

Determining Optimal Rates for Communication for Omniscience

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Abstract—This paper considers the communication for omniscience (CO) problem: A set of users observe a discrete memoryless multiple source and want to recover the entire multiple source via noise-free broadcast communications. We study the problem of how to determine an optimal rate vector that attains omniscience with the minimum sum-rate, the total number of communications. The results cover both asymptotic and non-asymptotic models where the transmission rates are real and integral, respectively. We propose a modified decomposition algorithm (MDA) and a sum-rate increment algorithm (SIA) for the asymptotic and non-asymptotic models, respectively, both of which determine the value of the minimum sum-rate and a corresponding optimal rate vector in polynomial time. For the coordinate saturation capacity (CoordSatCap) algorithm, a nesting algorithm in MDA and SIA, we propose to implement it by a fusion method and show by experimental results that this fusion method contributes to a reduction in computation complexity. Finally, we show that the separable convex minimization problem over the optimal rate vector set in the asymptotic model can be decomposed by the fundamental partition, the optimal partition of the user set that determines the minimum sum-rate, so that the problem can be solved more efficiently.

Index Terms—communication for omniscience, Dilworth truncation, mutual dependence, submodularity.

I. INTRODUCTION

Assume that there are a finite number of users in a system. Each of them observes a distinct component of a discrete multiple correlated source in private. The users are allowed to exchange their observations over public authenticated broadcast channels. We assume that these channels are noiseless so that all the transmissions are correctly heard, or received, by all users. The communications could be interactive and the rates of public communications are unconstrained. That is, there are no capacity upper bounds imposed on the broadcast links. The purpose is to attain *omniscience*, the state that each user obtains all the components in the entire multiple

source in the system. This problem is called *communication for omniscience (CO)*, which was originally formulated in [4]. The CO problem in [4] is based on an *asymptotic model* where the length of the observation sequence is allowed to approach infinity. Whereas the authors in [5]–[7] also study the *non-asymptotic model* where the number of observations is finite and the communication rates are restricted to be integral. In fact, the non-asymptotic model is important in a practical problem in peer-to-peer (P2P) wireless communications as described below.

The *finite linear source model* studied in [6] is an example of the non-asymptotic model, where the multiple random source is represented by a vector that belongs to a finite field and the users transmit linear combinations of their observations to obtain this vector. By assuming that each dimension in this vector represents a packet, the finite linear source model can describe the situation when a base station wants to disseminate a set of packets to a group of mobile clients: Each client only obtains a partial knowledge of the packet set due to the fading effects of the wireless channels. The omniscience of the packet set can be attained by letting the clients transmit linear combinations of packets, say, by some network coding scheme, e.g., [8], via the P2P channels, which could be more reliable than the retransmissions over base-to-peer (B2P) channels if the clients are geographically close to each other. The CO problem in this packet model is called *coded cooperative data exchange (CCDE)* which was independently proposed in [8]–[10] and further studied in [11]–[15]. In [15], [16], the idea of packet-splitting was introduced to CCDE. It allows each packet to be divided into a number of chunks so that the transmissions in CCDE refer to the linear combinations of chunks and the normalized transmission rates are fractional. This can be considered as an extension of the CCDE and finite linear source model towards the asymptotic model.

An optimization problem that naturally arises is how to attain omniscience with the least cost and the cost usually refers to the overall transmission rates, or sum-rate, e.g., the total number of linear combinations of packets that are transmitted by all clients in CCDE. It is shown in [4] that the Slepian-Wolf (SW) constraints [17] on all proper subsets of the user set determine the omniscience-achievability of a transmission rate vector. Hence, in [4], [7], [9], [10], [14], [15], [18]–[20], the problem of minimizing the sum-rate is formulated by linear programming (LP) and the combinatorial nature of this problem has also been revealed. Then, instead of solving the minimum sum-rate problem directly by the existing LP algorithms, the main issue is how to deal with

The primary results have been published in [1]. Part of the results have also been published in [2], [3].

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the exponentially growing number of constraints.

In the studies on the finite linear source model in [18], [21] and CCDE in [2], [3], [9], [10], [14], [15], [22], [23], the submodularity of the minimum sum-rate problem was revealed, which is essentially due to the submodularity of the entropy function.¹ In particular, it is shown in [2], [3], [21], [22] that, in a non-asymptotic model where the entropy function takes integer values,² all omniscience-achievable rate vectors that have the same sum-rate constitute a submodular base polyhedron. Since a rate vector in this submodular base polyhedron can be found by the Edmond greedy algorithm³ and the variation range of the minimum sum-rate in a non-asymptotic model is integral and bounded,⁴ the minimum sum-rate and a corresponding optimal rate vector are determined efficiently by the sum-rate adaption algorithms proposed in [15], [18]. However, it still remains unclear if all the results derived in [2], [3], [21], [22] for the non-asymptotic model also hold for the asymptotic one and if there exists an algorithm that can efficiently determine an optimal rate vector that attains omniscience by the minimum sum-rate in an asymptotic model where the variation range of the minimum sum-rate is continuous. On the other hand, the study in [16] shows that allowing packet-splitting in CCDE incurs less transmission costs in P2P communications. It could mean that the minimum sum-rate in the asymptotic model is no greater than the one in the non-asymptotic model in the same system, which makes it desirable to know the optimal solution for CO in the asymptotic model.

The importance of studying the CO problem is also highlighted by its dual relationship with the secret capacity, the maximum rate at which the secret key can be generated by the users in the system, in [4]: The secret capacity equals to the total amount of information in the multiple source subtracted by the minimum sum-rate (in an asymptotic model) for them to achieve omniscience. It is also pointed out in [4] that the secret capacity is upper bounded by a mutual dependence over the partitions of the users and this upper bound is shown to be tight in [27].⁵ The mutual dependence is also named as shared information in [28] for the secret generation problem and multivariate mutual information (MMI) in [19], [20], where the authors in [19] proposed this mutual dependence to be the generalization of Shannon's mutual information for multiple

random variables. Then, determining secret capacity, mutual dependence, shared information or MMI relies on the solution to the CO problem in the asymptotic model and vice versa. It is shown in [19] that the problem of obtaining the MMI reduces to the task of determining the value of the Dilworth truncation, which can be solved in strongly polynomial time due to the submodularity of the entropy function. However, for solving the CO problem, knowing the minimum sum-rate is not sufficient: We also need to know how to distribute the minimum sum-rate among the users so that omniscience is achievable. Therefore, it is required to determine an optimal rate vector that attains omniscience with the minimum sum-rate.

The work in this paper is based on the CO problem that is originally formulated in [4]. We consider the minimum sum-rate problem: how to attain omniscience with the minimum total number of communications. The work in this paper differs from [4], [27], [28] in that, in addition to characterizing the minimum sum-rate or discussing how to obtain the value of it, we are particularly interested in how to determine a corresponding optimal rate vector that attains omniscience. The results cover both asymptotic and non-asymptotic models. For the non-asymptotic model, we focus on the finite linear source model and CCDE. For solving the CO problem, we propose a modified decomposition algorithm (MDA) and a sum-rate increment algorithm (SIA) for the asymptotic and non-asymptotic models, respectively, both of which determine the value of the minimum sum-rate and a corresponding optimal rate vector in polynomial time. For the coordinate saturation capacity (CoordSatCap) algorithm, a nesting algorithm in the MDA and SIA algorithm, we propose to implement it by a fusion method and show by experimental results that this fusion method contributes to a reduction in computation complexity as compared to the CoordSatCap algorithm that is implemented in [15], [18]. We also derive results on the *fundamental partition* \mathcal{P}^* , the finest optimal partition of the user set that determines the value of the minimum sum-rate in the asymptotic model,⁶ which is also determined by the MDA algorithm. It is shown that, in CCDE, the omniscience of a packet set can be attained by splitting each packet into $|\mathcal{P}^*| - 1$ chunks. Finally, we reveal some decomposition properties of \mathcal{P}^* . We show that the separable convex minimization problem over the optimal rate vector set in the asymptotic model can be decomposed by \mathcal{P}^* so that the problem can be solved more efficiently.

A. Summary of Main Results

Our main results are summarized as follows.

1) We show that all omniscience-achievable rate vectors with the sum-rate equal to a given value form a base polyhedron. By observing the nonemptiness of the base polyhedron, we prove directly based on [26, Theorems 2.5(i) and 2.6(i)] that the minimum sum-rate in the asymptotic model is

¹In [24, Section 3], it is shown that the entropy function is the rank function of a polymatroid, which belongs to a subgroup of submodular functions. The entropy function reduces to the matrix rank function in the finite linear source model and the cardinality function in CCDE, both of which are submodular.

²Finite linear source model and CCDE are examples of non-asymptotic model with integer-valued entropy function.

³The Edmond greedy algorithm in [25] is a special case of the coordinate saturation capacity algorithm [26, Greedy Algorithm II in Section 3.2] for normalized submodular functions [26, Theorems 3.18 and 3.19]. The one implemented in [15], [18] is modified for the crossing and intersecting submodular functions, respectively.

⁴The value of the minimum sum-rate is real in the asymptotic model and integral in the non-asymptotic model. It is nonnegative and no greater than the total amount of information in the multiple source.

⁵The authors in [4] derived the results on secret capacity in a general setting: A subset of the users are active while the others are the helpers that assist the active users in generating the secret key. The author in [27] proved that the upper bound on secret capacity is tight when there is no helpers, i.e., all the users in the system are active.

⁶It is shown in [19] that the optimal partitions that give rise to the minimum sum-rate form a Dilworth truncation lattice [29] where the minimal/finest and maximal/coarsest minimizers exist. The name 'fundamental partition' was first used in [19] to denote the finest partition in this lattice.

determined by an optimization over the partitions of the user set. This result verifies the proof in [27] on the tightness of the lower bound on the minimum sum-rate for CO that was proposed in [4]. For the non-asymptotic model, we show that the minimum sum-rate is the least integer that is no less than the one in the asymptotic model, which provides theoretical proof to an observation in [15]: The difference in minimum sum-rate between the asymptotic and non-asymptotic models is no greater than one. Since the optimal rate vectors also form a base polyhedron, we use the integrality of the extreme points in this base polyhedron to show two results for the minimum sum-rate problem in the finite linear source model, or CCDE: (a) there exists an integral optimal rate vector, which is consistent with the results in [15], [18], and (b) there exists a fractional optimal rate vector that can be implemented by dividing each packet into $|\mathcal{P}^*| - 1$ chunks.

2) For determining an optimal rate vector in the asymptotic model, an MDA algorithm is proposed. It starts with a lower estimation on the minimum sum-rate and iteratively updates it by the finest minimizer of a Dilworth truncation problem until the minimum is reached and a corresponding optimal rate vector is determined. For the CoordSatCap algorithm, which is originally proposed in [26, Greedy Algorithm II in Section 3.2], for solving the Dilworth truncation problem in each iteration of the MDA algorithm, we propose a fusion method implementation (CoordSatCapFus) so that the submodular function minimization (SFM) in each iteration is solved over a merged or fused user set with the cardinality no greater than the original one. We show that the optimal solution returned by the MDA algorithm for the asymptotic model can also be utilized for solving the minimum sum-rate problem in the non-asymptotic model by no more than one additional call of the CoordSatCapFus algorithm. Independently, we also propose an SIA algorithm for determining an optimal rate vector as well as the minimum sum-rate for the non-asymptotic model. Both the MDA and SIA algorithms complete in polynomial time based on the existing SFM techniques. We run experiments to show that the fusion method CoordSatCapFus contributes to a reduction in computation complexity as compared to the CoordSatCap algorithm, which is implemented in [15], [18] for the finite linear source model, and the reduction is considerable when the number of users grows.

3) For the initial estimations in the MDA and SIA algorithms, we derive a lower bound (LB) on the minimum sum-rate for both asymptotic and non-asymptotic models that can be determined in linear time. We show that this lower bound can be used as the initial estimation of the minimum sum-rate searching algorithms, e.g., the MDA and SIA algorithm proposed in this paper for the asymptotic and non-asymptotic models, respectively. The observation that a lower bound can initiate the minimum sum-rate searching algorithm is also consistent with the results in [15], [18] for the finite linear source model.⁷ We run experiments to show that the proposed

LB in the non-asymptotic model is much tighter than the ones in [8], [11].

4) We also study the minimum weighted sum-rate problem in the optimal rate vector set, a problem that was originally formulated and studied for the finite linear source model in [15], [18]. It is shown that by choosing a proper linear ordering of the user indices the optimal rate vectors returned by the MDA and SIA algorithms also minimize a weighted sum-rate function in the optimal rate vector set for the asymptotic and non-asymptotic models, respectively. The result that the minimum weighted sum-rate problem can be solved by a proper linear ordering for the non-asymptotic model is consistent with the results in [15], [18]. Our study shows that this idea can also be applied to the asymptotic model.

5) We show that the fundamental partition \mathcal{P}^* is the minimal separator of a submodular function which gives rise to the decomposition property of \mathcal{P}^* in the asymptotic model: The separable convex function minimization problem over the optimal rate vector set can be broken into $|\mathcal{P}^*|$ subproblems, each of which formulates the separable convex function minimization problem in one element or user subset in \mathcal{P}^* . These subproblems can be solved separately so that the overall complexity is reduced.

B. Organization

In Section II, we present the system model for CO and describe the asymptotic and non-asymptotic models, the finite linear source model and CCDE. In Section III, we analyze the minimum sum-rate problem in both asymptotic and non-asymptotic models based on the concepts of submodularity and Dilworth truncation. In Section IV, a LB on the minimum sum-rate is proposed for the asymptotic and non-asymptotic models, which is used in Section V to initiate the MDA and SIA algorithms. The complexity of both algorithms is also discussed in Section V. In Section VII, we reveal the decomposition property of the fundamental partition in the asymptotic model.

II. SYSTEM MODEL

Let V with $|V| > 1$ be a finite set that contains the indices of all users in the system. We call V the *ground set*. Let $Z_V = (Z_i : i \in V)$ be a vector of discrete random variables indexed by V . For each $i \in V$, user i privately observes an n -sequence Z_i^n of the random source Z_i that is i.i.d. generated according to the joint distribution P_{Z_V} . We allow users exchange their sources directly so as to let all users in V recover the source sequence Z_V^n . The state that each user obtains the total information in the entire multiple source is called *omniscience*, and the process that users communicate with each other to attain omniscience is called *communication for omniscience (CO)* [4].

Let $\mathbf{r}_V = (r_i : i \in V)$ be a rate vector indexed by V . We call \mathbf{r}_V an *achievable rate vector* if the omniscience can be attained by letting users communicate with the rates designated by \mathbf{r}_V . Let r be the *sum-rate function* associated with \mathbf{r}_V such that

$$r(X) = \sum_{i \in X} r_i, \quad \forall X \subseteq V$$

⁷For the finite linear source model, or CCDE, it is suggested in [15, Appendix F] to adapt the sum-rate from either lower or upper bound to the minimum, while [18, Algorithms 3] is a binary search method starting with the initial lower and upper bounds on the minimum sum-rate.

with the convention that $r(\emptyset) = 0$. $r(V)$ is the sum-rate of \mathbf{r}_V over all users, or the total number of transmissions, in the system. For $X, Y \subseteq V$, let $H(Z_X)$ be the amount of randomness in Z_X measured by Shannon entropy [30] and $H(Z_X|Z_Y) = H(Z_{X \cup Y}) - H(Z_Y)$ be the conditional entropy of Z_X given Z_Y . In the rest of this paper, without loss of generality, we simplify the notation Z_X by X .

It is shown in [4] that an achievable rate vector must satisfy the Slepian-Wolf (SW) constraints [17]:

$$r(X) \geq H(X|V \setminus X), \quad \forall X \subsetneq V. \quad (1)$$

The interpretation of (1) is: To attain omniscience, the total amount of information sent from user set X should be at least equal to the total amount of information that is missing in $V \setminus X$. The set of all achievable rate vectors is [4]⁸

$$\mathcal{R}_{\text{CO}}(V) = \{\mathbf{r}_V \in \mathbb{R}^{|V|} : r(X) \geq H(X|V \setminus X), \forall X \subsetneq V\}.$$

We say that α is an *achievable sum-rate* if there exists an achievable rate vector $\mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V)$ such that $r(V) = \alpha$.

A. Asymptotic and Non-asymptotic Models

We consider both asymptotic and non-asymptotic models. In the asymptotic multiple random source model, we will study the CO problem by considering the asymptotic limits as the *block length* n goes to infinity. The communication rates in an asymptotic model could be real or fractional. The minimum sum-rate can be determined by the following linear programming (LP) [4, Proposition 1]

$$R_{\text{ACO}}(V) = \min\{r(V) : \mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V)\} \quad (2)$$

and the set of all optimal rate vectors is

$$\mathcal{R}_{\text{ACO}}^*(V) = \{\mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V) : r(V) = R_{\text{ACO}}(V)\}.$$

In the non-asymptotic model, the block length n is finite and the communication rates are required to be integral. The minimum sum-rate can be determined by the integer linear programming (ILP) [14], [15], [18]⁹

$$R_{\text{NCO}}(V) = \min\{r(V) : \mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V) \cap \mathbb{Z}^{|V|}\} \quad (3)$$

and the optimal rate vector set is

$$\mathcal{R}_{\text{NCO}}^*(V) = \{\mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V) \cap \mathbb{Z}^{|V|} : r(V) = R_{\text{NCO}}(V)\}.$$

The non-asymptotic model is exemplified by the finite linear source model and CCDE as explained as follows.

⁸The achievable rate region was originally given in [4] based on the SW constraints in a more general case: The omniscience problem in the active user set $A \subseteq V$ with the users in $V \setminus A$ serving as the helpers. The CO problem studied in this paper is the case when $A = V$.

⁹This ILP problem has been formulated in [14], [15] in terms of the cardinality function for CCDE and in [18] in terms of the rank function for the finite linear source model.

B. Finite Linear Source Model and CCDE

Let \mathbb{F}_q be a finite field. q is the order of \mathbb{F}_q such that $q = p^N$, where p is a prime number and N is a positive integer. In a finite linear multiple source model, we assume that each Z_i can be expressed by an $l(z_i)$ -dimensional column vector \mathbf{z}_i in the finite field $\mathbb{F}_q^{l(z_i)}$ such that

$$\mathbf{z}_i = A_i \mathbf{x},$$

where $\mathbf{x} \in \mathbb{F}_q^{l(\mathbf{x})}$ is some $l(\mathbf{x})$ -dimensional uniformly distributed random vector and $A_i \in \mathbb{F}_q^{l(z_i) \times l(\mathbf{x})}$ is an $l(z_i)$ -by- $l(\mathbf{x})$ matrix. For $X \subseteq V$, let $A_X = [A_i : i \in X]$. In the finite linear source model, the value of the entropy function at X reduces to the rank of A_X , i.e., $H(X) = \text{rank}(A_X)$ and $H(X|Y) = \text{rank}(A_{X \cup Y}) - \text{rank}(A_Y)$. Then, H is integer-valued, i.e., $H(X) \in \mathbb{Z}_+, \forall X \subseteq V$, and we assume that $H(V) = l(\mathbf{x})$. The users transmit linear combinations of \mathbf{z}_i s in order to attain the omniscience of \mathbf{x} .¹⁰ Therefore, the finite linear source model is an example of the non-asymptotic model where the value of the entropy function H is integral.

By realizing that each dimension in \mathbf{x} can represent a packet, the finite linear source model poses a practical problem in wireless communications: the omniscience, or recovery, of a packet set in peer-to-peer (P2P) wireless network. Let all the users in V be mobile clients that are geographically close to each other so that any client's broadcasts can be received losslessly by the others. Consider the problem of disseminating the packet set \mathbf{x} from a base station to all mobile clients in V . Due to the fading effects of wireless channels, each client may just obtain a partial knowledge of \mathbf{x} at the end of base-to-peer (B2P) transmissions, but the clients' knowledge could be complementary to each other. In this case, we can set free the base station and let the clients transmit linear combinations of packets, e.g., by some network coding scheme [8], so as to help each other recover \mathbf{x} . The omniscience problem in this packet model is how to let all users recover the packet set \mathbf{x} with the least number of transmissions and this problem, which was originally formulated in [8], is called the coded cooperative data exchange (CCDE). The concept of packet-splitting was also introduced to CCDE in [15], [16]. It extends the finite linear source model from a non-asymptotic setting towards an asymptotic one, which is explained by the following example.

Example II.1. *There are three users $V = \{1, 2, 3\}$ in the system. They observe respectively*

$$Z_1 = (W_a, W_b, W_c, W_d, W_e),$$

$$Z_2 = (W_a, W_b, W_f),$$

$$Z_3 = (W_c, W_d, W_f),$$

where each W_j is an independent uniformly distributed random bit. The purpose is to let all the users attain the omniscience of Z_V via communications. In the corresponding CCDE system (see Fig. 1), each W_j represents a packet so that the column vector \mathbf{z}_i denotes all the packets received by mobile client i after the B2P transmissions. All mobile clients in V

¹⁰In a finite linear source model, it is sufficient for the user i to transmit linear combinations of \mathbf{z}_i s to attain omniscience [6].

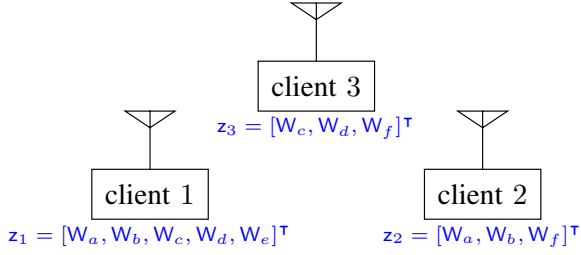


Fig. 1. The corresponding CCDE system for the CO problem in Example II.1, where W_j denotes a packet that belongs to a field \mathbb{F}_q . There are three clients that want to obtain six packets in $\mathbf{x} = [W_a, \dots, W_f]^T$. User i initially obtains \mathbf{z}_i . The users transmit linear combinations of \mathbf{z}_i s via lossless wireless broadcast channels to help the others recover all packets in \mathbf{x} .

transmit linear combinations of \mathbf{z}_i s over P2P channels in order to attain the omniscience of all packets in $\mathbf{x} = [W_a, \dots, W_f]^T$.

In this system, we have all the achievable rate vectors contained in

$$\begin{aligned} \mathcal{R}_{CO}(V) = \{ \mathbf{r}_V \in \mathbb{R}^{|V|} : r(\emptyset) = 0, \\ r(\{1\}) \geq H(\{1\}|\{2, 3\}) = 1, \\ r(\{2\}) \geq H(\{2\}|\{1, 3\}) = 0, \\ r(\{3\}) \geq H(\{3\}|\{1, 2\}) = 0, \\ r(\{1, 2\}) \geq H(\{1, 2\}|\{3\}) = 3, \\ r(\{1, 3\}) \geq H(\{1, 3\}|\{2\}) = 3, \\ r(\{2, 3\}) \geq H(\{2, 3\}|\{1\}) = 1 \}. \end{aligned}$$

One can show that the minimum sum-rate is $R_{ACO}(V) = \frac{7}{2}$ and the optimal rate vector set is $\mathcal{R}_{ACO}^*(V) = \{(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})\}$ for the asymptotic model. In CCDE, the rate vector $(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$ can be implemented by packet-splitting. Let the users divide each packets into two chunks of equal length, e.g., $\mathbf{z}_2 = (W_a^{(1)}, W_a^{(2)}, W_b^{(1)}, W_b^{(2)}, W_f^{(1)}, W_f^{(2)})$ where each W_j is split to $W_j^{(1)}$ and $W_j^{(2)}$. Let the users transmit the rate $(5, 1, 1)$ with each tuple denoting the number of linear combinations of the packet chunks. We have $(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$ and $\frac{7}{2}$ being the normalized rate vector and sum-rate, respectively. For the non-asymptotic model, we have the minimum sum-rate $R_{NCO}(V) = 4$ and the optimal rate vector set $\mathcal{R}_{NCO}^*(V) = \{(3, 0, 1), (2, 1, 1), (3, 1, 0)\}$.

We show an example of how to implement the rate vector $(2, 1, 1)$ in $\mathcal{R}_{NCO}^*(V)$ for the non-asymptotic model by a network coding scheme. The implementation of other optimal rate vectors in $\mathcal{R}_{NCO}^*(V)$ and $(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$ in $\mathcal{R}_{ACO}^*(V)$ can be derived in a similar manner. By letting user 1 transmit $W_b + W_f + W_c$ and W_e , user 2 transmit $W_a + W_f$ and user 3 transmit $W_d + W_f$, all the users are able to recover the whole packet set \mathbf{x} . For example, user 2 receives W_e , recovers W_c by subtracting message $W_b + W_f + W_c$ by $W_b + W_f$ and recovers W_d by subtracting message $W_d + W_f$ by W_f so that he/she obtains all the packets in \mathbf{x} . The corresponding transmission rate vector in this coding scheme is $(2, 1, 1)$.¹¹

For a fractional rate vector \mathbf{r}_V , let $k \in \mathbb{Z}_+$ be the least common multiple (LCM) of all denominators of r_i s, i.e., k is

the minimum nonnegative integer such that $k\mathbf{r}_V = (kr_i : i \in V)$ is integral. It means that \mathbf{r}_V can be implemented by k -packet-splitting, i.e., dividing each packet into k chunks, in CCDE. For example, in Example II.1, $k = 2$ is the LCM of the denominators of all dimensions in $\mathbf{r}_V = (\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$, which means $(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$ can be implemented by 2-packet-splitting. Therefore, in CCDE, we are not only interested in the existence of an integral optimal rate vector in $\mathcal{R}_{NCO}^*(V)$ for the non-asymptotic setting, but are also concerned whether there exists a fractional optimal rate vector in $\mathcal{R}_{ACO}^*(V)$ for the asymptotic setting and how large is the LCM k .

