CELLULAR STRUCTURES USING $\mathbf{U}_q$-TILTING MODULES

WITH ADDITIONAL NOTES TO THE PAPER AS AN APPENDIX

HENNING HAAHR ANDERSEN, CATHARINA STROPPEL, AND DANIEL TUBBENHAUER

Abstract. We use the theory of $\mathbf{U}_q$-tilting modules to construct cellular bases for centralizer algebras. Our methods are quite general and work for any quantum group $\mathbf{U}_q$ attached to a Cartan matrix and include the non-semisimple cases for q being a root of unity and ground fields of positive characteristic. Our approach also generalizes to certain categories containing infinite-dimensional modules. As applications, we give a new semisimplicty criterion for centralizer algebras, and recover the cellularity of several known algebras (with partially new cellular bases) which all fit into our general setup.

Contents

1. Introduction

Fix any field $\mathbb{K}$ and set $\mathbb{K}^* = \mathbb{K} - \{0,-1\}$ if $\mathrm{char}(\mathbb{K}) > 2$ and $\mathbb{K}^* = \mathbb{K} - \{0\}$ otherwise. Let $\mathbf{U}_q(\mathfrak{g})$ be the quantum group over $\mathbb{K}$ for a fixed, arbitrary parameter $q \in \mathbb{K}^*$ associated to a simple Lie algebra $\mathfrak{g}$. The main result in this paper is the following.

H.H.A. and D.T. were partially supported by the center of excellence grant "Centre for Quantum Geometry of Moduli Spaces (QGM)" from the "Danish National Research Foundation (DNRF)". C.S. was supported by the Max–Planck-Gesellschaft. D.T. was partially supported by a research funding of the "Deutsche Forschungsgemeinschaft (DFG)" during the last part of this work.

Theorem. (**Cellularity of endomorphism algebras.**) Let T be a $\mathbf{U}_q(\mathfrak{g})$-tilting module. Then $\mathrm{End}_{\mathbf{U}_q(\mathfrak{g})}(T)$ is a cellular algebra in the sense of Graham and Lehrer [38]. □

It is important to note that cellular bases are not unique. In particular, a single algebra can have many cellular bases. As a concrete application, see Section 5B, we construct (several) new cellular bases for the Temperley–Lieb algebra depending on the ground field and the choice of deformation parameter. These bases differ therefore for instance from the construction in [38, Section 6] of cellular bases for the Temperley–Lieb algebras. Moreover, we also show that some of our bases for the Temperley–Lieb algebra can be equipped with a $\mathbb{Z}$-grading which is in contrast to Graham and Lehrer's bases. Our bases also depend heavily on the characteristic of $\mathbb{K}$ (and on $q \in \mathbb{K}^*$). Hence, they see more of the characteristic (and parameter) depended representation theory, but are also more difficult to construct explicitly.

We stress that the cellularity itself can be deduced from general theory. Namely, any $\mathbf{U}_q(\mathfrak{g})$-tilting module T is a summand of a full $\mathbf{U}_q(\mathfrak{g})$-tilting module $\tilde{T}$. By [72, Theorem 6] $\mathrm{End}_{\mathbf{U}_q(\mathfrak{g})}(\tilde{T})$ is quasi-hereditary and comes equipped with an involution as we explain in Section 3C. Thus, it is cellular, see [55]. By their Theorem 4.3, this induces the cellularity of the idempotent truncation $\mathrm{End}_{\mathbf{U}_q(\mathfrak{g})}(T)$. In contrast, our approach provides the existence and a method of construction of many cellular bases. It generalizes to the infinite-dimensional Lie theory situation and has other nice consequences that we will explore in this paper. In particular, our results give a novel semisimplicity criterion for $\mathrm{End}_{\mathbf{U}_q(\mathfrak{g})}(T)$, see Theorem 4.13. This together with the Jantzen sum formula give rise to a new way to obtain semisimplicity criteria for these algebras (we explain and explore this in [9] where we recover semisimplicity criteria for several algebras using the results of this paper). Here a crucial fact is that the tensor product of $\mathbf{U}_q$-tilting modules is again a $\mathbf{U}_q$-tilting module, see [68]. This implies that our results also vastly generalize [94] to the non-semisimple cases (where our main theorem is non-trivial).

The framework. Given any simple, complex Lie algebra $\mathfrak{g}$, we can assign to it a *quantum deformation* $\mathbf{U}_v = \mathbf{U}_v(\mathfrak{g})$ of its universal enveloping algebra by deforming its Serre presentation. (Here v is a generic parameter and $\mathbf{U}_v$ is an $\mathbb{Q}(v)$-algebra.) The representation theory of $\mathbf{U}_v$ shares many similarities with the one of $\mathfrak{g}$. In particular, the category[1] $\mathbf{U}_v$-**Mod** is semisimple.

But one can spice up the story drastically: the quantum group $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ is obtained by specializing v to an arbitrary $q \in \mathbb{K}^*$. In particular, we can take q to be a root of unity[2]. In this case $\mathbf{U}_q$-**Mod** is not semisimple anymore, which makes the representation theory much more interesting. It has many connections and applications in different directions, e.g. the category has a neat combinatorics, is related to the corresponding almost-simple, simply connected algebraic group G over $\mathbb{K}$ with $\mathrm{char}(\mathbb{K})$ prime, see for example [4] or [60], to the representation theory of affine Kac–Moody algebras, see [50] or [87], and to (2+1)-TQFT's and the Witten–Reshetikhin–Turaev invariants of 3-manifolds, see for example [92].

Semisimplicity in light of our main result means the following. If we take $\mathbb{K} = \mathbb{C}$ and $q = \pm 1$, then our result says that the algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is cellular for any $\mathbf{U}_q$-module $T \in \mathbf{U}_q$-**Mod**

[1] For any algebra A we denote by A-**Mod** the category of finite-dimensional, left A-modules. If not stated otherwise, all modules are assumed to be finite-dimensional, left modules.

[2] In our terminology: The two cases $q = \pm 1$ are special and do not count as roots of unity. Moreover, for technical reasons, we always exclude $q = -1$ in case $\mathrm{char}(\mathbb{K}) > 2$.

because in this case all $\mathbf{U}_q$-modules are $\mathbf{U}_q$-tilting modules. This is no surprise: when T is a direct sum of simple $\mathbf{U}_q$-modules, then $\mathrm{End}_{\mathbf{U}_q}(T)$ is a direct sum of matrix algebras $M_n(\mathbb{K})$. Likewise, for any $\mathbb{K}$, if $q \in \mathbb{K}^* - \{1\}$ is not a root of unity, then $\mathbf{U}_q$-**Mod** is still semisimple and our result is (almost) standard. But even in the semisimple case we can say more: we get an Artin–Wedderburn basis as a cellular basis for $\mathrm{End}_{\mathbf{U}_q}(T)$, i.e. a basis realizing the decomposition of $\mathrm{End}_{\mathbf{U}_q}(T)$ into its matrix components, see Section 5A.

On the other hand, if $q = 1$ and $\mathrm{char}(\mathbb{K}) > 0$ or if $q \in \mathbb{K}^*$ is a root of unity, then $\mathbf{U}_q$-**Mod** is far from being semisimple and our result gives many interesting cellular algebras.

For example, if $G = \mathrm{GL}(V)$ for some n-dimensional $\mathbb{K}$-vector space V, then $T = V^{\otimes d}$ is a G-tilting module for any $d \in \mathbb{Z}_{\geq 0}$. By Schur–Weyl duality we have

$$\Phi_{\mathrm{SW}} \colon \mathbb{K}[S_d] \twoheadrightarrow \mathrm{End}_G(T) \quad \text{and} \quad \Phi_{\mathrm{SW}} \colon \mathbb{K}[S_d] \xrightarrow{\cong} \mathrm{End}_G(T), \text{ if } n \geq d, \tag{1}$$

where $\mathbb{K}[S_d]$ is the group algebra of the symmetric group S_d in d letters. We can realize this as a special case in our framework by taking $q = 1$, $n \geq d$ and $\mathfrak{g} = \mathfrak{gl}_n$ (although $\mathfrak{gl}_n$ is not a simple, complex Lie algebra, our approach works fine for it as well). On the other hand, by taking q arbitrary in $\mathbb{K}^* - \{1\}$ and $n \geq d$, the group algebra $\mathbb{K}[S_d]$ is replaced by the type A_{d-1} *Iwahori–Hecke algebra* $\mathcal{H}_d(q)$ over $\mathbb{K}$ and our theorem gives cellular bases for this algebra as well. Note that one underlying fact why (1) stays true in the non-semisimple case is that $\dim(\mathrm{End}_G(T))$ is independent of the characteristic of $\mathbb{K}$ (and of the parameter q in the quantum case), since T is a G-tilting module.

Of course, both $\mathbb{K}[S_d]$ and $\mathcal{H}_d(q)$ are known to be cellular (these cases were one of the main motivations of Graham and Lehrer to introduce the notion of cellular algebras), but the point we want to make is, that they fit into our more general framework.

The following known cellularity properties can also be recovered directly from our approach. And moreover: in most of the examples we either have no or only some mild restrictions on $\mathbb{K}$ and $q \in \mathbb{K}^*$.

- As sketched above: the algebras $\mathbb{K}[S_d]$ and $\mathcal{H}_d(q)$ and their quotients under Φ_{SW}.
- The *Temperley–Lieb algebras* $\mathcal{TL}_d(\delta)$ introduced in [88].
- Other less well-known endomorphism algebras for $\mathfrak{sl}_2$-related tilting modules appearing in more recent work, e.g. [5], [10] or [73].
- *Spider algebras* in the sense of [56].
- Quotients of the group algebras of $\mathbb{Z}/r\mathbb{Z} \wr S_d$ and its quantum version $\mathcal{H}_{d,r}(q)$, the *Ariki–Koike algebras* introduced in [12]. This includes the Ariki–Koike algebras themselves and thus, the Hecke algebras of type B. This also includes Martin and Saleur's *blob algebras* $\mathcal{BL}_d(q,m)$ [64] and *(quantized) rook monoid algebras* (also called *Solomon algebras*) $\mathcal{R}_d(q)$ in the spirit of [85].
- *Brauer algebras* $\mathcal{B}_d(\delta)$ introduced in [15] in the context of classical invariant theory, and related algebras, e.g. the *walled Brauer algebras* $\mathcal{B}_{r,s}(\delta)$ as in [54] and [91], and the *Birman–Murakami–Wenzl* algebras $\mathcal{BMW}_d(\delta)$, in the sense of [14] and [66].

Note our methods also apply for some categories containing infinite-dimensional modules. For example, with a little bit more care, one could allow T to be a not necessarily finite-dimensional $\mathbf{U}_q$-tilting module. Moreover, our methods also include the *BGG category* $\boldsymbol{\mathcal{O}}$, its *parabolic subcategories* $\boldsymbol{\mathcal{O}}^{\mathfrak{p}}$ and its quantum cousin $\boldsymbol{\mathcal{O}}_q$ from [6]. For example, using the "big projective tilting" in the principal block, we get a cellular basis for the coinvariant algebra of the Weyl group associated to $\mathfrak{g}$. In fact, we get a vast generalization of this, e.g. we can fit *generalized Khovanov arc algebras* (see e.g. [19]), $\mathfrak{sl}_n$*-web algebras* (see e.g. [62]), *cyclotomic Khovanov–Lauda and Rouquier algebras* of type A (see [52] and [53] or [74]), for which we obtain cellularity via the connection to cyclotomic quotients of the degenerate affine Hecke algebra, see [16], cyclotomic $\mathbb{W}_d$*-algebras* (see e.g. [33]) and *cyclotomic quotients of affine Hecke algebras* $\mathbf{H}^s_{\mathbb{K},d}$ (see e.g. [75]) into our framework as well, see Section 5A. However, we will for simplicity focus on the finite-dimensional world. Here we provide all necessary arguments in great detail, sometimes, for brevity, only in an extra file [8]. See also Remark 1.

Following Graham and Lehrer's approach, our cellular bases for $\mathrm{End}_{\mathbf{U}_q}(T)$ provide also $\mathrm{End}_{\mathbf{U}_q}(T)$-cell modules, the classification of simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules etc. We give an interpretation of this in our setting as well, see Section 4. For instance, we deduce a new criterion for semisimplicity of $\mathrm{End}_{\mathbf{U}_q}(T)$, see Theorem 4.13.

Remark 1. Instead of working with the infinite-dimensional algebra $\mathbf{U}_q$, we could also work with a finite-dimensional, quasi-hereditary algebra (with a suitable anti-involution). By using results summarized in [30, Appendix], our constructions will go through very much in the same spirit as for $\mathbf{U}_q$. However, using $\mathbf{U}_q$ has some advantages. For example, we can construct an abundance of cellular bases (for the explicit construction of our basis we need "weight spaces" such that e.g. (2) or Lemma 3.4 work). Having several cellular bases is certainly an advantage, although calculating these is in general a non-trivial task. (For example, getting an explicit understanding of the endomorphisms giving rise to the cellular basis is a tough challenge, but see [70] for some crucial steps in this direction.) As a direct consequence of the existence of many cellular bases: most of the algebras appearing in our list of examples above can be additionally equipped with a $\mathbb{Z}$-grading. The basis elements from Theorem 3.9 can be chosen such that our approach leads to a $\mathbb{Z}$-graded cellular basis in the sense of [41]. We make this more precise in case of the Temperley–Lieb algebras, but one could for instance also recover the $\mathbb{Z}$-graded cellular bases of the Brauer algebras from [34] from our approach. We stress that in both cases the cellular bases in [38, Sections 4 and 6] are not $\mathbb{Z}$-graded. To keep the paper within reasonable boundaries, we do not treat the graded setup in detail. ▲

Acknowledgements. We would like to thank Ben Cooper, Michael Ehrig, Matt Hogancamp, Johannes Kübel, Gus Lehrer, Paul Martin, Andrew Mathas, Volodymyr Mazorchuk, Steen Ryom-Hansen and Paul Wedrich for helpful comments and discussions, and the referees for further useful comments. H.H.A. would like to thank the Institut Mittag-Leffler for the hospitality he enjoyed there during the final stages of this work. C.S. is very grateful to the Max-Planck Institute in Bonn for the extraordinary support and excellent working conditions. A large part of her research was worked out during her stay there. D.T. would like to thank the dark Danish winter for very successfully limiting his non-work options and the Australian long blacks for pushing him forward.

2. Quantum groups, their representations and tilting modules

We briefly recall some facts we need in this paper. Details can be found e.g. in [7] and [47], or [30] and [48]. For notations and arguments adopted to our situation see [8]. See also [72] and [29] for the classical treatment of tilting modules (in the modular case). As in the introduction, we fix a field $\mathbb{K}$ over which we work throughout.

2A. **The quantum group $\mathbf{U}_q$.** Let Φ be a finite *root system* in an Euclidean space E. We fix a choice of *positive roots* $\Phi^+ \subset \Phi$ and *simple roots* $\Pi \subset \Phi^+$. We assume that we have n simple roots that we denote by $\alpha_1, \dots, \alpha_n$. For each $\alpha \in \Phi$, we denote by $\alpha^\vee \in \Phi^\vee$ the corresponding *coroot*. Then $\mathbf{A} = (\langle \alpha_i, \alpha_j^\vee \rangle)_{i,j=1}^n$ is called the *Cartan matrix*.

By the set of *(integral) weights* we mean $X = \{\lambda \in E \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha_i \in \Pi\}$. The *dominant (integral) weights* X^+ are those $\lambda \in X$ such that $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $\alpha_i \in \Pi$.

Recall that there is a *partial ordering* on X given by $\mu \leq \lambda$ if and only if $\lambda - \mu$ is an $\mathbb{Z}_{\geq 0}$-valued linear combination of the simple roots, that is, $\lambda - \mu = \sum_{i=1}^n a_i \alpha_i$ with $a_i \in \mathbb{Z}_{\geq 0}$.

We denote by $\mathbf{U}_q = \mathbf{U}_q(\mathbf{A})$ the *quantum enveloping algebra* attached to a Cartan matrix $\mathbf{A}$ and specialized at $q \in \mathbb{K}^*$, where we follow [7] with our conventions. Note $\mathbf{U}_q$ always means the quantum group over $\mathbb{K}$ defined via Lusztig's divided power construction. (Thus, we have generators K_i, E_i and F_i for all $i = 1, \dots, n$ as well as divided power generators.) We have a decomposition $\mathbf{U}_q = \mathbf{U}_q^- \mathbf{U}_q^0 \mathbf{U}_q^+$, with subalgebras generated by F's, K's and E's respectively (and some divided power generators, see e.g. their Section 1). Note we can recover the generic case $\mathbf{U}_v = \mathbf{U}_v(\mathbf{A})$ by choosing $\mathbb{K} = \mathbb{Q}(v)$ and $q = v$.

It is worth noting that $\mathbf{U}_q$ is a Hopf algebra, so its module category is a monoidal category with duals. We denote by $\mathbf{U}_q$-**Mod** the category of finite-dimensional $\mathbf{U}_q$-modules (of type 1, see [7, Section 1.4]). We consider only such $\mathbf{U}_q$-modules in what follows.

Recall that there is a contravariant, character-preserving *duality functor* $\mathcal{D}$ that is defined on the $\mathbb{K}$-vector space level via $\mathcal{D}(M) = M^*$ (the $\mathbb{K}$-linear dual of M) and an action of $\mathbf{U}_q$ on $\mathcal{D}(M)$ is defined as follows. Let $\omega \colon \mathbf{U}_q \to \mathbf{U}_q$ be the automorphism of $\mathbf{U}_q$ which interchanges E_i and F_i and interchanges K_i and K_i^{-1} (see e.g. [47, Lemma 4.6], which extends to our setup without difficulties). Then define $uf = m \mapsto f(\omega(S(u))m)$ for $u \in \mathbf{U}_q, f \in \mathcal{D}(M), m \in M$. Given any $\mathbf{U}_q$-homomorphism f between $\mathbf{U}_q$-modules, we also write $\mathrm{i}(f) = \mathcal{D}(f)$. This duality gives rise to the involution in our cellular datum from Section 3C.

Assumption 2.1. If q is a root of unity, then, to avoid technicalities, we assume that q is a primitive root of unity of odd order l. A treatment of the even case, that can be used to repeat everything in this paper in the case where l is even, can be found in [3]. Moreover, in case of type G_2 we additionally assume that l is prime to 3. ▲

For each $\lambda \in X^+$ there is a *Weyl* $\mathbf{U}_q$*-module* $\Delta_q(\lambda)$ and a *dual Weyl* $\mathbf{U}_q$*-module* $\nabla_q(\lambda)$ satisfying $\mathcal{D}(\Delta_q(\lambda)) \cong \nabla_q(\lambda)$. The $\mathbf{U}_q$-module $\Delta_q(\lambda)$ has a unique simple head $L_q(\lambda)$ which is the unique simple socle of $\nabla_q(\lambda)$. Thus, there is a (up to scalars) unique $\mathbf{U}_q$-homomorphism

$$c^\lambda \colon \Delta_q(\lambda) \to \nabla_q(\lambda) \quad \text{(mapping head to socle).} \tag{2}$$

This relies on the fact that $\Delta_q(\lambda)$ and $\nabla_q(\lambda)$ both have one-dimensional λ-weight spaces. The same fact implies that $\mathrm{End}_{\mathbf{U}_q}(L_q(\lambda)) \cong \mathbb{K}$ for all $\lambda \in X^+$, see [7, Corollary 7.4]. This last property fails for quasi-hereditary algebras in general when $\mathbb{K}$ is not algebraically closed.

Theorem 2.2. (**Ext-vanishing.**) We have for all $\lambda, \mu \in X^+$ that

$$\mathrm{Ext}^i_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \begin{cases} \mathbb{K}c^\lambda, & \text{if } i = 0 \text{ and } \lambda = \mu, \\ 0, & \text{else.} \end{cases}$$ □

We have to enlarge the category $\mathbf{U}_q$-**Mod** by non-necessarily finite-dimensional $\mathbf{U}_q$-modules to have enough injectives such that the $\mathrm{Ext}^i_{\mathbf{U}_q}$-functors make sense by using q-analogs arguments as in [48, Part I, Chapter 3]. However, $\mathbf{U}_q$-**Mod** has enough injectives in characteristic zero, see [1, Proposition 5.8] for a treatment of the non-semisimple cases.

Proof. Similar to the modular analog treated in [48, Proposition II.4.13] (a proof in our notation can be found in [8]). ■

2B. **Tilting modules and Ext-vanishing.** We say that a $\mathbf{U}_q$-module M has a Δ_q-*filtration* if there exists some $k \in \mathbb{Z}_{\geq 0}$ and a finite descending sequence of $\mathbf{U}_q$-submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_{k'} \supset \cdots \supset M_{k-1} \supset M_k = 0,$$

such that $M_{k'}/M_{k'+1} \cong \Delta_q(\lambda_{k'})$ for all $k' = 0, \ldots, k-1$ and some $\lambda_{k'} \in X^+$. A ∇_q-*filtration* is defined similarly, but using a finite ascending sequence of $\mathbf{U}_q$-submodules and $\nabla_q(\lambda)$'s instead of $\Delta_q(\lambda)$'s. We denote by $(M : \Delta_q(\lambda))$ and $(N : \nabla_q(\lambda))$ the corresponding multiplicities, which are well-defined by Corollary 2.3. Note that a $\mathbf{U}_q$-module M has a Δ_q-filtration if and only if its dual $\mathcal{D}(M)$ has a ∇_q-filtration.

A corollary of the Ext-vanishing theorem is the following, whose proof is left to the reader or can be found in [8]. (Note that the proof of Corollary 2.3 therein gives, in principle, a method to find and construct bases of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ and $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)$ respectively.)

Corollary 2.3. Let $M, N \in \mathbf{U}_q$-**Mod** and $\lambda \in X^+$. Assume that M has a Δ_q-filtration and N has a ∇_q-filtration. Then

$$\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))) = (M : \Delta_q(\lambda)) \quad \text{and} \quad \dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)) = (N : \nabla_q(\lambda)).$$

In particular, $(M : \Delta_q(\lambda))$ and $(N : \nabla_q(\lambda))$ are independent of the choice of filtrations. ■

Proposition 2.4. (**Donkin's Ext-criteria.**) The following are equivalent.

(a) An $M \in \mathbf{U}_q$-**Mod** has a Δ_q-filtration (respectively $N \in \mathbf{U}_q$-**Mod** has a ∇_q-filtration).
(b) We have $\mathrm{Ext}^i_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ (respectively $\mathrm{Ext}^i_{\mathbf{U}_q}(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$ and all $i > 0$.
(c) We have $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ (respectively $\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$. □

Proof. As in [48, Proposition II.4.16]. A proof in our notation can be found in [8]. ■

A $\mathbf{U}_q$-module T which has both, a Δ_q- and a ∇_q-filtration, is called a $\mathbf{U}_q$-*tilting module*. Following Donkin [29], we are now ready to define the *category of* $\mathbf{U}_q$-*tilting modules* that we denote by $\boldsymbol{\mathcal{T}}$. This category is our main object of study.

Definition 2.5. (**Category of $\mathbf{U}_q$-tilting modules.**) The category $\boldsymbol{\mathcal{T}}$ is the full subcategory of $\mathbf{U}_q$-**Mod** whose objects are given by all $\mathbf{U}_q$-tilting modules. ▲

From Proposition 2.4 we obtain directly an important statement.

Corollary 2.6. Let $T \in \mathbf{U}_q$-**Mod**. Then

$$T \in \boldsymbol{\mathcal{T}} \quad \text{if and only if} \quad \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) \quad \text{for all } \lambda \in X^+.$$

When $T \in \boldsymbol{\mathcal{T}}$, the corresponding higher Ext-groups vanish as well. ■

The indecomposable $\mathbf{U}_q$-modules in $\boldsymbol{\mathcal{T}}$, that we denote by $T_q(\lambda)$, are indexed by $\lambda \in X^+$. The $\mathbf{U}_q$-tilting module $T_q(\lambda)$ is determined by the property that it is indecomposable with λ as its unique maximal weight. In fact, $(T_q(\lambda) : \Delta_q(\lambda)) = 1$, and $(T_q(\lambda) : \Delta_q(\mu)) \neq 0$ only if $\mu \leq \lambda$. (Dually for ∇_q-filtrations.)

Note that the duality functor $\mathcal{D}$ from above restricts to $\boldsymbol{\mathcal{T}}$. Moreover, as a consequence of the classification of indecomposable $\mathbf{U}_q$-modules in $\boldsymbol{\mathcal{T}}$, we have $\mathcal{D}(T) \cong T$ for $T \in \boldsymbol{\mathcal{T}}$. In particular, we have for all $\lambda \in X^+$ that

$$(T : \Delta_q(\lambda)) = \dim(\mathrm{Hom}_{\mathbf{U}_q}(T, \nabla_q(\lambda))) = \dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)) = (T : \nabla_q(\lambda)).$$

It is known that $\boldsymbol{\mathcal{T}}$ is a Krull–Schmidt category, closed under finite direct sums, taking summands and finite tensor products (the latter is a non-trivial fact, see [68, Theorem 3.3]).

For a fixed $\lambda \in X^+$ we have $\mathbf{U}_q$-homomorphisms

$$\Delta_q(\lambda) \xhookrightarrow{\iota^\lambda} T_q(\lambda) \xrightarrow{\pi^\lambda} \!\!\!\!\rightarrow \nabla_q(\lambda),$$

where ι^λ is the inclusion of the first $\mathbf{U}_q$-submodule in a Δ_q-filtration of $T_q(\lambda)$ and π^λ is the surjection onto the last quotient in a ∇_q-filtration of $T_q(\lambda)$. Note that these are only defined up to scalars and we fix scalars in the following such that $\pi^\lambda \circ \iota^\lambda = c^\lambda$ (where c^λ is again the $\mathbf{U}_q$-homomorphism from (2)).

Remark 2. Let $T \in \boldsymbol{\mathcal{T}}$. An easy argument (based on Theorem 2.2) shows the following crucial fact:

(3) $$\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) \Rightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(\mathrm{coker}(\iota^\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \ker(\pi^\lambda))$$

for all $\lambda \in X^+$. Consequently, we see that any $\mathbf{U}_q$-homomorphism $g \colon \Delta_q(\lambda) \to T$ extends to a $\mathbf{U}_q$-homomorphism $\overline{g} \colon T_q(\lambda) \to T$ whereas any $\mathbf{U}_q$-homomorphism $f \colon T \to \nabla_q(\lambda)$ factors through $T_q(\lambda)$ via some $\overline{f} \colon T \to T_q(\lambda)$. ▲

Remark 3. In [8] it is described in detail how to compute $(T_q(\lambda) : \Delta_q(\mu))$ for $\lambda, \mu \in X^+$. This can be done algorithmically in case q is a complex, primitive l-th root of unity, i.e. one can use Soergel's version of the *affine parabolic Kazhdan–Lusztig polynomials*. For brevity, we do not recall the definition of these polynomials here, but refer to [84, Section 3] where the relevant polynomials are denoted $n_{y,x}$ (and where all the other relevant notions are defined). The main point for us is the following theorem due to Soergel [81, Theorem 5.12] (see also [84, Conjecture 7.1]): Suppose $\mathbb{K} = \mathbb{C}$ and q is a complex, primitive l-th root of unity. For each pair $\lambda, \mu \in X^+$ with λ being an l-regular $\mathbf{U}_q$-weight (that is, $T_q(\lambda)$ belongs to a regular block of $\boldsymbol{\mathcal{T}}$) we have (with $n_{\mu\lambda}$ equal to the relevant $n_{y,x}$)

$$(T_q(\lambda) : \Delta_q(\mu)) = n_{\mu\lambda}(1) = (T_q(\lambda) : \nabla_q(\mu)).$$

From this one obtains a method to find the indecomposable summands of $\mathbf{U}_q$-tilting modules with known characters (e.g. tensor products of minuscule representations). ▲

3. Cellular structures on endomorphism algebras

In this section we give our construction of cellular bases for endomorphism rings $\mathrm{End}_{\mathbf{U}_q}(T)$ of $\mathbf{U}_q$-tilting modules T and prove our main result, that is, Theorem 3.9.

The main tool is Theorem 3.1. The proof of the latter needs several ingredients which we establish in the form of separate lemmas collected in Section 3B.

3A. **The basis theorem.** As before, we consider the category $\mathbf{U}_q$-**Mod**. Moreover, we fix two $\mathbf{U}_q$-modules M, N, where we assume that M has a Δ_q-filtration and N has a ∇_q-filtration. Then, by Corollary 2.3, we have

$$\dim(\mathrm{Hom}_{\mathbf{U}_q}(M,N)) = \sum_{\lambda\in X^+}(M:\Delta_q(\lambda))(N:\nabla_q(\lambda)). \tag{4}$$

We point out that the sum in (4) is actually finite since $(M:\Delta_q(\lambda))\neq 0$ for only a finite number of $\lambda\in X^+$. (Dually, $(N:\nabla_q(\lambda))\neq 0$ for only finitely many $\lambda\in X^+$.)

Given $\lambda\in X^+$, we define for $(N:\nabla_q(\lambda))>0$ respectively for $(M:\Delta_q(\lambda))>0$ the two sets

$$\mathcal{I}^\lambda=\{1,\dots,(N:\nabla_q(\lambda))\}\quad\text{and}\quad\mathcal{J}^\lambda=\{1,\dots,(M:\Delta_q(\lambda))\}.$$

By convention, $\mathcal{I}^\lambda=\emptyset$ and $\mathcal{J}^\lambda=\emptyset$ if $(N:\nabla_q(\lambda))=0$ respectively if $(M:\Delta_q(\lambda))=0$.

We can fix a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M,\nabla_q(\lambda))$ indexed by $\mathcal{J}^\lambda$. We denote this fixed basis by $F^\lambda=\{f_j^\lambda\colon M\to\nabla_q(\lambda)\mid j\in\mathcal{J}^\lambda\}$. By Proposition 2.4 and (3), we see that all elements of F^λ factor through the $\mathbf{U}_q$-tilting module $T_q(\lambda)$, i.e. we have commuting diagrams

$$\begin{tikzcd} M \arrow[r, dotted, "\exists\,\overline{f}_j^\lambda"] \arrow[dr, "f_j^\lambda"'] & T_q(\lambda) \arrow[d, two heads, "\pi^\lambda"] \\ & \nabla_q(\lambda). \end{tikzcd}$$

We call $\overline{f}_j^\lambda$ a *lift* of f_j^λ. (Note that a lift $\overline{f}_j^\lambda$ is not unique.) Dually, we can choose a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda),N)$ as $G^\lambda=\{g_i^\lambda\colon\Delta_q(\lambda)\to N\mid i\in\mathcal{I}^\lambda\}$, which extends to give (a non-unique) lift $\overline{g}_i^\lambda\colon T_q(\lambda)\to N$ such that $\overline{g}_i^\lambda\circ\iota^\lambda=g_i^\lambda$ for all $i\in\mathcal{I}^\lambda$.

We can use this setup to define a basis for $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$ which, when $M=N$, turns out to be a cellular basis, see Theorem 3.9. For each $\lambda\in X^+$ and all $i\in\mathcal{I}^\lambda, j\in\mathcal{J}^\lambda$ set

$$c_{ij}^\lambda=\overline{g}_i^\lambda\circ\overline{f}_j^\lambda\in\mathrm{Hom}_{\mathbf{U}_q}(M,N).$$

Our main result here is now the following.

Theorem 3.1. (**Basis theorem.**) For any choice of F^λ and G^λ as above and any choice of lifts of the f_j^λ's and the g_i^λ's (for all $\lambda\in X^+$), the set

$$GF=\{c_{ij}^\lambda\mid\lambda\in X^+,\ i\in\mathcal{I}^\lambda,j\in\mathcal{J}^\lambda\}$$

is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$. □

Proof. This follows from Proposition 3.3 combined with Lemma 3.6 and Lemma 3.7. ■

The basis GF for $\mathrm{Hom}_{\mathbf{U}_q}(M, N)$ can be illustrated in a commuting diagram as

$$\begin{tikzcd}
 & \Delta_q(\lambda) \arrow[d, hook, "\iota^\lambda"'] \arrow[dr, "g_i^\lambda"] & \\
M \arrow[r, dotted, "\overline{f}_j^\lambda"] \arrow[dr, "f_j^\lambda"'] & T_q(\lambda) \arrow[r, dotted, "\overline{g}_i^\lambda"'] \arrow[d, two heads, "\pi^\lambda"] & N \\
 & \nabla_q(\lambda) &
\end{tikzcd} .$$

Since $\mathbf{U}_q$-tilting modules have both a Δ_q- and a ∇_q-filtration, we get as an immediate consequence a key result for our purposes.

Corollary 3.2. Let $T \in \boldsymbol{\mathcal{T}}$. Then GF is, for any choices involved, a basis of $\mathrm{End}_{\mathbf{U}_q}(T)$. ■

3B. **Proof of the basis theorem.** We first show that, given lifts $\overline{f}_j^\lambda$, there is a consistent choice of lifts $\overline{g}_i^\lambda$ such that GF is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, N)$.

Proposition 3.3. (**Basis theorem — dependent version.**) For any choice of F^λ and any choice of lifts of the f_j^λ's (for all $\lambda \in X^+$) there exist a choice of a basis G^λ and a choice of lifts of the g_i^λ's such that $GF = \{c_{ij}^\lambda \mid \lambda \in X^+,\ i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}$ is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, N)$. □

The corresponding statement with the roles of f's and g's swapped clearly holds as well.

Proof. We will construct GF inductively. For this purpose, let

$$0 = N_0 \subset N_1 \subset \cdots \subset N_{k-1} \subset N_k = N$$

be a ∇_q-filtration of N, i.e. $N_{k'+1}/N_{k'} \cong \nabla_q(\lambda_{k'})$ for some $\lambda_{k'} \in X^+$ and all $k' = 0, \dots, k-1$.

Let $k = 1$ and $\lambda_1 = \lambda$. Then $N_1 = \nabla_q(\lambda)$ and $\{c^\lambda \colon \Delta_q(\lambda) \to \nabla_q(\lambda)\}$ gives a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\lambda))$, where c^λ is again the $\mathbf{U}_q$-homomorphism chosen in (2). Set $g_1^\lambda = c^\lambda$ and observe that $\overline{g}_1^\lambda = \pi^\lambda$ satisfies $\overline{g}_1^\lambda \circ \iota^\lambda = g_1^\lambda$. Thus, we have a basis and a corresponding lift. This clearly gives a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, N_1)$, since, by assumption, we have that F^λ gives a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ and $\pi^\lambda \circ \overline{f}_j^\lambda = f_j^\lambda$.

Hence, it remains to consider the case $k > 1$. Set $\lambda_k = \lambda$ and observe that we have a short exact sequence of the form

$$0 \longrightarrow N_{k-1} \xhookrightarrow{\mathrm{inc}} N_k \xrightarrow{\mathrm{pro}} \nabla_q(\lambda) \longrightarrow 0. \tag{5}$$

By Theorem 2.2 (and the usual implication as in (3)) this leads to a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{U}_q}(M, N_{k-1}) \xhookrightarrow{\mathrm{inc}_*} \mathrm{Hom}_{\mathbf{U}_q}(M, N_k) \xrightarrow{\mathrm{pro}_*} \mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda)) \longrightarrow 0. \tag{6}$$

By induction, we get from (6) for all $\mu \in X^+$ a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_{k-1})$ consisting of g_i^μ's with lifts $\overline{g}_i^\mu$ such that

$$\{c_{ij}^\mu = \overline{g}_i^\mu \circ \overline{f}_j^\mu \mid \mu \in X^+,\ i \in \mathcal{I}_{k-1}^\mu, j \in \mathcal{J}^\mu\} \tag{7}$$

is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, N_{k-1})$ (here we use $\mathcal{I}_{k-1}^\mu = \{1, \dots, (N_{k-1} : \nabla_q(\mu))\}$). We define $g_i^\mu(N_k) = \mathrm{inc} \circ g_i^\mu$ and $\overline{g}_i^\mu(N_k) = \mathrm{inc} \circ \overline{g}_i^\mu$ for each $\mu \in X^+$ and each $i \in \mathcal{I}_{k-1}^\mu$.

We now have to consider two cases, namely $\lambda \neq \mu$ and $\lambda = \mu$. In the first case we see that $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), \nabla_q(\lambda)) = 0$, so that, by using (5) and the usual implication from (3),

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_{k-1}) \cong \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_k).$$

Thus, our basis from (7) gives a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_k)$ and also gives the corresponding lifts. On the other hand, if $\lambda = \mu$, then

$$(N_k : \nabla_q(\lambda)) = (N_{k-1} : \nabla_q(\lambda)) + 1.$$

By Theorem 2.2 (and the corresponding implication as in (3)), we can choose $g^\lambda \colon \Delta_q(\lambda) \to N_k$ such that $\mathrm{pro} \circ g^\lambda = c^\lambda$. Then any choice of a lift $\overline{g}^\lambda$ of g^λ will satisfy $\mathrm{pro} \circ \overline{g}^\lambda = \pi^\lambda$.

Adjoining g^λ to the basis from (7) gives a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N_k)$ which satisfies the lifting property. Note that we know from the case $k = 1$ that

$$\{\mathrm{pro} \circ \overline{g}^\lambda \circ \overline{f}^\lambda_j = \pi^\lambda \circ \overline{f}^\lambda_j \mid j \in \mathcal{J}^\lambda\}$$

is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$. Combining everything: we have that

$$\{c^\lambda_{ij} = \overline{g}^\lambda_i(N_k) \circ \overline{f}^\lambda_j \mid \lambda \in X^+,\ i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}$$

is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, N_k)$ (by enumerating $\overline{g}^\lambda_{(N:\nabla_q(\lambda))}(N_k) = \overline{g}^\lambda$ in the $\lambda = \mu$ case). ■

We assume in the following that we have fixed some choices as in Proposition 3.3.

Let $\lambda \in X^+$. Given $\varphi \in \mathrm{Hom}_{\mathbf{U}_q}(M, N)$, we denote by $\varphi_\lambda \in \mathrm{Hom}_{\mathbf{U}^0_q}(M_\lambda, N_\lambda)$ the induced $\mathbf{U}^0_q$-homomorphism (that is, $\mathbb{K}$-linear maps) between the λ-weight spaces M_λ and N_λ. In addition, we denote by $\mathrm{Hom}_{\mathbb{K}}(M_\lambda, N_\lambda)$ the $\mathbb{K}$-linear maps between these λ-weight spaces.

Lemma 3.4. For any $\lambda \in X^+$ the induced set $\{(c^\lambda_{ij})_\lambda \mid c^\lambda_{ij} \in GF\}$ is a linearly independent subset of $\mathrm{Hom}_{\mathbb{K}}(M_\lambda, N_\lambda)$. □

Proof. We proceed as in the proof of Proposition 3.3.

If $N = \nabla_q(\lambda)$ (this was $k = 1$ above), then $c^\lambda_{1j} = \pi^\lambda \circ \overline{f}^\lambda_j = f^\lambda_j$ and the c^λ_{1j}'s form a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$. By the q-Frobenius reciprocity from [7, Proposition 1.17] we have

$$\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda)) \cong \mathrm{Hom}_{\mathbf{U}^-_q\mathbf{U}^0_q}(M, \mathbb{K}_\lambda) \subset \mathrm{Hom}_{\mathbf{U}^0_q}(M, \mathbb{K}_\lambda) = \mathrm{Hom}_{\mathbb{K}}(M_\lambda, \mathbb{K}).$$

Hence, because $N_\lambda = \mathbb{K}$ in this case, we have the base of the induction.

