak{m}_0}/\mathfrak{m}_0)$ has infinite projective dimension, and therefore $A_{\mathfrak{m}_0}$ has infinite global dimension.*

Proof. By [B8, Lemmas 6.1 and 6.2], there are monomials $g, h \in S$ such that for each $n \geq 1$,

$$h^n \notin R \quad \text{and} \quad gh^n \in \mathfrak{m}_0 \subset R.$$

In particular, there is a vertex $i \in Q_0$ such that for each $n \geq 1$,

$$h^n \notin \bar{\tau}_\psi(e_i A e_i).$$

Let s_n be the cycle in $e_i A e_i$ satisfying $\bar{s}_n = gh^n$. Consider a projective resolution of $A_{\mathfrak{m}_0}/\mathfrak{m}_0$ over $A_{\mathfrak{m}_0}$,

$$\cdots \to P_1 \longrightarrow A_{\mathfrak{m}_0} \stackrel{\cdot 1}{\longrightarrow} A_{\mathfrak{m}_0}/\mathfrak{m}_0 \to 0.$$

Each s_n is in the zeroth syzygy module $\ker(\cdot 1) = \operatorname{ann}_{A_{\mathfrak{m}_0}}(A_{\mathfrak{m}_0}/\mathfrak{m}_0)$. Thus $\ker(\cdot 1)$ is not finitely generated over $A_{\mathfrak{m}_0}$ since $h^n \notin \bar{\tau}_\psi(e_i A e_i)$. Furthermore, the cycles s_n are pairwise commuting, and in particular there are an infinite number of independent commutation relations between them. It follows that $\operatorname{pd}_{A_{\mathfrak{m}_0}}(A_{\mathfrak{m}_0}/\mathfrak{m}_0) = \infty$. □

Lemma 6.2. *Let V be a simple $A_{\mathfrak{q}}$-module, and let $i \in Q_0$. Then*

$$\dim_k e_i V \leq 1.$$

Proof. Suppose V is a simple $A_{\mathfrak{q}}$-module. Then $e_i V$ is a simple $e_i A_{\mathfrak{q}} e_i$-module. Furthermore, the corner ring $e_i A_{\mathfrak{q}} e_i \cong \bar{\tau}_\psi(e_i A_{\mathfrak{q}} e_i) \subset B$ is a commutative k-algebra and k is algebraically closed. Therefore $\dim_k e_i V \leq 1$ by Schur's lemma. □

Lemma 6.3. *Let V be a simple $A_{\mathfrak{q}}$-module, and let $i \in Q_0$ be a vertex for which $e_i V \neq 0$. Suppose $s \in e_i A_{\mathfrak{q}} e_i$. Then $sV = 0$ if and only if $\bar{s} \in \mathfrak{q}$. Consequently, $\operatorname{ann}_R V = \mathfrak{m}_0$.*

Proof. (i) Suppose $s \in e_i A e_i$ satisfies $\bar{s} \in \mathfrak{q}$. We claim that $sV = 0$.

Indeed, let $v \in e_iV$ be nonzero. Then $\dim_k e_iV = 1$ by Lemma 6.2. Thus there is some $c \in k$ such that $(s - ce_i)e_iV = 0$. Assume to the contrary that c is nonzero. Then $\bar{s} - c \in S \setminus \mathfrak{q}$. Therefore

$$v = \frac{s - ce_i}{\bar{s} - c}\, v = \frac{1}{\bar{s} - c}\,(s - ce_i)v = 0,$$

contrary to our choice of v.

(ii) Conversely, suppose $s \in e_iAe_i$ satisfies $sV = 0$. Assume to the contrary that $\bar{s} \notin \mathfrak{q}$; then $\bar{s}^{-1} \in S_{\mathfrak{q}}$. Whence

$$e_iV = \frac{s}{\bar{s}}\, e_iV = \frac{1}{\bar{s}}\, sV = 0,$$

contrary to our choice of vertex i. $\square$

Definition 6.4. Let A be a ring with a complete set of orthogonal idempotents $\{e_1, \ldots, e_d\}$. We say an element $p \in e_jAe_i$ is *vertex invertible* if there is an element $p^* \in e_iAe_j$ such that

$$p^*p = e_i \quad \text{and} \quad pp^* = e_j.$$

Denote by $(e_jAe_i)^\circ$ the set of vertex invertible elements in e_jAe_i.

For an arrow $a \in Q_1^{\mathrm{t}}$, denote by δ_a the unique arrow with $\mathrm{h}(\delta_a) = \mathrm{t}(a)$; in particular, $\delta_a \in Q_1^*$.

Lemma 6.5. *A path $p \in A$ is vertex invertible in $A_{\mathfrak{q}}$ if and only if $x_D \nmid \bar{p}$ and the leftmost arrow subpath of p is not an arrow $\delta_a \in Q_1^*$ for which $x_D \mid \bar{a}$.*

Proof. (i) First suppose $x_D \mid \bar{p}$. Assume to the contrary that p has vertex inverse p^*. Then

$$p^* = \sum_{j=1}^{m} s_j^{-1} p_j \tag{7}$$

for some $s_j \in S \setminus \mathfrak{q}$ and $p_j \in e_{\mathrm{t}(p)}Ae_{\mathrm{h}(p)}$. In particular,

$$1 = \overline{pp^*} = \bar{p} \sum_j s_j^{-1} \bar{p}_j.$$

Whence

$$s_1 \cdots s_m = \bar{p} \sum_j (s_1 \cdots \hat{s}_j \cdots s_m)\, \bar{p}_j \in B.$$

Thus $x_D \mid s_1 \cdots s_m$ since $x_D \mid \bar{p}$. Therefore $x_D \mid s_j$ for some j. But then $s_j \in \mathfrak{q}$, a contradiction to our choice of s_j.

(ii) Now suppose the leftmost arrow subpath of p is an arrow $\delta_a \in Q_1^*$ for which $x_D \mid \bar{a}$. If p is a cycle, then a is the rightmost arrow subpath of p. Whence $x_D \mid \bar{p}$. Thus p is not vertex invertible by Claim (i).

So suppose p is not a cycle, and assume to the contrary that p has vertex inverse p^* given by (7). Since p is not a cycle, we have $\mathrm{h}(p) \neq \mathrm{t}(p)$. Thus each $p_j \in e_{\mathrm{t}(p)}Ae_{\mathrm{h}(p)}$ is a k-linear combination of nontrivial paths with tails at $\mathrm{h}(p)$. But since $\deg^+ \mathrm{h}(p) = 1$, each nontrivial path $q \in A$ with tail at $\mathrm{h}(p)$ satisfies $x_D \mid \bar{q}$. Therefore x_D divides each $\bar{p}_j$ (in B). Furthermore, x_D does not divide any s_j since $s_j \in S \setminus \mathfrak{q}$. Whence $x_D \mid \overline{p^*}$ in $BS_{\mathfrak{q}}$. Thus $x_D \mid \overline{p^*p}$ in $BS_{\mathfrak{q}}$, since $\bar{p} \in B$. Therefore $x_D \mid 1$ in $BS_{\mathfrak{q}}$. But then x_D is invertible in $BS_{\mathfrak{q}}$, a contradiction.

(iii) Finally suppose $x_D \nmid \bar{p}$, and the leftmost arrow subpath of $\bar{p}$ is not an arrow $\delta_a \in Q_1^*$ for which $x_D \mid \bar{a}$. Then there is a path $q \in e_{\mathrm{t}(p)} A e_{\mathrm{h}(p)}$ satisfying $x_D \nmid \bar{q}$, by Lemma 5.2. Whence pq is a cycle satisfying $x_D \nmid \overline{pq}$; that is, $\overline{pq} \in S \setminus \mathfrak{q}$. Furthermore, q has a vertex subpath i for which $e_i A e_i = S e_i$, by Lemma 4.1.4. Thus

$$p^* := q(\overline{pq})^{-1}$$

is in $A_{\mathfrak{q}}$. But then

$$p^*p = \frac{q}{\overline{pq}}\, p = \frac{\overline{qp}}{\overline{pq}}\, e_{\mathrm{t}(p)} = e_{\mathrm{t}(p)} \qquad \text{and} \qquad pp^* = p\, \frac{q}{\overline{pq}} = e_{\mathrm{h}(p)}\, \frac{\overline{pq}}{\overline{pq}} = e_{\mathrm{h}(p)}.$$

Therefore p is vertex invertible in $A_{\mathfrak{q}}$. □

Lemma 6.6. *Let V be a simple $A_{\mathfrak{q}}$-module.*

(1) If $a \in Q_1 \setminus Q_1^*$ satisfies $x_D \mid \bar{a}$, then $aV = 0$.
(2) If $\delta_a \in Q_1^*$ satisfies $x_D \mid \bar{a}$, then $\delta_a V = 0$.

Proof. Let $a \in Q_1$ be an arrow for which $x_D \mid \bar{a}$.

(i) First suppose $a \in Q_1 \setminus (Q_1^* \cup Q_1^{\mathrm{t}})$. We claim that $aV = 0$. Since $a \in Q_1 \setminus (Q_1^* \cup Q_1^{\mathrm{t}})$, there are paths

$$s \in e_{\mathrm{h}(a)} A e_{\mathrm{t}(a)} \quad \text{and} \quad t \in e_{\mathrm{t}(a)} A e_{\mathrm{h}(a)}$$

such that $x_D \nmid \bar{s}$ and $x_D \nmid \bar{t}$, by Lemma 5.2. In particular, $x_D \nmid \overline{st}$. Whence

$$\overline{st} \in S \setminus \mathfrak{q}.$$

Thus

$$a = \frac{st}{\overline{st}}\, a = \frac{s}{\overline{st}}\, ta \in A_{\mathfrak{q}} \mathfrak{q} e_{\mathrm{t}(a)}.$$

But $ta \in \mathfrak{q} e_{\mathrm{t}(a)} \cap e_{\mathrm{t}(a)} A e_{\mathrm{t}(a)}$. Therefore a annihilates V by Lemma 6.3.

