Ricci tensor on smooth metric measure space with boundary

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Abstract

The aim of this note is to study the measure-valued Ricci tensor on smooth metric measure space with boundary, which is a generalization of Bakry-Émery's modified Ricci tensor on weighted Riemannian manifold. As an application, we offer a new approach to study curvature-dimension condition of smooth metric measure space with boundary.

Keywords: metric measure space, curvature-dimension condition, boundary, Bakry-Émery theory.

1 Introduction

Let $M = (X, g, e^{-V} \text{Vol}_g)$ be a *n*-dimensional weighted Riemannian manifold (or smooth metric measure space) equipped with a metric tensor $g : [TM]^2 \mapsto C^{\infty}(M)$. The well-known Bakry-Émery's Bochner type formula

$$\Gamma_2(f) = \operatorname{Ricci}(\nabla f, \nabla f) + \operatorname{H}_V(\nabla f, \nabla f) + |\operatorname{H}_f|_{\operatorname{HS}}^2, \tag{1.1}$$

valid for any smooth function f, where $H_V = \nabla^2 V$ is the Hessian of V and $|H_f|_{HS}$ is the Hilbert-Schmidt norm of the Hessian H_f . The operator Γ_2 is defined by

$$\Gamma_2(f) := \frac{1}{2}L\Gamma(f,f) - \Gamma(f,Lf), \qquad \Gamma(f,f) := \frac{1}{2}L(f^2) - fLf$$

where $\Gamma(\cdot,\cdot) = g(\nabla\cdot,\nabla\cdot)$, and $L = \Delta - \nabla V$ is the Witten-Laplacian on M. It is known that $\Gamma_2 \geq K$ could characterize many important geometric and analysis properties of M.

The aim of this paper is to study the Bakry-Émery's Γ_2 -calculus on smooth metric measure space with boundary. It can be seen that smooth metric measure space with boundary is actually a non-smooth space, since the geodesics are not even C^2 in general (see e.g. [1]). Therefore, it will not be more difficult to study this

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problem in an abstract framework. In this paper, we will use the theory of (non-smooth) metric measure space with lower Ricci curvature bound, which was founded by Lott-Sturm-Villani, and systematically studied using different techniques which originally come from differential geometry, metric geometry, probability theory, etc.

We will see that the non-smooth Bochner inequality and the measure-valued Ricci tensor **Ricci**, which are introduced in [14] and [10] have precise representations on weighted Riemannian manifold $(\Omega, d_g, e^{-V} Vol_g)$ with boundary, where d_g is the intrinsic distance on $\Omega \subset X$ induced by the Riemannian metric g:

$$\mathbf{Ricci}_{\Omega} = \operatorname{Ricci}_{V} e^{-V} dVol_{g} + II e^{-V} d\mathcal{H}^{n-1}|_{\partial\Omega}$$
(1.2)

where $Ricci_V = Ricci + H_V$ is the Bakry-Émery Ricci tensor and II is the second fundamental form.

From [4,5] and [10] we know that $(\Omega, d_g, \operatorname{Vol}_g)$ is a $\operatorname{RCD}(K, \infty)$ space, or in other words, the Boltzman entropy is K-displacement convex, if and only if $\operatorname{\mathbf{Ricci}}_{\Omega} \geq K$. By (1.2) we know $\operatorname{\mathbf{Ricci}}_{\Omega} \geq K$ if and only if $\operatorname{Ricci} \geq K$ and $II \geq 0$. Then we immediately know (Ω, d_g) is locally convex if it is $\operatorname{RCD}(K, \infty)$. Even though this result could also be proved by combining the result of Ambrosio-Gigli-Savaré ([3,4]) and Wang (see e.g. Chapter 3, [17]). Our approach here is the first one totally 'inside' the framework of metric measure space.

In this paper, we will review the construction of measure-valued Ricci tensor and give a quick proof to our main formula. Then we end this note with some simple applications. More applications and generalizations will be studied in the future.

