STABLE SYSTEMS OF COMPETING LÉVY PARTICLES

CLAYTON BARNES AND ANDREY SARANTSEV

Abstract. Take a finite system of Lévy particles on the real line. Each particle moves as a Lévy process according to its current rank relative to other particles. We find a natural sufficient condition for convergence to a stationary distribution, which guarantees all particles move together, as opposed to eventually splitting into two or more groups. This extends the research of (Banner, Fernholz, Karatzas, 2005) to systems without jumps (Brownian particles).

1. Introduction and the Main Results

1.1. **Definition of systems of competing Lévy particles.** Assume the usual setting: a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{P})$, with the filtration satisfying the usual conditions. Fix $N \ge 2$, the number of particles. Take a Lévy process $L = (L(t),\, t \ge 0)$ in $\mathbb{R}^N$ which makes a.s. finitely many jumps over any finite time interval. This process can be characterized by a triple (g, A, Λ), where $g \in \mathbb{R}^N$ is a *drift vector*, A is an $N \times N$ positive definite symmetric *covariance matrix*, and Λ is a finite *Lévy measure* on $\mathbb{R}^N$. The process L behaves as an N-dimensional Brownian motion with drift vector g and covariance matrix A, with the possibility that it jumps. The times of jumps form a Poisson point process on the half-line $\mathbb{R}_+ := [0,\infty)$ with intensity $\lambda_0 = \Lambda(\mathbb{R}^N)$. Each jump has an independent displacement distributed according to the normalized measure $\lambda_0^{-1}\Lambda(\cdot)$ on $\mathbb{R}^N$. Now, consider an $\mathbb{R}^N$-valued continuous adapted process

$$X = (X(t),\, t \ge 0),\ X(t) = (X_1(t), \ldots, X_N(t))\,,$$

and rank components at every time $t \ge 0$: $X_{(1)}(t) \le \ldots \le X_{(N)}(t)$. For a vector $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, define the *ranking permutation* $\mathbf{p}_x$. This is a unique bijection $\{1, \ldots, N\} \to \{1, \ldots, N\}$ which has the following properties:

(a) $x_{\mathbf{p}_x(i)} \le x_{\mathbf{p}_x(j)}$ for $1 \le i < j \le N$;

(b) if $1 \le i < j \le N$ and $x_{\mathbf{p}_x(i)} = x_{\mathbf{p}_x(j)}$, then $\mathbf{p}_x(i) < \mathbf{p}_x(j)$.

Assume the process X satisfies the following system of SDE:

$$\mathrm{d}X_i(t) = \sum_{k=1}^N 1\left(\mathbf{p}_{X(t)}(k) = i\right)\,\mathrm{d}L_k(t),\ \ i = 1, \ldots, N. \tag{1}$$

Then X is called a *system of competing Lévy particles, governed by a triple* (g, A, Λ). Each component X_i is called the *ith named particle*. The following existence and uniqueness result is proved in Section 2.

Lemma 1.1. *For any initial condition $X(0) = x \in \mathbb{R}^N$, the system* (1) *has a unique weak solution. These solutions form a Feller continuous strong Markov family.*

For each $k = 1, \ldots, N$, the following process is called the *kth ranked particle*:

$$Y_k = (Y_k(t),\, t \ge 0),\ \ Y_k(t) \equiv X_{\mathbf{p}_{X(t)}(k)}(t).$$

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The process $Y = (Y_1, \ldots, Y_N)$ is called the *ranked system*, and is a Markov process on the wedge

$$\mathcal{W} := \{y = (y_1, \ldots, y_N) \in \mathbb{R}^N \mid y_1 \le \ldots \le y_N\}.$$

In this model, jumps of ranked particles are, in general, dependent. Assume we wish that ranked particles made independent jumps, with the jumps of the kth ranked particle governed by a finite Borel measure ν_k, $k = 1, \ldots, N$. Then we need to take

$$\Lambda = \sum_{k=1}^{N} \Lambda_k, \quad \Lambda_k := \delta_0 \otimes \delta_0 \otimes \ldots \delta_0 \otimes \nu_k \otimes \delta_0 \otimes \ldots \otimes \delta_0. \tag{2}$$

