$t-$ASPECT SUBCONVEXITY FOR $GL(2)-$L FUNCTIONS

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Abstract. Let f be a holomorphic cusp form for $SL_2(\mathbb{Z})$ of weight $k > 1$. In these notes, we follow Munshi [8] to prove the Burgess bound

$$L(1/2 + it, f) \ll_{f,\varepsilon} (1 + |t|)^{1/2-1/8+\varepsilon}.$$

1. Introduction

Let f be a holomorphic cusp form for $SL_2(\mathbb{Z})$ of weight $k > 1$. The L-series is given by,

$$L(s, f) = \sum_{n\geqslant 1} \lambda_f(n)n^{-s} \quad \text{for } Re(s) > 1.$$

This extends to an entire function on the whole complex plane $\mathbb{C}$. The convexity principle gives the bound $L(1/2 + it, f) \ll_f (1 + |t|)^{1/2}$, known as the convexity bound. The purpose of this paper is to prove the following bound.

Theorem 1.1. *Let f be a holomorphic cusp form for $SL_2(\mathbb{Z})$. Then we have,*

$$L(1/2 + it, f) \ll_{f,\varepsilon} (1 + |t|)^{1/2-1/8+\varepsilon}.$$

The first such bound was obtained by Good [2]. The result was extended to Maass cusp forms by Jutila [4]. t-aspect subconvexity for higher $GL(n)$ is largely unknown. Subconvex bounds for $GL(1)$ and $GL(2)$, uniformly in all aspects is known by the works of Michel-Venkatesh [7]. t-aspect subconvexity for self dual Hecke-Maass forms for $GL(3)$ was first established by Li [6]. Munshi [8] used a different method (that we follow and execute) to extend the result to all Hecke-Maass cusp forms. Recently, Singh [10] did similar calculations for t-aspect subconvexity for $GL(2)$ L-functions of holomorphic and Hecke-Maass cusp forms and claims to get the Weyl bound.

We have followed the ideas of Munshi [8] and use a modification of the circle method. In the present situation, Kloosterman's version of the circle method works best. Let,

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Then for any real number $Q > 0$, we have,

$$\delta(n) = 2Re \int_0^1 \sum_{1\leqslant q\leqslant Q<a\leqslant Q}\sum^{*} \frac{1}{aq} e\left(\frac{n\overline{a}}{q} - \frac{nx}{aq}\right) dx \tag{1.1}$$

for $n \in \mathbb{Z}$. Here $e(.) = e^{2\pi i.}$ and the $*$ on the inner sum means that $(a, q) = 1$. $\overline{a}$ is the multiplicative inverse of $a$mod q. There are well understood drawbacks of this circle method. It will turn out that this circle method in itself will not be sufficient, and we will have a apply a 'conductor lowering trick' as used by Munshi in his various works [8, 9].

Suppose $t > 2$. The approximate functional equation gives

$$L(1/2+it, f) \ll t^{\epsilon} \sup_{N \leqslant t^{1+\epsilon}} \frac{|S(N)|}{N^{1/2}} + t^{-2015}$$

where

$$S(N) := \sum_{n \geqslant 1} \lambda(n) n^{-it} V\left(\frac{n}{N}\right).$$

Let V be a smooth function supported on $[1, 2]$ satisfying $V^{(j)} \ll_j 1$. We further normalize V so that $\int_{\mathbb{R}} V(x)dx = 1$. We will apply (1.1) directly to $S(N)$ with a conductor lowering integral to separate the oscillations of $\lambda(n)$ and n^{-it}.

$$S(N) = \frac{1}{K} \int V\left(\frac{v}{K}\right) \sum_{n \geqslant 1} \sum_{m \geqslant 1} \lambda(n) m^{-it} \left(\frac{n}{m}\right)^{iv} V\left(\frac{n}{N}\right) U\left(\frac{m}{N}\right) \delta(n-m) dv. \tag{1.2}$$

where $t^{\varepsilon} < K < t$ is a parameter that will be optimized later. U is a smooth function which is supported on $[1/2, 5/2]$, with $U(x) = 1$ on $[1, 2]$ and satisfies $U^{(j)} \ll_j 1$. The extra integral introduced is

$$\frac{1}{K} \int \left(\frac{n}{m}\right)^{iv} V\left(\frac{v}{K}\right) dv.$$

For $n, m \in [N, 2N]$, integration by parts shows that the above integral is small if $|n-m| \gg Nt^{\varepsilon}/K$. This is the crucial 'trick' in the paper. As Munshi points out in the $SL_3(\mathbb{Z})$ case [8], introduction of this parameter K will seem to hurt us until the very last step, which we will justify in the proof sketch.

We can therefore write $S(N) = S^+(N) + S^-(N)$ where

$$\begin{aligned} S^{\pm}(N) =& \frac{1}{K} \int_0^1 \int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sum_{1 \leqslant q \leqslant Q < a \leqslant Q} \sideset{}{^*}\sum \frac{1}{aq} \sum_{n,m \geqslant 1} \lambda(n) n^{iv} m^{-i(t+v)} \\ & e\left(\pm \frac{(n-m)\bar{a}}{q} \mp \frac{(n-m)x}{aq}\right) V\left(\frac{n}{N}\right) U\left(\frac{m}{N}\right) dv dx. \end{aligned} \tag{1.3}$$

The analysis and bounds for $S^+(N)$ and $S^-(N)$ are similar. We therefore analyze only $S^+(N)$. We will justify later in Remark 3.2 that the natural choice for Q is $Q = (N/K)^{1/2}$ (and thus the lowering of conductor by $K^{1/2}$).

We will take

$$t^{3/4} \ll N < t^{1+\varepsilon} \text{ and } N^{1/2} \leqslant K \ll N^{1-\varepsilon} \tag{1.4}$$

In this range, we will establish the following bound.

Proposition 1.2. *For $t^{3/4} \ll N < t^{1+\epsilon}$, we have*

$$\frac{S^+(N)}{N^{1/2}} \ll t^{1/2+\varepsilon} \left(\frac{K^{1/2}}{N^{1/2}} + \frac{1}{K^{1/4}}\right). \tag{1.5}$$

Same bound holds for $S^-(N)$, and consequently for $S(N)$. The optimal choice of K is therefore $K = N^{2/3}$. With this choice of K, $S(N)/N^{1/2} \ll t^{1/2}/N^{1/6}$. For $N \ll t^{3/4}$, the trivial bound $S(N) \ll Nt^{\varepsilon}$ is sufficient. This follows by applying Cauchy's inequality to the n-sum followed by Lemma 2.2 (Ramanujan bound on average). Theorem 1.1 then follows from Lemma 2.2 and Proposition 1.2.

1.1. **Proof Sketch.** We briefly explain the steps of the proof and provide heuristics in this subsection. Temporarily assume Ramanujan conjecture $\lambda(n) \ll n^{\varepsilon}$. This is not a serious assumption, since at any step we can apply Cauchy inequality and use Lemma 2.2. The circle method is used to separate the sums on n and m, and we arrive at (1.3). Trivial estimate gives $S(N) \ll N^{2+\varepsilon}$. For simplicity, let $N \asymp t$ and $q \asymp Q$. So we are required to save N and a little more in a sum of the form

$$\int_K^{2K} \sum_{q \asymp Q} \ \sideset{}{^\star}\sum_{Q<a\leqslant Q+q} \sum_{n \asymp N} \lambda(n) n^{iv} e\left(\frac{n\bar{a}}{q} - \frac{nx}{aq}\right) \sum_{m \asymp N} m^{-i(t+v)} e\left(\frac{-m\bar{a}}{q} + \frac{mx}{aq}\right) dv.$$

The sum over m has 'conductor' $Qt \asymp N^{1/2}t/K^{1/2}$. Roughly speaking, the conductor takes into account the arithmetic modulus q, with the size $= (t+v)$ of oscillation of the analytic weight. If we assume $K \ll t^{1-\varepsilon}$, then the size of the oscillation is t, so the extra oscillation of m^{-iv} does not hurt us here. Poisson summation changes the length of summation to $Qt/N \asymp Q$, and contributes a factor of N along with a congruence condition mod q and an oscillatory integral. The oscillatory integral saves us $t^{1/2}$. In all, we will save $N/t^{1/2}$ in this step. So far the saving is independent of K. Next step is to apply Voronoi summation to the n-sum. We need to save $t^{1/2}$ in a sum of the form

$$\int_K^{2K} \sum_{q \asymp Q} \sum_{\substack{(m,q)=1 \\ |m| \ll Qt/N}} \left(\frac{(t+v)aq}{(x-ma)}\right)^{-i(t+v)} \sum_{n \asymp N} \lambda(n) e\left(\frac{nm}{q}\right) n^{iv} e\left(-\frac{nx}{aq}\right) dv,$$

where a is the unique multiplicative inverse of $m$mod q in the range $(Q, q+Q]$. Since the n-sum involves $GL(2)$ Fourier coefficients, the 'conductor' for the n-sum would be $(QK)^2$. The new length of sum would be $(QK)^2/N \asymp K$. Voronoi summation would contribute a factor of N/q, a dual additive twist and an oscillatory weight function. The oscillation in the weight function would save us $K^{1/2}$. In all, we will save $Q/K^{1/2} = N^{1/2}/K$. If K is large, we are actually making it worse. We are therefore left to save $t^{1/2}K/N^{1/2}$ in $S(N)$. Using stationary phase analysis, we will be able to save $K^{1/2}$ in the integral over v. At this point, K seems to be hurting more than helping. The final step is to get rid of the $GL(2)$ oscillations using Cauchy inequality and then change the structure using Poisson summation formula. After Cauchy, the sum roughly looks like,

$$\left[\sum_{n \ll K} \left| \sum_{\substack{q \asymp Q \\ |m| \asymp Qt/N \\ (m,q)=1}} e\left(-\frac{nm}{q}\right) \int_{-K}^{K} n^{-i\tau} g(q,m,\tau) d\tau \right|^2\right]^{1/2}.$$

where $g(q,m,\tau)$ is an oscillatory weight function of size $O(1)$. The next steps would be to open the absolute value squared and, apply Poisson to the n-sum and analyze the τ-integral. The τ-integral gives us a saving of $K^{1/2}$. After Cauchy and Poisson summation, we will save $N^{1/2}/K^{1/2}$ in the diagonal term and $K^{1/4}$ in the off-diagonal term. Saving over convexity bound in the diagonal terms is $N^{1/2}/K^{1/2}$. Saving over convexity from the off-diagonal terms is $K^{1/4}$. We will therefore get maximum saving when $N^{1/2}/K^{1/2} = K^{1/4}$, that is $K = N^{2/3}$. That gives us a saving of $N^{1/6}$ over the convexity bound of $t^{1/2+\varepsilon}$. Matching this with the trivial bound $N^{1/2}$ for $N \ll t^{3/4}$ gives us the Burgess bound.

