Joint probability distributions for projection probabilities of random orthonormal states

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September 2015

Abstract. A finite dimensional quantum system for which the quantum chaos conjecture applies has eigenstates, which show the same statistical properties than the column vectors of random orthogonal or unitary matrices. Here, we consider the different probabilities for obtaining a specific outcome in a projective measurement, provided the system is in one of its eigenstates. We then give analytic expressions for the joint probability density for these probabilities, with respect to the ensemble of random matrices. In the case of the unitary group, our results can be applied, also, to the phenomenon of universal conductance fluctuations, where the same mathematical quantities describe partial conductances in a two-terminal mesoscopic scattering problem with a finite number of modes in each terminal.

PACS numbers: 02.20.Hj, 73.23-b, 05.45.Mt

1. Introduction

Let \mathcal{H} be the Hilbert space of finite dimension $N = \dim(\mathcal{H})$ corresponding to some quantum system. Let S be a subspace in \mathcal{H} , with $K = \dim(S)$. We choose an orthonormal basis $\mathcal{B} = \{\varphi_j\}_{1 \leq j \leq N}$ in \mathcal{H} , with the first K elements lying in S. Then, the projection of a normalized state $\psi \in \mathcal{H}$ on the subspace S has the squared norm

$$t = \sum_{j=1}^{K} |\langle \varphi_j | \psi \rangle|^2 . \tag{1}$$

In a projective measurement, the observable to be measured is associated to a Hermitian operator \hat{A} , which always has an orthonormal basis of eigenstates. Assume \mathcal{B} is such a basis and S is the eigenspace corresponding to a K-fold degenerate eigenvalue of \hat{A} , then t as given in (1) is the probability that the outcome of the measurement of \hat{A} is that eigenvalue.

In this paper, we consider several orthonormal states $\psi_1, \dots \psi_R$ and the squared norms of their projections

$$t_{\xi} = \sum_{j=1}^{K} |\langle \varphi_j | \psi_{\xi} \rangle|^2 , \qquad 1 \le \xi \le R .$$
 (2)

We calculate the joined probability distribution of these quantities, provided the states $\psi_1, \ldots \psi_R$ a chosen at random from one of two invariant ensembles. In order to define these ensembles, we use the basis \mathcal{B} , which allows to write the quantum states as complex column vectors of length N, and the orthogonal (unitary) group as real (complex) unitary $N \times N$ matrices [1]. Then, the first ensemble is defined as the set of collections of R orthonormal column vectors equipped with the unique probability measure, which is invariant under orthogonal transformations. Similarly, the second ensemble is defined as the set of collections of R orthonormal column vectors equipped with the unique probability measure, which is invariant under unitary transformations.

Both, the orthogonal and the unitary group may be turned into an ensemble of random matrices, using the normalized Haar measure as probability measure [2, 3]. Unless states otherwise, we refer to these ensembles simply by their group names, O(N) and U(N). It then turns out that the two ensembles defined above can be obtained from O(N) and U(N), by selecting only R column vectors (we may always choose the first R column vectors) from a given group element. Below, we will no longer distinguish between the original ensembles of collections of R column vectors and the ensembles defined over the whole $N \times N$ matrices.

The groups O(N) and U(N) as invariant ensembles have become an important reference in the study of broad classes of quantum systems, first in nuclear physics [4], later in quantum chaos [5, 6] and mesoscopic quantum transport [7, 8], and more recently in quantum information; e.g. [9, 10, 11]. To give ye another example, a general definition of an entanglement measure for mixed states is based on the so called "convex roof" construction, where mutually orthogonal vectors are used to parameterize the different possibilities to write the density matrix in question as a mixture of pure states [12, 13]. Our results may thus be useful in any of these areas.

Averages over matrix elements of the orthogonal and the unitary group have been considered in different contexts. Averages over monomials are considered in [14, 15, 16]. The statistical properties of eigenvalues of $M \times M$ sub-matrices (M < N) in [17, 18]. Ensembles of scattering matrices may be derived from the unitary group, also [19]. In this area, there are connections to the distribution of transmission eigenvalues [20], and the probability distributions of individual scattering matrix elements in the so called Heidelberg model [21, 22].

The paper is organized as follows. In the next section, we introduce our notation and define the quantities of interest. There, we also treat the one-vector case, which has been solved much earlier. In section 3 we derive the general expressions for an arbitrary number of column vectors. In section 4 we treat the case R=2 in more detail. In that case, one can often evaluate all the integrals, to arrive at very simple expressions which contain only rational and/or algebraic functions. In this section, we also discuss two physical applications, the distribution of probabilities for measurement outcomes in a closed quantum system, and partial conductances in systems showing universal conductance fluctuations. Conclusions are provided in section 5.

2. Definitions and notation

We will denote the joint probability densities of the variables t_1, \ldots, t_R defined in (2) as $P_{NK}(t_1, \ldots, t_R)$ in the case the average is over the orthogonal group and as $\mathcal{P}_{NK}(t_1, \ldots, t_R)$ in the case of the unitary group. It is understood that the probability density is with respect to the flat measure $dt_1 \ldots dt_R$ in the real unit hyper-cube of dimension R. With the help of the Dirac delta function, we write the following formal expressions for these probability densities

$$P_{NK}(t_1,\ldots,t_R) = \left\langle \prod_{\xi=1}^R \delta\left(t_\xi - \sum_{j=1}^K w_{j\xi}^2\right) \right\rangle_{O(N)}, \qquad (3)$$

$$\mathcal{P}_{NK}(t_1, \dots, t_R) = \left\langle \prod_{\xi=1}^R \delta \left(t_{\xi} - \sum_{j=1}^K |w_{j\xi}|^2 \right) \right\rangle_{U(N)} . \tag{4}$$

Here, \boldsymbol{w} denotes a group element, i.e. a $N \times N$ matrix with real or complex entries. The matrix elements are denoted by $w_{j\xi}$, where we use Latin (Greek) indices for rows (columns). The angular brackets denote the ensemble average over the respective group, with respect to the respective normalized Haar measure [2]. We may write these ensemble averages as integrals over the flat space of all matrix elements, implementing the orthonormality conditions on the column vectors by additional delta functions. Originally, this idea is due to Ullah [23, 24] and more recently it had been used to calculate group averages of monomials [25, 16].

2.1. One point functions

In the case of the one point functions, we consider the probability density for the projection of only one random vector on the K-dimensional subspace. The result is known for a long time, see e.g. [4]. Nevertheless, we present our calculation in some detail, since it is different from the usual approach and since it introduces some techniques used again for the general case.

Orthogonal case For the orthogonal case, the desired probability density may be written as

$$P_{NK}(t) = \left\langle \delta \left(t - \sum_{j=1}^{K} w_j^2 \right) \right\rangle_{O(N)} = C_N \int d\Omega(\boldsymbol{w}) \, \delta \left(t - \sum_{j=1}^{K} w_j^2 \right)$$
$$= C_N \int d^N \boldsymbol{w} \, \delta (1 - \|\boldsymbol{w}\|^2) \, \delta \left(t - \sum_{j=1}^{K} w_j^2 \right) . \tag{5}$$

In this equation, $d\Omega(\boldsymbol{w})$ is the invariant measure on the unit hyper-sphere in \mathbb{R}^N , $d^N\boldsymbol{w}$ is the flat measure in \mathbb{R}^N , and $\|\boldsymbol{w}\|^2 = \sum_{j=1}^N w_j^2$. The normalization constant (to be determined below) is denoted by C_N , as it depends on N but not on K. We follow [24] to eliminate the delta-function which implements the normalization, and apply the transformation $w_j \to u_j = \sqrt{r} w_j$ for an arbitrary parameter r > 0. This gives

$$P_{NK}(t) \ r^{N/2-2} = C_N \int d^N \boldsymbol{u} \ \delta(r - \|\boldsymbol{u}\|^2) \ \delta(rt - \sum_{j=1}^K u_j^2) \ . \tag{6}$$

Multiplying both sides with e^{-r} and integrating r from 0 to ∞ then yields

$$P_{NK}(t) \Gamma(N/2-1) = C_N \int d^N \boldsymbol{u} \, \delta(\|\boldsymbol{u}\|^2 t - \sum_{j=1}^K u_j^2) e^{-\|\boldsymbol{u}\|^2}.$$
 (7)

The normalization constant, C_N , is obtained from the requirement that

$$1 = \int_0^1 dt \ P_{NK}(t) = C_N \int d^N \boldsymbol{w} \ \delta(1 - \|\boldsymbol{w}\|^2) \ . \tag{8}$$