2 Measure valued Ricci tensor and application

Let $M := (X, d, \mathfrak{m})$ be a complete, separable geodesic space. We define the local Lipschitz constant $\operatorname{lip}(f) : X \to [0, \infty]$ of a function f by

$$\operatorname{lip}(f)(x) := \begin{cases} \overline{\lim}_{y \to x} \frac{|f(y) - f(x)|}{\operatorname{d}(x,y)}, & x \text{ is not isolated} \\ 0, & \text{otherwise.} \end{cases}$$

We say that $f \in L^2(X, \mathfrak{m})$ is a Sobolev function in $W^{1,2}(M)$ if there exists a sequence of Lipschitz functions functions $\{f_n\} \subset L^2$, such that $f_n \to f$ and $\operatorname{lip}(f_n) \to G$ in L^2 for some $G \in L^2(X, \mathfrak{m})$. It is known that there exists a minimal function G in \mathfrak{m} -a.e. sense. We call this minimal G the minimal weak upper gradient (or weak gradient for simplicity) of the function f, and denote it by |Df|. It is known that the locality holds for |Df|, i.e. $|Df| = |Dg| \mathfrak{m}$ -a.e. on the set $\{x \in X : f(x) = g(x)\}$. If M is a Riemannian manifold, it is known that $|Df|_M = |\nabla f| = \operatorname{lip}(f)$ for any $f \in C^{\infty}$. Furthermore, let $\Omega \subset M$ be a domain such that $\partial \Omega$ is (n-1)-dimensional. Then we know $|Df|_{\Omega} = |\nabla f| \mathfrak{m}$ -a.e. (see Theorem 6.1, [8]). It can also be seen that the weighted measure $e^{-V}\mathfrak{m}$ does not change the value of weak gradients.

We equip $W^{1,2}(X, d, \mathfrak{m})$ with the norm

$$\|f\|_{W^{1,2}(X,\operatorname{d},\mathfrak{m})}^2 := \|f\|_{L^2(X,\mathfrak{m})}^2 + \||\operatorname{D}\! f|\|_{L^2(X,\mathfrak{m})}^2.$$

It is known that $W^{1,2}(X)$ is a Banach space, but not necessarily a Hilbert space. We say that $(X, \mathbf{d}, \mathbf{m})$ is an infinitesimally Hilbertian space if $W^{1,2}$ is a Hilbert space. Obviously, Riemannian manifolds (with or without boundary) are infinitesimally Hilbertian spaces.

On an infinitesimally Hilbertian space M, we have a natural pointwise bilinear map defined by

$$[W^{1,2}(M)]^2 \ni (f,g) \mapsto \langle \nabla f, \nabla g \rangle := \frac{1}{4} \Big(|D(f+g)|^2 - |D(f-g)|^2 \Big).$$

Then we can define the Laplacian by duality.

Definition 2.1 (Measure valued Laplacian, [10, 11]). The space $D(\Delta) \subset W^{1,2}(M)$ is the space of $f \in W^{1,2}(M)$ such that there is a measure μ satisfying

$$\int h \, \mathrm{d}\mu = -\int \langle \nabla h, \nabla f \rangle \, \mathrm{d}\mathfrak{m}, \forall h : M \mapsto \mathbb{R}, \text{ Lipschitz with bounded support.}$$

In this case the measure μ is unique and we shall denote it by Δf . If $\Delta f \ll m$, we denote its density by Δf .

We have the following proposition characterizing the curvature-dimensions conditions $\mathrm{RCD}(K,\infty)$ and $\mathrm{RCD}^*(K,N)$ through non-smooth Bakry-Émery theory. We say that a space is $\mathrm{RCD}(K,\infty)/\mathrm{RCD}^*(K,N)$ if it is a $\mathrm{CD}(K,\infty)/\mathrm{CD}^*(K,N)$ space which are defined by Lott-Sturm-Villani in [13,15,16] and Bacher-Sturm in [6], equipped with an infinitesimally Hilbertian Sobolev space. For more details, see [4] and [2].

We define $\operatorname{TestF}(M) \subset W^{1,2}(M)$, the set of test functions by

$$\operatorname{TestF}(M) := \Big\{ f \in \operatorname{D}(\Delta) \cap L^{\infty} : |\operatorname{D} f| \in L^{\infty} \ \text{ and } \ \Delta f \in W^{1,2}(M) \cap L^{\infty}(M) \Big\}.$$

It is known that $\operatorname{TestF}(M)$ is dense in $W^{1,2}(M)$ when M is $\operatorname{RCD}(K,\infty)$.

