Recognizing Matroids

Brahim Chaourar
Department of Mathematics and Statistics,
Al Imam University (IMSIU)
P.O. Box 90950, Riyadh 11623, Saudi Arabia
Correspondence address: P. O. Box 287574, Riyadh 11323, Saudi Arabia
email: bchaourar@hotmail.com

Abstract

Let E be a finite set and $\mathcal{P}$, $\mathcal{S}$, $\mathcal{L}$ three classes of subsets of E, and r a function defined on 2^E. In this paper, we give an algorithm for testing if the quadruple $(\mathcal{P}, \mathcal{S}, \mathcal{L}, r)$ is the locked structure of a given matroid, i.e., recognizing if $(\mathcal{P}, \mathcal{S}, \mathcal{L}, r)$ defines a matroid. This problem is intractable. Our algorithm improves the running time complexity of a previous algorithm due to Spinrad. We deduce a polynomial time algorithm for recognizing large classes of matroids called polynomially locked matroids and uniform matroids.

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1 Introduction

Sets and their characteristic vectors will not be distinguished. We refer to Oxley [5] and Schrijver [8] about matroids and polyhedra terminology and facts, respectively.

Given a matroid M defined on a finite set E. Suppose that M (and M^*) is 2-connected. A subset $L \subset E$ is called a locked subset of M if $M|L$ and $M^*|(E\backslash L)$ are 2-connected, and their corresponding ranks are at least 2, i.e., $min\{r(L), r^*(E\backslash L)\} \geq 2$. It is not difficult to see that if L is locked then both L and $E\backslash L$ are closed, respectively, in M and M^* (That is why we call them locked). We denote by $\mathcal{L}(M)$ and $\ell(M)$, respectively, the class of locked subsets of M and its cardinality, which is called the

locked number of M. Given a positive integer k, $\mathcal{L}_k$, the class of k-locked matroids, is the class of matroids M such that $\ell(M) \in O(|E|^k)$. M is 0-locked if $\mathcal{L}(M) = \varnothing$, i.e., $\ell(M) = 0$ and the class of such matroids is $\mathcal{L}_0$. For a given nonnegative integer k, $\mathcal{L}_k$ is also called a polynomially locked class of matroids, and its elements k-locked or polynomially locked matroids. It is not difficult to see that the class of lockeds subsets of a matroid M is the union of lockeds subsets of the 2-connected components of M. The locked structure of M is the quadruple ($\mathcal{P}(M)$, $\mathcal{S}(M)$, $\mathcal{L}(M)$, ρ), where $\mathcal{P}(M)$ and $\mathcal{S}(M)$ are, respectively, the class of parallel and coparallel closures, and ρ is the rank function restricted to $\mathcal{P}(M) \cup \mathcal{S}(M) \cup \mathcal{L}(M) \cup \{\varnothing, E\}$.
A matroid can be completely characterized by its locked structure through its bases polytope [3]. Chaourar [4] gave a new axiom system called the locked axioms for defining matroids based on this quadruple with an extension of the function ρ to 2^E. Let E be a finite set and $\mathcal{P}$, $\mathcal{S}$, $\mathcal{L}$ be three subclasses of 2^E, and r a function defined on $\mathcal{P}\cup\mathcal{S}\cup\mathcal{L}\cup\{\varnothing, E\}\cup\mathcal{P}^C\cup\mathcal{S}^C$, where $\mathcal{X}^C = \{E\backslash X$ such that $X \in \mathcal{X}\}$ and $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}\}$. In this paper, we give an algorithm for testing if the quadruple $(\mathcal{P}, \mathcal{S}, \mathcal{L}, r)$, called a basic quadruple, is the locked structure of a given matroid, i.e., recognizing if a basic quadruple defines a matroid. This problem is intractable (see [7]). A similar study has been done by Provan and Ball for testing if a given clutter Ω, defined on a finite set E, is the class of the bases of a matroid [6]. They provide an algorithm with running time complexity $O(|\Omega|^3|E|)$. Spinrad [9] improves the running time to $O(|\Omega|^2|E|)$. In this paper, we give an algorithm for matroid recognition with running time complexity $O((|E| + |\mathcal{L}|)^2 + |E||\mathcal{L}|log|\mathcal{L}|)$. This complexity bound is better than that of Spinrad's algorithm. This algorithm becomes polynomial on $|E|$ for recognizing polynomially locked matroids. Many hard problems (Kth best base of a matroid, maximum weight basis of a matroid, testing self duality of matroids, matroid isomorphism) has been proved polynomial for polynomially locked classes of matroids [1, 3, 4]. This motivates the interest of polynomially locked classes of matroids and their recognition. We also deduce a polynomial time algorithm for deciding if a basic quadruple defines a uniform matroid.
The remainder of this paper is organized as follows. In section 2, we describe the locked axioms system for defining matroids, then, in section 3, we give an algorithm for recognizing matroids based on a basic quadruple. Finally, we conclude in section 4.

