EXPONENTIAL DISTRIBUTION OF RETURN TIMES FOR WEAKLY MARKOV SYSTEMS

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Abstract. We introduce the concept of Weakly Markov dynamical systems. We study for these systems the asymptotic behaviour of the distribution of first return times to shrinking balls. We prove for almost all balls its convergence to the exponential law. We obtain this for sets of radii with relative Lebesgue measure converging very fast to one.

We show that several classes of smooth dynamical systems are Weakly Markov: expanding repellers, conformal IFSs, rational functions on $\widehat{\mathbb{C}}$ and many more.

For the conformal dynamical systems we in fact prove much more, namely that the convergence to the exponential law is along all radii. Up to our best knowledge any results achieving all radii have been hitherto known only in special cases (e.g. the invariant measure equivalent to an Ahlfors one). We obtain the aforementioned convergence along all radii by proving the Full Thin Annuli Property, the property interesting on its own.

Contents

1. Introduction

In this paper we deal with asymptotic statistics of return times to shrinking objects that are formed by ordinary open balls with radii converging to zero. Let (T, X, μ, ρ) be a metric

1991 *Mathematics Subject Classification.* Primary 37B20, Secondary 37A25.

Key words and phrases. return time, decay of correlations, exponential law.

The research of M. Urbański was supported in part by the NSF Grant DMS 1361677. A. Zdunik was supported in part by the NCN grant 2014/13/B/ST1/04551.

measure preserving dynamical system. By this we mean that (X, ρ) is a metric space and $T : X \to X$ is a Borel measurable map preserving a Borel probability measure μ on X. Given a set $U \subset X$ and $x \in X$ define

$$\tau_U(x) := \min\{n \geq 1 : T^n(x) \in U\},$$

and call it the (first) entry time to U.

The modern study of return times was initiated in the seminal papers of M. Boshernitzan, [Bos] and D. S. Ornstein and B. Weiss, [OW]. They looked at return times to shrinking balls (Boshernitzan) or to decreasing cylinders in a symbol space (Ornstein, Weiss). These papers triggered a growing interest in the statistics of return times reflected in numerous publications on the subject. Our paper also concerns such statistics; we focus on the convergence to the exponential law. Our approach is primarily of strong geometric flavour as we fully face the property of thin annuli and prove that this property actually always holds. Regarding the dynamics, we introduce and explore the concept of Weakly Markov systems which we apply to many natural classes of dynamical systems.

In what follows in the context of entrance and return times, U will be frequently an open ball in X, and we will denote the open ball of radius $r > 0$ centred at a point $x \in X$ by both $B(x, r)$ and $B_r(x)$ depending on the appropriate setting. Recall that given a finite measure μ and a measurable set A we denote by μ_A the normalized measure on A, i.e.. $\mu_A(F) := \frac{\mu(F)}{\mu(A)}$ for every measurable subset F of A.

Our main motivation and the main goal in this article are to identify a large rich class of metric measure preserving dynamical systems and large classes of families of positive radii $\mathcal{R} = \{R_x \subset (0, 1]\}$, such that $0 \in \overline{R}_x$, which are defined for μ–a.e. $x \in X$, and for which the following properties hold:

$$\lim_{R_x \ni r \to 0} \sup_{t \geq 0} \left| \mu\left(\left\{ z \in X \colon \tau_{B_r(x)}(z) > \frac{t}{\mu(B_r(x))} \right\}\right) - e^{-t} \right| = 0 \tag{1.1}$$

for μ–a.e. $x \in X$, i.e. the distributions of the normalized first entry time converge to the exponential one law, and

$$\lim_{R_x \ni r \to 0} \sup_{t \geq 0} \left| \mu_{B_r(x)}\left(\left\{ z \in B_r(x) \colon \tau_{B_r(x)}(z) > \frac{t}{\mu(B_r(x))} \right\}\right) - e^{-t} \right| = 0 \tag{1.2}$$

for μ–a.e. $x \in X$, i.e. the distributions of the normalized first return time converge to the exponential one law. Then formulas (1.1) and (1.2) are equivalent to saying that for every Borel set $F \subset [0, +\infty)$ with boundary of Lebesgue measure zero, we have that

$$\lim_{R_x \ni r \to 0} \mu\left(\left\{ z \in X \colon \tau_{B_r(x)}(z)\mu(B_r(x)) \in F \right\}\right) = \int_F e^{-t}\, dt \tag{1.3}$$

for μ–a.e. $x \in X$, and

$$\lim_{R_x \ni r \to 0} \mu_{B_r(x)}\left(\left\{ z \in B_r(x) \colon \tau_{B_r(x)}(z)\mu(B_r(x)) \in F \right\}\right) = \int_F e^{-t}\, dt \tag{1.4}$$

for μ–a.e. $x \in X$,

Our large class of metric measure preserving dynamical systems is that of *Weakly Markov* ones defined, somewhat lengthily but naturally, in the next section, i.e. Section 2, and it is motivated by the class of loosely Markov systems introduced and explored in [Ur]. This class captures systems (not necessarily conformal) such as expanding repellers, holomorphic endomorphisms of complex projective spaces, and Axiom A diffeomorphisms but also conformal ones that have no real non-conformal counterparts such as conformal graph directed Markov systems, conformal expanding repellers, rational functions of the Riemann sphere, and transcendental meromorphic functions. All this is described in detail in Section 4 devoted to examples. Having conformality in the system is not just to work in a more comfortable setting, it does have seminal qualitative impact on the range of radii for which our main theorems hold. In fact, up to our best knowledge, this is the first time that for so general classes of systems and invariant measures the convergence to the exponential law is proved to hold along all radii. We want to emphasize the previous sentence because — as far as we know — all the results achieving all radii have been hitherto proved only for very special systems, e.g. when the invariant measure is equivalent to an Ahlfors one.

1.1. **Subsets of radii.** We now describe the classes of radii for which we prove the above mentioned convergence to the exponential one law. As said, any family $\mathcal{R}$ appearing in formulas (1.1) – (1.4) will be commonly referred to as a class of radii. We call two such families $\mathcal{R} = \{R_x\}$ and $\mathcal{S} = \{S_x\}$ equivalent, if for μ–a.e. in X there exists $\eta_x > 0$ such that

$$R_x \cap (0, \eta_x] = S_x \cap (0, \eta_x].$$

Then any of the formulas (1.1) – (1.4) holds for $\mathcal{R}$ if and only if it holds for $\mathcal{S}$. So when talking about such families we really mean their equivalence classes. We define a few natural families (denote Leb for the Lebesgue measure).

- First one: $\mathcal{F} = \{T_x\}$, called *full*, for which $T_x = (0, 1]$ for all x.
- The next: $\mathcal{AF}$, called *almost full*, if $\mathrm{Leb}(T_x) = 1$ for all x.
- The third: $\mathcal{D}$, called *dense*, if

  $$\liminf_{r\to 0} \frac{\mathrm{Leb}(T_x \cap (0, r])}{r} = 1 \quad \text{for all } x$$

  or in other words – when 0 is the density point of T_x.
- Another one, denoted by $\mathcal{SD}$, called *super dense*, if for every $\alpha > 0$

  $$\lim_{r\to 0} \frac{\left|\frac{\mathrm{Leb}(T_x \cap (0,r])}{r} - 1\right|}{r^\alpha} = 0 \quad \text{for all } x.$$
- The last: $\beta\mathcal{T}$, called *β–thick*, $\beta > 0$, if

  $$\lim_{r\to 0} \frac{\left|\frac{\mathrm{Leb}(T_x \cap (0,r])}{r} - 1\right|}{r^{\ln^\beta(1/r)}} = 0 \quad \text{for all } x.$$

Trivially, (for any $\beta > 0$)

$$\mathcal{F} \subset \mathcal{AF} \subset \beta\mathcal{F} \subset \mathcal{SD} \subset \mathcal{D}.$$

1.2. **Main results.**

Theorem A. *If (T,X,μ,ρ) is Weakly Markov system, then for every $\beta>0$ there exists $\beta\mathcal{T}=\{T_x\}$, $x\in X$, a β–thick class of radii such that for μ–a.e. $x\in X$*

$$\lim_{T_x\ni r\to 0}\sup_{t\geq 0}\left|\mu\left(\left\{z\in X\colon \tau_{B_r(x)}(z)>\frac{t}{\mu(B_r(x))}\right\}\right)-e^{-t}\right|=0, \tag{1.5}$$

i.e. the distributions of the normalized first entry time converge to the exponential one law, and

$$\lim_{T_x\ni r\to 0}\sup_{t\geq 0}\left|\mu_{B_r(x)}\left(\left\{z\in B_r(x)\colon \tau_{B_r(x)}(z)>\frac{t}{\mu(B_r(x))}\right\}\right)-e^{-t}\right|=0, \tag{1.6}$$

i.e. the distributions of the normalized first return time converge to the exponential one law.

Remark 1.1. In fact, as we prove, there are even larger classes of radii for which Theorem A holds. See Theorem B and Remark 1.5.

We now introduce a crucial property of a measure, which we call the *Thin Annuli Property*. We define and discuss it now in two steps.

Definition 1.2. A function $\kappa\colon(0,1]\to\mathbb{R}_+$ will be called *subpolynomial* if it is monotone nonincreasing, and for every $\varepsilon>0$

$$\lim_{r\to 0}\kappa(r)r^{\varepsilon}=0. \tag{1.7}$$

Remark 1.3. Standard examples of subpolynomial functions include all positive constant functions and functions of the form $\kappa(r)=\alpha\ln^{\beta}(1/r)$, for $\alpha,\beta>0$.

Definition 1.4. Let μ be a finite Borel measure on a metric space X. Let $\mathcal{R}=\{R_x\}$, $x\in X$, be a class of radii defined μ–a.e. in X. The measure μ is said to have a *Thin Annuli Property* relative to $\mathcal{R}$ if for μ–almost every $x\in X$ there exists a subpolynomial function $\kappa_x:(0,1]\to\mathbb{R}_+$ such that

$$\lim_{R_x\ni r\to 0}\frac{\mu\left(B(x,r+r^{\kappa_x(r)})\setminus B(x,r)\right)}{\mu\left(B(x,r)\right)}=0. \tag{1.8}$$

Given $\beta>0$ we say that a measure μ satisfies the *β–Thick Thin Annuli Property* if it has the Thin Annuli Property with respect to some *β–thick* class of radii. We say that measure μ satisfies the *Thick Thin Annuli Property* if it satisfies the β–Thick Thin Annuli Property for every $\beta>0$. We analogously define the *Full Thin Annuli Property* and others.

The two main ingredients of the proof of Theorem A announced above, and interesting on their own, are the following.

Theorem B. *Let (T,X,μ,ρ) is Weakly Markov system. If $\mathcal{R}=\{R_x\}$, $x\in X$, is a class of radii defined μ–a.e. in X, and the measure μ has the Thin Annuli Property relative to $\mathcal{R}$, then both (1.1) and (1.1) hold, i.e. the distributions of the normalized first entry time and first return time converge to the exponential one law.*

Theorem C. *Every finite Borel measure μ in a Euclidean space $\mathbb{R}^d$, satisfies the Thick Thin Annuli Property, i.e. it satisfies the β–Thick Thin Annuli Property for every $\beta>0$.*

Remark 1.5. In fact we have even more: see Theorem 3.7 along with Theorem 3.5, Definition 1.2 (and Remark 1.3), and Remark 3.8.

Of course Theorem A is an immediate consequence of Theorem B and Theorem C.

The further natural question to ask is about the convergence to the exponential law along a full class of radii. Because of Theorem B the answer would be positive if we had a Weakly Markov system whose measure has the Full Thin Annuli Property.

We have discovered that this property is satisfied for a large class of systems. The only additional requirement is for the system to be generated by a countable (either finite or infinite) alphabet conformal iterated function system (IFS). This leads, via suitable inducing schemes, to several families of applications as shown in the last section.

Here is the fourth main result of our paper, also interesting on its own.

Theorem D (see Theorem 3.14)**.** *If $\mathcal{S} = \{\phi_e : X \to X\}_{e\in E}$ is a conformal geometrically irreducible IFS, then for every $\mu \in \mathcal{M}_E$, a large class of measures containing many Gibbs/equilibrium measures of Hölder continuous summable potentials on the symbol space $E^{\mathbb{N}}$, the projection measure $\mu\circ\pi^{-1}$ on $J_{\mathcal{S}}$ has the Full Thin Annuli Property, i.e.*

$$\lim_{r\to 0}\frac{\mu\circ\pi^{-1}\big(B(x,r)\setminus B(x,r+r^3)\big)}{\mu\circ\pi^{-1}\left(B(x,r)\right)} = 0 \qquad \textit{for} \;\; \mu\circ\pi^{-1}\text{–}\textit{a.e.} \;\; x\in J_{\mathcal{S}}.$$

We should immediately emphasize that in this theorem the conformal IFS $\mathcal{S}$ is not required to satisfy any kind of separation condition, nor even its weakest form known as the Open Set Condition. In other words, all kinds of overlaps are allowed. Also, the measures $\mu \in \mathcal{M}_E$ need not be Gibbs/equilibrium states nor even shift–invariant. These measures are just to satisfy two natural conditions formulated in Subsection 3.2. Theorem D via Theorem B leads to the convergence to the exponential distribution along all radii (full class) for Weakly Markov systems generated by conformal IFS themselves and Gibbs/equilibrium measures (now we do need them), and as a consequence of this, for systems generated by conformal graph directed Markov systems. We have the following.

Theorem E (see Theorem 4.12)**.** *Suppose that $\mathcal{S}$ is a finitely irreducible and geometrically irreducible conformal IFS satisfying the Strong Open Set Condition. If $f : E^{\mathbb{N}} \to \mathbb{R}$ is a summable Hölder continuous potential such that for some $\beta > 0$*

$$\sum_{e\in E}\exp\left(\inf\left(f|_{[e]}\right)\right)||\phi_e'||^{-\beta} < +\infty, \tag{1.9}$$

then the measure–preserving dynamical system $\left(T_{\mathcal{S}} : \mathring{J}_{\mathcal{S}} \to \mathring{J}_{\mathcal{S}}, \hat{\mu}_f\right)$ is Weakly Markov and satisfies the Full Thin Annuli Property. In consequence, the exponential one laws hold along all radii.

We emphasize that the above theorem is very general and allows us to prove, using a suitable inducing procedure, the exponential law for several naturally occurring classes of conformal systems, as seen in Section 4 devoted to applications and examples.