Here, the Dirac point mass measure at $x \in \mathbb{R}$ is denoted by δ_x. For the measure Λ_k in (2), the multiple ν_k is on the kth place. Under condition (2), the kth ranked particle makes jumps with intensity $\lambda_k := \nu_k(\mathbb{R})$, and the displacement during each jump is distributed according to a probability measure $\lambda_k^{-1}\nu_k(\cdot)$. If, in addition,

$$A = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_N^2), \tag{3}$$

then the components $L_1, \ldots, L_N$ of the process L are independent. In this case, the kth ranked particle moves as a one-dimensional Lévy process $L_k = (L_k(t), t \ge 0)$ with drift coefficient g_k, diffusion coefficient σ_k^2, and finite Lévy measure ν_k. That is, the process L_k behaves as a Brownian motion with drift coefficient g_k, diffusion coefficient σ_k^2, except that it makes jumps with intensity $\lambda_k := \nu_k(\mathbb{R})$, and the displacement during each jump is distributed according to a probability measure $\lambda_k^{-1}\nu_k(\cdot)$.

In particular, if $\Lambda = 0$ there are no jumps. If in addition the matrix A is given by (3), then we have a system of *competing Brownian particles* $X = (X_1, \ldots, X_N)$, and the governing equation (1) takes the form

$$\mathrm{d}X_i(t) = \sum_{k=1}^{N} 1\left(\mathbf{p}_{X(t)}(k) = i\right)\,\left(g_k\,\mathrm{d}t + \sigma_k\,\mathrm{d}W_i(t)\right),\ i = 1, \ldots, N, \tag{4}$$

where $W_1, \ldots, W_N$ are i.i.d. Brownian motions. This model was introduced in [2] for the purposes of financial modeling, and has been a subject of extensive research during the last decade: [3, 16, 17, 33, 34]. Further financial applications can be found in [8, 13, 14, 21, 23]. In particular, in [8] these models were used to explain a remarkable property of real-world markets: stability and linearity of the log-log plot of ranked market weights. We discuss this in more detail later in this section.

Systems of competing Brownian particles can also be viewed as an approximation of a nonlinear diffusion process, governed by a McKean-Vlasov equation. In fact, as the number N of particles goes to infinity, the system converges to this nonlinear diffusion process, [10, 18, 19, 32, 39]. In addition, these systems can serve as scaling limits of exclusion processes, [22].

Systems of competing Lévy particles were introduced in [38] for the case (2) and (3), with $\nu_1 = \ldots = \nu_N$. Systems of $N = 2$ competing Lévy particles (in the general case) were also studied in [35]. Here, we construct them in a more general case, when the measures ν_k, $k = 1, \ldots, N$, are not necessarily the same.

1.2. **Main result: Stability of the system.** The paper is devoted to sufficient stability conditions. Informally, the system above is called *stable* if all particles move roughly together, instead of splitting into two or more "clouds." More formally, we introduce the *centered system*

$\overline{X} = (\overline{X}_1, \ldots, \overline{X}_N)$, defined as

$$\overline{X}_k(t) := X_k(t) - \frac{1}{N}\sum_{j=1}^N X_j(t), \quad k = 1, \ldots, N, \quad t \ge 0.$$

This is a Markov process with state space which is the hyperplane

$$\Pi := \{(x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_1 + \ldots + x_N = 0\}. \tag{5}$$

Definition 1. A probability measure π on Π is called a *stationary distribution* for $\overline{X}$ if $\overline{X}(0) \sim \pi$ implies $\overline{X}(t) \sim \pi$. The system X is called *stable* if the process $\overline{X}$ has a unique stationary distribution π, and converges to this distribution in the total variation distance as $t \to \infty$. That is, denoting by $\mathbf{P}_x$ the probability given $\overline{X}(0) = x$, we have:

$$\sup_{E \subseteq \Pi} \left|\mathbf{P}_x(\overline{X}(t) \in E) - \pi(E)\right| \to 0 \text{ as } t \to \infty. \tag{6}$$

For competing Brownian particles as in (4), this problem was solved in [2, 3]. In this case the system is stable if and only if the average drift of the k bottom particles is strictly greater than the average drift of all particles, for each $k = 1, \ldots, N-1$:

$$\frac{1}{k}\left(g_1 + \ldots + g_k\right) > \frac{1}{N}\left(g_1 + \ldots + g_N\right), \quad k = 1, \ldots, N-1. \tag{7}$$

