Dynamics for a diffusive prey-predator model with different free boundaries[1]

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Abstract. To understand the spreading and interaction of prey and predator, in this paper we study the dynamics of the diffusive Lotka-Volterra type prey-predator model with different free boundaries. These two free boundaries, which may intersect each other as time evolves, are used to describe the spreading of prey and predator. We investigate the existence and uniqueness, regularity and uniform estimates, and long time behaviors of global solution. Some sufficient conditions for spreading and vanishing are established. When spreading occurs, we provide the more accurate limits of (u, v) as $t \to \infty$, and give some estimates of asymptotic spreading speeds of u, v and asymptotic speeds of g, h. Some realistic and significant spreading phenomena are found.

Keywords: Diffusive prey-predator model; Different free boundaries; Spreading and vanishing; Long time behavior; Asymptotic propagation.

AMS subject classifications (2000): 35K51, 35R35, 92B05, 35B40.

1 Introduction

The spreading and vanishing of multiple species is an important content in understanding ecological complexity. In order to study the spreading and vanishing phenomenon, many mathematical models have been established. In this paper we consider the diffusive Lotka-Volterra type prey-predator model with different free boundaries. It is a meaningful subject, because the following phenomenon will happen constantly in the real world:

There is some kind of species (the indigenous species, prey u) in a bounded area (initial habitat, for example, Ω_0), and at some time (initial time, $t = 0$) another kind of species (the new or invasive species, predator v) enters a part Σ_0 of Ω_0.

In general, both species have tendencies to emigrate from boundaries to obtain their respective new habitats. That is, as time t increases, Ω_0 and Σ_0 will evolve into expanding regions $\Omega(t)$ and $\Sigma(t)$ with expanding fronts $\partial\Omega(t)$ and $\partial\Sigma(t)$, respectively. The initial functions $u_0(x)$ and $v_0(x)$ will evolve into positive functions $u(t, x)$ and $v(t, x)$ governed by a suitable diffusive prey-predator system, $u(t, x)$ and $v(t, x)$ vanish on the moving boundaries $\partial\Omega(t)$ and $\partial\Sigma(t)$, respectively. We want to understand the dynamics/variations of these two species and free boundaries. For simplicity, we assume that the interaction between these two species obeys the Lokta-Volterra law, and restrict our problem to the one dimensional case. Moreover, we think that the left boundaries of $\Omega(t)$ and $\Sigma(t)$ are fixed and coincident. So, we can take $\Omega_0 = (0, g_0)$, $\Sigma_0 = (0, h_0)$ with $0 < h_0 \le g_0$, and

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$\Omega(t) = (0, g(t))$, $\Sigma(t) = (0, h(t))$. Based on the *deduction of free boundary conditions* given in [3, 41], we have the following free boundary conditions

$$g'(t) = -\beta u_x(t, g(t)), \quad h'(t) = -\mu v_x(t, h(t)),$$

where positive constants $\beta = d_1 k_1^{-1}$ and $\mu = d_2 k_2^{-1}$ can be considered as the *moving parameters*, d_1, d_2 and k_1, k_2 are, respectively, their diffusion coefficients and *preferred density levels* nearing free boundaries. Under the suitable rescaling, the model we are concerned here becomes the following free boundary problem

$$\begin{cases} u_t - du_{xx} = u(a - u - bv), & t > 0, \quad 0 < x < g(t), \\ v_t - v_{xx} = v(1 - v + cu), & t > 0, \quad 0 < x < h(t), \\ u_x(t, 0) = v_x(t, 0) = 0, & t \geq 0, \\ g'(t) = -\beta u_x(t, g(t)), \quad h'(t) = -\mu v_x(t, h(t)), & t \geq 0, \\ u(t, x) = 0 \ \text{ for } x \geq g(t), \quad v(t, x) = 0 \ \text{ for } x \geq h(t), & t \geq 0, \\ u(0, x) = u_0(x) \ \text{ in } [0, g_0], \quad v(0, x) = v_0(x) \ \text{ in } [0, h_0], \\ g(0) = g_0 \geq h_0 = h(0) > 0, \end{cases} \tag{1.1}$$

where $a, b, c, d, g_0, h_0, \beta$ and μ are given positive constants. The initial functions $u_0(x), v_0(x)$ satisfy

$$\begin{cases} u_0 \in C^2([0, g_0]), \ u_0'(0) = 0, \ u_0(x) > 0 \ \text{ in } [0, g_0), \ \ u_0(x) = 0 \ \text{ in } [g_0, \infty), \\ v_0 \in C^2([0, h_0]), \ v_0'(0) = 0, \ v_0(x) > 0 \ \text{ in } [0, h_0), \ \ v_0(x) = 0 \ \text{ in } [h_0, \infty). \end{cases} \tag{1.2}$$

Because the two free boundaries may intersect each other, it seems very difficult to understand the whole dynamics of this model. We shall see that the problem (1.1) possesses the multiplicity and complexity of spreading, vanishing and asymptotic propagation. The phenomena exhibited by these multiplicities and complexities seem closer to the reality.

Some related free boundary problems of competition-diffusion model with different free boundaries have been studied recently. In [14], Du and Wang discussed the following problem

$$\begin{cases} u_t - u_{xx} = u(1 - u - av), & t > 0, \quad -\infty < x < g(t), \\ v_t - dv_{xx} = rv(1 - v - bu), & t > 0, \quad h(t) < x < \infty, \\ u = 0 \ \text{ for } x \geq g(t), \quad v = 0 \ \text{ for } x \leq h(t), & t > 0, \\ g'(t) = -\beta u_x(t, g(t)), \quad h'(t) = -\mu v_x(t, h(t)), & t > 0 \\ u = u_0(x) \ \text{ in } (-\infty, g_0], \quad v = v_0(x) \ \text{ in } [h_0, \infty), & t = 0, \\ g(0) = g_0 < h_0 = h(0). \end{cases}$$

In this model, the competing species u and v occupied habitats $(-\infty, g_0]$ and $[h_0, \infty)$ at the initial time, respectively. They will move outward along free boundaries as time increases (u moves to right, v moves to left). When their habitats overlap, they obey the Lotka-Volterra competition law in the common habitat. Guo & Wu [18], Wang [39] and Wu [44] studied a two-species competition-diffusion model with two free boundaries, in there the left boundary conditions and free boundary conditions are the same as that of (1.1).

The same spreading mechanism as in [9] has been adopted in studying some two-species competition systems or prey-predator systems. The authors of [10, 15, 32, 33, 47] investigated a competition model in which the invasive species exists initially in a ball and invades into the new environment, while the resident species distributes in the whole space $\mathbb{R}^N$. In [41], Wang and Zhao studied a predator-prey model with double free boundaries in which the predator exists initially in a bounded interval and invades into the new environment from two sides, while the prey distributes in the whole line $\mathbb{R}$. In [17, 40], two competition species are assumed to spread along the same free boundary. Predator-prey models with homogeneous Dirichlet (Neumann, Robin) boundary conditions at the left side and free boundary at the right side can be found in [34, 36, 38, 45]. For traveling wave solutions of free boundary problems, see [4, 5, 43] for examples.

There have been many papers concerning the free boundary problems of single equation to describe the spreading mechanism of an invading species. Please refer to [6]-[9], [11]-[13], [16, 21, 22, 28, 35, 37] and the references therein.

This paper is organized as follows. In Section 2 we study the global existence, uniqueness, regularity and some estimates of (u, v, g, h). In section 3, we first recall some fundamental results from [3, 9] and then give some rough estimates, which will be used in the following two sections. Section 4 is concerned with the long time behaviors of (u, v), and Section 5 deals with conditions for spreading and vanishing. In Section 6, we provide some estimates of asymptotic speeds of $g(t)$, $h(t)$ and asymptotic spreading speeds of $u(t, x)$, $v(t, x)$. Finally, in section 7 we give a brief discussion.

2 Existence, uniqueness and estimates of global solution

In this section, we first prove the following local existence and uniqueness results. Then we give some uniform estimates and show that the solution exists globally in time t. The main ideas of this article and literature [18] are to straighten the free boundary and use the fixed point theorem. However, in [18] the authors considered a map of $(g(t), h(t))$ and used the contraction mapping theorem directly. In the present paper, based on the results of single equation, we shall use Schauder's fixed point theorem to get the existence of local solution and then prove the uniqueness.

In order to facilitate the writing, we denote

$$\Lambda = \big\{a, b, c, d, g_0, h_0, \beta, \mu, \|u_0\|_{W_p^2}, \|v_0\|_{W_p^2}\big\}.$$

For the given positive constants T, m and function $f(t)$, we set

$$\Delta_T^m = [0, T] \times [0, m], \quad D_T^f = \{0 \le t \le T,\ 0 \le x < f(t)\}, \quad D_\infty^f = \{t > 0,\ 0 \le x \le f(t)\}.$$

Theorem 2.1. *For any given (u_0, v_0) satisfying (1.2), $\alpha \in (0, 1)$ and $p > 3/(1-\alpha)$, there is a $T > 0$ such that the problem (1.1) admits a unique solution*

$$(u, v, g, h) \in W_p^{1,2}(D_T^g) \times W_p^{1,2}(D_T^h) \times [C^{1+\frac{\alpha}{2}}([0, T])]^2.$$

Moreover,

$$\|u\|_{W_p^{1,2}(D_T^g)} + \|v\|_{W_p^{1,2}(D_T^h)} + \|g\|_{C^{1+\frac{\alpha}{2}}([0,T])} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C, \tag{2.1}$$

where positive constants C and T depend only on Λ, α and p.

Proof. Some techniques of this proof have been inspired by [14, Theorem 2.1]. We will divide the proof into three steps.

Step 1: Transformation of the problem (1.1). Let

$$y = x/g(t), \quad w(t,y) = u\big(t, g(t)y\big), \quad z(t,y) = v\big(t, g(t)y\big),$$

then (1.1) is equivalent to

$$\begin{cases} w_t - \frac{d}{g^2(t)} w_{yy} - \frac{g'(t)}{g(t)} y w_y = w(a - w - bz), & 0 < t \le T, \quad 0 < y < 1, \\ w_y(t,0) = w(t,1) = 0, & 0 \le t \le T, \\ w(0,y) = u_0(g_0 y) := w_0(y), & 0 \le y \le 1, \\ g'(t) = -\beta \frac{1}{g(t)} w_y(t,1), & 0 \le t \le T, \\ g(0) = g_0, \end{cases} \tag{2.2}$$

$$\begin{cases} z_t - \frac{1}{g^2(t)} z_{yy} - \frac{g'(t)}{g(t)} y z_y = z(1 - z + cw), & 0 < t \le T, \quad 0 < y < \frac{h(t)}{g(t)}, \\ z_y(t,0) = z(t, \frac{h(t)}{g(t)}) = 0, & 0 \le t \le T, \\ z(0,y) = v_0(g_0 y) := z_0(y), & 0 \le y \le h_0/g_0, \\ h'(t) = -\mu \frac{1}{g(t)} z_y(t, s(t)), & 0 \le t \le T, \\ h(0) = h_0. \end{cases} \tag{2.3}$$

Step 2: Existence of the solution (w, z, g, h) *to* (2.2) *and* (2.3). Denote $m = 1 + h_0/g_0$ and define $\hat{z}_0(y) = z_0(y)$ for $0 \le y \le h_0/g_0$, $\hat{z}_0(y) = 0$ for $h_0/g_0 \le y \le m$. For $0 < T \ll 1$, we set

$$Z_T = \big\{ z \in C(\Delta_T^m) : \ z(0,y) = \hat{z}_0(y), \ \|z - \hat{z}_0\|_{C(\Delta_T^m)} \le 1 \big\}.$$

Then Z_T is a bounded and closed convex set of $C(\Delta_T^m)$. For the given $z \in Z_T$, we consider $z = z(y,t)$ as a coefficient. Since z satisfies

$$\|z\|_{C(\Delta_T^1)} \le \|z\|_{C(\Delta_T^m)} \le 1 + \|\hat{z}\|_{C(\Delta_T^m)} \le 1 + \|v_0\|_{L^\infty},$$

similarly to the arguments in the proof of [9, Theorem 2.1] ([41, Theorem 2.1]), by using the contraction mapping theorem we can prove that, when $0 < T \ll 1$, the problem (2.2) has a unique solution $(w(t,y), g(t))$ and $w \in W_p^{1,2}(\Delta_T^1) \hookrightarrow C^{\frac{1+\alpha}{2}, 1+\alpha}(\Delta_T^1)$, $g \in C^{1+\frac{\alpha}{2}}([0,T])$. Moreover,

$$\begin{aligned} g'(0) = -\beta g_0^{-1} w_y(0,1) = -\beta u_0'(g_0), \\ g' \ge 0, \ g \le g_0 + 1 \ \text{ in } [0,T]; \quad w > 0 \ \text{ in } [0,T] \times [0,1), \\ \|w\|_{W_p^{1,2}(\Delta_T^1)} + \|w\|_{C^{\frac{1+\alpha}{2}, 1+\alpha}(\Delta_T^1)} + \|g\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C_1, \end{aligned} \tag{2.4}$$

where C_1 depending only on $a, b, d, \beta, g_0, \alpha, p, \|u_0\|_{W_p^2}$ and $1 + \|v_0\|_{L^\infty}$. Besides, let $z_i \in Z_T$ and $(w_i, g_i) \in W_p^{1,2}(\Delta_T^1) \cap C^{1+\frac{\alpha}{2}}([0,T])$ be the unique solution of (2.2) with $z = z_i$. Similar to the following proof of the uniqueness we can get the estimate

$$\|w_1 - w_2\|_{C(\Delta_T^1)} + \|g_1 - g_2\|_{C^1([0,T])} \le \|z_1 - z_2\|_{C(\Delta_T^1)} \tag{2.5}$$

provided $0 < T \ll 1$, this implies that (w, g) depends continuously on z. We shall prove (2.5) in the appendix.