Applying the same trick as above, allows to evaluate the integral right away. It results in $C_N = \Gamma(N/2)/\pi^{N/2}$. Returning to $P_{NK}(t)$, we replace the last delta function by its Fourier representation.

$$P_{NK}(t) = \frac{N/2 - 1}{\pi^{N/2}} \int \frac{\mathrm{d}s}{2\pi} \int \mathrm{d}^{N} \boldsymbol{u} \, e^{-\|\boldsymbol{u}\|^{2}(1+\mathrm{i}st)} \prod_{j=1}^{K} e^{\mathrm{i}s \, u_{j}^{2}}$$

$$= \frac{N/2 - 1}{\pi^{N/2}} \int \frac{\mathrm{d}s}{2\pi} \int \mathrm{d}^{N} \boldsymbol{u} \prod_{j=K+1}^{N} e^{-u_{j}^{2}(1+\mathrm{i}st)} \prod_{j=1}^{K} e^{-u_{j}^{2}[1-\mathrm{i}s(1-t)]}$$

$$= \frac{N-2}{4\pi} \int \frac{\mathrm{d}s}{\sqrt{(1+\mathrm{i}st)^{N-K} [1-\mathrm{i}s(1-t)]^{K}}}.$$
(9)

This integral can be evaluated as explained in the appendix, with the result given in (A.1). Therefore,

$$P_{NK}(t) = I_{(N-K)/2,K/2}(t,1-t) = \frac{\Gamma(N/2) t^{K/2-1} (1-t)^{(N-K)/2-1}}{\Gamma((N-K)/2) \Gamma(K/2)}.$$
 (10)

It is easy to show that this result is in agreement with similar results on the statistics of random vector components reviewed in [4].

Unitary case For the unitary case, the probability density for the squared norm of the projection on a K-dimensional subspace may be written as

$$\mathcal{P}_{NK}(t) = \left\langle \delta \left(t - \sum_{j=1}^{K} |w_j|^2 \right) \right\rangle_{U(N)}$$
$$= \mathcal{C}_N \int d^{2N} \boldsymbol{w} \, \delta (1 - \|\boldsymbol{w}\|^2) \, \delta \left(t - \sum_{j=1}^{K} |w_j|^2 \right) . \tag{11}$$

The integral is now over the 2N dimensional space of real and imaginary parts of the complex components of the vector \boldsymbol{w} . The normalization restricts the integration to the unit hyper-sphere in this space. The sum in the second delta function, goes equally over the squares of real and imaginary parts of the vector \boldsymbol{w} . This simply means that $\mathcal{P}_{NK}(t) = P_{2N,2K}(t)$ such that

$$\mathcal{P}_{NK}(t) = \frac{N-1}{2\pi} \int \frac{ds}{(1+ist)^{N-K} (1-is(1-t))^K}.$$
 (12)

In this case, we may again use (A.1) to find

$$\mathcal{P}_{NK}(t) = I_{N-K,K}(t, 1-t) = \frac{\Gamma(N) t^{K-1} (1-t)^{N-K-1}}{\Gamma(N-K) \Gamma(K)}.$$
 (13)

3. General R point functions

In this section, we derive general integral expressions for the case of an arbitrary number of vectors. Again, we start with the orthogonal case and treat the unitary case afterwards. Let us adopt a few conventions which simplify the interpretation of the following expressions which often involve multiple integrals: (i) The symbol for integration together with the expression which denotes the integration measure form one unit, and the integrand then extends up to the next plus or minus sign. (ii) The symbol for multiple products, acts on the expression to its right, extending up to the next plus or minus sign. To restrict the symbol's scope otherwise, we use brackets surrounding the product term and the product symbol. The curly brackets below, in (14), are used in that way.

3.1. Orthogonal case

For the joint probability density as defined in (3) we write

$$P_{NK}(t_1, \dots, t_R) = C_{NR} \left\{ \prod_{\xi=1}^R \int d\Omega(\boldsymbol{w}_{\xi}) \, \delta\left(t_{\xi} - \sum_{j=1}^K w_{j\xi}^2\right) \right\} \prod_{\mu < \nu}^R \delta(\langle \boldsymbol{w}_{\mu} \, | \, \boldsymbol{w}_{\nu} \rangle) , \qquad (14)$$

where $d\Omega(\boldsymbol{w}_{\xi})$ denotes the uniform measure on the hyper-sphere in \mathbb{R}^{N} . The last product of delta functions implements the orthogonality condition between the column vectors \boldsymbol{w}_{ξ} of the elements of the orthogonal group.

Normalization Before treating the full expression, let us calculate the normalization constant.

$$C_{NR}^{-1} = \left\{ \prod_{\xi=1}^{R} \int d^{N} \boldsymbol{w}_{\xi} \, \delta(1 - \|\boldsymbol{w}_{\xi}\|^{2}) \right\} \prod_{\mu < \nu}^{R} \delta(\langle \boldsymbol{w}_{\mu} | \boldsymbol{w}_{\nu} \rangle)$$

$$= \left\{ \prod_{\xi=1}^{R} \int d^{N} \boldsymbol{u}_{\xi} \, r_{\xi}^{-N/2 + 1 + (R-1)/2} \, \delta(r_{\xi} - \|\boldsymbol{u}_{\xi}\|^{2}) \right\} \prod_{\mu < \nu}^{R} \delta(\langle \boldsymbol{u}_{\mu} | \boldsymbol{u}_{\nu} \rangle) , \quad (15)$$

from which it follows that

$$C_{NR}^{-1} \Gamma[(N-R+1)/2]^R = \left\{ \prod_{\xi=1}^R \int d^N \boldsymbol{u}_{\xi} e^{-\|\boldsymbol{u}_{\xi}\|^2} \right\} \prod_{\mu<\nu}^R \delta(\langle \boldsymbol{u}_{\mu} | \boldsymbol{u}_{\nu} \rangle)$$

$$= \left\{ \prod_{\mu<\nu}^R \int \frac{d\tau_{\mu\nu}}{2\pi} \right\} \left\{ \prod_{\xi=1}^R \int d^N \boldsymbol{u}_{\xi} e^{-\|\boldsymbol{u}_{\xi}\|^2} \right\} \prod_{\mu<\nu}^R e^{-i\tau_{\mu\nu}\langle \boldsymbol{u}_{\mu} | \boldsymbol{u}_{\nu} \rangle}$$

$$= \left\{ \prod_{\mu<\nu}^R \int \frac{d\tau_{\mu\nu}}{2\pi} \right\} \prod_{j=1}^N \int d^R \boldsymbol{u}_{j}' e^{-\langle \boldsymbol{u}_{j}' | \boldsymbol{D} \boldsymbol{u}_{j}' \rangle}. \tag{16}$$

In this expression and below we denote with $\mathbf{u}'_j = (u_{j1}, \dots, u_{jR})^T$ the j'th row vector of the orthogonal matrix \mathbf{u} , restricted to the first R components. Correspondingly, we

denote the scalar product between two row vectors u' and v' as $\langle u'|v'\rangle$. Hence, with

$$\mathbf{D} = 1 + \frac{\mathrm{i}}{2} \begin{pmatrix} 0 & \tau_{12} & \dots \\ \tau_{12} & 0 & \\ \vdots & & \ddots \end{pmatrix}$$

$$(17)$$

we find that

$$\langle \boldsymbol{u}'_{j} | \boldsymbol{D} \, \boldsymbol{u}'_{j} \rangle = \sum_{\mu\nu} u_{j\mu} \, D_{\mu\nu} \, u_{j\nu} = \| \boldsymbol{u}'_{j} \|^{2} + \mathrm{i} \sum_{\mu < \nu} \tau_{\mu\nu} \, u_{j\mu} u_{j\nu} \,.$$
 (18)

Here, $\|\boldsymbol{u}_j'\|^2 = \langle \boldsymbol{u}_j'|\boldsymbol{u}_j'\rangle$ denotes the squared norm of the row vector \boldsymbol{u}_j' . The integrals over the \boldsymbol{u}_j' in (16) are standard Gaussian integrals, which can be evaluated in terms of the determinant of \boldsymbol{D} . In this way, we obtain

$$C_{NR}^{-1} \Gamma[(N-R+1)/2]^R = \left\{ \prod_{\mu<\nu}^R \int \frac{d\tau_{\mu\nu}}{2\pi} \right\} \frac{\pi^{NR/2}}{\det(\mathbf{D})^{N/2}} .$$
 (19)

