UNIFORM REGULARITY FOR FREE-BOUNDARY NAVIER-STOKES EQUATIONS WITH SURFACE TENSION

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Abstract. We study the zero-viscosity limit of free boundary Navier-Stokes equations with surface tension in $\mathbb{R}^3$ thus extending the work of Masmoudi and Rousset [1] to take surface tension into account. Due to the presence of boundary layers, we are unable to pass to the zero-viscosity limit in the usual Sobolev spaces. Indeed, as viscosity tends to zero, normal derivatives at the boundary should blow-up. To deal with this problem, we solve the free boundary problem in the so-called Sobolev co-normal spaces (after fixing the boundary via a coordinate transformation). We prove estimates which are uniform in the viscosity. And after inviscid limit process, we get the local existence of free-boundary Euler equation with surface tension. One of the main differences between this work and the work [1] is our use of time-derivative estimates and certain properties of the Dirichlet-Neumann operator.

Contents

1. Introduction

The water-wave problem has been studied for several decades from several different points of view. First the local existence for free boundary problem of Navier-Stokes equation without surface tension was shown by Beale [8]. Allain [6] and Tani [7] proved local-existence for the free-boundary Navier-Stokes equation with surface tension in the case of two dimensions and three dimensions, respectively. Moreover, with surface tension, global regularity was also studied by Beale [8].

In the case where the fluid is assumed to be inviscid and irrotational, the problem can be reduced to the boundary. Recently, global regularity was achieved by S. Wu [11] and by Germain, Masmoudi and Shatah [12] for small data. In the general case (where the vorticity may be non-zero) local well-posedness was proven by a number of different authors first by Christodoulou and Lindblad [13] and Lindblad [14], then Coutand and Shkoller [16], Lannes [3], Shatah and Zheng [17], and Masmoudi and Rousset [1].

In this paper we consider the vanishing viscosity limit for the water wave problem when surface tension is taken into account. The inviscid limit problem of the free-surface Navier-Stokes equation *without* surface tension was studied by Masmoudi and Rousset in [1]. When surface tension is not taken into account, the boundary, h, has same regularity as the velocity, u,(say H^m). In the process of doing high order energy estimates, one loses half a derivative due to some commutators. That commutator comes from $D^m\nabla\varphi$, where φ is harmonic extension of h to the interior of the domain, which is $\frac{1}{2}$ more regular than h. The main idea of the paper [1] is to use Alinhac's good unknown which reduces the order of loss via a critical cancellation. A second important component of [1], which sets it apart from the rest of the works on the free-boundary Euler equations, is that the authors are forced to use spaces which only measure co-normal regularity. This is due to the presence of boundary layers which form during the process of sending the viscosity $\varepsilon > 0$ to zero. Indeed, because of the boundary layer, we expect that near the boundary u^ε behaves like

$$u^\varepsilon \sim u(t,x) + \sqrt{\varepsilon}U(t,y,\frac{z}{\sqrt{\varepsilon}}),$$

where u is the solution of the free-boundary Euler equation, U is a some profile, y is 2-d horizontal variable, and z is 1-d vertical variable. So, for high order Sobolev space, we cannot hope to get interval of time independent of ε, which is crucial to get stong compactness of solution sequences. Hence we consider a Sobolev co-normal space, in which we expect to maintain boundedness of the Lipschitz norm as well as boundedness of higher order co-normal derivatives on an interval of time independent of kinematic viscosity $\varepsilon > 0$.

Now let us consider the similar case with surface tension. We will still use Sobolev co-normal spaces like in [1] because boundary layers are still present. However, in this case the boundary is more regular so we will not need Alinhac's good unknown. Our main problem comes from the fact that the pressure term in the Euler equations becomes significantly less regular when surface tension is introduced. We thus encounter commutators with order $m+\frac{3}{2}$, which we cannot control. For this reason, we decided to do energy estimates using space *and* time derivatives. This helps because time derivatives actually count for $\frac{3}{2}$ space derivatives on the boundary (this is deduced by studying the properties of the Dirichlet to Neumann mapping). Using this fact, we can derive local existence for a time interval independent to ε. Finally, we deduce the solution of free-boundary Euler equation (subject to surface tension) as $\varepsilon \to 0$, using a strong compactness argument. Also note that upper index $(\cdot)^b$ means function value on free-surface, $(\cdot)|_{\text{free-surface}}$.

1.1. The Free-boundary Navier-Stokes equations.

We solve the incompressible free-boundary Navier-Stokes equations under the effect of gravity in an unbounded domain. Assume that above the free-surface of the fluid is a vacuum. The system we get is:

$$\partial_t u + u\cdot\nabla u + \nabla p = \varepsilon\Delta u, \quad (x,y,z)\in\Omega_t, \quad t>0, \quad \varepsilon>0, \tag{1}$$

$$\nabla\cdot u = 0, \quad (x,y,z)\in\Omega_t, \tag{2}$$

where Ω_t is free domain, occupied by fluid, and $\varepsilon > 0$ is kinematic viscosity. We write the fluid boundary as h, a function of horizontal variables x and y, so that

$$\Omega_t := \left\{(x, y, z) \in \mathbb{R}^3, \quad z < h(t, x, y)\right\},$$

Our first boundary condition is the moving boundary condition (or called kinematic boundary condition), which roughly says that the boundary moves with the fluid:

$$\partial_t h = u(t, x, y, h(t, x, y)) \cdot \mathbf{N}, \quad (x_1, x_2) \in \mathbb{R}^2, \tag{3}$$

where $\mathbf{N} = (-\partial_1 h, \partial_2 h, 1) := (-\nabla_y h, 1)$ is outward normal direction vector on the boundary. Our second boundary condition is the continuity of stress tensor on the boundary.

$$p^b \mathbf{n} - 2\varepsilon (Su)^b \mathbf{n} = gh\mathbf{n} - \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \mathbf{n}, \quad x \in \partial\Omega_t, \tag{4}$$

where $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$, outward unit normal vector, and Su is the symmetric part of the gradient of u

$$Su := \frac{(\nabla u) + (\nabla u)^T}{2}.$$

Note that g is gravitational constant. In this paper, we consider system (1)-(4) and its vanishing viscosity limit $\varepsilon \to 0$.

1.2. **Parametrization to a Fixed domain.** The first step is to rewrite the problem on the fixed domain $S := \{(x, y, z)|\ z < 0\}$. This can be done by a diffeomorphism $\Phi(t, \cdot)$,

$$\begin{aligned} \Phi(t, \cdot) : S = \mathbb{R}^2 \times (-\infty, 0) \to \Omega_t, \\ (x, y, z) \mapsto (x, y, \varphi(t, x, y, z)) := (x, y, Az + \eta(t, x, y, z)). \end{aligned} \tag{5}$$

We use function v and q to denote the velocity and pressure on the fixed domain S.

$$v(t, x, y, z) = u(t, \Phi(t, x, y, z)), \ q(t, x, y, z) = p(t, \Phi(t, x, y, z)). \tag{6}$$

There are several different choices for Φ and we have to decide which one is optimal for our purposes. We will need to find $\varphi(t, \cdot)$ so that $\Phi(t, \cdot)$ is a diffeomorphism (Surely, $\partial_z \varphi \geq 0$, because it is diffeomorphism). One easy option is to set $\varphi(t, x, y, z) = z + h(t, x, y)$. But it is more useful to take a function Φ which is actually *more* regular than h. If one thinks about using a harmonic extension, we see that it is possible for Φ to be $\frac{1}{2}$ of a derivative more regular than h. We take a smoothing diffeomorphism as was done in [1]:

$$\varphi(t, x, y, z) := Az + \eta(t, x, y, z).$$

To ensure that $\Phi(0, \cdot)$ is a diffeomorphism, A should be chosen so that

$$\partial_z \varphi(0, x, y, z) \geq 1, \quad \forall (x, y, z) \in S,$$

and η is given by the extension of h to the domain S, defined by

$$\hat{\eta}(\xi, z) = \chi(z\xi)\hat{h}(\xi).$$

We want to rewrite equations (1)-(5) on the moving domain Ω_t as equations on the fixed domain S. Using change of variables, we get

$$\begin{aligned} (\partial_i u)(t, x, y, \varphi) &= (\partial_i v - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z v)(t, x, y, z), \quad i = t, 1, 2, \\ (\partial_3 u)(t, x, y, \varphi) &= (\frac{1}{\partial_z \varphi} \partial_z v)(t, x, y, z). \end{aligned}$$

So we define the following operators on fixed domain S :

Definition 1. *We define the following differential operators.*

$$\partial_i^\varphi := \partial_i - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z, \quad i = t, 1, 2, \qquad \partial_3^\varphi := \frac{1}{\partial_z \varphi} \partial_z,$$
$$\nabla^\varphi := (\partial_1^\varphi, \partial_2^\varphi, \partial_3^\varphi), \quad \Delta^\varphi := \partial_{11}^\varphi + \partial_{22}^\varphi + \partial_{33}^\varphi.$$

With these definition, $\partial_i u \circ \Phi = \partial_i^\varphi v, \quad i = t, 1, 2, 3.$ *And each* ∂_i^φ*'s commute each other.*

Hence our equations in S are,

$$\partial_t^\varphi v + v\cdot\nabla^\varphi v + \nabla^\varphi q = \varepsilon\Delta^\varphi v, \quad \text{in} \quad (x,y,z)\in S, \quad \varepsilon>0, \tag{7}$$

$$\nabla^\varphi\cdot v = 0, \quad \text{in} \quad (x,y,z)\in S, \tag{8}$$

$$\partial_t h = v^b\cdot\mathbf{N}, \quad \text{on} \quad \{z=0\}, \tag{9}$$

$$q^b\mathbf{n} - 2\varepsilon(S^\varphi v)^b\mathbf{n} = gh\mathbf{n} - \nabla\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\mathbf{n}, \quad \text{on} \quad \{z=0\}, \tag{10}$$

In this paper, we use extended definition for normal vector $\mathbf{N}$.

Definition 2. *We define the folloiwng extended normal vector in S and on ∂S.*

$$\mathbf{N} := (-\partial_1\eta, -\partial_2\eta, 1) = (-\nabla_{x,y}\eta, 1), \quad \mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|}.$$

On the boundary $\{z=0\}$, $\mathbf{N}$ and $\mathbf{n}$ are just standard outward vectors on the boundary as we defined in (3).

1.3. **Functional Framework and Notations.** We introduce co-normal spaces and some function spaces that are tailored to our problem. From boundary layer behavior and boundary condition (4), we cannot perform normal derivative ∂_3 more than one on the boundary. So we redefine normal derivative so that it is equivalent to standard normal derivatives ∂_3, away from the boundary and vanishes on the boundary. We multiply function $g(z)$ to ∂_3 so that

$$g(0)=0 \quad \text{and} \quad \left|\frac{\mathrm{d}^k}{\mathrm{d}z^k}g(z)\right| \le \text{uniform bound } C_k,$$

for each $k\in\{0\}\cup\mathbb{N}$ and $z\le 0$. One possibility is to pick $g(z)=\frac{z}{1-z}$.

Definition 3. *We define Sobolev co-normal derivatives on S as:*

$$Z_1=\partial_1, \quad Z_2=\partial_2, \quad Z_3=\frac{z}{1-z}\partial_z, \quad Z^\alpha := Z^{(\alpha_1,\alpha_2,\alpha_3)} := Z_1^{\alpha_1}Z_2^{\alpha_2}Z_3^{\alpha_3}. \tag{11}$$

We also use the following symbol:

$$Z^m := \partial_t^k Z_1^{\alpha_1}Z_2^{\alpha_2}Z_3^{\alpha_3}, \quad \textit{for some} \quad k,\alpha_1,\alpha_2,\alpha_3\in\{0\}\cup\mathbb{N}, \quad k+|\alpha| := k+\alpha_1+\alpha_2+\alpha_3 = m. \tag{12}$$

There are many cases of $(\alpha_t,\alpha_1,\alpha_2,\alpha_3)$ for fixed m but we will sum all those cases later, so we do not have to distinguish each case. Using co-normal derivatives, we define Sobolev co-normal spaces, similar as standard Sobolev spaces. Here we use ∂S to denote upper boundary $\{z=0\}$.

$$\begin{aligned}
H^m_{co}(S) &:= \{f\in L^2(S)\,|\,Z_1^{\alpha_1}Z_2^{\alpha_2}Z_3^{\alpha_3}f\in L^2(S), \quad \alpha_1+\alpha_2+\alpha_3\le m\},\\
W^{m,\infty}_{co}(S) &:= \{f\in L^\infty(S)\,|\,Z_1^{\alpha_1}Z_2^{\alpha_2}Z_3^{\alpha_3}f\in L^\infty(S), \quad \alpha_1+\alpha_2+\alpha_3\le m\},\\
H^m_{tan}(S) &:= \{f\in L^2(S)\,|\,Z_1^{\alpha_1}Z_2^{\alpha_2}f\in L^2(S), \quad \alpha_1+\alpha_2\le m\},\\
W^{m,\infty}_{tan}(S) &:= \{f\in L^\infty(S)\,|\,Z_1^{\alpha_1}Z_2^{\alpha_2}f\in L^\infty(S), \quad \alpha_1+\alpha_2\le m\},\\
H^m_{co}(\partial S) &:= \{f\in L^2(\partial S)\,|\,Z_1^{\alpha_1}Z_2^{\alpha_2}f\in L^2(\partial S), \quad \alpha_1+\alpha_2\le m\},\\
W^{m,\infty}_{co}(\partial S) &:= \{f\in L^\infty(\partial S)\,|\,Z_1^{\alpha_1}Z_2^{\alpha_2}f\in L^\infty(\partial S), \quad \alpha_1+\alpha_2\le m\},
\end{aligned}$$

with following norms in each function spaces.

$$
\begin{aligned}
\|f\|^2_{H^m_{co}(S)} &:= \|f\|^2_m := \sum_{|\alpha|\le m} \|Z^\alpha f\|^2_{L^2(S)}, \\
\|f\|_{W^{m,\infty}_{co}(S)} &:= \|f\|_{m,\infty} := \sum_{|\alpha|\le m} \|Z^\alpha f\|_{L^\infty(S)}, \\
\|f\|^2_{H^m_{tan}(S)} &:= \|f\|^2_{m,tan} := \sum_{|\alpha|\le m, \alpha_3=0} \|Z^\alpha f\|^2_{L^2(S)}, \\
\|f\|_{W^{m,\infty}_{tan}(S)} &:= \|f\|_{m,\infty,tan} := \sum_{|\alpha|\le m, \alpha_3=0} \|Z^\alpha f\|_{L^\infty(S)}, \\
|f|^2_{H^m(\partial S)} &:= |f|^2_m := \sum_{|\alpha|\le m, \alpha_3=0} |Z^\alpha f|^2_{L^2(\partial S)}, \\
|f|_{W^{m,\infty}(\partial S)} &:= |f|_{m,\infty} := \sum_{|\alpha|\le m, \alpha_3=0} |Z^\alpha f|_{L^\infty(\partial S)}.
\end{aligned}
$$

Note that for boundary functions, co-normal space and standard Sobolev spaces are identical, since there is no Z_3. When $s = 0$, we write $\|\cdot\| := \|\cdot\|_{L^2}$. Also note that $H^m(S) \hookrightarrow H^m_{co}(S) \hookrightarrow H^m_{tan}(S)$ by property of $\frac{z}{1-z}$ and definition of H^m_{tan}.

Because we will consider time derivatives in energy estimate, it would be convenient to define function spaces containing time derivatives.

Definition 4. *Let v and h are smooth functions defined in S and on ∂S respectively. We define function spaces $X^{m,s}$, $Y^{m,s}$, $\mathcal{H}^{m,s}$, and $\mathcal{K}^{m,s}$.*

$$
\begin{aligned}
X^{m,s}(\partial S) &:= \{f \in L^2(\partial S) \,|\, \partial_t^k Z_1^{\alpha_1} Z_2^{\alpha_2} f \in H^s(\partial S), \quad k+\alpha_1+\alpha_2 \le m\}, \\
X^{m,s}(S) &:= \{f \in L^2(S) \,|\, \partial_t^k Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3} f \in H^s_{co}(S), \quad k+\alpha_1+\alpha_2+\alpha_3 \le m\}, \\
Y^{m,s}(\partial S) &:= \{f \in L^\infty(\partial S) \,|\, \partial_t^k Z_1^{\alpha_1} Z_2^{\alpha_2} f \in W^{s,\infty}(\partial S), \quad k+\alpha_1+\alpha_2 \le m\}, \\
Y^{m,s}(S) &:= \{f \in L^\infty(S) \,|\, \partial_t^k Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3} f \in W^{s,\infty}_{co}(S), \quad k+\alpha_1+\alpha_2+\alpha_3 \le m\}, \\
\mathcal{H}^{m,s}(S) &:= \{f \in L^2(S) \,|\, \partial_t^k Z_1^{\alpha_1} Z_2^{\alpha_2} \partial_3^{\alpha_3} f \in H^s(S), \quad k+\alpha_1+\alpha_2+\alpha_3 \le m\}, \\
\mathcal{K}^{m,s}(S) &:= \{f \in L^\infty(S) \,|\, \partial_t^k Z_1^{\alpha_1} Z_2^{\alpha_2} \partial_3^{\alpha_3} f \in W^{s,\infty}(S), \quad k+\alpha_1+\alpha_2+\alpha_3 \le m\}.
\end{aligned}
$$

Especially, to avoid confusion we use $|\cdot|$ for functions defined on the boundary ∂S and $\|\cdot\|$ for functions defined on the interior S. Using this norm symbols, we define norms of above function spaces $X^{m,s}$, $Y^{m,s}$, $\mathcal{H}^{m,s}$, and $\mathcal{K}^{m,s}$.

$$
\begin{aligned}
|h|^2_{X^{m,s}} &:= \sum_{(k,\alpha), k+|\alpha|\le m} |\partial_t^k D_h^\alpha h|^2_s, \quad \text{where } h \text{ is defined on } \quad \partial S, \\
\|v\|^2_{X^{m,s}} &:= \sum_{(k,\alpha), k+|\alpha|\le m} \|\partial_t^k Z^\alpha v\|^2_s, \quad \text{where } v \text{ is defined on } \quad S, \\
|h|_{Y^{m,s}} &:= \sum_{(k,\alpha), k+|\alpha|\le m} |\partial_t^k D_h^\alpha h|_{s,\infty}, \quad \text{where } h \text{ is defined on } \quad \partial S, \\
\|v\|_{Y^{m,s}} &:= \sum_{(k,\alpha), k+|\alpha|\le m} \|\partial_t^k Z^\alpha v\|_{s,\infty}, \quad \text{where } v \text{ is defined on } \quad S, \\
\|v\|^2_{\mathcal{H}^{m,s}} &:= \sum_{(k,\alpha), k+|\alpha|\le m} \|\partial_t^k \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3} v\|^2_{H^s}, \quad \text{where } v \text{ is defined in } \quad S, \\
\|v\|^2_{\mathcal{K}^{m,s}} &:= \sum_{(k,\alpha), k+|\alpha|\le m} \|\partial_t^k \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3} v\|^2_{W^{s,\infty}}, \quad \text{where } v \text{ is defined in } \quad S,
\end{aligned}
$$

where D_h^α means horizontal derivatives $\partial_1^{\alpha_1}\partial_2^{\alpha_2}$. This definition means $m+s$ order (including both time and space) Sobolev co-normal spaces which contains at least s number of spatial co-normal *derivatives.* $\mathcal{H}^{m,s}$ *is defined similar as* $X^{m,s}$*, except every derivatives are standard derivatives, not co-normal. Note that*

$$\mathcal{H}^{m,s} \hookrightarrow X^{m,s}, \quad \mathcal{K}^{m,s} \hookrightarrow Y^{m,s}, \quad X^{m+s,0} \hookrightarrow X^{m,s} \quad \text{and} \quad Y^{m+s,0} \hookrightarrow Y^{m,s},$$

by definition.

In co-normal spaces, there is no proper notion of half derivatives, so we define the following for convenience.

Definition 5. *We use the following definition for fractional order in Sobolev co-normal derivatives.*

$$[\frac{m}{2}] := \begin{cases} \frac{m}{2} \in \{0\} \cup \mathbb{N}, & m \ \text{ even}, \\ \frac{m-1}{2} \in \{0\} \cup \mathbb{N}, & m \ \text{ odd}. \end{cases}$$

1.4. **Main Result.** We state main results of this paper. The following theorem states the local energy estimate uniform in kinematic viscosity $\varepsilon > 0$.

Theorem 6. *For fixed sufficiently large* $m \geq 6$*, let initial data be given so that*

$$I_m(0) := \|v_0\|_{H^m(S)} + \sum_{k=1}^{m} \varepsilon^k \|v_0\|_{H^{m+k}(S)} + |h_0|_{H^{\frac{3}{2}m+1}(\partial S)} \leq R \tag{13}$$

and satisfy compatibility conditions

$$\Pi\big(S^{\varphi} v^{\varepsilon} \mathbf{n}\big)\big|_{t=0,z=0} = 0, \quad \Pi := I - \mathbf{n}^b \otimes \mathbf{n}^b. \tag{14}$$

Then for $\forall \varepsilon \in (0,1]$*, there exist time independent* $T > 0$ *and some* $C > 0$*, such that there exist a unique solution* $(v^{\varepsilon}, h^{\varepsilon})$ *on* $[0,T]$*, and the following energy estimate hold.*

$$\begin{aligned} \mathcal{N}_m(T) := \sup_{t\in[0,T]} &\Big(\|v(t)\|_{X^{m,0}}^2 + |h(t)|_{X^{m,1}}^2 + \|\partial_z v(t)\|_{X^{m-2,0}}^2 + \|\partial_z v(t)\|_{Y^{[\frac{m}{2}],0}}^2 \Big) \\ &+ \|\partial_z v\|_{L_T^4 X^{m-1,0}}^2 + \varepsilon \int_0^T \Big(\|\nabla v(t)\|_{X^{m,0}}^2 + \|\nabla \partial_z v(t)\|_{X^{m-2,0}}^2 \Big) dt < C. \end{aligned} \tag{15}$$

Using the result of Theorem 6, we send $\varepsilon > 0$ to zero to get a unique solution of free-boundary Euler equations with surface tension, i.e. vanishing viscosity limit.

Theorem 7. *Let us assume that*

$$\lim_{\varepsilon\to 0} \Big(\|v_0^{\varepsilon} - v_0\|_{L^2(S)} + \|h_0^{\varepsilon} - h_0\|_{H^1(\partial S)} \Big) = 0, \tag{16}$$

where $(v_0^{\varepsilon}, h_0^{\varepsilon})$ *and* (v_0, h_0) *satisfy the assumptions of Theorem 6. Then there exist* (v,h) *satisfying*

$$v \in L^{\infty}([0,T], H_{co}^m(S)), \ \partial_z v \in L^{\infty}([0,T], H_{co}^{m-2}(S)), \ h \in L^{\infty}([0,T], H_{co}^{m+1}(\mathbb{R}^2)) \tag{17}$$

and

$$\lim_{\varepsilon\to 0} \sup_{[0,T]} \Big(\|v^{\varepsilon} - v\|_{L^2(S)} + \|v^{\varepsilon} - v\|_{L^{\infty}(S)} + \|h^{\varepsilon} - h\|_{H^1(\mathbb{R}^2)} + \|h^{\varepsilon} - h\|_{W^{1,\infty}(\mathbb{R}^2)} \Big) = 0. \tag{18}$$

Moreover, (v,h) *is the unique solution of free-boundary Euler equation,*

$$\begin{aligned} &\partial_t^{\varphi} v + (v \cdot \nabla^{\varphi}) v + \nabla^{\varphi} q = 0, \quad \text{in} \quad S \\ &\nabla^{\varphi} \cdot v = 0, \quad \text{in} \quad S \\ &\partial_t h = v^b \cdot N, \quad \text{on} \quad \partial S \\ &q = gh - \eta \nabla \cdot \Big(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \Big), \quad \text{on} \quad \partial S. \end{aligned} \tag{19}$$

1.5. **Scheme of the Proof.** We briefly explain main idea of this paper in several steps.

Remark 8. *In this paper* $\Lambda(\cdot,\cdot)$ *denotes an increasing continuous function in all its arguments and* $\Lambda_0 = \Lambda(\frac{1}{c_0})$*. Both may vary from line to line. Also* C *means a constant independent to* ε *and also vary from line to line.*

1.5.1. *Energy estimate of v and h.* Let us apply $Z^m := Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$ to (7). Then we get energy estimate as following.

$$\begin{aligned} E_0(t) &:= \|v\|^2_{H^m_{co}} + |h|^2_{H^{m+1}} + \varepsilon \int_0^t \|\nabla v\|^2_{H^m_{co}} \\ &\leq C_0 + \Lambda(R) \int_0^t \Big(E_0(s) + \|\nabla v\|^2_{H^{m-1}_{co}} + |h|^2_{H^{m+\frac{3}{2}}} \Big) ds, \end{aligned} \tag{20}$$

where C_0 is some terms depending on the initial data, R contains E_0 and L^∞-type terms with order of $[\frac{m}{2}]$. The problem is that $|h|_{H^{m+\frac{3}{2}}}$, on the right hand side, cannot be controlled by E_0. This term comes from the pressure estimates involving the surface tension term. To estimate $|h|_{H^{m+\frac{3}{2}}}$, we use Dirichlet-Neumann estimates, time-derivatives, and a special decomposition of the pressure term. We decompose the pressure q into three pieces, $q = q^E + q^{NS} + q^S$. One piece, q^S (which is the pressure due to the surface tension term) sovles,

$$\Delta q^S = 0, \quad q^S|_{z=0} = -\nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}}. \tag{21}$$

Applying ∂_t on kinematic boundary condition (9), we get

$$h_{tt} = v_t^b \cdot \mathbf{N} + v^b \cdot \mathbf{N}_t.$$

And using Naiver-Stokes (7),

$$h_{tt} = -(\nabla q^S)^b \cdot N + v^b \cdot N_t + (\textit{other terms}).$$

Since $(q^S)^b \sim \Delta h$ by surface tension, we can get $h_{tt} \sim \nabla \Delta h$, so *heuristically*, $\partial_t h \sim \partial_x^{\frac{3}{2}} h$. Showing $|\partial_t h| \lesssim |\partial_x^{3/2} h|$ is easy by direct computation. However, what we need is to claim $|\partial_x^{3/2} h| \lesssim |\partial_t h|$. From Proposition 32, we have a kind of Garding's inequality which gives inequality with inverse direction. We will see that (roughly),

$$|Z^{m-1} h|^2_{L^2 H^{\frac{5}{2}}} \lesssim |Z^{m-1} \partial_t h|^2_{L^2 H^1} + \theta |Z^{m-1} \partial_t h(t)|^2_{L^2} + \sqrt{T} |Z^{m-1} \partial_t \nabla h(t)|^2_{L^2}. \tag{22}$$

Especially we note that above scheme $\partial_t h \sim \partial_x^{3/2} h$ is valid only in the sense of L^2 in time, not pointwise in time. Therefore, $|h|^2_{m+\frac{3}{2}}$ in the RHS of (20) is controlled by above (22). To control right hand sides of (22), we should perform energy estimate of which energy is gained by applying $Z^m := \partial_t Z^\alpha$, $(|\alpha| = m-1)$, $m-1$ spatial conormal derivatives and one time derivative. Then we get

$$\begin{aligned} E_1(t) &:= \|\partial_t v\|^2_{H^{m-1}_{co}} + |\partial_t h|^2_{H^m} + \varepsilon \int_0^t \|\nabla \partial_t v\|^2_{H^{m-1}_{co}} \\ &\leq C_0 + \Lambda(R) \int_0^t \Big(E_1(s) + \|\nabla \partial_t v\|^2_{H^{m-2}_{co}} + |\partial_t h|^2_{H^{m+\frac{1}{2}}} \Big) ds, \end{aligned}$$

where in this case, R contains E_0 and E_1 and L^∞-type terms up to $[\frac{m}{2}]$ order. For sufficiently small θ and time T, right hand side terms in (22) are absorbed or controlled by L^∞ type energy terms in $E_1(t)$. However, similar situation happens on the right hand side. To cotrol $|\partial_t h|_{H^{m+\frac{1}{2}}}$, we should use property of Dirichlet-Neumann operator (Proposition 32) again. Therefore, we need the energy of $E_2(t)$, by one more time derivative and one less space derivatives. For each steps, we should use dirichelt-Neumann operator estimate Proposition 32. This process will be repeated until the last step, $Z^m := \partial_t^{m-1} Z$.

$$\begin{aligned} E_2(t) &:= \|\partial_t^2 v\|^2_{H^{m-2}_{co}} + |\partial_t^2 h|^2_{H^{m-1}} + \varepsilon \int_0^t \|\nabla \partial_t^2 v\|^2_{H^{m-2}_{co}} \\ &\leq C_0 + \Lambda(R) \int_0^t \Big(E_2(s) + \big\|\nabla \partial_t^2 v\big\|^2_{H^{m-3}_{co}} + |\partial_t^2 h|^2_{H^{m-\frac{1}{2}}} \Big) ds, \end{aligned}$$

$$\vdots$$

$$\begin{aligned}
E_k(t) &:= \|\partial_t^k v\|^2_{H^{m-k}_{co}} + |\partial_t^k h|^2_{H^{m-k+1}} + \varepsilon \int_0^t \|\nabla \partial_t^k v\|^2_{H^{m-k}_{co}} \\
&\le C_0 + \Lambda(R) \int_0^t \Big(E_k(s) + \|\nabla \partial_t^k v\|^2_{H^{m-k-1}_{co}} + |\partial_t^k h|^2_{H^{m-k+\frac{3}{2}}} \Big) ds, \quad 1 \le k \le m-1,
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
E_{m-1}(t) &:= \|\partial_t^{m-1} v\|^2_{H^1_{co}} + |\partial_t^{m-1} h|^2_{H^2} + \varepsilon \int_0^t \|\nabla \partial_t^{m-1} v\|^2_{H^1_{co}} \\
&\le C_0 + \Lambda(R) \int_0^t \Big(E_{m-1}(s) + \big\|\nabla \partial_t^{m-1} v\big\|^2_{H^1_{co}} + |\partial_t^{m-1} h|^2_{H^{\frac{5}{2}}} \Big) ds.
\end{aligned}$$

If we sum the above m estimates, then $E_0 + E_1 + \cdots E_{m-2} + E_{m-1}$ controls every high order term of h, except $|\partial_t^{m-1} h|^2_{L^2 H^{\frac{5}{2}}}$ of the $m-1$ step estimate.

1.5.2. *Energy estimate for all-time derivatives.* To close energy estimate we consider $Z^m = \partial_t^m$ case. Then we get the following energy,

$$E^m(t) := \|\partial_t^m v\|^2_{L^2} + |\partial_t^m h|^2_{H^1} + \varepsilon \int_0^t \|\nabla \partial_t^m v\|^2_{L^2}.$$

In this step, we should claim that the commutator $|\partial_t^m h|_{H^{3/2}}$ *does not* appear. (We cannot apply ∂_t^{m+1} to our equation!) Since bad commutator $|\partial_t^m h|_{H^{3/2}}$ comes from pressure estimate, let us investigate pressure term in energy estimate.

When we apply $Z^m = \partial_t^m$, pressure term q^S defined in (21) in the energy estimate looks like

$$\int_0^t \int_S \partial_t^m v \cdot \nabla^\varphi \partial_t^m q^S \sim \int_0^t \int_{\partial S} \partial_t^m u \cdot \mathbf{N} \partial_t^m q - \int_0^t \int_S (\nabla^\varphi \cdot \partial_t^m v) \partial_t^m q^S, \quad \text{by integration by parts.} \tag{23}$$

We use kinematic boundary condition (9) and divergence free condition for v (8) to see

$$\begin{aligned}
&\nabla^\varphi \cdot \partial_t^m v \sim m \partial_t \mathbf{N} \partial_t^{m-1} \partial_z v + \text{low order terms}, \\
&\partial_t^m v \cdot \mathbf{N} \sim m \partial_t \mathbf{N} \partial_t^{m-1} v + \text{low order terms}.
\end{aligned}$$

So (23) becomes,

$$\sim \int_0^t \int_{\partial S} m \partial_t \mathbf{N} \partial_t^{m-1} v \partial_t^m q - \int_0^t \int_S m \partial_t \mathbf{N} \partial_t^{m-1} \partial_z v \partial_t^m q^S, \quad \text{by integration by parts.} \tag{24}$$

Meanwhile, from the equation (21), we cannot derive an estimate for $\|\partial_t^m q\|_{L^2}$ in the second term of the RHS. Instead we can estimate $\|\partial_t^{m-1} \partial_z q\|_{H^{m-1}}$. We perform integration by parts for ∂_z and ∂_t to exchange $\partial_t^m q$ into $\partial_t^{m-1} \partial_z q$.

$$(24) \sim \underbrace{\int_0^t \int_{\partial S} m \partial_t \mathbf{N} \partial_t^{m-1} v \partial_t^m q - \int_0^t \int_{\partial S} m \partial_t \mathbf{N} \partial_t^{m-1} v \partial_t^m q}_{\text{cancellation}} + \underbrace{\int_0^t \int_S m \partial_t \mathbf{N} \partial_t^m v \partial_t^{m-1} \partial_z q}_{(*)}. \tag{25}$$

We have cancellation for the boundary integrals those contain $\partial_t^m q$. Note that pressure part in $(*)$ has been changed into the same form as previous step, i.e. $Z^m = \partial_t Z^\alpha$, $(|\alpha| = m-1)$ case. So applying Dirichlet-Neumann estimate resolve $|\partial_t^{m-1} h|_{\frac{5}{2}}$ issue.

