VARIATION OF THE HOLOMORPHIC DETERMINANT BUNDLE

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Abstract. In this paper, we prove that the Grothendieck-Riemann-Roch formula in Deligne cohomology computing the determinant of the cohomology of a holomorphic vector bundle on the fibers of a proper submersion between abstract complex manifolds is invariant by deformation of the bundle.

1. Introduction

The Grothendieck-Riemann-Roch theorem is one of the cornerstones of modern algebraic geometry, and can be stated in its initial form as follows:

Theorem 1.1. [7] *For any smooth quasi-projective variety X over a field of characteristic zero, the morphism $\mathcal{F} \to \mathrm{ch}(\mathcal{F})\,\mathrm{Td}(X)$ from the Grothendieck group $\mathrm{K}(X)$ of coherent sheaves on X to the Chow ring $\mathrm{CH}(X)$ of X commutes with proper push-forward.*

Since Serre's fundamental papers on coherent sheaves [17] [16], it has become natural and useful to translate results from algebraic to analytic geometry. Concerning the Grothendieck-Riemann-Roch theorem, this has been the object of many researches from early sixties till eighties, starting with the case of analytic immersions [1] and pursuing with the index theorem for vector bundles and coherent analytic sheaves (see [2], [14], [21], [22]). The outcome of these works is O'Brian-Toledo-Tong's proof of the Grothendieck-Riemann-Roch theorem in Hodge cohomology for arbitrary proper holomorphic maps between complex manifolds [13]. By completely different methods, Levy [12] succeeded in proving the analogous statement in De Rham cohomology, where the Chern classes are constructed by means of locally-free resolutions in the category of real-analytic coherent sheaves.

From the middle of the eighties, some new ideas about the Grothendieck-Riemann-Roch theorem emerged after the seminal article of Quillen [15] introducing canonical hermitian metrics on determinant bundles associated with the cohomology of a vector bundle on the fibers of a holomorphic submersion (see [18, Chap. VI]). Building on initial results in the case of families of curves (see [15], [4], [8]), Bismut, Gillet and Soulé [5] proved that, for locally Kähler fibrations, the curvature of this determinant bundle is exactly given by the component of degree two of the Grothendieck-Riemann-Roch theorem at the level of differential forms. This theorem has been extended to all degrees in [6] provided that the higher direct images of the bundle are locally free. Quite recently, Bismut succeeded in removing the Kählerianity assumption on the morphism and obtained the following result:

Theorem 1.2. [3] *For any proper holomorphic submersion $f\colon X \to Y$ between complex manifolds and for any holomorphic vector bundle $\mathcal{E}$ on X such that the sheaves $\mathrm{R}^i f_* \mathcal{E}$ are locally* *free on Y, the Grothendieck-Riemann-Roch equality for the couple $(\mathcal{E}, f)$ holds in the Bott-Chern cohomology ring of Y.*

On abstract complex manifolds, the finest known cohomology theory where Chern classes exist for holomorphic vector bundles is Deligne-Beĭlinson cohomology. The ultimate goal of our program would be to prove the Grothendieck-Riemann-Roch theorem in this cohomology. The statement does not immediately make sense even for holomorphic vector bundles, because Chern classes must be defined for the direct images sheaves $\mathrm{R}^i f_* \mathcal{E}$. The problem of defining Chern classes of coherent sheaves in Deligne cohomology is solved on compact manifolds in [10]. The corresponding Grothendieck-Riemann-Roch theorem is proved only for projective morphisms between complex compact manifolds.

In this paper, we focus only on the component of degree two on the base of the Grothendieck-Riemann-Roch theorem in Deligne cohomology for holomorphic vector bundles. Our main result describes completely the variation of the determinant bundle:

