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LOCAL FORMULAS FOR EHRHART COEFFICIENTS FROM LATTICE TILES

MAREN H. RING AND ACHILL SCHÜRMANN

Abstract. As shown by McMullen and Morelli, the coefficients of the Ehrhart polynomial of a lattice polytope can be written as a weighted sum of facial volumes. The weights in such a local formula depend only on the outer normal cones of faces, but are far from being unique. In this paper, we develop a new class of such local formulas, based on polyhedral volume computations. We hereby also obtain a kind of geometric interpretation for the Ehrhart coefficients. Since our construction depends on the choice of fundamental domains for the sublattices of the given lattice, we obtain a great variety of possible local formulas. These can for instance be chosen to fit well with a given polyhedral symmetry group. In contrast to other constructions of local formulas, ours does not rely on a valuation property, respectively on triangulations of rational cones into simplicial or even unimodular ones.

1. Overview

Notation. We first fix some notation and recall some basic facts. Let V be a Euclidean space of dimension n with inner product $\langle\cdot,\cdot\rangle$ and let Λ be a lattice in V of rank n. For a linear subspace $S \subseteq V$, the set $\Lambda \cap S$ is a lattice in S, called the *induced lattice* in S. A (strict) *tiling* of a set $A \subseteq V$ is a family of subsets of A that cover A and have pairwise empty intersections. A *fundamental domain* for a sublattice $L \subseteq \Lambda$ is a connected and bounded subset $T \subseteq \operatorname{lin}(L)$ of the linear hull of L, such that the family of translations $\{x + T : x \in L\}$ is a tiling of $\operatorname{lin}(L)$.

The *affine hull* $\operatorname{aff}(A)$ of a subset $A \subseteq V$ is the smallest affine space containing A. We will mainly work with affine spaces $x + S$ which are translates of a linear subspace S by lattice vectors $x \in \Lambda$. Here, the sublattice $\Lambda \cap S$ in S is assumed to be of maximal possible rank $\dim S$. In these affine spaces the *relative volume* or *lattice volume* $\operatorname{vol}(A)$ of a set A is defined as the Lebesgue measure, normalized in a way that a fundamental domain of $\Lambda \cap S$ has volume 1.

For a polyhedron $\mathcal{P}$, we consider its *face lattice*, that is, the partially ordered set consisting of all faces of $\mathcal{P}$ with order given by inclusion, where we consider $\mathcal{P}$ as a face of itself. We denote the order by $\leq$ and write $f < g$ for faces f, g of $\mathcal{P}$ if we want to exclude the case $f = g$. The face lattice is a combinatorial lattice, since for every two faces f, g there exists a unique least upper bound $f \vee g$ called *join* and a unique greatest lower bound $f \wedge g$ called *meet*. $f \vee g$ ist the smallest face that contains both, f and g and $f \wedge g$ is given by the intersection $f \cap g$. Here, we formally consider the empty set as a face of $\mathcal{P}$. Since we never use it, we shorten notation by always implying $f \neq \varnothing$ whenever we talk about faces $f \leq \mathcal{P}$.

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A *rational cone* is a set that is defined by finitely many homogeneous inequalities with rational coefficients with respect to a lattice basis. A cone is called *pointed*, if it does not contain any nontrivial linear subspace. Let f be the face of a polyhedron $\mathcal{P}$. The (outer) *normal cone* N_f of P at f is defined as

$$N_f := \{x \in V : \langle x, y - s\rangle \leq 0 \ \forall y \in \mathcal{P}\}$$

for any vector s in the *relative interior* of f, i.e. the interior with respect to the affine hull $\mathrm{aff}(f)$. It can be shown that this definition does not depend on the choice of s. If $\mathcal{P}$ is full dimensional, then all normal cones are pointed cones. The *polar cone* $C^\vee$ of a cone C is defined as

$$C^\vee := \{x \in V : \langle x, y\rangle \leq 0 \ \forall y \in C\}.$$

It contains the linear subspace $C^\perp := \{x \in V : \langle x, y\rangle = 0 \ \forall y \in C\}$ that we call the *orthogonal space* of C. Given a face f of a polyhedron P, the polar cone $N_f^\vee$ of the normal cone N_f is in the literature often referred to as the *cone of feasible directions* (cf. [3], [5]). An example is given in Figure 1.

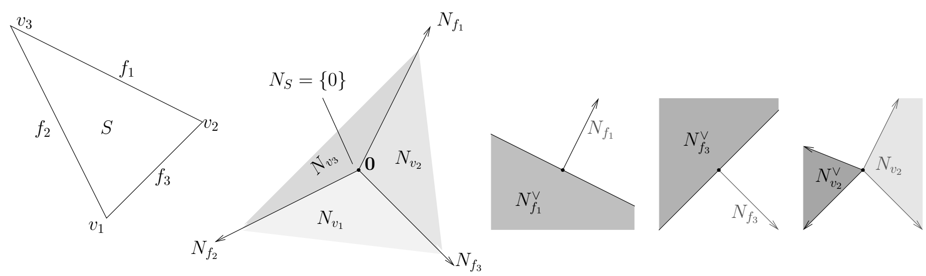

FIGURE 1. Triangle (simplex) S with faces; normal cones of S; polar cones of certain normal cones.

Local Ehrhart formulas. Let $\mathcal{P}$ be a *lattice polytope*, that is, $\mathcal{P} = \mathrm{conv}(v_1, \ldots, v_m)$ for some $v_1, \ldots, v_m \in \Lambda$. The *Ehrhart polynomial* $\mathrm{E}_\mathcal{P}$ of $\mathcal{P}$ is the function that maps a nonnegative integer $t \in \mathbb{Z}_{\geq 0}$ to the number of lattice points in the t-th dilate $t\mathcal{P}$ of $\mathcal{P}$. It was proven by Ehrhart [8] that this function is indeed a polynomial of degree $d := \dim(\mathcal{P})$:

$$\mathrm{E}_\mathcal{P}(t) := |t\mathcal{P} \cap \Lambda| = e_d t^d + e_{d-1} t^{d-1} + \cdots + e_1 t + e_0,$$

with $e_d, \ldots, e_0 \in \mathbb{Q}$. The Ehrhart polynomial plays an important role in areas such as combinatorics, integer linear programming and algebraic geometry. It is known that $e_0 = 1$, that the highest coefficient e_d is the relative volume of $\mathcal{P}$ and the second highest coefficient e_{d-1} equals half the sum of relative volumes of facets of $\mathcal{P}$. In dimension $d = 2$, this gives a full description of the polynomial, generally known as *Pick's formula* [11]. The knowledge about the remaining coefficients in higher dimensions is still very limited. In 1975, in the context of toric varieties, Danilov [7] asked whether it is possible to determine the i-th coefficient e_i as a weighted sum of relative volumes of the i-dimensional faces of $\mathcal{P}$, where the weights only depend on the normal cone of the faces. It was proven by McMullen [10] and Morelli [9] that such weights do indeed exist, though they are far from being unique. This gives rise to the following definition.

Definition 1. A real valued function μ on rational cones in V is called a *local formula for Ehrhart coefficients* (or *local formula* for short), if for any lattice polytope $\mathcal{P}$ with Ehrhart polynomial $\mathrm{E}_{\mathcal{P}}(t) = e_d t^d + e_{d-1} t^{d-1} + \cdots + e_1 t + e_0$, we have

$$e_i = \sum_{\substack{f \leq \mathcal{P} \\ \dim(f) = i}} \mu(N_f) \mathrm{vol}(f),$$

for all $i \in \{0, \ldots, d\}$.

Note that local formulas are sometimes also referred to as *McMullen's formulas*. The normal cones of the faces of a polytope do not change when taking a dilate by an integer $t \in \mathbb{Z}_{\geq 0}$. The relative volume of a face f however is homogeneous of degree $\dim(f)$, $\mathrm{vol}(tf) = t^{\dim(f)} \cdot \mathrm{vol}(f)$. For a function μ on rational cones in V, being a local formula for Ehrhart coefficients is thus equivalent to

$$|t\mathcal{P} \cap \Lambda| = \sum_{f \leq \mathcal{P}} \mu(N_f) \mathrm{vol}(tf), \tag{1}$$

for all lattice polytopes $\mathcal{P}$ and all $t \in \mathbb{Z}_{\geq 0}$. Since both sides of this equation are polynomials, it suffices to show equality for a finite number of values for t. We will use this fact to show that a function is a local formula by showing that equality holds for all sufficiently large t.

The word 'local' in the definition is justified, because the function μ only depends on the normal cone of the face. That means the only information taken from the face is its affine hull, respectively the class of parallel affine spaces it belongs to. In particular, μ does neither depend on the size or the shape of a face, nor on its boundary or other parts of the polytope. Advantages of such a local formula are immediate: Properties like positivity of the Ehrhart coefficients can be deduced from the values of μ, without computing the actual coefficients as attempted in [6]. Moreover, the values stay the same for all faces with the same normal cone, so that computations can be done for a whole class of polytopes at once. For example, the computations of μ for the regular permutohedron give the values of all generalized permutohedra as defined in [13].

Constructions of local formulas have been given by Pommersheim and Thomas [12] and by Berline and Vergne [5]. While the first is obtained from an expression for the Todd class of a toric variety, the second depends on the construction of certain differential operators. We note that Pommersheim and Thomas also take the normal cone as input for their local formula, while Berline and Vergne use the transverse cone of a face, which is an affine version of the cone of feasible directions modulo its contained linear subspace.

Main results. In this paper, we develop a new class of local formulas for Ehrhart coefficients, which is elementary in contrast to previous ones in the sense that all information is attained by considering a certain tiling of space and taking sums and differences of relative volumes. At this point it is unclear what the exact relation of our and other local formulas is, but our construction appears to allow a greater variety, for instance with irrational values (see Example 2). Moreover, we do not need a valuation property for our local formula and our construction does not rely on simplicial (cf. [12]) or even unimodular triangulations of cones (cf. [5]).

