SHARP ESTIMATES FOR PSEUDO-DIFFERENTIAL OPERATORS OF TYPE (1,1) ON TRIEBEL-LIZORKIN AND BESOV SPACES

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Abstract. Pseudo-differential operators of type $(1,1)$ and order m are continuous from $F_p^{s+m,q}$ to $F_p^{s,q}$ if $s>\max(0,d(1/p-1),d(1/q-1))$ for $0<p<\infty$, and from $B_p^{s+m,q}$ to $B_p^{s,q}$ if $s>\max(0,d(1/p-1))$ for $0<p\le\infty$. In this work we prove the sharpness of the conditions on s, and extend the F-boundedness result to $p=\infty$. We also prove that the operators map $F_\infty^{m,1}$ into bmo when $s=0$. The boundedness of the operators also hold for arbitrary s with Hörmander's condition on the twisted diagonal.

1. Introduction

Let $S(\mathbb{R}^d)$ denote the Schwartz space and $S'(\mathbb{R}^d)$ the space of tempered distributions. For $r>0$ let $\mathcal{E}(r)$ be the space of tempered distributions whose Fourier transforms are supported in $\{\xi : |\xi|\le 2r\}$. Let $\mathcal{D}$ denote the set of all dyadic cubes in $\mathbb{R}^d$ and $\mathcal{D}_k$ the subset of $\mathcal{D}$ consisting of the cubes with sidelength 2^{-k} for $k\in\mathbb{Z}$. For $f\in S(\mathbb{R}^d)$ the Fourier transform is defined by the formula

$$\widehat{f}(\xi)=\int_{\mathbb{R}^d} f(x)e^{-2\pi i\langle x,\xi\rangle}dx \ (\xi\in\mathbb{R}^d)$$

and denote by $f^\vee$ the inverse Fourier transform of f. We also extend these transforms to S'.

We recall the definitions of Besov sapces and Triebel-Lizorkin spaces from [5] and [15]. Let ϕ be a smooth function so that $\widehat{\phi}$ is supported in $\{\xi : 2^{-1}\le|\xi|\le 2\}$ and $\sum_{k\in\mathbb{Z}}\widehat{\phi_k}(\xi)=1$ for $\xi\neq 0$ where $\phi_k:=2^{kd}\phi(2^k\cdot)$. Let $\Phi:=1-\sum_{k=1}^\infty\phi_k$. Then we define convolution operators Π_0 and Π_k by $\Pi_0 f:=\Phi * f$ and $\Pi_k f:=\phi_k * f$. For $0<p,q\le\infty$ and $s\in\mathbb{R}$ the (inhomogeneous) Besov spaces $B_p^{s,q}$ are defined as a subspace of S' with (quasi-)norms

$$\|f\|_{B_p^{s,q}}:=\big\|\Pi_0 f\big\|_{L^p}+\big\|\{2^{sk}\Pi_k f\}_{k=1}^\infty\big\|_{l^q(L^p)}.$$

For $0<p<\infty$, $0<q\le\infty$, and $s\in\mathbb{R}$ we define (inhomogeneous) Triebel-Lizorkin spaces $F_p^{s,q}$ to be a subspace of S' with norms

$$\|f\|_{F_p^{s,q}}=\|\Pi_0 f\|_{L^p}+\big\|\{2^{sk}\Pi_k f\}_{k=1}^\infty\big\|_{L^p(l^q)}$$

for $p<\infty$ or for $p=q=\infty$, and

$$\|f\|_{F_\infty^{s,q}}:=\|\Pi_0 f\|_{L^\infty}+\sup_{l(P)<1}\Big(\frac{1}{|P|}\int_P\sum_{k=-\log_2 l(P)}^\infty 2^{skq}|\Pi_k f(x)|^q dx\Big)^{1/q}$$

where $l(P)$ means the side length of cube P and the supremum is taken over all dyadic cubes whose side length $l(Q)$ is less than 1. According to those norms, the spaces are quasi-Banach spaces (Banach spaces if $p\ge 1, q\ge 1$). Note that the spaces are a generalization of many standard function spaces such as L^p spaces, Sobolev spaces, and Hardy spaces. We

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recall $L^p = F_p^{0,2}$ for $1 < p < \infty$, $h^p = F_p^{0,2}$ for $0 < p < \infty$, $L_s^p = F_p^{s,2}$ for $s > 0, 1 < p < \infty$, and $bmo = F_\infty^{0,2}$.

A symbol a in Hörmander's class $\mathcal{S}_{\rho,\delta}^m$ is a smooth function defined on $\mathbb{R}^d \times \mathbb{R}^d$, satisfying that for all multi-indices α and β there exists a constant $c_{\alpha,\beta}$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le c_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|} \text{ for } (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

and the corresponding pseudo-differential operator $T_{[a]}$ is given by

$$T_{[a]}f(x) = \int_{\mathbb{R}^d} a(x,\xi)\widehat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}d\xi, \quad f \in S(\mathbb{R}^d).$$

Denote by $Op\mathcal{S}_{\rho,\delta}^m$ the class of pseudo-differential operators with symbols in $\mathcal{S}_{\rho,\delta}^m$.

It is well known that for $0 \le \delta \le \rho \le 1$ the operator $T_{[a]} \in Op\mathcal{S}_{\rho,\delta}^m$ maps S continuously into itself. Furthermore, unless $\delta = \rho = 1$ the adjoint operator of $T_{[a]} \in Op\mathcal{S}_{\rho,\delta}^m$ belongs to the same type of pseudo-differential operators (see [12, Appendix] and [7, p.94]) and thus $T_{[a]}$ extends via duality to a mapping from S' into itself. However when $\rho = \delta = 1$ this extension is not valid. In this case it was proved by Ching [2] that not all operators of order 0 are L^2 continuous. Afterwards Stein first proved that all operators in $Op(\mathcal{S}_{1,1}^0)$ are bounded on $H^s(= L_s^2)$ for $s > 0$ in his unpublished work and Meyer [11] improved this result by proving the continuity from L_{s+m}^p to L_s^p with $s > 0$ for $1 < p < \infty$. Let

$$\tau_{p,q} = \max\left(0, d(1/p-1), d(1/q-1)\right)$$

and

$$\tau_p = \max\left(0, d(1/p-1)\right).$$

Torres [16] and Johnsen [8] extended the continuity of $T_{[a]}$ to Triebel-Lizorkin spaces $F_p^{s,q}$ under the assumption $s > \tau_{p,q}$ for $p < \infty$. Let $0 < p < \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$. Suppose $m \in \mathbb{R}$ and $a \in \mathcal{S}_{1,1}^m$. Then

$$T_{[a]} : F_p^{s+m,q} \to F_p^{s,q} \quad \text{if} \quad s > \tau_{p,q}. \tag{1.1}$$

On the other hand, for $0 < p, q \le \infty$

$$T_{[a]} : B_p^{s+m,q} \to B_p^{s,q} \quad \text{if } s > \tau_p. \tag{1.2}$$

It is natural to ask whether the assumption $s > \tau_{p,q}$ in (1.1) can be replaced by the seemingly more natural condition $s > \tau_p$ as in (1.2). We also ask about boundedness results of $T_{[a]} \in Op\mathcal{S}_{1,1}^m$ when $p = \infty$.

In this paper, we prove that the q-dependence cannot be dropped in (1.1). In fact, we will prove that the assumption $s > \tau_{p,q}$ in (1.1) and $s > \tau_p$ in (1.2) are necessary conditions for the boundedness. Moreover, we shall extend the above F-boundedness results of $T_{[a]} \in \mathcal{S}_{1,1}^m$ to $p = \infty$ with $s > \tau_{\infty,q} = \tau_q$. In the last section, we present some other boundedness results for the case $s \le 0$.

Theorem 1.1 (Sharpness of $s > \tau_{p,q}$ and $s > \tau_p$). *Let $0 < p, q \le \infty$ and $m \in \mathbb{R}$.*

(1) Suppose $0 < p < \infty$ or $p = q = \infty$. If $s \le \tau_{p,q}$ then there exists $a \in \mathcal{S}_{1,1}^m$ so that $\left\|T_{[a]}\right\|_{F_p^{s+m,q} \to F_p^{s,q}} = \infty$.

(2) If $s \le \tau_p$ then there exists $a \in \mathcal{S}_{1,1}^m$ so that $\left\|T_{[a]}\right\|_{B_p^{s+m,q} \to B_p^{s,q}} = \infty$.

Theorem 1.2. *Let $m \in \mathbb{R}$, $0 < q < \infty$, and let μ be an integer such that $\mu \geq 1$. Suppose $a \in \mathcal{S}^m_{1,1}$. If $s > \tau_q$ then*

$$\sup_{P \in \mathcal{D}_\mu} \Big(\frac{1}{|P|} \int_P \sum_{k=\mu}^{\infty} 2^{skq} \big| \Pi_k T_{[a]} f(x) \big|^q \Big)^{1/q}$$

$$\lesssim \sup_{0 \leq k \leq \mu-1} \big\| 2^{k(s+m)} \Pi_k f \big\|_{L^\infty} + \sup_{R \in \mathcal{D}_\mu} \Big(\frac{1}{|R|} \int_R \sum_{k=\mu}^{\infty} 2^{(s+m)kq} \big| \Pi_k f(x) \big|^q \Big)^{1/q}$$

Here the implicit constant of the inequality is independent of μ.

Remark. We will give a short proof of the embedding $F^{s+m,q}_\infty \hookrightarrow F^{s+m,\infty}_\infty$ in Section 2, which shows that Theorem 1.2 implies $T_{[a]} : F^{s+m,q}_\infty \to F^{s,q}_\infty$ if $s > \tau_q$. Furthermore, Theorem 1.2 is sharp in the following sense.

Theorem 1.3. *Let $0 < q \leq \infty$ and $m \in \mathbb{R}$. Let μ be an integer such that $\mu \geq 1$. If $s \leq \tau_q$ then there exists $a \in \mathcal{S}^m_{1,1}$ and $f \in S'$ such that*

$$\sup_{0 \leq k \leq \mu-1} \big\| 2^{k(s+m)} \Pi_k f \big\|_{L^\infty} + \sup_{R \in \mathcal{D}_\mu} \Big(\frac{1}{|R|} \int_R \sum_{k=\mu}^{\infty} 2^{(s+m)kq} \big| \Pi_k f(x) \big|^q \Big)^{1/q} < \infty,$$

$$\sup_{P \in \mathcal{D}_\mu} \Big(\frac{1}{|P|} \int_P \sum_{k=\mu}^{\infty} 2^{skq} \big| \Pi_k T_{[a]} f(x) \big|^q \Big)^{1/q} = \infty.$$

Remark. Torres [16] applied atoms and molecules for $F^{s,q}_p$ to prove (1.1). Every $f \in F^{s,q}_p$ can be written as $f = \sum_Q s_Q A_Q$ where $\{s_Q\}_Q$ is a sequence of complex numbers in $f^{s+m,q}_p$ and A_Q's are atoms for $F^{s+m,q}_p$ (see [5] for more detail). Then he defined $\widetilde{T_{[a]}} f(x) := \sum_Q s_Q T_{[a]} A_Q$ and proved that $\widetilde{T_{[a]}}$ maps $F^{s+m,q}_p$ to $F^{s,q}_p$ by showing $T_{[a]}$ maps atoms for $F^{s+m,q}_p$ to molecules for $F^{s,q}_p$ for $0 < p \leq \infty$. Note that $\widetilde{T_{[a]}}$ agrees with $T_{[a]}$ on $F^{s,q}_p$ for $0 < p < \infty$ by the density of S in $F^{s,q}_p$ for $0 < p, q < \infty$. However, we should be careful to say this argument implies the boundedness of the operator $T_{[a]}$ when $p = \infty$ because $T_{[a]}$ is not continuous on S'. In fact, a rigorous definition of $T_{[a]}$ on S' was first given in [9] and we follow it on $F^{s,q}_\infty$, which coincides with

$$T_{[a]} f = \lim_{N \to \infty} \sum_{k=0}^{N} \sum_{j=0}^{N} T_{[a_{j,k}]} f$$

whenever the limit converges in S', where $a_{j,k}(x, \xi)$ is defined in (4.1).

This paper is organized as follows. We will give some maximal inequalities in Section 2 and key lemmas for the proof of Theorem 1.2 in Section 3. Then we shall prove Theorem 1.2 in Section 4. We provide basic settings for the proof of Theorem 1.1 and 1.3 in Section 5 and construct some counter examples to prove Theorem 1.1 and 1.3 in Section 6∼8. Finally, we discuss other mapping properties of $T_{[a]} \in Op\mathcal{S}^m_{1,1}$ for $s \leq 0$ in Section 9.

