arXiv:1611.00508v3 [math.DS] 1 Oct 2017

Co-existence of maximal and whiskered tori in the planetary three-body problem

Gabriella Pinzari[1, a)]

Università di Padova, Dipartimento di Matematica

We prove the *coexistence* of stable and unstable quasi–periodic kam tori in a region of phase space of the three-body problem. The proof goes along the production of two non smoothly related systems of canonical coordinates in the same region of the phase space, the possibility of which is foreseen, for "properly–degenerate" systems, by a theorem of Nekhorossev and Miščenko and Fomenko. The two coordinate systems are alternative to the classical reduction of the nodes by Jacobi, described, e.g., in Ref.[3] (III,§5, n. 4, p. 141).

PACS numbers: 02. Mathematical methods in physics

a) gabriella.pinzari@math.unipd.it

I. INTRODUCTION

At the end of the XIX century, motivated by the study of a three–body problem of celestial mechanics, H. Poincaré conjectured that a non-integrable system possesses, very close one to the other, stable and unstable trajectories, Ref.[31] (Vol III, Chapter 33, n. 397, p. 389). Numerical evidence of such an occurrence has been provided, since the 50s, as soon as computers could be used to simulate solutions of differential equations, by breakthrough papers by Fermi–Pasta–Ulam, Hénon–Heiles, Izrailev–Chirikov, ... The phenomenon was soon understood to be relevant for physics, since such papers revealed its appearance even in regular (e.g. analytic) systems. Notwithstanding efforts of the recent Aubry–Mather[36] theory, it seems that rigorous proofs of coexistence of order and chaos is still missing, at least for analytic physical systems with more than 2 degrees of freedom. In this paper, we address the question in the case of the celestial three–body problem; precisely, its *planetary* version. This is the 4 degrees of freedom problem of three point masses interacting through gravity, where one of the masses (the "star") is much larger than the two others (the "planets"). We move in the framework of Kolmogorov–Arnold–Moser (kam) theory; Refs[2,18,23] and Note[41]. kam theory has been successfully applied to problem of celestial mechanics it since the 60s. Under the point of view of kam theory, the question might be rephrased as wether one can prove coexistence, in a region of phase space, of quasi–periodic motions, both *maximal* and *whiskered*[42] possibly separated, according to Poincaré's picture and Aubry–Mather theory, by chaotic regions. This is precisely the question this paper is addressed to: we determine a connected region in the domain of analyticity of the planetary three–body problem Hamiltonian where such co–existence indeed occurs.

We provide a simple statement of the result. (more precise formulations will be given in Theorems II.1, III.1, and IV.1). We consider the following situation, which we shall refer to as *outer retrograde configuration* (ORC): *two planets describe almost co-planar orbits. The outer planet has a retrograde motion*[43]*.* Then we shall prove:

Theorem A

($\mathcal{S}$) *There exists an eight-dimensional open region $\widehat{\mathcal{D}}_{\mathcal{S}}$ of phase space contained in the analyticity domain of the Hamiltonian almost completely filled with a positive measure set of quasi–periodic motions with four incommensurate frequencies. The motions on such tori are in* ORC*;*

($\mathcal{U}$) *there exists a six-dimensional invariant surface $\widehat{\mathcal{D}}_{\mathcal{U}}$ of phase space almost-completely filled with unstable quasi–periodic motions with three frequencies, with motions in* ORC*;*

($\mathcal{SU}$) *the region $\widehat{\mathcal{D}}_{\mathcal{S}}$ and the surface $\widehat{\mathcal{D}}_{\mathcal{U}}$ have a non-empty intersection.*

We summarize the thesis in ($\mathcal{SU}$) by saying, with a rough expression, that the full-dimensional tori in ($\mathcal{S}$) and the six-dimensional tori in ($\mathcal{U}$) "co-exist". We underline that the interest of Theorem A relies on its address at the three–body problem Hamiltonian in an analyticity domain. It is possible to construct examples[44] of coexistence of maximal and hyperbolic tori in smooth and even analytic systems.

The proof of Theorem A is based on two tools, one (but not the most original) of which is kam theory. As it is usual in kam theory, by "almost-completely filled" one means "with relative density going to one as a small parameter goes to zero". Indeed, the statements of Theorems II.1, III.1 provide an estimate of how the densities two families of tori in $\widehat{\mathcal{D}}_{\mathcal{S}}$, $\widehat{\mathcal{D}}_{\mathcal{U}}$ increase as soon as certain small parameters, characteristic of the system, decrease. In the case ($\mathcal{S}$), the quasi–periodic motions are more and more dense in $\widehat{\mathcal{D}}_{\mathcal{S}}$ as soon as the maximum among the eccentricities and the mutual inclinations of the instantaneous Keplerian ellipses is small. I the case ($\mathcal{U}$), the density of quasi–periodic motions in $\widehat{\mathcal{D}}_{\mathcal{U}}$ increases with the ratio between the semi–major axes. Finally, motions that, accordingly to the thesis ($\mathcal{SU}$), belong to the intersection of $\widehat{\mathcal{D}}_{\mathcal{S}}$ and $\widehat{\mathcal{D}}_{\mathcal{U}}$, besides having small eccentricities, mutual inclination and semi–axes ratio, also need to have a mutual inclination, even though away from vanishing.

Let us now describe the main tool of proof. It is known that the *two-body problem*, i.e., the problem of the motions of two point masses interacting via a law proportional to their inverse squared distance, has, for an open set of initial conditions, *periodic motions* rather than, more generally, *quasi-periodic*. For this system, periodicity consists in the fact that the bounded motions evolve (according to Kepler's laws) on ellipses, and are governed by just *one* frequency ν proportional to $a^{-3/2}$ where a is the semi-major axis of the ellipse. This pretty remarkable fact unavoidably reflects – as already underlined by V. I. Arnold in his 1963's paper Ref.[3] – on the study of the dynamics of the so-called *planetary problem*, i.e., the problem of $(1 + N)$ point masses, one of which ("sun") is of "order one", while the remaining N ("planets") are of much smaller size, interacting through gravity. Indeed, after suitable rescaling, the planetary problem reduces to N uncoupled two–body problems when the masses of the planets are set to zero ("unperturbed problem"). The consequent lack of frequencies in the unperturbed problem was named by Arnold *proper degeneracy*. It represented a serious difficulty, if one wanted (as he was aiming to do) to apply Kolmogorov's theorem, Ref.[18] to the planetary problem.

At a technical level, the appearance of the proper degeneracy consists, we might say, of a "loss of frequencies", caused by the "too many" (or, better *Poisson non commuting*[46], see below) first integrals of motion. For such abundance, this kind of systems is often called *super–integrable*. Despite of the fact that the solutions of the two-body problem are known since Newton's times, a general, theoretical setting clearly explaining the phenomenon has been given only recently, thanks to the works by Nekhorossev and Miščenko and Fomenko, Refs.[21,24] (hereafter, nmf). The three authors proved a generalization of the best known Liouville-Arnold theorem, Ref.[1] which clearly relates the loss of frequencies to the existence of Poisson non commuting independent integrals. They proved that, to an integrable Hamiltonian system with n degrees of freedom which, in addition to n independent and commuting first integrals, affords additional $n_1 \leq n$ integrals which do not commute with *all* the integrals of the first family, one can associate canonical coordinates including only $n_0 := n - n_1$ action–angle couples $(\mathrm{I}, \varphi) = (\mathrm{I}_1, \cdots, \mathrm{I}_{n_0}, \varphi_1, \cdots, \varphi_{n_0})$ (analogous to the ones of Arnold–Liouville case), and, in addition, certain other couples $z = (p, q) = (p_1, \cdots, p_{n_1}, q_1, \cdots, q_{n_1})$, usually referred to as *degenerate* coordinates. The degenerate coordinates *are not uniquely defined*, and this is precisely the aspect that, in this paper, we shall exploit.

Indeed, a dynamical system that is close to a super-integrable system may be written as

$$\mathrm{H}(\mathrm{I}, \varphi, p, q) = \mathrm{h}(\mathrm{I}) + \mu f(\mathrm{I}, \varphi, p, q) \tag{1}$$

where $(\mathrm{I}, \varphi, p, q)$ is one of the various (as foreseen by nmf Theorem) sets of canonical coordinates associated with the unperturbed super-integrable term h. Now, while, given the I's, h is uniquely determined, the form of f, instead, strongly depends on the choice of coordinates. On the other hand, it is known since Arnold's paper Ref.[3] that, for system of the form (1), f may have a strategic importance.

As an outstanding example, let us recall just the case considered by Arnold in Ref.[3]. He wanted to prove (via an application of Kolmogorov's theorem) the existence of plenty of quasi–periodic, maximal tori, forming a positive measure set in phase space. He announced the result (known as "Arnold Theorem") at the 1962 ICM. Clearly, such result was going in the direction of the proof of stability of the Solar System, and for this Kolmogorov and Arnold were awarded, in 1965, of the Lenin prize. However, in order to obtain such result he was aware that he had to overcome the problem of the lack of frequencies in the unperturbed part (indeed succeeding in this), but this was not the only one. As for the choice of coordinates, Arnold considered, in the case of the planar problem, *Poincaré coordinates*, as described in Ref.[3] (Chapter III, §2, n.4). In term of such coordinates, the Hamiltonian of the planetary problem takes the form in (1), with n_0 equal to the number of planets N, $n_1 = N$ (so that the total number of degrees of freedom in $\mathbf{R}^2$ is $2N$), $(\mathrm{I},\ \varphi) := (\Lambda, \lambda) \subset \mathbf{R}^N \times \mathbf{T}^N$ (where $\mathbf{T} := \mathbf{R}/(2\pi\mathbf{Z})$) suitable action–angle couples related to the semi–major axis and the area spanned by the ellipse, $z = (p, q) := (\eta, \xi) \subset \mathbf{R}^N \times \mathbf{R}^N$ suitable degenerate coordinates related to the orientation of such ellipses, $\mathrm{h} = \mathrm{h}_{\mathrm{K}}$ the Keplerian Hamiltonian; μ a small a–dimensional parameter measuring the maximum planet/star mass ratio and, finally, $f(\mathrm{I}, \varphi, p, q) = f_{\mathrm{Poin}}(\Lambda, \lambda, \eta, \xi)$ a perturbing function related to the small mutual interactions among planets. Arnold observed that the average value $\overline{f_{\mathrm{Poin}}}(\Lambda, \eta, \xi)$ with respect to the λ's of the perturbing function $f_{\mathrm{Poin}}(\Lambda, \lambda, \eta, \xi)$ by symmetry reasons, has an *elliptic equilibrium point* for $(\eta, \xi) = 0$ (corresponding to circular motions of the planets around their sun), for all Λ. So he managed to construct, for degenerate systems of the form (1) with $\overline{f}(I, z)$ having an elliptic equilibrium in $z = 0$ for all I, a careful version of Kolmogorov Theorem, Ref.[3] (Fundamental Theorem, see Appendix A 2) which, after checking a certain non–degeneracy condition going back to A.N.Kolmogorov[18], allows to prove the existence of an invariant set with positive Lebesgue measure in phase space including only quasi-periodic motions. Arnold successfully applied his Fundamental Theorem to the case of the planar problem with $N = 2$ planets. However, while the extension to the planar problem with a generic number of planets revealed to be straightforward, Ref.[28] (see Ref.[11] for a previous result with a different strategy), the treatment of the problem in space contained strong extra-difficulties. Indeed, switching from planar to spatial Poincaré coordinates, the averaged perturbing function $\overline{f_{\mathrm{Poin}}}$ still exhibits an elliptic equilibrium in correspondence *circular and co–planar* motions, but such equilibrium is *degenerate*, in the sense that the eigenvalues of the quadratic part of $\overline{f_{\mathrm{Poin}}}$ verify, identically, two linear combinations with integer coefficients (known in the field as *secular resonances*). A fact strongly preventing, in principle, the possibility of checking Kolmogorov's condition. But this is not all: trying to compute, at least formally, Arnold's torsion with spatial Poincaré coordinates, one can prove (Ref.[7]) that such torsion identically vanishes, a fact already suspected by M. Herman, Ref.[15].

It may be argued that Arnold felt that a difficulty of this kind might appear, since, without explaining his motivations, in Ref.[3] (p. 141–42), he suggested to "change coordinates", without going further. Completion of the proof of his theorem revealed it to be more difficult than expected, and the story reached a conclusion only fifty years later, thanks to contributions by J. Laskar, P. Robutel, M. Herman, J.Féjoz, L. Chierchia and the author, Refs.[8,11,19,28,34]. Comprehensive reviews appeared in Refs.[9,12], to which papers we refer the interested reader. For the purposes of this paper we only mention that the solution Arnold had in mind, based on changing coordinates was considered, formally, in a particular case, by Malige, Robutel and Laskar, Ref.[20], and next completely achieved by the author, Ref.[28], published in Refs.[8,28]. The long proof of Arnold Theorem should give, we hope, an idea that, from a practical point of view, producing "good" canonical coordinates, which should: (i) leave the unperturbed part unvaried; (ii) overcome the degeneracies caused by SO(3) invariance and, eventually, (iii) preserve symmetries, parities, equilibria ... from which to depart in order to apply a perturbative scheme (e.g., in the case of Arnold Theorem, the Fundamental Theorem developed around the elliptic equilibrium), is other than "easy" or "straightforward".

In this paper, we make a different use the non-uniqueness of coordinates. For the three-body problem in the ORC configuration, we exhibit *two* sets of canonical coordinates, for a *common* region of the phase space which includes ORC motions, spatial and co–planar. The former of such two systems of coordinates is a modification- suited to the ORC configuration– of the rps coordinates proposed in Refs.[8,28]. The latter, called *perihelia reduction* has been proposed in Ref.[27], where it has been used to deal with the prograde configuration. The manifolds consisting of co–planar motions are invariant. Such invariant manifolds reduce to be equilibria for the respective averaged Hamiltonians, parametrized by the value of the remaining action coordinates. The crucial fact is that one can find a domain of such parametrizing actions of the first set of coordinates where the planar equilibrium has elliptic character and a domain of the second set, having non–empty intersection with the former, where the planar equilibrium is hyperbolic. Here is precisely the reason why we chose to focus on the case of *retrograde motion*: when the two planet turn in the same verse, the two natures (elliptic and hyperbolic) of the equilibrium does not coexist. This is also the main ingredient of the proof of Theorem A. We remark, as a conclusion, that the framework of the paper has nothing to do with the case studied in Ref.[17], where the authors found whiskered tori, in a completely different physical configuration (the mutual inclination is required to be higher than $\sim 40^{\rm o}$), without proving coexistence with maximal tori.

II. SET UP

In impulse-position coordinates $\mathcal{C}_{art} = (y, x) = (y^{(1)}, y^{(2)}, x^{(1)}, x^{(2)})$, the Hamiltonian of the three-body problem, after the translation invariance has been reduced according to the heliocentric method, is

$$\mathrm{H}(y,x) = \frac{|y^{(1)}|^2}{2\mathrm{m}_1} - \frac{\mathrm{m}_1\mathrm{M}_1}{|x^{(1)}|} + \frac{|y^{(2)}|^2}{2\mathrm{m}_2} - \frac{\mathrm{m}_2\mathrm{M}_2}{|x^{(2)}|} + \mu\Big(-\frac{m_1 m_2}{|x^{(1)} - x^{(2)}|} + \frac{y^{(1)}\cdot y^{(2)}}{m_0}\Big) \tag{2}$$

where m_0 is the mass of the star, μm_1, μm_2 those of the planets, with μ a very small number;

$$\mathrm{m}_i := \frac{m_0 m_i}{m_0 + \mu m_i} = m_i + \mathrm{O}(\mu) \qquad \mathrm{M}_i := m_0 + \mu m_i = m_0 + \mathrm{O}(\mu) \tag{3}$$

are the "reduced masses"; $y^{(i)} \in \mathbf{R}^3$, $x^{(i)} \in \mathbf{R}^3$, and the collision set

$$\Delta := \Big\{x^{(1)} = 0, \text{or } x^{(2)} = 0, \text{or } x^{(1)} = x^{(2)}\Big\}$$

is to be excluded. At this stage, H still exhibits six degrees of freedom, being no longer translation invariant but still SO(3) invariant. Along its trajectories, indeed, the three components of total angular momentum

$$\mathrm{C} = \mathrm{C}^{(1)} + \mathrm{C}^{(2)} \qquad \text{with} \qquad \mathrm{C}^{(i)} := x^{(i)} \times y^{(i)}$$

are preserved. As mentioned in the introduction, it is known since the classical reduction of the nodes by Jacobi (later rewritten in canonical form by Radau, Refs.[16,33]) that the complete reduction of its integrals would allow us to lower the number of degrees of freedom to *four*. Neglecting the expression inside parentheses, weighted by μ, the problem reduces to the sum of the Hamiltonians

$$\frac{|y^{(i)}|^2}{2\mathrm{m}_i} - \frac{\mathrm{m}_i\mathrm{M}_i}{|x^{(i)}|} \tag{4}$$

corresponding to the two-body problem interactions, after translation invariance has been carried on. As said, the integration of any such Hamiltonian leads to the properly degenerate, one-dimensional "Keplerian" Hamiltonian

$$\mathrm{h}_{\mathrm{k}}^{(i)}(\Lambda_i) := -\frac{\mathrm{m}_i^3\mathrm{M}_i^2}{2\Lambda_i^2}. \tag{5}$$

where Λ_i is related to the semi–major axis a_i of the Keplerian ellipse via

$$\Lambda_i = \mathrm{m}_i\sqrt{\mathrm{M}_i a_i}\ . \tag{6}$$

We emphasize the proper degeneracy using the following notion, already introduced in Ref.[27]. We call *Kepler map* any canonical change of coordinates

$$\mathrm{k} = (\Lambda_1, \Lambda_2, \ell_1, \ell_2, u, v) \in \mathcal{L}\times\mathbf{T}^2\times V \to (y^{(1)}_{\mathrm{k}}, y^{(2)}_{\mathrm{k}}, x^{(1)}_{\mathrm{k}}, x^{(2)}_{\mathrm{k}}) \in \mathbf{R}^{12} \tag{7}$$

such that

$$\frac{|y^{(i)}_{\mathrm{k}}|^2}{2\mathrm{m}_i} - \frac{\mathrm{m}_i\mathrm{M}_i}{|x^{(i)}_{\mathrm{k}}|} = \mathrm{h}^{(i)}_{\mathrm{k}}(\Lambda_i) \qquad i = 1, 2,$$

where $\mathcal{L}\subset\mathbf{R}^2$, $V\subset\mathbf{R}^8$ are open and connected sets, and $\mathrm{h}^{(i)}_{\mathrm{k}}$ is as in (5). Different Kepler maps are related by canonical changes

$$\mathrm{k} = (\Lambda_1, \Lambda_2, \ell_1, \ell_2, u, v) \to \mathrm{k}' = (\Lambda_1, \Lambda_2, \ell'_1, \ell'_2, u', v')$$

which leave the Λ_i's unvaried. In terms of any Kepler map the Hamiltonian (2) takes the aspect

$$\mathrm{H}_{\mathrm{k}} = \mathrm{h}_{\mathrm{k}}(\Lambda_1, \Lambda_2) + \mu f_{\mathrm{k}}(\Lambda_1, \Lambda_2, \ell_1, \ell_2, u, v)$$

where

$$\mathrm{h}_{\mathrm{k}}(\Lambda_1, \Lambda_2) = -\frac{\mathrm{m}_1^3\mathrm{M}_1^2}{2\Lambda_1^2} - \frac{\mathrm{m}_2^3\mathrm{M}_2^2}{2\Lambda_2^2}, \qquad f_{\mathrm{k}} = -\frac{m_1 m_2}{|x^{(1)}_{\mathrm{k}} - x^{(2)}_{\mathrm{k}}|} + \frac{y^{(1)}_{\mathrm{k}}\cdot y^{(2)}_{\mathrm{k}}}{m_0}. \tag{8}$$

A. Maximal tori

1. Generalities

The existence of a positive measure set of Lagrangian tori with maximal number of frequencies for the general planetary problem, in the regime of well spaced orbits, small eccentricities and small inclinations, has been established in the papers Refs.[3,8,11,19,27–29]. We refer to such technical papers for details, to Refs.[9,12] for reviews. However, the cases treated in the literature above, even though containing all the necessary information, are not perfectly suited to the proof of the $(\mathcal{S})$–part of Theorem A.

The papers in Refs.[3,8,11,19,28,34] deal with maximal, fully or partially reduced (see the discussion after Equation (25) and Note[39], for the definition of full, partial reduction in the particular case of ORC motions) quasi–periodic tori in the case when the planets revolve all in the same verse, and eccentricities and inclinations are small. The invariant set (so-called "*Kolmogorov-set*") is proved to fill almost completely the overall phase space, up to a residual set with measure going to zero with the planetary masses (once the mass of the sun has been fixed to 1), the maximum of eccentricities and inclinations. A (maybe optimal) estimate about the strength at which such measure goes to zero is provided in Refs.[8,28].

In Refs.[27,29] maximal, fully reduced quasi–periodic tori have been constructed out of the small eccentricities and inclination constraint. In such papers, the measure of the Kolmogorov-set has been found to increase while the masses decrease and the mutual semi–major axes ratios increase, independently of the values of eccentricities and inclinations. A suitable constraint on the semi–axes ratios is however imposed. Finally, the sense of revolution of the planets is the same for all of them.

The first study including the analysis of the retrograde case appeared in Refs.[25,26], where all relative equilibria giving rise to five-dimensional manifolds foliated by four–dimensional maximal tori are analyzed. However, the estimate of the Kolmogorov-set arising from that paper would hardly allow us to obtain the $(\mathcal{SU})$-part of Theorem A.

We then proceed to state the result as needed for the purposes of the paper, keeping proofs to the essential minimum. To this end, we need to introduce some technical tools. First of all, we remark that ORC configuration can be realized only if the planetary masses are tuned with the semi-major axes. Namely, if we denote as "1" and "2" the inner, outer planet; as G_1, G_2, with

$$\mathrm{G}_1 > \mathrm{G}_2 \tag{9}$$

their respective angular momenta as a_1, a_2, the semi-major axes of their respective instantaneous orbits around the sun; α_-, α_+, with $\alpha := \frac{a_1}{a_2}$ verifying

$$\alpha_- < \alpha < \alpha_+, \tag{10}$$

for suitable $0 < \alpha_- < \alpha_+ < 1$, then the following inequalities need to be satisfied:

$$\frac{m_1}{m_2}\sqrt{\alpha_-} > 1 \ , \qquad 0 < \mu < \overline{\mu}_* \ . \tag{11}$$

Indeed, since the motions are almost-circular, the angular momenta of the planets G_1, G_2 are close to Λ_1, Λ_2, which, by (6), are related to the semi-axes and the mass ratio via

$$\frac{\Lambda_1}{\Lambda_2} = \frac{\mathrm{m}_1}{\mathrm{m}_2}\sqrt{\frac{\mathrm{M}_1}{\mathrm{M}_2}}\sqrt{\alpha} \tag{12}$$

where m_i, M_i are as in (3). This equality does not conflict with (9) and (10) if one assumes that the inequality

$$k_- = \frac{\mathrm{m}_1}{\mathrm{m}_2}\sqrt{\frac{\mathrm{M}_1}{\mathrm{M}_2}}\sqrt{\alpha_-} > 1\ . \tag{13}$$

Whence, in view of the dependence of m_i, M_i upon μ specified in (3), the necessity of (11) (with $\overline{\mu}_*$ depending on m_0, m_1, m_2 and α_-) follows.

2. The canonical setting: the "partial reduction"

We next recall what is known as the *Jacobi's reduction of the nodes*. This is a procedure, due, in the case of two planets, to Jacobi, Radau, Refs.[16,33], but non–trivially extended, to their maximum generality, to any number of planets, by, A. Deprit, Ref.[10], eliminating, via a canonical change of coordinates, SO(3)–invariance of the system. The coordinates worked out by Deprit in Ref.[10], almost forgotten[47] for about twenty years, and about which Deprit himself seemed to be not much confident[48], were rediscovered (in a slightly different form, directly applicable to planetary systems) by the author during her PhD, Ref.[28]. It works via the production of a system of canonical coordinates well adapted to the rotation invariance of the system (2). These coordinates are denoted as

$$\mathrm{jrd} := (Z, \mathrm{G}, \mathrm{G}_1, \mathrm{G}_2, \Lambda_1, \Lambda_2, \zeta, \gamma, \gamma_1, \gamma_2, \ell_1, \ell_2) \tag{14}$$

jrd is a Kepler map in the sense given in Section II. Its definition is recalled in Appendix A 1; its canonical character is discussed in Appendix D.