What we though want to emphasize most is that for the conformal dynamical systems as above and in Theorem E the convergence to the exponential law is along all radii. This seems to be the first such general result — all (up to authors' best knowledge) the previous proofs of the convergence were given in very specific cases (under very strict assumptions on the invariant measure).

We end this introduction with a comment on the concept of Weakly Markov systems and some of its advantages. Indeed, the concept of Weakly Markov systems captures and extends that of Loosely Markov systems of [Ur] and of earliest works on the subject such as [STV]. One of the primary advantage of this approach is that now no transfer operators are involved, and merely the exponential decay of correlations is assumed along with two other standard hypotheses. This is especially more convenient when dealing with invertible systems as Axiom A diffeomorphisms, for which the method of transfer operators does not automatically apply.

2. Convergence to Exponential Distribution for Weakly Markov Systems

In this section we do two things. First, we define the class of Weakly Markov systems and then we prove Theorem B. We begin by recalling the following standard definition:

Definition 2.1. For a finite Borel measure μ on a metric space X, define the *lower and upper pointwise dimensions*, denoted respectively by $\underline{d}_\mu$ and $\overline{d}_\mu$, of the measure μ by

$$\underline{d}_\mu(z) = \liminf_{r\to 0} \frac{\ln\big(\mu(B_r(z))\big)}{\ln r}, \qquad \overline{d}_\mu(z) = \limsup_{r\to 0} \frac{\ln\big(\mu(B_r(z))\big)}{\ln r}.$$

Passing to the next concept we need, given $\xi \in (0,1]$ denote by $\mathcal{H}^\xi(X)$ the vector space of all real–valued Hölder continuous functions on a metric space (X,ρ) with exponent ξ, i.e. $f \in \mathcal{H}^\xi(X)$ if $f: X \to \mathbb{R}$ is bounded, continuous, and $v_\xi(f) < \infty$, where

$$v_\xi(f) := \inf\left\{H \geq 0\colon \forall_{x,y\in X}\, |f(x)-f(y)| \leq H\rho^\xi(x,y))\right\}. \tag{2.1}$$

The space $\mathcal{H}^\xi(X)$ is commonly endowed with the norm:

$$||f||_\xi := ||f||_\infty + v_\xi(f), \tag{2.2}$$

and then it becomes a Banach space.

Define further the first return of a set U to itself under the map T by

$$\tau(U) := \min_{x\in U} \tau_U(x).$$

Definition 2.2. We will call a metric measure preserving dynamical system (T, X, μ, ρ) *Weakly Markov*, if it satisfies the following three conditions (i) to (iii):

(i) *Exponential Decay of Correlations*: There exists $\gamma \in (0,1)$ and $C > 0$, which in general depends on ξ, such that for all $g \in \mathcal{H}^\xi$, all $f \in L_1(\mu)$ and every $n \in \mathbb{N}$, we have

$$|\mu(f\circ T^n \cdot g) - \mu(g)\cdot\mu(f)| \leq C\gamma^n ||g||_\xi \mu(|f|), \tag{2.3}$$

(ii) For μ–a.e. $x \in X$, we have that

$$0 < \underline{d}_\mu(x) \leq \overline{d}_\mu(x) < +\infty,$$

(iii) *No small returns*: $\displaystyle\liminf_{r\to 0} \frac{\tau\big(B_r(x)\big)}{-\ln(r)} > 0$ for μ–a.e. $x \in X$.

In addition, if measure μ also has the *thin annuli property* relative to a family $\mathcal{R}$ of radii, then we will call the system *Weakly Markov with thin annuli* relative to $\mathcal{R}$. If $\mathcal{R}$ is thick (resp. full) we will call the system *Weakly Markov with thick* (resp. *full*) *thin annuli.*

Remark 2.3. The *no small returns* property has been proved to hold for many dynamical systems; e.g. those considered in [STV] and also, as it is easy to check, for open transitive distance expanding maps and measures μ being Gibbs/equilibrium states of Hölder continuous potentials.

Remark 2.4. As it was mentioned in the introduction, the second named author introduced the concept of *Loosely Markov* systems in [Ur]. These systems are required to satisfy (ii), a stronger version of (i), and a Weak Partition Existence Condition, which implies (iii), as it was observed in [Ur]. Since we will also make use of this condition, we now formulate it below.

Definition 2.5. A system is said to satisfy Weak Partition Existence Condition if there exists a countable partition α with its entropy $\mathrm{h}_\mu(f,\alpha) > 0$ and such that for μ–a.e. $x \in X$ there exists $\chi(x) > 0$ such that

$$B\big(x, \exp(-\chi(x)n)\big) \subset \alpha^n(x) \tag{2.4}$$

for all integers $n \geq 0$ sufficiently large, where $\alpha^n := \bigvee_{j=0}^{n-1} T^{-j}(\alpha)$ denotes the n-th refinement of the partition α under the action of T, and $\alpha^n(x)$ denotes the only element of this partition containing x.

In order to prove Theorem B we will apply some results obtained in [HSV]. More precisely, we will make use of some two theorems proved there. First, that the distribution of the first return time into a fixed set is close to the exponential law if and only if the distributions of the first return time and first entry are close. Second, that we can bound this *closeness* by some explicit expressions. We will finish the proof by estimating those expressions.

Proof of Theorem B. Recall that (T, X, ρ, μ) is a Weakly Markov system. Let us start with some notation; we follow [HSV]. For a fixed set $U \subset X$ let us define

$$\begin{aligned} c(k,U) &:= \mu_U\left(\tau > k\right) - \mu\left(\tau > k\right), \\ c(U) &:= \sup_{k\in\mathbb{N}} \left|c(k,U)\right|. \end{aligned}$$

The first result from [HSV], valid in a fairly abstract context, is this:

Theorem 2.6. *For a transformation $T : X \to X$, preserving a probability measure μ on X, the distributions of both the first return time and first entry time differ from the exponential law by an expression which converges to 0 if both $\mu(U)$ and $c(U)$ go to 0. More precisely, for entry time*

$$\sup_{t\geq 0}\left|\mu\left(\left\{z \in X \colon \tau_U(z) > \frac{t}{\mu(U)}\right\}\right) - e^{-t}\right| \leq d(U), \tag{2.5}$$

and also for return time

$$\sup_{t\geq 0}\left|\mu_U\left(\left\{z\in U\colon \tau_U(z)>\frac{t}{\mu(U)}\right\}\right)-e^{-t}\right|\leq d(U), \tag{2.6}$$

where $d(U)=4\mu(U)+c(U)\big(1-\ln c(U)\big)$.

The second theorem (also from [HSV]) gives an estimate on the value of $c(U)$.

Theorem 2.7. *With the transformation as above:*

$$c(U)\leq \inf_{N\in\mathbb{N}}\left\{a_N(U)+b_N(U)+N\mu(U)\right\},$$

where

$$a_N(U)=\mu_U\left(\{\tau_U\leq N\}\right),$$

$$b_N(U)=\sup_{V\in\mathcal{B}}\left|\mu_U\big(T^{-N}V\big)-\mu\big(V\big)\right|=\sup_{V\in\mathcal{B}}\left|\frac{\mu\big(U\cap T^{-N}V\big)-\mu(U)\mu\big(V\big)}{\mu(U)}\right|$$

where $\mathcal{B}$ is the σ-algebra of Borel sets on X.

Remark. Note that for a fixed set U the number $a_N(U)$ grows to 1 as $N\to+\infty$, whereas $b_N(U)$ tends to 0 (provided that the system has some mixing properties). The tricky part is to find a number N such that b_N has already become small, but a_N and $N\cdot\mu(U)$ have not grown too big.

The proof of Theorem B is a consequence of those two theorems and the following lemma, which is our main technical result in this section.

Lemma 2.8. *If a system $(T,X,\mu,\mathcal{B},\rho)$ is Weakly Markov with Thin Annuli Property relative to a class $\mathcal{R}$ of radii, then for μ–almost all $x\in X$ and all radii $r>0$ there are integers $n_r(x)\geq 1$ such that*

$$\lim_{r\to 0}a_{n_r(x)}(B_r(x))=\lim_{R_x\ni r\to 0}b_{n_r(x)}(B_r(x))=\lim_{r\to 0}n_r(x)\cdot\mu(B_r(x))=0$$

for μ–almost all $x\in X$.

Proof. We will write B_r instead of $B_r(x)$, when dependence on x is clear. Similarly, put

$$n_r=n_r(x):=\mu(B_r)^{-\theta}.$$

Obviously, if $\theta<1$ we get $n_r\cdot\mu(B_r)\to 0$ instantly. So it remains to find θ such that both a_{n_r} and b_{n_r} will tend to 0.

Firstly, let us rewrite the *no small returns* assumption: there exist a Borel set $V\subset X$ of full μ measure and two measurable functions $\chi(x)$, $\rho_1(x)$, both positive μ–a.e., such that

$$B_r(x)\cap T^{-k}\big(B_r(x)\big)=\emptyset \tag{2.7}$$

for all $x\in V$, all radii $0<r<\rho_1(x)$ and all integers $1\leq k\leq\chi(x)\ln(1/r)$.

Secondly, the assumptions imposed on pointwise dimension imply that there exists a Borel set $W\subset V\subset X$, again of full measure μ, such that for all $x\in W$

$$r^{2\overline{d}_\mu(x)}\leq\mu\left(B_r(x)\right)\leq r^{\underline{d}_\mu(x)/2}, \tag{2.8}$$

for all radii $0 < r < \rho_2(x)$ with a certain measurable, positive μ–a.e. function $\rho_2 \leq \rho_1$. Now let us define a family of Lipschitz continuous functions approximating a characteristic function on a ball; depending on three parameters: radius $r > 0$, real number $\alpha > 0$, and $x \in X$, which will vary in the sequel; particularly, we will utilize various choices of $\alpha > 0$. First, auxiliary functions:

$$\phi_r^{(\alpha)}(t) := \begin{cases} 1 & \text{for } 0 \leq t \leq r \\ r^{-\alpha}(r + r^\alpha - t) & \text{for } r \leq t \leq r + r^\alpha \\ 0 & \text{for } t \geq r + r^\alpha \end{cases}.$$

The functions we are looking for are

$$g_{r,x}^{(\alpha)}(z) := \phi_r^{(\alpha)}(\rho(z, x)).$$

The Lipschitz constant of $g_{r,x}^{(\alpha)}$ is bounded from above by $r^{-\alpha}$ as metric ρ is 1–Lipschitz. In particular their Hölder norms (needed in the definition of exponential decay of correlations) are bounded above by (taking $\xi = 1$)

$$||g_{r,x}^{(\alpha)}||_\xi \leq 1 + r^{-\alpha} \approx r^{-\alpha}$$

for all $r > 0$ sufficiently small. Fix $x \in W \subset V$ and sufficiently small $r > 0$, put $g_r = g_{r,x}^{(\alpha)}$ and put $f_r := \mathbb{1}_{B_r}$. Note that

$$f_r \leq g_r.$$

Recall that

$$a_N(B_r) = \mu_{B_r}(\tau_{B_r} \leq N) = \mu_{B_r}\left(\bigcup_{n=1}^{N} T^{-n}(B_r)\right) \leq \sum_{n=1}^{N} \frac{\mu(B_r \cap T^{-n}(B_r))}{\mu(B_r)}.$$

As $x \in V$ we know that some first intersections are empty. Putting $\chi := \chi(x)$, this yields:

$$a_N(B_r) \leq \sum_{n=-\chi \ln(r)}^{N} \frac{\mu(B_r \cap T^{-n}(B_r))}{\mu(B_r)}.$$

The assumption (2.3) on decay of correlations gives

$$\begin{aligned} \mu\big(B_r \cap T^{-n}(B_r)\big) &= \mu(f_r \circ T^n \cdot f_r) \leq \mu(f_r \circ T^n \cdot g_r) \\ &\leq \mu(g_r) \cdot \mu(f_r) + C\gamma^n ||g_r||_\xi \mu(f_r) \\ &\leq \mu(f_r)\big(\mu(g_r) + C\gamma^n r^{-\kappa}\big). \end{aligned}$$

This allows us to rewrite the estimate on a_N and later to bound the sum's elements as simply as possible in the following way

$$\begin{aligned} a_N(B_r) &\leq \sum_{n=-\chi \ln(r)}^{N} \big(\mu(g_r) + C\gamma^n r^{-\alpha}\big) \leq N\mu(g_r) + Cr^{-\alpha} \sum_{n=-\chi \ln(r)}^{+\infty} \gamma^n \\ &= N\mu(g_r) + \frac{C}{1-\gamma} r^{-\alpha} \gamma^{-\chi \ln(r)} \\ &= N\mu(g_r) + Dr^{-\alpha - \chi \ln(\gamma)}. \end{aligned} \tag{2.9}$$

Now specify $\alpha > 0$ to be in $(0,1]$. Using (2.8) we estimate as follows.

$$\mu\big(g_r\big) \leq \mu\left(B(x, r + r^{\alpha})\right) \leq \mu\left(B(x, 2r^{\alpha})\right) \leq 2^{\underline{d}_\mu(x)/2} r^{\alpha \underline{d}_\mu(x)/2}. \tag{2.10}$$

Take $\theta > 0$ as small as needed in the course of the proof and fix $N = n_r = \mu(B_r)^{-\theta}$. Inserting (2.10) into (2.9), and using (2.8) again, we get

$$\begin{aligned} a_{n_r}(B_r) &\leq \mu(B_r)^{-\theta} \cdot 2^{\underline{d}_\mu(x)/2} r^{\alpha \underline{d}_\mu(x)/2} + D r^{-\alpha - \chi \ln(\gamma)} \leq \\ &\leq E r^{-2\theta \overline{d}_\mu(x)} r^{\alpha \underline{d}_\mu(x)/2} + D r^{-\alpha - \chi \ln(\gamma)}, \end{aligned} \tag{2.11}$$

with some positive constants D and E. Restrict further the choice of $\alpha > 0$ so that $\alpha < -\chi \ln(\gamma)$. Then fix any $\theta > 0$ so small that $2\theta \overline{d}_\mu(x) < \kappa \underline{d}_\mu(x)/2$. With these specifications both exponents of powers of r in formula (2.11) are positive; whence we arrive at the conclusion that

$$\lim_{r \to 0} a_{n_r}(B_r) = 0. \tag{2.12}$$

Note that our reasoning leading to this formula did not require any kind of the thin annuli property at all.

