ASYMPTOTICS FOR THE HEAT KERNEL ON H-TYPE GROUPS

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Abstract

We give sharp asymptotic estimates at infinity of all radial partial derivatives of the heat kernel on H-type groups. As an application, we give a new proof of the discreteness of the spectrum of some natural sub-Riemannian Ornstein-Uhlenbeck operators on these groups.

1. Introduction

Estimates at infinity for the heat kernel on the Heisenberg group or, more generally, H-type groups have attracted a lot of interest in the last decades (see, e.g., [2, 7, 10, 12, 15, 16]). In the context of H-type groups, in particular, some results were recently obtained by Eldredge [7] and Li [16] independently. In [7], Eldredge provides precise upper and lower bounds for the heat kernel p_s and its horizontal gradient $\nabla_{\mathcal{H}} p_s$. In [16], Li provides asymptotic estimates for the heat kernel p_s, as well as upper bounds for all its derivatives. Nevertheless, to the best of our knowledge, sharp asymptotic estimates at infinity for the derivatives of p_s are still missing. In this paper we address this problem by providing asymptotic expansions at infinity of the heat kernel and of all its derivatives.

Let G be an H-type group identified with $\mathbb{R}^{2n} \times \mathbb{R}^m$ via the exponential map, and denote by (x, t) its generic element, where $x \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}^m$. It is well known that the heat kernel p_s is a function of $R := |x|^2/4$ and $|t|$. Outside the region $\{(x, t) \in G : t = 0\}$, *any* derivative of $p_s(x, t)$ can thus be written as a finite linear combination with smooth coefficients of the functions

$$p_{s,k_1,k_2}(x, t) = \frac{\partial^{k_1}}{\partial R^{k_1}} \frac{\partial^{k_2}}{\partial |t|^{k_2}} p_s(x, t),$$

for suitable $k_1, k_2 \in \mathbb{N}$. We call these functions *radial partial derivatives* of p_s. Thus, everything can be reduced to finding asymptotic estimates at infinity of p_{s,k_1,k_2} for every $k_1, k_2 \in \mathbb{N}$; these will yield asymptotic estimates of every desired derivative of p_s.

We divide the paper in five sections. In the next section we fix the notation and recall some preliminary facts on H-type groups and the method of stationary phase. In the central Sections 3 and 4 the functions p_{s,k_1,k_2} are studied. In Section 3 we provide asymptotic estimates for p_{s,k_1,k_2} in the case $m = 1$, namely when G is a Heisenberg group; in Section 4 we extend the results of Section 3 to the more general class of H-type groups. This is done via a reduction to the case $m = 1$ when m is odd; a descent method is then applied in order to cover the case m even. The preliminary study of the case $m = 1$ is necessary except in a single case, for which the general case could be treated directly; nevertheless, we include both proofs for the sake of clarity. As the reader may see, our Theorem 4.2 and Corollary 4.15 cover the cases of [16, Theorems 1.4 and 1.5] and [7, Theorem 4.2] as particular instances, and imply [16, Theorems 1.1 and 1.2] and [7, Theorem 4.4] as easy corollaries, by means of formula (5.1). In Section 5 we show an interesting application of our estimates, providing a different proof of a theorem due to Inglis [14] which concerns the discreteness of the spectrum of some Ornstein-Uhlenbeck operators on G.

We emphasize that our methods are strongly related to those employed by Gaveau [10] and then Hueber and Müller [12] in the case of the Heisenberg group; some ideas are also

taken from the work of Eldredge [7]. In particular, we borrow from [10] and [12] the use of the method of *stationary phase*, though in a stronger form provided by Hörmander [11].

2. Preliminaries

2.1. H-type Groups.
An H-type group G is a 2-step stratified group whose Lie algebra $\mathfrak{g}$ is endowed with an inner product $(\,\cdot\,,\,\cdot\,)$ such that

1. if $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{h} = \mathfrak{z}^{\perp}$, then $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{z}$;
2. for every $Z \in \mathfrak{z}$, the map $J_Z \colon \mathfrak{h} \to \mathfrak{h}$,

$$(J_Z X, Y) = (Z, [X, Y]) \qquad \forall X, Y \in \mathfrak{h},$$

is an isometry whenever $(Z, Z) = 1$.

In particular, $\mathfrak{g}$ stratifies as $\mathfrak{h} \oplus \mathfrak{z}$. It is very convenient, however, to realize an H-type group G as $\mathbb{R}^{2n} \times \mathbb{R}^m$, for some $n, m \in \mathbb{N}$, via the exponential map. More precisely, we shall denote by (x, t) the elements of G, where $x \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}^m$. We denote by $(e_1, \ldots, e_{2n})$ and $(u_1, \ldots, u_m)$ the standard bases of $\mathbb{R}^{2n}$ and $\mathbb{R}^m$ respectively. Under this identification, the Haar measure dy is the Lebesgue measure. The maps $\{J_Z \colon Z \in \mathfrak{z}\}$ are identified with $2n \times 2n$ skew symmetric matrices $\{J_t \colon t \in \mathbb{R}^m\}$ which are orthogonal whenever $|t| = 1$. This identification endows $\mathbb{R}^{2n} \times \mathbb{R}^m$ with the group law

$$(x, t) \cdot (x', t') = \left(x + x', t + t' + \frac{1}{2} \sum_{k=1}^m (J_{u_k} x, x') u_k \right).$$

A basis of left-invariant vector fields for $\mathfrak{g}$ is

$$X_j = \partial_{x_j} + \frac{1}{2} \sum_{k=1}^m (J_{u_k} x, e_j) \partial_{t_k}, \quad j = 1, \ldots, 2n; \qquad T_k = \partial_{t_k}, \quad k = 1, \ldots, m.$$

In particular, $(X_j)_{1 \leq j \leq 2n}$ is a basis for the first layer $\mathfrak{h} \cong \mathbb{R}^{2n}$. If f is a sufficiently smooth function on G, its horizontal gradient will be the vector field $\nabla_{\mathcal{H}} f \coloneqq \sum_{j=1}^{2n} (X_j f) X_j$, and its sub-Laplacian $\mathcal{L} f \coloneqq -\sum_{j=1}^{2n} X_j^2 f$. We refer the reader to [3] for further details.

2.2. The Heat Kernel.
On an H-type group $G \simeq \mathbb{R}^{2n} \times \mathbb{R}^m$ the heat kernel $(p_s)_{s>0}$ has the form

$$p_s(x, t) = \frac{1}{(4\pi)^n (2\pi)^m s^{n+m}} \int_{\mathbb{R}^m} e^{\frac{i}{s}(\lambda, t) - \frac{|x|^2}{4s} |\lambda| \coth(|\lambda|)} \left(\frac{|\lambda|}{\sinh |\lambda|} \right)^n \, \mathrm{d}\lambda, \qquad (2.1)$$

for every $s > 0$ and every $(x, t) \in G$ (see [10] or [13] for the Heisenberg groups, [19] or [22] for H-type groups). For the sake of clarity, we shall sometimes stress the dependence of p_s on the dimension m of the centre of G by writing $p_s^{(m)}$ instead of p_s.

We begin by writing the heat kernel (2.1) in a more convenient form. Let $\mathcal{R}$ be an isometry such that $\mathcal{R}t = |t| u_1$, where u_1 is the first element of the canonical basis[1] of the centre of G, namely $\mathbb{R}^m$. Then make the change of variables $\lambda \mapsto \mathcal{R}^{-1} \lambda$ in (2.1), which gives

$$p_s(x, t) = \frac{1}{(4\pi)^n (2\pi)^m s^{n+m}} \int_{\mathbb{R}^m} e^{\frac{i}{s}(\lambda, u_1)|t| - \frac{|x|^2}{4s} |\lambda| \coth(|\lambda|)} \left(\frac{|\lambda|}{\sinh |\lambda|} \right)^n \, \mathrm{d}\lambda. \qquad (2.2)$$

It is now more evident that p_s depends only on $|x|$ and $|t|$. This leads us to the following definition.

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[1] The choice of u_1 is actually irrelevant.

Definition 2.1. Let $R = \frac{|x|^2}{4}$. For all $s > 0$ and for all $k_1, k_2 \in \mathbb{N}$, define

$$
\begin{aligned}
p_{s,k_1,k_2}(x,t) &:= \frac{\partial^{k_1}}{\partial R^{k_1}} \frac{\partial^{k_2}}{\partial |t|^{k_2}} p_s(x,t) = \frac{(-1)^{k_1} i^{k_2}}{(4\pi)^n (2\pi)^m s^{n+m+k_1+k_2}} \\
&\quad \times \int_{\mathbb{R}^m} e^{\frac{i}{s}|t|(\lambda,u_1) - \frac{|x|^2}{4s}|\lambda| \coth|\lambda|} \frac{|\lambda|^{n+k_1} \cosh(|\lambda|)^{k_1}}{\sinh(|\lambda|)^{n+k_1}} (\lambda, u_1)^{k_2} \, d\lambda.
\end{aligned}
\tag{2.3}
$$

Notice that p_s is a smooth function of R and $|t|$ by formula (2.2), so that the definition of p_{s,k_1,k_2} is meaningful on the whole of G. Moreover, consider a differential operator on G of the form

$$
X^\gamma = \frac{\partial^{|\gamma|}}{\partial x^{\gamma_1} \partial t^{\gamma_2}}
$$

for some $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^{2n} \times \mathbb{N}^m$. By means of Faà di Bruno's formula, the function $X^\gamma p_s$ can be written on $\{t \neq 0\}$ as a finite linear combination with smooth coefficients of the functions p_{s,k_1,k_2}, for suitable k_1 and k_2. Since $X^\gamma p_s$ is uniformly continuous, the value of $X^\gamma p_s(x,0)$ can then be recovered by continuity uniformly in $x \in \mathbb{R}^{2n}$. Therefore, one can obtain asymptotic estimates for $X^\gamma p_s$ by combining appropriately some given estimates of p_{s,k_1,k_2} (see also Remark 4.16). We shall see an application of this in Section 5.

Observe that it will be sufficient to study p_{1,k_1,k_2}, since

$$
p_{s,k_1,k_2}(x,t) = \frac{1}{s^{n+m+k_1+k_2}} p_{1,k_1,k_2}\left(\frac{x}{\sqrt{s}}, \frac{t}{s}\right)
$$

for every $s > 0$, $k_1, k_2 \in \mathbb{N}$ and $(x,t) \in G$. Hence, we shall focus only on p_{1,k_1,k_2}. Moreover, from now on we shall fix the integers $k_1, k_2 \geq 0$. Of course, the choice $k_1 = k_2 = 0$ gives the heat kernel p_s.

Remark 2.2. It is well known (see [6] or [3, Remark 3.6.7]) that there exist n and m for which $\mathbb{R}^{2n} \times \mathbb{R}^m$ cannot represent any H-type group. Nevertheless, (2.1) and hence (2.3) make sense for every positive $n, m \in \mathbb{N}$, and for such n and m we shall then study p_{s,k_1,k_2}.

Definition 2.3. (cf. [12]) For every $(x,t) \in G$, define[2]

$$
\omega := \frac{|t|}{R}, \qquad \delta := \sqrt{\frac{R}{\pi|t|}}, \qquad \kappa := 2\sqrt{\pi|t|R}.
$$

We shall split the asymptotic condition $(x,t) \to \infty$ into four cases, some of which depend on an arbitrary constant $C > 1$. In particular, the first one covers the case $|t|/|x|^2$ bounded, while the other three are a suitable splitting of the case $|t|/|x|^2 \to \infty$.

I. $(x,t) \to \infty$ and $\omega = 4|t|/|x|^2 \leq C$;

II. $\delta \to 0^+$ and $\kappa \to +\infty$;

III. $\delta \to 0^+$ and $\kappa \in [1/C, C]$;

IV. $\kappa \to 0^+$ and $|t| \to +\infty$.
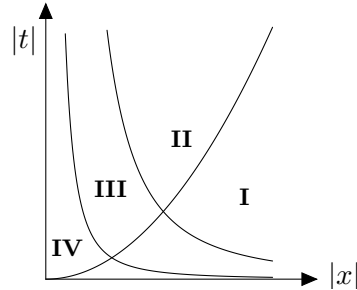
We shall describe the asymptotic behaviour of p_{1,k_1,k_2} in each of these four cases. The first two will both need the method of stationary phase (Theorem 2.7 below), while the other two can be treated through Taylor expansions.

In order to simplify the notation, we give some definitions.

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[2]Actually, ω is defined for $x \neq 0$ and δ for $t \neq 0$, but we shall not recall it again in the following.

Definition 2.4. Define the function $\theta\colon (-\pi, \pi) \to \mathbb{R}$ by

$$\theta(\lambda) := \begin{cases} \frac{2\lambda - \sin(2\lambda)}{2\sin^2(\lambda)}, & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases}$$

Lemma 2.5. [10, § 3, Lemma 3] θ is an odd, strictly increasing analytic diffeomorphism between $(-\pi, \pi)$ and $\mathbb{R}$.

Definition 2.6. For every $\omega \in \mathbb{R}$, set $y_\omega := \theta^{-1}(\omega)$. For every $(x, t) \in G$ define

$$d(x, t) := \begin{cases} |x|\frac{y_\omega}{\sin(y_\omega)} & \text{if } x \neq 0 \text{ and } t \neq 0, \\ |x| & \text{if } t = 0, \\ \sqrt{4\pi|t|} & \text{if } x = 0. \end{cases}$$

It is worth observing that $d(x, t)$ is the Carnot-Carathéodory distance between (x, t) and the origin with respect to the horizontal distribution generated by the vector fields $X_1, \ldots, X_{2n}$. See [2, §4] or [7, Theorem 3.5] for a proof and further details.

2.3. **The Method of Stationary Phase.** The main tool that we shall use is an easy corollary of Hörmander's theorem of stationary phase [11, Theorem 7.7.5], stated in a form convenient for our needs. We include a proof for the sake of clarity. Given an open set $V \subseteq \mathbb{R}^m$, we write $\mathcal{E}(V)$ for the space of C^∞ complex-valued functions on V, endowed with the topology of locally uniform convergence of all the derivatives. If f is a twice differentiable function on an open neighbourhood of 0, we write $P_{2,0}f$ for the Taylor polynomial of order 2 about 0 of f.

Theorem 2.7. *Let V be an open neighbourhood of 0 in $\mathbb{R}^m$, and let $\mathscr{F}, \mathscr{G}$ be bounded subsets of $\mathcal{E}(V)$ such that*

1. *$\operatorname{Im} f(\lambda) \geq 0$ for every $\lambda \in V$ and every $f \in \mathscr{F}$. Moreover, there exist $\eta > 0$ and $c_1 > 0$ such that $B(0, 2\eta) \subseteq V$ and $\operatorname{Im} f(\lambda) \geq c_1|\lambda|$ whenever $|\lambda| \geq \eta$ and $f \in \mathscr{F}$;*
2. *$\operatorname{Im} f(0) = f'(0) = 0$ and $\det f''(0) \neq 0$ for all $f \in \mathscr{F}$;*
3. *there exists $c_2 > 0$ such that $|f'(\lambda)| \geq c_2|\lambda|$ for all $|\lambda| \leq 2\eta$ and for all $f \in \mathscr{F}$;*
4. *there exists $c_3 > 0$ such that $|g(\lambda)| \leq c_3 e^{c_3|\lambda|}$ whenever $\lambda \in V$, for every $g \in \mathscr{G}$.*

Then, for every $k \in \mathbb{N}$,

$$\int_V e^{iRf(\lambda)}g(\lambda)\, \mathrm{d}\lambda = e^{iRf(0)}\sqrt{\frac{(2\pi i)^m}{R^m \det f''(0)}} \sum_{j=0}^k \frac{L_{j,f}g}{R^j} + O\left(\frac{1}{R^{\frac{m}{2}+k+1}}\right) \qquad (2.4)$$

as $R \to +\infty$, uniformly as $f \in \mathscr{F}$ and $g \in \mathscr{G}$, where

$$L_{j,f}g = i^{-j} \sum_{\mu=0}^{2j} \frac{(f''(0)^{-1}\partial, \partial)^{\mu+j}[(f - P_{2,0}f)^\mu g](0)}{2^{\mu+j}\mu!(\mu+j)!}.$$

In particular, $L_{0,f}g = g(0)$.

