

STRONG HAAGERUP INEQUALITIES WITH OPERATOR COEFFICIENTS

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ABSTRACT. We prove a Strong Haagerup inequality with operator coefficients. If for an integer d , \mathcal{H}_d denotes the subspace of the von Neumann algebra of a free group F_l spanned by the words of length d in the generators (but not their inverses), then we provide in this paper an explicit upper bound on the norm on $M_n(\mathcal{H}_d)$, which improves and generalizes previous results by Kemp-Speicher (in the scalar case) and Buchholz and Parcet-Pisier (in the non-holomorphic setting). Namely the norm of an element of the form $\sum_{i=(i_1, \dots, i_d)} a_i \otimes \lambda(g_{i_1} \dots g_{i_d})$ is less than $4^5 \sqrt{e}(\|M_0\|^2 + \dots + \|M_d\|^2)^{1/2}$, where M_0, \dots, M_d are $d+1$ different block-matrices naturally constructed from the family $(a_i)_{i \in I^d}$ for each decomposition of $I^d \simeq I^l \times I^{d-l}$ with $l = 0, \dots, d$.

It is also proved that the same inequality holds for the norms in the associated non-commutative L_p spaces when p is an even integer, $p \geq d$ and when the generators of the free group are more generally replaced by $*$ -free \mathcal{R} -diagonal operators. In particular it applies to the case of free circular operators. We also get inequalities for the non-holomorphic case, with a rate of growth of order $d+1$ as for the classical Haagerup inequality. The proof is of combinatorial nature and is based on the definition and study of a symmetrization process for partitions.

INTRODUCTION

Let F_r be the free group on r generators and $|\cdot|$ the length function associated to this set of generators and their inverses. The left regular representation of F_r on $\ell^2(F_r)$ is denoted by λ , and the C^* -algebra generated by $\lambda(F_r)$ is denoted by $C_\lambda^*(F_r)$. In [Haa79] (Lemma 1.4), Haagerup proved the following result, now known as the Haagerup inequality: for any function $f : F_r \rightarrow \mathbb{C}$ supported by the words of length d ,

$$(1) \quad \left\| \sum_{g \in F_r} f(g) \lambda(g) \right\|_{C_\lambda^*(F_r)} \leq (d+1) \|f\|_2.$$

This inequality has many applications and generalizations. It indeed gives a useful criterion for constructing bounded operators in $C_\lambda^*(F_r)$ since it implies in particular that for $f : F_r \rightarrow \mathbb{C}$

$$\left\| \sum_{g \in F_r} f(g) \lambda(g) \right\|_{C_\lambda^*(F_r)} \leq 2 \sqrt{\sum_{g \in F_r} (|g|+1)^4 |f(g)|^2},$$

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and the so-called Sobolev norm $\sqrt{\sum_{g \in F_r} (|g| + 1)^4 |f(g)|^2}$ is much easier to compute than the operator norm of $\lambda(f) = \sum f(g)\lambda(g)$. The groups for which the same kind of inequality holds for some length function (replacing the term $(d + 1)$ in (1) by some power of $(d + 1)$) are called groups with property RD [Jol90] and have been extensively studied; they play for example a role in the proof of the Baum-Connes conjecture for discrete cocompact lattices of $SL_3(\mathbb{R})$ [Laf98].

Another direction in which the Haagerup inequality was studied and extended is the theory of operator spaces. It concerns the same inequality when the function f is allowed to take operator values. This question was first studied by Haagerup and Pisier in [HP93], and the most complete inequality was then proved by Buchholz in [Buc99]. One of its interests is that it gives an explanation of the $(d + 1)$ term in the classical inequality. Indeed, in the operator valued case, the term $(d + 1)\|f\|_2$ is replaced by a sum of $d + 1$ different norms of f (which are all dominated by $\|f\|_2$ when f is scalar valued). More precisely if S denotes the canonical set of generators of F_r and their inverses, a function $f : F_r \rightarrow M_n(\mathbb{C})$ supported by the words of length d can be viewed as a family $(a_{(h_1, \dots, h_d)})_{(h_1, \dots, h_d) \in S^d}$ of matrices indexed by S^d in the following way: $a_{(h_1, \dots, h_d)} = f(h_1 h_2 \dots h_d)$ if $|h_1 \dots h_d| = d$ and $a_{(h_1, \dots, h_d)} = 0$ otherwise.

The family of matrices $a = (a_h)_{h \in S^d}$ can be seen in various natural ways as a bigger matrix, for any decomposition of $S^d \simeq S^l \times S^{d-l}$. If the a_h 's are viewed as operators on a Hilbert space H ($H = \mathbb{C}^n$), then let us denote by M_l the operator from $H \otimes \ell^2(S)^{\otimes d-l}$ to $H \otimes \ell^2(S)^{\otimes l}$ having the following block-matrix decomposition:

$$M_l = (a_{(s,t)})_{s \in S^l, t \in S^{d-l}}.$$

Then the generalization from [Buc99] is

Theorem 0.1 ([Buc99], Theorem 2.8). *Let $f : F_r \rightarrow M_n(\mathbb{C})$ supported by the words of length d and define $(a_h)_{h \in S^d}$ and M_l for $0 \leq l \leq d$ as above. Then*

$$\left\| \sum_{g \in W_d} f(g) \otimes \lambda(g) \right\|_{M_n \otimes C_\lambda^*(F_r)} \leq \sum_{l=0}^d \|M_l\|.$$

The same result has also been extended in [PP05] to the L_p -norms up to constants that are not bounded as $d \rightarrow \infty$. See also [RX06] and [JPX07].

More recently and in the direction of free probability, Kemp and Speicher [KS07] discovered the striking fact that, whereas the constant $(d + 1)$ is optimal in (1), when restricted to (scalar) functions supported by the set W_d^+ of words of length d in the generators g_1, \dots, g_r but *not their inverses* (it is the holomorphic setting in the vocabulary of [Kem05] and [KS07]), this constant $(d + 1)$ can be replaced by a constant of order \sqrt{d} .

Theorem 0.2 ([KS07], Theorem 1.4). *Let $f : F_r \rightarrow \mathbb{C}$ be a function supported on W_d^+ . Then*

$$\left\| \sum_{g \in W_d^+} f(g) \lambda(g) \right\|_{C_\lambda^*(F_r)} \leq \sqrt{e} \sqrt{d+1} \|f\|_2.$$

A similar result has been obtained when the operators $\lambda(g_1), \dots, \lambda(g_r)$ are replaced by free \mathcal{R} -diagonals elements: Theorem 1.3 in [KS07]. These results are

proved using combinatorial methods: to get bounds on operator norms the authors first get bounds for the norms in the non-commutative L_p -spaces for p even integers, and make p tend to infinity. For an even integer, the L_p -norms are expressed in terms of moments and these moments are studied using the free cumulants.

In this paper we generalize and improve these results to the operator-valued case. As for the generalization of the usual Haagerup inequality, the operator valued inequality we get gives an explanation of the term $\sqrt{d+1}$: for operator coefficients this term has to be replaced by the ℓ^2 combination of the norms $\|M_l\|$ introduced above. A precise statement is the following. We state the result for the free group F_∞ on countably many generators $(g_i)_{i \in \mathbb{N}}$, but it of course applies for a free group with finitely many generators.

Theorem 0.3. *For $d \in \mathbb{N}$, denote by $W_d^+ \subset F_\infty$ the set of words of length d in the g_i 's (but not their inverses). For $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ let $g_k = g_{k_1} \dots g_{k_d} \in W_d^+$.*

Let $a = (a_k)_{k \in \mathbb{N}^d}$ be a finitely supported family of matrices, and for $0 \leq l \leq d$ denote by $M_l = (a_{(k_1, \dots, k_l), (k_{l+1}, \dots, k_d)})$ the corresponding $\mathbb{N}^l \times \mathbb{N}^{d-l}$ block-matrix. Then

$$(2) \quad \left\| \sum_{k \in \mathbb{N}^d} a_k \otimes \lambda(g_k) \right\| \leq 4^5 \sqrt{e} \left(\sum_{l=0}^d \|M_l\|^2 \right)^{1/2}.$$

Note that even when $a_k \in \mathbb{C}$, this really is (up to the constant 4^5) an improvement of Theorem 0.2. Indeed it is always true that for any l , $\|M_l\|^2 \leq \text{Tr}(M_l^* M_l) = \sum_k |a_k|^2$. There is equality when $l = 0$ or d but the inequality is in general strict when $0 < l < d$. Thus if the a_k 's are scalars such that $\|(a_k)\|_2 = 1$ and $\|M_l\| \leq 1/\sqrt{d}$ for $0 < l < d$, the inequality in Theorem 0.3 becomes $\|\sum_{k \in \mathbb{N}^d} a_k \lambda(g_k)\| \leq 4^5 \sqrt{3e} \|(a_k)\|_2$. Since the reverse inequality $\|\sum_{k \in \mathbb{N}^d} a_k \lambda(g_k)\| \geq \|(a_k)\|_2$ always holds, we thus get that $\|\sum_{k \in \mathbb{N}^d} a_k \lambda(g_k)\| \simeq \|(a_k)\|_2$ with constants independent of d . An example of such a family is given by the following construction: if p is a prime number and $a_{k_1, \dots, k_d} = \exp(2i\pi k_1 \dots k_d/p)/p^{d/2}$ for any $k_i \in \{1, \dots, p\}$ and $a_k = 0$ otherwise then obviously $\sum_k |a_k|^2 = 1$, whereas a computation (see Lemma 3.5) shows that $\|M_l\|^2 \leq d/p$ if $0 < l < d$. It is thus enough to take $p \geq d^2$.

As in [KS07], the same arguments apply for the more general setting of \mathcal{R} -diagonal elements (\ast -free means that the C^* -algebras generated are free). Moreover we get significant results already for the L_p -norms for p even integers. Recall that on a C^* -algebra \mathcal{A} equipped with a trace τ , the p -norm of an element $x \in \mathcal{A}$ is defined by $\|x\|_p = \tau(|x|^p)^{1/p}$ for $1 \leq p < \infty$, and that for $p = \infty$ the L^∞ norm is just the operator norm. In the following the algebra $M_n \otimes \mathcal{A}$ will be equipped with the trace $\text{Tr} \otimes \tau$. The most general statement we get is thus:

Theorem 0.4. *Let c be an \mathcal{R} -diagonal operator and $(c_k)_{k \in \mathbb{N}}$ a family of \ast -free copies of c on a tracial C^* -probability space (\mathcal{A}, τ) . Let $(a_k)_{k \in \mathbb{N}^d}$ be as above a finitely supported family of matrices and $M_l = (a_{(k_1, \dots, k_l), (k_{l+1}, \dots, k_d)})$ for $0 \leq l \leq d$ the corresponding $\mathbb{N}^l \times \mathbb{N}^{d-l}$ block-matrix.*

For $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ denote $c_k = c_{k_1} \dots c_{k_d}$.

Then for any integer m ,

$$(3) \quad \left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m} \leq 4^5 \|c\|_2^{d-2} \|c\|_{2m}^2 e \sqrt{1 + \frac{d}{m}} \left(\sum_{l=0}^d \|M_l\|_{2m}^2 \right)^{1/2}.$$

For the operator norm,

$$(4) \quad \left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\| \leq 4^5 \|c\|_2^{d-2} \|c\|^2 \sqrt{e} \left(\sum_{l=0}^d \|M_l\|^2 \right)^{1/2}.$$

When the c_k 's are circular these inequalities are valid without the factor $4^5 \|c\|_2^{d-2} \|c\|^2$.

The outline of the proof of Theorem 0.4 is the same as the proof of Theorem 1.3 in [KS07]: we first prove the statement for the L_p -norms when $p = 2m$ is an even integer (letting $p \rightarrow \infty$ leads to the statement for the operator norm). This is done with the use of free cumulants that express moments in terms of non-crossing partitions (the definition of non-crossing partitions is recalled in part 1.2). More precisely to any integer n , any non-crossing partition π of the set $\{1, \dots, n\}$ and any family $b_1, \dots, b_n \in \mathcal{A}$ the free cumulant $\kappa_\pi[b_1, \dots, b_n] \in \mathbb{C}$ is defined (see [NS06] for a detailed introduction). When $\pi = 1_n$ is the partition into only one block, κ_π is denoted by κ_n . The free cumulants have the following properties:

- *Multiplicativity*: If $\pi = \{V_1, \dots, V_s\}$, $\kappa_\pi[b_1, \dots, b_n] = \prod_i \kappa_{|V_i|}[(b_k)_{k \in V_i}]$.
- *Moment-cumulant formula*: $\tau(b_1, \dots, b_n) = \sum_{\pi \in NC(n)} \kappa_\pi[b_1, \dots, b_n]$.
- *Characterization of freeness*: A family $(\mathcal{A}_i)_i$ of subalgebras is free iff all mixed cumulants vanish, i.e. for any n , any $b_k \in \mathcal{A}_{i_k}$ and any $\pi \in NC(n)$ then $\kappa_\pi[b_1, \dots, b_n] = 0$ unless $i_k = i_l$ for any k and l in a same block of π .

The first two properties characterize the free cumulants (and hence could be taken as a definition), whereas the third one motivates their use in free probability theory. Since the $*$ -distribution of an operator $c \in (\mathcal{A}, \tau)$ is characterized by the trace of the polynomials in c and c^* , the cumulants involving only c and c^* (that is the cumulants $\kappa_\pi[(b_i)]$ with $b_i \in \{c, c^*\}$ for any i) depend only on the $*$ -distribution of c .

In order to motivate the combinatorial study of certain non-crossing partitions in the first section, let us shortly sketch the proof of the main result. For details, see part 3.1. With the notation of Theorem 0.4 let $A = \sum a_k \otimes c_k$. For $k = (k(1), \dots, k(d)) \in \mathbb{N}^d$ set $\tilde{a}_k = a_{(k(d), \dots, k(1))}$ and $\tilde{c}_k = c_{k(d)} \dots c_{k(1)}$ so that $(\tilde{c}_k)^* = c_{k(1)}^* \dots c_{k(d)}^*$. Then $A^* = \sum_k \tilde{a}_k^* \otimes \tilde{c}_k^*$, and for $p = 2m$ the p -th power of the p -norm of A is just the trace $Tr \otimes \tau$ of $(AA^*)^m$, which can be expressed by linearity as the sum of the terms of the form $Tr(a_{k_1} \tilde{a}_{k_2}^* \dots a_{k_{2m-1}} \tilde{a}_{k_{2m}}^*) \otimes \tau(c_{k_1} \tilde{c}_{k_2}^* \dots c_{k_{2m-1}} \tilde{c}_{k_{2m}}^*)$. The expression $c_{k_1} \tilde{c}_{k_2}^* \dots c_{k_{2m-1}} \tilde{c}_{k_{2m}}^*$ is the product of $2dm$ terms of the form c_i or c_i^* (for $i \in \mathbb{N}$). Apply the moment-cumulant formula to its trace. Using the characterization of freeness with cumulants and then the multiplicativity of cumulants and the fact that cumulants only depend on the $*$ -distribution we thus get

$$\left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi \in NC(2dm)} \kappa_\pi[c_{d,m}] \underbrace{\sum_{(k_1, \dots, k_{2m}) \prec \pi} Tr(a_{k_1} \tilde{a}_{k_2}^* \dots \tilde{a}_{k_{2m}}^*)}_{\stackrel{\text{def}}{=} S(a, \pi, d, m)},$$

where for $k \in \mathbb{N}^{2dm}$ and $\pi \in NC(2dm)$ we write $k \prec \pi$ if $k_i = k_j$ whenever i and j belong to the same block of π and where

$$c_{d,m} = \overbrace{\underbrace{c, \dots, c}_d, \underbrace{c^*, \dots, c^*}_d, \dots, \underbrace{c, \dots, c}_d, \underbrace{c^*, \dots, c^*}_d}_{2m \text{ groups}}.$$

Up to this point we did not use the assumption that c is \mathcal{R} -diagonal. But as in [KS07], since the \mathcal{R} -diagonal operators are those operators for which the list of non-zero cumulants is very short (see part 3.1 for details), we get that the previous sum can be restricted to a sum over the partitions in the subset $NC^*(d, m) \subset NC(2dm)$, which is defined and extensively studied in part 1.2:

$$(5) \quad \left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi \in NC^*(d, m)} \kappa_\pi[c_{d,m}] S(a, \pi, d, m).$$

The term $\kappa_\pi[c_{d,m}]$ is easy to dominate (Lemma 3.1). When the a_k 's are scalars the second term $S(a, \pi, d, m)$ can be dominated by $\|(a_k)\|_{\ell_2}^{2m}$ (by the usual Cauchy-Schwarz inequality). This is what is done in the proof of [KS07]. But here the fact that we are dealing with operators and not scalars forces to derive a more sophisticated Cauchy-Schwarz type inequality that may control explicitly the expressions $S(a, \pi, d, m)$ in terms of norms of the operators M_l . This is one of the main technical results in this paper, Corollary 2.4. This Corollary states that

$$(6) \quad |S(a, \pi, d, m)| \leq \prod_{l=0}^d \|M_l\|_{2m}^{2m\mu_l}$$

for some non-negative μ_l with $\sum_l \mu_l = 1$. Moreover the μ_l are explicitly described by some combinatorial properties of π . This inequality is proved through a process of “symmetrization” of partitions. The basic observation is that if one applies a simple Cauchy-Schwarz inequality to $S(a, \pi, d, m)$ (Lemma 2.1), this corresponds on the level of partitions to a certain combinatorial operation of symmetrization that is studied in the part 1.1. This observation was already used implicitly in [Buc01], Lemma 2, in some special case: Buchholz indeed notices that for $d = 1$ and if π is a pairing (*i.e.* has blocks of size 2), this Cauchy-Schwarz inequality corresponds to some transformation of pairings (for which he does not give a combinatorial description), and that iterating this inequality eventually leads to an domination of the form (6) (for $d = 1$) but in which he does not compute the exponents μ_0 and μ_1 .