The jrd-coordinates are not well defined when the eccentricities or the mutual inclination between the planets' orbital planes vanish. Namely, when some of the following equalities is verified

$$\begin{aligned}&\mathrm{G}_1 = \Lambda_1 \quad \text{or} \quad \mathrm{G}_2 = \Lambda_2 \quad \text{or} \quad \mathrm{G} = \mathrm{G}_1 + \mathrm{G}_2 \quad \text{or} \quad \mathrm{G} = \mathrm{G}_1 - \mathrm{G}_2 \quad \text{or} \quad \mathrm{G} = \mathrm{G}_2 - \mathrm{G}_1\\ &\text{or} \quad Z = \mathrm{G} \quad \text{or} \quad Z = -\mathrm{G}\end{aligned} \tag{15}$$

To obviate to this problem, it is convenient, as in Refs.[8,28], to switch to a regularized version of the jrd-coordinates. It does not seem possible to find a system of coordinates where *all* of the above singularities are regularized and, simultaneously, the number of degrees of freedom is kept to four. It is however possible, as now we describe, to define new coordinates which are regularly defined (indeed, analytic) when three or four equalities in (15) hold. For example, in Refs.[8,28], a set of coordinates, named rps (see also Appendix A 2), similar in some respect to the well known Poincaré coordinates, was introduced, regularly defined on the manifold $\mathcal{M}_0 := \{\mathrm{G}_1 = \Lambda_1 \text{ or } \mathrm{G}_2 = \Lambda_2 \text{ or } \mathrm{G} = \mathrm{G}_1 + \mathrm{G}_2 \text{ or } \mathrm{G} = Z\}$, corresponding to orbit 1 circular or orbit 2 circular or orbits 1, 2 co–planar and co–rotating or C parallel to the third axis of a prefixed frame. In this section we are instead interested in looking at the singular manifold

$$\mathcal{M}_\pi^+ := \{\mathrm{G}_1 = \Lambda_1 \text{ or } \mathrm{G}_2 = \Lambda_2 \text{ or } \mathrm{G} = \mathrm{G}_1 - \mathrm{G}_2 \text{ or } Z = \mathrm{G}\} \ . \tag{16}$$

Then, by analogue arguments as in Refs.[8,28], we introduce the change of coordinates

$$(\Lambda_1, \Lambda_2, \mathrm{G}_1, \mathrm{G}_2, \mathrm{G}, Z, \ell_1, \ell_2, \gamma_1, \gamma_2, \gamma, \zeta) \to \mathrm{rps}_\pi^+ := (\Lambda_1, \Lambda_2, \lambda_1, \lambda_2, \eta_1, \eta_2, \xi_1, \xi_2, p_1, p_2, q_1, q_2), \tag{17}$$

via the formulae

$$\left\{\begin{array}{l} \Lambda_1 = \Lambda_1 \\ \Lambda_2 = \Lambda_2 \\ t_1 = \sqrt{\Lambda_1 - \mathrm{G}_1}\, e^{\mathrm{i}(\gamma_1+\gamma+\zeta)} \\ t_2 = -\mathrm{i}\sqrt{\Lambda_2 - \mathrm{G}_2}\, e^{\mathrm{i}(-\gamma_2+\gamma+\zeta)} \\ t_3 = -\mathrm{i}\sqrt{\mathrm{G} - \mathrm{G}_1 + \mathrm{G}_2}\, e^{\mathrm{i}(\gamma+\zeta)} \\ T = \sqrt{\mathrm{G} - Z}\, e^{\mathrm{i}\zeta} \end{array}\right. \qquad \left\{\begin{array}{l} \lambda_1 = \ell_1 + \gamma_1 + \gamma + \zeta \\ \lambda_2 = \ell_2 + \gamma_2 - \gamma - \zeta \\ t_1^* = -\mathrm{i}\sqrt{\Lambda_1 - \mathrm{G}_1}\, e^{-\mathrm{i}(\gamma_1+\gamma+\zeta)} \\ t_2^* = -\sqrt{\Lambda_2 - \mathrm{G}_2}\, e^{-\mathrm{i}(-\gamma_2+\gamma+\zeta)} \\ t_3^* = -\sqrt{\mathrm{G} - \mathrm{G}_1 + \mathrm{G}_2}\, e^{-\mathrm{i}(\gamma+\zeta)} \\ T^* = -\mathrm{i}\sqrt{\mathrm{G} - Z}\, e^{-\mathrm{i}\zeta} \end{array}\right. \tag{18}$$

and the definitions

$$\begin{array}{llll} t_1 := \dfrac{\eta_1 - \mathrm{i}\xi_1}{\sqrt{2}} & t_2 := \dfrac{\mathrm{i}\eta_2 - \xi_2}{\sqrt{2}} & t_3 := \dfrac{\mathrm{i}p_1 - q_1}{\sqrt{2}} & T := \dfrac{p_2 - \mathrm{i}q_2}{\sqrt{2}} \\ t_1^* := \dfrac{\eta_1 + \mathrm{i}\xi_1}{\sqrt{2}\mathrm{i}} & t_2^* := \dfrac{\mathrm{i}\eta_2 + \xi_2}{\sqrt{2}\mathrm{i}} & t_3^* := \dfrac{\mathrm{i}p_1 + q_1}{\sqrt{2}\mathrm{i}} & T^* := \dfrac{p_2 + \mathrm{i}q_2}{\sqrt{2}\mathrm{i}}. \end{array} \tag{19}$$

The coordinates at right hand of (17), as well as the ones at left hand side in (19), which are canonical, are useful both: the (17)'s have the advantage of being (as one immediately checks) *real*; the (19)'s turn to be useful for the computation of the Birkhoff normal form associated to $\overline{f_{\mathrm{rps}^+_\pi}}$ (discussed below).

Let us collect here the most remarkable properties of the perturbing function $f_{\mathrm{rps}^+_\pi}$ expressed in terms of the coordinates in (17)–(19).

- ⋆ The function $f_{\mathrm{rps}^+_\pi}$ is independent of (p_2, q_2), which are first integrals (in terms of the coordinates (19), we have that $f_{\mathrm{rps}^+_\pi}$ is independent of (T, T^*)). This allows to overcome the problem of the "identically vanishing frequency", mentioned (for the case of planets revolving in the same verse) in Ref.[3] (chap. III, §5, n. 3. Eq. (3.5.6), p. 140), that one would encounter, even in this case, using Poincaré coordinates.

- ⋆ The explicit expression of the function $f_{\mathrm{rps}^+_\pi}$ in terms of the complex coordinates (19) may be derived from the one of f_{rps} in terms of the corresponding complex coordinates of Refs.[8,28] (recalled in Equation (31) below) via[38]

$$f_{\mathrm{rps}^+_\pi}(\Lambda_1, \Lambda_2, \lambda_1, \lambda_2, t, t^*) = f_{\mathrm{rps}}(\Lambda_1, -\Lambda_2, \lambda_1, -\lambda_2, t, t^*). \tag{20}$$

  This relation allows to establish for $f_{\mathrm{rps}^+_\pi}$ analogue properties that have been established for f_{rps} in the quoted literature. In particular, the following two items are true.

- ⋆ Even though the change in (17) is not defined on the manifold $\mathcal{M}^+_\pi$ in (16), the change

$$\mathrm{rps}^+_\pi \to (y^{(1)}, y^{(2)}, x^{(1)}, x^{(2)})$$

  is analytic on a domain including $\mathcal{M}^+_\pi$, which, now, takes the form

$$\mathcal{M}^+_\pi = \Big\{(\Lambda, \lambda, t, T, t^*, T^*) : \ (t_1, t_1^*) = (0,0) \text{ or } (t_2, t_2^*) = (0,0) \text{ or } (t_3, t_3^*) = (0,0) \\ \text{or} \quad (T, T^*) = (0,0)\Big\}\ .$$

  In particular, $f_{\mathrm{rps}^+_\pi}$ expressed in such coordinates is *analytic* on a domain of the form $\mathcal{D} = \mathcal{A} \times \mathbf{T}^2 \times \mathcal{B}$, with $\mathcal{A}$, $\mathcal{B}$ a suitable open sets of $\mathbf{R}^2$, $\mathbf{R}^8$, respectively, and $0 \in \mathcal{B}$.

- ⋆ The change

$$(\Lambda, \lambda, \eta, \xi, p, q) \to (\Lambda, \lambda, \eta, \xi, -p, -q)$$

respectively:

$$(\Lambda, \lambda, t_1, t_2, t_3, T, t_1^*, t_2^*, t_3^*, T^*) \to (\Lambda, \lambda, t_1, t_2, t_3, T, t_1^*, t_2^*, -t_3^*, -T^*)$$

corresponds to change the sign of the third component of $y^{(j)}_{\mathrm{rps}_\pi^+}$, $x^{(j)}_{\mathrm{rps}_\pi^+}$. Due to the independence of (p_2, q_2) ((T, T^*), respectively), we have that $f_{\mathrm{rps}_\pi^+}$ is even in (p_1, q_1) ((t_3, t_3^*), respectively).

3. *An analyticity domain for* $f_{\mathrm{rps}_\pi^+}$

Without loss of generality, we fix the cyclic couple (p_2, q_2) to a prefixed value, e.g., $(0, 0)$. Then we establish, under the constraint (11), a real ten–dimensional domain $\mathcal{D}_\mathcal{S}$ such that $f_{\mathrm{rps}_\pi^+}$ is analytic in $\mathcal{D}_\mathcal{S}$.

Recall relation (6) relating the Λ_j–coordinates to semi–major axes a_j and reduced masses m_j, M_j in (3). Choosing $0 < \alpha_- < \alpha_+ < 1$ as in (10) and $0 < \Lambda_- < \Lambda_+$, we fix, for such coordinates the initial domain

$$\mathcal{L}_0 := \Big\{\Lambda = (\Lambda_1, \Lambda_2) : \ \Lambda_- \le \Lambda_2 \le \Lambda_+, k_-\Lambda_2 \le \Lambda_1 \le k_+\Lambda_2\Big\} \tag{21}$$

with

$$k_\pm := \frac{\mathrm{m}_1}{\mathrm{m}_2}\sqrt{\frac{\mathrm{M}_1}{\mathrm{M}_2}\alpha_\pm}. \tag{22}$$

with the $\alpha_\pm$ corresponding to the bound for the semi-axes ratio (10). We take $\overline{\mu}_*$ in (11) so small (depending on m_0, m_1, m_2, α_-) that k_- (hence, k_+) is greater than 1 for $0 \le \mu < \overline{\mu}_*$.

The coordinates $\lambda = (\lambda_1, \lambda_2)$ will be taken to run in the torus $\mathbf{T}^2$.

As for the coordinates $z = (\eta, \xi, p_1, q_1)$, we take a domain of the form

$$\mathcal{U}_\mathcal{S} := \Big\{z = (\eta, \xi, p_1, q_1) \in \mathbf{C}^6 : |z| \le \varepsilon\Big\} \tag{23}$$

with $\varepsilon < \varepsilon_0$, where ε_0 is so small with respect to $k_-\Lambda_-$ that the eccentricities are bounded away from the Levi-Civita value $0.6627\ldots$ (which ensures a convergent Taylor expansion around $z = 0$) and the angular momenta

$$\mathrm{G}_1 = \Lambda_1 - \mathrm{i}t_1t_1^* = \Lambda_1 - \frac{\eta_1^2 + \xi_1^2}{2}, \quad \mathrm{G}_2 = \Lambda_2 + \mathrm{i}t_2t_2^* = \Lambda_2 - \frac{\eta_2^2 + \xi_2^2}{2}$$

verify

$$\mathrm{G}_1 > \mathrm{G}_2 .$$

We finally let

$$\mathcal{D}_{\mathcal{S}} := \mathcal{L}_0 \times \mathcal{U}_{\mathcal{S}} \times \mathbf{T}^2 \ . \tag{24}$$

Letting $\Lambda := (\Lambda_1, \Lambda_2)$, $\lambda := (\lambda_1, \lambda_2)$ and proceeding as in Refs.[8,28], one proves that the map

$$(\Lambda, \lambda, z) \in \mathcal{D}_{\mathcal{S}} \to (y^{(1)}, y^{(2)}, x^{(1)}, x^{(2)})$$

is real-analytic. In particular, $\mathcal{D}_{\mathcal{S}}$ intersects the manifolds $\mathcal{M}^+_\pi$ defined in (16), along the sub–manifolds $\{(\Lambda, \lambda, \eta, \xi, p_1, q_1) \in \mathcal{D}_{\mathcal{S}} :\ (\eta_1, q_1) = (0, 0)$ or $(\eta_2, \xi_2) = (0, 0)$ or $(p_1, q_1) = (0, 0)$ or $(p_2, q_2) = (0, 0)\}$. Observe that ORC motions, according to their definition in the introduction, are consist of small neighborhoods of the manifolds $\{(\Lambda, \lambda, \eta, \xi, p_1, q_1) \in \mathcal{D}_{\mathcal{S}} : (\eta, \xi, p_1, q_1) = 0\} \subset \mathcal{M}^+_\pi$.

4. The conservation of G *and the "full reduction"*

The Hamiltonian (2), expressed in terms of the coordinates (17), still possesses a first integral, given by the Euclidean length G of C. For this reason, we shall refer to it as "partial reduction".

Following Refs.[8,28], it is possible to construct systems of canonical coordinates without extra-integrals ("full reduction").

In this section, we introduce an eight–dimensional set $\widehat{\mathcal{D}}_{\mathcal{S}}$ such that, up to a zero–measure set, $\mathcal{D}_{\mathcal{S}}$ may be parametrized by canonical coordinates in $\widehat{\mathcal{D}}_{\mathcal{S}} \times \{\mathrm{G} \in \mathbf{R}\} \times \mathbf{T}$.

It follows from the formulae in (19), that the integral G has the expression

$$\mathrm{G} = \Lambda_1 - \Lambda_2 - \mathrm{i}t_1 t_1^* - \mathrm{i}t_2 t_2^* - \mathrm{i}t_3 t_3^* = \Lambda_1 - \Lambda_2 - \frac{\eta_1^2 + \xi_1^2}{2} + \frac{\eta_2^2 + \xi_2^2}{2} + \frac{p_1^2 + q_1^2}{2}. \tag{25}$$

Due to the simplicity of Equation (25), proceeding analogously to Ref.[8], on the open set

$$\widetilde{\mathcal{D}}_{\mathcal{S}} = \mathcal{L}_0 \times \widetilde{\mathcal{U}}_{\mathcal{S}} \times \mathbf{T}^2 \subset \mathcal{D}_{\mathcal{S}} \tag{26}$$

where

$$\widetilde{\mathcal{U}}_{\mathcal{S}} := \mathcal{U}_{\mathcal{S}} \setminus \Big\{(\Lambda, \lambda, \eta, \xi, p_1, q_1) :\ ((\eta_1, \xi_1), (\eta_2, \xi_2), (p_1, q_1)) = \Big(0, 0, 0\Big)\Big\} \tag{27}$$

one can define an atlas of canonical coordinates

$$\mathcal{A} = \Big\{(\Lambda, \widetilde{\lambda}, \widetilde{\eta}, \widetilde{\xi}, \mathrm{G}, \widetilde{\mathrm{g}})\Big\} \tag{28}$$

including G as an action coordinate. Such atlas is composed of three charts

$$\phi^{(i)}_{\mathrm{red}^{\pm}_{\pi}}: \quad (\Lambda, \lambda, \eta, \xi, p_1, q_1) \in \widetilde{\mathcal{D}}^{(i)}_{\mathcal{S}} \to (\Lambda, \widetilde{\lambda}, \widetilde{\eta}, \widetilde{\xi}, \mathrm{G}, \widetilde{\mathrm{g}}) \qquad i = 1,\ 2,\ 3 \tag{29}$$

which are well defined on $\widetilde{\mathcal{D}}^{(1)}_{\mathcal{S}} = \mathcal{D}_{\mathcal{S}} \setminus \{(\eta_1, \xi_1) = (0,0)\}$ or $\widetilde{\mathcal{D}}^{(2)}_{\mathcal{S}} = \mathcal{D}_{\mathcal{S}} \setminus \{(\eta_2, \xi_2) = (0,0)\}$ or $\widetilde{\mathcal{D}}^{(3)}_{\mathcal{S}} = \mathcal{D}_{\mathcal{S}} \setminus \{(p_1, q_1) = (0,0)\}$ For the explicit expression of $\phi^{(i)}_{\mathrm{red}^{\pm}_{\pi}}$, see Note[39]. We then denote as $\widehat{\mathcal{D}}^{(i)}_{\mathcal{S}}$ the eight–dimensional projection set defined via

$$\phi^{(i)}_{\mathrm{red}^{\pm}_{\pi}}(\widetilde{\mathcal{D}}^{(i)}_{\mathcal{S}}) = \widehat{\mathcal{D}}^{(i)}_{\mathcal{S}} \times \{\mathrm{G} \in \mathbf{R}\} \times \{\widetilde{\mathrm{g}} \in \mathbf{T}\}$$

and, finally,

$$\widehat{\mathcal{D}}_{\mathcal{S}} := \widehat{\mathcal{D}}^{(1)}_{\mathcal{S}} \cup \widehat{\mathcal{D}}^{(2)}_{\mathcal{S}} \cup \widehat{\mathcal{D}}^{(3)}_{\mathcal{S}}\ . \tag{30}$$

5. Theorem on existence of maximal tori (on $\mathcal{D}_{\mathcal{S}}$ and $\widehat{\mathcal{D}}_{\mathcal{S}}$)

The next theorem establishes the existence of maximal tori in the domains $\mathcal{D}_{\mathcal{S}}$ and $\widehat{\mathcal{D}}_{\mathcal{S}}$.

Theorem II.1 *There exist two numbers $0 < \varepsilon_+ < \varepsilon_0$, $0 < \alpha_+ < 1$, such that, for any $0 < \varepsilon < \varepsilon_+$, $0 < \alpha_- < \alpha_+$ m_1, m_2, $\overline{\mu}_*$ so that (10), (11), (13) hold, $0 < \Lambda_- < \Lambda_+$, one can find $\mu_+(\varepsilon)$, with $\varepsilon^\sigma < \mu_+(\varepsilon) < \overline{\mu}_*$, with some $\sigma > 0$, such that, for any $0 < \mu < \mu_+(\varepsilon)$, inequality (13) is satisfied and, in the domain $\mathcal{D}_{\mathcal{S}}$ defined via (21), (23) and (24), there exists an invariant set $\mathcal{F}_{\varepsilon,\mu} \subset \mathcal{D}_{\mathcal{S}}$ with density $1 - \varepsilon^p - (\mu/\mu_+(\varepsilon))^q$ for some $0 < p, q < 1$ (in particular, with* ORC *motions forming a high density subset of it) which is foliated as*

$$\mathcal{F}_{\varepsilon,\mu} = \bigcup_{\omega} \mathcal{T}_{\omega,\varepsilon,\mu}$$

where $\mathcal{T}_{\omega,\varepsilon,\mu}$ is diffeomorphic to a 5–dimensional manifold (actually, a torus) on which the motions are quasi-periodic, in retrograde, outer configuration, with suitable ("diophantine") irrational frequencies. In turn, the manifolds $\mathcal{T}_{\omega,\varepsilon,\mu}$ completely included in $\widetilde{\mathcal{D}}_{\mathcal{S}}$ are foliated by 4–dimensional invariant, quasi–periodic tori in $\widehat{\mathcal{D}}_{\mathcal{S}}$, whose Lebesgue density in $\widehat{\mathcal{D}}_{\mathcal{S}}$ is $1 - \varepsilon^p - (\mu/\mu_+(\varepsilon))^q$.

Remark II.1 Theorem II.1 is an extension, to the ORC case, of the result, mentioned in the introduction, and proved between the 60s and 2011, at various stages, in Refs.[3,8,28,34], for the case of a planetary system with any number of planets revolving in the same verse.

Such extension works with exactly the same proof, but, however, notwithstanding relation (20), *is not a mere question of reversing signs.*

Indeed, accordingly to the theory developed by V. I. Arnold in Ref.[3] (see Appendix A 2 for some detail), in order to prove the thesis relatively to $\mathcal{D}_{\mathcal{S}}$ (respectively, $\widehat{\mathcal{D}}_{\mathcal{S}}$), one has to check the following two items:

- ⋆ that $\overline{f_{\mathrm{rps}^\pm_\pi}}$ (respectively, $\overline{f_{\mathrm{red}^\pm_\pi}}$) has an elliptic equilibrium at $(t, t^*) = (0, 0)$;
- ⋆ that the Birkhoff normal form of order four around such equilibrium exists and is non–degenerate, in the sense that the matrix of the second–order Birkhoff invariants is non–singular.

The basic difference between the prograde cases considered in Refs.[3,8,28,34] and the ORC configuration of the paper is that in the former cases the elliptic nature of the equilibrium is an immediate consequence of the conservation of G, which is not true in the latter.

In order to clarify the meaning of Theorem II.1 relatively to existing literature, we provide here the details of this assertion. For definiteness and simplicity, we reduce to the case (of our interest in the economy of the paper) of $N = 2$ planets, even though the argument is general.

The coordinates rps defined in Refs.[8,28] will be here denoted as

$$\mathrm{rps} := (\Lambda, \overline{\lambda}, \overline{\eta}, \overline{\xi}, \overline{p}, \overline{q}) = (\Lambda_1, \Lambda_2, \overline{\lambda}_1, \overline{\lambda}_2, \overline{\eta}_1, \overline{\eta}_2, \overline{\xi}_1, \overline{\xi}_2, \overline{p}_1, \overline{p}_2, \overline{q}_1, \overline{q}_2)$$

They are defined via the formulae

$$\begin{cases} \Lambda_1 = \Lambda_1 \\ \Lambda_2 = \Lambda_2 \\ \overline{t}_1 = \sqrt{\Lambda_1 - \mathrm{G}_1}\, e^{\mathrm{i}(\gamma_1+\gamma+\zeta)} \\ \overline{t}_2 = \sqrt{\Lambda_2 - \mathrm{G}_2}\, e^{\mathrm{i}(\gamma_2+\gamma+\zeta)} \\ \overline{t}_3 = \sqrt{\mathrm{G}_1 + \mathrm{G}_2 - \mathrm{G}}\, e^{\mathrm{i}(\gamma+\zeta)} \end{cases} \qquad \begin{cases} \overline{\lambda}_1 = \ell_1 + \gamma_1 + \gamma + \zeta \\ \overline{\lambda}_2 = \ell_2 + \gamma_2 + \gamma + \zeta \\ \overline{t}_1^* = -\mathrm{i}\sqrt{\Lambda_1 - \mathrm{G}_1}\, e^{-\mathrm{i}(\gamma_1+\gamma+\zeta)} \\ \overline{t}_2^* = -\mathrm{i}\sqrt{\Lambda_2 - \mathrm{G}_2}\, e^{-\mathrm{i}(\gamma_2+\gamma+\zeta)} \\ \overline{t}_3^* = -\mathrm{i}\sqrt{\mathrm{G}_1 + \mathrm{G}_2 - \mathrm{G}}\, e^{-\mathrm{i}(\gamma+\zeta)} \end{cases} \tag{31}$$

and

$$\overline{t}_1 := \frac{\overline{\eta}_1 - \mathrm{i}\overline{\xi}_1}{\sqrt{2}} \qquad \overline{t}_2 := \frac{\overline{\eta}_2 - \mathrm{i}\overline{\xi}_2}{\sqrt{2}} \qquad \overline{t}_3 := \frac{\overline{p}_1 - \mathrm{i}\overline{q}_1}{\sqrt{2}} \qquad \overline{T} := \frac{\overline{p}_2 - \mathrm{i}\overline{q}_2}{\sqrt{2}}$$

$$\overline{t}_1^* := \frac{\overline{\eta}_1 + \mathrm{i}\overline{\xi}_1}{\sqrt{2}\mathrm{i}} \qquad \overline{t}_2^* := \frac{\overline{\eta}_2 + \mathrm{i}\overline{\xi}_2}{\sqrt{2}\mathrm{i}} \qquad \overline{t}_3^* := \frac{\overline{p}_1 + \mathrm{i}\overline{q}_1}{\sqrt{2}\mathrm{i}} \qquad \overline{T}^* := \frac{\overline{p}_2 + \mathrm{i}\overline{q}_2}{\sqrt{2}\mathrm{i}}$$

with the cyclic couple $(\overline{T}, \overline{T}^*) = (T, T^*)$ being the same as in (18). In particular, the function G in (25), in terms of $(\Lambda, \overline{t}, \overline{t}^*)$ has an expression analogue to (25):

$$\mathrm{G} = \Lambda_1 + \Lambda_2 - \mathrm{i}\overline{t}_1\overline{t}_1^* - \mathrm{i}\overline{t}_2\overline{t}_2^* - \mathrm{i}\overline{t}_3\overline{t}_3^* \tag{32}$$

Let $\overline{f_{\rm rps}}$ be the λ–average of $f_{\rm rps}$. Due to the analyticity of this function on the sections $\{(t, t^*) = (0, 0)\}$, discussed in Refs.[8,28], we may consider its Taylor expansion

$$\overline{f_{\rm rps}}(\Lambda, \overline{t}, \overline{t}^*) = \sum_{a,a^*} c_{a,a^*}(\Lambda)\overline{t}_1^{a_1}\overline{t}_2^{a_2}\overline{t}_3^{a_3}\overline{t}_1^{*a_1^*}\overline{t}_2^{*a_2^*}\overline{t}_3^{*a_3^*}$$

in powers of $(\overline{t}, \overline{t}^*)$. Since $f_{\rm rps}$ Poisson–commutes with the function G in (32), so does $\overline{f_{\rm rps}}$. Then, only monomials with a_1, a_2, a_3, a_1^*, a_2^*, a_3^* verifying

$$a_1 + a_2 + a_3 = a_1^* + a_2^* + a_3^* \tag{33}$$

appear in the expansion above. In particular, the second–order term in such expansion has necessarily the form

$$\frac{1}{2}(t,\ \overline{\mathcal{Q}}(\Lambda)\overline{t}^*)\ ,$$

with $\mathcal{Q}(\Lambda)$ a suitable 3×3 symmetric matrix. The normal form associated to a quadratic form of this kind is of elliptic kind if and only if the algebraic eigenvalues of $\overline{\mathcal{Q}}$ are purely imaginary. But the reality condition, joint with relations (implied by (31))

$$\overline{\overline{t}} = \mathrm{i}\overline{t}^* \qquad \overline{\overline{t}^*} = \mathrm{i}\overline{t} \tag{34}$$

(with the upper bar denoting "complex conjugate") imply that $\overline{\mathcal{Q}}(\Lambda) = \mathrm{i}\widehat{\mathcal{Q}}$, with $\widehat{\mathcal{Q}}$ symmetric and real. Therefore, $\widehat{\mathcal{Q}}$ has only real eigenvalues, and hence $\overline{\mathcal{Q}}$ all purely imaginary ones, as anticipated.

