Translation-Invariant Estimates for Operators with Simple Characteristics

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Abstract

We prove L^2 estimates and solvability for a variety of simply characteristic constant coefficient partial differential equations $P(D)u = f$. These estimates

$$||u||_{L^2(D_r)} \leqslant C\sqrt{d_r d_s}||f||_{L^2(D_s)}$$

depend on geometric quantities — the diameters d_r and d_s of the regions D_r, where we estimate u, and D_s, the support of f — rather than weights. As these geometric quantities transform simply under translations, rotations, and dilations, the corresponding estimates share the same properties. In particular, this implies that they transform appropriately under change of units, and therefore are physically meaningful. The explicit dependence on the diameters implies the correct global growth estimates. The weighted L^2 estimates first proved by Agmon [1] in order to construct the generalized eigenfunctions for Laplacian plus potential in $\mathbb{R}^n$, and the more general and precise Besov type estimates of Agmon and Hörmander [2], are all simple direct corollaries of the estimate above.

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1 Introduction

Constant coefficient partial differential equations are translation invariant, so it is natural to seek estimates that share this property. For the Helmholtz equation, and other equations related to wave phenomena, L^2-norms are appropriate in bounded regions because they measure energy. For problems in all of $\mathbb{R}^n$, however, a solution with finite L^2-norm may radiate infinite power[1], and therefore not satisfy the necessary physical constraints. The solution provided by Agmon [1] was to introduce L^2_δ spaces where weights $(1+|x|^2)^{\frac{\delta}{2}}$ correctly enforced the finite transmission of power, but gave up the translation invariance, as well as scaling properties necessary for the estimates to make sense in physical units. Later work by Agmon and Hörmander [2] used Besov spaces to exactly characterize solutions that radiated finite power, but these spaces also relied on a weight and therefore broke the translation invariance that is intrinsically associated with both the physics and the mathematics of the underlying problem. Later work by Kenig, Ponce, and Vega [9] modified the Agmon-Hörmander norms to regain better scaling properties.

Our goal here is to offer L^2 estimates that enforce finite radiation of power without using weights that destroy translation invariance and scaling properties. The following theorem, which applies to a class of scalar pde's with constant coefficients and simple characteristics, summarizes our main results, which will be proved as Theorem 5.1 and Theorem 6.1.

Theorem. *Let $P(D)$ be a constant coefficient partial differential operator on $\mathbb{R}^n$. Assume that it is either*

1. *real, of second order, and with no real double characteristics, or*
2. *of N-th order, $N \geqslant 1$, with admissible symbol (Definition 6.10) and no complex double characteristics*

Then there exists a constant $C(P,n)$ such that, for every open bounded $D_s \subset \mathbb{R}^n$, and every $f \in L^2(D_s)$, there is a $u \in L^2_{loc}(\mathbb{R}^n)$ satisfying

$$P(D)u = f$$

and for any bounded domain $D_r \subset \mathbb{R}^n$

$$||u||_{L^2(D_r)} \leqslant C\sqrt{d_r d_s}||f||_{L^2(D_s)} \tag{1.1}$$

[1]Finite radiated power typically means that solutions decay fast enough at infinity. For outgoing solutions to the Helmholtz equation, radiated power can be expressed as the limit as $R \to \infty$ of the L^2 norm of the restriction of the solution to the sphere of radius R. It remains finite as long as solutions decay as $r^{-\frac{n-1}{2}}$ in n dimensions.

where d_j is the diameter of D_j, the supremum over all lines of the length of the intersection of the line with D_j; i.e.

$$d_j = \sup_{\text{lines } l} \mu_1(l \cap D_j).$$

If f is not compactly supported, but $\operatorname{supp} f \subset \overset{\infty}{\underset{j=1}{\cup}} B_j$ where each B_j has finite diameter b_j, then (1.1) becomes

$$||u||_{L^2(D)} \leqslant C\sqrt{d}\sum_{j=1}^{\infty} \sqrt{b_j}||f||_{L^2(B_j)}$$

which we may rewrite as

$$\sup_{D} \frac{1}{\sqrt{d}}||u||_{L^2(D)} \leqslant \sum_{j=1}^{\infty} \sqrt{b_j}||f||_{L^2(B_j)}. \tag{1.2}$$

In the special case that D is a ball with a fixed center and arbitrary radius; and the B_j include the ball of radius one and the dyadic spherical shells $2^j < |x| < 2^{j+1}$ for $j \geqslant 0$, these are the estimates of Agmon-Hörmander in [2]. The weighted L^2_δ estimates introduced by Agmon are also direct consequences of (1.1), so that these solutions do radiate finite power and are therefore physically meaningful. The solutions we construct are not necessarily unique, but include the *physically correct* solutions in all the cases we are aware of. For the Helmholtz equation, for example, the solution which satisfies the Sommerfeld radiation condition is among those which satisfy the estimate (1.1).

Our estimates do not include the uniform L^p estimates for the Helmholtz equation, shown below, which were derived in [16] and [10], and presented in [12] and [13].

Theorem (Uniform L^p estimates)**.** *Let $k > 0$ and $\frac{2}{n} \geqslant \frac{1}{p} - \frac{1}{q} \geqslant \frac{2}{n+1}$ for $n \geqslant 3$ and $1 > \frac{1}{p} - \frac{1}{q} \geqslant \frac{2}{3}$ for $n = 2$, where $\frac{1}{q} + \frac{1}{p} = 1$. There exists a constant $C(n,p)$, independent of k, such that, for smooth compactly supported u*

$$||u||_{L^q(\mathbb{R}^n)} \leqslant C(n,p)k^{n(\frac{1}{p}-\frac{1}{q})-2}||(\Delta + k^2)u||_{L^p(\mathbb{R}^n)} \tag{1.3}$$

The estimates for the Helmholtz equation in (1.3) share all the invariance properties of (1.1), and are stronger for small scatterers and applications to nonlinear problems. The dependence on the wavenumber k, however, is not as well-suited to applications where the sources are supported on sets that are several wavelengths in size and located far apart, nor do they have a direct physical interpretation in terms of power. Additionally, it seems reasonable that the estimate of the solution in the higher L^p norm indicates a gain in regularity. Our methods don't require, or make use of ellipticity, so we don't expect to recover these estimates.

Our methods make use of certain anisotropic norms introduced in [14] for the Helmholtz equation. Those estimates were scale and translation invariant, but, due to the anisotropy, not rotationally invariant. We show here that a consequence of these mixed norm estimates is (1.1), which is rotationally invariant and much simpler than the mixed norm estimates used to derive it. Because of the generality, the mixed norms we use here must be slightly different than those in in [14], and the techniques required to treat more general operators are substantially more complicated.

We treat only operators with simple characteristics because a *bona fide* real multiple characteristic (a real $\eta \in \mathbb{R}^n$ where the symbol $p(\eta)$ and $\nabla p(\eta)$ vanish simultaneously) will imply that our techniques cannot succeed. In Section 7, we show that estimates of the form (1.1) cannot hold for the Laplacian, which has a double characteristic at the origin.

For a single second order operator with real constant coefficients we will show in Theorem 5.1 that the absence of multiple characteristics is sufficient to conclude the estimate (1.1). Under some additional hypotheses, we will prove the same estimate for some higher order operators in Theorem 6.1. Additionally, we will prove the estimate (1.1) for the 4x4 Dirac system, and for a scalar 4th order equation where Hörmander's *uniformly simply characteristic* condition fails.

2 The Helmholtz case

We will illustrate our methods by outlining the proof of (1.1) for the outgoing solution to the Helmholtz equation below.

$$(\Delta + k^2)u = f \tag{2.1}$$

We will choose a direction Θ and write $x = t\Theta + x_{\Theta^\perp}$. We next Fourier transform in the $\Theta^\perp$ hyperplane to rewrite (2.1) as an ordinary differential equation. We use the notation $\mathscr{F}_{\Theta^\perp} u(t\Theta + \xi_{\Theta^\perp})$ to indicate this partial Fourier transform (see (3.2) below for a formal definition). If we set $g(t, \xi_{\Theta^\perp}) = \mathscr{F}_{\Theta^\perp} f(t\Theta + \xi_{\Theta^\perp})$ and $w(t, \xi_{\Theta^\perp}) = \mathscr{F}_{\Theta^\perp} u(t\Theta + \xi_{\Theta^\perp})$, then (2.1) becomes

$$(\partial_t^2 + k^2 - |\xi_{\Theta^\perp}|^2)w = g \tag{2.2}$$

We factor the second order operator as a product of first order operators

$$\left(\partial_t + i\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}\right)\left(\partial_t - i\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}\right) w = g$$

and define a solution $w = w_1 + w_2$ where w_1 and w_2 solve

$$\begin{cases} \left(\partial_t + i\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}\right) w_1 = \frac{ig}{2\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}}, \\ \left(\partial_t - i\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}\right) w_2 = \frac{-ig}{2\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}}. \end{cases} \tag{2.3}$$

The solutions w_1 and w_2 are given by the exact formulas

$$w_1(t, \xi_{\Theta^\perp}) = \frac{1}{2\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}} \int_{-\infty}^{t} e^{i\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}(t-s)}\ ig(s, \xi_{\Theta^\perp}) ds \tag{2.4}$$

$$w_2(t, \xi_{\Theta^\perp}) = \frac{1}{2\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}} \int_{t}^{\infty} e^{-i\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}(t-s)}\ ig(s, \xi_{\Theta^\perp}) ds \tag{2.5}$$

The square root $\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}$ is chosen so that it always has positive imaginary part for imaginary part of k positive, and extends continuously, as a function of k, to the real axis. This insures that the exponential in (2.4) and (2.5) is bounded by one[2] so that $\mathscr{F}_{\Theta^\perp} u = w = w_1 + w_2$ satisfies

$$|\mathscr{F}_{\Theta^\perp} u(t\Theta + \xi_{\Theta^\perp})| \leqslant \frac{||\mathscr{F}_{\Theta^\perp} f(s\Theta + \xi_{\Theta^\perp})||_{L^1(ds)}}{\sqrt{k^2 - |\xi_{\Theta^\perp}|^2}} \tag{2.6}$$

which would yield a simple estimate if the denominator had a lower bound.

In sections 5 and 6, we will construct Fourier multipliers that implement a partition of unity that decomposes f into a sum

$$f = f_1 + f_2 + \ldots + f_m \tag{2.7}$$

$$= f\phi_1 + f\phi_2(1-\phi_1) + f\phi_3(1-\phi_2)(1-\phi_1)\ldots + f\prod_{j=1}^{m}(1-\phi_j) \tag{2.8}$$

[2]This also selects the unique outgoing solution, which satisfies the Sommerfeld radiation condition.

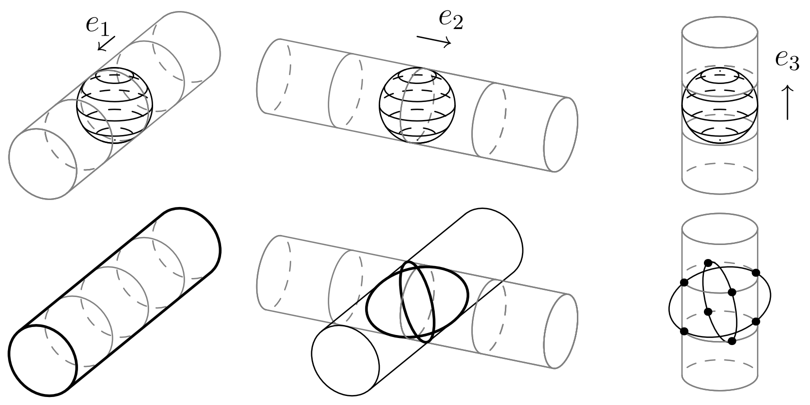

Figure 1: Partition of unity for $\Delta + k^2$.

such that, for each f_j, there is a direction Θ_j such that

$$\inf_{\operatorname{supp}\mathscr{F}_{\Theta_j^\perp} f} \sqrt{k^2 - \left|\xi_{\Theta_j^\perp}\right|^2} \;\geqslant\; \varepsilon k \tag{2.9}$$

and

$$\left\| \left\| \mathscr{F}_{\Theta_j^\perp} f_j(s\Theta_j + \xi_{\Theta_j^\perp}) \right\|_{L^1(ds)} \right\|_{L^2(d\xi_{\Theta_j^\perp})} \;\leqslant\; \left\| \left\| \mathscr{F}_{\Theta_j^\perp} f(s\Theta_j + \xi_{\Theta_j^\perp}) \right\|_{L^1(ds)} \right\|_{L^2(d\xi_{\Theta_j^\perp})}$$

which we write more compactly as

$$\left\| \mathscr{F}_{\Theta_j^\perp} f_j \right\|_{\Theta_j(1,2)} \;\leqslant\; \left\| \mathscr{F}_{\Theta_j^\perp} f \right\|_{\Theta_j(1,2)} \tag{2.10}$$

using norms which we will define precisely in (3.3).

We illustrate this decomposition for the 3-dimensional case in Figure 1. Let $\Theta_j = e_j$, $j = 1, 2, 3$, be an orthogonal basis. The cylinders illustrated in the top-row are the sets, denoted $\mathscr{B}_{\Theta_j,0}$, where the denominators $\sqrt{k^2 - \left|\xi_{\Theta_j^\perp}\right|^2}$ of (2.6) vanish. Each ϕ_j, and hence each f_j, vanishes in a neighborhood of $\mathscr{B}_{\Theta_j,0}$. The thick lines in the figures in the bottom row show the intersections $\mathscr{B}_{\Theta_1,0}$, $\mathscr{B}_{\Theta_1,0} \cap \mathscr{B}_{\Theta_2,0}$ and $\mathscr{B}_{\Theta_1,0} \cap \mathscr{B}_{\Theta_2,0} \cap \mathscr{B}_{\Theta_3,0}$, indicating the support of the $\prod_{j=1}^m (1 - \phi_j)$. To guarantee that the f_j sum to f, the intersection of (neighborhoods of) all the $\mathscr{B}_{\Theta_j,0}$ must be empty. We see in the figure that the intersection of the first three neighborhoods consists of neighborhoods of eight points, so we may add a fourth direction, for example $\Theta_4 = (e_1 + e_2 + e_3)/\sqrt{3}$ (not pictured), so that the corresponding cylinder $\mathscr{B}_{\Theta_4,0}$ does not intersect the eight points that are left.

Combining (2.6), (2.9), and (2.10)

$$||\mathscr{F}_{\Theta_j^\perp} u_j||_{\Theta_j(\infty,2)} \leqslant \frac{||\mathscr{F}_{\Theta_j^\perp} f||_{\Theta_j(1,2)}}{k\varepsilon} \tag{2.11}$$

where each of the u_j solves $(\Delta + k^2)u_j = f_j$. The estimates (2.11) estimate each u_j in a different norm, and the norms, which depend on a choice of the vectors Θ_j, are no longer rotationally invariant. They can, however, be combined to yield an estimate in a single norm that is rotationally and translationally invariant.

Lemma 2.1. *Let $D_s, D_r \subset \mathbb{R}^n$ be domains with diameters d_s and d_r, respectively. Let $\mathscr{F}_{\Theta^\perp} u \in \Theta(\infty, 2)$ and $f \in L^2_{loc}$. Assume that* $\operatorname{supp} f \subset D_s$. *Then $u_{|D_r} \in L^2$ and $\mathscr{F}_{\Theta^\perp} f \in \Theta(1,2)$. Moreover*

$$\begin{aligned} \|u\|_{L^2(D_r)} &\leqslant \sqrt{d_r}\, \|\mathscr{F}_{\Theta^\perp} u\|_{\Theta(\infty,2)}\,, \\ \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)} &\leqslant \sqrt{d_s}\, \|f\|_{L^2(D_s)}\,. \end{aligned} \tag{2.12}$$

Combining the lemma with (2.11) yields

$$\|u\|_{L^2(D_r)} \leqslant \frac{\sqrt{d_r d_s}}{\varepsilon k} \|f\|_{L^2(D_s)}\,. \tag{2.13}$$

We leave the proof of the lemma for the next section, after we have given the formal definitions of the norms.