2. $GL(2)$ Voronoi formula and Stationary phase method

2.1. **Voronoi summation formula for $SL_2(\mathbb{Z})$.** Suppose f is a holomorphic cusp form for $SL_2(\mathbb{Z})$ which is an eigenfunction for all Hecke operators with n^{th} Fourier coefficient $\lambda(n)$, normalized so that $\lambda(1) = 1$. In this subsection, we will mention two important results- a summation formula for Fourier coefficients twisted by an additive character, and a bound on the average size of these Fourier coefficients, both of which will play a crucial role in our analysis.

Let F be a smooth function compactly supported on $(0,\infty)$, and let $\tilde{F}(s)=\int_0^\infty g(x)x^{s-1}dx$ be its Mellin transform. An application of the functional equation of $L(s,f)$, followed by unwinding the integral and shifting the contour gives the Voronoi summation formula [5].

Lemma 2.1.

$$\sum_{n\geqslant 1}\lambda(n)e\left(n\frac{a}{q}\right)F(n)=\frac{1}{q}\sum_{n\geqslant 1}\lambda(n)e\left(-n\frac{\overline{a}}{q}\right)\int_0^\infty F(x)\left[2\pi i^k J_{k-1}\left(\frac{4\pi\sqrt{nx}}{q}\right)\right]dx. \tag{2.1}$$

For our calculations, we take a step back and use the following representation of J_{k-1} as an inverse Mellin transform,

$$J_{k-1}(x)=\frac{1}{2}\frac{1}{2\pi i}\int_{(\sigma)}\left(\frac{x}{2}\right)^{-s}\frac{\Gamma(s/2+(k-1)/2)}{\Gamma(1-s/2+(k-1)/2)}\quad \textit{for } 0<\sigma<1. \tag{2.2}$$

We would need to study the oscillation of the gamma factors more closely. Recall the Stirling's formula,

$$\Gamma(\sigma+i\tau)=\sqrt{2\pi}(i\tau)^{\sigma-1/2}e^{-\pi|\tau|/2}\left(\frac{|\tau|}{e}\right)^{i\tau}\left\{1+O\left(\frac{1}{|\tau|}\right)\right\}$$

as $|\tau|\to\infty$. Letting $\gamma(s)=\frac{\Gamma(s/2+(k-1)/2)}{\Gamma(1-s/2+(k-1)/2)}$, we get

$$\gamma(1+i\tau)=\left(\frac{|\tau|}{4e\pi}\right)^{i\tau}\Phi(\tau),\quad \textit{where}\quad \Phi'(\tau)\ll\frac{1}{|\tau|}. \tag{2.3}$$

We would also need the following bound, which gives Ramanujan conjecture on average. It follows from standard properties of Rankin-Selberg L-functions and is well known.

Lemma 2.2. *We have,*

$$\sum_{n\leqslant x}|\lambda(n)|^2\ll_{f,\varepsilon}x^{1+\varepsilon}.$$

2.2. **Stationary phase method.** We will need to estimate integrals of the type

$$\mathcal{I}=\int_a^b g(x)e(f(x))dx. \tag{2.4}$$

Let $\mathrm{supp}(g)\subset[a,b]$ and $g^{(j)}(x)\ll_{j,a,b}1$. Further suppose there is a $B>0$ such that for $x\in[a,b]$, $|f'(x)|\gg B$ and $f^{(j)}(x)\ll B^{1+\varepsilon}$ when $j>1$. Integration by parts j-times gives $\mathcal{I}\ll_{j,a,b,\varepsilon}B^{-j+\varepsilon}$.

In case $f'(x)=0$ at a unique point $x=x_0\in[a,b]$, there is an asymptotic expansion of the integral around x_0. x_0 is called the stationary phase. A sharp version useful for us can be found in [1, 3].

Lemma 2.3. *Suppose f and g are smooth real valued functions satisfying*

$$f^{(i)}(x)\ll\Theta_f/\Omega_f^i,\quad g^{(j)}(x)\ll 1/\Omega_g^j \tag{2.5}$$

for $i=2,3$ and $j=0,1,2$. Suppose $g(a)=g(b)=0$. Define

$$\mathcal{I}=\int_a^b g(x)e(f(x))dx.$$

(a) *Suppose f' and f'' do not vanish in $[a,b]$. Let $\Lambda = \min_{[a,b]} |f'(x)|$. Then we have*

$$\mathcal{I} \ll \frac{\Theta_f}{\Omega_f^2 \Lambda^3}\left(1 + \frac{\Omega_f}{\Omega_g} + \frac{\Omega_f^2}{\Omega_g^2}\frac{\Lambda}{\Theta_f/\Omega_f}\right). \tag{2.6}$$

(b) *Suppose f' changes sign from negative to positive at the unique point $x_0 \in (a,b)$. Let $\kappa = \min\{b - x_0, x_0 - a\}$. Further suppose that (2.5) holds for $i = 4$ and*

$$f^{(2)}(x) \gg \Theta_f/\Omega_f^2 \tag{2.7}$$

holds. Then

$$\mathcal{I} = \frac{g(x_0)e(f(x_0)+1/8)}{\sqrt{f''(x_0)}} + O\left(\frac{\Omega_f^4}{\Theta_f^2\kappa^3} + \frac{\Omega_f}{\Theta_f^{3/2}} + \frac{\Omega_f^3}{\Theta_f^{3/2}\Omega_g^2}\right). \tag{2.8}$$

We will also need a second derivative bound for integrals in two variables. Let

$$\mathcal{I}_{(2)} = \int_a^b \int_c^d g(x,y)e(f(x,y))dydx. \tag{2.9}$$

with f and g smooth real valued functions. Let $\mathrm{supp}(g) \subset (a,b) \times (c,d)$. Let r_1, r_2 be such that inside the support of the integral,

$$f^{(2,0)}(x,y) \gg r_1^2, \quad f^{(0,2)}(x,y) \gg r_2^2, \quad f^{(2,0)}(x,y)f^{(0,2)}(x,y) - \left[f^{(1,1)}(x,y)\right]^2 \gg r_1^2 r_2^2, \tag{2.10}$$

where $f^{(i,j)}(x,y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x,y)$. Then we have (see [11]),

$$\mathcal{I}_{(2)} \ll \frac{1}{r_1 r_2}.$$

Define the total variance of g by

$$var(g) := \int_a^b \int_c^d \left|\frac{\partial^2}{\partial x \partial y} g(x,y)\right| dydx.$$

Integration by parts along with the above bound gives us the following.

Lemma 2.4. *Suppose f, g, r_1, r_2 are as above and satisfy condition (2.10). Then we have*

$$\mathcal{I}_{(2)} \ll \frac{var(g)}{r_1 r_2}.$$

2.3. **An integral of interest.** Following Munshi [8], let W be a smooth real valued function with $\mathrm{supp}(W) \subset [a,b] \subset (0,\infty)$ and $W^{(j)}(x) \ll_{a,b,j} 1$. Define

$$W^{\dagger}(r,s) \int_0^\infty W(x)e(-rx)x^{s-1}dx \tag{2.11}$$

where $r \in \mathbb{R}$ and $s = \sigma + i\beta \in \mathbb{C}$. This integral is of the form (2.4) with

$$g(x) = W(x)x^{\sigma-1} \quad \textit{and} \quad f(x) = -rx + \frac{1}{2\pi}\beta \log x.$$

Then,

$$f'(x) = -r + \frac{1}{2\pi}\frac{\beta}{x} \quad \textit{and} \quad f^{(j)}(x) = (-1)^j (j-1)! \frac{1}{2\pi}\frac{\beta}{x^j}$$

for $j \geqslant 2$. The unique stationary phase occurs at $x_0 = \beta/2\pi r$. Note that we can write

$$f'(x) = \frac{\beta}{2\pi}\left(\frac{1}{x} - \frac{1}{x_0}\right) = r\left(\frac{x_0}{x} - 1\right). \tag{2.12}$$

Applying Lemma 2.3 appropriately to $W^{\dagger}(r,s)$, we get the following.

Lemma 2.5. *Let W be a smooth real valued function with $supp(W) \subset [a,b] \subset (0,\infty)$ and $W^{(j)}(x) \ll_{a,b,j} 1$. Let $r \in \mathbb{R}$ and $s = \sigma + i\beta \in \mathbb{C}$. We have*

$$W^{\dagger}(r,s) = \frac{\sqrt{2\pi}e(1/8)}{\sqrt{-\beta}} W\left(\frac{\beta}{2\pi r}\right)\left(\frac{\beta}{2\pi r}\right)^{\sigma}\left(\frac{\beta}{2\pi e r}\right)^{i\beta} + O_{a,b,\sigma}\left(\min\{|\beta|^{-3/2}, |r|^{-3/2}\}\right). \tag{2.13}$$

We also have

$$W^{\dagger}(r,s) = O_{a,b,j,\sigma}\left(\min\left\{\left(\frac{1+|\beta|}{|r|}\right)^{j}, \left(\frac{1+|r|}{|\beta|}\right)^{j}\right\}\right). \tag{2.14}$$

3. Application of dual summation formulas

3.1. Poisson summation to the m-sum.

The m-sum is given by

$$\sum_{m\geqslant 1} m^{-i(t+v)} e\left(\frac{-m\bar{a}}{q} + \frac{mx}{aq}\right) U\left(\frac{m}{N}\right) dvdx.$$

Breaking the m-sum into congruence classes modulo q, we get

$$\sum_{\alpha \bmod q} e\left(\frac{-\alpha\bar{a}}{q}\right) \sum_{m\in\mathbb{Z}} (\alpha + mq)^{-i(t+v)} e\left(\frac{(\alpha+mq)x}{aq}\right) U\left(\frac{\alpha+mq}{N}\right).$$

Poisson to the m-sum gets us

$$\sum_{\alpha \bmod q} e\left(\frac{-\alpha\bar{a}}{q}\right) \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} (\alpha + yq)^{-i(t+v)} e\left(\frac{(\alpha+yq)x}{aq}\right) U\left(\frac{\alpha+yq}{N}\right) e(-my)dy.$$

Making the change of variables $(\alpha + yq) \mapsto u$ and executing the complete character sum mod q, we arrive at

$$N^{1-i(t+v)} \sum_{\substack{m\in\mathbb{Z} \\ m\equiv\bar{a} \bmod q}} \int_{\mathbb{R}} U(u)u^{-i(t+v)} e\left(\frac{N(x-ma)}{aq}u\right) du. \tag{3.1}$$

The above integral equals

$$U^{\dagger}\left(\frac{N(ma-x)}{aq}, 1-i(t+v)\right). \tag{3.2}$$

Everything together,

$$\begin{aligned} S^{+}(N) = &\frac{1}{K}\int_0^1 \int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sum_{1\leqslant q \leqslant Q} \sideset{}{^*}\sum_{Q<a\leqslant Q} \frac{1}{aq} \sum_{n\geqslant 1} \lambda(n) n^{iv} V\left(\frac{n}{N}\right) \\ &e\left(\frac{n\bar{a}}{q} - \frac{nx}{aq}\right) N^{1-i(t+v)} \sum_{m\equiv\bar{a} \bmod q} U^{\dagger}\left(\frac{N(ma-x)}{aq}, 1-i(t+v)\right) dvdx. \end{aligned} \tag{3.3}$$

We can have $m = 0$ only when $q = 1$, in which case, $N(ma-x)/aq \ll N/Qq$, so its contribution to the sum will be negligible (as soon as Q has size).