Meanwhile, when we perform integration by parts in time, we should treat some terms evaluated at time 0 and t, without time integration. Since integrand of those term has one lesser time derivatives, we should treat (roughly)

$$\int_S \partial_t^{m-1} v \partial_t^{m-1} \partial_z q \bigg|_t \sim \|\partial_t^{m-1} v(t)\|^2 + |\partial_t^{m-1} h(t)|_{\frac{5}{2}}^2 + \text{(other terms)}, \tag{26}$$

without time integration. We want to apply Dirichlet-Neumann operator estimate to control right hand side, but (22) requires L^2 in time to change $\partial_x^{3/2}$ into ∂_t. Instead of above simple esimate, we use the fact that $\|\partial_t^{m-1} v(t)\|^2$ is not optimal. Instead of above (26), we split as

$$\int_S \partial_t^{m-1} v \partial_t^{m-1} \partial_z q \bigg|_t \sim \|\partial_t^{m-1} |\nabla| v(t)\|^2 + \||\nabla|^{-1} \partial_t^{m-1} \partial_z q(t)\|^2 + \text{(other terms)},$$

where $|\nabla|$ is Fourier muliplier. Then, the first term on the right hand side becomes $\|\partial_t^{m-1} \partial_z v(t)\|$. However, terms with normal derivative is not controlled by conormal Sobolev spaces, and following sections will explain that optimal regularity of $\partial_z v(t)$ is $L_T^\infty X^{m-2,0}$ or $L_T^4 X^{m-1,0}$. Therefore, we cannot close energy estimate with $\|\partial_t^{m-1} |\nabla_y| v(t)\|^2$ without time integration.
Instead, we sharply split above term as

$$\int_S \partial_t^{m-1} v \partial_t^{m-1} \partial_z q \bigg|_t \sim \|\partial_t^{m-1} |\nabla_y| v(t)\|^2 + \||\nabla_y|^{-1} \partial_t^{m-1} \partial_z q(t)\|^2 + \text{(other terms)}, \tag{27}$$

where $|\nabla_y|$ is horizontal Fourier muliplier. The first terms is controlled by $\|v\|_{X^{m-1,1}}^2$, so is absorbed by the conormal energy terms. However, $|\nabla_y|^{-1}$ could be insufficient to reduce order of the second term $\||\nabla_y|^{-1} \partial_t^{m-1} \partial_z q(t)\|^2$ because of ∂_z. So we use the fact that we can split pressure q into Euler part q^E, Navier-Stokes part q^{NS}, and Surface tension part q^S. These are defined in (52), (53), and (54), respectively. We can apply (26) for q^E because estimate of q^E is not harmful in terms of regularity of h. For q^{NS} and q^S, we use sharp split (27) and the fact that q^{NS} and q^S solve harmonic equation in the lower half plane. To explain this idea, let us consider simplified the model problem in the lower half plane which resembles q^S in (21).

$$\Delta g = 0, \quad g|_{z=0} = \Delta_y h, \quad \text{where} \quad z < 0, \tag{28}$$

where Δ_y is horizontal laplacian which is simplified term for surface tension. By horizontal Fourier transform, $\hat{g}$ solves

$$\begin{aligned} |\xi|^2 \hat{g} &= \partial_{zz} \hat{g}, \\ \hat{g} &= \hat{g}(\xi, z=0) e^{|\xi| z}, \\ \partial_z \hat{g} &= |\xi|^3 \hat{h} e^{\xi z} \end{aligned} \tag{29}$$

This implies ∂_z is changed into horizontal derivatives in the case of harmonic function. And we do not need full fourier multiplier $|\nabla|$ to reduce Sobolev order as following. We have

$$\begin{aligned} \||\nabla_y|^{-1} \partial_t^{m-1} \partial_z g\|^2 &\le \int_{-\infty}^0 \int_{\partial S} \frac{1}{|\xi|^2} |\xi|^6 |\partial_t^{m-1} \hat{h}|^2 e^{2|\xi| z} \\ &\lesssim \left| |\xi|^{3/2} \partial_t^{m-1} \hat{h} \right|_{L^2(\mathbb{R}^2)}^2 \\ &\lesssim |\partial_t^{m-1} h|_{\frac{3}{2}}^2 \sim |h|_{X^{m,\frac{1}{2}}}^2, \end{aligned}$$

which is one lesser order than estimate $\|\partial_t^{m-1} \partial_z g\|^2 \sim |h|_{X^{m,\frac{3}{2}}}^2$. We also have to perform similar estimate for q^{NS} to make $\varepsilon \|\nabla \partial_t^{m-1} v(t)\|$ as small dissipation type,

$$\||\nabla_y|^{-1} \partial_t^{m-1} \partial_z q^{NS}(T)\| \le \varepsilon T \int_0^T \|\nabla \partial_t^m v(t)\|^2 dt + \text{(good terms)}, \quad T \ll 1. \tag{30}$$

By combining $E_0(t), E_1(t), \cdots, E_{m-1}(t)$, and $E_m(t)$ properly (with some small weight for $E_m(t)$), we can close energy estimates in terms of regularity of h for sufficiently small $T \ll 1$.

1.5.3. *L^2-type normal estimate.* The problems from bad commutators in terms of h was resolved by above steps. However we will see another bad commutators which contain $\|\partial_z v\|_{X^{m-1,0}}$. This cannot be controlled, since conormal Sobolev space H^1_{co} is weaker than standard Sobolev space H^1. Therefore we should construct another energy estimate for ∇v. However equation of ∇v is not clear. Instead, we make an energy estimates of new variable:

$$S_n := \Pi(S^\varphi v\mathbf{n}), \quad \text{where} \quad \Pi := I - \mathbf{n}\otimes\mathbf{n}, \quad \text{tangential projection onto free-boundary.}$$

We can show that this quantity is equivalent to $\partial_z v$. And S_n has very good boundary condition $S_n|_{z=0} = 0$ by boundary condition (10). Therefore, if we get energy estimate for $\|S_n\|_{m-1}$, we can close energy estimate. Unfortunately, equation of S_n yields $(D^\varphi)^2 q$ as pressure term. Therefore, we cannot use advantage of divergence free of velocity in energy estimate and we cannot reduce the order of pressure. Finally, the optimal regularity of S_n will be $m-2$, not $m-1$. Considering pressure estimate, we get the estimate of S_n:

$$\|S_n(T)\|^2_{X^{m-2,0}} + \varepsilon\int_0^T \|\nabla S_n\|^2_{X^{m-2,0}} dt \leq C_0 + \Lambda(R)\int_0^T \left(E(t) + \|S_n\|^2_{X^{m-2,0}} + |h|^2_{X^{m-2,\frac{7}{2}}}\right)dt.$$

Since the last term $|h|^2_{X^{m-2,\frac{7}{2}}}$ is not in energy $E(t)$, we use Dirichlet-Neumann estimate again to close energy estimate in terms of h-regularity. We fail to get $m-1$ estimate for $\partial_z v$. But, above estimate can be used to control L^∞ type estimate.

1.5.4. *L^∞-type normal estimate.* Next, we consider estimates for L^∞-type terms, which is included in R above. In Sobolev conormal space, we cannot use standard Sobolev embedding, because of its normal derivative vanishes on the boundary. We use S_n instead of $\partial_z v$. The main difficulty is the commutator between Z_3 and Laplacian. We consider a thin layer near the boundary and reparameterize so that $\partial_z^\varphi\partial_z^\varphi$ look like ∂_{zz}. And then, we change the advection term as

$$\partial_t\rho + (w_y(t,y,0), zw_3(t,y,0))\cdot\nabla\rho - \varepsilon\partial_{zz}\rho = l.o.t.$$

We do not apply a simple maximum principle for convection-diffusion equations. We apply Duhamel's principle using Green's evolution kernel. Then we can conclude

$$\|\partial_z v\|^2_{Y^{[\frac{m}{2}],0}} \lesssim \|\partial_v(0)\|^2_{Y^{[\frac{m}{2}],0}} + \Lambda(R)\int_0^t \varepsilon\|\partial_{zz}v\|^2_{X^{m-2,0}},$$

for sufficiently large m.

1.5.5. *Vorticity estimate.* We still have to control $\|\partial_z v\|_{X^{m-1,0}}$, since we failed to control by S_n. We use critical idea of [1], vorticity estimate, which is also equivalent to $\partial_z v$. The biggest advantage of vorticity is that taking curl removes pressure term in Navier-Stokes equation. This makes us possible to get $m-1$ order estimate. However, because we are considering general vorticity data on the free-boundary, standard energy estimate yields high order boundary integral, which cannot be controlled. Instead, in [1], L^4_t type estimate for vorticity was developed, based on the heat equation with general boundary data and paradifferential calculus. This idea makes us possible to a get an estimate of $\|\omega\|_{L^4X^{m-1}}$, where $\omega := \nabla^\varphi\times v$. Although this is not L^∞_T type energy, but we are suffice to get $L^p_T X^{m-1,0}$, $(p>2)$ type estimate, because what we should control is $\|\nabla v\|_{L^2_T X^{m-1,0}}$ from the right hand side of (20), for instance.

1.5.6. *Uniform Existence, Uniqueness, and Vanishing viscosity limit.* At this point we have made all the necessary estimates to close the main energy estimate (15). In particular, the right hand side of the energy estimate is independent of kinematic viscosity ε, provided the energy remains bounded. So, using the preliminary existence result of A.Tani [7] and strong compactness arguments, we get local existence. For uniqueness, it is suffice to do L^2-estimate for the difference of two solutions and we conclude by Gronwall's inequality. For vanishing viscosity limit, we have L^2 weak convergence from compactness argument. Moreover, L^2 norm convergence is gained by zero order L^2 energy estimate. Finally we have we have strong L^2 convergence and the unique limit solves free-boundary Euler equation with surface tension.

1.6. **Comparing the problem with and without surface tension.** Surface tension is, overall, a regularizing force in the water wave problem; however, it introduces several (perhaps unexpected) difficulties. Here we want to elaborate upon the differences between the paper of Masmoudi-Rousset [1] (the case where no there is no surface tension) and our result (where surface tension is taken into account. In terms of the basic functional framework, both works use Sobolev co-normal spaces due to the presence of boundary layers. However, there are big differences between these two works. First, let us look at a scheme of [1] (no surface tension case). When we have no surface tension, m-order energy estimate contains $|h|_m$ as its energy. The main problem which the authors faced in [1] is the presence of certain high order commutators. To get around this problem, the authors made use of Alinhac's good unknown which allowed them to close the energy estimates. They use the good unknowns: $V^\alpha = Z^\alpha v - \partial_z^\varphi v Z^\alpha \eta$ and $Q^\alpha = Z^\alpha q - \partial_z^\varphi q Z^\alpha \eta$, because, with this new variable, the bad commutator $Z^\alpha N$ disappears.

Meanwhile, in the surface tension case (this paper), the m^{th}-order energy estimate has $|h|_{m+1}$ in its energy. So now one does not need Alinhac's good unknown. Nevertheless, we also lack $\frac{1}{2}$ order ($|h|_{m+\frac{3}{2}}$ appears in the commutators) because of the pressure. Because we use co-normal spaces,

$$\int_S Z^m v \cdot \nabla^\varphi Z^m q$$

make high order commutator about pressure q in S (which vanishes in case of standard Sobolev space derivatives D^m, by divergence free condition). Since $q^b \sim \Delta h$, $q \sim \partial_x^{\frac{3}{2}} h$. As mentioned in above scheme, it is bounded by taking time derivatives. The crucial point is that when we only take time derivatives of the equation the worst commutator does not show up. Meanwhile, lack of regularity of S_n also appear in our case. Therefore we should use the idea of vorticity estimate in [1] also.

Regarding L^∞ estimates for S_n, [1] requires $\varepsilon\|\partial_{zz} v\|_{L^\infty}$. But we do not need $\varepsilon\|\partial_{zz} v\|_{Y^{k,0}}$. This is because, $\varepsilon\|\partial_{zz} v\|_{L^\infty}$ appears by Alinhac's unknown which include $\partial_z v Z^\alpha \eta$.

2. Basic Propositions

2.1. **Basic propositions.** We construct some propositions to estimate commutators.

Proposition 9. *Let $u, v \in X^{m,0} \cap Y^{[\frac{m}{2}],0}(S)$. For $m \geq 2$, we get the following estimates.*

$$\begin{aligned}
\|Z^m(uv)\| &:= \|Z^m(uv)\|_{L^2(S)} \\
&\lesssim \|u\|_{X^{m,0}} \|v\|_{Y^{[\frac{m}{2}],0}} + \|v\|_{X^{m,0}} \|u\|_{Y^{[\frac{m}{2}],0}}, \\
\|[Z^m, u]v\| &:= \|Z^m(uv) - u(Z^m v)\|_{L^2(S)} \\
&\lesssim \|u\|_{X^{m,0}} \|v\|_{Y^{[\frac{m}{2}],0}} + \|v\|_{X^{m-1,0}} \|u\|_{Y^{[\frac{m}{2}],0}}, \\
\|[Z^m, u, v]\| &:= \|Z^m(uv) - (Z^m u)v - u(Z^m v)\|_{L^2(S)} \\
&\lesssim \|u\|_{X^{m-1,0}} \|v\|_{Y^{[\frac{m}{2}],0}} + \|v\|_{X^{m-1,0}} \|u\|_{Y^{[\frac{m}{2}],0}},
\end{aligned}$$

where Z^m is defined in (12) and function space X and Y are defined in Definition 4. Also note that $[\frac{m}{2}]$ is defined in Defintion 5.

Proof. We cannot use general Leibnitz Rule since $Z_3 = \frac{1}{1-z}\partial_z$, but every order of derivatives of $\frac{1}{1-z}$ is uniformly bounded for $z < 0$.

$$\begin{aligned}
\|Z^m(uv)\|_{L^2} &\leq \sum_{\substack{(k_1,\beta,k_2,\gamma),\\ k_1+|\beta|+k_2+|\gamma|\leq m}} \|\partial_t^{k_1}Z^\beta u\ \partial_t^{k_2}Z^\gamma v\|_{L^2} \\
&= \sum_{k_1+|\beta|\geq k_2+|\gamma|} \|\partial_t^{k_1}Z^\beta u\ \partial_t^{k_2}Z^\gamma v\|_{L^2} + \sum_{k_1+|\beta|<k_2+|\gamma|} \|\partial_t^{k_1}Z^\beta u\ \partial_t^{k_2}Z^\gamma v\|_{L^2} \\
&\leq \sum_{k_1+|\beta|\geq k_2+|\gamma|} \|\partial_t^{k_1}Z^\beta u\|_{L^2}\|\partial_t^{k_2}Z^\gamma v\|_{L^\infty} + \sum_{k_1+|\beta|<k_2+|\gamma|} \|\partial_t^{k_1}Z^\beta u\|_{L^\infty}\|\partial_t^{k_2}Z^\gamma v\|_{L^2} \\
&\lesssim \|u\|_{X^{m,0}}\|v\|_{Y^{[\frac{m}{2}],0}} + \|v\|_{X^{m,0}}\|u\|_{Y^{[\frac{m}{2}],0}}\,,
\end{aligned}$$

$$\begin{aligned}
\|[Z^m,u]v\|_{L^2} &\leq \sum_{\substack{(k_1,\beta,k_2,\gamma),\\ k_1+|\beta|+k_2+|\gamma|\leq m,\\ k_1+|\beta|\geq 1}} \|\partial_t^{k_1}Z^\beta u\ \partial_t^{k_2}Z^\gamma v\|_{L^2} \\
&= \sum_{\substack{k_1+|\beta|\geq k_2+|\gamma|,\\ k_1+|\beta|\geq 1}} \|\partial_t^{k_1}Z^\beta u\ \partial_t^{k_2}Z^\gamma v\|_{L^2} + \sum_{\substack{k_1+|\beta|\geq k_2+|\gamma|,\\ k_1+|\beta|\geq 1}} \|\partial_t^{k_1}Z^\beta u\ \partial_t^{k_2}Z^\gamma v\|_{L^2} \\
&\lesssim \|u\|_{X^{m,0}}\|v\|_{Y^{[\frac{m}{2}],0}} + \|v\|_{X^{m-1,0}}\|u\|_{Y^{[\frac{m}{2}],0}}\,,
\end{aligned}$$

$$\begin{aligned}
\|[Z^m,u,v]\|_{L^2} &\leq \sum_{\substack{(k_1,\beta,k_2,\gamma),\\ k_1+|\beta|+k_2+|\gamma|\leq m,\\ k_1+|\beta|\geq 1,\\ k_2+|\gamma|\geq 1}} \|\partial_t^{k_1}Z^\beta u\ \partial_t^{k_2}Z^\gamma v\|_{L^2} \\
&= \sum_{\substack{k_1+|\beta|\geq k_2+|\gamma|,\\ k_1+|\beta|\geq 1,\\ k_2+|\gamma|\geq 1}} \|\partial_t^{k_1}Z^\beta u\ \partial_t^{k_2}Z^\gamma v\|_{L^2} + \sum_{\substack{k_1+|\beta|\geq k_2+|\gamma|,\\ k_1+|\beta|\geq 1,\\ k_2+|\gamma|\geq 1}} \|\partial_t^{k_1}Z^\beta u\ \partial_t^{k_2}Z^\gamma v\|_{L^2} \\
&\lesssim \|u\|_{X^{m-1,0}}\|v\|_{Y^{[\frac{m}{2}],0}} + \|v\|_{X^{m-1,0}}\|u\|_{Y^{[\frac{m}{2}],0}}\,.
\end{aligned}$$

□

Remark 10. *The idea is that for each bilinear term, we put the L^2 norm on the term with higher derivatives and the L^∞ norm to low order term. In co-normal derivatives, there is no proper notion of fractional derivatives, so $Z_3^{1/2}$ does not make sense. We deal when m is even, so that $\frac{m}{2}$ is also a integer, but our result also work for odd m, because it suffices to give $\frac{m-1}{2}$ orders to L^∞ and $\frac{m+1}{2}$ orders to L^2.*

The following proposition is for anisotropic embeddings and trace properties in co-normal spaces.

Proposition 11. *1) For $s_1 \geq 0$, $s_2 \geq 0$ such that $s_1 + s_2 > 2$ and u such that $u \in H^{s_1}_{tan}(S)$, $\partial_z u \in H^{s_2}_{tan}(S)$, we have the anisotropic Sobolev embedding:*

$$\|u\|^2_{L^\infty} \lesssim \|\partial_z u\|_{H^{s_2}_{tan}}\|u\|_{H^{s_1}_{tan}}\,.$$

2) For $u \in H^1(S)$, we have the trace estimate :

$$|u(\cdot,0)|_{H^s(\mathbb{R}^2)} \lesssim \|\partial_z u\|^{\frac{1}{2}}_{H^{s_2}_{tan}}\|u\|^{\frac{1}{2}}_{H^{s_1}_{tan}}\,,$$

with $s_1 + s_2 = 2s \geq 0$, where H^m_{tan} is defined in Definition 3. Especially,

$$|u^b|_{H^{s+\frac{1}{2}}} \lesssim \|\partial_z u\|^2_{H^{s+1}_{tan}} + \|u\|^2_{H^s_{tan}}$$

Proof. For 1) and 2), see Proposition 2.2 in [1]. □

Proposition 12. *We state classical commutator estimate on $\mathbb{R}^2$. Let Λ is the Fourier mulitplier $(1+|\xi|^2)^{\frac{s}{2}}$. Then we have the following estimate.*

$$\begin{aligned}
|\Lambda^s(fg)| &\le C_s\big(|f|_{L^\infty(\mathbb{R}^2)}|g|_{H^s(\mathbb{R}^2)} + |g|_{L^\infty(\mathbb{R}^2)}|f|_{H^s(\mathbb{R}^2)}\big),\\
|[\Lambda^s, f]\nabla g|_{L^2(\mathbb{R}^2)} &\le C_s\big(|\nabla f|_{L^\infty(\mathbb{R}^2)}|g|_{H^s(\mathbb{R}^2)} + |\nabla g|_{L^\infty(\mathbb{R}^2)}|f|_{H^s(\mathbb{R}^2)}\big),\\
|uv|_{\frac{1}{2}} &\lesssim |u|_{1,\infty}|v|_{\frac{1}{2}},\\
|uv|_{-\frac{1}{2}} &\lesssim |u|_{1,\infty}|v|_{-\frac{1}{2}}.
\end{aligned} \tag{31}$$

Proof. First three estimates can be found in [1]. The last inequality comes from duality argument of third one. □

2.2. **Estimate of η.** In this subsection, we investigate regularity of mapping η, defined in (1.2) and (1.2). As explained before, the reason we choose such smoothing diffeomorphism is that regularity of η is $\frac{1}{2}$ better than h. This fact is crucial later, because this term can accommodate an extra $\frac{1}{2}$ derivative in bilinear estimates. For example, in the pressure estimates:

$$\int_S (\nabla\varphi)q \le \|\nabla\varphi\|_{\frac{1}{2}}\|q\|_{-\frac{1}{2}} \sim |\nabla\ h|_{L^2}\|q\|_{-\frac{1}{2}}.$$

We defined diffeomorphism so that at initial time, $\partial_z\varphi(0,y,z)\ge 1$. $\partial_z\varphi$ should be positive during our estimates, so our estimate is valid during on $[0,T^\varepsilon]$ such that

$$\partial_z\varphi(t,y,z)\ge c_0,\quad \forall t\in[0,T^\varepsilon] \tag{32}$$

for some c_0, where T^ε is time interval of solution with fixed $\varepsilon>0$.

Proposition 13. *For η, defined in (1.2), we obtain the following estimates.*

$$\begin{aligned}
\|\nabla\eta\|_{H^s(S)} &\le C_s|h|_{H^{s+\frac{1}{2}}(\partial S)},\\
\|\nabla\eta\|_{X^{m,0}} &\le C_s|h|_{X^{m,\frac{1}{2}}}.
\end{aligned}$$

Moreover, for L^∞ type estimates, we get

$$\begin{aligned}
\forall s\in\mathbb{N},\quad \|\eta\|_{W^{s,\infty}} &\le C_s|h|_{s,\infty},\\
\forall s\in\mathbb{N},\quad \|\eta\|_{Y^{m,0}} &\le C_s|h|_{Y^{m,0}}.
\end{aligned}$$

Note that we have standard Sobolev regularity, not co-normal one.

Proof. The first inequality is from [1], and $\left|\nabla\partial_t^k\eta\right|_{H^s(S)}\le C_s\left|\partial_t^k h\right|_{s+\frac{1}{2}}$ is also trivial by definition of η. So, by summing all cases, we get the second inequality. For L^∞ type estimates, the third inequality is from [1], and the last inequality is also gained in similar way. □

From the definition of ∂_i^φ in Definition 1, we should control Sobolev norm of fractional term $\frac{u}{\partial_z\varphi}$. The following lemma is useful to estimate such terms.

Lemma 14. *Assume that (32) holds. Then we have the following estimate for $u\in X^{m,0}\cap Y^{[\frac{m}{2}],0}(S)$ and $h\in X^{m,\frac{1}{2}}\cap Y^{[\frac{m}{2}],1}(S)$.*

$$\left\|Z^m\frac{u}{\partial_z\varphi}\right\|_{L^2}\lesssim\Lambda\Big(\frac{1}{c_0},\|u\|_{Y^{[\frac{m}{2}],0}}+|h|_{Y^{[\frac{m}{2}],1}}\Big)(\|u\|_{X^{m,0}}+|h|_{X^{m,\frac{1}{2}}}).$$

Proof. $F(x)=x/(A+x)$ is a smooth function of which all order derivatives are bounded when $A+x\ge c_0>0$. Since $\partial_z\varphi=A+\partial_z\eta$, $\frac{u}{\partial_z\varphi}=\frac{u}{A}-\frac{u}{A}F(\partial_z\eta)$. Therefore,

$$\begin{aligned}
\left\|Z^m\frac{u}{\partial_z\varphi}\right\|_{L^2} &= \left\|Z^m\Big(\frac{u}{A}-\frac{u}{A}F(\partial_z\eta)\Big)\right\|_{L^2}\lesssim\|u\|_{X^{m,0}}+\|Z^m(uF(\partial_z\eta))\|_{L^2}\\
&\lesssim \|u\|_{X^{m,0}}+\|u\|_{X^{m,0}}\|F(\partial_z\eta)\|_{Y^{[\frac{m}{2}],0}}+\|u\|_{Y^{[\frac{m}{2}],0}}\|F(\partial_z\eta)\|_{X^{m,0}},
\end{aligned}$$

where we used the first commutator estimate in Proposition 9. Moreover, using Proposition 9 again, we have,

$$\begin{aligned}
\|F(\partial_z\eta)\|_{X^{m,0}} &\lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla\eta\|_{Y^{[\frac{m}{2}],0}}\Big)\|\partial_z\eta\|_{X^{m,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}}\Big)|h|_{X^{m,\frac{1}{2}}},\\
\|F(\partial_z\eta)\|_{Y^{[\frac{m}{2}],0}} &\lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla\eta\|_{Y^{[\frac{m}{2}],0}}\Big) \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}}\Big),
\end{aligned}$$

where we used smoothing property of Proposition 13. □

2.3. **Dissipation term control.** From the definition (5), we define volume element $\mathrm{d}V_t := \partial_z\varphi dxdydz$ in S, i.e.

$$\int_{\Omega_t} f\mathrm{d}V = \int_S (f\circ\Phi)\ \partial_z\varphi\ \mathrm{d}x\mathrm{d}y\mathrm{d}z := \int_S (f\circ\Phi)\ \mathrm{d}V_t. \tag{33}$$

First, we compare L^2 type norms between ∇v and $\nabla^\varphi v$.

Lemma 15. *Assume that $\partial_z\varphi \geq c_0$ and $\|\nabla\varphi\|_{L^\infty} \leq \frac{1}{c_0}$ for some $c_0 > 0$, then we have,*

$$\|\nabla f\|^2_{X^{m,0}} \leq \Lambda\left(\frac{1}{c_0}, \|\nabla f\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\right)(\|\nabla^\varphi f\|^2_{X^{m,0}} + |h|^2_{X^{m,\frac{1}{2}}}).$$

Proof. From Definition 1,

$$\begin{aligned}
\|\partial_z f\|_{X^{m,0}} &= \|\partial_z\varphi\ \partial^\varphi_z f\|_{X^{m,0}} \lesssim \|\partial_z\varphi\|_{X^{m,0}}\|\partial^\varphi_z f\|_{Y^{[\frac{m}{2}],0}} + \|\partial^\varphi_z f\|_{X^{m,0}}\|\partial_z\varphi\|_{Y^{[\frac{m}{2}],0}}\\
&\lesssim \Lambda\left(\frac{1}{c_0}, \|\nabla f\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\right)(\|\partial^\varphi_z f\|_{X^{m,0}} + |h|_{X^{m,\frac{3}{2}}}),\\
\|\partial_{i=x,y} f\|_{X^{m,0}} &= \|\partial^\varphi_i f\|_{X^{m,0}} + \|\partial_i\varphi\ \partial^\varphi_z f\|_{X^{m,0}} \lesssim \|\partial^\varphi_i f\|_{X^{m,0}} + \|\partial_z\varphi\|_{X^{m,0}}\|\partial^\varphi_i f\|_{Y^{[\frac{m}{2}],0}} + \|\partial^\varphi_z f\|_{X^{m,0}}\|\partial_i\varphi\|_{Y^{[\frac{m}{2}],0}}\\
&\lesssim \|\partial^\varphi_i f\|_{X^{m,0}} + \Lambda\left(\frac{1}{c_0}, \|\nabla f\|_{Y^{[\frac{m}{2}],0}} + |\nabla h|_{Y^{[\frac{m}{2}],0}}\right)(\|\partial^\varphi_z f\|_{X^{m,0}} + |h|_{X^{m,\frac{1}{2}}}),
\end{aligned}$$

where we used Proposition 9 and 13. Combining above two estimate, we get the result. □

Using Lemma 15 and (33), we have a version of Korn's inequality for S^φ.

Proposition 16. *1) If $\partial_z\varphi \geq c_0$, $\|\nabla\varphi\|_{L^\infty} + \left\|\nabla^2\varphi\right\|_{L^\infty} \leq \frac{1}{c_0}$ for some $c_0 > 0$, then there exists $\Lambda_0 = \Lambda(\frac{1}{c_0}) > 0$ such that for every $v \in H^1(S)$, we have*

$$\|\nabla v\|^2_{L^2(S)} \leq \Lambda_0\Big(\int_S |S^\varphi v|^2\, dV_t + \|v\|^2\Big),$$

where $S^\varphi v := \frac{1}{2}(\nabla^\varphi v + (\nabla^\varphi v)^T)$.
2) For higher order case, we have the following estimate.

$$\|\nabla v\|^2_{X^{m,0}} \leq \Lambda_0\Big(\int_S |S^\varphi v|^2_{X^{m,0}}\,\mathrm{d}V_t + \|v\|^2_{X^{m,0}}\Big).$$

Proof. See Proposition 2.9 in [1] for the first estimate 1). For the second estimate 2), we apply same estimate for $\partial^k Z^\alpha v$. □

3. Equations of $(Z^m v, Z^m h, Z^m q)$

The aim of this section is to prove the following proposition.

Proposition 17. *Let positive integer $m \geq 2$. Then applying Z^m to the system (7)-(10), we get the following equations with respect to $(Z^m v, Z^m h, Z^m q)$ and commutator estimates.*

$$
\begin{aligned}
&(\partial_t^\varphi + v\cdot\nabla^\varphi)(Z^m v) + \nabla^\varphi(Z^m q) = 2\varepsilon\nabla^\varphi\cdot S^\varphi(Z^m v) - T^m(v) + C^m(q) + \varepsilon\Theta^m(v) - \varepsilon\mathcal{D}^m(S^\varphi v),\\
&\nabla^\varphi\cdot(Z^m v) = C^m(d),\\
&\partial_t(Z^m h) = (Z^m v^b)\cdot\mathbf{N} + v^b\cdot(Z^m\mathbf{N}) + C^m(KB),\\
&\Big\{Z^m q^b - gZ^m h - 2\varepsilon(S^\varphi(Z^m v))^b + \nabla\cdot\frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}}\Big\}\mathbf{N} = -C^m(B) + \varepsilon(\Theta^m(v))^b\mathbf{N},\\
&\qquad\qquad\qquad\qquad + \nabla\cdot\frac{\nabla h\langle\nabla h, \nabla Z^m h\rangle}{\sqrt{1+|\nabla h|^2}^3}\mathbf{N} + \nabla\cdot C^m(S)\mathbf{N},
\end{aligned}
$$

where

$$
\begin{aligned}
\|C^m(q)\| &\lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla q\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\Big)(\|\nabla q\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}}),\\
\|C^m(d)\| &\lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla v\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\Big)(\|\nabla v\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}}),\\
\|T^m(v)\| &\lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla v\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\Big)(\|\nabla v\|_{X^{m-1,0}} + \|v\|_{X^{m,0}} + |h|_{X^{m-1,\frac{1}{2}}}),\\
\|\Theta^m(v)\| &\lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla v\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\Big)(\|\nabla v\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}}),\\
\|C^m(KB)\| &\lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla v\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\Big)(\|v\|_{X^{m-1,0}} + \|\nabla v\|_{X^{m-1,0}} + |h|_{X^{m-1,1}}),\\
\|C^m(B)\| &\lesssim 2\varepsilon\ \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\Big)(|h|_{X^{m-1,1}} + |v^b|_{X^{m-1,1}}),\\
\|\nabla\cdot C^m(S)\mathbf{N}\| &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],2}} + \|\nabla v\|_{Y^{[\frac{m}{2}],1}}\Big)|h|_{X^{m-1,2}}.
\end{aligned}
$$

Proof. Proof of this proposition is given by (35)-(45), and (48)-(51), those are derived throughout following three subsections. ☐

3.1. **Commutator estimate.**

Proposition 18. *For $i = t, 1, 2, 3$, let us define $C_i^m(f)$ by*

$$
Z^m(\partial_i^\varphi f) := \partial_i^\varphi(Z^m f) + C_i^m(f). \tag{34}
$$

Then we have the following commutator estimate for $C_i^m(f)$.

$$
\|C_i^m(f)\| \lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla f\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\Big)\Big(\|\nabla f\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}}\Big).
$$

Proof. For $i = t, 1, 2$

$$
\begin{aligned}
Z^m\Big(\partial_i f - \frac{\partial_i\varphi}{\partial_z\varphi}\partial_z f\Big) &= \partial_i(Z^m f) - Z^m\Big(\frac{\partial_i\varphi}{\partial_z\varphi}\partial_i^\varphi f\Big)\\
&= \partial_i(Z^m f) - \Big(\Big[Z^m, \frac{\partial_i\varphi}{\partial_z\varphi}, \partial_z f\Big] + \Big(Z^m\frac{\partial_i\varphi}{\partial_z\varphi}\Big)\partial_z f + \frac{\partial_i\varphi}{\partial_z\varphi}\Big(Z^m\partial_z f\Big)\Big)\\
&= \partial_i^\varphi(Z^m f) \underbrace{-\Big(\Big[Z^m, \frac{\partial_i\varphi}{\partial_z\varphi}, \partial_z f\Big] + \Big(Z^m\frac{\partial_i\varphi}{\partial_z\varphi}\Big)\partial_z f + \frac{\partial_i\varphi}{\partial_z\varphi}\left[Z^m, \partial_z\right] f\Big)}_{:=C_i^m(f)}\\
&:= \partial_i^\varphi(Z^m f) + C_i^m(f).
\end{aligned}
$$

Now we estimate three terms of $C_i^m(f)$ using Propositions 9, 13, and Lemma 14.

$$\begin{aligned}
\left\|\left[Z^m, \frac{\partial_i\varphi}{\partial_z\varphi}, \partial_z f\right]\right\| &\lesssim \left\|\frac{\partial_i\varphi}{\partial_z\varphi}\right\|_{X^{m-1,0}} \|\partial_z f\|_{Y^{[\frac{m}{2}],0}} + \left\|\frac{\partial_i\varphi}{\partial_z\varphi}\right\|_{Y^{[\frac{m}{2}],0}} \|\partial_z f\|_{X^{m-1,0}} \\
&\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\partial_z f\|_{Y^{[\frac{m}{2}],0}}\Big)\Big(|h|_{X^{m-1,\frac{1}{2}}} + \|\partial_z f\|_{X^{m-1,0}}\Big), \\
\left\|\Big(Z^m \frac{\partial_i\varphi}{\partial_z\varphi}\Big)\partial_z f\right\| &\lesssim |\partial_z f|_{L^\infty} \left\|Z^m \frac{\partial_i\varphi}{\partial_z\varphi}\right\| \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + |\partial_z f|_{L^\infty}\Big)|h|_{X^{m,\frac{1}{2}}}, \\
\left\|\frac{\partial_i\varphi}{\partial_z\varphi}\left[Z^m, \partial_z\right] f\right\| &\lesssim \left|\frac{\partial_i\varphi}{\partial_z\varphi}\right|_{L^\infty} \|\sum_{k+|\beta|\leq m-1} c_\beta \partial_z(\partial_t^k Z^\beta f)\| \lesssim \Lambda\Big(\frac{1}{c_0}, \|\partial_z f\|_{Y^{[\frac{m}{2}],0}} + |h|_{1,\infty}\Big)\|\partial_z f\|_{X^{m-1,0}}.
\end{aligned}$$

By summing these three terms, we get the results. For $i = 3$,

$$\begin{aligned}
Z^m\Big(\frac{\partial_z f}{\partial_z\varphi}\Big) &= \left[Z^m, \frac{1}{\partial_z\varphi}, \partial_z f\right] + \Big(Z^m \frac{1}{\partial_z\varphi}\Big)\partial_z f + \frac{1}{\partial_z\varphi}\Big(Z^m\partial_z f\Big) \\
&= \partial_3^\varphi(Z^m)f + \left[Z^m, \frac{1}{\partial_z\varphi}, \partial_z f\right] + \Big(Z^m \frac{1}{\partial_z\varphi}\Big)\partial_z f + \frac{1}{\partial_z\varphi}\left[Z^m, \partial_z\right] f.
\end{aligned}$$

We just replace $\partial_i\varphi$ as 1, so the control is same as $i = t, 1, 2$ cases. □

3.2. **Interior Equation (7) and (8).** Now we apply Z^m to each terms in (7)-(10) and use Proposition 18 for each commutators.