For such a $(w(t,y), g(t))$, determined uniquely by the above, we put $w(t,y)$ zero extension to $[0,T] \times [1,m]$ and consider the problem (2.3). Set $s(t) = \frac{h(t)}{g(t)}$, and

$$\xi = y/s(t), \quad \phi(t,\xi) = z(t, s(t)\xi), \quad \psi(t,\xi) = w(t, s(t)\xi).$$

Then (2.3) is equivalent to the following problem

$$\begin{cases} \phi_t - \frac{1}{h^2(t)}\phi_{\xi\xi} - \frac{h'(t)}{h(t)}\xi\phi_\xi = \phi[1-\phi+c\psi(t,\xi)], & 0<t\le T, \quad 0<\xi<1, \\ \phi_\xi(t,0)=\phi(t,1)=0, & 0\le t\le T, \\ \phi(0,\xi)=v_0(h_0\xi):=\phi_0(\xi), & 0\le\xi\le 1, \\ h'(t)=-\mu\frac{1}{h(t)}\phi_\xi(t,1), & 0\le t\le T, \\ h(0)=h_0. \end{cases} \tag{2.6}$$

Similarly to the above, we can prove that, when $0<T\ll 1$, the problem (2.6) has a unique solution $(\phi(t,\xi),h(t))$ which depends continuously on (w,g), and thus continuously dependent on z. Let $\bar z(t,y)=\phi(t,\frac{y}{s(t)})$. Then $(\bar z(t,y),h(t))$ is the unique solution of (2.3), and $(\bar z(t,y),h(t))$ is continuous with respect to z. Moreover, the following hold:

(i) $h'(0)=-\mu v_0'(h_0)$, $h'\ge 0$, $h\le h_0+1$ and $h/g<1+h_0/g_0$ in $[0,T]$, $\bar z>0$ in $D_T^{h/g}$.

(ii) There exists a constant $C_2>0$ depending only on Λ, α and p, such that

$$\|\bar z\|_{W_p^{1,2}(D_T^{h/g})} + \|\bar z\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\bar D_T^{h/g})} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C_2. \tag{2.7}$$

Put $\bar z(t,\cdot)$ zero extension to $[h(t)/g(t),m]$ for each $t\in[0,T]$. Then $\bar z_y\in L^\infty(\Delta_T^m)$. In view of $h\in C^{1+\frac{\alpha}{2}}([0,T])$, we can verify that $\bar z\in C^{\frac{\alpha}{2},\alpha}(\Delta_T^m)$ and, upon using (2.7), that

$$\|\bar z_y\|_{L^\infty(\Delta_T^m)} + \|\bar z\|_{C^{\frac{\alpha}{2},\alpha}(\Delta_T^m)} \le C_3. \tag{2.8}$$

Define a map

$$\mathcal{G}:\ Z_T \longrightarrow C(\Delta_T^m), \quad \mathcal{G}(z)=\bar z.$$

From the above arguments we see that $\mathcal{G}$ is continuous in Z_T, and $z\in Z_T$ is a fixed point of $\mathcal{G}$ if and only if (w,z,g,h) solves (2.2) and (2.3) for $0<t\le T$, where (w,g) is the solution of (2.2), and (z,h) is the solution of (2.3) with the zero extension of $w(t,y)$ to $[0,T]\times[1,m]$. Estimation (2.8) indicates that $\mathcal{G}$ is compact.

Notice $\bar z(0,x)=\hat z_0(y)$. Using the mean value theorem and (2.8) we have

$$\|\bar z-\hat z_0\|_{C(\Delta_T^m)} \le \|\bar z\|_{C^{\frac{\alpha}{2},0}(\Delta_T^m)} T^{\frac{\alpha}{2}} \le C_3 T^{\frac{\alpha}{2}}.$$

Therefore, if we take $0<T\ll 1$, then $\mathcal{G}$ maps Z_T into itself. Hence, $\mathcal{G}$ has at least one fixed point $z\in Z_T$, i.e., (2.2) and (2.3) has at least one solution (w,z,g,h) in $[0,T]$. Moreover, from the above discussion we see that (w,z,g,h) satisfies

$$g,\, h\in C^{1+\frac{\alpha}{2}}([0,T]), \quad g'(t)\ge 0,\ \ h'(t)\ge 0 \ \text{ in } [0,T],$$
$$w\in W_p^{1,2}(\Delta_T^1)\cap C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T^1), \quad z\in W_p^{1,2}(\bar D_T^{h/g})\cap C^{\frac{1+\alpha}{2},1+\alpha}(\bar D_T^{h/g}).$$

Step 3: Existence and uniqueness of the solution (u,v,g,h) *to* (1.1). Define

$$u(t,x)=w(t,x/g(t)), \quad v(t,x)=z(t,x/g(t)).$$

Then (u,v,g,h) solves (1.1), and (u,v) satisfies

$$u\in W_p^{1,2}(D_T^g)\cap C^{\frac{1+\alpha}{2},1+\alpha}(\bar D_T^g), \quad v\in W_p^{1,2}(D_T^h)\cap C^{\frac{1+\alpha}{2},1+\alpha}(\bar D_T^h).$$

It is easy to see that (2.1) holds.

In the following we prove the uniqueness. Let (u_i, v_i, g_i, h_i), with $i = 1, 2$, be two solutions of (1.1), which are defined for $t \in [0, T]$ and $0 < T \ll 1$. We can think of that

$$
\begin{gathered}
g_0 \le g_i(t) \le g_0 + 1, \quad h_0 \le h_i(t) \le h_0 + 1 \quad \text{in } [0,T], \quad i = 1, 2, \\
\|u_i(t,x) - u_0(x)\|_{C(D_T^{g_i})} \le 1, \quad \|v_i(t,x) - v_0(x)\|_{C(D_T^{h_i})} \le 1, \quad i = 1, 2.
\end{gathered}
$$

Take $k = g_0 + h_0 + 1$. For each $t \in [0, T]$, define $u_i(t, \cdot) = 0$ in $[g_i(t), k]$ and $v_i(t, \cdot) = 0$ in $[h_i(t), k]$, $i = 1, 2$. Then $v_{ix} \in L^\infty(\Delta_T^k)$. Let

$$
w_i(t,y) = u_i\big(t, g_i(t)y\big), \quad z_i(t,y) = v_i\big(t, g_i(t)y\big), \quad 0 \le t \le T, \ 0 \le y \le 1.
$$

Then (w_i, g_i) solves (2.2) with $z = z_i$ and satisfies (2.4). Set $W = w_1 - w_2$, $G = g_1 - g_2$, we have

$$
\begin{cases}
W_t - \frac{d}{g_1^2(t)} W_{yy} - \frac{g_1'(t)}{g_1(t)} y W_y - (a - w_1 - w_2 - bz_1)W \\
\qquad = d\left(\frac{1}{g_1^2(t)} - \frac{1}{g_2^2(t)}\right) w_{2yy} + \left(\frac{g_1'(t)}{g_1(t)} - \frac{g_2'(t)}{g_2(t)}\right) y w_{2y} \\
\qquad\quad - b w_2 (z_1 - z_2), \qquad 0 < t \le T, \quad 0 < y < 1, \\
W_y(t,0) = W(t,1) = 0, \qquad 0 \le t \le T, \\
W(0,y) = 0, \qquad 0 \le y \le 1,
\end{cases} \tag{2.9}
$$

and

$$
\begin{cases}
G'(t) = -\beta \frac{1}{g_1(t)} W_y(t,1) - \beta\left(\frac{1}{g_1(t)} - \frac{1}{g_2(t)}\right) w_{2y}(t,1), \qquad 0 \le t \le T, \\
G(0) = 0.
\end{cases} \tag{2.10}
$$

Remember the facts $\|w_2\|_{W_p^{1,2}(\Delta_T^1)} \le C_1$, $g_0 \le g_i(t) \le g_0 + 1$, $|g_i'(t)| \le C_1$ and $\|z_i\|_{C(\Delta_T^1)} \le 1 + \|v_0\|_{L^\infty}$, $i = 1, 2$. We can apply the L^p estimate to (2.9) and use Sobolev's imbedding theorem to derive

$$
\|W\|_{C^{\frac{1+\alpha}{2}, 1+\alpha}(\Delta_T^1)} \le C_4 \big(\|z_1 - z_2\|_{C(\Delta_T^1)} + \|G\|_{C^1([0,T])} \big). \tag{2.11}
$$

Now we estimate $\|z_1 - z_2\|_{C(\Delta_T^1)}$. For any fixed $(t, y) \in \Delta_T^1$, we have

$$
\begin{aligned}
|z_1(t,y) - z_2(t,y)| &\le |v_1(t, g_1(t)y) - v_1(t, g_2(t)y)| + |v_1(t, g_2(t)y) - v_2(t, g_2(t)y)| \\
&\le \|v_{1x}\|_{L^\infty(\Delta_T^k)} \|G\|_{C([0,T])} + \|v_1 - v_2\|_{C(\Delta_T^k)},
\end{aligned}
$$

which implies,

$$
\|z_1 - z_2\|_{C(\Delta_T^1)} \le \|v_1 - v_2\|_{C(\Delta_T^k)} + \|v_{1x}\|_{L^\infty(\Delta_T^k)} \|G\|_{C([0,T])}.
$$

Combining this with (2.11), we get

$$
\|W\|_{C^{\frac{1+\alpha}{2}, 1+\alpha}(\Delta_T^1)} \le C_5 \big(\|v_1 - v_2\|_{C(\Delta_T^k)} + \|G\|_{C^1([0,T])} \big). \tag{2.12}
$$

Therefore, by use of (2.10),

$$
\begin{aligned}
\|G'\|_{C^{\frac{\alpha}{2}}([0,T])} &\le \beta \|g_1^{-1} W_y\|_{C^{\frac{\alpha}{2},0}(\Delta_T^1)} + \beta \|(g_1^{-1} - g_2^{-1}) w_{2y}\|_{C^{\frac{\alpha}{2},0}(\Delta_T^1)} \\
&\le C_6 \big(\|v_1 - v_2\|_{C(\Delta_T^k)} + \|G\|_{C^1([0,T])} \big).
\end{aligned} \tag{2.13}
$$

Recall $W(0,y)=0$, $G(0)=G'(0)=0$. Take advantage of the mean value theorem and (2.12), (2.13), it follows that

$$\|W\|_{C(\Delta_T^1)} \le T^{\frac{\alpha}{2}}\|W\|_{C^{\frac{\alpha}{2},0}(\Delta_T^1)} \le C_5T^{\frac{\alpha}{2}}\big(\|v_1-v_2\|_{C(\Delta_T^k)}+\|G\|_{C^1([0,T])}\big),$$
$$\|G\|_{C^1([0,T])} \le 2T^{\frac{\alpha}{2}}\|G'\|_{C^{\frac{\alpha}{2}}([0,T])} \le 2C_6T^{\frac{\alpha}{2}}\big(\|v_1-v_2\|_{C(\Delta_T^k)}+\|G\|_{C^1([0,T])}\big).$$

Make the zero extension of $w_i(t,\cdot)$ to $[1,k/g_0]$ for each $t\in[0,T]$. The above estimates lead to

$$\|W\|_{C(\Delta_T^{k/g_0})}+\|G\|_{C([0,T])} \le C_7T^{\frac{\alpha}{2}}\|v_1-v_2\|_{C(\Delta_T^k)} \tag{2.14}$$

provided $0<T\ll 1$. Moreover, because w_i satisfies (2.4), it is easy to show that

$$w_{iy}\in L^\infty(\Delta_T^{k/g_0}), \quad \|w_{iy}\|_{L^\infty(\Delta_T^{k/g_0})}\le C_1.$$

Now we estimate $\|u_1-u_2\|_{C(\Delta_T^k)}$. For any $(t,x)\in\Delta_T^k$, we have $0\le x\le k$ and

$$\begin{aligned}|u_1(t,x)-u_2(t,x)| &\le |w_1(t,g_1^{-1}(t)x)-w_2(t,g_1^{-1}(t)x)|+|w_2(t,g_1^{-1}(t)x)-w_2(t,g_2^{-1}(t)x)|\\ &\le \|w_1-w_2\|_{C(\Delta_T^{k/g_0})}+k\|w_{2y}\|_{L^\infty(\Delta_T^{k/g_0})}|g_1^{-1}(t)-g_2^{-1}(t)|\\ &\le \|w_1-w_2\|_{C(\Delta_T^{k/g_0})}+C_8\|g_1-g_2\|_{C([0,T])},\end{aligned}$$

where $C_8=mg_0^{-2}C_1$. This implies $\|u_1-u_2\|_{C(\Delta_T^k)}\le\|W\|_{C(\Delta_T^{k/g_0})}+C_8\|G\|_{C([0,T])}$. In consideration of (2.14), it follows that

$$\|u_1-u_2\|_{C(\Delta_T^k)}+\|g_1-g_2\|_{C([0,T])}\le C_9T^{\frac{\alpha}{2}}\|v_1-v_2\|_{C(\Delta_T^k)}.$$

Similarly,

$$\|v_1-v_2\|_{C(\Delta_T^k)}+\|h_1-h_2\|_{C([0,T])}\le C_{10}T^{\frac{\alpha}{2}}\|u_1-u_2\|_{C(\Delta_T^k)}.$$

Thus, when $0<T\ll 1$, we have $(u_1,v_1,g_1,h_1)=(u_2,v_2,g_2,h_2)$. The uniqueness is obtained and the proof is finished. $\square$

To show that the local solution obtained in Theorem 2.1 can be extended in time t, we need the following estimates, their proofs are similar to those of [41, Lemma 2.1], and the details will be omitted here.