(ii) Now suppose $a \in Q_1^{\mathrm{t}}$. Set $\delta := \delta_a \in Q_1^*$.

(ii.a) We first claim that $a\delta V = 0$. By assumption (B), $\deg^+ \mathrm{t}(\delta) \geq 2$ and $\deg^+ \mathrm{h}(a) \geq 2$. Thus there are paths

$$s \in e_{\mathrm{h}(a)} A e_{\mathrm{t}(\delta)} \quad \text{and} \quad t \in e_{\mathrm{t}(\delta)} A e_{\mathrm{h}(a)}$$

such that $x_D \nmid \bar{s}$ and $x_D \nmid \bar{t}$, by Lemma 5.2. Whence

$$\overline{st} \in S \setminus \mathfrak{q}.$$

Thus

$$a\delta = \frac{st}{\overline{st}}\, a\delta = \frac{s}{\overline{st}}\, ta\delta \in A_{\mathfrak{q}} \mathfrak{q} e_{\mathrm{t}(\delta)}.$$

Therefore $a\delta$ annihilates V by Lemma 6.3.

(ii.b) We claim that $aV = 0$. If $e_{\mathrm{t}(a)}V = 0$, then $aV = 0$, so suppose there is some nonzero $v \in e_{\mathrm{t}(a)}V$. Assume to the contrary that $av \neq 0$. Then, since V is simple and $\deg^+ \mathrm{t}(a) = 1$, there is some $p \in A_{\mathfrak{q}}$ such that

$$w := \delta pav \in e_{\mathrm{t}(a)}V$$

is nonzero. By Claim (ii.a), $aw = (a\delta)(pav) = 0$. Furthermore, $\dim_k e_{\mathrm{t}(a)}V = 1$ by Lemma 6.2. Thus, since $v, w \in e_{\mathrm{t}(a)}V$ are both nonzero, there is some $c \in k^*$ such that $cw = v$. But then

$$0 \neq av = acw = c(aw) = 0,$$

which is not possible.

(ii.c) Finally, we claim that $\delta V = 0$. Assume to the contrary that there is some $v \in e_{\mathrm{t}(\delta)}V$ such that $\delta v \neq 0$. By Claim (2.i), $a\delta v = 0$. But again a is the only arrow with tail at $\mathrm{t}(a)$, and δ is not vertex invertible by Lemma 6.5. Therefore V is not simple, a contradiction. □

For each $\mathfrak{q}_D \in \operatorname{Spec} S$ minimal over $\mathfrak{m}_0$, set

$$\epsilon_D := 1_A - \sum_{a \in Q_1^{\mathrm{t}} \,:\, x_D \mid \bar{a}} e_{\mathrm{t}(a)}.$$

Theorem 6.7. *Let $\mathfrak{q} = \mathfrak{q}_D \in \operatorname{Spec} S$ be minimal over $\mathfrak{m}_0 \in \operatorname{Max} R$. Suppose there are n arrows $a_1, \ldots, a_n \in Q_1^{\mathrm{t}}$ such that $x_D \mid \bar{a}_\ell$. Then there are precisely $n+1$ non-isomorphic simple $A_{\mathfrak{q}}$-modules:*

$$V_0 := A_{\mathfrak{q}}\epsilon_D / A_{\mathfrak{q}}\mathfrak{q}\epsilon_D \cong (S_{\mathfrak{q}}/\mathfrak{q})\,\epsilon_D, \tag{8}$$

and for each $1 \leq \ell \leq n$, a vertex simple

$$V_\ell := ke_{\mathrm{t}(a_\ell)} \cong (R_{\mathfrak{m}_0}/\mathfrak{m}_0)\,e_{\mathrm{t}(a_\ell)}. \tag{9}$$

Proof. Let V be a simple $A_{\mathfrak{q}}$-module. Let $a \in Q_1^{\mathrm{t}}$ be such that $x_D \mid \bar{a}$. Then either V is the vertex simple $V = ke_{\mathrm{t}(a)}$, or $e_{\mathrm{t}(a)}$ annihilates V, by Lemma 6.6.

So suppose $e_{\mathrm{t}(a)}V = 0$ for each $a \in Q_1^{\mathrm{t}}$ satisfying $x_D \mid \bar{a}$. We want to show that the sequence of left $A_{\mathfrak{q}}$-modules

$$0 \to A_{\mathfrak{q}}\mathfrak{q}\epsilon_D \longrightarrow A_{\mathfrak{q}}\epsilon_D \xrightarrow{g} V \to 0$$

is exact.

We first claim that g is onto. Indeed, since $V \neq 0$, there is a vertex summand e_i of ϵ_D for which $e_iV \neq 0$. Let e_j be an arbitrary vertex summand of ϵ_D. Then there is a path $p \in e_jAe_i$ satisfying $x_D \nmid \bar{p}$, by Lemma 5.2. Thus, since e_j is a summand of ϵ_D, p is vertex invertible by Lemma 6.5. Whence $e_jV \neq 0$ since $e_iV \neq 0$. Therefore g is onto by Lemma 6.2.

We now claim that the kernel of g is $A_{\mathfrak{q}}\mathfrak{q}\epsilon_D$. Let $b \in \epsilon_D A \epsilon_D$ be an arrow satisfying $bV = 0$. Then there is a path $p \in e_{\mathrm{t}(b)}Ae_{\mathrm{h}(b)}$ satisfying $x_D \nmid \bar{p}$, by Lemma 5.2. Thus, since $e_{\mathrm{t}(b)}$ and $e_{\mathrm{h}(b)}$ are vertex summands of ϵ_D, p is vertex invertible in $A_{\mathfrak{q}}$ by Lemma 6.5. Whence

$$b = (p^*p)b = p^*(pb) \in A_{\mathfrak{q}}\mathfrak{q}\epsilon_D.$$

Thus the $A_{\mathfrak{q}}\epsilon_D$-annihilator of V is $A_{\mathfrak{q}}\mathfrak{q}\epsilon_D$, by Lemma 6.2.

Therefore $V = V_0$. The simple modules $V_0, \ldots, V_n$ exhaust the possible simple $A_{\mathfrak{q}}$-modules, again by Lemma 6.2. □

If $p \in A_{\mathfrak{q}}$ is a concatenation of paths and vertex inverses of paths in A, then we call p a *path*.

Lemma 6.8. *Suppose $i \in Q_0$ satisfies $e_i\epsilon_D \neq 0$. Then for each $j \in Q_0$, the corner rings $e_jA_{\mathfrak{q}}e_i$ and $e_iA_{\mathfrak{q}}e_j$ are cyclic free $S_{\mathfrak{q}}$-modules. Consquently, $A_{\mathfrak{q}}e_i$ and $e_iA_{\mathfrak{q}}$ are free $S_{\mathfrak{q}}$-modules.*

Proof. Suppose e_i is a vertex summand of ϵ_D. Then either $e_iAe_i = Se_i$, or $i = \mathrm{t}(a)$ for some $a \in Q_1^{\mathrm{t}}$ with $x_D \nmid \bar{a}$, by Lemma 4.1.4. In the latter case, a is vertex invertible by Lemma 6.5, and $e_{\mathrm{h}(a)}Ae_{\mathrm{h}(a)} = Se_{\mathrm{h}(a)}$ by Lemma 4.1.4. Thus in either case we have

$$e_iA_{\mathfrak{q}}e_i = S_{\mathfrak{q}}e_i.$$

Therefore $A_{\mathfrak{q}}e_i$ and $e_iA_{\mathfrak{q}}$ are $S_{\mathfrak{q}}$-modules.

(i) We claim that for each $j \in Q_0$, $e_jA_{\mathfrak{q}}e_i$ is generated as an $S_{\mathfrak{q}}$-module by a single path; a similar argument holds for $e_iA_{\mathfrak{q}}e_j$.

(i.a) First suppose j is not the tail of an arrow $a \in Q_1^{\mathrm{t}}$ for which $x_D \mid \bar{a}$. Since $D \in \mathcal{S}'$ is a simple matching of Q', there is path s from i to j for which $x_D \nmid \bar{s}$ (that is, $\psi(s)$ is supported on $Q' \setminus D$). Thus s has a vertex inverse $s^* \in e_iA_{\mathfrak{q}}e_j$, by Lemma 6.5.

Let $t \in e_jA_{\mathfrak{q}}e_i$ be arbitrary. Then s^*t is in $e_iA_{\mathfrak{q}}e_i = S_{\mathfrak{q}}e_i$. Whence

$$t = ss^*t \in sS_{\mathfrak{q}}.$$

Therefore $e_jA_{\mathfrak{q}}e_i = sS_{\mathfrak{q}}$.

(i.b) Now suppose j is the tail of an arrow $a \in Q_1^{\mathrm{t}}$ for which $x_D \mid \bar{a}$; in particular, $j \neq i$. Since $D \in \mathcal{S}'$ is a simple matching of Q', there is path s from i to $\mathrm{t}(\delta_a)$ for which $x_D \nmid \bar{s}$. Thus s has a vertex inverse $s^* \in e_iA_{\mathfrak{q}}e_{\mathrm{t}(\delta_a)}$, again by Lemma 6.5.

Let $t \in e_jA_{\mathfrak{q}}e_i$ be arbitrary. Since $j \neq i$ and $\deg^+ j = 1$, there is some $r \in e_{\mathrm{t}(\delta_a)}A_{\mathfrak{q}}e_i$ satisfying $t = \delta_a r$. Whence

$$t = \delta_a r = \delta_a ss^*r \in \delta_a sS_{\mathfrak{q}}.$$

Therefore $e_jA_{\mathfrak{q}}e_i = \delta_a sS_{\mathfrak{q}}$.