In other words, these two groups of particles travel together, and do not separate into two distinct "clouds." For particles with jumps, it is natural to consider *effective drift coefficients*

$$m_k := g_k + f_k, \quad f_k := \int_{\mathbb{R}^N} z_k\, \Lambda(\mathrm{d}z), \quad k = 1, \ldots, N, \tag{8}$$

instead of drift coefficients $g_1, \ldots, g_N$. Indeed, the effective drift of the kth ranked particle is a combination of the *true drift* $g_k\,\mathrm{d}t$, corresponding to the diffusion part of the process, and of the drift created by jumps. On a unit time interval, the system makes on average $\lambda_0 = \Lambda(\mathbb{R}^N)$ jumps. During each jump, the displacements of the kth ranked particle, for $k = 1, \ldots, N$, form a vector distributed according to the probability measure $\lambda_0^{-1}\Lambda(\cdot)$. Therefore, the average displacement of the kth ranked particle is equal to

$$\int_{\mathbb{R}^N} z_k\, \left(\lambda_0^{-1}\,\Lambda(\mathrm{d}z)\right).$$

Multiplying this average displacement by the average quantity λ_0 of jumps, we get the term $\int_{\mathbb{R}^N} z_k\, \Lambda(\mathrm{d}z)$. This is the "average shift created by jumps." Adding it to the true drift coefficient g_k, we get (8). Assume that

$$\int_{\mathbb{R}^N} z_k^2\, \Lambda(\mathrm{d}z) < \infty \text{ for } \ k = 1, \ldots, N. \tag{9}$$

It is natural to guess that if a condition similar to (7) holds for competing Lévy particles:

$$\frac{1}{k}\left(m_1 + \ldots + m_k\right) > \frac{1}{N}\left(m_1 + \ldots + m_N\right), \quad k = 1, \ldots, N-1, \tag{10}$$

with effective drifts m_k instead of g_k, $k = 1, \ldots, N$, then the system is stable. That is, all particles stay together and do not split into clouds. The main result of this article is that this is indeed true.

Theorem 1.2. *Under conditions* (9) *and* (10)*, the system X is stable.*

The *gap process* is a process $Z = (Z(t),\, t \ge 0)$ in the positive orthant $\mathbb{R}_+^{N-1}$ defined by

$$Z(t) = (Z_1(t), \ldots, Z_{N-1}(t))\,, \quad Z_k(t) = Y_{k+1}(t) - Y_k(t), \quad k = 1, \ldots, N-1. \tag{11}$$

One can show similarly to [3] that this is a Markov process in $\mathbb{R}_+^{N-1}$. Since $Z(t)$ can be expressed as a continuous function of $\overline{X}(t)$, we have the following statement.

Corollary 1.3. *Under the conditions of Theorem* 1.2*, the gap process has a unique stationary distribution to which it converges as $t \to \infty$ in the total variation distance, similarly to* (6)*.*

From (8), after centering vectors g, f, m as in (16), we have:

$$\overline{m}_k = \overline{g}_k + \overline{f}_k, \quad k = 1, \ldots, N. \tag{12}$$

We can also rewrite (10) as

$$\min_{k=1,\ldots,N-1} \sum_{j=1}^{k} \overline{m}_j > 0. \tag{13}$$

For the case (2) of independent jumps, the expression (8) and the condition (9) take the form

$$m_k := g_k + f_k, \quad f_k := \int_{\mathbb{R}} z\, \nu_k(\mathrm{d}z), \quad \int_{\mathbb{R}} z^2\, \nu_k(\mathrm{d}z) < \infty \text{ for } k = 1, \ldots, N.$$

1.3. **Law of large numbers.** The next result follows from Theorem 1.2 and [12, Theorem 2.8].

Theorem 1.4. *Under the conditions of Theorem* 1.2*, for any bounded measurable function $f : \Pi \to \mathbb{R}$,*

$$\frac{1}{T}\int_0^T f(\overline{X}(t))\,\mathrm{d}t \to \int_{\Pi} f(z)\pi(\mathrm{d}z), \ \text{a.s. as } T \to \infty.$$

The next corollary states that, over the long-term, on average, each ranking permutation is realized the same amount of time. It follows from Theorem 1.4 in the same way that [2, Proposition 2.3] follows from [2, (2.18), (2.19), p.2303].

