REPRESENTATIONS OF FACTORIZABLE HOPF ALGEBRAS

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Abstract

Using the transmutation theory developed by Majid in [28], in this paper we give an explicit formula for the Müger centralizer in the category of representations of a quasitriangular Hopf algebra. A second formula for the Müger centralizer is also given in terms of the conjugacy classes introduced by Cohen and Westreich in [10]. This allows us to answer positively a question from [8]. More precisely we show that in the case of a factorizable Hopf algebra the centralizer of a normal fusion subcategory is also a normal fusion subcategory. As an application we give a structure theorem for factorizable Hopf algebras of dimension dq^n where d is an odd square free integer and q an odd prime number.

1. Introduction

The notion of centralizer in a braided fusion category was introduced by Müger in [30]. It was shown in [17, Theorem 8.21.4] that the centralizer of a nondegenerate fusion subcategory of a braided category is a categorical complement of the nondegenerate subcategory. This principle is the basis of many classification results of braided fusion categories, see for example [15, 18, 14] and references therein.

Despite its importance, there is no concrete formula for the centralizer of all fusion subcategories of a given fusion category. Only few cases

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are completely known in the literature. For instance, in the same paper [30], Müger described the centralizer of all fusion subcategories of the category of finite dimensional representations of a Drinfeld double of a finite abelian group. More generally, for the category of representations of a (twisted) Drinfeld double of an arbitrary finite group a similar formula was then given in [31]. For the braided center of Tambara-Yamagami categories, in [19], the centralizer was described by computing completely the S-matrix of the modular category. A different formula for braided equivariantized fusion categories was given by the author in [9].

Also, in the paper [8], the author has given some general formulae for the centralizer of certain subcategories (normal in the sense of [2]) of the category of representations of a factorizable Hopf algebras. In the present paper we extend these formulae for any subcategory of representations completing the study of the centralizer for these type of braided fusion categories. Moreover, in the case of the category of representations of a factorizable Hopf algebra, we show that the centralizer of any normal fusion subcategory is also normal, answering positively a question posed at the end of the paper [8].

Given a quasitriangular Hopf algebra (A, R) one can define the linear map

$$\phi_R: A^* \to A, \ f \mapsto (f \otimes \mathrm{id})(R_{21}R) = f(Q_1)Q_2$$

where $Q = R_{21}R$ is the monodromy matrix. It is well known, see [28] that ϕ_R sends Hopf subalgebras of A^* into left normal coideal subalgebras of A. We also define $\mathbf{K}_A := \phi_R(A^*)$. It was proven in [32] that in general, \mathbf{K}_A is a left normal coideal subalgebra of A. In this paper we show that if A is a semisimple Hopf algebra then \mathbf{K}_A is a normal Hopf subalgebra of A.

Given a braided fusion category \mathcal{C} and \mathcal{D} a fusion subcategory of \mathcal{C} , the notion of centralizer of \mathcal{D} was introduced in [15]. The centralizer

 \mathcal{D}' is defined as the fusion subcategory \mathcal{D}' of \mathcal{C} generated by all simple objects X of \mathcal{C} satisfying

$$c_{X, Y}c_{Y, X} = \mathrm{id}_{X \otimes Y}$$

for all objects $Y \in \mathcal{O}(\mathcal{D})$ (see also [30]).

We prove the following theorem which gives a description for the centralizer of a fusion subcategory of the category of representations of a quasitriangular Hopf algebra:

Theorem 1.1. Let (A, R) be a semisimple quasitriangular Hopf algebra and L be a left normal coideal subalgebras of A. Then

$$\operatorname{Rep}(A//L)' = \operatorname{Rep}(A//M)$$
 where $M = \phi_R((A//L)^*)$

Moreover, in this case one has that: $L\mathbf{K}_A = \mathbf{K}_A L = \phi_R((A//M)^*)$.

Recall that the quasitriangular Hopf algebra (A, R) is called *factorizable* if and only if the linear map ϕ_R is an isomorphism of vector spaces.

Let V_0, V_1, \ldots, V_r be a complete set of isomorphism classes of irreducible A-modules of a factorizable semisimple Hopf algebra A. Let also $\operatorname{Irr}(A) = \{\chi_0, \chi_1, \ldots, \chi_r\}$ be the set of irreducible characters afforded by these modules. For any $1 \leq j \leq r$ let $E_j \in \mathcal{Z}(A)$ be the associated central primitive idempotent of the irreducible character χ_j . Then $F_j := \phi_R^{-1}(E_j)$ is a primitive central idempotent in C(A) since $\phi_R : C(A) \to \mathcal{Z}(A)$ is an algebra isomorphism. Following [11] one can define the conjugacy classes \mathcal{C}^j of A as $\mathcal{C}^j := \Lambda \leftarrow F_j A^*$, where Λ is an idempotent integral of A and $a \leftarrow f = \langle f, a_1 \rangle a_2$ for all $a \in A$ and $f \in A^*$. It is well known that these conjugacy classes are the simple D(A)-submodules of the induced D(A)- module $k \uparrow_A^{D(A)}$, see [38].

Our second main result is the following:

Theorem 1.2. Let (A, R) be a semisimple factorizable Hopf algebra and L be a left normal coideal subalgebras of A. Then with the above

notations one has that:

$$\operatorname{Irr}(\operatorname{Rep}(A//L)') = \{\chi_i \mid \mathcal{C}^j \subseteq L\} = \{\chi_i \mid F_i(\Lambda_L) \neq 0\}.$$

Here Λ_L denotes the integral of L with $\epsilon(\Lambda_L) = 1$. Recall that Λ_L is unique up to a scalar element of k and satisfies $l\Lambda_L = \epsilon(l)\Lambda_L$ for any $l \in L$, see e.g [4].

We will also prove the following factorization result for factorizable Hopf algebras:

Theorem 1.3. Let A be a factorizable semisimple Hopf algebra and K be a normal Hopf subalgebra of A such that Rep(A//K) is a nondegenerate fusion category. Then there is another normal Hopf subalgebra K of A such that

$$A \simeq K \otimes L$$

as Hopf algebras. Moreover $\operatorname{Rep}(A//K)' = \operatorname{Rep}(A//L)$.

We should also notice that in Proposition 6.11, we correct a possible error in the results of [32, Theorem 4.8].

Shortly, this paper is organized as follows. In Section 2 we recall the basic notions of Hopf algebras and fusion categories that are used throughout this paper. In Section 3 we recall the main properties of quasitriangular Hopf algebras and their associated Drinfeld maps. In Section 4 we prove the first main result and its consequences. In Section 5 we prove the second main result of this paper. In Section 6 we prove the factorization result mentioned above in Theorem 1.3. In the last section of the paper we apply our main results to the category of representations of a Drinfeld double $D(\Bbbk G)$ of a finite group G. We show that using our first main result one can recover the centralizer formulae from [31]. As an application we also give in this section a structure theorem for factorizable Hopf algebras of dimension dq^n where d is a square free integer and q an odd prime number.

2. Preliminaries

Let A be a finite dimensional semisimple Hopf algebra over an algebraically closed field \mathbbm{k} of characteristic zero. Then A is also cosemisimple and $S^2 = \mathrm{id}$, [26]. The character ring $C(A) := \mathrm{K}_0(\mathcal{C}) \otimes_{\mathbbm{Z}} \mathbbm{k}$ is a semisimple subalgebra of A^* and it has a vector space basis given by the set $\mathrm{Irr}(A)$ of irreducible characters of A, see [37]. Moreover, $C(A) = \mathrm{Cocom}(A^*)$, the space of cocommutative elements of A^* . By duality, the character ring of A^* is a semisimple subalgebra of A and $C(A^*) = \mathrm{Cocom}(A)$. If M is an A-representation with character $\chi^* = \chi \circ S$. This induces an involution "*": $C(A) \to C(A)$ on C(A). Let also $m_A(\chi, \mu)$ be the usual multiplicity form on C(A). Recall that if M and N are A-representations affording characters $\chi, \mu \in C(A)$ respectively, then $m_A(\chi, \mu)$ is defined by $m_A(\chi, \mu) := \dim_{\mathbbm{R}} \mathrm{Hom}_A(M, N)$. We will use the notation G(A) for the set of grouplike elements of A and G(A) for the set of central grouplike elements of A.

Throughout of this paper we denote by Λ an idempotent integral of A and by t an idempotent integral of A^* . Moreover one has that $t(\Lambda) = \frac{1}{\dim_{\mathbb{K}}(A)}$. Recall also [25] that $\dim_{\mathbb{K}}(A)\Lambda = \sum_{d \in \operatorname{Irr}(A^*)} \epsilon(d)d$ is the regular character of A^* . Dually, $\dim_{\mathbb{K}}(A)t = \sum_{\chi \in \operatorname{Irr}(A)} \chi(1)\chi$ is the regular character of A.

2.1. **Left coideal subalgebras.** Let A be finite dimensional Hopf algebra over \mathbb{k} . Recall a *left* coideal subalgebra of A is a subalgebra L such that $\Delta(L) \subset A \otimes L$. Moreover L is called a *left normal coideal subalgebra* if L is closed under the left adjoint action of A, i.e, $a_1lS(a_2) \in L$ for any $l \in L$ and any $a \in A$.

