

# Precise characterisation of the minimiser of interaction energies

M. Kimura, P. van Meurs

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## Abstract

We consider both the minimisation of a class of nonlocal interaction energies over non-negative measures with unit mass and a class of singular integral equations of the first kind of Fredholm type. Our setting covers applications to dislocation pile-ups, contact problems, fracture mechanics and 1D log gases. Our main result shows that both the minimisation problems and the related singular integral equations have the same unique solution, which provides new regularity results on the minimiser of the energy and new positivity results on the solutions to singular integral equations.

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# 1 Introduction

We consider the minimisation problem of the energy

$$E(\rho) := \frac{1}{2} \int_{[0,1]} \int_{[0,1]} V(t-s) d\rho(s) d\rho(t) + \int_{[0,1]} U(t) d\rho(t), \quad (1)$$

over the space  $\mathcal{P}([0,1])$  of probability measures on the interval  $[0,1]$ . The potential  $U \in C([0,1])$  describes an externally applied field, and the lower semi-continuous interaction potential  $V : [-1,1] \rightarrow [0,\infty]$  models repulsive, nonlocal interactions. The main assumptions on  $U$  and  $V$  are that  $U$  is convex and that  $V$  is as in Figure 1 (i.e.,  $V$  is even on  $[-1,1]$ , and non-increasing and convex on  $(0,1]$ ). The precise assumptions on  $U$  and  $V$  are given in §1.1.

In a larger context, the minimisation problem of  $E$  fits to a variety of interacting particle systems where  $E$  is the mean-field limit and  $\rho$  is the (non-negative) particle density. See, e.g., [SST15] and the references therein for applications in statistical mechanics, models of collective behaviour of many-agent systems, granular media, self-assembly of nanoparticles, crystallization, and molecular dynamics simulations of matter. While the one-dimensional scenario in (1) does not encompass the full complexity of many such particle system, it does capture, for instance, the log-gases studied in [SS15] and the pile-ups of dislocations studied in [GPPS13, HCO10]. The endpoints of the interval  $[0,1]$  model impenetrable barriers for the particle density  $\rho$ . Due to the convexity assumptions on  $V$  and  $U$ , uniqueness of the minimiser  $\bar{\rho}$  of  $E$  is well-known. However, the regularity properties of  $\bar{\rho}$  remains elusive, which limits:

- the applicability of the result in [SS15] on crystallisation phenomena in log-gases;
- the extension of the discrete-to-continuum convergence results in [GPPS13] to convergence *rates*;
- the computation of the asymptotic expansions in [HCO10] to characterise boundary layers.

One of our objectives is therefore to establish satisfactory regularity properties of  $\bar{\rho}$ .

Another objective of our study is to find properties of solutions to the singular integral equation of the first kind of Fredholm type given by

$$\text{p.v.} \int_0^1 V'(t-s)\rho(s) ds + U'(t) = 0 \quad \text{for all } 0 < t < 1, \quad (2)$$

where solutions  $\rho$  are required to satisfy a Hölder-continuity condition such that the principle value integral is well-defined. We refer to [CFP03, LPV03] and the references therein for applications of (2) to fracture mechanics, and to ([Mus53, §102]) for applications to contact problems between two elastic bodies.

The first of our two main results, Theorem 1.6, states that (1) and (2) (equipped with the unit mass condition) have a unique solution, and that both solutions are equal. In particular, this implies that the minimiser  $\bar{\rho}$  of  $E$  is Hölder continuous and that the solution to (2) is non-negative, which are novel results on themselves. Furthermore, we derive two other useful characterisations for  $\bar{\rho}$  and show how its regularity depends on the regularity of  $V$  and  $U$ .

The second of our two main results, Theorem 6.6, extends Theorem 1.6 to the setting on the unbounded domain  $\mathbb{R}$  at which no barriers are present. Instead, the external potential  $U$  is assumed to be confining. It turns out that the minimiser to the corresponding energy  $E$  has compact support, which turns the corresponding singular integral equation to (2) into a free boundary problem. To avoid confusion between these two scenarios, we treat the scenario on  $\mathbb{R}$  separately in §1.5 and §6.

### 1.1 Assumptions on the potentials $V$ and $U$

The precise assumptions on the potentials  $V$  and  $U$  are as follows. We assume that  $V = V_a + V_{\text{reg}}$  for some fixed  $a \in [0, 1)$ , where  $V_a$  is the  $-a$ -homogeneous potential given by

$$V_a(t) := \begin{cases} -\log t & \text{if } a = 0 \\ t^{-a} & \text{if } 0 < a < 1 \end{cases} \quad \text{for all } 0 < t \leq 1, \quad (3)$$

and  $V_{\text{reg}}$  is the regular part which satisfies

$$V_{\text{reg}} \in \begin{cases} W^{2,1}(0, 1) & \text{if } 0 < a < 1, \\ W^{2,p_0}(0, 1) \text{ for some } 1 < p_0 < 2 & \text{if } a = 0. \end{cases} \quad (4a)$$

We further assume that

$$V''(t) \geq 0 \text{ for a.e. } t \in (0, 1), \quad V(1) = -V'(1) \geq 0, \quad (4b)$$

$$\exists c, \varepsilon > 0 \quad \forall t \in (0, \varepsilon) : \operatorname{ess\,inf}_{0 < s < t} V''(s) \geq \frac{c}{|t|^{2+a}}. \quad (4c)$$

We extend  $V$  to  $[-1, 1]$  as an even function by  $V(-r) := V(r)$ . Figure 1 illustrates  $V$ . The condition  $V'(1) \leq 0$ , in addition to convexity, ensures that there are no attractive forces between particles. Since adding a constant to  $V$  is equivalent to adding a constant to the energy  $E$ , we choose this constant such that  $V(1) = -V'(1)$  for convenience later on.

The main requirement on  $V$  is the splitting  $V = V_a + V_{\text{reg}}$ . With the addition of the mild, technical property (4c), we will show that the interaction term of  $E$  defines a norm with an inner product structure (see (11)). This norm is equivalent to the norm on the fractional Sobolev space  $H^{-(1-a)/2}$  defined in (10). It turns out that this is a natural space to consider for solutions to the integrated version of the singular integral equation (2). In particular, these solutions need not be non-negative. Moreover, this integrated singular integral equation has an explicit solution for  $V = V_a$ , i.e., when  $V_{\text{reg}} = 0$  (see §4). In the general case  $V_{\text{reg}} \not\equiv 0$ , this solution is given by an implicit expression (see (17)), which we employ to prove regularity of the solution.

For the external potential  $U : [0, 1] \rightarrow \mathbb{R}$  we assume that

$$U''(t) \geq 0 \text{ for a.e. } t \in (0, 1) \quad \text{and} \quad \begin{cases} U \in W^{2,1}(0, 1) & \text{if } 0 < a < 1, \\ U \in W^{2,p_0}(0, 1) & \text{if } a = 0, \end{cases} \quad (5)$$

where  $p_0$  is the same as in (4a).

To justify our choice of the normalisation of constants in (1), we note that any energy of the form

$$\tilde{E}(\tilde{\rho}) = \frac{1}{2} \int_{[t_1, t_2]} \int_{[t_1, t_2]} (V_a + \tilde{V}_{\text{reg}}) \left( \frac{t-s}{\alpha} \right) d\rho(s) d\rho(t) + \int_{[t_1, t_2]} \tilde{U}(t) d\tilde{\rho}(t),$$

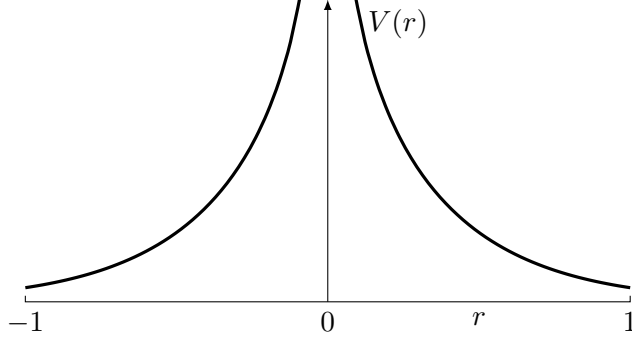


Figure 1: An example of  $V$  satisfying (4). It models dislocation walls (see (108)).

with  $-\infty < t_1 < t_2 < \infty$  and  $\alpha > 0$  can be cast, by an affine change of variables, into

$$\tilde{E}(\tilde{\rho}) = CE(\rho) + C',$$

for some constants  $C > 0$  and  $C' \in \mathbb{R}$ . Hence, the minimisation problem of  $\tilde{E}$  is equivalent to the minimisation problem of  $E$ .

Next we show that assumptions (4) and (5) give a proper meaning to (1). Since  $U \in C([0, 1])$ , the second term in (1) is finite for any  $\rho \in \mathcal{P}([0, 1])$ . Since  $V$  is lower semi-continuous and non-negative, the first term in (1) is well-defined with values in  $[0, \infty]$  for any  $\rho \in \mathcal{P}([0, 1])$ <sup>1</sup>. Likewise, the convolution

$$(V * \rho)(t) := \int_{[0,1]} V(t-s) d\rho(s) \quad \text{with } \rho \in \mathcal{P}([0, 1]) \quad (6)$$

is well-defined as a lower semi-continuous function on  $[0, 1]$ . In particular, we note that  $E(\rho) = \infty$  whenever  $\rho$  has an atom (i.e., a delta-peak). Therefore, we limit the minimisation of  $E$  over measures without atoms, which allows us to simplify notation by

$$E(\rho) = \frac{1}{2} \int_0^1 \int_0^1 V(t-s) d\rho(s) d\rho(t) + \int_0^1 U(t) d\rho(t) = \frac{1}{2} \int_0^1 (V * \rho) d\rho + \int_0^1 U d\rho.$$

We will prove that assumptions (4) and (5) are sufficient for the existence and uniqueness of the minimiser  $\bar{\rho}$  of  $E$ , and for characterising it by a variational inequality. However, if  $|U'|$  is large enough, then the support of  $\bar{\rho}$  may not reach both barriers at 0 and 1. This scenario requires a slightly different treatment, which we handle in detail in §6. Here, we impose the additional assumption

$$\sup_{(0,1)} |U'| \leq |V'(1)|, \quad (7)$$

which we show to be sufficient for the support of  $\bar{\rho}$  to reach both barriers at 0 and 1.

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<sup>1</sup>To prove this, take any approximation of  $V$  from below by continuous functions, and pass to the limit by using the Monotone Convergence Theorem.

## 1.2 Four characterisations of the minimiser $\bar{\rho}$ of $E$

While our objective is to show that the minimiser  $\bar{\rho}$  of  $E$  is the unique solution of (2) (for an appropriate solution concept), we present in this section four separate descriptions for  $\bar{\rho}$  (Problems 1.1–1.4) which are of interest on their own. In particular, Problem 1.3 is the integrated version of (2). In the next section we state that all four Problems 1.1–1.4 are equivalent, in the sense that they have the same unique solution  $\bar{\rho}$ .

Throughout this section,  $0 \leq a < 1$  is fixed, and  $E$  is defined as in (1) with the related potentials  $V$ ,  $V_a$ ,  $V_{\text{reg}}$  and  $U$  as in §1.1.

**Problem 1.1** (Minimisation). *Find the minimiser of  $E$  in  $\mathcal{P}([0, 1])$ .*

In view of (6), we define for any  $\rho \in \mathcal{P}([0, 1])$  the lower semi-continuous function

$$h_\rho := V * \rho + U : [0, 1] \rightarrow [0, \infty]. \quad (8)$$

**Problem 1.2** (Variational inequality). *Find  $\rho \in \mathcal{P}([0, 1])$  such that*

$$\int_{[0,1]} h_\rho d\mu \geq \int_{[0,1]} h_\rho d\rho \quad \text{for all } \mu \in \mathcal{P}([0, 1]) \text{ with } E(\mu) < \infty. \quad (9)$$

The following problem is an integrated version of the singular integral equation (2). As mentioned in §1.1, this problem is most naturally presented in a Hilbert space induced by the energy-norm related to  $E$ . With this aim, we define the fractional Sobolev space for  $s > 0$  by

$$H^{-s}(\mathbb{R}) := \left\{ \xi \in \mathcal{S}'(\mathbb{R}) : \int_{\mathbb{R}} (1 + \omega^2)^s |\widehat{\xi}(\omega)|^2 d\omega < \infty \right\}, \quad (10)$$

where  $\mathcal{S}'(\mathbb{R})$  is the space of tempered distributions, i.e., the dual of the Schwartz space  $\mathcal{S}(\mathbb{R})$ . With this interpretation, we define the subspace

$$H^{-s}(0, 1) := \left\{ \xi \in H^{-s}(\mathbb{R}) : \text{supp } \xi \subset [0, 1] \right\}.$$

In §3 we show that the energy norm  $\|\cdot\|_V$  induced by the inner product on  $L^2(0, 1)$  given by

$$(f, g)_V := \int_0^1 (V * f)g \quad (11)$$

for  $f, g \in L^2(0, 1)$ , is equivalent to the norm on  $H^{-(1-a)/2}(0, 1)$ . We write  $H_V(0, 1)$  for the Hilbert space  $H^{-(1-a)/2}(0, 1)$  equipped with  $(\cdot, \cdot)_V$ . In addition, we define the integral of  $\rho \in H_V(0, 1)$  (interpreted as a tempered distribution on  $\mathbb{R}$ ) as

$$\int_0^1 \rho := \langle \rho, \varphi \rangle,$$

where  $\varphi \in C_c^\infty(\Omega)$  is any test function which satisfies  $\varphi|_{(0,1)} = 1$ . We also define the convolution

$$V \tilde{*} \rho := \mathcal{F}^{-1}(\widehat{V} \widehat{\rho}), \quad (12)$$

where  $\mathcal{F}f = \widehat{f}$  is the Fourier transform (defined more precisely in §2). In §3 we show that  $V \tilde{*} \rho \in L^2(0, 1)$  and that 3.7 is consistent with (6). This motivates us to extend (8) by  $h_\rho := V \tilde{*} \rho + U \in L^2(0, 1)$  for  $\rho \in H_V(0, 1) \setminus \mathcal{P}([0, 1])$  without changing notation.

With these definitions, we are ready to state Problem 1.3. We emphasise that non-negativity of  $\rho$  is not required.

**Problem 1.3** (Weakly singular integral equation). *Find the solution  $(\rho, C)$  with  $C \in \mathbb{R}$  and  $\rho \in H_V(0, 1)$  with  $\int_0^1 \rho = 1$  to*

$$h_\rho = C \quad \text{a.e. on } (0, 1). \quad (13)$$

The final problem is motivated by the inversion formula for the weakly singular integral equation  $V_a * \rho = f$  on  $(0, 1)$ , which was first solved explicitly in [Car22]. To state it, we set

$$\phi(t) := [t(1-t)]^{\frac{1-a}{2}} \quad (14)$$

and recall that the usual  $\Gamma$ -function is defined as  $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t} dt$ . Furthermore, for any  $0 < b < 1$  and any  $f : [0, \infty) \rightarrow \mathbb{R}$  smooth enough, we recall that the Riemann-Liouville fractional integral  $I^b f$  and Riemann-Liouville fractional derivative  $D^b f$  are given by

$$(I^b f)(t) := \frac{1}{\Gamma(b)} \int_0^t \frac{f(s)}{(t-s)^{1-b}} ds \quad t > 0, \quad (15a)$$

$$(D^b f)(t) := \frac{1}{\Gamma(1-b)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^b} ds = \frac{d}{dt} (I^{1-b} f)(t) \quad t > 0. \quad (15b)$$

We further set  $D^1 f := f'$ . Furthermore, for Hölder-continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$ , we set

$$(Sf)(t) := \text{p.v.} \int_0^1 \frac{f(s)}{t-s} ds \quad \text{for all } 0 < t < 1. \quad (16)$$

The operator  $-\frac{1}{\pi}S$  is known as the *finite Hilbert transform*. In §2 we list further properties of  $I^b$ ,  $D^b$  and  $S$ , including the extension of their domains to Sobolev spaces. Lastly, we set  $\mathcal{M}([0, 1])$  as the space of finite, signed Borel measures on  $[0, 1]$ .

**Problem 1.4** (Implicit formula). *Find the solution  $(\rho, C)$  with  $C \in \mathbb{R}$  and  $\rho \in H_V(0, 1) \cap \mathcal{M}([0, 1])$  such that  $\rho([0, 1]) = 1$  to*

$$\rho = \begin{cases} \pi^{-2} \Gamma(a) \cos^2(\frac{a\pi}{2}) \left( \frac{1}{\phi} S(\phi g_\rho) + \pi \tan(\frac{a\pi}{2}) g_\rho \right) & \text{if } 0 < a < 1 \\ \pi^{-2} \frac{1}{\phi} \left[ S(\phi g_\rho) + \frac{1}{2 \log 2} \int_0^1 \frac{(C - V_{\text{reg}} * \rho - U)(s)}{\phi(s)} ds \right] & \text{if } a = 0 \end{cases} \quad \text{a.e. on } (0, 1), \quad (17)$$

where

$$g_\rho(t) := D^{1-a}(C - V_{\text{reg}} * \rho - U)(t), \quad \text{for a.e. } 0 < t < 1. \quad (18)$$

### 1.3 Equivalence of Problems 1.1–1.4 and properties of $\bar{\rho}$

Since Problems 1.1 and 1.2 do not rely on the bound on  $U'$  in (7), we state their equivalence separately in Theorem 1.5.

**Theorem 1.5** (Equivalence of Problems 1.1–1.2). *Let  $0 \leq a < 1$  and let the potential  $V$ ,  $V_{\text{reg}}$  and  $U$  satisfy (4) and (5). Then both Problem 1.1 and Problem 1.2 have a unique solution, and both solutions are equal.*

The first of the two main results in this paper is:

**Theorem 1.6** (Equivalence of Problems 1.1–1.4 and properties of the solution). *Let  $0 \leq a < 1$  and  $\phi$  be as in (14). If the potentials  $V$ ,  $V_{\text{reg}}$  and  $U$  satisfy (4), (5) and (7), then all four Problems 1.1–1.4 have a unique solution, and all these solutions are equal. The solution  $(\bar{\rho}, \bar{C})$  to all four problems satisfies*

$$\phi \bar{\rho} \in C^\beta([0, 1]), \quad \text{and} \quad (19a)$$

$$\bar{\rho} \in W_{\text{loc}}^{\ell, p_a}(0, 1), \quad (19b)$$

for any

$$0 < \beta < \begin{cases} \frac{1-a}{2} \wedge a & \text{if } 0 < a < 1 \\ 1 - p_0^{-1} & \text{if } a = 0 \end{cases} \quad \text{and any} \quad \begin{cases} 1 \leq p_a < \frac{1}{1-a} & \text{if } 0 < a < 1, \\ p_a = p_0 & \text{if } a = 0, \end{cases}$$

where  $1 < p_0 < 2$  is as in (5), and  $\ell \in \mathbb{N}_+$  is such that

$$V_{\text{reg}} \in W^{\ell+1,1}(0, 1) \quad \text{and} \quad U \in \begin{cases} W^{\ell+1,1}(0, 1) & \text{if } 0 < a < 1, \\ W^{\ell+1, p_0}(0, 1) & \text{if } a = 0. \end{cases} \quad (20)$$

Moreover,  $\text{supp } \bar{\rho} = [0, 1]$  and  $\bar{C} := \int_0^1 h_{\bar{\rho}}(t) \bar{\rho}(t) dt$ . Furthermore,  $(\bar{\rho}, \bar{C})$  satisfies (13) and (17) everywhere on  $[0, 1]$  and  $(0, 1)$  respectively. Finally, if  $\ell \geq 3$ , then  $\bar{\rho} > 0$  on  $(0, 1)$ .

We note that (20) is satisfied at least for  $\ell = 1$  because of the assumptions in (4) and (5). The role of (20) is solely to show that additional regularity on both potentials  $V$  and  $U$  results in additional regularity on  $\bar{\rho}$  (see (19b)).

## 1.4 Outline of the proof of Theorem 1.5 and Theorem 1.6

*Problem 1.1  $\longleftrightarrow$  Problem 1.2.* We show that the energy can be written as  $E(\rho) = \frac{1}{2}(\rho, \rho)_V - (\xi, \rho)_V$ , where the inner product is defined in (11). We rely on (5) to prove that  $\rho \mapsto \int_0^1 U d\rho$  is a bounded functional on  $H_V(0, 1)$ . With this quadratic form for the energy, it is easy to see (Theorem 2.1) that both Problems 1.1 and 1.2 attain a unique solution, and that both solutions are equal.

*Problem 1.2  $\longleftrightarrow$  Problem 1.3.* Given the solution  $\bar{\rho}$  to Problem 1.2, it is readily seen from (9) that  $h_{\bar{\rho}}$  is constant a.e. on  $\text{supp } \bar{\rho}$ . The novelty of our proof is the observation that  $V * \bar{\rho}$  is as regular as  $V_{\text{reg}}$  on  $(\text{supp } \bar{\rho})^c$ . In particular,  $(V * \bar{\rho})'' = V'' * \bar{\rho} \geq 0$  on  $(\text{supp } \bar{\rho})^c$ . Moreover, by (4c) we obtain in addition that  $(V * \bar{\rho})'' > 0$  close to the boundary of  $\text{supp } \bar{\rho}$ . We use this property to derive by contradiction with (9) that  $\text{supp } \bar{\rho} = [0, 1]$ . This argument yields that  $\bar{\rho}$  satisfies (13). We obtain the uniqueness of the solution to Problem 1.3 from  $V * \rho = 0 \implies \rho = 0$  (see Lemma 3.10).

*Problem 1.3  $\longleftrightarrow$  Problem 1.4.* With the explicit, classical solution in [Car22] to the weakly singular integral equation  $V_a * \rho = f$  on  $(0, 1)$ , we obtain that any solution to Problem 1.3 is also a solution to Problem 1.4 and vice versa. Since the right-hand side  $f$  contains  $V_{\text{reg}} * \bar{\rho}$ , we require that the solution by [Car22] is valid for a large enough class of functions  $f$ . Since such a statement appears to be missing in the literature, we establish it in Theorem 4.1 and Theorem 4.2.

*Regularity and positivity of  $\bar{\rho}$ .* From (17) in Problem 1.4 we obtain the regularity of  $\bar{\rho}$  as stated in (19). Then, it easily follows that  $\bar{\rho}$  is a classical solution to (13) and (17). Positivity is proven similarly to the argument which connects Problem 1.2 to Problem 1.3; if  $\bar{\rho}(t_0) = 0$  from some  $0 < t_0 < 1$ , then we can compute  $h_{\bar{\rho}}''(t_0) > 0$ , which contradicts (13). This computation requires  $\bar{\rho} \in C^2((0, 1))$ , which is guaranteed by (19b) when  $\ell \geq 3$ .

## 1.5 Extension to confining potentials $U$ on infinite domains

While the assumption (7) is sufficient for deriving that  $\bar{\rho}$  has full support, it has no physical motivation. In particular, for the three applications mentioned before §1.1, (7) is not satisfied. In those applications, the potential  $V$  and  $U$  are extended to  $\mathbb{R}$  such that the extended potential  $\tilde{U}$  has linear growth. This growth of  $\tilde{U}$  is sufficient for the support of the related minimiser  $\tilde{\rho}$  to be bounded, and thus no barriers are needed to confine  $\text{supp } \tilde{\rho}$ . This brings us to the second of our two main results as announced before §1.1.

