# FLOWS ON THE PSL(V)-HITCHIN COMPONENT

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ABSTRACT. In this article we define new flows on the Hitchin components for  $\mathrm{PSL}(n,\mathbb{R})$ . Special examples of these flows are associated to simple closed curves on the surface and give generalized twist flows. Other examples, so called eruption flows, are associated to pair of pants in S and capture new phenomena which are not present in the case when n=2. In a companion paper to this article [SZ17] two of the authors develop new tools to compute the Goldman symplectic form on the Hitchin component. Using this computation we determine a global Darboux coordinate system on the Hitchin component.

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### 1. Introduction

Given a closed oriented surface S of genus  $g \geq 2$  and a split real simple Lie group G, the Hitchin component is a connected component of the representation variety  $\operatorname{Hom}(\pi_1(S), G)/G$ . This was introduced by Hitchin in [Hit92], who proved that it is homeomorphic to a vector space of dimension  $\dim(G)(2g-2)$ . Hitchin components share many properties with the Teichmüller space of S, which is the Hitchin component for  $G = PSL(2, \mathbb{R})$ , and are therefore often referred to as higher Teichmüller spaces (where "higher" reflects the fact that G is a Lie group of higher rank). It is thus very natural to try to generalize well-known properties of Teichmüller space to Hitchin components, and even more interesting to investigate new phenomena which arise for Hitchin components for Lie groups G of higher rank, i.e. which are not locally isomorphic to  $PSL(2,\mathbb{R})$ . In this article we generalize some well known dynamical structure on Teichmüller space to Hitchin components for  $PSL(n,\mathbb{R})$ , which we denote by  $\operatorname{Hit}_n(S)$ . More precisely we define flows on the Hitchin component, which on the one hand generalize twist flows associated to simple closed curves, but also include as new phenomena flows determined by a pair of pants or a triangle in an ideal triangulation.

The Fenchel-Nielsen twist flow associated to a simple closed curve is one of the simplest way to move around in Teichmüller space. Wolpert showed that it is a Hamiltonian flow for the Weil-Petersson symplectic structure, and its Hamiltonian function is the length function of the simple closed curve. The twist flows associated to non-intersecting simple closed curves have Poisson commuting Hamiltonian functions, and thus the twist flows associated to 3g-3 pairwise non-intersecting simple closed curves (forming a pair of pants decomposition) give a half-dimensional family of Poisson commuting functions on the Teichmüller space [Wol82, Wol83].

Goldman later generalized these twist flows to a very general setting. He first introduced, for any reductive Lie group G, a natural symplectic structure on (the smooth part of) the  $\operatorname{Hom}(\pi_1(S),G)/G$  [Gol84]. We refer to this symplectic structure as the Goldman symplectic structure. He further considered the Hamiltonian flows associated to functions  $f_{\alpha}$  on the representation variety, which arise from invariant functions on G and an element  $\alpha \in \pi_1(S)$ . When  $\alpha$  corresponds to a simple closed curve, these Hamiltonian flows are generalized twist flows, i.e. they do not change (up to conjugation) the representation restricted to  $\pi_1(S \setminus \alpha)$  [Gol86]. Using this, Goldman proved that the Goldman symplectic form is a multiple of the Weil-Petersson symplectic structure restricted to  $\operatorname{Hit}_n(S)$  as a generalization of the Weil-Petersson symplectic structure on Teichmüller space.

The generalized twist flows are of course defined on  $\operatorname{Hit}_n(S)$ . However, the family of generalized twist flows associated to a pants decomposition of S do not provide a half-dimensional family of commuting Hamiltonian flows on  $\operatorname{Hit}_n(S)$  with Poisson commuting Hamiltonian functions; the dimension of the Hitchin component is too large. Thus, to get such a family, one has to consider new flows which are not

generalized twist flows. (This is also reflected in the fact that parametrizations of  $\operatorname{Hit}_n(S)$  involve not only invariants for the simple closed curves in a pants decomposition, but also invariants of the pairs of pants given by the pants decomposition, see [Gol90, Zha15a, Zha15b].) In this article we define such new flows.

Classical Fenchel-Nielsen twist flows are linked to the Fenchel-Nielsen coordinates of Teichmüller space. Fixing a pants decomposition of S, the Fenchel-Nielsen coordinate functions are the 3g-3 hyperbolic lengths  $\ell_i$ ,  $i=1\cdots 3g-3$  of the 3g-3 pants curves and the 3g-3 twists  $\tau_i$ ,  $i=1\cdots 3g-3$  along these pants curves. The lengths are canonical, but the twists are relative, and depend on the choice of a transversal to the pants curve. Wolpert proved [Wol82, Wol83] that the length and twist parameters give global Darboux coordinates for Teichmüller space, i.e. the symplectic form can be written as

(1.1) 
$$\omega = \sum_{i=1}^{3g-3} d\ell_i \wedge d\tau_i.$$

Explicit parametrizations of  $Hit_n(S)$  have been given by Bonahon-Dreyer [BD14, BD] (see also [Zha15a, Zha15b]). Fixing an ideal triangulation  $\mathcal{T}$  of the surface and some transversal data  $\mathcal{J}$ , which we here call a bridge system, they provide a realanalytic parametrization of  $Hit_n(S)$  by the interior of a convex polytope. Their parametrization is based on work by Fock-Goncharov [FG06] and uses results of Labourie [Lab06] and Guchard [Gui08], which allows one to identify a representation in  $\operatorname{Hit}_n(S)$  with an equivariant Frenet curve from the boundary  $\partial \pi_1(S) \cong S^1$  into the space of flags in  $\mathbb{R}^n$ . To every edge of the ideal triangulation, Bonahon-Dreyer associates n-1 edge-invariants. These edge-invariants are the logarithms of cross ratios of four flags which are associated to the edge via the Frenet curve, using the transversal data  $\mathcal{J}$ . To every triangle in the ideal triangulation, they associate  $\frac{(n-1)(n-2)}{2}$  triangle invariants. These triangle invariants are triple ratios of the three flags which are associated to the triangle via the Frenet curve. The cross ratios and triple ratios have been introduced by Fock-Goncharov to give a parametrization of the space of positive local systems on a surface with punctures, and are intimately related to Lusztig's total positivity in split real Lie groups [Lus94]. For every edge of the ideal triangulation which corresponds to a simple closed curve S, Bonahon and Dreyer determine n-1 equalities and n-1 inequalities which the coordinates have to satisfy. These cut out a convex polytope  $P_{\mathcal{T}}$  in a subspace  $W_{\mathcal{T}}$  of some higher dimensional vector space, which provides a real analytic parametrization of  $\operatorname{Hit}_n(S)$ . Note that the polytope is independent of the transversal data  $\mathcal{J}$  and depends only on  $\mathcal{T}$ , however the way it parametrizes the Hitchin component depends on  $\mathcal{J}$ .

In Section 4 we describe a reparametrization where we replace the Bonahon-Dreyer edge-invariant for the edges corresponding to simple closed curves by what we call the symplectic closed edge-invariant. Under this reparamerization, the closed leaf equalities and inequalities remain the same, and we can identify at every point the tangent space of  $\mathrm{Hit}_n(S)$  with the vector space  $W_{\mathcal{T}}$ . We then associate to every tangent vector  $\mu \in W_{\mathcal{T}} \cong T_{[\rho]}\mathrm{Hit}_n(S)$  a flow on  $\mathrm{Hit}_n(S)$ . We call these flows  $(\mathcal{T}, \mathcal{J})$ -parallel flows and prove:

**Theorem 1.1** (Theorem 5.8, Corollary 5.9). Let  $\mu \in W \cong T_{[\rho]} \operatorname{Hit}_n(S)$ . Then there exists a unique  $(\mathcal{T}, \mathcal{J})$ -parallel flow tangent to  $\mu$ . Furthermore, if  $\mu_1, \mu_2 \in W \cong T_{[\rho]} \operatorname{Hit}_n(S)$ , then the  $(\mathcal{T}, \mathcal{J})$ -parallel flows associated to  $\mu_1$  and  $\mu_2$  commute.

**Corollary 1.2** (Corollary 5.9). Any pair  $(\mathcal{T}, \mathcal{J})$  determines a global trivialization of the tangent bundle  $T\mathrm{Hit}_n(S)$ .

General  $(\mathcal{T}, \mathcal{J})$ -parallel flows have no immediate geometric description; they are defined as limits of combinations of what we call elementary flows on the space of (non-equivariant) Frenet curves. These elementary flows are associated to either an edge, in which case we call them elementary shearing flows, or to a triangle, in which case we call them elementary eruption flows. The elementary shearing and eruption flows in turn are quite geometric (see Section 3). In the case when n=3 these flows have a particularly nice description, see [WZ17]. In order to deform an equivariant Frenet curve and keep it equivariant, we have to perform infinitely many elementary flows at the same time. To prove that this infinite product of flows is well-defined, we have to prove convergence of partial products, which is where the technical issues arise.

In order to investigate the relation of  $(\mathcal{T}, \mathcal{J})$ -parallel flows with respect to the Goldman symplectic form, two of the authors develop in a companion paper [SZ17] a new approach to compute the Goldman symplectic form on  $\mathrm{Hit}_n(S)$ . They show in particular that the above trivialization is indeed symplectic, and deduce that all  $(\mathcal{T}, \mathcal{J})$ -parallel flows are Hamiltonian.

A case of particular interest is when the ideal triangulation  $\mathcal{T}$  is subordinate to a pants decomposition  $\mathcal{P}$  of the surface S, and  $\mathcal{J}$  is an appropriately chosen bridge system  $\mathcal{J}$ . In this case we define two special families of  $(\mathcal{T}, \mathcal{J})$ -parallel flows:

- (1) For every simple closed curve  $c \in \mathcal{P}$ , we consider n-1 twist flows, which are generalized twist flows, i.e. they do not change the representation restricted to the fundamental group of  $S \setminus c$  (up to conjugation).
- (2) For every pair of pants P given by  $\mathcal{P}$ , we define  $\frac{(n-1)(n-2)}{2}$  eruption flows, which deform the representation restricted to P, but do not change the representation restricted to the fundamental group of  $S \setminus P$ .

Note that the twist and eruption flows preserve the holonomy along all closed curves in  $\mathcal{P}$ . In particular, the eruption flows arise only when n>2 and are not present in Teichmüller space since holonomy of the boundary curves uniquely determines the hyperbolic structure on a pair of pants. We describe the twist flows and the eruption flows in Section 6.2. For the Hitchin component  $\mathrm{Hit}_3(S)$  these flows are explicitly described in [WZ17].

Taking all twist flows associated to the simple closed curves in the pants decomposition and all the eruption flows associated to the pairs of pants in the pants decomposition, we get a family of  $(n^2 - 1)(g - 1)$  commuting flows. In the companion paper [SZ17, Theorem 6.5] the authors consider the vector fields determined by the twist and eruption flows and prove that at every point they form a Lagrangian subspace (with respect to the Goldman symplectic form) of the tangent space to the Hit<sub>n</sub>(S). As a corollary (of Corollary 6.7), we have

**Corollary 1.3.** Let  $(\mathcal{T}, \mathcal{J})$  be an ideal triangulation and a bridge system which is subordinate to  $\mathcal{P}$ . Then the twist flows associated to all simple closed curves in  $\mathcal{P}$  and the eruption flows associated to all pairs of pants given by  $\mathcal{P}$  provide a half-dimensional family of commuting Hamiltonian flows whose Hamiltonian functions Poisson commute.

The computational tools developed in [SZ17] furthermore allow to determine precisely two further families of special vector fields, which are dual (with respect to the symplectic form) to the twist and eruption vector fields

- (1) a family of n-1 length vector fields associated to every simple closed curve in the pants decomposition,
- (2) a family of  $\frac{(n-1)(n-2)}{2}$  hexagon vector fields associated to every pair of pants in the pants decomposition.

Using Theorem 1.1 one can deduce that the twist vector fields, eruption vector fields, length vector fields and hexagon vector fields form a global Darboux basis on  $\operatorname{Hit}_n(S)$  (see [SZ17, Theorem 6.5]).

In Section 6.3 we use the behaviour of the  $(\mathcal{T}, \mathcal{J})$ -parallel flows associated to these vector fields to compute the Hamiltonian functions of these vector fields explicitly (see Theorem 6.5). This gives a global Darboux coordinate system for  $\operatorname{Hit}_n(S)$ , thus generalizing Wolpert's formula (1.1) to  $\operatorname{Hit}_n(S)$ . When n=3 and n=4, versions of this was known by Kim [Kim99] and H.T. Jung (in preparation).

Structure of the Paper: In Section 2.1 we review definitions of cross ratios, triple ratios, and Frenet curves. The elementary shear and eruption flows for Frenet curves are introduced in Section 3. In Section 4 we describe the (re)parametrization of the Hitchin component. Finally in Section 5 we define the  $(\mathcal{T}, \mathcal{J})$ -parallel flows and prove the technical convergence results. The twist flows and eruption flows for a pants decomposition are introduced in Section 6, where we also determine the Darboux coordinate system for  $\operatorname{Hit}_n(S)$ . The appendix contains a few proofs of technical statements.

The results in the companion paper [SZ17] rely only on Sections 1 through 5 of this article, and only Section 6 of this article relies on the results in the companion paper.

**Notation:** Throughout the paper we use the following notation.

- We denote the fundamental group  $\pi_1(S)$  of a closed oriented surface S of genus  $g \geq 2$  by  $\Gamma$ , and its boundary by  $\partial \Gamma$ . Note that  $\partial \Gamma$  is naturally homeomorphic to  $S^1$ , and the orientation of S introduces an orientation on  $\partial \Gamma$ .
- We denote by V an n-dimensional real vector space. We will be working with various basis of V and will not make a preferred choice. Therefore we work with  $\operatorname{PSL}(V)$  instead of  $\operatorname{PSL}(n,\mathbb{R})$  and denote the Hitchin component by  $\operatorname{Hit}_V(S)$ . We consider the Hitchin component as a connected component of the representation variety  $\operatorname{Hom}(\Gamma,\operatorname{PSL}(V))/\operatorname{PGL}(V)$ . The reader can also consider the representation variety  $\operatorname{Hom}(\Gamma,\operatorname{PSL}(V))/\operatorname{PSL}(V)$ . Then, when n is even, there will be two connected components homeomorphic to  $\operatorname{Hit}(V)$ , and our results apply to each component separately.

# 2. Projective invariants and Frenet curves

In this section we review some properties of the space of complete flags and recall facts about Frenet curves, which are special maps from  $S^1$  to the space of flags.

Let V be an n-dimensional real vector space. A flag F in V is a sequence of properly nested subspaces  $F^{(1)} \subset \cdots \subset F^{(n-1)}$ , where  $\dim (F^{(i)}) = i$ . We denote the set of flags in V by  $\mathcal{F}(V)$ . A pair of flags  $(F_1, F_2)$  is transverse if for all  $i = 1, \ldots, n-1$ ,  $F_1^{(i)} + F_2^{(n-i)} = V$ . Similarly, we call a triple of flags

 $(F_1, F_2, F_3)$  transverse if for all  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ , we have  $F_1^{(i_1)} + F_2^{(i_2)} + F_3^{(i_3)} = V$ . This is stronger than just requiring the triple of flags to be pairwise transverse. Let  $\mathcal{F}(V)^{(2)}$  and  $\mathcal{F}(V)^{(3)}$  denote the set of transverse pairs of flags and the set of transverse triples of flags respectively.

The space  $\mathcal{F}(V)$  can be equipped with a real-analytic structure so that the inclusion  $\mathcal{F}(V) \to \prod_{i=1}^{n-1} \operatorname{Gr}(i,V)$  given by  $F \mapsto (F^{(1)},\dots,F^{(n-1)})$  is a real-analytic embedding. Observe that  $\mathcal{F}(V)^{(2)} \subset \mathcal{F}(V)^2$  and  $\mathcal{F}(V)^{(3)} \subset \mathcal{F}(V)^3$  are open real-analytic submanifolds. Also, the space of lines through the origin in V is naturally identified with projective classes of vectors in V, denoted  $\mathbb{P}(V)$ . Similarly, the space of hyperplanes through the origin in V is naturally identified with the space of projective classes of covectors in  $V^*$ , denoted  $\mathbb{P}(V^*)$ . These identifications will be made throughout the paper without further comment.

2.1. **Projective invariants.** We recall now two projective invariants, namely the cross ratio and the triple ratio, that will play an important role in the rest of this paper. We begin with the cross ratio.

**Definition 2.1.** Let  $K_1, K_2 \in \mathbb{P}(V^*)$  and  $P_1, P_2 \in \mathbb{P}(V)$  so that  $K_i(P_j) \neq 0$  for all i, j = 1, 2. Then the *cross ratio* of  $(K_1, P_1, P_2, K_2)$  is

$$C(K_1, P_1, P_2, K_2) := \frac{K_1(P_2)K_2(P_1)}{K_1(P_1)K_2(P_2)}.$$

In the above notation, for i,j=1,2, choose a covector  $\alpha_i$  in the projective class  $K_i$  and a vector  $v_j$  in the projective class  $P_j$  to evaluate  $K_i(P_j) = \alpha_i(v_j)$ . Although  $K_i(P_j)$  itself is not well-defined (i.e. it depends on the choice of the covector  $\alpha_i$  and the vector  $v_j$ ), it is easy to verify that the condition  $K_i(P_j) \neq 0$  and the cross ratio  $C(K_1, P_1, P_2, K_2)$  do not depend on these choices. More geometrically,  $K_i(P_j) \neq 0$  exactly means that the line  $P_j \subset V$  does not lie in the hyperplane  $K_i \subset V$ .

Observe the definition of the cross ratio implies that

$$C(K_1, P_1, P_2, K_2) \cdot C(K_1, P_2, P_3, K_2) = C(K_1, P_1, P_3, K_2)$$

for any  $K_1, K_2 \in \mathbb{P}(V^*)$  and  $P_1, P_2, P_3 \in \mathbb{P}(V)$  so that  $K_i(P_j) \neq 0$  for all i = 1, 2, j = 1, 2, 3. In particular,  $C(K_1, P_1, P_2, K_2) \cdot C(K_1, P_2, P_1, K_2) = 1$ .

If we choose a basis  $k_{1,i}, \ldots, k_{n-1,i}$  for the hyperplane  $K_i$ , then there is a covector  $\alpha_i$  in the projective class  $K_i$  so that for any vector  $v_i$  in the projective class  $P_i$ ,

$$\alpha_i(v_j) = \det(k_{1,i}, \dots, k_{n-1,i}, v_j) =: K_i \wedge P_j.$$

As defined,  $K_i \wedge P_j$  depends on the choice of the vectors  $k_{1,i}, \ldots, k_{n-1,i}, v_j$ , but the expression

$$\frac{K_1 \wedge P_2 \cdot K_2 \wedge P_1}{K_1 \wedge P_1 \cdot K_2 \wedge P_2}$$

does not, and it is easy to verify that

$$C(K_1, P_1, P_2, K_2) = \frac{K_1 \wedge P_2 \cdot K_2 \wedge P_1}{K_1 \wedge P_1 \cdot K_2 \wedge P_2}.$$

One of the most important (but easily verified) properties of the cross ratio is its projective invariance. More precisely, if  $g \in \operatorname{PGL}(V)$ ,  $K_1, K_2 \in \mathbb{P}(V^*)$ , and  $P_1, P_2 \in \mathbb{P}(V)$  so that  $K_i(P_j) \neq 0$  for all i, j = 1, 2, then

$$C(K_1, P_1, P_2, K_2) = C(q \cdot K_1, q \cdot P_1, q \cdot P_2, q \cdot K_2).$$

We will apply the cross ratio most commonly to the following setting. Let  $(F_1, F_2, F_3, F_4)$  be a quadruple of flags in  $\mathcal{F}(V)$  so that  $(F_1, F_2, F_3), (F_1, F_3, F_4) \in \mathcal{F}(V)^{(3)}$ . For  $i = 1, \ldots, n-1$ , consider the hyperplane  $K_i(F_1, F_3) := F_1^{(i)} + F_3^{(n-i-1)} \subset V$  and define

$$C_{i}(F_{1}, F_{2}, F_{4}, F_{3}) := C(K_{i}(F_{1}, F_{3}), F_{2}^{(1)}, F_{4}^{(1)}, K_{i-1}(F_{1}, F_{3}))$$

$$= \frac{F_{1}^{(i)} \wedge F_{3}^{(n-i-1)} \wedge F_{4}^{(1)} \cdot F_{1}^{(i-1)} \wedge F_{3}^{(n-i)} \wedge F_{2}^{(1)}}{F_{1}^{(i)} \wedge F_{2}^{(n-i-1)} \wedge F_{2}^{(1)} \cdot F_{1}^{(i-1)} \wedge F_{2}^{(n-i)} \wedge F_{4}^{(1)}}$$

for all i = 1, ..., n - 1.

The second projective invariant that we recall is the triple ratio. When n=3 the triple ratio determines transverse triples of flags up to projective equivalence. Given a triple of transverse flags  $F_1, F_2, F_3$  with  $F_i = (P_i, K_i)$ , the triple ratio is defined by  $T(F_1, F_2, F_3) = \frac{K_1(P_2) \cdot K_2(P_3) \cdot K_3(P_1)}{K_1(P_3) \cdot K_3(P_2) \cdot K_2(P_1)}$ , where as before we choose a covector  $\alpha_i$  in the projective class  $K_i \in \mathbb{P}(V^*)$  and a vector  $v_j$  in the projective class  $P_j \in \mathbb{P}(V)$  to evaluate  $K_i(P_j) := \alpha_i(v_j)$ .

The definition of the triple ratio can then be generalized in the following way:

**Definition 2.2.** Let  $K_1, K_2, K_3 \in \mathbb{P}(V^*)$  be three hyperplanes in V so that  $K_1 \cap K_2 \cap K_3 \subset V$  has codimension 3. Also, let  $P_1, P_2, P_3 \in \mathbb{P}(V)$  be lines in V so that for all  $m = 1, 2, 3, P_m \subset K_m \setminus (K_{m-1} \cup K_{m+1})$ . (Arithmetic in the subscripts are done modulo 3.) The *triple ratio* of  $(K_1, P_1, K_2, P_2, K_3, P_3)$  is

$$T(K_1, P_1, K_2, P_2, K_3, P_3) = \frac{K_1(P_2) \cdot K_2(P_3) \cdot K_3(P_1)}{K_1(P_3) \cdot K_3(P_2) \cdot K_2(P_1)}$$

Just as we did in the case of cross ratios, for each i, j = 1, 2, 3, we choose a covector  $\alpha_i$  in the projective class  $K_i \in \mathbb{P}(V^*)$  and a vector  $v_j$  in the projective class  $P_j \in \mathbb{P}(V)$  to evaluate  $K_i(P_j) := \alpha_i(v_j)$ . Again, one can verify that the triple ratio does not depend on these choices, and is in fact a projective invariant. It is also clear from the definition of the triple ratio that

$$T(K_1, P_1, K_2, P_2, K_3, P_3) = T(K_2, P_2, K_3, P_3, K_1, P_1) = \frac{1}{T(K_3, P_3, K_2, P_2, K_1, P_1)}$$

We apply the triple ratio to give projective invariants of transverse triples of flags as follows. Let  $F_1, F_2, F_3$  be a transverse triple of flags in  $\mathcal{F}(V)$ , and let  $i_1, i_2, i_3 \in \mathbb{Z}^+$  such that  $i_1 + i_2 + i_3 = n$ . For all m = 1, 2, 3, let

$$K_m := F_m^{(i_m+1)} + F_{m-1}^{(i_{m-1}-1)} + F_{m+1}^{(i_{m+1}-1)},$$

and let  $P_m$  be a line so that  $P_m + F_m^{(i_m)} = F_m^{(i_m+1)}$ . Observe that  $K_1 \cap K_2 \cap K_3 = F_1^{(i_1-1)} + F_2^{(i_2-1)} + F_3^{(i_3-1)}$ , which has codimension 3 in V. Clearly,  $P_m \subset K_m$ , and the transversality of the triple of flags  $F_1, F_2, F_3$  imply that  $P_m$  does not lie in  $K_{m-1}$  and  $K_{m+1}$ . This allows us to define

$$T_{i_1,i_2,i_3}(F_1,F_2,F_3) := T(K_1,P_1,K_2,P_2,K_3,P_3).$$

As before, it is possible to give a formula for the triple ratio in terms of some determinants. To do so, consider an ordered basis of V associated to each of the flags  $F_1, F_2, F_3$ .

