# BLOW-UP PROFILE OF GROUND STATES FOR THE CRITICAL NEUTRON STARS WITH SINGULAR POTENTIALS

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ABSTRACT. We study the Chandrasekhar variational model for the neutron stars, with or without an external potential. We prove the existence of minimizers when the attractive interaction strength $\tau$ is strictly smaller than a critical value $\tau_c$, and investigate the blow-up phenomenon in the limit $\tau \nearrow \tau_c$.

## 1. Introduction

It is a fundamental fact that neutron and white dwarf stars *collapse* when their masses is bigger than a critical number. The maximum mass of stable stars, called the *Chandrasekhar limit*, was discovered by Chandrasekhar in 1930, which gained him the 1983 Nobel Prize for Physics. In this paper, we will study the details of the collapse phenomenon in a concrete, semiclassical model.

We consider the Thomas-Fermi ground state energy of a *neutron star*

$$e_\tau(1) := \inf\left\{\mathcal{E}_\tau(\rho) : \rho \geq 0, \rho \in L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3), \int_{\mathbb{R}^3} \rho(x)\mathrm{d}x = 1\right\} \tag{1.1}$$

with the energy functional

$$\mathcal{E}_\tau(\rho) := \int_{\mathbb{R}^3} j_m(\rho(x))\mathrm{d}x - \tau D(\rho,\rho) + \int_{\mathbb{R}^3} V(x)\rho(x)\mathrm{d}x. \tag{1.2}$$

Here $m > 0$ is a fixed parameter which refers to the mass of one particle and $j_m(\rho)$ is the semiclassical approximation for the relativistic kinetic energy of $q$ spin states neutrons at density $\rho$, namely

$$\begin{aligned} j_m(\rho) &= \frac{q}{(2\pi)^3}\int_{|p|<(6\pi^2\rho/q)^{1/3}} \sqrt{|p|^2+m^2}\mathrm{d}p \\ &= \frac{q}{16\pi^2}\left[\eta(2\eta^2+m^2)\sqrt{\eta^2+m^2} - m^4\ln\left(\frac{\eta+\sqrt{\eta^2+m^2}}{m}\right)\right], \end{aligned}$$

with $\eta = (6\pi^2\rho/q)^{1/3}$ ($q$ is fixed). The direct term $D(\rho,\rho)$ models the classical gravitational energy

$$D(\rho,\rho) = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\mathrm{d}x\mathrm{d}y.$$

Moreover, $V$ is a real value function, which stands for an external potential. The case $V = 0$ is allowed. In this case, the energy functional $\mathcal{E}_\tau^\infty(\rho)$ is translation-invariant, the corresponding gorund state energy is denoted by $e_\tau^\infty(1)$.

The rigorous derivation of the energy functional $\mathcal{E}_\tau(\rho)$ from many-body quantum theory has been done by Lieb and Yau in 1987 [10] (see also [9, 3] for another, new proof). Here one may interpret the strength of the attractive interaction as

$\tau = gN^{2/3}$ with $g$ the gravitational constant and $N$ the number of particles. This coupling constant $\tau > 0$ will play an essential role in our analysis. The collapse of big neutron stars boils down to the fact that $e_\tau(1) = -\infty$ if $\tau$ is bigger than a critical value $\tau_c$ (which corresponds to the Chandrasekhar limit).

From the simple inequality

$$|p| \leq \sqrt{|p|^2 + m^2} \leq |p| + m \tag{1.3}$$

and a standard scaling argument, we can see that $\tau_c$ is independent of $m$. In fact,

$$\tau_c = \frac{\gamma}{\sigma_f}$$

where $\gamma = \frac{3}{4}(6\pi^2/q)^{1/3}$ and $\sigma_f \approx 1.092$ is the optimal constant in the inequality

$$\sigma_f \|\rho\|_{L^1}^{2/3} \|\rho\|_{L^{4/3}}^{4/3} \geq D(\rho, \rho), \quad \forall \rho \in L^{4/3}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3). \tag{1.4}$$

It is well-known (see [11, Appendix A]) that (1.4) has an optimizing $Q$, it is unique up to scaling, dilation and translation. Such $Q$ can be chosen to be non-negative symmetric decreasing and satisfies

$$\sigma_f \int_{\mathbb{R}^3} Q(x)^{4/3} \mathrm{d}x = D(Q, Q) = \int_{\mathbb{R}^3} Q(x) \mathrm{d}x = 1. \tag{1.5}$$

Moreover, $Q$ solves the Lane-Emden equation

$$\frac{4}{3}\sigma_f Q(x)^{1/3} - (|\cdot|^{-1} \star Q)(x) + \frac{2}{3} \begin{cases} = 0 & \text{if } Q(x) > 0, \\ \geq 0 & \text{if } Q(x) = 0. \end{cases} \tag{1.6}$$

In the present paper, we will analyze the existence and the blow-up behavior of the minimizers for the variational problem $e_\tau(1)$ in (1.1) when $\tau \nearrow \tau_c$.

Our first result is

**Theorem 1** (Existence and non-existence of minimizers)**.** *Assume that $V$ satisfies*

*($V_1$) $0 \geq V \in L^4(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, and*
*($V_2$) $V$ vanishes at infinity, i.e. $|\{x : |V(x)| > a\}| < \infty$ for all $a > 0$.*

*Then the variational problem $e_\tau(1)$ in (1.1) has the following properties.*

*(i) If $\tau > \tau_c$ then $e_\tau(1) = -\infty$.*
*(ii) If $\tau = \tau_c$ then $e_\tau(1) = \inf_{x\in\mathbb{R}^3} V(x)$ but it has no minimizer.*
*(iii) If $0 < \tau < \tau_c$, then $e_\tau(1)$ has at least one minimizer.*

Here we focus on the case when $V$ is attractive and vanishes at infinity. We assume $V \in L^4(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ to ensure that the term $\int V\rho$ is meaningful when $\rho \in L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$. The value $\inf_{x\in\mathbb{R}^3} V(x)$ in (ii) should be interpreted properly as the essential infimum when $V$ is a general, measurable function.

In case $V \equiv 0$, the result is well-known (see [1, 10]). Furthermore, in this case, if $0 < \tau < \tau_c$ the minimizer is unique up to translations, and can be chosen to be radially symmetric decreasing by the rearrangement inequalities (see [10, Theorem 5]). For $V \not\equiv 0$, the existence result in Theorem 1 is non-trivial, and we have to use the concentration-compactness argument [12] to deal with the lack of compactness of the minimizing sequences.

Our next result concern the behavior of the minimizer $\rho$ of $e_\tau(1)$ as $\tau \nearrow \tau_c$. Here we consider a typical external potential of the form

$$V(x) = -\sum_{i=1}^{M} \frac{z_i}{|x - x_i|^{s_i}} \tag{1.7}$$

where $z_i > 0$, $x_i \in \mathbb{R}^3$, $x_i \neq x_j$ for $1 \leq i \neq j \leq M$, and $0 < s_i < \frac{3}{4}$ for all $1 \leq i \leq M$. Let $s = \max_{1 \leq i \leq M} s_i$, $z = \max_{1 \leq i \leq M}\{z_i : s_i = s\}$ and let

$$\mathcal{Z} = \{x_i : s_i = s\}$$

denote the locations of the most singular points of $V(x)$. We will show that $\rho$ blows up as $\tau \nearrow \tau_c$ and its blow-up profile is given by the optimizer $Q$ of (1.4).

**Theorem 2** (Blow-up profile of minimizers)**.** *Let $\rho_\tau$ be a non-negative minimizer of $e_\tau(1)$ in (1.1) for $0 < \tau < \tau_c$. Then for every sequence $\{\tau_n\}$ with $\tau_n \nearrow \tau_c$ as $n \to \infty$, there exists a subsequence of $\{\tau_n\}$ (still denoted by $\{\tau_n\}$ for simplicity) such that the following hold true.*

- *If $V$ is defined in (1.7), then there exists an $x_j \in \mathcal{Z}$, $1 \leq j \leq M$, such that*
$$\lim_{n\to\infty} (\tau_c - \tau_n)^{\frac{3}{1-s}} \rho_{\tau_n}\big((\tau_c - \tau_n)^{\frac{1}{1-s}} x + x_j\big) = \lambda^3 Q(\lambda x)$$
*strongly in $L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$ where*
$$\lambda = \left( sz \int_{\mathbb{R}^3} \frac{Q(x)}{|x|^s} \mathrm{d}x \right)^{\frac{1}{1-s}}.$$
- *If $V \equiv 0$, then there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ such that*
$$\lim_{n\to\infty} (\tau_c - \tau_n)^{3/2} \rho_{\tau_n}\big(x(\tau_c - \tau_n)^{1/2} + y_n\big) = \lambda^3 Q(\lambda x)$$
*strongly in $L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$ where*
$$\lambda = \frac{3}{4} m \sqrt{\frac{1}{\gamma} \int_{\mathbb{R}^3} Q(x)^{2/3} \mathrm{d}x}.$$