Corollary 1.5. *Under conditions of Theorem* 1.2*, for every permutation $\mathbf{q}$ on $\{1, \ldots, N\}$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T 1\left(\mathbf{p}_{X(t)} = \mathbf{q}\right)\,\mathrm{d}t = \frac{1}{N!}.$$

In particular, each particle X_i occupies each rank k on average $1/N$th of the time:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T 1\left(\mathbf{p}_{X(t)}(k) = i\right)\,\mathrm{d}t = \frac{1}{N}.$$

1.4. **Applications in financial mathematics.** Systems of competing Brownian particles were introduced in [2] for financial modeling. If $(X_1, \ldots, X_N)$ is such a system, the capitalizations (caps) of stocks are given by

$$S_1(t) = e^{X_1(t)}, \ldots, S_N(t) = e^{X_N(t)}. \tag{14}$$

A model of the stock market using competing Brownian particles captures the property observed in the real-world market: stocks with smaller caps have larger growth rates and larger volatilities. This can be modeled with decreasing sequences $g_1, \ldots, g_N$ and $\sigma_1^2, \ldots, \sigma_N^2$. Consider the *market weights:*

$$\mu_i(t) := \frac{S_i(t)}{S_1(t) + \ldots + S_N(t)}, \quad i = 1, \ldots, N, \quad t \ge 0.$$

Rank them from top to bottom:

$$\mu_{(k)}(t) := \mu_{\mathbf{p}_{X(t)}(N-k+1)}(t), \quad k = 1, \dots, N, \quad t \geq 0.$$

It was noted by [13, Chapter 5] that the double logarithmic plot of real-world stocks (called the *capital distribution curve*) is linear. In [8] it was shown that systems of competing Brownian particles can exhibit this property provided the system is in the stationary distribution. We simulated this capital distribution curve for market models of competing Lévy particles. The simulations in Figure 1 are of diffusions with a rank dependent drift and a pure jump process whose jump arrivals come from a rank dependent rate.