Proof. Take some $\tau \in C_c^\infty(\mathbb{R}^m)$ such that $\chi_{B(0,\eta)} \leq \tau \leq \chi_{B(0,2\eta)}$. Then split the integral as

$$\int_V e^{iRf(\lambda)}g(\lambda)\, \mathrm{d}\lambda = \int_V e^{iRf(\lambda)}g(\lambda)\tau(\lambda)\, \mathrm{d}\lambda + \int_V e^{iRf(\lambda)}g(\lambda)(1 - \tau(\lambda))\, \mathrm{d}\lambda$$

and apply [11, Theorem 7.7.5] to the first term, thanks to the first assumption in 1 and the assumptions 2 and 3: this represents the main contribution to the integral, and gives the right hand side of (2.4). The second term is instead negligible, since by the second

assumption in 1 and by 4 we get, if R is large enough,

$$\left| \int_V e^{iRf(\lambda)} g(\lambda)(1-\tau(\lambda)) \, d\lambda \right| \lesssim \int_{|\lambda| \geq \eta} e^{-R\operatorname{Im}f(\lambda)+c_3|\lambda|} \, d\lambda \lesssim \int_\eta^\infty e^{-Rc_1\rho + c_3\rho} \rho^{m-1} \, d\rho$$

$$= \int_\eta^\infty e^{-(c_1 R\rho - (1+c_3)\rho) - \rho} \rho^{m-1} \, d\rho$$

$$\lesssim e^{-(c_1 R - (1+c_3))\eta} \int_0^\infty e^{-\rho} \rho^{m-1} \, d\rho,$$

which is $O\left(e^{-Rc_1\eta}\right)$. The proof is complete. $\qquad\square$

Remark 2.8. Theorem 2.7 covers more cases than only oscillatory integrals. Indeed, assume we have an integral of the form

$$\int_V e^{-Rf(\lambda)} g(\lambda) \, d\lambda$$

where f is real. Under suitable assumptions, such integrals are usually treated via Laplace's method (see, e.g., [8] and [23]). In this case, one can use directly Theorem 2.7, by substituting $\operatorname{Im}f$ by f in the assumptions 1-4, thus getting

$$\int_V e^{-Rf(\lambda)} g(\lambda) \, d\lambda = \sqrt{\frac{(2\pi)^m}{R^m \det f''(0)}} \sum_{j=0}^k \frac{L_{j,f}g}{R^j} + O\left(\frac{1}{R^{\frac{m}{2}+k+1}}\right), \qquad (2.5)$$

with the obvious modifications on $L_{j,f}g$. Coherently, in such cases Theorem 2.7 will be referred to as *Laplace's method*.

3. The Heisenberg Group

In this section we deal with the case $m = 1$, namely when $G = \mathbb{H}^n$ is the Heisenberg group. The function p_{1,k_1,k_2} of Definition 2.1 here reads

$$p_{1,k_1,k_2}(x,t) = \frac{2(-1)^{k_1} i^{k_2}}{(4\pi)^{n+1}} \int_{\mathbb{R}} e^{i\lambda|t| - \frac{|x|^2}{4}\lambda \coth(\lambda)} \frac{\lambda^{n+k_1+k_2} \cosh(\lambda)^{k_1}}{\sinh(\lambda)^{n+k_1}} \, d\lambda.$$

Indeed, the absolute values of λ in the integral (2.3) can be removed by parity reasons. We begin by introducing some functions which greatly simplify the notation.

Definition 3.1. Define

$$h_{k_1,k_2}(R,t) := (-1)^{k_1} i^{k_2} \int_{\mathbb{R}} e^{i\lambda|t| - R\lambda \coth(\lambda)} \frac{\lambda^{n+k_1+k_2} \cosh(\lambda)^{k_1}}{\sinh(\lambda)^{n+k_1}} \, d\lambda = \int_{\mathbb{R}} e^{iR\varphi_\omega(\lambda)} a_{k_1,k_2}(\lambda) \, d\lambda,$$

where

$$a_{k_1,k_2}(\lambda) = \begin{cases} (-1)^{k_1} i^{k_2} \frac{\lambda^{n+k_1+k_2} \cosh(\lambda)^{k_1}}{\sinh(\lambda)^{n+k_1}} & \text{if } \lambda \notin \pi i\mathbb{Z}, \\ (-1)^{k_1} i^{k_2} \delta_{k_2,0} & \text{if } \lambda = 0, \end{cases}$$

$$\varphi_\omega(\lambda) = \begin{cases} \omega\lambda + i\lambda \coth(\lambda) & \text{if } \lambda \notin \pi i\mathbb{Z}, \\ i & \text{if } \lambda = 0. \end{cases} \qquad (3.1)$$

Notice that

$$p_{1,k_1,k_2}(x,t) = \frac{2}{(4\pi)^{n+1}} h_{k_1,k_2}(R,t)$$

for all $(x,t) \in \mathbb{H}^n$; hence we can reduce matters to studying $h_{k_1,k_2}(R,t)$. Observe moreover that $y_\omega = \theta^{-1}(\omega) \in [0,\pi)$, since $\omega \geq 0$.

It will be convenient to reverse the dependence relation between (R,ω) and (x,t): hence, we shall no longer consider R and ω as functions of (x,t), but rather as "independent variables". In this order of ideas, the formula $|t| = R\omega$ should sound as a definition.

Our intent will be to apply Theorem 2.7 to a function closely related to h_{k_1,k_2}; hence we shall find some stationary points of the phase of h_{k_1,k_2}, namely φ_ω. The lemma below is of fundamental importance.

Lemma 3.2. [10, § 3, Lemma 6] $\varphi'_\omega(\lambda) = \omega + \tilde{\theta}(i\lambda)$ *for all* $\lambda \notin \pi i\mathbb{Z}^*$, *where* $\tilde{\theta}$ *is the analytic continuation of* θ *to* $\mathrm{Dom}(\varphi_\omega)$. *In particular,* iy_ω *is a stationary point of* φ_ω.

3.1. I. Estimates for $(x,t) \to \infty$ **while** $4|t|/|x|^2 \leq C$.

Theorem 3.3. *Fix* $C > 0$. *If* $(x,t) \to \infty$ *while* $0 \leq \omega \leq C$, *then*

$$p_{1,k_1,k_2}(x,t) = \frac{1}{|x|} e^{-\frac{1}{4}d(x,t)^2} \Psi(\omega) \left[(-1)^{k_1+k_2} \frac{y_\omega^{n+k_1+k_2} \cos(y_\omega)^{k_1}}{\sin(y_\omega)^{n+k_1}} + O\left(\frac{1}{|x|^2}\right) \right] \qquad (3.2)$$

where

$$\Psi(\omega) = \begin{cases} \frac{1}{4^n \pi^{n+1}} \sqrt{\frac{\pi \sin(y_\omega)^3}{\sin(y_\omega) - y_\omega \cos(y_\omega)}}, & \text{if } \omega \neq 0, \\ \frac{(3\pi)^{1/2}}{4^n \pi^{n+1}}, & \text{if } \omega = 0. \end{cases}$$

It is worthwhile to stress that the above estimates may *not* be sharp when $\omega \to 0$ and $k_2 > 0$, as well as when $\omega \to \frac{\pi}{2}$ and $k_1 > 0$. In these cases indeed $y_\omega \to 0$ and $y_\omega \to \frac{\pi}{2}$, respectively, and the first term of the asymptotic expansion (3.2) may be smaller than the remainder. However, the *sharp* asymptotics of p_{1,k_1,k_2} when ω remains bounded turn out to be more involved, and for the moment we avoid to treat the complete picture of its asymptotic behaviour. The statement above is just a simplified version of Theorem 4.2 of Section 4.1, where the general case of H-type groups is completely described.

In this section we then limit ourselves to consider Theorem 3.3 in the stated form. Its proof mostly consists in a straightforward generalization of [10, Theorem 2 of § 3], but it can also be seen as Proposition 4.4 of Section 4.1 in the current setting of Heisenberg groups. Nevertheless, for the sake of completeness we give a brief sketch of the proof.

The main idea is to change the contour of integration in the integral defining h_{k_1,k_2} in order to meet a stationary point of φ_ω. Since $\mathrm{Im}\,\varphi_\omega(\lambda) = \omega\,\mathrm{Im}\,\lambda + \mathrm{Re}\,[\lambda \coth(\lambda)]$ for every $\lambda \notin \pi i\mathbb{Z}$, to make this change we need to deepen our knowledge of $\mathrm{Re}\,[\lambda \coth(\lambda)]$ and $|a_{k_1,k_2}|$; this is done in the following lemma, which we state without proof.

Lemma 3.4. *For all* $\lambda, y \in \mathbb{R}$ *such that* $|\lambda| > |y|$,

$$\mathrm{Re}[(\lambda + iy)\coth(\lambda + iy)] = \frac{\lambda\sinh(2\lambda) + y\sin(2y)}{2(\sinh(\lambda)^2 + \sin(y)^2)} > 0.$$

Moreover, for all $\lambda, y \in \mathbb{R}$ *such that either* $y \notin \pi\mathbb{Z}$ *or* $\lambda \neq 0$,

$$|a_{k_1,k_2}(\lambda + iy)| = \frac{|\lambda + iy|^{n+k_1+k_2}(\sinh(\lambda)^2 + \cos(y)^2)^{\frac{k_1}{2}}}{(\sinh(\lambda)^2 + \sin(y)^2)^{\frac{n+k_1}{2}}}.$$

In the following lemma we perform the change of the contour of integration in the definition of h_{k_1,k_2}. Its proof is a simple adaptation of that of [12, Lemma 1.4].

Lemma 3.5. *For all* $y \in [0, +\infty) \setminus \pi\mathbb{N}^*$

$$h_{k_1,k_2}(R,t) = \int_{\mathbb{R}} e^{iR\varphi_\omega(\lambda+iy)} a_{k_1,k_2}(\lambda + iy)\,\mathrm{d}\lambda + 2\pi i \sum_{\substack{k \in \mathbb{N}^* \\ k\pi \in [0,y]}} \mathrm{Res}\left(e^{iR\varphi_\omega} a_{k_1,k_2}, k\pi i\right).$$

Proof of Theorem 3.3. Define

$$\psi_\omega = \varphi_\omega(\cdot + iy_\omega) - \varphi_\omega(iy_\omega)$$

and observe that

$$\varphi_\omega(iy_\omega) = i\omega\,y_\omega + iy_\omega\cot(y_\omega) = i\frac{y_\omega^2}{\sin(y_\omega)^2},$$

since $\omega = \theta(y_\omega)$. Therefore, by Lemma 3.5 (recall that $0 \leq y_\omega < \pi$, so that there are no residues)

$$h_{k_1,k_2}(R,t) = e^{-\frac{1}{4}d(x,t)^2} \int_{\mathbb{R}} e^{iR\psi_\omega(\lambda)} a_{k_1,k_2}(\lambda + iy_\omega) \, d\lambda.$$

Our intent is to apply Theorem 2.7 to the bounded subsets $\mathscr{F} = \{\psi_\omega : \omega \in [0,C]\}$ and $\mathscr{G} = \{a_{k_1,k_2}(\cdot + iy_\omega) : \omega \in [0,C]\}$ of $\mathcal{E}(\mathbb{R})$. Therefore we first verify that the four conditions of its statement hold.

2. Lemmata 3.2 and 2.5 imply that $i\varphi_\omega''(iy_\omega) = -\theta'(-y_\omega) < 0$ for all $\omega \in \mathbb{R}_+$. From the definition of ψ_ω we then get

$$\psi_\omega(0) = \psi_\omega'(0) = 0, \qquad i\psi_\omega''(0) < 0. \tag{3.3}$$

3. Consider the mapping $\psi \colon \mathbb{R} \times (-\pi, \pi) \ni (\lambda, y) \mapsto \psi_{\theta(y)}(\lambda)$. By (3.3), $\partial_1 \psi(0, y) = 0$ and $i\partial_1^2 \psi(0, y) < 0$ for all $y \in [0, \pi)$; moreover, ψ is analytic thanks to Lemma 2.5. Therefore, by Taylor's formula we may find two constants $\eta > 0$ and $C' > 0$ such that $|\partial_1 \psi(\lambda, y)| \geq C'|\lambda|$ for all $\lambda \in [-2\eta, 2\eta]$ and for all $y \in [0, \theta^{-1}(C)]$.

1. Lemma 3.4 implies that

$$\operatorname{Im} \psi(\lambda, y) = \frac{\lambda \cosh(\lambda) \sinh(\lambda) - y \cot(y) \sinh(\lambda)^2}{\sinh(\lambda)^2 + \sin(y)^2}$$

for all $\lambda \in \mathbb{R}$ and for all $y \in (-\pi, \pi)$, $y \neq 0$; moreover, the mapping $(0, \pi) \ni y \mapsto y \cot(y)$ is strictly decreasing and tends to 1 as $y \to 0^+$. Therefore, if $\lambda \neq 0$ and $y \in [0, \pi)$, then

$$\operatorname{Im} \psi(\lambda, y) \geq \frac{\lambda \coth(\lambda) - 1}{1 + \frac{1}{\sinh(\lambda)^2}} > 0$$

since $\lambda \coth(\lambda) - 1 > 0$. Observe finally that, since $\frac{\lambda \coth(\lambda)-1}{1+\frac{1}{\sinh(\lambda)^2}} \sim |\lambda|$ for $\lambda \to \infty$, the second condition is also satisfied.

4. Just observe that $\mathscr{G}$ is bounded in $L^\infty(\mathbb{R})$.

By Theorem 2.7, we then get

$$\int_{\mathbb{R}} e^{iR\psi_\omega(\lambda)} a_{k_1,k_2}(\lambda + iy_\omega) \, d\lambda = \frac{(2\pi)(4\pi)^n}{|x|} \Psi(\omega) a_{k_1,k_2}(iy_\omega) + O\left(\frac{1}{|x|^3}\right)$$

for $R \to +\infty$, uniformly as ω runs through $[0, C]$. $\qquad \square$

From now on, we shall consider the case $\omega \to +\infty$. The method of stationary phase cannot be applied directly in this case, since $y_\omega \to \pi$, and $i\pi$ is a pole of the phase (as well as of the amplitude). Although it seems possible to adapt the techniques developed by Li [16] to this situation, our proof follows the idea presented by Hueber and Müller [12, Theorem 1.3 (i)]. We shall take advantage of this singularity to get the correct behaviour of h_{k_1,k_2}, by means of the residues obtained by Lemma 3.5.

3.2. **II. Estimates for $\delta \to 0^+$ and $\kappa \to +\infty$.** We state below the main result of this section.

Theorem 3.6. *For $\delta \to 0^+$ and $\kappa \to +\infty$*

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2} \pi^{k_1+k_2}}{4^n (\pi\delta)^{n+k_1-1} \sqrt{2\pi\kappa}} e^{-\frac{1}{4}d(x,t)^2} \left[1 + O\left(\frac{1}{\kappa} + \delta\right)\right].$$

The proof of Theorem 3.6 will be prepared by several lemmata. The first step will be to invoke Lemma 3.5, of which we keep the notation, to move the contour of integration beyond the singularity at πi; since at $2\pi i$ there is another one, it seems convenient to stop at $\frac{3\pi i}{2}$. We first notice that the integral on $\mathbb{R} + \frac{3\pi i}{2}$ may be neglected in some circumstances, as the following lemma shows. It is essentially [12, Lemma 1.4], so we omit the proof.

Lemma 3.7. *There exists a constant $C' > 0$ such that*

$$\left| \int_{\mathbb{R}} e^{iR\varphi_\omega \left(\lambda + \frac{3\pi i}{2} \right)} a_{k_1,k_2} \left(\lambda + \frac{3\pi i}{2} \right) \mathrm{d}\lambda \right| \le C' e^{-\frac{3\pi |t|}{2}}.$$

Hence, matters are reduced to the computation of the residue. First of all, define

$$r(\lambda) = \begin{cases} 1 + \frac{1}{\lambda} - \pi(1+\lambda)\cot(\pi\lambda), & \text{if } \lambda \notin \mathbb{Z}, \\ 0, & \text{if } \lambda = 0, \end{cases}$$

and observe that r is holomorphic on its domain. It will be useful to define also

$$\tilde{\varphi}_{k_1,k_2}(R,\xi) := \begin{cases} e^{Rr(-\xi)} \frac{(\pi\xi)^{n+k_1}\cos(\pi\xi)^{k_1}(1-\xi)^{n+k_1+k_2}}{\sin(\pi\xi)^{n+k_1}}, & \text{if } \xi \notin \mathbb{Z}, \\ 1, & \text{if } \xi = 0, \end{cases}$$

and

$$\varphi_{\delta,k_1,k_2}(s) := e^{-i(n+k_1-1)s}\tilde{\varphi}_{k_1,k_2}(0,\delta e^{is}) \tag{3.4}$$

whenever $\delta e^{is} \notin \mathbb{Z}^*$. The following lemma may be proved again on the lines of [12, Lemma 1.4].