2. Some maximal inequalities and embedding theorems for F-spaces

A crucial tool in theory of function spaces is maximal inequalities of Fefferman-Stein [4] and Peetre [14]. Denote by $\mathcal{M}$ the Hardy-Littlewood maximal operator and for $0 < t < \infty$

let $\mathcal{M}_t u = \big(\mathcal{M}(|u|^t)\big)^{1/t}$. Then for $0<r<p,q<\infty$

$$\Big\Vert \Big(\sum_k (\mathcal{M}_r u_k)^q\Big)^{1/q}\Big\Vert_{L^p} \lesssim \Big\Vert \Big(\sum_k |u_k|^q\Big)^{1/q}\Big\Vert_{L^p}. \tag{2.1}$$

Note that (2.1) also holds when $q=\infty$. Now for $k\in\mathbb{Z}$ and $\sigma>0$ we define the Peetre maximal operator $\mathfrak{M}_{\sigma,2^k}$ by

$$\mathfrak{M}_{\sigma,2^k}u(x) := \sup_{y\in\mathbb{R}^d}\frac{|u(x-y)|}{(1+2^k|y|)^\sigma}.$$

As shown in [14] one has the majorization

$$\mathfrak{M}_{d/r,2^k}u(x)\lesssim \mathcal{M}_r u(x), \tag{2.2}$$

if $u\in\mathcal{E}(2^k)$. Then via (2.1) the following maximal inequality holds. Suppose $0<p<\infty$ and $0<q\le\infty$. Then for $u_k\in\mathcal{E}(2^k)$

$$\Big\Vert \Big(\sum_k (\mathfrak{M}_{d/r,2^k} u_k)^q\Big)^{1/q}\Big\Vert_{L^p} \lesssim \Big\Vert \Big(\sum_k |u_k|^q\Big)^{1/q}\Big\Vert_{L^p} \quad \text{for } r<\min\{p,q\}. \tag{2.3}$$

When $p=\infty$ and $0<q<\infty$ the above maximal inequalities do not hold and we, instead, apply the following "$F_\infty^{s,q}$-variant" of the maximal inequalities. For $\epsilon\ge 0$, $r>0$, and $k\in\mathbb{Z}$, define a maximal operator

$$\begin{aligned}\mathcal{M}_r^{k,\epsilon}f(x) \;\; &:= \;\; \sup_{2^k l(Q)\le 1}\Big(\frac{1}{|Q|}\int_Q |f(x)|^r dx\Big)^{1/r}\\ &\quad + \sup_{2^{-k}l(Q)>1}\big(2^k l(Q)\big)^{-\epsilon}\Big(\frac{1}{|Q|}\int_Q |f(x)|^r dx\Big)^{1/r}.\end{aligned}$$

We observe that when $\epsilon=0$ we have $\mathcal{M}_r^{k,0}f(x)\sim\mathcal{M}_r f(x)$ and for $\epsilon\ge 0$

$$\mathcal{M}_r^{k,\epsilon}f(x)\lesssim \mathcal{M}_r f(x).$$

Then we have

Lemma 2.1. [13] *Let $0<r<q<\infty$ and $\epsilon>0$. Let $A>0$ and suppose that $u_k\in\mathcal{E}(A2^k)$ for all $k\in\mathbb{Z}$. Let $\mu\in\mathbb{Z}$.*

$$\sup_{P\in\mathcal{D}_\mu}\Big(\frac{1}{|P|}\int_P\sum_{k=\mu}^\infty\big(\mathcal{M}_r^{k,\epsilon}u_k(x)\big)^q dx\Big)^{1/q} \;\; \lesssim_A \;\; \sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=\mu}^\infty |u_k(x)|^q dx\Big)^{1/q}.$$

Here, the implicit constant of the inequality is independent of μ.

The pointwise estimate (2.2) does not hold when we replace $\mathcal{M}_r$ by $\mathcal{M}_r^{k,\epsilon}$, but we instead have the following estimate. Let $A>0$ and $u_k\in\mathcal{E}(A2^k)$. Then

$$\mathfrak{M}_{d/r,2^k}u_k(x)\lesssim_A \mathcal{M}_t^{k,d(1/r-1/t)}u_k(x) \quad \text{for} \quad r<t, \tag{2.4}$$

uniformly in k.

As an immediate consequence of the pointwise estimate and Lemma 2.1 we obtain the following maximal inequality.

Lemma 2.2. [13] *Let $0<r<q<\infty$ and $A>0$. Suppose that $u_k\in\mathcal{E}(A2^k)$ for all $k\in\mathbb{Z}$ and fix $\mu\in\mathbb{Z}$. Then*

$$\sup_{P\in\mathcal{D}_\mu}\Big(\frac{1}{|P|}\int_P\sum_{k=\mu}^\infty\big(\mathfrak{M}_{d/r,2^k}u_k(x)\big)^q dx\Big)^{1/q} \;\; \lesssim_A \;\; \sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=\mu}^\infty |u_k(x)|^q dx\Big)^{1/q}.$$

Here, the implicit constant of the inequality is independent of μ.

Note that Lemma 2.2 is sharp in the sense that if $r \geq q$ then there exists a sequence $\{u_k\}$ in $\mathcal{E}(2^k)$ for which the inequality does not hold.

We also observe that, as an application of Lemma 2.2, for $\mu \in \mathbb{Z}$, $q_1 < q_2 < \infty$, and $u := \{u_k\}_{k\in\mathbb{Z}}$ we have

$$\mathcal{V}_{\mu,q_2}[u] \lesssim \mathcal{V}_{\mu,q_1}[u], \tag{2.5}$$

provided that each u_k is defined as in Lemma 2.2, where

$$\mathcal{V}_{\mu,q}[u] := \sup_{P\in\mathcal{D},l(P)\leq 2^{-\mu}} \Big(\frac{1}{|P|}\int_P \sum_{k=-\log_2 l(P)}^{\infty} |u_k(x)|^q dx\Big)^{1/q}.$$

Indeed, for fixed $Q \in \mathcal{D}_k$ and $\sigma > 0$

$$\|\mathfrak{M}_{\sigma,2^k} u_k\|_{L^\infty(Q)} \lesssim_\sigma \inf_{y\in Q} \mathfrak{M}_{\sigma,2^k} u_k(y) \tag{2.6}$$

and then for $k \geq \mu$, $P \in \mathcal{D}_\mu$, and $\sigma > d/q_1$

$$\begin{aligned}
\|u_k\|_{L^\infty(P)} &\leq \sup_{Q\in\mathcal{D}_k, Q\subset P} \big\|\mathfrak{M}_{\sigma,2^k} u_k\big\|_{L^\infty(Q)} \\
&\lesssim_\sigma \sup_{Q\in\mathcal{D}_k, Q\subset P} \Big(\frac{1}{|Q|}\int_Q \big(\mathfrak{M}_{\sigma,2^k} u_k(y)\big)^{q_1} dy\Big)^{1/q_1} \lesssim \mathcal{V}_{\mu,q_1}[u]
\end{aligned}$$

where we used Lemma 2.2 in the last inequality. By applying $|u_k(x)|^{q_2} \lesssim \big(\mathcal{V}_{\mu,q_1}[u]\big)^{q_2-q_1} |u_k(x)|^{q_1}$ for $x \in P$, one can prove (2.5).

Furthermore, by using (2.6) and Lemma 2.2 we also obtain that for $0 < q < \infty$ and $\mu \in \mathbb{Z}$

$$\sup_{k\geq\mu} \|u_k\|_{L^\infty} \lesssim \mathcal{V}_{\mu,q}[u].$$

Then together with (2.5), this implies $F_\infty^{s,q_1} \hookrightarrow F_\infty^{s,q_2}$ for all $0 < q_1 < q_2 \leq \infty$.

3. Key Lemmas

Lemma 3.1. *Let $A > 1$, $0 < q < \infty$, $s \in \mathbb{R}$, and $\{u_j\}_{j=0}^\infty$ be a sequence of tempered distributions on $\mathbb{R}^d$ satisfying $Supp(\widehat{u_0}) \subset \{\xi : |\xi| \leq A\}$, $Supp(\widehat{u_j}) \subset \{\xi : 2^j/A \leq |\xi| \leq 2^j A\}$ for $j \geq 1$. Let $\mu \in \mathbb{Z}_+$ and $P \in \mathcal{D}_\mu$. Then there exists $h \in \mathbb{Z}$ such that*

$$\Big(\frac{1}{|P|}\int_P \sum_{n=\mu}^\infty 2^{snq}\Big|\Pi_n\Big(\sum_{k=0}^\infty u_k\Big)(x)\Big|^q dx\Big)^{1/q} \lesssim \sup_{R\in\mathcal{D}_\mu} \Big(\frac{1}{|P|}\int_P \sum_{k=\max(0,\mu-h)}^\infty 2^{skq}|u_k(x)|^q dx\Big)^{1/q}$$

where the implicit constant in the inequality is independent of μ and P.

Lemma 3.2. *Let $A > 1$, $0 < q < \infty$, $s > \tau_q$, and $\{u_k(x)\}_{k=0}^\infty$ be a sequence of tempered distributions on $\mathbb{R}^d$ satisfying $Supp(\widehat{u_j}) \subset \{\xi : |\xi| \leq 2^j A\}$ for $j \geq 0$. Let $\mu \in \mathbb{Z}_+$ and $P \in \mathcal{D}_\mu$. Then there exists $h \in \mathbb{Z}$ such that*

$$\Big(\frac{1}{|P|}\int_P \sum_{n=\mu}^\infty 2^{snq}\Big|\Pi_n\Big(\sum_{k=0}^\infty u_k\Big)(x)\Big|^q dx\Big)^{1/q} \lesssim \sup_{R\in\mathcal{D}_\mu} \Big(\frac{1}{|P|}\int_P \sum_{k=\max(0,\mu-h)}^\infty 2^{skq}|u_k(x)|^q dx\Big)^{1/q}$$

where the implicit constant in the inequality is independent of μ and P.

In the above Lemmas we regard $u_k \equiv 0$ when $k < 0$.

Remark. It has been observed in [10, Lemma11] that similar results hold, namely that the left hand side of the asserted inequalities in Lemma 3.1 and 3.2 are bounded by the supremum over arbitrary dyadic cubes, not over $R \in \mathcal{D}_\mu$. We provide a different proof by using Lemma 2.1 and 2.2, and improvements by deriving $R \in \mathcal{D}_\mu$.

We prove only Lemma 3.2 because Lemma 3.1 can be proved in a similar and simpler way. Our proof is based on the idea of [17, Corollary 2.11].

Proof of Lemma 3.2. We fix $P \in \mathcal{D}_\mu$. Let h be the smallest integer greater or equal to $\log_2 A$. The supports of $\widehat{\phi_n}$ and $\widehat{u_k}$ ensure that the left hand side of the desired inequality is less than

$$\Big(\frac{1}{|P|}\int_P \sum_{n=\mu}^{\infty} 2^{snq}\Big(\sum_{k=n-h}^{\infty} |\Pi_n u_k(x)|\Big)^q dx\Big)^{1/q}. \tag{3.1}$$

We choose $0 < r < 1$ such that $s > d(1/r-1) > \tau_q$, and then pick $0 < \epsilon < d/r$ so that $s > d(1/r-1)+\epsilon r$ and choose $\sigma > d/r$. For each $k \geq n-h$ we see that

$$\begin{aligned}
\big|\Pi_n u_k(x)\big| &\leq \int_{\mathbb{R}^d} \big(1+2^n|y|\big)^\sigma |\phi_n(y)| \frac{|u_k(x-y)|}{\big(1+2^n|y|\big)^\sigma} dy \\
&\leq \big\|(1+2^n|\cdot|)^\sigma \phi_n\big\|_{L^\infty} \Big(\sup_y \frac{|u_k(x-y)|}{(1+2^n|y|)^\sigma}\Big)^{1-r} \int_{\mathbb{R}^d} \frac{|u_k(x-y)|^r}{(1+2^n|y|)^{\sigma r}} dy \\
&\lesssim 2^{nd}\Big(\sup_y \frac{|u_k(x-y)|}{(1+2^n|y|)^\sigma}\Big)^{1-r} \int_{\mathbb{R}^d} \frac{|u_k(x-y)|^r}{(1+2^n|y|)^{\sigma r}} dy.
\end{aligned}$$

Let t satisfy $\epsilon = d(1/r-1/t)$. Then by (2.4)

$$\sup_y \frac{|u_k(x-y)|}{(1+2^n|y|)^\sigma} \leq 2^{(k-n)d/r}\mathfrak{M}_{d/r,2^k} u_k(x) \lesssim 2^{(k-n)d/r}\mathcal{M}_t^{k,\epsilon} u_k(x).$$