Now we turn to the task of estimating $b_{n_r}(B_r(x))$. For this we do need and we do use the Thin Annuli Property relative to $\mathcal{R}$. Let

$$\kappa_x : (0,1] \to (0, +\infty)$$

be the subpolynomial function resulting from the Thin Annuli Property of the system $(T, X, \mathcal{B}, \rho)$ relative to $\mathcal{R}$. The point $x \in W$ is as above and also respecting formula (1.8) of Definition 1.4. Set the parameter $\alpha > 0$ to be $\kappa_x(r)$ and put

$$g_r := g_{r,x}^{(\kappa_x(r))}.$$

Fix a Borel set H. Then

$$\begin{aligned} \mu(B_r) b_{n_r}(B_r) &= \big|\mu\big(B_r \cap T^{-N}(H)\big) - \mu(B_r)\mu\big(H\big)\big| = \big|\mu\big(\mathbb{1}_H \circ T^N \cdot f_r\big) - \mu(\mathbb{1}_H)\mu(f_r)\big| \leq \\ &\leq \big|\mu\big(\mathbb{1}_H \circ T^N \cdot f_r\big) - \mu\big(\mathbb{1}_H \circ T^N \cdot g_r\big)\big| + \big|\mu\big(\mathbb{1}_H \circ T^N \cdot g_r\big) - \mu(\mathbb{1}_H)\mu(g_r)\big| + \\ &+ \big|\mu(\mathbb{1}_H)\mu(g_r) - \mu(\mathbb{1}_H)\mu(f_r)\big|. \end{aligned}$$

So $b_{n_r}(B_r)$ is bounded by the supremum (over all Borel sets $H \subset X$) of the sum of the three terms.

The third expression bounding b_{n_r} is estimated easily:

$$\begin{aligned} \mu(B_r)^{-1}\big|\mu(\mathbb{1}_H)\mu(g_r) - \mu(\mathbb{1}_H)\mu(f_r)\big| &\leq \mu(B_r)^{-1}\big(\mu(g_r) - \mu(f_r)\big) \leq \\ &\leq \mu(B_r)^{-1}\big(\mu(B(x, r + r^{\kappa_x(r)}) - \mu(B(x,r)\big) = \\ &= \frac{\mu\left(B(x, r + r^{\kappa_x(r)}) \setminus B(x,r)\right)}{\mu(B_r)}. \end{aligned}$$

This tends to 0 as $R_x \ni r \to 0$ because of the Thin Annuli Property relative to $\mathcal{R}$ assumed to hold. The first term is bounded in the same way since

$$\big|\mu\big(\mathbb{1}_H \circ T^N \cdot f_r\big) - \mu\big(\mathbb{1}_H \circ T^N \cdot g_r\big)\big| \leq \big(\mu(g_r) - \mu(f_r)\big).$$

Dealing with the second term we may and we do use the exponential decay of correlations:

$$\left|\mu(\mathbb{1}_H \circ T^N \cdot g_r) - \mu(\mathbb{1}_H)\mu(g_r)\right| \leq C\gamma^N r^{-\kappa_x(r)}\mu(\mathbb{1}_H) \leq C\gamma^N r^{-\kappa_x(r)}.$$

The pointwise dimensions formula (2.8) gives $n_r = \mu(B_r)^{-\theta} \geq r^{-\theta \underline{d}_\mu(x)/2}$ and we get

$$\mu(B_r)^{-1}\left|\mu(\mathbb{1}_H \circ T^N \cdot g_r) - \mu(\mathbb{1}_H)\mu(g_r)\right| \leq Cr^{-\kappa_x(r)-2\overline{d}_\mu(x)}\gamma^{r^{-\theta \underline{d}_\mu(x)/2}} =$$
$$= Ce^{-\kappa_x(r)\ln(r)-2\overline{d}_\mu(x)\ln(r)+r^{-\theta \underline{d}_\mu(x)/2}\ln(\gamma)},$$

and the last term in this formula converges to zero as $r \to 0$ once we know that

$$\lim_{r\to 0} \kappa_x(r)\ln(r) r^{\theta \underline{d}_\mu(x)/2} = 0, \tag{2.13}$$

which indeed holds because κ_x is a subpolynomial function. We conclude that

$$\lim_{\mathcal{R}_x \ni r \to 0} b_{n_r}(B_r) = 0. \tag{2.14}$$

and this ends the proof of Lemma 2.8. □

The proof of Theorem B is thus complete. □

3. The Thin Annuli Property

3.1. **Thick Thin Annuli Property holds for All Finite Borel Measures.** Our main result in this subsection is Theorem C. We will need several technical auxiliary results, one of which, Theorem 3.7 is of high generality, interesting in itself, and entails Theorem C. The following well-known result comes from [BS].

Proposition 3.1. *Any Borel probability measure on $\mathbb{R}^n$ is* weakly diametrically regular, *i.e. for μ–almost every $x \in \mathbb{R}^n$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $0 < r < \delta$*

$$\mu(B(x, 2r)) \leq \mu(B(x, r))r^{-\varepsilon}. \tag{3.1}$$

And we start with a remarkable strengthening of that estimate.

Theorem 3.2. *Assume that μ is a Borel probability measure on $\mathbb{R}^d$ and fix any $\varepsilon > 0$. Then for μ–a.e. $x \in \mathbb{R}^d$ and every sufficiently small $r > 0$ (i.e. $0 < r \leq \delta(x)$ and $\delta(x) > 0$ μ–a.e.) we have*

$$\mu\left(B(x, 2r)\right) \leq \log_2^{2+\varepsilon}(1/r)\mu(B(x, r)). \tag{3.2}$$

Moreover,
if s and r are such that $[-\log_2(s)] = [-\log_2(r)]$ (i.e. for some k: $2^{-k-1} < r, s \leq 2^{-k}$), then

$$[-\log_2(r)]^{-1-\varepsilon}\mu\left(B(x, r)\right) \leq \mu\left(B(x, s)\right) \leq [-\log_2(r)]^{1+\varepsilon}\mu\left(B(x, r)\right). \tag{3.3}$$

Proof. Fix $(\alpha_n)_{n=1}^\infty$, a sequence of positive numbers converging to zero and define *bad* sets

$$Z_n := \{x \colon \mu(B(x, 2^{-n})) \cdot \alpha_n > \mu(B(x, 2^{-n-1}))\}.. \tag{3.4}$$

We will show that for every $n \geq 1$ we have that

$$\mu(Z_n) \leq M_d\alpha_n,$$

where M_d is the constant resulting from Besicovitch's Covering Theorem. Indeed, by virtue of this theorem we can cover Z_n by balls $B(z_i, 2^{-n-1})$, $i \in I$, centred at the set Z_n, in such a way that this covering has multiplicity bounded above by M_d. This leads to the estimate

$$\mu(Z_n) \le \sum_{i\in I} \mu(B(z_i, 2^{-n-1})) < \sum_{i\in I} \alpha_n \mu(B(z_i, 2^{-n})) \le \alpha_n M_d \mu(\mathbb{R}^d) \le \alpha_n M_d. \tag{3.5}$$

So, if in addition, $\sum_n \alpha_n < +\infty$, then by Borel–Cantelli Lemma μ–a.e. $x \in \mathbb{R}^d$ belongs only to finitely many sets Z_n.

Now take $\alpha_n = n^{-1-\varepsilon/2}$ (of course $\sum_n \alpha_n < +\infty$) and fix $x \in \mathbb{R}^d$ for which Borel–Cantelli Lemma holds. This means that there exists $N = N(x)$ such that for every $n \ge N$, we have

$$\mu(B(x, 2^{-n})) \le n^{1+\varepsilon/2} \mu(B(x, 2^{-n-1})). \tag{3.6}$$

Take any $0 < r \le 2^{-N-1}$ and put $k = [-\log_2(r)]$, i.e. $2^{-k-1} < r \le 2^{-k}$. Then

$$\begin{aligned} \mu(B(x,2r)) &\le \mu(B(x, 2^{-k+1})) \le (k-1)^{1+\varepsilon/2} \mu(B(x, 2^{-k})) \le \\ &\le k^{1+\varepsilon/2}(k-1)^{1+\varepsilon/2} \mu(B(x, 2^{-k-1})) \le k^{2+\varepsilon} \mu(B(x,r)) \le \\ &\le [-\log_2(r)]^{2+\varepsilon} \mu(B(x,r)) \le \log_2^{2+\varepsilon}\left(\frac{1}{r}\right) \mu(B(x,r)). \end{aligned} \tag{3.7}$$

□

Remark 3.3. By taking a different convergent series, e.g. $\frac{1}{n \log^2(n)}$ as α_n we could improve the above estimate (and therefore in Cor. 3.9) to $\log_2^2(1/r) \log^{2+\varepsilon}(\log(1/r))$.

Motivated by Theorem 3.2 and Remark 3.3 we introduce the following.

Definition 3.4. A monotone decreasing function $G : (0, +\infty) \to [0, +\infty)$ satisfying

$$G(r/2) \le \gamma G(r) \tag{3.8}$$

with some $\gamma \in [1, 2)$ and all $r > 0$ small enough, is called a *doubling bound* almost everywhere for a Borel probability measure μ on $\mathbb{R}^d$ if for μ–a.e. $x \in \mathbb{R}^d$, all sufficiently small $r > 0$ (i.e. $0 < r \le \delta(x)$ and $\delta(x) > 0$ μ–a.e.), we have that

$$\mu(B(x, 2r)) \le G(r) \mu(B(x,r)). \tag{3.9}$$

With this definition Theorem 3.2 and Remark 3.3 can be reformulated as follows.

Theorem 3.5. *For every $\varepsilon > 0$ the function*

$$(0, +\infty) \ni r \longmapsto \max\left\{0, \log_2^{2+\varepsilon}(1/r)\right\},$$

in fact any function of Remark 3.3, is a doubling bound *almost everywhere for any Borel probability measure μ on $\mathbb{R}^d$.*

Using Definition 3.4 and Theorem 3.5 will lead us to the following crucial technical estimate on the measures of annuli.

Lemma 3.6. *For every $x \in \mathbb{R}^d$ let $\kappa_x : (0, 1] \to (1, +\infty)$ be a subpolynomial function such that*

$$\underline{\kappa}_x := \inf_{r\in(0,1]} \kappa_x(r) > 1. \tag{3.10}$$

If μ is a Borel probability measure on $X = \mathbb{R}^d$, then for μ–a.e. $x \in X$ and every $A > 0$ the set of those radii $r > 0$ for which

$$\frac{\mu\left(B(x, r + r^{\kappa_x(r)}) \setminus B(x,r)\right)}{\mu\left(B(x,r)\right)} > A \tag{3.11}$$

has zero density at the point $r = 0$. In other words, if we denote by $Z_x(A)$ the set of all radii $r > 0$ that satisfy (3.11)*, then*

$$\lim_{r\to 0} \frac{l(Z_x(A) \cap [0,r])}{l([0,r])} = 0, \ \text{where } l \text{ is Lebesgue measure on } \mathbb{R}. \tag{3.12}$$

Moreover, let G be a doubling bound almost everywhere for μ, satisfying (3.7) with a constant $\gamma \in [1,2)$. Then the following, more precise estimate holds:

$$l(Z_x(A) \cap [0,r]) \le \frac{2}{\left(1 - \frac{\gamma}{2}\right)\ln(1+A)} r^{\kappa_x(r)} \ln G(r). \tag{3.13}$$

Proof. The first observation is that (3.12) follows from (3.13). Indeed, it suffices to take $G(r) = r^{-\alpha}$ for some $\alpha \in (0,1)$. We are therefore to prove (3.13) only. We do it now.

Since G is a doubling bound, there exists a Borel set $Y \subset \mathbb{R}^d$ such that both $\mu(Y) = 1$ and for every $x \in Y$ there exists $\delta_x > 0$ such that for all $r \in (0, 2\delta_x)$

$$\mu\big(B(x,2r)\big) \le G(r)\mu\big(B(x,r)\big). \tag{3.14}$$

Fix $x \in Y$ arbitrary. Then fix $r \in (0,\delta_x)$ arbitrary. Fix also $\eta > 0$ arbitrary. Then there exist an integer $n \ge 1$, and a sequence of n radii $r_j \in (0,\delta_x) \cap Z_x(A)$, $j = 1, 2 \dots, n$ such that

$$r \le r_1 \le r_1 + r_1^{\kappa_x(r_1)} < r_2 \le r_2 + r_2^{\kappa_x(r_2)} < r_3 \le \cdots \le r_n + r_n^{\kappa_x(r_n)} \le 2r \tag{3.15}$$

and

$$l\left(\big(Z_x(A) \cap [r, 2r]\big) \setminus \bigcup_{j=1}^{n} \left[r_j, r_j + r_j^{\kappa_x(r_j)}\right)\right) \le \eta. \tag{3.16}$$

In particular the annuli defined by radii r_j do not intersect. Since $r_j \in Z_x(A)$ for all $j = 1, 2 \dots, n$, for any $1 \le p \le n$, we have that

$$\frac{\mu\left(B\big(x, r_p + r_p^{\kappa_x(r_p)}\big)\right)}{\mu\left(B(x, r_p)\right)} > 1 + A. \tag{3.17}$$

Using this estimate n times we arrive at

$$\mu\left(B(x,r)\right) \le \mu\left(B(x,r_1)\right) \le \frac{\mu\left(B\big(x, r_1 + r_1^{\kappa_x(r_1)}\big)\right)}{1+A} \le \frac{\mu\left(B(x,r_2)\right)}{1+A} \le \cdots$$
$$\cdots \le \frac{\mu\left(B(x,r_n)\right)}{(1+A)^n} \le \frac{\mu\left(B(x,2r)\right)}{(1+A)^n}.$$

Applying further (3.14) yields

$$\mu(B(x,r)) \le \frac{\mu(B(x,r))G(r)}{(1+A)^n}.$$

This shows that

$$G(r) \geq (1+A)^n, \quad \text{giving the estimate:} \quad n \leq \frac{\ln G(r)}{\ln(1+A)}. \tag{3.18}$$

Now divide the interval $[r, 2r)$ into subintervals of length $(2r)^{\kappa_x(2r)}$, i.e. define:

$$I_1 = \left[r, r+(2r)^{\kappa_x(2r)}\right), \ \dots \ , I_k = \left[r+(k-1)(2r)^{\kappa_x(2r)}, r+k(2r)^{\kappa_x(2r)}\right), \ \dots$$

for all $k \geq 1$ until $(k+1)(2r)^{\kappa_x(r)} \geq 2r$.