III. MINIMUM SUM-RATE AND OPTIMAL RATE VECTOR

The fundamental problem in CO is how to obtain the value of the minimum sum-rate and an optimal rate vector: the value of $R_{ACO}(V)$ and a rate vector in $\mathcal{R}_{ACO}^*(V)$ for the asymptotic model and the value of $R_{NCO}(V)$ and a rate vector in $\mathcal{R}_{NCO}^*(V)$ for the non-asymptotic model. Although the minimum sum-rate problem can be formulated by LP (2) and ILP (3) for asymptotic and non-asymptotic settings, respectively, it is not efficient to directly solve them since the number of the constraints grow exponentially in $|V|$. In this section, we reveal the equivalence between the constant sum-rate achievable rate region and a base polyhedron. Directly based on the nonemptiness of this based polyhedron, we show that the minimum sum-rate in the asymptotic model is determined by an optimization problem over the partitions of the user set, which revisits the results in [4], [27]. We also show that the minimum sum-rate in the non-asymptotic model is the least integer that is no less than the one in the asymptotic model.

A. Preliminaries

We first describe submodularity and the related concepts as follows. For a set function $f: 2^V \mapsto \mathbb{R}$, the *polyhedron* and *base polyhedron* of f are respectively [26, Section 2.3] [31, Definition 9.7.1]

$$\begin{aligned} P(f, \leq) &= \{ \mathbf{r}_V \in \mathbb{R}^{|V|} : r(X) \leq f(X), \forall X \subseteq V \}, \\ B(f, \leq) &= \{ \mathbf{r}_V \in P(f, \leq) : r(V) = f(V) \}. \end{aligned}$$

In the same way, we can define $P(f, \geq)$ and $B(f, \geq)$. A set function f is *submodular* if the submodular inequality

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \quad (4)$$

holds for all $X, Y \subseteq V$; f is *supermodular* if $-f$ is submodular; f is *modular* if it is both submodular and supermodular [26, Section 2.3]. $P(f, \leq)$ and $B(f, \leq)$ are submodular polyhedron and base polyhedron, respectively, if f is submodular. A set function f is *intersecting submodular* if the submodular inequality (4) holds for all sets that are intersecting, i.e., all $X, Y \subseteq V$ such that $X \cap Y \neq \emptyset$ [26, Section 2.3]. Note, an intersecting submodular function f may or may not require (4) hold for all $X, Y \subseteq V$, which means a submodular function is also intersecting submodular, but not necessarily vice versa.

A set function f is the *rank function* of a *polymatroid* if it is (a) normalized: $f(\emptyset) = 0$; (b) monotonic: $f(X) \geq f(Y)$ for all $X, Y \subseteq V$ such that $Y \subseteq X$; (c) submodular [26, Section 2.2]. If f is a polymatroid rank function, the normality

¹¹The coding scheme that implements an achievable rate vector \mathbf{r}_V is not necessarily unique.

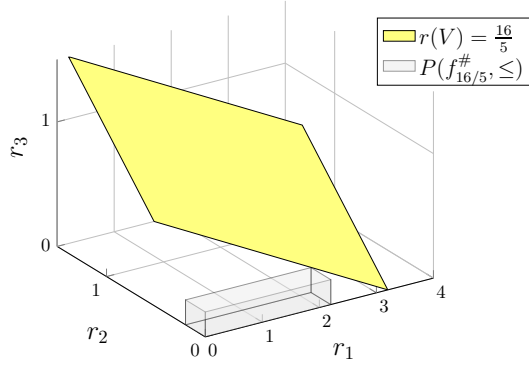


Fig. 2. For the system in Example II.1, when $\alpha = \frac{16}{5}$, the polyhedron $P(f_{16/5}^#, \leq)$ does not intersect with the plane $\{\mathbf{r}_V \in \mathbb{R}^3: r(V) = \frac{16}{5}\}$. Therefore, $B(f_{16/5}^#, \leq) = \{\mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V): r(V) = \frac{16}{5}\} = \emptyset$. It means that sum-rate $\frac{16}{5}$ is not achievable or there does not exist an achievable rate vector that has sum-rate equal to $\frac{16}{5}$.

and monotonicity ensure the nonnegativity of f , i.e., $f(X) \geq 0, \forall X \subseteq V$, and $B(f, \leq) \subseteq \mathbb{R}_+^{|V|}$ [26, Lemma 3.23]. It is shown in [24, Section 3] that the entropy function H is a polymatroid rank function. It is easy to see that r , the sum-rate function that is associated with a rate vector \mathbf{r}_V , is modular and $f(X) - r(X)$ is submodular/intersecting submodular if f is submodular/intersecting submodular.

B. Nonemptiness of the Base Polyhedron

For $\alpha \in \mathbb{R}_+$, we define

$$f_\alpha(X) = \begin{cases} H(X|V \setminus X) & X \subsetneq V \\ \alpha & X = V \end{cases}.$$

The base polyhedron of f_α

$$\begin{aligned} B(f_\alpha, \geq) &= \{\mathbf{r}_V \in P(f_\alpha, \geq): r(V) = f_\alpha(V)\} \\ &= \{\mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V): r(V) = \alpha\} \end{aligned}$$

contains all achievable rate vectors that have sum-rate equal to α . It is possible that $B(f_\alpha, \geq) = \emptyset$, which means that the sum-rate α is not achievable. Let $f_\alpha^\#$ be the dual set function of f_α that is defined as [26, Section 2.3]

$$\begin{aligned} f_\alpha^\#(X) &= f_\alpha(V) - f_\alpha(V \setminus X) \\ &= \alpha - f_\alpha(V \setminus X), \quad \forall X \subseteq V. \end{aligned}$$

Consider the constraint $r(X) \geq f_\alpha(X)$ in $B(f_\alpha, \geq)$. If we restrict the rate vector \mathbf{r}_V to satisfy $r(X) \geq f_\alpha(X)$ for some $X \subseteq V$ and $r(V) = \alpha$, then we necessarily put constraint

$$r(V \setminus X) = r(V) - r(X) \leq \alpha - f_\alpha(V \setminus X) = f_\alpha^\#(V \setminus X)$$

on set $V \setminus X$. By converting the constraints in $B(f_\alpha, \geq)$ in the same way for all $X \subseteq V$, we get the base polyhedron

$$B(f_\alpha^\#, \leq) = \{\mathbf{r}_V \in P(f_\alpha^\#, \leq): r(V) = f_\alpha^\#(V) = \alpha\}$$

such that $B(f_\alpha^\#, \leq) = B(f_\alpha, \geq)$ [26, Lemma 2.4].¹² Then, the set of all achievable rate vectors with sum-rate α is

¹²In [26, Lemma 2.4], $B(f_\alpha^\#, \leq) = B(f_\alpha, \geq)$ holds without the submodularity or intersecting submodularity of $f_\alpha^\#$.

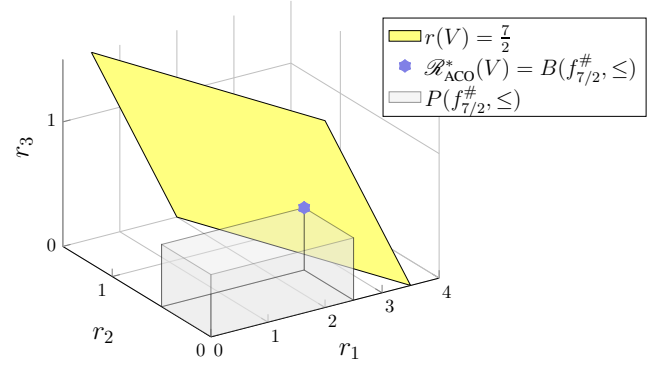


Fig. 3. For the system in Example II.1, when $\alpha = \frac{7}{2}$, the polyhedron $P(f_{7/2}^#, \leq)$ intersects with the plane $\{\mathbf{r}_V \in \mathbb{R}^3: r(V) = \frac{7}{2}\}$ at $\mathbf{r}_V = (\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$, i.e., $B(f_{7/2}^#, \leq) = \{\mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V): r(V) = \frac{7}{2}\} = \{(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})\}$. In this case, sum-rate $\frac{7}{2}$ is achievable and there is only one optimal rate vector for the asymptotic model.

described by $B(f_\alpha^\#, \leq)$. It is obvious that $B(f_\alpha^\#, \leq) \neq \emptyset$ if the polyhedron $P(f_\alpha^\#, \leq)$ intersects with the hyperplane $\{\mathbf{r}_V \in \mathbb{R}^{|V|}: r(V) = \alpha\}$. Also, the optimal rate sets can be described by the base polyhedra

$$\begin{aligned} \mathcal{R}_{\text{ACO}}^*(V) &= B(f_{R_{\text{ACO}}(V)}^\#, \leq); \\ \mathcal{R}_{\text{NCO}}^*(V) &= B(f_{R_{\text{NCO}}(V)}^\#, \leq) \cap \mathbb{Z}^{|V|}. \end{aligned}$$

Therefore, the minimum sum-rate, either $R_{\text{ACO}}(V)$ or $R_{\text{NCO}}(V)$, can be determined by studying the condition on α for the nonemptiness of $B(f_\alpha^\#, \leq)$, and a rate vector in the optimal rate set, either $\mathcal{R}_{\text{ACO}}^*(V)$ or $\mathcal{R}_{\text{NCO}}^*(V)$, can be determined by any algorithm that is able to search a base point \mathbf{r}_V in $B(f_{R_{\text{ACO}}(V)}^\#, \leq)$ or $B(f_{R_{\text{NCO}}(V)}^\#, \leq) \cap \mathbb{Z}^{|V|}$.¹³

Example III.1. For the system in Example II.1, we have $B(f_\alpha, \geq) = \{\mathbf{r} \in \mathcal{R}_{\text{CO}}(V): r(V) = \alpha\}$. For a fixed value of α , consider the constraint $r(\{1\}) \geq f_\alpha(\{1\}) = 1$ in $B(f_\alpha, \geq)$. Since we restrict the sum-rate to be $r(\{1, 2, 3\}) = \alpha$, we have constraint $r(\{2, 3\}) = \alpha - r(\{1\}) \leq \alpha - f_\alpha(\{1\}) = f_\alpha^\#(\{2, 3\}) = \alpha - 1$. If we convert the constraints one by one in $B(f_\alpha, \geq)$, we have the dual base polyhedron

$$\begin{aligned} B(f_\alpha^\#, \leq) &= \{\mathbf{r}_V \in \mathbb{R}^{|V|}: r(\emptyset) = 0, r(\{1\}) \leq \alpha - 1, \\ &\quad r(\{2\}) \leq \alpha - 3, r(\{3\}) \leq \alpha - 3, \\ &\quad r(\{1, 2\}) \leq \alpha, r(\{1, 3\}) \leq \alpha, \\ &\quad r(\{2, 3\}) \leq \alpha - 1, \\ &\quad r(\{1, 2, 3\}) = \alpha\} \end{aligned}$$

such that $B(f_\alpha^\#, \leq) = B(f_\alpha, \geq)$. We increase the value of α from 0. It can be shown that $B(f_\alpha^\#, \leq) = \emptyset$ when $\alpha < \frac{7}{2}$, e.g., when $\alpha = \frac{16}{5}$ in Fig. 2. When $\alpha = \frac{7}{2}$, we have $\mathcal{R}_{\text{ACO}}^*(V) = B(f_{7/2}^\#, \leq) = \{(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})\}$ as in Fig. 3. We keep increasing α after reaching $\frac{7}{2}$. It can be shown that $B(f_\alpha^\#, \leq) \cap \mathbb{Z}^{|V|} = \emptyset$ when $\alpha < 4$. When $\alpha = 4$, we have $\mathcal{R}_{\text{NCO}}^*(V) = B(f_4^\#, \leq) \cap \mathbb{Z}^{|V|} = \{(3, 0, 1), (2, 1, 1), (3, 1, 0)\}$ as shown in Fig. 4.

¹³A base (point) is a $|V|$ -dimension vector \mathbf{r}_V in $B(f, \leq)$, where f is a set function defined on the power set 2^V .

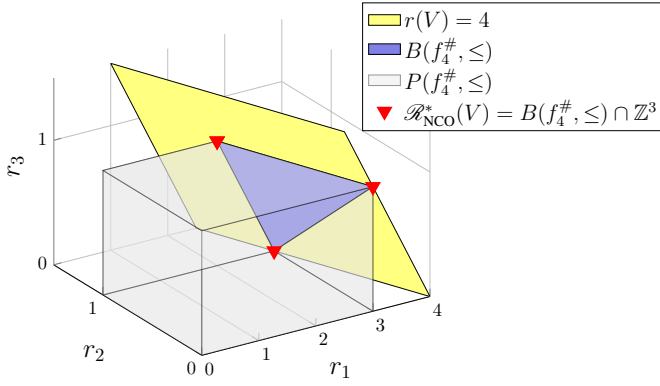


Fig. 4. For the system in Example II.1, when $\alpha = 4$, the polyhedron $P(f_4^\#, \leq)$ and the plane $\{\mathbf{r}_V \in \mathbb{R}^3: r(V) = 4\}$ intersect, i.e., $B(f_4^\#, \leq) = \{\mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V): r(V) = 4\} \neq \emptyset$. Also, $\mathcal{R}_{\text{NCO}}^*(V) = B(f_4^\#, \leq) \cap \mathbb{Z}^3 = \{(2, 1, 1), (3, 0, 1), (3, 1, 0)\}$. In this case, sum-rate 4 is achievable and there are three optimal rate vectors for the non-asymptotic model.

C. Minimum Sum-rate

The condition for $B(f_\alpha^\#, \leq) \neq \emptyset$ can be easily derived based on the intersecting submodularity of $f_\alpha^\#$.

Lemma III.2. For $\alpha \geq 0$, $f_\alpha^\#$ is intersecting submodular; If $\alpha \geq H(V)$, $f_\alpha^\#$ is submodular.

Proof: For function $f_\alpha^\#$, we have

$$f_\alpha^\#(X) + f_\alpha^\#(Y) - f_\alpha^\#(X \cup Y) - f_\alpha^\#(X \cap Y) = \begin{cases} H(X) + H(Y) \\ -H(X \cup Y) - H(X \cap Y) + \alpha - H(V) & X \cap Y = \emptyset \\ H(X) + H(Y) - H(X \cup Y) - H(X \cap Y) & \text{otherwise} \end{cases}$$

Then, $f_\alpha^\#(X) + f_\alpha^\#(Y) \geq f_\alpha^\#(X \cup Y) + f_\alpha^\#(X \cap Y)$, $\forall X, Y \subseteq V: X \cap Y \neq \emptyset$. Also, the inequality holds for all $X, Y \subseteq V$ when $\alpha \geq H(V)$. Lemma holds. ■

For $X \subseteq V$, denote by $\Pi(X)$ the set that contains all partitions of X . A partition \mathcal{P} of X is the set that satisfies: (a) $C \neq \emptyset$ for all $C \in \mathcal{P}$; (b) $C \cap C' = \emptyset$ for any distinctive $C, C' \in \mathcal{P}$; (c) $\bigcup_{C \in \mathcal{P}} C = X$. Denote $\Pi'(X) = \Pi(X) \setminus \{X\} = \{\mathcal{P} \in \Pi(X): |\mathcal{P}| > 1\}$. For a partition $\mathcal{P} \in \Pi(X)$, let

$$f_\alpha^\#[\mathcal{P}] = \sum_{C \in \mathcal{P}} f_\alpha^\#(C)$$

and $\hat{f}_\alpha^\#$ be the Dilworth truncation of $f_\alpha^\#$ that is defined as [32]

$$\hat{f}_\alpha^\#(X) = \min_{\mathcal{P} \in \Pi(X)} f_\alpha^\#[\mathcal{P}], \quad \forall X \subseteq V. \quad (5)$$

We have $\hat{f}_\alpha^\#$ being a submodular function due to the intersecting submodularity of $f_\alpha^\#$ [26, Theorems 2.5(i) and 2.6(i)]. It is shown in [33, Section 3] that, for a given value of α , the minimal/finest and maximal/coarsest partitions that minimize $\min_{\mathcal{P} \in \Pi(X)} f_\alpha^\#[\mathcal{P}]$ exist.¹⁴ We will show in the following context that a condition on the Dilworth truncation determines the nonemptiness of the base polyhedron $B(f_\alpha^\#, \leq)$, based on

¹⁴In [33, Section 3], it is shown that the minimizers of $\min_{\mathcal{P} \in \Pi(X)} f_\alpha^\#[\mathcal{P}]$ form a partition lattice, which is called the Dilworth truncation lattice, where the minimal/finest and maximal/coarsest minimizers uniquely exist.

which the value of the minimum sum-rate can be obtained by a maximization problem over the partition set $\Pi'(V)$. In Section V-A, we will show that the minimum sum-rate problem can be solved in polynomial time by the efficient algorithms for solving the *Dilworth truncation problem*, the minimization problem in (5).

Theorem III.3 ([26, Theorems 2.5(i) and 2.6(i)]¹⁵). $B(f_\alpha^\#, \leq)$ is nonempty, i.e., α is achievable, and $B(f_\alpha^\#, \leq) = B(\hat{f}_\alpha^\#, \leq)$ if and only if

$$\alpha = \hat{f}_\alpha^\#(V). \quad (6)$$

In Theorem III.3, $\hat{f}_\alpha^\#(V) = \max\{r(V): \mathbf{r}_V \in P(f_\alpha^\#, \leq)\}$ determines the maximum sum-rate of all rate vectors in polyhedron $P(f_\alpha^\#, \leq)$,¹⁶ while α is the sum-rate for all rate vectors in the hyperplane $\{\mathbf{r}_V \in \mathbb{R}^{|V|}: r(V) = \alpha\}$. There are two situations: if $\alpha > \hat{f}_\alpha^\#(V)$, $P(f_\alpha^\#, \leq)$ does not intersect with the hyperplane $\{\mathbf{r}_V \in \mathbb{R}^{|V|}: r(V) = \alpha\}$; if $\alpha = \hat{f}_\alpha^\#(V)$, $P(f_\alpha^\#, \leq)$ intersects with the hyperplane $\{\mathbf{r}_V \in \mathbb{R}^{|V|}: r(V) = \alpha\}$ at $B(f_\alpha^\#, \leq)$. In the latter case, $B(f_\alpha^\#, \leq) \neq \emptyset$. Theorem III.3 can also be interpreted by the principal sequence of partitions (PSP) in Appendix A.

Example III.4. For the system in Example II.1, it can be shown that: when $\alpha < \frac{7}{2}$, we have $\alpha > \hat{f}_\alpha^\#(V)$; when $\alpha \geq \frac{7}{2}$, we have $\alpha = \hat{f}_\alpha^\#(V)$.¹⁷ For example, in Fig. 2 when $\alpha = \frac{16}{5}$, one can show that $\max\{r(V): \mathbf{r}_V \in P(f_{16/5}^\#, \leq)\} = \hat{f}_{16/5}^\#(V) = \frac{13}{5} < \alpha$. So, $P(f_{16/5}^\#, \leq)$ does not intersect with hyperplane $\{\mathbf{r}_V \in \mathbb{R}^{|V|}: r(V) = \frac{16}{5}\}$, i.e., $B(f_{16/5}^\#, \leq) = \emptyset$.

In Fig. 4, when $\alpha = 4$, we have $\hat{f}_4^\#$ being

$$\begin{aligned} f_4^\#(\emptyset) &= 0, f_4^\#(\{1\}) = 3, f_4^\#(\{2\}) = 1, f_4^\#(\{3\}) = 1, \\ f_4^\#(\{1, 2\}) &= 4, f_4^\#(\{1, 3\}) = 4, f_4^\#(\{2, 3\}) = 3, \\ f_4^\#(\{1, 2, 3\}) &= 4 \end{aligned}$$

and the Dilworth truncation $\hat{f}_4^\#$ being

$$\begin{aligned} \hat{f}_4^\#(\emptyset) &= 0, \hat{f}_4^\#(\{1\}) = 3, \hat{f}_4^\#(\{2\}) = 1, \hat{f}_4^\#(\{3\}) = 1, \\ \hat{f}_4^\#(\{1, 2\}) &= 4, \hat{f}_4^\#(\{1, 3\}) = 4, \hat{f}_4^\#(\{2, 3\}) = 2, \\ \hat{f}_4^\#(\{1, 2, 3\}) &= 4. \end{aligned}$$

One can show that $\max\{r(V): \mathbf{r}_V \in P(f_4^\#, \leq)\} = \hat{f}_4^\#(V) = 4 = \alpha$ and $B(f_4^\#, \leq) = B(\hat{f}_4^\#, \leq) \neq \emptyset$.

By comparing the values of $f_4^\#$ and $\hat{f}_4^\#$, we can see that the Dilworth truncation tightens the constraints in the polyhedron $P(f_4^\#, \leq)$. For example, the inequality $r(\{2, 3\}) \leq f_4^\#(\{2, 3\}) = 3$ in $P(f_4^\#, \leq)$ can be tightened by $r(\{2\}) \leq f_4^\#(\{2\}) = 1$ and $r(\{3\}) \leq f_4^\#(\{3\}) = 1$ so that we have $r(\{2, 3\}) \leq \hat{f}_4^\#(\{2, 3\}) = 2$ in $P(\hat{f}_4^\#, \leq)$. It also explains that $\hat{f}_\alpha^\#(V)$ determines the maximum sum-rate over all rate vectors in the polyhedron $P(f_\alpha^\#, \leq)$.

¹⁵Theorem III.3 refers to the case when $\alpha < H(V)$ in particular, since, when $\alpha \geq H(V)$, $f_\alpha^\#$ is submodular according to Lemma III.2 and $\alpha = \hat{f}_\alpha^\#(V) = f_\alpha^\#(V)$ and $B(f_\alpha^\#, \leq) \neq \emptyset$ for sure [26, Theorem 2.3].

¹⁶ $\hat{f}_\alpha^\#(V) = r(V)$, $\forall \mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$ [26, Theorems 2.5(i) and 2.6(i)] and, for each $\mathbf{r}_V \in B(f_\alpha^\#, \leq)$, we have $r(V) = \max\{r(V): \mathbf{r}_V \in P(\hat{f}_\alpha^\#, \leq)\} = P(f_\alpha^\#, \leq)$ [26, Theorems 2.3 and 2.5(i)]. A detailed explanation can also be found in Appendix B.

¹⁷The two situations can be seen from the $\hat{f}_\alpha^\#(V)$ vs. α plot in Fig. 13.

Corollary III.5. For $\mathcal{P} \in \Pi'(V)$, define

$$\varphi(\mathcal{P}) = \sum_{C \in \mathcal{P}} \frac{H(V) - H(C)}{|\mathcal{P}| - 1}.$$

The minimum sum-rate in the asymptotic and non-asymptotic models are respectively

$$R_{\text{ACO}}(V) = \max_{\mathcal{P} \in \Pi'(V)} \varphi(\mathcal{P}), \quad (7a)$$

$$R_{\text{NCO}}(V) = \lceil \max_{\mathcal{P} \in \Pi'(V)} \varphi(\mathcal{P}) \rceil. \quad (7b)$$

Proof: Equation (6) in Theorem III.3 is equivalent to $\alpha \leq f_{\alpha}^{\#}[\mathcal{P}]$, $\forall \mathcal{P} \in \Pi(V)$, which can be converted to $\alpha \geq \varphi(\mathcal{P})$, $\forall \mathcal{P} \in \Pi'(V)$. It gives rise to the expressions of $R_{\text{ACO}}(V)$ and $R_{\text{NCO}}(V)$ in (7a) and (7b), respectively. ■

Remark III.6. It was first shown that $R_{\text{ACO}}(V)$ is lower bounded by $R_{\text{ACO}}(V) \geq \max_{\mathcal{P} \in \Pi'(V)} \varphi(\mathcal{P})$ in [4, Example 4] (see also Section III-E). In [19], [27], the authors proved the tightness of this lower bound, where the same result as (7a) in Corollary III.5 for the asymptotic model is derived. However, the equality in (7a) in [19], [27] was proved in a different way: Instead of showing the equivalence $B(f_{\alpha}^{\#}, \leq) = \{\mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V) : r(V) = \alpha\}$ and applying [26, Theorems 2.5(i) and 2.6(i)] to prove the nonemptiness of $B(f_{\alpha}^{\#}, \leq)$, the authors defined a polyhedron $P'(g'_{\alpha}, \leq) = \{\mathbf{r}_V \in \mathbb{R}^{|V|} : r(X) \leq g'_{\alpha}(X), \forall X \subseteq V, X \neq \emptyset\}$ for the submodular function $g'_{\alpha}(X) = \alpha - H(V) + H(X)$, $\forall X \subseteq V$ and applied [34, Theorem 48.3] to show (7a). As compared to the proof in [19, Section IV-B], our work in this section shows that (7a) straightforwardly follows from [26, Theorems 2.5(i) and 2.6(i)],¹⁸ which also leads to our new derivation of $R_{\text{NCO}}(V)$ in (7b). In Section III-D, we show that (7b) verifies an observation in [15, Section III-E] that the difference between $R_{\text{ACO}}(V)$ and $R_{\text{NCO}}(V)$ is bounded by one.

Corollary III.5 can be interpreted as follows. The minimum sum-rate can be determined by a maximization over all multi-way cuts of V . Any partition $\mathcal{P} \in \Pi'(V)$ can be considered as a multi-way cut of the user set V . For any $C \in \mathcal{P}$, the cut $\{C, V \setminus C\}$ imposes the SW constraint $r(V \setminus C) \geq H(V \setminus C|C) = H(V) - H(C)$. By applying this to each $C \in \mathcal{P}$, we have $\sum_{C \in \mathcal{P}} r(V \setminus C) = (|\mathcal{P}| - 1)r(V) \geq \sum_{C \in \mathcal{P}} (H(V) - H(C))$, which imposes requirement or lower bound

$$r(V) \geq \sum_{C \in \mathcal{P}} \frac{H(V) - H(C)}{|\mathcal{P}| - 1} = \varphi(\mathcal{P})$$

on the sum-rate for attaining omniscience. Here, $|\mathcal{P}| - 1$ is a normalization factor. Since the SW constraint applies to all the subsets of V , an achievable sum-rate must satisfy the highest requirement imposed by $\varphi(\mathcal{P})$ over all multi-way cuts, i.e., $\varphi(\mathcal{P})$ should be maximized over all $\mathcal{P} \in \Pi'(V)$. Therefore, we have (7a) and (7b). We call the minimal/finest maximizer of (7a) the *fundamental partition* and denote it by \mathcal{P}^* .