Following [33] there is a bijection between fusion subcategories of Rep(A) and Hopf subalgebras of A^* . Moreover by Takeuchi's results, see [36, Theorem 3.2], in the case of a finite dimensional Hopf algebra A one has a bijection between the set of Hopf subalgebras of A^* and the set of left normal coideal subalgebras of A. This bijection is given by

 $L \mapsto (A//L)^*$ with inverse given by

$$(B \xrightarrow{i} A^*) \mapsto L := A^{\operatorname{co} i^*}$$

Here A//L denotes the Hopf algebra quotient A/AL^+ where $L^+:=\ker(\epsilon)\cap L.$

Recall also [4] that there is a unique element $\Lambda_L \in L$ such that $l\Lambda_L = \epsilon(l)\Lambda_L$, see also [24]. Then Λ_L is called the integral of L and the left coideal subalgebra L is normal if and only if Λ_L is a central element of A. In this case the augmentation ideal AL^+ where $L^+ := \ker(\epsilon) \cap L$ can be written as $AL^+ = A(1 - \Lambda_L) = \operatorname{Ann}_A(\Lambda_L)$. Moreover one has that

$$(2.1) \quad (A//L)^* = \{ f \in A^* \mid f(al) = \epsilon(l)f(a) \text{ for all } a \in A, l \in L \}.$$

It follows that $\Lambda_L \rightharpoonup A^* = (A//L)^*$. Indeed one has $(\Lambda_L \rightharpoonup f)(al) = f(al\Lambda_L) = \epsilon(l)f(a\Lambda_L) = \epsilon(l)(\Lambda \rightharpoonup f)(a)$ for any $a \in A$. On the other hand clearly $\Lambda_L \rightharpoonup f = f$ for any $f \in (A//L)^*$.

Moreover, it can also be shown that

(2.2)
$$L = \{a \in a \mid gf(a) = f(1)g(a) \text{ for all } f \in (A//L)^*, g \in A^* \}$$

Since A is free as left L-module [35] it follows that the map

$$A \otimes_L \mathbb{k} \simeq A\Lambda_L, \ a \otimes_L 1 \mapsto a\Lambda_L$$

is an isomorphism of A-modules. Thus $A\Lambda_L$ is isomorphic to the regular A//L-module, see [5]. Moreover, by [5, Proposition 3.11] it follows that the regular character of the quotient Hopf algebra A//L is isomorphic to the induced module $A\otimes_L \Bbbk$.

Lemma 2.3. Let A be a semisimple Hopf algebra and L_1, L_2 be two left normal coideal subalgebras of A. Then $AL_1^+ \subseteq AL_2^+$ if and only if $L_1 \subseteq L_2$

Proof. If $L_1 \subseteq L_2$ then clearly $AL_1^+ \subseteq AL_2^+$. Conversely, suppose that $AL_1^+ \subseteq AL_2^+$. Let $\pi : A//L_1 \to A//L_2$ the canonical induced

projection. By duality this shows that $(A//L_2)^* \subseteq (A//L_1)^* \subseteq A^*$. Then Equation (2.2) shows that $L_1 \subseteq L_2$.

We denote by F_0, F_1, \ldots, F_r the central primitive idempotents of the character ring C(A). Without loss of generality we may suppose that $F_0 = t$ is the idempotent integral of A^* .

2.2. A result concerning factorization of normal Hopf subalgebras. In the proof of Theorem 1.3 we also need the following result. If L, K are two normal Hopf subalgebras of a semisimple Hopf algebra A with $L \cap K = k$ then $LK \simeq L \otimes K$ as Hopf algebras, see [3, Theorem 3.5].

Let L and M be two left normal coideal subalgebras of A. Then the following equalities hold in A^* , see [8]:

$$(2.4) (A//L)^* \cap (A//M)^* = (A//LM)^*.$$

and

$$(2.5) (A//L)^*, (A//M)^* >= (A//(L \cap M))^*.$$

The above relations can also be written as follows:

(2.6)
$$\operatorname{Rep}(A//L) \cap \operatorname{Rep}(A//M) = \operatorname{Rep}(A//LM)$$

(2.7)
$$\operatorname{Rep}(A//L) \vee \operatorname{Rep}(A//M) = \operatorname{Rep}(A//L \cap M)$$

Given two left normal coideal subalgebras L and M note that LM = ML since $lm = (l_1 mS(l_2))l_3 \in ML$ for all $l \in L$ and $m \in M$. Moreover any inclusion $M \subseteq L$ allows us to define the quotient

$$L//M := L//LM^+.$$

For any two left normal coideal subalgebras L,M of A one also has a canonical linear epimorphism

(2.8)
$$LM//M \xrightarrow{\pi} L//L \cap M, \ \widehat{lm} \mapsto \epsilon(m)\hat{l}.$$

If C(A) is commutative it follows by [8, Theorem 3.8] that

$$(2.9) |LM| = \frac{|L||M|}{|L \cap M|}$$

for any two left normal coideal subalgebras L and M of A. Moreover, in this case, the above canonical epimorphism from Equation (2.8) is in fact an isomorphism.

2.3. Left kernels and a Hopf-algebraic version of Brauer's theorem. Let M be an A-module and let $LKer_A(M)$ be the left kernel of M. Recall [4] that $LKer_A(M)$ is defined by:

(2.10) LKer₄(M) = {
$$a \in A | a_1 \otimes a_2 m = a \otimes m$$
, for all $m \in M$ }

Then by [4] it follows that $LKer_A(M)$ is the largest left coideal subalgebra of A that acts trivially on M. It is also a left normal coideal subalgebra.

Next theorem generalizes a well known result of Brauer in the representation theory of finite groups.

Theorem 2.11. [4, Theorem 4.2.1]. Suppose that M is a finite dimensional module over a semisimple Hopf algebra A. Then

$$(2.12) < M >= \operatorname{Rep}(A//\operatorname{LKer}_A(M))$$

where < M > is the fusion subcategory of Rep(A) generated by M.

3. Quasitriangular and factorizable Hopf algebras

Recall that a Hopf algebra A is called *quasitriangular* if A admits an R-matrix, i.e. an element $R \in A \otimes A$ satisfying the following properties:

- 1) $R\Delta(x) = \Delta^{\text{cop}}(x)R$ for all $x \in A$.
- 2) $(\Delta \otimes id)(R) = R^1 \otimes r^1 \otimes R^2 r^2$
- 3) $(id \otimes \Delta)(R) = R^1 r^1 \otimes r^2 \otimes R^2$.
- 4) $(id \otimes \epsilon)(R) = 1 = (\epsilon \otimes id)(R)$. Here $R = r = R^1 \otimes R^2 = r^1 \otimes r^2$.
- If (A, R) is a quasitriangular Hopf algebra then the category of finite

dimensional representations Rep(A) is a braided fusion category with the braiding given by

(3.1)
$$c_{M,N}: M \otimes N \to N \otimes M, \ m \otimes n \mapsto R_{21}(n \otimes m) = R^2 n \otimes R^1 m$$

for any two left A-modules $M, N \in \text{Rep}(A)$ (see [23]). Recall that $R_{21} := R^2 \otimes R^1$. Denote $Q := R_{21}R$. Recall that the monodromy of two objects $M, N \in \text{Rep}(A)$ is defined as: (3.2)

$$c_{N,M}c_{M,N}: M \otimes N \to N \otimes M, \ m \otimes n \mapsto R^2r^1m \otimes R^1r^2n = Q(m \otimes n)$$

The Drinfeld map $\phi_R: A^* \to A$ associated to (A, R) is defined by $\phi_R(f) = (f \otimes \mathrm{id})(Q) = f(Q^1)Q^2$, for any $f \in A^*$. One can also define the map $R\phi(f) = (\mathrm{id} \otimes f)(R_{21}R)$. Moreover by [34, Theorem 2.1], for any quasitriangular semisimple Hopf algebra (A, R), one has that

(3.3)
$${}_{R}\phi(f\chi) = {}_{R}\phi(f) {}_{R}\phi(\chi)$$

for all $f \in A^*$ and $\chi \in C(A)$. Also, by [34, Lemma 2.2] in this situation, the map R^{ϕ} maps the character ring C(A) to the center of $\mathcal{Z}(A)$ of A. By [32, Lemma 2.3] one also has that

$$(3.4) R\phi = S\phi_R s$$

where S and s are the antipodes of A and A^* respectively. Thus in the semisimple Hopf algebra case ϕ_R also maps the character ring C(A) of A to the center $\mathcal{Z}(A)$. Moreover by [11, Lemma 41.] one has that in the case of a semisimple Hopf algebra A the restrictions of R^{ϕ} and R^{ϕ} and R^{ϕ} to the character ring R^{ϕ} coincide, i.e.

$$\phi_R|_{C(A)} = {}_R\phi_{C(A)}.$$

Equation (3.3) gives that

(3.6)
$$\phi_R(\chi f) = \phi_R(\chi)\phi_R(f)$$

for all $f \in A^*$ and $\chi \in C(A)$. A quasitriangular Hopf algebra (A, R) is called *factorizable* if the Drinfeld map ϕ_R is bijective, i.e. an isomorphism of vector spaces. Thus the map $_R\phi|_{C(A)}:C(A)\to \mathcal{Z}(A)$ is an isomorphism of \mathbb{k} -algebras in the case of a factorizable Hopf algebra.

3.1. On the S-matrix for a quasitriangular Hopf algebra. Let (A, R) be a semisimple quasitriangular Hopf algebra. By [16] one has that the S-matrix $S = (s_{ij})$ of the modular tensor category Rep(A) is given by

$$(3.7) s_{ij} = \chi_i(\phi_R(\chi_{i^*}))$$

Moreover it is not difficult to see that for all $1 \le i, j \le s$, one has that $s_{ij} = s_{ji}$, and $s_{ij} = s_{i^*j^*}$ and $s_{ij^*} = s_{ji^*}$ (cf. [1, 34]). In this case one also has the following inequality $|s_{ij}| \le \chi_i(1)\chi_j(1)$ for any indices i, j.