In the ORC case, the situation is different, because (33) still holds, but (34) does not. The corresponding symmetric matrix $\widehat{\mathcal{Q}}(\Lambda)$ turns to have not only real entries and hence that $\overline{\mathcal{Q}}$ has purely imaginary eigenvalues is not a–priori guaranteed. Rather, it needs to be checked specifically (see equations (38)-(39) below).

Proof. Using (20) and the results of Refs.[8,28], we can assert that $\overline{f_{{\rm rps}^+_\pi}}$ is even around $(t, t^*) = (0, 0)$, because so is $\overline{f_{\rm rps}}$. Moreover, we can derive coefficients of the Taylor expansion for $\overline{f_{{\rm rps}^+_\pi}}$ around $(t, t^*) = (0, 0)$ from the corresponding ones for $\overline{f_{\rm rps}}$. Letting

$$t := (\hat{t}, t_3) := (t_1, t_2, t_3)\ ,\ \ t^* := (\hat{t}^*, t_3^*) := (t_1^*, t_2^*, t_3^*)$$

by the parity in (t_3, t_3^*) mentioned in II A 2, we can write such an expansion as

$$\overline{f_{\mathrm{rps}_\pi^+}} = C_0(\Lambda) + \frac{\mathrm{i}}{2}\widehat{t}^* \cdot \sigma(\Lambda)\widehat{t} + \frac{\mathrm{i}\varsigma(\Lambda)}{2} t_3^* t_3 + \mathrm{O}_4(t, t^*; \Lambda). \tag{35}$$

with (see Appendix A 2 for some detail)

$$\sigma(\Lambda_1, \Lambda_2) = \begin{pmatrix} \frac{\mathrm{s}}{\Lambda_1} & -\mathrm{i}\frac{\widetilde{\mathrm{s}}}{\sqrt{\Lambda_1\Lambda_2}} \\ -\mathrm{i}\frac{\widetilde{\mathrm{s}}}{\sqrt{\Lambda_1\Lambda_2}} & -\frac{\mathrm{s}}{\Lambda_2} \end{pmatrix} \qquad \varsigma(\Lambda_1, \Lambda_2) = -\Big(\frac{1}{\Lambda_1} - \frac{1}{\Lambda_2}\Big)\mathrm{s} \tag{36}$$

where

$$\mathrm{s} := -m_1 m_2 \frac{\alpha}{a_2} b_{3/2}^{(1)}(\alpha) \qquad \widetilde{\mathrm{s}} := m_1 m_2 \frac{\alpha}{a_2} b_{3/2}^{(2)}(\alpha) \tag{37}$$

and α, a_2 as in (6), (12). As usual, the $b_s^{(j)}(\alpha)$'s are the Laplace coefficients, defined via the Fourier expansion

$$\frac{1}{(1 - 2\alpha\cos\theta + \alpha^2)^s} = \sum_{k \in \mathbf{Z}} b_s^{(k)}(\alpha) e^{\mathrm{i}k\theta} \qquad \mathrm{i} := \sqrt{(-1)}\ .$$

In order to assert that the origin in (t, t^*) is an elliptic equilibrium for $\overline{f_{\mathrm{rps}_\pi^+}}$, we have to check that the eigenvalues of σ in (36) are real. The direct computation gives the eigenvalues

$$\sigma_1, \sigma_2 = \frac{\operatorname{tr}\sigma}{2} \pm \frac{1}{2}\sqrt{(\operatorname{tr}\sigma)^2 - 4\det\sigma}\ . \tag{38}$$

Since $\operatorname{tr}\sigma = \big(\frac{1}{\Lambda_1} - \frac{1}{\Lambda_2}\big)\mathrm{s}$ is real, we have to check that the discriminant

$$\Delta := (\operatorname{tr}\sigma)^2 - 4\det\sigma = (\frac{1}{\Lambda_1} - \frac{1}{\Lambda_2})^2\mathrm{s}^2 + \frac{4}{\Lambda_1\Lambda_2}(\mathrm{s}^2 - \widetilde{\mathrm{s}}^2)$$

is positive. Recalling that the Laplace coefficients verify

$$b_s^{(j)}(\beta) > b_s^{(j+1)}(\beta) \quad \text{for all} \quad s > 0, \quad j \in \mathbf{Z}, \quad 0 < |\beta| < 1,$$

(see Ref.[11] for a proof), one has

$$\mathrm{s}^2 - \widetilde{\mathrm{s}}^2 = (m_1 m_2 \frac{\alpha}{a_2})^2((b_{3/2}^{(1)}(\alpha))^2 - (b_{3/2}^{(2)}(\alpha))^2) > 0. \tag{39}$$

Then $\Delta > 0$, therefore, the equilibrium is elliptic. As in Ref.[8], the elliptic nature of the equilibrium in the partially reduced case also guarantees its ellipticity in the fully reduced case, at least in a small punctured neighborhood of such equilibrium. It will be also necessary to check the *full torsion* property, both in the partially and the totally reduced case (see Appendix A 2 for some detail). In conclusion, the proofs in Refs.[8,28] for both partially and totally reduced systems work also for the ORC, so that the thesis of Theorem II.1 follows.

III. WHISKERED TORI

To state with more precision and prove the $(\mathcal{U})$-part of Theorem A, we need some tool.

We begin with recalling the definition and the main properties of a set of canonical coordinates that have been presented in Ref.[27].

A. The reduction of perihelia

This section is not a repetition of the Appendix A; it contains the new coordinate system singular with respect to the jrd and it is the main contribution of the work, essential to deal with the retrograde motions (its use clarifies why passing from prograde to retrograde is not a mere change of sign: it allows treating retrograde motions and to obtain hyperbolic tori, possibly opening a way to attack Arnold diffusion Ref.[4] in a three body problem, as shown for instance by Theorem III.1).

We call *Perihelia reduction for the three-body problem* the Kepler map

$$\phi_{\rm p}: \qquad {\rm p} := (Z, \Theta, \chi, \Lambda, \zeta, \vartheta, \kappa, \ell) \in \mathbf{R}^6 \times \mathbf{T}^6 \to (y^{(1)}, y^{(2)}, x^{(1)}, x^{(2)}) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$$

where

$${\rm p} := (Z, \Theta, \chi, \Lambda, \zeta, \vartheta, \kappa, \ell)$$

with $\chi = ({\rm G}, {\rm G}_2)$, $\Lambda = (\Lambda_1, \Lambda_2)$, $\kappa = ({\rm g}, {\rm g}_2)$, $\ell = (\ell_1, \ell_2)$, defined as follows.

Assume that that the orbits $t \to x^{(i)}(t)$ generated by the Hamiltonians (4) are ellipses with non-vanishing eccentricity. Let a_i denote their semi–major axes. Let ${\rm C}^{(i)} := x^{(i)} \times y^{(i)}$ the $i^{\rm th}$ angular momentum; ${\rm P}^{(i)}$, with $|{\rm P}^{(i)}| = 1$, the direction of the $i^{\rm th}$ perihelion. Define the "nodes"

$$\nu_1 := k^{(3)} \times {\rm C}, \qquad {\rm n}_1 := {\rm C} \times {\rm P}^{(1)}, \qquad \nu_2 := {\rm P}^{(1)} \times {\rm C}^{(2)}, \quad {\rm n}_2 = {\rm C}^{(2)} \times {\rm P}^{(2)} \tag{40}$$

and assume that they do not vanish. Finally, for any three vectors u, v, $w \in \mathbf{R}^3$, with u, $v \perp w$, let $\alpha_w(u, v)$ denote the oriented angle formed from u to v, relative to the positive direction established by w. Then define p as

$$\begin{cases} Z := \mathrm{C}\cdot k^{(3)} \\ \Theta := \mathrm{C}\cdot \mathrm{P}^{(1)} = \mathrm{C}^{(2)}\cdot \mathrm{P}^{(1)} \\ \mathrm{G} := \|\mathrm{C}\| \\ \mathrm{G}_2 := \|\mathrm{C}^{(2)}\| \\ \Lambda_j := \mathrm{M}_j\sqrt{\mathrm{m}_j a_j} \end{cases} \qquad \begin{cases} \zeta := \alpha_{k^{(3)}}(k^{(1)}, \nu_1) \\ \vartheta := \alpha_{\mathrm{P}^{(1)}}(\mathrm{n}_1, \nu_2) \\ \mathrm{g} := \alpha_{\mathrm{C}}(\nu_1, \mathrm{n}_1) \\ \mathrm{g}_2 := \alpha_{\mathrm{C}^{(2)}}(\nu_2, \mathrm{n}_2) \\ \ell_j := \text{mean anomaly of } x^{(j)} \end{cases} \tag{41}$$

with $j = 1, 2$. Here, the "angles" in the right hand side column are conjugated (in the sense of the standard two–form) to the "actions" at left hand side. Recall that the mean anomaly ℓ_j is defined as the area of the elliptic sector from $\mathrm{P}^{(j)}$ to $x^{(j)}$ "normalized at 2π".

The canonical character of the coordinates (41) is briefly discussed in Appendix D. We recall, here, some properties of the p-coordinates, while we refer to Ref.[27] for more details.

- ⋆ The p-coordinates are canonical. Namely, they preserve the standard 2–form.

- ⋆ As well as jrd coordinates, the p-coordinates are well fitted to rotation invariance of the system. This means that, since Z, ζ, and G remain constant during the motion, the Hamiltonian

$$\mathrm{H}_{\mathrm{p}}(\Theta, \mathrm{G}_2, \Lambda, \vartheta, \mathrm{g}_2, \ell; \mathrm{G}) = -\sum_{j=1}^{2} \frac{\mathrm{m}_j^3 \mathrm{M}_j^2}{2\Lambda_j^2} + \mu f_{\mathrm{p}}(\Theta, \mathrm{G}_2, \Lambda, \vartheta, \mathrm{g}_2, \ell; \mathrm{G}).$$

will depend explicitly only on the eight coordinates

$$\mathrm{p}_{\mathrm{red}} := (\Theta, \mathrm{G}_2, \Lambda, \vartheta, \mathrm{g}_2, \ell)$$

(where $\Lambda = (\Lambda_1, \Lambda_2)$, $\ell = (\ell_1, \ell_2)$) and *parametrically* by G (but not on Z by SO(3) invariance). Consistently, we denote as

$$\phi_{\mathrm{p}_{\mathrm{red}}}: \qquad \mathrm{p}_{\mathrm{red}} \in \mathbf{R}^4 \times \mathbf{T}^4 \to (y^{(1)}, y^{(2)}, x^{(1)}, x^{(2)}) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$$

the immersion given by the restriction of ϕ_{p} to the (rotating, vertical angular momentum) manifold

$$\big\{Z = \mathrm{G}, \quad \mathrm{g} = 0\big\}$$

so that $\mathrm{H}_{\mathrm{p}} = \mathrm{H}_{\mathrm{p}_{\mathrm{red}}}$.

- ⋆ The $\mathrm{p}_{\mathrm{red}}$-coordinates defined above are well fitted to reflection transformation with respect to the second coordinate plane.

  That is, if we denote as $\mathcal{S}^-$ the sign-inversion of the (Θ, ϑ)'s coordinates

$$\mathcal{S}^-(\Theta, \mathrm{G}_2, \Lambda, \vartheta, \kappa, \ell) := (-\Theta, \mathrm{G}_2, \Lambda, 2k\pi - \vartheta, \kappa, \ell),$$

  and as $\mathcal{R}_2^-$ the reflection

$$\mathcal{R}_2^-(y_1^{(j)}, y_2^{(j)}, y_3^{(j)}, x_1^{(j)}, x_2^{(j)}, x_3^{(j)}) = (y_1^{(j)}, -y_2^{(j)}, y_3^{(j)}, x_1^{(j)}, -x_2^{(j)}, x_3^{(j)}) \qquad j = 1, 2,$$

  then

$$\mathcal{R}_2^- \circ \phi_{\mathrm{p}_{\mathrm{red}}} = \phi_{\mathrm{p}_{\mathrm{red}}} \circ \mathcal{S}^-. \tag{42}$$

  Relation (42) holds true even in the case $N \geq 3$, as discussed in Ref.[27] (Section 2.1). We remark that this feature is completely new, compared with the behavior of Deprit coordinates (see Appendix D) in the case $N \geq 3$. We mean that, while an analogue property also holds for the jrd coordinates in (14), in the case $N \geq 3$, no simple sign reversal $\mathcal{S}^-$ exists such that a relation like (42) is verified, replacing $\phi_{\mathrm{p}_{\mathrm{red}}}$ with Deprit coordinates and $\mathcal{R}_2^-$ with any reflection with respect to some coordinate plane.

- ⋆ Since the perturbing function in (2) does not change under the action of $\mathcal{R}_2^-$, the identity in (42) implies the parity property

$$f_{\mathrm{p}_{\mathrm{red}}}(-\Theta, \mathrm{G}_2, \Lambda, 2k\pi - \vartheta, \kappa, \ell) = f_{\mathrm{p}_{\mathrm{red}}}(\Theta, \mathrm{G}_2, \Lambda, \vartheta, \kappa, \ell). \tag{43}$$

  As a consequence, the sections

$$(\Theta, \vartheta) = (0, k\pi) \qquad k = 0, 1$$

  are *equilibria* to f_{p}, which will be called *planar equilibria* (since they correspond to motions on the plane $\{y^{(2)} = x^{(2)} = 0\}$ in configuration space). We may distinguish three different planar configurations:

$$\begin{aligned}
(\uparrow\uparrow) &:= \Big\{\mathrm{p}_{\mathrm{red}} : \ (\Theta, \vartheta) = (0, \pi) \quad \& \quad \sigma = -1\Big\};\\
(\downarrow\uparrow) &:= \Big\{\mathrm{p}_{\mathrm{red}} : \ (\Theta, \vartheta) = (0, \pi) \quad \& \quad \sigma = +1\Big\};\\
(\uparrow\downarrow) &:= \Big\{\mathrm{p}_{\mathrm{red}} : \ (\Theta, \vartheta) = (0, 0)\Big\}.
\end{aligned}$$

with $\sigma := \mathrm{sign}\,(\mathrm{G}_2 - \mathrm{G})$. The planar configurations above differ by the sense of rotation of the two planets: as one immediately sees from the formula in (45), the first/second arrow points up (down) if the inner/outer planet has a prograde (retrograde) motion in the sense of Note[43]. In particular, the invariant equilibrium manifold $(\uparrow\downarrow)$ corresponds to retrograde motion for the outer planet.

- $\star$ In the next section, we shall be concerned with the study of the invariant equilibrium manifold $(\uparrow\downarrow)$, in relation with the *secular system*

$$\overline{\mathrm{H}_{\mathrm{P}}}(\Theta, \mathrm{G}_2, \Lambda_1, \Lambda_2, \vartheta, \mathrm{g}_2; \mathrm{G}) = \mathrm{h}_{\mathrm{K}}(\Lambda) + \mu \overline{f_{\mathrm{P}}}(\Theta, \mathrm{G}_2, \Lambda_1, \Lambda_2, \vartheta, \mathrm{g}_2; \mathrm{G})$$

where $\overline{f_{\mathrm{P}}}$ is the (ℓ_1, ℓ_2)-averaged ("secular") perturbing function

$$\overline{f_{\mathrm{P}}}(\Theta, \mathrm{G}_2, \Lambda_1, \Lambda_2, \vartheta, \mathrm{g}_2; \mathrm{G}) = \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} f_{\mathrm{P}}(\Theta, \mathrm{G}_2, \Lambda_1, \Lambda_2, \vartheta, \mathrm{g}_2, \ell_1, \ell_2; \mathrm{G}) d\ell_1 d\ell_2 \,. \quad (44)$$

Observe that the secular system depends on two angles (g_2 and ϑ), and hence is *not integrable*. However, a deeper insight to $\overline{f_{\mathrm{P}}}$ reveals a close–to–integrability feature. Indeed, for all maps K of the form in (7), the double average of f_{K} coincides with the average of only the Newtonian part. Looking then at the semi–major axes expansion of such Newtonian term

$$\overline{f_{\mathrm{K}}} = -\frac{m_1 m_2}{a_2} \sum_{j=0}^{\infty} \overline{f_{\mathrm{K}}^{(j)}} \alpha^j \qquad \alpha = \frac{a_1}{a_2} = \frac{\mathrm{m}_2 \mathrm{M}_2}{\mathrm{m}_1 \mathrm{M}_1} \frac{\Lambda_1^2}{\Lambda_2^2}$$

it turns out that: (i) the former non–trivial term in the α–expansion above is the second–order term $\overline{f_{\mathrm{K}}^{(2)}}$ (see Appendix B, in particular: Equation (B2)). Moreover, (ii) for those K–maps, like P, such that some anomaly of perihelion of the outer planet $\widehat{\mathrm{g}}_2$ is among the (u, v)'s in (7), $\overline{f_{\mathrm{K}}^{(2)}}$, is actually *integrable*, being independent[40] of $\widehat{\mathrm{g}}_2$. In the next section we shall see that, the case of the p–map, under suitable conditions, the invariant manifold $(\uparrow\downarrow)$ has a hyperbolic character to $\overline{f_{\mathrm{P}}^{(2)}}$. This fact does not hold for the case of the equilibria $(\uparrow\uparrow)$, $(\downarrow\uparrow)$, which, on the contrary, are always *elliptic* (see Proposition III.1 and Remark III.1 below).

- $\star$ The planar equilibria mentioned in the previous item are *regular manifolds* for the p-coordinates, which, under suitable conditions on the remaining coordinates, the system is free to reach. This should be compared with the jrd coordinates in (14), where the number of degrees of freedom is still minimum, but planar configurations are *singular*.

Let us recall, indeed, that the jrd's loose their meaning when any of the vectors ν_1 or ν in (A1) vanishes. In the case of the planar configurations, one has, identically, $\nu = \mathrm{C}^{(2)} \times \mathrm{C}^{(1)} \equiv 0$. It is not so for p, whose regularity falls when any of the vectors ν_i or n_i ($i = 1, 2$) in (40) vanishes. None of such vectors *necessarily* vanishes in the case of planar configurations.

- ⋆ While, as also mentioned in Section II A, using regularized versions of jrd coordinates, one does not find a unique chart which includes *both* equilibria ($\uparrow\downarrow$) and ($\uparrow\uparrow$) or ($\downarrow\uparrow$), on the contrary, in terms of the p-coordinates this is possible. As an important example, the leading part in the semi–major axes ratio expansion of the secular perturbing function of the three–body problem (see Equation (49) below) presents a hyperbolic equilibrium at ($\uparrow\downarrow$), with a closed separatrix through such equilibrium encircling a stable equilibrium ($\uparrow\uparrow$) or ($\downarrow\uparrow$). Such a structure is clearly suited to the (non-trivial) study of *Arnold diffusion* for the problem.

- ⋆ A less pleasant aspect of the p-coordinates is the expression Euclidean length of the inner planet angular momentum $\|\mathrm{C}^{(1)}\|$, which, instead of being an action coordinate, depends on the angle ϑ and is given by

$$\|\mathrm{C}^{(1)}\| = \sqrt{\mathrm{G}^2 + \mathrm{G}_2^2 - 2\Theta^2 + 2\sqrt{\mathrm{G}^2 - \Theta^2}\sqrt{\mathrm{G}_2^2 - \Theta^2}\cos\vartheta}. \tag{45}$$

This has the following consequence. While, following Poincaré, one might consider also motions with vanishing outer's planet eccentricity (which are singular for the coordinates (41) because the perihelion axis becomes undefined) changing the quadruplet $(\Lambda_2, \ell_2, \mathrm{G}_2, \mathrm{g}_2)$ with

$$(\Lambda_2, \mathrm{l}_2, u_2, v_2) := (\Lambda_2, \ell_2 + \mathrm{g}_2, \sqrt{2(\Lambda_2 - \mathrm{G}_2)}\cos\mathrm{g}_2, -\sqrt{2(\Lambda_2 - \mathrm{G}_2)}\sin\mathrm{g}_2)$$

it seems no similar regularization exists for the vanishing of inner planet's eccentricity.

B. A domain of regularity including ($\uparrow\downarrow$)

In this section we establish a suitable domain which includes the equilibrium manifold ($\uparrow\downarrow$) where H_p is regular.

We check below that the following domain is suited to the scope:

$$\widehat{\mathcal{D}}_{\mathcal{U}}(\mathrm{G}) := \Big\{(\Lambda_1, \Lambda_2, \mathrm{G}_2, \Theta, \vartheta) :\ (\Lambda_1, \Lambda_2, \mathrm{G}_2) \in \mathcal{A}(\mathrm{G}), (\Theta, \vartheta) \in \mathcal{B}(\mathrm{G}_2, \mathrm{G})\Big\} \times \mathbf{T}^3.$$

where, if $\mathcal{L}_0$ is as in (21),

$$\begin{aligned}\mathcal{A}(\mathrm{G}) &:= \Big\{(\Lambda_1, \Lambda_2, \mathrm{G}_2) :\ (\Lambda_1, \Lambda_2) \in \mathcal{L}(\mathrm{G}), \mathrm{G}_2 \in \mathcal{G}(\Lambda_1, \Lambda_2, \mathrm{G})\Big\}\\ \mathcal{B}(\mathrm{G}_2, \mathrm{G}) &:= \Big\{(\Theta, \vartheta) :\ |\Theta| < \frac{1}{2}\min\{\mathrm{G}, \mathrm{G}_2\}, |\vartheta| < \frac{\pi}{2}\Big\}\end{aligned} \tag{46}$$

with

$$\begin{aligned}\mathcal{L}(\mathrm{G}) &:= \Big\{\Lambda = (\Lambda_1, \Lambda_2) :\ \Lambda \in \mathcal{L}_0, \quad \Lambda_1 > \mathrm{G} + \frac{2}{c}\sqrt{\alpha_+}\Lambda_2\Big\}\\ \mathcal{G}(\Lambda_1, \Lambda_2, \mathrm{G}) &:= \Big(\mathrm{G}_-, \mathrm{G}_+\Big), \qquad \mathrm{G}_- := \frac{2}{c}\sqrt{\alpha_+}\Lambda_2 \qquad \mathrm{G}_+ := \min\Big\{\Lambda_1 - \mathrm{G}, \Lambda_2\Big\}.\end{aligned}$$

where $\mathcal{L}_0$ is as in (21), while c is an arbitrarily fixed number in $(0, 1)$. We need to establish two kinds of conditions.

- ⋆ *Geometric conditions*

First of all, we need that the planets' eccentricities e_1, e_2 stay strictly confined in $(0, 1)$. Then the following inequalities are to be satisfied:

$$0 < \|\mathrm{C}^{(1)}_{\mathrm{p}}\| < \Lambda_1 \qquad 0 < \mathrm{G}_2 < \Lambda_2, \tag{47}$$

with $\|\mathrm{C}^{(1)}_{\mathrm{p}}\|$ as in (45). Note that $\|\mathrm{C}^{(1)}_{\mathrm{p}}\|$ may vanish only for

$$(\mathrm{G}_2, \vartheta) = (\mathrm{G}, \pi).$$

Since we deal with the equilibrium $(\uparrow\downarrow)$ (which holds for $(\Theta, \vartheta) = (0, 0)$), the occurrence of this equality is automatically excluded, limiting the values of the coordinates (Θ, ϑ) in the set $\mathcal{B}$ in (46) since in this case

$$\|\mathrm{C}^{(1)}_{\mathrm{p}}\|^2 \geq \frac{3}{4}\mathrm{G}^2. \tag{48}$$

Moreover, the two right inequalities in (47) are satisfied taking

$$\mathrm{G}_2 < \min\Big\{\Lambda_1 - \mathrm{G}, \Lambda_2\Big\} = \mathrm{G}_+$$

where we have used the triangular inequality $\|\mathrm{C}^{(1)}_{\mathrm{p}}\| \leq \|\mathrm{C}_{\mathrm{p}}\| + \|\mathrm{C}^{(2)}_{\mathrm{p}}\| = \mathrm{G} + \mathrm{G}_2$.