3 Mixed norms

We begin with the formal definition of the anisotropic norms we will use.

Definition 3.1. Let $\Theta \in \mathbb{S}^{n-1}(\mathbb{R}^n)$. We split any $x \in \mathbb{R}^n$ as

$$x = t\Theta + x_{\Theta^\perp} \tag{3.1}$$

where $t = x \cdot \Theta$ and $x_{\Theta^\perp} = x - (x \cdot \Theta)\Theta$. We split the dual variable ξ as

$$\xi = \tau\Theta + \xi_{\Theta^\perp}$$

The variables t and τ are dual, and so are $x_{\Theta^\perp}$ and $\xi_{\Theta^\perp}$.

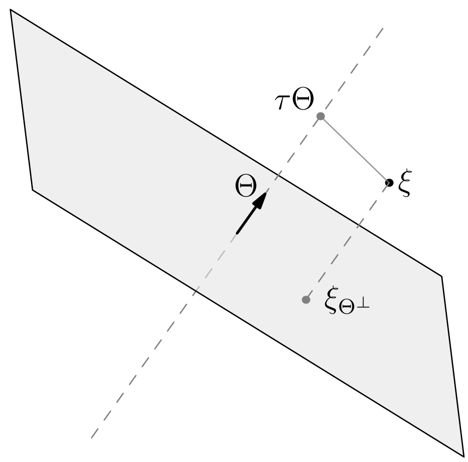

Figure 2: Splitting of $\xi = \tau\Theta + \xi_{\Theta^\perp}$.

Definition 3.2. By $\mathscr{F}_\Theta$ we denote the one-dimensional Fourier transform along the direction Θ. If $f \in \mathscr{S}(\mathbb{R}^n)$ then

$$\mathscr{F}_\Theta f(\tau\Theta + x_{\Theta^\perp}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\tau} f(t\Theta + x_{\Theta^\perp}) dt$$

using the notation of Definition 3.1. The Fourier transform in the orthogonal space $\Theta^\perp$ is denoted by $\mathscr{F}_{\Theta^\perp}$ and it acts by

$$\mathscr{F}_{\Theta^\perp} f(t\Theta + \xi_{\Theta^\perp}) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\Theta^\perp} e^{-ix_{\Theta^\perp}\cdot\xi_{\Theta^\perp}} f(t\Theta + x_{\Theta^\perp}) dx_{\Theta^\perp}. \tag{3.2}$$

The corresponding inverse transforms are denoted by $\mathscr{F}_\Theta^{-1}$ and $\mathscr{F}_{\Theta^\perp}^{-1}$.

Definition 3.3. We use $\Theta(p, q)$ we denote the space of $L^p(dt)$-valued $L^q(dx_{\Theta^\perp})$-functions (if the variable is $x = t\Theta + x_{\Theta^\perp}$). More precisely $f \in \Theta(p, q)$ if

$$\|f\|_{\Theta(p,q)} = \left(\int_{\Theta^\perp} \left(\int_{-\infty}^{\infty} |f(t\Theta + x_{\Theta^\perp})|^p \, dt \right)^{q/p} dx_{\Theta^\perp} \right)^{1/q} < \infty. \tag{3.3}$$

with obvious modifications for $p = \infty$ or $q = \infty$.

Remark 3.4. We make note of the fact that the order is important. For example, we will use the norm

$$\|f\|_{\Theta(1,\infty)} = \sup_{x_{\Theta^\perp} \in \Theta^\perp} \int_{-\infty}^{\infty} |f(t\Theta + x_{\Theta^\perp})| \, dt$$

in several lemmas. This is clearly not the same as

$$\int_{-\infty}^{\infty} \sup_{x_{\Theta^\perp}\in\Theta^\perp} |f(t\Theta + x_{\Theta^\perp})|\, dt$$

We convert estimates in these anisotropic norms to isotropic L^2 estimates with Lemma 2.1. We give the proof now.

Proof of Lemma 2.1. Let D_s^Θ, D_r^Θ be the projections of D_s and D_r onto the line $t \mapsto t\Theta$. We have

$$\begin{aligned}
\|u\|^2_{L^2(D_r)} &\leqslant \int_{D_r^\Theta}\int_{\Theta^\perp} |u(t\Theta + x_{\Theta^\perp})|^2\, dx_{\Theta^\perp} dt \\
&= \int_{D_r^\Theta}\int_{\Theta^\perp} |\mathscr{F}_{\Theta^\perp} u(t\Theta + \xi_{\Theta^\perp})|^2\, d\xi_{\Theta^\perp} dt \\
&\leqslant \int_{D_r^\Theta} dt \int_{\Theta^\perp} \sup_{t'} |\mathscr{F}_{\Theta^\perp} u(t'\Theta + \xi_{\Theta^\perp})|^2\, d\xi_{\Theta^\perp} \leqslant d_r \|\mathscr{F}_{\Theta^\perp} u\|^2_{\Theta(\infty,2)}
\end{aligned}$$

where we have used the Plancherel formula and the hypothesis that the diameter of D_r is at most d_r, which implies that D_r^Θ is contained in a union of intervals of length at most d_r. The proof of (2.12) is similar, and makes use of the fact that D_s^Θ is contained in a union of intervals of length less than d_s.

$$\int_{-\infty}^{\infty} |\mathscr{F}_{\Theta^\perp} f(t\Theta + \xi_{\Theta^\perp})|\, dt \leqslant \sqrt{d_s}\, \|\mathscr{F}_{\Theta^\perp} f(t\Theta + \xi_{\Theta^\perp})\|_{L^2(dt)}$$

The inequality (2.12) follows by taking the $L^2(d\xi_{\Theta^\perp})$-norm and using Fubini's theorem, and then the Plancherel formula. ☐

Remark 3.5. Let $\frac{1}{q} + \frac{1}{p} = 1$ and $q \geqslant 2$. An analogous argument shows that $\|u\|_{L^q(D_r)} \leqslant d_r^{1/q} \|\mathscr{F}_{\Theta^\perp} u\|_{\Theta(\infty,p)}$ and $\|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,p)} \leqslant d_s^{1/p} \|\mathscr{F} f\|_{L^p(\mathbb{R}^n)}$.

4 Fourier Multiplier Estimates

Definition 4.1. Let $\Psi : \mathbb{R}^n \to \mathbb{C}$ be locally integrable. We define the Fourier multiplier M_Ψ as the the operator

$$M_\Psi f = \mathscr{F}^{-1}\{\Psi \mathscr{F} f\}.$$

Because our estimates rely on decompositions of sources similar to (2.8) where $f_j = M_{\Psi_j} f$ must satisfy satisfy conditions similar to (2.9) and the estimate (2.10), we need to establish the boundedness of these Fourier multipliers on the mixed norms of the partial Fourier transforms of the sources, i.e. on $\|\mathscr{F}_{\Theta^\perp} f(t,\xi)\|_{\Theta(1,2)}$. Our first lemma tells us that $\left\|\mathscr{F}_\Theta^{-1}\Psi\right\|_{\Theta(1,\infty)} < \infty$ is enough to guarantee such a bound.

Lemma 4.2. *Let $\mathscr{F}_\Theta^{-1}\Psi \in \Theta(1,\infty)$. Then*

$$\|\mathscr{F}_{\Theta^\perp} M_\Psi f\|_{\Theta(1,p)} \leqslant \frac{1}{\sqrt{2\pi}} \left\|\mathscr{F}_\Theta^{-1}\Psi\right\|_{\Theta(1,\infty)} \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,p)}.$$

Proof. Write $\mathscr{F}_{\Theta^\perp} M_\Psi f = \mathscr{F}_\Theta^{-1}\{\Psi \mathscr{F} f\} = \frac{1}{\sqrt{2\pi}} \mathscr{F}_\Theta^{-1}\Psi *_t \mathscr{F}_{\Theta^\perp} f$. Then take the $L^1(dt)$-norm and use Young's inequality for convolutions. The result follows then by taking the $L^p(d\xi_{\Theta^\perp})$-norm. ☐

Our Fourier multipliers will not be Schwartz class functions. They will be smooth, but will always be constant in a direction ν, so the integrability properties necessary to verify that the $\Theta(1,\infty)$ norm is finite may be a bit subtle, and will depend on the relation between the direction ν of that coordinate and the direction Θ which defines the relevant norm. The estimates will be simplest when the directions ν and Θ coincide, or are perpendicular. Because second order operators have a convenient normal form, the decompositions in Section 5 will only require multipliers with ν and Θ either identical or perpendicular. Higher order operators do not admit such simple normal forms, so the decompositions are based on abstract algebraic properties, and we cannot, in general, restrict to these simple cases. The next proposition, and its corollary, tell us how to reduce the $\Theta(1,\infty)$ estimate for the norm of a multiplier that is constant in the ν direction, to the case where ν and Θ are either parallel or perpendicular.

We need a little notation first. Define $\nu_\perp$ to be a unit vector in the (Θ,ν) plane perpendicular to ν so that the pair $(\nu,\nu_\perp)$ is positively oriented , and define $\Theta_\perp$ analogously to be the unit vector in that plane perpendicular to Θ. Finally, let $\xi_{\perp\perp}$ denote the component of any $\xi \in \mathbb{R}^n$ perpendicular to the (Θ,ν) plane .

Proposition 4.3. *Let $\nu \in \mathbb{S}^{n-1}(\mathbb{R}^n)$ and $\psi \in \mathscr{S}(\nu^\perp)$. Define*

$$\Psi(\sigma\nu + \xi_{\nu^\perp}) = \psi(\xi_{\nu^\perp}) \qquad \forall \sigma \in \mathbb{R}.$$

If $\Theta \nparallel \nu$ and $\Theta \cdot \nu = \cos\alpha$, $\alpha \in (0,\pi)$ then,

$$\mathscr{F}_\Theta^{-1}\Psi(t\Theta + \xi_{\Theta^\perp}) = \frac{e^{i\ell t \cot\alpha}}{\sin\alpha} \mathscr{F}_{\nu_\perp}^{-1}\psi\Big(\frac{t}{\sin\alpha}\nu_\perp + \xi_{\perp\perp}\Big), \tag{4.1}$$

where $\ell = \xi_{\Theta^\perp} \cdot \Theta_\perp$ and $\xi_{\perp\perp}$ is the component of $\xi = t\Theta + \xi_{\Theta^\perp}$ perpendicular to the (Θ,ν) plane. If $\Theta \parallel \nu$ then

$$\mathscr{F}_\Theta^{-1}\Psi(t\Theta + \xi_{\Theta^\perp}) = \sqrt{2\pi}\delta_0(t)\psi(\xi_{\Theta^\perp}). \tag{4.2}$$

Proof. According to Definition 3.2

$$\mathscr{F}_\Theta^{-1}\Psi(t\Theta+\xi_{\Theta^\perp}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it\tau}\Psi(\tau\Theta+\xi_{\Theta^\perp})d\tau.$$

It is easy to check that

$$\begin{aligned}\Theta &= \cos\alpha\,\nu+\sin\alpha\,\nu_\perp,\\ \nu &= \cos\alpha\,\Theta+\sin\alpha\,\Theta_\perp,\\ \Theta_\perp &= \sin\alpha\,\nu-\cos\alpha\,\nu_\perp,\end{aligned}$$

and therefore that

$$\tau\Theta+\xi_{\Theta^\perp} = (\tau\cos\alpha+\ell\sin\alpha)\nu+(\tau\sin\alpha-\ell\cos\alpha)\nu_\perp+\xi_{\perp\perp}.$$

Because Ψ is constant in the direction ν and equal to ψ on $\nu^\perp$,

$$\begin{aligned}\mathscr{F}_\Theta^{-1}\Psi(t\Theta+\xi_{\Theta^\perp}) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it\tau}\psi\big((\tau\sin\alpha-\ell\cos\alpha)\nu_\perp+\xi_{\perp\perp}\big)d\tau\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it\left(\frac{\tau'}{\sin\alpha}+\ell\cot\alpha\right)}\psi(\tau'\nu_\perp+\xi_{\perp\perp})\frac{d\tau'}{\sin\alpha}\\ &= \frac{e^{i\ell t\cot\alpha}}{\sin\alpha}\mathscr{F}_{\nu_\perp}^{-1}\psi\big(\frac{t}{\sin\alpha}\nu_\perp+\xi_{\perp\perp}\big).\end{aligned}$$

If $\Theta \parallel \nu$, then $\Psi(t\Theta+\xi_{\Theta^\perp})$ is independent of t, so (4.2) follows from the fact that the one dimensional Fourier transform of the constant function is the Dirac delta. □

Corollary 4.4. *With the notation of Proposition 4.3 and $\Theta \nparallel \nu$ we have*

$$\left\|\mathscr{F}_\Theta^{-1}\Psi\right\|_{\Theta(1,\infty)} = \left\|\mathscr{F}_{\nu_\perp}^{-1}\psi\right\|_{\nu_\perp(1,\infty)}, \tag{4.3}$$

and therefore

$$\left\|\mathscr{F}_{\Theta^\perp}M_\Psi f\right\|_{\Theta(1,p)} \leqslant \left\|\mathscr{F}_{\nu_\perp}^{-1}\psi\right\|_{\nu_\perp(1,\infty)}\left\|\mathscr{F}_{\Theta^\perp}f\right\|_{\Theta(1,p)}. \tag{4.4}$$

If $\Theta \parallel \nu$, then

$$\left\|\mathscr{F}_{\Theta^\perp}M_\Psi f\right\|_{\Theta(1,p)} \leqslant \sup_{\Theta^\perp}|\psi|\left\|\mathscr{F}_{\Theta^\perp}f\right\|_{\Theta(1,p)}. \tag{4.5}$$

Remark 4.5. The $\nu_\perp(1,\infty)$ norm which appears in (4.3) is analogous to the $\Theta_\perp(1,\infty)$ norm , but is defined on functions of one fewer variable. Recall that ψ is defined on the $\nu^\perp$ hyperplane, and $\nu_\perp$ is a unit vector in that hyperplane perpendicular to Θ. Thus $\mathscr{F}_{\nu_\perp}^{-1}\psi$ is a function of $\tau\nu_\perp+\xi_{\perp\perp}$, and the $\nu_\perp(1,\infty)$ norm means the supremum over $\xi_{\perp\perp}$ of the $L^1(d\tau)$ norm.

Proof. We have $0 < \alpha < \pi$ and so $\sin\alpha > 0$. Hence

$$\int_{-\infty}^{\infty} \left|\mathscr{F}_{\Theta}^{-1}\Psi(t\Theta + \xi_{\Theta^\perp})\right| dt = \int_{-\infty}^{\infty} \left|\mathscr{F}_{\nu_\perp}^{-1}\psi(t'\nu_\perp + \xi_{\perp\perp})\right| dt'$$

by a change of variables. Then we can take the supremum over $\xi_{\Theta^\perp} \in \Theta^\perp$, which will give the same result as the supremum of $\xi_{\perp\perp}$ over $\nu^\perp \cap \nu_\perp^\perp$. The multiplier estimate follows from Lemma 4.2. For the second case note that $M_\Psi f = \mathscr{F}_{\Theta^\perp}^{-1}\{\psi(\xi_{\Theta^\perp})\mathscr{F}_{\Theta^\perp} f\}$. The claim follows directly. $\square$

5 Estimates for 2^{nd} order operators

We treat a second order constant coefficient partial differential operator $P(D)$, with no double characteristics, i.e. no simultaneous real root of $p(\xi) = 0$ and $\nabla p(\xi) = 0$. The main result of this section is:

Theorem 5.1. *Let $P(D)$ be a single real second order constant coefficient partial differential operator on $\mathbb{R}^n$ with no real double characteristics. Then there exists a constant $C(P, n)$ such that, for every open bounded $D_s \subset \mathbb{R}^n$, and every $f \in L^2(D_s)$, there is a $u \in L^2_{loc}(\mathbb{R}^n)$ satisfying*

$$P(D)u = f \tag{5.1}$$

such that, for any bounded domain $D_r \subset \mathbb{R}^n$

$$||u||_{L^2(D_r)} \leqslant C\sqrt{d_r d_s}||f||_{L^2(D_s)} \tag{5.2}$$

where d_j is the diameter of D_j, the supremum over all lines of the length of the intersection of the line with D_j; i.e.

$$d_j = \sup_{\text{lines } l} \mu_1(l \cap D_j).$$

We begin the proof by writing the second order operator in a simple normal form.