3.2. Transmutation theory. In this subsection we recall some elements from [28] and [32, Section 4]. We assume the reader is familiar with the notion of braided Hopf algebras in monoidal categories.

By [28, Example 9.4.9] given a quasitriangular Hopf algebra (A, R) one has a braided Hopf algebra \underline{A} in the category A-mod of left A-modules with $\underline{A} = A$ as algebras (thus as vector spaces too) and the following comultiplication structure

(3.8)
$$\underline{\Delta}(a) = a_1 S(R^{(2)}) \otimes ad_{R^{(1)}}(a_2).$$

By [28, Example 9.4.10] one also has a braided Hopf algebra \underline{A}^* algebra in the category of right A^* -comodules, i.e. left A-modules. One has that $\underline{A}^* = A^*$ as coalgebras and the multiplication given by

(3.9)
$$p_{\underline{q}} = (S(p_1)p_3 \otimes q_1)(R)p_2q_2$$

Moreover, \underline{A}^* is regarded as right A-comodule with the coadjoint structure

(3.10)
$$\rho(p) = p_2 \otimes p_1 S(p_3) \in \underline{A}^* \otimes A.$$

Note that this is equivalent with the following left A- action $h_{\text{-coad}}p = p(h_1?S(h_2))$.

In this context, the Drinfeld map $\phi_R : \underline{A}^* \to \underline{A}$ becomes a morphism of braided Hopf algebras. In other words ϕ_R is a left A-module map that transforms the product from Equation (3.9) into the product of A and the coproduct of \underline{A}^* into the coproduct from Equation (3.8). See [28, Propositions 2.1.14 and 7.4.3] for more details. By [32] it also follows that any subcoalgebra in \underline{A} is in fact a left normal coideal subalgebra of A. Moreover, by [32, Lemma 1.1] one has that $\phi_R(C)$ is a left normal coideal subalgebra for any subcoalgebra C of A^* .

3.3. Formulae for the Frobenius-Perron dimension and double centralizer. Let \mathcal{B} and \mathcal{D} be fusion subcategories of a braided fusion category \mathcal{C} . Following [14, Theorem 3.10] one has that

$$(3.11) \qquad \text{FPdim}(\mathcal{B} \cap \mathcal{D}')\text{FPdim}(\mathcal{D}) = \text{FPdim}(\mathcal{B}' \cap \mathcal{D})\text{FPdim}(\mathcal{B})$$

$$(3.12) \mathcal{D}'' = \mathcal{D} \vee \mathcal{C}'$$

In particular, for $\mathcal{B} = \mathcal{C}$ the first Equation (3.11) becomes

(3.13)
$$\operatorname{FPdim}(\mathcal{D})\operatorname{FPdim}(\mathcal{D}') = \operatorname{FPdim}(\mathcal{C})\operatorname{FPdim}(\mathcal{D} \cap \mathcal{C}')$$

4. Proof of the first main theorem on Müger centralizer

Let (A, R) be a semisimple quasitriangular Hopf algebra and $\mathcal{D} = \text{Rep}(A//L)$ be a fusion subcategory of Rep(A). In this section we prove the first main theorem mentioned in the introduction.

Through the rest of this paper we will use the following notation. Given a left normal coideal subalgebra L of A we denote by L' the left normal coideal subalgebra of A for which the following equality holds:

(4.1)
$$\operatorname{Rep}(A//L)' = \operatorname{Rep}(A//L').$$

We also use the following notation:

$$\phi_R((A//L)^*) = L^*.$$

Then the first main result Theorem 1.1 states that $L' = L^*$ if (A, R) is a semisimple quasitriangular Hopf algebra.

Lemma 4.3. Let (A, R) be a semisimple quasitriangular Hopf algebra and L, M be two left normal coideal subalgebras of A. Then the following assertions are equivalent:

- (1) $\operatorname{Rep}(A//M) \subseteq \operatorname{Rep}(A//L)'$
- (2) The equality

$$(4.4) Q(\Lambda_L \otimes \Lambda_M) = (\Lambda_L \otimes \Lambda_M)$$

(3) $M \supseteq L^{\star}$.

Proof. Recall that two fusion subcategories of Rep(A) centralize each other if and only if their regular representations centralize. Thus one needs to show that the two regular characters of A//L and A//M centralize each other if and only if Equation (4.4) holds and if and only if $M \supseteq L^*$. On the other hand, by the definition of the braiding in Rep(A), note that the two characters centralize each other if and only if $Q = R^2 r^1 \otimes R^1 r^2$ acts as identity on their tensor product $A//L \otimes A//M$. As noticed above one has $A//L \otimes A//M = k \uparrow_L^A \otimes k \uparrow_M^A = A\Lambda_L \otimes A\Lambda_M$. Since Λ_L and Λ_M are central elements of A it is clear that Q acts as identity on this subspace of $A \otimes A$ if and only if Equation (4.4) holds. This proves $i) \iff ii$.

It remains to show that $ii) \iff iii$). First we show that $ii) \implies iii$). As above the inclusion of fusion subcategories holds if and only if Q acts trivially on $A//L \otimes A//M$. Therefore we may suppose that

$$(4.5) Q^1 \Lambda_L \otimes Q^2 \Lambda_M = \Lambda_L \otimes \Lambda_M$$

Since $(A//L)^* = \Lambda_L \rightharpoonup A^*$ one has that $\phi_R(\Lambda_L \rightharpoonup f) = f(Q^1\Lambda_L)Q^2$ for any $f \in (A//L)^*$. From here, applying Equation (4.5) it follows

$$(4.6) \phi_R(\Lambda_L \to f)\Lambda_M = f(Q^1 \Lambda_L)Q^2 \Lambda_M = f(\Lambda_L)\Lambda_M.$$

On the other hand note that $\epsilon(\phi_R(\Lambda_L \to f)) = f(\Lambda_L)$. For $L^* := \phi_R((A//L)^*)$ it follows that $A(L^*)^+ \subseteq AM^+$. By Lemma 2.3 one has that $L^* \subseteq M$.

Note also that the converse $iii) \implies ii$) is clear from above: if $L^* \subseteq M$ then Equation (4.6) holds for any $f \in A^*$ which in turn implies that Equation (4.5) holds in $A \otimes A$.

Remark 4.7. Note that in the above case one also has the inclusion: $L\mathbf{K}_A \supseteq \phi_R((A//M)^*)$. Indeed, the inclusion $\operatorname{Rep}(A//L)' \supseteq \operatorname{Rep}(A//M)$ implies by centralising again that

$$\operatorname{Rep}(A//M)' \supseteq \operatorname{Rep}(A//L)'' = \operatorname{Rep}(A//L) \vee \operatorname{Rep}(A//\mathbf{K}_A) = \operatorname{Rep}(A//L\mathbf{K}_A).$$

By the above argument one has that $L\mathbf{K}_A \supseteq M^* = \phi_R((A//M)^*)$.

4.1. **Proof of Theorem 1.1.** We have to prove that $L' = L^*$ for any left normal coideal subalgebra L of A. Using the above notations the previous Lemma can be written as

$$L' \subseteq M \iff L^* \subseteq M.$$

This clearly implies that $L^* = L'$ by putting M = L' and $M = L^*$ above. Thus the first equality of Theorem 1.1 is proven. The second equality is proven by centralising once more the first equality as in the above remark.

Remark 4.8. By Equation (3.4) one has that $S_R \phi = \phi_R s$ where S and s are the antipodes of A and A^* respectively. Thus one can also write that

$$_R\phi((A//L)^*)\subseteq S(M)\iff \operatorname{Rep}(A//M)\subseteq \operatorname{Rep}(A//L)'.$$

5. Conjugacy classes and Müger centralizer

In this section we will prove the second main result.

5.1. Duality between the character ring and the center. Let A be a semisimple Hopf algebra over the ground field k. Let us denote by Irr(A) the set of irreducible characters of A. We suppose that $Irr(A) = \{\chi_0, \chi_1, \dots, \chi_r\}$. Without loss of generality we may suppose that $\chi_0 = \epsilon$. Let also E_0, E_1, \dots, E_r be the corresponding central primitive central idempotents in A. The evaluation form

(5.1)
$$C(A) \otimes \mathcal{Z}(A) \to k, \quad \chi \otimes a \mapsto \chi(a)$$

is nondegenerate. A pair of dual bases for this form is given by $\{\chi_i, \frac{1}{n_i}E_i\}$ since $\langle \chi_i, \frac{1}{n_j}E_j \rangle = \delta_{i,j}$ for any $1, \leq i, j \leq r$.

According to [11] in the case of a commutative ring C(A) there are is another pair of dual bases corresponding to this nondegenerate form. This pair of dual bases is given in terms of the conjugacy class sums as defined in [11]. Recall that the conjugacy class C^j is defined as $C^j = \Lambda \leftarrow F_j A^*$. One has that $\Delta C^j \subseteq A \otimes C^j$ and C^j is closed under the left adjoint action of A on itself. Thus C^j is a simple D(A)-submodule of $k \uparrow_A^{D(A)} \simeq A$, see [38] and [6]. Recall from [6] that the D(A)-module structure of A is given by: $(f \bowtie x).a = x_1 a S(x_2) \leftarrow S^{-1}f$, for any $a, x \in A$ and any $f \in A^*$.