Theorem 1.3. *Let $f\colon X \to Y$ be a proper holomorphic submersion between complex manifolds X, Y and let $(\mathcal{E}_t)_{t\in\Delta}$ be a holomorphic family of holomorphic vector bundles on X parameterized by the complex unit disc Δ. Then there exists a unique analytic curve α from Δ to $\mathrm{Pic}^0(Y)$ vanishing at 0 such that for any t in Δ,*

$$\mathrm{c}_1^{\mathrm{D}}(\alpha(t)) = [f_*\{[\mathrm{ch}^{\mathrm{D}}(\mathcal{E}_t) - \mathrm{ch}^{\mathrm{D}}(\mathcal{E}_0)]\,\mathrm{td}^{\mathrm{D}}(\mathrm{T}_{X/Y})\}]^{(2)}$$

in the rational Deligne cohomology group $\mathrm{H}^2_{\mathrm{D}}(Y, \mathbb{Q}(1))$. Besides, for any s and t in Δ,

$$\mathrm{c}_1^{\mathrm{D}}[\det \mathrm{R}f_*\, \mathcal{E}_s] - \mathrm{c}_1^{\mathrm{D}}[\det \mathrm{R}f_*\, \mathcal{E}_t] = \alpha(s) - \alpha(t)$$

in $\mathrm{Pic}(Y)$. In particular, the class $\mathrm{c}_1^{\mathrm{D}}[\det \mathrm{R}f_*\, \mathcal{E}_t] - [f_*\{\mathrm{ch}^{\mathrm{D}}(\mathcal{E}_t)\,\mathrm{td}^{\mathrm{D}}(\mathrm{T}_{X/Y})\}]^{(2)}$ in $\mathrm{H}^2_{\mathrm{D}}(Y, \mathbb{Q}(1))$ is independent of t.

Even if $\mathrm{Pic}(Y)$ is not a complex manifold, it is possible to give a precise definition of an analytic curve with values in $\mathrm{Pic}(Y)$ that matches with the usual one when the image of $\mathrm{H}^1(Y, \mathbb{Z}_Y)$ is discrete in $\mathrm{H}^1(Y, \mathcal{O}_Y)$.

On the one hand, Theorem 1.3 is motivated by Teleman's program on the classification of class VII surfaces (see [19], [20]). On the other hand, it is a significant step towards the general Grothendieck-Rieman-Roch theorem in Deligne cohomology, at least in degree two. For instance, we have the following result:

Theorem 1.4. *Let Y and F be complex manifolds such that F is compact, and let $p\colon Y \times F \to Y$ be the first projection. Then, for any $\mathcal{L}$ in $\mathrm{Pic}^0(Y \times F)$*

$$\mathrm{c}_1^{\mathrm{D}}[\det \mathrm{R}p_*\, \mathcal{L}] = [p_*\{\mathrm{ch}^{\mathrm{D}}(\mathcal{L})\,\mathrm{td}^{\mathrm{D}}(\mathrm{T}_{Y\times F/Y})\}]^{(2)}$$

in the rational Deligne cohomology group $\mathrm{H}^2_{\mathrm{D}}(Y, \mathbb{Q}(1))$.

The paper is organized as follows: in §2, we recall the theory of determinants for coherent analytic sheaves (see [11] and [5, §3]) and we prove in Proposition 2.4 a base change formula for determinant bundles. We also prove a folklore result (Lemma 2.2) saying that the first

Chern class of a coherent sheaf in Hodge cohomology is the same as the first Chern class of its determinant. In §3, we recall the basics of Deligne cohomology (see [9] and [23, Chap. 12]) and Lemma 3.1 is the main ingredient of the proof of Theorem 1.3. Then we discuss analytic curves with values in the Picard group of a complex manifold. Lastly, §4 is devoted to the proofs of Theorem 1.3 and Theorem 1.4.

Acknowledgments I wish to thank Andrei Teleman for suggesting the problem solved in this paper, and also Christophe Soulé for kindly explaining to me some of the material of [5].

2. Holomorphic determinant bundles

Let $\mathcal{F}$ be a coherent analytic sheaf on a connected complex manifold X. The determinant of $\mathcal{F}$, denoted by $\det \mathcal{F}$, is a holomorphic line bundle on X defined as follows:

– If $\mathcal{F}$ is torsion-free, there exists a Zariski-open subset U of X such that $\mathcal{F}$ is locally free on U and $X \setminus U$ has codimension at least two in X. Then the top exterior power of $\mathcal{F}_{|U}$ is a holomorphic line bundle on U, which extends uniquely to a line bundle $\det \mathcal{F}$ on X.