In §6 we start from a similar energy  $\tilde{E}$  as in (1) where the domain is extended to  $\mathbb{R}$  (i.e., no barriers are considered) and  $\tilde{U}$  has linear growth. Then, we state and prove our second main result, Theorem 6.6, which is the analogue of Theorem 1.6 for unbounded domains.

The proof of Theorem 6.6 starts by showing that  $\tilde{E}$  (the energy on infinite domains) has a unique minimiser  $\tilde{\rho}$ , which by the linear growth of  $\tilde{U}$  must have bounded support. On this bounded support, we show that Theorem 1.6 applies to obtain regularity and positivity of  $\tilde{\rho}$  on its support. However, we need additional arguments to characterise the two endpoints of  $\text{supp } \tilde{\rho}$ , which turns the equivalent of Problem 1.3 into a free boundary problem (see Problem 6.3). For this characterisation we rely on the regularity of  $\tilde{V} * \tilde{\rho}$  on  $(\text{supp } \tilde{\rho})^c$ . Moreover, this regularity allows us to prove that  $\tilde{\rho}$  is Hölder continuous on  $\mathbb{R}$ , which implies in particular that  $\tilde{\rho}$  equals 0 at the two endpoints of  $\text{supp } \tilde{\rho}$ . Such regularity allows for a classical version of Problem 6.3, given by Problem 6.5, in which the boundary conditions simplify to homogeneous Dirichlet boundary conditions.

## 1.6 Discussion

*Positivity of  $\bar{\rho}$ .* While Theorem 1.6 states that  $\ell \geq 3$  is sufficient for  $\bar{\rho} > 0$  on  $(0, 1)$ , we expect that  $\ell \geq 1$  is also sufficient. In the special case  $V_{\text{reg}} = 0$ , we can prove this. Indeed, by the linearity of (13), we can decompose its solution as  $\bar{\rho} = \varepsilon \rho_{\text{hg}} + (1 - \varepsilon) \rho_\varepsilon$ , where  $0 < \varepsilon < 1$  is arbitrary, and  $\rho_{\text{hg}}$  and  $\rho_\varepsilon$  are the unique solutions to  $V_a * \rho_{\text{hg}} = C_{\text{hg}}$  and  $V_a * \rho_\varepsilon + \frac{1}{1-\varepsilon} U = C_\varepsilon$  a.e. on  $(0, 1)$  respectively (guaranteed by Theorem 1.6 for the related choices of  $V_{\text{reg}}$  and  $U$ ). Here, a little care is needed because  $\frac{1}{1-\varepsilon} U$  need not satisfy (7); this can be fixed by taking  $\varepsilon > 0$  small enough. In Section 7 we show that  $\inf_{(0,1)} \rho_{\text{hg}} > 0$ . Since  $\rho_\varepsilon \geq 0$ , it follows that  $\inf_{(0,1)} \bar{\rho} > 0$ .

To extend this proof to the case  $V_{\text{reg}} \neq 0$ , one needs to prove that the solution to  $V * \rho = C$  as in Problem 1.3 is positive. We conjecture that this solution  $\rho$  is indeed positive, at least for a large subclass of potentials  $V$  which satisfy (4).

Finally, we note that this alternative proof method based on the linearity of (13) is not suited for the setting on unbounded domains (see §6 and Theorem 6.6). The reason for this is that the analogue of (13) on unbounded domains (see (89)) contains free boundaries.

*Weaker solution concept to Problem 1.4.* It may be possible to remove  $\rho \in \mathcal{M}([0, 1])$  from the solution concept in Problem 1.4 without losing the uniqueness of the solution. Our proof of the uniqueness of the solution to Problem 1.4 (see Step 5 of the proof of Theorem 1.6 in §5) relies on  $\rho \in \mathcal{M}([0, 1])$  to obtain that  $V_{\text{reg}} * \rho$  (appearing in (18)) has enough regularity. If we extend the solution concept to  $\rho \notin \mathcal{M}([0, 1])$ , then we either need stronger regularity conditions on  $V_{\text{reg}}$  or stronger regularity results than Theorems 4.1 and 4.2. We do not treat this extended solution concept any further.

*Problem 1.3  $\longleftrightarrow$  (2).* The singular integral equation (2) is covered by Theorem 1.6 under the additional assumption that, for  $0 < a < 1$ , (20) holds for  $\ell = 2$ . Formally, (2) is found



from Problem 1.3 by differentiating  $h_\rho(t) = C$  and by exchanging the order of integration and differentiation, i.e.,  $(V_a * \rho)' = V_a' * \rho$ . However, to justify this exchange, we need sufficient regularity on  $\rho$ . In [Man51, §4.2] it is shown that it is sufficient to show that  $\rho$  is Hölder continuous on compact subsets of  $(0, 1)$  with exponent greater than  $a$ . For  $a = 0$ , such regularity is implied by (19). For  $0 < a < 1$ , we further require that (20) holds for  $\ell = 2$ .

*Examples and applications.* In general, the equivalence between Problems 1.1–1.4 (and between Problems 6.1–6.5) and the properties of their solution is valuable both for developing efficient and accurate numerical solution methods and for proving connections to underlying interacting particle systems. In §7 we demonstrate the applicability of Theorem 1.6 and Theorem 6.6 by giving (partial) solutions to the three problems introduced before §1.1.

*Extension to higher dimensional domains.* Since most applications of nonlocal energies as in (1) and singular integral equations as in (2) require replacement of the interval  $(0, 1)$  by  $\Omega \subset \mathbb{R}^d$  with  $d \geq 2$ , we discuss the possibility of extending Theorem 1.5 and Theorem 1.6 to higher dimensional domains. Regarding Theorem 1.5, our proof method easily extends to higher dimensional domains. A somewhat similar proof method has recently appeared in [MRS16, Thm. 3.1]. This method relies on  $\widehat{V} \geq 0$ , but does not consider the Hilbert space generated by (11).

Regarding Theorem 1.5, the proof which connects Problem 1.2 to Problem 1.3 exploits the fact that for any measure  $\rho$ , any component of  $(0, 1) \setminus \text{supp } \rho$  is an open interval, which makes it easy to derive properties of  $V * \rho$  and its derivatives close to the endpoints of this interval. In higher dimensions, it is still true that  $\Delta(V * \rho) = (\Delta V) * \rho \geq 0$  on any component of  $\Omega \setminus \text{supp } \rho$  (given that the convexity condition on  $V$  is replaced by  $\Delta V \geq 0$  outside of the origin). The challenge lies in gaining control on the boundary of such a component and in showing that  $\text{supp } \bar{\rho}$  has full dimension. Even the latter is nontrivial, as for Coulomb interactions in 2D it is well-known that  $\text{supp } \bar{\rho}$  concentrates on  $\partial\Omega$ . Moreover, [LBCR14, MRS16] provide examples where the dimension of  $\text{supp } \bar{\rho}$  is smaller than  $d$  while  $\text{supp } \bar{\rho} \subset \subset \Omega$ .

Given that  $\text{supp } \bar{\rho}$  satisfies the higher-dimensional equivalent of Problem 1.3, the next challenge would be to prove regularity properties of  $\bar{\rho}$ . While in one dimension we can rely on the explicit solution formula for Carleman’s equations (see §4), in higher dimensions such a formula seems not available. One exception is thanks to [Kah81], where the singular integral equation  $V_a * \rho = g$  is solved on the unit ball in  $\mathbb{R}^2$  for  $g$  twice continuously differentiable. However, it remains elusive how to extend this result to general domains  $\Omega \subset \mathbb{R}^2$  or to  $d \geq 3$ .

## 1.7 Conclusion

Our main results given by Theorem 1.6 and Theorem 6.6 provide an equivalence between energy minimisation problems and singular integral equations, which establishes new regularity results on the minimiser of the energy and new positivity results on the solutions to singular integral equations. In particular, our results provide (partial) solutions to the three problems mentioned before §1.1. Furthermore, we plan to use our results in a future publication to prove convergence rates of the optimal particle configuration of the underlying particle system of the interaction energy  $E$  to its minimiser  $\bar{\rho}$  as the number of particles tends to  $\infty$ , and to solve the open problem on the discrete part of the boundary layer result in [GvMPS16].

The remainder of the paper is organised as follows. In §2 we set the notation and recall several results from textbooks. In §3 we define the Hilbert space  $H_V(0, 1)$  equipped with  $(\cdot, \cdot)_V$  and characterise its properties. In §4 we derive a precise regularity result for the unique solution of  $V_a * \rho = f$  on  $(0, 1)$ , which was firstly found in [Car22]. In §5 we prove

Theorem 1.5 and Theorem 1.6. In §6 we extend our setting on the interval  $[0, 1]$  to unbounded domains. Theorem 6.6 states the main result of this section. In §7 we treat several examples in the literature (including all three mentioned in the introduction) to which Theorem 1.6 and Theorem 6.6 give (partial) solutions. Appendices A and B contain computationally heavy proofs.

## 2 Preliminaries

In this section we list the symbols used throughout the paper, and cite several textbook results on which we rely in the subsequent sections.

$b \wedge c, b \vee c$	$\min\{b, c\}, \max\{b, c\}$	
$\ f\ _p$	$L^p$ -norm of $f$ on the domain of $f$	
$\widehat{f}, \mathcal{F}(f)$	Fourier transform of $f$ ; $\mathcal{F}(f)(\omega) = \widehat{f}(\omega) := \int_{\mathbb{R}} f(t)e^{-2\pi i \omega t} dt$	
$\mathbb{1}_A$	$\mathbb{1}_A(x)$ equals 1 if $x \in A$ and 0 if $x \notin A$	
$B_r(x)$	ball of radius $r$ centred at $x$	
$\Gamma$	$\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t} dt$ for any $\alpha > 0$	
$C^\alpha([0, 1])$	Hölder space; $0 < \alpha < 1$	
$C^{k,\alpha}([0, 1])$	$\{f \in C^\alpha([0, 1]) : f^{(k)} \in C^\alpha([0, 1])\}; k \in \mathbb{N}$	
$C_0^\alpha([0, 1])$	$\{f \in C^\alpha([0, 1]) : f(0) = f(1) = 0\}$	
$C_0^\alpha(\phi)$	weighted Hölder space	(25)
$H^s(\mathbb{R})$	fractional Sobolev space; $s \in \mathbb{R}$	(10)
$H_V(t_1, t_2)$	Hilbert space of functions $f : (t_1, t_2) \rightarrow \mathbb{R}$	(32), (35)
$\mathcal{L}$	Lebesgue measure on $\mathbb{R}$	
$L^p(\mu)$	weighted $L^p$ space for a non-negative measure $\mu$ on $\mathbb{R}$	
$\mathcal{M}([0, 1])$	space of finite, signed Borel measures on $[0, 1]$	
$\mathbb{N}_+, \mathbb{N}$	$\mathbb{N}_+ := \{1, 2, 3, \dots\}; \mathbb{N} = \{0\} \cup \mathbb{N}_+$	
$\mathcal{P}([0, 1])$	space of probability measures; $\mathcal{P}([0, 1]) \subset \mathcal{M}([0, 1])$	
$p_0$	regularity constant of $V_{\text{reg}}$ and $U$ when $a = 0$ ; $1 < p_0 < 2$	(4a)
$S$	singular integral operator	(16)
$\mathcal{S}(\mathbb{R})$	space of Schwartz functions	
$T_\# \rho$	push-forward of $\rho \in \mathcal{P}(\mathbb{R})$ by $T : \mathbb{R} \rightarrow \mathbb{R}$ ; $T_\# \rho(A) := \rho(T^{-1}(A))$	
$wL^p(0, 1)$	weak- $L^p$ space	(48)

We reserve the symbols  $C, C'$ , etc. for generic positive constants which we leave unspecified. Until Definition 3.9 we do not unify the notation of the two definitions of the convolution given in (6) and (12). In Lemma 3.7 we show that both definitions are consistent, and in Definition 3.9 we unify the notation.

Next we list several well-known results. The first one is a basic theorem in the calculus of variations (see, e.g., [KS80, Thm. 2.1]). It serves as the backbone of the proof of Theorem 1.5:

**Theorem 2.1** (Characterisation of minimiser). *Let  $X$  be a Hilbert space,  $K \subset X$  closed and convex,  $f \in X$ , and  $\mathbf{E} : X \rightarrow (-\infty, \infty]$  be given by  $\mathbf{E}(u) = \frac{1}{2}\|u\|^2 - (f, u)$ . Then  $\min_K \mathbf{E}$  has a unique minimiser in  $K$ , which is characterised by the unique solution  $u \in K$  of the variational*

inequality:

$$0 \leq (u - f, v - u) \quad \text{for all } v \in K.$$

The singular integral operator  $S$ , or finite Hilbert Transform  $-\frac{1}{\pi}S$ , plays a crucial role in connecting Problem 1.4 to the other problems. Moreover, we rely on detailed regularity properties of  $S$  to prove the regularity property (19). For any  $a, t_0 \in (0, 1)$  and any  $\alpha, \beta > 0$ , we first recall the identities

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(a\pi)}, \quad (21)$$

$$\int_0^t s^{\alpha-1}(t-s)^{\beta-1} ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}, \quad t > 0, \quad (22)$$

$$S\left(\frac{\mathbb{1}_{(0,t_0)}(t)}{t^{1-a}|t_0-t|^a}\right) = \frac{1}{t^{1-a}|t_0-t|^a} \begin{cases} \frac{\pi}{\tan(a\pi)} & \text{if } t < t_0, \\ \frac{\pi}{\sin(a\pi)} & \text{if } t > t_0, \end{cases} \quad 0 < t < 1, \quad (23)$$

$$S\left(\left[\frac{1-t}{t}\right]^{\frac{1-a}{2}}\right) = \frac{\pi}{\cos(\frac{a\pi}{2})} \left(1 - \left[\frac{1-t}{t}\right]^{\frac{1-a}{2}} \sin(\frac{a\pi}{2})\right), \quad 0 < t < 1, \quad (24)$$

which can be found, e.g., in [GR07, §8.334.3 and §3.191.1], [EK00, (2.47)] and [Kin09b, (12A.19)] respectively. Here and in the following, we abuse notation by writing  $S(f(t))$  instead of  $(Sf)(t)$  whenever convenient.

**Proposition 2.2** (Finite Hilbert transform on  $L^p$ ). *Let  $f \in L^p(0, 1)$  and  $g \in L^q(0, 1)$  with  $\frac{1}{p} + \frac{1}{q} = 1$  and  $p > 1$ . Then*

(i) [Kin09a, §4.20]  $S$  is a bounded linear operator from  $L^p(0, 1)$  to itself;

(ii) [Kin09a, (4.182)]:  $(Sf)(t) = \frac{d}{dt} \int_0^1 \log|t-s|f(s) ds$ ;

(iii) [Kin09a, §11.10.8]:  $\int_0^1 fSg = -\int_0^1 gSf$ ;

(iv) [Kin09a, (11.215)]:  $S(tf(t)) = tS(f(t)) - \int_0^1 f$ ;

(v) [Kin09a, (11.223)]: If  $f \in W^{k,p}(0, 1)$ , then

$$(Sf)^{(k)}(t) = (Sf^{(k)})(t) + \sum_{\ell=0}^{k-1} (k-\ell-1)! \left( \frac{f^{(\ell)}(1)}{(1-t)^{k-\ell}} - \frac{f^{(\ell)}(0)}{(-t)^{k-\ell}} \right).$$

For the following proposition we make use of weighted Hölder spaces. First, for any  $0 < \alpha < 1$  we set

$$C_0^\alpha([0, 1]) := \{f \in C^\alpha([0, 1]) : f(0) = f(1) = 0\}.$$

Then, we take  $\beta > 0$ , set  $\varphi(t) := t(1-t)$ , and define the weighted Hölder space by

$$C_0^\alpha(\varphi^\beta) := \{f : (0, 1) \rightarrow \mathbb{R} \mid \varphi^\beta f \in C_0^\alpha([0, 1])\}, \quad (25)$$

which is a Banach space with norm  $\|f\|_{C_0^\alpha(\varphi^\beta)} := \|\varphi^\beta f\|_{C^\alpha([0, 1])}$ ; see [GK92, §1.6].

**Proposition 2.3** (Finite Hilbert transform on  $C^\alpha$ ). *Let  $0 < \alpha < 1$ . Then*

(i) [GK92, Chap. 1, Thm. 6.2] For any  $\alpha < \beta < 1 + \alpha$ , the operator  $S$  is bounded on  $C_0^\alpha(\varphi^\beta)$ ;

- (ii) [GK92, Chap. 1, Thm. 6.3] For any  $f \in C^\alpha([0, 1])$ , the operator  $g \mapsto S(fg) - fSg$  is compact on  $C^\alpha([0, 1])$ .
- (iii) [Mus53, §29, (29.8)] For any  $f \in C^\alpha([0, 1])$  and any  $0 < \beta < 1$  there exist  $0 < \gamma < 1$  and  $g \in C_0^\gamma([0, 1])$  such that

$$(\varphi^\beta S(f\varphi^{-\beta}))(t) = \frac{\pi}{\tan(\beta\pi)} (f(1)t^\beta - f(0)(1-t)^\beta) + g(t), \quad \text{for all } 0 \leq t \leq 1.$$

The following proposition is a combination of two theorems. The well-posedness statement is a particular case of [Wid60, Thm. III], while the explicit solutions are taken from [Kin09a, (12.150), (12.153), (11.63)].

**Proposition 2.4** (A Cauchy integral equation). *If  $0 < a < 1$ , then for any  $\frac{2}{1+a} < p < \frac{2}{1-a}$ , it holds for all  $f \in L^p(0, 1)$  that the integral equation*

$$\pi \tan\left(\frac{a\pi}{2}\right)u - Su = f \quad \text{a.e. on } (0, 1) \quad (26)$$

*has a unique solution  $u$  in  $L^p(0, 1)$ . The solution is given by*

$$u(t) = \frac{\sin(a\pi)}{2\pi} f(t) + \frac{\cos^2(\frac{a\pi}{2})}{\pi^2} \left(\frac{t}{1-t}\right)^{\frac{1-a}{2}} S\left(\left(\frac{1-t}{t}\right)^{\frac{1-a}{2}} f(t)\right) \quad \text{for a.e. } 0 < t < 1.$$

*If  $a = 0$ , then for any  $1 < p < 2$  and any  $f \in L^p(0, 1)$  all solutions in  $L^p(0, 1)$  to the equation  $-Su = f$  (i.e., (26) with  $a = 0$ ) are given by*

$$u = \frac{1}{\pi^2 \phi} (S(\phi f) + C) \quad \text{a.e. on } (0, 1),$$

*where  $C \in \mathbb{R}$  is a parameter and  $\phi(t) = \sqrt{t(1-t)}$ .*

Since we are using both Sobolev spaces and Hölder spaces, Morrey's inequality is convenient to deduce Hölder continuity from Sobolev regularity:

**Proposition 2.5** (Morrey's inequality). *Let  $k \in \mathbb{N}$  and  $1 < p < \infty$ . Then  $W^{k+1,p}(0, 1)$  is continuously embedded in  $C^{k, 1-1/p}([0, 1])$ .*

In addition to the properties of  $S$ , we also rely on the regularity properties of the fractional operators  $I^a$  and  $D^a$  defined in (15); see Propositions 2.6 and 2.7. For convenience, we extend the definition of  $D^a$  to any starting value  $t_1 \in \mathbb{R}$  by

$$(D_{t_1}^a f)(t) := \frac{1}{\Gamma(1-a)} \frac{d}{dt} \int_{t_1}^t \frac{f(s)}{(t-s)^a} ds, \quad t > t_1. \quad (27)$$

We also recall the following formula for the fractional derivative of polynomials, which is a direct consequence of (22):

$$D^a \left( \sum_{k=0}^{\ell} \frac{b_k}{k!} t^k \right) = \sum_{k=0}^{\ell} \frac{b_k}{\Gamma(k+1-a)} t^{k-a}. \quad (28)$$

**Proposition 2.6** ([SKM93]; §2). *Let  $0 < a < 1$  and  $\ell \in \mathbb{N}$ . If  $f \in W^{\ell+1,1}(0,1)$ , then  $I^a(f - q_{\ell-1}) \in W^{\ell+1,1}(0,1)$  and  $D^a(f - q_{\ell-1}) \in W^{\ell,p}(0,1)$  for all  $1 \leq p < \frac{1}{a}$ , where  $q_{-1} = 0$  and*

$$q_{\ell-1}(t) := \sum_{k=0}^{\ell-1} \frac{f^{(k)}(0)}{k!} t^k, \quad \ell \geq 1$$

*is the  $(\ell - 1)$ -th order Taylor polynomial of  $f$  at 0.*

**Proposition 2.7** ([SKM93]; Thm. 2.4 and Thm. 2.6). *Let  $0 < a < 1$  and  $1 \leq p < \infty$ . Then  $I^a$  is a bounded linear operator from  $L^p(0,1)$  to itself. Moreover, for any  $f \in W^{1,1}(0,1)$ ,*

$$I^a D^a f = f \quad \text{a.e. on } (0,1).$$

### 3 The Hilbert space $H_V(0,1)$ induced by $(\cdot, \cdot)_V$

In this section we assume that the potentials  $V$ ,  $V_a$  and  $V_{\text{reg}}$  satisfy (4). We prove that the bilinear form  $(\cdot, \cdot)_V$  defined in (11) defines an inner product, and characterise the Hilbert space  $H_V(0,1)$  which it generates as the closure of  $L^2(0,1)$  (the precise definition is given in Corollary 3.4). The space  $H_V(0,1)$  provides a convenient functional framework for Problem 1.3. Moreover, we show that the operator  $\rho \mapsto V \tilde{*} \rho$  on  $H_V(0,1)$  is injective, and that it has a regularising effect on  $\rho$ .

Since we will make use of the Fourier transform, we rely on (4) to extend  $V$  to  $\mathbb{R}$  in the following manner:

$$\begin{aligned} V_{\text{reg}} = V - V_a \in \begin{cases} W_{\text{loc}}^{2,1}(\mathbb{R}) & \text{if } 0 < a < 1, \\ W_{\text{loc}}^{2,p_0}(\mathbb{R}) & \text{if } a = 0, \end{cases} \\ \exists b > 1 : \text{supp } V \subset [-b, b], \quad V \text{ even}, \quad V''(t) \geq 0 \text{ for a.e. } t \in \mathbb{R}. \end{aligned} \quad (29)$$

This extension induces the following properties on  $V$ , which we prove in Appendix A:

**Lemma 3.1** (Properties of the extended  $V$ ). *For any  $0 \leq a < 1$ , let  $V$ ,  $V_a$  and  $V_{\text{reg}}$  satisfy (4) and (29). Then*

- (i)  $V_{\text{reg}} \in C^1(\mathbb{R})$  and  $\begin{cases} V'_{\text{reg}} \in W^{1,1}(\mathbb{R}) & \text{if } 0 < a < 1, \\ V'_{\text{reg}} \in W^{1,p_0}(\mathbb{R}), V''_{\text{reg}} \in L^1(\mathbb{R}) & \text{if } a = 0; \end{cases}$
- (ii) *There exist  $V_k \in C([0, \infty))$  with  $0 \leq V_k \leq V$  and  $0 \leq V''_k \leq V''$  such that  $V_k \uparrow V$  pointwise on  $(0, \infty)$ ;*
- (iii)  $\exists C \geq c > 0 \forall \omega \in \mathbb{R} : c(1 + \omega^2)^{-\frac{1-a}{2}} \leq \widehat{V}(\omega) \leq C(1 + \omega^2)^{-\frac{1-a}{2}};$

It is straight-forward to extend  $(\cdot, \cdot)_V$  to  $\mathbb{R}$  by

$$(f, g)_V = \int_{\mathbb{R}} (V \tilde{*} f)g \quad (30)$$

for  $f, g \in L^2(\mathbb{R})$ .

