# Regularization by noise in (2x 2) hyperbolic systems of conservation law.

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#### Abstract

In this paper we study a non strictly systems of conservation law by stochastic perturbation. We show the existence and uniqueness of the solution. We do not assume that BV-regularity for the initial conditions. The proofs are based on the concept of entropy solution and in the characteristics method (in the influence of noise). This is the first result on the regularization by noise in hyperbolic systems of conservation law.

## 1 Introduction

A large number of problems in physics and engineering are modeled by systems of conservation laws

$$\partial_t u(t,x) + div(f(u(t,x))) = 0, \qquad (1.1)$$

here u = u(t, x) is called the conserved quantity, while F is the flux. Examples for hyperbolic systems of conservation laws include the shallow water equations of oceanography, the Euler equations of gas dynamics, the magnetohydrodynamics (MHD) equations of plasma physics, the equations of

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nonlinear elastodynamics and the Einstein equations of general relativity. When smooth initial data are considered, it is well known that the solution can develop shocks within finite time. Therefore, global solutions can only be constructed within a space of discontinuous functions. Moreover, when discontinuities are present, weak solutions may not be unique. A central issue is to regain uniqueness by imposing appropriate selection criteria. The well-posedness theorems within the class of entropy solutions, for the scalar case, were established by Kruzkov(see [18]). It is well known that the main techniques of abstract functional analysis do not apply to hyperbolic systems. Solutions cannot be represented as fixed points of continuous transformations, or in variational form, as critical points of suitable functionals. For the above reasons, the theory of hyperbolic conservation laws has largely developed by ad hoc methods. We refer to [6], [5] [9] and [28]. The well-posedness general system of conservation laws has been established only for initial data with sufficiently small total variation, see [6] and the references therein.

We consider the following systems of conservation law

$$\begin{cases}
\partial_t v(t,x) + Div(F(v)) = 0 \\
\partial_t u(t,x) + Div(vu) = 0.
\end{cases}$$
(1.2)

We point that in the  $L^1 \cap L^{\infty}$  setting this systems ill-posedness since the classical DiPerna-Lions-Ambrossio theory of uniqueness of distributional solutions for transport/ continuity equation does not apply when the drift has  $L^1 \cap L^2$  regularity, see [1] and [11]. Also see [2] and [10] for new developments in the theory. We point under strong assumption on the coefficients and initial conditions P. Le Floch in [22] solved this problem using Volpert multiplication of distributions. In contrast with its deterministic counterpart, the

singular stochastic continuity/transport equation with multiplicative noise is well-posed. The addition of a stochastic noise is often used to account for numerical, empirical or physical uncertainties. In [3, 4, 12, 13, 15, 25, 26], well-posedness and regularization by linear multiplicative noise for continuity/transport equations have been obtained. We refer to [26] for more details on the literature.

In this paper we study the influence of the noise in the hyperbolic systems (1.2). More precisely, we consider following stochastic systems of conserva-

tion law

$$\begin{cases} \partial_t v(t,x) + Div((F(v(t,x))) = 0, \\ \partial_t u(t,x) + Div((v(t,x) + \frac{dB_t}{dt}) \cdot u(t,x)) = 0, \\ v|_{t=0} = v_0, \ u_{t=0} = u_0. \end{cases}$$
 (1.3)

Here,  $(t,x) \in [0,T] \times \mathbb{R}$ ,  $\omega \in \Omega$  is an element of the probability space  $(\Omega, \mathbb{P}, \mathcal{F})$  and  $B_t$  is a standard Brownian motion in  $\mathbb{R}$ . The stochastic integration is to be understood in the Stratonovich sense. The Stratonovich form is the natural one for several reasons, including physical intuition related to the Wong-Zakai principle.

The main issue of this paper is to prove existence and uniqueness of entropy-weak solutions for the stochastic systems of the conservation law (1.3). We do not assume BV- regularity for the initial conditions. We use the entropy formulation of conservation law and we employ the stochastic characteristics in order to obtain a unique solution to the one-dimensional stochastic equation with a bounded measurable drift coefficient. We adapted the ideas in [26] and [27] in our context where the drift term in the continuity equation depend on time and it is bounded and integrable.