Lemma 3.8. *For every $\delta < 1$*

$$2\pi i \operatorname{Res}\left(e^{iR\varphi_\omega} a_{k_1,k_2}, \pi i\right) = \frac{(-1)^{k_2}\pi^{k_2+1}}{\delta^{n+k_1-1}} e^{-R-\pi|t|} \int_{-\pi}^{\pi} e^{\kappa\cos(s)+Rr(-\delta e^{is})} \varphi_{\delta,k_1,k_2}(s)\,\mathrm{d}s. \tag{3.5}$$

Therefore, it remains only to estimate the integral in (3.5), namely

$$H_{k_1,k_2}(R,t) := \int_{-\pi}^{\pi} e^{\kappa\cos(s)+Rr(-\delta e^{is})} \varphi_{\delta,k_1,k_2}(s)\,\mathrm{d}s = \int_{-\pi}^{\pi} e^{\kappa q_\delta(-is)} \varphi_{\delta,k_1,k_2}(s)\,\mathrm{d}s, \tag{3.6}$$

where

$$q(\delta,\zeta) = q_\delta(\zeta) := \cosh(\zeta) + \frac{\delta}{2} r(-\delta e^{-\zeta}). \tag{3.7}$$

Notice that we may apply Theorem 2.7 only when $\kappa \to +\infty$, and this is why we confined ourselves to the case where $\delta \to 0^+$ (and we shall assume $0 < \delta < 1$) and $\kappa \to +\infty$.

Again for technical convenience, we shall reverse the dependence relation between (δ, κ) and $(R, |t|)$, thus assuming that δ and κ are "independent variables". Indeed, δ and κ completely describe our problem, since

$$|t| = \frac{\kappa}{2\pi\delta}, \qquad R = \frac{\kappa\delta}{2},$$

and $|t| + R \to +\infty$ if $\delta \to 0^+$ and $\kappa \to +\infty$. We shall sometimes let δ assume complex values. The following lemma is essentially [12, Lemma 1.2]. We present a slightly shorter proof.

Lemma 3.9. *q is holomorphic on the set $\{(\delta,\zeta) \in \mathbb{C}\times\mathbb{C} \mid \delta e^{-\zeta} \notin \mathbb{Z}^*\}$. Moreover there exist two constants $\delta_1 \in (0,1)$ and $\eta_1 > 0$ such that for all $\delta \in \mathrm{B}_{\mathbb{C}}(0,\delta_1)$ there is a unique $\sigma_\delta \in \mathrm{B}_{\mathbb{C}}(0,\eta_1)$ such that $q'_\delta(\sigma_\delta) = 0$. Then the mapping $\mathrm{B}_{\mathbb{C}}(0,\delta_1) \ni \delta \mapsto \sigma_\delta$ is holomorphic and real on $(-\delta_1,\delta_1)$. Finally, $\sigma_\delta = O(\delta^2)$ and $q_\delta(\sigma_\delta) = 1 + O(\delta^2)$ for $\delta \to 0$.*

Proof. q is holomorphic since r is. Furthermore, $\partial_2 q(0,0) = 0$ and $\partial_2^2 q(0,0) = 1$. Therefore, the implicit function theorem (cf. [5, Proposition 6.1 of IV.5.6]) implies the existence of some δ_1 and η_1 as in the statement, the holomorphy of the mapping $\delta \mapsto \sigma_\delta$, and that $\frac{d}{d\delta}\sigma_\delta|_{\delta=0} = 0$. Notice also that $\sigma_0 = 0$, so that $\sigma_\delta = O(\delta^2)$ for $\delta \to 0$ by Taylor's formula.

Since q_δ is real on real numbers, $q'_\delta(\overline{\sigma_\delta}) = \overline{q'_\delta(\sigma_\delta)} = 0$; thus $\sigma_\delta = \overline{\sigma_\delta}$ for the uniqueness of σ_δ, and hence $\sigma_\delta \in \mathbb{R}$ for all $\delta \in (-\delta_1,\delta_1)$.

The last assertion follows from Taylor's formula, since $q_0(\sigma_0) = 1$ and $\frac{d}{d\delta}q_\delta(\sigma_\delta)|_{\delta=0} = \partial_1 q(0,0) + \partial_2 q(0,0)\frac{d}{d\delta}\sigma_\delta|_{\delta=0} = 0$. $\qquad\square$

The contour of integration can now be changed in order to apply the method of stationary phase. For the remainder of this section, we keep δ_1 and η_1 of Lemma 3.9 fixed.

Lemma 3.10. *Let $\tau \in C_c^\infty(\mathbb{R})$ such that $\chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]} \leq \tau \leq \chi_{[\pi, \pi]}$. Define, for all $\delta \in (-\delta_1, \delta_1)$, the path $\gamma_\delta(s) := s + i\sigma_\delta \tau(s)$, and*

$$F_\delta(s) := -iq_\delta(-i\gamma_\delta(s)) + iq_\delta(\sigma_\delta) \qquad and \qquad \psi_{\delta,k_1,k_2} := (\varphi_{\delta,k_1,k_2} \circ \gamma_\delta)\, \gamma_\delta'.$$

Then

$$H_{k_1,k_2}(R,t) = e^{\kappa\, q_\delta(\sigma_\delta)} \int_{-\pi}^{\pi} e^{i\kappa F_\delta(s)} \psi_{\delta,k_1,k_2}(s)\, \mathrm{d}s.$$

Proof of Theorem 3.6. We shall apply Theorem 2.7 to the bounded subsets $\mathscr{F} = \{F_\delta \colon \delta \in (0, \delta_2)\}$ and $\mathscr{G} = \{\psi_{\delta,k_1,k_2} \colon \delta \in (0, \delta_2)\}$ of $\mathcal{E}((-\pi, \pi))$, depending on some δ_2 to be fixed later. Hence we check that the four conditions of the statement are satisfied.

1. The mapping $F\colon (-\delta_1, \delta_1) \times \mathbb{R} \ni (\delta, s) \mapsto F_\delta(s)$ is of class C^∞, and $\partial_2^2 F(0,0) = i$; thus we may find $\delta_2 \in (0, \delta_1)$, $\eta_2 \in \left(0, \frac{\pi}{2}\right)$ and $C'' > 0$ such that $\operatorname{Im} \partial_2^2 F(\delta, s) \geq 2C''$ for all $\delta \in [-\delta_2, \delta_2]$ and for all $s \in [-2\eta_2, 2\eta_2]$. From Taylor's formula then

$$\operatorname{Im} F(\delta, s) = \int_0^s \partial_2^2 \operatorname{Im} F(\delta, \tau)(s - \tau)\, \mathrm{d}\tau \geq C'' s^2$$

   for all $s \in [-2\eta_2, 2\eta_2]$ and for all $\delta \in [-\delta_2, \delta_2]$. Since $\operatorname{Im} F(0, s) = 1 - \cos(s)$ for all $s \in [-\pi, \pi]$, by reducing δ_2 and C'' if necessary one may assume that $\operatorname{Im} F(\delta, s) \geq C'' \pi^2 \geq C'' s^2$ for all $s \in \mathbb{R}$ such that $2\eta_2 \leq |s| \leq \pi$ and for all $\delta \in [-\delta_2, \delta_2]$.
2. It is immediately seen that $F_\delta(0) = F_\delta'(0) = 0$ by definition.
3. For every $\delta \in [-\delta_2, \delta_2]$ and $s \in [-2\eta_2, 2\eta_2]$

$$|\partial_2 F(\delta, s)| \geq |\partial_2 \operatorname{Im} F(\delta, s)| = \left| \int_0^s \partial_2^2 \operatorname{Im} F(\delta, \tau)\, \mathrm{d}\tau \right| \geq 2C'' |s|.$$

4. Just observe that $\mathscr{G}$ is bounded in $L^\infty((-\pi, \pi))$.

By Theorem 2.7, then,

$$\int_{-\pi}^{\pi} e^{i\kappa F_\delta(s)} \tau_2(s) \psi_{\delta,k_1,k_2}(s)\, \mathrm{d}s = \sqrt{\frac{2\pi i}{\kappa F_\delta''(0)}} \psi_{\delta,k_1,k_2}(0) + O\left(\frac{1}{\kappa^{3/2}}\right).$$

It is then easily seen that $F_\delta''(0) = iq_\delta''(\sigma_\delta) = i(1 + O(\delta))$ and $\psi_{\delta,k_1,k_2}(0) = \varphi_{\delta,k_1,k_2}(i\sigma_\delta) = 1 + O(\delta)$ for $\delta \to 0^+$.

Now, by construction,

$$-R - \pi|t| + \kappa q_\delta(s) = iR\, \varphi_\omega(\pi i(1 - \delta e^{-s}))$$

for s in a neighbourhood of σ_δ. Take $\delta_3 \in (0, \delta_2]$ so that $(1 - \delta e^{-\sigma_\delta}) \in (-1, 1)$ for all $\delta \in [0, \delta_3]$, and fix $\delta \in (0, \delta_3)$ and $t \neq 0$. We shall prove that

$$y_\omega = \pi(1 - \delta e^{-\sigma_\delta}).$$

Indeed, y_ω is the unique element of $(-\pi, \pi)$ such that $\varphi_\omega'(iy_\omega) = 0$; furthermore, $\pi(1 - \delta e^{-\sigma_\delta}) \in (-\pi, \pi)$ for the choice of δ_3, and $-R\pi\delta e^{-\sigma_\delta}\varphi_\omega'(\pi i(1 - \delta e^{-\sigma_\delta})) = \kappa q_\delta'(\sigma_\delta) = 0$. Therefore, $y_\omega = \pi(1 - \delta e^{-\sigma_\delta})$. Finally, equality holds by analyticity whenever both sides are defined. It then follows that

$$-R - \pi|t| + \kappa q_\delta(\sigma_\delta) = iR\varphi_\omega(iy_\omega) = -\frac{1}{4}d(x,t)^2. \tag{3.8}$$

Finally observe that, by definition of κ and δ, and by Lemma 3.9,

$$-\frac{3\pi|t|}{2} + R + \pi|t| - \kappa q_\delta(\sigma_\delta) + \log\kappa \leq -\frac{\kappa}{2\pi\delta}\left[\frac{\pi}{2} - \pi\delta^2 + 2\pi\delta\left(1 + O\left(\delta^2\right)\right) - 2\pi\delta\frac{\log\kappa}{\kappa}\right],$$

which tends to $-\infty$ as $\delta \to 0^+$ and $\kappa \to +\infty$. This means that

$$e^{-\frac{3\pi|t|}{2}} = o\left(\frac{e^{-R-\pi|t|+\kappa q_\delta(\sigma_\delta)}}{\kappa}\right)$$

for $\kappa \to +\infty$, uniformly as δ runs through $(0, \delta_2]$. Our assertion is then a consequence of Lemmata 3.5 and 3.7. $\qquad \square$

3.3. III and IV. Estimates for $\delta \to 0^+$ and κ bounded.
Strictly speaking, cases **III** and **IV** have already been considered together by Hueber and Müller [12, Theorem 1.3 (ii)]. Despite this, we shall follow a different approach similar to that of Li [15], which will allow us to get slightly better results.

We first recall that, for all $\nu \in \mathbb{Z}$ and $\zeta \in \mathbb{C}$, the modified Bessel function I_ν of order ν is defined as

$$I_\nu(\zeta) = \sum_{k \in \mathbb{N}} \frac{\zeta^{2k+\nu}}{2^{2k+\nu} k! \, \Gamma(k+\nu+1)}.$$

If $s > 0$, then also

$$I_\nu(s) = \frac{1}{2\pi} \int_{-\pi}^\pi e^{s\cos(\xi)-i\nu\xi} \, \mathrm{d}\xi,$$

as one can verify from [9, 7.3.1 (2)] by applying the change of variables $\psi = \frac{\pi}{2} - \varphi$ and by taking into account the relationship [9, 7.2.2 (12)] between $I_\nu = I_{-\nu}$ and J_ν, and also the periodicity of the integrand. Notice that for $s > 0$ and $\nu \in \mathbb{Z}$, $I_\nu(s)$ is strictly positive unless $s = 0$ and $\nu \neq 0$. The main result of this section is the following.

Theorem 3.11. *Fix $C > 1$. If $\delta \to 0^+$ while $1/C \leq \kappa \leq C$, then*

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2} \pi^{k_1+k_2}}{4^n(\pi\delta)^{n+k_1-1}} e^{-\frac{1}{4}d(x,t)^2} e^{-\kappa} I_{n+k_1-1}(\kappa) \left[1 + O(\delta)\right]. \tag{3.9}$$

When $\kappa \to 0^+$ and $|t| \to +\infty$

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2} \pi^{k_1+k_2}}{4^n(n+k_1-1)!} |t|^{n+k_1-1} e^{-\frac{1}{4}d(x,t)^2} \left[1 + O\left(\frac{1}{|t|} + \kappa\right)\right]. \tag{3.10}$$

Lemma 3.12. *For every $N \in \mathbb{N}$*

$$H_{k_1,k_2}(R,t) = 2\pi \sum_{|\alpha| \leq N} I_{n+k_1-1-\alpha_2}(\kappa) \frac{\partial^\alpha \tilde{\varphi}_{k_1,k_2}(0,0)\kappa^{\alpha_1}}{2^{\alpha_1}\alpha!} \delta^{|\alpha|} + O\left(\delta^{N+1}\right)$$

for $\delta \to 0^+$, uniformly as κ runs through $[0, C]$.

Proof. By substituting (3.4) in (3.5) and by Taylor's formula applied to $\tilde{\varphi}_{k_1,k_2}$,

$$H_{k_1,k_2}(R,t) = \int_{-\pi}^\pi e^{\kappa\cos(s)} e^{-i(n+k_1-1)s} \tilde{\varphi}_{k_1,k_2}(R, \delta e^{is}) \, \mathrm{d}s$$

$$= \sum_{|\alpha| \leq N} \frac{\partial^\alpha \tilde{\varphi}_{k_1,k_2}(0,0)}{\alpha!} R^{\alpha_1} \delta^{\alpha_2} \int_{-\pi}^\pi e^{\kappa\cos(s)} e^{-i(n+k_1-1-\alpha_2)s} \, \mathrm{d}s + \mathcal{R}_{N+1}(\delta, \kappa)$$

$$= 2\pi \sum_{|\alpha| \leq N} I_{n+k_1-1-\alpha_2}(\kappa) \frac{\partial^\alpha \tilde{\varphi}_{k_1,k_2}(0,0)\kappa^{\alpha_1}}{2^{\alpha_1}\alpha!} \delta^{|\alpha|} + \mathcal{R}_{N+1}(\delta, \kappa),$$

where the last equality holds since $R = \frac{\delta\kappa}{2}$. Moreover, $\mathcal{R}_{N+1}(\delta, \kappa)$ is easily seen to be $O\left(\delta^{N+1}\right)$ for $\delta \to 0^+$ uniformly as κ runs through $[0, C]$. This completes the proof. $\qquad \square$

Proof of Theorem 3.11. Lemmata 3.7 and 3.8 imply that

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2} 2\pi^{k_2-n}}{4^{n+1}\delta^{n+k_1-1}} e^{-R-\pi|t|} H_{k_1,k_2}(R,t) + O\left(e^{-\frac{3\pi|t|}{2}}\right).$$

Moreover, recall that $\delta|t| = \frac{\kappa}{2\pi}$ and $R = \frac{\kappa\delta}{2}$; therefore, for every $N \in \mathbb{N}$,

$$e^{-\frac{3\pi|t|}{2}} = o\left(\delta^{N+2-n-k_1}e^{-R-\pi|t|}\right) \tag{3.11}$$

as $\delta \to 0^+$, uniformly as κ runs through $[1/C, C]$. By (3.8) and Lemma 3.9, the first assertion follows from Lemma 3.12 for $N = 0$.