– If $\mathcal{F}$ is a torsion sheaf, let Z be the maximal closed hypersurface contained in $\operatorname{supp} \mathcal{F}$, and let $(Z_i)_{i\in I}$ be the irreducible components of Z. For any index i, if $\mathcal{J}_{Z_i}$ denotes the ideal sheaf of Z_i, let m_i be the smallest integer m such that $\mathcal{J}_{Z_i}^m \mathcal{F}$ vanishes generically on Z_i. Then $\det \mathcal{F} = \bigotimes_i \mathcal{O}_X(m_i Z_i)$.

– If $\mathcal{F}$ is arbitrary and $\mathcal{F}_{\text{tors}}$ is its maximal torsion subsheaf, then $\mathcal{F}/\mathcal{F}_{\text{tors}}$ is torsion-free and $\det \mathcal{F} = \det(\mathcal{F}/\mathcal{F}_{\text{tors}}) \otimes \det \mathcal{F}_{\text{tors}}$.

The main property of determinants is the following: for any bounded complex $\mathcal{K}^\bullet$ of coherent sheaves on X, using additive notation for line bundles, we have a canonical isomorphism

$$\sum_i (-1)^i \det \mathcal{K}^i \simeq \sum_i (-1)^i \det \mathcal{H}^i(\mathcal{K}^\bullet).$$

For any bounded complex $\mathcal{K}^\bullet$ on X with coherent cohomology, the determinant of $\mathcal{K}^\bullet$ is defined by the formula $\det \mathcal{K}^\bullet = \sum_i (-1)^i \det \mathcal{H}^i(\mathcal{K}^\bullet)$. Two quasi-isomorphic bounded complexes with coherent cohomology have canonically isomorphic determinants.

Lemma 2.1. *Let $\varphi\colon X \to Y$ be a holomorphic map between connected complex manifolds. Then for any bounded complex $\mathcal{K}^\bullet$ of analytic sheaves on Y with coherent cohomology, if $\mathbb{L}\varphi^*$ denotes the derived pullback by φ, then $\varphi^* \det \mathcal{K}^\bullet \simeq \det(\mathbb{L}\varphi^* \mathcal{K}^\bullet)$.*

Proof. By dévissage, we are reduced to prove the lemma when $\mathcal{K}^\bullet$ is a single coherent sheaf in degree zero. For any Stein subset U of Y, let $\mathcal{E}^\bullet$ be a locally free resolution of $\mathcal{K}_{|U}$. Then we have canonical isomorphisms

$$\varphi^* \det \mathcal{K}_{|U} \simeq \varphi^*[\sum_i (-1)^i \det(\mathcal{E}^i)] = \sum_i (-1)^i \det(\varphi^* \mathcal{E}^i) = \det(\mathbb{L}\varphi^* \mathcal{K}_{|U})$$

which can be glued together to give a global isomorphism on X between $\varphi^* \det \mathcal{K}$ and $\det(\mathbb{L}\varphi^* \mathcal{K})$. $\square$

For any coherent sheaf $\mathcal{F}$ on X, we denote by $\mathrm{c}_i^{\mathrm{H}}(\mathcal{F})$ the Chern classes of $\mathcal{F}$ in $\mathrm{H}^i(X, \Omega_X^i)$ and by $\mathrm{ch}^{\mathrm{H}}(\mathcal{F})$ its Chern character in the total Hodge cohomology ring of X.

Lemma 2.2. *For any complex manifold X and any coherent analytic sheaf $\mathcal{F}$ on X, we have $\mathrm{c}_1^{\mathrm{H}}(\mathcal{F}) = \mathrm{c}_1^{\mathrm{H}}(\det \mathcal{F})$ in $\mathrm{H}^1(X, \Omega_X^1)$.*