3.2.1. *Pressure.*

$$Z^m(\nabla^\varphi q) = \nabla^\varphi(Z^m q) - (C_1^m(q), C_2^m(q), C_3^m(q)) := \nabla^\varphi(Z^m q) - C^m(q), \tag{35}$$

where $C^m(q) := (C_1^m(q), C_2^m(q), C_3^m(q))$. By Proposition 18,

$$\|C^m(q)\| \lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla q\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\Big)\Big(\|\nabla q\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}}\Big). \tag{36}$$

3.2.2. *Divergence-free.*

$$Z^m(\nabla^\varphi \cdot v) = \nabla^\varphi \cdot (Z^m v) - \sum_{i=1}^{3} C_i^m(v) = \nabla^\varphi \cdot (Z^m v) - C^m(d), \tag{37}$$

where $C^m(d) := \sum_{i=1}^3 C_i^m(v)$. By Proposition 18,

$$\|C^m(d)\| \lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla v\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\Big)\Big(\|\nabla v\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}}\Big). \tag{38}$$

3.2.3. *Transport term.* Using divergence free condition, we have

$$\partial_t^\varphi + (v \cdot \nabla^\varphi) = \partial_t + (v_y \cdot \nabla_y) + \frac{1}{\partial_z\varphi}\Big(v \cdot \mathbf{N} - \partial_t\varphi\Big)\partial_z,$$

where extended normal vector $\mathbf{N}$ is defined in Definition 2. Applying Z^m,

$$Z^m(\partial_t^\varphi + v \cdot \nabla^\varphi)v = (\partial_t^\varphi + v \cdot \nabla^\varphi)(Z^m v) + T^m(v), \tag{39}$$

where

$$T^m(v) := \sum_{i=1}^{2}\{\partial_i v \cdot Z^m v_i + [Z^m, v_i, \partial_i v]\} + [Z^m, V_z, \partial_z v] + (Z^m V_z) \cdot \partial_z v + V_z[Z^m, \partial_z]v,$$

where

$$V_z := \frac{1}{\partial_z\varphi}(v \cdot \mathbf{N} - \partial_t\varphi) \tag{40}$$

Using Proposition 18, we have

$$\|T^m(v)\| \lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla v\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\Big)\Big(\|\nabla v\|_{X^{m-1,0}} + \|v\|_{X^{m,0}} + |h|_{X^{m-1,\frac{1}{2}}}\Big). \tag{41}$$

3.2.4. *Diffusion.*

$$2\varepsilon Z^m \nabla^\varphi \cdot (S^\varphi v) = 2\varepsilon \nabla^\varphi \cdot Z^m (S^\varphi v) - \varepsilon \mathcal{D}^m (S^\varphi v),$$

where $\mathcal{D}^m (S^\varphi v)_i := 2C_j^m (S^\varphi v)_{ij}$ and

$$2Z^m(S^\varphi v) = Z^m(\partial_i^\varphi v_j + \partial_j^\varphi v_i) = 2S^\varphi(Z^m v) + (C_i^m(v_j) + C_j^m(v_i)) := 2S^\varphi(Z^m v) + \Theta^m(v),$$

where $\Theta^m(v)_{ij} := C_i^m(v_j) + C_j^m(v_i)$. Therefore,

$$2\varepsilon Z^m \nabla^\varphi \cdot (S^\varphi v) = 2\varepsilon \nabla^\varphi \cdot S^\varphi(Z^m v) + \varepsilon \Theta^m(v) - \varepsilon \mathcal{D}^m(S^\varphi v) \tag{42}$$

the estimate of $\Theta^m(v)$ is same as $C^m(v)$,

$$\|\Theta^m(v)\| \lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla v\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\Big)\Big(\|\nabla v\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}}\Big). \tag{43}$$

3.3. **Boundary Condition (9) and (10).** Note that $\alpha_3 = 0$, because we deal functions on boundary.

3.3.1. *Kinematic boundary condition.* Applying Z^m to (9),

$$\partial_t(Z^m h) - (Z^m v^b) \cdot \mathbf{N} - v^b \cdot (Z^m \mathbf{N}) - C^m(KB) = 0, \quad \text{where} \quad C^m(KB) := [Z^m, v^b, \mathbf{N}]. \tag{44}$$

$$\begin{aligned} \|C^m(KB)\| = \|[Z^m, v^b, \mathbf{N}]\| &\lesssim \Lambda\Big(|v^b|_{Y^{[\frac{m}{2}],0}} + |\mathbf{N}|_{Y^{[\frac{m}{2}],0}}\Big)\Big(|v^b|_{X^{m-1,0}} + |\mathbf{N}|_{X^{m-1,0}}\Big) \\ &\lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla v\|_{Y^{[\frac{m}{2}],0}} + |h|_{Y^{[\frac{m}{2}],1}}\Big)(\|v\|_{X^{m-1,0}} + \|\nabla v\|_{X^{m-1,0}} + |h|_{X^{m-1,1}}), \end{aligned} \tag{45}$$

where we used trace inequality Proposition 11.

3.3.2. *Continuity of Stress tensor.*

Lemma 19. *We have the following estimate to control $\nabla v(\cdot, 0) := (\nabla v)^b := (\nabla v)|_{z=0}$ by v^b.*

$$|\nabla v(\cdot,0)|_{X^{m,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|v\|_{Y^{[\frac{m}{2}],1}}\Big)\Big(|h|_{X^{m,1}} + |v(\cdot,0)|_{X^{m,1}}\Big).$$

Proof. We divide ∇v into normal part and tangential part. For tangential derivatives $\partial_1 v$ and $\partial_2 v$, results are obvious, since tangential derivatives commute with $\nabla v(\cdot, 0)$. So, we only focus on $\partial_z v$. Firstly, from the divergence free condition $\nabla^\varphi \cdot v = 0$,

$$\partial_z v \cdot \mathbf{n} = \frac{1}{|\mathbf{N}|}(A + \partial_z \eta)(\partial_1 v_1 + \partial_2 v_2), \quad \text{where } \mathbf{N} \text{ is defined in Definition 2.}$$

Using Proposition 9, 13, and 14, we get normal part estimate.

$$\begin{aligned} |\partial_z v \cdot \mathbf{n}|_{X^{m,0}} &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|v\|_{Y^{[\frac{m}{2}],1}}\Big)\Big(|\partial_z \eta(\cdot,0)|_{X^{m,0}} + |v(\cdot,0)|_{X^{m,1}}\Big) \\ &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|v\|_{Y^{[\frac{m}{2}],1}}\Big)\Big(|h|_{X^{m,1}} + |v(\cdot,0)|_{X^{m,1}}\Big), \quad \text{on} \quad \partial S. \end{aligned} \tag{46}$$

To estimate tangential part, note that

$$2S^\varphi v \mathbf{N} = \frac{1}{\partial_z \varphi}\Big(1 + |\nabla h|^2\Big)(\partial_z v) - \Big(\partial_1 h \partial_1 v + \partial_2 h \partial_2 v\Big) + \begin{pmatrix} \partial_1 v \cdot \mathbf{N} \\ \partial_2 v \cdot \mathbf{N} \\ 0 \end{pmatrix} \mathbf{N} + \frac{1}{\partial_z \varphi}\Big(\partial_z v \cdot \mathbf{N}\Big)\mathbf{N}, \text{ on } \partial S.$$

Now, let define $\Pi = I - \mathbf{n} \otimes \mathbf{n}$ (tangential projection). Then, from compatibility condition (10), we have $\Pi(S^\varphi v \mathbf{N}) = 0$. Therefore,

$$\Pi \partial_z v(\cdot,0) = \Pi\Big(\frac{\partial_z \varphi}{1 + |\nabla h|^2}\Big\{\Big(\partial_1 h \partial_1 v + \partial_2 h \partial_2 v\Big) - \begin{pmatrix} \partial_1 v \cdot \mathbf{N} \\ \partial_2 v \cdot \mathbf{N} \\ 0 \end{pmatrix} \mathbf{N} - \frac{1}{\partial_z \varphi}\Big(\partial_z v \cdot \mathbf{N}\Big)\mathbf{N}\Big\}\Big), \text{ on } \partial S.$$

Using Proposition 9, 13, 14, and (46), we get tangential part estimate,

$$|\Pi \partial_z v(\cdot,0)|_{X^{m,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|v\|_{Y^{[\frac{m}{2}],1}}\Big)\Big(|h|_{X^{m,1}} + |v(\cdot,0)|_{X^{m,1}}\Big), \quad \text{on} \quad \partial S. \tag{47}$$

Combining (46) and (47), we get the result. $\square$

Now we return to the Stress-continuity condition (10). Applying Z^m,

$$\begin{aligned} &\Big\{ Z^m q^b - gZ^m h - 2\varepsilon\Big(S^\varphi(Z^m v)\Big)^b - \varepsilon\Big(\Theta^m(v)\Big)^b \Big\}\mathbf{N} \\ &+ \Big(q^b - gh - 2\varepsilon(S^\varphi v)^b + \nabla\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)Z^m\mathbf{N} + \Big(\nabla\cdot Z^m\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)\mathbf{N} + C^m(B) = 0, \end{aligned} \tag{48}$$

where

$$\begin{aligned} C^m(B) &:= -C^m(B)_1 + C^m(B)_2 \\ &= -2\varepsilon[Z^m, (S^\varphi v)^b, \mathbf{N}] + \Big[Z^m, q^b - gh + \nabla\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}, \mathbf{N}\Big]. \end{aligned}$$

Using Proposition 9, 11, and Lemma 19,

$$\begin{aligned} \|C^m(B)_1\| &= 2\varepsilon\|[Z^m, (S^\varphi v)^b, \mathbf{N}]\| \\ &\lesssim 2\varepsilon\ \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\Big)(|h|_{X^{m-1,1}} + |\nabla v(\cdot,0)|_{X^{m-1,0}}) \\ &\lesssim 2\varepsilon\ \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\Big)\Big(|h|_{X^{m-1,1}} + |v^b|_{X^{m-1,1}}\Big), \qquad (49) \\ \|C^m(B)_2\| &= 2\varepsilon\|[Z^m, (S^\varphi v)^b\mathbf{n}\cdot\mathbf{n}, \mathbf{N}]\| \\ &\lesssim 2\varepsilon\ \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\Big)\Big(|h|_{X^{m-1,1}} + |v^b|_{X^{m-1,1}}\Big), \end{aligned}$$

where the way of estimate for $\|C^m(B)_2\|$ is same as $\|C^m(B)_1\|$. We also should estimate $Z^m\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}$. We define $C^m(S)$ as

$$Z^m\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} := \frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla h\langle\nabla h, \nabla Z^m h>}{\sqrt{1+|\nabla h|^2}^3} + C^m(S), \tag{50}$$

which is consist of low order polynomials in terms of h. Take a term in this $C^m(S)$, then we take L^2 norm for the highest order, and L^∞ to others. For large $m \geq 2$, L^∞ can be controlled by the highest order terms by Sobolev embedding. Therefore,

$$\|\nabla\cdot C^m(S)\mathbf{N}\| \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],2}} + \|\nabla v\|_{Y^{[\frac{m}{2}],1}}\Big)|h|_{X^{m-1,2}}. \tag{51}$$

4. Pressure Estimates

From Proposition 17, we should estimate pressure q. From (7), (8), and (10), pressure $q(t,x,y,z)$ solves,

$$\begin{aligned} -\Delta^\varphi q &= \nabla^\varphi\cdot(v\cdot\nabla^\varphi v) \quad \text{in} \quad S, \\ q^b &:= q|_{z=0} = q(t,x,y,0) = gh + 2\varepsilon(S^\varphi v)^b\mathbf{n}\cdot\mathbf{n} - \nabla\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}. \end{aligned}$$

We decompose pressure q into $q = q^E + q^{NS} + q^S$, where
q^E solves

$$\begin{aligned} &-\Delta^\varphi q^E = \nabla^\varphi\cdot\Big(v\cdot\nabla^\varphi v\Big) \quad \text{in} \quad S, \\ &q^E|_{z=0} = gh, \end{aligned} \tag{52}$$

and q^{NS} solves

$$\begin{aligned} &-\Delta^\varphi q^{NS} = 0, \quad \text{in} \quad S, \\ &q^{NS}|_{z=0} = 2\varepsilon(S^\varphi v)\mathbf{n}\cdot\mathbf{n}, \end{aligned} \tag{53}$$

and q^S solves

$$\begin{aligned} -\Delta^\varphi q^S &= 0, \quad \text{in} \quad S, \\ q^S|_{z=0} &= -\nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}}. \end{aligned} \tag{54}$$

To find expand Δ^φ, we should calculate gradient form. Using Definition 1,

$$\nabla^\varphi f = \begin{pmatrix} \partial_1 f - \frac{\partial_1\varphi}{\partial_z\varphi}\partial_z f \\ \partial_2 f - \frac{\partial_2\varphi}{\partial_z\varphi}\partial_z f \\ \frac{\partial_z f}{\partial_z \varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{\partial_1\varphi}{\partial_z\varphi} \\ 0 & 1 & -\frac{\partial_2\varphi}{\partial_z\varphi} \\ 0 & 0 & \frac{1}{\partial_z\varphi} \end{pmatrix} \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_z f \end{pmatrix} := \frac{1}{\partial_z\varphi} P^* \nabla f, \tag{55}$$

where

$$P := \begin{pmatrix} \partial_z\varphi & 0 & 0 \\ 0 & \partial_z\varphi & 0 \\ -\partial_1\varphi & -\partial_2\varphi & 1 \end{pmatrix}. \tag{56}$$

To calculate divergence, using Definition 1,

$$\nabla^\varphi \cdot v = \frac{1}{\partial_z\varphi} \nabla \cdot (Pv). \tag{57}$$

Therefore, by (55) and (57), we get the following form for Δ^φ.

$$\Delta^\varphi f = \nabla^\varphi \cdot (\nabla^\varphi f) = \frac{1}{\partial_z\varphi}\nabla \cdot (P\nabla^\varphi f) = \frac{1}{\partial_z\varphi}\nabla\cdot(E\nabla f), \tag{58}$$

where

$$E := \frac{1}{\partial_z\varphi} PP^* = \begin{pmatrix} \partial_z\varphi & 0 & -\partial_1\varphi \\ 0 & \partial_z\varphi & -\partial_2\varphi \\ -\partial_1\varphi & -\partial_2\varphi & \frac{1+(\partial_1\varphi)^2+(\partial_2\varphi)^2}{\partial_z\varphi} \end{pmatrix}. \tag{59}$$

We state two lemmas about elliptic equation with Dirichlet boundary condition.

Lemma 20. *Let $m \geq 6$ and ρ solves the following system,*

$$-\nabla\cdot(E\nabla\rho) = \nabla\cdot F, \quad \textit{in} \quad S, \quad \rho|_{z=0} := \rho(t,x,y,0) = 0,$$

where matrix E and P are defined in (59) and (56), respectively. Then we have the following estimate.

$$\begin{aligned} \|\nabla\rho\|_{X^{m-1,0}} + \|\nabla^2\rho\|_{X^{m-2,0}} \quad &\lesssim \quad \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|\nabla\cdot F\|_{H^1_{tan}} + \|F\|_{H^2_{tan}}\Big) \\ &\times \quad (\|F\|_{X^{m-1,0}} + \|\nabla\cdot F\|_{X^{m-2,0}} + |h|_{X^{m-2,1}}), \end{aligned}$$

where function spaces $X^{m,s}$, $Y^{m,s}$, and H^s_{tan} are defined in Definition 3 and 4, respectively.

Proof. By variational argument with homogeneous boundary condition, we have

$$\|\nabla\rho\| \leq \Lambda\Big(\frac{1}{c_0}, |h|_{1,\infty}\Big)\|F\|_{L^2}. \tag{60}$$

If we apply Z^m to the equation, then divergence structure is broken, since Z_3 and ∂_z do not commute. Therefore, we redefine,

$$\tilde{Z}_3 f := Z_3 f + (2z+1)f. \tag{61}$$

With this new definition, it is easy to check $\tilde{Z}_3\partial_z = \partial_z Z_3$. We apply

$$\tilde{Z}^{m-1} = \partial_t^k Z_1^{\alpha_1} Z_2^{\alpha_2} \tilde{Z}_3^{\alpha_3}, (k+\alpha_1+\alpha_2+\alpha_3 = m-1)$$

to the equation.

$$\nabla\cdot(Z^{m-1}(E\nabla\rho)) = \nabla\cdot(Z^{m-1}F + (\tilde{Z}^{m-1} - Z^{m-1})F_h - (\tilde{Z}^{m-1} - Z^{m-1})(E\nabla\rho)_h), \tag{62}$$

where sub index h means horizontal component, i.e. $F_h := (F_1, F_2, 0)$. Again, considering commutators between Z^{m-1} and E,

$$\begin{aligned}\nabla\cdot(E\cdot\nabla(Z^{m-1}\rho)) = \nabla\cdot\Big(&Z^{m-1}F + (\tilde{Z}^{m-1} - Z^{m-1})F_h - (\tilde{Z}^{m-1} - Z^{m-1})(E\nabla\rho)_h\Big)\\ &+\nabla\cdot\tilde{C},\end{aligned}$$

where

$$\tilde{C} := -E[Z^{m-1},\nabla]\rho - \Big(\sum_{\substack{k_1+\beta+k_2+\gamma=m-1,\\ k_1+\beta\neq 0}} c_{k_1,\beta,k_2,\gamma}\ \partial_t^{k_1}Z^\beta E\ \partial_t^{k_2}Z^\gamma\rho\Big).$$

Since $Z^{m-1}\rho$ is zero on the boundary, using (60),

$$\begin{aligned}\|\nabla\rho\|_{X^{m-1,0}} \le \Lambda\Big(\frac{1}{c_0}, |h|_{1,\infty}\Big)\Big(&\|F\|_{X^{m-1,0}} + \|E\nabla\rho\|_{X^{m-2,0}} + \big\|E[Z^{m-1},\nabla]\rho\big\|\\ &+\Big\|\sum_{\substack{k_1+\beta+k_2+\gamma=m-1,\\ k_1+\beta\neq 0}} c_{k_1,\beta,k_2,\gamma}\ \partial_t^{k_1}Z^\beta E\ \partial_t^{k_2}Z^\gamma\rho\Big\|\Big).\end{aligned}\tag{63}$$

Three terms on the right hand side can be estimated using Proposition 9.

$$\begin{aligned}&\|E\nabla\rho\|_{X^{m-2,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],0}} + \|\nabla\rho\|_{Y^{[\frac{m-1}{2}],0}}\Big)\Big(\|\nabla\rho\|_{X^{m-2,0}} + |h|_{X^{m-2,1}}\Big),\\ &\big\|E[Z^{m-1},\nabla]\rho\big\| \lesssim \Big\|E\sum_{|\beta|\le m-2} c_{\alpha,\beta}\partial_z\Big(Z^\beta\rho\Big)\Big\| \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],0}}\Big)\|\nabla\rho\|_{X^{m-2,0}},\\ &\Big\|\sum_{\substack{k_1+\beta+k_2+\gamma=m-1,\\ k_1+\beta\neq 0}} c_{k_1,\beta,k_2,\gamma}\ \partial_t^{k_1}Z^\beta E\ \partial_t^{k_2}Z^\gamma\rho\Big\| \lesssim \Lambda\Big(|h|_{Y^{[\frac{m-2}{2}],1}} + \|\nabla\rho\|_{Y^{[\frac{m-1}{2}],0}}\Big)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\times\Big(\|\nabla\rho\|_{X^{k-1,0}} + |h|_{X^{k-1,1}}\Big).\end{aligned}\tag{64}$$

Putting (64) to the right hand side of (63), we have

$$\|\nabla\rho\|_{X^{m-1,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + \|\nabla\rho\|_{Y^{[\frac{m-1}{2}],0}}\Big)\Big(\|F\|_{X^{m-1,0}} + \|\nabla\rho\|_{X^{m-2,0}} + |h|_{X^{m-2,1}}\Big).\tag{65}$$

On the right hand side, we use induction for $\|\nabla\rho\|_{X^{m-2,0}}$ until zero order term $\|\nabla\rho\|$ and $\|\nabla\rho\|_{L^\infty}$ appear. To estimate $\|\nabla^2\rho\|_{X^{m-2,0}}$, we are suffice to focus on $\|\partial_{zz}\rho\|_{X^{m-2,0}}$. We apply $Z^{m-1}\partial_z$ to the equation to the system again, and make the equation with the form of (62). We follow the same argument as above to get the result.

$$\begin{aligned}\big\|\nabla^2\rho\big\|_{X^{m-2,0}} \ &\lesssim\ \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|\nabla\cdot F\|_{H^1_{tan}} + \|F\|_{H^2_{tan}}\Big)\\ &\times\ \ (\|F\|_{X^{m-1,0}} + \|\nabla\cdot F\|_{X^{m-2,0}} + |h|_{X^{m-2,1}}).\end{aligned}$$

□

For homogeneous elliptic problem, *i.e* source $F=0$, we can derive standard Sobolev regularity.

Lemma 21. *Let $m\ge 6$ and ρ solves the following system,*

$$-\nabla\cdot(E\nabla\rho) = 0, \quad in \quad S, \quad \rho|_{z=0} := \rho(t,x,y,0) = f^b,$$

Then we have the following estimate.

$$\|\nabla\rho\|_{\mathcal{H}^{m-1,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + |f^b|_{Y^{[\frac{m-1}{2}],0}}\Big)\Big(|h|_{X^{m-1,\frac12}} + |f^b|_{X^{m-1,\frac12}}\Big),$$

where function space $\mathcal{H}^{m,s}$ is defined in Definition 4.

Proof. We decompose $\rho = \rho^H + \rho^r$, where ρ^H absorbs boundary data and ρ^r solves

$$-\nabla \cdot (E\nabla\rho^r) = \nabla \cdot (E\nabla\rho^H), \quad \rho^r|_{z=0} = 0.$$

We choose ρ^H as $\hat{\rho}^H(\xi, z) = \chi(z\xi)\hat{f}^b$, where $\hat{\rho}$ is Fourier transform with respect to horizontal variables x, y. ξ is corresponding two dimensional frequency variable. Using Proposition 13, we get

$$\|\nabla\rho^H\|_{\mathcal{H}^{m-1,0}} \lesssim |f^b|_{X^{m-1,\frac{1}{2}}}, \quad \text{and} \quad \|\rho^H\|_{\mathcal{K}^{m-1,0}} \lesssim |f^b|_{Y^{m-1,0}}, \tag{66}$$

where $\mathcal{H}^{m,s}$ and $\mathcal{K}^{m,s}$ are defined in Definition 4.

We use similar approach as we did in Lemma 20 for standard Sobolev regularity and we get

$$\begin{aligned}
\|\nabla\rho^r\|_{\mathcal{H}^{m-1,0}} \quad &\lesssim \quad \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|\nabla\cdot(E\nabla\rho^H)\|_{H^1_{tan}} + \|E\nabla\rho^H\|_{H^2_{tan}}\Big) \\
&\quad\quad \times(\|E\nabla\rho^H\|_{\mathcal{H}^{m-1,0}} + |h|_{X^{m-2,1}}) \\
&\lesssim \quad \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|E\nabla\rho^H\|_{H^2}\Big)(\|E\nabla\rho^H\|_{\mathcal{H}^{m-1,0}} + |h|_{X^{k-1,1}}),
\end{aligned}$$

where $\|E\nabla\rho^H\|_{\mathcal{H}^{m-1,0}}$ on the right hand side can be controlled using (66) and Proposition 9,

$$\|E\nabla\rho^H\|_{\mathcal{H}^{m-1,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],0}} + |f^b|_{Y^{[\frac{m-1}{2}],0}}\Big)(|h|_{X^{m-1,\frac{1}{2}}} + |f^b|_{X^{m-1,\frac{1}{2}}}).$$

Consequently, this yields estimate for $\nabla\rho^r$,

$$\|\nabla\rho^r\|_{\mathcal{H}^{m-1,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + |f^b|_{Y^{[\frac{m-1}{2}],0}}\Big)(|h|_{X^{m-1,\frac{1}{2}}} + |f^b|_{X^{m-1,\frac{1}{2}}}). \tag{67}$$

We put (66) and (67) together.

□

We use above Lemma 20 and 21 to estimate q^E, q^{NS} and q^S, those are defined in (52), (53), and (54).

Proposition 22. *Let $m \geq 6$ and q^E solves (52), we have the following estimate.*

$$\begin{aligned}
&\|\nabla q^E\|_{X^{m-1,0}} + \|\nabla^2 q^E\|_{X^{m-2,0}} \\
\lesssim \quad &\Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|\nabla v\|_2 + \|v\|_3\Big)(\|v\|_{X^{m-1,0}} + \|\nabla v\|_{X^{m-2,0}} + |h|_{X^{m-1,1}}).
\end{aligned}$$

Proof. We split $q^E = q_1^E + q_2^E$, where

$$\begin{aligned}
q_1^E \quad &= \quad 0, \quad q_1^E|_{z=0} = gh \\
q_2^E \quad &= \quad \nabla\cdot(P(v\cdot\nabla^\varphi v)) = \partial_z\varphi\nabla^\varphi v\cdot\nabla^\varphi v, \quad q_2^E|_{z=0} = 0.
\end{aligned}$$

Using Lemma 21

$$\|\nabla q_1^E\|_{\mathcal{H}^{m-1,0}} \quad \lesssim \quad \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3\Big)(|h|_{X^{m-1,\frac{1}{2}}}).$$

For q_2^E, using Lemma 20 and $\nabla^\varphi\cdot(v\cdot\nabla^\varphi v) = \nabla^\varphi v : (\nabla^\varphi v)^T$, which comes from divergence free condition,

$$\begin{aligned}
&\|\nabla q_2^E\|_{X^{m-1,0}} + \|\nabla^2 q_2^E\|_{X^{m-2,0}} \\
\lesssim \quad &\Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|\nabla v\|_{H^1_{tan}} + \|P(v\cdot\nabla^\varphi v)\|_{H^2_{tan}}\Big) \\
&\times(\|\nabla\cdot P(v\cdot\nabla^\varphi v)\|_{X^{m-1,0}} + \|P(v\cdot\nabla^\varphi v)\|_{X^{m-1,0}} + |h|_{X^{m-2,1}}) \\
\lesssim \quad &\Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|\nabla v\|_2 + \|v\|_3\Big)(\|v\|_{X^{m-1,0}} + \|\nabla v\|_{X^{m-2,0}} + |h|_{X^{m-2,1}}).
\end{aligned}$$

This yields estimate for $\|\nabla q^E\|_{X^{m-1,0}}$.

□

Proposition 23. *Let $m \geq 6$ and q^{NS} solves (53), we have the following estimate.*

$$\begin{aligned}\|\nabla q^{NS}\|_{\mathcal{H}^{m-1,0}} &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|\nabla v\|_{Y^{[\frac{m-1}{2}],0}}\Big) \\ &\quad \times(|h|_{X^{m-1,\frac{1}{2}}} + \varepsilon|h|_{X^{m-1,\frac{3}{2}}} + \varepsilon\|\nabla v\|_{X^{m-1,1}} + \varepsilon\|v\|_{X^{m-1,2}}).\end{aligned}$$

Proof. Applying Lemma 21 with $f^b = 2\varepsilon(S^\varphi v)^b \mathbf{n}\cdot\mathbf{n}$,

$$\begin{aligned}\|\nabla q^{NS}\|_{\mathcal{H}^{m-1,0}} &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \Big|\varepsilon(S^\varphi v)^b \mathbf{n}\cdot\mathbf{n}\Big|_{Y^{[\frac{m-1}{2}],0}}\Big) \\ &\quad \times(|h|_{X^{m-1,\frac{1}{2}}} + \Big|\varepsilon(S^\varphi v)^b \mathbf{n}\cdot\mathbf{n}\Big|_{X^{m-1,\frac{1}{2}}}) \\ &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|\nabla v\|_{Y^{[\frac{m-1}{2}],0}}\Big) \\ &\quad \times(|h|_{X^{m-1,\frac{1}{2}}} + \varepsilon\,|h|_{X^{m-1,\frac{3}{2}}} + \varepsilon|(\nabla v)^b|_{X^{m-1,\frac{1}{2}}}) \\ &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|\nabla v\|_{Y^{[\frac{m-1}{2}],0}}\Big) \\ &\quad \times(|h|_{X^{m-1,\frac{1}{2}}} + \varepsilon\,|h|_{X^{m-1,\frac{3}{2}}} + \varepsilon\,\|\nabla v\|_{X^{m-1,1}} + \varepsilon\,\|v\|_{X^{m-1,2}}),\end{aligned}$$

where we used Proposition 9 and 11 in the first step and used Lemma 19 in the second step. □

Proposition 24. *Let $m \geq 6$ and q^S solves (54), we have the following estimate.*

$$\|\nabla q^S\|_{\mathcal{H}^{m-1,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{2,\infty} + |h|_3 + |h|_{Y^{[\frac{m-1}{2}],2}}\Big)\Big(|h|_{X^{m-1,\frac{1}{2}}} + \Big|\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big|_{X^{m-1,\frac{3}{2}}}\Big).$$

Proof. Applying Lemma 21, $f^b = -\nabla\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}$,

$$\begin{aligned}\|\nabla q^S\|_{\mathcal{H}^{m-1,0}} &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \Big|\nabla\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big|_{Y^{[\frac{m-1}{2}],0}}\Big) \\ &\quad \times\Big(|h|_{X^{m-1,\frac{1}{2}}} + \Big|\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big|_{X^{m-1,\frac{3}{2}}}\Big) \\ &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{2,\infty} + |h|_3 + |h|_{Y^{[\frac{m-1}{2}],2}}\Big)\Big(|h|_{X^{m-1,\frac{1}{2}}} + \Big|\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big|_{X^{m-1,\frac{3}{2}}}\Big).\end{aligned}$$

□

We should also estimate L^∞-type for pressure. Since we have standard Sobolev regularity for q^{NS} and q^S, we can use standard Sobolev embedding. For q^E, we use anisotropic embedding Proposition 11.

Proposition 25. *Let $m \geq 6$. We have the following L^∞ type estimate for $q := q^E + q^{NS} + q^S$.*

$$\|\nabla q\|_{Y^{[\frac{m}{2}],0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|\nabla v\|_2 + \|v\|_3\Big)(\|\nabla v\|_{X^{[\frac{m}{2}]-1,0}} + |h|_{X^{[\frac{m}{2}]-1,1}}). \tag{68}$$

Proof. Using anisotropic Sobolev embedding and Proposition 22,

$$\begin{aligned}\|\nabla q^E\|^2_{Y^{[\frac{m}{2}],0}} &\lesssim \|\nabla^2 q^E\|^2_{X^{[\frac{m}{2}]+1,0}} + \|\nabla q^E\|^2_{X^{[\frac{m}{2}]+2,0}} \\ &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{[\frac{m}{2}]-1}{2}],1}} + |h|_{2,\infty} + |h|_3 + \|\nabla v\|_2 + \|v\|_3\Big) \\ &\quad \times(\|\nabla v\|_{X^{[\frac{m}{2}]-1,0}} + |h|_{X^{[\frac{m}{2}]-1,1}}).\end{aligned} \tag{69}$$

For q^{NS} and q^S, we can use standard Sobolev embedding from the regularity of Proposition 23 and 24.

$$\begin{aligned}\|\nabla q\|_{Y^{[\frac{m}{2}],0}} &\lesssim \|\nabla q^E\|_{Y^{[\frac{m}{2}],0}} + \|\nabla q^{NS}\|_{Y^{[\frac{m}{2}],0}} + \|\nabla q^S\|_{Y^{[\frac{m}{2}],0}} \\ &\lesssim \|\nabla^2 q^E\|^2_{X^{[\frac{m}{2}]+1,0}} + \|\nabla q^E\|^2_{X^{[\frac{m}{2}]+2,0}} + \|\nabla q^{NS}\|_{X^{[\frac{m}{2}],2}} + \|\nabla q^S\|_{X^{[\frac{m}{2}],2}}.\end{aligned}$$

By (69) and Proposition 23 and 24, we end the proof with $m \geq 6$. □

5. Energy Estimates for $Z^m \neq \partial_t^m$

In this section, we perform energy estimate of the system derived in Proposition 17 in S. As defined in (33), We will use notation $\mathrm{d}V_t$ for volume element in S,

$$\int_{\Omega_t} f \mathrm{d}V := \int_S f \circ \Phi dV_t, \quad \text{where} \quad dV_t = \partial_z \varphi(t,x,y,z) \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{70}$$

The following lemma gives integration by parts rule in fixed domain S, with our new derivatives in Definition 1 and volume element $\mathrm{d}V_t$

Lemma 26. *Let f and g are functions defined on S. Then, we have the following integration by parts rules in S.*

$$\begin{aligned}
\int_S \partial_i^\varphi f g dV_t &= -\int_S f \partial_i^\varphi g dV_t + \int_{z=0} f g N_i dy, \qquad i = 1,2,3, \\
\int_S \partial_t^\varphi f g dV_t &= \partial_t \int_S f g dV_t - \int_S f \partial_t^\varphi g dV_t - \int_{z=0} f g \partial_t h,
\end{aligned}$$

where $\mathbf{N} = (N_1, N_2, N_3)$.