Lemma 5.2. *After an orthogonal change of coordinates and a rescaling:*

$$P(D) = \sum \epsilon_j \left(\frac{\partial}{\partial x_j}\right)^2 + 2\sum \alpha_j \frac{\partial}{\partial x_j} + B \tag{5.3}$$

where each ϵ_j equals one of $0, 1, -1$; $\alpha_j \in \mathbb{R}$, and $B \in \mathbb{R}$.

Proof. This is a statement about the principal (second order) part P_2 of the operator. $P_2(\xi)$ is a real quadratic form with eigenvalues $-\lambda_i$ and eigenvectors e_j. If we introduce coordinates

$$x = \sum x_j e_j$$

then

$$P_2(D) = \sum \lambda_j \left(\frac{\partial}{\partial x_j}\right)^2$$

After the rescaling

$$x_j = \sqrt{\lambda_j} y_j \qquad \frac{\partial}{\partial y_j} = \sqrt{\lambda_j}\frac{\partial}{\partial x_j}$$

the second order part takes the desired form in (5.3). □

Next, we dismiss the simple cases.

Proposition 5.3. *If some $\epsilon_j = 0$ and the corresponding $\alpha_j \neq 0$, then Theorem 5.1 is true.*

Proof. Without loss of generality, we may assume that $j = 1$. We do a partial Fourier transform in the $x_1^\perp$ plane, i.e. with $\tilde{x} = (x_2, \ldots, x_n)$ and $\xi = (\xi_2, \ldots, \xi_n)$. We let Θ_1 denote the unit vector in the x_1 direction, and let $w = \mathscr{F}_{\Theta^\perp} u$ and $g = \frac{1}{2}\mathscr{F}_{\Theta^\perp} f$, i.e.

$$w(x_1, \xi) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} e^{-i\xi\cdot\tilde{x}} u(x_1, \tilde{x}) dx_2 \ldots dx_n$$

Then w satisfies

$$\alpha_1 \frac{\partial w}{\partial x_1} + Q(\xi) w = g(x_1, \xi) \tag{5.4}$$

where $Q(\xi) = -\frac{1}{2}\sum_{j=2}^n \epsilon_j \xi_j^2 + i\sum_{j=2}^n \alpha_j \xi_j + \frac{1}{2}B$. We may write an explicit formula for w:

$$w(x_1, \xi) = \frac{1}{\alpha_1}\begin{cases} \int_{-\infty}^{x_1} e^{Q(\xi)\frac{x_1 - s}{\alpha_1}} g(s, \xi) ds & \text{if } \frac{\Re Q(\xi)}{\alpha_1} > 0 \\ -\int_{x_1}^{\infty} e^{Q(\xi)\frac{x_1 - s}{\alpha_1}} g(s, \xi) ds & \text{if } \frac{\Re Q(\xi)}{\alpha_1} < 0 \end{cases} \tag{5.5}$$

Our formula insures that, on the domain of integration,

$$|e^{Q(\xi)\frac{x-s}{\alpha_1}}| < 1 \tag{5.6}$$

and therefore, for each fixed ξ, that

$$||w(\cdot,\xi)||_{L^\infty} \leqslant \frac{1}{\alpha_1}||g(\cdot,\xi)||_{L^1}$$

so that, squaring and integrating with respect to ξ gives

$$||w||_{L^2(d\xi,L^\infty(dx_1))} \leqslant \frac{1}{\alpha_1}||g||_{L^2(d\xi,L^1(dx_1))} \tag{5.7}$$

or, using the notation of mixed norms

$$||\mathscr{F}_{\Theta_1^\perp}u||_{\Theta_1(\infty,1)} \leqslant \frac{1}{2\alpha_1}||\mathscr{F}_{\Theta_1^\perp}f||_{\Theta_1(1,2)}$$

with Θ_1 equal to the unit vector in the x_1 direction. This combines with Lemma 2.1 to yield the estimate (5.2). $\square$

The proof of Theorem 5.1 will use partitions of unity and coordinate changes to reduce to a case very similar to (5.4) and (5.5) and prove estimates of the form in (2.13). Our main proof will prove Theorem 5.1 in the case that no ϵ_j in (5.3) is zero. We have already treated the case where some $\epsilon_j = 0$ and the corresponding $\alpha_j \neq 0$. If, for one or more values of j, $\epsilon_j = \alpha_j = 0$, then the PDE in (5.1) is independent of the x_j variables. In this case, we may obtain the inequality (2.13) from the corresponding inequality in the lower dimensional case. We record this in the proposition below.

Proposition 5.4. *Let $x = (x_1, \tilde{x}, y)$, and suppose that, for each y,*

$$||u(\cdot,\cdot,y)||_{L^2(d\tilde{x},L^\infty(dx_1))} \leqslant C||f(\cdot,\cdot,y)||_{L^2(d\tilde{x},L^1(dx_1))} \tag{5.8}$$

then

$$||u||_{L^2(d\tilde{x}dy,L^\infty(dx_1))} \leqslant C||f(\cdot,\cdot,y)||_{L^2(d\tilde{x}dy,L^1(dx_1))} \tag{5.9}$$

Proof. Just square both sides of (5.8) and integrate with respect to y. $\square$

Henceforth, we will assume that no $\epsilon_j = 0$, and complete the squares in (5.3) to rewrite that equation as

$$P(D) = \sum_{j=1}^{n} \epsilon_j(\frac{\partial}{\partial x_j} - \beta_j)^2 + b \qquad ; \quad \epsilon_j = \pm 1$$

where the $\beta_j = -\epsilon_j\alpha_j$ from (5.3) and $b = B - \sum \epsilon_j\beta_j^2$.

Proposition 5.5. *$P(D)$ has a real double characteristic iff $b = 0$ and $\beta = \overline{0}$.*

Proof.

$$p(\eta) = \sum \epsilon_j (i\eta_j - \beta_j)^2 + b \tag{5.10}$$
$$dp = \sum 2i\epsilon_j (i\eta_j - \beta_j) d\eta_j$$

so that

$$dp = 0 \quad \iff \quad \text{every } \eta_j = -i\beta_j$$

but, as the β_j are real, this can only happen if

$$\eta_j = \beta_j = 0$$

If p vanishes as well, we must also have $b = 0$. ☐

We now begin the proof of Theorem 5.1 in earnest. We intend to use partial Fourier transforms, as defined in (3.2). To this end, we will choose special directions $\Theta \in \mathbb{R}^n$ (the unit vectors Θ_k in the coordinate directions will suffice for the proof of Theorem 5.1) and express $x \in \mathbb{R}^n$ as

$$x = t\Theta + x_{\Theta^\perp}$$

as in (3.1) and write the dual variable η as

$$\eta = \tau\Theta + \xi \qquad \text{with } \xi \in \Theta^\perp$$

In these coordinates, we consider $p(\eta)$ as a polynomial $p(\tau;\xi)$ in τ with coefficients depending on ξ. We will arrive at the estimate (5.2) as long as the roots of p are simple. When $\Theta = \Theta_k$, $\xi = (\eta_1, \ldots \eta_{k-1}, \eta_{k+1}, \ldots, \eta_n)$. If we define

$$Q_k(\xi) := \sum_{j \neq k} \epsilon_j (i\eta_j - \beta_j)^2 + b \tag{5.11}$$

then by (5.10) we have $p(\tau, \xi) = \epsilon_k (i\tau - \beta_k)^2 + Q_k(\xi)$ and its roots are

$$\tau_\pm = -i\beta_k \pm \sqrt{\epsilon_k Q_k(\xi)}$$

and they are simple as long as

$$Q_k(\xi) \neq 0.$$

Proposition 5.6. *Suppose that*

$$\operatorname{supp} \widehat{f}(\eta) \subset \{\eta \in \mathbb{R}^n \big| \; |Q_k(\xi)| > \varepsilon\} \tag{5.12}$$

Then there exists u solving

$$P(D)u = f$$

satisfying

$$||\mathscr{F}_{\Theta_k^\perp} u||_{\Theta_k(\infty,2)} \leqslant \frac{1}{\sqrt{\varepsilon}} ||\mathscr{F}_{\Theta_k^\perp} f||_{\Theta_k(1,2)} \tag{5.13}$$

Proof. With $x = t\Theta_k + x_{\Theta^\perp}$, we again use the partial Fourier transform

$$\mathscr{F}_{\Theta_k^\perp} u(t,\xi) = \frac{1}{(2\pi)^{(n-1)/2}} \int u(t, x_{\Theta_k^\perp}) e^{-i\xi\cdot x_{\Theta_k^\perp}} dx_{\Theta_k^\perp}$$

Letting $w := \mathscr{F}_{\Theta_k^\perp} u(t,\xi)$ and $g = \mathscr{F}_{\Theta^\perp} f(t,\xi)$, we see that

$$\epsilon_k \left(\frac{d}{dt} - \beta_1\right)^2 w + Q_k(\xi) w = g$$

which factors as

$$\left(\frac{d}{dt} - (\beta_1 + \sqrt{\epsilon_k Q_k})\right)\left(\frac{d}{dt} - (\beta_1 - \sqrt{\epsilon_k Q_k})\right) w = \epsilon_k g$$

so that we can write a solution formula analogous to that in (2.2) through (2.5); i.e.

$$w = \frac{\epsilon_k}{\sqrt{Q_k}} \left(\int e^{(\beta_1 + \sqrt{\epsilon_k Q_k})(t-s)} g(s,\xi) ds - \int e^{(\beta_1 - \sqrt{\epsilon_k Q_k})(t-s)} g(s,\xi) ds \right)$$

where the limits of integration in the first integral are $-\infty < s < t$ for those ξ that satisfy $\Re(\beta_1 + \sqrt{\epsilon_k Q_k}) > 0$ and $t < s < \infty$ for ξ with $\Re(\beta_1 + \sqrt{\epsilon_k Q_k}) < 0$. The limits in the the second integral are chosen similarly, based on the real part of $\beta_1 - \sqrt{\epsilon_k Q_k}$. We may choose either set of limits if the real part is zero.

We now obtain the estimate (5.13) just as in (5.6) through (5.7). □

Our next step is to show that any compactly supported $f \in L^2$ can be decomposed into a sum of sources, each of which will satisfy (5.12) for some Θ_k. To accomplish this, we let $\phi(t) \in C_0^\infty(\mathbb{R})$ be a positive bump function, equal to 0 for $|t| < 1$ and 1 for $|t| > 2$. We let $\phi_\varepsilon(t) = \phi(\frac{t}{\varepsilon})$. Again writing $\eta \in \mathbb{R}^n$ as

$$\eta = \tau\Theta_k + \xi$$

it is natural to define the multiplier

$$\Phi_k(\eta) = \phi_\varepsilon(|Q_k(\xi)|)$$

which will equal 0 near the set where Q_k is small. It is, however, more convenient to define

$$\Phi_k(\eta) = \phi_\varepsilon(\Re Q_k) + \phi_\varepsilon(\Im Q_k) - \phi_\varepsilon(\Re Q_k)\phi_\varepsilon(\Im Q_k) \tag{5.14}$$

which equals 0 if both $\Re Q_k < \varepsilon$ and $\Im Q_k < \varepsilon$, and equals 1 if either or both is greater than 2ε. We decompose f as

$$\begin{aligned}\widehat{f} &= \widehat{f}_1 + \widehat{f}_2 + \widehat{f}_3 + \ldots \widehat{f_{n+1}} \\ &= \Phi_1 \widehat{f} + \Phi_2(1-\Phi_1)\widehat{f} + \Phi_3(1-\Phi_2)(1-\Phi_1)\widehat{f} + \ldots + \prod_{j=1}^{n}(1-\Phi_j)\widehat{f} \qquad (5.15)\end{aligned}$$

and solve

$$P(D)u_k = f_k \qquad (5.16)$$

which will guarantee that, for all $k = 1 \ldots n$, f_k will satisfy the hypothesis (5.12) of Proposition 5.6 with direction vector Θ_k. We will use that proposition to construct and estimate the u_k. To estimate the solution to $P(D)u_{n+1} = f_{n+1}$ we will need the following:

Lemma 5.7. *Let*

$$Z^{\varepsilon}_{Q_k} = \{\eta \in \mathbb{R}^n \big| \, |\Re Q_k(\xi)| < \varepsilon \text{ and } |\Im Q_k(\xi)| < \varepsilon\} \qquad (5.17)$$

then $\bigcap_{k=1}^{n} Z^{\varepsilon}_{Q_k}$ *is bounded with diameter less than* $4\sqrt{2n\varepsilon}$. *Moreover, if* P *has no double characteristics, and* ε *is chosen small enough,*

$$\bigcap_{k=1}^{n} Z^{\varepsilon}_{Q_k} \cap Z^{\varepsilon}_{P} = \varnothing \qquad (5.18)$$

where Z^{ε}_{P} *is defined similarly as* $Z^{\varepsilon}_{Q_k}$.

Before we begin the proof we record one simple lemma, which we will use here and again in the proof of Proposition 5.13.

Lemma 5.8. *Suppose that* $q(t) = t^2 - B$ *and* $Z^q_\delta = \{t \in \mathbb{R} \big| |q(t)| < \delta\}$, *then*

$$\mu\left(Z^q_\delta\right) \leqslant 4 \min\left(\sqrt{\delta}, \frac{\delta}{\sqrt{B}}\right) \qquad (5.19)$$

Proof. If $B < -\delta$, Z^q_δ is empty, so assume that is not the case and $t \in Z^{\varepsilon}_\delta$. Then

$$\max(0, B-\delta) \leqslant t^2 < B + \delta$$

so $\pm t$ belongs to the interval $\left[\sqrt{\max(0, B-\delta)}, \sqrt{B+\delta}\right]$, which has length

$$\begin{aligned}\sqrt{B+\delta} - \sqrt{\max(0,B-\delta)} &= \frac{2\delta}{\sqrt{B+\delta} + \sqrt{\max(0,B-\delta)}} \\ &\leqslant \frac{2\delta}{\max(\sqrt{B}, \sqrt{\delta})} \qquad (5.20)\end{aligned}$$

∎

Proof of Lemma 5.7. If $\eta \in \bigcap_{k=1}^{n} Z^{\varepsilon}_{Q_k}$, we will show that, each coordinate, η_m belongs to the union of two intervals, with total length at most $4\sqrt{2\varepsilon}$, so that the diameter of the set is no more than $\sqrt{n}$ times $4\sqrt{2\varepsilon}$. For $\eta \in \bigcap_{k=1}^{n} Z^{\varepsilon}_{Q_k}$,

$$\left|\sum_{k\neq m} Q_k\right| \leqslant (n-1)\varepsilon$$

and

$$\begin{aligned}
\sum_{k\neq m} Q_k &= \sum_{k\neq m}\left[\sum_{j\neq k}\epsilon_j(i\eta_j-\beta_j)^2+b\right]\\
&= (n-2)\left[\sum_{j\neq m}\epsilon_j(i\eta_j-\beta_j)^2+b\right]+(n-1)\epsilon_m(i\eta_m-\beta_m)^2+b\\
&= (n-2)Q_m+(n-1)\epsilon_m(i\eta_m-\beta_m)^2+b,
\end{aligned}$$

so

$$|(i\eta_m-\beta_m)^2 \pm b/(n-1)| \leqslant (n-1)\varepsilon+(n-2)\varepsilon < 2\varepsilon.$$

The real part of $(i\eta_j-\beta_j)^2 \pm b/(n-1)$ is $-\eta_m^2+B$ with $B=\beta_m^2 \pm b/(n-1)$, so we may invoke Lemma 5.8 with $\delta = 2\varepsilon$ to conclude that η_m belongs to set with diameter at most $4\sqrt{2\varepsilon}$.