Full expression Returning to the full expression (14), we start again by removing the delta functions implementing the normalization of the column vectors. From

$$P_{NK}(t_1, \dots, t_R) = C_{NR} \left\{ \prod_{\xi=1}^R \int d^N \boldsymbol{w}_{\xi} \, \delta(1 - \|\boldsymbol{w}_{\xi}\|^2) \, \delta\left(t_{\xi} - \sum_{j=1}^K w_{j\xi}^2\right) \right\} \, \prod_{\mu < \nu}^R \delta(\langle \boldsymbol{w}_{\mu} | \boldsymbol{w}_{\nu} \rangle)$$

$$= C_{NR} \left\{ \prod_{\xi=1}^R \int d^N \boldsymbol{u}_{\xi} \, r^{-N/2 + 2 + (R-1)/2} \, \delta(r_{\xi} - \|\boldsymbol{u}_{\xi}\|^2) \, \delta\left(r_{\xi} t_{\xi} - \sum_{j=1}^K u_{j\xi}^2\right) \right\}$$

$$\times \prod_{\mu < \nu}^R \delta(\langle \boldsymbol{u}_{\mu} | \boldsymbol{u}_{\nu} \rangle) \,, \tag{20}$$

it follows that

$$P_{NK}(t_{1},...,t_{R}) \Gamma[(N-R-1)/2]^{R} = C_{NR} \left\{ \prod_{\mu<\nu}^{R} \int \frac{d\tau_{\mu\nu}}{2\pi} \right\}$$

$$\times \left\{ \prod_{\xi=1}^{R} \int \frac{ds_{\xi}}{2\pi} \int d^{N} \boldsymbol{u}_{\xi} e^{-\|\boldsymbol{u}_{\xi}\|^{2}(1+is_{\xi}t_{\xi})} e^{is_{\xi} \sum_{j=1}^{K} u_{j\xi}^{2}} \right\} \prod_{\mu<\nu}^{R} e^{-i\tau_{\mu\nu} \langle \boldsymbol{u}_{\mu} | \boldsymbol{u}_{\nu} \rangle}$$

$$= C_{NR} \left\{ \prod_{\mu<\nu}^{R} \int \frac{d\tau_{\mu\nu}}{2\pi} \right\} \left\{ \prod_{\xi=1}^{R} \int \frac{ds_{\xi}}{2\pi} \right\}$$

$$\times \left\{ \prod_{j=1}^{K} \int d^{R} \boldsymbol{u}_{j}' e^{-\sum_{\xi=1}^{R} u_{j\xi}^{2}[1-is_{\xi}(1-t_{\xi})]-i\sum_{\mu<\nu}^{R} u_{j\mu} \tau_{\mu\nu} u_{j\nu}} \right\}$$

$$\times \left\{ \prod_{j=K+1}^{N} \int d^{R} \boldsymbol{u}_{j}' e^{-\sum_{\xi=1}^{R} u_{j\xi}^{2}(1+is_{\xi}t_{\xi})-i\sum_{\mu<\nu}^{R} u_{j\mu} \tau_{\mu\nu} u_{j\nu}} \right\} .$$

$$(21)$$

Then, with the help of the matrices

$$\mathbf{A} = 1 + i \begin{pmatrix} s_1 t_1 & \tau_{12}/2 & \dots \\ \tau_{12}/2 & s_2 t_2 & \\ \vdots & & \ddots \end{pmatrix}, \quad \mathbf{B} = 1 + i \begin{pmatrix} -s_1 (1 - t_1) & \tau_{12}/2 & \dots \\ \tau_{12}/2 & -s_2 (1 - t_2) & \\ \vdots & & \ddots \end{pmatrix}, (22)$$

we may write that

$$P_{NK}(t_1, \dots, t_R) \Gamma[(N - R - 1)/2]^R = C_{NR} \left\{ \prod_{\mu < \nu}^R \int \frac{\mathrm{d}\tau_{\mu\nu}}{2\pi} \right\} \left\{ \prod_{\xi=1}^R \int \frac{\mathrm{d}s_{\xi}}{2\pi} \right\}$$

$$\times \left\{ \prod_{j=K+1}^N \int \mathrm{d}^R \boldsymbol{u}_j' \, \mathrm{e}^{-\langle \boldsymbol{u}_j' | \boldsymbol{A} \boldsymbol{u}_j' \rangle} \right\} \left\{ \prod_{j=1}^K \int \mathrm{d}^R \boldsymbol{u}_j' \, \mathrm{e}^{-\langle \boldsymbol{u}_j' | \boldsymbol{B} \boldsymbol{u}_j' \rangle} \right\}$$

$$= C_{NR} \left\{ \prod_{\xi=1}^R \int \frac{\mathrm{d}s_{\xi}}{2\pi} \right\} \left\{ \prod_{\mu < \nu}^R \int \frac{\mathrm{d}\tau_{\mu\nu}}{2\pi} \right\} \frac{\pi^{NR/2}}{\det(\boldsymbol{A})^{(N-K)/2} \det(\boldsymbol{B})^{K/2}} . (23)$$

In view of the result for the normalization constant C_{NR} in (19) and the fact that at $s = \mathbf{o}$, it holds that A = B = D. Thus, we define

$$Z(\mathbf{s}) = \left\{ \prod_{\mu < \nu}^{R} \int \frac{\mathrm{d}\tau_{\mu\nu}}{2\pi} \right\} \frac{1}{\det(\mathbf{A})^{(N-K)/2} \det(\mathbf{B})^{K/2}}, \qquad (24)$$

which allows to write

$$P_{NK}(t_1, \dots, t_R) = \frac{[(N - R - 1)/2]^R}{Z(\mathbf{o})} \left\{ \prod_{\xi=1}^R \int \frac{\mathrm{d}s_{\xi}}{2\pi} \right\} Z(\mathbf{s}) . \tag{25}$$

One vector case Strictly speaking, (25) applies for R > 1, only. However, for R = 1, $\mathbf{A} = 1 + \mathrm{i} s t$ and $\mathbf{B} = 1 - \mathrm{i} s (1 - t)$, such that (24) may be interpreted as

$$Z(s) = \frac{1}{(1+ist)^{(N-K)/2} \left[1-is(1-t)\right]^{K/2}}, \qquad Z(0) = 1.$$
 (26)

This yields

$$P_{NK}(t) = \frac{N-2}{2} \int \frac{\mathrm{d}s}{2\pi} \, \frac{1}{(1+\mathrm{i}\,st)^{(N-K)/2} \, [1-\mathrm{i}s(1-t)]^{K/2}} \,, \tag{27}$$

in agreement with (9).

3.2. Unitary case

For the joint probability density as defined in (4) we write

$$\mathcal{P}_{NK}(t_1,\ldots,t_R) = \mathcal{C}_{NR} \left\{ \prod_{\xi=1}^R \int d\Omega_2(\boldsymbol{w}_{\xi}) \, \delta\left(t_{\xi} - \sum_{j=1}^K |w_{j\xi}|^2\right) \right\} \prod_{\mu<\nu}^R \delta^2(\langle \boldsymbol{w}_{\mu} \, | \, \boldsymbol{w}_{\nu} \rangle) . (28)$$

Here, $d\Omega_2(\boldsymbol{w}_{\xi})$ denotes the uniform measure on the hyper-sphere in \mathbb{R}^{2N} . The last product of delta functions implements the orthogonality conditions between the column vectors \boldsymbol{w}_{ξ} of the elements of the unitary group. These are two-dimensional because for $\langle \boldsymbol{w}_{\mu} | \boldsymbol{w}_{\nu} \rangle$ to be zero, the real and the imaginary part must be zero.