Lemma 2.1. *Let $T\in(0,\infty)$ and (u,v,g,h) be a solution of (1.1) defined in $[0,T]$. Then*

$$\begin{aligned}
&0<u(t,x)\le\max\{a,\ \|u_0\|_\infty\}:=M_1, && \forall\ 0\le t\le T,\ 0\le x<g(t),\\
&0<v(t,x)\le\max\{1+cM_1,\ \|v_0\|_\infty\}:=M_2, && \forall\ 0\le t\le T,\ 0\le x<h(t),\\
&0\le g'(t)\le 2\beta\max\left\{M_1\sqrt{a/(2d)},\ -\min\nolimits_{[0,\,g_0]}u_0'(x)\right\}, && \forall\ 0\le t\le T,\\
&0\le h'(t)\le 2\mu\max\left\{M_2\sqrt{(1+cM_1)/2},\ -\min\nolimits_{[0,\,h_0]}v_0'(x)\right\}, && \forall\ 0\le t\le T.
\end{aligned}$$

Theorem 2.2. *The problem (1.1) admits a unique global solution (u,v,g,h) and $g'(t),h'(t)>0$,*

$$(u,v,g,h)\in C^{1+\frac{\alpha}{2},2+\alpha}(D_\infty^g)\times C^{1+\frac{\alpha}{2},2+\alpha}(D_\infty^h)\times[C^{1+\frac{1+\alpha}{2}}([0,\infty))]^2. \tag{2.15}$$

Moreover, there exists a positive constant C, depends only on Λ, such that

$$\|u(t,\cdot)\|_{C^1([0,\,g(t)])}\le C,\quad \forall\ t\ge1;\qquad \|g'\|_{C^{\alpha/2}([1,\infty))}\le C, \tag{2.16}$$
$$\|v(t,\cdot)\|_{C^1([0,\,h(t)])}\le C,\quad \forall\ t\ge1;\qquad \|h'\|_{C^{\alpha/2}([1,\infty))}\le C. \tag{2.17}$$

Proof. Thanks to the estimate (2.1) and Lemma 2.1, we can extend the unique local solution (u, v, g, h) obtained in Theorem 2.1 to a global solution and

$$u \in C^{\frac{1+\alpha}{2},1+\alpha}(D^g_\infty), \quad v \in C^{\frac{1+\alpha}{2},1+\alpha}(D^h_\infty), \quad g,\, h \in C^{1+\frac{\alpha}{2}}([0,\infty)), \tag{2.18}$$

see [14, 18] for the details. Moreover, make use of [25, Lemma 2.6], it can be deduced that $g'(t) > 0$, $h'(t) > 0$ for $t > 0$. Note that $g,\, h \in C^{1+\alpha/2}([0,\infty))$. It is easy to verify that, for any given $T, k > 0$, $u, v \in C^{\alpha/2,\,\alpha}(\Delta^k_T)$. Using this fact and (2.18), we can prove (2.15) by the similar way to that of [37, Theorem 2.1]. The details are omitted here.

Now we prove (2.16). It suffices to show that

$$\|u(t,\cdot)\|_{C^1([0,\,g(t)])} \le C, \quad \forall\, t \ge 1; \qquad \|g'\|_{C^{\alpha/2}([n+1,n+3])} \le C', \quad \forall\, n \ge 0. \tag{2.19}$$

In fact, the second estimate implies that $g'(t)$ is bounded in $[1,\infty)$. This combined with (2.19) allows us to derive $|g'(t+\sigma) - g'(t)| \le C\sigma^{\alpha/2}$ for some constant $C > 0$ and all $t \ge 1$, $\sigma \ge 0$. Hence (2.16) holds.

When $g_\infty < \infty$, similarly to the arguments of [37, Theorem 2.1] we can obtain (2.19). In the following we consider the case $g_\infty = \infty$. For the integer $n \ge 0$, let $u^n(t,x) = u(n+t,x)$. Then u^n satisfies

$$\begin{cases} u^n_t - du^n_{xx} - f^n(t,x)u^n = 0, & 0 < t \le 3,\ 0 < x < g(n+t), \\ u^n_x(t,0) = 0, \quad u^n(t, g(n+t)) = 0, & 0 \le t \le 3, \\ u^n(0,x) = u(n,x), & 0 \le x \le g(n), \end{cases} \tag{2.20}$$

where

$$f^n(t,x) = a - u(n+t,x) - bv(n+t,x).$$

According to Lemma 2.1, we know that u^n and f^n are bounded uniformly on n, and $g(n+t) \le g(n+1) + M(t-1) \le g(n+1) + 2M$ for $1 \le t \le 3$, where

$$M = 2\beta \max\left\{\sqrt{\frac{a}{2d}}\max\{a,\, \|u_0\|_\infty\},\ -\min_{[0,\,g_0]} u_0'(x)\right\}.$$

As $g_\infty = \infty$, there exists an $n_0 \ge 0$ such that

$$g(n_0+1) > 2M + 2, \quad g(n_0) > 3.$$

In the same way as the proof of [37, Theorem 2.1] we can show that

$$\|u(t,\cdot)\|_{C^1([0,\,g(t)])} \le C, \quad \forall\, 1 \le t \le n_0 + 3; \qquad \|g'\|_{C^{\alpha/2}([n+1,n+3])} \le C, \quad \forall\, n \le n_0. \tag{2.21}$$

Choose $p \gg 1$. For any integer $0 \le k \le g(n+1) - 3$, we can apply the interior L^p estimate (cf. [25, Theorem 7.20]) to the problem (2.20) and derive that there exists a positive constant C independent of k and n such that

$$\|u^n\|_{W^{1,2}_p([1,3]\times[k,k+2])} \le C, \quad \forall\, k, n \ge 0.$$

By the embedding theorem, $\|u^n\|_{C^{\frac{1+\alpha}{2},1+\alpha}([1,3]\times[k,k+2])} \le C$, which leads to $\|u^n\|_{C^{0,1}([1,3]\times[k,k+2])} \le C$ for all $n \ge n_0$ and $0 \le k \le g(n+1) - 3$. Since these intervals $[k, k+2]$ overlap and C is independent of k, it follows that $\|u^n\|_{C^{0,1}([1,3]\times[0,[g(n+1)]-1])} \le C$. Therefore

$$\|u^n\|_{C^{0,1}([1,3]\times[0,g(n+1)-2])} \le \|u^n\|_{C^{0,1}([1,3]\times[0,[g(n+1)]-1])} \le C, \quad \forall\, n \ge n_0, \tag{2.22}$$

here $[g(n+1)]$ is the integral part of $g(n+1)$. Notice that

$$g(n+1)-2 \ge g(n+t)-2M-2 \ge g(n+t_0)-2M-2 \ge g(n_0+1)-2M-2>0$$

for all $n \ge n_0$ and $1 \le t \le 3$. Make use of the estimate (2.22) we get

$$\|u^n(t,\cdot)\|_{C^1([0,\,g(n+t)-2M-2])} \le C, \quad \forall\, n \ge n_0,\ 1 \le t \le 3.$$

This leads to

$$\|u\|_{C^{0,1}(E_n)} \le C, \quad \forall\, n \ge n_0, \tag{2.23}$$

where

$$E_n=\{(t,x):\, n+1 \le t \le n+3,\ 0 \le x \le g(t)-2M-2\}, \quad n \ge n_0.$$

Since these rectangles E_n overlap and C is independent of n, it follows from (2.23) that

$$\|u(t,\,\cdot)\|_{C^1([0,\,g(t)-2M-2])} \le C, \quad \forall\, t \ge n_0+1. \tag{2.24}$$

In the following we shall show that

$$\|u(t,\cdot)\|_{C^1([g(t)-2M-2,\,g(t)])} \le C, \quad \forall\, t \ge n_0+1; \qquad \|g'\|_{C^{\alpha/2}([n+1,n+3])} \le C, \quad \forall\, n \ge n_0. \tag{2.25}$$

Once this is done, using (2.21) and (2.24) we can derive (2.19).

Let $y=g(t)-x$ and $w(t,y)=u(t,g(t)-y)$. Then $w(t,y)$ satisfies

$$\begin{cases} w_t-dw_{yy}+g'(t)w_y-F(t,y)w=0, & 0<t<\infty,\ 0<y<g(t),\\ w(t,0)=0, \quad w_y(t,g(t))=0, & 0 \le t<\infty,\\ w(0,y)=u(0,g_0-y), & 0 \le y \le g_0, \end{cases}$$

where $F(t,y)=a-u(t,g(t)-y)-bv(t,g(t)-y)$. Similar to the above, for the integer $n \ge n_0$, let $w^n(t,y)=w(n+t,y)$. Then w^n satisfies

$$\begin{cases} w^n_t-dw^n_{yy}+g'(n+t)w^n_y-F^n(t,y)w^n=0, & 0<t \le 3,\ 0<y<g(n+t),\\ w^n(t,0)=0, \quad w^n_y(t,g(n+t))=0, & 0 \le t \le 3,\\ w^n(0,y)=w(n,y), & 0 \le y \le g(n), \end{cases} \tag{2.26}$$

where $F^n(t,y)=F(n+t,y)$. It follows from Lemma 2.1 that w^n, $g'(n+t)$ and F^n are bounded uniformly on n. Remember

$$g(n+t)-(2M+2) \ge g(1+n_0)-(2M+2)>0, \quad \forall\, n \ge n_0,\ 1 \le t \le 3.$$

For $\Omega=[1,3]\times[0,2M+2]$, applying the interior L^p estimate to (2.26) and embedding theorem we have that $\|w^n\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Omega)} \le C$ for all $n \ge n_0$. Hence, $\|w\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Omega_n)} \le C$, where $\Omega_n=[n+1,n+3]\times[0,2M+2]$, $n \ge n_0$. This fact combined with

$$g'(t)=-\mu u_x(t,g(t))=\beta w_y(t,0),$$

allows us to derive the second estimate of (2.25).

Obviously, $\|w\|_{C^{0,1}(\Omega_n)} \le C$. Since these rectangles Ω_n overlap and C is independent of n, it follows that $\|w\|_{C^{0,1}([n_0+1,\infty)\times[0,2M+2])} \le C$. Notice that $0 \le y \le 2M+2$ is equivalent to $g(t)-2M-2 \le x \le g(t)$, and $u_x(t,x) = -w_y(t,y)$, the first estimate of (2.25) is followed.

The estimate (2.17) can be proved by the similar way. $\square$

It is worth stressing that, in general, the smoothness of the solution cannot be further promoted because of nonlinear source terms $u(a-u-bv)$ and $v(1-v+cu)$ are only Hölder continuous in D^g_∞ and D^h_∞, respectively. For example, if $h(t) < g(t)$ in $[0,T]$ for some $T>0$, then $v_x(t,h(t)) < 0$ and $v_x(t,x) \equiv 0$ for $x > h(t)$. Therefore, $v_x(t,x)$ is not continuous at $x = h(t)$, so is $(u(a-u-bv))_x(t,x)$.

3 Preliminaries

To establish the long time behaviors of (u,v) and conditions for spreading and vanishing, in this section we will state some known results.

We first consider the logistic equation with a free boundary

$$\begin{cases} z_t - dz_{xx} = z(\theta - z), & t>0, \quad 0<x<\rho(t), \\ z_x(t,0)=0, \quad z(t,\rho(t))=0, & t \ge 0, \\ \rho'(t) = -\gamma z_x(t,\rho(t)), & t \ge 0, \\ \rho(0)=\rho_0, \quad z(0,x)=z_0(x), & 0 \le x \le \rho_0, \end{cases} \tag{3.1}$$

where d, θ, γ and ρ_0 are positive constants. Utilize the results of [9], the problem (3.1) has a unique global solution and $\lim_{t\to\infty}\rho(t) = \rho_\infty$ exists. Moreover, the following facts are true:

(a) If $\rho_0 \ge \frac{\pi}{2}\sqrt{d/\theta}$, then $\rho_\infty = \infty$ for all $\gamma > 0$;

(b) If $\rho_0 < \frac{\pi}{2}\sqrt{d/\theta}$, then there exists a positive constant $\gamma(d,\theta,\rho_0,z_0)$ such that $\rho_\infty = \infty$ if $\gamma > \gamma(d,\theta,\rho_0,z_0)$, while $\rho_\infty < \infty$ if $\gamma \le \gamma(d,\theta,\rho_0,z_0)$. By use of the comparison principle we can see that $\gamma(d,\theta,\rho_0,z_0)$ is decreasing in θ, ρ_0 and $z_0(x)$;

(c) If $\rho_\infty = \infty$, then $\lim_{t\to\infty} z(t,x) = \theta$ uniformly in the compact subset of $[0,\infty)$.

Denote

$$\beta^* = \gamma(d,a,g_0,u_0), \quad \mu^* = \gamma(1,1,h_0,v_0), \quad \mu_* = \gamma(1,1+ac,h_0,v_0). \tag{3.2}$$

Next, we consider the problem

$$\begin{cases} dq'' - kq' + q(\theta - q) = 0, \quad 0<y<\infty, \\ q(0)=0, \quad q'(0)=k/\nu, \quad q(\infty)=\theta, \\ k \in (0, 2\sqrt{\theta d}); \quad q'(y)>0, \quad 0<y<\infty, \end{cases} \tag{3.3}$$

where ν, d, θ and k are constants.