Observe that $r_p^{\kappa_x(r_p)} \leq (2r)^{\kappa_x(2r)}$, since $r_p \leq 2r$ and the function κ_x is nonincreasing. So, if $r_p \in I_k$ then the interval $\left[r_p, r_p + r_p^{\kappa_x(r_p)}\right)$ is contained in $I_k \cup I_{k+1}$.

So, the union

$$\bigcup_{j=1}^{n} \left[r_j, r_j + r_j^{\kappa_x(r_j)}\right)$$

is contained in a union of at most $2n \leq \frac{2\ln G(r)}{\ln(1+A)}$ intervals of the form I_k. Therefore,

$$l\left(\bigcup_{j=1}^{n} \left[r_j, r_j + r_j^{\kappa_x(r_j)}\right)\right) \leq (2r)^{\kappa_x(2r)} \cdot \frac{2\ln G(r)}{\ln(1+A)} = \frac{2}{\ln(1+A)}(2r)^{\kappa_x(2r)} \ln G(r) \tag{3.19}$$

Along with (3.16), this gives

$$l\big(Z_x(A) \cap [r, 2r]\big) \leq \eta + \frac{2}{\ln(1+A)}(2r)^{\kappa_x(2r)} \ln G(r).$$

Since $\eta > 0$ was arbitrary, this in turn gives

$$l\big(Z_x(A) \cap [r, 2r]\big) \leq \frac{2}{\ln(1+A)}(2r)^{\kappa_x(r)} \ln G(r).$$

By summing this estimate and recalling that the function κ_x is monotone decreasing while the function G satisfies (3.8), we get

$$\begin{aligned} l(Z_x(A) \cap [0, r]) &\leq \sum_{j=1}^{\infty} l\left(Z_x(A) \cap \left[\frac{r}{2^j}, \frac{r}{2^{j-1}}\right]\right) \leq \frac{2}{\ln(1+A)} \sum_{j=1}^{\infty} \left(\frac{r}{2^{j-1}}\right)^{\kappa_x(r/2^{j-1})} \ln G\left(\frac{r}{2^j}\right) \leq \\ &\leq \frac{2}{\ln(1+A)} r^{\kappa_x(r)} \ln G(r) \sum_{j=1}^{\infty} (\gamma/2)^{j-1} = \frac{2}{\left(1-\frac{\gamma}{2}\right)\ln(1+A)} r^{\kappa_x(r)} \ln G(r). \end{aligned}$$

□

As a consequence of this lemma we get the following first main result of this section.

Theorem 3.7. *Let $g : (0, +\infty) \to (0, +\infty)$ be a function such that*

$$\lim_{r \to 0} g(r) = +\infty$$

and

$$\frac{g(r)}{g(s)} \leq \left(\frac{s}{r}\right)^{\alpha}$$

for every $\alpha > 0$, every $s > 0$ sufficiently small, and every $0 < r \leq s$.

Let μ is a Borel probability measure on $X=\mathbb{R}^d$ and let G be a doubling bound almost everywhere for μ. For every $x\in\mathbb{R}^d$ let $\kappa_x:(0,1]\to(1,+\infty)$ be a subpolynomial function such that

$$\underline{\kappa}_x:=\inf_{r\in(0,1)}\kappa_x(r)>1. \tag{3.20}$$

Then the measure μ has the Thin Annuli Property with respect to some class of radii $\mathcal{R}=\big\{\{R_x\}\big\}_{x\in X}$ for which

$$\lim_{R_x\ni r\to 0}\frac{\left|\frac{l(R_x\cap(0,r])}{r}-1\right|}{g(r)r^{\kappa_x(r)-1}\ln G(r)}=0. \tag{3.21}$$

In addition, the subpolynomial functions witnessing this Thin Annuli Property are just the functions κ_x introduced above in the hypotheses of this theorem.

Proof. We first shall prove the following.

Claim 1^0: There exists a constant $Q\ge 1$ such that

$$g(r)r^{\kappa_x(r)}\ln G(r)\le Qg(s)s^{\kappa_x(s)}\ln G(s)$$

for every $s>0$ sufficiently small and every $0<r\le s$.

Proof. The formula of this claim is equivalent to the following:

$$\frac{g(r)}{g(s)}\cdot\frac{\ln G(r)}{\ln G(s)}\le Q\frac{s^{\kappa_x(s)}}{r^{\kappa_x(r)}}.$$

Since the function κ_x is monotone decreasing, we have that

$$\left(\frac{s}{r}\right)^{\underline{\kappa}_x}\le\left(\frac{s}{r}\right)^{\kappa_x(s)}\le\frac{s^{\kappa_x(s)}}{r^{\kappa_x(r)}}.$$

It therefore suffices to show that

$$\frac{g(r)}{g(s)}\cdot\frac{\ln G(r)}{\ln G(s)}\le Q\left(\frac{s}{r}\right)^{\underline{\kappa}_x}.$$

And in order to have this it suffices to know that

$$\frac{g(r)}{g(s)}\le\left(\frac{s}{r}\right)^{\underline{\kappa}_x/2}$$

and

$$\frac{\ln G(r)}{\ln G(s)}\le Q\left(\frac{s}{r}\right)^{\underline{\kappa}_x/2}. \tag{3.22}$$

The former follows directly (for $s>0$ small enough) from our hypotheses while for proving the latter fix a unique integer $k\ge 0$ such that

$$2^kr\le s \quad\text{and}\quad 2^{k+1}r>s.$$

Then

$$G(r)\le\gamma^{k+1}G(2^{k+1}r)\le G(s).$$

Therefore $\ln G(r) \le (k+1)\ln\gamma + \ln G(s)$. Hence

$$\frac{\ln G(r)}{\ln G(s)} \le 1 + \ln\gamma \frac{k+1}{\ln G(s)}.$$

Thus in order to have (3.22) it suffices to know that

$$1 + \ln\gamma \frac{k+1}{\ln G(s)} \le Q \cdot 2^{\frac{1}{2}\kappa_x k}.$$

But as $\inf \ln G > 0$, this inequality clearly holds for a sufficiently large constant $Q \ge 1$, all integers $k \ge 0$ and all $s > 0$ sufficiently small. The claim is proved. □

Passing to the actual proof of Theorem 3.7, we note that by Lemma 3.6, the estimate (3.13), and by the fact that $g(r) \to \infty$ as $r \to \infty$, there exists $(r_n)_{n=1}^\infty$, a strictly decreasing sequence of positive radii converging to 0 such that

$$l(Z_x(1/n) \cap [0,r]) \le 2^{-n} Q^{-1} g(r) r^{\kappa_x(r)} \ln G(r) \tag{3.23}$$

for all integers $n \ge 1$ and all radii $r \in (0, r_n]$. For $x \in X$ define

$$Z_x := \bigcup_{n=1}^\infty Z_x(1/n) \cap (r_{n+1}, r_n]$$

and then

$$R_x := (0,1) \setminus Z_x.$$

For every $r \in (0, r_1]$ let $n = n_r \ge 1$ be the unique integer such that $r_{n+1} < r \le r_n$. Using Claim 1^0, we then estimate

$$\begin{aligned}
l(Z_x \cap (0,r]) &= \sum_{k=n+1}^\infty l\big(Z_x \cap (r_{k+1}, r_k]\big) + l\big(Z_x \cap (r_{n+1}, r]\big) \\
&= \sum_{k=n+1}^\infty l\big(Z_x(1/k) \cap (r_{k+1}, r_k]\big) + l\big(Z_x(1/n) \cap (r_{n+1}, r]\big) \\
&\le \sum_{k=n+1}^\infty 2^{-k} Q^{-1} g(r_k) r_k^{\kappa_x(r_k)} \ln G(r_k) + 2^{-n} Q^{-1} g(r) r^{\kappa_x(r)} \ln G(r) \\
&\le Q \sum_{k=n+1}^\infty 2^{-k} Q^{-1} g(r) r^{\kappa_x(r)} \ln G(r) + 2^{-n} g(r) r^{\kappa_x(r)} \ln G(r) \\
&= \sum_{k=n_r}^\infty 2^{-k} g(r) r^{\kappa_x(r)} \ln G(r) = 2^{-n_r+1} g(r) r^{\kappa_x(r)} \ln G(r).
\end{aligned}$$

Therefore, since $\lim_{r\to 0} n_r = +\infty$, we get that

$$\lim_{r\to 0} \frac{l(Z_x \cap (0,r])}{g(r) r^{\kappa_x(r)} \ln G(r)} \le \lim_{r\to 0} 2^{-n_r+1} = 0.$$

The proof is complete. □

Remark 3.8. Note that any function g of the form

$$g(r) = \ln^{(k)}(1/r), \quad k \in \mathbb{N},$$

i.e. any iterate of the logarithmic function, satisfies the hypotheses of Theorem 3.7.

By taking $g(r)$ as in the remark above and the functions $G(r)$ and $\kappa_x(r)$, $x \in X$, being respectively of the form $r \mapsto \log_2^{2+\varepsilon}(1/r)$ and $r \mapsto \alpha \ln^\beta(1/r)$, we get the following.

Corollary 3.9. *Every finite Borel measure μ on $X = \mathbb{R}^d$ for every $\beta > 0$ has the Thin Annuli Property with respect to some class of radii $\mathcal{R}(\beta) = \big\{\{R_x(\beta)\}\big\}_{x \in X}$ for which*

$$\lim_{R_x(\beta) \ni r \to 0} \frac{\left|\frac{l(R_x(\beta) \cap (0,r])}{r} - 1\right|}{r^{\ln^\beta(1/r)} \ln\ln(1/r)} = 0. \tag{3.24}$$

As the last thing in this section we observe that this corollary directly entails Theorem C.

3.2. **Full Thin Annuli Property holds for (essentially all) Conformal IFSs.** In this subsection we work in the setting of Iterated Function Systems. Some notations and basic definitions follow. For a full introduction see e.g. [MauU4].

But first let us comment that in the Section 4 (examples) we will show how to use the facts proved in this section as a tool to prove the Full Thin Annuli Property for many other conformal dynamical systems with their natural invariant measures.

The Iterated Function System $\mathcal{S}$ (denoted commonly as IFS) is given as a collection of one-to-one contractions $\phi_e \colon X \to X$ with Lipschitz constant $\leq \lambda$, where $X \subset \mathbb{R}^d$ ($d \geq 1$) is a compact set.

$$\mathcal{S} = \{\phi_e : X \to X\}_{e \in E}.$$

The set of indices E is called an *alphabet* and may be finite or infinite. We call the IFS $\mathcal{S}$ *finite* if the alphabet E is finite. The set of all *words* E^* is defined as

$$E^* = \bigcup_{n=1}^{\infty} E^n$$

The above union defining E^* is disjoint and for every $\omega \in E^*$ we denote by $|\omega|$ the unique integer n such that $\omega \in E^n$; we call $|\omega|$ the *length* of ω. For $\omega \in E^\infty$ and $n \in \mathbb{N}$, we write

$$\omega|_n := \omega_1\omega_2 \ldots \omega_n \in E^n.$$

For each $n \geq 1$ and $\omega \in E^n$ we define

$$\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} \colon X \to X..$$

Those functions are contractions so the intersection $\bigcap_{n \in \mathbb{N}} \phi_{\omega|_n}(X)$ is a singleton and we denote its only element by $\pi(\omega)$. In this way we have defined a map

$$\pi \colon E^\infty \to X.,$$

The map π is called the *coding map*, and the set

$$J = J_{\mathcal{S}} = \pi(E^\infty)$$

is called the *limit set* $\mathcal{S}$.

Intending to pass to geometry and following [MauU4], we call a IFS *conformal* if for some $d \in \mathbb{N}$, the following conditions are satisfied:

(a) X is a compact connected subset of $\mathbb{R}^d$, and $X = \overline{\mathrm{Int}(X)}$.
(b) There exists an open connected set $W \supset X$ such that for every $e \in E$, the map ϕ_e extends to a C^1 conformal diffeomorphism from W into W with Lipschitz constant $\leq \lambda$.
(c) There are two constants $L \geq 1$ and $\alpha > 0$ such that for every $e \in E$ and every pair of points $x, y \in X$,
$$\left| \frac{|\phi_e'(y)|}{|\phi_e'(x)|} - 1 \right| \leq L \|y - x\|^\alpha,$$
where $|\phi_\omega'(x)|$ denotes the scaling of the derivative, which is a linear similarity map.

Remark 3.10. If $d \geq 2$ and a family $\mathcal{S} = \{\phi_e\}_{e \in E}$ satisfies the conditions (a) and (b), then it also satisfies condition (c) with $\alpha = 1$. When $d = 2$ this is due to the well–known Koebe's distortion theorem When $d \geq 3$ it is due to [MauU4] depending heavily on Liouville's representation theorem for conformal mappings (see [IM] for a detailed development leading up to the strongest current version).

Remark 3.11. We do emphasize that, unlike to [MauU4], in the above definition, and in all the results of this subsection we do not need and we do not require any separation condition whatsoever. In particular even its weakest form
$$\phi_a(\mathrm{Int}(X)) \cap \phi_b(\mathrm{Int}(X)) = \emptyset.$$
for all $a, b \in E$ such that $a \neq b$, known as the Open Set Condition, is not assumed to hold. We also do emphasize that we do not impose any form of boundary regularity, in particular no Cone Condition of [MauU4].

Definition 3.12. We say that the system $\mathcal{S}$ is *geometrically irreducible* if the limit set $J_\mathcal{S}$ is not contained in any proper, i.e. of dimension $\leq d-1$, real analytic sub-manifold; precisely: is not contained in a conformal image of any affine hyperspace or geometric round sphere of dimension $\leq d - 1$.

Throughout this whole Subsection 3.2 we assume that the system $\mathcal{S}$ is geometrically irreducible. For the sake of brevity we denote
$$D(\omega) := \mathrm{diam}(\phi_\omega(X))$$
for all $\omega \in E^*$. The Bounded Distortion Property tells us that
$$Q^{-1} D(\omega) D(\tau) \leq D(\omega\tau) \leq Q D(\omega) D(\tau) \tag{3.25}$$
for all $\omega, \tau \in E^*$ and some constant $Q \geq 1$. In this section we consider a (really large) class, called $\mathcal{M}_E$, of Borel probability measures μ on the symbol space E^∞, determined by the following two requirements:

(A) Weak Independence:
$$P^{-1} \mu([\omega]) \mu([\tau]) \leq \mu([\omega\tau]) \leq P \mu([\omega]) \mu([\tau])$$
for some constant $P \geq 1$ and all $\omega, \tau \in E^*$.