Let $f, g \in \text{TestF}(M)$. We know (see [14]) that the measure $\Gamma_2(f, g)$ is well-defined by

$$\Gamma_2(f,g) = \frac{1}{2} \Delta \langle \nabla f, \nabla g \rangle - \frac{1}{2} (\langle \nabla f, \nabla \Delta g \rangle + \langle \nabla g, \nabla \Delta f \rangle) \mathfrak{m},$$

and we put $\Gamma_2(f) := \Gamma_2(f, f)$. Then we have the following Bochner inequality on metric measure space, which can be regarded as variant definitions of $RCD(K, \infty)$ and $RCD^*(K, N)$ conditions.

Proposition 2.2 (Bakry-Émery condition, [4, 5], [9]). Let $M = (X, d, \mathfrak{m})$ be an infinitesimally Hilbertian space satisfying Sobolev-to-Lipschitz property (see [5] or [12] for the definition). Then it is a RCD*(K, N) space with $K \in \mathbb{R}$ and $N \in [1, \infty]$ if and only if

$$\Gamma_2(f) \ge \left(K|\mathrm{D}f|^2 + \frac{1}{N}(\Delta f)^2\right)\mathfrak{m}$$

for any $f \in \text{TestF}(M)$.

Let $f \in \text{TestF}(M)$. We define the Hessian $H_f : {\nabla g : g \in \text{TestF}(M)}^2 \mapsto L^0(M)$ by

$$2H_f(\nabla g, \nabla h) = \langle \nabla g, \nabla \langle \nabla f, \nabla h \rangle \rangle + \langle \nabla h, \nabla \langle \nabla f, \nabla g \rangle \rangle - \langle \nabla f, \nabla \langle \nabla g, \nabla h \rangle \rangle$$

for any $g, h \in \text{TestF}(M)$. Using the estimate obtained in [14], it can be seen that H_f can be extended to a symmetric $L^{\infty}(M)$ -bilinear map on $L^2(TM)$ (see [10] for the definition) and continuous with values in $L^0(M)$, see Theorem 3.3.8 in [10] for a proof. On Riemannian manifolds (with boundary), it can be seen that H_f coincides with the usual Hessian $\nabla^2 f$, \mathfrak{m} -a.e., and the Hilbert-Schimidt norms are also identified.

Furthermore, we have the following proposition.

Proposition 2.3 (See [10]). Let M be an infinitesimally Hilbertian space satisfying Sobolev-to-Lipschitz property. Then M is $RCD(K, \infty)$ if and only if

$$\mathbf{Ricci}(\nabla f, \nabla f) \ge K |\mathrm{D}f|^2 \, \mathfrak{m}$$

for any $f \in \text{TestF}(M)$, where

$$\mathbf{Ricci}(\nabla f, \nabla f) := \Gamma_2(f) - |\mathbf{H}_f|_{\mathrm{HS}}^2 \mathfrak{m}.$$

Now we introduce our main theorem.

Theorem 2.4 (Measure-valued Ricci tensor). Let $M = (X, g, e^{-V} \operatorname{Vol}_g)$ be a n-dimensional weighted Riemannian manifold and $\Omega \subset M$ be a submanifold with (n-1)-dimensional smooth orientable boundary. Then the measure valued Ricci tensor on $(\Omega, d_{\Omega}, e^{-V} \operatorname{Vol}_g)$ can be computed as

$$\mathbf{Ricci}_{\Omega}(\nabla g, \nabla g) = \mathbf{Ricci}_{V}(\nabla g, \nabla g) e^{-V} dVol_{g} + II(\nabla g, \nabla g) e^{-V} d\mathcal{H}^{n-1}|_{\partial\Omega}$$
 (2.1)

for any $g \in C_c^{\infty}$ with $g(N, \nabla g) = 0$, where N is the outwards normal vector field on $\partial \Omega$, and Ricci_V is the usual Bakry-Émery Ricci tensor on M.