**Definition 2.3.** Let  $F \in \mathcal{F}(V)$ . An ordered basis  $\{f_1, \ldots, f_n\}$  of V is associated to F if  $F^{(i)} = \operatorname{Span}_{\mathbb{R}}(f_1, \ldots, f_i)$  for all  $i = 1, \ldots, n-1$ .

For m = 1, 2, 3, let  $\{f_{m,1}, \ldots, f_{m,n}\}$  be an ordered basis of V that is associated to  $F_m$ . Since  $F_1, F_2, F_3$  is a transverse triple of flags, we know that for all  $i_1, i_2, i_3 \in \mathbb{Z}^+$ so that  $i_1 + i_2 + i_3 = n$ , the quantity

$$F_1^{(i_1)} \wedge F_2^{(i_2)} \wedge F_3^{(i_3)} := \det(f_{1,1}, \dots, f_{1,i_1}, f_{2,1}, \dots, f_{2,i_2}, f_{3,1}, \dots, f_{3,i_3})$$

is non-zero. Of course,  $F_1^{(i_1)} \wedge F_2^{(i_2)} \wedge F_3^{(i_3)}$  depends on the choice of the three bases, but one can verify that the ratio

$$\frac{F_1^{(i_1+1)} \wedge F_2^{(i_2)} \wedge F_3^{(i_3-1)} \cdot F_1^{(i_1-1)} \wedge F_2^{(i_2+1)} \wedge F_3^{(i_3)} \cdot F_1^{(i_1)} \wedge F_2^{(i_2-1)} \wedge F_3^{(i_3+1)}}{F_1^{(i_1+1)} \wedge F_2^{(i_2-1)} \wedge F_3^{(i_3)} \cdot F_1^{(i_1)} \wedge F_2^{(i_2+1)} \wedge F_3^{(i_3-1)} \cdot F_1^{(i_1-1)} \wedge F_2^{(i_2)} \wedge F_3^{(i_3+1)}}$$

does not, and in fact evaluates to  $T_{i_1,i_2,i_3}(F_1,F_2,F_3)$ . An important and easily verified application of these triple ratios is to give a parameterization of  $\mathcal{F}(V)^{(3)}/\mathrm{PGL}(V)$ . More precisely, we have the following proposition, which was used to great effectiveness by Fock-Goncharov [FG06].

**Proposition 2.4.** The map 
$$\mathcal{F}(V)^{(3)}/\text{PGL}(V) \to (\mathbb{R} \setminus \{0\})^{\frac{(n-1)(n-2)}{2}}$$
 given by  $[F_1, F_2, F_3] \mapsto (T_{i_1, i_2, i_3}(F_1, F_2, F_3))_{i_1 + i_2 + i_3 = n}$ 

is a real-analytic diffeomorphism.

Fock-Goncharov [FG06, Section 9] used the triple ratios and cross ratios to parameterize the space of positive *n*-tuples of flags, which we will now define.

## Definition 2.5.

- Let  $(F,G) \in \mathcal{F}(V)^{(2)}$  and let  $f_1,\ldots,f_n$  be a basis of V so that  $F^{(i)} \cap G^{(n-i+1)} = [f_i]$ . A unipotent  $u \in \operatorname{Stab}_{\operatorname{PSL}(V)}(F)$  is totally positive with respect to the pair (F,G) if in the basis  $f_1,\ldots,f_n$ , it is represented by an upper-triangular matrix where all the minors are positive, except those that are forced to be zero by u being upper-triangular.
- Let  $(F_1, \ldots F_k)$  be a k-tuple of flags in  $\mathcal{F}(V)$  (with  $k \geq 3$ ). This k-tuple is positive if  $(F_1, F_2) \in \mathcal{F}(V)^{(2)}$  and there are unipotent  $u_1, \ldots, u_{k-2} \in$  $\operatorname{Stab}_{\operatorname{PSL}(V)}(F_1)$  that are totally positive with respect to  $(F_1, F_2)$ , so that  $F_i = \left(\prod_{j=1}^{i-2} u_j\right) \cdot F_2$  for all  $i = 3, \ldots, k$ . Denote the space of positive k-tuples of flags in  $\mathcal{F}(V)$  by  $\mathcal{F}(V)^k$

Remark 2.6. One can verify that  $(F_1, \ldots, F_k) \in \mathcal{F}(V)_+^k$  if and only if  $(F_2, \ldots, F_k, F_1) \in \mathcal{F}(V)_+^k$  $\mathcal{F}(V)_{+}^{k}$ . Thus, positivity of a k-tuple of flags depends only on the cyclic order (and not the total order) on the k-tuple. Using this observation, Fock-Goncharov defined a map  $\xi: S^1 \to \mathcal{F}(V)$  to be *positive* if for any  $k \geq 3$  and any

$$x_1 < x_2 < \dots < x_k < x_1$$

in  $S^1$ , we have  $(\xi(x_1), \dots, \xi(x_k)) \in \mathcal{F}(V)_+^k$ . The notion of a positive map will not be central to the results in this paper, but it will be useful to keep this notion in mind to compare with the notion of Frenet curves which we will define later.

Fock-Goncharov proved the following characterization of a positive k-tuple of flags (this is a consequence of Theorem 9.1(a) in [FG06]).

**Theorem 2.7.** [FG06] Let  $(F_1, \ldots F_k)$  be a k-tuple of flags in  $\mathcal{F}(V)$ . This k-tuple is positive if and only if for any  $a,b,c \in \{1,\ldots,k\}$  that are pairwise distinct and for any  $i, j, k \in \mathbb{Z}^+$  so that i + j + k = n, we have  $T_{i,j,k}(F_a, F_b, F_c) > 0$ .

Remark 2.8. Geometrically, the condition  $T_{i,j,k}(F_a, F_b, F_c) > 0$  can be described in the following way. Let

$$\begin{split} K_a &:= F_a^{(i+1)} + F_b^{(j-1)} + F_c^{(k-1)}, \\ K_b &:= F_a^{(i-1)} + F_b^{(j+1)} + F_c^{(k-1)}, \\ K_c &:= F_a^{(i-1)} + F_b^{(j-1)} + F_c^{(k+1)}, \end{split}$$

and let  $P_a, P_b, P_c \in \mathbb{P}(V)$  so that

$$\begin{split} P_a + F_a^{(i)} &= F_a^{(i+1)}, \\ P_b + F_b^{(j)} &= F_b^{(j+1)}, \\ P_c + F_c^{(k)} &= F_c^{(k+1)}. \end{split}$$

Then  $T_{i,j,k}(F_a, F_b, F_c) > 0$  if and only if one of the four connected components of  $\mathbb{RP}^n \setminus (K_a \cup K_b \cup K_c)$  contains all of  $P_a$ ,  $P_b$  and  $P_c$  in its boundary (see Figure 1).

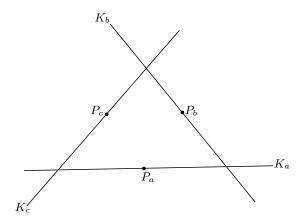


FIGURE 1. A positive triple of flags in  $\mathcal{F}(V)$  when n=3.

To describe the parameterization of  $\mathcal{F}(V)_+^k/\mathrm{PGL}(V)$  by cross ratios and triple ratios, consider an oriented planar polygon M with k vertices, and choose a triangulation  $\mathcal{T}$  of M so that the vertices of the triangles in the triangulation are exactly the vertices of M (see Figure 2). For any vertex v of M, let  $F(v) \in \mathcal{F}(V)$  be a flag associated to the vertex v.

Notation 2.9.

- (1) For each interior edge e of  $\mathcal{T}$ , let  $v_{e,1}$  and  $v_{e,2}$  be the endpoints of e respectively, and let  $w_{e,1}$  and  $w_{e,2}$  be the two vertices of M so that the triangles with vertices  $v_{e,1}$ ,  $v_{e,2}$ ,  $w_{e,1}$  and  $v_{e,1}$ ,  $v_{e,2}$ ,  $w_{e,2}$  are both triangles of  $\mathcal{T}$ , and  $v_{e,1} < w_{e,1} < v_{e,2} < w_{e,2} < v_{e,1}$  according to the clockwise cyclic ordering on  $\partial M$  (induced by the orientation on M, see Figure 2).
- (2) For each triangle T of  $\mathcal{T}$ , let  $v_{T,1}$ ,  $v_{T,2}$  and  $v_{T,3}$  be the three vertices of T so that  $v_{T,1} < v_{T,2} < v_{T,3} < v_{T,1}$  according to the clockwise cyclic ordering on  $\partial M$  (see Figure 2).

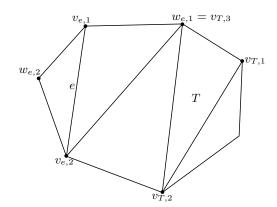


FIGURE 2. The oriented planar polygon M, where the orientation induces a clockwise orientation on  $\partial M$ .

**Proposition 2.10.** [FG06, Theorem 9.1(a)] Let M be an oriented planar k-gon  $(k \geq 3)$  and  $\mathcal{T}$  a triangulation of M as described above. Let  $T_1, \ldots, T_{k-2}$  be the triangles given by  $\mathcal{T}$  and  $e_1, \ldots, e_{k-3}$  be the interior edges of  $\mathcal{T}$ . The map  $\mathcal{F}(V)_+^k/\mathrm{PGL}(V) \to (\mathbb{R}^+)^{\frac{(n-1)(n-2)(k-2)}{2}} \times (\mathbb{R}^+)^{(n-1)(k-3)}$  given by

$$[F(v_j)]_{j \in [1,k]} \mapsto \left( \left( T_{i_1,i_2,i_3} \left( F(v_{T_j,1}), F(v_{T_j,2}), F(v_{T_j,3}) \right) \right)_{j \in [1,k-2]; i_1+i_2+i_3=n}, \left( -C_i \left( F(v_{e_j,1}), F(w_{e_j,1}), F(w_{e_j,2}), F(v_{e_j,2}) \right) \right)_{j \in [1,k-3]; i \in [1,n-1]} \right)$$

is a real analytic diffeomorphism.

Furthermore, the positivity of a k-tuple of flags ensures a strong transversality condition, which we state as the following proposition. (A weaker version of this is stated as [FG06] Proposition 9.4, although one can prove the following proposition using a similar argument.) See Appendix A for the proof.

**Proposition 2.11.** [FG06] Let  $F_1, \ldots, F_k$  be a positive k-tuple of flags in  $\mathcal{F}(V)$ . For any non-negative integers  $n_1, \ldots, n_k$  so that  $\sum_{i=1}^k n_i = d \leq n$ , we have that

$$\dim\left(\sum_{j=1}^k F(v_j)^{(n_j)}\right) = d.$$

2.2. **Frenet curves.** Next, we will describe a special class of continuous maps  $\xi: S^1 \to \mathcal{F}(V)$ , and describe their relationship with the the  $\mathrm{PSL}(V)$ -Hitchin component,  $\mathrm{Hit}_V(S)$ . For the rest of this article, we will fix once and for all a cyclic orientation on  $S^1$ .

**Definition 2.12.** A continuous curve  $\xi: S^1 \to \mathcal{F}(V)$  is Frenet if for all  $n_1, \ldots, n_k \in \mathbb{Z}^+$  so that  $\sum_{j=1}^k n_j = d \leq n$ , the following conditions hold.

(1) For all  $x_1, \ldots, x_k \in S^1$  pairwise distinct, the subspace

$$\sum_{j=1}^{k} \xi^{(n_j)}(x_j) \subset V$$

is of dimension d.

(2) Let  $x \in S^1$ . For all sequences  $\{(x_{i,1},\ldots,x_{i,k})\}_{i=1}^{\infty}$  of pairwise distinct k-tuples in  $S^1$  so that  $\lim_{i\to\infty} x_{i,j} = x$  for all  $j=1,\ldots,k$ , we have

$$\lim_{i \to \infty} \sum_{j=1}^{k} \xi^{(n_j)}(x_{i,j}) = \xi^{(d)}(x).$$

In particular, Frenet curves are injective. Denote the space of Frenet curves from  $S^1$  to  $\mathcal{F}(V)$  by  $\mathcal{FR}(V)$ . This space admits a topology so that a sequence  $\{\xi_i\}_{i=1}^{\infty}$  in  $\mathcal{FR}(V)$  converges to  $\xi \in \mathcal{FR}(V)$  if and only if  $\lim_{i\to\infty} \xi_i(p) = \xi(p)$  for all  $p \in S^1$ . With this topology, the continuous action of  $\mathrm{PGL}(V)$  on  $\mathcal{F}(V)$  induces a continuous action of  $\mathrm{PGL}(V)$  on  $\mathcal{FR}(V)$ . Equip  $\mathcal{FR}(V)/\mathrm{PGL}(V)$  with the quotient topology.

Remark 2.13. Observe that PGL(V) acts transitively and freely on the set

$$\{(F, G, P) \in \mathcal{F}(V)^{(2)} \times \mathbb{P}(V) : F^{(i)} + G^{(n-i-1)} + P = V \text{ for all } i = 1, \dots, n-1\}.$$

Hence, if we choose a triple of distinct points  $x, y, z \in S^1$  and a representative  $\xi_0$  for some  $[\xi_0] \in \mathcal{FR}(V)/\mathrm{PGL}(V)$ , then any  $[\xi] \in \mathcal{FR}(V)/\mathrm{PGL}(V)$ , has a unique representative  $\xi$  so that  $\xi(x) = \xi_0(x)$ ,  $\xi(y) = \xi_0(y)$  and  $\xi^{(1)}(z) = \xi_0^{(1)}(z)$ .

The next theorem is a well-known result about cross ratios and triple ratios of points along a Frenet curve. This was by Fock-Goncharov [FG06, Theorem 9.1] in the setting of positive maps, and a similar argument can be used to prove this in the setting of Frenet curves (see for example [Zha15b, Proposition 2.5.7] or [LM09, Appendix B]).

**Theorem 2.14.** Let  $\xi: S^1 \to \mathcal{F}(V)$  be a Frenet curve.

(1) For all  $x_1, x_2, x_3 \in S^1$  that are pairwise distinct, and for all  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ , we have

$$T_{i_1,i_2,i_3}(\xi(x_1),\xi(x_2),\xi(x_3)) > 0.$$

(2) For all  $x_1, x_2, x_3, x_4 \in S^1$  in this cyclic (or reverse cyclic) order, and for all i = 1, ..., n-1, we have

$$C_i(\xi(x_1), \xi(x_2), \xi(x_4), \xi(x_3)) < 0.$$

Frenet curves were first related to  $\mathrm{PSL}(V)$ -Hitchin representations by Labourie, who proved that for every  $\mathrm{PSL}(V)$ -Hitchin representation  $\rho$ , there exists a unique  $\rho$ -equivariant Frenet curve  $\xi:\partial\Gamma\to\mathcal{F}(V)$  (this makes sense because  $\partial\Gamma$ , the Gromov boundary of  $\Gamma$ , is topologically a circle). Guichard later proved the converse of this statement. Their combined work gives the following crucial result.

**Theorem 2.15.** [Gui08, Theorem 1], [Lab06, Theorem 1.4] Let  $\rho : \Gamma \to \mathrm{PSL}(V)$  be any representation. Then  $[\rho] \in \mathrm{Hit}_V(S)$  if and only if there is a  $\rho$ -equivariant Frenet curve  $\xi_{\rho} : \partial \Gamma \to \mathcal{F}(V)$ . Furthermore, if such a Frenet curve exists, then it is necessarily unique, and if  $\xi_{\rho} = \xi_{\rho'}$ , then  $\rho = \rho'$  necessarily.

Theorem 2.15 allows to identify points in  $\mathrm{Hit}_V(S)$  with projective classes of equivariant Frenet curves. This was used by Bonahon-Dreyer to give a real analytic parameterization of  $\mathrm{Hit}_V(S)$  by a convex polytope in some higher dimensional vector space. We will describe this parameterization in Section 4.4.

Remark 2.16. Theorem 2.7 and Theorem 2.14 imply that all Frenet curves are positive maps, although the converse is not true, because positive maps are not

required to be continuous. However, if we replace "Frenet curves" in the statement of Theorem 2.15 with "positive maps", then that is a theorem of Fock-Goncharov [FG06, Theorem 1.15]. In particular, even though positive maps  $\partial\Gamma \to \mathcal{F}(V)$  are not necessarily Frenet curves, the two notions agree if the maps are required to be  $\rho$ -equivariant for some representation  $\rho: \Gamma \to \mathrm{PSL}(V)$ .

### 3. Deforming Frenet curves

The goal of this section is to describe two families of flows, elementary eruption flows and elementary shearing flows on  $\mathcal{FR}(V)$ , which descend to flows on  $\mathcal{FR}(V)/PGL(V)$ . In Section 5 we will compose infinitely many such flows together to construct commuting flows on  $Hit_V(S)$ .

Let  $\xi: S^1 \to \mathcal{F}(V)$  be a Frenet curve, and let  $x_1 < x_2 < x_3 < x_1$  be a triple of points in  $S^1$ . Also, for m=1,2,3, let  $\{f_{m,1},\ldots,f_{m,n}\}$  be an ordered basis of Vassociated to the flag  $F_m := \xi(x_m)$ . Since  $(F_1, F_2, F_3) \in \mathcal{F}(V)^{(3)}$ , the basis

$$B_{F_1,F_2,F_3}^{i_1,i_2,i_3} := \{f_{1,1},\ldots,f_{1,i_1},f_{2,1},\ldots,f_{2,i_2},f_{3,1},\ldots,f_{3,i_3}\}$$

is an ordered basis of V for all non-negative integers  $i_1, i_2, i_3$  so that  $i_1 + i_2 + i_3 = n$ . In the case when  $i_3 = 0$ , we will write  $B^{i_1, i_2, i_3}_{F_1, F_2, F_3}$  simply as  $B^{i_1, i_2}_{F_1, F_2}$ .

3.1. Elementary eruption flow. In this section we define the elementary eruption flows.

Let  $(F_1, F_2, F_3) \in \mathcal{F}(V)^{(3)}$ , and let  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ . For m = 1, 2, 3, let  $b_{F_{m-1}, F_m, F_{m+1}}^{i_{m-1}, i_{m}, i_{m+1}}(t) \in \mathrm{PSL}(V)$  be the projective transformation which, when written in the basis  $B_{F_1,F_2,F_3}^{i_1,i_2,i_3}$ , is given by the matrix

$$b^{i_1,i_2,i_3}_{F_1,F_2,F_3}(t) := e^{\frac{(-i_2+i_3)t}{3n}} \cdot \left[ \begin{array}{ccc} \mathrm{id}_{i_1} & 0 & 0 \\ 0 & e^{\frac{t}{3}} \cdot \mathrm{id}_{i_2} & 0 \\ 0 & 0 & e^{-\frac{t}{3}} \cdot \mathrm{id}_{i_3} \end{array} \right],$$

$$b_{F_2,F_3,F_1}^{i_2,i_3,i_1}(t) := e^{\frac{(-i_3+i_1)t}{3n}} \cdot \left[ \begin{array}{ccc} e^{-\frac{t}{3}} \cdot \mathrm{id}_{i_1} & 0 & 0 \\ 0 & \mathrm{id}_{i_2} & 0 \\ 0 & 0 & e^{\frac{t}{3}} \cdot \mathrm{id}_{i_3} \end{array} \right],$$

$$b_{F_3,F_1,F_2}^{i_3,i_1,i_2}(t) := e^{\frac{(-i_1+i_2)t}{3n}} \cdot \left[ \begin{array}{ccc} e^{\frac{t}{3}} \cdot \mathrm{id}_{i_1} & 0 & 0 \\ 0 & e^{-\frac{t}{3}} \cdot \mathrm{id}_{i_2} & 0 \\ 0 & 0 & \mathrm{id}_{i_2} \end{array} \right].$$

Then for all m = 1, 2, 3, define

$$a^{i_m,i_{m+1},i_{m-1}}_{F_m,F_{m+1},F_{m-1}} := b^{i_{m-1},i_m,i_{m+1}}_{F_{m-1},F_m,F_{m+1}}(t) b^{i_{m+1},i_{m-1},i_m}_{F_{m+1},F_{m-1},F_m}(-t).$$

(The subscripts are to be read modulo 3.) One can compute that

$$(3.1) \quad a^{i_m,i_{m+1},i_{m-1}}_{F_m,F_{m+1},F_{m-1}}(t) = \left[ \begin{array}{ccc} e^{-\frac{i_mt}{n}} \cdot \mathrm{id}_{i_{m-1}} & 0 & 0 \\ 0 & e^{\frac{(n-i_m)t}{n}} \cdot \mathrm{id}_{i_m} & 0 \\ 0 & 0 & e^{-\frac{i_mt}{n}} \cdot \mathrm{id}_{i_{m+1}} \end{array} \right]$$

when written as a matrix in the basis  $B^{i_{m-1},i_m,i_{m+1}}_{F_{m-1},F_m,F_{m+1}}$ . Using these projective transformations, we define a continuous flow on the space of Frenet curves.

Notation 3.1. Let  $p,q \in S^1$  be a pair of distinct points. Then let  $\overline{(p,q)}$  denote the open subinterval of  $S^1$  from p to q in the cyclic order of  $S^1$ . Similarly, let  $\overline{(p,q)} := \overline{(p,q)} \cup \{q\}, \overline{[p,q)} := \overline{(p,q)} \cup \{p\}$  and  $\overline{[p,q]} := \overline{(p,q)} \cup \{p\} \cup \{q\}$ .

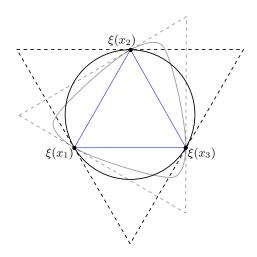


FIGURE 3. When n=3, the dark conic is  $\xi^{(1)}(S^1) \subset \mathbb{RP}^2$  and the dark dotted lines are  $\xi^{(2)}(x_1)$ ,  $\xi^{(2)}(x_2)$  and  $\xi^{(2)}(x_3)$ . After applying an elementary eruption at  $x_1, x_2, x_3, \xi_t^{(1)}(S^1)$  is the light curve and  $\xi_t^{(2)}(x_1)$ ,  $\xi_t^{(2)}(x_2)$  and  $\xi_t^{(2)}(x_3)$  are the light dotted lines.

**Definition 3.2.** Let  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1+i_2+i_3=n$ , and let  $x_1 < x_2 < x_3 < x_1$  be a triple of points in  $S^1$ . The  $(i_1, i_2, i_3)$ -elementary eruption flow with respect to the triple  $(x_1, x_2, x_3)$  is the continuous flow

$$\left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t: \mathcal{FR}(V) \to \mathcal{FR}(V)$$

defined by

$$\left(\epsilon_{x_{1},x_{2},x_{3}}^{i_{1},i_{2},i_{3}}\right)_{t}(\xi) := \xi_{t}(p) = \begin{cases} b_{1}(t) \cdot \xi(p) & \text{if } p \in \overline{[x_{2},x_{3}]} \\ b_{2}(t) \cdot \xi(p) & \text{if } p \in \overline{[x_{3},x_{1}]} \\ b_{3}(t) \cdot \xi(p) & \text{if } p \in \overline{[x_{1},x_{2}]} \end{cases}$$

where  $b_m(t) := b_{\xi(x_m),\xi(x_{m+1}),\xi(x_{m-1})}^{i_m,i_{m+1},i_{m-1}}(t)$  for all m = 1, 2, 3.

It is clear from the definition that  $(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3})_t = (\epsilon_{x_2,x_3,x_1}^{i_2,i_3,i_1})_t = (\epsilon_{x_3,x_1,x_2}^{i_3,i_1,i_2})_t$ . In the case when n=3 and  $i_1=i_2=i_3=1$ , the elementary eruption flow is (up to projective transformations) the eruption flow defined in [WZ17, Section 4.2.] and admits a nice geometric interpretation (see Figure 3).