In our more general setting it also appears that repeating this operation in an appropriate way turns every non-crossing partition $\pi \in NC^*(d, m)$ into one very simple and fully symmetric partition for which the expression $S(a, \pi, d, m)$ is exactly the $(2m)$ -power of the $(2m)$ -norm of one of the M_l 's. This is stated and proved in Corollary 1.4 and Lemma 2.2. One important feature of our study of the symmetrization operation on $NC^*(d, m)$ is the fact that we are able to determine some combinatorial invariants of this operation (see part 1.3). This allows to keep track of the exponents of the $\|M_l\|_{2m}$ that progressively appear during the symmetrization process, and to compute the coefficients μ_l in (6).

The second technical result that we prove and use is a finer study of $NC^*(d, m)$. The main conclusion is Theorem 1.5 which expresses that partitions in $NC^*(d, m)$

have mainly blocks of size 2 and that $NC^*(d, m)$ is not very far from the set $NC(m)^{(d)}$ of non-decreasing chains of non-crossing partitions on m (in the sense that there is a natural surjection $NC^*(d, m) \rightarrow NC(m)^{(d)}$ such that the fiber of any point has a cardinality dominated by a term not depending on d). This combinatorial result is then generalized in Theorem 1.13 and Lemma 1.14, and then used to transform the sum in (5) into a sum over $NC(m)^{(d)}$ for which the combinatorics are well known by [Ede80].

We prove also the following results, which are extensions to the non-holomorphic case of the previous results and their proofs. Let c be an \mathcal{R} -diagonal operator and $(c_k)_{k \in \mathbb{N}}$ a family of $*$ -free copies of c on a tracial C^* -probability space (\mathcal{A}, τ) . For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{1, *\}^d$ and $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ denote $c_{k, \varepsilon} = c_{k_1}^{\varepsilon_1} \dots c_{k_d}^{\varepsilon_d}$. The result is an extension of Haagerup's inequality for the space generated by the $c_{k, \varepsilon}$ for the k, ε satisfying $k_i = k_{i+1} \Rightarrow \varepsilon_i = \varepsilon_{i+1}$, i.e. for which $\lambda(g)_{k, \varepsilon}$ has length d . Denote by I_d the set of such (k, ε) .

Theorem 0.5. *Let $(a_{(k, \varepsilon)})_{(k, \varepsilon) \in (\mathbb{N} \times \{1, *\})^d}$ be a finitely supported family such that $a_{(k, \varepsilon)} = 0$ for $(k, \varepsilon) \notin I_d$. For $0 \leq l \leq d$, let M_l be the matrix formed as above from $(a_{(k, \varepsilon)})$ for the decomposition $(\mathbb{N} \times \{1, *\})^d = (\mathbb{N} \times \{1, *\})^l \times (\mathbb{N} \times \{1, *\})^{d-l}$. Then for any $p \in 2\mathbb{N} \cup \{\infty\}$*

$$\left\| \sum_{(k, \varepsilon) \in (\mathbb{N} \times \{1, \varepsilon\})^d} a_{k, \varepsilon} \otimes c_{k, \varepsilon} \right\|_p \leq 4^5 \|c\|_p^2 \|c\|_2^{d-2} (d+1) \max_{0 \leq l \leq d} \|M_l\|_p.$$

Similarly for self-adjoint operators we have:

Theorem 0.6. *Let μ be a symmetric compactly supported probability measure on \mathbb{R} , and c a self-adjoint element of a tracial C^* -algebra distributed as μ .*

Let $(c_k)_{k \in \mathbb{N}}$ be self-adjoint free copies of c and $(a_{k_1, \dots, k_d})_{k_1, \dots, k_d \in \mathbb{N}}$ be a finitely supported family of matrices such that $a_{k_1, \dots, k_d} = 0$ if $k_i = k_{i+1}$ for some $1 \leq i < d$. Then for any $p \in 2\mathbb{N} \cup \{\infty\}$

$$(7) \quad \left\| \sum_{(k_1, \dots, k_d) \in \mathbb{N}^d} a_{k_1, \dots, k_d} \otimes c_{k_1} \dots c_{k_d} \right\|_p \leq 4^5 \|c\|_p^2 \|c\|_2^{d-2} (d+1) \max_{0 \leq l \leq d} \|M_l\|_p.$$

For the case of the semicircular law and scalar coefficient a_k , this result is not new. It is due to Bożejko [Boż91], and was reproved using combinatorial methods by Biane and Speicher, Theorem 5.3.4 of [BS98]. Our proof is a generalization of their proof and uses it. Note also that the condition that $a_{k_1, \dots, k_d} = 0$ if $k_i = k_{i+1}$ for some i is crucial to get (7): indeed if $a_{k_1, \dots, k_d} = 0$ except for $a_{1, \dots, 1} = 1$ then we have the equality $\|\sum_{k \in \mathbb{N}^d} a_k \otimes c_{k_1} \dots c_{k_d}\|_p = \|c_1^d\|_p = \|c\|_{dp}^d$, whereas $\max_l \|M_l\|_p = 1$ and if μ is not a Bernoulli measure $\|c\|_p^2 \|c\|_2^{d-2} (d+1) = o(\|c\|_{dp}^d)$ when $d \rightarrow \infty$. The inequality (7) thus does not hold for this choice of (a_k) , even up to a constant.

These results are of some interest since they prove a new version of Haagerup's inequality in a broader setting, but they are still unsatisfactory since one would expect to be able to replace the term $(d+1) \max_{0 \leq l \leq d} \|M_l\|$ by $\sum_{l=0}^d \|M_l\|$.

The paper is organized as follows: the first part only deals with combinatorics of non-crossing partitions. In the second part we use the results of the first part to get inequalities for the expressions $S(a, \pi, d, m)$. In the third and last part we finally prove the main results stated above.

Although some definitions are recalled, the reader will be assumed to be familiar with the basics of free probability theory and more precisely to its combinatorial aspect (non-crossing partitions, free cumulants, \mathcal{R} -diagonal operators...). For more on this see [NS06]. For the vocabulary of non-commutative L^p spaces nothing more than the definitions of the p -norm, the Cauchy-Schwarz inequality $|\tau(ab)| \leq \|a\|_2 \|b\|_2$ and the fact that $\|x\|_\infty = \lim_{p \rightarrow \infty} \|x\|_p$ will be used.

1. SYMMETRIZATION OF NON-CROSSING PARTITIONS

For any integer n , we denote by $[n]$ the interval $\{1, 2, \dots, n\}$, which we identify with $\mathbb{Z}/n\mathbb{Z}$ and which is endowed with the natural cyclic order: for $k_1, \dots, k_p \in [n]$ we say that $k_1 < k_2 < \dots < k_p$ for the cyclic order if there are integers l_1, \dots, l_p such that $l_1 < l_2 < \dots < l_p$, $k_i = l_i \bmod n$ and $l_p - l_1 \leq n$. In other words, if the elements of $[n]$ are represented on the vertices of a regular polygon with n vertices labelled by elements of $[n]$ as in Figure 1, then we say that $k_1 < k_2 < \dots < k_p$ if the sequence k_1, \dots, k_p can be read on the vertices of the regular polygon by following the circle clockwise for at most one full circle.

If π is a partition of $[n]$, and $i \in [n]$, the element of the partition π to which i belongs is denoted by $\pi(i)$. We also write $i \sim_\pi j$ if i and j belong to the same block of the partition π .

If the elements of $[n]$ are represented on the vertices of a regular polygon with n vertices, a partition π of $[n]$ is then represented on the regular polygon by drawing a path between i and j if $i \sim_\pi j$. See Figure 1 for an example.

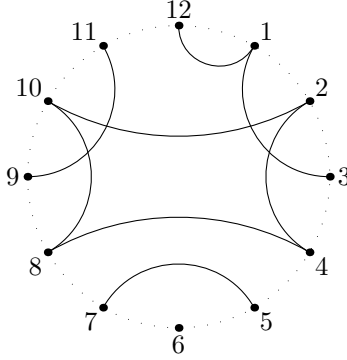


FIGURE 1. A graphical representation of the partition $\{\{1, 3, 12\}, \{2, 4, 8, 10\}, \{5, 7\}, \{6\}, \{9, 11\}\}$.

1.1. Definitions and first observation. We introduce the operations P_k on the set of partitions of an even number $n = 2N$. This definition is motivated by Lemma 2.1.

Definition 1.1. Let $k \in [2N]$, and I_k the subinterval of $[2N]$ of length N and ending with k , $I_k = \{k - N + 1, k - N + 2, \dots, k\}$ and $s_k^{(N)}$ (or simply s_k when no confusion is possible) the symmetry $s_k(i) = 2k + 1 - i$ (note that s_k is an involution of $[2N]$ that exchanges I_k and $[2N] \setminus I_k$). For a partition π of $[2N]$, $s_k(\pi)$ is the symmetric of π : $A \in s_k(\pi)$ if $s_k^{-1}(A) = s_k(A) \in \pi$. In other words $i \sim_{s_k(\pi)} j$ if and only if $s_k(i) \sim_\pi s_k(j)$.

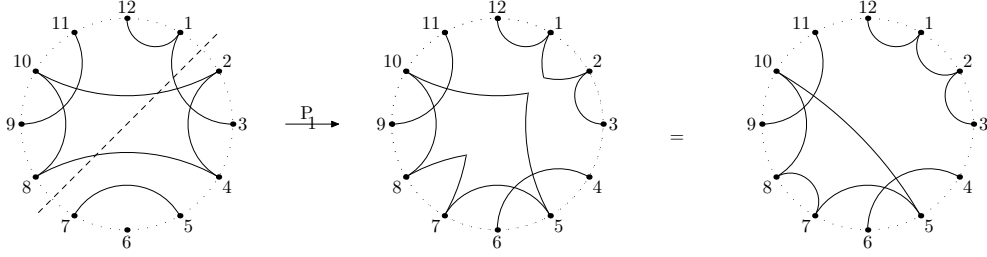


FIGURE 2. The operation P_1 on the partition $\{\{1, 3, 12\}, \{2, 4, 8, 10\}, \{5, 7\}, \{6\}, \{9, 11\}\}$.

For any partition π of $[2N]$, we denote by $P_k(\pi)$ the partition of $[2N]$ that we view as a symmetrization of π around k , and which is formally defined by the following: if one denotes $\pi' = P_k(\pi)$, then

- (8) for $i, j \in I_k$ $i \sim_{\pi'} j$ if and only if $i \sim_{\pi} j$
- (9) for $i, j \in [2N] \setminus I_k$ $i \sim_{\pi'} j$ if and only if $s_k(i) \sim_{\pi} s_k(j)$
- (10) for $i \in I_k, j \notin I_k$ $i \sim_{\pi'} j$ if and only if $i \sim_{\pi} s_k(j)$ and $\exists l \notin I_k, i \sim_{\pi} l$.

It is straightforward to check that this indeed defines a partition of $[2N]$, and that it is symmetric with respect to k , that is $s_k(\pi') = \pi'$.

The operation P_k is perhaps more easily described graphically: represent π on a regular polygon as above, and draw the symmetry line going through the middle of the segment $[k, k+1]$. A graphical representation of $P_k(\pi)$ is then obtained by erasing all the half-polygon not containing k and replacing it by the mirror-image of the half-polygon containing k . See Figure 2 for an example.

The following lemma expresses the fact that applying sufficiently many times appropriate operators P_k , one can make a partition symmetric with respect to all the s_k 's. See Figure 3 to see an example of this symmetrization process.

Lemma 1.1. *Let m be a positive integer.*

Let $k \in \mathbb{N}$ such that $2^k \geq m$. Then for any partition π of $[2m]$, the partition $\pi_k = P_{2^k} P_{2^{k-1}} \dots P_2 P_1 P_m(\pi)$ is one of the four following partitions:

$$\begin{aligned} \pi_k = 0_{2m} &= \{\{j\}, j \in [2m]\} \\ \pi_k = c_m &= \{\{2j; 2j+1\}, j \in [m]\} \\ \pi_k = r_m &= \{\{2j-1; 2j\}, j \in [m]\} \\ \pi_k = 1_{2m} &= \{[2m]\} \end{aligned}$$

Proof. Let $A = I_m \cap \pi(1) \setminus \{1\}$ and $B = ([2m] \setminus I_m) \cap \pi(1)$. The four cases correspond respectively to the four following cases:

- (1) $A = B = \emptyset$.
- (2) $A = \emptyset$ and $B \neq \emptyset$.
- (3) $A \neq \emptyset$ and $B = \emptyset$.
- (4) $A \neq \emptyset$ and $B \neq \emptyset$.

In the first case, it is straightforward to prove by induction on k that π_k includes the blocks $\{i\}$ for any $i \in \{1, \dots, 2^{k+1}\}$.

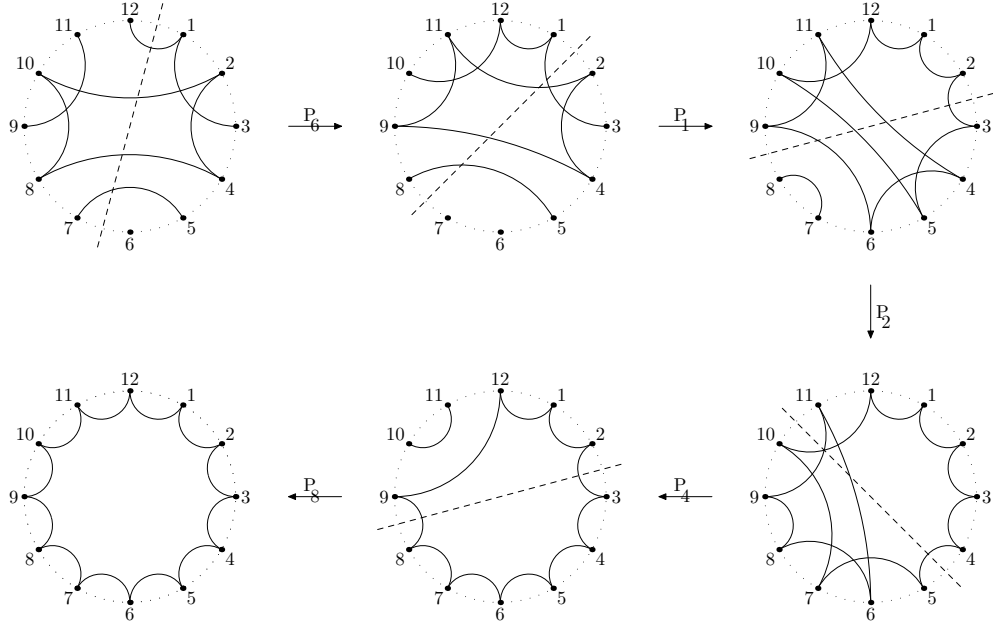


FIGURE 3. The symmetrization process starting from the partition $\{\{1, 3, 12\}, \{2, 4, 8, 10\}, \{5, 7\}, \{6\}, \{9, 11\}\}$.

If $A = \emptyset$ and $B \neq \emptyset$, then $P_m(\pi)$ includes the block $\{0, 1\}$ and this implies that $P_1 P_m(\pi)$ includes the blocks $\{0, 1\}$ and $\{2, 3\}$, which in turn implies that $P_2 P_1 P_m(\pi)$ includes the blocks $\{0, 1\}, \{2, 3\}$ and $\{4, 5\}$... More generally π_k includes the blocks $\{0, 1\}, \{2, 3\}, \dots, \{2^{k+1}, 2^{k+1} + 1\}$ (this can be proved by induction). For $2^{k+1} \geq 2m$ this is exactly $\pi_k = c_m$. We leave the details to the reader.