Proof. This can be derived directly from standard integration by parts in Ω_t. See [1] for more detail. □

Corollary 27. *Let $v(t,\cdot)$ is a vector field on S, such that $\nabla^\varphi \cdot v = 0$. Then, for every smooth functions f, g and smooth vector field u, w, we have the following estimates.*

$$\begin{aligned}
\int_S (\partial_t^\varphi f + v \cdot \nabla^\varphi f) f dV_t &= \frac{1}{2}\partial_t \int_S |f|^2 dV_t - \frac{1}{2}\int_{z=0} (\partial_t h - v \cdot \mathbf{N}) dy, \\
\int_S (\Delta^\varphi f) g dV_t &= -\int_S \nabla^\varphi f \cdot \nabla^\varphi g dV_t + \int_{z=0} \nabla^\varphi f \cdot \mathbf{N} g dy, \\
\int_S \nabla^\varphi \cdot (S^\varphi u) \cdot w dV_t &= -\int_S S^\varphi u \cdot S^\varphi w dV_t + \int_{z=0} (S^\varphi u \mathbf{N}) \cdot w dy.
\end{aligned}$$

Proof. This comes from Lemma 27. See also [1]. □

Lemma 28. *Let v, h, and φ are smooth solutions of the system (7)-(10). Then we have the following energy identity.*

$$\frac{d}{dt}\Big(\int_S |v|^2 dV_t + g\int_{z=0} |h|^2 dy + 2\int_{\partial S} (\sqrt{1+|\nabla h|^2} - 1) dy \Big) + 4\varepsilon \int_S |S^\varphi v|^2 dV_t = 0. \tag{71}$$

Proof. Using (7), (8), (9), and Corollary 27, we have,

$$\frac{d}{dt}\int_S |v|^2 dV_t = 2\int_S \nabla^\varphi \cdot (2\varepsilon S^\varphi v - q) \cdot v dV_t$$

We use the last equality in Corollary 27 and (10) to get,

$$\begin{aligned}
\frac{d}{dt}\int_S |v|^2 dV_t + 4\varepsilon \int_S |S^\varphi v|^2 dV_t &= 2\int_{\partial S} (2\varepsilon S^\varphi v - qI)\mathbf{N} \cdot v dy \\
&= 2\int_{\partial S} \Big(-gh\mathbf{N} \cdot v + \nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \mathbf{N} \cdot v \Big) dy \\
&= -g\frac{d}{dt}\int_{\partial S} |h|^2 dy - 2\int_{\partial S} \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \nabla h_t dy \\
&= -g\frac{d}{dt}\int_{\partial S} |h|^2 dy - 2\frac{d}{dt}\int_{\partial S} (\sqrt{1+|\nabla h|^2} - 1) dy.
\end{aligned}$$

□

Before we perform energy estimate, we use define the $\Lambda_{m,\infty}$ which contain L^∞ type norms and L^2 type norms with finite order.

$$\Lambda_{m,\infty}(t) := \Lambda\Big(\frac{1}{c_0}, |h(t)|_{Y^{[\frac{m}{2}],1}} + \|\nabla v(t)\|_{Y^{[\frac{m}{2}],0}} + |h(t)|_3 + \|v(t)\|_3 + \|\nabla v(t)\|_2\Big), \tag{72}$$

The following proposition gives high order energy estimates for $Z^m \neq \partial_t^m$ in the system of Proposition 17.

Proposition 29. *Assume that Let $m \geq 6$ and (v, φ, q, h) is a smooth solution on $[0,T]$ of the system (7)-(10). Also assume that*

$$\partial_z\varphi \geq c_0, \quad |h|_{2,\infty} \leq \frac{1}{c_0}, \quad \forall t \in [0,T]. \tag{73}$$

Then, from the higher order system Proposition 17, we get the following high order energy estimate.

$$\begin{aligned}
&\|v(T)\|^2_{X^{m-1,1}} + \|h(T)\|^2_{X^{m-1,2}} + \varepsilon\int_0^T \|\nabla v(t)\|^2_{X^{m-1,1}}dt \\
&\leq \Lambda(\frac{1}{c_0})\sum_{Z^m\neq\partial_t^m}(\|(Z^m v)(0)\|_{L^2} + |(Z^m h)(0)|_{H^1}) \\
&\quad + \int_0^T \Lambda_{m,\infty}(t)\Big(\|v(t)\|^2_{X^{m,0}} + \|\nabla v(t)\|^2_{X^{m-1,0}} + \varepsilon\|\nabla v(t)\|^2_{X^{m,0}} + \varepsilon\|v(t)\|_{X^{m-1,2}} + |h(t)|^2_{X^{m-1,\frac{5}{2}}}\Big)dt.
\end{aligned}$$

Remark 30. *In this estimate, we do not treat $Z^m = \partial_t^m$ case. Therefore, function space of h is $X^{m-1,\frac{5}{2}}(S)$, instead of $X^{m,\frac{3}{2}}(S)$.*

Proof. Let $Z^m \neq \partial_t^m$. From the system in Proposition 17 and Corollary 27, we get,

$$\begin{aligned}
&\frac{d}{dt}\int_S |Z^m v|^2\,dV_t + 4\varepsilon\int_S |S^\varphi(Z^m v)|^2\,dV_t \\
&= 2\int_{\partial S}(2\varepsilon S^\varphi(Z^m v)^b - (Z^m q)^b I)\mathbf{N}\cdot(Z^m v)^b \mathrm{d}x\mathrm{d}y + R_S + R_C,
\end{aligned} \tag{74}$$

where

$$R_S := 2\varepsilon\int_S\{\nabla^\varphi\cdot\Theta^m(v) - D^m(S^\varphi v)\}\cdot Z^m v dV_t, \tag{75}$$

$$R_C := 2\int_S\{C^m(q) - C^m(T)\}(Z^m v) + C^m(d)(Z^m q)dV_t. \tag{76}$$

And we use the third equation in Proposition 17 (the continuity of stress tensor condition) for the boundary integral term on the right hand side. From the boundary condition,

$$\begin{aligned}
&2\int_{\partial S}(2\varepsilon S^\varphi(Z^m v)^b - (Z^m q)^b I)\mathbf{N}\cdot(Z^m v)^b \\
=&2\int_{\partial S}\Big\{-gZ^m h + \Big(\nabla\cdot\Big(\frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla h\langle\nabla h, \nabla Z^m h\rangle}{\sqrt{1+|\nabla h|^2}^3} + C^m(S)\Big)\Big)\Big\}\mathbf{N}\cdot(Z^m v)^b \\
&+ 2\int_{\partial S}\Big\{(q-gh)I - 2\varepsilon(S^\varphi v)^b + \nabla\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}I\Big\}(Z^m\mathbf{N})\cdot(Z^m v)^b + R_B, \\
=&2\int_{\partial S}\Big\{-gZ^m h + \Big(\nabla\cdot\Big(\frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla h\langle\nabla h, \nabla Z^m h\rangle}{\sqrt{1+|\nabla h|^2}^3} + C^m(S)\Big)\Big)\Big\} \\
&\times\Big(\partial_t Z^m h - v^b\cdot(Z^m\mathbf{N}) + C^m(KB)\Big) + R_B \\
=&-g\frac{\mathrm{d}}{\mathrm{dt}}\int_{\partial S}|Z^m h|^2 - \Big\{\frac{d}{dt}\int_{\partial S}\frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}} - \int_{\partial S}\frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}^3}\langle\nabla h, \nabla\partial_t h\rangle\Big\} \\
&+ R_S + R_C + R_B + P_1 + P_2 + P_3,
\end{aligned} \tag{77}$$

where R_S and R_C are defined in (75) and (76), and R_B, P_1, P_2, and P_3 are defined by

$$
\begin{aligned}
R_B &:= 2\int_{\partial S}(C^m(B)-\varepsilon(\Theta^m(v))^b\mathbf{N})\cdot(Z^m v)^b \\
P_1 &:= 2\int_{\partial S}\Big(\frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}}-\frac{\nabla h\langle\nabla h,\nabla Z^m h\rangle}{\sqrt{1+|\nabla h|^2}^3}+C^m(S)\Big)\cdot\nabla\Big(v^b\cdot(Z^m\mathbf{N})+C^m(KB)\Big) \\
&\quad +2\int_{\partial S}\Big(\frac{\nabla h\langle\nabla h,\nabla Z^m h\rangle}{\sqrt{1+|\nabla h|^2}^3}-C^m(S)\Big)\cdot\nabla\Big(\partial_t(Z^m h)\Big), \\
P_2 &:= 2\int_{\partial S}\left\{q-gh-2\varepsilon(S^\varphi v)+\nabla\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right\}(Z^m\mathbf{N})\cdot(Z^m v), \\
P_3 &:= 2g\int_{\partial S}Z^m h\Big(v^b\cdot Z^m\mathbf{N}+C^m(KB)\Big).
\end{aligned}
\tag{78}
$$

Now we estimate R_S, R_C, R_B, P_1, P_2, and P_3.

1) Estimate of R_B

$$
\begin{aligned}
|R_B| &= 2\Big|\int_{\partial S}(C^m(B)-\varepsilon(\Theta^m(v))^b\mathbf{N})\cdot(Z^m v)^b\mathrm{d}x\mathrm{d}y\Big| \\
&\lesssim \Big|C^m(B)-\varepsilon(\Theta^m(v))^b\mathbf{N}\Big|_{L^2(\partial S)}|(Z^m v)^b|_{L^2(\partial S)} \\
&\lesssim \varepsilon\Lambda\Big(\frac{1}{c_0},|h|_{Y^{[\frac{m}{2}],1}}+\|\nabla v\|_{Y^{[\frac{m}{2}],0}}\Big)(|h|_{X^{m-1,1}}+|v^b|_{X^{m,\frac{1}{2}}})\|v\|_{X^m} \\
&\lesssim \varepsilon\Lambda\Big(\frac{1}{c_0},|h|_{Y^{[\frac{m}{2}],1}}+\|\nabla v\|_{Y^{[\frac{m}{2}],0}}\Big)(|h|^2_{X^{m-1,1}}+|v^b|^2_{X^{m,\frac{1}{2}}}+\|v\|^2_{X^m}),
\end{aligned}
\tag{79}
$$

by Proposition 9 and 17.

2) Estimate of P_2 Note that

$$
\begin{aligned}
P_2 &:= 2\int_{\partial S}\Big\{q-gh-2\varepsilon(S^\varphi v)+\nabla\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big\}(Z^m\mathbf{N})\cdot(Z^m v) \\
&= 2\int_{\partial S}\big\{q^{NS}|_{z=0}I-2\varepsilon(S^\varphi v)\big\}(Z^m\mathbf{N})\cdot(Z^m v) \\
&= 4\varepsilon\int_{\partial S}\{(S^\varphi v)\mathbf{n}\cdot\mathbf{n}I-(S^\varphi v)\}(Z^m\mathbf{N})\cdot(Z^m v).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|P_2| &= \Big|2\int_{\partial S}\Big\{q-gh-2\varepsilon(S^\varphi v)+\nabla\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big\}(Z^m\mathbf{N})\cdot(Z^m v)\mathrm{d}x\mathrm{d}y\Big| \\
&\lesssim 2\varepsilon\Big|Z^m\mathbf{N}\Big|_{-\frac{1}{2}}\Big|\{(S^\varphi v)\mathbf{n}\cdot\mathbf{n}I-(S^\varphi v)\}^b(Z^m v)^b\Big|_{\frac{1}{2}} \\
&\lesssim \varepsilon\,|h|_{X^{m,\frac{1}{2}}}\,|v^b|_{X^{m,\frac{1}{2}}}\Big|(S^\varphi v)\mathbf{n}\cdot\mathbf{n}I-(S^\varphi v)\Big|_{1,\infty} \\
&\lesssim \varepsilon\Lambda\Big(\frac{1}{c_0},|h|_{Y^{[\frac{m}{2}],1}}+\|\nabla v\|_{Y^{[\frac{m}{2}],0}}\Big)|h|_{X^{m,\frac{1}{2}}}|v^b|_{X^{m,\frac{1}{2}}},
\end{aligned}
\tag{80}
$$

by Proposition 9.

3) Estimate of P_3 Since,

$$
P_3 := 2g\int_{\partial S}Z^m h\Big(v^b\cdot Z^m\mathbf{N}+C^m(KB)\Big),
$$

$$
\begin{aligned}
|P_3| &\leq \Big| \int_{\partial S} Z^m h (v^b \cdot Z^m \mathbf{N} + C^m(KB)) \Big| \\
&\leq |Z^m h(t)|_{L^2(\partial S)} \Big\{ |v^b \cdot Z^m \mathbf{N}|_{L^2(\partial S)} + |C^m(KB)(t)|_{L^2(\partial S)} \Big\} \\
&\lesssim |h|_{X^{m,0}} \left\{ |v^b|_{L^\infty} |h|_{X^{m,1}} + \|C^m(KB)\| \right\} \\
&\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\Big) \Big(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m,1}}^2 \Big),
\end{aligned}
\tag{81}
$$

by Proposition 9 and $\|C^m(KB)\|$ estimate in Proposition 17.

4) Estimate of R_C

$$
\begin{aligned}
|R_C| &\lesssim \Lambda_0 \Big(\|C^m(d)\|_{L^2} \|Z^m q(t)\|_{L^2} + \|T^m(v)\|_{L^2} \|Z^m v(t)\|_{L^2} + \|C^m(q)\|_{L^2} \|Z^m v(t)\|_{L^2} \Big) \\
&\lesssim \Lambda_0 \Big(\|C^m(d)\| \|q(t)\|_{X^{m,0}} + \|T^m(v)\| \|v(t)\|_{X^{m,0}} + \|C^m(q)\| \|v(t)\|_{X^{m,0}} \Big) \\
&\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}} + \|\nabla q\|_{Y^{[\frac{m}{2}],0}}\Big) \Big(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m,\frac{1}{2}}}^2 + \|\nabla q\|_{X^{m-1,0}}^2 \Big).
\end{aligned}
\tag{82}
$$

5) Estimate of R_S From definition of R_S,

$$
\begin{aligned}
R_S &= 2\varepsilon \int_S \{ \nabla^\varphi \cdot \Theta^m(v) - D^m(S^\varphi v) \} \cdot Z^m v dV_t \\
&= -2\varepsilon \int_S \Theta^m(v) : \nabla^\varphi (Z^m v) + 2\varepsilon \int_{\partial S} \Theta^m(v) N \cdot (Z^m v)^b + 2\varepsilon \sum_{i,j} \int_S C_j^m (S^\varphi v)_{ij} (Z^m v)_i dV_t,
\end{aligned}
$$

We have only two types of integrals. ($m = m_1 + m_2$ and both are non-zero.)

$$
\begin{aligned}
I_1 &:= \int_S \partial_z (Z^m v_i) Z^{m_1} (S^\varphi v)_{ij} Z^{m_2} \Big(\frac{\partial_i \varphi}{\partial_z \varphi} \Big), \\
I_2 &:= \int_{\partial S} (Z^m v_i) Z^{m_1} (S^\varphi v)_{ij} Z^{m_2} \Big(\frac{\partial_i \varphi}{\partial_z \varphi} \Big).
\end{aligned}
$$

By $L^\infty L^2 L^2$ Holder inequality, we get,

$$
\begin{aligned}
|I_1| &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\Big) \Big(\|\nabla v\|_{X^{m-1,0}}^2 + \|S^\varphi v\|_{X^{m-1,0}}^2 + |h|_{X^{m-1,\frac{1}{2}}} \Big), \\
|I_2| &\lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\Big) \Big(\|v\|_{X^m}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m-1,\frac{1}{2}}} \Big).
\end{aligned}
$$

Therefore, we get

$$
|R_S| \lesssim \varepsilon \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\Big) \Big(\|v\|_{X^m}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + \|S^\varphi v\|_{X^{m-1,0}}^2 + |h|_{X^{m-1,\frac{1}{2}}}^2 \Big). \tag{83}
$$

6) Estimate of P_1

Let us decompose $P_1 = P_{1,1} + P_{1,2}$, where

$$
\begin{aligned}
P_{1,1} &:= 2 \int_{\partial S} \Big(\frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \langle \nabla h, \nabla Z^m h \rangle}{\sqrt{1 + |\nabla h|^2}^3} + C^m(S) \Big) \cdot \nabla \Big(v^b \cdot (Z^m \mathbf{N}) + C^m(KB) \Big) \\
P_{1,2} &:= 2 \int_{\partial S} \Big(\frac{\nabla h \langle \nabla h, \nabla Z^m h \rangle}{\sqrt{1 + |\nabla h|^2}^3} - C^m(S) \Big) \cdot \nabla \Big(\partial_t (Z^m h) \Big).
\end{aligned}
\tag{84}
$$

Let treat $P_{1,2}$ first. We split $P_{1,2} := P_{1,2,1} + P_{1,2,2}$ again, where

$$
\begin{aligned}
P_{1,2,1} &:= 2\int_{\partial S} \frac{\langle \nabla h, \nabla Z^m h\rangle}{\sqrt{1+|\nabla h|^2}^3} \langle \nabla h, \nabla Z^m \partial_t h\rangle := P_{1,2,1,1} + P_{1,2,1,2} + P_{1,2,1,3} \\
&:= \frac{d}{dt}\int_{\partial S} \frac{\langle \nabla h, \nabla Z^m h\rangle^2}{\sqrt{1+|\nabla h|^2}^3} - 2\int_{\partial S} \frac{\langle \nabla h, \nabla Z^m h\rangle}{\sqrt{1+|\nabla h|^2}^3} \langle \nabla \partial_t h, \nabla Z^m h\rangle \\
&\quad + 3\int_{\partial S} \frac{\langle \nabla h, \nabla \partial_t h\rangle}{\sqrt{1+|\nabla h|^2}^3} \langle \nabla h, \nabla Z^m h\rangle^2, \\
P_{1,2,2} &= -2\int_{\partial S} C^m(S)\cdot \nabla \partial_t Z^m h.
\end{aligned}
\tag{85}
$$

$P_{1,2,1,1} := \frac{d}{dt}\int_{\partial S} \frac{\langle \nabla h, \nabla Z^m h\rangle^2}{\sqrt{1+|\nabla h|^2}^3} \mathrm{d}x\mathrm{d}y$ should be absorbed by the energy on the left hand side under the assumption that $|h|_{1,\infty}$ is bounded. Controls of $P_{1,2,1,2}$ and $P_{1,2,1,3}$ are as following,

$$
|P_{1,2,1,2} + P_{1,2,1,3}| \le \Lambda\big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\big)\, |h|^2_{X^{m,1}}\,. \tag{86}
$$

To estimate $P_{1,2,2}$, we use integration by part in horizontal variables to get,

$$
\begin{aligned}
|P_{1,2,2}| &\le \Big|\int_{\partial S} C^m(S)\cdot \nabla(\partial_t Z^m h)\Big| = \Big|\int_{\partial S} \nabla\cdot C^m(S)\partial_t Z^m h\Big| \\
&\le \|\nabla\cdot C^m(S)\| |\partial_t h|_{X^{m-1,1}} \\
&\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\big) |h|_{X^{m,1}}\big(\|v\|_{X^{m,0}} + |h|_{X^{m,1}}\big).
\end{aligned}
\tag{87}
$$

For $P_{1,1}$, by definition (84), it can be controlled by

$$
\begin{aligned}
|P_{1,1}| &\le \Big|\frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla h\langle \nabla h, \nabla Z^m h\rangle}{\sqrt{1+|\nabla h|^2}^3} + C^m(S)\Big|_{\frac{1}{2}} \Big|v^b\cdot(Z^m\mathbf{N}) + C^m(KB)\Big|_{\frac{1}{2}} \\
&\le \Lambda\big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\big)\Big(\|v\|^2_{X^{m,0}} + \|\nabla v\|^2_{X^{m-1,0}} + |h|^2_{X^{m-1,\frac{5}{2}}}\Big),
\end{aligned}
\tag{88}
$$

by Proposition 9, 11, 12, and $C^m(KB)$ estimate in Proposition 17.
We combine (84), (85), (86), (87), and (88), to get

$$
\begin{aligned}
P_1 &\le \frac{d}{dt}\int_{\partial S} \frac{\langle \nabla h, \nabla Z^m h\rangle^2}{\sqrt{1+|\nabla h|^2}^3} \\
&\quad + \Lambda\big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}\big)\Big(\|v\|^2_{X^{m,0}} + \|\nabla v\|^2_{X^{m-1,0}} + |h|^2_{X^{m-1,\frac{5}{2}}}\Big).
\end{aligned}
\tag{89}
$$

7) Estimate of $\Big|\int_{\partial S} \frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}^3}\langle \nabla h, \nabla \partial_t h\rangle\Big|$ **.**

$$
\begin{aligned}
\Big|\int_{\partial S} \frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}^3}\langle \nabla h, \nabla \partial_t h\rangle\Big| &\le \Lambda(\frac{1}{c_0})\int_{\partial S} |\nabla Z^m h|^2 \Big|\langle \nabla h, \nabla\partial_t h\rangle\Big| dA \\
&\le \Lambda(\frac{1}{c_0})\,\big|\langle \nabla h, \nabla(v^b\cdot\mathbf{N})\rangle\big|_{L^\infty} |h|^2_{X^{m,1}} \le \Lambda\Big(\frac{1}{c_0}, |h|_{2,\infty} + \|\nabla v\|_{1,\infty}\Big)|h|^2_{X^{m,1}}.
\end{aligned}
\tag{90}
$$

We collect (74), (77), and (79)-(90), to get, (with small $\varepsilon \ll 1$,)

$$\begin{aligned}
&\frac{d}{dt}\int_S |Z^m v|^2\, dV_t + 4\varepsilon\int_S |S^\varphi(Z^m v)|^2\, dV_t + g\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial S}|Z^m h|^2 \\
&+\frac{d}{dt}\Big(\int_{\partial S}\frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}} - \int_{\partial S}\frac{\langle \nabla h, \nabla Z^m h\rangle^2}{\sqrt{1+|\nabla h|^2}^3}\Big) \le \Lambda\big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}} + \|\nabla q\|_{Y^{[\frac{m}{2}],0}}\big) \\
&\times\Big(\|v\|^2_{X^{m,0}} + \|\nabla v\|^2_{X^{m-1,0}} + \varepsilon\|\nabla v\|^2_{X^{m,0}} + |h|^2_{X^{m-1,\frac{5}{2}}} + \|\nabla q\|^2_{X^{m-1,0}}\Big).
\end{aligned} \tag{91}$$

Note that

$$\int_{\partial S}\frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}} - \int_{\partial S}\frac{\langle \nabla h, \nabla Z^m h\rangle^2}{\sqrt{1+|\nabla h|^2}^3} \ge \int_{\partial S}\frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}}\Big(\frac{1}{1+|\nabla h|^2}\Big) \ge \Lambda(c_0)\int_{\partial S}\frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}}. \tag{92}$$

We consider every $Z^m \neq \partial_t^m$ and sum for all the result. Then we use (92) and Proposition 16, 11, and integrate in time for $t \in [0,T]$, to get the result,

$$\begin{aligned}
&\|v(T)\|^2_{X^{m-1,1}} + \|h(T)\|^2_{X^{m-1,2}} + \varepsilon\int_0^T \|\nabla v(t)\|^2_{X^{m-1,1}} dt \\
&\le \Lambda(\frac{1}{c_0})\sum_{Z^m\neq\partial_t^m}(\|(Z^m v)(0)\|_{L^2} + |(Z^m h)(0)|_{H^1}) \\
&\quad + \Lambda(\frac{1}{c_0})\int_0^T \Lambda\big(\frac{1}{c_0}, |h(t)|_{Y^{[\frac{m}{2}],1}} + \|\nabla v(t)\|_{Y^{[\frac{m}{2}],0}} + \|\nabla q(t)\|_{Y^{[\frac{m}{2}],0}}\big) \\
&\qquad \times\Big(\|v(t)\|^2_{X^{m,0}} + \|\nabla v(t)\|^2_{X^{m-1,0}} + \varepsilon\|\nabla v(t)\|^2_{X^{m,0}} + |h(t)|^2_{X^{m-1,\frac{5}{2}}} + \|\nabla q(t)\|^2_{X^{m-1,0}}\Big) dt.
\end{aligned}$$

We use pressure estimates Proposition 22, 23, 24, and 25 to finish our proof.

□

In Proposition 29, we should control $|h|_{X^{m-1,\frac{5}{2}}}$. In the next section we claim that this term can be controlled by $|\partial_t h|_{X^{m-1,1}} \le |h|_{X^{m,1}}$.

6. Dirichlet-Neumann Operator estimate on the boundary

In this section, we claim that, on the boundary $\partial_x^{3/2}h$ can be controlled by $\partial_t h$. We start with this section with a lemma which is needed to prove the next proposition.

Lemma 31. *There exists $c > 0$ such that for every $h \in W^{1,\infty}(\mathbb{R}^2)$,*

$$(G[h]f^b, f^b) \ge c(1 + \|h\|_{W^{1,\infty}(\mathbb{R}^2)})^{-2}\left\|\frac{|\nabla|}{(1+|\nabla|)^{1/2}}f^b\right\|^2_{L^2(\mathbb{R}^2)}, \quad \forall f^b \in H^{\frac{1}{2}}(\mathbb{R}^2), \tag{93}$$

where $G[h]f^b$ means Dirichlet-Neumann operator,

$$G[h]f^b := (\nabla f)^b \cdot \mathbf{N},$$

where $\mathbf{N} = (-\nabla h, 1)$ and f solves harmonics equation, $\Delta f = 0$.

Proof. See Proposition 3.4 of [2]. □

We can apply above lemma for $f = p^S$, since $\Delta p^S = \Delta^\varphi q^S = 0$ in (54).

Proposition 32. *Assume that (v, φ, q, h) is a smooth solution on $t \in [0,T]$ of the system (7)-(10) and*

$$\partial_z\varphi \ge c_0, \quad \forall t \in [0,T]. \tag{94}$$

Then h enjoys the following estimate.

$$\int_0^T |Z^{m-1}\nabla h|_{\frac{3}{2}}^2 dt \le \Lambda\Big(\frac{1}{c_0}, \|h(0)\|_{X^{m,1}}\Big) + \Lambda_{m,\infty}(T)\Big\{\theta|Z^{m-1}\partial_t h(T)|_{L^2}^2 + \sqrt{T}|Z^{m-1}\nabla\partial_t h(T)|_{L^2}^2\Big\}$$
$$+ (1+\sqrt{T})\int_0^T \Lambda_{m,\infty}(t)\Big(\|v\|_{X^{m-1,1}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m,1}}^2 + \varepsilon\|\nabla v\|_{X^{m-1,1}}^2 + \varepsilon\|v\|_{X^{m-1,2}}^2\Big)dt, \tag{95}$$

where $\Lambda_{m,\infty}(t)$ is defined in (72) and $\theta > 0$ is sufficiently small.

Proof. From kinematic boundary condition (9), $h_t = v^b \cdot \mathbf{N}$, we get $\partial_{tt} h = v_t^b \cdot \mathbf{N} + v^b \cdot \mathbf{N}_t$. We apply Z^{m-1} to this equation, where $\alpha_3 = 0$, because we are on the boundary, i.e. we do not apply any normal derivatives.

$$\begin{aligned}
\partial_{tt}\Big(Z^{m-1}h\Big) &= \Big(Z^{m-1}v_t^b\Big)\cdot\mathbf{N} + v_t^b\cdot\Big(Z^{m-1}\mathbf{N}\Big) + \big[Z^{m-1}, v_t^b, \mathbf{N}\big] + \Big(Z^{m-1}v^b\Big)\cdot\mathbf{N}_t \\
&\quad + v^b\cdot\Big(Z^{m-1}\mathbf{N}_t\Big) + \big[Z^{m-1}, v^b, \mathbf{N}_t\big] \\
&:= \Big\{ - Z^{m-1}(v\cdot\nabla^\varphi v)^b - Z^{m-1}(\nabla^\varphi q^E + \nabla^\varphi q^{NS} + \nabla^\varphi q^S)^b + 2\varepsilon Z^{m-1}(\nabla^\varphi\cdot S^\varphi v)^b\Big\}\cdot\mathbf{N} \\
&\quad + (I_1+I_2+I_3+I_4+I_5) \\
&:= G[h]V^b + (I_1+I_2+I_3+I_4+I_5) + (J_1+J_2+J_3),
\end{aligned} \tag{96}$$

where V^b, $I_{1,2,3,4,5}$, and $J_{1,2,3}$ are defined by

$$\begin{aligned}
V^b &:= Z^{m-1}\nabla\cdot\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big), \\
I_1 &:= v_t^b\cdot\Big(Z^{m-1}\mathbf{N}\Big), \\
I_2 &:= \big[Z^{m-1}, v_t^b, \mathbf{N}\big], \\
I_3 &:= \Big(Z^{m-1}v^b\Big)\cdot\mathbf{N}_t, \\
I_4 &:= v^b\cdot\Big(Z^{m-1}\mathbf{N}_t\Big), \\
I_5 &:= \big[Z^{m-1}, v^b, \mathbf{N}_t\big], \\
J_1 &:= -Z^{m-1}(v\cdot\nabla^\varphi v)^b\cdot\mathbf{N}, \\
J_2 &:= -Z^{m-1}(\nabla^\varphi q^E + \nabla^\varphi q^{NS})^b\cdot\mathbf{N}, \\
J_3 &:= 2\varepsilon Z^{m-1}(\nabla^\varphi\cdot S^\varphi v)^b\cdot\mathbf{N}.
\end{aligned} \tag{97}$$

We take dot product with V^b and $\int_0^T\int_{\partial S}$ to the equation (96), i.e,

$$\begin{aligned}
\int_0^T (G[h]V^b, V^b)dt \le \int_0^T\int_{\partial S} Z^{m-1}h_{tt}V^b dt \\
+ \int_0^T\int_{\partial S}(I_1+I_2+I_3+I_4+I_5+J_1+J_2+J_3)V^b dt.
\end{aligned} \tag{98}$$

We estimate upper bound of RHS terms and lower bound of LHS.

1) Estimate of $\int_0^T\int_{\partial S} Z^{m-1}h_{tt}V^b$.

$$\int_0^T\int_{\partial S} Z^{m-1}h_{tt}\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt = -\int_0^T\int_{\partial S} Z^{m-1}\nabla h_{tt}\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt \tag{99}$$

$$= -\Big[\int_{\partial S} Z^{m-1}\nabla h_t \cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)\Big]_0^T + \int_0^T \int_{\partial S} Z^{m-1}\nabla h_t \cdot Z^{m-1}\partial_t(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt$$
$$\leq \Lambda\Big(\frac{1}{c_0}, \|h(0)\|_{X^{m,1}}\Big) - \int_{\partial S} Z^{m-1}\nabla h_t \cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)\Big|_{t=T}$$
$$+ \int_0^T \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}})\|\nabla h_t\|^2_{X^{m-1,0}}dt$$

To estimate $\int_{\partial S} Z^{m-1}\nabla h_t \cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)\Big|_{t=T}$ on the RHS, we compute

$$\int_{\partial S} Z^{m-1}\nabla h_t \cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)\Big|_{t=T}$$
$$= \int_{\partial S} Z^{m-1}\nabla h_t(T)\Big\{Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)(0) + \int_0^T Z^{m-1}\partial_t\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt\Big\}$$
$$= -\int_{\partial S} Z^{m-1}h_t(T)Z^{m-1}\nabla\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)(0) + \int_{\partial S} Z^{m-1}\nabla h_t(T)\int_0^T Z^{m-1}\partial_t\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt$$
$$\leq \Lambda\Big(\frac{1}{c_0}, \|h(0)\|_{X^{m,1}}\Big) + \theta|Z^{m-1}h_t(T)|^2_{L^2}$$
$$+ |Z^{m-1}\nabla\partial_t h(T)|_{L^2(\partial S)}\Big|\int_0^T Z^{m-1}\nabla\partial_t\nabla h(t)dt\Big|_{L^2(\partial S)}.$$

Using Jensen's inequality for the last term, we get

$$\int_{\partial S} Z^{m-1}\nabla h_t \cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)\Big|_{t=T}$$
$$\leq \Lambda\Big(\frac{1}{c_0}, \|h(0)\|_{X^{m,1}}\Big) + \theta|Z^{m-1}h_t(T)|^2_{L^2}$$
$$+ \sqrt{T}|Z^{m-1}\nabla h_t(T)|_{L^2}\Big\{\int_0^T \Big|Z^{m-1}\partial_t\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)\Big|^2_{L^2}dt\Big\}^{1/2}$$
$$\leq \Lambda\Big(\frac{1}{c_0}, \|h(0)\|_{X^{m,1}}\Big) + \theta|Z^{m-1}h_t(T)|^2_{L^2} + 2\sqrt{T}|Z^{m-1}\nabla h_t(T)|^2_{L^2}$$
$$+ \sqrt{T}\int_0^T \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}})|Z^{m-1}\nabla h_t(t)|^2_{L^2}dt,$$

where sufficiently small θ came from Young's inequality.