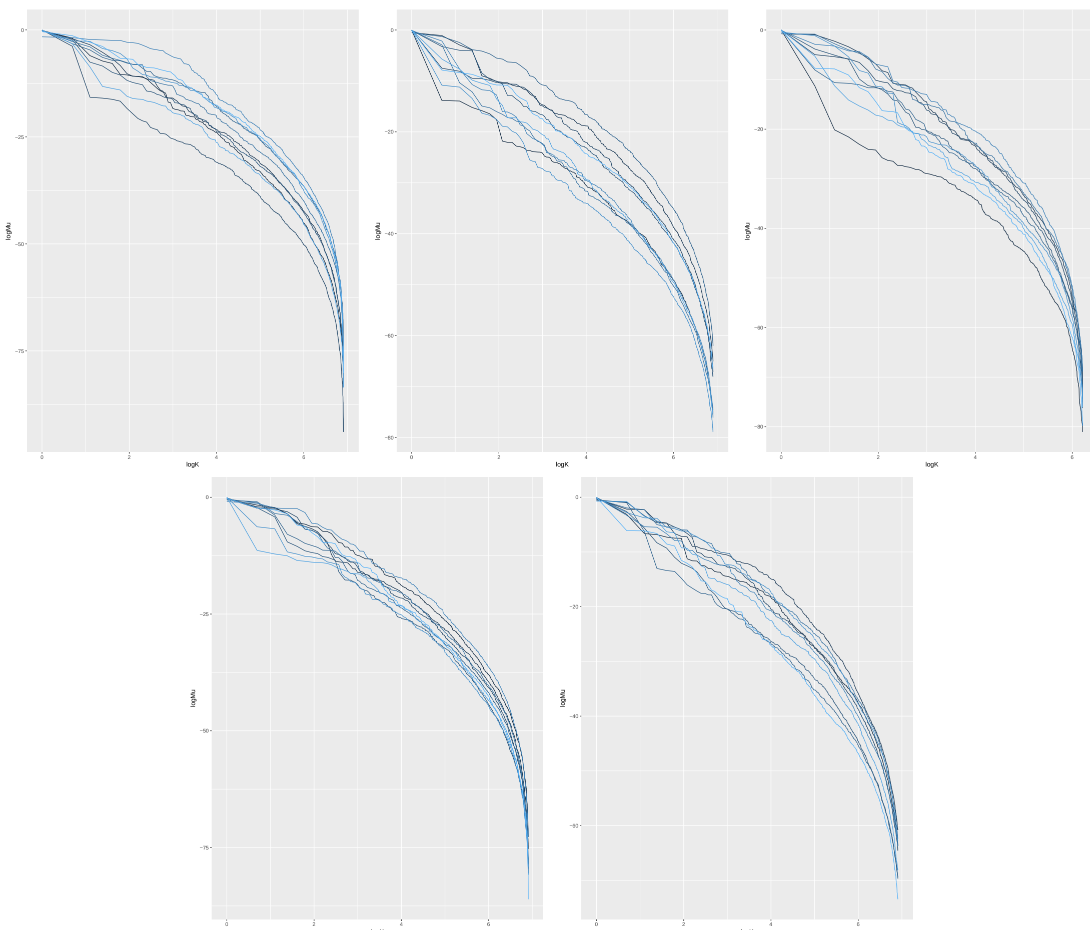

FIGURE 1. Each graph displays ten simulations of the capital distribution curve for the specified settings. In the following we let k = number of stocks, N = number of subdivisions in a unit time interval, T = time. That is, NT is the total number of discrete time steps in our simulation. Clockwise from the top left: (a) $k = N = 1000, T = 200$ with the lowest ranked particle jumping a unit in the positive direction, with jump times incoming at rate $1/2$; (b) the same as (a), but the lowest ranked particle has an additional drift of 5; (c) $k = 500, N = 1000, T = 200$ with the lowest particle jumping with the step size of independent Exp(1) incoming at a unit rate; (d) $k = N = 1000, T = 200$ with both the lowest and second to lowest particles jumping independently with a unit step incoming at a rate of $1/2$; (e) $k = N = 1000, T = 200$, same dynamics as in (d), except that the bottom particle has an additional drift of 5.

The motivation to study systems of competing Lévy particles, as opposed to systems of competing Brownian particles, comes from financial mathematics. In practice, stocks do not exactly have Brownian dynamics. Instead, they experience occasional large swings, so they can be modeled by Lévy processes more accurately. Financial models with jumps also have other useful properties (such as incompleteness) which Brownian models lack, see [9].

1.5. **Review of related results.** Theorem 1.2 generalizes the results of [38, Theorem 1.2(a), Theorem 1.3(a)], where stability was proved for systems of competing Lévy particles under conditions (2) and (3), and

$$\nu_1 = \ldots = \nu_N = \nu, \quad \int_{\mathbb{R}} z\,\nu(\mathrm{d}z) = 0, \quad \int_{\mathbb{R}} |z|\,\nu(\mathrm{d}z) < \frac{1}{N}\min\left(g_2 - g_1, \ldots, g_N - g_{N-1}\right). \tag{15}$$

As the reader can see, the conditions (15) are much more restrictive than (10). Let us also mention our article [35], where we studied stability of systems of two competing Lévy particles, as well as the explicit rate of exponential convergence of the gap process to its stationary distribution.

A related question is to explicitly find the stationary gap distribution π. For competing Brownian particles, under a so-called *skew-symmetry condition*, π has the form of a product of exponentials, see [3, 30]. In other cases, an explicit form is not known. This stationary gap distribution satisfies a certain hard-to-solve integro-differential equation, which is called *basic adjoint relationship*, see [3, 42]. There are some tail estimates for π in [36]. However, for competing Lévy particles, the stationary gap distribution does not have a product form, see [15]. This makes finding an explicit form of π even more difficult than for competing Brownian particles.

One can also try to improve statements about convergence. For competing Brownian particles, if the system is stable, then the convergence of $Z(t)$ to π is actually exponentially fast: the TV distance at time t from π is estimated as $Ce^{-\varkappa t}$ for some positive constants C and $\varkappa$. This follows from the main result of [7]; see also [36, Proposition 4.1]. However, it is hard to find or estimate $\varkappa$. It was done only for $N = 2$ competing Brownian particles in [25]. See also related work [16] for systems of $N \geq 2$ competing Brownian particles, with $A = cI_N$, for $c > 0$ and I_N the identity $N \times N$ matrix. For $N = 2$ competing Lévy particles, such estimates were done in [35]. Finally, let us mention the papers [1, 15, 24, 26], which study stability for reflected diffusions with jumps.

2. Proofs

2.1. **Notation.** The dot product of two vectors $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ from $\mathbb{R}^d$ is denoted by $a \cdot b = a_1 b_1 + \ldots + a_d b_d$. The exponential distribution on $\mathbb{R}_+$ with rate λ (and mean λ^{-1}) is denoted by $\mathrm{Exp}(\lambda)$. For a vector $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we denote by $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_N)$ its centered version:

$$\overline{x}_i := x_i - \frac{1}{N}\sum_{i=1}^N x_i, \quad i = 1, \ldots, N. \tag{16}$$

For a vector $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and a permutation $\mathbf{q}$ on $\{1, \ldots, N\}$, we denote by $x_{\mathbf{q}}$ the vector x with permuted components:

$$x_{\mathbf{q}} = \left((x_{\mathbf{q}})_1, \ldots, (x_{\mathbf{q}})_N\right), \quad (x_{\mathbf{q}})_k := x_{\mathbf{q}(k)}, \quad k = 1, \ldots, N.$$