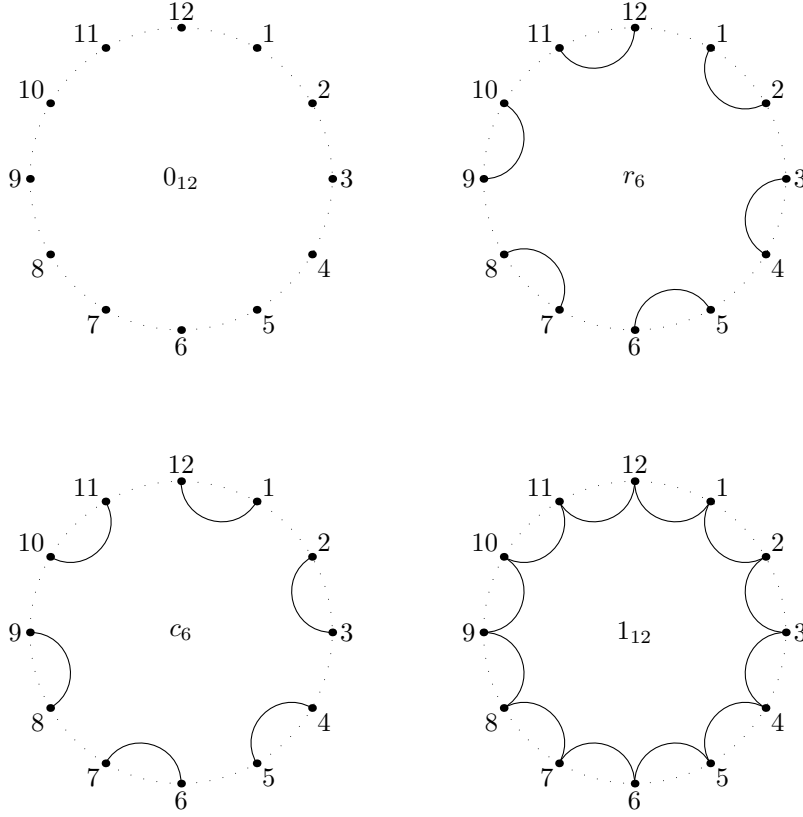
In the same way, in the third case it is easy to prove by induction on k that π_k includes the blocks $\{2l - 1, 2l\}$ for $l \in \{1, \dots, 2^k\}$.

The fourth case follows from a similar proof by induction that $\pi_k(1)$ contains $\{0, 1, 2, \dots, 2^{k+1} + 1\}$. The details are not provided. \square

Although $P_k(\pi)$ is defined for any partition π , we will be mainly interested in the case when π is a non-crossing partition, and more precisely when $\pi \in NC^*(d, m)$.

1.2. Study of $NC^*(d, m)$. We first recall the definition of a non-crossing partition. A partition π of $[N]$ is called non-crossing if for any distinct $i < j < k < l \in [N]$, $i \sim_\pi k$ and $j \sim_\pi l$ implies $i \sim_\pi j$ (in this definition either take for $<$ the usual order on $\{1, \dots, N\}$ or the cyclic order since it gives to the same notion). More intuitively π is non-crossing if and only if there is a graphical representation of π (on a regular polygon with n vertices as explained in the beginning of section 1) such that the paths lie inside the polygon and only intersect (possibly) at the vertices of the regular polygon. For example the partitions of Figures 1, 2 are crossing, whereas the partitions in Figures 4, 5, 6 are all non-crossing. The set of non-crossing partitions of $[N]$ is denoted by $NC(N)$. The cardinality of $NC(N)$ is known to be equal to the Catalan number $(2N)!/(N!(N+1)!)$ (see [Kre72]), but we will only use that it is less than 4^{N-1} .

Following [KS07], we introduce the subset $NC^*(d, m)$ of $NC(2dm)$.

FIGURE 4. The partitions 0_{12} , r_6 , c_6 and 1_{12} .

In the following, for a real number x one denotes by $\lfloor x \rfloor$ the biggest integer smaller than or equal to x .

Divide the set $[2dm]$ into $2m$ intervals $J_1 \dots J_{2m}$ of size d : the first one is $J_1 = \{1, 2, \dots, d\}$, and the k -th is $J_k = \{(k-1)d + 1, \dots, kd\}$.

To each element of $[2dm]$ we assign a label in $\{1, \dots, d\}$ in the following way: in any interval J_k of size d as above, the elements are labelled from 1 to d if k is odd and from d to 1 if k is even. We shall denote by A_k the set of elements labelled by k .

Definition 1.2. A non-crossing partition π of $[2dm]$ belongs to $NC^*(d, m)$ if each block of the partition has an even cardinality, and if within each block, two consecutive elements i and j belong to intervals of size d of different parity. Formally, the last condition means that $\lfloor (i-1)/d \rfloor \neq \lfloor (j-1)/d \rfloor \pmod 2$ or equivalently $k(i) \neq k(j) \pmod 2$ when $i \in J_{k(i)}$ and $j \in J_{k(j)}$.

Here are some first elementary properties of $NC^*(d, m)$:

Lemma 1.2. *If $d = 1$, a non-crossing partition $\pi \in NC(2m)$ belongs to $NC^*(1, m)$ if and only if it has blocks of even cardinality.*

A non-crossing partition of $[2dm]$ is in $NC^*(d, m)$ if and only if it has blocks of even cardinality and it connects only elements with the same labels (i.e. it is finer than the partition $\{A_1, \dots, A_d\}$).

Proof. The first statement is a particular case of the second statement, which we now prove. For any $i \in [2dm]$ denote by $k(i)$ the integer such that $i \in J_{k(i)}$: $k(i) = 1 + \lfloor (i-1)/d \rfloor$. Let $\pi \in NC^*(d, m)$. Then by the definition of $NC^*(d, m)$ every block of π contains as many elements i such that $k(i)$ is odd than elements i such that $k(i)$ is even. We have to prove that if s and t are two consecutive elements of a block of π , then s and t have the same labellings. Assume for example that s belongs to an odd interval, i.e. $k(s)$ is odd, and denote by $l(s)$ the label of s . Then $s = (k(s)-1)d + l(s)$. In the same way, $k(t)$ is then even and if $l(t)$ is the label of t , we have that $t = k(t)d + 1 - l(t)$. Hence the number of elements $i \in \{s+1, \dots, t-1\}$ such that $k(i) = 1 + \lfloor (i-1)/d \rfloor$ is odd is equal to $d - l(s) + d \cdot (k(t) - k(s) - 1)/2$, and the number of elements i such that $k(i)$ is even is equal to $d - l(t) + d \cdot (k(t) - k(s) - 1)/2$. But since π is non-crossing, the interval $\{s+1, \dots, t-1\}$ is a union of blocks of π and therefore contains as many elements i such that $k(i)$ is odd than elements i such that $k(i)$ is even. This implies $l(s) = l(t)$. The proof is the same if $k(s)$ is even.

Now assume that $\pi \in NC(dm)$ has blocks of even cardinality and that π is finer than the partition $\{A_1, \dots, A_d\}$. Let s and t be two consecutive elements of a block of π . Then there is i such that $s, t \in A_i$. Since π is non-crossing and π is finer than $\{A_1, \dots, A_d\}$, the set $\{s+1, \dots, t-1\} \cap A_i$ is a union of blocks of π , and in particular it has an even cardinality. But $\{s+1, \dots, t-1\} \cap A_i$ is the set of elements labelled by i in the union of the intervals J_k for $k(s) < k < k(t)$ (for the cyclic order). Hence its cardinality is $k(t) - k(s) - 1$. Hence $k(t) - k(s)$ is odd. Since s and t are arbitrary, this proves that $\pi \in NC^*(d, m)$. \square

Thus to any $\pi \in NC^*(d, m)$ we can assign d partitions $\pi|_{A_1}, \dots, \pi|_{A_d}$, which are the restrictions of π to A_1, \dots, A_d respectively. It is immediate that for any $i \in \{1, \dots, d\}$, $\pi|_{A_i} \in NC^*(1, m)$. See Figure 5 for an example. To study $NC^*(d, m)$, we thus begin with the study of $NC^*(1, m)$.

The first lemma shows that if k is a multiple of d , then P_k maps $NC^*(d, m)$ into itself:

Lemma 1.3. *If $k \in \mathbb{N}$ and $\pi \in NC(2N)$ then $P_k(\pi) \in NC(2N)$.*

If $k \in \mathbb{N}$ then for any $\pi \in NC^(d, m)$, $P_{kd}(\pi) \in NC^*(d, m)$.*

Moreover if $\pi \in NC^(d, m)$, then for any $i \in \{1, \dots, d\}$:*

$$P_{kd}(\pi)|_{A_i} = P_k(\pi|_{A_i}).$$

Sketch of Proof. The first statement is obvious from the graphical point of view: if there are no crossing, the symmetrization map will not produce one.

The second statement follows from the characterization of Lemma 1.2: it is not hard to check that if π has blocks of even cardinality then $P_{kd}(\pi)$ also has. The fact that $P_{kd}(\pi)$ is finer than $\{A_1, \dots, A_d\}$ if π is follows from the fact that $s_{kd}(A_i) = A_i$ for any k and $1 \leq i \leq d$.

The third statement follows from the fact that $s_{kd}^{(dm)}$ is characterized by the properties that for any $1 \leq j \leq 2m$, $s_{kd}^{(dm)}(J_i) = J_{s_k^{(m)}(i)}$ and $s_{kd}^{(dm)}(A_i) = A_i$ for $1 \leq i \leq d$. \square

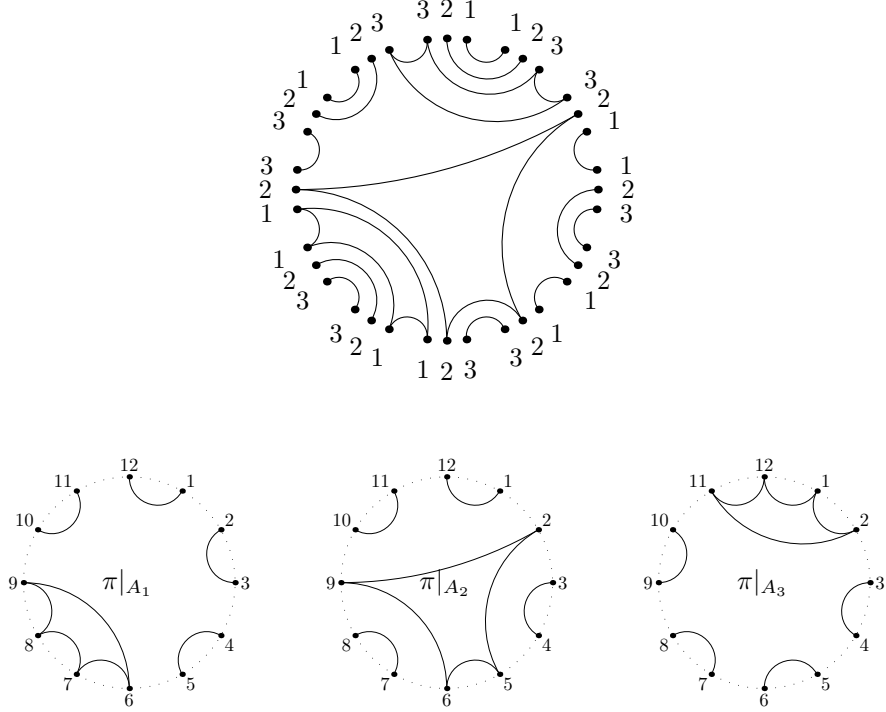


FIGURE 5. A graphical representation of a partition π in $NC^*(3, 6)$ and the corresponding restrictions $\pi|_{A_1}$, $\pi|_{A_2}$ and $\pi|_{A_3}$.

We have the following corollary of Lemma 1.1.

Corollary 1.4. *Let $\pi \in NC^*(d, m)$. Then for $2^k \geq m$, the partition $\pi_k = P_{2^k d} P_{2^{k-1} d} \dots P_{2d} P_d P_m(\pi)$ is one of the $2d + 1$ partitions $\sigma_l^{(d, m)}$ for $l = 0, 1, \dots, d$ and $\tilde{\sigma}_l^{(d, m)}$ for $l = 1, 2, \dots, d$ defined by:*

$$\sigma_l^{(d, m)}|_{A_i} = \begin{cases} c_m & \text{if } 1 \leq i \leq l \\ r_m & \text{if } l < i \leq d, \end{cases}$$

$$\tilde{\sigma}_l^{(d, m)}|_{A_i} = \begin{cases} c_m & \text{if } 1 \leq i < l \\ 1_{2m} & \text{if } i = l \\ r_m & \text{if } l < i \leq d. \end{cases}$$

Moreover for any integer i , $P_{id}(\pi) = \pi$ when π is one of the partitions $\sigma_l^{(d, m)}$ for $l = 0, 1, \dots, d$ and $\tilde{\sigma}_l^{(d, m)}$ for $l = 1, 2, \dots, d$.

Proof. Let k and π as above. By Lemma 1.3, $\pi_k|_{A_i} = P_{2^k} P_{2^{k-1}} \dots P_2 P_1 P_m(\pi|_{A_i})$, which is by Lemma 1.1 one of 0_{2m} , r_m , c_m and 1_{2m} . But since 0_{2m} does not have blocks of even sizes, only the three r_m , c_m and 1_{2m} are possible.

Let $1 \leq i < j \leq d$. If $\pi_k|_{A_i} = r_m$ or 1_{2m} then in particular $i \sim_{\pi_k} 2d + 1 - i$. Since π_k is non-crossing, $j \approx_{\pi_k} 1 - j$, which implies that $\pi_k|_{A_j} \neq c_m, 1_{2m}$. Thus $\pi_k|_{A_j} = r_m$. In the same way if $\pi_k|_{A_j} = c_m$ or 1_{2m} then $\pi_k|_{A_i} = c_m$. This concludes the proof.

Similarly, the second claim follows from the fact (easy to verify) that $P_i(\pi) = \pi$ for any $i \in [2m]$ when $\pi = 1_{2m}$, r_m or c_m . \square

An important subset of $NC^*(d, m)$ is the subset $NC_2^*(d, m)$ of partitions in $NC^*(d, m)$ with blocks of size 2. As explained in part 3.1 of [KS07], $NC_2^*(d, m)$ is naturally in bijection with the non-decreasing chains (for the natural lattice structure on $NC(m)$) of length d of non-crossing partitions of $[m]$. Let us denote by $NC(m)^{(d)}$ this set of non-decreasing chains in $NC(m)$, for the order of refinement, given by $\pi \leq \pi'$ if π' is finer than π . The bijective map $NC_2^*(d, m) \rightarrow NC(m)^{(d)}$ extends naturally to a (of course non-bijective) map $NC^*(d, m) \rightarrow NC(m)^{(d)}$ which is of interest. We now describe the construction of this map.

Let $\pi \in NC^*(1, m)$, that is a non-crossing partition of $[2m]$ with blocks of even size. Then $\Phi(\pi)$ is the partition of $[m]$ defined by the fact that $\sim_{\Phi(\pi)}$ is the transitive closure of the relation that relates k and l if $2k \sim_{\pi} 2l$ or $2k-1 \sim_{\pi} 2l$ or $2k \sim_{\pi} 2l-1$ or $2k-1 \sim_{\pi} 2l-1$. That is $\Phi(\pi)$ is the partition obtained by identifying the $2k-1$ and $2k$ in $[2m]$ to get $k \in [m]$.

If $\pi \in NC^*(d, m)$, we define the map \mathcal{P} by $\mathcal{P}(\pi) = (\Phi(\pi|_{A_1}), \dots, \Phi(\pi|_{A_d}))$. See Figure 6.

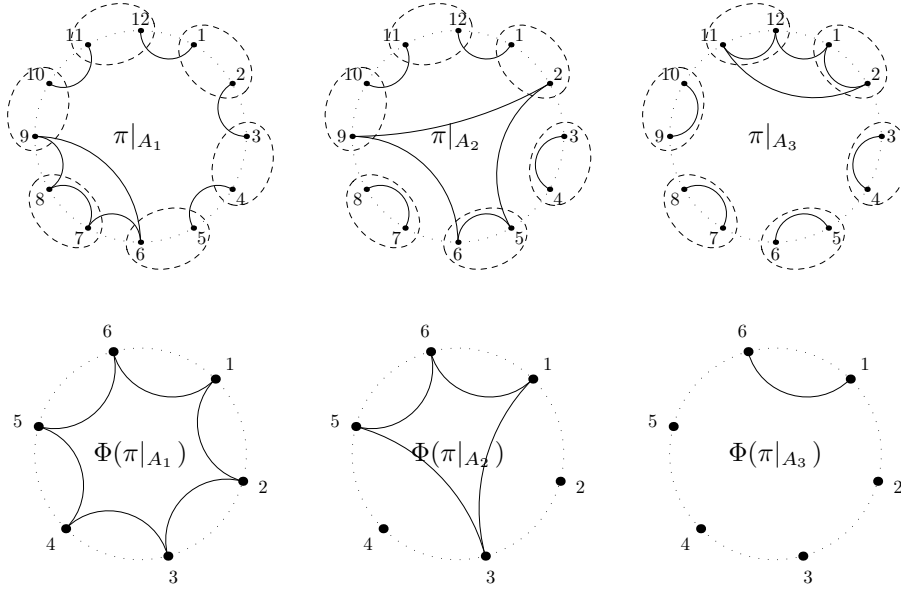


FIGURE 6. The map \mathcal{P} for the partition $\pi \in NC^*(3, 6)$ of Figure 5.

The map \mathcal{P} is a good tool to make a finer study of $NC^*(d, m)$.