Given  $\xi \in \mathcal{FR}(V)$ , the flow  $\left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t$  deforms the three subintervals of  $\xi(S^1)$  determined by  $x_1,x_2,x_3$  by three different projective transformations. These three projective transformations are chosen to ensure that  $\xi_t := \left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t(\xi)$  is Frenet (see Theorem 3.4) and so that the flow  $\left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t$  has the effect of "only changing" the triple ratio  $T_{i_1,i_2,i_3}\left(\xi_t(x_1),\xi_t(x_2),\xi_t(x_3)\right)$  (see Proposition 3.3).

We prove now that  $(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3})_t$  is well-defined. First, let us observe that  $\xi_t: S^1 \to S^1$  $\mathcal{F}(V)$  is indeed a continuous curve, i.e. for all m=1,2,3 and for all  $l=1,\ldots,n-1,$ 

$$b_{m-1}(t) \cdot \xi^{(l)}(x_m) = b_{m+1}(t) \cdot \xi^{(l)}(x_m).$$

This is clear when  $l \leq i_m$ . On the other hand, for any  $l > i_m$  and any  $v \in \xi^{(l)}(x_m)$ , write

$$v = \sum_{j=1}^{i_1} \alpha_{1,j} f_{1,j} + \sum_{j=1}^{i_2} \alpha_{2,j} f_{2,j} + \sum_{j=1}^{i_3} \alpha_{3,j} f_{3,j},$$

where  $\alpha_{m,j} \in \mathbb{R}$ . A direct computation gives that

$$b_{m-1}(t) \cdot v = e^{\frac{(-i_m + i_{m+1})t}{3n}} \left( \sum_{j=1}^{i_{m-1}} \alpha_{m-1,j} f_{m-1,j} + e^{\frac{t}{3}} \sum_{j=1}^{i_m} \alpha_{m,j} f_{m,j} + e^{-\frac{t}{3}} \sum_{j=1}^{i_{m+1}} \alpha_{m+1,j} f_{m+1,j} \right),$$

$$b_{m+1}(t) \cdot v = e^{\frac{(-i_{m-1} + i_m)t}{3n}} \left( e^{\frac{t}{3}} \sum_{j=1}^{i_{m-1}} \alpha_{m-1,j} f_{m-1,j} + e^{-\frac{t}{3}} \sum_{j=1}^{i_m} \alpha_{m,j} f_{m,j} + \sum_{j=1}^{i_{m+1}} \alpha_{m+1,j} f_{m+1,j} \right).$$

Hence,  $b_{m-1}(t) \cdot v$  and  $b_{m+1}(t) \cdot v$  both lie in the subspace of V spanned by the  $i_m + 1$  vectors

$$f_{m,1}, \ldots, f_{m,i_m}, e^{\frac{t}{3}} \sum_{j=1}^{i_{m-1}} \alpha_{m-1,j} f_{m-1,j} + \sum_{j=1}^{i_{m+1}} \alpha_{m+1,j} f_{m+1,j}.$$

In particular,  $b_{m-1}(t) \cdot \xi^{(l)}(x_m) = b_{m+1}(t) \cdot \xi^{(l)}(x_m)$ .

To prove that  $\xi_t$  is Frenet, we need the following proposition, which describes how the certain cross ratios and triple ratios along the image of  $\xi_t$  change with t.

**Proposition 3.3.** Let  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ , let  $x_1 < x_2 < x_3 < x_1$  be a triple of points in  $S^1$ , and let  $\xi_t := (\epsilon_{x_1, x_2, x_3}^{i_1, i_2, i_3})_t(\xi)$ .

(1) Let 
$$\delta(j_1, j_2, j_3) = \begin{cases} 1 & \text{if } (i_1, i_2, i_3) = (j_1, j_2, j_3) \\ 0 & \text{otherwise} \end{cases}$$
. Then

$$T_{j_1,j_2,j_3}\big(\xi_t(x_1),\xi_t(x_2),\xi_t(x_3)\big) = e^{t\delta(j_1,j_2,j_3)} \cdot T_{j_1,j_2,j_3}\big(\xi(x_1),\xi(x_2),\xi(x_3)\big)$$

for all  $j_1, j_2, j_3 \in \mathbb{Z}^+$  so that  $j_1 + j_2 + j_3 = n$ .

(2) Let  $p_1, p_2, p_3 \in S^1$  be a triple of points so that  $p_1 < p_2 < p_3 \le x_1 < x_2 < x_3 \le x_1 < x_2 < x_2 < x_3 \le x_1 < x_2 < x_2 < x_3 \le x_1 < x_2 < x_2 < x_3 < x_3$  $x_3 \leq p_1$ . Then

$$T_{j_1,j_2,j_3}\big(\xi_t(p_1),\xi_t(p_2),\xi_t(p_3)\big) = T_{j_1,j_2,j_3}\big(\xi(p_1),\xi(p_2),\xi(p_3)\big)$$

for all  $j_1, j_2, j_3 \in \mathbb{Z}^+$  so that  $j_1 + j_2 + j_3 = n$ . (3) Let  $p_1, p_2, p_3 \in S^1$  be a triple of points so that  $p_1 < p_2 < p_3 \le x_1 < x_2 < x_3 < x_4 < x_2 < x_4 < x_4 < x_5 < x_4 < x_5 <$  $x_3 \leq p_1$ . Let  $p_4 \in \overline{(p_3, x_1)} \cup \overline{(x_3, p_1)} \cup \{x_2\}$ . Then for all  $i = 1, \ldots, n-1$ ,  $C_i(\xi_t(p_1), \xi_t(p_2), \xi_t(p_4), \xi_t(p_3)) = C_i(\xi(p_1), \xi(p_2), \xi(p_4), \xi(p_3)).$ 

*Proof.* Let  $a_m(t) := a_{\xi(x_m),\xi(x_{m+1}),\xi(x_{m-1})}^{i_{m,i_{m+1},i_{m-1}}}(t)$ , let  $b_m(t) := b_{\xi(x_m),\xi(x_{m+1}),\xi(x_{m-1})}^{i_{m,i_{m+1},i_{m-1}}}(t)$ , and note that for all m = 1, 2, 3,

$$b_{m+1}(t)^{-1} \cdot \left( \xi_t(x_{m-1}), \xi_t(x_m), \xi_t(x_{m+1}) \right) = \left( \xi(x_{m-1}), \xi(x_m), a_m(t) \cdot \xi(x_{m+1}) \right).$$

(1) Suppose first that  $(j_1, j_2, j_3) \neq (i_1, i_2, i_3)$ . Then  $i_{m+1} \geq j_{m+1} + 1$  for some m = 1, 2, 3. By (3.1),  $a_m(t)$  acts on  $\xi^{(j_{m+1}+1)}(x_{m+1}) \subset \xi^{(i_{m+1})}(x_{m+1})$  as scaling by some  $\lambda := e^{-\frac{i_m t}{n}}$ . Since the triple ratio is a projective invariant, we have

$$T_{j_{1},j_{2},j_{3}}(\xi_{t}(x_{1}),\xi_{t}(x_{2}),\xi_{t}(x_{3}))$$

$$= T_{j_{m-1},j_{m},j_{m+1}}(\xi(x_{m-1}),\xi(x_{m}),a_{m}(t)\cdot\xi(x_{m+1}))$$

$$= \frac{\xi^{(j_{m-1}+1)}(x_{m-1})\wedge\xi^{(j_{m})}(x_{m})\wedge\xi^{(j_{m+1}-1)}(x_{m+1})\cdot\lambda^{j_{m+1}-1}}{\xi^{(j_{m-1}+1)}(x_{m-1})\wedge\xi^{(j_{m}-1)}(x_{m})\wedge\xi^{(j_{m+1})}(x_{m+1})\cdot\lambda^{j_{m+1}}} \cdot \frac{\xi^{(j_{m-1}-1)}(x_{m-1})\wedge\xi^{(j_{m}+1)}(x_{m})\wedge\xi^{(j_{m+1})}(x_{m+1})\cdot\lambda^{j_{m+1}-1}}{\xi^{(j_{m-1})}(x_{m-1})\wedge\xi^{(j_{m}+1)}(x_{m})\wedge\xi^{(j_{m+1}-1)}(x_{m+1})\cdot\lambda^{j_{m+1}-1}} \cdot \frac{\xi^{(j_{m-1})}(x_{m-1})\wedge\xi^{(j_{m}-1)}(x_{m})\wedge\xi^{(j_{m+1}+1)}(x_{m+1})\cdot\lambda^{j_{m+1}+1}}{\xi^{(j_{m-1}-1)}(x_{m-1})\wedge\xi^{(j_{m})}(x_{m})\wedge\xi^{(j_{m+1}+1)}(x_{m+1})\cdot\lambda^{j_{m+1}+1}}$$

$$= T_{j_{m-1},j_{m},j_{m+1}}(\xi(x_{m-1}),\xi(x_{m}),\xi(x_{m+1}))$$

$$= T_{j_{1},j_{2},j_{3}}(\xi(x_{1}),\xi(x_{2}),\xi(x_{3})).$$

On the other hand,

$$T_{i_{1},i_{2},i_{3}}\left(\xi_{t}(x_{1}),\xi_{t}(x_{2}),\xi_{t}(x_{3})\right)$$

$$= T_{i_{1},i_{2},i_{3}}\left(\xi(x_{1}),\xi(x_{2}),a_{2}(t)\cdot\xi(x_{3})\right)$$

$$= \frac{\xi^{(i_{1}+1)}(x_{1})\wedge\xi^{(i_{2})}(x_{2})\wedge\xi^{(i_{3}-1)}(x_{3})\cdot e^{-\frac{i_{2}(i_{3}-1)t}{n}}}{\xi^{(i_{1}+1)}(x_{1})\wedge\xi^{(i_{2}-1)}(x_{2})\wedge\xi^{(i_{3})}(x_{3})\cdot e^{-\frac{i_{2}i_{3}t}{n}}}\cdot \frac{\xi^{(i_{1}-1)}(x_{1})\wedge\xi^{(i_{2}+1)}(x_{2})\wedge\xi^{(i_{3})}(x_{3})\cdot e^{-\frac{i_{2}i_{3}t}{n}}}{\xi^{(i_{1})}(x_{1})\wedge\xi^{(i_{2}+1)}(x_{2})\wedge\xi^{(i_{3}-1)}(x_{3})\cdot e^{-\frac{i_{2}i_{3}t}{n}}\cdot \frac{\xi^{(i_{1})}(x_{1})\wedge\xi^{(i_{2}+1)}(x_{2})\wedge\xi^{(i_{3}-1)}(x_{3})\cdot e^{-\frac{i_{2}i_{3}t}{n}}e^{\frac{(n-i_{2})t}{n}}}{\xi^{(i_{1}-1)}(x_{1})\wedge\xi^{(i_{2})}(x_{2})\wedge\xi^{(i_{3}+1)}(x_{3})\cdot e^{-\frac{i_{2}i_{3}t}{n}}e^{-\frac{i_{2}t}{n}}}$$

$$= e^{t}\cdot T_{i_{1},i_{2},i_{3}}\left(\xi(x_{1}),\xi(x_{2}),\xi(x_{3})\right).$$

- (2) Note that for all  $m = 1, 2, 3, \xi_t(p_m) = b_2(t) \cdot \xi(p_m)$ . Since the triple ratio is a projective invariant, (2) follows immediately.
- (3) Observe that  $C_i(\xi_t(p_1), \xi_t(p_2), \xi_t(p_4), \xi_t(p_3))$  depends only on  $\xi_t^{(1)}(p_4)$  (and not on the rest of the flag  $\xi_t(p_4)$ ). By the definition of  $b_2(t)$ , it is clear that  $\xi_t^{(1)}(p_4) = b_2(t) \cdot \xi^{(1)}(p_4)$ . Since  $\xi_t(p_m) = b_2(t) \cdot \xi(p_m)$  for m = 1, 2, 3, the projective invariance of the cross ratio implies (3).

We can now prove the following theorem, which implies that the elementary eruption flow is well-defined.

**Theorem 3.4.** Let  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ , and let  $x_1 < x_2 < x_3 < x_1$  be a triple of points in  $S^1$ . Then  $\xi_t := (\epsilon_{x_1, x_2, x_3}^{i_1, i_2, i_3})_t(\xi)$  is a Frenet curve for all  $t \in \mathbb{R}$ .

*Proof.* First, we will prove that  $\xi_t$  satisfies property (1) of Definition 2.12. We have already established that  $\xi_t$  is continuous. Choose pairwise distinct points  $p_1, \ldots, p_k \in S^1$  and  $n_1, \ldots, n_k \in \mathbb{Z}^+$  so that  $\sum_{j=1}^k n_j = d \leq n$ . Then the set

 $\{p_1, \ldots, p_k\} \cup \{x_1, x_2, x_3\}$  consists of at most k+3 points, and admits a cyclic ordering induced by the chosen cyclic ordering on  $S^1$ . Hence, we can think of these points as the vertices of a planar polygon M inscribed in the circle  $S^1$ . Choose a triangulation  $\mathcal{T}$  of M so that the vertices of  $\mathcal{T}$  are the vertices of M, and the triangle with vertices  $x_1, x_2, x_3$  is a triangle in  $\mathcal{T}$ .

For each interior edge e of  $\mathcal{T}$ , let  $v_{e,1}, v_{e,2}, w_{e,1}, w_{e,2}$  be the vertices defined in Notation 2.9(1) and for each triangle T of  $\mathcal{T}$ , let  $v_{T,1}, v_{T,2}$  and  $v_{T,3}$  be the vertices defined in Notation 2.9(2). By Proposition 2.14, we know that

• For every interior edge e of  $\mathcal{T}$  and for all  $i = 1, \ldots, n-1$ ,

$$C_i(\xi(v_{e,1}), \xi(w_{e,1}), \xi(w_{e,2}), \xi(v_{e,2})) < 0.$$

• For every triangle T of the triangulation  $\mathcal{T}$  and for all  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ ,

$$T_{i_1,i_2,i_3}(\xi(v_{T,1}),\xi(w_{T,2}),\xi(w_{T,3})) > 0.$$

It then follows from Proposition 3.3 that the two statements about the cross ratio and triple ratio above also holds if we replace  $\xi$  with  $\xi_t$ . Hence, by Proposition 2.10, the curve  $\xi_t$  is positive, and Proposition 2.11 allows us to conclude that

$$\dim\left(\sum_{j=1}^k \xi_t^{(n_j)}(p_j)\right) = d.$$

Next, we will show that  $\xi_t$  satisfies property (2) of Definition 2.12. Let  $x \in S^1$ , and consider any sequence  $\{(p_{i,1}, \ldots, p_{i,k})\}_{i=1}^{\infty}$  of pairwise distinct k-tuples in  $S^1$ , so that  $\lim_{i\to\infty} p_{i,j} = x \in S^1$  for all  $j=1,\ldots,k$ . If  $x \in \overline{(x_{m+1},x_{m-1})}$  for some m=1,2,3, then  $p_{i,j} \in \overline{(x_{m+1},x_{m-1})}$  for sufficiently large i and for all  $j=1,\ldots,k$ . Let  $b_m(t) := b_{\xi(x_m),\xi(x_{m+1}),\xi(x_{m-1})}^{i_m,i_{m+1},i_{m-1}}(t)$  for all m=1,2,3. Since  $\xi_t(p) = b_m(t) \cdot \xi(p)$  for all  $p \in \overline{(x_{m+1},x_{m-1})}$ ,

$$\lim_{t \to \infty} \sum_{j=1}^{k} \xi_t^{(n_j)}(p_{i,j}) = b_m(t) \cdot \lim_{t \to \infty} \sum_{j=1}^{k} \xi^{(n_j)}(p_{i,j}) = b_m(t) \cdot \xi^{(d)}(x) = \xi_t^{(d)}(x).$$

Now, suppose that  $x = x_m$  for some m = 1, 2, 3. Observe that  $\xi_t(y) = b_{m+1}(t) \cdot \xi(y)$  for all  $y \in \overline{(x_{m-1}, x_m]}$ , and  $\xi_t(y) = b_{m-1}(t) \cdot \xi(y)$  for all  $y \in \overline{[x_m, x_{m+1})}$ . Recall that  $a_m(t) := b_{m-1}(t)b_{m+1}(t)^{-1}$  fixes the flag  $\xi(x_m)$ . For sufficiently large i and for all  $j = 1, \ldots, k$ , we know that  $p_{i,j} \in \overline{(x_{m+1}, x_{m-1})}$ . Let

$$A:=\{j:p_{i,j}\in\overline{(x_{m-1},x_m]}\text{ for sufficiently large }i\}$$

and let

$$B := \{j : p_{i,j} \in \overline{(x_m, x_{m+1})} \text{ for sufficiently large } i\}.$$

By taking subsequences, we may assume that  $A \cup B = \{1, \dots, k\}$  is a disjoint union. Then

$$\lim_{i \to \infty} \sum_{j=1}^{k} \xi_{t}^{(n_{j})}(p_{i,j}) = b_{m+1}(t) \cdot \lim_{i \to \infty} \sum_{j \in A} \xi^{(n_{j})}(p_{i,j}) + b_{m-1}(t) \cdot \lim_{i \to \infty} \sum_{j \in B} \xi^{(n_{j})}(p_{i,j})$$

$$= b_{m+1}(t) \cdot \lim_{i \to \infty} \left( \sum_{j \in A} \xi^{(n_{j})}(p_{i,j}) + a_{m}(t) \cdot \sum_{j \in B} \xi^{(n_{j})}(p_{i,j}) \right)$$

Since  $\xi$  is Frenet,

$$\lim_{i \to \infty} \sum_{j \in A} \xi^{(n_j)}(p_{i,j}) \subset \xi^{(d)}(x)$$

and

$$\lim_{i \to \infty} a_m(t) \cdot \sum_{j \in B} \xi^{(n_j)}(p_{i,j}) \subset a_m(t) \cdot \xi^{(d)}(x) = \xi^{(d)}(x),$$

which implies that

$$\lim_{i \to \infty} \left( \sum_{j \in A} \xi^{(n_j)}(p_{i,j}) + a_m(t) \cdot \sum_{j \in B} \xi^{(n_j)}(p_{i,j}) \right) = \xi^{(d)}(x)$$

because  $\sum_{j \in A} \xi^{(n_j)}(p_{i,j}) + a_m(t) \cdot \sum_{j \in B} \xi^{(n_j)}(p_{i,j})$  has dimension d. Thus, we now know

that

$$\lim_{i \to \infty} \sum_{j=1}^{k} \xi_t^{(n_j)}(p_{i,j}) = b_{m+1}(t) \cdot \xi^{(d)}(x) = \xi_t^{(d)}(x).$$

We establish now some basic properties of elementary eruption flows.

**Proposition 3.5.** Let  $i_1, i_2, i_3, j_1, j_2, j_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n = j_1 + j_2 + j_3$ . Also, let  $x_1 < x_2 < x_3 < x_1$  be a triple of points in  $S^1$ . Then the following hold:

- (1) The flow  $\left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t$  on  $\mathcal{FR}(V)$  descends to a flow on  $\mathcal{FR}(V)/\mathrm{PGL}(V)$ . We denote the flow on  $\mathcal{FR}(V)/\mathrm{PGL}(V)$  also by  $\left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t$  and call it the  $(i_1,i_2,i_3)$ -elementary eruption flow (associated to  $x_1,x_2,x_3$ ).
- (2) For all  $t, t' \in \mathbb{R}$ ,

$$\left(\epsilon_{x_{1},x_{2},x_{3}}^{i_{1},i_{2},i_{3}}\right)_{t}\circ\left(\epsilon_{x_{1},x_{2},x_{3}}^{j_{1},j_{2},j_{3}}\right)_{t'}=\left(\epsilon_{x_{1},x_{2},x_{3}}^{j_{1},j_{2},j_{3}}\right)_{t'}\circ\left(\epsilon_{x_{1},x_{2},x_{3}}^{i_{1},i_{2},i_{3}}\right)_{t}$$

as flows on  $\mathcal{FR}(V)/PGL(V)$ .

(3) Let  $y_1, y_2, y_3 \in S^1$  be pairwise distinct points that lie in the closure in  $S^1$  of a connected component of  $S^1 \setminus \{x_1, x_2, x_3\}$ . For all  $t, t' \in \mathbb{R}$ ,

$$\left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t \circ \left(\epsilon_{y_1,y_2,y_3}^{j_1,j_2,j_3}\right)_{t'} = \left(\epsilon_{y_1,y_2,y_3}^{j_1,j_2,j_3}\right)_{t'} \circ \left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t$$

as flows on  $\mathcal{FR}(V)/PGL(V)$ .

*Proof.* (1) Observe that for any triple of transverse flags  $F_1, F_2, F_3 \in \mathcal{F}(V)$  and any  $g \in \mathrm{PGL}(V)$ , we have that

$$gb_{F_1,F_2,F_3}^{i_1,i_2,i_3}g^{-1} = b_{g \cdot F_1,g \cdot F_2,g \cdot F_3}^{i_1,i_2,i_3}.$$

Let  $\xi: S^1 \to \mathcal{F}(V)$  be a Frenet curve. For all  $p \in S^1$  and  $g \in PGL(V)$ ,

$$\begin{split} g \cdot \left( \epsilon_{x_1, x_2, x_3}^{i_1, i_2, i_3} \right)_t (\xi) (p) &= \begin{cases} g b_{\xi(x_1), \xi(x_2), \xi(x_3)}^{i_1, i_2, i_3} (t) \cdot \xi(p) & \text{if } p \in \overline{[x_2, x_3]} \\ g b_{\xi(x_2), \xi(x_3), \xi(x_1)}^{i_2, i_3, i_1} (t) \cdot \xi(p) & \text{if } p \in \overline{[x_3, x_1]} \\ g b_{\xi(x_3), \xi(x_1), \xi(x_2)}^{i_3, i_1, i_2} (t) \cdot \xi(p) & \text{if } p \in \overline{[x_1, x_2]} \end{cases} \\ &= \begin{cases} b_{\xi(x_1), \xi(x_2), \xi(x_2), \xi(x_3)}^{i_1, i_2} (t) g \cdot \xi(p) & \text{if } p \in \overline{[x_2, x_3]} \\ g_{\xi(x_1), g, \xi(x_2), g, \xi(x_3), g, \xi(x_1)} (t) g \cdot \xi(p) & \text{if } p \in \overline{[x_3, x_1]} \\ b_{\xi(x_1, x_2, x_3), \xi(x_1), g, \xi(x_2)}^{i_3, i_1} (t) g \cdot \xi(p) & \text{if } p \in \overline{[x_1, x_2]} \\ &= \left( \epsilon_{x_1, x_2, x_3}^{i_1, i_2, i_3} \right)_t (g \cdot \xi)(p). \end{cases} \end{split}$$

(2) Let  $\xi_1 := (\epsilon_{x_1, x_2, x_3}^{i_1, i_2, i_3})_t \circ (\epsilon_{x_1, x_2, x_3}^{j_1, j_2, j_3})_{t'} (\xi)$  and let  $\xi_2 := (\epsilon_{x_1, x_2, x_3}^{j_1, j_2, j_3})_{t'} \circ (\epsilon_{x_1, x_2, x_3}^{i_1, i_2, i_3})_t (\xi)$ . By Proposition 3.3(1) and Proposition 2.4, we know that there is a projective transformation  $g \in \operatorname{PGL}(V)$  so that  $g \cdot \xi_1(x_m) = \xi_2(x_m)$  for all m = 1, 2, 3. Let  $\eta_1 := (\epsilon_{x_1, x_2, x_3}^{j_1, j_2, j_3})_{t'} (\xi)$ , let  $\eta_2 := (\epsilon_{x_1, x_2, x_3}^{i_1, i_2, i_3})_t (\xi)$ , and let

$$b_{m,\xi}(t) := b_{\xi(x_m),\xi(x_{m+1}),\xi(x_{m-1})}^{i_{m,i_{m+1},i_{m-1}}}(t),$$

$$b'_{m,\xi}(t') := b_{\xi(x_m),\xi(x_{m+1}),\xi(x_{m-1})}^{j_{m,j_{m+1},j_{m-1}}}(t'),$$

$$b_{m,\eta_1}(t) := b_{\eta_1(x_m),\eta_1(x_{m+1}),\eta_1(x_{m-1})}^{i_{m,i_{m+1},i_{m-1}}}(t),$$

$$b'_{m,\eta_2}(t') := b_{\eta_2(x_m),\eta_2(x_{m+1}),\eta_2(x_{m-1})}^{j_{m,j_{m+1},j_{m-1}}}(t').$$

Fix m = 1, 2, 3, and observe that for all  $x \in \overline{[x_{m+1}, x_{m-1}]}$ ,

$$\xi_1(x) = b_{m,\eta_1}(t)b'_{m,\xi}(t') \cdot \xi(x)$$
 and  $\xi_2(x) = b'_{m,\eta_2}(t')b_{m,\xi}(t) \cdot \xi(x)$ .