Throughout of this paper, we fix a stochastic basis with a d-dimensional Brownian motion  $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}, (B_t))$ .

## 1.1 One Example

We consider the systems

$$\begin{cases} \partial_t v(t,x) + Div(\frac{1}{2}v^2(t,x)) = 0, \\ \partial_t u(t,x) + Div((v(t,x)u(t,x))) = 0, \\ v|_{t=0} = v_0, \ u_{t=0} = u_0, \end{cases}$$
(1.4)

here v is the velocity and u the density of the particles. This system has applications in cosmology, the model describes the evolution of matter in the last stage of the expansion of the universe as cold dust moving under gravity alone and the laws are governed by the system (1.4). Clearly the eigenvalues are equal  $\lambda_1 = \lambda_2 = v$ . Thus the system (1.4) is not strictly

hyperbolic. The first equation of (1.4)-Burgers equation is known to develop singularities in finite time even if the initial data  $v_0$  is smooth, and it is not at all obvious to solve the second equation. One question that remains is a well-posedness theory and large time behaviour of solution. In [17] and [30] the authors proved existence of weak solutions via  $\delta$ - shock for Riemann initial condition. Another approach of nonconservative product can be found in the work of J.F. Colombeau [?] .

#### 1.2 Scalar case.

We point that recently there has been an interest in studying the effect of stochastic forcing on nonlinear conservation laws driven by space-time white noise, see [7, 8, 14, 16]. For other hand, in [23] and [24] the authors introduced the theory of pathwise solutions to study the stochastic conservation law driven by continuous noise.

#### 1.3 Possible extensions.

We point that our approach can be apply to other class of non-coupled systems like

$$\begin{cases}
\partial_t v(t,x) = P(v) \\
\partial_t u(t,x) + Div(f(v)u) = 0.
\end{cases}$$
(1.5)

where the P is some differential operator. For instance for the systems of the Hamilton-Jacobi and the continuity equations, see [29]. The problem of coupled systems is much more complicated and we shall consider it in future investigations.

## 1.4 Hypothesis

We assume the following conditions

**Hypothesis 1.1.** The flux F satisfies

$$F \in C^1 \tag{1.6}$$

and the initial condition holds

$$v_0 \in L^{\infty}(\mathbb{R}) \cap L^1(\mathbb{R}), u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}).$$
 (1.7)

### 2 Existence

#### 2.1 Definition of solutions

**Definition 2.1.** Let  $\eta \in C^1(\mathbb{R})$  be a convex function. If there exist  $q \in C^1(\mathbb{R})$  such that for all v

$$\eta'(v)F'(v) = q'(v)$$

then  $\eta, q$  is called an entropy-entropy flux pair of the conservation law

$$\partial_t v(t, x) + Div(f(v)) = 0, \ v(t, 0) = v_0(x).$$

**Definition 2.2.** The stochastic process  $v \in L^{\infty}([0,T] \times \mathbb{R}) \cap L^{\infty}([0,T], L^{1}(\mathbb{R}))$  and  $u \in L^{\infty}([0,T], L^{2}(\Omega \times \mathbb{R})) \cap L^{1}([0,T] \times \Omega \times \mathbb{R})$  are called a entropy weak solution of the stochastic hyperbolic systems (1.3) when:

• v is entropy solution of the conservation law

$$\partial_t v(t, x) + Div(F(v)) = 0, \ v(t, 0) = v_0(x).$$

That is, if for every entropy flux pair  $\eta$ , q we have

$$\partial_t \eta(v) + Div(q(v)) \le 0$$

in the sense of distribution.