2 The Locked Axioms for a Matroid

Given a finite set E, $M = (E, \mathcal{P}, \mathcal{S}, \mathcal{L}, r)$ is a locked system defined on E if:
(L1) $E \neq \varnothing$,
(L2) $\mathcal{P}$ and $\mathcal{S}$ are partitions of E,
(L3) For any $(P, S) \in \mathcal{P} \times \mathcal{S}$, if $P \cap S \neq \varnothing$ then $|P| = 1$ or $|S| = 1$,
(L4) $\mathcal{L}$ is a class of nonempty and proper subsets of E such that $\mathcal{L} \cap \mathcal{P} = \mathcal{L} \cap \mathcal{S} = \varnothing$,

(L5) For any $(X, L) \in (\mathcal{P} \cup \mathcal{S}) \times \mathcal{L}$, $X \cap L = \varnothing$ or $X \subset L$,
(L6) r is a nonegative integer function defined on $\mathcal{P} \cup \mathcal{S} \cup \mathcal{L} \cup \{\varnothing, E\} \cup \mathcal{P}^C \cup \mathcal{S}^C$, where $\mathcal{X}^C = \{E \backslash X$ such that $X \in \mathcal{X}\}$ and $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}\}$,
(L7) $r(\varnothing) = 0$ and $r(E) \geq r(X)$ for any $X \subseteq E$,
(L8) $r(P) = min\{1, r(E)\}$ for any $P \in \mathcal{P}$,
(L9) $r(E \backslash P) = min\{|E \backslash P|, r(E)\}$ for any $P \in \mathcal{P}$,
(L10) $r(S) = min\{|S|, r(E)\}$ for any $S \in \mathcal{S}$,
(L11) $r(E \backslash S) = min\{|E \backslash S|, r(E) + 1 - |S|\}$ for any $S \in \mathcal{S}$,
(L12) $r(L) \geq max\{2, r(E) + 2 - |E \backslash L|\}$ for any $L \in \mathcal{L}$,
(L13) r is increasing on $\mathcal{P} \cup \mathcal{L} \cup \{\varnothing, E\}$,
(L14) r is submodular on $\mathcal{P} \cup \mathcal{S} \cup \mathcal{L} \cup \{\varnothing, E\}$,
(L15) For any $X \notin \mathcal{P} \cup \mathcal{S} \cup \mathcal{L} \cup \{\varnothing, E\}$, one of the following holds:

> (P1) There exists $L \in \mathcal{L}$ such that $L \subset X$, $r(X) = r(L) + r(X \backslash L)$, and $X \backslash L$ verifies (P1) or (P2),
>
> (P2) There exists $P \in \mathcal{P}$ such that $P \cap X \neq \varnothing$, $r(X) = r(P) + r(X \backslash P)$, and $X \backslash P$ verifies (P1) or (P2),
>
> (P3) There exists $L \in \mathcal{L}$ such that $X \subset L$, $r(X) = r(L) + r(X \cup (E \backslash L)) - r(E)$, and $X \cup (E \backslash L$ verifies (P3) or (P4),
>
> (P4) There exists $S \in \mathcal{S}$ such that $(E \backslash S) \cup X \neq E$, $r(X) = r(E \backslash S) + r(X \cup S) + |S \cap X| - r(E)$, and $X \cup S$ verifies (P3) or (P4),