⋆ *Non-collision conditions*

We have to exclude possible encounters of the planets with the sun and each other. Collisions of the inner planet with the sun are excluded by (46). Indeed, using (48), with $\Lambda_1^+ := k_+\Lambda_2^+$,

$$1 - e_1^2 = \frac{\|\mathrm{C}_\mathrm{P}^{(1)}\|^2}{\Lambda_1^2} \geq \frac{3}{4}\frac{\mathrm{G}^2}{(\Lambda_1^+)^2}$$

whence the minimum distance of the inner planet with the sun $a_1(1-e_1)$ is positive. In order to avoid planetary collisions, it is typical to ensure the following inequality:

$$a_1(1+e_1) < c^2 a_2(1-e_2)$$

with $0 < c < 1$. A sufficient condition for it is

$$\mathrm{G}_2 \geq \frac{2}{c}\sqrt{\alpha_+}\Lambda_2 = \mathrm{G}_-.$$

Indeed, if this inequality is satisfied, one has

$$a_1(1+e_1) < 2a_1 < \frac{a_2}{2}\frac{\mathrm{G}_2^2 c^2}{\Lambda_2^2} = \frac{a_2}{2}(1-e_2^2)c^2 < a_2(1-e_2)c^2.$$

C. Bifurcations to instability in the secular problem

The manifold $(\uparrow\downarrow)$ defined in the previous section is invariant to f_P. Then, it is invariant to the "secular" (i.e., averaged) perturbing function in (44). It is possible (see Appendix B) to write $\overline{f_\mathrm{P}}$ as

$$\overline{f_\mathrm{P}} = -\frac{m_1 m_2}{a_2}\Big(1 + \alpha^2(\mathrm{P}_0 + \mathrm{P}) + \alpha^3\widehat{\mathrm{P}}\Big) \tag{49}$$

where P_0 is independent of (Θ, ϑ) and P vanishes for $(\Theta, \vartheta) = (0, 0)$. Moreover, as mentioned in the Section III A, both P_0 and P do not depend on g_2. Therefore, it is quite natural to study the character of $(\uparrow\downarrow)$ to P, accordingly to the values of $(\Lambda_1, \Lambda_2, \mathrm{G}_2)$. To proceed, we need to recall the definition of the sets $\mathcal{A}(\mathrm{G})$, $\mathcal{B}(\mathrm{G}_2, \mathrm{G})$ in (46).

Proposition III.1 *For any fixed* $\mathrm{G} \in \mathbf{R}_+$*, there exist suitable domains* $\mathcal{A}_\mathcal{U}(\mathrm{G}) \subset \mathcal{A}(\mathrm{G})$, $\mathcal{B}_\mathcal{U}(\mathrm{G}) \subset \cap_{\mathrm{G}_2 \in \Pi_{\mathrm{G}_2}\mathcal{A}(\mathrm{G})}\mathcal{B}(\mathrm{G}_2, \mathrm{G})$ *depending only on* G *such that* $(\Theta, \vartheta) = (0, 0)$ *is a hyperbolic equilibrium point for* $\mathrm{P}_\mathcal{U}$*, the restriction of* P *in* (49) *to* $\mathcal{A}_\mathcal{U} \times \mathcal{B}_\mathcal{U}$*. More precisely, there exist two functions* ω, Ω *of* Λ_1, Λ_2, G_2 *and the parameter* G*, with* $\omega > 0$ *such that, in terms of the canonical coordinates*[37]

$$p_0 := \frac{\Theta - \omega\vartheta}{\sqrt{2\omega}}, \qquad q_0 := \frac{\Theta + \omega\vartheta}{\sqrt{2\omega}} \tag{50}$$

one has

$$\mathrm{P}_{\mathcal{U}} = \Omega p_0 q_0 + \mathrm{O}(p_0, q_0; \Lambda_2, \Lambda_2, \mathrm{G}_2, \mathrm{G})^4.$$

We shall prove the proposition with

$$\begin{aligned}
\Omega &:= -\frac{3}{4}\frac{\Lambda_2^3}{\Lambda_1^2 \mathrm{G}_2^4}\sqrt{(5\Lambda_1^2 \mathrm{G} - (\mathrm{G}+\mathrm{G}_2)^2(4\mathrm{G}+\mathrm{G}_2))(\mathrm{G}_2-\mathrm{G})} \\
\omega &:= \mathrm{G}\mathrm{G}_2\sqrt{\frac{\mathrm{G}_2-\mathrm{G}}{5\Lambda_1^2\mathrm{G} - (\mathrm{G}+\mathrm{G}_2)^2(4\mathrm{G}+\mathrm{G}_2)}} \\
\mathcal{A}_{\mathcal{U}}(\mathrm{G}) &:= \Big\{(\Lambda_1, \Lambda_2) \in \mathcal{L}_{\mathcal{U}}(\mathrm{G}), \quad \mathrm{G}_2 \in \mathcal{G}_{\mathcal{U}}(\Lambda_1, \Lambda_2, \mathrm{G})\Big\} \\
\mathcal{B}_{\mathcal{U}}(\mathrm{G}) &:= \Big\{(\Theta, \vartheta):\ |\Theta| < \frac{\mathrm{G}}{2}, |\vartheta| < \frac{\pi}{2}\Big\}
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
\mathcal{L}_{\mathcal{U}}(\mathrm{G}) &:= \Big\{\Lambda = (\Lambda_1, \Lambda_2) \in \mathcal{L}_0:\ 5\Lambda_1^2\mathrm{G} - (\mathrm{G} + \frac{2}{c}\sqrt{\alpha_+}\Lambda_1)^2(4\mathrm{G} + \frac{2}{c}\sqrt{\alpha_+}\Lambda_1) > 0, \\
&\qquad \Lambda_2 > \mathrm{G}, \Lambda_1 > \max\{\mathrm{G} + \frac{2}{c}\sqrt{\alpha_+}\Lambda_2, 2\mathrm{G}\}\Big\} \\
\mathcal{G}_{\mathcal{U}}(\Lambda_1, \Lambda_2, \mathrm{G}) &:= \Big(\overline{\mathrm{G}}_-, \overline{\mathrm{G}}_+\Big)
\end{aligned} \tag{52}$$

where $\mathcal{L}_0$ is as in (21) and, if $\mathrm{G}^\star(\Lambda_1, \mathrm{G})$ is the unique positive root of the cubic polynomial $\mathrm{G}_2 \to 5\Lambda_1^2\mathrm{G} - (\mathrm{G}+\mathrm{G}_2)^2(4\mathrm{G}+\mathrm{G}_2)$, then

$$\overline{\mathrm{G}}_- := \max\{\frac{2}{c}\sqrt{\alpha_+}\Lambda_2, \mathrm{G}\} \qquad \overline{\mathrm{G}}_+ := \min\{\Lambda_2, \mathrm{G}^\star\}. \tag{53}$$

Implicitly, we shall prove that

$$\overline{\mathrm{G}}_- < \overline{\mathrm{G}}_+. \tag{54}$$

Proof. The expansion of P in (B4) around $(0,0)$ is

$$\mathrm{P} = -\frac{1}{8}\frac{\Lambda_2^3}{\Lambda_1^2\mathrm{G}_2^5} \times \Big[\frac{3}{\mathrm{G}}a(\Lambda_1, \mathrm{G}_2; \mathrm{G})\Theta^2 + 3\mathrm{G}\mathrm{G}_2^2 b(\mathrm{G}_2; \mathrm{G})\vartheta^2 + o_2(\Theta, \vartheta)\Big]$$

where

$$a(\Lambda_1, \mathrm{G}_2; \mathrm{G}) := 5\Lambda_1^2\mathrm{G} - (\mathrm{G}+\mathrm{G}_2)^2(4\mathrm{G}+\mathrm{G}_2) \quad \text{and} \quad b(\mathrm{G}_2; \mathrm{G}) := \mathrm{G} - \mathrm{G}_2. \tag{55}$$

Both $\mathrm{G}_2 \to a(\Lambda_1, \mathrm{G}_2; \mathrm{G})$ and $\mathrm{G}_2 \to b(\mathrm{G}_2; \mathrm{G})$, as functions of G_2 decrease monotonically from a positive value (respectively, $\mathrm{G}(5\Lambda_1^2 - 4\mathrm{G}^2)$ and G) to $-\infty$ as G_2 increases from $\mathrm{G}_2 = 0$ to $\mathrm{G}_2 = +\infty$. The function $a(\Lambda_1, \mathrm{G}_2; \mathrm{G})$ changes its sign for G_2 equal to a suitable unique positive value $\mathrm{G}^\star(\Lambda_1, \mathrm{G})$, while $b(\mathrm{G}_2; \mathrm{G})$ does it for $\mathrm{G}_2 = \mathrm{G}$. We note that (i) inequality

$G < \min\{G_+, G^\star\}$ follows immediately from the assumptions (52) (in particular, the two last ones) and (ii), more generally, that $G^\star \le G$ is equivalent to $\Lambda_1 \le 2G$. Since, for our purposes, we have to exclude $G^\star = G$ (otherwise, $a(\Lambda_1, G_2; G)$ and $b(G_2; G)$ would be simultaneously positive and simultaneously negative, and no hyperbolicity would be possible), we distinguish two cases.

(a) $G > \frac{2}{c}\sqrt{\alpha_+}\Lambda_2$ and $G + \frac{2}{c}\sqrt{\alpha_+}\Lambda_2 < \Lambda_1 < 2G$. In this case $G^\star < G$. We show that no such $\mathcal{G}_{\mathcal{U}}$ can exist in this case. In fact, since $G^\star < G$, in order that the interval $(G^\star, G)$ and the set $\mathcal{G}$ have a non-empty intersection, one should have, necessarily, $G_+ = \sup \mathcal{G} > G^\star$, hence, in particular, $\Lambda_1 - G > G^\star$. Using the definition of $G^\star$, this would imply $\Lambda_1 > 2G$, which is a contradiction.

(b) $\Lambda_1 > \max\{2G, G + \frac{2}{c}\sqrt{\alpha_+}\Lambda_2\}$. In this case $G < G^\star < \Lambda_1 - G$. In order that the interval $(G, G^\star)$ and the set $\mathcal{G}$ have a non-empty intersection, we need

$$G_- < G^\star \qquad \text{and} \qquad G_+ > G \tag{56}$$

and such intersection will be given by the interval $\mathcal{G}_{\mathcal{U}}$ as in (52). Note that the definition of $\overline{G}_+$ does not include $\Lambda_1 - G$ in the brackets because, as noted, $G^\star < \Lambda_1 - G$. But (56) are equivalent to (52).

Remark III.1

- ⋆ The analysis worked out in Proposition III.1 would have worked word-by-word (without affecting the results of Theorems III.1, IV.1 below) using, instead of the coordinates in (41), the coordinates

$$\widetilde{p} := (Z, \widetilde{\Theta}, \chi, \Lambda, \zeta, \widetilde{\vartheta}, \widetilde{\kappa}, \ell)$$

where $\widetilde{\Theta}$, $\widetilde{\vartheta}$, $\widetilde{\kappa} = (\widetilde{g}, \widetilde{g}_2)$ are defined, in terms of the coordinates (14), as

$$\widetilde{\Theta} = \sqrt{2(G - G_1 + G_2)}\cos\gamma_1, \quad \widetilde{\vartheta} = -\sqrt{2(G - G_1 + G_2)}\sin\gamma_1,$$
$$\widetilde{g} = \gamma + \gamma_1, \quad \widetilde{g}_2 = \gamma_1 + \gamma_2$$

while the remaining ones are the same (of those with the same names) as in (41). We remark however that the $\widetilde{p}$-coordinates do not exhibit the advantages mentioned in Section III A above for the p's. In particular, the $\widetilde{p}$'s are regular around $(\uparrow\downarrow)$, but not around $(\uparrow\uparrow)$ or $(\downarrow\uparrow)$.

- $\star$ The "bifurcation" towards the hyperbolic behavior that Proposition III.1 talks about does not appear in the case of the equilibria $(\uparrow\uparrow)$ and $(\downarrow\uparrow)$, which, in contrast, are always *elliptic*. The equilibrium $(\uparrow\uparrow)$ has been worked out in Ref.[27], in the more general situation with $N \geq 2$ planets.

  Indeed, in such cases, one obtains an expansion analogous to (49)-(B4), with the coefficients a, b in (55) to be replaced by

$$\widehat{a} = 5\Lambda_1^2 \mathrm{G} - (\mathrm{G} - \mathrm{G}_2)^2(4\mathrm{G} - \mathrm{G}_2), \qquad \widehat{b} = \mathrm{G} + \mathrm{G}_2.$$

  Clearly, $\widehat{b}$ is positive for all G_2 and $\widehat{a}$ is so for $\mathrm{G}_2 \geq 4\mathrm{G}$. On the other hand, when $\mathrm{G}_2 < 4\mathrm{G}$, inequality $|\mathrm{G} - \mathrm{G}_2| < \Lambda_1$ implies

$$\widehat{a} \geq \Lambda_1^2(\mathrm{G} + \mathrm{G}_2) > 0.$$

  Therefore, $\widehat{a}$ and $\widehat{b}$ have always the same (positive) sign.

D. Three-dimensional whiskered tori

Let $\mathcal{A}_{\mathcal{U}}$, $\mathcal{B}_{\mathcal{U}}$ be as in Proposition III.1. On the domain

$$\mathcal{D}_{\mathcal{U}} := \mathcal{A}_{\mathcal{U}} \times \mathcal{B}_{\mathcal{U}} \times \mathbf{T}^3$$

consider the "averaged, α-truncated" Hamiltonian

$$\mathrm{H}_0(\Lambda_1, \Lambda_2, \mathrm{G}_2, \Theta, \vartheta; \mathrm{G}) := \mathrm{h}_{\mathrm{k}}(\Lambda_1, \Lambda_2) - \mu \frac{m_1 m_2}{a_2}\Big(1 + \alpha^2(\mathrm{P}_0 + \mathrm{P})\Big) \tag{57}$$

where h_{k} is as in (8), P_0, P as in (49).

H_0 possesses, by Proposition III.1, a hyperbolic fixed point at $(\Theta, \vartheta) = (0, 0)$ and hence a family of three-dimensional tori having equation

$$\mathcal{T}_{\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star} = \Big\{(\Lambda_1, \Lambda_2, \mathrm{G}_2) = (\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star), \quad (\ell_1, \ell_2, \mathrm{g}_2) \in \mathbf{T}^3, \quad (\Theta, \vartheta) = (0, 0)\Big\}$$

parametrized by $(\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star) \in \mathcal{A}_{\mathcal{U}}(\mathrm{G})$. The tori $\mathcal{T}_{\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star}$ may also be viewed as the intersection of two four-dimensional manifolds, the *whiskers*, surrounded by orbits of H_0 which approach/leave the torus at an exponential rate, with velocity $\Omega^\star := \Omega(\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star; \mathrm{G})$. Their equation, closely to the equilibrium, is

$$\begin{aligned} \mathcal{W}^{\mathrm{s,loc}}_{\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star, \varepsilon} &= \Big\{(\Lambda_1, \Lambda_2, \mathrm{G}_2) = (\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star), (\ell_1, \ell_2, \mathrm{g}_2) \in \mathbf{T}^3, \quad q_0 = 0, |p_0| < \varepsilon\Big\} \\ \mathcal{W}^{\mathrm{u,loc}}_{\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star, \varepsilon} &= \Big\{(\Lambda_1, \Lambda_2, \mathrm{G}_2) = (\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star), (\ell_1, \ell_2, \mathrm{g}_2) \in \mathbf{T}^3, \quad p_0 = 0, |q_0| < \varepsilon\Big\} \end{aligned} \tag{58}$$

where p_0, q_0 are as in (50), with ω replaced by $\omega^* := \omega(\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star; \mathrm{G})$. More generally, closely to each fixed torus $\mathcal{T}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star}$, the trajectories of H_0 are partially hyperbolic and evolve in the full (eight-)dimensional set

$$\mathcal{W} = \bigcup_{(\overline{\Lambda}_1,\overline{\Lambda}_2,\overline{\mathrm{G}}_2)\in\mathcal{A}(\mathrm{G}),|\overline{\zeta}|<\varepsilon/2} \mathcal{W}_{\overline{\Lambda}_1,\overline{\Lambda}_2,\overline{\mathrm{G}}_2,\overline{\zeta},\varepsilon}$$

foliated by the four-dimensional invariant manifolds

$$\mathcal{W}_{\overline{\Lambda}_1,\overline{\Lambda}_2,\overline{\mathrm{G}}_2,\overline{\zeta},\varepsilon} = \Big\{(\Lambda_1, \Lambda_2, \mathrm{G}_2) = (\overline{\Lambda}_1, \overline{\Lambda}_2, \overline{\mathrm{G}}_2), \quad (\ell_1, \ell_2, \mathrm{g}_2) \in \mathbf{T}^3, \quad p_0 q_0 = \overline{\zeta}, \quad |(p_0, q_0)| < \varepsilon\Big\}$$

for any $(\overline{\Lambda}_1, \overline{\Lambda}_2, \overline{\mathrm{G}}_2) \in \mathcal{A}(\mathrm{G})$ and $|\overline{\zeta}| < \varepsilon/2$, and the local whiskers in (58) correspond to

$$\mathcal{W}^{\mathrm{s,loc}}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star,\varepsilon} := \mathcal{W}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star,0,\varepsilon} \cap \{q_0 = 0\}, \qquad \mathcal{W}^{\mathrm{u,loc}}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star,\varepsilon} := \mathcal{W}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star,0,\varepsilon} \cap \{p_0 = 0\}.$$

It turns out that this description persists in the full system, at the expenses, for what concerns the surviving tori, of switching to a suitable "Cantor-like" subset of $\mathcal{A}_{\mathcal{U}}(\mathrm{G})$ and, for the dynamics around each preserved torus, to change the neighborhood $\mathcal{W}$ with a lower-dimensional set $\widetilde{\mathcal{W}}$.

Indeed, we have the following result, consisting of a more precise statement of the ($\mathcal{U}$)-part of Theorem A.

Theorem III.1 *Let $\overline{\mu}_*$ be as in Theorem II.1. There exists a number α_+ such that for any $\alpha_- < \alpha_+$, $0 < \Lambda_- < \Lambda_+$, and any $\alpha_- < \alpha < \alpha_+$, one can find $\alpha^\rho < \mu_*(\alpha) < \overline{\mu}_*$ for some $\rho > 0$ such that, for any $0 < \mu < \mu_*(\alpha)$ the following holds. There exists a set $\mathcal{A}^{\mathrm{nr}}_{\mathcal{U}}(\mathrm{G}) \subset \mathcal{A}_{\mathcal{U}}(\mathrm{G})$ with density $1 - \alpha^{p'} - (\mu/\mu_*(\alpha))^{q'}$ for some $0 < p', q' < 1$, such that, for any $(\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star) \in \mathcal{A}^{\mathrm{nr}}_{\mathcal{U}}(\mathrm{G})$ one can find a three-dimensional invariant manifold*

$$\widetilde{\mathcal{T}}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star} \subset \mathcal{A}_{\mathcal{U}} \times \Big\{(0,0)\Big\} \times \mathbf{T}^3 =: \widehat{\mathcal{D}}_{\mathcal{U}} \tag{59}$$

which is $(\alpha, \mu/\mu_*(\alpha))$-close to $\mathcal{T}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star}$, made of quasi-periodic, motions with irrational (diophantine) frequencies. Moreover, for any $\widetilde{\mathcal{T}}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star}$ there exist a positive number ε and a five-dimensional set $\widetilde{\mathcal{W}}$ including $\widetilde{\mathcal{T}}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star}$ which foliates as

$$\widetilde{\mathcal{W}} = \bigcup_{|\zeta|<\varepsilon/2} \widetilde{\mathcal{W}}_{\zeta,\varepsilon}$$

where any $\widetilde{\mathcal{W}}_{\zeta,\varepsilon}$ is an invariant manifold having the form

$$\widetilde{\mathcal{W}}_{\zeta,\varepsilon} = \widetilde{\mathcal{W}}_{\overline{\Lambda}_1,\overline{\Lambda}_2,\overline{\mathrm{G}}_2,\zeta,\varepsilon}$$

for suitable $\overline{\Lambda}_1$, $\overline{\Lambda}_2$, $\overline{\mathrm{G}}_2$ *depending on* $(\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star,\ \zeta,\ \mu,\ \alpha,\ \varepsilon)$*, and* $\widetilde{\mathcal{W}}_{\overline{\Lambda}_1,\overline{\Lambda}_2,\overline{\mathrm{G}}_2,\zeta,\varepsilon}$ $(\alpha, \mu/\mu_*(\alpha))$*-close to* $\mathcal{W}_{\overline{\Lambda}_1,\overline{\Lambda}_2,\overline{\mathrm{G}}_2,\overline{\zeta},\varepsilon}$*. In particular,* $\widetilde{\mathcal{W}}_{\zeta,\varepsilon}$ *includes suitable four-dimensional sets ("whiskers")* $\widetilde{\mathcal{W}}^{\mathrm{s,loc}}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star,\varepsilon}$*,* $\widetilde{\mathcal{W}}^{\mathrm{u,loc}}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star,\varepsilon}$ $(\alpha, \mu/\mu_*(\alpha))$*-close to the unperturbed whiskers* $\mathcal{W}^{\mathrm{s,loc}}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star,\varepsilon}$*,* $\mathcal{W}^{\mathrm{u,loc}}_{\Lambda_1^\star,\Lambda_2^\star,\mathrm{G}_2^\star,\varepsilon}$*.*

The proof of Theorem III.1, that will be detailed in a further publication, consists of joining a well-established technique, aimed at eliminating the "proper degeneracy" of the problem (firstly proposed by V. I. Arnold, Ref.[3] for the case of maximal kam tori; see also, Ref.[6]) to a kam scheme for partially hyperbolic systems, as proposed, e.g., in Refs.[5,35]. Let us briefly describe the main steps.

Idea of proof One starts with the Hamiltonian of the system written in p-coordinates:

$$\begin{aligned}\mathrm{H_p} &= \mathrm{h_k}(\Lambda_1, \Lambda_2) + \mu\Big(\frac{y_{\mathrm{p}}^{(1)} \cdot y_{\mathrm{p}}^{(2)}}{m_0} - \frac{m_1 m_2}{\|x_{\mathrm{p}}^{(1)} - x_{\mathrm{p}}^{(2)}\|}\Big)\\ &= \mathrm{h_k}(\Lambda_1, \Lambda_2) + \mu f_{\mathrm{p}}(\Lambda_1, \Lambda_2, \mathrm{G}_2, \Theta, \ell_1, \ell_2, \mathrm{g}_2, \vartheta; \mathrm{G}) \end{aligned} \tag{60}$$

Then, one proceeds in five steps.

Step 1 Let s, r be small positive numbers such that $\mathrm{H_p}$ is real-analytic on the (r, s)-complex neighborhood of $\mathcal{D}_{\mathcal{U}}$

$$(\mathcal{D}_{\mathcal{U}})_{r,s} := (\mathcal{A}_{\mathcal{U}})_r \times (\mathcal{B}_{\mathcal{U}})_r \times \mathbf{T}^3_s.$$

Let $\mathcal{L}^{\mathrm{nr}}_{\mathcal{U}} \subset \mathcal{L}_{\mathcal{U}}$ be the (Λ_1, Λ_2) pre-images under the frequency map $(\Lambda_1, \Lambda_2) \to \omega_{\mathrm{k}}(\Lambda_1, \Lambda_2)$, where $\omega_{\mathrm{k}} := \partial \mathrm{h_k}$, of (γ, τ)-diophantine numbers, for some small γ and large τ, and let $\mathcal{A}^{\mathrm{nr}}_{\mathcal{U}} \subset \mathcal{A}_{\mathcal{U}}$, $\mathcal{D}^{\mathrm{nr}}_{\mathcal{U}} \subset \mathcal{D}_{\mathcal{U}}$ be correspondingly defined.