Proof. We can assume that $\mathcal{F}$ is either a torsion sheaf or a torsion-free sheaf. Besides, it is enough to prove that $\mathrm{c}_1^{\mathrm{H}}(\mathcal{F}_{|U}) = \mathrm{c}_1^{\mathrm{H}}(\det \mathcal{F}_{|U})$ where U is a Zariski open subset of X such that $\mathrm{codim}_X(X \setminus U) \geq 2$. Therefore, we have to deal with two different cases:
– First case: the sheaf $\mathcal{F}$ is a torsion sheaf whose support is a smooth hypersurface. By dévissage, it is possible to assume without loss of generality that $\mathcal{J}_Z \mathcal{F} = 0$, so that $\det \mathcal{F} = \mathcal{O}_X(Z)$. Then, using the Grothendieck-Riemann-Roch theorem in Hodge cohomology for immersions [13], we get $\mathrm{c}_1^{\mathrm{H}}(\mathcal{F}) = [Z]_{\mathrm{H}} = \mathrm{c}_1^{\mathrm{H}}(\mathcal{O}_X(Z)) = \mathrm{c}_1^{\mathrm{H}}(\det \mathcal{F})$.
– Second case: the sheaf $\mathcal{F}$ is locally free. Then we know that $\mathrm{c}_1^{\mathrm{H}}(\mathcal{F}) = \mathrm{c}_1^{\mathrm{H}}(\det \mathcal{F})$. □

Let $f \colon X \to Y$ be a proper holomorphic submersion between two connected complex manifolds X and Y, and let $\mathcal{E}$ be a locally free sheaf on X. By Grauert-Riemenschneider's theorem, the bounded complex $\mathrm{R}f_* \,\mathcal{E}$ has coherent cohomology.

Definition 2.3. The determinant of the cohomology $\lambda(\mathcal{E}, f)$ attached to the couple $(\mathcal{E}, f)$ is the class of $\det(\mathrm{R}f_* \,\mathcal{E})$ in $\mathrm{Pic}(Y)$.

We now state and prove a base change theorem for the determinant of the cohomology. Let T be a complex manifold and $u \colon T \to Y$ be a closed immersion, and consider the cartesian diagram

$$\begin{tikzcd} Z \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ T \arrow[r, "u"'] & Y \end{tikzcd}$$

Proposition 2.4. *For any vector bundle $\mathcal{E}$ on X, $u^*\lambda(\mathcal{E}, f) = \lambda(v^*\mathcal{E}, g)$ in $\mathrm{Pic}(T)$.*

Proof. In the bounded derived category of Y, by using the projection formula twice, we have

$$\begin{aligned} \mathrm{R}f_* \,\mathcal{E} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} u_*\mathcal{O}_T &\simeq \mathrm{R}f_* \,(\mathcal{E} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} f^*u_*\mathcal{O}_T) \simeq \mathrm{R}f_* \,(\mathcal{E} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} v_*g^*\mathcal{O}_T) \simeq \mathrm{R}f_* \,\mathrm{R}v_* \,(v^*\mathcal{E} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Z} g^*\mathcal{O}_T) \\ &\simeq \mathrm{R}u_* \,\mathrm{R}g_* \,(v^*\mathcal{E} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Z} g^*\mathcal{O}_T) \simeq u_*[\mathrm{R}g_* \,(v^*\mathcal{E})]. \end{aligned}$$

This proves that for any integer i

$$\mathcal{H}^i \,(\mathbb{L}u^* \,\mathrm{R}f_*\mathcal{E}) \simeq \mathrm{R}^i g_*(v^*\mathcal{E}).$$

Then we can conclude using Lemma 2.1. □

3. Deligne cohomology

For any complex manifold X and any nonnegative integer p, the Deligne complex $\mathbb{Z}_{\mathrm{D},X}(p)$ is the complex

$$\mathbb{Z}_X \xrightarrow{(2i\pi)^p} \mathcal{O}_X \xrightarrow{\partial} \Omega_X^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_X^{p-1},$$

where the sheaf $\mathbb{Z}_X$ sits in degree zero. The integral Deligne cohomology groups of X are defined by the formula $\mathrm{H}^k_{\mathrm{D}}(X, \mathbb{Z}(p)) = \mathbb{H}^k(X, \mathbb{Z}_{\mathrm{D,X}}(p))$. Similar definitions hold for the rational Deligne complex $\mathbb{Q}_{\mathrm{D},X}(p)$ as well as for the rational Deligne cohomology groups $\mathrm{H}^k_{\mathrm{D}}(X, \mathbb{Q}(p))$.