The main result in this section is that partitions in $NC^*(d, m)$ are not far from belonging to $NC_2^*(d, m)$:

Theorem 1.5. *For any $\sigma \in NC_2^*(d, m)$ there are less than 4^{2m} partitions $\pi \in NC^*(d, m)$ such that $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$.*

Moreover for such a π , the partition σ is finer than π and the number of blocks of π of size 2 is greater than $dm - 2m$, and every block has size at most $2m$.

Remark. The remarkable feature of $NC^*(d, m)$ illustrated in this Theorem is that the bounds we get on the number of $\pi \in NC^*(d, m)$ such that $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$ and on the elements of $[2dm]$ that do not belong to a block of size 2 of $\pi \in NC^*(d, m)$ do not depend on d .

In particular since the cardinality of $NC_2^*(d, m)$ is equal to the Fuss-Catalan number $1/m \binom{m(d+1)}{m-1}$ which is less than $e^m(d+1)^m$ (Corollary 3.2 in [KS07]) the first statement of the Theorem implies that the cardinality of $NC^*(d, m)$ is less than $(16e(d+1))^m$.

This Theorem will follow from a series of lemmas. Here is the first one, which treats the case $d = 1$:

Lemma 1.6. *Let $\sigma \in NC_2^*(1, m)$ and $\pi \in NC^*(1, m)$ such that $\Phi(\pi) = \Phi(\sigma)$. Then σ is finer than π .*

More precisely if $\pi \in NC^(1, m)$ and if $\{k_1 < k_2 \dots < k_p\}$ is a block of $\Phi(\pi)$, then for any i , $2k_i \sim_\pi 2k_{i+1} - 1$ (with the convention $k_{p+1} = k_1$).*

Proof. The first statement follows easily from the second one. We thus focus on the second statement. At least as far as partitions in $NC_2^*(1, m)$ are concerned, this is explained in the discussion preceding Corollary 3.2 in [KS07]. The proof is the same for a general $\pi \in NC^*(1, m)$, but for completeness we still provide a proof.

It is clear that $\Phi(\pi)(k) = \{k\}$ implies that $2k \sim_\pi 2k - 1$. Thus to prove the statement we have to prove that if k and l are consecutive and distinct elements of a block of $\Phi(\pi)$ then $2k \sim_\pi 2l - 1$.

The first element in $\pi(2k)$ after $2k$ is odd, that is of the form $2p - 1$, because $2k$ is even and the parity alternates in blocks of π . We claim that $p = l$. Note that we necessarily have $k < l \leq p$ (again for the cyclic order) because $k \sim_{\Phi(\pi)} p$. Suppose that $k < l < p$. We get to a contradiction: indeed since $l \sim_{\Phi(\pi)} k$ and $\{2l - 1, 2l\} \subset \{2k + 1, 2k + 2, \dots, 2p - 2\}$ there is at least one $j \in \{2k + 1, 2k + 2, \dots, 2p - 2\}$ and $i \in \{2p - 1, 2p, \dots, 2k\}$ such that $i \sim_\pi j$. But by definition of p , $j \approx_\pi 2k$ and $j \approx_\pi 2p - 1$. This contradicts the fact that π is non-crossing. \square

We can now check that \mathcal{P} is well-defined:

Lemma 1.7. *The map \mathcal{P} from $NC^*(d, m)$ takes values in $NC(m)^{(d)}$.*

Proof. Let $\pi \in NC^*(d, m)$; we have to prove that if $1 \leq i < j \leq d$ then $\Phi(\pi|_{A_j})$ is finer than $\Phi(\pi|_{A_i})$.

Let $\{k_1 < k_2 \dots < k_p\}$ be a block of $\Phi(\pi|_{A_i})$. Suppose that $\Phi(\pi|_{A_j})(k_1) \not\subseteq \{k_1, k_2 \dots k_p\}$. Then there exist $1 \leq s \leq p$ and $l \notin \{k_1, k_2 \dots k_p\}$ such that k_s and l are consecutive elements of $\Phi(\pi|_{A_j})(k_1)$ (for the cyclic order). If $1 \leq t \leq p$ is such that $k_t < l < k_{t+1}$ (with again the convention $k_{p+1} = k_1$), we have by Lemma 1.6 that $2k_t \sim_{\pi|_{A_i}} 2k_{t+1} - 1$ and $2k_s \sim_{\pi|_{A_j}} 2l - 1$, which contradicts the fact that π is non-crossing. This shows that $\Phi(\pi|_{A_j})(k_1) \subseteq \{k_1, k_2 \dots k_p\} = \Phi(\pi|_{A_i})(k_1)$. Since k_1 was arbitrary, the proof is complete. \square

Here is a last elementary lemma concerning general non-crossing partitions:

Lemma 1.8. *Let $N \in \mathbb{N}$ and $\pi \in NC(N)$ with α blocks. Then the number of $k \in [N]$ such that $k \sim_\pi k + 1$ is greater or equal to $N - 2(\alpha - 1)$.*

Proof. For $\pi \in NC(N)$, let us denote by $c(\pi)$ the number of $k \in [N]$ such that $k \sim_\pi k+1$. We prove by induction on α that if $\pi \in NC(N)$ has α blocks, then $c(\pi) \geq N - 2(\alpha - 1)$. If $\alpha = 1$, this is clear since $c(\pi) = N$.

Assume that the statement of the lemma is true for all N and all $\pi \in NC(N)$ with α blocks. Take $\pi \in NC(N)$ with $\alpha + 1$ blocks. Since π is non-crossing there is a block of π , say A , which is an interval of size S . If $\pi|_{[N] \setminus A}$ is regarded as an element of $NC(N - S)$ then $c(\pi) \geq S - 1 + c(\pi|_{[N] \setminus A}) - 1$. By the induction hypothesis $c(\pi|_{[N] \setminus A}) \geq N - S - 2(\alpha - 1)$, which implies $c(\pi) \geq N - 2\alpha$ and thus concludes the proof. \square

The next Lemma is the main result of this section, and Theorem 1.5 will easily follow from it:

Lemma 1.9. *Let $\sigma \in NC_2^*(d, m)$. Then there is a subset A of $[2dm]$ of size greater than $2dm - 4m$, which is a union of blocks of σ , and such that for any $\pi \in NC^*(d, m)$ with $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$ and any $k \in A$, $\pi(k) = \sigma(k)$.*

Proof. For any $1 \leq j \leq d$, denote by $\sigma_j = \Phi(\sigma|_{A_j})$. Denote by $\sigma_{d+1} = 0_m$. Fix now $1 \leq i \leq d$ and $\{k_1 < k_2 < \dots < k_p\}$ a block of σ_i . As usual we take the convention that $k_{p+1} = k_1$. We claim that if $k_s \sim_{\sigma_{i+1}} k_{s+1}$ then for any $\pi \in NC^*(d, m)$ with $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$, $\pi(2dk_s - i + 1) = \{2dk_s - i + 1, 2dk_{s+1} - 2d + i\}$ ($= \sigma(2dk_s - i + 1)$) by Lemma 1.6).

Let us first check that this claim implies the Lemma. By Lemma 1.7, σ_{i+1} is finer than σ_i and in particular its restriction to $\{k_1, k_2, \dots, k_p\}$ makes sense. By Lemma 1.8, the number of s 's in $\{1, \dots, p\}$ such that $k_s \sim_{\sigma_{i+1}} k_{s+1}$ is greater than $p - 2(|\sigma_{i+1}|_{\{k_1, k_2, \dots, k_p\}} - 1)$ where $|\sigma|$ is the number of blocks of σ . Thus summing over all blocks of σ_i we get at least $2m - 4(|\sigma_{i+1}| - |\sigma_i|)$ elements k in A_i such that $\pi(k) = \sigma(k)$ for any $\pi \in NC^*(d, m)$ with $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$. This allows to conclude the proof since

$$\sum_{i=1}^d (2m - 4(|\sigma_{i+1}| - |\sigma_i|)) = 2md - 4|\sigma_{d+1}| + 4|\sigma_1| > 2md - 4m.$$

Note that A is constructed as a union of blocks of σ .

We now only have to prove the claim. Assume that $k_s \sim_{\sigma_{i+1}} k_{s+1}$ and take $\pi \in NC^*(d, m)$ such that $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$. By Lemma 1.6 applied to $\Phi(\sigma|_{A_i}) = \sigma_i$, $2dk_s - i + 1 \sim_\pi 2dk_{s+1} - 2d + i$. Thus we only have to prove that if $k_s \sim_{\sigma_{i+1}} k_{s+1}$ there is no $k \in \{k_1, \dots, k_p\} \setminus \{k_{s+1}\}$ such that $2dk_s - i + 1 \sim_\pi 2dk - 2d + i$.

But if $k_s \sim_{\sigma_{i+1}} k_{s+1}$ then $i \neq d$ (because $\sigma_{d+1} = 0_m$) and by Lemma 1.7, k_s and k_{s+1} are consecutive elements in $\sigma_{i+1}(k_s)$. Thus by Lemma 1.6, $2dk_s - i \sim_\pi 2dk_{s+1} - 2d + i + 1$. The condition that π is non-crossing implies the claim since for $k \in \{k_1, \dots, k_p\} \setminus \{k_{s+1}\}$,

$$2dk_s - i < 2dk_s - i + 1 < 2dk_{s+1} - 2d + i + 1 < 2dk - 2d + i,$$

that is $(2dk_s - i + 1, 2dk - 2d + i)$ and $(2dk_s - i, 2dk_{s+1} - 2d + i + 1)$ are crossing. \square

We can now prove the Theorem.

Proof of Theorem 1.5. Let $\sigma \in NC_2^*(d, m)$. If $\pi \in NC^*(d, m)$ satisfies $\mathcal{P}(\pi) = \sigma$ then Lemma 1.6 applied to $\sigma|_{A_i}$ and $\pi|_{A_i}$ for $i = 1, \dots, d$ proves that σ is finer than π , and 1.9 implies that π has at least $dm - 2m$ blocks of size 2. The fact that

every block of π has size at most m just follows from the definition of $NC^*(d, m)$: π is indeed finer than $\{A_1, \dots, A_d\}$ with $|A_j| = 2m$.

We now prove the first statement of Theorem 1.5. Let A be the subset of $[2dm]$ given by Lemma 1.9. Then there is an injection:

$$\begin{aligned} \{\pi \in NC^*(d, m), \mathcal{P}(\pi) = \mathcal{P}(\sigma)\} &\rightarrow NC([2dm] \setminus A) \\ \pi &\mapsto \pi|_{[2dm] \setminus A} \end{aligned}$$

In particular since there are less than 4^N non-crossing partitions on $[N]$, the first statement of the Theorem follows with 4^{2m} replaced by 4^{4m} because $[2dm] \setminus A$ has cardinality less than $4m$. To get the 4^{2m} just replace $[2dm] \setminus A$ by a set B that contains exactly one element of $\sigma(k)$ for any $k \in [2dm] \setminus A$. Then B has cardinality less than $2m$ because $[2dm] \setminus A$ is a union of blocks (=pairs) of σ , and the previous map is still an injection since $\pi \in NC^*(d, m)$ and $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$ implies that σ is finer than π . \square

1.3. Invariant of the P_k 's. Motivated by Lemma 2.1 we are interested in invariants of the operations P_{kd} on $NC^*(d, m)$. For $\pi \in NC^*(1, m)$ let $B(\pi)$ be the number of blocks in $\Phi(\pi)$. This is the fundamental observation:

Lemma 1.10. *For any $\pi \in NC^*(1, m)$,*

$$B(\pi) = \frac{1}{2} (B(P_k(\pi)) + B(P_{k+m}(\pi))).$$

This Lemma is a consequence of the following description, which proves that for any k , the set of blocks of $\Phi(\pi)$ but one is in bijection with the set of blocks of π that do not contain k and that begin with an odd element (after k for the cyclic order):

Lemma 1.11. *Let $k \in [2m]$ and $\pi \in NC^*(1, m)$. Then $B(\pi) - 1$ is equal to the number of $l \in [2m] \setminus \{k\}$ such that l is odd and such that for any $l' \sim_\pi l$, $l \leq l' < k$ (for the cyclic order).*

Proof. Indeed the set of odd l 's different from k such that $l' \sim_\pi l \Rightarrow l \leq l' < k$ (for the cyclic order) is in bijection with the blocks of $\Phi(\pi)$ that do not contain $\lfloor (k+1)/2 \rfloor$.

The direct map consists in mapping to any such l the block $\Phi(\pi)(\lfloor (l+1)/2 \rfloor)$ and the reverse map gives to any block A of $\Phi(\pi)$ not containing $\lfloor (k+1)/2 \rfloor$ the smallest l greater than k (again for the cyclic order) such that $\lfloor (l+1)/2 \rfloor \in A$. The reader can check using Lemma 1.6 that these maps are indeed inverses of each other. \square

Proof of Lemma 1.10. We use Lemma 1.11 with $k+1$ instead of k . For any $\pi \in NC^*(1, m)$ we denote by $F(\pi, k)$ the set of odd $l \in [2m] \setminus \{k+1\}$ such that $l' \sim_\pi l \Rightarrow l \leq l' < k+1$. We know that $|F(\pi, k)| = B(\pi) - 1$. Moreover let us decompose $F(\pi, k)$ as the disjoint union of $F_1(\pi, k)$ and $F_2(\pi, k)$ defined by: $l \in F_1(\pi, k)$ if and only if $l \in F(\pi, k)$ and $\pi(l) \subset I_{k+m}$; and $F_2(\pi, k)$ is the set of $l \in F(\pi, k)$ such that $\pi(l) \cap I_l \neq \emptyset$.

If $l \in I_{k+m}$ then $l \in F(P_{k+m}(\pi), k)$ if and only if $l \in F(\pi, k)$ because if $k+1 \leq l' < l$, then $l' \sim_{P_{k+m}(\pi)} l$ if and only if $l' \sim_\pi l$.

Take now $l \notin I_{k+m}$. By definition of $F(\cdot, k)$, l is in $F(P_{k+m}(\pi), k)$ if and only if l is odd and l is the first element (after $k+1$ for the cyclic order) of a block of $P_{k+m}(\pi)$ contained in I_k , which is equivalent to the fact that $s_k(l) = 2k+1-l$ is

even and is the last element of a block of π contained in I_{k+m} . Such a block then has first element odd, and thus belongs to $F_1(\pi, k)$ except if it is equal to $k+1$. To summarize, we have thus proved that

$$(11) \quad |F(P_{k+m}(\pi), k)| = |F(\pi, k) \cap I_{k+m}| + |F_1(\pi, k)| + 1$$

if $k+1$ is odd and $\pi(k+1) \subset I_{k+m} = \{k+1, k+2, \dots, k+m\}$, and

$$(12) \quad |F(P_{k+m}(\pi), k)| = |F(\pi, k) \cap I_{k+m}| + |F_1(\pi, k)|$$

otherwise.

We now compute $|F(P_k(\pi), k)|$. If $l \in I_k$ then as above $l \in F(P_k(\pi), k)$ if and only if $l \in F(\pi, k)$. If $l \notin I_k$ then $l \in F(P_k(\pi), k)$ if and only if l is odd and l is the first element strictly after $k+1$ (in the cyclic order) of a block of $P_k(\pi)$ not containing $k+1$. By construction of $P_k(\pi)$ this is equivalent to the fact that $s_k(l) = 2k+1-l$ is even, belongs to I_k , is different from k and is the last element before k in a block of π . The first element (strictly after k in the cyclic order) of such a block is then in $F_2(\pi, k)$ except if it is equal to $k+1$. Reciprocally, if l' is the last element of a block containing an element of $F_2(\pi, k)$ then $l' = s_k(l') \in F(P_k(\pi), k)$ except if $l' = k$. The same is true if $\pi(k+1) \not\subset I_{k+m}$, $k+1$ is odd and if l' denotes the last element in $\pi(k+1)$. Thus

$$\begin{aligned} |F(P_k(\pi), k)| &= |F(\pi, k) \cap I_k| + |F_2(\pi, k)| - 1_{k \text{ is even}} + 1_{k \text{ is even and } \pi(k+1) \not\subset I_{k+m}} \\ &= |F(\pi, k) \cap I_k| + |F_2(\pi, k)| - 1_{k \text{ is even and } \pi(k+1) \subset I_{k+m}}. \end{aligned}$$

Summing this last equality with (11) or (12) yields

$$\begin{aligned} |F(P_k(\pi), k)| + |F(P_{k+m}(\pi), k)| &= \\ |F(\pi, k) \cap I_k| + |F_2(\pi, k)| + |F(\pi, k) \cap I_{k+m}| + |F_1(\pi, k)| &= 2|F(\pi, k)|. \end{aligned}$$

This concludes the proof since by Lemma 1.11 for any $\sigma \in NC^*(1, m)$, $|F(\sigma, k)| = B(\sigma) - 1$. \square

1.4. Study of $NC(d, m)$. Another relevant subset of $NC(2dm)$ is the set $NC(d, m)$ of partitions π with blocks of even cardinality and that connect only elements of different intervals J_k . In other words for all $i, j \in [2dm]$, $i \sim_\pi j$ if $i, j \in J_k$.