(B) There exists $\beta > 0$ such that $\displaystyle\sum_{e \in E} \frac{\mu([e])}{\mathrm{diam}^\beta(\phi_e(X))} < +\infty.$

Remark 3.13. All Gibbs measures, on the symbol space E^∞, introduced and considered in [MauU3] are weakly independent, i.e. enjoy the property (A). It is easy to have abundance of such measures satisfying the property (B); among them are the Gibbs states of all (geometrically most significant) potentials $E^\infty \ni \omega \mapsto t\log\left|\phi'_{\omega_1}(\pi(\sigma(\omega)))\right| \in \mathbb{R}$, where $t \geq 0$ is sufficiently large.

The main result of this subsection follows.

Theorem 3.14. *If $\mathcal{S} = \{\phi_e : X \to X\}_{e\in E}$ is a geometrically irreducible conformal IFS, then for every $\mu \in \mathcal{M}_E$ the measure $\mu\circ\pi^{-1}$ on $J_\mathcal{S}$ has the* thin annuli property *with $\kappa = 3$ (in fact this is true for any $\kappa > 1$). In other words:*

$$\lim_{r\to 0}\frac{\mu\circ\pi^{-1}\left(B(x,r+r^3)\setminus B(x,r)\right)}{\mu\circ\pi^{-1}\left(B(x,r)\right)} = 0 \qquad \mu\circ\pi^{-1}\text{–}a.e.$$

In order to ease notation, let us denote by $R(x,r,r^3)$ the annulus centred at x with inner radius $r > 0$ and outer radius $r + r^3$, i.e.

$$R(x,r,r^3) := B(x,r+r^3)\setminus B(x,r).$$

The proof of Theorem 3.14 consists of several steps listed below, and it has been influenced by the techniques of [DFSU]. For the sake of brevity we denote

$$\hat{\mu} := \mu\circ\pi^{-1}.$$

Lemma 3.15. *There exist constants $\rho > 0$, $H < \infty$ and a finite set $F \subset E^*$ such that for any $x \in J_\mathcal{S}$, any radius $0 < r < \rho$, and any finite word $\omega \in E^*$, with diameter $D(\omega) \geq Hr^3$, there exists a word $\tau \in F$ such that $\pi([\omega\tau])$ does not intersect $R(x,r,r^3)$. In symbols:*

$$\pi([\omega\tau])\cap R(x,r,r^3) = \emptyset.$$

Lemma 3.16. *There exist $\alpha > 0$, $C < \infty$ and $\rho > 0$ such that for all $0 < r < \rho$, $x \in J_\mathcal{S}$ and any finite word $\omega \in E^*$, with diameter $D(\omega) \geq r^2$, we have*

$$\mu\left([\omega]\cap\pi^{-1}(R(x,r,r^3))\right) \leq Cr^\alpha\mu([\omega]). \tag{3.26}$$

Lemma 3.17. *For any numbers $0 < A < B$ define the set*

$$T_A^B := \left\{\omega \in E^\mathbb{N} \colon \forall_{k\in\mathbb{N}}\, D(\omega|_k) \notin (A,B)\right\}. \tag{3.27}$$

Then there exists $C < \infty$ for which $\mu(T_A^B) \leq C\left(\frac{A}{B}\right)^\beta \ln\left(\frac{\operatorname{diam} X}{A}\right)$, where β is the constant from condition (B) from the definition of the space $\mathcal{M}_E$.

Lemma 3.18. *Let ν be an arbitrary Borel probability measure defined on some bounded Borel set $X \subset \mathbb{R}^d$. Let F be a measurable subset of X. Define*

$$S(F,c,\rho) = \{x \in X \colon \nu(B(x,\rho)\cap F) > c\nu(B(x,\rho))\nu(F)\}. \tag{3.28}$$

Then for any numbers $c, \rho > 0$ we have $\nu(S(F,c,\rho)) \leq M/c$, where M is some constant depending only on the space X.

Proof of Lemma 3.15. Assume without loss of generality that $E = \mathbb{N}$. Seeking a contradiction suppose that there exist a sequence $(r_n)_{n=1}^\infty \searrow 0$, a sequence $x_n \in J_\mathcal{S}$, $n \in \mathbb{N}$, and a sequence of finite words $\omega^{(n)} \in E^*$ with diameters satisfying

$$D(\omega^{(n)}) \geq nr_n^3 \tag{3.29}$$

such that for every $\tau \in \{1, 2, \dots, n\}^n$ the "cylinder" $\pi([\omega^{(n)}\tau])$ intersects $R(x_n, r_n, r_n^3)$. Let us denote

$$R_n := R(x_n, r_n, r_n^3) \quad \text{and} \quad S_n := \partial B(x_n, r_n) = \{x \in \mathbb{R}^d : ||x - x_n|| = r_n\}.$$

Take then any sequence of similarities T_n, $n \geq 1$, for which $0 \in T_n(\pi([\omega^{(n)}]))$ and $|T_n'| = (D(\omega^{(n)}))^{-1}$ for all $n \geq 1$. Note that $(T_n \circ \phi_{\omega^{(n)}})_{n=1}^\infty$ is a bounded equicontinuous sequence of conformal maps with derivatives uniformly bounded from above and uniformly separated from zero. Therefore, applying Ascoli-Arzela Theorem and passing to an appropriate subsequence we will have that the sequence $(T_n \circ \phi_{\omega^{(n)}})_{n=1}^\infty$ converges uniformly on X to a conformal map $U : X \to \mathbb{R}^d$. Now, working with the one-point (Alexandrov) compactification $\hat{\mathbb{R}}^d$ of $\mathbb{R}^d$, with ∞ as the compactifying point, endowing $\hat{\mathbb{R}}^d$ with spherical metric, and then the collection $\mathcal{K}_d$ of non-empty compact subsets of $\hat{\mathbb{R}}^d$ with the corresponding Hausdorff metric d_H, we see that the collection Γ of all geometric spheres of $\hat{\mathbb{R}}^d$, including the spheres containing infinity (hyperplanes) and singletons, forms a compact subset of $\mathcal{K}_d$. Since $T_n(S_n) \in \Gamma$, passing to a subsequence, we can therefore assume without loss of generality that $T_n(S_n)$ converges in the Hausdorff metric d_H to some element $Q \in \Gamma$. Depending on actual sizes of $D(\omega^{n)})$, the limit object Γ may be either a circle – the case if $D(\omega^{(n)}) \asymp r_n$, a point in $\mathbb{R}^d$ – the case if $D(\omega^{(n)})/r_n \to \infty$, or a line in $\mathbb{R}^d$ – which is so if $D(\omega^{(n)})/r_n \to 0$. In all three cases the ratio of the outer and inner radii of the annulus $T_n(R_n)$ converges to one, as $\frac{r+r^3}{r} \to 1$ when $r \to 0$.

In the first two cases, this immediately implies that also $\lim_{n\to\infty} T_n(R_n) = Q$. In the third case, we need to use additionally (3.29), to conclude that both circles bounding the annulus $R(x_n, r_n, r_n + r_n^3)$, after rescaling by $(D(\omega^{n)}))^{-1} \leq \frac{1}{nr_n^3}$ tend to the same line in $\mathbb{R}^d$.

So, finally, in all three cases we may conclude that

$$\lim_{n\to\infty} T_n(R_n) = Q. \tag{3.30}$$

Observe also that by Definition 3.12 for every $M \in \Gamma$ there exists a point $w_M \in J_{\mathcal{S}}$ such that $\text{dist}\,(w_M, M) > 0$. Writing the $w_{U^{-1}(Q)} = \pi(\xi) \in J_{\mathcal{S}}$, $\xi \in E^{\mathbb{N}}$, we have that

$$\text{dist}\,(\pi(\xi), U^{-1}(Q)) > 0.$$

We therefore conclude that there exists $k \geq 1$ such that

$$\text{dist}\,\big(\pi([\xi|_k]), U^{-1}(Q)\big) > 0. \tag{3.31}$$

Consider now only integers $n \geq k$ so large that all letters forming $\xi|_k$ belong to $\{1, 2, \dots, n\}$. By our contrary hypothesis

$$\phi_{\omega^{(n)}}\big(\phi_{\xi|_k}(J_{\mathcal{S}})\big) \cap R_n = \phi_{\omega^{(n)}\xi|_k}(J_{\mathcal{S}}) \cap R_n \neq \emptyset.$$

Fix an arbitrary $z_n \in J_{\mathcal{S}}$ such that $\phi_{\omega^{(n)}\xi|_k}(z_n) \in R_n$. Passing to a subsequence we may assume without loss of generality that $\lim_{n\to\infty} z_n = z \in X$ for some point $z \in \overline{J}_{\mathcal{S}}$. Then, invoking also (3.30), we get that

$$\lim_{n\to\infty} T_n \circ \phi_{\omega^{(n)}}(\phi_{\xi|_k}(z)) = U(\phi_{\xi|_k}(z)) \in Q.$$

Hence $\phi_{\xi|_k}(z) \in U^{-1}(Q)$, and as $\phi_{\xi|_k}(z) \in \overline{\pi([\xi|_k])}$, this contradicts (3.31) and finishes the proof of our lemma. $\square$

We will also use Proposition 3.1 (proved in [BS]) in this section. Recall that it is an immediate consequence of, much stronger, Theorem 3.2.

Proof of Lemma 3.16. Take ρ, H and the set F given by Lemma 3.15. First of all observe that, because the lengths of all words in F are uniformly bounded above, by taking an iterate of the system $\mathcal{S}$ we may assume that $F \subset E$ (instead of E^*). Fix $x \in J_{\mathcal{S}}$, $0 < r < \rho$, and denote $R := R(x, r, r^3)$.

We will in fact prove a stronger fact; namely that with no restrictions on $D(\omega)$

$$\mu\left([\omega] \cap \pi^{-1}(R)\right) \leq \left(\frac{Hr^3}{D(\omega)}\right)^\alpha \mu([\omega]), \tag{3.32}$$

for all $\omega \in E^*$. This will trivially prove the lemma as its hypotheses require that $D(\omega) \geq r^2$. So, we now focus on the proof of (3.32). First note that if $D(\omega) \leq Hr^3$, then inequality (3.32) is trivial. Also for all $n \geq 1$ big enough and all $\omega \in E^n$ we have $D(\omega) \leq Hr^3$. Now let us work *from the bottom upwards*. Take a cylinder $[\omega]$ such that (3.32) is already proven for all subcylinders $[\omega e]$, $e \in E$. We have

$$\mu([\omega] \cap \pi^{-1}(R)) = \sum_{e \in E} \mu([\omega e] \cap \pi^{-1}(R))$$

and, applying Lemma 3.15, we may drop at least one element of this sum, say $b \in E$, to get

$$\begin{aligned}
\mu([\omega] \cap \pi^{-1}(R)) &= \sum_{E \ni a \neq b} \mu([\omega a] \cap \pi^{-1}(R)) \leq \\
&\leq \sum_{E \ni a \neq b} \left(\frac{Hr^3}{D(\omega a)}\right)^\alpha \mu([\omega a]) = \\
&= \left(Hr^3\right)^\alpha \sum_{E \ni a \neq b} \frac{\mu([\omega a])}{(D(\omega a))^\alpha},
\end{aligned}$$

where we used the estimate (3.32) for every cylinder $[\omega a]$. In order to prove the required inequality we need to have

$$\sum_{E \ni a \neq b} \frac{\mu([\omega a])}{(D(\omega a))^\alpha} \leq \frac{\mu([\omega])}{(D(\omega))^\alpha} = \sum_{a \in E} \frac{\mu([\omega a])}{(D(\omega))^\alpha},$$

where the equality sign trivially holds. Simplifying this gives

$$\sum_{E \ni a \neq b} \left(\left(\frac{D(\omega)}{D(\omega a)}\right)^\alpha - 1\right) \mu([\omega a]) \leq \mu([\omega b]).$$

Applying Bounded Distortion Property (3.25) and Weak Independence of μ, i.e. condition (A), we see that it is thus enough to prove that

$$\sum_{E \ni a \neq b} \left(\left(\frac{QD(\omega)}{D(\omega)D(a)}\right)^\alpha - 1\right) P\mu([\omega])\mu([a]) \leq P^{-1}\mu([\omega])\mu([b]).$$

Recall that b was chosen from a finite set so $P^{-2}\mu([b])$ is bounded away from zero, say $P^{-2}\mu([b]) > \delta$ for some fixed $\delta > 0$. Simplifying again, we see that it is enough to prove

$$\sum_{E\ni a\neq b}\left(\left(\frac{Q}{D(a)}\right)^{\alpha}-1\right)\mu([a]) \leq \delta.$$

Therefore, it is enough to have

$$\sum_{E\ni a\neq b}\frac{\mu([a])}{D(a)^{\alpha}} \leq Q^{-\alpha}\delta + \sum_{E\ni a\neq b}\mu([a]).$$

But since, by Assumption (B), the series on the left-hand side of this formula converges for all $\alpha > 0$ small enough. Its sum tends to $\sum_{E\ni a\neq b}\mu([a])$ as $\alpha\to 0$ and using a dominated convergence theorem we get that this formula will hold for all $\alpha > 0$ small enough. Thus the proof is complete. $\square$

Proof of Lemma 3.17. First, divide T_A^B into disjoint subsets (for $k = 0, 1\ldots$)

$$T_A^B(k) = \left\{\omega\in E^{\mathbb{N}} \colon D(\omega|_{k+1}) \leq A < B \leq D(\omega|_k)\right\}.$$

Recall that $s = \sup_{e\in E}\{|\phi_e'|\} < 1$. For any cylinder $D(\omega|_k) \leq s^k \operatorname{diam}(X)$, so for any $n \geq N := \log_s\left(\frac{A}{\operatorname{diam} X}\right)$ we have $D(\omega|_n) \leq A$ and $T_A^B(n) = \emptyset$. This allows us to write

$$\mu(T_A^B) \leq \sum_{n=0}^{N}\mu\left(T_A^B(n)\right). \tag{3.33}$$

Now, fix $0 \leq k \leq N$ and $\omega \in E^{\mathbb{N}}$. If $D(\omega|_k) < B$, then $\mu\left(T_A^B(k)\cap[\omega|_k]\right) = 0$. If $D(\omega|_k) \geq B$, then

$$\mu\left(T_A^B(k)\cap[\omega|_k]\right) = \sum_{e}\mu([\omega|_k e]),$$

where the sum is taken over those $e\in E$ for which $D(\omega|_k e) \leq A$. Applying the Weak Independence of μ, i.e. condition (A) and Bounded Distortion (3.25), we further get

$$\mu\left(T_A^B(k)\cap[\omega|_k]\right) \leq \sum_{D(\omega|_k a)\leq A} P\mu([\omega|_k])\mu([a]) \leq \sum_{Q^{-1}D(\omega|_k)D(a)\leq A} P\mu([\omega|_k])\mu([a]),$$

and using the fact that $D(\omega|_k) \geq B$, this gives

$$\mu(T_A^B(k)\cap[\omega|_k]) \leq P\mu([\omega|_k])\sum_{D(a)\leq QA/B}\mu([a]). \tag{3.34}$$