We perform a similar calculation to establish (5.18). The absence of real double characteristics means that either b or some β_j in (5.10) is nonzero. For $\eta \in \bigcap_{k=1}^{n} Z^{\varepsilon}_{Q_k}$,

$$\begin{aligned}
\left|\sum_{k=1}^{n} Q_k\right| &\leqslant n\varepsilon\\
\left|(n-1)\sum_{k=1}^{n}\epsilon_k(i\eta_k-\beta_k)^2+nb\right| &\leqslant n\varepsilon\\
|(n-1)p(\eta)+b| &\leqslant n\varepsilon\\
|p(\eta)| &\geqslant \frac{|b|-n\varepsilon}{n-1}\\
&\geqslant \varepsilon
\end{aligned}$$

as long as $|b|>0$ and ε is chosen sufficiently smaller than $|b|$. If $b=0$, then some $\beta_k \neq 0$ and

$$\begin{aligned} p - Q_k &= \epsilon_k (i\eta_k - \beta_k)^2 \\ |p| &\geqslant |i\eta_k - \beta_k|^2 - |Q_k| \\ |p| &\geqslant \beta_k^2 - \varepsilon \\ &\geqslant \varepsilon \end{aligned}$$

for ε sufficiently smaller than β_k^2. □

Proposition 5.6 gives us the estimates

$$||\mathscr{F}_{\Theta_k^\perp} u_k||_{\Theta_k(\infty,2)} \leqslant \frac{1}{\sqrt{\varepsilon}} ||\mathscr{F}_{\Theta_k^\perp} f_k||_{\Theta_k(1,2)}$$

for $k = 1 \ldots n$. To estimate u_{n+1}, we prove

Proposition 5.9. *Suppose that* $\operatorname{supp} \widehat{f}$ *has diameter at most* d *, and further that* $|P(\eta)| > \varepsilon$ *on* $\operatorname{supp} \widehat{f}$*. Then*

$$u := \mathscr{F}^{-1}\left(\frac{\widehat{f}}{P}\right) \qquad \text{solves} \qquad P(D)u = f$$

and, for any unit vector Θ*,*

$$||\mathscr{F}_{\Theta^\perp} u||_{\Theta(\infty,2)} \leqslant \frac{d}{\varepsilon} ||\mathscr{F}_{\Theta^\perp} f||_{\Theta(1,2)} \tag{5.21}$$

Proof. We write $\eta = \tau\Theta + \xi$, and inverse Fourier transform in the Θ direction, obtaining

$$|\mathscr{F}_{\Theta^\perp} u(t,\xi)| \leqslant \left| \mathscr{F}_\Theta^{-1} \left\{ \frac{\chi_{\operatorname{supp} \hat{f}}}{P} \right\} *_t \mathscr{F}_{\Theta^\perp} f \right| \leqslant \frac{d}{\varepsilon} \int |\mathscr{F}_{\Theta^\perp} f(t',\xi)| dt'$$

so for each fixed ξ,

$$||\mathscr{F}_{\Theta^\perp} u(\cdot,\xi)||_{L^\infty} \leqslant \frac{d}{\varepsilon} ||\mathscr{F}_{\Theta^\perp} f(\cdot,\xi)||_{L^1(dt)}$$

Taking $L^2(d\xi)$ norms of both sides yields

$$||\mathscr{F}_{\Theta^\perp} u||_{\Theta(\infty,2)} \leqslant \frac{d}{\varepsilon} ||\mathscr{F}_{\Theta^\perp} f||_{\Theta(1,2)}$$

□

To complete the proof of Theorem 5.1, we need only show that, for $k = 1 \ldots n+1$,

$$||\mathscr{F}_{\Theta_k^\perp} f_k||_{\Theta_k(1,2)} \leqslant C||\mathscr{F}_{\Theta_k^\perp} f||_{\Theta_k(1,2)} \tag{5.22}$$

We will then apply Lemma 2.1 to conclude that each u_k satisfies

$$||u_k||_{L^2(D_r)} \leqslant \sqrt{d_r}||\mathscr{F}_{\Theta_k^\perp} u_k||_{\Theta(\infty,2)}$$

and

$$||\mathscr{F}_{\Theta_k^\perp} f||_{\Theta_k(1,2)} \leqslant C\sqrt{d_s}||f||_{L^2(D_s)}$$

Recalling that $u = \sum u_k$ in (5.16) will then finish the proof of Theorem 5.1. *Note that we can't apply Lemma 2.1 directly to the f_k because their supports need not be contained in the support of f.*

In order to establish (5.22) for f_k defined as in (5.15), we need to estimate $||\mathscr{F}_{\Theta_k^\perp} M_{\Psi_j} f||_{\Theta_k(1,2)}$ for all j and k. The case $j = k$ is the simplest.

Lemma 5.10. *Let Ψ_k denote either $\phi_\varepsilon(\Re Q_k)$ or $\phi_\varepsilon(\Im Q_k)$. Then,*

$$||\mathscr{F}_{\Theta_k^\perp} M_{\Psi_k} f||_{\Theta_k(1,2)} \leqslant ||\mathscr{F}_{\Theta_k^\perp} f||_{\Theta_k(1,2)} \tag{5.23}$$

Proof. Recall that , writing $\eta = \tau\Theta_k + \xi$ with $\xi \in \Theta_k^\perp$,

$$Q_k(\eta) = Q_k(\tau\Theta_k + \xi) = Q_k(\xi)$$

so that Q_k, and therefore Ψ_k does not depend on τ. Hence

$$\begin{array}{rcl} \mathscr{F}_{\Theta^\perp} M_{\Psi_k} f &=& \Psi_k(\xi)\mathscr{F}_{\Theta^\perp} f(t,\xi) \\ ||\mathscr{F}_{\Theta^\perp} M_{\Psi_k} f||_{\Theta_k}(1,2) &\leqslant& ||\Psi_k(\xi)||_{L^\infty}\, ||\mathscr{F}_{\Theta^\perp} f||_{\Theta_k}(1,2) \end{array}$$

and (5.23) now follows on noting that $|\phi_\varepsilon| \leqslant 1$. $\square$

According to Lemma 4.2, we may establish (5.22) for $j \neq k$ by proving that, $||\mathscr{F}_{\Theta_k}^{-1}\Psi_j||_{\Theta_k(1,\infty)}$ is bounded. We address this in the next few lemmas.

Lemma 5.11. *Let $q(t)$ be a real valued function of $t \in \mathbb{R}$, and let*

$$\begin{array}{rcl} \Phi(t) &=& \phi_\varepsilon(q(t)) \\ Z_q^\varepsilon &=& \{t \in \mathbb{R} \big| \; |q(t)| < \varepsilon\} \end{array} \tag{5.24}$$

Suppose that

$$\mu(Z_q^\varepsilon) \leqslant \mu_1 \qquad \sup_{t\in Z_q^\varepsilon}\left|\frac{dq}{dt}\right| \leqslant M_1 \qquad \sup_{t\in Z_q^\varepsilon}\left|\frac{d^2q}{dt^2}\right| \leqslant M_2 \tag{5.25}$$

then

$$||\check{\Phi}||_{L^1} \leqslant 2\mu_1 \left[\frac{M_1}{\varepsilon} + \sqrt{\frac{M_2}{\varepsilon}} \right] \tag{5.26}$$

where $\check{\Phi}$ *denotes the (one dimensional) inverse Fourier transform of* Φ.

Proof.

$$\left|\check{\Phi}(\tau)\right| = \left| \frac{1}{\sqrt{2\pi}} \int e^{-it\tau} \phi_\varepsilon(q(t))dt \right| \leqslant \mu(Z_q^\varepsilon) \leqslant \mu_1$$

Two integrations by parts yield

$$\left|\check{\Phi}(\tau)\right| = \left| \frac{-1}{\tau^2\sqrt{2\pi}} \int e^{it\tau} \left[\phi'_\varepsilon\, q'' + \phi''_\varepsilon\, (q')^2\right] dt \right| \leqslant \frac{1}{\tau^2} \left[\frac{M_2}{\varepsilon} + \left(\frac{M_1}{\varepsilon}\right)^2 \right]$$

so that

$$\left|\check{\Phi}(\tau)\right| \leqslant \mu_1 \begin{cases} 1, & \tau \leqslant \left[\frac{M_2}{\varepsilon} + \left(\frac{M_1}{\varepsilon}\right)^2\right]^{\frac{1}{2}} \\ \frac{\left[\frac{M_2}{\varepsilon} + \left(\frac{M_1}{\varepsilon}\right)^2\right]}{\tau^2}, & \tau \geqslant \left[\frac{M_2}{\varepsilon} + \left(\frac{M_1}{\varepsilon}\right)^2\right]^{\frac{1}{2}} \end{cases}$$

which implies that

$$\int \left|\check{\Phi}(\tau)\right| d\tau \leqslant 2\mu_1 \left[\frac{M_2}{\varepsilon} + \left(\frac{M_1}{\varepsilon}\right)^2 \right]^{\frac{1}{2}} \leqslant 2\mu_1 \left[\left(\frac{M_2}{\varepsilon}\right)^{\frac{1}{2}} + \frac{M_1}{\varepsilon} \right]$$

□

An immediate corollary is:

Corollary 5.12. *Let* $Q(t, \xi)$ *be a real valued function of* $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, *and let*

$$\begin{aligned} \Phi(t\Theta + \xi) &= \phi_\varepsilon(Q(t, \xi)) \\ Z_Q^\varepsilon(\xi) &= \{t \in \mathbb{R} \big|\ |Q(t, \xi)| < \varepsilon\} \end{aligned} \tag{5.27}$$

Suppose that

$$\mu(Z_Q^\varepsilon) \leqslant \mu_1(\xi) \qquad \sup_{t \in Z_Q^\varepsilon} \left|\frac{dQ}{dt}\right| \leqslant M_1(\xi) \qquad \sup_{t \in Z_Q^\varepsilon} \left|\frac{d^2Q}{dt^2}\right| \leqslant M_2(\xi) \tag{5.28}$$

then

$$||\mathscr{F}_\Theta^{-1}\Phi||_{\Theta(1,\infty)} \leqslant \sup_\xi \mu_1(\xi) \left[\frac{M_1(\xi)}{\varepsilon} + \sqrt{\frac{M_2(\xi)}{\varepsilon}} \right] \tag{5.29}$$

Finally, we specialize to $\Theta = \Theta_j$ and estimate the quantities on the right hand side of (5.29).

Lemma 5.13. *Let $Q(t,\xi) = \Re Q_k(t\Theta_j + \xi)$ or $Q(t,\xi) = \Im Q_k(t\Theta_j + \xi)$, with*

$$\mu(Z_Q^\varepsilon) =: \mu_1(\xi) \qquad \sup_{t \in Z_Q^\varepsilon} \left| \frac{dQ}{dt} \right| =: M_1(\xi) \qquad \sup_{t \in Z_Q^\varepsilon} \left| \frac{d^2Q}{dt^2} \right| =: M_2(\xi) \quad (5.30)$$

then

$$\sup_\xi \mu_1(\xi) \left[\frac{M_1(\xi)}{\varepsilon} + \sqrt{\frac{M_2(\xi)}{\varepsilon}} \right] \leqslant 9\sqrt{2} \qquad (5.31)$$

Proof. We write $\eta = \sigma\Theta_k + t\Theta_j + \xi$ where ξ is orthogonal to both Θ_k and Θ_j. Recall from (5.11) that Q_k does not depend on σ, and assume for convenience that $\epsilon_j = +1$; First let

$$Q = \Re Q_k(t,\xi,\sigma) = t^2 - C(\xi)$$

where

$$C(\xi) = \sum_{l \neq j,k} \epsilon_l \left(\xi_l^2 - \beta_l^2 \right) - b + \beta_j^2$$

so that we may conclude from Lemma 5.8 that

$$\mu_1(\xi) \leqslant 4 \min\left(\sqrt{\varepsilon}, \frac{\varepsilon}{\sqrt{|C(\xi)|}} \right)$$

$$\left| \frac{dQ}{dt} \right| = 2|t| \leqslant 2\sqrt{C(\xi) + \varepsilon}$$

and

$$\left| \frac{d^2Q}{dt^2} \right| = 2$$

so that

$$\mu_1 \frac{M_1}{\varepsilon} \leqslant 8 \min\left(\sqrt{1 + \frac{C(\xi)}{\varepsilon}}, \sqrt{1 + \frac{\varepsilon}{C(\xi)}} \right) \leqslant 8\sqrt{2}$$

and

$$\mu_1 \sqrt{\frac{M_2}{\varepsilon}} \leqslant \sqrt{2}$$

We next treat the case $Q = \Im Q_k = 2\beta_j t + 2\sum_{l \neq j,k} \epsilon_l \beta_l \xi_l$. In this case

$$\mu_1 = \frac{\varepsilon}{2|\beta_j|}$$

$$\frac{dQ}{dt} = 2\beta_j$$

and

$$\frac{d^2Q}{dt^2} = 0$$

so that

$$\mu_1 \frac{M_1}{\varepsilon} = 1$$

and

$$\mu_1 \sqrt{\frac{M_2}{\varepsilon}} = 0$$

$\square$

The combination of Lemma 5.11, Corollary 5.12, and Lemma 5.13 gives us the hypothesis necessary to invoke Lemma 4.2 and conclude that

Corollary 5.14. *For $j \neq k$, $||\mathscr{F}^{-1}_{\Theta_k} \Psi_j||_{\Theta_k(1,\infty)} \leqslant 18$*

and consequently that (5.22) holds with $C = 19^{n+1}$ – because we use products (with $n+1$ factors) of these multipliers and identity minus these multipliers for our cutoffs.

We can now finish the

Proof of Theorem 5.1. We have shown that multiplication by $\phi_\varepsilon(\Re Q_k)$ and $\phi_\varepsilon(\Im Q_k)$ preserve bounds on $||\mathscr{F}_{\Theta_k^\perp} f||_{\Theta_k(1,2)}$. Hence let us start with

$$\begin{aligned} u &= \sum_k u_k \\ ||u||_{L^2(D_r)} &\leqslant \sum_k ||u_k||_{L^2(D_r)} \\ &\leqslant \sum_k ||u_k||_{L^2(S_1)} \end{aligned}$$

where S_1 is a strip bounded by the two planes $\Theta_k \cdot x = s_1$ and $\Theta_k \cdot x = s_2$, with $|s_2 - s_1| \leqslant d_r$

$$
\begin{aligned}
&= \sum_k ||\mathscr{F}_{\Theta_k^\perp} u_k||_{\Theta_k(2,2)(S_1)} \\
&\leqslant \sum_k d_r^{\frac{1}{2}} ||\mathscr{F}_{\Theta_k^\perp} u_k||_{\Theta_k(\infty,2)(S_1)} \\
&\leqslant C_1(P,n)\ d_r^{\frac{1}{2}} \sum_k ||\mathscr{F}_{\Theta_k^\perp} f_k||_{\Theta_k(1,2)} \\
&\leqslant C_2(P,n)\ d_r^{\frac{1}{2}} \sum_k ||\mathscr{F}_{\Theta_k^\perp} f||_{\Theta_k(1,2)} \\
&\leqslant C_3(P,n)\ d_r^{\frac{1}{2}} \sum_k ||\chi_{S_2} \mathscr{F}_{\Theta_k^\perp} f||_{\Theta_k(1,2)}
\end{aligned}
$$

where the C_i are constants depending only on P and the dimension n, and S_2 is a strip containing D_s defined analogously to S_1.

$$
\begin{aligned}
&\leqslant C_3(P,n)\ d_r^{\frac{1}{2}} \sum_k d_s^{\frac{1}{2}} ||\chi_{S_2} \mathscr{F}_{\Theta_k^\perp} f||_{\Theta_k(2,2)} \\
&\leqslant C_4(P,n)\ (d_r d_s)^{\frac{1}{2}} ||f||_{L^2}
\end{aligned}
$$

and Theorem 5.1 is proved. $\square$

6 Estimates for higher order operators

In this section we will consider an N^{th} order constant coefficient partial differential operator $P(D)$ on $\mathbb{R}^n$, $D = -i\nabla$. We refer to polynomials which satisfy the three conditions of Definition 6.10 as *admissible*. For these admissible polynomials, we will prove the same estimate as we did for second order operators in Theorem 5.1. We will again use partial Fourier transforms and solve ordinary differential equations, using partitions of unity to decompose our source into a sum of sources, each of which has support suited to that particular direction, so that the solution to the ODE satisfies the same estimates as in the previous section.