Now we estimate other terms involving I_k and J_k.

2) Estimate of $\int_0^T \int_{\partial S} I_1 \nabla \cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt$. Using 9,

$$\int_0^T \int_{\partial S} v_t^b \cdot (Z^{m-1}\mathbf{N})\nabla \cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt \leq \int_0^T \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{2,\infty})|h|^2_{X^{m-2,2}}dt.$$

3) Estimate of $\int_0^T \int_{\partial S} I_2 \nabla \cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt$. Using 9,

$$\int_0^T \int_{\partial S} \big[Z^{m-1}, v_t^b, \mathbf{N}\big]\,\nabla \cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt$$
$$\leq \int_0^T \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})\Big(\|v\|^2_{X^{m-1,1}} + \|\nabla v\|^2_{X^{m-1,0}} + |h|^2_{X^{m-2,2}}\Big)dt,$$

We used trace estimate Lemma 11.

4) Estimate of $\int_0^T\int_{\partial S} I_3\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt$.

$$\begin{aligned}&\int_0^T\int_{\partial S}\Big(Z^{m-1}v^b\Big)\cdot\mathbf{N}_t\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt\\&\le\int_0^T\Lambda(\frac{1}{c_0},|h|_{Y^{[\frac{m}{2}],1}}+\|\nabla v\|_{2,\infty})\Big(\|v\|^2_{X^{m-1,1}}+\|\nabla v\|^2_{X^{m-1,0}}+|h|^2_{X^{m-2,2}}\Big)dt,\end{aligned}$$

similarly as above.

5) Estimate of $\int_0^T\int_{\partial S} I_4\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt$.

$$\int_0^T\int_{\partial S}v^b\cdot\Big(Z^{m-1}\mathbf{N}_t\Big)\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt\le\int_0^T\Lambda(\frac{1}{c_0},|h|_{Y^{[\frac{m}{2}],1}}+\|\nabla v\|_{2,\infty})|h|^2_{X^{m-2,2}}dt.$$

6) Estimate of $\int_0^T\int_{\partial S} I_5\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt$.

$$\begin{aligned}&\int_0^T\int_{\partial S}\big[Z^{m-1},v_t^b,\mathbf{N}\big]\,\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt\\&\le\int_0^T\Lambda(\frac{1}{c_0},|h|_{Y^{[\frac{m}{2}],1}}+\|\nabla v\|_{2,\infty})\Big(\|v\|^2_{X^{m-1,1}}+\|\nabla v\|^2_{X^{m-1,0}}+|h|^2_{X^{m-2,2}}\Big)dt.\end{aligned}$$

7) Estimate of $\int_0^T\int_{\partial S} J_1\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt$. Using Lemma 11, we can replace v^b into v by giving $|\cdot|_{H^{-1/2}}$. Hence,

$$\begin{aligned}&\int_0^T\int_{\partial S}-Z^{m-1}(v\cdot\nabla^\varphi v)^b\cdot\mathbf{N}\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt\\&\le\int_0^T\Big|Z^{m-1}(v\cdot\nabla^\varphi v)^b\Big|_{-\frac12}\Big|\mathbf{N}\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)\Big|_{\frac12}dt\\&\le\int_0^T\Lambda(\frac{1}{c_0},|h|_{Y^{[\frac{m}{2}],1}}+\|\nabla v\|_{2,\infty})(\|v\|_{X^{m-1,1}}+\|\nabla v\|_{X^{m-1,0}})|h|_{X^{m-1,\frac52}}dt\\&\lesssim\int_0^T\Lambda(\frac{1}{c_0},|h|_{Y^{[\frac{m}{2}],1}}+\|\nabla v\|_{2,\infty})(\|v\|^2_{X^{m-1,1}}+\|\nabla v\|^2_{X^{m-1,0}})dt\\&\quad+\theta\int_0^T\Lambda(\frac{1}{c_0},|h|_{Y^{[\frac{m}{2}],1}}+\|\nabla v\|_{2,\infty})|\nabla h|^2_{X^{m-1,\frac32}}dt,\end{aligned}$$

where θ is sufficiently small constant from Young's inequality.

8) Estimate of $\int_0^T\int_{\partial S} J_2\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt$. We perform estimate same as J_1 case. Hence, using pressure estimate in section 4,

$$\begin{aligned}&\int_0^T\int_{\partial S}-Z^{m-1}(\nabla^\varphi q^E+\nabla^\varphi q^{NS})^b\cdot\mathbf{N}\nabla\cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt\\&\lesssim\int_0^T\Lambda(\frac{1}{c_0},|h|_{Y^{[\frac{m}{2}],1}}+\|\nabla v\|_{2,\infty})(|h|^2_{X^{m-1,2}}+\|v\|^2_{X^{m-1,1}}+\varepsilon\|\nabla v\|^2_{X^{m-1,1}}+\varepsilon\|v\|^2_{X^{m-1,2}})dt\\&+\theta\int_0^T\Lambda(\frac{1}{c_0},|h|_{Y^{[\frac{m}{2}],1}}+\|\nabla v\|_{2,\infty})|\nabla h|^2_{X^{m-1,\frac32}}dt.\end{aligned}$$

9) Estimate of $\int_0^T \int_{\partial S} J_3 \nabla \cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt$. Similarly as above estimate, we get

$$\begin{aligned}
&\int_0^T \int_{\partial S} 2\varepsilon Z^{m-1}(\nabla^\varphi \cdot S^\varphi v)^b \cdot \mathbf{N}\nabla \cdot Z^{m-1}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)dt \\
&\leq \int_0^T \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(\varepsilon\|\nabla v\|^2_{X^{m-1,1}} + \varepsilon\|v\|^2_{X^{m-1,2}})dt \\
&+ \theta \int_0^T \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})|\nabla h|^2_{X^{m-1,\frac{3}{2}}} dt.
\end{aligned}$$

10) Estimate of Dirichlet-Neumann operator term. The left hand side of (98),

$$\Big(1 + |h|_{W^{1,\infty}}\Big)^{-2} \left|\frac{|\nabla|}{\Big(1+|\nabla|\Big)^{1/2}} V^b\right|^2_{L^2} \leq \Big(G[h]V^b, V^b\Big). \tag{100}$$

Note that, ($\mathcal{F}$ means Fourier Transform with respect to horizontal direction.)

$$\begin{aligned}
\Big|\frac{|\nabla|}{(1+|\nabla|)^{1/2}} V^b\Big|_{L^2(\partial S)} &\geq \Big|\frac{|\xi|^2}{(1+|\xi|)^{1/2}}\mathcal{F}\Big(Z^{m-1}\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)\Big|_{L^2(\partial S)} \\
&= \Big|\Big(\frac{1+|\xi|^2}{(1+|\xi|)^{1/2}} - \frac{1}{(1+|\xi|)^{1/2}}\Big)\mathcal{F}\Big(Z^{m-1}\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)\Big|_{L^2(\partial S)} \\
&\gtrsim \Big|Z^{m-1}\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big|_{H^{\frac{3}{2}}(\partial S)} - \Big|Z^{m-1}\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big|_{H^{-\frac{1}{2}}(\partial S)}.
\end{aligned} \tag{101}$$

Now, let Λ^s be horizontal Fourier multiplier by $(1+|\xi|^2)^{s/2}$. We split $\Big|Z^{m-1}\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big|_{H^{\frac{3}{2}}(\partial S)}$ into highest order term and other low order terms. Then we can rewrite the first term in the RHS as

$$\begin{aligned}
&\Big|Z^{m-1}\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big|_{H^{\frac{3}{2}}(\partial S)} \\
&\geq \Big|\frac{Z^{m-1}\nabla h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla\langle\nabla h, Z^{m-1}\nabla h\rangle}{\sqrt{1+|\nabla h|^2}^3}\Big|_{H^{\frac{3}{2}}(\partial S)} - |C^{m-1}(S)|_{H^{\frac{3}{2}}(\partial S)} \\
&\geq \Big|\frac{\Lambda^{3/2}Z^{m-1}\nabla h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla\langle\nabla h, \Lambda^{3/2}Z^{m-1}\nabla h\rangle}{\sqrt{1+|\nabla h|^2}^3}\Big|_{L^2(\partial S)} \\
&\quad - \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}})|Z^{m-1}\nabla h|_{H^1} - |C^{m-1}(S)|_{H^{\frac{3}{2}}(\partial S)} \\
&\geq \Lambda(c_0)\Big\{\int_{\partial S}\Big|(1+|\nabla h|^2)|\Lambda^{3/2}Z^{m-1}\nabla h| - |\nabla h|^2|\Lambda^{3/2}Z^{m-1}\nabla h|\Big|\mathrm{d}x\mathrm{d}y\Big\} \\
&\quad - \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}})|Z^{m-1}\nabla h|_{H^1} - |C^{m-1}(S)|_{H^{\frac{3}{2}}(\partial S)} \\
&\geq \Lambda(c_0)|Z^{m-1}\nabla h|_{\frac{3}{2}} - \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}})|Z^{m-1}\nabla h|_{H^1} - |C^{m-1}(S)|_{H^{\frac{3}{2}}(\partial S)}.
\end{aligned} \tag{102}$$

From (100), (101), and (102), we get

$$\int_0^T |Z^{m-1}\nabla h|^2_{\frac{3}{2}} dt \lesssim \sup_{t\in[0,T]} \Lambda(\frac{1}{c_0}, |h(t)|_{Y^{[\frac{m}{2}],1}}) \int_0^T \Big((G[h]V^b, V^b) + |h|^2_{X^{m,1}}\Big)dt. \tag{103}$$

We apply estimates **1) - 9)** to (98) with sufficiently small θ (from Young's inequaltiy) and Cauchy inequality to finish the proof. $\square$

In the next section, we estimate for $Z^m = \partial_t^m$ case. By summing with above estimate, we get the estimate for norm $\|\cdot\|_{X^{m,0}}$.

7. Energy Estimate for $Z^m = \partial_t^m$ case

In this section, we treat $Z^m = \partial_t^m$ case. If we apply Proposition 29, we will see $|\partial_t^m h|_{\frac{3}{2}}$ on the right hand side of energy estimate. However, from elliptic problem of pressure, $\|\partial_t^m q\|$ is not controllable. Therefore, we perform integration by part for time and z to change $\partial_t^m q$ into $\partial_t^{m-1}\partial_z q$. Boundary integral from these integration by parts vanish by cancellation. To see this, we will investigate two terms about pressure and surface tension, which generate high order commutators. We start with a simple proposition in the case of $Z^m = \partial_t^m$.

Proposition 33. *When $Z^m = \partial_t^m$, $C^m(f)$ can be estimated as follow.*

$$\|C^m(f)\| \leq \Lambda\big(\frac{1}{c_0}, |h(s)|_{Y^{[\frac{m}{2}],2}} + \|\nabla f(s)\|_{Y^{[\frac{m}{2}],0}}\big)(|h|_{X^{m,\frac{1}{2}}} + \|\partial_t^{m-1}\partial_z f\|)$$

Proof. Since, ∂_t commutes with ∂_z, we get the following.

$$\partial_t^m(\partial_i^\varphi f) = \partial_i^\varphi(\partial_t^m f) + C_i^m(f),$$

$$C_i^m(f) = -\Big[\partial_t^m, \frac{\partial_i\varphi}{\partial_z\varphi}, \partial_z f\Big] - \Big(\partial_t^m \frac{\partial_i\varphi}{\partial_z\varphi}\Big)\partial_z f.$$

From Proposition 9,

$$\begin{aligned}\Big\|\Big[\partial_t^m, \frac{\partial_i\varphi}{\partial_z\varphi}, \partial_z f\Big]\Big\| &\leq \Big\|\partial_t^{m-1}\frac{\partial_i\varphi}{\partial_z\varphi}\Big\|\Big\|\partial_t^{[\frac{m}{2}]}\partial_z f\Big\| + \Big\|\partial_t^{[\frac{m}{2}]}\frac{\partial_i\varphi}{\partial_z\varphi}\Big\|\Big\|\partial_t^{m-1}\partial_z f\Big\| \\ &\leq \Lambda\big(\frac{1}{c_0}, |h(s)|_{Y^{[\frac{m}{2}],2}} + \|\nabla f(s)\|_{Y^{[\frac{m}{2}],0}}\big)\big(\big|\partial_t^{m-1}h\big|_{\frac{1}{2}} + \big\|\partial_t^{m-1}\partial_z f\big\|\big), \\ \Big\|\Big(\partial_t^m\frac{\partial_i\varphi}{\partial_z\varphi}\Big)\partial_z f\Big\| &\leq \|\partial_z f\|_{L^\infty}\Big\|\partial_t^m\frac{\partial_i\varphi}{\partial_z\varphi}\Big\| \leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],2}})\|\partial_z f\|_{L^\infty}|h|_{X^{m,\frac{1}{2}}}.\end{aligned}$$

We use Lemma 14 in the last step. □

Proposition 34. *Assume that Let $m \geq 6$ and (v, φ, q, h) is a smooth solution on $[0,T]$ of the system (7)-(10). Also assume that*

$$\partial_z\varphi \geq c_0, \quad \forall t \in [0,T]. \tag{104}$$

For $Z^m = \partial_t^m$, we have the following energy estimate.

$$\begin{aligned}&\|\partial_t^m v(T)\|^2_{L^2(S)} + |\partial_t^m h(T)|^2_{H^1} + \varepsilon\int_0^T \|\nabla\partial_t^m v(t)\|^2_{L^2(S)}\,dt \\ &\leq \Lambda_{m,\infty}(0)\big(\|v(0)\|^2_{X^{m-1,1}} + |h(0)|^2_{X^{m-1,2}} + \varepsilon\|\nabla v(0)\|^2_{X^{m-1,1}}\big) \\ &\quad + \Lambda_{m,\infty}(T)\big(\|v(T)\|^2_{X^{m-1,1}} + |h(T)|^2_{X^{m-1,2}}\big) \\ &\quad + \int_0^T \Lambda_{m,\infty}(s)\big(\|v\|^2_{X^{m,0}} + \|\nabla v\|^2_{X^{m-1,0}} + |h|^2_{X^{m-1,2}} + \varepsilon\|\nabla v\|^2_{X^{m-1,1}} + |\partial_t^{m-1}h|^2_{H^{\frac{5}{2}}}\big)dt.\end{aligned} \tag{105}$$

Proof. From the analysis of previous section, there are two terms which generate highest order terms. **First term** is $-\int_S C^m(d)\partial_t^m q\partial_z\varphi \mathrm{d}x\mathrm{d}y\mathrm{d}z$ which corresponds to (82). The **Second term** is boundary integral

$$-\int_{\partial S}\partial_t^m q^b C^m(KB)$$

of (88). Except these two terms, we do not see any high order commutator $|\partial_t^m h|_{\frac{3}{2}}$. Therefore, similar as in proof of Proposition 63, we have

$$\frac{d}{dt}\int_S |\partial_t^m v|^2\, dV_t + 4\varepsilon \int_S |S^\varphi(\partial_t^m v)|^2\, dV_t + g\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial S} |\partial_t^m h|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial S} \frac{|\partial_t^m \nabla h|^2}{\sqrt{1+|\nabla h|^2}}$$
$$\leq \Lambda\big(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}} + \|\nabla q\|_{Y^{[\frac{m}{2}],0}}\big)\Big(\|v\|^2_{X^{m,0}} + \|\nabla v\|^2_{X^{m-1,0}} + \varepsilon\|\nabla v\|^2_{X^{m,0}} + |h|^2_{X^{m,1}}\Big)$$
$$+ \Lambda(\frac{1}{c_0})\Big(\underbrace{-\int_S C^m(d)\partial_t^m q\partial_z\varphi}_{\textbf{First term}} + \underbrace{\int_{\partial S} \partial_t^m q^b C^m(KB)}_{\textbf{Second term}}\Big). \tag{106}$$

Let us analyze **First term**. Using divergence free condition, we can expand $\partial_z\varphi C^m(d)$ as follow.

$$\begin{aligned}\partial_z\varphi C^m(d) &= [\partial_t^m, \mathbf{N}, \cdot\partial_z v] + [\partial_t^m, \partial_z\eta, \partial_1 v_1 + \partial_2 v_2] \\ &:= C^m(d)_1 + C^m(d)_2 + C^m(d)_3 + C^m(d)_4 + C^m(d)_5,\end{aligned} \tag{107}$$

where

$$\begin{aligned}
C^m(d)_1 &:= m\partial_t\mathbf{N}\cdot\partial_t^{m-1}\partial_z v,\\
C^m(d)_2 &:= m\partial_t\partial_z\eta\partial_t^{m-1}(\partial_1 v_1 + \partial_2 v_2),\\
C^m(d)_3 &:= m\partial_t^{m-1}\mathbf{N}\cdot\partial_t\partial_z v,\\
C^m(d)_4 &:= m\partial_t^{m-1}\partial_z\eta\partial_t(\partial_1 v_1 + \partial_2 v_2),\\
C^m(d)_5 &:= \sum_{l=2}^{m-2} C_m^l\Big(\partial_t^l\mathbf{N}\cdot\partial_t^{m-l}\partial_z v + \partial_t^l\partial_z\eta\cdot\partial_t^{m-l}(\partial_1 v_1 + \partial_2 v_2)\Big).
\end{aligned} \tag{108}$$

We use (107) to integrand $C^m(d)\partial_z\varphi$ for $i = 1,2,3,4,5$. Since $\|\partial_t^m q\|$ cannot be controlled, we should perform integration by part in times. And then $\partial_t C^m(d)_1, \partial_t C^m(d)_2 \sim \partial_t^m\nabla v$, so we should also perform integration by part in space.

1) Estimate of $-\int_0^T\int_S C^m(d)_1\partial_t^m q ds$. We perfrom integration by part in z and also in t again. When we split $q = q^E + q^{NS} + q^S$, q^E and q^{NS} parts are okay, since their estimates contain only low order terms. The only problem is estimate including q^S. Inspired by harmonic extension, we consider horizontal fourier multiplier $|\nabla_y|$ to estimate terms with q^S. Using pressure estimates (22), (23), and $\varepsilon \ll 1$, we get

$$\begin{aligned}
&-\int_0^T\int_S C^m(d)_1\partial_t^m q dt = -\int_0^T\int_S m\partial_t\mathbf{N}\cdot\partial_t^{m-1}\partial_z v\partial_t^m q dt.\\
&= -\int_0^T\int_{\partial S} m\partial_t\mathbf{N}\partial_t^{m-1}v^b\partial_t^m q^b dt + \int_0^T\int_S m\partial_z\partial_t\mathbf{N}\partial_t^{m-1}v\partial_t^m q dt + \int_0^T\int_S m\partial_t\mathbf{N}\partial_t^{m-1}v\partial_t^m\partial_z q dt\\
&\leq -\int_0^T\int_{\partial S} m\partial_t\mathbf{N}\partial_t^{m-1}v^b\partial_t^m q^b dt\\
&\quad + \Lambda_{m,\infty}(0)\|\partial_t^{m-1}v(0)\|\|\partial_t^{m-1}\partial_z q^E(0)\| + \Lambda_{m,\infty}(T)\|\partial_t^{m-1}v(T)\|\|\partial_t^{m-1}\partial_z q^E(T)\|\\
&\quad + \Lambda_{m,\infty}(0)\||\nabla_y|\partial_t^{m-1}v(0)\|\||\nabla_y|^{-1}\partial_t^{m-1}\partial_z(q^S + q^{NS})(0)\|\\
&\quad + \Lambda_{m,\infty}(T)\||\nabla_y|\partial_t^{m-1}v(T)\|\||\nabla_y|^{-1}\partial_t^{m-1}\partial_z(q^S + q^{NS})(T)\|\\
&\quad + \int_0^T \Lambda_{m,\infty}(t)\big(\|v\|^2_{X^{m,0}} + \|\nabla v\|^2_{X^{m-1,0}} + |h|^2_{X^{m-1,2}} + \varepsilon\|\nabla v\|^2_{X^{m-1,1}} + |\partial_t^{m-1}h|^2_{H^{\frac{5}{2}}}\big)dt,
\end{aligned} \tag{109}$$

where we performed integration by parts in time in the last step. Using estimates of q^E, we have

$$\begin{aligned}
&\leq -\int_0^T \int_{\partial S} m\partial_t \mathbf{N}\partial_t^{m-1}v^b\partial_t^m q^b dt \\
&\quad + \Lambda_{m,\infty}(0)\||\nabla_y|\partial_t^{m-1}v(0)\|\||\nabla_y|^{-1}\partial_t^{m-1}\partial_z(q^S+q^{NS})(0)\| \\
&\quad + \Lambda_{m,\infty}(T)\||\nabla_y|\partial_t^{m-1}v(T)\|\||\nabla_y|^{-1}\partial_t^{m-1}\partial_z(q^S+q^{NS})(T)\| \\
&\quad + \Lambda_{m,\infty}(0)\big(\|v(0)\|^2_{X^{m-1,0}}+|h(0)|^2_{X^{m-1,2}}\big)+\Lambda_{m,\infty}(T)\big(\|v(T)\|^2_{X^{m-1,0}}+|h(t)|^2_{X^{m-1,2}}\big) \\
&\quad + \int_0^T \Lambda_{m,\infty}(t)\big(\|v\|^2_{X^{m,0}}+\|\nabla v\|^2_{X^{m-1,0}}+|h|^2_{X^{m-1,2}}+\varepsilon\|\nabla v\|^2_{X^{m-1,1}}+|\partial_t^{m-1}h|^2_{H^{\frac{5}{2}}}\big)dt
\end{aligned} \tag{110}$$

It is easy to control $\||\nabla_y|\partial_t^{m-1}v\|\leq \|v\|_{X^{m-1,1}}$ by definition. We treat $\||\nabla_y|^{-1}\partial_t^{m-1}\partial_z q^S\|$ and $\||\nabla_y|^{-1}\partial_t^{m-1}\partial_z q^{NS}\|$ using horizontal Fourier multiplier. Let us write $\hat{q}^S := (\Lambda_y q^S)(\xi,z)$, where Λ_y horizontal fourier multiplier. Note that q^{NS} and q^S are harmonic,

$$\begin{aligned}
|\xi|^2\hat{q}^S &= \partial_{zz}\hat{q}^S, \\
\hat{q}^S &= \hat{q}^S(\xi,\hat{h})e^{|\xi|z}, \\
\partial_z\hat{q}^S &= |\xi|\mathcal{F}\Big(\nabla\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)e^{|\xi|z}\leq \Lambda_{m,\infty}|\xi|^3\hat{h}e^{|\xi|z},
\end{aligned} \tag{111}$$

where $\mathcal{F}$ means horizontal Fourier transform. This implies ∂_z is changed into horizontal derivatives in the case of harmonic function. And we do not need full Fourier multiplier $|\nabla|$ to reduce order and

$$\begin{aligned}
\||\nabla_y|^{-1}\partial_t^{m-1}\partial_z q^S\|^2 &= \||\nabla_y|^{-1}\partial_t^{m-1}\partial_z\hat{q}^S\|^2 \\
&= \int_{-\infty}^0\int_{\mathbb{R}^2}\frac{1}{|\xi|^2}|\xi|^6|\partial_t^{m-1}\hat{q}^S|^2e^{2|\xi|z} \\
&\leq \Lambda_{m,\infty}\int_{\mathbb{R}^2}\frac{1}{2\xi|\xi|^2}|\xi|^6|\partial_t^{m-1}\hat{h}|^2 \\
&\leq \Lambda_{m,\infty}\||\xi|^{3/2}\partial_t^{m-1}\hat{h}\|^2\leq \Lambda_{m,\infty}|h|^2_{X^{m-1,\frac{3}{2}}}.
\end{aligned} \tag{112}$$

We treat q^{NS} similar as q^S case since q^{NS} also solves harmonic equation. First, $\||\nabla_y|\partial_t^{m-1}v\|\leq \|v\|_{X^{m-1,1}}$. Horizontal operator $|\nabla_y|^{-1}$ reduce order by 1 of boundary data. For $t=0$, we get

$$\||\nabla_y|^{-1}\partial_t^{m-1}\partial_z q^{NS}(0)\|\leq \varepsilon\Lambda_{m,\infty}(0)\big(|h(0)|_{X^{m-1,2}}+\|\nabla\partial_t^{m-1}v(0)\|\big). \tag{113}$$

For $t=T$, since the $\varepsilon\|\nabla\partial_t^{m-1}v\|$ is dissipation type, we derive time integration by

$$\begin{aligned}
\||\nabla_y|^{-1}\partial_t^{m-1}\partial_z q^{NS}(T)\| &\leq \varepsilon\Lambda_{m,\infty}(T)\big(|h(T)|_{X^{m-1,2}}+\|\nabla\partial_t^{m-1}v(T)\|\big) \\
&\leq \varepsilon\Lambda_{m,\infty}(T)\big(|h(T)|_{X^{m-1,2}}+\|\nabla\partial_t^{m-1}v(0)\|+\int_0^T\|\nabla\partial_t^m v(t)\|dt\big).
\end{aligned} \tag{114}$$

Therefore,

$$\begin{aligned}
&\Lambda_{m,\infty}(T)\||\nabla_y|^1\partial_t^{m-1}v(T)\|\||\nabla_y|^{-1}\partial_t^{m-1}\partial_z q^{NS}(T)\| \\
&\leq \varepsilon\Lambda_{m,\infty}(T)\|\partial_t^{m-1}v(T)\|_{X^{0,1}}\big(|h(T)|_{X^{m-1,2}}+\|\nabla\partial_t^{m-1}v(0)\|+\int_0^T\|\nabla\partial_t^m v(t)\|dt\big) \\
&\leq \varepsilon\|\nabla\partial_t^{m-1}v(0)\|^2+\varepsilon\Lambda_{m,\infty}(T)\big(\|v(T)\|^2_{X^{m-1,1}}+|h(T)|^2_{X^{m-1,2}}\big)+T\int_0^T\|\nabla\partial_t^m v(t)\|^2dt.
\end{aligned} \tag{115}$$

We apply (112), (113), (114), and (115) to (110) to gain

$$
\begin{aligned}
(116) \le & -\underbrace{\int_0^T \int_{\partial S} m\partial_t \mathbf{N} \partial_t^{m-1} v^b \partial_t^m q^b dt}_{(*)} \\
& + \Lambda_{m,\infty}(0)\big(\|v(0)\|^2_{X^{m-1,1}} + |h(0)|^2_{X^{m-1,2}} + \varepsilon \|\nabla v(0)\|^2_{X^{m-1,1}}\big) \\
& + \Lambda_{m,\infty}(T)\big(\|v(T)\|^2_{X^{m-1,1}} + |h(T)|^2_{X^{m-1,2}}\big) + T\varepsilon \int_0^T \|\nabla \partial_t^m v(t)\|^2 dt \\
& + \int_0^T \Lambda_{m,\infty}(t)\big(\|v\|^2_{X^{m,0}} + \|\nabla v\|^2_{X^{m-1,0}} + |h|^2_{X^{m-1,2}} + \varepsilon \|\nabla v\|^2_{X^{m-1,1}} + |\partial_t^{m-1} h|^2_{H^{\frac{5}{2}}}\big) dt.
\end{aligned}
\tag{116}
$$

2) Estimate of $-\int_0^T \int_S C^m(d)_2 \partial_t^m q dt$. This estimate is very similar as above $C^m(d)_1$ case. However, $C^m(d)_2$ contains horizontal derivatives $\partial_1 v,\ \partial_2 v$, instead of $\partial_z v$, so we perform integration by part in t and horizontal x or y. Also we treat pressure term similar as we did in (110) and (116). Therefore we get same estimate with (116) but without $(*)$.

$$
\begin{aligned}
& -\int_0^T \int_S C^m(d)_2 \partial_t^m q dt \\
& \le +\Lambda_{m,\infty}(0)\big(\|v(0)\|^2_{X^{m-1,1}} + |h(0)|^2_{X^{m-1,2}} + \varepsilon \|\nabla v(0)\|^2_{X^{m-1,1}}\big) \\
& \quad + \Lambda_{m,\infty}(T)\big(\|v(T)\|^2_{X^{m-1,1}} + |h(T)|^2_{X^{m-1,2}}\big) + T\varepsilon \int_0^T \|\nabla \partial_t^m v(t)\|^2 dt \\
& \quad + \int_0^T \Lambda_{m,\infty}(s)\big(\|v\|^2_{X^{m,0}} + \|\nabla v\|^2_{X^{m-1,0}} + |h|^2_{X^{m-1,2}} + \varepsilon \|\nabla v\|^2_{X^{m-1,1}} + |\partial_t^{m-1} h|^2_{H^{\frac{5}{2}}}\big) dt.
\end{aligned}
\tag{117}
$$

3) Estimate of $-\int_0^T \int_S C^m(d)_i \partial_t^m q dt$ for $i = 3, 4, 5$. Since η is $\frac{1}{2}$ better regularity than h, we just perform integration by part in time to derive similar estimate as (117). Of course we treat pressure similar way as we did in (110). For $i = 3, 4, 5$,

$$
\begin{aligned}
& -\int_0^t \int_S C^m(d)_i \partial_t^m q dt \\
& \le \Lambda_{m,\infty}(0)\big(\|v(0)\|^2_{X^{m-1,1}} + |h(0)|^2_{X^{m-1,2}} + \varepsilon \|\nabla v(0)\|^2_{X^{m-1,1}}\big) \\
& \quad + \Lambda_{m,\infty}(T)\big(\|v(T)\|^2_{X^{m-1,1}} + |h(T)|^2_{X^{m-1,2}}\big) + T\varepsilon \int_0^T \|\nabla \partial_t^m v(t)\|^2 dt \\
& \quad + \int_0^T \Lambda_{m,\infty}(s)\big(\|v\|^2_{X^{m,0}} + \|\nabla v\|^2_{X^{m-1,0}} + |h|^2_{X^{m-1,2}} + \varepsilon \|\nabla v\|^2_{X^{m-1,1}} + |\partial_t^{m-1} h|^2_{H^{\frac{5}{2}}}\big) dt.
\end{aligned}
\tag{118}
$$

Now let us analyze underbraced **Second term** in (106). From (44),

$$
\begin{aligned}
C^m(KB) &= [Z^m, v^b, \mathbf{N}] \\
&= m\partial_t \mathbf{N} \partial_t^{m-1} v^b + m\partial_t^{m-1} \mathbf{N} \partial_t v^b + \sum_{\ell=2}^{m-2} \beta_\ell \partial_t^\ell \mathbf{N} \partial_t^{m-\ell} v^b,
\end{aligned}
\tag{119}
$$

where β_ℓ is some combinatoric positive integers.

$$
\begin{aligned}
&\int_0^T \int_{\partial S} \partial_t^m q^b C^m(KB)dt \\
&= \int_0^T \int_{\partial S} \partial_t^m q^b \Big(m\partial_t \mathbf{N} \partial_t^{m-1} v^b + m\partial_t^{m-1}\mathbf{N}\partial_t v^b + \sum_{\ell=2}^{m-2} \beta_\ell \partial_t^\ell \mathbf{N} \partial_t^{m-\ell} v^b \Big) dt \\
&= \underbrace{\int_0^T \int_{\partial S} \partial_t^m q^b m\partial_t \mathbf{N}\partial_t^{m-1} v^b dt}_{(*)} \\
&\quad + \int_0^T \int_{\partial S} \partial_t^m q^b \Big(m\partial_t^{m-1}\mathbf{N}\partial_t v^b + \sum_{\ell=2}^{m-2} \beta_\ell \partial_t^\ell \mathbf{N}\partial_t^{m-\ell} v^b \Big) dt.
\end{aligned}
\tag{120}
$$

For the first term in the last line, we perform integration by parts in z and t as we did in estimate of **First term**. We use simple estimate

$$
\int_{\partial S} \partial_t^{m-1}\partial_z q^b \partial_t^{m-1}\mathbf{N}\partial_t v^b \le |\partial_t^{m-1}\mathbf{N}\partial_t v^b|_{\frac{1}{2}} |\partial_t^{m-1}\partial_z q^b|_{-\frac{1}{2}},
$$

and use Proposition 11, 12 and pressure estimates (22), (23), (24) to gain

$$
\begin{aligned}
&\int_0^T \int_{\partial S} \partial_t^m q^b \partial_t^{m-1}\mathbf{N}\partial_t v^b \\
&\le \Lambda_{m,\infty}(0)|\partial_t^{m-1}h(0)|_{\frac{3}{2}} \|\partial_t^{m-1}\nabla q^E(0)\| + \Lambda_{m,\infty}(T)|\partial_t^{m-1}h(T)|_{\frac{3}{2}} \|\partial_t^{m-1}\nabla q^E(T)\| \\
&\quad + \Lambda_{m,\infty}(0)\||\nabla_y|\partial_t^{m-1}h(0)\| \||\nabla_y|^{-1}\partial_t^{m-1}\partial_z(q^S+q^{NS})(0)\| \\
&\quad + \Lambda_{m,\infty}(T)\||\nabla_y|\partial_t^{m-1}h(T)\| \||\nabla_y|^{-1}\partial_t^{m-1}\partial_z(q^S+q^{NS})(T)\| \\
&\quad + \int_0^T \Lambda_{m,\infty}(t)\big(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m-1,2}}^2 + \varepsilon\|\nabla v\|_{X^{m-1,1}}^2 + |\partial_t^{m-1}h|_{H^{\frac{5}{2}}}^2 \big) dt.
\end{aligned}
\tag{121}
$$

This form is nearly same as (109), except $v(0), v(T)$ are changed into $h(0), h(T)$. So we can use same estimates of (116).