By Assumption (B) we may write:

$$+\infty > Z := \sum_{a\in E}\frac{\mu([a])}{D(a)^{\beta}} \geq \sum_{D(a)\leq QA/B}\frac{\mu([a])}{D(a)^{\beta}} \geq \sum_{D(a)\leq QA/B}\frac{\mu([a])}{(QA/B)^{\beta}} =$$
$$= \sum_{D(a)\leq QA/B}\mu([a])\left(\frac{B}{QA}\right)^{\beta}.$$

Combining this estimate with (3.34) gives

$$\mu\left(T_A^B(k)\cap[\omega|_k]\right)\le P\mu([\omega|_k])\cdot Z\left(\frac{QA}{B}\right)^\beta,$$

and summing over all cylinders $[\omega|_k]$, this gives $\mu(T_A^B(k))\le C(A/B)^\beta$ with some constant C. Finally applying (3.33), we get

$$\mu(T_A^B)\le\log_s\left(\frac{A}{\operatorname{diam}X}\right)\cdot C\left(\frac{A}{B}\right)^\beta,$$

which finishes the proof. □

Proof of Lemma 3.18. Set

$$S:=S(F,c,\rho).$$

By Besicovitch's Covering Theorem there exists a covering of S with balls $B(x_i,\rho)$, $i\in I$, all centred at S, with finite multiplicity M_d depending only on the dimension d. The following estimate uses first, the definition of S and then the bounded by M_d multiplicity of covering.

$$\nu(S)\le\sum_{i\in I}\nu(B(x_i,\rho))\le\sum_{i\in I}\frac{\nu(B(x_i,\rho)\cap F)}{c\nu(F)}\le\frac{M_d\nu(F)}{c\nu(F)}=\frac{M_d}{c}.$$ □

Proof of Theorem 3.14. We will show that for $\hat\mu$ almost every $x\in J_{\mathcal S}$ and all sufficiently small radii $r>0$ we have that

$$\hat\mu(R(x,r,r^3))\le C\hat\mu(B(x,r))r^\gamma$$

for some $\gamma>0$. First, using notation from Lemmas 3.17 and 3.18 define

$$T_n:=T_{4^{-n}}^{2^{-n}},\quad n\ge1\text{ and denote }\hat T_n=\pi(T_n).$$

Then

$$S_n:=S(\hat T_n,n^2,4\cdot2^{-n}).$$

Lemma 3.18 gives that $\hat\mu(S_n)\le M/n^2$ and so $\sum_n\hat\mu(S_n)<\infty$. Thus the Borel–Cantelli Lemma applies to tell us that for $\hat\mu$ almost every $x\in J_{\mathcal S}$ there exists an integer $K(x)\ge1$ such that $x\notin S_k$ for all $k\ge K(x)$. Fix $x\in J_{\mathcal S}$ with such property, i.e. an arbitrary x produced by the Borel–Cantelli Lemma. For any $n\ge1$ define the set

$$C_n=\left\{[\omega]\in E^*\colon D(\omega)\le2^{-n}<D(\omega|_{|\omega|-1})\right\}.\tag{3.35}$$

Now, take any $0<r\le2^{-(K(x)+1)}$. Define $n\ge1$ so as to satisfy the inequalities $2^{-n-1}<r\le2^{-n}$. Then

$$n\ge K(x).\tag{3.36}$$

Denote the annulus $R(x,r,r^3)$ by R and cover $\pi^{-1}(R)$ by cylinders from C_n. We estimate the measure

$$\begin{aligned}\hat\mu(R)=\mu\circ\pi^{-1}(R)&\le{\sum_{[\omega]}}^*\mu([\omega]\cap\pi^{-1}(R))\\&\le\underbrace{{\sum_{[\omega]\subset T_n}}^*\mu([\omega]\cap\pi^{-1}(R))}_{I}+\underbrace{{\sum_{[\omega]\cap T_n=\emptyset}}^*\mu([\omega]\cap\pi^{-1}(R))}_{II},\end{aligned}$$

where the * indicates that the corresponding sum above is taken over all cylinders $[\omega] \in C_n$ intersecting $\pi^{-1}(B(x, r+r^3))$. Recall that for such cylinders $D(\omega) < 2r$, and as $r+r^3 \le 2r$, the cylinder $[\omega]$ is contained in the set $\pi^{-1}(B(x, 4r))$. So

$$I \le \sum_{[\omega]\subset T_n}^{*} \mu([\omega]) \le \mu\left(T_n \cap \pi^{-1}(B(x, 4r))\right) \le \mu\left(T_n \cap \pi^{-1}(B(x, 4 \cdot 2^{-n}))\right).$$

Now, first straightforward from the definition of S_n, and from the fact that, because of (3.36), $x \notin S_n$, then by applying Lemma 3.17, we get that

$$\begin{aligned}
I &\le n^2 \mu\left(\pi^{-1}(B(x, 4 \cdot 2^{-n}))\right) \mu(T_n) \\
&\le n^2 \hat{\mu}\left(B(x, 4 \cdot 2^{-n})\right) C \left(\frac{4^{-n}}{2^{-n}}\right)^{\beta} \ln\left(\frac{\operatorname{diam} X}{4^{-n}}\right) \\
&\le n^2 \hat{\mu}\left(B(x, 4 \cdot 2^{-n})\right) \cdot \widehat{C} n 2^{-n\beta} \\
&\le \widetilde{C} \hat{\mu}\left(B(x, 8r)\right) r^{\beta/2}
\end{aligned}$$

with appropriate constants $\widehat{C}$ and $\widetilde{C}$. Finally we apply the estimate of Proposition 3.1 with $\varepsilon = \beta/4$ to get

$$I \le \widetilde{C} \hat{\mu}\left(B(x, r)\right) r^{-\varepsilon} r^{\beta/2} \le \widetilde{C} \hat{\mu}\left(B(x, r)\right) r^{\beta/4}$$

which completes the estimate of the first sum, i.e. the one labelled by I..

Now, observe that if $[\omega] \cap T_n = \emptyset$, then $D(\omega) \ge 4^{-n} \ge r^2$ and so we may first apply Lemma 3.16, and then Proposition 3.1 with $\varepsilon = \alpha/2$ to estimate as follows:

$$\begin{aligned}
II &\le \sum_{[\omega]\cap T_n=\emptyset}^{*} C r^{\alpha} \mu([\omega]) \le C r^{\alpha} \hat{\mu}(B(x, 4r)) \\
&\le C r^{\alpha} \hat{\mu}(B(x, r)) r^{-\varepsilon} \\
&\le C r^{\alpha/2} \hat{\mu}(B(x, r)).
\end{aligned}$$

This completes the upper estimate of II and finishes the entire proof. □

4. Applications and Examples: Exponential One Laws

There are several good examples of systems for which we may apply our theorems and prove the exponential limit laws for return (entry) times. Due to space constraints we cannot show them in full detail. We will, however, list the applications and comment extensively on three of them. The detailed description will be published in a separate paper. We divided the application into two separate groups (non–conformal and conformal ones).

A. **Non–Necessarily–Conformal Systems, for which we apply Theorem A, proving the exponential law along *most* radii.**

1. Expanding Repellers. These general systems will be discussed in detail, and the theorem will be formulated in Subsection 4.1.
2. Axiom A Diffeomorphisms — the techniques from the previous example are easily adapted to provide an easy and straghtforward proof of the following.

Theorem 4.1. *Let $f : M \to M$ be an Axiom A diffeomorphism on a smooth manifold M. Let $\Omega \subset M$ be its non-wandering set. Further, let $\varphi : \Omega \to \mathbb{R}$ be a Hölder continuous potential, and let (see [Bow]) μ_φ be the unique equilibrium (Gibbs) measure associated to this potential. Then the system (Ω, μ_φ, f) is Weakly Markov. Consequently, Theorem A holds for this system.*

(Wydaje mi sie, ze sformulowanie tego twierdzenia pochodzi z jakiejs starej wersji naszej pracy i jest bledne. Na przyklad bez zalozenia, ze $f|_\Omega : \Omega \to \Omega$ jest topologicznie tranzytywne, moze byc wiecej niz jeden stan rownowagi na Ω) The proof is very simple due to the fact that we have not touched in the definition of Weakly Markov Systems the concept of the Perron–Frobenius operator at all, but used only exponential decay of correlations. Employing the method of Perron–Frobenius operator routinely requires a painful, and somewhat odd, procedure of making an invertible system non-invertible. Our method allows us to avoid this.

3. Holomorphic Endomorphisms of Complex Projective Spaces — the properties of equilibrium measures obtained in [UZ] coupled with a good inducing scheme proved in [SUZ2] (see that paper for necessary definitions) allow us to prove the following.

Theorem 4.2. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$, $k \geq 1$, be a holomorphic endomorphism of a complex projective space $\mathbb{P}^k$ of degree $d \geq 1$. Let $\varphi : J(f) \to \mathbb{R}$ be an admissible potential, and let μ_φ be its unique equilibrium state. Then $(J(f), f, \mu_\varphi)$ forms a Weakly Markov system. Consequently, Theorem A holds for this system.*

("Admissible potential" brzmi troche tajemniczo. Czy nie moglibysmy raczej napisac ze ϕ ma byc Holderoswki z dostatecznie mala roznica $\sup(\phi) - \inf(\phi)$ **odsylajac czytelnika do** [UZ] **jesli chce sie dowiedziec jak mala ma byc ta roznica?)**

B. **Conformal Systems, for which we apply Theorem E, proving the exponential law along *all* radii.**

4. Conformal Iterated Function Systems — in Subsection 4.2 we show that most IFSs satisfy the mild assumptions of Theorem E, thus giving several important examples.
5. Conformal Graph Directed Markov Systems — these are a useful generalization of IFSs (see [MauU4]). All of our results pertaining to IFSs are in fact extendable to GDMSs. In particular Theorem 4.12 (and Subsection 4.2) holds for them.
6. Conformal Parabolic IFSs (and GDMSs) — these cover such examples as Parabolic Cantor Sets, Apollonian packing systems, etc. (see [MauU2]). The result follows.

Theorem 4.3. *Suppose that $\mathcal{S}$ is a finite irreducible and geometrically irreducible parabolic conformal GDMS satisfying the Strong Open Set Condition. Fix a real number t for which the measure μ_t (given by the appropriate geometric potential) is finite. Then the corresponding measure–preserving dynamical system satisfies the Full Thin Annuli Property and the exponential one laws hold.*

This may be further applied for parabolic rational functions assuming that

$$h_f > \frac{2p}{p+1}, \tag{4.1}$$

where h_f is the Hausdorff dimension of the Julia set $J(f)$ and p denotes the maximal number of petals around parabolic periodic points of f. We then have

Theorem 4.4. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a parabolic rational function whose Julia set is not contained in any real analytic curve. Assume that (4.1) holds. Then the measure μ_{h_f} (unique probability f-invariant measure absolutely continuous w.r.t. the Hausdorff measure) satisfies the Full Thin Annuli Property and the exponential one laws hold.*

7. Conformal Expanding Repellers — for the conformal counterpart to the first example, described in detail in Subsection 4.1, we are now able to prove the exponential law along all radii.

   Theorem 4.5. *Let $T : J \to J$ be a conformal expanding repeller such that $J \subset \mathbb{R}^d$ is not contained in any real analytic submanifold of dimension $\leq d-1$. Let $\psi : J \to \mathbb{R}$ be a Hölder continuous potential and, see [PU], let μ_ψ be the corresponding equilibrium (Gibbs) state. Then the measure–preserving dynamical system is Weakly Markov with the Full Thin Annuli Property, so the exponential one laws hold.*

8. Complex Rational Maps and Hölder Continuous Potentials with a Pressure Gap — Subsection 4.3 is devoted to this application, and the result is formulated there.
9. Dynamically Semi–Regular Meromorphic Functions — this subclass (introduced in [MayU]) of meromorphic maps leads to a rich thermodynamical formalism. The inducing method shown for the previous example works also in this case.

   Theorem 4.6. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function whose Julia set is not contained in a real analytic curve (for instance, if $\mathrm{HD}(J_f) > 1$). Assume it is dynamically semi-regular, assume that $t > \rho_f/\alpha$, where ρ_f is the order of the function f. If $h : J_f \to \mathbb{R}$ is a weakly Hölder continuous potential (for definitions see [MayU]), then the system $\big(f, \mu_t\big)$ is Weakly Markov with Full Thin Annuli Property, and so the exponential one laws hold.*

We now provide a more detailed description and proofs for some of the applications listed above, in parts A and B.

4.1. **Expanding Repellers.** In this class of examples conformality is not assumed.

Definition 4.7. Let U be an open subset of $\mathbb{R}^d$, $d \geq 1$. Let J be a compact subset of U. Let $T : U \to \mathbb{R}^d$ be a $C^{1+\epsilon}$–differentiable map. The map T is called an expanding repeller, if the following conditions are satisfied:

(1) $T(J) = J$,

(2) for every $z \in J$ the derivative $T'(z) : \mathbb{R}^d \to \mathbb{R}^d$ is invertible and the norm of its inverse is smaller than 1.

(3) there exists an open set V such that $\overline{V} \subset U$ and

$$J = \bigcap_{k=0}^{\infty} T^{-n}(V).$$

(4) the map $T|_J : J \to J$ is topologically transitive.

Note that T is not required to be one-to-one; in fact usually it is not. Abusing notation slightly we frequently refer to the set J alone as an expanding repeller. In order to use a uniform terminology we also call J the limit set of T.

In this section, as well as in the next, we will need the classical concepts of topological pressure, variational principle, and equilibrium states. We bring them up now. Let X be a compact metrizable space, $T : X \to X$ be a continuous map, and $\varphi : X \to \mathbb{R}$ be a continuous function. We denote by $\mathrm{P}(\varphi)$ its topological pressure with respect to the dynamical system given by the map $T : X \to X$, see for example [PU] for the definition and properties. The most important of these properties is the following formula, commonly referred to as the Variational Principle.

$$\mathrm{P}(\varphi) = \sup\left\{\mathrm{h}_\mu(T) + \int_X \varphi \, d\mu\right\}, \tag{4.2}$$

where the supremum is taken over all Borel probability T–invariant measures on X. Any such measure for which the supremum is attained is called the equilibrium state of φ.