It follows from the definitions that  $b_{m,\xi}(t), b'_{m,\xi}(t'), b_{m,\eta_1}(t)$  and  $b'_{m,\eta_2}(t')$  fix  $\xi^{(1)}(x_m)$ . In particular,  $\xi^{(1)}(x_m) = \xi_1^{(1)}(x_m) = \xi_2^{(1)}(x_m)$ . Moreover, the product

$$b'_{m,\eta_2}(t')b_{m,\xi}(t)b'_{m,\xi}(t')^{-1}b_{m,\eta_1}(t)^{-1}$$

maps  $\xi_1(x_{m-1})$  to  $\xi_2(x_{m-1})$  and  $\xi_1(x_{m+1})$  to  $\xi_2(x_{m+1})$ . By Remark 2.13,

$$b'_{m,\eta_2}(t')b_{m,\xi}(t)b'_{m,\xi}(t')^{-1}b_{m,\eta_1}(t)^{-1} = g$$

for all m = 1, 2, 3, so  $g \cdot \xi_1(x) = \xi_2(x)$  for all  $x \in S^1$ . This proves (2).

(3) We can assume without loss of generality that  $y_1 < y_2 < y_3 \le x_1 < x_2 < x_3 \le y_1$  in the cyclic ordering on  $S^1$ . Let  $\xi_1 := \left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t \circ \left(\epsilon_{y_1,y_2,y_3}^{j_1,j_2,j_3}\right)_{t'} (\xi)$  and  $\xi_2 := \left(\epsilon_{y_1,y_2,y_3}^{i_1,j_2,j_3}\right)_{t'} \circ \left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t (\xi)$ . Applying Proposition 3.3 and Proposition 2.10, we deduce that there is some  $g \in \operatorname{PGL}(V)$  so that  $g \cdot \xi_1(x_m) = \xi_2(x_m)$  and  $g \cdot \xi_1(y_m) = \xi_2(y_m)$  for all m = 1, 2, 3.

Let  $\eta_1 := (\epsilon_{y_1, y_2, y_3}^{j_1, j_2, j_3})_{t'}(\xi)$  and let  $\eta_2 := (\epsilon_{x_1, x_2, x_3}^{i_1, i_2, i_3})_t(\xi)$ . For all  $x \in \overline{[x_1, x_2]}$ , observe that

$$\xi_1(x) = b_{\eta_1(x_3),\eta_1(x_1),\eta_1(x_2)}^{i_3,i_1,i_2}(t)b_{\xi(y_2),\xi(y_3),\xi(y_1)}^{j_2,j_3,j_1}(t') \cdot \xi(x)$$

and

$$\xi_2(x) = b_{\eta_2(y_2), \eta_2(y_3), \eta_2(y_1)}^{j_2, j_3, j_1}(t') b_{\xi(x_3), \xi(x_1), \xi(x_2)}^{i_3, i_1, i_2}(t) \cdot \xi(x).$$

Similarly,

$$\xi_1^{(1)}(x_3) = b_{\eta_1(x_3),\eta_1(x_1),\eta_1(x_2)}^{i_3,i_1,i_2}(t) b_{\xi(y_2),\xi(y_3),\xi(y_1)}^{j_2,j_3,j_1}(t') \cdot \xi^{(1)}(x_3)$$

and

$$\xi_2^{(1)}(x_3) = b_{\eta_2(y_2),\eta_2(y_3),\eta_2(y_1)}^{j_2,j_3,j_1}(t')b_{\xi(x_3),\xi(x_1),\xi(x_2)}^{i_3,i_1,i_2}(t) \cdot \xi^{(1)}(x_3).$$

The same argument as in the proof of (2) then implies  $g \cdot \xi_1(x) = \xi_2(x)$  for all  $x \in \overline{[x_1, x_2]}$ . Repeating a similar argument for each of the intervals  $\overline{[x_2, x_3]}$ ,  $\overline{[y_1, y_2]}$ ,  $\overline{[y_2, y_3]}$ ,  $\overline{[y_3, x_1]}$  and  $\overline{[x_3, y_1]}$  shows that  $g \cdot \xi_1(x) = \xi_2(x)$  for all  $x \in S^1$ .

We abuse terminology and refer to the descended flows on  $\mathcal{FR}(V)/PGL(V)$  (Proposition 3.5(1)) as elementary eruption flows as well.

Finally, we state the following fact as a lemma as we will need it in Section 5.

**Lemma 3.6.** Let  $\xi: S^1 \to \mathcal{F}(V)$  be a Frenet curve, let  $x_1 < x_2 < x_3 < x_1$  be a triple of points in  $S^1$  in this cyclic order, and let  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1+i_2+i_3=n$ . Then there is a representative  $\xi_t$  of  $\left(\epsilon_{x_1, x_2, x_3}^{i_1, i_2, i_3}\right)_t [\xi] \in \mathcal{FR}(V)/\mathrm{PGL}(V)$  so that

$$\xi_t(p) = \begin{cases} a_3(t)^{-1} \cdot \xi(p) & \text{if } p \in \overline{[x_2, x_3]} \\ \xi(p) & \text{if } p \in \overline{[x_3, x_1]} \\ a_1(t) \cdot \xi(p) & \text{if } p \in \overline{[x_1, x_2]} \end{cases}$$

where  $a_m(t):=a^{i_m,i_{m+1},i_{m-1}}_{\xi(x_m),\xi(x_{m+1}),\xi(x_{m-1})}(t)\in \mathrm{PGL}(V)$  for m=1,2,3. Furthermore, if  $\{f_1,\ldots,f_n\}$  is a basis of V associated to  $\xi(x_m)$ , then  $a_m(t)$  is represented in this basis by an upper triangular matrix where the first  $i_m$  entries down the diagonal are  $e^{\frac{(n-i_m)t}{n}}$  and the last  $n-i_m$  entries down the diagonal are  $e^{-\frac{i_mt}{n}}$ .

*Proof.* Define

$$\xi_t := b_{\xi(x_2),\xi(x_3),\xi(x_1)}^{i_2,i_3,i_1}(t)^{-1} \cdot \left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t(\xi).$$

It is clear from the definitions that  $\xi_t$  is given by the formula in the lemma. As a consequence of the computation after Definition 3.2, we see that  $a_m(t)$  fixes the flag  $\xi(x_m)$ . This means that in the basis  $\{f_1,\ldots,f_n\}$ ,  $a_m(t)$  is represented by an upper triangular matrix. Also, by (3.1), we see that  $a_m(t)$  has exactly two eigenvalues;  $e^{\frac{(n-i_m)t}{n}}$  with multiplicity  $i_m$  and  $e^{-\frac{i_mt}{n}}$  with multiplicity  $n-i_m$ . Furthermore,  $f_1,\ldots,f_{i_m}$  spans the eigenspace of  $a_m(t)$  with eigenvalue  $e^{\frac{(n-i_m)t}{n}}$ . This implies the lemma.

More informally,  $\xi_t$  is the unique representative of  $(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3})_t[\xi]$  so that  $\xi_t$  and  $\xi$  agree on  $[x_3,x_1]$ .

3.2. **Elementary shearing flow.** In this section we define the elementary shearing flows.

Let  $F_1, F_2 \in \mathcal{F}(V)$  be a pair of transverse flags. Then for all  $i = 1, \ldots, n-1$ , let  $b_{F_1, F_2}^{i, n-i}(t) \in \mathrm{PSL}(V)$  be the projective transformation which, when written in the basis  $B_{F_1, F_2}^{i, n-i}$ , is given by

$$b^{i,n-i}_{F_1,F_2}(t) := e^{\frac{(2n-3i)t}{6n}} \cdot \left[ \begin{array}{cc} e^{\frac{t}{6}}\mathrm{id}_i & 0 \\ 0 & e^{-\frac{2t}{6}} \cdot \mathrm{id}_{n-i} \end{array} \right] = \left[ \begin{array}{cc} e^{\frac{(n-i)t}{2n}}\mathrm{id}_i & 0 \\ 0 & e^{-\frac{it}{2n}} \cdot \mathrm{id}_{n-i} \end{array} \right].$$

**Definition 3.7.** Let i = 1, ..., n-1, and let  $x_1, x_2 \in S^1$  be distinct. The (i, n-i)-elementary shearing flow with respect to the pair  $(x_1, x_2)$  is the continuous flow

$$(\psi_{x_1,x_2}^{i,n-i})_t: \mathcal{FR}(V) \to \mathcal{FR}(V)$$

defined by

$$\xi_t(p) = \begin{cases} b(-t) \cdot \xi(p) & \text{if } p \in \overline{[x_2, x_1]} \\ b(t) \cdot \xi(p) & \text{if } p \in \overline{[x_1, x_2]} \end{cases}$$

where  $\xi_t := (\psi_{x_1, x_2}^{i, n-i})_t(\xi)$ , and  $b(t) := b_{\xi(x_1), \xi(x_2)}^{i, n-i}(t)$ .

It is clear that b(t) fixes both  $\xi(x_1), \xi(x_2) \in \mathcal{F}(V)$ , so  $\xi_t$  is continuous. Also, note that  $(\psi_{x_1,x_2}^{i,n-i})_t = (\psi_{x_2,x_1}^{n-i,i})_t$ . Given  $\xi \in \mathcal{FR}(V)$ ,  $(\psi_{x_1,x_2}^{i,n-i})_t$  deforms the two subsegments of  $\xi(S^1)$  given by  $x_1, x_2$  using two different projective transformations, which are chosen to ensure that  $\xi_t := (\psi_{x_1,x_2}^{i,n-i})_t(\xi)$  is Frenet (see Theorem 3.10), and that  $(\psi_{x_1,x_2}^{i,n-i})_t$  "only changes" the cross ratio  $C_i(\xi_t(x_1),\xi_t(x_2),\xi_t(x_4),\xi_t(x_3))$  (see Proposition 3.9).

Remark 3.8. In the case when n=3 and i=1,2, these are (up to projective transformations) the elementary shearing and bulging flows described in [WZ17, Section 4.1], see also [Gol13].

We will now prove, just as we did in the case of elementary eruption flows, that  $\xi_t$  is again a Frenet curve. To do so, we need the analog of Proposition 3.3 for shearing flows, which we state as the following proposition.

**Proposition 3.9.** Let i = 1, ..., n-1, let  $x_1 < x_2 < x_3 < x_4 < x_1$  be points in  $S^1$  in this cyclic order, and let  $\xi_t := (\psi_{x_1, x_3}^{i, n-i})_t(\xi)$ .

(1) Let 
$$\delta(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$
. Then for all  $j = 1, \dots, n-1$ ,

$$C_j(\xi_t(x_1), \xi_t(x_2), \xi_t(x_4), \xi_t(x_3)) = e^{t\delta(j)} \cdot C_j(\xi(x_1), \xi(x_2), \xi(x_4), \xi(x_3)).$$

(2) Let  $p_1, p_2, p_3, p_4 \in S^1$  be a quadruple of points so that either  $p_1 < p_2 < p_3 < p_4 \le x_1 < x_3 \le p_1$  or  $p_1 < p_2 < p_3 \le x_1 < x_3 \le p_4 < p_1$ . Then for all  $m = 1, \ldots, n - 1$ ,

$$C_i(\xi_t(p_1), \xi_t(p_2), \xi_t(p_4), \xi_t(p_3)) = C_i(\xi(p_1), \xi(p_2), \xi(p_4), \xi(p_3)).$$

(3) Let  $p_1, p_2, p_3 \in S^1$  be a triple of points so that  $p_1 < p_2 < p_3 \le x_1 < x_3 \le p_1$ . Then for all  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ , we have

$$T_{i_1,i_2,i_3}(\xi_t(p_1),\xi_t(p_2),\xi_t(p_3)) = T_{i_1,i_2,i_3}(\xi(p_1),\xi(p_2),\xi(p_3)).$$

Proof. (1) Let  $\{f_1,\ldots,f_n\}$  be a basis of V so that  $\operatorname{Span}_{\mathbb{R}}(f_i)=\xi(x_1)^{(i)}\cap\xi(x_3)^{(n-i+1)}$ , and let  $b(t):=b^{i,n-i}_{\xi(x_1),\xi(x_3)}(t)$ . Note that  $b(t)\cdot f_k=e^{\frac{(n-i)t}{2n}}\cdot f_k$  for all  $k=1,\ldots,i$  and  $b(t)\cdot f_k=e^{-\frac{it}{2n}}\cdot f_k$  for all  $k=i+1,\ldots,n$ . For m=2,4, let  $v_m\in\xi^{(1)}(x_m)$  be a non-zero vector, and write

$$v_m = \sum_{k=1}^n \alpha_{m,k} f_k.$$

Then we can compute that for all j = 1, ..., n-1

$$C_j(\xi_t(x_1), \xi_t(x_2), \xi_t(x_4), \xi_t(x_3))$$

$$= C_i(\xi(x_1), b(t) \cdot \xi(x_2), b(t)^{-1} \cdot \xi(x_4), \xi(x_3))$$

$$= \frac{\det(f_1, \dots, \hat{f}_{j+1}, \dots, f_n, b(t)^{-1} \cdot v_4)}{\det(f_1, \dots, \hat{f}_{j+1}, \dots, f_n, b(t) \cdot v_2)} \cdot \frac{\det(f_1, \dots, \hat{f}_{j}, \dots, f_n, b(t) \cdot v_2)}{\det(f_1, \dots, \hat{f}_{j}, \dots, f_n, b(t)^{-1} \cdot v_4)}$$

$$= \frac{\alpha_{4,j+1} \det(f_1, \dots, \hat{f}_{j+1}, \dots, f_n, b(t)^{-1} \cdot f_{j+1})}{\alpha_{2,j+1} \det(f_1, \dots, \hat{f}_{j+1}, \dots, f_n, b(t) \cdot f_{j+1})} \cdot \frac{\alpha_{2,j} \det(f_1, \dots, \hat{f}_{j}, \dots, f_n, b(t) \cdot f_j)}{\alpha_{4,j} \det(f_1, \dots, \hat{f}_{j}, \dots, f_n, b(t)^{-1} \cdot f_j)}$$

If  $j \neq i$ , then b(t) scales  $f_i$  and  $f_{i+1}$  by the same amount, so

$$C_{j}(\xi_{t}(x_{1}), \xi_{t}(x_{2}), \xi_{t}(x_{4}), \xi_{t}(x_{3}))$$

$$= \frac{\alpha_{4,j+1} \det(f_{1}, \dots, \hat{f}_{j+1}, \dots, f_{n}, f_{j+1})}{\alpha_{2,j+1} \det(f_{1}, \dots, \hat{f}_{j+1}, \dots, f_{n}, f_{j+1})} \cdot \frac{\alpha_{2,j} \det(f_{1}, \dots, \hat{f}_{j}, \dots, f_{n}, f_{j})}{\alpha_{4,j} \det(f_{1}, \dots, \hat{f}_{j}, \dots, f_{n}, f_{j})}$$

$$= \frac{\det(f_{1}, \dots, \hat{f}_{j+1}, \dots, f_{n}, v_{4})}{\det(f_{1}, \dots, \hat{f}_{j+1}, \dots, f_{n}, v_{2})} \cdot \frac{\det(f_{1}, \dots, \hat{f}_{j}, \dots, f_{n}, v_{2})}{\det(f_{1}, \dots, \hat{f}_{j}, \dots, f_{n}, v_{4})}$$

$$= C_{j}(\xi(x_{1}), \xi(x_{2}), \xi(x_{4}), \xi(x_{3})).$$

On the other hand,

$$C_{i}(\xi_{t}(x_{1}), \xi_{t}(x_{2}), \xi_{t}(x_{4}), \xi_{t}(x_{3}))$$

$$= \frac{\alpha_{4,i+1} \det(f_{1}, \dots, \hat{f}_{i+1}, \dots, f_{n}, e^{\frac{it}{2n}} f_{i+1})}{\alpha_{2,i+1} \det(f_{1}, \dots, \hat{f}_{i+1}, \dots, f_{n}, e^{-\frac{it}{2n}} f_{i+1})} \cdot \frac{\alpha_{2,i} \det(f_{1}, \dots, \hat{f}_{i}, \dots, f_{n}, e^{\frac{(n-i)t}{2n}} f_{i})}{\alpha_{4,i} \det(f_{1}, \dots, \hat{f}_{i}, \dots, f_{n}, e^{\frac{(i-n)t}{2n}} f_{i})}$$

$$= e^{t} \cdot \frac{\det(f_{1}, \dots, \hat{f}_{i+1}, \dots, f_{n}, v_{4})}{\det(f_{1}, \dots, \hat{f}_{i+1}, \dots, f_{n}, v_{2})} \cdot \frac{\det(f_{1}, \dots, \hat{f}_{i}, \dots, f_{n}, v_{2})}{\det(f_{1}, \dots, \hat{f}_{i}, \dots, f_{n}, v_{4})}$$

$$= e^{t} \cdot C_{i}(\xi(x_{1}), \xi(x_{2}), \xi(x_{4}), \xi(x_{3})).$$

To prove (2) and (3), simply observe that in either case,  $\xi_t(p_m) = b(t) \cdot \xi(p_m)$  for all  $m = 1, \ldots, 4$ . The projective invariance of the cross ratio and triple ratio immediately gives (2) and (3).

With Proposition 3.9 we can now prove the following theorem using the same argument (with obvious modifications) as the proof of Theorem 3.4. We omit the proof to avoid repetition.

**Theorem 3.10.** Let i = 1, ..., n-1 and let  $x_1, x_2 \in S^1$  be a pair of distinct points. Then  $\xi_t := (\psi_{x_1, x_2}^{i, n-i})_t(\xi)$  is a Frenet curve for all  $t \in \mathbb{R}$ .

Similarly, we also have the analog of Proposition 3.5, which we state as the next proposition. We omit its proof as it is essentially a repetition of the proof of Proposition 3.5 with some simple modifications.

**Proposition 3.11.** Let  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$  and let  $i, j = 1, \ldots, n-1$ . Also, let  $x_1, x_2 \in S^1$  be distinct. Then the following hold:

- (1) For any  $i=1,\ldots,n-1$ , the flow  $\left(\psi_{x_1,x_2}^{i,n-i}\right)_t$  on  $\mathcal{FR}(V)$  descends to a flow on  $\mathcal{FR}(V)/\mathrm{PGL}(V)$ , which we also denote by  $\left(\psi_{x_1,x_2}^{i,n-i}\right)_t$ .
- (2) Let  $y_1, y_2$  be points that lie in the closure in  $S^1$  of a connected component of  $S^1 \setminus \{x_1, x_2\}$ . For all  $t, t' \in \mathbb{R}$ ,

$$\left(\psi_{x_{1},x_{2}}^{i,n-i}\right)_{t}\circ\left(\psi_{y_{1},y_{2}}^{j,n-j}\right)_{t'}=\left(\psi_{y_{1},y_{2}}^{j,n-j}\right)_{t'}\circ\left(\psi_{x_{1},x_{2}}^{i,n-i}\right)_{t}$$

as flows on  $\mathcal{FR}(V)/PGL(V)$ .

(3) Let  $y_1, y_2, y_3$  be points that lie in the closure in  $S^1$  of a connected component of  $S^1 \setminus \{x_1, x_3\}$ . For all  $t, t' \in \mathbb{R}$ ,

$$\left(\psi_{x_{1},x_{3}}^{i,n-i}\right)_{t} \circ \left(\epsilon_{y_{1},y_{2},y_{3}}^{i_{1},i_{2},i_{3}}\right)_{t'} = \left(\epsilon_{y_{1},y_{2},y_{3}}^{i_{1},i_{2},i_{3}}\right)_{t'} \circ \left(\psi_{x_{1},x_{3}}^{i,n-i}\right)_{t}$$

as flows on  $\mathcal{FR}(V)/\mathrm{PGL}(V)$ .

Finally, we state the following analog of Lemma 3.6 for elementary shearing flows. The proof is again a direct adaption of the proof of Lemma 3.6.

**Lemma 3.12.** Let  $\xi: S^1 \to \mathcal{F}(V)$  be a Frenet curve and let  $x_1, x_2 \in S^1$  be a pair of distinct points. Then there is a representative  $\xi_t$  of  $(\psi_{x_1, x_2}^{i, n-i})_t [\xi] \in \mathcal{FR}(V)/\mathrm{PGL}(V)$  so that

$$\xi_t(p) = \begin{cases} \xi(p) & \text{if } p \in \overline{[x_2, x_1]} \\ b(t)^2 \cdot \xi(p) & \text{if } p \in \overline{[x_1, x_2]} \end{cases}$$

where  $b(t) := b_{\xi(x_1),\xi(x_2)}^{i,n-i}(t) \in PGL(V)$ . Furthermore, if  $\{f_1,\ldots,f_n\}$  is a basis of V associated to  $\xi(x_1)$ , then  $b(t)^2$  is represented by an upper triangular matrix where the first i entries down the diagonal are  $e^{\frac{(n-i)t}{n}}$  and the last n-i entries down the diagonal are  $e^{\frac{-it}{n}}$ .

## 4. Parametrizing the Hitchin component $Hit_V(S)$

In this section we describe a parametrization of  $\operatorname{Hit}_V(S)$ . This is a modification of an earlier parameterization given by Bonahon-Dreyer [BD14] that we will also describe. In Section 5, we will define a family of flows in  $\operatorname{Hit}_V(S)$ , called  $(\mathcal{T}, \mathcal{J})$ -parallel flows, which have an easy description in terms of this new parameterization.

For the rest of this article, we will take  $\partial\Gamma$  to be the domain of all Frenet curves we consider. By Theorem 2.15 there is a natural embedding  $\mathrm{Hit}_V(S) \subset \mathcal{FR}(V)/\mathrm{PGL}(V)$  as the subset of  $\mathrm{PGL}(V)$ -orbits of Frenet curves that are  $\rho$ -equivariant for some representation  $\rho:\Gamma\to\mathrm{PSL}(V)$ . This point of view was used by Bonahon-Dreyer [BD14] to construct their parameterization of  $\mathrm{Hit}_V(S)$ . To describe their parameterization, we need to fix some topological choices, which we describe in Section 4.1, and define some projective invariants associated to these topological choices, which we describe in Section 4.2.

4.1. Ideal triangulations of S. We identify geodesics in the universal cover  $\widetilde{S}$  of the topological surface S with unordered pairs of distinct points  $\{x,y\}$  in  $\partial\Gamma$ , and denote the set of geodesics in  $\widetilde{S}$  by  $\mathcal{G}(\widetilde{S})$ . Note that there is an obvious  $\Gamma$ -action on  $\mathcal{G}(\widetilde{S})$ , so we can consider the quotient  $\mathcal{G}(S) := \mathcal{G}(\widetilde{S})/\Gamma$ . Recall that if we choose a negatively curved metric  $\Sigma$  on S, then this induces a complete, negatively curved metric  $\widetilde{\Sigma}$  on  $\widetilde{S}$  so that the action of  $\Gamma$  on  $\widetilde{\Sigma}$  by deck transformations is by isometries. This induces a natural identification of  $\partial\Gamma$  with the visual boundary  $\partial\widetilde{\Sigma}$  of  $\widetilde{\Sigma}$ . In particular,  $\mathcal{G}(\widetilde{S})$  is canonically homeomorphic to the space of geodesics (in the Riemannian geometry sense) in  $\widetilde{\Sigma}$ , and  $\mathcal{G}(S)$  is canonically homeomorphic to the space of geodesics in  $\Sigma$ .

