EXTENSION PROPERTIES OF PLANAR UNIFORM DOMAINS

VYRON VELLIS

Abstract. The classical bi-Lipschitz and quasisymmetric Schönflies theorems in the plane [Tuk80, BA56] are generalized in this paper for all planar uniform domains. Specifically, we show that if $U \subset \mathbb{R}^2$ is a uniform domain, then it has the following two extension properties: (1) every bi-Lipschitz map $f : \partial U \to \mathbb{R}^2$ that can be extended homeomorphically to $\mathbb{R}^2$, can also be extended bi-Lipschitzly to $\mathbb{R}^2$ and (2) if ∂U is relatively connected, then every quasisymmetric map $f : \partial U \to \mathbb{R}^2$ that can be extended homeomorphically to $\mathbb{R}^2$, can also be extended quasisymmetrically to $\mathbb{R}^2$. In higher dimensions, we show that if U is the exterior of a uniformly disconnected set in $\mathbb{R}^n$, then every bi-Lipschitz embedding $f : \partial U \to \mathbb{R}^n$ extends to a bi-Lipschitz homeomorphism of $\mathbb{R}^{n+1}$. The same is also true for quasisymmetric embeddings under the additional assumption that ∂U is relatively connected.

Contents

1. Introduction

Let X be a metric space, $E \subset X$ and $f : E \to X$ be a map in a class $\mathscr{F}$. When can f be extended to a mapping $F : X \to X$ in the same class? We are interested in the above extension question for the classes of bi-Lipschitz maps and quasisymmetric maps. Questions related to quasisymmetric extensions have been considered by

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Beurling and Ahlfors [BA56], Ahlfors [Ahl63, Ahl64], Carleson [Car74], Tukia and Väisälä [TV82, TV81, TV84], Väisälä [Väi86], Kovalev and Onninen [KO11] and Fujino [Fuj16]. Results related to bi-Lipschitz extension appear in the work of Tukia [Tuk80, Tuk81], David and Semmes [DS91], MacManus [Mac95] and Alestalo and Väisälä [AV97].

Tukia and Väisälä [TV84] showed that if $X = \mathbb{R}^p$ or $X = \mathbb{S}^p$ and $n > p$, then any quasisymmetric mapping $f : X \to \mathbb{R}^n$ extends to a quasisymmetric homeomorphism of $\mathbb{R}^n$ when f is locally close to being a similarity, and every bi-Lipschitz mapping $f : X \to \mathbb{R}^n$ extends to a bi-Lipschitz mapping of $\mathbb{R}^n$ when f is close to being an isometry. Later, Väisälä [Väi86] extended these results to all compact, C^1 or piecewise linear $(n-1)$-manifolds X in $\mathbb{R}^n$. Similar results appeared recently in the work of Azzam, Badger and Toro [ABT15]. The requirements on the embedding f in these three papers, ensured the homeomorphic extension of f to $\mathbb{R}^n$.

In this article we looka at the extension problem from a different perspective: assuming that there is a homeomorphic extension, when can we extend the mapping in question to a quasisymmetric or bi-Lipschitz homeomorphism? Given a metric space X we say that $E \subset X$ has the *quasisymmetric extension property* (resp. *bi-Lipschitz extension property*) in X or *QSEP* in short (resp. *BLEP*) if every quasisymmetric (resp. bi-Lipschitz) embedding $f : E \to X$ that can be extended as a homeomorphism of X can also be extended as a quasisymmetric (resp. bi-Lipschitz) homeomorphism of X.

When $X = \mathbb{R}$ or $X = \mathbb{S}^1$, trivially every subset of X has the BLEP in X but the same is not true in the quasisymmetric class. If $E = \{0\} \cup \{e^{-n!}\}_{n\geq 2}$, then $f : E \to \mathbb{R}$ with $f(x) = (-\log x)^{-1}$ is monotone and quasisymmetric but can not be extended quasisymmetrically in any open set containing the point 0 [Hei01, p. 89]. Thus, more regularity for sets E must be assumed. Trotsenko and Väisälä [TV99] introduced the notion of *relative connectedness*, a weak version of uniform perfectness, and as a corollary of their main theorem, *if $E \subset \mathbb{R}^n$ is not relatively connected, then there exists a quasisymmetric embedding $f : E \to \mathbb{R}^n$ that can be extended homeomorphically to $\mathbb{R}^n$ but not quasisymmetrically*; see §10.1. Conversely, we showed in [Vel16] that if $E \subset \mathbb{R}$ is relatively connected, then it has the QSEP in $\mathbb{R}$.

On the other hand, for each $n \geq 2$ there exists a relatively connected, compact and countable set $E_n \subset \mathbb{R}^n$ and a bi-Lipschitz embedding $f : E_n \to \mathbb{R}^n$ that admits a homeomorphic extension to $\mathbb{R}^n$ but not a quasisymmetric extension [Vel16, Theorem 5.1]. These examples show that in dimensions $n \geq 2$ relative connectedness does not suffice for either the QSEP or the BLEP and the geometry of the complement of E comes into play.

It follows from the celebrated work of Ahlfors [Ahl63], Beurling and Ahlfors [BA56] and Tukia [Tuk80] that $\mathbb{R}$ and $\mathbb{S}^1$ have both extension properties in $\mathbb{R}^2$. In this paper we extend their results to boundaries of planar *uniform domains*, a broad family of domains in $\mathbb{R}^2$ whose local geometry resembles that of the disk and of the upper half-plane. Uniform domains were already known to be extension domains for Sobolev spaces [Jon81] and, more generally, Newtonian spaces [BS07].

Theorem 1.1. *Let $U \subset \mathbb{R}^2$ be a c-uniform domain and $f : \partial U \to \mathbb{R}^2$ be an embedding that can be extended homeomorphically to $\overline{U}$.*

(1) If f is L-bi-Lipschitz, then f extends to an L'-bi-Lipschitz homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $L' > 1$ depending only on L and c.

(2) If ∂U is C-relatively connected and f is η-quasisymmetric, then f extends to an η'-quasisymmetric homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ with η' depending only on η, c and C.

The second part of Theorem 1.1 can be viewed as a converse of a boundary quasiconformal extension result of Väisälä [Väi85]: if $U, U' \subset \mathbb{R}^2$ are uniform domains and $f : U \to U'$ is a quasiconformal homeomorphism that can be extended homeomorphically to $\overline{U}$, then f can be extended quasisymmetrically to $\overline{U}$; see Lemma 2.10.

Roughly speaking, uniformity is a combination of two other notions: a domain is uniform if every pair of points can be joined by a curve whose length is comparable to the distance between the points (*quasiconvexity*) and the curve does not go too close to the boundary of the domain (*John property*); see §2.5 for precise definition. The assumption of uniformity of U is somewhat necessary for both extensions as neither quasiconvexity nor John property alone are sufficient; see §10.

In $\mathbb{R}^3$, Theorem 1.1 fails in both cases as there exists a bi-Lipschitz embedding of $\mathbb{S}^2$ into $\mathbb{R}^3$ that can be extended homeomorphically to $\mathbb{R}^3$ but not quasisymmetrically [Tuk80, §15].

As a corollary, we obtain a sufficient condition for sets E to satisfy the QSEP and the BLEP in $\mathbb{R}^2$. The arguments apply verbatim in the case that $E \subset \mathbb{S}^2$.

Corollary 1.2. *If $E \subset \mathbb{R}^2$ is such that each component of $\mathbb{R}^2 \setminus \overline{E}$ is uniform with the same constant, then E has the BLEP in $\mathbb{R}^2$. If additionally E is relatively connected, then it has the QSEP in $\mathbb{R}^2$.*

The tameness of Cantor sets in $\mathbb{R}^2$ implies that in Theorem 1.1 the assumption of homeomorphic extension of f to $\mathbb{R}^2 \setminus E$ can be dropped when E is totally disconnected. However, in higher dimensions, due to the existence of wild Cantor sets, an increase in dimension is needed. For simple examples of wild Cantor sets we refer to Daverman [Dav07].

Moreover, in the plane, the complement of a closed set $E \subset \mathbb{R}^2$ with empty interior is uniform if and only if E is *uniformly disconnected* [Mac99] but this is not true in $\mathbb{R}^n$ when $n \geq 3$. Uniform disconnectedness is in a sense the opposite of uniform perfectness: for each point x there exists an "isolated island" $E' \subset E$ of practically any diameter whose distance from the rest of E is at least a fixed multiple of its diameter. In dimensions $n \geq 3$, uniform disconnectedness of E can be used as a natural analogue of uniformity of $\mathbb{R}^n \setminus E$.

Theorem 1.3. *Let $n \geq 3$ be an integer, let E be a c-uniformly disconnected subset of $\mathbb{R}^n$ and let $f : E \subset \mathbb{R}^n$.*

(1) If f is L-bi-Lipschitz, then it extends to an L'-bi-Lipschitz homeomorphism $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $L' > 1$ depending only on L, c and n.
(2) If E is C-relatively connected and f is η-quasisymmetric, then f extends to an η'-quasisymmetric homeomorphism $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with η' depending only on η, c, C and n.

In the statement of Theorem 1.3, E is identified with the set $E \times \{0\} \subset \mathbb{R}^{n+1}$.

In [Väi98], Väisälä asked if the *Klee trick* holds true in the quasisymmetric class, i.e., if $E \subset \mathbb{R}^n$ is compact and $f : E \to \mathbb{R}^m$ is a quasisymmetric embedding, is there a quasisymmetric homeomorphism $F : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ that extends f? Since all uniformly perfect and uniformly disconnected sets quasisymmetrically embed in $\mathbb{R}$

[DS91], Theorem 1.3 provides an affirmative answer for this class of sets. However, the general case remains open.

1.1. **Organization of the paper.** In §2 we review the notions of quasisymmetric maps, uniform domains, relatively connected sets, uniformly disconnected sets and Whitney-type decompositions.

In §5 and §7 we reduce the proof of Theorem 1.1 to the case where U is the complement of a compact perfect set, and the proof of Theorem 1.3 to the case where E is compact and perfect. In §6, given a uniform domain $U \subset \mathbb{R}^2$ and a bi-Lipschitz embedding $f : \partial U \to \mathbb{R}^2$ (resp. uniform domain $U \subset \mathbb{R}^2$ with relatively connected boundaty and a quasisymmetric embedding $f : \partial U \to \mathbb{R}^2$) that can be extended homeomorphically to $\mathbb{R}^2$, we extend f bi-Lipschitzly (resp. quasisymmetrically) to $\mathbb{R}^2 \setminus U$. After that reduction, it suffices to extend f to a map $f : \overline{U} \to \overline{U'}$ where U' is a uniform domain.

The extension of f to $\overline{U}$ follows Carleson's method [Car74]. The main idea is the construction of two combinatorially equivalent Whitney-type decompositions $\mathscr{Q}$ and $\mathscr{Q}'$ for U and U' respectively. That is, $\mathscr{Q}$ (resp. $\mathscr{Q}'$) is a family of mutually disjoint open subsets of U (resp. U') such that the union of their closures is the whole U (resp. U'), the diameter of each element of $\mathscr{Q}$ (resp. $\mathscr{Q}'$) is comparable to its distance to ∂U (resp. $\partial U'$) and there exists a homeomorphism of $\overline{U}$ onto $\overline{U'}$ that maps each element of $\mathscr{Q}$ onto exactly one element of $\mathscr{Q}'$. Moreover, the boundary of every domain in $\mathscr{Q}$ and $\mathscr{Q}'$ is a finite union of L-bi-Lipschitz circles whose mutual distances and diameters are bounded below by a constant $d > 0$. We show in §4 that such domains possess both the BLEP and the QSEP.

The main novelty in this approach is that, unlike a quasidisk, the boundary of a uniform domain may have uncountably many components. Nevertheless, we show in §3 that the boundary of a uniform domain satisfies a weak form of uniform connectedness: given a point $x \in \partial U$, for any $r > 0$ there exists a closed set $A \subset \partial U$ containing x whose distance from $\partial U \setminus A$ is at least a constant multiple of r.

In §9, using the results of §3, we construct the decompositions $\mathscr{Q}$ and $\mathscr{Q}'$ and we prove Theorem 1.1. Towards the construction we distinguish two cases: one for the part of U around non-degenerate components of ∂U, which we treat in §8, and another for the rest of U which we treat in §9. In the first case the decomposition resembles that of the exterior of a quasidisk (although extra care has to be taken for all of the components of ∂U around the quasidisk) while in the second U resembles the exterior of a uniformly disconnected set.

The proof of Theorem 1.3 relies on a uniformization result for Cantor sets with bounded geometry that generalizes a 2-dimensional result of MacManus [Mac99] to higher dimensions. Namely, in §11 we show that a compact set $E \subset \mathbb{R}^n$ is uniformly perfect and uniformly disconnected if and only if there exists a quasiconformal homeomorphism of $\mathbb{R}^{n+1}$ mapping E onto the standard middle-third Cantor set $\mathcal{C} \subset \mathbb{R}$.

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2. Preliminaries

A set E with one point is called a *degenerate* set. A non-degenerate compact connected set is called a *continuum*.

For the rest of the paper, for all integers $m < n$, we identify sets $E \subset \mathbb{R}^m$ with sets $E \times \{0\}^{n-m} \subset \mathbb{R}^n$ via the natural embedding of $\mathbb{R}^m$ into $\mathbb{R}^n$

$$(x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, 0, \dots, 0).$$

2.1. **Mappings.** A homeomorphism $f\colon D \to D'$ between two domains in $\mathbb{R}^n$ is called K-*quasiconformal* for some $K \geq 1$ if, for all $x \in D$, f satisfies the distortion inequality

$$\limsup_{r\to 0} \frac{\sup_{y\in\partial B^n(x,r)} |f(x)-f(y)|}{\inf_{y\in\partial B^n(x,r)} |f(x)-f(y)|} \leq K.$$

An embedding f of a metric space X into a metric space Y is said to be η-*quasisymmetric* if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that for all $x, a, b \in X$ with $x \neq b$

$$\frac{d_Y(f(x), f(a))}{d_Y(f(x), f(b))} \leq \eta\left(\frac{d_X(x,a)}{d_X(x,b)}\right)$$

where d_X and d_Y are the metrics of X and Y respectively. An η-quasisymmetric map with $\eta(t) = C\max\{t^\alpha, t^{1/\alpha}\}$ for some $C > 1$ and $\alpha > 1$ is known in literature as *power quasisymmetric map*.

For doubling connected metric spaces it is known that the quasisymmetric condition is equivalent to a weaker (but simpler) condition known in literature as *weak quasisymmetry*. Recall that a metric space is C-*doubling* ($C > 1$) if every ball of radius r can be covered by at most C balls of radius at most $r/2$.

Lemma 2.1 ([WZ17, Theorem 4.1])**.** *Suppose that X and Y are C-doubling and c-uniformly perfect metric spaces. Suppose also that $f : X \to Y$ is an embedding for which there are constants $h > 0$ and $H \geq 1$ such that for all $x, a, b \in X$,*

$$d_X(x,a) \leq h d_X(x,b) \qquad \text{implies} \qquad d_Y(f(x), f(a)) \leq H d_Y(f(x), f(b)).$$

Then, f is η-quasisymmetric for some η depending only on c, C, h and H.

A quasisymmetric mapping between two domains in $\mathbb{R}^n$ is quasiconformal. The converse holds true for uniform domains; see Lemma 2.10. For a systematic treatment of quasiconformal mappings see [Väi71].

A map $f\colon X \to Y$ between metric spaces is L-*bi-Lipschitz* for some $L \geq 1$ if

$$L^{-1} d_X(x,y) \leq d_Y(f(x), f(y)) \leq L d_X(x,y)$$

for all $x, y \in X$. Note that an L-bi-Lipschitz mapping is L^2t-quasisymmetric.

A weaker notion of bi-Lipschitz mappings is that of *bounded length distortion* (*BLD*) mappings. A mapping $f\colon X \to Y$ between metric spaces is L-BLD for some $L \geq 1$ if

$$L^{-1}\ell(\gamma) \leq \ell(f(\gamma)) \leq \ell(\gamma)$$

for all paths $\gamma : [0,1] \to X$. Here and for the rest, ℓ denotes the length of a path. Clearly, L-bi-Lipschitz mappings are L-BLD mappings but BLD mappings need not be bi-Lipschitz even if they are homeomorphisms. However, BLD homeomorphisms between quasiconvex spaces are bi-Lipschitz.

Lemma 2.2. *Let $f : X \to Y$ be an L-BLD homeomorphism between two c-quasiconvex metric spaces. Then f is Lc-bi-Lipschitz.*

A mapping $f\colon X \to Y$ between metric spaces is a (λ, L)-*quasisimilarity* for some $\lambda > 0$ and $L \geq 1$ if $L^{-1}\lambda d_X(x,y) \leq d_Y(f(x), f(y)) \leq L\lambda d_X(x,y)$ for all $x, y \in X$. Note that $(\lambda, 1)$-quasisimilarities are similarities, $(1, L)$-quasisimilarities are L-bi-Lipschitz and $(1,1)$-quasisimilarites are isometries.

A simple curve $\Gamma \subset \mathbb{R}^2$ is a K-*quasicircle* with $K \geq 1$ if $\Gamma = f(\mathbb{S}^1)$ or $\Gamma = f(\mathbb{R})$ for some K-quasiconformal $f : \mathbb{R}^2 \to \mathbb{R}^2$. A simply connected domain $D \subset \mathbb{R}^2$ is called a K-*quasidisk* if ∂D is a K-quasicircle. A geometric characterization of quasicircles was given by Ahlfors [Ahl63] in terms of the the bounded turning property; see §2.5.

A curve $\Gamma \subset \mathbb{R}^2$ is called an L-*chordarc circle* with $L \geq 1$ if $\Gamma = f(\mathbb{S}^1)$ for some (λ, L)-quasisimilarity $f : \mathbb{R}^2 \to \mathbb{R}^2$. A simply connected domain $D \subset \mathbb{R}^2$ is called an L-*chordarc disk* if ∂D is an L-chordarc circle.

2.2. **Relative distance.** For two non-degenerate closed sets $E, E' \subset \mathbb{R}^n$ define the *relative distance*

$$\text{dist}^*(E,E') = \frac{\text{dist}(E,E')}{\min\{\operatorname{diam} E, \operatorname{diam} E'\}}$$

where $\text{dist}(E',E') = \min\{|x-y| : x \in E, y \in E'\}$. If both E and E' have infinite diameter we set $\text{dist}^*(E,E') = 0$.

If $E, E' \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity, then $d^*(f(E), f(E')) = d^*(E,E')$. In general, if $f : E \cup E' \to Y$ is η-quasisymmetric, then

$$\frac{1}{2}\phi\left(\frac{\text{dist}(E,E')}{\operatorname{diam} E}\right) \leq \frac{\text{dist}(f(E), f(E'))}{\operatorname{diam} f(E)} \leq \eta\left(2\frac{\text{dist}(E,E')}{\operatorname{diam} E}\right) \tag{2.1}$$

where $\phi(t) = (\eta(t^{-1}))^{-1}$; see for example [Tys98, p. 532].

2.3. **Relatively connected sets.** Relatively connected sets were first introduced by Trotsenko and Väisälä [TV99] in the study of spaces for which every quasisymmetric mapping is power quasisymmetric. A metric space X is called *c-relatively connected* for some $c \geq 1$ if for any $x \in X$ and any $r > 0$ either $\overline{B}(x,r) = \{x\}$ or $\overline{B}(x,r) = X$ or $\overline{B}(x,r) \setminus B(x, r/c) \neq \emptyset$. The definition given in [TV99] is equivalent to the one above quantitatively [TV99, Theorem 4.11].

A connected space is c-relatively connected for any $c > 1$. Relative connectedness is a weak form of the well known notion of uniform perfectness. A metric space X is *c-uniformly perfect* for some $c > 1$ if for all $x \in X$, $\overline{B}(x,r) \neq X$ implies $\overline{B}(x,r) \setminus B(x,r/c) \neq \emptyset$. The difference between the two notions is that relatively connected sets allow isolated points. In particular, if E is c-uniformly perfect, then it is c'-relatively connected for all $c' > c$, and if E is c-relatively connected and perfect, then it is $(2c+1)$-uniformly perfect [TV99, Theorem 4.13].

The connection between relative connectedness and quasisymmetric mappings is illustrated in the following theorem from [TV99].

Lemma 2.3 ([TV99, Theorem 6.20])**.** *A subset E of a metric space X is relatively connected if and only if every quasisymmetric map $f : E \to X$ is power quasisymmetric.*

It easily follows from its definition that the image of a relatively connected (resp. uniformly perfect) space under a quasisymmetric mapping is relatively connected (resp. uniformly perfect) quantitatively. We conclude the discussion on relatively connected sets with the following remark.

Remark 2.4. *Suppose that X is a c-uniformly perfect metric space and $E \subset X$ is compact. Then* $\operatorname{dist}(E, E \setminus X) \leq c \operatorname{diam} E$.

2.4. **Uniformly disconnected sets.** In [DS97], David and Semmes introduced a scale-invariant version of total disconnectedness towards a uniformization of all metric spaces that are quasisymmetric to the standard middle-third Cantor set $\mathcal{C}$. A metric space X is *c-uniformly disconnected* for some $c \geq 1$ if for all $x \in X$ and all positive $r < \frac{1}{4} \operatorname{diam} X$, there exists $E \subset X$ containing x such that $\operatorname{diam} E \leq r$ and $\operatorname{dist}(E, X \setminus E) \geq r/c$.

Theorem 2.5 ([DS97, Proposition 15.11])**.** *A metric space is quasisymmetrically homeomorphic to $\mathcal{C}$ if and only if it is compact, doubling, uniformly disconnected and uniformly perfect.*

This result was later improved by MacManus [Mac99] for sets in $\mathbb{R}^2$; see §11. In the same article, MacManus found an elegant connection between planar uniform domains and uniformly disconnected sets: *a set $E \subset \mathbb{R}^2$ with empty interior is uniformly disconnected if and only if its complement is uniform* [Mac99, Theorem 1.1]. In higher dimensions only the necessity is true.

It is easy to check that if X is a c-uniformly disconnected space and $f : X \to Y$ is η-quasisymmetric, then $f(X)$ is c'-uniformly disconnected with c' depending only on η and c.

2.5. **Uniform domains.** A domain $U \subset \mathbb{R}^n$ is said to be *c-uniform* for some $c \geq 1$ if for all $x, y \in U$, there exists a curve $\gamma \subset U$ joining x with y such that

(1) $\ell(\gamma) \leq c|x - y|$ and
(2) for all $z \in \gamma$, $\operatorname{dist}(z, \partial U) \geq c^{-1} \min\{|x - z|, |y - z|\}$.

A curve γ as in the above definition is called a *c-cigar curve*. The definition above is equivalent to the classical definition of Martio and Sarvas [MS79] quantitatively; see Theorem 2.10 in [Väi88b].

Metric spaces for which, for every two points there exists curve satisfying the first property of uniformity are called *c-quasiconvex*. If in the definition of quasiconvexity the length of curves is replaced by diameter, then the space is called *c-bounded turning*. Metric spaces for which, for every two points there exists curve satisfying the second condition are called *c-John spaces*.

A simple curve $\Gamma \subset \mathbb{R}^2$ is a K-quasicircle if and only if it is c-bounded turning with c and K being related quantitatively [Ahl63]. A simple curve $\Gamma \subset \mathbb{R}^2$ is an L-chordarc circle if and only if it is c-quasiconvex with c and L being related quantitatively [JK82]. Finally, a simply connected domain $D \subset \mathbb{R}^2$ is c-uniform if and only if it is a K-quasidisk (with K and c quantitatively related) and D is a c-John domain if and only if its complement is C-bounded turning (with c and C quantitatively related) [NV91].

Two remarks are in order.

Remark 2.6. *It is easy to check that all curves in the definition of uniform domains can be chosen to be simple. For the rest of the paper, all cigar curves are assumed to be simple.*

Remark 2.7. *Let $U \subset \mathbb{R}^n$ be a c-uniform domain, $x, y \in U$ and γ be a c-cigar curve joining x, y. Then,*

$$\mathrm{dist}(\gamma, \partial U) \geq (2c)^{-1} \min\{\mathrm{dist}(x, \partial U), \mathrm{dist}(y, \partial U)\}.$$

Indeed, set $\epsilon = \min\{\mathrm{dist}(x, \partial U), \mathrm{dist}(y, \partial U)\}$ and let $z \in \gamma$. If $z \in \overline{B}^n(x, \epsilon/2) \cup \overline{B}^n(y, \epsilon/2)$, then $\mathrm{dist}(z, \partial U) \geq \epsilon/2$. If z is in the exterior of these balls, then $\mathrm{dist}(z, \partial U) \geq c^{-1} \min\{|z-x|, |z-y|\} \geq (2c)^{-1}\epsilon$.

The following proposition describes the geometry of uniform domains. For the proof see Corollary 2.33 in [MS79], Theorem 2 and Lemma 3 in [Geh82] and Theorem 1.1 in [Her87].

Proposition 2.8. *Let U be a c-uniform domain.*

1. *(Boundary components) Each component of ∂U is either a point or a K-quasicircle for some $K > 1$ depending only on c.*
2. *(Relative distance) If A_1, A_2 are non-degenerate components of ∂U, then* $\mathrm{dist}^*(A_1, A_2) \geq (2c)^{-2}$.
3. *(Porosity) For every $x \in \overline{U}$ and every $0 < r \leq \frac{1}{4} \mathrm{diam}\, U$, there exists $x' \in \partial B^2(x, r)$ such that $B^2(x, r/c) \subset U$.*

The porosity of ∂U implies that if U is bounded then, there exists a point $x \in U$ such that $B^2(x, \frac{1}{4c} \mathrm{diam}\, U) \subset U$.

Although Proposition 2.8 provides a lot of information about the boundaries of uniform domains, it fails to characterize them. Namely, if $E \subset \mathbb{R}$ is a Cantor set with $\mathcal{H}^1(E) > 0$, then $\mathbb{R}^2 \setminus E$ trivially satisfies all three properties of Proposition 2.8 but it is not uniform. If U is finitely connected and satisfies properties (1) and (2), then it is uniform. We record this observation as a remark.