We first restrict to the case that the considered rational cones are *pointed* and the lattice polytopes are full dimensional. For a pointed rational cone C, we first assign a subset of V that we want to call *region of* C, denoted by $R(C)$. The construction of this region is quite involved, a thorough description is given in Section 2. The most important property will be the following: Let $\mathcal{P}$ be a full dimensional lattice polytope and $f \le \mathcal{P}$ a face. For $t \in \mathbb{Z}_{>0}$ we define the set $\mathcal{X}(tf)$ of all *feasible lattice points in* tf as the finite set of lattice points in the dilated face tf that are 'far enough' from its boundary (a precise definition can be found in Section 2). Then we have

Theorem 1 (Tiling)**.** *Let $\mathcal{P} \subseteq V$ be a full dimensional lattice polytope. There exists a $t_0 \in \mathbb{Z}_{>0}$ such that for each $t \ge t_0$ we have a tiling of V into translated regions of the form*

$$\{x + R(N_f) : \ f \le \mathcal{P},\, x \in \mathcal{X}(tf)\}.$$

See Figure 2 (left) for an example in dimension 2 (cf. Section 3, Example 3).

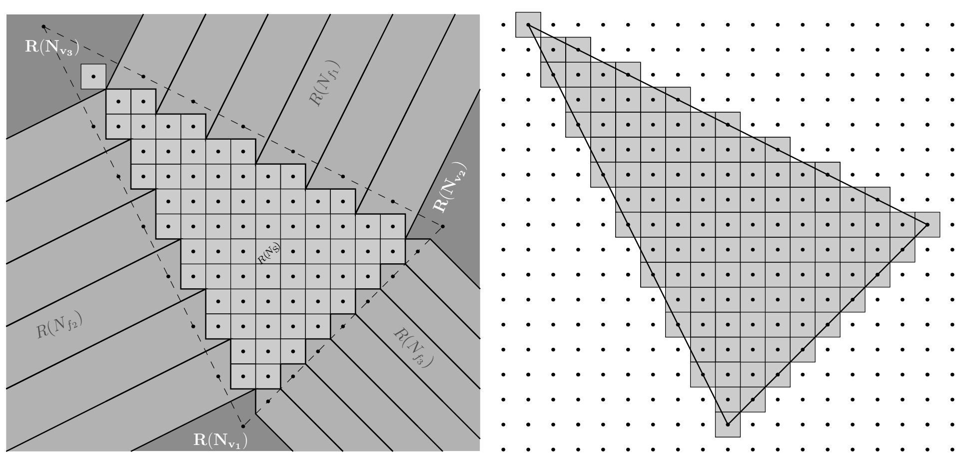

Figure 2. **Left:** Tiling of $\mathbb{R}^2$ by regions corresponding to the normal cones of the triangle S; **right:** The (fundamental) domain complex of $4 \cdot S$.

Using the region $R(C)$, we want to determine the value $\mu(C)$. To give an intuition how this can be achieved, we interpret the number of lattice points in $t\mathcal{P}$ as the volume of all translates of a fundamental domain T of Λ around the lattice points in $\mathcal{P}$:

$$|\Lambda \cap t\mathcal{P}| = \sum_{x \in \Lambda \cap t\mathcal{P}} \operatorname{vol}(x + T) = \operatorname{vol}\underbrace{\big((\Lambda \cap t\mathcal{P}) + T\big)}_{=:DC} \tag{2}$$

The first equation holds, since by definition $\operatorname{vol}(T) = 1$ for any fundamental domain T of Λ, and the second equation follows from $(x + T) \cap (y + T) = \varnothing$ for all $x, y \in \Lambda$ with $x \ne y$. We call the set $DC := (\Lambda \cap t\mathcal{P}) + T$ a *(fundamental) domain complex* of $t\mathcal{P}$ (cf. Figure 2, right). By taking the volume of the respective part of the domain complex in each region of the tiling in Theorem 1 (cf. Figure 2, left), we get

$$|\Lambda \cap t\mathcal{P}| = \operatorname{vol}(DC) = \sum_{f \le \mathcal{P}} \sum_{x \in \mathcal{X}(tf)} \underbrace{\operatorname{vol}\big((x + R(N_f)) \cap DC\big)}_{(*)} \tag{3}$$

It turns out (cf. Section 5) that $(*)$ can be defined only in terms of the cone N_f which leads to the definition of v_C, the *DC-volume* in $R(C)$, for pointed rational cones C:

$$v_C := \mathrm{vol}\big(R(C) \cap ((C^\vee \cap \Lambda) + T)\big). \tag{4}$$

For an illustration, see Figure 3.

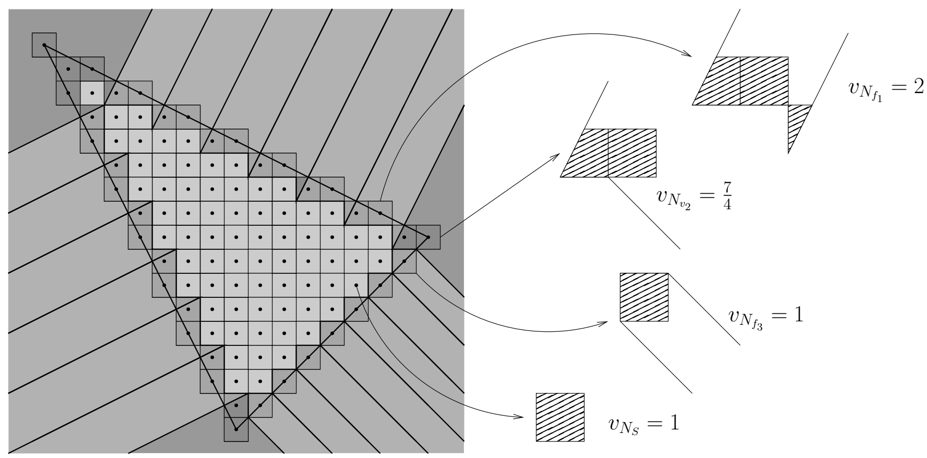

FIGURE 3. Values for the DC-volume v_C for normal cones $C = N_f$ of certain faces f of the triangle S.

Equation (3) thus yields

$$|\Lambda \cap t\mathcal{P}| = \sum_{f \leq \mathcal{P}} \sum_{x \in \mathcal{X}(tf)} v_{N_f} = \sum_{f \leq \mathcal{P}} v_{N_f} \cdot |\mathcal{X}(tf)| \tag{5}$$

This already looks a lot like a local formula, especially since $|\mathcal{X}(tf)|$ behaves like $\mathrm{vol}(tf)$ in the limit $t \to \infty$. In fact, $|\mathcal{X}(tf)|$ equals $|\Lambda \cap tf|$ minus some lower order terms. To achieve exactness, we use $v_C = v_{N_f}$ together with a *correction volume* defined by

$$w_K^C := \mathrm{vol}(R(C) \cap (K^\perp \cap C^\vee)) \tag{6}$$

for faces $K < C$. Exemplary values with illustrations are given in Figure 4. Note that unlike v_C, the correction term w_K^C measures a volume in $K^\perp$, which is only full-dimensional if K is the *trivial cone* $C_0 := \{0\}$. Note that here we have $N_{\mathcal{P}} = \{0\}$, since for the moment we are still assuming $\mathcal{P}$ to be full dimensional.

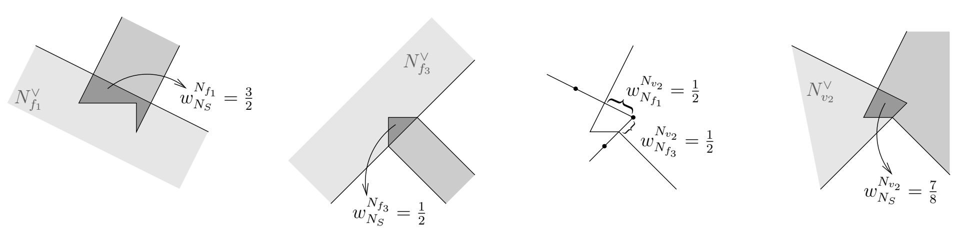

FIGURE 4. Values for the correction volume w_K^C with $C = N_f$ and $K = N_g$ for certain faces $f < g$ of S.

Using these notations, the function μ can be defined on pointed rational cones in V. We define it by induction on the dimension of the cone, starting with the trivial cone $C_0 = \{0\}$ by setting

$$\mu(C_0) := 1. \tag{7}$$

For a pointed rational cone $C \subseteq V$ with $\dim(C) \geq 1$, we define

$$\mu(C) := v_C - \sum_{K<C} w_K^C \cdot \mu(K). \tag{8}$$

For a rational cone $C \subseteq V$ that is not pointed, but contains a maximal nontrivial linear subspace U, we can consider the pointed cone $C' := C \cap U^\perp$ in $U^\perp$, where we consider $U^\perp$ as a Euclidean space equipped with the induced inner product and the lattice $\Lambda \cap U^\perp$. We can then construct $R(C') \subseteq U^\perp$ and set

$$\mu(C) := \mu(C').$$

That leads us to the main result of this work:

Theorem 2 (Local Formula)**.** *The function μ on rational cones in V as defined in Equations (7) and (8) is a local formula for Ehrhart coefficients.*

The given construction for local formulas has several nice properties. It is more basic than previous constructions in the sense that it is based on basic notions from polyhedral geometry. In a way, it hereby also gives a geometric meaning to the coefficients of the Ehrhart polynomial.

Another nice property of the construction is the freedom of choice of a fundamental domain in each occurring sublattice. An interesting observation is for example that this construction also allows irrational values, which can be achieved simply by taking a fundamental domain and shifting it by an irrational vector. This shows that the range of this construction is wider than the one of others previously known. It is yet to be determined, how extensive this variety actually is, whether for example hitherto existing constructions can be described in terms of our construction using certain fundamental domains. In [6] it is shown that all local formulas that are invariant under the standard action of the symmetric group must agree on certain polytopes that are invariant with respect to this action themselves. As described in Section 2, a natural choice for fundamental domains are the so-called *Dirichlet-Voronoi cells* that depend on a chosen inner product. Given the lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$, and taking the Dirichlet-Voronoi cell given by the standard inner product, one gets a fundamental domain and thus a local formula that is symmetric about the origin. This principle can be extended to other symmetries and, as we will discuss in Section 3, it leads to new possibilities to exploit symmetries in given polytopes.

Our paper is organized as follows. In Section 2 we give a construction of regions for pointed rational cones based on a choice of fundamental domains. We introduce an important class of examples coming from Dirichlet-Voronoi cells of lattices. In Section 3 we then give several descriptive, two-dimensional examples with concrete values for μ. Moreover, we describe a canonical way to exploit symmetries using certain Dirichlet-Voronoi cells. In Section 4, we give a proof for Theorem 1, showing that a tiling of V can be gained from translates of regions corresponding to normal cones of a full dimensional lattice polytope. In Section 5 we close with a proof of Theorem 2, showing that the constructed functions μ on rational cones are indeed local formulas.