Step 2 Properly-degenerate averaging theory, Ref.[6] based on Ref.[32] allows us to conjugate, in the domain $(\mathcal{D}^{\mathrm{nr}}_{\mathcal{U}})_{r/2,s/6}$, the Hamiltonian $\mathrm{H_p}$ in (60) to

$$\begin{aligned}\overline{\mathrm{H}}_{\mathrm{p}} &= \mathrm{h_k}(\Lambda_1, \Lambda_2) + \mu \overline{f_{\mathrm{p}}}(\Lambda_1, \Lambda_2, \mathrm{G}_2, \Theta, \mathrm{g}_2, \vartheta; \mathrm{G}) + \mu^2 \overline{\overline{f_{\mathrm{p}}}}(\Lambda_1, \Lambda_2, \mathrm{G}_2, \Theta, \mathrm{g}_2, \vartheta; \mathrm{G})\\ &\quad + \mu e^{-K} \overline{\overline{\overline{f_{\mathrm{p}}}}}(\Lambda_1, \Lambda_2, \mathrm{G}_2, \Theta, \ell_1, \ell_2, \mathrm{g}_2, \vartheta; \mathrm{G})\end{aligned}$$

where $\overline{f_{\mathrm{p}}}$ is as in (44), $\overline{\overline{f_{\mathrm{p}}}}$, $\overline{\overline{\overline{f_{\mathrm{p}}}}}$ are bounded as $\mu \to 0$, and K is some arbitrarily large parameter.

Step 3 Choosing μ and K^{-1} suitably small with respect to α and using (49), one rewrites $\overline{\mathrm{H}}_{\mathrm{p}}$ as

$$\overline{\mathrm{H}}_{\mathrm{p}} = \mathrm{H}_0 + \mu\alpha^3 \overline{\mathrm{H}}_1$$

with H_0 as in (57) and $\overline{\mathrm{H}}_1$ uniformly bounded for $\mu < \overline{\mu}_*$ and $\alpha \in (\alpha_-, \alpha_+)$.

Step 4 Normal form theory, Ref.[22] allows to find a system of canonical coordinates (Λ_1, Λ_2, G_2, p, $\widehat{\ell}_1$, $\widehat{\ell}_2$, $\widehat{\mathrm{g}}_2$, q) ("normal coordinates") which sees the $\Lambda_1, \Lambda_2, \mathrm{G}_2$ unchanged, sends $(\Theta, \vartheta) = (0, 0)$ to $(p, q) = (0, 0)$ and casts $\overline{\mathrm{H}}_\mathrm{p}$ to

$$\begin{aligned}\widehat{\mathrm{H}}_\mathrm{p}(\Lambda_1, \Lambda_2, \mathrm{G}_2, \Theta, \ell_1, \ell_2, \mathrm{g}_2, \vartheta; \mathrm{G}, \mu) &= \widehat{\mathrm{H}}_0(\Lambda_1, \Lambda_2, \mathrm{G}_2, \Theta, \vartheta; \mathrm{G}, \mu)\\ &\quad + \mu\alpha^3\widehat{\mathrm{H}}_1(\Lambda_1, \Lambda_2, \mathrm{G}_2, \Theta, \ell_1, \ell_2, \mathrm{g}_2, \vartheta; \mathrm{G}, \mu)\end{aligned}$$

with

$$\widehat{\mathrm{H}}_0(\Lambda_1, \Lambda_2, \mathrm{G}_2, \Theta, \vartheta; \mathrm{G}, \mu) = \mathrm{h}_\mathrm{k}(\Lambda_1, \Lambda_2) - \mu\frac{m_1 m_2}{a_2}\Big(1 + \alpha^2(\mathrm{P}_0 + \widehat{\mathrm{P}})\Big) \tag{61}$$

and $\widehat{\mathrm{P}}$ a function of $\Lambda_1, \Lambda_2, \mathrm{G}_2$ and $\zeta := pq$ only.

Step 5 A suitable quantitative kam theory for small perturbation of partially hyperbolic Hamiltonians (obtained, e.g., with a "two-scale" – in the sense of Refs.[3,6] – scheme adapted to the algorithm in Refs.[5,35]) can be applied to (61) to obtain the thesis of Theorem III.1.

We remark that the parity of f_p (see Equation (43)) easily implies that the tori $\widetilde{\mathcal{T}}_{\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star}$ not only stay $(\alpha, \mu/\mu_*(\alpha))$-close to the unperturbed tori $\mathcal{T}_{\Lambda_1^\star, \Lambda_2^\star, \mathrm{G}_2^\star}$, but verify the inclusion in (59).

IV. CO-EXISTENCE OF TORI

In this section, we aim to prove that the stable and unstable motions constructed in the previous sections (Theorem II.1 and Theorem III.1 above) have a common domain of existence.

More precisely, we shall prove the following:

Theorem IV.1 *Assume that all the hypotheses of Theorems II.1- III.1 are verified and, in addition,* $\alpha_+ < \frac{c^2}{16}$*. Then there exist universal numbers* $1 < \underline{k} < \overline{k}$ *and and a positive constant* μ_0*, depending on* m_0*,* m_1*,* m_2*,* α_-*,* α_+*, such that, if*

$$\frac{\alpha_-}{\alpha_+} < \frac{\underline{k}^2}{\overline{k}^2}\ , \quad \max\Big\{\frac{\overline{k}}{\sqrt{\alpha_+}},\ \frac{1}{\sqrt{\alpha_-}}\Big\} < \frac{m_1}{m_2} < \frac{\underline{k}}{\sqrt{\alpha_-}}\ , \quad \mu < \min\{\mu_+(\varepsilon),\ \mu_*(\alpha),\ \mu_0\} \tag{62}$$

the sets $\widehat{\mathcal{D}}_\mathcal{S}$ *and* $\widehat{\mathcal{D}}_\mathcal{U}$ *have a non–empty intersection.*

Since Theorem IV.1 will be obtained intersecting the results of Theorem II.1 and Theorem III.1, the two families of tori tend to fill their respective ambient spaces as α and ε

decrease and μ is small with respect to α, ε. However, as one might expect, the estimated conditions for existence of the maximal tori and for the unstable ones depend on the coordinates and might be incompatible for all μ, α for which one of the two kinds of tori is shown to exist. Not to mention the completely different behavior of the two families of tori in relation to the smallness of parameter ε: due to the singularity of the coordinates (41) when the inner planet's eccentricity vanishes (see the last item in Section III A; the discussion below Equation (45)), nothing is known about the fate of the whiskered tori when ε approaches zero. On the contrary, the situation with the eccentricities of the two planets and their mutual inclination simultaneously small is the main condition of existence of the stable tori mentioned in Theorem II.1.

Our purpose is to prove that, instead, if the assumptions of Theorems II.1, III.1 are simultaneously satisfied, then the sets $\widehat{\mathcal{D}}_{\mathcal{S}}$, $\widehat{\mathcal{D}}_{\mathcal{U}}$ have a non–empty intersection.

To this end, we observe that we may change $\widehat{\mathcal{D}}_{\mathcal{S}}$ with the set $\widehat{\mathcal{D}}^{\star}_{\mathcal{S}}$, analogously defined, but disregarding to exclude the manifold inside parentheses in (27). Indeed, since $\widehat{\mathcal{D}}_{\mathcal{U}}$ includes only phase point with non–vanishing eccentricity of the inner planet, namely $(\eta_1, \xi_1) \neq 0$, then $\widehat{\mathcal{D}}^{\star}_{\mathcal{S}} \cap \widehat{\mathcal{D}}_{\mathcal{U}} \neq \emptyset$ implies $\widehat{\mathcal{D}}_{\mathcal{S}} \cap \widehat{\mathcal{D}}_{\mathcal{U}} \neq \emptyset$. As a first step, we write $\widehat{\mathcal{D}}^{\star}_{\mathcal{S}}$ in terms of the coordinates $(\Lambda_1, \Lambda_2, \mathrm{G}_2)$, (Θ, ϑ), $(\ell_1, \ell_2, \mathrm{g}_2)$ in (41). To be sure that this is possible, we assume, without loss of generality, that all those points where some of nodes in (40) vanish have been eliminated from $\widehat{\mathcal{D}}^{\star}_{\mathcal{S}}$. One readily sees, using (25), the definition of (η_2, ξ_2) in (18)–(19) and the triangular inequality, that, in terms of the coordinates (41), $\widehat{\mathcal{D}}^{\star}_{\mathcal{S}}$ becomes

$$\widehat{\mathcal{D}}^{\star}_{\mathcal{S}} := \Big\{ \Big((\Lambda_1, \Lambda_2, \mathrm{G}_2),\ (\Theta, \vartheta),\ (\ell_1, \ell_2, \mathrm{g}_2) \Big) \in \mathcal{A}_{\mathcal{S}} \times \mathcal{B}_{\mathcal{S}} \times \mathbf{T}^3 \Big\}$$

where, if $\mathcal{L}_0$ is as in (21) and

$$\mathcal{L}_{\mathcal{S}}(\mathrm{G}) := \Big\{ \Lambda = (\Lambda_1, \Lambda_2) \in \mathcal{L}_0 :\ |\Lambda_1 - \Lambda_2 - \mathrm{G}| < \varepsilon \Big\}, \quad \mathcal{G}_{\mathcal{S}}(\Lambda_2) := \Big\{ \mathrm{G}_2 :\ 0 < \Lambda_2 - \mathrm{G}_2 < \varepsilon \Big\} \tag{63}$$

then

$$\mathcal{A}_{\mathcal{S}} := \Big\{ (\Lambda_1, \Lambda_2, \mathrm{G}_2) :\ (\Lambda_1, \Lambda_2) \in \mathcal{L}_{\mathcal{S}}, \mathrm{G}_2 \in \mathcal{G}_{\mathcal{S}}(\Lambda_2) \Big\} \tag{64}$$

while

$$\mathcal{B}_{\mathcal{S}} := \Big\{ (\Theta, \vartheta) :\ |(\Theta, \vartheta)| < \varepsilon \Big\},$$

possibly diminishing ε, if necessary. Recalling the definition of $\widehat{\mathcal{D}}_{\mathcal{U}}$ in (59), since, obviously, $(0, 0) \in \mathcal{B}_{\mathcal{S}}$, all we have to do is to check that the intersection $\mathcal{A}_{\mathcal{S}} \cap \mathcal{A}_{\mathcal{U}}$ is non-empty. We do this below.

Proof of Theorem IV.1. We prove the theorem with

$$\underline{k} = \frac{1}{4}\sqrt{\frac{3}{10}(69 + 11\sqrt{33})} \sim 1.57, \quad \overline{k} = 2. \tag{65}$$

Recalling the definition of $\mathcal{A}_{\mathcal{U}}$ in (51)-(52) and the definition of $\mathcal{A}_{\mathcal{S}}$ in (63)-(64), non emptyness of $\mathcal{A}_{\mathcal{S}} \cap \mathcal{A}_{\mathcal{U}}$ is equivalent to the two conditions

$$\mathcal{L}_{\mathcal{S}}(\mathrm{G}) \cap \mathcal{L}_{\mathcal{U}}(\mathrm{G}) \neq \emptyset$$

and

$$\mathcal{G}_{\mathcal{S}}(\Lambda_2) \cap \mathcal{G}_{\mathcal{U}}(\Lambda_1, \Lambda_2, \mathrm{G}) \neq \emptyset \quad \forall\ (\Lambda_1, \Lambda_2) \in \mathcal{L}_{\mathcal{S}}(\mathrm{G}) \cap \mathcal{L}_{\mathcal{U}}(\mathrm{G}).$$

It will be enough to check that

$$\mathcal{L}_{\mathcal{S}}(\mathrm{G}) \cap \mathcal{L}_{\mathcal{U}}(\mathrm{G}) \cap \mathcal{L}_{\mathcal{S}\mathcal{U}}(\mathrm{G}) \neq \emptyset \tag{66}$$

and

$$\mathcal{G}_{\mathcal{S}}(\Lambda_2) \cap \mathcal{G}_{\mathcal{U}}(\Lambda_1, \Lambda_2, \mathrm{G}) \neq \emptyset \quad \forall\ (\Lambda_1, \Lambda_2) \in \mathcal{L}_{\mathcal{S}}(\mathrm{G}) \cap \mathcal{L}_{\mathcal{U}}(\mathrm{G}) \cap \mathcal{L}_{\mathcal{S}\mathcal{U}}(\mathrm{G}), \tag{67}$$

where, if $\overline{\mathrm{G}}_{\pm}$ are as in (53), $\mathcal{L}_{\mathcal{S}\mathcal{U}}$ is defined as

$$\mathcal{L}_{\mathcal{S}\mathcal{U}} := \Big\{(\Lambda_1, \Lambda_2) :\ \overline{\mathrm{G}}_+ = \Lambda_2\Big\}. \tag{68}$$

Note that (67) is certainly satisfied provided (66) is since, in fact, for $(\Lambda_1, \Lambda_2) \in \mathcal{L}_{\mathcal{S}}(\mathrm{G}) \cap \mathcal{L}_{\mathcal{U}}(\mathrm{G}) \cap \mathcal{L}_{\mathcal{S}\mathcal{U}}(\mathrm{G})$,

$$\mathcal{G}_{\mathcal{S}}(\Lambda_2) \cap \mathcal{G}_{\mathcal{U}}(\Lambda_1, \Lambda_2, \mathrm{G}) = \Big\{\mathrm{G}_2 :\ \max\{\overline{\mathrm{G}}_-, \Lambda_2 - \varepsilon\} < \mathrm{G}_2 < \Lambda_2\Big\},$$

which is well-defined by (53)-(54).

On the other hand, in view of the definition of $\overline{\mathrm{G}}_+$ in (53), and of $\mathrm{G}^\star$ a few lines above, $\mathcal{L}_{\mathcal{S}\mathcal{U}}$ in (68) is equivalently defined as

$$\mathcal{L}_{\mathcal{S}\mathcal{U}} = \Big\{(\Lambda_1, \Lambda_2) :\ 5\Lambda_1^2\mathrm{G} - (\mathrm{G} + \Lambda_2)^2(4\mathrm{G} + \Lambda_2) > 0\Big\}. \tag{69}$$

Therefore, in view of this definition and the definitions of $\mathcal{L}_{\mathcal{S}}$, $\mathcal{L}_{\mathcal{U}}$ in (52) and (63), one sees that the set on the left hand side in (66) is determined by inequalities

$$\Lambda_- < \Lambda_2 < \Lambda_+$$
$$k_-\Lambda_2 < \Lambda_1 < k_+\Lambda_2$$

$$
\begin{aligned}
&5\Lambda_1^2\mathrm{G} - (\mathrm{G} + \frac{2}{c}\sqrt{\alpha_+}\Lambda_1)^2(4\mathrm{G} + \frac{2}{c}\sqrt{\alpha_+}\Lambda_1) > 0\\
&\Lambda_2 > \mathrm{G}\\
&\Lambda_1 > \max\{\mathrm{G} + \frac{2}{c}\sqrt{\alpha_+}\Lambda_2, 2\mathrm{G}\}\\
&|\Lambda_1 - \Lambda_2 - \mathrm{G}| < \varepsilon\\
&5\Lambda_1^2\mathrm{G} - (\mathrm{G} + \Lambda_2)^2(4\mathrm{G} + \Lambda_2) > 0
\end{aligned}
$$

where $k_\pm$ are defined in (22). We observe that no phase point with $\Lambda_1 - \Lambda_2 - \mathrm{G} < 0$ will ever satisfy the last inequality and $\Lambda_2 > \mathrm{G}$, therefore we may limit to $\Lambda_1 - \Lambda_2 - \mathrm{G} \geq 0$. Moreover, inequality $\Lambda_1 > 2\mathrm{G}$ is implied by $\Lambda_2 > \mathrm{G}$ and (69). Then, we divide such inequalities in three groups, so as to rewrite the set (66) as the intersection of the sets

$$
\begin{aligned}
\widehat{\mathcal{L}}_1 &:= \Big\{(\Lambda_1, \Lambda_2):\ \Lambda_- < \Lambda_2 < \Lambda_+, \Lambda_2 > \mathrm{G}, \Lambda_1 > 2\mathrm{G},\\
&\qquad\qquad \max\{k_-\Lambda_2, (\mathrm{G} + \Lambda_2)\sqrt{\frac{4\mathrm{G} + \Lambda_2}{5\mathrm{G}}}\} < \Lambda_1 < k_+\Lambda_2\Big\}\\
\widehat{\mathcal{L}}_2 &:= \Big\{(\Lambda_1, \Lambda_2):\ 0 < \Lambda_1 - \Lambda_2 - \mathrm{G} < \varepsilon, \Lambda_1 > \mathrm{G} + \frac{2}{c}\sqrt{\alpha_+}\Lambda_2, \Lambda_2 > \mathrm{G}\Big\}\\
\widehat{\mathcal{L}}_3 &:= \Big\{(\Lambda_1, \Lambda_2):\ 5\Lambda_1^2\mathrm{G} - (\mathrm{G} + \frac{2}{c}\sqrt{\alpha_+}\Lambda_1)^2(4\mathrm{G} + \frac{2}{c}\sqrt{\alpha_+}\Lambda_1) > 0, \Lambda_1 > 2\mathrm{G}\Big\}
\end{aligned}
$$

We now aim to choose the parameters $\Lambda_\pm$, $k_\pm$, and α_+ so as to find a non-empty intersection of the sets a above.

Let us denote as $\mathcal{C}$ the curve (see Figure 1), in the (Λ_2, Λ_1)-plane, having equation

$$
\mathcal{C}: \qquad \Lambda_1 = (\mathrm{G} + \Lambda_2)\sqrt{\frac{4\mathrm{G} + \Lambda_2}{5\mathrm{G}}}. \tag{70}
$$

Let

$$
\Lambda_2 = k\Lambda_1
$$

be any straight line through the origin. The straight line intersecting $\mathcal{C}$ at the point $(\underline{\Lambda_2}, \underline{\Lambda_1}) = (\mathrm{G}, 2\mathrm{G})$ has $\overline{k} = 2$ and intersects this curve, also in the higher point

$$
(\overline{\Lambda_2}, \overline{\Lambda_1}) = \left(\frac{1}{2}(13 + \sqrt{185}), (13 + \sqrt{185})\right)\mathrm{G}.
$$

Any other line with $k > \overline{k}$ has a lower intersection $(\underline{\Lambda_2}', \underline{\Lambda_1}')$ with $\underline{\Lambda_2}' < \mathrm{G}$ and $\underline{\Lambda_1}' < 2\mathrm{G}$ and a higher intersection $(\overline{\Lambda_2}', \overline{\Lambda_1}')$ with $\overline{\Lambda_2}' > \overline{\Lambda_2}$ and $\overline{\Lambda_1}' > \overline{\Lambda_1}$.

The last straight line, in the plane (Λ_2, Λ_1), through the origin intersecting $\mathcal{C}$ is the tangent line, and it is easy to compute (see Appendix C for the detail of the computations)

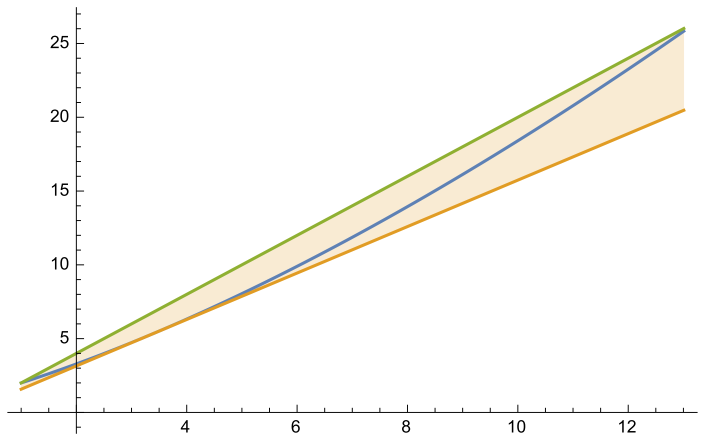

FIG. 1. The blue curve is $\mathcal{C}$; the orange line has slope $\underline{k}$, the green one has slope $\overline{k}$ (MATHEMATICA).

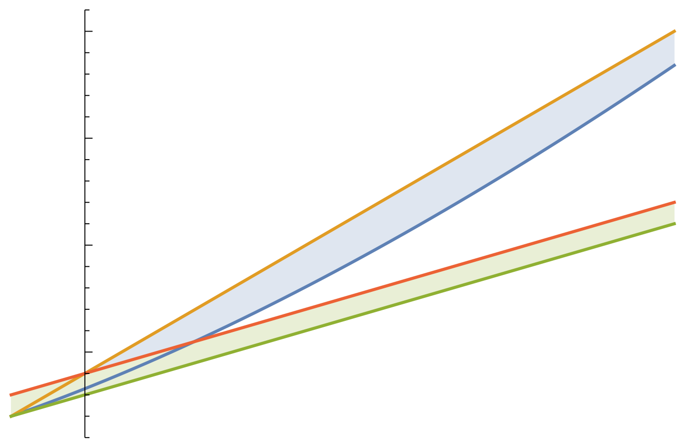

FIG. 2. The blue strip corresponds to the set $\mathcal{L}_1$, the green one to $\mathcal{L}_2$ (MATHEMATICA).

that such a tangent line has slope has slope $\underline{k}$ as in (65) (Figure 1). Observe that the middle assumption in (62) and a suitably small value of μ_0, depending on m_0, m_1, m_2, $\alpha_\pm$ (recall that $\underline{k}$, $\overline{k}$ have been fixed as in (65)) immediately imply

$$k_- < \underline{k}\ , \quad k_+ > \overline{k}\ .$$

We then conclude that, as soon as we choose $\Lambda_- < \underline{\Lambda_2}$, $\Lambda_+ > \overline{\Lambda_2}$, we have the inclusion

$$\widehat{\mathcal{L}}_1 \supset \mathcal{L}_1 := \Big\{(\Lambda_1, \Lambda_2) :\ (\mathrm{G} + \Lambda_2)\sqrt{\frac{4\mathrm{G} + \Lambda_2}{5\mathrm{G}}} < \Lambda_1 \leq 2\Lambda_2\Big\}.$$

Let us now turn to $\widehat{\mathcal{L}}_2$. Since we are assuming $\alpha_+ < \frac{c^2}{16}$, we conclude that the strip

$$\mathcal{L}_2 := \Big\{(\Lambda_1, \Lambda_2) :\ 0 < \Lambda_1 - \Lambda_2 - \mathrm{G} < \varepsilon, \Lambda_2 > \mathrm{G}\Big\}$$

is all included in the region

$$\widetilde{\mathcal{L}}_2 = \left\{(\Lambda_1, \Lambda_2) : \ \Lambda_1 > \mathrm{G} + \frac{2}{c}\sqrt{\alpha_+}\Lambda_2, \Lambda_2 > \mathrm{G}\right\}$$

and this allows us to conclude

$$\widehat{\mathcal{L}}_2 = \mathcal{L}_2 \cap \widetilde{\mathcal{L}}_2 = \mathcal{L}_2.$$

Since the sets $\mathcal{L}_1$ and $\mathcal{L}_2$ have a non-empty intersection, independent of α_+ (see Figure 2), a fortiori, $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ have one:

$$\widehat{\mathcal{L}}_1 \cap \widehat{\mathcal{L}}_2 \supset \mathcal{L}_1 \cap \mathcal{L}_2 \neq \emptyset.$$

Observe, in particular, that $\mathcal{L}_1 \cap \mathcal{L}_2$ (hence, $\widehat{\mathcal{L}}_1 \cap \widehat{\mathcal{L}}_2$) has non-empty intersection with any strip $\mathbf{R} \times \left[2\mathrm{G}, y\right]$, with $y > 2\mathrm{G}$ (see Figure 3). On the other hand, it is immediate to check that $\widehat{\mathcal{L}}_3$ includes the horizontal strip

$$\mathcal{L}_3 := \left\{(\Lambda_1, \Lambda_2) : \ 2\mathrm{G} < \Lambda_1 < \frac{\mathrm{G}}{\frac{2}{c}\sqrt{\alpha_+}}, \Lambda_2 \in \mathbf{R}\right\} \qquad 0 < \alpha_+ < \frac{c^2}{16}$$

and so we conclude

$$\mathcal{L}_{\mathcal{S}}(\mathrm{G}) \cap \mathcal{L}_{\mathcal{U}}(\mathrm{G}) \cap \mathcal{L}_{\mathcal{SU}}(\mathrm{G}) = \widehat{\mathcal{L}}_1 \cap \widehat{\mathcal{L}}_2 \cap \widehat{\mathcal{L}}_3 \supset \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3 \neq \emptyset,$$

which is what we wanted to prove.

ACKNOWLEDGMENTS

I wish to thank the anonymous Referees for their thoughtful and careful remarks, which helped me a lot to improve in a substantial way the presentation of the result. Thanks also to A. Celletti, A. Giorgilli and V. Kaloshin for their interest.

This research was partially supported by H2020 Excellent Science (Grant 677793 StableChaoticPlanetM).

Figures 1, 2, and 3 were produced with Mathematica.