Moreover, for $j,n \in \mathbb{Z}$ let $E_j^n(x) = \{y : 2^{j-1} < 2^n|x-y| \leq 2^j\}$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|u_k(x-y)|^r}{(1+2^n|y|)^{\sigma r}} dy = \int_{\mathbb{R}^d} \frac{|u_k(y)|^r}{(1+2^n|x-y|)^{\sigma r}} dy \\
\leq\ & \sum_{j=-\infty}^{n-k} \frac{1}{(1+2^j)^{\sigma r}} \int_{E_j^n(x)} |u_k(y)|^r dy + \sum_{j=n-k+1}^{\infty} \frac{1}{(1+2^j)^{\sigma r}} \int_{E_j^n(x)} |u_k(y)|^r dy \\
\leq\ & \sum_{j=-\infty}^{n-k} \frac{2^{(j-n)d}}{(1+2^j)^{\sigma r}} \frac{1}{2^{(j-n)d}} \int_{E_j^n(x)} |u_k(y)|^r dy + \sum_{j=n-k+1}^{\infty} \frac{2^{(j-n)(d+\epsilon r)}}{(1+2^j)^{\sigma r}} \frac{1}{2^{(j-n)(d+\epsilon r)}} \int_{E_j^n(x)} |u_k(y)|^r dy \\
\lesssim\ & 2^{-nd}\Big(\sup_{\substack{V \ni x,\\ l(V) \leq 2^{-k}}} \frac{1}{|V|}\int_V |u_k(y)|^r dy + 2^{-n\epsilon r} \sup_{\substack{V \ni x,\\ l(V) > 2^{-k}}} \frac{1}{|V|^{1+\epsilon r/d}} \int_V |u_k(y)|^r dy\Big) \\
\lesssim\ & 2^{-nd}\Big[\Big(\sup_{\substack{V \ni x,\\ l(V) \leq 2^{-k}}} \frac{1}{|V|}\int_V |u_k(y)|^t dy\Big)^{r/t} + 2^{-n\epsilon r}\Big(\sup_{\substack{V \ni x,\\ l(V) > 2^{-k}}} \frac{1}{|V|^{1+\epsilon t/d}} \int_V |u_k(y)|^t dy\Big)^{r/t}\Big] \\
\lesssim\ & 2^{-nd}2^{(k-n)\epsilon r}\big(\mathcal{M}_t^{k,\epsilon} u_k(x)\big)^r
\end{aligned}$$

for $k \geq n-h$, where the fourth inequality follows by Hölder's inequality. By putting these together we obtain

$$\big|\Pi_n u_k(x)\big| \lesssim 2^{(k-n)d(1/r-1)}2^{(k-n)\epsilon r}\mathcal{M}_t^{k,\epsilon} u_k(x). \tag{3.2}$$

Now choose $\delta > 0$ sufficiently small so that $0 < \delta < s - d(1/r-1) - r\epsilon$. Then by using Hölder's inequality for $q > 1$ and embedding $l^q \hookrightarrow l^1$ for $q \le 1$, we see that

$$\sum_{n=\mu}^{\infty} 2^{snq}\Big(\sum_{k=n-h}^{\infty} \big|\Pi_n u_k(x)\big|\Big)^q \lesssim \sum_{n=\mu}^{\infty} 2^{snq} 2^{-\delta nq} \sum_{k=n-h}^{\infty} 2^{\delta kq}\big|\Pi_n u_k(x)\big|^q$$

and by (3.2) and the choice of δ this is bounded by

$$\sum_{k=\mu-h}^{\infty} 2^{qkd(1/r-1)} 2^{qk\epsilon r} 2^{qk\delta}\big(\mathcal{M}_t^{k,\epsilon} u_k(x)\big)^q \sum_{n=\mu}^{k+h} 2^{qn(s-d(1/r-1)-\delta-r\epsilon)} \lesssim \sum_{k=\mu-h}^{\infty} 2^{skq}\big(\mathcal{M}_t^{k,\epsilon} u_k(x)\big)^q.$$

Finally it follows that

$$(3.1) \quad \lesssim_h \quad \sup_{R\in\mathcal{D}_\mu} \Big(\frac{1}{|R|}\int_R \sum_{k=\max(0,\mu-h)}^{\infty} 2^{skq}|u_k(x)|^q dx\Big)^{1/q}$$

by Lemma 2.1.

□

4. **Proof of Theorem 1.2**

We fix $\mu \ge 1$ and $P \in \mathcal{D}_\mu$. The idea of our proof is based on the paradifferential technique, introduced by Bony [1]. For $j,k \ge 0$ let

$$a_{j,k}(x,\xi) = \phi_j * a(\cdot,\xi)(x)\widehat{\phi_k}(\xi) \tag{4.1}$$

where we use Φ, instead of ϕ_0 when $j=0$ or $k=0$. Then by using this we decompose $T_{[a]}$ as

$$T_{[a]}f = \mathfrak{S}_{[a]}^{near} f + \mathfrak{S}_{[a]}^{far} f$$

where

$$\mathfrak{S}_{[a]}^{near} f := \sum_{k,j:|j-k|\le 2} T_{[a_{j,k}]}f \tag{4.2}$$

$$\mathfrak{S}_{[a]}^{far} f := \sum_{k,j:|j-k|\ge 3} T_{[a_{j,k}]}f. \tag{4.3}$$

We first consider $\mathfrak{S}_{[a]}^{far}$ part for which we actually do not need the condition $s > \tau_q$. Write

$$\mathfrak{S}_{[a]}^{far} f = \sum_{k=3}^{\infty}\sum_{j=0}^{k-3} T_{[a_j,k]}f + \sum_{j=3}^{\infty}\sum_{k=0}^{j-3} T_{[a_j,k]}f := \sum_{k=3}^{\infty} T_{[b_k]}f + \sum_{j=3}^{\infty} T_{[c_j]}f$$

and observe that $b_k, c_j \in \mathcal{S}_{1,1}^m$ uniformly in k and j, and furthermore

$$Supp(\widehat{T_{[b_k]}f}) \subset \{\xi : 2^{k-2} \le |\xi| \le 2^{k+2}\},$$

$$Supp(\widehat{T_{[c_j]}f}) \subset \{\xi : 2^{j-2} \le |\xi| \le 2^{j+2}\}.$$

Then due to Lemma 3.1 with $h=2$ we see that

$$\Big(\frac{1}{|P|}\int_P\sum_{k=\mu}^{\infty}2^{skq}\Big|\Pi_k\Big(\sum_{n=3}^{\infty}T_{[b_n]}f\Big)(x)\Big|^q dx\Big)^{1/q}$$

$$\lesssim \sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=\max(3,\mu-2)}^{\infty}2^{skq}\big|T_{[b_k]}f(x)\big|^q dx\Big)^{1/q}. \tag{4.4}$$

Let $\widetilde{\Pi}_k=\Pi_{k-1}+\Pi_k+\Pi_{k+1}$ for $k\geq 3$ so that $\widetilde{\Pi}_k\Pi_k=\Pi_k$. We claim that for $\sigma>0$

$$\big|T_{[b_k]}f(x)\big| \lesssim_\sigma 2^{km}\mathfrak{M}_{\sigma,2^k}\widetilde{\Pi}_kf(x). \tag{4.5}$$

Then we choose $\sigma>d/q$ and this proves that, by Lemma 2.2,

$$\begin{aligned}(4.4) &\lesssim \sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\sum_{k=\max(3,\mu-2)}^{\infty}2^{(s+m)kq}\big|\widetilde{\Pi}_kf(x)\big|^q\Big)^{1/q}\\ &\lesssim \sup_{0\leq k\leq \mu-1}\big\|2^{k(s+m)}\Pi_kf\big\|_{L^\infty}+\sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=\mu}^{\infty}2^{(s+m)kq}\big|\Pi_kf(x)\big|^q\Big)^{1/q}\end{aligned}$$

where we used a triangle inequality for $\widetilde{\Pi}_k=\Pi_{k-1}+\Pi_k+\Pi_{k+1}$. To see (4.5) let K_k be the kernel of $T_{[b_k]}$, which is defined by

$$K_k(x,y)=\int_{\mathbb{R}^d}b_k(x,\xi)e^{2\pi i<x-y,\xi>}d\xi.$$

By using integration by parts and the fact $b_k\in\mathcal{S}_{1,1}^m$ uniformly in k, we have a size estimate

$$|K_k(x,y)|\lesssim_M 2^{k(m+d)}\frac{1}{(1+2^k|x-y|)^M}$$

for all $M>0$. We choose $M>\sigma+d$ and then we can prove (4.5) with a standard computation.

Similarly, we have

$$\Big(\frac{1}{|P|}\int_P\sum_{k=\mu}^{\infty}2^{skq}\Big|\Pi_k\Big(\sum_{j=3}^{\infty}T_{[c_j]}f\Big)(x)\Big|^q dx\Big)^{1/q}$$

$$\lesssim \sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{j=\max(3,\mu-2)}^{\infty}2^{sjq}\big|T_{[c_j]}f(x)\big|^q dx\Big)^{1/q}. \tag{4.6}$$

For the estimation of this part we claim that for each $j\geq 3$

$$|T_{[a_{j,k}]}f(x)| \lesssim_N 2^{km}2^{-(j-k)N}\mathfrak{M}_{\sigma,2^k}\widetilde{\Pi}_kf(x) \tag{4.7}$$

for any $N>0$ and then (4.6) is less than

$$\sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{j=\max(3,\mu-2)}^{\infty}2^{sjq}\Big(\sum_{k=0}^{j-3}2^{km}2^{-(j-k)N}\mathfrak{M}_{\sigma,2^k}\widetilde{\Pi}_kf(x)\Big)^q dx\Big)^{1/q}$$

where $\widetilde{\Pi}_k$'s are suitable inhomogeneous dyadic frequency decomposition, as before, so that $\widetilde{\Pi}_k\Pi_k=\Pi_k$ for $k\geq 0$. We choose $N>s$ and $\epsilon>0$ satisfying $s+\epsilon<N$. Using Hölder's

inequality if $q>1$ and embedding $l^q \hookrightarrow l^1$ if $q \le 1$ this expression is bounded by a constant times

$$\sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R \sum_{j=\max(3,\mu-2)}^{\infty} 2^{(s+\epsilon)jq}\sum_{k=0}^{j-3}2^{kq(m-\epsilon)}2^{-(j-k)Nq}\big(\mathfrak{M}_{\sigma,2^k}\widetilde{\Pi}_k f(x)\big)^q dx\Big)^{1/q}$$

and we exchange the sums over j and over k, and split it by using

$$\sum_{k=\max(0,\mu-5)}^{\infty}\ \sum_{j=k+3}^{\infty}+\sum_{k=0}^{\mu-4}\ \sum_{j=\mu-2}^{\infty}$$

where we consider only the first part if $\mu\le 3$. Then Lemma 2.2 proves that the term corresponding to the first sum is controlled by

$$\begin{aligned}
&\sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R \sum_{k=\max(0,\mu-5)}^{\infty}2^{(m+s)kq}\big|\Pi_k f(x)\big|^q dx\Big)^{1/q}\\
\le\ &\sup_{0\le k\le\mu-1}\big\|2^{k(s+m)}\Pi_k f\big\|_{L^\infty}+\sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=\mu}^{\infty}2^{(s+m)kq}\big|\Pi_k f(x)\big|^q\Big)^{1/q}
\end{aligned}$$

Moreover, if $\mu\ge 4$ then the other part is bounded by a constant times

$$\begin{aligned}
&\sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=0}^{\mu-4}2^{(k-\mu)(N-\epsilon-s)q}2^{k(m+s)q}\big(\mathfrak{M}_{\sigma,2^k}\widetilde{\Pi}_k f(x)\big)^q dx\Big)^{1/q}\\
\lesssim\ &\sup_{0\le k\le\mu-3}\big\|2^{k(m+s)}\Pi_k f\big\|_{L^\infty}\le\sup_{0\le k\le\mu-1}\big\|2^{k(m+s)}\Pi_k f\big\|_{L^\infty}
\end{aligned}$$

because $N-\epsilon-s>0$.