The following observation is very simple but, in view of Theorem 0.5 or 0.6, it is the motivation for the introduction of $NC(d, m)$:

Lemma 1.12. *Let $\pi \in NC(2dm)$ with blocks of even cardinality. Then $\pi \in NC(d, m)$ if and only if π does not connect two consecutive elements of a same subinterval J_i . In other words, $i \sim_\pi i+1$ only if i is a multiple of d .*

Proof. The *only if* part of the proof is obvious. The converse follows from the fact that a non-crossing partition always contains an interval (if π is non-crossing with blocks of even size, and $s < t \in J_i$ with $s \sim_\pi t$ and $t \neq s+1$, apply this fact to $\pi|_{\{s, s+1, \dots, t-1\}}$). \square

The purpose of this section is to generalize Theorem 1.5. Namely we prove

Theorem 1.13. *The cardinality of $NC(d, m)$ is less than $(4d+4)^{2m}$.*

Moreover for any $\pi \in NC(d, m)$ the number of blocks of π of size 2 is greater than $(d-2)m$.

The idea of the proof is similar to the proof of Theorem 1.5: we try to reduce to the subset of $NC(d, m)$ consisting of partitions into pairs. For this we introduce the map $Q = Q^{(N)}$ from the set of non-crossing partitions of $[2N]$ into blocks of even sizes to the set of non-crossing partitions of $[2N]$ into pairs. The map Q has the property that if $\pi \in NC(2N)$ has blocks of even sizes, then $Q(\pi)$ is finer than π and any block $\{k_1, \dots, k_{2p}\}$ of π with $1 \leq k_1 < \dots < k_{2p} \leq 2N$ becomes p blocks of $Q(\pi)$, namely $\{k_1, k_2\}, \dots, \{k_{2p-1}, k_{2p}\}$. It is straightforward to check that this indeed defines a non-crossing partition of $[2N]$ into pairs. Note that unlike in the rest of the paper here the element $1 \in [2N]$ plays a specific role in the definition of Q and we abandon the cyclic symmetry of $[2N]$. But this has the advantage to allow to define an order relation on the set of pairs of elements of $[2N]$: we will say that a pair (i, j) covers a pair (k, l) if $1 \leq i < k < l < j \leq 2N$.

A noteworthy property of Q is that if $\sigma = Q(\pi)$ then two blocks (=pairs) of σ cannot be contained in the same block of π if one covers the other. In other words if $1 \leq i < k < l < j \leq 2N$ with $i \sim_\sigma j$ and $k \sim_\sigma l$ then $i \sim_\pi k$.

Following the notation of section 3.1 in [KS07], the image $Q(NC(d, m))$ is denoted by $\mathcal{J}(d, m)$; it is the set of partitions of π into pairs that do not connect elements of a same subinterval J_k for $k = 1, \dots, 2m$. We are not aware of any nice combinatorial description of $\mathcal{J}(d, m)$ as for $NC_2^*(d, m)$, but a precise bound for its cardinality is known: by the proof of Theorem 5.3.4 in [BS98], the cardinality of $\mathcal{J}(d, m)$ is equal to $\tau(T_d(s)^{2m})$ where T_d is the d -th Tchebycheff polynomial and s is a semicircular element of variance 1 in a tracial C^* -algebra (A, τ) . In particular since $\|T_d(s)\| = d + 1$ we have that $|\mathcal{J}(d, m)| \leq (d + 1)^{2m}$. Theorem 1.13 will thus follow from the following more general statement:

Lemma 1.14. *Suppose that $[2N]$ is divided into k non-empty intervals S_1, \dots, S_k and let σ be a non-crossing partition of $[2N]$ into pairs that do not connect elements of a same subinterval S_i . Then there are at most 4^{k-2} non-crossing partitions π of $[2N]$ that do not connect elements of a same subinterval S_i and such that $Q(\pi) = \sigma$. Moreover for such a π there are at most $2k - 4$ elements $i \in [2N]$ for which $\pi(i)$ is not a pair.*

Proof. We prove this statement by induction on N . For simplicity of notation we will assume that the intervals S_1, \dots, S_k are ordered, i.e. that if $i \in S_s$ and $j \in S_t$ with $s < t$ then $i < j$.

If $N = 1$ and σ is as above then $\sigma = 1_2$, $k = 2$, and there is only one $\pi \in NC(2)$ with $Q(\pi) = \sigma$. This proves the assertion for $N = 1$.

Assume that the above statement holds for $1, 2, \dots, N - 1$ and take σ as above. Consider the set $\{\{s_i, t_i\}, i = 1 \dots p\}$ of outermost blocks (=pairs) of σ , i.e. the set of pairs of σ that are not being covered by another block of σ . If we order the s_i 's and t_i 's so that $s_i < t_i$ and $s_i < s_{i+1}$ then we have that $s_1 = 1$, $s_{i+1} = t_i + 1$ and $t_p = 2N$.

By the property of Q mentioned above, a partition $\pi \in NC(2N)$ that does not connect elements of the same interval S_j (for $j = 1, \dots, k$) satisfies $Q(\pi) = \sigma$ if and only if the following properties are satisfied:

- For any $1 \leq i \leq p$, $\{s_i + 1, \dots, t_i - 1\}$ is a union of blocks of π , the non-crossing partition $\pi|_{\{s_i+1, \dots, t_i-1\}}$ does not connect elements of the same subinterval $S_j \cap \{s_i + 1, \dots, t_i - 1\}$ for $j = 1, \dots, k$, and $Q(\pi|_{\{s_i+1, \dots, t_i-1\}}) = \sigma|_{\{s_i+1, \dots, t_i-1\}}$.

- Any block of $\pi|_{\{s_1, t_1, s_2, t_2, \dots, s_p, t_p\}}$ is a union of pairs $\{s_i, t_i\}$ and does not contain 2 elements of a same interval S_j .

Define $k_+(i)$ and $k_-(i)$ for $1 \leq i \leq p$ by $s_i \in S_{k_-(i)}$ and $t_i \in S_{k_+(i)}$. Then for any $1 \leq i \leq p$, $k_-(i) < k_+(i)$ and for $i < p$, $k_+(i) \leq k_-(i+1)$.

Since $\{s_i + 1, \dots, t_i - 1\}$ intersects at most $k_+(i) - k_-(i) + 1$ different intervals S_j , we have by the induction hypothesis that the number of non-crossing partitions of $\{s_i + 1, \dots, t_i - 1\}$ that satisfy the first point above is at most $4^{k_+(i) - k_-(i) - 1}$, and for such a partition at most $2(k_+(i) - k_-(i) - 1)$ elements of $\{s_i + 1, \dots, t_i - 1\}$ do not belong to a pair.

Moreover the set of non-crossing partitions of $\{s_1, t_1, s_2, t_2, \dots, s_p, t_p\}$ that satisfy the second point is in bijection with the set of non-crossing partitions of $\{s_i, i = 1 \dots p\}$ such that $s_i \sim s_{i+1}$ if $k_+(i) = k_-(i+1)$. Its cardinality is in particular less than (or equals) the number of non-crossing partitions of $[p]$, which is less than 4^{p-1} . Therefore the total number of non-crossing partitions π of $[2N]$ that do not connect elements of a same subinterval S_j and such that $Q(\pi) = \sigma$ is less than

$$4^{p-1} \prod_{i=1}^p 4^{k_+(i) - k_-(i) - 1} \leq 4^{k-2}.$$

We used the inequality $\sum_{i=1}^p k_+(i) - k_-(i) - 1 \leq k - 1 - p$.

To prove that for such a π at most $2k - 4$ elements of $[2N]$ do not belong to a pair of π , note that for an element $j \in [2N]$ the block $\pi(j)$ is not a pair either if $j \in \{s_1, t_1, \dots, s_p, t_p\}$ or if j belongs to a block of $\pi|_{\{s_{i+1}, \dots, t_i - 1\}}$ which is not a pair for some $1 \leq i \leq p$. If $k_+(i) < k_-(i+1)$ for some i then we are done since $2p + \sum_{i=1}^p 2k_+(i) - 2k_-(i) - 2 \leq 2k - 4$. To conclude the proof we thus have to check that if $k_+(i) = k_-(i+1)$ for any $1 \leq i < p$ then there are at least 2 elements of $\{s_1, t_1, \dots, s_p, t_p\}$ that belong to a pair of π . But this amounts to showing that a non-crossing partition of $[p]$ such that $i \sim i+1$ for any $1 \leq i < p$ contains at least one singleton, which is clear. \square

The following Lemma is also an easy extension of Lemma 1.1. Remember that the partitions $\sigma_l^{(d,m)}$ and $\tilde{\sigma}_l^{(d,m)}$ are defined in Corollary 1.4:

Lemma 1.15. *Fix integers d and m .*

For any $k \in [2m]$ and $\pi \in NC(d, m)$ the partition $P_{kd}(\pi)$ also belongs to $NC(d, m)$.

Let $k \in \mathbb{N}$ such that $2^k \geq m$. Then for any partition $\pi \in NC(d, m)$, the partition $\pi_k = P_{2^k} P_{2^{k-1}} \dots P_2 P_1 P_m(\pi)$ is one of the $2d + 1$ partitions $\sigma_l^{(d,m)}$ for $0 \leq l \leq d$ or $\tilde{\sigma}_l^{(d,m)}$ for $1 \leq l \leq d$.

Proof. The first point is straightforward.

The proof of the second point is the same as Lemma 1.1: depending on the fact that $\{1, 2, \dots, dm\} \cap \pi(i) \setminus \{i\}$ and $\{dm + 1, \dots, 2dm\} \cap \pi(i)$ are empty or not for $i = 1, \dots, d$, we prove by induction on k that π_k has the right properties. The details are left to the reader. \square

2. INEQUALITIES

For any partition π of $[2N]$, and any $k = (k_1, \dots, k_{2N}) \in \mathbb{N}^{2N}$, we write $k \prec \pi$ if for any $i, j \in [2N]$ such that $i \sim_\pi j$, $k_i = k_j$.

Let $a = (a_k)_{k \in \mathbb{N}^N}$ be a finitely supported family of matrices. For any $k = (k_1, \dots, k_N) \in \mathbb{N}^N$ let $\tilde{a}_k = a_{(k_N, k_{N-1}, \dots, k_1)}$.

For such a and for a partition π of $[2N]$, we denote by $S(a, \pi, N, 1)$ the following quantity:

$$(13) \quad S(a, \pi, N, 1) = \sum_{k, l \in \mathbb{N}^N, (k, l) \prec \pi} \text{Tr}(a_k \tilde{a}_l^*).$$

More generally for integers m, d , for a finitely supported family of matrices $a = (a_k)_{k \in \mathbb{N}^d}$ and a partition π of $[2dm]$, we define

$$(14) \quad S(a, \pi, d, m) = \sum_{k_1, \dots, k_{2m} \in \mathbb{N}^d, (k_1, \dots, k_{2m}) \prec \pi} \text{Tr}(a_{k_1} \tilde{a}_{k_2}^* a_{k_3} \dots a_{k_{2m-1}} \tilde{a}_{k_{2m}}^*).$$

In this equation and in the rest of the paper an element $k = (k_1, \dots, k_{2m}) \in (\mathbb{N}^d)^{2m}$ is identified with an element of \mathbb{N}^{2dm} . Therefore the expression $k \prec \pi$ has a meaning for $\pi \in NC(2dm)$.

The following application of the Cauchy-Schwarz inequality is what motivates the introduction of the operations P_k on the partitions of $[2N]$. The same use of the Cauchy-Schwarz inequality has been made in the second part of [Buc01].

Lemma 2.1. *For a partition π of $[2N]$ and a finitely supported family of matrices $a = (a_k)_{k \in \mathbb{N}^N}$,*

$$|S(a, \pi, N, 1)| \leq (S(a, P_0(\pi), N, 1))^{1/2} (S(a, P_N(\pi), N, 1))^{1/2}.$$

More generally for a partition π of $[2dm]$, for a finitely supported family of matrices $a = (a_k)_{k \in \mathbb{N}^d}$ and any integer i

$$(15) \quad |S(a, \pi, d, m)| \leq (S(a, P_{di}(\pi), d, m))^{1/2} (S(a, P_{(m+i)d}(\pi), d, m))^{1/2}.$$

Proof. The second statement for $i = 0$ follows from the first one by replacing N by dm . Indeed for any $k = (k_1, \dots, k_m) \in (\mathbb{N}^d)^m \simeq \mathbb{N}^{dm}$, denote $\beta_k = a_{k_1} \tilde{a}_{k_2}^* a_{k_3} \dots a_{k_m}$ if m is odd and $\beta_k = a_{k_1} \tilde{a}_{k_2}^* a_{k_3} \dots \tilde{a}_{k_m}^*$ if m is even. We claim that $S(a, \pi, d, m) = S(\beta, \pi, dm, 1)$. We give a proof when m is odd, the case when m is even is similar. It is enough to prove that if $k = (k_1, \dots, k_m) \in (\mathbb{N}^d)^m$ then $\tilde{\beta}_k^* = \tilde{a}_{k_1}^* a_{k_2} \dots \tilde{a}_{k_m}^*$. But if $r : \mathbb{N}^d \rightarrow \mathbb{N}^d$ denotes the map $r(s_1, \dots, s_d) = (s_d, \dots, s_1)$ we have that

$$\begin{aligned} \tilde{\beta}_k^* &= \beta_{r(k_m), \dots, r(k_1)}^* = (a_{r(k_m)} \dots \tilde{a}_{r(k_2)}^* a_{r(k_1)})^* \\ &= a_{r(k_1)}^* \tilde{a}_{r(k_2)} \dots a_{r(k_m)}^* \\ &= \tilde{a}_{k_1}^* a_{k_2} \dots \tilde{a}_{k_m}^*. \end{aligned}$$

For a general i the following argument based on the trace property allow to reduce to the case $i = 0$: for a partition π of $[2dm]$ and any $n \in [2dm]$ denote $\tau_n(\pi)$ the partition such that $s \sim_{\tau_n(\pi)} t$ if and only if $s + n \sim_{\pi} t + n$, so that $P_{n+k}(\pi) = (\tau_n^{-1} \circ P_k \circ \tau_n)(\pi)$ for any integer k . Moreover by the trace property $S(a, \pi, d, m) = S(a, \tau_{di}(\pi), d, m)$ if n is even and $S(a, \pi, d, m) = S(\tilde{a}^*, \tau_{di}(\pi), d, m)$ if i is even (here \tilde{a}^* denotes the family $(\tilde{a}_k^*)_{k \in \mathbb{N}^d}$). Therefore if one assumes that the inequality (15) is satisfied for any π and any a but only for $i = 0$, then we can deduce it for a general i in the following way. Denote $b = (a_k)_{k \in \mathbb{N}^d}$ if i is even and

$b = (\tilde{a}_k^*)_{k \in \mathbb{N}^d}$ if i is odd and :

$$\begin{aligned}
|S(a, \pi, d, m)|^2 &= |S(b, \tau_{di}(\pi), d, m)|^2 \\
&\leq S(b, P_0(\tau_{di}(\pi)), d, m) S(b, P_{dm}(\tau_{di}(\pi)), d, m) \\
&= S(b, \tau_{di}(P_{di}(\pi)), d, m) S(b, \tau_{di}(P_{dm+di}(\pi)), d, m) \\
&= S(a, P_{di}(\pi), d, m) S(a, P_{(m+i)d}(\pi), d, m)
\end{aligned}$$

We now prove the first statement. We take the same notation as in Definition 1.1.

Let us clarify the notation for the rest of the proof. In the whole proof, for a set X we see a $k \in \mathbb{N}^X$ as a function from X to \mathbb{N} , and for an integer N we will identify \mathbb{N}^N with $\mathbb{N}^{[N]}$. In particular, if X and Y are disjoint subsets of a set Z , and if $k \in \mathbb{N}^X$ and $l \in \mathbb{N}^Y$, $[k, l]$ will denote the element of $\mathbb{N}^{X \cup Y}$ corresponding to the function on $X \cup Y$ that has k as restriction to X and l as restriction to Y .