**Lemma 3.2.** *The bilinear form  $(\cdot, \cdot)_V$  in (30) is an inner product on  $L^2(\mathbb{R})$ .*

*Proof.* Since  $V \in L^1(\mathbb{R})$ , it follows from Young's inequality that  $f \mapsto V \tilde{*} f$  defines a bounded linear operator on  $L^2(\mathbb{R})$  to itself, and thus (30) is well-defined. Except for positivity, it is readily checked that (30) satisfies all other properties of an inner product. To show positivity, i.e.,  $\|f\|_V \geq 0$  and  $f = 0 \Leftrightarrow \|f\|_V = 0$ , we use  $\widehat{V} > 0$  (Lemma 3.1.(iii)) to estimate

$$\|f\|_V^2 = \int_{\mathbb{R}} (V \tilde{*} f) f = \int_{\mathbb{R}} \widehat{V} |\widehat{f}|^2 \geq 0 \quad (31)$$

and to deduce that  $\|f\|_V = 0 \Leftrightarrow \widehat{f} = 0 \Leftrightarrow f = 0$ .  $\square$

Lemma 3.2 shows that

$$H_V(\mathbb{R}) := \overline{L^2(\mathbb{R})}^{\|\cdot\|_V} \quad (32)$$

is a Hilbert space. Proposition 3.3 characterises  $H_V(\mathbb{R})$  as a fractional Sobolev space defined in (10).

**Proposition 3.3.** *Let  $0 \leq a < 1$ . Then  $H_V(\mathbb{R}) \cong H^{-(1-a)/2}(\mathbb{R})$ .*

*Proof.* It is enough to show that the norms on  $H_V(\mathbb{R})$  and  $H^{-(1-a)/2}(\mathbb{R})$  are equivalent for all  $f \in L^2(\mathbb{R})$ . With  $c, C > 0$  as in Lemma 3.1.(iii) and the characterisation (31), we show equivalence of the norms by

$$\begin{aligned} c\|f\|_{H^{-(1-a)/2}(\mathbb{R})}^2 &= \int c(1 + \omega^2)^{-\frac{1-a}{2}} |\widehat{f}(\omega)|^2 d\omega \leq \int \widehat{V}(\omega) |\widehat{f}(\omega)|^2 d\omega = \|f\|_V^2 \\ &\leq \int C(1 + \omega^2)^{-\frac{1-a}{2}} |\widehat{f}(\omega)|^2 d\omega = C\|f\|_{H^{-(1-a)/2}(\mathbb{R})}^2. \quad \square \end{aligned}$$

**Corollary 3.4** (Properties of  $H_V(\mathbb{R})$ ). *For  $\xi, \eta \in H_V(\mathbb{R})$ , we have*

$$(\xi, \eta)_V = \int_{\mathbb{R}} (T\xi)(T\eta), \quad (33)$$

where  $T$  is given by

$$T\xi = \mathcal{F}^{-1}(v\widehat{\xi}), \quad \text{where } v := \sqrt{\widehat{V}}. \quad (34)$$

The linear operator  $T$  is an isometry from  $H_V(\mathbb{R})$  to  $L^2(\mathbb{R})$ , which is self-adjoint in  $L^2(\mathbb{R})$ . Moreover, for any bounded interval  $\Omega = (t_1, t_2)$ , the following space is a Hilbert space

$$H_V(\Omega) := \{\xi \in H_V(\mathbb{R}) : \text{supp } \xi \subset \overline{\Omega}\}, \quad (35)$$

which is characterised by

$$H_V(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_V}. \quad (36)$$

**Remark 3.5.** From (12) and (34) we observe that  $T^2 f = V \tilde{*} f$ . The operator  $T$  itself is given formally by  $Tf = (\mathcal{F}^{-1}v) \tilde{*} f$ , where  $\mathcal{F}^{-1}v$  need not be a function. Indeed, by Lemma 3.1.(iii) we have that  $\widehat{V} \in L^\infty(\mathbb{R}) \setminus L^1(\mathbb{R})$ . Hence,  $v \notin L^1(\mathbb{R}) \cup L^2(\mathbb{R})$ , and thus  $\mathcal{F}^{-1}v$  may not be a function.

The operator  $T$  is also examined in [GvMPS16], but for different assumptions on  $V$ . We refer to [GvMPS16, Lem. A.2] in several steps of the proof of Corollary 3.4.

**Remark 3.6.** We note that  $H_V(0, 1)$  is independent of the extension of  $V$  from  $[-1, 1]$  to  $\mathbb{R}$ . Indeed, by using (36) to approximate  $\xi \in H_V(0, 1)$  by  $(\varphi_n) \subset C_c^\infty(0, 1)$ , we observe from (11) that  $\|\varphi_n\|_V$  is independent of the extension of  $V$  for all  $n$ , and hence  $\|\xi\|_V$  is also independent of the extension of  $V$ .

*Proof of Corollary 3.4.* We start by proving the asserted properties of  $T$ . By Proposition 3.3 the Fourier transform is well-defined on  $H_V(\mathbb{R})$ . We recall from [GvMPS16, Lem. A.2] that  $T\xi$  is real-valued for  $\xi \in H_V(\mathbb{R})$  (this follows from (34) by the Hermitian symmetry of the Fourier transform and  $v$  being real-valued and even). It is readily seen that  $T$  is self-adjoint in  $L^2(\mathbb{R})$  by computing it in Fourier space.

By (31) and Proposition 3.3 we characterise

$$H_V(\mathbb{R}) \cong \left\{ \xi \in \mathcal{S}'(\mathbb{R}) : \|\xi\|_V^2 = \int_{\mathbb{R}} \widehat{V} |\widehat{\xi}|^2 < \infty \right\}.$$

Hence, for any  $\xi \in H_V(\mathbb{R})$ ,  $\widehat{\xi} \in L^2(\widehat{V})$  can be treated as a complex-valued function. Moreover

$$\|\xi\|_V^2 = \int_{\mathbb{R}} \widehat{V} |\widehat{\xi}|^2 = \int_{\mathbb{R}} |v\widehat{\xi}|^2 = \int_{\mathbb{R}} |\widehat{T\xi}|^2 = \|T\xi\|_{L^2(\mathbb{R})}^2,$$

which shows that  $T$  is isometric from  $H_V(\mathbb{R})$  to  $L^2(\mathbb{R})$ . Thus, the right-hand side of (33) is well-defined, and from Lemma 3.1.(iii) we easily see that  $T^{-1}f = \mathcal{F}^{-1}(\widehat{f}/v)$ .

Next we show that  $H_V(\Omega)$  is a Hilbert space by showing that it is a closed subspace of  $H_V(\mathbb{R})$ . It is trivial that  $H_V(\Omega)$  is a subspace, and closedness follows from

$$\{\xi \in \mathcal{S}'(\mathbb{R}) : \text{supp } \xi \subset \overline{\Omega}\}$$

being closed in  $\mathcal{S}'(\mathbb{R})$ , and  $\mathcal{S}'(\mathbb{R}) \supset H^{-(1-a)/2}(\mathbb{R}) \supset H_V(\Omega)$ .

Next we prove (36) by using standard approximation arguments. Without loss of generality we set  $\Omega = (0, 1)$ . Let

$$H_{V,c}(0, 1) := \{\xi \in H_V(0, 1) : \text{supp } \xi \subset (0, 1)\}.$$

We show that

$$\forall \xi \in H_{V,c}(0, 1) \exists (f_n) \subset C_c^\infty(0, 1) : \|f_n - \xi\|_V \xrightarrow{n \rightarrow \infty} 0, \quad \text{and} \quad (37a)$$

$$\forall \xi \in H_V(0, 1) \exists (\xi_n) \subset H_{V,c}(0, 1) : \|\xi_n - \xi\|_V \xrightarrow{n \rightarrow \infty} 0, \quad (37b)$$

from which (36) follows by a diagonal argument.

To prove (37a), we set  $\eta_n$  as the usual mollifier, and take  $f_n := \eta_n * \xi$ . Since  $\xi$  is a distribution with compact support in  $(0, 1)$ , it follows from basic theory on distributions and the Fourier transform (see, e.g., [RY07, Thm. 2.7 and Lem. 2.10]) that  $f_n \in C_c^\infty(0, 1)$  for all  $n \in \mathbb{N}$  large enough. Moreover,  $\widehat{\eta_n} \rightarrow 1$  as  $n \rightarrow \infty$  uniformly on bounded sets, and  $\sup_{\omega \in \mathbb{R}} |\widehat{\eta_n}(\omega)| = \sup_{\omega \in \mathbb{R}} |\widehat{\eta_1}(\omega)| < \infty$ . Using these properties, we compute for any  $R > 0$

$$\begin{aligned} \|f_n - \xi\|_V^2 &= \int_{\mathbb{R}} \widehat{V} |\widehat{\eta_n * \xi} - \widehat{\xi}|^2 = \int_{\mathbb{R}} \widehat{V} |\widehat{\eta_n} - 1|^2 |\widehat{\xi}|^2 \\ &\leq \left( \max_{[-R, R]} |\widehat{\eta_n} - 1|^2 \right) \int_{-R}^R \widehat{V} |\widehat{\xi}|^2 + \sup_{\omega \in \mathbb{R}} (\widehat{\eta_n}(\omega) - 1)^2 \int_{[-R, R]^c} \widehat{V} |\widehat{\xi}|^2 \\ &\leq \left( \max_{[-R, R]} |\widehat{\eta_n} - 1|^2 \right) \|\xi\|_V^2 + C \int_{[-R, R]^c} \widehat{V} |\widehat{\xi}|^2. \end{aligned} \quad (38)$$

Since  $\xi \in H_{V,c}(0,1) \subset H_V(\mathbb{R})$ , it holds that  $\widehat{V}|\widehat{\xi}|^2 \in L^1(\mathbb{R})$ , and thus the second term in the right-hand side of (38) converges to 0 as  $R \rightarrow \infty$ . We conclude (37a) by first passing to the limit  $n \rightarrow \infty$  in (38), and then  $R \rightarrow \infty$ .

In preparation for proving (37b), we introduce the sequence of dilation operators  $\tau_\varepsilon : H_V(\mathbb{R}) \rightarrow H_V(\mathbb{R})$  parametrised by  $0 \leq \varepsilon < \frac{1}{2}$  and given by

$$\tau_\varepsilon \xi(\omega) = \mathcal{F}^{-1}((1-2\varepsilon)e^{-2\pi i \varepsilon \omega} \widehat{\xi}((1-2\varepsilon)\omega)),$$

where we have used the characterisation  $\widehat{\xi} \in L^2(\widehat{V})$ . By construction,

$$\text{supp } \xi \subset [0,1] \implies \text{supp}(\tau_\varepsilon \xi) \subset [\varepsilon, 1-\varepsilon], \quad (39)$$

and, for  $f \in C_c(\mathbb{R})$ , it is easy to see that

$$\|\tau_\varepsilon f - f\|_{L^2(\mathbb{R})}^2 \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (40)$$

We claim that  $\tau_\varepsilon$  is a bounded linear operators from  $H^s(\mathbb{R})$  to itself with operator norm bounded by 1 for all  $0 \leq s \leq \frac{1}{2}$ . Indeed

$$\begin{aligned} \|\tau_\varepsilon f\|_{H^s(\mathbb{R})}^2 &= \int_{\mathbb{R}} (1+\omega^2)^s |\widehat{\tau_\varepsilon f}|^2 d\omega = (1-2\varepsilon)^2 \int_{\mathbb{R}} (1+\omega^2)^s |\widehat{f}((1-2\varepsilon)\omega)|^2 d\omega \\ &= (1-2\varepsilon) \int_{\mathbb{R}} \left(1 + \frac{k^2}{(1-2\varepsilon)^2}\right)^s |\widehat{f}(k)|^2 dk \\ &\leq (1-2\varepsilon)^{1-2s} \int_{\mathbb{R}} (1+k^2)^s |\widehat{f}(k)|^2 dk \leq \|f\|_{H^s(\mathbb{R})}^2, \end{aligned}$$

which proves the claim. Then, by Proposition 3.3, we conclude that the norm of  $\tau_\varepsilon$  as an operators from  $H_V(\mathbb{R})$  to itself is bounded by some  $\varepsilon$ -independent constant.

Finally, we prove (37b). Let  $\xi \in H_V(0,1)$ , and set  $\xi_\varepsilon := \tau_\varepsilon \xi$  for  $0 < \varepsilon < \frac{1}{2}$ . By (39),  $\xi_\varepsilon \in H_{V,c}(0,1)$  for all  $\varepsilon > 0$ . Regarding the convergence, we take any  $\delta > 0$ , and observe from  $\widehat{\xi} \in L^2(\widehat{V})$  that there exists  $\varphi \in \mathcal{S}(\mathbb{R})$  with  $\widehat{\varphi} \in C_c^\infty(\mathbb{R})$  such that

$$\|\xi - \varphi\|_V^2 = \int_{\mathbb{R}} \widehat{V} |\widehat{\xi} - \widehat{\varphi}|^2 < \delta^2.$$

Then, we estimate

$$\|\tau_\varepsilon \xi - \xi\|_V \leq \|\tau_\varepsilon(\xi - \varphi)\|_V + \|\tau_\varepsilon \varphi - \varphi\|_V + \|\varphi - \xi\|_V \leq C\|\varphi - \xi\|_V + \|\tau_\varepsilon \varphi - \varphi\|_V.$$

The first term is bounded by  $C\delta$ . For the second term, we use  $\|\cdot\|_V \leq C\|\cdot\|_{L^2(\mathbb{R})}$  and (40) to show that it is arbitrarily small in  $\varepsilon$ . Since  $\delta > 0$  is arbitrary, we conclude (37b), which completes the proof of (36).  $\square$

Reflecting back on Remark 3.5, Corollary 3.4 allows us to extend the operator  $f \mapsto V \tilde{*} f$  to  $H_V(0,1)$  by

$$V \tilde{*} \xi := T^2 \xi \quad \text{for all } \xi \in H_V(0,1).$$

Lemma 3.7 states the consistency of the operator  $T^2$  and the convolution defined in (6):



**Lemma 3.7.** *For any  $0 \leq a < 1$  and any  $\rho, \mu \in H_V(0, 1) \cap \mathcal{P}([0, 1])$  it holds that*

$$V * \rho = V \tilde{*} \rho \quad \text{a.e. on } (0, 1), \text{ and} \quad (41)$$

$$(\mu, \rho)_V = \int_0^1 (V * \mu) d\rho. \quad (42)$$

**Remark 3.8.** *While any  $\rho \in H_V(0, 1) \cap \mathcal{P}([0, 1])$  has no atoms, we cannot exclude Cantor parts. Indeed, [Fal03, Thm. 4.13] guarantees for all  $0 \leq a < 1$  the existence of a Cantor measure  $\rho \in \mathcal{P}([0, 1])$  with  $\|\rho\|_{H^{-(1-a)/2}(\mathbb{R})} < \infty$ . Hence, we cannot treat the elements of  $H_V(0, 1) \cap \mathcal{P}([0, 1])$  as functions.*

*Proof of Lemma 3.7.* Let  $\rho, \mu \in H_V(0, 1) \cap \mathcal{P}([0, 1])$  be arbitrary. We extend  $\rho, \mu$  to  $\mathbb{R}$  with zero extension, and extend  $V$  to  $\mathbb{R}$  as in (29). We first introduce regularisations of  $V$ ,  $\rho$  and  $\mu$ . Let  $V_k$  as given by Lemma 3.1.(ii), for any  $k \in \mathbb{N}$ , and let  $\rho_\varepsilon := \eta_\varepsilon * \rho \in \mathcal{S}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$  and  $\mu_\delta := \eta_\delta * \mu \in \mathcal{S}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ , where  $\eta_\varepsilon$  is the usual mollifier. It is easy to verify that

$$\int_{\mathbb{R}} \varphi d\rho_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \varphi d\rho \quad \text{for all } \varphi \in C_b(\mathbb{R}), \text{ and } v\widehat{\rho}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} v\widehat{\rho} \quad \text{in } L^2(\mathbb{R}),$$

where  $v$  is define in (34).  $\mu_\delta$  satisfies analogous convergence properties. As a consequence of Lemma 3.1.(ii) and (29), we obtain that  $V_k \in C_c(\mathbb{R})$ . Moreover, since both  $V_k$  and  $V - V_k$  are convex on  $(0, \infty)$  and even on  $\mathbb{R}$ , we obtain from (111) that  $0 \leq \widehat{V}_k \leq \widehat{V}$ . Moreover, applying the Dominated Convergence Theorem, we obtain

$$\|\widehat{V} - \widehat{V}_k\|_\infty = \sup_{\omega \in \mathbb{R}} \left| \int_{\mathbb{R}} (V - V_k)(t) \cos(2\pi t\omega) dt \right| = \int_{\mathbb{R}} (V - V_k) \xrightarrow{k \rightarrow \infty} 0. \quad (43)$$

Since  $V_k$ ,  $\rho_\varepsilon$  and  $\mu_\delta$  are all continuous and integrable, we obtain from the classical definition of convolution that

$$V_k * \rho_\varepsilon = V_k \tilde{*} \rho_\varepsilon \quad \text{on } \mathbb{R}, \text{ and } \int_{\mathbb{R}} (V_k * \mu_\delta) \rho_\varepsilon = \int_{\mathbb{R}} \widehat{V}_k \widehat{\mu_\delta} \widehat{\rho_\varepsilon}. \quad (44)$$

Next we pass to the limit in (44); first  $\delta \rightarrow 0$ , then  $\varepsilon \rightarrow 0$ , and finally  $k \rightarrow \infty$ . Regarding  $V_k * \rho_\varepsilon$ , we note that for all  $t \in \mathbb{R}$ , the mapping  $s \mapsto V_k(t - s)$  is bounded and continuous. Hence,

$$(V_k * \rho_\varepsilon)(t) = \int_{\mathbb{R}} V_k(t - s) d\rho_\varepsilon(s) \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathbb{R}} V_k(t - s) d\rho(s) = (V_k * \rho)(t) \quad \text{for all } t \in \mathbb{R}. \quad (45)$$

Then, we use the Monotone Convergence Theorem to conclude that

$$(V_k * \rho_\varepsilon)(t) \xrightarrow{k \rightarrow \infty} (V * \rho)(t) \quad \text{for all } t \in [0, 1].$$

Regarding  $V_k \tilde{*} \rho_\varepsilon$ , we take any test function  $\varphi \in \mathcal{S}(\mathbb{R})$ , and compute

$$\int_{\mathbb{R}} (V_k \tilde{*} \rho_\varepsilon) \varphi = \int_{\mathbb{R}} \widehat{V}_k v \widehat{\rho_\varepsilon} \frac{\widehat{\varphi}}{v} \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \widehat{V}_k \widehat{\rho} \widehat{\varphi}.$$

Using  $0 \leq \widehat{V}_k \leq \widehat{V}$  and the pointwise convergence of  $\widehat{V}_k$  to  $\widehat{V}$ , we conclude from the Dominated Convergence Theorem that

$$\int_{\mathbb{R}} \widehat{V}_k \widehat{\rho} \widehat{\varphi} \xrightarrow{k \rightarrow \infty} \int_{\mathbb{R}} \widehat{V} \widehat{\rho} \widehat{\varphi} = \int_{\mathbb{R}} (T^2 \rho) \varphi.$$

Since  $\varphi$  is arbitrary, we conclude by the uniqueness of limits that (41) holds.