For the pointwise estimate (4.7) we also use a size estimate of the kernel of $T_{[a_{j,k}]}$. Let $W_{j,k}(x,y)$ be the kernel of $T_{[a_{j,k}]}$. For $r>0$ define $a^r(x,\xi):=a(x,\xi)\gamma(x/2^r)$ where γ is a smooth function which is identically 1 near a neighborhood of the origin and compactly supported, and let $W^r_{j,k}(x,y)$ be the kernel of $T_{[a^r_{j,k}]}$. Note that $a^r_{j,k}$ belongs to $\mathcal{S}^{1,1}_m$ uniformly in r,j,k. For $k\ge 1$ and $j\ge 3$ write

$$W^r_{j,k}(x,y)\ =\ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}a^r(z,\xi)e^{-2\pi i\langle z,\eta\rangle}dz\Big)\widehat{\phi_j}(\eta)\widehat{\phi_k}(\xi)2^{2\pi i\langle x,\eta\rangle}e^{2\pi i\langle x-y,\xi\rangle}d\xi d\eta$$

and then by using integration by parts in z and in ξ, we obtain, for multi-indices α,β

$$\begin{aligned}
&|W^r_{j,k}(x,y)|\\
\lesssim\ &\frac{1}{|x-y|^{|\beta|}}\int_{\mathbb{R}^d}\Big|\int_{|\eta|\sim 2^j}\frac{1}{\eta^\alpha}\widehat{\phi_j}(\eta)e^{-2\pi i\langle z-x,\eta\rangle}d\eta\Big|dz\int_{|\xi|\sim 2^k}\big(1+|\xi|\big)^{m-|\beta|+|\alpha|}|\widehat{\phi_k}(\xi)|d\xi\\
\lesssim\ &2^{k(m-|\beta|+|\alpha|+d)}\frac{1}{|x-y|^{|\beta|}}\Big(\int_{|x-z|<2^{-j}}\Big|\int_{|\eta|\sim 2^j}\frac{1}{\eta^\alpha}\widehat{\phi_j}(\eta)e^{-2\pi i\langle z-x,\eta\rangle}d\eta\Big|dz\\
&+\int_{|x-z|>2^{-j}}\Big|\int_{|\eta|\sim 2^j}\frac{1}{\eta^\alpha}\widehat{\phi_j}(\eta)e^{-2\pi i\langle z-x,\eta\rangle}d\eta\Big|dz\Big)\\
\lesssim\ &2^{k(m+d)}2^{-(j-k)|\alpha|}\frac{1}{\big(2^k|x-y|\big)^{|\beta|}}
\end{aligned}$$

where in the third inequality the integration corresponding to $|x-z|<2^{-j}$ is clearly bounded by $2^{-j|\alpha|}$ and the other one is also dominated by $2^{-j|\alpha|}$ by applying again integration by parts in η. We also obtain the same kernel estimate for $k=0$ and $j\ge 3$ similarly.

Since α and β are arbitrary, we could get

$$2^{-k(m+d)}|W^r_{j,k}(x,y)| \lesssim 2^{-(j-k)N}\frac{1}{(1+2^k|x-y|)^M},$$

uniformly in r. Here the restriction of the support of $a(x,\xi)$ in x-variable is used only for integration by parts and we can drop this condition by the following approximation.

$$\begin{aligned}\big|W_{j,k}(x,y)-W^r_{j,k}(x,y)\big| &= \Big|\int_{\mathbb{R}^d\times\mathbb{R}^d} a(z,\xi)\phi_j(x-z)\big(1-\gamma(z/2^r)\big)\widehat{\phi_k}(\xi)e^{2\pi i\langle x-y,\xi\rangle}d\xi dz\Big| \\ &\lesssim \int_{\mathbb{R}^d}(1+|\xi|)^m|\widehat{\phi_k}(\xi)|d\xi\int_{\mathbb{R}^d}|\phi_j(x-z)|\big|1-\gamma(z/2^r)\big|dz \\ &\lesssim 2^{k(m+d)}\int_{\mathbb{R}^d}|\phi_j(x-z)|\big|1-\gamma(z/2^r)\big|dz\end{aligned}$$

and thus

$$\lim_{r\to\infty}2^{-k(m+d)}\big|W_{j,k}(x,y)-W^r_{j,k}(x,y)\big|=0$$

uniformly in k and j. This yields

$$|W_{j,k}(x,y)| \lesssim 2^{k(m+d)}2^{-(j-k)N}\frac{1}{(1+2^k|x-y|)^M}$$

and by using the idea in (4.5) we obtain (4.7). This completes the proof for $\mathfrak{S}^{far}_{[a]}$ part.

We now turn to the expression $\mathfrak{S}^{near}_{[a]}$. For $k\geq 0$ define

$$d_k(x,\xi) := \sum_{j=k-2}^{k+2} a_{j,k}(x,\xi)$$

assuming $a_{j,k}(x,\xi)=0$ for $j<0$. Then $d_k\in\mathcal{S}^m_{1,1}$ uniformly in k, and

$$\mathfrak{S}^{near}_{[a]}f=\sum_{k=0}^{\infty}T_{[d_k]}f.$$

Since the support of $\widehat{T_{d_k}f}$ is contained in $\{\xi:|\xi|\leq 2^{k+3}\}$, we apply Lemma 3.2 with the assumption $s>\tau_q$ and then we obtain

$$\Big(\frac{1}{|P|}\int_P\sum_{k=\mu}^{\infty}2^{skq}\big|\Pi_k\mathfrak{S}^{near}_{[a]}f(x)\big|^q dx\Big)^{1/q} \lesssim \sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=\max(0,\mu-3)}^{\infty}2^{skq}\big|T_{[d_k]}f(x)\big|^q dx\Big)^{1/q}$$

By the same argument we used to get (4.5) we establish the pointwise estimate

$$\big|T_{[d_k]}f_k(x)\big| \lesssim 2^{km}\mathfrak{M}_{\sigma,2^k}\widetilde{\Pi}_kf(x) \tag{4.8}$$

for $\sigma>d/q$ and then use Lemma 2.2 to gain the bound

$$\begin{aligned}&\sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=\max(0,\mu-4)}^{\infty}2^{(s+m)kq}\big|\Pi_kf(x)\big|^q dx\Big)^{1/q} \\ &\lesssim \sup_{0\leq k\leq\mu-1}\big\|2^{k(m+s)}\Pi_kf\big\|_{L^\infty}+\sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=\mu}^{\infty}2^{(m+s)kq}\big|\Pi_kf(x)\big|^q dx\Big)^{1/q},\end{aligned}$$

which completes the proof.

5. Basic settings

We introduce some notations and settings which we shall employ to prove Theorem 1.1 and 1.3 in Section 6∼8. In what follows let η denote Schwartz functions so that $\eta \geq 0$, $\eta(x) \geq c$ on $\{x : |x| \leq 1/100\}$ for some $c > 0$. Fix a real-valued Schwartz function Γ so that $\widehat{\Gamma}$ is supported in $\{\xi : 1 < |\xi| < 2\}$ and $\Gamma(x) \geq 1$ for $|x| \leq 2^{-M+1}$ for some $M > 0$. Define $\Gamma_k(x) := 2^{kd}\Gamma(2^k x)$. Put $\Lambda \in S(\mathbb{R}^d)$ such that $\widehat{\Lambda}$ is supported on $\{\xi : 9/8 < |\xi| < 15/8\}$, $\widehat{\Lambda} \geq 0$, and $\widehat{\Lambda}(\xi) \geq 1$ on $\{\xi : 5/4 \leq |\xi| \leq 7/4\}$ and let $\Lambda_k(x) = 2^{kd}\Lambda(2^k x)$. For $k \geq 2$ let $\widetilde{\phi}_k := \phi_{k-1} + \phi_k + \phi_{k+1}$ and $t_k := 10k$. Let $e_1 := (1, 0, \dots, 0) \in \mathbb{Z}^d$ and $\mathbf{v}_k = (3/2)2^k e_1$. For the proof of Theorem 1.1 and 1.3 we may assume $m = 0$.

6. Proof of Theorem 1.1 (1)

6.1. **The case $1 < p < \infty$ and $1 < q \leq \infty$ or the case $p = q = \infty$.** In this case we simply apply the idea in [2]. Assume $s \leq 0$ and $L > 0$ is sufficiently large. Let $\{\zeta_k\}$ be a sequence of positive numbers. We define $a \in \mathcal{S}^0_{1,1}$ by

$$a(x,\xi) := \sum_{k=10}^{\infty} \widehat{\widetilde{\phi}_k}(\xi) e^{2\pi i \langle 2^k e_1, x\rangle}. \tag{6.1}$$

and

$$f_L(x) := \eta(x) \sum_{k=10}^{L} \zeta_{t_k} e^{-2\pi i \langle 2^{t_k} e_1, x\rangle}. \tag{6.2}$$

Then since $\big|\phi_{t_k} * \big(\eta e^{-2\pi i\langle 2^k e_1, \cdot\rangle}\big)(x)\big| \lesssim \mathcal{M}\eta(x)$, we have

$$\|f_L\|_{F^{s,q}_p} \lesssim \Big(\sum_{k=10}^{L} 2^{st_k q}\zeta_{t_k}^q\Big)^{1/q} \big\|\mathcal{M}\eta\big\|_{L^p} \lesssim \Big(\sum_{k=10}^{L} \zeta_{t_k}^q\Big)^{1/q}$$

On the other hand, since $\widehat{\widetilde{\phi}_k}(\xi) = 1$ for $2^{k-1} \leq |\xi| \leq 2^{k+1}$ we obtain

$$T_{[a]} f_L(x) = \eta(x) \sum_{k=10}^{L} \zeta_{t_k}$$

and thus

$$\big\|T_{[a]} f_L\big\|_{F^{s,q}_p} \sim \sum_{k=10}^{L} \zeta_{t_k}.$$

By choosing a sequence $\{\zeta_k\}$ satisfying $\sum_k \zeta_{t_k} = \infty$ and $\sum_k \zeta_{t_k}^q < \infty$, like $\zeta_k = k^{-1}$, we are done.

The same idea can be applied to the case $p = q = \infty$.

6.2. **Construction of pseudo-differential operators for the case $\min(p, q) \leq 1$.** We use apply the random construction technique by Christ and Seeger [3]. Suppose L is a sufficiently large integer. Let $\{g_k\}_{k=M}^{L}$ be a family of Schwartz functions and define

$$f_L(x) := \sum_{k=M}^{L} 2^{-st_k} \Lambda_{t_k} * \big(g_{t_k} e^{-2\pi i\langle \mathbf{v}_{t_k}, \cdot\rangle}\big)(x). \tag{6.3}$$

By definition we have

$$\|f_L\|_{F_p^{s,q}} = \Big[\int_{\mathbb{R}^d}\Big(\sum_{k=0}^{\infty}\big(2^{sk}|\phi_k * f_L(x)|\big)^q\Big)^{p/q}dx\Big]^{1/p}.$$

The support conditions on $\widehat{\phi_k}$ and $\widehat{\Lambda_{t_k}}$ ensures that

$$\phi_k * f_L(x) = \begin{cases} 2^{-st_n}\phi_{t_n} * \Lambda_{t_n} * \big(g_{t_n}e^{-2\pi i\langle \mathbf{v}_{t_n},\cdot\rangle}\big)(x), & k=t_n,\ M\le n\le L \\ 2^{-st_n}\phi_{t_n+1} * \Lambda_{t_n} * \big(g_{t_n}e^{-2\pi i\langle \mathbf{v}_{t_n},\cdot\rangle}\big)(x), & k=t_n+1,\ M\le n\le L \\ 0, & otherwise. \end{cases}$$

and thus

$$\begin{aligned} \|f_L\|_{F_p^{s,q}} &\lesssim \Big[\int_{\mathbb{R}^d}\Big(\sum_{n=M}^{L}\big|\phi_{t_n} * \Lambda_{t_n} * \big(g_{t_n}e^{-2\pi i\langle \mathbf{v}_{t_n},\cdot\rangle}\big)(x)\big|^q\Big)^{p/q}dx\Big]^{1/p} \\ &\quad + \Big[\int_{\mathbb{R}^d}\Big(\sum_{n=M}^{L}\big|\phi_{t_n+1} * \Lambda_{t_n} * \big(g_{t_n}e^{-2\pi i\langle \mathbf{v}_{t_n},\cdot\rangle}\big)(x)\big|^q\Big)^{p/q}dx\Big]^{1/p} \\ &\lesssim_\sigma \Big[\int_{\mathbb{R}^d}\Big(\sum_{n=M}^{L}\big(\mathfrak{M}_{\sigma,2^{t_n}}g_{t_n}(x)\big)^q\Big)^{p/q}dx\Big]^{1/p} \qquad (6.4) \end{aligned}$$

for any $\sigma>0$. Now we define

$$a(x,\xi) := \sum_{k=1}^{\infty}\widehat{\Lambda}(\xi/2^k)e^{2\pi i\langle \mathbf{v}_k,x\rangle}. \tag{6.5}$$

It is clear that $a(x,\xi)\in\mathcal{S}_{1,1}^0(\mathbb{R}^d)$. We observe that

$$\begin{aligned} T_{[a]}f_L(x) &= \sum_{k=1}^{\infty}\Lambda_k * f_L(x)e^{2\pi i\langle \mathbf{v}_k,x\rangle} \\ &= \sum_{n=M}^{L}2^{-st_n}\Lambda_{t_n} * \Lambda_{t_n} * \big(g_{t_n}e^{-2\pi i\langle \mathbf{v}_{t_n},\cdot\rangle}\big)(x)e^{2\pi i\langle \mathbf{v}_{t_n},x\rangle} := \sum_{n=M}^{L}d_{t_n}(x). \end{aligned}$$

and

$$\begin{aligned} \Big\|\sum_{k=M}^{L}d_{t_n}\Big\|_{F_p^{s,q}} &= \Big\|\Big(\sum_{l=0}^{\infty}2^{slq}\Big|\sum_{n=M}^{L}\phi_l * d_{t_n}\Big|^q\Big)^{1/q}\Big\|_{L^p} \\ &\gtrsim \Big\|\Big[\sum_{l=M}^{L-M}\Big(2^{st_l}\Big|\sum_{n=M}^{L}(\phi_{t_l}+\phi_{t_l+1}) * d_{t_n}\Big|\Big)^q\Big]^{1/q}\Big\|_{L^p}. \qquad (6.6) \end{aligned}$$