As for (3.10), observe first that $\kappa \to 0^+$ and $|t| \to +\infty$ is equivalent to saying $\delta, \kappa \to 0^+$ and $\delta = o(\kappa)$. Then Lemma 3.12 with $N = n + k_1 - 1$ and an easy development of the Bessel function in a neighbourhood of 0 imply that

$$p_{1,k_1,k_2}(x,t) = \frac{\pi^{k_1+k_2}(-1)^{k_2}}{4^n(\pi\delta)^{n+k_1-1}}e^{-\pi|t|-R}\left[\kappa^{n+k_1-1}\frac{I_{n+k_1-1}^{(n+k_1-1)}(0)}{(n+k_1-1)!} + O(\kappa^{n+k_1})\right.$$

$$\left. + \sum_{1\le|\alpha|\le n+k_1-1} O\left(I_{n+k_1-1-\alpha_2}(\kappa)\kappa^{\alpha_1}\delta^{|\alpha|}\right) + O(\delta^{n+k_1})\right] + O\left(e^{-\frac{3\pi|t|}{2}}\right).$$

Since $\delta = o(\kappa)$, one has $\delta^{\alpha_2+\alpha_1-1} = O(\kappa^{\alpha_2+\alpha_1-1})$ for every $\alpha \neq 0$. Therefore,

$$\sum_{1\le|\alpha|\le n+k_1-1} O\left(I_{n+k_1-1-\alpha_2}(\kappa)\kappa^{\alpha_1}\delta^{|\alpha|}\right) = \sum_{1\le|\alpha|\le n+k_1-1} O\left(\kappa^{n+k_1-2+2\alpha_1}\delta\right)$$

$$= O\left(\kappa^{n+k_1-2}\delta\right).$$

Since $\frac{\kappa}{2\pi\delta} = |t|$ and $I_{n+k_1-1}^{(n+k_1-1)}(0) = \frac{1}{2^{n+k_1-1}}$, we get

$$p_{1,k_1,k_2}(x,t) = \frac{\pi^{k_1+k_2}(-1)^{k_2}}{4^n(n+k_1-1)!}e^{-\pi|t|-R}|t|^{n+k_1-1}\left[1 + O\left(\frac{1}{|t|} + \kappa + \delta\right)\right] + O\left(e^{-\frac{3\pi|t|}{2}}\right).$$

Finally, $\delta = o\left(\frac{1}{|t|}\right)$ since $\delta|t| = \frac{\kappa}{2\pi}$; moreover

$$e^{-\frac{3\pi|t|}{2}} = o\left(e^{-\pi|t|-R}|t|^{n+k_1-2}\right)$$

since $R \to 0^+$ and $|t| \to +\infty$. The assertion follows. $\qquad\square$

The estimates in cases **II**, **III**, and **IV** can be put together. This is done in the following corollary, which will turn out to be fundamental later on. Define first, for $\zeta \in \mathbb{C}$ and $\nu \in \mathbb{Z}$,

$$\tilde{I}_\nu(\zeta) := \sum_{k\ge0} \frac{\zeta^{2k}}{2^{2k+\nu}k!\Gamma(k+\nu+1)}.$$

From now on we shall use the following abbreviation. We keep the notation of Lemma 3.9.

Definition 3.13. For $\delta \in B_{\mathbb{C}}(0, \delta_1)$, define $\rho(\delta) := q_\delta(\sigma_\delta)$.

By Lemma 3.9, ρ is a holomorphic function such that $\rho(0) = 1$ and $\rho'(0) = 0$, so that $\rho(\delta) = 1 + O(\delta^2)$ as $\delta \to 0$.

Corollary 3.14. *When $(x,t) \to \infty$ and $\delta \to 0^+$*

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2}\pi^{k_1+k_2}}{2^{n-k_1+1}}|t|^{n+k_1-1}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa\rho(\delta)}\tilde{I}_{n+k_1-1}\left(\kappa\rho(\delta)\right)\left[1 + g(|x|,|t|)\right],$$

where

$$g(|x|,|t|) = \begin{cases} O\left(\delta + \frac{1}{\kappa}\right) & \text{if } \delta \to 0^+ \text{ and } \kappa \to +\infty, \\ O(\delta) & \text{if } \delta \to 0^+ \text{ and } \kappa \in [1/C, C], \\ O\left(\frac{1}{|t|} + \kappa\right) & \text{if } \delta \to 0^+ \text{ and } \kappa \to 0^+ \end{cases} \tag{3.12}$$

for every $C > 1$.

Proof. **1.** Assume first that $\kappa \to +\infty$. Since $I_\nu(s) = \frac{e^s}{\sqrt{2\pi s}}\left[1 + O\left(\frac{1}{s}\right)\right]$ for $s \to +\infty$, $\nu \in \mathbb{Z}$ (cf. [9, 7.13.1 (5)]),

$$\tilde{I}_\nu(s) = \frac{e^s}{s^\nu\sqrt{2\pi s}}\left[1 + O\left(\frac{1}{s}\right)\right] \quad \text{for } s \to +\infty. \tag{3.13}$$

Therefore, Theorem 3.6 implies that

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2}\pi^{k_1+k_2}}{4^n(\pi\delta)^{n+k_1-1}\sqrt{2\pi\kappa}}e^{-\frac{1}{4}d(x,t)^2}\left[1 + O\left(\frac{1}{\kappa} + \delta\right)\right]$$

$$= \frac{(-1)^{k_2}\pi^{k_1+k_2}\tilde{I}_{n+k_1-1}\left(\kappa\rho(\delta)\right)}{2^{n-k_1+1}}|t|^{n+k_1-1}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa\rho(\delta)}$$

$$\times \left[1 + O\left(\frac{1}{\kappa\rho(\delta)}\right)\right]\left[1 + O\left(\frac{1}{\kappa} + \delta\right)\right]$$

$$= \frac{(-1)^{k_2}\pi^{k_1+k_2}\tilde{I}_{n+k_1-1}\left(\kappa\rho(\delta)\right)}{2^{n-k_1+1}}|t|^{n+k_1-1}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa\rho(\delta)}\left[1 + O\left(\frac{1}{\kappa} + \delta\right)\right],$$

since $\rho(\delta) = 1 + O(\delta^2)$ and $\frac{2|t|}{\kappa} = \frac{1}{\pi\delta}$.

2. Assume now that $\kappa \in [1/C, C]$ for some $C > 1$. Then, by Theorem 3.11,

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2}\pi^{k_1+k_2}}{4^n(\pi\delta)^{n+k_1-1}}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa}I_{n+k_1-1}(\kappa)\left[1 + O\left(\delta\right)\right]$$

$$= \frac{(-1)^{k_2}\pi^{k_1+k_2}}{2^{n-k_1+1}}|t|^{n+k_1-1}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa\rho(\delta)}\tilde{I}_{n+k_1-1}(\kappa\rho(\delta))\left[1 + O\left(\delta^2\right)\right]\left[1 + O\left(\delta\right)\right]$$

$$= \frac{(-1)^{k_2}\pi^{k_1+k_2}}{2^{n-k_1+1}}|t|^{n+k_1-1}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa\rho(\delta)}\tilde{I}_{n+k_1-1}(\kappa\rho(\delta))\left[1 + O\left(\delta\right)\right],$$

where the second equality holds since $I_{n+k_1-1}(\kappa\rho(\delta)) - I_{n+k_1-1}(\kappa) = O(\kappa(\rho(\delta)-1)) = O(\delta^2)$ uniformly as κ runs through $[1/C, C]$ by Taylor's formula.

3. Finally, if $\kappa \to 0^+$ then

$$\tilde{I}_{n+k_1-1}(\kappa) = \tilde{I}_{n+k_1-1}(0) + O(\kappa) = \frac{1}{2^{n+k_1-1}(n+k_1-1)!} + O(\kappa)$$

by the definition of $\tilde{I}_{n+k_1-1}$. Combining this estimate with Theorem 3.11 yields the assertion. $\square$

4. H-type Groups

In this section we deal with the general case $m \geq 1$. In particular, we prove a refined version of Theorem 3.3, and extend Theorems 3.6 and 3.11: this is done through Theorems 4.2, 4.13 and 4.14 respectively. Theorem 4.2 treats the case **I** and is still inspired by [10, Theorem 2 of § 3]. The asymptotic estimates in the other three cases are first obtained in the case m odd, "reducing" to the case $m = 1$; the case m even is then achieved through a descent method.

The first step in order to apply the method of stationary phase is to extend the integrand to a meromorphic function on $\mathbb{C}^m$. If $m > 1$, such extension is no longer automatic as when $m = 1$. A natural way consists in taking advantage of the parity of the functions that appear, as in [7]. Indeed, any continuous branch of $\lambda \mapsto \sqrt{\lambda^2}$ is a holomorphic function which coincides with $\lambda \mapsto \pm|\lambda|$ on $\mathbb{R}^m$; therefore, whenever g is an even holomorphic function defined on a symmetric open subset of $\mathbb{C}$, the function $\lambda \mapsto g(\sqrt{\lambda^2})$ is well-defined, holomorphic, and coincides with $\lambda \mapsto g(|\lambda|)$ on $\mathbb{R}^m$. Hence, we are led to the following definition, which is the analogue of Definition 3.1. We shall use the same notation as before, without stressing the (new) dependence on m.

Definition 4.1. Define

$$h_{k_1,k_2}(R,t) = \int_{\mathbb{R}^m} e^{iR\varphi_\omega(\lambda)} a_{k_1,k_2}(\lambda) \, d\lambda$$

where

$$a_{k_1,k_2}(\lambda) = \begin{cases} (-1)^{k_1} i^{k_2} \frac{\sqrt{\lambda^2}^{n+k_1} \cosh(\sqrt{\lambda^2})^{k_1}}{\sinh(\sqrt{\lambda^2})^{n+k_1}} (\lambda, u_1)^{k_2} & \text{if } \sqrt{\lambda^2} \notin i\pi\mathbb{Z}^*, \\ (-1)^{k_1} i^{k_2} \delta_{k_2,0} & \text{if } \lambda = 0, \end{cases}$$

$$\varphi_\omega(\lambda) = \begin{cases} \omega\,(\lambda, u_1) + i\sqrt{\lambda^2} \coth(\sqrt{\lambda^2}) & \text{if } \sqrt{\lambda^2} \notin i\pi\mathbb{Z}^*, \\ i & \text{if } \lambda = 0. \end{cases} \tag{4.1}$$

Define also

$$a_{k_1,k_2,\omega}(\lambda) := a_{k_1,k_2}(\lambda + iy_\omega u_1). \tag{4.2}$$

Observe again that

$$p_{1,k_1,k_2}(x,t) = \frac{1}{(4\pi)^n (2\pi)^m} h_{k_1,k_2}(R,t)$$

for all $(x,t) \in \mathbb{R}^{2n} \times \mathbb{R}^m$, and that $y_\omega = \theta^{-1}(\omega) \in [0,\pi)$, since $\omega \geq 0$.

4.1. **I. Estimates for $(x,t) \to \infty$ while $4|t|/|x|^2 \leq C$.** The main result of this section is Theorem 4.2 below. As already said, the main ingredient of its proof is the method of stationary phase (cf. Proposition 4.4), which is already employed in [10, Theorem 2 of § 3] to treat the case $n = m = 1$ and $k_1 = k_2 = 0$.

The novelty of considering all the derivatives of the heat kernel p_1 (in other words, all the cases $k_1 \geq 0$ and $k_2 \geq 0$) introduces additional complexity to the developments, since the choice $k = 0$ in (2.4) may not give the sharp asymptotic behaviour of p_{1,k_1,k_2} at infinity, while ω remains bounded. In particular, this happens in the cases $\omega \to 0$ and $k_2 > 0$, or $\omega \to \frac{\pi}{2}$ and $k_1 > 0$. If ω remains bounded and away from 0 and $\frac{\pi}{2}$, the first term is instead enough.

Theorem 4.2. Fix $\varepsilon, C > 0$. If $(x,t) \to \infty$ while $0 \leq \omega \leq C$, then

$$p_{1,k_1,k_2}(x,t) = \frac{1}{|x|^m} e^{-\frac{1}{4}d(x,t)^2} \Psi(\omega) \Upsilon(x,t)$$

where

$$\Psi(\omega) = \begin{cases} \frac{1}{4^n \pi^{n+m}} \sqrt{\frac{(2\pi)^m y_\omega^{m-1} \sin(y_\omega)^3}{2\omega^{m-1}(\sin(y_\omega) - y_\omega \cos(y_\omega))}}, & \text{if } \omega \neq 0, \\ \frac{(3\pi)^{m/2}}{4^n \pi^{n+m}}, & \text{if } \omega = 0, \end{cases} \tag{4.3}$$

and

1. if $\varepsilon \leq \omega \leq \frac{\pi}{2} - \varepsilon$ or $\frac{\pi}{2} + \varepsilon \leq \omega \leq C$,

$$\Upsilon(x,t) = (-1)^{k_1+k_2} \frac{y_\omega^{n+k_1+k_2} \cos(y_\omega)^{k_1}}{\sin(y_\omega)^{n+k_1}} + O\left(\frac{1}{|x|^2}\right); \tag{4.4}$$

2. if $\omega \to 0$ and k_2 is even,

$$\Upsilon(x,t) = \sum_{j=0}^{k_2/2} c_{k_1,k_2,j} \frac{\omega^{k_2-2j}}{|x|^{2j}} + O\left(\sum_{j=0}^{k_2/2} \frac{\omega^{k_2-2j+1}}{|x|^{2j}} + \frac{1}{|x|^{k_2+2}}\right); \tag{4.5}$$

3. if $\omega \to 0$, k_2 is odd and $|t| \to \infty$,

$$\Upsilon(x,t) = \sum_{j=0}^{(k_2-1)/2} c_{k_1,k_2,j} \frac{\omega^{k_2-2j}}{|x|^{2j}} + O\left(\sum_{j=0}^{(k_2+1)/2} \frac{\omega^{k_2-2j+1}}{|x|^{2j}}\right); \tag{4.6}$$

4. *if $\omega \to 0$, k_2 is odd and $0 \le |t| \le C$*

$$\Upsilon(x,t) = c_{k_1,k_2+1,(k_2+1)/2} \frac{|t|}{|x|^{k_2+1}} + O\left(\frac{|t|}{|x|^{k_2+3}}\right); \tag{4.7}$$

5. *if $\omega \to \frac{\pi}{2}$ and k_1 is even,*

$$\Upsilon(x,t) = \sum_{j=0}^{k_1/2} b_{k_1,k_2,j} \frac{\left(\omega - \frac{\pi}{2}\right)^{k_1-2j}}{|x|^{2j}} + O\left(\sum_{j=0}^{k_1/2} \frac{\left(\omega - \frac{\pi}{2}\right)^{k_1-2j+1}}{|x|^{2j}} + \frac{1}{|x|^{k_1+2}}\right); \tag{4.8}$$

6. *if $\omega \to \frac{\pi}{2}$ and k_1 is odd,*

$$\Upsilon(x,t) = \sum_{j=0}^{(k_1-1)/2} b_{k_1,k_2,j} \frac{\left(\omega - \frac{\pi}{2}\right)^{k_1-2j}}{|x|^{2j}} + \frac{b_{k_1,k_2,(k_1+1)/2}}{|x|^{k_1+1}}$$

$$+ O\left(\sum_{j=0}^{(k_1-1)/2} \frac{\left(\omega - \frac{\pi}{2}\right)^{k_1-2j+1}}{|x|^{2j}} + \frac{\omega - \frac{\pi}{2}}{|x|^{k_1+1}} + \frac{1}{|x|^{k_1+3}}\right). \tag{4.9}$$

The coefficients $c_{k_1,k_2,j}$ and $b_{k_1,k_2,j}$ are explicitly given by (4.15), (4.17) and (4.18).

The remainder of this section is devoted to the proof of Theorem 4.2. Since it is quite involved, we split this section into two parts: in the first one we apply the method of stationary phase, while in the second one we find the asymptotics of the development given by Theorem 2.7 which are required to get the sharp developments (4.5)–(4.9). These proofs go through several lemmata.

Remark 4.3. Notice that any pair of terms in the sums appearing in the developments (4.5), (4.6), (4.8), and (4.9) are not comparable with each other under the stated asymptotic condition. Therefore, these developments cannot be simplified. Observe moreover that for k_1 and k_2 fixed the coefficients $b_{k_1,k_2,j}$ (resp. $c_{k_1,k_2,j}$) have the same sign; thus, no cancellation can occur, and our developments are indeed *sharp*. A more detailed description will be given in Section 4.1.2.

Finally, notice that it is possible to obtain even more precise expansions if one does not develop the terms $L_{j,\psi_\omega} a_{k_1,k_2,\omega}$ which appear in Proposition 4.4 below. In particular, in the cases when $\omega \to 0^+$ and $k_2 = 0$, or $\omega \to \frac{\pi}{2}$ and $k_1 = 0$, the explicit computation of $L_{0,\psi_\omega} a_{k_1,k_2,\omega} = a_{k_1,k_2}(iy_\omega u_1)$ leads to better remainders than those in (4.5) and (4.8) respectively.

4.1.1. *Application of the Method of Stationary Phase.* As already said, Proposition 4.4 below is essentially an easy generalization of Theorem 3.3.