Proposition 3.1. ([3, 9]) *For any given $\nu, d, \theta > 0$, the problem* (3.3) *has a unique solution $(q(y),k)$. Denote $k = k(\nu,d,\theta)$. Then $k(\nu,d,\theta)$ is strictly increasing in ν and θ, respectively. Moreover,*

$$\lim_{\frac{\theta\nu}{d}\to\infty} \frac{k(\nu,d,\theta)}{\sqrt{\theta d}} = 2, \qquad \lim_{\frac{\theta\nu}{d}\to 0} \frac{k(\nu,d,\theta)}{\sqrt{\theta d}}\,\frac{d}{\theta\nu} = \frac{1}{\sqrt{3}}. \tag{3.4}$$

Let $w(t)$ be the unique solution of

$$w' = w(a-w), \quad t>0; \quad w(0) = \|u_0\|_{L^\infty}.$$

Then $w(t) \to a$ as $t \to \infty$. The comparison principle leads to

$$\limsup_{t\to\infty} u(t,x) \le a \ \text{ uniformly in } [0,\infty).$$

Similarly,

$$\limsup_{t\to\infty} v(t,x) \le 1+ac \ \text{ uniformly in } [0,\infty).$$

Consequently, for any given $0<\varepsilon \ll 1$, there exists $T \gg 1$ such that

$$\begin{cases} u[a-b(1+ac+\varepsilon)-u] \le u(a-u-bv) \le u(a-u) & \text{in } \ [T,\infty)\times[0,\infty),\\ v(1-v) \le v(1-v+cu) \le v[1+c(a+\varepsilon)-v] & \text{in } \ [T,\infty)\times[0,\infty). \end{cases} \tag{3.5}$$

In view of [9, Theorem 4.2] and the comparison principle, it can be deduced that

$$\limsup_{t\to\infty} \frac{g(t)}{t} \le k(\beta,d,a) := \bar{k}_\beta, \tag{3.6}$$

$$\liminf_{t\to\infty} \frac{h(t)}{t} \ge k(\mu,1,1) := \underline{k}_\mu, \tag{3.7}$$

and

$$\limsup_{t\to\infty} \frac{h(t)}{t} \le k(\mu,1,1+c(a+\varepsilon)),$$
$$\liminf_{t\to\infty} \frac{g(t)}{t} \ge k(\beta,d,a-b(1+ac+\varepsilon)) \quad \text{if} \ \ a>b(1+ac).$$

The arbitrariness of ε yields that

$$\limsup_{t\to\infty} \frac{h(t)}{t} \le k(\mu,1,1+ac) := \bar{k}_\mu, \tag{3.8}$$

$$\liminf_{t\to\infty} \frac{g(t)}{t} \ge k(\beta,d,a-b(1+ac)) := \underline{k}_\beta \quad \text{if} \ \ a>b(1+ac). \tag{3.9}$$

4 Long time behavior of (u,v)

This section concerns with the limits of $(u(t,x),\, v(t,x))$ as $t\to\infty$. We first give a lemma.

Lemma 4.1. *Let d, C, μ and m_0 be positive constants, $w \in W_p^{1,2}((0,T)\times(0,m(t)))$ for some $p>1$ and any $T>0$, and $w_x \in C([0,\infty)\times[0,m(t)])$, $m\in C^1([0,\infty))$. If (w,m) satisfies*

$$\begin{cases} w_t - dv_{xx} \ge -Cw, & t>0, \quad 0<x<m(t),\\ w\ge 0, & t>0, \quad x=0,\\ w=0, \quad m'(t) \ge -\mu w_x, & t>0, \quad x=m(t),\\ w(0,x)=w_0(x) \ge, \not\equiv 0, & x\in(0,m_0),\\ m(0)=m_0, \end{cases}$$

and

$$\lim_{t\to\infty} m(t) = m_\infty < \infty, \qquad \lim_{t\to\infty} m'(t) = 0,$$
$$\|w(t,\cdot)\|_{C^1[0,\,m(t)]} \le M, \quad \forall\ t>1$$

for some constant $M>0$. Then

$$\lim_{t\to\infty} \max_{0\le x\le m(t)} w(t,x) = 0.$$

Proof. Firstly, the maximum principle gives $w(t,x)>0$ for $t>0$ and $0<x<m(t)$. Follow the proof of [40, Theorem 2.2] word by word we can prove this lemma and the details are omitted. □

Theorem 4.1. *If $g_\infty<\infty$ ($h_\infty<\infty$), then*

$$\lim_{t\to\infty}\max_{0\le x\le g(t)} u(t,x)=0 \quad \left(\lim_{t\to\infty}\max_{0\le x\le h(t)} v(t,x)=0\right).$$

Proof. Notice Lemma 2.1 and Theorem 2.2, the conclusion can be deduced by Lemma 4.1 directly. □

When $g_\infty<\infty$ (resp., $h_\infty<\infty$), we say that the species u (resp., v) vanishes eventually. When $g_\infty=\infty$ (resp., $h_\infty=\infty$), we say that the species u (resp., v) spreads successfully.

Theorem 4.2. (i) *If $g_\infty<\infty$ and $h_\infty=\infty$, then*

$$\lim_{t\to\infty} v(t,x)=1 \ \text{ uniformly in the compact subset of } [0,\infty).$$

(ii) *If $h_\infty<\infty$ and $g_\infty=\infty$, then*

$$\lim_{t\to\infty} u(t,x)=a \ \text{ uniformly in the compact subset of } [0,\infty).$$

Proof. We only prove (i) as (ii) can be proved by the similar way. Firstly, utilize the comparison principle and conclusions about the logistic equation, it is easy to get

$$\liminf_{t\to\infty} v(t,x)\ge 1 \ \text{ uniformly in the compact subset of } [0,\infty). \tag{4.1}$$

For any given $0<\varepsilon\ll 1$. Note that $\lim_{t\to\infty}\max_{0\le x\le g(t)} u(t,x)=0$ (Theorem 4.1) and $u(t,x)=0$ for $x>g(t)$. There exists $T\gg 1$ such that $u(t,x)<\varepsilon$ for all $t\ge T$ and $x\in[0,\infty)$. Thus, v satisfies

$$\begin{cases} v_t-v_{xx}\le v(1+c\varepsilon-v), & t\ge T,\ \ 0<x<h(t),\\ v_x(t,0)=0,\ \ v(t,h(t))=0, & t\ge T. \end{cases}$$

This implies $\limsup_{t\to\infty} v(t,x)\le 1+c\varepsilon$ uniformly in $[0,\infty)$. And so,

$$\limsup_{t\to\infty} v(t,x)\le 1 \ \text{ uniformly in } [0,\infty).$$

Combining (4.1), the desired result is obtained immediately. □

When u (resp., v) vanishes eventually and v (resp., u) spreads successfully, our model formally reduces to the single species model. In such a case, the speed of $h(t)$ (resp., $g(t)$) is the same as the one given in [9], the sharp estimates of $v(t,x)$ and $h(t)$ ($u(t,x)$ and $g(t)$) as those investigated by [12, 48].

In the following we deal with the case that both two species spread successfully. We first give a local result.

Theorem 4.3. *Assume that $g_\infty=h_\infty=\infty$.*

(i) *For the weak predation case $b<\min\{a,\,1/c\}$, we denote*

$$A=\frac{a-b}{1+bc},\quad B=\frac{1+ac}{1+bc}.$$

Then

$$\lim_{t\to\infty} u(t,x) = A, \quad \lim_{t\to\infty} v(t,x) = B$$

uniformly in any compact subset of $[0,\infty)$*;*

(ii) *For the strong predation case* $b \geq a$*, we have*

$$\lim_{t\to\infty} v(t,x) = 1, \quad \lim_{t\to\infty} u(t,x) = 0$$

uniformly in any compact subset of $[0,\infty)$.

Applying Propositions 2.1-2.3 of [40], we can prove Theorem 4.3 by the similar arguments to those of [41, Theorems 4.3 and 4.4]. The details are omitted here.

Now, we are going to study the more accurate limits of (u,v) as $t\to\infty$. Let $\underline{k}_\mu$ and $\underline{k}_\beta$ be given by (3.7) and (3.9), respectively.

Theorem 4.4. *Suppose* $g_\infty = h_\infty = \infty$*. For the weak predation case* $b < \min\{a,\, 1/c\}$*, if we further assume* $a > b(1+ac)$*, then for each* $0 < k_0 < \min\{\underline{k}_\beta,\, \underline{k}_\mu\}$*, there hold:*

$$\lim_{t\to\infty} \max_{[0,\,k_0 t]} |u(t,\cdot) - A| = 0, \quad \lim_{t\to\infty} \max_{[0,\,k_0 t]} |v(t,\cdot) - B| = 0. \tag{4.2}$$

Proof. Some ideas in this proof are inspired by [39, Theorem 7]. To facilitate writing, for $\tau \geq 0$, we introduce the following free boundary problem

$$\begin{cases} z_t - D z_{xx} = z(\theta - z), & t > \tau, \quad 0 < x < s(t), \\ z_x(t,0) = 0, \quad z(t,s(t)) = 0, & t \geq \tau, \\ s'(t) = -\nu z_x(t,s(t)), & t \geq \tau, \\ s(\tau) = s_0, \quad z(\tau,x) = z_0(x), & 0 \leq x \leq s_0, \end{cases} \tag{4.3}$$

and set $\Gamma = (\tau, D, \theta, \nu, s_0)$, where D, θ, ν and s_0 are positive constants. For any given constant $T \geq 0$ and function $f(t)$, we define

$$\Omega_T^f = \{(t,x):\, t \geq T,\ 0 \leq x \leq f(t)\}.$$

Recall $0 < k_0 < \min\{\underline{k}_\beta,\, \underline{k}_\mu\}$. Take advantage of (3.7) and (3.9), there exist $0 < \sigma_0 \ll 1$ and $t_\sigma \gg 1$ such that

$$\begin{gathered} k_\sigma := k_0 + \sigma < \min\{\underline{k}_\beta,\, \underline{k}_\mu\}, \quad \forall\ 0 < \sigma \leq \sigma_0, \\ g(t) > k_\sigma t, \quad h(t) > k_\sigma t, \quad \forall\ t \geq t_\sigma,\ 0 < \sigma \leq \sigma_0. \end{gathered}$$

The following proof will be divided into five steps. The method used here is the cross-iteration scheme. In order to construct iteration sequences, in the first four steps, we will prove that, for any fixed $0 < \sigma < \sigma_0/5$,

$$\liminf_{t\to\infty} \min_{[0,\,k_{5\sigma} t]} v(t,\cdot) \geq 1 := \underline{v}_1, \tag{4.4}$$

$$\limsup_{t\to\infty} \max_{[0,\,k_{4\sigma} t]} u(t,\cdot) \leq a - b := \bar{u}_1, \tag{4.5}$$

$$\limsup_{t\to\infty} \max_{[0,\,k_{3\sigma} t]} v(t,\cdot) \leq 1 + c\bar{u}_1 := \bar{v}_1, \tag{4.6}$$

and

$$\begin{cases} \liminf\limits_{t\to\infty} \min\limits_{[0,\,k_{2\sigma}t]} u(t,\cdot) \geq a - b\bar{v}_1 := \underline{u}_1, \\ \liminf\limits_{t\to\infty} \min\limits_{[0,\,k_\sigma t]} v(t,\cdot) \geq 1 + c\underline{u}_1 := \underline{v}_2, \end{cases} \tag{4.7}$$

respectively. In the last step, we will construct four sequences $\{\bar{u}_i\}$, $\{\bar{v}_i\}$, $\{\underline{u}_i\}$ and $\{\underline{v}_i\}$, and derive the desired conclusion.

Step 1: As $h_\infty = \infty$, we can find a $t_1 \gg 1$ so that $h(t_1) > \pi/2$. Let (z_1, s_1) be the unique solution of (4.3) with $\Gamma = (t_1, 1, 1, \mu, h(t_1))$ and $z_0(x) = v(t_1, x)$. Then $h(t) \geq s_1(t)$, $v(t,x) \geq z_1(t,x)$ in $\Omega_{t_1}^{s_1}$ by the comparison principle. And $s_1(\infty) = \infty$ since $s_1(t_1) > \pi/2$. Make use of [48, Theorem 3.1] (see also [12, 16]), we get

$$\lim_{t\to\infty} (s_1(t) - k_1 t) = \varsigma_1 \in \mathbb{R}, \quad \lim_{t\to\infty} \|z_1(t,x) - q_1(k_1 t + \varsigma_1 - x)\|_{L^\infty([0,s_1(t)])} = 0, \tag{4.8}$$

where $(q_1(y), k_1)$ is the unique solution of (3.3) with $(\nu, d, \theta) = (\mu, 1, 1)$, i.e., $k_1 = k(\mu, 1, 1) = \underline{k}_\mu$. Note $0 < k_{5\sigma} < \underline{k}_\mu = k_1$, it is easy to see that $s_1(t) - k_{5\sigma}t \to \infty$ and $\min_{[0,\,k_{5\sigma}t]}(k_1 t + \varsigma_1 - x) \to \infty$ as $t \to \infty$. Owing to $q_1(y) \nearrow 1$ as $y \nearrow \infty$, we have $\min_{x\in[0,\,k_{5\sigma}t]} q_1(k_1 t + \varsigma_1 - x) \to 1$ as $t \to \infty$. It then follows, upon using (4.8), that $\min_{[0,\,k_{5\sigma}t]} z_1(t,\cdot) \to 1$ as $t \to \infty$. Thus, (4.4) holds because of $v \geq z_1$ in $\Omega_{t_1}^{s_1}$.