For $m \neq 0$, we have $N(ma-x)/aq \asymp N|m|/q$. Bounds on $U^\dagger$ give

$$U^\dagger\left(\frac{N(ma-x)}{aq}, 1-i(t+v)\right) \ll_j \left(\frac{1+|t+v|}{N|m|q^{-1}}\right)^j \tag{3.4}$$

Thus we get arbitrary saving for $|m| > (1+|t+v|)q/N$. If we make sure $v < t$, that is $K < t$, we'll have arbitrary saving for $|m| \gg qt^{1+\epsilon}/N$. Noting the condition $m \equiv \bar{a} \text{mod } q$ and rearranging the sums in $S^+(N)$,

$$\begin{aligned} S^+(N) =& \frac{N}{K}\int_0^1\int_{\mathbb{R}} N^{-i(t+v)} V\left(\frac{v}{K}\right) \sum_{1\leqslant q\leqslant Q} \sum_{\substack{1\leqslant |m| \ll \frac{qt^{1+\epsilon}}{N} \\ (m,q)=1}} \frac{1}{aq} U^\dagger\left(\frac{N(ma-x)}{aq}, 1-i(t+v)\right) \\ & \sum_{n\geqslant 1} \lambda(n) n^{iv} V\left(\frac{n}{N}\right) e\left(\frac{nm}{q} - \frac{nx}{aq}\right) dv dx \end{aligned} \tag{3.5}$$

where $a \in (Q, q+Q]$ is the unique multiplicative inverse of $m \text{mod } q$.

Remark 3.1. *Trivial bound here gives $S^+(N) \ll Nt^{1+\epsilon}$. We need to save t and a bit more.*

We next split the $q-$sum into dyadic segments $(C, 2C]$

$$S^+(N) = \frac{N}{K} \sum_{1\leqslant C\leqslant Q} S(N,C)$$

where

$$\begin{aligned} S(N,C) =& \int_0^1\int_{\mathbb{R}} N^{-i(t+v)} V\left(\frac{v}{K}\right) \sum_{C< q\leqslant 2C} \sum_{\substack{1\leqslant |m| \ll \frac{qt^{1+\epsilon}}{N} \\ (m,q)=1}} \frac{1}{aq} U^\dagger\left(\frac{N(ma-x)}{aq}, 1-i(t+v)\right) \\ & \sum_{n\geqslant 1} \lambda(n) n^{iv} V\left(\frac{n}{N}\right) e\left(\frac{nm}{q} - \frac{nx}{aq}\right) dv dx. \end{aligned} \tag{3.6}$$

3.2. **Voronoi summation to the n-sum.** Applying Lemma 2.1 to the n-sum gets us

$$\begin{aligned} \sum_{n\geqslant 1} \lambda_f(n) e\left(n\frac{m}{q}\right) F(n) =& \frac{\pi i^k}{q} \sum_{n\geqslant 1} \lambda_f(n) e\left(-n\frac{a}{q}\right) \int_0^\infty y^{iv} V\left(\frac{y}{N}\right) e\left(\frac{-xy}{aq}\right) \\ & \times \frac{1}{2\pi i}\int_{(\sigma)} \left(\frac{2\pi\sqrt{ny}}{q}\right)^{-s} \frac{\Gamma(s/2+(k-1)/2)}{\Gamma(1-s/2+(k-1)/2)} ds dy \end{aligned} \tag{3.7}$$

where $F(y) = y^{iv} V(\frac{y}{N}) e(\frac{-xy}{aq})$. We want to be able to interchange integrals. For this, we use the complex Stirling approximation

$$|\Gamma(z)| = \sqrt{2\pi} e^{-\sigma} |z|^{\sigma-1/2} e^{-\tau \arg(z)} \left(1 + O\left(\frac{1}{|z|}\right)\right)$$

for $\arg(z) < \pi$ and $|z| \to \infty$. For

$$\gamma(s) = (2\pi)^{-s} \frac{\Gamma(s/2+(k-1)/2)}{\Gamma(1-s/2+(k-1)/2)}$$

we have

$$|\gamma(s)| \sim (2\pi)^{-\sigma} e^{1-\sigma} |\tau|^{\sigma-1} \quad \text{as } |\tau| \to \infty$$

Looking at the pole free regions of the $\Gamma-$factors in the definition of $\gamma(s)$, we get

$$|\gamma(s)| \ll 1 + |\tau|^{\sigma-1} \quad \text{for } \sigma > 1-k \tag{3.8}$$

We cannot apply Fubini theorem to interchange integrals right away since the integral is not absolutely convergent for $0 < \sigma < 1$. But if we assume that $k > 1$, we can shift the integral to the line $\sigma = -1/2$ without picking any residues and the integral would be absolutely convergent, allowing us to apply Fubini and interchange integrals.

$$\begin{aligned}\sum_{n\geqslant 1}\lambda_f(n)e\left(n\frac{m}{q}\right)F(n) =& \frac{\pi i^k}{q}\sum_{n\geqslant 1}\lambda_f(n)e\left(-n\frac{a}{q}\right)\frac{1}{2\pi i}\int_{(-1/2)}\left(\frac{\sqrt{n}}{q}\right)^{-s}\gamma(s)\\ &\times\int_0^\infty y^{-s/2+iv}V\left(\frac{y}{N}\right)e\left(\frac{-xy}{aq}\right)dyds\\ =&\frac{\pi i^k N^{1+iv}}{q}\sum_{n\geqslant 1}\lambda_f(n)e\left(-n\frac{a}{q}\right)\frac{1}{2\pi i}\int_{(-1/2)}\left(\frac{\sqrt{nN}}{q}\right)^{-s}\gamma(s)\\ &\times\int_0^\infty y^{-s/2+iv}V(y)e\left(\frac{-xN}{aq}y\right)dyds\\ =&\frac{\pi i^k N^{1+iv}}{q}\sum_{n\geqslant 1}\lambda_f(n)e\left(-n\frac{a}{q}\right)\frac{1}{2\pi i}\int_{(-1/2)}\left(\frac{\sqrt{nN}}{q}\right)^{-s}\gamma(s)\\ &\times V^\dagger\left(\frac{xN}{aq},1-s/2+iv\right)ds\end{aligned}$$

The bound on $V^\dagger$ gives

$$V^\dagger\left(\frac{xN}{aq},1-s/2+iv\right) \ll_j \min\left\{1,\left(\frac{1+|Nx/aq|}{|v-\tau/2|}\right)^j\right\} \tag{3.9}$$

We can therefore shift the integral from $\sigma = -1/2$ to $\sigma = M$ for any large M by choosing $j = M+1$ (which kills the growth of $\gamma(s)$). We'll thus get saving for large n.

Remark 3.2. *Using the above bound on $V^\dagger$, we get*

$$\left(\frac{\sqrt{nN}}{q}\right)^{-s}\gamma(s)V^\dagger\left(\frac{Nx}{aq},1-\frac{s}{2}+iv\right) \ll_j \left(\frac{\sqrt{nN}}{q}\right)^{-M}(1+|\tau|^{M-1})\min\left\{1,\left(\frac{1+|Nx/aq|}{|v-\tau/2|}\right)^j\right\}$$

Since $v \asymp K$, the better bound on $V^\dagger$ would be $O(1)$ when $|\tau| \leqslant 8K$. In that case,

$$\begin{aligned}\int_{|\tau|\leqslant 8K}\left(\frac{\sqrt{nN}}{q}\right)^{-s}\gamma(s)V^\dagger\left(\frac{Nx}{aq},1-\frac{s}{2}+iv\right) &\ll \int_{|\tau|\leqslant 8K}\left(\frac{\sqrt{nN}}{q}\right)^{-M}|\tau|^{M-1}d\tau\\ &\ll \left(\frac{\sqrt{nN}}{qK}\right)^{-M}\end{aligned}$$

We'll thus get arbitrary saving for $n \gg Q^2K^2t^\epsilon/N$. On the other hand, when $|\tau| > 8K$, we have the bound $V^\dagger \ll_j (N/aq|\tau|)^j$. Taking $j = M+1$,

$$\int_{|\tau|>8K} \left(\frac{\sqrt{nN}}{q}\right)^{-s} \gamma(s) V^\dagger\left(\frac{Nx}{aq}, 1-\frac{s}{2}+iv\right) \ll \int_{|\tau|>8K} \left(\frac{\sqrt{nN}}{q}\right)^{-M} |\tau|^{M-1} \left(\frac{N}{aq|\tau|}\right)^{M+1} d\tau$$
$$\ll \left(\frac{\sqrt{nN}}{q}\right)^{-M} \left(\frac{N}{aq}\right)^{M+1}$$
$$= \left(\frac{an^{1/2}}{N^{1/2}}\right)^{-M} \left(\frac{N}{aq}\right)^{2}$$

We'll thus get arbitrary saving for $n \gg Nt^\epsilon/Q^2$. It makes sense to choose Q so that the two bounds on n are equal. Therefore set $Q = (N/K)^{1/2}$. We'll get arbitrary saving for $n \gg Kt^\epsilon$.