¹⁸It is clear that $P(f_{\alpha}^{\#}, \leq) = P'(g'_{\alpha}, \leq)$ since we always have $r(\emptyset) = 0$. In this sense, the proof in [19, Section IV-B] is essentially the same as the proof of (7a) in this paper. However, the proof of Corollary (III.5) is much simpler than [19, Section IV-B].

Example III.7. For the system in Example II.1, by applying (7a) and (7b), we have $R_{\text{ACO}}(V) = \frac{7}{2}$ and $R_{\text{NCO}}(V) = 4$, which are consistent with the results in Examples II.1 and III.1. In addition, we have $\mathcal{P}^* = \{\{1\}, \{2\}, \{3\}\}$ being the fundamental partition.

D. Related Results

Based on Corollary III.5, $R_{\text{NCO}}(V) = \lceil R_{\text{ACO}}(V) \rceil$ so that $R_{\text{NCO}}(V) \geq R_{\text{ACO}}(V)$, i.e., the minimum sum-rate in the non-asymptotic model is no less than the one in the asymptotic model, which is consistent with the results in [15], [16].

Based on (7a) and (7b) in Corollary III.5, it is straightforward that $R_{\text{ACO}}(V) + 1 > R_{\text{NCO}}(V)$. There is an observation in [15, Section III-E] that the maximum difference between $R_{\text{ACO}}(V)$ and $R_{\text{NCO}}(V)$ is one transmission. Whereas the result $R_{\text{NCO}}(V) - R_{\text{ACO}}(V) < 1$ based on Corollary III.5 provides theoretical proof to this observation. It states that the maximum difference between $R_{\text{ACO}}(V)$ and $R_{\text{NCO}}(V)$ is strictly less than one. Besides, we also have the following results.

Theorem III.8. For all $\alpha \geq R_{\text{ACO}}(V)$, $\hat{f}_{\alpha}^{\#}$ is a polymatroid rank function and $B(\hat{f}_{\alpha}^{\#}, \leq) = B(f_{\alpha}^{\#}, \leq) = \{\mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V) : r(V) = \alpha\} \neq \emptyset$.

Proof: According to the definition of function $f_{\alpha}^{\#}$ and the Dilworth truncation $\hat{f}_{\alpha}^{\#}$, for all $\alpha \in \mathbb{R}_+$, we have $\hat{f}_{\alpha}^{\#}$ being normalized, i.e., $\hat{f}_{\alpha}^{\#}(\emptyset) = 0$, and submodular [26, Theorem 2.5(i)]. The remaining task is to prove the monotonicity of $\hat{f}_{\alpha}^{\#}$ when $\alpha \geq R_{\text{ACO}}(V)$. According to (7a), we have $\alpha \geq R_{\text{ACO}}(V) \geq \varphi(\{\{i\}, V \setminus \{i\}\}) = 2H(V) - H(\{i\}) - H(V \setminus \{i\})$ for all $i \in V$. Then, when $\alpha \geq R_{\text{ACO}}(V)$,

$$\begin{aligned} \hat{f}_{\alpha}^{\#}(\{i\}) &= f_{\alpha}^{\#}(\{i\}) \\ &= \alpha - H(V) + H(\{i\}) \\ &\geq H(V) - H(V \setminus \{i\}) \geq 0, \end{aligned}$$

where the last inequality is due to the monotonicity of the entropy function H . So, for all $\alpha \geq R_{\text{ACO}}(V)$ and $i \in V$, we have $\hat{f}_{\alpha}^{\#}(\{i\}) \geq \hat{f}_{\alpha}^{\#}(\emptyset)$ and $\hat{f}_{\alpha}^{\#}(X) \geq \hat{f}_{\alpha}^{\#}(Y)$ for all $X, Y \subseteq V$ such that $i \in Y \subseteq X$, i.e., $\hat{f}_{\alpha}^{\#}$ is monotonic. According to Theorem III.3, when $\alpha \geq R_{\text{ACO}}(V)$, we have $\alpha = \hat{f}_{\alpha}^{\#}(V)$ and $B(\hat{f}_{\alpha}^{\#}, \leq) = B(f_{\alpha}^{\#}, \leq) \neq \emptyset$. ■

Theorem III.8 is important in proving the existence of a fractional and an integral rate vector in $\mathcal{R}_{\text{ACO}}^*(V)$ and $\mathcal{R}_{\text{NCO}}^*(V)$, respectively, in the finite linear source model and CCDE.

Corollary III.9. In a finite linear source model,

- (a) there exists an integral optimal rate vector in $\mathcal{R}_{\text{NCO}}^*(V)$.¹⁹
- (b) there exists a fractional optimal rate vector in $\mathcal{R}_{\text{ACO}}^*(V)$ that can be implemented by $(|\mathcal{P}^*| - 1)$ -packet-splitting in CCDE.

¹⁹In fact, in a finite linear source model, there exists an integral rate vector in $B(\hat{f}_{\alpha}^{\#}, \leq) = \{\mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V) \cap \mathbb{Z}^{|V|} : r(V) = \alpha\}$ for all $\alpha \in \mathbb{Z}_+$ such that $\alpha \geq R_{\text{NCO}}(V)$, which can be proved in the same way as Corollary III.9(a). Corollary III.9(a) is consistent with the results derived in [15], [18].

Proof: Recall that in a finite linear source model, the entropy function H is integer-valued. Then, since $f_{R_{\text{NCO}}(V)}^\#$ is integer-valued, based on Theorem III.8, $\hat{f}_{R_{\text{NCO}}(V)}^\#$ is an integer-valued polymatroid rank function. Denote by $\text{EX}(\hat{f}_{R_{\text{NCO}}(V)}^\#)$ the set of all extreme points, or vertices, in $B(\hat{f}_{R_{\text{NCO}}(V)}^\#, \leq)$. According to [26, Theorem 3.22], all $\mathbf{r}_V \in \text{EX}(\hat{f}_{R_{\text{NCO}}(V)}^\#)$ are integral and belong to $B(\hat{f}_{R_{\text{NCO}}(V)}^\#, \leq) \cap \mathbb{Z}^{|V|} = \mathcal{R}_{\text{NCO}}^*(V)$.

On the other hand, in a finite linear source model, we have $R_{\text{ACO}}(V) = \sum_{C \in \mathcal{P}^*} \frac{H(V) - H(C)}{|\mathcal{P}^*| - 1}$ according to Corollary III.5, i.e., $R_{\text{ACO}}(V)$ is a fractional number with denominator $|\mathcal{P}^*| - 1$. So, $R_{\text{ACO}}(V)(|\mathcal{P}^*| - 1)$ is integral. Then, $(|\mathcal{P}^*| - 1)\hat{f}_{R_{\text{ACO}}(V)}^\#$ is an integer-valued polymatroid rank function. According to [26, Theorem 3.22], all $\mathbf{r}_V \in \text{EX}((|\mathcal{P}^*| - 1)\hat{f}_{R_{\text{ACO}}(V)}^\#)$ are integral. For all $\mathbf{r}_V \in \text{EX}((|\mathcal{P}^*| - 1)\hat{f}_{R_{\text{ACO}}(V)}^\#)$, $\frac{1}{|\mathcal{P}^*| - 1}\mathbf{r}_V \in \text{EX}(\hat{f}_{R_{\text{ACO}}(V)}^\#) \subseteq B(\hat{f}_{R_{\text{ACO}}(V)}^\#, \leq) = \mathcal{R}_{\text{ACO}}^*(V)$ is a fractional optimal rate vector with $|\mathcal{P}^*| - 1$ being the LCM of the denominators of all dimensions. Corollary holds. ■

Example III.10. Consider the system in Example II.1. We have $R_{\text{ACO}}(V) = \frac{7}{2}$, $R_{\text{NCO}}(V) = 4$ and $\mathcal{P}^* = \{\{1\}, \{2\}, \{3\}\}$. From Fig. 4, it can be seen that the set of extreme points in $B(\hat{f}_{R_{\text{NCO}}(V)}^\#, \leq)$ is

$$\text{EX}(\hat{f}_{R_{\text{NCO}}(V)}^\#) = \{(2, 1, 1), (3, 1, 0), (3, 0, 1)\}.$$

All $\mathbf{r}_V \in \text{EX}(\hat{f}_{R_{\text{NCO}}(V)}^\#)$ are the integral optimal rate vectors in $\mathcal{R}_{\text{NCO}}^*(V)$. From Fig. 3, it can be seen that

$$\text{EX}(\hat{f}_{R_{\text{ACO}}(V)}^\#) = \left\{\left(\frac{5}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\} = B(\hat{f}_{R_{\text{ACO}}(V)}^\#, \leq).$$

$(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$ can be implemented by $(|\mathcal{P}^*| - 1)$ -packet-splitting.

Example III.11. Consider a different system compared to previous examples where $V = \{1, \dots, 5\}$ and each user observes respectively

$$\begin{aligned} Z_1 &= (W_b, W_c, W_d, W_h, W_i), \\ Z_2 &= (W_e, W_f, W_h, W_i), \\ Z_3 &= (W_b, W_c, W_e, W_j), \\ Z_4 &= (W_a, W_b, W_c, W_d, W_f, W_g, W_i, W_j), \\ Z_5 &= (W_a, W_b, W_c, W_f, W_i, W_j), \end{aligned}$$

where W_j is an independent uniformly distributed random bit. In this system, we have $R_{\text{ACO}}(V) = \frac{13}{2}$, $R_{\text{NCO}}(V) = 7$ and $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$. One can show that all rate vectors $\mathbf{r}_V \in \text{EX}(\hat{f}_{R_{\text{NCO}}(V)}^\#) \subseteq B(\hat{f}_{R_{\text{NCO}}(V)}^\#, \leq) \cap \mathbb{Z}^{|V|} = \mathcal{R}_{\text{NCO}}^*(V)$ are integral, e.g., $\mathbf{r}_V = (0, 1, 1, 5, 0)$. For $\mathcal{R}_{\text{ACO}}^*(V) = B(\hat{f}_{R_{\text{ACO}}(V)}^\#, \leq)$, the extreme point set is

$$\begin{aligned} \text{EX}(\hat{f}_{R_{\text{ACO}}(V)}^\#) &= \left\{ \left(1, \frac{1}{2}, \frac{1}{2}, 2, \frac{5}{2}\right), \left(2, \frac{1}{2}, \frac{1}{2}, 1, \frac{5}{2}\right), \right. \\ &\quad \left(1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0\right), \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 4, 0\right), \\ &\quad \left.\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 1, 3\right) \right\}. \end{aligned}$$

We have $|\mathcal{P}^*| - 1 = 2$ and all rate vectors in $\text{EX}(\hat{f}_{R_{\text{ACO}}(V)}^\#)$ can be implemented by 2-packet-splitting.

It is shown in [15, Section III-D] that $(|V| - 1)$ -packet-splitting is sufficient to achieve the minimum sum-rate $R_{\text{ACO}}(V)$ in a CCDE system with high probability.²⁰ However, Corollary III.9(b) states that $(|\mathcal{P}^*| - 1)$ -packet-splitting with $|\mathcal{P}^*| - 1 \leq |V| - 1$ is sufficient to achieve the minimum sum-rate $R_{\text{ACO}}(V)$ in a CCDE system for sure.

Remark III.12. The proof of Corollary III.9 states that determining the integral and fractional optimal rate vectors in $\mathcal{R}_{\text{NCO}}^*(V)$ and $\mathcal{R}_{\text{ACO}}^*(V)$, respectively, is equivalent to searching the extreme points in $B(\hat{f}_{R_{\text{NCO}}(V)}^\#, \leq)$ and $B(\hat{f}_{R_{\text{ACO}}(V)}^\#, \leq)$. In Section V-C, we show the extreme points can be determined by the MDA and SIA algorithms for the asymptotic and non-asymptotic models, respectively.

E. Secrecy Capacity and Mutual Dependence

The CO problem was first formulated in [4] based on the study on secret capacity $\mathcal{C}_S(A)$, the largest rate that the secret key can be generated by the users in A . It is assumed that the active users form a subset $A \subseteq V$ and the others in $V \setminus A$ are the helpers that assist the active users generate the secret key. The problem studied in this paper is the case when $A = V$, for which the following results are derived in [4]. The duality relationship between $\mathcal{C}_S(V)$ and $R_{\text{ACO}}(V)$ has been revealed in [4, Theorem 1]:

$$R_{\text{ACO}}(V) = H(V) - \mathcal{C}_S(V). \quad (8)$$

Let

$$I(V) = \min_{\mathcal{P} \in \Pi'(V)} \frac{D(P_{Z_V} \| \prod_{C \in \mathcal{P}} P_{Z_C})}{|\mathcal{P}| - 1}, \quad (9)$$

where $D(\cdot \| \cdot)$ is the Kullback-Leibler divergence and we have $D(P_{Z_V} \| \prod_{C \in \mathcal{P}} P_{Z_C}) = \sum_{C \in \mathcal{P}} H(C) - H(V)$. It is shown in [4, Example 4] that $\mathcal{C}_S(V)$ is upper bounded by

$$\mathcal{C}_S(V) \leq I(V) \quad (10)$$

and $D(P_{Z_V} \| \prod_{C \in \mathcal{P}} P_{Z_C})$ is interpreted as the *mutual dependence* for partition $\mathcal{P} \in \Pi'(V)$. Then, the minimum sum-rate is necessarily lower bounded by $R_{\text{ACO}}(V) \geq H(V) - I(V) = \max_{\mathcal{P} \in \Pi'(V)} \varphi(\mathcal{P})$. There is also a conjecture in [4, Example 4] that the upper bound in (10) is tight.

By realizing that $|\mathcal{P}| - 1$ is a normalization factor, the author in [27] proposed $\frac{D(P_{Z_V} \| \prod_{C \in \mathcal{P}} P_{Z_C})}{|\mathcal{P}| - 1}$ to be the mutual dependence measure for $\mathcal{P} \in \Pi'(V)$. The tightness of the upper bound in (10) is shown in [27, Theorem 1]. In [19], $I(V)$ is proposed as the *multivariate mutual information (MMI)* measure in Z_V so that the duality relationship (8) is given in terms of $I(V)$ in [7], [19], [20] as

$$R_{\text{ACO}}(V) = H(V) - I(V), \quad (11)$$

Note, $I(V)$ is also called the *shared information* in [28]. The interpretation of (11) is: the minimum sum-rate $R_{\text{ACO}}(V)$ must be the amount of information that is not mutual to the users in

²⁰The authors in [15] study the more general CCDE system where each user can only communicate with a subset of V . In the CCDE problem considered in this paper, we assume that each user can communicate losslessly with all other users, which is a special case of the general CCDE system in [15]. The authors in [15] call the model studied in this paper a *clique* system.

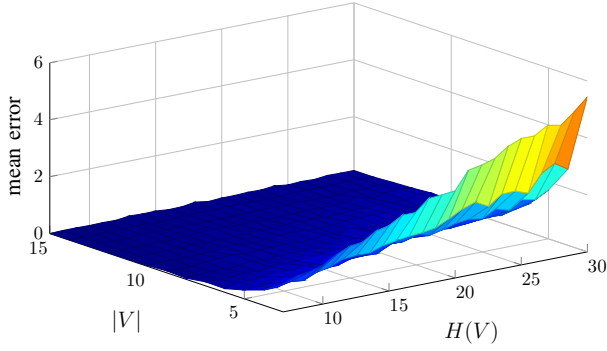


Fig. 5. The average error incurred by (12), the lower bound on $R_{\text{NCO}}(V)$ that is proposed in [8], in Experiment IV.3.

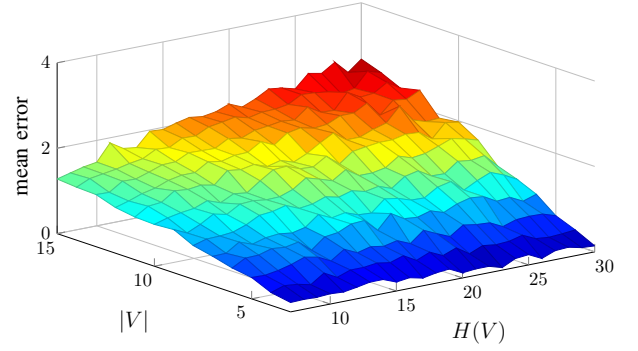


Fig. 6. The average error incurred by (13), the lower bound on $R_{\text{NCO}}(V)$ that is proposed in [11], in Experiment IV.3.

V . The fundamental partition defined in [7], [19], [20] refers to the finest/minimal minimizer of (9), the same as in this paper. The dual relationship between $R_{\text{ACO}}(V)$ and $\mathcal{C}_S(V)$, or $I(V)$, makes it more significant to study the minimum sum-rate problem in CO: Determining the secret capacity $\mathcal{C}_S(V)$ or MMI/shared information $I(V)$ relies on the efficient algorithms for solving the minimum sum-rate problem and vice versa.

IV. LOWER BOUND ON MINIMUM SUM-RATE

The existing algorithms for solving the minimum sum-rate problem in [15], [18] for the finite linear source model start with an estimation of the minimum sum-rate. In this section, we propose lower bounds (LBs) on $R_{\text{ACO}}(V)$ and $R_{\text{NCO}}(V)$ that can be obtained in $O(|V|)$ time. In Section V, we will show that the LBs on $R_{\text{ACO}}(V)$ and $R_{\text{NCO}}(V)$ can be used as an initial guess to start the MDA and SIA algorithms for searching the exact value of $R_{\text{ACO}}(V)$ and $R_{\text{NCO}}(V)$ in the asymptotic and non-asymptotic models, respectively.

Proposition IV.1. *The minimum sum-rate is lower bounded by*

$$R_{\text{ACO}}(V) \geq \max_{i \in V} \{ \varphi(\{\{i\}, V \setminus \{i\}\}), \varphi(\{\{m\} : m \in V\}) \},$$

$$R_{\text{NCO}}(V) \geq \lceil \max_{i \in V} \{ \varphi(\{\{i\}, V \setminus \{i\}\}), \varphi(\{\{m\} : m \in V\}) \} \rceil.$$

Proof: The LBs on $R_{\text{ACO}}(V)$ and $R_{\text{NCO}}(V)$ are obtained by (7a) and (7b), respectively, by partitions $\mathcal{P} = \{\{i\}, V \setminus \{i\}\}$ for all $i \in V$ and partition $\mathcal{P} = \{\{m\} : m \in V\}$. ■

The LBs in Proposition IV.1 can be obtained in $O(|V|)$ time.

Remark IV.2. *If $|V| = 3$, the LBs in Proposition IV.1 are tight for both asymptotic and non-asymptotic models: If $|V| = 3$, $\{\{i\}, V \setminus \{i\}\}$ for all $i \in V$ and $\{\{m\} : m \in V\}$ constitute all partitions in $\Pi'(V)$. The tightness of the lower bound on $R_{\text{ACO}}(V)$ when $|V| = 3$ for the asymptotic model is consistent with the result in [4, Example 3].*

The LB on $R_{\text{NCO}}(V)$ has also been proposed in [8], [11] for the finite linear source model. In [8], it is shown that

$$R_{\text{NCO}}(V) \geq H(V) - \min_{i \in V} H(\{i\}). \quad (12)$$

In addition, if $H(\{i\}) = b$ for all $i \in V$, $R_{\text{NCO}}(V) \geq H(V) - b + 1$, which can be explained by the LB

$$R_{\text{NCO}}(V) \geq \left\lceil \sum_{i \in V} \frac{H(V) - H(\{i\})}{|V| - 1} \right\rceil \quad (13)$$

$$= \lceil \varphi(\{\{i\} : i \in V\}) \rceil$$

proposed in [11] in that $\lceil \varphi(\{\{i\} : i \in V\}) \rceil = \lceil \frac{|V|}{|V|-1} (H(V) - b) \rceil = H(V) - b + 1$ in the case when $H(\{i\}) = b, \forall i \in V$. But,

$$\max_{i \in V} \varphi(\{\{i\}, V \setminus \{i\}\}) \geq \max_{i \in V} \{ H(V) - H(\{i\}) \}$$

$$= H(V) - \min_{i \in V} H(\{i\}).$$

The LB on $R_{\text{NCO}}(V)$ in Proposition IV.1 is tighter than the ones in [8], [11].

Experiment IV.3. *We generate a number of CCDE systems as follows. The number of packets $H(V)$ varies from 6 to 30, while the number of users $|V|$ varies from 3 to 15. For each combination of $H(V)$ and $|V|$, we repeat the procedure below for 20 times.*

- randomly generate the packet sets $\mathbf{z}_i = A_i \mathbf{x}$ for all $i \in V$ subject to the condition $l(\mathbf{x}) = H(V)$;
- compute the LBs on $R_{\text{NCO}}(V)$ based on [8], [11] and Proposition IV.1.

We obtain the error as the absolute difference between the LB and $R_{\text{NCO}}(V)$. We plot the average error incurred by the LBs on $R_{\text{NCO}}(V)$ in [8], [11] and Proposition IV.1 over repetitions in Figs. 5, 6 and 7, respectively. It can be seen that the LB on $R_{\text{NCO}}(V)$ in Proposition IV.1 is much tighter than the ones in [8], [11]. In addition, the error in Fig. 7 is zero for $|V| = 3$ according to Remark IV.2.

V. ALGORITHMS FOR THE MINIMUM SUM-RATE PROBLEM

The remaining problem is to discuss how to efficiently solve the maximization problems in (7a) and (7b) in Corollary III.5 for the asymptotic and non-asymptotic settings, respectively, and determine a corresponding optimal rate vector. For this purpose, we propose the MDA and SIA algorithms in this section.

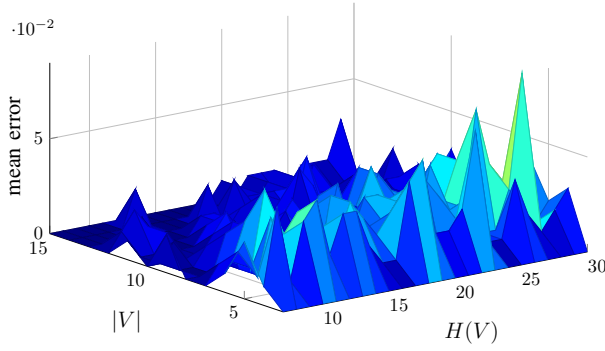


Fig. 7. The average error incurred by the lower bound on $R_{NCO}(V)$ in Proposition IV.1, in Experiment IV.3. The error is zero when $|V| = 3$ according to Remark IV.2.

Algorithm 1: Modified Decomposition Algorithm (MDA)

input : the ground set V , an oracle that returns the value of $H(X)$ for a given $X \subseteq V$
output: α that equals to $R_{ACO}(V)$, \mathcal{P}^* which is the fundamental partition and a rate vector \mathbf{r}_V in the optimal rate set $\mathcal{R}_{ACO}^*(V) = B(\hat{f}_{R_{ACO}(V)}^\#, \leq)$

- 1 initialize α according to Proposition IV.1:
 $\alpha \leftarrow \max_{i \in V} \{ \varphi(\{\{i\}, V \setminus \{i\}\}), \varphi(\{\{m\} : m \in V\}) \};$
- 2 find the minimal/finest minimizer \mathcal{P}^* of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ and a rate vector $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq);$
- 3 $\hat{\alpha} \leftarrow \varphi(\mathcal{P}^*);$
- 4 **while** $\hat{\alpha} \neq \alpha$ **do**
- 5 $\alpha \leftarrow \hat{\alpha};$
- 6 find the minimal/finest minimizer \mathcal{P}^* of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ and a rate vector $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq);$
- 7 $\hat{\alpha} \leftarrow \varphi(\mathcal{P}^*);$
- 8 **end**
- 9 return α , \mathcal{P}^* and \mathbf{r}_V ;

A. Modified Decomposition Algorithm

The MDA algorithm is given in Algorithm 1 for solving the minimum sum-rate problem in the asymptotic model. The optimality of the MDA algorithm is summarized in Theorem V.1 below. The proof is in Appendix A-A.