Proof. By integration by part formula (or Green's formula) on Riemannian manifold, we know

$$\int g(\nabla f, \nabla g) e^{-V} dVol_{g} = -\int f \Delta_{V} g e^{-V} dVol_{g} + \int_{\partial \Omega} f g(N, \nabla g) e^{-V} d\mathcal{H}^{n-1}|_{\partial \Omega}$$

for any $f, g \in C_c^{\infty}$, where $\Delta_V := (\Delta - \nabla V)$ and N is the outwards normal vector field, $\mathcal{H}^{n-1}|_{\partial\Omega}$ is the (n-1)-dimensional Hausdorff measure on $\partial\Omega$. From the discussions before we know

$$\int \langle \nabla f, \nabla g \rangle_{\Omega} e^{-V} dVol_{g} = -\int f \Delta_{V} g e^{-V} dVol_{g} + \int_{\partial \Omega} f g(N, \nabla g) e^{-V} d\mathcal{H}^{n-1}|_{\partial \Omega}.$$

Therefore we know $g \in D(\Delta_{\Omega})$ and we obtain the following formula concerning the measure-valued Laplacian

$$\Delta_{\Omega} g = \Delta_{V} g e^{-V} dVol_{g} - g(N, \nabla g) e^{-V} d\mathcal{H}^{n-1}|_{\partial \Omega}.$$

Therefore for any $g \in C_c^{\infty}$ with $g(N, \nabla g) = 0$ on $\partial \Omega$, we know $g \in \text{TestF}(\Omega)$.

Now we can compute the measure-valued Bakry-Émery tensor. Let $g \in C_c^{\infty}$ with $g(N, \nabla g) = 0$ on $\partial \Omega$. We have

$$\begin{aligned} \mathbf{Ricci}_{\Omega}(\nabla g, \nabla g) &= \frac{1}{2} \Delta_{\Omega} |\mathrm{D}g|_{\Omega}^{2} - \langle \nabla g, \nabla \Delta_{\Omega} g \rangle_{\Omega} \, e^{-V} \mathrm{dVol}_{\mathrm{g}} - \|\mathrm{Hess}_{g}\|_{\mathrm{HS}}^{2} \, e^{-V} \mathrm{dVol}_{\mathrm{g}} \\ &= \frac{1}{2} \Delta_{V} |\nabla g|^{2} \, e^{-V} \mathrm{dVol}_{\mathrm{g}} - \mathrm{g}(\nabla g, \nabla \Delta_{V} g) \, e^{-V} \mathrm{dVol}_{\mathrm{g}} - \|\mathrm{Hess}_{g}\|_{\mathrm{HS}}^{2} \, e^{-V} \mathrm{dVol}_{\mathrm{g}} \\ &- \frac{1}{2} \mathrm{g}(N, \nabla |\nabla g|^{2}) \, e^{-V} \mathrm{d}\mathcal{H}^{n-1}|_{\partial \Omega} \\ &= \mathrm{Ricci}(\nabla g, \nabla g) \, e^{-V} \mathrm{dVol}_{\mathrm{g}} + \mathrm{H}_{V}(\nabla g, \nabla g) \, e^{-V} \mathrm{Vol}_{\mathrm{g}} \\ &- \frac{1}{2} \mathrm{g}(N, \nabla |\nabla g|^{2}) \, e^{-V} \mathrm{d}\mathcal{H}^{n-1}|_{\partial \Omega} \\ &= \mathrm{Ricci}_{V}(\nabla g, \nabla g) \, e^{-V} \mathrm{dVol}_{\mathrm{g}} - \frac{1}{2} \mathrm{g}(N, \nabla |\nabla g|^{2}) \, e^{-V} \mathrm{d}\mathcal{H}^{n-1}|_{\partial \Omega}, \end{aligned}$$

where we use Bochner formula at the third equality and $Ricci_V = Ricci + H_V$ is the Bakry-Émery Ricci tensor on weighted Riemannian manifold w.r.t the weight e^{-V} .