For any given $0 < \varepsilon \ll 1$, there exists $t_2 \gg 1$ such that

$$g(t) > k_{5\sigma}t, \quad v(t,x) \geq 1 - \varepsilon := \lambda_\varepsilon, \quad \forall\ t \geq t_2,\ 0 \leq x \leq k_{5\sigma}t.$$

Step 2: Obviously, $a - b\lambda_\varepsilon > a - b > (a-b)/(1+bc)$. Make use of Theorem 4.3(i), it follows that $u(t,0) \leq a - b\lambda_\varepsilon$ in $[t_3, \infty)$ for some $t_3 > t_2$. Thus, u satisfies

$$\begin{cases} u_t - du_{xx} \leq u(a - b\lambda_\varepsilon - u), & t \geq t_3, \quad 0 < x < k_{5\sigma}t, \\ u \leq M_1, & t \geq t_3,\ \ 0 < x \leq k_{5\sigma}t, \\ u(t,0) \leq a - b\lambda_\varepsilon, \end{cases}$$

where M_1 is given by Lemma 2.1. We will show that

$$\limsup_{t\to\infty} \max_{[0,\,k_{4\sigma}t]} u(t,\cdot) \leq a - b\lambda_\varepsilon. \tag{4.9}$$

Once this is done, (4.5) is obtained immediately because $\varepsilon > 0$ is arbitrary. To prove (4.9), we choose $0 < \delta \ll 1$ and define

$$f(t,x) = a - b\lambda_\varepsilon + M_1 \mathrm{e}^{\delta k_{5\sigma} t_3} \mathrm{e}^{\delta(x - k_{5\sigma}t)}, \quad t \geq t_3, \quad 0 \leq x \leq k_{5\sigma}t.$$

Evidently,

$$\max_{[0,\,k_{4\sigma}t]} f(t,\cdot) \leq a - b\lambda_\varepsilon + M_1 \mathrm{e}^{\delta k_{5\sigma} t_3} \mathrm{e}^{-\delta\sigma t} \to a - b\lambda_\varepsilon$$

as $t \to \infty$, and

$$\begin{cases} f(t,0) > a - b\lambda_\varepsilon,\ f(t, k_{5\sigma}t) > M_1, & t \geq t_3, \\ f(t_3, x) > M_1, & 0 \leq x \leq k_{5\sigma}t_3. \end{cases}$$

It is easy to verify that, when $\delta(k_{5\sigma} + d\delta) \leq a - b$,

$$f_t - df_{xx} \geq f(a - b\lambda_\varepsilon - f).$$

The comparison principle gives $u(t,x) \le f(t,x)$ for all $t \ge t_3$ and $0 \le x \le k_{5\sigma}t$. Thus we have (4.9) and then obtain (4.5).

There exists $t_4 > t_3$ such that

$$h(t) > k_{4\sigma}t, \quad u(t,x) \le \bar{u}_1 + \varepsilon := \bar{u}_1^\varepsilon < 1, \quad \forall\, t \ge t_4,\ 0 \le x \le k_{4\sigma}t.$$

Step 3: The condition $a > b$ implies $1 + c\bar{u}_1^\varepsilon > (1+ac)/(1+bc)$. Similarly to Step 2, by use of Theorem 4.3(i), there exists $t_5 > t_4$ such that $v(t,0) \le 1 + c\bar{u}_1^\varepsilon$ in $[t_5, \infty)$. Thus, v satisfies

$$\begin{cases} v_t - v_{xx} \le v(1 + c\bar{u}_1^\varepsilon - v), & t \ge t_5, \quad 0 < x < k_{4\sigma}t, \\ v(t,0) \le 1 + c\bar{u}_1^\varepsilon, \quad v \le M_2, & t \ge t_5,\ 0 < x \le k_{4\sigma}t, \end{cases}$$

where M_2 is given by Lemma 2.1. In the same way as Step 2, it can be proved that

$$\limsup_{t\to\infty} \max_{[0,\, k_{\bar\sigma}t]} v(t,\cdot) \le 1 + c\bar{u}_1,$$

where $\bar\sigma = 7\sigma/2$. So, (4.6) holds.

Take $t_6 > t_5$ such that

$$g(t) > k_{\bar\sigma}t, \quad v(t,x) \le \bar{v}_1 + \varepsilon := \bar{v}_1^\varepsilon, \quad \forall\, t \ge t_6,\ 0 \le x \le k_{\bar\sigma}t. \tag{4.10}$$

Step 4: It is easy to see that $a - b\bar{v}_1^\varepsilon > a - b(1+ac)$ since $0 < \varepsilon \ll 1$. So,

$$k(\beta, d, a - b\bar{v}_1^\varepsilon) > k(\beta, d, a - b(1+ac)) = \underline{k}_\beta > k_{3\sigma}.$$

Owing to $k(\beta, d, a - b\bar{v}_1^\varepsilon) \to 0$ as $\beta \to 0$ (cf. (3.4)), we can take $0 < \beta^* < \beta$ so that $k(\beta^*, d, a - b\bar{v}_1^\varepsilon) = k_{3\sigma}$. In this way, we get a function $q(y)$, where $(q(y), k_{3\sigma})$ is the unique solution of (3.3) with $(\nu, d, \theta) = (\beta^*, d, a - b\bar{v}_1^\varepsilon)$. Because of $g(t) > k_{3\sigma}t$ for all $t \ge t_6$, we can find a function $\tilde{u} \in C^2([0, k_{3\sigma}t_6])$ satisfying $\tilde{u}'(0) = \tilde{u}(k_{3\sigma}t_6) = 0$, $\tilde{u}(x) > 0$ in $[0, k_{3\sigma}t_6)$ and

$$\tilde{u}(x) \le u(t_6, x), \quad \forall\, x \in [0, k_{3\sigma}t_6].$$

Let (z_2, s_2) be the unique solution of (4.3) with $\Gamma = (t_6, d, a - b\bar{v}_1^\varepsilon, \beta^*, k_{3\sigma}t_6)$ and $z_0(x) = \tilde{u}(x)$. Then, using [48, Theorem 3.1], we have

$$s_2(t) - k_{3\sigma}t \to \varsigma_2 \in \mathbb{R}, \quad \|z_2(t,x) - q(k_{3\sigma}t + \varsigma_2 - x)\|_{L^\infty([0, s_2(t)])} \to 0 \tag{4.11}$$

as $t \to \infty$. As $\bar\sigma > 3\sigma$, in consideration of (4.10) and the first limit of (4.11), we can think of $g(t) > s_2(t)$, $v(t,x) \le \bar{v}_1^\varepsilon$ for all $t \ge t_6$ and $0 \le x \le s_2(t)$. As a consequence, u satisfies

$$u_t - du_{xx} \ge u(a - b\bar{v}_1^\varepsilon - u) \quad \text{in } \Omega_{t_6}^{s_2}.$$

Note that $z_{2x}(t,0) = u_x(t,0) = 0$, $z_2(t, s_2(t)) = 0 < u(t, s_2(t))$ in $[t_6, \infty)$ and $z_2(t_6, x) = \tilde{u}(x) \le u(t_6, x)$ in $[0, k_{3\sigma}t_6]$, it is deduced that $u \ge z_2$ in $\Omega_{t_6}^{s_2}$ by the comparison principle.

Since $q(y) \nearrow a - b\bar{v}_1^\varepsilon$ as $y \nearrow \infty$, we see that $\lim_{t\to\infty} \min_{x\in[0,\, k_{2\sigma}t]} q(k_{3\sigma}t + \varsigma_2 - x) = a - b\bar{v}_1^\varepsilon$. Apply (4.11) once again, it follows that $s_2(t) - k_{2\sigma}t \to \infty$ and $\min_{[0,\, k_{2\sigma}t]} z_2(t,\cdot) \to a - b\bar{v}_1^\varepsilon$ as $t \to \infty$. This gives the first inequality of (4.7) because $u \ge z_2$ in $\Omega_{t_6}^{s_2}$ and $\varepsilon > 0$ is arbitrary.

Similarly, we can prove the second inequality of (4.7).

Step 5: Five positive constants $\underline{v}_1$, $\bar{u}_1$, $\bar{v}_1$, $\underline{u}_1$ and $\underline{v}_2$ have been obtained. Now we define

$$\bar{u}_i = a - b\underline{v}_i, \quad \bar{v}_i = 1 + c\bar{u}_i, \quad \underline{u}_i = a - b\bar{v}_i, \quad \underline{v}_{i+1} = 1 + c\underline{u}_i, \quad i = 2, 3, \cdots.$$

Then (cf. the proof of [41, Theorem 4.3])

$$\lim_{i\to\infty}\bar u_i=\lim_{i\to\infty}\underline{u}_i=A,\quad \lim_{i\to\infty}\bar v_i=\lim_{i\to\infty}\underline{v}_i=B.$$

Repeating the above process we can show that

$$\begin{aligned}&\underline{u}_i\le\liminf_{t\to\infty}\min_{[0,\,k_0t]}u(t,\cdot),\quad \limsup_{t\to\infty}\max_{[0,\,k_0t]}u(t,\cdot)\le\bar u_i,\quad \forall\ i\ge1.\\ &\underline{v}_i\le\liminf_{t\to\infty}\min_{[0,\,k_0t]}v(t,\cdot),\quad \limsup_{t\to\infty}\max_{[0,\,k_0t]}v(t,\cdot)\le\bar v_i,\quad \forall\ i\ge1.\end{aligned}$$

The proof is finished. $\square$

5 Conditions for spreading and vanishing

In this section we will give some conditions to identify the spreading and vanishing of u, v. Throughout this section, the positive constants β^*, μ^* and μ_* are given by (3.2).

Theorem 5.1. (i) *If $g_0<\frac{\pi}{2}\sqrt{d/a}$ and $\beta\le\beta^*$, then $g_\infty<\infty$;*

(ii) *If either $h_0\ge\pi/2$, or $h_0<\pi/2$ and $\mu>\mu^*$, then $h_\infty=\infty$;*

(iii) *If $u_0(x)\le a$, $h_0<\frac{\pi}{2}\sqrt{1/(1+ac)}$ and $\mu<\mu_*$, then $h_\infty<\infty$.*

This theorem can be proved directly by the comparison principle. We omit the details.

Theorem 5.2. *Assume that $u_0(x)\le a$, $h_0<\frac{\pi}{2}\sqrt{1/(1+ac)}$ and $\mu<\mu_*$. If either $g_0>\frac{\pi}{2}\sqrt{d/a}$, or $g_0<\frac{\pi}{2}\sqrt{d/a}$ and $\beta>\beta^*$, then $g_\infty=\infty$.*

Proof. First, there exists $0<\varepsilon\ll1$ such that either

(a) $g_0>\frac{\pi}{2}\sqrt{d/(a-b\varepsilon)}$, or

(b) $g_0<\frac{\pi}{2}\sqrt{d/(a-b\varepsilon)}$ and $\beta>\beta_\varepsilon:=\gamma(d,a-b\varepsilon,g_0,u_0)$.

As $u_0(x)\le a$, $h_0<\frac{\pi}{2}\sqrt{1/(1+ac)}$ and $\mu<\mu_*$, using Theorem 5.1(iii) and Theorem 4.1, successively, we have $h_\infty<\infty$ and $\lim_{t\to\infty}\max_{0\le x\le h(t)}v(t,x)=0$. Note that $v(t,x)=0$ for $x\ge h(t)$. We can find a $T\gg1$ so that $0\le v(t,x)\le\varepsilon$ for all $t\ge T$ and $x\ge0$. Hence, (u,g) satisfies

$$\begin{cases}u_t-du_{xx}\ge u(a-b\varepsilon-u),\ t\ge T,\quad 0<x<g(t),\\ u_x(t,0)=u(t,g(t))=0,\qquad t\ge T,\\ g'(t)=-\beta u_x(t,g(t)),\qquad t\ge T.\end{cases}$$

Consequently, $g_\infty=\infty$ because at least one of conditions (a) and (b) holds. $\square$

The conclusions of Theorem 5.2 show that, if one of g_0 and β (the initial habitat and moving parameter of the prey) is "suitably large", both h_0 and μ (the initial habitat and moving parameter of the predator) are "suitably small", the prey will spread successfully, while the predator will vanishes eventually. However, the predator always able to successfully spread if either $h_0\ge\pi/2$, or $h_0<\pi/2$ and $\mu>\mu^*$. A natural question arises: does the prey always die out eventually if the predator spreads successfully? Intuitively, if the predator spreads faster enough than the prey, the prey would have no chance to survive eventually even its initial population and initial habitat size are large.

In the following, we will give two results to answer the above question. The first one indicates that if the predator spreads slowly and the prey's initial habitat is much larger than that of the predator, the prey will spread successfully and its territory always cover that of the predator no matter whether the latter successful spread. The second one illustrates that for the strong predation case $b > a$, if the prey spreads slowly and the predator spreads quickly, the prey will vanish eventually and the predator will spread successfully.

In view of Lemma 2.1, we have

$$0 < h(t) \le K\mu t + h_0, \quad \forall\, t > 0,$$

where

$$K = 2\max\left\{M_2\sqrt{(1+cM_1)/2},\ -\min\nolimits_{[0,\,h_0]} v_0'(x)\right\},$$
$$M_2 = \max\{1+cM_1,\ \|v_0\|_\infty\}, \quad M_1 = \max\{a,\ \|u_0\|_\infty\}.$$

Theorem 5.3. *Let d, a, b, c and β be fixed. Then there exists $0 < \bar{\mu} < \sqrt{2da}/K$ such that, when*

$$0 < \mu < \bar{\mu}, \quad g_0 - h_0 > \frac{2d\pi}{\sqrt{2da - K^2\mu^2}} := L(\mu),$$

we have $g(t) \ge K\mu t + h_0 + L(\mu)$ for all $t \ge 0$, which leads to $g(t) > h(t)$ for all $t \ge 0$ and $g(t) \to \infty$ as $t \to \infty$.

Moreover, if $h_0 \ge \pi/2$, we also have $h_\infty = \infty$ for all $\mu > 0$.