Theorem V.1. *The MDA algorithm outputs the minimum sum-rate $R_{ACO}(V)$, the fundamental partition \mathcal{P}^* and an optimal rate vector $\mathbf{r}_V \in \mathcal{R}_{ACO}^*(V)$ for the asymptotic model. The estimation sequence of $R_{ACO}(V)$, i.e., the value of α in each iteration, converges monotonically upward to $R_{ACO}(V)$.* ■

Example V.2. *We apply the MDA algorithm to the system in Example III.11. We initiate $\alpha = \max_{i \in V} \{ \varphi(\{\{i\}, V \setminus \{i\}\}), \varphi(\{\{i\} : i \in V\}) \} = \frac{23}{4}$ as the LB on $R_{ACO}(V)$ in Proposition IV.1, and have the following results.*

- When $\alpha = \frac{23}{4}$, we have $\mathcal{P}^* = \{\{4, 5\}, \{1\}, \{2\}, \{3\}\}$ being the minimal minimizer of the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_{23/4}^\#[\mathcal{P}]$ and $\mathbf{r}_V = (\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{15}{4}, 0)$ being a vector in $B(\hat{f}_{23/4}^\#, \leq)$. We get $\hat{\alpha} = \varphi(\mathcal{P}^*) = \frac{19}{3}$. Since $\hat{\alpha} \neq \alpha$, the iteration continues;
- When $\alpha = \frac{19}{3}$, we have $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$ being

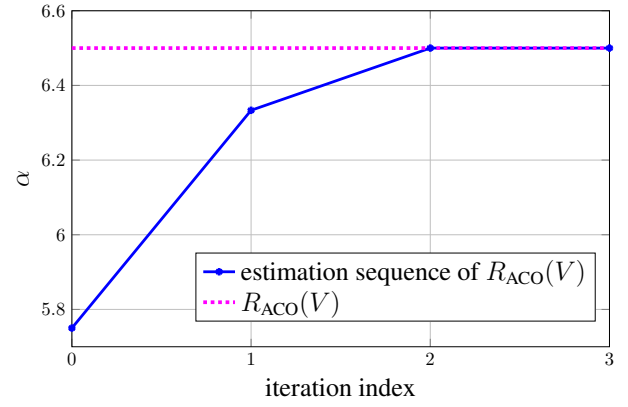


Fig. 8. The estimation sequence of $R_{ACO}(V)$, i.e., the value of α in each iteration, when the MDA algorithm is applied to the system in Example III.11. According to Theorem V.1, it converges monotonically upward to $R_{ACO}(V)$.

the minimal minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_{19/3}^\#[\mathcal{P}]$ and $\mathbf{r}_V = (1, \frac{1}{3}, \frac{1}{3}, \frac{13}{3}, 0) \in B(\hat{f}_{19/3}^\#, \leq)$. Since $\hat{\alpha} = \varphi(\mathcal{P}^*) = \frac{13}{2} \neq \alpha$, the iteration continues;

- When $\alpha = \frac{13}{2}$, we have $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$ being the minimal minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_{13/2}^\#[\mathcal{P}]$ and $\mathbf{r}_V = (1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0) \in B(\hat{f}_{13/2}^\#, \leq)$. Since $\hat{\alpha} = \varphi(\mathcal{P}^*) = \frac{13}{2} = \alpha$, the iteration terminates.

At the output, $\alpha = \frac{13}{2}$ and $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$ coincide with the minimum sum-rate $R_{ACO}(V)$ and the fundamental partition in Example III.11, respectively. In addition, $\mathbf{r}_V = (1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0) \in B(\hat{f}_{13/2}^\#, \leq) = \mathcal{R}_{ACO}^*(V)$ is an optimal rate vector. In fact, $(1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0) \in EX(\hat{f}_{R_{ACO}(V)}^\#)$ is one of the extreme points in the base polyhedron $B(\hat{f}_{R_{ACO}(V)}^\#, \leq)$, where $EX(\hat{f}_{R_{ACO}(V)}^\#)$ is shown in Example III.11. Fig. 8 shows that the value of α in each iteration of the MDA algorithm converges monotonically upwards to the minimum sum-rate $R_{ACO}(V)$.

In the next subsection, we show how to solve the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ in steps 2 and 6 in the MDA algorithm by a fusion implementation of the coordinate-wise saturation capacity algorithm, where a rate vector $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$ is also returned.

B. Coordinate-wise Saturation Capacity Algorithm by Fusion Method

There exist several algorithms for solving the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ in the literature. For example, the fusion set method is proposed in [31], [33] for determining the PSP of electronic networks; The coordinate-wise saturation capacity (CoordSatCap) algorithm in [26, Section 3.2] has been applied to determine the PSP of a network in [35, Algorithm A], [36, Section 2] and the strength of a network in [37, Section 3].²¹

In this paper, we consider the CoordSatCap algorithm, which determines not only the minimum and the minimizer of

²¹One can verify that the methods in [35, Algorithm A], [36, Section 2] and [37, Section 3] are in fact the CoordSatCap algorithm that is presented as Greedy Algorithm II in [26, Section 3.2]. The studies in [35]–[37] are based on the cut function of a network.

the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$, but also a rate vector \mathbf{r}_V in the base polyhedron $B(\hat{f}_\alpha^\#, \leq)$. In this section, we first describe the CoordSatCap algorithm and then show how to implement it by a fusion method.

We introduce some related definitions as follows. For $X \subseteq V$, let $\chi_X = (r_i : i \in V)$ be the *characteristic vector* of the subset X such that $r_i = 1$ if $i \in X$ and $r_i = 0$ if $i \notin X$. The notation $\chi_{\{i\}}$ is simplified by χ_i . Let $\Phi = (\phi_1, \dots, \phi_{|V|})$, where $\phi_i \in V$ and $\phi_i \neq \phi_{i'}$ for all $i, i' \in \{1, \dots, |V|\}$ such that $i \neq i'$. We call Φ a *linear ordering* of V . For example, $\Phi = (2, 3, 1, 4)$ is a linear ordering of $V = \{1, \dots, 4\}$.

For $U \subseteq \mathcal{P}$ where \mathcal{P} is some partition in $\Pi(V)$, denote

$$\tilde{U} = \sqcup_{X \in U} X$$

where \sqcup is the disjoint union, i.e., $\tilde{U} \subseteq V$ is a fusion of all the subsets in U . For example, for $\mathcal{P} = \{\{1, 3\}, \{2, 4\}, \{5\}, \{6\}, \{7\}\} \in \Pi(\{1, \dots, 7\})$ and $U = \{\{1, 3\}, \{2, 4\}, \{5\}, \{6\}\} \subseteq \mathcal{P}$, we have $\tilde{U} = \{1, \dots, 6\}$.

For a rate vector $\mathbf{r}_V \in P(f_\alpha^\#, \leq)$,

$$\hat{\xi}_i = \max\{\xi : \mathbf{r}_V + \xi \chi_i \in P(f_\alpha^\#, \leq)\}$$

is called the *saturation capacity* in dimension i [26, Section 2.3]. Here, $\hat{\xi}_i \in \mathbb{R}_+$ denotes the maximum increment in r_i such that the resulting rate vector $\mathbf{r}_V + \hat{\xi}_i \chi_i$ is still in the polyhedron $P(f_\alpha^\#, \leq)$. The saturation capacity can be determined by solving the minimization problem [26, Section 2.3]

$$\begin{aligned} \max\{\xi : \mathbf{r}_V + \xi \chi_i \in P(f_\alpha^\#, \leq)\} = \\ \min\{f_\alpha^\#(X) - r(X) : i \in X \subseteq V\}. \end{aligned} \quad (14)$$

Here, $f_\alpha^\#$ is submodular over all $X \subseteq V$ such that $i \in X$ because of the intersecting submodularity of $f_\alpha^\#$, i.e., the minimization problem in (14) is an SFM one. An SFM problem can be solved in polynomial time [38]–[43] (see also Section V-D). The minimizers of an SFM problem form a lattice, where the minimal/smallest and maximal/largest minimizer exist [26, Lemma 2.1]. In Step 3 of Algorithm 2, we obtain the minimal/smallest minimizer.

1) *Coordinate-wise Saturation Capacity (CoordSatCap) Algorithm*: The main purpose of the CoordSatCap algorithm in Algorithm 2 is to determine a rate vector, or base point, in $B(\hat{f}_\alpha^\#, \leq)$. The idea is to start with a point $\mathbf{r}_V \in P(f_\alpha^\#, \leq)$ and increase each dimension of \mathbf{r}_V in order by the saturation capacity. Finally, we have \mathbf{r}_V still in $P(f_\alpha^\#, \leq)$ but reaching saturation in each dimension, i.e., $\mathbf{r}_V + \epsilon \chi_i \notin P(f_\alpha^\#, \leq)$ for all $\epsilon > 0$ and $i \in V$, which means $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$ necessarily. Based on the tight sets of this base point, the minimizers of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ can be determined, i.e., the Dilworth truncation problem is solved accordingly.²² See also Appendix B for the detailed explanation of the CoordSatCap algorithm.

²²The definition of tight set and related explanations are in Appendix B, where we also present a brief proof/explanation that the CoordSatCap algorithm outputs a rate vector $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$ and the minimal minimizer \mathcal{P}^* of the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ based on the studies in [25], [26], [37].

Algorithm 2: Coordinate-wise Saturation Capacity (CoordSatCap) Algorithm [26]

input : the ground set V , an oracle that returns the value of $H(X)$ for a given $X \subseteq V$, α which is an estimation of $R_{\text{ACO}}(V)$

output: \mathbf{r}_V which is a rate vector in $B(\hat{f}_\alpha^\#, \leq)$ and \mathcal{P}^* which is the minimal/finest minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$

- 1 initiate \mathbf{r}_V so that $\mathbf{r}_V \in P(f_\alpha^\#, \leq)$ and $\mathcal{P}^* = \{\{i\} : i \in V\}$ and choose a linear ordering $\Phi = (\phi_1, \dots, \phi_{|V|})$;
- 2 for $i = 1$ to $|V|$ do
- 3 determine the saturation capacity

$$\hat{\xi}_{\phi_i} \leftarrow \min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$$
 and the minimal/smallest minimizer \hat{X}_{ϕ_i} ;
- 4 $\mathbf{r}_V \leftarrow \mathbf{r}_V + \hat{\xi}_{\phi_i} \chi_{\phi_i}$;
- 5 merge/fuse all subsets in \mathcal{P}^* that intersect with \hat{X}_{ϕ_i} in to one subset $\tilde{\mathcal{X}} = \sqcup_{X \in \mathcal{X}} X$:

$$\mathcal{X} \leftarrow \{C \in \mathcal{P}^* : C \cap \hat{X}_{\phi_i} \neq \emptyset\};$$

$$\mathcal{P}^* \leftarrow (\mathcal{P}^* \setminus \mathcal{X}) \sqcup \{\tilde{\mathcal{X}}\};$$
- 6 endfor
- 7 return \mathbf{r}_V and \mathcal{P}^* ;

The following lemma shows one way to initiate a rate vector \mathbf{r}_V such that $\mathbf{r}_V \in P(f_\alpha^\#, \leq)$. The proof is in Appendix B-B.

Lemma V.3. For any α such that $0 \leq \alpha \leq H(V)$, $\mathbf{r}_V = (\alpha - H(V))\chi_V \in P(f_\alpha^\#, \leq)$. ■

In Algorithm 2, the linear ordering Φ matters when we want to minimize a weighted sum-rate objective function in the optimal rate set, which will be discussed in Section VI. We remark that for the minimum (equal-weight) sum-rate problem for both asymptotic and non-asymptotic models that is considered in this section, any linear ordering Φ of the user indices can be chosen.

Example V.4. Consider the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ for $\alpha = \frac{23}{4}$ in Example V.2. We apply the CoordSatCap algorithm by initiating $\mathbf{r}_V = (\alpha - H(V))\chi_V = (-\frac{17}{4}, \dots, -\frac{17}{4})$ according to Lemma V.3 and $\mathcal{P}^* = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$. The linear ordering is set to $\Phi = (4, 5, 3, 2, 1)$.

- For $\phi_1 = 4$, we have $\hat{\xi}_4 = 8$ and $\hat{X}_4 = \{4\}$ being the minimum and minimal minimizer of $\min\{f_{23/4}^\#(X) - r(X) : 4 \in X \subseteq V\}$, respectively. By executing $\mathbf{r}_V \leftarrow \mathbf{r}_V + 8\chi_4$, we update \mathbf{r}_V to $(-\frac{17}{4}, -\frac{17}{4}, -\frac{17}{4}, \frac{15}{4}, -\frac{17}{4})$. Also, $\mathcal{X} = \{\{4\}\}$ and $\tilde{\mathcal{X}} = \{C \in \mathcal{P}^* : C \cap \{4\} \neq \emptyset\} = \{4\}$. By executing $\mathcal{P}^* = (\mathcal{P}^* \setminus \mathcal{X}) \sqcup \{\tilde{\mathcal{X}}\}$, we still have $\mathcal{P}^* = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$;
- For $\phi_2 = 5$, we have $\hat{\xi}_5 = \frac{17}{4}$ and $\hat{X}_5 = \{4, 5\}$. \mathbf{r}_V is updated to $(-\frac{17}{4}, -\frac{17}{4}, -\frac{17}{4}, \frac{15}{4}, 0)$. Since $\mathcal{X} = \{C \in \mathcal{P}^* : C \cap \{4, 5\} \neq \emptyset\} = \{\{4\}, \{5\}\}$ and $\tilde{\mathcal{X}} = \{4, 5\}$, by executing $\mathcal{P}^* = (\mathcal{P}^* \setminus \mathcal{X}) \sqcup \{\tilde{\mathcal{X}}\}$, we update \mathcal{P}^* to $\{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$;
- For $\phi_3 = 3$, we have $\hat{\xi}_3 = 4$ and $\hat{X}_3 = \{2\}$. \mathbf{r}_V is updated to $(-\frac{17}{4}, -\frac{1}{4}, -\frac{17}{4}, \frac{15}{4}, 0)$. $\mathcal{P}^* = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$ after executing $\mathcal{P}^* = (\mathcal{P}^* \setminus \mathcal{X}) \sqcup \{\tilde{\mathcal{X}}\}$

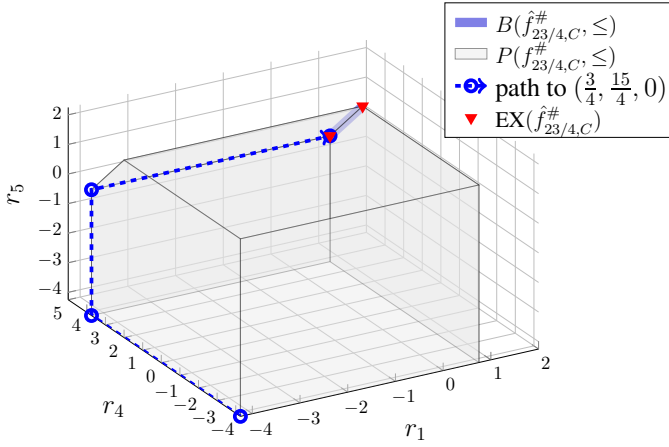


Fig. 9. For $\alpha = \frac{23}{4}$ and $C = \{1, 4, 5\}$, the figure shows the polyhedron $P(f_{\alpha,C}^{\#}, \leq)$ and base polyhedron $B(f_{\alpha,C}^{\#}, \leq)$, where $f_{\alpha,C}^{\#}$ is the reduction of $f_{\alpha}^{\#}$ on C . Note, in this case, the base polyhedron $B(f_{\alpha,C}^{\#}, \leq)$ reduces to a line segment with two vertices, $(\frac{3}{4}, \frac{15}{4}, 0)$ and $(\frac{3}{4}, 2, \frac{7}{4})$. This figure also shows the adaptation of the rate vector \mathbf{r}_C resulted from the CoordSatCapFus algorithm in Example V.4: the path $(-\frac{17}{4}, \frac{15}{4}, -\frac{17}{4}) \rightarrow (-\frac{17}{4}, \frac{15}{4}, 0) \rightarrow (\frac{3}{4}, \frac{15}{4}, 0)$. We have the rate vector $(\frac{3}{4}, \frac{15}{4}, 0) \in B(f_{\alpha,C}^{\#}, \leq)$ at the end.

$\{\hat{\mathcal{X}}\};$

- For $\phi_4 = 3$, we have $\hat{\xi}_3 = 4$ and $\hat{X}_3 = \{3\}$. \mathbf{r}_V is updated to $(-\frac{17}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{15}{4}, 0)$. $\mathcal{P}^* = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$ after executing $\mathcal{P}^* = (\mathcal{P}^* \setminus \mathcal{X}) \sqcup \{\mathcal{X}\}$;
- For $\phi_5 = 1$, we have $\hat{\xi}_1 = 5$ and $\hat{X}_1 = \{1\}$. \mathbf{r}_V is updated to $(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{15}{4}, 0)$. $\mathcal{P}^* = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$ after executing $\mathcal{P}^* = (\mathcal{P}^* \setminus \mathcal{X}) \sqcup \{\mathcal{X}\}$.

At the output, we have $\mathcal{P}^* = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$ and $\mathbf{r}_V = (\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{15}{4}, 0)$ being the minimal minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_{23/4}^{\#}[\mathcal{P}]$ and a base point in $B(\hat{f}_{23/4}^{\#}, \leq)$, respectively.

Example V.5. Consider the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_{\alpha}^{\#}[\mathcal{P}]$ for $\alpha = \frac{13}{2}$ in Example V.2. By applying the CoordSatCap algorithm in the same way as in Example V.4, one can show that, at the output, $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$ and $\mathbf{r}_V = (1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0)$, which are the minimal minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_{13/2}^{\#}[\mathcal{P}]$ and a rate vector in $B(\hat{f}_{13/2}^{\#}, \leq)$, respectively.

For $C \subseteq V$, we call $f_{\alpha,C}^{\#}: 2^C \mapsto \mathbb{R}$ with $f_{\alpha,C}^{\#}(X) = f_{\alpha}^{\#}(X)$ for all $X \subseteq C$ the reduction of $f_{\alpha}^{\#}$ on C [26, Section 3.1(a)]. The polyhedron and base polyhedron of $f_{\alpha,C}^{\#}$ are respectively

$$P(f_{\alpha,C}^{\#}, \leq) = \{\mathbf{r}_C \in \mathbb{R}^{|C|}: r(X) \leq f_{\alpha,C}^{\#}(X), X \subseteq C\},$$

$$B(f_{\alpha,C}^{\#}, \leq) = \{\mathbf{r}_C \in P(f_{\alpha,C}^{\#}, \leq): r(C) = f_{\alpha,C}^{\#}(C)\}.$$

In Fig. 9, we show $P(f_{\alpha,C}^{\#}, \leq)$ and $B(\hat{f}_{\alpha,C}^{\#}, \leq)$ when $\alpha = \frac{23}{4}$ and $C = \{1, 4, 5\}$, where we can see the the path to $\mathbf{r}_C = (\frac{3}{4}, \frac{15}{4}, 0)$ as a result of the CoordSatCap algorithm in Example V.4. In Fig. 10, we show $P(f_{\alpha,C}^{\#}, \leq)$ and $B(\hat{f}_{\alpha,C}^{\#}, \leq)$

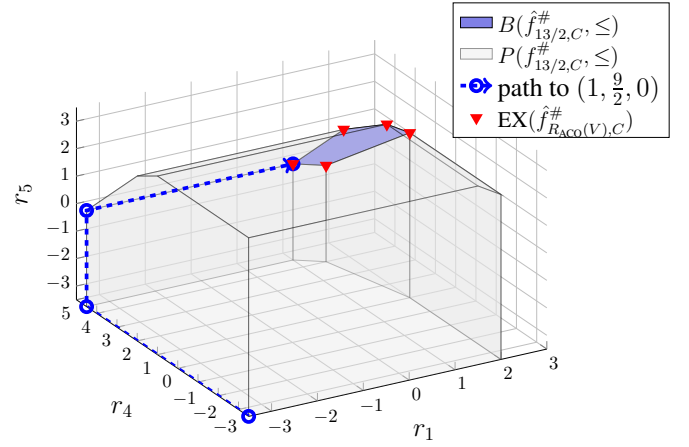


Fig. 10. For $\alpha = \frac{13}{2}$ and $C = \{1, 4, 5\}$, the figure shows the polyhedron $P(f_{\alpha,C}^{\#}, \leq)$ and base polyhedron $B(\hat{f}_{\alpha,C}^{\#}, \leq)$. In this case, the base polyhedron is a 2-dimensional polygon with five vertices that constitutes the extreme points set $\text{EX}(\hat{f}_{\text{RACO}(V),C}^{\#}) = \{(1, 2, \frac{5}{2}), (2, 1, \frac{5}{2}), (1, \frac{9}{2}, 0), (\frac{3}{2}, 4, 0), (\frac{3}{2}, 1, 3)\}$. This figure also shows the adaptation of the rate vector \mathbf{r}_C resulted from the CoordSatCapFus algorithm in Example V.5: the path $(-\frac{7}{2}, -\frac{7}{2}, -\frac{7}{2}) \rightarrow (-\frac{7}{2}, \frac{9}{2}, -\frac{7}{2}) \rightarrow (-\frac{7}{2}, \frac{9}{2}, 0) \rightarrow (1, \frac{9}{2}, 0)$. We have the rate vector $(1, \frac{9}{2}, 0) \in B(\hat{f}_{\alpha,C}^{\#}, \leq)$ at the end.

when $\alpha = \frac{13}{2}$ and $C = \{1, 4, 5\}$, where we can see the the path to $\mathbf{r}_C = (1, \frac{9}{2}, 0)$ as a result of the CoordSatCap algorithm in Example V.5.

2) *A Fusion Method Implementation:* In the CoordSatCap algorithm, the saturation capacity ξ_{ϕ_i} in dimension ϕ_i is determined by solving problem $\min\{f_{\alpha}^{\#}(X) - r(X): \phi_i \in X \subseteq V\}$, where each element in V is the index of a user in the system. In this section, we show that this problem can be solved over a merged user set where each non-singleton element denotes a super user, i.e., the CoordSatCap algorithm can be implemented by a fusion method. The validity of this fusion method is based on Lemma V.6 and Lemma V.7 below with the proofs in Appendix B-C and Appendix B-D, respectively.

Lemma V.6. Let the CoordSatCap algorithm start with a rate vector $\mathbf{r}_V \in P(f_{\alpha}^{\#}, \leq)$ such that $\mathbf{r}_V \leq \mathbf{0}$, where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^{|V|}$. We have

$$\min\{f_{\alpha}^{\#}(X) - r(X): \phi_i \in X \subseteq V\} = \min\{f_{\alpha}^{\#}(X) - r(X): \phi_i \in X \subseteq V_i\}, \quad (15)$$

where $V_i = \{\phi_1, \dots, \phi_i\}$ and the minimal minimizer $\hat{X}_{\phi_i} \subseteq V_i$ for all $i \in \{1, \dots, |V|\}$. ■

The equality (15) in Lemma V.6 was originally derived in the proof of [26, Theorem 3.19], based on which the authors in [15], [18] apply $\min\{f_{\alpha}^{\#}(X) - r(X): \phi_i \in X \subseteq V_i\}$ in the CoordSatCap algorithm for solving the non-asymptotic minimum sum-rate problem in the finite linear source model and CCDE. However, the proof of [26, Theorem 3.19] does not show that $\hat{X}_{\phi_i} \subseteq V_i$ for all i .

Lemma V.7. In the CoordSatCap algorithm,

$$\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\} = \quad (16)$$

$$\min\{f_\alpha^\#(\tilde{U}) - r(\tilde{U}) : U \subseteq \mathcal{P}^*, \phi_i \in \tilde{U}\} \quad (17)$$

for all $i \in \{1, \dots, |V|\}$. Let \hat{X}_{ϕ_i} and $U_{\phi_i}^*$ be the minimal minimizer of the (16) and (17), respectively. $\tilde{\mathcal{X}} = \tilde{U}_{\phi_i}^*$, where $\mathcal{X} = \{C \in \mathcal{P}^* : C \cap \hat{X}_{\phi_i} \neq \emptyset\}$. ■

Both minimization problems, (16) and (17), in Lemma V.7 are SFM problems due to the intersecting submodularity of $f_\alpha^\#$. But, \mathcal{P}^* is a fused user set since some of the users have been merged into a super user set, which is treated as one dimension in the SFM problem (17). So, $|\mathcal{P}^*| \leq |V|$. We will show in Section V-D that minimizing over the fused user set contributes to a reduction in computation complexity.

Remark V.8. Lemma V.7 is based on the fact: If \hat{X}_{ϕ_i} is a non-singleton minimal minimizer of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$ for some ϕ_i , then, for any partition \mathcal{P} that crosses \hat{X}_{ϕ_i} , there exists a \mathcal{P}' that does not cross \hat{X}_{ϕ_i} such that $f_\alpha^\#[\mathcal{P}'] < f_\alpha^\#[\mathcal{P}]$.²³ It means that, to determine the minimal/finest minimizer of the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi'(V)} f_\alpha^\#[\mathcal{P}]$, we just need to consider all the partitions in $\Pi(V)$ that do not cross \hat{X}_{ϕ_i} .²⁴ See Appendix B-E for the explanation.

Example V.9. In Exmaple V.4, we have $\hat{X}_5 = \{4, 5\}$ when $\alpha = \frac{23}{4}$. For $\mathcal{P} = \{\{1, 4\}, \{2, 3\}, \{5\}\}$ that crosses $\{4, 5\}$, we have $\mathcal{P}' = \{\{1, 4, 5\}, \{2, 3\}\}$ that does not cross $\{4, 5\}$ such that $f_\alpha^\#[\mathcal{P}'] = \frac{15}{2} < f_\alpha^\#[\mathcal{P}] = \frac{37}{4}$. For $\mathcal{P} = \{\{1, 4\}, \{2, 3, 5\}\}$ that crosses $\{4, 5\}$, we have $\mathcal{P}' = \{\{4, 5\}, \{1\}, \{2, 3\}\}$ that does not cross $\{4, 5\}$ such that $f_\alpha^\#[\mathcal{P}'] = \frac{29}{4} < f_\alpha^\#[\mathcal{P}] = \frac{17}{2}$. One can show that for each partition \mathcal{P} that crosses $\{4, 5\}$ there always exists a partition \mathcal{P}' that does not cross $\{4, 5\}$ such that $f_\alpha^\#[\mathcal{P}'] < f_\alpha^\#[\mathcal{P}]$. Therefore, we just need to consider all partitions that do not cross $\{4, 5\}$ to determine the minimal minimizer of $\min_{\mathcal{P} \in \Pi'(V)} f_\alpha^\#[\mathcal{P}]$. It is equivalent to considering all partitions of $\{\{4, 5\}, \{1\}, \{2\}, \{3\}\}$, which is the value of \mathcal{P}^* when $\phi_2 = 5$ in Exmaple V.4.