For smaller values of n, we take $\sigma = 1$. Note that the γ factor will then be bounded.

$$\sum_{n\geqslant 1} \lambda_f(n) e\left(n\frac{m}{q}\right) F(n) = \pi i^k N^{1/2+iv} \sum_{n \ll Q^2K^2/N} \frac{\lambda_f(n)}{n^{1/2}} e\left(-n\frac{a}{q}\right) \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\sqrt{nN}}{q}\right)^{-i\tau} \times \gamma(1+i\tau) V^\dagger\left(\frac{xN}{aq}, 1/2 - i\tau/2 + iv\right) d\tau \tag{3.10}$$

Assuming $K \ll t^{1-\epsilon}$, we get arbitrary saving for $|\tau| > Nt^\epsilon/QC$ due to bounds on $V^\dagger$. Thus we can restrict the integral to $\tau \in [-Nt^\epsilon/QC, Nt^\epsilon/QC]$ by defining a smooth partition of unity on this set. Let W_J for $J \in \mathcal{J}$ be smooth bump functions satisfying $x^l W_J^{(l)} \ll_l 1$ for all $l \geqslant 0$. For $J = 0$, let the support of W_0 be in $[-1, 1]$ and for $J > 0$ (resp. $J < 0$), let the support of W_J be in $[J, 4J/3]$ (resp $[4J/3, J]$). Finally, we require that

$$\sum_{J \in \mathcal{J}} W_J(x) = 1 \quad \text{for} \quad x \in [-Nt^\epsilon/QC, Nt^\epsilon/QC]$$

The precise definition of the functions W_J will not be needed. We note that we need only $O(\log(t))$ such $J \in \mathcal{J}$. We can write the integral appearing in Voronoi summation as

$$\int_{\mathbb{R}} \left(\frac{\sqrt{nN}}{q}\right)^{-i\tau} \gamma(1+i\tau) V^\dagger\left(\frac{xN}{aq}, 1/2 - i\tau/2 + iv\right) d\tau =$$
$$\sum_{J\in\mathcal{J}} \int_{\mathbb{R}} \left(\frac{\sqrt{nN}}{q}\right)^{-i\tau} \gamma(1+i\tau) V^\dagger\left(\frac{xN}{aq}, 1/2 - i\tau/2 + iv\right) W_J(\tau) d\tau + O(t^{-20150})$$

Combining everything, we write $S(N, C)$ as

$$S(N,C) = \frac{i^k N^{1/2-it} K}{2} \sum_{J\in\mathcal{J}} \sum_{n \ll Q^2K^2/N} \frac{\lambda_f(n)}{n^{1/2}} \sum_{C < q \leqslant 2C} \sum_{\substack{(m,q)=1 \\ 1 \leqslant |m| \ll \frac{qt^{1+\epsilon}}{N}}} e\left(\frac{-na}{q}\right) \frac{1}{aq} \int_{\mathbb{R}} \left(\frac{\sqrt{nN}}{q}\right)^{-i\tau} \times \gamma(1+i\tau) W_J(\tau) \mathcal{I}^{**}(q, m, \tau) d\tau + O(t^{-2015}) \tag{3.11}$$

where

$$\mathcal{I}^{**}(q,m,\tau) = \int_0^1 \int_{\mathbb{R}} V(v) U^{\dagger}\left(\frac{N(ma-x)}{aq}, 1-i(t+Kv)\right) V^{\dagger}\left(\frac{Nx}{aq}, \frac{1}{2}-\frac{i\tau}{2}+iKv\right) dvdx$$

Remark 3.3. *We can trivially bound $I^{**}(q,m,\tau)$ by $O(1)$, and the τ-integral is over the interval $[-Nt^{\varepsilon}/QC, Nt^{\varepsilon}/QC]$. Trivial bound on $S(N,C)$ will imply $S(N,C) \ll K^{5/2}t^{1+\varepsilon}/N^{1/2}$. So $S(N) \ll N^{1/2}K^{3/2}t^{1+\varepsilon}$. We need to save $N^{1/2}K^{3/2}$ and a bit more.*

4. Analysis of the integrals

We next analyze the integral $\mathcal{I}^{**}(q,m,\tau)$. Application of Lemma 2.5 to $U^{\dagger}$ gives us

$$U^{\dagger} = \frac{e^{i\pi/4}(t+Kv)^{1/2}aq}{(2\pi)^{1/2}N(x-ma)} U\left(\frac{(t+Kv)aq}{2\pi N(x-ma)}\right)\left(\frac{(t+Kv)aq}{2\pi eN(x-ma)}\right)^{-i(t+Kv)} + O(t^{-3/2}).$$

Therefore,

$$\begin{aligned}\mathcal{I}^{**}(\tau) = \frac{c_1 aq}{N}\int_0^1\int_{\mathbb{R}} V(v)V^{\dagger}\left(\frac{Nx}{aq}, \frac{1}{2}-\frac{i\tau}{2}+iKv\right)\frac{(t+Kv)^{1/2}}{(x-ma)}U\left(\frac{(t+Kv)aq}{2\pi N(x-ma)}\right)\\ \times\left(\frac{(t+Kv)aq}{2\pi eN(x-ma)}\right)^{-i(t+Kv)} dvdx + O(t^{-3/2+\epsilon})\end{aligned}$$

where $c_1 = e^{i\pi/4}/\sqrt{2\pi}$. We next apply Lemma 2.5 to $V^{\dagger}$.

$$\begin{aligned}V^{\dagger} = \frac{2\sqrt{\pi}e^{-i\pi/4}}{(4\pi)^{1/2}}\left(\frac{aq}{Nx}\right)^{1/2} V\left(\frac{(2Kv-\tau)aq}{4\pi Nx}\right)\left(\frac{(2Kv-\tau)aq}{4\pi eNx}\right)^{i(Kv-\tau/2)}\\ + O\left(\min\left\{\left(\frac{aq}{Nx}\right)^{3/2}, \frac{1}{|\tau/2-Kv|^{3/2}}\right\}\right).\end{aligned}$$

The integral then becomes

(4.1)
$$\begin{aligned}\mathcal{I}^{**}(q,m,\tau) = c_2\left(\frac{aq}{N}\right)^{3/2}\int_0^1\int_{\mathbb{R}} V(v)\left(\frac{1}{x}\right)^{1/2} V\left(\frac{(2Kv-\tau)aq}{4\pi Nx}\right)\left(\frac{(2Kv-\tau)aq}{4\pi eNx}\right)^{i(Kv-\tau/2)}\\ \times\frac{(t+Kv)^{1/2}}{(x-ma)}U\left(\frac{(t+Kv)aq}{2\pi N(x-ma)}\right)\left(\frac{(t+Kv)aq}{2\pi eN(x-ma)}\right)^{-i(t+Kv)} dvdx\\ + O(E^{**}+t^{-3/2+\epsilon})\end{aligned}$$

with $c_2 = 1/(2\pi)^{1/2}$ and since $uU(u) \ll 1$,

$$E^{**} = \frac{1}{t^{1/2}}\int_0^1\int_1^2 \min\left\{\left(\frac{aq}{Nx}\right)^{3/2}, \frac{1}{|\tau/2-Kv|^{3/2}}\right\} dvdx$$

(We note that more generally $u^jU(u) \ll_j 1$, but using this does not improve the error term.)

4.1. **Analysis of the error term E^{**}.** The first term is smaller than the second if and only if

$$\frac{\tau}{2K} - \frac{Nx}{aqK} < v < \frac{\tau}{2K} + \frac{Nx}{aqK}.$$

If $|\tau| \geqslant 10K$, this interval does not intersect $[1,2]$ unless $Nx/aq \asymp |\tau|$. For this, we use the trivial bound $O(1)$ for the inner integral over v. And if $|\tau| < 10K$, the inner integral is bounded by the length of the interval, which is $2Nx/aqK$. Hence the contribution where the first term is smaller than the second is of the order

$$\frac{1}{t^{1/2}}\int_0^1 \left(\frac{aq}{Nx}\right)^{1/2} \frac{1}{K}\mathbf{1}_{|\tau|<10K} dx + \frac{1}{t^{1/2}}\int_0^1 \left(\frac{aq}{Nx}\right)^{1/2} \frac{1}{|\tau|}\mathbf{1}_{|\tau|\geqslant 10K} dx.$$

This is bounded by

$$O\left(\frac{Q}{t^{1/2}N^{1/2}K}\min\left\{1, \frac{10K}{|\tau|}\right\} t^{\epsilon}\right).$$

Next we estimate the contribution to E^{**} when the second term is smaller. This would be

$$\frac{1}{t^{1/2}} \underset{|\tau/2-Kv|>Nx/aq}{\int_0^1\int_1^2} \frac{1}{|\tau-Kv|^{3/2}} dvdv \ll \frac{1}{t^{1/2}}\int_0^1 \left(\frac{aq}{Nx}\right)^{1/2+\epsilon}\int_1^2 \frac{1}{|\tau/2-Kv|^{1-\epsilon}} dvdx$$

$$\ll t^{\epsilon}\frac{Q}{t^{1/2}N^{1/2}K}\min\left\{1, \frac{10K}{|\tau|}\right\}.$$

The total error term therefore is

$$E^{**} + t^{-3/2+\epsilon} \ll t^{\epsilon}\frac{Q}{t^{1/2}N^{1/2}K}\min\left\{1, \frac{10K}{|\tau|}\right\} + t^{-3/2+\epsilon}, \tag{4.2}$$

and we can write

(4.3)

$$\begin{aligned}\mathcal{I}^{**}(q,m,\tau) = c_2\left(\frac{aq}{N}\right)^{3/2}\int_0^1\int_{\mathbb{R}} V(v)\left(\frac{1}{x}\right)^{1/2} V\left(\frac{(2Kv-\tau)aq}{4\pi Nx}\right)\left(\frac{(2Kv-\tau)aq}{4\pi eNx}\right)^{i(Kv-\tau/2)}\\ \times\frac{(t+Kv)^{1/2}}{(x-ma)}U\left(\frac{(t+Kv)aq}{2\pi N(x-ma)}\right)\left(\frac{(t+Kv)aq}{2\pi eN(x-ma)}\right)^{-i(t+Kv)} dvdx\\ + O\left(\frac{t^{\epsilon}}{t^{1/2}K^{3/2}}\min\left\{1, \frac{10K}{|\tau|}\right\} + t^{-3/2+\epsilon}\right).\end{aligned}$$