The main difference here is that we don't have a simple normal form as we did in Lemma 5.2, so we cannot explicitly choose directions and construct cutoffs. We need to rely on algebraic properties of the discriminant to guarantee that we can find a finite decomposition of the source analogous to the one we used in (2.8). Additionally, the order of the ODE can depend on the direction. In the second order case we dismissed these cases easily in propositions 5.3 and 5.4 because we could represent them explicitly. In the higher order case, we choose our directions to avoid these cases.

Theorem 6.1. *Let $P : \mathbb{C}^n \to \mathbb{C}$ be a degree $N \geqslant 1$ admissible polynomial. Then there is a constant $C = C(P, n)$ such that for every bounded domain $D_s \subset \mathbb{R}^n$ and every $f \in L^2(D_s)$ there is a $u \in L^2_{loc}(\mathbb{R}^n)$ satisfying*

$$P(D)u = f \tag{6.1}$$

Moreover for any bounded domain $D_r \subset \mathbb{R}^n$

$$\|u\|_{L^2(D_r)} \leqslant C\sqrt{d_r d_s}\,\|f\|_{L^2(D_s)} \tag{6.2}$$

where d_ℓ is the diameter of D_ℓ.

We will prove Theorem 6.1 by reducing the solution of the equation (6.1) to solving a set of parameterized ODE's, just as we wrote the solution to (2.1) in terms of solutions to (2.3). To accomplish this, we must choose a set of directions Θ_k and build a partition of unity on the Fourier side so that the denominators of the source terms in each of these model problems are strictly positive, just as was explained after (2.6). These two ingredients will then imply the final estimate.

We choose a direction $\Theta \in \mathbb{S}^{n-1}(\mathbb{R}^n)$ and Fourier transform (6.1) along the $\Theta^\perp$ hyperplane to obtain the *ordinary differential equation*

$$P(-i(\Theta \cdot \nabla)\Theta + \xi_{\Theta^\perp})\mathscr{F}_{\Theta^\perp} u = \mathscr{F}_{\Theta^\perp} f$$

in the direction Θ, which we solve for each $\xi_{\Theta^\perp}$. The next lemma gives the estimate we seek in the case that ODE is first order.

Lemma 6.2. *Let $\Theta \in \mathbb{S}^{n-1}$ and let $q : \Theta^\perp \to \mathbb{C}$ be measurable. Assume that $g \in \Theta(1,2)$, or that $g \in \Theta(\infty,2)$ and $\inf_{\Theta^\perp} |\Im q| > 0$. Then there is $w \in \Theta(\infty,2)$ satisfying $(-i\partial_t - q(\xi_{\Theta^\perp}))w = g$ and*

$$\|w\|_{\Theta(\infty,2)} \leqslant \|g\|_{\Theta(1,2)}\,, \tag{6.3}$$

or in the second case

$$\|w\|_{\Theta(\infty,2)} \leqslant \frac{1}{\inf_{\Theta^\perp} |\Im q|}\,\|g\|_{\Theta(\infty,2)} \tag{6.4}$$

Remark 6.3. It is also true that $\|w\|_{\Theta(\infty,p)} \leqslant \|g\|_{\Theta(1,p)}$ and $\|w\|_{\Theta(\infty,p)} \leqslant \|g\|_{\Theta(\infty,p)} / (\inf |\Im q|)$ with $1 \leqslant p \leqslant \infty$.

Proof. The general solution to $(-i\partial_t - q(\xi_{\Theta^\perp}))w = g$ is

$$w(t\Theta + \xi_{\Theta^\perp}) = i \int_{t_0}^{t} e^{iq(\xi_{\Theta^\perp})(t-t')} g(t'\Theta + \xi_{\Theta^\perp}) dt', \quad t_0 \in \mathbb{R} \cup \{\pm\infty\}.$$

If $\Im q(\xi_{\Theta^\perp}) = 0$ we may set t_0 as we please and the claim follows. If $\Im q < 0$ set $t_0 = \infty$, and then $|\exp(iq(\xi_{\Theta^\perp})(t - t'))| \leqslant 1$ for $t' \in [t, t_0]$. Similarly, if $\Im q > 0$ set $t_0 = -\infty$.

Now (6.3) follows by estimating the integral on the right by the L^∞ norm of the exponential times the L^1 norm of g for each fixed $\xi_{\Theta^\perp}$, and then taking L^2 norms in the $\Theta^\perp$ hyperplane. The inequality (6.4) follows in the same way, but using the L^1 norm of the exponential times the L^∞ norm of g instead of the other way around.

□

In general the differential equation $P(-i(\Theta \cdot \nabla)\Theta + \xi_{\Theta^\perp})\mathscr{F}_{\Theta^\perp} u = \mathscr{F}_{\Theta^\perp} f$ is not first order in $\partial_t = \Theta \cdot \nabla$, but we can factor it into a product of first order operators of the form $(-i\partial_t - q(\xi_{\Theta^\perp}))$, and then use a partial fractions expansion to express its solution as a sum of solutions to first order ODE's.

Definition 6.4. For $\xi_{\Theta^\perp}$ fixed, let $p : \mathbb{C} \to \mathbb{C}$ be the polynomial in τ

$$p(\tau) = P(\tau\Theta + \xi_{\Theta^\perp}).$$

Then $p'(\tau) = \Theta \cdot \nabla P(\tau\Theta + \xi_{\Theta^\perp})$.

Lemma 6.5. *Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $N \geqslant 1$. Assume that its roots τ_j are simple and that its leading coefficient is p_N. Then*

$$\frac{1}{p(\tau)} = \sum_{j=1}^{N} \frac{1}{(\tau - \tau_j) p_N \prod_{k \neq j} (\tau_j - \tau_k)} = \sum_{j=1}^{N} \frac{1}{(\tau - \tau_j) p'(\tau_j)}. \tag{6.5}$$

Proof. $p'(\tau_j) = \lim_{\tau \to \tau_j} p(\tau)/(\tau - \tau_j)$ since $p(\tau_j) = 0$. □

If, for some direction Θ_k, $|p'(\tau_j(\xi_{\Theta_k^\perp}))| > \varepsilon > 0$ for all $\xi_{\Theta_k^\perp}$ and for all j in the support of $\mathscr{F}_{\Theta_k^\perp} f_k$, then we can define u_{kj} as solutions to

$$\big(- i\partial_t - \tau_j(\xi_{\Theta_k^\perp})\big) \mathscr{F}_{\Theta_k^\perp} u_{kj} = \mathscr{F}_{\Theta_k^\perp} f_k / p'(\tau_j)$$

that satisfy (6.3) with $w = \mathscr{F}_{\Theta_k^\perp} u_{kj}$ and $g = \mathscr{F}_{\Theta_k^\perp} f_k / p'(\tau_j)$. Then $u_k = \sum_j u_{kj}$ will solve $P(D)u_k = f_k$.

We must find a finite set of directions Θ_k, and split $f = \sum_k f_k$ such that $\mathscr{F} f_k(\xi) = 0$ whenever $\xi_{\Theta^\perp}$ is such that $|p'(\tau_j)| \leqslant \varepsilon$ for any j, as was done in (5.15). Thus we need to define the sets where $p'(\tau_j)$ becomes small. In reading the definition below, recall that $\xi_{\Theta^\perp}$ is the component of ξ perpendicular to Θ and $\tau_j = \tau_j(\xi_{\Theta^\perp})$, $j = 1, \ldots, N$ are the roots of $p(\tau)$.

Definition 6.6. Given $\Theta \in \mathbb{S}^{n-1}$ and $\varepsilon \geqslant 0$ let

$$\begin{aligned} \mathscr{B}_{\Theta,\varepsilon} &= \{\xi \in \mathbb{R}^n \mid \min_{P(\tau_j\Theta + \xi_{\Theta^\perp})=0} |\Theta \cdot \nabla P(\tau_j \Theta + \xi_{\Theta^\perp})| \leqslant \varepsilon\} \\ &= \{\xi \in \mathbb{R}^n \mid \min_{p(\tau_j)=0} |p'(\tau_j)| \leqslant \varepsilon\} \end{aligned} \tag{6.6}$$

where the minimum is taken with respect to $\tau_j \in \mathbb{C}$. If, for some Θ, $\deg p = 0$ we adopt the convention that $\min_{p(\tau_j)=0} |p'(\tau_j)| := 0$.

Proposition 6.7. *Let $P : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $N \geqslant 1$ with principal term P_N. Assume that $P_N(\Theta) \neq 0$. Let $\mathscr{F}_{\Theta^\perp} f \in \Theta(1,2)$ be such that $\mathscr{F} f(\xi) = 0$ for all $\xi \in \mathscr{B}_{\Theta,\varepsilon}$. Then there exists u solving $P(D)u = f$ and*

$$\|\mathscr{F}_{\Theta^\perp} u\|_{\Theta(\infty,2)} \leqslant \frac{N}{\varepsilon} \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)} . \tag{6.7}$$

Remark 6.8. The mixed norm estimate is also true for any $1 \leqslant p \leqslant \infty$: $\|\mathscr{F}_{\Theta^\perp} u\|_{\Theta(\infty,p)} \leqslant N\varepsilon^{-1} \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,p)}$.

Proof. The roots of p are simple when $\xi_{\Theta^\perp} \in \mathbb{R}^n \backslash \mathscr{B}_{\Theta,\varepsilon}$ so we have

$$\frac{1}{p(\tau)} = \sum_{p(\tau_j)=0} \frac{1}{(\tau - \tau_j)p'(\tau_j)}$$

there according to Lemma 6.5. For each $\xi_{\Theta^\perp}$, order the roots τ_j lexicographically by $j = 1, \ldots, N$, i.e. $\Re\tau_j \leqslant \Re\tau_{j+1}$ and $\Im\tau_j < \Im\tau_{j+1}$ if the real parts are equal. The maps $\xi_{\Theta^\perp} \mapsto \tau_j(\xi_{\Theta^\perp})$ are measurable since the coefficients of $p(\tau)$ are polynomials in the $\xi_{\Theta^\perp}$.

The assumption on $\mathscr{F} f$ implies that

$$\|\mathscr{F}_{\Theta^\perp} f / p'(\tau_j)\|_{\Theta(1,2)} \leqslant \varepsilon^{-1} \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)} < \infty.$$

for any root $\tau_j = \tau_j(\xi_{\Theta^\perp})$ of $p(\tau_j) = 0$. Let $u_j \in \Theta(\infty, 2)$ be the solution to

$$\big(-i\partial_t - \tau_j(\xi_{\Theta^\perp})\big)\mathscr{F}_{\Theta^\perp} u_j = \frac{\mathscr{F}_{\Theta^\perp} f}{p'(\tau_j)}$$

given by Lemma 6.2. It satisfies the norm estimate

$$\|\mathscr{F}_{\Theta^\perp} u_j\|_{\Theta(\infty,2)} \leqslant \varepsilon^{-1} \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)} .$$

The claim follows by setting $u = \sum_{j=1}^N u_j$ and recalling the partial fraction decomposition of Lemma 6.5. $\square$

We now focus on the second task, splitting an arbitrary source function f into a sum $f = f_1 + f_2 + \ldots + f_m$ with directions $\Theta = \Theta_1, \Theta_2, \ldots, \Theta_m$ such that $\mathscr{F} f_k(\xi) = 0$ when $\xi \in \mathscr{B}_{\Theta_k,\varepsilon}$. Proposition 6.7 would then imply the existence of a solution to $P(D)u_k = f_k$. Linearity then implies that $u = u_1 + u_2 + \ldots + u_m$ solves the original problem $P(D)u = f$.

The partial fraction expansion in Lemma 6.5 cannot hold if $P(D)$ has a double characteristic, even a complex double characteristic. Unlike in Theorem 5.1, the algebraic techniques we use here rely on properties of the discriminant which involves the multiplicities of all the roots, including the the complex ones. Hence we require that P be what algebraic geometers call a *nonsingular* polynomial.

Definition 6.9. A polynomial $P : \mathbb{C}^n \to \mathbb{C}$ is *nonsingular* if, given $\xi \in \mathbb{C}^n$, $P(\xi) = 0$ implies that $|\nabla P(\xi)| \neq 0$.

The sets $\mathscr{B}_{\Theta,\varepsilon}$ are difficult to deal with for a general polynomial P, but the sets $\mathscr{B}_{\Theta,0}$ are algebraic sets, and this will enable us to prove that the intersection of finitely many of them is empty. In order to conclude that the intersections of the $\mathscr{B}_{\Theta,\varepsilon}$ are empty, we will assume that each $\mathscr{B}_{\Theta,\varepsilon}$ is contained in a tubular neighborhood of $\mathscr{B}_{\Theta,0}$.

Additionally, we require a compactness hypothesis on projections of two of the sets $\mathscr{B}_{\Theta,0}$ to insure that the cut-off function Ψ associated with $\mathscr{B}_{\Theta,\varepsilon}$ is a Fourier multiplier as in Lemma 4.2.

Definition 6.10. Let $P : \mathbb{C}^n \to \mathbb{C}$ be a degree $N \geqslant 1$ nonsingular polynomial with principal term P_N. It is *admissible* if

1. for any $\Theta \in \mathbb{S}^{n-1}(\mathbb{R}^n) \backslash P_N^{-1}(0)$ and $r_0 > 0$ there is $\varepsilon > 0$ such that
$$\mathscr{B}_{\Theta,\varepsilon} \subset \overline{B}(\mathscr{B}_{\Theta,0}, r_0),$$
2. there are non-parallel vectors $\Theta_1, \Theta_2 \in \mathbb{S}^{n-1}(\mathbb{R}^n) \backslash P_N^{-1}(0)$ such that $\mathscr{B}_{\Theta_1,0} \cap (\Theta_1)^\perp$ and $\mathscr{B}_{\Theta_2,0} \cap (\Theta_2)^\perp$ are compact,

where $\mathscr{B}_{\Theta,\varepsilon}$ is defined in (6.6).

We suspect that Condition 1 is true for any nonsingular polynomial. It has been straightforward to verify in the examples we have considered. Another way to state this condition is as follows: let $\mathscr{D}$ be the set of $\xi \in \Theta^\perp$ where $p(\tau)$, whose coefficients are polynomials of ξ, has a double root, i.e. $p(\tau_0) = p'(\tau_0) = 0$. Let $r_0 > 0$. Then we require that there is $\varepsilon > 0$ such that if $\xi \in \Theta^\perp$, $d(\xi, \mathscr{D}) > r_0$, then $|p'(\tau_0)| > \varepsilon$ for all roots $\tau = \tau_0$ of $p(\tau) = 0$.

Condition 2 is likely only a technical requirement. Requiring it could be avoided if a theorem similar to Corollary 5.12 and Lemma 5.13 could be proven for higher order operators. Moreover this condition is always satisfied in $\mathbb{R}^2$ because each $\mathscr{B}_{\Theta,0}$ is a finite set of lines in the direction Θ in this case.

A key point in our proof is the observation that $\mathscr{B}_{\Theta,0}$ is an algebraic variety which can be defined by the vanishing of a certain discriminant. We first show that there is an infinite sequence of directions Θ_k such that the intersection $\cap_k \mathscr{B}_{\Theta_k,0}$ is empty. Because the $\mathscr{B}_{\Theta_k,0}$ are algebraic varieties, Hilbert's basis theorem then guarantees that the intersection of a finite subset of the $\mathscr{B}_{\Theta_k,0}$ is empty.