*Here $Q$ is the unique non-negative radial function which satisfies (1.5) and (1.6).*

Note that when $V$ is defined by (1.7) or $V \equiv 0$, then the existence of minimizer $\rho_\tau$ for all for $0 < \tau < \tau_c$ has been proved in Theorem 1. Our proof is based on a detailed analysis on the Euler-Lagrange equation associated to the minimizer of $e_\tau(1)$ when $\tau$ tends to $\tau_c$. As a by-product of our proof, we also obtain the asymptotic behavior of the energy

$$\lim_{\tau \nearrow \tau_c} \frac{e_\tau(1)}{(\tau_c - \tau)^{\frac{s}{s-1}}} = \left(1 - \frac{1}{s}\right) \left( sz \int_{\mathbb{R}^3} \frac{Q(x)}{|x|^s} \mathrm{d}x \right)^{\frac{1}{1-s}} \tag{1.8}$$

if $V$ is defined in (1.7), and

$$\lim_{\tau \nearrow \tau_c} \frac{e_\tau^\infty(1)}{(\tau_c - \tau)^{\frac{1}{2}}} = \frac{3}{2} m \sqrt{\frac{1}{\gamma} \int_{\mathbb{R}^3} Q(x)^{2/3} \mathrm{d}x}. \tag{1.9}$$

if $V \equiv 0$.

Our work is inspired by the recent studies in [5, 13, 15] on the mass concentration of pseudo-relativistic boson stars described by the Hartree equation, and the works in [4, 6, 2, 14] on the concentration of the Bose-Einstein condensate described by the 2D focusing Gross-Pitaevskii equation. Here our model is a semiclassical model and the Lane-Emden equation (1.6) is different from the Hartree-type equations in the mentioned papers. This requires new ideas to prove both non-existence and blow-up results. We hope that our study can be served as a first step toward to the understanding of the blow-up phenomenon of neutron stars from a rigorous mathematical approach.

**Organization of the paper.** In Section 2 we prove the Theorem 1 which gives the existence and non-existence of minimizers of $e_\tau(1)$ in the case $V \not\equiv 0$. In Section 3, we prove the Theorem 2 which give the blow-up profiles of minimizers of $e_\tau(1)$.

## 2. Existence of Minimizers

In this section, we prove the existence and non-existence of minimizers of $e_\tau(1)$ as stated in Theorem 1. The exsistence and non-exsitence of minimizer of $e_\tau^\infty(1)$ when $V = 0$ is well-known result (see e.g, [1, 10]). Here we consider the case $V \not\equiv 0$ which satisfies conditions $(V_1)-(V_2)$.

For $0 < \tau < \tau_c$, let $\{\rho_n\}$ be a minimizing sequence for $e_\tau(1)$, i.e.

$$\lim_{k\to\infty} \mathcal{E}_\tau(\rho_n) = e_\tau(1), \text{ with } \{\rho_n\} \in L^1(\mathbb{R}^3)\cap L^{4/3}(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} \rho_n(x)\mathrm{d}x = 1 \text{ for all } k.$$

Firstly, we see that all the terms of functional $\mathcal{E}_\tau$ in (1.2) are finites. Indeed, the conditions $(V_1)-(V_2)$ imply that the potential term $\int_{\mathbb{R}^3} V(x)\rho_n(x)\mathrm{d}x$ is well-defined. On the other hand, since $\sqrt{|p|^2+m^2} \le |p| + m$, we have $j_m(\rho_n) \le \gamma\rho_n^{4/3} + m\rho_n$, which show that kinetic-energy is well-defined. Moreover, since $\rho_n \in L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$ we have $\rho_n \in L^r(\mathbb{R}^3)$ for any $1 \le r \le \frac{4}{3}$, then it follows from Hardy-Littlewood-Sobolev inequality that the direct term $D(\rho_n, \rho_n)$ is well-defined.

We have already seen that the energy $\mathcal{E}_\tau$ is well-defined on $L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$. Thus $e_\tau(1) > -\infty$ for $0 < \tau < \tau_c$. Next, we collect some basic facts.

**Lemma 3** (Binding inequality)**.** *Assume that $V \not\equiv 0$ satisfies conditions $(V_1)-(V_2)$, then for any $0 < \alpha < 1$ we have*

$$e_\tau(1) \le e_\tau(\alpha) + e_\tau^\infty(1-\alpha). \tag{2.1}$$

*Proof.* For a contradiction we assume that there exists a $\delta > 0$ such that

$$e_\tau(1) > e_\tau(\alpha) + e_\tau^\infty(1-\alpha) + \delta$$

for some $0 < \alpha < 1$. Then there exists a ground state $\rho_0$ (resp. $\rho_\infty$) with $\int_{\mathbb{R}^3} \rho_0(x)\mathrm{d}x = \alpha$ (resp. $\int_{\mathbb{R}^3} \rho_\infty(x)\mathrm{d}x = 1-\alpha$) such that $e_\tau(\alpha) > \mathcal{E}_\tau(\rho_0) - \delta/3$ (resp. $e_\tau^\infty(1-\alpha) > \mathcal{E}_\tau^\infty(\rho_\infty) - \delta/3$). By a density argument, we can assume that $\rho_0(x)$ and $\rho_\infty(x)$ are compactly supported. Denote by $R > 0$ the radius of a ball in $\mathbb{R}^6$ which contains the supports of $\rho_0(x)$ and $\rho_\infty(x)$. We define a translated operator by

$$\widetilde{\rho}_\infty(x) := \rho_\infty(x - 3R).$$

Moreover, we define a trial density operator $\rho_\alpha := \rho_0 + \widetilde{\rho}_\infty$. By construction, we have $\int_{\mathbb{R}^3} \rho_\alpha(x)\mathrm{d}x = 1$, and $\rho_0\tilde{\rho}_\infty = 0$. Hence

$$j_m(\rho_\alpha) = j_m(\rho_0) + j_m(\tilde{\rho}_\infty) = j_m(\rho_0) + j_m(\rho_\infty).$$

This implies that

$$\frac{\delta}{3} + \mathcal{E}_\tau(\rho_0) + \mathcal{E}_\tau^\infty(\rho_\infty) < e_\tau(1) \le \mathcal{E}_\tau(\rho_\alpha) \le \mathcal{E}_\tau(\rho_0) + \mathcal{E}_\tau^\infty(\rho_\infty),$$

where we have used the translation invariance of $\mathcal{E}_\tau^\infty$.

This is actually a contradiction. Hence we must have (2.1). $\square$

**Lemma 4** (Coercivity of $\mathcal{E}_\tau$)**.** *Assume that $V \not\equiv 0$ satisfies conditions $(V_1) - (V_2)$, then for $0 < \tau < \tau_c$, the energy $\mathcal{E}_\tau$ is coercive on $L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$, i.e. we have*

$$\mathcal{E}_\tau(\rho) \to \infty \ \ as \ \int_{\mathbb{R}^3} \rho(x)^{4/3} \mathrm{d}x \to \infty.$$

*In particular, all minimizing sequences for $\mathcal{E}_\tau$ on $L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$ are bounded.*

*Proof.* For any $\rho \in L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$ and $1 > \epsilon > 0$ we have

$$\mathcal{E}_\tau(\rho) \geq \epsilon \int_{\mathbb{R}^3} \rho(x)^{4/3} \mathrm{d}x + (1-\epsilon) e_{\frac{\tau}{1-\epsilon}}(1).$$

Since $0 < \tau < \tau_c$ we can chose $\epsilon$ small such that $\frac{\tau}{1-\epsilon} < \tau_c$, and hence $e_{\frac{\tau}{1-\epsilon}}(1) > -\infty$. This implies that $\mathcal{E}_\tau(\rho) \to \infty$ as $\int_{\mathbb{R}^3} \rho(x)^{4/3} \mathrm{d}x \to \infty$. $\square$

We deduce from Lemma 4 that the minimizing sequence $\rho_n$ is bounded in $L^{4/3}(\mathbb{R}^3)$, and hence there exists a subsequence of $\rho_n$, still denote by $\rho_n$, such that $\rho_n \rightharpoonup \rho_0$ weakly in $L^{4/3}(\mathbb{R}^3)$.

We now apply the the following adaptation concentration-compactness lemma.

**Lemma 5.** *Let $\{\rho_n\}_{n \geq 1}$ be a sequence in $L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$ satisfying $\rho_n \geq 0$, $\int_{\mathbb{R}^3} \rho_n(x) \mathrm{d}x = 1$ and $\rho_n \rightharpoonup \rho_0$ weakly in $L^{4/3}(\mathbb{R}^3)$. Then there exist a subsequence $\{\rho_{n_k}\}_{k \geq 1}$ satisfying one of the three following possibilities*

*(i) (compactness) $\rho_{n_k}$ is tight, i.e. for all $\epsilon > 0$, there exists $R < \infty$ such that*

$$\int_{B(0,R)} \rho_{n_k}(x) \mathrm{d}x \geq 1 - \epsilon.$$

*(ii) (vanishing)* $\lim_{k\to\infty} \int_{B(0,R)} \rho_{n_k}(x) \mathrm{d}x = 0$ *for all* $R < \infty$.