2.2. **Proof of Lemma 1.1.** Systems of competing Brownian particles (that is, without jumps, with $\Lambda = 0$), have weak existence and uniqueness in law; it follows from [5]. Now, let us construct a system of competing Lévy particles by *piecing out*. That is, we construct it as a continuous process until the first jump, then construct this jump; starting from the destination of the jump, we construct the second continuous piece of this process, until the second jump, etc. Take a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ with:

(a) for every $x \in \mathbb{R}^N$, infinitely many i.i.d. copies $X^{(x,n)} = \left(X_1^{(x,n)}, \ldots, X_N^{(x,n)}\right)$, $n = 1, 2, \ldots$ of a system of N competing Brownian particles with drift vector g and covariance matrix A, starting from $X^{(x,n)}(0) = x$;

(b) infinitely many i.i.d. copies of an exponential random variable: $\eta_1, \eta_2, \ldots \sim \mathrm{Exp}(\lambda_0)$;

(c) infinitely many i.i.d. copies of an $\mathbb{R}^N$-valued random variable: $\zeta^{(1)}, \zeta^{(2)}, \ldots \sim \lambda_0^{-1}\Lambda(\cdot)$.

We assume all these random objects are independent. Fix a starting point $x^{(0)} \in \mathbb{R}^N$, and let $X(0) = x^{(0)}$. Let $\tau_k := \eta_1 + \ldots + \eta_k$, $k = 1, 2, \ldots$; $\tau_0 := 0$. Define the system $X = (X(t), t \ge 0)$ on the time interval $(\tau_k, \tau_{k+1}]$ for each $k = 0, 1, \ldots$, using induction by k. Assume it is already defined on the time interval $[0, \tau_k]$. For $t \in (\tau_k, \tau_{k+1})$, let $X(t) = X^{(x^{(k)}, k+1)}(t - \tau_k)$, where $x^{(k)} := X(\tau_k)$. Then let $\overline{x}^{(k+1)} = X^{(x^{(k)}, k+1)}(\eta_{k+1})$, and define

$$x^{(k+1)} = \left(x_1^{(k+1)}, \ldots, x_N^{(k+1)}\right), \quad x_i^{(k+1)} := \overline{x}_i^{(k+1)} + \zeta_{q(i)}^{(k+1)}, \quad q := p_{x^{(k+1)}}^{-1}.$$

Next, let $X(\tau_{k+1}) := x^{(k+1)}$. Thus, we constructed the system X on time interval $(\tau_k, \tau_{k+1}]$. Since $\tau_k \to \infty$ a.s. as $k \to \infty$, this completes the construction of the system X.

2.3. **Proof of Theorem 1.2.** Our first observation would be that the centered system $\overline{X}$ also forms a Feller continuous strong Markov process. Indeed, that X is a Feller continuous strong Markov process follows from standard results on piecing out, see for example [37, Theorem 2.4, Theorem 5.3, Example 1]. Next, $\overline{X}$ is a continuous function of X. Finally, the dynamics of $\overline{X}(s)$ at time $s \ge t$ depends only on $\overline{X}(t)$, for every $t \ge 0$. That is, additional information about the previous history of $\overline{X}$ before time t, as well as additional information about the values of $X(s)$, $s \le t$, which is not contained in the values of the current process, does not give any new information.

The following is the key lemma.

Lemma 2.1. *There exists constants $b, k, r > 0$ such that the function*

$$V(x) := \left[\|x\|^2 + 1\right]^{1/2} \tag{17}$$

satisfies for all $x \in \Pi$:

$$\mathcal{L}_{\overline{X}} V(x) \le -k + b 1_{B_r^{\Pi}}(x), \quad B_r^{\Pi} := \{x \in \Pi \mid \|x\| \le r\}.$$

Assuming we proved Lemma 2.1, let us complete the proof of Theorem 1.2. Apply [29, Theorem 4.4, Theorem 5.1]. We first need to check that:

(a) The process $\overline{X}$ is a T-process in the terminology of [28, Subsection 3.2]. We need to check the following: Let $Q^t(x, \cdot)$ be the transition kernel of a (centered) system of competing Brownian particles with the same drift and diffusion coefficients, but without any jumps. Then

$$P^t(x, \cdot) \ge e^{-\lambda_0 t} Q^t(x, \cdot). \tag{18}$$

Indeed, with probability $e^{-\lambda_0 t}$ the system of competing Lévy particles will not make any jumps during the time interval $[0, t]$, and will behave as a system of competing Brownian particles.