Estimating the second term in the last line of (120) is exactly same as above because $|\sum_{\ell=2}^{m-2} \beta_\ell \partial_t^\ell \mathbf{N}\partial_t^{m-\ell} v^b|_{\frac{1}{2}}$ does not include high order bad commutators. Finally,

$$
\begin{aligned}
(120) &\le \underbrace{\int_0^T \int_{\partial S} \partial_t^m q^b m\partial_t \mathbf{N}\partial_t^{m-1} v^b dt}_{(*)} \\
&\quad + \Lambda_{m,\infty}(0)\big(\|v(0)\|_{X^{m-1,1}}^2 + |h(0)|_{X^{m-1,2}}^2 + \varepsilon\|\nabla v(0)\|_{X^{m-1,1}}^2 \big) \\
&\quad + \Lambda_{m,\infty}(T)\big(\|v(T)\|_{X^{m-1,1}}^2 + |h(T)|_{X^{m-1,2}}^2 \big) + T\varepsilon \int_0^T \|\nabla \partial_t^m v(t)\|^2 dt \\
&\quad + \int_0^T \Lambda_{m,\infty}(t)\big(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m-1,2}}^2 + \varepsilon\|\nabla v\|_{X^{m-1,1}}^2 + |\partial_t^{m-1}h|_{H^{\frac{5}{2}}}^2 \big) dt.
\end{aligned}
\tag{122}
$$

Now we apply (116), (117), (120), (122), and (25) to (106) to get estimate. Especially uncontrollable boundary integral $\int_0^T \int_{\partial S} \partial_t^m q^b m\partial_t \mathbf{N}\partial_t^{m-1} v^b dt$ vanishes from $(*)$ terms in (116) and (122) with opposite sign. Dissipation type terms $T\varepsilon \int_0^T \|\nabla \partial_t^m v(t)\|^2 dt$ in (116), (117), and (120) are absorbed by dissipation $\varepsilon \int_0^T \|\nabla \partial_t^m v(t)\|^2 dt$ which is in the left hand side of (106) for sufficiently small $T \ll 1$. This finishes proof. □

8. Normal derivative estimate

From Proposition 29, we should control $\|\partial_z v\|_{X^{m-1,0}}$, since $H^m \hookrightarrow H^m_{co}$. However, it is hard to estimate $\partial_z v$ directly. Instead, we estimate S_n, which is tangential part of $S^\varphi v\mathbf{n}$, i.e.

$$S_n := \Pi(S^\varphi v\mathbf{n}) \quad \text{where} \quad \Pi = I - \mathbf{n}\otimes\mathbf{n}, \tag{123}$$

where I is identity matrix and $\mathbf{n}$ is defined in Definition 2. First, we show that S_n is equivalent to $\partial_z v$ in the space $X^{m-1,0}$. It is clear that

$$\|S_n\|_{X^{m-1,0}} \leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(|h|_{X^{m-1,1}} + \|v\|_{X^{m,0}} + \|\nabla v\|_{X^{m-1,0}}), \tag{124}$$

from above definition (123) using Proposition 9 and 14. The following two lemmas show how to control $\partial_z v$ using S_n.

Lemma 35. *We have the following normal part estimate of $\partial_z v$.*

$$\|\partial_z v\cdot\mathbf{n}\|_{X^{m-1,0}} \leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(|h|_{X^{m-1,1}} + \|v\|_{X^{m-1,1}}). \tag{125}$$

Proof. From divergence free condition, we have,

$$\partial_z v\cdot\mathbf{n} = \frac{1}{\sqrt{1+|\nabla_y\varphi|^2}}\partial_z\varphi(\partial_1 v_1 + \partial_2 v_2).$$

Applying Z^{m-1} and using Proposition 9, we get

$$\|Z^{m-1}(\partial_z v\cdot\mathbf{n})\| \leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(|h|_{X^{m-1,1}} + \|v\|_{X^{m-1,1}}).$$

□

Lemma 36. *We have the following estimate.*

$$\|\partial_z v\|_{X^{m-1,0}} \leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(\|v\|_{X^{m,0}} + |h|_{X^{m-1,1}} + \|S_n\|_{X^{m-1,0}}).$$

Proof. By definition,

$$2S^\varphi v\mathbf{n} := (\nabla u)\mathbf{n} + (\nabla u)^T\mathbf{n} = (\nabla u)\mathbf{n} + g^{ij}(\partial_j v\cdot\mathbf{n})\partial_{y^i}.$$

And from divergence free condition, we get

$$\partial_N u = \frac{1+|\nabla_y\varphi|^2}{\partial_z\varphi}\partial_z v - \partial_1\varphi\partial_1 v - \partial_2\varphi\partial_2 v. \tag{126}$$

Therefore,

$$\|\partial_z v\|_{X^{m-1,0}} \leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(\|v\|_{X^{m,0}} + |h|_{X^{m-1,1}} + \|\partial_z v\cdot\mathbf{n}\|_{X^{m-1,0}} + \|S^\varphi v\mathbf{n}\|_{X^{m,0}})$$

and

$$S^\varphi v\mathbf{n} = S_n + (\mathbf{n}\otimes\mathbf{n})(S^\varphi v\mathbf{n}).$$

We use Lemma 35 to get

$$\|\partial_z v\|_{X^{m-1,0}} \leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(\|v\|_{X^{m,0}} + |h|_{X^{m-1,1}} + \|S_n\|_{X^{m-1,0}}).$$

□

To get an estimate for S_n we make an equation for S_n. First, we take ∇^φ to (7).

$$\partial_t^\varphi\nabla^\varphi v + (v\cdot\nabla^\varphi)\nabla^\varphi v + (\nabla^\varphi v)^2 + (D^\varphi)^2 q - \varepsilon\Delta^\varphi\nabla^\varphi v = 0, \tag{127}$$

where $(D^\varphi)^2$ is Hessian matrix. We also take symmetric part of above (127). Then by adding both equations, we get

$$\partial_t^\varphi S^\varphi v + (v\cdot\nabla^\varphi)S^\varphi v + \frac{1}{2}((\nabla v)^2 + ((\nabla v)^T)^2) + (D^\varphi)^2 q - \varepsilon\Delta^\varphi(S^\varphi v) = 0.$$

By taking tangential operator, Π, which was defined in (123),

$$\partial_t^\varphi S_n + (v\cdot\nabla^\varphi)S_n - \varepsilon\Delta^\varphi(S_n) = F_S, \tag{128}$$

where F_S is commutator,

$$
\begin{aligned}
&F_S = F_S^1 + F_S^2 + F_S^3,\\
&F_S^1 = -\frac{1}{2}\Pi((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^T)^2)\mathbf{n} + (\partial_t \Pi + v\cdot\nabla^\varphi \Pi)S^\varphi v\mathbf{n} + \Pi S^\varphi v(\partial_t \mathbf{n} + v\cdot\nabla^\varphi \mathbf{n}),\\
&F_S^2 = -2\varepsilon\partial_i^\varphi \Pi \partial_i^\varphi (S^\varphi v\mathbf{n}) - 2\varepsilon\Pi(\partial_i^\varphi (S^\varphi v)\partial_i^\varphi \mathbf{n}) - \varepsilon(\Delta^\varphi \Pi)S^\varphi v\mathbf{n} - \varepsilon\Pi S^\varphi v\Delta^\varphi \mathbf{n},\\
&F_S^3 = -\Pi((D^\varphi)^2 q)\mathbf{n}.
\end{aligned} \tag{129}
$$

We apply Z^k to get higher order estimate. We need $\|S_n\|_{X^{m-1,0}}$, however, from pressure estimate, optimal regularity is $k = m-2$. This is because,

$$
\|F_S^3\|_{m-2} \sim \|D^2 q\|_{m-2} \sim |h|_{X^{m-2,\frac{7}{2}}}.
$$

To estimateFor $Z^{m-2}F_S^1$, using Propositions 9,

$$
\begin{aligned}
\left\|Z^{m-2}F_S^1\right\|_{L^2(S)} &\leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(\|\nabla v\|_{X^{m-2,0}} + |h|_{X^{m-2,1}} + \|v\|_{X^{m-2,0}})\\
&\leq \Lambda_{m,\infty}(\|S_n\|_{X^{m-2,0}} + |h|_{X^{m,1}} + \|v\|_{X^{m-2,0}}).
\end{aligned} \tag{130}
$$

where we used Lemma 36.
Similary, for $Z^{m-2}F_S^2$,

$$
\begin{aligned}
\left\|Z^{m-2}F_S^2\right\|_{L^2(S)} &\leq \varepsilon\Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(\|\partial_{zz} v\|_{X^{m-2,0}} + \|\partial_z v\|_{X^{m-1,0}} + |h|_{X^{m-2,\frac{5}{2}}})\\
&\leq \varepsilon\Lambda_{m,\infty}(\|\nabla S_n\|_{X^{m-2,0}} + \|S_n\|_{X^{m-1,0}} + |h|_{X^{m-2,\frac{5}{2}}}).
\end{aligned} \tag{131}
$$

For $Z^{m-2}F_S^3$, using pressure estimates Proposition 22, 23, and 24, and Lemma 36,

$$
\left\|Z^{m-2}F_S^3\right\|_{L^2(S)} \leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(\|S_n\|_{X^{m-2,0}} + |h|_{X^{m-2,\frac{7}{2}}} + \|v\|_{X^{m-2,0}}). \tag{132}
$$

Now we apply Z^{m-2} to (128) to get,

$$
\partial_t^\varphi Z^{m-2}S_n + (v\cdot\nabla^\varphi)Z^{m-2}S_n - \varepsilon\Delta^\varphi Z^{m-2}S_n = Z^{m-2}(F_S) + C_S,
$$

where C_S is commutator. We divide C_S into,

$$
C_S^1 = [Z^\alpha v_y]\cdot\nabla_y S_n + [Z^\alpha, V_z]\partial_z S_n := C_{S_y} + C_{S_z}\ \ ,\quad C_S^2 = -\varepsilon[Z^\alpha, \Delta^\varphi]S_n, \tag{133}
$$

where V_z is defined in (40). From (14), $S_n|_{\partial S} = 0$ and $\partial_t^k Z^\alpha S_n|_{\partial S} = 0$ hold. Therefore, we get the following energy estimate.

$$
\frac{1}{2}\frac{d}{dt}\int_S |Z^{m-2}S_n|^2 dV_t + \varepsilon\int_S |\nabla^\varphi Z^{m-2}S_n|^2 dV_t = \int_S (Z^{m-2}F_S + C_S)\cdot Z^{m-2}S_n dV_t. \tag{134}
$$

Estimate of C_{S_y} is easy. Using Proposition 9,

$$
\left\|C_{S_y}\right\| \leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(\|S_n\|_{X^{m-2,0}} + \|v\|_{X^{m-1,0}}). \tag{135}
$$

To control C_{S_z} we give ∂_z to V_z by integration by part. From the commutator, we have to control the terms like,

$$
\|Z^\beta V_z \partial_z Z^\gamma S_n\|, \tag{136}
$$

where $|\beta| + |\gamma| \leq m-2$, $|\gamma| \leq m-3$ or equivalently $|\beta| \neq 0$. We interchange ∂_z and Z_3 by

$$
Z^\beta V_z \partial_z Z^\gamma S_n = \frac{1-z}{z} Z^\beta V_z Z_3 Z^\gamma S_n.
$$

Then by commutation between $\frac{1-z}{z}$ and Z^β, we encounter the terms of the following terms,

$$
c_{\tilde\beta} Z^{\tilde\beta}\Big(\frac{1-z}{z}V_z\Big) Z_3 Z^\gamma S_n,
$$

where $c_{\tilde{\beta}}$ is sufficiently nice and bounded function, and $|\tilde{\beta}| \leq |\beta|$.
If $\tilde{\beta} = 0$,

$$\|c_{\tilde{\beta}} Z^{\tilde{\beta}}\Big(\frac{1-z}{z}V_z\Big)Z_3 Z^\gamma S_n\| \lesssim \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})(\|S_n\|_{X^{m-2,0}} + |h|_{X^{m-1,0}}).$$

If $\tilde{\beta} \neq 0$,

$$\Big\|c_{\tilde{\beta}} Z^{\tilde{\beta}}\Big(\frac{1-z}{z}V_z\Big)Z_3 Z^\gamma S_n\Big\| \lesssim \Big\|Z\Big(\frac{1-z}{z}V_z\Big)\Big\|_{Y^{[\frac{m}{2}],0}}\|S_n\|_{X^{m-2,0}} + \|S_n\|_{Y^{[\frac{m}{2}],0}}\Big\|Z\Big(\frac{1-z}{z}V_z\Big)\Big\|_{X^{m-3,0}}.$$

First, we see that,

$$\Big\|Z\Big(\frac{1-z}{z}V_z\Big)\Big\|_{Y^{[\frac{m}{2}],0}} \lesssim \|V_z\|_{Y^{[\frac{m}{2}]+1,0}} + \|\partial_z V_z\|_{Y^{[\frac{m}{2}]+1,0}}$$

and,

$$\begin{aligned}\Big\|Z\Big(\frac{1-z}{z}V_z\Big)\Big\|_{X^{m-3,0}} &\lesssim \|\nabla V_z\|_{X^{m-3,0}} + \Big\|\frac{1}{z(1-z)}V_z\Big\|_{X^{m-3,0}}\\ &\lesssim \Big\|\frac{1-z}{z}ZV_z\Big\|_{X^{m-3,0}} + \Big\|\frac{1}{z(1-z)}V_z\Big\|_{X^{m-3,0}}.\end{aligned}$$

So we should estimate the terms those have forms of,

$$\Big\|\frac{1-z}{z}Z^\xi Z V_z\Big\| \quad \text{and} \quad \Big\|\frac{1}{z(1-z)}Z^\xi V_z\Big\|, \tag{137}$$

where $|\xi| \leq m-3$. To estimate these two types of terms, we use the following lemma.

Lemma 37. *If $f(0) =$, we have the following inequalites,*

$$\begin{aligned}\int_{-\infty}^0 \frac{1}{z^2(1-z)^2}|f(z)|^2\,dz &\lesssim \int_{-\infty}^0 |\partial_z f(z)|^2\,dz,\\ \int_{-\infty}^0 \Big(\frac{1-z}{z}\Big)^2|f(z)|^2\,dz &\lesssim \int_{-\infty}^0 \Big(|f(z)|^2 + |\partial_z f(z)|^2\Big)dz.\end{aligned}$$

Proof. This estimate is Lemma 8.4 in [1]. □

To estimate two types in (137), we use above lemma to get

$$\begin{aligned}\Big\|\frac{1-z}{z}Z^\xi ZV_z\Big\|^2 &\lesssim \Big\|Z^\xi ZV_z\Big\|^2 + \Big\|\partial_z(Z^\xi ZV_z)\Big\|^2,\\ \Big\|\frac{1}{z(1-z)}Z^\xi ZV_z\Big\|^2 &\lesssim \Big\|\partial_z(Z^\xi ZV_z)\Big\|^2.\end{aligned} \tag{138}$$

Therefore,

$$\|C_{S_z}\| \lesssim \|ZV_z\|_{X^{m-3,0}} + \|\partial_z ZV_z\|_{X^{m-3,0}} + \|\partial_z V_z\|_{X^{m-3,0}} \tag{139}$$

Combining (135) and (139) we have

$$\begin{aligned}\|C_S^1\| &\leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})\Big(\|S_n\|_{X^{m-2,0}} + \|v\|_{X^{m-2,0}} + \|\partial_z V_z\|_{X^{m-2,0}}\Big)\\ &\leq \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}})\Big(\|S_n\|_{X^{m-2,0}} + \|v\|_{X^{m-1,0}} + |h|_{X^{m-2,1}}\Big).\end{aligned} \tag{140}$$

For C_S^2, we expand C_S^2 as following.

$$\begin{aligned}\varepsilon Z^{m-2}(\Delta^\varphi S_n) &= \varepsilon Z^{m-2}(\frac{1}{\partial_z\varphi}\nabla\cdot(E\nabla S_n)) = \varepsilon\frac{1}{\partial_z\varphi}Z^{m-2}(\nabla\cdot(E\nabla S_n)) + C_{S_1}^2\\ &= \varepsilon\frac{1}{\partial_z\varphi}\nabla\cdot Z^{m-2}(E\nabla S_n) + C_{S_2}^2 + C_{S_1}^2\\ &= \varepsilon\frac{1}{\partial_z\varphi}\nabla\cdot(E\nabla Z^\alpha S_n) + C_{S_3}^2 + C_{S_2}^2 + C_{S_1}^2\\ &= \varepsilon\Delta^\varphi(Z^\alpha S_n) + C_S^2,\end{aligned} \tag{141}$$

where

$$
\begin{aligned}
C_S^2 &:= C_{S_1}^2 + C_{S_2}^2 + C_{S_3}^2, \\
C_{S_1}^2 &:= \varepsilon \left[Z^\beta, \frac{1}{\partial_z \varphi} \right] \nabla \cdot \Big(E \nabla S_n \Big), \\
C_{S_2}^2 &:= \varepsilon \frac{1}{\partial_z \varphi} [Z^\alpha, \nabla] \cdot \Big(E \nabla S_n \Big) \\
C_{S_3}^2 &:= \varepsilon \frac{1}{\partial_z \varphi} \nabla \cdot \Big([Z^\alpha, E\nabla] \, S_n \Big).
\end{aligned}
\tag{142}
$$

1) Estimate of $C_{S_1}^2$

We need to estimate the following types of terms,

$$
\varepsilon \int_S Z^\beta \Big(\frac{1}{\partial_z \varphi} \Big) Z^{\tilde{\gamma}} (\nabla \cdot (E \nabla S_n)) \cdot Z^{m-3} S_n dV_t,
$$

where $|\beta| + |\tilde{\gamma}| = m-2$, $\beta \neq 0$. Then again, by commutator between $Z^{\tilde{\gamma}}$ and ∇, the forms becomes like the following forms.

$$
\varepsilon \int_S Z^\beta \Big(\frac{1}{\partial_z \varphi} \Big) \partial_i Z^\gamma ((E \nabla S_n)_j) \cdot Z^{m-2} S_n dV_t,
$$

where $|\tilde{\gamma}| \leq |\gamma|$. Now we preform integrate by part, to get

$$
\begin{aligned}
&\left| \varepsilon \int_S Z^\beta \Big(\frac{1}{\partial_z \varphi} \Big) \partial_i Z^\gamma ((E \nabla S_n)_j) \cdot Z^{m-2} S_n dV_t \right| \\
&\leq \left| \varepsilon \int_S \partial_i Z^\beta (\frac{1}{\partial_z \varphi}) Z^\gamma ((E \nabla S_n)_j) \cdot Z^{m-2} S_n dV_t \right| + \left| \varepsilon \int_S Z^\beta \Big(\frac{1}{\partial_z \varphi} \Big) Z^\gamma ((E \nabla S_n)_j) \cdot \partial_i Z^{m-2} S_n dV_t \right|.
\end{aligned}
$$

Using Proposition 9 and Holder inequality, we get,

$$
\left| \int_S C_{S_1}^2 \cdot Z^{m-3} S_n dV_t \right| \lesssim \varepsilon \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}) (\|\nabla Z^{m-2} S_n\|^2 + \|S_n\|_{X^{m-2,0}}^2 + |h|_{X^{m-1,1}}^2).
\tag{143}
$$

2) Estimate of $C_{S_2}^2$

We need to estimate following types.

$$
\varepsilon \int_S \partial_z Z^\beta \Big(E \nabla S_n \Big) \cdot Z^{m-2} S_n dV_t,
$$

where $\beta \leq m-3$. Then, by integration by parts again, we can get the same estimate as like $C_{S_1}^2$

$$
\left| \int_S C_{S_2}^2 \cdot Z^{m-2} S_n dV_t \right| \leq \varepsilon \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}) (\|\nabla Z^{m-2} S_n\|^2 + \|S_n\|_{X^{m-2,0}}^2 + |h|_{X^{m-1,1}}^2).
\tag{144}
$$

3) Estimate of $C_{S_3}^2$

We give $\nabla \cdot$ to $Z^{m-2} S_n$ by integration by parts, then easily,

$$
\begin{aligned}
&\Big| \int_S C_{S_3}^2 \cdot Z^{m-2} S_n dV_t \Big| \leq \varepsilon \| [Z^{m-2}, E\nabla] S_n \| \| \nabla Z^{m-2} S_n \| \\
&\leq \varepsilon \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}) (\|\nabla Z^{m-2} S_n\|^2 + \|S_n\|_{X^{m-2,0}}^2 + |h|_{X^{m-1,1}}^2).
\end{aligned}
\tag{145}
$$

Combining above three estimates (143), (144), and (145), we have the following.

$$
\Big| \int_S C_S^2 \cdot Z^{m-2} S_n dV_t \Big| \leq \varepsilon \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}) (\|\nabla Z^{m-2} S_n\|^2 + \|S_n\|_{X^{m-2,0}}^2 + |h|_{X^{m-1,1}}^2).
\tag{146}
$$

Now we use (130), (131), (132), (140), and (146) to estimate the right hand side of (134). Especially we use Young's inequality to $\varepsilon \|\nabla S_n\|_{X^{m-2,0}}$ in (131), i.e.

$$
\varepsilon \|Z^{m-2} S_n\| \|\nabla S_n\|_{X^{m-2,0}} \lesssim \theta \varepsilon \|\nabla S_n\|_{X^{m-2,0}}^2 + \frac{1}{\theta} \|S_n\|_{X^{m-2,0}}^2.
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_S |Z^{m-2}S_n|^2 dV_t + 2\varepsilon\int_S |\nabla^\varphi Z^{m-2}S_n|^2 dV_t \\
&\le \Lambda(\frac{1}{c_0}, |h|_{Y^{[\frac{m}{2}],1}} + \|\nabla v\|_{Y^{[\frac{m}{2}],0}}) \\
&\quad\times\Big\{\|S_n\|^2_{X^{m-2,0}} + |h|^2_{X^{m-2,1}} + \|v\|^2_{X^{m-1,0}} + \varepsilon\Big(\theta\|\nabla S_n\|^2_{X^{m-2,0}} + |h|^2_{X^{m-2,\frac{7}{2}}} + |h|^2_{X^{m-1,1}}\Big)\Big\},
\end{aligned}
\tag{147}
$$

for sufficiently small $\theta \ll 1$. We integrate for time from 0 to T and sum for all Z^{m-2}. Dissipation type term $\theta\|\nabla S_n\|^2_{X^{m-2,0}}$ is absorbed by left hand side for sufficienlty small $\theta \ll 1$, under assumption of $\sup_{t\in[0,T]}\Lambda(t) \le C$. And then we use Proposition 16 to get the following result.

Proposition 38. *Assume that $m \ge 6$ and (v, φ, q, h) is a smooth solution on $[0, T]$ of the system (7)-(10). Also assume that*

$$
\partial_z\varphi \ge c_0, \quad \forall t \in [0,T] \quad \textit{and} \quad \sup_{t\in[0,T]} \Lambda_{m,\infty}(t) \le C, \tag{148}
$$

for some uniform constant C. Then we have the following L^2 type estimate for S_n.

$$
\begin{aligned}
\|S_n(T)\|^2_{X^{m-2,0}} + \varepsilon\int_0^T \|\nabla S_n(t)\|^2_{X^{m-2,0}}\,dt &\le \Lambda(\frac{1}{c_0})\sum_{\forall Z^{m-2}} \|(Z^{m-2}S_n)(0)\|^2 \\
&+ \int_0^T \Lambda_{m,\infty}(t)\big(\|S_n\|^2_{X^{m-2,0}} + \|v\|^2_{X^{m-1,0}} + |h|^2_{X^{m,1}} + |h|^2_{X^{m-2,\frac{7}{2}}}\big)dt.
\end{aligned}
$$

9. L^∞ type estimate

In this section, we estimate high order L^∞ type norm. In (72), we can use standard Sobolev embedding for $|h(t)|_{Y^{[\frac{m}{2}],1}}$ for sufficiently large m. Howver, $\|\nabla v\|_{Y^{[\frac{m}{2}],0}}$ cannot be controlled by Sobolev embedding, since conormal space is weaker than standard Sobolev space. Therefore, as we did in previous section, we use S_n instead of $\partial_z v$ directly. Although S_n was not sufficient smooth to get $m-1$ order, we can use energy terms $\|S_n\|_{X^{m-3,0}}$ to control L^∞ type terms. The following lemmas show that equivalent relation between S_n and $\partial_z v$ in L^∞ sense.

Lemma 39. *We have the following estimate for normal part of $\partial_z v$.*

$$
\|\partial_z v\cdot\mathbf{n}\|_{Y^{k,0}} \le \Lambda(\frac{1}{c_0}, |h|_{Y^{k,1}})\|v\|_{Y^{k+1,0}}.
$$

Proof. From divergence free condition,

$$
\partial_z v\cdot\mathbf{N} = \partial_z\varphi(\partial_1 v_1 + \partial_2 v_2).
$$

We take $\|\cdot\|_{Y^{k,0}}$ and use Proposition 11 to get,

$$
\|\partial_z v\cdot\mathbf{n}\|_{Y^{k,0}} \le \Lambda(\frac{1}{c_0}, |h|_{Y^{k,1}})\|v\|_{Y^{k+1,0}}.
$$

□

Proposition 40. *We have the following estimate.*

$$
\|\partial_z v\|_{Y^{k,0}} \le \Lambda(\frac{1}{c_0}, |h|_{Y^{k,1}})(\|v\|_{Y^{k+1,0}} + \|S_n\|_{Y^{k,0}}).
$$

Proof. We have,

$$
2S^\varphi v\mathbf{n} = (\nabla u)\mathbf{n} + (\nabla u)^T\mathbf{n} = (\nabla u)\mathbf{n} + g^{ij}(\partial_j v\cdot\mathbf{n})\partial_{y^i}
$$

and divergence free condition gives,

$$
\partial_N u = \frac{1 + |\nabla_y\varphi|^2}{\partial_z\varphi}\partial_z v - \partial_1\varphi\partial_1 v - \partial_2\varphi\partial_2 v.
$$

So we obtain,

$$
\|\partial_z v\|_{Y^{k,0}} \le \Lambda(\frac{1}{c_0}, |h|_{Y^{k,1}})(\|v\|_{Y^{k+1,0}} + \|S^\varphi v\mathbf{n}\|_{Y^{k,0}}). \tag{149}
$$

And, since

$$S^{\varphi}v\mathbf{n} = S_n + (\mathbf{n}\otimes\mathbf{n})(S^{\varphi}v\mathbf{n}),$$

we get

$$\|S^{\varphi}v\mathbf{n}\|_{Y^{k,0}} \le \|S_n\|_{Y^{k,0}} + \Lambda(\frac{1}{c_0}, |h|_{Y^{k,1}})\|\partial_z v\cdot\mathbf{n}\|_{Y^{k,0}} + \|v\|_{Y^{k,0}}.$$

Using Lemma 39 and (149),

$$\|\partial_z v\|_{Y^{k,0}} \le \Lambda(\frac{1}{c_0}, |h|_{Y^{k,1}})(\|v\|_{Y^{k+1,0}} + \|S_n\|_{Y^{k,0}}).$$

□

Above proposition implies that we suffice to estimate $\|S_n\|_{Y^{[\frac{m}{2}],0}}$, instead of $\|\partial_z v\|_{Y^{[\frac{m}{2}],0}}$. So we use equation for S_n with Dirichlet boundary condition. Main difficulty in this section is commutator between $Z_3 = \frac{z}{1-z}\partial_z$ and Δ^{φ}. This commutator was not a problem in basic L^2-type energy estimate of v and S_n, because the highest order commutator, which looks like $\sim \varepsilon Z^{\alpha}\partial_z S_n$ can be absorbed into dissipation term in the energy. But, in L^{∞} estimate, we use the following maximal principle for convection-diffusion equation. For S_n equation,

$$\partial_t^{\varphi} S_n + (v\cdot\nabla^{\varphi})S_n - \varepsilon\Delta^{\varphi}S_n = F_S.$$

We have the following L^{∞}-type estimate which does not have dissipation in energy term.

$$\|S_n(t)\|_{L^{\infty}} \le \|S_n(0)\|_{L^{\infty}} + \int_0^t \|F_S(s)\|_{L^{\infty}}\mathrm{d}s.$$

Note that, standard Sobolev embedding does not hold for co-normal space in general. This is because of behavior of near the boundary. Away from the boundary, however, $\frac{z}{1-z}$ is not zero, and its all order derivative for z is always uniformly bounded. Hence, we divide co-normal function into two parts, one is supported near the boundary and another is supported away from the boundary. Then the second part is easy to control by standard Sobolev embedding. For the first part, we deform the coordinate so that locally ∂_{zz}^{φ} look like ∂_{zz}. Then ∂_z commute with ∂_{zz}, so it does not generate any harmful (which has 1-more order than L^{∞}-type energy) commutator. This clever idea is introduced in [15]. We introduce this system briefly and use similar arguments to get the result. First, we start with very simple lemma, which means, co-normal Sobolev space is equivalent to standard Sobolev away from the boundary .

Lemma 41. *For any smooth cut-off function $\bar{\chi}$ such that $\bar{\chi} = 0$ in a vicinity of $z = 0$, we have for $m > k + 3/2$:*

$$\|\bar{\chi}f\|_{W^{k,\infty}} \lesssim \|f\|_{H^m_{co}}.$$

We decompose $Z^k S_n$ as

$$\|Z^k S_n\|_{L^{\infty}} \le \|\chi Z^k S_n\|_{L^{\infty}} + \|v\|_{Y^{k+1,0}}.$$

To estimate second term $\|v\|_{Y^{k+1,0}}$, we use the following proposition.

Proposition 42. *We have the following estimate.*

$$\|v\|_{Y^{k,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla v\|_{Y^{1,0}} + \|v\|_{X^{k+2,0}} + |h|_{X^{k+1,1}} + \|S_n\|_{X^{k+1,0}}\Big). \tag{150}$$

Proof. Using anisotropic Sobolev embedding in Proposition 11,

$$\|Z^k v\|^2_{L^{\infty}} \lesssim \|\partial_z v\|_{X^{k+1,0}}\|v\|_{X^{k+2,0}} \le \|\partial_z v\|^2_{X^{k+1,0}} + \|v\|^2_{X^{k+2,0}}$$

and using lemme of previous section,

$$\|\partial_z v\|_{X^{k+1,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, \|\nabla v\|_{Y^{\frac{k+2}{2},0}}\Big)\Big(\|v\|_{X^{k+2,0}} + |h|_{X^{k+1,1}} + \|S_n\|_{X^{k+1,0}}\Big).$$

Then using induction for $\|\nabla v\|_{Y^{\frac{k+2}{2},0}}$, until it become 1. And notice that $\|v\|_{X^{k+2,0}}$ is absorbed by estimate of $\|\partial_z v\|_{X^{k+1,0}}$. □

Note that above proposition means that we suffice to estimate $Z^k S_n$ only near the boundary, so now we introduce modified coordinate which was introduced in [15] and [1]. Let us define transformation Ψ,

$$\Psi(t,\cdot) : S = \mathbb{R}^2 \times (-\infty, 0) \to \Omega_t, \tag{151}$$
$$x = (y,z) \mapsto \begin{pmatrix} y \\ h(t,y) \end{pmatrix} + z\mathbf{n}^b(t,y),$$

where $\mathbf{n}^b$ is unit normal at the boundary, $(-\nabla h, 1)/|N|$. To show that this is diffeomorphism near the boundary, we check

$$D\Psi(t,\cdot) = \begin{pmatrix} 1 & 0 & -\partial_1 h \\ 0 & 1 & -\partial_2 h \\ \partial_1 h & \partial_2 h & 1 \end{pmatrix} + \begin{pmatrix} -z\partial_{11}h & -z\partial_{12}h & 0 \\ -z\partial_{21}h & -z\partial_{22}h & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is diffeomorphism near the boundary since norm of second matrix is controlled by $|h|_{2,\infty}$. So, we restrict $\Psi(t,\cdot)$ on $\mathbb{R}^2 \times (-\delta, 0)$ so that it is diffeomorphism.(δ is depend on c_0. Of course, think that above support separation was done by $\chi(z) = \kappa(\frac{z}{\delta(c_0)})$. Now we write laplacian Δ^φ with respect to Riemannian metric of above parametrization. Riemannian metric becomes,

$$g(y,z) = \begin{pmatrix} \tilde{g}(y,z) & 0 \\ 0 & 1 \end{pmatrix}, \tag{152}$$

where $\tilde{g}$ is 2×2 block matrix. And with this metric, laplacian becomes,

$$\Delta_g f = \partial_{zz} f + \frac{1}{2}\partial_z(\ln|g|)\partial_z f + \Delta_{\tilde{g}} f, \tag{153}$$

where

$$\Delta_{\tilde{g}} f = \frac{1}{|\tilde{g}|^{\frac{1}{2}}} \sum_{1\le i,j\le 2} \partial_{y^i}(\tilde{g}^{ij}|\tilde{g}|^{\frac{1}{2}}\partial_{y^j} f), \tag{154}$$

where $\tilde{g}^{ij}$ is inverse matrix element of $\tilde{g}$. We now solve problem in domain of Ψ. We restrict $S^\varphi v$ near the boundary and parametrize them via Ψ. Let us define

$$S^\chi := \chi(z) S^\varphi v, \tag{155}$$

where $\chi(z) = \kappa(\frac{z}{\delta(c_0)}) \in [0,1]$ where κ is smooth and compactly supported near the boundary, taking value 1 there. Equation for S^χ is

$$\partial_t^\varphi S^\chi + (v\cdot\nabla^\varphi)S^\chi - \varepsilon\Delta^\varphi S^\chi = F_{S^\chi}, \tag{156}$$

where

$$F_{S^\chi} = F^\chi + F_v,$$
$$F^\chi = (V_z\partial_z\chi)S^\varphi v - \varepsilon\nabla^\varphi\chi\cdot\nabla^\varphi S^\varphi v - \varepsilon\Delta^\varphi\chi S^\varphi v,$$
$$F_v = -\chi(D^\varphi)^2 q - \frac{\chi}{2}((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^t)^2).$$

Note that F^χ is supported away from the boundary. We rewrite this function on our new frame by taking $\Phi^{-1}\circ\Psi$. We define

$$S^\Psi(t,y,z) = S^\chi(t, \Phi^{-1}\circ\Psi) \tag{157}$$

and S^Ψ solves

$$\partial_t S^\Psi + w\cdot\nabla S^\Psi - \varepsilon(\partial_{zz}S^\Psi + \frac{1}{2}\partial_z(\ln|g|)\partial_z S^\Psi + \Delta_{\tilde{g}} S^\Psi) = F_{S^\chi}(t, \Phi^{-1}\circ\Psi), \tag{158}$$

where

$$w = \bar{\chi}(D\Psi)^{-1}(v(t,\Phi^{-1}\circ\Psi) - \partial_t\Psi),$$

where $\bar{\chi}$ is slightly larger support so that $\bar{\chi}S^\Psi = S^\Psi$ and as like S^χ, S^Ψ is also only supported near the boundary. In this frame S_n correspond to S_n^Ψ, which is defined as following.

$$S_n^\Psi(t,y,z) = \Pi^b(t,y)S^\Psi\mathbf{n}^b(t,y) = \Pi^b(t,y)S^\chi(t,\Phi^{-1}\circ\Psi)\mathbf{n}^b(t,y), \tag{159}$$

where $\Pi^b = I - \mathbf{n}^b \otimes \mathbf{n}^b$.(tangential operator at the boundary, so they are independent to z.) Then equation for S_n^Ψ becomes,

$$\partial_t S_n^\Psi + w \cdot \nabla S_n^\Psi - \varepsilon(\partial_{zz} + \frac{1}{2}\partial_z(\ln|g|)\partial_z)S_n^\Psi = F_n^\Psi, \tag{160}$$

where

$$F_n^\Psi = \Pi^b F_{S_\chi}\mathbf{n}^b + F_n^{\Psi,1} + F_n^{\Psi,2},$$

where

$$F_n^{\Psi,1} = ((\partial_t + w_y \cdot \nabla_y)\Pi^b)S^\Psi \mathbf{n}^b + \Pi^b S^\Psi(\partial_t + w_y \cdot \nabla_y)\mathbf{n}^b,$$
$$F_n^{\Psi,2} = -\varepsilon \Pi^b(\Delta_{\tilde{g}} S^\Psi)\mathbf{n}^b,$$

with zero-boundary condition at $z = 0$. Note that $S_n = S_n^\Psi$ on the boundary. We will estimate S_n^Ψ instead of S_n. to validate this, we should show that equivalence of these two terms. Firstly, by definition of S_n^Ψ,

$$\|S_n^\Psi\|_{Y^{k,0}} \le \Lambda(|h|_{Y^{k+1,0}})\|\Pi^b S^\varphi v \mathbf{n}^b\|_{Y^{k,0}} \tag{161}$$

and since $|\Pi - \Pi^b| + |\mathbf{n} - \mathbf{n}^b| = O(z)$ near the boundary $z = 0$,

$$\|S_n^\Psi\|_{Y^{k,0}} \le \Lambda(\frac{1}{c_0}, \|S_n\|_{Y^{k,0}} + \|v\|_{Y^{k+1,0}}). \tag{162}$$

Now, we apply anisotropic Sobolev embedding to the last term,

$$\|S_n^\Psi\|_{Y^{k,0}} \le \Lambda(\frac{1}{c_0}, \|S_n\|_{Y^{k,0}} + \|\partial_z v\|_{X^{k+2,0}} + \|v\|_{X^{k+3,0}}).$$

For $\|\partial_z v\|_{X^{k+2,0}}$, we use Lemma 8.2 inductively,(to reduce the order of $\|\nabla v\|_{Y^{\frac{k}{2},0}}$, to get

$$\|S_n^\Psi\|_{Y^{k,0}} \le \Lambda(\frac{1}{c_0}, \|v\|_{X^{k+3,0}} + \|S_n\|_{X^{k+2,0}} + |h|_{X^{k+2,0}} + \|S_n\|_{Y^{k,0}}). \tag{163}$$

Since we choose sufficiently smaller k than m, this estimate is okay. For opposite direction, we can do similarly to get

$$\|S_n\|_{Y^{k,0}} \le \Lambda(\frac{1}{c_0}, \|v\|_{X^{k+3,0}} + \|S_n\|_{X^{k+2,0}} + |h|_{X^{k+2,0}} + \|S_n^\Psi\|_{Y^{k,0}}). \tag{164}$$

So, we finish equivalence argument.
Now we should apply Z^k to the system (9.17). Applying tangential derivative (Z_1, Z_2) is not that harmful, but commutator between Z_3 and Laplacian is still a problem. Critical observation in [1] is the following Lemma. (Lemma 9.6 in [1]).