For any locally-free sheaf $\mathcal{E}$ on X, we will denote by $\mathrm{c}_i^{\mathrm{D}}(\mathcal{E})$ the rational Chern classes of $\mathcal{E}$ in $\mathrm{H}^{2i}_{\mathrm{D}}(X, \mathbb{Q}(i))$, and by $\mathrm{ch}^{\mathrm{D}}(\mathcal{E})$ the Chern character of $\mathcal{E}$. Recall that $\mathrm{H}^2_{\mathrm{D}}(X, \mathbb{Z}(1))$ is the Picard group of X, and that the kernel of the first Chern class morphism

$$\mathrm{c}_1^{\mathrm{D}} \colon \mathrm{Pic}(X) \simeq \mathrm{H}^2_{\mathrm{D}}(X, \mathbb{Z}(1)) \to \mathrm{H}^2_{\mathrm{D}}(X, \mathbb{Q}(1)) \tag{1}$$

is exactly the maximal torsion subgroup of $\mathrm{Pic}(X)$.

There is a natural cup-product in Deligne cohomology (cf [9]). Besides, for any nonnegative integer p, the morphism $\partial \colon \Omega_X^{p-1} \to \Omega_X^p$ induces a morphism from $\mathbb{Z}_{\mathrm{D,X}}(p)$ to $\Omega_X^p[-p]$. Hence for any nonnegative integer k, we obtain a map from $\mathrm{H}^{k+p}_{\mathrm{D}}(X, \mathbb{Z}(p))$ to $\mathrm{H}^k(X, \Omega^p_X)$ which is compatible with cup-products on both sides.

We now give the key lemma of the proof of Theorem 1.3.

Lemma 3.1. *Let Δ be the complex unit disc, let X be a complex manifold and let α be a class in $\mathrm{H}^2_{\mathrm{D}}(X \times \Delta, \mathbb{Q}(1))$ whose image in $\mathrm{H}^1(X \times \Delta, \Omega^1_{X\times\Delta})$ vanishes. Then there exists a class β in $\mathrm{H}^2_{\mathrm{D}}(X, \mathbb{Q}(1))$ such that $\alpha = \mathrm{pr}_1^* \beta$.*

Proof. Let δ be the morphism obtained by the composition

$$\mathbb{Q}_{\mathrm{D},X\times\Delta}(1) \to \Omega^1_{X\times\Delta}[-1] \to \mathcal{O}_X \boxtimes \Omega^1_\Delta[-1].$$

Then we have an exact sequence

$$0 \longrightarrow \mathrm{pr}_1^{-1}\, \mathbb{Q}_{\mathrm{D},X}(1) \longrightarrow \mathbb{Q}_{\mathrm{D},X\times\Delta}(1) \xrightarrow{\delta} \mathcal{O}_X \boxtimes \Omega^1_\Delta[-1] \longrightarrow 0$$

which yields the long exact sequence

$$\mathrm{H}^2_{\mathrm{D}}(X, \mathbb{Q}(1)) \xrightarrow{\mathrm{pr}_1^*} \mathrm{H}^2_{\mathrm{D}}(X \times \Delta, \mathbb{Q}(1)) \xrightarrow{\delta} \mathrm{H}^1(X \times \Delta, \mathcal{O}_X \boxtimes \Omega^1_\Delta).$$

The lemma follows. □

To end this section, we discuss the notion of analytic curve in the Picard group of a complex manifold.

For an arbitrary X, the group $\mathrm{Pic}^0(X) = \mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \mathbb{Z}_X)$ is generally not a complex manifold since $\mathrm{H}^1(X, \mathbb{Z}_X)$ is not always discrete in $\mathrm{H}^1(X, \mathcal{O}_X)$. However, we can give the following definition:

Definition 3.2. For any complex manifold X, a curve $\gamma \colon \Delta \to \mathrm{Pic}(X)$ is analytic if the curve $t \to \gamma(t) - \gamma(0)$ can be lifted to a holomorphic map with values in $\mathrm{H}^1(X, \mathcal{O}_X)$.