Note also that $A = \bigoplus_{j=1}^r \mathcal{C}^j$ since $f \mapsto \Lambda \leftarrow f$ is a bijective map and therefore it preserves direct sums. One can also define the corresponding class sum

(5.2)
$$\mathbf{C}_j = \Lambda - (\dim A) F_j.$$

Note that $\mathbf{C}_j \in \mathcal{Z}(A)$ and since $\dim_{\mathbb{k}} \mathcal{Z}(A) = \dim_{\mathbb{k}} C(A)$ it follows that $C^j \cap \mathcal{Z}(A) = \mathbb{k}\mathbf{C}_j$.

Note that

$$\dim_{\mathbb{k}}(A)\Lambda = \sum_{i=0}^{\tau} \mathbf{C}_{i}.$$

Remark 5.3. By the Class Equation for semisimple Hopf algebras, see [27], one has that the value $n_j := \frac{\dim_{\mathbb{R}} A^*}{\dim_{\mathbb{R}} (A^* F_j)}$ is an integer. Moreover as in [11, Equation (11)] one can write that $F_j(\Lambda) = \frac{1}{n_j}$.

Then the second pair of dual bases is given by $\{F_i, \frac{n_i}{\dim_{\mathbb{R}}(A)} \mathbf{C}_i\}$, see for instance [11, Equation (17)]. Thus $\langle F_i, \frac{n_j}{\dim_{\mathbb{R}}(A)} \mathbf{C}_j \rangle = \delta_{i,j}$

5.2. **Decomposition of the integral.** Let L be a left normal coideal subalgebra of a semisimple Hopf algebra A with a commutative character ring. Since L is also a D(A)-submodule of A one can write that $L = \bigoplus_{j \in \mathcal{I}_L} \mathcal{C}^j$ for some subset $\mathcal{I}_L \subset \{0, 1, \dots, r\}$. Then for the decomposition of the idempotent integral Λ_L of L we write $\Lambda_L = \sum_{j \in \mathcal{I}'_L} \Lambda_j$ for some $\mathcal{I}'_L \subseteq \mathcal{I}_L$ with $\Lambda_j \in \mathcal{C}^j$ nonzero components. It follows that for any $a \in A$ one has that $\epsilon(a)\Lambda_L = a_1\Lambda_L Sa_2 = \sum_{j \in \mathcal{I}_L} a_1\Lambda_j Sa_2$. Thus $a_1\Lambda_j S(a_2) = \epsilon(a)\Lambda_j$ which shows that $\Lambda_j \in \mathcal{C}^j$ are central elements. Moreover since $\mathcal{C}^j \cap \mathcal{Z}(A)$ has dimension 1 one then can write $\Lambda_j = \alpha_j \mathbf{C}_j$ and

(5.4)
$$\Lambda_L = \sum_{j \in \mathcal{I}_L'} \alpha_j \mathbf{C}_j$$

Then it follows that $L = \Lambda_L - A^* = \bigoplus_{j \in \mathcal{I}'_L} \mathbf{C}_j - A^* = \bigoplus_{j \in \mathcal{I}'_L} \mathcal{C}^j$ which shows that $\mathcal{I}'_L = \mathcal{I}_L$.

Lemma 5.5. (see also [6, Theorem 5.13].) Suppose that A is a semisimple Hopf algebra with a commutative character ring C(A). Then F_j coincides to the functional $p_{C^j} \in A^*$ defined as the unique functional that coincides to ϵ on C^j and it is equal to zero on the other conjugacy classes C_l with $l \neq j$.

Proof. One has the following equality $\langle F_j, \Lambda - F_l f \rangle = \langle F_l f, \Lambda_1 \rangle \langle F_j \Lambda_2 \rangle = \langle F_l f, \Lambda_2 \rangle \langle F_j, \Lambda_1 \rangle = \delta_{j,l} \epsilon(\Lambda - F_l f)$. Note that in the above computations we used the cocommutativity of Λ .

We shall use the notation $\lambda_L \in (A//L)^*$ for the idempotent integral of the Hopf algebra $(A//L)^*$. Clearly $\lambda_L \in C((A//L)^*) \subset C(A^*)$.

Lemma 5.6. Let L be a left normal coideal subalgebra of a semisimple Hopf algebra A. If we write $\lambda_L = \sum_{j \in \mathcal{J}_L} F_j$ then $\mathcal{I}_L = \mathcal{J}_L$. Moreover, $j \in \mathcal{J}_L \iff F_j(\Lambda_L) \neq 0$.

Proof. Note that [13, Lemma 1.1] $\Lambda \leftarrow \lambda_L = \frac{\dim_{\mathbb{R}}(L)}{\dim_{\mathbb{R}}(A)} \Lambda_L$. If $j \in \mathcal{J}_L$ then $\mathcal{C}^j = \Lambda \leftarrow F_j A^* = \Lambda \leftarrow F_j \lambda_L A^* = \Lambda \leftarrow \lambda_L F_j A^* = \Lambda_L \leftarrow F_j A^* \subseteq L$ which shows that $j \in \mathcal{I}_L$. Thus $\mathcal{J}_L \subseteq \mathcal{I}_L$.

Suppose now that $j \in \mathcal{I}_L$ where $L = \bigoplus_{j \in \mathcal{I}_L} \mathcal{C}^j$. Then by Lemma 5.5 one has that $F_j(\Lambda_L) = \alpha_j \epsilon(\mathbf{C}_j) \neq 0$. Thus if $j \in \mathcal{I}_L$ then $F_j(\Lambda_L) \neq 0$. On the other hand, by [13, Lemma 1.1] one has that $\frac{\dim_{\mathbb{R}}(L)}{\dim_{\mathbb{R}}(A)} \Lambda_L = \Lambda \leftarrow \lambda_L$. Since $\epsilon(\mathbf{C}_j) = \frac{\dim_{\mathbb{R}}(A)}{n_j}$ it follows that if $0 \neq F_j(\Lambda_L) = (\lambda_L F_j)(\Lambda)$ then $\lambda_L F_j \neq 0$ i.e, $j \in \mathcal{J}_L$. This shows that $\mathcal{I}_L \subseteq \mathcal{J}_L$.

Remark 5.7. With the above notations, by Equation (5.4) one has that

$$\alpha_{j}\epsilon(\mathbf{C}_{j}) = F_{j}(\Lambda_{L}) = F_{j}(\frac{\dim_{\mathbb{K}}(A)}{\dim_{\mathbb{K}}(L)}(\Lambda - \lambda_{L})) = \frac{\dim_{\mathbb{K}}(A)}{\dim_{\mathbb{K}}(L)}(\lambda_{L}F_{j})(\Lambda) =$$

$$= \frac{\dim_{\mathbb{K}}(A)}{\dim_{\mathbb{K}}(L)}F_{j}(\Lambda) = \frac{\dim_{\mathbb{K}}(A)}{n_{j}\dim_{\mathbb{K}}(L)}.$$

On the other hand note that Equation (5.2) gives that $\epsilon(\mathbf{C}_j) = \dim_{\mathbb{K}}(A)F_j(\Lambda) = \frac{\dim_{\mathbb{K}}(A)}{n_j}$. Thus $\alpha_j = \frac{1}{\dim_{\mathbb{K}}(L)}$ and one has that

(5.8)
$$\Lambda_L = \frac{1}{\dim_{\mathbb{k}}(L)} \sum_{j \in \mathcal{I}_L} \mathbf{C}_j$$

Let A be a semisimple Hopf algebra with commutative character ring C(A). Then $\{F_j\}$ form a \mathbb{k} -linear basis for C(A) and for any character $\chi \in C(A)$ one can write $\chi = \sum_{j=0}^r \alpha_{\chi,j} F_j$ with $\alpha_{\chi,j} \in \mathbb{k}$.

5.3. **A first relation.** It is well known that any central primitive idempotent can be written as:

$$E_i = \frac{\chi_i(1)}{\dim_{\mathbb{K}}(A)}(\dim_{\mathbb{K}}(A)\Lambda \leftarrow \chi_{i^*}) = \frac{\chi_i(1)}{\dim_{\mathbb{K}}(A)}(\sum_{j=0}^r \mathbf{C}_j \leftarrow \chi_{i^*}).$$

Note that since $\chi_{i^*} = \sum_{j=0}^r \alpha_{i^*j} F_j$ one has that $\mathbf{C}_j \leftarrow \chi_{i^*} = \dim_{\mathbb{R}}(A) \Lambda \leftarrow F_j \chi_{i^*} = \alpha_{i^*j} \mathbf{C}_j$. Thus one obtains that

$$\Lambda_L = \sum_{\chi_i \in \operatorname{Irr}(\operatorname{Rep}(A//L))} E_i = \sum_{\chi_i \in \operatorname{Irr}(\operatorname{Rep}(A//L))} \frac{\chi_i(1)}{\dim_{\mathbb{K}}(A)} \sum_{j=0}^r \alpha_{i^{\star}j} \mathbf{C}_j =$$

$$= \sum_{j=0}^{r} \left(\sum_{i \in \operatorname{Irr}(\operatorname{Rep}(A//L))} \frac{\chi_{i}(1)}{\dim_{\mathbb{K}}(A)} \alpha_{i \star j} \right) \mathbf{C}_{j}$$

Comparing the two equations for Λ_L one has that

$$\sum_{\chi_i \in \operatorname{Irr}(\operatorname{Rep}(A//L))} \chi_i(1) \alpha_{i^* j} = \frac{\dim_{\Bbbk}(A)}{\dim_{\Bbbk}(L)} \text{ if } j \in \mathcal{J}_L$$

and

$$\sum_{\chi_i \in \operatorname{Irr}(\operatorname{Rep}(A//L))} \chi_i(1) \alpha_{i^*j} = 0 \text{ if } j \notin \mathcal{J}_L.$$

Proposition 5.9. If A is a semisimple Hopf algebra with commutative character ring and $\chi \in C(A)$. Then one has $\chi \in C(A//L) \iff \chi F_i = \chi(1)F_i$ for all $j \in \mathcal{J}_L$.