An orientation on a geodesic  $\{x_1, x_2\}$  is an ordering of the unordered pair  $\{x_1, x_2\}$ . We will denote the two orientations of  $\{x_1, x_2\}$  by  $(x_1, x_2)$  and  $(x_2, x_1)$ . If  $\{x_1, x_2\}$  is equipped with the orientation  $(x_1, x_2)$ , we refer to  $x_1$  and  $x_2$  as the backward and forward endpoints of  $\{x_1, x_2\}$  respectively. Since  $\partial \Gamma$  carries a cyclic ordering induced by the orientation of S, we say that  $z \in \partial \Gamma$  lies on the right (resp. left) of  $(x_1, x_2)$  if  $x_2 < z < x_1$  (resp.  $x_1 < z < x_2$ ).

We say that two geodesics  $\{x_1, x_2\}, \{y_1, y_2\} \in \mathcal{G}(\tilde{S})$  intersect transversely if  $x_1 < y_1 < x_2 < y_2 < x_1$  or  $x_1 < y_2 < x_2 < y_1 < x_1$  lie in  $\partial \Gamma$  in this cyclic order. A maximal geodesic lamination of  $\tilde{S}$  is a maximal,  $\Gamma$ -invariant collection of geodesics in  $\mathcal{G}(\tilde{S})$  so that no pair of geodesics in this collection intersect transversely. The elements in a maximal geodesic lamination are called edges. A maximal geodesic lamination  $\tilde{T}$  of  $\tilde{S}$  is an ideal triangulation if for any  $\{x_1, x_2\} \in \tilde{T}$ , one of the following hold:

- There is some  $z \in \partial \Gamma$  so that  $\{x_1, z\}$  and  $\{x_2, z\}$  are in  $\widetilde{\mathcal{T}}$ .
- There is some  $\gamma \in \Gamma$  whose fixed points in  $\partial \Gamma$  are  $x_1$  and  $x_2$ .

If the former holds,  $\{x_1, x_2\}$  is said to be an *isolated edge*. If the latter holds,  $\{x_1, x_2\}$  is said to be a *closed edge* (see Figure 4). The  $\Gamma$ -invariance of ideal triangulations of  $\widetilde{S}$  allows us to define ideal triangulations  $\mathcal{T}$  on S as quotients of ideal triangulations  $\widetilde{\mathcal{T}}$  on  $\widetilde{S}$  by  $\Gamma$ .

An ideal triangle of an ideal triangulation  $\widetilde{\mathcal{T}}$  is a triple  $\{x_1, x_2, x_3\}$  so that  $\{x_1, x_2\}, \{x_2, x_3\}$  and  $\{x_3, x_1\}$  are edges in  $\widetilde{\mathcal{T}}$ . We refer to  $x_1, x_2, x_3 \in \partial \Gamma$  as the vertices, and  $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_1\} \in \widetilde{\mathcal{T}}$  as the edges of the triangle  $\{x_1, x_2, x_3\}$ .

Note that  $\Gamma$  acts on the set of ideal triangles of  $\widetilde{\mathcal{T}}$ , so we can define an *ideal triangle* of  $\mathcal{T}$  to be a  $\Gamma$ -orbit of ideal triangles of  $\widetilde{\mathcal{T}}$ .

If we choose a negatively curved metric  $\Sigma$  on S, then the identification of  $\mathcal{G}(\widetilde{S})$  with the space of geodesics on  $\widetilde{\Sigma}$  realizes maximal geodesic laminations of  $\widetilde{S}$ , ideal triangulations of  $\widetilde{S}$ , and ideal triangulations of S as maximal geodesic laminations of  $\widetilde{\Sigma}$ , ideal triangulations of  $\widetilde{\Sigma}$ , and ideal triangulations of S respectively in the Riemannian geometry sense. As such, it is classically known that a maximal geodesic lamination  $\widetilde{T}$  of  $\widetilde{S}$  is an ideal triangulation if and only if T is finite. Similarly, ideal triangles of  $\widetilde{T}$  or T are realized as Riemannian ideal triangles in  $\widetilde{\Sigma}$  or  $\Sigma$  respectively.

For the rest of this article, we fix the following notation for any ideal triangulation  $\widetilde{\mathcal{T}}$  of  $\widetilde{S}$ .

Notation 4.1.

- (1) The set of ideal triangles of  $\widetilde{\mathcal{T}}$  and  $\mathcal{T}$  are denoted by  $\widetilde{\Theta} = \widetilde{\Theta}_{\widetilde{\mathcal{T}}}$  and  $\Theta = \Theta_{\mathcal{T}} := \widetilde{\Theta}/\Gamma$  respectively.
- (2) The  $\Gamma$ -orbit of the ideal triangle  $\{x_1, x_2, x_3\} \in \widetilde{\Theta}$  is denoted by  $[x_1, x_2, x_3] \in \Theta$ . Similarly, the  $\Gamma$ -orbit of the edge  $\{x_1, x_2\} \in \widetilde{\mathcal{T}}$  is denoted by  $[x_1, x_2] \in \mathcal{T}$ .
- (3) The set of isolated edges in  $\widetilde{\mathcal{T}}$  is denoted by  $\widetilde{\mathcal{Q}} = \widetilde{\mathcal{Q}}_{\widetilde{\mathcal{T}}}$ . Observe that  $\widetilde{\mathcal{Q}}$  is  $\Gamma$ -invariant, so  $\mathcal{Q} = \mathcal{Q}_{\mathcal{T}} := \widetilde{\mathcal{Q}}/\Gamma$  well-defined.
- (4) The set of closed edges in  $\widetilde{\mathcal{T}}$  is denoted by  $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}_{\widetilde{\mathcal{T}}}$ . The set  $\widetilde{\mathcal{P}}$  are  $\Gamma$ -invariant and  $\mathcal{P} = \mathcal{P}_{\mathcal{T}} := \widetilde{\mathcal{P}}/\Gamma$  is well-defined.

Note that  $\mathcal{T}=\mathcal{Q}\cup\mathcal{P}$ . By the Gauss-Bonnet theorem, if  $g\geq 2$  is the genus of S, then  $|\Theta|=4g-4$ ,  $|\mathcal{Q}|=6g-6$  and  $1\leq |\mathcal{P}|\leq 3g-3$ . In the case when  $|\mathcal{P}|=3g-3$ ,  $\mathcal{P}$  is known as a pants decomposition of S.

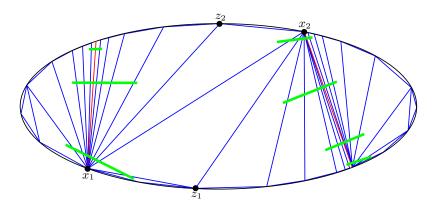


FIGURE 4. The blue lines represent isolated edges in  $\widetilde{\mathcal{Q}}$ , the red lines represent closed edges in  $\widetilde{\mathcal{P}}$ , and the green lines represent bridges in  $\widetilde{\mathcal{J}}$ .

**Definition 4.2.** Let  $\widetilde{\mathcal{T}}$  be an ideal triangulation of S, let  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$  be a closed edge, and let  $\gamma \in \Gamma$  be a primitive element whose fixed points are  $x_1$  and  $x_2$ .

• Let  $\partial\Gamma_1$  and  $\partial\Gamma_2$  be the connected components of  $\partial\Gamma\setminus\{x_1,x_2\}$ . A bridge across  $\{x_1,x_2\}$  is an unordered pair of triangles  $\{T_1,T_2\}\subset\widetilde{\Theta}$  so that for all

- m = 1, 2, one of the vertices of  $T_m$  is  $x_1$  or  $x_2$ , while the other two vertices lie in  $\partial \Gamma_m$  (see Figure 4).
- A bridge system compatible with  $\widetilde{\mathcal{T}}$ , denoted  $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_{\widetilde{\mathcal{T}}}$ , is a minimal  $\Gamma$ -invariant collection of bridges so that for every closed leaf  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$ , there is a bridge  $\{T_1, T_2\}$  across  $\{x_1, x_2\}$  that lies in  $\widetilde{\mathcal{J}}$ . We also say that  $\mathcal{J} = \mathcal{J}_{\mathcal{T}} := \widetilde{\mathcal{J}}/\Gamma$  is a bridge system compatible with  $\mathcal{T} = \widetilde{\mathcal{T}}/\Gamma$ .

If we choose a negatively curved metric  $\Sigma$  on S, then the bridge system  $\mathcal{J}_{\mathcal{T}}$  can be realized as a collection of "short" geodesic segments, one for each simple closed curve in  $\mathcal{P}_{\mathcal{T}}$ . Each geodesic segment in  $\mathcal{J}_{\mathcal{T}}$  intersects a unique simple closed geodesic in  $\mathcal{P}_{\mathcal{T}}$  transversely, and each simple closed geodesic in  $\mathcal{P}_{\mathcal{T}}$  intersects a unique geodesic segment in  $\mathcal{J}_{\mathcal{T}}$  transversely.

4.2. Edge and triangle invariants. Let  $\mathcal{T}$  be an ideal triangulation of S and  $\mathcal{J}$  a compatible bridge system. Given  $\xi:\partial\Gamma\to\mathcal{F}(V)$  a Frenet curve, we associate invariants to the edges and triangles of the triangulation  $\mathcal{T}$ . These invariants were introduced by Fock-Goncharov [FG06], and are based on the cross ratios and triple ratios described in Section 2.1.

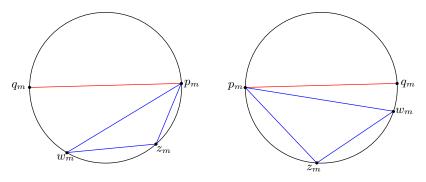


FIGURE 5. The red line is the closed edge  $\{x_1, x_2\}$ , and the blue triangle is  $T_m$ . Observe that  $\{p_m, w_m\}$  and  $\{q_m, z_m\}$  intersect transversely.

4.2.1. The edge invariants. Let  $\{x_1,x_2\} \in \widetilde{\mathcal{P}}$  and let  $J = \{T_1,T_2\} \in \widetilde{\mathcal{J}}$  be a bridge across  $\{x_1,x_2\}$ , so that the vertices of  $T_1$  and  $T_2$  that are neither  $x_1$  nor  $x_2$  lie to the right and left of the oriented edge  $(x_1,x_2)$  respectively. In this case, we say  $T_1$  (resp.  $T_2$ ) lies to the right (resp. left) of  $(x_1,x_2)$ . We set, for  $m=1,2,\,T_m=:\{p_m,z_m,w_m\}$ , where  $p_m$  is the vertex of  $T_m$  that lies in  $\{x_1,x_2\},\,q_m\in\{x_1,x_2\}$  is the point that is not  $p_m$ , and  $\{z_m,q_m\}$  intersects  $\{w_m,p_m\}$  transversely (see Figure 5). For any  $i=1,\ldots,n-1$ , the i-th edge invariant along the closed edge  $\{x_1,x_2\}\in\widetilde{\mathcal{P}}$  is the function  $\sigma_{x_1,x_2,J}^{i,n-i}:\mathcal{FR}(V)/\mathrm{PGL}(V)\to\mathbb{R}$  define by

$$\sigma_{x_1,x_2,J}^{i,n-i}[\xi] := \log\Big(-C_i\big(\xi(x_1),\xi(z_2),\xi(z_1),\xi(x_2)\big)\Big).$$

By Theorem 2.14(2) and the projective invariance of the cross ratio,  $\sigma_{x_1,x_2,J}^{i,n-i}$  is well-defined. Observe that  $\sigma_{x_1,x_2,J}^{i,n-i} = \sigma_{x_2,x_1,J}^{n-i,i}$ .

Let  $\{x_1, x_2\} \in \widetilde{\mathcal{Q}}$  be an isolated edge and let  $z_1, z_2 \in \partial \Gamma$  be the two points so that  $\{x_1, x_2, z_m\} \in \widetilde{\Theta}$  for m = 1, 2, and  $z_1$  and  $z_2$  lie the the right and left of  $(x_1, x_2)$ 

respectively (see Figure 4). For any  $i = 1, \ldots, n-1$ , the i-th edge invariant along the isolated edge  $\{x_1, x_2\} \in \widetilde{\mathcal{Q}}$  is the function  $\sigma_{x_1, x_2}^{i, n-i} : \mathcal{FR}(V)/\mathrm{PGL}(V) \to \mathbb{R}$  by

$$\sigma_{x_1,x_2}^{i,n-i}[\xi] := \log\Big(-C_i\big(\xi(x_1),\xi(z_2),\xi(z_1),\xi(x_2)\big)\Big).$$

The invariant  $\sigma_{x_1,x_2}^{i,n-i}$  is well-defined, and  $\sigma_{x_1,x_2}^{i,n-i}=\sigma_{x_2,x_1}^{n-i,i}$ .

4.2.2. The triangle invariants. For each triangle  $T = \{x_1, x_2, x_3\} \in \widetilde{\Theta}$  so that  $x_1 < x_2 < x_3 < x_1$  in this cyclic order along  $\partial \Gamma$ , and for each  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ , we define the  $(i_1, i_2, i_3)$ -triangle invariant to be the function  $\tau_{x_1,x_2,x_3}^{i_1,i_2,i_3}: \mathcal{FR}(V)/\mathrm{PGL}(V) \to \mathbb{R}$  defined by

$$\tau_{x_1, x_2, x_3}^{i_1, i_2, i_3}[\xi] := \log \left( T_{i_1, i_2, i_3} (\xi(x_1), \xi(x_2), \xi(x_3)) \right).$$

The invariant  $\tau_{x_1,x_2,x_3}^{i_1,i_2,i_3}$  is well-defined because of Theorem 2.14 and the fact that the triple ratio is a projective invariant. Moreover,  $\tau_{x_1,x_2,x_3}^{i_1,i_2,i_3} = \tau_{x_2,x_3,x_1}^{i_2,i_3,i_1} = \tau_{x_3,x_1,x_2}^{i_3,i_1,i_2}$ . The next proposition states that the edge and triangle invariants determine the

PGL(V)-orbit of a Frenet curve.

**Proposition 4.3.** Let  $[\xi_1], [\xi_2] \in \mathcal{FR}(V)/PGL(V)$ . Then  $[\xi_1] = [\xi_2]$  if and only if

- $\sigma_{x_1,x_2,J}^{i,n-i}[\xi_1] = \sigma_{x_1,x_2,J}^{i,n-i}[\xi_2]$  for all  $i = 1,\ldots,n-1$ , for all  $\{x_1,x_2\} \in \widetilde{\mathcal{P}}$ , and for all  $J \in \widetilde{\mathcal{J}}$  across  $\{x_1, x_2\}$ .
- $\sigma_{x_1,x_2}^{i,n-i}[\xi_1] = \sigma_{x_1,x_2}^{i,n-i}[\xi_2]$  for all  $i = 1, \ldots, n-1$  and for all  $\{x_1, x_2\} \in \widetilde{\mathcal{Q}}$ .  $\tau_{x_1,x_2,x_3}^{i_1,i_2,i_3}[\xi_1] = \tau_{x_1,x_2,x_3}^{i_1,i_2,i_3}[\xi_2]$  for all  $i_1,i_2,i_3 \in \mathbb{Z}^+$  so that  $i_1+i_2+i_3=n$  and for all  $\{x_1, x_2, x_3\} \in \Theta$ .

Remark 4.4. Note that in Proposition 4.3 we do not assume any equivariance properties for the Frenet curves. Without the equivariance properties, the analog of Proposition 4.3 for positive maps does not hold. In the case of equivariant Frenet curves, Proposition 4.3 has been proved by Bonahon-Dreyer [BD14, Theorem 2].

In other words, Proposition 4.3 states that the map

$$\Phi: \quad \mathcal{FR}(V)/\mathrm{PGL}(V) \quad \to \quad \mathbb{R}^{|\widetilde{\mathcal{P}}|\cdot|\widetilde{\mathcal{J}}|\cdot(n-1)} \times \mathbb{R}^{|\widetilde{\mathcal{Q}}|\cdot(n-1)} \times \mathbb{R}^{|\widetilde{\Theta}|\cdot\frac{(n-1)(n-2)}{2}} \\ [\xi] \quad \mapsto \quad (\Sigma_1, \Sigma_2, \Sigma_3)$$

is injective, where

$$\begin{split} & \Sigma_1 &:= & \left(\sigma_{x_1, x_2, J}^{i, n-i}[\xi]\right)_{i=1, \dots, n-1; \{x_1, x_2\} \in \widetilde{\mathcal{P}}, J \in \widetilde{\mathcal{J}} \text{ across } \{x_1, x_2\}} \\ & \Sigma_2 &:= & \left(\sigma_{x_1, x_2}^{i, n-i}[\xi]\right)_{i=1, \dots, n-1; \{x_1, x_2\} \in \widetilde{\mathcal{Q}}} \\ & \Sigma_3 &:= & \left(\tau_{x_1, x_2, x_3}^{i_1, i_2, i_3}[\xi]\right)_{i_1, i_2, i_3 \in \mathbb{Z}^+; i_1 + i_2 + i_3 = n; \{x_1, x_2, x_3\} \in \widetilde{\Theta}} \,. \end{split}$$

The proof of Proposition 4.3 will be given in Section 4.3.

The map  $\Phi$  is not surjective, but Bonahon-Dreyer [BD14] classified the image of  $\Phi$  restricted to projective classes of Frenet curves that are  $\rho$ -equivariant for some representation  $\rho:\Gamma\to \mathrm{PSL}(V)$ . We will describe this in Section 4.4.

4.3. Combinatorial description of a pair of distinct vertices of  $\widetilde{\mathcal{T}}$ . The proof of Proposition 4.3 consists of two steps. In the first step we show that the Frenet curve is determined by its restriction to the vertices of the triangulation  $\widetilde{\mathcal{T}}$ . This follows from the following lemma, whose proof is given in Appendix B

**Lemma 4.5.** Let V denote the set of vertices of  $\widetilde{T}$ , and for  $j = 0, 1, ..., \infty$ , let  $\xi_j \in \mathcal{FR}(V)$  be a Frenet curve. If  $\lim_{j \to \infty} \xi_j|_{V} = \xi_0|_{V}$ , then  $\lim_{j \to \infty} \xi_j = \xi_0$ .

In the second step, we show that the assumptions of Proposition 4.3 imply that we can find Frenet curves  $\xi_1$  and  $\xi_2$  representing  $[\xi_1]$  and  $[\xi_2]$  respectively such that  $\xi_1|_{\mathcal{V}} = \xi_2|_{\mathcal{V}}$  on the set of vertices  $\mathcal{V}$  of  $\widetilde{\mathcal{T}}$ .

In order to control the Frenet curve on an arbitrary vertex of  $w_0$  of  $\widetilde{\mathcal{T}}$  we consider a triangle  $\{x_0, y_0, z_0\} \in \widetilde{\Theta}$  and consider the pair of distinct vertices  $z_0, w_0$  of  $\widetilde{\mathcal{T}}$ . For any such pair of distinct vertices we consider the set

$$\mathcal{E}'_{z_0, w_0} := \{ \{x, y\} \in \widetilde{\mathcal{T}} : x < z_0 < y < w_0 < x \}.$$

We view this set as giving us a combinatorial description of the pair of vertices  $z_0, w_0$ . The properties we describe here for the set  $\mathcal{E}'_{z_0, w_0}$  and the set  $\mathcal{E}_{z_0, w_0}$  defined below will also be used in Section 5.3 and Section 5.4.

Note that if  $\{z_0, w_0\} \in \mathcal{T}$  is an edge of the triangulation, then  $\mathcal{E}'_{z_0, w_0}$  is empty. We orient both components of  $\partial \Gamma \setminus \{z_0, w_0\}$  from  $z_0$  to  $w_0$ . This induces an ordering on  $\mathcal{E}'_{z_0, w_0}$  by  $\{x, y\} \leq \{x', y'\}$  if x and x' (hence y and y') lie in the same connected component of  $\partial \Gamma \setminus \{z_0, w_0\}$ , x weakly precedes x', and y weakly precedes y'.

Observe that  $\mathcal{E}'_{z_0,w_0}$  does not have a minimum (in the ordering described above) if and only if there is some vertex  $p_0$  of  $\widetilde{\mathcal{T}}$  so that  $\{z_0,p_0\}\in\widetilde{\mathcal{P}}$ , and there is a sequence  $\{z_i\}_{i=1}^{\infty}$  of vertices of  $\widetilde{\mathcal{T}}$  that converges to  $z_0$ , and  $\{z_i,p_0\}\in\mathcal{E}'_{z_0,w_0}$  for all i. Similarly,  $\mathcal{E}'_{z_0,w_0}$  does not have a maximum if and only if there is some vertex  $q_0$  of  $\widetilde{\mathcal{T}}$  so that  $\{w_0,q_0\}\in\widetilde{\mathcal{P}}$ , and there is a sequence  $\{w_i\}_{i=1}^{\infty}$  of vertices of  $\widetilde{\mathcal{T}}$  that converges to  $w_0$ , and  $\{w_i,q_0\}\in\mathcal{E}'_{z_0,w_0}$  for all i. Then define

$$\mathcal{E}_{z_0,w_0} := \begin{cases} & \mathcal{E}'_{z_0,w_0} & \text{if } \mathcal{E}'_{z_0,w_0} \text{ has a max and a min} \\ & \mathcal{E}'_{z_0,w_0} \cup \left\{ \{z_0,p_0\} \right\} & \text{if } \mathcal{E}'_{z_0,w_0} \text{ has a max but no min} \\ & \mathcal{E}'_{z_0,w_0} \cup \left\{ \{w_0,q_0\} \right\} & \text{if } \mathcal{E}'_{z_0,w_0} \text{ has a min but no max} \\ & \mathcal{E}'_{z_0,w_0} \cup \left\{ \{z_0,p_0\}, \{w_0,q_0\} \right\} & \text{if } \mathcal{E}'_{z_0,w_0} \text{ has neither a max nor a min} \end{cases}$$

and observe that  $\mathcal{E}_{z_0,w_0}$  has an obvious ordering (see Figure 6).

It is easy to see that there are only finitely many closed edges (possibly none) in  $\widetilde{\mathcal{P}}$  that lie in  $\mathcal{E}_{z_0,w_0}$ . Let  $l_1,\ldots,l_k$  denote these closed edges, enumerated according to the ordering on  $\mathcal{E}_{z_0,w_0}$ . Observe that for any  $s=1,\ldots,k$ , if  $e\in\mathcal{E}_{z_0,w_0}$  shares a common vertex x with some  $l_s$  and satisfies  $e< l_s$ , then every edge e' satisfying  $e< e' < l_s$  also has x as a vertex. Similarly, if  $e\in\mathcal{E}_{z_0,w_0}$  shares a common vertex x with some  $l_s$  and  $l_s< e$ , then every edge e' satisfying  $l_s< e'< e$  also has x as a vertex. Thus, if we define

$$\mathcal{E}_{z_0,w_0,s} = \mathcal{E}_s := \{ e \in \mathcal{E}_{z_0,w_0} : e \text{ shares a vertex with } l_s \},$$

$$\mathcal{F}_s^- := \{ e \in \mathcal{E}_{z_0,w_0} : e < e' \text{ for all } e' \in \mathcal{E}_s \},$$

$$\mathcal{F}_s^+ := \{ e \in \mathcal{E}_{z_0,w_0} : e > e' \text{ for all } e' \in \mathcal{E}_s \},$$

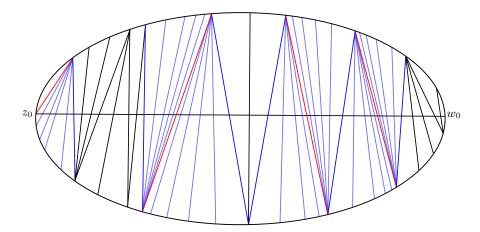


FIGURE 6. The geodesics drawn above that intersect  $\{z_0, w_0\}$  transversely represent the edges in  $\mathcal{E}_{z_0, w_0}$ . The red geodesics are edges in  $\widetilde{\mathcal{P}}$ , the colored geodesics are edges in  $\mathcal{E}_j$  and the black geodesics are edges in  $\mathcal{E}_{j,j+1}$  for some j.

then  $\mathcal{E}_{z_0,w_0} = \mathcal{F}_s^- \cup \mathcal{E}_s \cup \mathcal{F}_s^+$  is a disjoint union. Note that  $\mathcal{E}_s$  is infinite, but has a well-defined minimum and maximum. We further define

$$\mathcal{E}_{z_0, w_0, s, s+1} = \mathcal{E}_{s, s+1} := \begin{cases} \mathcal{F}_1^- & \text{if } s = 0\\ \mathcal{F}_s^+ \cap \mathcal{F}_{s+1}^- & \text{if } 0 < s < k\\ \mathcal{F}_k^+ & \text{if } s = k \end{cases}$$

and note that  $\mathcal{E}_{z_0,w_0} = \bigcup_{s=1}^k \mathcal{E}_s \cup \bigcup_{s=0}^k \mathcal{E}_{s,s+1}$  is a disjoint union. Note also that  $\mathcal{E}_{s,s+1}$  is finite (possibly empty) for all  $s = 0, \ldots, k$ . Hence, each  $\mathcal{E}_{s,s+1}$  has a minimum and maximum if it is non-empty. In this setting, we will use the following notation in the rest of this article.