(b) All compact sets are petite for the discrete-time Markov chain $(\overline{X}(n))_{n \ge 0}$. See the definition of *petite* in [28, Subsection 4.1]. First, from (18) we have: $P^t(x, A) > 0$ for all $t \ge 0$, $x \in \Pi$, and $A \subseteq \Pi$ of positive Lebesgue measure. Therefore, the Markov chain $(\overline{X})_{n \ge 0}$ is irreducible and aperiodic (see definitions in [28]): This was shown in [36, Lemma 2.3]. Thus, all compact sets are petite: This was shown in [27, Chapter 6] for discrete-time Markov chains.

Applying [29, Theorem 4.2, Theorem 5.1], we complete the proof for the process $\overline{X}$.

Proof of Lemma 2.1. The generator $\overline{\mathcal{L}}$ consists of the continuous part $\overline{\mathcal{A}}$ and the jump part $\overline{\mathcal{N}}$: for $f \in C^2(\Pi)$,

$$\overline{\mathcal{L}}f(x) := \overline{\mathcal{A}}f(x) + \overline{\mathcal{N}}f(x), \tag{19}$$

$$\overline{\mathcal{A}}f(x) := g(x)\cdot\nabla f(x) + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N\rho_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i\partial x_j}, \tag{20}$$

$$\overline{\mathcal{N}}f(x) := \int_{\Pi}\left[f(y) - f(x)\right]\,\mu_x(\mathrm{d}y), \tag{21}$$

where for $x \in \Pi$ μ_x is the push-forward of the measure Λ with respect to the mapping

$$F_x : \Pi \to \mathbb{R}^N,\quad F_x : w \mapsto x + \overline{w}_{\mathbf{p}_x^{-1}}. \tag{22}$$

Therefore, we can rewrite the integral in the right-hand side of (21) as

$$\overline{\mathcal{N}}f(x) := \int_{\mathbb{R}^N}\left[f(F_x(w)) - f(x)\right]\,\Lambda(\mathrm{d}w). \tag{23}$$

Now, plug in $f = V$ from (17) into (19). In [2, Appendix, Proof of (2.18)] (with notation slightly different than here), the expression $\overline{\mathcal{A}}V$ is already calculated: This is the coefficient near $\mathrm{d}t$ in [2, Appendix, Proof of (2.18), (A.13)]. In our notation, we have:

$$\overline{\mathcal{A}}V(x) := \frac{x\cdot G(x)}{V(x)} + \frac{1}{V(x)}(1 - N^{-1})\sum_{k=1}^N\sigma_k^2 - \frac{1}{V^3(x)}\sum_{k=1}^N\sigma_k^2x^2_{\mathbf{p}_x(k)}, \tag{24}$$

where we define:

$$G(x) := \sum_{k=1}^N x_{\mathbf{p}_x(k)}\overline{g}_k,\quad x \in \Pi.$$

We can rewrite (24) as

$$\overline{\mathcal{A}}V(x) := \frac{x\cdot G(x)}{V(x)} + \delta_*(x),\quad \lim_{\substack{x\in\Pi\\ \|x\|\to\infty}}|\delta_*(x)| = 0. \tag{25}$$

Consider then the expression inside of the integral in (23) for $f := V$ from (17). The function V is infinitely differentiable on $\mathbb{R}^N$, and

$$\nabla V(z) = \frac{z}{V(z)},\quad \frac{\partial^2 V}{\partial z_i\partial z_j} = \frac{\delta_{ij}}{V(z)} - \frac{z_iz_j}{V^3(z)},\ i, j = 1, \ldots, N. \tag{26}$$

Therefore,

$$\lim_{\|z\|\to\infty}\left|\frac{\partial^2 V(z)}{\partial z_i\partial z_j}\right| = 0,\quad i, j = 1, \ldots, N. \tag{27}$$

We can write the following Taylor decomposition for all $z, u \in \mathbb{R}^N$:

$$V(z+u) - V(z) = \frac{z\cdot u}{V(z)} + \theta(z, u), \tag{28}$$

where the error term $\theta(z,u)$ is given by the following expression for some $\eta(z,u)\in[0,1]$:

$$\theta(z,u) := \frac{\eta^2(z,u)}{2}\sum_{i=1}^N\sum_{j=1}^N u_iu_j\frac{\partial^2 V(z)}{\partial z_i\partial z_j}. \tag{29}$$

From (27) and (29), we have:

$$\lim_{\|z\|\to\infty}\theta(z,u) = 0 \text{ for every } u\in\mathbb{R}^N. \tag{30}$$