(ii) Finally, we claim that $e_jA_{\mathfrak{q}}e_i$ is a free $S_{\mathfrak{q}}$-module; a similar argument holds for $e_iA_{\mathfrak{q}}e_j$. By Claim (i), there is a path s such that

$$e_jA_{\mathfrak{q}}e_i = sS_{\mathfrak{q}}.$$

Furthermore, the $S_{\mathfrak{q}}$-module homomorphism

$$S_{\mathfrak{q}} \to sS_{\mathfrak{q}}, \quad t \mapsto st,$$

is an isomorphism since $S_{\mathfrak{q}}$ and $\bar{s}$ belong to the domain $\operatorname{Frac} B$, and $\bar{\tau}_\psi$ is injective. $\square$

Lemma 6.9. *The $A_{\mathfrak{q}}$-module V_0 satisfies*

$$\operatorname{pd}_{A_{\mathfrak{q}}}(V_0) \le \operatorname{pd}_{S_{\mathfrak{q}}}(S_{\mathfrak{q}}/\mathfrak{q}).$$

Proof. Consider a minimal free resolution of $S_{\mathfrak{q}}/\mathfrak{q}$ over $S_{\mathfrak{q}}$,

$$\cdots \to S_{\mathfrak{q}}^{\oplus n_1} \to S_{\mathfrak{q}} \to S_{\mathfrak{q}}/\mathfrak{q} \to 0.$$

Set $\epsilon := \epsilon_D$. By Lemma 6.8, $A_{\mathfrak{q}}\epsilon$ is a free $S_{\mathfrak{q}}$-module. Thus $A_{\mathfrak{q}}\epsilon$ is a flat $S_{\mathfrak{q}}$-module, that is, the functor $A_{\mathfrak{q}}\epsilon \otimes_{S_{\mathfrak{q}}} -$ is exact. Therefore the sequence of left $A_{\mathfrak{q}}$-modules

$$\cdots \to A_{\mathfrak{q}}\epsilon \otimes S_{\mathfrak{q}}^{\oplus n_1} \to A_{\mathfrak{q}}\epsilon \otimes S_{\mathfrak{q}} \to A_{\mathfrak{q}}\epsilon \otimes S_{\mathfrak{q}}/\mathfrak{q} \to 0 \tag{10}$$

is exact. Each term is a projective $A_{\mathfrak{q}}$-module since

$$A_{\mathfrak{q}}\epsilon \otimes_{S_{\mathfrak{q}}} \left(S_{\mathfrak{q}}^{\oplus n_i}\right) \cong (A_{\mathfrak{q}}\epsilon)^{\oplus n_i}.$$

Furthermore, there is a left $A_{\mathfrak{q}}$-module isomorphism

$$V_0 = A_{\mathfrak{q}}\epsilon / A_{\mathfrak{q}}\mathfrak{q}\epsilon \cong A_{\mathfrak{q}}\epsilon \otimes_{S_{\mathfrak{q}}} S_{\mathfrak{q}}/\mathfrak{q}.$$

Therefore (10) is a projective resolution of V_0 over $A_{\mathfrak{q}}$ of length at most $\operatorname{pd}_{S_{\mathfrak{q}}}(S_{\mathfrak{q}}/\mathfrak{q})$. $\square$

Lemma 6.10. *The local ring $S_{\mathfrak{q}}$ is regular.*

Proof. S is normal since S is isomorphic to the center of the (noetherian) NCCR A'. In particular, the singular locus of $\operatorname{Max} S$ has codimension at least 2. Furthermore, the zero locus $\mathcal{Z}(\mathfrak{q})$ in $\operatorname{Max} S$ has codimension 1, by Theorem 5.7.1. Therefore $\mathcal{Z}(\mathfrak{q})$ contains a smooth point of $\operatorname{Max} S$. $\square$

Proposition 6.11. *Let $\mathfrak{q} \in \operatorname{Spec} S$ be minimal over $\mathfrak{m}_0$. Then each simple $A_{\mathfrak{q}}$-module has projective dimension 1. Consequently, for each simple $A_{\mathfrak{q}}$-module V,*

$$\operatorname{pd}_{A_{\mathfrak{q}}}(V) = \operatorname{ht}_S(\mathfrak{q}).$$

Proof. Recall the classification of simple $A_{\mathfrak{q}}$-modules given in Theorem 6.7.

(i) Let V_0 be the simple $A_{\mathfrak{q}}$-module defined in (8). Then

$$1 \overset{\text{(I)}}{\le} \operatorname{pd}_{A_{\mathfrak{q}}}(V_0) \overset{\text{(II)}}{\le} \operatorname{pd}_{S_{\mathfrak{q}}}(S_{\mathfrak{q}}/\mathfrak{q}) \overset{\text{(III)}}{=} \operatorname{ht}_S(\mathfrak{q}) \overset{\text{(IV)}}{=} 1.$$

Indeed, (I) holds since V_0 is clearly not a direct summand of a free $A_{\mathfrak{q}}$-module; (II) holds by Lemma 6.9; (III) holds by Lemma 6.10; and (IV) holds by Theorem 5.7.1.

(ii) Fix $1 \le \ell \le n$, and let V_ℓ be the vertex simple $A_{\mathfrak{q}}$-module defined in (9). Set $a := a_\ell$. We claim that V_ℓ has minimal projective resolution

$$0 \to A_{\mathfrak{q}}e_{\mathrm{h}(a)} \xrightarrow{\cdot a} A_{\mathfrak{q}}e_{\mathrm{t}(a)} \xrightarrow{\cdot 1} ke_{\mathrm{t}(a)} = V_\ell \to 0. \tag{11}$$

(ii.a) We first claim that $\cdot a$ is injective. Suppose $b \in A_{\mathfrak{q}}e_{\mathrm{h}(a)}$ is nonzero. Then $\bar{\tau}_\psi(ba) = \bar{b} \cdot \bar{a} \neq 0$ since B is an integral domain. Whence $ba \neq 0$ since $\bar{\tau}_\psi$ is injective. Therefore $\cdot a$ is injective.

(ii.b) We now claim that $\operatorname{im}(\cdot a) = \ker(\cdot 1)$. Since $aV = 0$, we have $\operatorname{im}(\cdot a) \subseteq \ker(\cdot 1)$. To show the reverse inclusion, suppose $g \in \ker(\cdot 1)$; then $gV = 0$. We may write

$$g = \sum_j s_j^{-1} p_j,$$

where each $p_j \in Ae_{\operatorname{t}(a)}$ is a path and $s_j \in S \setminus \mathfrak{q}$. If p_j is nontrivial, then $p_j = p_j' a$ for some path p_j' since $\deg^+ \operatorname{t}(a) = 1$. Whence

$$p_j V_\ell = p_j' a V_\ell = 0.$$

It thus suffices to suppose that each p_j is trivial, $p_j = e_{\operatorname{t}(a)}$. But then $g = s^{-1} e_{\operatorname{t}(a)}$ for some $s \in S \setminus \mathfrak{q}$. Therefore

$$e_{\operatorname{t}(a)} V_\ell = sgV_\ell = 0,$$

a contradiction.

(ii.c) Finally, (11) is minimal since V_ℓ is clearly not a direct summand of a free $A_{\mathfrak{q}}$-module. □

Lemmas 6.12, 6.14, and Proposition 6.13 are not specific to homotopy algebras.

Lemma 6.12. *Suppose S is a depiction of R. Let $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathfrak{q} \in \iota_{S/R}^{-1}(\mathfrak{p})$. If $\operatorname{ht}_S(\mathfrak{q}) = 1$, then $\operatorname{ght}_R(\mathfrak{p}) = 1$.*

Proof. Assume to the contrary that $\operatorname{ght}_R(\mathfrak{p}) = 0$. Then there is a depiction S' of R and a prime ideal $\mathfrak{q}' \in \iota_{S'/R}^{-1}(\mathfrak{p})$ such that $\operatorname{ht}_{S'}(\mathfrak{q}') = 0$. Whence $\mathfrak{q}' = 0$ since S' is an integral domain. But then $\mathfrak{q}' \cap R = 0 \neq \mathfrak{q} \cap R = \mathfrak{p}$, a contradiction. Therefore

$$\operatorname{ht}_S(\mathfrak{q}) = 1 \leq \operatorname{ght}_R(\mathfrak{p}) \leq \operatorname{ht}_S(\mathfrak{q}).$$

□

Recall that an ideal I of an integral domain S is a projective S-module if and only if I is invertible, i.e., there is a fractional ideal J such that $IJ = S$. In this case, I is a finitely generated rank one S-module [C, Theorem 19.10].

Proposition 6.13. *Let B be an integral domain, and let $A = [A^{ij}] \subset M_d(B)$ be a tiled matrix ring with cycle algebra S. Set $Q_0 := \{1, \ldots, d\}$. Suppose that*

(1) S is a regular local ring.
(2) There is some $i \in Q_0$ such that
  (a) $A^i = S$;
  (b) for each $j \in Q_0$, A^{ij} is an invertible ideal of S; and
  (c) for each $j \in Q_0$, either $(e_i A e_j)^\circ \neq \emptyset$, or there is some $\ell \in Q_0$ and $b \in e_j A e_\ell$ satisfying

$$e_j A = bA \oplus k e_j \quad \text{and} \quad (e_i A e_\ell)^\circ \neq \emptyset.$$

Then

$$\operatorname{gldim} A \leq \dim S.$$

Proof. Suppose the hypotheses hold, and set $n := \dim S$. Let V be a left A-module. We claim that

$$\operatorname{pd}_A(V) \leq n.$$

It suffices to show that there is a projective resolution $P_\bullet$ of V,

$$\cdots \longrightarrow P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} V \to 0,$$

for which $\ker \delta_{n-1}$ is a projective A-module [R, Proposition 8.6.iv].