Assume now $k > 1$. The construction of $\{c^\mu_{ij}(N_k)\}_{\mu,i,j}$ in the proof of Proposition 3.3 shows that this set consists of two separate parts: one being the basis from (7) coming from a basis for $\mathrm{Hom}_{\mathbf{U}_q}(M, N_{k-1})$ and the second part (which only occurs when $\lambda = \mu$) coming from a basis from Hom $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N_k)$.

By (6) there is a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{K}}(M_\lambda, (N_{k-1})_\lambda) \xhookrightarrow{\mathrm{inc}_*} \mathrm{Hom}_{\mathbb{K}}(M_\lambda, (N_k)_\lambda) \xrightarrow{\mathrm{pro}_*} \mathrm{Hom}_{\mathbb{K}}(M_\lambda, \mathbb{K}) \longrightarrow 0.$$

Thus, we can proceed as in the proof of Proposition 3.3. ■

We need another piece of notation: we define for each $\lambda \in X^+$

$$\mathrm{Hom}_{\mathbf{U}_q}(M, N)^{\leq\lambda} = \{\varphi \in \mathrm{Hom}_{\mathbf{U}_q}(M, N) \mid \varphi_\mu = 0 \text{ unless } \mu \leq \lambda\}.$$

In words: a $\mathbf{U}_q$-homomorphism $\varphi \in \mathrm{Hom}_{\mathbf{U}_q}(M,N)$ belongs to $\mathrm{Hom}_{\mathbf{U}_q}(M,N)^{\leq\lambda}$ if and only if φ vanishes on all $\mathbf{U}_q$-weight spaces M_μ with $\mu \not\leq \lambda$. In addition to the notation above, we use the evident notation $\mathrm{Hom}_{\mathbf{U}_q}(M,N)^{<\lambda}$. We arrive at the following.

Lemma 3.5. For any fixed $\lambda \in X^+$ the sets

$$\{c_{ij}^\mu \mid c_{ij}^\mu \in GF,\ \mu \leq \lambda\} \quad \text{and} \quad \{c_{ij}^\mu \mid c_{ij}^\mu \in GF,\ \mu < \lambda\}$$

are bases of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)^{\leq\lambda}$ and $\mathrm{Hom}_{\mathbf{U}_q}(M,N)^{<\lambda}$ respectively. □

Proof. As c_{ij}^μ factors through $T_q(\mu)$ and $T_q(\mu)_\nu = 0$ unless $\nu \leq \mu$ (which follows using the classification of indecomposable $\mathbf{U}_q$-tilting modules), we see that $(c_{ij}^\mu)_\nu = 0$ unless $\nu \leq \mu$. Moreover, by Lemma 3.4, each $(c_{ij}^\mu)_\mu$ is non-zero. Thus, $c_{ij}^\mu \in \mathrm{Hom}_{\mathbf{U}_q}(M,N)^{\leq\lambda}$ if and only if $\mu \leq \lambda$. Now choose any $\varphi \in \mathrm{Hom}_{\mathbf{U}_q}(M,N)^{\leq\lambda}$. By Proposition 3.3 we may write

$$\varphi = \sum_{\mu,i,j} a_{ij}^\mu c_{ij}^\mu, \quad a_{ij}^\mu \in \mathbb{K}. \tag{8}$$

Choose $\mu \in X^+$ maximal with the property that there exist $i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda$ such that $a_{ij}^\mu \neq 0$.

We claim that $a_{ij}^\nu (c_{ij}^\nu)_\mu = 0$ whenever $\nu \neq \mu$. This is true because, as observed above, $(c_{ij}^\nu)_\mu = 0$ unless $\mu \leq \nu$, and for $\mu < \nu$ we have $a_{ij}^\nu = 0$ by the maximality of μ. We conclude $\varphi_\mu = \sum_{i,j} a_{ij}^\mu (c_{ij}^\mu)_\mu$ and thus, $\varphi_\mu \neq 0$ by Lemma 3.4. Hence, $\mu \leq \lambda$, which gives by (8) that $\varphi \in \mathrm{span}_{\mathbb{K}}\{c_{ij}^\mu \mid c_{ij}^\mu \in GF,\ \mu \leq \lambda\}$ as desired. This shows that $\{c_{ij}^\mu \mid c_{ij}^\mu \in GF,\ \mu \leq \lambda\}$ spans $\mathrm{Hom}_{\mathbf{U}_q}(M,N)^{\leq\lambda}$. Since it is clearly a linearly independent set, it is a basis.

The second statement follows analogously, so the details are omitted. ■

We need the following two lemmas to prove that all choices in Proposition 3.3 lead to bases of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$. As before we assume that we have, as in Proposition 3.3, constructed $\{g_i^\lambda, i \in \mathcal{I}^\lambda\}$ and the corresponding lifts $\overline{g}_i^\lambda$ for all $\lambda \in X^+$.

Lemma 3.6. Suppose that we have other $\mathbf{U}_q$-homomorphisms $\tilde{g}_i^\lambda \colon T_q(\lambda) \to N$ such that $\tilde{g}_i^\lambda \circ \iota^\lambda = g_i^\lambda$. Then the following set is also a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$:

$$\{\tilde{c}_{ij}^\lambda = \tilde{g}_i^\lambda \circ \overline{f}_j^\lambda \mid \lambda \in X^+,\ i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}.$$ □

Proof. As $(\overline{g}_i^\lambda - \tilde{g}_i^\lambda) \circ \iota^\lambda = 0$, we see that $\overline{g}_i^\lambda - \tilde{g}_i^\lambda \in \mathrm{Hom}_{\mathbf{U}_q}(T_q(\lambda),N)^{<\lambda}$. Hence, we have $c_{ij}^\lambda - \tilde{c}_{ij}^\lambda \in \mathrm{Hom}_{\mathbf{U}_q}(M,N)^{<\lambda}$. Thus, by Lemma 3.5, there is a unitriangular change-of-basis matrix between $\{c_{ij}^\lambda\}_{\lambda,i,j}$ and $\{\tilde{c}_{ij}^\lambda\}_{\lambda,i,j}$. ■

Now assume that we have chosen another basis $\{h_i^\lambda \mid i \in \mathcal{I}^\lambda\}$ of the spaces $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda),N)$ for each $\lambda \in X^+$ and the corresponding lifts $\overline{h}_i^\lambda$ as well.

Lemma 3.7. The following set is also a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$:

$$\{d_{ij}^\lambda = \overline{h}_i^\lambda \circ \overline{f}_j^\lambda \mid \lambda \in X^+,\ i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}.$$ □

Proof. Write $g_i^\lambda = \sum_{k=1}^{(N:\nabla_q(\lambda))} b_{ik}^\lambda h_k^\lambda$ with $b_{ik}^\lambda \in \mathbb{K}$ and set $\tilde{g}_i^\lambda = \sum_{k=1}^{(N:\nabla_q(\lambda))} b_{ik}^\lambda \overline{h}_k^\lambda$. Then the $\tilde{g}_i^\lambda$'s are lifts of the g_i^λ's. Hence, by Lemma 3.6, the elements $\tilde{g}_i^\lambda \circ \overline{f}_j^\lambda$ form a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$.

Thus, this proves the lemma, since, by construction, $\{d_{ij}^{\lambda}\}_{\lambda,i,j}$ is related to this basis by the invertible change-of-basis matrix $(b_{ik}^{\lambda})_{i,k=1;\lambda\in X^+}^{(N:\nabla_q(\lambda))}$. ■

In total, we established Proposition 3.3.

3C. **Cellular structures on endomorphism algebras of $\mathbf{U}_q$-tilting modules.** This subsection finally contains the statement and proof of our main theorem. We keep on working over a field $\mathbb{K}$ instead of a ring as for example Graham and Lehrer [38] do. (This avoids technicalities, e.g. the theory of indecomposable $\mathbf{U}_q$-tilting modules over rings is much more subtle than over fields. See e.g. [29, Remark 1.7].)

Definition 3.8. (**Cellular algebras.**) Suppose A is a finite-dimensional $\mathbb{K}$-algebra. A *cell datum* is an ordered quadruple $(\mathcal{P},\mathcal{I},\mathcal{C},\mathrm{i})$, where $(\mathcal{P},\leq)$ is a finite poset, $\mathcal{I}^{\lambda}$ is a finite set for all $\lambda\in\mathcal{P}$, i is a $\mathbb{K}$-linear anti-involution of A and $\mathcal{C}$ is an injection

$$\mathcal{C}\colon \coprod_{\lambda\in\mathcal{P}}\mathcal{I}^{\lambda}\times\mathcal{I}^{\lambda}\to A,\ (i,j)\mapsto c_{ij}^{\lambda}.$$

The whole data should be such that the c_{ij}^{λ}'s form a basis of A with $\mathrm{i}(c_{ij}^{\lambda})=c_{ji}^{\lambda}$ for all $\lambda\in\mathcal{P}$ and all $i,j\in\mathcal{I}^{\lambda}$. Moreover, for all $a\in A$ and all $\lambda\in\mathcal{P}$ we have

$$ac_{ij}^{\lambda}=\sum_{k\in\mathcal{I}^{\lambda}}r_{ik}(a)c_{kj}^{\lambda}\ (\mathrm{mod}\ A^{<\lambda})\quad\text{for all } i,j\in\mathcal{I}^{\lambda}. \tag{9}$$

Here $A^{<\lambda}$ is the subspace of A spanned by the set $\{c_{ij}^{\mu}\mid\mu<\lambda\text{ and } i,j\in\mathcal{I}(\mu)\}$ and the scalars $r_{ik}(a)\in\mathbb{K}$ are supposed to be independent of j.

An algebra A with such a quadruple is called a *cellular algebra* and the c_{ij}^{λ} are called a *cellular basis* of A (with respect to the $\mathbb{K}$-linear anti-involution i). ▲

Let us fix $T\in\boldsymbol{\mathcal{T}}$ in the following. We will now construct cellular bases of $\mathrm{End}_{\mathbf{U}_q}(T)$ in the semisimple as well as in the non-semisimple case.

To this end, we need to specify the cell datum. Set

$$(\mathcal{P},\leq)=(\{\lambda\in X^+\mid(T:\nabla_q(\lambda))=(T:\Delta_q(\lambda))\neq 0\},\leq),$$

where $\leq$ is the usual partial ordering on X^+, see at the beginning of Section 2A. Note that $\mathcal{P}$ is finite since T is finite-dimensional. Moreover, motivated by Theorem 3.1, for each $\lambda\in\mathcal{P}$ define $\mathcal{I}^{\lambda}=\{1,\dots,(T:\nabla_q(\lambda))\}=\{1,\dots,(T:\Delta_q(\lambda))\}=\mathcal{J}^{\lambda}$.

Recalling that we write $\mathrm{i}(\cdot)=\mathcal{D}(\cdot)$ (for $\mathcal{D}$ being the duality functor from Section 2A that exchanges Weyl and dual Weyl $\mathbf{U}_q$-modules and fixes all $\mathbf{U}_q$-tilting modules), the assignment $\mathrm{i}\colon\mathrm{End}_{\mathbf{U}_q}(T)\to\mathrm{End}_{\mathbf{U}_q}(T),\phi\mapsto\mathcal{D}(\phi)$ is clearly a $\mathbb{K}$-linear anti-involution. Choose any basis G^{λ} of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda),T)$ as above and any lifts $\overline{g}_i^{\lambda}$. Then $\mathrm{i}(G^{\lambda})$ is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(T,\nabla_q(\lambda))$ and $\mathrm{i}(\overline{g}_i^{\lambda})$ is a lift of $\mathrm{i}(g_i^{\lambda})$. By Corollary 3.2 we see that

$$\{c_{ij}^{\lambda}=\overline{g}_i^{\lambda}\circ\mathrm{i}(\overline{g}_j^{\lambda})=\overline{g}_i^{\lambda}\circ\overline{f}_j^{\lambda}\mid\lambda\in\mathcal{P},\ i,j\in\mathcal{I}^{\lambda}\}$$

is a basis of $\mathrm{End}_{\mathbf{U}_q}(T)$. Finally let $\mathcal{C}\colon\mathcal{I}^{\lambda}\times\mathcal{I}^{\lambda}\to\mathrm{End}_{\mathbf{U}_q}(T)$ be given by $(i,j)\mapsto c_{ij}^{\lambda}$.

Now we are ready to state and prove our main theorem.

Theorem 3.9. (**A cellular basis for** $\mathrm{End}_{\mathbf{U}_q}(T)$.) The quadruple $(\mathcal{P},\mathcal{I},\mathcal{C},\mathrm{i})$ defined above is a cell datum for $\mathrm{End}_{\mathbf{U}_q}(T)$. □

Proof. As mentioned above, the sets $\mathcal{P}$ and $\mathcal{I}^\lambda$ are finite for all $\lambda \in \mathcal{P}$. Moreover, i is a $\mathbb{K}$-linear anti-involution of $\mathrm{End}_{\mathbf{U}_q}(T)$ and the c_{ij}^λ's form a basis of $\mathrm{End}_{\mathbf{U}_q}(T)$ by Corollary 3.2. Because the functor $\mathcal{D}(\cdot)$ is contravariant, we see that

$$\mathrm{i}(c_{ij}^\lambda) = \mathrm{i}(\overline{g}_i^\lambda \circ \mathrm{i}(\overline{g}_j^\lambda)) = \overline{g}_j^\lambda \circ \mathrm{i}(\overline{g}_i^\lambda) = c_{ji}^\lambda.$$

Thus, only the condition (9) remains to be proven. For this purpose, let $\varphi \in \mathrm{End}_{\mathbf{U}_q}(T)$. Since $\varphi \circ \overline{g}_i^\lambda \circ \iota^\lambda = \varphi \circ g_i^\lambda \in \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$, we have coefficients $r_{ik}^\lambda(\varphi) \in \mathbb{K}$ such that

$$\varphi \circ g_i^\lambda = \sum_{k \in \mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) g_k^\lambda, \tag{10}$$

because we know that the g_i^λ's form a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$. But this implies then that $\varphi \circ \overline{g}_i^\lambda - \sum_{k \in \mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) \overline{g}_k^\lambda \in \mathrm{Hom}_{\mathbf{U}_q}(T_q(\lambda), T)^{<\lambda}$, so that

$$\varphi \circ \overline{g}_i^\lambda \circ \overline{f}_j^\lambda - \sum_{k \in \mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) \overline{g}_k^\lambda \circ \overline{f}_j^\lambda \in \mathrm{Hom}_{\mathbf{U}_q}(T, T)^{<\lambda} = \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda},$$

which proves (9). The theorem follows. ■

4. The cellular structure and $\mathrm{End}_{\mathbf{U}_q}(T)$-Mod

The goal of this section is to present the representation theory of cellular algebras for $\mathrm{End}_{\mathbf{U}_q}(T)$ from the viewpoint of $\mathbf{U}_q$-tilting theory. In fact, most of the results in this section are not new and have been proved for general cellular algebras, see e.g. [38, Section 3]. However, they take a nice and easy form in our setup. The last theorem, the semisimplicity criterion from Theorem 4.13, is new and has potentially many applications, see e.g. [9].

4A. Cell modules for $\mathrm{End}_{\mathbf{U}_q}(T)$.

We study now the representation theory for $\mathrm{End}_{\mathbf{U}_q}(T)$ via the cellular structure we have found for it. We denote its module category by $\mathrm{End}_{\mathbf{U}_q}(T)$-**Mod**.

Definition 4.1. (**Cell modules.**) Let $\lambda \in \mathcal{P}$. The *cell module* associated to λ is the left $\mathrm{End}_{\mathbf{U}_q}(T)$-module given by $C(\lambda) = \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$. The right $\mathrm{End}_{\mathbf{U}_q}(T)$-module given by $C(\lambda)^* = \mathrm{Hom}_{\mathbf{U}_q}(T, \nabla_q(\lambda))$ is called the *dual cell module* associated to λ. ▲

The link to the definition of cell modules from [38, Definition 2.1] is given via our choice of basis $\{g_i^\lambda\}_{i \in \mathcal{I}^\lambda}$. In this basis the action of $\mathrm{End}_{\mathbf{U}_q}(T)$ on $C(\lambda)$ is given by

$$\varphi \circ g_i^\lambda = \sum_{k \in \mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) g_k^\lambda, \quad \varphi \in \mathrm{End}_{\mathbf{U}_q}(T), \tag{11}$$

see (10). Here the coefficients are the same as those appearing when we consider the left action of $\mathrm{End}_{\mathbf{U}_q}(T)$ on itself in terms of the cellular basis $\{c_{ij}^\lambda\}_{i,j \in \mathcal{I}^\lambda}^{\lambda \in \mathcal{P}}$, that is,

$$\varphi \circ c_{ij}^\lambda = \sum_{k \in \mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) c_{kj}^\lambda \ (\mathrm{mod}\ \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}), \quad \varphi \in \mathrm{End}_{\mathbf{U}_q}(T). \tag{12}$$

In a completely similar fashion: the dual cell module $C(\lambda)^*$ has a basis consisting of $\{f_j^\lambda\}_{j \in \mathcal{I}^\lambda}$ with $f_j^\lambda = \mathrm{i}(g_j^\lambda)$. In this basis the right action of $\mathrm{End}_{\mathbf{U}_q}(T)$ is given via

$$f_j^\lambda \circ \varphi = \sum_{k \in \mathcal{I}^\lambda} r_{kj}^\lambda(\mathrm{i}(\varphi)) f_k^\lambda, \quad \varphi \in \mathrm{End}_{\mathbf{U}_q}(T). \tag{13}$$

We can use the unique $\mathbf{U}_q$-homomorphism from (2) and the duality functor $\mathcal{D}(\cdot)$ to define the following *cellular pairing* in the spirit of Graham and Lehrer [38, Definition 2.3].

Definition 4.2. (**Cellular pairing.**) Let $\lambda \in \mathcal{P}$. Then we denote by ϑ^λ the $\mathbb{K}$-bilinear form $\vartheta^\lambda \colon C(\lambda) \otimes C(\lambda) \to \mathbb{K}$ determined by the property

$$\mathrm{i}(h) \circ g = \vartheta^\lambda(g,h)c^\lambda, \quad g,h \in C(\lambda) = \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T).$$

We call ϑ^λ the *cellular pairing* associated to $\lambda \in \mathcal{P}$. ▲

Lemma 4.3. The cellular pairing ϑ^λ is well-defined, symmetric and contravariant. □

Proof. That ϑ^λ is well-defined follows directly from the uniqueness of c^λ.

Applying i to the defining equation of ϑ^λ gives

$$\vartheta^\lambda(g,h)\mathrm{i}(c^\lambda) = \mathrm{i}(\vartheta^\lambda(g,h)c^\lambda) = \mathrm{i}(\mathrm{i}(h) \circ g) = \mathrm{i}(g) \circ h = \vartheta^\lambda(h,g)c^\lambda,$$

and thus, $\vartheta^\lambda(g,h) = \vartheta^\lambda(h,g)$, because $c^\lambda = \mathrm{i}(c^\lambda)$. (Recall that $c^\lambda \colon \Delta_q(\lambda) \to \nabla_q(\lambda)$ is unique up to scalars. Hence, we can fix scalars accordingly such that $c^\lambda = \mathrm{i}(c^\lambda)$.) Similarly, contravariance of $\mathcal{D}(\cdot)$ gives

$$\vartheta^\lambda(\varphi \circ g, h) = \vartheta^\lambda(g, \mathrm{i}(\varphi) \circ h), \quad \varphi \in \mathrm{End}_{\mathbf{U}_q}(T),\ g,h \in C(\lambda),$$

which shows contravariance of the cellular pairing. ■

Proposition 4.4. Let $\lambda \in \mathcal{P}$. Then $T_q(\lambda)$ is a summand of T if and only if $\vartheta^\lambda \neq 0$. □

Proof. (See also [2, Proposition 1.5].) Assume $T \cong T_q(\lambda) \oplus \mathrm{rest}$. We denote by $\overline{g} \colon T_q(\lambda) \to T$ and by $\overline{f} \colon T \to T_q(\lambda)$ the corresponding inclusion and projection respectively. As usual, set $g = \overline{g} \circ \iota^\lambda$ and $f = \pi^\lambda \circ \overline{f}$. Then we have $f \circ g \colon \Delta_q(\lambda) \hookrightarrow T_q(\lambda) \hookrightarrow T \twoheadrightarrow T_q(\lambda) \twoheadrightarrow \nabla_q(\lambda) = c^\lambda$ (mapping head to socle), giving $\vartheta^\lambda(g, \mathrm{i}(f)) = 1$. This shows that $\vartheta^\lambda \neq 0$.

Conversely, assume that there exist $g,h \in C(\lambda)$ with $\vartheta^\lambda(g,h) \neq 0$. Then the commuting "bow tie diagram", i.e.

$$\begin{tikzcd}
\Delta_q(\lambda) \arrow[d, hook, "\iota^\lambda"'] \arrow[dr, "g"] & & \\
T_q(\lambda) \arrow[r, dotted, "\overline{g}"'] & T \arrow[r, dotted, "\overline{\mathrm{i}(h)}"] \arrow[dr, "\mathrm{i}(h)"'] & T_q(\lambda), \arrow[d, two heads, "\pi^\lambda"] \\
& & \nabla_q(\lambda)
\end{tikzcd}$$

shows that $\overline{\mathrm{i}(h)} \circ \overline{g}$ is non-zero on the λ-weight space of $T_q(\lambda)$, because $\mathrm{i}(h) \circ g = \vartheta^\lambda(g,h)c^\lambda$. Thus, $\overline{\mathrm{i}(h)} \circ \overline{g}$ must be an isomorphism (because $T_q(\lambda)$ is indecomposable and has therefore only invertible or nilpotent elements in $\mathrm{End}_{\mathbf{U}_q}(T_q(\lambda))$) showing that $T \cong T_q(\lambda) \oplus \mathrm{rest}$. ■

In view of Proposition 4.4, it makes sense to study the set

$$\mathcal{P}_0 = \{\lambda \in \mathcal{P} \mid \vartheta^\lambda \neq 0\} \subset \mathcal{P}. \tag{14}$$

Hence, if $\lambda \in \mathcal{P}_0$, then we have $T \cong T_q(\lambda) \oplus \mathrm{rest}$ for some $\mathbf{U}_q$-tilting module called rest. Note also that $\mathrm{End}_{\mathbf{U}_q}(T)$ is quasi-hereditary if and only if $\mathcal{P} = \mathcal{P}_0$, see e.g. [38, Remark 3.10].

4B. **The structure of $\mathrm{End}_{\mathbf{U}_q}(T)$ and its cell modules.** Recall that, for any $\lambda \in \mathcal{P}$, we have that $\mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda}$ and $\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}$ are two-sided ideals in $\mathrm{End}_{\mathbf{U}_q}(T)$ (this follows from (9) and its right-handed version obtained by applying i), as in any cellular algebra. In our case we can also see this as follows. If $\varphi \in \mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda}$, then $\varphi_\mu = 0$ unless $\mu \leq \lambda$. Hence, for any $\varphi, \psi \in \mathrm{End}_{\mathbf{U}_q}(T)$ we have $(\varphi \circ \psi)_\mu = \varphi_\mu \circ \psi_\mu = 0 = \psi_\mu \circ \varphi_\mu = (\psi \circ \varphi)_\mu$ unless $\mu \leq \lambda$. As a consequence, $\mathrm{End}_{\mathbf{U}_q}(T)^{\lambda} = \mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda}/\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}$ is an $\mathrm{End}_{\mathbf{U}_q}(T)$-bimodule.

Recall that, for any $g \in C(\lambda)$ and any $f \in C(\lambda)^*$, we denote by $\overline{g}\colon T_q(\lambda) \to T$ and $\overline{f}\colon T \to T_q(\lambda)$ a choice of lifts which satisfy $\overline{g} \circ \iota^\lambda = g$ and $\pi^\lambda \circ \overline{f} = f$, respectively.

Lemma 4.5. Let $\lambda \in \mathcal{P}$. Then the pairing map

$$\langle \cdot, \cdot \rangle^\lambda \colon C(\lambda) \otimes C(\lambda)^* \to \mathrm{End}_{\mathbf{U}_q}(T)^{\lambda}, \quad \langle g, f \rangle^\lambda = \overline{g} \circ \overline{f} + \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda},$$

with $g \in C(\lambda)$, $f \in C(\lambda)^*$, is an isomorphism of $\mathrm{End}_{\mathbf{U}_q}(T)$-bimodules. □

Proof. First we note that $\overline{g} \circ \overline{f} + \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}$ does not depend on the choices for the lifts $\overline{f}, \overline{g}$, because the change-of-basis matrix from Lemma 3.6 is unitriangular (and works for swapped roles of f's and g's as well). This makes the pairing well-defined.

Note that the pairing $\langle \cdot, \cdot \rangle^\lambda$ takes, by birth, the basis $(g_i^\lambda \otimes f_j^\lambda)_{i,j \in \mathcal{I}^\lambda}$ of $C(\lambda) \otimes C(\lambda)^*$ to the basis $\{c_{ij}^\lambda + \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}\}_{i,j \in \mathcal{I}^\lambda}$ of $\mathrm{End}_{\mathbf{U}_q}(T)^{\lambda}$ (where the latter is a basis by Lemma 3.5).

So we only need to check that $\langle \varphi \circ g_i^\lambda, f_j^\lambda \circ \psi \rangle^\lambda = \varphi \circ c_{ij}^\lambda \circ \psi \pmod{\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}}$ for any $\varphi, \psi \in \mathrm{End}_{\mathbf{U}_q}(T)$. But this is a direct consequence of (11), (12) and (13). ■

The next lemma is straightforward by Lemma 4.5. Details are left to the reader.

Lemma 4.6. We have the following.

(a) There is an isomorphism of $\mathbb{K}$-vector spaces $\mathrm{End}_{\mathbf{U}_q}(T) \cong \bigoplus_{\lambda \in \mathcal{P}} \mathrm{End}_{\mathbf{U}_q}(T)^{\lambda}$.
(b) If $\varphi \in \mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda}$, then we have $r_{ik}^\mu(\varphi) = 0$ for all $\mu \not\leq \lambda, i, k \in \mathcal{I}(\mu)$. Equivalently, $\mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda} C(\mu) = 0$ unless $\mu \leq \lambda$. ■

In the following we assume that $\lambda \in \mathcal{P}_0$ as in (14). Define m_λ via

$$T \cong T_q(\lambda)^{\oplus m_\lambda} \oplus T', \tag{15}$$

where T' is a $\mathbf{U}_q$-tilting module containing no summands isomorphic to $T_q(\lambda)$.

Choose now a basis of $C(\lambda) = \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$ as follows. For $i = 1, \ldots, m_\lambda$ let $\overline{g}_i^\lambda$ be the inclusion of $T_q(\lambda)$ into the i-th summand of $T_q(\lambda)^{\oplus m_\lambda}$ and set $g_i^\lambda = \overline{g}_i^\lambda \circ \iota^\lambda$. Then extend $\{g_1^\lambda, \ldots, g_{m_\lambda}^\lambda\}$ to a basis of the cell module $C(\lambda)$ by adding an arbitrary basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T')$. Thus, in our usual notation, we have $c_{ij}^\lambda = \overline{g}_i^\lambda \circ \overline{f}_j^\lambda$ with $\overline{f}_j^\lambda = \mathrm{i}(\overline{g}_j^\lambda)$.

In particular, $\overline{f}_j^\lambda$ projects onto the j-th summand in $T_q(\lambda)^{\oplus m_\lambda}$ for $j = 1, \ldots, m_\lambda$. Thus, the c_{ii}^λ's for $i \leq m_\lambda$ are idempotents in $\mathrm{End}_{\mathbf{U}_q}(T)$ corresponding to the i-th summand in $T_q(\lambda)^{\oplus m_\lambda}$. Since $\lambda \in \mathcal{P}_0$ (which implies $1 \leq m_\lambda$), c_{11}^λ is always such an idempotent. This is crucial for the following lemma, which will play an important role in the proof of Proposition 4.8.

Lemma 4.7. In the above notation:

(a) $c_{i1}^\lambda \circ g_1^\lambda = g_i^\lambda$ for all $i \in \mathcal{I}^\lambda$,
(b) $c_{ij}^\lambda \circ g_1^\lambda = 0$ for all $i, j \in \mathcal{I}^\lambda$ with $j \neq 1$. □

Proof. We have $\overline{f}_1^\lambda \circ g_1^\lambda = \overline{f}_1^\lambda \circ \overline{g}_1^\lambda \circ \iota^\lambda = \iota^\lambda$. This implies $c_{i1}^\lambda \circ g_1^\lambda = \overline{g}_i^\lambda \circ \iota^\lambda = g_i^\lambda$. Next, if $j \neq 1$, then $\overline{f}_j^\lambda \circ g_1^\lambda = 0$, since $\overline{f}_j^\lambda$ is zero on $T_q(\lambda)$. Thus, $c_{ij}^\lambda \circ g_1^\lambda = 0$ for all $i, j \in \mathcal{I}^\lambda$ with $j \neq 1$. ■

Proposition 4.8. (**Homomorphism criterion.**) Let $\lambda \in \mathcal{P}_0$ and fix $M \in \mathrm{End}_{\mathbf{U}_q}(T)$-**Mod**. Then there is an isomorphism of $\mathbb{K}$-vector spaces

$$\mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\lambda), M) \cong \{m \in M \mid \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda} m = 0 \text{ and } c_{11}^\lambda m = m\}. \tag{16}$$ □

Proof. Let $\psi \in \mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\lambda), M)$. Then $\psi(g_1^\lambda)$ belongs to the right-hand side, because, by item (b) of Lemma 4.6, we get $\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda} C(\lambda) = 0$, and we have $c_{11}^\lambda \circ g_1^\lambda = g_1^\lambda$ by item (a) of Lemma 4.7. Conversely, if $m \in M$ belongs to the right-hand side in (16), then we may define $\psi \in \mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\lambda), M)$ by $\psi(g_i^\lambda) = c_{i1}^\lambda m$, $i \in \mathcal{I}^\lambda$. Moreover, the fact that this definition gives an $\mathrm{End}_{\mathbf{U}_q}(T)$-homomorphism follows from (10), (11) and (12) via a direct computation, since $\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda} m = 0$. Clearly these two operations are mutually inverse. ■

Corollary 4.9. Let $\lambda \in \mathcal{P}_0$. Then $C(\lambda)$ has a unique simple head, denoted by $L(\lambda)$. □

Proof. Set $\mathrm{Rad}(\lambda) = \{g \in C(\lambda) \mid \vartheta^\lambda(g, C(\lambda)) = 0\}$. As the cellular pairing ϑ^λ from Definition 4.2 is contravariant by Lemma 4.3, we see that $\mathrm{Rad}(\lambda)$ is an $\mathrm{End}_{\mathbf{U}_q}(T)$-submodule of $C(\lambda)$. Since $\vartheta^\lambda \neq 0$ for $\lambda \in \mathcal{P}_0$, we have $\mathrm{Rad}(\lambda) \subsetneq C(\lambda)$. We claim that $\mathrm{Rad}(\lambda)$ is the unique maximal proper $\mathrm{End}_{\mathbf{U}_q}(T)$-submodule of $C(\lambda)$.

Let $g \in C(\lambda) - \mathrm{Rad}(\lambda)$. Moreover, choose $h \in C(\lambda)$ with $\vartheta^\lambda(g, h) = 1$. Then $\mathrm{i}(h) \circ g = c^\lambda$ so that $\overline{\mathrm{i}(h)} \circ g = \iota^\lambda \ (\mathrm{mod}\ \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda})$. Therefore,

$$g' = \overline{g}' \circ \overline{\mathrm{i}(h)} \circ g \ (\mathrm{mod}\ \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}), \quad \text{for all } g' \in C(\lambda).$$

This implies $C(\lambda) = \mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda} g$. Thus, any proper $\mathrm{End}_{\mathbf{U}_q}(T)$-submodule of $C(\lambda)$ is contained in $\mathrm{Rad}(\lambda)$ which implies the desired statement. ■

Corollary 4.10. Let $\lambda \in \mathcal{P}_0, \mu \in \mathcal{P}$ and assume that $\mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\lambda), M) \neq 0$ for some $\mathrm{End}_{\mathbf{U}_q}(T)$-module M isomorphic to a subquotient of $C(\mu)$. Then we have $\mu \leq \lambda$. In particular, all composition factors $L(\lambda)$ of $C(\mu)$ satisfy $\mu \leq \lambda$. □

Proof. By Proposition 4.8 the assumption in the corollary implies the existence of an element $m \in M$ with $c_{11}^\lambda m = m$. But if $\mu \not\leq \lambda$, then c_{11}^λ vanishes on the $\mathbf{U}_q$-weight space T_μ and hence, $c_{11}^\lambda g$ kills the highest weight vector in $\Delta_q(\mu)$ for all $g \in C(\mu)$. This makes the existence of such an $m \in M$ impossible unless $\mu \leq \lambda$. ■

4C. **Simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules and semisimplicity of $\mathrm{End}_{\mathbf{U}_q}(T)$.** Let $\lambda \in \mathcal{P}_0$. Note that Corollary 4.9 shows that $C(\lambda)$ has a unique simple head $L(\lambda)$. We then arrive at the following classification of all simple modules in $\mathrm{End}_{\mathbf{U}_q}(T)$-**Mod**.

Theorem 4.11. (**Classification of simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules.**) The set $\{L(\lambda) \mid \lambda \in \mathcal{P}_0\}$ forms a complete set of pairwise non-isomorphic, simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules. □

Proof. We have to show three statements, namely that the $L(\lambda)$'s are simple, that they are pairwise non-isomorphic and that every simple $\mathrm{End}_{\mathbf{U}_q}(T)$-module is one of the $L(\lambda)$'s.

Because the first statement follows directly from the definition of $L(\lambda)$ (see Corollary 4.9), we start by showing the second. Thus, assume that $L(\lambda) \cong L(\mu)$ for some $\lambda, \mu \in \mathcal{P}_0$. Then

$$\mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\lambda), C(\mu)/\mathrm{Rad}(\mu)) \neq 0 \neq \mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\mu), C(\lambda)/\mathrm{Rad}(\lambda)).$$

By Corollary 4.10, we get $\mu \leq \lambda$ and $\lambda \leq \mu$ from the left and right-hand side. Thus, $\lambda = \mu$.

Suppose that $L \in \mathrm{End}_{\mathbf{U}_q}(T)$-**Mod** is simple. Then we can choose $\lambda \in \mathcal{P}$ minimal such that (recall that $\mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda}$ is a two-sided ideal)

$$\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}L = 0 \quad \text{and} \quad \mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda}L = L. \tag{17}$$

We claim that $\lambda \in \mathcal{P}_0$. Indeed, if not, then, by Proposition 4.4, we see that $T_q(\lambda)$ is not a summand of T. Hence, in our usual notation, all $\overline{f}_j^\lambda \circ \overline{g}_{i'}^\lambda$ vanishes on the λ-weight space. It follows that $c_{ij}^\lambda c_{i'j'}^\lambda$ also vanish on the λ-weight space for all $i, j, i', j' \in \mathcal{I}^\lambda$. This means that we have $\mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda}\mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda} \subset \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}$ making (17) impossible.

For $\lambda \in \mathcal{P}_0$ we see by Lemma 4.7 that

$$c_{i1}^\lambda c_{1j}^\lambda = c_{ij}^\lambda \ (\mathrm{mod}\ \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}). \tag{18}$$

Hence, by (17), there exist $i, j \in \mathcal{I}^\lambda$ such that $c_{ij}^\lambda L \neq 0$. By (18) we also have $c_{i1}^\lambda L \neq 0 \neq c_{1j}^\lambda L$. This in turn (again by (18)) ensures that $c_{11}^\lambda L \neq 0$. Take then $m \in c_{11}^\lambda L - \{0\}$ and observe that $c_{11}^\lambda m = m$. Hence, by Proposition 4.8, there is a non-zero $\mathrm{End}_{\mathbf{U}_q}(T)$-homomorphism $C(\lambda) \to L$. The conclusion follows now from Corollary 4.9. ■

Recall from Section 4B the notation m_λ (the multiplicity of $T_q(\lambda)$ in T) and the choice of basis for $C(\lambda)$ (in the paragraphs before Lemma 4.7). Then we get the following connection between the decomposition of T as in (15) and the simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules $L(\lambda)$.

Theorem 4.12. (**Dimension formula.**) If $\lambda \in \mathcal{P}_0$, then $\dim(L(\lambda)) = m_\lambda$. □

Note that this result is implicit in [38] and has also been observed in e.g. [37] and [82].

Proof. We use the notation from Section 4B. Since T' has no summands isomorphic to $T_q(\lambda)$, we see that $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T') \subset \mathrm{Rad}(\lambda)$ (see the proof of Corollary 4.9). On the other hand, $g_i^\lambda \notin \mathrm{Rad}(\lambda)$ for $1 \leq i \leq m_\lambda$ because for these i we have $f_i^\lambda \circ g_i^\lambda = c^\lambda$ by construction. Thus, the statement follows. ■

Theorem 4.13. (**Semisimplicity criterion.**) The cellular algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple if and only if T is a semisimple $\mathbf{U}_q$-module. □

Proof. Note that the $T_q(\lambda)$'s are simple if and only if $T_q(\lambda) \cong \Delta_q(\lambda) \cong L_q(\lambda) \cong \nabla_q(\lambda)$. Hence, T is semisimple as a $\mathbf{U}_q$-module if and only if $T = \bigoplus_{\lambda \in \mathcal{P}_0} \Delta_q(\lambda)^{\oplus m_\lambda}$ with m_λ as in Section 4B.

Thus, we see that, if T decomposes into simple $\mathbf{U}_q$-modules, then $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple by the Artin–Wedderburn theorem (since $\mathrm{End}_{\mathbf{U}_q}(T)$ will decompose into a direct sum of matrix algebras in this case).

On the other hand, if $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple, then we know, by Corollary 4.9, that the cell modules $C(\lambda)$ are simple, i.e. $C(\lambda) = L(\lambda)$ for all $\lambda \in \mathcal{P}_0$. Then

$$T \cong \bigoplus_{\lambda \in \mathcal{P}_0} T_q(\lambda)^{\oplus m_\lambda}, \quad m_\lambda = \dim(L(\lambda)) = \dim(C(\lambda)) = \dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)) \tag{19}$$

by Theorem 4.12. Assume now that there exists a summand $T_q(\lambda')$ of T as in (19) with $T_q(\lambda') \ncong \Delta_q(\lambda')$ and choose $\lambda' \in \mathcal{P}_0$ minimal with this property.

Then there exists a $\mu < \lambda'$ such that $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), T_q(\lambda')) \neq 0$. Choose also μ minimal among those. By our usual construction this then gives in turn a non-zero $\mathbf{U}_q$-homomorphism

$\overline{g}\circ\overline{f}\colon T_q(\lambda')\to T_q(\mu)\to T_q(\lambda')$. By (19), we can extend $\overline{g}\circ\overline{f}$ to an element of $\mathrm{End}_{\mathbf{U}_q}(T)$ by defining it to be zero on all other summands.

Clearly, by construction, $(\overline{g}\circ\overline{f})C(\mu')=0$ for $\mu'\in\mathcal{P}_0$ with $\mu'\neq\lambda'$ and $\mu'\not\leq\mu$. If $\mu'\leq\mu$, then consider $\varphi\in C(\mu')$. Then $(\overline{g}\circ\overline{f})\circ\varphi=0$ unless φ has some non-zero component $\varphi'\colon\Delta_q(\mu')\to T_q(\lambda')$. This forces $\mu'=\mu$ by minimality of μ. But since $\Delta_q(\mu')\cong T_q(\mu')$, by minimality of λ', we conclude that $\overline{f}\circ\varphi=0$ (otherwise $T_q(\mu')$ would be a summand of $T_q(\lambda')$).