Remark 2.9. *Let $U \subset \mathbb{R}^2$ be a c-uniform domain and $D \subset U$ be a K-quasidisk such that* $\mathrm{dist}^*(\overline{D}, \partial U) \geq d > 0$*. Then $U \setminus \overline{D}$ is c'-uniform with c' depending only on c, K and d.*

A quasiconformal homeomorphism between uniform domains of $\mathbb{R}^n$ is quasisymmetric quantitatively.

Lemma 2.10 ([Väi85, Theorem 5.6])**.** *Let U, U' be c-uniform domains in $\overline{\mathbb{R}^n}$ and $f : U \to U'$ be a K-quasiconformal homeomorphism. Then f is η-quasisymmetric with η depending only on K, c and n.*

We conclude the discussion on uniform domains with two results on the invariance of uniformity under quasisymmetric mappings. The first result says that uniform domains are preserved under quasisymmetric mappings while the second result roughly says that complements of uniform domains are preserved under quasisymmetric mappings.

Lemma 2.11 ([GO79, Corollary 3])**.** *If $U \subset \mathbb{R}^2$ is c-uniform and $f : U \to \mathbb{R}^2$ is η-quasisymmetric, then $f(U)$ is c'-uniform with c' depending only on c and η.*

Lemma 2.12 ([Väi88a, Theorem 5.6])**.** *If $E \subset \mathbb{R}^2$ is closed, $\mathbb{R}^2 \setminus E$ is c-uniform and $f : E \to \mathbb{R}^2$ is η-quasisymmetric, then $\mathbb{R}^2 \setminus f(E)$ is c'-uniform with c' depending only on c and η.*

2.6. **Whitney-type decompositions.** Let D be a proper open subset of $\mathbb{R}^2$. A collection of sets $\mathscr{Q}$ is called a (L,c)-*Whitney-type decomposition for* D for some $c>1$ and $L>1$, if the following properties hold true.

(1) The elements of $\mathscr{Q}$ are L-chordarc disks and are mutually disjoint.
(2) $D=\bigcup_{Q\in\mathscr{Q}}\overline{Q}$.
(3) For all $Q\in\mathscr{Q}$, $c^{-1}\operatorname{diam}Q\leq\operatorname{dist}(Q,\partial D)\leq c\operatorname{diam}Q$.
(4) If $Q_1,Q_2\in\mathscr{Q}$ are such that $\overline{Q_1}\cap\overline{Q_2}\neq\emptyset$ then either $\overline{Q_1}$ and $\overline{Q_2}$ intersect only at a point, or their intersection is an arc Γ satisfying
$$\operatorname{diam}\Gamma\geq c^{-1}\max\{\operatorname{diam}Q_1,\operatorname{diam}Q_2\}.$$

It is well known that every proper open subset of $\mathbb{R}^2$ has a Whitney-type decomposition [Ste70, Theorem IV.1.1].

Two Whitney-type decompositions $\mathscr{Q}$, $\mathscr{Q}'$ of open sets D, D', respectively, are called *combinatorially equivalent* if there exists a homeomorphism $f:D\to D'$ such that for each $Q\in\mathscr{Q}$ there exists $Q'\in\mathscr{Q}$ with $f(Q)=Q'$.

3. Separation of boundary components for planar uniform domains

For this section, fix an unbounded c-uniform domain $U\subset\mathbb{R}^2$ with bounded boundary. The goal of this section is to break the boundary of ∂U into sets that are contained in chord-arc disks and are far from the boundary of those disks.

3.1. **Square thickening.** Here we show that, given a continuum $E\subset\mathbb{R}^2$ and some $\epsilon>0$, there exists a chordarc disk D containing E so that each point of ∂D is of distance roughly $\epsilon>0$ from E.

We first review some terminology from [Mac99]. Let $\epsilon>0$. We define the square grid
$$\mathcal{G}_\epsilon=\{[m\epsilon,(m+1)\epsilon]\times[n\epsilon,(n+1)\epsilon]:m,n\in\mathbb{Z}\}$$
and the 1-skeleton of the grid
$$\mathcal{G}_\epsilon^1=\{e: e\text{ is an edge of some square }S\in\mathcal{G}_\epsilon\}$$
Given a bounded set $W\subset\mathbb{R}^2$ let W^ϵ be the union of the elements of $\mathcal{G}_\epsilon$ that intersect W. Let $\mathcal{T}_\epsilon(W)=(W^{4\epsilon})^\epsilon$.

Lemma 3.1. *There exists a decreasing homeomorphism $L:(0,+\infty)\to(1,+\infty)$ with the following property. If $E\subset\mathbb{R}^2$ is a continuum and $\epsilon>0$, there exists an $L(\epsilon)$-chordarc disk $D_\epsilon\subset\mathbb{R}^2$ containing E such that, for all $x\in\partial D$,*
$$\epsilon\operatorname{diam}E\leq\operatorname{dist}(x,E)\leq 8\epsilon\operatorname{diam}E.$$

Proof. If $\epsilon\geq 3$, then the set $\gamma_E=\{x\in\mathbb{R}^2\colon\operatorname{dist}(x,E)=\epsilon\operatorname{diam}E\}$ is the boundary of an L_0-chordarc disk with L_0 being a universal constant [Bro72, Lemma 1].

For the rest of the proof we fix $\epsilon<3$ and set $\delta=\epsilon\operatorname{diam}E$. By Lemma 2.1 in [Mac99], $\mathcal{T}_\delta(E)$ is the closure of a domain whose boundary consists of at most N_1/ϵ disjoint Jordan curves, each of which is a subset of $\mathcal{G}_\delta$. Here, the number $N_1>1$ is a universal constant. Moreover, the distance from any boundary point of $\mathcal{T}_\delta(E)$ to E is less than 8δ and greater than δ. Let D_ϵ be the domain bounded by the outermost component of $\partial\mathcal{T}_\delta(E)$, that is, D_ϵ is the exterior of the unbounded component of $\mathbb{R}^2\setminus\mathcal{T}_\delta(E)$. Then, $\delta\leq\operatorname{dist}(x,E)\leq 8\delta$ for all $x\in\partial D_\epsilon$.

Notice now that, for some universal $N_2>1$, there are at most N_2/ϵ squares of $\mathcal{G}_\delta$ intersecting an 8δ-neighborhood $N(E,8\delta)$. Therefore, there are at most $(N_2/\epsilon)^{N_2/\epsilon}$

different ways to form D. As each Jordan curve consisting of edges of $\mathcal{G}^1_\delta$ is a chordarc circle, ∂D is L-bi-Lipschitz for some constant $L > 1$ depending only on ϵ. $\square$

As $\epsilon \to \infty$, the disk D_ϵ constructed in Lemma 3.1 resembles a big disk and $L(\epsilon) \to 1$. On the other hand, if E is not the closure of a chordarc disk, $L(\epsilon)$ may increase without control as $\epsilon \to 0$. Nevertheless, we show in the next lemma that if E is the closure of a quasidisk, then the disk D_ϵ given in Lemma 3.1 is always a K'-quasidisk with K' depending only on K.

Lemma 3.2. *Suppose that $E \subset \mathbb{R}^2$ is the closure of a bounded K-quasidisk and let D_ϵ be the $L(\epsilon)$-chordarc disk of Lemma 3.1. Then, for all $\epsilon > 0$, D_ϵ is a K'-quasidisk with K' depending only on K.*

Proof. Fix $\epsilon > 0$. As with the proof of Lemma 3.1, we may assume that $\epsilon < 3$. Let $\delta = \epsilon \operatorname{diam} E$, $D = D_\epsilon$, $\Gamma = \partial D$ and $\gamma = \partial E$. Since γ is a K-quasicircle, it satisfies the c-bounded turning property for some $c > 1$ depending only on K. We show that Γ is $136c$-bounded turning and the lemma follows.

Let $x_1, x_2 \in \Gamma$. Since, Γ is a polygonal curve with edges in $\mathcal{G}^1_\delta$, it is enough if we assume that x, y are in non-adjacent edges of Γ. Therefore, $|x_1 - x_2| \geq \delta$. Contrary to our claim assume that there exist $x_3, x_4 \in \Gamma$ such that x_3 and x_4 are in different components of $\Gamma \setminus \{x_1, x_2\}$ and $\min\{|x_3 - x_1|, |x_4 - x_1|\} \geq 68c|x_1 - x_2|$. For each $i \in \{1, 2, 3, 4\}$ let x_i' be the point in γ closest to x_i. Then, $|x_i - x_i'| \in (\delta, 8\delta)$ for each $i = 1, 2, 3, 4$ and x_3' and x_4' are in different components of $\gamma \setminus \{x_1', x_2'\}$. Therefore, for $i = 3, 4$, $|x_1' - x_i'| \geq |x_1 - x_i| - 16\delta \geq 34c|x_1 - x_2|$ while $2c|x_1 - x_2'| \leq 2c|x_1 - x_2| + 32c\delta \leq 34c|x_1 - x_2|$. Therefore, $\min\{|x_1' - x_3'|, |x_1' - x_4'|\} > c|x_1' - x_2'|$ and the c-bounded turning property for γ is violated. $\square$

Remark 3.3. *In addition to Lemma 3.2, note that for each $M > 0$ there exists $L(M) > 1$ such that any subarc γ of ∂D_ϵ whose endpoints x, y satisfy $|x - y| \leq M\epsilon$, is an $L(M)$-bi-Lipschitz arc.*

3.2. **Local separation in the boundary of U.** Here, given a compact $A \subset U$ that is disjoint from $\overline{\partial U \setminus A}$ and some $\epsilon > 0$ sufficiently small, we construct a chordarc circle that separates the two sets A and $\partial U \setminus A$ and its distance from ∂U is at least a fixed multiple of ϵ.

Lemma 3.4. *Suppose that $A \subset \partial U$ is compact and disjoint from $\overline{\partial U \setminus A}$ and let $\epsilon \leq (32c)^{-1} \operatorname{dist}(A, \partial U \setminus A)$. There exists $C > 1$ and $L > 1$ depending only on c and $\epsilon(\operatorname{diam} A)^{-1}$, and there exists an L-chordarc disk Δ with the following properties.*

1. *$\partial\Delta \subset U$ and $\overline{\Delta} \cap \partial U = A$,*
2. *$\operatorname{dist}(z, \partial\Delta) \leq 8\epsilon$ for all $z \in A$,*
3. *$C^{-1} \operatorname{diam} A \leq \operatorname{dist}(z', \partial U)$ for all $z' \in \partial\Delta$.*

Proof. As in the proof of Lemma 3.1, consider the thickening $\mathcal{T}_\epsilon(A)$. Then, $\partial\mathcal{T}_\epsilon(A)$ consists of at most N_0 components in U, each being an L'-chordarc circle with N_0 and L' depending only on c and $\epsilon(\operatorname{diam} A)^{-1}$. The choice of ϵ implies that $\mathcal{T}_\epsilon(A) \cap \partial U = A$. Let D be the closure of $\mathbb{R}^2 \setminus V$ where V is the unbounded component of $\mathbb{R}^2 \setminus \mathcal{T}_\epsilon(A)$. Observe that $\operatorname{dist}(x, A) \leq 8\epsilon$ for all $x \in \partial D$.

We claim that $D \cap \partial = A$. Contrary to the claim, assume that there exists $x \in (\partial U \setminus A) \cap D$. Note that $\operatorname{dist}(x, \partial D) > \frac{1}{2} \operatorname{dist}(A, \partial U \setminus A)$ as otherwise $\operatorname{dist}(x, A) <$

$\operatorname{dist}(A, \partial U \setminus A)$. Let $y \in \overline{U}$ be exterior to D and satisfying $\operatorname{dist}(y, A) \geq \operatorname{dist}(A, \partial U \setminus A)$. Let γ be a c-cigar curve joining x and y in U and let $z \in \partial D \cap \gamma$. Then,

$$\begin{aligned} \operatorname{dist}(A, \partial U \setminus A) &> 16c\epsilon \geq 2c \operatorname{dist}(z, A) \geq 2\min\{|x-z|, |y-z|\} \\ &\geq 2\min\{\operatorname{dist}(x, \partial D), \operatorname{dist}(y, \partial D)\} \geq \operatorname{dist}(A, \partial U \setminus A) \end{aligned}$$

which is a contradiction and the claim follows.

If D has only one component, then set $\Delta = D$ and the proof is complete.

Let $D_1, \dots, D_n$ be the components of D. Then, $n \leq N_0$ and $\operatorname{dist}(D_i, D_j) \geq \delta_0 \operatorname{diam} A$ for some $\delta_0 > 0$ and $N_0 > 1$ depending only on c and $\epsilon(\operatorname{diam} A)^{-1}$. Set $D^{(0)} = D$ and $U^{(0)} = U \setminus \overline{D}$. By Remark 2.9, $U^{(0)}$ is c_0-uniform for some $c_0 > 1$ depending only on c and $\epsilon(\operatorname{diam} A)^{-1}$.

Inductively, suppose that for some $0 \leq i \leq n-1$, $D^{(i)}$ is a union of $n-i$ many L_i-chord-arc disks $D_1^{(i)}, \dots, D_{n-i}^{(i)}$ such that

(1) $\partial D^{(i)} \subset U$,
(2) $\operatorname{dist}(\partial D^{(i)}, \partial U) \geq C_i \operatorname{diam} A$ and $\operatorname{dist}(D_j^{(i)}, D_{j'}^{(i)}) \geq \delta_i \operatorname{diam} A$,
(3) $U^{(i)} = U \setminus \overline{D^{(i)}}$ is c_i-uniform

for some $C_i > 0$, $\delta_i > 0$, $c_i > 1$ and $L_i > 1$ depending only on c, $\epsilon(\operatorname{diam} A)^{-1}$ and i. Let $x \in \partial D_1^{(i)}$, $y \in \partial D_2^{(i)}$ and $\gamma \subset U^{(i)}$ be a simple c_i-cigar curve joining x_1 with x_2. Applying Lemma 3.1 with $E = \gamma$ and $\epsilon = (32c_i)^{-1} \min\{\delta_i, C_i\}$ we have an L'-chordarc disk D' containing γ. Set $D_1^{(i+1)} = D_1^{(i)} \cup D'$, $D_j^{(i+1)} = D_{j+1}^{(i)}$ for $j = 2, \dots, n-i-1$, $D^{(i+1)} = \bigcup_{j=1}^{n-i-1} D_j^{(i+1)}$ and $U^{(i+1)} = U \setminus \overline{D^{(i+1)}}$. Then,

(1) each $D_j^{(i+1)}$ is a L_{i+1}-chordarc disk with $\partial D_j^{(i+1)} \subset U$,
(2) $\operatorname{dist}(\partial D^{(i+1)}, \partial U) \geq d_{i+1} \operatorname{diam} A$,
(3) $\operatorname{dist}(D_j^{(i+1)}, D_{j'}^{(i+1)}) \geq \delta_{i+1} \operatorname{diam} A$ and
(4) $U^{(i+1)}$ is c_{i+1}-uniform

for some $d_{i+1} > 0$, $\delta_{i+1} > 0$, $c_{i+1} > 1$ and $L_{i+1} > 1$ depending only on d_i, δ_i, c_i and L_i.

Set $\Delta = D^{(n-1)}$ and note that Δ satisfies the desired properties with constants depending only on c and $\epsilon(\operatorname{diam} A)^{-1}$. $\square$

The chordarc disk Δ constructed in the proof of Lemma 3.4 is denoted by $V(A, U, \epsilon)$.

Remark 3.5. *The construction of $V(A, U, \epsilon)$ involves creating curves in a neighborhood of A. Therefore, if A and A' are mutually disjoint compact subsets of ∂U such that $A \cap \overline{\partial U \setminus A} = A' \cap \overline{\partial U \setminus A'} = \emptyset$ and $\operatorname{dist}^*(A, A')$ is big compared to $\operatorname{diam} A$ and $\operatorname{diam} A'$, then $V(A, U, \epsilon) \cap V(A', U, \epsilon') = \emptyset$ for all $\epsilon \leq (32c)^{-1} \operatorname{dist}(A, \partial U \setminus A)$ and $\epsilon' \leq (32c)^{-1} \operatorname{dist}(A', \partial U \setminus A')$.*

3.3. **A weak form of uniform disconnectedness.** Here we consider a separation of ∂U that resembles uniform disconnectedness. Given $x \in \partial U$ and $r > 0$ we construct in Proposition 3.9 an L-chordarc disk Δ that contains x such that every point of $\partial\Delta$ has distance from ∂U at least a fixed multiple of r and

(1) either $\operatorname{diam} \Delta$ is comparable to r,
(2) or Δ contains a component of ∂U whose diameter is at least a fixed multiple of $\operatorname{diam} \Delta$.

If the first condition was satisfied for all x and r, then ∂U would be uniformly disconnected.

Lemma 3.6. *There exists $C > 1$ depending only on c such that for every non-degenerate component A of ∂U and for every positive $r \leq C^{-2}\operatorname{diam} A$ there exists $A' \subset \partial U$ containing A and a bounded Jordan domain $D \subset \mathbb{R}^2$ such that, $\partial D \subset U$, $D \cap \partial U = A'$ and*

$$C^{-1}\operatorname{dist}(z, A) \leq r \leq C\operatorname{dist}(z, \partial U) \text{ for all } z \in \gamma. \tag{3.1}$$

Proof. By Proposition 2.8, A is a bounded K-quasicircle with K depending only on c. Therefore, A satisfies the c_1-bounded turning property for some $c_1 > 1$ depending only on c. Set $c_2 = \max\{c, c_1\}$.

Fix now $r \leq (4c_2)^{-2}\operatorname{diam} A$. Find ordered points $x_1, \dots, x_n$ on A such that $r/2 \leq |x_i - x_{i+1}| \leq r$ for all $i = 1, \dots, n$ with the convention $x_{n+1} = x_1$. For each $i = 1, \dots n$, join x_i to x_{i+1} with a c-cigar curve γ_i. On each γ_i, $i = 1, \dots, n$, let $z_i \in \gamma_i$ be a point such that $\min\{|z_i - x_i|, |z_i - x_{i+1}|\} \geq |x_i - x_{i+1}|/2 \geq r/4$. Join each z_i to z_{i+1} with a c-cigar curve γ_i'. As before, we conventionally set $z_{n+1} = z_1$. Then, $|z_i - z_{i+1}| \leq 2c_2\epsilon$ and $\operatorname{diam}\gamma_i' \leq 2(c_2)^2 r$. The upper bound of r implies that $\gamma_i \cup \gamma_i' \cup \gamma_{i+1} \cup A(x_i, x_{i+1})$ is contractible in $\mathbb{R}^2 \setminus A$ for any i. In particular, $\gamma' = \bigcup \gamma_i'$ separates A from ∞. Let $\gamma \subset \gamma'$ be a simple closed curve homotopic to γ in U.

For the proof of (3.1) fix $z \in \gamma$ and $i \in \{1, \dots, n\}$ such that $z \in \gamma_i'$. Then, $\operatorname{dist}(z, A) \leq |z - x_i| \leq \operatorname{diam}\gamma_i' + \operatorname{diam}\gamma_i \leq (2c_2)^2 r$. On the other hand, by Remark 2.7, $\operatorname{dist}(z, \partial U) \geq (c_2)^{-1}\min\{\operatorname{dist}(z_i, \partial U), \operatorname{dist}(z_i, \partial U)\} \geq (c_2)^{-2} r/2$. Thus, the lemma holds with $C = (4c_2)^2$. □

Remark 3.7. *Note that* $\operatorname{diam}\gamma_i \cup \gamma_i' \cup \gamma_{i+1} \cup A(x_i, x_{i+1}) \leq Cr$. *Therefore, if $A_1 \subset A'$ and $A_1 \neq A$, then* $\operatorname{diam} A_1 \leq Cr \leq C^{-1}\operatorname{diam} A$.

Given a non-degenerate component A of ∂U and $r < C^{-2}\operatorname{diam} A$ we set $N_1(A, r)$ to be a set $A' \subset \partial U$ as in the statement of Lemma 3.6. Moreover, if γ is a simple closed curve as in Lemma 3.6 associated to $A' = N_1(A, r)$, let $D_1(A, r)$ be the $L(r)$-chordarc disk containing γ as in Lemma 3.1 with $\epsilon = r/24$. Although $L(r)$ may be large, arguments similar to that of Lemma 3.2 show that $D_1(A, r)$ is a K'-quasidisk for some $K' > 1$ depending only on c.

The next lemma provides us with a different kind of a neighborhood where the radius r is big compared to the diameters of the components of ∂U in a r-neighborhood of A.

Lemma 3.8. *Let A be a component of ∂U, $x \in A$ and $r > 8\operatorname{diam} A$ is such that every component A' of $\mathbb{R}^2 \setminus U$ intersecting $B^2(x, r)$ satisfies $\operatorname{diam} A \leq c'r$ for some $c' > 1$. Then, there exists $C' > 1$ depending only on c, c' and there exists a simple closed curve γ separating A from ∞ satisfying*

$$\frac{r}{2(C')^2} \leq \frac{\operatorname{dist}(z, \partial U)}{2C'} \leq \frac{r}{2} \leq \operatorname{diam}\gamma \leq C'r \text{ for all } z \in \gamma. \tag{3.2}$$

Proof. The proof follows closely that of Lemma 2.2 in [Mac99].

Let $A_1, \cdots, A_n$ be the components of $\partial U \setminus A$ intersecting $\partial B^2(x, r)$ such that $\operatorname{diam} A_i \geq (16c)^{-1} r$. By Lemma 4.6, n is bounded above by a constant depending only on c, c'. For each $i = 1, \dots, n$ let D_i be the Jordan domain given by Lemma 3.6 for A_i and $r_i = (2C)^{-1}\min\{\operatorname{diam} A_i, r\}$. Note that

$$(32Cc)^{-1} r \leq r_i \leq (2C)^{-1} c' r.$$

Let V be the component of $B^2(x,r)\setminus\bigcup\overline{D}_i$ that contains x. By the uniformity of U and the choice of r_i, there exists at least one nontrivial component in $\partial B^2(x,r)\cap V$. Suppose that $V\cap\partial B^2(x,r)=\Gamma_1\cup\Gamma_2\cup\cdots$ where each Γ_i is an open subarc of $\partial B^2(x,r)$. If $\operatorname{diam}\Gamma_i<(2c)^{-1}r$, then replace Γ_i by a c_1-cigar curve Γ_i' joining the endpoints of Γ_i.

Assume now that $\operatorname{diam}\Gamma_i\geq(2c)^{-1}r$. Let $y_1,\dots,y_{n_i}$ be consecutive points on Γ_i such that y_1 and y_{n_i} are the endpoints of Γ_i and $(8c)^{-1}r\leq|y_j-y_{j+1}|\leq(4c)^{-1}r$. Set $w_1=y_1$, $w_{n_i}=y_{n_i}$ and if $\operatorname{dist}(y_j,\partial U)>(32c)^{-1}r$ for some $j=2,\dots,n_i-1$ set $w_j=y_j$. Otherwise, take $z_j\in\partial U$ such that $|y_j-z_j|=\operatorname{dist}(y_j,\partial U)$. By the porosity of ∂U there exists $w_j\in U\cap\partial B^2(z_j,(16c)^{-1}r)$ satisfying the third conclusion of Proposition 2.8. Then, $|w_j-w_{j+1}|\leq 6(16c)^{-1}r$. For each $j=2,\dots,n_i$ let γ_j be a c-cigar curve in U joining w_{j-1} with w_j and let $\Gamma_i'=\bigcup_{j=1}^{n_i}\gamma_j$. The distance estimates above imply that Γ_i' is homotopic to Γ_i in $\mathbb{R}^2\setminus\{x\}$. Replace Γ_i with Γ_i'.

Thus, we obtain a closed curve

$$\Gamma=(\partial V\cap B^2(x,r))\cup\bigcup_{i\in\mathbb{N}}\Gamma_i'$$

that is homotopic to $\partial B^2(x,r)$ in $\mathbb{R}^2\setminus\{x\}$. Take $\gamma\subset\Gamma$ to be a simple closed curve that is homotopic to Γ in $\mathbb{R}^2\setminus\{x\}$. □

For the rest of the paper, Lemma 3.8 is applied with $c'=C^2$ where C is as in Lemma 3.6. Given a component A of ∂U and $r>8\operatorname{diam}A$, if γ is as in Lemma 3.8, then we denote by $N_2(A,r)$ the subset of ∂U that is enclosed by γ. Moreover, applying Lemma 3.1 for $E=\gamma$ and $\epsilon=(3C')^{-1}$ (C' is as in Lemma 3.8), there exists an L-chordarc disk $D_2(A,r)$ that contains $N_2(A,r)$ with L depending only on c.

Lemma 3.6 and Lemma3.8 combined yield the next proposition.

Proposition 3.9. *Let $x\in\partial U$, let A_x be the component of ∂U that contains x and let $r>0$.*

(1) If $\overline{B}(x,r)$ intersects a non-degenerate component A of ∂U with diameter at least C^2r, then x is contained in a set $N_1(A,r)$.
(2) If $r\leq 8\operatorname{diam}A_x$, then x is contained in a set $N_1(A_x,\frac{1}{8C^2}r)$.
(3) If $r>8\operatorname{diam}A_x$ and $\overline{B}(x,r)$ intersects only components of ∂U with diameters less than C^2r, then x is contained in a set $N_2(A_x,r)$.

Given a non-degenerate component A of ∂U, $N_1(A,r)$ is always defined when r is sufficiently small compared to $\operatorname{diam}A$. On the other hand, $N_2(A,r)$ is not defined for r which are small compared to $\operatorname{diam}A$, and even when r is large, it still may not be defined.

The properties of the sets $N_i(A,r)$ and $D_i(A,r)$ are summarized in the next lemma.

Lemma 3.10. *Suppose that A is a component of ∂U and $r>0$. There exists $c'>1$ depending only on c and there exists c'' depending only on c and r with the following properties.*

(1) Every component of $\mathbb{R}^2\setminus N_1(A,r)$ is c'-uniform. If ∂U is c-relatively connected, then each component of $\mathbb{R}^2\setminus N_1(A,r)$ has c'-relatively connected boundary.