(i) We first claim that there is a projective resolution $P_\bullet$ of V so that for each $\alpha \geq 1$,

$$\ker \delta_\alpha = Ae_i \ker \delta_\alpha. \tag{12}$$

Indeed, fix $j \in Q_0$, and recall assumption (2.c). If $p \in (e_iAe_j)^\circ$, then

$$e_j \ker \delta_\alpha = p^*p \ker \delta_\alpha = p^* e_i p \ker \delta_\alpha \subseteq Ae_i \ker \delta_\alpha.$$

Otherwise there is some $\ell \in Q_0$ and $b \in e_jAe_\ell$ such that $e_jA = bA \oplus ke_j$ and $(e_iAe_\ell)^\circ \neq \emptyset$. Let $p \in (e_iAe_\ell)^\circ$. Since the sum $e_jA = bA \oplus ke_j$ is direct, we may choose $P_\bullet$ so that for each $\alpha \geq 1$,

$$\delta_\alpha \mid_{e_jP_\alpha} = b \cdot \delta_\alpha \mid_{e_\ell P_\alpha} .$$

Furthermore, for nonzero $q \in e_\ell A$, $bq \neq 0$ since B is an integral domain. Thus

$$e_j \ker \delta_\alpha = b \ker \delta_\alpha.$$

Whence

$$e_j \ker \delta_\alpha = b \ker \delta_\alpha = bp^* e_i p \ker \delta_\alpha \subseteq Ae_i \ker \delta_\alpha.$$

Therefore in either case,

$$e_j \ker \delta_\alpha \subseteq Ae_i \ker \delta_\alpha.$$

(ii) Fix a projective resolution $P_\bullet$ of V satisfying (12). We claim that the left A-module $Ae_i \ker \delta_{n-1}$ is projective.

The right A-module e_iA is projective, hence flat. Thus, setting $\otimes := \otimes_A$, the complex of S-modules

$$\cdots \longrightarrow e_iA \otimes P_2 \xrightarrow{1\otimes\delta_2} e_iA \otimes P_1 \xrightarrow{1\otimes\delta_1} e_iA \otimes P_0 \xrightarrow{1\otimes\delta_0} e_iA \otimes V \to 0 \tag{13}$$

is exact. Each term $e_iA \otimes P_\ell$ is a free S-module since

$$e_iA \otimes P_\ell \cong e_iA \otimes \bigoplus_j (Ae_j)^{\oplus n_j} \cong \bigoplus_j (e_iA \otimes Ae_j)^{\oplus n_j}$$

$$\cong \bigoplus_j (e_iAe_j)^{\oplus n_j} \cong \bigoplus_j (A^{ij})^{\oplus n_j} \overset{(\text{I})}{\cong} \bigoplus_j S^{\oplus n_j},$$

where (I) holds by assumption (2.b). Furthermore, $e_iA \otimes V$ is an S-module since $e_iAe_i \cong S$ by assumption (2.a). Therefore (13) is a free resolution of an S-module.

But $\operatorname{gldim} S = \dim S = n$ by assumption (1). Therefore the nth syzygy module of (13) is a free S-module,

$$\ker(1 \otimes \delta_{n-1}) \cong S^{\oplus m}.$$

Since e_iA is a flat right A-module, the sequence

$$0 \to e_iA \otimes \ker \delta_{n-1} \longrightarrow e_iA \otimes P_{n-1} \stackrel{1\otimes\delta_{n-1}}{\longrightarrow} e_iA \otimes P_{n-2}$$

is exact. Whence

$$e_iA \otimes \ker \delta_{n-1} \cong \ker(1 \otimes \delta_{n-1}) \cong S^{\oplus m}.$$

Therefore

$$Ae_i \ker \delta_{n-1} \cong Ae_iA \otimes \ker \delta_{n-1} \cong Ae_iS^{\oplus m} \stackrel{(\text{I})}{\cong} A(e_iAe_i)^{\oplus m} \cong (Ae_i)^{\oplus m},$$

where (I) holds by assumption (2.a), proving our claim.

(iii) Finally, $\ker \delta_{n-1}$ is a projective left A-module by Claims (i) and (ii). Therefore ${}_AV$ has projective dimension at most n. □

Lemma 6.14. *Suppose S is a noetherian integral domain and a k-algebra, and R is a subalgebra of S. Let $\mathfrak{p} \in \operatorname{Spec} R$. If $\mathfrak{t} \in \operatorname{Spec}(SR_{\mathfrak{p}})$ is a minimal prime over $\mathfrak{p}R_{\mathfrak{p}}$, then the ideal $\mathfrak{t} \cap S \in \operatorname{Spec} S$ is a minimal prime over $\mathfrak{p}$.*

Proof. Suppose that $\mathfrak{t} \cap S$ is not a minimal prime over $\mathfrak{p}$. We want to show that $\mathfrak{t}$ is not a minimal prime over $\mathfrak{p}R_{\mathfrak{p}}$. Since $\mathfrak{t} \cap S$ is not minimal, there is some $\mathfrak{q} \in \operatorname{Spec} S$, minimal over $\mathfrak{p}$, such that

$$\mathfrak{p} \subseteq \mathfrak{q} \subset \mathfrak{t} \cap S. \tag{14}$$

(i) We claim that $\mathfrak{q} \cap R = \mathfrak{p}$. Assume to the contrary that there is some $a \in (\mathfrak{t} \cap R) \setminus \mathfrak{p}$. Then $a^{-1} \in R_{\mathfrak{p}}$. Whence $1 = aa^{-1} \in \mathfrak{t}SR_{\mathfrak{p}} = \mathfrak{t}$, contrary to the fact that $\mathfrak{t}$ is prime. Therefore

$$\mathfrak{t} \cap R \subseteq \mathfrak{p}. \tag{15}$$

Consequently,

$$\mathfrak{p} \subseteq \mathfrak{q} \cap R \stackrel{(\text{I})}{\subseteq} \mathfrak{t} \cap R \stackrel{(\text{II})}{\subseteq} \mathfrak{p},$$

where (I) holds by (14) and (II) holds by (15). Thus $\mathfrak{q} \cap R = \mathfrak{p}$, proving our claim.

(ii) Now fix $a \in (\mathfrak{t} \cap S) \setminus \mathfrak{q}$, and assume to the contrary that $a \in \mathfrak{q}R_{\mathfrak{p}}$. Then there is some $b \in \mathfrak{q}$ and $c \in R \setminus \mathfrak{p}$ such that $a = bc^{-1}$. In particular, $ac = b \in \mathfrak{q}$. Whence $c \in \mathfrak{q}$ since $c \in R \subseteq S$ and $\mathfrak{q}$ is prime. Thus

$$c \in \mathfrak{q} \cap R \stackrel{(\text{I})}{=} \mathfrak{p},$$

where (I) holds by Claim (i). But $c \notin \mathfrak{p}$, a contradiction. Whence $a \in \mathfrak{t} \setminus \mathfrak{q}R_{\mathfrak{p}}$. Thus

$$\mathfrak{p}R_{\mathfrak{p}} \subseteq \mathfrak{q}R_{\mathfrak{p}} \subset \mathfrak{t}.$$

Furthermore, $\mathfrak{q}R_{\mathfrak{p}}$ is a prime ideal of $SR_{\mathfrak{p}}$. Therefore $\mathfrak{t}$ is a not a minimal prime over $\mathfrak{p}$. □

Again let A be a nonnoetherian homotopy algebra satisfying assumptions (A) and (B). Recall that the center and cycle algebra of $A_{\mathfrak{m}_0} := A \otimes_R R_{\mathfrak{m}_0}$ are isomorphic to $R_{\mathfrak{m}_0}$ and $SR_{\mathfrak{m}_0}$ respectively.

Theorem 6.15. *$A_{\mathfrak{m}_0}$ is a noncommutative desingularization of its center. Furthermore, for each $\mathfrak{t} \in \operatorname{Spec}(SR_{\mathfrak{m}_0})$ minimal over $\mathfrak{t} \cap R_{\mathfrak{m}_0}$,*

$$\operatorname{gldim} A_{\mathfrak{t}} = \dim(SR_{\mathfrak{m}_0})_{\mathfrak{t}} = \dim S_{\mathfrak{t} \cap S}.$$

Proof. By Lemma 6.14 (with $\mathfrak{p} = \mathfrak{m}_0$), it suffices to consider prime ideals $\mathfrak{q} \in \operatorname{Spec} S$ which are minimal over $\mathfrak{m}_0$.

(i) $A_{\mathfrak{m}_0}$ *is cycle regular.* Let $\mathfrak{q} \in \operatorname{Spec} S$ be minimal over $\mathfrak{m}_0$, and let V be a simple $A_{\mathfrak{q}}$-module. The hypotheses of Proposition 6.13 hold: condition (1) holds by Lemma 6.10; (2.a) holds by Lemma 4.1.4; (2.b) holds by Lemma 6.8; and (2.c) holds by Lemma 6.5. Thus

$$1 \overset{\text{(I)}}{\leq} \operatorname{gldim} A_{\mathfrak{q}} \overset{\text{(II)}}{\leq} \dim S_{\mathfrak{q}} = \operatorname{ht}_S(\mathfrak{q}) \overset{\text{(III)}}{=} 1 \overset{\text{(IV)}}{=} \operatorname{ght}_R(\mathfrak{m}_0) \overset{\text{(V)}}{=} \operatorname{pd}_{A_{\mathfrak{q}}}(V).$$

Indeed, (I) and (V) hold by Proposition 6.11; (II) holds by Proposition 6.13; (III) holds by Theorem 5.7.3; and (IV) holds by Lemma 6.12. Therefore $A_{\mathfrak{m}_0}$ is cycle regular.

(ii) $A_{\mathfrak{m}_0}$ *is a noncommutative desingularization.* By [B3, Corollary 2.14.1], the (noncommutative) function fields of A and R, and hence $A_{\mathfrak{m}_0}$ and $R_{\mathfrak{m}_0}$, are Morita equivalent,

$$A \otimes_R \operatorname{Frac} R \sim \operatorname{Frac} R.$$

(iii) Finally, suppose $\mathfrak{q} \in \operatorname{Spec} S$ is minimal over $\mathfrak{q} \cap R$. We claim that $\operatorname{gldim} A_{\mathfrak{q}} = \dim S_{\mathfrak{q}}$. By Theorem 5.7.2, either $\mathfrak{q} = \mathfrak{q}_D$ for some $D \in \mathcal{S}'$, or $\mathfrak{q} = 0$. The case $\mathfrak{q} = \mathfrak{q}_D$ was shown in Claim (i), so suppose $\mathfrak{q} = 0$.