Proposition 4.4. *Fix $C > 0$ and let $k \in \mathbb{N}$. Then, if $(x,t) \to \infty$ while $0 \le \omega \le C$,*

$$p_{1,k_1,k_2}(x,t) = \frac{1}{|x|^m} e^{-\frac{1}{4}d(x,t)^2} \Psi(\omega) \left[\sum_{j=0}^{k} \frac{4^j L_{j,\psi_\omega} a_{k_1,k_2,\omega}}{|x|^{2j}} + O\left(\frac{1}{|x|^{2k+2}}\right)\right] \tag{4.10}$$

where Ψ is defined by (4.3).

In the same way as in Section 3.1, we begin by finding some stationary points of the phase of h_{k_1,k_2}, namely φ_ω.

Lemma 4.5. [7, Formula (5.7)] *For all λ such that $\sqrt{\lambda^2} \notin i\pi\mathbb{Z}^*$,*

$$\varphi'_\omega(\lambda) = \omega u_1 + \lambda \frac{\tilde{\theta}(i\sqrt{\lambda^2})}{\sqrt{\lambda^2}}$$

where $\tilde{\theta}$ is the analytic continuation of θ to $\mathrm{Dom}(\varphi_\omega)$. In particular, $iy_\omega u_1$ is a stationary point of φ_ω.

We then change the contour of integration in the integral defining h_{k_1,k_2} in order to meet a stationary point of φ_ω. This is done in the following lemma, which is the analogue of Lemma 3.5.

Lemma 4.6. *For every* $y \in [0,\pi)$

$$h_{k_1,k_2}(R,t) = \int_{\mathbb{R}^m} e^{iR\varphi_\omega(\lambda+iyu_1)} a_{k_1,k_2}(\lambda+iyu_1)\,\mathrm{d}\lambda.$$

Proof. The theorem is proved in a similar fashion to [7, Lemma 5.4]. It may be useful to observe that for every $\lambda \in \mathbb{C}^m$ such that either $\mathrm{Im}\sqrt{\lambda^2} \notin \pi\mathbb{Z}$ or $\mathrm{Re}\sqrt{\lambda^2} \neq 0$, we have

$$|a_{k_1,k_2}(\lambda)| = \frac{|\lambda|^{n+k_1}\left(\sinh\left(\mathrm{Re}\sqrt{\lambda^2}\right)^2 + \cos\left(\mathrm{Im}\sqrt{\lambda^2}\right)^2\right)^{k_1/2}}{\left(\sinh\left(\mathrm{Re}\sqrt{\lambda^2}\right)^2 + \sin\left(\mathrm{Im}\sqrt{\lambda^2}\right)^2\right)^{(n+k_1)/2}}|(\lambda,u_1)|^{k_2},$$

by Lemma 3.4, since $|\sqrt{\lambda^2}| = |\lambda|$. Moreover, a_{k_1,k_2} is bounded on the set $\{\lambda+iyu_1 \colon \lambda \in \mathbb{R}^m, y \in [0,C']\}$ for every $C' \in (0,\pi)$. $\qquad\square$

Proof of Proposition 4.4. Define

$$\psi_\omega = \varphi_\omega(\,\cdot\, + iy_\omega u_1) - \varphi_\omega(iy_\omega u_1)$$

and observe that, since $\sqrt{(iy_\omega u_1)^2} = \pm iy_\omega$ and $\omega = \theta(y_\omega)$, $\varphi_\omega(iy_\omega u_1) = i\frac{y_\omega^2}{\sin(y_\omega)^2}$. Therefore, by Lemma 4.6

$$h_{k_1,k_2}(R,t) = e^{-\frac{1}{4}d(x,t)^2} \int_{\mathbb{R}} e^{iR\psi_\omega(\lambda)} a_{k_1,k_2}(\lambda+iy_\omega u_1)\,\mathrm{d}\lambda.$$

We shall apply Theorem 2.7 to the bounded subsets $\mathscr{F} = \{\psi_\omega \colon \omega \in [0,C]\}$ and $\mathscr{G} = \{a_{k_1,k_2,\omega} \colon \omega \in [0,C]\}$ of $\mathcal{E}(\mathbb{R}^m)$.

  2. Elementary computations show that

$$-i\psi_\omega''(0) = \theta'(y_\omega)u_1 \otimes u_1 + \frac{\omega}{y_\omega}\sum_{j=2}^m u_j \otimes u_j, \qquad (4.11)$$

  so that $\det(-i\psi_\omega''(0)) = \theta'(y_\omega)\left(\frac{\theta(y_w)}{y_w}\right)^{m-1} > 0$. The conditions $\psi_\omega(0) = \psi_\omega'(0) = 0$ hold by construction.

  3. Consider the mapping $\psi \colon \mathbb{R}^m \times (-\pi,\pi) \ni (\lambda,y) \mapsto \psi_{\theta(y)}(\lambda)$. Then, by the preceding arguments, there is $c > 0$ such that $\partial_1 \psi(0,y) = 0$ and $-i\partial_1^2 \psi(0,y) \geq c(\,\cdot\,,\,\cdot\,)$ for all $y \in [0,\pi)$; moreover, ψ is analytic by Lemma 2.5. Therefore, by Taylor's formula we may find two constants $\eta > 0$ and $C' > 0$ such that $|\partial_1 \psi(\lambda,y)| \geq C'|\lambda|$ for all $\lambda \in B_{\mathbb{R}^m}(0,2\eta)$ and for all $y \in [0,\theta^{-1}(C)]$.

  1. Combining [7, Lemmata 5.3 and 5.7], we infer that there is a constant $C'' > 0$ such that

$$\mathrm{Im}\,\psi(\lambda,y) = y\theta(y) + \mathrm{Re}\left[\sqrt{(\lambda+iyu_1)^2}\coth\sqrt{(\lambda+iyu_1)^2}\right] - \frac{y^2}{\sin^2 y} \geq C''|\lambda|$$

  whenever $|\lambda| \geq \eta$ and $0 \leq y \leq \theta^{-1}(C)$.

  4. Just observe that $\mathscr{G}$ is bounded in $L^\infty(\mathbb{R}^m)$.

By Theorem 2.7, then,

$$\int_{\mathbb{R}^m} e^{iR\psi_\omega(\lambda)} a_{k_1,k_2}(\lambda+iy_\omega u_1)\,\mathrm{d}\lambda = \frac{(2\pi)^m(4\pi)^n}{|x|^m}\Psi(\omega)\sum_{j=0}^k \frac{4^j L_{j,\psi_\omega} a_{k_1,k_2,\omega}}{|x|^{2j}} + O\left(\frac{1}{|x|^{m+2k+2}}\right)$$

for $R \to +\infty$, uniformly as ω runs through $[0,C]$. $\qquad\square$

4.1.2. *Further Developments and Completion of the Proof of Theorem 4.2.* We begin by recalling that, for every $j \in \mathbb{N}$,

$$L_{j,\psi_\omega} a_{k_1,k_2,\omega} = i^{-j} \sum_{\mu=0}^{2j} \frac{(\psi_\omega''(0)^{-1}\partial, \partial)^{\mu+j}[(\psi_\omega - P_{2,0}\psi_\omega)^\mu a_{k_1,k_2,\omega}](0)}{2^{\mu+j}\mu!(\mu+j)!}. \qquad (4.12)$$

Thus, the point 1 of Theorem 4.2 follows immediately by taking $k = 0$ in Proposition 4.4, since

$$L_{0,\psi_\omega} a_{k_1,k_2,\omega} = a_{k_1,k_2,\omega}(0) = a_{k_1,k_2}(iy_\omega u_1).$$

As for the other developments, observe that by (4.11)

$$(\psi_\omega''(0)^{-1}\partial, \partial)^{\mu+j}[(\psi_\omega - P_{2,0}\psi_\omega)^\mu a_{k_1,k_2,\omega}](0)$$

$$= \sum_{|\alpha|=\mu+j} \frac{(\mu+j)!}{\alpha!} \frac{1}{(i\theta'(y_\omega))^{\alpha_1}} \left(\frac{y_\omega}{i\omega}\right)^{|\alpha|-\alpha_1} \partial^{2\alpha}[(\psi_\omega - P_{2,0}\psi_\omega)^\mu a_{k_1,k_2,\omega}](0). \qquad (4.13)$$

where

$$\partial^{2\alpha}[(\psi_\omega - P_{2,0}\psi_\omega)^\mu a_{k_1,k_2,\omega}](0)$$

$$= \sum_{\substack{\beta \leq 2\alpha, \\ |\beta| \geq 3\mu}} \frac{(2\alpha)!}{\beta!(2\alpha-\beta)!} \partial^\beta[(\psi_\omega - P_{2,0}\psi_\omega)^\mu](0) \partial^{2\alpha-\beta} a_{k_1,k_2}(iy_\omega u_1). \qquad (4.14)$$

The sum above is restricted to $|\beta| \geq 3\mu$ since $\psi_\omega(\lambda) - P_{2,0}\psi_\omega(\lambda)$ is infinitesimal of order at least 3 for $\lambda \to 0$. Observe moreover that, since $|2\alpha - \beta| = 2|\alpha| - |\beta| \leq 2j - \mu$, we have $|2\alpha - \beta| \leq 2j$ and $|2\alpha - \beta| = 2j$ if and only if $\mu = 0$ and $\beta = 0$. We first consider the case $\omega \to 0$.

Lemma 4.7. *For every $j \in \mathbb{N}$ such that $2j \leq k_2$, define*

$$c_{k_1,k_2,j} := (-1)^{k_1+k_2} \frac{3^{k_2-j}k_2!}{2^{k_2-2j}(k_2-2j)!j!}. \qquad (4.15)$$

Then

$$4^j L_{j,\psi_\omega} a_{k_1,k_2,\omega} = c_{k_1,k_2,j}\omega^{k_2-2j} + O\left(\omega^{k_2-2j+1}\right)$$

for $\omega \to 0$.

Proof. Recall that a_{k_1,k_2} is an analytic function on its domain, and observe that[3]

$$a_{k_1,k_2}(\lambda) = (-1)^{k_1} i^{k_2}\lambda_1^{k_2} + O\left(|\lambda|^{k_2+2}\right)$$

for $\lambda \to 0$. Therefore, for every $h = 0, \ldots, k_2$ we have

$$a_{k_1,k_2}^{(h)}(\lambda) = (-1)^{k_1} i^{k_2} \frac{k_2!}{(k_2-h)!}\lambda_1^{k_2-h} u_1^{\otimes h} + O\left(|\lambda|^{k_2-h+2}\right) \qquad (4.16)$$

as $\lambda \to 0$.

We now consider (4.14). If $|2\alpha - \beta| < 2j$, then by (4.16)

$$\partial^\beta[(\psi_\omega - P_{2,0}\psi_\omega)^\mu](0)\partial^{2\alpha-\beta} a_{k_1,k_2}(iy_\omega u_1) = O\left(y_\omega^{k_2-|2\alpha-\beta|}\right) = O\left(y_\omega^{k_2-2j+1}\right)$$

for $\omega \to 0$. Otherwise, let $|2\alpha - \beta| = 2j$, so that $\mu = 0$ and $\beta = 0$. If $\alpha \neq ju_1$, then (4.16) implies that

$$\partial^{2\alpha} a_{k_1,k_2}(iy_\omega u_1) = O\left(y_\omega^{k_2-2j+2}\right) = O\left(y_\omega^{k_2-2j+1}\right),$$

while, if $\alpha = ju_1$,

$$\partial_1^{2j} a_{k_1,k_2}(iy_\omega u_1) = (-1)^{k_1+k_2} i^{-2j} \frac{k_2!}{(k_2-2j)!} y_\omega^{k_2-2j}.$$

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[3]Here and in the following, λ_1 stands for (λ, u_1).

From this and the fact that

$$\theta'(0) = \lim_{\omega \to 0} \frac{\omega}{y_\omega} = \frac{2}{3}$$

we get the asserted estimate. □

Lemma 4.7 above gives the expansions 2 and 3 of Theorem 4.2. Indeed, it allows us to choose k in Proposition 4.4 as

2. $k = k_2/2$ if k_2 is even, since in this case the last term of the sum in (4.10) is

$$\frac{c_{k_1,k_2,k_2/2}}{|x|^{k_2}} + O\left(\frac{\omega}{|x|^{k_2}}\right)$$

which is bigger than the remainder.

3. $k = (k_2 - 1)/2$ if k_2 is odd and $|t| \to \infty$, since in this case the last term of the sum in (4.10) is

$$c_{k_1,k_2,(k_2-1)/2} \frac{\omega}{|x|^{k_2-1}} + O\left(\frac{\omega^2}{|x|^{k_2-1}}\right) = c_{k_1,k_2,(k_2-1)/2} \frac{|t|}{|x|^{k_2+1}} + O\left(\frac{|t|^2}{|x|^{k_2+3}}\right)$$

which is bigger than the remainder, since $|t| \to \infty$.

The case 4 of Theorem 4.2, that is the case when k_2 is odd, $\omega \to 0$ and $|t|$ is bounded, has to be treated in a different way, since $\omega/|x|^{k_2-1}$ may be comparable with the remainder $1/|x|^{k_2+1}$ or even smaller. Thus, the development given above may not be sharp in this case. To overcome this difficulty, we make use of the following lemma. For the reader's convenience, we also consider k_2 even and a stronger statement than that we need (see Remark 4.16).

Lemma 4.8. *Let $N \in \mathbb{N}$. Then, when $\omega \to 0$,*

$$p_{1,k_1,k_2}(x,t) = \sum_{h=0}^{N} \frac{1}{(2h+1)!} |t|^{2h+1} p_{1,k_1,k_2+2h+1}(x,0) + O\left(|t|^{2N+3} p_{1,k_1,k_2+2N+3}(x,0)\right)$$

if k_2 odd; if k_2 is even

$$p_{1,k_1,k_2}(x,t) = \sum_{h=0}^{N} \frac{1}{(2h)!} |t|^{2h} p_{1,k_1,k_2+2h}(x,0) + O\left(|t|^{2N+2} p_{1,k_1,k_2+2N+2}(x,0)\right).$$

Proof. Assume that k_2 is odd. Then

$$(4\pi)^n (2\pi)^m \left| p_{1,k_1,k_2}(x,t) - \sum_{h=0}^{N} \frac{1}{(2h+1)!} |t|^{2h+1} p_{1,k_1,k_2+2h+1}(x,0) \right|$$

$$= \left| \int_{\mathbb{R}^m} e^{-\frac{|x|^2}{4} |\lambda| \coth |\lambda|} \frac{|\lambda|^{n+k_1} \cosh(|\lambda|)^{k_1}}{\sinh(|\lambda|)^{n+k_1}} (\lambda, u_1)^{k_2} \left\{ e^{i|t|(\lambda,u_1)} - \sum_{h=0}^{N} \frac{[i|t|(\lambda,u_1)]^{2h+1}}{(2h+1)!} \right\} d\lambda \right|$$

$$\leq \frac{|t|^{2N+3}}{(2N+3)!} \int_{\mathbb{R}^m} e^{-\frac{|x|^2}{4} |\lambda| \coth |\lambda|} \frac{|\lambda|^{n+k_1} \cosh(|\lambda|)^{k_1}}{\sinh(|\lambda|)^{n+k_1}} (\lambda, u_1)^{k_2+2N+3} d\lambda$$

$$= \frac{(4\pi)^n (2\pi)^m}{(2N+3)!} |t|^{2N+3} |p_{1,k_1,k_2+2N+3}(x,0)|.$$

The first assertion is then proved. The proof in the case k_2 even is analogous. □

Thus, the case $\omega \to 0$ while $|t|$ remains bounded when k_2 is odd can be related to the same case when k_2 is even, which is completely described by Lemma 4.7. Observe that the expansion appearing in Theorem 4.2, 4, is obtained with the choice $N = 0$ in Lemma 4.8.

We finally consider the case $\omega \to \frac{\pi}{2}$, which as above provides the expansions 5 and 6 of Theorem 4.2.