Proof. This proof is similar to that of [39, Theorem 6]. For the completeness and convenience to reader, we shall give the details. Denote $\sigma = K\mu$. For these t satisfying $g(t) > \sigma t + h_0$, we define

$$y = x - \sigma t - h_0, \quad \varphi(t,y) = u(t,x), \quad \eta(t) = g(t) - \sigma t - h_0.$$

Then $\varphi(t,y) > 0$ for $t \ge 0$ and $0 \le y < \eta(t)$. Note that $v(t,x) = 0$ for $x \ge h(t)$, and $y \ge 0$ implies $x \ge h(t)$, it follows that φ satisfies

$$\begin{cases} \varphi_t - d\varphi_{yy} - \sigma\varphi_y = \varphi(a-\varphi), & t > 0,\ 0 < y < \eta(t),\\ \varphi(t,0) = u(t, \sigma t + h_0), \quad \varphi(t,\eta(t)) = 0, & t \ge 0,\\ \varphi(0,y) = u_0(y+h_0), & 0 \le y \le g_0 - h_0. \end{cases}$$

Let λ be the principal eigenvalue of

$$\begin{cases} -d\phi'' - \sigma\phi' - a\phi = \lambda\phi, \quad 0 < x < L,\\ \phi(0) = 0 = \phi(L). \end{cases} \tag{5.1}$$

Then the following relation (between λ and L) holds:

$$\frac{\pi}{L} = \frac{\sqrt{4d(a+\lambda) - \sigma^2}}{2d}.$$

Choose $\lambda = -a/2$ and define

$$L_\sigma = \frac{2d\pi}{\sqrt{2da - \sigma^2}}, \quad \phi(y) = \mathrm{e}^{-\frac{\sigma}{2d}y}\sin\frac{\pi}{L_\sigma}y \quad \text{with } \ 0 < \sigma < \sqrt{2da}.$$

Then (L_σ, ϕ) satisfies (5.1) with $\lambda = -a/2$ and $L = L_\sigma$. Assume $g_0 - h_0 > L_\sigma$ and set

$$\delta_\sigma = \min\left\{\inf_{(0,\,L_\sigma)}\frac{\varphi(0,y)}{\phi(y)},\ \frac{a}{2}\inf_{(0,\,L_\sigma)}\frac{1}{\phi(y)}\right\}, \quad \psi(y) = \delta_\sigma\phi(y).$$

Then $0 < \delta_\sigma < \infty$. It is easy to verify that $\psi(y) \le \varphi(0,y)$ in $[0, L_\sigma]$ and $\psi(y)$ satisfies

$$\begin{cases} -d\psi'' - \sigma\psi' \le \psi(a-\psi), & 0 < x < L_\sigma, \\ \psi(0) = 0 = \psi(L_\sigma). \end{cases}$$

Take a maximal $\bar\sigma \in (0, \sqrt{2da})$ so that

$$\sigma < \beta\delta_\sigma \frac{\pi}{L_\sigma} \exp\left(-\frac{\sigma L_\sigma}{2d}\right), \quad \forall\, \sigma \in (0, \bar\sigma). \tag{5.2}$$

For any given $\sigma \in (0, \bar\sigma)$, we claim that $\eta(t) > L_\sigma$ for all $t \ge 0$, which implies

$$g(t) \ge \sigma t + h_0 + L_\sigma \to \infty.$$

In fact, note $\eta(0) = g_0 - h_0 > L_\sigma$, if our claim is not true, then we can find a $t_0 > 0$ such that $\eta(t) > L_\sigma$ for all $0 \le t < t_0$ and $\eta(t_0) = L_\sigma$. Therefore, $\eta'(t_0) \le 0$, i.e, $g'(t_0) \le \sigma$. On the other hand, by the comparison principle, we have $\varphi(t,y) \ge \psi(y)$ in $[0,t_0] \times [0, L_\sigma]$. Particularly, $\varphi(t_0,y) \ge \psi(y)$ in $[0, L_\sigma]$. Due to $\varphi(t_0, L_\sigma) = \varphi(t_0, \eta(t_0)) = 0 = \psi(L_\sigma)$, it derives that

$$\varphi_y(t_0, L_\sigma) \le \psi'(L_\sigma) = -\delta_\sigma \frac{\pi}{L_\sigma} \exp\left(-\frac{\sigma L_\sigma}{2d}\right).$$

It follows, upon using $u_x(t_0, g(t_0)) = \varphi_y(t_0, \eta(t_0))$, that

$$\sigma \ge g'(t_0) = -\beta\varphi_y(t_0, \eta(t_0)) = -\beta\varphi_y(t_0, L_\sigma) \ge \beta\delta_\sigma \frac{\pi}{L_\sigma} \exp\left(-\frac{\sigma L_\sigma}{2d}\right).$$

It is in contradiction with (5.2).

Take $\bar\mu = \bar\sigma/K$, $L(\mu) = L_\sigma$. Then $0 < \mu < \bar\mu$ is equivalent to $0 < \sigma < \bar\sigma$, and $g_0 - h_0 > L(\mu)$ is equivalent to $g_0 - h_0 > L_\sigma$.

At last, when $h_0 \ge \pi/2$, we have $h_\infty = \infty$ for any $\mu > 0$ by Theorem 5.1(ii). The proof is complete. $\square$

In consideration of (3.4), it is easy to see that

$$\lim_{\beta \to 0} k(\beta, d, a) = 0, \quad \lim_{\mu \to \infty} k(\mu, 1, 1) = 2.$$

By the monotonicity of $k(\nu, d, \theta)$ in ν, there exist $\bar\beta$, $\bar\mu > 0$ such that $k(\beta, d, a) < k(\mu, 1, 1)$ for all $0 < \beta \le \bar\beta$ and $\mu \ge \bar\mu$. Therefore, $(0, \bar\beta] \times [\bar\mu, \infty) \subset \mathcal{A}$, where

$$\mathcal{A} = \{(\beta, \mu) : \beta, \mu > 0,\ k(\beta, d, a) < k(\mu, 1, 1)\}.$$

Theorem 5.4. *Assume that $(\beta, \mu) \in \mathcal{A}$. If $b > a$ and $h_\infty = \infty$, then $g_\infty < \infty$.*

Proof. Firstly, because of $b > a$, there exists $0 < \varepsilon \ll 1$ such that $a < b(1 - \varepsilon)$.

There exists $t_1 \gg 1$ such that $h(t_1) > \pi/2$. Let (z_1, s_1) be the unique solution of (4.3) with $\Gamma = (t_1, 1, 1, \mu, h(t_1))$ and $z_0(x) = v(t_1, x)$. Then $s_1(\infty) = \infty$, $h(t) \ge s_1(t)$, $v(t,x) \ge z_1(t,x)$ in $\Omega_{t_1}^{s_1}$. Moreover, make use of [48, Theorem 3.1] (see also [12, 16]) we have that, as $t \to \infty$,

$$s_1(t) - k_1 t \to \varsigma_1 \in \mathbb{R}, \quad \|z_1(t,x) - q_1(k_1 t + \varsigma_1 - x)\|_{L^\infty([0, s_1(t)])} \to 0, \tag{5.3}$$

where $(q_1(y), k_1)$ is the unique solution of (3.3) with $(\nu, d, \theta) = (\mu, 1, 1)$, i.e., $k_1 = k(\mu, 1, 1)$.

Assume on the contrary that $g_\infty = \infty$. Let (z_2, s_2) be the unique solution of (4.3) with $\Gamma = (0, d, a, \beta, g_0)$ and $z_0(x) = u_0(x)$. Then $z_2(t,x) \ge u(t,x)$, $s_2(t) \ge g(t)$ for all $t \ge 0$ and $0 \le x \le g(t)$. Similarly to the above, $s_2(t) - k(\beta, d, a)t \to \varsigma_2 \in \mathbb{R}$ as $t \to \infty$.

Because of $(\beta, \mu) \in \mathcal{A}$, we have $k_1 > k(\beta, d, a)$. This implies $s_1(t) - g(t) \ge s_1(t) - s_2(t) \to \infty$ and $\min_{0 \le x \le g(t)} q_1(k_1 t + \varsigma_1 - x) \to 1$ as $t \to \infty$. Thus, upon using (5.3), $\lim_{t\to\infty} \min_{0 \le x \le g(t)} z_1(t,x) = 1$. There exists $t_2 > t_1$ such that $z_1(t,x) > 1 - \varepsilon$ for all $t \ge t_2$ and $0 \le x \le g(t)$. Consequently, $v(t,x) > 1 - \varepsilon$, and hence $a - u - bv < a - b(1-\varepsilon) - u < 0$ for all $t \ge t_2$ and $0 \le x \le g(t)$. Take advantage of [20, Lemma 3.2], it follows that $g_\infty < \infty$. $\square$

6 Estimates of asymptotic spreading speeds of u, v and asymptotic speeds of g, h

The authors of [26] and [30], by means of the construction of the appropriate and elaborate upper and lower solutions, established some interesting results for the asymptotic spreading speeds of solution to the following Cauchy problem

$$\begin{cases} u_t - du_{xx} = u(a - u - bv), & t > 0, \quad x \in \mathbb{R}, \\ v_t - v_{xx} = v(1 - v + cu), & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x \in \mathbb{R}, \\ 0 \le u_0(x) \le a, \quad 0 \le v_0(x) \le 1 + ac, & x \in \mathbb{R}. \end{cases} \tag{6.1}$$

Their conclusions show that the prey and predator may have different asymptotic spreading speeds.

Definition 6.1. *Let $w(t,x)$ be a nonnegative function for $t > 0$ and $x \in [0, \infty)$. A number $c_* > 0$ is called the asymptotic spreading speed of $w(t,x)$ if*

(a) $\lim\limits_{t\to\infty} \sup\limits_{x \ge (c_*+\varepsilon)t} w(t,x) = 0$ *for any given $\varepsilon > 0$;*

(b) $\lim\limits_{t\to\infty} \inf\limits_{0 \le x \le (c_*-\varepsilon)t} w(t,x) > 0$ *for any given $0 < \varepsilon < c_*$.*

The asymptotic spreading speed gives the observed phenomena imagining an observer moves to the right at a fixed speed [42], and it describes the speed at which the geographic range of the new population expands in population dynamics [19]. Thus the asymptotic spreading of prey and predator are useful and important in understanding the interspecies action between the prey and predator. The background of prey-predator system implies that the predator has a negative effect on the prey, while the prey has a positive effect on the predator (see [29] for some biological results). Intuitively, we guess (believe) that the asymptotic propagation of prey (asymptotic spreading speed of u and asymptotic speed of g) may be slower than the case of no predator, and that of the predator (asymptotic spreading speed of v and asymptotic speed of h) may be faster than the case of no prey. However, our results indicate that this is not necessarily right.

The other related works on the asymptotic spreading speeds of evolutionary systems, please refer to [2, 23, 24, 27, 31, 42, 46] and the references cited therein. In some evolutionary systems, the nonexistence of constant asymptotic spreading speed has been observed, see Berestycki et al. [1] for some examples.

In this section we study the asymptotic spreading speeds of u, v and asymptotic speeds of g, h. Assume $a > b(1 + ac)$. In consideration of (3.5), using the known results of (3.1) and comparison principle, we see that both prey and predator must spread successfully as long as their moving parameters are suitably large. That is, there are β_1, $\mu_1 > 0$ such that $g_\infty = h_\infty = \infty$ for all $\beta \geq \beta_1$ and $\mu \geq \mu_1$.

Throughout this section we assume $a > b(1+ac)$, which is equivalent to $bc < 1$ and $a > b/(1-bc)$. Denote

$$c_1 = 2\sqrt{da}, \quad c_2 = 2\sqrt{1+ac}, \quad c_3 = 2\sqrt{da - db(1+ac)},$$
$$c_4 = 2\sqrt{da - db}, \qquad c_5 = 2\sqrt{(1+ac)(1-bc)}.$$

Theorem 6.1. *For any given $0 < \varepsilon \ll 1$, there exist β_ε, μ_ε, $T \gg 1$ such that, when $\beta \geq \beta_\varepsilon$ and $\mu \geq \mu_\varepsilon$,*

$$u(t,x) = 0 \quad \text{for } t \geq T, \;\; x \geq (c_1 + \varepsilon)t, \tag{6.2}$$

$$v(t,x) = 0 \quad \text{for } t \geq T, \;\; x \geq (c_2 + \varepsilon)t, \tag{6.3}$$

$$\liminf_{t\to\infty} \min_{0 \leq x \leq (c_3 - \varepsilon)t} u(t,x) \geq a - b(1+ac), \quad \liminf_{t\to\infty} \min_{0 \leq x \leq (2-\varepsilon)t} v(t,x) \geq 1. \tag{6.4}$$

Proof. According to the first limit of (3.4), it follows that

$$\lim_{\beta\to\infty} \bar{k}_\beta = c_1, \quad \lim_{\mu\to\infty} \bar{k}_\mu = c_2, \quad \lim_{\beta\to\infty} \underline{k}_\beta = c_3, \quad \lim_{\mu\to\infty} \underline{k}_\mu = 2,$$

where $\bar{k}_\beta$, $\underline{k}_\mu$, $\bar{k}_\mu$ and $\underline{k}_\beta$ are given by (3.6)-(3.9), respectively. Note that (3.6)-(3.9), for any given $0 < \varepsilon \ll 1$, there exist $\beta_\varepsilon \gg 1$ and $\mu_\varepsilon \gg 1$ such that

$$c_3 - \varepsilon/2 < \underline{k}_\beta \leq \liminf_{t\to\infty} \frac{g(t)}{t}, \quad \limsup_{t\to\infty} \frac{g(t)}{t} \leq \bar{k}_\beta < c_1 + \varepsilon/4, \quad \forall\, \beta \geq \beta_\varepsilon, \tag{6.5}$$

$$2 - \varepsilon/2 < \underline{k}_\mu \leq \liminf_{t\to\infty} \frac{h(t)}{t}, \quad \limsup_{t\to\infty} \frac{h(t)}{t} \leq \bar{k}_\mu < c_2 + \varepsilon/2, \quad \forall\, \mu \geq \mu_\varepsilon. \tag{6.6}$$