Definition 6.11. Let $P : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $N \geqslant 1$. Write P_N for its principal term. For any $\xi \in \mathbb{C}^n$ and $\Theta \in \mathbb{C}^n$ such that $P_N(\Theta) \neq 0$ we define

$$\Delta(\Theta, \xi) = \mathrm{disc}_\tau(P(\tau\Theta + \xi)) := \big(P_N(\Theta)\big)^{2(N-1)} \prod_{i<j} (\tau_i - \tau_j)^2, \tag{6.8}$$

where $\{\tau_j(\Theta, \xi) \mid j = 1, \ldots, N\}$ are the roots of $P(\tau\Theta + \xi) = 0$. If $N = 1$ we set $\mathrm{disc}_\tau(a_1\tau + a_0) = a_1$.

Remark 6.12. The discriminant of a polynomial P is a polynomial in the coefficients of P. Hence we can extend Δ to the set $\mathbb{C}^n \times \mathbb{C}^n$ by analytic continuation, and therefore it is well-defined without the assumption that $P_N(\Theta) \neq 0$. We point out, however, that the discriminant of a degree N polynomial, with the high-order coefficients equal to zero, is not the same as the discriminant of the resulting lower degree polynomial. See for example the introduction of Gel'fand, Kapranov and Zelevinsky [7].

Remark 6.13. We have $\Delta(\Theta, \xi) = \Delta(\Theta, \xi + r\Theta)$ for any $r \in \mathbb{C}$. This follows from the fact that the roots of $P(\tau\Theta + \xi + r\Theta) = P((\tau + r)\Theta + \xi)$ are just the roots of $P(\tau\Theta + \xi)$, all translated by r, so the discriminant remains the same.

Remark 6.14. We have $\Delta(\lambda\Theta, \xi) = \lambda^{N(N-1)}\Delta(\Theta, \xi)$ because $P(\tau\lambda\Theta + \xi)$ has roots $\tau_j = r_j/\lambda$ where $P(r_j\Theta + \xi) = 0$, and the principal term will be $\big(\lambda^N P_N(\Theta)\big)\tau^N$.

Definition 6.15. Let $\Theta \in \mathbb{C}^n$ and $P : \mathbb{C}^n \to \mathbb{C}$ be a degree $N \geqslant 1$ polynomial. Then the *algebraic* tangent set (in the direction Θ) is defined as

$$\overline{\mathscr{D}_\Theta} = \{\xi \in \mathbb{C}^n \mid \Delta(\Theta, \xi) = 0\}. \tag{6.9}$$

The *real* tangent set is $\mathscr{D}_\Theta = \overline{\mathscr{D}_\Theta} \cap \mathbb{R}^n$.

Figure 1 on page 6 illustrates the example $P(\xi) = |\xi|^2 - 1$ with $\Theta \in \{e_1, e_2, e_3\}$. We have then

$$P(\tau\Theta + \xi) = (\Theta \cdot \Theta)\tau^2 + 2(\Theta \cdot \xi)\tau + \xi \cdot \xi - 1$$

and

$$\Delta(\Theta, \xi) = (\Theta \cdot \xi)^2 - (\Theta \cdot \Theta)(\xi \cdot \xi - 1).$$

Homogeneity is easy to see in this example and a simple calculation demonstrates that $\Delta(\Theta, \xi + r\Theta) = \Delta(\Theta, \xi)$ for all $r \in \mathbb{C}$, as expected.

We can study the sets $\mathscr{D}_\Theta$ as a proxy for the sets $\mathscr{B}_{\Theta,\varepsilon}$, defined in (6.6), that are actually used.

Lemma 6.16. *Let $P : \mathbb{C}^n \to \mathbb{C}$ be a degree $N \geqslant 1$ polynomial and $\Theta \in \mathbb{S}^{n-1}(\mathbb{R}^n)$ such that $P_N(\Theta) \neq 0$. Then*

$$\mathscr{D}_\Theta = \{\xi \in \mathbb{R}^n \mid \exists \tau_0 \in \mathbb{C} : p(\tau_0) = p'(\tau_0) = 0\} = \mathscr{B}_{\Theta,0}.$$

Proof. If $N \geqslant 2$ this follows from the definition of $\mathscr{B}_{\Theta,0}$ in (6.6) and the fact that τ_0 is a double root of p if and only if $p(\tau_0) = 0$ and $p'(\tau_0) = 0$. If $N = 1$ then $\xi \in \mathscr{D}_\Theta$ iff the first order coefficient of p vanishes, which is the same condition as $\xi \in \mathscr{B}_{\Theta,0}$. This is impossible since $P_N(\Theta) \neq 0$. $\square$

We will show that if P is nonsingular then the intersection $\cap_{\Theta \in \mathbb{S}^{n-1}} \mathscr{D}_\Theta$ is empty. In other words, we show that, given any $\xi \in \mathbb{R}^n$, there is some direction Θ such that the line $\tau \mapsto \tau\Theta + \xi$ is not tangent to the characteristic manifold $P^{-1}(0)$ at any point.

Lemma 6.17. *Assume that $P : \mathbb{C}^n \to \mathbb{C}$ is nonsingular. Let $\xi \in \mathbb{C}^n$. Then there is $\Theta \in \mathbb{S}^{n-1}(\mathbb{R}^n)$ such that $P_N(\Theta) \neq 0$ and $\Delta(\Theta, \xi) \neq 0$.*

Proof. We keep the second variable ξ fixed in this proof, and suppress the dependence on ξ, writing $\Delta(\Theta) = \Delta(\Theta, \xi)$. We view $P(\tau\Theta + \xi)$ as a polynomial $p(\tau, \Theta)$ in τ and Θ.

According to [8], Appendix 1.2., Δ is a polynomial in $\Theta \in \mathbb{C}$ and $\Delta \not\equiv 0$ if $p(\tau, \Theta)$ is square-free. A nontrivial complex polynomial cannot vanish identically on $\mathbb{R}^n$, and thus neither on $\mathbb{R}^n \backslash P_N^{-1}(0)$.

Hence, if $p(\tau, \Theta)$ has no square factor, there is a $\Theta \in \mathbb{R}^n$ such that $P_N(\Theta) \neq 0$ and $\Delta(\Theta) \neq 0$. Because Δ, as pointed out in Remark 6.14, is a homogeneous function of Θ, we may scale Θ so it has unit length, and the lemma follows in this case.

Next, we show that if $p(\tau, \Theta)$ has a square factor, then $P(z)$, viewed as a polynomial of $z \in \mathbb{C}^n$ has a square factor, which contradicts the assumption that P is nonsingular. Suppose that $p(\tau, \Theta) = \big(S_1(\tau, \Theta)\big)^2 S_2(\tau, \Theta)$. If we choose $\tau = \lambda$ and $\Theta = (z - \xi)/\lambda$, then, for any $z \in \mathbb{C}$,

$$\begin{aligned} P(z) &= P\left(\lambda \frac{z-\xi}{\lambda} + \xi\right) \\ &= p(\lambda, (z-\xi)/\lambda) \\ &= \big(S_1(\lambda, (z-\xi)/\lambda)\big)^2 S_2(\lambda, (z-\xi)/\lambda) \end{aligned}$$

so that, unless $S_1(\lambda, \Theta)$ is independent of Θ, $P(z)$ must have a square factor, which is a contradiction. Suppose now that S_1 is independent of Θ. It is a non-constant polynomial, so there is $\tau_0 \in \mathbb{C}$ such that $S_1(\tau_0) = 0$. If $\tau_0 \neq 0$, choosing $\lambda = \tau_0$ implies that $P \equiv 0$. If $\tau_0 = 0$, then $P(\tau\Theta + \xi)$ vanishes to at least second order at the point ξ in every direction $\Theta \in \mathbb{C}^n$. This means that ξ is a singular point of P, again contradicting the hypothesis that P is nonsingular. Hence $p(\tau, \Theta)$ has no square factors and thus Δ is not identically zero. □

Proposition 6.18. *Let $P : \mathbb{C}^n \to \mathbb{C}$ be a nonsingular polynomial of degree $N \geqslant 1$ with principal term P_N. Then there is a finite set of directions $\Theta_1, \ldots, \Theta_m \in \mathbb{S}^{n-1}(\mathbb{R}^n) \backslash P_N^{-1}(0)$ such that*

$$\bigcap_{k=1}^{m} \overline{\mathscr{D}_{\Theta_k}} = \varnothing.$$

Proof. We recall a few facts from algebra. A ring R is *Noetherian* if every ideal is finitely generated. Another characterization is that every increasing sequence of ideals stabilizes at a finite index. In other words, if $I_1 \subset I_2 \subset \ldots$ are ideals in R, then there is $m < \infty$ such that $I_\ell = I_m$ for all $\ell \geqslant m$.

The ring of complex numbers is Noetherian: its only ideals are $\{0\}$ and $\mathbb{C}$. Hilbert's basis theorem says that polynomial rings over Noetherian rings are also Noetherian. If $V \subset \mathbb{C}^n$ is an affine variety then $V = \mathbb{V}(\mathbb{I}(V))$, where

$$\begin{aligned} &\mathbb{V}(I) = \{\xi \in \mathbb{C}^n \mid f(\xi) = 0 \quad \forall f \in I\}, \\ &\mathbb{I}(V) = \{f \in \mathbb{C}[\xi_1, \ldots, \xi_n] \mid f(\xi) = 0 \quad \forall \xi \in V\}. \end{aligned}$$

Now we begin the proof. Let $\Theta_1, \Theta_2, \ldots \in \mathbb{S}^{n-1}(\mathbb{R}^n) \backslash P_N^{-1}(0)$ be a sequence that's dense in the surface measure inherited from the Lebesgue measure of $\mathbb{R}^n$. Set

$$V_\ell := \{\xi \in \mathbb{C}^n \mid \Delta(\Theta, \xi) = 0, \text{ for } \Theta = \Theta_1, \Theta_2, \ldots, \Theta_\ell\} = \bigcap_{k=1}^{\ell} \overline{\mathscr{D}_{\Theta_k}}.$$

We have $V_1 \supset V_2 \supset V_3 \supset \ldots$ and hence $\mathbb{I}(V_1) \subset \mathbb{I}(V_2) \subset \mathbb{I}(V_3) \subset \ldots$ etc. By Hilbert's basis theorem there is a finite m such that $\mathbb{I}(V_\ell) = \mathbb{I}(V_m)$ for all $\ell \geqslant m$. This implies that $V_\ell = \mathbb{V}(\mathbb{I}(V_\ell)) = \mathbb{V}(\mathbb{I}(V_m)) = V_m$ for $\ell \geqslant m$.

If $V_m = \varnothing$ we are done. If not, then there is $\xi_* \in V_m$, such that

$$\Delta(\Theta_k, \xi_*) = 0$$

for all $k \in \mathbb{N}$. Because $\{\Theta_k\}$ is dense in $\mathbb{S}^{n-1}(\mathbb{R}^n)$ and the discriminant is a continuous function, we see that $\Delta(\Theta, \xi_*) = 0$ for all $\Theta \in \mathbb{S}^{n-1}(\mathbb{R}^n)$, which contradicts Lemma 6.17. $\square$

Proposition 6.19. *Let $P : \mathbb{C}^n \to \mathbb{C}$ be a nonsingular polynomial of degree $N \geqslant 1$. Let $\Theta_k \in \mathbb{S}^{n-1}(\mathbb{R}^n)$ be a finite sequence of non-parallel vectors such that $\cap_k \mathscr{D}_{\Theta_k} = \varnothing$.*

If $\mathscr{D}_{\Theta_k} \cap \Theta_k^\perp$ is compact for $k = 1, 2$ then there is $r_0 > 0$ such that

$$\bigcap_k \overline{B}(\mathscr{D}_{\Theta_k}, 2r_0) = \varnothing. \tag{6.10}$$

Moreover, there are smooth $\Psi_k : \mathbb{R}^n \to [0, 1]$ such that Ψ_k are bounded Fourier multipliers acting on $\mathscr{F}_{\Theta^\perp}^{-1}\Theta(1, 2)$ for every $\Theta \in \mathbb{S}^{n-1}(\mathbb{R}^n)$, satisfying

$$\sum_k \Psi_k \equiv 1 \tag{6.11}$$

and $\Psi_k \equiv 0$ in $B(\mathscr{D}_{\Theta_k}, r_0)$.

Proof. If $\mathscr{D}_{\Theta_1}$ is empty, then so is any neighborhood of it, hence the intersection in (6.10) is empty. If not, there are at least two linearly independent Θ_k. Then the intersection $\mathscr{D}_{\Theta_1} \cap \mathscr{D}_{\Theta_2}$ is compact because our assumption that the first two $\mathscr{D}_{\Theta_k} \cap \Theta_k^\perp$ are compact implies that the orthogonal projections of any point in $\mathscr{D}_{\Theta_1} \cap \mathscr{D}_{\Theta_2}$ onto two different codimension 1 subspaces, $\Theta_1^\perp$ and $\Theta_2^\perp$, are bounded. Therefore, a closed neighborhood of finite radius about the intersection is compact too. Hence $\overline{B}(\mathscr{D}_{\Theta_1}, 1) \cap \overline{B}(\mathscr{D}_{\Theta_2}, 1)$ is compact. We will use this below.

Assume, contrary to the claim, that for any $r_0 > 0$ the intersection $\cap_k \overline{B}(\mathscr{D}_{\Theta_k}, 2r_0)$ is non-empty. Then there is a sequence $\xi^1, \xi^2, \ldots \in \mathbb{R}^n$ such that $\sup_k d(\xi^\ell, \mathscr{D}_{\Theta_k})$ approaches zero. By the compactness of $\overline{B}(\mathscr{D}_{\Theta_1}, 1) \cap \overline{B}(\mathscr{D}_{\Theta_2}, 1)$ we may assume that ξ^ℓ converges to some ξ. Then $\xi \in \mathscr{D}_{\Theta_k}$ for all k since the latter are closed sets. This contradicts the assumption that the intersection of the $\mathscr{D}_{\Theta_k}$ is empty and establishes (6.10).

Let $\psi_k : \Theta_k^\perp \to [0,1]$ be smooth and such that $\psi_k(\xi_{\Theta_k^\perp}) = 0$ if $d(\xi_{\Theta_k^\perp}, \mathscr{D}_{\Theta_k}) \leqslant r_0$ and $\psi_k(\xi_{\Theta_k^\perp}) = 1$ if $d(\xi_{\Theta_k^\perp}, \mathscr{D}_{\Theta_k}) \geqslant 2r_0$. Set

$$\Psi_1(\xi) = \psi_1(\xi_{\Theta_1^\perp}), \qquad \Psi_{k+1}(\xi) = \psi_{k+1}(\xi_{\Theta_{k+1}^\perp}) \prod_{\ell=1}^{k} \big(1 - \psi_\ell(\xi_{\Theta_\ell^\perp})\big) \tag{6.12}$$

where $\xi_{\Theta_\ell^\perp} = \xi - (\xi \cdot \Theta_\ell)\Theta_\ell \in \Theta_\ell^\perp$. Then $\Psi_k : \mathbb{R}^n \to [0,1]$ smoothly and $\Psi_k \equiv 0$ on $\overline{B}(\mathscr{D}_{\Theta_k}, r_0)$.