Lemma 43. *Consider ρ a smooth solution of*

$$\partial_t \rho + w \cdot \nabla \rho = \varepsilon \partial_{zz}\rho + \mathcal{H}, \ -1 < z < 0, \ \rho(t, y, 0) = 0, \ \rho(t = 0) = \rho_0 \tag{165}$$

for some smooth vector field w such that w_3 vanishes on the boundary. Assume that ρ and $\mathcal{H}$ are compactly supported in z. Then we have the estimate:

$$\|Z_i\rho(t)\|_\infty \lesssim \|Z_i\rho_0\|_\infty + \|\rho_0\|_\infty + \int_0^t \Big((\|w\|_{E^{2,\infty}} + \|\partial_{zz}w_3\|_{L^\infty})(\|\rho\|_{1,\infty} + \|\rho\|_4) + \|\mathcal{H}\|_{1,\infty}\Big), \ i = 1, 2, 3.$$

We should generalize this to high order, since we need k-order L^∞ -type estimate. Let us first introduce rewriting of system (160) to circumvent difficulty. We set

$$\rho(t, y, z) := |g|^{\frac{1}{4}} S_n^\Psi \tag{166}$$

and then ρ solves,

$$\partial_t \rho + w \cdot \nabla \rho - \varepsilon \partial_{zz}\rho = |g|^{\frac{1}{4}}(F_n^\Psi + F_g) \doteq \mathcal{H}, \tag{167}$$

where

$$F_g = \frac{\rho}{|g|^{\frac{1}{2}}}\Big(\partial_t + w \cdot \nabla - \varepsilon \partial_{zz}\Big)|g|^{\frac{1}{4}},$$

which shows that $\varepsilon \partial_z \ln|g|\partial_z$ is removed. And trivially, $Z_3 S_n^\Psi$ and ρ are equivalent, i.e

$$\|\rho\|_{Y^{k,0}} \le \Lambda(|h|_{Y^{k+1,0}})\|S_n^\Psi\|_{Y^{k,0}}, \ \ \|S_n^\Psi\|_{Y^{k,0}} \le \Lambda(|h|_{Y^{k+1,0}})\|\rho\|_{Y^{k,0}}. \tag{168}$$

Hence, instead of S_n^Ψ, we estimate ρ. Also note that equation of ρ is applicable above lemma. Now we extend above lemma to high order. Since we split horizontal components and vertical component, we use y to denote two horizontal coodinates.

Lemma 44. *(High order version)Consider ρ a smooth solution of*

$$\partial_t\rho + w\cdot\nabla\rho = \varepsilon\partial_{zz}\rho + \mathcal{H},\ -1<z<0,\ \rho(t,y,0)=0,\ \rho(t=0)=\rho_0 \tag{169}$$

for some smooth vector field w such that vertical component w_3 vanishes on the boundary. Assume that ρ and $\mathcal{H}$ are compactly supported in z. Then for any integer k, we have the following estimate:

$$\|Z^k\rho\|_{L^\infty} \lesssim \|\rho_0\|_{Y^{k,0}} + \int_0^t \Big(\Big(\|\partial_z w_y\|_{Y^{k,0}} + \|\partial_{zz}w_3\|_{Y^{k,0}}\Big)\|\rho\|_{X^{k+3,0}} + \|\mathcal{H}\|_{Y^{k,0}}\Big)d\tau,$$

where w_y is horizontal component (w_1,w_2).

Proof. Applying co-normal derivatives to equation generate bad commutator which come from between Z_3 and Laplacian. For simplicity, we use notations $\vec{y}=(x,y)$ and horizontal derivative $\nabla_{\vec{y}} := (\partial_x,\partial_y)$ in this proof. We rewrite equation as

$$\partial_t\rho + z\partial_z w_3(t,\vec{y},0)\partial_z\rho + w_{\vec{y}}(t,\vec{y},0)\cdot\nabla_{\vec{y}}\rho - \varepsilon\partial_{zz}\rho = \mathcal{H}-\mathcal{R} \doteq G, \tag{170}$$

where ($w_3=0$ on the boundary)

$$\mathcal{R} = (w_y(t,y,z)-w_y(t,y,0))\cdot\nabla_y\rho + (w_3(t,y,z)-z\partial_z w_3(t,y,0))\partial_z\rho.$$

We use evolution operator $S(t,\tau)$ for homogeneous solution of above system. Let,

$$\rho(t,y,z) = S(t,\tau)f_0(y,z),\ \ f(\tau,y,z)=f_0(y,z)\,(\mathit{initial\,condition}) \tag{171}$$

solves

$$\partial_t\rho + z\partial_z w_3(t,y,0)\partial_z\rho + w_y(t,y,0)\cdot\nabla_y\rho - \varepsilon\partial_{zz}\rho = 0,\ -1<z<0,\ t>\tau,\ \rho(t,y,0)=0. \tag{172}$$

For the full non-homogeneous system, by Duhamel's formula,

$$\rho(t) = S(t,0)\rho_0 + \int_0^t S(t,\tau)G(\tau)d\tau. \tag{173}$$

Now, suppose that ρ is compactly supported in z,(near the boundary) and $z<0$, then

$$\|Z_3^k\rho\|_{L^\infty} \lesssim \sum_{i,j=1}^k |z^j\partial_z^i\rho|_{L^\infty}.$$

Since z is near boundary, we don't have to consider relatively small index j, which means

$$\|Z_3^k\rho\|_{L^\infty} \lesssim \sum_{j=1}^k\sum_{i=1}^j |z^j\partial_z^i\rho|_{L^\infty}. \tag{174}$$

To estimate each terms on right hand side, we should control each $|z^j\partial_z^i\rho|_{L^\infty}$.

Lemma 45. *For evolution operator S as above, we have following estimate.*

$$|z^j\partial_z^i S(t,\tau)\rho_0|_{L^\infty} \lesssim \|\rho_0\|_{L^\infty} + \sum_{i_1+i_2=i}\|z^{j-i_1}\partial_z^{i_2}\rho_0\|_{L^\infty}. \tag{175}$$

Proof. Basically we follow the method of Lemma 9.6 in [1]. Let $\rho(t,y,z)=S(t,\tau)\rho_0(y,z)$ solves homogeneous system of (9.31). We extend this variables to whole space by

$$\tilde\rho(t,y,z)=\rho(t,y,z),\ z>0,\ \ \tilde\rho(t,y,z)=-\rho(t,y,-z),\ z<0. \tag{176}$$

So that $\tilde\rho$ solves

$$\partial_t\tilde\rho + z\partial_z w_3(t,y,0)\partial_z\tilde\rho + w_y(t,y,0)\cdot\nabla_y\tilde\rho - \varepsilon\partial_{zz}\tilde\rho = 0,\ z\in\mathbb{R}, \tag{177}$$

with initial condition $\tilde\rho(\tau,y,z)=\tilde\rho_0(y,z)$.
By introducing $\mathcal{E}$, which solves,

$$\partial_t\mathcal{E} = w_y(t,\mathcal{E},0),\ \ \mathcal{E}(\tau,\tau,y)=y$$

and define

$$g(t,y,z)=\rho(t,\mathcal{E}(t,y,z),z).$$

Then g solves,

$$\partial_t g+z\gamma(t,y)\partial_z g-\varepsilon\partial_{zz}g=0,\ \ z\in\mathbb{R},\ \ g(\tau,y,z)=\tilde{\rho}_0(y,z), \tag{178}$$

where

$$\gamma(t,y)=\partial_z w_3(t,\mathcal{E}(t,y,z),0).$$

By using Fourier transform, we get explicit form of the solution,

$$g(t,y,z)=\int_{\mathbb{R}}k(t,\tau,y,z-z')\tilde{\rho}_0(y,e^{-\Gamma(t)}z')dz', \tag{179}$$

where

$$k(t,\tau,y,z-z')=\frac{1}{\sqrt{4\pi\varepsilon\int_\tau^t e^{2\varepsilon(\Gamma(t)-\Gamma(s))}ds}}exp\Big(-\frac{(z-z')^2}{4\varepsilon\int_\tau^t e^{2\varepsilon(\Gamma(t)-\Gamma(s))}ds}\Big),\ \int_{\mathbb{R}}k(t,\tau,y,z)dz=1$$

$$\Gamma(t)=\int_\tau^t\gamma(s,y)ds.$$

We note that,

$$\begin{aligned}z^j\partial_z^i g&=\int_{\mathbb{R}}\Big(z^j\partial_z^i k(t,\tau,z-z')\Big)\tilde{\rho}_0(y,e^{-\Gamma(t)}z')dz' \\ &=\int_{\mathbb{R}}\Big((z^j-z'^j)\partial_z^i k(t,\tau,y,z-z')+(-1)^i z'^j\partial_{z'}^i k(t,\tau,y,z-z')\Big)\tilde{\rho}_0(y,e^{-\Gamma(t)}z')dz'.\end{aligned} \tag{180}$$

Now, since k is Gaussian,

$$\int_{\mathbb{R}}|(z^j-z'^j)\partial_z^i k|dz'\lesssim 1. \tag{181}$$

So, using integration by parts on the 2nd term, we can deduce

$$\begin{aligned}\|z^j\partial_z^i g\|_{L^\infty}&\lesssim\|\tilde{\rho}_0\|_{L^\infty}+\Big\|\int_{\mathbb{R}}\partial_{z'}^{i-1}k(t,\tau,y,z-z')\{jz'^{j-1}\tilde{\rho}_0(y,e^{-\Gamma(t)}z') \\ &\quad+z'^j\partial_{z'}\tilde{\rho}_0(y,e^{-\Gamma(t)}z')e^{-\Gamma(t)}\}dz'\Big\|_{L^\infty} \\ &\lesssim\cdots \\ &\lesssim\|\tilde{\rho}_0\|_{L^\infty}+\Big\|\int_{\mathbb{R}}k(t,\tau,y,z-z') \\ &\qquad\times\Big\{\sum_{i_1+i_2=i}(z')^{j-i_1}\partial_{z'}^{i_2}\tilde{\rho}_0(y,e^{-\Gamma(t)}z')e^{-i_2\Gamma(t)}\Big\}dz'\Big\|_{L^\infty} \\ &\lesssim\|\tilde{\rho}_0\|_{L^\infty}+\sum_{i_1+i_2=i}\Big\|\int_{\mathbb{R}}k(t,\tau,y,z-z')\Big\{(z')^{j-i_1}\partial_{z'}^{i_2}\tilde{\rho}_0(y,e^{-\Gamma(t)}z')\Big\}dz'\Big\|_{L^\infty}.\end{aligned} \tag{182}$$

By relation of ρ and g, we get

$$\begin{aligned}\|z^j\partial_z^i\rho\|_{L^\infty}&\le\|z^j\partial_z^i\tilde{\rho}\|_{L^\infty}\lesssim\|\tilde{\rho}_0\|_{L^\infty}+\sum_{i_1+i_2=i}\|z^{j-i_1}\partial_z^{i_2}\tilde{\rho}_0\|_{L^\infty} \\ &\le\|z^j\partial_z^i\tilde{\rho}\|_{L^\infty}\lesssim\|\rho_0\|_{L^\infty}+\sum_{i_1+i_2=i}\|z^{j-i_1}\partial_z^{i_2}\rho_0\|_{L^\infty}.\end{aligned} \tag{183}$$

□

Now we apply Z_3^k to Duhamel's formula to get

$$Z_3^k\rho(t)=Z_3^k(S(t,\tau)\rho_0)+\int_0^t Z_3^k(S(t,\tau)G(\tau))d\tau. \tag{184}$$

Using above Lemma 9.7 twice on the right hand side,

$$\|Z_3^k\rho(t)\|_{L^\infty}\lesssim\sum_{i=1}^k\sum_{j=1}^i\left\{\|\rho_0\|_{L^\infty}+\sum_{j+1+j_2=j}\|z^{j-j_1}\partial_z^{j_2}\rho_0\|_{L^\infty}\right\} \tag{185}$$

$$+\sum_{i=1}^{k}\sum_{j=1}^{i}\int_0^t \left\{ \|G\|_{L^\infty} + \sum_{j+1+j_2=j} \|z^{j-j_1}\partial_z^{j_2} G\|_{L^\infty} \right\} d\tau.$$

Using the fact that ρ (also ρ_0) and G are compactly supported in z, we have

$$\|Z_3^k \rho(t)\|_{L^\infty} \lesssim \|\rho_0\|_{W^{k,\infty}} + \int_0^t \|G\|_{W^{k,\infty}} d\tau.$$

We also note that other tangential derivatives cases also holds.(This is easier than Z_3 case.) Hence

$$\|Z^k \rho(t)\|_{L^\infty} \lesssim \|\rho_0\|_{Y^{k,0}} + \int_0^t \|G\|_{Y^{k,0}} d\tau. \tag{186}$$

Let us estimate $\|\mathcal{R}\|_{Y^{k,0}}$. Since ρ is compactly supported in z (near the boundary), using Taylor's series and inserting function $\zeta(z) := z(z+1)$ (inserting this function is very useful, because existence of $\zeta(z)$ enables us to control using co-normal derivatives of ρ), we have

$$\begin{aligned}
\|\mathcal{R}\|_{Y^{k,0}} &\leq \|(w_y(t,y,z) - w_y(t,y,0)) \cdot \nabla_y \rho\|_{Y^{k,0}} \\
&\quad + \|(w_3(t,y,z) - z\partial_z w_3(t,y,0))\partial_z \rho\|_{Y^{k,0}} \\
&\leq \|\partial_z w_y\|_{Y^{k,0}} \|\zeta(z)\rho\|_{Y^{k+1,0}} + \|\partial_{zz} w_3\|_{Y^{k,0}} \|\zeta^2(z)\partial_z \rho\|_{Y^{k,0}} \\
&\leq \Big(\|\partial_z w_y\|_{Y^{k,0}} + \|\partial_{zz} w_3\|_{Y^{k,0}} \Big) \Big(\|\zeta(z)\rho\|_{Y^{k+1,0}} + \|\zeta^2(z)\partial_z \rho\|_{Y^{k,0}} \Big).
\end{aligned} \tag{187}$$

Using anisotropic Sobolev embedding for co-normal derivatives,

$$\begin{aligned}
\|\zeta(z)\rho\|_{Y^{k+1,0}} &\lesssim \|\zeta(z)\rho\|_{X^{k+1,2}} + \|\partial_z(\zeta(z)\rho)\|_{X^{k+1,1}} \\
&\leq \|\zeta(z) Z^{k+3}\rho\|_{L^2} + \|\partial_z(\zeta(z) Z^{k+2}\rho)\|_{L^2} \\
&\lesssim \|\rho\|_{X^{k+3,0}} + \|\zeta'(z)\rho\|_{X^{k+2,0}} + \|\rho\|_{X^{k+3,0}} \\
&\lesssim \|\rho\|_{X^{k+3,0}}, \\
\|\zeta^2(z)\partial_z \rho\|_{Y^{k,0}} &= \|\zeta(z) Z_3 \rho\|_{Y^{k,0}}.
\end{aligned} \tag{188}$$

and $\zeta(z)$ is nice bounded function for all order of derivatives, so at result,

$$\|\mathcal{R}\|_{Y^{k,0}} \lesssim \Big(\|\partial_z w_y\|_{Y^{k,0}} + \|\partial_{zz} w_3\|_{Y^{k,0}} \Big) \|\rho\|_{X^{k+3,0}}. \tag{189}$$

Combining with (9.45), we finish the proof. $\square$

Now, we are ready to get energy estimate for $\|S_n\|_{Y^{k,0}}$.

Proposition 46. *Let us define non-dissipation type energy E_{m-1} as*

$$E_{m-1} := \Lambda\Big(\frac{1}{c_0}, \|v\|_{X^{m-1,0}} + |h|_{X^{m-1,1}} + \|\partial_z v\|_{X^{m-3,0}} + \|\partial_z v\|_{Y^{[\frac{m-1}{2}],0}}\Big). \tag{190}$$

We have the following estimate for $\|S_n\|_{Y^{[\frac{m}{2}],0}}$ for sufficiently large $m \geq 6$.

$$\|S_n(t)\|^2_{Y^{[\frac{m}{2}],0}} \leq \|S_n(0)\|^2_{Y^{[\frac{m}{2}],0}} + \int_0^T E^2_{m-1}(t) dt + \varepsilon \int_0^T \|\nabla S_n\|^2_{X^{m-2,0}} dt, \tag{191}$$

Note that E_{m-2} is equivalent to alternate energy

$$\mathcal{Q}_{m-1} := \|v\|_{X^{m-1,0}} + |h|_{X^{m-1,1}} + \|S_n\|_{X^{m-3,0}} + \|S_n\|_{Y^{[\frac{m-1}{2}],0}},$$

when $\sup_{t\in[0,T]} \Lambda_{m,\infty}(t) \leq C$, for some uniform constant C.

Proof. We already transformed S_n equation into equivalent-ρ equation system (9.26). From the result of Lemma 9.6, we should estimate the following four terms. Here, $k = [\frac{m}{2}]$, and m is sufficiently large.

$$\|\rho\|_{X^{k+3,0}}, \quad \|\partial_z w_y\|_{Y^{k,0}}, \quad \|\partial_{zz} w_3\|_{Y^{k,0}}, \quad \|\mathcal{H}\|_{Y^{k,0}}.$$

1), 2) $\|\rho\|_{X^{k+3,0}}$ and $\|\partial_z w_y\|_{Y^{k,0}}$ are trivially controlled by E_{m-2} by definition of ρ.
3) $\|\partial_{zz} w_3\|_{Y^{k,0}}$: By definition of w,

$$w = \bar{\chi}(D\Psi)^{-1}(v(t, \Phi^{-1} \circ \Psi) - \partial_t \Psi).$$

$$\|\partial_{zz}w_3\|_{Y^{k,0}} \le \|\partial_{zz}\Big(\bar\chi(D\Phi)^{-1}\partial_t\Psi\Big)\|_{Y^{k,0}} + \|\partial_{zz}\Big(\bar\chi(D\Phi)^{-1}v(t,\Phi^{-1}\circ\Psi)\Big)_3\|_{Y^{k,0}}. \tag{192}$$

For the first term,

$$\|\partial_{zz}\Big(\bar\chi(D\Phi)^{-1}\partial_t\Psi\Big)\|_{Y^{k,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{Y^{k,0}} + |\partial_t h|_{Y^{k,0}}\Big) \lesssim E_{m-2}.$$

For the second term, by using commutators,

$$\|\partial_{zz}\Big(\bar\chi(D\Psi)^{-1}v_3(t,\Phi^{-1}\circ\Psi)\Big)\|_{Y^{k,0}} \le \|\bar\chi\partial_{zz}\Big((D\Psi(t,y,0))^{-1}v(t,\Phi^{-1}\circ\Psi)\Big)_3\|_{Y^{k,0}} + E_{m-2}\|\partial_z v\|_{Y^{k,0}}$$

$$= \|\bar\chi\partial_{zz}\Big(v(t,\Phi^{-1}\circ\Psi)\cdot\mathbf{n}^b\Big)\|_{Y^{k,0}} + E_{m-2},$$

where we used

$$\Big((D\Psi(t,y,0))^{-1}f\Big)_3 = f\cdot\mathbf{n}^b.$$

Main difficulty is two normal derivatives. Meanwhile, by definition, $v(t,\Phi^{-1}\circ\Psi) = u(t,\Psi) \doteq u^\Psi$ and u is divergence free is zero. In the local coordinate, divergence free condition gives,

$$\partial_z u^\Psi\cdot\mathbf{n}^b = -\frac{1}{2}\partial_z(\ln|g|)u^\Psi\cdot\mathbf{n}^b - \nabla_{\tilde g}\cdot u_y^\Psi, \tag{193}$$

which means one normal derivative is replaced by tangential derivative. So,

$$\|\bar\chi\partial_{zz}\Big(v(t,\Phi^{-1}\circ\Psi)\cdot\mathbf{n}^b\Big)\|_{Y^{k,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, \|\partial_z u^\Psi\|_{Y^{k,0}} + |h|_{Y^{k+3,0}}\Big) \lesssim E_{m-2}.$$

In conclusion, $\|\partial_{zz}w_3\|_{Y^{k,0}} \lesssim E_{m-2}$.
4) $\|\mathcal{H}\|_{Y^{k,0}}$: We have

$$\|F_n^\Psi\|_{Y^{k,0}} \le \|F_n^{\Psi,1}\|_{Y^{k,0}} + \|F_n^{\Psi,2}\|_{Y^{k,0}} + \|\Pi^b F_{S^\chi}\mathbf{n}^b\|_{Y^{k,0}}. \tag{194}$$

For the first term,

$$\|F_n^{\Psi,1}\|_{Y^{k,0}} \lesssim \Lambda\Big(\frac{1}{c_0}, |\partial_t h|_{Y^{k+1,0}} + |h|_{Y^{k+2,0}} + \|w\|_{Y^{k,0}}\Big)\|\partial_z v\|_{Y^{k,0}} \lesssim E_{m-2},$$

where E_{m-2} is defined in (190). For second term,

$$\|F_n^{\Psi,2}\|_{Y^{k,0}} \lesssim \varepsilon E_{m-2}\|S_n\|_{Y^{k+2,0}} \lesssim E_{m-2}\Big(\varepsilon\|S_n\|_{X^{k+2,0}} + \varepsilon\|\partial_z S_n\|_{X^{k+1,0}}\Big).$$

Note that right hand side can be bounded by dissipation type energy. For third term,

$$\|\Pi^b F_{S^\chi}\mathbf{n}^b\|_{Y^{k,0}} \lesssim E_{m-2}\Lambda\Big(\frac{1}{c_0}, 1 + \varepsilon\|S_n\|_{Y^{k+2,0}} + \|\Pi^b((D^\varphi)^2 q)\mathbf{n}^b\|_{Y^{k,0}}\Big).$$

$\varepsilon\|S_n\|_{Y^{k+2,0}}$ was treated as second term, and $\|\Pi^b((D^\varphi)^2 q)\mathbf{n}^b\|_{Y^{k,0}}$ can be estimated since $\Pi^b\partial_z \sim Z_3$.

$$\|\Pi^b((D^\varphi)^2 q)\mathbf{n}^b\|_{Y^{k,0}} \lesssim \Lambda\Big(\frac{1}{c_0}\Big)\|\nabla q\|_{Y^{k+1,0}} \lesssim E_{m-2}\Big(1 + \varepsilon\|S_n\|_{X^{k+4,0}}\Big).$$

where last term can be treated similarly as above by anisotropic Sobolev embedding. We performed all the estimates to apply Lemma 44 and we get finally

$$\begin{aligned}\|S_n(T)\|^2_{Y^{[\frac{m}{2}],0}} &\le \|S_n(0)\|^2_{Y^{[\frac{m}{2}],0}} + \varepsilon\int_0^T E_{m-2}(t)\|\nabla S_n(t)\|_{X^{m-2,0}}dt \\ &\le \|S_n(0)\|^2_{Y^{[\frac{m}{2}],0}} + \int_0^T E^2_{m-2}(t)dt + \varepsilon\int_0^T \|\nabla S_n\|^2_{X^{m-2,0}}dt.\end{aligned} \tag{195}$$

$\square$

10. Vorticity estimate

From above sections, we checked that regularity for $\partial_z v$ is $L^\infty X^{m-3,0}$. However, we should control $X^{m-1,0}$ in space. Using critical idea of [1], we claim that we have a regularity of $\partial_z v$ in $L^4_T X^{m-1,0}$. We use vorticity to control $\partial_z v$, instead of S_n. Let us define vorticity:

$$\omega := \nabla^\varphi \times v = (\nabla \times u) \circ \Phi. \tag{196}$$

From (126), we get

$$\|Z^\alpha \partial_z v(t)\| \leq \Lambda_{m,\infty}(t)(\|v(t)\|_m + |h(t)|_{m-\frac{1}{2}} + \|\omega(t)\|_{m-1}), \quad |\alpha| = m-1. \tag{197}$$

This means that we are suffice to control vorticity instead of $\partial_z v$. Using vorticity, we can neglect pressure effect, because $\nabla^\varphi \times \nabla^{\ \varphi} q = \nabla \times \nabla p = 0$. However, main difficulty in estimating vorticity is that vorticity ω does not vanish on the free boundary. To analyze vorcitiy estimate, we divide the equation of vorticity into homogeneous and non-homogeneous parts and apply critical $H^{\frac{1}{4}}$ estimate type estimate from [1]. Applying $\nabla^\varphi \times$ to the Navier-Stokes equation gives,

$$\partial_t^\varphi \omega + (v \cdot \nabla^\varphi)\omega - (\omega \cdot \nabla^\varphi) v = \varepsilon \triangle^\varphi \omega. \tag{198}$$

Applying Z^{m-1} yields,

$$\partial_t^\varphi Z^{m-1}\omega + (v \cdot \nabla^\varphi) Z^{m-1}\omega - \varepsilon \triangle^\varphi Z^{m-1}\omega = F, \tag{199}$$

where

$$F := Z^{m-1}(\omega \cdot \nabla^\varphi v) + \mathcal{C}_S, \quad \mathcal{C}_S := \mathcal{C}_S^1 + \mathcal{C}_S^2 + \mathcal{C}_S^3, \tag{200}$$

with

$$\mathcal{C}_S^1 = [Z^{m-1} v_y] \cdot \nabla_y \omega + [Z^{m-1}, V_z]\partial_z \omega := \mathcal{C}_{S_y}^1 + \mathcal{C}_{S_z}^1, \tag{201}$$

$$\mathcal{C}_S^2 = \varepsilon [Z^{m-1}, \Delta^\varphi]\omega. \tag{202}$$

And for boundary data we have, by Lemma 19,

$$|(Z^{m-1}\omega)^b(t)| \leq \Lambda_{\infty,m}(t)(|v^b(t)|_{X^{m,0}} + |h(t)|_{X^{m,0}}). \tag{203}$$

Using trace theorem,

$$|(Z^{m-1}\omega)^b(t)| \leq \Lambda_{m,\infty}(t)\Big(\|\nabla v(t)\|_{X^{m,0}}^{\frac{1}{2}} \|v(t)\|_{X^{m,0}}^{\frac{1}{2}} + \|v(t)\|_{X^{m,0}} + |h(t)|_{X^{m,0}}\Big).$$

Because of the first term on the RHS, the only way to control this boundary terms is to treat this term as dissipation, i.e.

$$\sqrt{\varepsilon}\int_0^T |(Z^{m-1}\omega)^b|^2 dt \leq \sqrt{\varepsilon}\int_0^T \Lambda_{m,\infty}(t)\big(\|\nabla v\|_{X^{m,0}}\|v\|_{X^{m,0}} + \|v\|_{X^{m,0}}^2 + |h|_{X^{m,0}}^2\big)dt. \tag{204}$$

Using Young's inequality,

$$\sqrt{\varepsilon}\int_0^T |(Z^{m-1}\omega)^b|^2 \leq \varepsilon \int_0^T \|\nabla v\|_{X^{m,0}}^2 dt + \int_0^T \Lambda_{m,\infty}(t)\big(\|v\|_{X^{m,0}}^2 + |h|_{X^{m,0}}^2\big)dt.$$

Now, we split vorticity into two parts.

$$Z^{m-1}\omega := \omega_h^{m-1} + \omega_{nh}^{m-1}.$$

I) ω_{nh}^{m-1} solves nonhomogeneous equation:

$$\partial_t^\varphi \omega_{nh}^{m-1} + (v \cdot \nabla^\varphi)\omega_{nh}^{m-1} - \varepsilon \triangle^\varphi \omega_{nh}^{m-1} = F, \tag{205}$$

with non-zero initial data and zero-boundary condition,

$$(\omega_{nh}^{m-1})^b = 0, \ \ (\omega_{nh}^{m-1})_{t=0} = \omega(0). \tag{206}$$

Note that F on the right hand side is defined in (200), (201), and (202).
II) ω_h^{m-1} solves homogeneous equation :

$$\partial_t^\varphi \omega_h^{m-1} + (v \cdot \nabla^\varphi)\omega_h^{m-1} - \varepsilon \triangle^\varphi \omega_h^{m-1} = 0, \tag{207}$$

with zero initial data and nonhomogeneous boundary condition,

$$(\omega_h^{m-1})^b = (Z^{m-1}\omega)^b, \ \ (\omega_h^{m-1})_{t=0} = 0. \tag{208}$$

We state the energy estimates for these two vorticity terms ω_{nh}^{m-1} and ω_h^{m-1}.

10.1. **Non-homogeneous estimate.** We estimate ω_{nh}^{m-1} using (205) and (206). Similar as Proposition 10.2 in [1], we get the following Proposition.

Proposition 47. *Non-homogeneous part ω_{nh}^{m-1} estimate.*

$$\begin{aligned}\|\omega_{nh}^{m-1}(T)\|^2 + \varepsilon\int_0^T \|\nabla^\varphi \omega_{nh}^{m-1}\|^2 dt \le \Lambda_0\Big(\|\omega_{nh}^{m-1}(0)\|^2\Big) + \varepsilon\int_0^T \Lambda_{m,\infty}(t)\|\nabla S_n\|_{X^{m-2,0}}^2 dt \\ + \int_0^T \Lambda_{m,\infty}(t)\Big(\|v\|_{X^{m,0}}^2 + \|\omega\|_{m-1}^2 + |h|_{X^{m,1}}^2\Big)ds.\end{aligned} \tag{209}$$

10.2. **Homogeneous estimate.** We estimate ω_h^α. We use critical estimate Proposition 10.4 in [1]. Instead of conormal derivatives Z^α of [1], we can change Z^α into space-time conormal derivatives Z^{m-1} version.