By definition of second fundamental form, we have

$$II(\nabla g, \nabla g) = g(\nabla_{\nabla g} N, \nabla g) = g(\nabla g(N, \nabla g), \nabla g) - \frac{1}{2}g(N, \nabla |\nabla g|^2).$$

However, we assume that $g(N, \nabla g) = 0$ on $\partial \Omega$. Hence $g(\nabla_{\nabla g} N, \nabla g) = -\frac{1}{2}g(N, \nabla |\nabla g|^2)$. Finally, we obtain

$$\mathbf{Ricci}_{\Omega}(\nabla g, \nabla g) = \mathbf{Ricci}_{V}(\nabla g, \nabla g) \, d\mathbf{Vol}_{g} + II(\nabla g, \nabla g) \, e^{-V} d\mathcal{H}_{n-1|_{\partial\Omega}}$$
(2.2)

for any
$$g \in C_c^{\infty}$$
 with $g(N, \nabla g) = 0$.

In the next corollary we will see that the space $\{g:g\in C_c^{\infty}, g(N,\nabla g)=0\}\subset \operatorname{TestF}(\Omega)$ is big enough to characterize the Ricci curvature and the mean curvature.

Corollary 2.5 (Rigidity: convexity of the boundary). Let $(\Omega, d_{\Omega}, e^{-V} \operatorname{Vol}_{g})$ be a space as in Theorem 2.4. Then it is $\operatorname{RCD}(K, \infty)$ if and only if $\partial\Omega$ is convex and $\operatorname{Ricci}_{V} \geq K$ on Ω .

Proof. If Ω is $\mathrm{RCD}(K,\infty)$, then from Proposition 2.3 we know $\mathrm{Ricci}_{\Omega}(\nabla g, \nabla g) \geq K|\nabla g|^2\mathrm{Vol}_g$ for any $g \in \mathrm{TestF}(\Omega)$. By Theorem 2.4 we know $\mathrm{Ricci}_V(\nabla g, \nabla g) \geq K|\nabla g|^2$ and $II(\nabla g, \nabla g) \geq 0$ for any $g \in C_c^{\infty}$ with $\mathrm{g}(N, \nabla g) = 0$.

On one hand, for any $g \in C_c^{\infty}(\Omega)$ with support inside Ω , we know $g \in \text{TestF}$. Applying Theorem 2.4 with any of these g, we know $\text{Ricci}_V(\nabla g, \nabla g) \geq K|\nabla g|^2$, hence $\text{Ricci}_V \geq K$. On the other hand, for any $g \in C_c^{\infty}(\partial\Omega)$. By Cauchy–Kovalevskaya theorem we know the Cauchy problem:

- 1) f = g on $\partial \Omega$,
- 2) $g(\nabla f, N) = 0$ on $\partial \Omega$

has a local analytical solution \bar{g} . Furthermore, by multiplying an appropriate smooth cut-off function we can assume further that $\bar{g} \in C_c^{\infty}(\Omega)$ and $\bar{g} \in \text{TestF}(\Omega)$. Applying Theorem 2.4 with \bar{g} , we know $II(\nabla g, \nabla g) \geq 0$. Since g is arbitrary, we know $II \geq 0$.

Conversely, if $\partial\Omega$ is convex we know Ω is locally convex in the ambient space X (see e.g. [7]). Combining with $\mathrm{Ricci}_V \geq 0$ we know Ω is locally $\mathrm{RCD}(K, \infty)$. By local to global property of $\mathrm{RCD}(K, \infty)$ condition (see e.g. [15]), we prove the result.

Remark 2.6. In this corollary, we only study the manifolds with boundary which can be regarded as a submanifold with orientable boundary. Since the problem we are considering is local, it is not more restrictive than general case.

Remark 2.7. In [3] Ambrosio-Gigli-Savaré identify the gradient flow of Boltzman entropy with the (Neumann) heat flow. In [4] they prove the exponential contraction of heat flows in Wasserstein distance. Combining the result of Wang (see Theorem 3.3.2 in [17]) we can also prove this result.

Corollary 2.8. A N-dimensional Riemannian manifold with boundary is $RCD(K, \infty)$ if and only if it is $RCD^*(K, N)$.

The next corollary characterize the Ricci-flat space as a metric measure space.

Corollary 2.9. Let M and Ω be as above. Then Ω is a Ricci flat space, i.e. $\mathbf{Ricci}_{\Omega} = 0$, if and only if it is a minimal hypersurface with zero Ricci curvature inside.

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