2. Construction of Regions

The construction of regions that we use to define the local formula μ is based on the choice of fundamental domains, not only for Λ, but for all occurring induced sublattices. Different fundamental domains form different regions and ultimately result in different local formulas.

By definition, fundamental domains (for the lattice Λ considered as an additive group acting on V by translation) have the property that every Λ-orbit of V meets T in exactly one point. Besides being bounded and connected, we further require our fundamental domains to contain 0, which we obtain by translating an arbitrary fundamental domain by minus the unique lattice point it contains. We make this assumption to simplify notation considerably, but it can actually be omitted, which we will briefly discuss in Section 3 after Example 2.

Example. An important family of examples of fundamental domains are *Dirichlet-Voronoi cells.* Given a space V and an inner product $\langle\cdot,\cdot\rangle$ with induced norm $\|\cdot\|$, the Dirichlet-Voronoi cell of a sublattice $L \subseteq \Lambda$ is defined as

$$\mathrm{DV}(L, \langle\cdot,\cdot\rangle) := \{x \in \operatorname{lin}(L) : \|x\| \leq \|x - a\| \text{ for all } a \in L\}.$$

In this definition, it is not yet a fundamental domain of the lattice L, since it is closed and thus translates by lattice points can intersect on the boundary.

However, by considering the Dirichlet-Voronoi cell "half open", it can be seen as a fundamental domain of the lattice. In Figure 5, two different Dirichlet-Voronoi cells in $\mathbb{R}^2$ with lattice $\mathbb{Z}^2$ are given. They correspond to the standard inner product and the inner product $\langle x, y\rangle = x^t G y$ defined by the Gram matrix $G = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, respectively.

Figure 5. Square and hexagonal Dirichlet-Voronoi cells of $\mathbb{Z}^2$.

For a pointed rational cone C in V, let $L(C) := \Lambda \cap C^\perp$ be the induced lattice in $C^\perp$ and let $T(C) \subseteq C^\perp$ be a fundamental domain of $L(C)$. If we consider the domain complex as illustrated in Figure 2 on the right, we see that the structure of the domain complex is periodic with respect to lattice translations from $L(C)$. To obtain these periodicities, we build the regions as part of the strip $\operatorname{lin}(C) + T(C)$. For each face K of C, we further cut out all translates of regions $R(K)$ that 'fit properly' into $C^\vee$. A thorough definition of the inductive construction of the regions is given below. The definition is somewhat technical. As an example, we give a detailed picture for two rational cones of different dimension in $\mathbb{R}^2$ in Figure 6.

Construction. Let C be a pointed rational cone in V. If $C = C_0 = \{0\}$ is the trivial cone, we set

$$R(C_0) := T(C_0).$$

Otherwise, if $\dim(C) \geq 1$, we assume we have constructed all regions $R(K)$ for faces $K < C$. Let X_K^C be the set of all points x in $L(K)$ that fulfill the conditions:

(I) $[x + (R(K) \cap \mathrm{int}(K^\vee))] \subseteq \mathrm{int}(C^\vee)$ and

(II) $(x + R(K)) \cap (x' + R(K')) = \varnothing$ for all $K' < C$, with K' incomparable to K and $x' \in L(K')$.

Then we define

$$R(C) := \left(V \setminus \bigcup_{K<C} \left(X_K^C + R(K) \right) \right) \cap \left(T(C) + \mathrm{lin}(C) \right). \tag{9}$$

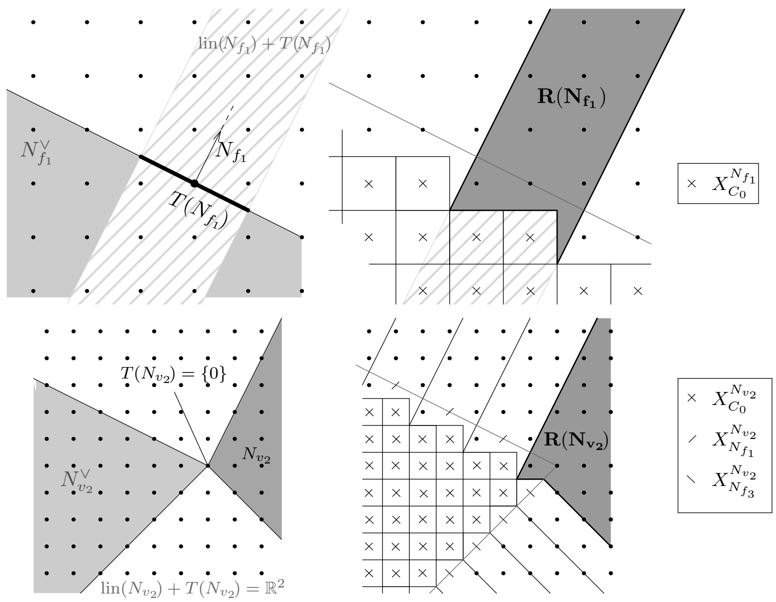

Figure 6. Construction of the regions $R(N_{f_1})$ and $R(N_{v_2})$ for the 1-dim. cone N_{f_1} (above) and the 2-dim. cone N_{v_2} (below).

One property of $R(C)$ is that it contains the fundamental domain $T(C)$, which we will show in Lemma 6.

Note that $R(C)$ is a piecewise linear set, respectively a union of finitely many convex polyhedra (a *polyhedral complex*). Therefore the values v_C and w_K^C (cf. (4) and (6)) in the definition of $\mu(C)$ in (8) are obtained from volume computations of finitely many convex polyhedra. In a sense, we hereby also obtain a kind of geometric interpretation for the Ehrhart coefficients.

3. Computations and symmetry

In this section we will give some easy examples that illustrate how diverse our construction of local formulas is and how symmetries can be taken advantage of.

In the following examples, a lattice polytope $\mathcal{P}$ is given as the convex hull of specific lattice points. But since translation by lattice points and dilation by a positive integer do not change the values of μ, we will not show a coordinate system in our figures. In fact, since the tiling from Theorem 1 demands a certain dilation and since the structure of the tiling becomes clearer for larger dilations, in the following examples we give all figures of $\mathcal{P}$ dilated by a factor of at least 3.

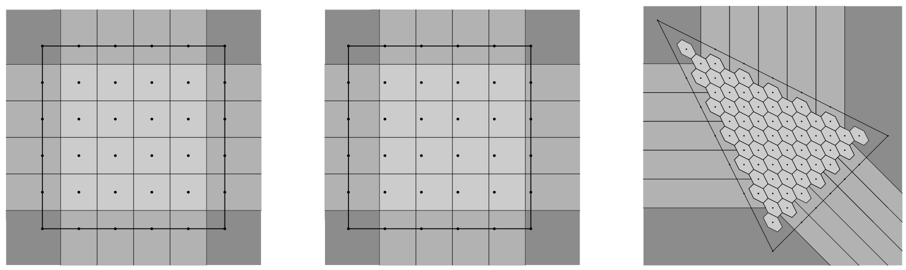

FIGURE 7. Tiling of $\mathbb{R}^2$ by regions from Examples 1, 2 and 4, respectively.

Example 1. Let $V = \mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$ and $\mathcal{P} = H := \operatorname{conv}\{(0,0),(1,0),(1,1),(0,1)\}$, the unit square. Let $\langle\cdot,\cdot\rangle$ be the standard inner product. We take a Dirichlet-Voronoi cell as a fundamental domain of $\mathbb{Z}^2$ and also of all sublattices. Then $D := \mathrm{DV}(\mathbb{Z}^2, \langle\cdot,\cdot\rangle)$ is a square as shown on the left of Figure 5 and for a one dimensional subspace $U \subseteq \mathbb{R}^2$, $D(U) := \mathrm{DV}(U \cap \mathbb{Z}^2, \langle\cdot,\cdot\rangle)$ is the segment with vertices being midpoints between the origin and its two neighboring lattice points. From Theorem 1 we get the regions as shown on the left of Figure 7. Since the symmetries of the Dirichlet-Voronoi cell and of H are the same, we get the same values for all faces with the same dimension for reasons that are discussed below. The values of μ for the normal cones N_f of faces $f \leq H$ are then:

$\dim(f)$	2	1	0
$\mu(N_f)$	1	$\frac{1}{2}$	$\frac{1}{4}$

(10)

That yields the Ehrhart coefficients $e_2 = 1$, $e_1 = 2$ and $e_0 = 1$.

Example 2. As above, let $V = \mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$ and $\mathcal{P} = H$ be the unit square. We again take the standard inner product, but as a fundamental domain we take D_η instead of D, which is a translate by $(\eta, 0)$ of the fundamental domain from the previous example, $D_\eta := D + (\eta, 0)$, where $\eta \in \mathbb{R}$ is any real number with $-\frac{1}{2} < \eta < \frac{1}{2}$. For the sublattice $\operatorname{lin}((0,1)) \cap \mathbb{Z}^2$ we take the usual Dirichlet-Voronoi cell and for the sublattice $\operatorname{lin}((1,0)) \cap \mathbb{Z}^2$ we again take the translate of the Dirichlet-Voronoi cell by $(\eta, 0)$. We then get a tiling by regions as shown in the middle of Figure 7. The resulting (possible irrational) values for μ are:

face f	T	f_1, f_2	f_3, f_4	v_1, v_2	v_3, v_4
$\mu(N_f)$	1	$\frac{1}{2} + \eta$	$\frac{1}{2} - \eta$	$\frac{1}{4} + \frac{1}{2}\eta$	$\frac{1}{4} - \frac{1}{2}\eta$

(11)

This example shows that irrational values are actually possible, showing a difference to all previous constructions of local formulas which have rational values only.

Remark. It is actually possible to drop the assumption made in Section 2, that 0 is contained in each fundamental domain. The parts that change in the proofs are mainly that $\mathcal{X}(tf)$ is not necessarily contained in tf, but only in the affine span

$\text{aff}(tf)$ and the chosen radii in the proof of Lemma 3 might get larger, but are still finite.

The values for μ in Example 2 are therefore also applicable for $\eta \in \mathbb{R}$ arbitrary, which in particular allows the values to be negative even for the easy example of the unit square.