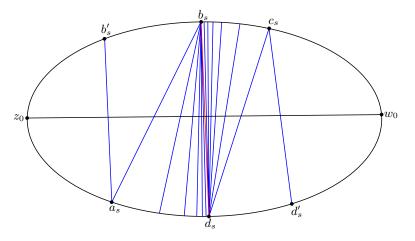


FIGURE 7. The lines in  $\mathcal{E}_s \cup \{\{a_s, b_s'\}\} \cup \{\{c_s, d_s'\}\}$ , with the closed edge  $l_s$  in red.

Notation 4.6. Let  $z_0, w_0 \in \partial \Gamma$  be a pair of distinct vertices of  $\widetilde{\mathcal{T}}$ , and decompose

$$\mathcal{E}_{z_0,w_0} = \bigcup_{s=1}^k \mathcal{E}_s \cup \bigcup_{s=0}^k \mathcal{E}_{s,s+1}$$

as described above.

- Let  $\{a_s, b_s\}$  and  $\{c_s, d_s\}$  be the minimum and maximum of  $\mathcal{E}_s$  respectively, so that  $b_s$  and  $d_s$  are vertices of  $l_s$  (see Figure 7).
- If  $s=1,\ldots,k-1$ , or s=k and  $\mathcal{E}_{k,k+1}$  is non-empty, let  $d_s'\in\partial\Gamma$  be the point so that  $\{c_s,d_s'\}$  is the successor of  $\{c_s,d_s\}$ . Similarly, if  $s=2,\ldots,k$ , or s=1 and  $\mathcal{E}_{0,1}$  is non-empty, let  $b_s'\in\partial\Gamma$  be the point so that  $\{a_s,b_s'\}$  is the predecessor of  $\{a_s,b_s\}$ . If s=k and  $\mathcal{E}_{k,k+1}$  is empty, let  $d_k':=w_0$  and if s=1 and  $\mathcal{E}_{0,1}$  is empty, let  $b_0':=z_0$  (see Figure 7).
- Suppose that  $\mathcal{E}_{s,s+1}$  is non-empty. If s=0, let  $b'_{0,1}:=z_0$  and if s>0, let  $b'_{s,s+1}$  be the vertex of the predecessor of the minimum of  $\mathcal{E}_{s,s+1}$  that is not a vertex of the minimum of  $\mathcal{E}_{s,s+1}$ . Then denote the minimum of  $\mathcal{E}_{s,s+1}$  by  $\{a_{s,s+1},b_{s,s+1}\}$ , so that its predecessor is  $\{a_{s,s+1},b'_{s,s+1}\}$ . Similarly, if s=k, let  $d'_{k,k+1}:=w_0$  and if s< k, let  $d'_{s,s+1}$  be the vertex of the successor of the maximum of  $\mathcal{E}_{s,s+1}$  that is not a vertex of the maximum of  $\mathcal{E}_{s,s+1}$ . Then denote the maximum of  $\mathcal{E}_{s,s+1}$  by  $\{c_{s,s+1},d_{s,s+1}\}$  so that its successor is  $\{c_{s,s+1},d'_{s,s+1}\}$  (see Figure 8).

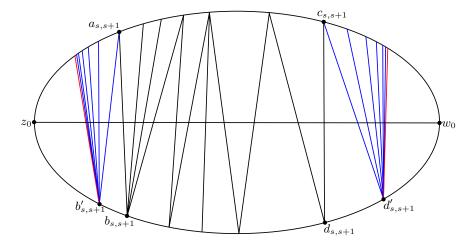


FIGURE 8. The red lines are  $l_s$  and  $l_{s+1}$ , the blue lines lie in  $\mathcal{E}_s$  and  $\mathcal{E}_{s+1}$ , and the black lines lie in  $\mathcal{E}_{s,s+1}$ .

With this, we are ready to prove Proposition 4.3.

Proof of Proposition 4.3. Let  $\{x_0, y_0, z_0\} \in \widetilde{\Theta}$  be any ideal triangle, and let  $\xi_1$  and  $\xi_2$  be representatives of  $[\xi_1]$  and  $[\xi_2]$  so that  $\xi_1(x_0) = \xi_2(x_0)$ ,  $\xi_1(y_0) = \xi_2(y_0)$  and  $\xi_1^{(1)}(z_0) = \xi_2^{(1)}(z_0)$ . By Lemma 4.5, it is sufficient to prove that  $\xi_1|_{\mathcal{V}}$  and  $\xi_2|_{\mathcal{V}}$  agree on the vertices of  $\widetilde{\mathcal{T}}$ . Since

$$T_{i_1,i_2,i_3}(\xi_1(x_0),\xi_1(y_0),\xi_1(z_0)) = T_{i_1,i_2,i_3}(\xi_2(x_0),\xi_2(y_0),\xi_2(z_0)),$$

Proposition 2.10 implies that  $\xi_1(z_0) = \xi_2(z_0)$ .

Now, let  $w_0 \in \partial \Gamma$  be any vertex of  $\widetilde{\mathcal{T}}$  that is not  $x_0$   $y_0$  or  $z_0$ . By relabelling the vertices of  $\{x_0, y_0, z_0\}$  if necessary, we may assume without loss of generality that  $\{x_0, y_0\}$  is the minimal element of  $\mathcal{E}_{z_0, w_0}$ . Decompose

$$\mathcal{E}_{z_0,w_0} = \bigcup_{s=1}^k \mathcal{E}_{z_0,w_0,s} \cup \bigcup_{s=0}^k \mathcal{E}_{z_0,w_0,s,s+1} = \bigcup_{s=1}^k \mathcal{E}_s \cup \bigcup_{s=0}^k \mathcal{E}_{s,s+1}.$$

Note that if  $\mathcal{E}_{s,s+1}$  is non-empty, then  $b'_{s,s+1}=d_s$ ,  $a_{s,s+1}=c_s$ ,  $b_{s,s+1}=d'_s$ ,  $d_{s,s+1}=b'_{s+1}$ ,  $c_{s,s+1}=a_{s+1}$ , and  $d'_{s,s+1}=b_{s+1}$ . On the other hand, if  $\mathcal{E}_{s,s+1}$  is empty, then  $d_s=b'_{s+1}$ ,  $c_s=a_{s+1}$ , and  $d'_s=b_{s,s+1}$ . Since we have already established that  $\xi_1$  and  $\xi_2$  agree on  $z_0=b'_{0,1}$  and on the points in  $\{x_0,y_0\}=\{a_{0,1},b_{0,1}\}$ , to prove the proposition, it is sufficient to prove the following statements.

- (1) Let s = 0, ..., k, and suppose that  $\mathcal{E}_{s,s+1}$  is non-empty. If  $\xi_1(p) = \xi_2(p)$  for all  $p = a_{s,s+1}, b_{s,s+1}, b'_{s,s+1}$ , then  $\xi_1(p) = \xi_2(p)$  for all  $p = c_{s,s+1}, d_{s,s+1}, d'_{s,s+1}$ .
- (2) Let s = 1, ..., k. If  $\xi_1(p) = \xi_2(p)$  for all  $p = a_s, b_s, b'_s$ , then  $\xi_1(p) = \xi_2(p)$  for all  $p = c_s, d_s, d'_s$ .

Since  $\mathcal{E}_{s,s+1}$  is finite for  $s=0,\ldots,k$ , Proposition 2.10 immediately implies (1). To prove (2), observe that  $\mathcal{E}_s \cup \{a_s,b_s'\} \cup \{c_s,d_s'\}$  can be written as  $\mathcal{E}_{s,1} \cup \mathcal{E}_{s,2} \cup \{l_s\}$ , where  $\mathcal{E}_{s,1} := \{e \in \mathcal{E}_s : e < l_s\} \cup \{a_s,b_s'\}$  and  $\mathcal{E}_{s,2} := \{e \in \mathcal{E}_s : e > l_s\} \cup \{c_s,d_s'\}$ . For m=1,2, let  $\mathcal{V}_{s,m}$  be the vertices of the edges in  $\mathcal{E}_{s,m}$ . Proposition 2.10 implies that there is some  $g_m \in \operatorname{PGL}(V)$  so that  $g_m \cdot \xi_1(\mathcal{V}_{s,m}) = \xi_2(\mathcal{V}_{s,m})$ .

Since  $\xi_1(p) = \xi_2(p)$  for  $p = a_s, b_s, b'_s$ , we see that  $g_1 = \text{id}$ . Let  $l_s = \{x_s, y_s\}$ . The Frenet property of  $\xi_1$  and  $\xi_2$  implies that  $\xi_1(x_s) = \xi_2(x_s)$  and  $\xi_1(y_s) = \xi_2(y_s)$ , which allows us to conclude that  $g_2$  fixes both  $\xi_1(x_s)$  and  $\xi_2(y_s)$ . The fact that  $\sigma_{b_s,d_s,J}^{i,n-i}[\xi_1] = \sigma_{b_s,d_s,J}^{i,n-i}[\xi_2]$  for some bridge J across  $\{b_s,d_s\}$  then implies that  $g_2 = \text{id}$ . Thus,  $\xi_1(p) = \xi_2(p)$  for  $p = c_s, d_s, d'_s$ .

4.4. The Bonahon-Dreyer parameterization of  $\mathrm{Hit}_V(S)$ . In this section, we will describe the real-analytic parameterization of the  $\mathrm{Hit}_V(S)$  given by Bonahon-Dreyer [BD14]. Their work builds on a coordinate system on the space of framed local systems on the surface with at least one hole or one puncture constructed by Fock-Goncharov [FG06].

Let  $\rho:\Gamma\to \mathrm{PSL}(V)$  be a representation in  $\mathrm{Hit}_V(S)$  and  $\xi:\partial\Gamma\to\mathcal{F}(V)$  a  $\rho$ -equivariant Frenet curve. The Bonahon-Dreyer parametrization is based on the edge invariants and the triangle invariants associated to  $\xi$ .

First note that since  $\xi$  is  $\rho$ -equivariant we have that, for any  $i=1,\ldots,n-1$ ,  $\sigma_{x_1,x_2,J_1}^{i,n-i}=\sigma_{x_1,x_2,J_2}^{i,n-i}$  for any closed edge  $\{x_1,x_2\}\in\widetilde{\mathcal{P}}$  and any  $J_1,J_2\in\widetilde{\mathcal{J}}$  that crosses  $\{x_1,x_2\}$ . We will thus denote  $\sigma_{x_1,x_2}^{i,n-i}:=\sigma_{x_1,x_2,J_1}^{i,n-i}$ . Moreover  $\sigma_{x_1,x_2}^{i,n-i}=\sigma_{\gamma\cdot x_1,\gamma\cdot x_2}^{i,n-i}$  for all  $\gamma\in\Gamma$ ,  $i=1,\ldots,n-1$  and  $\{x_1,x_2\}\in\widetilde{\mathcal{T}}$ , and  $\tau_{i_1,i_2,i_3}^{i_1,i_2,i_3}=\tau_{\gamma\cdot x_1,\gamma\cdot x_2,\gamma\cdot x_3}^{i_1,i_2,i_3}$  for all  $\gamma\in\Gamma$ ,  $\{x_1,x_2,x_3\}\in\widetilde{\Theta}$  and  $i_1,i_2,i_3\in\mathbb{Z}^+$  so that  $i_1+i_2+i_3=n$ . Thus, the map  $\Phi:\mathcal{FR}(V)/\mathrm{PGL}(V)\to\mathbb{R}^{|\widetilde{\mathcal{P}}|\cdot|\widetilde{\mathcal{J}}|\cdot(n-1)}\times\mathbb{R}^{|\widetilde{\mathcal{Q}}|\cdot(n-1)}\times\mathbb{R}^{|\widetilde{\mathcal{Q}}|\cdot(n-1)}$  described in the previous section, when restricted to  $\mathrm{Hit}_V(S)$ , gives the map

$$\Phi_{\mathrm{Hit}_V(S)}: \mathrm{Hit}_V(S) \to \mathbb{R}^{|\mathcal{T}| \cdot (n-1)} \times \mathbb{R}^{|\Theta| \cdot \frac{(n-1)(n-2)}{2}}$$

The image of this map is not surjective, but subject to certain linear equations and inequalities. These relations are associated to closed edges and have been determined by Bonahon-Dreyer [BD14] Section 4.

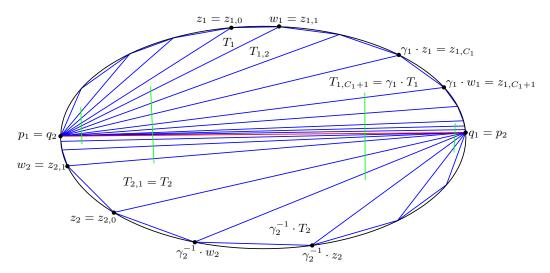


FIGURE 9. The red line is  $\{x_1, x_2\}$  and the blue lines are isolated edges in  $\widetilde{\mathcal{Q}}$ , and the green lines are bridges in  $\widetilde{\mathcal{J}}$ , and the second bridge from the left is  $\{T_1, T_2\}$ .

Given a closed edge  $\{x_1, x_2\} \in \mathcal{P}$  and a bridge  $J = \{T_1, T_2\} \in \widetilde{\mathcal{J}}$  across  $\{x_1, x_2\}$  we introduce the following notation to label the triangles on the left and right of  $(x_1, x_2)$ . This notation will also be used in Section 5.

Notation 4.7. Let  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$  be a closed edge and  $J = \{T_1, T_2\} \in \widetilde{\mathcal{J}}$  a bridge across  $\{x_1, x_2\}$  so that the vertices of  $T_1$  and  $T_2$  lie to the right and left of  $(x_1, x_2)$  respectively. For m = 1, 2, let  $p_m$  be the vertex of  $T_m$  that is either  $x_1$  or  $x_2$ , and let  $q_m \in \{x_1, x_2\}$  be the point that is not  $p_m$ . Let  $\gamma_m \in \Gamma$  be the primitive group element with  $p_m, q_m$  as its repelling and attracting fixed points respectively. Also, let  $\{T_m =: T_{m,1}, T_{m,2}, \ldots, T_{m,C_m}\} \subset \widetilde{\Theta}$  denote the set of ideal triangles so that

- $T_{m,c}$  has  $p_m$  as a common vertex for all  $c=1,\ldots,C_m$ ,
- $T_{m,c}$  and  $T_{m,c+1}$  share a common edge for all  $c=1,\ldots,C_m-1$ ,
- $\gamma_m \cdot T_{m,1}$  shares a common edge with  $T_{m,C_m}$ .

Then for any  $k \in \mathbb{Z}$  and  $c \in \{1, \ldots, C_m\}$ , denote  $T_{m,kC_m+c} := \gamma_m^k \cdot T_{m,c}$ , and note that  $T_{m,l}$  shares a common edge with  $T_{m,l+1}$  for all  $l \in \mathbb{Z}$ . Let  $e_{m,l}$  denote the common edge of  $T_{m,l}$  and  $T_{m,l+1}$ , and let  $z_{m,l}$  be the vertex of  $e_{m,l}$  that is not  $p_m$ . For convenience, we will also denote  $z_m := z_{m,0}$  and  $w_m := z_{m,1}$ . See Figure 9 for the case when  $p_1 = q_2$ .

With this notation,  $T_{m,l+1}=\{p_m,z_{m,l},z_{m,l+1}\}$  and  $T_m=\{p_m,z_m,w_m\}$  for m=1,2.

Using his description of  $\mathrm{Hit}_V(S)$  via Frenet curves, Labourie [Lab06, Theorem 1.5] proved that for any  $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ ,  $\rho(\gamma)$  is diagonalizable over  $\mathbb R$  with eigenvalues having pairwise distinct absolute values. Bonahon-Dreyer then explicitly computed the eigenvalue data of  $\rho(\gamma_m)$  in terms of the triangle and edge invariants. More concretely, let

$$|\lambda_1(\rho(\gamma_m))| > \cdots > |\lambda_n(\rho(\gamma_m))|$$

be the absolute values of the eigenvalues of  $\rho(\gamma_m)$ , and let

(4.1) 
$$\ell_{\rho}^{i}(\gamma_{m}) := \log \left| \frac{\lambda_{i}(\rho(\gamma_{m}))}{\lambda_{i+1}(\rho(\gamma_{m}))} \right|.$$

Bonahon-Dreyer computed that

$$\ell_{\rho}^{i}(\gamma_{m}) = \begin{cases} -\sum_{c=1}^{C_{m}} \left( \sigma_{p_{m}, z_{m,c}}^{i, n-i}[\xi] + \sum_{j+k=n-i} \tau_{p_{m}, z_{m,c}, z_{m,c-1}}^{i, j, k}[\xi] \right) & \text{if } p_{m} = x_{m} \\ \sum_{c=1}^{C_{m}} \left( \sigma_{p_{m}, z_{m,c}}^{i, n-i}[\xi] + \sum_{j+k=n-i} \tau_{p_{m}, z_{m,c-1}, z_{m,c}}^{i, j, k}[\xi] \right) & \text{if } p_{m} = x_{3-m} \end{cases}$$

As an immediate consequence, the edge invariants and triangle invariants have to satisfy the following inequalities, called the *closed leaf inequalities* associated to the closed edge  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$ :

(1) If  $p_m = x_m$ , then for all i = 1, ..., n - 1,

$$\sum_{c=1}^{C_m} \left( \sigma_{p_m, z_{m,c}}^{i, n-i}[\xi] + \sum_{j+k=n-i} \tau_{p_m, z_{m,c}, z_{m,c-1}}^{i, j, k}[\xi] \right) < 0.$$

(2) If  $p_m = x_{3-m}$ , then for all i = 1, ..., n-1,

$$\sum_{c=1}^{C_m} \left( \sigma_{p_m, z_{m,c}}^{i, n-i}[\xi] + \sum_{j+k=n-i} \tau_{p_m, z_{m,c-1}, z_{m,c}}^{i, j, k}[\xi] \right) > 0.$$

There are n-1 such inequalities for each closed edge in  $\mathcal{P}$ .

Also, since  $\gamma_1 = \gamma_2$  if  $p_1 = p_2$  and  $\gamma_1 = \gamma_2^{-1}$  if  $p_1 \neq p_2$ , the edges invariants and triangle invariants have to satisfy the following equalities, called the *closed leaf* equalities associated to the closed edge  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$ :

(1) If  $p_1 = x_1$  and  $p_2 = x_2$ , then for all i = 1, ..., n - 1,

$$\sum_{c=1}^{C_1} \left( \sigma_{p_1, z_{1,c}}^{i, n-i}[\xi] + \sum_{j+k=n-i} \tau_{p_1, z_{1,c}, z_{1,c-1}}^{i, j, k}[\xi] \right) = \sum_{c=1}^{C_2} \left( \sigma_{p_2, z_{2,c}}^{n-i, i}[\xi] + \sum_{j+k=i} \tau_{p_2, z_{2,c}, z_{2,c-1}}^{n-i, j, k}[\xi] \right).$$

(2) If  $p_1 = x_2$  and  $p_2 = x_1$ , then for all i = 1, ..., n - 1,

$$\sum_{c=1}^{C_1} \left( \sigma_{p_1,z_{1,c}}^{i,n-i}[\xi] + \sum_{j+k=n-i} \tau_{p_1,z_{1,c-1},z_{1,c}}^{i,j,k}[\xi] \right) = \sum_{c=1}^{C_2} \left( \sigma_{p_2,z_{2,c}}^{n-i,i}[\xi] + \sum_{j+k=i} \tau_{p_2,z_{2,c-1},z_{2,c}}^{n-i,j,k}[\xi] \right).$$

(3) If  $p_1 = p_2 = x_1$ , then for all i = 1, ..., n - 1,

$$\sum_{c=1}^{C_1} \left( \sigma_{p_1,z_{1,c}}^{i,n-i}[\xi] + \sum_{j+k=n-i} \tau_{p_1,z_{1,c},z_{1,c-1}}^{i,j,k}[\xi] \right) = -\sum_{c=1}^{C_2} \left( \sigma_{p_2,z_{2,c}}^{i,n-i}[\xi] + \sum_{j+k=n-i} \tau_{p_2,z_{2,c-1},z_{2,c}}^{i,j,k}[\xi] \right).$$

(4) If  $p_1 = p_2 = x_2$ , then for all i = 1, ..., n - 1,

$$\sum_{c=1}^{C_1} \left( \sigma_{p_1,z_{1,c}}^{i,n-i}[\xi] + \sum_{j+k=n-i} \tau_{p_1,z_{1,c-1},z_{1,c}}^{i,j,k}[\xi] \right) = -\sum_{c=1}^{C_2} \left( \sigma_{p_2,z_{2,c}}^{i,n-i}[\xi] + \sum_{j+k=n-i} \tau_{p_2,z_{2,c},z_{2,c-1}}^{i,j,k}[\xi] \right).$$

There are n-1 such identities for each closed edge in  $\mathcal{P}$ . These are sums of invariants on the left versus sums of invariants on the right of the closed edge.

Observe that the closed leaf equalities and inequalities are associated to the closed edge  $\{x_1, x_2\}$  since they do not depend on the choice of bridge  $\{T_1, T_2\} \in \widetilde{\mathcal{J}}$  because of the  $\rho$ -equivariance of  $\xi$ .

Notation 4.8. Let  $W_{\mathcal{T}} \subset \mathbb{R}^{|\mathcal{T}| \cdot (n-1)} \times \mathbb{R}^{|\Theta| \cdot \frac{(n-1)(n-2)}{2}}$  be the vector subspace cut out by the closed leaf equalities, and let  $P_{\mathcal{T}} \subset W_{\mathcal{T}}$  be the convex polytope cut out by the closed leaf inequalities.

The following theorem of Bonahon-Dreyer states that the closed leaf equalities and inequalities are the only relations between the shear and triangle parameters.

**Theorem 4.9.** [BD14, Theorem 17] The edge and triangle invariants give a real analytic diffeomorphism from  $\operatorname{Hit}_V(S)$  to  $P_{\mathcal{T}}$ .

In particular, this classifies the image of  $\Phi|_{\mathrm{Hit}_V(S)}$  and gives a real analytic parametrization of  $\mathrm{Hit}_V(S)$ .

Remark 4.10. Note that  $P_{\mathcal{T}}$  only depends on the triangulation  $\mathcal{T}$ , and not on a choice of associated bridge system  $\mathcal{J}$ . However the explicit identification of  $\mathrm{Hit}_V(S)$  with P depends on the choice of  $\mathcal{J}$ .