• For any  $\varphi \in C_0^{\infty}(\mathbb{R})$ , the real valued process  $\int u(t,x)\varphi(x)dx$  has a continuous modification which is an  $\mathfrak{F}_t$ -semimartingale, and for all  $t \in [0,T]$ , we have  $\mathbb{P}$ -almost surely

$$\int_{\mathbb{R}} u(t,x)\varphi(x)dx = \int_{\mathbb{R}} u_0(x)\varphi(x) dx + \int_0^t \int_{\mathbb{R}} u(s,x)v(t,x)\partial_x\varphi(x) dxds 
+ \int_0^t \int_{\mathbb{R}} u(s,x) \partial_x\varphi(x) dx \circ dB_s.$$
(2.8)

**Remark 2.3.** Using the same idea as in Lemma 13 [15], one can write the problem (2.8) in Itô form as follows, a stochastic process  $u \in L^{\infty}([0,T], L^2(\Omega \times$ 

 $\mathbb{R}$ ))  $\cap L^1([0,T] \times \Omega \times \mathbb{R})$  is solution of the SPDE (2.8) iff for every test function  $\varphi \in C_0^{\infty}(\mathbb{R})$ , the process  $\int u(t,x)\varphi(x)dx$  has a continuous modification which is a  $\mathfrak{F}_t$ -semimartingale and satisfies the following Itô's formulation

$$\int_{\mathbb{R}} u(t,x)\varphi(x)dx = \int_{\mathbb{R}} u_0(x)\varphi(x) \ dx + \int_0^t \int_{\mathbb{R}} u(s,x) \ v(t,x)\partial_x\varphi(x) \ dxds$$
$$+ \int_0^t \int_{\mathbb{R}} u(s,x) \ \partial_x\varphi(x) \ dx \ dB_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u(s,x) \ \partial_x^2\varphi(x) \ dx \ ds.$$

#### 2.2 Existence.

We shall prove existence of solutions under hypothesis 1.1.

**Lemma 2.4.** Assume that hypothesis 1.1 holds. Then there exists entropyweak solution of the hyperbolic systems (1.3).

*Proof. Step 1: Conservation law*. According to the classical theory of conservation law, see for instance [9], we have that there exists a unique entropy solution of the conservation law

$$\partial_t v(t,x) + Div(F(v)) = 0, \ v(t,0) = v_0(x).$$

If the the initial condition  $v_0 \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$  then the solution  $v \in L^{\infty}([0,T] \times \mathbb{R}) \cap L^{\infty}([0,T],L^1(\mathbb{R}))$ .

Step 2: Primitive of v. It easy to see that for any test function  $\varphi \in C_0^{\infty}(\mathbb{R})$  we have

$$\int_{\mathbb{R}} v(t,x)\varphi(x)dx = \int_{\mathbb{R}} v_0(x)\varphi(x) \ dx + \int_0^t \int_{\mathbb{R}} F(v(s,x))\partial_x\varphi(x) \ dxds.$$

since any entropy solution is also a weak solution.

Let  $\{\rho_{\varepsilon}\}_{\varepsilon}$  be a family of standard symmetric mollifiers. Then we obtain

$$\int_{\mathbb{R}} v(t,y)\rho_{\varepsilon}(x-y)dy = \int_{\mathbb{R}} v_0(y)\rho_{\varepsilon}(x-y)dy + \int_0^t \int_{\mathbb{R}} F(v(s,y))\partial_y\rho_{\varepsilon}(x-y)dyds.$$

Integrating we get

$$\int_0^z v_{\varepsilon}(t,x)dx = \int_0^z v_0^{\varepsilon}(x)dz + \int_0^t (F(v) * \rho_{\varepsilon})(z)ds.$$

We denoted  $\int_0^z v_{\varepsilon}(t,x)dx = \bar{v}_{\varepsilon}(t,x)$ .