*(iii) (dichotomy) there exist $\alpha = \int_{\mathbb{R}^3} \rho_0(x) \mathrm{d}x \in (0,1)$ and sequence $\{R_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+$ with $R_k \to \infty$ such that for $\rho_k^{(1)} = \rho_{n_k} \mathbf{1}_{B(0,R_k)}$, $\rho_k^{(2)} = \rho_{n_k} \mathbf{1}_{\mathbb{R}^3 \setminus B(0,2R_k)}$, we have*

$$\lim_{k\to\infty} \int_{\mathbb{R}^3} \rho_k^{(1)} = \alpha, \quad \lim_{k\to\infty} \int_{\mathbb{R}^3} \rho_k^{(2)} = 1 - \alpha,$$

$$\lim_{k\to\infty} \int_{R_k \leq |x| \leq 2R_k} \rho_{n_k}(x) \mathrm{d}x = 0.$$

We will not detail the proof which is an adaptation of ideas by Lions [12], where one introduces another sequence of concentration functions

$$Q_n(t) = \int_{B(0,t)} \rho_n(x) \mathrm{d}x,$$

then there exist a subsequence $(n_k)_{k \geq 1}$ and a nondecreasing nonnegative function $Q$ such that $Q_{n_k}(t) \to Q(t)$ as $k \to \infty$ for all $t > 0$. The $\alpha$ in iii) is defined

$$\alpha = \lim_{t\to\infty} Q(t) = \int_{\mathbb{R}^3} \rho_0(x) \mathrm{d}x \in [0,1]$$

since $\rho_n \rightharpoonup \rho_0$ weakly in $L^{4/3}(\mathbb{R}^3)$.

Invoking Lemma 5, we conclude that a suitable subsequence $\{\rho_{n_k}\}$ satisfies either i), ii), or iii). We rule out ii) and iii) as follows.

Vanishing does not occur. Suppose that $\{\rho_{n_k}\}$ exhibits property ii), we deduce from ii) and the weak limit $\rho_{n_k} \rightharpoonup \rho_0$ in $L^{4/3}(\mathbb{R}^3)$ that $\int_{B(0,R)} \rho_0(x)\mathrm{d}x = 0$ for all $R < \infty$. This implies that $\rho_0 = 0$ almost everywhere in $\mathbb{R}^3$. Then we infer from the weak limit $\rho_{n_k} \rightharpoonup 0$ in $L^{4/3}(\mathbb{R}^3)$ and the conditions $(V_1) - (V_2)$ that we must have

$$\int_{\mathbb{R}^3} V(x)\rho_{n_k}(x)\mathrm{d}x \to 0$$

as $n \to \infty$. Thus, we obtain that

$$e_\tau(1) = \lim_{k\to\infty} \mathcal{E}_\tau(\rho_{n_k}) \geq e_\tau^\infty(1). \tag{2.2}$$

It is well-known result that $e_\tau^\infty(1)$ possess unique minimizer, up to translation. Since $V \not\equiv 0$ we can chose a minimizer $\rho_\infty$ of $e_\tau^\infty(1)$ such that $\int_{\mathbb{R}^3} V(x)\rho_\infty(x)\mathrm{d}x < 0$. Then we have

$$e_\tau^\infty(1) = \mathcal{E}_\tau^\infty(\rho_\infty) \geq e_\tau(1) - \int_{\mathbb{R}^3} V(x)\rho_\infty(x)\mathrm{d}x > e_\tau(1),$$

which contradicts (2.2). Hence ii) cannot occur.

Dichotomy does not occur. Let us suppose that iii) is true for $\{\rho_{n_k}\}$. Then there exists $\alpha \in (0,1)$ and sequence $\{R_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ with $R_k \to \infty$ such that for $\rho_k^{(1)} = \rho_{n_k}\mathbf{1}_{B(0,R_k)}$, $\rho_k^{(2)} = \rho_{n_k}\mathbf{1}_{\mathbb{R}^3\setminus B(0,2R_k)}$

$$\lim_{k\to\infty} \int_{\mathbb{R}^3} \rho_k^{(1)} = \alpha, \quad \lim_{k\to\infty} \int_{\mathbb{R}^3} \rho_k^{(2)} = 1-\alpha. \tag{2.3}$$

$$\lim_{k\to\infty} \int_{R_k \leq |x| \leq 2R_k} \rho_{n_k}(x)\mathrm{d}x = 0.$$

By the same arguments in [12, p.122] we can prove that

$$D(\rho_{n_k}, \rho_{n_k}) \leq D(\rho_k^{(1)}, \rho_k^{(1)}) + D(\rho_k^{(2)}, \rho_k^{(2)}) + o(1).$$

On the other hand, since $V$ satisfies $(V_1) - (V_2)$ we have

$$\int_{\mathbb{R}^3} V(x)\rho_{n_k}(x)\mathrm{d}x \geq \int_{\mathbb{R}^3} V(x)\rho_k^{(1)}(x)\mathrm{d}x - o(1).$$

In addition, we have

$$\int_{\mathbb{R}^3} j_m(\rho_{n_k})\mathrm{d}x \geq \int_{\mathbb{R}^3} j_m(\rho_k^{(1)})\mathrm{d}x + \int_{\mathbb{R}^3} j_m(\rho_k^{(2)})\mathrm{d}x$$

Combinning these inequalities we find

$$\mathcal{E}_\tau(\rho_{n_k}) \geq \mathcal{E}_\tau(\rho_k^{(1)}) + \mathcal{E}_\tau^\infty(\rho_k^{(2)}) - o(1). \tag{2.4}$$

Since $\{\rho_k^{(1)}\}$ and $\{\rho_k^{(2)}\}$ satisfy (2.3), we infer from (2.4) that

$$e_\tau(1) = \lim_{k\to\infty} \mathcal{E}_\tau(\rho_{n_k}) \geq e_\tau(\alpha) + e_\tau^\infty(1-\alpha)$$

using the continuity of $e_\tau(\lambda)$ and $e_\tau^\infty(\lambda)$ in $0 < \lambda < 1$.

The above inequality, together with (2.1), implies that

$$e_\tau(1) = e_\tau(\alpha) + e_\tau^\infty(1-\alpha). \tag{2.5}$$

Moreover, $\rho_k^{(1)}$ and $\rho_k^{(2)}$ are minimizing sequences for $e_\tau(\alpha)$ and $e_\tau^\infty(1-\alpha)$ respectively. Note that, it follows from a simple scaling $\rho(x) \mapsto \rho((1-\alpha)^{-1/3}x)$ that $e_\tau^\infty(1-\alpha) = e_{(1-\alpha)^{2/3}\tau}^\infty(1)$. Since $(1-\alpha)^{2/3}\tau < \tau < \tau_c$, we deduce from [10, Theorem 5] that $e_\tau^\infty(1-\alpha)$ has a unique minimizer, up to translations. On the other hand, it follows from the weakly convergence $\rho_{n_k} \rightharpoonup \rho_0$ in $L^{4/3}(\mathbb{R}^3)$ that $\rho_k^1 \to \rho_0$ in $L^1(\mathbb{R}^3)$ as $k \to \infty$. In fact, $\rho_k^{(1)} \to \rho_0$ strongly in $L^r(\mathbb{R}^3)$ for $1 \le r < 4/3$ because of $L^{4/3}(\mathbb{R}^3)$ boundedness. By the Hardy-Littlewood-Sobolev inequality we have

$$\lim_{k\to\infty} D(\rho_k^{(1)}, \rho_k^{(1)}) = D(\rho_0, \rho_0).$$

We infer from the convergence $\rho_k^1 \to \rho_0$ in $L^r(\mathbb{R}^3)$ for $1 \le r < 4/3$, and the conditions $(V_1) - (V_2)$ that

$$\lim_{k\to\infty} \int_{\mathbb{R}^3} V(x)\rho_k^{(1)}(x)\mathrm{d}x = \int_{\mathbb{R}^3} V(x)\rho_0(x)\mathrm{d}x.$$

In addition that the convex functional $\int_{\mathbb{R}^3} j_m(\rho(x))\mathrm{d}x$ being strongly lower semicontinuous on $L^{4/3}$ by Fatou's lemma is weakly lower semicontinuous and we have

$$\liminf_{k\to\infty} \int_{\mathbb{R}^3} j_m(\rho_k^{(1)}(x))\mathrm{d}x \ge \int_{\mathbb{R}^3} j_m(\rho_0(x))\mathrm{d}x.$$

Hence, we conclude

$$e_\tau(\alpha) = \lim_{k\to\infty} \mathcal{E}_\tau(\rho_k^{(1)}) \ge \mathcal{E}_\tau(\rho_0) \ge e_\tau(\alpha).$$

This implies that $\rho_0 > 0$ is a minimizer of $e_\tau(\alpha)$, and it satisfies the following variational equation

$$\sqrt{\eta_0(x)^2 + m^2} - \tau(|\cdot|^{-1} \star \rho_0)(x) + V(x) - \mu = 0 \tag{2.6}$$

where $\eta_0 = (6\pi^2\rho_0/q)^{1/3}$ and $\mu$ is a real-valued Lagrange multiplier.