Similarly, we prove (42) starting from (44). From the same argument as in (45) we obtain from the Dominated Convergence Theorem that

$$\int_{\mathbb{R}} (V_k * \mu_\delta) \rho_\varepsilon \xrightarrow{\delta \rightarrow 0} \int_{\mathbb{R}} (V_k * \mu) \rho_\varepsilon.$$

Since  $V_k * \mu \in C_b(\mathbb{R})$  and  $\rho$  has no atoms, we further obtain

$$\int_{\mathbb{R}} (V_k * \mu) \rho_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \int_0^1 (V_k * \mu) d\rho.$$

Since  $0 \leq V_k \nearrow V$  as  $k \rightarrow \infty$ , we conclude by the Monotone Convergence Theorem that

$$\int_0^1 (V_k * \mu) d\rho \xrightarrow{k \rightarrow \infty} \int_0^1 (V * \mu) d\rho.$$

Regarding the right-hand side of (44), we rewrite it as

$$\int_{\mathbb{R}} \widehat{V_k} \widehat{\mu_\delta} \overline{\widehat{\rho_\varepsilon}} = \int_{\mathbb{R}} \frac{\widehat{V_k}}{\widehat{V}} (v \widehat{\mu_\delta}) (\overline{v \widehat{\rho_\varepsilon}})$$

Since  $0 \leq \widehat{V_k}/\widehat{V} \leq 1$ , we obtain

$$\int_{\mathbb{R}} \frac{\widehat{V_k}}{\widehat{V}} (v \widehat{\mu_\delta}) (\overline{v \widehat{\rho_\varepsilon}}) \xrightarrow[\varepsilon \rightarrow 0]{\delta \rightarrow 0} \int_{\mathbb{R}} \widehat{V_k} \widehat{\mu} \overline{\widehat{\rho}}.$$

Since  $\widehat{V} > 0$  and (43) imply that  $\|(\widehat{V} - \widehat{V_k})/\widehat{V}\|_{L^\infty(K)} \rightarrow 0$  as  $k \rightarrow \infty$  for any compact  $K \subset \mathbb{R}$ , we obtain

$$\int_{\mathbb{R}} \widehat{V_k} \widehat{\mu} \overline{\widehat{\rho}} = \int_{\mathbb{R}} \left(1 - \frac{\widehat{V} - \widehat{V_k}}{\widehat{V}}\right) \widehat{V} \widehat{\mu} \overline{\widehat{\rho}} \xrightarrow{k \rightarrow \infty} \int_{\mathbb{R}} \widehat{V} \widehat{\mu} \overline{\widehat{\rho}} = (\mu, \rho)_V. \quad \square$$

Lemma 3.7 motivates the following notational convention:

**Definition 3.9** (Convolution with  $V$ ). *Let  $0 \leq a < 1$  and  $\rho \in H_V(0, 1) \cup \mathcal{P}([0, 1])$ . If  $\rho \in \mathcal{P}([0, 1])$ , then we interpret  $V * \rho$  as the lower semi-continuous function defined in (6). If  $\rho \in H_V(0, 1)$ , then  $V * \rho := T^2 \rho \in L^2(0, 1)$ .*

We end this section with two further properties of the convolution with  $V$ :

**Lemma 3.10.** *Let  $0 \leq a < 1$ . If  $\xi \in H_V(0, 1)$  satisfies  $V * \xi = 0$  a.e. on  $(0, 1)$ , then  $\xi = 0$ .*

*Proof.* By (36) there exists  $(\varphi_n) \subset C_c^\infty(0, 1)$  such that  $\varphi_n \rightarrow \xi$  in  $H_V(0, 1)$  as  $n \rightarrow \infty$ . From Corollary 3.4, we then obtain that  $T\varphi_n \rightarrow T\xi$  in  $L^2(\mathbb{R})$ . Using these observations, we compute

$$0 = \int_0^1 (V * \xi) \varphi_n = (T^2 \xi, \varphi_n)_{L^2(\mathbb{R})} = (T\xi, T\varphi_n)_{L^2(\mathbb{R})} \xrightarrow{n \rightarrow \infty} (T\xi, T\xi)_{L^2(\mathbb{R})} = \|\xi\|_V^2. \quad \square$$

**Lemma 3.11.** *It holds for all  $\rho \in H_V(0, 1) \cap \mathcal{M}([0, 1])$  that*

$$\begin{cases} V_{\text{reg}} * \rho \in W^{2,1}(0, 1) & \text{if } 0 < a < 1, \\ V_{\text{reg}} * \rho \in W^{2,p_0}(0, 1) & \text{if } a = 0. \end{cases}$$

*Proof.* We set  $p = 1$  if  $0 < a < 1$ , and  $p = p_0$  if  $a = 0$ . Since  $\rho \in \mathcal{P}([0, 1])$  and  $V_{\text{reg}}^{(k)} \in L^p(-1, 1)$  for  $k = 0, 1, 2$ , we obtain by the generalised convolution inequality (see, e.g., [Bog07, Prop. 3.9.9]) that

$$\|V_{\text{reg}}^{(k)} * \rho\|_{L^p(0,1)} \leq \|V_{\text{reg}}^{(k)}\|_{L^p(0,1)} \rho([0, 1]) < \infty, \quad (46)$$

and thus it remains to check that  $V_{\text{reg}}^{(k)} * \rho = (V_{\text{reg}} * \rho)^{(k)}$  a.e. on  $(0, 1)$ . To check this, we take any  $\varphi \in C_c^\infty(0, 1)$ , and compute, using that  $V_{\text{reg}}$  is even,

$$\begin{aligned} \langle \varphi, V_{\text{reg}}^{(k)} * \rho \rangle &= (-1)^k \langle V_{\text{reg}}^{(k)} * \varphi, \rho \rangle = (-1)^k \langle V_{\text{reg}} * \varphi^{(k)}, \rho \rangle \\ &= (-1)^k \langle \varphi^{(k)}, V_{\text{reg}} * \rho \rangle = \langle \varphi, (V_{\text{reg}} * \rho)^{(k)} \rangle. \quad \square \end{aligned}$$

## 4 Regularity of the solutions to Carleman's equations

Carleman [Car22] was the first to give explicit solutions for the integral equation

$$\int_{t_1}^{t_2} \frac{u(s)}{|t-s|^a} ds = f(t), \quad \text{for a.e. } t_1 < t < t_2 \quad (47)$$

for any  $0 \leq a < 1$ , where  $-\infty < t_1 < t_2 < \infty$  are given. The family of equations (47) parametrised by  $a$ , are called *Carleman's equations*. However, no precise solution concept for  $u$  is given, and the minimal requirements on the regularity of the data  $f$  are not specified. Since it is not readily verified that the expression for  $u$  (see (49) and (63)) indeed satisfies the integral equation, the minimal requirements on  $f$  are not easily obtained. The aim of this section is to find sufficient requirements on  $f$  for which (47) has a unique solution  $u$  in a specific function space (see Theorem 4.1 and Theorem 4.2). Our proof reverses the steps of the constructive solution method of (47) in [EK00, §2.6], and justifies all these steps for the assumed regularity on  $f$ .

We note that the left-hand side in (47) equals  $(V_a * u)(t)$ . It is therefore natural to consider the Hilbert space  $H_{V_a}(t_1, t_2)$  as in (35). We will also use the fractional derivative  $D_{t_1}^a$  with starting value  $t_1$  (see (27)).

### 4.1 The case $0 < a < 1$

We introduce for  $1 \leq p < \infty$  the weak  $L^p$  space [Gra04] by

$$wL^p(0, 1) := \left\{ f : (0, 1) \rightarrow \mathbb{R} \text{ measurable} : \text{ess sup}_{y>0} y^p \mathcal{L}(\{|f| > y\}) < \infty \right\}, \quad (48)$$

where  $\{|f| > y\} \subset (0, 1)$  is the upper-level set of  $|f|$ . We only use the basic properties

$$L^p(0, 1) \subset wL^p(0, 1) \subset L^q(0, 1) \quad \text{for all } 1 \leq q < p$$

and that  $wL^p(0, 1) \setminus L^p(0, 1)$  contains functions with a  $|t|^{-1/p}$ -type singularity.

**Theorem 4.1** (Explicit solution). *Let  $0 < a < 1$ ,  $-\infty < t_1 < t_2 < \infty$  and  $f \in W^{2,1}(t_1, t_2)$ . Then (47) has a unique solution  $u \in H_{V_a}(t_1, t_2)$ . It is given by*

$$\begin{aligned} u(t) &= \frac{\cos^2(\frac{a\pi}{2})}{\pi^2} \left( \pi \tan(\frac{a\pi}{2}) g(t) + \frac{1}{\phi(t)} \int_{t_1}^{t_2} \frac{\phi(s)g(s)}{t-s} ds \right), \\ \phi(t) &:= [(t-t_1)(t_2-t)]^{\frac{1-a}{2}}, \quad g(t) := \Gamma(a)(D_{t_1}^{1-a}f)(t) = \frac{d}{dt} \int_{t_1}^t \frac{f(s)}{(t-s)^{1-a}} ds \end{aligned} \quad (49)$$

for any  $t_1 < t < t_2$ . Moreover,  $u$  satisfies the following regularity property:

$$\phi u \in C^\beta([t_1, t_2]) \quad \text{for any } 0 < \beta < \frac{1-a}{2} \wedge a. \quad (50)$$

*Proof of Theorem 4.1.* By a linear transformation of variables, it is enough to consider  $t_1 = 0$  and  $t_2 = 1$ . To prove that (47) attains at most one solution in  $H_{V_a}(0, 1)$ , we take any  $u_1, u_2 \in H_{V_a}(0, 1)$  which satisfy (47), note that the difference  $u := u_1 - u_2$  satisfies  $V_a * u = 0$  a.e. on  $(0, 1)$ , and conclude from Lemma 3.10 that  $u = 0$ .

Next we show that  $u$  given by (49) satisfies (50). We first prove this for constant functions  $f \equiv C$ . For such  $f$ , we obtain from (28) that  $g(t) = Ct^{-(1-a)}$ , and compute

$$\phi(t)u(t) = \frac{C}{\pi^2} \cos^2\left(\frac{a\pi}{2}\right) \left( \pi \tan\left(\frac{a\pi}{2}\right) \left(\frac{1-t}{t}\right)^{\frac{1-a}{2}} + S\left(\left(\frac{1-t}{t}\right)^{\frac{1-a}{2}}\right) \right), \quad (51)$$

which, by (24), is constant in  $t$ . Hence,  $\phi u \in C^\beta([0, 1])$  for any  $0 < \beta < 1$ .

Since the right-hand side of (49) is linear in  $f$ , we use the result above for constant functions to restrict the proof of (50) to those  $f \in W^{2,1}(0, 1)$  with  $f(0) = 0$ . For such  $f$  we obtain from Proposition 2.6 that

$$g = \Gamma(a)D^{1-a}f \in W^{1,p}(0, 1) \quad \text{for any } 1 \leq p < \frac{1}{1-a}.$$

Then, we obtain from Proposition 2.5 that  $g \in C^\alpha([0, 1])$  for any  $0 < \alpha < a$ . To conclude (50), we observe from (49) that  $u$  has the structure  $u = C_1g + C_2\frac{1}{\phi}S(\phi g)$  with  $\phi \in C^{(1-a)/2}([0, 1])$ . Splitting  $S(\phi g) = (S(\phi g) - \phi Sg) + \phi Sg$ , we obtain from Proposition 2.3.(ii) that  $(S(\phi g) - \phi Sg) \in C^\beta([0, 1])$  for any  $0 < \beta < \frac{1-a}{2} \wedge a$ . Recalling the definition of the weighted Hölder space in (25), we obtain by Proposition 2.3.(i) and  $g \in C^\beta([0, 1]) \subset C_0^\beta(\phi)$  that  $Sg \in C_0^\beta(\phi)$ , and thus  $\phi Sg \in C_0^\beta([0, 1])$ . In conclusion

$$\phi u = \underbrace{C_1\phi g}_{\in C^\beta([0, 1])} + \underbrace{C_2(S(\phi g) - \phi Sg)}_{\in C^\beta([0, 1])} + \underbrace{C_2\phi Sg}_{\in C_0^\beta([0, 1])} \in C^\beta([0, 1]) \quad \text{for any } 0 < \beta < \frac{1-a}{2} \wedge a.$$

In preparation for proving the existence of a solution to (47), we list several observations on the regularity of  $u$  and  $g$ . From (50) we obtain

$$u \in wL^{\frac{2}{1-a}}(0, 1) \cap L_{\text{loc}}^\infty(0, 1), \quad (52)$$

where  $wL^{2/(1-a)}(0, 1)$  is defined in (48),

$$\frac{u(t)}{t^a} = \frac{(\phi u)(t)}{\phi(t)t^a} \in wL^{\frac{2}{1+a}}(0, 1), \quad (53)$$

and, using Proposition 2.3.(iii),

$$S\left(\frac{u(t)}{t^a}\right) = S\left(\frac{(1-t)^a(\phi u)(t)}{[t(1-t)]^{(a+1)/2}}\right) \in wL^{\frac{2}{1+a}}(0, 1) \cap L_{\text{loc}}^\infty(0, 1). \quad (54)$$

Regarding  $g$ , we obtain from Proposition 2.6 that

$$\tilde{g}(t) := t^{1-a}g(t) \in L^\infty(0, 1). \quad (55)$$

Next we prove the existence of a solution to (47) for any  $f \in W^{2,1}(0,1)$ . We show that  $u$  given by (49) satisfies (47). Our proof follows in reversed order the computation for the solution of (47) outlined in [EK00, §2.6].

We set  $\tilde{u}(t) := t^{1-a}u(t)$  and obtain from (49) that

$$\begin{aligned}\tilde{u}(t) &= \frac{\sin(\frac{a\pi}{2})\cos(\frac{a\pi}{2})}{\pi}t^{1-a}g(t) + \frac{\cos^2(\frac{a\pi}{2})t^{1-a}}{\pi^2}\frac{1}{\phi(t)}S(\phi g)(t) \\ &= \frac{\sin(a\pi)}{2\pi}\tilde{g}(t) + \frac{\cos^2(\frac{a\pi}{2})}{\pi^2}\left(\frac{t}{1-t}\right)^{\frac{1-a}{2}}S\left(\left(\frac{1-t}{t}\right)^{\frac{1-a}{2}}\tilde{g}(t)\right).\end{aligned}$$

Then, we observe from Proposition 2.4 and (55) that  $\tilde{u}$  satisfies

$$\pi \tan(\frac{a\pi}{2})\tilde{u}(t) - S\tilde{u}(t) = \tilde{g}(t) = \Gamma(a)t^{1-a}D^{1-a}f(t) \quad \text{for a.e. } 0 < t < 1. \quad (56)$$

Relying on (53), we use Proposition 2.2.(iv) to rewrite (56) as

$$\begin{aligned}D^{1-a}f(t) &= \frac{1}{\Gamma(a)}\left(\pi \tan(\frac{a\pi}{2})u(t) - \frac{1}{t^{1-a}}S(t^{1-a}u(t))\right) \\ &= \frac{\pi(1 - \cos(a\pi))}{\Gamma(a)\sin(a\pi)}u(t) - \frac{t^a}{\Gamma(a)}S\left(\frac{u(t)}{t^a}\right) + \frac{1}{\Gamma(a)t^{1-a}}\int_0^1 \frac{u(s)}{s^a}ds. \quad (57)\end{aligned}$$

We observe from (52) and (54) that all three terms in the right-hand side of (57) are in  $L^1(0,1)$ . Then, we use Proposition 2.7 to apply  $I^{1-a}$  to all four terms in both sides of (57). This yields, using (28),

$$I^{1-a}D^{1-a}f = f \quad \text{and} \quad I^{1-a}\left(\frac{1}{\Gamma(a)t^{1-a}}\right) = 1.$$

Regarding the other two terms in (57), we use (21) to obtain

$$\frac{\pi}{\Gamma(a)\sin(a\pi)}I^{1-a}u(t) = \int_0^t \frac{u(s)}{(t-s)^a}ds$$

and

$$I^{1-a}\left(\frac{t^a}{\Gamma(a)}S\left(\frac{u(t)}{t^a}\right)\right) = \frac{\sin(a\pi)}{\pi}\int_0^t \frac{s^a}{(t-s)^a}S\left(\frac{u(s)}{s^a}\right)ds.$$

Hence, applying  $I^{1-a}$  to (57) yields

$$\begin{aligned}f(t) &= \int_0^t \frac{u(s)}{(t-s)^a}ds + \left[ -\cos(a\pi)\int_0^t \frac{u(s)}{(t-s)^a}ds - \frac{\sin(a\pi)}{\pi}\int_0^t \frac{s^a}{(t-s)^a}S\left(\frac{u(s)}{s^a}\right)ds \right. \\ &\quad \left. + \int_0^1 \frac{u(s)}{s^a}ds \right] \quad \text{for a.e. } 0 < t < 1. \quad (58)\end{aligned}$$

To complete our proof of the existence of solutions to (47), it remains to show that the expression within brackets in (58) equals

$$\int_t^1 \frac{u(s)}{|t-s|^a}ds. \quad (59)$$

With this aim, we fix  $0 < t < 1$  arbitrarily, and focus on the second term within these brackets. In preparation for applying Proposition 2.2.(iii), we regularise the integrand by replacing  $(t-s)^{-a}$  by  $|t-s|^{-a}\mathbb{1}_{(0,t-\varepsilon)}(s)$  for  $0 < \varepsilon < t$ , which we interpret as a function of  $s$  on  $(0,1)$ . Since  $|t-s|^{-a}\mathbb{1}_{(0,t-\varepsilon)}(s)$  converges in  $L^p(0,1) \cap L^\infty(0,\frac{t}{2})$  for any  $1 \leq p < \frac{1}{a}$  to  $|t-s|^{-a}\mathbb{1}_{(0,t)}(s)$  as  $\varepsilon \rightarrow 0$ , we obtain from (54) that

$$\int_0^1 \frac{s^a \mathbb{1}_{(0,t-\varepsilon)}(s)}{|t-s|^a} S\left(\frac{u(s)}{s^a}\right) ds \xrightarrow{\varepsilon \rightarrow 0} \int_0^t \frac{s^a}{(t-s)^a} S\left(\frac{u(s)}{s^a}\right) ds. \quad (60)$$

Since  $[s \mapsto |t-s|^{-a}\mathbb{1}_{(0,t-\varepsilon)}(s)] \in L^\infty(0,1)$  for any  $\varepsilon \in (0,t)$ , we can apply Proposition 2.2.(iii),(iv) to obtain

$$\begin{aligned} \int_0^1 \frac{s^a \mathbb{1}_{(0,t-\varepsilon)}(s)}{|t-s|^a} S\left(\frac{u(s)}{s^a}\right) ds &= - \int_0^1 \frac{u(s)}{s^a} S\left(\frac{s^a \mathbb{1}_{(0,t-\varepsilon)}(s)}{|t-s|^a}\right) ds \\ &= - \int_0^1 s^{1-a} u(s) S\left(\frac{\mathbb{1}_{(0,t-\varepsilon)}(s)}{s^{1-a}|t-s|^a}\right) ds + \left[ \int_0^1 \frac{u(s)}{s^a} ds \right] \left[ \int_0^{t-\varepsilon} \frac{1}{s^{1-a}|t-s|^a} ds \right]. \end{aligned} \quad (61)$$

Next we pass to the limit  $\varepsilon \rightarrow 0$  in the right-hand side of (61). For the second term we obtain with (21) and (22) that

$$\int_0^{t-\varepsilon} \frac{1}{s^{1-a}|t-s|^a} ds \xrightarrow{\varepsilon \rightarrow 0} \frac{\pi}{\sin(a\pi)}.$$

For the first term, we split the integration domain  $(0,1)$  in  $(0, \frac{1+t}{2})$  and  $(\frac{1+t}{2}, 1)$ . On  $(0, \frac{1+t}{2})$ , we obtain from (50) that  $s^{1-a}u(s)$  is bounded, and from Proposition 2.2.(i) that the other term in the integrand converges in  $L^p(0, \frac{1+t}{2})$  for any  $1 \leq p < \frac{1}{a} \wedge \frac{1}{1-a}$ . On  $(\frac{1+t}{2}, 1)$ , we obtain that

$$S\left(\frac{\mathbb{1}_{(0,t-\varepsilon)}(s)}{s^{1-a}|t-s|^a}\right) = \int_0^{t-\varepsilon} \frac{r^{-(1-a)}(t-r)^{-a}}{s-r} dr \xrightarrow{\varepsilon \rightarrow 0} S\left(\frac{\mathbb{1}_{(0,t)}(s)}{s^{1-a}|t-s|^a}\right) \quad \text{for } \frac{1+t}{2} < s < 1$$

is a regular integral, where the convergence is uniformly in  $s \in (\frac{1+t}{2}, 1)$ . Using (23), we further rewrite

$$S\left(\frac{\mathbb{1}_{(0,t)}(s)}{s^{1-a}|t-s|^a}\right) = \frac{1}{s^{1-a}|t-s|^a} \begin{cases} \frac{\pi}{\tan(a\pi)} & \text{if } s < t \\ \frac{\pi}{\sin(a\pi)} & \text{if } s > t \end{cases} \quad \text{for a.e. } 0 < s < 1.$$

In conclusion, passing to the limit  $\varepsilon \rightarrow 0$  in the right-hand side of (61) yields, together with (60),

$$\begin{aligned} \int_0^t \frac{s^a}{(t-s)^a} S\left(\frac{u(s)}{s^a}\right) ds \\ = - \frac{\pi}{\tan(a\pi)} \int_0^t \frac{u(s)}{|t-s|^a} ds - \frac{\pi}{\sin(a\pi)} \int_t^1 \frac{u(s)}{|t-s|^a} ds + \frac{\pi}{\sin(a\pi)} \int_0^1 \frac{u(s)}{s^a} ds. \end{aligned}$$

Substituting this expression in (58), we obtain that the term within brackets equals (59), which completes the proof.  $\square$

## 4.2 The case $a = 0$

**Theorem 4.2** (Explicit solution). *Let  $-\infty < t_1 < t_2 < \infty$  be such that  $t_2 - t_1 \neq 4$ , and let  $f \in C^{1,\alpha}([t_1, t_2])$  for some  $0 < \alpha < 1$ . Then*

$$\int_{t_1}^{t_2} -\log|t-s| u(s) ds = f(t), \quad \text{for a.e. } t_1 < t < t_2 \quad (62)$$

has a unique solution  $u \in H_{1-\log|\cdot|}(t_1, t_2)$ . It is given by

$$u(t) = \frac{1}{\pi^2 \phi(t)} \left( \int_{t_1}^{t_2} \frac{\phi(s) f'(s)}{t-s} ds - \frac{1}{\log(\frac{1}{4}(t_2 - t_1))} \int_{t_1}^{t_2} \frac{f(s)}{\phi(s)} ds \right) \quad (63)$$

for any  $t_1 < t < t_2$ , where  $\phi(t) := \sqrt{(t_2 - t)(t - t_1)}$ . Moreover,  $u$  satisfies the following regularity property:

$$\phi u \in C^\alpha([t_1, t_2]). \quad (64)$$

*Proof.* We follow a similar proof strategy as for Theorem 4.1. In fact, the proof is analogous for the uniqueness of  $u$ . Also, the regularity property (64) follows by an analogous argument after observing from (63) that  $\phi u = C_1 S(\phi f') + C_2$  with  $f' \in C^\alpha([0, 1])$ . Therefore, we omit further details, and focus on proving the existence of a solution  $u$  to (62).

Unlike the case  $0 < a < 1$ , a linear change of variables to transform  $(t_1, t_2)$  to  $(0, 1)$  slightly changes the integral equation (62). Setting

$$\begin{aligned} \tilde{t} &:= \frac{t - t_1}{t_2 - t_1} =: \mathcal{T}(t) & t &= \mathcal{T}^{-1}(\tilde{t}) = (t_2 - t_1)\tilde{t} + t_1, \\ \tilde{u}(\tilde{t}) &:= (t_2 - t_1)u(\mathcal{T}^{-1}(\tilde{t})) & u(t) &= \frac{\tilde{u}(\mathcal{T}(t))}{t_2 - t_1}, \\ \tilde{f}(\tilde{t}) &:= f(\mathcal{T}^{-1}(\tilde{t})) & f(t) &= \tilde{f}(\mathcal{T}(t)), \\ \tilde{\phi}(\tilde{t}) &:= \frac{\phi(\mathcal{T}^{-1}(\tilde{t}))}{t_2 - t_1} = \sqrt{\tilde{t}(1 - \tilde{t})} & \phi(t) &= (t_2 - t_1)\tilde{\phi}(\mathcal{T}(t)), \end{aligned} \quad (65)$$

we find that (62) and (63) transform respectively to

$$\int_0^1 -\log|\tilde{t} - \tilde{s}| \tilde{u}(\tilde{s}) d\tilde{s} = \tilde{f}(\tilde{t}) + \log|t_2 - t_1| \int_0^1 \tilde{u}(\tilde{s}) d\tilde{s}, \quad \text{for a.e. } 0 < \tilde{t} < 1, \quad (66)$$

and

$$\tilde{u}(\tilde{t}) = \frac{1}{\pi^2 \tilde{\phi}(\tilde{t})} \left( S(\tilde{\phi} \tilde{f}')(\tilde{t}) - \frac{1}{\log(\frac{1}{4}(t_2 - t_1))} \int_0^1 \frac{\tilde{f}(\tilde{s})}{\tilde{\phi}(\tilde{s})} d\tilde{s} \right) \quad \text{for any } 0 < \tilde{t} < 1. \quad (67)$$

It remains to prove that  $\tilde{u}$  given by (67) satisfies (66). We observe from Proposition 2.4 that  $\tilde{u}$  given by (67) satisfies

$$-S\tilde{u} = \tilde{f}' \quad \text{a.e. on } (0, 1). \quad (68)$$

Using Proposition 2.2.(ii) and integrating (68) from  $\tilde{r}$  to  $\tilde{t}$  for given  $\tilde{r}, \tilde{t} \in (0, 1)$ , we obtain

$$\int_0^1 -\log|\tilde{t} - \tilde{s}| \tilde{u}(\tilde{s}) d\tilde{s} - \tilde{f}(\tilde{t}) = \int_0^1 -\log|\tilde{r} - \tilde{s}| \tilde{u}(\tilde{s}) d\tilde{s} - \tilde{f}(\tilde{r}). \quad (69)$$

Next we divide both sides of (69) by  $\pi \tilde{\phi}(\tilde{r})$  and integrate  $\tilde{r}$  over  $(0, 1)$ . We claim that the result is given by (66). This is easy to see for the left-hand side, since it does not depend on  $\tilde{r}$  and  $\int_0^1 1/(\pi \tilde{\phi}) = 1$ . The computation of the right-hand side is left to Appendix B.  $\square$

## 5 Proofs of Theorem 1.5 and Theorem 1.6

In this section we prove Theorem 1.5 and Theorem 1.6. An outline of these proofs is given in §1.4.