Normalization Before treating the full expression, let us calculate the normalization constant.

$$\mathcal{C}_{NR}^{-1} = \left\{ \prod_{\xi=1}^{R} \int d^{2N} \boldsymbol{w}_{\xi} \, \delta(1 - \|\boldsymbol{w}_{\xi}\|^{2}) \right\} \prod_{\mu < \nu}^{R} \delta^{2}(\langle \boldsymbol{w}_{\mu} | \boldsymbol{w}_{\nu} \rangle)$$

$$= \left\{ \prod_{\xi=1}^{R} \int d^{2N} \boldsymbol{u}_{\xi} \, r_{\xi}^{-N+1+R-1} \, \delta(r_{\xi} - \|\boldsymbol{u}_{\xi}\|^{2}) \right\} \prod_{\mu < \nu}^{R} \delta^{2}(\langle \boldsymbol{u}_{\mu} | \boldsymbol{u}_{\nu} \rangle) , \qquad (29)$$

from which it follows

$$\mathcal{C}_{NR}^{-1} \Gamma(N - R + 1)^{R} = \left\{ \prod_{\xi=1}^{R} \int d^{2N} \boldsymbol{u}_{\xi} e^{-\|\boldsymbol{u}_{\xi}\|^{2}} \right\} \prod_{\mu < \nu}^{R} \delta^{2}(\langle \boldsymbol{u}_{\mu} | \boldsymbol{u}_{\nu} \rangle)$$

$$= \left\{ \prod_{\mu < \nu}^{R} \int \frac{d^{2} \tau_{\mu\nu}}{4\pi^{2}} \right\} \left\{ \prod_{\xi=1}^{R} \int d^{2N} \boldsymbol{u}_{\xi} e^{-\|\boldsymbol{u}_{\xi}\|^{2}} \right\} \prod_{\mu < \nu}^{R} e^{-i\operatorname{Im}(\tau_{\mu\nu}\langle \boldsymbol{u}_{\mu} | \boldsymbol{u}_{\nu} \rangle)}, (30)$$

where we have used the Fourier representation of the two-dimensional delta function with complex argument, defined in (A.10). As in the orthogonal case, it will again prove convenient to change from column vectors with Greek indices to row vectors with Latin indices. The notations for vectors, scalar products and the vector norm are analogous to the orthogonal case. However, the coefficients are now complex such that $\langle \boldsymbol{u}_j' | \boldsymbol{u}_k' \rangle = \sum_{\xi=1}^R u_{j\xi}^* u_{k\xi}$. With this, we may write

$$\sum_{\xi=1}^{R} \|\boldsymbol{u}_{\xi}\|^{2} + i \sum_{\mu<\nu}^{R} \operatorname{Im}(\tau_{\mu\nu} \langle \boldsymbol{u}_{\mu} | \boldsymbol{u}_{\nu} \rangle) = \sum_{j=1}^{N} \left[\|\boldsymbol{u}_{j}'\|^{2} + \frac{1}{2} \sum_{\mu<\nu}^{R} (\tau_{\mu\nu} u_{j\mu}^{*} u_{j\nu} - \tau_{\mu\nu}^{*} u_{j\mu} u_{j\nu}^{*}) \right]$$

$$= \sum_{j=1}^{N} \left[\|\boldsymbol{u}_{j}'\|^{2} + \frac{1}{2} \sum_{\mu<\nu}^{R} u_{j\mu}^{*} \tau_{\mu\nu} u_{j\nu} - \frac{1}{2} \sum_{\nu<\mu}^{R} u_{j\mu}^{*} \tau_{\nu\mu}^{*} u_{j\nu} \right] = \langle \boldsymbol{u}_{j}' | \boldsymbol{G} \boldsymbol{u}_{j}' \rangle , \quad (31)$$

where

$$G = 1 + \frac{1}{2} \begin{pmatrix} 0 & \tau_{12} & \dots \\ -\tau_{12}^* & 0 & \\ \vdots & & \ddots \end{pmatrix} = 1 + \frac{i}{2} \begin{pmatrix} 0 & -i\tau_{12} & \dots \\ i\tau_{12}^* & 0 & \\ \vdots & & \ddots \end{pmatrix} .$$
 (32)

We thus find

$$\mathcal{C}_{NR}^{-1} \Gamma(N-R+1)^R = \left\{ \prod_{\mu<\nu}^R \int \frac{\mathrm{d}\tau_{\mu\nu}}{4\pi^2} \right\} \prod_{j=1}^N \int \mathrm{d}^{2R} \boldsymbol{u}_j' \, \mathrm{e}^{-\langle \boldsymbol{u}_j' | \boldsymbol{G} \, \boldsymbol{u}_j' \rangle} \\
= \left\{ \prod_{\mu<\nu}^R \int \frac{\mathrm{d}\tau_{\mu\nu}}{4\pi^2} \right\} \frac{\pi^{NR}}{\det(\boldsymbol{G})^N} . \tag{33}$$

Full expression Returning to the full expression (28), we again remove first the delta functions implementing the normalization of the column vectors. Hence,

$$\mathcal{P}_{NK}(t_{1},\ldots,t_{R}) = \mathcal{C}_{NR} \left\{ \prod_{\xi=1}^{R} \int d^{2N} \boldsymbol{w}_{\xi} \, \delta(1 - \|\boldsymbol{w}_{\xi}\|^{2}) \, \delta\left(t_{\xi} - \sum_{j=1}^{K} |w_{j\xi}|^{2}\right) \right\} \, \prod_{\mu < \nu}^{R} \delta^{2}(\langle \boldsymbol{w}_{\mu} | \boldsymbol{w}_{\nu} \rangle)$$

$$= \mathcal{C}_{NR} \left\{ \prod_{\xi=1}^{R} \int d^{2N} \boldsymbol{u}_{\xi} \, r^{-N+2+(R-1)} \, \delta(r_{\xi} - \|\boldsymbol{u}_{\xi}\|^{2}) \, \delta\left(r_{\xi} t_{\xi} - \sum_{j=1}^{K} |u_{j\xi}|^{2}\right) \right\}$$

$$\times \prod_{\mu < \nu}^{R} \delta^{2}(\langle \boldsymbol{u}_{\mu} | \boldsymbol{u}_{\nu} \rangle) , \qquad (34)$$

from which it follows that

$$\mathcal{P}_{NK}(t_1,\ldots,t_R) \; \Gamma(N-R)^R = \mathcal{C}_{NR} \left\{ \prod_{\mu<\nu}^R \int \frac{\mathrm{d}^2 \tau_{\mu\nu}}{4\pi^2} \right\}$$

$$\times \left\{ \prod_{\xi=1}^{R} \int \frac{\mathrm{d}s_{\xi}}{2\pi} \int \mathrm{d}^{2N} \boldsymbol{u}_{\xi} \, \mathrm{e}^{-\|\boldsymbol{u}_{\xi}\|^{2}(1+\mathrm{i}s_{\xi}t_{\xi})} \, \mathrm{e}^{\mathrm{i}s_{\xi} \sum_{j=1}^{K} |u_{j\xi}|^{2}} \right\} \, \prod_{\mu<\nu}^{R} \mathrm{e}^{-\mathrm{i}\operatorname{Im}(\tau_{\mu\nu}\langle\boldsymbol{u}_{\mu}|\boldsymbol{u}_{\nu}\rangle)}$$

$$= \mathcal{C}_{NR} \left\{ \prod_{\mu<\nu}^{R} \int \frac{\mathrm{d}^{2}\tau_{\mu\nu}}{4\pi^{2}} \right\} \left\{ \prod_{\xi=1}^{R} \int \frac{\mathrm{d}s_{\xi}}{2\pi} \right\}$$

$$\times \left\{ \prod_{j=1}^{K} \int \mathrm{d}^{2R} \boldsymbol{u}_{j}' \, \mathrm{e}^{-\sum_{\xi=1}^{R} |u_{j\xi}|^{2}[1-\mathrm{i}s_{\xi}(1-t_{\xi})]-\mathrm{i}/2\sum_{\mu<\nu}^{R} (u_{j\mu}^{*}\tau_{\mu\nu}u_{j\nu}-u_{j\mu}\tau_{\mu\nu}^{*}u_{j\nu}^{*})} \right\}$$

$$\times \left\{ \prod_{j=K+1}^{N} \int \mathrm{d}^{2R} \boldsymbol{u}_{j}' \, \mathrm{e}^{-\sum_{\xi=1}^{R} |u_{j\xi}|^{2}(1+\mathrm{i}s_{\xi}t_{\xi})-\mathrm{i}/2\sum_{\mu<\nu}^{R} (u_{j\mu}^{*}\tau_{\mu\nu}u_{j\nu}-u_{j\mu}\tau_{\mu\nu}^{*}u_{j\nu}^{*})} \right\} . (35)$$