From (27) we get: $C_* := \max_{z,i,j}\left|\frac{\partial^2 V(z)}{\partial z_i\partial z_j}\right| < \infty$. Therefore,

$$|\theta(z,u)| \le \frac{C_*}{2}\|u\|^2. \tag{31}$$

Letting $z := x$ and $u := \overline{w}_{\mathbf{p}_x^{-1}}$ in (28), we get:

$$V(F_x(w)) - V(x) = \frac{x\cdot\overline{w}_{\mathbf{p}_x^{-1}}}{V(x)} + \theta\left(x, \overline{w}_{\mathbf{p}_x^{-1}}\right). \tag{32}$$

From (31) and $\|\overline{w}_{\mathbf{p}_x^{-1}}\| = \|\overline{w}\| \le \|w\|$, we have:

$$\left|\theta\left(x,\overline{w}_{\mathbf{p}_x^{-1}}\right)\right| \le \frac{C_*}{2}\|w\|^2. \tag{33}$$

Combine (9), (30), (32), (33), and apply Lebesgue dominated convergence theorem:

$$\overline{\mathcal{N}}V(x) = \frac{1}{V(x)}\int_{\mathbb{R}^N} x\cdot\overline{w}_{\mathbf{p}_x^{-1}}\,\Lambda(\mathrm{d}w) + \delta(x), \quad \lim_{\substack{\|x\|\to\infty\\ x\in\Pi}}\delta(x) = 0. \tag{34}$$

Rewrite the expression inside the integral in (34) as follows:

$$x\cdot\overline{w}_{\mathbf{p}_x^{-1}} = \sum_{i=1}^N x_i\overline{w}_{\mathbf{p}_x^{-1}(i)} = \sum_{k=1}^N x_{\mathbf{p}_x(k)}\overline{w}_k. \tag{35}$$

Integrating (35) with respect to $\Lambda(\mathrm{d}w)$, we get:

$$\int_{\mathbb{R}^N} x\cdot\overline{w}_{\mathbf{p}_x^{-1}}\,\Lambda(\mathrm{d}w) = \sum_{k=1}^N x_{\mathbf{p}_x(k)}\overline{f}_k, \tag{36}$$

where $\overline{f}_k$ are obtained from centering f_k from (8), as in (16). Combining (34), (36), we get:

$$\overline{\mathcal{N}}V(x) = \frac{1}{V(x)}\sum_{k=1}^N x_{\mathbf{p}_x(k)}\overline{f}_k + \delta(x). \tag{37}$$

As calculated in [2, p.2302, (2.17)] (summation by parts):

$$x\cdot G(x) = -\sum_{k=1}^{N-1}\left(x_{\mathbf{p}_x(k+1)} - x_{\mathbf{p}_x(k)}\right)\sum_{j=1}^k\overline{g}_j. \tag{38}$$

Similarly, we can rewrite the sum in (37) as

$$\sum_{k=1}^N x_{\mathbf{p}_x(k)}\overline{f}_k = -\sum_{k=1}^{N-1}\left(x_{\mathbf{p}_x(k+1)} - x_{\mathbf{p}_x(k)}\right)\sum_{j=1}^{k-1}\overline{g}_j \tag{39}$$

Combine (12), (25), (34), (37), (38), (39). Letting $\delta^*(x) := \delta(x) + \delta_*(x)$, we have:

$$\overline{\mathcal{L}}V(x) = -\frac{1}{V(x)}\sum_{k=1}^{N-1}\left(x_{\mathbf{p}_x(k+1)} - x_{\mathbf{p}_x(k)}\right)\sum_{j=1}^{k-1}\overline{m}_j + \delta^*(x). \tag{40}$$

As in [2, p.2302], we have:

$$\sum_{k=1}^{N-1}\left(x_{\mathbf{p}_x(k+1)} - x_{\mathbf{p}_x(k)}\right)\sum_{j=1}^{k-1}\overline{m}_j \le -k\|x\|,\quad k := -N^{-1/2}\min_{1\le k\le N-1}\sum_{j=1}^{k}\overline{m}_j > 0. \tag{41}$$

Since $V(x)/\|x\| \to 1$, and $\delta^*(x) \to 0$ as $\|x\| \to \infty$ for $x \in \Pi$, from (41) and (40) it is straightforward to complete the proof of Lemma 2.1.

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Department of Mathematics, University of Washington
E-mail address: clayton.barnes@math.washington.edu

Department of Statistics and Applied Probability, University of California, Santa Barbara
E-mail address: sarantsev@pstat.ucsb.edu