Let us denote by A the union of the blocks of π that are contained in $I_N = \{1, \dots, N\}$, by B the union of the blocks of π that are contained in $[2N] \setminus I_N = \{N+1, \dots, 2N\} = I_{2N}$ and by C the rest of $[2N]$. In the following equations, s will vary in \mathbb{N}^A , t in $\mathbb{N}^{I_N \setminus A}$, u in \mathbb{N}^B and v in $\mathbb{N}^{I_{2N} \setminus B}$. For such s, t, u and v and with the previous notation, $[s, t, u, v] \prec \pi$ if and only if $s \prec \pi|_A$, $[t, v] \prec \pi|_C$ and $u \prec \pi|_B$. For $k \in \mathbb{N}^{I_{2N}}$ (i.e. k is a function $k : I_{2N} \rightarrow \mathbb{N}$), we will also abusively denote $\tilde{a}_k \stackrel{\text{def}}{=} \tilde{a}_{(k(N+1), \dots, k(2N))}$. With this notation the definition in (13) becomes

$$\begin{aligned}
S(a, \pi, N, 1) &= \sum_{\substack{s \in \mathbb{N}^A, t \in \mathbb{N}^{I_N \setminus A}, u \in \mathbb{N}^B, v \in \mathbb{N}^{I_{2N} \setminus B} \\ [s, t, u, v] \prec \pi}} \text{Tr}(a_{[s, t]} \tilde{a}_{[u, v]}^*) \\
&= \sum_{\substack{t, v \\ [t, v] \prec \pi|_C}} \text{Tr} \left(\left(\sum_{s \prec \pi|_A} a_{[s, t]} \right) \left(\sum_{u \prec \pi|_B} \tilde{a}_{[u, v]} \right)^* \right).
\end{aligned}$$

Thus

$$|S(a, \pi, N, 1)| \leq \sum_{[t, v] \prec \pi|_C} \left| \text{Tr} \left(\left(\sum_{s \prec \pi|_A} a_{[s, t]} \right) \left(\sum_{u \prec \pi|_B} \tilde{a}_{[u, v]} \right)^* \right) \right|.$$

Applying the Cauchy-Schwarz inequality for the trace, we get

$$|S(a, \pi, N, 1)| \leq \sum_{[t, v] \prec \pi|_C} \left\| \sum_{s \prec \pi|_A} a_{[s, t]} \right\|_2 \left\| \sum_{u \prec \pi|_B} \tilde{a}_{[u, v]} \right\|_2.$$

The classical Cauchy-Schwarz inequality yields

$$|S(a, \pi, N, 1)| \leq (1)^{1/2} (2)^{1/2}$$

where

$$\begin{aligned} (1) &= \sum_{[t,v] \prec \pi|_C} \left\| \sum_{s \prec \pi|_A} a_{[s,t]} \right\|_2^2 \\ (2) &= \sum_{[t,v] \prec \pi|_C} \left\| \sum_{u \prec \pi|_B} \tilde{a}_{[u,v]} \right\|_2^2. \end{aligned}$$

We claim that $(1) = S(a, P_N(\pi), N, 1)$ and $(2) = S(a, P_0(\pi), N, 1)$. We only prove the first equality, the second is proved similarly (or follows from the first). But

$$\begin{aligned} (1) &= \sum_{[t,v] \prec \pi|_C} \left\| \sum_{s \prec \pi|_A} a_{[s,t]} \right\|_2^2 \\ &= \sum_{[t,v] \prec \pi|_C} \text{Tr} \left(\left(\sum_{s \prec \pi|_A} a_{[s,t]} \right) \cdot \left(\sum_{s \prec \pi|_A} a_{[s,t]} \right)^* \right) \\ &= \text{Tr} \left(\sum_{[t,v] \prec \pi|_C} \sum_{s \prec \pi|_A} \sum_{s' \prec \pi|_A} a_{[s,t]} a_{[s',t]}^* \right) \\ &= \text{Tr} \left(\sum_{[t,v] \prec \pi|_C} \sum_{s \prec \pi|_A} \sum_{s' \prec \pi|_A} a_{[s,t]} \tilde{a}_{r([s',t])}^* \right), \end{aligned}$$

where on the last line for any $k = (k_1, \dots, k_N) \in \mathbb{N}^{I_N}$, $r(k) \in \mathbb{N}^{I_N}$ is defined by $r(k) = (k_N, k_{N-1}, \dots, k_1)$.

By definition of B , for any $j \in I_{2N} \setminus B$ there is $i \in I_N \setminus A$ such that $i \sim_\pi j$. Thus for any $t \in \mathbb{N}^{I_N \setminus A}$ there is exactly one or zero $v \in \mathbb{N}^{I_{2N} \setminus B}$ such that $[t, v] \prec \pi_C$, depending whether $t \prec \pi_{I_N \setminus A}$ or not.

The claim that $(1) = S(a, P_N(\pi), N, 1)$ thus follows from the observation that for $k, l \in \mathbb{N}^N$, $(k, l) \prec P_N(\pi)$ if and only there are $s, s' \in \mathbb{N}^A$ and $t \in \mathbb{N}^{I_N \setminus A}$ such that $k = [s, t]$, $l = r([s', t])$ and $s \prec \pi|_A$, $s' \prec \pi|_A$ and $t \prec \pi|_{I_N \setminus A}$. \square

We now have to observe that the quantities $S(a, \sigma_l^{(d,m)}, d, m)$ for $l = 0, \dots, d$ and $S(a, \tilde{\sigma}_l^{(d,m)}, d, m)$ for $l = 0, \dots, d$ have simple expressions.

A (finitely supported) family of matrices $a = (a_k)_{k \in \mathbb{N}^d}$ can be made in various natural ways into a bigger matrix, for any decomposition of $\mathbb{N}^d \simeq \mathbb{N}^l \times \mathbb{N}^{d-l}$. If the a_k 's are viewed as operators on a Hilbert space H ($H = \mathbb{C}^\alpha$ if the a_k 's are in $M_\alpha(\mathbb{C})$), then let us denote by M_l the operator from $H \otimes \ell^2(\mathbb{N})^{\otimes d-l}$ to $H \otimes \ell^2(\mathbb{N})^{\otimes l}$ having the following block-matrix decomposition:

$$(a_{[s,t]})_{s \in \mathbb{N}^{\{1, \dots, l\}}, t \in \mathbb{N}^{\{l+1, \dots, d\}}}.$$

Note that since (a_k) has finite support, the above matrix has only finitely many nonzero entries, and hence corresponds to a finite rank operator. In particular, it belongs to $S_p(H \otimes \ell^2(\mathbb{N})^{\otimes d-l}; H \otimes \ell^2(\mathbb{N})^{\otimes l})$ for any $p \in (0, \infty]$.

Lemma 2.2. *Let $d, m, a = (a_k)_{k \in \mathbb{N}^d}$ and M_l as above, and σ_l and $\tilde{\sigma}_l$ defined in Corollary 1.4. Then for $l \in \{0, 1, \dots, d\}$:*

$$S(a, \sigma_l^{(d,m)}, d, m) = \|M_l\|_{S_{2m}(H \otimes \ell^2(\mathbb{N})^{\otimes d-l}; H \otimes \ell^2(\mathbb{N})^{\otimes l})}^{2m}.$$

Moreover for $l \in \{1, \dots, d\}$

$$S(a, \tilde{\sigma}_l^{(d,m)}, d, m) \leq \|M_l\|_{S_{2m}(H \otimes \ell^2(\mathbb{N})^{\otimes d-l}; H \otimes \ell^2(\mathbb{N})^{\otimes l})}^{2m}.$$

Remark. It is also true that

$$S(a, \tilde{\sigma}_l^{(d,m)}, d, m) \leq \|M_{l-1}\|_{S_{2m}(H \otimes \ell^2(\mathbb{N})^{\otimes d-l}; H \otimes \ell^2(\mathbb{N})^{\otimes l})}^{2m},$$

but we will only use the inequality stated in the lemma. This inequality follows from the one stated by conjugating by the rotation $k \in [2dm] \mapsto k + d$.

Proof. We fix $l \in \{0, \dots, d\}$. For any $s = (s_1, \dots, s_l) \in \mathbb{N}^l$ we denote by $A_s = (a_{s,t})_{t \in \mathbb{N}^{d-l}}$ viewed as a row matrix. As an operator, A_s thus acts from $H \otimes \ell^2(\mathbb{N})^{\otimes d-l}$ to H . For $s, s' \in \mathbb{N}^l$, if $r(k_1, \dots, k_d) = (k_d, \dots, k_1)$

$$A_s A_{s'}^* = \sum_{t \in \mathbb{N}^{d-l}} a_{s,t} a_{s',t}^* = \sum_{t \in \mathbb{N}^{d-l}} a_{s,t} \tilde{a}_{r(s',t)}^*.$$

Hence for $s^{(1)}, s^{(2)}, \dots, s^{(m)} \in \mathbb{N}^l$, if $s^{(m+1)} = s^{(1)}$,

$$\prod_{i=1}^m A_{s^{(i)}} A_{s^{(i+1)}}^* = \sum_{t^{(1)}, \dots, t^{(m)} \in \mathbb{N}^{d-l}} a_{s^{(1)}, t^{(1)}} \tilde{a}_{r(s^{(2)}, t^{(1)})}^* a_{s^{(2)}, t^{(2)}} \tilde{a}_{r(s^{(3)}, t^{(2)})}^* \dots \tilde{a}_{r(s^{(1)}, t^{(m)})}^*.$$

But for $k \in \mathbb{N}^{[2dm]}$, $k \prec \sigma_l^{(d,m)}$ if and only if there exist $s^{(1)}, s^{(2)}, \dots, s^{(m)} \in \mathbb{N}^l$ and $t^{(1)}, t^{(2)}, \dots, t^{(m)} \in \mathbb{N}^{d-l}$ such that for all i , $(k_{2di+1}, k_{2di+2}, \dots, k_{2di+d}) = (s^{(i)}, t^{(i)})$ and $(k_{2di+2d}, k_{2di+2d-1}, \dots, k_{2di+d+1}) = (s^{(i+1)}, t^{(i)})$. Thus summing over $s^{(1)}, s^{(2)}, \dots, s^{(m)} \in \mathbb{N}^l$ in the preceding equation leads to

$$\sum_{s^{(1)}, s^{(2)}, \dots, s^{(m)} \in \mathbb{N}^l} \prod_{i=1}^m A_{s^{(i)}} A_{s^{(i+1)}}^* = \sum_{(k_1, \dots, k_{2m}) \prec \sigma_l^{(d,m)}} a_{k_1} \tilde{a}_{k_2}^* a_{k_3} \dots a_{k_{2m-1}} \tilde{a}_{k_{2m}}^*.$$

Taking the trace and using the trace property we get

$$\begin{aligned} S(a, \sigma_l^{(d,m)}, d, m) &= \sum_{s^{(1)}, s^{(2)}, \dots, s^{(m)} \in \mathbb{N}^l} \text{Tr} \left(\prod_{i=1}^m A_{s^{(i)}}^* A_{s^{(i)}} \right) \\ &= \text{Tr} \left[\left(\sum_{s \in \mathbb{N}^l} A_s^* A_s \right)^m \right] \\ &= \text{Tr} [(M_l^* M_l)^m] \end{aligned}$$

where the last identity follows from the fact that $M_l = \sum A_s \otimes e_{s1}$. This concludes the proof for $\sigma_l^{(d,m)}$. For $\tilde{\sigma}_l^{(d,m)}$ with $1 \leq l \leq d$, the same kind of computations yield to

$$S(a, \tilde{\sigma}_l^{(d,m)}, d, m) = \sum_{s_l \in \mathbb{N}} \text{Tr} \left[\left(\sum_{s \in \mathbb{N}^{l-1}} A_{(s, s_l)}^* A_{(s, s_l)} \right)^m \right].$$

To conclude we only have to use Lemma 2.3 below. \square

Lemma 2.3. *Let $X_1, X_2 \dots X_N$ be matrices. Then for any integer $m \geq 1$*

$$\sum_{i=1}^N \text{Tr}((X_i^* X_i)^m) \leq \text{Tr}((\sum_{i=1}^N X_i^* X_i)^m).$$

Proof. This is a general inequality for the non-commutative L_p -norms. Indeed, for any $\alpha, N \in \mathbb{N}$, and $p \in [2, \infty]$, the map

$$T : M_{N,1}(M_\alpha(\mathbb{C})) \rightarrow M_N(M_\alpha(\mathbb{C}))$$

$$\begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} \mapsto \begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_N \end{pmatrix}$$

is a contraction for all p -norms. For $p = 2$, this is easy because T is an isometry. For $p = \infty$ this is also obvious. For a general $p \in (2, \infty)$ the claim follows by interpolation.

Applied for $p = 2m$, this concludes the proof since for an integer m ,

$$\left\| \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} \right\|_{2m}^{2m} = \text{Tr}((\sum_{i=1}^N X_i^* X_i)^m)$$

and

$$\left\| \begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_N \end{pmatrix} \right\|_{2m}^{2m} = \sum_{i=1}^N \text{Tr}((X_i^* X_i)^m).$$

□

We are now able to state and prove the main result of this section. Recall that for a partition π of $NC^*(1, m)$, $B(\pi)$ was defined in part 1.3 as the number of blocks of the partition $\Phi(\pi)$ (the map Φ was defined after Corollary 1.4).

Corollary 2.4. *Let $\pi \in NC^*(d, m)$. Then if a and M_l are as in Lemma 2.2,*

$$|S(a, \pi, d, m)| \leq \prod_{l=0}^d \|M_l\|_{S_{2m}(H \otimes \ell^2(\mathbb{N}) \otimes d-l; H \otimes \ell^2(\mathbb{N}) \otimes l)}^{2m\mu_l}$$

where $\mu_l = (B(\pi|_{A_{l+1}}) - B(\pi|_{A_l})) / (m - 1)$ where we take the convention that $B(\pi|_{A_0}) = 1$ and $B(\pi|_{A_{d+1}}) = m$.

Proof. The idea is, as in Lemma 2 and Corollary 3 of [Buc01], to iterate the inequality of Lemma 2.1, except that here the combinatorial invariants of the map $\pi \mapsto (P_{kd}(\pi), P_{kd+md}(\pi))$ (Lemma 1.10) allow us to precisely determine the exponents of each $\|M_l\|_{2m}$. In the rest of the proof since no confusion is possible, we will simply denote $\sigma_l = \sigma_l^{(d,m)}$ and $\tilde{\sigma}_l = \tilde{\sigma}_l^{(d,m)}$, and S will denote the set $\{\sigma_l, 0 \leq l \leq d\} \cup \{\tilde{\sigma}_l, 0 \leq l \leq d\}$. Fix $\pi \in NC^*(d, m)$.

Maybe the clearest way to write out a proof is using the basic vocabulary of probability theory (for a reference see for example [GS92]). Let us consider the (homogeneous) Markov chain $(\pi_n)_{n \geq 0}$ on (the finite state space) $NC^*(d, m)$ given by $\pi_0 = \pi$ and $\pi_{n+1} = P_{id}(\pi_n)$ where i is uniformly distributed in $[2m]$ and independent from $(\pi_k)_{0 \leq k \leq n}$ (note that $\pi_{n+1} \in NC^*(d, m)$ if $\pi_n \in NC^*(d, m)$ by Lemma 1.3). Corollary 1.4 implies that the sequence $(\pi_n)_n$ is almost surely eventually equal to one of

the σ_l or $\tilde{\sigma}_l$. Its second statement indeed expresses that if $\pi_n \in S$ then $\pi_N = \pi_n$ for all $N \geq n$; it suffices therefore to prove that $p_n \stackrel{\text{def}}{=} \mathbb{P}(\pi_n \notin S) \rightarrow 0$ as $n \rightarrow \infty$. But if k is fixed with $2^{k-2} \geq m$, its first statement implies that $p_k \leq 1 - (1/2m)^k = c < 1$ for any starting state π_0 . From the equality $p_{n+k} = p_n \mathbb{P}(\pi_{n+k} \notin S | \pi_n \notin S)$ and the Markov property we get that $p_{n+k} \leq c p_n$ for any integer $n \in \mathbb{N}$, from which we deduce that $p_n \leq c^{\lfloor n/k \rfloor} \rightarrow 0$ as $n \rightarrow \infty$.

Let us denote $\lambda_l(\pi) = \mathbb{P}(\lim_n \pi_n = \sigma_l)$ and $\tilde{\lambda}_l(\pi) = \mathbb{P}(\lim_n \pi_n = \tilde{\sigma}_l)$ for $0 \leq l \leq d$ (take $\tilde{\lambda}_0(\pi) = 0$); note that $\sum_l \lambda_l(\pi) + \tilde{\lambda}_l(\pi) = 1$.

Lemma 1.10 and the last statement of Lemma 1.3 show that for any $i \in \{1, \dots, d\}$ the sequence $B(\pi_n|_{A_i})$ is a martingale. In particular since $\pi_0 = \pi$, $B(\pi|_{A_i}) = \mathbb{E}[B(\pi_n|_{A_i})]$ for any $n \geq 0$. Letting $n \rightarrow \infty$ we get

$$\begin{aligned} B(\pi|_{A_i}) &= \sum_{l=0}^d \lambda_l(\pi) B(\sigma_l|_{A_i}) + \sum_{l=1}^d \tilde{\lambda}_l(\pi) B(\tilde{\sigma}_l|_{A_i}) \\ &= \sum_{l=0}^d \left(\lambda_l(\pi) + \tilde{\lambda}_l(\pi) \right) (1 + (m-1)1_{l < i}) \\ &= 1 + (m-1) \sum_{0 \leq l < i} \lambda_l(\pi) + \tilde{\lambda}_l(\pi). \end{aligned}$$

We used the fact that $B(\sigma_l|_{A_i}) = B(\tilde{\sigma}_l|_{A_i}) = 1 + (m-1)1_{l < i}$. This follows from the observations that since $\Phi(c_m) = \Phi(1_{2m}) = 1_m$, $B(c_m) = |1_m| = 1$ and that since $\Phi(r_m) = 0_m$, $B(r_m) = m$. Subtracting the equalities above for i and $i+1$ gives

$$(16) \quad (\lambda_i(\pi) + \tilde{\lambda}_i(\pi))(m-1) = B(\pi|_{A_{i+1}}) - B(\pi|_{A_i})$$

with the convention that $B(\pi|_{A_0}) = 1$ and $B(\pi|_{A_{d+1}}) = m$.