Hence, the non-zero element $\overline{g}\circ\overline{f}\in\mathrm{End}_{\mathbf{U}_q}(T)$ kills all $C(\mu')$ for $\mu'\in\mathcal{P}_0$. This contradicts the semisimplicity of $\mathrm{End}_{\mathbf{U}_q}(T)$: as noted above, $C(\lambda)=L(\lambda)$ for all $\lambda\in\mathcal{P}_0$ which implies $\mathrm{End}_{\mathbf{U}_q}(T)\cong\bigoplus_{\lambda\in\mathcal{P}_0}C(\lambda)^{\oplus k_\lambda}$ for some $k_\lambda\in\mathbb{Z}_{\geq 0}$. ■

5. Cellular structures: examples and applications

In this section we provide many examples of cellular algebras arising from our main theorem. This includes several renowned examples where cellularity is known (but usually proved case by case spread over the literature and with cellular bases which differ in general from ours), and also new ones. In the first subsection we give a full treatment of the semisimple case and describe how to obtain all the examples from the introduction using our methods. In the second subsection we focus on the Temperley–Lieb algebras $\mathcal{TL}_d(\delta)$ and give a detailed account how to apply our results to these.

5A. **Cellular structures using $\mathbf{U}_q$-tilting modules: several examples.** In the following let ω_i for $i=1,\dots,n$ denote the fundamental weights (of the corresponding type).

5A.1. *The semisimple case.* Suppose the category $\mathbf{U}_q$-**Mod** is semisimple, that is, q is not a root of unity in $\mathbb{K}^*-\{1\}$ or $q=\pm 1\in\mathbb{K}$ with $\mathrm{char}(\mathbb{K})=0$.

In this case $\boldsymbol{\mathcal{T}}=\mathbf{U}_q$-**Mod** and any $T\in\boldsymbol{\mathcal{T}}$ has a decomposition $T\cong\bigoplus_{\lambda\in X^+}\Delta_q(\lambda)^{\oplus m_\lambda}$ with the multiplicities $m_\lambda=(T:\Delta_q(\lambda))$. This induces an Artin–Wedderburn decomposition

$$\mathrm{End}_{\mathbf{U}_q}(T)\cong\bigoplus_{\lambda\in X^+}M_{m_\lambda}(\mathbb{K}) \tag{20}$$

into matrix algebras. A natural choice of basis for $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda),T)$ is

$$G^\lambda=\{g_1^\lambda,\dots,g_{m_\lambda}^\lambda\mid g_i^\lambda\colon\Delta_q(\lambda)\hookrightarrow T\text{ is the inclusion into the i-th summand}\}.$$

Then our cellular basis consisting of the c_{ij}^λ's as in Section 3C (no lifting is needed in this case) is an Artin–Wedderburn basis, i.e., a basis that realizes the decomposition (20) in the following sense. The basis element c_{ij}^λ is the matrix $\mathbf{E}_{ij}^\lambda$ (in the λ-summand on the right-hand side in (20)) which has all entries zero except one entry equals 1 in the i-th row and j-th column. Note that, as expected in this case, $\mathrm{End}_{\mathbf{U}_q}(T)$ has, by the Theorem 4.11 and Theorem 4.12, one simple $\mathrm{End}_{\mathbf{U}_q}(T)$-module $L(\lambda)$ of dimension m_λ for all summands $\Delta_q(\lambda)$ of T.

5A.2. *The symmetric group and the Iwahori–Hecke algebra.* Let us fix $d\in\mathbb{Z}_{\geq 0}$ and let us denote by S_d the symmetric group in d letters and by $\mathcal{H}_d(q)$ its associated Iwahori–Hecke algebra. We note that $\mathbb{K}[S_d]\cong\mathcal{H}_d(1)$. Moreover, let $\mathbf{U}_q=\mathbf{U}_q(\mathfrak{gl}_n)$. The vector representation of $\mathbf{U}_q$, which we denote by $V=\mathbb{K}^n=\Delta_q(\omega_1)$, is a $\mathbf{U}_q$-tilting module (since ω_1 is minimal in X^+). Set $T=V^{\otimes d}$, which is again a $\mathbf{U}_q$-tilting module. Quantum Schur–Weyl duality (see

[32, Theorem 6.3] for surjectivity and use Ext-vanishing for the fact that $\dim(\mathrm{End}_{\mathbf{U}_q}(T))$ is obtained via base change from $\mathbb{Z}[v,v^{-1}]$ to $\mathbb{K}$ for all $\mathbb{K}$ and $q\in\mathbb{K}^*$) states that

$$\Phi_{q\mathrm{SW}}\colon \mathcal{H}_d(q)\twoheadrightarrow \mathrm{End}_{\mathbf{U}_q}(T) \quad\text{and}\quad \Phi_{q\mathrm{SW}}\colon \mathcal{H}_d(q)\xrightarrow{\cong}\mathrm{End}_{\mathbf{U}_q}(T),\ \text{if } n\geq d. \tag{21}$$

Thus, our main result implies that $\mathcal{H}_d(q)$, and in particular $\mathbb{K}[S_d]$, are cellular for any $q\in\mathbb{K}^*$ and any field $\mathbb{K}$ (by taking $n\geq d$).

In this case the cell modules for $\mathrm{End}_{\mathbf{U}_q}(T)$ are usually called Specht modules $S^\lambda_{\mathbb{K}}$ and our Theorem 4.12 gives the following.

- If $q=1$ and $\mathrm{char}(\mathbb{K})=0$, then the dimension $\dim(S^\lambda_{\mathbb{K}})$ is equal to the multiplicity of the simple $\mathbf{U}_1$-module $\Delta_1(\lambda)\cong L_1(\lambda)$ in $V^{\otimes d}$ for all $\lambda\in\mathcal{P}^0$. These numbers are given by known formulas (e.g. the hook length formula).
- If $q=1$ and $\mathrm{char}(\mathbb{K})>0$, then the dimension of the simple head $D^\lambda_{\mathbb{K}}$ of $S^\lambda_{\mathbb{K}}$ is the multiplicity with which $T_1(\lambda)$ occurs as a summand in $V^{\otimes d}$ for all $\lambda\in\mathcal{P}_0$, see also [37]. It is a wide open problem to determine these numbers. (See however [70].)
- If q is a complex, primitive root of unity, then we can compute the dimension of the simple $\mathcal{H}_d(q)$-modules by using the algorithm as in [8]. In particular, this connects with the LLT algorithm from [57].
- If q is a root of unity and $\mathbb{K}$ is arbitrary, then not much is known. Still, our methods apply and we get a way to calculate the dimensions of the simple $\mathcal{H}_d(q)$-modules, if we can decompose T into its indecomposable summands.

5A.3. *The Temperley–Lieb algebra and other $\mathfrak{sl}_2$-related algebras.* Let $\mathbf{U}_q=\mathbf{U}_q(\mathfrak{sl}_2)$ and let T be as in Section 5A.2 with $n=2$. For any $d\in\mathbb{Z}_{\geq 0}$ we have $\mathcal{TL}_d(\delta)\cong\mathrm{End}_{\mathbf{U}_q}(T)$ by Schur–Weyl duality, where $\mathcal{TL}_d(\delta)$ is the Temperley–Lieb algebra in d-strands with parameter $\delta=q+q^{-1}$. This works for all $\mathbb{K}$ and all $q\in\mathbb{K}^*$ (this can be deduced from, for example, [32, Theorem 6.3]). Hence, $\mathcal{TL}_d(\delta)$ is always cellular. We discuss this case in more detail in Section 5B.

Furthermore, if we are in the semisimple case, then $\Delta_q(i)$ is a $\mathbf{U}_q$-tilting module for all $i\in\mathbb{Z}_{\geq 0}$ and so is $T=\Delta_q(i_1)\otimes\cdots\otimes\Delta_q(i_d)$. Thus, we obtain that $\mathrm{End}_{\mathbf{U}_q}(T)$ is cellular.

The algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is known to give a diagrammatic presentation of the (tensor) category of $\mathbf{U}_q$-modules, see [73], and can be used to define the colored Jones polynomial.

If $q\in\mathbb{K}$ is a root of unity and l is the order of q^2, then, for any $0\leq i<l$, $\Delta_q(i)$ is a $\mathbf{U}_q$-tilting module (since its simple) and so is $T=\Delta_q(i)^{\otimes d}$. The endomorphism algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is cellular. This reproves parts of [5, Theorem 1.1] using our general approach.

In characteristic 0: Another family of $\mathbf{U}_q$-tilting modules was studied in [10]. For any $d\in\mathbb{Z}_{\geq 0}$, fix any $\lambda_0\in\{0,\dots,l-2\}$ and consider $T=T_q(\lambda_0)\oplus\cdots\oplus T_q(\lambda_d)$ where λ_k is the unique integer $\lambda_k\in\{kl,\dots,(k+1)l-2\}$ linked to λ_0. We again obtain that $\mathrm{End}_{\mathbf{U}_q}(T)$ is cellular. Note that $\mathrm{End}_{\mathbf{U}_q}(T)$ can be identified with the so-called (type A) zig-zag algebra A_d, see [10, Proposition 3.9], introduced in [44]. These algebras are naturally graded making $\mathrm{End}_{\mathbf{U}_q}(T)$ into a graded cellular algebra in the sense of [41] and are special examples arising from the family of generalized Khovanov arc algebras whose cellularity is studied in [19].

5A.4. *Spider algebras.* Let $\mathbf{U}_q=\mathbf{U}_q(\mathfrak{sl}_n)$ (or, alternatively, $\mathbf{U}_q(\mathfrak{gl}_n)$). One has for any $q\in\mathbb{K}^*$ that all $\mathbf{U}_q$-representations $\Delta_q(\omega_i)$ are $\mathbf{U}_q$-tilting modules (because the ω_i's are minimal in X^+). Hence, for any $k_i\in\{1,\dots,n-1\}$, $T=\Delta_q(\omega_{k_1})\otimes\cdots\otimes\Delta_q(\omega_{k_d})$ is a $\mathbf{U}_q$-tilting module. Thus, $\mathrm{End}_{\mathbf{U}_q}(T)$ is cellular. These algebras are related to type A_{n-1} spider algebras as in

[56], are connected to the Reshetikhin–Turaev $\mathfrak{sl}_n$-link polynomials and give a diagrammatic description of the representation theory of $\mathfrak{sl}_n$, see [23], providing a link from our work to low-dimensional topology and diagrammatic algebra. Note that cellular bases (which, in this case, coincide with our cellular bases) of these were found in [36, Theorem 2.57].

More general: In any type we have that $\Delta_q(\lambda)$ are $\mathbf{U}_q(\mathfrak{g})$-tilting modules for minuscule $\lambda \in X^+$, see [48, Part II, Chapter 2, Section 15]. Moreover, if q is a root of unity "of order l big enough" (ensuring that the ω_i's are in the closure of the fundamental alcove), then the $\Delta_q(\omega_i)$ are $\mathbf{U}_q(\mathfrak{g})$-tilting modules by the linkage principle (see [3, Corollaries 4.4 and 4.6]). So in these cases we can generalize the above results to other types.

Still more generally: we may take (for any type and any $q \in \mathbb{K}^*$) arbitrary $\lambda_j \in X^+$ (for $j = 1, \dots, d$) and obtain a cellular structure on $\mathrm{End}_{\mathbf{U}_q}(T)$ for $T = T_q(\lambda_1) \otimes \cdots \otimes T_q(\lambda_d)$.

5A.5. *The Ariki–Koike algebra and related algebras.* Take $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ (which can be easily fit into our context) with $m_1 + \cdots + m_r = m$ and let V be the vector representation of $\mathbf{U}_1(\mathfrak{gl}_m)$ restricted to $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{g})$. This is again a $\mathbf{U}_1$-tilting module and so is $T = V^{\otimes d}$. Then we have a cyclotomic analog of (21), namely

$$\Phi_{\mathrm{cl}} \colon \mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d] \twoheadrightarrow \mathrm{End}_{\mathbf{U}_1}(T) \quad \text{and} \quad \Phi_{\mathrm{cl}} \colon \mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d] \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_1}(T), \text{ if } m \geq d, \tag{22}$$

where $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$ is the group algebra of the complex reflection group $\mathbb{Z}/r\mathbb{Z} \wr S_d \cong (\mathbb{Z}/r\mathbb{Z})^d \rtimes S_d$, see [65, Theorem 9]. Thus, we can apply our main theorem and obtain a cellular basis for these quotients of $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$. If $m \geq d$, then (22) is an isomorphism (see Lemma 11 loc. cit.) and we obtain that $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$ itself is a cellular algebra for all r, d. In the extremal case $m_1 = m - 1$ and $m_2 = 1$, the resulting quotient of (22) is known as Solomon's algebra introduced in [85] (also called the algebra of the inverse semigroup or the rook monoid algebra) and we obtain that Solomon's algebra is cellular. In the extremal case $m_1 = m_2 = 1$, the resulting quotient is a specialization of the blob algebra $\mathcal{BL}_d(1, 2)$ (in the notation used in [77]). To see this, note that both algebras are quotients of $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$. The kernel of the quotient to $\mathcal{BL}_d(1, 2)$ is described explicitly by Ryom-Hansen in [77, (1)] and is by [65, Lemma 11] contained in the kernel of Φ_{cl} from (22). Since both algebras have the same dimensions, they are isomorphic.

Let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$. We get in the quantized case (for $q \in \mathbb{C} - \{0\}$ not a root of unity)

$$\Phi_{\mathrm{qcl}} \colon \mathcal{H}_{d,r}(q) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_q}(T) \quad \text{and} \quad \Phi_{\mathrm{qcl}} \colon \mathcal{H}_{d,r}(q) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_q}(T), \text{ if } m \geq d, \tag{23}$$

where $\mathcal{H}_{d,r}(q)$ is the Ariki–Koike algebra introduced in [12]. A proof of (23) can for example be found in [78, Theorem 4.1]. Thus, as before, our main theorem applies and we obtain: the Ariki–Koike algebra $\mathcal{H}_{d,r}(q)$ is cellular (by taking $m \geq d$), the quantized rook monoid algebra $\mathcal{R}_d(q)$ from [39] is cellular and the blob algebra $\mathcal{BL}_d(q, m)$ is cellular (which follows as above). Note that the cellularity of $\mathcal{H}_{d,r}(q)$ was obtained in [28], the cellularity of the quantum rook monoid algebras and of the blob algebra can be found in [67] and in [76] respectively.

In fact, (23) is still true in the non-semisimple cases, see [43, Theorem 1.10 and Lemma 2.12] as long as $\mathbb{K}$ satisfies a certain separation condition (which implies that the algebra in question has the right dimension, see [11]). Again, our main theorem applies.

5A.6. *The Brauer algebras and related algebras.* Consider $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ where $\mathfrak{g}$ is either an orthogonal $g = \mathfrak{o}_{2n}$ and $g = \mathfrak{o}_{2n+1}$ or the symplectic $g = \mathfrak{sp}_{2n}$ Lie algebra. Let $V = \Delta_q(\omega_1)$ be the quantized version of the corresponding vector representation. In both cases, V is a

$\mathbf{U}_q$-tilting module (for type B and $q = 1$ this requires $\mathrm{char}(\mathbb{K}) \neq 2$, see [46, Page 20]) and hence, so is $T = V^{\otimes d}$. We first take $q = 1$ and set $\delta = 2n$ in case $\mathfrak{g} = \mathfrak{o}_{2n}$, and $\delta = 2n + 1$ in case $\mathfrak{g} = \mathfrak{o}_{2n+1}$ and $\delta = -2n$ in case $\mathfrak{g} = \mathfrak{sp}_{2n}$ respectively. Then (see [26, Theorem 1.4] and [31, Theorem 1.2] for infinite $\mathbb{K}$, or [35, Theorem 5.5] for $\mathbb{K} = \mathbb{C}$)

$$\Phi_{\mathrm{Br}} \colon \mathcal{B}_d(\delta) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_1}(T) \quad \text{and} \quad \Phi_{\mathrm{Br}} \colon \mathcal{B}_d(\delta) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_1}(T), \text{ if } n > d, \tag{24}$$

where $\mathcal{B}_d(\delta)$ is the Brauer algebra in d strands (for $\mathfrak{g} \neq \mathfrak{o}_{2n}$ the isomorphism in (24) already holds for $n = d$). Thus, we get cellularity of $\mathcal{B}_d(\delta)$ by observing that in characteristic p we can always assume that n is large because $\mathcal{B}_d(\delta) = \mathcal{B}_d(\delta + p)$.

Similarly, let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{gl}_n)$, $q \in \mathbb{K}^*$ arbitrary and $T = \Delta_q(\omega_1)^{\otimes r} \otimes \Delta_q(\omega_{n-1})^{\otimes s}$. By [27, Theorem 7.1 and Corollary 7.2] we have

$$\Phi_{\mathrm{wBr}} \colon \mathcal{B}^n_{r,s}([n]) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_q}(T) \quad \text{and} \quad \Phi_{\mathrm{wBr}} \colon \mathcal{B}^n_{r,s}([n]) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_q}(T), \text{ if } n \geq r + s. \tag{25}$$

Here $\mathcal{B}^n_{r,s}([n])$ is the quantized walled Brauer algebra for $[n] = q^{1-n} + \cdots + q^{n-1}$. Since T is a $\mathbf{U}_q$-tilting module, we get from (25) cellularity of $\mathcal{B}^n_{r,s}([n])$ and of its quotients under Φ_{wBr}.

The walled Brauer algebra $\mathcal{B}^n_{r,s}(\delta)$ over $\mathbb{K} = \mathbb{C}$ for arbitrary parameter $\delta \in \mathbb{Z}$ appears as the centralizer of $\mathrm{End}_{\mathfrak{gl}(m|n)}(T)$ for $T = V^{\otimes r} \otimes (V^*)^{\otimes s}$ where V is the vector representation of the superalgebra $\mathfrak{gl}(m|n)$ with $\delta = m - n$. That is, we have

$$\Phi_{\mathrm{s}} \colon \mathcal{B}^n_{r,s}(\delta) \twoheadrightarrow \mathrm{End}_{\mathfrak{gl}(m|n)}(T) \text{ and } \Phi_{\mathrm{s}} \colon \mathcal{B}^n_{r,s}(\delta) \xrightarrow{\cong} \mathrm{End}_{\mathfrak{gl}(m|n)}(T), \text{ if } (m+1)(n+1) \geq r+s, \tag{26}$$

see [18, Theorem 7.8]. It can be shown that T is a $\mathfrak{gl}(m|n)$-tilting module and thus, our main theorem applies and hence, by (26), $\mathcal{B}^n_{r,s}(\delta)$ is cellular. Similarly for the quantized version.

Quantizing the Brauer case, taking $q \in \mathbb{K}^*$, $\mathfrak{g}$, $V = \Delta_q(\omega_1)$ and T as before (without the restriction $\mathrm{char}(\mathbb{K}) \neq 2$ for type B) gives us a cellular structure on $\mathrm{End}_{\mathbf{U}_q}(T)$. The algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is a quotient of the Birman–Murakami–Wenzl algebra $\mathcal{BMW}_d(\delta)$ (for appropriate parameters), see [58, (9.6)] for the orthogonal case (which works for any $q \in \mathbb{C} - \{0, \pm 1\}$) and [40, Theorem 1.5] for the symplectic case (which works for any $q \in \mathbb{K}^* - \{1\}$ and infinite $\mathbb{K}$). Again, taking $n \geq d$ (or $n > d$), we recover the cellularity of $\mathcal{BMW}_d(\delta)$.

5A.7. *Infinite-dimensional modules — highest weight categories.* Observe that our main theorem does not use the specific properties of $\mathbf{U}_q$-**Mod**, but works for any $\mathrm{End}_{A\text{-}\mathbf{Mod}}(T)$ where T is an A-tilting module for some finite-dimensional, quasi-hereditary algebra A over $\mathbb{K}$ or $T \in \mathcal{C}$ for some highest weight category $\mathcal{C}$ in the sense of [24]. For the explicit construction of our basis we however need a notion like "weight spaces" such that Lemma 3.4 makes sense.

The most famous example of such a category is the BGG category $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ attached to a complex semisimple or reductive Lie algebra $\mathfrak{g}$ with a corresponding Cartan $\mathfrak{h}$ and fixed Borel subalgebra $\mathfrak{b}$. We denote by $\Delta(\lambda) \in \mathcal{O}$ the Verma module attached to $\lambda \in \mathfrak{h}^*$. In the same vein, pick a parabolic $\mathfrak{p} \supset \mathfrak{b}$ and denote for any $\mathfrak{p}$-dominant weight λ the corresponding parabolic Verma module by $\Delta^{\mathfrak{p}}(\lambda)$. It is the unique quotient of the Verma module $\Delta(\lambda)$ which is locally $\mathfrak{p}$-finite, i.e. contained in the parabolic category $\mathcal{O}^{\mathfrak{p}} = \mathcal{O}^{\mathfrak{p}}(\mathfrak{g}) \subset \mathcal{O}$ (see e.g. [45]).

There is a contravariant, character preserving duality functor ${}^{\vee} \colon \mathcal{O}^{\mathfrak{p}} \to \mathcal{O}^{\mathfrak{p}}$ which allows us to set $\nabla^{\mathfrak{p}}(\lambda) = \Delta^{\mathfrak{p}}(\lambda)^{\vee}$. Hence, we can play the same game again since the $\mathcal{O}$-tilting theory works in a very similar fashion as for $\mathbf{U}_q$-**Mod** (see [45, Chapter 11] and the references therein). In particular, we have indecomposable $\mathcal{O}$-tilting modules $T(\lambda)$ for any $\lambda \in \mathfrak{h}^*$. Similarly for $\mathcal{O}^{\mathfrak{p}}$ giving an indecomposable $\mathcal{O}^{\mathfrak{p}}$-tilting module $T(\lambda)$ for any $\mathfrak{p}$-dominant $\lambda \in \mathfrak{h}^*$.

We give a few examples where our approach leads to cellular structures on interesting algebras. For this purpose, let $\mathfrak{p} = \mathfrak{b}$ and $\lambda = 0$. Then $T(0)$ has Verma factors of the form $\Delta(w.0)$ (for $w \in W$, where W is the Weyl group associated to $\mathfrak{g}$). Each of these appears with multiplicity 1. Hence, $\dim(\mathrm{End}_{\mathcal{O}}(T(0))) = |W|$ by the analog of (4). Then we have $\mathrm{End}_{\mathcal{O}}(T(0)) \cong S(\mathfrak{h}^*)/S_+^W$. The algebra $S(\mathfrak{h}^*)/S_+^W$ is called the coinvariant algebra. (For the notation, the conventions and the result see [83] — this is Soergel's famous Endomorphismensatz.) Hence, our main theorem implies that $S(\mathfrak{h}^*)/S_+^W$ is cellular, which is no big surprise since all finite-dimensional, commutative algebras are cellular, see [55, Proposition 3.5].

There is also a quantum version of this result: replace $\mathcal{O}$ by its quantum cousin $\mathcal{O}_q$ from [6] (which is the analog of $\mathcal{O}$ for $\mathbf{U}_q(\mathfrak{g})$). This works over any field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$ and any $q \in \mathbb{K}^* - \{1\}$ (which can be deduced from Section 6 therein). There is furthermore a characteristic p version of this result: consider the G-tilting module $T(p\rho)$ in the category of finite-dimensional G-modules (here G is an almost simple, simply connected algebraic group over $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = p$). Its endomorphism algebra is isomorphic to the corresponding coinvariant algebra over $\mathbb{K}$, see [4, Proposition 19.8].

Returning to $\mathbb{K} = \mathbb{C}$, we can generalize the example of the coinvariant algebra. To this end, note that, if T is an $\mathcal{O}^{\mathfrak{p}}$-tilting module, then so is $T \otimes M$ for any finite-dimensional $\mathfrak{g}$-module M, see [45, Proposition 11.1 and Section 11.8] (and the references therein). Thus, $\mathrm{End}_{\mathcal{O}^{\mathfrak{p}}}(T \otimes M)$ is cellular by our main theorem.

A special case is: $\mathfrak{g}$ is of classical type, $T = \Delta^{\mathfrak{p}}(\lambda)$ is simple (hence, $\mathcal{O}^{\mathfrak{p}}$-tilting), V is the vector representation of $\mathfrak{g}$ and $M = V^{\otimes d}$. Let first $\mathfrak{g} = \mathfrak{gl}_n$ with standard Borel $\mathfrak{b}$ and parabolic $\mathfrak{p}$ of block size $(n_1, \dots, n_\ell)$. Then one can find a certain $\mathfrak{p}$-dominant weight λ_{I}, called Irving-weight, such that $T = \Delta^{\mathfrak{p}}(\lambda_{\mathrm{I}})$ is $\mathcal{O}^{\mathfrak{p}}$-tilting. Moreover, $\mathrm{End}_{\mathcal{O}^{\mathfrak{p}}}(T \otimes V^{\otimes d})$ is isomorphic to a sum of blocks of cyclotomic quotients of the degenerate affine Hecke algebra $\mathcal{H}_d/\Pi_{i=1}^{\ell}(x_i - n_i)$, see [17, Theorem 5.13]. In the special case of level $\ell = 2$, these algebras can be explicitly described in terms of generalizations of Khovanov's arc algebra (which Khovanov introduced in [51] to give an algebraic structure underlying Khovanov homology and which categorifies the Temperley–Lieb algebra $\mathcal{TL}_d(\delta)$) and have an interesting representation theory, see [19], [20], [21] and [22]. A consequence of this is that, using the results from [79, Theorem 6.9] and [80, Theorem 1.1], one can realize the walled Brauer algebra from Section 5A.6 for arbitrary parameter $\delta \in \mathbb{Z}$ as endomorphism algebras of some $\mathcal{O}^{\mathfrak{p}}$-tilting module and hence, using our main theorem, deduce cellularity again.

If $\mathfrak{g}$ is of another classical type, then the role of the (cyclotomic quotients of the) degenerate affine Hecke algebra is played by (cyclotomic quotients of) degenerate BMW algebras or so-called (cyclotomic quotients of) $\mathbb{W}_d$-algebras (also called Nazarov–Wenzl algebras). These are still poorly understood and technically quite involved, see [13]. In [33] special examples of level $\ell = 2$ quotients were studied and realized as endomorphism algebras of some $\mathcal{O}^{\mathfrak{p}}(\mathfrak{so}_{2n})$-tilting module $\Delta^{\mathfrak{p}}(\underline{\delta}) \otimes V \in \mathcal{O}^{\mathfrak{p}}(\mathfrak{so}_{2n})$ where V is the vector representation of $\mathfrak{so}_{2n}$, $\underline{\delta} = \frac{\delta}{2}\sum_{i=1}^{n} \epsilon_i$ and $\mathfrak{p}$ is a maximal parabolic subalgebra of type A (Theorem B loc. cit.). Hence, our theorem implies cellularity of these algebras. Soergel's theorem is therefore just a shadow of a rich world of endomorphism algebras whose cellularity can be obtained from our approach.

Our methods also apply to (parabolic) category $\mathcal{O}^{\mathfrak{p}}(\hat{\mathfrak{g}})$ attached to an affine Kac–Moody algebra $\hat{\mathfrak{g}}$ over $\mathbb{K}$ and related categories. In particular, one can consider a (level-dependent) quotient $\hat{\mathfrak{g}}_\kappa$ of $\mathbf{U}(\hat{\mathfrak{g}})$ and a category, denoted by $\mathbf{O}_{\mathbb{K},\tau}^{\nu,\kappa}$, attached to it (we refer the reader to

[75, Sections 5.2 and 5.3] for the details). Then there is a subcategory $\mathbf{A}^{\nu,\kappa}_{\mathbb{K},\tau} \subset \mathbf{O}^{\nu,\kappa}_{\mathbb{K},\tau}$ and a $\mathbf{A}^{\nu,\kappa}_{\mathbb{K},\tau}$-tilting module $\mathbf{T}_{\mathbb{K},d}$ defined in Section 5.5 loc. cit. such that

$$\Phi_{\mathrm{aff}} \colon \mathbf{H}^{s}_{\mathbb{K},d} \to \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{\mathbb{K},\tau}}(\mathbf{T}_{\mathbb{K},d}) \quad \text{and} \quad \Phi_{\mathrm{aff}} \colon \mathbf{H}^{s}_{\mathbb{K},d} \xrightarrow{\cong} \mathrm{End}_{\mathbf{A}^{\nu,\kappa}_{\mathbb{K},\tau}}(\mathbf{T}_{\mathbb{K},d}),\ \text{if } \nu_p \geq d, p = 1, \ldots N,$$

see [75, Theorem 5.37 and Proposition 8.1]. Here $\mathbf{H}^{s}_{\mathbb{K},d}$ denotes an appropriate cyclotomic quotient of the affine Hecke algebra. Again, our main theorem applies for $\mathbf{H}^{s}_{\mathbb{K},d}$ in case $\nu_p \geq d$.

5A.8. *Graded cellular structures.* A striking property which arises in the context of (parabolic) category $\mathcal{O}$ (or $\mathcal{O}^{\mathfrak{p}}$) is that all the endomorphism algebras from Section 5A.7 can be equipped with a $\mathbb{Z}$-grading as in [86] arising from the Koszul grading of category $\mathcal{O}$ (or of $\mathcal{O}^{\mathfrak{p}}$). We might choose our cellular basis compatible with this grading and obtain a grading on the endomorphism algebras turning them into graded cellular algebras in the sense of [41, Definition 2.1].

For the cyclotomic quotients this grading is non-trivial and in fact is the type A KL–R grading in the spirit of Khovanov and Lauda and independently Rouquier (see [52] and [53] or [74]), which can be seen as a grading on cyclotomic quotients of degenerate affine Hecke algebras, see [16]. See [21] for level $\ell = 2$ and [42] for all levels where the authors construct explicit graded cellular bases. For gradings on (cyclotomic quotients of) $\mathbb{W}_d$-algebras see [33, Section 5] and for gradings on Brauer algebras see [34] or [59].

In the same spirit, it should be possible to obtain the higher level analogs of the generalizations of Khovanov's arc algebra, known as $\mathfrak{sl}_n$-web (or, alternatively, $\mathfrak{gl}_n$-web) algebras (see [62] and [61]), from our setup as well using the connections from cyclotomic KL–R algebras to these algebras in [89] and [90]. Although details still need to be worked out, this can be seen as the categorification of the connections to the spiders from Section 5A.4: the spiders provide the setup to study the corresponding Reshetikhin–Turaev $\mathfrak{sl}_n$-link polynomials; the $\mathfrak{sl}_n$-web algebras provide the algebraic setup to study the Khovanov–Rozansky $\mathfrak{sl}_n$-link homologies. This would emphasize the connection between our work and low-dimensional topology.

5B. **(Graded) cellular structures and the Temperley–Lieb algebras: a comparison.** Finally we want to present one explicit example, the Temperley–Lieb algebras, which is of particular interest in low-dimensional topology and categorification. Our main goal is to construct new (graded) cellular bases, and use our approach to establish semisimplicity conditions, and construct and compute the dimensions of its simple modules in new ways.

We start by briefly recalling the necessary definitions. The reader unfamiliar with these algebras might consider for example [38, Section 6] (or [8], where we recall the basics in detail using the usual Temperley–Lieb diagrams and our notation).

Fix $\delta = q + q^{-1}$ for $q \in \mathbb{K}^*$.[3] Recall that the *Temperley–Lieb algebra* $\mathcal{TL}_d(\delta)$ in d strands with parameter δ is the free diagram algebra over $\mathbb{K}$ with basis consisting of all possible non-intersecting tangle diagrams with d bottom and top boundary points modulo boundary preserving isotopy and the local relation for evaluating circles given by the parameter[4] δ.

Recall from Section 5A.3 (whose notation we use now) that, by quantum Schur–Weyl duality, we can use Theorem 3.9 to obtain cellular bases of $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ (we fix the

[3] The $\mathfrak{sl}_2$ case works with any $q \in \mathbb{K}^*$, including even roots of unity, see e.g. [10, Definition 2.3].

[4] We point out that there are two different conventions about circle evaluations in the literature: evaluating to δ or to $-\delta$. We use the first convention because we want to stay close to the cited literature.

isomorphism coming from quantum Schur–Weyl duality from now on). The aim now is to compare our cellular bases to the one given by Graham and Lehrer in [38, Theorem 6.7], where we point out that we do not obtain their cellular basis: our cellular basis depends for instance on whether $\mathcal{TL}_d(\delta)$ is semisimple or not. In the non-semisimple case, at least for $\mathbb{K} = \mathbb{C}$, we obtain a non-trivially $\mathbb{Z}$-graded cellular basis in the sense of [41, Definition 2.1], see Proposition 5.8.

Before stating our cellular basis, we provide a criterion which tells precisely whether $\mathcal{TL}_d(\delta)$ is semisimple or not. Recall that there is a known criteria for which Weyl modules $\Delta_q(i)$ are simple, see e.g. [10, Proposition 2.7].

Proposition 5.1. (**Semisimplicity criterion for $\mathcal{TL}_d(\delta)$.**) We have the following.

(a) Let $\delta \neq 0$. Then $\mathcal{TL}_d(\delta)$ is semisimple if and only if $[i] = q^{1-i} + \cdots + q^{i-1} \neq 0$ for all $i = 1, \dots, d$ if and only if q is not a root of unity with $d < l = \mathrm{ord}(q^2)$, or $q = 1$ and $\mathrm{char}(\mathbb{K}) > d$.

(b) Let $\mathrm{char}(\mathbb{K}) = 0$. Then $\mathcal{TL}_d(0)$ is semisimple if and only if d is odd (or $d = 0$).

(c) Let $\mathrm{char}(\mathbb{K}) = p > 0$. Then $\mathcal{TL}_d(0)$ is semisimple if and only if $d \in \{1, 3, 5, \dots, 2p-1\}$ (or $d = 0$). □

Proof. (a): We want to show that $T = V^{\otimes d}$ decomposes into simple $\mathbf{U}_q$-modules if and only if $d < l$, or $q = 1$ and $\mathrm{char}(\mathbb{K}) > d$, which is clearly equivalent to the non-vanishing of the $[i]$'s.

Assume that $d < l$. Since the maximal $\mathbf{U}_q$-weight of $V^{\otimes d}$ is d and since all Weyl $\mathbf{U}_q$-modules $\Delta_q(i)$ for $i < l$ are simple, we see that all indecomposable summands of $V^{\otimes d}$ are simple.

Otherwise, if $l \leq d$, then $T_q(d)$ (or $T_q(d-2)$ in the case $d \equiv -1 \bmod l$) is a non-simple, indecomposable summand of $V^{\otimes d}$ (note that this arguments fails if $l = 2$, i.e. $\delta = 0$).

The case $q = 1$ works similarly, and we can now use Theorem 4.13 to finish the proof of (a).

(b): Since $\delta = 0$ if and only if $q = \pm\sqrt[2]{-1}$, we can use the linkage from e.g. [10, Theorem 2.23] in the case $l = 2$ to see that $T = V^{\otimes d}$ decomposes into a direct sum of simple $\mathbf{U}_q$-modules if and only if d is odd (or $d = 0$). This implies that $\mathcal{TL}_d(0)$ is semisimple if and only if d is odd (or $d = 0$) by Theorem 4.13.

(c): If $\mathrm{char}(\mathbb{K}) = p > 0$ and $\delta = 0$ (for $p = 2$ this is equivalent to $q = 1$), then we have $\Delta_q(i) \cong L_q(i)$ if and only if $i = 0$ or $i \in \{2ap^n - 1 \mid n \in \mathbb{Z}_{\geq 0}, 1 \leq a < p\}$. In particular, this means that for $d \geq 2$ we have that either $T_q(d)$ or $T_q(d-2)$ is a simple $\mathbf{U}_q$-module if and only if $d \in \{3, 5, \dots, 2p-1\}$. Hence, using the same reasoning as above, we see that $T = V^{\otimes d}$ is semisimple if and only if $d \in \{1, 3, 5, \dots, 2p-1\}$ (or $d = 0$). By Theorem 4.13 we see that $\mathcal{TL}_d(0)$ is semisimple if and only if $d \in \{1, 3, 5, \dots, 2p-1\}$ (or $d = 0$). ■

Example 5.2. We have that $[k] \neq 0$ for all $k = 1, 2, 3$ is satisfied if and only if q is not a fourth or a sixth root of unity. By Proposition 5.1 we see that $\mathcal{TL}_3(\delta)$ is semisimple as long as q is not one of these values from above. The other way around is only true for q being a sixth root of unity (the conclusion from semisimplicity to non-vanishing of the quantum numbers above does not work in the case $q = \pm\sqrt[2]{-1}$). ▲

Remark 4. The semisimplicity criterion for $\mathcal{TL}_d(\delta)$ was already already found, using quite different methods, in [95, Section 5] in the case $\delta \neq 0$, and in the case $\delta = 0$ in [63, Chapter 7] or [71, above Proposition 4.9]. For us it is an easy application of Theorem 4.13. ▲

A direct consequence of Proposition 5.1 is that the Temperley–Lieb algebra $\mathcal{TL}_d(\delta)$ for $q \in \mathbb{K}^* - \{1\}$ not a root of unity is semisimple (or $q = \pm 1$ and $\mathrm{char}(\mathbb{K}) = 0$), regardless of d.

5B.1. *Temperley–Lieb algebra: the semisimple case.* Assume $q \in \mathbb{K}^* - \{1\}$ is not a root of unity (or $q = \pm 1$ and $\mathrm{char}(\mathbb{K}) = 0$). Thus, we are in the semisimple case.

Let us compare our cell datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ to the one of Graham and Lehrer (indicated by a subscript GL) from [38, Section 6]. They have the poset $\mathcal{P}_{\mathrm{GL}}$ consisting of all length-two partitions of d, and we have the poset $\mathcal{P}$ consisting of all $\lambda \in X^+$ such that $\Delta_q(\lambda)$ is a factor of T. The two sets are clearly the same: an element $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}_{\mathrm{GL}}$ corresponds to $\lambda_1 - \lambda_2 \in \mathcal{P}$. Similarly, an inductive reasoning shows that $\mathcal{I}_{\mathrm{GL}}$ (standard fillings of the Young diagram associated to λ) is also the same as our $\mathcal{I}$ (to see this one can use the facts listed in [10, Section 2]). One directly checks that the $\mathbb{K}$-linear anti-involution i_{GL} (turning diagrams upside-down) is also our involution i. Thus, except for $\mathcal{C}$ and $\mathcal{C}_{\mathrm{GL}}$, the cell data agree.

In order to state how our cellular bases for $\mathcal{TL}_d(\delta)$ look like, recall that the so-called *generalized Jones–Wenzl projectors* $JW_{\vec{\epsilon}}$ are indexed by d-tuples (with $d > 0$) of the form $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_d) \in \{\pm 1\}^d$ such that $\sum_{j=1}^{k} \epsilon_j \geq 0$ for all $k = 1, \dots, d$, see e.g. [25, Section 2]. In case $\vec{\epsilon} = (1, \dots, 1)$, one recovers the usual Jones–Wenzl projectors introduced by Jones in [49] and then further studied by Wenzl in [93].

Now, in [25, Proposition 2.19 and Theorem 2.20] it is shown that there exist non-zero scalars $a_{\vec{\epsilon}} \in \mathbb{K}$ such that $JW'_{\vec{\epsilon}} = a_{\vec{\epsilon}} JW_{\vec{\epsilon}}$ are well-defined idempotents forming a complete set of mutually orthogonal, primitive idempotents in $\mathcal{TL}_d(\delta)$. (The authors of [25] work over $\mathbb{C}$, but as long as $q \in \mathbb{K}^* - \{1\}$ is not a root of unity their arguments work in our setup as well.) These project to the summands of $T = V^{\otimes d}$ of the form $\Delta_q(i)$ for $i = \sum_{j=1}^{k} \epsilon_j$. In particular, the usual Jones–Wenzl projectors project to the highest weight summand $\Delta_q(d)$ of $T = V^{\otimes d}$.