(2) *Every component of* $D_1(A,r)\setminus N_1(A,r)$ *is* c''*-uniform. If* ∂U *is* c*-relatively connected, then each component of* $D_1(A,r)\setminus N_1(A,r)$ *has* c'*-relatively connected boundary.*
(3) *If* $N_2(A,r)$ *is defined, then all the components of* $\mathbb{R}^2\setminus N_2(A,r)$ *are* c'*-uniform. If* ∂U *is* c*-relatively connected, then each component of* $\mathbb{R}^2\setminus N_2(A,r)$ *has* c'*-relatively connected boundary.*
(4) *If* $N_2(A,r)$ *is defined, then all the components of* $D_2(A,r)\setminus N_2(A,r)$ *are* c'*-uniform. If* ∂U *is* c*-relatively connected, then each component of* $D_2(A,r)\setminus N_2(A,r)$ *has* c'*-relatively connected boundary.*

Proof. We show (1) and (2). The proofs of (3) and (4) are similar. As every quasidisk is uniform with relatively connected boundary, it is enough to show (1) for the unbounded component and (2) for the component of $D_1(A,r)\setminus N_1(A,r)$. whose boundary contains $\partial D_1(A,r)$. For the rest of the proof, C is the constant of Lemma 3.6.

To prove (1), let U' be the unbounded component of $\mathbb{R}^2\setminus N_1(A,r)$. To show uniformity of U', fix $x,y\in(\mathbb{R}^2\setminus N_1(A,r))\cap U$. If $x,y\in\mathbb{R}^2\setminus D_1(A,r)$, then uniformity follows from the fact that $D_1(A,r)$ is a K'-quasidisk for some K' depending only on c. If $x\in\overline{D_1(A,r)}$ and $y\in\mathbb{R}^2\setminus D_1(A,r)$, then join x to a point $z\in\partial D_1(A,r)$ with a c-cigar curve $\gamma_1\subset D_1(A,r)$ using uniformity of U and then z to y with a c'-cigar curve $\gamma_2\subset\mathbb{R}^2\setminus D_1(A,r)$ using uniformity of $\mathbb{R}^2\setminus D_1(A,r)$. Since $\text{dist}(z,N_1(A,r))>d|x-z|$ for some d depending only on c, the curve $\gamma=\gamma_1\cup\gamma_2$ is c''-cigar for some c'' depending only on c. Finally, if $x,y\in\overline{D_1(A,r)}$, then $x,y\in U$ and we use uniformity of U.

Suppose now that ∂U is c-relatively connected. Let $x\in\partial U'$ and $R>0$, and assume that $\overline{B}^2(x,R)\cap\partial U'\setminus\{x\}\neq\emptyset$ and $\partial U'\setminus\overline{B}^2(x,R)\neq\emptyset$. The second assumptiom implies that $R<2\operatorname{diam}A$. If $R\leq 8Cr$, then $\overline{B}^2(x,(8C^2)^{-1}R)\cap\partial U'=\overline{B}^2(x,(8C^2)^{-1}R)\cap\partial U$ and relative connectedness is satisfied with $c'=8C^2c$. Suppose now that $8rC<R<2\operatorname{diam}A$. Then, if $\overline{B}^2(x,R/8)$ intersects A we have $A\setminus\overline{B}^2(x,R/8)\neq\emptyset$ and relative connectedness is satisfied with $c'=8$.

To prove (2), let U'' be the bounded domain with boundary $\partial D_1(A,r)\cup N_1(A,r)$. The uniformity of U'' follows from (1) and Remark 2.9. If ∂U is relatively connected, for the relative connectedness of $\partial U''$ we work as above. □

3.4. **Total separation of ∂U.** Fix $\epsilon\in(0,\operatorname{diam}\partial U)$. For each point $x\in\partial U$ let $D_{i_x}(A_x,r_x)$ be as in §3.3 where $i_x\in\{1,2\}$, $r_x\in\{80c\epsilon,10c\epsilon\}$ and A_x is a component of ∂U. Set $\gamma_x=\partial D_{i_x}(A_x,r_x)$ and note that $\text{dist}(\gamma_x,\partial U)\geq 10\epsilon$.

Define $G=\bigcup_{x\in\partial U}\gamma_x$. Then, $\text{dist}(G,\partial U)\geq 10\epsilon$. The boundary of $\mathcal{T}_\epsilon(G)$ is a finite disjoint union of polygonal Jordan curves each of which has edges in $\mathcal{G}^1_\epsilon$ and is at least distance ϵ from ∂U. Define

$$\mathscr{G}=\{\overline{D}\colon D \text{ is a bounded Jordan domain whose boundary is a component of } \mathcal{T}_\epsilon(G)\}.$$

Note that two elements of $\mathscr{G}$ are either disjoint or one is contained in the other. An element D of $\mathscr{G}$ is called *minimal* if for all $D'\in\mathscr{G}$, $D'\subset D$ implies $D'=D$.

For each component A of ∂U let D_A be the minimal element of $\mathscr{G}$ that contains A and D'_A be the *maximal element* in $\{D_{A'}\colon A' \text{ is a component of } \partial U\}$ that contains A, that is, if $D'_A\subset D_{A'}$ for some component A' of ∂U, then $D_{A'}=D'_A$. Let

$D_1, \dots, D_n$ be the elements of the set $\{D'_A : A \text{ is a component of } \partial U\}$ and let $A_i = \partial U \cap D_i$. Note the following.

(1) By the doubling property of $\mathbb{R}^2$, $n \leq \epsilon^{-2}(\operatorname{diam} \partial U)^2$.
(2) Each D_i has diameter at least ϵ and, by the c-relative connectedness of ∂U, each A_i has size at least ϵ/M for some $M > 1$ depending only on c.
(3) For all $i \neq j$, $\operatorname{dist}(D_i, D_j) \geq \epsilon$.

The size of each D_i can be estimated from the following lemma.

Lemma 3.11. *Let $\epsilon \in (0, \operatorname{diam} \partial U)$ and $D_1, \dots, D_n$ be as above.*

(1) For all $i = 1, \dots, n$

$$\epsilon + \sup_A \operatorname{diam} A \leq \operatorname{diam} D_i \leq 80c^2(\epsilon + \sup_A \operatorname{diam} A)$$

where the supremum is taken over all components A of A_i.

(2) Each D_i is an L-chordarc disk for some L depending only on $\epsilon^{-1} \operatorname{diam} D_i$ and c.

Proof. The lower bound of the first claim follows from the fact that for each $z \in \partial D_i$, $\operatorname{dist}(z, \partial U) \geq \epsilon$. For the upper bound note that for each $x \in A_i$ we have $\operatorname{diam} D_i \leq \operatorname{diam} D_{i_x}(A_x, r_x) \leq c((80c)\epsilon + \operatorname{diam} A_x)$. The second claim follows from the first claim and Lemma 3.1. □

4. Quasisymmetric and bi-Lipschitz extension for a class of finitely connected domains

The classical Schönflies theorem states that every embedding of $\mathbb{S}^1$ in $\mathbb{R}^2$ extends to a homeomorphism of $\mathbb{R}^2$. Beurling and Ahlfors, in their celebrated paper [BA56], proved the quasisymmetric version of Schönflies theorem and, later, Tukia proved the bi-Lipschitz version.

Theorem 4.1. [BA56, Tuk80] *If $f : \mathbb{S}^1 \to \mathbb{R}^2$ is η_1-quasisymmetric (resp. L_1-bi-Lipschitz), then it extends η_2-quasisymmetrically (resp. L_2-bi-Lipschitzly) to $\mathbb{R}^2$ with η_2 depending only on η_1 (resp. L_2 depending only on L_1).*

Remark 4.2. *In the quasisymmetric case (resp. bi-Lipschitz case) of Theorem 4.1, $\mathbb{S}^1$ can be replaced by any K-quasicircle (resp. λ-chordarc circle) with η' depending only on η and K (resp. L' depending only on L and λ).*

In §4.1 we use Carleson's method to show that $\mathbb{S}^1$, in the bi-Lipschitz case of Theorem 4.1, can be replaced by quasicircles. In §4.2 we extend Theorem 4.1 for the class of finitely connected uniform domains.

4.1. Bi-Lipschitz extensions to quasidisks.

The main result of §4.1 is the following lemma.

Lemma 4.3. *Let D be a K-quasidisk and let $f : \partial D$ be an L-bi-Lipschitz embedding. Then, there exists an L'-bi-Lipschitz extension $f_D : \overline{D} \to \mathbb{R}^2$.*

For the rest of §4.1, for two positive quantities A, B we write $A \lesssim B$ if there exists constant C^*, depending only on K and L such that $A \leq C^* B$. We write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$. For the case that D is unbounded we make the following observation that is also used in §7; the proof is straightforward and is left to the reader. Given $x_0 \in \mathbb{R}^2$, define $I_{x_0} : \mathbb{R}^2 \setminus \{x_0\} \to \mathbb{R}^2 \setminus \{0\}$ to be the inversion map given by

$$I_{x_0}(x) = \frac{x - x_0}{|x - x_0|^2}.$$

Remark 4.4. *Let $E \subset \mathbb{R}^n$ be non-degenerate, $x_0 \in E$ and $f : E \to \mathbb{R}^n$ be L-bi-Lipschitz. Then, the map $g : I_{x_0}(E \setminus \{x_0\}) \to \mathbb{R}^n$ given by*

$$g = I_{f(x_0)} \circ f \circ I_{x_0}^{-1}|_{I_{x_0}(E\setminus\{x_0\})}$$

is L'-bi-Lipschitz with L' depending only on L.

Proof of Lemma 4.3. By Remark 4.4, we only need to prove the lemma in the case that D is bounded. After rescaling, we may further assume that $\operatorname{diam} D = \operatorname{diam} D' = 1$. Set $\Gamma_\infty = \partial D$. Then, $\Gamma'_\infty = f(\partial D)$ is a bounded K'-quasicircle. Let $D' \subset \mathbb{R}^2$ be the bounded domain that is bounded by Γ'_∞. Then, both D and D' are c-uniform domains for some $c > 1$ depending only on K and L. Moreover, we assume that both Γ_∞ and Γ'_∞ are C-bounded turning for some C depending only on K and, for simplicity, we assume $C = c$.

Fix an orientation for Γ_∞. Through f, an orientation for Γ'_∞ is also defined. Given a set of points $\{p_1, \dots p_n\} \subset \Gamma_\infty$ we say that p_i and p_j are *neighbors in the set* $\{p_1, \dots p_n\}$ if one of the two subarcs of $\Gamma_\infty \setminus \{p_i, p_j\}$ contains no point from $\{p_1, \dots, p_n\}$. Such a subarc is denoted by $\Gamma_\infty(p_i, p_j)$. Moreover, we say that p_i *is on the right of* p_j *in the set* $\{p_1, \dots p_n\}$ if p_i and p_j are neighbors and, under the orientation of Γ_∞, p_j and p_i are the starting and ending, respectively, points of $\Gamma_\infty(p_i, p_j)$. In opposite case, we say that p_i *is on the left of* p_j.

Fix points $x_0 \in D$ and $x'_0 \in D'$ such that

$$\operatorname{dist}(x_0, \Gamma_\infty) \geq (4c)^{-1} \text{ and } \operatorname{dist}(x'_0, \Gamma'_\infty) \geq (4c)^{-1}.$$

The existence of these points follows from Proposition 2.8.

Recall the definitions of square thickenings $\mathcal{T}_\delta(E)$ from §3.1. For each $m \in \mathbb{N}$, let D_m (resp. D'_m) be the component of $\mathcal{T}_{\epsilon_m}(\Gamma_\infty)$ (resp. $\mathcal{T}_{\epsilon_m}(\Gamma'_\infty)$) that contains x_0 (resp. x'_0) where $\epsilon_m = 2^{-lm}$ and l is an integer such that $2^l \geq 2c$. For each $m \in \mathbb{N}$ set $\Gamma_m = \partial D_m$. Choosing l appropriately, we may assume that, for each $m \in \mathbb{N}$, $D_m \subset D_{m+1} \subset D$ and $\operatorname{dist}(D_m, \Gamma_{m+1}) \geq 2^{-l(m+1)}$ and similarly for the domains D'_m.

Choose points $x_1, \dots, x_k \in \Gamma_\infty$, following its orientation, such that

$$16c^3 \leq |x_i - x_{i+1}| \leq 32c^3$$

with the convention $x_{n+1} = x_1$. Note that $k \leq N$ for some $N \in \mathbb{N}$ depending only on c and C.

For each $i \in \{1, \dots, k\}$ let $\hat{y}_i \in \Gamma_1$ be a point closest to x_i and join x_i to $\hat{y}_i$ with a c-cigar curve σ_i. For each σ_i we construct a broken line γ_i as follows. For each $z \in \gamma_i$ let $\Sigma(z)$ be the union of all squares in $\mathcal{G}_{2^{-l(z)}}$ that contain z where $l(z)$ is the smallest integer such that $2^{-l(z)} \leq \epsilon_1$ and $2^{-l(z)} \leq \frac{1}{6}\operatorname{dist}(z, \Gamma_\infty)$. Let γ_i be a subarc in the boundary of $\bigcup_{z\in\sigma_i} \Sigma(z)$ that connects x_i with Γ_1 and is entirely contained (except for its endpoints) in $D \setminus D_1$. Denote by y_i the endpoint of γ_i which is on Γ_1. Note that the broken lines γ_i are mutually disjoint and that $\operatorname{dist}(\gamma_i, \gamma_j) \gtrsim 1$ when $i \neq j$.

Next, for each $n \in \{2, 3, \cdots\}$, we modify γ_i close to its intersection points with Γ_n. We start with Γ_2.

Let T_2 be the union of all squares in $\mathcal{G}_{\epsilon_2/4}$ that intersect with Γ_2. Note that ∂T_2 consists of exactly two Jordan curves; one contained in D_2, the other contained in $D_1 \setminus \overline{D_2}$. Let p_i and q_i be the points of $\gamma_i \cap \partial T_2$ such that the part of γ_i joining x_i with p_i is entirely in D_2 while the part of γ_i joining y_i with q_i is entirely in $D_1 \setminus \overline{D_2}$. Let $\hat{q}_i$ be the *flat vertex* (i.e. $\hat{q}_i$ is the common vertex of two co-linear edges) on the component of ∂T_2 containing q_i that is closest to p_i and let τ_1 be the shorter in diameter subarc of T_2 joining q_i with $\hat{q}_i$. Let t_i be the flat vertex of Γ_1 closest to $\hat{q}_i$ and τ_2 be the line segment $[t_i, \hat{q}_i]$. Let $\hat{p}_i$ be the flat vertex on the component of ∂T_2 containing p_i that is closest to t_i and let τ_3 be the line segment $[t_i, \hat{p}_i]$ Finally, let τ_4 be the shorter in diameter subarc of T_2 joining q_i with $\hat{q}_i$. Replace $\gamma_i(\hat{p}_i, q_i)$ with $\bigcup_{i=1}^4 \tau_i$ and note that γ_i intersects Γ_2 Note that the two curves γ_i and Γ_2 intersect orthogonally and their intersection is only one point t_i which we denote for the rest by y_{i1}.

Similarly, we modify γ_i close to its intersection points with Γ_k. We denote by $y_{i1^{n-1}}$ the unique intersection point of Γ_n and γ_i.

We proceed inductively. Assume that for some $m \in \mathbb{N}$, we have defined points $x_w \in \Gamma_\infty$, curves γ_w and points y_{w1^l} where $w \in \mathbb{N}^m$ is a finite word formed from m letters in $\mathbb{N}$ and $l \in \mathbb{N} \cup \{0\}$. We denote by $|w|$ the number of letters the word w has. Conventionally, $|\emptyset| = 0$.

Fix x_w and x_u such that $|w| = |u| = m$ and x_w is on the left of x_u in the collection $\{x_v : |v| = m\}$. Choose points x_{wi} in $\Gamma_\infty^{(w)} = \Gamma_\infty(x_w, x_u)$, with $i = 1, \dots, N_w + 1$, following the orientation of Γ_∞ such that $x_{w1} = x_w$, $x_{w(N_{wi}+1)} = x_u$ and for each $i = 1, \dots, N_{wi}$

$$4c(2c)^{3-2m} \le |x_{wi} - x_{w(i+1)}| \le 8c(2c)^{3-2m}.$$

Note that $N_w \le N_0$ where N_0 depends only on c. Without loss of generality, we assume for the rest that $N = N_0$.

For $i =\in \{2, \dots, N_{wi}\}$ and $m \ge |w| + 2$, let $\hat{y}_{wi}$ be a point of $\Gamma_{|w|+1}$ closest to x_{wi} and let σ_{wi} be a c-cigar curve joining x_{wi} with $\hat{y}_{wi}$. Construct γ_{wi} as before and let y_{wi} be the point on $\Gamma_{|w|+1}$ such that the part of γ_{wi} connecting x_{wi} with y_{wi} is entirely in $D \setminus D_{|w|+1}$. As before, for each $k \in \mathbb{N}$, we modify γ_{wi} close to its intersection points with $\Gamma_{|w|+1+k}$ and denote with y_{wi1^k} the unique intersection point of γ_{wi} and $\Gamma_{|w|+1+k}$.

Let $\mathcal{W}$ be the set of finite words w formed by letters $\{1, \dots, N\}$ for which x_w has been defined. Let also $\mathcal{W}_k$ be the set of words $w \in \mathcal{W}$ whose length is k. Again, numbers ϵ_m have been chosen so that

$$\operatorname{dist}(\gamma_w, \gamma_u) \gtrsim \min\{\operatorname{diam} \gamma_w, \gamma_u\} \simeq \min\{\epsilon_w, \epsilon_u\}.$$

Fix $w \in \mathcal{W}$ and let x_u be on the left of x_w in the collection $\{x_v : |v| = |w|\}$. Define $\mathcal{Q}_w$ to be the Jordan domain bounded by γ_w, γ_u, $\Gamma_{|w|}$ and $\Gamma_{|w|+1}$. Set $\mathscr{Q} = \{\mathcal{Q}_w : w \in \mathcal{W}\}$. Then, for each $\mathcal{Q}_w$ there exists $l_w \in \mathbb{N}$ such that $\partial \mathcal{Q}_w$ is a poygonal arc with edges in $\mathcal{G}^1_{2^{-l_w}}$ and $\operatorname{diam} \mathcal{Q}_w \simeq 2^{-l_w}$. Moreover, the distance of each $\mathcal{Q}_w$ from Γ_∞ is comparable to its distance from x_w. Therefore,

(1) each $\mathcal{Q}_w \in \mathscr{Q}$ is an L_1-chordarc disk with $L_1 \simeq 1$,
(2) $\operatorname{dist}(\mathcal{Q}_w, x_w) \simeq \operatorname{dist}(\mathcal{Q}_w, \partial U) \simeq \operatorname{diam} \mathcal{Q}_w$.

Using now the points $x'_w = f(x_w)$ and working as above, we obtain a Whitney-type decomposition $\mathscr{Q}'$ of D' that is combinatorially equivalent to $\mathscr{Q}$ and satisfies properties (1) and (2) above. Moreover, for each $w \in \mathcal{W}$,

$$\operatorname{diam} \mathcal{Q}_w \simeq \operatorname{diam} \Gamma_\infty^{(w)} \simeq \operatorname{diam} f(\Gamma_\infty^{(w)}) \simeq \operatorname{diam} \mathcal{Q}'_w.$$

We can now extend f to the 1-skeleton of the decomposition

$$f : \overline{\bigcup_{w\in\mathcal{W}} \partial\mathcal{Q}_w} \to \overline{\bigcup_{w\in\mathcal{W}} \partial\mathcal{Q}_w}$$

so that $f|_{\partial\mathcal{Q}_w} : \partial\mathcal{Q}_w \to \partial\mathcal{Q}'_w$ is an L_2-bi-Lipschitz homeomorphism. By Theorem 4.1, each $f|_{\partial\mathcal{Q}_w}$ can be extended to an L_3-bi-Lipschitz $f : \mathcal{Q}_w \to \mathcal{Q}'_w$. Therefore, the map $f : D \to D'$ is L_4-BLD and, by Lemma 2.2, f is L_5-bi-Lipschitz with constants $L_2, \dots, L_5$ depending only on L and K. □

4.2. **Extension for a class of finitely connected domains.** Let $L \geq 1$, $K \geq 1$ and $d \geq 1$. Denote by $\mathscr{QC}(K, d)$ (resp. $\mathscr{CA}^*(L, d)$) the collection of planar bounded domains $U \subset \mathbb{R}^2$ whose boundary consists of mutually disjoint K-quasicircles (resp. L-chordarc circles) with mutual distances and diameters bounded below by $d^{-1} \operatorname{diam} U$. Let also $\mathscr{CA}(L, d)$ be the collection of bounded domains $U \subset \mathbb{R}^2$ whose boundary consists of mutually disjoint L-chordarc circles with mutual distances bounded below by $d^{-1} \operatorname{diam} U$. Note that $\mathscr{CA}^*(L, d) \subset \mathscr{CA}(L, d)$ and $\mathscr{CA}^*(L, d) \subset \mathscr{QC}(L^2, d)$.

The following proposition, which is the main result of this section, generalizes Theorem 4.1 and is a special case of Theorem 1.1.

Proposition 4.5. *Let $U \subset \mathbb{R}^2$ be a bounded domain and $f : \partial U \to \mathbb{R}^2$ be an embedding that can be extended homeomorphically to $\overline{U}$.*

1. *If $U \in \mathscr{QC}(K, d)$ and f is η_1-quasisymmetric, then it extends to an η_2-quasisymmetric embedding of $\overline{U}$ with η_2 depending only on η_1, K and d.*
2. *If $U \in \mathscr{CA}(L, d)$ and f is L_1-bi-Lipschitz, then it extends to an L_2-bi-Lipschitz embedding of $\overline{U}$ with L_2 depending only on L_1, K and d.*

We first show that domains in $\mathscr{QC}(K, d)$ and $\mathscr{CA}(L, d)$ are finitely connected quantitatively. Although this result follows almost immediately from the doubling property, with a little more effort, one can show the following stronger statement.

Lemma 4.6. *For each $n \in \mathbb{N}$, $c > 1$ and $d > 1$ there exists $N > 1$ depending only on n, c and d that satisfies the following property. If $U_1, \dots, U_m \subset \mathbb{R}^n$ are disjoint c-uniform domains of mutual relative distances at most d, then $m \leq N$.*

Proof. Let $U_1, \dots, U_m \subset \mathbb{R}^n$ be mutually disjoint c-uniform domains of mutual relative distances at most d. The proof is divided in two cases.

Case 1. Assume first that at least one of the U_i is bounded. In particular, assume that U_m is bounded and that it has the smallest diameter among the U_1. Applying a dilation, we may further assume that $\operatorname{diam} U_m = 1$. Fix a point $z_0 \in U_m$. Since $\operatorname{dist}^*(U_i, U_m) \leq d$, every domain U_i intersects $B^n(z_0, 2d)$.

We claim that for each $i \in \{1, \dots, m\}$, there exists $x_i \in U_i$ such that

$$B^n(x_i, (4c)^{-1}) \subset U_i \cap B^n(z_0, 4d).$$

To prove the claim, suppose first that U_i is contained entirely in $B^n(z_0, 4d)$. Then, the claim follows from the third assertion of Proposition 2.8 and the fact that $\operatorname{diam} U_i \geq 1$. Suppose now that U_i has a point $y_i \in \mathbb{R}^n \setminus B^n(z_0, 4d)$. Then, there exists a point $y'_i \in U_i \cap B^n(z_0, 2d)$ and a c-cigar curve γ_i joining y_i with y'_i. Fix a point $x_i \in \gamma_i \cap \partial B^n(z_0, 3d)$ and note that the John property of γ_i implies that $B^n(x_i, d/c) \subset U_i$ which completes the proof of the claim.

By the doubling property of $\mathbb{R}^n$, there exists some $c_0 > 1$ depending only on n such that the ball $B^n(z_0, 4d)$ can contain at most $N = c_0(4d/(4c)^{-1})^n = c_0(16cd)^n$ mutually disjoint balls of radius at least $(4c)^{-1}$. Therefore, $m \leq c_0(16cd)^n$.

Case 2. Assume that all U_i are unbounded. Fix a point $z_0 \in \mathbb{R}^n$ and let $r > 0$ be such that $B^n(z_0, r) \cap U_i \neq \emptyset$ for all $i \in \{1, \dots, m\}$. As with *Case 1*, we can show that, for each $i \in \{1, \dots, m\}$, there exists $x_i \in U_i$ such that $B^n(x_i, r(2c)^{-1}) \subset U_i \cap B^n(z_0, 2r)$. Now, the doubling property of $\mathbb{R}^n$ yields $m \leq c_0(4c)^n$. ☐

Corollary 4.7. *For each $c > 1$ and $d > 1$ there exists $N > 1$ depending only on L and d such that every domain in $\mathscr{QC}(c, d) \cup \mathscr{CA}(c, d)$ has at most N boundary components.*

For the rest of §4.2, set $\mathcal{U}_0 = (-1, 1) \times (-1, 1)$ and for each $m \in \mathbb{N}$ and $k \in \{1, \dots, m\}$ set

$$\mathcal{S}_{m,k} = [\frac{4k - 2m - 3}{2m + 1}, \frac{4k - 2m - 1}{2m + 1}] \times [\frac{-1}{2m + 1}, \frac{1}{2m + 1}]$$

and

$$\mathcal{U}_m = \mathcal{U}_0 \setminus \bigcup_{k=1}^{m} \mathcal{S}_{m,k}.$$

Lemma 4.8. *Each $U \in \mathscr{QC}(K, d)$ (resp. $\mathscr{CA}^*(L, d)$) is η-quasisymmetric (resp. $(L', \operatorname{diam} U)$-quasisimilar) to $\mathcal{U}_m$ for some $0 \leq m \leq N$ with η and N depending only on K and d (resp. N and L' depending only on L and d).*

For the proof of the lemma recall that a *dyadic n-cube* $D \subset \mathbb{R}^n$ is an n-cube of the form $D = [i_1 2^k, (i_1 + 1)2^k] \times \cdots \times [i_n 2^k, (i_n + 1)2^k]$ where $k, i_1, \dots, i_n \in \mathbb{Z}$. If $n = 2$, D is called a dyadic square.