We first note that (2.6) imply that $\rho_0$ is compactly supported. If not, by letting $|x| \to \infty$ one has that $\mu \ge m$, since $(|\cdot|^{-1}\star\rho_0)(x)$ and $V(x)$ tend to zero in (2.6). We then would have by (2.6), $\rho_0(x) \ge C(|\cdot|^{-1} \star \rho_0)^3(x)$, where constant $C$ is positive. For sufficiently large $|x|$, we see that $\rho_0(x) \ge C|x|^{-3}$. This implies that $\rho_0$ is not integrable, contradicting the fact that $\int_{\mathbb{R}^3} \rho_0(x)\mathrm{d}x = \alpha$.

By the same argument we can prove that the minimizer of $e_\tau^\infty(1-\alpha)$ is compactly supported (see also [11, Appendix A]).

**Lemma 6** (Strict binding inequality)**.** *Assume that $V$ satisfies $(V_1) - (V_2)$, then for $0 < \alpha < 1$ as above, we have*

$$e_\tau(1) < e_\tau(\alpha) + e_\tau^\infty(1-\alpha),$$

*Proof.* We assume that $e_\tau^\infty(1-\alpha)$ possess a minimizer $\rho_\infty$. Denote by $R > 0$ the radius of a ball in $\mathbb{R}^6$ which contains the supports of $\rho_0(x)$ and $\rho_\infty(x)$. We define a translated operator by

$$\tilde{\rho}_\infty(x) := \rho_\infty(x - 3R).$$

Moreover, we define a trial density operator $\rho_\alpha := \rho_0 + \tilde{\rho}_\infty$. By construction, we have $\int_{\mathbb{R}^3} \rho_\alpha(x)\mathrm{d}x = 1$, $\rho_0\tilde{\rho}_\infty = 0$. Hence

$$j_m(\rho_\alpha) = j_m(\rho_0) + j_m(\tilde{\rho}_\infty) = j_m(\rho_0) + j_m(\rho_\infty),$$

and

$$\int_{\mathbb{R}^3} V(x)\rho_\alpha(x)\mathrm{d}x \le \int_{\mathbb{R}^3} V(x)\rho_0(x)\mathrm{d}x.$$

On the other hand, we notice that when $|x - y| > 5R$ we have $\rho_0(x)\tilde{\rho}_\infty(y) = 0$. Thus

$$-D(\rho_\alpha, \rho_\alpha) + D(\rho_0, \rho_0) + D(\tilde{\rho}_\infty, \tilde{\rho}_\infty) = -2D(\rho_0, \tilde{\rho}_\infty) \le -\frac{\alpha(1-\alpha)}{5R} < 0.$$

We conclude that

$$e_\tau(1) \le \mathcal{E}_\tau(\rho_\alpha) < \mathcal{E}_\tau(\rho_0) + \mathcal{E}_\tau^\infty(\rho_\infty) = e_\tau(\alpha) + e_\tau^\infty(1-\alpha),$$

where we have used the translation invariance of $\mathcal{E}_\tau^\infty$. □

The Lemma 6 and (2.5) give us a contradiction. Therefore iii) of Lemma 5 is ruled out. We conclude that there exists a subsequence $\{\rho_{n_k}\}$ such that i) of Lemma 5 is true. Then we have

$$1 \ge \int_{\mathbb{R}^3} \rho_0(x)\mathrm{d}x \ge \int_{|x|\le R} \rho_0(x)\mathrm{d}x = \lim_{k\to\infty} \int_{|x|\le R} \rho_{n_k}(x)\mathrm{d}x \ge 1-\epsilon,$$

for every $\epsilon > 0$ and suitable $R = R(\epsilon) < \infty$. This implies that $\int_{\mathbb{R}^3} \rho_0(x)\mathrm{d}x = 1$.

Now we prove that $\rho_0$ is indeed a minimizer of $e_\tau(1)$. We first deduce from the norm preservation that

$$\lim_{n\to\infty} D(\rho_n,\rho_n) = D(\rho_0,\rho_0),$$

by the same arguments in [12, p.125]. On the other hand, it follows from the weakly convergence $\rho_n \rightharpoonup \rho_0$ in $L^{4/3}(\mathbb{R}^3)$ and the conditions $(V_1)-(V_2)$ that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} V(x)\rho_n(x)\mathrm{d}x = \int_{\mathbb{R}^3} V(x)\rho_0(x)\mathrm{d}x.$$

In addition the convex functional $\int_{\mathbb{R}^3} j_m(\rho(x))\mathrm{d}x$ being strongly lower semicontinuous on $L^{4/3}$ by Fatou's lemma is weakly lower semicontinuous and we have

$$\liminf_{k\to\infty} \int_{\mathbb{R}^3} j_m(\rho_n(x))\mathrm{d}x \ge \int_{\mathbb{R}^3} j_m(\rho_0(x))\mathrm{d}x.$$

We conclude that

$$e_\tau(1) = \lim_{n\to\infty} \mathcal{E}_\tau(\rho_n) \ge \mathcal{E}_\tau(\rho_0) \ge e_\tau(1),$$

which implies that $\rho_0$ is a minimizer of $e_\tau(1)$.

To prove that there is no minimizer for (1.1) as soon as $\tau \ge \tau_c$ and $V \not\equiv 0$ satisfies conditions $(V_1)-(V_2)$, we proceed as follow. Let $Q$ be a minimizer in (1.4). Since $\sqrt{|p|^2+m^2} \le |p| + m^2/2|p|$, we find that $j_m(\rho) \le \gamma\rho^{4/3} + \frac{9}{16}m^2\gamma^{-1}\rho^{2/3}$. Using this we have, for $\ell > 0$ and $x_0 \in \mathbb{R}^3$,

$$\begin{aligned}\mathcal{E}_\tau(\ell^3 Q(\ell(x-x_0)) \le \left(1-\frac{\tau}{\tau_c}\right)\ell\gamma \int_{\mathbb{R}^3} Q(x)^{4/3}\mathrm{d}x + \frac{9m^2}{16\ell\gamma}\int_{\mathbb{R}^3} Q(x)^{2/3}\mathrm{d}x \\ + \int_{\mathbb{R}^3} V(\ell^{-1}x + x_0)Q(x)\mathrm{d}x. \qquad (2.7)\end{aligned}$$

Since the function $x \mapsto Q(x)$ has compact support (see e.g, [11, Appendix A]), the convergence

$$\lim_{\ell\to\infty} \int_{\mathbb{R}^3} V(\ell^{-1}x+x_0)Q(x)\mathrm{d}x = V(x_0)$$

holds for almost every $x_0 \in \mathbb{R}^3$ (see e.g, [8]). Hence, it follows from (2.7) that, for $\tau = \tau_c$ and $V$ satisfies $(V_1)-(V_2)$,

$$e_\tau(1) \le \lim_{\ell\to\infty} \mathcal{E}_\tau(\ell^3 Q(\ell x)) = \inf_{x\in\mathbb{R}^3} V(x).$$

We argue that there does not exist a minimizer for $e_\tau(1)$ with $\tau=\tau_c$ by contradiction as follows. We suppose that $\rho\in L^1(\mathbb{R})\cap L^{4/3}(\mathbb{R})$ is a minimizer for $e_\tau(1)$ with $\tau=\tau_c$. It follows from the strict inequality $\sqrt{|p|^2+m^2}>|p|$ that

$$\inf_{x\in\mathbb{R}^3} V(x)\geq \mathcal{E}_\tau(u)>\mathcal{E}_\tau(u)|_{m>0}\geq \inf_{x\in\mathbb{R}^3} V(x).$$

which is a contradiction. Hence no minimizer exists for $e_\tau(1)$ if $\tau=\tau_c$.

For $\tau>\tau_c$, it follows easily from (2.7) that

$$e_\tau(1)\leq \lim_{\ell\to\infty}\mathcal{E}_\tau(\ell^3 Q(\ell x))=-\infty$$

This implies that $e_\tau(1)$ is unbounded from below for any $\tau>\tau_c$, and hence the non-existence of minimizers of $e_\tau(1)$ is therefore proved.

## 3. Blow-up Behavior

In this section, we prove the blow-up profile of minimizers of $e_\tau(1)$ when $\tau_n\nearrow\tau_c$, as stated in Theorem 2.