Appendix A: Regularization, second-order expansion, full torsion

1. The Jacobi-Radau-Deprit coordinates

We fix a domain $\mathcal{D}_{\mathrm{jrd}} \subset \mathbf{R}^{12}$ in phase space as follows. Let $(k^{(1)}, k^{(2)}, k^{(3)})$ be a prefixed orthonormal frame in $\mathbf{R}^3$. For $(y^{(1)}, y^{(2)}, x^{(1)}, x^{(2)}) \in \mathcal{D}_{\mathrm{jrd}}$, we assume the following. Let

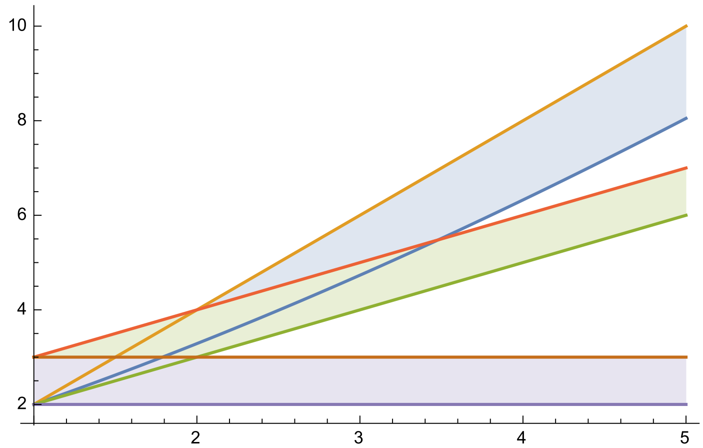

FIG. 3. $\mathcal{L}_1$: the blue region; $\mathcal{L}_2$: the green region; $\mathcal{L}_3$: the violet region (MATHEMATICA).

$\mathrm{C}^{(i)} = x^{(i)} \times y^{(i)}$, $i = 1, 2$, be the angular momenta of the two planets; $\mathrm{C} := \mathrm{C}^{(1)} + \mathrm{C}^{(2)}$ the total angular momentum integral, and assume that the "nodes"

$$\nu_1 := k^{(3)} \times \mathrm{C}\ , \quad \nu := \mathrm{C} \times \mathrm{C}^{(1)} = \mathrm{C}^{(2)} \times \mathrm{C}^{(1)} \tag{A1}$$

do not vanish. We fix the same notations as in Section III A: we assume that the orbits $t \to x^{(i)}(t)$ generated by the Hamiltonians (4) are ellipses with non-vanishing eccentricity. Then we denote as $\mathrm{P}^{(i)}$ the unit vectors pointing in the directions of the perihelia; as a_i the semi–major axes; and, for three vectors u, v, w with u, $v \perp w$, denote as $\alpha_w(u, v)$ the angle formed by u to v relatively to the positive (counterclockwise) orientation established by w. Then we define the jrd-coordinates (14) via the following formulae:

$$\begin{cases} Z := \mathrm{C} \cdot k^{(3)} \\ \mathrm{G} := \|\mathrm{C}\| \\ \mathrm{G}_1 := \|\mathrm{C}^{(1)}\| \\ \mathrm{G}_2 := \|\mathrm{C}^{(2)}\| \\ \Lambda_j := \mathrm{M}_j\sqrt{\mathrm{m}_j a_j} \end{cases} \qquad \begin{cases} \zeta := \alpha_{k^{(3)}}(k^{(1)}, \nu_1) \\ \gamma := \alpha_{\mathrm{C}}(\nu_1, \nu) \\ \gamma_1 := \alpha_{\mathrm{C}^{(1)}}(\nu, \mathrm{P}^{(1)}) \\ \gamma_2 := \alpha_{\mathrm{C}^{(2)}}(\nu, \mathrm{P}^{(2)}) \\ \ell_j := \text{mean anomaly of } x^{(j)} \end{cases} \tag{A2}$$

Such coordinates consist of a modification (useful for applications to celestial mechanics) of a set of canonical coordinates, available indeed for any number $N \geq 2$ of particles, discovered by A. Deprit, Ref.[10]. A discussion of their canonical character may be found in Appendix D.

2. The outer retrograde configuration: regularization, second-order expansion, full torsion

As in the case treated in Refs.[8,28], the proof of Theorem II.1 relies on applying twice (in the partially reduced and fully reduced case, as now we describe) the following kam theorem for properly–degenerate systems. Such theorem goes back to V.I.Arnold[3] (Fundamental Theorem), notwithstanding its refinement given in[6,28]. Its statement is as follows (assumptions $i) + iii)$ are often referred to as "full torsion" or Kolmogorov condition).

Theorem (V.I.Arnold, 1963) *Let*

$$\mathrm{H}(I,\varphi,u,v)=\mathrm{h}(I)+\mu f(I,\varphi,u,v)$$

be real–analytic for $(I,\varphi,u,v)\in\mathcal{P}:=V\times\mathbf{T}^{n_0}\times B^{2n_1}_{\varepsilon}$, *where* $V\subset\mathbf{R}^{n_0}$ *is open and connected,* $\mathbf{T}:=\mathbf{R}/(2\pi\mathbf{Z})$ *and* B^m_r *is the m–dimensional ball with radius r centered at* $0\in\mathbf{R}^m$, $0<\varepsilon<\varepsilon_0$, $0<\mu<\mu_0$. *Assume that*

i) the frequency map $I\to\omega_0(I):=\partial_I\mathrm{h}(I)$ *is a diffeomorphism on V with non–singular Hessian matrix* $\partial^2_I\mathrm{h}(I)$;

ii) the φ–averaged perturbing function $\overline{f}(I,u,v):=\frac{1}{(2\pi)^{n_0}}\int_0^{2\pi}f(I,\varphi,u,v)d\varphi$ *is in Birkhoff normal form of order at least 4:*

$$\overline{f}(I,u,v)=\sum_{h=0}^{2}\sum_{(i_1\cdots i_h)\in\{1,\cdots,n_1\}^h}c^{(h)}_{i_1\cdots i_h}(I)\tau_{i_1}\cdots\tau_{i_h}\qquad\tau_j=\frac{u_j^2+v_j^2}{2}+\mathrm{O}_6(u,v;I)$$

iii) the matrix $\mathrm{T}_{ij}(I):=c^{(2)}_{ij}(I)$ *("torsion") is non–singular on V.*

Then one can find numbers $0<p,q<1$, $0<\epsilon_*<\varepsilon_0$ *and, for any* $0<\varepsilon<\varepsilon_*$ *a number* $0<\mu_*(\varepsilon)<\mu_0$ *such that, for* $0<\mu<\mu_*(\varepsilon)$ *there exists an invariant set* $\mathcal{F}_{\varepsilon,\mu}\subset\mathcal{P}$ *with measure at least* $(1-(\varepsilon/\varepsilon_*)^p-(\mu/\mu_\star(\varepsilon))^q))\,\mathrm{meas}\,\mathcal{P}$ *which foliates in* (n_0+n_1)*–dimensional quasi–periodic tori surrounded of quasi–periodic motions with Diophantine frequencies.*

a) Application to the partially reduced system

As a first step, one looks at the Hamiltonian expressed in the coordinates rps^+_π in (17), which, as discussed in Section II A, is (p_2,q_2)–independent:

$$\mathrm{H}_{\mathrm{rps}^+_\pi}(\Lambda,\lambda,\eta,\xi,p_1,q_1)=\mathrm{h}_{\mathrm{k}}(\Lambda)+\mu f_{\mathrm{rps}^+_\pi}(\Lambda,\lambda,\eta,\xi,p_1,q_1)\ .\tag{A3}$$

As discussed in Section II A, the λ–average $\overline{f_{\rm rps^+_\pi}}$ has an elliptic equilibrium $z = (\eta, \xi, p_1, q_1) = 0$ for all Λ. Proceeding as in Refs.[8,28], one sees that the first order Birkhoff invariants relatively to such equilibrium are not resonant up to any finite order, provided the semi–major axes are chosen to be well–separated. Then it is possible to conjugate ${\rm H}_{\rm rps^+_\pi}$ to a new Hamiltonian

$$
{\rm H}_{\rm b} = {\rm h}_{\rm k}(\Lambda) + \mu f_{\rm b}(\Lambda, \lambda', \eta', \xi', p_1', q_1') \tag{A4}
$$

such that the new λ'–averaged perturbing function $\overline{f_{\rm b}}$ is in Birkhoff normal form of order at least 4, as claimed by condition *ii*). Condition *i*) is trivially satisfied. To check condition *iii*), one proceeds as follows. We shall regard (abusively) the function $f_{\rm rps^+_\pi}$ in (A3) as a function of the coordinates $(\Lambda, \lambda, \eta, \xi, p_1, q_1)$ in (17) or of their complexified canonical version $(\Lambda, \lambda, t, t^*)$ in (19), where $t = (t_1, t_2, t_3)$, $t^* = (t_1^*, t_2^*, t_3^*)$. The use of this latter set is particularly convenient for the computation of the first order Birkhoff invariants, done in Section II A, and of the matrix ${\rm T}_{ij}$, done below. We shall derive such quantities from the corresponding ones for the function $f_{\rm rps}$, worked out in Refs.[8,28], making use of (20).

It follows from (20) that the matrix $\sigma(\Lambda_1, \Lambda_2)$ and the function $\zeta(\Lambda_1, \Lambda_2)$ in (35) are related to the corresponding quantities in the second-order expansion of

$$
\overline{f_{\rm rps}} = \overline{C}_0(\Lambda) + \frac{\rm i}{2}\widehat{\overline{t}}^* \cdot \overline{\sigma}(\Lambda)\widehat{\overline{t}} + \frac{\rm i}{2}\overline{t}_3^* \cdot \overline{\varsigma}(\Lambda)\overline{t}_3 + {\rm O}_4(\overline{t}, \overline{t}^*), \qquad \widehat{\overline{t}} := (\overline{t}_1, \overline{t}_2), \widehat{\overline{t}}^* := (\overline{t}_1^*, \overline{t}_2^*) \tag{A5}
$$

via the relations

$$
\sigma(\Lambda_1, \Lambda_2) = \overline{\sigma}(\Lambda_1, -\Lambda_2) \qquad \varsigma(\Lambda_1, \Lambda_2) = \overline{\varsigma}(\Lambda_1, -\Lambda_2). \tag{A6}
$$

On the other hand, it follows from Refs.[8,28] that

$$
\overline{\sigma}(\Lambda_1, \Lambda_2) = \begin{pmatrix} \frac{\rm s}{\Lambda_1} & \frac{\widetilde{\rm s}}{\sqrt{\Lambda_1\Lambda_2}} \\ \frac{\widetilde{\rm s}}{\sqrt{\Lambda_1\Lambda_2}} & \frac{\rm s}{\Lambda_2} \end{pmatrix} \qquad \overline{\varsigma}(\Lambda_1, \Lambda_2) = -\Big(\frac{1}{\Lambda_1} + \frac{1}{\Lambda_2}\Big){\rm s}, \tag{A7}
$$

with ${\rm s}$, $\widetilde{\rm s}$ as in (37). Therefore, in view of (A6) and the fact that ${\rm s}$, $\widetilde{\rm s}$ are even in Λ_2, we have (36).

We now turn to check that the *torsion matrix* T associated with $\overline{f_{\rm rps^+_\pi}}$ in (35) is non-singular. We recall that T is the matrix of the coefficients of the second-order term

$$
\sum_{i,j=1}^{3} {\rm T}_{ij}(\Lambda_1, \Lambda_2)\tau_i'\tau_j' \tag{A8}
$$

with

$$\tau_1' = \frac{(\eta_1')^2 + (\xi_1')^2}{2} \qquad \tau_2' = \frac{(\eta_2')^2 + (\xi_2')^2}{2} \qquad \tau_3' = \frac{(p_1')^2 + (q_1')^2}{2}$$

of the Birkhoff normal form $\overline{f_{\rm b}}$ associated with $\overline{f_{{\rm rps}^+_\pi}}$. They are computed taking the projections over $\tau_i\tau_j$, where

$$\tau_1 = {\rm i} t_1 t_1^* \qquad \tau_2 = -{\rm i} t_2 t_2^* \qquad \tau_3 = -{\rm i} t_3 t_3^* \tag{A9}$$

of the quartic part in the Taylor expansion of $\overline{f_{{\rm rps}^+_\pi}}$.

We also recall that the corresponding torsion matrix $\overline{\rm T}$ is the second-order term

$$\sum_{i,j=1}^{3} \overline{\rm T}_{ij} \overline{\tau}_i' \overline{\tau}_j'$$

where

$$\overline{\tau}_1' = \frac{\overline{\eta_1'}^2 + \overline{\xi_1'}^2}{2} \qquad \overline{\tau}_2' = \frac{\overline{\eta_2'}^2 + \overline{\xi_2'}^2}{2} \qquad \overline{\tau}_3' = \frac{\overline{p_1'}^2 + \overline{q_1'}^2}{2}$$

associated with $\overline{f_{\rm rps}}$ in (A5) is given by

$$\overline{\rm T} = m_1 m_2 \frac{\alpha^2}{a_2} \begin{pmatrix} \frac{3}{4\Lambda_1^2} & -\frac{9}{4\Lambda_1\Lambda_2} & \frac{3}{\Lambda_1^2} \\ -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{\Lambda_2^2} & \frac{9}{4\Lambda_1\Lambda_2} \\ \frac{3}{\Lambda_1^2} & \frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{4\Lambda_1^2} \end{pmatrix} (1 + {\rm O}(a_2^{-5/4}))$$

(compare, Ref.[8] (Equation (8.6))).

We observe that, in the case treated in[8,28],

$$\overline{\tau}_1 = {\rm i}\overline{t}_1 \overline{t}_1^* \qquad \overline{\tau}_2 = {\rm i}\overline{t}_2 \overline{t}_2^* \qquad \overline{\tau}_3 = {\rm i}\overline{t}_3 \overline{t}_3^*. \tag{A10}$$

By (20), (A9), and (A10), the matrix T in (A8) is obtained from $\overline{\rm T}$ by changing the signs of the entries $(1,3)$, $(2,3)$, $(3,1)$, and $(3,2)$, while leaving all the other entries unvaried.

Then we find

$${\rm T} = m_1 m_2 \frac{\alpha^2}{a_2} \begin{pmatrix} \frac{3}{4\Lambda_1^2} & -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{\Lambda_1^2} \\ -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{\Lambda_2^2} & -\frac{9}{4\Lambda_1\Lambda_2} \\ -\frac{3}{\Lambda_1^2} & -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{4\Lambda_1^2} \end{pmatrix} (1 + {\rm O}(a_2^{-5/4})). \tag{A11}$$

We then conclude, as in Refs.[8,28], that, for small $1/a_2$, T is non-singular since, surprisingly, T has the same determinant as $\overline{\rm T}$ (compare, Ref.[8] (Equation (8.7)))

$$\det {\rm T} = \det \overline{\rm T} = -\frac{27}{16\Lambda_1^4\Lambda_2^2}\Big(m_1 m_2 \frac{\alpha^2}{a_2}\Big)^3 (1 + {\rm O}(a_2^{-5/4})) = -\frac{27}{16}\frac{m_2}{m_1 m_0^3}\frac{a_1^4}{a_2^7}(1 + {\rm O}(\mu) + {\rm O}(a_2^{-5/4})).$$

b) Application to the fully reduced systems

A fully reduced system is obtained applying to the system (A3) any of the transformations mentioned in Note[39]. Let us focus, for simplicity, to the reduction in (29) with $i = 3$, explicitly described in Equation (D19). Let $\mathrm{H}_{\mathrm{red}^+_\pi}(\Lambda, \widetilde\lambda, \widetilde\eta, \widetilde\xi; \mathrm{G}) = \mathrm{h}_{\mathrm{k}}(\Lambda) + \mu f_{\mathrm{red}^+_\pi}(\Lambda, \widetilde\lambda, \widetilde\eta, \widetilde\xi; \mathrm{G})$ be the g–independent Hamiltonian after such transformation, which one regards as a function of $(\Lambda, \widetilde\lambda, \widetilde\eta, \widetilde\xi)$, for any fixed value of the first integral G. It is well defined for $\widetilde\varrho(\Lambda, \mathrm{G})^2 := \mathrm{G} - \Lambda_1 + \Lambda_2 > 0$ and $|\frac{\widetilde\eta_1^2+\widetilde\xi_1^2}{2} - \frac{\widetilde\eta_2^2+\widetilde\xi_2^2}{2}| < \widetilde\varrho(\Lambda, \mathrm{G})^2$. As in Ref.[8] (Section 10), we then fix a domain with $|\frac{\widetilde\eta_1^2+\widetilde\xi_1^2}{2} - \frac{\widetilde\eta_2^2+\widetilde\xi_2^2}{2}| < \widetilde\varrho(\Lambda, \mathrm{G})^2 < \varepsilon^2$. Since g is a cyclic coordinate, the Hamiltonian $\mathrm{H}_{\mathrm{red}^+_\pi}$ can be equivalently obtained letting, into the transformation (D19), $\mathrm{g} = 0$, namely:

$$\mathrm{H}_{\mathrm{red}^+_\pi}(\Lambda, \widetilde\lambda, \widetilde\eta, \widetilde\xi; \mathrm{G}) = \mathrm{H}_{\mathrm{rps}^+_\pi}(\Lambda, \widetilde\lambda, \widetilde\eta, \widetilde\xi, \sqrt{2(\mathrm{G} - \Lambda_1 + \Lambda_2 + \frac{\widetilde\eta_1^2+\widetilde\xi_1^2}{2} - \frac{\widetilde\eta_2^2+\widetilde\xi_2^2}{2}}), 0) \ . \tag{A12}$$

Since the degree in $(\widehat{\widetilde t}, \widetilde t^*)$ and (t_3, t_3^*), respectively, $a_1 + a_2 + a_1^* + a_2^*$ and $a_3 + a_3^*$, are separately even, one has that $\widetilde\lambda$–averaged perturbing function $\overline{f_{\mathrm{red}^+_\pi}}$ has an equilibrium point for $\widetilde z := (\widetilde\eta, \widetilde\xi) = 0$ which, for ε suitably small, turns to be elliptic (for large ε it hyperbolic). Then, as in the partially reduced case considered above, it is possible to apply Arnold's Fundamental Theorem (see above), as soon as one checks assumption *iii*). At the lowest order in ε, its Birkhoff invariants can be equivalently computed applying the transformation (D19) *directly to the Hamiltonian* (A4) (because the normalization of the quadratic form (35) and the Birkhoff transformation leading to (A4) leave the function (25) unvaried). Then, analogously to Ref.[8] (Equation (10.7)) the entries $\widehat{\mathrm{T}}_{ij}$ of the 2×2 second order Birkhoff invariants matrix associated to $\overline{f_{\mathrm{red}^+_\pi}}$ are related to the T_{hk} in (A11) via

$$\widetilde{\mathrm{T}}_{ij} = \mathrm{T}_{ij} + \sigma_i \mathrm{T}_{i3} + \sigma_j \mathrm{T}_{j3} + \sigma_i\sigma_j \mathrm{T}_{33} \ , \qquad i, \ j = 1, \ 2,$$

with $\sigma_1 = -\sigma_2 = 1$. Using this formula to compute the $\widetilde{\mathrm{T}}_{ij}$'s jointly with (A11), one checks that $\det \widetilde{\mathrm{T}} \neq 0$ for well–spaced a_1, a_2. Further details are omitted.

Observe that the procedure described above is not convenient when one aims to obtain, in the "planar limit" , a fully reduced system. Indeed, analogously to Arnold's observation in the case of prograde motions, Ref.[3] (chapter III, §5, n.4, p. 141), the planar limit of the system (A12) is obtained letting $\mathrm{G} - \Lambda_1 + \Lambda_2 + \frac{\widetilde\eta_1^2+\widetilde\xi_1^2}{2} - \frac{\widetilde\eta_2^2+\widetilde\xi_2^2}{2} = 0$. In such limit, one obtains the Hamiltonian $\mathrm{H}_{\mathrm{rps}^+_\pi}|_{(p_1,q_1)=(0,0)}$, which coincides with the one that would be obtained using Poincaré coordinates. It still possesses the integral G in (25) with $(p_1, q_1) = (0, 0)$.

For such scope it is more convenient reduce one of the couples (η_1, ξ_1) or (η_2, ξ_2), as mentioned in Note[39]. In this case, some extra–work (described, e.g., in Ref.[29]) will be needed in order to find an equilibrium and be able to apply Arnold's Fundamental Theorem, due to the fact that $\overline{f_{\mathrm{rps}_\pi^+}}$ is not even (η_1, ξ_1) or (η_2, ξ_2). The procedure has however the advantage that the four degrees of freedom quasi–periodic motions that one obtains, in the limit $(\widetilde{p}_1, \widetilde{q}_1) \to 0$ (which is well defined), reduce to three degrees of freedom quasi–periodic maximal tori for the planar problem.

Appendix B: Semi–axes expansion

The maps jrd and p in (14), (41), respectively, are examples of *Kepler maps* (see Section II).

Kepler maps share a number of properties, which we recall here. For definiteness, we restrict to the case of our interest of $N = 2$ planets. First of all, for all k, the indirect part of the perturbing function does not contribute to the average $\overline{f_{\mathrm{k}}}$, and hence one has just to average the Newtonian potential:

$$\overline{f_{\mathrm{k}}} = -\frac{m_1 m_2}{(2\pi)^2}\int_{\mathbf{T}^2}\frac{d\ell_1 d\ell_2}{\|x^{(1)}_{\mathrm{k}} - x^{(2)}_{\mathrm{k}}\|}.$$

Exploiting the homogeneity of the Newtonian potential, and the fact that $x^{(i)}_{\mathrm{k}}$ is proportional to the semi-major axis a_i, it then becomes natural to de-homogenize with respect to a_2 and next to expand such an average

$$\overline{f_{\mathrm{k}}} = -\frac{m_1 m_2}{a_2}\Big(\overline{f_{\mathrm{k}}}^{(0)} + \overline{f_{\mathrm{k}}}^{(1)} + \cdots\Big) \tag{B1}$$

where $\overline{f_{\mathrm{k}}}^{(j)}$ is proportional to α^j, with the semi-major axes ratio ratio α suitably small.

Due to the fact that, for any Kepler map, the curve $\ell_2 \to x^{(2)}_{\mathrm{k}}$ (where ℓ_2 is the mean anomaly of the body 2) is an orbit of the two-body problem, one readily sees that, in this expansion the two former terms are trivial:

$$\overline{f_{\mathrm{k}}}^{(0)} = 1 \qquad \overline{f_{\mathrm{k}}}^{(1)} \equiv 0. \tag{B2}$$

Less trivial (see, Ref.[27] (Appendix B)) is to prove that the next (and first meaningful) term is given by

$$\overline{f_{\mathrm{k}}}^{(2)} = -\frac{\alpha^2}{8}\frac{\Lambda_2^3}{\Lambda_1^2\|\mathrm{C}^{(2)}_{\mathrm{k}}\|^5}\Big[5(3(\mathrm{P}^{(1)}_{\mathrm{k}}\cdot \mathrm{C}^{(2)}_{\mathrm{k}})^2 - \|\mathrm{C}^{(2)}_{\mathrm{k}}\|^2)\Lambda_1^2$$

$$- 3(4(\mathrm{P}_{\mathrm{k}}^{(1)} \cdot \mathrm{C}_{\mathrm{k}}^{(2)})^2 - \|\mathrm{C}_{\mathrm{k}}^{(2)}\|^2)\|\mathrm{C}_{\mathrm{k}}^{(1)}\|^2 + 3(\mathrm{C}_{\mathrm{k}}^{(1)} \times \mathrm{C}_{\mathrm{k}}^{(2)} \cdot \mathrm{P}_{\mathrm{k}}^{(1)})^2\Big].$$

In the case of $\mathrm{k} = \mathrm{p}$, we obtain the expression

$$\begin{aligned}
\overline{f\mathrm{p}}^{(2)} = -\frac{\alpha^2}{8}\frac{\Lambda_2^3}{\Lambda_1^2 \mathrm{G}_2^5}\Big[&5(3\Theta^2 - \mathrm{G}_2^2)\Lambda_1^2 \\
&- 3(4\Theta^2 - \mathrm{G}_2^2)\Big(\mathrm{G}^2 + \mathrm{G}_2^2 - 2\Theta^2 + 2\sqrt{(\mathrm{G}_2^2 - \Theta^2)(\mathrm{G}^2 - \Theta^2)}\cos\vartheta\Big) \\
&+ 3(\mathrm{G}_2^2 - \Theta^2)(\mathrm{G}^2 - \Theta^2)\sin^2\vartheta\Big].
\end{aligned} \tag{B3}$$

Therefore, collecting (B1), (B2) and (B3), one has (49) with

$$\alpha^2(\mathrm{P}_0 + \mathrm{P}) := \overline{f\mathrm{p}}^{(2)}, \quad \alpha^3\widehat{\mathrm{P}} := \sum_{j=3}^{\infty}\overline{f\mathrm{p}}^{(j)}$$

and, in turn,

$$\begin{aligned}
\mathrm{P}_0 &:= -\frac{1}{8}\frac{\Lambda_2^3}{\Lambda_1^2\mathrm{G}_2^3}\Big(-5\Lambda_1^2 + 3(\mathrm{G} + \mathrm{G}_2)^2\Big) \\
\mathrm{P} &:= -\frac{1}{8}\frac{\Lambda_2^3}{\Lambda_1^2\mathrm{G}_2^5}\Big[15\Lambda_1^2\Theta^2 - 3(4\Theta^2 - \mathrm{G}_2^2)\Big(\mathrm{G}^2 + \mathrm{G}_2^2 - 2\Theta^2 + 2\sqrt{(\mathrm{G}_2^2 - \Theta^2)(\mathrm{G}^2 - \Theta^2)}\cos\vartheta\Big) \\
&\quad - 3\mathrm{G}_2^2(\mathrm{G} + \mathrm{G}_2)^2 + 3(\mathrm{G}_2^2 - \Theta^2)(\mathrm{G}^2 - \Theta^2)\sin^2\vartheta\Big].
\end{aligned} \tag{B4}$$

Appendix C: Check of non–empty intersection

We switch to the homogenized variables

$$x := \frac{\Lambda_2}{\mathrm{G}} \qquad y = \frac{\Lambda_1}{\mathrm{G}}$$

so that the curve $\mathcal{C}$ in (70) becomes

$$\widehat{\mathcal{C}}: \qquad y = (1 + x)\sqrt{\frac{4 + x}{5}}.$$

We look for a straight line through the origin $y = \underline{k}x$ with $\underline{k} > 0$ which is tangent to $\widehat{\mathcal{C}}$ at some point (a, b), with $a > 0$.