Proof of Lemma 4.8. The lemma is trivial if U is simply connected. Suppose now that $U = D_0 \setminus (\overline{D_1} \cup \cdots \cup \overline{D_m})$ where

(1) $D_1, \dots, D_m$ are mutually disjoint subsets of D_0,
(2) each D_i, $i \in \{0, \dots, m\}$, is a K-quasidisk with $\operatorname{diam} D_i \geq d^{-1} \operatorname{diam} U$
(3) $\operatorname{dist}(\partial D_i, \partial D_j) \geq d^{-1} \operatorname{diam} U$ for each $i, j \in \{0, \dots, m\}$ with $i \neq j$.

By Corollary 4.7, $m \leq N$ for some $N \in \mathbb{N}$ depending only on L and d.

Assume first that $U \in \mathscr{CA}^*(L, d)$. Applying a $(\operatorname{diam} U, L_1)$-quasisimilarity, with L_1 depending only on L, we may assume that $D_0 = \mathcal{U}_0$. By Lemma 3.1, there exists $L_2 > 1$ depending only on L and d such that for each $i = 1, \dots, m$ there exists an L_2-chordarc disk D_i' containing D_i such that $(24d)^{-1} \leq \operatorname{dist}(z, \partial D_i) \leq (3d)^{-1}$. Moreover, each D_i contains a dyadic square S_i with side-length 2^{-m_0} where m_0 is the smallest integer such that $2^{-m_0} \leq \min\{(4Ld)^{-1}, \log(2N^{-1})/\log 2\}$. Note that both $\operatorname{dist}(\partial D_i, S_i)$ and $\operatorname{diam} S_i$ are bounded below by δ for some $\delta \in (0, 1)$ depending only on L and d.

There exists L_3 depending only on L, d such that for each $i = 1, \dots, m$ there exists an L_3-bi-Lipschitz mapping $f_i : \partial D_i' \cup \partial D_i \to \partial D_i' \cup \partial S_i$ with $f_i|_{\partial D_i'} = \mathrm{Id}$ and $f_i(\partial D_i) = \partial S_i$. By the Annulus Theorem in the LIP category (see Theorem 3.4 in [Väi77]), each f_i can be extended to an L_4-bi-Lipschitz mapping $f_i : \overline{D_i' \setminus D_i} \to \overline{D_i' \setminus S_i}$ with L_4 depending only on L and d. Moving the squares S_i around and properly dialating them, we can map $D_0 \setminus (\bigcup_{i=1}^n S_i)$ bi-Lipschitzly onto $\mathcal{U}_n$. Since there are at most $(2^{2m_0+2})!$ different configurations for the position of the squares $S_1, \dots, S_n$ inside D_0, the bi-Lipschitz constant of the latter map depends at most on L and d.

The proof of the quasisymmetric case is almost identical. The only difference is that, now, we use the Annulus Theorem in the LQC category (see Theorem 3.12 in [TV81]) to obtain η-quasisymmetric extensions of the mappings f_i with η depending only on K and d. $\square$

Proof of Proposition 4.5. Since the embedding f in both cases can be extended homeomorphically to $\overline{U}$, there exists a domain $U' \subset \mathbb{R}^2$ such that $\partial U' = f(\partial U)$ and f can be extended to a homeomorphism of $\overline{U}$ onto $\overline{U'}$.

Suppose first that $U \in \mathscr{QC}(K,d)$ and that f is η_1-quasisymmetric. By Lemma 4.8, we may assume that $U = U' = \mathcal{U}_m$ where $m \leq N$ for some N depending only on K and d. Moreover, applying a λ-bi-Lipschitz homeomorphism of $\mathcal{U}_m$ onto itself with $\lambda > 1$ depending only on N, we may assume that f maps $\partial\mathcal{S}_{m,k}$ onto $\partial\mathcal{S}_{m,k}$.

If $m = 0$, the claim follows from Theorem 4.1 while if $m = 1$, it follows from the Annulus theorem in the LQC category. Assume for the rest that $m \geq 2$.

Let $\mathcal{S}'_0 = [\frac{1/2}{2m+1} - 1, 1 - \frac{1/2}{2m+1}]^2$ and for each $k = 1, \dots, m$ let

$$\mathcal{S}'_{m,k} = [\frac{4k-2m-7/2}{2m+1}, \frac{4k-2m-1/2}{2m+1}] \times [-\frac{3/2}{2m+1}, \frac{3/2}{2m+1}]$$

so that $\mathcal{S}_{k,m} \subset \mathcal{S}'_{k,m} \subset \mathcal{S}'_0 \subset (-1,1)^2$ for each $k = 1, \dots, m$. Extend f to $\partial\mathcal{S}'_0$ and to each $\mathcal{S}'_{k,m}$ with identity and note that the new embedding, which we still denote by f, is η'_1-quasisymmetric with η'_1 depending only on η_1 and d. Applying the Annulus theorem in the LQC category on the interior of each $\mathcal{S}'_{k,m} \setminus \mathcal{S}_{k,m}$ we obtain an η_2-quasisymmetric extension $F : U \to U'$ with η_2 depending only on K, d and η.

Suppose now that $U \in \mathscr{CA}(L,d)$ and $f : \partial U \to \mathbb{R}^2$ is L_1-bi-Lipschitz and can be extended homeomorphically to $\overline{U}$. If U is simply connected, then the claim follows from Theorem 4.1. Assume now that U is not simply connected. As before, there exists $N \in \mathbb{N}$ depending only on L and d such that $\mathbb{R}^2 \setminus U$ has at most N bounded components $D_1, \dots, D_m$. Moreover, there exists $U' \in \mathscr{CA}(L,d)$ such that $\partial U' = f(\partial U)$ and f extends to a homeomorphism from U onto U'.

For each $i = 1, \dots, m$, set D'_i to be the bounded component of $\mathbb{R}^2 \setminus U'$ such that $\partial D'_i = f(\partial D_i)$. Let k_i be the maximal integer such that

$$\operatorname{diam} D_i \leq 2^{-k_i-3}(L_1 d)^{-1} d \operatorname{diam} U.$$

If $k_i \leq 1$, then $\operatorname{diam} D_i \geq \frac{1}{32L_1 d} \operatorname{diam} U$ and set $\tilde{D}_i = D_i$.

Suppose that $k_i \geq 2$. Fix a point $x \in \partial D_i$ and let $x' = f(x) \in \partial D_i$. Let $B_i = B^2(x, 2L_1 \operatorname{diam}\partial D_i)$, $\tilde{D}_i = B^2(x', 2L_1 \operatorname{diam}\partial D_i)$, $B'_i = B^2(x, 2^{k_i} L_1 \operatorname{diam}\partial D_i)$ and $\tilde{D}'_i = B^2(x', 2^{k_i} L_1 \operatorname{diam}\partial D_i)$. Note that $D_i \subset B_i \subset \tilde{D}_i$ and $\tilde{D}_i \cap (\partial U \setminus \partial D_i) = \emptyset$ and similarly for D'_i. Moreover, $\operatorname{diam}\tilde{D}_i \geq \frac{1}{32L_1 d} \operatorname{diam} U$ and

$$\min\{\operatorname{dist}(\tilde{D}_i, \partial U \setminus \partial D_i), \operatorname{dist}(\tilde{D}'_i, \partial U' \setminus \partial D'_i)\} \geq (4Ld)^{-1} \operatorname{diam} U.$$

Therefore, $\tilde{U} = U \setminus \bigcup_{i=1}^m \tilde{D}_i \in \mathscr{CA}^*(L,d')$ for some d' depending only on d and L_1. For each $i = 1, \dots, m$ define $f|_{\tilde{D}_i \setminus \partial B_i}$ by translation and apply the Annulus Theorem in the LIP category [Väi77, Theorem 3.4] to extend f to $B_i \setminus D_i$ L'_1-bi-Lipschitz with $L'_1 > 1$ depending only on L, d and L_1.

Applying Lemma 4.8 and the Annulus Theorem in the LIP category, we obtain the extension of f to $\tilde{U}$ as in the first part of Proposition 4.5. $\square$

4.3. **A higher dimensional extension.** It is well known that both cases of Theorem 4.1 are false in $\mathbb{R}^3$ due to the existence of a Lipschitz embedding of $\mathbb{S}^2$ into $\mathbb{R}^3$ that can be extended homeomorphically to $\mathbb{R}^3$ but not quasisymmetrically; see [Tuk80, §15].

In this subsection we work with a much simpler setting. For $d > 1$ denote by $\mathscr{C}_n(d)$ the collection of domains $U \subset \mathbb{R}^n$ whose boundary components are boundaries of n-cubes of mutual distances bounded below by $d^{-1} \operatorname{diam} U$.

Proposition 4.9. *Let $U \in \mathscr{C}_n(d)$ and $f : \partial U \to \mathbb{R}^n$ be an L-bi-Lipschitz map that is a similarity on each component of ∂U and that extends homeomorphically to $\overline{U}$. Then f extends L'-bi-Lipschitzly to $\overline{U}$ with L' depending only on L, d and n.*

For the proof of Proposition 4.9, given a set $A \subset \mathbb{R}^n$ and $\delta > 0$, we define the *δ-neighborhood* of A in $\mathbb{R}^n$ by $N^n_d(A) = \bigcup_{x\in A} B^n(x, \delta)$.

Proof of Proposition 4.9. We only give a sketch of the proof as it is similar to that of Proposition 4.5. Since f extends to $\overline{U}$, there exists a domain $U' \subset \mathbb{R}^n$ whose boundary is a union of disjoint cubes such that f maps ∂U on $\partial U'$ and any homeomorphic extension to $\overline{U}$ maps U on U'.

Firstly, by the doubling property of $\mathbb{R}^n$, there exists $N \in \mathbb{N}$ depending only on n and d such that ∂U has at most N components. In particular, $U = D_0 \setminus \bigcup_{i=1}^m \overline{D_i}$ where D_i are open n-cubes and $m \le N$.

Secondly, applying the Annulus Theorem in the LIP category, we obtain a small $\delta > 0$ and an L_1-bi-Lipschitz map $F : \mathbb{R}^n \to \mathbb{R}^n$ such that

(1) F is the identity in $U \setminus N_\delta(\partial U)$,
(2) F maps ∂D_0 on a dyadic cube D_0' of side-length 2^{k_0+1} where k_0 is the minimal integer such that $\operatorname{diam} D_0 \le 2^{k_0-1}$,
(3) F maps each cube D_i is mapped to a dyadic cube of side-length 2^{k_i+1} where $k_i \in \mathbb{Z}$ is the maximal integer such that $2^{k_i} \le \min\{\frac{1}{N}2^{k_0}, \frac{1}{16}\operatorname{diam} D_i\}$.

Here, δ and L_1 depend only on n, L and d.

If $\operatorname{diam} D_i'$ is very small compared to $\operatorname{diam} D_0'$, then, as in the proof of Proposition 4.5, we can replace D_i' with a new dyadic cube which we still denote by D_i' whose side-length is comparable to that of D_0 but no more than $\frac{1}{N}2^{k_0}$.

Applying a uniformization result like Lemma 4.8, we may assume that $U = U'$. The rest of the proof follows from applying the Annulus Theorem in the LIP category m times. $\square$

5. First reduction: perfect boundary

In this section, we reduce the proof of Theorem 1.1 to the case that ∂U is a perfect set, and the proof of Theorem 1.3 to the case that E is perfect.

Let $E \subset \mathbb{R}^n$ be a closed set. For each isolated point $x \in E$ let $\hat{(x)} \in E \setminus \{x\}$ be a point of smallest distance to x and E_x be the image of the standard middle-third Cantor set $\mathcal{C}$ under a similarity with scaling factor $\frac{1}{10}|x - \hat{(x)}|$ such that E_x contains x. If $x \in E$ is not isolated, set $E_x = \{x\}$. Set $\hat{E} = \bigcup_{x\in E} E_x$ and note that $\hat{E}$ is closed.

Lemma 5.1. *For each $c \geq 1$ there exists $c' \geq 1$ depending only on c that satisfies the following properties.*

(1) *If $E \subset \mathbb{R}^n$ is c-relatively connected, then $\hat{E}$ is c'-uniformly perfect.*
(2) *If $E \subset \mathbb{R}^n$ is c-uniformly disconnected, then $\hat{E}$ is c'-uniformly disconnected.*
(3) *If $U \subset \mathbb{R}^2$ is c-uniform and $U' \subset U$ is the domain with $\partial U' = \hat{\partial U}$, then U' is c'-uniform.*

Proof. The proof of the first claim is similar to that of Lemma 3.3 in [Vel16]. Let $x \in \hat{E}$ and $r > 0$. From the fact that $\hat{E}$ is perfect, we have $\{x\} \subsetneq \overline{B}(x,r) \cap \hat{E}$. Suppose that $\hat{E} \setminus \overline{B}(x,r) \neq \emptyset$. If $x \in E$ and is not isolated in E,

$$\emptyset \neq E \cap (\overline{B}^n(x,r) \setminus B^n(x,r/c)) \subset \hat{E} \cap (\overline{B}^n(x,r) \setminus B^n(x,r/c)).$$

Suppose $x \in E_z$ for some isolated point $z \in E$. If $r > 2c\operatorname{dist}(z, E \setminus \{z\})$, then $\emptyset \neq (E \setminus \{z\}) \cap \overline{B}^n(z,r/2) \subset \hat{E} \cap \overline{B}^n(x,r)$. Therefore,

$$\emptyset \neq E \cap (\overline{B}^n(z,r/2) \setminus B^n(z,(2c)^{-1}r)) \subset \hat{E} \cap (\overline{B}^n(x,r) \setminus B^n(x,(4c)^{-1}r)).$$

If $r \leq 2c\operatorname{dist}(z, E \setminus \{z\})$, then $(20c)^{-1}r \leq \frac{1}{10}\operatorname{dist}(z, E \setminus \{z\})$. The relative connectedness of $\mathcal{C}$ gives

$$\emptyset \neq E_z \cap (\overline{B}^n(x,r) \setminus B^n(z,(20c_0)^{-1}r)) \subset \hat{E} \cap (\overline{B}^n(x,r) \setminus B^n(x,(20c_0)^{-1}r))$$

for some $c_0 > 1$ depending only on c.

To show the second claim, let $x \in \hat{E}$ and $0 < r < \frac{1}{4}\operatorname{diam}\hat{E}$ and let $z \in E$ be the unique point of E such that $x \in E_z$. If z is an accumulation point, then $z = x$. Let E' be the subset of E containing x with $\operatorname{diam} E' \leq r$ and $\operatorname{dist}(E', E \setminus E') \geq c^{-1}r$. Then $\operatorname{diam}\hat{E}' \leq \frac{11}{10}r$ and $\operatorname{dist}(\hat{E} \setminus \hat{E}') \geq \frac{9}{10}c^{-1}r$.

Assume now that z is isolated point. Since $\mathcal{C}$ is c_0-uniformly disconnected, the claim of the lemma follows with $c' = c_0$ if $r < \frac{1}{8}\operatorname{diam} E_z$. Also, by uniform disconnectedness of $\mathcal{C}$, if $r < 100\operatorname{diam} E_z$, then the claim of the lemma is true for $c' = c_0/400$. If $100\operatorname{diam} E_z \geq \frac{1}{4}\operatorname{diam}\hat{E}$, then we are done. Assume the opposite and let $r > 100\operatorname{diam} E_z$. By uniform disconnectedness of E, there exists $E' \subset \overline{E}$ containing z such that $\operatorname{diam} E' \leq r/2$ and $\operatorname{dist}(E', \overline{E} \setminus E') \geq (2c)^{-1}r$. Then $x \in \hat{E}'$, $\operatorname{diam}\hat{E}' \leq r$ and $\operatorname{dist}(\hat{E}', \hat{E} \setminus \hat{E}') \geq (4c)^{-1}r$.

For the third claim let E be the set of isolated points of ∂U. Then $U' = U \setminus \hat{E}$. The uniformity of U' follows from the fact that $\hat{E}$ is uniformly disconnected and therefore a NUD set (nullset for uniform domains) in the sense of Väisälä [Väi88b]; see Theorem 1 and Corollary 2 in [Mac99]. $\square$

Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^n$ be a mapping. We extend f to $\hat{f} : \hat{E} \to \mathbb{R}^2$ as follows. If f is η-quasisymmetric, then for any isolated point $x \in E$ and any $y \in E_x$ define

$$\hat{f}|_{E_x}(y) = f(x) + \frac{1}{\eta(1)}\frac{|f(x) - f(\hat{(x)})|}{|x - \hat{(x)}|}(y - x).$$

If f is L-bi-Lipschitz, then for any isolated point $x \in E$ and any $y \in E_x$ define

$$\hat{f}|_{E_x}(y) = f(x) + \frac{1}{L^2}\frac{|f(x) - f(\hat{(x)})|}{|x - \hat{(x)}|}(y - x).$$

Lemma 5.2. *Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^n$ be η-quasisymmetric (resp. L-bi-Lipschitz). Then $\hat{f}$ is η'-quasisummetric (resp. L'-bi-Lipschitz) with η' depending only on η (resp. L' depending only on L).*

Proof. We first show the claim for bi-Lipschitz mappings. Given two distinct points $x, y \in \hat{E}$, there exist unique $x_1, y_1 \in E$ such that $x \in E_{x_1}$ and $y \in E_{y_1}$. If $x_1 = y_1$ there is nothing to prove as $\hat{f}$ is affine on E_{x_1}. Suppose that $x_1 \neq y_1$ and note that

$$|x-y| \geq \max\{\frac{9}{10}|x_1 - \hat{(}x_1)|, \frac{9}{10}|y_1 - \hat{(}y_1)|, |x-x_1|, |y-y_1|\}.$$

Then, $|f(x)-f(y)| \leq |f(x_1)-f(y_1)| + L^{-1}(|x-x_1|+|y-y_1|) \leq 5L|x-y|$ and $|f(x)-f(y)| \geq L^{-1}|x_1-y_1| - L^{-1}(|x-x_1|+|y-y_1|) \geq (2L)^{-1}|x-y|$.

The proof in the case that f is quasisymmetric is similar to that of Lemma 3.3 in [Vel16]. Let $x, y, z \in E^*$ be three distinct points with $x \in E_{x_1}$, $y \in E_{y_1}$ and $z \in E_{z_1}$ for some $x_1, y_1, z_1 \in E$. If $x_1 = y_1 = z_1$, then x, y, z are in the same E_{x_1} where $\hat{f}$ is affine. If $x_1 \neq z_1$ and $x_1 = y_1$, then the prerequisites of Lemma 2.29 in [Sem96] are satisfied (see also Remark 3.2 in [Vel16]) for $A = E \setminus \{x_1\}$, $A^* = E \cup E_{x_1}$ and $H = \hat{f}|_{A^*}$. Thus, $\hat{f}|_{E\cup E_{x_1}}$ is η'-quasisymmetric for some η' depending only on η. Hence,

$$\frac{|\hat{f}(x)-\hat{f}(y)|}{|\hat{f}(x)-\hat{f}(z)|} \leq C_1 \frac{|\hat{f}(x)-\hat{f}(y)|}{|\hat{f}(x)-\hat{f}(z_1)|} \leq C_1\eta'\left(\frac{|x-y|}{|x-z_1|}\right) \leq C_1\eta'\left(C_2\frac{|x-y|}{|x-z|}\right)$$

for some $C_1, C_2 > 1$ depending only on η. Similarly for $x_1 = z_1 \neq y_1$. If x_1, y_1, z_1 are distinct, then by Remark 3.2 in [Vel16],

$$\frac{|\hat{f}(x)-\hat{f}(y)|}{|\hat{f}(x)-\hat{f}(z)|} \leq C_3 \frac{|\hat{f}(x_1)-\hat{f}(y_1)|}{|\hat{f}(x_1)-\hat{f}(z_1)|} \leq C_3\eta\left(\frac{|x_1-y_1|}{|x_1-z_1|}\right) \leq C_3\eta\left(C_4\frac{|x-y|}{|x-z|}\right)$$

for some $C_3, C_4 > 1$ depending only on η. Therefore, $\hat{f}$ is quasisymmetric. □

6. Extension to the complements of quasicircle domains

A K-*quasicircle domain* $U \subset \mathbb{R}^2$ is a planar domain such that every component of ∂U is either a point or a K-quasicircle. The next proposition, which is the main result of this section, reduces the proof of Theorem 1.1 to extending f to $\overline{U}$.

Proposition 6.1. *Let $U \subset \mathbb{R}^2$ be a K-quasicircle domain and let $f : \partial U \to \mathbb{R}^2$ be an embedding which can be extended homeomorphically to $\overline{U}$.*

(1) If f is L-bi-Lipschitz, then it extends L'-bi-Lipschitzly to $\mathbb{R}^2 \setminus U$ with L' depending only on L and K.

(2) If ∂U is c-relatively connected and f is η-quasisymmetric, then it extends η'-quasisymmetrically to $\mathbb{R}^2 \setminus U$ with η' depending only on η, c and K.

Note that Proposition 6.1 and Lemma 2.12 yield the next corollary.

Corollary 6.2. *Let $U \subset \mathbb{R}^2$ be a c-uniform domain (resp. c-uniform domain with C-relatively connected boundary) and let $f : \partial U \to \mathbb{R}^2$ be an L-bi-Lipschitz (resp. η-quasisymmetric) embedding that can be extended homeomorphically to $\mathbb{R}^2$. Then, for some c' depending only on c and L (resp. depending only on c, C and η), there exists a c'-uniform domain $U' \subset \mathbb{R}^2$ and a homeomorphic extension $F : \overline{U} \to \overline{U'}$.*

Proof. Assume first that U is c-uniform with C-relatively connected boundary and that $f : \partial U \to \mathbb{R}^2$ is an η-quasisymmetric embedding that can be extended homeomorphically to $\mathbb{R}^2$. Set $E = \mathbb{R}^2 \setminus U$. By the second part of Proposition 6.1, there exists an η'-quasisymmetric extension $g : E \to \mathbb{R}^2$ of f. Since $\mathbb{R}^2 \setminus E$ is c-uniform, by Lemma 2.12, $\mathbb{R}^2 \setminus g(E)$ is c'-uniform for some c' depending only on c and η', thus only on c and η. Since f admits a homeomorphic extension to $\overline{U}$ it follows that g admits a homeomorphic extension to $\mathbb{R}^2$.

The bi-Lipschitz case follows similarly from Lemma 2.12 and the first part of Proposition 6.1. □

In §6.1 we prove the quasisymmetric case of Proposition 6.1 while in §6.2 we prove the bi-Lipschitz case of Proposition 6.1.

6.1. **Quasisymmetric case of Proposition 6.1.** For the rest of §6.1, given a K-quasidisk D in $\mathbb{R}^2 \setminus \overline{U}$ and an η-quasisymmetric $f : \partial D \to \mathbb{R}^2$, we denote by $f_D : \overline{D} \to \mathbb{R}^2$ the η^*-quasisymmetric extension of f given by the Beurling-Ahlfors extension; see Theorem 4.1 and Remark 4.2. Here, η^* depends only on η and K.

To prove the quasisymmetric case of Proposition 6.1, we use the following lemma.

Lemma 6.3. *Let $E \subset \mathbb{R}^2$ be a closed set and $D \subset \mathbb{R}^2$ be a K-quasidisk such that $D \cap E = \emptyset$ and $\partial D \cup E$ is c-uniformly perfect. Suppose that $f : \partial D \cup E \to \mathbb{R}^2$ is an η-quasisymmetric embedding that can be extended homoeomorphically on $\overline{D} \cup E$. Then the map $F : \overline{D} \cup E \to \mathbb{R}^2$ defined by*

$$F|_{\overline{D}} = f_D \qquad \text{and} \qquad F|_E = f|_E$$

is η' quasisymmetric for some η' depending only on η, K and c.

Assuming Lemma 6.3, the proof of the quasisymmetric case of Proposition 6.1 is as follows.

Proof of Proposition 6.1(2). Fix a K-quasicircle domain $U \subset \mathbb{R}^2$ with c-relatively connected boundary. Let $f : \partial U \to \mathbb{R}^2$ be a quasisymmetric embedding that can be extended homeomorphically to $\overline{U}$. Applying the arguments of §5 we may assume that ∂U is uniformly perfect. Extend f to $F : \mathbb{R}^2 \setminus U \to \mathbb{R}^2$ by setting $F|_{\overline{D}} = f_D$ for every component D of $\mathbb{R}^2 \setminus \overline{U}$. The proof of quasisymmetry of F is given in 2 steps.

First, iterating Lemma 6.3 three times, it is easy to see that if D_1, D_2, D_3 are three components $\mathbb{R}^2 \setminus \overline{U}$, then that the restriction of F on $\partial U \cup \bigcup_{i=1}^3 \overline{D_i}$ is η'' quasisymmetric for some η'' depending only on η, K and c.

To show that the map F is η'-quasisymmetric take points $x, a, b \in \mathbb{R}^2 \setminus U$. Find components D_1, D_2, D_3 of $\mathbb{R}^2 \setminus \overline{U}$ such that $x, a, b \in \partial U \cup \bigcup_{i=1}^3 \overline{D_i}$; if less than 3 components exist then the proof is a double iteration of Lemma 6.3. The quasisymmetry of F follows now from the quasisymmetry of F restricted on $\partial U \cup \bigcup_{i=1}^3 \overline{D_i}$. □

The next lemma is used in the proof of Lemma 6.3. For the rest of §6.1, for two positive quantities A, B we write $A \lesssim B$ if there exists constant C^*, depending only on c, K and η, such that $A \leq C^* B$. We write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$. Furthermore, for a point $z \in \mathbb{R}^2$ we denote by $\pi(z)$ the radial projection of z on $\mathbb{S}^1$.