We suppose that $V$ satisfies (1.7), and we define $0<s=\max_{1\leq i\leq M}s_i<\frac{3}{4}$. Let $\tau_n\nearrow\tau_c$ as $n\to\infty$ and let $\rho_n:=\rho_{\tau_n}$ be a non-negative minimizer for $e_{\tau_n}(1)$.

We start with two preliminary lemmas.

**Lemma 7.** *There exist positive constants $M_1>M_2$ independent of $\tau_n$ such that, for $n$ sufficiently large,*

$$-M_1(\tau_c-\tau_n)^{\frac{s}{s-1}}\leq e_{\tau_n}(1)\leq -M_2(\tau_c-\tau_n)^{\frac{s}{s-1}} \tag{3.1}$$

*if $V$ is defined in* (1.7)*, and*

$$M_1(\tau_c-\tau_n)^{\frac{1}{2}}\leq e^\infty_{\tau_n}(1)\leq M_2(\tau_c-\tau_n)^{\frac{1}{2}} \tag{3.2}$$

*if $V=0$.*

*Proof.* We start with the proof of the upper bound in (3.1). If $V$ is defined in (1.7), it follows from (2.7) that, for $\ell>0$,

$$e_{\tau_n}(1)\leq\left(1-\frac{\tau}{\tau_c}\right)\ell\gamma\int_{\mathbb{R}^3}Q(x)^{4/3}\mathrm{d}x+\frac{9m^2}{16\ell\gamma}\int_{\mathbb{R}^3}Q(x)^{2/3}\mathrm{d}x-\ell^s\int_{\mathbb{R}^3}\frac{Q(x)}{|x|^s}\mathrm{d}x.$$

By taking $\ell=C(\tau_c-\tau_n)^{\frac{1}{s-1}}$, for some suitable positive constant $C$, we arrive at the desired upper bound in (3.1). In the case $V=0$, the term $-\ell^s\int_{\mathbb{R}^3}\frac{Q(x)}{|x|^s}\mathrm{d}x$ does not appear in the above estimation, hence the desired upper bound in (3.2) follows by taking $\ell=C(\tau_c-\tau_n)^{-\frac{1}{2}}$.

Next we prove the lower bound in (3.1). For every $1\leq i\leq M$ and for some $L>0$ small, it follows from Hölder's inequality and the definiton of $s$ that

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{\rho_n(x)}{|x-x_i|^{s_i}}\mathrm{d}x&\leq\int_{|x-x_i|\leq L}\frac{\rho_n(x)}{|x-x_i|^{s_i}}\mathrm{d}x+\int_{|x-x_i|\geq L}\frac{\rho_n(x)}{|x-x_i|^{s_i}}\mathrm{d}x\\&\leq L^{\frac{3-4s_i}{4}}\left(\int_{\mathbb{R}^3}\rho_n(x)^{4/3}\mathrm{d}x\right)^{3/4}+\frac{1}{L^{s_i}}\\&\leq L^{\frac{3-4s}{4}}\left(\int_{\mathbb{R}^3}\rho_n(x)^{4/3}\mathrm{d}x\right)^{3/4}+\frac{1}{L^{s}}.\end{aligned} \tag{3.3}$$

We deduce from (3.3) and (1.4) that

$$\begin{aligned} e_{\tau_n}(1) &\geq \left(1-\frac{\tau_n}{\tau_c}\right)\int_{\mathbb{R}^3}\rho_n(x)^{4/3}\mathrm{d}x - ML^{\frac{3-4s}{4}}\left(\int_{\mathbb{R}^3}\rho_n(x)^{4/3}\mathrm{d}x\right)^{3/4} - \frac{M}{L^s} \\ &\geq -C\frac{L^{3-4s}}{(\tau_c-\tau_n)^3} - \frac{M}{L^s}. \end{aligned} \tag{3.4}$$

Hence, the desired lower bound in (3.1) follows by taking $L = (\tau_c - \tau_n)^{\frac{1}{1-s}}$ for $n$ sufficiently large.

We finish the lemma by proving the lower bound in (3.2). From (1.4) and the upper bound of $e^{\infty}_{\tau_n}(1)$ in (3.2) we see that

$$M_2(\tau_c-\tau_n)^{1/2} \geq \left(1-\frac{\tau_n}{\tau_c}\right)\int_{\mathbb{R}^3} j_m(\rho_n(x))\mathrm{d}x,$$

which implies that

$$\int_{\mathbb{R}^3} j_m(\rho_n(x))\mathrm{d}x \leq M_2\tau_c(\tau_c-\tau_n)^{-1/2}.$$

Let

$$\begin{aligned} \tilde{j}_m(\rho) &= \frac{q}{(2\pi)^3}\int_{|p|<(6\pi^2\rho/q)^{1/3}} \frac{1}{\sqrt{|p|^2+m^2}}\mathrm{d}p \\ &= \frac{q}{4\pi^2}\left[\eta\sqrt{\eta^2+m^2} - m^2\ln\left(\frac{\eta+\sqrt{\eta^2+m^2}}{m}\right)\right], \end{aligned}$$

it follows from operator inequality

$$\sqrt{|p|^2+m^2} \geq |p| + \frac{m^2}{2\sqrt{|p|^2+m^2}}$$

that

$$\int_{\mathbb{R}^3} j_m(\rho_n(x))\mathrm{d}x \geq \gamma\rho_n^{4/3} + \frac{m^2}{2}\int_{\mathbb{R}^3}\tilde{j}_m(\rho_n(x))\mathrm{d}x.$$

From Hölder's inequality and the fact that $j_m(\rho_n)\tilde{j}_m(\rho_n) \geq \frac{q^2}{32\pi^4}\eta_n^6$ where $\eta_n = (6\pi^2\rho_n/q)^{1/3}$, we have

$$\int_{\mathbb{R}^3} j_m(\rho_n(x))\mathrm{d}x\int_{\mathbb{R}^3}\tilde{j}_m(\rho_n(x))\mathrm{d}x \geq \left(\int_{\mathbb{R}^3}\sqrt{j_m(\rho_n(x))\tilde{j}_m(\rho_n(x))}\mathrm{d}x\right)^2 = \frac{9}{8}.$$

From (1.4) and the above inequality we have

$$\begin{aligned} e^{\infty}_{\tau_n}(1) = \mathcal{E}^{\infty}_{\tau_n}(\rho_n) &\geq \frac{m^2}{2}\int_{\mathbb{R}^3}\tilde{j}_m(\rho(x))\mathrm{d}x \geq \frac{9m^2}{16\int_{\mathbb{R}^3} j_m(\rho(x))\mathrm{d}x} \\ &\geq \frac{9m^2}{16M_2\tau_c(\tau_c-\tau_n)^{-\frac{1}{2}}} = M_1(\tau_c-\tau_n)^{\frac{1}{2}}. \end{aligned}$$

□

**Lemma 8.** *There exist a positive constant $0 < K_1 < K_2$ independent of $\tau_n$ such that, for $n$ sufficiently large,*

$$K_1(\tau_c-\tau_n)^{\frac{1}{s-1}} \leq D(\rho_n,\rho_n) \leq K_2(\tau_c-\tau_n)^{\frac{1}{s-1}}. \tag{3.5}$$

*if $V$ is defined in* (1.7), *and*

$$K_1(\tau_c-\tau_n)^{-\frac{1}{2}} \le D(\rho_n,\rho_n) \le K_2(\tau_c-\tau_n)^{-\frac{1}{2}}. \tag{3.6}$$

*if $V=0$.*

*Proof.* We start with the proof of the upper bound in (3.5). From (1.4) we see that it suffices to find an upper bound for $\int_{\mathbb{R}^3}\rho_n(x)^{4/3}\mathrm{d}x$. This follows from the upper bound of $e_{\tau_n}(1)$ in (3.1), and (3.4) where we had chosen $L = M^{\frac{1}{s}}M_2{}^{-\frac{1}{s}}(\tau_c-\tau_n)^{\frac{1}{1-s}}$ for $n$ sufficiently large. The upper bound in (3.6) follows easily from (1.4) and the upper bound of $e^{\infty}_{\tau_n}(1)$ in (3.2).