On the other hand Lemma 2.1 implies that the sequence $M_n = \log |S(a, \pi_n, d, m)|$ is a submartingale. As above letting $n \rightarrow \infty$ in the inequality $M_0 \leq \mathbb{E}[M_n]$ yields

$$\log |S(a, \pi, d, m)| \leq \sum_{l=0}^d \lambda_l(\pi) \log |S(a, \sigma_l, d, m)| + \sum_{l=1}^d \tilde{\lambda}_l(\pi) \log |S(a, \tilde{\sigma}_l, d, m)|.$$

If we denote simply by $\|M_l\|_{2m}$ the quantity $\|M_l\|_{S_{2m}(H \otimes \ell^2(\mathbb{N})^{\otimes d-l}; H \otimes \ell^2(\mathbb{N})^{\otimes l})}$, then by Lemma 2.2 this inequality becomes

$$|S(a, \pi, d, m)| \leq \prod_{l=0}^d \|M_l\|_{2m}^{2m(\lambda_l(\pi) + \tilde{\lambda}_l(\pi))}.$$

This inequality, combined with (16), concludes the proof. \square

3. MAIN RESULT

We are now able to prove the main results of this paper. We first treat the “holomorphic” setting (Theorems 0.3 and 0.4) for which the results we get are completely satisfactory.

3.1. Holomorphic setting. It is a generalization to operator coefficients of the main result of [KS07]. When the coefficients a_k are taken to be scalars, the techniques of our Theorem 0.4 give a new proof and an improvement of the theorem 1.3 of [KS07]. In [KS07], Kemp and Speicher introduce free Poisson variables to get an upper bound, whereas our proof is more combinatorial and lies in the study of $NC^*(d, m)$ that is done in part 1.2. We refer to [NS06] or to the paper [KS07] for definitions and facts on free cumulants and \mathcal{R} -diagonal operators. We just recall that the $*$ -distribution of a variable c in a C^* -probability space is characterized by its free cumulants, which are the family of complex numbers $\kappa_n[c^{\varepsilon_1}, \dots, c^{\varepsilon_n}]$, for $n \in \mathbb{N}$ and $\varepsilon_i \in \{1, *\}$. Moreover the \mathcal{R} -diagonal operators are exactly the operators c for which the cumulants $\kappa_n[c^{\varepsilon_1}, \dots, c^{\varepsilon_n}]$ vanish except if n is even and if 1's and *'s alternate in the sequence $\varepsilon_1, \dots, \varepsilon_n$. Since the family $\lambda(g_1), \dots, \lambda(g_r)$ (where g_1, \dots, g_r are the generators of the free group F_r) form an example of $*$ -free \mathcal{R} -diagonal operators, Theorem 0.3 is a particular case of Theorem 0.4, that is why we do not include a proof.

Proof of Theorem 0.4. The start of the proof is the same as in the proof of Theorem 1.3 of [KS07], and was sketched in the Introduction. Fix $p = 2m \in 2\mathbb{N}$.

As in (14), if $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ denote by $\tilde{a}_k = a_{(k_d, \dots, k_1)}$ and $\tilde{c}_k = c_{(k_d, \dots, k_1)} = c_{k_d} \dots c_{k_1}$. First develop the norms:

$$\begin{aligned} \left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} &= \sum_{k_1, \dots, k_{2m} \in \mathbb{N}^d} \text{Tr}(a_{k_1} a_{k_2}^* \dots a_{k_{2m}}^*) \tau(c_{k_1} c_{k_2}^* \dots c_{k_{2m}}^*) \\ &= \sum_{k_1, \dots, k_{2m} \in \mathbb{N}^d} \text{Tr}(a_{k_1} \tilde{a}_{k_2}^* \dots \tilde{a}_{k_{2m}}^*) \tau(c_{k_1} \tilde{c}_{k_2}^* \dots \tilde{c}_{k_{2m}}^*). \end{aligned}$$

Take $k_1, \dots, k_{2m} \in \mathbb{N}^d$; if $k_l = (k_l(1), k_l(2), \dots, k_l(d))$ then

$$c_{k_1} \tilde{c}_{k_2}^* \dots \tilde{c}_{k_{2m}}^* = c_{k_1(1)} c_{k_1(2)} \dots c_{k_1(d)} c_{k_2(1)}^* \dots c_{k_2(d)}^* \dots c_{k_{2m}(1)}^* \dots c_{k_{2m}(d)}^*$$

and by the fundamental property of cumulants:

$$\tau(c_{k_1} \tilde{c}_{k_2}^* \dots \tilde{c}_{k_{2m}}^*) = \sum_{\pi \in NC(2dm)} \kappa_\pi[c_{k_1(1)}, \dots, c_{k_1(d)}, c_{k_2(1)}^*, \dots, c_{k_2(d)}^*, \dots, c_{k_{2m}(1)}^*, \dots, c_{k_{2m}(d)}^*].$$

Denote $k = (k_1, \dots, k_{2m}) \in (\mathbb{N}^d)^{2m} \simeq \mathbb{N}^{2dm}$. Since freeness is characterized by the vanishing of mixed cumulants (Theorem 11.16 in [NS06]), $\kappa_\pi[c_{k_1(1)}, \dots, c_{k_{2m}(d)}^*]$ is non-zero only if $k \prec \pi$, and in this case we claim that it is equal to $\kappa_\pi[c_{d,m}]$ where

$$(17) \quad c_{d,m} = \overbrace{c, \dots, c, c^*, \dots, c^*, \dots, c, \dots, c, c^*, \dots, c^*}^{2m \text{ groups}}.$$

Relabel indeed the sequence $k_1(1), \dots, k_{2m}(d)$ by k_1, \dots, k_{2dm} , and denote also by $\varepsilon_1, \dots, \varepsilon_{2dm}$ the corresponding sequence of 1's and *'s, in such a way that $\kappa_\pi[c_{k_1(1)}, \dots, c_{k_{2m}(d)}^*] = \kappa_\pi[(c_{k_i}^{\varepsilon_i})_{1 \leq i \leq 2dm}]$ and $\kappa_\pi[c_{d,m}] = \kappa_\pi[(c_{k_i}^{\varepsilon_i})_{1 \leq i \leq 2dm}]$. By the definition of κ_π , we have

$$\kappa_\pi[(c_{k_i}^{\varepsilon_i})_{1 \leq i \leq 2dm}] = \prod_{V \in \pi} \kappa_{|V|}[(c_{k_i}^{\varepsilon_i})_{i \in V}]$$

where the products runs over by the blocks of π . Similarly

$$\kappa_\pi[c_{d,m}] = \prod_{V \in \pi} \kappa_{|V|}[(c^{\varepsilon_i})_{i \in V}].$$

Our claim thus follows from the observation that if $k \prec \pi$ then for any block V of π there is an index s such that $k_i = s$ for all $i \in V$, and the equality $\kappa_{|V|}[(c_s^{\varepsilon_i})_{i \in V}] = \kappa_{|V|}[(c^{\varepsilon_i})_{i \in V}]$ expresses just the fact that c and c_s have the same $*$ -distribution and therefore the same cumulants.

The next claim is that since c is \mathcal{B} -diagonal, $\kappa_\pi[c_{d,m}]$ is non-zero only if $\pi \in NC^*(d, m)$. Since with the previous notation $\kappa_\pi[c_{d,m}] = \prod_{V \in \pi} \kappa_{|V|}[(c^{\varepsilon_i})_{i \in V}]$, this amounts to showing that if there is a block V of π which is not of even cardinality or for which 1's and $*$'s do not alternate in the sequence $(\varepsilon_i)_{i \in V}$, then $\kappa_{|V|}[(c^{\varepsilon_i})_{i \in V}] = 0$. But this is exactly the definition of \mathcal{R} -diagonal operators. Thus we get

$$\left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi \in NC^*(d, m)} \kappa_\pi[c_{d,m}] \sum_{(k_1, \dots, k_{2m}) \prec \pi} \text{Tr}(a_{k_1} \tilde{a}_{k_2}^* \dots \tilde{a}_{k_{2m}}^*),$$

or with the notation introduced in (14)

$$(18) \quad \left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi \in NC^*(d, m)} \kappa_\pi[c_{d,m}] S(a, \pi, d, m).$$

Up to this point we have mainly reproduced the beginning of the proof of Theorem 1.3 of [KS07] (the authors of [KS07] only deal with scalar a_k 's but there is no other difference).

We can now use the study of $NC^*(d, m)$ that we did in part 1.2. Recall in particular that there is a map $\mathcal{P} : NC^*(d, m) \rightarrow NC(m)^{(d)}$ the properties of which are summarized in Theorem 1.5.

Take $(\sigma_1, \dots, \sigma_d) \in NC(m)^{(d)}$ and denote $\mu_l = (|\sigma_{l+1}| - |\sigma_l|)/(m-1)$ where $|\sigma|$ denotes the number of blocks of σ with the convention $|\sigma_0| = 1$ and $|\sigma_{d+1}| = m$. If $\pi \in NC^*(d, m)$ and $\mathcal{P}(\pi) = (\sigma_1, \dots, \sigma_d)$ then by Corollary 2.4, $|S(a, \pi, d, m)| \leq \prod_{l=0}^d \|M_l\|_{2m}^{2m\mu_l}$.

Thus by the first part of Theorem 1.5, we have that

$$\begin{aligned} & \left| \sum_{\pi \in NC^*(d, m), \mathcal{P}(\pi) = (\sigma_1, \dots, \sigma_d)} \kappa_\pi[c_{d,m}] S(a, \pi, d, m) \right| \\ & \leq 4^{2m} \prod_{l=0}^d \|M_l\|_{2m}^{2m\mu_l} \max_{\mathcal{P}(\pi) = (\sigma_1, \dots, \sigma_d)} |\kappa_\pi[c_{d,m}]|. \end{aligned}$$

But by the second statement of Theorem 1.5 and Lemma 3.1 below (recall that for $\tau(c) = \kappa_1[c] = 0$ since c is \mathcal{R} -diagonal)

$$|\kappa_\pi[c_{d,m}]| \leq \|c\|_2^{2dm} \left(\frac{16\|c\|_{2m}}{\|c\|_2} \right)^{4m},$$

which implies

$$(19) \quad \left| \sum_{\pi \in NC^*(d, m), \mathcal{P}(\pi) = (\sigma_1, \dots, \sigma_d)} \kappa_\pi[c_{d, m}] S(a, \pi, d, m) \right| \leq 4^{10m} \prod_{l=0}^d \|M_l\|_{2m}^{2m\mu_l} \|c\|_2^{2dm} \left(\frac{\|c\|_{2m}}{\|c\|_2} \right)^{4m}.$$

But by Theorem 3.2 in [Ede80], for any non-negative integers s_0, \dots, s_d such that $\sum_i s_i = m - 1$, the number of $(\sigma_1, \dots, \sigma_d) \in NC(m)^{(d)}$ such that $|\sigma_{l+1}| - |\sigma_l| = s_l$ for any $0 \leq l \leq d$ (with the conventions $|\sigma_0| = 1$ and $|\sigma_{d+1}| = m$) is equal to $(1/m) \binom{m}{s_0} \binom{m}{s_1} \dots \binom{m}{s_d}$. Thus from (18) we deduce

$$\left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} \leq 4^{10m} \|c\|_2^{2dm} \left(\frac{\|c\|_{2m}}{\|c\|_2} \right)^{4m} \sum_{s_0 + \dots + s_d = m-1} (1/m) \binom{m}{s_0} \binom{m}{s_1} \dots \binom{m}{s_d} \prod_{l=0}^d \|M_l\|_{2m}^{2ms_l/(m-1)}.$$

Denote for simplicity $\gamma_l = \|M_l\|_{2m}^{2m/(m-1)}$. Since the number of $s_0, \dots, s_d \in \mathbb{N}$ such that $s_0 + \dots + s_d = m - 1$ is equal to $\binom{m+d-1}{d}$, this inequality becomes

$$\left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} \leq 4^{10m} \|c\|_2^{2dm} \left(\frac{\|c\|_{2m}}{\|c\|_2} \right)^{4m} \binom{m+d-1}{d} \sup_{s_0 + \dots + s_d = m-1} (1/m) \binom{m}{s_0} \binom{m}{s_1} \dots \binom{m}{s_d} \prod_{l=0}^d \gamma_l^{s_l}.$$

Now use the fact that for any integers N and n , $\binom{N}{n} \leq (N/n)^n (N/(N-n))^{N-n}$ with the convention $(N/0)^0 = 1$. For a fixed N , this can be proved by induction on $n \leq N/2$ using the fact that $x \in \mathbb{R}^+ \mapsto x \log(1 + 1/x)$ is increasing. Thus

$$\binom{m+d-1}{d} \leq \binom{m+d}{d} \leq \left(1 + \frac{m}{d}\right)^d \left(1 + \frac{d}{m}\right)^m.$$

But since \log is concave, if $s_0 + \dots + s_d = m - 1$,

$$\begin{aligned} \prod_{l=0}^d \left(\frac{m}{m-s_l} \right)^{m-s_l} &= \exp \left((md+1) \sum_0^d \frac{m-s_l}{md+1} \log(m/(m-s_l)) \right) \\ &\leq \exp \left((md+1) \log \left(\sum_0^d m/(md+1) \right) \right) \\ &= \exp((md+1) \log(1 + (m-1)/(md+1))) \leq \exp(m) \end{aligned}$$

and

$$\begin{aligned} \prod_{l=0}^d \left(\frac{m\gamma_l}{s_l} \right)^{s_l} &= \exp \left((m-1) \sum_0^d \frac{s_l}{m-1} \log(m\gamma_l/s_l) \right) \\ &\leq \exp \left((m-1) \log(m/(m-1)) \sum_0^l \gamma_l \right) \\ &= (\gamma_0 + \dots \gamma_l)^{m-1} \left(\frac{m}{m-1} \right)^{m-1} \end{aligned}$$

But $(m/(m-1))^{m-1} \leq m$ for any $m \geq 1$. This leads to

$$(20) \quad \left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} \leq 4^{10m} \|c\|_2^{2dm} \left(\frac{\|c\|_{2m}}{\|c\|_2} \right)^{4m} \left(1 + \frac{m}{d} \right)^d \left(1 + \frac{d}{m} \right)^m \exp(m) (\gamma_0 + \dots \gamma_l)^{m-1}.$$

Noting that since $2m/(m-1) \geq 2$,

$$(\gamma_0 + \dots \gamma_l)^{m-1} = \|(\|M_l\|_{2m})_l\|_{\ell^{2m/(m-1)}(\{0, \dots, d\})}^{2m} \leq \|(\|M_l\|_{2m})_l\|_{\ell^2(\{0, \dots, d\})}^{2m}$$

and taking the $2m$ -th root in (20) one finally gets

$$\left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m} \leq 4^5 \sqrt{e(1+d/m)} \left(1 + \frac{m}{d} \right)^{d/2m} \|c\|_2^d \left(\frac{\|c\|_{2m}}{\|c\|_2} \right)^2 \|(\|M_l\|_{2m})_l\|_{\ell^2(\{0, \dots, d\})}.$$

To conclude for the case $m < \infty$, just note that $(1 + \frac{m}{d})^{d/m} \leq e$.

Letting $m \rightarrow \infty$ and noting that $(1 + \frac{m}{d})^{d/m} \rightarrow 1$ concludes the proof for the operator norm.

When the c_k 's are circular, since $\kappa_\pi[c_{d,m}] = 1$ if $\pi \in NC_2^*(d, m)$ and $\kappa_\pi[c_{d,m}] = 0$ otherwise, we can replace (19) by

$$\left| \sum_{\pi \in NC^*(d, m), \mathcal{P}(\pi) = (\sigma_1, \dots, \sigma_d)} \kappa_\pi[c_{d,m}] S(a, \pi, d, m) \right| \leq \prod_{l=0}^d \|M_l\|_{2m}^{2m\mu_l}.$$

Following the rest of the arguments we get the claimed results. \square

We still have to prove this Lemma that was used in the above proof.

Lemma 3.1. *Let $\pi \in NC(n)$ a non-crossing partition that has at least K blocks of size 2 and in which all blocks have a size at most N .*

Let c_1, \dots, c_n be elements of a tracial C^ -probability space (\mathcal{A}, τ) that are centered: $\tau(c_k) = 0$ for all k . Let $m_p = \max_k \|c_k\|_p$ for $p = 2, N$. Then*

$$(21) \quad |\kappa_\pi[c_1, \dots, c_n]| \leq m_2^{2K} (16m_N)^{n-2K}.$$

Proof. Since both $\pi \mapsto \kappa_\pi$ and the right-hand side of (21) are multiplicative, we only have to prove (21) when $\pi = 1_n$ with $n \leq N$. Then as usual κ_π is denoted by κ_n . If $n = 1$ it is obvious since $\kappa_1(c_1) = \phi(c_1) = 0$.