Based on Lemmas V.6 and V.7, the CoordSatCap algorithm in Algorithm 2 is equivalent to the CoordSatCapFus algorithm in Algorithm 3. Steps 4, 5 and 7 in Algorithm 3 are explained as follows.

According to Lemmas V.6 and V.7, the saturation capacity can be obtained by $\hat{\xi}_{\phi_i} = \min\{f_\alpha^\#(\tilde{U}) - r(\tilde{U}) : \{\phi_i\} \in U \subseteq \mathcal{P}_i^*\}$. Here, $\mathcal{P}_i^* = \mathcal{P}_{i-1}^* \sqcup \{\{\phi_i\}\} \in \Pi(V_i)$ with $\mathcal{P}_{i-1}^* \in \Pi(V_{i-1})$ being the partition obtained at the end of iteration $i - 1$. For the minimal minimizer $U_{\phi_i}^*$ of this problem, we have $\hat{X}_{\phi_i} \subseteq \tilde{\mathcal{X}} = \tilde{U}_{\phi_i}^*$, where \hat{X}_{ϕ_i} is the minimal minimizer of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V_i\}$ and $\mathcal{X} = \{C \in \mathcal{P}_i^* : C \cap \hat{X}_{\phi_i} \neq \emptyset\}$. This explains steps 4 and 5

²³A multi-way cut or partition $\mathcal{P} \in \Pi'(V)$ does not cross a set $X \subsetneq V$ if there exist $C \in \mathcal{P}$ such that $X \subseteq C$. For example, for $X = \{1, 3\}$, $\{\{1, 3, 4\}, \{2\}\}$ and $\{\{1, 3\}, \{2\}, \{4\}\}$ are the partitions that do not cross X , while $\{\{1, 2\}, \{3\}, \{4\}\}$ and $\{\{1\}, \{2, 3, 4\}\}$ are the partitions that cross X .

²⁴In addition, the fundamental partition \mathcal{P}^* must be a multi-way cut of V that does not cross \hat{X}_{ϕ_i} . See Appendix B-E.

Algorithm 3: Coordinate-wise Saturation Capacity Algorithm by Fusion Method (CoordSatCapFus)

input : the ground set V , an oracle that returns the value of $H(X)$ for a given $X \subseteq V$, α which is an estimation of $R_{\text{ACO}}(V)$

output: \mathbf{r}_V which is a rate vector in $B(\hat{f}_\alpha^\#, \leq)$ and $\mathcal{P}_{|V|}^*$ which is the minimal/finest minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$

- 1 let $\mathbf{r}_V \leftarrow (\alpha - H(V))\chi_V$ so that $\mathbf{r}_V \in P(f_\alpha^\#, \leq)$ and choose a linear ordering $\Phi = (\phi_1, \dots, \phi_{|V|})$;
- 2 initiate $r_{\phi_1} \leftarrow f_\alpha^\#(\{\phi_1\})$ and $\mathcal{P}_1^* \leftarrow \{\{\phi_1\}\}$;
- 3 **for** $i = 2$ **to** $|V|$ **do**
- 4 $\mathcal{P}_i^* \leftarrow \mathcal{P}_{i-1}^* \sqcup \{\{\phi_i\}\}$;
- 5 determine the saturation capacity

$$\hat{\xi}_{\phi_i} \leftarrow \min\{f_\alpha^\#(\tilde{U}) - r(\tilde{U}) : \{\phi_i\} \in U \subseteq \mathcal{P}_i^*\}$$
 and the minimal/smallest minimizer $U_{\phi_i}^*$;
- 6 $\mathbf{r}_V \leftarrow \mathbf{r}_V + \hat{\xi}_{\phi_i}\chi_{\phi_i}$;
- 7 merge/fuse all subsets in $U_{\phi_i}^*$ into one subset

$$\tilde{U}_{\phi_i}^* = \sqcup_{X \in U_{\phi_i}^*} X;$$

$$\mathcal{P}_i^* \leftarrow (\mathcal{P}_i^* \setminus U_{\phi_i}^*) \sqcup \{\tilde{U}_{\phi_i}^*\};$$
- 8 **endfor**
- 9 return \mathbf{r}_V and $\mathcal{P}_{|V|}^*$;

in Algorithm 3. The equality $\tilde{\mathcal{X}} = \tilde{U}_{\phi_i}^*$ also makes the update of \mathcal{P}_i^* easier: We do not need to obtain \mathcal{X} as we did in step 5 in Algorithm 2. Since $(\mathcal{P}_i^* \setminus U_{\phi_i}^*) \sqcup \{\tilde{U}_{\phi_i}^*\} = (\mathcal{P}_i^* \setminus \mathcal{X}) \sqcup \{\tilde{\mathcal{X}}\}$, simply do $\mathcal{P}_i^* \leftarrow (\mathcal{P}_i^* \setminus U_{\phi_i}^*) \sqcup \{\tilde{U}_{\phi_i}^*\}$, which does not require the determination of \mathcal{X} . This explains step 7 in Algorithm 3.

Since the saturation capacity in the CoordSatCapFus algorithm is always obtained by a minimization problem over a fused user set $\mathcal{P}^* \in \Pi(V_i)$ for all $i \in \{1, \dots, |V|\}$, we call the CoordSatCapFus algorithm a fusion method implementation of the CoordSatCap algorithm.

Example V.10. Consider the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ when $\alpha = \frac{23}{4}$ in Example V.2. We apply the CoordSatCapFus algorithm by initiating $\mathbf{r}_V = (\alpha - H(V))\chi_V = (-\frac{17}{4}, \dots, -\frac{17}{4})$. The linear ordering is set to $\Phi = (4, 5, 3, 2, 1)$.

- For $\phi_1 = 4$, we assign $r_4 = f_{23/4}^\#(\{4\}) = \frac{15}{4}$ so that $\mathbf{r}_V = (-\frac{17}{4}, -\frac{17}{4}, -\frac{17}{4}, \frac{15}{4}, -\frac{17}{4})$ and let $\mathcal{P}_1^* = \{\{4\}\}$;
- For $\phi_2 = 5$, we have $\mathcal{P}_2^* = \{\{4\}, \{5\}\}$ and consider the problem $\min\{f_\alpha^\#(\tilde{U}) - r(\tilde{U}) : \{5\} \in U \subseteq \mathcal{P}_2^*\}$. Since $\{U : \{5\} \in U \subseteq \mathcal{P}_2^*\} = \{\{\{5\}\}, \{\{4\}, \{5\}\}\}$ and

$$f_{23/4}^\#(\{5\}) - r(\{5\}) = 6,$$

$$f_{23/4}^\#(\{4, 5\}) - r(\{4, 5\}) = \frac{17}{4},$$

we have $\hat{\xi}_5 = \frac{17}{4}$ and $U_{\phi_2}^* = \{\{4\}, \{5\}\}$. \mathbf{r}_V and \mathcal{P}_2^* are updated to $(-\frac{17}{4}, -\frac{17}{4}, -\frac{17}{4}, \frac{15}{4}, 0)$ and $\{\{4, 5\}\}$, respectively;

- For $\phi_3 = 2$, we have $\mathcal{P}_3^* = \{\{2\}, \{4, 5\}\}$ and consider the problem $\min\{f_\alpha^\#(\tilde{U}) - r(\tilde{U}) : \{2\} \in U \subseteq \mathcal{P}_3^*\}$. Since

$\{U: \{2\} \in U \subseteq \mathcal{P}_3^* = \{\{\{2\}\}, \{\{2\}, \{4, 5\}\}\}$ and

$$f_{23/4}^\#(\{2\}) - r(\{2\}) = 4,$$

$$f_{23/4}^\#(\{2, 4, 5\}) - r(\{2, 4, 5\}) = \frac{25}{4},$$

we have $\hat{\xi}_2 = 4$ and $U_2^* = \{\{2\}\}$. \mathbf{r}_V and \mathcal{P}_3^* are updated to $(-\frac{17}{4}, -\frac{1}{4}, -\frac{17}{4}, \frac{15}{4}, 0)$ and $\{\{2\}, \{4, 5\}\}$, respectively;

- In the same way, we can show the followings at the end of each iteration.

– For $\phi_4 = 3$, $\mathbf{r}_V = (-\frac{1}{4}, -\frac{1}{4}, -\frac{17}{4}, \frac{15}{4}, 0)$ and $\mathcal{P}_4^* = \{\{2\}, \{3\}, \{4, 5\}\}$;

– For $\phi_5 = 1$, $\mathbf{r}_V = (\frac{3}{4}, -\frac{1}{4}, -\frac{17}{4}, \frac{15}{4}, 0)$ and $\mathcal{P}_5^* = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$.

At the end, we have $\mathbf{r} = (\frac{3}{4}, -\frac{1}{4}, -\frac{17}{4}, \frac{15}{4}, 0)$ and $\mathcal{P}_5^* = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$, which are the same as in Example V.4. It can be seen that the CoordSatCapFus outputs the same results as in Example V.5 for the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_{13/2}^\#[\mathcal{P}]$.

C. Solutions for the Finite Linear Source Model

As discussed in Section II-B, in a finite linear source model, we are particularly interested in the existence of the fractional and integral optimal rate vectors in $\mathcal{R}_{ACO}^*(V)$ and $\mathcal{R}_{NCO}^*(V)$, respectively. As pointed out in Remark III.12, since the extreme points in $\mathcal{R}_{ACO}^*(V) = B(\hat{f}_{R_{ACO}(V)}^\#, \leq)$ and $\mathcal{R}_{NCO}^*(V) = B(\hat{f}_{R_{NCO}(V)}^\#, \leq) \cap \mathbb{Z}^{|V|}$ are fractional and integral, respectively, in a finite linear source model, the problem reduces to determining a rate vector in $\text{EX}(\hat{f}_{R_{ACO}(V)}^\#)$ and $\text{EX}(\hat{f}_{R_{NCO}(V)}^\#)$. For this purpose, we have the following theorem with the proof in Appendix C.

Theorem V.11. For all $\alpha \geq R_{ACO}(V)$, the CoordSatCap algorithm returns \mathbf{r}_V that is an extreme point, or a vertex, in $B(\hat{f}_\alpha^\#, \leq)$, i.e., $\mathbf{r}_V \in \text{EX}(\hat{f}_\alpha^\#)$. ■

According to Theorem V.11 and the proof of Corollary III.9, we have the following results straightforwardly.

Corollary V.12. For a finite linear source model,

- (a) the MDA algorithm outputs $\mathbf{r}_V \in \mathcal{R}_{ACO}^*(V)$ such that $(|\mathcal{P}^*| - 1)\mathbf{r}_V$ is integral, i.e., \mathbf{r}_V can be implemented by $(|\mathcal{P}^*| - 1)$ -packet-splitting in CCDE;
- (b) with input $\alpha = R_{NCO}(V)$, the CoordSatCapFus algorithm returns $\mathbf{r}_V \in \mathcal{R}_{NCO}^*(V)$ which is integral. ■

If the value of $R_{ACO}(V)$ can be determined by the MDA algorithm, we know automatically $R_{NCO}(V) = \lceil R_{ACO}(V) \rceil$. Then, according to Corollary V.12, we can determine an integral optimal rate vector in $\mathcal{R}_{NCO}^*(V)$ for the non-asymptotic model in a finite linear source model by no more than one additional call of the CoordSatCapFus algorithm.²⁵

²⁵If $R_{ACO}(V) = R_{NCO}(V)$, $R_{ACO}(V)$ is integral necessarily and the rate vector in $\mathcal{R}_{ACO}^*(V)$ returned by the MDA algorithm is also an integral rate vector in $\mathcal{R}_{NCO}^*(V)$; If $R_{ACO}(V) < R_{NCO}(V)$, an integral optimal rate vector in $\mathcal{R}_{NCO}^*(V)$ can be determined by an extra call CoordSatCapFus($V, H, \lceil R_{ACO}(V) \rceil$). Therefore, in a finite linear source model, the non-asymptotic minimum sum-rate problem can be solved by no more than one extra call of the CoordSatCapFus algorithm after obtaining $R_{ACO}(V)$.

Algorithm 4: sum-rate increment algorithm (SIA) for solving the non-asymptotic minimum sum-rate problem in the finite linear source model

input : the ground set V , an oracle that returns the value of $H(X)$ for a given $X \subseteq V$

output: a rate vector \mathbf{r}_V in the optimal rate set $\mathcal{R}_{NCO}^*(V) = B(\hat{f}_{R_{NCO}(V)}^\#, \leq) \cap \mathbb{Z}^{|V|}$ and α which equals to $R_{NCO}(V)$

- 1 initialize α according to Proposition IV.1:
 $\alpha \leftarrow \lceil \max_{i \in V} \{\varphi(\{\{i\}, V \setminus \{i\}\}), \varphi(\{\{m\} : m \in V\})\} \rceil$;
- 2 determine a rate vector $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$ by solving the problem $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$;
- 3 **while** $r(V) \neq \alpha$ **do**
- 4 $\alpha \leftarrow \alpha + 1$;
- 5 determine a rate vector $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$ by solving the problem $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$;
- 6 **end**
- 7 return \mathbf{r}_V and α ;

Example V.13. The optimal rate vector $\mathbf{r}_V = (1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0) \in \mathcal{R}_{ACO}^*(V)$ determined by the MDA algorithm in Example V.2 is an extreme point in $B(\hat{f}_{R_{ACO}(V)}^\#, \leq) = \mathcal{R}_{ACO}^*(V)$, i.e., $\mathbf{r}_V = (1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0) \in \text{EX}(\hat{f}_{R_{ACO}(V)}^\#)$, where $\text{EX}(\hat{f}_{R_{ACO}(V)}^\#)$ is shown in Example III.11. Recall that the fundamental partition in this system is $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$ so that $|\mathcal{P}^*| = 3$. Therefore, $\mathbf{r}_V = (1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0)$ can be implemented by 2-packet-splitting.

Since $R_{ACO}(V) = \frac{13}{2}$, we have $R_{NCO}(V) = \lceil R_{ACO}(V) \rceil = 7$. By setting the linear ordering $\Phi = (4, 5, 2, 3, 1)$, we call CoordSatCapFus($V, H, 7$) and have $\mathbf{r}_V = (0, 1, 1, 5, 0)$ and $\mathcal{P}^* = \{\{1, 2, 3, 4, 5\}\}$ at the output. One can show that $\mathbf{r}_V = (0, 1, 1, 5, 0) \in \text{EX}(\hat{f}_7^\#) \subsetneq B(\hat{f}_7^\#, \leq) \cap \mathbb{Z}^{|V|} = \mathcal{R}_{NCO}^*(V)$.²⁶

On the other hand, we can also adopt a proper sum-rate adaptation method to solve the non-asymptotic minimum sum-rate problem in the finite linear source model. This idea was originally proposed in [15], [18]. The method is to iteratively update α , the estimation of the minimum sum-rate $R_{NCO}(V)$, on an integer set in \mathbb{Z}_+ until it reaches $R_{NCO}(V)$. The implementation of this method requires: (a) a method that can check if a sum-rate α is achievable; (b) an algorithm that can determine a rate vector $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq) = \{\mathbf{r}_V \in \mathcal{R}_{CO}(V) : r(V) = \alpha\}$ if α is achievable. It is fortunate that the CoordSatCap and CoordSatCapFus algorithms can complete both tasks.

Corollary V.14. For a sum-rate α , let \mathbf{r}_V be the rate vector returned by the CoordSatCapFus, or CoordSatCap, algorithm: α is achievable if and only if $r(V) = \alpha$; If α is achievable, we have $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq) = \{\mathbf{r}_V \in \mathcal{R}_{CO}(V) : r(V) = \alpha\}$ being an achievable rate vector with sum-rate α . ■

The proof of Corollary V.14 is in Appendix C. According to Corollary V.14, we can start with a lower estimation α of $R_{NCO}(V)$, e.g., the LB in Proposition IV.1, and increase α by

²⁶The result $\mathcal{P}^* = \{\{1, 2, 3, 4, 5\}\}$ is consistent with property (a) of the PSP in Theorem A.2 in Appendix A: Since $R_{NCO}(V) > R_{ACO}(V) = \alpha_1$, $\{V\}$ is the only minimizer of the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_{R_{NCO}(V)}^\#[\mathcal{P}]$.

one until it is achievable. The first achievable α necessarily equals to $R_{\text{NCO}}(V)$. Since $R_{\text{NCO}}(V) \in \{0, 1, \dots, H(V)\}$, we can adjust α to $R_{\text{NCO}}(V)$ within a finite number of iterations. This idea is implemented by the SIA algorithm in Algorithm 4, where the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_{\alpha}^{\#}[\mathcal{P}]$ in steps 2 and 5 can be solved by the CoordSatCap or CoordSatCapFus algorithm.

Example V.15. We apply the SIA algorithm to the system in Example III.11. We initiate $\alpha = \lceil \max_{i \in V} \{\varphi(\{\{i\}, V \setminus \{i\}\}), \varphi(\{\{m\}: m \in V\})\} \rceil = 6$ according to Proposition IV.1. We implement the CoordSatCapFus algorithm for solving the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_{\alpha}^{\#}[\mathcal{P}]$.

- For $\alpha = 6$, we have $\mathbf{r}_V = (1, 0, 0, 4, 0)$ and $\mathcal{P}^* = \{\{4, 5\}, \{1\}, \{2\}, \{3\}\}$ returned by the CoordSatCapFus algorithm. Since $r(V) = 5 < \alpha$, we update α to 7 and continue the iteration;
- For $\alpha = 7$, we have $\mathbf{r}_V = (0, 1, 1, 5, 0)$ and $\mathcal{P}^* = \{\{1, 2, 3, 4, 5\}\}$ returned by the CoordSatCapFus algorithm. Since $r(V) = 7 = \alpha$, the iteration terminates.

At the output, we have $\mathbf{r}_V = (0, 1, 1, 5, 0) \in \mathcal{R}_{\text{NCO}}^*(V)$, which is consistent with the result in Example V.13. Here, the SIA algorithm solves the non-asymptotic minimum sum-rate problem in the finite linear source model without obtaining the value of $R_{\text{ACO}}(V)$.

Note, in the SIA algorithm, the updates of α do not require the minimal/finest minimizer of the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_{\alpha}^{\#}[\mathcal{P}]$.²⁷ On the other hand, the sum-rate adaptation method is not unique. For solving the non-asymptotic minimum sum-rate problem in CCDE, the authors in [15], [18] proposed efficient algorithms to update α to $R_{\text{NCO}}(V)$, where the CoordSatCap algorithm based on Lemma V.6 is implemented as a subroutine. Since the CoordSatCap and CoordSatCapFus algorithms accomplish the same tasks in Corollary V.14, we can replace the CoordSatCap algorithm by the CoordSatCapFus algorithm in the sum-rate adaptation algorithms in [15], [18]. In the next subsection, we will show the advantage of this replacement: the reduction in complexity.

D. Complexity

Let δ be the computation complexity of evaluating the value of a submodular function $f: 2^V \mapsto \mathbb{R}$.²⁸ We denote $O(\text{SFM}(|V|))$ the complexity of solving the *submodular function minimization (SFM) problem* $\min\{f(X): X \subseteq V\}$, which is strongly polynomial [38]–[43]. For example, the SFM algorithm proposed in [44] completes in $O(|V|^5 \cdot \delta + |V|^6)$ time, which is the most efficient SFM algorithm in the literature to the best of our knowledge. Also, the minimizers of an SFM problem form a lattice, where the minimal/smallest and

maximal/largest minimizers exist [26, Lemma 2.1]. It is shown in [26, Section 7.1] that the minimal and maximal minimizers can be determined by the minimum-norm point in the base polyhedron and the minimum-norm point can be determined by the SFM algorithm in [43]. It means that the minimal minimizer of an SFM problem, as required by step 3 in the CoordSatCap algorithm and step 5 in the CoordSatCapFus algorithm, can be determined at the same time when the SFM is solved.

Although the SFM algorithms in [38]–[43] vary in computation complexity, the exact completion time of an SFM algorithm depends on $|V|$. We call $|V|$ the *size of the SFM problem* $\min\{f(X): X \subseteq V\}$. In this section, we study the complexity of the MDA and SIA algorithms proposed in Sections V in terms of the size of the SFM problem and completion time (in seconds), respectively. It should be noted that, in this paper, we assume that the value of the entropy function H at a given subset $X \subseteq V$ can be evaluated by an oracle call, which takes X as an input and outputs $H(X)$, and δ refers to the complexity upper bound of this oracle call.

1) *CoordSatCapFus vs. CoordSatCap*: The main subroutine of the MDA and SIA algorithms is the CoordSatCap or CoordSatCapFus algorithm, and the core part of the CoordSatCap and CoordSatCapFus algorithms is the SFM problem that determines the saturation capacity $\hat{\xi}_{\phi_i}$. Consider the CoordSatCap algorithm where the saturation capacity is determined by

$$\min\{f_{\alpha}^{\#}(X) - r(X): \phi_i \in X \subseteq V_i\}. \quad (18)$$

The size of this SFM problem is $|V_i| - 1$. The SFM problem $\min\{f_{\alpha}^{\#}(\tilde{U}) - r(\tilde{U}): \{\phi_i\} \in \tilde{U} \subseteq \mathcal{P}_i^*\}$ in the CoordSatCapFus algorithm is over \mathcal{P}_i^* , a fused user set of V_i , where each non-singleton subset $X \in \mathcal{P}_i^*$ is treated as a super user that corresponds to one dimension in \mathcal{P}_i^* . Since $|\mathcal{P}_i^*| - 1 \leq |V_i| - 1$, the computation complexity of the CoordSatCapFus algorithm is no greater than that of the CoordSatCap algorithm.

Example V.16. For $\phi_3 = 2$ in Example V.10, we have $\mathcal{P}_3^* = \{\{4, 5\}, \{2\}\}$, where $\{4, 5\}$ forms one dimension in \mathcal{P}_3^* , so that the size of the SFM problem $\min\{f_{\alpha}^{\#}(\tilde{U}) - r(\tilde{U}): \{\phi_3\} \in \tilde{U} \subseteq \mathcal{P}_3^*\}$ is $|\mathcal{P}_3^*| - 1 = |\{\{4, 5\}, \{2\}\}| - 1 = 1$. Suppose that we solve the problem $\min\{f_{\alpha}^{\#}(X) - r(X): \phi_3 \in X \subseteq V_3\}$ instead. Then, the size of this SFM problem is $|V_3| - 1 = |\{2, 4, 5\}| - 1 = 2$, which is greater than $|\mathcal{P}_3^*| - 1$.

However, the complexity of the CoordSatCapFus algorithm in the worst case is the same as that of the CoordSatCap algorithm, which is $O(|V| \cdot \text{SFM}(|V|))$. The worst case is when $\mathcal{P}_i^* = \{\{\phi_1\}, \dots, \{\phi_i\}\}$ for all $i \in V$, which happens when the components in \mathbf{Z}_V are mutually independent.

2) *MDA algorithm*: The MDA algorithm with the CoordSatCapFus being the subroutine completes in $O(|V|^2 \cdot \text{SFM}(|V|))$ time. We remark that $O(|V|^2 \cdot \text{SFM}(|V|))$ is the complexity upper bound for two reasons. On one hand, the complexity of the CoordSatCapFus algorithm is upper bounded by $O(|V| \cdot \text{SFM}(|V|))$; On the other hand, the number of calls of the CoordSatCapFus algorithm in the MDA algorithm is upper bounded by $|V|$. Then, the complexity of solving the non-asymptotic minimum sum-rate problem in the finite linear source model by no more than one additional call

²⁷It means that, when applied to the SIA algorithm, the initiation and updates of \mathcal{P}^* in the CoordSatCap algorithm and \mathcal{P}_i^* in the CoordSatCapFus algorithm are not required. In addition, the determination of the minimal minimizers, $U_{\phi_i}^*$ and \tilde{X}_{ϕ_i} in the CoordSatCapFus and CoordSatCap algorithms, respectively, is not required.

²⁸We assume that the value of $f(X)$ for any $X \subseteq V$ can be obtained by an oracle call and δ refers to the upper bound on the computation time of this oracle call.

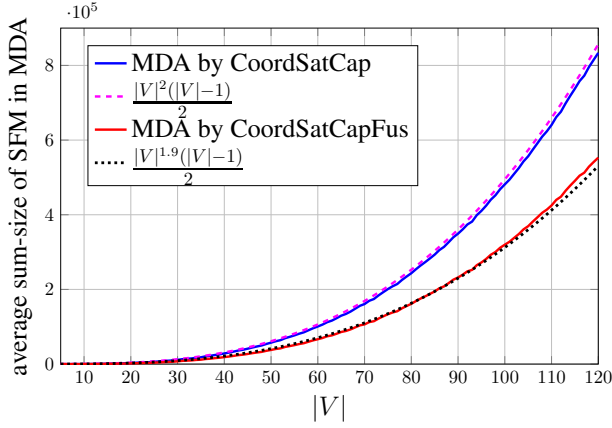


Fig. 11. The results in Experiment V.17. $H(V)$ is fixed to 50 and the number of users $|V|$ varies from 5 to 120. MDA by CoordSatCapFus refers to the MDA algorithm with the CoordSatCapFus algorithm being the subroutine, while MDA by CoordSatCap refers to the MDA algorithm with the CoordSatCap algorithm where the saturation capacity is determined by problem (18) being the subroutine. The sum-size of SFM refers to the total size of SFM algorithm in one call of the MDA algorithm. The sum-size averaged over 20 repetitions is presented in this figure. The average sum-size of SFM in one call of the MDA algorithm is upper bounded by $\frac{|V|^2(|V|-1)}{2}$.

of the CoordSatCapFus algorithm, as proposed in Section V-C, is upper bounded by $O((|V| + 1) \cdot |V| \cdot \text{SFM}(|V|))$.