Proof. We may suppose that $\chi = \chi_M$ is the character of an A-module M. If $\chi \in C(A//L)$ then $\chi \lambda_L = \chi(1)\lambda_L$ and Lemma 5.6 implies that $\chi F_j = \chi(1)F_j$, for any $j \in \mathcal{J}_L$. Conversely if $\chi F_j = \chi(1)F_j$ for all $j \in \mathcal{J}_L$ then $\chi \lambda_L = \lambda_L \chi(1)$ and,

$$\chi(\Lambda_L) = \frac{\dim_{\mathbb{k}}(A)}{\dim_{\mathbb{k}}(L)}\chi(\lambda_L \rightharpoonup \Lambda) = \frac{\dim_{\mathbb{k}}(A)}{\dim_{\mathbb{k}}(L)}\chi\lambda_L(\Lambda) = \frac{\dim_{\mathbb{k}}(A)}{\dim_{\mathbb{k}}(L)}\chi(1)\lambda_L(\Lambda).$$

On the other hand

$$\lambda_L(\Lambda) = \frac{\dim_{\mathbb{k}}(L)}{\dim_{\mathbb{k}}(A)} \sum_{\alpha \in \operatorname{Irr}(A//L)} \alpha(1)\alpha(\Lambda) = \frac{\dim_{\mathbb{k}}(L)}{\dim_{\mathbb{k}}(A)}$$

Thus $\chi(\Lambda_L) = \chi(1)$ which shows that the restriction of M to L is trivial. It follows that $M \in \text{Rep}(A//L)$.

Remark 5.10. Suppose as above that $L = \bigoplus_{j \in \mathcal{I}_L} C^j$. Then $S(L) = \bigoplus_{j \in \mathcal{I}_L} S(C^j)$ and $S(C^j)$ are conjugacy classes for the other left adjoint action, $a.l = a_2 l S(a_1)$.

5.4. A generalization of the second main result to quasitriangular semisimple Hopf algebras. Suppose that (A, R) is a semisimple quasitriangular Hopf algebra. Define as above $s_{ij} := \operatorname{tr}_{V_i \otimes V_j}(Q) = (\chi_i \otimes \chi_j)(Q) = \langle \chi_j, \phi_R(\chi_i) \rangle$. It follows from [15] that one has $|s_{ij}| \leq \chi_i(1)\chi_j(1)$ and V_i, V_j centralize each other if and only if $s_{ij} = |s_{ij}| \leq \chi_i(1)\chi_j(1)$

 $\chi_i(1)\chi_j(1)$. In this case we write $V_i \perp V_j$ or $\chi_i \perp \chi_j$ at the level of characters.

On the other hand $\phi_R: C(A) \to \mathcal{Z}(A)$ is an algebra map and we may suppose that

$$\phi_R(F_j) = \sum_{s \in \mathcal{A}_j} E_s.$$

Recall that E_i 's are the central primitive idempotents of A and F_j 's are the (central) primitive idempotents of C(A). Without loss of generality we may suppose that $F_0 = t$, the idempotent integral of A^* . Then $\phi_R(F_0)$ is an integral for \mathbf{K}_A since $\phi_R(f)\phi_R(F_0) = \phi_R(fF_0) = f(1)\phi_R(F_0) = \epsilon(\phi_R(f))\phi_R(F_0)$ for any $f \in A^*$.

Note that the set \mathcal{A}_j is empty if and only if $\phi_R(F_j) = 0$. Denote by $J \subseteq \{0, 1, ..., r\}$ the set of all indices j with \mathcal{A}_j not a empty set. Since $\phi_R(1) = 1$ we obtain in this way a partition for the set of indices of all irreducible representations $\{0, 1, ..., r\} = \bigsqcup_{j \in \mathcal{J}} \mathcal{A}_j$.

For any index $0 \le l \le r$ we denoted by j_l the unique index $j \in J$ such that $l \in \mathcal{A}_j$.

Lemma 5.11. Let (A, R) be a quasitriangular Hopf algebra and V_i, V_m be two irreducible A-representations. Then, with the above notations the following assertions are equivalent:

- (1) V_i and V_m centralize each other in Rep(A).
- (2) $\chi_m F_{j_i} = \chi_m(1) F_{j_i}$.
- (3) $\chi_i F_{j_m} = \chi_i(1) F_{j_m}$.
- (4) $C^{j_i} \subseteq \mathrm{LKer}_A(V_m)$.
- (5) $C^{j_m} \subseteq LKer_A(V_i)$.

Proof. For any character $\chi \in C(A)$ write $\chi = \sum_{j=0}^{r} \alpha_{\chi,j} F_j$. Then one has that $\phi_R(\chi) = \sum_{j=0}^{r} \alpha_{\chi,j} \phi_R(F_j) = \sum_{j=0}^{r} \alpha_{\chi,j} (\sum_{s \in \mathcal{A}_j} E_s)$. With these formulae note that

(5.12)
$$s_{im} = \langle \chi_i, \phi_R(\chi_m) \rangle = \langle \chi_i, \sum_{j=0}^r \sum_{s \in \mathcal{A}_l} \alpha_{\chi_m, j} E_s \rangle = \chi_i(1) \alpha_{\chi_m, j_i}$$

where j_i as above, is the unique index $j \in J$ with $i \in A_j$. Therefore we see that V_i centralise V_m if and only if:

$$\alpha_{\chi_m,j_i} = \chi_m(1) \iff \chi_m F_{j_i} = \chi_m(1) \iff \mathcal{C}^{j_i} \subseteq \mathrm{LKer}_A(V_m).$$

The last equivalence follows from [12, Theorem 3.6]. The rest of the equivalences follow from the symmetry property of the centralizer. \Box

Remark 5.13. The above lemma also shows that if V_m centralizes V_i then V_m centralize all V_r with $r \in A_{j_i}$.

Proposition 5.14. Suppose that (A, R) is a semisimple quasitriangular Hopf algebra and L is a left normal coideal subalgebra of A. With the above notations one has

$$\operatorname{Rep}(A//L)' = \{ \chi_m \mid \mathcal{C}^{j_m} \subseteq L \}$$

Proof. We have the following equalities:

$$\operatorname{Rep}(A//L)' = \bigcap_{\chi_i \in \operatorname{Rep}(A//L)} \langle \chi_i \rangle' = \bigcap_{\chi_i \in \operatorname{Rep}(A//L)} \{ \chi_m \mid \mathcal{C}^{j_m} \subseteq \operatorname{LKer}_A(\chi_i) \} =$$

$$= \{ \chi_m \mid \mathcal{C}^{j_m} \subseteq \bigcap_{\chi_i \in \operatorname{Rep}(A//L)} \operatorname{LKer}_A(\chi_i) \} = \{ \chi_m \mid \mathcal{C}^{j_m} \subseteq L \}$$

5.5. **Description of K**_A. Suppose that $\phi_R(t) = \sum_{s \in A_0} E_s$, where t is an idempotent integral of A^* . Then it follows that

$$\operatorname{Rep}(A)' = \operatorname{Rep}(A//\mathbb{k})' = \{\chi_m \mid \mathcal{C}^{j_m} \subseteq \mathbb{k}\} = \{\chi_m \mid \mathcal{C}^{j_m} = \mathbb{k}\} = \{\chi_m \mid m \in \mathcal{A}_0\}$$

On the other hand note that

$$\operatorname{Rep}(A) = \operatorname{Rep}(A//\mathbf{K}_A)' = \{\chi_m \mid \mathcal{C}^{j_m} \subseteq \mathbf{K}_A\}$$

which implies that

(5.15)
$$\mathbf{K}_A = \bigoplus_{j \in J} \mathcal{C}^j.$$

Putting together the two descriptions of the centralizer of a fusion subcategory one obtains the following corollary:

Corolarry 5.16. Let (A, R) be a quasitriangular semisimple Hopf algebra and L be a left normal coideal subalgebra of A. With the above notations, if V_m is an irreducible representation of A we have that the following assertions are equivalent:

- (1) $V_m \in \mathcal{O}(\operatorname{Rep}(A//L)')$.
- (2) $\mathcal{C}^{j_m} \subseteq L$
- (3) $\phi_R((A//L)^*) \subseteq LKer_A(V_m)$

Proof. The definition of the left kernel of an A-module is recalled in subsection 2.3. Since A is quasitriangular one has that $\operatorname{Rep}(A//L)' = \operatorname{Rep}(A//L^*)$. Then note that the first main result can be written as $\operatorname{Irr}(\operatorname{Rep}(A//L)') = \{\chi_j \mid \operatorname{LKer}_A(V_j) \supseteq L^*\}$. Thus for any $j \ge 0$ one has the equivalences:

$$V_m \in \mathcal{O}(\operatorname{Rep}(A//L)') \iff \mathcal{C}^{j_m} \subseteq L \iff \operatorname{LKer}_A(V_i) \supseteq L^*.$$

5.6. The factorizable case. If A is factorizable then each set A_j contains a single element since $\phi_R : C(A) \to \mathcal{Z}(A)$ is a bijection. Without loss of generality, after a permutation of the indices, we may suppose that $A_j = \{j\}$ for any index $0 \le j \le r$.