Remark 4.1. *The error term in the above estimate for I^{**} saves a further $t^{1/2}K^{3/2}$. The main term saves $K^{1/2}t^{1/2}$. So we need to save K and a bit more. Note that at this point K seems to be hurting us rather than helping us. Moreover, if K had no size, we would get the bound $S(N) \ll N^{1+\varepsilon}$, which would get us the convexity bound.*

4.2. **Analysis of integral over v.** The integral is given by

$$\begin{aligned}I_1 = c_2\left(\frac{aq}{N}\right)^{3/2}\int_0^1\int_{\mathbb{R}} V(v)\left(\frac{1}{x}\right)^{1/2} V\left(\frac{(2Kv-\tau)aq}{4\pi Nx}\right)\left(\frac{(2Kv-\tau)aq}{4\pi eNx}\right)^{i(Kv-\tau/2)}\\ \times\frac{(t+Kv)^{1/2}}{(x-ma)}U\left(\frac{(t+Kv)aq}{2\pi N(x-ma)}\right)\left(\frac{(t+Kv)aq}{2\pi eN(x-ma)}\right)^{-i(t+Kv)} dvdx\end{aligned}$$

Due to the argument of U, the integral vanishes if $m>0$. Trivial estimate gives

$$I_1 \ll \left(\frac{aq}{N}\right)^{3/2}\int_0^1\int_{\mathbb{R}}\frac{(t+Kv)^{1/2}}{x^{1/2}(x-ma)}V(v)V\left(\frac{(2Kv-\tau)aq}{4\pi Nx}\right)U\left(\frac{(t+Kv)aq}{2\pi N(x-ma)}\right)dvdx$$

The length of the integral over v is restricted due to the weight functions, respectively given by $1, -Nm/Kq$ and Nx/aqK. $Nx/aqK < -Nm/Kq$, so we can restrict the length of integral over v to Nx/aqK. We restrict the integral over x to $[0,1/K]$ and estimate the resulting integral trivially.

$$\begin{aligned}
&\left(\frac{aq}{N}\right)^{3/2}\int_0^{1/K}\int_{\mathbb{R}}\frac{(t+Kv)^{1/2}}{x^{1/2}(x-ma)}V(v)V\left(\frac{(2Kv-\tau)aq}{4\pi Nx}\right)U\left(\frac{(t+Kv)aq}{2\pi N(x-ma)}\right)dvdx\\
&\ll \left(\frac{aq}{N}\right)^{1/2}\frac{1}{t^{1/2}}\int_0^{1/K}\frac{1}{x^{1/2}}\frac{Nx}{aqK}dx\\
&\ll \frac{1}{t^{1/2}K^{3/2+1}}\left(\frac{N}{aq}\right)^{1/2} = E
\end{aligned}$$

We write $I_1(\tau)=I_2(\tau)+O(E)$, where $I_2(\tau)$ is

$$\begin{aligned}
I_2 = c_2\frac{1}{t^{1/2}}\left(\frac{aq}{N}\right)^{3/2}\int_{1/K}^1\int_{\mathbb{R}}\frac{t^{1/2}(t+Kv)^{1/2}}{(x-ma)x^{1/2}}V(v)V\left(\frac{(2Kv-\tau)aq}{4\pi Nx}\right)\left(\frac{(2Kv-\tau)aq}{4\pi eNx}\right)^{i(Kv-\tau/2)}\\
\times U\left(\frac{(t+Kv)aq}{2\pi N(x-ma)}\right)\left(\frac{(t+Kv)aq}{2\pi eN(x-ma)}\right)^{-i(t+Kv)}dvdx
\end{aligned}$$

where an extra $t^{1/2}$ is multiplied to balance the size of the function. Set

$$f(v) = -\frac{t+Kv}{2\pi}\log\left(\frac{(t+Kv)aq}{2\pi eN(x-ma)}\right)+\frac{2Kv-\tau}{4\pi}\log\left(\frac{(2Kv-\tau)aq}{4\pi eNx}\right)$$

and

$$g(v) = \frac{t^{1/2}(t+Kv)^{1/2}aq}{N(x-ma)}V(v)V\left(\frac{(2Kv-\tau)aq}{4\pi Nx}\right)U\left(\frac{(t+Kv)aq}{2\pi N(x-ma)}\right)$$

So that

$$I_2 = c_2\frac{1}{t^{1/2}}\left(\frac{aq}{N}\right)^{1/2}\int_{1/K}^1\frac{1}{x^{1/2}}\int_{\mathbb{R}}g(v)e(f(v))dvdx$$

Then

$$f'(v) = -\frac{K}{2\pi}\log\left(\frac{2(t+Kv)x}{(2Kv-\tau)(x-ma)}\right),\quad f^{(j)}(v) = -\frac{(j-1)!(-K)^j}{2\pi(t+Kv)^{j-1}}+\frac{(j-1)!(-2K)^j}{4\pi(2Kv-\tau)^{j-1}}$$

The stationary phase is given by

$$v_0 = -\frac{(2t+\tau)x-\tau ma}{2Kma}$$

In support of the integral, we have

$$f^{(j)}(v) \asymp \frac{Nx}{aq}\left(\frac{Kaq}{Nx}\right)^j$$

for $j \geqslant 2$, and for $j \geqslant 0$

$$g^{(j)}(v) \ll \left(1 + \frac{Kaq}{Nx}\right)^j$$

We shall apply the sharp version of stationary phase method due to Huxley[3] (as given in Lemma 3 of Munshi[8]):

We can write

$$f'(v) = \frac{K}{2\pi}\log\left(1 + \frac{K(v_0 - v)}{(t + Kv)}\right) - \frac{K}{2\pi}\log\left(1 + \frac{2K(v_0 - v)}{(2Kv - \tau)}\right)$$

In the support of the integral, we have $0 \leqslant 2Kv - \tau \ll N/aq \ll t^{1+\epsilon}/Q$ (since $N/t^{1+\epsilon} < q$ and $a \asymp Q$). Therefore

$$f''(v) = -\frac{K^2}{2\pi(t + Kv)} + \frac{K^2}{2\pi(Kv - \tau/2)}$$

is positive on the support of the integral for large enough t. So f' changes sign from negative to positive at v_0. Support of the integral is contained in $[1, 2]$ due the weight function $V(v)$. If $v_0 \notin [0.5, 2.5]$, then v_0 is not in the support of the integral and $|v_0 - v| > 0.5$. In the support of the integral, we will have

$$|f'(v)| \gg K^{1-\epsilon} \min\left\{1, \frac{Kaq}{Nx}\right\}$$

Applying the first statement of Lemma (2.3) with

$$\Theta_f = \frac{Nx}{aq}, \quad \Omega_f = \frac{Nx}{Kaq}, \quad \Omega_g = \min\left\{1, \frac{Nx}{Kaq}\right\}, \quad \Lambda = K^{1-\epsilon}\min\left\{1, \frac{Kaq}{Nx}\right\}$$

we obtain the bound

$$\int_{\mathbb{R}} g(x)e(f(x))dx \ll \frac{\Theta_f}{\Omega_f^2\Lambda^3}\left(1 + \frac{\Omega_f}{\Omega_g} + \frac{\Omega_f^2}{\Omega_g^2}\frac{\Lambda}{\Theta_f/\Omega_f}\right)t^\epsilon \tag{4.4}$$

On the other hand, if $v_0 \in [0.5, 2.5]$, then treating the integral as one over the finite range $[0.1, 4]$ (so that $\kappa > 0.4$) and applying the second part of Lemma (2.3), we get

$$I = \frac{g(x_0)e(f(x_0) + 1/8)}{\sqrt{f''(x_0)}} + O\left(\left(\frac{\Omega_f^4}{\Theta_f^2} + \frac{\Omega_f}{\Theta_f^{3/2}} + \frac{\Omega_f^3}{\Theta_f^{3/2}\Omega_g^2}\right)t^\epsilon\right) \tag{4.5}$$

For the range $x \in [1/K, 1]$, we use the bound in lemma (2.3). In the case there is no stationary phase, we will use the first statement of lemma (2.3). We have,

$$\Theta_f = \frac{Nx}{aq}, \quad \Omega_f = \frac{Nx}{aqK}, \quad \Lambda = K^{1-\varepsilon}\min\left\{1, \frac{Kaq}{Nx}\right\}, \quad \Omega_g = \min\left\{1, \frac{Nx}{aqK}\right\}. \tag{4.6}$$

Next is the contribution of $x \in [1/K, 1]$ when there is no stationary phase. When $x < aqK/N$, $\Lambda = K$ and $\Omega_g = \Omega_f$. In that case, the contribution is

$$\left(\frac{2\pi aq}{Nt}\right)^{1/2}\int_{1/K}^{\max\{\frac{1}{K}, \frac{Kaq}{N}\}} \frac{1}{x^{1/2}}\frac{aq}{NKx}dx \ll \frac{1}{t^{1/2}K^2}.$$

This is always smaller than the contribution of the bound E. When $x > aqK/N$, $\Lambda = K^2aq/Nx$ and $\Omega_g = 1$. In that case, the contribution is $1/K^3t^{1/2}$, which is better than above. We next calculate the contribution of the error term when there is a stationary phase. For that we have $\kappa > 0.4$. One can calculate that for both $x < aqK/N$ and $x > aqK/N$, the contribution is $1/K^2t^{1/2}$.