Proposition 5.3. (**(New) cellular bases.**) The datum given by the quadruple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ for $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$. Moreover, $\mathcal{C} \neq \mathcal{C}_{\mathrm{GL}}$ for all $d > 1$ and all choices involved in the definition of $\mathrm{im}(\mathcal{C})$. In particular, there is a choice such that all generalized Jones–Wenzl projectors $JW'_{\vec{\epsilon}}$ are part of $\mathrm{im}(\mathcal{C})$. □

Proof. That we get a cell datum as stated follows from Theorem 3.9 and the discussion above.

That our cellular basis $\mathcal{C}$ will never be $\mathcal{C}_{\mathrm{GL}}$ for $d > 1$ is due to the fact that Graham and Lehrer's cellular basis always contains the identity (which corresponds to the unique standard filling of the Young diagram associated to $\lambda = (d, 0)$).

In contrast, let $\lambda_k = (d - k, k)$ for $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$. Then

$$T = V^{\otimes d} \cong \Delta_q(d) \oplus \bigoplus_{0 < k \leq \lfloor \frac{d}{2} \rfloor} \Delta_q(d - 2k)^{\oplus m_{\lambda_k}} \tag{27}$$

for some multiplicities $m_{\lambda_k} \in \mathbb{Z}_{>0}$, we see that for $d > 1$ the identity is never part of any of our bases: all the $\Delta_q(i)$'s are simple $\mathbf{U}_q$-modules and each c_{ij}^k factors only through $\Delta_q(k)$. In particular, the basis element c_{11}^λ for $\lambda = \lambda_d$ has to be (a scalar multiple) of $JW_{(1,\dots,1)}$.

As in Section 5A.1 we can choose for $\mathcal{C}$ an Artin–Wedderburn basis of $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$. Hence, by the above, the corresponding basis consists of the projectors $JW_{\vec{\epsilon}}$. ■

Note the following classification result (see for example [71, Corollary 5.2] for $\mathbb{K} = \mathbb{C}$).

Corollary 5.4. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$-modules $L(\lambda)$, where $\lambda = (\lambda_1, \lambda_2)$ is a length-two partition of d. Moreover, $\dim(L(\lambda)) = |\mathrm{Std}(\lambda)|$, where $\mathrm{Std}(\lambda)$ is the set of all standard tableaux of shape λ. □

Proof. This follows directly from Proposition 5.3 and Theorem 4.11 and Theorem 4.12 because we have $m_\lambda = |\mathrm{Std}(\lambda)|$. ■

5B.2. *Temperley–Lieb algebra: the non-semisimple case.* Let us assume that we have fixed $q \in \mathbb{K}^* - \{1, \pm\sqrt[2]{-1}\}$ to be a critical value such that $[k] = 0$ for some $k = 1, \dots, d$. Then, by Proposition 5.1, the algebra $\mathcal{TL}_d(\delta)$ is no longer semisimple. In particular, to the best of our knowledge, there is no diagrammatic analog of the Jones–Wenzl projectors in general.

Proposition 5.5. (**(New) cellular basis — the second.**) The datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ with $\mathcal{C}$ as in Theorem 3.9 for $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$. Moreover, $\mathcal{C} \neq \mathcal{C}_{\mathrm{GL}}$ for all $d > 1$ and all choices involved in the definition of our basis. Thus, there is a choice such that all generalized, non-semisimple Jones–Wenzl projectors are part of $\mathrm{im}(\mathcal{C})$. □

Proof. As in the proof of Proposition 5.3 and left to the reader. ■

Hence, directly from Proposition 5.5 and Theorem 4.11 and Theorem 4.12, we obtain:

Corollary 5.6. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$-modules $L(\lambda)$, where $\lambda = (\lambda_1, \lambda_2)$ is a length-two partition of d. Moreover, $\dim(L(\lambda)) = m_\lambda$, where m_λ is the multiplicity of $T_q(\lambda_1 - \lambda_2)$ as a summand of $T = V^{\otimes d}$. ■

Note that we can do better: one gets a decompositions

$$\boldsymbol{\mathcal{T}} \cong \boldsymbol{\mathcal{T}}_{-1} \oplus \boldsymbol{\mathcal{T}}_0 \oplus \boldsymbol{\mathcal{T}}_1 \oplus \cdots \oplus \boldsymbol{\mathcal{T}}_{l-3} \oplus \boldsymbol{\mathcal{T}}_{l-2} \oplus \boldsymbol{\mathcal{T}}_{l-1}, \tag{28}$$

where the blocks $\boldsymbol{\mathcal{T}}_{-1}$ and $\boldsymbol{\mathcal{T}}_{l-1}$ are semisimple if $\mathbb{K} = \mathbb{C}$. (This follows from the linkage principle. For notation and the statement see [10, Section 2].)

Fix $\mathbb{K} = \mathbb{C}$. As explained in [10, Section 3.5] each block in the decomposition (28) can be equipped with a non-trivial $\mathbb{Z}$-grading coming from the zig-zag algebra from [44]. Hence, we have the following.

Lemma 5.7. The $\mathbb{C}$-algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ can be equipped with a non-trivial $\mathbb{Z}$-grading. Thus, $\mathcal{TL}_d(\delta)$ over $\mathbb{C}$ can be equipped with a non-trivial $\mathbb{Z}$-grading. □

Proof. The second statement follows directly from the first using quantum Schur–Weyl duality. Hence, we only need to show the first.

Note that $T = V^{\otimes d}$ decomposes as in (27), but with $T_q(k)$'s instead of $\Delta_q(k)$'s, and we can order this decomposition by blocks. Each block carries a $\mathbb{Z}$-grading coming from the zig-zag algebra, as explained in [10, Section 3]. In particular, we can choose the basis elements c_{ij}^λ in such a way that we get the $\mathbb{Z}$-graded basis obtained in Corollary 4.23 therein. Since there is no interaction between different blocks, the statement follows. ■

Recall from [41, Definition 2.1] that a $\mathbb{Z}$-graded cell datum of a $\mathbb{Z}$-graded algebra is a cell datum for the algebra together with an additional *degree function* $\deg\colon \coprod_{\lambda \in \mathcal{P}} \mathcal{I}^\lambda \to \mathbb{Z}$, such that $\deg(c_{ij}^\lambda) = \deg(i) + \deg(j)$. For us the choice of $\deg(\cdot)$ is as follows.

If $\lambda \in \mathcal{P}$ is in one of the semisimple blocks, then we simply set $\deg(i) = 0$ for all $i \in \mathcal{I}^\lambda$.

Assume that $\lambda \in \mathcal{P}$ is not in the semisimple blocks. It is known that every $T_q(\lambda)$ has precisely two Weyl factors. The g_i^λ that map $\Delta_q(\lambda)$ into a higher $T_q(\mu)$ should be indexed by a 1-colored i whereas the g_i^λ mapping $\Delta_q(\lambda)$ into $T_q(\lambda)$ should have 0-colored i. Similarly for the f_j^λ's. Then the degree of the elements $i \in \mathcal{I}^\lambda$ should be the corresponding color. We get the following. (Here $\mathcal{C}$ is as in Theorem 3.9.)

Proposition 5.8. (**Graded cellular basis.**) The datum $(\mathcal{P},\mathcal{I},\mathcal{C},\mathrm{i})$ supplemented with the function $\deg(\cdot)$ from above is a $\mathbb{Z}$-graded cell datum for the $\mathbb{C}$-algebra $\mathcal{TL}_d(\delta)\cong\mathrm{End}_{\mathbf{U}_q}(T)$.□

Proof. The hardest part is cellularity which directly follows from Theorem 3.9. That the quintuple $(\mathcal{P},\mathcal{I},\mathcal{C},\mathrm{i},\deg)$ gives a $\mathbb{Z}$-graded cell datum follows from the construction. ■

Remark 5. Our grading and the one found by Plaza and Ryom-Hansen in [69] agree (up to a shift of the indecomposable summands). To see this, note that our algebra is isomorphic to the algebra $K_{1,n}$ studied in [19] which is by (4.8) therein and [21, Theorem 6.3] a quotient of some particular cyclotomic KL–R algebra (the compatibility of the grading follows for example from [42, Corollary B.6]). The same holds, by construction, for the grading in [69]. ▲

References

[1] H.H. Andersen. Tensor products of quantized tilting modules. *Comm. Math. Phys.*, 149(1):149–159, 1992.

[2] H.H. Andersen. Filtrations and tilting modules. *Ann. Sci. École Norm. Sup. (4)*, 30(3):353–366, 1997. doi:10.1016/S0012-9593(97)89924-7.

[3] H.H. Andersen. The strong linkage principle for quantum groups at roots of 1. *J. Algebra*, 260(1):2–15, 2003. doi:10.1016/S0021-8693(02)00618-X.

[4] H.H. Andersen, J.C. Jantzen, and W. Soergel. Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. *Astérisque*, (220):321, 1994.

[5] H.H. Andersen, G. Lehrer, and R. Zhang. Cellularity of certain quantum endomorphism algebras. *Pacific J. Math.*, 279(1-2):11–35, 2015. URL: http://arxiv.org/abs/1303.0984, doi:10.2140/pjm.2015.279.11.

[6] H.H. Andersen and V. Mazorchuk. Category $\mathcal{O}$ for quantum groups. *J. Eur. Math. Soc. (JEMS)*, 17(2):405–431, 2015. URL: http://arxiv.org/abs/1105.5500, doi:10.4171/JEMS/506.

[7] H.H. Andersen, P. Polo, and K.X. Wen. Representations of quantum algebras. *Invent. Math.*, 104(1):1–59, 1991. doi:10.1007/BF01245066.

[8] H.H. Andersen, C. Stroppel, and D. Tubbenhauer. Additional notes for the paper "Cellular structures using $\mathbf{U}_q$-tilting modules". 2015. URL: http://pure.au.dk/portal/files/100562565/cell_tilt_proofs_1.pdf, http://www.math.uni-bonn.de/ag/stroppel/cell-tilt-proofs_neu.pdf, http://www.math.uni-bonn.de/people/dtubben/cell-tilt-proofs.pdf.

[9] H.H. Andersen, C. Stroppel, and D. Tubbenhauer. Semisimplicity of Hecke and (walled) Brauer algebras. *J. Aust. Math. Soc.*, 103(1):1–44, 2017. URL: http://arxiv.org/abs/1507.07676, doi:10.1017/S1446788716000392.

[10] H.H. Andersen and D. Tubbenhauer. Diagram categories for $\mathbf{U}_q$-tilting modules at roots of unity. *Transform. Groups*, 22(1):29–89, 2017. URL: https://arxiv.org/abs/1409.2799, doi:10.1007/s00031-016-9363-z.

[11] S. Ariki. Cyclotomic q-Schur algebras as quotients of quantum algebras. *J. Reine Angew. Math.*, 513:53–69, 1999. doi:10.1515/crll.1999.063.

[12] S. Ariki and K. Koike. A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z})\wr\mathfrak{S}_n$ and construction of its irreducible representations. *Adv. Math.*, 106(2):216–243, 1994. doi:10.1006/aima.1994.1057.

[13] S. Ariki, A. Mathas, and H. Rui. Cyclotomic Nazarov–Wenzl algebras. *Nagoya Math. J.*, 182:47–134, 2006. URL: http://arxiv.org/abs/math/0506467.

[14] J.S. Birman and H. Wenzl. Braids, link polynomials and a new algebra. *Trans. Amer. Math. Soc.*, 313(1):249–273, 1989. doi:10.2307/2001074.

[15] R. Brauer. On algebras which are connected with the semisimple continuous groups. *Ann. of Math. (2)*, 38(4):857–872, 1937. doi:10.2307/1968843.

[16] J. Brundan and A. Kleshchev. Blocks of cyclotomic Hecke algebras and Khovanov–Lauda algebras. *Invent. Math.*, 178(3):451–484, 2009. URL: http://arxiv.org/abs/0808.2032, doi:10.1007/s00222-009-0204-8.

[17] J. Brundan and A. Kleshchev. Schur–Weyl duality for higher levels. *Selecta Math. (N.S.)*, 14(1):1–57, 2008. URL: http://arxiv.org/abs/math/0605217, doi:10.1007/s00029-008-0059-7.

[18] J. Brundan and C. Stroppel. Gradings on walled Brauer algebras and Khovanov's arc algebra. *Adv. Math.*, 231(2):709–773, 2012. URL: http://arxiv.org/abs/1107.0999, doi:10.1016/j.aim.2012.05.016.

[19] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra I: cellularity. *Mosc. Math. J.*, 11(4):685–722, 821–822, 2011. URL: http://arxiv.org/abs/0806.1532.

[20] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra II: Koszulity. *Transform. Groups*, 15(1):1–45, 2010. URL: http://arxiv.org/abs/0806.3472, doi:10.1007/s00031-010-9079-4.

[21] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra III: category $\mathcal{O}$. *Represent. Theory*, 15:170–243, 2011. URL: http://arxiv.org/abs/0812.1090, doi:10.1090/S1088-4165-2011-00389-7.

[22] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra IV: the general linear supergroup. *J. Eur. Math. Soc. (JEMS)*, 14(2):373–419, 2012. URL: http://arxiv.org/abs/0907.2543, doi:10.4171/JEMS/306.

[23] S. Cautis, J. Kamnitzer, and S. Morrison. Webs and quantum skew Howe duality. *Math. Ann.*, 360(1-2):351–390, 2014. URL: http://arxiv.org/abs/1210.6437, doi:10.1007/s00208-013-0984-4.

[24] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.

[25] B. Cooper and M. Hogancamp. An exceptional collection for Khovanov homology. *Algebr. Geom. Topol.*, 15(5):2659–2707, 2015. URL: http://arxiv.org/abs/1209.1002, doi:10.2140/agt.2015.15.2659.

[26] R. Dipper, S. Doty, and J. Hu. Brauer algebras, symplectic Schur algebras and Schur–Weyl duality. *Trans. Amer. Math. Soc.*, 360(1):189–213 (electronic), 2008. URL: http://arxiv.org/abs/math/0503545, doi:10.1090/S0002-9947-07-04179-7.

[27] R. Dipper, S. Doty, and F. Stoll. The quantized walled Brauer algebra and mixed tensor space. *Algebr. Represent. Theory*, 17(2):675–701, 2014. URL: http://arxiv.org/abs/0806.0264, doi:10.1007/s10468-013-9414-2.

[28] R. Dipper, G. James, and A. Mathas. Cyclotomic q-Schur algebras. *Math. Z.*, 229(3):385–416, 1998. doi:10.1007/PL00004665.

[29] S. Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993. doi:10.1007/BF02571640.

[30] S. Donkin. *The q-Schur algebra*, volume 253 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998. doi:10.1017/CBO9780511600708.

[31] S. Doty and J. Hu. Schur–Weyl duality for orthogonal groups. *Proc. Lond. Math. Soc. (3)*, 98(3):679–713, 2009. URL: http://arxiv.org/abs/0712.0944, doi:10.1112/plms/pdn044.

[32] J. Du, B. Parshall, and L. Scott. Quantum Weyl reciprocity and tilting modules. *Comm. Math. Phys.*, 195(2):321–352, 1998. doi:10.1007/s002200050392.

[33] M. Ehrig and C. Stroppel. Nazarov–Wenzl algebras, coideal subalgebras and categorified skew Howe duality. 2013. URL: http://arxiv.org/abs/1310.1972.

[34] M. Ehrig and C. Stroppel. Koszul Gradings on Brauer Algebras. *Int. Math. Res. Not. IMRN*, (13):3970–4011, 2016. URL: http://arxiv.org/abs/1504.03924, doi:10.1093/imrn/rnv267.

[35] M. Ehrig and C. Stroppel. Schur–Weyl duality for the Brauer algebra and the ortho-symplectic Lie superalgebra. *Math. Z.*, 284(1-2):595–613, 2016. URL: http://arxiv.org/abs/1412.7853, doi:10.1007/s00209-016-1669-y.

[36] B. Elias. Light ladders and clasp conjectures. 2015. URL: http://arxiv.org/abs/1510.06840.

[37] K. Erdmann. Tensor products and dimensions of simple modules for symmetric groups. *Manuscripta Math.*, 88(3):357–386, 1995. doi:10.1007/BF02567828.

[38] J.J. Graham and G. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996. doi:10.1007/BF01232365.

[39] T. Halverson and A. Ram. q-rook monoid algebras, Hecke algebras, and Schur–Weyl duality. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 283(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 6):224–250, 262–263, 2001. URL: http://arxiv.org/abs/math/0401330, doi:10.1023/B:JOTH.0000024623.99412.13.

[40] J. Hu. BMW algebra, quantized coordinate algebra and type C Schur–Weyl duality. *Represent. Theory*, 15:1–62, 2011. URL: http://arxiv.org/abs/0708.3009, doi:10.1090/S1088-4165-2011-00369-1.

[41] J. Hu and A. Mathas. Graded cellular bases for the cyclotomic Khovanov–Lauda–Rouquier algebras of type A. *Adv. Math.*, 225(2):598–642, 2010. URL: http://arxiv.org/abs/0907.2985, doi:10.1016/j.aim.2010.03.002.

[42] J. Hu and A. Mathas. Quiver Schur algebras for linear quivers. *Proc. Lond. Math. Soc. (3)*, 110(6):1315–1386, 2015. URL: http://arxiv.org/abs/1110.1699, doi:10.1112/plms/pdv007.

[43] J. Hu and F. Stoll. On double centralizer properties between quantum groups and Ariki–Koike algebras. *J. Algebra*, 275(1):397–418, 2004. doi:10.1016/j.jalgebra.2003.10.026.

[44] R.S. Huerfano and M. Khovanov. A category for the adjoint representation. *J. Algebra*, 246(2):514–542, 2001. URL: https://arxiv.org/abs/math/0002060, doi:10.1006/jabr.2001.8962.

[45] J.E. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, volume 94 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008. doi:10.1090/gsm/094.

[46] J.C. Jantzen. Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formen. *Bonn. Math. Schr.*, (67):v+124, 1973.

[47] J.C. Jantzen. *Lectures on quantum groups*, volume 6 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.

[48] J.C. Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[49] V.F.R. Jones. Index for subfactors. *Invent. Math.*, 72(1):1–25, 1983. doi:10.1007/BF01389127.

[50] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. I, II, III, IV. *J. Amer. Math. Soc.*, 6,7(4,2):905–947, 949–1011, 335–381, 383–453, 1993,1994.

[51] M. Khovanov. A functor-valued invariant of tangles. *Algebr. Geom. Topol.*, 2:665–741, 2002. URL: http://arxiv.org/abs/math/0103190, doi:10.2140/agt.2002.2.665.

[52] M. Khovanov and A.D. Lauda. A diagrammatic approach to categorification of quantum groups I. *Represent. Theory*, 13:309–347, 2009. URL: http://arxiv.org/abs/0803.4121, doi:10.1090/S1088-4165-09-00346-X.

[53] M. Khovanov and A.D. Lauda. A diagrammatic approach to categorification of quantum groups II. *Trans. Amer. Math. Soc.*, 363(5):2685–2700, 2011. URL: http://arxiv.org/abs/0804.2080, doi:10.1090/S0002-9947-2010-05210-9.

[54] K. Koike. On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters. *Adv. Math.*, 74(1):57–86, 1989. doi:10.1016/0001-8708(89)90004-2.

[55] S. König and C. Xi. On the structure of cellular algebras. In *Algebras and modules, II (Geiranger, 1996)*, volume 24 of *CMS Conf. Proc.*, pages 365–386. Amer. Math. Soc., Providence, RI, 1998.

[56] G. Kuperberg. Spiders for rank 2 Lie algebras. *Comm. Math. Phys.*, 180(1):109–151, 1996. URL: http://arxiv.org/abs/q-alg/9712003.

[57] A. Lascoux, B. Leclerc, and J.-Y. Thibon. Hecke algebras at roots of unity and crystal bases of quantum affine algebras. *Comm. Math. Phys.*, 181(1):205–263, 1996.

[58] G. Lehrer and R. Zhang. The second fundamental theorem of invariant theory for the orthogonal group. *Ann. of Math. (2)*, 176(3):2031–2054, 2012. URL: http://arxiv.org/abs/1102.3221, doi:10.4007/annals.2012.176.3.12.

[59] G. Li. A KLR grading of the Brauer algebras. 2014. URL: http://arxiv.org/abs/1409.1195.

[60] G. Lusztig. Modular representations and quantum groups. In *Classical groups and related topics (Beijing, 1987)*, volume 82 of *Contemp. Math.*, pages 59–77. Amer. Math. Soc., Providence, RI, 1989. doi:10.1090/conm/082/982278.

[61] M. Mackaay. The $\mathfrak{sl}_n$-web algebras and dual canonical bases. *J. Algebra*, 409:54–100, 2014. URL: http://arxiv.org/abs/1308.0566, doi:10.1016/j.jalgebra.2014.02.036.

[62] M. Mackaay, W. Pan, and D. Tubbenhauer. The $\mathfrak{sl}_3$-web algebra. *Math. Z.*, 277(1-2):401–479, 2014. URL: http://arxiv.org/abs/1206.2118, doi:10.1007/s00209-013-1262-6.

[63] P. Martin. *Potts models and related problems in statistical mechanics*, volume 5 of *Series on Advances in Statistical Mechanics*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1991. doi:10.1142/0983.

[64] P. Martin and H. Saleur. The blob algebra and the periodic Temperley-Lieb algebra. *Lett. Math. Phys.*, 30(3):189–206, 1994. URL: http://arxiv.org/abs/hep-th/9302094, doi:10.1007/BF00805852.

[65] V. Mazorchuk and C. Stroppel. $G(\ell, k, d)$-modules via groupoids. *J. Algebraic Combin.*, 43(1):11–32, 2016. URL: http://arxiv.org/abs/1412.4494, doi:10.1007/s10801-015-0623-0.

[66] J. Murakami. The Kauffman polynomial of links and representation theory. *Osaka J. Math.*, 24(4):745–758, 1987.

[67] R. Paget. Representation theory of q-rook monoid algebras. *J. Algebraic Combin.*, 24(3):239–252, 2006. doi:10.1007/s10801-006-0010-y.

[68] J. Paradowski. Filtrations of modules over the quantum algebra. In *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 93–108. Amer. Math. Soc., Providence, RI, 1994.

[69] D. Plaza and S. Ryom-Hansen. Graded cellular bases for Temperley–Lieb algebras of type A and B. *J. Algebraic Combin.*, 40(1):137–177, 2014. URL: http://arxiv.org/abs/1203.2592, doi:10.1007/s10801-013-0481-6.

[70] S. Riche and G. Williamson. Tilting modules and the p-canonical basis. 2015. https://hal-clermont-univ.archives-ouvertes.fr/hal-01249796/document. URL: http://arxiv.org/abs/1512.08296.

[71] D. Ridout and Y. Saint-Aubin. Standard modules, induction and the structure of the Temperley–Lieb algebra. *Adv. Theor. Math. Phys.*, 18(5):957–1041, 2014. URL: http://arxiv.org/abs/1204.4505.

[72] C.M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208(2):209–223, 1991. doi:10.1007/BF02571521.

[73] D.E.V. Rose and D. Tubbenhauer. Symmetric webs, Jones–Wenzl recursions, and q-Howe duality. *Int. Math. Res. Not. IMRN*, (17):5249–5290, 2016. URL: http://arxiv.org/abs/1501.00915, doi:10.1093/imrn/rnv302.

[74] R. Rouquier. 2-Kac–Moody algebras. 2008. URL: http://arxiv.org/abs/0812.5023.

[75] R. Rouquier, P. Shan, M. Varagnolo, and E. Vasserot. Categorifications and cyclotomic rational double affine Hecke algebras. *Invent. Math.*, 204(3):671–786, 2016. URL: http://arxiv.org/abs/1305.4456, doi:10.1007/s00222-015-0623-7.

[76] S. Ryom-Hansen. Cell structures on the blob algebra. *Represent. Theory*, 16:540–567, 2012. URL: http://arxiv.org/abs/0911.1923, doi:10.1090/S1088-4165-2012-00424-1.

[77] S. Ryom-Hansen. The Ariki–Terasoma–Yamada tensor space and the blob algebra. *J. Algebra*, 324(10):2658–2675, 2010. URL: http://arxiv.org/abs/math/0505278, doi:10.1016/j.jalgebra.2010.08.018.

[78] M. Sakamoto and T. Shoji. Schur–Weyl reciprocity for Ariki–Koike algebras. *J. Algebra*, 221(1):293–314, 1999. doi:10.1006/jabr.1999.7973.

[79] A. Sartori. The degenerate affine walled Brauer algebra. *J. Algebra*, 417:198–233, 2014. URL: http://arxiv.org/abs/1305.2347, doi:10.1016/j.jalgebra.2014.06.030.

[80] A. Sartori and C. Stroppel. Walled Brauer algebras as idempotent truncations of level 2 cyclotomic quotients. *J. Algebra*, 440:602–638, 2015. URL: http://arxiv.org/abs/1411.2771, doi:10.1016/j.jalgebra.2015.06.018.

[81] W. Soergel. Character formulas for tilting modules over Kac–Moody algebras. *Represent. Theory*, 2:432–448 (electronic), 1998. doi:10.1090/S1088-4165-98-00057-0.

[82] W. Soergel. Character formulas for tilting modules over quantum groups at roots of one. In *Current developments in mathematics, 1997 (Cambridge, MA)*, pages 161–172. Int. Press, Boston, MA, 1999.

[83] W. Soergel. Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. *J. Amer. Math. Soc.*, 3(2):421–445, 1990. doi:10.2307/1990960.

[84] W. Soergel. Kazhdan–Lusztig polynomials and a combinatoric[s] for tilting modules. *Represent. Theory*, 1:83–114 (electronic), 1997. doi:10.1090/S1088-4165-97-00021-6.

[85] L. Solomon. The Bruhat decomposition, Tits system and Iwahori ring for the monoid of matrices over a finite field. *Geom. Dedicata*, 36(1):15–49, 1990. doi:10.1007/BF00181463.

[86] C. Stroppel. Category $\mathcal{O}$: gradings and translation functors. *J. Algebra*, 268(1):301–326, 2003. doi:10.1016/S0021-8693(03)00308-9.

[87] T. Tanisaki. Character formulas of Kazhdan–Lusztig type. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, volume 40 of *Fields Inst. Commun.*, pages 261–276. Amer. Math. Soc., Providence, RI, 2004.

[88] H.N.V. Temperley and E.H. Lieb. Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem. *Proc. Roy. Soc. London Ser. A*, 322(1549):251–280, 1971.

[89] D. Tubbenhauer. $\mathfrak{sl}_3$-web bases, intermediate crystal bases and categorification. *J. Algebraic Combin.*, 40(4):1001–1076, 2014. URL: https://arxiv.org/abs/1310.2779, doi:10.1007/s10801-014-0518-5.

[90] D. Tubbenhauer. $\mathfrak{sl}_n$-webs, categorification and Khovanov–Rozansky homologies. 2014. URL: http://arxiv.org/abs/1404.5752.

[91] V.G. Turaev. Operator invariants of tangles, and R-matrices. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(5):1073–1107, 1135, 1989. Translation in Math. USSR-Izv. 35:2 (1990), 411-444.

[92] V.G. Turaev. *Quantum invariants of knots and* 3*-manifolds*, volume 18 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, revised edition, 2010. doi:10.1515/9783110221848.

[93] H. Wenzl. On sequences of projections. *C. R. Math. Rep. Acad. Sci. Canada*, 9(1):5–9, 1987.

[94] B.W. Westbury. Invariant tensors and cellular categories. *J. Algebra*, 321(11):3563–3567, 2009. URL: http://arxiv.org/abs/0806.4045, doi:10.1016/j.jalgebra.2008.07.004.

[95] B.W. Westbury. The representation theory of the Temperley–Lieb algebras. *Math. Z.*, 219(4):539–565, 1995. doi:10.1007/BF02572380.

H.H.A.: h.haahr.andersen@gmail.com

C.S.: stroppel@math.uni-bonn.de

D.T.: dtubben@math.uni-bonn.de

ADDITIONAL NOTES FOR THE PAPER "CELLULAR STRUCTURES USING $\mathbf{U}_q$-TILTING MODULES"

HENNING HAAHR ANDERSEN, CATHARINA STROPPEL, AND DANIEL TUBBENHAUER

Abstract. This eprint contains additional notes for the paper "Cellular structures using $\mathbf{U}_q$-tilting modules". We recall some basic notions about representation and tilting theory for $\mathbf{U}_q(\mathfrak{g})$, and give some proofs are omitted in the published version.

Contents

1. Introduction

In this note we first recall some facts, notions and notations about the representation theory of quantum enveloping algebras attached to some Cartan datum. (In particular, results that are useful to understand the construction in [6].) This is done in Section 2 and Section 3, where we stress that almost all results are known, but, to the best of our knowledge, were never collected in one document before.

Second, we give a more detailed construction of the cellular bases for the Temperley–Lieb algebras given in [6, Section 6B], which we also use to deduce semi-simplicity criteria as well as dimension formulas for the simple modules of the Temperley–Lieb algebras. This is done in Section 4. Again, no of the results are new, but might be helpful to understand the novel cellular bases obtained in [6, Section 6B].

We stress that we throughout have (almost no) restriction on the underlying field or the quantum parameter q.

Additional remarks. We hope that this note provides an easier access to the basic facts on tilting modules adapted to the special quantum group case than currently available (spread over different articles) in the literature. The paper [6] – as well as [5] – follow the setup here.

We might change this note in the future by adding extra material or by improving the exposition.

The first two sections of this note can be read without knowing any results or notation from [6], but Section 4 depends on the construction from [6] in the sense that we elaborate the arguments given therein (we only recall the main results). We hope that all of this together will make [6] (and [5]) reasonably self-contained.

2. Quantum groups and their representations

In the present section we recall the definitions and results about quantum groups and their representation theory in the semisimple and the non-semisimple case. From now on fix a field $\mathbb{K}$ and set $\mathbb{K}^* = \mathbb{K} - \{0, -1\}$, if $\mathrm{char}(\mathbb{K}) > 2$, and $\mathbb{K}^* = \mathbb{K} - \{0\}$, otherwise.

2A. **The quantum groups $\mathbf{U}_v$ and $\mathbf{U}_q$.** Let Φ be a finite *root system* in an Euclidean space E. We fix a choice of *positive roots* $\Phi^+ \subset \Phi$ and *simple roots* $\Pi \subset \Phi^+$. We assume that we have n simple roots that we denote by $\alpha_1, \dots, \alpha_n$. For each $\alpha \in \Phi$, we denote by $\alpha^\vee \in \Phi^\vee$ the corresponding *coroot*, and we let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ be the half-sum of all positive roots. Then $\mathbf{A} = (\langle \alpha_i, \alpha_j^\vee\rangle)_{i,j=1}^n$ is called the *Cartan matrix*.

As usual, we need to symmetrize $\mathbf{A}$ and we do so by choosing for $i = 1, \dots, n$ minimal $d_i \in \mathbb{Z}_{>0}$ such that $(d_i a_{ij})_{i,j=1}^n$ is symmetric. (The Cartan matrix $\mathbf{A}$ is already symmetric in most of our examples. Thus, $d_i = 1$ for all $i = 1, \dots, n$.)

By the set of *(integral) weights* we mean $X = \{\lambda \in E \mid \langle \lambda, \alpha_i^\vee\rangle \in \mathbb{Z} \text{ for all } \alpha_i \in \Pi\}$. The *dominant (integral) weights* X^+ are those $\lambda \in X$ such that $\langle \lambda, \alpha_i^\vee\rangle \geq 0$ for all $\alpha_i \in \Pi$.

The *fundamental weights*, denoted by $\omega_i \in X$ for $i = 1, \dots, n$, are characterized by

$$\langle \omega_i, \alpha_j^\vee\rangle = \delta_{ij} \quad \text{for all } j = 1, \dots, n.$$

Recall that there is a *partial ordering* on X given by $\mu \leq \lambda$ if and only if $\lambda - \mu$ is an $\mathbb{Z}_{\geq 0}$-valued linear combination of the simple roots, that is, $\lambda - \mu = \sum_{i=1}^n a_i\alpha_i$ with $a_i \in \mathbb{Z}_{\geq 0}$.

Example 2.1. One of the most important examples is the standard choice of a Cartan datum $(\mathbf{A}, \Pi, \Phi, \Phi^+)$ associated with the Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}$ for $n \geq 1$. Here $E = \mathbb{R}^{n+1}/(1, \dots, 1)$ (which we identify with $\mathbb{R}^n$ in calculations) and $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n\}$, where the ε_i's denote the standard basis of E. The positive roots are $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n+1\}$ with maximal root $\alpha_0 = \varepsilon_1 - \varepsilon_{n+1}$. Moreover,

$$\rho = \tfrac{1}{2}\sum_{i=1}^{n+1}(n - 2(i-1))\varepsilon_i = \sum_{i=1}^{n+1}(n - i + 1)\varepsilon_i - \tfrac{1}{2}(n, \dots, n).$$

(Seen as a $\mathfrak{sl}_{n+1}$-weight, i.e. we can drop the $-\frac{1}{2}(n, \dots, n)$.)

The set of fundamental weights is $\{\omega_i = \varepsilon_1 + \dots + \varepsilon_i \mid 1 \leq i \leq n\}$. For explicit calculations one often identifies

$$\lambda = \sum_{i=1}^n a_i\omega_i \in X^+$$

with the partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$ given by $\lambda_k = \sum_{i=k}^n a_i$ for $k = 1, \dots, n$. ▲

As some piece of notation, for $a \in \mathbb{Z}$ and $b, d \in \mathbb{Z}_{\geq 0}$, $[a]_d$ denotes the *a-quantum integer* (with $[0]_d = 0$), $[b]_d!$ denotes the *b-quantum factorial*. That is,

$$[a]_d = \frac{v^{ad} - v^{-ad}}{v^d - v^{-d}}, \quad [a] = [a]_1 \quad \text{and} \quad [b]_d! = [1]_d \cdots [b-1]_d [b]_d, \quad [b]! = [b]_1!$$

(with $[0]_d! = 1$, by convention) and

$$\begin{bmatrix} a \\ b \end{bmatrix}_d = \frac{[a]_d [a-1]_d \cdots [a-b+2]_d [a-b+1]_d}{[b]_d!}, \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}_1$$

denotes the (a,b)-*quantum binomial*. Observe that $[-a]_d = -[a]_d$.

Next, we assign an algebra $\mathbf{U}_v = \mathbf{U}_v(\mathbf{A})$ to a given Cartan matrix $\mathbf{A}$. Abusing notation, we also write $\mathbf{U}_v(\mathfrak{g})$ etc. if no confusion can arise. Here and throughout, v always means a generic parameter, while $q \in \mathbb{K}^*$ will always mean a specialization (to e.g. a root of unity).

Definition 2.2. (**Quantum enveloping algebra — generic**.) Given a Cartan matrix $\mathbf{A}$, then the *quantum enveloping algebra* $\mathbf{U}_v = \mathbf{U}_v(\mathbf{A})$ associated to it is the associative, unital $\mathbb{Q}(v)$-algebra generated by $K_1^{\pm 1}, \dots, K_n^{\pm 1}$ and $E_1, F_1, \dots, E_n, F_n$, where n is the size of $\mathbf{A}$, subject to the relations

$$\begin{gathered} K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\ K_i E_j = v^{d_i a_{ij}} E_j K_i, \quad K_i F_j = v^{-d_i a_{ij}} F_j K_i, \\ E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}, \\ \sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^r E_j E_i^s = 0, \quad \text{if} \quad i \neq j, \\ \sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^r F_j F_i^s = 0, \quad \text{if} \quad i \neq j, \end{gathered}$$

with the quantum numbers as above. ▲

It is worth noting that $\mathbf{U}_v$ is a Hopf algebra with coproduct Δ given by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i.$$

The antipode S and the counit ε are given by

$$\begin{gathered} S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1}, \\ \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1. \end{gathered}$$

We want to "specialize" the generic parameter v of $\mathbf{U}_v$ to be, for example, a root of unity $q \in \mathbb{K}^*$. In order to do so, let $\mathscr{A} = \mathbb{Z}[v, v^{-1}]$.

Definition 2.3. (**Lusztig's $\mathscr{A}$-form $\mathbf{U}_{\mathscr{A}}$.**) Define for all $j \in \mathbb{Z}_{\geq 0}$ the *j-th divided powers*

$$E_i^{(j)} = \frac{E_i^j}{[j]_{d_i}!} \quad \text{and} \quad F_i^{(j)} = \frac{F_i^j}{[j]_{d_i}!}.$$

Then $\mathbf{U}_{\mathscr{A}} = \mathbf{U}_{\mathscr{A}}(\mathbf{A})$ is defined as the $\mathscr{A}$-subalgebra of $\mathbf{U}_v$ generated by K_i, K_i^{-1}, $E_i^{(j)}$ and $F_i^{(j)}$ for $i = 1, \dots, n$ and $j \in \mathbb{Z}_{\geq 0}$. ▲

Lusztig's $\mathscr{A}$-form originates in [25] and is designed to allow specializations.

Definition 2.4. (**Quantum enveloping algebras — specialized.**) Fix $q \in \mathbb{K}^*$. Consider $\mathbb{K}$ as an $\mathscr{A}$-module by specializing v to q. Define

$$\mathbf{U}_q = \mathbf{U}_q(\mathbf{A}) = \mathbf{U}_{\mathscr{A}} \otimes_{\mathscr{A}} \mathbb{K}.$$

Abusing notation, we will usually abbreviate $E_i^{(j)} \otimes 1 \in \mathbf{U}_q$ with $E_i^{(j)}$. Analogously for the other generators of $\mathbf{U}_q$. ▲

Note that we can recover the generic case $\mathbf{U}_v$ by choosing $\mathbb{K} = \mathbb{Q}(v)$ and $q = v$.

Example 2.5. In the $\mathfrak{sl}_2$ case and the datum $\mathbf{A}$ as in Example 2.1 above, the $\mathbb{Q}(v)$-algebra $\mathbf{U}_v(\mathfrak{sl}_2) = \mathbf{U}_v(\mathbf{A})$ is generated by K and K^{-1} and E, F subject to the relations

$$KK^{-1} = K^{-1}K = 1,$$
$$EF - FE = \frac{K - K^{-1}}{v - v^{-1}},$$
$$KE = v^2 EK \quad \text{and} \quad KF = v^{-2} FK.$$

We point out that $\mathbf{U}_v(\mathfrak{sl}_2)$ already contains the divided powers since no quantum number vanishes in $\mathbb{Q}(v)$. Let q be a complex, primitive third root of unity. Thus, $q + q^{-1} = [2] = -1$, $q^2 + 1 + q^{-2} = [3] = 0$ and $q^3 + q^1 + q^{-1} + q^{-3} = [4] = 1$. More generally,

$$[a] = i \in \{0, +1, -1\}, \quad i \equiv a \bmod 3.$$

Hence, $\mathbf{U}_q(\mathfrak{sl}_2)$ is generated by $K, K^{-1}, E, F, E^{(3)}$ and $F^{(3)}$ subject to the relations as above. (Here $E^{(3)}, F^{(3)}$ are extra generators since $E^3 = [3]!E^{(3)} = 0$ because of $[3] = 0$.) This is precisely the convention used in [18, Chapter 1], but specialized at q. ▲

It is easy to check that $\mathbf{U}_{\mathscr{A}}$ is a Hopf subalgebra of $\mathbf{U}_v$, see [23, Proposition 4.8]. Thus, $\mathbf{U}_q$ inherits a Hopf algebra structure from $\mathbf{U}_v$.