We first claim that for each $i \in Q_0$,

$$e_i A_{\mathfrak{q}} e_i = (\operatorname{Frac} S) e_i. \tag{16}$$

Indeed, let $g \in \operatorname{Frac} S$ be arbitrary. Fix $j \in Q_0$ for which $e_j A e_j = S e_j$. Since S is a domain,

$$e_j A_{\mathfrak{q}} e_j = S_{\mathfrak{q}} e_j = (\operatorname{Frac} S) e_j. \tag{17}$$

Thus there is an element $s \in e_j A_{\mathfrak{q}} e_j$ satisfying $\bar{s} = g$.

Now fix a cycle $t_2 e_j t_1 \in e_i A_{\mathfrak{q}} e_i$ that passes through j. Then $t_1 t_2 \in e_j A_{\mathfrak{q}} e_j$ has a vertex inverse $(t_1 t_2)^*$ by (17). Thus the element

$$s' := t_2 (t_1 t_2)^* s t_1 \in e_i A_{\mathfrak{q}} e_i$$

satisfies $\bar{s}' = \bar{s} = g$. Therefore (16) holds.

We now claim that for each $i, j \in Q_0$, there is a $(\operatorname{Frac} S)$-module isomorphism[9]

$$e_j A_{\mathfrak{q}} e_i \cong \operatorname{Frac} S. \tag{18}$$

[9]In general, $\bar{\tau}_\psi(e_j A e_i)$ is not contained in $\operatorname{Frac} S$; otherwise (18) would trivially hold.

Let $s \in e_j A_{\mathfrak{q}} e_i$ be arbitrary, and fix a cycle $t_2 e_j t_1 \in e_i A_{\mathfrak{q}} e_i$ that passes through j. Then $t_1 t_2$ has a vertex inverse $(t_1 t_2)^*$ by (16). Furthermore, $st_2 \in e_j A_{\mathfrak{q}} e_j$. Thus

$$s = (t_1 t_2)^* s(t_2 t_1) \in (\operatorname{Frac} S)t_1.$$

Whence $e_j A_{\mathfrak{q}} e_i \subseteq (\operatorname{Frac} S)t_1$. Conversely, (16) implies $e_j A_{\mathfrak{q}} e_i \supseteq (\operatorname{Frac} S)t_1$. Thus

$$e_j A_{\mathfrak{q}} e_i = (\operatorname{Frac} S)t_1.$$

Furthermore, the $(\operatorname{Frac} S)$-module homomorphism

$$\operatorname{Frac} S \to (\operatorname{Frac} S)t_1, \quad s \mapsto st_1,$$

is an isomorphism since $\bar{t}_1$ and $\operatorname{Frac} S$ are in the domain $\operatorname{Frac} B$, and $\bar{\tau}_\psi$ is injective. Therefore (18) holds.

It follows from (16) and (18) that

$$A_{\mathfrak{q}} \cong M_d(\operatorname{Frac} S).$$

Thus $A_{\mathfrak{q}}$ is a semisimple algebra. Therefore

$$\operatorname{gldim} A_{\mathfrak{q}} = 0 = \dim(\operatorname{Frac} S) = \dim S_{\mathfrak{q}}.$$

□

7. Local endomorphism rings

Recall that A is a nonnoetherian homotopy algebra satisfying assumptions (A) and (B) given in Section 4, unless stated otherwise. For $a \in Q_1$, recall the ideal

$$\mathfrak{m}_a := \bar{\tau}_\psi(e_{\mathrm{t}(a)} A a) \subset S$$

from Proposition 5.5. Given a simple matching $D \in \mathcal{S}'$ for which $\mathfrak{q} := \mathfrak{q}_D$ is a minimal prime over $\mathfrak{m}_0$, set

$$\mathfrak{m}_D := \bigcap_{a \in Q_1^{\mathrm{t}} \,:\, x_D | \bar{a}} \mathfrak{m}_a \quad \text{and} \quad \tilde{R} := (k + \mathfrak{m}_D)_{\mathfrak{m}_D} + \mathfrak{q} S_{\mathfrak{q}}.$$

Lemma 7.1. *Let $D \in \mathcal{S}'$ be a simple matching for which $\mathfrak{q} := \mathfrak{q}_D$ is a minimal prime over $\mathfrak{m}_0$, and let $a \in Q_1$. If $x_D \mid \bar{a}$, then*

$$\mathfrak{m}_a S_{\mathfrak{q}} = \mathfrak{q} S_{\mathfrak{q}} = \sigma S_{\mathfrak{q}}.$$

We note that the relation $\mathfrak{m}_a S_{\mathfrak{q}} = \mathfrak{q} S_{\mathfrak{q}}$ is nontrivial since if $\bar{a} \neq x_D$, then $\mathfrak{q} \not\subseteq \mathfrak{m}_a$ in general; that is, there may be a cycle s for which $x_D \mid \bar{s}$ but $\bar{a} \nmid \bar{s}$.

Proof. Suppose $x_D \mid \bar{a}$. Then

$$\sigma S_{\mathfrak{q}} \subseteq \bar{\tau}_\psi(e_{\mathrm{t}(a)} A a) S_{\mathfrak{q}} = \mathfrak{m}_a S_{\mathfrak{q}} \subseteq \mathfrak{q} S_{\mathfrak{q}} \overset{(\mathrm{I})}{=} \sigma S_{\mathfrak{q}},$$

where (I) holds by Proposition 5.6. □

Proposition 7.2. *Let $D \in \mathcal{S}'$ be a simple matching for which $\mathfrak{q} := \mathfrak{q}_D$ is a minimal prime over $\mathfrak{m}_0$. The center $Z(A_{\mathfrak{q}})$ of $A_{\mathfrak{q}}$ is isomorphic to the subalgebra*

$$\tilde{R} := (k + \mathfrak{m}_D)_{\mathfrak{m}_D} + \mathfrak{q}S_{\mathfrak{q}} = \bigcap_{a \in Q_1^{\mathrm{t}}} \bar{\tau}_{\psi}(e_{\mathrm{t}(a)} A_{\mathfrak{q}} e_{\mathrm{t}(a)}) \subset S_{\mathfrak{q}} \cong Z(A'_{\mathfrak{q}}).$$

Proof. Set

$$Q_1^{\mathrm{t}} \cap D := Q_1^{\mathrm{t}} \cap \psi^{-1}(D) = \{a \in Q_1^{\mathrm{t}} \ : \ x_D \mid \bar{a}\}.$$

We claim that

$$\begin{aligned}
Z(A_{\mathfrak{q}}) &\overset{\text{(I)}}{\cong} \textstyle\bigcap_{i \in Q_0} \bar{\tau}_{\psi}(e_i A_{\mathfrak{q}} e_i) \\
&\overset{\text{(II)}}{=} \textstyle\bigcap_{a \in Q_1^{\mathrm{t}}} \bar{\tau}_{\psi}(e_{\mathrm{t}(a)} A_{\mathfrak{q}} e_{\mathrm{t}(a)}) \\
&\overset{\text{(III)}}{=} \textstyle\bigcap_{a \in Q_1^{\mathrm{t}}} \left((k + \mathfrak{m}_a)_{\mathfrak{q} \cap (k + \mathfrak{m}_a)} + \mathfrak{m}_a S_{\mathfrak{q}} \right) \\
&\overset{\text{(IV)}}{=} \textstyle\bigcap_{a \in Q_1^{\mathrm{t}} \cap D} \left((k + \mathfrak{m}_a)_{\mathfrak{m}_a} + \mathfrak{q} S_{\mathfrak{q}} \right) \\
&\overset{\text{(V)}}{=} \textstyle\bigcap_{a \in Q_1^{\mathrm{t}} \cap D} (k + \mathfrak{m}_a)_{\mathfrak{m}_a} + \mathfrak{q} S_{\mathfrak{q}} \\
&\overset{\text{(VI)}}{=} (k + \cap_{a \in Q_1^{\mathrm{t}} \cap D} \mathfrak{m}_a) \left(\left(k + \cap_{a \in Q_1^{\mathrm{t}} \cap D} \mathfrak{m}_a\right) \setminus \cup_{a \in Q_1^{\mathrm{t}} \cap D} \mathfrak{m}_a \right)^{-1} + \mathfrak{q} S_{\mathfrak{q}} \\
&= (k + \mathfrak{m}_D)_{\mathfrak{m}_D} + \mathfrak{q} S_{\mathfrak{q}} \\
&= \tilde{R}.
\end{aligned}$$

Indeed, (I) holds by Lemma 4.1.2 and (II) holds by Lemma 4.1.4.

To show (III), suppose $a \in Q_1^{\mathrm{t}}$. Recall the notation $A^i := \bar{\tau}_{\psi}(e_i A e_i)$. Then

$$A^{\mathrm{t}(a)} = k + \mathfrak{m}_a \quad \text{and} \quad A^{\mathrm{h}(a)} = S.$$

Thus by the definition of cyclic localization,

$$\begin{aligned}
\bar{\tau}_{\psi}\left(e_{\mathrm{t}(a)} A_{\mathfrak{q}} e_{\mathrm{t}(a)}\right) &= A^{\mathrm{t}(a)}_{\mathfrak{q} \cap A^{\mathrm{t}(a)}} + \sum_{\substack{qp \in e_{\mathrm{t}(a)} A e_{\mathrm{t}(a)} \\ \text{a nontrivial cycle}}} \bar{q}\, A^{\mathrm{h}(p)}_{\mathfrak{q} \cap A^{\mathrm{h}(p)}}\, \bar{p} \\
&= (k + \mathfrak{m}_a)_{\mathfrak{q} \cap (k + \mathfrak{m}_a)} + \sum_{\substack{q \in e_{\mathrm{t}(a)} A e_{\mathrm{h}(a)} \\ \text{a path}}} \bar{q}\, S_{\mathfrak{q}}\, \bar{a} \\
&= (k + \mathfrak{m}_a)_{\mathfrak{q} \cap (k + \mathfrak{m}_a)} + \mathfrak{m}_a S_{\mathfrak{q}}.
\end{aligned}$$

To show (IV), note that for $a \in Q_1^{\mathrm{t}}$,

$$\mathfrak{m}_a \subseteq \mathfrak{q} \quad \text{if and only if} \quad a \in \psi^{-1}(D).$$

Furthermore, if $\mathfrak{m}_a \subseteq \mathfrak{q}$, then $\mathfrak{m}_a S_{\mathfrak{q}} = \mathfrak{q} S_{\mathfrak{q}}$ by Lemma 7.1. Otherwise if $\mathfrak{m}_a \not\subseteq \mathfrak{q}$, then $\mathfrak{m}_a S_{\mathfrak{q}} = S_{\mathfrak{q}}$.