With all of this, we summarize the analysis in the following Lemma. Let

$$B(C,\tau) = \frac{t^{\varepsilon}}{t^{1/2}K^{3/2}} \min\left\{1, \frac{10K}{|\tau|}\right\} + \frac{1}{t^{1/2}K^{5/2}}\left(\frac{N}{QC}\right)^{1/2}. \tag{4.7}$$

Note that,

$$\int_{-Nt^{\varepsilon}/QC}^{Nt^{\varepsilon}/QC} B(C,\tau)d\tau \ll \frac{K}{t^{1/2}K^{3/2}} + \frac{1}{t^{1/2}K^{5/2}}\left(\frac{N}{QC}\right)^{3/2}. \tag{4.8}$$

Putting everything together, we have

Lemma 4.2. *Suppose $C < q \leqslant 2C$, with $1 \ll C \leqslant (N/K)^{1/2}$ and K satisfies $1 \leqslant K \ll t^{1-\epsilon}$. Suppose $t > 2$ and $|\tau| \ll N^{1/2}K^{1/2}t^{\epsilon}$. We have*

$$\mathcal{I}^{**}(q,m,\tau) = \mathcal{I}_1(q,m,\tau) + \mathcal{I}_2(q,m,\tau)$$

where

$$\mathcal{I}_1(q,m,\tau) = \frac{c_4}{(t+\tau/2)^{1/2}K}\left(-\frac{(t+\tau/2)q}{2\pi eNm}\right)^{3/2-i(t+\tau/2)} V\left(-\frac{(t+\tau/2)q}{2\pi Nm}\right)\int_0^1 V\left(\frac{\tau}{2K} - \frac{(t+\tau/2)x}{Kma}\right)dx$$

for some absolute constant c_4 and

$$\mathcal{I}_2(q,m,\tau) := \mathcal{I}^{**}(q,m,\tau) - \mathcal{I}_1(q,m,\tau) = O(B(C,\tau)t^{\epsilon})$$

with $B(C,\tau)$ as defined in (4.7).

Consequently, we have the following decomposition of $S(N,C)$.

Lemma 4.3.

$$S(N,C) = \sum_{J\in\mathcal{J}}\{S_{1,J}(N,C) + S_{2,J}(N,C)\} + O(t^{-2015})$$

where

$$S_{l,J}(N,C) = \frac{i^k N^{1/2-it}K}{2}\sum_{n\ll Q^2K^2/N}\frac{\lambda_f(n)}{n^{1/2}}\sum_{C<q\leqslant 2C}\sum_{\substack{(m,q)=1\\ 1\leqslant|m|\ll\frac{qt^{1+\epsilon}}{N}}} e\left(\frac{-na}{q}\right)\frac{1}{aq}\mathcal{I}_{l,J}(q,m,n)$$

and

$$\mathcal{I}_{l,J}(q,m,n) = \int_{\mathbb{R}}\left(\frac{\sqrt{nN}}{q}\right)^{-i\tau}\gamma(1+i\tau)\,W_J(\tau)\mathcal{I}_l(q,m,\tau)d\tau$$

with $\mathcal{I}_l(q,m,\tau)$ as defined in the previous lemma.

Remark 4.4. *The saving due to $\mathcal{I}_1(q,m,\tau)$ is still $t^{1/2}K^{1/2}$, same as the main term before this analysis. The saving due to $\mathcal{I}_2(q,m,\tau)$ is $t^{1/2}K^{9/4}/N^{1/4}$. In all, we need to save* $\max\{K, t^{1/4}/K^{3/4}\}$ *and a bit more.*

5. Application of Cauchy and Poisson summation- I

In this section, we will estimate

$$S_2(N,C) := \sum_{J\in\mathcal{J}} S_{2,J}(N,C)$$

Here, we'll not apply any cancellation over the τ-integral. Dividing the n-sum into dyadic segments and using the bound $\gamma(1+i\tau) \ll 1$, we get

(5.1)

$$S_2(N,C) \ll t^{\epsilon}N^{1/2}K\int_{-\frac{(NK)^{1/2}t^{\epsilon}}{C}}^{\frac{(NK)^{1/2}t^{\epsilon}}{C}} \sum_{\substack{1\leqslant L\ll Kt^{\epsilon}\\ dyadic}} \sum_n \frac{|\lambda_f(n)|}{n^{1/2}} U\left(\frac{n}{L}\right)\Bigg|\sum_{\substack{C<q\leqslant 2C\\ 1\leqslant |m|\ll \frac{qt^{1+\epsilon}}{N}}} \sum_{(m,q)=1} e\left(\frac{-na}{q}\right)\frac{1}{aq^{1-i\tau}}\mathcal{I}_2(q,m,\tau)\Bigg| d\tau$$

Applying Cauchy to the $n-$sum and using the Ramanujan bound on average (Lemma 2.2), we get

$$S_2(N,C) \ll t^{\epsilon}N^{1/2}K\int_{-\frac{(NK)^{1/2}t^{\epsilon}}{C}}^{\frac{(NK)^{1/2}t^{\epsilon}}{C}} \sum_{\substack{1\leqslant L\ll Kt^{\epsilon}\\ dyadic}} L^{1/2}[S_2(N,C,L,\tau)]^{1/2}d\tau \tag{5.2}$$

where

$$\begin{aligned} S_2(N,C,L,\tau) &= \sum_n \frac{1}{n}U\left(\frac{n}{L}\right) \mid \sum_{\substack{C<q\leqslant 2C\\ 1\leqslant |m|\ll \frac{qt^{1+\epsilon}}{N}}} \sum_{(m,q)=1} e\left(\frac{-na}{q}\right)\frac{1}{aq^{1-i\tau}}\mathcal{I}_2(q,m,\tau) \mid^2 \\ &= \sum_n \frac{1}{n}U\left(\frac{n}{L}\right) \sum_{\substack{C<q\leqslant 2C\\ 1\leqslant |m|\ll \frac{qt^{1+\epsilon}}{N}}} \sum_{(m,q)=1} e\left(\frac{-na}{q}\right)\frac{1}{aq^{1-i\tau}}\mathcal{I}_2(q,m,\tau) \\ &\quad \times \sum_{\substack{C<q'\leqslant 2C\\ 1\leqslant |m'|\ll \frac{q't^{1+\epsilon}}{N}}} \sum_{(m',q')=1} e\left(\frac{na'}{q'}\right)\frac{1}{a'q'^{1+i\tau}}\overline{\mathcal{I}_2(q',m',\tau)} \\ &= \sum_{\substack{C<q\leqslant 2C\\ 1\leqslant |m|\ll \frac{qt^{1+\epsilon}}{N}}} \sum_{(m,q)=1} \sum_{\substack{C<q'\leqslant 2C\\ 1\leqslant |m'|\ll \frac{q't^{1+\epsilon}}{N}}} \sum_{(m',q')=1} \frac{1}{aq^{1-i\tau}}\frac{1}{a'q'^{1+i\tau}}\mathcal{I}_2(q,m,\tau)\overline{\mathcal{I}_2(q',m',\tau)}T \end{aligned}$$

where we set

$$T = \sum_n \frac{1}{n}U\left(\frac{n}{L}\right)e\left(\frac{-na}{q}\right)e\left(\frac{na'}{q'}\right)$$

We break the $n-$sum modulo qq' to get

$$T = \sum_{\beta \bmod qq'} e\left(\frac{\beta(a'q-aq')}{qq'}\right)\sum_{l\in\mathbb{Z}}\frac{1}{\beta+lqq'}U\left(\frac{\beta+lqq'}{L}\right)$$

Applying Poisson summation formula to $l-$sum,

$$T = \sum_{\beta \bmod qq'} e\left(\frac{\beta(a'q-aq')}{qq'}\right)\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\frac{1}{\beta+yqq'}U\left(\frac{\beta+yqq'}{L}\right)e(-ny)dy$$

Change variables $w = (\beta + yqq')/L$ to get

$$T = \frac{1}{qq'} \sum_{\beta \bmod qq'} e\left(\frac{\beta(a'q - aq')}{qq'}\right) \sum_{n \in \mathbb{Z}} e\left(\frac{n\beta}{qq'}\right) \int_{\mathbb{R}} \frac{1}{w} U(w) e\left(\frac{-nLw}{qq'}\right) dw$$

Integration by parts will give arbitrary saving for $n \gg C^2 t^\epsilon / L$. Thus,

$$T = \frac{1}{qq'} \sum_{n \ll \frac{C^2 t^\epsilon}{L}} \left[\sum_{\beta \bmod qq'} e\left(\frac{\beta(a'q - aq')}{qq'}\right) e\left(\frac{n\beta}{qq'}\right) \right] \int_{\mathbb{R}} \frac{1}{w} U(w) e\left(\frac{-nLw}{qq'}\right) dw + O(t^{-2015})$$

Plugging this in the expression for $S_2(N, C, L, \tau)$, we get

$$S_2(N, C, L, \tau) \ll \frac{K}{NC^4} B(C, \tau)^2 \sum_{\substack{C < q \leqslant 2C \\ 1 \leqslant |m| \ll \frac{qt^{1+\epsilon}}{N}}} \sum_{(m,q)=1} \sum_{\substack{C < q' \leqslant 2C \\ 1 \leqslant |m'| \ll \frac{q' t^{1+\epsilon}}{N}}} \sum_{(m',q')=1} \sum_{n \ll \frac{C^2 t^\epsilon}{L}} |\mathfrak{C}| + O(t^{-2015})$$

where

$$\mathfrak{C} = \sum_{\beta \bmod qq'} e\left(\frac{\beta(a'q - aq')}{qq'}\right) e\left(\frac{n\beta}{qq'}\right)$$

Note that $\mathfrak{C} = qq'\delta(n \equiv aq' - a'q \mod qq')$. Plugging that into the above expression and rearranging the sums, we get

Lemma 5.1.

$$S_2(N, C, L, \tau) \ll \frac{K}{NC^2} B(C, \tau)^2 \sum_{n \ll \frac{C^2 t^\epsilon}{L}} \sum_{\substack{C < q \leqslant 2C \\ 1 \leqslant |m| \ll \frac{qt^{1+\epsilon}}{N}}} \sum_{(m,q)=1} \sum_{\substack{C < q' \leqslant 2C \\ 1 \leqslant |m'| \ll \frac{q' t^{1+\epsilon}}{N}}} \sum_{(m',q')=1} \delta(n \equiv aq' - a'q \mod qq') + O(t^{-2015})$$

We have to analyze the cases $n = 0$ and $n \neq 0$ separately. When $n = 0$, the congruence condition above gives $q = q'$ and $a = a'$. For a given m, this fixes m' up to a factor of $t^{1+\varepsilon}/N$. Moreover, in the case $Q^2 < K$, that is, $K > N^{1/2}$, we'll have only $n = 0$ for $L > C^2$. Therefore for $n \neq 0$, we will let L go up to $\min\{C^2, K\}$.

We note that the congruence condition implies $q | (n - aq')$ and $q' | (n + a'q)$. Since a and a' lie in an interval of length q, fixing n, q and q' fixes both a and a'. That saves q, q' in the m, m'-sums respectively.

Remark 5.2. *We haven't used the conditions $(a, q) = 1$ and $(a', q') = 1$. But we can show that these conditions give us a saving of at most a power of* $\log t$.