(L16) For any $(L_1, L_2) \in \mathcal{L}^2$, if $L_1 \cap L_2 \neq \varnothing$ and $L_1 \cap L_2 \notin \mathcal{L}$ then $L_1 \cap L_2$ verifies (P1) or (P2) of (L15),
(L17) For any $(L_1, L_2) \in \mathcal{L}^2$, if $L_1 \cup L_2 \neq E$ and $L_1 \cup L_2 \notin \mathcal{L}$ then $L_1 \cup L_2$ verifies (P3) or (P4) of (L15),
Without loss of generality, we can replace axioms (L8)-(L11) by the following axioms respectively:
(LL8) $r(P) = 1$ for any $P \in \mathcal{P}$,
(LL9) $r(E \backslash P) = r(E)$ for any $P \in \mathcal{P}$,
(LL10) $r(S) = |S|$ for any $S \in \mathcal{S}$,
(LL11) $r(E \backslash S) = r(E) + 1 - |S|$ for any $S \in \mathcal{S}$.

Chaourar [4] proved that the locked axioms define uniquely a matroid. So, recognition of matroids is equivalent to recognize if a basic quadruple is a locked system.

3 An efficient algorithm for matroid recognition

We give now the running time complexity for testing each of the locked axioms. (L1) can be tested in $O(1)$. (L2) can be tested in $O(|E|^2)$. (L3) can be tested in

$O(|E|^2)$. (L4) and (L5) can be tested in $O(|E||\mathcal{L}|)$. We need the following lemma for (L6).

Lemma 3.1. *We can replace axiom (L6) by the following axiom:*
(LL6) r is a nonegative integer function defined on $\mathcal{L} \cup \{E\}$.

Proof. Axioms (LL6) and (L7)-(L11) imply axiom (L6). □

It follows that (LL6) can be tested in $O(|\mathcal{L}|)$. We need the following lemma for (L7).

Lemma 3.2. *We can replace axiom (L7) by the following axiom:*
(LL7) $r(\varnothing) = 0$.

Proof. Axioms (LL7) and (L13) imply axiom (L7). □

It follows that (LL7) can be tested in $O(1)$. (L8)-(L11) can be tested in $O(|E|)$. (L12) can be tested in $O(|\mathcal{L}|)$. We need the following lemma for (L13).

Lemma 3.3. *(L13) can be tested in $O(|E||\mathcal{L}|log|\mathcal{L}|)$.*

Proof. We can construct a lattice (ordered by inclusion) for elements of $\mathcal{P} \cup \mathcal{L} \cup \{\varnothing, E\}$. The root is the empty set and the sink is the ground set. Adjacent vertices to the root are the elements of $\mathcal{P}$ because of axioms (L4) and (L5). After sorting the elements of $\mathcal{L}$ according to their cardinalities, we can complete the lattice. We can test the axiom (L13) at each step of the lattice construction. □

(L14) can be tested in $O((|E| + |\mathcal{L}|)^2)$.
We need the following lemma for (L15).

Lemma 3.4. *Axiom (L15) is equivalent to the following axiom:*
(LL15) For any $X \notin \mathcal{P} \cup \mathcal{S} \cup \mathcal{L} \cup \{\varnothing, E\}$, one of the following holds:

> *(PP12) There exist $\{L_1, ..., L_p\} \subseteq \mathcal{L}$ and $\{P_1, ..., P_q\} \subseteq \mathcal{P}$ such that $L_i \subset X, i = 1, ..., p$, $P_j \cap X \neq \varnothing, j = 1, ..., q$, and $r(X) = r(L_1) + ... + r(L_p) + r(P_1) + ... + r(P_q)$ with p and q nonegative integers such that $p + q \geq 2$,*
>
> *(PP34) There exist $\{L_1, ..., L_p\} \subseteq \mathcal{L}$ and $\{S_1, ..., S_q\} \subseteq \mathcal{S}$ such that $L_i \supset X, i = 1, ..., p$, $(E \backslash S_j) \cup X \neq E, j = 1, ..., q$, and $r(X) = r(L_1) + ... + r(L_p) + r(E \backslash S_1) + ... + r(E \backslash S_q) - |S_1 \cap X| + ... + |S_q \cap X| - (p + q)r(E)$ with p and q nonegative integers such that $p + q \geq 2$.*