(V) holds since for $a \in Q_1^{\mathrm{t}} \cap D$,

$$\mathfrak{m}_a (k + \mathfrak{m}_a)_{\mathfrak{m}_a} \subseteq \mathfrak{q} S_{\mathfrak{q}}.$$

Finally, to show (vi), recall that each $\mathfrak{m}_a$ is generated over S by the $\bar{\tau}_\psi$-images of a set of nontrivial cycles, and thus by a set of nonconstant monomials in S. Therefore for any $a, b \in Q_1^{\mathrm{t}}$, we have $(k + \mathfrak{m}_a) \cap \mathfrak{m}_b = \mathfrak{m}_a \cap \mathfrak{m}_b$. $\square$

Definition 7.3. We say two arrows $a, b \in Q_1$ are *coprime* if $\bar{a}$ and $\bar{b}$ are coprime in B; that is, the only common factors of $\bar{a}$ and $\bar{b}$ in B are the units.

Lemma 7.4. *Suppose the arrows in Q_1^{t} are pairwise coprime, and let $a \in Q_1^{\mathrm{t}}$. Consider a simple matching $D \in \mathcal{S}'$ for which $x_D \mid \bar{a}$. Set $\mathfrak{q} := \mathfrak{q}_D$ and $i := \mathrm{t}(a)$. Then*

$$Z(A_\mathfrak{q}) = \tilde{R}\,\mathbf{1} = A_\mathfrak{q}^i\,\mathbf{1} \cong e_i A_\mathfrak{q} e_i.$$

Proof. Suppose the arrows in Q_1^{t} are pairwise coprime. Then each arrow in $Q_1^{\mathrm{t}} \setminus \{a\}$ is vertex invertible in $A_\mathfrak{q}$ by Lemma 6.5. Thus for each $j \in Q_0 \setminus \{i\}$,

$$e_j A_\mathfrak{q} e_j = S_\mathfrak{q} e_j,$$

by Lemma 4.1.4. The lemma then follows by Proposition 7.2. $\square$

In the following two lemmas, let B be an integral domain, and let $A = [A^{ij}] \subset M_d(B)$ be a tiled matrix ring. Fix $i, j, k \in \{1, \ldots, d\}$. For $p \in e_i A e_j$, denote by $\bar{p}$ the element of B satisfying $p = \bar{p} e_{ij}$.

Lemma 7.5. *Suppose*

$$A^{ij} \neq 0, \qquad A^{ji} \neq 0, \tag{19}$$

and

$$A^i \mathbf{1}_d = Z(A). \tag{20}$$

Then for each $f \in \operatorname{Hom}_{Z(A)}(e_j A e_i, e_k A e_i)$, there is some $h \in \operatorname{Frac} B$ such that for each $p \in e_j A e_i$, we have $\overline{f(p)} = h\bar{p}$.

Proof. Let $f \in \operatorname{Hom}_{Z(A)}(e_j A e_i, e_k A e_i)$. By assumption (19), there is some $0 \neq q \in e_i A e_j$. By assumption (20), for $p_1, p_2 \in e_j A e_i$,

$$\bar{q}\,\bar{p}_1 f(p_2) = \overline{p_1 q} f(p_2) = f((p_1 q) p_2) = f(p_1(q p_2)) = f((p_2 q) p_1) = \overline{p_2 q} f(p_1) = \bar{q}\,\bar{p}_2 f(p_1).$$

Thus, since B is an integral domain,

$$\bar{p}_1 f(p_2) = \bar{p}_2 f(p_1).$$

In particular, if p_1 and p_2 are nonzero, then

$$\frac{\overline{f(p_1)}}{\bar{p}_1} = \frac{\overline{f(p_2)}}{\bar{p}_2} =: h \in \operatorname{Frac} B.$$

Therefore for each $p \in e_j A e_i$, we have $\overline{f(p)} = h\bar{p}$. $\square$

Lemma 7.6. *Suppose (19) and (20) hold. If there is some $p \in e_jAe_i$ such that for each $f \in \operatorname{Hom}_{Z(A)}(e_jAe_i, e_kAe_i)$, there is some $r \in e_kAe_j$ satisfying*

$$f(p) = rp, \tag{21}$$

then

$$\operatorname{Hom}_{Z(A)}(e_jAe_i, e_kAe_i) \cong e_kAe_j.$$

Similarly, if there is some $p \in e_iAe_j$ such that for each $f \in \operatorname{Hom}_{Z(A)}(e_iAe_j, e_iAe_k)$, there is some $r \in e_jAe_k$ satisfying $f(p) = pr$, then

$$\operatorname{Hom}_{Z(A)}(e_iAe_j, e_iAe_k) \cong e_jAe_k.$$

Proof. Fix $f \in \operatorname{Hom}_{Z(A)}(e_jAe_i, e_kAe_i)$. By Lemma 7.5, there is some $h \in \operatorname{Frac} B$ such that for each $p \in e_jAe_i$, we have

$$\overline{f(p)} = h\bar{p}. \tag{22}$$

Let p' be as in (21). Then there is some $r \in e_kAe_j$ such that $f(p') = rp'$. Whence $\bar{r} = h$ by (22), since B is an integral domain. Thus $r = he_{kj}$. Therefore for each $p \in e_jAe_i$, we have $f(p) = rp$ by (22). Consequently, there is a surjective $Z(A)$-module homomorphism

$$\begin{array}{ccc} e_kAe_j & \twoheadrightarrow & \operatorname{Hom}_{Z(A)}(e_jAe_i, e_kAe_i) \\ r & \mapsto & (p \mapsto rp). \end{array} \tag{23}$$

To show injectivity, suppose $r, r' \in e_kAe_j$ are sent to the same homomorphism in $\operatorname{Hom}_{Z(A)}(e_jAe_i, e_kAe_i)$. Then for each $p \in e_jAe_i$,

$$rp = r'p.$$

But $e_jAe_i \neq 0$ by assumption (19). Whence $r = r'$ since B is an integral domain. Therefore (23) is an isomorphism.

Similarly, there is a $Z(A)$-module isomorphism

$$\begin{array}{ccc} e_jAe_k & \xrightarrow{\sim} & \operatorname{Hom}_{Z(A)}(e_iAe_j, e_iAe_k) \\ r & \mapsto & (p \mapsto pr). \end{array}$$

□

Again let A be a nonnoetherian homotopy algebra satisfying assumptions (A) and (B). Furthermore, suppose the arrows in Q_1^{t} are pairwise coprime. Fix $a \in Q_1^{\mathrm{t}}$, and consider a simple matching $D \in \mathcal{S}'$ such that $x_D \mid \bar{a}$. Set $\mathfrak{q} := \mathfrak{q}_D$ and $i := \mathrm{t}(a)$.

Lemma 7.7. *If $j \in Q_0$ is a vertex distinct from i and $f \in \operatorname{Hom}_{\tilde{R}}(e_jA_{\mathfrak{q}}e_i, e_iA_{\mathfrak{q}}e_i)$, then*

$$\overline{f(e_jA_{\mathfrak{q}}e_i)} \subseteq \mathfrak{m}_0\tilde{R}.$$

Proof. Fix a vertex $j \neq i \in Q_0$ and an $\tilde{R}$-module homomorphism $f : e_jA_{\mathfrak{q}}e_i \to e_iA_{\mathfrak{q}}e_i$. We may apply Lemma 7.5 to f: assumption (19) holds since there is a path between any two vertices of Q, and assumption (20) holds by Lemma 7.4. Thus there is some $h \in \operatorname{Frac} B$ such that for each $p \in e_jA_{\mathfrak{q}}e_i$, we have

$$\overline{f(p)} = h\bar{p}. \tag{24}$$

Assume to the contrary that there is some $p \in e_jA_{\mathfrak{q}}e_i$ such that $f(p) = ce_i + q$, where $0 \neq c \in k$ and $\bar{q} \in \mathfrak{m}_0\tilde{R}$. By (24),

$$h\bar{p} = \overline{f(p)} = c + \bar{q}.$$

Whence $h = (c + \bar{q})\bar{p}^{-1}$.

By assumption (A), there is a path $t' \in e_jAe_{\mathrm{h}(a)}$ such that (i) $x_D \nmid \bar{t}'$, and (ii) $t'a$ is not a scalar multiple of p. Set $t := t'a$. Then

$$c\bar{t}\bar{p}^{-1} + \bar{q}\bar{t}\bar{p}^{-1} = (c + \bar{q})\bar{t}\bar{p}^{-1} = h\bar{t} \overset{\text{(I)}}{=} \overline{f(t)} \in \tilde{R} \overset{\text{(II)}}{=} \bar{\tau}_{\psi}(e_iA_{\mathfrak{q}}e_i), \tag{25}$$

where (I) holds by (24) and (II) holds by Lemma 7.4. Furthermore, $\tilde{R}$ is a unique factorization domain since it is the localization of a subalgebra of the polynomial ring B on a multiplicatively closed subset. Thus, since $c \neq 0$, (25) implies

$$\bar{t}\bar{p}^{-1} \in \bar{\tau}_{\psi}(e_iA_{\mathfrak{q}}e_i). \tag{26}$$

Now every element $g \in \bar{\tau}_{\psi}(e_iA_{\mathfrak{q}}e_i)$ is of the form

$$g = d + \sum_{\ell=1}^{m} x_D^{n_\ell} u_\ell v_\ell^{-1}, \tag{27}$$

where $d \in k$, and u_ℓ, v_ℓ are monomials in B not divisible by x_D. Moreover, for each ℓ we have $n_\ell \geq 1$, by Lemma 6.5. The element $\bar{t}\bar{p}^{-1}$ is of the form (27), with $m \geq 1$ since t is not a scalar multiple of p. But each $n_\ell \leq 0$ since $x_D \nmid \bar{t}'$, contrary to (26). $\square$

Proposition 7.8. *For each $j, k \in Q_0$,*

$$\operatorname{Hom}_{\tilde{R}}(e_jA_{\mathfrak{q}}e_i, e_kA_{\mathfrak{q}}e_i) \cong e_kA_{\mathfrak{q}}e_j \quad \text{and} \quad \operatorname{Hom}_{\tilde{R}}(e_iA_{\mathfrak{q}}e_j, e_iA_{\mathfrak{q}}e_k) \cong e_jA_{\mathfrak{q}}e_k.$$

Proof. Suppose the hypotheses hold. We claim that $A_{\mathfrak{q}}$ satisfies the assumptions of Lemma 7.6, with $i = \mathrm{t}(a)$ and arbitrary $j, k \in Q_0$.