Using $\mathcal{I}_2(q, m, \tau) \ll B(C, \tau)$, we get

$$S_2(N, C, L, \tau) \ll t^\epsilon \frac{Kt^2 B(C, \tau)^2}{N^3} \left[\underbrace{1}_{n=0} + \underbrace{\frac{C^2}{L}}_{n \neq 0} \right]$$

so that

$$S_2(N, C, L, \tau)^{1/2} \ll t^\epsilon \frac{K^{1/2} t B(C, \tau)}{N^{3/2}} \left[1 + \frac{C}{L^{1/2}} \right]$$

Therefore,

$$S_2(N, C) \ll t^\epsilon N^{1/2} K \int_{-\frac{(NK)^{1/2} t^\epsilon}{C}}^{\frac{(NK)^{1/2} t^\epsilon}{C}} \left[\sum_{\substack{1 \leqslant L \ll Kt^\epsilon \\ dyadic}} L^{1/2} . \frac{K^{1/2} t B(C, \tau)}{N^{3/2}} + \sum_{\substack{1 \leqslant L \ll \min\{C^2, K\} t^\epsilon \\ dyadic}} \frac{K^{1/2} t C B(C, \tau)}{N^{3/2}} \right] d\tau$$

If $K \geqslant N^{1/2}$, then the contribution of the second term is smaller than that of the first. So we neglect the second term. Summing over L, using (4.8) (and noting $N \asymp t^{1+\varepsilon}$), we get

$$S_2(N,C) \ll t^{\epsilon}\frac{K^2 t}{N}\left(\frac{1}{t^{1/2}K^{1/2}} + \frac{1}{t^{1/2}K^{5/2}}\left(\frac{N}{QC}\right)^{3/2}\right)$$

Multiplying by $N^{1/2}/K$ and summing over C dyadically,

$$\frac{S_2(N)}{N^{1/2}} \ll t^{1/2+\varepsilon}\left(\frac{K^{1/2}}{N^{1/2}} + \frac{N^{1/4}}{K^{3/4}}\right) \tag{5.3}$$

where $K \geqslant N^{1/2}$.

6. Application of Cauchy and Poisson summation- II

$$\mathcal{I}_1(q,m,\tau) = \frac{c_4}{(t+\tau/2)^{1/2}K}\left(-\frac{(t+\tau/2)q}{2\pi eNm}\right)^{3/2-i(t+\tau/2)} V\left(-\frac{(t+\tau/2)q}{2\pi Nm}\right)\int_0^1 V\left(\frac{\tau}{2K} - \frac{(t+\tau/2)x}{Kma}\right)dx$$

$$S_{1,J}(N,C) = \frac{i^k N^{1/2-it}K}{2}\sum_{\substack{1\leqslant L\ll Kt^{\epsilon}\\ Ldyadic}}\sum_n \frac{\lambda_f(n)}{n^{1/2}}U\left(\frac{n}{L}\right)\sum\sum_{\substack{C<q\leqslant 2C,(m,q)=1\\ 1\leqslant |m|\ll qt^{1+\epsilon}/N}} e\left(\frac{-na}{q}\right)\frac{1}{aq}$$
$$\times\int_{\mathbb{R}}\left(\frac{\sqrt{nN}}{q}\right)^{-i\tau}\gamma(1+i\tau)W_J(\tau)I_1(q,m,\tau)d\tau$$

Using the two, rearranging $q,m-$sums and integral, taking absolute values and using Cauchy, we get

$$|S_{1,J}(N,C)| \leqslant N^{1/2}K\sum_{\substack{1\leqslant L\ll Kt^{\epsilon}\\ L-dyadic}}\left(\sum_n |\lambda_f(n)|^2 U\left(\frac{n}{L}\right)\right)^{1/2}[S_{1,J}(N,C,L)]^{1/2} \tag{6.1}$$

where

(6.2)

$$S_{1,J}(N,C,L) = \sum_n \frac{1}{n}U\left(\frac{n}{L}\right)\ |\int_{\mathbb{R}}(\sqrt{nN})^{-i\tau}\gamma(1+i\tau)\sum\sum_{\substack{C<q\leqslant 2C,(m,q)=1\\ 1\leqslant |m|\ll qt^{1+\epsilon}/N}} e\left(\frac{-na}{q}\right)\frac{1}{aq^{1-i\tau}}W_J(\tau)I_1(q,m,\tau)d\tau\ |^2$$

Opening $|...|^2$ and rearranging sums and integrals

$$S_{1,J}(N,C,L) = \iint_{\mathbb{R}^2}(\sqrt{N})^{-i\tau+i\tau'}\gamma(1+i\tau)\overline{\gamma(1+i\tau')}W_J(\tau)W_J(\tau')$$
$$\times\sum\sum_{\substack{C<q\leqslant 2C,(m,q)=1\\ 1\leqslant |m|\ll qt^{1+\epsilon}/N}}\sum\sum_{\substack{C<q'\leqslant 2C,(m',q')=1\\ 1\leqslant |m'|\ll q't^{1+\epsilon}/N}}\frac{1}{aq^{1-i\tau}}\frac{1}{aq'^{1+i\tau'}}I_1(q,m,\tau)\overline{I_1(q',m',\tau')}\mathbf{T}d\tau d\tau'$$

where

$$\mathbf{T} = \sum_n n^{-1+\frac{-i\tau+i\tau'}{2}}U\left(\frac{n}{L}\right)e\left(\frac{n(a'q-aq')}{qq'}\right)$$

Analyzing $\mathbf{T}$: Breaking the sum modulo qq',

$$\mathbf{T} = \sum_{\beta(qq')} e\left(\frac{\beta(a'q-aq')}{qq'}\right)\sum_{l\in\mathbb{Z}}(\beta+qq'l)^{-1+\frac{-i\tau+i\tau'}{2}}U\left(\frac{\beta+qq'l}{L}\right)$$

applying Poisson summation to the $l-$sum,

$$\mathbf{T} = \sum_{\beta(qq')} e\left(\frac{\beta(a'q-aq')}{qq'}\right)\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}(\beta+qq'y)^{-1+\frac{-i\tau+i\tau'}{2}}U\left(\frac{\beta+qq'y}{L}\right)e(-ny)dy$$

and changing variables $w=(\beta+qq'y)/L$,

$$\begin{aligned}\mathbf{T} &= \frac{1}{qq'}\sum_{\beta(qq')} e\left(\frac{\beta(a'q-aq')}{qq'}\right)\sum_{n\in\mathbb{Z}}e\left(\frac{n\beta}{qq'}\right)L^{(-i\tau+i\tau')/2}U^{\dagger}\left(\frac{nL}{qq'},-\frac{i\tau}{2}+\frac{i\tau'}{2}\right)\\ &= \frac{L^{-i\tau/2+i\tau'/2}}{qq'}\sum_{n\in\mathbb{Z}}\mathfrak{C}U^{\dagger}\left(\frac{nL}{qq'},-\frac{i\tau}{2}+\frac{i\tau'}{2}\right)\end{aligned}$$

with $\mathfrak{C}$ as before. Since $|\tau-\tau'|\ll(NK)^{1/2}t^{\epsilon}/C$, the bound on $U^{\dagger}$ gives arbitrary saving for $|n|\gg C(NK)^{1/2}t^{\epsilon}/L$. We therefore get

Lemma 6.1.

$$S_{1,J}(N,C,L)\ll\frac{K}{NC^4}\sum\sum_{\substack{C<q\leqslant 2C,(m,q)=1\\ 1\leqslant|m|\ll qt^{1+\epsilon}/N}}\sum\sum_{\substack{C<q'\leqslant 2C,(m',q')=1\\ 1\leqslant|m'|\ll q't^{1+\epsilon}/N}}\sum_{|n|\ll C(NK)^{1/2}t^{\epsilon}/L}|\mathfrak{C}||\mathfrak{K}|+O(t^{-2015}) \tag{6.3}$$

where

$$\mathfrak{K}=\iint_{\mathbb{R}^2}(NL)^{-i\tau/2+i\tau'/2}\gamma(1+i\tau)\overline{\gamma(1+i\tau')}\frac{1}{q^{-i\tau}q'^{i\tau}}W_J(\tau)W_J(\tau')\mathcal{I}_1(q,m,\tau)\overline{\mathcal{I}_1(q',m',\tau')}U^{\dagger}\left(\frac{nL}{qq'},-\frac{i\tau}{2}+\frac{i\tau'}{2}\right)d\tau d\tau' \tag{6.4}$$

Using the expression for $\mathcal{I}_1(q,m,\tau)$ as given in lemma (4.2), we get the expression

$$\begin{aligned}\mathfrak{K}=\frac{|c_4|^2}{K^2}\iint_{\mathbb{R}^2}&\gamma(1+i\tau)\overline{\gamma(1+i\tau')}W_J(q,m.\tau)\overline{W_J(q',m',\tau')}\frac{(LN)^{-i\tau/2+i\tau'/2}}{q^{-i\tau}q'^{i\tau'}}\left(-\frac{(t+\tau/2)q}{2\pi eNm}\right)^{-i(t+\tau/2)}\\ &\left(-\frac{(t+\tau'/2)q'}{2\pi eNm'}\right)^{i(t+\tau'/2)}U^{\dagger}\left(\frac{nL}{qq'},-\frac{i\tau}{2}+\frac{i\tau'}{2}\right)d\tau d\tau'\end{aligned} \tag{6.5}$$

where

$$W_J(q,m,\tau)=\frac{1}{(t+\tau/2)^{1/2}}W_J(\tau)\left(-\frac{(t+\tau/2)q}{2\pi eNm}\right)^{3/2}V\left(-\frac{(t+\tau/2)q}{2\pi Nm}\right)\int_0^1 V\left(\frac{\tau}{2K}-\frac{(t+\tau/2)x}{Kma}\right)dx$$

Since $u^{3/2}V(u)\ll 1$ and $\tau\ll J\ll t^{1-\epsilon}$, it follows that

$$\frac{\partial}{\partial\tau}W_J(q,m,\tau)\ll\frac{1}{t^{1/2}|\tau|} \tag{6.6}$$

We also note that the x-integral inside the expression of $W_J(q,m,\tau)$ contributes a factor of the size of its length, which is $\ll Kma/(t+\tau)$. Since $m\ll Ct^{1+\varepsilon}/N$ and $\tau\ll t$, the contribution is $\ll KCQt^{\varepsilon}/N$. Therefore $W_J(q,m,\tau)\ll K^{1/2}C/t^{1/2}N^{1/2}$.