Since $\widehat{d_{t_n}}$ is supported in $\{\xi : |\xi| < (27/16)2^{t_n+1}\}$ and $\widehat{\phi_{t_l}}$ is in $\{\xi : 2^{t_l-1} \le |\xi| \le 2^{t_l+1}\}$, $\phi_{t_l} * d_{t_n}$ vanishes unless $l \le n$. Therefore

$$(6.6) \quad = \quad \Big\| \Big[\sum_{l=M}^{L-M} \Big(2^{st_l} \Big| \sum_{n=l}^{L} (\phi_{t_l} + \phi_{t_l+1}) * d_{t_n} \Big| \Big)^q \Big]^{1/q} \Big\|_{L^p}$$

$$\gtrsim \quad \Big\| \Big[\sum_{l=M}^{L-M} \Big(2^{st_l} \Big| \sum_{n=l+1}^{L} (\phi_{t_l} + \phi_{t_l+1}) * d_{t_n} \Big| \Big)^q \Big]^{1/q} \Big\|_{L^p} \tag{6.7}$$

$$- \Big\| \Big(\sum_{l=M}^{L-M} \big(2^{st_l} \big| (\phi_{t_l} + \phi_{t_l+1}) * d_{t_l} \big| \big)^q \Big)^{1/q} \Big\|_{L^p}. \tag{6.8}$$

By an elementary computation, (6.8) is bounded by

$$\Big[\int_{\mathbb{R}^d} \Big(\sum_{l=M}^{L} \big(\mathfrak{M}_{\sigma,2^{t_l}} g_{t_l}(x) \big)^q \Big)^{p/q} dx \Big]^{1/p}$$

and now we consider (6.7). When $2^{t_l-1} \le |\xi| \le 2^{t_l+2}$ and $l+1 \le n$, we see that

$$(5/4)2^{t_n} < \big(3/2 - 1/2^8\big)2^{t_n} \le |\xi - \mathbf{v}_{t_n}| \le \big(3/2 + 1/2^8\big)2^{t_n} < (7/4)2^{t_n}$$

in which $\widehat{\Lambda_{t_n}}(\xi - \mathbf{v}_{t_n}) = 1$. Hence we write

$$\widehat{d_{t_n}}(\xi) = 2^{-st_n} \widehat{\Lambda_{t_n}}(\xi - \mathbf{v}_{t_n}) \widehat{\Lambda_{t_n}}(\xi - \mathbf{v}_{t_n}) \widehat{g_{t_n}}(\xi) = 2^{-st_n} \widehat{g_{t_n}}(\xi)$$

and this gives

$$(\phi_{t_l} + \phi_{t_l+1}) * d_{t_n}(x) = 2^{-st_n} (\phi_{t_l} + \phi_{t_l+1}) * g_{t_n}(x) \tag{6.9}$$

for $l+1 \le n$. Finally we obatin

$$\begin{aligned}
(6.7) \quad &= \quad \Big\| \Big(\sum_{l=M}^{L-M} 2^{st_l q} \Big| \sum_{n=l+1}^{L} 2^{-st_n} \big(\phi_{t_l} + \phi_{t_l+1} \big) * g_{t_n} \Big|^q \Big)^{1/q} \Big\|_{L^p} \\
&\gtrsim \quad \Big\| \Big[\sum_{l=M}^{L-M} 2^{st_l q} \Big(\mathfrak{M}_{\sigma,2^{t_l}} \Big(\sum_{n=l+1}^{L} 2^{-st_n} \big(\phi_{t_l} + \phi_{t_l+1} \big) * g_{t_n} \Big) \Big)^q \Big]^{1/q} \Big\|_{L^p} \\
&\gtrsim \quad \Big\| \Big(\sum_{l=M}^{L-M} 2^{st_l q} \Big| \sum_{n=l+1}^{L} 2^{-st_n} \Gamma_{t_l} * \big(\phi_{t_l} + \phi_{t_l+1} \big) * g_{t_n} \Big|^q \Big)^{1/q} \Big\|_{L^p} \\
&= \quad \Big\| \Big(\sum_{l=M}^{L-M} \Big| \sum_{k=1}^{L-l} 2^{-st_n} \Gamma_{t_l} * g_{t_{n+l}} \Big|^q \Big)^{1/q} \Big\|_{L^p} \qquad (6.10)
\end{aligned}$$

by (2.3) with $\sigma > \max(d/p, d/q)$. So far we have established

$$\big\| f_L \big\|_{F_p^{s,q}} \quad \lesssim \quad \Big[\int_{\mathbb{R}^d} \Big(\sum_{n=M}^{L} \big(\mathfrak{M}_{\sigma,2^{t_n}} g_{t_n}(x) \big)^q \Big)^{p/q} dx \Big]^{1/p} \tag{6.11}$$

and

$$\begin{aligned}
\big\| T_{[a]} f_L \big\|_{F_p^{s,q}} \quad &\gtrsim \quad \Big\| \Big(\sum_{l=M}^{L-M} \Big| \sum_{n=1}^{L-l} 2^{-st_n} \Gamma_{t_l} * g_{t_{n+l}} \Big|^q \Big)^{1/q} \Big\|_{L^p} \qquad (6.12) \\
&\quad - \Big[\int_{\mathbb{R}^d} \Big(\sum_{n=M}^{L} \big(\mathfrak{M}_{\sigma,2^{t_n}} g_{t_n}(x) \big)^q \Big)^{p/q} dx \Big]^{1/p}.
\end{aligned}$$

6.3. **The case $0 < q \le 1$ and $q \le p < \infty$.** Assume $s \le d(1/q-1)$. For each $k \in \{0,1,\dots\}$, let $\mathcal{Q}(k)$ be the set of all dyadic cubes of side-length 2^{-t_k-M} in $[0,1]^d$ and $\mathcal{Q} = \bigcup_{k\in\mathbb{Z}} \mathcal{Q}(k)$. Let Ω be a probability space with probability measure λ. Let $\{\theta_Q\}$ be a family of independent random variables indexed by $Q \in \mathcal{Q}$, each of which takes the value 1 with probability $1/L$ and the value 0 with probability $1-1/L$ and χ_Q denote the characteristic function of Q. Consider random functions

$$h_k^{\omega}(x) := \sum_{Q\in\mathcal{Q}(k)} \theta_Q(\omega)\chi_Q(x)$$

for $\omega \in \Omega$. Then the following result holds due to Christ and Seeger [3].

Lemma 6.1. [3] *Suppose $0 < p,q < \infty$ and $\sigma > \max(d/p, d/q)$. Then*

$$\Big(\int_\Omega \Big\| \Big(\sum_{k=1}^{L} \big(\mathfrak{M}_{\sigma,2^{t_k}} h_k^{\omega}\big)^q\Big)^{1/q}\Big\|_{L^p}^p d\lambda\Big)^{1/p} \lesssim_{p,q} 1.$$

We note that one of the key idea in the proof of Lemma 6.1 is the pointwide estimate

$$\mathfrak{M}_{\sigma,2^{t_k}} h_k^{\omega}(x) \lesssim_\sigma \mathcal{M}_r h_k^{\omega}(x) \quad \text{for } \sigma > d/r. \tag{6.13}$$

To obtain lower bounds we benefit from the technique of Khintchine's inequality.

Lemma 6.2. *(Khintchine's inequality) Let $\{r_n(t)\}_{n=1}^\infty$ be the Rademacher functions and $\{a_n\}_{n=1}^\infty$ be a sequence of complex numbers. Suppose $0 < p < \infty$. Then*

$$\Big(\int_0^1 \Big|\sum_{n=1}^\infty a_n r_n(t)\Big|^p dt\Big)^{1/p} \sim \Big(\sum_{n=1}^\infty |a_n|^2\Big)^{1/2}.$$

Let $\{r_Q\}$ be a family of the Rademacher functions, defined on $[0,1]$, indexed by $Q \in \mathcal{Q}$ and define random functions

$$h_k^{\omega,t}(x) := \sum_{Q\in\mathcal{Q}(k)} r_Q(t)\theta_Q(\omega)\chi_Q(x)$$

for $\omega \in \Omega, t \in [0,1]$. Then by using $\mathfrak{M}_{\sigma,2^{t_k}} h_k^{\omega,t}(x) \le \mathfrak{M}_{\sigma,2^{t_k}} h_k^{\omega}(x)$ we establish

$$\Big(\int_0^1\int_\Omega \Big\| \Big(\sum_{k=1}^{L} (\mathfrak{M}_{\sigma,2^{t_k}} h_k^{\omega,t})^q\Big)^{1/q}\Big\|_{L^p}^p d\lambda dt\Big)^{1/p} \lesssim_{p,q} 1. \tag{6.14}$$

for $0 < p,q < \infty$ and $\sigma > \max(d/p, d/q)$.

Now we apply (6.11) with $g_{t_k}(x) = h_k^{\omega,t}$ and (6.14) to obtain

$$\Big(\int_0^1\int_\Omega \|f_L^{\omega,t}\|_{F_p^{s,q}}^p d\lambda dt\Big)^{1/p} \lesssim \Big(\int_0^1\int_\Omega \Big\| \Big(\sum_{k=M}^{L} (\mathfrak{M}_{\sigma,2^{t_k}} h_k^{\omega,t})^q\Big)^{1/q}\Big\|_{L^p}^p d\lambda dt\Big)^{1/p} \lesssim 1.$$

Thus, due to (6.12) it suffices to show that for $s+d-d/q \le 0$

$$\begin{aligned} &\Big(\int_0^1\int_\Omega \Big\| \Big(\sum_{l=M}^{L-M} \Big|\sum_{n=1}^{L-l} 2^{-st_n}\Gamma_{t_l} * h_{n+l}^{\omega,t}\Big|^q\Big)^{1/q}\Big\|_{L^p}^p d\lambda dt\Big)^{1/p} \\ &\quad \gtrsim L^{-(s+d-d/q)/(2d)}\big(\log L\big)^{1/2}. \end{aligned} \tag{6.15}$$

By Hölder's inequality for $p/q > 1$ and Khintchine's inequality we see that

$$\begin{aligned}
&\Big(\int_0^1 \int_\Omega \Big\| \Big(\sum_{l=M}^{L-M} \Big| \sum_{n=1}^{L-l} 2^{-st_n} \Gamma_{t_l} * h^{\omega,t}_{n+l} \Big|^q \Big)^{1/q} \Big\|_{L^p}^p d\lambda dt \Big)^{1/p} \\
\geq\ & \Big(\int_0^1 \int_\Omega \int_{[0,1]^d} \sum_{l=M}^{L-M} \Big| \sum_{n=1}^{L-l} 2^{-st_n} \Gamma_{t_l} * h^{\omega,t}_{n+l}(x) \Big|^q dx d\lambda dt \Big)^{1/q} \\
=\ & \Big(\int_{[0,1]^d} \int_\Omega \sum_{l=M}^{L-M} \int_0^1 \Big| \sum_{n=1}^{L-l} \sum_{Q\in\mathcal{Q}(n+l)} 2^{-st_n} r_Q(t) \theta_Q(\omega) \Gamma_{t_l} * \chi_Q(x) \Big|^q dt d\lambda dx \Big)^{1/q} \\
\gtrsim\ & \Big[\int_{[0,1]^d} \int_\Omega \sum_{l=M}^{L-M} \Big(\sum_{n=1}^{L-l} \sum_{Q\in\mathcal{Q}(n+l)} 2^{-2st_n} \theta_Q(\omega) \big| \Gamma_{t_l} * \chi_Q(x) \big|^2 \Big)^{q/2} d\lambda dx \Big]^{1/q}.
\end{aligned}$$

For each $P \in \mathcal{Q}(l)$ and $n \geq 0$ let $\mathcal{V}_n(l,P) = \{Q \in \mathcal{Q}(l+n) : Q \subset P\}$. Then the last expression is bounded below by

$$\Big[\sum_{l=M}^{L-M} \sum_{P\in\mathcal{Q}(l)} \int_P \int_\Omega \Big(\sum_{n=1}^{L-l} 2^{-2st_n} \sum_{Q\in\mathcal{V}_n(l,P)} \theta_Q(\omega) \big| \Gamma_{t_l} * \chi_Q(x) \big|^2 \Big)^{q/2} d\lambda dx \Big]^{1/q}. \tag{6.16}$$