Then we have:

Lemma 3.3. (i) *If $\gamma \colon \Delta \to \mathrm{Pic}(X)$ is an analytic curve, then γ is entirely determined by $\gamma(0)$ and by the curve $\mathrm{c}_1^{\mathrm{D}} \circ \gamma$ from Δ to $\mathrm{H}^2_{\mathrm{D}}(X, \mathbb{Q}(1))$, where $\mathrm{c}_1^{\mathrm{D}}$ is given by* (1)*.*

(ii) *For any class α in $\mathrm{Pic}(X \times \Delta)$ the curve $\gamma \colon \Delta \to \mathrm{Pic}(X)$ defined by $\gamma(t) = \alpha_{|X\times\{t\}}$ is analytic.*

Proof. (i) Assume that γ is analytic, that $\gamma(0)=0$ and that for any t in Δ, the image of $\gamma(t)$ in $\mathrm{H}^2_{\mathrm{D}}(X,\mathbb{Q}(1))$ vanishes. If $\tilde{\gamma}$ is a lift of γ from Δ to $\mathrm{H}^1(X,\mathcal{O}_X)$ such that $\tilde{\gamma}(0)=0$, then $\tilde{\gamma}(\Delta)$ lies in the image of $\mathrm{H}^1(X,\mathbb{Q}_X)$ in $\mathrm{H}^1(X,\mathcal{O}_X)$. Therefore $\tilde{\gamma}$ vanishes since it has countable image.
(ii) Let $(U_i)_{i\in I}$ be a Stein cover of X such that all finite intersections of the U_i's are contractible. Then α can be written as a Čhech cohomology class in $\mathrm{H}^2_{\mathrm{D}}(X\times\Delta,\mathbb{Z}(1))$ for the covering $(U_i\times\Delta)_{i\in I}$. This means that we can find a one-cochain $u_{ij}(z,t)$ with values in $\mathcal{O}_{X\times\Delta}$ and a two-cochain $c_{ijk}(z,t)$ with values in $\mathbb{Z}_{X\times\Delta}$ such that α is represented by the couple (u_{ij},c_{ijk}). Then we have $u_{ij}+u_{jk}+u_{ki}=c_{ijk}$ on $U_i\cap U_j\cap U_k$. Since $c_{ijk}(z,t)=c_{ijk}(z,0)$, we get that for any t in Δ, $\alpha_{|X\times t}-\alpha_{|X\times 0}$ is represented by the couple $(u_{ij}(z,t)-u_{ij}(z,0),0)$. The one cochain $(u_{ij}(z,t)-u_{ij}(z,0))$ is a one-cocycle with values in $\mathcal{O}_X$ depending holomorphically on the variable t. This proves the result. □

4. Proof of the theorems

Proof of Theorem 1.3

Let us consider the line bundle $\lambda(\mathcal{E},f\times\mathrm{id})$ on $Y\times\Delta$. By Proposition 2.4, its restriction to a slice $Y\times\{t\}$ is $\lambda(\mathcal{E}_t,f)$. Using Lemma 3.3, we obtain that the curve $t\to\lambda(\mathcal{E}_t,f)$ is analytic.

Let us now consider the class α in $\mathrm{H}^2_{\mathrm{D}}(Y\times\Delta,\mathbb{Q}(1))$ defined by

$$\alpha=\mathrm{c}_1^{\mathrm{D}}(\lambda(\mathcal{E},f\times\mathrm{id}))-\big[(f\times\mathrm{id})_*\big(\mathrm{ch}^{\mathrm{D}}(\mathcal{E})\,\mathrm{td}^{\mathrm{D}}(\mathrm{T}_{X\times\Delta/Y\times\Delta})\big)\big]^{(2)}.$$

By Lemma 2.2,

$$\mathrm{c}_1^{\mathrm{H}}(\lambda(\mathcal{E},f\times\mathrm{id}))=\sum_p(-1)^p\,\mathrm{c}_1^{\mathrm{H}}\,[\mathrm{R}^p(f\times\mathrm{id})_*\,\mathcal{E}]$$

in $\mathrm{H}^1(Y\times\Delta,\Omega^1_{Y\times\Delta})$. Therefore, thanks to the Grothendieck-Riemann-Roch theorem in Hodge cohomology [13], α maps to zero in $\mathrm{H}^1(Y\times\Delta,\Omega^1_{Y\times\Delta})$. By Lemma 3.1, we obtain that for all t in Δ, $\alpha_{|Y\times\{t\}}=\alpha_{|Y\times\{0\}}$ in $\mathrm{H}^2_{\mathrm{D}}(Y,\mathbb{Q}(1))$.