Example 3. Let $V = \mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$ and $\mathcal{P} = S$, the triangle (simplex) shown on the left of Figure 1. We can think of S as the triangle that is the convex hull of the vertices $v_1 = (1,0)$, $v_2 = (2,1)$ and $v_3 = (0,2)$. On the left hand side of Figure 2, one can see the tiling of $4 \cdot S$ by regions that we get, if we choose the Dirichlet-Voronoi cell with respect to the standard inner product in $\mathbb{R}^2$. Figures 3 and 4 show some values for the DC-volume and the correction volume which determine the values for μ:

face f	S	f_1	f_2	f_3	v_1	v_2	v_3
$\mu(N_f)$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{4}$

(12)

Symmetry. In Example 1 it was possible to use the standard inner product and get the same values for each face in the same dimension. This principle can easily be generalized using suitable Dirichlet-Voronoi cells.

Let $\mathcal{P}$ be a lattice polytope and $\mathcal{G}$ a subgroup of all lattice symmetries of $\mathcal{P}$, i.e. $\mathcal{G}$ is a finite matrix group with $A \cdot \mathcal{P} := \{A \cdot x : x \in \mathcal{P}\} = \mathcal{P}$ and $A \cdot \Lambda = \Lambda$ for all $A \in \mathcal{G}$. Then we can define a $\mathcal{G}$-invariant inner product by taking

$$\langle x, y \rangle_{\mathcal{G}} := x^t G y \qquad \text{for all } x, y \in V, \tag{13}$$

with the Gram matrix G given by

$$G := \frac{1}{|\mathcal{G}|} \sum_{A \in \mathcal{G}} A^t A. \tag{14}$$

Let $\|\cdot\|_{\mathcal{G}}$ be the induced norm and let D be the Dirichlet-Voronoi cell for Λ given by the inner product,

$$D := \text{DV}(\Lambda, \langle \cdot, \cdot \rangle_{\mathcal{G}}) = \{x \in V : \|x\|_{\mathcal{G}} \leq \|x - p\|_{\mathcal{G}} \text{ for all } p \in \Lambda\}.$$

Then D is invariant under the action of $\mathcal{G}$: Let $x \in D$, then for $A \in \mathcal{G}$ we have

$$\|Ax\|_{\mathcal{G}} = \|x\|_{\mathcal{G}} \leq \|x - p\|_{\mathcal{G}} = \|Ax - Ap\|_{\mathcal{G}} \qquad \text{for all } p \in \Lambda.$$

Since $A\Lambda = \Lambda$, we get $AD \subseteq D$ for all $A \in \mathcal{G}$. Substituting A by A^{-1}, we get $A^{-1}D \subseteq D$ which yields $D \subseteq AD$ and hence $AD = D$. Similarly, we see that for all faces f in the same $\mathcal{G}$-orbit the normal cones and Dirichlet-Voronoi cells in $\Lambda \cap N_f^{\perp}$ are mapped onto each other. Hence, the used regions are invariant under the action of $\mathcal{G}$ and μ is constant on $\mathcal{G}$-orbits.

Example 4. Again, let $V = \mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$ and $\mathcal{P} = S$ as in the previous example. One might notice that S is invariant under the action of the group $\mathcal{G} = \langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right) \rangle$ of order 3. An invariant inner product $\langle x, y \rangle_{\mathcal{G}} = x^t G y$ is defined by the Gram matrix $G = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. The Dirichlet-Voronoi cell corresponding to $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ is the hexagon shown on the right of Figure 5 and the resulting tiling is given on the right of Figure 7.

Since all faces of the same dimension lie in the same orbit, the computation of μ can be reduced:

$\dim(f)$	2	1	0
$\mu(N_f)$	1	$\frac{1}{2}$	$\frac{1}{3}$

4. Proof of Theorem 1 (Tiling)

Given a full dimensional lattice polytope $\mathcal{P}$, we can choose fundamental domains for all relevant sublattices of Λ and construct all regions $R(N_f)$ for the normal cones N_f of faces $f \leq \mathcal{P}$. We want to show that it is possible to take translated copies of the regions to form a tiling of space, where the region $R(N_f)$ is translated by lattice points x in a dilation of the face f. The set $\mathcal{X}(tf) \subseteq tf \cap \Lambda$ of these lattice points in tf for some sufficiently large integer $t \in \mathbb{Z}_{>0}$ is yet to be defined, it stands in strong relation to the sets X_K^C that we used in the construction of regions in Section 2. The aim of this section is to find the right definition of the sets $\mathcal{X}(tf)$ and to prove Theorem 1, which we recall here:

Theorem 1 (Tiling)**.** *Let $\mathcal{P} \subseteq V$ be a full dimensional lattice polytope. There exists a $t_0 \in \mathbb{Z}_{>0}$ such that for each $t \geq t_0$ we have a tiling of V into translated regions of the form*

$$\{x + R(N_f) : \ f \leq \mathcal{P}, \, x \in \mathcal{X}(tf)\}.$$

For an example of such a tiling, see the left of Figure 2. An intermediate result of the proof is the following: If we start with just one pointed rational cone C, we get a tiling of space by taking translates of the regions $R(K)$ for all $K \leq C$. This result is given in Lemma 2. But before we start with that, we need another technical observation about a certain periodicity in the construction of the regions, namely Lemma 1.

As in the preceding section, for a pointed rational cone C, we write $L(C) := \Lambda \cap C^\perp$ for the induced lattice in the orthogonal space of C and let $T(C)$ be an arbitrary but fixed fundamental domain of $L(C)$.

Lemma 1. *Let C be a pointed rational cone in V. Then for all $y \in L(C)$ we have*

$$y + \left(V \setminus \bigcup_{K<C} \left(X_K^C + R(K) \right) \right) = V \setminus \bigcup_{K<C} \left(X_K^C + R(K) \right).$$

In other words, the construction of the region $R(C)$ in Equation (9) is invariant under translation of points in the lattice $L(C)$.

Proof. Let $y \in L(C)$. Since translation by y is a bijection, it commutes with unions and complements:

$$y + \left(V \setminus \bigcup_{K<C} \left(X_K^C + R(K) \right) \right) = \left(V \setminus \bigcup_{K<C} \left(X_K^C + y + R(K) \right) \right)$$

In order to proof the lemma, we only need to show that X_K^C is invariant under translation by y, i.e. $y + X_K^C = X_K^C$ for all faces $K < C$.

Thus, let $K < C$ be a face of C and $x \in X_K^C$, i.e. x meets conditions (I) and (II). We want to show that $x + y \in X_K^C$. Since $y \in C^\perp$ and $C^\perp + C^\vee = C^\vee$, we have that

$$x + (R(K) \cap \operatorname{int}(K^\vee)) \subseteq \operatorname{int}(C^\vee) \Leftrightarrow x + y + (R(K) \cap \operatorname{int}(K^\vee)) \subseteq \operatorname{int}(C^\vee)$$

which means that $x + y$ meets condition (I).

Now let $K' < C$ be a face of C such that K and K' are incomparable and let $x' \in L(K')$ (cf. Condition (II)). Since $y \in L(C) \subseteq L(K')$, we get:

$$\begin{aligned}
(x + y + R(K)) \cap (x' + R(K')) &= (x + y + R(K)) \cap (x' + y - y + R(K')) \\
&= y + \left((x + R(K)) \cap (\underbrace{x' - y}_{\in L(K')}) + R(K') \right) \\
&= \varnothing.
\end{aligned}$$

The last step follows from the fact that x meets Condition (II) and the whole equation shows that $x + y$ also meets condition (II). Altogether, we have shown that $x + y \in X_K^C$ for all $x \in X_K^C$ and all $y \in L(C)$. That means $y + X_K^C \subseteq X_K^C$. Conversely, let $x \in X_K^C$, then $x = x - y + y$. As with $y \in L(C)$ we also have $-y \in L(C)$, we can use that $x - y \in X_K^C$ and hence $x \in y + X_K^C$. We thus get $X_K^C \subseteq y + X_K^C$ which finishes the proof. ☐

Lemma 2. *For any pointed rational cone C we have a tiling*

$$\{x + R(C) : \ x \in L(C)\} \ \cup \ \{x + R(K) : \ K < C, \ x \in X_K^C\}$$

of V consisting of lattice point translates of regions corresponding to C and its faces.

Proof. In general, for a lattice L in V (not necessarily of full rank) and subsets $A, B \subseteq V$ with the properties that $B + L = V$ and $A + L = A$ we have that

$$L + (A \cap B) = A. \tag{15}$$

To show this, we show both inclusions:

- $\subseteq$: $L + (A \cap B) \subseteq L + A = A$
- $\supseteq$: Let $x \in A$. Since $V = L + B$, we can write $x = l + b$ with $l \in L$ and $b \in B$. Then $b = x - l \in A + L = A$ and hence $x \in L + (A \cap B)$.

From (15) and using Lemma 1 we can deduce:

$$\begin{aligned}
L(C) + R(C) &= \underbrace{L(C)}_{L} + \left[\underbrace{\left(V \setminus \bigcup_{K<C} (X_K^C + R(K)) \right)}_{A} \cap \underbrace{(T(C) + \operatorname{lin}(C))}_{B} \right] \\
&= V \setminus \bigcup_{K<C} (X_K^C + R(K)).
\end{aligned} \tag{16}$$

Thus, we get

$$V = (L(C) + R(C)) \cup \bigcup_{K<C} \left(X_K^C + R(K) \right).$$

Since $R(C) \subseteq T(C) + \operatorname{lin}(C)$, we know that the translates of $R(C)$ by points in $L(C)$ do not intersect. For each $K < C$, the set X_K^C is a subset of $L(K)$, so the same argument shows that the sets $\{x + R(K) : \ x \in X_K^C\}$ have pairwise empty intersections. Now we only need to show that for two faces $K, K' < N$ the sets of the form $x + R(K)$ and $y + R(K')$ with $x \in X_K^C$ and $y \in X_{K'}^C$ do not intersect. If K or K' is a face of the other one, say $K' < K$, we have:

$$\begin{aligned}
X_K^C + R(K) &\subseteq L(K) + R(K) \\
&\overset{(16)}{=} V \setminus \left(\bigcup_{M<K} (X_M^K + R(M)) \right) \\
&\subseteq V \setminus \left(X_{K'}^K + R(K') \right) \\
&\subseteq V \setminus \left(X_{K'}^C + R(K') \right).
\end{aligned}$$

The last inclusion follows from the property that $X_{K'}^C \subseteq X_{K'}^K$ whenever $K' < K < C$. If K and K' are incomparable in the face lattice, then $X_K^C + R(K)$ and $X_{K'}^C + R(K')$ do not intersect by construction of X_K^C, Property II. $\square$

For a pointed rational cone C, Lemma 2 yields a tiling of space into copies of translated regions $R(K)$, for $K \leq C$. In particular, given a full dimensional lattice polytope $\mathcal{P}$, we have a tiling of V for each normal cone N_v with v a vertex of $\mathcal{P}$. In Figure 8, these tilings are given for the triangle S that is shown in Figure 1 and was formally introduced in Section 3. Comparing these tilings to the tiling in Figure 2 on the left, that we want to construct for Theorem 1, one might already get an idea of how to achieve this goal: For all vertices v of $\mathcal{P}$, we take the tilings from Lemma 2 applied to all N_v and translate each by tv. In this joint tiling, we disregard all translates of regions $R(N_f)$, with $f \leq \mathcal{P}$ not a vertex, that do not fit.