The intersections between $\widehat{\mathcal{C}}$ and any straight line through the origin $y = kx$ are ruled by a complete cubic equation, given by

$$x^3 + (6 - 5k^2)x^2 + 9x + 4 = 0. \tag{C1}$$

In order that such an equation has a double solution $x = a$ for $k = \underline{k}$, one needs that, when $k = \underline{k}$, it can factorized as

$$(x - a)^2(x - c) = 0 \tag{C2}$$

Therefore, equating the respective coefficients of (C1) and (C2) one finds the equations

$$\begin{cases} -(c+2a) = 6 - 5\underline{k}^2 \\ 2ac + a^2 = 9 \\ -a^2 c = 4 \end{cases}$$

The two last equations allow us to eliminate c so as to obtain the equation for a

$$a^3 - 9a - 8 = 0$$

which has the following three roots:

$$a_0 = -1, \qquad a_\pm = \frac{1 \pm \sqrt{33}}{2}.$$

The only admissible (positive) value is then

$$a = a_+ = \frac{1+\sqrt{33}}{2}$$

and it provides the values

$$c = \frac{-17+\sqrt{33}}{32}, \qquad \underline{k} = \frac{1}{4}\sqrt{\frac{3}{10}(69+11\sqrt{33})}.$$

Appendix D: On the canonical character of Jacobi–Radau–Deprit and p–coordinates

In this section we discuss, in a unified way, the canonical character of the jrd coordinates in (A2) and the p–coordinates in (41). In both cases, we reduce to the discussion to the case $N = 2$, as needed in the paper. Both such sets of coordinates are defined for a general number of particles. The generalization of jrd can be deduced from the original Deprit's paper, Ref.[10]. See also Ref.[28], for a direct, inductive approach. As for the generalization of p, one can look at Ref.[27].

The proof of the canonical character of jrd and p will be based on the Delaunay coordinates, that here we recall, and a simple lemma.

⋆ Delaunay coordinates (see Ref.[14]), six for every body, here denoted as

$$\mathrm{d} = (\Lambda_j, \mathrm{G}_j, \mathrm{H}_j, \ell_j, \overline{\mathrm{g}}_j, \mathrm{h}_j)$$

are defined as follows. The coordinates Λ_j, G_j, ℓ_j are as in (A2), while, if

$$n_j := k^{(3)} \times \mathrm{C}^{(j)} \tag{D1}$$

and $\alpha_w(u,v)$ as said in Section A 1, then

$$\mathrm{g}_j = \alpha_{\mathrm{C}^{(j)}}(n_j, \mathrm{P}^{(j)}) \ , \qquad \mathrm{h}_j = \alpha_{k^{(3)}}(k^{(1)}, n_j) \ , \quad \mathrm{H}_j = \mathrm{C}^{(j)} \cdot k^{(3)} = \mathrm{C}^{(j)}_3 \ , \tag{D2}$$

with $(k^{(1)}, k^{(2)}, k^{(3)})$ a prefixed orthonormal frame in $\mathbf{R}^3$.

⋆ Let $R_i(\alpha)$ denotes the rotation matrices

$$R_1(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \quad R_2(\alpha) := \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$
$$R_3(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{D3}$$

and e_i the unit vector in $\mathbf{R}^3$ having as $j^{\rm th}$ component δ_{ji}, the Kronecker symbol. Then

Lemma D.1 *If $y(\alpha, \overline{y})$, $x(\alpha, \overline{x}) \in \mathbf{R}^3$, $\overline{\mathrm{C}}(\overline{x}, \overline{y})$, $\mathrm{C}(\alpha, \overline{x}, \overline{y})$, are the following functions of $\alpha \in \mathbf{T}$, $\overline{y}$, $\overline{x} \in \mathbf{R}^3$:*

$$y = R_i(\alpha)\overline{y} \ , \quad x = R_i(\alpha)\overline{x} \ , \quad \overline{\mathrm{C}}(\overline{x}, \overline{y}) := \overline{x} \times \overline{y}$$
$$\mathrm{C}(\alpha, \overline{x}, \overline{y}) := x(\alpha, \overline{x}) \times y(\alpha, \overline{y}) = R_i(\alpha)\overline{\mathrm{C}}(\overline{x}, \overline{y})$$

then

$$y \cdot dx = (\mathrm{C} \cdot e_i) d\alpha + \overline{y} \cdot d\overline{x} = (\overline{\mathrm{C}} \cdot e_i) d\alpha + \overline{y} \cdot d\overline{x} \ .$$

Now we proceed with proving the canonical character of jrd and p. Since the couples (Λ_j, ℓ_j) are in common to Delaunay, jrd and p and all the other do not depend on such couples, we just need to check that the changes

$$(\mathrm{H}_1, \mathrm{H}_2, \mathrm{G}_1, \mathrm{G}_2, \mathrm{h}_1, \mathrm{h}_2, \overline{\mathrm{g}}_1, \overline{\mathrm{g}}_2) \to (Z, \mathrm{G}, \mathrm{G}_1, \mathrm{G}_2, \zeta, \gamma, \gamma_1, \gamma_2) \to (Z, \mathrm{G}, \Theta, \mathrm{G}_2, \zeta, \mathrm{g}, \vartheta, \mathrm{g}_2) \tag{D4}$$

are canonical. We shall prove that

Theorem D.1 *The changes of coordinates in (D4) preserve the standard* 1*–form:*

$$\sum_{i=1}^{2}(\mathrm{H}_i d\mathrm{h}_i + \mathrm{G}_i d\overline{\mathrm{g}}_i) = Zd\zeta + \mathrm{G}d\gamma + \mathrm{G}_1 d\gamma_1 + \mathrm{G}_2 d\gamma_2 = Zd\zeta + \mathrm{G}d\mathrm{g} + \Theta d\vartheta + \mathrm{G}_2 d\mathrm{g}_2 \ . \quad \text{(D5)}$$

We shall use many times the following definitions.

Definition D.1 *If* $n \perp n' \in \mathbf{R}^3$*, we denote as* $\mathrm{F} \sim (n, \cdot, n')$ *the orthonormal frame* $\mathrm{F} = (\frac{n}{|n|}, \frac{n' \times n}{|n' \times n|}, \frac{n'}{|n'|})$.

Definition D.2 *We denote as*

$$\mathrm{F} \to \mathrm{F}'$$

any couple $(\mathrm{F}, \mathrm{F}')$ *of orthonormal frames, with* $\mathrm{F} = (i, j, k)$*,* $\mathrm{F}' = (i', j', k')$*, such that* $i' \parallel \pm k \times k'$.

Observe that, in such case, the transformation of coordinates which relates the coordinates X' relatively to F' to the coordinates X relatively to F is

$$\mathrm{X} = R_3(\psi) R_1(\iota) \mathrm{X}' \quad \text{(D6)}$$

where ι, the "mutual inclination between F and F'", is the convex angle between k and k', while ψ, called "longitude of the node of F' with respect to F, is defined by $\psi := \alpha_k(i, i')$.

Proof. Let F_0 be a prefixed reference frame, and let $\mathrm{F}_{\mathrm{d}}^{(j)} \sim (n_j, \cdot, \mathrm{C}^{(j)})$, with n_j as in (D1). Then we have

$$\mathrm{F}_0 \to \mathrm{F}_{\mathrm{d}}^{(j)} \ . \quad \text{(D7)}$$

Let $\mathrm{P}^{(j)}$ be the coordinates of the j^{th} perihelion with respect to F_0, and denote as $\mathrm{Q}^{(j)} := \widehat{\mathrm{C}}^{(j)} \times \mathrm{P}^{(j)}$, with $\widehat{\mathrm{C}}^{(j)} = \frac{\mathrm{C}^{(j)}}{|\mathrm{C}^{(j)}|}$, so that $(\mathrm{P}^{(j)}, \mathrm{Q}^{(j)}, \widehat{\mathrm{C}}^{(j)})$ is an orthonormal triple in $\mathbf{R}^3$. The coordinates of such vectors relatively to $\mathrm{F}_{\mathrm{d}}^{(j)}$, are

$$\mathrm{P}_{\mathrm{d}}^{(j)} = R_3(\overline{\mathrm{g}}_j) e_1 \qquad \mathrm{Q}_{\mathrm{d}}^{(j)} = R_3(\overline{\mathrm{g}}_j) e_2 \ .$$

Therefore, by (D6), (D7) and the definitions in (D2), we have

$$\mathrm{P}^{(j)} = R_3(\mathrm{h}_j) R_1(\overline{i}_j) R_3(\overline{\mathrm{g}}_j) e_1 \ , \qquad \mathrm{Q}^{(j)} = R_3(\mathrm{h}_j) R_1(\overline{i}_j) R_3(\overline{\mathrm{g}}_j) e_2 \ ,$$

where $\cos \overline{i}_j = \frac{\mathrm{H}_j}{\mathrm{G}_j}$. Then in view of Lemma D.1, we obtain (using $e_1 \cdot R_3(\gamma_j)(e_1 \times e_2) = e_1 \cdot e_3 = 0$ and $\mathrm{P}^{(j)} \times \mathrm{Q}^{(j)} = \widehat{\mathrm{C}}^{(j)}$)

$$\mathrm{Q}^{(j)} \cdot d\mathrm{P}^{(j)} = \widehat{\mathrm{C}}^{(j)} \cdot e_3 d\mathrm{h}_j + d\overline{\mathrm{g}}_j \ .$$

Multiplying by $\mathrm{G}_j = |\mathrm{C}^{(j)}|$ and recognizing that $\mathrm{G}_j \widehat{\mathrm{C}}^{(j)} \cdot e_3 = \mathrm{H}_j$, we then have

$$\sum_{i=1}^{2} (\mathrm{H}_i d\mathrm{h}_i + \mathrm{G}_i d\overline{\mathrm{g}}_i) = \sum_{j=1}^{2} |\mathrm{C}^{(j)}| \mathrm{Q}^{(j)} \cdot d\mathrm{P}^{(j)} \ . \tag{D8}$$

Now we compute the right hand side of this equation, using the jrd and p–coordinates. To this end, we need to express $\mathrm{P}^{(j)}$ and $\mathrm{Q}^{(j)}$ in terms of such two sets. To accomplish this, we observe that, in the sense of Definition D.2,

- ⋆ In the case of jrd, we have the "tree" of changes of frames,

$$\begin{array}{ccc} \mathrm{F}_0 \ \rightarrow & \mathrm{F}_* \rightarrow & \mathrm{F}^{(1)}_{\mathrm{jrd}} \\ & \downarrow & \\ & \mathrm{F}^{(2)}_{\mathrm{jrd}} & \end{array} \tag{D9}$$

  where $\mathrm{F}^{(j)}_{\mathrm{jrd}} \sim (\nu, \cdot, \mathrm{C}^{(j)})$, while $\mathrm{F}_* \sim (\nu_1, \cdot, \mathrm{C})$ is the *invariable frame.*

- ⋆ In the case of p, we have the "chain"

$$\mathrm{F}_0 \rightarrow \mathrm{F}_* \rightarrow \mathrm{G}^{(1)}_{\mathrm{p}} \rightarrow \mathrm{F}^{(2)}_{\mathrm{p}} \rightarrow \mathrm{G}^{(2)}_{\mathrm{p}} \tag{D10}$$

  where F_0, F_* are as in the previous item, while

$$\mathrm{G}^{(1)}_{\mathrm{p}} \sim (\mathrm{n}_1, \cdot, \mathrm{P}^{(1)}) \ , \qquad \mathrm{F}^{(2)}_{\mathrm{p}} \sim (\nu_2, \cdot, \mathrm{C}^{(2)}) \ , \qquad \mathrm{G}^{(2)}_{\mathrm{p}} \sim (\mathrm{n}_2, \cdot, \mathrm{P}^{(2)}) \ .$$

Therefore,

- ⋆ Recognizing (by the analysis of the triangle formed by $\mathrm{C}^{(1)}$, $\mathrm{C}^{(2)}$ and $\mathrm{C} = \mathrm{C}^{(1)} + \mathrm{C}^{(2)}$) that the inclinations i, i_1, i_2 between F_0 and F_*,F_* and $\mathrm{F}^{(1)}_{\mathrm{jrd}}$, F_* and $\mathrm{F}^{(2)}_{\mathrm{jrd}}$ are given by

$$\cos i = \frac{Z}{\mathrm{G}} \ , \quad \cos i_1 = \frac{\mathrm{G}_1^2 + \mathrm{G}^2 - \mathrm{G}_2^2}{2\mathrm{G}\mathrm{G}_1} \ , \quad \cos i_2 = \frac{\mathrm{G}_2^2 + \mathrm{G}^2 - \mathrm{G}_1^2}{2\mathrm{G}\mathrm{G}_2} \tag{D11}$$

  while the longitudes of the nodes are, respectively, ζ, γ, $\gamma + \pi$, we find the formulae

$$\begin{aligned} \mathrm{P}^{(j)} &= R_3(\zeta) R_1(i) R_3(\gamma) R_1(\sigma_j i_j) R_3(\gamma_j) e_1 \\ \mathrm{Q}^{(j)} &= R_3(\zeta) R_1(i) R_3(\gamma) R_1(\sigma_j i_j) R_3(\gamma_j) e_2 \end{aligned} \tag{D12}$$

  where $\sigma_1 = -\sigma_2 = 1$.

⋆ From the analysis of the mutual inclinations and the longitude of the nodes between any two consecutive frames in the chain (D10) and the definitions in (41), we find the following formulae:

$$\begin{aligned}
&\mathrm{C} = \mathrm{G}R_3(\zeta)R_1(i)e_3\ , \quad \mathrm{P}^{(1)} = R_3(\zeta)R_1(i)R_3(\mathrm{g})R_1(\iota_1)e_3\\
&\mathrm{C}^{(2)} = \mathrm{G}_2R_3(\zeta)R_1(i)R_3(\mathrm{g})R_1(\iota_1)R_3(\vartheta)R_1(\iota_2)e_3\\
&\mathrm{P}^{(2)} = R_3(\zeta)R_1(i)R_3(\mathrm{g})R_1(\iota_1)R_3(\vartheta)R_1(\iota_2)R_3(\mathrm{g}_2-\pi/2)e_1\\
&\mathrm{Q}^{(2)} = R_3(\zeta)R_1(i)R_3(\mathrm{g})R_1(\iota_1)R_3(\vartheta)R_1(\iota_2)R_3(\mathrm{g}_2)e_1
\end{aligned} \tag{D13}$$

where i is as in the previous item, while

$$\cos\iota_1 = \frac{\Theta}{\mathrm{G}}\ , \quad \cos\iota_2 = \frac{\Theta}{\mathrm{G}_2}\ .$$

The expression of $\mathrm{Q}^{(1)}$ is a bit more complicated, since it is to be derived from the previous formulae via

$$\mathrm{Q}^{(1)} = \widehat{\mathrm{C}}^{(1)}\times\mathrm{P}^{(1)} = \frac{\mathrm{C}-\mathrm{C}^{(2)}}{|\mathrm{C}-\mathrm{C}^{(2)}|}\times\mathrm{P}^{(1)}$$

However, this will be not really needed for the purpose of this section.

We are now ready to compute the right hand side of (D8), in terms of jrd and p. To this scope, we shall use Lemma D.1.

⋆ Using the formulae in (D12), iterate applications of Lemma D.1 and linear algebra, we obtain

$$\mathrm{Q}^{(j)}\cdot d\mathrm{P}^{(j)} = \widehat{\mathrm{C}}^{(j)}\cdot e_3d\zeta + \widehat{\mathrm{C}}^{(j)}\cdot\nu di + \widehat{\mathrm{C}}^{(j)}\cdot kd\gamma + \mathrm{f}_j\cdot e_1di_j + d\gamma_j$$

with $\nu := R_3(\zeta)e_1 = (\cos\zeta, \sin\zeta, 0)$, $k := R_3(\zeta)R_1(\iota)e_3$, $\mathrm{f}_j := R_1(\sigma_j i_j)e_3$. Multiplying by $\mathrm{G}_j = |\mathrm{C}^{(j)}|$, summing over $j = 1$, 2 and recognizing that k has the direction of C, ν is orthogonal to C and $\mathrm{G}_1\mathrm{f}_1 + \mathrm{G}_2\mathrm{f}_2 = \mathrm{G}e_3$, we immediately obtain, after some cancellation,

$$\sum_{j=1}^{(2)}|\mathrm{C}^{(j)}|\mathrm{Q}^{(j)}\cdot d\mathrm{P}^{(j)} = Zd\zeta + \mathrm{G}d\gamma + \mathrm{G}_1d\gamma_1 + \mathrm{G}_2d\gamma_2\ . \tag{D14}$$

- $\star$ Using the formulae in (D13), defining $\widehat{\mathrm{C}}_1^{(1)}$, $\widehat{\mathrm{C}}_2^{(1)}$, $\widehat{\mathrm{C}}_1^{(2)}$, $\widehat{\mathrm{C}}_2^{(2)}$, $\widehat{\mathrm{C}}_3^{(2)}$ via

$$\begin{aligned}\widehat{\mathrm{C}}^{(1)} &= R_3(\zeta)R_1(i)\widehat{\mathrm{C}}_1^{(1)} = R_3(\zeta)R_1(i)R_3(\mathrm{g})R_1(i_1)\widehat{\mathrm{C}}_2^{(1)}\\ \widehat{\mathrm{C}}^{(2)} &= R_3(\zeta)R_1(i)\widehat{\mathrm{C}}_1^{(2)} = R_3(\zeta)R_1(i)R_3(\mathrm{g})R_1(i_1)\widehat{\mathrm{C}}_2^{(2)}\\ &= R_3(\zeta)R_1(i)R_3(\mathrm{g})R_1(i_1)R_3(\vartheta)R_1(i_2)\widehat{\mathrm{C}}_3^{(2)}\end{aligned}$$

and applying iteratively Lemma D.1, we obtain

$$\begin{aligned}\mathrm{Q}^{(1)}\cdot d\mathrm{P}^{(1)} &= \widehat{\mathrm{C}}^{(1)}\cdot e_3 d\zeta + \widehat{\mathrm{C}}_1^{(1)}\cdot e_1 di + \widehat{\mathrm{C}}_1^{(1)}\cdot e_3 d\mathrm{g} + \widehat{\mathrm{C}}_2^{(1)}\cdot e_1 di_1\\ \mathrm{Q}^{(2)}\cdot d\mathrm{P}^{(2)} &= \widehat{\mathrm{C}}^{(2)}\cdot e_3 d\zeta + \widehat{\mathrm{C}}_1^{(2)}\cdot e_1 di + \widehat{\mathrm{C}}_1^{(2)}\cdot e_3 d\mathrm{g} + \widehat{\mathrm{C}}_2^{(2)}\cdot e_1 di_1\\ &\quad + \widehat{\mathrm{C}}_2^{(2)}\cdot e_3 d\vartheta + \widehat{\mathrm{C}}_3^{(2)}\cdot e_1 di_2 + \widehat{\mathrm{C}}_3^{(2)}\cdot e_3 d\mathrm{g}_2\end{aligned}$$

We multiply, as above, the first equation by $|\mathrm{C}^{(1)}|$, the second by $|\mathrm{C}^{(2)}|$, and take the sum of the two. The sum of the first three respective terms gives, analogously to the previous item, $Zd\zeta + \mathrm{G}d\mathrm{g}$. As for the remaining terms, we recognize that $\widehat{\mathrm{C}}_3^{(2)} = e_3$ so that $\widehat{\mathrm{C}}_3^{(2)}\cdot e_1 = 0$, $\widehat{\mathrm{C}}_3^{(2)}\cdot e_3 = 1$; $\widehat{\mathrm{C}}_2^{(2)}\cdot e_3 = \Theta$; $|\mathrm{C}^{(1)}|\widehat{\mathrm{C}}_2^{(1)}\cdot e_1 + |\mathrm{C}^{(2)}|\widehat{\mathrm{C}}_2^{(2)}\cdot e_1 = (\mathrm{C}^{(1)} + \mathrm{C}^{(2)})\cdot R_3(\zeta)R_1(i)R_3(\mathrm{g})R_1(i_1)e_1 = \mathrm{C}\cdot R_3(\zeta)R_1(i)R_3(\mathrm{g})R_1(i_1)e_1 = \mathrm{G}e_3\cdot R_3(\mathrm{g})e_1 = 0$. We finally obtain

$$\sum_{j=1}^{2}|\mathrm{C}^{(j)}|\mathrm{Q}^{(j)}\cdot d\mathrm{P}^{(j)} = Zd\zeta + \mathrm{G}d\mathrm{g} + \Theta d\vartheta + \mathrm{G}_2 d\mathrm{g}_2\ . \tag{D15}$$

The collection of (D8), (D14) and (D15) proves Theorem D.1.

REFERENCES

[1]V. I. Arnold. A theorem of Liouville concerning integrable problems of dynamics. *Sibirsk. Mat. Ž.*, 4:471–474, 1963.

[2]V. I. Arnold. Proof of a theorem by A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian. *Russian Math. Survey*, 18:13–40, 1963.

[3]V.I. Arnold. Small denominators and problems of stability of motion in classical and celestial mechanics. *Russian Math. Surveys*, 18(6): 85–191, 1963.

[4]V. I. Arnold. Instability of dynamical systems with many degrees of freedom. *Dokl. Akad. Nauk SSSR*, 156:9–12, 1964.

[5]L. Chierchia and G. Gallavotti. Drift and diffusion in phase space. *Ann. Inst. H. Poincaré Phys. Théor.*, 60(1): 144, 1994.

[6]L. Chierchia and G. Pinzari. Properly–degenerate KAM theory (following V.I. Arnold). *Discrete Contin. Dyn. Syst. Ser. S*, 3(4): 545–578, 2010.

[7]L. Chierchia and G. Pinzari. Planetary Birkhoff normal forms. *J. Mod. Dyn.*, 5(4): 623–664, 2011.

[8]L. Chierchia and G. Pinzari. The planetary N-body problem: symplectic foliation, reductions and invariant tori. *Invent. Math.*, 186(1): 1–77, 2011.

[9]L. Chierchia and G. Pinzari. Metric stability of the planetary n–body problem. *Proceedings of the International Congress of Mathematicians*, 2014.

[10]A. Deprit. Elimination of the nodes in problems of n bodies. *Celestial Mech.*, 30(2): 181–195, 1983.

[11]J. Féjoz. Démonstration du 'théorème d'Arnold' sur la stabilité du système planétaire (d'après Herman). *Ergodic Theory Dynam. Systems*, 24(5): 1521–1582, 2004.

[12]J. Féjoz. On "Arnold's theorem" in celestial mechanics –a summary with an appendix on the poincaré coordinates. *Discrete and Continuous Dynamical Systems*, 33: 3555–3565, 2013.

[13]S. Ferrer and C. Osácar. Harrington's Hamiltonian in the stellar problem of three bodies: reductions, relative equilibria and bifurcations. *Celestial Mech. Dynam. Astronom.*, 58(3): 245–275, 1994.

[14]G. Gallavotti. The elements of mechanics. Texts and Monographs in Physics. 1983

[15]M. R. Herman. Torsion du problème planétaire, edited by J. Féjoz in 2009. Available in the electronic 'Archives Michel Herman' at http://www.college-de-france.fr/default/EN/all/equ_dif/archives_michel_herman.htm.

[16]C. G. J. Jacobi. Sur l'élimination des noeuds dans le problème des trois corps. *Astronomische Nachrichten*, Bd XX: 81–102, 1842.

[17]W. H. Jefferys and J. Moser. Quasi-periodic solutions for the three-body problem. *Astronom. J.*, 71: 568–578, 1966.

[18]A.N. Kolmogorov. On the Conservation of Conditionally Periodic Motions under Small Perturbation of the Hamiltonian. *Dokl. Akad. Nauk SSR*, 98: 527–530, 1954.

[19]J. Laskar and P. Robutel. Stability of the planetary three-body problem. I. Expansion of the planetary Hamiltonian. *Celestial Mech. Dynam. Astronom.*, 62(3): 193–217, 1995.