Again, we may define matrices

$$\mathbf{E} = 1 + i \begin{pmatrix} s_1 t_1 & -i\tau_{12}/2 & \dots \\ i\tau_{12}^*/2 & s_2 t_2 & \\ \vdots & & \ddots \end{pmatrix}, \quad \mathbf{F} = 1 + i \begin{pmatrix} -s_1(1 - t_1) & -i\tau_{12}/2 & \dots \\ i\tau_{12}^*/2 & -s_2(1 - t_2) & \\ \vdots & & \ddots \end{pmatrix}, (36)$$

such that

$$\mathcal{P}_{NK}(t_{1},\ldots,t_{R}) \Gamma(N-R)^{R} = \mathcal{C}_{NR} \left\{ \prod_{\mu<\nu}^{R} \int \frac{\mathrm{d}^{2}\tau_{\mu\nu}}{4\pi^{2}} \right\} \left\{ \prod_{\xi=1}^{R} \int \frac{\mathrm{d}s_{\xi}}{2\pi} \right\}$$

$$\times \left\{ \prod_{j=K+1}^{N} \int \mathrm{d}^{2R} \boldsymbol{u}_{j}' \, \mathrm{e}^{-\langle \boldsymbol{u}_{j}' | \boldsymbol{E} \, \boldsymbol{u}_{j}' \rangle} \right\} \left\{ \prod_{j=1}^{K} \int \mathrm{d}^{2R} \boldsymbol{u}_{j}' \, \mathrm{e}^{-\langle \boldsymbol{u}_{j}' | \boldsymbol{F} \, \boldsymbol{u}_{j}' \rangle} \right\}$$

$$= \mathcal{C}_{NR} \left\{ \prod_{\xi=1}^{R} \int \frac{\mathrm{d}s_{\xi}}{2\pi} \right\} \left\{ \prod_{\mu<\nu}^{R} \int \frac{\mathrm{d}^{2}\tau_{\mu\nu}}{4\pi^{2}} \right\} \frac{\pi^{NR}}{\det(\boldsymbol{E})^{N-K} \det(\boldsymbol{F})^{K}} . \tag{37}$$

In view of the result for the normalization constant C_{NR} in (33), and because E = F = G, at s = 0, we define

$$Z(\mathbf{s}) = \left\{ \prod_{\mu < \nu}^{R} \int \frac{\mathrm{d}^{2} \tau_{\mu\nu}}{4\pi^{2}} \right\} \frac{1}{\det(\mathbf{E})^{N-K} \det(\mathbf{F})^{K}}, \tag{38}$$

which allows to write

$$\mathcal{P}_{NK}(t_1,\dots,t_R) = \frac{(N-R)^R}{Z(\mathbf{o})} \left\{ \prod_{\xi=1}^R \int \frac{\mathrm{d}s_{\xi}}{2\pi} \right\} Z(\mathbf{s}) . \tag{39}$$

One vector case In a similar manner as for the orthogonal group, we may write for R = 1,

$$Z(s) = \frac{1}{(1+ist)^{N-K} [1-is(1-t)]^K}, \qquad Z(0) = 1,$$
(40)

such that

$$\mathcal{P}_{NK}(t) = (N-1) \int \frac{\mathrm{d}s}{2\pi} \, \frac{1}{(1+\mathrm{i}\,st)^{N-K} \, [1-\mathrm{i}s(1-t)]^K} \,, \tag{41}$$

in agreement with (12).

4. Two-point functions, examples and applications

In this section, we consider the case of R=2 column vectores, and thus the joint probability density $P_{NK}(t_1,t_2)$. That is the probability density of two probabilities t_1 and t_2 , where t_1 (t_2) is the probability for finding the system in a given K dimensional subspace when it is prepared in one (another) eigenstate. In practice, each realization of the system leads to one unitary matrix, representing the eigenstates, and each choice of two eigenstates leads to a pair of probabilities (t_1, t_2). Averaging over many different systems then yields the joint probability density $P_{NK}(t_1, t_2)$.

Again, we will treat the orthogonal group and the unitary group in separate subsections. In both cases it will be convenient to use the abbreviations $\alpha_j = 1 + is_j t_j$ and $\beta_j = 1 - is_j (1 - t_j)$.

4.1. Orthogonal case

Starting again from the general expression (25), we find for R=2 that

$$\det(\mathbf{A}) = (1 + is_1 t_1) (1 + is_2 t_2) + \tau^2 / 4 = \alpha_1 \alpha_2 + \tau^2 / 4,$$

$$\det(\mathbf{B}) = [1 - is_1 (1 - t_1)] [1 - is_2 (1 - t_2)] + \tau^2 / 4 = \beta_1 \beta_2 + \tau^2 / 4,$$
 (42)

and

$$Z(s) = \int \frac{d\tau}{2\pi} \frac{1}{(\alpha_1 \alpha_2 + \tau^2/4)^{(N-K)/2} (\beta_1 \beta_2 + \tau^2/4)^{K/2}}.$$
 (43)

Here, it turns out to be more convenient to postpone the integration over τ , and evaluate the integral over s_2 first.

$$\int \frac{\mathrm{d}s_2}{2\pi} Z(\mathbf{s}) = \int \frac{\mathrm{d}\tau}{2\pi} \frac{1}{\alpha_1^{(N-K)/2} \beta_1^{K/2}} \int \frac{\mathrm{d}s_2}{2\pi} \frac{1}{(\alpha_1' + \mathrm{i}s_2 t_2)^{(N-K)/2} [\beta_1' - \mathrm{i}s_2 (1 - t_2)]^{K/2}}
= \frac{2}{N-2} \frac{1}{\alpha_1^{(N-K)/2} \beta_1^{K/2}} \int \frac{\mathrm{d}\tau}{2\pi} \frac{I_{(N-K)/2, K/2}(t_2, 1 - t_2)}{[\beta_1' t_2 + \alpha_1' (1 - t_2)]^{N/2 - 1}}
= \frac{2 P_{NK}(t_2)}{N-2} \frac{1}{\alpha_1^{(N-K)/2} \beta_1^{K/2}} \int \frac{\mathrm{d}\tau}{2\pi} \frac{1}{[\beta_1' t_2 + \alpha_1' (1 - t_2)]^{N/2 - 1}}, \tag{44}$$

where $\alpha'_1 = 1 + \tau^2/(4\alpha_1)$, $\beta'_1 = 1 + \tau^2/(4\beta_1)$, and where we have used (A.7) and (A.5). We then find that

$$\int \frac{\mathrm{d}s_2}{2\pi} Z(s) = \frac{2 P_{NK}(t_2)}{N - 2} \frac{1}{\alpha_1^{(N-K)/2} \beta_1^{K/2}} \int \frac{\mathrm{d}\tau}{2\pi} \frac{1}{[1 + \tau^2 (t_2/\beta_1 + (1 - t_2)/\alpha_1)/4]^{N/2 - 1}}
= \frac{2 P_{NK}(t_2)}{N - 2} \frac{1}{\alpha_1^{(N-K)/2} \beta_1^{K/2}} \frac{1}{\sqrt{\pi}} \frac{\Gamma[(N-3)/2]}{\sqrt{t_2/\beta_1 + (1 - t_2)/\alpha_1} \Gamma(N/2 - 1)}
= \frac{\Gamma[(N-3)/2] P_{NK}(t_2)}{\sqrt{\pi} \Gamma(N/2)} \frac{1}{\alpha_1^{(N-K-1)/2} \beta_1^{(K-1)/2}} \frac{1}{\sqrt{t_2 \alpha_1 + (1 - t_2) \beta_1}}, (45)$$

where we used the integration formula (A.8) for real Lorentzian integrals. From this it follows

$$P_{NK}(t_1, t_2) = \frac{\Gamma[(N-3)/2] P_{NK}(t_2)}{\sqrt{\pi} \Gamma(N/2)} \frac{(N-3)^2/4}{Z(\mathbf{o})}$$

$$\times \int \frac{\mathrm{d}s_1}{2\pi} \, \frac{1}{\alpha_1^{(N-K-1)/2} \, \beta_1^{(K-1)/2} \sqrt{1 + \mathrm{i}s_1(t_1 + t_2 - 1)}} \,. \tag{46}$$

With

$$Z(\mathbf{o}) = \int \frac{d\tau}{2\pi} \frac{1}{(1+\tau^2/4)^{N/2}} = \frac{\Gamma[(N-1)/2]}{\sqrt{\pi} \Gamma(N/2)},$$
(47)

we therefore obtain

$$P_{NK}(t_1, t_2) = P_{NK}(t_2) \frac{N-3}{4\pi} \int \frac{\mathrm{d}s_1}{\alpha_1^{(N-K-1)/2} \beta_1^{(K-1)/2} \sqrt{1 + \mathrm{i}s_1(t_1 + t_2 - 1)}} . \tag{48}$$