Lemma 6.4. *Suppose that D is a K-quasidisk and E is a closed set such that $E \cap D = \emptyset$. Suppose also that $F : E \cup \overline{D} \to \mathbb{R}^2$ is an embedding such that the restrictions $F|_{\overline{D}}$ and $F_{E\cup\partial D}$ are η-quasisymmetric. If $x \in \overline{D}$ and $y \in E$, then there exists $x' \in \partial D$ such that $|x - y| \simeq |x' - y|$ and $|F(x) - F(y)| \simeq |F(x') - F(y)|$.*

Proof. Assume first that D is bounded. Applying quasisymmetric homeomorphisms of $\mathbb{R}^2$, we may assume that $D = F(D) = \mathbb{B}^2$. Let x and y be as in the statement of Lemma 6.4. We consider four possible cases.

CASE I. Suppose that $\mathrm{dist}(y,\mathbb{S}^1) \geq 1/10$. By the quasisymmetry of $F|_{E\cup\mathbb{S}^1}$, $\mathrm{dist}(F(y),\mathbb{S}^1) \gtrsim 1$. Let $x' \in \mathbb{S}^1$ be a point such that $|x-x'| = 1$. Then, $|x-y| \simeq |x'-y|$ and by the quasisymmetry of $F|_{E\cup\mathbb{S}^1}$ we have

$$|F(y)-F(x')| \simeq \mathrm{dist}(F(y),\mathbb{S}^1) \simeq |F(y)-F(x')|.$$

CASE II. Suppose that $\mathrm{dist}(y,\mathbb{S}^1) < \eta^{-1}(1/4)/10$ and $\mathrm{dist}(x,\mathbb{S}^1) \geq \eta^{-1}(1/4)/10$. Let $x' \in \mathbb{S}^1$ such that such that $|x-x'| \geq \eta^{-1}(1/4)/10$ and $|y-x'| \geq \eta^{-1}(1/4)/10$. Then, $|x-y| \simeq 1 \simeq |x'-y|$. Moreover, the quasisymmetry of $F|_{E\cup\mathbb{S}^1}$ implies that $|F(y)-F(x')| \simeq 1$ while the quasisymmetry of $F|_{\overline{\mathbb{B}^2}}$ gives $|F(x)-F(x')| \simeq 1$. Thus,

$$|F(y)-F(x)| \simeq 1 \simeq |F(y)-F(x')|.$$

CASE III. Suppose that $\mathrm{dist}(y,\mathbb{S}^1) < \eta^{-1}(1/4)/10$, $\mathrm{dist}(x,\mathbb{S}^1) < \eta^{-1}(1/4)/10$. We consider two subcases.

CASE III(1). Suppose that $\max\{|x-\pi(x)|,|y-\pi(y)|\} \leq \eta^{-1}(1/4)|\pi(x)-\pi(y)|$. Then, the quasisymmetry of $F|_{\overline{\mathbb{B}^2}}$ gives

$$\mathrm{dist}(F(x),\mathbb{S}^1) \simeq |F(x)-F(\pi(x))| \leq |F(\pi(x))-F(\pi(y))|/4$$

while the quasisymmetry of $F|_{E\cup\mathbb{S}^1}$ gives

$$\mathrm{dist}(F(y),\mathbb{S}^1) \simeq |F(y)-F(\pi(y))| \leq |F(\pi(x))-F(\pi(y))|/4.$$

Set $x' = \pi(x)$ and note that $|x-y| \simeq |x'-y|$ and

$$|F(x)-F(y)| \simeq |F(x')-F(\pi(y))| \simeq |F(x')-F(y)|.$$

CASE III(2). Suppose that $\max\{|x-\pi(x)|,|y-\pi(y)|\} \geq \eta^{-1}(1/4)|\pi(x)-\pi(y)|$. Choose a point $x' \in \mathbb{S}^1$ such that

$$|\pi(x)-x'| = \max\{|x-\pi(x)|,|y-\pi(y)|\}$$

and $\pi(x)$ is contained in the smaller subarc of $\mathbb{S}^1$ joining $\pi(y)$ and x'. It is easy to see that

$$|x-y| \simeq |x-\pi(x)| + |\pi(x)-\pi(y)| + |\pi(y)-y| \simeq |\pi(x)-x'| \simeq |y-x'|.$$

On the other hand,

$$\begin{aligned}|F(x)-F(y)| &\simeq |F(x)-\pi(F(x))| + |\pi(F(x))-\pi(F(y))| + |F(y)-\pi(F(y))|\\ &\simeq |F(x)-F(\pi(x))| + |\pi(F(x))-\pi(F(y))| + |F(y)-F(\pi(y))|\\ &\simeq |F(x')-F(\pi(x))| + |\pi(F(x))-\pi(F(y))| + |F(y)-F(x')|\\ &\simeq |\pi(F(x))-\pi(F(y))| + |F(y)-F(x')|.\end{aligned}$$

We conclude this case and the proof by showing that

$$|\pi(F(x))-\pi(F(y))| \lesssim |F(y)-F(x')|.$$

Indeed,

$$\begin{aligned}|\pi(F(x))-\pi(F(y))| &\leq |\pi(F(x))-F(x)| + |F(x)-F(\pi(x))|\\ &\quad + |F(\pi(y))-F(\pi(x))|\\ &\quad + |\pi(F(y))-F(y)| + |F(y)-F(\pi(y))|\\ &\lesssim |F(x)-F(\pi(x))| + |F(\pi(y))-F(\pi(x))| + |F(y)-F(\pi(y))|\\ &\lesssim |F(x)-F(x')| + |F(\pi(y))-F(x')| + |F(y)-F(x')|\\ &\lesssim |F(y)-F(x')|.\end{aligned}$$

Suppose now that D is unbounded. As before, we may assume that $D = F(D) = \mathbb{R} \times (0, +\infty)$. The proof is virtually the same with the difference that $\operatorname{dist}(y, \mathbb{R})$ and $\operatorname{dist}(x, \mathbb{R})$ do not matter and we only need to consider Case III(1) and Case III(2). $\square$

Remark 6.5. *With similar reasoning we can show that if D is bounded, $x \in \overline{D}$ and $y \in E$ and $\operatorname{dist}(y, \partial D) \leq 3 \operatorname{diam} D$, then there exists $y' \in \partial D$ such that $|x - y| \simeq |x - y'|$ and $|F(x) - F(y)| \simeq |F(x) - F(y')|$.*

We conclude now §6.1 by proving Lemma 6.3.

Proof of Lemma 6.3. Assume that D is bounded; the proof in the case that D is unbounded is similar. Setting $\eta_1(t) = \max\{\eta^*(t), \eta(t)\}$, we may assume that $F|_{\overline{D}}$ and $F|_{E\cup\partial D}$ are η_1-quasisymmetric.

Note that $\overline{D} \cup f(E)$ is c'-uniformly perfect for some c' depending only on η and c. Moreover, both $\overline{D} \cup E$ and $F(\overline{D} \cup E)$ are C_0-doubling for some universal $C_0 > 1$. Therefore, by Lemma 2.1, it suffices to show that there exists $H \geq 1$ such that for all $x, a, b \in \overline{D} \cup E$ we have

$$|x - a| \leq |x - b| \qquad \text{implies} \qquad |F(x) - F(a)| \leq H|F(x) - F(b)|. \tag{6.1}$$

Fix now $x, a, b \in \overline{D} \cup E$ with $|x - a| \leq |x - b|$. The proof of Lemma 6.3 is case study with respect to the position of points x, a, b in $\overline{D} \cup E$.

Case 1. If $x, a, b \in \partial D \cup E$ or $x, a, b \in \overline{D}$, then (6.1) holds with $H = \eta_1(1)$.

Case 2. Suppose that $a \in \overline{D}$ and $x, b \in E$. Applying Lemma 6.4, there exists $a' \in \partial D$ such that $|x - a| \simeq |x - a'|$ and $|F(x) - F(a)| \simeq |F(x) - F(a')|$. Apply now the quasisymmetry of $F|_{E\cup\partial D}$ for the points a', x, b.

Case 3. Suppose that $a, x \in \overline{D}$ and $b \in E$.

Case 3.1. Assume that $\operatorname{dist}(b, \partial D) \geq \operatorname{diam} D$. Then, by the quasisymmetry of $F|_{E\cup\partial D}$,

$$|F(x) - F(b)| \simeq \operatorname{dist}(F(b), \mathbb{S}^1) \gtrsim \operatorname{diam} D \gtrsim |F(x) - F(a)|.$$

Case 3.2. Assume that $\operatorname{dist}(b, \partial D) \leq \operatorname{diam} D$. As in Remark 6.5, choose a point $b' \in \partial D$ such that $|x-b'| \simeq |x-b|$ and $|F(x)-F(b')| \simeq |F(x)-F(b)|$. Then, apply quasisymmetry of $F|_{\overline{D}}$ on the points x, a, b'.

Case 4. Suppose that $a, b \in \overline{D^2}$ and $x \in E$. As in Lemma 6.4, choose points $a', b' \in \partial D$ such that $|x - a| \simeq |x - a'|$, $|x - b| \simeq |x - b'|$, $|x - a| \simeq |F(x) - F(a')|$, $|F(x) - F(b)| \simeq |F(x) - F(b')|$. Equation (6.1) follows now applying the quasisymmetry of $F|_{E\cup\partial D}$ on the the points x, a', b'.

Case 5. Suppose that $x \in \overline{D}$ and $a, b \in E$.

Case 5.1. Assume that $\operatorname{dist}(a, \partial D) \geq 3 \operatorname{diam} D$. Then, $\operatorname{dist}(b, \partial D) \geq \operatorname{diam} D$. Choose any point $x' \in \partial D$ and note that $|x - a| \simeq |x' - a|$, $|x - b| \simeq |x' - b|$ and, by the quasisymmetry of $F|_{E\cup\partial D}$, $|F(x) - F(a)| \simeq |F(x') - F(a)|$, $|F(x) - F(b)| \simeq |F(x') - F(b)|$. Equation (6.1) follows now applying the quasisymmetry of $F|_{E\cup\partial D}$ on the the points x', a, b.

Case 5.2. Assume that $\operatorname{dist}(a, \partial D) < 3 \operatorname{diam} D$. Choose a point $a' \in \partial D$ such that $|x-a| \simeq |x-a'|$ and $|F(x)-F(a)| \simeq |F(x)-F(a')|$. This case is now reduced to Case 3.

Case 6. Suppose that $x, b \in \overline{D}$ and $a \in E$. Note that $\operatorname{dist}(a, \partial D) \leq 2 \operatorname{diam} D$. As in Remark 6.5, choose a point $a' \in \partial D$ such that $|x - a'| \simeq |x - a|$ and $|F(x) - F(a')| \simeq |F(x) - F(a)|$. Then, apply the quasisymmetry of $F|_{\overline{D}}$ on points x, a', b.

Case 7. Suppose that $b \in \overline{D}$ and $a, x \in E$. As in Lemma 6.4, choose a point $b' \in \partial D$ such that $|x-b'| \simeq |x-b|$ and $|F(x)-F(b')| \simeq |F(x)-F(b)|$. Then, apply the quasisymmetry of $F|_{E\cup\partial D}$ on points x, a, b'. □

6.2. **Bi-Lipschitz case of Proposition 6.1.** The proof of the bi-Lipschitz case of Proposition 6.1 is almost the same as in the quasisymmetric case with one notable difference: instead of using the Beurling-Ahlfors quasisymmetric extension, for each component D of $\mathbb{R}^2 \setminus \overline{U}$, we use the extension $f_D : \overline{D} \to \mathbb{R}^2$ given by Lemma 4.3.

Proof of Proposition 6.1(2). The proof is similar to the quasisymmetric case so we only outline the steps of the proof.

Step 1. We show that if D is a K-quasidisk, $E \subset \mathbb{R}^2$ is a closed set disjoint from D and $F : \overline{D} \cup E \to \mathbb{R}^2$ is an embedding such that the restrictions $F|_{\overline{D}}$ and $F|_{\partial D\cup E}$ are L-bi-Lipschitz, then F is L'-bi-Lipschitz for some L' depending only on L and K. To prove this claim, fix $x, y \in \overline{D} \cup E$ and consider the only nontrivial case $x \in \overline{D}$, $y \in \partial D \cup E$. Lemma 6.4 can be used to reduce this setting to either $x, y \in \overline{D}$ or $x, y \in \partial D \cup E$.

Step 2. To prove the proposition, fix $x, y \in \mathbb{R}^2 \setminus U$, find two components D_1, D_2 of $\mathbb{R}^2 \setminus \overline{U}$ such that $x_1, x_2 \in \partial U \cup \overline{D_1} \cup \overline{D_2}$ and use Step 1 twice. □

7. Second reduction: bounded boundary

In this section, we reduce the proof of Theorem 1.1 to the case that U is the complement of a compact set, and the proof of Theorem 1.3 to the case that E is compact.

7.1. **Uniform domains.** Assume for the rest of §7.1 that $U \subset \mathbb{R}^2$ is c-uniform and that $f : \partial U \to \mathbb{R}^2$ is L-bi-Lipschitz (resp. f is η-quasisymmetric and ∂U is c-relatively connected) that admits a homeomorphic extension to $\overline{U}$. Assume, moreover, that Theorem 1.1 holds for unbounded uniform domains with bounded boundary.

To simplify the exposition, we use complex coordinates for the rest of §7.1.

By Corollary 6.2, there exist $c' > 1$ depending only on c and L (resp. on c and η) and a c'-uniform domain U' such that f extends to a homeomorphism between U and U'. There are three cases to consider.

Case 1: Suppose that U is bounded. In this case, U' is bounded. By the porosity of ∂U and $\partial U'$, there exist points $x_0 \in U$ and $x_0' \in U'$ such that

$$B^2(x_0, (4c)^{-1}\operatorname{diam} U) \subset U \qquad \text{and} \qquad B^2(x_0', (4c')^{-1}\operatorname{diam} U') \subset U'.$$

Applying similarity mappings we may assume that $x_0 = x_0' = 0$ and $\operatorname{diam} U = \operatorname{diam} U' = 1$.

Assume first that f is L-bi-Lipschitz. The domain $U\setminus\{x_0\}$ is c_1-uniform for some c_1 depending only on c and the map $f_1 : \partial U \cup \{x_0\} \to \partial U \cup \{x_0'\}$ with $f_1|_{\partial U} = f$ and $f_1(x_0) = x_0'$ is L_1-bi-Lipschitz for some $L_1 > 1$ depending only on L and c. Moreover, f_1 admits a homeomorphic extension to $\overline{U}$.

Recall the definitions of inversion maps I_{x_0} from §4.1. Set $V = I_0(U \setminus \{0\})$, $V' = I_0(U' \setminus \{0\})$ and $g : V \to V'$ with $g = I_0 \circ f \circ I_0|_V$. Then, V is an unbounded c_1-uniform domain, $\partial 1$ is bounded. By Remark 4.4, g is a bi-Lipschitz embedding defined on the boundary of an unbounded uniform domain with bounded boundary and the extension of g follows by our assumption. Taking inversions again, we obtain an L'-bi-Lipschitz extension of f to U with L' depending only on L and c.

Assume now that f is η-quasisymmetric. The inversion map $I_0\colon \mathbb{R}^2\setminus\{0\}\to\mathbb{R}^2\setminus\{0\}$ is 1-quasiconformal while the restrictions of I_0 on $B^2(0,2)\setminus B^2(0,(8c)^{-1})$ and on $B^2(0,2)\setminus B^2(0,(8c')^{-1})$ are L_2-bi-Lipschitz for some L_2 depending only on c and η. As in the bi-Lipschitz case, the domain $V=I_0(U\setminus\{0\})$ is an unbounded c_1-uniform domain and ∂V is bounded and c_1-relatively connected. Furthermore, $g=I_0\circ f\circ I_0|_{\partial V}$ is an η_1-quasisymmetric embedding that can be extended as a homeomorphism of V. Here, c_1 and η_1 depending only on c and η. By our assumption, there exists an η_1'-quasisymmetric extension $G:V\to V'$. Let $F:U\to U'$ with $F=I_0\circ G\circ I_0|_U$. Then, F is K-quasiconformal for some K depending only on c and η and, by Lemma 2.10, F is η'-quasisymmetric with η' depending only on c and η.

Case 2: Suppose that U is unbounded and ∂U contains an unbounded component. By Proposition 2.8, ∂U contains an unbounded quasicircle Γ, all other components of ∂U are bounded and U is contained in one of the two components of $\mathbb{R}^2\setminus\Gamma$. Fix $z_0\in\Gamma$ and let $z_0'=f(z_0)$.

The bi-Lipschitz case is similar to Case 1. Let $r>0$ and let x_0 be a point on $\partial B^2(z_0,r)$ such that $B^2(x_0,r/c)\subset U$. Similarly define a point $x_0'\in\partial B^2(z_0',r)$. The rest is as in Case 1.

Assume now that f is η-quasisymmetric. Applying an η_0-quasisymmetric homeomorphism of $\mathbb{R}^2$ we may assume that $\Gamma=f(\Gamma)=\mathbb{R}$, $z_0=z_0'=0$ and that U and U' are subsets of the upper half-plane. Here η_0 depends only on c and η. For each $k\in\mathbb{N}$ let $z_k=(12c^2)^{k-1}$ and let γ_k be a c-cigar curve in U joining z_k with $-z_k$. Note that for $k\geq 2$, $\gamma_k\subset B^2(z_0,3c(12c^2)^{k-1})\setminus B^2(z_0,(2c)^{-1}(12c^2)^{k-1})$ and, therefore, $\operatorname{dist}(\gamma_k,\gamma_{k+1})\geq(4c)^{-1}(12c^2)^k$.

For each $k\in\mathbb{N}$ let U_k be the domain bounded by γ_k and $[-z_k,z_k]$, and set $E_k=\partial U\cap\overline{U_k}$ and $E_k'=f(E_k)$. Each U_k is bounded and it is easy to check that each U_k is c'-uniform with c'-relatively connected boundary for some $c'>1$ depending only on c. Note also that $\operatorname{diam}E_k'\leq C|f(z_k)-f(-z_k)|$ for some C depending only on c and η. Define

$$\gamma_k'=[f(z_k),f(z_k)-2i\operatorname{diam}E_k']\cup[f(-z_k),f(-z_k)-2i\operatorname{diam}E_k']\cup\sigma_1\cup\sigma_2$$

where σ_1, σ_2 are circular arcs of $\partial B^2(f(z_k),\operatorname{diam}E_k)$, $\partial B^2(f(-z_k),\operatorname{diam}E_k)$ respectively so that $\gamma_k\cup f([-z_k,z_k])$ is the boundary of a K-quasidisk D_k which contains E_k in its closure. Here K depends only on c and η.

Applying the quasisymmetric extension property of relatively connected subsets of quasicircles [Vel16], we extend f to an η_1-quasisymmetric $f_k:\partial U_k\to\mathbb{R}^2$ with $f_k(\gamma_k)=\gamma_k'$. Here η_1 depends only on η and c. The extension F_k of f_k to each U_k follows from Case 1. Since $U=\bigcup_{k\in\mathbb{N}}U_k$, by standard converging arguments [Hei01, Corollary 10.30], $\{F_k\}$ subconverges to a mapping $F:U\to\mathbb{R}^2$ with $F|_{\partial U}=f$ that is η'-quasisymmetric.

Case 3: Suppose that U is unbounded and all components of ∂U are bounded. Fix $x\in\partial U$ and let A_x be the component of ∂U containing x. Let $r_1>8\operatorname{diam}A_x$. By Proposition 3.9, A_x is contained in a neighborhood $N_{i_1}(A_1,r_1)$ where $i_1\in\{1,2\}$ and A_1 is a component of ∂U. Let U_1 be the subset of U with boundary $N_{i_1}(A_1,r_1)$. Let $r_2>8\operatorname{diam}N_{i_1}(A_1,r_1)$. Inductivelly, having defined r_k and $N_{i_k}(A_k,r_k)$, let $r_{k+1}>8\operatorname{diam}N_{i_k}(A_k,r_k)$ and arguments similar to that of Proposition 3.9 show that $N_{i_k}(A_k,r_k)$ is contained in a neighborhood $N_{i_{k+1}}(A_{k+1},r_{k+1})$ where $i_{k+1}\in\{1,2\}$ and A_{k+1} is a component of ∂U.

For each $k \in \mathbb{N}$ let $U_k \subset \mathbb{R}^2$ be the unbounded domain with $\partial U_k = N_{i_k}(A_k, r_k)$. By Lemma 3.10, each U_k is c'-uniform and each ∂U_k is c'-relatively connected for some $c' > 1$ depending only on c. By our assumption, there exists a mapping $F_k : U_k \to \mathbb{R}^2$ that extends $f|_{\partial U_k}$ which is η'-quasisymmetric (resp. L'-bi-Lipschitz) with η' depending only on c and η (resp. $L' \geq 1$ depending only on c and L). As in Case 2, $\{F_k\}$ subconverges to a mapping $F : U \to \mathbb{R}^2$ with $F|_{\partial U} = f$ that is η'-quasisymmetric (resp. L'-bi-Lipschitz).

7.2. **Reduction for Theorem 1.3.** Assume that $n \geq 3$ and that $E \subset \mathbb{R}^n$ $(n \geq 3)$ is unbounded and c-uniformly disconnected. Assume also that $f : E \to \mathbb{R}^n$ is L-bi-Lipschitz (resp. E is c-uniformly perfect and f is η-quasisymmetric) and that Theorem 1.3 holds for bounded sets.

Suppose that E is unbounded. Fix $x \in E$. For each $k \in \mathbb{N}$ let E_k be a subset of E containing x such that $\operatorname{diam} E_k \leq 2^k$ and $\operatorname{dist}(E_k, E \setminus E_k) \geq c^{-1}2^k$. Note that each E_k is c-uniformly disconnected as the property is preserved on subsets.

Suppose that E is uniformly perfect and that f is η-quasisymmetric; the bi-Lipschitz case is identical. By c-uniform perfectness, $\operatorname{diam} E_k \geq c^{-2}2^k$. We show that each E_k is c^2-uniformly perfect. Let $y \in E_k$ and $r > 0$. Since E_k is perfect, either $\overline{B}^n(y,r) \cap E_k = E_k$ or $E_k \setminus \overline{B}^n(y,r) \neq \emptyset$. Assume the latter. Then, $r \leq 2^k$ and by uniform discontinuity, $\overline{B}^n(y, c^{-1}r) \cap E_k = \overline{B}^n(y, c^{-1}r) \cap E$. By c-uniform perfectness of E, $(\overline{B}^n(y, c^{-1}r) \setminus B^n(y, c^{-2}r)) \cap E_k = (\overline{B}^n(y, c^{-1}r) \setminus B^n(y, c^{-2}r)) \cap E \neq \emptyset$.

Therefore, each E_k is c^2-uniformly disconnected and c^2-uniformly perfect. By our assumption, each $f|_{E_k}$ extends to a mapping $F_k : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that is η'-quasisymmetric. with η' depending only on c and η. As in §7.1, $\{F_k\}$ subconverges to a mapping $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $F|_E = f$ that is η'-quasisymmetric.

8. Whitney-type decompositions around quasidisks

Let $D, D' \subset \mathbb{R}^2$ be Jordan domains, A, A' be unions of disjoint closed quasidisks in D, D' respectively and $\Delta_\infty, \Delta'_\infty$ be closed quasidisks contained in A, A' respectively. Also, let $f : A \to A'$ be an η-quasisymmetric homeomorphism with $f(\Delta_\infty) = \Delta'_\infty$. For the rest of this section, we assume that there exist $c > 1$ and $C > 1$ with the following properties.

(I) A and A' are compact with c-uniform complements.

(II) The Jordan curves $\Gamma_\infty = \partial\Delta_\infty$ and $\Gamma'_\infty = \partial\Delta'_\infty$ are c-bounded turning and
$$\operatorname{diam}\Gamma_\infty = \operatorname{diam}\Gamma'_\infty = 1$$

(III) For all $z \in \partial D$ and all $z' \in \partial D'$
$$C^{-1} \leq \operatorname{dist}(z, A) \leq \operatorname{dist}(z, \Delta_\infty) \leq (8c)^{-3}$$
$$C^{-1} \leq \operatorname{dist}(z', A') \leq \operatorname{dist}(z', \Delta'_\infty) \leq C.$$

For some $L > 1$ and $c_1 > 1$ depending only on c, C and η, we construct two families of sets $\mathscr{Q}$ and $\mathscr{Q}'$ with the following properties.

(P1): The family $\mathscr{Q}$ is a (L, c_1)-Whitney-type decomposition of $D \setminus \Delta_\infty$ and $\mathscr{Q}$ is a (L, c_1)-Whitney-type decomposition of $D' \setminus \Delta'_\infty$.

(P2): For all $Q \in \mathscr{Q}$, $c_1^{-1} \operatorname{diam} Q \leq \operatorname{dist}(\partial Q, A) \leq c_1 \operatorname{diam} Q$. Similarly for $\mathscr{Q}'$.

(P3): There exists homeomorphism $g : \overline{D} \to \overline{D'}$ such that $f|_{\partial A} = g|_{\partial A}$ and for each $Q \in \mathscr{Q}$, $g(Q) \in \mathscr{Q}'$.

The construction of $\mathscr{Q}$ is very similar to that of §6.2. However, the construction of $\mathscr{Q}'$ in this setting is more complicated for two reasons. The first is that, unlike §6.2, the map f is assumed to be only quasisymmetric. The second reason is that A may have infinite (even uncountably many) components and we need to make sure that the boundaries of all Whitney domains properly avoid all the components of $A \setminus \Delta'_\infty$ around Δ'_∞.

For the rest of §8, given two positive quantities a, b we write $a \lesssim b$ if there exists constant C_0, depending only on c, C and η, such that $a \leq C_0 b$. We write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

In §8.1 we construct $\mathscr{Q}$. In §8.2 we perform some preliminary steps towards the construction of $\mathscr{Q}'$ while the actual construction is given in §8.3 in an inductive manner. In §8.4 we record some observations which imply the desired properties of $\mathscr{Q}$ and $\mathscr{Q}'$.