The basic concept associated with repellers which will be used in this section is given by the following definition.

Definition 4.8. A finite cover $\mathcal{R} = \{R_1, \ldots, R_q\}$ of X is said to be a Markov partition of the space X for the mapping T if the following conditions are satisfied.

(a) $R_i = \overline{\mathrm{Int} R_i}$ for all $i = 1, 2, \ldots, q$.

(b) $\mathrm{Int} R_i \cap \mathrm{Int} R_j = \emptyset$ for all $i \neq j$.

(c) $\mathrm{Int} R_j \cap T(\mathrm{Int} R_i) \neq \emptyset \implies R_j \subset T(R_i)$ for all $i, j = 1, 2, \ldots, q$.

The elements of a Markov partition will be called cells in the sequel. The basic theorem about Markov partitions follows. Its proof can be found for instance in [PU].

Theorem 4.9. *Any expanding repeller $T : J \to J$ admits Markov partitions of arbitrarily small diameters.*

We shall now state and prove the result of this subsection.

Theorem 4.10. *Let $T : J \to J$ be an expanding repeller, let $\psi : J \to \mathbb{R}$ be a Hölder continuous potential, and let μ_ψ be the corresponding equilibrium (Gibbs) state. Then the measure–preserving dynamical system (T, μ_ψ) is Weakly Markov. In particular, the exponential one laws of Theorem A hold.*

Proof. We shall check that the system satisfies the requirements of Definition 2.2 defining Weakly Markov systems. Property (i) of this definition for the dynamical system (T, μ_ψ) has been proved in [PU]. Property (ii) also has been proved therein. For property (iii) we also use [PU], namely Markov partitions discussed above and their basic properties. We aim to show that these partitions fulfil the requirements of Definition 2.5, i.e. the Weak Partition Existence Condition.

Towards this end fix $\delta > 0$ so small that for every $x \in X$ and every $n \geq 0$ there exists $T_x^{-n} : B(T^n(x), 4\delta) \to \mathbb{R}^d$, a unique continuous branch of T^{-n} sending $T^n(x)$ to x. Theorem 4.9 guarantees us the existence of $\mathcal{R} = \{R_1, \dots, R_q\}$, a Markov partition of T with all cells of diameter smaller than δ. It is not hard to see and it was proved in [PU] that any two distinct elements of $\mathcal{R}$ intersect along a set of μ_ψ measure zero. So, we can treat $\mathcal{R}$ as an ordinary partition. Its entropy $\mathrm{h}_{\mu_\psi}(T, \mathcal{R})$ is finite since the partition $\mathcal{R}$ is finite, and this entropy is positive for all $\delta > 0$ small enough since $\mathrm{h}_{\mu_\psi}(T) > 0$.

We now shall check that formula (2.4) holds. Fix one element $\xi \in R_1$. Now fix $R > 0$ so small that

$$B(\xi, 2R) \subset R_1. \tag{4.3}$$

Since $\mu_\psi\big(B(\xi, R)\big) > 0$ (as μ_ψ has full topological support in J), it follows from ergodicity of μ_ψ and Birkhoff's Ergodic Theorem that for μ_ψ–a.e. $z \in J$ there exists an infinite increasing sequence $(n_j)_{j=1}^\infty$ of positive integers such that

$$T^{n_j}(z) \in B(\xi, R)$$

for all $j \geq 1$ and

$$\lim_{j\to\infty} \frac{n_{j+1}}{n_j} = 1.$$

So, there exists a constant $A \geq 1$ such that

$$n_{j+1} \leq A n_j$$

for all $j \geq 1$. One consequence of such choice of z is that

$$z \notin \bigcup_{n=0}^\infty T^{-n}\left(\bigcup_{i=1}^q \partial\mathcal{R}_i\right).$$

In particular all elements $\mathcal{R}^n(z)$, $n \geq 0$, are well-defined and $\mathcal{R}$ being a Markov partition yields

$$\mathcal{R}^n(z) = T_z^{-n}(\mathcal{R}(T^n(z))). \tag{4.4}$$

Now fix an arbitrary integer $k > n_1$. Then there exists a unique integer $j \geq 2$ such that

$$n_{j-1} < k \leq n_j.$$

We then have $k > A^{-1} n_j$, and with $L \geq 1$ being a Lipschitz constant of T, looking up at (4.4) and (4.3), we get that

$$\begin{aligned} \mathcal{R}^k(z) &\supset \mathcal{R}^{n_j}(z) = T_z^{-n_j}(\mathcal{R}(T^{n_j}(z))) = T_z^{-n_j}(R_1) \supset T_z^{-n_j}\big(B(T^{n_j}(z), R)\big) \\ &\supset B(z, L^{-n_j}R) \supset B(z, L^{-Ak}R) \\ &= B\big(z, R\exp\big(-A\log Lk\big)\big). \end{aligned} \tag{4.5}$$

Also $B\big(z, R\exp\big((-A\log L)k\big)\big) \supset B\big(z, \exp\big(-2A\log Lk\big)\big)$ for all $k \geq 1$ large enough. So,

$$\mathcal{R}^k(z) \supset B\big(z, \exp\big(-2A\log Lk\big)\big) \tag{4.6}$$

for all such k, say $k \geq k(z) \geq n_1$. On the other hand, obviously there exists some $\chi^*(z) > 0$ so large that

$$\mathcal{R}^k(z) \supset B\big(z, \exp(-\chi^*(z)k)\big)$$

for all $k = 0, 1, \ldots, k(z) - 1$. In conjunction with (4.6) this gives that

$$\mathcal{R}^k(z) \supset B\Big(z, \exp\big(-\max\big\{2A\log L, \chi^*(z)\big\}k\big)\Big)$$

for all $k \geq 0$, and formula (2.4) is proved. The proof of Theorem 4.10 is thus complete. $\square$

4.2. **Conformal IFSs, with application to harmonic measure.** In this subsection we deal with conformal systems. The ultimate difference between this section and the previous is that now we will be able to establish the convergence to the exponential one law, i.e. formulas (1.1)–(1.4) for Full classes of radii and not merely Thick ones.

Up to our best knowledge, this is the first time that the convergence to the exponential law is proved to hold for such general systems and measures along all radii. Basically, only previously known situation is for μ being any finite Borel measure satisfying Ahlfors property with exponent $h > d - 1$ **(I think it should be explained/recall what d is)**. Then, as a straight volume argument shows, this measure satisfies the Full Thin Annuli Property at each point of its topological support.

We apply our results about the exponential distribution of statistics of return times, namely Theorem B for Weakly Markov systems and also the thin annuli property (Theorem D) for conformal IFSs to obtain Theorem 4.12 (i.e. Theorem E from the introduction), proving the statistics of exponential one law for dynamical systems given by conformal IFSs.

Let us also note here that after suitable changes all the result pertaining to IFSs may be generalized to Graph Directed Markov Systems.

Here we use all the notations of Subsection 3.2. This time, however, we assume in addition that the Open Set Condition, in fact the Strong Open Set Condition of [MauU4] holds. The Open Set Condition means that

$$\phi_a\left(\mathrm{Int}(X)\right) \cap \phi_b\left(\mathrm{Int}(X)\right) = \emptyset \tag{4.7}$$

whenever $a, b \in E$ with $a \neq b$. The Strong Open Set Condition requires that in addition

$$J_{\mathcal{S}} \cap \mathrm{Int}(X) \neq \emptyset.$$

Let $f : E^{\mathbb{N}} \to \mathbb{R}$ be a Hölder continuous function, called the potential. We assume that f is summable, meaning that

$$\sum_{e \in E} \exp\left(\sup(f|_{[e]})\right) < +\infty.$$

It is well known (see [MauU4] or [MauU3]) that the following limit, called the topological pressure of f, exists.

$$\mathrm{P}(f) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in E^n} \exp\left(\sup(f|_{[\omega]})\right)$$

It was proved in [MauU3] that there exists a unique shift–invariant equilibrium measure μ_f, satisfying additionally the Gibbs property, i.e.

$$C_f^{-1} \leq \frac{\mu_f([\omega|_n])}{\exp\left(S_n f(\omega) - \mathrm{P}(f)n\right)} \leq C_f$$

with some constant $C_f \geq 1$ for every $\omega \in E^{\mathbb{N}}$ and every integer $n \geq 1$, where

$$S_n(g) = g_n(\omega) := \sum_{j=0}^{n-1} g \circ \sigma^j.$$

Let us record the following basic properties of the Gibbs state μ_f.

Fact 1. *If $f : E^{\mathbb{N}} \to \mathbb{R}$ is a summable Hölder continuous potential, then the unique Gibbs state μ_f is ergodic and its topological support is equal to $E^{\mathbb{N}}$. In addition μ_f enjoys the Weak Independence Property (A).*

For ergodicity see e.g. [MauU4] while the Weak Independence Property follows immediately from the definition of μ_f.

To avoid problems with boundary intersections (and non-unique coding) we introduce

$$\mathring{J}_{\mathcal{S}} := J_{\mathcal{S}} \setminus \bigcup_{\omega \in E^*} \phi_\omega(\partial X) \quad \text{and} \quad \mathring{E}^{\mathbb{N}} := \pi_{\mathcal{S}}^{-1}\big(\mathring{J}\big)$$

and notice that for every $z \in \mathring{J}_{\mathcal{S}}$ there exists a unique $\omega(z) \in E^{\mathbb{N}}$ such that $z = \pi(\omega(z))$. Moreover, $\omega(z) \in \mathring{E}^{\mathbb{N}}$ and we simply denote it by $\pi^{-1}(z)$. Note that

$$\sigma\big(\mathring{E}^{\mathbb{N}}\big) \subset \mathring{E}^{\mathbb{N}}$$

and this restricted shift map induces a map $T_{\mathcal{S}} : \mathring{J}_{\mathcal{S}} \to \mathring{J}_{\mathcal{S}}$ by the formula

$$T_{\mathcal{S}}(z) = \pi \circ \sigma(\pi^{-1}(z)) \in \mathring{J}_{\mathcal{S}},$$

so that the diagram

$$\begin{array}{ccc} \mathring{E}^{\mathbb{N}} & \xrightarrow{\ \sigma\ } & \mathring{E}^{\mathbb{N}} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathring{J}_{\mathcal{S}} & \xrightarrow[T_{\mathcal{S}}]{} & \mathring{J}_{\mathcal{S}} \end{array}$$

commutes and the map $\pi : \mathring{E}^{\mathbb{N}} \to \mathring{J}_{\mathcal{S}}$ is a continuous bijection. Denote

$$\hat{\mu}_f := \mu_f \circ \pi_{\mathcal{S}}^{-1}.$$

The following observation we deduce directly from Fact 1.

Observation 4.11. *Suppose that $\mathcal{S}$ is a conformal IFS satisfying the Strong Open Set Condition. If $f \colon E^{\mathbb{N}} \to \mathbb{R}$ is a summable Hölder continuous potential, then*

$$\mu_f\big(\mathring{E}^{\mathbb{N}}\big) = 1 \quad \textit{and} \quad \hat{\mu}_f\big(\mathring{J}_{\mathcal{S}}\big) = 1.$$

Moreover, the measure–preserving dynamical systems $\left(\sigma, \mu_f\right)$ and $\left(T_{\mathcal{S}}, \hat{\mu}_f\right)$ are isomorphic.

This first main result of this subsection is the following.

Theorem 4.12. *Let $\mathcal{S}$ be a geometrically irreducible conformal IFS satisfying the Strong Open Set Condition. If f is a summable Hölder continuous potential such that*

$$\sum_{e\in E} \exp\big(\inf\big(f|_{[e]}\big)\big)||\phi'_e||_\infty^{-\beta} < +\infty \tag{4.8}$$

for some $\beta > 0$, then the dynamical system $\big(T_{\mathcal{S}}, \hat{\mu}_f\big)$ is Weakly Markov and satisfies the Full Thin Annuli Property. In consequence, the exponential one laws of (1.1) and (1.2) hold along all radii.

Proof. Property (i) of being Weakly Markov (Definition 2.2) for the dynamical system $\big(\sigma, \mu_f\big)$ has been proved in [MauU4]. For the system $\big(T_{\mathcal{S}}, \hat{\mu}_f\big)$ it then follows from the fact that the projection $\pi_{\mathcal{S}} : E_A^{\mathbb{N}} \to J_{\mathcal{S}}$ is Hölder continuous. Property (ii) has been also proved in [MauU4]. By virtue of Remark 2.4, in order to prove property (iii), it is enough to check that the Weak Partition Existence Condition holds. As the proof closely resembles the one of property (iii) in Theorem 4.10, we do it briefly. Let

$$\alpha := \{[e]\}_{e\in E}$$

be the partition of $\mathring{E}^{\mathbb{N}}$ into cylinders of length one and let

$$\pi(\alpha) := \{\pi([e])\}_{e\in E} = \{\phi_e(J_{\mathcal{S}})\}_{e\in E}.$$

Then

$$\alpha_\sigma^n = \{[\omega] : \omega \in E^n\}$$

and

$$\pi(\alpha)_T^n = \pi(\alpha_\sigma^n) = \{\phi_\omega(\mathring{J}_{\mathcal{S}}) : \omega \in E^n\}.$$

We know from [MauU4] that $\mathrm{h}_{\mu_f}(\sigma, \alpha) = \mathrm{h}_{\mu_f}(\sigma) \in (0, +\infty)$, and so, by isomorphism, $\mathrm{h}_{\hat{\mu}_f}(T, \pi(\alpha)) \in (0, +\infty)$. We also know that for μ_f–a.e. $\omega \in \mathring{E}^{\mathbb{N}}$, say $\omega \in F \subset \mathring{E}^{\mathbb{N}}$ with $\mu_f(F) = 1$, the limit

$$\chi_{\mu_f}(\omega) := -\lim_{n\to\infty} \frac{1}{n} \log\big|\phi'_{\omega|_n}(\pi(\sigma^n(\omega)))\big|$$

exists, is equal to

$$\chi_{\mu_f} := \int_{\mathring{E}^{\mathbb{N}}} \log\big|\phi'_1(\pi(\sigma(\omega)))\big|\, d\mu_f$$

and belongs to $(0, +\infty)$. Fix $u \in V$, then fix $\xi \in \mathrm{Int}(X_u)$, and finally fix $R > 0$ so small that

$$B(\xi, 2R) \subset \mathrm{Int}(X_u).$$

From this point the proof follows almost identically to the one Theorem 4.10 — starting at equation (4.3). For example, the inclusions (4.5) in this setting become (using the Distortion Property and the definition of χ_{μ_f} instead of the Lipschitz constant).

$$\begin{aligned}
&\pi(\alpha)_T^k(\pi(\omega)) \supset \pi(\alpha)_T^{n_j}(\pi(\omega)) = \phi_{\omega|_{n_j}}\big(\mathrm{Int}\big(X_{t(\sigma^{n_j}(\omega))}\big)\big) \supset \phi_{\omega|_{n_j}}\big(B(\xi, 2R)\big) \\
&\supset \phi_{\omega|_n}\big(B(\pi(\sigma^{n_j}(\omega)), R)\big) \supset B\big(\pi(\omega), K^{-1}R\big|\phi'_{\omega|_{n_j}}(\pi(\sigma^{n_j}(\omega)))\big|\big) \\
&\supset B\big(\pi(\omega), \exp(-2\chi_{\mu_f} n_j)\big) \supset B\big(\pi(\omega), \exp(-2A\chi_{\mu_f} k)\big),
\end{aligned} \tag{4.9}$$

thus ending the proof in the same fashion. $\square$

Remark 4.13. Note that if the system $\mathcal{S}$ of Theorem 4.12 is finite, then the hypothesis (4.8) is automatically satisfied and can be removed from its assumptions.