As a conclusion, we can find a $\tau_1 \gg 1$ such that, for all $t \geq \tau_1$, $\beta \geq \beta_\varepsilon$ and $\mu \geq \mu_\varepsilon$,

$$(c_3 - \varepsilon)t < g(t) < (c_1 + \varepsilon/2)t, \quad (2-\varepsilon)t < h(t) < (c_2 + \varepsilon)t. \tag{6.7}$$

Obviously, (6.2) and (6.3) hold. Similarly to Step 1 in the proof of Theorem 4.4, we can prove (6.4). The proof is finished. $\square$

Theorem 6.2. *Suppose $da < 1$. Then the following hold:*

(i) *For any given $0 < k_0 < c_3$, (4.2) holds as long as β and μ are suitably large.*

(ii) *There exists $\mu_0 \gg 1$ such that, when $\mu > \mu_0$,*

$$\limsup_{\beta\to\infty} \limsup_{t\to\infty} \frac{g(t)}{t} \leq c_4. \tag{6.8}$$

(iii) *For any given $\varepsilon > 0$, there exist β_ε, μ_ε, $T \gg 1$ such that, when $\beta \geq \beta_\varepsilon$ and $\mu \geq \mu_\varepsilon$,*

$$u(t,x) = 0 \quad \text{for} \;\; t \geq T, \;\; x \geq (c_4 + \varepsilon)t, \tag{6.9}$$

$$\lim_{t\to\infty} \sup_{x \geq (2+\varepsilon)t} v(t,x) = 0, \tag{6.10}$$

$$\lim_{t\to\infty} \max_{(c_1+\varepsilon)t \leq x \leq (2-\varepsilon)t} |v(t,x) - 1| = 0. \tag{6.11}$$

(iv) *There exists $\beta_0 \gg 1$ such that, when $\beta > \beta_0$,*

$$\lim_{\mu\to\infty} \lim_{t\to\infty} \frac{h(t)}{t} = 2. \tag{6.12}$$

Proof. Take advantage of (6.5) and (6.6), we have

$$\liminf_{\beta\to\infty}\liminf_{t\to\infty}\frac{g(t)}{t}\geq c_3,\quad \liminf_{\mu\to\infty}\liminf_{t\to\infty}\frac{h(t)}{t}\geq 2.$$

Since $da<1$, it is obvious that

$$2>c_1>c_4>c_3.$$

The conclusion (i) can be proved by the same way as that of Theorem 4.4.

(ii) Choose $0<\varepsilon\ll 1$ such that $c_1+\varepsilon<2-\varepsilon$. Then, in view of (6.7), we have

$$g(t)<(c_1+\varepsilon/2)t<(c_1+\varepsilon)t<(2-\varepsilon)t<h(t),\quad \forall\ \beta\geq\beta_\varepsilon,\ \mu\geq\mu_\varepsilon,\ t\geq\tau_1. \tag{6.13}$$

The second conclusion of (6.4) shows that for any given $0<\delta\ll 1$, there exists $\tau_2>\tau_1$ such that $v(t,x)\geq 1-\delta$ for all $t\geq\tau_2$ and $0\leq x\leq(2-\varepsilon)t$. Combining this with (6.13), we see that, when $\beta\geq\beta_\varepsilon$ and $\mu\geq\mu_\varepsilon$, u satisfies

$$\begin{cases} u_t-du_{xx}\leq u[a-b(1-\delta)-u], & t\geq\tau_2,\ 0<x<g(t),\\ u_x(t,0)=0,\quad u(t,g(t))=0, & t\geq\tau_2,\\ g'(t)=-\beta u_x(t,g(t)), & t\geq\tau_2.\end{cases}$$

It follows that

$$\limsup_{\beta\to\infty}\limsup_{t\to\infty}\frac{g(t)}{t}\leq 2\sqrt{da-db(1-\delta)},$$

and then (6.8) holds because $\delta>0$ is arbitrary.

(iii) The result (6.9) is a direct consequence of (6.8).

Now we prove (6.10). By virtue of $c_4<2$ and (6.8), there exist $\tau_*\gg 1$, $\beta_*\gg 1$ such that $g(t)<2t$ for all $t\geq\tau_*$ and $\beta\geq\beta_*$. This implies $u(t,x)=0$ for all $t\geq\tau_*$, $x\geq 2t$ and $\beta\geq\beta_*$. Define

$$s(t)=\max\{h(t),\,(2+\varepsilon)t\}\quad\text{for}\quad t\geq\tau_*.$$

Note that $v(t,x)=0$ for $x\geq h(t)$ and $v_x(t,h(t))<0$, it is not hard to see that v satisfies, in the weak sense,

$$\begin{cases} v_t-v_{xx}\leq v(1-v), & t\geq\tau_*,\quad 2t\leq x<s(t),\\ v(t,x)\leq M_2, & t\geq\tau_*,\quad 2t\leq x<s(t),\\ v(t,s(t))=0, & t\geq\tau_*,\end{cases}$$

where M_2 is given by Lemma 2.1. Define

$$\xi(t,x)=M_2\mathrm{e}^{s(\tau_*)-2\tau_*}\mathrm{e}^{2t-x},\quad t\geq\tau_*,\ 2t\leq x<s(t).$$

Clearly,

$$\sup_{x\geq(2+\varepsilon)t}\xi(t,x)\leq M_2\mathrm{e}^{s(\tau_*)-2\tau_*}\mathrm{e}^{-\varepsilon t}\to 0$$

as $t\to\infty$, and

$$\xi(t,2t)>M_2,\ \xi(t,s(t))>0,\quad t\geq\tau_*,$$

$$\xi(\tau_*,x)>M_2,\quad 2\tau_*\leq x<s(\tau_*).$$

It is easy to verify that

$$\xi_t - \xi_{xx} \geq \xi(1-\xi), \quad t \geq \tau_*, \ 2t \leq x < s(t).$$

By the comparison principle, $v(t,x) \leq \xi(t,x)$ for all $t \geq \tau_*$ and $2t \leq x < s(t)$. The limit (6.10) is obtained.

In the following we prove (6.11). Based on (6.13), we see that v satisfies

$$\begin{cases} v_t - v_{xx} = v(1-v), & t \geq \tau_1, \ (c_1+\varepsilon/2)t \leq x < h(t), \\ v(t,(c_1+\varepsilon/2)t) \leq M_2, \ v(t,h(t)) = 0, & t \geq \tau_1, \\ v(\tau_1,x) \leq M_2, & (c_1+\varepsilon/2)\tau_1 \leq x < h(\tau_1), \end{cases}$$

where M_2 is given by Lemma 2.1. Define

$$\varphi(t,x) = 1 + M_2 \mathrm{e}^{h(\tau_1)} \mathrm{e}^{(c_1+\varepsilon/2)t - x}, \quad t \geq \tau_1, \ (c_1+\varepsilon/2)t \leq x < h(t).$$

The direct calculations yield

$$\varphi_t - \varphi_{xx} \geq \varphi(1-\varphi), \quad t \geq \tau_1, \ (c_1+\varepsilon/2)t \leq x < h(t),$$

and

$$\varphi(t,(c_1+\varepsilon/2)t) > 1 + M_2, \quad \varphi(t,h(t)) \geq 1, \quad \forall \ t \geq \tau_1,$$
$$\varphi(\tau_1,x) \geq 1 + M_2, \quad \forall \ (c_1+\varepsilon/2)\tau_1 \leq x \leq h(\tau_1).$$

By the comparison principle, $v(t,x) \leq \varphi(t,x)$ for $t \geq \tau_1$ and $(c_1+\varepsilon/2)t \leq x < h(t)$. According to (6.13), we have $(c_1+\varepsilon)t < h(t)$ for all $t \geq \tau_1$ and $\beta \geq \beta_\varepsilon, \ \mu \geq \mu_\varepsilon$. And so

$$\max_{x \geq (c_1+\varepsilon)t} v(t,x) = \max_{(c_1+\varepsilon)t \leq x \leq h(t)} v(t,x) \leq \max_{(c_1+\varepsilon)t \leq x \leq h(t)} \varphi(t,x) = 1 + M_2 \mathrm{e}^{h(\tau_1)} \mathrm{e}^{-\varepsilon t/2}, \tag{6.14}$$

which implies $\limsup\limits_{t\to\infty} \max\limits_{x \geq (c_1+\varepsilon)t} v(t,x) \leq 1$. This combined with the second inequality of (6.4) allows us to derive (6.11).

(iv) For any given $0 < \sigma \ll 1$ and $\beta \geq \beta_\varepsilon$. Let $(q(y),k)$ be the unique solution of (3.3) with $(\nu,d,\theta) = (\mu,1,1+\sigma)$. Then $q'(y) > 0$, $q(y) \to 1+\sigma$ as $y \to \infty$ and $\lim_{\mu\to\infty} k = 2\sqrt{1+\sigma}$. Combining these facts with (6.13) and (6.14), we can find three constants $\mu_0 > \mu_\varepsilon$, $\tau_0 > \tau_1$, $y_0 \gg 1$ such that, for all $\mu \geq \mu_0$,

$$k > c_1 + \varepsilon, \quad h(t) > (c_1+\varepsilon)t, \quad \forall \ t \geq \tau_0,$$
$$v(t,x) < (1 + M_2 \mathrm{e}^{h(\tau_1)} \mathrm{e}^{-\varepsilon t/2}) q(y), \quad \forall \ t \geq \tau_0, \ x \geq (c_1+\varepsilon)t, \ y \geq y_0.$$

Denote $K = M_2 \mathrm{e}^{h(\tau_1)}$ and define

$$\begin{aligned} \bar{h}(t) &= kt + \varrho K \big(\mathrm{e}^{-\varepsilon\tau_0/2} - \mathrm{e}^{-\varepsilon t/2}\big) + y_0 + h(\tau_0), \quad t \geq \tau_0, \\ \bar{v}(t,x) &= \big(1 + K\mathrm{e}^{-\varepsilon t/2}\big) q(\bar{h}(t) - x), \quad t \geq \tau_0, \quad (c_1+\varepsilon)t \leq x \leq \bar{h}(t), \end{aligned}$$

where ϱ is a positive constant to be determined. Obviously,

$$\bar{h}(\tau_0) > h(\tau_0), \quad \bar{v}(\tau_0,x) \geq v(\tau_0,x), \quad \forall \ (c_1+\varepsilon)\tau_0 \leq x \leq h(\tau_0),$$
$$\bar{v}(t,\bar{h}(t)) = 0, \quad \bar{v}(t,(c_1+\varepsilon)t) > v(t,(c_1+\varepsilon)t), \quad \forall \ t \geq \tau_0.$$

In the same way as the arguments of [21, Lemma 3.5] we can verify that, when ϱ is suitably large,

$$\bar{v}_t-\bar{v}_{xx}\geq\bar{v}(1-\bar{v}),\quad \forall\, t\geq\tau_0,\ \ (c_1+\varepsilon)t\leq x<\bar{h}(t),$$
$$\bar{h}'(t)\geq-\mu\bar{v}_x(t,\bar{h}(t)),\quad \forall\, t\geq\tau_0.$$

Because v satisfies $v_t-v_{xx}=v(1-v)$ for $t\geq\tau_0$ and $(c_1+\varepsilon)t\leq x<h(t)$, by the comparison principle we have $v(t,x)\leq\bar{v}(t,x)$ and $h(t)\leq\bar{h}(t)$ for all $t\geq\tau_0$ and $(c_1+\varepsilon)t\leq x<h(t)$. Hence,

$$\limsup_{t\to\infty}\frac{h(t)}{t}\leq k,\quad \limsup_{\mu\to\infty}\limsup_{t\to\infty}\frac{h(t)}{t}\leq\limsup_{\mu\to\infty}k=2\sqrt{1+\sigma}.$$

The arbitrariness of σ leads to

$$\limsup_{\mu\to\infty}\limsup_{t\to\infty}\frac{h(t)}{t}\leq 2.$$

This together with the first inequality of (6.6) derive (6.12), and the proof is finished. □

Theorem 6.3. *If $d[a-b(1+ac)]>1+ac$, we have the following conclusions:*

(i) *There exists $\beta_0\gg1$ such that, when $\beta>\beta_0$,*

$$\liminf_{\mu\to\infty}\liminf_{t\to\infty}\frac{h(t)}{t}\geq c_5.$$

(ii) *For any given $\varepsilon>0$, there exist β_ε, $\mu_\varepsilon\gg1$ such that*

$$\liminf_{t\to\infty}\min_{0\leq x\leq(c_1-\varepsilon)t}u(t,x)>0,\qquad \liminf_{t\to\infty}\min_{0\leq x\leq(c_5-\varepsilon)t}v(t,x)\geq(1+ac)(1-bc)>0 \tag{6.15}$$

provided $\beta\geq\beta_\varepsilon$, $\mu\geq\mu_\varepsilon$.