4.2. Unitary case

For the unitary group, we start from the general expression (39) and for R=2, we find

$$\det(\mathbf{E}) = (1 + is_1 t_1) (1 + is_2 t_2) + |\tau|^2 / 4 = \alpha_1 \alpha_2 + |\tau|^2 / 4,$$

$$\det(\mathbf{F}) = [1 - is_1 (1 - t_1)] [1 - is_2 (1 - t_2)] + |\tau|^2 / 4 = \beta_1 \beta_2 + |\tau|^2 / 4, \quad (49)$$

and therefore

$$Z(s) = \int \frac{\mathrm{d}^2 \tau}{4\pi^2} \frac{1}{(\alpha_1 \,\alpha_2 + |\tau|^2/4)^{N-K} \,(\beta_1 \,\beta_2 + |\tau|^2/4)^K} \,. \tag{50}$$

Again, we will evaluate the integral over s_2 before that over τ . Thus,

$$\int \frac{\mathrm{d}s_2}{2\pi} Z(\mathbf{s}) = \int \frac{\mathrm{d}^2 \tau}{4\pi^2} \frac{1}{\alpha_1^{N-K} \beta_1^K} \int \frac{\mathrm{d}s_2}{2\pi} \frac{1}{(\alpha_1' + \mathrm{i}s_2 t_2)^{N-K} [\beta_1' - \mathrm{i}s_2 (1 - t_2)]^K}
= \frac{1}{N-1} \frac{1}{\alpha_1^{N-K} \beta_1^K} \int \frac{\mathrm{d}^2 \tau}{4\pi^2} \frac{I_{N-K,K}(t_2, 1 - t_2)}{[\beta_1' t_2 + \alpha_1' (1 - t_2)]^{N-1}}
= \frac{\mathcal{P}_{NK}(t_2)}{N-1} \frac{1}{\alpha_1^{N-K} \beta_1^K} \int \frac{\mathrm{d}^2 \tau}{4\pi^2} \frac{1}{[\beta_1' t_2 + \alpha_1' (1 - t_2)]^{N-1}} ,$$
(51)

where $\alpha'_1 = 1 + |\tau|^2/(4\alpha_1)$, $\beta'_1 = 1 + |\tau|^2/(4\beta_1)$, and where we have used (A.7) and (A.5) once more. Hence,

$$\int \frac{\mathrm{d}s_2}{2\pi} Z(\mathbf{s}) = \frac{\mathcal{P}_{NK}(t_2)}{N-1} \frac{1}{\alpha_1^{N-K} \beta_1^K} \int \frac{\mathrm{d}^2 \tau}{4\pi^2} \frac{1}{[1+|\tau|^2 (t_2/\beta_1 + (1-t_2)/\alpha_1)/4]^{N-1}}
= \frac{\mathcal{P}_{NK}(t_2)}{N-1} \frac{1}{\alpha_1^{N-K} \beta_1^K} \frac{1}{\pi (N-2)} \frac{1}{t_2/\beta_1 + (1-t_2)/\alpha_1}
= \frac{\mathcal{P}_{NK}(t_2)}{\pi (N-1)(N-2)} \frac{1}{\alpha_1^{N-K-1} \beta_1^{K-1}} \frac{1}{t_2 \alpha_1 + (1-t_2) \beta_1},$$
(52)

where we have used the integration formula (A.9) for complex Lorentzian integrals. From this it follows

$$\mathcal{P}_{NK}(t_1, t_2) = \frac{\mathcal{P}_{NK}(t_2)}{\pi (N-1)(N-2)} \frac{(N-2)^2}{Z(\mathbf{o})} \int \frac{\mathrm{d}s_1}{2\pi} \frac{1}{\alpha_1^{N-K-1} \beta_1^{K-1} \left[1 + \mathrm{i}s_1(t_1 + t_2 - 1)\right]} . (53)$$

With

$$Z(\mathbf{o}) = \int \frac{\mathrm{d}^2 \tau}{4\pi^2} \frac{1}{(1+|\tau|^2/4)^N} = \frac{1}{\pi (N-1)},$$
 (54)

we finally obtain

$$\mathcal{P}_{NK}(t_1, t_2) = \mathcal{P}_{NK}(t_2) \frac{N-2}{2\pi} \int \frac{\mathrm{d}s_1}{\alpha_1^{N-K-1} \beta_1^{K-1} \left[1 + \mathrm{i}s_1(t_1 + t_2 - 1)\right]} . (55)$$

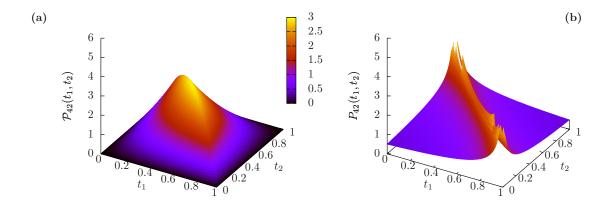


Figure 1. Joint probability distributions for N=4 and R=K=2. Panel (a) shows the unitary case, (57); panel (b) shows the orthogonal case, (61). The color box shown in panel (a) is also valid for panel (b).

4.3. Examples and applications

Small dimensions Here, we choose the dimensions as N=4 and R=K=2. The unitary case is much simpler than the orthogonal one, as the evaluation of the remaining integral can be done by a straight forward application of the residue theorem. Namely, from (55) we find

$$\mathcal{P}_{42}(t_1, t_2) = \frac{\mathcal{P}_{42}(t_2)}{\pi} \int \frac{\mathrm{d}s}{(1 + it_1 s) (1 - i(1 - t_1) s) (1 + i(t_1 + t_2 - 1) s)} .(56)$$

The integrand has three simple poles on the imaginary line, one pole above the point i, the other pole below the point -i, and the third pole below -i (above i) for $t_1 + t_2 < 1$ $(t_1 + t_2 > 1)$. With $\mathcal{P}_{42}(t_2) = 6 t_2 (1 - t_2)$ obtained from (13), we find

$$\mathcal{P}_{42}(t_1, t_2) = 12 \begin{cases} t_1 t_2 & : t_1 + t_2 < 1 \\ (1 - t_1) (1 - t_2) & : t_1 + t_2 > 1 \end{cases}$$
 (57)

In the orthogonal case, the calculation is more involved. According to (48) we have

$$P_{42}(t_1, t_2) = \frac{P_{42}(t_2)}{4\pi} \int \frac{\mathrm{d}s}{\sqrt{(1 + \mathrm{i}t_1 s) (1 - \mathrm{i}(1 - t_1) s) (1 + \mathrm{i}(t_1 + t_2 - 1) s)}} .(58)$$

Via the variable substitution $s \to \phi$ with $\tan \phi = 2 [t_1(1-t_1) s + i (t_1-1/2)]$, we arrive at

$$P_{42}(t_1, t_2) = \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{(t_1 t_2 - t_\mathrm{s}) \cos^2 \phi + \mathrm{i} t_\mathrm{s} \sin \phi \cos \phi}},$$
 (59)

where $t_s = (t_1 + t_2 - 1)/2$ and where we have used that $P_{42}(t) = 1$, cf. (10). Standard manipulations of the trigonometric expressions then lead to

$$P_{42}(t_1, t_2) = \frac{1}{\sqrt{2\sqrt{t_1 t_2(1 - t_1)(1 - t_2)}(1 + \cos \alpha)}} \frac{1}{\pi} \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1 - 2(1 + \cos \alpha)^{-1} \sin^2 \phi}}, (60)$$

where $2\cos\alpha = [t_1t_2 + (1-t_1)(1-t_2)]/\sqrt{t_1t_2(1-t_1)(1-t_2)}$. The remaining integral is identical to the complete elliptic integral of the first kind [26]. This allows us to write $P_{42}(t_1, t_2)$ in the following form

$$P_{42}(t_1, t_2) = \frac{1}{\pi(a+b)} K\left[\frac{2\sqrt{ab}}{a+b}\right], \tag{61}$$

where $a = \sqrt{t_1 t_2}$ and $b = \sqrt{(1 - t_1)(1 - t_2)}$.

Figure 1 shows the joint probability distributions for N=4 and K=2, for the unitary case in panel (a), and the orthogonal case in panel (b). The distribution function looks rather unspectacular in the unitary case. However, the orthogonal case shows some peculiarities worthwhile mentioning: First, there is a square root singularity in the (t_1, t_2) plane as one approaches the line $t_1 + t_2 = 1$, and second, the distribution function approaches finite values on the border of the (t_1, t_2) unit square. This is surprising, because anywhere outside the unit square, the distribution function must be equal to zero, of course. We have verified both analytic results with the help of random matrix simulations [27].