Proof. Repeating recursively axiom (L15) gives the lemma. □

It follows that axiom (LL15) gives a way on how to compute the function r outside $\mathcal{P}\cup\mathcal{S}\cup\mathcal{L}\cup\{\varnothing, E\}\cup\mathcal{P}^C\cup\mathcal{S}^C$, where $\mathcal{X}^C = \{E\backslash X$ such that $X \in \mathcal{X}\}$ and $\mathcal{X} \in \{\mathcal{P}, \mathcal{S}\}$. So we do not need to test it and we have to force it.
(L16) and (L17) can be tested in $O(|\mathcal{L}|^2)$ by using (LL15).
We can summarize all the previous steps as follows.

Theorem 3.5. *We can decide if a basic quadruple $(\mathcal{P}, \mathcal{S}, \mathcal{L}, r)$ is a locked system or not in $O((|E| + |\mathcal{L}|)^2 + |E||\mathcal{L}|log|\mathcal{L}|)$.*

This algorithm improves the running time complexity of that given by Spinrad's algorithm because if the answer is yes, i.e., the given clutter form the class of bases of a matroid, then its running time complexity is $O(|\mathcal{B}|^2|E|)$ where $\mathcal{B}$ is the class of bases, and $|\mathcal{B}| > |\mathcal{P}| + |\mathcal{S}| + |\mathcal{L}| > |E| + |\mathcal{L}|$ because the facets of the bases polytope are completely described by $\mathcal{P} \cup \mathcal{S} \cup \mathcal{L} \cup \{E\}$ [3] and the number of extreme points is greater than the number of facets. Furthermore, Spinrad's algorithm has in the input a clutter which is not a basic structure as in our algorithm (basic quadruple).
A consequence of Theorem 3.5 is the following corollary about recognition of polynomially locked classes of matroids.

Corollary 3.6. *Let k be a nonnegative integer and a basic quadruple Q. We can decide if Q defines a matroid in $\mathcal{L}_k$ or not as follows:*
(1) If $k \geq 2$ then $\mathcal{L}_k$ recognition can be done in $O(|E|^{2k})$;
(2) $\mathcal{L}_1$ recognition can be done in $O(|E|^2log|E|)$;
(3) $\mathcal{L}_0$ recognition can be done in $O(|E|^2)$.

Another consequence is recognizing uniform matroids in polynomial time. We need the following theorem [3] for this purpose.

Theorem 3.7. *A matroid M is uniform if and only if one of the following properties holds:*
(i) $\ell(M) = 0$ and $|\mathcal{P}(M)| = |E| = |\mathcal{S}(M)|$;
(ii) $|\mathcal{P}(M)| = 1$;
(iii) $|\mathcal{S}(M)| = 1$;
(iv) $r(M) = |E|$;
(v) $r(M) = 0$.

It follows that:

Corollary 3.8. *We can decide if a basic quadruple is an uniform matroid or not in $O(|E|^2)$.*

Proof. Testing (i) of Theorem 3.7 can be done in $O(|E|^2)$ by using Corollary 3.6. (ii) of Theorem 3.7 is equivalent to: $|\mathcal{P}(M)| = 1$, $|\mathcal{S}(M)| = |E|$ and $|\ell(M)| = 0$. So we can test (ii) in $O(|E|^2)$. We can use a similar argument for (iii). (iv) of Theorem 3.7 is equivalent to: $r(M) = |E|$, $|\mathcal{S}(M)| = |E| = |\mathcal{P}(M)|$ and $|\ell(M)| = 0$. So we can test (iv) in $O(|E|^2)$. Finally, a similar argument can be used for (v). $\square$

4 Conclusion

We have used a new system of axioms for defining a matroid based essentially on locked subsets to describe an algorithm for matroid recognition if the input is a basic quadruple. We have deduced a polynomial time algorithm for polynomially locked classes of matroids recognition. A second consequence is a polynomial time algorithm to decide if a basic quadruple is a uniform matroid. Future investigations can be improving the running time complexity of all problems treated in this paper, i.e., matroids recognition in general, polynomially locked matroids recognition, and uniform matroids recognition in particular.

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