Now we will pass to deal with measures that are of more geometric flavour. Define the function $\zeta\colon E^{\mathbb{N}} \to \mathbb{R}$

$$\zeta(\omega) := \log\left|\phi'_{\omega_1}(\pi_{\mathcal{S}}(\sigma(\omega)))\right|$$

and for every $t \in \mathbb{R}$ we consider the potential $t\zeta\colon E^{\infty} \to \mathbb{R}$. Furthermore, we set

$$\mathrm{P}(t) := \mathrm{P}(t\zeta).$$

We recall (e.g. from [MauU1]) the following definition.

$$\gamma_{\mathcal{S}} := \inf\left\{s \in \mathbb{R}\colon \sum_{e\in E} ||\phi'_e||_{\infty}^{s} < +\infty\right\}.$$

Note that if the alphabet E is finite, then $\gamma_{\mathcal{S}} = -\infty$ and if E is infinite, then $\gamma_{\mathcal{S}} \geq 0$. The proof of the following statement can be found in [MauU4].

Proposition 4.14. *If $\mathcal{S}$ is a conformal IFS, then for every $s \geq 0$ we have that*

$$\gamma_{\mathcal{S}} = \inf\left\{s \in \mathbb{R} : \mathrm{P}(s) < +\infty\right\}.$$

Let us abbreviate $\mu_t := \mu_{t\zeta}$. As an immediate consequence of Theorem 4.12 we get the following.

Corollary 4.15. *Suppose that $\mathcal{S}$ is a geometrically irreducible conformal IFS satisfying the Strong Open Set Condition. Fix a real number $t > \gamma_{\mathcal{S}}$. Then the corresponding dynamical system $\left(T_{\mathcal{S}} : \mathring{J}_{\mathcal{S}} \to \mathring{J}_{\mathcal{S}}, \hat{\mu}_t\right)$ is Weakly Markov and satisfies the Full Thin Annuli Property.*

Remark 4.16. In the setting of Corollary 4.15, $\mathrm{H}_{h_{\mathcal{S}}}(J_{\mathcal{S}})$, the $h_{\mathcal{S}}$–dimensional Hausdorff measure of $J_{\mathcal{S}}$ is finite while the corresponding packing measure $\mathrm{P}_{h_{\mathcal{S}}}(J_{\mathcal{S}})$ is positive (see [MauU4]). If either one of these two measures is both finite and positive, then this measure is equivalent to the measure $\hat{\mu}_{h_{\mathcal{S}}}$ (which then does exist!) with uniformly bounded Radon–Nikodym derivatives. Thus the Full Thin Annuli Property holds respectively for $\mathrm{H}_{h_{\mathcal{S}}}$ or $\mathrm{P}_{h_{\mathcal{S}}}$ (or both) restricted to $J_{\mathcal{S}}$. This is always the case when the system $\mathcal{S}$ is finite.

It is well known that the harmonic measure of the limit set of a finite conformal IFS, which satisfies the Strong Separation Condition, is equivalent, with uniformly bounded densities, to a Gibbs/equilibrium measure. Thus as an immediate consequence of Theorem 4.12, we get the following important property of the harmonic measure.

Corollary 4.17. *Suppose that $\mathcal{S}$ is a conformal IFS in the complex plane $\mathbb{C}$ satisfying the Strong Separation Condition. Then the harmonic measure of its limit set satisfies the Full Thin Annuli Property.*

In fact this corollary is a consequence of Theorem 4.12 under the additional assumption of geometrical irreducibility. In the real–analytic case, still IFS, it follows, from easy to prove, upper estimates of the harmonic measure of a ball by its radius raised to some positive power.

4.3. **Example of a fine inducing scheme: application of IFSs for Rational Maps on the Riemann sphere.**

We will now show how to prove the exponential laws for rational maps on $\widehat{\mathbb{C}}$ and a class of equilibrium measures. To do this we build a suitable IFS using an inducing technique and then applying Theorem 4.12.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree larger than 1. Let $\varphi : J(f) \to \mathbb{R}$ be a Hölder continuous function. As before $\mathrm{P}(\varphi)$ denotes the topological pressure with respect to the dynamical system generated by the map $f : J(f) \to J(f)$. M. Lyubich proved in [Ly] that in our context of rational functions each continuous function admits an equilibrium state. It was shown in [DU] that if φ (being Hölder continuous) has a pressure gap, i.e. if

$$\mathrm{P}(\varphi) > \frac{1}{n}\sup(S_n\varphi) \tag{4.10}$$

for some $n \geq 1$, then there exists a unique equilibrium measure for φ, again denoted by μ_φ.

In [SUZ] several strong stochastic properties of measure μ_φ have been deduced from a special inducing scheme. The induced map forms a conformal IFS, satisfying the Strong Separation Condition, in particular the Strong Open Set Condition.

Before proving Theorem 4.19 we formulate a technical result; see in particular [SUZ] Proposition 11, for a similar statement.

Proposition 4.18. *If μ is a finite Borel measure in $\widehat{\mathbb{C}}$, then for every $\delta > 0$ there exists a finite partition $\alpha = \{U_i\}_{i\in I}$ of $\widehat{\mathbb{C}}$ with $\mathrm{diam}(U_i) < \delta$ (the diameter with respect to the spherical metric) for all $i \in I$, and such that*

$$\mu\left(\bigcup_{j\in I} B(\partial U_j, r)\right) \leq r^{1/2} \tag{4.11}$$

for all sufficiently small $r > 0$. In fact, the number $1/2$ in formula (4.11) can be replaced by any positive number smaller than 1.

Theorem 4.19. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be an arbitrary rational map of degree larger than 1 whose Julia set is not contained in a real analytic curve (for instance, if $\mathrm{HD}(J(f)) > 1$). Let $\varphi : J(f) \to \mathbb{R}$ be a Hölder continuous function with a pressure gap. Then $(J(f), f, \mu_\varphi)$ forms a Weakly Markov system with the Full Thin Annuli Property. Consequently, the exponential one laws hold for the dynamical system $\big(f\colon J(f) \to J(f), \mu_\varphi\big)$ along all radii.*

Proof. In order to check that the required properties hold, we refer to appropriate results in [SUZ]. First, we shall check that the system is Weakly Markov (Definition 2.2).

Item (i) of this definition follows from Theorem 56 in [SUZ]. Item (ii), i.e. positive lower pointwise dimension, holds in the current one-dimensional setting, since the limit under consideration exists μ_φ–a.e. and is equal to the Hausdorff dimension of the measure μ_φ, which is a positive number. By virtue of Remark 2.4, in order to prove item (iii), it is enough to check that the Weak Partition Existence Condition (2.4) holds.

Proposition 4.18 provides a partition α with elements of arbitrarily small diameter, satisfying the estimate (4.11). If $\max\{\mathrm{diam}(U_i) : i \in I\}$ is sufficiently small, then $h_{\mu_\varphi}(f, \alpha) > 0$ as can be immediately seen by combining Shannon–Breiman–McMillan Theorem together with a local entropy formula in [KB].

Fix $\beta > 0$ (later it will be needed to be sufficiently large) and for every $n \geq 1$ put

$$A_n := f^{-n}\Big(\bigcup_{i\in I} B(\partial U_i, e^{-\beta n})\Big).$$

Using the estimate (4.11) and f–invariance of measure μ_φ, we see that

$$\mu_\varphi(A_n) \leq e^{-(\beta/2)n}$$

for every $n \geq 1$ provided that $\beta > 0$ is sufficiently large. Since the series $\sum_{n\geq 1} e^{-(\beta/2)n}$ converges, Borel–Cantelli Lemma thus applies and it tells us that for μ_φ–a.e. $x \in J(f)$ there exists an integer $N = N(x) \geq 1$ such that for all integers $n \geq N$

$$B\big(f^n(x), e^{-\beta n}\big) \subset \alpha(f^n(x)). \tag{4.12}$$

Keep such an x and assume in addition that

$$x \notin \bigcup_{n=0}^{\infty} \bigcup_{i\in I} f^{-n}(\partial U_i).$$

The set A of such all points $x \in J(f)$ is of full measure, i.e. $\mu_\varphi(A) = 1$. For all integers $n \geq N = N(x)$ denote by $\alpha_N^n(x)$ the only element of the partition $\bigvee_{k=N}^n f^{-k}(\alpha)$, containing the point x. Similarly, denote by $\alpha_0^{N-1}(x)$ the only element of the partition $\bigvee_{k=0}^{N-1} f^{-k}(\alpha)$, containing the point x. It follows from (4.12) that

$$\alpha_N^n(x) \supset \bigcap_{k=N}^{n} f^{-k}\big(B(f^k(x), e^{-k\beta})\big).$$

Note also that

$$f^{-k}\big(B(f^k(x), e^{-k\beta})\big) \supset B\big(x, e^{-k(\beta+\Delta)}\big),$$

where e^Δ is a Lipschitz constant, with respect to the spherical metric, of the map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Thus,

$$\alpha_N^n(x) \supset \bigcap_{k=N}^{n} B\big(x, e^{-k(\beta+\Delta)}\big) = B\big(x, e^{-n(\beta+\Delta)}\big)$$

Finally, since $\alpha_0^n(x) = \alpha_0^{N-1}(x) \vee \alpha_N^n(x)$ and since $x \notin \bigcup_{n=0}^\infty \bigcup_{i\in I} f^{-n}(\partial U_i)$, there exists $\rho(x) > 0$ such that $B(x, \rho(x)) \subset \alpha_0^{N-1}(x)$. Thus, $\alpha_0^n(x)$ contains the ball $B\big(x, e^{-n(\beta+\Delta)}\big)$ for every integer $n \geq 1$ large enough (depending on x), and so $\alpha_0^n(x) \supset B\big(x, C(x)e^{-n(\beta+\Delta)}\big)$ for every integer $n \geq 1$, where $C(x)$ is some positive finite constant depending on x. Hence, the Weak Partition Existence Condition holds and property (iii) of Definition 2.2 is established.

The Full Thin Annuli Property is a consequence of the above mentioned fine inducing procedure, see [SUZ], Section 3. We follow the notation of [SUZ], especially Section 8 of that paper. The fine inducing construction leads to a conformal IFS, satisfying the Strong Separation Condition, and such that the limit set of this system is of full μ_φ measure. We denote this system by $\mathcal{S}$. We recall briefly the way this induced system is constructed. For a properly chosen topological disc U, the system $\mathcal{S}$ is defined by a family of conformal univalent homeomorphisms $\phi_e : U \to D_e$, $e \in E$, where E is some countable set and $\overline{D}_e \subset U$ for every $e \in E$. Each map ϕ_e, $e \in E$, is just, a suitably chosen, holomorphic branch of the inverse of some iterate of f, say $f^{N(e)}$, mapping U onto D_e. As usual, denote the corresponding projection from $E^{\mathbb{N}}$ to $\widehat{\mathbb{C}}$ by $\pi_{\mathcal{S}}$. The iterated function system $\mathcal{S}$, together with the summable Hölder potential

$$\overline{\varphi} = S_{N(e)}\varphi \circ \pi_{\mathcal{S}} - \mathrm{P}(\varphi)N(e) : E^{\mathbb{N}} \to \mathbb{R},$$

arising naturally from the inducing procedure, admits an (invariant) equilibrium state which is equivalent to the initial measure μ_φ. We claim that the IFS $\mathcal{S}$ together with the (induced) potential $\overline{\varphi}$, satisfies the hypotheses of Theorem 4.12, with f therein being replaced by $\overline{\varphi}$. We shall sketch the argument here, referring to appropriate estimates in [SUZ]. The estimate which we need to verify is the following (see (4.8))

$$\sum_{e\in E} \exp\big(\inf\,(\overline{\varphi}|_{[e]})\big)||\phi_e'||_\infty^{-\beta} < +\infty \tag{4.13}$$

with some $\beta > 0$. Note, however, that $\exp\big(\inf\,(\overline{\varphi}|_{[e]})\big)$ is multiplicatively comparable to $\mu_\varphi(D_e)$ independently of e; thus, in order to verify (4.13), it is enough to check that

$$\int |F'|^\beta d\mu_\varphi < \infty \quad \text{for some} \quad \beta > 0, \tag{4.14}$$

where the map F is defined on each set D_e just as $(\phi_e)^{-1}$.

Now, by the definition of F and the system $\mathcal{S}$, we have that $F|_{D_e} = f^{N(e)}|_{D_e}$. Moreover, (see e.g. the formula (3.1) in [SUZ]) we know that

$$\mu_\varphi\left(\bigcup_{e:\ N(e)\geq n} D_e\right) \leq 2e^{-n\gamma}$$

for every integer $n \geq 1$ and some $\gamma > 0$. Using the trivial estimate $|F'| \leq ||f'||^{N(e)}$, (4.14) follows immediately. Finally, the system $\mathcal{S}$ is geometrically irreducible since the Julia set is not contained in a real analytic curve.

Therefore, we are in position to apply Theorem 4.12, and the measure μ_φ has the Full Thin Annuli Property. The proof is complete. □

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