Proposition 48. *Let us assume that*

$$\sup_{[0,T]} \Lambda_{m,\infty}(t) + \int_0^T \Big(\varepsilon\|\nabla v\|_{X^{m,0}}^2 + \varepsilon\|\nabla S_n\|_{X^{m-2,0}}^2\Big) \le M.$$

Then there exists $\Lambda(M)$ such that

$$\|\omega_h^{m-1}\|_{L^4(0,T;L^2)}^2 \le \Lambda(M)\int_0^T \Big(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_m^2\Big)dt + \varepsilon\int_0^T \|\nabla v\|_{X^{m,0}}^2\, dt. \tag{210}$$

Finally combining Proposition 47 and 48, we gain the following vorticity estimate.

Proposition 49. *Let us assume that*

$$\sup_{[0,T]} \Lambda_{m,\infty}(t) + \int_0^T \Big(\varepsilon\|\nabla v\|_{X^{m,0}}^2 + \varepsilon\|\nabla S_n\|_{X^{m-2,0}}^2\Big) \le M.$$

Then there exists $\Lambda(M)$ such that

$$\begin{aligned}\|\omega^{m-1}\|_{L^4(0,T;L^2)}^2 \le \Lambda_0\Big(\|\omega^{m-1}(0)\|^2\Big) + \Lambda(M)\int_0^T \Big(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_m^2\Big)dt \\ + \varepsilon\int_0^T \Lambda_{m,\infty}(t)\big(\|\nabla S_n\|_{X^{m-2,0}}^2 + \|\nabla v\|_{X^{m,0}}^2\big)dt.\end{aligned} \tag{211}$$

11. Uniform Regularity and Local Existence

To get local existence of free-boundary Navier-Stokes equations with viscosity ε, we use local existence theory by A.Tani [7], and combine our propositions to get uniform regularity. From Proposition 29, 34, 38, 46, and 49, we get required initial data:

$$\|v(0)\|_{X^{m,0}} + \varepsilon\|\nabla v(0)\|_{X^{m-1,1}} + |h(0)|_{X^{m,1}} + \|\nabla v(0)\|_{X^{m-1,0}} + \|\partial_z v(0)\|_{Y^{[\frac{m}{2}],0}}. \tag{212}$$

Since above terms include some terms like

$$\|(\partial_t^k v)(0)\|_{X^{m-k,0}} \quad \text{or} \quad |(\partial_t^k h)(0)|_{X^{m-k,1}}, \tag{213}$$

we can control (212) by initial data (v_0, h_0) with only spatial derivatives. From (7) and pressure estimate (22), (23), (24),

$$\|(\partial_t^m v)(0)\|_{L^2(S)} \le \Lambda\Big(\frac{1}{c_0}, \|(\partial_t^{m-1} v)(0)\|_{H^1(S)} + \varepsilon\|(\partial_t^{m-1} v)(0)\|_{H^2(\partial S)} + |(\partial_t^{m-1} h)(0)|_{H^{\frac{3}{2}}(\partial S)}\Big). \tag{214}$$

Similarly, for initial data of h, using (9),

$$|(\partial_t^m h)(0)|_{H^1(\partial S)} \le \Lambda\Big(\frac{1}{c_0}, \|(\partial_t^{m-1} v)(0)\|_{H^1(S)} + |(\partial_t^{m-1} h)(0)|_{H^{\frac{5}{2}}(\partial S)}\Big) \tag{215}$$

We apply same method for the terms with ∂_t^{m-1} on the RHS of (214) and (215) inductively to get

$$\begin{aligned}(212) &\leq \Lambda\Big(\frac{1}{c_0}, \|v_0\|_{H^m(S)} + \sum_{k=1}^{m} \varepsilon^k \|v_0\|_{H^{m+k}(S)} + |h_0|_{H^{\frac{3}{2}m+1}(\partial S)}\Big) \\ &:= \Lambda\Big(\frac{1}{c_0}, I_m(0)\Big),\end{aligned} \tag{216}$$

where initial data satisfy compatibility conditions,

$$\Pi\big(S^\varphi v^\varepsilon \mathbf{n}\big)\big|_{t=0,z=0} = 0, \quad \Pi := I - \mathbf{n}\otimes\mathbf{n}, \tag{217}$$

for $m \geq 6$ ($\frac{m}{2}+3 \leq m$). Note that our initial data should be standard Sobolev space not only in conormal space by some terms like (213).

First we regularize v_0^ε by parameter δ, so that $v_0^{\varepsilon,\delta} \in H^{l+1}(S)$, where $l \in (\frac{1}{2}, 1)$. Then by [7], for this initial condition, there is a time interval $T^{\varepsilon,\delta}$ such that on $[0, T^{\varepsilon,\delta}]$, we have a unique solution $v^{\varepsilon,\delta} \in W_2^{l+2,\frac{l}{2}+1}\Big([0,T^{\varepsilon,\delta}]\times S\Big) = L^2\Big([0,T^{\varepsilon,\delta}], H^{l+2}(S)\Big) \cap H^{\frac{l}{2}+1}\Big([0,T^{\varepsilon,\delta}], L^2(S)\Big)$. For convenience, we use v instead of $v^{\varepsilon,\delta}$. Then by parabolic regularity, for $T \in [0, T^{\varepsilon,\delta}]$,

$$\begin{aligned}\mathcal{N}_m(T) := &\sup_{t\in[0,T]} \Big(\|v(t)\|^2_{X^{m,0}} + |h(t)|^2_{X^{m,1}} + \|\partial_z v(t)\|^2_{X^{m-2,0}} + \|\partial_z v(t)\|^2_{Y^{[\frac{m}{2}],0}}\Big) \\ &+ \|\partial_z v\|^2_{L^4_T X^{m-1,0}} + \varepsilon \int_0^T \Big(\|\nabla v(t)\|^2_{X^{m,0}} + \|\nabla\partial_z v(t)\|^2_{X^{m-2,0}}\Big) dt < \infty.\end{aligned} \tag{218}$$

Let us suppose $\mathcal{N}_m(T_0) < \infty$ for some T_0, then using Stoke's regularity on $[\frac{T_0}{2}, T_0]$, we know that $v(T_0) \in H^{l+1}(S)$, so by considering this as initial condition again, we know that it can be extended to some $T_1 > T_0$. We have to show that this extension is uniform in ε and δ using our propositions.

We also define equivalent quantity

$$\mathcal{E}_m(T) := \sup_{t\in[0,T]} \mathcal{Q}_m(t) + \mathcal{D}_m(T) + \|\omega\|_{L^4_T X^{m-1,0}}, \tag{219}$$

where

$$\begin{aligned}\mathcal{Q}_m(t) &:= \|v(t)\|^2_{X^{m,0}} + |h(t)|^2_{X^{m,1}} + \|S_n(t)\|^2_{X^{m-2,0}} + \|S_n(t)\|^2_{Y^{[\frac{m}{2}],0}}, \\ \mathcal{D}_m(T) &:= \varepsilon \int_0^T \Big(\|\nabla v(t)\|^2_{X^{m,0}} + \|\nabla S_n(t)\|^2_{X^{m-2,0}}\Big) dt.\end{aligned} \tag{220}$$

We know that $\mathcal{N}_m(T)$ and $\mathcal{E}_m(T)$ are equivalent,

$$\mathcal{N}_m(T) \leq \Lambda\Big(\frac{1}{c_0}, \mathcal{E}_m(T)\Big) \quad \text{and} \quad \mathcal{E}_m(T) \leq \Lambda\Big(\frac{1}{c_0}, \mathcal{N}_m(T)\Big).$$

To derive uniform time interval, we choose R and c_0 so that, $\frac{1}{c_0} << R$, and define,

$$T_*^{\varepsilon,\delta} = \sup\left\{T \in [0,1] \mid \mathcal{E}_m(t) \leq R,\ |h(t)|_{2,\infty} \leq \frac{1}{c_0},\ \partial_z\varphi(t) \geq c_0,\ \forall t \in [0,T]\right\}. \tag{221}$$

Now, let us combine Propositions to close energy estimate. For convenience we define

$$\begin{aligned}\mathcal{M}_m(T) := &\sup_{t\in[0,T]} \Big(\|v(t)\|^2_{X^{m-1,1}} + |h(t)|^2_{X^{m-1,2}} + \|\partial_z v(t)\|^2_{X^{m-2,0}} + \|\partial_z v(t)\|^2_{Y^{[\frac{m}{2}],0}}\Big) \\ &+ \|\partial_z v\|^2_{L^4_T X^{m-1,0}} + \varepsilon \int_0^T \Big(\|\nabla v(t)\|^2_{X^{m,0}} + \|\nabla\partial_z v(t)\|^2_{X^{m-2,0}}\Big) dt < \infty.\end{aligned} \tag{222}$$

For Proposition 29, we apply Proposition 32 to $|h|_{X^{m-1,\frac{5}{2}}}$ for sufficiently small $\theta \ll 1$ and $T \ll 1$ on the right hand side of (95) so that $\Lambda_{m,\infty}(T)\Big\{\theta|Z^{m-1}\partial_t h(T)|^2_{L^2} + \sqrt{T}|Z^{m-1}\nabla\partial_t h(T)|^2_{L^2}\Big\}$ is absorbed by energy of left hand sides except $Z^{m-1} = \partial_t^{m-1}$ case.

$$\begin{aligned} &\|v(T)\|^2_{X^{m-1,1}} + |h(T)|^2_{X^{m-1,2}} + \varepsilon \int_0^T \|\nabla v(t)\|^2_{X^{m,0}} dt \\ &\lesssim \Lambda(\frac{1}{c_0}, I_m(0)) + (\theta + T)|\partial_t^m h|^2_{H^1} + \int_0^T \Lambda(\frac{1}{c_0}, \mathcal{Q}_m(t))dt \end{aligned} \tag{223}$$

For $Z^m = \partial_t^m$ case, we multiply suffciently small constant $\gamma \ll 1$ to Proposition 34 and apply Propositino 32 to $|\partial_t^{m-1} h|_{H^{\frac{5}{2}}}$ to get

$$\begin{aligned} &\gamma\|\partial_t^m v(T)\|^2 + \gamma|\partial_t^m h(T)|^2_{H^1} + \gamma\varepsilon \int_0^T \|\nabla \partial_t^m v(t)\|^2 dt \\ &\lesssim \Lambda(\frac{1}{c_0}, I_m(0)) + (\theta + \sqrt{T})|\partial_t^m h(T)|^2_{H^1} \\ &\quad + \gamma\Lambda_0(R)\Big(\|v(T)\|^2_{X^{m-1,1}} + |\partial_t^{m-1} h|^2_{H^2}\Big) + \gamma \int_0^T \Lambda(\frac{1}{c_0}, \mathcal{Q}_m(t))dt, \end{aligned} \tag{224}$$

where we used notation $\Lambda_0(R) := \Lambda(\frac{1}{c_0}, R)$. We choose $\theta, T, \gamma \ll 1$ and $\theta + \sqrt{T} \ll \gamma$ so that $(\theta + R)|\partial_t^m h|^2_{H^1}$ on the RHS of (223) is absorbed by $\gamma|\partial_t^m h(T)|^2_{H^1}$ in the LHS of (224), and $(\theta + \sqrt{T})|\partial_t^m h(T)|^2_{H^1} + \gamma\Lambda_0(R)\Big(\|v(T)\|^2_{X^{m-1,1}} + |\partial_t^{m-1} h|^2_{H^2}\Big)$ in the RHS of (224) is absorbed by $\|v(T)\|^2_{X^{m-1,1}} + |h(T)|^2_{X^{m-1,2}}$ in the LHS of (223).

Now, we combine (223), (224), and (49) with sufficiently small $\gamma, T \ll 1$. Therefore

$$\begin{aligned} &\|v(T)\|^2_{X^{m,0}} + |h(T)|^2_{X^{m,1}} + \varepsilon \int_0^T \|\nabla v(t)\|^2_{X^{m-1,1}} dt \\ &\lesssim \Lambda(\frac{1}{c_0}, I_m(0)) + \int_0^T \Lambda(\frac{1}{c_0}, \mathcal{Q}_m(t))dt. \end{aligned} \tag{225}$$

Applying Proposition 32 to Proposition 38,

$$\begin{aligned} &\|S_n(T)\|^2_{X^{m-2,0}} + \varepsilon \int_0^T \|\nabla S_n(t)\|^2_{X^{m-2,0}} dt \leq \Lambda(\frac{1}{c_0}, I_m(0)) \\ &\quad + \Lambda_0(R)(\theta + \sqrt{T})|\partial_t^m h|^2_{H^1} + \int_0^T \Lambda(\frac{1}{c_0}, \mathcal{Q}_m(t))dt, \quad \theta, T \ll 1. \end{aligned} \tag{226}$$

From (49),

$$\begin{aligned} \|\omega\|^2_{L^4_T X^{m-1,0}} &\leq \Lambda(\frac{1}{c_0}, I_m(0)) + \Lambda(\frac{1}{c_0}, \mathcal{Q}_m(t))dt \\ &\quad + \varepsilon \int_0^T \Lambda_{m,\infty}(t)\big(\|\nabla S_n\|^2_{X^{m-2,0}} + \|\nabla v\|^2_{X^{m,0}}\big)dt. \end{aligned} \tag{227}$$

We combine (225), (226), and (227) with sufficiently small $\theta, T \ll 1$ then finally we get

$$\mathcal{E}_m(T) \leq \Lambda\Big(\frac{1}{c_0}, I_m(0)\Big) + \Lambda_0(R)T, \quad \Lambda_0(R) := \Lambda(\frac{1}{c_0}, R). \tag{228}$$

Now, we calculate conditions in (221).

$$|h(t)|_{2,\infty} \leq |h(0)|_{2,\infty} + \Lambda_0(R)T \quad \forall t \in [0, T], \tag{229}$$

and since we've chosen A in diffeomorphism to be 1, at initial time,

$$\partial_z \varphi(t) \geq 1 - \int_0^t \|\partial_t \nabla \eta\|_{L^\infty} \geq 1 - \Lambda_0(R)T, \quad \forall t \in [0, T]. \tag{230}$$

From (228),(229), and (230), we see that right hand side is independent to ε and δ, so are possible to choose $R = \Lambda\Big(|h_0|_{2,\infty}, I_m(0)\Big)$ which satisfies that there exist T_*(independent to ε, δ) such that

$\forall t \in [0, T_*]$,

$$\mathcal{N}_m(t) \leq \frac{R}{2}, \quad |h(t)|_{2,\infty} \leq \frac{1}{2c_0}, \quad \partial_z \varphi(t) \geq c_0 + \frac{1-c_0}{2} > c_0. \tag{231}$$

This implies $T_* < T_*^{\varepsilon,\delta}$, hence T_* implies there exist uniform time, independent to ε, δ. Since $\Psi_m(T_*)$ is uniformly bounded in δ, we can pass the limit, $\delta \to 0$, by strong compactness argument.

12. Uniqueness

12.1. **Uniqueness for Navier-Stokes.** We prove uniqueness of Theorem 7. As usual, we consider two solution sets $(v_1^\varepsilon, \varphi_1^\varepsilon, q_1^\varepsilon), (v_2^\varepsilon, \varphi_2^\varepsilon, q_2^\varepsilon)$ with same initial condition and proper compatibility conditions. Then on the interval $[0, T^\varepsilon]$, we have uniform bounds of energy,

$$\Psi_m^i(T^\varepsilon) \leq R, \quad i = 1, 2.$$

Let us define,

$$\bar{v}^\varepsilon := v_1^\varepsilon - v_2^\varepsilon, \;\; \bar{h}^\varepsilon := h_1^\varepsilon - h_2^\varepsilon, \;\; \bar{q}^\varepsilon := q_1^\varepsilon - q_2^\varepsilon. \tag{232}$$

We will make system of equations for $(\bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{q}^\varepsilon)$ and do energy estimate. By divergence free condition, $\nabla^{\varphi_i} \cdot v_i^\varepsilon = 0$,

$$\Big(\partial_t + v_{y,i}^\varepsilon \cdot \nabla_y + V_{z,i}^\varepsilon \partial_z\Big) v_i^\varepsilon + \nabla^{\varphi_i} q_i^\varepsilon - \varepsilon \Delta^{\varphi_i} v_i^\varepsilon = 0.$$

Then we have equation about $(\bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{q}^\varepsilon)$. First for Navier-Stokes,

$$\Big(\partial_t + v_{y,1}^\varepsilon \cdot \nabla_y + V_{z,1}^\varepsilon \partial_z\Big) \bar{v}^\varepsilon + \nabla^{\varphi_1} \bar{q}^\varepsilon - \varepsilon \Delta^{\varphi_1} \bar{v}^\varepsilon = F, \tag{233}$$

where

$$F = (v_{y,2}^\varepsilon - v_{y,1}^\varepsilon) \cdot \nabla_y v_2^\varepsilon + (V_{z,2}^\varepsilon - V_{z,1}^\varepsilon) \partial_z v_2^\varepsilon - \Big(\frac{1}{\partial_z \varphi_2^\varepsilon} - \frac{1}{\partial_z \varphi_1^\varepsilon}\Big)\Big(P_1^* \nabla q_2^\varepsilon\Big) + \frac{1}{\partial_z \varphi_2^\varepsilon}\Big((P_2 - P_1)^* \nabla q_2^\varepsilon\Big)$$
$$+\varepsilon\Big(\frac{1}{\partial_z \varphi_2^\varepsilon} - \frac{1}{\partial_z \varphi_1^\varepsilon}\Big) \nabla \cdot (E_1 \nabla v_2^\varepsilon) + \varepsilon \frac{1}{\partial_z \varphi_2^\varepsilon} \nabla \cdot \Big((E_2 - E_1) \nabla v_2^\varepsilon\Big).$$

For divergence-free condition,

$$\nabla^{\varphi_1} \cdot \bar{v}^\varepsilon = -\Big(\frac{1}{\partial_z \varphi_2^\varepsilon} - \frac{1}{\partial_z \varphi_1^\varepsilon}\Big) \nabla \cdot \Big(P_1 v_2^\varepsilon\Big) - \frac{1}{\partial_z \varphi_2^\varepsilon} \nabla \cdot \Big((P_2 - P_1) v_2^\varepsilon\Big). \tag{234}$$

For Kinematic boundary condition,

$$\partial_t \bar{h}^\varepsilon - (v^\varepsilon)_{y,1}^b \cdot \nabla h + \Big((v_{z,1}^\varepsilon)^b - (v_{z,2}^\varepsilon)^b\Big) = -\Big((v_{y,2}^\varepsilon)^b - (v_{y,1}^\varepsilon)^b\Big) \cdot \nabla h_2^\varepsilon. \tag{235}$$

Continuity of stress tensor condition becomes,

$$\bar{q}^\varepsilon \mathbf{n}_1 - 2\varepsilon \Big(S^{\varepsilon_1} \bar{v}^\varepsilon\Big) \mathbf{n}_1 = g \bar{h}^\varepsilon - \eta \nabla \cdot \Big(\frac{\nabla \bar{h}^\varepsilon}{\sqrt{1 + |\nabla h_1^\varepsilon|^2}}\Big) \tag{236}$$

$$+2\varepsilon \Big(\Big(S^{\varphi_1} - S^{\varphi_2}\Big) v_2^\varepsilon\Big) \mathbf{n}_1 + 2\varepsilon \Big(S^{\varphi_2} v_2^\varepsilon\Big)\Big(\mathbf{n}_1 - \mathbf{n}_2\Big) - \eta \nabla \cdot \Big(\left\{\frac{1}{\sqrt{1 + |\nabla h_1^\varepsilon|^2}} - \frac{1}{\sqrt{1 + |\nabla h_2^\varepsilon|^2}}\right\} \nabla h_2^\varepsilon\Big).$$

Using above 4 equations, we get L^2- energy estimate,(since initial condition is zero, no initial term appear on right hand side)

$$\|\bar{v}^\varepsilon(t)\|_{L^2}^2 + \big|\bar{h}^\varepsilon(t)\big|_{H^1}^2 + \varepsilon \int_0^t \|\nabla \bar{v}^\varepsilon\|_{L^2}^2 \, ds \leq \Lambda(R) \int_0^t \Big(\|\bar{v}^\varepsilon(s)\|_{L^2}^2 + \big|\bar{h}^\varepsilon(s)\big|_{H^{\frac{3}{2}}}^2\Big) ds. \tag{237}$$

We skip detail calculation, since we can use our previous energy estimates basically. But, in above equations for $(\bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{q}^\varepsilon)$, right hand side does not have low order than L^2 energy. However we have uniform bound of m-order energy, so we can extract bad high order terms into $\Lambda(R)$. To estimate $\big|\bar{h}^\varepsilon(s)\big|_{H^{\frac{3}{2}}}$, we don't have to take time derivatives as like in previous section, since we already have bounded high order energy. And moreover, we don't need uniform estimate in ε, since we are dealing about for fixed ε. So, we estimate $\varepsilon \big|\bar{h}^\varepsilon(s)\big|_{H^{\frac{3}{2}}}^2$.

Lemma 50. *For every $m \in \mathbb{N}$, $\varepsilon \in (0,1)$, we have the estimate*

$$\varepsilon \left|h(t)\right|_{m+\frac{1}{2}}^2 \leq \varepsilon \left|h(0)\right|_{m+\frac{1}{2}}^2 + \varepsilon \int_0^t |v^b|_{m+\frac{1}{2}}^2 + \int_0^t \Lambda_{1,\infty}\Big(\|v\|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \Big) ds, \tag{238}$$

where

$$\Lambda_{1,\infty} = \Lambda\Big(|\nabla h|_{L^\infty} + \|v\|_{1,\infty} \Big).$$

Proof. See proposition 3.4 in [1]. $\square$

This is true for our case, since it comes from Kinematic boundary condition. We can also apply this lemma, to $\bar{h}^\varepsilon$ case,(surely, $\bar{h}^\varepsilon(0) = 0$) and then combine with above L^2 estimate, then we get the following estimate.(non-uniform in ε)

$$\|\bar{v}^\varepsilon(t)\|_{L^2}^2 + \left|\bar{h}^\varepsilon(t)\right|_{H^1}^2 + \varepsilon \left|\bar{h}^\varepsilon(t)\right|_{H^{\frac{3}{2}}}^2 + \varepsilon \int_0^t \|\nabla \bar{v}^\varepsilon\|_{L^2}^2 \, ds \leq \frac{\Lambda(R)}{\varepsilon} \int_0^t \Big(\|\bar{v}^\varepsilon(s)\|_{L^2}^2 + \varepsilon \left|\bar{h}^\varepsilon(s)\right|_{H^{\frac{3}{2}}}^2 \Big) ds. \tag{239}$$

Then we can use Gronwall's inequality to get uniqueness. So finish uniqueness part of theorem 6.

12.2. **Uniqueness for Euler.** Since our estimate in above subsection(uniqueness for Navier-Stokes) is not uniform in ε, result cannot be applied to Euler equation. As like in Navier-Stokes case, let we have two solutions $(v_1, h_1, q_1), (v_2, h_2, q_2)$ with same initial condition. Suppose,

$$\sup_{[0,T]} \Big(\|v_i\|_m + \|\partial_z v_i\|_{m-1} + \|\partial_z v_i\|_{[\frac{m}{2}],\infty} + |h_i|_{m+1} \Big) \leq R, \quad i = 1, 2. \tag{240}$$

(This is true from result in Theorem 6) Define $\bar{v} \doteq v_1 - v_2$, $\ \bar{h} \doteq h_1 - h_2$, $\ \bar{q} \doteq q_1 - q_2$ and we write equation of $(\bar{v}, \bar{h}, \bar{q})$, as before. Euler equation becomes,

$$\Big(\partial_t + v_{y,1} \cdot \nabla_y + V_{z,1} \partial_z \Big) \bar{v} + \nabla^{\varphi_1} \bar{q} = F', \tag{241}$$

where

$$F' = (v_{y,2} - v_{y,1}) \cdot \nabla_y v_2 + (V_{z,2} - V_{z,1}) \partial_z v_2 - \Big(\frac{1}{\partial_z \varphi_2} - \frac{1}{\partial_z \varphi_1} \Big) \Big(P_1^* \nabla q_2 \Big) + \frac{1}{\partial_z \varphi_2} \Big((P_2 - P_1)^* \nabla q_2 \Big).$$

For divergence-free condition,

$$\nabla^{\varphi_1} \cdot \bar{v} = -\Big(\frac{1}{\partial_z \varphi_2} - \frac{1}{\partial_z \varphi_1} \Big) \nabla \cdot \Big(P_1 v_2 \Big) - \frac{1}{\partial_z \varphi_2} \nabla \cdot \Big((P_2 - P_1) v_2 \Big). \tag{242}$$

For Kinematic boundary condition,

$$\partial_t \bar{h} - v_{y,1}^b \cdot \nabla h + \Big(v_{z,1}^b - v_{z,2}^b \Big) = -\Big(v_{y,2}^b - v_{y,1}^b \Big) \cdot \nabla h_2. \tag{243}$$

Continuity of stress tensor condition becomes,

$$\bar{q} \mathbf{n}_1 = g \bar{h} - \eta \nabla \cdot \Big(\frac{\nabla \bar{h}}{\sqrt{1 + |\nabla h_1|^2}} \Big) - \eta \nabla \cdot \Big(\Big\{ \frac{1}{\sqrt{1 + |\nabla h_1|^2}} - \frac{1}{\sqrt{1 + |\nabla h_2|^2}} \Big\} \nabla h_2 \Big). \tag{244}$$

By performing basic L^2-estimate, as similarly above, (we skip detail here)

$$\|\bar{v}(t)\|_{L^2}^2 + \left|\bar{h}(t)\right|_{H^1}^2 \leq \Lambda(R) \int_0^t \Big(\|\bar{v}(s)\|_{H^1}^2 + \left|\bar{h}(s)\right|_{H^{\frac{3}{2}}}^2 \Big) ds. \tag{245}$$

We should control $\|v\|_1$ on right hand side. But, since there are no dissipation on left hand side, we cannot make it absorbed. Instead, we use vorticity. Let's define vorticity $\omega = \nabla^\varphi \times v$ (which is equivalent to $\omega = (\nabla \times u)(t, \Phi)$). We have

$$\begin{aligned} \omega \times \mathbf{n} &= \frac{1}{2} \Big(D^\varphi v \mathbf{n} - (D^\varphi v)^T \mathbf{n} \Big) \\ &= S^\varphi v \mathbf{n} - (D^\varphi v)^T \mathbf{n} = \frac{1}{2} \partial_{\mathbf{n}} u - g^{ij} \Big(\partial_j v \cdot \mathbf{n} \Big) \partial_{y^i}. \end{aligned}$$

Hence, it is suffice to estimate ω instead of $\partial_z v$, $i.e$

$$\|\partial_z v\|_{L^2} \leq \Lambda(R)\Big(\|\omega\|_{L^2} + \|v\|_1 + |h|_{\frac{3}{2}}\Big). \tag{246}$$

To estimate ω, we use vorticity equation.

$$\Big(\partial_t^{\varphi_i} + v_i \cdot \nabla^{\varphi_i}\Big)\omega_i = \Big(\omega_i \cdot \nabla^{\varphi_i}\Big)v_i. \tag{247}$$

L^2 energy estimate of $\bar{\omega}$ is

$$\|\bar{\omega}(t)\|_{L^2}^2 \leq \Lambda(R)\int_0^t \Big(|\bar{h}(s)|_1^2 + \|\bar{v}(s)\|_1^2 + \|\partial_z \bar{v}(s)\|_{L^2}^2 + \|\bar{\omega}(s)\|_{L^2}^2\Big)ds. \tag{248}$$

We also should control $|h|^2_{L^2H^{\frac{3}{2}}}$. As similar to Dirichlet-Neumann estimate we can control this by $|\partial_t h|^2_{L^2L^2}$. And, from kinematic boundary condition of $\bar{h}$, we easily get

$$\big|\partial_t \bar{h}(t)\big|_{L^2}^2 \leq \Lambda(R)\Big(\|\nabla \bar{v}\|_{L^2}^2 + |\bar{h}|_{H^1}^2\Big). \tag{249}$$

Then we can use Gronwall's inequality to get uniqueness. So finish uniqueness part of theorem 6 and theorem 7.

13. Inviscid limit

In this section we send ε to zero, and get a unique solution of free boundary Euler equation. For $\varepsilon \in (0,1]$ and $T \leq T_*$, we have uniform energy bound,

$$\mathcal{N}_m(T) \leq C. \tag{250}$$

So we have uniform boundness for v^ε in $L^\infty\Big([0,T], H^m_{co}\Big)$ and h^ε in $L^\infty\Big([0,T], H^{m+1}\Big)$. And, by Rellich-Kondrachov theorem, we have, for each t, compactness of $v^\varepsilon(t)$ in $H^{m-1}_{co,loc}$ and $h^\varepsilon(t)$ in H^m_{loc}. And from our energy function, we have a uniform boundness of $\partial_t v^\varepsilon(t)$ in $H^{m-1}_{co,loc}$ and of $\partial_t h^\varepsilon(t)$ in H^m_{loc} for $\forall t \in [0,T]$. Now, we have subsequence v^{ε_n}, h^{ε_n}, such that

$$v^{\varepsilon_n} \to v, \ \textit{strongly in}\ C([0,T], H^{m-1}_{co,loc}), \tag{251}$$

$$h^{\varepsilon_n} \to h, \ \textit{strongly in}\ C([0,T], H^m_{loc}).$$

About pressure, from pressure section, we have boundness of ∇q^ε in $L^2([0,T]\times S)$, so get some q such that,

$$\nabla q^{\varepsilon_n} \rightharpoonup \nabla q, \ \textit{weakly in}\ L^2([0,T]\times S)$$

and limit functions $(v, h, \nabla q)$ satisfy

$$\sup_{t\in[0,T]}\Big(\|v(t)\|^2_{H^m_{co}} + |h(t)|^2_{H^{m+1}} + \|\partial_z v(t)\|^2_{H^{m-2}_{co}} + \|\partial_z v\|^2_{W_{co}^{[\frac{m}{2}],\infty}}\Big) + \|\partial_z v\|^2_{L^4_T H^{m-1}_{co}} \leq R, \tag{252}$$

Now, we can pass to the limit and get the fact that $(v, h, \nabla q)$ is a weak solution of Euler equation.(interior). For boundary condition, first we can assume that the trace(boundary function),

$$v^{\varepsilon_n}(t, y, z=0) \rightharpoonup v^b \text{ weakly in } L^2([0,T]\times S) \tag{253}$$

for some v^b. In kinematic boundary condition, v^b is linear and we have strong convergence of h, so kinematic boundary condition is satisfied weakly surely. Next, for continuity of stress tensor condition, by bounded lipschitz norm of v^{ε_n}, $2\varepsilon(Su)\mathbf{n} \to 0$ in weak limit process. And, limit of surface tension part is trivial by stong convergence of h. Hence, in the weak sense,

$$q^b = gh - \eta\nabla\cdot\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big). \tag{254}$$

Hence we have a weak solution (v, h) which is strong $H^{m-1}_{co,loc}\times H^m_{co,loc}$ - convergence of $(v^{\varepsilon_n}, h^{\varepsilon_n})$.(and global for weak convergence in $L^2 \times H^1$) Moreover this limit is unique by previous section. Meanwhile, we can get strong convergence (non-local) in $L^2\times H^1$. To get this, we just investigate norm convergence. For $t \in [0,T]$, using basic L^2-energy estimate and uniform boundness of high order energy,

$$\Big(\|v^{\varepsilon_n} J^{\varepsilon_n}(t)\|_{L^2}^2 + g\,|h^{\varepsilon_n}(t)|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h^{\varepsilon_n}(t)|^2} - 1|_{L^1} \Big) \tag{255}$$
$$-\Big(\|v_0^{\varepsilon_n} J_0^{\varepsilon_n}\|_{L^2}^2 + g\,|h_0^{\varepsilon_n}|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h_0^{\varepsilon_n}|^2} - 1|_{L^1} \Big) \leq \varepsilon_n \Lambda(R) \to 0,\ as\ \varepsilon_n \to 0,$$

where $J^\varepsilon \doteq (\partial_z \varphi^\varepsilon)^{1/2}$ and ε_n on the right hand side come from dissipation of energy estimate. We assume that $\|v_0^\varepsilon - v_0\|_{L^2} \to 0,\ \|h_0^\varepsilon - h_0\|_{H^1} \to 0$ in statement of theorem 1.3 and

$$\|\partial_z \varphi^\varepsilon|_{t=0} - \partial_z \varphi|_{t=0}\|_{L^2} \lesssim |h_0^\varepsilon - h_0|_{H^{\frac{1}{2}}} \leq \Lambda(R)\,|h_0^\varepsilon - h_0|_{L^2}^{\frac{1}{2}}.$$

This implies

$$\begin{aligned} &\lim_{\varepsilon\to 0} \Big(\|v_0^{\varepsilon_n} J_0^{\varepsilon_n}\|_{L^2}^2 + g\,|h_0^{\varepsilon_n}|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h_0^{\varepsilon_n}|^2} - 1|_{L^1} \Big) \\ &= \|v_0 J_0\|_{L^2}^2 + g\,|h_0|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h_0|^2} - 1|_{L^1}. \end{aligned} \tag{256}$$

Lastly, using energy conservation in Euler equation ($\varepsilon = 0$, in basic L^2-estimate), we get

$$\|v_0 J_0\|_{L^2}^2 + g\,|h_0|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h_0|^2} - 1|_{L^1} = \|vJ\|_{L^2}^2 + g\,|h|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h|^2} - 1|_{L^1}. \tag{257}$$

Finally we get norm convergence.

$$\begin{aligned} &\lim_{\varepsilon\to 0} \Big(\|v^{\varepsilon_n} J^{\varepsilon_n}(t)\|_{L^2}^2 + g\,|h^{\varepsilon_n}(t)|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h^{\varepsilon_n}(t)|^2} - 1|_{L^1} \Big) \\ &= \|vJ\|_{L^2}^2 + g\,|h|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h|^2} - 1|_{L^1}. \end{aligned} \tag{258}$$

With weak convergence, this implies strong convergence to (vJ, h) in $L^2 \times H^1$. And strong convergence of h means $(v^\varepsilon, h^\varepsilon) \to (v, h)$ strongly in $L^2 \times H^1$. (without J) L^∞-type convergence can be done by L^2 convergence, uniform energy boundness, and anisotropic embedding.

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