Lemma 4.9. *Define, for $j \in \mathbb{N}$ such that $2j \leq k_1$,*

$$b_{k_1,k_2,j} := (-1)^{k_2} \frac{k_1!}{2^{k_1-2j}(k_1-2j)!j!} \left(\frac{\pi}{2}\right)^{n+k_1+k_2}, \tag{4.17}$$

and, when k_1 is odd,

$$b_{k_1,k_2,(k_1+1)/2} := (-1)^{k_2} \frac{(k_1+1)!}{[(k_1+1)/2]!} \left(\frac{\pi}{2}\right)^{n+k_1+k_2-1} \left(n + k_1 + k_2 + \frac{\pi^2}{24}(k_1+2) + \frac{3}{2}(m-1)\right). \tag{4.18}$$

Then, for $\omega \to \frac{\pi}{2}$, if $2j \leq k_1$

$$4^j L_{j,\psi_\omega} a_{k_1,k_2,\omega} = b_{k_1,k_2,j} \left(\omega - \frac{\pi}{2}\right)^{k_1-2j} + O\left(\left(\omega - \frac{\pi}{2}\right)^{k_1-2j+1}\right)$$

while if k_1 is odd, then

$$2^{k_1+1} L_{(k_1+1)/2,\psi_\omega} a_{k_1,k_2,\omega} = b_{k_1,k_2,(k_1+1)/2} + O\left(\omega - \frac{\pi}{2}\right).$$

Proof. By elementary computations,

$$a_{k_1,k_2,\pi/2}(\lambda) = (-1)^{k_1} i^{k_2-n} \left(i\frac{\pi}{2}\right)^{n+k_1+k_2} \lambda_1^{k_1}$$
$$+ (-1)^{k_1} i^{k_2-n} \left(i\frac{\pi}{2}\right)^{n+k_1+k_2-1} \left((n+k_1+k_2)\lambda_1^{k_1+1} + \frac{k_1}{2}\lambda_1^{k_1-1}(\lambda^2 - \lambda_1^2)\right) + O\left(|\lambda|^{k_1+2}\right). \tag{4.19}$$

Therefore, since $a_{k_1,k_2,\pi/2}$ is analytic on its domain, we infer that, for every $h = 0, \ldots, k_1$ we have

$$a_{k_1,k_2,\pi/2}^{(h)}(\lambda) = (-1)^{k_1} i^{k_2-n} \left(i\frac{\pi}{2}\right)^{n+k_1+k_2} \frac{k_1!}{(k_1-h)!} \lambda_1^{k_1-h} u_1^{\otimes h} + O\left(|\lambda|^{k_1-h+1}\right) \tag{4.20}$$

as $\lambda \to 0$.

Consider first j such that $2j \leq k_1$. Then, arguing as in the proof of Lemma 4.7 and taking into account (4.20) and the fact that

$$y_\omega - \frac{\pi}{2} = \frac{1}{2}\left(\omega - \frac{\pi}{2}\right) + O\left[\left(\omega - \frac{\pi}{2}\right)^2\right]$$

when $\omega \to \pi/2$, the first assertion follows.

Let now k_1 be odd, so that $(k_1+1)/2$ is an integer. We shall prove that

$$2^{k_1+1} L_{(k_1+1)/2,\psi_{\pi/2}} a_{k_1,k_2,\pi/2} = b_{k_1,k_2,(k_1+1)/2}.$$

The estimate in the statement is then a consequence of this by Taylor expansion.

Since $(\psi_{\pi/2}''(0)^{-1}\partial, \partial)^{\mu+(k_1+1)/2}$ is a differential operator of degree $2\mu + k_1 + 1$ while $[(\psi_\omega - P_{2,0}\psi_\omega)^\mu a_{k_1,k_2,\omega}]$ is infinitesimal of degree $3\mu + k_1$ at 0, the only terms in the sum (4.12) (with $j = (k_1+1)/2$) which are not zero are clearly those for which

$$2\mu + k_1 + 1 \geq 3\mu + k_1,$$

namely $\mu \leq 1$. Consider first $\mu = 0$. Then, since $\theta'(y_{\pi/2}) = 2$, by (4.13)

$$(\psi_{\pi/2}''(0)^{-1}\partial, \partial)^{(k_1+1)/2} a_{k_1,k_2,\pi/2}(0) = i^{-(k_1+1)/2} \sum_{|\alpha|=(k_1+1)/2} \frac{[(k_1+1)/2]!}{2^{\alpha_1}\alpha!} \partial^{2\alpha} a_{k_1,k_2,\pi/2}(0).$$

Observe that, by (4.19), $\partial^{2\alpha} a_{k_1,k_2,\pi/2}(0) \neq 0$ only if $\alpha = ((k_1-1)/2)u_1 + u_h$ for some $h = 1, \ldots, m$. For the choice $h = 1$,

$$\partial_1^{k_1+1} a_{k_1,k_2,\pi/2}(0) = (-1)^{k_1} i^{k_2-n} \left(i\frac{\pi}{2}\right)^{n+k_1+k_2-1} (k_1+1)!(n+k_1+k_2)$$

while, for $h = 2, \ldots, m$,

$$\partial_1^{k_1-1} \partial_h^2 a_{k_1,k_2,\pi/2}(0) = (-1)^{k_1} i^{k_2-n} \left(i\frac{\pi}{2}\right)^{n+k_1+k_2-1} k_1!$$

so that

$$(\psi_{\pi/2}''(0)^{-1}\partial, \partial)^{(k_1+1)/2} a_{k_1,k_2,\pi/2}(0)$$
$$= (-1)^{k_1} \frac{i^{k_2-n-\frac{k_1+1}{2}}}{2^{\frac{k_1+1}{2}}} \left(i\frac{\pi}{2}\right)^{n+k_1+k_2-1} (k_1+1)!(n+k_1+k_2+m-1).$$

Consider now $\mu = 1$. Then by (4.13)

$$(\psi_{\pi/2}''(0)^{-1}\partial, \partial)^{(k_1+3)/2} \left[(\psi_{\pi/2} - P_{2,0}\psi_{\pi/2})a_{k_1,k_2,\pi/2}\right](0)$$
$$= i^{-(k_1+3)/2} \sum_{|\alpha|=(k_1+3)/2} \frac{[(k_1+3)/2]!}{2^{\alpha_1}\alpha!} \partial^{2\alpha} \left[(\psi_{\pi/2} - P_{2,0}\psi_{\pi/2})a_{k_1,k_2,\pi/2}\right](0).$$

Since

$$\psi_{\pi/2}'''(0) = \pi u_1 \otimes u_1 \otimes u_1 + \frac{2}{\pi}\sum_{h=2}^{m}(u_1 \otimes u_h \otimes u_h + u_h \otimes u_1 \otimes u_h + u_h \otimes u_h \otimes u_1),$$

we deduce that the only α for which we get a non-zero term in the above sum are $u_1(k_1 + 1)/2 + u_h$ for $h = 1, \ldots, m$. Now,

$$\partial_1^{k_1+3} \left[(\psi_{\pi/2} - P_{2,0}\psi_{\pi/2})a_{k_1,k_2,\pi/2}\right](0) = \frac{(k_1+3)!}{3!}(-1)^{k_1}i^{k_2-n}\pi \left(i\frac{\pi}{2}\right)^{n+k_1+k_2},$$

while, for $h = 2, \ldots, m$,

$$\partial_1^{k_1+1}\partial_h^2 \left[(\psi_{\pi/2} - P_{2,0}\psi_{\pi/2})a_{k_1,k_2,\pi/2}\right](0) = \frac{2}{\pi}(-1)^{k_1}i^{k_2-n}\left(i\frac{\pi}{2}\right)^{n+k_1+k_2}(k_1+1)!.$$

Therefore,

$$(\psi_{\pi/2}''(0)^{-1}\partial, \partial)^{(k_1+3)/2} \left[(\psi_{\pi/2} - P_{2,0}\psi_{\pi/2})a_{k_1,k_2,\pi/2}\right](0)$$
$$= (-1)^{k_1}i^{k_2-\frac{k_1+1}{2}} \frac{(k_1+1)!}{2^{(k_1+3)/2}}i^{-n}\left(i\frac{\pi}{2}\right)^{n+k_1+k_2-1}(k_1+3)\left[\frac{\pi^2}{12}(k_1+2) + m - 1\right]$$

from which one gets the asserted estimate. □

Theorem 4.2 is now completely proved. In the following table we summarize the asymptotic behaviour, without remainders, of $\Upsilon(x,t)$.

Asymptotic behaviour of $\Upsilon(x,t)$ in case **I**: principal part

$\begin{array}{c}\varepsilon \leq \omega \leq \pi/2 - \varepsilon \\ \text{or} \\ \pi/2 + \varepsilon \leq \omega \leq C\end{array}$			$(-1)^{k_1+k_2}\dfrac{y_\omega^{n+k_1+k_2}\cos(y_\omega)^{k_1}}{\sin(y_\omega)^{n+k_1}}$
$\omega \to 0$	k_2 even		$\displaystyle\sum_{j=0}^{k_2/2} c_{k_1,k_2,j}\frac{\omega^{k_2-2j}}{|x|^{2j}}$
	k_2 odd	$|t| \to \infty$	$\displaystyle\sum_{j=0}^{(k_2-1)/2} c_{k_1,k_2,j}\frac{\omega^{k_2-2j}}{|x|^{2j}}$
		$0 \leq |t| \leq C$	$c_{k_1,k_2+1,(k_2+1)/2}\dfrac{|t|}{|x|^{k_2+1}}$
$\omega \to \frac{\pi}{2}$	k_1 even		$\displaystyle\sum_{j=0}^{k_1/2} b_{k_1,k_2,j}\frac{\left(\omega - \frac{\pi}{2}\right)^{k_1-2j}}{|x|^{2j}}$
	k_1 odd		$\displaystyle\sum_{j=0}^{(k_1-1)/2} b_{k_1,k_2,j}\frac{\left(\omega - \frac{\pi}{2}\right)^{k_1-2j}}{|x|^{2j}} + \frac{b_{k_1,k_2,(k_1+1)/2}}{|x|^{k_1+1}}$

The Other Cases. We now consider the case $\omega \to +\infty$. We begin by showing that, when m is odd, matters can be reduced to the case $m = 1$.

Lemma 4.10. *When m is odd, $m \geq 3$,*

$$p_{1,k_1,k_2}^{(m)}(x,t) = \sum_{k=1}^{\frac{m-1}{2}} \frac{c_{m,k}(-1)^k}{(2\pi)^{\frac{m-1}{2}}} \sum_{r=0}^{k_2} \binom{k_2}{r}\frac{(-1)^r(m-1-k)_r}{|t|^{m-1-k+r}} p_{1,k_1,k_2+k-r}^{(1)}(x,|t|), \qquad (4.21)$$

where

$$c_{m,k} = \frac{(m-k-2)!}{2^{\frac{m-1}{2}-k}\left(\frac{m-1}{2}-k\right)!(k-1)!}$$

and $(m-1-k)_r = (m-1-k)\cdots(m-1-k+r-1)$ is the Pochhammer symbol[4].

Proof. Let m be odd, $m \geq 3$. We first pass to polar coordinates in (2.3) for $k_2 = 0$, and get

$$p_{1,k_1,0}^{(m)}(x,t) = \frac{(-1)^{\frac{m-1}{2}}}{(2\pi)^m(4\pi)^n}\int_0^\infty \int_{S^{m-1}} e^{i\rho|t|(\sigma,u_1)}\,\mathrm{d}\sigma\, e^{-R\rho\coth(\rho)}a_{k_1,m-1}(\rho)\,\mathrm{d}\rho$$

where $\mathrm{d}\sigma$ is the $(m-1)$-dimensional (Hausdorff) measure on S^{m-1} and $a_{k_1,m-1}$ is the function defined in (3.1). Since the Bessel function is an elementary function when m is odd, one can prove that (see e.g. [7, equation (6.5)] and references therein)[5]

$$\int_{S^{m-1}} e^{i\rho|t|(\sigma,u_1)}\,\mathrm{d}\sigma = 2(2\pi)^{\frac{m-1}{2}}\mathrm{Re}\,\frac{e^{i\rho|t|}}{(\rho|t|)^{m-1}}\sum_{k=1}^{\frac{m-1}{2}} c_{m,k}(-i|t|\rho)^k.$$

This yields

$$p_{1,k_1,0}^{(m)}(x,t) = \sum_{k=1}^{\frac{m-1}{2}}\frac{c_{m,k}(-1)^k}{(2\pi)^{\frac{m-1}{2}}}\frac{1}{|t|^{m-1-k}}p_{1,k_1,k}^{(1)}(x,|t|)$$

which gives (4.21), since $p_{1,k_1,k_2}^{(m)}(x,t) = \frac{\partial^{k_2}}{\partial|t|^{k_2}}p_{1,k_1,0}^{(m)}(x,t)$ by definition. $\qquad\square$

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[4]See, e.g., [9].

[5]This is why we had to restrict to the case $k_2 = 0$; otherwise, we would get the additional term $(\sigma,u_1)^{k_2}$ in the integral on the sphere.

Corollary 4.11. *Let m be odd. Then, when $(x,t) \to \infty$ and $\delta \to 0^+$*

$$p_{1,k_1,k_2}^{(m)}(x,t) = \frac{(-1)^{k_2} \pi^{k_1+k_2}}{2^{n-k_1+1+\frac{m-1}{2}}} |t|^{n+k_1-1-\frac{m-1}{2}} e^{-\frac{1}{4}d(x,t)^2} e^{-\kappa\rho(\delta)} \tilde{I}_{n+k_1-1}(\kappa\rho(\delta)) \left[1 + g(|x|,|t|)\right],$$
(4.22)

where g satisfies the estimates (3.12).

Proof. If $m = 1$, the statement reduces to Corollary 3.14. Suppose then $m \geq 3$. Since $p_{1,k_1,r}^{(1)} \asymp p_{1,k_1,k_2}^{(1)}$ for every $0 \leq r \leq k_2$ by Corollary 3.14, the principal term in (4.21) corresponds to $r = 0$, $k = \frac{m-1}{2}$. Hence

$$p_{1,k_1,k_2}^{(m)}(x,t) = \frac{(-1)^{\frac{m-1}{2}}}{(2\pi)^{\frac{m-1}{2}}} |t|^{-\frac{m-1}{2}} p_{1,k_1,k_2+\frac{m-1}{2}}^{(1)}(x,t) \left[1 + O\left(\frac{1}{|t|}\right)\right].$$
(4.23)

Now substitute the estimate given by Corollary 3.14 into (4.23). The remainder g in (4.22) still satisfies (3.12), since (3.12) is satisfied by $1/|t|$. $\square$

Let now m be even, $m \geq 2$. We start by a descent method, in the same spirit of [7]: indeed, observe that the Fourier inversion formula yields

$$p_{1,k_1,0}^{(m)}(x,t) = \int_{\mathbb{R}} p_{1,k_1,0}^{(m+1)}(x,(t,t_{m+1})) \, \mathrm{d}t_{m+1},$$

so that, by differentiating under the integral sign,

$$p_{1,k_1,k_2}^{(m)}(x,t) = \int_{\mathbb{R}} \frac{\partial^{k_2}}{\partial |t|^{k_2}} p_{1,k_1,0}^{(m+1)}(x,(t,t_{m+1})) \, \mathrm{d}t_{m+1}.$$

Observe that $|(t,t_{m+1})| = |t|\sqrt{1 + \frac{t_{m+1}^2}{|t|^2}}$. Therefore, if we define $\mathfrak{J}^{k_2} := \{h \in \mathbb{N}^{k_2} : \sum_{j=1}^{k_2} jh_j = k_2\}$, Faà di Bruno's formula applied twice[6] leads to

$$p_{1,k_1,k_2}^{(m)}(x,t) = \sum_{h \in \mathfrak{J}^{k_2}} \frac{k_2!}{h!} \int_{\mathbb{R}} p_{1,k_1,|h|}^{(m+1)}(x,(t,t_{m+1})) \, F_h(t,t_{m+1}) \, \mathrm{d}t_{m+1}$$

where

$$F_h(t,t_{m+1}) = \prod_{j=1}^{k_2} \left(\sum_{\ell_1+2\ell_2=j} \frac{2^{\ell_1}}{\ell!} (-1)^{|\ell|} \left(-\frac{1}{2}\right)_{|\ell|} |t|^{1-j} \left(1 + \frac{t_{m+1}^2}{|t|^2}\right)^{\frac{1}{2}-|\ell|} \right)^{h_j}.$$

Since $F_{(k_2,0,\dots,0)} = \left(1 + \frac{t_{m+1}^2}{|t|^2}\right)^{-k_2/2}$ while $F_h = O\left(\frac{1}{|t|}\left(1 + \frac{t_{m+1}^2}{|t|^2}\right)^{-1/2}\right)$ otherwise, we have proved the following lemma.