Step 3: Regularization. We define the family of regularized coefficients given by

$$v^{\epsilon}(t,.) = (v(t,x) *_{x} \rho_{\varepsilon})(t,.).$$

Clearly we observe that, for every  $\varepsilon > 0$ , any element  $v^{\varepsilon}$ ,  $u_0^{\varepsilon}$  are smooth (in space) and with bounded derivatives of all orders. We observe that to study the stochastic continuity equation (SCE) (2.8) is equivalent to study the stochastic transport equation given by (regularized version):

$$\begin{cases}
du^{\varepsilon}(t,x) + \nabla u^{\varepsilon}(t,x) \cdot \left(v^{\varepsilon}(t,x)dt + \circ dB_{t}\right) + \operatorname{div}b^{\varepsilon}(x) u^{\varepsilon}(t,x)dt = 0, \\
u^{\varepsilon}\big|_{t=0} = u_{0}^{\varepsilon}
\end{cases}$$
(2.9)

Following the classical theory of H. Kunita [19, Theorem 6.1.9] we obtain that

$$u^{\varepsilon}(t,x) = u_0^{\varepsilon}(\psi_t^{\varepsilon}(t,x))J\psi_t^{\varepsilon}(t,x),$$

is the unique solution to the regularized equation (2.9), where  $\phi_t^{\varepsilon}$  is the flow associated to the following stochastic differential equation (SDE):

$$dX_t = v^{\varepsilon}(t, X_t) dt + dB_t, X_0 = x,$$

and  $\psi_t^{\varepsilon}$  is the inverse of  $\phi_t^{\varepsilon}$ .

Step 4: Itô Formula . Applying the Itô formula to  $\bar{v}_{\varepsilon}(t,X_t^{\epsilon})$  we deduce

$$\bar{v}_{\varepsilon}(t, X_{t}^{\epsilon}) = \int_{0}^{X_{t}^{\epsilon}} u_{0}^{\epsilon}(x) dx + \int_{0}^{t} (F(v) * \rho_{\varepsilon})(s, X_{s}^{\epsilon}) ds + \int_{0}^{t} v_{\varepsilon}^{2}(s, X_{s}^{\epsilon}) ds$$
$$+ \int_{0}^{t} v_{\varepsilon}(s, X_{s}^{\epsilon}) dB_{s} + \frac{1}{2} \int_{0}^{t} (\partial_{x} v_{\varepsilon})(s, X_{s}^{\epsilon}) ds$$

Step 5: Boundeness. We observe that

$$\|\bar{v}_{\varepsilon}(t, X_{t}^{\epsilon})\|_{L^{\infty}(\Omega \times [0, T] \times \mathbb{R})} \leq \|v\|_{L^{\infty}([0, T], L^{1}(\mathbb{R}))},$$

$$\|\int_{0}^{X_{t}^{\epsilon}} v_{0}^{\epsilon}(x) dx\|_{L^{\infty}(\Omega \times [0, T] \times \mathbb{R})} \leq \|v_{0}\|_{L^{1}(\mathbb{R})},$$

$$\|\int_{0}^{t} (F(v) * \rho_{\varepsilon})(s, X_{s}^{\epsilon}) ds\|_{L^{\infty}(\Omega \times [0, T] \times \mathbb{R})} \leq C \|F(v)\|_{L^{\infty}},$$

$$\|\int_{0}^{t} v_{\varepsilon}^{2}(s, X_{s}^{\epsilon}) ds\|_{L^{\infty}(\Omega \times [0, T] \times \mathbb{R})} \leq C \|v\|_{L^{2}([0, T], L^{\infty}(\mathbb{R}))}^{2}.$$

Step 6: Estimation on Jacobain.

We denote

$$\mathcal{E}\left(\int_0^t v_{\epsilon}(s, X_s) dB_s\right) = \exp\bigg\{\int_0^t v_{\epsilon}(s, X_s^{\epsilon}) dB_s - \frac{1}{2}\int_0^t v_{\epsilon}^2(s, X_s^{\epsilon}) ds\bigg\},$$

We note that  $\partial_x X_t$  satisfies

$$\partial_x X_t = \exp\bigg\{\int_0^t (\partial_x v_\epsilon)(s, X_s) \ ds\bigg\}.$$

From steps 4-5 we have

$$\mathbb{E}|\partial_x X_t|^{-1} \le C \mathbb{E} \mathcal{E} \left( \int_0^t v_{\epsilon}(s, X_s) dB_s \right).$$

We observe that the processes  $\mathcal{E}\left(\int_0^t v_{\epsilon}(s, X_s) dB_s\right)$ , is martingale with expectation equal to one. The we conclude that

$$\mathbb{E}|\partial_x X_t|^{-1} \le C.$$

Step 7: Passing to the limit.