Now let us prove the lower bound in (3.5) (since the proof of the lower bound in (3.6) is analogous). For any $b$ such that $0\le b\le\tau_n$, we have

$$e_b(1) \le \mathcal{E}_b(\rho_n) = e_{\tau_n}(1) + (\tau_n-b)D(\rho_n,\rho_n), \tag{3.7}$$

and

$$e_{\tau_n}(1) \le \mathcal{E}_{\tau_n}(\rho_b) = e_b(1) + (b-\tau_n)D(\rho_b,\rho_b).$$

From the above two inequalities we obtain that

$$D(\rho_n,\rho_n) \ge D(\rho_b,\rho_b). \tag{3.8}$$

From (3.7) and (3.1), we deduce that there exist two positive constants $M_1>M_2$ such that for any $0<b<\tau_n<\tau_c$,

$$D(\rho_n,\rho_n) \ge \frac{e_b(1)-e_{\tau_n}(1)}{\tau_n-b} \ge \frac{-M_1\,(\tau_c-b)^{\frac{s}{s-1}} + M_2(\tau_c-\tau_n)^{\frac{s}{s-1}}}{\tau_n-b}.$$

Choosing $b=\tau_n-\beta(\tau_c-\tau_n)$ with $\beta>0$, we arrive at

$$D(\rho_n,\rho_n) \ge (\tau_c-\tau_n)^{\frac{1}{s-1}}\frac{-M_1\,(1+\beta)^{\frac{s}{s-1}}+M_2}{\beta}.$$

The last fraction is positive for $\beta$ large enough. For $\tau_n$ closes to $\tau_c$, then there exists a positive constant $K_1$ such that

$$D(\rho_n,\rho_n) \ge K_1(\tau_c-\tau_n)^{\frac{s}{s-1}}.$$

$\square$

*Remark* 9. When $V$ is defined in (1.7), it follows from (3.5) that $\int_{\mathbb{R}^3}\rho_n(x)^{4/3}\mathrm{d}x$ is large for $n$ sufficiently large. Hence, by taking $L=\left(\int_{\mathbb{R}^3}\rho_n(x)^{4/3}\mathrm{d}x\right)^{-1}$ in (3.3), we obtain that there exists a constant $C>0$ such that

$$\int_{\mathbb{R}^3} V(x)\rho_n(x)\mathrm{d}x \ge -C\left(\int_{\mathbb{R}^3}\rho_n(x)^{4/3}\mathrm{d}x\right)^s \ge -C(\tau_c-\tau_n)^{\frac{s}{s-1}}, \tag{3.9}$$

for $n$ sufficiently large.

Now we are ready to complete the proof of Theorem 2.

*Proof of Theorem 2.* First, we focus on the case when $V$ is defined by (1.7).

Let $\epsilon_n := (\tau_c-\tau_n)^{\frac{1}{1-s}} > 0$, we see that $\epsilon_n\to 0$ as $n\to\infty$. For every $1\le i\le M$ we define $w_n^{(i)}(x)=\epsilon_n^3\rho_n(\epsilon_n x + x_i)$ be non-negative $L^1$-normalize of $\rho_n$. It follows from (1.4) and the upper bound of $e_{\tau_n}(1)$ in (3.1) that there exists a positive constant $M_2$ such that

$$\sum_{i=1}^{M}\int_{\mathbb{R}^3}\frac{z_i}{|x-x_i|^{s_i}}\rho_n(x)\mathrm{d}x = -\int_{\mathbb{R}^3}V(x)\rho_n(x)\mathrm{d}x \ge M_2\epsilon_n^{-s}.$$

From this, we deduce that there exists an $1 \leq j \leq M$ such that

$$\epsilon_n^{s-s_j} \int_{\mathbb{R}^3} \frac{z_j}{|x|^{s_j}} w_n^{(j)}(x)\mathrm{d}x = \epsilon_n^s \int_{\mathbb{R}^3} \frac{z_j}{|x-x_j|^{s_j}} \rho_n(x)\mathrm{d}x \geq \frac{M_2}{M} \tag{3.10}$$

which implies that $s_j = s = \max_{1\leq i\leq M} s_i$. Otherwise, if $s_j < s$ then $\epsilon_n^{s-s_j} \to 0$ as $n \to \infty$, which contradicts (3.10).

Now, for such $j$, we deduce from (3.5) that

$$D(w_n^{(j)}, w_n^{(j)}) > 0, \tag{3.11}$$

and there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^3} w_n^{(j)}(x)^{4/3}\mathrm{d}x = \epsilon_n \int_{\mathbb{R}^3} \rho_n(x)^{4/3}\mathrm{d}x \leq C.$$

Thus $\{w_n^{(j)}\}$ is bounded in $L^{4/3}(\mathbb{R}^3)$, and hence there exists a subsequence of $w_n^{(j)}$, still denote by $w_n^{(j)}$, such that $w_n^{(j)} \rightharpoonup w$ weakly in $L^{4/3}(\mathbb{R}^3)$. Since $\rho_n$ is a non-negative minimizer of $e_{\tau_n}(1)$, it satisfies the following variational equations

$$\sqrt{\eta_n(x)^2+m^2} - \tau_n(|\cdot|^{-1} \star \rho_n)(x) + V(x) - \mu_n \begin{cases} = 0 & \text{if } \rho_n(x) > 0, \\ \geq 0 & \text{if } \rho_n(x) = 0. \end{cases} \tag{3.12}$$

where $\eta_n = (6\pi^2\rho_n/q)^{1/3}$ and $\mu_n$ is Lagrange multiplier. In fact,

$$\mu_n = \int_{\mathbb{R}^3} \sqrt{\eta_n(x)^2+m^2}\rho_n(x)\mathrm{d}x - 2\tau_n D(\rho_n,\rho_n) + \int_{\mathbb{R}^3} V(x)\rho_n(x)\mathrm{d}x \tag{3.13}$$

We see that $w_n^{(j)}$ is a non-negative solution to

$$\sqrt{\zeta_n^j(x)^2+m^2} - \tau_n(|\cdot|^{-1} \star w_n^{(j)})(x) + \epsilon_n V(\epsilon_n x + x_j) - \epsilon_n\mu_n \begin{cases} = 0 & \text{if } w_n^{(j)}(x) > 0, \\ \geq 0 & \text{if } w_n^{(j)}(x) = 0. \end{cases} \tag{3.14}$$

where $\zeta_n^j = (6\pi^2 w_n^{(j)}/q)^{1/3}$.

From the fact that

$$j_m(\rho_n(x)) \leq \sqrt{\eta_n(x)^2+m^2}\rho_n(x) \leq \frac{4}{3} j_m(\rho_n(x)) \tag{3.15}$$

we have

$$e_{\tau_n}(1) - \tau_n D(\rho_n,\rho_n) \leq \mu_n \leq \frac{4}{3}e_{\tau_n}(1) - \frac{2}{3}\tau_n D(\rho_n,\rho_n) - \frac{1}{3}\int_{\mathbb{R}^3} V(x)\rho_n(x)\mathrm{d}x.$$

Hence we deduce from (3.1), (3.5) and (3.9) that $\epsilon_n\mu_n$ is bounded uniformly and strictly negative as $n \to \infty$. By passing to a subsequence if necessary, we can thus assume that $\epsilon_n\mu_n$ converges to some number $-\alpha < 0$ as $n \to \infty$.

Passing (3.14) to weak limit, we have $w$ be a non-negative solution to

$$\frac{4}{3}\gamma w(x)^{1/3} - \tau_c(|\cdot|^{-1} \star w)(x) + \alpha \begin{cases} = 0 & \text{if } w(x) > 0, \\ \geq 0 & \text{if } w(x) = 0. \end{cases}$$

By a simple rescaling we see that, $Q(x) = \lambda^{-3} w(\lambda^{-1}x + y_0)$ is a non-negative solution of (1.6). Here $\lambda = \frac{3\alpha}{2\tau_c}$ and $y_0 \in \mathbb{R}^3$.

Now we claim that there exists a positive constant $R_0$ such that

$$\liminf_{n\to\infty} \int_{B(0,R_0)} w_n^{(j)}(x)\mathrm{d}x > 0. \tag{3.16}$$

On the contrary, we assume that for any $R > 0$ there exists a sequence of $\{\tau_n\}$, still denoted by $\{\tau_n\}$, such that

$$\lim_{n\to\infty} \int_{B(0,R)} w_n^{(j)}(x)\mathrm{d}x = 0$$

then by the same arguments in [12, p.124] we can prove that $D(w_n^{(j)}, w_n^{(j)}) \to 0$ as $n \to \infty$. This contradicts (3.11), hence (3.16) holds true. It follows from (3.16) and the weak limit $w_n^{(j)} \rightharpoonup w$ in $L^{4/3}(\mathbb{R}^3)$ that

$$\int_{B(0,R_0)} w(x)\mathrm{d}x = \lim_{n\to\infty} \int_{B(0,R_0)} w_n^{(j)}(x)\mathrm{d}x > 0$$

which implies that $w > 0$ in $\mathbb{R}^3$. Hence $Q > 0$ in $\mathbb{R}^3$, and it solves equation

$$\frac{4}{3}\sigma_f Q(x)^{1/3} - (|\cdot|^{-1} \star Q)(x) + \frac{2}{3} = 0,$$

which implies that

$$\frac{2}{3}\sigma_f \|Q\|_{L^{4/3}}^{4/3} - D(Q,Q) + \frac{1}{3}\|Q\|_{L^1} = 0 \tag{3.17}$$