Hence, for a face $f \leq \mathcal{P}$, the correct way of defining $\mathcal{X}(tf) \subseteq \Lambda \cap tf$, the *set of all feasible lattice points in* tf, is the following:

$$\mathcal{X}(tf) := \bigcap_{v \text{ vertex of } f} X_{N_f}^{N_v} + tv. \tag{17}$$

By setting $X_{N_v}^{N_v} = \{0\}$ for a vertex v of $\mathcal{P}$, this definition is also valid for $\mathcal{X}(tv)$ and yields $\mathcal{X}(tv) = tv$ as desired.

Before we start with the proof of Theorem 1, we need the following lemma:

Lemma 3. *Let C be a pointed rational cone in V and $R(C)$ the corresponding region. Then $P(C) := R(C) \cap C^\vee$ is bounded.*

This property is very important for the proof of Theorem 1, since without $R(N_{v_i}) \cap N_{v_i}^\vee$ being bounded, there is no chance to fit the tilings from Lemma 2 into one tiling. Though the statement of Lemma 3 might seem rather apparent, its proof is quite technical. Assuming Lemma 3 for the moment, we now give the proof of Theorem 1. To shorten notation, we will write $R(f)$, $P(f)$ and X_g^f instead of $R(N_f)$, $P(N_f)$ and $X_{N_g}^{N_f}$, respectively, but keep in mind that these sets do not depend on the faces, but only the normal cones.

Proof of Theorem 1. Let $\mathcal{P}$ be a full-dimensional lattice polytope with vertices $v_1, \ldots, v_m \in \Lambda$. We start by specifying t_0. Lemma 3 yields that $P(f)$ is bounded for each $f \leq \mathcal{P}$. Thus, for any $f, g \leq \mathcal{P}$ that do not intersect, there is a $t_{fg} \in \mathbb{Z}_{>0}$ such that $(P(f) + t_{fg} \cdot f) \cap (P(g) + t_{fg} \cdot g) = \varnothing$. We set

$$t_0 = \max\{t_{fg} : f, g \leq \mathcal{P} \text{ and } f \cap g = \varnothing\}.$$

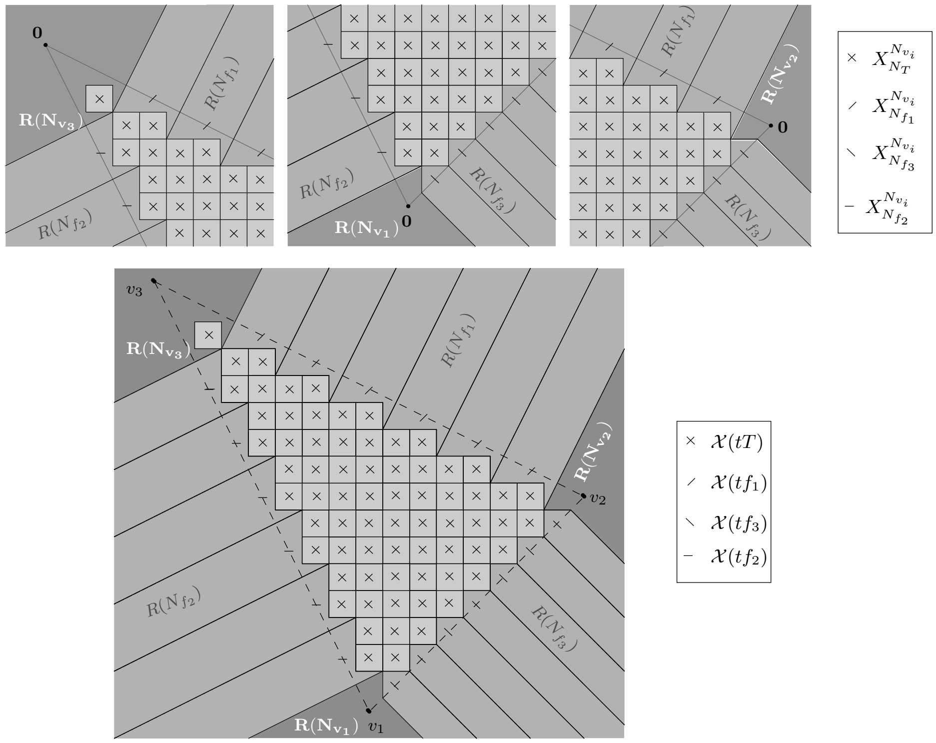

FIGURE 8.

This yields that $(R(f)+t\cdot f)\cap(R(g)+t\cdot g)=\varnothing$ for non-intersecting $f,g\leq\mathcal{P}$ and all $t\geq t_0$.

First, we show that the translated regions

$$\{x+R(f):\ f\leq\mathcal{P},\, x\in\mathcal{X}(tf)\} \tag{18}$$

are pairwise disjoint and in a second step we prove that they cover the whole space. Let $f,g\leq\mathcal{P}$ be arbitrary faces of $\mathcal{P}$, let $t\in\mathbb{Z}_{\geq 0}$ with $t\geq t_0$ and let $x\in\mathcal{X}(tf)$ and $y\in\mathcal{X}(tg)$. For $f\cap g=\varnothing$ it follows from the construction of t_0 that $(x+R(f))\cap(y+R(g))=\varnothing$. Otherwise, we find $j\in\{1,\dots,m\}$ such that $v_j\in f\cap g$. Then, since $x\in\mathcal{X}(tf)$ and $y\in\mathcal{X}(tg)$, we have $(x-tv_j)\in X_f^{v_j}$ and $(y-tv_j)\in X_g^{v_j}$. By Lemma 2 for N_{v_j} we get that the sets $(x-tv_j)+R(f)$ and $(y-tv_j)+R(g)$ do not intersect, which then also holds for the translations $x+R(f)$ and $y+R(g)$.

It remains to show that (18) is indeed a covering of the whole space. To this end, let $p\in V$ be an arbitrary point. Lemma 2 yields that

$$V=\bigcup_{\substack{f\leq\mathcal{P}\\ v_i\in f}}\Big(X_f^{v_i}+tv_i+R(f)\Big)$$

for each vertex v_i of $\mathcal{P}$. Hence, for each $i\in\{1,\dots,m\}$ we find $f_i\leq\mathcal{P}$ with $v_i\in f_i$ and $x_i\in X_{f_i}^{v_i}$ such that $p\in(x_i+tv_i+R(f_i))$.

Let v_i be a vertex, such that f_i is smallest in dimension. Without loss of generality we can assume $i=1$, hence, $p\in(x_1+tv_1+R(f_1))$. We want to show that

$x_1 + tv_1 \in \mathcal{X}(tf_1)$, because then $x_1 + tv_1 + R(f_1)$ is an element of the set in (18) and contains p.

Let's assume this is not the case. After possibly renumbering, we can assume that $(x_1 + tv_1) \notin (X_{f_1}^{v_2} + tv_2)$. In particular, we have $v_2 \in f_1$.

But then we can find $f_2 \leq \mathcal{P}$ with $v_2 \in f_2$ and $x_2 \in X_{f_2}^{v_2}$ such that $p \in (x_2 + tv_2 + R(f_2))$. This yields

$$p \in (x_2 + tv_2 + R(f_2)) \cap (x_1 + tv_1 + R(f_1)),$$

and thus

$$(x_2 + R(f_2)) \cap (\underbrace{x_1 + tv_1 - tv_2}_{\in L(f_1)} + R(f_1)) \neq \varnothing, \tag{19}$$

which contradicts $x_2 \in X_{f_2}^{v_2}$ by Property (II), unless f_1 and f_2 are comparable.

The case $f_2 \subsetneq f_1$ is not possible, since $\dim(f_2) \geq \dim(f_1)$ by assumption on the minimality of the dimension of f_1. The case $f_1 = f_2$ is not possible either, since $x_2 + tv_2 + R(f_1)$ and $x_1 + tv_1 + R(f_1)$ can only intersect if $x_2 + tv_2 = x_1 + tv_1$, in which case $x_1 + tv_1 \in (X_{f_2}^{v_2} + tv_2)$.

We are left with the case $f_1 \subseteq f_2$ to be excluded. We now can consider $X_{f_2}^{f_1}$ and have the inclusion $X_{f_2}^{v_2} \subseteq X_{f_2}^{f_1}$ (since we only add conditions when going from $X_{f_2}^{f_1}$ to $X_{f_2}^{v_2}$). Since $x_2 \in X_{f_2}^{v_2}$, we also have $x_2 \in X_{f_2}^{f_1}$. Then the sets $(x_1 + tv_1 - tv_2 + R(f_1))$ and $(x_2 + R(f_2))$ are part of the tiling that we get by applying Lemma 2 to N_{f_1}. But, as we see in equation (19), the two sets do intersect, which is a contradiction. □

For proving Lemma 3, we use the following linear algebra fact which we will use to assume a certain distance between points on faces of a cone. Its proof is straightforward using standard arguments and is omitted here.

Lemma 4. *Let $V_1, V_2 \subseteq V$ be subspaces of V with intersection $U := V_1 \cap V_2$. Then for all $r > 0$ there exists $\delta > 0$ such that for all $x \in V_1$*

$$\operatorname{dist}(x, U) > \delta \quad \Rightarrow \quad \operatorname{dist}(x, V_2) > r.$$

We prove Lemma 3 inductively. The idea is easy: The regions are constructed by cutting something off. So a rather obvious approach to show that $R(C) \cap C^\vee$ is bounded, is to show that every point in $C^\vee$ which is far enough away from 0 is contained in a region that is in the complement of $R(C)$. The complement consists of all regions corresponding to faces $K < C$ and translated by a lattice point in $L(K)$ that fulfills the properties (I) and (II) for being in X_K^C. Therefore, we consider two cases, the first being that a point is away from the boundary of $C^\vee$. Then it will be easy to see that there is a fundamental domain $T(C_0) = R(C_0)$ of Λ whose translate contains the point and is in the complement of $R(C)$. If a point p is close to the boundary and far enough away from $C^\perp$, we can show that close to that point we find a lattice point x on the boundary and a region $R(K)$, $K < C$, such that $x + R(K)$ is in the complement of $R(C)$. We then face the obstacle that it is still unclear whether p is in that particular region, since our regions are not even convex. The solution is to ensure that not only $x + R(K)$ is in the complement, but also all regions that cover the area around it.