Moreover, it is known that all three algebras—$\mathbf{U}_v$, $\mathbf{U}_{\mathscr{A}}$ and $\mathbf{U}_q$—have a triangular decomposition

$$\mathbf{U}_v = \mathbf{U}_v^- \mathbf{U}_v^0 \mathbf{U}_v^+, \qquad \mathbf{U}_{\mathscr{A}} = \mathbf{U}_{\mathscr{A}}^- \mathbf{U}_{\mathscr{A}}^0 \mathbf{U}_{\mathscr{A}}^+, \qquad \mathbf{U}_q = \mathbf{U}_q^- \mathbf{U}_q^0 \mathbf{U}_q^+,$$

where $\mathbf{U}_v^-, \mathbf{U}_{\mathscr{A}}^-, \mathbf{U}_q^-$ denote the subalgebras generated only by the F_i's (or, in addition, the divided powers for $\mathbf{U}_{\mathscr{A}}^-$ and $\mathbf{U}_q^-$) and $\mathbf{U}_v^+, \mathbf{U}_{\mathscr{A}}^+, \mathbf{U}_q^+$ denote the subalgebras generated only by the E_i's (or, in addition, the divided powers for $\mathbf{U}_{\mathscr{A}}^+$ and $\mathbf{U}_q^+$). The Cartan part $\mathbf{U}_v^0$ is as usual generated by K_i, K_i^{-1} for $i = 1, \dots, n$. For the Cartan part $\mathbf{U}_{\mathscr{A}}^0$ one needs to be a little bit more careful, since it is generated by

$$\tilde{K}_{i,t} = \begin{bmatrix} K_i \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i v^{d_i(1-s)} - K_i^{-1} v^{-d_i(1-s)}}{v^{d_i s} - v^{-d_i s}} \tag{1}$$

for $i = 1, \dots, n$ and $t \in \mathbb{Z}_{\geq 0}$ in addition to the generators K_i, K_i^{-1}. Similarly for $\mathbf{U}_q^0$.

Roughly: the triangular decomposition can be proven by ordering F's to the left and E's to the right using the relations from Definition 2.2. (The hard part here is to show linear independence.) Details can, for example, be found in [18, Chapter 4, Section 17] for the generic case, and in [25, Theorem 8.3(iii)] for the other cases.

Note that, if $q = 1$, then $\mathbf{U}_q$ modulo the ideal generated by $\{K_i - 1 \mid i = 1, \dots, n\}$ can be identified with the hyperalgebra of the semisimple algebraic group G over $\mathbb{K}$ associated to the Cartan matrix, see [19, Part I, Chapter 7.7].

2B. **Representation theory of $\mathbf{U}_v$: the generic, semisimple case.** Let $\lambda \in X$ be a $\mathbf{U}_v$-weight. As usual, we identify λ with a *character of* $\mathbf{U}_v^0$ (an algebra homomorphism to $\mathbb{Q}(v)$) via

$$\lambda\colon \mathbf{U}_v^0 = \mathbb{Q}(v)[K_1^{\pm}, \dots, K_n^{\pm}] \to \mathbb{Q}(v), \quad K_i^{\pm} \mapsto v^{\pm d_i \langle \lambda, \alpha_i^{\vee} \rangle}, \quad i = 1, \dots, n.$$

Abusing notation, we use the same symbols for the $\mathbf{U}_v$-weights λ and the characters λ.

Moreover, if $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$, then this can be viewed as a character of $\mathbf{U}_v^0$ via

$$\underline{\epsilon}\colon \mathbf{U}_v^0 = \mathbb{Q}(v)[K_1^{\pm}, \dots, K_n^{\pm}] \to \mathbb{Q}(v), \quad K_i^{\pm} \mapsto \pm\epsilon_i, \quad i = 1, \dots, n.$$

This extends to a character of $\mathbf{U}_v$ by setting $\underline{\epsilon}(E_i) = \underline{\epsilon}(F_i) = 0$.

Every finite-dimensional $\mathbf{U}_v$-module M can be decomposed into

$$M = \bigoplus_{\lambda, \underline{\epsilon}} M_{\lambda, \underline{\epsilon}}, \qquad (2)$$
$$M_{\lambda, \underline{\epsilon}} = \{m \in M \mid um = \lambda(u)\underline{\epsilon}(u)m, u \in \mathbf{U}_v^0\}$$

where the direct sum runs over all $\lambda \in X$ and all $\underline{\epsilon} \in \{\pm 1\}^n$, see [18, Chapter 5, Section 2].

Set $M_1 = \bigoplus_{\lambda} M_{\lambda, (1, \dots, 1)}$ and call a $\mathbf{U}_v$-module M a $\mathbf{U}_v$*-module of type* 1 if $M_1 = M$.

Example 2.6. If $\mathfrak{g} = \mathfrak{sl}_2$, then the $\mathbf{U}_v(\mathfrak{sl}_2)$-modules of type 1 are precisely those where K has eigenvalues v^k for $k \in \mathbb{Z}$ whereas type -1 means that K has eigenvalues $-v^k$. ▲

Given a $\mathbf{U}_v$-module M satisfying (2), we have $M \cong \bigoplus_{\underline{\epsilon}} M_1 \otimes \underline{\epsilon}$. Thus, morally it suffices to study $\mathbf{U}_v$-modules of type 1, which we will do in this paper:

Assumption 2.7. From now on, all appearing $\mathbf{U}_v$-modules are assumed to be of type 1 and we omit to mention this in the following. Similarly for $\mathbf{U}_q$-modules later on. ▲

Proposition 2.8. (**Semisimplicity: the generic case.**) The category $\mathbf{U}_v$-**Mod** consisting of finite-dimensional $\mathbf{U}_v$-modules is semisimple. □

Proof. This is [4, Corollary 7.7] or [18, Theorem 5.17]. ■

The simple modules in $\mathbf{U}_v$-**Mod** can be constructed as follows. For each $\lambda \in X^+$ set

$$\nabla_v(\lambda) = \mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v} \mathbb{Q}(v)_\lambda,$$

called the *dual Weyl* $\mathbf{U}_v$-module associated to $\lambda \in X^+$. Here $\mathbb{Q}(v)_\lambda$ is the one-dimensional $\mathbf{U}_v^- \mathbf{U}_v^0$-module determined by the character λ (and extended to $\mathbf{U}_v^- \mathbf{U}_v^0$ via $\lambda(F_i) = 0$) and $\mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v}(\cdot)$ is the induction functor from [4, Section 2], i.e. the functor

$$\mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v}\colon \mathbf{U}_v^- \mathbf{U}_v^0\text{-}\mathbf{Mod} \to \mathbf{U}_v\text{-}\mathbf{Mod},\ M' \mapsto \mathcal{F}(\mathrm{Hom}_{\mathbf{U}_v^- \mathbf{U}_v^0}(\mathbf{U}_v, M'))$$

obtained by using the standard embedding of $\mathbf{U}_v^- \mathbf{U}_v^0 \hookrightarrow \mathbf{U}_v$. Here the functor $\mathcal{F}$—as given in [4, Section 2.2]—assigns to an arbitrary $\mathbf{U}_v$-module M the $\mathbf{U}_v$-module

$$\mathcal{F}(M) = \left\{ m \in \textstyle\bigoplus_{\lambda \in X} M_\lambda \mid E_i^{(r)} m = 0 = F_i^{(r)} m \quad \text{for all } i \in \mathbb{Z}_{\geq 0} \text{ and for } r \gg 0 \right\}.$$

(Which thus, defines $\mathcal{F}(M)$ for $M = \mathrm{Hom}_{\mathbf{U}_v^-\mathbf{U}_v^0}(\mathbf{U}_v, M')$.)

It turns out that the $\nabla_v(\lambda)$ for $\lambda \in X^+$ form a complete set of non-isomorphic, simple $\mathbf{U}_v$-modules, see [18, Theorem 5.10]. Moreover, all $M \in \mathbf{U}_v$-**Mod** have a $\mathbf{U}_v$-*weight space decomposition*, cf. (2), i.e.:

$$M = \bigoplus_{\lambda \in X} M_\lambda = \bigoplus_{\lambda \in X} \{m \in M \mid um = \lambda(u)m, u \in \mathbf{U}_v^0\}. \tag{3}$$

Remark 1. One can show that the category $\mathbf{U}_v(\mathfrak{g})$-**Mod** is equivalent to the well-studied category of finite-dimensional $\mathbf{U}(\mathfrak{g})$-modules, where $\mathbf{U}(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. ▲

By construction, the $\mathbf{U}_v$-modules $\nabla_v(\lambda)$ satisfy the *Frobenius reciprocity*, that is, we have

$$\mathrm{Hom}_{\mathbf{U}_v}(M, \nabla_v(\lambda)) \cong \mathrm{Hom}_{\mathbf{U}_v^-\mathbf{U}_v^0}(M, \mathbb{Q}(v)_\lambda) \quad \text{for all } M \in \mathbf{U}_v\text{-}\mathbf{Mod}. \tag{4}$$

Moreover, if we let $\mathrm{ch}(M)$ denote the *(formal) character of* $M \in \mathbf{U}_v$-**Mod**, that is,

$$\mathrm{ch}(M) = \sum_{\lambda \in X} (\dim(M_\lambda)) y^\lambda \in \mathbb{Z}[X][y].$$

(Recall that the group algebra $\mathbb{Z}[X]$, where we regard X to be the free abelian group generated by the dominant (integral) $\mathbf{U}_v$-weights X^+, is known as the *character ring*.) Then we have

$$\mathrm{ch}(\nabla_v(\lambda)) = \chi(\lambda) \in \mathbb{Z}[X][y] \quad \text{for all } \lambda \in X^+. \tag{5}$$

Here $\chi(\lambda)$ is the so-called *Weyl character*, which completely determines the simple $\mathbf{U}_v$-modules. In fact, $\chi(\lambda)$ is the classical character obtained from Weyl's character formula in the non-quantum case (cf. Remark 1). A proof of the equation from (5) can be found in [4, Corollary 5.12 and the following remark], see also [18, Theorem 5.15].

In addition, we have a contravariant, character-preserving *duality functor*

$$\mathcal{D} \colon \mathbf{U}_v\text{-}\mathbf{Mod} \to \mathbf{U}_v\text{-}\mathbf{Mod} \tag{6}$$

that is defined on the $\mathbb{Q}(v)$-vector space level via $\mathcal{D}(M) = M^*$ (the $\mathbb{Q}(v)$-linear dual of M) and an action of $\mathbf{U}_v$ on $\mathcal{D}(M)$ is defined by

$$uf = m \mapsto f(\omega(S(u))m), \quad m \in M, u \in \mathbf{U}_v, f \in \mathcal{D}(M).$$

Here $\omega \colon \mathbf{U}_v \to \mathbf{U}_v$ is the automorphism of $\mathbf{U}_v$ which interchanges E_i and F_i and interchanges K_i and K_i^{-1}, see for example [18, Lemma 4.6]. Note that the $\mathbf{U}_v$-weights of M and $\mathcal{D}(M)$ coincide. In particular, we have $\mathcal{D}(\nabla_v(\lambda)) \cong \Delta_v(\lambda)$, where the latter $\mathbf{U}_v$-module is called the *Weyl* $\mathbf{U}_v$-module associated to $\lambda \in X^+$. Thus, the Weyl and dual Weyl $\mathbf{U}_v$-modules are related by duality, since clearly $\mathcal{D}^2 \cong \mathrm{id}_{\mathbf{U}_v\text{-}\mathbf{Mod}}$.

Example 2.9. If we have $\mathfrak{g} = \mathfrak{sl}_2$, then the dominant (integral) $\mathfrak{sl}_2$-weights X^+ can be identified with $\mathbb{Z}_{\geq 0}$.

The i-th Weyl module $\Delta_v(i)$ is the $i+1$-dimensional $\mathbb{Q}(v)$-vector space with a basis given by $m_0, \dots, m_i$ and an $\mathbf{U}_v(\mathfrak{sl}_2)$-action defined by

$$K m_k = v^{i-2k} m_k, \quad E^{(j)} m_k = \begin{bmatrix} i-k+j \\ j \end{bmatrix} m_{k-j} \quad \text{and} \quad F^{(j)} m_k = \begin{bmatrix} k+j \\ j \end{bmatrix} m_{k+j}, \tag{7}$$

with the convention that $m_{<0} = m_{>i} = 0$. For example, for $i = 3$ we can visualize $\Delta_v(3)$ as

$$\begin{tikzcd} m_3 \arrow[loop above, "v^{-3}"] \arrow[r, shift left, "{[1]}"] & m_2 \arrow[loop above, "v^{-1}"] \arrow[l, shift left, "{[3]}"] \arrow[r, shift left, "{[2]}"] & m_1 \arrow[loop above, "v^{+1}"] \arrow[l, shift left, "{[2]}"] \arrow[r, shift left, "{[3]}"] & m_0, \arrow[loop above, "v^{+3}"] \arrow[l, shift left, "{[1]}"] \end{tikzcd} \tag{8}$$

$$\text{Character: } y^{-3} + y^{-1} + y^{1} + y^{3},$$

where the action of E points to the right, the action of F to the left and K acts as a loop.

Note that the $\mathbf{U}_v(\mathfrak{sl}_2)$-action from (7) is already defined by the action of the generators $E, F, K^{\pm 1}$. For $\mathbf{U}_q(\mathfrak{sl}_2)$ the situation is different, see Example 2.13. ▲

2C. **Representation theory of $\mathbf{U}_q$: the non-semisimple case.** As before in Section 2A, we let q denote a fixed element of $\mathbb{K}^*$.

Let $\lambda \in X$ be a $\mathbf{U}_q$-weight. As above, we can identify λ with a character of $\mathbf{U}^0_{\mathscr{A}}$ via

$$\lambda\colon \mathbf{U}^0_{\mathscr{A}} \to \mathscr{A}, \quad K_i^{\pm} \mapsto v^{\pm d_i \langle \lambda, \alpha_i^{\vee}\rangle}, \quad \tilde{K}_{i,t} \mapsto \begin{bmatrix} \langle \lambda, \alpha_i^{\vee}\rangle \\ t \end{bmatrix}_{d_i}, \quad i = 1, \dots, n,\ t \in \mathbb{Z}_{\geq 0},$$

which then also gives a character of $\mathbf{U}^0_q$. Here we use the definition of $\tilde{K}_{i,t}$ from (1). Abusing notation again, we use the same symbols for the $\mathbf{U}_q$-weights λ and the characters λ.

It is still true that any finite-dimensional $\mathbf{U}_q$-module M is a direct sum of its $\mathbf{U}_q$-weight spaces, see [4, Theorem 9.2]. Thus, if we denote by $\mathbf{U}_q$-**Mod** the category of finite-dimensional $\mathbf{U}_q$-modules, then we get the same decomposition as in (3), but replacing $\mathbf{U}^0_v$ by $\mathbf{U}^0_q$.

Hence, in complete analogy to the generic case discussed in Section 2B, we can define the *(formal) character* $\chi(M)$ of $M \in \mathbf{U}_q$-**Mod** and the *(dual) Weyl* $\mathbf{U}_q$-module $\Delta_q(\lambda)$ (or $\nabla_q(\lambda)$) associated to $\lambda \in X^+$.

Using this notation, we arrive at the following which explains our main interest in the root of unity case. Note that we do not have any restrictions on the characteristic of $\mathbb{K}$ here.

Proposition 2.10. (**Semisimplicity: the specialized case.**) We have:

$$\mathbf{U}_q\text{-}\mathbf{Mod} \text{ is semisimple } \Leftrightarrow \begin{cases} q \in \mathbb{K}^* - \{1\} \text{ is not a root of unity,} \\ q = \pm 1 \in \mathbb{K} \text{ with } \mathrm{char}(\mathbb{K}) = 0. \end{cases}$$

Moreover, if $\mathbf{U}_q$-**Mod** is semisimple, then the $\nabla_q(\lambda)$'s for $\lambda \in X^+$ form a complete set of pairwise non-isomorphic, simple $\mathbf{U}_q$-modules. □

Proof. For semisimplicity at non-roots of unity, or $q = \pm 1, \mathrm{char}(\mathbb{K}) = 0$ see [4, Theorem 9.4] (and additionally [24, Section 33.2] for $q = -1$). To see the converse: (most of) the $\nabla_q(\lambda)$'s are not semisimple in general (compare to Example 2.13). ■

Remark 2. In particular, if $\mathbb{K} = \mathbb{C}$, $q = 1$ and the Cartan datum comes from a simple Lie algebra $\mathfrak{g}$, then, $\mathbf{U}_1$-**Mod** is equivalent to the well-studied category of finite-dimensional $\mathbf{U}(\mathfrak{g})$-modules. This is as in the generic case, cf. Remark 1. ▲

Thus, Proposition 2.10 motivates the study of the case where q is a root of unity.

Assumption 2.11. If we want q to be a root of unity, then, to avoid technicalities, we assume that q is a primitive root of unity of odd order l (a treatment of the even case, that can be used to repeat everything in this paper in the case where l is even, can be found in [2]). Moreover, if we are in type G_2, then we, in addition, assume that l is prime to 3. ▲

In the root of unity case, by Proposition 2.10, our main category $\mathbf{U}_q$-**Mod** under study is no longer semisimple. In addition, the $\mathbf{U}_q$-modules $\nabla_q(\lambda)$ are in general not simple anymore, but they have a unique *simple socle* that we denote by $L_q(\lambda)$. By duality (note that the functor $\mathcal{D}(\cdot)$ from (6) carries over to $\mathbf{U}_q$-**Mod**), these are also the unique *simple heads* of the $\Delta_q(\lambda)$'s.

Proposition 2.12. (**Simple $\mathbf{U}_q$-modules: the non-semisimple case.**) The socles $L_q(\lambda)$ of the $\nabla_q(\lambda)$'s are simple $\mathbf{U}_q$-modules $L_q(\lambda)$'s for $\lambda \in X^+$. They form a complete set of pairwise non-isomorphic, simple $\mathbf{U}_q$-modules in $\mathbf{U}_q$-**Mod**. □

Proof. See [4, Corollary 6.2 and Proposition 6.3]. ■

Example 2.13. With the same notation as in Example 2.9 but for q being a complex, primitive third root of unity, we have $[3] = 0$ and we can thus visualize $\Delta_q(3)$ as

$$\begin{tikzcd}
m_3 \arrow[loop above, "q^{-3}"] \arrow[r, shift left, "+1"] \arrow[rrr, bend right, leftrightarrow, "+1"'] & m_2 \arrow[loop above, "q^{-1}"] \arrow[l, shift left, "0"] \arrow[r, shift left, "-1"] & m_1 \arrow[loop above, "q^{+1}"] \arrow[l, shift left, "-1"] \arrow[r, shift left, "0"] & m_0, \arrow[loop above, "q^{+3}"] \arrow[l, shift left, "+1"]
\end{tikzcd} \tag{9}$$

Character: $y^{-3} + y^{-1} + y^{1} + y^{3}$,

where the action of E points to the right, the action of F to the left and K acts as a loop. In contrast to Example 2.9, the picture in (9) also shows the actions of the divided powers $E^{(3)}$ and $F^{(3)}$ as a long arrow connecting m_0 and m_3 (recall that these are additional generators of $\mathbf{U}_q(\mathfrak{sl}_2)$, see Example 2.5). Note also that, again in contrast to (8), some generators act on these basis vectors as zero. We also have $F^{(3)}m_1 = 0$ and $E^{(3)}m_2 = 0$. Thus, the $\mathbb{C}$-span of $\{m_1, m_2\}$ is now stable under the action of $\mathbf{U}_q(\mathfrak{sl}_2)$.

In particular, $L_q(3)$ is the $\mathbf{U}_q(\mathfrak{sl}_2)$-module obtained from $\Delta_q(3)$ as in (9) by taking the quotient of the $\mathbb{C}$-span of the set $\{m_1, m_2\}$. The latter can be seen to be isomorphic to $L_q(1)$.

We encourage the reader to work out its dual case $\nabla_q(3)$. Here the result, using the same conventions as before:

$$\begin{tikzcd}
m_3 \arrow[loop above, "q^{-3}"] \arrow[r, shift right, "0"'] \arrow[rrr, bend right, leftrightarrow, "+1"'] & m_2 \arrow[loop above, "q^{-1}"] \arrow[l, shift right, "+1"'] \arrow[r, shift right, "-1"'] & m_1 \arrow[loop above, "q^{+1}"] \arrow[l, shift right, "-1"'] \arrow[r, shift right, "+1"'] & m_0, \arrow[loop above, "q^{+3}"] \arrow[l, shift right, "0"']
\end{tikzcd}$$

Character: $y^{-3} + y^{-1} + y^{1} + y^{3}$,

Note that $\nabla_q(3)$ has the same character as $\Delta_q(3)$, but one can check that they are not equivalent. This has no analog in the generic $\mathfrak{sl}_2$ case.

It turns out that $L_q(1)$ is a $\mathbf{U}_q$-submodule of $\Delta_q(3)$ and $L_q(3)$ is a $\mathbf{U}_q$-submodule of $\nabla_q(3)$ and these can be visualized as

$$L_q(1) \cong \begin{tikzcd} m_2 \arrow[loop above, "q^{-1}"] \arrow[r, shift left, "-1"] & m_1 \arrow[loop above, "q^{+1}"] \arrow[l, shift left, "-1"] \end{tikzcd} \quad \text{and} \quad L_q(3) \cong \begin{tikzcd} m_3^* \arrow[loop above, "q^{-3}"] \arrow[r, shift left, "+1"] & m_0^*, \arrow[loop above, "q^{+3}"] \arrow[l, shift left, "+1"] \end{tikzcd}$$

where for $L_q(3)$ the displayed actions are via $E^{(3)}$ (to the right) and $F^{(3)}$ (to the left). Note that $L_q(1)$ and $L_q(3)$ have both dimension 2. Again, this has no analogon in the generic $\mathfrak{sl}_2$ case where all simple $\mathbf{U}_v$-modules $L_v(i) \cong \Delta_v(i) \cong \nabla_v(i)$ have different dimensions. ▲

A non-trivial fact (which relies on the q-version of the so-called *Kempf's vanishing theorem*, see [32, Theorem 5.5]) is that the characters of the $\nabla_q(\lambda)$'s are still given by Weyl's character formula as in (5). (By duality, similar for the $\Delta_q(\lambda)$'s.) In particular, $\dim(\nabla_q(\lambda)_\lambda) = 1$ and $\dim(\nabla_q(\lambda)_\mu) = 0$ unless $\mu \leq \lambda$. (Again similar for the $\Delta_q(\lambda)$'s.)

Example 2.14. We have calculated the characters of some (dual) Weyl $\mathbf{U}_v$-modules in Example 2.9, and in case of $\mathbf{U}_q$ in Example 2.13. They agree, although the modules behave completely different. ▲

On the other hand, the characters of the $L_q(\lambda)$'s are only known if $\mathrm{char}(\mathbb{K}) = 0$ (and "big enough" l). In that case, certain *Kazhdan–Lusztig polynomials* determine the character $\mathrm{ch}(L_q(\lambda))$, see for example [36, Theorem 6.4 and 7.1] and the references therein.

3. Tilting modules

In the present section we recall a few facts from the theory of $\mathbf{U}_q$-tilting modules. In the semisimple case all $\mathbf{U}_q$-modules in $\mathbf{U}_q$-**Mod** are $\mathbf{U}_q$-tilting modules. Hence, the theory of $\mathbf{U}_q$-tilting modules is kind of redundant in this case. In the non-semisimple case however the theory of $\mathbf{U}_q$-tilting modules is extremely rich and a source of neat combinatorics. For brevity, we only provide some of the proofs. For more details see for example [13].

3A. $\mathbf{U}_q$-modules with a Δ_q- and a ∇_q-filtration.

As recalled above Proposition 2.12, the $\mathbf{U}_q$-module $\Delta_q(\lambda)$ has a unique simple head $L_q(\lambda)$ which is the unique simple socle of $\nabla_q(\lambda)$. Thus, there is a (up to scalars) unique $\mathbf{U}_q$-homomorphism

$$c^\lambda \colon \Delta_q(\lambda) \to \nabla_q(\lambda) \quad \text{(mapping head to socle).} \tag{10}$$

To see this: by Frobenius reciprocity from (4)—to be more precise, the q-version of it which can be found in [4, Proposition 2.12]—we have

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\lambda)) \cong \mathrm{Hom}_{\mathbf{U}_q^- \mathbf{U}_q^0}(\Delta_q(\lambda), \mathbb{K}_\lambda)$$

which gives $\dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\lambda))) = 1$. This relies on the fact that $\Delta_q(\lambda)$ and $\nabla_q(\lambda)$ both have one-dimensional λ-weight spaces. The same fact implies that $\mathrm{End}_{\mathbf{U}_q}(L_q(\lambda)) \cong \mathbb{K}$ for all $\lambda \in X^+$, see [4, Corollary 7.4]. (Note that this last property fails for quasi-hereditary algebras in general when $\mathbb{K}$ is not algebraically closed.)

Theorem 3.1. (**Ext-vanishing.**) We have for all $\lambda, \mu \in X^+$ that

$$\mathrm{Ext}^i_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \begin{cases} \mathbb{K}c^\lambda, & \text{if } i = 0 \text{ and } \lambda = \mu, \\ 0, & \text{else.} \end{cases} \qquad \square$$

Although the category $\mathbf{U}_q$-**Mod** has enough injectives in characteristic zero, see [1, Proposition 5.8] for a treatment of the non-semisimple cases, this does not hold in general. Hence, in the following, we will use the extension functors $\mathrm{Ext}^i_{\mathbf{U}_q}$ in the usual sense by passing to the injective completion of $\mathbf{U}_q$-**Mod**. One can find the precise definition of this completion in [22, Definition 6.1.1] (where it is called indization). In this framework one can then work as usual thanks to [22, Theorem 8.6.5 and Corollary 15.3.9 and its proof], and so our formal manipulations in the following make sense.

Proof. Denote by $\boldsymbol{\mathcal{W}}^0$ and $\boldsymbol{\mathcal{W}}^{0-}$ the categories of integrable $\mathbf{U}_q^0$ and $\mathbf{U}_q^0\mathbf{U}_q^-$-modules respectively. Then, for any $\mathbf{U}_q^0$-module M:

$$M \in \boldsymbol{\mathcal{W}}^0 \Leftrightarrow M = \bigoplus_{\lambda \in X} M_\lambda.$$

Similarly, for any $\mathbf{U}_q^0\mathbf{U}_q^-$-module M':

$$M' \in \boldsymbol{\mathcal{W}}^{0-} \Leftrightarrow M' \in \boldsymbol{\mathcal{W}}^0 \text{ and } \begin{Bmatrix} \text{for all } m' \in M' \text{ there exists } r \in \mathbb{Z}_{\geq 0} \\ \text{such that } F_i^{(r)} m' = 0 \text{ for all } i = 1, \dots, n \end{Bmatrix} \text{ holds.}$$

Moreover, let $\boldsymbol{\mathcal{W}}$ denote the category of integrable $\mathbf{U}_q$-modules[1].

Below we will need a certain induction functor. To this end, recall the functor $\mathcal{F}$ which to an arbitrary $\mathbf{U}_q^0$-module $M \in \boldsymbol{\mathcal{W}}^0$ assigns

$$\mathcal{F}(M) = \{m \in \textstyle\bigoplus_{\lambda \in X} M_\lambda \mid F_i^{(r)} m = 0 \quad \text{for all } i \in \mathbb{Z}_{\geq 0} \text{ and for } r \gg 0\},$$

see [4, Section 2.2]. Then set

$$\operatorname{Ind}_{\boldsymbol{\mathcal{W}}^0}^{\boldsymbol{\mathcal{W}}^{0-}} : \boldsymbol{\mathcal{W}}^0 \to \boldsymbol{\mathcal{W}}^{0-}, \ M \mapsto \mathcal{F}(\operatorname{Hom}_{\boldsymbol{\mathcal{W}}^0}(\mathbf{U}_q^0\mathbf{U}_q^-, M)). \tag{11}$$

(Obtained by using the standard embedding of $\mathbf{U}_q^0 \hookrightarrow \mathbf{U}_q^0\mathbf{U}_q^-$, see [4, Section 2.4].)

Recall from [4, Section 2.11] that this functor is exact and that

$$\operatorname{Ind}_{\boldsymbol{\mathcal{W}}^0}^{\boldsymbol{\mathcal{W}}^{0-}}(M) = \bigoplus_{\lambda \in X} (M_\lambda \otimes \mathbb{K}[\mathbf{U}_q^-]_{-\lambda}).$$

Here $\mathbb{K}[\mathbf{U}_q^-]$ is the quantum coordinate algebra for $\mathbf{U}_q^-$ (see [4, Section 1.8]). Note in particular that the weights $\lambda \in X$ of $\mathbb{K}[\mathbf{U}_q^-]$ satisfy $\lambda \geq 0$ with $\lambda = 0$ occurring with multiplicity 1.

If $\lambda \in X$, then we denote by $\mathbb{K}_\lambda \in \boldsymbol{\mathcal{W}}^0$ the corresponding one-dimensional $\mathbf{U}_q^0$-module. This modules extends to $\mathbf{U}_q^0\mathbf{U}_q^-$ by letting all $F_i^{(r)}$'s act trivially for $r > 0$ and we, by abuse of notation, denote this $\mathbf{U}_q^0\mathbf{U}_q^-$-module also by $\mathbb{K}_\lambda$.

Claim3.1. We claim that

$$\operatorname{Ext}^i_{\boldsymbol{\mathcal{W}}^{0-}}(\mathbb{K}_0, \mathbb{K}_\lambda) \cong \begin{cases} \mathbb{K}, & \text{if } i = 0 \text{ and } \lambda = 0, \\ 0, & \text{if } i > 0 \text{ and } \lambda \not< 0, \end{cases} \tag{12}$$

for all $\lambda \in X$.

Proof of Claim3.1. The $i = 0$ part of this claim is clear. To check the $i > 0$ part, we construct an injective resolution of $\mathbb{K}_\lambda$ as follows.

We set $I_0(\lambda) = \operatorname{Ind}_{\boldsymbol{\mathcal{W}}^0}^{\boldsymbol{\mathcal{W}}^{0-}}(\mathbb{K}_\lambda)$. Note that $\mathbb{K}_\lambda$ is a $\mathbf{U}_q^0\mathbf{U}_q^-$-submodule of $I_0(\lambda)$. Thus, we may define the quotient $Q_1(\lambda) = I_0(\lambda)/Q_0(\lambda)$ by setting $Q_0(\lambda) = \mathbb{K}_\lambda$.

This pattern can be repeated: define for $k > 0$ recursively

$$I_k(\lambda) = \operatorname{Ind}_{\boldsymbol{\mathcal{W}}^0}^{\boldsymbol{\mathcal{W}}^{0-}}(Q_k(\lambda)), \quad \text{with } Q_k(\lambda) = I_{k-1}(\lambda)/Q_{k-1}(\lambda)$$

[1]We need to go to the categories of integrable modules due to the fact that the injective modules we use are usually infinite-dimensional. Furthermore, we take $\mathbf{U}_q^0\mathbf{U}_q^-$ here instead of $\mathbf{U}_q^-\mathbf{U}_q^0$, since we want to consider $\mathbf{U}_q^0\mathbf{U}_q^-$ as a left $\mathbf{U}_q^0$-module for the induction functor.

and obtain

$$0 \hookrightarrow \mathbb{K}_\lambda \hookrightarrow I_0(\lambda) \longrightarrow I_1(\lambda) \longrightarrow \cdots . \tag{13}$$

All $\mathbf{U}_q^0$-modules in $\boldsymbol{\mathcal{W}}^0$ are clearly injective and the functor from (11) takes injective $\mathbf{U}_q^0$-modules to injective $\mathbf{U}_q^0\mathbf{U}_q^-$-modules (see [4, Corollary 2.13]). Thus, (13) is an injective resolution of $\mathbb{K}_\lambda$ in $\boldsymbol{\mathcal{W}}^{0-}$. Moreover, by the above observation on the weights of $\mathbb{K}[\mathbf{U}_q^-]$, we get

$$I_0(\lambda)_\mu = 0 \quad \text{for all } \mu \not\geq 0,$$
$$I_k(\lambda)_\mu = 0 \quad \text{for all } \mu \not> 0,\ k > 0.$$

It follows that $\mathrm{Hom}_{\boldsymbol{\mathcal{W}}^{0-}}(\mathbb{K}_0, I_k(\lambda)) = 0$ for $k > 0$ which shows the second line in (12).

Note now that

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}^{0-}}(\mathbb{K}_\mu, \mathbb{K}_\lambda) \cong \mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}^{0-}}(\mathbb{K}_0, \mathbb{K}_{\lambda-\mu}) \tag{14}$$

for all $i \in \mathbb{Z}_{\geq 0}$ and all $\lambda, \mu \in X$.

Let $M \in \boldsymbol{\mathcal{W}}^{0-}$ be finite-dimensional such that no weight of M is strictly bigger than $\lambda \in X$. Then (12) and (14) imply

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}^{0-}}(M, \mathbb{K}_\lambda) = 0 \quad \text{for all } k > 0. \tag{15}$$

We are now aiming to prove the Ext-vanishing theorem. Recall that $\nabla_q(\lambda) = \mathrm{Ind}^{\boldsymbol{\mathcal{W}}}_{\boldsymbol{\mathcal{W}}^{0-}}\mathbb{K}_\lambda$. From the q-version of Kempf's vanishing theorem—see [32, Theorem 5.5]—we get

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}^{0-}}(\Delta_q(\lambda), \mathbb{K}_\mu). \tag{16}$$

Thus, the Ext-vanishing follows for $\mu \not< \lambda$ from (15). So let $\mu < \lambda$. Recall from above that the character-preserving duality functor $\mathcal{D}(\cdot)$ as in (6) satisfies $\mathcal{D}(\nabla_q(\lambda)) \cong \Delta_q(\lambda)$ and $\mathcal{D}(\Delta_q(\lambda)) \cong \nabla_q(\lambda)$ for all $\lambda \in X^+$. This gives

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}}(\Delta_q(\mu), \nabla_q(\lambda)).$$

Thus, we can conclude as before, since now $\lambda \not< \mu$. Finally, if $i = 0$, then (16) implies

$$\mathrm{Hom}_{\boldsymbol{\mathcal{W}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \mathrm{Hom}_{\boldsymbol{\mathcal{W}}^{0-}}(\Delta_q(\lambda), \mathbb{K}_\mu) = \begin{cases} \mathbb{K}, & \text{if } \lambda = \mu, \\ 0, & \mu \not\leq \lambda. \end{cases}$$

If $\mu < \lambda$, then we apply $\mathcal{D}$ as before which finally shows that

$$\mathrm{Hom}_{\boldsymbol{\mathcal{W}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \begin{cases} \mathbb{K}c^\lambda, & \lambda = \mu, \\ 0, & \text{else}, \end{cases}$$

for all $\lambda, \mu \in X^+$. This proves the statement since $\mathbf{U}_q$-**Mod** is a full subcategory of $\boldsymbol{\mathcal{W}}$. ■

Definition 3.2. (**Δ_q- and ∇_q-filtration**.) We say that a $\mathbf{U}_q$-module M has a *Δ_q-filtration* if there exists some $k \in \mathbb{Z}_{\geq 0}$ and a finite descending sequence of $\mathbf{U}_q$-submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_{k'} \supset \cdots \supset M_{k-1} \supset M_k = 0,$$

such that $M_{k'}/M_{k'+1} \cong \Delta_q(\lambda_{k'})$ for all $k' = 0, \ldots, k-1$ and some $\lambda_{k'} \in X^+$.

A *∇_q-filtration* is defined similarly, but using $\nabla_q(\lambda)$ instead of $\Delta_q(\lambda)$ and a finite ascending sequence of $\mathbf{U}_q$-submodules, that is,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{k'} \subset \cdots \subset M_{k-1} \subset M_k = M,$$

such that $M_{k'+1}/M_{k'} \cong \nabla_q(\lambda_{k'})$ for all $k' = 0, \dots, k-1$ and some $\lambda_{k'} \in X^+$. ▲

We denote by $(M : \Delta_q(\lambda))$ and $(N : \nabla_q(\lambda))$ the corresponding multiplicities, which are well-defined by Corollary 3.4 below. Clearly, a $\mathbf{U}_q$-module M has a Δ_q-filtration if and only if its dual $\mathcal{D}(M)$ has a ∇_q-filtration.

Example 3.3. The simple $\mathbf{U}_q$-module $L_q(\lambda)$ has a Δ_q-filtration if and only if $L_q(\lambda) \cong \Delta_q(\lambda)$. In that case we have also $L_q(\lambda) \cong \nabla_q(\lambda)$ and thus, $L_q(\lambda)$ has a ∇_q-filtration as well. ▲

A corollary of the Ext-vanishing Theorem 3.1 is:

Corollary 3.4. Let $M, N \in \mathbf{U}_q\textbf{-Mod}$ and $\lambda \in X^+$. Assume that M has a Δ_q-filtration and N has a ∇_q-filtration. Then

$$\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))) = (M : \Delta_q(\lambda)) \quad \text{and} \quad \dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)) = (N : \nabla_q(\lambda)).$$

In particular, $(M : \Delta_q(\lambda))$ and $(N : \nabla_q(\lambda))$ are independent of the choice of filtrations. ■

Note that the proof of Corollary 3.4 below gives a method to find and construct bases of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ and $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)$, respectively.

Proof. Let k be the length of the Δ_q-filtration of M. If $k = 1$, then

$$\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))) = (M : \Delta_q(\lambda)) \tag{17}$$

follows from the uniqueness of c^λ from (10). Otherwise, we take the short exact sequence

$$0 \longrightarrow M' \hookrightarrow M \twoheadrightarrow \Delta_q(\mu) \longrightarrow 0$$

for some $\mu \in X^+$. Since both sides of (17) are additive with respect to short exact sequences by Theorem 3.1, the claim in for the Δ_q's follows by induction.

Similarly for the ∇_q's, by duality. ■

Fix two $\mathbf{U}_q$-modules M, N, where we assume that M has a Δ_q-filtration and N has a ∇_q-filtration. Then, by Corollary 3.4, we have

$$\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, N)) = \sum_{\lambda \in X^+} (M : \Delta_q(\lambda))(N : \nabla_q(\lambda)). \tag{18}$$

We point out that the sum in (18) is actually finite since $(M : \Delta_q(\lambda)) \neq 0$ for only a finite number of $\lambda \in X^+$. (Dually, $(N : \nabla_q(\lambda)) \neq 0$ for only finitely many $\lambda \in X^+$.)

In fact, following Donkin [12] who obtained the result below in the modular case, we can state two useful consequences of the Ext-vanishing Theorem 3.1.

Proposition 3.5. (**Donkin's Ext-criteria.**) The following are equivalent.

(a) An $M \in \mathbf{U}_q\textbf{-Mod}$ has a Δ_q-filtration (respectively $N \in \mathbf{U}_q\textbf{-Mod}$ has a ∇_q-filtration).
(b) We have $\mathrm{Ext}^i_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ (respectively $\mathrm{Ext}^i_{\mathbf{U}_q}(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$ and all $i > 0$.
(c) We have $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ (respectively $\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$. □

Proof. As usual: we are lazy and only show the statement about the Δ_q-filtrations and leave the other to the reader.

Suppose the $\mathbf{U}_q$-module M has a Δ_q-filtration. Then, by the results from Theorem 3.1, $\mathrm{Ext}^i_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ for all $\lambda \in X^+$ and all $i > 0$—which shows that (a) implies (b).