8.1. **Decomposition around the preimage.** The construction of $\mathscr{Q}$ is almost identical to the construction in the proof of Lemma 4.3 so we only outline the steps.

Fix an orientation on Γ_∞. As in the proof of Lemma 4.3, two points p_i and p_j are *neighbors* in the collection $\{p_k : k = 1, \dots, n\}$ if one of the two components of $\Gamma_\infty \setminus \{p_i, p_j\}$ contains no point in $\{p_k : k = 1, \dots, n\}$. Furthermore, p_i is on the left (resp. right) of p_j in $\{p_k : k = 1, \dots, n\}$ if p_i and p_j are neighbors in $\{p_k : k = 1, \dots, n\}$ and, following the orientation of Γ_∞, p_i and p_j are the starting and ending points (resp. the ending and starting points) of $\Gamma_\infty(p_i, p_j)$.

Since Γ_∞ is c-bounded turning, by assumption (III) we have that

$$C^{-1} \leq \operatorname{dist}(w, \partial D) \leq 3c(8c)^{-3} \leq (32c^2)^{-1}$$

for all $w \in \Gamma_\infty$ [VW16, Lemma 3.4].

Set $D_1 = D$ and for each $m = 2, 3, \dots$ apply Lemma 3.6 on A with $\epsilon_m = (2c)^{-2m} C^{-1}$ and obtain a chordarc disk D_m. Let $l_1 \in \mathbb{N}$ be the smallest integer such that $2^{-l_1} \leq 2^{-6} C^{-1}$ and let Δ_1 be the chordarc disk containing D with $\epsilon = 2^{-l_1} C^{-1}$. Note that $\Delta_1 \cap A = D \cap A$. The constant l_1 will be chosen in §9.1 towards the proof of Theorem 1.1 but, in any case, it will be bounded above by a constant depending only on c and η.

For each $m = 2, 3, \dots$ let $l_m \in \mathbb{N}$ be the smallest integer such that $2^{-l_m} \leq \frac{1}{16}(2c)^{-2m-1} C^{-1}$ and let Δ_m be as in Lemma 3.1, where $E = \partial D_m$ and $\epsilon = 2^{-l_m}$. Choose points $x_1, \dots, x_n \in \Delta_\infty$ following the orientation of Γ_∞ such that

$$(32)^{-1} \operatorname{diam} \Delta_\infty \leq |x_i - x_{i+1}| \leq (16)^{-1}$$

with the convention $x_{n+1} = x_1$. Note that $k \leq N_0$ for some $N \in \mathbb{N}$ depending only on c. For each $i \in \{1, \dots, k\}$ we construct a broken line γ_k as in the proof of Lemma 4.3, that connects ∂D with x_i and intersects each Γ_k at exactly one point, denoted by $y_{i1^{k-1}}$.

Proceeding inductively, we define a set $\mathcal{W}$ of words formed from letters $\{1, \dots, N\}$ (N depending only on c), points $x_w \in \Gamma_\infty$ satisfying

$$\frac{c}{16C}(2c)^{-2|w|} \leq |x_{wi} - x_{w(i+1)}| \leq \frac{c}{8C}(2c)^{-2|w|}.$$

and broken lines γ_w joining x_w with $\partial\Delta_{|w|}$ intersecting each $\partial\Delta_{|w|+k}$, $k \geq 0$, only once.

As in the proof of Lemma 4.3, two points p_i and p_j are *neighbors* in the collection $\{p_k : k = 1,\dots,n\}$ if one of the two components of $\Gamma_\infty \setminus \{p_i, p_j\}$ contains no point in $\{p_k : k = 1,\dots,n\}$. Furthermore, p_i is on the left (resp. right) of p_j in $\{p_k : k = 1,\dots,n\}$ if p_i and p_j are neighbors in $\{p_k : k = 1,\dots,n\}$ and, following the orientation of Γ_∞, p_i and p_j are the starting and ending points (resp. the ending and starting points) of $\Gamma_\infty(p_i,p_j)$. The number $(8c)^3$ in assumption (III) is chosen so that for all $w,u \in \mathcal{W}$,

$$\operatorname{dist}(\gamma_w,\gamma_u) \gtrsim \min\{\operatorname{diam}\gamma_w, \operatorname{diam}\gamma_u\}.$$

Finally, given $w \in \mathcal{W}$, points $x_u, x_w \in \Gamma_\infty$ (with x_w being on the left of x_w in the collection $\{x_v : |v| = |w|\}$) define $\mathcal{Q}_w$ to be the Jordan domain bounded by γ_w, γ_u, $\Delta_{|w|}$ and $\Delta_{|w|+1}$. As in Lemma 4.3, the following remark holds true.

Remark 8.1. *For each $w \in \mathcal{W}$,*

(1) *each $\mathcal{Q}_w$ is an L_1-chordarc disk for some $L_1 \simeq 1$,*
(2) $\operatorname{dist}(\mathcal{Q}_w, x_w) \simeq \operatorname{dist}(\mathcal{Q}_w, \partial U) \simeq \operatorname{diam}\mathcal{Q}_w$.

If $A \cap \mathcal{Q}_w \neq \emptyset$ set $A_w = A \cap \mathcal{Q}_w$. Otherwise, let $z_w \in \mathcal{Q}_w$ be a point such that $\operatorname{dist}(z_w, \partial\mathcal{Q}_w) \geq \frac{1}{L}\operatorname{diam}\mathcal{Q}_w$ and set $A_w = \overline{B}^2(z_w, \frac{1}{2L}\operatorname{diam}\mathcal{Q}_w)$.

Remark 8.2. *If A is c-uniformly perfect, then* $\operatorname{diam}\mathcal{Q}_w \lesssim \operatorname{diam} A_w$ *for all $w \in \mathcal{W}$.*

8.2. **Preliminary steps for the construction of $\mathscr{Q}'$.** As with D, we first define Δ_1'. Fix $l_1' \in \mathbb{N}$ and set $\epsilon = 2^{-l_1'-3}$. As with the integer l_1, an integer l_1' will be chosen in the proof of Theorem 1.1. Let Δ_1' be as in Lemma 3.1 with $E = \overline{D}$.

For each $w \in \mathcal{W}$ let $x_w' = f(x_w)$. The notion of neighbor points follows from the orientation of Γ_∞' induced by f. For the rest of §8, two words $w,u \in \mathcal{W}$ with $|w| = |u|$ are called *neighbors* if x_w' and x_u' are neighbors in the collection $\{x_v' : v \in \mathcal{W}_k\}$. Similarly, if $w,u \in \mathcal{W}_k$ we say that w is at the left (resp. right) of u if x_w' is at the left (resp. right) of x_u' in the collection $\{x_v' : v \in \mathcal{W}_k\}$.

For each $w \in \mathcal{W}$, set $A_w' = f(A_w)$ (could be empty for some words $w \in \mathcal{W}$). For each $w \in \mathcal{W}$ such that $A_w \neq \emptyset$, set $V_w = \mathbb{R}^2 \setminus \overline{V_w'}$, where V_w' is the unbounded component of $\mathbb{R}^2 \setminus \mathcal{T}_{d_w}(A_w')$, $d_w = 2^{-n_w}$ and n_w is the smallest integer such that $2^{-n_w} \leq (32c')^{-1}\operatorname{dist}(A_w', A' \setminus A_w' \cup \partial\Delta_1')$. If $A_w = \emptyset$, set $V_w = \emptyset$.

The quasisymmetry of f along with Remark 8.1 imply that $d_w \simeq \operatorname{diam} V_w$ (when $A_w' \neq \emptyset$). We show in the next lemma, that if we remove the extra sets V_w from $\mathbb{R}^2 \setminus \Delta_\infty$ we still get a uniform domain.

Lemma 8.3. *There exists $c' > 1$ depending only on c and η satisfying the following properties.*

(1) *Each V_w is a disjoint union of at most c' many c'-chordarc disks of mutual distances and diameters bounded below by $(c')^{-1}\operatorname{dist}(A_w', \partial Y)$;*
(2) *For any $w,u \in \mathcal{W}$ with $w \neq u$, $\operatorname{dist}(V_w, V_u) \geq \frac{1}{c'}\max\{\operatorname{diam} A_w', \operatorname{diam} A_u'\}$. In particular, $V_w \cap V_u = \emptyset$.*
(3) $V = \Delta_1 \setminus (\Delta_\infty \cup \bigcup_{w\in\mathcal{W}} \overline{V_w})$ *is c'-uniform.*

Proof. The first claim follows from Lemma 3.4 while the second claim follows almost immediately from the definition of the domains V_w. To show the third claim, fix $x,y \in V$. For each $w \in \mathcal{W}$ such that $A_w' \neq \emptyset$, we define $V_w^* = \mathcal{T}_{d_w/2}(V_w)$.

Let τ be a c-cigar arc in $\Delta_1' \setminus A'$ joining x with y. Since γ does not get too close to Δ_∞ and its length is comparable to $|x-y|$, there exists some $N' \in \mathbb{N}$ depending only on c such that γ intersects at most N' components of $\bigcup_{w\in\mathcal{W}} V_w$.

Suppose that γ does not intersect any V_w^*. Let $z \in \gamma$ and let $w \in \mathcal{W}$ be such that V_w is nonempty and closest to z among all nonempty V_u, $u \in \mathcal{W}$. Then,

$$\operatorname{dist}(z, \partial V) = \operatorname{dist}(z, V_w) \gtrsim \operatorname{dist}(z, A'_w) \geq \operatorname{dist}(z, A') \gtrsim \min\{|x-z|, |y-z|\}.$$

Therefore, γ is a c_1-cigar curve in V for some c_1 depending only on c and η.

Suppose now that γ intersects some component H of V_w^*. We replace all the pieces of γ inside H by a subarc on the boundary of H. Since this procedure is performed at most N' times, working as above, we can show that the final curve γ is a c_2-cigar curve in V for some c_2 depending only on c and η. □

To reduce the use of constants, we assume for the rest that V is c-uniform.

8.2.1. *Domes.* For each $x, y \in \Gamma'_\infty$ fix a c-cigar curve $\tau'_{x,y}$ that joins x with y. As in §8.1, for each $z \in \tau'_{x,y}$ let $\Sigma(z)$ be the union of all squares in $\mathcal{G}_{2^{-l(z)}}$ that contain z where $l(z)$ is the smallest integer such that $2^{-l(z)} \leq (16c^2)^{-1}\operatorname{dist}(z, \partial U'_2)$. Let $\tau_{x,y}$ be an arc on the boundary of $\bigcup_{z\in\tau'_{x,y}}\Sigma(z)$ that connects x with y such that $\tau'_{x,y}\setminus\{x,y\}$ is contained in the Jordan domain bounded by Γ'_∞ and $\tau_{x,y}$.

Fix $k \in \mathbb{N}$ and $w \in \mathcal{W}$. Suppose that w_1 and w_2 are the left and right, respectively, neighbors of $w1^k$ in $\mathcal{W}_{|w|+k}$. Define $R_w^{(k)}$ to be the Jordan domain bounded by $\tau_{x'_{w_1}, x'_{w_2}}$ and Γ_∞.

Note that for each $k, l \in \mathbb{N}$ and $w \in \mathcal{W}$, $R_w^{(k+l)} = R_{w1^l}^{(k)}$. Moreover, each $\partial R_w^{(k)}$ is a c'-cigar curve for some $c' \simeq 1$. Thus, for each $w \in \mathcal{W}$ and $k \in \mathbb{N}$ there exists a point $p_w^{(k)} \in \partial R_w^{(k)}$ which is a midpoint of an edge of $\partial R_w^{(k)}$ such that $\operatorname{dist}(p_w^{(k)}, \partial V) \gtrsim \operatorname{diam} R_w^{(k)}$. Subdividing each edge of $\partial R_w^{(k)}$ into 2 edges we may assume that $p_w^{(k)}$ is a flat vertex of $\partial R_w^{(k)}$. Recall that z is a flat vertex of a polygon P if the two edges of P with z as their common point are co-linear.

The next lemma follows from a straightforward application of the quasisymmetry of f and the uniformity of V. The proof is left to the reader.

Lemma 8.4. *Given a small positive number $\delta_0 \in (0,1)$, there exists k_0 depending only on c, C, η and δ_0 such that if $k \geq k_0$ and $w \in \mathcal{W}$, then the following hold.*

(1) $\operatorname{diam} R_w^{(k)} \leq \delta_0$ *and* $x'_w \in \partial R_w^{(k)}$.
(2) If $u \in \mathcal{W}$ with $|u| = |w|$, then $\operatorname{dist}(R_w^{(k)}, R_u^{(k)}) \geq (1-\delta_0)|x'_w - x'_u|$.
(3) If $u \in \mathcal{W}$ with $|u| < |w|$, then $\operatorname{dist}(R_w^{(k)}, V_u) \geq (1-\delta_0)\operatorname{dist}(x'_w, V_u)$.
(4) If $uv \in \mathcal{W}$, $|u| = |w|$ and u is not a neighbor of w, then

$$\operatorname{dist}(R_w^{(k)}, V_{uv}) \geq (1-\delta_0)\operatorname{dist}(x'_w, V_{uv}).$$

(5) If $l \in \mathbb{N}$, then $\operatorname{diam} R_w^{(k+l)} \leq \delta_0 \operatorname{diam} R_w^{(k)}$.
(6) If $l \in \mathbb{N}$ and $u \in \mathcal{W}$ with $|u| \geq |w| + k$, then

$$\operatorname{diam} R_w^{(l)} \leq (1-\delta_0)\operatorname{dist}(p_w^{(l)}, R_u^{(l)}).$$

We specify δ_0 in §8.3.1, §8.3.2 and §8.3.3. For each $w \in \mathcal{W}$ set $R_w = R_w^{(k_0)}$ and $p_w = p_w^{(k)}$. We call the domain R_w *dome* and the point p_w, the *peak of* R_w. To simplify the notation, we write $\tau_w = \tau_{x'_{w_1}, x'_{w_2}}$ where x'_{w_1} (resp. x'_{w_2}) is the neighbor of $x_{w1^{k_0}}$ at its left (resp. right) in $\mathcal{W}_{|w|+k_0}$. We call the points x'_{w_1} and x'_{w_2} the left and right, respectively, endpoints of τ_w. In what follows, we only consider domes R_w for words w satisfying $|w| = lk_0 + 1$ for some $l \in \mathbb{N}\cup\{0\}$.

Before proceeding to the construction of $\mathscr{Q}'$, we make one final modification to the domes R_w. Given a word $w \in \mathcal{W}_{lk_0+1}$ and a word $u = w1^{mk_0} \in \mathcal{W}$, note that τ_w intersects τ_u. By modifying τ_w as in §8.1, we may assume that the two polygonal arcs τ_w and τ_u intersect only at p_u.

8.3. **Decomposition around the image.** Here we construct $\mathscr{Q}' = \{\mathcal{Q}'_w\}$ in an inductive manner. In §8.3.1 we construct the chord-arc disks $\mathcal{Q}_1, \dots, \mathcal{Q}'_N$. In §8.3.2 and in §8.3.3 we perform the inductive step.

8.3.1. *Construction of $\mathscr{Q}'$: Step 0.* Given $i \in \{1, \dots, N\}$, we define a simple polygonal path $\sigma_{i,i+1}$ that joins R_i with R_{i+1} as follows. Apply Lemma 3.6 on Δ'_∞ with

$$r = (cC)^{-2} \min_{i=1,\dots,N} \operatorname{dist}(p_i, \Delta'_\infty)$$

and obtain a Jordan domain $\hat{D}_1$ containing Δ'_∞. Now apply Lemma 3.1 on $\hat{D}_1$ with $\epsilon = 2^{-l(1)}$, where $l(1)$ is the smallest positive integer such that

$$2^{-l(1)} \leq \frac{1}{16} \operatorname{dist}(\partial \hat{D}_1, \partial V \cup \{p_1, \dots, p_N\}).$$

Thus, we obtain a chord-arc disk D'_1 containing $\hat{D}_1$. For each $i = 1, \dots, N$, there exists a subarc $\sigma_{i,i+1}$ of $\partial D'_1$ such that

(1) except of its endpoints, $\sigma_{i,i+1}$ is in $\Delta'_1 \setminus \bigcup R_i$;
(2) one of its endpoints is on τ_i between the peak of R_i and the right endpoint of τ_i and the other endpoint is on τ_{i+1} between the peak of R_{i+1} and the left endpoint of τ_{i+1}.

Choosing δ_0 sufficiently small in Lemma 8.4, we may assume that $\bigcup_{i=1}^N V_i$ is contained in the open annulus $T_\emptyset$ whose boundary is $\partial\Delta'_0$ and a polygonal Jordan curve which is the union of the curves $\sigma_{i,i+1}$ and subarcs of τ_i. For each $i = 1, \dots, n$, define $\tilde{T}_{i,i+1}$ to be the bounded Jordan domain that does not contain Δ_∞ and whose boundary is the union of a subarc of R_i, a subarc of R_{i+1}, a subarc of Γ_∞ and $\sigma_{i,i+1}$.

Note that $T_\emptyset$ contains every set V_i and at most C_1 components of $\bigcup_{w\in\mathcal{W},|w|\geq 2} V_w$ for some $C_1 > 1$ depending only on c, C and η. Suppose that $H_1 \dots, H_m$ are components of $T_\emptyset \cap \bigcup_{w\in\mathcal{W},|w|\geq 2} V_w$. There exists $m_1 > l'_1 + 4$, $m_1 \simeq 1$, and polygonal curves s_j with edges in $\mathcal{G}^1_{2^{-m_1}}$, $j = 1, \dots, m$ joining H_j with $\Delta'_1 \setminus T_\emptyset$ such that,

(1) except for its endpoints, each s_j is entirely in $\tilde{T}_0$;
(2) $\operatorname{dist}(s_j, \partial T_\emptyset \setminus \sigma_{i,i+1}) \geq 2^{-m_1}$ and $\operatorname{dist}(s_j, s_{j'}) \geq 2^{-m_1}$ when $j \neq j'$;
(3) if H_j is a component of V_w and x'_w is on $\Gamma'_\infty \cap \partial R_i$, then s_j joins V_j with a point on τ_i other than y_i;
(4) if H_j is a component of V_w and x'_w is on Γ'_∞ between the right endpoint of τ_i and the left endpoint of τ_{i+1}, then s_j joins V_j with a point on $\sigma_{i,i+1}$.

Let $\mathsf{Q}'_\emptyset = T_\emptyset \setminus \mathcal{T}_{2^{-m_1}}(\bigcup_j H_j \cup \bigcup_j s_j)$. Given $i = 1, \dots, N$, if $H_{j_1}, \dots, H_{j_l}$ are all the components of $T_\emptyset \cap \bigcup_{w\in\mathcal{W},|w|\geq 2} V_w$ connected to R_i as above, then set

$$\mathcal{R}_i = R_i \cup \mathcal{T}_{2^{-m_1}}(\bigcup_{n=1}^{l} (H_{j_n} \cup s_{j_n})).$$

Similarly, if $H_{j_1}, \dots, H_{j_l}$ are all the components of $T_\emptyset \cap \bigcup_{w\in\mathcal{W},|w|\geq 2} V_w$ connected to $\tilde{T}_{i,i+1}$ for some $i = 1, \dots, N$, then set

$$T_{i,i+1} = \tilde{T}_{i,i+1} \cup \mathcal{T}_{2^{-m_1}}(\bigcup_{n=1}^{l}(H_{j_n} \cup s_{j_n})).$$

Finally, in the preimage side, define $\mathsf{Q}_\emptyset$ to be the interior of $\bigcup_{i=1}^N \mathcal{Q}_i$.

Subdividing $\mathsf{Q}'_\emptyset$ we obtain Jordan domains $\mathcal{Q}'_i$ with the following properties.

(1a): Domains $\mathcal{Q}'_i$ are mutually disjoint and the union of their closures is all of $\overline{\mathsf{Q}'_\emptyset}$.

(1b): There exists some positive integer $m(0) \simeq 1$ such that each $\partial\mathcal{Q}'_i$ is a polygonal curve with edges in $\mathcal{G}^1_{2^{-m(0)}}$ and $\operatorname{dist}(\partial\mathcal{Q}'_i, \partial V \setminus \partial\Delta'_1) \geq 2^{-m(0)}$.

(1c): Collection $\{\mathcal{Q}_i\}$ is combinatorially equivalent to $\{\mathcal{Q}'_i\}$ in the following sense: if $g : (A \cap \mathsf{Q}_\emptyset) \cup \partial\mathsf{Q}_\emptyset \to (A' \cap \mathsf{Q}'_\emptyset) \cup \partial\mathsf{Q}'_\emptyset$ is a homeomorphism such that $g|_{A\cap\mathsf{Q}_\emptyset} = f|_{A\cap\mathsf{Q}_\emptyset}$, then g extends to a homeomorphism $G : \mathsf{Q}_\emptyset \to \mathsf{Q}'_\emptyset$ with $G(\mathcal{Q}_i) = \mathcal{Q}'_i$.

Note that

$$\overline{\Delta'_0} = \overline{\Delta'_\infty} \cup \bigcup_{i=1}^N \overline{\mathcal{Q}_i}' \cup \bigcup_{i=1}^N \overline{\mathcal{R}_i} \cup \bigcup_{i=1}^N \overline{T_{i,i+1}}.$$

For the induction step, we consider the following two possible cases.

8.3.2. *Construction of $\mathscr{Q}'$: Decomposition in $\mathcal{R}_w$.* Let $w \in \mathcal{W}_{lk_0+1}$ with l being a nonnegative integer. Let also w_1 and w_2 be the left and right, respectively, endpoints of τ_w.

We work as in §8.3.1 to obtain a polygonal path $\sigma_{w_1,w1^{k_0}}$ joining R_{w_1} and $R_{w1^{k_0}}$. Set $r = (cC)^{-2}\min\{\operatorname{dist}(p_{w_1}, \Delta'_\infty), \operatorname{dist}(p_{w1^{k_0}}, \Delta'_\infty)\}$, and let $\hat{\Omega}$ be the domain obtained by Lemma 3.6 for Δ_∞ and r. Let Ω be the chord-arc disk obtained by Lemma 3.1 for $\hat{\Omega}$ and $\epsilon = 2^{-l}$ where l is the smallest integer such that

$$\epsilon \leq \frac{1}{16}\operatorname{dist}(\partial\hat{\Omega}, (R_w \cap \partial V) \cup \{p_{w_1}, p_{w1^{k_0}}, p_{w_2}\}).$$

Let now $\sigma_{w_1,w1^{k_0}}$ be a subarc of $\partial\Omega$ such that its first endpoint is on τ_{w_1} between its peak and its right endpoint, and its second endpoint is on $\tau_{w1^{k_0}}$ between its peak and its left endpoint. Modifying the curve on its intersection points with τ_w we may assume that the curve is contained in R_w, Similarly we obtain a curve $\sigma_{w1^{k_0},w_2}$ that does not intersect with $\sigma_{w_1,w1^{k_0}}$.

Define $\tilde{T}_{w_1,w1^{k_0}}$ to be the bounded Jordan domain that does not contain Δ_∞ and whose boundary is the union of $\sigma_{w_1,w1^{k_0}}$, a subarc of R_{w_1}, a subarc of $R_{w1^{k_0}}$ and a subarc of Γ_∞. Similarly we define $\tilde{T}_{w1^{k_0},w_2}$. Define now

$$\mathsf{Q}'_w = \mathcal{R}_w \setminus (\overline{R_{w_1}} \cup \overline{R_{w1^{k_0}}} \cup \overline{R_{w_2}} \cup \overline{\tilde{T}_{w_1,w1^{k_0}}} \cup \overline{\tilde{T}_{w1^{k_0},w_2}}).$$

Choosing δ_0 sufficiently small in Lemma 8.4, we may assume that if $u \in \mathcal{W}$ with $|u| \leq |w| + k_0$ and $V_u \subset \mathcal{R}_w$, then V_u is contained in Q'_w.

As in §8.3.1, if Q'_w contains a component H of V_u for some $|u| > (l+1)k_0 + 1$, we construct a polygonal curve s_H joining H with the appropriate domain from the list: R_{w_1}, $R_{w1^{k_0}}$, R_{w_2}, $\tilde{T}_{w_1,w1^{k_0}}$, $\tilde{T}_{w1^{k_0},w_2}$. Then remove a thickening of s_H and H from Q'_w. After all possible removals, we denote the new Jordan domain again by Q'_w. Furthermore, in the process of adding these thickenings, we obtain new domains $T_{w_1,w1^{k_0}}$ and $T_{w1^{k_0},w_2}$ in place of $\tilde{T}_{w_1,w1^{k_0}}$ and $\tilde{T}_{w1^{k_0},w_2}$ respectively.

Similarly, the domes R_{w_1}, $R_{w1^{k_0}}$ and R_{w_2} are replaced by new domains $\tilde{R}_{w_1}$, $\mathcal{R}_{w1^{k_0}}$ and $\tilde{R}_{w_2}$, respectively. Further modifications on the left of $\tilde{R}_{w_1}$ and on the right of $\tilde{R}_{w_2}$ will give us the final domains $\mathcal{R}_{w_1}$ and $\mathcal{R}_{w_2}$; see §8.3.3. Finally, in the preimage side, define Q_w to be the interior of $\bigcup \mathcal{Q}_u$ where the union is taken over all words $u \in \mathcal{W}$ such that $lk_0 + 1 < |u| \leq (l+1)k_0 + 1$ and x_u is contained in the smaller subarc of $\Gamma_\infty \setminus \{x_{w_1}, x_{w_2}\}$.

Now, as in §8.3.1, we subdivide Q_w into Jordan domains $\mathcal{Q}'_u$, where u is as above, that satisfy the following properties.

(2a): Domains $\mathcal{Q}'_u$ are mutually disjoint and the union of their closures is all of $\overline{\mathsf{Q}'_w}$.

(2b): There exists some positive integer $m(w)$ with $\operatorname{diam} \mathsf{Q}'_w \simeq 2^{-m(w)}$ such that each $\partial \mathcal{Q}'_u$ is a polygonal curve with edges in $\mathcal{G}^1_{2^{-m(w)}}$ and $\operatorname{dist}(\partial \mathcal{Q}'_u, \partial V) \geq 2^{-m(w)}$.