Making the change of variables  $y = \psi_t^{\varepsilon}(x)$  we have that

$$\int_{\mathbb{R}} \mathbb{E}[|u^{\varepsilon}(t,x)|^2] dx = \int_{\mathbb{R}} |u_0^{\varepsilon}(y)|^2 \mathbb{E}|J\phi_t^{\varepsilon}|^{-1} dy.$$

From step 6 we have

$$\int_{\mathbb{R}} \mathbb{E}[|u^{\varepsilon}(t,x)|^2] dx \le C. \tag{2.10}$$

Therefore, the sequence  $\{u^{\varepsilon}\}_{{\varepsilon}>0}$  is bounded in  $L^{\infty}([0,T],L^2(\Omega\times\mathbb{R}))\cap L^1([0,T]\times\Omega\times\mathbb{R})$ . Then there exists a convergent subsequence, which we denote also by  $u^{\varepsilon}$ , such that converge weakly in  $L^{\infty}([0,T],L^2(\Omega\times\mathbb{R}))$  to some process  $u\in L^{\infty}([0,T],L^2(\Omega\times\mathbb{R}))\cap L^1([0,T]\times\Omega\times\mathbb{R})$ .

Now, if  $u^{\varepsilon}$  is a solution of (2.9), it is also a weak solution, that is, for any test function  $\varphi \in C_0^{\infty}(\mathbb{R})$ ,  $u^{\varepsilon}$  satisfies (written in the Itô form):

$$\int_{\mathbb{R}} u^{\varepsilon}(t,x)\varphi(x)dx = \int_{\mathbb{R}} u_{0}^{\varepsilon}(x)\varphi(x) dx + \int_{0}^{t} \int_{\mathbb{R}} u^{\varepsilon}(s,x) v^{\varepsilon}(s,x)\partial_{x}\varphi(x) dxds 
+ \int_{0}^{t} \int_{\mathbb{R}} u^{\varepsilon}(s,x) \partial_{x}\varphi(x) dx dB_{s} + \frac{1}{2} \int_{0}^{t} \int_{\mathbb{R}} u^{\varepsilon}(s,x) \partial_{x}^{2}\varphi(x) dx ds.$$

Thus, for prove existence of the SCE (1.3) is enough to pass to the limit in the above equation along the convergent subsequence found. This is made through of the same arguments of [15, theorem 15].

## 3 Uniqueness.

In this section, we shall present a uniqueness theorem for the SPDE (1.3)

**Theorem 3.1.** Under the conditions of hypothesis 1.1, uniqueness holds for entropy -weak solutions of the hyperbolic problem (1.3).

*Proof. Step 1: Set of solutions.* The uniqueness of the conservation law

$$\partial_t v(t, x) + Div(F(v)) = 0, \ v(t, 0) = v_0(x).$$

follows from the classical theory of entropy solutions.

Step 2: We remark that the set of solutions of equation (2.8) is a linear subspace of  $L^{\infty}([0,T]\times R, L^2(\Omega))\cap L^1([0,T]\times \Omega\times \mathbb{R})$ , because the stochastic continuity equation is linear, and the integrability conditions is a linear constraint. Therefore, it is enough to show that a u with initial condition  $u_0=0$  vanishes identically.