Note that $1 - \psi_k \in C_0^\infty(\Theta_k^\perp)$ for $k = 1, 2$ and $\xi \mapsto \psi_k(\xi_{\Theta_k^\perp})$ is constant in the direction of Θ_k. Thus, given any $\Theta \in \mathbb{S}^{n-1}$, Corollary 4.4 implies that

$$\|\mathscr{F}_{\Theta^\perp} M_{1-\psi_k} f\|_{\Theta(1,2)} \leqslant \big\|\mathscr{F}_{\Theta_{k\perp}}^{-1}\{1 - \psi_k\}\big\|_{\Theta_{k\perp}(1,\infty)} \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)} \tag{6.13}$$

for some direction $\Theta_{k\perp}$ in the (Θ, Θ_k)-plane perpendicular to Θ_k when $\Theta \nparallel \Theta_k$, and

$$\|\mathscr{F}_{\Theta^\perp} M_{1-\psi_k} f\|_{\Theta(1,2)} \leqslant \sup_{\Theta_k^\perp} |1 - \psi_k| \, \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)} \tag{6.14}$$

when $\Theta \parallel \Theta_k$. Recall that in the first case the $\Theta_{k\perp}(1,\infty)$-norm is taken in the $n-1$ dimensional space $\Theta_k^\perp$. In both cases the multiplier norm, which we denote by $C_k = C_k(\Theta, \Theta_k)$, is finite since $1 - \psi_k$ is smooth and compactly supported in $\Theta_k^\perp$, so in particular $\mathscr{F}_{\Theta_{k\perp}}^{-1}\{1 - \psi_k\}$ is a Schwartz test function.

Thus, by (6.12), (6.13) and (6.14)

$$\begin{aligned}
\|\mathscr{F}_{\Theta^\perp} M_{\Psi_1} f\|_{\Theta(1,2)} &\leqslant \|\mathscr{F}_{\Theta^\perp} f + M_{1-\psi_1} f\|_{\Theta(1,2)} \leqslant (1 + C_1) \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)}\,, \\
\|\mathscr{F}_{\Theta^\perp} M_{\Psi_2} f\|_{\Theta(1,2)} &\leqslant C_1 \|\mathscr{F}_{\Theta^\perp} M_{\psi_2} f\|_{\Theta(1,2)} \leqslant C_1(1 + C_2) \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)}\,.
\end{aligned}$$

We cannot apply the same argument to $M_{\Psi_3}, M_{\Psi_4}, \ldots$ because the multipliers $1 - \psi_k$ are not necessarily compactly supported in $\Theta_k^\perp$. Instead we note that $K = \operatorname{supp}(1 - \psi_1)(1 - \psi_2) \subset \mathbb{R}^n$ is compact. So $\Psi_{k+1} \in C_0^\infty(B(K, 1))$ for $k \geqslant 2$. Lemma 4.2 then implies that

$$\big\|\mathscr{F}_{\Theta^\perp} M_{\Psi_{k+1}} f\big\|_{\Theta(1,2)} \leqslant \frac{1}{\sqrt{2\pi}} \big\|\mathscr{F}_{\Theta}^{-1} \Psi_{k+1}\big\|_{\Theta(1,\infty)} \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)}$$

where the first norm is finite since $\mathscr{F}_{\Theta}^{-1} \Psi_{k+1} \in \mathscr{S}(\mathbb{R}^n)$. So the multipliers are bounded in all directions: there are finite $C_k' = C_k'(\Theta, \Theta_1, \ldots, \Theta_k)$ such that $\|\mathscr{F}_{\Theta^\perp} M_{\Psi_k} f\|_{\Theta(1,2)} \leqslant C_k' \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)}$ for all k and any $\Theta \in \mathbb{S}^{n-1}$.

For the last claim sum the Ψ_k all up to get

$$\sum_k \Psi_k(\xi) = 1 - \prod_k \big(1 - \psi_k(\xi_{\Theta_k^\perp})\big).$$

Since $\operatorname{supp}(1 - \psi_k) \subset \overline{B}(\mathscr{D}_{\Theta_k}, 2r_0)$ and the intersection of the latter is empty, the product vanishes everywhere. ☐

We now have all the necessary ingredients for the proof of the main theorem of this section.

Proof of Theorem 6.1. Let P be admissible of degree $N \geqslant 1$ and P_N its principal term. Then propositions 6.18 and 6.19 imply the existence of a finite set of directions $\Theta_k \in \mathbb{S}^{n-1}(\mathbb{R}^n)\backslash P_N^{-1}(0)$, $k = 1, \ldots, m$, an associated partition of unity Ψ_k and a constant $r_0 > 0$.

Set $f_k = M_{\Psi_k} f$. Then $f = \sum_k f_k$, $\mathscr{F} f_k(\xi) = 0$ when $d(\xi, \mathscr{D}_{\Theta_k}) \leqslant r_0$, and

$$\|\mathscr{F}_{\Theta^\perp} f_k\|_{\Theta(1,2)} \leqslant C_{k,\Theta} \|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)} \leqslant C_{k,\Theta}\sqrt{d_s}\, \|f\|\, L^2(D_s)$$

for any $\Theta \in \mathbb{S}^{n-1}$ by Proposition 6.19 and Lemma 2.1.

By Condition 1 of the admissibility definition in 6.10 there is $\varepsilon > 0$ such that $\mathscr{F} f_k = 0$ on $\mathscr{B}_{\Theta_k,\varepsilon}$. Let u_k be the solution to $P(D)u_k = f_k$ given by Proposition 6.7. We have

$$\left\|\mathscr{F}_{\Theta_k^\perp} u_k\right\|_{\Theta_k(\infty,2)} \leqslant \frac{N}{\varepsilon} \left\|\mathscr{F}_{\Theta_k^\perp} f_k\right\|_{\Theta_k(1,2)}$$

by that same proposition. The theorem follows by setting $u = \sum_k u_k$ since

$$\|u_k\|_{L^2(D_r)} \leqslant \sqrt{d_r} \left\|\mathscr{F}_{\Theta_k^\perp} u_k\right\|_{\Theta_k(\infty,2)}$$

by Lemma 2.1. □

Remark 6.20. The same proof gives $\|u\|_{L^q(D_r)} \leqslant C d_r^{1/q} d_s^{1/p} \|\mathscr{F} f\|_{L^p(\mathbb{R}^n)}$ if $p \leqslant 2 \leqslant q$ and $p^{-1} + q^{-1} = 1$.

7 Examples

We describe estimates for a few specific PDE's below. Some of the estimates follow directly from Theorem 5.1 or Theorem 6.1. Others illustrate how the method can be applied in different settings.

Example 7.1. The inhomogeneous Helmholtz equation $(\Delta + k^2)u = f$ is the motivating example for this work. The equation is rotation and translation invariant, and scales simply under dilations. Estimates in weighted norms typically share none of these properties[3]. For this reason, the dependence of the estimate on wavenumber k, which is the physically relevant parameter,

[3]Homogeneous weights, e.g. $\|\, |x|^\delta f\|_{L^2}$, retains scaling properties at the cost of allowing singularities at the origin. They are invariant under rotations about the origin, but not about any other point

is not clear. However, an estimate that comes from Theorem 5.1 or Theorem 6.1, with $k = 1$, i.e.

$$\|u\|_{L^2(D_r)} \leqslant C_1\sqrt{d_r d_s}\,\|f\|_{L^2(D_s)}$$

for f with $\operatorname{supp} f \subset D_s \subset \mathbb{R}^n$, immediately implies

$$\|u\|_{L^2(D_r)} \leqslant C\frac{\sqrt{d_r d_s}}{k}\,\|f\|_{L^2(D_s)}$$

by simply noting that $U(x) = u(kx)$ satisfies

$$(\Delta + k^2)U = k^2 f(kx)$$

and using the fact that the diameters scale as distance (i.e. $d \mapsto kd$) and L^2 norms like distance to the power $\frac{n}{2}$.

A second advantage is that *diameter* in Theorems 5.1 and 6.1 means the length of the intersection of any line with D_r or D_s. This is particularly appropriate for a source that is supported on a union of small sets that are far apart[4]. In weighted norms, the parts of the source that are far from the *origin* at which the weights are based, will have large norm because of their location, yet their contribution to the solution u or its far field (asymptotics used in scattering theory and inverse problems) is no larger than it would be if it were located at the origin. Insisting that our estimates share all the invariance properties of the underlying PDE eliminates these artificial differences between the physics and the mathematics[5].

Estimates of the L^q norms of u in in terms of L^p norms of $\mathscr{F}f$ are sometimes useful as well [3]. For $p^{-1} + q^{-1} = 1$, $p \leqslant 2 \leqslant q$, our methods give

$$\|u\|_{L^q(D_r)} \leqslant Ck^{-1} d_r^{1/q} d_s^{1/p}\,\|\mathscr{F}f\|_{L^p(\mathbb{R}^n)}\,.$$

Example 7.2. The Bilaplacian is a fourth order PDE that arises in the theory of elasticity and in the modelling of fluid flow (Stokes flow). We include a spectral parameter λ and an external force f:

$$(\Delta^2 - \lambda^2)u = f.$$

Let us show that the admissibility conditions for Theorem 6.1 given by Definition 6.10 are satisfied.

[4]Locating well-separated sources and scatterers is one of the most well-studied applied inverse problems modelled by the Helmholtz equation[5].

[5]Honesty demands that we acknowledge that our domain dependent estimates provide semi-norms, rather than norms, so we are not ready to give up weighted norms and Besov type norms entirely.

Assume $\lambda > 0$ and write $\Delta^2 - \lambda^2 = P(D)$, and so $P(\xi) = |\xi|^4 - \lambda^2$. Let $\Theta \in \mathbb{S}^{n-1}$ and for $\xi_{\Theta^\perp} \in \Theta^\perp$ write

$$p(\tau) = P(\tau\Theta + \xi_{\Theta^\perp}) = (\tau^2 + |\xi_{\Theta^\perp}|^2 - \lambda)(\tau^2 + |\xi_{\Theta^\perp}|^2 + \lambda).$$

The roots $\tau = \tau_j$ are easily seen to be

$$\tau_1 = \sqrt{\lambda - |\xi_{\Theta^\perp}|^2}, \qquad \tau_3 = i\sqrt{\lambda + |\xi_{\Theta^\perp}|^2},$$
$$\tau_2 = -\sqrt{\lambda - |\xi_{\Theta^\perp}|^2}, \qquad \tau_4 = -i\sqrt{\lambda + |\xi_{\Theta^\perp}|^2},$$

where the square root has been chosen to return a non-negative real part and mapping the negative real axis to the imaginary axis in the upper half-plane.

The derivative in the direction Θ is given by $p'(\tau) = 4(\tau^2 + |\xi_{\Theta^\perp}|^2)\tau$. Hence

$$|p'(\tau_j)| = 4\lambda\sqrt{\lambda - |\xi_{\Theta^\perp}|^2}$$

for $j = 1, 2$, and

$$|p'(\tau_j)| = 4\lambda\sqrt{\lambda + |\xi_{\Theta^\perp}|^2}$$

for $j = 3, 4$. Note that in the latter case $|p'(\tau_j)| \geqslant 4\lambda^{3/2}$ for all $\xi_{\Theta^\perp}$.

If $\varepsilon < 4\lambda^{3/2}$ we see that

$$\mathscr{B}_{\Theta,\varepsilon} = \left\{\xi \in \mathbb{R}^n \,\middle|\, \lambda - \frac{\varepsilon^2}{16\lambda^2} \leqslant |\xi_{\Theta^\perp}|^2 \leqslant \lambda + \frac{\varepsilon^2}{16\lambda^2}\right\},$$
$$\mathscr{D}_\Theta = \left\{\xi \in \mathbb{R}^n \,\middle|\, |\xi_{\Theta^\perp}|^2 = \lambda\right\}. \tag{7.1}$$

For any $r_0 > 0$, set $\varepsilon < 4\lambda^{5/4} r_0^{1/2}$. For any $\xi \in \mathscr{B}_{\Theta,\varepsilon}$ set $\zeta = (\xi \cdot \Theta)\Theta + \lambda^{1/2}\xi_{\Theta^\perp}/|\xi_{\Theta^\perp}|$. Then $\zeta \in \mathscr{D}_\Theta$ and

$$|\xi - \zeta| = \big||\xi_{\Theta^\perp}| - \lambda^{1/2}\big| = \frac{\big||\xi_{\Theta^\perp}|^2 - \lambda\big|}{|\xi_{\Theta^\perp}| + \lambda^{1/2}} \leqslant \frac{\varepsilon^2/(16\lambda^2)}{\lambda^{1/2}} < r_0$$

and so $\mathscr{B}_{\Theta,\varepsilon} \subset B(\mathscr{D}_\Theta, r_0)$ whenever $\varepsilon < \min(4\lambda^{3/2}, 4\lambda^{5/4} r_0^{1/2})$. Condition 1 in Definition 6.10 is thus satisfied. Condition 2 is an easy consequence of (7.1). Combining the estimate from Theorem 6.1 with a scaling argument similar to the previous example yields

$$\|u\|_{L^2(D_r)} \leqslant C\frac{\sqrt{d_r d_s}}{\lambda^{\frac{3}{2}}} \|f\|_{L^2(D_s)}.$$

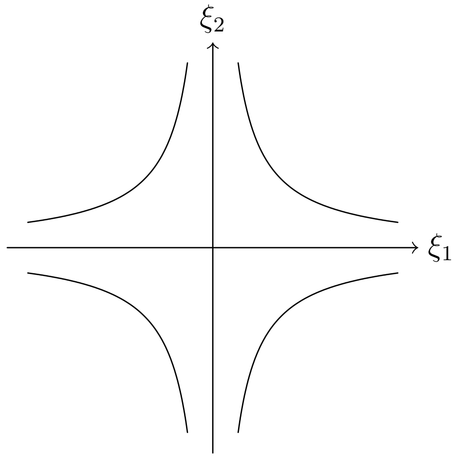

Figure 3: Characteristic variety of $D_1^2 D_2^2 - 1$.

Example 7.3. The operator $P(D) = D_1^2 D_2^2 - 1$ is not *simply characteristic*, and its zeros are not *uniformly simple*, as defined in definitions 4.2 and 6.2 by Agmon and Hörmander [2] or Section 14.3.1 in Hörmander's book [8]. This is because the characteristic variety $P^{-1}(0)$ has two different branches approaching a common asymptote (Figure 3). Thus the Besov style estimates established using uniform simplicity do not apply to this operator. We show below that the conditions in Definition 6.10 are satisfied, so that the estimate of Theorem 6.1 holds. As we remarked in the introduction, the Besov style estimates of [2] are a specialization of (1.2), and therefore a consequence of Theorem 6.1.

It is straightforward to check that $P(\xi)$ is nonsingular. We will verify the conditions in Definition 6.10 for $\Theta \in \mathbb{S}^{n-1}$, with $\Theta_1 \neq 0$ and $\Theta_2 \neq 0$, and calculate $|\Theta \cdot \nabla P(\xi)|$ for every ξ in $P(\xi) = 0$ with a fixed $\xi_{\Theta^\perp}$ component. A glance at Figure 3 shows that there will between two and four real ξ's satisfying $P(\xi) = 0$ that have the same $\xi_{\Theta^\perp}$ component. We begin by parameterizing the complex characteristic variety

$$P^{-1}(0) = \{(s, s^{-1}), (s, -s^{-1}) \in \mathbb{C}^2 \mid s \in \mathbb{C}, s \neq 0\}.$$

Next we project each point in the variety, $\xi = (s, \pm s^{-1})$, onto $\Theta^\perp = \{b(-\Theta_2, \Theta_1) \mid b \in \mathbb{R}\}$. Its $\Theta^\perp$ component is[6]

[6]Not all complex roots will project to $\Theta^\perp$ embedded in the reals. But we are only interested in the part of the characteristic variety that does.

$$\xi_{\Theta^\perp} = \big(-\Theta_2 s \pm \Theta_1 s^{-1}\big)(-\Theta_2,\Theta_1) =: b(-\Theta_2,\Theta_1). \tag{7.2}$$

To verify conditions about $\mathscr{B}_{\Theta,\varepsilon}$, we want to parameterize the points on the variety $P^{-1}(0)$ in terms of their $\xi_{\Theta^\perp}$ component, which is parameterized by b. So we use (7.2) to solve for $s = s(b)$. The four (complex) roots $\xi = (s(b), \pm s(b)^{-1})$ of P on the line defined by $\xi_{\Theta^\perp} = b(-\Theta_2,\Theta_1)$ are

$$\begin{aligned}
\xi^{(1)} &= \left(\frac{-b+\sqrt{b^2+4\Theta_1\Theta_2}}{2\Theta_2}, \frac{b+\sqrt{b^2+4\Theta_1\Theta_2}}{2\Theta_1}\right),\\
\xi^{(2)} &= \left(\frac{-b-\sqrt{b^2+4\Theta_1\Theta_2}}{2\Theta_2}, \frac{b-\sqrt{b^2-4\Theta_1\Theta_2}}{2\Theta_1}\right),\\
\xi^{(3)} &= \left(\frac{-b+\sqrt{b^2-4\Theta_1\Theta_2}}{2\Theta_2}, \frac{b+\sqrt{b^2-4\Theta_1\Theta_2}}{2\Theta_1}\right),\\
\xi^{(4)} &= \left(\frac{-b-\sqrt{b^2-4\Theta_1\Theta_2}}{2\Theta_2}, \frac{b-\sqrt{b^2-4\Theta_1\Theta_2}}{2\Theta_1}\right).
\end{aligned}$$

The derivative in the direction Θ at any root ξ having $\xi_1\xi_2 = \pm 1$ is

$$\Theta \cdot \nabla P(\xi) = \pm 2(\Theta_2\xi_1 + \Theta_1\xi_2).$$

Hence, after simplification,

$$\begin{aligned}
\Theta \cdot \nabla P(\xi^{(1)}) &= 2\sqrt{b^2+4\Theta_1\Theta_2},\\
\Theta \cdot \nabla P(\xi^{(2)}) &= -2\sqrt{b^2+4\Theta_1\Theta_2},\\
\Theta \cdot \nabla P(\xi^{(3)}) &= -2\sqrt{b^2-4\Theta_1\Theta_2},\\
\Theta \cdot \nabla P(\xi^{(4)}) &= 2\sqrt{b^2-4\Theta_1\Theta_2}.
\end{aligned}$$

So $|\Theta\cdot\nabla P(\xi)| \geqslant 2\,\big|b^2 - 4\,|\Theta_1\Theta_2|\big|^{1/2}$ at any root ξ with $\xi_{\Theta^\perp} = b(-\Theta_2,\Theta_1)$.