*Proof of Theorem 1.5.* We start by showing that Theorem 2.1 applies with  $H_V(0, 1)$  as the Hilbert space,  $H_V(0, 1) \cap \mathcal{P}([0, 1])$  as the closed convex subset, and  $[\xi \mapsto \int_0^1 U \xi]$  as the linear term. Convexity of  $H_V(0, 1) \cap \mathcal{P}([0, 1])$  is obvious, and closedness follows by interpreting any  $H_V(0, 1)$ -converging sequence  $(\xi_n) \subset H_V(0, 1) \cap \mathcal{P}([0, 1])$  as a converging sequence of distributions on  $\mathbb{R}$  with support in  $[0, 1]$ , for which non-negativity is conserved in the limit, and the unit integral condition follows by testing with any  $\varphi \in C_c^\infty(\mathbb{R})$  with  $\varphi|_{(0,1)} = 1$ . For the linear term, we use (5) to extend  $U$  to  $\mathbb{R}$  such that  $U \in H^{(1-a)/2}(\mathbb{R}) \cap C_c(\mathbb{R})$ . Then,  $\langle U, f \rangle := \int_{\mathbb{R}} U f$  extends as a bounded linear functional on  $L^2(\mathbb{R})$  to a bounded linear functional on  $H^{-(1-a)/2}(\mathbb{R}) + \mathcal{M}(\mathbb{R})$ . Using Corollary 3.4, we find in particular that  $\langle U, \cdot \rangle$  is a bounded linear functional on  $H_V(0, 1)$ , and that there exists  $\xi \in H_V(0, 1)$  such that  $\langle U, \eta \rangle = (\xi, \eta)_V$  for all  $\eta \in H_V(0, 1)$ . We conclude that Theorem 2.1 applies.

By Theorem 2.1 we obtain that both Problem 1.1 and the variational inequality given by

$$0 \leq (\rho + \xi, \mu - \rho)_V, \quad \text{for all } \mu \in \mathcal{P}([0, 1]) \cap H_V(0, 1), \quad (70)$$

with solution concept  $\rho \in H_V(0, 1) \cup \mathcal{P}([0, 1])$ , have the same unique solution  $\bar{\rho}$ .

It is left to show the equivalence with Problem 1.2. We first show that any solution  $\rho \in H_V(0, 1) \cup \mathcal{P}([0, 1])$  of (70) satisfies (9). We expand

$$0 \leq (\rho + \xi, \mu - \rho)_V = (\rho, \mu)_V - (\rho, \rho)_V + (\xi, \mu - \rho)_V. \quad (71)$$

Interpreting  $\rho, \mu$  as measures with no atoms, we use (42) to write

$$(\rho, \mu)_V - (\rho, \rho)_V = \int_0^1 V * \rho d\mu - \int_0^1 V * \rho d\rho.$$

Moreover, since  $\langle U, \cdot \rangle$  is extended to  $\mathcal{M}([0, 1])$ , its extension to the subset  $\mathcal{P}([0, 1])$  is simply given by  $\langle U, \tilde{\mu} \rangle = \int_{[0,1]} U d\tilde{\mu}$ . Hence,

$$(\xi, \mu - \rho)_V = \langle U, \mu \rangle - \langle U, \rho \rangle = \int_0^1 U d\mu - \int_0^1 U d\rho.$$

Collecting our results, we obtain that (71) implies (9).

To show that any solution  $\rho \in \mathcal{P}([0, 1])$  to Problem 1.2 is also a solution to (70), we separate two cases. If  $\rho \in H_V(0, 1)$ , then the argument above readily shows that  $\rho$  satisfies (70). If  $\rho \notin H_V(0, 1)$ , then

$$\int_0^1 h_\rho d\rho \geq \int_0^1 V * \rho d\rho = \infty.$$

However, for  $\mu = \mathcal{L}|_{(0,1)}$ , we find with  $V_k$  as in Lemma 3.1.(ii) that

$$\int_0^1 h_\rho d\mu = \lim_{k \rightarrow \infty} \int_0^1 \int_0^1 V_k(t-s) d\rho(s) dt + \int_0^1 U \leq \int_{-1}^1 V + \int_0^1 U < \infty,$$

which contradicts (9). □



*Proof of Theorem 1.6.* By Theorem 1.5, Problem 1.2 has a unique solution  $\bar{\rho} \in \mathcal{P}([0, 1]) \cap H_V(0, 1)$ . We recall from (6) and Corollary 3.4 that  $h_{\bar{\rho}} = V * \bar{\rho} + U \in L^2(0, 1)$  is lower semi-continuous on  $[0, 1]$ .

*Step 1:  $\bar{\rho}$  satisfies Problem 1.3.* In this step we prove by contradiction that

$$\exists C > 0 : h_{\bar{\rho}} = C \quad \text{a.e. on } (0, 1). \quad (72)$$

Suppose  $h_{\bar{\rho}}$  is not constant a.e. on  $(0, 1)$ , i.e.

$$0 \leq m := \operatorname{ess\,inf}_{(0,1)} h_{\bar{\rho}} < \sup_{(0,1)} h_{\bar{\rho}} \leq \infty. \quad (73)$$

By (73) and  $h_{\bar{\rho}} \in L^2(0, 1)$ , we have

$$\forall \varepsilon > 0 \exists g \in \mathcal{P}([0, 1]) \cap L^2(0, 1) : \int_0^1 h_{\bar{\rho}} g < m + \varepsilon. \quad (74)$$

We reach a contradiction between (74) and (9) by showing that

$$\int_0^1 h_{\bar{\rho}} d\bar{\rho} > m. \quad (75)$$

With this aim, we consider the superlevel set

$$A := \{h_{\bar{\rho}} > m\}.$$

By (73),  $A \neq \emptyset$ , and since  $h_{\bar{\rho}}$  is lower semi-continuous,  $A$  is open. We take  $(r, s) \subset A$  to be any open component<sup>2</sup>. If  $\bar{\rho}((r, s)) > 0$ , then (75) holds, and the contradiction is reached. Hence, we assume  $\bar{\rho}((r, s)) = 0$ , and thus  $(r, s) \cap \operatorname{supp} \bar{\rho} = \emptyset$ . By (4) and (5) we then find that

$$h_{\bar{\rho}} \in C([r, s]) \cap W_{\operatorname{loc}}^{2,1}(r, s), \quad \text{and} \quad h_{\bar{\rho}}'' = V'' * \bar{\rho} + U'' \geq 0 \quad \text{on } (r, s), \quad (76)$$

where continuity up to the boundary holds by the following argument. For  $r < t < s$  we split

$$h_{\bar{\rho}}(t) = U(t) + \int_0^r V(t - \tau) d\bar{\rho}(\tau) + \int_s^1 V(t - \tau) d\bar{\rho}(\tau),$$

where the first two terms are continuous for  $t > r$ , and the third term is increasing for  $t \leq s$ . Hence,  $\limsup_{t \uparrow s} h_{\bar{\rho}}(t) \leq h_{\bar{\rho}}(s)$ , and together with  $h_{\bar{\rho}}$  being lower semi-continuous on  $[0, 1]$ , we obtain that  $h_{\bar{\rho}}$  is left-continuous at  $t = s$ . A similar argument shows that  $h_{\bar{\rho}}$  is right-continuous at  $t = r$ , and thus (76) follows.

We separate three cases to complete the contradiction between (74) and (9):

1. Let  $r = 0$  and  $s = 1$ . This contradicts with  $A \cap \operatorname{supp} \bar{\rho} = \emptyset$ .
2. Let  $0 < r < s < 1$ . Since  $r, s \notin A$ , it holds that  $(V * \bar{\rho})(r), (V * \bar{\rho})(s) \leq m$ . By definition of  $A$ , we also have that  $h_{\bar{\rho}} > m$  on  $(r, s)$ . However, by the convexity of  $h_{\bar{\rho}}$  (see (76)) and  $h_{\bar{\rho}} \in C(\bar{A})$  we find that  $h_{\bar{\rho}} \leq m$  on  $[r, s]$ , and a contradiction is reached.
3. Let  $r = 0$  and  $s < 1$ ; the case  $r > 0$  and  $s = 1$  can be dealt with analogously. Given  $\varepsilon > 0$  is as in (4c), we obtain

$$h_{\bar{\rho}}'(t) = (V' * \bar{\rho})(t) + U'(t) > |V'(1)| - \sup_{(0,1)} |U'| \geq 0, \quad \text{for all } s - \varepsilon < t < s. \quad (77)$$

We obtain the desired contradiction similarly to case 2 above.

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<sup>2</sup> If  $r = 0$  or  $s = 1$ , then also the boundary of the interval may be included in the open component

This concludes the proof of (72).

*Step 2:*  $\bar{\rho} = [0, 1]$ . We proof  $\bar{\rho} = [0, 1]$  by a small modification to the argument in Step 1. Suppose that the open set  $[0, 1] \setminus \text{supp } \bar{\rho}$  is non-empty, and set  $(r, s)$  as one of its components<sup>3</sup>. Then (76) holds, and  $h_{\bar{\rho}}'' > 0$  close to the endpoints of  $(r, s)$  (by (4c)) implies that  $h_{\bar{\rho}}$  is not constant a.e. on  $(r, s)$ , which contradicts (72).

*Step 3:*  $\bar{\rho}$  is a solution to Problem 1.4, satisfies (19a) and satisfies (17) everywhere on  $(0, 1)$ . We rewrite (72) as

$$C = h_{\bar{\rho}} = V_a * \bar{\rho} + V_{\text{reg}} * \bar{\rho} + U \quad \text{a.e. on } (0, 1). \quad (78)$$

*Step 3a:*  $0 < a < 1$ . Since Lemma 3.11 implies that  $V_{\text{reg}} * \bar{\rho} + U \in W^{2,1}(0, 1)$ , Theorem 4.1 states that

$$V_a * \rho = C - V_{\text{reg}} * \bar{\rho} - U \quad \text{a.e. on } (0, 1) \quad (79)$$

has a unique solution  $\tilde{\rho}$  in  $H_{V_a}(0, 1) \cong H_V(0, 1)$  (cf. Proposition 3.3). Subtracting (78) from (79), we find  $V_a * (\tilde{\rho} - \bar{\rho}) = 0$ . Then, by Lemma 3.10, we obtain  $\bar{\rho} = \tilde{\rho}$  a.e. on  $(0, 1)$ . Together with (49) and (50), we obtain that  $\bar{\rho}$  has a continuous representative which satisfies the regularity property (19a) and the implicit relation (17), i.e.,

$$\bar{\rho} = \tilde{\rho} = C_1 g_{\bar{\rho}} + C_2 \frac{1}{\phi} S(\phi g_{\bar{\rho}}) \quad \text{on } (0, 1) \quad (80)$$

for explicit constants  $C_1, C_2 > 0$ . Hence,  $\bar{\rho}$  is a function, and  $(\bar{\rho}, C)$  is a solution to Problem 1.4 for which (17) holds everywhere on  $(0, 1)$ . Moreover, we obtain from  $\bar{\rho} \in L^1(0, 1)$  that  $C = \int_0^1 C \bar{\rho} = \int_0^1 h_{\bar{\rho}} \bar{\rho} =: \bar{C}$ .

*Step 3b:*  $a = 0$ . Since the argument is similar to Step 3a, we focus on the differences. To prove that (79) has a unique solution  $\tilde{\rho}$  in  $H_{1-\log|\cdot|} \cong H_V(0, 1)$ , we obtain from Lemma 3.11 that the right-hand side of (79) is in  $W^{2,p_0}(0, 1)$ , find by Proposition 2.5 that  $W^{2,p_0}(0, 1) \subset C^{1,1-1/p_0}([0, 1])$ , and conclude with Theorem 4.2 that (79) has a unique solution  $\tilde{\rho} \in H_V(0, 1)$ . Again,  $\bar{\rho} = \tilde{\rho}$  a.e. on  $(0, 1)$ , and from (63) and (64) we conclude that  $(\bar{\rho}, C)$  is a solution to Problem 1.4 for which (17) holds everywhere on  $(0, 1)$ , and that  $C = \int_0^1 h_{\bar{\rho}} \bar{\rho} = \bar{C}$ .

*Step 4:*  $(\bar{\rho}, \bar{C})$  is the unique solution to Problem 1.3. Let  $(\rho_i, C_i)$  for  $i = 1, 2$  be two solutions to Problem 1.3. Then

$$V * (\rho_1 - \rho_2) = C_1 - C_2 \quad \text{a.e. on } (0, 1). \quad (81)$$

To show that  $\rho_1 = \rho_2$ , we consider the auxiliary energy  $\tilde{E}(\rho) := \frac{1}{2} \|\rho\|_V^2$ . We observe that  $\tilde{E}$  is of the same form as (1), and that the assumptions (4), (5) and (7) are all satisfied. Hence, Theorem 1.5 provides the unique minimiser  $\tilde{\rho} \in \mathcal{P}([0, 1]) \cap H_V(0, 1)$  of  $\tilde{E}$ , and Step 1 above yields that  $V * \tilde{\rho} = \tilde{C}$  a.e. on  $(0, 1)$ , where  $\tilde{C} := \|\tilde{\rho}\|_V^2 > 0$ . Setting  $\alpha := (C_1 - C_2)/\tilde{C}$ , we obtain that  $V * (\alpha \tilde{\rho}) = C_1 - C_2$  a.e. on  $(0, 1)$ . Then, by (81) and Lemma 3.10, we find that  $\alpha \tilde{\rho} = \rho_1 - \rho_2$ . Hence  $\alpha = \int_0^1 \alpha d\tilde{\rho} = \int_0^1 d(\rho_1 - \rho_2) = 0$ , and thus  $C_1 = C_2$ . By (81) and Lemma 3.10 we then also have  $\rho_1 = \rho_2$ .

*Step 5:*  $(\bar{\rho}, \bar{C})$  is the unique solution to Problem 1.4. By Step 4 it is enough to show that any solution  $(\rho, C)$  to Problem 1.4 satisfies Problem 1.3. We let  $(\rho, C)$  be such a solution, and assume  $0 < a < 1$  (the proof for  $a = 0$  is analogous). Since  $(\rho, C)$  satisfies (17), it also

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<sup>3</sup>See footnote 2 on page 25

satisfies (49) in Theorem 4.1 with  $t_1 = 0$ ,  $t_2 = 1$  and  $f := C - V_{\text{reg}} * \rho - U$ . By Lemma 3.11,  $f \in W^{2,1}(0,1)$ , and thus the solution  $\tilde{\rho}$  to  $V_a * \tilde{\rho} = f$  a.e. on  $(0,1)$  is given by

$$\tilde{\rho} = \pi^{-2} \Gamma(a) \cos^2\left(\frac{a\pi}{2}\right) \left(\frac{1}{\phi} S(\phi g_\rho) + \pi \tan\left(\frac{a\pi}{2}\right) g_\rho\right) = \rho \quad \text{a.e. on } (0,1).$$

Hence,

$$V_a * \rho = V_a * \tilde{\rho} = C - V_{\text{reg}} * \rho - U \quad \text{a.e. on } (0,1),$$

which implies that  $(\rho, C)$  satisfies Problem 1.3.

*Step 6:*  $(\bar{\rho}, \bar{C})$  satisfies (13) everywhere on  $[0,1]$ . For  $t, s \in [0,1]$  we estimate by Hölder's inequality

$$|(V * \bar{\rho})(t) - (V * \bar{\rho})(s)| \leq \int_{\mathbb{R}} |V(r-t) - V(r-s)| \bar{\rho}(r) dr \leq \|V(\cdot - t + s) - V\|_p \|\bar{\rho}\|_q,$$

where  $\frac{2}{a+1} < p < \frac{1}{a}$  and  $q := \frac{p}{p-1} < \frac{2}{1-a}$  is the conjugate power. Since  $V \in wL^{1/a}(\mathbb{R})$ , we obtain by (19a) that the right-hand side converges to 0 as  $|t-s| \rightarrow 0$ . Hence,  $h_{\bar{\rho}} \in C([0,1])$ , and thus (13) holds everywhere on  $[0,1]$ .

*Step 7:*  $\bar{\rho}$  satisfies (19b). We first treat the case  $0 < a < 1$ . Since  $\bar{\rho} \in L^1(0,1)$  we obtain from (20) that

$$f_{\bar{\rho}} := \bar{C} - V_{\text{reg}} * \bar{\rho} - U \in W^{\ell+1,1}(0,1).$$

Setting  $q$  as the  $(\ell-1)$ -th order Taylor polynomial of  $f_{\bar{\rho}}$  at 0, we rewrite (80) as

$$\bar{\rho} = C_1 g_{\bar{\rho}} + C_2 \frac{1}{\phi} S(\phi(g_{\bar{\rho}} - D^{1-a}q)) + C_2 \frac{1}{\phi} S(\phi D^{1-a}q). \quad (82)$$

From Proposition 2.6 we obtain that the first term in the right-hand side of (82) is contained in  $W_{\text{loc}}^{\ell,p}(0,1)$  for any  $1 \leq p < \frac{1}{1-a}$ . Regarding the third term, we expand

$$S(\phi D^{1-a}q)(t) = \sum_{k=0}^{\ell-1} \frac{f_{\bar{\rho}}^{(k)}(0)}{\Gamma(k+a)} S\left((1-t)^{\frac{1-a}{2}} t^{k-\frac{1-a}{2}}\right).$$

Since Proposition 2.2 implies that  $S(t^\alpha(1-t)^\beta) \in C_{\text{loc}}^\infty(0,1)$  for any  $\alpha, \beta > -1$ , the third term in the right-hand side of (82) is contained in  $C_{\text{loc}}^\infty(0,1)$ .

Regarding the second term in the right-hand side of (82), we obtain from Proposition 2.6 that  $g_{\bar{\rho}} - D^{1-a}q \in W^{\ell,p}(0,1)$  for any  $1 \leq p < \frac{1}{1-a}$ . Relying on the claim that  $S(\phi f) \in W_{\text{loc}}^{\ell,r}(0,1)$  for any  $f \in W^{\ell,r}(0,1)$  with  $1 < r < \frac{1}{1-a}$ , we conclude (19b) from (82).

Next we prove the claim that  $f \in W^{\ell,r}(0,1)$  with  $1 < r < \frac{1}{1-a}$  implies  $S(\phi f) \in W_{\text{loc}}^{\ell,r}(0,1)$ . We take  $[t_1, t_2] \subset (0,1)$  arbitrary, and choose  $\psi \in W^{\ell,r}(0,1)$  such that  $\psi|_{(t_1, t_2)} = (\phi f)|_{(t_1, t_2)}$ . Then

$$(S(\phi f))(t) = (S\psi)(t) + \int_0^{t_1} \frac{(\phi f - \psi)(s)}{t-s} ds + \int_{t_2}^1 \frac{(\phi f - \psi)(s)}{t-s} ds, \quad t_1 < t < t_2. \quad (83)$$

Since for any  $t_1 < t < t_2$  the map  $s \mapsto (t-s)^{-1}$  is in  $C^\infty([0, t_1] \cup [t_2, 1])$ , the two integrals in the right-hand side of (83) are in  $C^\infty((t_1, t_2))$  as functions of  $t$ . By Proposition 2.2.(i),(v) we obtain  $S\psi \in W_{\text{loc}}^{\ell,r}(0,1)$ . Since  $0 < t_1 < t_2 < 1$  are arbitrary, we conclude that  $S(\phi f) \in W_{\text{loc}}^{\ell,r}(0,1)$ , which completes the proof of (19b).

The case  $a = 0$  follows by a similar argument. The only difference is that (20) and  $\bar{\rho} \in wL^2(0, 1) \subset L^{p_0}(0, 1)$  imply

$$f_{\bar{\rho}} = \bar{C} - V_{\text{reg}} * \bar{\rho} - U \in W^{\ell+1, p_0}(0, 1).$$

This yields  $g_{\bar{\rho}} = f'_{\bar{\rho}} \in W^{\ell, p_0}(0, 1)$ , and (19b) follows by the same argument as in the case  $0 < a < 1$  without subtracting any Taylor polynomial.

*Step 8:*  $\bar{\rho} > 0$  on  $(0, 1)$ . We reason by contradiction. Assume that there exists  $0 < t_0 < 1$  such that  $\bar{\rho}(t_0) = 0$ . We set  $0 < r < t_0 \wedge (1 - t_0)$ , and compute for any  $t \in B_r(t_0) = (t_0 - r, t_0 + r)$

$$\begin{aligned} 0 = h''_{\bar{\rho}}(t) &= (V * \bar{\rho})''(t) + U''(t) \\ &= \int_{B_r(t_0)^c} V''(t - s) \bar{\rho}(s) ds + \frac{d^2}{dt^2} \left[ \int_{t-t_0-r}^{t-t_0+r} V(s) \bar{\rho}(t - s) ds \right] + U''(t). \end{aligned}$$

To compute the derivative explicitly, we use (19b) and  $\ell \geq 3$  to obtain  $\bar{\rho} \in C^2(0, 1)$ . This yields

$$\begin{aligned} &\frac{d^2}{dt^2} \left[ \int_{t-t_0-r}^{t-t_0+r} V(s) \bar{\rho}(t - s) ds \right] \\ &= \frac{d}{dt} \left[ V(t - t_0 + r) \bar{\rho}(t_0 - r) - V(t - t_0 - r) \bar{\rho}(t_0 + r) + \int_{t-t_0-r}^{t-t_0+r} V(s) \bar{\rho}'(t - s) ds \right] \\ &= V'(t - t_0 + r) \bar{\rho}(t_0 - r) - V'(t - t_0 - r) \bar{\rho}(t_0 + r) \\ &\quad + V(t - t_0 + r) \bar{\rho}'(t_0 - r) - V(t - t_0 - r) \bar{\rho}'(t_0 + r) + \int_{t-t_0-r}^{t-t_0+r} V(s) \bar{\rho}''(t - s) ds. \end{aligned}$$

Setting  $t = t_0$  and using that  $V$  is even, we find

$$\begin{aligned} 0 = h''_{\bar{\rho}}(t_0) &= \int_{B_r(t_0)^c} V''(t_0 - s) \bar{\rho}(s) ds + r^2 V'(r) \frac{\bar{\rho}(t_0 + r) + \bar{\rho}(t_0 - r)}{r^2} \\ &\quad + 2r V(r) \frac{\bar{\rho}'(t_0 + r) - \bar{\rho}'(t_0 - r)}{2r} + \int_{-r}^r V(s) \bar{\rho}''(t_0 - s) ds + U''(t_0). \end{aligned} \quad (84)$$

Next we establish a contradiction by showing that the limit  $r \rightarrow 0$  in (84) yields a positive value. By (5),  $U''(t_0) \geq 0$ . By (4c) and Step 2, the integral over  $B_r(t_0)^c$  is non-negative and increasing along any sequence  $r_k \downarrow 0$  for  $k$  large enough. For the second and the third term in the right-hand side of (84), we observe from  $\bar{\rho} \in C^2(0, 1)$  and  $\bar{\rho}(t_0) = 0$  that

$$\begin{aligned} \frac{\bar{\rho}(t_0 + r) + \bar{\rho}(t_0 - r)}{r^2} &= \frac{\bar{\rho}(t_0 + r) - 2\bar{\rho}(t_0) + \bar{\rho}(t_0 - r)}{r^2} \xrightarrow{r \rightarrow 0} \bar{\rho}''(t_0), \\ \frac{\bar{\rho}'(t_0 + r) - \bar{\rho}'(t_0 - r)}{2r} &\xrightarrow{r \rightarrow 0} \bar{\rho}''(t_0). \end{aligned}$$

Since both  $r^2 V'(r)$  and  $r V(r)$  converge to 0 as  $r \rightarrow 0$ , we conclude that the second and the third term in the right-hand side of (84) also converge to 0 as  $r \rightarrow 0$ . Lastly, since  $V \in L^1(-1, 1)$  and  $\bar{\rho}'' \in C(0, 1)$ , we obtain that the fourth term in the right-hand side of (84) converges to 0 as  $r \rightarrow 0$ .