Experiment V.17. Let $H(V)$ be fixed to 50 and the number of users $|V|$ vary from 5 to 120. For each value of $|V|$, we repeat the following procedure for 20 times:

- randomly generate a finite linear source model with the column vector $z_i = A_i x$ for all $i \in V$ subject to the condition $l(x) = H(V)$;
- solve the asymptotic minimum sum-rate problem by applying the MDA algorithm as follows:
 - MDA by CoordSatCap: Algorithm 1, where the Dilworth truncation problem in steps 2 and 6 is solved by the CoordSatCap algorithm with the saturation capacity determined by (18);
 - MDA by CoordSatCapFus: Algorithm 1, where the Dilworth truncation problem in steps 2 and 6 is solved by the CoordSatCapFus algorithm.

We sum up the sizes of the SFM algorithm in each run of the MDA algorithm. This sum-size is averaged over the 20 repetitions and shown in Fig. 11. Note, the average sum-size of the MDA by CoordSatCap is upper bounded by $\frac{|V|^2(|V|-1)}{2}$.²⁹ It can be shown that there is a reduction from $\frac{|V|^2(|V|-1)}{2}$ to $\frac{|V|^{1.9}(|V|-1)}{2}$ in the average sum-size of the SFM problem by implementing the CoordSatCapFus algorithm. This reduction could be considerable when $|V|$ is large. For example, in Fig. 11, when $|V| = 25$, the average sum-size of SFM is 6037.5 for MDA by CoordSatCap and 3831.4 for MDA by CoordSatCapFus.

The authors in [45] proposed a divide-and-conquer (DC) algorithm for solving the asymptotic minimum sum-rate prob-

²⁹This refers to the average size of the SFM algorithm in the MDA algorithm. The computation complexity of the MDA algorithm is still $O(|V|^2 \cdot \text{SFM}(|V|))$.

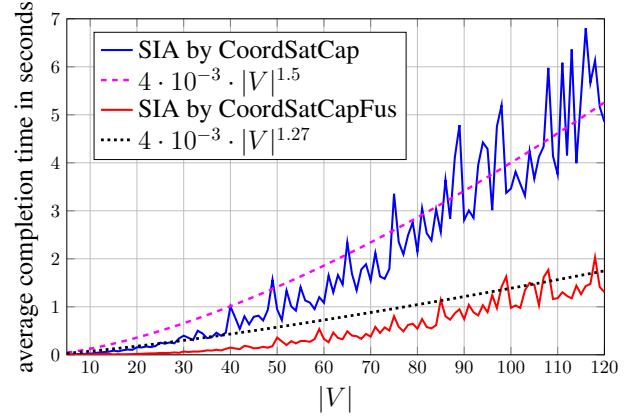


Fig. 12. The results of Experiment V.18. $H(V)$ is fixed to 50 and the number of users $|V|$ varies from 5 to 120. SIA by CoordSatCapFus refers to the SIA algorithm with the CoordSatCapFus algorithm being the subroutine, while SIA by CoordSatCap refers to the SIA algorithm with the CoordSatCap algorithm being the subroutine. The run-time in seconds of each call of the SIA algorithm is recorded and averaged over 20 repetitions. We implemented the minimum norm algorithm in [43] as the SFM algorithm. The SIA algorithm is run in MATLAB R2013a.

lem. This algorithm finds the fundamental partition \mathcal{P}^* and recursively breaks each element in \mathcal{P}^* to singletons by calling the decomposition algorithm (DA) algorithm in [46, Section 3] [33, Algorithm II]. The DC algorithm completes in $O(|V|^3 \cdot \text{SFM}(|V|))$ time. The detailed description of the DC algorithm is in Appendix D, where we also show that the recursive splitting of the subsets in \mathcal{P}^* is not necessary since the asymptotic minimum sum-rate problem can be solved at the same time when the fundamental partition \mathcal{P}^* is determined.

3) SIA algorithm: The authors in [15, Appendix F] [18, Section III-C] show that the complexity of adapting the estimation α on an integer set to the minimum sum-rate $R_{\text{NCO}}(V)$ in a finite linear source model grows logarithmically in $H(V)$. Therefore, the SIA algorithm completes in $O(\log H(V) \cdot |V| \cdot \text{SFM}(|V|))$ time. To show the actual run-time, or completion time in seconds, of the SIA algorithm, we do the following experiment.

Experiment V.18. Let $H(V)$ be fixed to 50 and the number of users $|V|$ vary from 5 to 120. For each value of $|V|$, we repeat the following procedure for 20 times:

- randomly generate a finite linear source model with the column vector $z_i = A_i x$ for all $i \in V$ subject to the condition $l(x) = H(V)$;
- solve the non-asymptotic minimum sum-rate problem by applying the SIA algorithm as follows:
 - SIA by CoordSatCap: Algorithm 4, where the Dilworth truncation problem in steps 2 and 5 is solved by the CoordSatCap algorithm with the saturation capacity determined by (18);
 - SIA by CoordSatCapFus: Algorithm 4, where the Dilworth truncation problem in steps 2 and 5 is solved by the CoordSatCapFus algorithm.

We implement the minimum-norm point algorithm proposed in [43] for solving the SFM problems in the CoordSatCap and CoordSatCapFus algorithms. The SIA algorithm is written in

MATLAB and run in MATLAB R2013a. We do the experiment on a desktop computer with Intel Core i7-3770 processor, 8Gb RAM and 64-bit Windows 7 Enterprise operating system. The run-time in seconds in each call of the SIA algorithm is recorded and averaged over repetitions. The results are shown in Fig. 12. The run-time of SIA by CoordSatCap is comparable to $4 \cdot 10^{-3} \cdot |V|^{1.5}$. With the fusion method, SIA by CoordSatCapFus reduces it to $4 \cdot 10^{-3} \cdot |V|^{1.27}$.

The authors in [15, Appendix G] show that the SIA by CoordSatCap method based on the minimum-norm algorithm [43] completes in $4 \cdot 10^{-3} \cdot |V|^{1.85}$ seconds on average, which is slower than the result in Experiment V.18. The main reason that can cause this run-time reduction is that we do the experiment on a dataset and computer that are different from those in [15, Appendix G]. In addition, the LB that we used in the SIA is tighter than the one in [15, Appendix F] may be another reason that results in a faster run-time. On the other hand, the complexity of the minimum-norm algorithm is still unknown and may vary with different data processing softwares [43]. Therefore, while the average run-time just shows an example on how faster the SIA algorithm completes in practice, the complexity of the SIA algorithm is still $O(\log H(V) \cdot |V| \cdot \text{SFM}(|V|))$, i.e., no matter how good the run-time is, it cannot be used to characterise the complexity of the SIA algorithm. However, based on Figs. 11 and 12, we can see clearly that the fusion method in CoordSatCapFus algorithm contributes to a considerable reduction in computation complexity when the number of users $|V|$ grows.

It should be noted that finite linear source model is used in Experiments V.17 and V.18 since the optimality of the output rate vector can be verified by random linear network coding (RLNC) according to [11, Theorem 6]. The results of the MDA algorithm in Experiment V.17 is checked by packet-splitting and applying RLNC to the packet chunks.

VI. MINIMUM WEIGHTED SUM-RATE PROBLEM

The minimum weighted sum-rate problem, the problem of minimizing a weighted sum-rate in the optimal rate vector set, has been considered in CO in [15], [18] for the finite linear source mode. In this section, we show how to solve the minimum weighted sum-rate problem in the asymptotic and non-asymptotic models by choosing a proper linear ordering in the CoordSatCapFus or CoordSatCap algorithm.

Let $\mathbf{w}_V = (w_i: i \in V) \in \mathbb{R}_+^{|V|}$ be a weight vector and $\mathbf{w}_V^\top \mathbf{r}_V = \sum_{i \in V} w_i r_i$ be the weighted sum-rate of \mathbf{r}_V . The minimum weighted sum-rate problem in the asymptotic model and non-asymptotic models are respectively

$$\begin{aligned} \min\{\mathbf{w}_V^\top \mathbf{r}_V : \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}, \\ \min\{\mathbf{w}_V^\top \mathbf{r}_V : \mathbf{r}_V \in \mathcal{R}_{\text{NCO}}^*(V)\}. \end{aligned}$$

We say that $\Phi = (\phi_1, \dots, \phi_{|V|})$ is a linear ordering w.r.t. \mathbf{w}_V if $w_{\phi_1} \leq w_{\phi_2} \leq \dots \leq w_{\phi_{|V|}}$. For a given weight vector \mathbf{w}_V , a linear ordering Φ w.r.t. \mathbf{w}_V can be chosen, for which we have Theorem VI.1 below. The proof is in Appendix E.

Theorem VI.1. *For a weight vector $\mathbf{w}_V \in \mathbb{R}_+^{|V|}$, by fixing Φ to be the linear ordering w.r.t. \mathbf{w}_V in the CoordSatCapFus*

or CoordSatCap algorithm, the optimal rate \mathbf{r}_V returned by the MDA algorithm is the minimizer of $\min\{\mathbf{w}_V^\top \mathbf{r}_V : \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^(V)\}$. In the finite linear source model and CCDE, the optimal rate \mathbf{r}_V returned by CoordSatCapFus or CoordSatCap algorithm with input $\alpha = R_{\text{NCO}}(V)$ and linear ordering w.r.t. \mathbf{w}_V is the minimizer of $\min\{\mathbf{w}_V^\top \mathbf{r}_V : \mathbf{r}_V \in \mathcal{R}_{\text{NCO}}^*(V)\}$. ■*

Based on Theorem VI.1, for a finite linear source model, the minimum weighted sum-rate problem can be solved by the SIA algorithm if we choose a proper linear ordering in the CoordSatCapFus or CoordSatCap algorithm. This result is consistent with the ones in [15], [18]. Note, the SIA algorithm adapts the sum-rate by starting with a LB that is tighter than the ones in [15], [18].³⁰

Example VI.2. *It can be shown that the optimal rate vector $\mathbf{r}_V = (1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0) \in \mathcal{R}_{\text{ACO}}^*(V)$ determined by the MDA algorithm based on the linear ordering $\Phi = (4, 5, 2, 3, 1)$ in Example V.2 is the minimizer of $\min\{\mathbf{w}_V^\top \mathbf{r}_V : \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$ where $\mathbf{w}_V \in \mathbb{R}_+^{|V|}$ could be any weight vector such that $w_4 \leq w_5 \leq w_2 \leq w_3 \leq w_1$, e.g., $\mathbf{w}_V = (2, 0.85, 1.1, 0.11, 0.13)$. The optimal rate vector $\mathbf{r}_V = (0, 1, 1, 5, 0) \in \mathcal{R}_{\text{NCO}}^*(V)$ based on the linear ordering $\Phi = \{4, 5, 2, 3, 1\}$ in Examples V.13 and V.15 is the minimizer of $\min\{\mathbf{w}_V^\top \mathbf{r}_V : \mathbf{r}_V \in \mathcal{R}_{\text{NCO}}^*(V)\}$ where $\mathbf{w}_V \in \mathbb{R}_+^{|V|}$ could be any weight vector such that $w_4 \leq w_5 \leq w_2 \leq w_3 \leq w_1$.*

Note, for a weight vector $\mathbf{w}_V \in \mathbb{R}_+^{|V|}$ such that all dimensions w_i are equal, e.g., $\mathbf{w}_V = \mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{|V|}$, the minimum weighted sum-rate problem reduces to the minimum sum-rate problem. In addition, if the problem is just to determine an optimal rate vector in $\mathcal{R}_{\text{ACO}}^*(V)$ or $\mathcal{R}_{\text{NCO}}^*(V)$, the linear ordering Φ in the CoordSatCap and CoordSatCapFus algorithms can be arbitrarily chosen.

VII. FUNDAMENTAL PARTITION: MINIMAL SEPARATORS

The fundamental partition \mathcal{P}^* is not only the optimizer for the asymptotic minimum sum-rate problem, but also an essential solution to many problems. In network strength or optimal attack problems [35]–[37], [47], the fundamental partition is an optimal way for an attacker to disconnect a network, i.e., decomposing the network into the fundamental partition requires the least effort on breaking the connections/edges between nodes. The authors in [46] proposed a novel clustering criterion, which is called minimum average cost (MAC) clustering, based on the submodularity of the similarity measures that is generally used in clustering problems. The objective function in MAC is defined as the clustering cost averaged over the incremental number of clusters. Based on the MAC, the authors in [48] proposed an information-theoretic clustering (info-clustering) framework where the MMI is used as the similarity measure and the purpose is to search a clustering solution such that the intra-cluster MMI is maximized while the inter-cluster MMI is minimized. In both MAC clustering and info-clustering, the optimal clustering is the fundamental

³⁰The authors in [15] suggested LB $H(V) - \min_{i \in V} H(\{i\})$ as the initial guess of the minimum sum-rate, which is shown to be looser than the one proposed in Proposition IV. In [15, Algorithms 3], the rate adaptation is done in the region $\{0, 1, \dots, H(V)\}$.

partition \mathcal{P}^* . In CO, beyond being the optimizer of the minimum sum-rate problem (7a), the fundamental partition \mathcal{P}^* has practical interpretation or usefulness in other aspects. In this section, we show that \mathcal{P}^* is the set of the minimal separators of a submodular function which makes the estimation of the value of function $\hat{f}_{\text{RACO}}^\#(V)$ and the separable convex minimization problem over $\mathcal{R}_{\text{ACO}}^*(V)$ decomposable.

For a normalized submodular set function $f: 2^V \mapsto \mathbb{R}$, a nonempty proper subset $X \subsetneq V$ is a *separator* of f if $f(X) + f(V \setminus X) = f(V)$ [26, Section 3.3].³¹ A submodular set function f is called *separable* if there exists a separator of f . For each separable submodular set function, there exists a unique set of minimal separators as defined below.

Definition VII.1 (minimal separators [26, Theorem 3.38]). *For a separable submodular set function $f: 2^V \mapsto \mathbb{R}$, a partition $\mathcal{P} \in \Pi(V)$ is the set of minimal separators if, for all $X \in \mathcal{P}$, X is a separator and any $X' \subsetneq X$ such that $X' \neq \emptyset$ is not a separator of f .*

Theorem VII.2. $\hat{f}_{\text{RACO}}^\#$ is a separable submodular function and the fundamental partition \mathcal{P}^* is the set of minimal separators of $\hat{f}_{\text{RACO}}^\#$. ■

The proof of Theorem VII.2 is in Appendix F.

A. Properties of Minimal/Finest Separators

For any $X, Y \subseteq V$ such that $X \cap Y = \emptyset$, let $\mathbf{r}_X \oplus \mathbf{r}_Y = \mathbf{r}_{X \cup Y}$ be the direct sum of \mathbf{r}_X and \mathbf{r}_Y . For example, for $\mathbf{r}_{\{1,3\}} = (r_1, r_3) = (3, 0.7)$ and $\mathbf{r}_{\{2,5,6\}} = (r_2, r_5, r_6) = (2.4, 2, 4)$, $\mathbf{r}_{\{1,3\}} \oplus \mathbf{r}_{\{2,5,6\}} = \mathbf{r}_{\{1,2,3,5,6\}} = (3, 2.4, 0.7, 2, 4)$.

Lemma VII.3 (properties of minimal separators [26, Theorems 3.32 and 3.38, Lemma 3.37]). *For the fundamental partition \mathcal{P}^* as the set of minimal separators of $\hat{f}_{\text{RACO}}^\#$, the followings hold.*

- (a) $\hat{f}_{\text{RACO}}^\#(X) = \sum_{C \in \mathcal{P}^*} \hat{f}_{\text{RACO}}^\#(X \cap C)$ for all $X \subseteq V$;
- (b) The dimension of $B(\hat{f}_{\text{RACO}}^\#(V), \leq)$ is $|V| - |\mathcal{P}^*|$ and

$$B(\hat{f}_{\text{RACO}}^\#(V), \leq) = \{\oplus_{C \in \mathcal{P}^*} \mathbf{r}_C : \mathbf{r}_C \in B(\hat{f}_{\text{RACO}}^\#(V), C, \leq), C \in \mathcal{P}^*\},$$

where $\hat{f}_{\text{RACO}}^\#(V), C$ is the Dilworth truncation of $\hat{f}_{\text{RACO}}^\#(V), C$, the reduction of $\hat{f}_{\text{RACO}}^\#$ on C ;

- (c) Let \mathbf{r}_V be any rate vector in $B(\hat{f}_{\text{RACO}}^\#(V), \leq)$. For any $C, C' \in \mathcal{P}^*$ such that $C \neq C'$,

$$\mathbf{r}_V + \epsilon(\chi_i - \chi_j) \notin B(\hat{f}_{\text{RACO}}^\#(V), \leq)$$

for all $\epsilon > 0$, $i \in C$ and $j \in C'$. ■

Based on property (a) in Lemma VII.3, by using the fundamental partition \mathcal{P}^* , we can break the task of evaluating

the value of $\hat{f}_{\text{RACO}}^\#$ at any subset $X \subseteq V$ into subtasks: get the values of $\hat{f}_{\text{RACO}}^\#$ at $C \cap X$ for all $C \in \mathcal{P}^*$ and sum them up. Here, each value of $\hat{f}_{\text{RACO}}^\#(C \cap X)$ can be obtained by applying the CoordSatCap or CoordSatCapFus algorithm. By doing so, the complexity of evaluating $\hat{f}_{\text{RACO}}^\#(X)$ is reduced from $O(|X| \cdot \text{SFM}(|X|))$ to $O(\eta \cdot \text{SFM}(\eta))$ where $\eta = \max\{|C \cap X| : C \in \mathcal{P}^*\}$. Property (b) means that a separable submodular function results in a separable base polyhedron, which gives rise to property (c) [26, Lemma 3.41]. Property (c) is an important result in CO in that it makes the separable convex minimization problem over the optimal rate vector set $\mathcal{R}_{\text{ACO}}^*(V)$ decomposable. In the following context, we first show the examples of properties (a) and (b) and then discuss the decomposability of the separable convex minimization problem based on property (c).

Example VII.4. For the system in Example III.11, we know that the minimum sum-rate $R_{\text{ACO}}(V) = \frac{13}{2}$ and the fundamental partition $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$ for the asymptotic model by the MDA algorithm. Consider the value of the Dilworth truncation function $\hat{f}_{\text{RACO}}^\#$ at $X = \{1, 2, 3, 5\}$. Based on property (a) in Lemma VII.3, we have

$$\begin{aligned} \hat{f}_{13/2}^\#(\{1, 2, 3, 5\}) &= \sum_{C \in \mathcal{P}^*} \hat{f}_{13/2}^\#(\{1, 2, 3, 5\} \cap C) \\ &= \hat{f}_{13/2}^\#(\{1, 5\}) + \hat{f}_{13/2}^\#(\{2\}) + \hat{f}_{13/2}^\#(\{3\}) \\ &= 4 + \frac{1}{2} + \frac{1}{2} \\ &= 5. \end{aligned}$$

Here, the value of $\hat{f}_{13/2}^\#(\{1, 2, 3, 5\})$ can be obtained in $O(4 \cdot \text{SFM}(4))$ time, while the value of $\sum_{C \in \mathcal{P}^*} \hat{f}_{13/2}^\#(\{1, 2, 3, 5\} \cap C)$ can be obtained in $O(2 \cdot \text{SFM}(2))$ time. Consider the base polyhedron $B(\hat{f}_{13/2}^\#, \leq)$ for the system in Example III.11. According to property (b) in Lemma VII.3, the dimension of $B(\hat{f}_{13/2}^\#, \leq)$ is $|V| - |\mathcal{P}^*| = 2$.

We can visualize property (b) in Lemma VII.3 via Figs. 9 and 10. In Fig. 9, when $\alpha = \frac{23}{4}$ and $C = \{1, 4, 5\}$, $\hat{f}_{\alpha, C}^\#$ is separable with the minimal separator set being $\mathcal{P}^* = \{\{1\}, \{4, 5\}\}$ so that the dimension of $B(\hat{f}_{\alpha, C}^\#, \leq)$ is $|C| - |\mathcal{P}^*| = 1$, i.e., $B(\hat{f}_{\alpha, C}^\#, \leq)$ is a 1-dimension line segment, and $\mathbf{r}_{\{4, 5\}} \oplus \mathbf{r}_1 \in B(\hat{f}_{23/4, C}^\#, \leq)$ for $\mathbf{r}_1 = \frac{3}{4}$ and all $\mathbf{r}_{\{4, 5\}} \in B(\hat{f}_{23/4, \{4, 5\}}^\#, \leq)$. In Fig. 10, when $\alpha = \frac{13}{2}$ and $C = \{1, 4, 5\}$, $\hat{f}_{\alpha, C}^\#$ is nonseparable so that the dimension of $B(\hat{f}_{13/2, C}^\#, \leq)$ is $|C| - 1 = 2$, i.e., $B(\hat{f}_{13/2, C}^\#, \leq)$ is a 2-dimension polygon on the plane $\{\mathbf{r}_C \in \mathbb{R}^{|C|} : r(C) = \hat{f}_{13/2}^\#(C) = \frac{11}{2}\}$.³²

³¹In [26, Section 3.3], the author defined the connectivity of a submodular system that is denoted by two tuple: the power set 2^V and the rank function $f: 2^V \mapsto \mathbb{R}$ that is submodular. $(2^V, f)$ is called connected if there does not exist a nonempty subset $X \subsetneq V$ such that $f(X) + f(V \setminus X) = f(V)$. Then, X is a 'separator' if a submodular system is disconnected. The name 'separator' is also used in [49, Section 3]. In this paper, without introducing the concept of the submodular system, we define the separator and separability w.r.t. a submodular set function.

³²A nonseparable function $f: 2^V \mapsto \mathbb{R}$ can be considered as a separable function with the minimal separator set being $\mathcal{P}^* = \{V\}$ so that, according to property (b) in Lemma VII.3, the dimension of $B(f, \leq)$ is $|V| - 1$. Also, for a nonseparable function f , there does not exist $\mathcal{P} \in \Pi(V)$ such that $f(X) = \sum_{C \in \mathcal{P}} f(X \cap C), \forall X \subseteq V$ or $B(f, \leq) = \{\oplus_{C \in \mathcal{P}} \mathbf{r}_C : \mathbf{r}_C \in B(f, \leq), C \in \mathcal{P}\}$. The latter means that if we determine $\mathbf{r}_C \in B(f, \leq)$ for all $C \in \mathcal{P}$, the direct sum $\mathbf{r}_V = \oplus_{C \in \mathcal{P}} \mathbf{r}_C$ does not necessarily belong to $B(f, \leq)$.

B. Separable Convex Function Minimization over $\mathcal{R}_{\text{ACO}}^*(V)$

We call $g: \mathbb{R}^{|V|} \mapsto \mathbb{R}$ a *separable convex function* if $g(\mathbf{r}_V) = \sum_{i \in V} g_i(r_i)$ where $g_i: \mathbb{R} \mapsto \mathbb{R}$ is convex for all $i \in V$. For the minimization problem $\min\{g(\mathbf{r}_V): \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$ where g is a separable convex function, the local optimality w.r.t. the elementary transform $\chi_i - \chi_j$ implies global optimality.

Theorem VII.5 ([26, Theorem 20.3]). *For a separable convex function g , \mathbf{r}_V^* is the minimizer of $\min\{g(\mathbf{r}_V): \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$ if and only if, for all $i, j \in V$ and $\epsilon > 0$ such that $\mathbf{r}_V^* + \epsilon(\chi_i - \chi_j) \in \mathcal{R}_{\text{ACO}}^*(V)$,*

$$g(\mathbf{r}_V^*) \leq g(\mathbf{r}_V^* + \epsilon(\chi_i - \chi_j)). \quad \blacksquare$$

Corollary VII.6. *For all $C \in \mathcal{P}^*$, let \mathbf{r}_C^* be the minimizer of the separable convex minimization problem $\min\{g(\mathbf{r}_C): \mathbf{r}_C \in B(\hat{f}_{\text{RACO}(V),C}^\#, \leq)\}$. $\mathbf{r}_V^* = \oplus_{C \in \mathcal{P}^*} \mathbf{r}_C^*$ is the minimizer of $\min\{g(\mathbf{r}_V): \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$.* \blacksquare

In Corollary VII.6, $B(\hat{f}_{\text{RACO}(V),C}^\#, \leq)$ is the projection of the optimal rate vector set $\mathcal{R}_{\text{ACO}}^*(V) = B(\hat{f}_{\text{RACO}(V)}^\#, \leq)$ on the subset C and \mathbf{r}_C is the projection of some optimal rate vector $\mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)$ on C . The proof of Corollary VII.6 is in Appendix F based on property (c) in Lemma VII.3 and Theorem VII.5. Based on Corollary VII.6, the minimization problem $\min\{g(\mathbf{r}_V): \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$ can be divided to $|\mathcal{P}^*|$ minimization problems

$$\min\{g(\mathbf{r}_C): \mathbf{r}_C \in B(\hat{f}_{\text{RACO}(V),C}^\#, \leq)\}, \quad (19)$$

each of which has a lower dimension than the original one. On the other hand, there exist many algorithms in the literature that efficiently solve the minimization problem (19), e.g., the algorithms in [50], [51]. We show an example of Corollary VII.6 below, where g is a quadratic function.