Proposition 5.17. Suppose that (A, R) is a semisimple factorizable Hopf algebra. Then for any irreducible A-module V_i one has that:

$$\mathrm{LKer}_A(V_i) = \bigoplus_{\{j \mid \chi_i \perp \chi_j\}} \mathcal{C}^j$$

Proof. Since $J=\{0,1,\cdots,r\}$ in this case, by [12, Theorem 3.6] one has that

$$\chi_i F_j = \chi_i(1) F_j \iff \chi_i \perp \chi_j \iff \mathcal{C}^j \subseteq \mathrm{LKer}_A(V_i).$$

It follows that in this case one has $\mathrm{LKer}_A(V_i) = \bigoplus_{\{m \mid \chi_i \perp \chi_m\}} \mathcal{C}^{j_m}$.

5.7. **Proof of Theorem 1.2.** It follows from the more general Proposition 5.14 and Lemma 5.6.

Remark 5.18. Note that using Lemma 5.6 our second main result now generalizes [8, Theorem 1.4] from normal Hopf subalgebras L to left normal coideal subalgebras L of A. More precisely we have shown that:

$$\operatorname{Rep}(A//L)' = \{\chi_j \mid F_j(\Lambda_L) \neq 0\} = \{\chi_j \mid j \in \mathcal{I}_L\}$$

for any left normal coideal subalgebra L of A.

Define $C_{V_i} := C_{\chi_i} \subset A^*$ as the subcoalgebra of A^* generated by χ_j .

5.8. On the dimension of conjugacy classes.

Proposition 5.19. Suppose (A, R) is a semisimple quasitriangular Hopf algebra and let L be a left normal coideal subalgebra of A. With the above notations if $C^j \subseteq L$ with $j \in \mathcal{J}$ then

$$\dim_{\mathbb{k}}(V_m) | \frac{\dim_{\mathbb{k}}(A)}{\dim_{\mathbb{k}}(L^*)} = \dim_{\mathbb{k}}(L) \frac{\dim_{\mathbb{k}}(A)}{\dim_{\mathbb{k}}(\mathbf{K}_A L)} = \frac{\dim_{\mathbb{k}}(A)}{\dim_{\mathbb{k}}(\mathbf{K}_A \cap L)}$$

for any irreducible A-representation V_m such that $j_m = j$.

Proof. By the second criterion, one has that in this situation any V_m with $j_m = j$ is an irreducible representation of $A//L^*$. Since this is a quasitriangular Hopf algebra it follows that $\dim_{\mathbb{R}}(V_m)|\dim_{\mathbb{R}}(A//L^*)$. The other equalities follow from the Equation (3.13) which gives the dimension of the centralizer of a fusion subcategory.

Corolarry 5.20. Suppose that A is a factorizable Hopf algebra. Let L be a left normal coideal subalgebra of A and suppose that $C^j \subseteq L$. Then $\dim_{\mathbb{K}}(V_j)|\dim_{\mathbb{K}}(L)$.

Proof. As above one has that V_j is an irreducible representation of $A//L^*$. Since this is a semisimple quasitriangular Hopf algebra it follows that $\dim_{\mathbb{k}}(V_j)$ divides $\dim_{\mathbb{k}}(A//L^*) = \dim_{\mathbb{k}}(L)$.

Corolarry 5.21. With the above notations, if moreover A//L is a factorizable Hopf algebra then $\dim_{\mathbb{K}}(V_i)^2 | \dim_{\mathbb{K}}(L)$.

6. Proof of the Hopf algebra factorization

In this section we will prove the factorization result from Theorem 1.3. We will show first that in the case of factorizable Hopf algebras the map ϕ_R sends normal Hopf subalgebras to normal Hopf subalgebras.

Proposition 6.1. Let (A, R) be a factorizable semisimple Hopf algebra and $\chi = \chi_M$ be the character of an A-module M. Let $E_M := \sum_{\{\chi_i \mid (\chi_i, \chi) > 0\}} E_i$ be its associated central idempotent in A. If C_{χ} denotes the subcoalgebra of A^* then $\phi_R(C_{\chi}) = \Lambda \leftarrow \phi_R^{-1}(E_M)A^*$.

Proof. One has by [12, Lemma 4.2.i)] that $\phi_R(C_{\chi_j}) = \mathcal{C}^j = \Lambda \leftarrow F_j A^*$. Note that $C_{\chi} = \bigoplus_{\{\chi_i \mid (\chi_i, \chi) > 0\}} C_{\chi_i}$. Then the result follows since " \leftarrow " is an isomorphism of vector spaces.

Remark 6.2. One can write a similar result for the other Drinfeld map $_R\phi$. Since $S[_R\phi(C_{\chi_j})] = \phi_R(C_{\chi_{j^*}})$ it follows that $_R\phi(C_{\chi_j}) = S^{-1}(\Lambda \leftarrow F_{j^*}A^*) = F_jA^* \rightharpoonup \Lambda$ since A is semisimple and $S^2 = \mathrm{id}$.

Theorem 6.3. Let L be a left normal coideal subalgebra of a factorizable semisimple Hopf algebra A. Then $\mathcal{I}_{L^*} = \{j \mid \chi_j \in \operatorname{Irr}(A//L)\},$ i.e.

$$L^{\star} = \bigoplus_{\{j \mid \chi_j \in Irr(A//L)\}} \mathcal{C}^j$$

where by λ_L we denote an idempotent integral in $(A//L)^*$.

Proof. One has that $(A//L)^* = \bigoplus_{\chi_j \in Irr(A//L)} C_{\chi_j}$. Then by the previous proposition one can write

$$L^* = \phi_R((A//L)^*) = \bigoplus_{\chi_j \in \operatorname{Irr}(A//L)} \phi_R(C_j) = \bigoplus_{\chi_j \in \operatorname{Irr}(A//L)} C_j.$$

This shows the first equality above which also implies that $\mathcal{I}_{L^*} = \{j \mid \chi_j \in \operatorname{Irr}(A//L)\}.$

Remark 6.4. Let A be a semisimple factorizable Hopf algebra and L be a left normal coideal subalgebra of A. Since $L^{\star\star} = L$ we also have

 $\mathcal{I}_L = \{j \mid \chi_j \in \operatorname{Irr}(\operatorname{Rep}(A//L^*))\}.$ Therefore we can write

$$\Lambda_{L^{\star}} = \frac{1}{\dim_{\mathbb{k}}(L^{\star})} \sum_{j \in \mathcal{I}_{L^{\star}}} \mathbf{C}_{j} = \sum_{j \in \mathcal{I}_{L}} E_{j},$$

and

$$\Lambda_L = \frac{1}{\dim_{\mathbb{K}}(L)} \sum_{j \in \mathcal{I}_L} \mathbf{C}_j = \sum_{j \in \mathcal{I}_{L^*}} E_j.$$

On the other hand we also have:

$$\lambda_L = \sum_{j \in \mathcal{I}_L} F_j = \frac{\dim_{\mathbb{k}}(L)}{\dim_{\mathbb{k}}(A)} \sum_{j \in \mathcal{I}_{L^*}} \chi_j(1) \chi_j,$$

and

$$\lambda_{L^{\star}} = \sum_{j \in \mathcal{I}_{L^{\star}}} F_j = \frac{\dim_{\Bbbk}(L^{\star})}{\dim_{\Bbbk}(A)} \sum_{j \in \mathcal{I}_L} \chi_j(1) \chi_j.$$

Remark 6.5. As already mentioned, one can also consider the second Drinfeld map $_R\phi: A^* \to A$. By Equation (3.4) one has that $S_R\phi = \phi_R s$ where S and s are the antipodes of A and A^* respectively. This shows that $_R\phi((A//L)^*) = S^{-1}(\phi_R((A//L)^*)) = S^{-1}(L^*)$. Thus $_R\phi$ sends subcoalgebras of A^* into right coideal subalgebras.

Theorem 6.6. Let A be a semisimple factorizable Hopf algebra. If L is a normal Hopf subalgebra of A then L^* is also a normal Hopf subalgebra of A.

Proof. Using Remark 6.5 one has to show the equality $\phi_R((A//L)^*) = R\phi((A//L)^*)$. On the other hand by Theorem 6.3 and the remark right before it one has that $\phi_R((A//L)^*) = \Lambda \leftarrow \lambda_L A^*$ and $R\phi((A//L)^*) = A^*\lambda_L \rightharpoonup \Lambda$. Since Λ is cocommutative and λ_L is a central element (see [29]) the result follows.

Remark 6.7. In particular our result implies that $\mathbf{K}_A := \phi_R(A^*)$ is a normal Hopf algebra in the case of a semisimple Hopf algebra A. Note that in [32] it was shown that in general \mathbf{K}_A is a left normal coideal subalgebra.

Corolarry 6.8. If A is a semisimple quasitriangular Hopf algebra and simple then A is either factorizable of or triangular.

Proof. Note that in this case either $\mathbf{K}_A = A^*$ and A is factorizable or $\mathbf{K}_A = \mathbb{k}$ and A is triangular.

Remark 6.9. Moreover, using Theorem 1.1 the last result shows that the centralizer of any normal fusion subcategory of Rep(A) is also a normal fusion subcategory, answering positively a question from the end of the paper [8].

Lemma 6.10. Suppose that (A, R) is a semisimple quasitriangular Hopf algebra and L a left normal coideal subalgebra of A. Then with the above notations one has that: $\phi_R(\lambda_L) = \Lambda_{L^*}$.