Asymmetric cases Let us start again with the unitary case. According to (55) $\mathcal{P}_{NK}(t_1, t_2) = \mathcal{P}_{NK}(t_2) I_{N-K-1,K-1}(t_1, t_2)$, with

$$I_{mk}(t_1, t_2) = \frac{m+k}{2\pi} \int \frac{\mathrm{d}s}{(1+\mathrm{i}st_1)^m (1-\mathrm{i}s(1-t_1))^k (1+\mathrm{i}st_s)},$$
 (62)

being the integral left to be evaluated, and $t_s = t_1 + t_2 - 1$. Still, the integrand has three poles on the imaginary axis, at i/t_1 (of order m), another at $-i/(1-t_1)$ (of order k), and a third simple pole at i/t_s . Applying the residue theorem, we find the following: For $t_s < 0$, the pole at i/t_1 is the only pole in the upper half plane, such that

$$I_{mk}(t_1, t_2) = i(m+k) \operatorname{Res} \left[\alpha_1^{-m} \beta_1^{-k} (1 + ist_s)^{-1}, i/t_1 \right].$$
 (63)

For $t_s > 0$, the pole at $-i/(1-t_1)$ is the only pole in the lower half plane, such that

$$I_{mk}(t_1, t_2) = -i(m+k) \operatorname{Res} \left[\alpha_1^{-m} \beta_1^{-k} (1 + ist_s)^{-1}, -i/(1 - t_1) \right].$$
 (64)

We calculate these residues with the help of a computer algebra system [28]. This allows to obtain exact analytic expressions even for large integers N and K. Here, we choose N=6 and K=2 and obtain

$$\mathcal{P}_{62}(t_1, t_2) = 80 \begin{cases} t_1 t_2 S_{31}(t_1, t_2) & : t_1 + t_2 < 1 \\ (1 - t_1)^3 (1 - t_2)^3 & : t_1 + t_2 > 1 \end{cases},$$
 (65)

where $S_{31}(t_1, t_2) = t_1 t_2(t_1 t_2 + 9) + 3 t_1^2 (1 - t_2) + 3 t_2^2 (1 - t_1) - 6 (t_1 + t_2) + 3$. This probability density is shown in figure 2, panel (a).

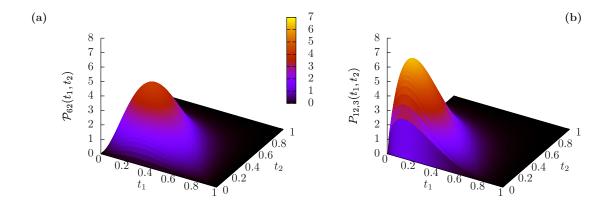


Figure 2. Panel (a) shows the joint probability density for N=6 and K=2 for the unitary case, (65); panel (b) shows the joint probability density for N=12, K=3 for the orthogonal case, (70). The color box shown in panel (a) is also valid for panel (b).

Now, let us consider the orthogonal case. Starting from (48), we apply the substitution $s_1 \to z = \sqrt{1 + is_1(t_1 + t_2 - 1)}$ to obtain

$$P_{NK}(t_1, t_2) = P_{NK}(t_2) J_{(N-K-1)/2, (K-1)/2}(t_1, t_2) , \qquad (66)$$

where

$$J_{mk}(t_1, t_2) = \frac{m + k - 1/2}{i\pi} \frac{t_s^{m+k-1}}{t_1^m (1 - t_1)^k} \int \frac{dz}{[z^2 - (1 - t_2)/t_1]^m [t_2/(1 - t_1) - z^2]^k} . \tag{67}$$

Here, we can again apply the residue theorem, but only when m and k are both integers. This means that K must be odd and N must be even. Below, we will assume that this is the case.

For $t_s = t_1 + t_2 - 1 > 0$, the integration path comes from infinity, following the diagonal in the lower right quadrant of the complex plane, approaching the origin it leaves the diagonal upwards to cross the real axis at z = 1. Then the path continues towards infinity again, approaching the diagonal in the upper right quadrant. For $m+k \geq 1$, this integration path can be considered as a closed loop without changing the value of the integral. Moreover, since $t_s > 0$ implies $t_2/(1-t_1) > 1 > (1-t_2)/t_1$, the only pole inside this loop is at $b = \sqrt{t_2/(1-t_1)}$. Therefore, we find with $a = \sqrt{(1-t_2)/t_1}$:

$$J_{mk}^{+}(t_1, t_2) = (m + k - 1/2) \frac{-2 t_s^{m+k-1}}{t_1^m (1 - t_1)^k} \operatorname{Res}[(z^2 - a^2)^{-m} (b^2 - z^2)^{-k}, b],$$
 (68)

where the minus sign comes from the fact that the orientation of the path is mathematically negative. For $t_{\rm s} < 0$ the orientation of the integration path changes sign. Furthermore, $t_{\rm s} < 0$ implies b < 1 < a such that the only pole within the

integration path is now at a. Therefore, we find for this case

$$J_{mk}^{-}(t_1, t_2) = (m + k - 1/2) \frac{2 t_s^{m+k-1}}{t_1^m (1 - t_1)^k} \operatorname{Res}[(z^2 - a^2)^{-m} (b^2 - z^2)^{-k}, a].$$
 (69)

As an example, let us choose N = 12 and K = 3. Using [28] again, we find

$$P_{12,3}(t_1, t_2) = \frac{72}{7\pi} \begin{cases} \sqrt{t_1 t_2} \, \mathcal{S}_{12,3} & : t_1 + t_2 < 1 \\ 16 \, (1 - t_1)^{7/2} \, (1 - t_2)^{7/2} & : t_1 + t_2 > 1 \end{cases} , \tag{70}$$

where $S_{12,3} = 16 t_1^3 t_2^3 - 56 t_1^2 t_2^3 + 70 t_1 t_2^3 - 35 t_2^3 - 56 t_1^3 t_2^2 + 196 t_1^2 t_2^2 - 245 t_1 t_2^2 + 105 t_2^2 + 70 t_1^3 t_2 - 245 t_1^2 t_2 + 280 t_1 t_2 - 105 t_2 - 35 t_1^3 + 105 t_1^2 - 105 t_1 + 35$. This probability density is shown in figure 2, panel (b).

For the orthogonal case, we choose values for N and K which are approximately twice as large as in the unitary case. The reason is that for the one-point functions it holds that $\mathcal{P}_{NK}(t) = P_{2N,2K}(t)$ as discussed before (12). Thus one could have expected that the probability densities $\mathcal{P}_{62}(t_1, t_2)$ and $P_{12,3}(t_1, t_2)$ would at least look similar. Note though that we choose $P_{12,3}(t_1, t_2)$ instead of $P_{12,4}(t_1, t_2)$ because in this case it is much easier to evaluate the integral in (48), as explained above. Comparing the two cases in figure 2, one can clearly see that $P_{12,3}(t_1, t_2)$, shown in panel (b), has much steeper slopes as t_1 or t_2 tend to zero, than $\mathcal{P}_{62}(t_1, t_2)$, shown in panel (a).

Measurement outcomes for orthonormal states At last, we would like to mention two instances, where our results can describe statistical properties of real physical systems. The first example, discussed here, is about the probabilities of a measurement outcome for different eigenstates of a quantum system, for which the quantum chaos conjecture applies [29, 30, 31, 32]. When applied to the eigenstates of a Hamiltonian quantum system, the conjecture states that in the semiclassical limit these eigenstates will have the same statistical properties as the column vectors of elements from O(N) or U(N), depending on whether the system has an anti-unitary symmetry (usually related to time-reversal) or not [5, 6]. We will call the Hamiltonian of such a quantum system a "quantum-chaotic" Hamiltonian.

In this situation, we consider the K-fold degenerate measurement outcome of a projective measurement, as described in (1) and (2). Then, our results describe the joint probability distribution for the probabilities t_{ξ} . In order to compare experimental data or numerical simulations with our results, one needs to perform an average over samples of different eigenstates and different systems. Samples of different quantum-chaotic systems may be obtained by changing some parameter of the Hamiltonian taking care to remain in a region where the quantum chaos conjecture still applies.

Our results can be used also, if the system is prepared in a mixture of eigenstates of the quantum chaotic Hamiltonian. That might be a thermal state with canonical distribution. Such a mixture may be described by the density matrix

$$\varrho = \sum_{\xi=1}^{R} p_{\xi} |\psi_{\xi}\rangle\langle\psi_{\xi}|, \qquad \sum_{\xi=1}^{R} p_{\xi} = 1.$$
 (71)

Then, the probability of the above measurement outcome is given by

$$\bar{t} = \operatorname{tr}(\hat{P}_1 \,\varrho) = \sum_{\xi=1}^R p_\xi \, t_\xi \,, \tag{72}$$

where \hat{P}_1 denotes the projector on the K-dimensional eigenspace, corresponding to that measurement outcome. In order to calculate the probability distribution of \bar{t} , for fixed but arbitrary occupation probabilities p_{ξ} , the joint probability density for the individual t_{ξ} would be the ideal starting point.