Since (b) clearly implies (c), we only need to show that (c) implies (a).

To this end, suppose the $\mathbf{U}_q$-module M satisfies $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ for all $\lambda \in X^+$. We inductively, with respect to the filtration (by simples $L_q(\lambda)$) length $\ell(M)$ of M, construct the Δ_q-filtration for M.

So, by Proposition 2.12, we can assume that $M = L_q(\lambda)$ for some $\lambda \in X^+$.

Consider the short exact sequence

$$0 \longrightarrow \ker(\mathrm{pro}^\lambda) \hookrightarrow \Delta_q(\lambda) \xrightarrow{\mathrm{pro}^\lambda} L_q(\lambda) \longrightarrow 0. \tag{19}$$

By Theorem 3.1 we get from (19) a short exact sequence for all $\mu \in X^+$ of the form

$$0 \longleftarrow \mathrm{Hom}_{\mathbf{U}_q}(\ker(\mathrm{pro}^\lambda), \nabla_q(\mu)) \twoheadleftarrow \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \hookleftarrow \mathrm{Hom}_{\mathbf{U}_q}(L_q(\lambda), \nabla_q(\mu)) \longleftarrow 0.$$

By Theorem 3.1, $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu))$ is zero if $\mu \neq \lambda$ and one-dimensional if $\mu = \lambda$. By construction, $\mathrm{Hom}_{\mathbf{U}_q}(L_q(\lambda), \nabla_q(\lambda))$ is also one-dimensional. Thus, $\mathrm{Hom}_{\mathbf{U}_q}(\ker(\mathrm{pro}^\lambda), \nabla_q(\mu)) = 0$ for all $\mu \in X^+$ showing that $\ker(\mathrm{pro}^\lambda) = 0$. This, by (19), implies $\Delta_q(\lambda) \cong L_q(\lambda)$.

Now assume that $\ell(M) > 1$. Choose $\lambda \in X^+$ minimal such that $\mathrm{Hom}_{\mathbf{U}_q}(M, L_q(\lambda)) \neq 0$. As before in (19), we consider the projection $\mathrm{pro}^\lambda \colon \Delta_q(\lambda) \twoheadrightarrow L_q(\lambda)$ and its kernel $\ker(\mathrm{pro}^\lambda)$.

Note now that $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ implies $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \ker(\mathrm{pro}^\lambda)) = 0$:

Assume the contrary. Then we can find a composition factor $L_q(\mu)$ for $\mu < \lambda$ of $\ker(\mathrm{pro}^\lambda)$ such that $\mathrm{Ext}^1_{\mathbf{U}_q}(M, L_q(\mu)) \neq 0$. Then the exact sequence

$$\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\mu)/L_q(\mu)) \longrightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(M, L_q(\mu)) \neq 0 \longrightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\mu)) = 0$$

implies that $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\mu)/L_q(\mu)) \neq 0$. Since $\mu < \lambda$, this gives a contradiction to the minimality of λ.

Hence, any non-zero $\mathbf{U}_q$-homomorphism $\mathrm{pro} \in \mathrm{Hom}_{\mathbf{U}_q}(M, L_q(\lambda))$ lifts to a surjection

$$\overline{\mathrm{pro}} \colon M \twoheadrightarrow \Delta_q(\lambda).$$

By assumption and Theorem 3.1 we have $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\mu)) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu))$ for all $\mu \in X^+$. Thus, we have $\mathrm{Ext}^1_{\mathbf{U}_q}(\ker(\overline{\mathrm{pro}}), \nabla_q(\mu)) = 0$ for all $\mu \in X^+$ and we can proceed by induction (since $\ell(\ker(\overline{\mathrm{pro}})) < \ell(M)$, by construction). ■

Example 3.6. Let us come back to our favorite example, i.e. q being a complex, primitive third root of unity for $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{sl}_2)$. The simple $\mathbf{U}_q$-module $L_q(3)$ does neither have a Δ_q- nor a ∇_q-filtration (compare Example 2.13 with Example 3.3). This can also be seen with Proposition 3.5, because $\mathrm{Ext}^1_{\mathbf{U}_q}(L_q(3), L_q(1))$ is not trivial: by Example 2.13 from above we have $\Delta_q(1) \cong L_q(1) \cong \nabla_q(1)$, but

$$0 \longrightarrow L_q(1) \hookrightarrow \Delta_q(3) \twoheadrightarrow L_q(3) \longrightarrow 0$$

does not split. Analogously, $\mathrm{Ext}^1_{\mathbf{U}_q}(L_q(1), L_q(3)) \neq 0$, by duality. ▲

3B. **$\mathbf{U}_q$-tilting modules.** A $\mathbf{U}_q$-module T which has both, a Δ_q- and a ∇_q-filtration, is called a *$\mathbf{U}_q$-tilting module*. Following Donkin [12], we are now ready to define the *category of $\mathbf{U}_q$-tilting modules* that we denote by $\boldsymbol{\mathcal{T}}$. This category is our main object of study.

Definition 3.7. (**Category of $\mathbf{U}_q$-tilting modules.**) The category $\mathcal{T}$ is the full subcategory of $\mathbf{U}_q$-**Mod** whose objects are given by all $\mathbf{U}_q$-tilting modules. ▲

From Proposition 3.5 we obtain directly an important statement.

Corollary 3.8. Let $T \in \mathbf{U}_q$-**Mod**. Then

$$T \in \mathcal{T} \quad \text{if and only if} \quad \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) \quad \text{for all } \lambda \in X^+.$$

When $T \in \mathcal{T}$, the corresponding higher Ext-groups vanish as well. ■

Recall the contravariant, character preserving functor $\mathcal{D}\colon \mathbf{U}_q\text{-}\mathbf{Mod} \to \mathbf{U}_q\text{-}\mathbf{Mod}$ from (6). Clearly, by Corollary 3.8, $T \in \mathcal{T}$ if and only if $\mathcal{D}(T) \in \mathcal{T}$. Thus, $\mathcal{D}(\cdot)$ restricts to a functor $\mathcal{D}\colon \mathcal{T} \to \mathcal{T}$. In fact, we show below in Corollary 3.12, that the functor $\mathcal{D}(\cdot)$ restricts to (a functor isomorphic to) the identity functor on objects of $\mathcal{T}$.

Example 3.9. The $L_q(\lambda)$ are $\mathbf{U}_q$-tilting modules if and only if $\Delta_q(\lambda) \cong L_q(\lambda) \cong \nabla_q(\lambda)$.

Coming back to our favourite example, the case $\mathfrak{g} = \mathfrak{sl}_2$ and q is a complex, primitive third root of unity: a direct computation using similar reasoning as in Example 2.13 (that is, the appearance of some actions equals zero as in (9)) shows that $L_q(i)$ is a $\mathbf{U}_q$-tilting module if and only if $i = 0, 1$ or $i \equiv -1 \bmod 3$. More general: if q is a complex, primitive l-th root of unity, then $L_q(i)$ is a $\mathbf{U}_q$-tilting module if and only if $i = 0, \dots, l-1$ or $i \equiv -1 \bmod l$. ▲

Proposition 3.10. $\mathcal{T}$ is a Krull–Schmidt category, closed under duality $\mathcal{D}(\cdot)$ and under finite direct sums. Furthermore, $\mathcal{T}$ is closed under finite tensor products. □

Proof. That $\mathcal{T}$ is Krull–Schmidt is immediate. By [6, Corollary 3.8] we see that $\mathcal{T}$ is closed under duality $\mathcal{D}(\cdot)$ and under finite direct sums.

Only that $\mathcal{T}$ is closed under finite tensor products remains to be proven. By duality, this reduces to show the statement that, given $M, N \in \mathbf{U}_q$-**Mod** where both have a ∇_q-filtration, then $M \otimes N$ has a ∇_q-filtration. In addition, this reduces further to the following claim.

Claim3.10.1. We have:

$$\nabla_q(\lambda) \otimes \nabla_q(\mu) \quad \text{has a } \nabla_q\text{-filtration for all } \lambda, \mu \in X^+. \tag{20}$$

In this note we give a proof of (20) in type A where it is true that the ω_i's are minuscule. The idea of the proof goes back to [37]. (We point out, this case and the arguments used here are enough for most of the examples considered in [6].) For the general case the only known proofs of (20) rely on crystal bases, see [28, Theorem 3.3] or alternatively [21, Corollary 1.9].

Claim3.10.2. Is suffices to show

$$\nabla_q(\lambda) \otimes \nabla_q(\omega_i) \quad \text{has a } \nabla_q\text{-filtration for all } \lambda \in X^+ \text{ and all } i = 1, \dots, n. \tag{21}$$

(Note that our proof of the fact that (21) implies (20) works in all types.)

Proof of Claim3.10.2. To see that (21) implies (20) we shall work with the the $\mathbb{Q}_{\geq 0}$-version of the partial ordering $\leq$ on X given by $\mu \leq_{\mathbb{Q}} \lambda$ if and only if $\lambda - \mu$ is a $\mathbb{Q}_{\geq 0}$-valued linear combination of the simple roots, that is, $\lambda - \mu = \sum_{i=1}^n a_i \alpha_i$ with $a_i \in \mathbb{Q}_{\geq 0}$. Clearly $\mu \leq_{\mathbb{Q}} \lambda$ implies $\mu \leq \lambda$. Note that $0 \leq_{\mathbb{Q}} \omega_i$ for all $i = 1, \dots, n$ which means that 0 is the unique minimal $\mathbf{U}_q$-weight in X^+ with respect to $\leq_{\mathbb{Q}}$.

Assume now that (21) holds. We shall prove (20) by induction with respect to $\leq_{\mathbb{Q}}$. For $\lambda = 0$ we have $\nabla_q(\lambda) \cong \mathbb{K}$ and there is nothing to prove.

So let $\lambda \in X^+ - \{0\}$ and assume that (20) holds for all $\mu <_{\mathbb{Q}} \lambda$. Note that there exists a fundamental $\mathbf{U}_q$-weight ω such that $\mu = \lambda - \omega$. This means that, by (21), we have a short exact sequence of the form

$$0 \longrightarrow M \hookrightarrow \nabla_q(\mu) \otimes \nabla_q(\omega) \twoheadrightarrow \nabla_q(\lambda) \longrightarrow 0. \tag{22}$$

Here the $\mathbf{U}_q$-module M has a ∇_q-filtration. By induction, $\nabla_q(\lambda') \otimes \nabla_q(\mu)$ has a ∇_q-filtration for all $\lambda' \in X^+$ and so, by (21), has $\nabla_q(\lambda') \otimes \nabla_q(\mu) \otimes \nabla_q(\omega)$. Moreover, the ∇_q-factors of M have the form $\nabla_q(\nu)$ for $\nu <_{\mathbb{Q}} \lambda$. Hence, by the induction hypothesis, we have that $\nabla_q(\lambda') \otimes M$ has a ∇_q-filtration for all $\lambda' \in X^+$. Thus, tensoring (22) with $\nabla_q(\lambda')$ from the left gives a ∇_q-filtration for the two leftmost terms. Therefore, also the third has a ∇_q-filtration (by Proposition 3.5). This shows that (21) implies (20).

Proof of Claim3.10.1 in types A. Assume that the fundamental $\mathbf{U}_q$-weights are minuscule. By the above, it remains to show (21). For this purpose, recall that

$$\nabla_v(\lambda) = \mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v} \mathbb{K}_\lambda.$$

By the tensor identity (see [4, Proposition 2.16]) this implies

$$\nabla_q(\lambda) \otimes \nabla_q(\omega_i) \cong \mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v} (\mathbb{K}_\lambda \otimes \nabla_q(\omega_i))$$

for all $i = 1, \ldots, n$. Now take a filtration of $\mathbb{K}_\lambda \otimes \nabla_q(\omega_i)$ of the form

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{k'} \subset \cdots \subset M_{k-1} \subset M_k = \mathbb{K}_\lambda \otimes \nabla_q(\omega_i), \tag{23}$$

such that for all $k' = 0, \ldots, k-1$ we have $M_{k'+1}/M_{k'} \cong \mathbb{K}_{\lambda_{k'+1}}$ for some $\lambda_{k'} \in X^+$. Thus, the set $\{\lambda_{k'} \mid k' = 1, \ldots, k\}$ is the set of $\mathbf{U}_q$-weights of $\mathbb{K}_\lambda \otimes \nabla_q(\omega_i)$. But the $\mathbf{U}_q$-weights of $\nabla_q(\omega_i)$ are of the form $\{w(\omega_i) \mid w \in W\}$ where W is the Weyl group associated to $\mathbf{U}_q$. Hence, $\lambda_{k'} = \lambda + w_{k'}(\omega_i)$ for some $w_{k'} \in W$. We get[2]

$$\langle \lambda_{k'}, \alpha_j^\vee \rangle = \langle \lambda, \alpha_j^\vee \rangle + \langle \omega_i, w_{k'}^{-1}(\alpha_j^\vee) \rangle \geq 0 + (-1) = -1$$

for all $j = 1, \ldots, n$. Said otherwise, $\lambda_{k'} + \rho \in X^+$. Hence, the q-version of Kempf's vanishing theorem (see [32, Theorem 5.5]) shows that we can apply the functor $\mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v}(\cdot)$ to (23) to obtain a ∇_q-filtration of $\nabla_q(\lambda) \otimes \nabla_q(\omega_i)$. Thus, we obtain (21). ■

In particular, for $\mathfrak{g}$ of type A, the proof of Proposition 3.10 gives us the special case that $T = \Delta_q(\omega_{i_1}) \otimes \cdots \otimes \Delta_q(\omega_{i_d})$ is a $\mathbf{U}_q$-tilting module for any $i_k \in \{1, \ldots, n\}$. Moreover, the proof of Proposition 3.10 generalizes: using similar arguments, one can prove that, given the vector representation $V = \Delta_q(\omega_1)$ and $\mathfrak{g}$ of type A, C or D, then $T = V^{\otimes d}$ is a $\mathbf{U}_q$-tilting module. Even more generally, the arguments also generalize to show that, given the $\mathbf{U}_q$-module $V = \Delta_q(\lambda)$ with $\lambda \in X^+$ minuscule, then $T = V^{\otimes d}$ is a $\mathbf{U}_q$-tilting module.

Next, we come to the indecomposables of $\boldsymbol{\mathcal{T}}$. These $\mathbf{U}_q$-tilting modules, that we denote by $T_q(\lambda)$, are indexed by the dominant (integral) $\mathbf{U}_q$-weights $\lambda \in X^+$ (see Proposition 3.11

[2] Here we need that the ω_i's are minuscule because we need that $\langle \omega_i, w_{k'}^{-1}(\alpha_j^\vee) \rangle \geq -1$.

below). The $\mathbf{U}_q$-tilting module $T_q(\lambda)$ is determined by the property that it is indecomposable with λ as its unique maximal weight. Then λ appears in fact with multiplicity one.

The following classification is, in the modular case, due to Ringel [31] and Donkin [12].

Proposition 3.11. (**Classification of the indecomposable $\mathbf{U}_q$-tilting modules.**) For each $\lambda \in X^+$ there exists an indecomposable $\mathbf{U}_q$-tilting module $T_q(\lambda)$ with $\mathbf{U}_q$-weight spaces $T_q(\lambda)_\mu = 0$ unless $\mu \leq \lambda$. Moreover, $T_q(\lambda)_\lambda \cong \mathbb{K}$.

In addition, given any indecomposable $\mathbf{U}_q$-tilting module $T \in \boldsymbol{\mathcal{T}}$, then there exists $\lambda \in X^+$ such that $T \cong T_q(\lambda)$.

Thus, the $T_q(\lambda)$'s form a complete set of non-isomorphic indecomposables of $\boldsymbol{\mathcal{T}}$, and all indecomposable $\mathbf{U}_q$-tilting modules $T_q(\lambda)$ are uniquely determined by their maximal weight $\lambda \in X^+$, that is,

$$\{\text{indecomposable } \mathbf{U}_q\text{-tilting modules}\} \xleftrightarrow{1:1} X^+. \qquad \square$$

Proof. We start by constructing $T_q(\lambda)$ for a given, fixed $\lambda \in X^+$.

If the Weyl $\mathbf{U}_q$-module $\Delta_q(\lambda)$ is a $\mathbf{U}_q$-tilting module, then we simply define $T_q(\lambda) = \Delta_q(\lambda)$.

Otherwise, by Theorem 3.1, we can choose a $\mathbf{U}_q$-weight $\mu_2 \in X^+$ minimal such that $\dim(\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\mu_2), \Delta_q(\lambda))) = m_2 \neq 0$ (note that all appearing Ext's are finite-dimensional). Then there is a non-splitting extension

$$0 \longrightarrow \Delta_q(\lambda) = M_1 \hookrightarrow M_2 \twoheadrightarrow \Delta_q(\mu_2)^{\oplus m_2} \longrightarrow 0.$$

Note the important fact that necessarily $\mu_2 < \lambda$. This follows from the universal property of $\Delta_q(\lambda)$ saying that

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), M) = \{m \in M_\lambda \mid E_i^{(r)} m = 0 \text{ for all } i = 1, \ldots, n,\ r \in \mathbb{Z}_{\geq 0}\}$$

for any $\mathbf{U}_q$-module M (here M_λ again denotes the λ-weight space of M). This is the dual of the (q-version of the) Frobenius reciprocity, i.e. the dual of (4).

If M_2 is a $\mathbf{U}_q$-tilting module, then we set $T_q(\lambda) = M_2$. Otherwise, by Theorem 3.1 again, we can choose $\mu_3 \in X^+$ minimal with $\dim(\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\mu_3), M_2)) = m_3 \neq 0$ and we get a non-split extension

$$0 \longrightarrow M_2 \hookrightarrow M_3 \twoheadrightarrow \Delta_q(\mu_3)^{\oplus m_3} \longrightarrow 0.$$

Again $\mu_3 < \lambda$ and also $\mu_3 < \mu_2$.

And hence, we can continue as above and obtain a filtration of the form

$$\cdots \supset M_3 \supset M_2 \supset M_1 \supset M_0 = 0 \tag{24}$$

which is a Δ_q-filtration by construction, since we have $M_{k'+1}/M_{k'} \cong \Delta_q(\mu_{k'+1})^{\oplus m_{k'+1}}$ for all $k' = 0, 1, 2, \ldots$, where we use $\mu_1 = \lambda$ and $m_1 = 1$.

Thus, because there are only finitely many $\mu < \lambda$ (with $\mu \in X^+$), this process stops at some point giving a $\mathbf{U}_q$-module M_k. The $\mathbf{U}_q$-module M_k has a ∇_q-filtration, since otherwise there would, by Proposition 3.5, exist a $\mu_{k+1} \in X^+$ with $\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\mu_{k+1}), M_k) \neq 0$. Moreover, we have constructed a Δ_q-filtration for M_k in (24) which shows that M_k is a $\mathbf{U}_q$-tilting module.

To show that M_k is indecomposable, let us denote $T = M_k$, $U = M_{k-1}$, $m = m_k$ and $\mu = \mu_k$ for short. By the above we have

$$0 \longrightarrow U \hookrightarrow T \twoheadrightarrow \Delta_q(\mu)^{\oplus m} \longrightarrow 0,$$

$$\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), T) = 0 \text{ for all } \nu \in X^+, \quad \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), U) = 0 \text{ for all } \nu \in X^+, \nu \neq \mu,$$

and with m minimal satisfying these properties. Note that U is the largest $\mathbf{U}_q$-submodule of T such that $\mathrm{Hom}_{\mathbf{U}_q}(U, \Delta_q(\mu))$.

Assume that we have a decomposition $T = T_1 \oplus T_2$. This thus induces a decomposition $U = U_1 \oplus U_2$. By induction, U is indecomposable and so we can assume we can assume without loss of generality that $U_1 = U$ and $U_2 = 0$. Thus, $T/U \cong T_1/U_1 \oplus T_2 \cong \Delta_q(\mu)^{\oplus m}$. By the Krull–Schmidt property we get $T_1/U_1 \cong \Delta_q(\mu)^{\oplus j}$, $T_2 \cong \Delta_q(\mu)^{\oplus(m-j)}$ for some $j \leq m$ and we have a short exact sequence

$$0 \longrightarrow U \hookrightarrow T_1 \twoheadrightarrow \Delta_q(\mu)^{\oplus j} \longrightarrow 0. \tag{25}$$

Now, since $\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), \Delta_q(\mu)) = 0$ for $\nu \geq \mu$, we have

$$\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), T) \cong \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), T_1 \oplus T_2) \cong \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), T_1)$$

for any $\nu \geq \mu$. Hence, by (25) and the minimality of m we obtain $m = j$ which in turn implies $T_2 = 0$. This means that $T = M_k$ is indecomposable, and setting $T_q(\lambda) = T$ we are done.

We have to show that any indecomposable $\mathbf{U}_q$-tilting module is isomorphic to some $T_q(\lambda)$. To this end let us suppose that $T \in \boldsymbol{\mathcal{T}}$ is indecomposable. Choose any maximal $\mathbf{U}_q$-weight λ of T. Then we have $\mathrm{Hom}_{\mathbf{U}_v^- \mathbf{U}_v^0}(T, \mathbb{K}_\lambda) \neq 0$. By the Frobenius reciprocity (or, to be more precise, the q-version of it) from (4), we get a non-zero $\mathbf{U}_q$-homomorphism $f \colon T \to \nabla_q(\lambda)$. By duality, we also get a non-zero $\mathbf{U}_q$-homomorphism $g \colon \Delta_q(\lambda) \to T$ with $f \circ g \neq 0$. Consider now the diagram

$$\begin{tikzcd} \Delta_q(\lambda) \arrow[r, hook, "\iota^\lambda"] \arrow[dr, "g"'] & T_q(\lambda) \arrow[r, two heads, "\pi^\lambda"] & \nabla_q(\lambda) \\ & T \arrow[ur, "f"'] & \end{tikzcd} \tag{26}$$

where ι^λ is the inclusion of the first $\mathbf{U}_q$-submodule in a Δ_q-filtration of $T_q(\lambda)$ and π^λ is the surjection onto the last quotient of in a ∇_q-filtration of $T_q(\lambda)$. Since both path in the diagram (26) are non-zero, we can scale everything by some non-zero scalars in $\mathbb{K}$ such that (26) commutes—which we assume in the following. (To see this, recall that there is an (up to scalars) unique $\mathbf{U}_q$-homomorphism $c^\lambda \colon \Delta_q(\lambda) \to \nabla_q(\lambda)$.)

As in the proof of Proposition 3.5, we see that

$$\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) \Rightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(\mathrm{coker}(\iota^\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \ker(\pi^\lambda)) \tag{27}$$

holds. Here $\ker(\pi^\lambda)$ and $\mathrm{coker}(\iota^\lambda)$ are the corresponding kernel and co-kernel respectively.

Thus, we see that the $\mathbf{U}_q$-homomorphism g extends to an $\mathbf{U}_q$-homomorphism $\overline{g} \colon T_q(\lambda) \to T$ whereas f factors through T via $\overline{f} \colon T \to T_q(\lambda)$. Then the composition $\overline{f} \circ \overline{g}$ is an isomorphism since it is so on $T_q(\lambda)_\lambda$. Hence, $T_q(\lambda)$ is a summand of T which shows $T \cong T_q(\lambda)$ since we have assumed that T is indecomposable.

Next, suppose that $T_1 \in \boldsymbol{\mathcal{T}}$ satisfies the characteristic properties of $T_q(\lambda)$. Consider the short exact sequences

$$0 \longrightarrow \Delta_q(\lambda) \xhookrightarrow{\iota^\lambda} T_q(\lambda) \longrightarrow\!\!\!\!\rightarrow \operatorname{coker}(\iota^\lambda) \longrightarrow 0,$$

$$0 \longrightarrow \Delta_q(\lambda) \xhookrightarrow{\iota} T_1 \longrightarrow\!\!\!\!\rightarrow \operatorname{coker}(\iota) \longrightarrow 0,$$

where the cokernels have Δ_q-flags. Thus, by Corollary 3.8, we have $\operatorname{Ext}^1_{\mathbf{U}_q}(\operatorname{coker}(\iota^\lambda), T_1) = 0$, and so the restriction map

$$\operatorname{Hom}_{\mathbf{U}_q}(T_q(\lambda), T_1) \longrightarrow \operatorname{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T_1)$$

is surjective. In particular, the "identity map" $\Delta_q(\lambda) \to \operatorname{im}(\iota)$ has a preimage $f\colon T_q(\lambda) \to T_1$. Similarly, we find a preimage $g\colon T_1 \to T_q(\lambda)$ of $\Delta_q(\lambda) \to \operatorname{im}(\iota^\lambda)$. The composition $g \circ f$ is an endomorphism of the indecomposable $\mathbf{U}_q$-module $T_q(\lambda)$, and thus an isomorphism since it is not nilpotent. Hence, we get $T_1 \cong T_q(\lambda)$.

The other statements are direct consequences of the first three which finishes the proof. ■

Remark 3. For a fixed $\lambda \in X^+$ we have $\mathbf{U}_q$-homomorphisms

$$\Delta_q(\lambda) \xhookrightarrow{\iota^\lambda} T_q(\lambda) \xrightarrow{\pi^\lambda}\!\!\!\!\rightarrow \nabla_q(\lambda)$$

where ι^λ is the inclusion of the first $\mathbf{U}_q$-submodule in a Δ_q-filtration of $T_q(\lambda)$ and π^λ is the surjection onto the last quotient in a ∇_q-filtration of $T_q(\lambda)$. Note that these are only defined up to scalars. One can fix scalars such that $\pi^\lambda \circ \iota^\lambda = c^\lambda$ (where c^λ is again the $\mathbf{U}_q$-homomorphism from (10)). This is done in [6] and crucial for the construction of the cellular basis therein. ▲

Remark 4. Let $T \in \boldsymbol{\mathcal{T}}$. An easy argument shows (see also the proof of Proposition 3.5) the following crucial fact:

$$\operatorname{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) = 0 = \operatorname{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) \Rightarrow \operatorname{Ext}^1_{\mathbf{U}_q}(\operatorname{coker}(\iota^\lambda), T) = 0 = \operatorname{Ext}^1_{\mathbf{U}_q}(T, \ker(\pi^\lambda))$$

for all $\lambda \in X^+$. Consequently, we see that any $\mathbf{U}_q$-homomorphism $g\colon \Delta_q(\lambda) \to T$ extends to a $\mathbf{U}_q$-homomorphism $\overline{g}\colon T_q(\lambda) \to T$ whereas any $\mathbf{U}_q$-homomorphism $f\colon T \to \nabla_q(\lambda)$ factors through $T_q(\lambda)$ via some $\overline{f}\colon T \to T_q(\lambda)$. ▲

Corollary 3.12. We have $\mathcal{D}(T) \cong T$ for $T \in \boldsymbol{\mathcal{T}}$, that is, all $\mathbf{U}_q$-tilting modules T are self-dual. In particular, we have for all $\lambda \in X^+$ that

$$(T : \Delta_q(\lambda)) = \dim(\operatorname{Hom}_{\mathbf{U}_q}(T, \nabla_q(\lambda))) = \dim(\operatorname{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)) = (T : \nabla_q(\lambda)). \qquad \square$$

Proof. By the Krull–Schmidt property it suffices to show the statement for the indecomposable $\mathbf{U}_q$-tilting modules $T_q(\lambda)$. Since $\mathcal{D}$ preserves characters, we see that $\mathcal{D}(T_q(\lambda))$ has λ as unique maximal weight, therefore $\mathcal{D}(T_q(\lambda)) \cong T_q(\lambda)$ by Proposition 3.11. Moreover, the leftmost and the rightmost equalities follow directly from Corollary 3.4. Finally

$$(T_q(\lambda) : \Delta_q(\lambda)) = (\mathcal{D}(T_q(\lambda)) : \mathcal{D}(\Delta_q(\lambda))) = (\mathcal{D}(T_q(\lambda)) : \nabla_q(\lambda)) = (T_q(\lambda) : \nabla_q(\lambda))$$

by definition and $\mathcal{D}(T_q(\lambda)) \cong T_q(\lambda)$ from above, which settles also the middle equality. ■

Example 3.13. Let us go back to the $\mathfrak{sl}_2$ case again. Then we obtain the family $(T_q(i))_{i\in\mathbb{Z}_{\geq 0}}$ of indecomposable $\mathbf{U}_q$-tilting modules as follows.

Start by setting $T_q(0) \cong \Delta_q(0) \cong L_q(0) \cong \nabla_q(0)$ and $T_q(1) \cong \Delta_q(1) \cong L_q(1) \cong \nabla_q(1)$. Then we denote by $m_0 \in T_q(1)$ any eigenvector for K with eigenvalue q. For each $i > 1$ we define $T_q(i)$ to be the indecomposable summand of $T_q(1)^{\otimes i}$ which contains the vector $m_0 \otimes \cdots \otimes m_0 \in T_q(1)^{\otimes i}$. The $\mathbf{U}_q(\mathfrak{sl}_2)$-tilting module $T_q(1)^{\otimes i}$ is not indecomposable if $i > 1$: by Proposition 3.11 we have $(T_q(1)^{\otimes i} : \Delta_q(i)) = 1$ and

$$T_q(1)^{\otimes i} \cong T_q(i) \oplus \bigoplus_{k<i} T_q(k)^{\oplus \mathrm{mult}_k} \quad \text{for some } \mathrm{mult}_k \in \mathbb{Z}_{\geq 0}.$$

In the case $l = 3$, we have for instance $T_q(1)^{\otimes 2} \cong T_q(2) \oplus T_q(0)$ since the tensor product $T_q(1) \otimes T_q(1)$ looks as follows (abbreviation $m_{ij} = m_i \otimes m_j$):

$$\begin{tikzcd}
\otimes \arrow[r, dash, dotted] \arrow[d, dash, dotted] & m_1 \arrow[loop above, "q^{-1}"] \arrow[r, shift left, "1"] & m_0 \arrow[loop above, "q^{+1}"] \arrow[l, shift left, "1"] \\
m_1 \arrow[loop left, "q^{-1}"] \arrow[d, shift left, "1"] & m_{11} \arrow[loop left, "q^{-2}"] \arrow[r, shift left, "1"] \arrow[d, shift left, "1"] & m_{01} \arrow[loop right, "q^{0}"] \arrow[l, shift left, "1"] \arrow[d, shift left, "1"] \\
m_0 \arrow[loop left, "q^{+1}"] \arrow[u, shift left, "1"] & m_{10} \arrow[loop left, "q^{0}"] \arrow[r, shift left, "1"] \arrow[u, shift left, "1"] \arrow[ur, dash, dotted, "+" description] & m_{00} \arrow[loop right, "q^{+2}"] \arrow[l, shift left, "1"] \arrow[u, shift left, "1"]
\end{tikzcd}.$$

By construction, the indecomposable $\mathbf{U}_q(\mathfrak{sl}_2)$-module $T_q(2)$ contains m_{00} and therefore has to be the $\mathbb{C}$-span of $\{m_{00}, q^{-1}m_{10} + m_{01}, m_{11}\}$ as indicated above. The remaining summand is the one-dimensional $\mathbf{U}_q$-tilting module $T_q(0) \cong L_q(0)$ from before. ▲

The following is interesting in its own right.

Corollary 3.14. Let $\mu \in X^+$ be a minuscule $\mathbf{U}_q$-weight. Then $T = \Delta_q(\mu)^{\otimes d}$ is a $\mathbf{U}_q$-tilting module for any $d \in \mathbb{Z}_{\geq 0}$ and $\dim(\mathrm{End}_{\mathbf{U}_q}(T))$ is independent of the field $\mathbb{K}$ and of $q \in \mathbb{K}^*$, and is given by

$$\dim(\mathrm{End}_{\mathbf{U}_q}(T)) = \sum_{\lambda \in X^+} (T : \Delta_q(\lambda))^2 = \sum_{\lambda \in X^+} (T : \nabla_q(\lambda))^2. \tag{28}$$

In particular, this holds for $\Delta_q(\omega_1)$ being the vector representation of $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ for $\mathfrak{g}$ of type A, C or D. □

Proof. Since $\mu \in X^+$ is minuscule: $\Delta_q(\mu) \cong L_q(\mu)$ is a simple $\mathbf{U}_q$-tilting module for any field $\mathbb{K}$ and any $q \in \mathbb{K}^*$. Thus, by Proposition 3.10 we see that T is a $\mathbf{U}_q$-tilting module for any $d \in \mathbb{Z}_{\geq 0}$. Hence, by Corollary 3.4—in particular by (18)—and Corollary 3.12, we have the equality in (28). Now use the fact that the character of $\Delta_q(\mu)$ and $\nabla_q(\lambda)$ is as in the classical case, which implies the statement. ■

3C. **The characters of indecomposable $\mathbf{U}_q$-tilting modules.** In this section we describe how to compute $(T_q(\lambda) : \Delta_q(\mu))$ for all $\lambda, \mu \in X^+$ (which can be done algorithmically in the case where q is a complex, primitive l-th root of unity). As an application, we illustrate how to decompose tensor products of $\mathbf{U}_q$-tilting modules. This shows that, in principle, our cellular basis for endomorphism rings $\mathrm{End}_{\mathbf{U}_q}(T)$ of $\mathbf{U}_q$-tilting modules T (as defined in [6, Section 3]) can be made more or less explicit.

We start with some preliminaries. Given an abelian category $\boldsymbol{\mathcal{A}b}$, we denote its *Grothendieck group* by $G_0(\boldsymbol{\mathcal{A}b})$ and its *split Grothendieck group* by $K_0^{\oplus}(\boldsymbol{\mathcal{A}b})$. We point out that the notation of the split Grothendieck group also makes sense for a given additive category that satisfies the Krull–Schmidt property where we use the same notation. (We refer the reader unfamiliar with these and the notation we use to [27, Section 1.2].)

Recall that G_0 and K_0 are $\mathbb{Z}$-modules and one might ask for $\mathbb{Z}$-basis of them. Moreover, if the categories in question are monoidal, then G_0 and K_0 inherit the structure of $\mathbb{Z}$-algebras.

The category $\mathbf{U}_q$-**Mod** is abelian and we can consider $G_0(\mathbf{U}_q\text{-}\mathbf{Mod})$. In contrast, $\boldsymbol{\mathcal{T}}$ is not abelian (see Example 3.9), but it is additive and satisfies the Krull–Schmidt property, so we can consider $K_0(\boldsymbol{\mathcal{T}})$. Since both $\mathbf{U}_q$-**Mod** and $\boldsymbol{\mathcal{T}}$ are closed under tensor products, $G_0(\mathbf{U}_q\text{-}\mathbf{Mod})$ and $K_0^{\oplus}(\boldsymbol{\mathcal{T}})$ get a—in fact isomorphic—induced $\mathbb{Z}$-algebra structure.

Moreover, by Proposition 2.10 and Proposition 2.12, a $\mathbb{Z}$-basis of $G_0(\mathbf{U}_q\text{-}\mathbf{Mod})$ is given by isomorphism classes $\{[\Delta_q(\lambda)] \mid \lambda \in X^+\}$. On the other hand, a $\mathbb{Z}$-basis of $K_0^{\oplus}(\boldsymbol{\mathcal{T}})$ is, by Proposition 3.11, spanned by isomorphism classes $\{[T_q(\lambda)]_{\oplus} \mid \lambda \in X^+\}$.

Corollary 3.15. The inclusion of categories $\iota\colon \boldsymbol{\mathcal{T}} \to \mathbf{U}_q$-**Mod** induces an isomorphism

$$[\iota]\colon K_0^{\oplus}(\boldsymbol{\mathcal{T}}) \to G_0(\mathbf{U}_q\text{-}\mathbf{Mod}), \quad [T_q(\lambda)]_{\oplus} \mapsto [T_q(\lambda)], \quad \lambda \in X^+$$

of $\mathbb{Z}$-algebras. □

Proof. The set $B = \{[T_q(\lambda)] \mid \lambda \in X^+\}$ forms a $\mathbb{Z}$-basis of $K_0^{\oplus}(\boldsymbol{\mathcal{T}})$ by Proposition 3.11 and it is clear that $[\iota]$ is a well-defined $\mathbb{Z}$-algebra homomorphism.

Moreover, we have

$$[T_q(\lambda)] = [\Delta_q(\lambda)] + \sum_{\mu<\lambda\in X^+} (T_q(\mu) : \Delta_q(\mu))[\Delta_q(\mu)] \in G_0(\mathbf{U}_q\text{-}\mathbf{Mod}) \tag{29}$$

with $T_q(0) \cong \Delta_q(0)$ by Proposition 3.11. Hence, $[\iota](B)$ is also a $\mathbb{Z}$-basis of $K_0(\mathbf{U}_q\text{-}\mathbf{Mod})$ since the $\Delta_q(\lambda)$'s form a $\mathbb{Z}$-basis and the claim follows. ■

In Section 2B we have met Weyl's character ring $\mathbb{Z}[X]$. Further, recall that $\mathbb{Z}[X]$ carries an action of the Weyl group W associated to the Cartan datum (see below). Thus, we can look at the invariant part of this action, denoted by $\mathbb{Z}[X]^W$.

We obtain the following (known) categorification result.

Corollary 3.16. The tilting category $\boldsymbol{\mathcal{T}}$ (naively) categorifies $\mathbb{Z}[X]^W$, that is,

$$K_0^{\oplus}(\boldsymbol{\mathcal{T}}) \cong \mathbb{Z}[X]^W \quad \text{as } \mathbb{Z}\text{-algebras.}$$

□

Proof. It is known that there is an isomorphism $K_0(\mathfrak{g}\text{-}\mathbf{Mod}) \xrightarrow{\cong} \mathbb{Z}[X]^W$ given by sending finite-dimensional $\mathfrak{g}$-modules to their characters (which can be regarded as elements in $\mathbb{Z}[X]^W$).

Now the characters $\chi(\Delta_q(\lambda))$ of the $\Delta_q(\lambda)$'s are (as mentioned below Example 2.13) the same as in the classical case. Thus, we can adopt the isomorphism from $K_0(\mathfrak{g}\text{-}\mathbf{Mod})$ to $\mathbb{Z}[X]^W$ from above. Details can, for example, be found in [8, Chapter VIII, §7.7].

Then the statement follows from Corollary 3.15. ■

For each simple root $\alpha_i \in \Pi$ let s_i be the reflection

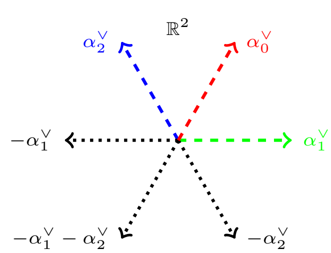
$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \text{for } \lambda \in E,$$

in the hyperplane $H_{\alpha_i^\vee} = \{x \in E \mid \langle x, \alpha_i^\vee \rangle = 0\}$ orthogonal to α_i. These reflections s_i generate a group W, called *Weyl group*, associated to our Cartan datum.

For any fixed $l \in \mathbb{Z}_{\geq 0}$, the *affine Weyl group* $W_l \cong W \ltimes l\mathbb{Z}\Pi$ is the group generated by the reflections $s_{\beta,r}$ in the *affine hyperplanes* $H^l_{\beta^\vee,r} = \{x \in E \mid \langle x, \beta^\vee \rangle = lr\}$ for $\beta \in \Phi$ and $r \in \mathbb{Z}$. Note that, if $l = 0$, then $W_0 \cong W$.