In conclusion, by taking the limit  $r \rightarrow 0$  in (84), we obtain that the right-hand side is positive, and the contradiction is reached. Hence, we conclude that  $\bar{\rho}(t) > 0$  for any  $t \in (0, 1)$ .  $\square$

## 6 Extension to confining potentials $U$ on $\mathbb{R}$

As motivated in §1.5, we treat here the case in which either one or both barriers at 0 and 1 are replaced by the assumption that  $U$  has at least linear growth. In §6.1 we establish the equivalent of Theorem 1.6 (given by Theorem 6.6) for the case where both barriers are removed. In §6.2 we discuss how to modify Theorem 6.6 to allow for the inclusion of one or two barriers.

### 6.1 Counterpart of Theorem 1.6 on $\mathbb{R}$

Removing the barriers at 0 and 1 from the energy  $E$  results in

$$\tilde{E} : \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty], \quad \tilde{E}(\rho) = \frac{1}{2} \int_{\mathbb{R}} \tilde{V} * \rho \, d\rho + \int_{\mathbb{R}} \tilde{U} \, d\rho, \quad (85)$$

where  $\tilde{V} = V_a + \tilde{V}_{\text{reg}}$ , with  $V_a$  as in (3) for a fixed  $a \in [0, 1)$ , and

$$\tilde{V}_{\text{reg}} \in \begin{cases} W_{\text{loc}}^{2,1}(\mathbb{R}) & \text{if } 0 < a < 1, \\ W_{\text{loc}}^{2,p_0}(\mathbb{R}) \text{ for some } p_0 > 1 & \text{if } a = 0 \end{cases} \quad (86a)$$

such that

$$\tilde{V} \in L^1(\mathbb{R}), \quad \tilde{V}''(t) \geq 0 \text{ for a.e. } t \in \mathbb{R}, \quad \tilde{V} \text{ even}, \quad (86b)$$

$$\exists c, \varepsilon > 0 \, \forall |\tilde{r}| < \varepsilon : \text{ess inf}_{|\tilde{s}| < |\tilde{r}|} \tilde{V}''(\tilde{s}) \geq \frac{c}{|\tilde{r}|^{2+a}}. \quad (86c)$$

We assume on the external potential  $\tilde{U}$  that

$$\tilde{U} \in \begin{cases} W_{\text{loc}}^{2,1}(\mathbb{R}) & \text{if } 0 < a < 1, \\ W_{\text{loc}}^{2,p_0}(\mathbb{R}) & \text{if } a = 0, \end{cases} \quad (87a)$$

$$\tilde{U} \geq \tilde{U}(0) = 0, \quad \tilde{U}''(t) \geq 0 \text{ for a.e. } t \in \mathbb{R}, \quad \exists 0 < c, C \, \forall \tilde{t} \in \mathbb{R} : \tilde{U}(\tilde{t}) \geq c(|\tilde{t}| - C). \quad (87b)$$

Next we motivate the choice for these assumptions on  $\tilde{V}$  and  $\tilde{U}$ . The condition  $\tilde{V} \in L^1(\mathbb{R})$  in (86b) together with the convexity imply that  $\tilde{V}(\tilde{t}) \geq 0 \geq \tilde{V}'(\tilde{t})$  for all  $\tilde{t} > 0$ , which is similar to the assumptions on  $V$  in (4). Given that  $\tilde{U}$  is a convex, confining potential, the further assumptions in (87) are minimal. Indeed, since the interaction part of  $\tilde{E}$  is translation invariant, it is not restrictive to assume that  $\tilde{U}$  is minimal at  $t = 0$ . Moreover, since adding a constant to  $\tilde{U}$  amounts to adding a constant to  $\tilde{E}$ , it is not restrictive to assume  $\tilde{U}(0) = 0$ .

Next we state the counterpart of Problems 1.1–1.4 for given  $0 \leq a < 1$ .

**Problem 6.1** (Minimisation). *Find the minimiser of  $\tilde{E}$  (defined in (85)) in  $\mathcal{P}(\mathbb{R})$ .*

Similar to (8), we define for any  $\rho \in \mathcal{P}(\mathbb{R})$

$$\tilde{h}_\rho(\tilde{t}) := \int_{\mathbb{R}} \tilde{V}(\tilde{t} - \tilde{s}) \, d\rho(\tilde{s}) + \tilde{U}(\tilde{t}) \in [0, \infty] \quad \text{for all } \tilde{t} \in \mathbb{R}.$$

We note that we cannot construct the space  $H_{\tilde{V}}(\mathbb{R})$  as in §3, because the compact support condition in (29) is not satisfied. We will side-step the construction of  $H_{\tilde{V}}(\mathbb{R})$  by first showing that the solution  $\tilde{\rho}$  to Problem 6.1 has finite support. This property allows us to modify the tails of  $\tilde{V}$  such that (29) is satisfied without losing the minimality of  $\tilde{\rho}$ . Then, by Proposition 3.3 we can identify the related Hilbert space by  $H^{-(1-a)/2}(\mathbb{R})$ , which does not depend on the choice of the regularisation of the tails of  $\tilde{V}$ . This motivates the following problem:

**Problem 6.2** (Variational inequality). *Find  $\rho \in \mathcal{P}(\mathbb{R})$  such that*

$$\int_{\mathbb{R}} \tilde{h}_\rho d\mu \geq \int_{\mathbb{R}} \tilde{h}_\rho d\rho \quad \text{for all } \mu \in \mathcal{P}(\mathbb{R}) \cap H^{-(1-a)/2}(\mathbb{R}). \quad (88)$$

To state Problem 6.3, we make use of the functional setting introduced before Problem 1.3. Furthermore, we denote for functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  the left and right limit at  $\tilde{t} \in \mathbb{R}$ , if it exists, respectively by

$$f(\tilde{t}-) := \lim_{\tilde{s} \uparrow \tilde{t}} f(\tilde{s}), \quad f(\tilde{t}+) := \lim_{\tilde{s} \downarrow \tilde{t}} f(\tilde{s}).$$

**Problem 6.3** (Weakly singular integral equation with free boundary). *Find the solution  $(\rho, t_1, t_2, C)$  with  $C \in \mathbb{R}$ ,  $-\infty < t_1 < t_2 < \infty$ ,  $\rho \in H^{-(1-a)/2}(t_1, t_2)$  and  $\int_{t_1}^{t_2} \rho = 1$  to*

$$\begin{cases} \tilde{h}_\rho(\tilde{t}) = C & \text{a.e. on } (t_1, t_2), \\ \tilde{h}'_\rho(t_1-) \leq 0 \leq \tilde{h}'_\rho(t_2+). \end{cases} \quad (89)$$

We note that it is not restrictive to assume the support of  $\tilde{\rho}$  in Problem 6.3 to be finite. Indeed, if  $\tilde{\rho} \in H^{-(1-a)/2}(\mathbb{R})$  satisfies (89) (without the boundary conditions), then we observe from the following inclusion of levelsets,

$$\text{supp } \rho \subset \overline{\{\tilde{h}_\rho = C\}} \subset \overline{\{\tilde{U} \leq C\}},$$

that  $\text{supp } \rho$  is bounded due to (87).

It is possible to weaken the solution concept of Problem 6.3 by requiring that  $\tilde{h}_\rho(\tilde{t}) = C$  only holds  $\tilde{\rho}$ -a.e. In that case, one needs to put boundary conditions similar to (89) at the endpoints of each open component of  $(\text{supp } \rho)^c$ . Since this is both conceptionally and notationally heavier than Problem 6.3, and since the minimiser in Problem 6.1 turns out to have a connected support, we do not treat this extended solution concept in any more detail.

On the other hand, we can also seek classical solutions to Problem 6.3 in  $C_0^\beta([t_1, t_2])$  for some  $\beta > 0$ . This solution concept coincides with that in [Mus53]. Indeed, in Theorem 6.6 we show that the weak solution to Problem 6.3 coincides with the classical one. The benefit of working with classical solutions is that the boundary conditions in (89) turn into homogeneous Dirichlet boundary conditions (which are already included in the space  $C_0^\beta([t_1, t_2])$ ). We state the classical version of Problem 6.3 in Problem 6.5. We first state Problem 6.4 to remain consistent with the ordering of Problems 1.1–1.4.

Similarly to the extension  $D_{t_1}^a$  in (27) of  $D^a$  to an interval  $[t_1, t_2]$ , we extend the singular integral operator  $S$  defined in (16) by

$$(S_{t_1}^{t_2} f)(\tilde{t}) := \int_{t_1}^{t_2} \frac{f(\tilde{s})}{\tilde{t} - \tilde{s}} d\tilde{s} \quad \text{for all } \tilde{t} \in (t_1, t_2).$$

**Problem 6.4** (Implicit formula). *Find the solution  $(\rho, t_1, t_2, C)$  with  $C \in \mathbb{R}$ ,  $-\infty < t_1 < t_2 < \infty$ ,  $\rho \in \mathcal{M}([t_1, t_2]) \cap H^{-(1-a)/2}(t_1, t_2)$ ,  $\tilde{h}'_\rho(t_1-) \leq 0 \leq \tilde{h}'_\rho(t_2+)$  and  $\rho([t_1, t_2]) = 1$  to*

$$\rho = \begin{cases} \pi^{-2} \Gamma(a) \cos^2\left(\frac{a\pi}{2}\right) \left( \frac{1}{\tilde{\phi}} S_{t_1}^{t_2}(\tilde{\phi} \tilde{g}_\rho) + \pi \tan\left(\frac{a\pi}{2}\right) \tilde{g}_\rho \right) & \text{if } 0 < a < 1 \\ \pi^{-2} \frac{1}{\tilde{\phi}} \left[ S_{t_1}^{t_2}(\tilde{\phi} \tilde{g}_\rho) + \frac{1}{2 \log 2} \int_0^1 \frac{(C - \tilde{V}_{\text{reg}} * \rho - \tilde{U})(\tilde{s})}{\tilde{\phi}(\tilde{s})} d\tilde{s} \right] & \text{if } a = 0 \end{cases} \quad \text{a.e. on } (t_1, t_2), \quad (90)$$

where, for a.e.  $t_1 < \tilde{t} < t_2$ ,

$$\tilde{g}_\rho(\tilde{t}) := D_{t_1}^{1-a}(C - \tilde{V}_{\text{reg}} * \rho - \tilde{U})(\tilde{t}), \quad (91)$$

$$\tilde{\phi}(\tilde{t}) := [(\tilde{t} - t_1)(t_2 - \tilde{t})]^{\frac{1-a}{2}}. \quad (92)$$

**Problem 6.5** (Classical weakly singular integral equation with free boundary). *Find the solution  $(\rho, t_1, t_2, C)$  with  $C \in \mathbb{R}$ ,  $-\infty < t_1 < t_2 < \infty$ ,  $\rho \in C_0^\beta([t_1, t_2])$  for some  $\beta > 0$  and  $\int_{t_1}^{t_2} \rho = 1$  to*

$$\tilde{h}_\rho(\tilde{t}) = C \quad \text{on } [t_1, t_2].$$

With Problems 6.1–6.5 being defined, we can state the second of the two main results in this paper:

**Theorem 6.6** (Equivalence of Problems 6.1–6.5 and properties of the solution). *Let  $0 \leq a < 1$ , let  $\tilde{\phi}$  be as in (92) and  $\tilde{E}$  be as in (85) with the related  $\tilde{V}$ ,  $\tilde{V}_{\text{reg}}$ ,  $\tilde{U}$  and  $1 < p_0 < 2$  defined in (86) and (87). Then all five Problems 6.1–6.5 have a unique solution, and all these solutions are equal. The solution  $(\tilde{\rho}, t_1, t_2, \tilde{C})$  to these problems satisfies*

$$\tilde{\rho} \in C_0^\beta([t_1, t_2]) \quad \text{and} \quad \tilde{\rho} \in W_{\text{loc}}^{\ell, p_a}(t_1, t_2), \quad (93)$$

for some  $\beta > 0$  and any

$$\begin{cases} 1 \leq p_a < \frac{1}{1-a} & \text{if } 0 < a < 1, \\ p_a = p_0 & \text{if } a = 0. \end{cases}$$

where  $1 < p_0 < 2$  is as in (87), and  $\ell \in \mathbb{N}_+$  is such that

$$\tilde{V}_{\text{reg}} \in W^{\ell+1,1}(\mathbb{R}) \quad \text{and} \quad \tilde{U} \in \begin{cases} W_{\text{loc}}^{\ell+1,1}(\mathbb{R}) & \text{if } 0 < a < 1, \\ W_{\text{loc}}^{\ell+1, p_0}(\mathbb{R}) & \text{if } a = 0. \end{cases}$$

Moreover,  $\text{supp } \tilde{\rho} = [t_1, t_2]$  and  $\tilde{C} := \int_{t_1}^{t_2} \tilde{h}_{\tilde{\rho}}(\tilde{t}) \tilde{\rho}(\tilde{t}) d\tilde{t}$ . Furthermore,  $(\tilde{\rho}, \tilde{C})$  satisfies (90) everywhere on  $[t_1, t_2]$ . Finally, if  $\ell \geq 3$ , then  $\tilde{\rho} > 0$  on  $(t_1, t_2)$ .

We employ Theorem 1.6 to prove Theorem 6.6. However, Theorem 1.6 relies on the assumption given by (7) to ensure that the support of  $\bar{\rho}$  contains the barriers at 0 and 1. This assumption does not fit to the setting of Theorem 6.6. Therefore, we first establish an alternative assumption to (7) under which Theorem 1.6 holds:

**Lemma 6.7** (Alternative assumption for Theorem 1.6). *Given the setting of Theorem 1.5, let  $\bar{\rho}$  be the solution to Problems 1.1 and 1.2. If  $\{0, 1\} \subset \text{supp } \bar{\rho}$ , then Theorem 1.6 also holds when (7) is not satisfied.*

*Proof.* We note that (7) is solely used in (77) in Step 1 of the proof of Theorem 1.6, which treats those intervals  $(r, s) \subset A = \{h_{\bar{\rho}} > m\}$  in Step 1 for which either  $r = 0$  or  $s = 1$ . Since  $\bar{\rho}$  contains no atoms and  $\{0, 1\} \in \text{supp } \bar{\rho}$ , we obtain that  $\bar{\rho}((r, s)) > 0$  if  $r = 0$  or  $s = 1$ . This contradicts with (75), and thus the argument in the third out of the three cases containing (77) can be omitted.  $\square$

*Proof of Theorem 6.6. Step 1: Problem 6.1 has a unique solution  $\tilde{\rho}$ .* The proof of [GPPS13, Thm. 2], with minor adaptations, implies that Problem 6.1 has a unique solution. Moreover, the singularity of  $\tilde{V}$  implies that  $\tilde{\rho}$  cannot have atoms, and thus

$$\int_{[s_1, s_2]} f d\tilde{\rho} = \int_{(s_1, s_2)} f d\tilde{\rho}$$

for all  $s_1 < s_2$  and all non-negative,  $\tilde{\rho}$ -measurable functions  $f$ .

*Step 2:  $\text{supp } \tilde{\rho}$  is bounded.* The argument in [MRS16, §2.2], with minor adaptations, implies that  $\text{supp } \tilde{\rho}$  is bounded. We set  $t_1$  and  $t_2$  respectively as the minimum and maximum of  $\text{supp } \tilde{\rho}$ , and put  $L := t_2 - t_1 > 0$ .

*Step 3: translation to Theorem 1.6.* Let  $t_1$  and  $t_2$  be as in Step 2, and let  $\mathcal{T}$  (defined in (65)) be the corresponding affine map for the coordinate transformation.

*Step 3a:  $0 < a < 1$ .* Changing coordinates, we rewrite

$$\begin{aligned} \tilde{E}(\tilde{\rho}) &= \frac{1}{2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} \tilde{V}(\tilde{t} - \tilde{s}) d\tilde{\rho}(\tilde{s}) d\tilde{\rho}(\tilde{t}) + \int_{t_1}^{t_2} \tilde{U}(\tilde{t}) d\tilde{\rho}(\tilde{t}) \\ &= \frac{1}{2} \int_0^1 \int_0^1 \tilde{V}(L(t - s)) d(\mathcal{T}_\# \tilde{\rho})(s) d(\mathcal{T}_\# \tilde{\rho})(t) + \int_0^1 \tilde{U}(\mathcal{T}^{-1}(t)) d(\mathcal{T}_\# \tilde{\rho})(t). \end{aligned} \quad (94)$$

Since  $\tilde{V}(Lt) = V_a(Lt) + \tilde{V}_{\text{reg}}(Lt) = L^{-a}V_a(t) + \tilde{V}_{\text{reg}}(Lt)$ , we can further rewrite (94) as

$$\begin{aligned} \tilde{E}(\tilde{\rho}) &= L^{-a} \left[ \frac{1}{2} \int_0^1 \int_0^1 [V_a(t - s) + L^a \tilde{V}_{\text{reg}}(L(t - s))] d(\mathcal{T}_\# \tilde{\rho})(s) d(\mathcal{T}_\# \tilde{\rho})(t) \right. \\ &\quad \left. + \int_0^1 L^a \tilde{U}(Lt + t_1) d(\mathcal{T}_\# \tilde{\rho})(t) \right] = L^{-a} [E(\mathcal{T}_\# \tilde{\rho}) + c], \end{aligned} \quad (95)$$

where  $E$  is as in (1) with

$$V_{\text{reg}}(t) := L^a \tilde{V}_{\text{reg}}(Lt) - 2c, \quad U(t) := L^a \tilde{U}(Lt + t_1), \quad (96)$$

and  $c \in \mathbb{R}$  is a constant such that  $V(1) = -V'(1)$ .

*Step 3b:  $a = 0$ .* We argue analogously to Step 3a. Since  $\tilde{V}(Lt) = V_0(t) - \log L + \tilde{V}_{\text{reg}}(Lt)$ , the corresponding computation in (94) and (95) yields

$$\tilde{E}(\tilde{\rho}) = E(\mathcal{T}_\# \tilde{\rho}) + c - \frac{1}{2} \log L, \quad (97)$$

where  $E$  is as in (1) with

$$V_{\text{reg}}(t) := \tilde{V}_{\text{reg}}(Lt) - 2c \quad \text{and} \quad U(t) := \tilde{U}(Lt + t_1),$$

and  $c \in \mathbb{R}$  is a constant such that  $V(1) = -V'(1)$ . Since these definitions are consistent with putting  $a = 0$  in (95) and (96) (except for an additive constant to  $\tilde{E}$ ), we refer in the remainder of the proof to (95) and (96) for all  $0 \leq a < 1$ .

*Step 4: Properties of  $\tilde{\rho}$  for given  $-\infty < t_1 < t_2 < \infty$ .* Since  $\tilde{\rho}$  is the unique minimiser of  $\tilde{E}$ , it follows from (95) that  $E$  also has a unique minimiser, which is given by  $\bar{\rho} := \mathcal{T}_\# \tilde{\rho}$ . It is easy to see that the transformations in (96) transfer the properties of  $\tilde{V}$ ,  $\tilde{V}_{\text{reg}}$  and  $\tilde{U}$  in (86) and (87) to those listed in (4) and (5). Moreover, by Step 2 we obtain that  $0, 1 \in \text{supp } \bar{\rho}$ . Hence, Lemma 6.7 implies that Theorem 1.6 applies to  $\bar{\rho}$ . Thus, by using that  $\bar{\rho} = \mathcal{T}_\# \tilde{\rho}$  with  $\mathcal{T} : \mathbb{R} \rightarrow \mathbb{R}$  invertible, we obtain that the combination of  $\tilde{\rho} = (\mathcal{T}^{-1})_\# \bar{\rho}$  and  $\tilde{C} = \int_{\mathbb{R}} h_{\tilde{\rho}} d\tilde{\rho}$  is the unique solution to the problems:



- find the solution  $\rho \in \mathcal{P}([t_1, t_2]) \cap H^{-(1-a)/2}(t_1, t_2)$  to the variational inequality

$$\int_{\mathbb{R}} \tilde{h}_\rho d\mu \geq \int_{\mathbb{R}} \tilde{h}_\rho d\rho \quad \text{for all } \mu \in \mathcal{P}([t_1, t_2]) \cap H^{-(1-a)/2}(t_1, t_2);$$

- find the solution  $(\rho, C)$  with  $C \in \mathbb{R}$ ,  $\rho \in \mathcal{P}(\mathbb{R}) \cap H^{-(1-a)/2}(t_1, t_2)$  and  $\text{supp } \rho = [t_1, t_2]$  to

$$\tilde{h}_\rho(\tilde{t}) = C \quad \rho\text{-a.e.};$$

- Find the solution  $(\rho, C)$  with  $C \in \mathbb{R}$ ,  $\rho \in \mathcal{P}(\mathbb{R}) \cap H^{-(1-a)/2}(t_1, t_2)$  and  $\text{supp } \rho = [t_1, t_2]$  to (90).