Step 1: Primitive of the solution. We define  $V(t,x) = \int_{-\infty}^{x} u(t,y) \, dy$ . We consider a nonnegative smooth cut-off function  $\eta$  supported on the ball of radius 2 and such that  $\eta = 1$  on the ball of radius 1. For any R > 0, we introduce the rescaled functions  $\eta_R(\cdot) = \eta(\frac{\cdot}{R})$ . Let be  $\varphi \in C_0^{\infty}(\mathbb{R})$ , we have

$$\int_{\mathbb{R}} V(t,x) \varphi(x) \eta_R(x) dx = -\int_{\mathbb{R}} u(t,x) \theta(x) \eta_R(x) dx - \int_{\mathbb{R}} V(t,x) \theta(x) \partial_x \eta_R(x) dx ,$$

where  $\theta(x) = \int_{-\infty}^{x} \varphi(y) dy$ . By definition of the solution u, taking as test function  $\theta(x)\eta_R(x)$  we deduce that

$$\int_{\mathbb{R}} V(t,x) \, \eta_R(x) \varphi(x) dx = -\int_0^t \int_{\mathbb{R}} \partial_x V(s,x) \, v(s,x) \eta_R(x) \varphi(x) \, dx ds$$

$$-\int_0^t \int_{\mathbb{R}} \partial_x V(s,x) \, \eta_R(x) \varphi(x) \, dx \circ dB_s - \int_0^t \int_{\mathbb{R}} \partial_x V(s,x) \, v(s,x) \partial_x \eta_R(x) \theta(x) \, dx ds$$

$$-\int_0^t \int_{\mathbb{R}} \partial_x V(s,x) \, \partial_x \eta_R(x) \theta(x) \, dx \circ dB_s - \int_{\mathbb{R}} V(t,x) \theta(x) \partial_x \eta_R(x) dx. \tag{3.11}$$

Since  $V \in L^{\infty}([0,T],L^{1}(\mathbb{R}))$  taking the limit as  $R \to \infty$  we get

$$\int_{\mathbb{R}} V(t, x) \varphi(x) dx =$$

$$- \int_{0}^{t} \int_{\mathbb{R}} \partial_{x} V(s, x) v(s, x) \varphi(x) dx ds - \int_{0}^{t} \int_{\mathbb{R}} \partial_{x} V(s, x) \varphi(x) dx \circ dB_{s}.$$
(3.12)

Step 2: Smoothing. Let  $\{\rho_{\varepsilon}(x)\}_{\varepsilon}$  be a family of standard symmetric mollifiers. For any  $\varepsilon > 0$  and  $x \in \mathbb{R}^d$  we use  $\rho_{\varepsilon}(x - \cdot)$  as test function and we obtain

$$\int_{\mathbb{R}} V(t,y) \rho_{\varepsilon}(x-y) \, dy = -\int_{0}^{t} \int_{\mathbb{R}} \left( v(s,y) \partial_{y} V(s,y) \right) \rho_{\varepsilon}(x-y) \, dy ds$$
$$-\int_{0}^{t} \int_{\mathbb{R}} \partial_{y} V(s,y) \, \rho_{\varepsilon}(x-y) \, dy \circ dB_{s}$$

We put  $V_{\varepsilon}(t,x) = (V * \rho_{\varepsilon})(x)$ ,  $v_{\varepsilon}(t,x) = (v * \rho_{\varepsilon})(t,x)$  and  $(vV)_{\varepsilon}(t,x) = (v \cdot V * \rho_{\varepsilon})(x)$ . Then have

$$V_{\varepsilon}(t,x) + \int_{0}^{t} v \varepsilon(s,x) \partial_{x} V_{\varepsilon}(s,x) ds + \int_{0}^{t} \partial_{x} V_{\varepsilon}(s,x) \circ dB_{s}$$
$$= \int_{0}^{t} (\Re_{\epsilon}(V,v))(x,s) ds,$$

where  $\mathcal{R}_{\epsilon}(V, v) = v_{\epsilon} \partial_x V_{\epsilon} - (v \partial_x V)_{\epsilon}$ .

Step 3: Method of Characteristic.