We analyze the integral $\mathfrak{K}$ in two cases, when $n = 0$ and when $n \neq 0$. For $n = 0$, the expression for $\mathfrak{C}$ gives $q = q'$, and the bound on $U^\dagger$ gives us arbitrary saving for $|\tau - \tau'| \gg t^\epsilon$. In this case,

$$\mathfrak{K} \ll \frac{|c_4|^2}{K^2} \int_{|\tau| \ll (NK)^{1/2}/C} \gamma(1+i\tau)|^2 W_J(q,m,\tau) \int_{|\tau'-\tau| \ll t^\varepsilon} W_J(q,m',\tau') d\tau' d\tau \ll \frac{t^\epsilon C}{K^{1/2}N^{1/2}t} =: B^*(C,0)$$

When $n \neq 0$,

$$U^\dagger\left(\frac{nL}{qq'}, 1 - \frac{i\tau}{2} + \frac{i\tau'}{2}\right) = \frac{c_5}{(\tau-\tau')^{1/2}} U\left(\frac{(\tau-\tau')qq'}{4\pi nL}\right)\left(\frac{(\tau-\tau')qq'}{4\pi enL}\right)^{-i\tau/2 + i\tau'/2} + O\left(\min\left\{\frac{1}{|\tau-\tau'|^{3/2}}, \frac{C^3}{(|n|L)^{3/2}}\right\}\right) \tag{6.7}$$

for some absolute constant c_5.

Contribution of the error term towards $\mathfrak{K}$ is of the order of

$$\frac{t^\epsilon}{K^2} \iint_{[J,4J/3]^2} \frac{1}{t} \min\left\{\frac{1}{|\tau-\tau'|^{3/2}}, \frac{C^3}{(|n|L)^{3/2}}\right\} d\tau d\tau'$$

When the second term is smaller,

$$\frac{t^\epsilon}{K^2} \iint_{\substack{[J,4J/3]^2 \\ |\tau-\tau'| \ll |n|L/C^2}} \frac{1}{t}\frac{C^3}{(|n|L)^{3/2}} d\tau d\tau' \ll \frac{1}{K^{3/2}t}\frac{N^{1/2}}{(|n|L)^{1/2}} t^\epsilon \tag{6.8}$$

When the first term is smaller,

$$\frac{t^\epsilon}{K^2} \iint_{\substack{[J,4J/3]^2 \\ |\tau-\tau'| \gg |n|L/C^2}} \frac{1}{t}\frac{1}{|\tau-\tau'|^{3/2}} d\tau d\tau' \ll \frac{t^\epsilon}{K^2 t}\frac{C}{(|n|L)^{1/2}} \iint_{[J,4J/3]^2} \frac{1}{|\tau-\tau'|^{1-\epsilon}} d\tau d\tau' \ll \frac{1}{K^{3/2}t}\frac{N^{1/2}}{(|n|L)^{1/2}} t^\epsilon \tag{6.9}$$

The error contribution (for $n \neq 0$) is

$$B^*(C,n) = \frac{1}{K^{3/2}t}\frac{N^{1/2}}{(|n|L)^{1/2}} t^\epsilon$$

We finally analyze the main term. Striling's formula is

$$\Gamma(\sigma + i\tau) = \sqrt{2\pi}(i\tau)^{\sigma-1/2} e^{-\pi|\tau|/2}\left(\frac{|\tau|}{e}\right)^{i\tau}\left\{1 + O\left(\frac{1}{|\tau|}\right)\right\}$$

as $|\tau| \to \infty$. That gives

$$\gamma(1+i\tau) = \left(\frac{|\tau|}{4\pi e}\right)^{i\tau}\Phi(\tau), \quad \textit{where } \Phi'(\tau) \ll \frac{1}{|\tau|} \tag{6.10}$$

By Fourier inversion, we write

$$\left(\frac{4\pi nL}{(\tau-\tau')qq'}\right)^{1/2} U\left(\frac{(\tau-\tau')qq'}{4\pi nL}\right) = \int_{\mathbb{R}} U^\dagger(r,1/2) e\left(\frac{(\tau-\tau')qq'}{4\pi nL} r\right) dr$$

We conclude that for some constant c_6 (depending on the sign of n)

$$\mathfrak{K} = \frac{c_6}{K^2}\left(\frac{qq'}{|n|L}\right)^{1/2}\int_{\mathbb{R}} U^{\dagger}(r,1/2)\iint_{\mathbb{R}^2} g(\tau,\tau')e(f(\tau,\tau'))d\tau d\tau' dr + O(B^*(C,n)) \tag{6.11}$$

where

$$\begin{aligned} 2\pi f(\tau,\tau') =& \tau\log\left(\frac{\tau}{4\pi e}\right) - \tau'\log\left(\frac{\tau'}{4\pi e}\right) - \frac{(\tau-\tau')}{2}\log(LN) + \tau\log q - \tau'\log q' \\ & - (t+\tau/2)\log\left(-\frac{(t+\tau/2)q}{2\pi eNm}\right) + (t+\tau'/2)\log\left(-\frac{(t+\tau'/2)q'}{2\pi eNm'}\right) \\ & \frac{(\tau-\tau')}{2}\log\left(\frac{(\tau-\tau')qq'}{4\pi enL}\right) + \frac{(\tau-\tau')qq'}{2nL}r \end{aligned}$$

and

$$g(\tau,\tau') = \Phi(\tau)\overline{\Phi(\tau')}W_J(q,m,\tau)W_J(q',m',\tau')$$

We intend to use the second derivative bound as given in Lemma 2.4. For that, we need the following

$$2\pi\frac{\partial^2}{\partial\tau^2}f(\tau,\tau') = \frac{1}{4}\left(\frac{4}{\tau} - \frac{1}{(t+\tau/2)} + \frac{2}{(\tau'-\tau)}\right), \quad 2\pi\frac{\partial^2}{\partial\tau'^2}f(\tau,\tau') = \frac{1}{4}\left(\frac{-4}{\tau'} + \frac{1}{(t+\tau'/2)} + \frac{2}{(\tau'-\tau)}\right)$$

and

$$2\pi\frac{\partial^2}{\partial\tau'\partial\tau}f(\tau,\tau') = \frac{-1}{4}\left(\frac{2}{\tau'-\tau}\right)$$

Also, by explicit computation,

$$4\pi^2\left[\frac{\partial^2}{\partial\tau^2}f(\tau,\tau')\frac{\partial^2}{\partial\tau'^2}f(\tau,\tau') - \left(\frac{\partial^2}{\partial\tau'\partial\tau}f(\tau,\tau')\right)^2\right] = -\frac{1}{2\tau\tau'} + O\left(\frac{1}{tJ}\right)$$

for τ,τ' such that $g(\tau,\tau') \neq 0$. So the conditions of lemma 4 of Munshi [8] hold with $r_1 = r_2 = 1/J^{1/2}$. To calculate the total variation of $g(\tau,\tau')$, recall that $\Phi'(\tau) \ll |\tau|^{-1}$ and $W_J'(q,m,\tau) \ll t^{-1/2}|\tau|^{-1}$. So $var(g) \ll t^{-1+\epsilon}$. So the double integral in (6.11) over τ,τ' is bounded by $O(Jt^{-1+\epsilon})$. Integrating trivially over r using the rapid decay of the Fourier transform, we get that total contribution of the leading term in (6.11) towards $\mathfrak{K}$ is bounded by

$$O\left(\frac{1}{K^2}\frac{C}{(|n|L)^{1/2}}\frac{(NK)^{1/2}}{C}t^{-1+\epsilon}\right) = O(B^*(C,n))$$

Putting everything together, we get the final bound

$$\begin{aligned} S_{1,J}(N,C,L) \ll& \frac{t^{\epsilon}K}{NC^2}\Bigg[\underbrace{\sum\sum_{\substack{C<q\leqslant 2C,(m,q)=1 \\ 1\leqslant m\ll qt^{1+\epsilon}/N}}}_{n=0}\left(\frac{t}{N}\right)B^*(C,0) + \sum_{\substack{|n|\ll\frac{C(NK)^{1/2}t^{\epsilon}}{L} \\ n\neq 0}}\sum_{C<q\leqslant 2C}\sum_{C<q'\leqslant 2C}\left(\frac{t}{N}\right)^2 B^*(C,n)\Bigg] \\ =& \frac{t^{\epsilon}K}{NC^2}\left[\frac{C^3t}{N^{5/2}K^{1/2}} + \frac{C^{1/2}(NK)^{1/4}}{L}\frac{C^2t}{N^{3/2}K^{3/2}}\right] \end{aligned}$$

That gives

$$S_{1,J}(N,C) \leqslant t^{\epsilon} N^{1/2} K \sum_{\substack{1 \leqslant L \ll K t^{\epsilon} \\ dyadic}} L^{1/2} \frac{K^{1/2}}{N^{1/2} C} \left[\frac{C^{3/2} t^{1/2}}{N^{5/4} K^{1/4}} + \frac{C^{1/4} (NK)^{1/8}}{L^{1/2}} \frac{C t^{1/2}}{N^{3/4} K^{3/4}} \right]$$
$$\ll t^{\epsilon} K^{3/2} \left(\frac{K^{1/4} C^{1/2} t^{1/2}}{N^{5/4}} + \frac{C^{1/4} t^{1/2}}{(NK)^{5/8}} \right)$$

Multiplying by $N^{1/2}/K$ and summing over the dyadic range $C \ll Q$, we get

$$\frac{S_1(N)}{N^{1/2}} \ll t^{1/2+\varepsilon} \left(\frac{K^{1/2}}{N^{1/2}} + \frac{1}{K^{1/4}} \right) \tag{6.12}$$

Finally, from equations (5.3) and (6.12), it follows that for $N \ll t^{1+\epsilon}$ and $K \gg N^{1/2}$,

$$\frac{S(N)}{N^{1/2}} \ll t^{1/2+\varepsilon} \left(\frac{K^{1/2}}{N^{1/2}} + \frac{N^{1/4}}{K^{3/4}} + \frac{K^{1/2}}{N^{1/2}} + \frac{1}{K^{1/4}} \right).$$

The optimal choice for K occurs at $K = N^{2/3}$ and we get Proposition 1.2.

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