We note that  $t_1$  and  $t_2$  are fixed by Step 2. Moreover,  $\tilde{\phi}\tilde{\rho}$  is Hölder continuous on  $[t_1, t_2]$ ,  $\tilde{\rho}$  satisfies the second of the two regularity properties in (93) and  $\text{supp } \tilde{\rho} = [t_1, t_2]$ . Furthermore,  $(\tilde{\rho}, \tilde{C})$  satisfies (89) and (90) everywhere on  $[t_1, t_2]$  and  $(t_1, t_2)$  respectively (in Step 8 below we prove that the boundary points  $\{t_1, t_2\}$  can be included). Lastly, if  $\ell \geq 3$ , then  $\tilde{\rho} > 0$  on  $(t_1, t_2)$ .

*Step 5:  $\tilde{\rho}$  is the unique solution of Problem 6.2.* First, we show that  $\tilde{\rho}$  satisfies Problem 6.2. We use a different dilation map than  $\mathcal{T}$  on a larger interval  $(r_1, r_2) \supset (t_1, t_2)$  such that

$$\tilde{U}(r_1) \wedge \tilde{U}(r_2) \geq \tilde{C} \quad \text{and} \quad \tilde{U}'(r_1) < 0 < \tilde{U}'(r_2). \quad (98)$$

We set  $\mathcal{R}(\tilde{t}) := (\tilde{t} - r_1)/(r_2 - r_1)$  as the dilation map. Then, as in Step 3, we introduce

$$\tilde{E}(\tilde{\rho}) = (r_2 - r_1)^{-a} (E^*(\mathcal{R}_\# \tilde{\rho}) + c),$$

where  $E^*$  is as in (1) with

$$V_{\text{reg}}^*(t) := \tilde{V}_{\text{reg}}((r_2 - r_1)t) - 2c \quad \text{and} \quad U^*(t) := \tilde{U}((r_2 - r_1)t + r_1),$$

and  $c \in \mathbb{R}$  is such that  $V^*(1) = -(V^*)'(1)$ . Again, we find that  $\rho^* := \mathcal{R}_\# \tilde{\rho}$  is the unique minimiser of  $E^*$  and that  $V_{\text{reg}}^*$  and  $U^*$  satisfy (4) and (5). Hence, Theorem 1.5 applies to  $E^*$ , and thus  $\rho^*$  satisfies the related Problem 1.2. Hence,  $\tilde{\rho} := \mathcal{R}_\#^{-1} \rho^*$  satisfies

$$\int_{r_1}^{r_2} \tilde{h}_{\tilde{\rho}} d\mu \geq \int_{\mathbb{R}} \tilde{h}_{\tilde{\rho}} d\tilde{\rho} \quad \text{for all } \mu \in \mathcal{P}([r_1, r_2]) \cap H^{-(1-a)/2}(r_1, r_2). \quad (99)$$

Since  $\text{supp } \tilde{\rho} = [t_1, t_2] \subset (r_1, r_2)$  and  $\tilde{h}_{\tilde{\rho}} = \tilde{C}$  on  $[t_1, t_2]$ , we obtain from (99) that  $\tilde{h}_{\tilde{\rho}} \geq \tilde{C}$  a.e. on  $[r_1, r_2]$ . Moreover, by (98) we have for all  $\tilde{t} \in [r_1, r_2]^c$  that

$$\tilde{h}_{\tilde{\rho}}(\tilde{t}) \geq \tilde{U}(\tilde{t}) > \tilde{U}(r_1) \wedge \tilde{U}(r_2) \geq \tilde{C}.$$

Hence,  $\tilde{\rho}$  satisfies Problem 6.2.

Next we prove the uniqueness of solutions to Problem 6.2. We will use a sequence of approximating energies

$$\tilde{E}_k(\rho) = \frac{1}{2} \int_{\mathbb{R}} \tilde{V}_k * \rho d\rho + \int_{\mathbb{R}} \tilde{U} d\rho, \quad k \in \mathbb{N}_+,$$

where the potentials  $\tilde{V}_k : \mathbb{R} \rightarrow [0, \infty]$  are chosen such that  $\tilde{V}_k$  satisfies (29),  $\tilde{V}_k|_{(-k, k)} = \tilde{V}|_{(-k, k)}$ , and  $\tilde{V}_k \leq \tilde{V}_{k+1} \leq \tilde{V}$  for all  $k \in \mathbb{N}_+$ . By the Monotone Convergence Theorem, it is easy to see that  $\tilde{E}_k(\rho) \rightarrow \tilde{E}(\rho)$  as  $k \rightarrow \infty$  pointwise for all  $\rho \in \mathcal{P}(\mathbb{R})$ .

To prove the uniqueness of solutions to Problem 6.2, let  $\rho \in \mathcal{P}(\mathbb{R}) \cap H^{-(1-a)/2}(\mathbb{R})$  be a solution. Using (88) with  $\mu = \tilde{\rho}$ , we obtain by the Monotone Convergence Theorem that

$$0 \leq \int_{\mathbb{R}} \tilde{h}_{\rho} d(\tilde{\rho} - \rho) = \lim_{k \rightarrow \infty} \int_{\mathbb{R}} (\tilde{V}_k * \rho + \tilde{U}) d(\tilde{\rho} - \rho).$$

Since  $\tilde{V}_k$  satisfies (29) for any  $k \in \mathbb{N}_+$ , we can apply Corollary 3.4 to  $\tilde{V}_k$  to obtain

$$\begin{aligned} \int_{\mathbb{R}} (\tilde{V}_k * \rho + \tilde{U}) d(\tilde{\rho} - \rho) &= (\rho, \tilde{\rho} - \rho)_{\tilde{V}_k} + \int_{\mathbb{R}} \tilde{U} d(\tilde{\rho} - \rho) \\ &\leq \frac{1}{2}(\tilde{\rho}, \tilde{\rho})_{\tilde{V}_k} - \frac{1}{2}(\rho, \rho)_{\tilde{V}_k} + \int_{\mathbb{R}} \tilde{U} d(\tilde{\rho} - \rho) = \tilde{E}_k(\tilde{\rho}) - \tilde{E}_k(\rho) \xrightarrow{k \rightarrow \infty} \tilde{E}(\tilde{\rho}) - \tilde{E}(\rho). \end{aligned}$$

By Step 1, we conclude  $\rho = \tilde{\rho}$ .

*Step 6:*  $(\tilde{\rho}, t_1, t_2, \tilde{C})$  satisfies Problems 6.3 and 6.4. By Step 4 it is enough to show that  $(\tilde{\rho}, \tilde{C})$  satisfies the boundary conditions in Problems 6.3 and 6.4 at  $t_1$  and  $t_2$ . We focus on proving  $\tilde{h}'_{\tilde{\rho}}(t_1-) \leq 0$ ; the boundary condition at  $t_2$  follows from a similar argument.

By the argument in Step 1 of the proof of Theorem 1.6 it follows that  $\tilde{h}_{\tilde{\rho}} \in C(\mathbb{R}) \cup C^1((-\infty, t_1))$  and  $\tilde{h}_{\tilde{\rho}}'' \geq 0$  on  $(-\infty, t_1)$ . Moreover, by the Monotone Convergence Theorem, we obtain that  $\tilde{h}'_{\tilde{\rho}}(t_1-)$  is well-defined as a value in  $(-\infty, \infty]$ . Since  $\tilde{h}_{\tilde{\rho}} \geq \tilde{C}$  on  $\mathbb{R}$  and  $\tilde{h}_{\tilde{\rho}}(t_1) = \tilde{C}$ , it must hold that  $\tilde{h}'_{\tilde{\rho}}(t_1-) \leq 0$ .

*Step 7:* Both Problems 6.3 and 6.4 have a unique solution. We start with Problem 6.3. Let  $(\rho, s_1, s_2, C)$  be a solution to Problem 6.3. We prove that  $\rho$  is a solution to Problem 6.2, and conclude by the statement of Step 5. We redefine  $\mathcal{R}$  from Step 5 with  $r_i = s_i$ , and obtain, analogously to Steps 3 and 4, that  $(\mathcal{R}_{\#}\rho, (s_2 - s_1)^a C - 2c)$  equals the solution  $(\bar{\rho}, \bar{C})$  to Problem 1.3. In particular, from  $\mathcal{R}_{\#}\rho = \bar{\rho} \geq 0$  we obtain  $\rho \geq 0$ . Similar to (76), we deduce that  $\tilde{h}_{\rho}'' \geq 0$  on  $[s_1, s_2]^c$ . Together with the boundary condition  $\tilde{h}'_{\rho}(s_1-) \leq 0 \leq \tilde{h}'_{\rho}(s_2+)$ , we conclude that  $\tilde{h}_{\rho} \geq C$  on  $\mathbb{R}$ , and thus  $\rho$  is a solution to Problem 6.2.

Next we treat Problem 6.4. Let  $(\rho, s_1, s_2, C)$  be a solution to Problem 6.4. By the previous paragraph, it is enough to show that  $(\rho, s_1, s_2, C)$  satisfies Problem 6.3. We follow the argument above to conclude that  $(\mathcal{R}_{\#}\rho, (s_2 - s_1)^a C - 2c)$  satisfies Problem 1.4. From Step 6 in the proof of Theorem 1.6 it then follows that  $(\mathcal{R}_{\#}\rho, (s_2 - s_1)^a C - 2c)$  also satisfies Problem 1.3. Translating back to the interval  $[s_1, s_2]$  and recalling the boundary conditions in Problem 6.4, we conclude that  $(\rho, s_1, s_2, C)$  satisfies Problem 6.3.

*Step 8:*  $\tilde{\rho} \in C_0^{\beta}([t_1, t_2])$ . Let  $(\tilde{\rho}, t_1, t_2, \tilde{C})$  be the solution to Problem 6.4. As in Step 3, we use  $\mathcal{T}$  to dilate  $\tilde{\rho}$  to the interval  $[0, 1]$  by  $\bar{\rho} := \mathcal{T}_{\#}\tilde{\rho}$ , and set  $E, V, V_{\text{reg}}$  and  $U$  accordingly. We recall from Step 4 that  $\bar{\rho}$  satisfies all statements in Theorem 1.6. Moreover, the boundary condition in Problem 6.4 translates to

$$h'_{\bar{\rho}}(0-) \leq 0 \leq h'_{\bar{\rho}}(1+). \quad (100)$$

Hence, it is enough to show that  $\bar{\rho} = \mathcal{T}_{\#}\tilde{\rho} \in C_0^{\beta}([0, 1])$  for some  $\beta > 0$ . We obtain this regularity by analysing the implicit expression (17) with  $\phi$  and  $g_{\bar{\rho}}$  as in (14) and (18). We also use that (100) implies

$$\liminf_{t \downarrow 0} \bar{\rho}(t) = 0 \quad \text{and} \quad \liminf_{t \uparrow 1} \bar{\rho}(t) = 0. \quad (101)$$

*Step 8a:*  $a = 0$ . Starting from (17), we recall from Step 7 of the proof of Theorem 1.6 that  $g_{\bar{p}} \in W^{1,p_0}(0,1) \subset C^{1-1/p_0}([0,1])$ . Hence,  $g_{\bar{p}}/\phi \in wL^2(0,1)$ , and thus we can use Proposition 2.2.(iv) to rewrite

$$S(\phi(t)g_{\bar{p}}(t)) = S\left(t(1-t)\frac{g_{\bar{p}}(t)}{\phi(t)}\right) = t(1-t)S\left(\frac{g_{\bar{p}}(t)}{\phi(t)}\right) + \int_0^1 (t+s-1)\frac{g_{\bar{p}}(s)}{\phi(s)} ds.$$

Substituting this expression in (17), we obtain

$$\pi^2 \bar{p}(t) = \phi(t)S\left(\frac{g_{\bar{p}}}{\phi}\right)(t) + \frac{C_2 t + C_1}{\phi(t)}$$

for some constants  $C_1, C_2 \in \mathbb{R}$ . Since  $g_{\bar{p}} \in C^{1-1/p_0}([0,1])$ , we can use Proposition 2.3.(iii) to rewrite the right-hand side as

$$\pi^2 \bar{p}(t) = \psi(t) + \frac{C_2 t + C_1}{\phi(t)}$$

for some  $\psi \in C_0^\beta([0,1])$  with some  $0 < \beta < 1$ . From (101) we obtain  $C_1 = C_2 = 0$ , and thus  $\bar{p} \in C_0^\beta([0,1])$  for some  $\beta > 0$ .

*Step 8b:*  $0 < a < 1$ . We employ a similar argument as in Step 8a. However, the computations are more involved, because  $g_{\bar{p}}$  may not be Hölder continuous. We proceed as in Step 7 of the proof of Theorem 1.6 with  $\ell = 1$ ; we set  $q = f_{\bar{p}}(0)$  and note that  $\hat{g}_{\bar{p}} := g_{\bar{p}} - D^{1-a}q$  is Hölder continuous. We write (17) as

$$C_1 \bar{p} = \pi \tan\left(\frac{a\pi}{2}\right) \hat{g}_{\bar{p}} + \frac{1}{\phi} S(\phi \hat{g}_{\bar{p}}) + \pi \tan\left(\frac{a\pi}{2}\right) D^{1-a}q + \frac{1}{\phi} S(\phi D^{1-a}q), \quad (102)$$

where  $C_1 > 0$  is an explicit constant.

Next we show that the right-hand side in (102) is continuous up to the boundary points of  $[0,1]$  with value 0. For the last two terms we recall from the last part of the proof of Theorem 4.1 that they equal  $C_2/\phi(t)$  for some constant  $C_2 \in \mathbb{R}$ . We rewrite the first two terms similarly as in Step 8a. Setting  $\phi^*(t) := t(1-t)/\phi(t) = [t(1-t)]^{(1+a)/2}$ , we compute

$$\begin{aligned} \frac{S(\phi \hat{g}_{\bar{p}})(t)}{\phi(t)} &= \phi^*(t)S\left(\frac{\hat{g}_{\bar{p}}}{\phi^*}\right)(t) + \frac{C_4 t + C_3}{\phi(t)} \\ &= -\pi \tan\left(\frac{a\pi}{2}\right) \left( \hat{g}_{\bar{p}}(1)t^{\frac{1+a}{2}} - \hat{g}_{\bar{p}}(0)(1-t)^{\frac{1+a}{2}} \right) + \psi_1(t) + \frac{C_4 t + C_3}{\phi(t)} \end{aligned}$$

for some constants  $C_3, C_4 \in \mathbb{R}$  and some function  $\psi_1 \in C_0^\beta([0,1])$  with  $0 < \beta < 1$ . Hence, the first two terms in the right-hand side of (102) equal  $\psi_2(t) + (C_4 t + C_3)/\phi(t)$  for some  $\psi_2 \in C_0^\beta([0,1])$  with a possibly smaller value for  $\beta > 0$ .

In conclusion, we can rewrite (102) as

$$C_1 \bar{p}(t) = \psi_2(t) + \frac{C_4 t + C_2 + C_3}{\phi(t)}.$$

From (101) we obtain  $C_4 = C_2 + C_3 = 0$ , and thus  $\bar{p} \in C_0^\beta([0,1])$ .

As a corollary of Step 8, we obtain that (90) also holds at the boundary points  $\{t_1, t_2\}$ .

*Step 9:*  $(\tilde{\rho}, t_1, t_2, \tilde{C})$  is the unique solution to Problem 6.5. In the previous steps we have already shown that  $(\tilde{\rho}, t_1, t_2, \tilde{C})$  satisfies Problem 6.5. To prove the uniqueness, we take any solution  $(\rho, s_1, s_2, C)$  to Problem 6.5, and follow the argument in Step 7. The only difference with Step 7 is that we can rely no more on the boundary conditions in Problem 6.3 to prove that  $\tilde{h}_\rho \geq C$  on  $\mathbb{R}$ . Instead, we obtain from  $\rho \in C_0^\beta([s_1, s_2])$  that  $\tilde{h}_\rho \in C(\mathbb{R})$ . Moreover, from  $\rho \geq 0$  with  $\text{supp } \rho \subset [s_1, s_2]$  we obtain, similar to the proof of Step 1 in Theorem 1.6, that  $\tilde{h}'_\rho \leq 0$  on  $(-\infty, s_1)$ . Hence,  $\tilde{h}_\rho \geq C$  on  $(-\infty, s_1)$ . A similar argument shows that  $\tilde{h}_\rho \geq C$  on  $(s_2, \infty)$ , and thus  $\tilde{h}_\rho \geq C$  on  $\mathbb{R}$ . Hence,  $\rho$  satisfies Problem 6.2, which completes the proof.  $\square$

## 6.2 One or two barriers

The setting of Theorem 6.6 does not include barriers, while several applications do (see §7). In this section we sketch how Theorem 6.6 can be extended to allow for the presence of barriers *without* an assumption such as (7) which guarantees that the support of the minimiser stretches until the barriers. With this aim, we extend the definition of  $\tilde{E}$  in (85) to closed intervals  $[T_1, T_2]$  with barriers as

$$\tilde{E}_{[T_1, T_2]} : \mathcal{P}([T_1, T_2]) \rightarrow [0, \infty], \quad \tilde{E}_{[T_1, T_2]} := \tilde{E}|_{\mathcal{P}([T_1, T_2])}, \quad (103)$$

where  $-\infty \leq T_1 < T_2 \leq \infty$  are given. We interpret  $T_1 = -\infty$  as the scenario with no barrier at the left end. Similarly, the choice  $T_2 = \infty$  corresponds to the absence of a barrier at the right end. In particular, we note that  $\tilde{E}_{\mathbb{R}} = \tilde{E}$  and  $\tilde{E}_{[0, 1]} = E$  as in (1).

The key to the extension of Theorem 6.6 to  $\tilde{E}_{12} := \tilde{E}_{[T_1, T_2]}$  is given by the combination of Lemma 6.7 and Step 1 of the proof of Theorem 1.6. To see this, let  $(\tilde{\rho}_{12}, t_1, t_2, \tilde{C}_{12})$  be the minimiser of  $\tilde{E}_{12}$  (see Theorem 1.5 and Steps 1 and 2 of the proof of Theorem 6.6 for the existence, uniqueness and compactness of the support of the minimiser  $\tilde{\rho}_{12}$ ). While  $T_1 \leq t_1 < t_2 \leq T_2$ , it depends on the choice of  $V$  and  $U$  whether equality holds, and thus we cannot apply Lemma 6.7 directly to obtain that  $\tilde{\rho}_{12}$  satisfies the properties in Theorem 1.6. We separate four cases:

1.  $T_1 = t_1$  and  $t_2 = T_2$ . As in Step 3 of the proof of Theorem 6.6, we consider  $\mathcal{T}_\# \tilde{\rho}_{12}$ . We note that  $\{0, 1\} \in \text{supp}(\mathcal{T}_\# \tilde{\rho}_{12})$ , and thus Lemma 6.7 applies to  $\mathcal{T}_\# \tilde{\rho}_{12}$ . Then, with  $E$  as in (97) and  $\bar{\rho}$  as in Theorem 1.6, we obtain that  $\tilde{\rho}_{12}$  satisfies similar properties through the translation  $\tilde{\rho}_{12} = (\mathcal{T}^{-1})_\# \bar{\rho}$ ;
2.  $T_1 < t_1$  and  $t_2 < T_2$ . We claim that by the displacement convexity of the energy  $\tilde{E}$ ,  $\tilde{\rho}_{12}$  equals the minimiser  $\tilde{\rho}$  of  $\tilde{E}$ . Then, Theorem 6.6 applies with  $\tilde{\rho}$  replaced by  $\tilde{\rho}_{12}$ . We sketch the proof of this claim below this list;
3.  $T_1 = t_1$  and  $t_2 < T_2$ . Again, by the displacement convexity of the energy  $\tilde{E}$ , we obtain that  $\tilde{\rho}_{12}$  is also the minimiser of  $\tilde{E}_1 := \tilde{E}_{[T_1, \infty)}$ . While  $\tilde{E}_1$  is not directly covered by Theorem 1.6 and Theorem 6.6, it is not difficult to see that both proofs can be combined. Indeed, in the interior of  $\text{supp } \tilde{\rho}_{12}$  the proofs are the same, and the two endpoints can be treated independently of each other. In this manner, we obtain that  $\tilde{\rho}_{12}$  satisfies all properties of Theorem 6.6 applied to  $\tilde{E}_1$ , except for the Hölder continuity property in (93) around  $T_1$ . Also, the boundary conditions in Problems 6.3 and 6.4 at  $T_1$  need to be replaced with those in Problems 1.3 and 1.4, and in Problem 6.5 the Hölder continuity around  $T_1$  needs to be replaced with (19a) (we refer to [Mus53] for a detailed treatment of classical solutions with singularities at the boundary points);

4.  $T_1 < t_1$  and  $t_2 = T_2$ . Analogously to Case 3, we find that  $\tilde{\rho}_{12}$  is Hölder continuous at  $t_1$  and that its possible singularity at  $T_2$  can be controlled as in (19a).

It remains to prove the claim on the displacement convexity of  $\tilde{E}$ . In [GPPS13, Proof of Thm. 2] it is proven that  $\tilde{E}$  is convex along geodesics in the Wasserstein space  $(\mathcal{P}_2(\mathbb{R}), W_2)$ . We refer to a detailed treatment of this kind of convexity and Wasserstein spaces to [AGS08]. In our one-dimensional setting, the geodesics in  $(\mathcal{P}_2(\mathbb{R}), W_2)$  have a convenient representation (see, e.g., [Vil03, (2.52)]). In particular, it follows that along the geodesic  $\mu_\alpha$  which connects two measures  $\mu_0$  and  $\mu_1$  with compact support, the two endpoints of  $\text{supp } \mu_\alpha$  vary continuously and monotonically for  $0 \leq \alpha \leq 1$ . Thus, in Case 2 above, we can reason by contradiction; if  $\tilde{\rho}_{12} \neq \tilde{\rho}$ , then the first part of the geodesic connecting  $\tilde{\rho}_{12}$  to  $\tilde{\rho}$  remains in  $\mathcal{P}([T_1, T_2])$ , which contradicts the minimality of  $\tilde{\rho}_{12}$  of  $\tilde{E}_{12}$ . A similar reasoning can be applied to complete the corresponding convexity argument in Case 3.

While the approach above covers all cases, it can in practice be difficult to determine for given  $\tilde{E}_{12}$  which of the four cases is satisfied. We claim that  $\tilde{U}'(T_1) \geq 0$  and  $t_2 < T_2$  is a sufficient condition for the conclusion of Case 3. Indeed, with  $\tilde{U}'(T_1) \geq 0$  we can complete the argument in Step 1 of the proof of Theorem 1.6 to deduce that  $T_1 \in \text{supp } \tilde{\rho}$ . Then, we rely on  $t_2 < T_2$  to identify  $\tilde{\rho}_{12}$  as the minimiser of  $\tilde{E}_1$ . Similarly,  $\tilde{U}'(T_2) \leq 0$  and  $T_1 < t_1$  is a sufficient condition for the conclusion of Case 4 to hold.

## 7 Applications of Theorem 1.6 and Theorem 6.6

In this section we apply Theorem 1.6 and Theorem 6.6 to improve previous results in the literature, including all three examples mentioned in the introduction. We start from the general form of the energy  $E_{[T_1, T_2]}$  as in (103), where we remove the tildes for convenience, and use the notation as in Theorem 1.6. In §7.1 we evaluate explicitly the integrals in the expression for the minimiser  $\bar{\rho} \in \mathcal{P}([T_1, T_2])$  in Problems 1.4 and 6.4 for specific choices of  $V$  and  $U$ . In §7.2 we strengthen the results of [GPPS13] and [SS15].