[20] F. Malige, P. Robutel, and J. Laskar. Partial reduction in the n-body planetary problem using the angular momentum integral. *Celestial Mech. Dynam. Astronom.*, 84(3): 283–316, 2002.

[21] A. S. Miščenko and A. T. Fomenko. Integration of Hamiltonian systems with noncommutative symmetries. *Trudy Sem. Vektor. Tenzor. Anal.*, (20): 5–54, 1981.

[22] J. Moser. The analytic invariants of an area-preserving mapping near a hyperbolic fixed point. *Comm. Pure Appl. Math.*, 9: 673–692, 1956.

[23] J. Moser. On invariant curves of area-preserving mappings of an annulus. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 1962:1–20, 1962.

[24] N. N. Nehorošev. Action-angle variables, and their generalizations. *Trudy Moskov. Mat. Obšč.*, 26: 181–198, 1972.

[25] J. F. Palacián, F. Sayas, and P. Yanguas. Regular and singular reductions in the spatial three-body problem. *Qualitative Theory of Dynamical Systems*, 12(1): 143–182, 2013.

[26] J. F. Palacián, F. Sayas, and P. Yanguas. Flow reconstruction and invariant tori in the spatial three-body problem. *J. Differential Equations*, 258(6): 2114–2159, 2015.

[27] G. Pinzari. Perihelia reduction and global Kolmogorov tori in the planetary problem. *Memoirs American Mathematical Society.* in press.

[28] G. Pinzari. *On the Kolmogorov set for many–body problems.* PhD thesis, Università Roma Tre, April 2009.

[29] G. Pinzari. Aspects of the planetary Birkhoff normal form. *Regul. Chaotic Dyn.*, 18(6): 860–906, 2013.

[30] G. Pinzari.A first integral to the partially averaged Newtonian potential of the three-body problem. In revision. Draft: arXiv:1607.03056

[31] H. Poincaré. Les méthodes nouvelles de la mécanique céleste, 1899. Gauthier-Villars, Paris.

In the referenced text, Poincaré draws the following imaginary picture of the phase portrait of a non–integrable system closely to a homoclinic point: "*Que l'on cherche à se représenter la figure formée par ces courbes et leurs intersections en nombre infini dont chacune correspond à une solution doublement asymptotique, ces intersections forment une sorte de treillis, de tissu, de réseau à mailles infiniment serrées; chacune des deux courbes ne doit jamais se recouper elle-mme, mais elle doit se replier sur elle-mme d'une manire trs complexe pour venir recouper une infinité de fois toutes les mailles du réseau. On sera*

frappé de la complexité de cette figure, que je ne cherche mme pas à tracer. Rien n'est plus propre à nous donner une idée de la complication du problme des trois corps et en général de tous les problmes de Dynamique o il n'y a pas d'intégrale uniforme et o les séries de Bohlin sont divergentes."

[32] J. Pöschel. Nekhoroshev estimates for quasi-convex Hamiltonian systems. *Math. Z.*, 213(2): 187–216, 1993.

[33] R. Radau. Sur une transformation des équations différentielles de la dynamique. *Ann. Sci. Ec. Norm. Sup.*, 5: 311–375, 1868.

[34] P. Robutel. Stability of the planetary three-body problem. II. KAM theory and existence of quasiperiodic motions. *Celestial Mech. Dynam. Astronom.*, 62(3): 219–261, 1995.

[35] E. Valdinoci. Families of whiskered tori for a-priori stable/unstable Hamiltonian systems and construction of unstable orbits. *Math. Phys. Electron. J.*, 6: Paper 2, 31 pp. (electronic), 2000.

[36] Aubry–Mather theory establishes the existence of quasi–periodic motions filling completely invariant sets with *any* prescribed frequency. It concerns Hamiltonian systems

$$\begin{cases} \dot x = \partial_p \mathrm{H}(p,x) \\ \dot p = -\partial_x \mathrm{H}(p,x) \end{cases}$$

where $\mathrm{H}(p,x)\ :\ \mathbf{R}^n\times\mathcal{M}\ \to\ \mathbf{R}$ is a C^2–smooth Hamiltonian, $\mathcal{M}\subset\mathbf{R}^n$ is a n–dimensional compact surface, the Hessian $\partial^2_{p,p}\mathrm{H}$ is *convex*, H is *super–linear in p*, i.e., $\lim_{|p|\to\infty}\mathrm{H}(p,x)/|p|=+\infty$, and the flow is *complete*, i.e., for any $(p_0,x_0)\in\mathcal{M}\subset\mathbf{R}^n$, the Hamiltonian flow $(p(t),x(t))$ with initial datum (p_0,x_0) exists at *any* time $t\geq 0$.

[37] As well known, it is possible to find functions $\widetilde\lambda_2$, $\widetilde\lambda_2$, $\widetilde{\mathrm{g}}_2$ such that the change $(\Lambda_1,\Lambda_2,\mathrm{G}_2,\Theta,\lambda_1,\lambda_2,\mathrm{g}_2,\vartheta)\to(\Lambda_1,\Lambda_2,\mathrm{G}_2,p_0,\widetilde\lambda_1,\widetilde\lambda_2,\widetilde{\mathrm{g}}_2,q_0)$ is canonical. We omit the details.

[38] Identity (20) may be checked writing explicitly f_{rps}, $f_{\mathrm{rps}^+_\pi}$ via f_{jrd}.

a) The expression of f_{rps} in terms of f_{jrd} has been widely discussed in Ref.[28] (Section 4) or Ref.[8] (Appendix A). Referring to such literature for full details, let us recall the main steps.

One starts with writing the expressions of $x^{(j)}_{\mathrm{jrd}}$, $y^{(j)}_{\mathrm{jrd}}$, which are

$$\begin{cases} x^{(j)}_{\mathrm{jrd}} = R_3(\zeta)R_1(i)R_3(\gamma)R_1(\sigma_j i_j)R_3(\gamma_j)x^{(j)}_{\mathrm{orb}}(\Lambda_j,\mathrm{G}_j,\ell_j) \\ y^{(j)}_{\mathrm{jrd}} = R_3(\zeta)R_1(i)R_3(\gamma)R_1(\sigma_j i_j)R_3(\gamma_j)y^{(j)}_{\mathrm{orb}}(\Lambda_j,\mathrm{G}_j,\ell_j) \end{cases}$$

where $\sigma_1=-\sigma_2=1$, $R_i\in SO(3)$ are as in (D3), i, i_1, $i_2\in(0,\pi)$ are as in (D11),

$x^{(j)}_{\rm orb}(\Lambda_j, {\rm G}_j, \ell_j)$, $y^{(j)}_{\rm orb}(\Lambda_j, {\rm G}_j, \ell_j) \in \mathbf{R}^2 \times \{0\}$ as in Ref.[28] (Section 4) or Ref.[8] (Appendix A). Due to SO(3)–invariance of $f_{\rm k}$, we may take the freedom of multiplying such expressions by an arbitrary $\mathcal{R} \in {\rm SO}(3)$. We let

$$\widehat{x}^{(j)}_{\rm jrd} := \mathcal{R}\, x^{(j)}_{\rm jrd}\ , \qquad \widehat{y}^{(j)}_{\rm jrd} := \mathcal{R}\, y^{(j)}_{\rm jrd}$$

with

$$\mathcal{R} = R_3(\gamma+\zeta)R_1(-i_1)R_3(-\gamma)R_1(-i)R_3(-\zeta)$$

Then one rewrites

$$\begin{cases} \widehat{x}^{(1)}_{\rm jrd} = x^{(1)}_{\rm pl}(\Lambda_1, {\rm G}_1, \ell_1, \gamma_1+g+\zeta) \\ \widehat{x}^{(2)}_{\rm jrd} = R_{313}(\gamma+\zeta, -i_{12})x^{(2)}_{\rm pl}(\Lambda_2, {\rm G}_2, \ell_2, \gamma_2+g+\zeta) \end{cases}$$
$$\begin{cases} \widehat{y}^{(1)}_{\rm jrd} = y^{(1)}_{\rm pl}(\Lambda_1, {\rm G}_1, \ell_1, \gamma_1+g+\zeta) \\ \widehat{y}^{(2)}_{\rm jrd} = R_{313}(\gamma+\zeta, -i_{12})y^{(2)}_{\rm pl}(\Lambda_2, {\rm G}_2, \ell_2, \gamma_2+g+\zeta) \end{cases} \tag{D16}$$

with $i_{12} := i_1 + i_2 \in (0, \pi)$ and

$$R_{313}(\gamma+\zeta, i_{12}) := R_3(\gamma+\zeta)R_1(i_{12})R_3(-(\gamma+\zeta))$$
$$x^{(j)}_{\rm pl}(\Lambda_j, {\rm G}_j, \ell_j, \gamma_j+g+\zeta) := R_3(\gamma_j+g+\zeta)x^{(j)}_{\rm orb}(\Lambda_j, {\rm G}_j, \ell_j)$$

Since $x^{(j)}_{\rm pl}$ and $y^{(j)}_{\rm pl} \in \mathbf{R}^2 \times \{0\}$ and $\widehat{x}^{(j)}_{\rm jrd}$, $\widehat{y}^{(j)}_{\rm jrd}$ appear in $f_{\rm jrd}$ only via their lengths or their mutual scalar product, we may replace, in (D16), such vectors with their projections $\overline{x}^{(j)}_{\rm pl}$, $\overline{y}^{(j)}_{\rm pl}$, $\overline{x}^{(j)}_{\rm jrd}$ and $\overline{y}^{(j)}_{\rm jrd}$ along $\mathbf{R}^2$ and the matrix $R_{313}(\gamma+\zeta, i_{12}) \in {\rm SO}(3)$ with its upper left (2×2) truncation $\overline{R}_{313}(\gamma+\zeta, i_{12})$, which is given by (see Ref.[8] (Eq. (A26), with $\psi := -(\gamma+\zeta)$)):

$$\overline{R}_{313} = \begin{pmatrix} 1-\sin^2(\gamma+\zeta)(1-\cos i_{12}) & \sin(\gamma+\zeta)\cos(\gamma+\zeta)(1-\cos i_{12}) \\ \sin(\gamma+\zeta)\cos(\gamma+\zeta)(1-\cos i_{12}) & 1-\cos^2(\gamma+\zeta)(1-\cos i_{12}) \end{pmatrix} \tag{D17}$$

The corresponding expressions in terms of rps are then obtained replacing the quantities that appear in these formulae as it follows inverting (31):

$$\begin{cases} {\rm G}_j = \Lambda_j - {\rm i}\overline{t}_j\overline{t}^*_j \\ {\rm G} = \Lambda_1+\Lambda_2 - {\rm i}\overline{t}_1\overline{t}^*_1 - {\rm i}\overline{t}_2\overline{t}^*_2 - {\rm i}\overline{t}_3\overline{t}^*_3 \\ 1-\cos i_{12} = \frac{(\Lambda_1+\Lambda_2-{\rm i}\overline{t}_1\overline{t}^*_1-{\rm i}\overline{t}_2\overline{t}^*_2)^2-(\Lambda_1+\Lambda_2-{\rm i}\overline{t}_1\overline{t}^*_1-{\rm i}\overline{t}_2\overline{t}^*_2-{\rm i}\overline{t}_3\overline{t}^*_3)^2}{2(\Lambda_1-{\rm i}\overline{t}_1\overline{t}^*_1)(\Lambda_2-{\rm i}\overline{t}_2\overline{t}^*_2)} \end{cases}, \quad \begin{cases} \gamma+\zeta = \arg\frac{\overline{t}_3}{\sqrt{{\rm i}\overline{t}_3\overline{t}^*_3}} \\ \gamma_j+\gamma+\zeta = \arg\frac{\overline{t}_j}{\sqrt{{\rm i}\overline{t}_j\overline{t}^*_j}} \\ \ell_j = \lambda_j - \arg\frac{\overline{t}_j}{\sqrt{{\rm i}\overline{t}_j\overline{t}^*_j}} \end{cases}$$

where we have used $\cos i_{12} = \frac{{\rm G}^2-{\rm G}_1^2-{\rm G}_2^2}{2{\rm G}_1{\rm G}_2}$.

b) To obtain the expressions of $f_{{\rm rps}^\pm_\pi}$ via $f_{\rm jrd}$, one rewrites

$$\begin{cases} \widehat{x}^{(1)}_{\rm jrd} = x^{(1)}_{\rm pl}(\Lambda_1, {\rm G}_1, \ell_1, \gamma_1+g+\zeta) \\ \widehat{x}^{(2)}_{\rm jrd} = S_{313}(\gamma+\zeta, -i_{12})x^{(2)}_{\rm pl}(\Lambda_2, {\rm G}_2, -\ell_2, -\gamma_2+g+\zeta) \end{cases}$$

$$\begin{cases} \widehat{y}^{(1)}_{\rm jrd} = y^{(1)}_{\rm pl}(\Lambda_1, {\rm G}_1, \ell_1, \gamma_1 + g + \zeta) \\ \widehat{y}^{(2)}_{\rm jrd} = S_{313}(\gamma+\zeta, -i_{12}) y^{(2)}_{\rm pl}(\Lambda_2, {\rm G}_2, -\ell_2, -\gamma_2 + g + \zeta) \end{cases} \tag{D18}$$

where

$$S_{313} := R_3(\gamma+\zeta)R_1(i_{12})R_3(\gamma+\zeta)\Pi_2^- \qquad \Pi_2^- := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Again, one substitutes, in (D18), $\widehat{x}^{(j)}_{\rm jrd}$, $\widehat{y}^{(j)}_{\rm jrd}$, $x^{(j)}_{\rm pl}$, $y^{(j)}_{\rm pl}$, S_{313} with their projections $\overline{x}^{(j)}_{\rm jrd}$, $\overline{y}^{(j)}_{\rm jrd}$, $\overline{x}^{(j)}_{\rm pl}$, $\overline{y}^{(j)}_{\rm pl}$, $\overline{S}_{313}$. In particular,

$$\overline{S}_{313} = \begin{pmatrix} 1 - \sin^2(\gamma+\zeta)(1+\cos i_{12}) & \sin(\gamma+\zeta)\cos(\gamma+\zeta)(1+\cos i_{12}) \\ \sin(\gamma+\zeta)\cos(\gamma+\zeta)(1+\cos i_{12}) & 1 - \cos^2(\gamma+\zeta)(1+\cos i_{12}) \end{pmatrix}$$

Inverting (18) and using $1+\cos i_{12} = \frac{({\rm G}_1-{\rm G}_2)^2 - {\rm G}^2}{{\rm G}_1(-{\rm G}_2)}$, we now have

$$\begin{cases} {\rm G}_1 = \Lambda_1 - {\rm i}t_1t_1^* \\ {\rm G}_2 = \Lambda_2 + {\rm i}t_2t_2^* \\ {\rm G} = \Lambda_1 - \Lambda_2 - {\rm i}t_1t_1^* - {\rm i}t_2t_2^* - {\rm i}t_3t_3^* \\ 1+\cos i_{12} = \frac{(\Lambda_1-\Lambda_2-{\rm i}t_1t_1^*-{\rm i}t_2t_2^*)^2 - (\Lambda_1-\Lambda_2-{\rm i}t_1t_1^*-{\rm i}t_2t_2^*-{\rm i}t_3t_3^*)^2}{2(\Lambda_1-{\rm i}t_1t_1^*)(-\Lambda_2-{\rm i}t_2t_2^*)} \end{cases}, \quad \begin{cases} \gamma+\zeta = \arg\frac{t_3}{\sqrt{{\rm i}t_3t_3^*}} \\ \gamma_1+\gamma+\zeta = \arg\frac{t_1}{\sqrt{{\rm i}t_1t_1^*}} \\ -\gamma_2+\gamma+\zeta = \arg\frac{t_2}{\sqrt{{\rm i}t_2t_2^*}} \\ \ell_1 = \lambda_1 - \arg\frac{t_1}{\sqrt{{\rm i}t_1t_1^*}} \\ \ell_2 = \lambda_2 + \arg\frac{t_2}{\sqrt{{\rm i}t_2t_2^*}} \end{cases}$$

Using the parity of $x^{(j)}_{\rm pl}$, $y^{(j)}_{\rm pl}$ with respect to Λ_j, ${\rm G}_j$, in view of the formulae obtained, relation (20) follows.

[39]One can take the transformation (29) with $i=3$ being defined via

$$\begin{cases} \Lambda_i = \Lambda_i \\ \lambda_1 = \widetilde{\lambda}_1 + \widetilde{\rm g} \\ \lambda_2 = \widetilde{\lambda}_2 - \widetilde{\rm g} \\ \eta_1 = \widetilde{\eta}_1\cos\widetilde{\rm g} + \widetilde{\xi}_1\sin\widetilde{\rm g} \\ \xi_1 = \widetilde{\xi}_1\cos\widetilde{\rm g} - \widetilde{\eta}_1\sin\widetilde{\rm g} \\ \eta_2 = \widetilde{\eta}_2\cos\widetilde{\rm g} - \widetilde{\xi}_1\sin\widetilde{\rm g} \\ \xi_2 = \widetilde{\xi}_2\cos\widetilde{\rm g} + \widetilde{\eta}_1\sin\widetilde{\rm g} \\ p_1 = \sqrt{2({\rm G} - \Lambda_1 + \Lambda_2 + \frac{\widetilde{\eta}_1^2+\widetilde{\xi}_1^2}{2} - \frac{\widetilde{\eta}_2^2+\widetilde{\xi}_2^2}{2})}\cos\widetilde{\rm g} \\ q_1 = \sqrt{2({\rm G} - \Lambda_1 + \Lambda_2 + \frac{\widetilde{\eta}_1^2+\widetilde{\xi}_1^2}{2} - \frac{\widetilde{\eta}_2^2+\widetilde{\xi}_2^2}{2})}\sin\widetilde{\rm g} \end{cases} \tag{D19}$$

The transformations with $i = 1, 2$ are similarly obtained.

[40]This rather peculiar fact deserves some comment. In the 60s, Harrington observed that the second–order term in the semi–axes ratio expansion of the doubly averaged Newtonian potential

after Jacobi reduction of the nodes (see Equation (14)) is independent of the anomaly of the perihelion of the outer planet γ_2; see Ref.[13] and references therein. The independence of $\overline{f_{\rm K}^{(2)}}$ of $\widehat{\rm g}_2$, mentioned in the text, is a generalization of Harrington's remark and, in turn, admits a generalization to all $\overline{f_{\rm K}^{(j)}}$'s, with $j \geq 2$. It has been recently understood that this strange occurrence is caused by the existence of a first integral to the partial average (with respect to ℓ_2) of the Newtonian potential. We do not enter in such details, for which we refer to Ref.[30].

[41]By Liouville-Arnold Theorem, Ref.[1], the phase space of a system verifying its assumptions foliates (modulo a diffeomorphism) as the union of n-dimensional sub-manifolds $\{{\rm I}_0\} \times \mathbf{T}^n$, invariant for the h-flow, where the motion has linear law:

$$\mathrm{I} = \mathrm{I}_0, \qquad \varphi = \varphi_0 + \omega_0(\mathrm{I}_0)(t - t_0), \qquad \mathrm{I}_0 \in \mathcal{I}, \qquad \varphi_0 \in \mathbf{T}^n$$

with $\omega_0(\mathrm{I}) := \partial_{\mathrm{I}}\mathrm{h}(\mathrm{I})$. The continuation, under suitable assumptions, of many of such motions to non-integrable systems which however have the "close to integrable" form

$$\mathrm{H}(I, \varphi) = \mathrm{h}(\mathrm{I}) + \mu f(\mathrm{I}, \varphi) \qquad (0 < \mu \ll 1)$$

is the precisely the scope of kam theory. More precisely, kam theory ensures the possibility of continuing to the full systems at least those motions where $\partial_{\mathrm{I}}^2\mathrm{h}$ does not vanish identically (the so–called Kolmogorov condition, or possible weakened versions of it) and the frequencies ω satisfy the Diophantine inequality:

$$|\omega \cdot k| \geq \frac{\gamma}{|k|^\tau} \qquad \forall k \in \mathbf{Z}^n \setminus \{0\} \quad \text{for suitable} \quad \gamma,\ \tau > 0\ .$$

kam theory originated with Kolmogorov's pioneering 1954 paper Ref.[18]. The literature concerning such a theory is so wide that we give up any attempt at completeness.

[42]A motion $t \to q(t) = (x_1(t), x_2(t), \cdots)$ of a system of N particles in $\mathbf{R}^\nu$ is *quasi-periodic* with $m \geq 2$ frequencies if, for all $t \geq 0$, $q(t)$ has the form

$$q(t) = \widehat{q}(\omega_1 t, \ldots, \omega_m t)$$

for some 2π–periodic function of m arguments and suitable $\omega := (\omega_1, \ldots, \omega_n) \in \mathbf{R}^m \setminus \mathbf{Q}^m$. The case $m = 1$ or $m \geq 2$ and $\omega \in \mathbf{Q}^m$ corresponds to a *periodic* motion. In the former case of ω_i's not rationally related, the motion evolves on the m–dimensional manifold $\mathcal{T} := \bigcup_{t\geq 0}\{q(t)\}$ called *quasi–periodic torus*.

A quasi–periodic torus $\mathcal{T}$ is said

– *maximal* if it is not possible to find quasi–periodic tori with $m' > m$ frequencies;

– *whiskered* if it is not maximal and, moreover, there exist two non–empty manifolds $\mathcal{W}^+ = \bigcup_{t\geq 0}\{q(t)\}$, $\mathcal{W}^- = \bigcup_{t\leq 0}\{q(t)\}$ made of motions of the system such that $\mathcal{T} = \mathcal{W}^+ \cap \mathcal{W}^-$ and

$$\lim_{t\to\pm\infty} \mathrm{dist}(q(t), \mathcal{T}) = 0 \quad \forall\, q \in \mathcal{W}^\pm \ .$$

Note that our definition of maximal torus is a bit different from the standard one (according to which, a torus is maximal when the number of frequencies equals the number of degrees of freedom). This is because for systems which (like the N–body problem, and hence the case considered in the paper) possess non–commuting first integrals, the maximum number of not identically vanishing frequencies may be less than the number of degrees of freedom.

[43]A celestial body is said to have a *retrograde* (as opposite to *direct*, or *prograde*) motion if its angular momentum $\mathrm{C}(t) = x(t) \times \mathrm{m}\dot{x}(t)$ has a negative projection along the direction of the (constant) total angular momentum $\mathrm{C}_{\mathrm{tot}} = \sum_{\mathrm{m}} \mathrm{C}(t)$ of the smallest closed system the celestial body is part of. The retrograde motion is frequently observed in the dynamics of gravitational systems. In our solar system, many natural satellites of Jupiter and Saturn (e.g., Carme, Anake, Pasiphae groups for the former; Phoebe, Skathi, Skoll, and many others, for the latter) have such kind of motion.

[44]An example in the C^∞ class is as follows. Let H_1, H_2 be integrable Hamiltonians of class at least C^∞ on a common phase space $\mathcal{P}$, with $H_2 < 0 < H_1$. Assume that H_1 has only maximal tori, while H_2 only hyperbolic tori on $\mathcal{P}$. Let $x \to \chi_1(x)$, $x \to \chi_2(x)$ be real functions of class C^∞, with $\chi_1 \equiv 0$ for $x \geq 0$, $\chi_2 \equiv 0$ for $x \leq 0$. Then the Hamiltonian, of class C^∞, defined as $H := \chi_1(H_1)H_1 + \chi_2(H_2)H_2$ has maximal and hyperbolic tori on $\mathcal{P}$.

[45]The i^{th} instantaneous ellipse through a given initial datum $(\overline{y}^{(i)}, \overline{x}^{(i)}) \in \mathbf{R}^3 \times \mathbf{R}^3$ relatively to mass parameters m_i, M_i is defined as the orbit generated by the Hamiltonian (4) with initial datum $(\overline{y}^{(i)}, \overline{x}^{(i)})$, provided that, in correspondence of such initial datum, the energy (4) is negative.

[46]A family $\{\mathrm{F}_i\}_{i=1,\dots,n}$ is called *independent* if the gradients $\{\partial \mathrm{F}_i\}_{i=1,\dots,n}$ are *linearly independent* at each point of phase space. Moreover, F and F′ are called *Poisson-commuting* if the *Poisson parentheses* $\{\mathrm{F}, \mathrm{F}'\} := \sum_{i=1}^n (\partial_{y_i}\mathrm{F}\partial_{x_i}\mathrm{F}' - \partial_{x_i}\mathrm{F}\partial_{y_i}\mathrm{F}')$ vanish.

[47]The only work known to the author where Deprit's coordinates are mentioned, up to Ref.[28], is Ref.[13]. However, such case is not really exhaustive, because it deals with the three–body problem, in which case Deprit's coordinates are the same as the classical Jacobi reduction of the nodes,

Ref.[16].

[48]In this respect, Deprit declared: "Whether the new phase variables [...] are practical in the General Theory of Perturbation is an open question. At least, for planetary theories, the answer is likely to be in the negative [...]", Ref.[10] (p. 194).