If $n = 2$, then $K = 1$ and $\kappa_2(c_k, c_l) = \tau(c_k c_l) - \tau(c_k)\tau(c_l) = \tau(c_k c_l)$. By the Cauchy-Schwarz inequality we get $|\kappa_2(c_k, c_l)| \leq m_2^2$.

We now focus on the case $n > 2$, and then $K = 0$. This is essentially done in the proof of Lemma 4.3 in [KS07] but we have to replace the inequality $|\tau(c_{k_1} \dots c_{k_l})| \leq m_\infty^l$ by Hölder's inequality $|\tau(c_{k_1} \dots c_{k_l})| \leq m_N^l$ for any $l \leq n \leq N$. Following the proof of Lemma 4.3 in [KS07], we thus get that

$$\kappa_n[c_1, \dots, c_n] \leq 4^{n-1} \sum_{\sigma \in NC(n)} m_n^n \leq 4^{2n} m_N^n.$$

□

3.2. Non-holomorphic setting. Here we consider Theorems 0.5 and 0.6. We only sketch their proofs. The idea is the same as in the holomorphic setting, except that here the relevant subset of non-crossing partitions is the set $NC(d, m)$ introduced and studied in part 1.4.

Sketch of proof of Theorem 0.6. We will use that if c has a symmetric distribution, then c has vanishing odd cumulants. This means that $\kappa_\pi[c, \dots, c] = 0$ unless π has only blocks of even cardinality. To check this, by the multiplicativity of free cumulants, we have to prove that $\kappa_n[c, \dots, c] = \kappa_{1_n}[c, \dots, c] = 0$ if n is odd. But this is clear: since $-c$ and c have the same distribution, $\kappa_n[c, \dots, c] = \kappa_n[-c, \dots, -c]$. On the other hand since κ_n is n -linear, $\kappa_n[-c, \dots, -c] = (-1)^n \kappa_n[c, \dots, c]$.

Take $(c_k)_{k \in \mathbb{N}}$ and $(a_k)_{k \in \mathbb{N}^d}$ as in Theorem 0.6 and define \tilde{a}_k and c_{k_1, \dots, k_d} as in the proof of Theorem 0.4. Assume for simplicity that c_k is normalized by $\|c_k\|_2 = 1$. Denote by I the set of $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ such that for any $1 \leq i < d$ $k_i \neq k_{i+1}$. Then for $p = 2m$ we have that

$$\left\| \sum_{k \in I} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{k_1, \dots, k_{2m} \in I} \text{Tr}(a_{k_1} \tilde{a}_{k_2}^* \dots \tilde{a}_{k_{2m}}^*) \tau(c_{k_1} c_{k_2} \dots c_{k_{2m}}).$$

Expanding the moment $\tau(c_{k_1} \dots c_{k_{2m}})$ using cumulants we get

$$\tau(c_{k_1} c_{k_2} \dots c_{k_{2m}}) = \sum_{\pi \in NC(2dm)} \kappa_\pi[c_{k_1(1)}, \dots, c_{k_1(d)}, c_{k_2(1)}, \dots, c_{k_2(d)}, \dots, c_{k_{2m}(d)}].$$

By freeness of the family $(c_k)_{k \in \mathbb{N}}$, by the assumption on the vanishing of odd moments and by Lemma 1.12 such a cumulant is equal to 0 except if $\pi \in NC(d, m)$ and $(k_1, \dots, k_{2m}) \prec \pi$, in which case it is equal to $\kappa_\pi[c, c, \dots, c]$. We get

$$\left\| \sum_{k \in I} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi \in NC(d, m)} \kappa_\pi[c, \dots, c] S(a, \pi, d, m).$$

But by Lemma 1.15, Lemma 2.2 and an iteration of Lemma 2.1 we get that for any $\pi \in NC(d, m)$

$$S(a, \pi, d, m) \leq \max_{0 \leq l \leq d} \|M_l\|_{2m}^{2m}.$$

On the other hand (remembering that $\|c\|_2 = 1$), Theorem 1.13 and Lemma 3.1 imply that for $\pi \in NC(d, m)$,

$$|\kappa_\pi[c, \dots, c]| \leq (16\|c\|_{2m})^{4m}.$$

This yields

$$\left\| \sum_{k \in I} a_k \otimes c_k \right\|_{2m}^{2m} \leq \sum_{\pi \in NC(d, m)} (16 \|c\|_{2m})^{4m} \max_{0 \leq l \leq d} \|M_l\|_{2m}^{2m}.$$

But by Theorem 1.13 $NC(d, m)$ has cardinality less than $4^{2m}(d+1)^{2m}$. Taking the $2m$ -th root in the preceding equation we thus get

$$\left\| \sum_{k \in I} a_k \otimes c_k \right\|_{2m} \leq 4^5 (d+1) \|c\|_{2m}^2 \max_{0 \leq l \leq d} \|M_l\|_{2m}.$$

This proves Theorem 0.6 for the case when $p \in 2\mathbb{N}$. For $p = \infty$ just make $p \rightarrow \infty$. \square

For Theorem 0.5 the proof is the same except that we have to be slightly more careful in the beginning. Recall that I_d is the set of $(k_1, \varepsilon_1, \dots, k_d, \varepsilon_d) \in (\mathbb{N} \times \{1, *\})^d$ such that $\lambda(g_{k_1})^{\varepsilon_1} \dots \lambda(g_{k_d})^{\varepsilon_d}$ corresponds to an element of length d in the free group F_∞ . For a family of matrices $(a_{k, \varepsilon})_{(k, \varepsilon) \in I_d}$ denote by

$$\check{a}_{k, \varepsilon} = a_{(k_d, \dots, k_1), (\bar{\varepsilon}_d, \dots, \bar{\varepsilon}_1)}$$

where $\bar{*} = 1$ and $\bar{1} = *$. The motivation for this notation is the following: for $(k, \varepsilon) \in I_d$ denote by $c_{k, \varepsilon} = c_{k_1}^{\varepsilon_1} \dots c_{k_d}^{\varepsilon_d}$, so that if $\check{c}_{k, \varepsilon}$ is defined as $\check{a}_{k, \varepsilon}$, we have that $\check{c}_{k, \varepsilon}^* = c_{k, \varepsilon}$.

For $k = (k_1, \dots, k_{2m}) \in (\mathbb{N}^d)^{2m}$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{2m}) \in (\{1, *\})^{2m}$ and $\pi \in NC(2dm)$ with blocks of even cardinality we will also write $(k, \varepsilon) \prec \pi$ if $k_i = k_j$ for all $i \sim_\pi j$ and if in addition for each block $\{i_1 < \dots < i_{2p}\}$ of π , 1's and *'s alternate in the sequence $\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_{2p}}$.

Last we denote, for $\pi \in NC(d, m)$

$$\tilde{S}(a, \pi, d, m) = \sum_{(k, \varepsilon) \prec \pi} \text{Tr}(a_{k_1, \varepsilon_1} \check{a}_{k_2, \varepsilon_2}^* a_{k_3, \varepsilon_3} \dots \check{a}_{k_{2m}, \varepsilon_{2m}}^*).$$

The proofs of Lemma 2.1 and Lemma 2.2 still apply with this notation:

Lemma 3.2. *Let $\pi \in NC(d, m)$, and take a finitely supported family of matrices $a = (a_{k, \varepsilon})_{(k, \varepsilon) \in I_d}$ as above. For any integer i*

$$\left| \tilde{S}(a, \pi, d, m) \right| \leq \left(\tilde{S}(a, P_{di}(\pi), d, m) \right)^{1/2} \left(\tilde{S}(a, P_{(m+i)d}(\pi), d, m) \right)^{1/2}.$$

Lemma 3.3. *Let $d, m, a = (a_{k, \varepsilon})_{(k, \varepsilon) \in I_d}$ and M_l be as in Theorem 0.5, and σ_l and $\tilde{\sigma}_l$ as defined in Corollary 1.4. Then for $l \in \{0, 1, \dots, d\}$:*

$$\tilde{S}(a, \sigma_l^{(d, m)}, d, m) = \|M_l\|_{S_{2m}(H \otimes \ell^2(\mathbb{N})^{\otimes d-l}; H \otimes \ell^2(\mathbb{N})^{\otimes l})}^{2m}.$$

Moreover for $l \in \{1, \dots, d\}$

$$\tilde{S}(a, \tilde{\sigma}_l^{(d, m)}, d, m) \leq \|M_l\|_{S_{2m}(H \otimes \ell^2(\mathbb{N})^{\otimes d-l}; H \otimes \ell^2(\mathbb{N})^{\otimes l})}^{2m}.$$

We leave the proofs to the reader.

Sketch of the proof of Theorem 0.5. Use the same notation as above. Take $m \in \mathbb{N}$. Then as for the self-adjoint case we expand the $2m$ -norm as follows:

$$\left\| \sum_{(k,\varepsilon) \in I_d} a_{k,\varepsilon} \otimes c_{k,\varepsilon} \right\|_{2m}^{2m} = \sum_{(k_1,\varepsilon_1), \dots, (k_{2m},\varepsilon_{2m}) \in I_d} \text{Tr}(a_{k_1,\varepsilon_1} \check{a}_{k_2,\varepsilon_2}^* \dots \check{a}_{k_{2m},\varepsilon_{2m}}^*) \tau(c_{k_1,\varepsilon_1} c_{k_2,\varepsilon_2} \dots c_{k_{2m},\varepsilon_{2m}}).$$

By the freeness, the definition of I_d , Lemma 1.12 and the fact that the c_k 's are \mathcal{R} -diagonal, the expression of the moment $\tau(c_{k_1,\varepsilon_1} \dots c_{k_{2m},\varepsilon_{2m}})$ becomes simply

$$\tau(c_{k_1,\varepsilon_1} \dots c_{k_{2m},\varepsilon_{2m}}) = \sum_{\pi \in NC(d,m)} 1_{(k,\varepsilon) \prec \pi} \kappa_\pi[c_{k_1(1)}^{\varepsilon_1(1)}, \dots, c_{k_{2m}(d)}^{\varepsilon_{2m}(d)}].$$

Where if $(k, \varepsilon) \prec \pi$ and $\alpha_n(c) = \kappa_{2n}[c, c^*, c, c^*, \dots, c, c^*] = \kappa_{2n}[c^*, c, c^*, c, \dots, c^*, c]$ we have that

$$\kappa_\pi[c_{k_1(1)}^{\varepsilon_1(1)} \dots c_{k_{2m}(d)}^{\varepsilon_{2m}(d)}] = \prod_{V \text{ block of } \pi} \alpha_{|V|/2}(c).$$

In particular this quantity (which we will abusively denote by $\kappa_\pi(c)$) does not depend on (k, ε) . We therefore get

$$\left\| \sum_{k \in I} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi \in NC(d,m)} \kappa_\pi[c] \tilde{S}(a, \pi, d, m).$$

From this point the proof of Theorem 0.6 applies except that we use Lemma 3.3 and an iteration of Lemma 3.2 instead of Lemma 2.2 and an iteration of Lemma 2.1. \square

3.3. Lower bounds. Here we get some lower bounds on the norms we investigated before. For example the following minoration is classical:

Lemma 3.4. *Let $(c_k)_{k \in \mathbb{N}}$ be circular $*$ -free elements with $\|c\|_1 = 1$. Then for any finitely supported family of matrices $(a_{k_1, \dots, k_d})_{k_1, \dots, k_d \in \mathbb{N}}$ the following inequality holds:*

$$\left\| \sum_{k_1, \dots, k_d \in \mathbb{N}} a_{k_1, \dots, k_d} \otimes c_{k_1} \dots c_{k_d} \right\| \geq \max_{0 \leq l \leq d} \|M_l\|.$$

Proof. We use the following (classical) realization of free circular elements on a Fock space. Let $H = H_1 \oplus H_2$ be a Hilbert space with an orthonormal basis given by $(e_k)_{k \in \mathbb{N}} \cup (f_k)_{k \in \mathbb{N}}$ ((e_k) is a basis of H_1 and (f_k) of H_2). Let $\mathcal{F}(H) = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} H^{\otimes n}$ be the full Fock space constructed on H and for $k \in \mathbb{N}$ $s(k)$ (resp. $\tilde{s}(k)$) the operator of creation by e_k (resp. f_k). Define finally $c_k = s_k + \tilde{s}_k^*$. It is well-known that $(c_k)_{k \in \mathbb{N}}$ form of $*$ -free family of circular variables for the state $\langle \cdot, \Omega \rangle$ which is tracial on the C^* -algebra generated by the c_k 's.

Let K be the Hilbert space on which the a_k 's act ($K = \mathbb{C}^\alpha$ if $a_k \in M_\alpha(\mathbb{C})$). Then if P_k denotes the orthogonal projection from $\mathcal{F}(H) \rightarrow H_2^{\otimes k}$, for $0 \leq l \leq d$ the operator $(\text{id} \otimes P_l) \circ \sum_{k_1, \dots, k_d \in \mathbb{N}} a_{k_1, \dots, k_d} \otimes c_{k_1} \dots c_{k_d} \Big|_{K \otimes H_1^{\otimes d-l}}$ corresponds to M_l if it is viewed as an operator from $K \otimes H_1^{\otimes d-l} \simeq K \otimes \ell^2(\mathbb{N})^{\otimes d-l}$ to $K \otimes H_2^{\otimes l} \simeq K \otimes \ell^2(\mathbb{N})^{\otimes l}$ for the identification $H_1 \simeq \ell^2$ and $H_2 \simeq \ell^2$ with the orthonormal bases (e_k) and (f_k) .

This proves the Lemma. \square

We also prove the following Lemma which was stated in the introduction.

Lemma 3.5. *Let p be a prime number and define $a_{k_1, \dots, k_d} = \exp(2i\pi k_1 \dots k_d/p)$ for any $k_i \in \{1, \dots, p\}$.*

Then $\|(a_k)\|_2 = p^{d/2}$ and for any $1 \leq l \leq d-1$ the matrix M_l defined by $M_l = (a_{(k_1, \dots, k_l), (k_{l+1}, \dots, k_d)}) \in M_{p^l, p^{d-l}}(\mathbb{C})$ satisfies $\|M_l\| \leq p^{d/2} \sqrt{(d-1)/p}$.

Proof. Since $\|M_l\|^2 = \|M_l M_l^*\|$ we compute the matrix $M_l M_l^* \in M_{p^l, p^l}(\mathbb{C})$.

For any $s = (s_1, \dots, s_l)$ and $t = (t_1, \dots, t_l) \in \{1, \dots, p\}^l$ the s, t -th entry of $M_l M_l^*$ is equal to

$$\sum_{(k_{l+1}, \dots, k_d) \in \{1, \dots, p\}^{d-l}} \exp(2i\pi(s_1 \dots s_l - t_1 \dots t_l)k_{l+1} \dots k_d/p).$$

If $s_1 \dots s_l = t_1 \dots t_l \pmod p$ then this quantity is equal to p^{d-l} whereas otherwise, $\omega = \exp(2i\pi(s_1 \dots s_l - t_1 \dots t_l)/p)$ is a primitive p -th root of 1, and it is straightforward to check that for such an ω ,

$$\begin{aligned} \sum_{(k_{l+1}, \dots, k_d) \in \{1, \dots, p\}^{d-l}} \omega^{k_{l+1} \dots k_d} &= \sum_{k_{l+1}, \dots, k_{d-1}} \sum_{k_d=1}^p (\omega^{k_{l+1} \dots k_{d-1}})^{k_d} \\ &= \sum_{k_{l+1}, \dots, k_{d-1}} p \mathbf{1}_{k_{l+1} \dots k_{d-1} = 0 \pmod p} \\ &= p(p^{d-l-1} - (p-1)^{d-l-1}). \end{aligned}$$

We therefore have that

$$M_l M_l^* = (p^{d-l} - p(p-1)^{d-l-1}) \mathbf{1}_{s, t \in [p]^l} + p(p-1)^{d-l-1} \mathbf{1}_{s_1 \dots s_l = t_1 \dots t_l}_{s, t \in [p]^l}.$$

The norm of an $N \times N$ matrix with entries all equal to 1 is N .

Moreover if $[p]^l = \{(s_1, \dots, s_l)\}$ is decomposed depending on the value of $s_1 \dots s_l$ modulo p , the matrix $\mathbf{1}_{s_1 \dots s_l = t_1 \dots t_l}_{s, t \in [p]^l}$ is a block-diagonal matrix with blocks having all entries equal to 1. Its norm is therefore equal to

$$\begin{aligned} \max_{i \in [p]} |\{(s_1, \dots, s_l) \in [p]^l, s_1 \dots s_l = i \pmod p\}| \\ = |\{(s_1, \dots, s_l) \in [p]^l, s_1 \dots s_l = 0\}| = p^l - (p-1)^l. \end{aligned}$$

By the triangle inequality the norm of $M_l M_l^*$ is thus less than

$$\begin{aligned} p^{l+1}(p^{d-l-1} - (p-1)^{d-l-1}) + p(p-1)^{d-l-1}(p^l - (p-1)^l) \\ = p^d - p(p-1)^{d-1} \leq (d-1)p^{d-1} \end{aligned}$$

□

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