---

[6]Applied once, it yields

$$\frac{\partial^{k_2}}{\partial |t|^{k_2}} p_{1,k_1,0}^{(m+1)}(x,(t,t_{m+1})) = \sum_{h \in \mathfrak{J}^{k_2}} \frac{k_2!}{h!} p_{1,k_1,|h|}^{(m+1)}(x,(t,t_{m+1})) \prod_{j=1}^{k_2} \left(\frac{1}{j!} \frac{\partial^j}{\partial |t|^j} \sqrt{|t|^2 + t_{m+1}^2}\right)^{h_j},$$

and then

$$\frac{\partial^j}{\partial |t|^j} \sqrt{|t|^2 + t_{m+1}^2} = \sum_{\ell_1+2\ell_2=j} \frac{j!}{\ell!} (-1)^{|\ell|} \left(-\frac{1}{2}\right)_{|\ell|} (|t|^2 + t_{m+1}^2)^{\frac{1}{2}-|\ell|} (2|t|)^{\ell_1}.$$

Lemma 4.12. *When m is even, $m \geq 2$,*

$$p_{1,k_1,k_2}^{(m)}(x,t) = \int_{\mathbb{R}} \left(1 + \frac{t_{m+1}^2}{|t|^2}\right)^{-\frac{k_2}{2}} p_{1,k_1,k_2}^{(m+1)}(x,(t,t_{m+1}))\, \mathrm{d}t_{m+1}$$

$$+ O\left[\frac{1}{|t|} \max_{0 \leq r < k_2} \int_{\mathbb{R}} \left(1 + \frac{t_{m+1}^2}{|t|^2}\right)^{-\frac{1}{2}} p_{1,k_1,r}^{(m+1)}(x,(t,t_{m+1}))\, \mathrm{d}t_{m+1}\right].$$

As a consequence of Lemma 4.12, matters can be reduced to finding the asymptotic expansions of the integrals

$$\int_{\mathbb{R}} \left(1 + \frac{t_{m+1}^2}{|t|^2}\right)^{\alpha} p_{1,k_1,r}^{(m+1)}(x,(t,t_{m+1}))\, dt_{m+1} \tag{4.24}$$

when $\alpha \in \mathbb{R}$ and $0 \leq r \leq k_2$. From these, it will also be proved that the remainder in Lemma 4.12 is indeed smaller than the principal part, which *a priori* is not obvious.

With this aim, we define the function $\sigma \colon \mathbb{R} \ni s \mapsto \sqrt{1+s^2}$, and write $t' = (t,t_{m+1}) \in \mathbb{R}^{m+1}$. It is straightforward to check that $|t'| = |t|\sigma\left(\frac{t_{m+1}}{|t|}\right)$. Thus, define

$$\delta(s) := \frac{\delta}{\sqrt{\sigma(s)}}, \qquad \kappa(s) := \kappa\sqrt{\sigma(s)} = 2\pi|t|\delta\sqrt{\sigma(s)}.$$

Obviously, $\delta(0) = \delta$ and $\kappa(0) = \kappa$. If we put a prime on the quantities introduced in Definition 2.3 relative to t', moreover,

$$\delta' = \delta\left(\frac{t_{m+1}}{|t|}\right), \qquad \kappa' = \kappa\left(\frac{t_{m+1}}{|t|}\right).$$

In cases **II**, **III** and **IV**, $|t| \to \infty$ and $\delta \to 0^+$. By substituting (4.22) into (4.24) and by the change of variable $\frac{t_{m+1}}{|t|} \mapsto s$ in the integral

$$(4.24) = \frac{(-1)^r \pi^{r+k_1}}{2^{n-k_1+1+\frac{m}{2}}} |t|^{n+k_1-1-\frac{m}{2}+1} e^{-\frac{1}{4}d(x,t)^2} e^{-\kappa\rho(\delta)} \mathscr{I}_{2\alpha+n+k_1-1-\frac{m}{2}},$$

where

$$\mathscr{I}_\beta = \int_{\mathbb{R}} \sigma(s)^\beta e^{-|t|\pi(\sigma(s)-1)} \tilde{I}_{n+k_1-1}\left(\kappa(s)\rho\left(\delta(s)\right)\right) \left[1 + g(|x|,|t|\sigma(s))\right]\, \mathrm{d}s, \tag{4.25}$$

and g satisfies the estimates (3.12). Therefore, matters can be reduced to finding some asymptotic estimates of the integrals $\mathscr{I}_\beta$.

4.2. II. Estimates for $\delta \to 0^+$ and $\kappa \to +\infty$.

Theorem 4.13. *For $\delta \to 0^+$ and $\kappa \to +\infty$*

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2}\pi^{k_1+k_2}}{4^n(\pi\delta)^{n+k_1-\frac{m+1}{2}}\sqrt{2\pi\kappa^m}} e^{-\frac{1}{4}d(x,t)^2} \left[1 + O\left(\delta + \frac{1}{\kappa}\right)\right].$$

Proof. When m is odd, the theorem is obtained by combining Theorem 3.6 with (4.23). Therefore, we only consider m even. By the preceding arguments, it will be sufficient to study $\mathscr{I}_\beta$ in (4.25).

Since the argument of the modified Bessel function tends to $+\infty$, we use the development (3.13), which gives

$$\mathscr{I}_\beta = \frac{(2\pi)^{-n-k_1} e^{\kappa\rho(\delta)}}{\delta^{n+k_1-\frac{1}{2}}|t|^{n+k_1-\frac{1}{2}}} \int_{\mathbb{R}} e^{-|t|\varphi_\delta(s)} \frac{\sigma(s)^{\beta-\frac{1}{4}-\frac{n+k_1-1}{2}}}{\rho\left(\delta(s)\right)^{n+k_1-\frac{1}{2}}}$$

$$\times \left[1 + O\left(\frac{1}{\delta|t|\sqrt{\sigma(s)}}\right)\right] \left[1 + g(|x|,|t|\sigma(s))\right]\, \mathrm{d}s$$

where

$$\varphi_\delta(s) = \pi[\sigma(s) - 1] + 2\pi\delta \left[\rho(\delta) - \sqrt{\sigma(s)}\rho(\delta(s))\right].$$

We first study the principal part of the integral, to which we apply Laplace's method (see Remark 2.8) with

$$\mathscr{F} = \{\varphi_\delta \colon \delta \in [0, \delta_2]\}, \qquad \mathscr{G} = \left\{\frac{\sigma(\cdot)^{\beta - \frac{1}{4} - \frac{n+k_1-1}{2}}}{\rho(\delta(\cdot))^{n+k_1-\frac{1}{2}}} \colon \delta \in [0, \delta_2]\right\}$$

for some δ_2, smaller than the δ_1 of Lemma 3.9, to be determined.

2. It is easily seen that $\varphi_\delta(0) = 0$. Moreover

$$\varphi_\delta'(s) = \pi\frac{s}{\sigma(s)}\left[1 - \delta(s)\rho(\delta(s)) + \frac{\delta^2}{\sigma(s)^{\frac{3}{2}}}\rho'(\delta(s))\right], \tag{4.26}$$

so that $\varphi_\delta'(0) = 0$ and $\varphi_\delta''(0) = \pi(1 - \delta\rho(\delta) + \delta^2\rho'(\delta))$. Observe that there is $\delta_2 > 0$, which we may choose smaller than δ_1, such that

$$1 - \delta(s)\rho(\delta(s)) + \frac{\delta^2}{\sigma(s)^{\frac{3}{2}}}\rho'(\delta(s)) \geq \frac{1}{2} \tag{4.27}$$

for every s and every $\delta \in [0, \delta_2]$. Therefore, $\varphi_\delta''(0) \geq \frac{\pi}{2}$ for every $\delta \in [0, \delta_2]$.

3. By (4.26) and (4.27), for $s \in \mathbb{R}$ and $\delta \in (0, \delta_2)$,

$$|\varphi_\delta'(s)| \geq \frac{\pi}{2\sigma(s)}|s|. \tag{4.28}$$

In particular, $|\varphi_\delta'(s)| \geq \frac{\pi}{2\sigma(2)}|s|$ for every $s \in [-2, 2]$.

1. Observe that $\varphi_\delta'(s) = \text{sign}(s)|\varphi_\delta'(s)|$ by (4.26); then, by (4.28),

$$\varphi_\delta(s) = \int_0^s \text{sign}(s)|\varphi_\delta'(u)|\,\mathrm{d}u = \left|\int_0^s |\varphi_\delta'(u)|\,\mathrm{d}u\right| \geq \frac{\pi}{2\sigma(s)}\left|\int_0^s |u|\,\mathrm{d}u\right| \geq \frac{\pi s^2}{4\sigma(s)}$$

for every $s \in \mathbb{R}$, since σ is even and increasing on $[0, \infty)$.

4. By definition of σ and since ρ is continuous in zero, we get $g(s) \lesssim |s|^{\beta - \frac{1}{4} - \frac{n+k_1-1}{2}}$ for $s \to \infty$, uniformly in $g \in \mathscr{G}$.

By Theorem 2.7, then,

$$\int_{\mathbb{R}} e^{-|t|\varphi_\delta(s)}\frac{\sigma(s)^{\beta - \frac{1}{4} - \frac{n+k_1-1}{2}}}{\rho(\delta(s))^{n+k_1-\frac{1}{2}}}\,\mathrm{d}s = \sqrt{\frac{2}{|t|(1 - \delta\rho(\delta) + \delta^2\rho'(\delta))}}\left[1 + O\left(\frac{1}{|t|}\right)\right]$$

$$= \sqrt{\frac{2}{|t|}}\left[1 + O\left(\delta + \frac{1}{|t|}\right)\right].$$

The remainder can be treated similarly, and with the same arguments as above one gets

$$\int_{\mathbb{R}} e^{-|t|\varphi_\delta(s)}\frac{\sigma(s)^{\beta - \frac{1}{4} - \frac{n+k_1-1}{2}}}{\rho(\delta(s))^{n+k_1-\frac{1}{2}}}\left[O\left(\frac{1}{\delta|t|\sqrt{\sigma(s)}}\right) + O\left(\frac{1}{\kappa} + \delta\right)\right]\,\mathrm{d}s$$

$$= \sqrt{\frac{2}{|t|}}\left[1 + O\left(\delta + \frac{1}{|t|}\right)\right]O\left(\frac{1}{\delta|t|} + \frac{1}{\kappa} + \delta\right) = \sqrt{\frac{2}{|t|}}O\left(\frac{1}{\kappa} + \delta\right)$$

since $\frac{1}{\delta|t|} = \frac{2\pi}{\kappa} = O\left(\frac{1}{\kappa}\right)$ and $1/\sqrt{\sigma(s)} \leq 1$ for every $s \in \mathbb{R}$. The proof is complete. $\square$

4.3. **III and IV. Estimates for $\delta \to 0^+$ and κ bounded.** These two cases can be treated together and the principal part of $p_{1,k_1,k_2}^{(m)}$ is easy to get. The remainders are more tricky, since when passing from the m-dimensional variable t to the $(m+1)$-dimensional variable t' the asymptotic conditions in **II**, **III** and **IV** do not correspond to those in **II'**, **III'**, **IV'** (these symbols standing for the cases relative to $m+1$); on the contrary, they mix together according to the values of the additional variable t_{m+1}.

Theorem 4.14. *Fix $C > 1$. If $\delta \to 0^+$ while $1/C \leq \kappa \leq C$, then*

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2} \pi^{k_1+k_2}}{4^n (\pi\delta)^{n+k_1-\frac{m+1}{2}} \kappa^{\frac{m-1}{2}}} e^{-\frac{1}{4}d(x,t)^2} e^{-\kappa} I_{n+k_1-1}(\kappa) \left[1 + O\left(\delta\right)\right].$$

When $\kappa \to 0^+$ and $|t| \to +\infty$

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2} \pi^{k_1+k_2}}{2^{2n+\frac{m-1}{2}} (n+k_1-1)!} |t|^{n+k_1-1-\frac{m-1}{2}} e^{-\frac{1}{4}d(x,t)^2} \left[1 + O\left(\kappa + \frac{1}{|t|}\right)\right].$$

Proof. The theorem holds when m is odd by Theorem 3.11 combined with (4.23). When m is even, we shall apply Laplace's method to $\mathscr{I}_\beta$. We first deal with the principal part. Define first

$$\varphi(s) = \pi\sigma(s) - \pi,$$

so that Theorem 2.7 will be applied to

$$\mathscr{F} = \{\varphi\}, \qquad \mathscr{G} = \{\sigma(\cdot)^\beta \tilde{I}_{n+k_1-1}(\kappa(\cdot)\rho(\delta(\cdot))) : \delta \in [0,\delta_1), \kappa \in [0,C]\}$$

where δ_1 is that of Lemma 3.9.

    2. Notice that $\varphi(0) = 0$, that $\varphi'(s) = \pi\frac{s}{\sigma(s)}$, and that $\varphi''(0) = \pi$.

    1. Observe that $\varphi(s) = \pi\frac{s^2}{1+\sqrt{1+s^2}} \geq \pi\frac{s^2}{2+|s|}$, for every $s \in \mathbb{R}$.

    3. It is easily seen that $|\varphi'(s)| \geq \frac{\pi}{\sigma(1)}|s|$ for every $s \in [-1,1]$.

    4. Recall that by (3.13)

$$\tilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s))) \lesssim e^{\kappa(s)\rho(\delta(s))} \lesssim e^{\kappa\sqrt{\sigma(s)}}$$

    as $s \to \infty$, uniformly as $\kappa \in [0,C]$ and $\delta \in [0,\delta_1)$. Hence, there is a constant $c_1 > 0$ such that $|\sigma(s)^\beta \tilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))| \leq c_1 e^{c_1|s|}$.

Therefore, by Theorem 2.7

$$\int_{\mathbb{R}} e^{-|t|\varphi(s)} \sigma(s)^\beta \tilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s))) \, ds = \sqrt{\frac{2}{|t|}} \tilde{I}_{n+k_1-1}(\kappa\rho(\delta)) \left[1 + O\left(\frac{1}{|t|}\right)\right]$$

uniformly in κ and δ. Since $\tilde{I}_{n+k_1-1}(\kappa\rho(\delta)) - \tilde{I}_{n+k_1-1}(\kappa) = O(\kappa\rho(\delta) - \kappa) = O(\delta^2)$ uniformly as $\kappa \in [0,C]$ by Taylor's formula, we are done with the principal part. We now deal with the remainders, namely

$$\mathscr{I}_\beta' = \int_{\mathbb{R}} e^{-|t|\varphi(s)} \sigma(s)^\beta \tilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s))) \, g(|x|, |t|\sigma(s)) \, ds$$

where

$$g(|x|, |t|\sigma(s)) = \begin{cases} O\left(\delta(s) + \frac{1}{\kappa(s)}\right) & \text{if } \delta(s) \to 0^+ \text{ and } \kappa(s) \to +\infty, \\ O(\delta(s)) & \text{if } \delta(s) \to 0^+ \text{ and } \kappa(s) \in [1/C', C'], \\ O\left(\frac{1}{|t|\sigma(s)} + \kappa(s)\right) & \text{if } \delta(s) \to 0^+ \text{ and } \kappa(s) \to 0^+ \end{cases}$$

for every $C' > 1$. Since $\delta(s) \leq \delta$ for every $s \in \mathbb{R}$, we may find some positive constants C'', $\delta_2 \leq \delta_1$, where δ_1 is that of Lemma 3.9, and $\kappa_2 \leq \kappa_1$ such that

$$|g(|x|, |t|\sigma(s))| \leq \begin{cases} C''\left(\delta(s) + \frac{1}{\kappa(s)}\right) & \text{when } \delta \leq \delta_2, \ \kappa(s) \geq \kappa_1, \\ C''\delta(s) & \text{when } \delta \leq \delta_2, \ \kappa_2 \leq \kappa(s) \leq \kappa_1, \\ C''\left(\frac{1}{|t|\sigma(s)} + \kappa(s)\right) & \text{when } \delta \leq \delta_2, \ \kappa(s) \leq \kappa_2. \end{cases}$$

We shall split the integrals accordingly. Notice first that we may assume also that $\kappa_2 \leq 1/(2C) \leq 2C \leq \kappa_1$, and, up to taking a smaller δ_2, that

$$\varphi(s) - 2\pi\delta\sqrt{\sigma(s)}\rho(\delta(s)) \geq \frac{1}{2}|s|$$

whenever $|s| \geq 2$ and $\delta \in [0, \delta_2)$.