(2c): Collection $\{\mathcal{Q}_u\}$ is combinatorially equivalent to $\{\mathcal{Q}'_u\}$ in the following sense: if $g : (A \cap \mathsf{Q}_w) \cup \partial \mathsf{Q}_w \to (A' \cap \mathsf{Q}'_w) \cup \partial \mathsf{Q}'_w$ is a homeomorphism such that $g|_{A \cap \mathsf{Q}_w} = f|_{A \cap \mathsf{Q}_w}$, then g extends to a homeomorphism $G : \mathsf{Q}_w \to \mathsf{Q}'_w$ with $G(\mathcal{Q}_u) = \mathcal{Q}'_u$.

Note that

$$\overline{\mathcal{R}_w} = \overline{\tilde{R}_{w_1}} \cup \overline{\mathcal{R}_{w1^{k_0}}} \cup \overline{\tilde{R}_{w_2}} \cup \overline{T_{w_1, w1^{k_0}}} \cup \overline{T_{w1^{k_0}, w_2}} \cup \bigcup \overline{\mathcal{Q}'_u}$$

where domains $\mathcal{Q}'_u$ are as in (2a)–(2c).

8.3.3. *Construction of $\mathscr{Q}'$: Decomposition in T_{w_1,w_2}.* Let l be a nonnegative integer and $w_1, w_2 \in \mathcal{W}_{lk_0+1}$ so that w_1 is on the left of w_2 in $\mathcal{W}_{jk_0+1}$ and T_{w_1,w_2}, $\mathcal{R}_{w_1}$ and $\mathcal{R}_{w_2}$ have been defined by the previous steps. The construction in this case is similar to that of §8.3.2 and we only sketch the steps.

Consider words $u_1, \dots, u_n \in \mathcal{W}_{(l+1)k_0+1}$ such that u_1 is at the right of $w_1 1^{k_0}$, u_i is at the left of u_{i+1} for $i = 1, \dots, n-1$ and u_n is at the left of $w_2 1^{k_0}$ in $\mathcal{W}_{(l+1)k_0+1}$. Choosing δ_0 is small enough in Lemma 8.4, we may assume that each dome R_{u_i} is contained in T_{w_1,w_2} and $\operatorname{diam} R_{u_i} \leq \frac{1}{2} \operatorname{dist}(R_{u_i}, \partial T_{w_1,w_2})$ when $i = 2, \dots, n-1$. As in §8.3.1 and §8.3.2, we join each R_{u_i} with $R_{u_{i+1}}$, $i = 1, \dots, k-1$, with a polygonal arc $\sigma_{u_i,u_{i+1}}$ contained in T_{w_1,w_2} that, except for its endpoints, does not intersect ∂V, $\partial T_{w_1,w_2}$, ∂R_{u_j} ($j = 1, \dots, n$). We also assume that the polygonal arcs $\sigma_{u_i,u_{i+1}}$, $i = 1, \dots, n-1$, are mutually disjoint.

For each $i = 2, \dots, n-1$, let $\tilde{T}_{u_i,u_{i+1}}$ be the bounded Jordan domain that does not contain Δ_∞ and is bounded by a subarc of R_{u_i}, a subarc of $R_{u_{i+1}}$, $\sigma_{u_i,u_{i+1}}$ and a subarc of Γ_∞. Let

$$\mathsf{Q}'_{w_1,w_2} = T_{w_1,w_2} \setminus \bigcup_{i=1}^{n} \overline{R_{u_i}} \cup \bigcup_{i=1}^{n-1} \overline{\tilde{T}_{u_i,u_{i+1}}}.$$

As in §8.3.1 and §8.3.2, if H is a component of V_w with $|w| > (l+1)k_0 + 1$, then we construct a polygonal curve s_H joining H with the appropriate R_{u_i} or the appropriate $\tilde{T}_{u_i,u_{i+1}}$ and remove a thickening of s_H and H from $\tilde{T}_{w_1,w_2}$. This way we obtain a new domain which we still denote by Q'_{w_1,w_2}. We also obtain new domains $T_{u_i,u_{i+1}}$ in place of $\tilde{T}_{u_i,u_{i+1}}$ for $i = 1, \dots, n-1$, new domains $\mathcal{R}_{u_i}$ in place of R_{u_i} for $i = 2, \dots, n-1$, and new domains $\tilde{R}_{u_1}$ and $\tilde{R}_{u_n}$ in place of R_{u_1} and R_{u_n}. Further modifications on the left of $\tilde{R}_{u_1}$ and on the right of $\tilde{R}_{u_n}$ will yield the final sets $\mathcal{R}_{u_1}$ and $\mathcal{R}_{u_n}$; see §8.3.2. Finally, in the preimage side, define Q_{w_1,w_2}

to be the interior of $\bigcup \overline{\mathcal{Q}_u}$ where the union is taken over all words $u \in \mathcal{W}$ such that $lk_0 + 1 < |u| \leq (l+1)k_0 + 1$ and x_u is contained in the smaller subarc of $\Gamma_\infty \setminus \{x_{u_1}, x_{u_k}\}$.

As in previous sections, we subdivide Q_{w_1,w_2} into domains $\mathcal{Q}'_u$, where u is as above, that satisfy the following properties.

(3a): Domains $\mathcal{Q}'_u$ are mutually disjoint and the union of their closures is all of $\overline{\mathsf{Q}'_{w_1,w_2}}$.

(3b): There exists some positive integer $m(w_1,w_2)$ with $\operatorname{diam} \mathsf{Q}'_{w_1,w_2} \simeq 2^{-m(w_1,w_2)}$ such that each $\partial \mathcal{Q}'_u$ is a polygonal curve with edges in $\mathcal{G}^1_{2^{-m(w_1,w_2)}}$ and $\operatorname{dist}(\partial \mathcal{Q}'_u, \partial V) \geq 2^{-m(w_1,w_2)}$.

(3c): Collection $\{\mathcal{Q}_u\}$ is combinatorially equivalent to $\{\mathcal{Q}'_u\}$ in the following sense: if $g : (A \cap \mathsf{Q}_{w_1,w_2}) \cup \partial \mathsf{Q}_{w_1,w_2} \to (A' \cap \mathsf{Q}'_{w_1,w_2}) \cup \partial \mathsf{Q}'_{w_1,w_2}$ is a homeomorphism such that $g|_{A\cap \mathsf{Q}_{w_1,w_2}} = f|_{A\cap \mathsf{Q}_{w_1,w_2}}$, then g extends to a homeomorphism $G : \mathsf{Q}_{w_1,w_2} \to \mathsf{Q}'_{w_1,w_2}$ with $G(\mathcal{Q}_u) = \mathcal{Q}'_u$.

Note that

$$\overline{T_{w_1,w_2}} = \overline{\tilde{R}_{u_1}} \cup \overline{\tilde{R}_{u_n}} \cup \bigcup_{i=2}^{n-1} \overline{\mathcal{R}_{u_i}} \cup \bigcup_{i=1}^{n-1} \overline{T_{u_i,u_{i+1}}} \cup \bigcup \overline{\mathcal{Q}'_u}$$

where $\mathcal{Q}'_u$ are as in (3a)–(3c) above.

8.4. **Concluding remarks.** Proceeding inductively, we obtain a collection of sets $\mathscr{Q}' = \{\mathcal{Q}'_w : w \in \mathcal{W}\}$. We make the following three observations.

Firstly, by properties (1c), (2c) and (3c), $\mathscr{Q}'$ is combinatorially equivalent to $\mathscr{Q}$ in the sense that there exists homeomorphism $g : \overline{D} \to \overline{D'}$ such that $f|_{\partial A} = g|_{\partial A}$ and for each $\mathcal{Q}_w \in \mathscr{Q}$, $g(\mathcal{Q}_w) \in \mathcal{Q}'_w$. Secondly,sSince there is a finite number of different combinations for domains Q'_w, Q'_{w_1,w_2} and a finite number of different combinations that these domains can be cut into pieces, it follows that each $\mathcal{Q}'_w$ is an L-chord-arc disk for some L depending only on c and η. Thirdly, by their construction, each $\mathcal{Q}'_w$ satisfies

$$\operatorname{dist}(\partial \mathcal{Q}'_w, A) \lesssim \operatorname{dist}(\partial \mathcal{Q}'_w, \Delta_\infty) \simeq \operatorname{diam} \mathcal{Q}'_w.$$

On the other hand, by (1b), (2b) and (3b), we have that $\operatorname{dist}(\partial \mathcal{Q}'_w, A) \gtrsim \operatorname{diam} \mathcal{Q}'_w$.

These three observations, in conjunction with Remark 8.1, show that the two collections $\mathscr{Q}$ and $\mathscr{Q}'$ have the desired properties (P1)–(P3).

9. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We focus on the quasisymmetric case; the proof in the bi-Lipschitz case is similar and is given in §9.3.

By §5 and §7, we may assume for the rest of this section that $U \subset \mathbb{R}^2$ is an unbounded c-uniform domain and that ∂U is compact and C-uniformly perfect. By Corollary 6.2, we assume that $f : \mathbb{R}^2 \setminus U \to \mathbb{R}^2$ is an η-quasisymmetric map that can be extended homeomorphically to $\mathbb{R}^2$. By Lemma 2.12 and the invariance of uniformly perfect sets under quasisymmetric maps, the domain $U' = \mathbb{R}^2 \setminus f(\mathbb{R}^2 \setminus U)$ is c'-uniform and $\partial U'$ is C'-relatively connected for some c' depending only on c and η, and some C' depending only on C and η. For simplicity, we assume for the rest that $C = C' = c' = c$.

Fix $x_0 \in \partial U$ and $x'_0 \in \partial U'$. Let $\mathcal{D}_0 = \mathbb{R}^2 \setminus \overline{B}^2(x_0, 2c \operatorname{diam} \partial U)$ and $\mathcal{D}'_0 = \mathbb{R}^2 \setminus \overline{B}^2(x'_0, 2c \operatorname{diam} \partial U')$. In §9.1 we apply the results of §3 and §8 to construct two Whitney-type decompositions, one in $U \setminus \mathcal{D}_0$ and another in $U' \setminus \mathcal{D}'_0$ that are combinatorially equivalent. The difference here is that the two decompositions consist of domains in $\mathscr{QC}(K,d)$ and not chord-arc disks. Specifically, we show the following proposition.

Proposition 9.1. *There exist $K > 1$, $d > 1$, $C > 1$ depending only on c and η and two families of domains $\mathscr{D}, \mathscr{D}' \subset \mathscr{QC}(K,d)$ with the following properties.*

1. *The domains in $\mathscr{D}$ are mutually disjoint and $U \setminus \mathcal{D}_0 = \bigcup_{D\in\mathscr{D}} \overline{D}$. Similarly for $\mathscr{D}'$.*
2. *For all $D \in \mathscr{D}$, $C^{-1} \operatorname{diam} D \leq \operatorname{dist}(D, \partial U) \leq C \operatorname{diam} D$. Similarly for $\mathscr{D}'$.*
3. *For each $D \in \mathscr{D}$, there are at most C elements in $\mathscr{D}$ whose boundary intersects that of D. If $\Gamma = \partial D \cap \partial D' \neq \emptyset$ for $D, D' \in \mathscr{D}$, then Γ is an L-bi-Lipschitz arc and $\operatorname{diam}\Gamma \geq C^{-1} \max\{\operatorname{diam} D, \operatorname{diam} D\}$. Similarly for $\mathscr{D}'$.*
4. *There exists homeomorphism $g : \overline{U \setminus \mathcal{D}_0} \to \overline{U' \setminus \mathcal{D}'_0}$ such that $f|_{\partial U} = g|_{\partial U}$ and $g(D) \in \mathscr{D}'$ for each $D \in \mathscr{D}$.*

In §9.1 we construct the families $\mathscr{D}$ and $\mathscr{D}'$ while in §9.2 we give the proof of Theorem 1.1.

9.1. **Decompositions for planar uniform domains.** We describe the steps in the construction of the families $\mathscr{D}$ and $\mathscr{D}'$. Here and for the rest of §9.1 we write $E = \mathbb{R}^2 \setminus U$ and $E' = \mathbb{R}^2 \setminus U'$. To reduce the use of constants and simplify the exposition, we assume that all constants in the lemmas and propositions in §3 are equal to c for both U and U'. For two positive constants a, b we write $a \lesssim b$ if $a \leq Cb$ for some constant $C > 1$depending at most on c and η. We write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

Step 1. We apply the construction of §3.4 on ∂U inside $U \setminus \mathcal{D}_0$ with $\epsilon = (80c)^{-3} \operatorname{diam} E$ and obtain Jordan domains $D_1, \dots D_n$. Set $\mathcal{D}_\emptyset = (U\setminus\mathcal{D}_0)\setminus\bigcup_{i=1}^n \overline{D_i}$ and note that $\operatorname{diam}\mathcal{D}_\emptyset \geq \operatorname{diam}\partial U$.

For each $i = 1, \dots, N$ let $E_i = D_i \cap E$ and $E'_i = f(E_i)$. Applying Lemma 3.4 repeatedly on each set E'_i, we obtain Jordan domains $D'_1 = V(E'_1, (U'\setminus\mathcal{D}'_0), r_1)$ and

$$D'_i = V(E'_i, (U' \setminus \mathcal{D}'_0) \setminus \bigcup_{k=1}^{i-1} D'_k, r_i) \quad \text{for } i = 2, \dots, n$$

where

$$r_i = (32c)^{-1} \min\{\operatorname{diam} E'_i, \operatorname{dist}(E'_i, E' \setminus E'_i)\} \quad \text{for } i = 1, 2, \dots, n.$$

Set $\mathcal{D}'_\emptyset = (U' \setminus \mathcal{D}'_0) \setminus \bigcup_{i=1}^n \overline{D'_i}$. Since E is relatively connected, by Remark 2.4, r_i is comparable to $\operatorname{dist}(E'_i, E' \setminus E'_i)$.

Step 2. Fix $i \in \{1, \dots, n\}$ and let

$$\epsilon'_i = (8c)^{-3} \min\{\operatorname{dist}(E'_i, \partial D'_i), \operatorname{diam} E'_i\}.$$

Observe that, although $U' \cap D'_i$ may not be c-uniform like U', the condition of c-uniformity for E' holds true in the scale of ϵ'_i. That is, for all $x, y \in E'_i$ with $|x-y| \leq \epsilon'_i$, there exists c-cigar curve γ joining them in $U \cap D'_i$. In fact, ϵ'_i has been chosen in such a way that both Lemma 3.6 and Lemma 3.8 can be applied as if $U\cap D'_i$ was c-uniform. Therefore, we can apply on each E'_i inside D'_i the construction of §3.4 with ϵ'_i and obtain Jordan domains $D'_{i1}, \dots, D'_{in_i}$. Set $\mathcal{D}'_i = D'_i \setminus \bigcup_{j=1}^{n_i} D'_{ij}$.

In each D_i, we apply now the second part of Step 1. Fix $i \in \{1, \dots, n\}$ and let $E'_{ij} = E' \cap D'_{ij}$ and $E_{ij} = f^{-1}(A'_{ij})$. By Lemma 3.10, $D_i \cap U$ is c'-uniform for some c' depending only on c. Applying Lemma 3.4 repeatedly on each set E_{ij}, we obtain Jordan domains $D_{i1} = V(E_{i1}, U \cap D_i, (32c)^{-1})$ and

$$D_{ij} = V(E_{ij}, (U \cap D_i) \setminus \bigcup_{k=1}^{j-1} D'_{ik}, (32c)^{-1}) \quad \text{for } j = 2, \dots, n_i.$$

Set $\mathcal{D}_i = (U \cap D_i) \setminus \bigcup_{j=1}^{n_i} \overline{D_{ij}}$ and note that $\operatorname{diam} \mathcal{D}'_i \geq \epsilon'_0 \operatorname{diam} D_i$ with ϵ'_0 depending only on c.

Inductive assumption. Suppose that from *Step* $2k$ we have obtained mutually disjoint bounded Jordan domains $D_w \subset U$, mutually disjoint Jordan domains $D'_w \subset U'$, boundary sets $E_w = D_w \cap E$ and boundary sets $E'_w = D'_w \cap E' = f(E_w)$ such that, for some $M > 1$ depending only on c and η,

(I1): $M^{-1} \operatorname{diam} E_w \leq \operatorname{dist}(E_w, E \setminus E_w) \leq M \operatorname{dist}(\partial D_w, E \setminus E_w)$;

(I2): $M^{-1} \operatorname{dist}(z, E_w) \leq \min\{\operatorname{dist}(E_w, E \setminus E_w), \operatorname{diam} E_w\} \leq M \operatorname{dist}(z, E_w)$ for all $z \in \partial D_w$;

(I3): if $\operatorname{dist}(z, A) \geq (8c)^{-3} \operatorname{diam} A$ for some $z \in \partial D_w$ and some component A of E_w, then $\operatorname{dist}(A, \partial D_w) \geq M^{-1} \operatorname{diam} A$.

Similarly for E', E'_w and D'_w. Also, since E is uniformly perfect, we have that $\operatorname{dist}(E_w, E \setminus E_w) \leq M_0 \operatorname{diam} E_w$ for some $M_0 > 1$ depending only on c.

Fix now a a Jordan domain D_w, $w = i_1 \cdots i_{2k}$. We distinguish two cases.

9.1.1. *Case 1.* For all components A of E_w, there exists $z \in \partial D_w$ such that $\operatorname{diam} A \leq (8c)^3 \operatorname{dist}(A, z)$.

Remark 9.2. *By (I3) and the quasisymmetry of f,*

$$\operatorname{diam} f(A) \lesssim \operatorname{dist}(f(A), E' \setminus E'_w).$$

By Lemma 3.11,

$$\operatorname{diam} E'_w \lesssim \operatorname{diam} D'_w \lesssim c'_2 \operatorname{dist}(\operatorname{diam} E'_w, E' \setminus \operatorname{diam} E'_w).$$

By quasisymmetry of f,

$$\operatorname{diam} E_w \lesssim \operatorname{dist}(\operatorname{diam} E_w, E \setminus \operatorname{diam} E_w).$$

Step $2k+1$. Applying the construction of §3.4 on E_w with

$$\epsilon_w = (80c)^{-3} \min\{\operatorname{dist}(E_w, \partial D_w), \operatorname{diam} E_w\}$$

we obtain Jordan domains $D_{wi} \subset D_w$ with $i \leq n_w$. Define $E_w = D_{wi} \cap E$, $E'_{wi} = f(E_{wi})$ and $\mathcal{D}_w = D_w \setminus \bigcup_i D_{wi}$. In each D'_w apply the second part of Step 1 and obtain Jordan domains D'_{wi}. Set $\mathcal{D}'_w = D'_w \setminus \bigcup_i D'_{wi}$.

Step $2k+2$. In each D'_{wi} apply the construction of §3.4 on E'_{wi} with

$$\epsilon'_{wi} = (80c)^{-3} \min\{\operatorname{dist}(E'_{wi}, \partial D'_{wi}), \operatorname{diam} E'_{wi}\}.$$

Again, ϵ'_{wi} has been chosen in such a way that both Lemma 3.6 and Lemma 3.8 can be applied as if $U \cap D'_{wi}$ was c-uniform. Thus, we obtain Jordan domains $D'_{wij} \subset D'_{wi}$. Set $E'_{wij} = D'_{wij} \cap E'$, $E_{wij} = f^{-1}(E'_{wij})$ and $\mathcal{D}'_{wi} = D'_{wi} \setminus \bigcup_j D'_{wij}$.

In each D_{wi} we apply the second part of Step 1. Applying Lemma 3.4 repeatedly on each set E_{wij}, we obtain Jordan domains D_{wij}. Set $\mathcal{D}_{wi} = D_{wi} \setminus \bigcup_j D_{wij}$.

Combining Lemma 3.10, Lemma 3.11, Remark 9.2 and using the fact that E is uniformly perfect, we obtain the next corollary which completes the induction step for Case 1.

Corollary 9.3. *There exists $N_2 > 1$, $d > 1$ and $K > 1$ depending only on c and η with the following properties.*

(1) $n_w \leq N_2$.
(2) Each D_{wi} is an K-quasidisk.
(3) For all $i \in \{1,\dots,n_w\}$, $\operatorname{diam} D_{wi} \geq d^{-1} \operatorname{diam} D_w$.
(4) For all $i \in \{1,\dots,n_w\}$, $\operatorname{dist}(\partial D_w, \partial D_{wi}) \geq d^{-1} \operatorname{diam} D_w$.
(5) For all $i, i' \in \{1,\dots,n_w\}$, $\operatorname{dist}(\partial D_{wi}, \partial D_{wi'}) \geq d^{-1} \operatorname{diam} D_w$.
(6) Properties (I1), (I2) and (I3) hold for E, E_{wi} and D_{wi} with constant d.

By Remark 9.2 and the fact that $n \leq N$, the same is true for domains D'_{wi}. Similarly, the conclusions of Corollary 9.3, hold true for domains D_{wij} and D'_{wij}.

9.1.2. *Case 2.* There exists a component $\overline{\Delta_\infty}$ of E_w such that

$$\operatorname{dist}(z, \Delta_\infty) \leq (8c)^{-3} \operatorname{diam} \Delta_\infty \text{ for all } z \in \partial D_w.$$

By (I1), $\operatorname{dist}(D_w, \Delta_\infty) \gtrsim \operatorname{diam} E_w \geq \operatorname{diam} \Delta_\infty$. Set $\Delta'_\infty = f(\Delta_\infty)$ and $E'_w = f(E_w)$. We show that that $\operatorname{dist}(z, \Delta'_\infty)$ is comparable to $\operatorname{diam} \Delta'_\infty$ for each $z \in \partial D'_w$.

Lemma 9.4. *There exists $C > 1$ depending only on c and η such that*

$$C^{-1} \operatorname{diam} \Delta'_\infty \leq \operatorname{dist}(\Delta'_\infty, z) \leq C \operatorname{diam} \Delta'_\infty \textit{ for all } z \in \partial D'_w.$$

Proof. Note that $\operatorname{diam} E_w \leq \operatorname{diam} D_w \leq (2(8c)^{-3}+1) \operatorname{diam} \Delta_\infty$. By quasisymmetry of f, $\operatorname{diam} E'_w \lesssim \operatorname{diam} \Delta'_\infty$ [Hei01, Proposition 10.8] and the right inequality follows from (I1) and (I2) for E', E'_w, D'_w. The left inequality follows from (I1) and (I2) for E', E'_w, D'_w and the fact that $\operatorname{diam} \Delta'_\infty \leq \operatorname{diam} E'_w$. □

By Lemma 9.4, all assumptions of §8 are satisfied and by setting $D = D_w$, $D' = D'_w$, $A = E_w$ and $A' = E'_w$ Step $2k+1$ and Step $2k+2$ are as follows

Step $2k+1$ and Step $2k+2$. We replace D_w with Δ_1 where l_1 is chosen so that $\operatorname{dist}(\Delta_1, D_{w'}) \geq \frac{1}{2} \operatorname{dist}(D_w, D_{w'})$ for all $w' = j_1 \cdots j_{2k} \neq w$. Define $\{\mathcal{Q}_u\}_{u \in \mathcal{W}}$, $\{\mathcal{Q}'_u\}_{u \in \mathcal{W}}$, $\{A_u\}_{u \in \mathcal{W}}$ and $\{A'_u\}_{u \in \mathcal{W}}$ as in §8. Set $D_{wu} = Q_u$ and $D'_{wu} = Q'_u$.

If $D_{wu} \cap \partial U = \emptyset$, then set $\mathcal{D}_{wu} = D_{wu}$ and $\mathcal{D}'_{wu} = D'_{wu}$. If $D_{wu} \cap \partial U \neq \emptyset$ then set $\mathcal{D}_{wu} = D_{wu} \setminus \mathcal{T}_{\delta_w}(E \cap D_{wu})$ and $\mathcal{D}'_{wu} = D'_{wu} \setminus \mathcal{T}_{\delta'_w}(E' \cap D'_{wu})$ where

$$\delta_w = \frac{1}{10} \operatorname{dist}(E \cap D_{wu}, \partial D_{wu}) \qquad \text{and} \qquad \delta_w = \frac{1}{10} \operatorname{dist}(E' \cap D'_{wu}, \partial D'_{wu}).$$

It is straightforward to check that Corollary 9.3 holds true in Case 2 as well. This completes the inductive step and the construction of the two decompositions $\mathscr{D}$ and $\mathscr{D}'$.

9.2. **Proof of Theorem 1.1 in the quasisymmetric case.** Suppose that $U \subset \mathbb{R}^2$ is an unbounded c-uniform domain with bounded c-uniformly perfect boundary. Let $\mathscr{D} = \{\mathcal{D}_w\}$ and $\mathscr{D}' = \{\mathcal{D}'_w\}$ be the decompositions of Proposition 9.1.

Given $\mathcal{D}_w, \mathcal{D}_u \in \mathscr{D}$ whose boundaries intersect in a non-degenerate set, let $\Gamma_{w,u} = \mathcal{D}_w \cap \mathcal{D}_u$ and $\Gamma'_{w,u} = \mathcal{D}'_w \cap \mathcal{D}'_u$. Then, $\Gamma_{w,u}, \Gamma'_{w,u}$ are L-bi-Lipschitz arcs and $\operatorname{diam} \Gamma_{w,u} \gtrsim \operatorname{diam} \partial\mathcal{D}_w$ and $\operatorname{diam} \Gamma'_{w,u} \gtrsim \operatorname{diam} \partial\mathcal{D}'_w$. Define a homeomorphism $g : \bigcup_{\mathcal{D}_w \in \mathscr{D}} \partial\mathcal{D}_w \to \bigcup_{\mathcal{D}'_w \in \mathscr{D}'} \partial\mathcal{D}'_w$ so that $g|_{\Gamma_{w,u}}$ maps $\Gamma_{w,u}$ onto $\Gamma'_{w,u}$ by arc-length parametrization and can be homeomorphically extended to each $\mathcal{D}_w$. Note that

$g|_{\mathcal{D}_w}$ is a (λ_w, Λ)-quasisimilarity with $\Lambda \simeq 1$ and with $\lambda_w = \frac{\operatorname{diam} \mathcal{D}'_w}{\operatorname{diam} \mathcal{D}_w}$. By Proposition 4.5, each $g|_{\mathcal{D}_w}$ extends to a (λ_w, Λ')-quasisimilarity $F_w : \mathcal{D}_w \to \mathcal{D}'_w$ with $\Lambda' \simeq 1$.