Proof of Lemma 3. Let C be a rational pointed cone. We prove the lemma by induction on $n := \dim(C)$. We want to show that there exists a certain bounded set that contains $R(C) \cap C^\vee$. The bounded sets we want to consider are cylinders around the linear space $C^\perp$ with radius $r_C \in \mathbb{R}_{>0}$:

$$\mathrm{Cyl}(r_C, C) := \left(C^\perp + B_{r_C}(0)\right) \cap (T(C) + \mathrm{lin}(C)),$$

where $B_r(x)$ is the open ball around $x \in V$ with radius $r \in \mathbb{R}_{>0}$. Since $T(C) \subseteq C^\perp$, we can alternatively describe $\mathrm{Cyl}(r_C, C)$ as

$$\mathrm{Cyl}(r_C, C) = T(C) + (B_{r_C}(0) \cap \mathrm{lin}(C)).$$

The case $\mathbf{n = 0}$ is simply noticing that $R(C_0) = R(C_0) \cap C_0^\vee$ is a fundamental domain of the lattice $\Lambda = L(C_0)$ and as such it is by definition bounded.

For $\mathbf{n > 0}$, let $C \subseteq V$ be a pointed rational cone of dimension $n > 0$ and we assume that $R(K) \cap K^\vee \subseteq \mathrm{Cyl}(r_K, K)$ for all faces $K < C$ with suitable $r_k \in \mathbb{R}_{>0}$.

By construction, $R(C) \subseteq (T(C) + \mathrm{lin}(C))$. What we now need to show is that $R(C) \cap C^\vee \subseteq \left(C^\perp + B_{r_C}(0)\right)$ for some $r_C \in \mathbb{R}_{>0}$. We thus need to find an r_C that is 'big enough'. In detail we choose r_C defined by the following constraints that might seem technical, but it has the advantage to be constructive.

Construction of r_C:

For each face $K < C$ we have $R(K) \cap K^\vee \subseteq \mathrm{Cyl}(r_K, K)$ by the inductive hypothesis. We define a second radius r'_K by

$$r'_K = r_K + 2 \cdot \max_{M<K} r_M.$$

We recall that by Lemma 2 we have a tiling of the space

$$V = (L(K) + R(K)) \cup \bigcup_{M<K} \left(X_M^K + R(M)\right).$$

So there exists an $s \in \mathbb{R}_{>0}$ such that

$$U(K) := (R(K) \cap K^\vee) \cup \bigcup_{M<K} \bigcup_{x \in X_M^K \cap B_s(0)} x + (R(M) \cap M^\vee) \tag{20}$$

contains $\mathrm{Cyl}(r'_K, K) \cap K^\vee$. Since $U(K)$ is a finite union of bounded sets, $U(K)$ itself is bounded and we can find $u_K \in \mathbb{R}_{>0}$ such that $U(K) \subseteq B_{u_K}(0)$.

Then for $K_1, K_2 < C$ with $K_1 \vee K_2 = C$ implies $K_1^\perp \cap K_2^\perp = C^\perp$ and we can apply Lemma 4: We find $\lambda_C \in \mathbb{R}_{>0}$ such that $\mathrm{dist}(x, K_2^\perp) > u_{K_1} + u_{K_2}$ for all $x \in K_1^\perp$ with $\mathrm{dist}(x, C^\perp) > \lambda_C$. Let $t_K \in \mathbb{R}_{>0}$ be a radius such that the fundamental domain $T(K) \subset B_{t_K}(0)$ and set

$$h_C := \lambda_C + \max_{K<C}\{t_K\}. \tag{21}$$

Then we define

$$r_C := \max_{K<C} \sqrt{h_C^2 + (r'_K)^2}. \tag{22}$$

To show that $R(C) \cap C^\vee \subseteq \left(C^\perp + B_{r_C}(0)\right)$, we show that for each point $p \in C^\vee$ with $\mathrm{dist}(p, C^\perp) > r_C$ there is a face $K < C$ and a lattice point $x \in X_K^C$ such that $p \in (x + R(K)) \subseteq V \setminus R(C) \subseteq V \setminus \left(R(C) \cap C^\vee\right)$.

So let $p \in C^\vee$ with $\mathrm{dist}(p, C^\perp) \geq r_C$. We consider two cases. Figuratively speaking, we consider the case of p being far away from the boundary of $C^\vee$ (in which case we can just find a translate of $R(C_0)$ that covers it), and the case p

being close to the boundary (where we use a translate of $U(K)$ as defined in (20) to cover it by a translated region).

Case 1: $\forall K < C$ with $\dim(K) \geq 1$, we have $\operatorname{dist}(p, K^\perp) \geq r'_K$.

Let $x \in \Lambda$ with $p \in x + R(C_0)$. Since $R(C_0) = T(C_0) \subseteq B_{r_{C_0}}(0)$, we get

$$\operatorname{dist}(x + T(C_0), K^\perp) \geq r'_K - 2r_{C_0} \geq r_K > 0$$

for all $K < C$ with $\dim(K) \geq 1$. Hence, $(x + R(C_0)) \subseteq \operatorname{int}(C^\vee)$, which means $x \in X^C_{C_0}$ and $p \in (x + R(C_0)) \subseteq V \setminus (R(C) \cap C^\vee)$.

Case 2: $\exists K < C$, $\dim(K) \geq 1$ with $\operatorname{dist}(p, K^\perp) < r'_K$.

Let $K < C$ be the face of C with maximal dimensionsuch that $\operatorname{dist}(p, K^\perp) < r'_K$. Define $y := p|_{K^\perp}$ to be the orthogonal projection of p onto $K^\perp$ and let $x \in L(K)$ with $y \in x + T(K)$.

As **first step** we want to show that $x \in X^C_K$.

From $\operatorname{dist}(p, K^\perp) < r'_K$ by the Pythagorean theorem we can deduce

$$\operatorname{dist}(y, C^\perp)^2 + \operatorname{dist}(p, K^\perp)^2 = \operatorname{dist}(p, C^\perp)^2$$

ans thus

$$\operatorname{dist}(y, C^\perp)^2 = \operatorname{dist}(p, C^\perp)^2 - \operatorname{dist}(p, K^\perp)^2 > h_C^2 + (r'_K)^2 - (r'_K)^2 = h_C.$$

The last line follows from the premise that $\operatorname{dist}(p, C^\perp) \geq r_C$ and from the definition of r_C in Equation (4). Since we have now shown that $\operatorname{dist}(y, C^\perp)^2 > h_C$, we have by definition of h_C in (21) that $\operatorname{dist}(x, C^\perp) > \lambda_C$ and thus by definition of λ_C

$$\operatorname{dist}(x, M^\perp) > u_K + u_M \tag{23}$$

for all $M < C$ with $K \vee M = C$.

We now want to show the same result for all $M < C$ such that K and M are incomparable and $K \vee M < C$. By maximality of K we get

$$\operatorname{dist}(p, (K \vee M)^\perp) > r'_{K\vee M} > r_{K\vee M}.$$

Let $z \in L(M)$. With exactly the same computations as above (with $K \vee M$ instead of C) we get

$$\begin{aligned} &\operatorname{dist}(y, (K \vee M)^\perp)^2 > h_{K\vee M} \\ \Rightarrow\ &\operatorname{dist}(x, (K \vee M)^\perp) > \lambda_{K\vee M} \\ \Rightarrow\ &\operatorname{dist}(x, M^\perp) > u_K + u_M. \end{aligned} \tag{24}$$

That means we have

$$\operatorname{dist}(x, M^\perp) > u_K + u_M \tag{25}$$

for all $M < C$ incomparable to K. Since $u_M > 0$ and $x + (R(K) \cap K^\vee) \subseteq B_{u_K}(x)$ (in other words $x + (R(K) \cap K^\vee)$ has a positive distance to all other faces of $C^\vee$), we immediately get that x fulfills Property (I) for being in X^C_K: $x + (R(K) \cap \operatorname{int}(K^\vee)) \subseteq C^\vee$.

Regarding Property (II) of X^C_K, from Equation (25) we get

$$(x + (R(K) \cap K^\vee)) \cap (z + (R(M) \cap M^\vee)) = \varnothing$$

for all $M < C$ with $K \vee M = C$. Since $(x + R(K)) \setminus (x + (R(K) \cap K^\vee)) \subseteq x + T(K) + K$, as well as $(z + R(M)) \setminus (z + (R(M) \cap M^\vee)) \subseteq z + T(M) + M$ and K, M are both normal cones of different faces of $C^\vee$, we also have

$$(x + R(K)) \cap (z + R(M)) = \varnothing$$

for all $z \in L(M)$. Hence, x also has Property (II) for being in X_K^C and we have $x \in X_K^C$, which finishes step one.

Now we have that $x + U(K)$ covers $x + (\mathrm{Cyl}(r'_K, K) \cap K^\vee)$, where the latter contains p. If $p \in (x + (R(K) \cap K^\vee))$, we are done, since we have just shown that $x \in X_K^C$ and hence $(x + R(K)) \subseteq V\backslash(R(C) \cap C^\vee)$. Otherwise, we have $p \in (a + R(K_1))$ for some $K_1 < K$ and $a \in X_{K_1}^K$.

The **second step** is to show that then $a \in X_{K_1}^C$ which yields $(a + R(K_1)) \subseteq V\backslash(R(C) \cap C^\vee)$.

Again, we need to consider different cases. Firstly, we observe that

$$\big(a + (R(K_1) \cap K_1^\vee)\big) \cap \big(b + (R(M) \cap M^\vee)\big) = \varnothing$$

for all M incomparable to K (both for $K \vee M < C$ and $K \vee M = C$) and all $b \in L(M)$: We have $(a + (R(K_1) \cap K_1^\vee)) \subseteq B_{u_K}(x)$ and $(b + (R(M) \cap M^\vee)) \subseteq B_{u_M}(b)$ and as we have seen in (25), we have $\mathrm{dist}(x, M^\perp)) > u_K + u_M$.