For $x \in P$ and $Q \in \mathcal{V}_n(l,P)$ we have $\Gamma_{t_l} * \chi_Q(x) \geq 2^{-t_n d}$ because $\Gamma(x) \geq 1$ for $|x| \leq 2^{-M}$. This yields that (6.16) is greater than

$$\Big[\sum_{l=M}^{L-M} 2^{-t_l d} \sum_{P\in\mathcal{Q}(l)} \int_\Omega \Big(\sum_{n=1}^{L-l} 2^{-2(s+d)t_n} \sum_{Q\in\mathcal{V}_n(l,P)} \theta_Q(\omega) \Big)^{q/2} d\lambda \Big]^{1/q}. \tag{6.17}$$

and, by Minkowski's inequality with $2/q > 1$

$$\begin{aligned}
&\sum_{P\in\mathcal{Q}(l)} \int_\Omega \Big(\sum_{n=1}^{L-l} 2^{-2(s+d)t_n} \sum_{Q\in\mathcal{V}_n(l,P)} \theta_Q(\omega) \Big)^{q/2} d\lambda \\
\geq\ & \Big(\sum_{n=1}^{L-l} 2^{-2(s+d)t_n} \Big[\sum_{P\in\mathcal{Q}(l)} \int_\Omega \Big(\sum_{Q\in\mathcal{V}_n(l,P)} \theta_Q(\omega) \Big)^{q/2} d\lambda \Big]^{2/q} \Big)^{q/2}.
\end{aligned} \tag{6.18}$$

For $P \in \mathcal{Q}(l)$ and $R \in \mathcal{V}_n(l,P)$, let $\Omega(P,R,l,n)$ be the event that $\theta_R(\omega) = 1$, but $\theta_{R'}(\omega) = 0$ for $R' \in \mathcal{V}_n(l,P) \setminus \{R\}$. We observe that the probability of this event is $\lambda(\Omega(P,R,l,n)) \geq 1/L\big(1 - 1/L\big)^{2^{t_n d}}$. Therefore

$$\int_\Omega \Big(\sum_{Q\in\mathcal{V}_n(l,P)} \theta_Q(\omega) \Big)^{q/2} d\lambda \geq \sum_{R\in\mathcal{V}_n(l,P)} \lambda(\Omega(P,R,l,n)) \geq 2^{t_n d} \frac{1}{L} \Big(1 - \frac{1}{L} \Big)^{2^{t_n d}},$$

which implies

$$(6.18) \quad \geq \quad 2^{t_l d} \frac{1}{L} \Big(\sum_{n=1}^{L-l} 2^{-2(s+d-d/q)t_n} \Big(1 - \frac{1}{L} \Big)^{(2/q)2^{t_n d}} \Big)^{q/2}.$$

Finally we obtain

$$(6.17) \geq \Big[\frac{1}{L}\sum_{l=M}^{L-M}\Big(\sum_{n=1}^{L-l}2^{-2(s+d-d/q)t_n}\Big(1-\frac{1}{L}\Big)^{(2/q)2^{t_n d}}\Big)^{q/2}\Big]^{1/q}$$

$$\geq \Big[\frac{1}{L}\sum_{l=\lfloor L/3\rfloor}^{\lfloor L/2\rfloor}\Big(\sum_{n=1}^{\lfloor (1/10d)\log_2 L\rfloor}2^{-2(s+d-d/q)t_n}\Big(1-\frac{1}{L}\Big)^{(2/q)L}\Big)^{q/2}\Big]^{1/q}$$

$$\gtrsim L^{-(s+d-d/q)/(2d)}\big(\log L\big)^{1/2}$$

for sufficiently large $L>0$, and this proves (6.15).

6.4. **The case $0<p\leq 1$ and $p\leq q$.** Suppose $s\leq d/p-d$ and we could apply (6.11) and (6.12) as in the previous case. Pick a nonnegative smooth function g supported in a ball of radius 2^{-M}, centered at the origin, and define

$$g_k(x) := 2^{kd/p}g(2^k x). \tag{6.19}$$

Then for all $\sigma>0$

$$\mathfrak{M}_{\sigma,2^{t_n}}g_{t_n}(x) \leq 2^{dt_n/p}\frac{1}{(1+2^{t_n}|x|)^{\sigma}}\sup_{|y|\leq 2^{-M}}(1+|y|)^{\sigma}|g(y)|$$

$$\lesssim 2^{dt_n/p}\frac{1}{(1+2^{t_n}|x|)^{\sigma}}. \tag{6.20}$$

We choose $\sigma>d/p$. By (6.11) and $l^p\hookrightarrow l^q$, we have the upper bound

$$\|f_L\|_{F_p^{s,q}} \lesssim \Big[\int_{\mathbb{R}^d}\Big(\sum_{n=M}^{L}2^{dt_nq/p}\frac{1}{(1+2^{t_n}|x|)^{\sigma q}}\Big)^{p/q}dx\Big]^{1/p}$$

$$\leq \Big(\int_{\mathbb{R}^d}\sum_{n=M}^{L}2^{dt_n}\frac{1}{(1+2^{t_n}|x|)^{\sigma p}}dx\Big)^{1/p}\lesssim L^{1/p}.$$

For the lower bound we see that

$$\Big\|\Big(\sum_{l=M}^{L-M}\Big|\sum_{n=1}^{L-l}2^{-st_n}\Gamma_{t_l}*g_{t_{n+l}}\Big|^q\Big)^{1/q}\Big\|_{L^p}$$

$$\geq \Big(\sum_{l=M}^{L-M}\int_{2^{-t_l-M-1}\leq|x|\leq 2^{-t_l-M}}\Big|\sum_{n=1}^{L-l}2^{-st_n}\Gamma_{t_l}*g_{t_{n+l}}(x)\Big|^p dx\Big)^{1/p}. \tag{6.21}$$

If $|y|\leq 2^{-M}$ and $|x|\leq 2^{-t_l-M}$ then $|2^{t_l}x-y|\leq 2^{-M+1}$ in which $\Gamma(2^{t_l}x-y)\geq 1$. Thus

$$\Gamma_{t_l}*g_{t_{n+l}}(x) = 2^{dt_n/p}2^{dt_l/p}\int_{|y|\leq 2^{-M}}\Gamma(2^{t_l}x-y)g(2^{t_n}y)dy$$

$$\geq 2^{dt_n/p}2^{dt_l/p}2^{-t_nd}\|g\|_{L^1}. \tag{6.22}$$

and this gives

$$(6.21) \gtrsim \Big[\sum_{l=M}^{L-M}\Big(\sum_{n=1}^{L-l}2^{-t_n(s+d-d/p)}\Big)^p\Big]^{1/p}\gtrsim L^{1/p}L^{-(s+d-d/p)}\log L$$

By (6.12),

$$\big\|a(x,\cdot)f_L\big\|_{F_p^{s,q}} \gtrsim L^{1/p}L^{-(s+d-d/p)}\log L$$

and we are done by letting $L \to \infty$ since $s + d - d/p \le 0$.

7. Proof of Theorem 1.1 (2)

7.1. **The case** $0 < p \le 1$**.** Define $a(x,\xi)$ and f_L be as (6.5) and (6.3) with (6.19), and suppose $s \le d(1/p-1)$. Corresponding to (6.11) we have the analogous estimate

$$\big\|f_L\big\|_{B_p^{s,q}} \lesssim \Big(\sum_{n=M}^{L}\big\|\mathfrak{M}_{\sigma,2^{t_n}}g_{t_n}\big\|_{L^p}^q\Big)^{1/q} \tag{7.1}$$

for $\sigma > d/p$ and (6.20) yields

$$(7.1) \lesssim \Big(\sum_{n=M}^{L}2^{dqt_n/p}\Big\|\frac{1}{(1+2^{t_n}|\cdot|)^{\sigma}}\Big\|_{L^p}^q\Big)^{1/q} \lesssim L^{1/q}.$$

Furthermore, similar to (6.12) we have

$$\begin{aligned}\big\|T_{[a]}f_L\big\|_{B_p^{s,q}} &\gtrsim \Big(\sum_{l=M}^{L-M}\Big\|\sum_{n=1}^{L-l}2^{-st_n}\Gamma_{t_l}*g_{t_{n+l}}\Big\|_{L^p}^q\Big)^{1/q}\\ &\ge \Big(\sum_{l=M}^{L-M}\Big(\int_{2^{-t_l-M-1}\le|x|\le 2^{-t_l-M}}\Big|\sum_{n=1}^{L-l}2^{-st_n}\Gamma_{t_l}*g_{t_{n+l}}(x)\Big|^p dx\Big)^{q/p}\Big)^{1/q}. \end{aligned} \tag{7.2}$$

By (6.22),

$$(7.2) \gtrsim \Big(\sum_{l=M}^{L-M}\Big(\sum_{n=1}^{L-l}2^{-t_n(s+d-d/p)}\Big)^q\Big)^{1/q} \gtrsim L^{1/q}L^{-(s+d-d/p)}\log L$$

for $s + d - d/p \le 0$. This shows that the condition $s > \tau_p$ is necessary.

7.2. **The case** $1 < p < \infty$**.** When $1 < p < \infty$ and $1 < q \le \infty$, the example in subsection 6.1 can be applied. Indeed, when $a \in \mathcal{S}_{1,1}^m$ and f_L are defined as (6.1) and (6.2) we have $\big\|f_L\big\|_{B_p^{s,q}} \lesssim \big\|\{\zeta_{t_k}\}_{k=M}^L\big\|_{l^q}$ and $\big\|T_{[a]}f_L\big\|_{B_p^{s,q}} \sim \sum_{k=M}^L \zeta_{t_k}$. Then we choose $\zeta_k = 1/k$ and let $L \to \infty$.

Now suppose $0 < q \le 1 < p < \infty$. Let

$$g_L(x) := \sum_{n=M}^{L} b_n 2^{-t_n d(1-1/p)}\phi_{t_n}(x)$$

and define

$$f_L(x) := 2^{-st_L}\Lambda_{t_L} * \big(g_L e^{-2\pi i\langle \mathbf{v}_{t_L},\cdot\rangle}\big)(x).$$

Then we observe that

$$\begin{aligned}\big\|f_L\big\|_{B_p^{s,q}} &\lesssim \big\|\phi_{t_L}*\Lambda_{t_L}*\big(g_L e^{-2\pi i\langle \mathbf{v}_{t_L},\cdot\rangle}\big)\big\|_{L^p} + \big\|\phi_{t_L-1}*\Lambda_{t_L}*\big(g_L e^{-2\pi i\langle \mathbf{v}_{t_L},\cdot\rangle}\big)\big\|_{L^p}\\ &\lesssim \big\|g_L\big\|_{L^p}\end{aligned}$$

by Young's inequality. When $1 < p < 2$, by Littlewood-Paley theory and $l^p \hookrightarrow l^2$

$$\begin{aligned}\big\|g_L\big\|_{L^p} &\lesssim \Big\|\Big(\sum_{n=M}^{L} b_n^2 2^{-2t_n d(1-1/p)}|\phi_{t_n}(x)|^2\Big)^{1/2}\Big\|_{L^p}\\ &\le \Big(\sum_{n=M}^{L} b_n^p 2^{-t_n d(p-1)}\big\|\phi_{t_n}\big\|_{L^p}^p\Big)^{1/p} \sim \Big(\sum_{n=M}^{L} b_n^p\Big)^{1/p}.\end{aligned}$$

(Here we may also use orthogonality for $p=2$ and triangle inequality for $p=1$, and then apply interpolation.)

When $2\le p<\infty$, by Hausdorff-Young's inequality,

$$\big\Vert g_L\big\Vert_{L^p} \lesssim \Big\Vert \sum_{n=M}^{L} b_n 2^{-t_n d(1-1/p)}\widehat{\phi_{t_n}}\Big\Vert_{L^{p'}} = \Big(\sum_{n=M}^{L} b_n^{p'} 2^{-t_n d}\big\Vert \widehat{\phi_{t_n}}\big\Vert_{L^{p'}}^{p'}\Big)^{1/p'} \sim \Big(\sum_{k=M}^{L} b_k^{p'}\Big)^{1/p'}$$

where $1/p+1/p'=1$. Thus for $1<p<\infty$ we have obtained

$$\big\Vert f_L\big\Vert_{F_p^{s,q}} \lesssim \Big(\sum_{n=M}^{L} b_n^{\widetilde{p}}\Big)^{1/\widetilde{p}} \tag{7.3}$$

where $\widetilde{p}=\min\big(p, \frac{p}{p-1}\big)>1$.