Example 3.17. Here the prototypical example to keep in mind. We consider $\mathfrak{g} = \mathfrak{sl}_3$ with the Cartan datum from Example 2.1, i.e.:

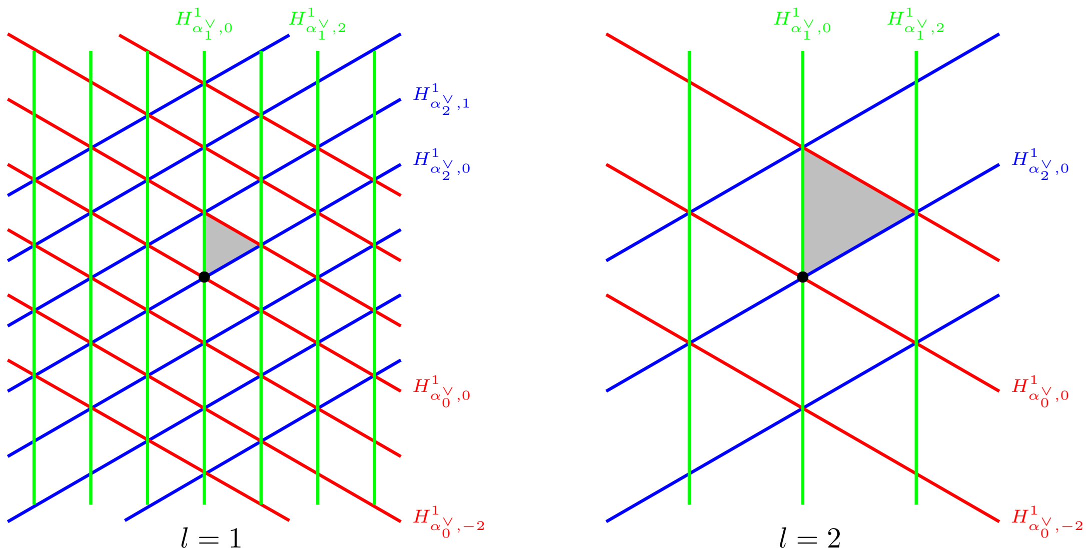
$$E = \mathbb{R}^3/(1,1,1)(\cong \mathbb{R}^2), \qquad \begin{aligned} \alpha_1 &= (1,-1,0) = \alpha_1^\vee, \\ \alpha_2 &= (0,1,-1) = \alpha_2^\vee, \\ \alpha_0^\vee &= (1,0,-1) = \alpha_1^\vee + \alpha_2^\vee, \end{aligned} \qquad \mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$



where we—for simplicity—have identified the roots and coroots. Choosing $l = 1$ or $l = 2$ gives then the following hyperplanes:

$$\begin{aligned} l = 1\colon \quad & H^1_{\alpha_1^\vee,r} = \{(a,b,c) \in E \mid a - b = r\}, \quad H^1_{\alpha_2^\vee,r} = \{(a,b,c) \in E \mid b - c = r\}, \\ l = 2\colon \quad & H^2_{\alpha_1^\vee,r} = \{(a,b,c) \in E \mid a - b = 2r\}, \quad H^2_{\alpha_2^\vee,r} = \{(a,b,c) \in E \mid b - c = 2r\}. \end{aligned}$$

Using the isomorphism $E = \mathbb{R}^3/(1,1,1) \cong \mathbb{R}^2$ (which we will in later $\mathfrak{sl}_3$ examples), these can be illustrated via the classical picture of the hyperplane arrangement for $\mathfrak{sl}_3$:



In these pictures we have additionally chosen an origin and a fundamental alcove (as defined in Definition 3.18 below). Note that both hyperplane arrangements are combinatorial the same, but the precise coordinates of the lattice points within the regions differs. (Every second hyperplane $H^1_{\alpha_i^\vee,r}$ is omitted in case $l = 2$.)

The affine Weyl group W_l is now generated by the reflections in these hyperplanes. ▲

For $\beta \in \Phi$ there exists $w \in W$ such that $\beta = w(\alpha_i)$ for some $i = 1, \ldots, n$. We set $l_\beta = l_i$ where $l_i = \frac{l}{\gcd(l,d_i)}$. Using this, we have the *dot-action* of W_l on the $\mathbf{U}_q$-weight lattice X via

$$s_{\beta,r}.\lambda = s_\beta(\lambda + \rho) - \rho + l_\beta r \beta.$$

Note that the case $l = 1$ recovers the usual action of the affine Weyl group W_1 on X.

Definition 3.18. (**Alcove combinatorics.**) The *fundamental alcove* $\mathcal{A}_0$ is

(30) $$\mathcal{A}_0 = \{\lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < l, \text{ for all } \alpha \in \Phi^+\} \subset X^+.$$

Its *closure* $\overline{\mathcal{A}}_0$ is given by

(31) $$\overline{\mathcal{A}}_0 = \{\lambda \in X \mid 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq l, \text{ for all } \alpha \in \Phi^+\} \subset X^+ - \rho.$$

The *non-affine walls* of $\mathcal{A}_0$ are

$$\check{\partial}\mathcal{A}_0^i = \overline{\mathcal{A}}_0 \cap (H_{\alpha_i^\vee,0} - \rho), i = 1, \ldots, n, \quad \check{\partial}\mathcal{A}_0 = \bigcup_{i=1}^{n} \check{\partial}\mathcal{A}_0^i.$$

Let α_0 denote the maximal short root. The set

$$\hat{\partial}\mathcal{A}_0 = \overline{\mathcal{A}}_0 \cap (H_{\alpha_0^\vee,1} - \rho)$$

is called the *affine* wall of $\mathcal{A}_0$. We call the union of all these walls the *boundary* $\partial\mathcal{A}_0$ of $\mathcal{A}_0$. More generally, an *alcove* $\mathcal{A}$ is a connected component of

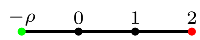
$$E - \bigcup_{r\in\mathbb{Z},\beta\in\Phi} (H_{\beta^\vee,r} - \rho).$$

We denote the set of alcoves by $\mathcal{A}\mathrm{l}$. ▲

Note that the affine Weyl group W_l acts simply transitively on $\mathcal{A}\mathrm{l}$. Thus, we can associate $1 \in W_l \mapsto \mathcal{A}(1) = \mathcal{A}_0 \in \mathcal{A}\mathrm{l}$ and in general $w \in W_l \mapsto \mathcal{A}(w) \in \mathcal{A}\mathrm{l}$.

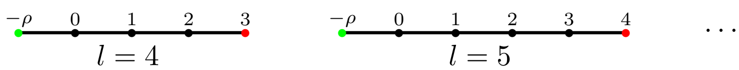
Example 3.19. In the case $\mathfrak{g} = \mathfrak{sl}_2$ we have $\rho = \omega_1 = 1$. Consider for instance again $l = 3$. Then $k \in \mathbb{Z}_{\geq 0} = X^+$ is contained in the fundamental alcove $\mathcal{A}_0$ if and only if $0 < k+1 < 3$.

Moreover, $-\rho \in \check{\partial}\mathcal{A}_0$ and $2 \in \hat{\partial}\mathcal{A}_0$ are on the walls. Thus, $\overline{\mathcal{A}}_0$ can be visualized as



where the affine wall on the right is indicated in red and the non-affine wall on the left is indicated in green.

The picture for bigger l is easy to obtain, e.g.:



as we encourage the reader to verify. ▲

Example 3.20. Let us leave our running $\mathfrak{sl}_2$ example for a second and do another example which is graphically more interesting.

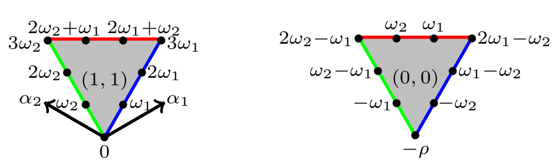
In the case $\mathfrak{g} = \mathfrak{sl}_3$ we have $\rho = \alpha_1 + \alpha_2 = \omega_1 + \omega_2 \in X^+$ and $\alpha_0 = \alpha_1 + \alpha_2$. Now consider again $l = 3$. The condition (30) means that $\mathcal{A}_0$ consists of those $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ for which

$$0 < \langle \lambda_1\omega_1 + \lambda_2\omega_2 + \omega_1 + \omega_2, \alpha_i^\vee \rangle < 3 \quad \text{for } i = 1, 2, 0.$$

Thus, $0 < \lambda_1 + 1 < 3$, $0 < \lambda_2 + 1 < 3$ and $0 < \lambda_1 + \lambda_2 + 2 < 3$. Hence, only the $\mathbf{U}_q(\mathfrak{sl}_3)$-weight $\lambda = (0,0) \in X^+$ is in $\mathcal{A}_0$. In addition, we have by condition (31) that

$$\check{\partial}\mathcal{A}_0 = \{-\rho, -\omega_1, -\omega_2, \omega_1 - \omega_2, \omega_2 - \omega_1\}, \qquad \hat{\partial}\mathcal{A}_0 = \{\omega_1, \omega_2, 2\omega_1 - \omega_2, 2\omega_2 - \omega_1\}.$$

Hence, $\overline{\mathcal{A}}_0$ can be visualized as (displayed without the $-\rho$ shift on the left)



where, as before, the affine wall at the top is indicated in red, the hyperplane orthogonal to α_1 on the left in green and the hyperplane orthogonal to α_2 on the right in blue. See also Example 3.17, where we again stress that the precise coordinates of points in the alcoves or on their boundaries depend on l. ▲

We say $\lambda \in X^+ - \rho$ is *linked* to $\mu \in X^+$ if there exists $w \in W_l$ such that $w.\lambda = \mu$. We note the following theorem, called *the linkage principle*, where we, by convention, set $T_q(\lambda) = \Delta_q(\lambda) = \nabla_q(\lambda) = L_q(\lambda) = 0$ for $\lambda \in \check{\partial}\mathcal{A}_0$.

Theorem 3.21. (**The linkage principle.**) All composition factors of $T_q(\lambda)$ have maximal weights μ linked to λ. Moreover, $T_q(\lambda)$ is a simple $\mathbf{U}_q$-module if $\lambda \in \overline{\mathcal{A}}_0$.

If λ is linked to an element of $\mathcal{A}_0$, then $T_q(\lambda)$ is a simple $\mathbf{U}_q$-module if and only if $\lambda \in \mathcal{A}_0$. $\square$

Proof. This is a slight reformulation of [2, Corollaries 4.4 and 4.6]. ■

The linkage principle gives us now a decomposition into a direct sum of categories

$$\boldsymbol{\mathcal{T}} \cong \bigoplus_{\lambda \in \mathcal{A}_0} \boldsymbol{\mathcal{T}}_\lambda \oplus \bigoplus_{\lambda \in \partial\mathcal{A}_0} \boldsymbol{\mathcal{T}}_\lambda,$$

where each $\boldsymbol{\mathcal{T}}_\lambda$ consists of all $T \in \boldsymbol{\mathcal{T}}$ whose indecomposable summands are all of the form $T_q(\mu)$ for $\mu \in X^+$ lying in the W_l-dot orbit of $\lambda \in \mathcal{A}_0$ (or of $\lambda \in \partial\mathcal{A}_0$). We call these categories *blocks* to stress that they are homologically unconnected—although they might be decomposable. Moreover, if $\lambda \in \mathcal{A}_0$, then we call $\boldsymbol{\mathcal{T}}_\lambda$ an *l-regular* block, while the $\boldsymbol{\mathcal{T}}_\lambda$'s with $\lambda \in \partial\mathcal{A}_0$ are called *l-singular* blocks. (We say for short just regular and singular blocks in what follows.)

In fact, by Proposition 3.11, the $\mathbf{U}_q$-weights labeling the indecomposable $\mathbf{U}_q$-tilting modules are only the dominant (integral) weights $\lambda \in X^+$. Let $\mathrm{d}\mathcal{C} = \{x \in E \mid \langle x, \beta^\vee \rangle \geq 0, \beta \in \Phi\}$. Then these $\mathbf{U}_q$-weights correspond blockwise precisely to the alcoves

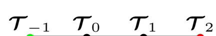
$$\mathcal{A}\mathrm{l}^+ = \mathcal{A}\mathrm{l} \cap \mathrm{d}\mathcal{C},$$

contained in the dominant chamber $\mathrm{d}\mathcal{C}$. That is, they correspond to the set of coset representatives of minimal length in $\{wW_0 \mid w \in W_1\}$. In formulas,

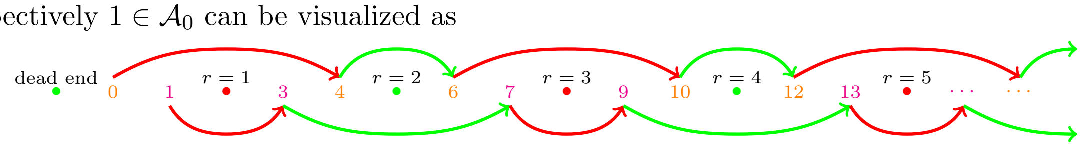
$$T_q(w.\lambda) \in \boldsymbol{\mathcal{T}}_\lambda \rightsquigarrow \mathcal{A}(w) \in \mathcal{A}\mathrm{l}^+ \rightsquigarrow wW_0 \subset W_1, \tag{32}$$

for all $\lambda \in \mathcal{A}_0$.

Example 3.22. In our pet example with $\mathfrak{g} = \mathfrak{sl}_2$ and $l = 3$ we have, by Theorem 3.21 and Example 3.19 a block decomposition

$\boldsymbol{\mathcal{T}} \cong$ 

(Taking direct sums of the categories on the right-hand side.) The W_l-dot orbit of $0 \in \mathcal{A}_0$ respectively $1 \in \mathcal{A}_0$ can be visualized as

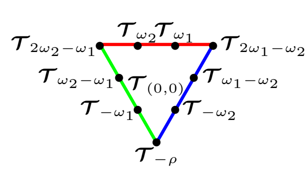


Compare also to [7, (2.4.1)].

It turns out that, for $\mathbb{K} = \mathbb{C}$, both singular blocks $\boldsymbol{\mathcal{T}}_{-1}$ and $\boldsymbol{\mathcal{T}}_2$ are semisimple (in particular, these blocks decompose further), see Example 3.27 or [7, Lemma 2.25].

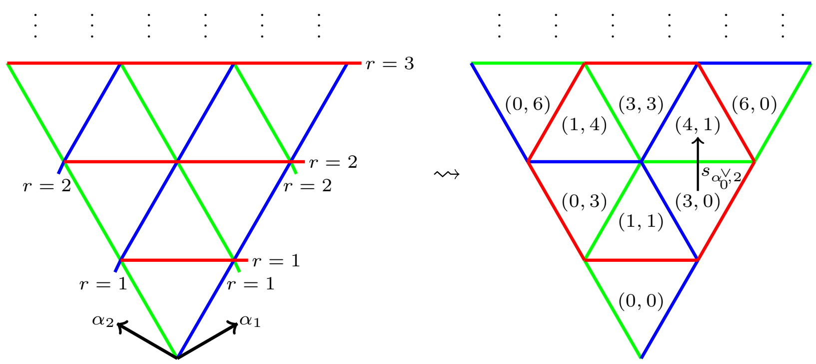
All of this generalizes as already indicated in Example 3.19. ▲

Example 3.23. In the $\mathfrak{sl}_3$ case with $l = 3$ we have the block decomposition

$\boldsymbol{\mathcal{T}} \cong$ 

(Again, taking direct sums of the categories on the right-hand side.) Note that the singular blocks are not necessarily semisimple anymore, even when $\mathbb{K} = \mathbb{C}$.

The W_l-dot orbit in $\mathcal{AC}^+$ of the regular block $\boldsymbol{\mathcal{T}}_{(0,0)}$ looks as follows.



Here we reflect either in a red (that is, $\alpha_0 = (1,1)$), green (that is, $\alpha_1 = (2,-1)$) or blue (that is, $\alpha_2 = (-1,2)$) hyperplane, and the r measures the hyperplane-distance from the origin (both indicated in the left picture above). In the right picture we have indicated the linkage (we have also displayed one of the dot-reflections).

Theorem 3.21 means now that $T_q((1,1))$ satisfies

$$(T_q((1,1)) : \Delta_q(\mu)) \neq 0 \quad \Rightarrow \quad \mu \in \{(0,0),(1,1)\}$$

and $T_q((3,3))$ satisfies

$$(T_q((3,3)) : \Delta_q(\mu)) \neq 0 \quad \Rightarrow \quad \mu \in \{(0,0),(1,1),(3,0),(0,3),(4,1),(1,4),(3,3)\}.$$

We calculate the precise values later in Example 3.25. ▲

In order to get our hands on the multiplicities, we need Soergel's version of the *(affine) parabolic Kazhdan–Lusztig polynomials*, which we denote by

$$n_{\mu\lambda}(t) \in \mathbb{Z}[v, v^{-1}], \quad \lambda, \mu \in X^+ - \rho. \tag{33}$$

For brevity, we do not recall the definition of these polynomials—which can be computed algorithmically—here, but refer to [34, Section 3] where the relevant polynomial is denoted $n_{y,x}$ for $x, y \in W_l$ (which translates by (32) to our notation). The main point for us is the following theorem due to Soergel.

Theorem 3.24. (**Multiplicity formula.**) Suppose $\mathbb{K} = \mathbb{C}$ and q is a complex, primitive l-th root of unity. For each pair $\lambda, \mu \in X^+$ with λ being an l-regular $\mathbf{U}_q$-weight (that is, $T_q(\lambda)$ belongs to a regular block of $\boldsymbol{\mathcal{T}}$) we have

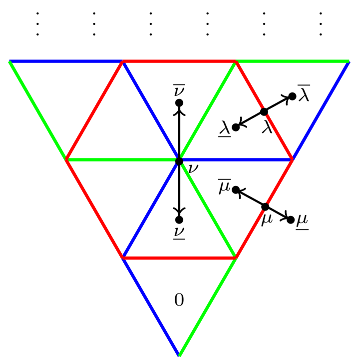
$$(T_q(\lambda) : \Delta_q(\mu)) = (T_q(\lambda) : \nabla_q(\mu)) = n_{\mu\lambda}(1).$$

In particular, if $\lambda, \mu \in X^+$ are not linked, then $n_{\mu\lambda}(v) = 0$. □

Proof. This follows from [33, Theorem 5.12], see also [34, Conjecture 7.1]. ■

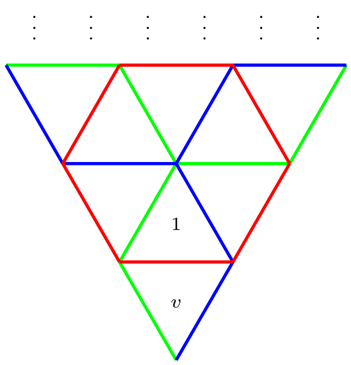
In addition to Theorem 3.24, we are going to describe now an algorithmic way to compute $(T_q(\lambda) : \Delta_q(\mu))$ for all $T_q(\lambda)$ lying in a singular blocks of $\boldsymbol{\mathcal{T}}$. We point out that Theorem 3.26 below is valid for $q \in \mathbb{K}$ being a primitive l-th root of unity, where $\mathbb{K}$ is—in contrast to Theorem 3.24—an arbitrary field.

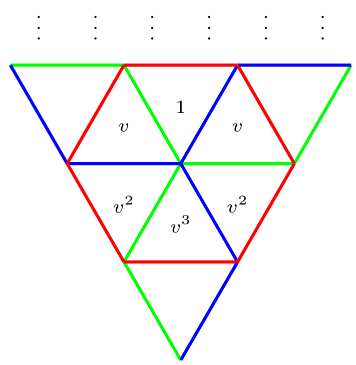
Assume in the following that $\lambda \in X^+$ is not l-regular. Set $W_\lambda = \{w \in W_l \mid w.\lambda = \lambda\}$. Then we can find a unique l-regular $\mathbf{U}_q$-weight $\overline{\lambda} \in W_l.0$ such that λ is in the closure of the alcove containing $\overline{\lambda}$ and $\overline{\lambda}$ is maximal in $W_\lambda.\overline{\lambda}$. Similarly, we find a can find a unique l-regular $\mathbf{U}_q$-weight $\underline{\lambda} \in W_l.0$ such that λ is in the closure of the alcove containing $\underline{\lambda}$ and $\underline{\lambda}$ is minimal in $W_\lambda.\overline{\lambda}$. Some examples in the $\mathfrak{g} = \mathfrak{sl}_3$ case are



We stress that, in the μ case above, Theorem 3.26 is not valid: recall that in those cases $T_q(\mu) = \Delta_q(\mu) = L_q(\mu) = \nabla_q(\mu) = 0$ and thus, we do not have to worry about these.

Example 3.25. Back to Example 3.23: For $\nu = \omega_1 + \omega_2 = (1,1)$ we have $n_{\nu\nu}(v) = 1$ and $n_{\nu(0,0)}(v) = v$, as shown in the left picture below. Similarly, for $\xi = 3\omega_1 + 3\omega_2 = (3,3)$ the only non-zero parabolic Kazhdan–Lusztig polynomials are $n_{\xi\xi}(v) = 1$, $n_{\xi(1,4)}(v) = v = n_{\xi(4,1)}(v)$, $n_{\xi(0,3)}(v) = v^2 = n_{\xi(3,0)}(v)$ and $n_{\xi\nu}(v) = v^3$ as illustrated on the right below.

ν:


ξ:


Therefore, we have, by Theorem 3.24, that $(T_q(\nu) : \Delta_q(\mu)) = 1$ if $\mu \in \{(0,0),(1,1)\}$ and $(T_q(\nu) : \Delta_q(\mu)) = 0$ if $\mu \notin \{(0,0),(1,1)\}$. We encourage the reader to work out $(T_q(\xi) : \Delta_q(\mu))$ by using the above patterns and Example 3.23. For all patterns in rank 2 see [35]. ▲

We are aiming to show the following Theorem.

Theorem 3.26. (**Multiplicity formula—singular case.**) We have

$$(T_q(\lambda) : \Delta_q(\mu)) = (T_q(\overline{\lambda}) : \Delta_q(\overline{\mu}))$$

for all $\mu \in W_l.\lambda \cap X^+$.

We consider the translation functors $\mathcal{T}_{\xi}^{\xi'}\colon \boldsymbol{\mathcal{T}}_{\xi} \to \boldsymbol{\mathcal{T}}_{\xi'}$ for various $\xi, \xi' \in X^+$ in the proof. The reader unfamiliar with these can for example consider [19, Part II, Chapter 7]. We only stress here that $\mathcal{T}_{\xi}^{\xi'}\colon \boldsymbol{\mathcal{T}}_{\xi} \to \boldsymbol{\mathcal{T}}_{\xi'}$ is the biadjoint of $\mathcal{T}_{\xi'}^{\xi}\colon \boldsymbol{\mathcal{T}}_{\xi'} \to \boldsymbol{\mathcal{T}}_{\xi}$.

Proof. In order to prove Theorem 3.26, we have to show some intermediate steps. We start with the following two claims.

Claim3.26a. We have:

$$[\Delta_q(\lambda') : L_q(\underline{\lambda})] = 1 \quad \text{for all} \quad \lambda' \in W_\lambda.\overline{\lambda}. \tag{34}$$

Moreover, for all $\lambda' \in W_\lambda.\overline{\lambda}$:

$$\text{there is a unique} \quad \varphi \in \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\overline{\lambda})) \quad \text{with} \quad [\mathrm{Im}(\varphi) : L_q(\underline{\lambda})] = 1. \tag{35}$$

Here uniqueness is meant up to scalars.

Proof of Claim3.26a. We start by showing (34). We have $\mathcal{T}_{\overline{\lambda}}^{\lambda}(\Delta_q(\lambda')) \cong \Delta_q(\lambda)$. In addition, for any $\lambda'' \in W_l.\overline{\lambda} \cap X^+$, we have $\mathcal{T}_{\overline{\lambda}}^{\lambda}(L_q(\lambda'')) \cong L_q(\lambda)$ if and only if $\lambda'' = \underline{\lambda} \in X^+$.

Next, we show (35). We use descending induction. If $\lambda' = \overline{\lambda}$, then (35) is clear. So assume $\lambda' < \overline{\lambda}$ and denote by $\mathcal{A}'$ the alcove containing λ'. Choose an upper wall H of $\mathcal{A}'$ such that the corresponding reflection s_H belongs to W_λ. Then $\lambda'' = s_H.\lambda' > \lambda'$. Thus, by induction, there exists an (up to scalars) unique non-zero $\mathbf{U}_q$-homomorphism $\psi\colon \Delta_q(\lambda'') \to \Delta_q(\overline{\lambda})$ with $[\mathrm{Im}(\psi) : L_q(\underline{\lambda})] = 1$. We claim now that for all $\lambda' \in W_\lambda.\overline{\lambda}$:

$$\text{there exists a unique} \quad \tilde{\varphi} \in \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\lambda'')) \quad \text{with} \quad [\mathrm{Im}(\tilde{\varphi}) : L_q(\underline{\lambda})] = 1. \tag{36}$$

Again uniqueness is meant up to scalars.

Because (36) implies that $\varphi = \psi \circ \tilde{\varphi}$ is the (up to scalars) unique non-zero $\mathbf{U}_q$-homomorphism we are looking for, it remains to show (36). To this end, choose $\nu \in H$. Then we have a short exact sequence

$$0 \longrightarrow \Delta_q(\lambda'') \hookrightarrow \mathcal{T}_{\nu}^{\overline{\lambda}}\Delta_q(\nu) \twoheadrightarrow \Delta_q(\lambda') \longrightarrow 0.$$

This sequence does not split since $\mathcal{T}_{\nu}^{\overline{\lambda}}\Delta_q(\nu)$ has simple head $L_q(\lambda')$. Thus, the inclusion

$$\begin{aligned} \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\lambda'')) &\hookrightarrow \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \mathcal{T}_{\nu}^{\overline{\lambda}}\Delta_q(\nu)) \\ &\cong \mathrm{Hom}_{\mathbf{U}_q}(\mathcal{T}_{\overline{\lambda}}^{\nu}\Delta_q(\lambda'), \Delta_q(\nu)) \\ &\cong \mathrm{End}_{\mathbf{U}_q}(\Delta_q(\nu)) \cong \mathbb{K} \end{aligned}$$

is an equality. So we can pick any non-zero $\mathbf{U}_q$-homomorphism $\tilde{\varphi} \in \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\lambda''))$ which will be unique up to scalars. Then $L_q(\lambda')$ is a composition factor of $\mathrm{Im}(\tilde{\varphi})$. This implies that $\mathcal{T}_{\overline{\lambda}}^{\nu}\tilde{\varphi} \in \mathrm{End}_{\mathbf{U}_q}(\Delta_q(\nu))$ is non-zero and thus, an isomorphism. In particular, $L_q(\underline{\lambda})$ is a composition factor of $\mathrm{Im}(\tilde{\varphi})$, because $\mathcal{T}_{\overline{\lambda}}^{\nu} L_q(\lambda') \neq 0$. Hence, (36) follows and thus, (35) holds.

Claim3.26b. We keep the notation from before.

$$\text{We have} \quad (T_q(\overline{\lambda}) : \Delta_q(\lambda')) = 1 \quad \text{for all} \quad \lambda' \in W_\lambda.\overline{\lambda}. \tag{37}$$

Proof of Claim3.26b. By (35) we have $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'),\Delta_q(\overline{\lambda}))\cong\mathbb{K}$. This together with

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'),T_q(\overline{\lambda}))\supset\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'),\Delta_q(\overline{\lambda}))\cong\mathbb{K}$$

implies (37).

Claim3.26c. Our last claim is:

$$\text{We have}\quad \mathcal{T}_{\lambda}^{\overline{\lambda}}T_q(\lambda)=T_q(\overline{\lambda}). \tag{38}$$

Proof of Claim3.26c. We have $\mathcal{T}_{\lambda}^{\overline{\lambda}}T_q(\lambda)=T_q(\overline{\lambda})\oplus\mathrm{rest}$ where rest is some $\mathbf{U}_q$-tilting module with $\mathbf{U}_q$-weights $<\overline{\lambda}$. However, applying $\mathcal{T}_{\overline{\lambda}}^{\lambda}(\cdot)$, we get

$$T_q(\lambda)^{\oplus|W_\lambda|}\cong\mathcal{T}_{\overline{\lambda}}^{\lambda}T_q(\lambda)\oplus\mathcal{T}_{\overline{\lambda}}^{\lambda}(\mathrm{rest}).$$

However, by (37), we also have

$$\mathcal{T}_{\overline{\lambda}}^{\lambda}T_q(\overline{\lambda})\cong T_q(\lambda)^{\oplus|W_\lambda|}.$$

Thus, $\mathcal{T}_{\overline{\lambda}}^{\lambda}(\mathrm{rest})=0$. This implies rest $=0$:

Suppose the contrary. Then there exists $\tilde{\lambda}\in X^+$ with

$$0\neq\mathrm{Hom}_{\mathbf{U}_q}(L_q(\tilde{\lambda}),\mathrm{rest})\subset\mathrm{Hom}_{\mathbf{U}_q}(L_q(\tilde{\lambda}),\mathcal{T}_{\lambda}^{\overline{\lambda}}T_q(\lambda))\cong\mathrm{Hom}_{\mathbf{U}_q}(\mathcal{T}_{\overline{\lambda}}^{\lambda}L_q(\tilde{\lambda}),T_q(\lambda)).$$

But then $0\neq\mathcal{T}_{\overline{\lambda}}^{\lambda}L_q(\tilde{\lambda})\subset\mathcal{T}_{\overline{\lambda}}^{\lambda}(\mathrm{rest})$. This is a contradiction. Hence, (38) follows.

We are now ready to prove the theorem itself. For this purpose, note that we get

$$(T_q(\lambda):\Delta_q(w.\lambda))=(T_q(\overline{\lambda}):\Delta_q(w.\overline{\lambda})))\quad\text{for all } w\in W_l \text{ with } w.\lambda\in X^+.$$

from (38). This in turn implies the statement of the theorem by the linkage principle. ■

Since the polynomials from (33) can be computed inductively, we can use Theorem 3.24 and Theorem 3.26 in the case $\mathbb{K}=\mathbb{C}$ to explicitly calculate the decomposition of a tensor product of $\mathbf{U}_q$-tilting modules $T=T_q(\lambda_1)\otimes\cdots\otimes T_q(\lambda_d)$ into its indecomposable summands:

- Calculate, by using Theorem 3.24 and Theorem 3.26, $(T_q(\lambda_i):\Delta_q(\mu))$ for $i=1,\dots,d$.
- This gives the multiplicities of T, by the Corollary 3.15 and the fact that the characters of the $\Delta_q(\lambda)$'s are as in the classical case.
- Use (29) to recursively compute the decomposition of T (starting with any maximal $\mathbf{U}_q$-weight of T).

Example 3.27. Let us come back to our favourite case, that is, $\mathfrak{g}=\mathfrak{sl}_2$, $\mathbb{K}=\mathbb{C}$ and $l=3$. In the regular cases we have $T_q(k)\cong\Delta_q(k)$ for $k=0,1$ and the parabolic Kazhdan–Lusztig polynomials are

$$n_{jk}(v)=\begin{cases}1, & \text{if } j=k,\\ v, & \text{if } j<k \text{ are separated by precisely one wall},\\ 0, & \text{else,}\end{cases}$$

for $k>1$. By the above we obtain $T_q(k)\cong\Delta_q(k)$ for $k\in\mathbb{Z}_{\geq 0}$ singular, hence, the two singular blocks $\boldsymbol{\mathcal{T}}_{-1}$ and $\boldsymbol{\mathcal{T}}_2$ are semisimple.

In Example 3.13 we have already calculated $T_q(1)\otimes T_q(1)\cong T_q(2)\oplus T_q(0)$. Let us go one step further now: $T_q(1)\otimes T_q(1)\otimes T_q(1)$ has only $(T_q(1)^{\otimes 3}:\Delta_q(3))=1$ and $(T_q(1)^{\otimes 3}:\Delta_q(1))=2$ as non-zero multiplicities. This means that $T_q(3)$ is a summand of $T_q(1)\otimes T_q(1)\otimes T_q(1)$. Since $T_q(3)$ has only $(T_q(3):\Delta_q(3))=1$ and $(T_q(3):\Delta_q(1))=1$ as non-zero multiplicities (by the calculation of the periodic Kazhdan–Lusztig polynomials), we have

$$T_q(1)\otimes T_q(1)\otimes T_q(1)\cong T_q(3)\oplus T_q(1)\in \boldsymbol{\mathcal{T}}_1. \tag{39}$$

Moreover, we have (as we, as usual, encourage the reader to work out)

$$T_q(1)\otimes T_q(1)\otimes T_q(1)\otimes T_q(1)\cong (T_q(4)\oplus T_q(0))\oplus(T_q(2)\oplus T_q(2)\oplus T_q(2))\in \boldsymbol{\mathcal{T}}_0\oplus\boldsymbol{\mathcal{T}}_2.$$

To illustrate how this decomposition depends on l: Assume now that $l>3$. Then, which can be verified similarly as in Example 3.19, the $\mathbf{U}_q$-tilting module $T_q(3)$ is in the fundamental alcove $\mathcal{A}_0$. Thus, by Theorem 3.21, $T_q(3)$ is simple as in the generic case. Said otherwise, we have $T_q(3)\cong\Delta_q(3)$. Hence, in the same spirit as above, we obtain as in the generic case

$$T_q(1)\otimes T_q(1)\otimes T_q(1)\cong T_q(3)\oplus(T_q(1)\oplus T_q(1))\in \boldsymbol{\mathcal{T}}_3\oplus\boldsymbol{\mathcal{T}}_1 \tag{40}$$

in contrast to the decomposition in (39). ▲

4. Cellular structures: examples and applications

4A. **Cellular structures using $\mathbf{U}_q$-tilting modules.** The main result of [6] is the following. To state it, we need to specify the cell datum. Set

$$(\mathcal{P},\leq)=(\{\lambda\in X^+\mid (T:\nabla_q(\lambda))=(T:\Delta_q(\lambda))\neq 0\},\leq),$$

where $\leq$ is the usual partial ordering on X^+, see at the beginning of Section 2. Note that $\mathcal{P}$ is finite since T is finite-dimensional. For each $\lambda\in\mathcal{P}$ define

$$\mathcal{I}^\lambda=\{1,\dots,(T:\nabla_q(\lambda))\}=\{1,\dots,(T:\Delta_q(\lambda))\}=\mathcal{J}^\lambda,$$

and let $\mathrm{i}\colon \mathrm{End}_{\mathbf{U}_q}(T)\to\mathrm{End}_{\mathbf{U}_q}(T),\phi\mapsto\mathcal{D}(\phi)$ denote the $\mathbb{K}$-linear anti-involution induced by the duality functor $\mathcal{D}(\cdot)$. For $\overline{f}_j^\lambda$ and $\overline{g}_i^\lambda$ as in [6, Section 3A] set

$$c_{ij}^\lambda=\overline{g}_i^\lambda\circ\mathrm{i}(\overline{g}_j^\lambda)=\overline{g}_i^\lambda\circ\overline{f}_j^\lambda,\quad \text{for } \lambda\in\mathcal{P},\ i,j\in\mathcal{I}^\lambda.$$

Finally let $\mathcal{C}\colon\mathcal{I}^\lambda\times\mathcal{I}^\lambda\to\mathrm{End}_{\mathbf{U}_q}(T)$ be given by $(i,j)\mapsto c_{ij}^\lambda$. Now we are ready to state the main result from [6].

Theorem 4.1. ([6, Theorem 3.9]) The quadruple $(\mathcal{P},\mathcal{I},\mathcal{C},\mathrm{i})$ is a cell datum for $\mathrm{End}_{\mathbf{U}_q}(T)$.■

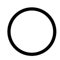
We also use the following consequences of Theorem 4.1. First note that each cellular algebra gives rise to a construction of simple modules which we denote by $L(\lambda)$ for $\lambda\in\mathcal{P}_0\subset X^+$ in case of $\mathrm{End}_{\mathbf{U}_q}(T)$. (The precise definition can be found in [6, Section 4].) Then:

Theorem 4.2. ([6, Theorem 4.12]) If $\lambda\in\mathcal{P}_0$, then $\dim(L(\lambda))=m_\lambda$, where m_λ is the multiplicity of the indecomposable tilting module $T_q(\lambda)$ in T. ■

Theorem 4.3. ([6, Theorem 4.13]) The cellular algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple if and only if T is a semisimple $\mathbf{U}_q$-module. ■

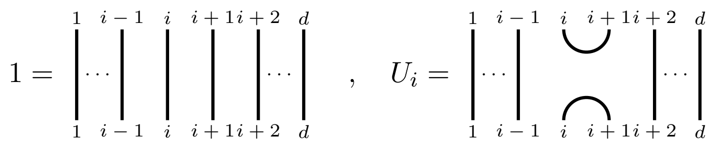
4B. **(Graded) cellular structures and the Temperley–Lieb algebras: a comparison.** We want to present one explicit example, the Temperley–Lieb algebras, which is of particular interest in low-dimensional topology and categorification. Our main goal is to construct new (graded) cellular bases, and use our approach to establish semisimplicity conditions, and construct and compute the dimensions of its simple modules in new ways.

Fix $\delta = q + q^{-1}$ for $q \in \mathbb{K}^*$.[3] Recall that the *Temperley–Lieb algebra* $\mathcal{TL}_d(\delta)$ in d strands with parameter δ is the free diagram algebra over $\mathbb{K}$ with basis consisting of all possible non-intersecting tangle diagrams with d bottom and top boundary points modulo boundary preserving isotopy and the local relation for evaluating circles given by the parameter[4] δ:



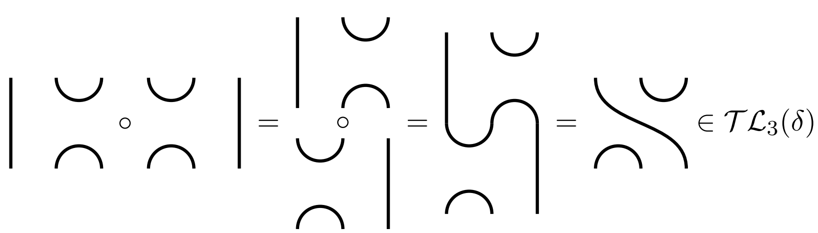
$$= \delta = q + q^{-1} \in \mathbb{K}.$$

The algebra $\mathcal{TL}_d(\delta)$ is locally generated by



for $i = 1, \dots, d-1$ called *identity* 1 and *cap-cup* U_i (which takes place between the strand i and $i+1$). For simplicity, we suppress the boundary labels in the following.

The multiplication $y \circ x$ is giving by stacking diagram y on top of diagram x. For example



Recall from [6, 5A.3] (whose notation we use now; in particular, $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{sl}_2)$) that, by quantum Schur–Weyl duality, we can use Theorem 4.1 to obtain a cellular basis of $\mathcal{TL}_d(\delta)$. The aim now is to compare our cellular bases to the one given by Graham and Lehrer in [14, Theorem 6.7], where we point out that we do not obtain their cellular basis: our cellular basis depends for instance on whether $\mathcal{TL}_d(\delta)$ is semisimple or not. In the non-semisimple case, at least for $\mathbb{K} = \mathbb{C}$, we obtain a non-trivially $\mathbb{Z}$-graded cellular basis in the sense of [15, Definition 2.1], see Proposition 4.21.

Before stating our cellular basis, we provide a criterion which tells precisely whether $\mathcal{TL}_d(\delta)$ is semisimple or not. Recall that the following known criteria whether Weyl modules $\Delta_q(i)$

[3] The $\mathfrak{sl}_2$ case works with any $q \in \mathbb{K}^*$, including even roots of unity, see e.g. [7, Definition 2.3].