It remains to compute $\alpha_{|Y\times\{t\}}$. If we denote by $[\,.\,]_{\mathrm{D}}$ the Deligne cohomology class of an analytic cycle, for any class β in the rational Deligne cohomology ring of $X\times\Delta$, we have

$$\begin{aligned}\{(f\times\mathrm{id})_*\,\beta\}_{|Y\times\{t\}}&=\mathrm{pr}_{1*}\{(f\times\mathrm{id})_*\,\beta\,.\,[Y\times\{t\}]_{\mathrm{D}}\}\\&=\mathrm{pr}_{1*}(f\times\mathrm{id})_*\{\beta\,.\,(f\times\mathrm{id})^*\,[Y\times\{t\}]_{\mathrm{D}}\}\\&=f_*\ \mathrm{pr}_{1*}\{\beta\,.\,[X\times\{t\}]_{\mathrm{D}}\}\\&=f_*(\beta_{|X\times\{t\}}).\end{aligned}$$

Therefore we get $\alpha_{|Y\times\{t\}}=\mathrm{c}_1^{\mathrm{D}}(\lambda(\mathcal{E}_t,f))-[f_*\{\mathrm{ch}^{\mathrm{D}}(\mathcal{E}_t)\,\mathrm{td}^{\mathrm{D}}(\mathrm{T}_{X/Y})\}]^{(2)}$. Remark now that for all t in Δ,

$$\mathrm{c}_1^{\mathrm{D}}(\lambda(\mathcal{E}_t,f))-\mathrm{c}_1^{\mathrm{D}}(\lambda(\mathcal{E}_0,f))=[f_*\{[\mathrm{ch}^{\mathrm{D}}(\mathcal{E}_t)-\mathrm{ch}^{\mathrm{D}}(\mathcal{E}_0)]\,\mathrm{td}^{\mathrm{D}}(\mathrm{T}_{X/Y})\}]^{(2)}.$$

Thus the curve $t\to[f_*\big([\mathrm{ch}^{\mathrm{D}}(\mathcal{E}_t)-\mathrm{ch}^{\mathrm{D}}(\mathcal{E}_0)]\,\mathrm{td}^{\mathrm{D}}(\mathrm{T}_{X/Y})\big)]^{(2)}$ in $\mathrm{H}^2_{\mathrm{D}}(Y,\mathbb{Q}(1))$ can be lifted to the analytic curve $\alpha\colon t\to\lambda(\mathcal{E}_t,f)-\lambda(\mathcal{E}_0,f)$ in $\mathrm{Pic}^0(Y)$. This finishes the proof.

□

Proof of Theorem 1.4

It is a direct consequence of Theorem 1.3. Indeed, since $\mathcal{L}$ lies in $\mathrm{Pic}^0(Y \times F)$, there exists a holomorphic family of holomorphic line bundles joining $\mathcal{O}_{Y\times F}$ to $\mathcal{L}$. Thanks to Theorem 1.3, we are reduced to the case $\mathcal{L} = \mathcal{O}_{Y\times F}$. In this case, Theorem 1.4 is straightforward since

$$\det \mathrm{R}p_*(\mathcal{O}_{Y\times F}) \simeq \det\, [\mathcal{O}_Y \overset{\mathbb{L}}{\otimes}_{\mathbb{C}_Y} \mathrm{R}\Gamma(F, \mathcal{O}_F)] \simeq \mathcal{O}_Y$$

and $p_*[\mathrm{td}^{\mathrm{D}}\,(\mathrm{T}_{Y\times F/Y})] = p_*[1 \boxtimes \mathrm{td}^{\mathrm{D}}\,(F)]$ is concentrated in degree zero.

□

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