We consider the stochastic flow

$$dX_t^{\epsilon} = v^{\epsilon}(t, X_t^{\epsilon}) dt + dB_t, \qquad X_0 = x.$$

Using the same arguments that in steps 3-5-6 of the existence proof we have

$$\mathbb{E}|JX_{t-s}^{\epsilon}|^2 \le C. \tag{3.13}$$

Applying the Itô-Wentzell-Kunita formula to  $V_{\varepsilon}(t,X_t^{\epsilon})$  , see Theorem 8.3 of [20], we have

$$V_{\varepsilon}(t, X_t^{\epsilon}) = \int_0^t (\Re_{\epsilon}(V, v))(X_s^{\epsilon}, s) ds.$$

Hence

$$V_{\varepsilon}(t,x) = \int_{0}^{t} (\mathcal{R}_{\epsilon}(V,v))(X_{t-s}^{-1,\epsilon},s)ds.$$

Multiplying by the test functions  $\varphi$  and integrating in  $\mathbb{R}$  we obtain

$$\int V_{\varepsilon}(t,x) \ \varphi(x)dx = \int_{0}^{t} \int \left( \mathcal{R}_{\epsilon}(V,v) \right) (X_{t-s}^{-1,\epsilon},s) \ \varphi(x) \ dx \ ds.$$
 (3.14)

Doing the change of variable we obtain

$$\int_0^t \int \left( \mathcal{R}_{\epsilon}(V, v) \right) (X_{t-s}^{-1, \epsilon}, s) \, \varphi(x) \, dx \, ds = \int_0^t \int \left( \mathcal{R}_{\epsilon}(V, v) \right) (x, s) \, J X_{t-s}^{\epsilon} \varphi(X_{t-s}^{\epsilon}) \, dx \, ds.$$

$$(3.15)$$

Step 4: Convergence of the commutator. Now, we observe that  $\mathcal{R}_{\epsilon}(V, b)$  converge to zero in  $L^2([0, T] \times \mathbb{R})$ . In fact, we have

$$(v \partial_x V)_{\varepsilon} \to v \partial_x V \text{ in } L^2([0,T] \times \mathbb{R}),$$

and by the dominated convergence theorem we obtain

$$v_{\epsilon}\partial_x V_{\varepsilon} \to v \,\partial_x V \, in \, L^2([0,T] \times \mathbb{R}).$$

Step 5: Conclusion. From step 3 we have

$$\int V_{\varepsilon}(t,x) \varphi(x) dx = \int_{0}^{t} \int (\mathcal{R}_{\epsilon}(V,v))(x,s) JX_{t-s}^{\epsilon} \varphi(X_{t-s}^{\epsilon}) dx ds, \quad (3.16)$$

Using Hölder's inequality we obtain

$$\mathbb{E} \left| \int_0^t \int \left( \mathcal{R}_{\epsilon}(V, v) \right) (x, s) \ J X_{t-s}^{\epsilon} \varphi(X_{t-s}^{\epsilon}) \ dx \ ds \right|$$

$$\leq \left(\mathbb{E}\int_0^t \int |(\mathcal{R}_{\epsilon}(V,v))(x,s)|^2 dx ds\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^t \int |JX_{t-s}^{\epsilon}\varphi(X_{t-s}^{\epsilon})|^2 dx ds\right)^{\frac{1}{2}}$$

From step 4 we deduce

$$\left(\mathbb{E}\int_0^t \int |(\mathcal{R}_{\epsilon}(V,v))(x,s)|^2 dx ds\right)^{\frac{1}{2}} \to 0.$$

From estimation (3.13) we obtain

$$\left(\mathbb{E}\int_0^t \int |JX_{t-s}^{\epsilon}\varphi(X_{t-s}^{\epsilon})|^2 dx ds\right)^{\frac{1}{2}}$$

$$\leq C\left(\int_0^t \int_{\mathbb{R}} |\varphi(x)|^2 dx ds\right)^{\frac{1}{2}} \leq C \int_{\mathbb{R}} |\varphi(x)|^2 dx.$$

Passing to the limit in equation (3.16) we conclude that V=0. Then we deduce that u=0.

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