Secondly, we are left with the case $M < K$ and M, K_1 incomparable. But since we have $a \in X_{K_1}^K$, we get from Property (II) that $(a + (R(K_1) \cap K_1^\vee)) \cap (b + (R(M) \cap M^\vee)) = \varnothing$ for all $M < K$ with M, K_1 incomparable and all $b \in L(M)$. Hence, $a \in X_{K_1}^C$ and $p \in (a + R(K_1)) \subseteq V \backslash (R(C) \cap C^\vee)$ as we wanted to show.

Hence, we have shown that $R(C) \cap C^\vee$ is bounded. $\square$

5. Proof of Theorem 2 (Local formula)

Assume we have chosen and fixed fundamental domains for all sublattices $L \subseteq \Lambda$. We recall the definition of the function μ on rational cones that was given in Section 1. We first set

$$\mu(C_0) := v_{C_0} = 1 \tag{26}$$

for the trivial cone $C_0 = \{0\}$. For a pointed rational cone $C \subseteq V$ with $\dim(C) \geq 1$ we then define by induction on the dimension

$$\mu(C) := v_C - \sum_{K < C} w_K^C \cdot \mu(K). \tag{27}$$

Here, v_C is the DC-volume defined in (4) and w_K^C is the correction volume from (6).

For a rational cone $C \subseteq V$ that is not pointed, but contains a maximal nontrivial linear subspace U, we can consider the pointed cone $C' := C \cap U^\perp$ in $U^\perp$, where we consider $U^\perp$ as a Euclidean space equipped with the induced inner product and the lattice $\Lambda \cap U^\perp$. We can then construct $R(C') \subseteq U^\perp$ and set

$$\mu(C) := \mu(C').$$

Theorem 2. *The function μ on rational cones in V as defined in* (26) *and* (27) *is a local formula for Ehrhart coefficients.*

That is, for every lattice polytope $\mathcal{P}$ with Ehrhart polynomial $\mathrm{E}_{\mathcal{P}}(t) = e_d t^d + e_{d-1} t^{d-1} + \cdots + e_1 t + e_0$, $t \in \mathbb{Z}_{\geq 0}$, we have

$$e_i = \sum_{\substack{f \leq \mathcal{P} \\ \dim(f) = i}} \mu(N_f) \mathrm{vol}(f),$$

for all $i \in \{0, \dots, d\}$.

As discussed in Section 1, this is equivalent to

$$|\Lambda \cap t\mathcal{P}| = \sum_{f \le \mathcal{P}} \mu(N_f)\mathrm{vol}(tf)$$

for all lattice polytopes $\mathcal{P}$.

Before we prove Theorem 2, we make some simplifications. Since neither the Ehrhart polynomial, nor the function μ, nor the relative volumes of faces change when we translate $\mathcal{P}$ by a lattice point, we can without loss of generality assume that $0 \in \mathcal{P}$. Then $\mathrm{lin}(\mathcal{P}) = \mathrm{aff}(\mathcal{P})$ and each normal cone N_f with f a face of $\mathcal{P}$ contains the orthogonal space $P^{\perp}$ as a maximal linear subspace. The definition thus yields $\mu(N_f) = \mu(N_f \cap \mathrm{lin}(\mathcal{P}))$ for all faces f of $\mathcal{P}$. By considering $\mathcal{P} \subseteq \mathrm{lin}(\mathcal{P})$, we can hence assume, without loss of generality, that $\mathcal{P}$ is full dimensional and that all normal cones are pointed. As before, to shorten notation, for faces $f \le \mathcal{P}$ we henceforth write $R(f)$ instead of $R(N_f)$, also $L(f)$ instead of $L(N_f)$ for the sublattice $\Lambda \cap N_f^{\perp} \subseteq \Lambda$ and $T(f)$ for the fundamental domain $T(N_f)$ in $L(f)$. We also write v_f and w_f^g instead of v_{N_f} and $w_{N_f}^{N_g}$ for faces $g < f \le \mathcal{P}$. But we keep in mind, that these objects do not depend on the face f itself, but only on the normal cone N_f. Note that $N_f \le N_g$ if and only if $g \le f$.

Proof of Theorem 2. To make the structure of the proof easier to grasp, we delay some steps into Lemmas that we will state and prove afterwards.

Let $\mathcal{P}$ be a full dimensional lattice polytope. Recall that T is a fundamental domain of Λ. Since the relative volume is normalized, such that every fundamental domain has volume 1, we have the following equation for every $t \in \mathbb{Z}_{\ge 0}$:

$$|t\mathcal{P} \cap \Lambda| = \mathrm{vol}((t\mathcal{P} \cap \Lambda) + T). \tag{28}$$

Instead of counting the (discrete) number of lattice points in $t\mathcal{P}$, we thus can compute the (continuous) volume of fundamental domains around each lattice point in $t\mathcal{P}$. Following the notation of Section 1, the right hand side of Equation (28) is the volume of the domain complex of $t\mathcal{P}$.

Let $\mathcal{P}$ be a polytope and $t \in \mathbb{Z}_{>0}$ big enough, such that we have a tiling of V by regions as in Theorem 1:

$$\{x + R(N_f) : \ f \le \mathcal{P},\, x \in \mathcal{X}(tf)\},$$

with $\mathcal{X}(tf)$ the set of feasible lattice points in tf as defined in Section 4, Equation (17). Then, as we will show in Lemma 5 below, we can divide the volume of the domain complex into the parts in each region, which equals v_f (the DC-volume in R(F)):

$$\mathrm{vol}((t\mathcal{P} \cap \Lambda) + T) = \sum_{f \le \mathcal{P}} |\mathcal{X}(tf)| \cdot v_f.$$

Thus with the definition of $\mu(N_f)$ solved for v_f we get

$$\begin{aligned} |\Lambda \cap t\mathcal{P}| &= \sum_{f \le \mathcal{P}} |\mathcal{X}(tf)| \cdot v_f \\ &= \sum_{f \le \mathcal{P}} \left[|\mathcal{X}(tf)| \cdot \left(\mu(f) + \sum_{h > f} w_h^f \cdot \mu(h) \right) \right] \end{aligned}$$

We can now expand the product and combinatorially rearrange the sum to get

$$\begin{aligned} |\Lambda \cap t\mathcal{P}| &= \sum_{f \leq \mathcal{P}} \left[|\mathcal{X}(tf)| \cdot \mu(f) + |\mathcal{X}(tf)| \cdot \sum_{h > f} w_h^f \cdot \mu(h) \right] \\ &= \sum_{f \leq \mathcal{P}} \underbrace{\left[\sum_{g \leq f} |\mathcal{X}(tg)| \cdot w_f^g \right]}_{=:V(tf)} \cdot \mu(f) \end{aligned}$$

In the last line the expression w_f^f for the correction volume appears that technically has not been defined yet – we simply set $w_f^f := 1$ for faces $f \leq \mathcal{P}$.

By Lemma 8 below we can now use that

$$V(tf) = \mathrm{vol}(tf),$$

which yields

$$|t\mathcal{P} \cap \Lambda| = \sum_{f \leq \mathcal{P}} \mathrm{vol}(tf) \cdot \mu(f) \tag{29}$$

for all $t \in \mathbb{Z}_{\geq 0}$ with $t > t_0$ for a certain $t_0 \in \mathbb{Z}_{\geq 0}$. By Ehrhart's Theorem [8], we know that $\mathrm{E}_{\mathcal{P}}(t) = |t\mathcal{P} \cap \Lambda|$ is a polynomial in t, as is the right hand side of Equation (29). Since these polynomials agree for infinitely many t, we have equality and get

$$\mathrm{E}_{\mathcal{P}}(t) = \sum_{f \leq \mathcal{P}} \mu(N_f) \mathrm{vol}(tf)$$

for all $t \in \mathbb{Z}_{\geq 0}$. That shows that for each choice of fundamental domains, the resulting function μ is a local formula for Ehrhart coefficients.

□

Lemma 5. *We have*

$$\mathrm{vol}((t\mathcal{P} \cap \Lambda) + T) = \sum_{f \leq \mathcal{P}} |\mathcal{X}(tf)| \cdot v_f$$

for all $t \in \mathbb{Z}_{\geq 0}$ big enough in the sense of Theorem 1.

Proof. We recall the definition of $\mathcal{X}(tf)$ as

$$\mathcal{X}(tf) := \bigcap_{v \text{ vertex of } f} X_{N_f}^{N_v} + tv,$$

where $X_{N_f}^{N_v}$ is the set of lattice that we constructed in Section 2. For an illustration of the sets $X_{N_f}^{N_v}$ in a triangle see Figure 6 and for $\mathcal{X}(tf)$ see Figure 8.

Let $t \in \mathbb{Z}_{\geq 0}$ be big enough, such that by Theorem 1 we have a tiling of V into regions:

$$\{x + R(N_f) : \ f \leq \mathcal{P}, \, x \in \mathcal{X}(tf)\}.$$

To compute the Volume $\mathrm{vol}((t\mathcal{P} \cap \Lambda) + T)$ we can thus compute the volume in each region and add everything up:

$$\mathrm{vol}((t\mathcal{P} \cap \Lambda) + T) = \sum_{f \leq \mathcal{P}} \sum_{x \in \mathcal{X}(tf)} \mathrm{vol}\big((x + R(f)) \cap ((t\mathcal{P} \cap \Lambda) + T) \big). \tag{30}$$

Hence, for $f \leq \mathcal{P}$ it suffices to show that the volume on the right hand side of the equation equals the DC-volume of N_f, namely that

$$\operatorname{vol}\big((x + R(f)) \cap ((t\mathcal{P} \cap \Lambda) + T) \big) = v_f$$

for all $x \in \mathcal{X}(tf)$. Then Equation (30) yields

$$\begin{aligned}\operatorname{vol}((t\mathcal{P} \cap \Lambda) + T) &= \sum_{f \leq \mathcal{P}} \sum_{x \in \mathcal{X}(tf)} v_f \\ &= \sum_{f \leq \mathcal{P}} |\mathcal{X}(tf)| \cdot v_f\end{aligned}$$

as desired.