On the other hand, let $a\in\mathcal{S}_{1,1}^0$ be defined as (6.5). Then

$$T_{[a]}f_L(x) = 2^{-st_L}\Lambda_{t_L}\ast\Lambda_{t_L}\ast\big(g_L e^{-2\pi i\langle \mathbf{v}_{t_L},\cdot\rangle}\big)(x)e^{2\pi i\langle \mathbf{v}_{t_L},x\rangle}$$

and

$$\big\Vert T_{[a]}f_L\big\Vert_{B_p^{s,q}} = 2^{-st_L}\Big(\sum_{l=0}^{\infty}2^{slq}\big\Vert \phi_l\ast\big[\Lambda_{t_L}\ast\Lambda_{t_L}\ast\big(g_L e^{-2\pi i\langle \mathbf{v}_{t_L},\cdot\rangle}\big)e^{2\pi i\langle \mathbf{v}_{t_L},\cdot\rangle}\big]\big\Vert_{L^p}^q\Big)^{1/q}.$$

Let

$$m_0(x)=\Lambda_0\ast\Lambda_0\ast\big(g_L e^{-2\pi i\langle \mathbf{v}_0,\cdot\rangle}\big)(x)e^{2\pi i\langle \mathbf{v}_0,x\rangle}$$

and

$$\begin{aligned} m_k(x) &= \Lambda_{t_k}\ast\Lambda_{t_k}\ast\big(g_L e^{-2\pi i\langle \mathbf{v}_{t_k},\cdot\rangle}\big)(x)e^{2\pi i\langle \mathbf{v}_{t_k},x\rangle} \\ &\quad -\Lambda_{t_{k-1}}\ast\Lambda_{t_{k-1}}\ast\big(g_L e^{-2\pi i\langle \mathbf{v}_{t_{k-1}},\cdot\rangle}\big)(x)e^{2\pi i\langle \mathbf{v}_{t_{k-1}},x\rangle}\end{aligned}$$

for $k\ge 1$. Then we see that

$$\Lambda_{t_L}\ast\Lambda_{t_L}\ast\big(g_L e^{-2\pi i\langle \mathbf{v}_{t_L},\cdot\rangle}\big)(x)e^{2\pi i\langle \mathbf{v}_{t_L},x\rangle} = \sum_{k=0}^{L} m_k(x)$$

and

$$\widehat{m_k}(\xi) = \widehat{g_L}(\xi)\Big(\big(\widehat{\Lambda}\big(\xi/2^{t_k}-\mathbf{v}_0\big)\big)^2-\big(\widehat{\Lambda}\big(\xi/2^{t_{k-1}}-\mathbf{v}_0\big)\big)^2\Big) := \widehat{g_L}(\xi)\widehat{\Psi_{t_k}}(\xi)$$

for $k\ge1$. If $|\xi|<2^{-13}2^{t_k}$ then $5/4<\big|\xi/2^{t_k}-\mathbf{v}_0\big|<7/4$ and $5/4<\big|\xi/2^{t_{k-1}}-\mathbf{v}_0\big|<7/4$ for which $\widehat{\Psi_k}(\xi)=0$. Moreover, if $|\xi|>(27/8)2^{t_k}$ then $15/8<\big|\xi/2^{t_k}-\mathbf{v}_0\big|$ and $15/8<\big|\xi/2^{t_{k-1}}-\mathbf{v}_0\big|$ and thus $\widehat{\Psi_k}(\xi)=0$. Therefore we have $Supp(\widehat{m_k})\subset\{2^{-13}2^{t_k}\le|\xi|\le(27/8)2^{t_k}\}$ and this yields that for $s\le 0$

$$\begin{aligned} \big\Vert T_{[a]}f_L\big\Vert_{B_p^{s,q}} &= 2^{-st_L}\Big(\sum_{l=0}^{\infty}2^{slq}\Big\Vert\sum_{k=0}^{L}\phi_l\ast m_k\Big\Vert_{L^p}^q\Big)^{1/q} \\ &\ge \Big(\sum_{l=M}^{L-1}\big\Vert\phi_{t_l}\ast(m_l+m_{l+1})\big\Vert_{L^p}^q\Big)^{1/q} = \Big(\sum_{l=M}^{L-1}\big\Vert\phi_{t_l}\ast(\Psi_{t_l}+\Psi_{t_{l+1}})\ast g_L\big\Vert_{L^p}^q\Big)^{1/q} \\ &= \Big(\sum_{l=M}^{L-1}b_l^q 2^{-t_l d(1-1/p)q}\big\Vert\phi_{t_l}\ast(\Psi_{t_l}+\Psi_{t_{l+1}})\ast\phi_{t_l}\big\Vert_{L^p}^q\Big)^{1/q} \sim \Big(\sum_{l=M}^{L-1}b_l^q\Big)^{1/q}. \end{aligned} \tag{7.4}$$

We are done by choosing a sequence $\{b_n\}$ so that (7.3) $\lesssim 1$ uniformly in L , but (7.4) diverges as $L\to\infty$.

7.3. **The case $p = \infty$.** First of all, when $p = q = \infty$ it is done by Theorem 1.1 (1) and the fact that $B_\infty^{s,\infty} = F_\infty^{s,\infty}$. Now we assume $0 < q < \infty$ and prove that if $a \in \mathcal{S}_{1,1}^0$ is defined in (6.1) then

$$\|T_{[a]}\|_{B_\infty^{s,q} \to B_\infty^{s,q}} = \infty \tag{7.5}$$

if $s \le 0$.

For sufficiently large $L > 0$ we define

$$g_L(x) := 2^{-st_L} e^{-2\pi i \langle 2^{t_L} e_1, x\rangle} \sum_{k=10}^{L-10} \phi(2^{t_k}(x - t_k)).$$

We observe that $Supp(\widehat{g_L}) \subset \{\xi : 2^{t_L}(1 - 1/2^{99}) \le |\xi| \le 2^{t_L}(1 + 1/2^{99})\}$ and thus

$$\|g_L\|_{B_\infty^{0,q}} = \|\phi_{t_L} * g_L\|_{L^\infty} \lesssim \|g_L\|_{L^\infty} \lesssim 1.$$

On the other hand, we see that

$$T_{[a]} g_L(x) = 2^{-st_L} \sum_{k=10}^{L-10} \phi(2^{t_k}(x - t_k))$$

and

$$\|T_{[a]} g_L\|_{B_\infty^{s,q}} = 2^{-st_L} \Big(\sum_{k=10}^{L-10} 2^{st_k q} \big\| \phi * \phi\big(2^{t_k}(\cdot - t_k)\big)\big\|_{L^\infty}^q \Big)^{1/q} \sim \begin{cases} L^{1/q} & s = 0 \\ 2^{-st_L} & s < 0 \end{cases}$$

By letting $L \to \infty$ we prove (7.5).

Remark. When $1 < q < \infty$ or $0 < q < 1$ then the examples in subsection 7.1 and 7.2 can be also applied.

8. Proof of Theorem 1.3

8.1. **The case $1 < q < p = \infty$.** Assume $s \le 0$ and $L > 0$. Let $\{\nu_k\}$ be a sequence of positive numbers. We define

$$a(x,\xi) := \sum_{k=10}^{\infty} \widehat{\widetilde{\phi}_k}(\xi) e^{2\pi i \langle 2^{-k} e_1, \xi\rangle} e^{-2\pi i \langle 2^k e_1, x\rangle},$$

$$f_L(x) := \sum_{k=10}^{L} \nu_{t_k} \eta(x - 2^{-t_k} e_1) e^{2\pi i \langle 2^{t_k} e_1, x\rangle}.$$

Clearly, $a \in \mathcal{S}_{1,1}^0$ and

$$\|f_L\|_{F_\infty^{0,q}} \lesssim \Big(\sum_{k=10}^{L} \nu_{t_k}^q \Big)^{1/q}.$$

We see that $T_{[a]} f_L(x) = \eta(x) \sum_{k=10}^{L} \nu_{t_k}$ and this gives

$$\|T_{[a]} f_L\|_{F_\infty^{s,q}} \gtrsim \sum_{k=10}^{L} \nu_{t_k}.$$

Choose a sequence $\{a_k\}$ so that $\sum_k \nu_{t_k} = \infty$ and $\sum_k \nu_{t_k}^q < \infty$.

8.2. **The case $0 < q \leq 1$ and $p = \infty$.** We apply the idea in subsection 6.2 and 6.3. We fix $\mu \geq 1$. We may assume $M > \mu$ and let L be sufficiently large integer so that $L \gg M$, and $s \leq d(1/q - 1)$. For each $k \in \{0, 1, \dots\}$, let $\mathcal{R}^\mu(k)$ be the set of all dyadic cubes of side-length $2^{-t_k - M}$ in $[0, 2^{-\mu}]^d$ and $\mathcal{R}^\mu = \bigcup_{k\in\mathbb{Z}} \mathcal{R}^\mu(k)$. Let Ω be a probability space with probability measure λ. Let $\{\theta_Q\}$ be a family of independent random variables indexed by $Q \in \mathcal{R}^\mu$, each of which takes the value 1 with probability $1/L$ and the value 0 with probability $1 - 1/L$ and χ_Q denote the characteristic function of Q. Let $\{r_Q\}$ be a family of the Rademacher functions, defined on $[0, 1]$, indexed by $Q \in \mathcal{R}^\mu$ and define random functions

$$h_k^{\omega,t,\mu}(x) := \sum_{Q\in\mathcal{R}^\mu(k)} r_Q(t)\theta_Q(\omega)\chi_Q(x)$$

for $\omega \in \Omega, t \in [0, 1]$.

Now define

$$g_L^{\omega,t,\mu}(x) := \sum_{k=M}^{L} 2^{-st_k} \Lambda_{t_k} * \big(h_k^{\omega,t,\mu} e^{-2\pi i \langle \mathbf{v}_k, \cdot\rangle}\big)(x)$$

and $a \in \mathcal{S}^0_{1,1}$ as in (6.5). Then our claim is that

$$\Big[\int_0^1 \int_\Omega \sup_{R\in\mathcal{D}_\mu} \Big(\frac{1}{|R|}\int_R \sum_{k=\mu}^{\infty} 2^{skq}\big|\Pi_k g_L^{\omega,t,\mu}(x)\big|^q dx\Big) d\lambda dt\Big]^{1/q} \lesssim 1 \tag{8.1}$$

$$\begin{aligned} &\Big[\int_0^1 \int_\Omega \sup_{R\in\mathcal{D}_\mu} \Big(\frac{1}{|R|}\int_R \sum_{k=\mu}^{\infty} 2^{skq}\big|\Pi_k T_{[a]} g_L^{\omega,t,\mu}(x)\big|^q dx\Big) d\lambda dt\Big]^{1/q} \\ \gtrsim\;& L^{-(s+d-d/q)/2d}\big(\log L\big)^{1/2} \end{aligned} \tag{8.2}$$

We observe that the left hand side of (8.1) is less than a constant times

$$\Big[\int_0^1 \int_\Omega \sup_{R\in\mathcal{D}_\mu} \Big(\frac{1}{|R|}\int_R \sum_{k=M}^{L} \big(\mathfrak{M}_{\sigma,2^{t_k}} h_k^{\omega,t,\mu}(x)\big)^q dx\Big) d\lambda dt\Big]^{1/q}$$

by the idea of (6.4). Then by (6.13) and the boundedness of Hardy-Littlewood maximal operator, the above expression is dominated by

$$\begin{aligned} &\Big[\int_\Omega \sup_{R\in\mathcal{D}_\mu} \Big(\frac{1}{|R|}\sum_{k=M}^{L}\sum_{Q\in\mathcal{R}^\mu(k)} \theta_Q(\omega)|Q|\Big) d\lambda\Big]^{1/q} \\ \sim\;& \Big[2^{\mu d}\sum_{k=M}^{L} 2^{-kd} \sum_{Q\in\mathcal{R}^\mu(k)} \lambda(\{\theta_Q = 1\})\Big]^{1/q} \lesssim 1. \end{aligned}$$

This proves (8.1).

Now we write, as before, $T_{[a]} g_L^{\omega,t,\mu}(x) = \sum_{k=M}^{L} d_{t_k}(x)$ where

$$d_{t_k}(x) := 2^{-st_j} \Lambda_{t_k} * \Lambda_{t_k} * \big(h_k^{\omega,t,\mu} e^{-2\pi i\langle \mathbf{v}_k, \cdot\rangle}\big)(x) e^{2\pi i < \mathbf{v}_k, x>}.$$

Then we observe that the left hand side of the inequality (8.2) is bounded below by

$$\Big[\int_0^1\int_\Omega \sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=M}^{L}2^{st_kq}\Big|\sum_{n=k}^{L}(\phi_{t_k}+\phi_{t_k+1})*d_{t_n}(x)\Big|^q dx\Big)d\lambda dt\Big]^{1/q}$$

$$\gtrsim \Big[\int_0^1\int_\Omega \sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=M}^{L}2^{st_kq}\Big|\sum_{n=k+1}^{L}(\phi_{t_k}+\phi_{t_k+1})*d_{t_n}(x)\Big|^q dx\Big)d\lambda dt\Big]^{1/q} \tag{8.3}$$

$$-\Big[\int_0^1\int_\Omega \sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=M}^{L}2^{st_kq}\big|(\phi_{t_k}+\phi_{t_k+1})*d_{t_k}(x)\big|^q dx\Big)d\lambda dt\Big]^{1/q} \tag{8.4}$$

Clearly, (8.4) $\lesssim 1$ uniformly in L by the idea in (8.1).