Example VII.7. *For the system in Example III.11, consider the quadratic programming $\min\{\sum_{i \in V} \frac{r_i^2}{w_i}: \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$ where $\mathbf{w}_V \in \mathbb{R}_+^{|V|}$ is a weight vector.³³ The objective function $\sum_{i \in V} \frac{r_i^2}{w_i}$ is separable convex. For this system, we have $\mathcal{R}_{\text{ACO}}(V) = \frac{13}{2}$ and $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$.*

According to Corollary VII.6, we can determine the minimizers of

$$\min\{\sum_{i \in C} \frac{r_i^2}{w_i}: \mathbf{r}_C \in B(\hat{f}_{\text{RACO}(V),C}^\#, \leq)\} \quad (20)$$

for all $C \in \mathcal{P}^$ and combine the results by obtaining the direct sum of them. In fact, we just need to solve the problem (20) for $C = \{1, 4, 5\}$ since both $B(\hat{f}_{\text{RACO}(V),\{2\}}^\#, \leq)$ and $B(\hat{f}_{\text{RACO}(V),\{3\}}^\#, \leq)$ are singletons that only contain $r_2 = \frac{1}{2}$ and $r_3 = \frac{1}{2}$, respectively. For problem (20) when $C = \{1, 4, 5\}$, it can be shown that $\mathbf{r}_{\{1,4,5\}}^* = (\frac{3}{2}, 2, 2)$ when $\mathbf{w}_{\{1,4,5\}} = (1, 1, 1)$ and $\mathbf{r}_{\{1,4,5\}}^* = (1, 2, \frac{5}{2})$ when $\mathbf{w}_{\{1,4,5\}} = (1, 3, 4)$. Therefore, the minimizer of $\min\{\sum_{i \in V} \frac{r_i^2}{w_i}: \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$ is $\mathbf{r}_V^* = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2)$ when $\mathbf{w}_V = \mathbf{1}$ and $\mathbf{r}_V^* = (1, \frac{1}{2}, \frac{1}{2}, 2, \frac{5}{2})$ when $\mathbf{w}_V = (1, 2, 5, 3, 4)$.*

It is shown in [54] that the problem $\min\{\sum_{i \in V} \frac{r_i^2}{w_i}: \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^(V)\}$ can be solved in $O(|V|^2 \cdot \text{SFM}(|V|))$ time, where $|V| = 5$ for the system in Example III.11. But, if we compute the minimizer of (20) for each subset C in the fundamental partition \mathcal{P}^* , the problem can be solved in $O(\eta^2 \cdot \text{SFM}(\eta))$ time, where $\eta = \max\{|C|: C \in \mathcal{P}^*\} = 3$. Therefore, the separate computation of the minimizer of $\min\{g(\mathbf{r}_C): \mathbf{r}_C \in \mathcal{R}_{\text{ACO}}^*(V)\}$ for all $C \in \mathcal{P}^*$ based on Corollary VII.6 reduces the computation complexity.*

In Example VII.7, the minimizer of the quadratic programming problem $\min\{\sum_{i \in V} \frac{r_i^2}{w_i}: \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$ is called the lexicographical optimizer in [55] since it lexicographically dominates any other rate vectors in the submodular base polyhedron $B(\hat{f}_{\text{RACO}(V)}^\#, \leq) = \mathcal{R}_{\text{ACO}}^*(V)$.³⁴ It is also the optimizer of many other optimization problems in $\mathcal{R}_{\text{ACO}}^*(V)$ [51], [57], e.g., $\min\{\sum_{i \in V} e^{r_i + w_i}: \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$, $\max\{\sum_{i \in V} w_i \ln r_i: \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$ and $\min\{\sum_{i \in V} r_i \log \frac{r_i}{w_i}: \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$. The authors in [21] proposed an integral rate incremental method for solving the fairness problem $\min\{\sum_{i \in V} r_i \log r_i: \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$, where the objective function is equivalent to $\sum_{i \in V} r_i \log \frac{r_i}{w_i}$ when $\mathbf{w} = \mathbf{1}$. However, this method is not able to provide a solution to $\min\{\sum_{i \in V} r_i \log r_i: \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$ for the asymptotic model, since the step size of each increment is uncertain when the rates are real numbers.

VIII. CONCLUSION

We proposed the MDA and SIA algorithms for searching an optimal rate vector that attains omniscience with the minimum sum-rate in the asymptotic and non-asymptotic models, respectively. We also proposed a fusion method to solve the Dilworth truncation problem, a subroutine in the MDA and SIA algorithms, and ran experiment to show that this fusion method contributes to a reduction in computational complexity. We showed that the minimum weighted sum-rate problem in both asymptotic and non-asymptotic models can be solved by choosing a proper linear ordering in the MDA and SIA algorithms, respectively. We proved the existence of a fractional optimal rate vector in the finite linear source model which can be implemented by $(|\mathcal{P}^*| - 1)$ -packet-splitting in CCDE. In addition, we revealed the decomposition property of the fundamental partition \mathcal{P}^* in the asymptotic model, where we showed that the tasks of evaluating the Dilworth truncation function and minimizing a separable parametric convex function over the optimal rate vector set can be decomposed into subtasks so that the overall complexity is reduced.

To solve a CO problem in practice, there still remains one problem: What to send in each transmission. It is shown in [18] that, for an optimal transmission rate vector, the coding scheme for attaining omniscience in the finite linear source model can be designed based on a simultaneous matrix completion algorithm [58], which completes in $O(|V|^4 \cdot \gamma \cdot \log(|V| \cdot H(V)))$ time with γ denoting the complexity of the matrix rank

³³This quadratic programming problem is also called the resource allocation problem under submodular constraints in [52], [53].

³⁴We refer the reader to [55], [56] for the detailed definition and explanation of the lexicographical domination and its properties in the submodular base polyhedron.

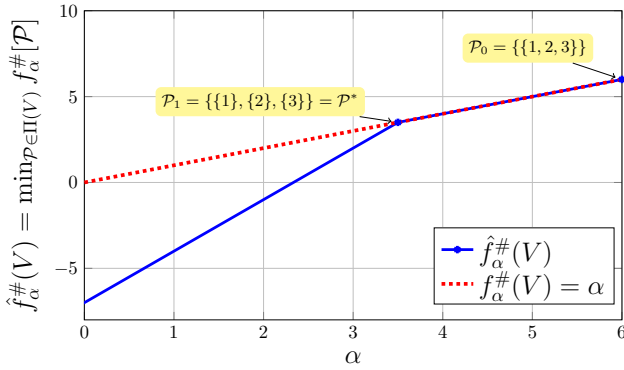


Fig. 13. The value of $\hat{f}_\alpha^\#(V)$ as a function of α for the system in Example II.1. $\alpha_1 = R_{\text{ACO}}(V) = \frac{7}{2}$ and $\mathcal{P}_1 = \mathcal{P}^* = \{\{1\}, \{2\}, \{3\}\}$ according to Corollary A.3.

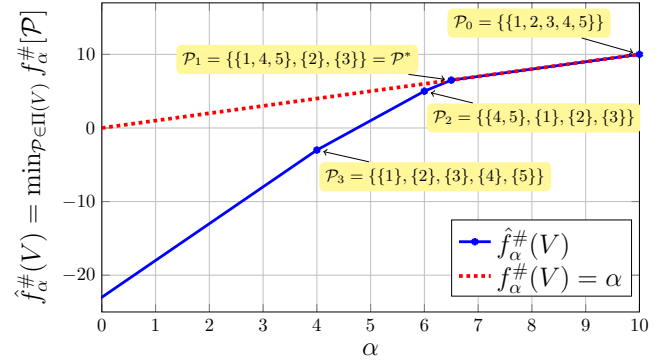


Fig. 14. The value of $\hat{f}_\alpha^\#(V)$ as a function of α for the system in Example III.11. $\alpha_1 = R_{\text{ACO}}(V) = \frac{13}{2}$ and $\mathcal{P}_1 = \mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$ according to Corollary A.3.

function. But, it is still worth discussing if there exists other less complex algorithms for the coding design.

APPENDIX A PRINCIPAL SEQUENCE OF PARTITIONS (PSP)

We define the pairwise relationship between two partitions in $\Pi(V)$ as follows.

Definition A.1 (order \preceq). For $\mathcal{P}, \mathcal{P}' \in \Pi(V)$, we denote

- $\mathcal{P} \preceq \mathcal{P}'$ if, for all $X \in \mathcal{P}$, $\exists X' \in \mathcal{P}'$ such that $X \subseteq X'$;
- $\mathcal{P} = \mathcal{P}'$ if $\mathcal{P} \preceq \mathcal{P}'$ and $\mathcal{P} \succeq \mathcal{P}'$;
- $\mathcal{P} \prec \mathcal{P}'$ if $\mathcal{P} \preceq \mathcal{P}'$ and $\mathcal{P} \neq \mathcal{P}'$.

In other words, $\mathcal{P} \preceq \mathcal{P}'$ if \mathcal{P} is finer than \mathcal{P}' and $\mathcal{P} \prec \mathcal{P}'$ if \mathcal{P} is strictly finer than \mathcal{P}' . For example, for $\mathcal{P} = \{\{1, 2\}, \{3\}, \{4\}\}$ and $\mathcal{P}' = \{\{1, 2, 3\}, \{4\}\}$, we have $\mathcal{P} \preceq \mathcal{P}'$. In fact, $\mathcal{P} \prec \mathcal{P}'$.

Theorem A.2 (PSP [46, Sections 2.2 and 3]³⁵, [33, Definition 3.8]). $\hat{f}_\alpha^\#(V) = \min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ is a piecewise linear nondecreasing curve in α with $p \leq |V| - 1$ critical/turning points

$$H(V) = \alpha_0 > \alpha_1 > \alpha_2 > \dots > \alpha_p \geq 0$$

that have the following properties.

(a) Denote \mathcal{P}_j the finest/minimal minimizers of the Dilworth truncation problem $\min_{\mathcal{P} \in \Pi(V)} f_{\alpha_j}^\#[\mathcal{P}]$. All \mathcal{P}_j s form a partition chain/sequence $\mathcal{C}_\mathcal{P}$:

$$\{V\} = \mathcal{P}_0 \succ \mathcal{P}_1 \succ \dots \succ \mathcal{P}_p = \{\{i\} : i \in V\}.$$

If $\alpha_j > \alpha > \alpha_{j+1}$ for some $j \in \{0, \dots, p-1\}$, the minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ is uniquely \mathcal{P}_j ; If $\alpha < \alpha_p$, the minimizer is uniquely $\mathcal{P}_p = \{\{i\} : i \in V\}$;

(b) The gradient of $\hat{f}_\alpha^\#(V)$ is decreasing in α : The gradient is $|\mathcal{P}_p| = |V|$ initially; It changes to $|\mathcal{P}_{j-1}|$ after each critical value α_j and finally decreases to 1 after α_1 . ■

Corollary A.3. $\alpha_1 = R_{\text{ACO}}(V)$ and $\mathcal{P}_1 = \mathcal{P}^*$, i.e., the parameters in PSP that correspond to the first critical point α_1

³⁵The authors in [46] discuss the problem that is called β -minimum average clustering (β -MAC). It can be shown that the minimum sum-rate problem considered in this paper is a β -MAC problem when $\beta = 1$.

constitute the solutions to the asymptotic minimum sum-rate problem.

Proof: According to Theorem III.3, the base polyhedron $B(f_\alpha^\#, \leq)$ is nonempty if and only if $\alpha = \hat{f}_\alpha^\#(V)$. In other words, $B(f_\alpha^\#, \leq) \neq \emptyset$ if and only if the value of α falls in the segment of the piecewise linear $\hat{f}_\alpha^\#(V)$ vs. α curve where the $\hat{f}_\alpha^\#(V)$ and $f_\alpha^\#(V) = \alpha$ overlap, which, based on property (b) in Theorem A.2, is when $\alpha \geq \alpha_1$. Then, the minimum sum-rate $R_{\text{ACO}}(V)$ is the smallest value of α such that $B(f_\alpha^\#, \leq) \neq \emptyset$, which is α_1 . All maximizers of (7a) and $\{V\}$ constitute the set of all minimizers of $\min_{\mathcal{P} \in \Pi(V)} f_{\alpha_1}^\#[\mathcal{P}]$. So, the minimal minimizer \mathcal{P}_1 is the minimal maximizer of (7a), i.e., \mathcal{P}_1 equals to the fundamental partition \mathcal{P}^* . ■

The proof of Corollary A.3 is exemplified below.

Example A.4. We show the plot $\hat{f}_\alpha^\#(V)$ in α for the systems in Examples II.1 and III.11 in Figs. 13 and 14, respectively. It can be seen from both figures that $\hat{f}_\alpha^\#(V)$ is an increasing piecewise linear function in α . We discuss Fig. 14 based on Theorem A.2 and Corollary A.3 as follows.

In addition to α_0 and \mathcal{P}_0 , there are three critical points α_j with \mathcal{P}_j , the minimal minimizers of $\min_{\mathcal{P} \in \Pi(V)} f_{\alpha_j}^\#[\mathcal{P}]$, being

$$\begin{aligned} \alpha_0 &= 10, & \mathcal{P}_0 &= \{\{1, 2, 3, 4, 5\}\}; \\ \alpha_1 &= \frac{13}{2}, & \mathcal{P}_1 &= \{\{1, 4, 5\}, \{2\}, \{3\}\}; \\ \alpha_2 &= 6, & \mathcal{P}_2 &= \{\{4, 5\}, \{1\}, \{2\}, \{3\}\}; \\ \alpha_3 &= 4, & \mathcal{P}_3 &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}. \end{aligned}$$

We have the partition sequence $\mathcal{C}_\mathcal{P}$: $\mathcal{P}_0 \succ \mathcal{P}_1 \succ \mathcal{P}_2 \succ \mathcal{P}_3$. The gradient is: 5 when $\alpha \in [0, \alpha_3]$; 4 when $\alpha \in [\alpha_3, \alpha_2]$; 3 when $\alpha \in [\alpha_2, \alpha_1]$; 1 when $\alpha \in [\alpha_1, \alpha_0]$. In addition, one can show that the minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ is uniquely \mathcal{P}_3 when $\alpha \in [0, \alpha_3]$, \mathcal{P}_2 when $\alpha \in (\alpha_3, \alpha_2)$, \mathcal{P}_1 when $\alpha \in (\alpha_2, \alpha_1)$ and \mathcal{P}_0 when $\alpha \in (\alpha_1, \alpha_0]$. Here, α_1 and \mathcal{P}_1 coincide with the minimum sum-rate $R_{\text{ACO}}(V) = \frac{13}{2}$ and the fundamental partition $\mathcal{P}^* = \{\{4, 5\}, \{1\}, \{2\}, \{3\}\}$ in Example III.11, respectively.

In Fig. 14, we also plot the line $f_\alpha^\#(V) = \alpha$. It can be seen that $f_\alpha^\#(V)$ overlaps with $\hat{f}_\alpha^\#(V)$, i.e., $\alpha = \hat{f}_\alpha^\#(V)$, when $\alpha \in [\alpha_1, \alpha_0]$. So, $B(f_\alpha^\#, \leq) \neq \emptyset$ when $\alpha \geq \alpha_1$. According to

Theorem III.3. $\alpha_1 = \frac{13}{2}$, the minimal value of α in the region $[\alpha_1, \alpha_0]$, is the minimum sum-rate $R_{ACO}(V)$.

Consider the region when $\alpha < \alpha_1 = \frac{13}{2}$ in Fig. 14. We have $f_\alpha^\#(V) = \alpha > \hat{f}_\alpha^\#(V)$. As discussed in Section III-C, the polyhedron $P(f_\alpha^\#, \leq)$ does not intersect with the hyperplane $\{\mathbf{r}_V \in \mathbb{R}^{|V|} : r(V) = \alpha\}$, i.e., $B(f_\alpha^\#, \leq) = \emptyset$, in this region. It means that the minimum sum-rate α is too low for attaining the omniscience in V . On the contrary, when $\alpha \geq \alpha_1 = \frac{13}{2}$, $P(f_\alpha^\#, \leq)$ intersects with the hyperplane $\{\mathbf{r}_V \in \mathbb{R}^{|V|} : r(V) = \alpha\}$, i.e., $B(f_\alpha^\#, \leq) \neq \emptyset$. It can be seen that the PSP provides another interpretation of Theorem III.3.

A. Proof of Theorem V.1

We have the following properties for φ and the partitions \mathcal{P}_j s in the PSP.

Lemma A.5 (property of $\varphi(\mathcal{P})$ [33, Theorem 3.14], [46, Section 3]). *The followings hold for $\varphi(\mathcal{P})$ and $\mathcal{P}_j \in \mathcal{C}_\mathcal{P}$ in the PSP for $j \in \{1, \dots, p\}$.*

- (a) $\alpha_1 = \varphi(\mathcal{P}_1)$;
- (b) For any j such that $1 < j \leq p$, let $\alpha = \varphi(\mathcal{P}_j)$. Then, $\alpha_j < \alpha < \alpha_1$.

Based on Lemma A.5 and Theorem A.2, consider two situations for a partition $\mathcal{P}_j \in \mathcal{C}_\mathcal{P}$ where $j \in \{1, \dots, p\}$:

- If $j = 1$, then $\alpha_1 = \varphi(\mathcal{P}_j)$;
- If $j > 1$, then $\alpha = \varphi(\mathcal{P}_j)$ satisfies $\alpha_j < \alpha < \alpha_1$. Let $\mathcal{P}_{j'}$ be the minimal minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$. Then, $\mathcal{P}_1 \succ \mathcal{P}_{j'} \succ \mathcal{P}_j$.

They suggest a recursive method for determining \mathcal{P}_1 and α_1 . Consider the iteration

$$\alpha^{(n+1)} = \varphi(\mathcal{P}^{(n)}),$$

where $\mathcal{P}^{(n)}$ is the minimal minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_{\alpha^{(n)}}^\#[\mathcal{P}]$. Let the iteration start with $\alpha^{(0)} \leq \alpha_1$ and terminate when $\alpha^{(n+1)} = \alpha^{(n)}$. We necessarily have $\{\alpha^{(n)}\}$ and $\{\mathcal{P}^{(n)}\}$ converge to α_1 and \mathcal{P}_1 , respectively. Let the recursion terminate at N th iteration, where we have $\alpha^{(N+1)} = \alpha^{(N)}$. Here, $N \leq |V| - 1$ necessarily since, according to Theorem A.2, we have at most $|V| - 1$ critical values. According to Lemma A.5(b), $\alpha^{(n+1)} > \alpha^{(n)}$ and $\mathcal{P}^{(n+1)} \succ \mathcal{P}^{(n)}$ for all $n \in \{0, \dots, N\}$. Recall that $\alpha_1 = R_{ACO}(V)$ and $\mathcal{P}_1 = \mathcal{P}^*$ (Corollary A.3). The MDA algorithm exactly implements the recursion above with α initiated as the LB on $R_{ACO}(V)$ in Proposition IV.1. Therefore, Theorem V.1 holds. ■

The authors in [46, Section 3] [33, Algorithm II] proposed a decomposition algorithm (DA) for determining all partitions \mathcal{P}_j s and the corresponding critical values α_j s in the PSP. The MDA algorithm can be considered as an adapted version of the DA algorithm for determining just \mathcal{P}_1 and α_1 . Hence the name MDA.

APPENDIX B

TIGHT SETS AND PROOF OF OPTIMALITY OF COORDSATCAP ALGORITHM

According to Lemma III.2, $f_\alpha^\# : 2^V \mapsto \mathbb{R}$ is an intersecting submodular set function for all $\alpha \in \mathbb{R}_+$. In addition, $f_\alpha^\#$

is normalized, i.e., $f_\alpha^\#(\emptyset) = 0$, and $P(f_\alpha^\#, \leq) = P(\hat{f}_\alpha^\#, \leq)$ [26, Theorems 2.5(i) and 2.6(i)] where $\hat{f}_\alpha^\#$ is submodular [31, Theorem 12.2.4], [46, Theorem 7]. Consider the maximum sum-rate problem

$$\begin{aligned} \max\{r(V) : \mathbf{r}_V \in P(f_\alpha^\#, \leq)\} \\ = \max\{r(V) : \mathbf{r}_V \in P(\hat{f}_\alpha^\#, \leq)\}. \end{aligned} \quad (21)$$

The maximizers of $\max\{r(V) : \mathbf{r}_V \in P(\hat{f}_\alpha^\#, \leq)\}$ form the base polyhedron $B(\hat{f}_\alpha^\#, \leq)$ [26, Theorem 2.3]. Then, problem (21) is solved if a rate vector $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$ is determined. On the other hand, there is a relationship between (21) and the Dilworth truncation problem [37, Theorem 3.1]³⁶:

$$\max\{r(V) : \mathbf{r}_V \in P(f_\alpha^\#, \leq)\} = \min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}].$$

Then, for a rate vector $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$, its sum-rate $r(V)$ also determines the value of $\hat{f}_\alpha^\#(V) = \min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$.

Consider the CoordSatCap algorithm, which is originally proposed as Greedy Algorithm II in [26, Section 3.2]. Since the algorithm starts with a rate vector $\mathbf{r}_V \in P(f_\alpha^\#, \leq)$, it can be shown by induction that, we have $\mathbf{r}_V \in P(f_\alpha^\#, \leq)$ and $\mathbf{r}_V + \hat{\xi}_{\phi_i} \chi_{\phi_i} \in P(f_\alpha^\#, \leq)$ for all $i \in \{1, \dots, |V|\}$ (also refer to [26, proof of Theorem 3.19]). Since CoordSatCap algorithm outputs a point $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$ [26, Theorem 3.19], we finally have a rate vector $\mathbf{r}_V \in P(f_\alpha^\#, \leq)$ with saturation reached in each dimension, i.e., for all $i \in \{1, \dots, |V|\}$ and $\epsilon > 0$, $\mathbf{r}_V + \epsilon \chi_{\phi_i} \notin P(f_\alpha^\#, \leq)$ [26, Theorem 2.3]. It means $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$ with the sum-rate $r(V)$ being the maximum of $\max\{r(V) : \mathbf{r}_V \in P(f_\alpha^\#, \leq)\}$ or the minimum of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$. The minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ can be determined by the minimizers \hat{X}_{ϕ_i} for all i as follows.

Let \hat{X}_{ϕ_i} be the minimizer of the saturation capacity problem $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$ for $i \in \{1, \dots, |V|\}$ in the CoordSatCap algorithm. For i , let \mathbf{r}_V be the rate vector after executing $\mathbf{r}_V \leftarrow \mathbf{r}_V + \hat{\xi}_{\phi_i} \chi_{\phi_i}$. Then, \hat{X}_{ϕ_i} is \mathbf{r}_V -tight, i.e., $r(\hat{X}_{\phi_i}) = f_\alpha^\#(\hat{X}_{\phi_i})$ [26, proof of Theorem 3.19]. At the end of the CoordSatCap algorithm, we have $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$ and it can be shown by induction that \hat{X}_{ϕ_i} is \mathbf{r}_V -tight for all $i \in \{1, \dots, |V|\}$. Due to the intersecting submodularity of $f_\alpha^\#$, \hat{X}_{ϕ_i} s satisfy property: If $\hat{X}_{\phi_i} \cap \hat{X}_{\phi_j} \neq \emptyset$ for $i \neq j$, then $\hat{X}_{\phi_i} \cap \hat{X}_{\phi_j}$ and $\hat{X}_{\phi_i} \cup \hat{X}_{\phi_j}$ are also \mathbf{r}_V -tight [37, Section 3], [46, Section 4.2], [15, Lemma 6]. Then, consider the following process:

- Let $\mathcal{P}^* \leftarrow \{\hat{X}_{\phi_i} : i \in V\}$;
- Repeatedly merge any two elements $\hat{X}_{\phi_i}, \hat{X}_{\phi_j} \in \mathcal{P}^*$ such that $\hat{X}_{\phi_i} \cap \hat{X}_{\phi_j} \neq \emptyset$, i.e., do $\mathcal{P}^* \leftarrow (\mathcal{P}^* \setminus \{\hat{X}_{\phi_i}, \hat{X}_{\phi_j}\}) \cup \{\hat{X}_{\phi_i} \cup \hat{X}_{\phi_j}\}$, until there are no such elements left.

Since $\phi_i \in \hat{X}_{\phi_i}$ for all $i \in \{1, \dots, |V|\}$, we finally have \mathcal{P}^* being a partition of V , i.e. $\mathcal{P}^* \in \Pi(V)$ and each element in \mathcal{P}^* is \mathbf{r}_V -tight, i.e., $r(C) = f_\alpha^\#(C), \forall C \in \mathcal{P}^*$. Then, $r(V) =$

³⁶In [37], Theorem 3.1 is based on the polyhedron $P'(g'_\alpha, \leq) = \{\mathbf{r}_V \in \mathbb{R}^{|V|} : r(X) \leq g'_\alpha(X), \forall X \subseteq V, X \neq \emptyset\}$ with the submodular function $g'_\alpha(X) = \alpha - H(V) + H(X), \forall X \subseteq V$, a similar approach as in [15, Section III-E] that is explained in Remark III.6. Note, g'_α is not normalized. Therefore, it is easy to verify that $P(f_\alpha^\#, \leq) = P'(g'_\alpha, \leq)$, since $r(\emptyset) = 0$ always, and the maximum and maximizers of $\max\{r(V) : \mathbf{r}_V \in P(f_\alpha^\#, \leq)\}$ and $\max\{r(V) : \mathbf{r}_V \in P'(g'_\alpha, \leq)\}$ coincide.

$\sum_{C \in \mathcal{P}^*} r(C) = r[\mathcal{P}^*] = f_\alpha^\#[\mathcal{P}^*]$. Recall that $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$ at the end of the CoordSatCap algorithm, which means

$$r[\mathcal{P}^*] = \max\{r(V) : \mathbf{r}_V \in P(f_\alpha^\#, \leq)\} = \min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}].$$

Then, \mathcal{P}^* is a minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ and \mathcal{P}^* is the minimal minimizer if \hat{X}_{ϕ_i} is the minimal minimizer of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$ for all $i \in \{1, \dots, |V|\}$. The process above is equivalent to steps 6 and 7 in the CoordSatCap algorithm. There is another explanation of the optimality of the CoordSatCap algorithm in [46, Section 4.2].