We recall the definition of the DC-volume v_f as:

$$v_f = \operatorname{vol}\big(R(f) \cap ((N_f^\vee \cap \Lambda) + T)\big).$$

Let $x \in \mathcal{X}(tf)$. Since $(t\mathcal{P} \cap \Lambda) \subseteq (x + N_f^\vee) \cap \Lambda$, we have

$$\big(x + R(f)\big) \cap \big((t\mathcal{P} \cap \Lambda) + T\big) \subseteq \big(x + R(f)\big) \cap \big(((x + N_f^\vee) \cap \Lambda) + T\big) \tag{31}$$

We first want to show that we have equality in (31). To do so, we show that

$$(x + R(f)) \cap (y + T) = \varnothing \tag{32}$$

for all $y \in ((x+N_f^\vee)\cap\Lambda)\setminus(t\mathcal{P}\cap\Lambda)$. $y \notin (t\mathcal{P}\cap\Lambda)$ means there is a vertex v of $\mathcal{P}$ with $y \notin (N_v^\vee + tv)$. By considering t large enough, we can ensure that Equation (32) holds for all y with $y \in \bigcap_{v \in f}(N_v^\vee + tv)$ and $y \notin (N_v^\vee + tv)$ for a vertex $v \in \mathcal{P}$ that is not a vertex of f. So let $y \in ((x + N_f^\vee) \cap \Lambda)$ with $y \notin (N_v^\vee + tv)$ for a vertex v of f. Then there is a facet F of $\mathcal{P}$ with $v \in F$ and $y \notin (N_F^\vee + tv)$. But then $(y - tv) + T \nsubseteq \operatorname{int}(N_F^\vee)$ and hence, $(y - tv + T) \subseteq L(F) + R(F)$ (cf. Equation (16) in the proof of Lemma 2). Since $x \in \mathcal{X}(tf)$, we have $(x - tv) \in X_{N_f}^{N_v}$ and thus $(x - tv + R(f)) \cap (y - tv + T) \neq \varnothing$, which is equivalent to (32). Hence, equality in Equation (31) follows.

Since $x \in \mathcal{X}(tf) \subseteq \Lambda$, we have $(x + N_f^\vee) \cap \Lambda = x + (N_f^\vee \cap \Lambda)$ and hence,

$$\begin{aligned}\operatorname{vol}\big((x + R(f)) \cap ((t\mathcal{P} \cap \Lambda) + T)\big) &= \operatorname{vol}\big((x + R(f)) \cap ((x + N_f^\vee) \cap \Lambda) + T\big) \\ &= \operatorname{vol}\big(R(f) \cap ((N_f^\vee \cap \Lambda) + T)\big) \\ &= v_f,\end{aligned}$$

as we wanted to show.

$\square$

Lemma 6 gives a general property of the regions and independent of a concrete polytope, so we use the general notation of cones.

Lemma 6. *For each pointed cone C the fundamental domain $T(C)$ in the linear space $C^\perp$ is contained in the region $R(C)$.*

Proof. We show the statement inductively. Let $C_0 = \{0\}$ be the 0-dimensional cone. Then by construction $R(C_0) = T(C_0)$ and the assertion holds. Now, let C be a 1-dimensional pointed cone. Then the corresponding region $R(C)$ is given by

$$R(C) = \big(V \setminus \big(X_{C_0}^C + R(C_0)\big)\big) \cap (T(C) + \operatorname{lin}(C))\,.$$

Since obviously $T(C) \subseteq (T(C) + \operatorname{lin}(C))$, we only need to show that $T(C) \subseteq V \setminus \left(X_{C_0}^C + R(C_0)\right)$. Therefore, let $p \in T(C)$. Then there is exactly one $x \in \Lambda$ with $p \in (x + R(C_0))$. Since $p \notin \operatorname{int}(C^\vee)$, we have $x + R(C_0) \nsubseteq C^\vee$ and thus $x \notin X_{C_0}^C$ by property (I) in the construction of $X_{C_0}^C$. Hence,

$$p \in (x + T(C_0)) \subseteq \left(V \setminus \left(X_{C_0}^C + R(C_0)\right)\right).$$

Now, let C be any pointed cone with $\dim(C) > 1$. We now have

$$R(C) = \left(V \setminus \bigcup_{K<C} \left(X_K^C + R(K)\right)\right) \cap (T(C) + \operatorname{lin}(C)).$$

Again, $T(C) \subseteq (T(C) + \operatorname{lin}(C))$. Let $p \in T(C)$. Assume we have a face $K < C$ and a lattice point $x \in L(K)$ with $p \in x + R(K)$. We again want to show that then $x \notin X_K^C$. Since $\dim(C) > 1$ we find another face $M < C$ incomparable to K with $\dim(M) = 1$ and $K \vee M = C$. The inclusion $T(C) \subseteq C^\perp \subseteq M^\perp$ yields $T(C) \subseteq L(M) + T(M)$. By induction we can assume $T(M) \subseteq R(M)$ and hence, $T(C) \subseteq L(M) + R(M)$. So for $p \in T(C)$, if $p \in (x + R(K))$ for some $x \in L(K)$, then there exists $y \in L(M)$, such that $p \in (x + R(K)) \cap (y + R(M))$ and hence, $x \notin X_K^C$ by Property (II) in the construction of X_K^C. Hence, we have shown, that $p \in \left(V \setminus \bigcup_{K<C} \left(X_K^C + R(K)\right)\right)$ for any $p \in T(C)$ and thus $T(C) \subseteq R(C)$. $\square$

Lemma 7. *There exists a $t_0 \in \mathbb{Z}_{>0}$ such that for each $t \geq t_0$ the dilation of a face $f < \mathcal{P}$ by t satisfies*

$$tf \subseteq \bigcup_{g \leq f} (\mathcal{X}(tg) + R(g)).$$

Proof. For t_0 big enough, Theorem 1 yields that

$$tf \subseteq \bigcup_{g \leq \mathcal{P}} (\mathcal{X}(tg) + R(g)) \tag{33}$$

for all $t \geq t_0$. We need to show that in (33) the translated regions of faces g of $\mathcal{P}$ with $g \nleq f$ do not intersect with tf. We can divide these faces into four groups: the faces g with $f \cap g = \varnothing$; the faces g with $f \cap g \neq \varnothing$ and f, g are incomparable; the face $g = \mathcal{P}$; and the faces g with $f < g < P$.

By choosing t_0 big enough (cf. proof of Theorem 1), we can ensure that $(x + R(g)) \cap tf = \varnothing$ for all $g \leq \mathcal{P}$ that do not intersect f and all $x \in \mathcal{X}(tg)$.

For the second case, let $g \leq \mathcal{P}$ with f, g incomparable and there exists a vertex v of $\mathcal{P}$ with $v \in f \cap g$. Let $x \in \mathcal{X}(tg)$. Then $(x - tv) \in X_g^v$. In particular (by Property (I)), that means $((x - tv) + (R(g) \cap \operatorname{int}(N_g^\vee)) \subseteq \operatorname{int}(N_v^\vee)$. But since $tf - tv$ is on the boundary of $N_v^\vee$ and not in $N_g^\perp$, we get that $(x - tv + R(g)) \cap (tf - tv) = \varnothing$ and hence, $(x + R(g)) \cap tf = \varnothing$.

For the case $g = \mathcal{P}$ we note that $R(g) = T$ and then $(x + R(g)) \cap tf = \varnothing$ follows by exactly the same arguments as in the second case.

In the fourth case, we consider g with $f < g < \mathcal{P}$. Then there exists a facet F of $\mathcal{P}$ with $f \subseteq g \cap F$ and g, F incomparable. By Lemma 6 we have that $T(F) \subseteq R(F)$. Let v be a vertex of $\mathcal{P}$ with $v \in f$ and let $x \in \mathcal{X}(tg)$. Then $(x - tv) \in X_g^v$ and thus, by Property (II) we have $(y + R(F)) \cap (x - tv + R(g)) = \varnothing$ for all $y \in L(F)$ and in particular

$$(y + T(N_F)) \cap (x - tv + R(g)) = \varnothing \tag{34}$$

for all $y \in L(N_F)$. Since $f \subseteq F$, we have $tf \subseteq tv + L(F) + T(F)$ which, together with (34) shows that $(x + R(g)) \cap tf = \varnothing$.

Hence, we have shown that for all faces g with $g \not\leq f$ the intersection $tf \cap (\mathcal{X}(tg) + R(g))$ is empty and hence, $tf \subseteq \bigcup_{g \leq f} (\mathcal{X}(tg) + R(g))$ as we wanted to show. ◻

Lemma 8. *There exists a $t_0 \in \mathbb{Z}_{>0}$ such that for each $t \geq t_0$ and every face $f < \mathcal{P}$*

$$\mathrm{vol}(tf) = \sum_{g \leq f} |\mathcal{X}(tg)| \cdot w_f^g.$$

In other words, the volume of tf is given by the number of feasible lattice points in tf, $\mathcal{X}(tf)$, plus the correction volumes for each face $g < f$.

Proof. We recall that w_f^g is defined in (6) by

$$w_f^g := \mathrm{vol}\left(R(g) \cap N_f^{\perp} \cap N_g^{\vee}\right).$$

From Lemma 7 we can deduce

$$\begin{aligned} \mathrm{vol}(tf) &= \mathrm{vol}\left(\bigcup_{g \leq f} ((\mathcal{X}(tg) + R(g)) \cap tf)\right) \\ &= \sum_{g \leq f} \sum_{x \in \mathcal{X}(tg)} \mathrm{vol}\left((x + R(g)) \cap tf\right). \end{aligned}$$

For $g \leq f$ we have

$$\mathrm{vol}\left((x + R(g)) \cap tf\right) = \mathrm{vol}\left(R(g) \cap (-x + tf)\right)$$

and since $(-x + tf) \subseteq (N_f^{\perp} \cap N_g^{\vee})$ and $[(N_f^{\perp} \cap N_g^{\vee}) \backslash (-x + tf)] \cap R(g) = \varnothing$, we get

$$\mathrm{vol}\left(R(g) \cap (-x + tf)\right) = w_f^g$$

and hence,

$$\begin{aligned} \mathrm{vol}(tf) &= \sum_{g \leq f} \sum_{x \in \mathcal{X}(tg)} \mathrm{vol}\left((x + R(g)) \cap tf\right) \\ &= \sum_{g \leq f} |\mathcal{X}(tg)| \cdot w_f^g \end{aligned}$$

as we wanted to show. ◻

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Institute for Mathematics, University of Rostock, 18051 Rostock, Germany
E-mail address: maren.ring@uni-rostock.de
E-mail address: achill.schuermann@uni-rostock.de