Proof. Since λ_L is a character and ϕ_R is multiplicative when one variable is a character, one has that $\phi_R((\Lambda_L \rightharpoonup f)\lambda_L) = \phi_R(\Lambda_L \rightharpoonup f)\phi_R(\lambda_L)$. Thus one has to show that $\phi_R(\Lambda_L \rightharpoonup f)\phi_R(\lambda_L) = \epsilon(\phi_R(\Lambda_L \rightharpoonup f))\phi_R(\lambda_L)$ This follows since λ_L is an integral in $(A//L)^*$ and $\epsilon(\phi_R(f)) = f(1)$. \square

6.1. **Description of an intersection.** Let L be a left normal coideal subalgebra of a semisimple factorizable Hopf algebra A. Denote $B := (A//L)^*$ and $B' := (A//L^*)^*$. We have the following result concerning the description of the intersection of the two Hopf subalgebras B, B' of A^* , compare to [32, Theorem 4.8].

Theorem 6.11. Suppose that (A, R) is a factorizable Hopf algebra and L is a left normal coideal subalgebra of A. Then

$$\phi_R(B \cap B') = L \cap L^*$$

Proof. Let $M := \phi_R(B \cap B')$. From the first main result above we know that

$$\operatorname{Rep}((B \cap B')^*)' = \operatorname{Rep}(A//M)$$

On the other hand we know from Equation (2.4) that $B \cap B' = (A//L)^* \cap (A//L^*)^* = (A//LL^*)^*$. Thus $\text{Rep}(A/M) = \text{Rep}(A//LL^*)'$ which gives that

$$\operatorname{Rep}(A/M) = \operatorname{Rep}(A//LL^{\star})' = (\operatorname{Rep}(A//L) \cap \operatorname{Rep}(A//L^{\star}))' =$$

 $= \operatorname{Rep}(A//L)' \vee \operatorname{Rep}(A//L^{\star})' = \operatorname{Rep}(A//L \cap L^{\star}).$

Thus $M = L \cap L^*$ as desired.

6.2. Proof of the factorization result.

Proof. Since $\operatorname{Rep}(A//K)$ is a nondegenerate fusion category it follows that $\operatorname{Rep}(A//K) \cap \operatorname{Rep}(A//L) = \operatorname{Vec}$ and $\operatorname{Rep}(A//K) \vee \operatorname{Rep}(A//L) \simeq \operatorname{Rep}(A)$. This gives by the previous formulae that KL = A and $K \cap L = \mathbb{k}$. Thus $A \simeq K \otimes L$ as Hopf algebras by the factorization result mentioned in Subsection 2.2.

Theorem 6.13. Let A be a factorizable semisimple Hopf algebra and K be a normal Hopf subalgebra of A such that Rep(A//K) is a nondegenerate fusion category. Then the two normal Hopf subalgebras L and K from Theorem 1.3 are also factorizable Hopf algebras.

Proof. Since $A \simeq K \otimes L$ as Hopf algebras we have canonical Hopf projections $\pi_K : A \to K$ and $\pi_L : A \to L$. They induces Hopf algebra isomorphisms $A//L \xrightarrow{\pi_L} K$ and $A//L \xrightarrow{\pi_L} K$.

We construct the R-matrices $R_K := (\pi_K \otimes \pi_K)(R)$ and $R_L := (\pi_L \otimes \pi_L)(R)$ Then we consider the composition $L^* \xrightarrow{\pi_L^*} (A//K)^* \xrightarrow{\phi_R|_{(A//K)^*}} L$. Clearly this map is bijective. We will show that this map coincides with the Drinfeld's map $\phi_R^L = f_{Q_L} : L^* \to L$, $f \mapsto f(Q_L^1)Q_L^2$ where $Q_L := R_L^{21}R_L$. Indeed if $g \in L^*$ then clearly $(\phi_R| \circ \pi_L^*)(g) = g(\pi_L(R^2r^1))R^1r^2 = (g \otimes \mathrm{id})(R_L^{21}R_L)$. On the other hand note that $\phi_R((A//K)^*) = L \subset A$ via $x \hookrightarrow 1_K \otimes x$. Thus $\pi_L(\phi_R(f)) = \phi_R(f)$ for any $f \in (A//K)^*$. This shows that $\phi_R| \circ \pi_L^* = f_{Q_L}$ and thus L is factorizable. The proof for K factorizable is similar.

Remark 6.14. One can see that A//K and A//L are factorizable Hopf algebras with the quotient R-matrices since their associated fusion subcategories are nondegenerate. Then the Hopf isomorphism π_L , and π_K from above give also that the Hopf subalgebras K and L are

also factorizable Hopf algebras with the R-matrices transposed by these isomorphism.

6.3. On the commutativity for normal Hopf subalgebras. Let (A, R) be a finite dimensional quasitriangular Hopf algebra. Recall that there are two Hopf algebra maps $f_R: A^{*cop} \to A$ and $f_{R_{21}}: A^* \to A^{op}$ defined by: $p \mapsto p(R^1)R^2$ and $p \mapsto p(R^2)R^1$ for all $p \in A^*$.

For a Hopf algebra map $\pi: A \to B$ define as usually: $A^{co \pi} := \{a \in A \mid a_1 \otimes \pi(a_2) = a \otimes 1\}$ and $= {}^{co \pi}A := \{a \in A \mid a_2 \otimes \pi(a_1) = a \otimes 1\}.$

Next Proposition is basically contained in [32, Proposition 3.10].

Proposition 6.15. Let $q: A \to B$ be a surjective Hopf algebra map and suppose that (A, R) is quasitriangular. Then the following assertions are equivalent:

- (1) $q: A \to B$ is a normal map.
- (2) $f_{R_{21}}(B^*) \subseteq (A^{co \pi})'$
- (3) $f_R(B^*) \subset ({}^{co \pi}A)'$

where for any subalgebra $S \subseteq A$ we denote by S' the centralizer of S in A.

Proof. The equivalence between i) and ii) is contained [32, Proposition 3.10]. The equivalence between i) and iii) follows from the previous equivalence using the fact that (A^{cop}, R_{21}) is a also quasitriangular, see [23, Lemma 7.2.3].

Proposition 6.16. Suppose that (A, R) is a semisimple quasitriangular Hopf algebra and L is a normal Hopf subalgebra of A. Then ml = lm for any $l \in L$ and $m \in L^*$.

Proof. Since $q:A\to A//L$ is normal one can apply the previous proposition. First note that $\phi_R=f_{R_{21}}\star f_R$. Suppose that $m=\phi_R(g)$ with $g\in (A//L)^*$. Then $m=f_{R_{21}}(g_1)f_R(g_2)$ and note that by Proposition 6.15 both terms of the above product commute with l. Thus ml=lm.

Corolarry 6.17. Suppose that (A, R) is a semisimple factorizable Hopf algebra and L is a normal Hopf subalgebra of A. Then

- (1) The image $L^* := \phi_R((A//L)^*)$ is also a normal Hopf subalgebra of A.
- (2) One has that ml = lm for any $l \in L$ and $m \in L^*$.
- (3) One has that pq = qp for all $p \in (A//L)^*$ and any $q \in (A//L^*)^*$.

Proof. The last item follows by applying ϕ_R^{-1} to the equality ml = lm.

7. Example: the Drinfeld double $D(\Bbbk G)$ of a group

In this section we consider the case $A = D(\Bbbk G)$, the Drinfeld double of some finite group G. One has the following formula for the R-matrix of A: $R = \sum_{g \in G} (\epsilon \bowtie g) \otimes (p_g \bowtie 1)$. Thus $Q = \sum_{g,h \in G} (p_g \bowtie h) \otimes gp_h = \sum_{g,h \in G} (p_g \bowtie h) \otimes (p_{ghg^{-1}} \bowtie g)$. After the identification $D(\Bbbk G)^* \simeq \Bbbk^G \otimes \Bbbk G^{op}$ as algebras, the Drinfeld map is given by (7.1)

$$_R\phi:D(G)^*\to D(G),\ x\otimes p_q\mapsto (x\otimes p_q)(Q^1)Q^2=p_{xqx^{-1}}\bowtie x=xp_q.$$

7.1. Left normal coideal subalgebras of $D(\Bbbk G)$. Let H and L be two normal subgroups of G such that they commute point-wise, i.e [H, L] = 1. Let also $\lambda : M \times H \to k^*$ be a bicharacter, i.e. a function satisfying the following two properties

(7.2)
$$\lambda(mm', h) = \lambda(m, h)\lambda(m', h)$$

(7.3)
$$\lambda(m, hh') = \lambda(m, h)\lambda(m, h')$$

for all $m, m' \in M$ and $h, h' \in H$. Following [7] we define the following subspace of $D(\Bbbk G)$

$$C(M,H,\lambda) := \bigoplus_{h \in H} C_{\lambda}^{M}(h) \bowtie h$$

where $C_{\lambda}^{M}(h) = \{ f \in k^{G} \mid f(mx) = \lambda(m,h)f(x) \text{ for all } m \in M, x \in G \}$. Note that that $C_{\lambda}^{M}(h)$ is a right coideal subalgebra of k^{G} .

Theorem 7.4. ([7]) Let G be a finite group. Then all left coideal subalgebras of $D(\Bbbk G)$ are of the type $C(M, H, \lambda)$ as above. Moreover $C(M, H, \lambda)$ is a left normal coideal subalgebra if and only if the bicharacter λ is G-invariant, i.e.

(7.5)
$$\lambda(x^{-1}mx, h) = \lambda(m, xhx^{-1})$$

for all $x \in G$.

A basis for $C(M, H, \lambda)$ is given by $f_s^{M,h} := \sum_{m \in M} \lambda(m, h) p_{ms}$ where s runs over a set of coset representatives for $H \subset G$. When M is implicitly understood we will shortly write f_s^h for $f_s^{M,h}$. Note also that f_{ms}^h is a scalar of f_s^h for any $m \in M$. From this we deduce that: $\dim_k C(M, H, \lambda) = |H||G:M|$.