Now we have explicit descriptions of the sets that appear in Definition 6.10 and can verify the hypotheses of Theorem 6.1; namely,

$$\begin{aligned}
\mathscr{D}_\Theta &= \{\xi \in \mathbb{R}^2 \mid \xi_{\Theta^\perp} = b(-\Theta_2,\Theta_1), \quad b^2 - 4\,|\Theta_1\Theta_2| = 0\},\\
\mathscr{B}_{\Theta,\varepsilon} &= \{\xi \in \mathbb{R}^2 \mid \xi_{\Theta^\perp} = b(-\Theta_2,\Theta_1), \quad \big|b^2 - 4\,|\Theta_1\Theta_2|\big| \leqslant \varepsilon^2/4\}
\end{aligned}$$

as long as we choose $\varepsilon^2 < 16\,|\Theta_1\Theta_2|$. For any $r_0 > 0$ let $\varepsilon^2 \leqslant 8r_0\sqrt{|\Theta_1\Theta_2|}$. Then if $\xi \in \mathscr{B}_{\Theta,\varepsilon}$, we have (with $\xi_{\Theta^\perp} = b(-\Theta_2,\Theta_1)$)

$$d(\xi,\mathscr{D}_\Theta) \leqslant \left|b - 2\sqrt{|\Theta_1\Theta_2|}\right| = \frac{|b^2 - 4\,|\Theta_1\Theta_2||}{b + 2\sqrt{|\Theta_1\Theta_2|}} \leqslant \frac{\varepsilon^2}{8\sqrt{|\Theta_1\Theta_2|}} \leqslant r_0$$

for $b \geqslant 0$, and similarly $d(\xi, \mathscr{D}_\Theta) \leqslant \left|b + 2\sqrt{|\Theta_1\Theta_2|}\right| \leqslant r_0$ for $b \leqslant 0$. Hence $\mathscr{B}_{\Theta,\varepsilon} \subset \overline{B}(\mathscr{D}_\Theta, r_0)$ for any $r_0 > 0$ if $\varepsilon^2 \leqslant 8r_0\sqrt{|\Theta_1\Theta_2|}$ and $\varepsilon^2 < 16\,|\Theta_1\Theta_2|$, so we have verified Condition 1, and Condition 2 is automatic in two dimensions, so we are finished.

Example 7.4. The Faddeev operator is ubiquitous in the area of inverse problems. Its solution enables the construction of the so-called *Complex Geometric Optics* solutions to the Laplace equation that are used to prove uniqueness for many inverse scattering and inverse boundary value problems. See [6] for an early application to scattering theory, Sylvester and Uhlmann [15] and Nachmann [11] for its application to solving the *Calderón problem* [4], and [17] for a review of more recent developments in that area.

The simplest form, as introduced by Calderón is

$$(\Delta + 2\zeta \cdot \nabla)\, u = f \tag{7.3}$$

with $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$. It has complex coefficients, but setting $v = e^{i\Im\zeta\cdot x}u$ and $g = e^{i\Im\zeta\cdot x}f$ results in

$$(\Delta + 2\Re\zeta \cdot \nabla)\, v = g$$

which has real coefficients. Moreover, u and v have the same L^p norms, as do f and g. The symbol and its gradient are

$$\begin{aligned} P(\xi) &= -\xi \cdot (\xi - 2i\Re\zeta) \\ \nabla P &= 2(-\xi + i\Re\zeta) \end{aligned}$$

so ∇P has no real zeros. Thus P has no real double characteristics and Theorem 5.1 applies. Because the equation, and the estimates, dilate simply, scaling again gives the exact dependence on ζ.

$$||v||_{L^2(D_r)} \leqslant C\frac{\sqrt{d_r d_s}}{|\Re\zeta|}||g||_{L^2(D_s)} \tag{7.4}$$

with $\operatorname{supp} f \subset D_s$. Here d_r, d_s are the diameters of the open sets D_r, D_s. We may, of course, replace v by u and g by f.

In some applications, the condition $\zeta \cdot \zeta = 0$ is replaced by $\zeta \cdot \zeta = \lambda$. As the gradient of P is still nowhere vanishing, Theorem 5.1 still applies, and the estimates still scale, but it is not clear how the estimates depend on the ratio $\frac{\Re\zeta}{\sqrt{\lambda}}$. A direct calculation shows that (7.4) still holds. In addition, Remark 3.5 also applies here, so we have for $\frac{1}{p}+\frac{1}{q}=1, p \leqslant 2 \leqslant q$,

$$\|u\|_{L^q(D_r)} \leqslant \frac{d_r^{1/q} d_s^{1/p}}{|\Re\zeta|} \|\mathscr{F} f\|_{L^p(\mathbb{R}^n)} .$$

Equation (7.3) has a special direction. We expect a solution to decay exponentially in the direction $\Theta = \Re\zeta/|\Re\zeta|$, so an anisotropic estimate is natural here. Taking the Fourier transform in the $\Theta^\perp$ hyperplane reduces (7.3) to an ordinary differential equation which can be factored into the product of two first order operators. Then using (6.3) for one of the factors and (6.4) for the other gives the estimate

$$\|\mathscr{F}_{\Theta^\perp} u\|_{\Theta(\infty,2)} \leqslant \frac{\|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)}}{\inf_{\xi_{\Theta^\perp} \in \Theta^\perp} \left| |\Re\zeta| + \sqrt{|\Im\zeta + \xi_{\Theta^\perp}|^2 - \lambda} \right|} \leqslant \frac{\|\mathscr{F}_{\Theta^\perp} f\|_{\Theta(1,2)}}{|\Re\zeta|} \tag{7.5}$$

when $\lambda \in \mathbb{R}$. This estimate implies (7.4) by (2.12).

Theorems 5.1 and 6.1 apply to scalar valued PDE's only, but the method can be applied to systems. The next proposition could be substantially more general, but it is enough to establish estimates for the Dirac system.

Proposition 7.5. *Consider a constant coefficient first order system*

$$\boldsymbol{A}(D) = \sum_{j=1}^n \boldsymbol{A}_j \frac{\partial}{\partial x_j} + \boldsymbol{B}$$

with $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n, \boldsymbol{B} \in \mathbb{C}^{n\times n}$ and suppose that, for some k,

$$\boldsymbol{M}(\xi) = \boldsymbol{A}_k^{-1}\left(\sum_{j\neq k} \boldsymbol{A}_j \xi_j + \boldsymbol{B}\right)$$

is normal for all ξ[7] . Then, there is a constant C, such that for every f, there exists u solving

$$\boldsymbol{A}(D)u \quad = \quad f$$

and

$$||\mathscr{F}_{\Theta^\perp} u||_{\Theta(\infty,2)} \quad \leqslant \quad C||\mathscr{F}_{\Theta^\perp} f||_{\Theta(1,2)} \tag{7.6}$$

[7]Equivalently, for some k and all j, $\boldsymbol{A}_k^{-1}\boldsymbol{A}_j$ and $\boldsymbol{A}_k^{-1}\boldsymbol{B}$ are normal

where Θ is the unit vector in the kth coordinate direction, and consequently, for f supported in D_r and any D_s,

$$||u||_{L^2(D_s)} \leqslant C\sqrt{d_r d_s}||f||_{L^2(D_r)} \tag{7.7}$$

where d_i is the diameter of D_i and C is a constant that depends only $\boldsymbol{A}(D)$ and the dimension n.

Proof. We take the partial Fourier transform in the $\Theta^\perp$ hyperplane, and note that the vector $\tilde{u} := \mathscr{F}_{\Theta^\perp} u$ must satisfy

$$\frac{\partial}{\partial x_k}\tilde{u} + \boldsymbol{M}(\xi)\tilde{u} \quad = \quad \boldsymbol{A}_k^{-1}\tilde{f} \tag{7.8}$$

and simply write the solution

$$\begin{aligned} \tilde{u}(t,\xi) \quad &= \quad \int_{-\infty}^{t} e^{\boldsymbol{M}(\xi)(t-s)}P^{+}\boldsymbol{A}_k^{-1}\tilde{f}(s,\xi)ds && (7.9) \\ &- \quad \int_{t}^{\infty} e^{\boldsymbol{M}(\xi)(t-s)}P^{-}\boldsymbol{A}_k^{-1}\tilde{f}(s,\xi)ds && (7.10) \end{aligned}$$

where $P^+(\xi)$ is the orthogonal projection onto the $\Re\lambda \geqslant 0$ eigenspace of $\boldsymbol{M}(\xi)$ and $P^-(\xi)$ is the orthogonal projection onto the $\Re\lambda < 0$ eigenspace. The projections need not be continuous functions of ξ, but they need only be measurable for the formula to make sense. The fact that $\boldsymbol{M}(\xi)$ is normal guarantees that the sum of the projections is the identity, and therefore that $\tilde{u}$ really does solve (7.8). The estimate (7.6) follows immediately from the formula (7.9) and the fact that the orthogonal projections have norm one or zero. $\square$

Example 7.6. The 4x4 Dirac operator may be written as a prolongation of the curl operator

$$\boldsymbol{D} = \begin{pmatrix} \nabla\times & -\nabla \\ \nabla\cdot & 0 \end{pmatrix}. \tag{7.11}$$

Alternatively, we may express the first order system as

$$\boldsymbol{D} - i\omega\boldsymbol{I} = \sum_{j=1}^{3} \boldsymbol{A}_j \frac{\partial}{\partial x_j} - i\omega\boldsymbol{I}$$

where $\boldsymbol{I}$ is the 4×4 identity matrix and

$$\boldsymbol{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{A}_1 = \begin{pmatrix} 0 & \boldsymbol{P} \\ \boldsymbol{P} & 0 \end{pmatrix}, \quad \boldsymbol{A}_2 = \begin{pmatrix} 0 & -\boldsymbol{I} \\ \boldsymbol{I} & 0 \end{pmatrix}, \quad \boldsymbol{A}_3 = \begin{pmatrix} \boldsymbol{P} & 0 \\ 0 & -\boldsymbol{P} \end{pmatrix}.$$

It is straightforward to verify that each $\boldsymbol{A}_j$ is skew, $\boldsymbol{A}_j^2 = -\boldsymbol{I}$ and that $\boldsymbol{A}_i\boldsymbol{A}_j = \pm\boldsymbol{A}_k$ when all three indices are different. These facts guarantee the hypotheses of Proposition 7.5, and hence the estimate (7.6) for a solution u of

$$(\boldsymbol{D} - i\omega\boldsymbol{I})\, u = f.$$

Example 7.7 (Non-Example)**.** We show that the estimates (1.1) do not hold for the Laplacian in 3 dimensions, which has a double characteristic. Suppose that f is compactly supported and

$$\Delta u = f$$

In 3 dimensions,

$$u(x) = \int \frac{f(y)}{|x-y|}dy + H(x)$$

where H is a harmonic polynomial. For compactly supported f, the estimates (7.4) would imply that u grows no faster than $1/|x|$, so H must be zero. We choose f to be identically one on the ball of radius A centered at the origin. In this case, $||f||_{L^2(A)}$ is $\sqrt{\frac{4}{3}\pi A^3}$. We next compute $||u||_{L^2(B_R(c))}$, with $R >> A$ and $|c| = 2|R|$. For $x \in B_R(c)$

$$|u(x)| \quad > \quad \frac{1}{2}\frac{\int f(y)}{R}$$

so that

$$||u||_{L^2(B_R(c))} \quad \geqslant \quad \frac{1}{2}\frac{(\frac{4}{3}\pi A^3)(\sqrt{\frac{4}{3}\pi R^3})}{R} \quad = \frac{1}{2}(\frac{4}{3}\pi)^{\frac{3}{2}}A^3R^{\frac{1}{2}}$$

Estimate (1.1) would imply that

$$\frac{1}{2}(\frac{4}{3}\pi)^{\frac{3}{2}}A^3R^{\frac{1}{2}} \leqslant C\sqrt{AR}A^{\frac{3}{2}} = CA^{\frac{5}{2}}R^{\frac{1}{2}}$$

which is impossible for large A.

8 Conclusions

We have introduced a technique for proving some simple, translation invariant estimates, which scale naturally, and can therefore be directly interpreted for physical systems and remain meaningful in any choice of units. Such estimates are necessary because because physical principles dictate that the fields should store finite energy in a bounded region (i.e. solutions should be locally L^2) and radiate finite power, which implies that they should decay at least as fast as $r^{-\frac{n-1}{2}}$ near infinity. We have replaced weighted norms by estimates on bounded regions which depend on the diameter of these regions. Because the estimates depend on natural geometric quantities, which rotate, dilate and translate in natural ways, the estimates themselves have the same symmetries as the underlying PDE models. The L^2 estimates are based on anisotropic estimates that are analogous to those that hold for a parameterized ODE, so it is reasonable to expect them to hold for all simply characteristic PDE, but we have not proven any theorems in that generality, nor produced examples to show that more restrictions are necessary. Indeed, we expect that these estimates are true for many more PDE's and systems than we have covered here.

Theorem 5.1 can certainly be extended to allow first order terms with complex coefficients using the change of dependent variable in the line following (7.3), but we do not know if we can allow other complex coefficients as well.

Theorem 6.1 includes many technical assumptions that we doubt are necessary. The hypothesis that the characteristic variety is non-singular over $\mathbb{C}^n$ rather than $\mathbb{R}^n$ is clearly not necessary, but we don't know of a simple replacement. The admissibility conditions in Definition 6.10 were chosen to facilitate the proof, and enforce a certain uniform behavior outside compact sets, somewhat similar to Agmon-Hörmander's *uniformly simple* hypothesis. In two dimensions, where Condition 2 is automatically satisfied, we are not aware of any nonsingular polynomial $P : \mathbb{C}^2 \to \mathbb{C}$ for which Condition 1 does not hold.

We have only given one example of a system of PDE's. The estimates for the Dirac system were particularly easy because, for any direction Θ, the resulting model system was normal (this is the equivalent of a non-vanishing discriminant for a single high order equation). Other interesting systems of PDE, e.g. Maxwell's equations, do no have this property.

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