We prove that $Q$ is indeed minimizer in (1.4). Let $S$ be a minimizer in (1.4) with $\|S\|_{L^1} = 1$, and define $s(x) = \epsilon_n^{-3} S(\epsilon_n^{-1} x)$, we then have

$$\epsilon_n \mathcal{E}_{\tau_n}(v) \leq \left(1 - \frac{\tau_n}{\tau_c}\right)\gamma \int_{\mathbb{R}^3} S(x)^{4/3}\mathrm{d}x + \frac{9m^2\epsilon_n^2}{16\gamma}\int_{\mathbb{R}^3} S(x)^{2/3}\mathrm{d}x + \epsilon_n \int_{\mathbb{R}^3} V(\epsilon_n x)S(x)\mathrm{d}x$$

On the other hand, since $\mu_n$ satisfies (3.14), we deduce from inequality 3.15 that

$$\begin{aligned}
\epsilon_n \mathcal{E}_{\tau_n}(\rho_n) &= \int_{\mathbb{R}^3} j_{m\epsilon_n}(w_n^{(j)}(x))\mathrm{d}x - \tau_n D(w_n^{(j)}, w_n^{(j)}) + \epsilon_n \int_{\mathbb{R}^3} V(\epsilon_n x + x_j) w_n^{(j)}(x)\mathrm{d}x \\
&\geq \frac{3}{4}\int_{\mathbb{R}^3} w_n^{(j)}\sqrt{\zeta_n^2 + m^2\epsilon_n^2} - \tau_n D(w_n^{(j)}, w_n^{(j)}) + \epsilon_n \int_{\mathbb{R}^3} V(\epsilon_n x + x_j) w_n^{(j)}(x)\mathrm{d}x \\
&= \frac{3}{4}\epsilon_n\mu_n + \frac{1}{2}\tau_n D(w_n^{(j)}, w_n^{(j)}) + \frac{1}{4}\epsilon_n \int_{\mathbb{R}^3} V(\epsilon_n x + x_j) w_n^{(j)}(x)\mathrm{d}x.
\end{aligned}$$

By the assumption that $\rho_n$ is a minimizer of $e_{\tau_n}(1)$, we have $\mathcal{E}_{\tau_n}(\rho_n) \leq \mathcal{E}_{\tau_n}(v)$ and hence

$$\liminf_{n\to\infty} \epsilon_n \mathcal{E}_{\tau_n}(\rho_n) \leq \liminf_{n\to\infty} \epsilon_n \mathcal{E}_{\tau_n}(v).$$

From the above estimates and the fact that $D(\rho,\rho)$ is weakly lower semicontinuous, we deduce that

$$D(Q,Q) = \frac{1}{\lambda}D(\omega,\omega) \leq \frac{2}{3\alpha}\liminf_{n\to\infty} \tau_n D(w_n^{(j)}, w_n^{(j)}) \leq 1.$$

On the other hand, from (1.4), (3.17) and the fact that $\|Q\|_{L^1} \leq 1$, we have

$$\begin{aligned}
D(Q,Q) &= \frac{2}{3}\sigma_f \|Q\|_{L^{4/3}}^{4/3} + \frac{1}{3}\|Q\|_{L^1} \geq \left(\sigma_f \|Q\|_{L^{4/3}}^{4/3} \|Q\|_{L^1}^{1/2}\right)^{2/3} \\
&\geq \left(\sigma_f \|Q\|_{L^{4/3}}^{4/3} \|Q\|_{L^1}^{2/3}\right)^{2/3} \geq D(Q,Q)^{2/3}.
\end{aligned}$$

This implies that $D(Q,Q) \geq 1$, and hence $D(Q,Q) = 1$. From this, it is easy to see that $\sigma_f \|Q\|_{L^{4/3}}^{4/3} = \|Q\|_{L^1} = 1$. Thus $Q$ is indeed minimizer in (1.4). Since $\|Q\|_{L^p} =$

$\|Q^*\|_{L^p}$ for all $1 \le p \le \infty$, it follows from (1.4) and the Riesz's rearrangement inequality (see [8, Theorem 3.7]) that

$$1 = \sigma_f \|Q^*\|_{L^1}^{2/3} \|Q^*\|_{L^{4/3}}^{4/3} \ge D(Q^*, Q^*) \ge D(Q, Q) = 1.$$

The equality occurs only if $Q(x) = Q^*(x - y)$ for some $y \in \mathbb{R}^3$ (see [8, Theorem 3.9]). Thus, up to translation, $Q$ is positive radially symmetric decreasing function which satisfies (1.5) and (1.6).

We note that $\|w\|_{L^1} = \|Q\|_{L^1} = 1$. From the norm preservation and the same arguments in [12, p.125] we conclude that

$$\lim_{n\to\infty} D(w_n^{(j)}, w_n^{(j)}) = D(w, w).$$

We deduce from this convergence and the inequality

$$\epsilon_n \mathcal{E}_{\tau_n}(\epsilon_n^{-3} w_n^{(j)}(\epsilon_n^{-1}(x - x_j))) = \epsilon_n \mathcal{E}_{\tau_n}(\rho_n) \le \epsilon_n \mathcal{E}_{\tau_n}(\epsilon_n^{-3} w(\epsilon_n^{-1} x))$$

that

$$\begin{aligned}
&\limsup_{n\to\infty} \gamma \int_{\mathbb{R}^3} w_n^{(j)}(x)^{4/3} \mathrm{d}x \le \limsup_{n\to\infty} \int_{\mathbb{R}^3} j_{m\epsilon_n}(w_n^{(j)}(x)) \mathrm{d}x \\
&\le \limsup_{n\to\infty} \int_{\mathbb{R}^3} j_{m\epsilon_n}(w(x)) \mathrm{d}x = \gamma \int_{\mathbb{R}^3} w(x)^{4/3} \mathrm{d}x.
\end{aligned}$$

On the other hand, since $w_n^{(j)} \rightharpoonup w$ weakly in $L^{4/3}(\mathbb{R}^3)$, by Fatou's Lemma we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} w_n^{(j)}(x)^{4/3} \mathrm{d}x \ge \int_{\mathbb{R}^3} w(x)^{4/3} \mathrm{d}x.$$

Therefore we have proved that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} w_n^{(j)}(x)^{4/3} \mathrm{d}x = \int_{\mathbb{R}^3} w(x)^{4/3} \mathrm{d}x.$$

which implies that $w_n^{(j)} \to w$ strongly in $L^{4/3}(\mathbb{R}^3)$. Thus, up to subsequence, we have $w_n^{(j)} \to w$ pointwise almost everywhere in $\mathbb{R}^3$. Using this pointwise convergence, we deduce from the Brezis-Lieb refinement of Fatou's lemma (see, e.g., [8, Theorem 1.9]) that

$$\int_{\mathbb{R}^3} w_n^{(j)}(x) \mathrm{d}x = \int_{\mathbb{R}^3} w(x) \mathrm{d}x + \int_{\mathbb{R}^3} |w_n^{(j)}(x) - w(x)| \mathrm{d}x + o(1),$$

where $o(1) \to 0$ as $n \to \infty$. Therefore $\int_{\mathbb{R}^3} w_n^{(j)}(x) \mathrm{d}x = \int_{\mathbb{R}^3} w(x) \mathrm{d}x = 1$ implies that $w_n \to w$ strongly in $L^1(\mathbb{R}^3)$.

We have thus shown that there exists a subsequence of $\{\tau_n\}$, still denoted by $\{\tau_n\}$, such that

$$\epsilon_n^3 \rho_n(\epsilon_n x + x_j) = w_n^{(j)}(x) \to w(x) = \lambda^3 Q(\lambda(x - y_0))$$

strongly in $L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$, where $Q$ is positive radially symmetric decreasing and optimizes in (1.4) and $\lambda > 0$, $y_0 \in \mathbb{R}^3$.

To complete the proof of Theorem 2, we determine the exact values of $\lambda$ and $y_0$. Since $\rho_n(x) = \epsilon_n^{-3} w_n^{(j)}(\epsilon_n^{-1}(x - x_j))$ is a minimizer of $e_{\tau_n}(1)$ we have

$$\begin{aligned} e_{\tau_n}(1) &= \left( \int_{\mathbb{R}^3} j_m(\rho_n(x))\mathrm{d}x - \tau_c D(\rho_n, \rho_n) \right) + (\tau_c - \tau_n) D(\rho_n, \rho_n) + \int_{\mathbb{R}^3} V(x)\rho_n(x)\mathrm{d}x \\ &\geq \int_{\mathbb{R}^3} \tilde{j}_m(\rho_n(x))\mathrm{d}x + \epsilon_n^{1-s} D(\rho_n, \rho_n) + \int_{\mathbb{R}^3} V(x)\rho_n(x)\mathrm{d}x \\ &= \epsilon_n \int_{\mathbb{R}^3} \tilde{j}_{m\epsilon_n}(w_n^{(j)}(x))\mathrm{d}x + \epsilon_n^{-s} D(w_n^{(j)}, w_n^{(j)}) + \int_{\mathbb{R}^3} V(\epsilon_n x + x_j) w_n^{(j)}(x)\mathrm{d}x, \end{aligned} \tag{3.18}$$

where we have used (1.4) for the term in the brackets.