7.2. The irreducible $D(\Bbbk G)$ -representations. The set of irreducible D(G)-representations are parameterized by pairs (a,χ) where a runs through a set of representative group elements of the conjugacy classes of G and χ is an irreducible character of $C_G(a)$. The associated simple $D(\Bbbk G)$ -representation to (a,χ) is given by $\Bbbk G \otimes_{\Bbbk C_G(a)} M_{\chi}$. The action of $\Bbbk G$ is just the usual left action. The action of k is given by

$$p_x(g \otimes_{C_G(a)} m) = \delta_{x,gag^{-1}}(g \otimes_{C_G(a)} m).$$

It is well known that $\operatorname{Rep}(G) \subseteq \operatorname{Rep}(D(\Bbbk G))$ and the irreducible representations coming from a=1 are in fact the simple objects $M \in \operatorname{Rep}(G)$ with $p_x.m = \delta_{x,1}m$. for all $x \in G$ and $m \in M$.

7.3. The parameterization from [31]. Fusion subcategories of the category Rep(D(G)) are parameterized by triples (M, H, B) as in [31]. Here M and H are normal subgroups of G that commutes element-wise and $B: H \times K \to \mathbb{k}^*$ is a bicharacter.

The category $\mathcal{S}(M, H, \lambda)$ associated to such a triple consists of the simple objects $S_{(a,\chi)}$ with the property that $a \in M$ and $\lambda(a,h) = \frac{\chi(h)}{\chi(1)}$ for all $h \in H$. It was shown in [31] that $\operatorname{FPdim}(\mathcal{S}(M, H, \lambda)) = |M|[G:H]$.

Conversely given $\mathcal{D} \subseteq \operatorname{Rep}(D(\Bbbk G))$ one obtains $H_{\mathcal{D}}$ by intersection with $\operatorname{Rep}(G)$, i.e. $\operatorname{Rep}(G) \cap \mathcal{D} = \operatorname{Rep}(G/H_{\mathcal{D}})$. Moreover $K_{\mathcal{D}}$ can be obtained as the restriction to $\mathbb{k}^{G cop}$ of all objects of \mathcal{D} , i.e. $K_{\mathcal{D}} = \langle gag^{-1} \mid (a, \chi) \in \mathcal{D} \text{ for some } \chi \in \operatorname{Irr}(C_G(a)) \rangle$. Moreover the bicharacter $B_{\mathcal{D}} : K_{\mathcal{D}} \times H_{\mathcal{D}} \to \mathbb{k}^*$ is given $B_{\mathcal{D}}(g^{-1}ag, h) = \frac{\chi(ghg^{-1})}{\chi(1)}$.

Proposition 7.6. With the above notations one has that:

$$\operatorname{Rep}(D(G)//C(M, H, \lambda)) = \mathcal{S}(M, H, \lambda^{-1})$$

Proof. Let $\mathcal{D} := \text{Rep}(D(G)//C(M, H, \lambda))$. One defines $H_{\mathcal{D}}$ by the formulae

$$\operatorname{Rep}(G/H_{\mathcal{D}}) = \mathcal{D} \cap \operatorname{Rep}(G)$$

and $K_{\mathcal{D}}$ is obtained by the restriction to $k^{G cop}$ of all objects of \mathcal{D} . The bilinear bicharacter $B_{\mathcal{D}}: K_{\mathcal{D}} \times H_{\mathcal{D}} \to \mathbb{k}^*$ is given $B_{\mathcal{D}}(g^{-1}ag, h) := \frac{\chi(ghg^{-1})}{\chi(1)}$. Note that an object $(a, \chi) \in \text{Rep}(D(G)//C(M, H, \lambda))$ if and only if $C(M, H, \lambda)$ acts trivially on (a, χ) . This can be written as $(f_s^h \bowtie h)(g \otimes_{kC_G(a)} v) = f_s^h(1)(g \otimes_{kC_G(a)} v)$ for any $s, g \in G$ and $v \in M_{\chi}$. Thus $(f_s^h \bowtie h)(g \otimes_{kC_G(a)} v) = \delta_{s,M}\lambda(s^{-1},h)(g \otimes_{kC_G(a)} v)$. Note that the left hand side of the above equation can be written as

$$\sum_{m \in M} \lambda(m,h) p_{ms}(hg \otimes_{kC_G(a)} v) = \sum_{m \in M} \delta_{ms, (hg)a(hg)^{-1}} \lambda(m,h) (hg \otimes_{kC_G(a)} v)$$

Equating the two terms one obtains that

$$\sum_{m \in M} \delta_{ms, (hg)a(hg)^{-1}} \lambda(m, h) (hg \otimes_{kC_G(a)} v) = \delta_{s,M} \lambda(s^{-1}, h) (g \otimes_{kC_G(a)} v)$$

If $s \notin M$ this equation implies that $(hg)a(hg)^{-1} \notin Ms$ which can happen if and only if $a \in M$. On the other hand if $s \in M$ then the above equation can be rewritten as

$$\lambda((hg)a(hg)^{-1}s^{-1},h)(hg\otimes_{kC_G(a)}v) = \lambda(s^{-1},h)(g\otimes_{kC_G(a)}v)$$

Using the properties of λ this shows that $\lambda(gag^{-1}, h)(hg \otimes_{kC_G(a)} v) = (g \otimes_{kC_G(a)} v)$ which implies that $\lambda(gag^{-1}, h)(g^{-1}hg)v = v$ for any $g \in G$

and $v \in M_{\chi}$. Thus $\frac{\chi(g^{-1}hg)}{\chi(1)} = \lambda(ga^{-1}g^{-1}, h) = \lambda(a, ghg^{1-})^{-1}$. Using the above reconstruction of a triple from a fusion subcategory of Rep(D(G)) the proof is finished.

Using the formula for ϕ_R from Equation (7.1) a direct computation gives

(7.7)
$$\phi_R((D(G)//C(M, H, \lambda^{\text{op}-1})^*)) = C(H, M, \lambda).$$

Then Theorem 1.1 implies that

(7.8)
$$S(M, H, \lambda)' = S(H, M, \lambda)$$

which recovers the centralizer result from [31, Lemma 3.11].

7.4. On factorizable semisimple Hopf algebras of dimension dq^n . A natural number m is called almost square-free if it is of the form dq^n where d is a square free integer and q a prime number not dividing d. Modular categories of almost square free dimension were recently intensively studied, see e.g [20, 21, 22] and the references therein.

Theorem 7.9. Suppose that A is a factorizable semisimple Hopf algebra of dimension dq^n where d is a square free integer and q an odd prime number not dividing d. Then $A \simeq \Bbbk G \otimes B$ where G is an abelian group of order d and B is a factorizable Hopf algebra of dimension q^n .

Proof. Since A is factorizable it follows by [16] that $\dim_{\mathbb{K}}(V)^2 | \dim_{\mathbb{K}}(A)$ for any irreducible A-representation V. Thus possible dimensions for irreducible A-representations are 1 and q^i with $1 \le i \le n/2$. Moreover if n = 1 then all the dimension are 1 and therefore A is a commutative Hopf algebra. Suppose $n \ge 2$ and write:

$$\dim_{\mathbb{K}}(A) = a_1 1^2 + \sum_{i} a_{q^i}(q^i)^2.$$

where a_s represents the number of non-isomorphic simple representations of dimension s. One dimensional conjugacy classes of A are of the form kg for some central grouplike element $g \in \bar{G}(A)$. Moreover, since A is factorizable, note that for any $0 \le j \le r$ one has that

$$\dim_{\mathbb{k}}(\mathcal{C}^j) = \dim_{\mathbb{k}}(F_j A^*) = \dim_{\mathbb{k}}(\phi_R(F_j A^*)) = \dim_{\mathbb{k}} E_j A^* = \chi_j(1)^2.$$

Thus all conjugacy classes of dimension strictly greater than 1 have dimension q^{2i} for some $1 \le i \le n/2$.

Since by [22, Theorem 4.7] the braided category Rep(A) is nilpotent it follows by results of [14, Theorem 1.1] that one has

$$\operatorname{Rep}(A) \simeq \mathcal{C}_1 \boxtimes \mathcal{C}_2$$

where C_1 is a fusion category of dimension d and C_2 is a fusion subcategory of dimension q^n .

One has that $C_1 = \operatorname{Rep}(A//L_1)$ for a left normal coideal subalgebra L_1 of A with $\dim_{\mathbb{k}}(L_1) = q^n$. It also follows that $C_2 = \operatorname{Rep}(A//L_2)$ for a left normal coideal subalgebra L_2 with $\dim_{\mathbb{k}}(L_2) = d$. By Corollary 5.20 L_2 cannot contain conjugacy classes of dimension q^{2i} since q does not divide d. Therefore L_2 is sum of conjugacy classes generated by central grouplike elements. Thus L_2 is a normal (in fact central) Hopf subalgebra of A and therefore $L_1 = L_2^*$ is also a normal Hopf subalgebra of A. By the factorization result mentioned above one has that $A \simeq L_1 \otimes L_2$ as Hopf algebras. Moreover $L_2 \simeq \mathbb{k}G$ for some abelian group of order d since L_2 is commutative. \square

Remark 7.10. The above theorem shows that in the case q odd prime number, the study of factorizable Hopf algebras of dimension dq^n reduces to the study of factorizable Hopf algebras of prime power dimension q^n .

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