(iii) *There exists $\mu_0\gg1$ such that, when $\mu>\mu_0$,*

$$\lim_{\beta\to\infty}\lim_{t\to\infty}\frac{g(t)}{t}=2\sqrt{da}=c_1.$$

(iv) *For any given $0<k_0<c_5$, (4.2) holds as long as β and μ are suitably large.*

Proof. The assumption $d[a-b(1+ac)]>1+ac$ implies $c_3>c_2$. Choose $\varepsilon>0$ is so small that $c_3-\varepsilon>c_2+\varepsilon$, then, by (6.7), we have

$$h(t)<(c_2+\varepsilon)t<(c_3-\varepsilon)t<g(t),\ \ \forall\,\beta\geq\beta_\varepsilon,\ \mu\geq\mu_\varepsilon,\ t\geq\tau_1. \tag{6.16}$$

For any given $0<\delta\ll1$, there exists $0<\sigma_0\ll1$ such that $2\sqrt{(1+ac)(1-bc)-c\sigma}>c_5-\delta$ for all $0<\sigma\leq\sigma_0$, where $c_5=2\sqrt{(1+ac)(1-bc)}$. For such a fixed σ, combining (6.16) with the first inequality of (6.4), we have that there exists $\tau_3>\tau_1$ such that (v,h) satisfies

$$\begin{cases} v_t-v_{xx}\geq v\big[1+c(a-b(1+ac)-\sigma)-v\big], & t\geq\tau_3,\ \ 0<x<h(t),\\ v_x(t,0)=0,\ \ v(t,h(t))=0, & t\geq\tau_3,\\ h'(t)=-\mu v_x(t,h(t)), & t\geq\tau_3\end{cases}$$

for all $\beta\geq\beta_\varepsilon$, $\mu\geq\mu_\varepsilon$. Let $(\underline{v},\underline{h})$ be the unique solution of

$$\begin{cases} \underline{v}_t-\underline{v}_{xx}=\underline{v}\big[1+c(a-b(1+ac)-\sigma)-\underline{v}\big], & t\geq\tau_3,\ \ 0<x<\underline{h}(t),\\ \underline{v}_x(t,0)=0,\ \ \underline{v}(t,\underline{h}(t))=0, & t\geq\tau_3,\\ \underline{h}'(t)=-\mu\underline{v}_x(t,\underline{h}(t)), & t\geq\tau_3,\\ \underline{h}(\tau_3)=h(\tau_3),\ \ \underline{v}(\tau_3,x)=v(\tau_3,x), & 0\leq x\leq\underline{h}(\tau_3).\end{cases}$$

Then $h(t) \geq \underline{h}(t)$, $v(t,x) \geq \underline{v}(t,x)$ for all $t \geq \tau_3$ and $0 \leq x \leq \underline{h}(t)$ by the comparison principle. Make use of [48, Theorem 3.1], it follows that, for any given $0 < \varepsilon \ll 1$,

$$\lim_{\mu\to\infty}\lim_{t\to\infty}\frac{\underline{h}(t)}{t} = 2\sqrt{(1+ac)(1-bc)-c\sigma} > c_5-\delta,$$
$$\lim_{t\to\infty}\max_{0\leq x\leq(c_5-\delta-\varepsilon)t}\left|\underline{v}(t,x)-[(1+ac)(1-bc)-c\sigma]\right| = 0.$$

Consequently,

$$\liminf_{\mu\to\infty}\liminf_{t\to\infty}\frac{h(t)}{t} \geq c_5, \quad \liminf_{t\to\infty}\min_{0\leq x\leq(c_5-\varepsilon)t} v(t,x) \geq (1+ac)(1-bc)$$

since δ and σ are arbitrary. The conclusion (i) and the second inequality of (6.15) are obtained.

Now we prove the first inequality of (6.15). Notice $v \leq M_2$ and the first inequality of (6.4), there exists $\tau_4 \gg 1$ such that $v(t,x) \leq \frac{2M_2}{a-b(1+ac)}u(t,x)$ for all $t \geq \tau_4$ and $0 \leq x \leq (c_3-\varepsilon)t$. In view of (6.16) we see that $v(t,x) = 0$ when $x \geq (c_3-\varepsilon)t$. Denote $r = \frac{2M_2}{a-b(1+ac)}$, then (u,g) satisfies

$$\begin{cases} u_t - du_{xx} \geq u(a-u-bru), & t\geq\tau_4, \quad 0<x<g(t), \\ u_x(t,0)=0, \quad u(t,g(t))=0, & t\geq\tau_4, \\ g'(t) = -\beta u_x(t,g(t)), & t\geq\tau_4. \end{cases}$$

Similarly to the above, we can show that

$$\liminf_{\beta\to\infty}\liminf_{t\to\infty}\frac{g(t)}{t} \geq 2\sqrt{ad} = c_1, \quad \liminf_{t\to\infty}\min_{0\leq x\leq(c_1-\varepsilon)t} u(t,x) \geq a/(1+br). \tag{6.17}$$

The above second inequality implies the first one of (6.15).

The conclusion (iii) can be derived from the first inequality of (6.17) and the second one of (6.5).

Since $c_5 < c_1$, the proof of (iv) is the same as that of Theorem 4.4. The proof of Theorem 6.3 is complete. $\square$

7 Discussion–biological significance of the conclusions

In this paper, we investigated a free boundary problem which describes the expanding of prey and predator in a one-dimensional habitat. In this model, the prey occupying the interval $[0, g(t)]$, while the predator with the territory $[0, h(t)]$ at time t. Here, the two free boundaries $x = g(t)$ and $x = h(t)$ may intersect each other as time evolves. They describe the spreading fronts of prey and predator, respectively. Our aim is to study its dynamics. Because these two free boundaries may intersect each other, it seems very difficult to understand the whole dynamics of this model.

(A) Concerning the long time behaviors of solution, we established some realistic and more sophisticated results.

(I) If the prey (predator) species can not spread into $[0,\infty)$, then it will die out in the long run.

(II) When both two species spread successfully. For the weak predation case $b < a$, under the condition $a > b(1+ac)$, we find an important expanding phenomenon: If an observer were to move to the right at a fixed speed less than $\min\{\underline{k}_\beta, \underline{k}_\mu\}$, it will be observed that the two species will stabilize at the unique positive equilibrium state, while, if we observe the two species in front of the curves $x = g(t)$ and $x = h(t)$, we could see nothing. This is different from the Cauchy problem (6.1) because in (6.1) the two species become positive for all x once t is positive.

(B) Main results about the spreading and vanishing show the following important phenomena, these look more realistic and may play an important role in the understanding of ecological complexity.

(I) When one of the initial habitat and moving parameter of the predator is "suitably large", the predator is always able to successfully spread.

(II) If one of the initial habitat and moving parameter of the prey is "suitably large", but both the initial habitat and moving parameter of the predator are "suitably small", the former will spread successfully, while the latter will vanishes eventually.

(III) When prey's initial habitat is much larger than that of predator, and predator spreads slowly, the prey will spread successfully and its territory always cover that of the predator, whether or not the latter spreads successfully.

(IV) In the case of strong predation, if the prey spreads slowly and the predator does quickly, the former will vanish eventually (it will be eaten up by the latter) and the predator will spread successfully.

(C) The conclusions regarding the asymptotic propagations reveal the complicated and realistic spreading phenomena of prey and predator.

(I) When (1.1) is uncoupled ($b = c = 0$), the prey and predator satisfy

$$\begin{cases} w_t - dw_{xx} = w(a - w), & t > 0, \quad 0 < x < \gamma(t), \\ w_x(t,0) = w(t,\gamma(t)) = 0, & t \geq 0, \\ g_0'(t) = -\beta w_x(t,\gamma(t)), & t \geq 0, \\ w(0,x) = u_0(x), & 0 \leq x \leq \gamma = \gamma(0) \end{cases} \tag{7.1}$$

and

$$\begin{cases} z_t - z_{xx} = z(1 - z), & t > 0, \quad 0 < x < \zeta(t), \\ v_x(t,0) = z(t,\zeta(t)) = 0, & t \geq 0, \\ \zeta'(t) = -\mu v_x(t,\zeta(t)), & t \geq 0, \\ z(0,x) = v_0(x), & 0 \leq x \leq h_0 = \zeta(0), \end{cases} \tag{7.2}$$

respectively. By use of (3.4),

$$\lim_{\beta\to\infty}\lim_{t\to\infty}\frac{\gamma(t)}{t} = 2\sqrt{da}, \quad \lim_{\mu\to\infty}\lim_{t\to\infty}\frac{\zeta(t)}{t} = 2.$$

Similarly to the discussion in Section 6 we can show that the asymptotic spreading speed of w is $2\sqrt{da}$ when β is sufficient large, and that of z is 2 when μ is sufficient large.

For the case that $a > b(1 + ac)$ and $da < 1$. The conclusions of Theorems 6.1 and 6.2 show that the asymptotic spreading speed of predator is 2 and that of prey is between $2\sqrt{da - db(1 + ac)}$ and $2\sqrt{da - db}$ when β and μ are sufficiently large. Moreover,

$$\lim_{\mu\to\infty}\lim_{t\to\infty}\frac{h(t)}{t} = 2 \quad \text{for} \quad \beta > \beta_0,$$

$$2\sqrt{da - db(1 + ac)} \leq \liminf_{\beta\to\infty}\liminf_{t\to\infty}\frac{g(t)}{t} \leq \limsup_{\beta\to\infty}\limsup_{t\to\infty}\frac{g(t)}{t} \leq 2\sqrt{da - db}.$$

These illustrate that the prey is not helpful to the predator's asymptotic propagation, while the predator could decrease that of the prey. The reason is that prey's ability to diffuse and grow is weaker than that of the predator.

In the case of $a > b(1+ac)$ and $d[a-b(1+ac)] \geq 1+ac$. The conclusions of Theorems 6.1 and 6.3 indicate that the asymptotic spreading speed of prey is $2\sqrt{da}$ and that of predator is between $2\sqrt{(1+ac)(1-bc)}$ and $2\sqrt{(1+ac)}$ when β and μ are sufficiently large. Moreover,

$$\lim_{\beta\to\infty}\lim_{t\to\infty}\frac{g(t)}{t} = 2\sqrt{da} \quad \text{for} \quad \mu > \mu_0,$$

$$2\sqrt{(1+ac)(1-bc)} \leq \liminf_{\mu\to\infty}\liminf_{t\to\infty}\frac{h(t)}{t} \leq \limsup_{\mu\to\infty}\limsup_{t\to\infty}\frac{h(t)}{t} \leq 2\sqrt{1+ac}.$$

Since $2\sqrt{(1+ac)(1-bc)} > 2$ in this case, we see that the prey accelerates the asymptotic propagation of predator, while the predator has no effect on that of the prey. The reason is that the prey spreads faster and provides predator with more food.

(II) In the case of $a > b(1+ac)$ and $da < 1$. If an observer were to move to the right at a fixed speed less than $2\sqrt{da-db(1+ac)}$, it will be observed that the two species will stabilize at the unique positive equilibrium state; When the observer do this with a fixed speed $k \in (2\sqrt{da-db},\, 2)$, he can only watch the predator; When the observer do this with a fixed speed greater than 2, he can not find anything because the two species have not arrived in his horizon.

(III) For the case $a > b(1+ac)$ and $d[a-b(1+ac)] > 1+ac$. When we are to move to the right at a fixed speed less than $2\sqrt{(1+ac)(1-bc)}$, we shall observe that the two species will stabilize at the unique positive equilibrium state; When we do this with a fixed speed $k \in (2\sqrt{(1+ac)},\, 2\sqrt{da})$, we can only see the prey; When we do this with a fixed speed greater than $2\sqrt{da}$, we could see nothing because the two species are not in our sight.

(D) A great deal of previous mathematical investigation on the spreading of population has been based on the traveling wave fronts of prey-predator system (6.1). A striking difference between our free boundary problem (1.1) and the Cauchy problem (6.1) is that the spreading fronts in (1.1) are given explicitly by two functions $x = g(t)$ and $x = h(t)$, beyond them respectively the population densities of prey and predator are zero, while in (6.1), the two species become positive for all x once t is positive. Secondly, (6.1) guarantees successful spreading of the two species for any nontrivial initial populations $u(0,x)$ and $v(0,x)$, regardless of their initial sizes and supporting area, but the dynamics of (1.1) possesses the multiplicity and complexity of spreading and vanishing. The phenomena exhibited by these multiplicities and complexities seem closer to the reality.

Appendix. Proof of (2.5)

Set $W = w_1 - w_2$, $G = g_1 - g_2$, then W, G satisfy (2.9), (2.10) and the estimate (2.11) holds. Using (2.10) and (2.11) we have

$$\begin{aligned}
\|G'\|_{C^{\frac{\alpha}{2}}([0,T])} &\leq \beta\|g_1^{-1}W_y\|_{C^{\frac{\alpha}{2},0}(\Delta_T^1)} + \beta\|(g_1^{-1}-g_2^{-1})w_{2y}\|_{C^{\frac{\alpha}{2},0}(\Delta_T^1)} \\
&\leq C_{11}\big(\|W\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T^1)} + \|G\|_{C^1([0,T])}\big) \\
&\leq C_{12}\big(\|z_1 - z_2\|_{C(\Delta_T^1)} + \|G\|_{C^1([0,T])}\big).
\end{aligned} \tag{A.1}$$

Recall $W(0,y) = 0$, $G(0) = G'(0) = 0$. Take advantage of the mean value theorem and (2.11), (A.1), it follows that

$$\|W\|_{C(\Delta_T^1)} \leq T^{\frac{\alpha}{2}}\|W\|_{C^{\frac{\alpha}{2},0}(\Delta_T^1)} \leq C_4 T^{\frac{\alpha}{2}}\big(\|z_1 - z_2\|_{C(\Delta_T^1)} + \|G\|_{C^1([0,T])}\big),$$

$$\|G\|_{C^1([0,T])} \leq 2T^{\frac{\alpha}{2}}\|G'\|_{C^{\frac{\alpha}{2}}([0,T])} \leq 2C_{12}T^{\frac{\alpha}{2}}\big(\|z_1 - z_2\|_{C(\Delta_T^1)} + \|G\|_{C^1([0,T])}\big).$$

Thus we have

$$\|W\|_{C(\Delta_T^1)} + \|G\|_{C^1([0,T])} \le (C_4 + 2C_{12})T^{\frac{\alpha}{2}} \big(\|z_1 - z_2\|_{C(\Delta_T^1)} + \|G\|_{C^1([0,T])}\big).$$

If we choose $T > 0$ such that $(C_4 + 2C_{12})T^{\frac{\alpha}{2}} \le 1/2$, then

$$\|W\|_{C(\Delta_T^1)} + \|G\|_{C^1([0,T])} \le \|z_1 - z_2\|_{C(\Delta_T^1)},$$

which is exactly (2.5).

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