The quantum chaos conjecture has been extended to classical wave systems, also. Examples are two-dimensional microwave cavities [33] and one-dimensional microwave networks [34], elastomechanic systems [35], and acoustic waves [36], among others. In those cases, the main difference is that the t_{ξ} will describe intensities rather than probabilities.

Universal conductance fluctuations In this area, one is interested in the distribution of the conductance of charge carriers through mesoscopic structures [37, 38]. The theoretical description is based on the Landauer-Büttiker formula [39], and a maximum entropy principle [19], which predicts the transport properties to be statistical in nature, and described by appropriate ensembles of scattering matrices [7, 8]. At present, we can make contact with the statistics of conductances in the unitary case, only. Again, this is the case which describes systems without time-reversal symmetry.

Hence, consider a two terminal scattering problem, with K modes in one lead and N-K modes in the other lead. Then, according to the Landauer-Büttiker formalism, the scattering matrix \mathbf{S} is a $N \times N$ -matrix where the off-diagonal $K \times (N-K)$ dimensional block-matrix $\boldsymbol{\tau}$ contains the amplitudes for transitions from the modes of one lead to those of the other lead. In terms of the elementary conductance unit $g_0 = 2e^2/h$, the conductance is given by the simple formula $g = \operatorname{tr}(\boldsymbol{\tau}^{\dagger}\boldsymbol{\tau})$. In the universal regime, i.e. when the maximum entropy principle applies, \boldsymbol{S} may be taken as a random matrix from U(N) provided with the Haar measure, which is the reason that our results apply. Usually, the statistical properties of the conductance are computed from the distribution of the eigenvalues of $\boldsymbol{\tau}^{\dagger}\boldsymbol{\tau}$. However, it is also possible to express g as a sum of partial conductances

$$g = \sum_{\xi=1}^{N-K} \sum_{j=1}^{K} |\tau_{j\xi}|^2 = \sum_{\xi=1}^{N-K} t_{\xi} . \tag{73}$$

In distinction to the eigenvalues of $\tau^{\dagger} \tau$, the partial conductances t_{ξ} may be directly measurable by mode-selective measurements. Also, just as in the case of the closed systems discussed above, there might be situations where the total conductance is given by a weighted sum of the partial conductances for instance when the modes in one lead are occupied with different probabilities according to some temperature profile. In such a case, g can no longer be expressed in terms of the eigenvalues of $\tau^{\dagger} \tau$.

In [40], the authors collected known results and presented a new approach to calculate the distribution of the conductance in two-terminal quantum transport with an arbitrary number of modes in each lead. As an illustration, we consider the case of a scattering system with two modes in each lead, which corresponds to our unitary case with N = 4 and K = 2, considered above. In this case, the quantities t_1 and t_2 are the partial conductances such that $g = t_1 + t_2$. Then, starting from the probability density given in (57) we can recover p(g) from [40] as

$$p(g) = \int dt_1 dt_2 \, \delta(g - t_1 - t_2) \, \mathcal{P}_{42}(t_1, t_2) = \int_0^1 dt \, \mathcal{P}_{42}(t, g - t)$$

$$= \begin{cases} 2 g^3 &: 0 \le g \le 1 \\ 2 (2 - g)^3 &: 1 \le g \le 2 \end{cases}$$
(74)

5. Conclusions

In this paper, we considered the partial sums of absolute values squared of a random orthonormal basis with R elements in a N dimensional vector space. We derived general expressions for the joint probability density of these partial sums, and explained how these results can be related to experiments. Distinguishing between the vectors being real (orthogonal case) or complex (unitary case), the general results are given in (25) and (39), respectively. They still involve R(R+1)/2 integrals, but for R=2, we could eventually evaluate all integrals and arrive at explicit analytic expressions.

Obviously, the joint probability distributions are important only as long as correlations between the partial sums are important. Otherwise the one-point functions would be enough to describe their statistical properties. For small dimensions, as the ones considered in our examples, such correlations are present. However, for increasing N and K, it is natural to expect that correlations become less important.

Acknowledgments

We acknowledge financial support from grant CONACyT CB-2009/129309.

Appendix: Integration formulas

Integral related to the one vector case

$$I_{mk}(a,b) = (m+k-1) \int \frac{\mathrm{d}s}{2\pi} \frac{1}{(1+\mathrm{i}a\,s)^m \,(1-\mathrm{i}b\,s)^k} = \frac{\Gamma(m+k)}{\Gamma(m)\,\Gamma(k)} \,\frac{a^{k-1}\,b^{m-1}}{(a+b)^{m+k-1}} \,. \quad (A.1)$$

This integral is well defined for m,k being integers or half integers, with $m+k \geq 2$ and 0 < a,b < 1. To prove this integration formula, we first use integration by parts to demonstrate

$$I_{mk}(a,b) = \frac{a m}{b (k-1)} I_{m+1,k-1}(a,b) . (A.2)$$

This yields, in the case of integer k:

$$I_{mk}(a,b) = \frac{\Gamma(m+k-1)}{\Gamma(m)\Gamma(k)} \left(\frac{a}{b}\right)^{k-1} I_{n-1,1}(a,b) . \tag{A.3}$$

For k being a half integer (2k: odd) we find instead

$$I_{mk}(a,b) = \frac{\Gamma(m+k-1/2)}{\Gamma(m)\Gamma(k)} \left(\frac{a}{b}\right)^{k-1/2} I_{n-1/2,1/2}(a,b) . \tag{A.4}$$

In the first case, (A.3), we directly use the residue theorem to evaluate $I_{n-1,1}(a,b)$, while in the second case, (A.4), we first apply the variable transformation $s \to z = \sqrt{1 - ibs}$, before applying the residue theorem. In both cases, the results lead to the same formula as given in (A.1). For the orthogonal and the unitary case, we find respectively

$$P_{NK}(t) = I_{(N-K)/2,K/2}(t,1-t)$$
, $\mathcal{P}_{NK}(t) = I_{N-K,K}(t,t-1)$. (A.5)

A generalization of the integral above

For the case R=2, we need a generalized version of the integration formula in (A.1). This is

$$I_{mk}(\alpha, a; \beta, b) = (m + k - 1) \int \frac{\mathrm{d}s}{2\pi} \frac{1}{(\alpha + \mathrm{i}a\,s)^m (\beta - \mathrm{i}b\,s)^k}, \tag{A.6}$$

for complex parameters α, β with real parts larger than one. With the help of simple algebraic manipulations it can be shown that

$$I_{mk}(\alpha, a; \beta, b) = I_{mk}(a, b) \left(\frac{a+b}{\beta a + \alpha b}\right)^{m+k-1}.$$
 (A.7)

Lorentzian integrals

With $m \geq 1$, we obtain

$$\frac{1}{2\pi} \int \frac{d\tau}{(1+c\tau^2)^m} = \frac{\Gamma(m-1/2)}{2\sqrt{\pi c} \Gamma(m)}.$$
 (A.8)

This formula can be obtained from the residue theorem. For the unitary case, we also need the following two-dimensional (complex) version:

$$\frac{1}{4\pi^2} \int \frac{\mathrm{d}^2 \tau}{(1+c|\tau|^2)^m} = \frac{1}{4\pi^2} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}r \, \frac{r}{(1+c\,r^2)^m} = \frac{1}{2\pi} \left. \frac{(1+cr^2)^{1-m}}{2c\,(1-m)} \right|_{r=0}^{\infty}$$

$$= \frac{1}{4\pi\,c\,(m-1)} . \tag{A.9}$$

Delta function for complex arguments

For $w \in C$, we mat write

$$\delta^{2}(w) = \delta(\operatorname{Re}(w)) \, \delta(\operatorname{Im}(w)) = \int \frac{\mathrm{d}x \, \mathrm{d}y}{4\pi^{2}} \, \mathrm{e}^{-\mathrm{i}[y \operatorname{Re}(w) + x \operatorname{Im}(w)]} = \int \frac{\mathrm{d}x \, \mathrm{d}y}{4\pi^{2}} \, \mathrm{e}^{-\mathrm{i}\operatorname{Im}[(x + \mathrm{i}y) \, w]}$$
$$= \int \frac{\mathrm{d}^{2}z}{4\pi^{2}} \, \mathrm{e}^{-\mathrm{i}\operatorname{Im}(zw)} \,. \tag{A.10}$$

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