4.5. Parametrization using symplectic closed edge invariants. In this section we consider a reparametrization of  $\mathrm{Hit}_V(S)$  (see Theorem 4.17), based on the Bonahon-Dreyer parametrization. We replace the edge invariants associated to the closed edges in  $\mathcal{P}$  by a new invariant, which we call the symplectic closed-edge invariants. We will see in Section 6 that one can use the symplectic closed-edge invariants to give an easy description of the Goldman symplectic structure on  $\mathrm{Hit}_V(S)$ .

Let  $u_m \in \mathrm{PSL}(V)$  be the unique unipotent projective transformation that fixes the flag  $\xi(p_m)$  and sends the flag  $\xi(z_m)$  to  $\xi(q_m)$  (see Notation 4.7). The first step to define the new invariants is to prove the following proposition.

**Proposition 4.11.** Let  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$ , let  $u_1, u_2 \in \mathrm{PSL}(V)$  be as defined above, let  $w_1, w_2 \in \partial \Gamma$  be as defined in Notation 4.7, and let  $\xi : \partial \Gamma \to \mathcal{F}(V)$  be any Frenet curve (without any equivariance assumptions). For all  $i = 1, \ldots, n-1$ , we have

$$C_i(\xi(x_1), u_2 \cdot \xi(w_2), u_1 \cdot \xi(w_1), \xi(x_2)) < 0.$$

*Proof.* By Theorem 2.7 and Theorem 2.14,  $\xi$  is positive, so for m=1,2, there are totally positive unipotent matrices  $v_{1,m}$  and  $v_{2,m}$  that fix  $\xi(p_m)$  so that  $\xi(w_m)=v_{1,m}\cdot\xi(z_m)$  and  $\xi(q_m)=v_{1,m}v_{2,m}\cdot\xi(z_m)$ . This implies that  $u_m=v_{1,m}v_{2,m}$ , so  $u_m\cdot\xi(w_m)=v_{1,m}v_{2,m}v_{1,m}\cdot\xi(z_m)$ . In particular,  $\xi(p_m)$ ,  $\xi(z_m)$ ,  $\xi(q_m)$ ,  $u_m\cdot\xi(w_m)$  is a positive quadruple of flags, so Proposition 2.10 tells us that

$$C_i(\xi(p_m), \xi(z_m), u_m \cdot \xi(w_m), \xi(q_m)) < 0$$

for all  $i = 1, \ldots, n - 1$ . Hence,

$$= \frac{C_i(\xi(x_1), u_2 \cdot \xi(w_2), u_1 \cdot \xi(w_1), \xi(x_2))}{C_i(\xi(x_1), \xi(z_2), \xi(z_1), \xi(x_2)) \cdot C_i(\xi(x_1), u_2 \cdot \xi(w_2), \xi(z_2), \xi(x_2))}{C_i(\xi(x_1), u_1 \cdot \xi(w_1), z_1, \xi(x_2))} < 0.$$

With this, we can make the following definition.

**Definition 4.12.** Let  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$  and let  $J = \{T_1, T_2\} \in \widetilde{\mathcal{J}}$  be a bridge across  $\{x_1, x_2\}$  so that  $T_1$  and  $T_2$  lie to the right and left of  $(x_1, x_2)$  respectively. Also, let  $u_1, u_2 \in \mathrm{PSL}(V)$  be as defined above and let  $w_1, w_2 \in \partial \Gamma$  be as defined in Notation 4.7. For  $i = 1, \ldots, n-1$ , the function  $\alpha_{x_1, x_2, J}^{i, n-i} : \mathcal{FR}(V)/\mathrm{PGL}(V) \to \mathbb{R}$  defined by

$$\alpha_{x_1,x_2,J}^{i,n-i}[\xi] := \log\left(-C_i(\xi(x_1), u_2 \cdot \xi(w_2), u_1 \cdot \xi(w_1), \xi(x_2))\right)$$

is called a symplectic closed-edge invariant.

The projective invariance of the cross ratio implies that  $\alpha_{x_1,x_2,J}^{i,n-i}[\xi]$  does not depend on the choice of representative  $\xi$  in  $[\xi]$ , so Proposition 4.11 implies that  $\alpha_{x_1,x_2,J}^{i,n-i}[\xi]$  is indeed well-defined. Note also that  $\alpha_{x_1,x_2,J}^{i,n-i}[\xi] = \alpha_{x_2,x_1,J}^{n-i,i}[\xi]$ . The main advantage that the symplectic closed-edge invariant has over the Bonahon-Dreyer closed edge invariant is the following observation, which we state as a lemma.

**Lemma 4.13.** Let  $F_1, F_2, F_3, F_4, F'_4 \in \mathcal{F}(V)$  so that

$$(F_1, F_2, F_3), (F_1, F_3, F_4), (F_1, F_3, F_4) \in \mathcal{F}(V)^{(3)},$$

and let  $P_1, P_2, P_2' \in \mathbb{P}(V)$  be so that  $P + F_1^{(i)} + F_3^{(n-i-1)} = V$  for all  $i = 1, \ldots, n-1$  and  $P = P_1, P_2, P_2'$ . Let  $u, v, v', w, w' \in \mathrm{PSL}(V)$  be the unipotent projective transformations so that

$$u \cdot F_1 = F_1$$
 and  $u \cdot F_2 = F_3$ ,  
 $v \cdot F_1 = F_1$  and  $v \cdot F_4 = F_3$ ,  
 $v' \cdot F_1 = F_1$  and  $v' \cdot F'_4 = F_3$ ,  
 $w \cdot F_3 = F_3$  and  $w \cdot F_4 = F_1$ ,  
 $w' \cdot F_3 = F_3$  and  $w' \cdot F'_4 = F_1$ .

- (1)  $C_i(F_1, u \cdot P_1, v \cdot P_2, F_3) = C_i(F_1, u \cdot P_1, v' \cdot P'_2, F_3)$  for all i = 1, ..., n-1 if and only if there is a unipotent projective transformation  $g \in PSL(V)$  that fixes  $F_1$  and satisfies  $g \cdot F_4 = F'_4$  and  $g \cdot P_2 = P'_2$ .
- (2)  $C_i(F_1, u \cdot P_1, w \cdot P_2, F_3) = C_i(F_1, u \cdot P_1, w' \cdot P'_2, F_3)$  for all i = 1, ..., n-1 if and only if there is a unipotent projective transformation  $g \in PSL(V)$  that fixes  $F_3$  and satisfies  $g \cdot F_4 = F'_4$  and  $g \cdot P_2 = P'_2$ .

*Proof.* We will only prove (1); the proof of (2) is identical. Suppose first that such a unipotent  $g \in \mathrm{PSL}(V)$  exists, then v = v'g because both v'g and v are unipotent projective transformations that fix  $F_1$  and send  $F_4$  to  $F_3$ . In particular, this means that  $v \cdot P_2 = v'g \cdot P_2 = v' \cdot P_2'$ , which implies that

$$C_i(F_1, u \cdot P_1, v \cdot P_2, F_3) = C_i(F_1, u \cdot P_1, v' \cdot P_2', F_3)$$

for all i = 1, ..., n - 1

Conversely, suppose that  $C_i(F_1, u \cdot P_1, v \cdot P_2, F_3) = C_i(F_1, u \cdot P_1, v' \cdot P'_2, F_3)$  for all  $i = 1, \ldots, n-1$ . An easy computation verifies that  $v \cdot P_2 = v' \cdot P'_2$ . Define  $g := v'^{-1}v$ , and it is clear that  $g \cdot F_1 = F_1$ ,  $g \cdot F_4 = F'_4$  and  $g \cdot P_2 = P'_2$ . Since v' and v are both unipotent and fix  $F_1$ , this implies that g is also unipotent.  $\square$ 

As an immediate consequence, we have the following lemma.

**Lemma 4.14.** Let  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$ , let  $J = \{T_1, T_2\} \in \widetilde{\mathcal{J}}$  be a bridge across  $\{x_1, x_2\}$  so that  $T_1$  and  $T_2$  lie to the right and left of  $(x_1, x_2)$  respectively, and let  $p_1, p_2, q_1, q_2, z_1, z_2 \in \partial \Gamma$  be as defined in Notation 4.7. Suppose that  $\xi$  and  $\xi'$  are a pair of Frenet curves so that  $\xi(x_1) = \xi'(x_1), \xi(x_2) = \xi'(x_2), \xi(z_2) = \xi'(z_2), \xi^{(1)}(w_2) = \xi'^{(1)}(w_2)$ . Then  $\alpha_{x_1, x_2, J}^{i, n-i}[\xi] = \alpha_{x_1, x_2, J}^{i, n-i}[\xi']$  for all  $i = 1, \ldots, n-1$  if and only if there is some unipotent projective transformation  $v \in \mathrm{PSL}(V)$  so that  $v \cdot \xi(p_1) = \xi(p_1), v \cdot \xi(z_1) = \xi'(z_1)$  and  $v \cdot \xi^{(1)}(w_1) = \xi'^{(1)}(w_1)$ .

Moreover, we can determine explicitly how  $\alpha_{x_1,x_2,J}^{i,n-i}$  changes under the *i*-th elementary shearing flow about the closed edge  $\{x_1,x_2\}$ .

**Lemma 4.15.** Let  $\xi : \partial \Gamma \to \mathcal{F}(V)$  be a Frenet curve, let  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$ , and let  $\xi_t := (\psi_{x_1, x_2}^{i, n-i})_t(\xi)$ . Then for all  $i = 1, \ldots, n-1$ ,

$$\alpha_{x_1,x_2,J}^{i,n-i}[\xi_t] = \alpha_{x_1,x_2,J}^{i,n-i}[\xi] + t.$$

Proof. For m=1,2, let  $u_m\in \mathrm{PSL}(V)$  be the unipotent group element that fixes  $\xi(p_m)$  and sends  $\xi(z_m)$  to  $\xi(q_m)$ . Similarly, let  $u_m(t)$  be the unipotent group element that fixes  $\xi_t(p_m)$  and sends  $\xi_t(z_m)$  to  $\xi_t(q_m)$ . Let  $b(t):=b_{\xi(x_1),\xi(x_2)}^{i,n-i}(t)$ , and consider the product  $u_1(t)b(t)^{-1}u_1^{-1}$ . By definition,  $b(t)^{-1}\cdot\xi(z_1)=\xi_t(z_1)$ ,  $b(t)^{-1}\cdot\xi(w_1)=\xi_t(w_1)$ , and b(t) fixes both  $\xi(x_1)=\xi_t(x_1)$  and  $\xi(x_2)=\xi_t(x_2)$ . Thus,  $u_1(t)b(t)^{-1}u_1^{-1}$  fixes  $\xi(p_1)$  and

$$\begin{array}{rcl} u_1(t)b(t)^{-1}u_1^{-1} \cdot \xi(q_1) & = & u_1(t)b(t)^{-1} \cdot \xi(z_1) \\ & = & u_1(t) \cdot \xi_t(z_1) \\ & = & \xi_t(q_1) \\ & = & \xi(q_1). \end{array}$$

This implies that in the basis  $\{f_1,\ldots,f_n\}$  of V so that  $\operatorname{Span}_{\mathbb{R}}(f_i)=\xi^{(i)}(p_1)\cap \xi^{(n-i+1)}(q_1)$ , both  $u_1(t)b(t)^{-1}u_1^{-1}$  and  $b(t)^{-1}$  are represented by diagonal matrices and both  $u_1(t)$  and  $u_1$  are represented by upper-triangular unipotent matrices. Hence,  $u_1(t)b(t)^{-1}u_1^{-1}=b(t)^{-1}$ , so we can conclude that

$$u_1(t) \cdot \xi_t(w_1) = u_1(t)b(t)^{-1} \cdot \xi(w_1)$$
  
=  $u_1(t)b(t)^{-1}u_1^{-1} \cdot (u_1 \cdot \xi(w_1))$   
=  $b(t)^{-1} \cdot (u_1 \cdot \xi(w_1)).$ 

Similarly, we also have that  $u_2(t) \cdot \xi_t(w_2) = b(t) \cdot (u_2 \cdot \xi(w_2))$ . This allows us to conclude that

$$\alpha_{x_1,x_2,J}^{i,n-i}[\xi_t] = \log\left(-C_i(\xi_t(x_1), u_2(t) \cdot \xi_t(w_2), u_1(t) \cdot \xi_t(w_1), \xi_t(x_2))\right)$$

$$= \log\left(-C_i(\xi(x_1), b(t) \cdot (u_2 \cdot \xi(w_2)), b(t)^{-1} \cdot (u_1 \cdot \xi(w_1)), \xi(x_2))\right)$$

$$= \log\left(-e^t \cdot C_i(\xi(x_1), u_2 \cdot \xi(w_2), u_1 \cdot \xi(w_1), \xi(x_2))\right)$$

$$= \alpha_{x_1, x_2, J}^{i,n-i}[\xi] + t.$$

where the second last equality is the same computation that we did in the proof of Proposition 3.9 (1).

We will now prove that replacing the Bonahon-Dreyer edge invariants along closed edges by the symplectic closed-edge invariants gives a parameterization of  $\operatorname{Hit}_{V}(S)$ . When  $[\xi] \in \operatorname{Hit}_{V}(S)$ , the equivariance of  $\xi$  implies that for any  $\{x_{1}, x_{2}\} \in \widetilde{\mathcal{P}}$ ,  $\alpha_{x_{1}, x_{2}, J}^{i, n-i}[\xi]$  does not depend on the choice of bridge J across  $\{x_{1}, x_{2}\}$ . In that case, we will denote  $\alpha_{x_{1}, x_{2}}^{i, n-i}[\xi] := \alpha_{x_{1}, x_{2}, J}^{i, n-i}[\xi]$ 

**Lemma 4.16.** Choose any  $[\xi_0] \in \operatorname{Hit}_V(S)$  and let  $\operatorname{Hit}_V(S)^{[\xi_0]}$  denote the set of projective classes of Frenet curves  $[\xi] \in \operatorname{Hit}_V(S)$  so that all the triangle invariants and edge invariants along isolated edges of  $\widetilde{\mathcal{T}}$  agree for  $[\xi_0]$  and  $[\xi]$ . Then the map  $A: \operatorname{Hit}_V(S)^{[\xi_0]} \to \mathbb{R}^{|\mathcal{P}| \cdot (n-1)}$  given by

$$A: [\xi] \mapsto \left(\alpha_{x_1, x_2}^{i, n-i}[\xi]\right)_{[x_1, x_2] \in \mathcal{P}, i \in [1, n-1]}$$

is a real-analytic diffeomorphism.

*Proof.* It follows immediately from Theorem 4.9 that if  $\xi_t$  is a one-parameter family of Frenet curves corresponding to a real-analytic family of Hitchin representations  $\rho_t: \Gamma \to \mathrm{PSL}(V)$ , then for any  $p \in \partial \Gamma$ ,  $\xi_t(p)$  is a real-analytic path in  $\mathcal{F}(V)$ . This implies the real-analyticity of A. The surjectivity of A follows immediately from Lemma 4.15. It is thus sufficient to argue that A is injective.

Let  $[\xi] \neq [\xi'] \in \operatorname{Hit}_V(S)^{[\xi_0]}$ , then by Theorem 4.9, there is some  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$  and some  $i = 1, \ldots, n-1$  so that  $\sigma_{x_1, x_2}^{i, n-i}[\xi] \neq \sigma_{x_1, x_2}^{i, n-i}[\xi']$ . Let  $\{T_1, T_2\}$  be a bridge across  $\{x_1, x_2\}$  so that  $T_1$  and  $T_2$  lie to the right and left of  $(x_1, x_2)$  respectively, and let  $p_1, p_2, q_1, q_2, z_1, z_2, w_1, w_2 \in \partial \Gamma$  be as defined in Notation 4.7. Since all the triangle invariants and edge invariants along isolated edges for  $[\xi]$  and  $[\xi']$  agree, Proposition 2.10 and the continuity of Frenet curves imply that there are representatives  $\xi$  and  $\xi'$  of  $[\xi]$  and  $[\xi']$  respectively, and some  $g \in \operatorname{PSL}(V)$  so that

•  $\xi(x_1) = \xi'(x_1), \ \xi(x_2) = \xi'(x_2), \ \xi(z_2) = \xi'(z_2), \ \xi(w_2) = \xi'(w_2),$ •  $g \cdot \xi(x_1) = \xi(x_1), \ g \cdot \xi(x_2) = \xi(x_2), \ g \cdot \xi(w_1) = \xi'(w_1) \ \text{and} \ g \cdot \xi(z_1) = \xi'(z_1).$ 

Assume for contradiction that  $\alpha_{x_1,x_2}^{i,n-i}[\xi] = \alpha_{x_1,x_2}^{i,n-i}[\xi']$  for all  $i=1,\ldots,n-1$ . By Lemma 4.14, we see that there is some unipotent  $u\in \mathrm{PSL}(V)$  so that  $u\cdot \xi(p_1)=\xi(p_1), u\cdot \xi(z_1)=\xi'(z_1)$  and  $u\cdot \xi^{(1)}(w_1)=\xi'^{(1)}(w_1)$ , which implies that u=g. In the basis  $\{f_1,\ldots,f_n\}$  so that  $[f_i]=\xi(p_1)^{(i)}\cap \xi(q_1)^{(n-i+1)}$ , we see that g is represented by a diagonal matrix, but u is represented by an upper-triangular unipotent matrix. Since g=u, this means that  $g=u=\mathrm{id}$ . However, this implies that  $\sigma_{x_1,x_2}^{i,n-i}[\xi]=\sigma_{x_1,x_2}^{i,n-i}[\xi']$  for all  $i=1,\ldots,n-1$ , which contradicts Theorem 4.9.

The next theorem is follows immediately from Lemma 4.16 and Theorem 4.9.

Theorem 4.17. The map

$$\Omega = \Omega_{\mathcal{T},\mathcal{J}}: \quad \mathrm{Hit}_V(S) \quad \to \quad \mathbb{R}^{|\mathcal{Q}| \cdot (n-1)} \times \mathbb{R}^{|\mathcal{P}| \cdot (n-1)} \times \mathbb{R}^{|\Theta| \cdot \frac{(n-1)(n-2)}{2}} \\ [\xi] \quad \mapsto \quad (\Sigma_1, \Sigma_2, \Sigma_3),$$

with

$$\begin{split} & \Sigma_1 &:= & \left(\sigma_{x_1, x_2}^{i, n-i}[\xi]\right)_{i=1, \dots, n-1; [x_1, x_2] \in \mathcal{Q}} \\ & \Sigma_2 &:= & \left(\alpha_{x_1, x_2}^{i, n-i}[\xi]\right)_{i=1, \dots, n-1; [x_1, x_2] \in \mathcal{P}} \\ & \Sigma_3 &:= & \left(\tau_{x_1, x_2, x_3}^{i_1, i_2, i_3}[\xi]\right)_{i_1, i_2, i_3 \in \mathbb{Z}^+; i_1 + i_2 + i_3 = n; [x_1, x_2, x_3] \in \Theta} \,. \end{split}$$

is a real-analytic diffeomorphism onto  $P_{\mathcal{T}}$  as defined in Notation 4.8.

## 5. Deforming PSL(V)-Hitchin representations

In this section we use the elementary eruption and shearing flows defined in Section 3 to construct eruption and shearing flows on  $\mathrm{Hit}_V(S) \subset \mathcal{FR}(V)/\mathrm{PGL}(V)$ . The idea to do this is straight forward: choose an ideal triangulation  $\mathcal{T}$  on S, and perform elementary eruption flows and shearing flows on the triangles and edges of  $\widetilde{\mathcal{T}}$  in a " $\Gamma$ -invariant" way to obtain a flow in  $\mathrm{Hit}_V(S)$ . However, the  $\Gamma$ -orbits of edges in  $\widetilde{\mathcal{T}}$  and triangles in  $\widetilde{\Theta}$  are infinite, so defining these flows in a  $\Gamma$ -invariant way involves taking the product of infinitely many elementary eruption and shearing flows. In general, such products do not converge. The main goal of this section is thus to resolve these convergence issues. As a result we can define eruption and shearing flows, and more general  $(\mathcal{T}, \mathcal{J})$ -parallel flows on  $\mathrm{Hit}_V(S)$ . By Sun-Zhang [SZ17, Corollary 5.4], these flows are in fact Hamiltonian flows with respect to the Goldman symplectic form on  $\mathrm{Hit}_V(S)$ .

5.1.  $(\mathcal{T}, \mathcal{J})$ -parallel flows on  $\mathrm{Hit}_V(S)$ . Fix an ideal triangulation  $\mathcal{T}$  on S and a compatible bridge system  $\mathcal{J}$ . Recall that

$$W = W_{\mathcal{T}} \subset \mathbb{R}^{|\mathcal{Q}| \cdot (n-1)} \times \mathbb{R}^{|\mathcal{P}| \cdot (n-1)} \times \mathbb{R}^{|\Theta| \cdot \frac{(n-1)(n-2)}{n}}$$

the linear subspace cut out by the closed leaf equalities defined in Notation 4.8. The image of the diffeomorphism  $\Omega: \mathrm{Hit}_V(S) \to \mathbb{R}^{|\mathcal{Q}| \cdot (n-1)} \times \mathbb{R}^{|\mathcal{P}| \cdot (n-1)} \times \mathbb{R}^{|\Theta| \cdot \frac{(n-1)(n-2)}{2}}$  given by Theorem 4.17 is an open subset of W. Thus, for every  $[\rho] \in \mathrm{Hit}_V(S)$  we identify W with  $T_{[\rho]}\mathrm{Hit}_V(S)$ .

Having fixed  $\mathcal{T}$  and  $\mathcal{J}$ , we assign to every vector  $\mu \in W$  a flow on  $\mathrm{Hit}_V(S)$  that is defined by performing the elementary shearing and eruption flows on  $\mathcal{FR}(V)/\mathrm{PGL}(V)$  in a " $\Gamma$ -invariant" way. Every vector  $\mu \in W$  can be denoted by

$$\begin{split} \mu &= \left( \left( \mu_{x_1, x_2}^{i, n-i} \right)_{i \in [1, n-1]; [x_1, x_2] \in \mathcal{Q}}, \left( \mu_{x_1, x_2}^{i, n-i} \right)_{i \in [1, n-1]; [x_1, x_2] \in \mathcal{P}}, \\ &\qquad \qquad \left( \mu_{x_1, x_2, x_3}^{i_1, i_2, i_3} \right)_{i_1, i_2, i_3 \in \mathbb{Z}^+; i_1 + i_2 + i_3 = n; [x_1, x_2, x_3] \in \Theta} \right), \end{split}$$

where we also denote  $\mu^{i_1,i_2,i_3}_{x_1,x_2,x_3} = \mu^{i_2,i_3,i_1}_{x_2,x_3,x_1} = \mu^{i_3,i_1,i_2}_{x_3,x_1,x_2}$  and  $\mu^{i,n-i}_{x_1,x_2} = \mu^{n-i,i}_{x_2,x_1}$ . In this notation,  $\mu^{i_1,i_2,i_3}_{x_1,x_2,x_3} = \mu^{i_1,i_2,i_3}_{\gamma\cdot x_1,\gamma\cdot x_2,\gamma\cdot x_3}$  and  $\mu^{i,n-i}_{x_1,x_2} = \mu^{i,n-i}_{\gamma\cdot x_1,\gamma\cdot x_2}$  for any  $\gamma\in\Gamma$ . For the rest of this section we will fix such a vector  $\mu\in W$ .

For any  $o \in \mathcal{T} \cup \Theta$ , define the flow  $(\phi_o^{\mu})_t$  on  $\mathcal{FR}(V)/\mathrm{PGL}(V)$  as follows.

• If  $o = \{x_1, x_2\} \in \widetilde{\mathcal{T}}$  is an edge, let

$$(\phi^{\mu}_{o})_{t} := \prod_{i=1}^{n-1} \left(\psi^{i,n-i}_{x_{1},x_{2}}\right)_{\mu^{i,n-i}_{x_{1},x_{2}}\cdot t} = \prod_{i=1}^{n-1} \left(\psi^{n-i,i}_{x_{2},x_{1}}\right)_{\mu^{n-i,i}_{x_{2},x_{1}}\cdot t},$$

where  $(\psi_{x_1,x_2}^{i,n-i})_t$  is the (i,n-i)-elementary shearing flow with respect to  $(x_1,x_2)$ .