Consider first case **III**, where $\kappa \in [1/C, C]$. We split

$$\mathscr{I}'_\beta = \int_{\kappa(s) \leq \kappa_1} + \int_{\kappa(s) \geq \kappa_1} = \mathscr{I}'_{\beta,1} + \mathscr{I}'_{\beta,2}.$$

Observe that $\kappa(s) \geq \kappa_1$ if and only if $|s| \geq \sqrt{\frac{\kappa_1^4}{\kappa^4} - 1} =: s_{1,\kappa} \geq 2$. Since

$$\tilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))e^{-|t|\varphi(s)} = O\left(e^{|t|[2\pi\delta\sqrt{\sigma(s)}\rho(\delta(s))-\varphi(s)]}\right) = O\left(e^{-\frac{1}{2}|t||s|}\right)$$

as $s \to \infty$, and since $\delta = O\left(\frac{1}{\kappa}\right) = O(1)$ in case **III**, we get

$$|\mathscr{I}'_{\beta,2}| \leq C'\left(\delta + \frac{1}{\kappa}\right)\int_{|s| \geq s_{1,\kappa}} \sigma(s)^{\beta-\frac{1}{2}}\tilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))e^{-|t|\varphi(s)}\,\mathrm{d}s = O\left(e^{-\frac{s_{1,\kappa}}{4}|t|}\right),$$

which is negligible relative to $\frac{1}{|t|^{3/2}}$. By Laplace's method, moreover,

$$|\mathscr{I}'_{\beta,1}| \leq C'\delta\int_{|s| \leq s_{1,\kappa}} \sigma(s)^{\beta-\frac{1}{2}}\tilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))e^{-|t|\varphi(s)}\,\mathrm{d}s = O\left(\delta\frac{1}{\sqrt{|t|}}\right)$$

with the same arguments as above. This concludes the study of case **III**.

Consider now case **IV**, where $\kappa \to 0^+$. We split

$$\mathscr{I}'_\beta = \int_{\kappa(s) \leq \kappa_2} + \int_{\kappa_2 \leq \kappa(s) \leq \kappa_1} + \int_{\kappa(s) \geq \kappa_1} = \mathscr{I}'_{\beta,1} + \mathscr{I}'_{\beta,2} + \mathscr{I}'_{\beta,3}.$$

Observe that $\kappa(s) \geq \kappa_2$ if and only if $s \geq \sqrt{\frac{\kappa_2^4}{\kappa^4} - 1} =: s_{2,\kappa}$, and $s_{1,\kappa} \geq s_{2,\kappa} \geq 2$ if κ is sufficiently small. Exactly as above, we get

$$|\mathscr{I}'_{\beta,3}| \leq C'\left(\delta + \frac{1}{\kappa}\right)\int_{|s| \geq s_{1,\kappa}} \sigma(s)^{\beta-\frac{1}{2}}\tilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))e^{-|t|\varphi(s)}\,\mathrm{d}s = O\left(\frac{1}{\kappa}e^{-\frac{s_{1,\kappa}}{4}|t|}\right)$$

which is negligible relative to $\frac{1}{|t|^{3/2}}$. Then

$$|\mathscr{I}'_{\beta,2}| \leq C'\delta\int_{s_{2,\kappa} \leq |s| \leq s_{1,\kappa}} \sigma(s)^{\beta-\frac{1}{2}}\tilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))e^{-|t|\varphi(s)}\,\mathrm{d}s = O\left(\delta\,e^{-\frac{s_{2,\kappa}}{4}|t|}\right),$$

which is negligible relative to $\frac{1}{|t|^{3/2}}$ in case **IV**. Finally,

$$|\mathscr{I}'_{\beta,1}| \leq C'\int_{|s| \leq s_{2,\kappa}} \sigma(s)^\beta\tilde{I}_{n+k_1-1}(\kappa(s)\rho(\delta(s)))e^{-|t|\varphi(s)}\left(\sqrt{\sigma(s)}\kappa + \frac{1}{\sigma(s)|t|}\right)\,\mathrm{d}s$$

$$= O\left[\frac{1}{\sqrt{|t|}}\left(\frac{1}{|t|} + \kappa\right)\right],$$

by Laplace's method as above. The proof is complete. $\qquad\square$

We can finally state the following corollary, which is the natural extension of Corollary 3.14.

Corollary 4.15. *For $(x,t) \to \infty$ and $\delta \to 0^+$*

$$p_{1,k_1,k_2}(x,t) = \frac{(-1)^{k_2}\pi^{k_1+k_2}}{2^{n-k_1+1+\frac{m-1}{2}}}|t|^{n+k_1-\frac{m+1}{2}}e^{-\frac{1}{4}d(x,t)^2}e^{-\kappa\rho(\delta)}\tilde{I}_{n+k_1-1}(\kappa\rho(\delta))\left[1+g(|x|,|t|)\right],$$

where

$$g(|x|,|t|) = \begin{cases} O\left(\delta + \frac{1}{\kappa}\right) & \text{if } \delta \to 0^+ \text{ and } \kappa \to +\infty, \\ O(\delta) & \text{if } \delta \to 0^+ \text{ and } \kappa \in [1/C, C], \\ O\left(\frac{1}{|t|} + \kappa\right) & \text{if } \delta \to 0^+ \text{ and } \kappa \to 0^+ \end{cases}$$

for every $C > 0$.

We have not been able to find a single function which displays the asymptotic behaviour of $p_{1,k_1,k_2}(x,t)$ as $(x,t) \to \infty$, though we showed that the exponential decrease is the same in the four cases. This is also the same decrease found by Eldredge [7, Theorems 4.2 and 4.4], when $k_1 = k_2 = 0$ and for the horizontal gradient, and Li [16, Theorems 1.4 and 1.5], when $k_1 = k_2 = 0$. Notice that in [16, Theorem 1.5 and the following Remark (1)] the remainders for $k_1 = k_2 = 0$ seem to be better than the one we put in Corollary 4.15, but they reduce to ours when developing the estimates in a more convenient form in cases **II** and **IV**, as we did in Theorems 4.13 and 4.14.

Remark 4.16. Our sharp estimates for p_{1,k_1,k_2} can be used to obtain asymptotic estimates of all the derivatives of the heat kernel p_1. Indeed, Faà di Bruno's formula leads to

$$\frac{\partial^{|\gamma|}}{\partial x^{\gamma_1}\partial t^{\gamma_2}}p_1(x,t)$$

$$= \gamma_1!\gamma_2!\sum_{\eta,\mu,\beta}\frac{|\mu|!2^{|\mu_1|-|\gamma_1|}}{\eta!\mu!\beta!}\left[\prod_{h=1}^{|\mu|}\left(\frac{\left(\frac{1}{2}\right)_h}{h!}\right)^{\beta_h}\right]x^{\eta_1}\text{sign}(t)^{\mu_1}|t|^{|\beta|-|\gamma_2|}p_{1,|\eta|,|\beta|}(x,t), \quad (4.29)$$

where the sum is extended to all $\eta = (\eta_1,\eta_2) \in \mathbb{N}^{2n}\times\mathbb{N}^{2n}$, $\mu = (\mu_1,\mu_2) \in \mathbb{N}^m\times\mathbb{N}^m$ and $\beta \in \mathbb{N}^{|\mu|}$ such that

$$\gamma_1 = \eta_1 + 2\eta_2, \qquad \gamma_2 = \mu_1 + 2\mu_2, \qquad \sum_{h=1}^{|\mu|}h\beta_h = |\mu|.$$

However, the sharp asymptotic expansions we explicitly provided in Theorems 4.2, 4.13 and 4.14 may not be enough to get directly *sharp* asymptotic estimates of any desired derivative of p_1: some cancellations among the principal terms may indeed occur in (4.29). Nevertheless, by inspecting case by case, the interested reader could consider as many terms of the expansions given by Theorem 2.7 or Lemma 3.12 as necessary. In the case when $t \to 0$, one may also make use of Lemma 4.8 before expanding each term: a suitable choice for N gets rid of the negative powers of $|t|$ appearing in (4.29). Despite this, our estimates for p_{1,k_1,k_2} lead to the *sharp* behaviour at infinity of $\nabla_{\mathcal{H}}p_s$ and $\mathcal{L}p_s$, as we shall see in the next section.

5. SUB-RIEMANNIAN ORNSTEIN UHLENBECK OPERATORS

For every $s > 0$ consider the operator on $L^2(p_s)$ given by

$$\mathcal{L}^{p_s} = \mathcal{L} - \frac{\nabla_{\mathcal{H}}p_s}{p_s}\cdot\nabla_{\mathcal{H}}\colon C_c^\infty \to L^2(p_s)$$

which arises from the Dirichlet form $\varphi \mapsto \int_G |\nabla_{\mathcal{H}}\varphi(y)|^2 p_s(y)\,dy$. For a fixed time $s > 0$, $\mathcal{L}^{p_s}$ can be considered as a sub-Riemannian version of the classical Ornstein-Uhlenbeck operator

(see [1, 17]). Arguing as Strichartz ([21, Theorem 2.4]) it is not hard to see that $\mathcal{L}^{p_s}$ with domain $C_c^\infty(G)$ is essentially self-adjoint on $L^2(p_s)$, for every $s > 0$. Let us then consider its closure, which we still denote by $\mathcal{L}^{p_s}$.

Theorem 5.1. *$\mathcal{L}^{p_s}$ has purely discrete spectrum for all $s > 0$.*

Theorem 5.1 is indeed due to Inglis [14], whose proof relies on super Poincaré inequalities. Instead, we reduce matters to studying a Schrödinger-type operator by conjugating $\mathcal{L}^{p_s}$ with the isometry $U_s \colon L^2(p_s) \to L^2$ defined by $U_s f = f\sqrt{p_s}$ (see e.g. [4, 18]). More precisely, we consider the operator $U_s \mathcal{L}^{p_s} U_s^{-1} \colon L^2 \to L^2$. Simple computations then lead to $U_s \mathcal{L}^{p_s} U_s^{-1} = \mathcal{L} + V_s$, where V_s is the multiplication operator[7] given by the function

$$V_s = -\frac{1}{4}\frac{|\nabla_{\mathcal{H}}p_s|^2}{p_s^2} - \frac{1}{2}\frac{\mathcal{L}p_s}{p_s} = -\frac{1}{4}\frac{\sum_{j=1}^{2n}(X_j p_s)^2}{p_s^2} + \frac{1}{2}\frac{\sum_{j=1}^{2n}X_j^2 p_s}{p_s}.$$

The main ingredient of the proof is due to Simon [20, Theorem 2]. Given a potential V and $M > 0$, we define $\Omega_M := \{g \in G \colon V(g) \leq M\}$. For a subset E of G, we write $|E|$ to denote its measure with respect to dy.

Theorem 5.2. *Let V be a potential bounded from below such that $|\Omega_M| < \infty$ for every $M > 0$. Then there exists a self-adjoint extension of $\mathcal{L} + V$ with purely discrete spectrum.*

In order to apply Proposition 5.2, some estimates of the potential are needed; this is done in the following proposition.

Proposition 5.3. *When $(x, t) \to \infty$, $V_s(x, t) \asymp s^{-2}d(x, t)^2$ for every $s > 0$.*

Proof. Since $V_s(x, t) = \frac{1}{s}V_1\left(\frac{x}{\sqrt{s}}, \frac{t}{s}\right)$, it will be sufficient to consider V_1 only. For every $(x, t) \in G$

$$|\nabla_{\mathcal{H}}p_1|^2(x, t) = R\,p_{1,1,0}(x, t)^2 + R\,p_{1,0,1}(x, t)^2, \tag{5.1}$$

while

$$\mathcal{L}p_1(x, t) = -R\,p_{1,2,0}(x, t) - n\,p_{1,1,0}(x, t) - R\,p_{1,0,2}(x, t) + \frac{R}{|t|}(m - 1)p_{1,0,1}(x, t).$$

Hence

$$V_1 = -\frac{R}{4}\frac{p_{1,1,0}^2 + p_{1,0,1}^2}{p_{1,0,0}^2} + \frac{R}{2}\frac{p_{1,2,0} + p_{1,0,2} + \frac{n}{R}p_{1,1,0} - \frac{m-1}{|t|}p_{1,0,1}}{p_{1,0,0}}.$$

In order to find the asymptotics for the potential, it turns out that only the principal term of p_{1,k_1,k_2} is necessary, and therefore, for the sake of simplicity, we shall avoid an explicit treatment of the remainders. If one is interested in a more detailed description of the behaviour of the potential, however, it is enough to use the remainders that we found in the previous sections.

I. If ω runs through $[0, C]$ for some $C > 0$, then both $\frac{y_\omega}{\sin(y_\omega)}$ and $\frac{y_\omega}{\omega}$ are positive and bounded both from above and from below. Hence,

$$V_1(x, t) \sim -\frac{R}{4}\frac{y_\omega^2}{\sin(y_\omega)^2} + \frac{R}{2}\frac{y_\omega^2}{\sin(y_\omega)^2} = \frac{R}{4}\frac{y_\omega^2}{\sin(y_\omega)^2} \asymp d(x, t)^2$$

thanks to Theorem 4.2.

II. Let $\delta \to 0^+$ and $\kappa \to +\infty$. Then $\frac{1}{R\delta} = o\left(\frac{1}{\delta^2\sqrt{\kappa}}\right)$, and $\kappa + \frac{|t|}{\sqrt{\kappa}} = o(|t|)$. Therefore, by Theorem 4.13,

$$V_1(x, t) \sim -\frac{R}{4}\frac{1}{\delta^2} + \frac{R}{2}\frac{1}{\delta^2} = \frac{\pi}{4}|t| \asymp d(x, t)^2.$$

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[7]With a slight abuse of notation, we do not distinguish between a multiplication operator by a function and the function itself.

III. Let $\delta \to 0^+$ and $\kappa \in [1/C, C]$ for some $C > 1$. Then $\delta \asymp R$. Elementary computations yield

$$I'_\nu(\zeta) = \frac{I_{\nu-1}(\zeta) + I_{\nu+1}(\zeta)}{2},$$

so that

$$(2I_{\nu-1}I_{\nu+1} - I_\nu^2)'(\zeta) = I_{\nu-2}(\zeta)I_{\nu+1}(\zeta) + I_{\nu-1}(\zeta)I_{\nu+2}(\zeta)$$

for all $\nu \in \mathbb{Z}$ and for all $\zeta \in \mathbb{C}$. Thus, $2I_{n-1}I_{n+1} - I_n^2$ is strictly increasing on $[0, \infty)$, hence strictly positive on $(0, \infty)$. Therefore, by Theorem 4.14

$$V_1(x,t) \sim -\frac{R}{4} \frac{\frac{1}{\delta^2} I_n(\kappa)^2}{I_{n-1}(\kappa)^2} + \frac{R}{2} \frac{\frac{1}{\delta^2} I_{n+1}(\kappa) + \frac{n}{R\delta} I_n(\kappa)}{I_{n-1}(\kappa)}$$

$$= \frac{\pi|t|}{4} \frac{\frac{2n}{\kappa} I_n(\kappa)I_{n-1}(\kappa) + 2I_{n-1}(\kappa)I_{n+1}(\kappa) - I_n(\kappa)^2}{I_{n-1}(\kappa)^2} \asymp d(x,t)^2.$$

IV. Finally, let $\kappa \to 0^+$ and $|t| \to +\infty$. Then $|t| = o\left(\frac{1}{R}\right)$, so that

$$V_1(x,t) \sim -\frac{R}{4} \frac{\frac{\pi^2}{(n!)^2}|t|^2 + \frac{\pi^2}{[(n-1)!]^2}}{\frac{1}{[(n-1)!]^2}} + \frac{R}{2} \frac{\frac{\pi^2}{(n+1)!}|t|^2 + \frac{\pi^2}{(n-1)!} + \frac{n\pi}{R\,n!}|t| + \frac{(m-1)\pi}{(n-1)!|t|}}{\frac{1}{(n-1)!}}$$

$$\sim \frac{\pi}{2}|t| \asymp d(x,t)^2,$$

thanks to Theorem 4.14 again. $\qquad\square$

Remark 5.4. The estimates provided by Eldredge [7] are not sufficient to prove Proposition 5.3, not even with some precise estimates of $\mathcal{L}p_1/p_1$. Indeed, as the proof above shows, in cases **I**, **II** and **III** one has $\mathcal{L}p_1/p_1 \asymp |\nabla_{\mathcal{H}}p_1|^2/p_1^2$, so that no lower control of V_1 can be inferred. On the other hand, the upper bounds of the derivatives of p_s explicitly provided by Li [16] are not enough to describe the behaviour at infinity of V_s.

Proof of Theorem 5.1. Since V_s is continuous and diverges at infinity by Proposition 5.3, the assumptions of Theorem 5.2 are fulfilled and this ensures the existence of a self-adjoint extension $(T_s, \mathscr{D}_s)$ of $(\mathcal{L} + V_s, C_c^\infty)$ with purely discrete spectrum. Since the multiplication by the square root of p_s, which we called U_s, preserves C_c^∞, $U_s^{-1}\mathscr{D}_s \supseteq C_c^\infty$; therefore, $(U_s^{-1}T_s U_s, U_s^{-1}\mathscr{D}_s)$ is a self-adjoint extension – with purely discrete spectrum – of $(\mathcal{L}^{p_s}, C_c^\infty)$, which is essentially self-adjoint. The result follows. $\qquad\square$

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