Indeed, assumption (19) holds since there is a path between any two vertices of Q, and assumption (20) holds by Lemma 7.4.

To show that the third assumption (21) holds, fix $j, k \in Q_0$. Consider a path $p \in e_jAe_i$ for which $x_D^2 \nmid \bar{p}$; such a path exists by assumption (A), and since D is a simple matching of A'. Let $f \in \operatorname{Hom}_{\tilde{R}}(e_jA_{\mathfrak{q}}e_i, e_kA_{\mathfrak{q}}e_i)$ be arbitrary. We want to show that there is an $r \in e_kA_{\mathfrak{q}}e_j$ such that $f(p) = rp$.

Write $f(p) = \sum_\ell c_\ell q_\ell$ as an $\tilde{R}$-linear combination of paths $q_\ell \in e_k A e_i$. To show that $f(p) = rp$, it suffices to show that for each path q_ℓ, there is a path r_ℓ such that

$$q_\ell = r_\ell p,$$

since then we may take $r = \sum_\ell c_\ell r_\ell$. It therefore suffices to assume that $f(p) = q$ is a single path.

Let p^+ and q^+ be lifts of p and q to the covering quiver Q^+ with coincident tails, $\mathrm{t}(p^+) = \mathrm{t}(q^+) \in Q_0^+$. Let $s \in e_k A e_j$ be a path for which s^+ has no cyclic subpaths in Q^+ and

$$\mathrm{t}(s^+) = \mathrm{h}(p^+) \quad \text{and} \quad \mathrm{h}(s^+) = \mathrm{h}(q^+).$$

Then by [B2, Lemma 4.3], there is some $n \in \mathbb{Z}$ such that

$$\overline{sp} = \bar{q}\sigma^n.$$

(i) First suppose $n \leq 0$. Set

$$r := \sigma_k^n s.$$

Then $\overline{rp} = \bar{q}$. Thus $rp = q$ since $\bar{\tau}_\psi$ is injective.

(ii) So suppose $n \geq 1$; without loss of generality we may assume $n = 1$.

(ii.a) Further suppose $i \neq k$ or $i = k \neq j$. Then q is a nontrivial path: if $i \neq k$, then q is clearly nontrivial, and if $i = k \neq j$, then q is nontrivial by Lemma 7.7.

Since $\deg^+ i = 1$, x_D divides the $\bar{\tau}_\psi$-image of each nontrivial path in Ae_i. Whence $x_D \mid \bar{q}$. Thus $x_D^2 \mid \bar{q}\sigma = \overline{sp}$. But $x_D^2 \nmid \bar{p}$ by our choice of p. Therefore $x_D \mid \bar{s}$. Consequently, s factors into paths $s = s_3 s_2 s_1$, where s_2 is a subpath of a unit cycle satisfying $x_D \mid \bar{s}_2$. Let b be one of the two paths for which bs_2 is a unit cycle. Then $x_D \nmid \bar{b}$ since $x_D \mid \bar{s}_2$. Thus b has vertex inverse

$$b^* \in e_{\mathrm{t}(s_3)} A_{\mathfrak{q}} e_{\mathrm{h}(s_1)},$$

by Lemma 6.5. Set

$$r := s_3 b^* s_1.$$

Then, since $\overline{b^*} = \bar{b}^{-1}$, we have

$$\overline{rp} = \overline{s_3 b^* s_1 p} = \bar{b}^{-1}\bar{s}_3\bar{s}_1\bar{p} = \frac{\bar{s}_2}{\sigma}\bar{s}_3\bar{s}_1\bar{p} = \frac{\overline{sp}}{\sigma} = \bar{q}.$$

Therefore $rp = q$ since $\bar{\tau}_\psi$ is injective, proving our claim.

(ii.b) Finally, suppose $i = j = k$. Then $rp = f(p)$ holds by taking $p = e_i$ and $r = f(e_i)$. $\square$

Theorem 7.9. *Suppose the arrows in Q_1^{t} are pairwise coprime. Let $\mathfrak{q} \in \operatorname{Spec} S$ be a minimal prime over $\mathfrak{q} \cap R = \mathfrak{m}_0$. Then there is some $i \in Q_0$ for which*

$$A_{\mathfrak{q}} \cong \operatorname{End}_{Z(A_{\mathfrak{q}})}(A_{\mathfrak{q}} e_i).$$

Furthermore, $A_{\mathfrak{q}} e_i$ is a reflexive $Z(A_{\mathfrak{q}})$-module.

Proof. Suppose the hypotheses hold. By Theorem 5.7.2, there is some $D \in \mathcal{S}'$ such that $\mathfrak{q} = \mathfrak{q}_D$. Since the arrows in Q_1^{t} are pairwise coprime, there is a unique arrow $a \in Q_1^{\mathrm{t}}$ for which $x_D \mid \bar{a}$. Set

$$i := \mathrm{t}(a) \quad \text{ and } \quad \epsilon := \epsilon_D = 1_A - e_i.$$

For brevity, denote $\operatorname{Hom}_{\tilde{R}}(-,-)$ by ${}_{\tilde{R}}(-,-)$. There are algebra isomorphisms

$$\begin{aligned} A_{\mathfrak{q}} & \cong \begin{bmatrix} e_i A_{\mathfrak{q}} e_i & e_i A_{\mathfrak{q}} \epsilon \\ \epsilon A_{\mathfrak{q}} e_i & \epsilon A_{\mathfrak{q}} \epsilon \end{bmatrix} \\ & \stackrel{(\text{I})}{\cong} \begin{bmatrix} {}_{\tilde{R}}(e_i A_{\mathfrak{q}} e_i, e_i A_{\mathfrak{q}} e_i) & {}_{\tilde{R}}(\epsilon A_{\mathfrak{q}} e_i, e_i A_{\mathfrak{q}} e_i) \\ {}_{\tilde{R}}(e_i A_{\mathfrak{q}} e_i, \epsilon A_{\mathfrak{q}} e_i) & {}_{\tilde{R}}(\epsilon A_{\mathfrak{q}} e_i, \epsilon A_{\mathfrak{q}} e_i) \end{bmatrix} \\ & \stackrel{(\text{II})}{\cong} \operatorname{End}_{Z(A_{\mathfrak{q}})}(e_i A_{\mathfrak{q}} e_i \oplus \epsilon A_{\mathfrak{q}} e_i) \\ & = \operatorname{End}_{Z(A_{\mathfrak{q}})}(A_{\mathfrak{q}} e_i), \end{aligned}$$

where (I) holds by Proposition 7.8 and (II) holds by Lemma 7.4.

Furthermore, $A_{\mathfrak{q}} e_i$ is a reflexive $Z(A_{\mathfrak{q}})$-module:

$$\begin{aligned} {}_{Z(A_{\mathfrak{q}})}\big({}_{Z(A_{\mathfrak{q}})}(A_{\mathfrak{q}} e_i, Z(A_{\mathfrak{q}})), Z(A_{\mathfrak{q}})\big) & \stackrel{(\text{I})}{\cong} ((A_{\mathfrak{q}} e_i, e_i A_{\mathfrak{q}} e_i), e_i A_{\mathfrak{q}} e_i) \\ & \stackrel{(\text{II})}{=} (e_i A_{\mathfrak{q}}, e_i A_{\mathfrak{q}} e_i) \\ & \stackrel{(\text{III})}{=} A_{\mathfrak{q}} e_i, \end{aligned}$$

where (I) holds by Lemma 7.4, and (II), (III) hold by Proposition 7.8. □

Theorem 7.10. *Let A be a nonnoetherian homotopy algebra satisfying assumptions (A) and (B), and suppose the arrows in Q_1^{t} are pairwise coprime. Then $A_{\mathfrak{m}_0}$ is a nonnoetherian NCCR.*

Proof. $A_{\mathfrak{m}_0}$ is nonnoetherian and an infinitely generated module over its nonnoetherian center by [B6, Section 3]; has a normal Gorenstein cycle algebra $SR_{\mathfrak{m}_0}$ by Proposition 5.9; is cycle regular by Theorem 6.15; and for each prime $\mathfrak{q} \in \operatorname{Spec}(SR_{\mathfrak{m}_0})$ minimal over $\mathfrak{m}_0$, the cyclic localization $A_{\mathfrak{q}}$ is an endomorphism ring of a reflexive $Z(A_{\mathfrak{q}})$-module by Theorem 7.9. □