### 7.1 Explicit formulas for $\bar{\rho}$

When  $V_{\text{reg}} = 0$  in  $E_{[T_1, T_2]}$ , Problems 1.4 and 6.4 provide an explicit expression for its minimiser  $(\bar{\rho}, t_1, t_2, \bar{C})$ . In this section we simplify this expression for several examples in the literature. This includes an example from [HCO10] for which the explicit expression for  $\bar{\rho}$  was not known yet.

The case  $a = 0$  is studied in [HL55] in the context of dislocation densities. For  $U = 0$  and  $[T_1, T_2] = [0, 1]$  it is found that  $[t_1, t_2] = [0, 1]$  and

$$\bar{\rho}(t) = \frac{1}{\pi \sqrt{t(1-t)}},$$

and for  $U(t) = t$  with  $[T_1, T_2] = [0, \infty)$  it is found that  $[t_1, t_2] = [0, 2]$  with

$$\bar{\rho}(t) = \frac{1}{\pi} \sqrt{\frac{2-t}{t}}.$$

We omit showing that both formulas also follow from the explicit expressions in Problems 1.4 and 6.4, and from the discussion in §6.2.

In the case  $0 < a < 1$ , the easiest setting is given by  $[T_1, T_2] = [0, 1]$  and  $U = 0$ . With (51) in the proof of Theorem 4.1 we have shown that  $[t_1, t_2] = [0, 1]$  and

$$\bar{\rho}(t) = \frac{\bar{C}}{\pi} \cos\left(\frac{a\pi}{2}\right) [t(1-t)]^{-\frac{1-a}{2}}.$$

To find  $\bar{C}$ , we use (22) to compute

$$1 = \int_0^1 \bar{\rho} = \frac{\bar{C}}{\pi} \cos\left(\frac{a\pi}{2}\right) \int_0^1 [t(1-t)]^{-\frac{1-a}{2}} dt = \frac{\bar{C}}{\pi} \cos\left(\frac{a\pi}{2}\right) \frac{\Gamma(\frac{1+a}{2})^2}{a\Gamma(a)}.$$

We conclude that

$$\bar{\rho}(t) = \frac{a\Gamma(a)}{\Gamma(\frac{1+a}{2})^2} [t(1-t)]^{-\frac{1-a}{2}} \quad \text{and} \quad \bar{C} = \frac{a\pi\Gamma(a)}{\Gamma(\frac{1+a}{2})^2 \cos(\frac{a\pi}{2})}.$$

Next we consider the setting in [HCO10]:  $0 < a < 1$ ,  $[T_1, T_2] = [0, \infty)$  and  $U(t) = \gamma t$  with  $\gamma > 0$ . In this setting, an explicit formula for  $\bar{\rho}$  was still missing. Here we derive this formula (see (106)). Since  $U'(T_1) = \gamma \geq 0$  and  $T_2 = \infty$ , we obtain from the argument in the last paragraph of §6.2 that  $t_1 = 0$  and that  $\bar{\rho}$  satisfies the conclusion from Case 3 in §6.2. We use this property to solve  $(\bar{\rho}, t_2, \bar{C})$  from the related version of Problem 6.4.

Leaving  $t_2$  and  $\bar{C}$  unknown, we compute  $\bar{\rho}$  from (90). With this aim, we compute

$$g_{\bar{\rho}}(t) = D^{1-a}(\bar{C} - \gamma t) = \frac{1}{\Gamma(a)} \frac{d}{dt} \int_0^t \frac{\bar{C} - \gamma s}{(t-s)^{1-a}} ds = \frac{\bar{C}t^{-(1-a)} - \frac{\gamma}{a}t^a}{\Gamma(a)}.$$

Then, we obtain from (90) that

$$\bar{\rho}(t) = \frac{\gamma \cos^2(\frac{a\pi}{2})}{a\pi^2} \left[ \int_0^{t_2} \frac{\phi(s)}{\phi(t)} \frac{\frac{a}{\gamma} \bar{C} s^{-(1-a)} - s^a}{t-s} ds + \pi \tan(\frac{a\pi}{2}) \left( \frac{a}{\gamma} \frac{\bar{C}}{t^{1-a}} - t^a \right) \right], \quad (104)$$

where  $\phi(t) = [t(t_2 - t)]^{(1-a)/2}$ . The contribution in the right-hand side of the terms involving the constant  $\bar{C}$  are computed in the previous example, and result in  $\bar{C} \cos(\frac{a\pi}{2})/(\pi\phi)$ . To solve the remaining Cauchy-integral, we use formulas [Kin09b, (12A.24)], [GR07, (9.100), (9.101)] and (21) to compute

$$\begin{aligned} \int_0^{t_2} \frac{\phi(s)}{\phi(t)} \frac{-s^a}{t-s} ds &= \frac{-1}{\phi(t)} \int_0^{t_2} \frac{s^{\frac{1+a}{2}} (t_2 - s)^{\frac{1-a}{2}}}{t-s} ds \\ &= \Gamma\left(\frac{1+a}{2}\right) \Gamma\left(\frac{1-a}{2}\right) \frac{\frac{1-a}{2} t_2 - t}{\phi(t)} + \pi \tan\left(\frac{a\pi}{2}\right) \frac{(t_2 - t)^{\frac{1-a}{2}} t^{\frac{1+a}{2}}}{\phi(t)} \\ &= \frac{\pi}{\cos(\frac{a\pi}{2})} \frac{\frac{1-a}{2} t_2 - t}{\phi(t)} + \pi \tan\left(\frac{a\pi}{2}\right) t^a. \end{aligned}$$

Substituting these findings in (104), we obtain

$$\bar{\rho}(t) = \frac{\gamma \cos(\frac{a\pi}{2})}{a\pi} \frac{\bar{C} + \frac{1-a}{2} t_2 - t}{\phi(t)}. \quad (105)$$

Next we compute  $\overline{C}$ . From (93) we obtain  $\overline{\rho}(t_2) = 0$ . Then, we obtain from (105) that  $\overline{C} = \frac{1+a}{2}t_2$ , and thus

$$\overline{\rho}(t) = \frac{\gamma \cos(\frac{a\pi}{2})}{a\pi} (t_2 - t)^{\frac{1+a}{2}} t^{-\frac{1-a}{2}}.$$

We determine  $t_2$  from the unit mass condition. Using (22), we compute

$$1 = \int_0^{t_2} \overline{\rho}(t) dt = \gamma \frac{\cos(\frac{a\pi}{2})}{2\pi a^2 \Gamma(a)} \Gamma(\frac{1+a}{2})^2 t_2^{1+a}.$$

Gathering our computations, we have

$$\overline{\rho}(t) = \gamma \frac{\cos(\frac{a\pi}{2})}{\pi a} (t_2 - t)^{\frac{1+a}{2}} t^{-\frac{1-a}{2}}, \quad \overline{C} = \frac{1+a}{2}t_2, \quad t_2 = \left[ \frac{\gamma}{2\pi} \frac{\cos(\frac{a\pi}{2})}{a^2 \Gamma(a)} \Gamma(\frac{1+a}{2})^2 \right]^{-\frac{1}{1+a}}. \quad (106)$$

## 7.2 Improved regularity

In this section we use Theorem 1.6 and Theorem 6.6 to strengthen the results of [SS15] and [GPPS13].

The setting in [SS15] is given by  $a = 0$ ,  $[T_1, T_2] = \mathbb{R}$ ,  $V_{\text{reg}} \equiv 0$  and  $U$  a confining potential. The energy  $E_{\mathbb{R}}$  and its minimiser  $\overline{\rho}$  (possibly with a disconnected support) are the starting point in [SS15] to derive crystallisation phenomena in 1D log-gases. Instead of proving that  $\overline{\rho}$  satisfies the following sufficient properties for their further results,

$$\overline{\rho} \in C^{\frac{1}{2}}(\mathbb{R}); \quad (107a)$$

$$\text{supp } \overline{\rho} \text{ is a finite union of } K \in \mathbb{N} \text{ compact intervals } [t_1^k, t_2^k]; \quad (107b)$$

$$\exists c > 0 \forall k = 1, \dots, K \forall t_1^k \leq t \leq t_2^k : \overline{\rho}(t) \geq c \sqrt{(t_2^k - t)(t - t_1^k)}, \quad (107c)$$

the authors of [SS15] assume that  $U$  is chosen such that these properties hold. The motivation for this implicit assumption is that it is satisfied for the specific choice  $U(t) = \frac{1}{2}t^2$ , in which case  $K = 1$ ,  $[t_1, t_2] = [-2, 2]$  and  $\overline{\rho}(t) = \frac{1}{2\pi} \sqrt{4 - t^2}$  is Wigner's "semi-circle law", which indeed satisfies (107). However, no further discussion is given on the class of potentials  $U$  for which (107) holds.

Theorem 6.6 helps in finding such a class of potentials  $U$ . Indeed, if  $U$  satisfies (87), then (107b) is satisfied for  $K = 1$ , i.e.,  $\text{supp } \overline{\rho} = [t_1, t_2]$ . While the conditions (107a) and (107c) cannot be obtained directly from Theorem 6.6, we discuss next to which extend one can prove weaker versions of these statements. Such weaker versions may still be enough for proving the main results in [SS15] with minor modifications. A detailed analysis of the proof of these weaker statements and their application to [SS15] is beyond our scope.

Regarding (107a), (93) guarantees that  $\overline{\rho}$  is  $\beta$ -Hölder continuous for *some*  $\beta > 0$ . To get an explicit value for  $\beta$ , we require a refinement of Proposition 2.3.(iii). However, even if such a refinement can be proven, our proof of (93) requires  $\beta < 1 - 1/p_0$ , which is always smaller than  $\frac{1}{2}$  since  $1 < p_0 < 2$  is required, regardless of the regularity of  $U$  in (87).

Regarding (107c), a refined version of Proposition 2.3.(iii) may characterise  $\overline{\rho}$  close to the endpoints  $t_1$  and  $t_2$ . Together with the positivity property given by Theorem 6.6, this may give a sufficiently strong alternative to assumption (107c).

Next we state an improvement of [GPPS13, Thm. 2]. The setting is given by  $a = 0$ ,  $[T_1, T_2] = [0, \infty)$ ,  $U(t) = t$  and

$$V(t) := t \coth t - \log |2 \sinh t|. \quad (108)$$

Figure 1 illustrates  $V$ . For this setting, [GPPS13, Thm. 2] states that Problem 6.1 has a unique solution  $\bar{\rho}$  in  $\mathcal{P}([0, \infty))$ , but no further properties of  $\bar{\rho}$  were sought.

Here, we provide such properties by applying Theorem 6.6 in the sense of Case 3 in §6.2 (see also the last paragraph of §6.2). With this aim, we first prove that  $V$  satisfies (86). The conditions in (86b) can be proven by direct computations (see, e.g., the appendices of [GPPS13, vM15]). To prove (86a), we set  $\varphi(t) := \frac{\sinh t}{t}$  and rewrite

$$V_{\text{reg}}(t) = t \frac{\cosh t}{\sinh t} - \log |2 \sinh t| + \log |t| = \frac{\cosh t}{\varphi(t)} - \log |2\varphi(t)|. \quad (109)$$

It is easy to see that  $\varphi$  is analytic and that  $\varphi \geq 1$ . Then, we obtain from (109) that  $V_{\text{reg}} \in C^\infty(\mathbb{R})$ . This proves (86a), and (86c) follows readily.

Thus, Theorem 6.6 applies for any  $\ell \in \mathbb{N}$  and any  $1 < p_0 < 2$ . In particular,  $\text{supp } \bar{\rho} = [0, t_2]$ ,  $\bar{\rho}(t_2) = 0$ ,  $\bar{\rho} \in C^\beta([\frac{1}{2}t_2, t_2]) \cap C^\infty((0, t_2))$  for some  $\beta > 0$  and  $\bar{\rho} > 0$  on  $(0, t_2)$ . Furthermore, from (19a) we obtain that  $\bar{\rho}(t) \leq C/\sqrt{t}$  for some  $C > 0$  and all  $t \in (0, t_2]$ . A further application of this improved regularity result on  $\bar{\rho}$  is outlined in §1.7.

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## A Proof of Lemma 3.1

*Proof of Lemma 3.1.* Lemma 3.1.(i) follows from the local regularity property in (29) and  $V_{\text{reg}}(t) = -V_a(t)$  for all  $|t| \geq b$ . Lemma 3.1.(ii) is satisfied for the sequence of convex functions given by  $V_k(t) := \int_t^\infty k \wedge (-V')(s) ds$  for  $t > 0$ , with even extension to the negative half-line.

Next we prove Lemma 3.1.(iii). We start with the upper bound on  $\widehat{V}$ . Since  $V \in L^1(\mathbb{R})$ , we have  $\widehat{V} \in L^\infty(\mathbb{R})$ , and hence it is sufficient to prove the decay of the tails of  $|\widehat{V}(\omega)|$  for all  $|\omega| > 1$ . We employ the splitting  $V = V_a + V_{\text{reg}}$ . By interpreting  $V_a$  as a tempered distribution, we obtain from [EB54, 1.3.(1)] and [RY07, §5.2 Example 4] that

$$\widehat{V}_a(\omega) = \begin{cases} \frac{1}{2|\omega|} & \text{if } a = 0, \\ \frac{2 \sin(\frac{a\pi}{2}) \Gamma(1-a)}{|2\pi\omega|^{1-a}} & \text{if } 0 < a < 1. \end{cases} \quad \text{for } |\omega| > 0, \quad (110)$$

where the distribution  $|\cdot|^{-1}$  is defined by

$$\langle \varphi, |\cdot|^{-1} \rangle := - \int_{\mathbb{R}} \varphi'(\omega) \text{sign}(\omega) \log |\omega| d\omega, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}).$$



From (110) we find for all  $0 \leq a < 1$  a constant  $C_a > 0$  such that

$$\widehat{V}_a(\omega) \leq C_a(1 + \omega^2)^{-\frac{1-a}{2}} \quad \text{for all } |\omega| \geq 1.$$

Regarding  $V_{\text{reg}}$ , we have by Lemma 3.1.(i) that  $V_{\text{reg}}'' \in L^1(\mathbb{R})$  for any  $0 \leq a < 1$ , and thus  $\mathcal{F}(V_{\text{reg}}'') \in L^\infty(\mathbb{R})$ . By applying basic results from the theory on tempered distributions (see, e.g., [RY07, Chap. 5], we conclude from

$$|4\pi^2\omega^2\widehat{V_{\text{reg}}}(\omega)| = |\widehat{V_{\text{reg}}''}(\omega)| \leq C$$

that

$$|\widehat{V_{\text{reg}}}(\omega)| \leq C|\omega|^{-2} \leq \tilde{C}(1 + \omega^2)^{-\frac{1-a}{2}} \quad \text{for all } |\omega| \geq 1.$$

Together with (110) and  $V = V_a + V_{\text{reg}}$  we conclude that the upper bound on  $\widehat{V}$  in Lemma 3.1.(iii) holds.

Next we prove the lower bound on  $\widehat{V}$  in Lemma 3.1.(iii). We use the identity

$$\begin{aligned} \widehat{V}(\omega) &= \int_{\mathbb{R}} V(t) \cos(2\pi\omega t) dt = 2 \sum_{\ell=0}^{\infty} \int_{\frac{\ell}{\omega}}^{\frac{\ell+1}{\omega}} V(t) \cos(2\pi\omega t) dt \\ &= \frac{2}{\omega} \sum_{\ell=0}^{\infty} \int_0^1 V\left(\frac{\ell+\xi}{\omega}\right) \cos(2\pi\xi) d\xi = \frac{2}{\omega} \sum_{\ell=0}^{\infty} - \int_0^1 \frac{1}{\omega} V'\left(\frac{\ell+\xi}{\omega}\right) \frac{1}{2\pi} \sin(2\pi\xi) d\xi \quad (111) \\ &= \frac{1}{\pi\omega^2} \sum_{\ell=0}^{\infty} - \int_0^1 \left[ V'\left(\frac{\ell+\xi}{\omega}\right) - V'\left(\frac{\ell+\frac{1}{2}}{\omega}\right) \right] \sin(2\pi\xi) d\xi \\ &= \frac{1}{\pi\omega^2} \sum_{\ell=0}^{\infty} \left( \int_0^{\frac{1}{2}} \left[ \int_{\xi}^{\frac{1}{2}} \frac{1}{\omega} V''\left(\frac{\ell+\eta}{\omega}\right) d\eta \right] \sin(2\pi\xi) d\xi \right. \\ &\quad \left. - \int_{\frac{1}{2}}^1 \left[ \int_{\frac{1}{2}}^{\xi} \frac{1}{\omega} V''\left(\frac{\ell+\eta}{\omega}\right) d\eta \right] \sin(2\pi\xi) d\xi \right) \\ &= \frac{1}{\pi\omega^3} \sum_{\ell=0}^{\infty} \int_0^{\frac{1}{2}} \left[ \int_0^{\frac{1}{2}} V''\left(\frac{\ell+\xi+\zeta}{\omega}\right) d\zeta \right] \sin(2\pi\xi) d\xi. \quad (112) \end{aligned}$$

We first prove that  $\widehat{V}$  is positive. In view of (4c) we set  $\lambda(t) := \text{ess inf}_{0 < s < t} V''(s)$ . Continuing from (112), we use (4c) to estimate

$$\begin{aligned} \widehat{V}(\omega) &\geq \frac{1}{\pi\omega^3} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \lambda\left(\frac{\xi+\eta}{\omega}\right) \sin(2\pi\xi) d\xi d\eta \\ &\geq \frac{1}{\pi\omega^3} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{4}} \lambda\left(\frac{\xi+\eta}{\omega}\right) 4\xi d\xi d\eta \\ &\geq \frac{4}{\pi} \int_0^{\frac{1}{2\omega}} \int_0^{\frac{1}{4\omega}} \lambda(x+y)x dx dy > 0, \end{aligned}$$

and thus the lower bound in Lemma 3.1.(iii) holds on any bounded interval. To bound the

tails of  $\widehat{V}$  from below, we set  $c$  and  $\varepsilon$  as in (4c), and estimate for any  $\omega \geq \frac{2}{\varepsilon}$

$$\begin{aligned}\widehat{V}(\omega) &\geq \frac{1}{\pi\omega^3} \sum_{\ell=0}^{\infty} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \lambda\left(\frac{\ell+\xi+\zeta}{\omega}\right) \sin(2\pi\xi) d\xi d\zeta \\ &\geq \frac{1}{\pi\omega^2} \sum_{\ell=0}^{\infty} \frac{1}{\omega} \lambda\left(\frac{\ell+1}{\omega}\right) \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \sin(2\pi\xi) d\xi d\zeta \\ &= \frac{1}{2\pi^2\omega^2} \sum_{\ell=0}^{\infty} \frac{1}{\omega} \lambda\left(\frac{\ell+1}{\omega}\right) \geq \frac{1}{2\pi^2\omega^2} \int_{\frac{1}{\omega}}^{\varepsilon} \lambda(s) ds \geq \frac{c}{2\pi^2\omega^2} \int_{\frac{1}{\omega}}^{\varepsilon} \frac{1}{s^{2+a}} ds \geq \tilde{c}(1+\omega^2)^{-\frac{1-a}{2}},\end{aligned}$$

which completes the proof of Lemma 3.1.(iii).  $\square$

## B Computation of the constant in Theorem 4.2

Here we perform the computation which shows that (69) implies (66). More precisely, after removing the tildes for convenience, it is left to show that

$$-\frac{1}{\pi} \int_0^1 \int_0^1 \frac{\log|r-s|}{\phi(r)} u(s) ds dr - \frac{1}{\pi} \int_0^1 \frac{f(s)}{\phi(s)} ds = \log(t_2 - t_1) \int_0^1 u(s) ds, \quad (113)$$

where

$$u(t) = \frac{1}{\pi^2 \phi(t)} \left( S(\phi f')(t) - \frac{1}{\log(\frac{1}{4}(t_2 - t_1))} \int_0^1 \frac{f}{\phi} \right), \quad \phi(t) = \sqrt{t(1-t)}, \quad f \in C^{1,\alpha}([0,1]).$$

We start by computing the right-hand side of (113). Since  $f' \in L^p(0,1)$  for any  $p \geq 1$ , we obtain from Proposition 2.2.(iii) and (23) that

$$\int_0^1 \frac{1}{\phi} S(\phi f') = - \int_0^1 S\left(\frac{1}{\phi}\right) \phi f' = 0.$$

Since (22) implies that  $\int_0^1 \frac{1}{\phi} = \pi$ , we obtain that the right-hand side of (113) satisfies

$$\log(t_2 - t_1) \int_0^1 u = \frac{-\log(t_2 - t_1)}{\pi \log(\frac{1}{4}(t_2 - t_1))} \int_0^1 \frac{f}{\phi}. \quad (114)$$

Next we compute the first term in the left-hand side of (113). Using Fubini's Theorem, we rewrite

$$\int_0^1 \int_0^1 \frac{\log|r-s|}{\phi(r)} u(s) ds dr = \int_0^1 \left( \int_0^1 \frac{\log|r-s|}{\phi(r)} dr \right) u(s) ds. \quad (115)$$

With Proposition 2.2.(ii) and (23) we obtain that the term in parentheses satisfies

$$\frac{d}{ds} \int_0^1 \frac{\log|r-s|}{\phi(r)} dr = \left( S \frac{1}{\phi} \right)(s) = 0.$$

Hence, the term in parentheses in (115) is constant in  $s$ . We evaluate it at  $s = \frac{1}{2}$ :

$$\begin{aligned} \int_0^1 \frac{\log|r-\frac{1}{2}|}{\phi(r)} dr &= 2 \int_{\frac{1}{2}}^1 \frac{\log|r-\frac{1}{2}|}{\sqrt{r(1-r)}} dr = 2 \int_0^1 \frac{\log \frac{t}{2}}{\sqrt{1-t^2}} dt \\ &= 2 \int_0^1 \frac{\log t}{\sqrt{1-t^2}} dt - 2 \log 2 \int_0^1 \frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

Both integrals are evaluated respectively in [GR07, 4.241.7] and (22). This yields

$$\int_0^1 \frac{\log |r - \frac{1}{2}|}{\phi(r)} dr = -2\pi \log 2.$$

Collecting the computations above, we rewrite the left-hand side of (113) as

$$\begin{aligned} -\frac{1}{\pi} \int_0^1 \int_0^1 \frac{\log |r - s|}{\phi(r)} u(s) ds dr - \frac{1}{\pi} \int_0^1 \frac{f}{\phi} &= 2 \log 2 \int_0^1 u - \frac{1}{\pi} \int_0^1 \frac{f}{\phi} \\ &= \left( -\frac{2 \log 2}{\pi \log(\frac{1}{4}(t_2 - t_1))} - \frac{1}{\pi} \right) \int_0^1 \frac{f}{\phi} = \frac{-\log(t_2 - t_1)}{\pi \log(\frac{1}{4}(t_2 - t_1))} \int_0^1 \frac{f}{\phi}, \end{aligned}$$

and conclude from (114) that (113) holds.

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