Define $F : U \to U'$ with $F|_{\mathcal{D}_w} = F_w$. By a theorem of Väisälä on removability of singularities [Väi71, Theorem 35.1], F is K-quasiconformal with K depending only on Λ', thus only on c and η. By Lemma 2.10, F is η' quasisymmetric for some η' depending only on c and η.

9.3. **Proof of Theorem 1.1 in the bi-Lipschitz case.** The proof of Theorem 1.1 in the case that f is L-bi-Lipschitz is similar. The sets $\mathscr{D}$ and $\mathscr{D}$ are constructed as in the quasisymmetric case. Since ∂U may not be uniformly perfect, in Proposition 9.1, property (4) holds with min instead of max and the families $\mathscr{D}, \mathscr{D}' \subset \mathscr{CA}(L_0, d)$ for some $L_0 > 1$ and $d > 1$ depending only on L and c. Moreover, in Corollary, 9.3, the domains D_{wi} are L'_0-chord-arc disks and property (3) may not hold.

Since f is L-bi-Lipschitz, there exists $c_0 > 1$ depending only on c and L such that $c_0^{-1} \operatorname{diam} \mathcal{D}'_w \leq \operatorname{diam} \mathcal{D}_w \leq c_0 \operatorname{diam} \mathcal{D}'_w$ for all $\mathcal{D}_w \in \mathscr{D}$. Therefore, applying Proposition 4.5 and defining F as above, we note that $F|_{\mathcal{D}_w}$ is L_1-bi-Lipschitz for all $\mathcal{D}_w \in \mathscr{D}$ with L_1 depending only on c and L. Thus, F is L_1-BLD and, by Lemma 2.2, F is L'-bi-Lipschitz with L' depending only on c and L.

10. The assumptions of Theorem 1.1

We discuss the assumptions of Theorem 1.1 and their necessity. In §10.1, applying a result of Trotsenko and Väisälä [TV99], we show that relative connectedness is necessary for the QSEP in all dimensions. In §10.2, we show that uniformity is somewhat necessary for the QSEP or the BLEP in the plane, as neither the John property nor quasiconvexity of U, alone, is sufficient in Theorem 1.1.

10.1. **Relative connectedness.** Let $E \subset \mathbb{R}^n$ be a closed set that is not relatively connected. In the proof of Theorem 6.6 in [TV99], a quasisymmetric map $f : E \to \mathbb{R}^n$ is constructed that is not power quasisymmetric. It follows then from Lemma 2.3 that f can not be extended quasisymmetrically to $\mathbb{R}^n$. We present the construction here again to illustrate why the map f can be extended homeomorphically to $\mathbb{R}^n$.

Since E is not relatively connected, for each $i \in \mathbb{N}$, there exists E_i containing at least two points so that $\operatorname{dist}^*(E_i, E \setminus E_i) \geq i$. We assume for the rest that $\operatorname{diam} E_i \leq \operatorname{diam} E \setminus E_i$. Set $d_i = \operatorname{dist}^*(E_i, E \setminus E_i)$. We may assume that $4 < d_1 < d_2 < \cdots$. The conditions on E_i imply one of the following three cases.

Case 1. There exists subsequence $i_1 < i_2 < \cdots$ with $E_{i_1} \supset E_{i_2} \supset \cdots$. For simplicity, write $E_{i_k} = E_k$ and $d_{i_k} = d_k$. Then $\{x_0\} = \bigcap_{k \in \mathbb{N}} E_k$ for some $x_0 \in E$. Set $E^0 = E \setminus E_1$ and, for each $k \in \mathbb{N}$, set $E^k = E_k \setminus E_{k+1}$. Note that E is a disjoint union of the sets E^k. Define $f : E \to \mathbb{R}^n$ by $f(x_0) = x_0$, $f|_{E^0}(x) = x$ and, for each $k \in \mathbb{N}$, $f|_{E^k}(x) = s_k x$ with

$$s_k = \frac{e^{d_1 + \cdots + d_k}}{(1 + d_1) \cdots (1 + d_k)}.$$

Case 2. There exists subsequence $i_1 < i_2 < \cdots$ with $E_{i_1} \subset E_{i_2} \subset \cdots$. For simplicity, write $E_{i_k} = E_k$ and $d_{i_k} = d_k$. Set $E^0 = E_1$ and, for each $i \in \mathbb{N}$, set $E^k = E_{k+1} \setminus E_k$. Note that E is a disjoint union of the sets E^k. Define $f : E \to \mathbb{R}^n$ by $f|_{E^0}(x) = x$ and for, each $k \in \mathbb{N}$, $f|_{E^k}(x) = s_k x$ with

$$s_k = \frac{e^{e^{d_1} + \cdots + e^{d_k}}}{e^{k + d_1 + \cdots + d_k}}.$$

Case 3. There exists subsequence $i_1 < i_2 < \cdots$ with $E_{i_1}, E_{i_2}, \dots$ being mutually disjoint. For simplicity, write $E_{i_k} = E_k$, $d_{i_k} = d_k$ and $x_{i_k} = x_k$. Set $E^0 = E \setminus \bigcup_{k\in\mathbb{N}} E_k$ and, for each $k \in \mathbb{N}$, set $E^k = E_k$. Note that E is a disjoint union of the sets E^k. Define $f : E \to \mathbb{R}^n$ by $f|_{E^0}(x) = x$ and, for each $k \in \mathbb{N}$, $f|_{E^k}(x) = x_k + s_k(x - x_k)$ with

$$s_k = \frac{e^{d_k}}{(1 + d_k)}.$$

Following the proof of [TV99, Theorem 6.6], in each case f is quasisymmetric but not power quasisymmetric. Thus, it only remains to show that, in each case, f extends to a self homeomorphism of $\mathbb{R}^n$. Assume the first case. For each $i \in \mathbb{N}$, set $B_i = B^n(x_i, 2\operatorname{diam} E_i)$, set $B_i' = B^n(x_i, \frac{1}{2}d_i \operatorname{diam} E_i)$ and set $A_i = B_i \setminus B_{i+1}'$. On $\mathbb{R}^n \setminus B_1$, set $F(x) = x$ and, for each $i \in \mathbb{N}$, set $F|_{\overline{A_i}}(x) = s_i x$. By [TV99, (6.8)], for each $i \in \mathbb{N}$, $F(\partial B_{i+1}')$ is contained in a ball with boundary $F(\partial B_i)$, $F(B_{i+1})$ is contained in a ball with boundary $F(\partial B_{i+1}')$ and both $\operatorname{dist}(F(\partial B_i), F(\partial B_{i+1}'))$ and $\operatorname{dist}(F(\partial B_{i+1}), F(\partial B_{i+1}'))$ are nonzero. Therefore, F can be extended homeomorphically to each $B_i' \setminus B_i$.

The other two cases are similar; see [TV99, (6.14)] and [TV99, (6.18)].

10.2. **Uniformity.** We construct a compact, countable, relatively connected set $E \subset \mathbb{R}^2$ whose complement is both John and quasiconvex but E fails to have either the QSEP or the BLEP. For the rest of §10.2 we use complex coordinates.

For each $n \in \mathbb{N}$ divide the interval $[0, 2^{-n(n-1)/2}]$ into 2^n intervals of equal length and let A be the set of endpoints of all the intervals produced. Set

$$E = A \cup (iA) \cup (-A) \cup (-iA).$$

Note that E is a relatively connected, compact and countable. Moreover, since the coordinate projections of E both have measure zero, E is quasiconvex; see [GM85, Lemma 2.5, Lemma 2.7] or [HH08, Theorem A].

Define now $f : E \to E$ with

$$f|_{A\cup(iA)}(z) = -i\overline{z} \quad \text{and} \quad f|_{(-A)\cup(-iA)}(z) = z.$$

Clearly f is 2-bi-Lipschitz and, as E is totally disconnected, f extends to a homeomorphism of $\mathbb{C}$. We show that f can not be extended quasisymmetrically to $\mathbb{C}$.

Suppose that there exists an η-quasisymmetric extension $F : \mathbb{C} \to \mathbb{C}$. For each point $a \in A$, let $x_a = (-a, 0)$, $y_a = (0, a)$ and $\gamma_a = \{(-at, a(1-t)) : t \in [0,1]\}$. For all $a \in A$, γ_a is 2-bounded turning and 2-cigar in $\mathbb{R}^2 \setminus E$; that is, $\operatorname{diam} \gamma_a \leq 2|x_r - y_r|$ and $\operatorname{dist}(z, E) \geq \frac{1}{2}\min\{|x_r - z|, |y_r - z|\}$ for all $z \in \gamma_a$. Since F is η-quasisymmetric, there exists $C > 1$ depending only on η such that $F(\gamma_a)$ is C-bounded turning and C-cigar in $\mathbb{R}^2 \setminus E$ for all $a \in A$.

Let $n \in \mathbb{N}$ such that $2^{-n} \leq (3C)^{-2}$. Let $a' = 2^{-n(n+1)/2} \in A$ and $a = m2^{-n}a' \in A$ with m being the smallest integer bigger than $2C$. The C-bounded turning condition of $F(\gamma_a)$ implies that $F(\gamma_a) \cap (\{0\} \times \mathbb{R}) = F(\gamma_a) \cap (\{0\} \times [-a', a'])$. However, for any point $F(\gamma_a) \cap (\{0\} \times \mathbb{R})$ we have $\operatorname{dist}(z, E) \leq 2^{-n}a' \leq (3C)^{-2}a' \leq (2C)^{-1}a \leq (2C)^{-1}\min\{|z - F(x_a)|, |z - F(y_a)|\}$ and the John property of $F(\gamma_a)$ is violated.

Remark 10.1. *It turns out that the John property and the quasiconvexity of $\mathbb{R}^2 \setminus E$ are not necessary for bi-Lipschitz or quasisymmetric extensions to $\mathbb{R}^2 \setminus E$. For example, let $E = (0, +\infty) \times (-1, 1)$ and note that E is not John while $\mathbb{R}^2 \setminus \overline{E}$ is not quasiconvex. Nevertheless, any η-quasisymmetric (resp. L-bi-Lipschitz) embedding* *of E or $\mathbb{R}^2 \setminus E$ into $\mathbb{R}^2$ extends to an η'-quasisymmetric (resp. L'-bi-Lipschitz) homeomorphism of $\mathbb{R}^2$. The proofs are left to the reader.*

11. Uniformization of Cantor sets with bounded geometry

In [DS97], David and Semmes characterized the metric spaces that are quasisymmetrically homeomorphic to the standard middle-third Cantor set $\mathcal{C} \subset \mathbb{R}$: a metric space X is quasisymmetric homeomorphic to $\mathcal{C}$ if and only if it is compact, doubling, uniformly disconnected and uniformly perfect.

For planar sets, MacManus [Mac99] proved a slightly stronger statement: *for a compact set $E \subset \mathbb{R}^2$ there exists a quasisymmetric mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ with $F(\mathcal{C}) = E$ if and only if E is uniformly perfect and uniformly disconnected.* However, the same is not true in dimensions $n \geq 3$ due to the existence of wild Cantor sets in $\mathbb{R}^n$ that are uniformly perfect and uniformly disconnected [Dav07, pp. 70–75]. By increasing the dimension by 1, MacManus' result generalizes to dimensions $n \geq 3$.

Theorem 11.1. *For a compact set $E \subset \mathbb{R}^n$ there exists a quasisymmetric map $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $F(\mathcal{C}) = E$ if and only if E is uniformly perfect and uniformly disconnected.*

One direction of Theorem 11.1 is clear. Namely, if there exists quasisymmetric homeomorphism $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ mapping $\mathcal{C}$ onto E, then E is c-uniformly perfect and c-uniformly disconnected with c depending only on η. For the converse, we use the fact that there exists a quasisymmetric homeomorphism $f : \mathcal{C} \to E$. Our goal is to extend this mapping quasisymmetrically to $\mathbb{R}^{n+1}$.

Consider the set of finite words $\mathcal{W}$ formed from the letters $\{1, 2\}$ and denote by $\emptyset$ the empty word. The length of a word $|w|$ is the number of letters that the word contains with $|\emptyset| = 0$. Define $\mathcal{W}^N$ to be the set of words in $\mathcal{W}$ whose length is exactly N. Let $I_\emptyset = [0, 1]$ and given $I_w = [a, b]$ let $I_{w1} = [a, a + \frac{1}{3}(b - a)]$, $I_{w2} = [b - \frac{1}{3}(b - a), b]$. For each $w \in \mathcal{W}$, let $\mathcal{C}_w = I_w \cap \mathcal{C}$.

Lemma 11.2. *Let X be a metric space and $f : \mathcal{C} \to X$ be an η-quasisymmetric homeomorphism. There exists $k \in \mathbb{N}$ depending only on η with the following property. For any $m \in \mathbb{N}$ there exist sets $\mathscr{E}_1, \dots, \mathscr{E}_k$ whose elements are sets $f(\mathcal{C}_w)$ with $w \in \mathcal{W}^m$ such that*

1. *$\mathscr{E}_i \cap \mathscr{E}_j = \emptyset$ when $i \neq j$ and $\bigcup_{i=1}^k \mathscr{E}_i = \{f(\mathcal{C}_w) \colon w \in \mathcal{W}^m\}$;*
2. *for any $i \in \{1, \dots, k\}$ and any $f(\mathcal{C}_w), f(\mathcal{C}_{w'}) \in \mathscr{E}_i$ with $w \neq w'$ we have*

$$\operatorname{dist}(f(\mathcal{C}_w), f(\mathcal{C}_{w'})) \geq 5 \max\{\operatorname{diam} f(\mathcal{C}_w), \operatorname{diam} f(\mathcal{C}_{w'})\}.$$

Proof. Set $d = (\eta^{-1}(1/10))^{-1}$. We prove the lemma for k being the integer part of$d^{\log 2/\log 3} + 1$.

By quasisymmetry of f and (2.1), property (2) of the lemma is satisfied if $\operatorname{dist}(\mathcal{C}_w, \mathcal{C}_{w'}) \geq d3^{-m}$. Note that, for each $w \in \mathcal{W}^m$, there exist at most k words $w_1, \dots, w_l \in \mathcal{W}^m$ such that $\operatorname{dist}(\mathcal{C}_w, \mathcal{C}_{w_i}) < d3^{-m}$. Let now $\mathcal{C}'_1, \dots, \mathcal{C}'_{2^m}$ be an enumeration of $\{\mathcal{C}_w \colon w \in \mathcal{W}^m\}$ such that for all $1 \leq i < j \leq 2^m$, $\mathcal{C}'_i$ lies to the left of $\mathcal{C}'_j$. For each $i = 1, \dots, k$ define A_i to be the integers in $\{1, \dots, 2^m\}$ that are of the form $i + rk$ with $r \in \mathbb{N} \cup \{0\}$ and set $\mathscr{E}_i = \{f(C'_j) \colon j \in A_i\}$. It is now straightforward to verify that the sets $\mathscr{E}_j$ satisfy the properties (1) and (2) of the lemma. ☐

We are now ready to establish Theorem 11.1.

Proof of Theorem 11.1. Let E be a compact, c-uniformly perfect and c-uniformly disconnected subset of $\mathbb{R}^n$. By Theorem 2.5, there exists an η-quasisymmetric homeomorphism $f : \mathcal{C} \to E$ with η depending only on n and c. The first step of the proof is the construction of a bi-Lipschitz mapping $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that unlinks E. The second step is the construction of a quasiconformal mapping $G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that maps the unlinked image $\Phi(E)$ onto $\mathcal{C}$. The composition $G \circ \Phi$ is the desired map F of Theorem 11.1.

Without loss of generality, we assume that $\operatorname{diam} E = 1$. For the rest of the proof we write $E_w = f(\mathcal{C}_w)$.

Let k be the number obtained by Lemma 11.2 for the set E. Let also N be the smallest positive integer such that $3^{-N} \leq \min\{\eta^{-1}(1/16), \eta^{-1}((5k)^{-1})\}$. Then, for any $w, w' \in \mathcal{W}$ with $E_w \cap E_{w'} = \emptyset$ and any $u \in \mathcal{W}^N$, we have

$$\delta' \operatorname{diam} E_w \leq \operatorname{diam} E_{wu} \leq \delta \operatorname{diam} E_w, \tag{11.1}$$

$$\operatorname{dist}(E_w, E_w) \geq (\eta(1))^{-1} \max\{\operatorname{diam} E_w, \operatorname{diam} E_{w'}\} \tag{11.2}$$

with $\delta = \min\{\frac{1}{16}, (5k)^{-1}\}$ and $\delta' = (2\eta(3^N))^{-1}$.

Let $\mathscr{E}_1^0, \dots, \mathscr{E}_k^0$ be the sets of Lemma 11.2 corresponding to $m = N$. Define $\phi_1 : E \to \mathbb{R}$ such that

$$\phi_1|_{E_w}(x) = 5(i-1)\delta \quad \text{for all } x \in E_w$$

where $w \in \mathcal{W}^N$, $E_w \in \mathscr{E}_i^0$ and $i = 1, \dots, k$. Inductively, suppose that we have defined $\phi_j : E \to \mathbb{R}$ such that $\phi_j|_{E_w}$ is constant whenever $w \in \mathcal{W}^{jN}$. For each $w \in \mathcal{W}^{jN}$ let $\mathscr{E}_1^w, \dots, \mathscr{E}_k^w$ be the sets of 11.2 corresponding to $E = E_w$ and $m = N$. Define $\phi_{j+1} : E \to \mathbb{R}$ such that

$$\phi_{j+1}|_{E_{wu}}(x) = \phi_j|_{E_w}(x) + 5(i-1)\delta \operatorname{diam} E_w \quad \text{for all } x \in E_{wu}$$

where $w \in \mathcal{W}^{jN}$, $u \in \mathcal{W}^N$, $E_{wu} \in \mathscr{E}_i^w$ and $i = 1, \dots, k$. Then, for all $x \in E$, $|\phi_i(x) - \phi_j(x)| \leq \delta^{\max\{i,j\}}$. Therefore, the mappings ϕ_j converge uniformly to a mapping $\phi : E \to \mathbb{R}$.

We claim that ϕ is Lipschitz. Indeed, let $x, y \in E$ and let $m_0 \in \mathbb{N}$ be the greatest integer m such that $x, y \in E_w$ with $w \in \mathcal{W}^{mN}$. In particular, suppose that $x, y \in E_{w_0}$ with $w_0 \in \mathcal{W}^{m_0 N}$. By (11.1), (11.2) and the maximality of m_0,

$$|\phi(x) - \phi(y)| \leq \operatorname{diam} E_{w_0} \leq \eta(1)(\delta')^{-1}|x-y|.$$

and the claim follows.

Fix $x_0 \in E$, $B_0 = B^n(x_0, 5 \operatorname{diam} E)$ and set $\phi|_{\mathbb{R}^n \setminus B_0} \equiv 0$. Then, the map

$$\phi : (\mathbb{R}^n \setminus B_0) \cup E \to \mathbb{R}$$

is L-Lipschitz for some L depending only on η and, by Kirszbraun Theorem, there exists an L-Lipschitz extension of ϕ to $\mathbb{R}^n$ which we also denote by ϕ. Then, the mapping $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by $\Phi(x, z) = (x, \phi(x) + z)$ is L'-bi-Lipschitz with L' depending only on L.

For each $m = 0, 1, \dots$ and each $w \in \mathcal{W}^{mN}$ fix $x_w \in E_w$ and set

$$\mathsf{K}_w = x_w + [-2 \operatorname{diam} E_w, 2 \operatorname{diam} E_w]^n.$$

Note that if $w \in \mathcal{W}^{mN}$ and $u \in \mathcal{W}^N$, then $\mathsf{K}_{wu} \subset \mathsf{K}_w$ and $\operatorname{dist}(\mathsf{K}_{wu}, \partial \mathsf{K}_w) \geq \frac{1}{2} \operatorname{diam} E_w$. However, if $w, w' \in \mathcal{W}^{mN}$ are distinct, then K_w may intersect $\mathsf{K}_{w'}$

which is why we lift different sets to different heights. For each $m = 0, 1, \dots$ and each $w \in \mathcal{W}^{mN}$ define

$$\mathcal{K}_w = \mathsf{K}_w \times [\phi_m(x_w) - 2\operatorname{diam} E_w, \phi_m(x_w) + 2\operatorname{diam} E_w].$$

From the definition of the functions ϕ_w, it follows that for all $m \in \mathbb{N}$, for all distinct $w, w' \in \mathcal{W}^{mN}$ and for all $u \in \mathcal{W}^N$,

$$\operatorname{dist}(\mathcal{K}_w, \mathcal{K}_{w'}) \geq \max\{\operatorname{diam} E_w, \operatorname{diam} E_{w'}\}; \tag{11.3}$$

$$\mathcal{K}_{wu} \subset \mathcal{K}_w \text{ and } \operatorname{dist}(\mathcal{K}_{wu}, \partial\mathcal{K}_w) \geq \frac{1}{2}\operatorname{diam} E_w; \tag{11.4}$$

$$\mathcal{K}_w \cap \Phi(E) = \Phi(E_w) \text{ and } \operatorname{dist}(\Phi(E_w), \partial\mathcal{K}_w) \geq \frac{1}{2}\operatorname{diam} E_w. \tag{11.5}$$

For each $m = 0, 1, \dots$ and $w \in \mathcal{W}^{mN}$, let z_w be the centre of I_w and define

$$\mathcal{Q}_w = [z_w - \frac{5}{6}3^{-mN}, z_w + \frac{5}{6}3^{-mN}] \times [-\frac{5}{6}3^{-mN}, \frac{5}{6}3^{-mN}]^n.$$

For each $w \in \mathcal{W}^{mN}$, let $g_w : \partial\mathcal{K}_w \to \partial\mathcal{Q}_w$ be a sense-preserving similarity map. By Proposition 4.9, there exists $\Lambda > 1$ depending only on η, and there exists $G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that

(1) G is the identity outside of B_0 and
(2) for all $w \in \mathcal{W}^{mN}$, the restriction of G on $\mathcal{K}_w \setminus (\bigcup_{u\in\mathcal{W}^N} \mathcal{K}_{wu})$ extends g_w and is a $(\frac{\operatorname{diam}\mathcal{Q}_w}{\operatorname{diam}\mathcal{K}_w}, \Lambda)$-quasisimilarity that maps $\mathcal{K}_w \setminus (\bigcup_{u\in\mathcal{W}^N} \mathcal{K}_{wu})$ onto $\mathcal{Q}_w \setminus (\bigcup_{u\in\mathcal{W}^N} \mathcal{Q}_{wu})$.

Therefore, by a theorem of Väisälä on removability of singularities [Väi71, Theorem 35.1], G is K-quasiconformal with K depending only on η and n. Set $F = G \circ \Phi$ and note that F extends f and that $F(E_w) = \mathcal{C}_w$. □

11.1. **Proof of Theorem 1.3.** Let $E \subset \mathbb{R}^n$ be closed and c-uniformly disconnected. By §5 and §7, the proof is reduced to the case that E is compact and perfect. Hence, by Brouwer's topological characterization of Cantor sets [Kec95, Theorem 7.4] we may assume that E is a Cantor set.

Suppose first that E is c-uniformly perfect and that $f : E \to \mathbb{R}^n$ is η-quasisymmetric. By Theorem 11.1, we may assume that $f : \mathcal{C} \to \mathcal{C}$. By Theorem 1.1 and the tameness of planar totally disconnected sets [Moi77, §10], f extends to an η_1-quasisymmetric homeomorphism $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ with η_1 depending only on η. By the Tukia-Väisälä extension theorem [TV82], F_1 extends to an η'-quasisymmetric $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with η' depending only on η and n.

Suppose now that $f : E \to \mathbb{R}^n$ is L-bi-Lipschitz. Denote $E' = f(E)$ and $E'_w = f(E_w)$. By choosing N sufficiently large in the proof of Theorem 11.1, we may assume that the right inequality of (11.1) and the inequality (11.2) are satisfied for both E and E'. Following the construction of Φ, we can construct cubes $\mathcal{K}'_w$, corresponding to the sets E'_w with $w \in \mathcal{W}^{mN}$, and an L_2-bi-Lipschitz map $\Phi' : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that satisfy properties (11.3), (11.4) and (11.5).

For each $w \in \mathcal{W}^{mN}$, let $g_w : \partial\mathcal{K}_w \to \partial\mathcal{K}'_w$ be a similarity mapping. By Proposition 4.9, there exist $\lambda > 0$ and $\Lambda > 1$ depending only on c and L and there exists $G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that

(1) G is the identity outside of B_0 and

(2) for all $w \in \mathcal{W}^{mN}$, the restriction of G on $\mathcal{K}_w \setminus (\bigcup_{u\in\mathcal{W}^N} \mathcal{K}_{wu})$ extends g_w and is a (λ, Λ)-quasisimilarity that maps $\mathcal{K}_w \setminus (\bigcup_{u\in\mathcal{W}^N} \mathcal{K}_{wu})$ onto $\mathcal{Q}_w \setminus (\bigcup_{u\in\mathcal{W}^N} \mathcal{Q}_{wu})$.

Therefore, G is BLD and, by Lemma 2.2, G is L_3-bi-Lipschitz for some L_3 depending only on L. Thus, $F = (\Phi')^{-1} \circ G \circ \Phi$ is an L'-bi-Lipschitz extension of f for some L' depending only on L and c and extends f.

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Department of Mathematics, 341 Mansfield Road U1009, University of Connecticut, Storrs, CT 06269-1009

E-mail address: vyron.vellis@uconn.edu