A. Edmond Greedy Algorithm

It is shown in [26, Theorem 3.18] that, for a normalized submodular set function f , the CoordSatCap algorithm reduces to the Edmond greedy algorithm [25]: for $i = 1$ to $|V|$, do

$$r_{\phi_i} = f(V_i) - f(V_{i-1}),$$

where $V_i = \{\phi_1, \dots, \phi_i\}$ and $V_0 = \emptyset$. The resulting \mathbf{r}_V is a point in $B(f, \leq)$ [26, Corollary 3.17]. The Edmond greedy algorithm is modified as: For $i = 1$ to $|V|$, do

$$r_{\phi_i} \leftarrow r_{\phi_i} + \min\{f(X) - r(X) : \phi_i \in X \subseteq V_i\}$$

for solving the non-asymptotic minimum sum-rate problem in the finite linear source model and CCDE in [15], [18], which is exactly the CoordSatCap algorithm based on Lemma V.6.

B. Proof of Lemma V.3

Since the entropy function H is a polymatroid rank function [24], we have $\mathbf{0} \in P(H, \leq)$, i.e., $0 \leq H(X), \forall X \subseteq V$. Consider the rate vector $\mathbf{r}_V = (\alpha - H(V))\chi_V$. For $0 \leq \alpha \leq H(V)$, we have $\alpha - H(V) \leq 0$. Then, for $X = \emptyset$, we have $r(\emptyset) = 0 = f_\alpha^\#(\emptyset)$; for all $X \subseteq V$ such that $X \neq \emptyset$, we have

$$\begin{aligned} r(X) &= (\alpha - H(V))|X| \\ &\leq \alpha - H(V) \\ &\leq \alpha - H(V) + H(X) = f_\alpha^\#(X) \end{aligned}$$

So, $\mathbf{r}_V = (\alpha - H(V))\chi_V \in P(f_\alpha^\#, \leq)$. ■

C. Proof of Lemma V.6

Recall that the entropy function H is monotonic [24], i.e., $H(X) \leq H(Y), \forall X, Y \subseteq V : X \subseteq Y$. Then, $f_\alpha^\#(X) \leq f_\alpha^\#(Y), \forall X, Y \subseteq V : \emptyset \neq X \subseteq Y$. If $\mathbf{r}_V \in P(f_\alpha^\#, \leq)$ such that $r_i \leq 0, \forall i \in V$ initially, we have

$$\begin{aligned} f_\alpha^\#(X) - r(X) - (f_\alpha^\#(X \cap V_i) - r(X \cap V_i)) &= \\ f_\alpha^\#(X) - f_\alpha^\#(X \cap V_i) - r(X \setminus (X \cap V_i)) &\geq 0 \end{aligned}$$

holds for all $X \subseteq V$ such that $\phi_i \in X$. Therefore, equality (15) holds for all $i \in \{1, \dots, |V|\}$. Let \hat{X} be a minimizer of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V_i\}$. Due to the intersecting submodularity of $f_\alpha^\#$, whenever there is a minimizer \hat{Y} of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$ such that $\hat{Y} \not\subseteq V_i$, we have $\hat{Y} \cap \hat{X} \subseteq V_i$ and $\phi_i \in \hat{Y} \cap \hat{X} \neq \emptyset$. So, $\hat{Y} \cap \hat{X}$ is

also the minimizer of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$.³⁷ Therefore, if \hat{X}_{ϕ_i} is the minimal minimizer of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V_i\}$, it is also the minimal minimizer of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$. ■

D. Proof of Lemma V.7

For all $i \in \{1, \dots, |V|\}$, we have $r(C) = f_\alpha^\#(C), \forall C \in \mathcal{P}^* : |C| > 1$ in the CoordSatCap algorithm (due to the previous updates of \mathbf{r}_V and \mathbf{r}_V -tightness as stated in Appendix B) and, since $\mathbf{r}_V \in P(f_\alpha^\#, \leq)$, $r(C') + f_\alpha^\#(C \setminus C') \geq r(C') + r(C \setminus C') = f_\alpha^\#(C)$, i.e., $r(C') \geq f_\alpha^\#(C) - f_\alpha^\#(C \setminus C'), \forall C' \subsetneq C : C' \neq \emptyset, |C'| > 1$.

For any $X \subseteq V$, let $\mathcal{Y} = \{C \in \mathcal{P}^* : C \cap X \neq \emptyset\}$. We have

$$\begin{aligned} f_\alpha^\#(X) - r(X) - f_\alpha^\#(\tilde{\mathcal{Y}}) + r(\tilde{\mathcal{Y}}) &= \\ f_\alpha^\#(X) - f_\alpha^\#(\tilde{\mathcal{Y}}) + r(\tilde{\mathcal{Y}} \setminus X) &= \\ f_\alpha^\#(X) - f_\alpha^\#(\tilde{\mathcal{Y}}) + \sum_{C \in \mathcal{Y}} r(C \setminus (C \cap X)) &\geq \\ f_\alpha^\#(X) - f_\alpha^\#(\tilde{\mathcal{Y}}) + \sum_{C \in \mathcal{Y}} (f_\alpha^\#(C) - f_\alpha^\#(C \cap X)) &\geq 0, \end{aligned}$$

where the last inequality is due to the intersecting submodularity of $f_\alpha^\#$. The minimality of \tilde{U}_i^* over all $X \subseteq V$ such that $\phi_i \in X$ and $\tilde{\mathcal{X}} = \tilde{U}_i^*$ can be proved by the induction below.

Consider the set $\mathcal{X} = \{C \in \mathcal{P}^* : C \cap \hat{X}_{\phi_i} \neq \emptyset\}$. For all $C \in \mathcal{X} : |C| = 1$, we have $C \subseteq \hat{X}_{\phi_i}$ and $r(C \cup \hat{X}_{\phi_i}) + \hat{\xi}_{\phi_i} = r(\hat{X}_{\phi_i}) + \hat{\xi}_{\phi_i} = f_\alpha^\#(\hat{X}_{\phi_i})$; For all $C \in \mathcal{X} : |C| > 1$, we have $r(C) = f_\alpha^\#(C)$ and $r(\hat{X}_{\phi_i}) + \hat{\xi}_{\phi_i} = f_\alpha^\#(\hat{X}_{\phi_i})$ so that $r(C \cup \hat{X}_{\phi_i}) + \hat{\xi}_{\phi_i} = f_\alpha^\#(C \cup \hat{X}_{\phi_i})$ (This is also due to the \mathbf{r}_V -tightness as stated in Appendix B). By induction, we have $r(\hat{\mathcal{X}}) + \hat{\xi}_{\phi_i} = f_\alpha^\#(\hat{\mathcal{X}})$ and $\hat{\mathcal{X}} = \tilde{U}_i^*$, where \tilde{U}_i^* is the minimal minimizer of $\min\{f_\alpha^\#(\tilde{U}) - r(\tilde{U}) : U \subseteq \mathcal{P}^*, \phi_i \in \tilde{U}\}$. ■

E. Fundamental Partition \mathcal{P}^* noncrossing \hat{X}_{ϕ_i}

For $\alpha \in [0, R_{\text{ACO}}(V)]$,³⁸ let \hat{X}_{ϕ_i} be the minimal minimizer of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$ that is non-singleton. According to Theorem A.2(a), the minimal minimizer of $\min_{\mathcal{P} \in \Pi(V)} f_\alpha^\#[\mathcal{P}]$ must be a \mathcal{P}_j for some $j \in \{1, \dots, p\}$ in the PSP and, according to Appendix B, \mathcal{P}_j must be a multi-way cut that does not cross \hat{X}_{ϕ_i} . We also have $\alpha \geq f_\alpha^\#[\mathcal{P}_j]$ which is equivalent to $\alpha \leq \varphi(\mathcal{P}_j)$. For a \mathcal{P} that is crossing \hat{X}_{ϕ_i} , we have $f_\alpha^\#[\mathcal{P}_j] < f_\alpha^\#[\mathcal{P}]$, which is equivalent to

$$\varphi(\mathcal{P}) < (1 - \theta)\alpha + \theta\varphi(\mathcal{P}_j) \leq \varphi(\mathcal{P}_j),$$

where $\theta = \frac{|\mathcal{P}_j| - 1}{|\mathcal{P}| - 1}$. It means that for any partition \mathcal{P} that is crossing \hat{X}_{ϕ_i} there always exist a partition \mathcal{P}_j for some $j \in \{1, \dots, p\}$ in the PSP that is not crossing \hat{X}_{ϕ_i} such that $\varphi(\mathcal{P}) < \varphi(\mathcal{P}_j)$. Since $\mathcal{P}_j \preceq \mathcal{P}_1 = \mathcal{P}^*$ for all $j \in \{1, \dots, p\}$, the fundamental partition does not cross \hat{X}_{ϕ_i} , necessarily.

³⁷ Let \mathbf{r}_V be any rate vector in $P(f_\alpha^\#, \leq)$. The following property can be shown by the intersecting submodularity of $f_\alpha^\#$ [37, Section 3], [46, Section 4.2], [15, Lemma 6]: If \hat{X} and \hat{Y} are two distinctive minimizers of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$, then $\hat{X} \cap \hat{Y}$ and $\hat{X} \cup \hat{Y}$ are also the minimizers of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$. Note, this property is the same as the tight set argument in Appendix B for the proof of the optimality of the CoordSatCap algorithm.

³⁸ In the MDA algorithm, we always have $\alpha \in [0, R_{\text{ACO}}(V)]$.

APPENDIX C

PROOFS OF THEOREM V.11 AND COROLLARY V.14

According to Theorem III.8, $\hat{f}_\alpha^\#$ is a polymatroid rank function for all $\alpha \geq R_{\text{ACO}}(V)$. In this case, we have $B(\hat{f}_\alpha^\#, \leq) \subseteq \mathbb{R}_+^{|V|}$ so that $\mathbf{0} \leq \mathbf{r}'_V, \forall \mathbf{r}'_V \in \text{EX}(\hat{f}_\alpha^\#)$ [26, Lemma 3.23]. Based on Lemma V.3, for all $\alpha \leq H(V)$, if we start with a rate vector $\mathbf{r}_V \leq \mathbf{0} \leq \mathbf{r}'_V, \forall \mathbf{r}'_V \in \text{EX}(\hat{f}_\alpha^\#)$, we necessarily have $\mathbf{r}_V \in \text{EX}(\hat{f}_\alpha^\#)$ at the output of the CoordSatCap algorithm. This proves Theorem V.11. A similar proof can be found in [46, Section 4.2]. ■

As explained in Appendix B, for a given value of $\alpha \geq 0$, the CoordSatCap algorithm outputs $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq)$, where the sum-rate $r(V) = \hat{f}_\alpha^\#(V) = \min_{\mathcal{P} \in \Pi(V)} \hat{f}_\alpha^\#[\mathcal{P}]$. According to Theorem III.3, we have $r(V) = \alpha$ being the necessary and sufficient condition for a sum-rate α to be achievable. Also, if α is achievable, we have the output rate vector $\mathbf{r}_V \in B(\hat{f}_\alpha^\#, \leq) = B(\hat{f}_\alpha^\#, \leq) = \{\mathbf{r}_V \in \mathcal{R}_{\text{CO}}(V) : r(V) = \alpha\}$. Since the CoordSatCap and CoordSatCapFus algorithms return the same results, corollary also holds for the CoordSatCapFus algorithm. This proves Corollary V.14. ■

APPENDIX D

DIVIDE-AND-CONQUER (DC) ALGORITHM [45]

The authors in [45] proposed a divide-and-conquer (DC) algorithm (as shown in Algorithm 5) for solving the asymptotic minimum sum-rate problem based on the results as follows. For a subset $X \subseteq V$, let $R_{\text{ACO}}(X)$ and \mathcal{P}_X^* be the minimum sum-rate and the fundamental partition, respectively, for the asymptotic minimum sum-rate problem for attaining the omniscience in X . Then, for all optimal rate vectors $\mathbf{r}_X \in \mathcal{R}_{\text{ACO}}^*(X)$, we have $r(C) = R_{\text{ACO}}(X) - H(X) - H(C), \forall C \in \mathcal{P}_X^*$ [45, Lemma 1].³⁹ Then, for all $C \in \mathcal{P}_X^*$ such that $|C| = 1$, the optimal rate of the only user in C is determined; for all $C \in \mathcal{P}_X^*$ such that $|C| > 1$, the optimal rate can be determined by recursively solving the omniscience problem in C . Therefore, by knowing the fundamental partition \mathcal{P}^* for the omniscience in V , an optimal rate vector in $\mathcal{R}_{\text{ACO}}^*(V)$ can be solved by recursively breaking the non-singleton subset in the \mathcal{P}^* until the rates of all individual users are determined. This idea is implemented by the DC algorithm, where $R_{\text{ACO}}(X)$ and \mathcal{P}_X^* are determined by the DA algorithm in [46, Section 3]. In step 3 in the DC algorithm, Δr is the excessive rate, the additional rate that should be transmitted by X in addition to $R_{\text{ACO}}(V)$. Δr can be transmitted by any users in X after the omniscience is attained in X [45, Lemma 3].

Example D.1. By applying the DC algorithm to the system in Example III.11, we first determine that the minimum sum-rate $R_{\text{ACO}}(V) = \frac{13}{2}$ and fundamental partition $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$. We also know that we must have $r(\{1, 4, 5\}) = \frac{11}{2}$, $r(\{2\}) = \frac{1}{2}$ and $r(\{3\}) = \frac{1}{2}$ for attaining the omniscience in V with the minimum sum-rate $R_{\text{ACO}}(V) = \frac{13}{2}$. By recursively calling the DC algorithm, the individual rates in $\{1, 4, 5\}$ are determined as $r_1 = 1$,

³⁹Here, $R_{\text{ACO}}(X) - H(X) + H(C)$ corresponds to $\hat{f}_{R_{\text{ACO}}(V)}^\#(C) = R_{\text{ACO}}(V) - H(V) + H(C)$ in that X is the ground set for the omniscience problem in X .

Algorithm 5: divide-and-conquer (DC) algorithm [45]

input : a subset $X \subseteq V$, and $r(X)$ if $X \neq V$
output: a rate vector $\mathbf{r}_X \in \mathcal{R}_{\text{ACO}}^*(X)$

- 1 $(R_{\text{ACO}}(X), \mathcal{P}_X^*) \leftarrow \text{DA}(\emptyset, X)$;
- 2 **forall** $C \in \mathcal{P}_X^*$ **do** $r(C) \leftarrow R_{\text{ACO}}(X) - H(X) + H(C)$;
- 3 **if** $X \neq V$ **then** $\Delta r \leftarrow r(X) - R_{\text{ACO}}(X)$, choose any $C \in \mathcal{P}_X^*$ and let $r(C) \leftarrow r(C) + \Delta r$;
- 4 **forall** $C \in \mathcal{P}_X^* : |C| > 1$ **do** $\mathbf{r}_C \leftarrow \text{DC}(C, r(C))$;

$r_4 = \frac{5}{2}$ and $r_5 = 0$ so that we have the optimal rate vector $\mathbf{r}_V = (1, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, 0) \in \mathcal{R}_{\text{ACO}}^*(V)$ at the output.

Since the DA algorithm completes in $O(|V|^2 \cdot \text{SFM}(|V|))$ time [46, Theorem 5], the complexity of the DC algorithm is $O(|V|^3 \cdot \text{SFM}(|V|))$. However, the recursive splitting of the non-singleton subset in the fundamental partition \mathcal{P}^* in the DC algorithm is not necessary since an optimal rate vector $\mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)$ is obtained when \mathcal{P}^* is determined by the DA algorithm.⁴⁰

APPENDIX E

PROOF OF THEOREM VI.1

According to Theorem V.11, the MDA algorithm returns an optimal rate vector $\mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)$ that is an extreme point in $B(\hat{f}_{R_{\text{ACO}}(V)}^\#, \leq)$, i.e., $\mathbf{r}_V \in \text{EX}(\hat{f}_{R_{\text{ACO}}(V)}^\#)$. Based on [26, Corollary 3.17], for linear ordering w.r.t. \mathbf{w}_V , the output \mathbf{r}_V is also the minimizer of $\min\{\mathbf{w}_V^\top \mathbf{r}_V : \mathbf{r}_V \in \mathcal{R}_{\text{ACO}}^*(V)\}$, where $\mathcal{R}_{\text{ACO}}^*(V) = B(\hat{f}_{R_{\text{ACO}}(V)}^\#, \leq)$. It is shown in the proof of Corollary V.12 that, for a finite linear source model, when the input $\alpha = R_{\text{NCO}}(V)$, the CoordSatCapFus algorithm outputs a rate vector $\mathbf{r}_V \in \text{EX}(\hat{f}_{R_{\text{NCO}}(V)}^\#)$, which, according to [26, Corollary 3.17] is the minimizer of $\min\{\mathbf{w}_V^\top \mathbf{r}_V : \mathbf{r}_V \in \mathcal{R}_{\text{NCO}}^*(V)\}$ where $\mathcal{R}_{\text{NCO}}^*(V) = B(\hat{f}_{R_{\text{NCO}}(V)}^\#, \leq)$. Since the outputs of the CoordSatCapFus and CoordSatCap algorithms are the same, it also applies to the CoordSatCap algorithm. ■

APPENDIX F

PROOFS OF THEOREM VII.2 AND COROLLARY VII.6

The proof of Theorem VII.2 relies on the method for determining the minimal separators of a submodular function that is proposed in [26, Lemma 3.41] [49, Theorems 3.1 and 3.2].⁴¹

Lemma F.1 ([26, Lemma 3.41]). *For any rate vector $\mathbf{r}_V \in B(\hat{f}_{R_{\text{ACO}}(V)}^\#, \leq)$, let \hat{X}_i be the minimal minimizers of $\min\{\hat{f}_{R_{\text{ACO}}(V)}^\# - r(X) : i \in X \subseteq V\}$. Initiate the minimal*

⁴⁰In fact, the DC algorithm utilizes $R_{\text{ACO}}(V)$ and \mathcal{P}^* determined by the DA algorithm in [46, Section 3] while discards the optimal rate vector $\mathbf{r}_V \in B(\hat{f}_{R_{\text{ACO}}(V)}^\#, \leq)$. This is not surprising since the study in [46] aims to solve a clustering problem, where the optimal partition \mathcal{P}^* is of the most interest. Therefore, although a rate vector $\mathbf{r}_V \in B(\hat{f}_{R_{\text{ACO}}(V)}^\#, \leq)$ is returned as an auxiliary result, it is not explicitly stated in [46] that $B(\hat{f}_{R_{\text{ACO}}(V)}^\#, \leq) = \mathcal{R}_{\text{ACO}}^*(V)$ so that this rate vector is the solution to the asymptotic minimum sum-rate problem.

⁴¹This method was first proposed in [49, Theorems 3.1 and 3.2] for polymatroid rank functions and then generalized to [26, Lemma 3.41] for submodular functions. Here, the minimal separators also correspond to the principal structure of a submodular system in [59].

minimizer set $\mathcal{P}^* = \{\hat{X}_i : i \in V\}$ and repeatedly merge any two distinctive elements $\hat{X}_i, \hat{X}_j \in \mathcal{P}^*$ such that $\hat{X}_i \cap \hat{X}_j \neq \emptyset$, i.e., do $\mathcal{P}^* \leftarrow (\mathcal{P}^* \setminus \{\hat{X}_i, \hat{X}_j\}) \cup \{\hat{X}_i \cup \hat{X}_j\}$, until there are no such elements left. The resulting \mathcal{P}^* is the set of minimal separators of $\hat{f}_{R_{ACO}(V)}^\#$.

Lemma F.2. Let Φ be any linear ordering of V . For all $\alpha \in \mathbb{R}_+$, $i \in \{1, \dots, |V|\}$ and $\mathbf{r}_V \in P(f_\alpha^\#, \leq)$, if $\hat{\xi}_{\phi_i}$ and \hat{X}_{ϕ_i} are the minimum and minimal minimizer of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$, respectively, then $\hat{\xi}_{\phi_i}$ and \hat{X}_{ϕ_i} are also the minimum and minimal minimizer of $\min\{\hat{f}_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$, respectively.

Proof. Let $\mathbf{r}'_V = \mathbf{r}_V + \hat{\xi}_{\phi_i} \chi_{\phi_i}$. Then, we have $\mathbf{r}'_V \in P(f_\alpha^\#, \leq)$ and \hat{X}_{ϕ_i} is \mathbf{r}'_V -tight, i.e., $f_\alpha^\#(\hat{X}_{\phi_i}) = r'(\hat{X}_{\phi_i})$ [26, proof of Theorem 3.19]. Since $P(f_\alpha^\#, \leq) = P(\hat{f}_\alpha^\#, \leq)$, $\mathbf{r}'_V \in P(\hat{f}_\alpha^\#, \leq)$ and we have $r'(\hat{X}_{\phi_i}) \leq \hat{f}_\alpha^\#(\hat{X}_{\phi_i})$ [26, Theorems 2.5(i) and 2.6(i)]. On the other hand, $\hat{f}_\alpha^\#(X) \leq f_\alpha^\#(X)$ for all $X \subseteq V$ by the definition of the Dilworth truncation. Then, $\hat{f}_\alpha^\#(\hat{X}_{\phi_i}) \leq f_\alpha^\#(\hat{X}_{\phi_i}) = r'(\hat{X}_{\phi_i})$. Therefore, $r'(\hat{X}_{\phi_i}) = \hat{f}_\alpha^\#(\hat{X}_{\phi_i})$, which means $\hat{\xi}_{\phi_i}$ is also the minimum of $\min\{\hat{f}_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$.

We prove that \hat{X}_{ϕ_i} is the minimal minimizer of $\min\{\hat{f}_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$ by contradiction. Assume that \hat{X}_{ϕ_i} is not the minimal minimizer, i.e., there exists $X \subsetneq \hat{X}_{\phi_i}$ such that $\phi_i \in X$ and $\hat{f}_\alpha^\#(X) - r(X) = \hat{\xi}_{\phi_i}$. Then, $r'(X) = \hat{f}_\alpha^\#(X) < f_\alpha^\#(X)$, where the last inequality is strict due to the fact that X is not the minimizer of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$. According to the definition of the Dilworth truncation, $\hat{f}_\alpha^\#(X) < f_\alpha^\#(X)$ means that there exists a $\mathcal{P}^* \in \Pi'(X)$ such that

$$r'(X) = \hat{f}_\alpha^\#(X) = f_\alpha^\#[\mathcal{P}^*] = r'[\mathcal{P}^*].$$

On one hand, we have $r'(C) \leq f_\alpha^\#(C), \forall C \in \mathcal{P}^*$ since $\mathbf{r}'_V \in P(f_\alpha^\#, \leq)$. On the other hand, $f_\alpha^\#(C) - r'(C) = \sum_{C' \in \mathcal{P}^* \setminus \{C\}} (r'(C') - f_\alpha^\#(C')) \leq 0$, i.e., $r'(C) \geq f_\alpha^\#(C)$ for all $C \in \mathcal{P}^*$. Then, $r'(C) = f_\alpha^\#(C)$ for all $C \in \mathcal{P}^*$. Let $\hat{C} \in \mathcal{P}^*$ such that $\phi_i \in \hat{C}$. We have $f_\alpha^\#(\hat{C}) = r'(\hat{C})$, i.e., $f_\alpha^\#(\hat{C}) - r(\hat{C}) = \hat{\xi}_{\phi_i}$. Here, $\hat{C} \subsetneq X$ necessarily, which means \hat{X}_{ϕ_i} is not the minimal minimizer of $\min\{f_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$. This contradicts the given condition. Therefore, we must have \hat{X}_{ϕ_i} being the minimal minimizer of $\min\{\hat{f}_\alpha^\#(X) - r(X) : \phi_i \in X \subseteq V\}$. ■

Recall that we have $\mathbf{r}_V \in B(\hat{f}_{R_{ACO}(V)}^\#, \leq)$ and the fundamental partition \mathcal{P}^* at the output of the CoordSatCap algorithm by inputting $\alpha = R_{ACO}(V)$. For the base point $\mathbf{r}_V \in B(\hat{f}_{R_{ACO}(V)}^\#, \leq)$, \hat{X}_i is the minimal minimizer of $\min\{\hat{f}_{R_{ACO}(V)}^\#(X) - r(X), i \in X \subseteq V\}$ for all $i \in V$ according to Lemma F.2. If we implement the method in Lemma F.1 over all \hat{X}_i s, we have the set of minimal separators of $\hat{f}_{R_{ACO}(V)}^\#$ the same as the fundamental partition \mathcal{P}^* . This proves Theorem VII.2. ■

Corollary VII.6 is proved as follows. For all $C \in \mathcal{P}^*$, $g(\mathbf{r}_C^*) \leq g(\mathbf{r}_C^* + \epsilon(\chi_i - \chi_j)), \forall i, j \in C$ according to Theorem VII.5. On the other hand, according to property (c) in Lemma VII.3, if $i \in C$ and $i \in C'$ for any $C, C' \in \mathcal{P}^*$ such that $C \neq C'$, $\mathbf{r}_V + \epsilon(\chi_i - \chi_j) \notin B(\hat{f}_{R_{ACO}(V)}^\#, \leq) = \mathcal{R}_{ACO}(V)$ for all $\epsilon > 0$. Then, we have $g(\mathbf{r}_V^*) \leq$

$g(\mathbf{r}_V^* + \epsilon(\chi_i - \chi_j))$ for all $i, j \in V$ and $\epsilon > 0$ such that $\mathbf{r}_V + \epsilon(\chi_i - \chi_j) \in B(\hat{f}_{R_{ACO}(V)}^\#, \leq) = \mathcal{R}_{ACO}(V)$. Therefore, according to Theorem VII.5, $\mathbf{r}_V^* = \bigoplus_{C \in \mathcal{P}^*} \mathbf{r}_C^*$ is the minimizer of $\min\{g(\mathbf{r}_V) : \mathbf{r}_V \in B(\hat{f}_{R_{ACO}(V)}^\#, \leq) = \mathcal{R}_{ACO}(V)\}$. ■

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