[4] We point out that there are two different conventions about circle evaluations in the literature: evaluating to δ or to $-\delta$. We use the first convention because we want to stay close to the cited literature.

are simple, see e.g. [7, Proposition 2.7] or [4, Corollary 4.6]:

$$q \neq \pm 1\colon \quad \Delta_q(i) \text{ is a simple } \mathbf{U}_q\text{-module} \Leftrightarrow \begin{cases} q \text{ is not a root of unity,} \\ q^{2l} = 1 \text{ and } (i < l \text{ or } i \equiv -1 \bmod l). \end{cases}$$

$$q = \pm 1\colon \quad \Delta_q(i) \text{ is a simple } \mathbf{U}_q\text{-module} \Leftrightarrow \begin{cases} \mathrm{char}(\mathbb{K}) = 0, \\ \mathrm{char}(\mathbb{K}) = p \text{ and } (i < p \text{ or } i \equiv -1 \bmod p). \end{cases}$$

We use this criteria to prove the following.

Proposition 4.4. (**Semisimplicity criterion for $\mathcal{TL}_d(\delta)$.**) We have the following.

(a) Let $\delta \neq 0$. Then $\mathcal{TL}_d(\delta)$ is semisimple if and only if $[i] = q^{1-i} + \cdots + q^{i-1} \neq 0$ for all $i = 1, \ldots, d$ if and only if q is not a root of unity with $d < l = \mathrm{ord}(q^2)$, or $q = 1$ and $\mathrm{char}(\mathbb{K}) > d$.

(b) Let $\mathrm{char}(\mathbb{K}) = 0$. Then $\mathcal{TL}_d(0)$ is semisimple if and only if d is odd (or $d = 0$).

(c) Let $\mathrm{char}(\mathbb{K}) = p > 0$. Then $\mathcal{TL}_d(0)$ is semisimple if and only if $d \in \{1, 3, 5, \ldots, 2p-1\}$ (or $d = 0$). □

Proof. (a): We want to show that $T = V^{\otimes d}$ decomposes into simple $\mathbf{U}_q$-modules if and only if $d < l$, or $q = 1$ and $\mathrm{char}(\mathbb{K}) > d$, which is clearly equivalent to the non-vanishing of the $[i]$'s.

Assume that $d < l$. Since the maximal $\mathbf{U}_q$-weight of $V^{\otimes d}$ is d and since all Weyl $\mathbf{U}_q$-modules $\Delta_q(i)$ for $i < l$ are simple, we see that all indecomposable summands of $V^{\otimes d}$ are simple.

Otherwise, if $l \leq d$, then $T_q(d)$ (or $T_q(d-2)$ in the case $d \equiv -1 \bmod l$) is a non-simple, indecomposable summand of $V^{\otimes d}$ (note that this arguments fails if $l = 2$, i.e. $\delta = 0$).

The case $q = 1$ works similarly, and we can now use Theorem 4.3 to finish the proof of (a).

(b): Since $\delta = 0$ if and only if $q = \pm\sqrt[2]{-1}$, we can use the linkage from e.g. [7, Theorem 2.23] in the case $l = 2$ to see that $T = V^{\otimes d}$ decomposes into a direct sum of simple $\mathbf{U}_q$-modules if and only if d is odd (or $d = 0$). This implies that $\mathcal{TL}_d(0)$ is semisimple if and only if d is odd (or $d = 0$) by Theorem 4.3.

(c): If $\mathrm{char}(\mathbb{K}) = p > 0$ and $\delta = 0$ (for $p = 2$ this is equivalent to $q = 1$), then we have $\Delta_q(i) \cong L_q(i)$ if and only if $i = 0$ or $i \in \{2ap^n - 1 \mid n \in \mathbb{Z}_{\geq 0}, 1 \leq a < p\}$. In particular, this means that for $d \geq 2$ we have that either $T_q(d)$ or $T_q(d-2)$ is a simple $\mathbf{U}_q$-module if and only if $d \in \{3, 5, \ldots, 2p-1\}$. Hence, using the same reasoning as above, we see that $T = V^{\otimes d}$ is semisimple if and only if $d \in \{1, 3, 5, \ldots, 2p-1\}$ (or $d = 0$). By Theorem 4.3 we see that $\mathcal{TL}_d(0)$ is semisimple if and only if $d \in \{1, 3, 5, \ldots, 2p-1\}$ (or $d = 0$). ■

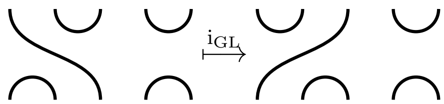
Example 4.5. We have that $[k] \neq 0$ for all $k = 1, 2, 3$ is satisfied if and only if q is not a fourth or a sixth root of unity. By Proposition 4.4 we see that $\mathcal{TL}_3(\delta)$ is semisimple as long as q is not one of these values from above. The other way around is only true for q being a sixth root of unity (the conclusion from semisimplicity to non-vanishing of the quantum numbers above does not work in the case $q = \pm\sqrt[2]{-1}$). ▲

Remark 5. The semisimplicity criterion for $\mathcal{TL}_d(\delta)$ was already already found, using quite different methods, in [39, Section 5] in the case $\delta \neq 0$, and in the case $\delta = 0$ in [26, Chapter 7] or [30, above Proposition 4.9]. For us it is an easy application of Theorem 4.3. ▲

A direct consequence of Proposition 4.4 is that the Temperley–Lieb algebra $\mathcal{TL}_d(\delta)$ for $q \in \mathbb{K}^* - \{1\}$ not a root of unity is semisimple (or $q = \pm 1$ and $\mathrm{char}(\mathbb{K}) = 0$), regardless of d.

4B.1. *Temperley–Lieb algebra: the semisimple case.* Assume that $q \in \mathbb{K}^* - \{1\}$ is not a root of unity (or $q = \pm 1$ and $\mathrm{char}(\mathbb{K}) = 0$). Thus, we are in the semisimple case.

Let us compare our cell datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ to the one of Graham and Lehrer (indicated by a subscript GL) from [14, Section 6]. To this end, let us recall Graham and Lehrer's cell datum $(\mathcal{P}_{\mathrm{GL}}, \mathcal{I}_{\mathrm{GL}}, \mathcal{C}_{\mathrm{GL}}, \mathrm{i}_{\mathrm{GL}})$. The $\mathbb{K}$-linear anti-involution i_{GL} is given by "turning pictures upside down". For example



For the insistent reader: more formally, the $\mathbb{K}$-linear anti-involution i_{GL} is the unique $\mathbb{K}$-linear anti-involution which fixes all U_i's for $i = 1, \dots, d-1$.

The data $\mathcal{P}_{\mathrm{GL}}$ and $\mathcal{I}_{\mathrm{GL}}$ are given combinatorially: $\mathcal{P}_{\mathrm{GL}}$ is the set $\Lambda^+(2,d)$ of all Young diagrams with d nodes and at most two rows. For example, the elements of $\Lambda^+(2,3)$ are

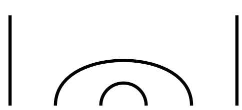
(41) $$\lambda = \begin{array}{|c|c|c|}\hline & & \\ \hline \end{array} \quad , \quad \mu = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},$$

where we point out that we use the English notation for Young diagrams. Now $\mathcal{I}_{\mathrm{GL}}^{\lambda}$ is the set of all standard tableaux of shape λ, denoted by $\mathrm{Std}(\lambda)$, that is, all fillings of λ with numbers $1, \dots, d$ (non-repeating) such that the entries strictly increase along rows and columns. For example, the elements of $\mathrm{Std}(\mu)$ for μ as in (41) are

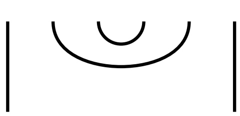
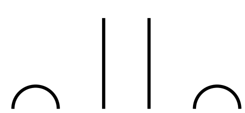
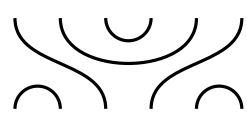
(42) $$t_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad , \quad t_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

The set $\mathcal{P}_{\mathrm{GL}}$ is a poset where the order $\leq$ is the so-called dominance order on Young diagrams. In the "at most two rows case" this is $\mu \leq \lambda$ if and only if μ has fewer columns (an example is (41) with the same notation).

The only thing missing is thus the parametrization of the cellular basis. This works as follows: fix $\lambda \in \Lambda^+(2,d)$ and assign to each $t \in \mathrm{Std}(\lambda)$ a "half diagram" x_t via the rule that one "caps off" the strands whose numbers appear in the second row with the biggest possible candidate to the left of the corresponding number (going from left to right in the second row). Note that this is well-defined due to planarity. For example,

(43) $s = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \rightsquigarrow x_s =$  , $t = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \rightsquigarrow x_t =$ 

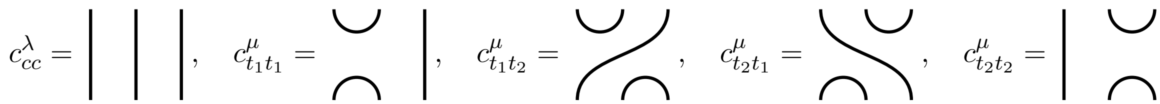
Then one obtains c_{st}^{λ} by "turning x_s upside down and stacking it on top of x_t". For example,

$c_{st}^{\lambda} = \mathrm{i}_{\mathrm{GL}}(x_s) \circ x_t =$  $\circ$  $=$ 

for $\lambda \in \Lambda^+(2,6)$ and $s, t \in \mathrm{Std}(\lambda)$ as in (43). The map $\mathcal{C}_{\mathrm{GL}}$ sends $(s,t) \in \mathcal{I}_{\mathrm{GL}}^{\lambda} \times \mathcal{I}_{\mathrm{GL}}^{\lambda}$ to c_{st}^{λ}.

Theorem 4.6. ([14, Theorem 6.7]) The quadruple $(\mathcal{P}_{\mathrm{GL}}, \mathcal{I}_{\mathrm{GL}}, \mathcal{C}_{\mathrm{GL}}, \mathrm{i}_{\mathrm{GL}})$ is a cell datum for the algebra $\mathcal{TL}_d(\delta)$. ■

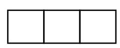
Example 4.7. For $\mathcal{TL}_3(\delta)$ we have five basis elements, namely



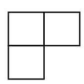
where we use the notation from (41) and (42) (and the “canonical” filling c for λ). ▲

Let us now compare the cell datum of Graham and Lehrer with our cell datum. We have the poset $\mathcal{P}_{\mathrm{GL}}$ consisting of all $\lambda \in \Lambda^+(2,d)$ in Graham and Lehrer's case and the poset $\mathcal{P}$ consisting of all $\lambda \in X^+$ such that $\Delta_q(\lambda)$ is a factor of T in our case.

The two sets are the same: an element $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}_{\mathrm{GL}}$ corresponds to $\lambda_1 - \lambda_2 \in \mathcal{P}$. This is clearly an injection of sets. Moreover, $\Delta_q(i) \otimes \Delta_q(1) \cong \Delta_q(i+1) \oplus \Delta_q(i-1)$ for $i > 0$ shows surjectivity. Two easy examples are

$$\lambda = (\lambda_1, \lambda_2) = (3, 0) =$$  $$\in \mathcal{P}_{\mathrm{GL}} \rightsquigarrow \lambda_1 - \lambda_2 = 3 \in \mathcal{P},$$

and

$$\mu = (\mu_1, \mu_2) = (2, 1) =$$  $$\in \mathcal{P}_{\mathrm{GL}} \rightsquigarrow \mu_1 - \mu_2 = 1 \in \mathcal{P}\ ,$$

which fits to the decomposition as in (40).

Similarly, an inductive reasoning shows that there is a factor $\Delta_q(i)$ of T for any standard filling for the Young diagram that gives rise to i under the identification from above. Thus, $\mathcal{I}_{\mathrm{GL}}$ is also the same as our $\mathcal{I}$.

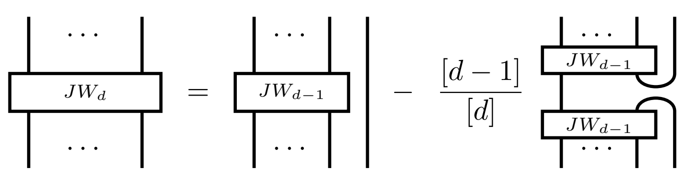
As an example, we encourage the reader to compare (41) and (42) with (40).

To see that the $\mathbb{K}$-linear anti-involution i_{GL} is also our involution i, we note that we build our basis from a “top” part g^{λ}_i and a “bottom” part f^{λ}_j and i switches top and bottom similarly as the $\mathbb{K}$-linear anti-involution i_{GL}.

Thus, except for $\mathcal{C}$ and $\mathcal{C}_{\mathrm{GL}}$, the cell data agree.

In order to state how our cellular basis for $\mathcal{TL}_d(\delta)$ looks like, recall the following definition(s) of the *(generalized) Jones–Wenzl projectors.*

Definition 4.8. (**Jones–Wenzl projectors.**) The d-th *Jones–Wenzl projector*, which we denote by $JW_d \in \mathcal{TL}_d(\delta)$, is recursively defined via the recursion rule

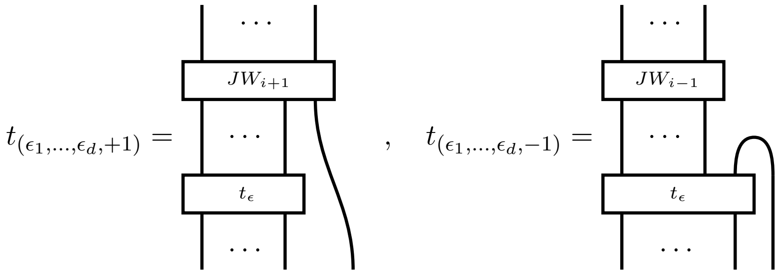


where we assume that JW_1 is the identity diagram in one strand. ▲

Note that the projector JW_d can be identified with the projection of $T = V^{\otimes d}$ onto its maximal weight summand. These projectors were introduced by Jones in [20] and then further studied by Wenzl in [38]. In fact, they can be generalized as follows.

Definition 4.9. (**Generalized Jones–Wenzl projectors**.) Given any d-tuple (with $d > 0$) of the form $\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_d) \in \{\pm 1\}^d$ such that $\sum_{j=1}^{k} \epsilon_j \geq 0$ for all $k = 1, \ldots, d$. Set $i = \sum_{j=1}^{d} \epsilon_j$.

We define recursively two certain “half-diagrams” $t_{(\epsilon_1,\dots,\epsilon_d,\pm 1)}$ via



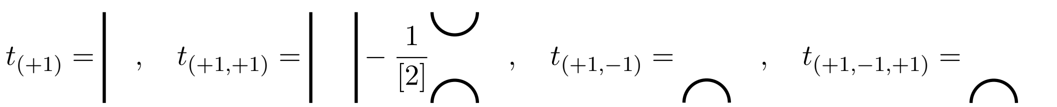
where $t_{(+1)} \in \mathcal{TL}_1(\delta)$ is defined to be the identity element. By convention, $t_{(\epsilon_1,\dots,\epsilon_d,-1)} = 0$ if $i-1<0$. Note that $t_{(\epsilon_1,\dots,\epsilon_d,\pm 1)}$ has always $d+1$ bottom boundary points, but $i \pm 1$ top boundary points.

Then we assign to any such $\vec{\epsilon}$ a *generalized Jones–Wenzl “projector”* $JW_{\vec{\epsilon}} \in \mathcal{TL}_d(\delta)$ via

$$JW_{\vec{\epsilon}} = \mathrm{i}(t_{\vec{\epsilon}}) \circ t_{\vec{\epsilon}},$$

where i is, as above, the $\mathbb{K}$-linear anti-involution that “turns pictures upside down”. ▲

Example 4.10. We point out again that the $t_{\vec{\epsilon}}$'s are “half-diagrams”. For example,



where we can read-off the top boundary points by summing the ϵ_i's. ▲

Note that the Jones–Wenzl projectors are special cases of the construction in Definition 4.9, i.e. $JW_d = JW_{(+1,\dots,+1)}$. Moreover, while we think about the Jones–Wenzl projectors as projecting to the maximal weight summand of $T = V^{\otimes d}$, the generalized Jones–Wenzl projectors $JW_{\vec{\epsilon}}$ project to the summands of $T = V^{\otimes d}$ of the form $\Delta_q(i)$ where i is as above $i = \sum_{j=1}^{d} \epsilon_j$. To be more precise, we have the following.

Proposition 4.11. (**Diagrammatic projectors**.) There exist non-zero scalars $a_{\vec{\epsilon}} \in \mathbb{K}$ such that $JW'_{\vec{\epsilon}} = a_{\vec{\epsilon}} JW_{\vec{\epsilon}}$ are well-defined idempotents forming a complete set of mutually orthogonal, primitive idempotents in $\mathcal{TL}_d(\delta)$. □

Proof. That they are well-defined follows from the fact that no (appearing) quantum number vanishes in the semisimple case, cf. Proposition 4.4.

The other statements can be proven as in [11, Proposition 2.19 and Theorem 2.20] (beware that they call these projectors higher Jones–Wenzl projectors), since their arguments work – mutatis mutandis – in the semisimple case as well. ■

One can also show that the sum of the $JW'_{\vec{\epsilon}}$'s for fixed $i = \sum_{j=1}^{d} \epsilon_j$ are central. These should be thought of as being the projectors to the isotypic $\Delta_q(i)$-components of $T = V^{\otimes d}$.

Example 4.12. Recall from Example 3.27 that we have the following decompositions.

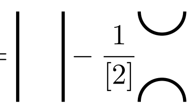
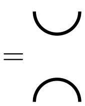
$$V^{\otimes 1} = \Delta_q(1), \quad V^{\otimes 2} \cong \Delta_q(2) \oplus \Delta_q(0), \quad V^{\otimes 3} \cong \Delta_q(3) \oplus \Delta_q(1) \oplus \Delta_q(1). \tag{44}$$

Moreover, there are the following $\vec{\epsilon}$ vectors.

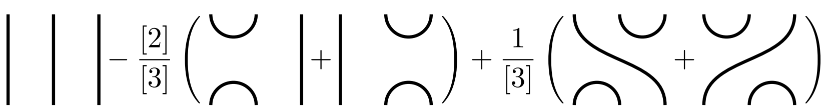
$$\vec{\epsilon}_1 = (+1), \quad \vec{\epsilon}_2 = (+1,+1), \quad \vec{\epsilon}_3 = (+1,-1),$$
$$\vec{\epsilon}_4 = (+1,+1,+1), \quad \vec{\epsilon}_5 = (+1,+1,-1), \quad \vec{\epsilon}_6 = (+1,-1,+1).$$

(We point out that $(+1,-1,-1)$ does not satisfy the sum property from Definition 4.9.)

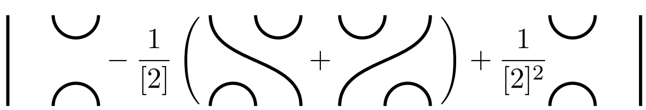
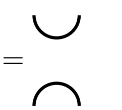
By construction, $JW'_{\vec{\epsilon}_1} = JW_{\vec{\epsilon}_1}$ is the identity strand in one variable and hence, is the projector on the unique factor in (44). Moreover, we have

$$JW_2 = JW'_{\vec{\epsilon}_2} = JW_{\vec{\epsilon}_2} =$$

$$, \quad JW_{\vec{\epsilon}_3} =$$


where $JW_{\vec{\epsilon}_2}$ is the projection onto $\Delta_q(2)$ and $JW_{\vec{\epsilon}_3}$ is the (up to scalars) projector onto $\Delta_q(0)$ as in (44), respectively. Furthermore, we have

$$JW_3 = JW'_{\vec{\epsilon}_4} = JW_{\vec{\epsilon}_4} =$$


is the projection to the $\Delta_q(3)$ summand in (44). The other two (up to scalars) projectors are

$$JW_{\vec{\epsilon}_5} =$$

$$, \quad JW_{\vec{\epsilon}_6} =$$


as we invite the reader to check. Note that their sum (up to scalars) is the projector on the isotypic component $\Delta_q(1) \oplus \Delta_q(1)$ in (44). ▲

Proposition 4.13. (**(New) cellular bases.**) The datum given by the quadruple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ for $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$. Moreover, $\mathcal{C} \neq \mathcal{C}_{\mathrm{GL}}$ for all $d > 1$ and all choices involved in the definition of $\mathrm{im}(\mathcal{C})$. In particular, there is a choice such that all generalized Jones–Wenzl projectors $JW'_{\vec{\epsilon}}$ are part of $\mathrm{im}(\mathcal{C})$. □

Proof. That we get a cell datum as stated follows from Theorem 4.1 and the discussion above.

That our cellular basis $\mathcal{C}$ will never be $\mathcal{C}_{\mathrm{GL}}$ for $d > 1$ is due to the fact that Graham and Lehrer's cellular basis always contains the identity (which corresponds to the unique standard filling of the Young diagram associated to $\lambda = (d, 0)$).

In contrast, let $\lambda_k = (d-k, k)$ for $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$. Then

$$T = V^{\otimes d} \cong \Delta_q(d) \oplus \bigoplus_{0 < k \leq \lfloor \frac{d}{2} \rfloor} \Delta_q(d-2k)^{\oplus m_{\lambda_k}} \tag{45}$$

for some multiplicities $m_{\lambda_k} \in \mathbb{Z}_{>0}$, we see that for $d > 1$ the identity is never part of any of our bases: all the $\Delta_q(i)$'s are simple $\mathbf{U}_q$-modules and each c_{ij}^k factors only through $\Delta_q(k)$. In particular, the basis element c_{11}^λ for $\lambda = \lambda_d$ has to be (a scalar multiple) of $JW_{(+1,\dots,+1)}$.

As in [6, 5A.1] we can choose for $\mathcal{C}$ an Artin-Wedderburn basis of $\mathrm{End}_{\mathbf{U}_q}(T) \cong \mathcal{TL}_d(\delta)$.

By our construction, all basis elements c_{ij}^k are block matrices of the form

$$\begin{pmatrix} \mathbf{M}_d & 0 & \cdots & 0 \\ 0 & \mathbf{M}_{d-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{M}_\varepsilon \end{pmatrix}$$

with $\varepsilon = 0$ if d is even and $\varepsilon = 1$ if d is odd (where we regard V as decomposed as in (45), the indices should indicate the summands and $\mathbf{M}_{d-2k}$ is of size $m_k \times m_k$).

Clearly, the block matrices of the form $\mathbf{E}_{ii}^k$ for $i = 1, \dots, m_k$ with only non-zero entry in the i-th column and row of $\mathbf{M}_k$ form a set of mutually orthogonal, primitive idempotents. Hence, by Proposition 4.11, these have to be the generalized Jones–Wenzl projectors $JW'_{\vec{\epsilon}}$ for $k = \sum_{j=1}^k \epsilon_j$ up to conjugation. ■

Example 4.14. Let us consider $\mathcal{TL}_3(\delta)$ as in Example 4.7 for any $q \in \mathbb{K}^* - \{1, \pm\sqrt[2]{-1}\}$ that is not a critical value as in Example 4.5. Then $\mathcal{TL}_3(\delta)$ is semisimple by Proposition 4.4.

In particular, we have a decomposition of $V^{\otimes 3}$ as in (44). Fix the same order as therein. Identifying λ, μ with $3, 1$, we can choose five basis elements

$$c_{cc}^\lambda = \mathbf{E}_{11}^3, \quad c_{t_1t_1}^\mu = \mathbf{E}_{11}^1, \quad c_{t_1t_2}^\mu = \mathbf{E}_{12}^1, \quad c_{t_2t_1}^\mu = \mathbf{E}_{21}^1, \quad c_{t_2t_2}^\mu = \mathbf{E}_{22}^1,$$

where we use the notation from (41) and (42) (and the "canonical" filling c for λ) again.

Note that c_{cc}^λ corresponds to the Jones–Wenzl projector $JW_3 = JW'_{(+1+1+1)}$, $c_{t_1t_1}^\mu$ corresponds to $JW'_{(+1+1-1)}$ and $c_{t_2t_2}^\mu$ corresponds to $JW'_{(+1-1+1)}$. Compare to Example 4.12. ▲

Note the following classification result (see for example [30, Corollary 5.2] for $\mathbb{K} = \mathbb{C}$).

Corollary 4.15. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$-modules $L(\lambda)$, where $\lambda = (\lambda_1, \lambda_2)$ is a length-two partition of d. Moreover, $\dim(L(\lambda)) = |\mathrm{Std}(\lambda)|$, where $\mathrm{Std}(\lambda)$ is the set of all standard tableaux of shape λ. □

Proof. This follows directly from Proposition 4.13, the classification of simple modules for $\mathrm{End}_{\mathbf{U}_q}(T)$, see [6, Theorem 4.11], and Theorem 4.2 because we have $m_\lambda = |\mathrm{Std}(\lambda)|$. ■

Example 4.16. The Temperley–Lieb algebra $\mathcal{TL}_3(\delta)$ in the semisimple case has

$$\dim\left(L\left(\,\square\square\square\,\right)\right) = 1, \qquad \dim\left(L\left(\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}\right)\right) = 2.$$

Compare to (42). ▲

4B.2. *Temperley–Lieb algebra: the non-semisimple case.* Let us assume that we have fixed $q \in \mathbb{K}^* - \{1, \pm\sqrt[2]{-1}\}$ to be a critical value such that $[k] = 0$ for some $k = 1, \dots, d$. Then, by Proposition 4.4, the algebra $\mathcal{TL}_d(\delta)$ is no longer semisimple. In particular, to the best of our knowledge, there is no diagrammatic analog of the Jones–Wenzl projectors in general.

Proposition 4.17. (**(New) cellular basis — the second.**) The datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ with $\mathcal{C}$ as in Theorem 4.1 for $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$. Moreover, $\mathcal{C} \neq \mathcal{C}_{\mathrm{GL}}$ for all $d > 1$ and all choices involved in the definition of our basis. Thus, there is a choice such that all generalized, non-semisimple Jones–Wenzl projectors are part of $\mathrm{im}(\mathcal{C})$. □

Proof. As in the proof of Proposition 4.13 and left to the reader. ■

Hence, directly from Proposition 4.17, the classification of simple modules for $\mathrm{End}_{\mathbf{U}_q}(T)$, see [6, Theorem 4.11], and Theorem 4.2, we obtain:

Corollary 4.18. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$-modules $L(\lambda)$, where $\lambda = (\lambda_1, \lambda_2)$ is a length-two partition of d. Moreover, $\dim(L(\lambda)) = m_\lambda$, where m_λ is the multiplicity of $T_q(\lambda_1 - \lambda_2)$ as a summand of $T = V^{\otimes d}$. ■

Example 4.19. If q is a complex, primitive third root of unity, then $\mathcal{TL}_3(\delta)$ still has the same indexing set of its simples as in Example 4.16, but now both are of dimension one, since we have a decomposition of $T = V^{\otimes 3}$ as in (39). ▲

Remark 6. In the case $\mathbb{K} = \mathbb{C}$ we can give a dimension formula, namely

$$\dim(L(\lambda)) = m_\lambda = \begin{cases} |\mathrm{Std}(\lambda)|, & \text{if } \lambda_1 - \lambda_2 \equiv -1 \bmod l, \\ \sum_{\mu = w.\lambda, \mu \geq \lambda \in \Lambda^+(2,d)} (-1)^{\ell(w)} |\mathrm{Std}(\mu)|, & \text{if } \lambda_1 - \lambda_2 \not\equiv -1 \bmod l, \end{cases}$$

where $w \in W_l$ is the affine Weyl group and $\ell(w)$ is the length of a reduced word $w \in W_l$. This matches the formulas from, for example, [3, Proposition 6.7] or [30, Corollary 5.2]. ▲

Note that we can do better: as in Example 3.22 one gets a decomposition

$$\boldsymbol{\mathcal{T}} \cong \boldsymbol{\mathcal{T}}_{-1} \oplus \boldsymbol{\mathcal{T}}_0 \oplus \boldsymbol{\mathcal{T}}_1 \oplus \cdots \oplus \boldsymbol{\mathcal{T}}_{l-3} \oplus \boldsymbol{\mathcal{T}}_{l-2} \oplus \boldsymbol{\mathcal{T}}_{l-1}, \tag{46}$$

where the blocks $\boldsymbol{\mathcal{T}}_{-1}$ and $\boldsymbol{\mathcal{T}}_{l-1}$ are semisimple if $\mathbb{K} = \mathbb{C}$. Compare also to [7, Lemma 2.25].

Fix $\mathbb{K} = \mathbb{C}$. As explained in [7, Section 3.5] each block in the decomposition (46) can be equipped with a non-trivial $\mathbb{Z}$-grading coming from the zig-zag algebra from [17]. Hence, we have the following.

Lemma 4.20. The $\mathbb{C}$-algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ can be equipped with a non-trivial $\mathbb{Z}$-grading. Thus, $\mathcal{TL}_d(\delta)$ over $\mathbb{C}$ can be equipped with a non-trivial $\mathbb{Z}$-grading. □

Proof. The second statement follows directly from the first using quantum Schur–Weyl duality. Hence, we only need to show the first.

Note that $T = V^{\otimes d}$ decomposes as in (45), but with $T_q(k)$'s instead of $\Delta_q(k)$'s, and we can order this decomposition by blocks. Each block carries a $\mathbb{Z}$-grading coming from the zig-zag algebra, as explained in [7, Section 3]. In particular, we can choose the basis elements c_{ij}^λ in such a way that we get the $\mathbb{Z}$-graded basis obtained in Corollary 4.23 therein. Since there is no interaction between different blocks, the statement follows. ■

Recall from [15, Definition 2.1] that a $\mathbb{Z}$-graded cell datum of a $\mathbb{Z}$-graded algebra is a cell datum for the algebra together with an additional *degree function* $\deg\colon \coprod_{\lambda \in \mathcal{P}} \mathcal{I}^\lambda \to \mathbb{Z}$, such that $\deg(c_{ij}^\lambda) = \deg(i) + \deg(j)$. For us the choice of $\deg(\cdot)$ is as follows.

If $\lambda \in \mathcal{P}$ is in one of the semisimple blocks, then we simply set $\deg(i) = 0$ for all $i \in \mathcal{I}^\lambda$.

Assume that $\lambda \in \mathcal{P}$ is not in the semisimple blocks. It is known that every $T_q(\lambda)$ has precisely two Weyl factors. The g_i^λ that map $\Delta_q(\lambda)$ into a higher $T_q(\mu)$ should be indexed by a 1-colored i whereas the g_i^λ mapping $\Delta_q(\lambda)$ into $T_q(\lambda)$ should have 0-colored i. Similarly for the f_j^λ's. Then the degree of the elements $i \in \mathcal{I}^\lambda$ should be the corresponding color. We get the following. (Here $\mathcal{C}$ is as in Theorem 4.1.)

Proposition 4.21. (**Graded cellular basis.**) The datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ supplemented with the function $\deg(\cdot)$ from above is a $\mathbb{Z}$-graded cell datum for the $\mathbb{C}$-algebra $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$.□

Proof. The hardest part is cellularity which directly follows from Theorem 4.1. That the quintuple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i}, \deg)$ gives a $\mathbb{Z}$-graded cell datum follows from the construction. ■

Example 4.22. Let us consider $\mathcal{TL}_3(\delta)$ as in Example 4.14, namely q being a complex, primitive third root of unity. Then $\mathcal{TL}_3(\delta)$ is non-semisimple by Proposition 4.4. In particular, we have a decomposition of $V^{\otimes 3}$ different from (44), namely as in (39). In this case $\mathcal{P} = \{1, 3\}$, $\mathcal{I}^3 = \{1, 3\}$ and $\mathcal{I}^1 = \{1\}$. By our choice from above $\deg(i) = 0$ if $i = 1 \in \mathcal{I}^1$ or $i = 3 \in \mathcal{I}^3$, and $\deg(i) = 1$ if $i = 1 \in \mathcal{I}^3$. As in Example 4.14 (if we use the ordering induced by the decomposition from (39)), we can choose basis elements as $c^3_{11} = \mathbf{E}^3_{11}, c^3_{12} = \mathbf{E}^1_{12}, c^3_{21} = \mathbf{E}^1_{21}, c^3_{22} = \mathbf{E}^1_{22}, c^1_{11} = \mathbf{E}^1_{33}$, where we use the notation from (41) and (42) again. These are of degrees $0, 1, 1, 2$ and 0. ▲

Remark 7. Our grading and the one found by Plaza and Ryom-Hansen in [29] agree (up to a shift of the indecomposable summands). To see this, note that our algebra is isomorphic to the algebra $K_{1,n}$ studied in [9] which is by (4.8) therein and [10, Theorem 6.3] a quotient of some particular cyclotomic KL–R algebra (the compatibility of the grading follows for example from [16, Corollary B.6]). The same holds, by construction, for the grading in [29]. ▲

References

[1] H.H. Andersen. Tensor products of quantized tilting modules. *Comm. Math. Phys.*, 149(1):149–159, 1992.

[2] H.H. Andersen. The strong linkage principle for quantum groups at roots of 1. *J. Algebra*, 260(1):2–15, 2003. doi:10.1016/S0021-8693(02)00618-X.

[3] H.H. Andersen, G. Lehrer, and R. Zhang. Cellularity of certain quantum endomorphism algebras. *Pacific J. Math.*, 279(1-2):11–35, 2015. URL: http://arxiv.org/abs/1303.0984, doi:10.2140/pjm.2015.279.11.

[4] H.H. Andersen, P. Polo, and K.X. Wen. Representations of quantum algebras. *Invent. Math.*, 104(1):1–59, 1991. doi:10.1007/BF01245066.

[5] H.H. Andersen, C. Stroppel, and D. Tubbenhauer. Semisimplicity of Hecke and (walled) Brauer algebras. *J. Aust. Math. Soc.*, 103(1):1–44, 2017. URL: http://arxiv.org/abs/1507.07676, doi:10.1017/S1446788716000392.

[6] H.H. Andersen, C. Stroppel, and D. Tubbenhauer. Cellular structures using $\mathbf{U}_q$-tilting modules. *Pacific J. Math.*, 292(1):21–59, 2018. URL: http://arxiv.org/abs/1503.00224, doi:10.2140/pjm.2018.292.21.

[7] H.H. Andersen and D. Tubbenhauer. Diagram categories for $\mathbf{U}_q$-tilting modules at roots of unity. *Transform. Groups*, 22(1):29–89, 2017. URL: https://arxiv.org/abs/1409.2799, doi:10.1007/s00031-016-9363-z.

[8] N. Bourbaki. *Lie groups and Lie algebras. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[9] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra I: cellularity. *Mosc. Math. J.*, 11(4):685–722, 821–822, 2011. URL: http://arxiv.org/abs/0806.1532.

[10] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra III: category $\mathcal{O}$. *Represent. Theory*, 15:170–243, 2011. URL: http://arxiv.org/abs/0812.1090, doi:10.1090/S1088-4165-2011-00389-7.

[11] B. Cooper and M. Hogancamp. An exceptional collection for Khovanov homology. *Algebr. Geom. Topol.*, 15(5):2659–2707, 2015. URL: http://arxiv.org/abs/1209.1002, doi:10.2140/agt.2015.15.2659.

[12] S. Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993. doi:10.1007/BF02571640.

[13] S. Donkin. *The q-Schur algebra*, volume 253 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998. doi:10.1017/CBO9780511600708.

[14] J.J. Graham and G. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996. doi:10.1007/BF01232365.

[15] J. Hu and A. Mathas. Graded cellular bases for the cyclotomic Khovanov–Lauda–Rouquier algebras of type *A*. *Adv. Math.*, 225(2):598–642, 2010. URL: http://arxiv.org/abs/0907.2985, doi:10.1016/j.aim.2010.03.002.

[16] J. Hu and A. Mathas. Quiver Schur algebras for linear quivers. *Proc. Lond. Math. Soc. (3)*, 110(6):1315–1386, 2015. URL: http://arxiv.org/abs/1110.1699, doi:10.1112/plms/pdv007.

[17] R.S. Huerfano and M. Khovanov. A category for the adjoint representation. *J. Algebra*, 246(2):514–542, 2001. URL: https://arxiv.org/abs/math/0002060, doi:10.1006/jabr.2001.8962.

[18] J.C. Jantzen. *Lectures on quantum groups*, volume 6 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.

[19] J.C. Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[20] V.F.R. Jones. Index for subfactors. *Invent. Math.*, 72(1):1–25, 1983. doi:10.1007/BF01389127.

[21] M. Kaneda. Based modules and good filtrations in algebraic groups. *Hiroshima Math. J.*, 28(2):337–344, 1998.

[22] M. Kashiwara and P. Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006. doi:10.1007/3-540-27950-4.

[23] G. Lusztig. Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra. *J. Amer. Math. Soc.*, 3(1):257–296, 1990. doi:10.2307/1990988.

[24] G. Lusztig. *Introduction to quantum groups*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition. doi:10.1007/978-0-8176-4717-9.

[25] G. Lusztig. Quantum groups at roots of 1. *Geom. Dedicata*, 35(1-3):89–113, 1990. doi:10.1007/BF00147341.

[26] P. Martin. *Potts models and related problems in statistical mechanics*, volume 5 of *Series on Advances in Statistical Mechanics*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1991. doi:10.1142/0983.

[27] V. Mazorchuk. *Lectures on algebraic categorification*. QGM Master Class Series. European Mathematical Society (EMS), Zürich, 2012. URL: http://arxiv.org/abs/1011.0144, doi:10.4171/108.

[28] J. Paradowski. Filtrations of modules over the quantum algebra. In *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 93–108. Amer. Math. Soc., Providence, RI, 1994.

[29] D. Plaza and S. Ryom-Hansen. Graded cellular bases for Temperley–Lieb algebras of type A and B. *J. Algebraic Combin.*, 40(1):137–177, 2014. URL: http://arxiv.org/abs/1203.2592, doi:10.1007/s10801-013-0481-6.

[30] D. Ridout and Y. Saint-Aubin. Standard modules, induction and the structure of the Temperley–Lieb algebra. *Adv. Theor. Math. Phys.*, 18(5):957–1041, 2014. URL: http://arxiv.org/abs/1204.4505.

[31] C.M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208(2):209–223, 1991. doi:10.1007/BF02571521.

[32] S. Ryom-Hansen. A q-analogue of Kempf's vanishing theorem. *Mosc. Math. J.*, 3(1):173–187, 260, 2003. URL: http://arxiv.org/abs/0905.0236.

[33] W. Soergel. Character formulas for tilting modules over Kac–Moody algebras. *Represent. Theory*, 2:432–448 (electronic), 1998. doi:10.1090/S1088-4165-98-00057-0.

[34] W. Soergel. Kazhdan–Lusztig polynomials and a combinatoric[s] for tilting modules. *Represent. Theory*, 1:83–114 (electronic), 1997. doi:10.1090/S1088-4165-97-00021-6.

[35] C. Stroppel. Untersuchungen zu den parabolischen Kazhdan–Lusztig-Polynomen für affine Weyl-Gruppen. Diploma Thesis (1997), 74 pages (German). URL: http://www.math.uni-bonn.de/ag/stroppel/arbeit_Stroppel.pdf.

[36] T. Tanisaki. Character formulas of Kazhdan–Lusztig type. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, volume 40 of *Fields Inst. Commun.*, pages 261–276. Amer. Math. Soc., Providence, RI, 2004.

[37] J.P. Wang. Sheaf cohomology on G/B and tensor products of Weyl modules. *J. Algebra*, 77(1):162–185, 1982. doi:10.1016/0021-8693(82)90284-8.

[38] H. Wenzl. On sequences of projections. *C. R. Math. Rep. Acad. Sci. Canada*, 9(1):5–9, 1987.

[39] B.W. Westbury. The representation theory of the Temperley–Lieb algebras. *Math. Z.*, 219(4):539–565, 1995. doi:10.1007/BF02572380.