By using (6.9) with $g_{t_n}=h_{t_n}^{\omega,t,\mu}$ and the method in (6.10) with Lemma 2.2 we obtain that (8.3) is greater than a constant times

$$\begin{aligned}
&\Big[\int_0^1\int_\Omega \sup_{R\in\mathcal{D}_\mu}\Big(\frac{1}{|R|}\int_R\sum_{k=M}^{L}\Big|\sum_{n=1}^{L-k}2^{-st_n}\Gamma_{t_k}*h_{t_{n+k}}^{\omega,t,\mu}(x)\Big|^q dx\Big)d\lambda dt\Big]^{1/q}\\
\geq\ &\Big[2^{\mu d}\int_{[0,2^{-\mu}]^d}\int_\Omega\sum_{k=M}^{L}\Big(\int_0^1\Big|\sum_{n=1}^{L-k}\sum_{Q\in\mathcal{R}^\mu(n+k)}r_Q(t)\theta_Q(\omega)2^{-st_n}\Gamma_{t_k}*\chi_Q(x)\Big|^q\Big)dtd\lambda dx\Big]^{1/q}\\
\sim\ &\Big[2^{\mu d}\sum_{k=M}^{L}\int_{[0,2^{-\mu}]^d}\int_\Omega\Big(\sum_{n=1}^{L-k}\sum_{Q\in\mathcal{R}^\mu(n+k)}\theta_Q(\omega)2^{-2st_n}\big|\Gamma_{t_k}*\chi_Q(x)\big|^2\Big)^{q/2}d\lambda dx\Big]^{1/q}.
\end{aligned}$$

For each $P\in\mathcal{R}^\mu(k)$ let $\mathcal{V}_n(k,P):=\{Q\in\mathcal{R}^\mu(k+n):Q\subset P\}$. Then we see that the last expression is

$$\begin{aligned}
\geq\ &\Big[2^{\mu d}\sum_{k=M}^{L}\sum_{P\in\mathcal{R}^\mu}\int_P\int_\Omega\Big(\sum_{n=1}^{L-k}\sum_{Q\in\mathcal{V}_n(k,P)}\theta_Q(\omega)2^{-2st_n}\big|\Gamma_{t_k}*\chi_Q(x)\big|^2\Big)^{q/2}d\lambda dx\Big]^{1/q}\\
\gtrsim\ &\Big[2^{\mu d}\sum_{k=M}^{L}2^{-t_kd}\sum_{P\in\mathcal{R}^\mu(k)}\int_\Omega\Big(\sum_{n=1}^{L-k}2^{-2st_n}2^{-2t_nd}\sum_{Q\in\mathcal{V}_n(k,P)}\theta_Q(\omega)\Big)^{q/2}d\lambda\Big]^{1/q}\\
\gtrsim\ &\Big[2^{\mu d}\sum_{k=M}^{L}2^{-t_kd}\Big(\sum_{n=1}^{L-k}2^{-2st_n}2^{-2t_nd}\Big[\sum_{P\in\mathcal{R}^\mu(k)}\int_\Omega\Big(\sum_{Q\in\mathcal{V}_n(k,P)}\theta_Q(\omega)\Big)^{q/2}d\lambda\Big]^{2/q}\Big)^{q/2}\Big]^{1/q}\\
\gtrsim\ &\Big[2^{\mu d}\sum_{k=M}^{L}2^{-t_kd}\Big(\sum_{n=1}^{L-k}2^{-2st_n}2^{-2t_nd}2^{2t_nd/q}\Big[\sum_{P\in\mathcal{R}^\mu(k)}1/L\big(1-1/L\big)^{2^{t_nd}}\Big]^{2/q}\Big)^{q/2}\Big]^{1/q}\\
=\ &\Big[\frac{1}{L}\sum_{k=M}^{L}\Big[\sum_{n=1}^{L-k}2^{-2t_n(s+d-d/q)}\Big(1-\frac{1}{L}\Big)^{(2/q)2^{t_nd}}\Big]^{q/2}\Big]^{1/q}\\
\gtrsim\ &L^{-(s+d-d/q)/2d}(\log L)^{1/2}
\end{aligned}$$

by following the process to get (6.15). This proves (8.2).

9. **Other mapping properties**

Hörmander [6] proved that $T_{[a]}\in Op\mathcal{S}_{1,1}^m$ maps H^{s+m} to H^s with arbitrary $s\in\mathbb{R}$ if a satisfies the twisted diagonal condition

$$\widehat{a}(\eta,\xi)=0\quad\text{where}\quad C\big(|\eta+\xi|+1\big)\leq|\xi| \tag{9.1}$$

for some $C > 1$. This means that the partially Fourier transformed symbol $\widehat{a}(\eta,\xi) := \big(a(\cdot,\xi)\big)^{\wedge}(\eta)$ vanishes in a conical neighborhood of a non-compact part of the twisted diagonal $\{(\eta,\xi) : \xi+\eta = 0\}$.

In Theorem 1.2 the condition $s > \tau_q$ is used just to apply Lemma 3.2 for $T_{[d_k]}$ because the Fourier transform of $T_{[d_k]}f$ is supported in a ball. We observe that if (9.1) holds, then $T_{[d_k]}f$ has a Fourier support in an annulus so that we apply Lemma 3.1, instead of Lemma 3.2. Therefore we can drop the restriction on s with (9.1).

Corollary 9.1. *Let $0 < q < \infty$, $s \in \mathbb{R}$ and $m \in \mathbb{R}$. Suppose $a \in \mathcal{S}^m_{1,1}$. If (9.1) holds with $C > 1$, then $T_{[a]}$ maps $F^{s+m,q}_{\infty}$ into $F^{s,q}_{\infty}$.*

We also prove the following boundedness result of $T_{[a]} \in Op\mathcal{S}^m_{1,1}$ when $s = 0$.

Theorem 9.2. *Let $m \in \mathbb{R}$ and suppose $a \in \mathcal{S}^m_{1,1}$. Then $T_{[a]}$ maps $F^{m,1}_{\infty}$ into $bmo(= F^{0,2}_{\infty})$.*

Proof. Suppose $f \in F^{m,1}_{\infty}$. Let $\mathfrak{S}^{far}_{[a]}$ and $\mathfrak{S}^{far}_{[a]}$ be defined as in (4.2) and (4.3). Then we already proved in Section 4 that $\big\|\mathfrak{S}^{far}_{[a]}f\big\|_{F^{0,1}_{\infty}} \lesssim \|f\|_{F^{m,1}_{\infty}}$ and the embedding $F^{0,1}_{\infty} \hookrightarrow bmo$ yields that $\|\mathfrak{S}^{far}_{[a]}f\|_{bmo} \lesssim \|f\|_{F^{m,1}_{\infty}}$.

Thus it remains to prove

$$\|\mathfrak{S}^{near}_{[a]}f\|_{bmo} \lesssim \|f\|_{F^{m,1}_{\infty}}.$$

By definition

$$\begin{aligned}\|\mathfrak{S}^{near}_{[a]}f\|_{bmo} &= \sup_{l(Q)\geq 1}\frac{1}{|Q|}\int_Q \big|\mathfrak{S}^{near}_{[a]}f(x)\big|dx \\ &\quad + \sup_{l(Q)<1}\frac{1}{|Q|}\int_Q \big|\mathfrak{S}^{near}_{[a]}f(x) - \big(\mathfrak{S}^{near}_{[a]}f\big)_Q\big|dx\end{aligned}$$

where $(\mathfrak{S}^{near}_{[a]}f)_Q$ means the average of $\mathfrak{S}^{near}_{[a]}f$ over Q.

By using (4.8) we obtain that for $l(Q) \geq 1$ and $\sigma > d$

$$\frac{1}{|Q|}\int_Q \big|\mathfrak{S}^{near}_{[a]}f(x)\big|dx \lesssim \frac{1}{|Q|}\int_Q \sum_{k=0}^{\infty} 2^{km}\mathfrak{M}_{\sigma,2^k}\widetilde{\Pi}_k f(x)dx$$

$$\lesssim \sup_{0\leq k\leq 2}\big\|2^{km}\mathfrak{M}_{\sigma,2^k}\widetilde{\Pi}_k f\big\|_{L^\infty} + \frac{1}{|Q|}\sum_{P\in\mathcal{D}_2, P\subset Q}\int_P \sum_{k=3}^{\infty} 2^{km}\mathfrak{M}_{\sigma,2^k}\widetilde{\Pi}_k f(x)dx$$

By the L^∞ boundedness of $\mathfrak{M}_{\sigma,2^k}$, triangle inequality, and embedding $F^{m,1}_{\infty} \hookrightarrow F^{m,\infty}_{\infty}$ the supremum part is dominated by a constant times

$$\sup_{0\leq k\leq 3}\big\|2^{km}\Pi_k f\big\|_{L^\infty} \lesssim \|f\|_{F^{m,1}_{\infty}}.$$

Moreover, by using Lemma 2.2 and triangle inequality we have

$$\frac{1}{|Q|}\sum_{P\in\mathcal{D}_2, P\subset Q}\int_P \sum_{k=3}^{\infty} 2^{km}\mathfrak{M}_{\sigma,2^k}\widetilde{\Pi}_k f(x)dx \lesssim \sup_{R\in\mathcal{D}_2}\int_R \sum_{k=2}^{\infty} 2^{km}\big|\Pi_k f(x)\big|dx \lesssim \|f\|_{F^{m,1}_{\infty}}.$$

For the other part we assume $l(Q) < 1$. Then

$$\begin{aligned}\frac{1}{|Q|}\int_Q \big|\mathfrak{S}^{near}_{[a]}f(x) - \big(\mathfrak{S}^{near}_{[a]}f\big)_Q\big|dx &\leq \frac{1}{|Q|}\int_Q \sum_{k=0}^{\infty}\Big|T_{[d_k]}f(x) - \frac{1}{|Q|}\int_Q T_{[d_k]}f(y)dy\Big|dx \\ &\leq I + II\end{aligned}$$

where

$$I := \|T_{[d_0]}f\|_{L^\infty} + \frac{1}{|Q|}\int_Q \sum_{k=-\log_2 l(Q)+1}^{\infty} \big|T_{[d_k]}f(x)\big|dx$$

$$II := \frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \sum_{k=1}^{-\log_2 l(Q)} \Big|T_{[d_k]}f(x) - T_{[d_k]}f(y)\Big|dxdy.$$

It is easy to see that $I \lesssim \|f\|_{F^{m,1}_\infty}$ by using the above argument.

In order to get the estimate of II, let U_k be the kernel of $T_{[d_k]}$. Then for $x, y \in Q$

$$\begin{aligned} \big|T_{d_k}f(x) - T_{d_k}f(y)\big| &= \Big|\int_{\mathbb{R}^d} \big(U_k(x,z) - U_k(y,z)\big)\widetilde{\Pi}_k f(z)dz\Big| \\ &\le \|\widetilde{\Pi}_k f\|_{L^\infty}\int_{\mathbb{R}^d}\big|U_k(x,z) - U_k(y,z)\big|dz. \end{aligned}$$

We claim that for $k \le -\log_2 l(Q)$

$$\int_{\mathbb{R}^d}\big|U_k(x,z) - U_k(y,z)\big|dz \lesssim 2^{km}2^k l(Q). \tag{9.2}$$

Then

$$II \lesssim \sup_{k\ge 1} 2^{km}\|\widetilde{\Pi}_k f\| \sup_{l(Q)<1}\Big(\sum_{k=1}^{-\log_2 l(Q)} 2^k l(Q)\Big) \lesssim \|f\|_{F^{m,\infty}_\infty} \lesssim \|f\|_{F^{m,1}_\infty}.$$

To see (9.2) write

$$\begin{aligned} \big|U_k(x,z) - U_k(y,z)\big| \le\ & \Big|\int_{|\xi|\sim 2^k}\big(d_k(x,\xi) - d_k(y,\xi)\big)e^{2\pi i\langle x-z,\xi\rangle}d\xi\Big| \\ & + \Big|\int_{|\xi|\sim 2^k} d_k(y,\xi)e^{2\pi i\langle x-z,\xi\rangle}\big(1 - e^{2\pi i\langle y-x,\xi\rangle}\big)d\xi\Big|. \end{aligned}$$

Then by an elementary computation we obtain that if $x, y \in Q$ then for $M > 0$

$$\big|U_k(x,z) - U_k(y,z)\big| \lesssim_M \begin{cases} 2^{km}2^{kd}2^k l(Q) \\ 2^{km}2^{kd}2^k l(Q)\big(2^k|x-z|\big)^{-M}. \end{cases}$$

The first one follows from $|x-y| \le \sqrt{d}l(Q)$ and the second one from integration by parts in ξ variable M times and $2^k|x-y| \le \sqrt{d}2^k l(Q) \lesssim 1$. It follows that for $x, y \in Q$

$$\int_{|x-z|\le 2^{-k}} \big|U_k(x,z) - U_k(y,z)\big|dz \lesssim 2^{km}2^k l(Q)$$

and

$$\int_{|x-z|>2^{-k}} \big|U_k(x,z) - U_k(y,z)\big|dz \lesssim_M 2^{km}2^k l(Q)$$

for $M > d$, which proves (9.2). □

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