7.1. **Examples.**

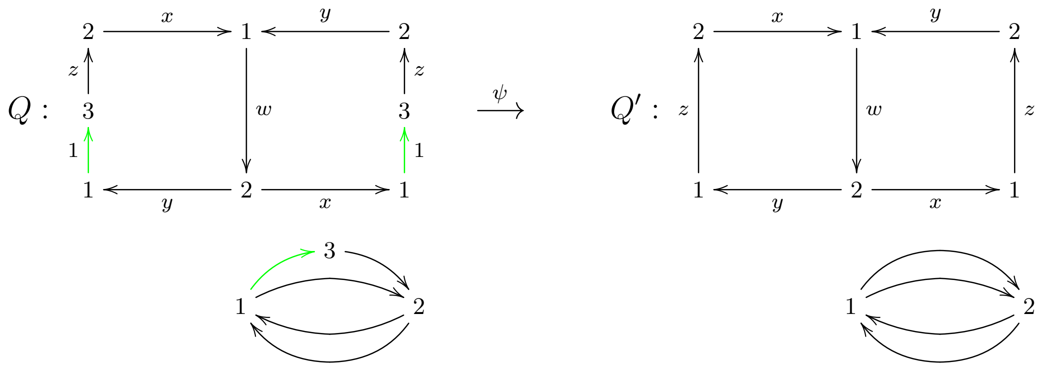
Example 7.11. Set

$$B := k\left[x,y,z,w\right], \quad S := k\left[xz,yz,xw,yw\right] \cong k\left[a,b,c,d\right]/(ad-bc),$$

and

$$I := (x,y)S, \quad J := (z,w)S, \quad \mathfrak{m}_0 := zI, \quad R := k + \mathfrak{m}_0.$$

Consider the contraction of homotopy algebras given in Figure 3. Each arrow is labeled by its $\bar{\tau}_\psi/\bar{\tau}$-image in B. The center and cycle algebra of A are R and S respectively.



$$A = \begin{bmatrix} S & I & zI \\ J & S & zS \\ S & I & R \end{bmatrix} \cong \operatorname{End}_R(Ae_3) \qquad A' = \begin{bmatrix} S & I \\ J & S \end{bmatrix} \cong \operatorname{End}_S(A'e_i).$$

FIGURE 3. (Example 7.11.) The homotopy algebra A is a nonnoetherian NCCR. The quivers Q and Q' on the top line are each drawn on a torus, and the contracted arrow of Q is drawn in green.

In this example, the maximal ideal $\mathfrak{m}_0 \in \operatorname{Max} R$ at the origin is a height one prime ideal of S.[10] Therefore $\mathfrak{m}_0$ itself is the only minimal prime of S over $\mathfrak{m}_0$. Furthermore, the cyclic localization of A at $\mathfrak{m}_0$ is

$$A_{\mathfrak{m}_0} = \left\langle \begin{bmatrix} S_{\mathfrak{m}_0} & I & zI \\ J & S_{\mathfrak{m}_0} & zS \\ S & I & R_{\mathfrak{m}_0} \end{bmatrix} \right\rangle = \begin{bmatrix} S_{\mathfrak{m}_0} & IS_{\mathfrak{m}_0} & zIS_{\mathfrak{m}_0} \\ JS_{\mathfrak{m}_0} & S_{\mathfrak{m}_0} & zS_{\mathfrak{m}_0} \\ S_{\mathfrak{m}_0} & IS_{\mathfrak{m}_0} & R_{\mathfrak{m}_0} + \mathfrak{m}_0 S_{\mathfrak{m}_0} \end{bmatrix},$$

with center $Z(A_{\mathfrak{m}_0}) \cong R_{\mathfrak{m}_0} + \mathfrak{m}_0 S_{\mathfrak{m}_0}$.

Example 7.12. Set

$$B := k\left[x, y, z, w\right], \qquad S := k\left[xz, yz, xw, yw\right],$$

and

$$I := (x, y)S, \qquad J := (z, w)S, \qquad \mathfrak{m}_0 := zwI^2, \qquad R := k + \mathfrak{m}_0.$$

Consider the contraction of homotopy algebras given in Figure 1. As in Example 7.11, the center and cycle algebra of A are R and S respectively.

The minimal primes in S over $\mathfrak{m}_0$ are

$$\mathfrak{q}_1 := zI \quad \text{and} \quad \mathfrak{q}_2 := wI,$$

[10] Note that the ideals xzS and yzS, each of which is properly contained in zI, are not prime since $(xw) \cdot (yz) \in xzS$ and $(xz) \cdot (yw) \in yzS$.

each of height 1. The cyclic localizations of A at $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are

$$A_{\mathfrak{q}_1} = \begin{bmatrix} S_{\mathfrak{q}_1} & IS_{\mathfrak{q}_1} & \mathfrak{q}_1 S_{\mathfrak{q}_1} & S_{\mathfrak{q}_1} \\ wS_{\mathfrak{q}_1} & S_{\mathfrak{q}_1} & zS_{\mathfrak{q}_1} & wS_{\mathfrak{q}_1} \\ S_{\mathfrak{q}_1} & IS_{\mathfrak{q}_1} & (k+\mathfrak{q}_1)_{\mathfrak{q}_1} + \mathfrak{q}_1 S_{\mathfrak{q}_1} & S_{\mathfrak{q}_1} \\ S_{\mathfrak{q}_1} & IS_{\mathfrak{q}_1} & \mathfrak{q}_1 S_{\mathfrak{q}_1} & S_{\mathfrak{q}_1} \end{bmatrix} \cong \operatorname{End}_{Z(A_{\mathfrak{q}_1})}(A_{\mathfrak{q}_1} e_3)$$

and

$$A_{\mathfrak{q}_2} = \begin{bmatrix} S_{\mathfrak{q}_2} & IS_{\mathfrak{q}_2} & S_{\mathfrak{q}_2} & \mathfrak{q}_2 S_{\mathfrak{q}_2} \\ zS_{\mathfrak{q}_2} & S_{\mathfrak{q}_2} & zS_{\mathfrak{q}_2} & wS_{\mathfrak{q}_2} \\ S_{\mathfrak{q}_2} & IS_{\mathfrak{q}_2} & S_{\mathfrak{q}_2} & \mathfrak{q}_2 S_{\mathfrak{q}_2} \\ S_{\mathfrak{q}_2} & IS_{\mathfrak{q}_2} & S_{\mathfrak{q}_2} & (k+\mathfrak{q}_2)_{\mathfrak{q}_2} + \mathfrak{q}_2 S_{\mathfrak{q}_2} \end{bmatrix} \cong \operatorname{End}_{Z(A_{\mathfrak{q}_2})}(A_{\mathfrak{q}_2} e_4),$$

with respective centers

$$Z(A_{\mathfrak{q}_1}) \cong (k+\mathfrak{q}_1)_{\mathfrak{q}_1} + \mathfrak{q}_1 S_{\mathfrak{q}_1} \qquad \text{and} \qquad Z(A_{\mathfrak{q}_2}) \cong (k+\mathfrak{q}_2)_{\mathfrak{q}_2} + \mathfrak{q}_2 S_{\mathfrak{q}_2}.$$

(Note that $wS_{\mathfrak{q}_1} = JS_{\mathfrak{q}_1}$ since $z = w\,\frac{xz}{xw}$, and similarly $zS_{\mathfrak{q}_2} = JS_{\mathfrak{q}_2}$.) In contrast to Example 7.11, A itself is not an endomorphism ring, although its cyclic localizations are.

Acknowledgments. The author would like to thank Kenny Brown for useful discussions, and an anonymous referee for helpful comments. This article was completed while the author was a research fellow at the Heilbronn Institute for Mathematical Research at the University of Bristol.

References

[AB] M. Auslander and D. A. Buchsbaum, Homological dimension in noetherian rings, Proc. Nat. Acad. Sci. U.S.A. **42** (1956).
[AB2] ______ Homological dimension in local rings, Trans. Amer. Math. Soc. **85** (1957) 390-405.
[BKM] K. Baur, A. King, R. Marsh, Dimer models and cluster categories of Grassmannians, Proc. London Math. Soc. (2016).
[B1] C. Beil, Cyclic contractions of dimer algebras always exist, arXiv:1703.04450.
[B2] ______, Homotopy dimer algebras and cyclic contractions, in preparation.
[B3] ______, Morita equivalences and Azumaya loci from Higgsing dimer algebras, J. Algebra **453** (2016) 429-455.
[B4] ______, Noetherian criteria for dimer algebras, in preparation.
[B5] ______, Nonnoetherian geometry, J. Algebra Appl. **15** (2016).
[B6] ______, On the central geometry of nonnoetherian dimer algebras, in preparation.
[B7] ______, On the noncommutative geometry of square superpotential algebras, J. Algebra **371** (2012) 207-249.
[B8] ______, The central nilradical of nonnoetherian dimer algebras, in preparation.

[BFHMS] S. Benvenuti, S. Franco, A. Hanany, D. Martelli, J. Sparks, An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals, J. High Energy Phys. **06** (2005) 064.

[BD] D. Berenstein and M. Douglas, Seiberg duality for quiver gauge theories, (2002) arXiv:hep-th/0207027.

[Bo] R. Bocklandt, Consistency conditions for dimer models, Glasgow Math. J. **54** (2012) 429-447.

[Br] N. Broomhead, Dimer models and Calabi-Yau algebras, Memoirs AMS (2012) 1011.

[BH] K. Brown and C. Hajarnavis, Homologically homogeneous rings, Trans. Amer. Math. Soc. **281** (1984) 197-208.

[D] B. Davison, Consistency conditions for brane tilings, J. Algebra **338** (2011) 1-23.

[C] P. L. Clark, Commutative Algebra, unpublished. Available at: http://math.uga.edu/∼pete/MATH8020.html.

[G] D. Gulotta, Properly ordered dimers, R-charges, and an efficient inverse algorithm, J. High Energy Phys. (2008).

[IU] A. Ishii and K. Ueda, Dimer models and the special McKay correspondence, Geometry and Topology **19** (2015) 3405-3466.

[R] J. Rotman, *An introduction to homological algebra*, Springer, 2009.

[S] J.-P. Serre, Sur la dimension homologique des anneaux et des modules noethériens. (French) Proceedings of the international symposium on algebraic number theory, Tokyo and Nikko (1955) 175-189. Science Council of Japan, Tokyo (1956).

[V] M. Van den Bergh, Non-commutative crepant resolutions, The legacy of Niels Henrik Abel, Springer, Berlin (2004) 749-770.

Institut für Mathematik und Wissenschaftliches Rechnen, Universität Graz, Heinrichstrasse 36, 8010 Graz, Austria.

E-mail address: charles.beil@uni-graz.at