Note that

$$\begin{aligned} &\int_{\mathbb{R}^3} \tilde{j}_{m\epsilon_n}(w_n^{(j)}(x))\mathrm{d}x \\ =&\frac{q}{4\pi^2} \int_{\mathbb{R}^3} \left[ \zeta_n^j(x)\sqrt{\zeta_n^j(x)^2 + m^2\epsilon_n^2} - m^2\epsilon_n^2 \ln\left( \frac{\zeta_n^j(x) + \sqrt{\zeta_n^j(x)^2 + m^2\epsilon_n^2}}{m\epsilon_n} \right) \right], \end{aligned}$$

by Fatou's Lemma, we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} \tilde{j}_{m\epsilon_n}(w_n^{(j)}(x))\mathrm{d}x \geq \frac{q}{4\pi^2} \int_{\mathbb{R}^3} \zeta(x)^2\mathrm{d}x = \frac{9}{8\gamma\lambda} \int_{\mathbb{R}^3} Q(x)^{2/3}\mathrm{d}x, \tag{3.19}$$

where $\zeta = (6\pi^2 w/q)^{1/3}$.

By the Hardy–Littlewood–Sobolev inequality, we have

$$\lim_{n\to\infty} D(w_n^{(j)}, w_n^{(j)}) = D(w, w) = \lambda D(Q, Q) = \lambda. \tag{3.20}$$

On the other hand, we have

$$\begin{aligned} \lim_{n\to\infty} \epsilon_n^s \int_{\mathbb{R}^3} V(\epsilon_n x + x_j) w_n^{(j)}(x)\mathrm{d}x &= -z_j \int_{\mathbb{R}^3} \frac{w(x)}{|x|^s}\mathrm{d}x \\ = -\lambda^s z_j \int_{\mathbb{R}^3} \frac{Q(x + y_0)}{|x|^s}\mathrm{d}x &\geq -\lambda^s z \int_{\mathbb{R}^3} \frac{Q(x)}{|x|^s}\mathrm{d}x \end{aligned} \tag{3.21}$$

since $Q$ is a radial decreasing function, and $z = \max_{1\leq i\leq M} z_i$.

It follows from (3.18), (3.20) and (3.21) that

$$\liminf_{n\to\infty} \frac{e_{\tau_n}(1)}{\epsilon_n^{-s}} \geq \lambda - \lambda^s z \int_{\mathbb{R}^3} \frac{Q(x)}{|x|^s}\mathrm{d}x.$$

Thus

$$\liminf_{n\to\infty} \frac{e_{\tau_n}(1)}{\epsilon_n^{-s}} \geq \inf_{\tilde{\lambda}>0} \left( \tilde{\lambda} - \tilde{\lambda}^s z \int_{\mathbb{R}^3} \frac{Q(x)}{|x|^s}\mathrm{d}x \right) = \left(1 - \frac{1}{s}\right) \left( sz \int_{\mathbb{R}^3} \frac{Q(x)}{|x|^s}\mathrm{d}x \right)^{\frac{1}{1-s}}. \tag{3.22}$$

To see the matching upper bound in (3.22), one simply takes

$$\rho_n(x) = (\tilde{\lambda}\epsilon_n^{-1})^3 Q(\tilde{\lambda}\epsilon_n^{-1}(x - x_i))$$

as trial state for $\mathcal{E}_{\tau_n}$, where $\tilde{\lambda} > 0$ and $x_i \in \mathcal{Z}$, i.e. $s_i = s$, $z_i = z$. We use (1.4) and the fact that $j_m(\rho_n) \leq \gamma \rho_n^{4/3} + \frac{9}{16} m^2 \gamma^{-1} \rho_n^{2/3}$ to obtain

$$\begin{aligned} e_{\tau_n}(1) &\leq \frac{9m^2}{16\gamma} \int_{\mathbb{R}^3} \rho_n(x)^{2/3} \mathrm{d}x + (\tau_c - \tau_n) D(\rho_n, \rho_n) + \int_{\mathbb{R}^3} V(x) \rho_n(x) \mathrm{d}x \\ &= \frac{9m^2 \epsilon_n}{16\gamma\tilde{\lambda}} \int_{\mathbb{R}^3} Q(x)^{2/3} \mathrm{d}x + \tilde{\lambda} \epsilon_n^{-s} + \int_{\mathbb{R}^3} V(\epsilon_n \tilde{\lambda}^{-1} x + x_i) Q(x) \mathrm{d}x. \end{aligned}$$

This implies that

$$\limsup_{n\to\infty} \frac{e_{\tau_n}(1)}{\epsilon_n^{-s}} \leq \tilde{\lambda} - \tilde{\lambda}^s z \int_{\mathbb{R}^3} \frac{Q(x)}{|x|^s} \mathrm{d}x.$$

Thus, taking the infimum over $\tilde{\lambda} > 0$ we see that

$$\limsup_{n\to\infty} \frac{e_{\tau_n}(1)}{\epsilon_n^{-s}} \leq \left(1 - \frac{1}{s}\right) \left(sz \int_{\mathbb{R}^3} \frac{Q(x)}{|x|^s} \mathrm{d}x\right)^{\frac{1}{1-s}}. \tag{3.23}$$

From (3.22) and (3.23) we conclude that

$$\lim_{n\to\infty} \frac{e_{\tau_n}(1)}{\epsilon_n^{-s}} = \left(1 - \frac{1}{s}\right) \left(sz \int_{\mathbb{R}^3} \frac{Q(x)}{|x|^s} \mathrm{d}x\right)^{\frac{1}{1-s}}.$$

and

$$\lambda = \left(sz \int_{\mathbb{R}^3} \frac{Q(x)}{|x|^s} \mathrm{d}x\right)^{\frac{1}{1-s}}, \quad z_j = \max_{1\leq i\leq M: s_i = s} z_i = z.$$

We note that the limit of $e_{\tau_n}(1)/\epsilon_n^{-s}$ is independent of the subsequence $\{\tau_n\}$. Therefore, we have the convergence of the whole family in (1.8).

Now we come to the case when $V \equiv 0$. In this case, we put $\epsilon_n := (\tau_c - \tau_n)^{\frac{1}{2}}$, and we define $\tilde{w}_n(x) = \epsilon_n^3 \rho_n(\epsilon_n x)$. It follows from (3.6) that $D(\tilde{w}_n, \tilde{w}_n) > 0$. By the same arguments in [12, p.124] we can prove that, there exists a sequence $\{x_n\} \subset \mathbb{R}^3$ and a positive constant $R_1$ such that

$$\liminf_{n\to\infty} \int_{B(x_n, R_1)} \tilde{w}_n(x) \mathrm{d}x > 0.$$

By the same arguments as we have done before for the case $V \not\equiv 0$ , we can prove that there exists a subsequence of $\{\tau_n\}$, still denoted by $\{\tau_n\}$, and a sequence $\{x_n\} \subset \mathbb{R}^3$ such that

$$w_n(x) := \tilde{w}_n(x + x_n) = \epsilon_n^3 \rho_n(\epsilon_n x + \epsilon_n x_n) \to w(x) = \lambda^3 Q(\lambda x)$$

strongly in $L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$, where $Q$ is positive radially symmetric decreasing and optimizes inequality (1.4), and $\lambda > 0$. It remains to compute the exact value of $\lambda$, which is the consequence of the computation $\lim_{n\to\infty} \frac{e^{\infty}_{\tau_n}(1)}{\epsilon_n}$. The lower bound follows from (3.18), (3.19) and (3.20), while the upper bound is done by taking the trial state

$$\rho_n(x) = (\tilde{\lambda}\epsilon_n^{-1})^3 Q(\tilde{\lambda}\epsilon_n^{-1} x),$$

and by optimizing the quantity $\limsup_{n\to\infty} \frac{e^{\infty}_{\tau_n}(1)}{\epsilon_n}$ over $\tilde{\lambda} > 0$. Here $\tilde{\lambda} > 0$ and $Q$ is positive radially symmetric decreasing and optimizes inequality (1.4).

In summary, we conclude that

$$\lim_{n\to\infty} \frac{e^{\infty}_{\tau_n}(1)}{\epsilon_n} = \frac{3}{2} m \sqrt{\frac{1}{\gamma} \int_{\mathbb{R}^3} Q(x)^{2/3} \mathrm{d}x}, \quad \lambda = \frac{3}{4} m \sqrt{\frac{1}{\gamma} \int_{\mathbb{R}^3} Q(x)^{2/3} \mathrm{d}x}.$$

Since the limit of $e^{\infty}_{\tau_n}(1)/\epsilon_n$ is independent of the subsequence $\{\tau_n\}$, we have the convergence of the whole family in (1.9).

The proof is complete. $\square$

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