• If  $o = \{x_1, x_2, x_3\} \in \Theta$  is an ideal triangle, let

$$\begin{split} (\phi_o^\mu)_t &:= \prod_{i_1+i_2+i_3=n} \left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_{\substack{\mu_{x_1,x_2,x_3}^{i_1,i_2,i_3} \\ \mu_{x_1,x_2,x_3}^{i_1,i_2,i_3} \cdot t}} \\ &= \prod_{i_1+i_2+i_3=n} \left(\epsilon_{x_2,x_3,x_1}^{i_2,i_3,i_1}\right)_{\substack{\mu_{x_2,x_3,x_1}^{i_3,i_1} \cdot t \\ \mu_{x_2,x_3,x_1,x_2}^{i_3,i_1,i_2} \cdot t}}, \end{split}$$

where  $\left(\epsilon_{x_1,x_2,x_3}^{i_1,i_2,i_3}\right)_t$  is the  $(i_1,i_2,i_3)$ -elementary eruption flow with respect to  $(x_1, x_2, x_3).$ 

By Proposition 3.5 and Proposition 3.11, we see that  $(\phi_o^{\mu})_t$  is a commuting product, and is well-defined for all  $t \in \mathbb{R}$ .

More informally,  $(\phi_a^{\mu})_t$  is the product of all possible elementary shearing and eruption flows associated to o, where the speed of each of these elementary shearing and eruption flows are rescaled according to  $\mu$ .

**Lemma 5.1.** Let  $\xi: \partial \Gamma \to \mathcal{F}(V)$  be Frenet, and let  $\{x_1, x_2, x_3\} \in \widetilde{\Theta}$  so that  $x_1 < x_2 < x_3 < x_1$ . Also, let  $\{f_1, \ldots, f_n\}$  be a basis of V associated to  $\xi(x_1)$ .

(1) Suppose that  $o = \{x_1, x_2, x_3\}$ . There is a representative  $\xi_t$  of  $(\phi_o^{\mu})_t [\xi]$  so that

$$\xi_t(x) = \begin{cases} a'_o(t) \cdot \xi(x) & \text{if } x \in \overline{[x_2, x_3]} \\ \xi(x) & \text{if } x \in \overline{[x_3, x_1]} \\ a_o(t) \cdot \xi(x) & \text{if } x \in \overline{[x_1, x_2]} \end{cases},$$

where

$$a_{\xi,o}(t) = a_o(t) = \prod_{i+i+k=n} a_{\xi(x_1),\xi(x_2),\xi(x_3)}^{i_1,i_2,i_3} \left(\mu_{x_1,x_2,x_3}^{i_1,i_2,i_3}t\right) \in PSL(V),$$

where
$$a_{\xi,o}(t) = a_o(t) = \prod_{i+j+k=n} a^{i_1,i_2,i_3}_{\xi(x_1),\xi(x_2),\xi(x_3)} \left(\mu^{i_1,i_2,i_3}_{x_1,x_2,x_3}t\right) \in \mathrm{PSL}(V),$$

$$a'_{\xi,o}(t) = a'_o(t) = \prod_{i+j+k=n} a^{i_3,i_1,i_2}_{\xi(x_3),\xi(x_1),\xi(x_2)} \left(-\mu^{i_3,i_1,i_2}_{x_3,x_1,x_2}t\right) \in \mathrm{PSL}(V).$$

In particular,  $a_o(t)$  is represented by an upper-triangular matrix

$$\begin{bmatrix} \lambda_1 & * & \dots & * & * \\ 0 & \lambda_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1} & * \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

in the basis  $\{f_1, \dots, f_n\}$ , where  $\frac{\lambda_k}{\lambda_{k+1}} = \exp\left(\sum_{i_2+i_3=n-k} \mu_{x_1, x_2, x_3}^{k, i_2, i_3} \cdot t\right)$  for all k = 1, ..., n-1  $(\frac{\lambda_{n-1}}{\lambda_n} = 1)$ . Furthermore,  $a_o(t)$  fixes  $\xi^{(1)}(x_3)$ . (2) Suppose that  $o = \{x_1, x_2\}$ . There is a representative  $\xi_t$  of  $(\phi_o^{\mu})_t$   $[\xi]$  so that

$$\xi_t(x) = \begin{cases} \xi(x) & \text{if } x \in \overline{[x_2, x_1]} \\ a_o(t) \cdot \xi(x) & \text{if } x \in \overline{[x_1, x_2]} \end{cases},$$

where

$$a_{\xi,o}(t) = a_o(t) = \prod_{i=1}^{n-1} b^{i,n-i}_{\xi(x_1),\xi(x_2)} \left( 2\mu^{i,n-i}_{x_1,x_2} t \right).$$

In particular,  $a_o(t)$  is represented by an upper triangular matrix

$$\begin{bmatrix} \lambda_1 & * & \dots & * & * \\ 0 & \lambda_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1} & * \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

in the basis  $\{f_1,\ldots,f_n\}$ , where  $\frac{\lambda_k}{\lambda_{k+1}}=\exp\left(\mu_{x_1,x_2}^{k,n-k}\cdot t\right)$  for all  $k=1,\ldots,n-1$ 1. Furthermore, if  $\operatorname{Span}_{\mathbb{R}}\{f_n,\ldots,f_{n-i+1}\}=\xi^{(i)}(x_2)$  for all  $i=1,\ldots,n-1$ , then the matrix representing  $a_o(t)$  in the basis  $\{f_1,\ldots,f_n\}$  is diagonal. *Proof.* This is a consequence of Lemma 3.6, Lemma 3.12, and a straight forward computation.  $\Box$ 

In order to define the flows on  $\mathrm{Hit}_V(S)$  we will consider "semi-elementary" flows, which are products of elementary flows on larger and larger collection of triangles and edges. The  $(\mathcal{T}, \mathcal{J})$ -parallel flows on  $\mathrm{Hit}_V(S)$  will then be defined as limits of these semi-elementary flows. To define the semi-elementary flows we introduce a way of grouping the triangles and edges of  $\widetilde{\mathcal{T}}$ . For this we introduce the following.

**Definition 5.2.** For any closed edge  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$  and any bridge  $J = \{T_1, T_2\} \in \widetilde{\mathcal{J}}$  across  $\{x_1, x_2\}$  we define (using the Notation introduced in Notation 4.7), for m = 1, 2,

$$\widetilde{\Theta}(J, T_m) := \{T_{m,1}, T_{m,2}, \dots, T_{m,C_m}\}$$

and

$$\widetilde{\mathcal{T}}(J, T_m) := \{e_{m,1}, e_{m,2}, \dots, e_{m,C_m}\}.$$

Each  $\widetilde{\Theta}(J, T_m)$  is called a *closed edge subset* of  $\widetilde{\Theta}$  (see Figure 10). Let  $\mathcal{D}$  denote the collection of closed edge subsets of  $\widetilde{\Theta}$ .

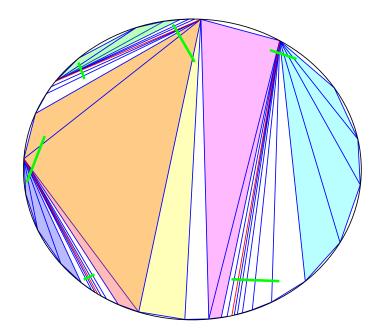


FIGURE 10. The red lines are closed edges in  $\widetilde{\mathcal{P}}$ , the blue lines are isolated edges in  $\widetilde{\mathcal{Q}}$ , and the green lines are bridges in  $\widetilde{\mathcal{J}}$ . There are 6 different closed edge subsets that are represented by colored regions, namely turquoise, purple, blue, green, yellow+orange, and red+orange. The turquoise closed edge subset is adjacent to the purple one, the yellow+orange one is adjacent to the red+orange one, and the red+orange one is adjacent to both the blue one and the yellow+orange one. The turquoise and purple closed edge subsets are opposite, and so are the blue and red+orange ones.

We a notion of adjacency between the closed edge subsets in  $\mathcal{D}$ .

### Definition 5.3.

- A pair of distinct triangles  $T, T' \in \widetilde{\Theta}$  are adjacent if T and T' share a common edge in  $\widetilde{\mathcal{T}}$ , or  $\{T, T'\}$  is a bridge in  $\widetilde{\mathcal{J}}$ .
- A pair of distinct closed edge subsets  $\Theta(J_1, T_1), \Theta(J_2, T_2) \in \mathcal{D}$  are adjacent if there exist triangles  $T \in \widetilde{\Theta}(J_1, T_1)$  and  $T' \in \widetilde{\Theta}(J_2, T_2)$  so that T and T' are adjacent (see Figure 10).
- A pair of distinct closed edge subsets  $\widetilde{\Theta}(J_1, T_1)$ ,  $\widetilde{\Theta}(J_2, T_2) \in \mathcal{D}$  are opposite if  $J_1 = J_2 = \{T_1, T_2\}$  (see Figure 10).

Being opposite defines an equivalence relation on  $\mathcal{D}$  where each equivalence class contains exactly two points in  $\widetilde{\mathcal{D}}$ . Furthermore, if two points in  $\mathcal{D}$  are opposite, then they are adjacent. Adjacency allows us to define the *closed edge graph*.

**Definition 5.4.** The *closed edge graph* is the graph whose vertex set is  $\mathcal{D}$ , and two vertices are joined by an edge if they are adjacent in the sense of Definition 5.3.

In order to define an exhaustion of the closed edge graph, we choose a closed edge subset  $p_0 \in \mathcal{D}$  as base point, and let  $K_1 := \{p_0\}$ . Then for all  $i \in \mathbb{Z}^+$ , let

$$K_i' := K_i \cup \{p \in \mathcal{D} : p \text{ is adjacent to some } p' \in K_i\},$$
  
 $K_{i+1} := K_i' \cup \{p \in \mathcal{D} : p \text{ is opposite to some } p' \in K_i'\}.$ 

This iteratively defines a nested sequence  $(K_1, K_2, ...)$  of subsets of  $\mathcal{D}$ . Furthermore, it is clear that  $\bigcup_{i=1}^{\infty} K_i = \mathcal{D}$ .

For each  $i \in \mathbb{Z}^+$ , let  $N_i := \{T \in \widetilde{\Theta} : T \in p \text{ for some } p \in K_i\}$  and define

$$M_i := \{ o \in \widetilde{\mathcal{T}} \cup \widetilde{\Theta} : o \in N_i \text{ or } o \text{ is an edge of an ideal triangle in } N_i \}.$$

Note that each  $M_j$  is a finite set, and  $(M_1, M_2, ...)$  is a nested sequence whose union is  $\widetilde{\mathcal{Q}} \cup \widetilde{\Theta}$ . (Recall that  $\widetilde{\mathcal{Q}}$  is the set of isolated edges in  $\widetilde{\mathcal{T}}$ .) This allows us to define the semi-elementary flows mentioned at the start of this section.

**Definition 5.5.** Let  $M_j \subset \widetilde{\mathcal{Q}} \cup \widetilde{\Theta}$  be as defined above. Then the  $(\mu, M_j)$ -semi elementary flow is defined by

$$\left(\phi_{M_j}^{\mu}\right)_t := \prod_{o \in M_j} (\phi_o^{\mu})_t : \mathcal{FR}(V)/\mathrm{PGL}(V) \to \mathcal{FR}(V)/\mathrm{PGL}(V).$$

Since  $M_j$  is a finite set for all  $j \in \mathbb{Z}^+$ , so Proposition 3.5 and Proposition 3.11 again ensure that each  $\phi^{\mu}_{M_j}$  is a commuting product, and is well-defined for all  $t \in \mathbb{R}$ . Using this define now the flow

$$\left(\phi_{\mathcal{Q},\Theta}^{\mu}\right)_t: \mathrm{Hit}_V(S) \to \mathrm{Hit}_V(S)$$

by

$$\left(\phi^{\mu}_{\mathcal{Q},\Theta}\right)_t[\xi] := \lim_{j \to \infty} \left(\phi^{\mu}_{M_j}\right)_t[\xi].$$

We prove (Proposition 5.23) that  $\left(\phi_{\mathcal{Q},\Theta}^{\mu}\right)_t$  is a well-defined flow and does not depend on the choice of  $p_0$ . Note that the flow  $\left(\phi_{\mathcal{Q},\Theta}^{\mu}\right)_t$  does not involve the invariants along the closed leaves in  $\mathcal{P}$ . We emphasize, that, as a flow on  $\mathrm{Hit}_V(S)$ ,  $\left(\phi_{\mathcal{Q},\Theta}^{\mu}\right)_t$  is not defined for all t, but only as long as the closed leaf inequalities are satisfied, see Theorem 5.8.

We now define the shearing flow along a closed edge  $c = [y_1, y_2] \in \mathcal{P}$ . Choose an enumeration  $\{o_1, o_2, \dots\}$  of  $\Gamma \cdot \{y_1, y_2\}$ . The flow  $(\phi_c^{\mu})_t : \operatorname{Hit}_V(S) \to \operatorname{Hit}_V(S)$  is defined to be

$$\left(\phi_{c}^{\mu}\right)_{t}\left[\xi\right]=\prod_{l=1}^{\infty}\left(\phi_{o_{l}}^{\mu}\right)_{t}\left[\xi\right]=\lim_{r\rightarrow\infty}\prod_{l=1}^{r}\left(\phi_{o_{l}}^{\mu}\right)_{t}\left[\xi\right].$$

By Proposition 3.5 and Proposition 3.11,  $\prod_{l=1}^{j} \left(\phi_{o_{l}}^{\mu}\right)_{t}$  is again a commuting product for all  $j \in \mathbb{Z}^{+}$ , and is a well-defined flow on  $\mathcal{F}(V)/\operatorname{PGL}(V)$  for all  $t \in \mathbb{R}$ . The fact that  $(\phi_{c}^{\mu})_{t}$  is well-defined for all  $t \in \mathbb{R}$  is a consequence of Proposition 5.13.

Remark 5.6. The flows  $(\phi_c^{\mu})_t$  are examples of generalized twist flows which have been studied by Goldman in the context of representation varieties of surface groups into reductive Lie groups [Gol86].

Using the flows  $(\phi_c^{\mu})_t$  and  $(\phi_{\mathcal{Q},\Theta}^{\mu})_t$ , we associate flows on  $\mathrm{Hit}_V(S)$  to any  $\mu \in W$ .

**Definition 5.7.** For any  $\mu \in W$ , define the  $(\mathcal{T}, \mathcal{J})$ -parallel flow associated to  $\mu$ ,  $\phi_t^{\mu} : \operatorname{Hit}_V(S) \to \operatorname{Hit}_V(S)$  to be

$$\phi_t^\mu := \left(\prod_{c \in \mathcal{P}} \left(\phi_c^\mu\right)_t\right) \circ \left(\phi_{\mathcal{Q},\Theta}^\mu\right)_t.$$

**Theorem 5.8.** Let  $\Omega$  be the real-analytic diffeomorphism from  $\mathrm{Hit}_V(S)$  onto the convex polytope  $P_{\mathcal{T}}$  as defined in Theorem 4.17. For any  $[\xi] \in \mathrm{Hit}_V(S)$  and  $\mu \in W$ , let

 $I_{[\xi],\mu} := \{t \in \mathbb{R} : \Omega[\xi] + t \cdot \mu \text{ satisfy the closed leaf inequalities}\}$ 

For any  $t \in I_{[\xi],\mu}$ , let

$$[\xi_t] := \Omega^{-1} (\Omega[\xi] + t\mu).$$

Then  $\phi_t^{\mu}[\xi] = [\xi_t].$ 

Theorem 5.8 implies that the finite product used to define  $\phi_t^{\mu}$  in Definition 5.7 is a commuting product. A corollary of Theorem 5.8 is the following.

Corollary 5.9. Every pair of  $(\mathcal{T}, \mathcal{J})$ -parallel flows on  $\mathrm{Hit}_V(S)$  commute, and the space of  $(\mathcal{T}, \mathcal{J})$ -parallel flows on  $\mathrm{Hit}_V(S)$  is naturally in bijection with  $T_{[\xi]}\mathrm{Hit}_V(S)$ . In particular, the pair  $(\mathcal{T}, \mathcal{J})$  determines a trivialization of  $T\mathrm{Hit}_V(S)$ .

In Sun-Zhang [SZ17, Theorem 5.3], we prove that this trivialization of  $T\mathrm{Hit}_V(S)$  is in fact symplectic with respect to the Goldman symplectic form on  $\mathrm{Hit}_V(S)$ . Combining this with the fact that any pair of  $(\mathcal{T}, \mathcal{J})$ -parallel flows commute yields the following corollary (see [SZ17, Corollary 5.4] for a proof).

Corollary 5.10. Every  $(\mathcal{T}, \mathcal{J})$ -parallel flow is a Hamiltonian flow.

In order to prove Theorem 5.8 we consider the flows  $(\phi_c^{\mu})_t$  for a closed edge  $c = [y_1, y_2] \in \mathcal{P}$  and the flows  $(\phi_{\mathcal{Q}, \Theta}^{\mu})$  separately. The special cases of Theorem 5.8 where we replace  $\phi_t^{\mu}$  with  $(\phi_c^{\mu})_t$  for some  $c \in \mathcal{P}$  and  $(\phi_{\mathcal{Q}, \Theta}^{\mu})_t$  are established in Proposition 5.13 and Proposition 5.23 respectively. The strategy for the proofs of these two propositions is the same: We choose for any  $[\xi] \in \operatorname{Hit}_V(S)$  a suitable representative  $\xi_j$  of  $\prod_{l=1}^j (\phi_{o_l}^{\mu})_t [\xi]$  or  $(\phi_{M_j}^{\mu})_t [\xi]$ , and show that  $\{\xi_j\}_{j=1}^{\infty}$  converges to a  $\rho$ -equivariant Frenet curve  $\xi_0$  on the vertices of  $\widetilde{\mathcal{T}}$ . We then use the Frenet

property of  $\xi_j$  to conclude that the sequence  $\{\xi_j\}_{j=1}^\infty$  converges to  $\xi_0$  in  $\mathcal{FR}(V)$ , see Lemma 4.5. There is one difference which makes the proof of Proposition 5.23 more delicate: In the case of a closed edge c, even though  $(\phi_c^\mu)_t$  is a composition of infinitely many elementary shearing flows, every edge in  $\widetilde{\mathcal{T}}$  has a neighborhood that is deformed by only finitely many of these. In the case of  $(\phi_{\mathcal{Q},\Theta}^\mu)_t$  however, neighborhoods about the closed edges are deformed by infinitely many elementary eruption and shearing flows.

Let us state two propositions, the first is an immediate consequence of the definition of  $\phi_t^{\mu}$  and Lemma 5.1.

**Proposition 5.11.** Let  $[\xi] \in \operatorname{Hit}_V(S)$  and let  $\mu \in W$ . Choose any  $\{x, y, z\} \in \widetilde{\Theta}$  so that x < y < z < x, and choose a representative  $\xi$  of  $[\xi]$ . For all  $t \in I_{[\xi],\mu}$ , choose the representative  $\xi_t$  of  $\phi_t^{\mu}[\xi]$  so that  $\xi_t(x) = \xi(x)$ ,  $\xi_t(y) = \xi(y)$  and  $\xi_t^{(1)}(z) = \xi^{(1)}(z)$ . Let  $w \in \partial \Gamma$  be the point so that  $\{y, x, w\} \in \widetilde{\Theta}$  with y < x < w < y.

(1) For all  $t \in I_{[\xi],\mu}$ ,

$$\left(\xi_t(y), \xi_t(x), \xi_t^{(1)}(w)\right) = \prod_{i=1}^{n-1} b_{\xi(x), \xi(y)}^{i, n-i} \left(2\mu_{x, y}^{i, n-i} t\right) \cdot \left(\xi(y), \xi(x), \xi^{(1)}(w)\right).$$

(2) For all  $t \in I_{[\xi],\mu}$ ,

$$\left(\xi_t(y), \xi_t(z), \xi_t^{(1)}(x)\right) = \prod_{i+j+k=n} a_{\xi(y),\xi(z),\xi(x)}^{j,k,i} \left(\mu_{y,z,x}^{j,k,i}t\right) \cdot \left(\xi(y), \xi(z), \xi^{(1)}(x)\right).$$

The second proposition is the analogous statement for bridges, and is established in the process of proving Theorem 5.8 (see Remark 5.15 and Remark 5.17). Let  $\{x_1, x_2\} \in \widetilde{T}$  and let  $J = \{T_1, T_2\}$  be any bridge across  $\{x_1, x_2\}$  so that  $T_1$  and  $T_2$  lie to the right and left of  $(x_1, x_2)$  respectively. For m = 1, 2 and  $c \in \mathbb{Z}$ , let  $p_m$ ,  $q_m$ ,  $z_m$ ,  $w_m$ ,  $z_{m,c}$ ,  $T_{m,c}$ ,  $e_{m,c}$  be as defined in Notation 4.7. Also, let  $[\xi] \in \mathrm{Hit}_V(S)$ , let  $\xi$  be a representative in  $[\xi]$ , let  $\mu \in W$ , and let  $t \in I_{[\xi],\mu}$ . For each triangle in  $T_{m,c}$  (see Definition 5.2), define (5.1)

$$a_{\xi,T_{m,c}}(t) := \begin{cases} \prod_{\substack{i+j+k=n \\ i+j+k=n}} a_{\xi(p_m),\xi(z_{m,c}),\xi(z_{m,c-1})}^{i,j,k} \left(\mu_{p_m,z_{m,c},z_{m,c-1}}^{i,j,k}t\right) & \text{if } p_m = x_m \\ \prod_{\substack{i+j+k=n \\ i+j+k=n}} a_{\xi(p_m),\xi(z_{m,c-1}),\xi(z_{m,c})}^{i,j,k} \left(-\mu_{p_m,z_{m,c-1},z_{m,c}}^{i,j,k}t\right) & \text{if } p_m = x_{3-m} \end{cases}$$

and for each edge  $e_{m,c}$ , define

$$(5.2) a_{\xi,e_{m,c}}(t) := \begin{cases} \prod_{i=1}^{n-1} b_{\xi(p_m),\xi(z_{m,c})}^{i,n-i} \left( 2\mu_{p_m,z_{m,c}}^{i,n-i} t \right) & \text{if } p_m = x_m \\ \prod_{i=1}^{n-1} b_{\xi(p_m),\xi(z_{m,c})}^{i,n-i} \left( -2\mu_{p_m,z_{m,c}}^{i,n-i} t \right) & \text{if } p_m = x_{3-m} \end{cases}$$

Using this, we set

$$(5.3) a_m(t) = a_{\xi,m}(t) := a_{\xi,T_{m,1}}(t)a_{\xi,e_{m,1}}(t)\dots a_{\xi,T_{m,C_m}}(t)a_{\xi,e_{m,C_m}}(t),$$

where  $C_m = |\widetilde{\Theta}(J, T_m)| = |\widetilde{T}(J, T_m)|$  is as defined in Notation 4.7. It follows from Lemma 5.1 that  $a_m(t)$  fixes  $\xi(p_m)$ . In other words, if  $\{f_1, \ldots, f_n\}$  be a basis of V so that  $\operatorname{Span}_{\mathbb{R}}(f_i) = \xi(x_1)^{(i)} \cap \xi(x_2)^{(n-i+1)}$ , then  $a_m(t)$  is upper triangular if  $p_m = x_1$ 

and lower triangular if  $p_m = x_2$ . Let  $u_m(t) = u_{\xi,m}(t)$  be the unipotent projective transformation that fixes  $\xi(p_m)$  so that

$$a_m(t) = u_m(t)h_m(t)$$

for some projective transformation  $h_m(t) = h_{\xi,m}(t)$  that fixes both  $\xi(x_1)$  and  $\xi(x_2)$ .

**Proposition 5.12.** Let  $[\xi] \in \text{Hit}_V(S)$ , let  $\mu \in W$ , and let  $t \in I_{[\xi],\mu}$ , Also, let  $\xi$  be a representative of  $[\xi]$ .

(1) The infinite product

(5.4) 
$$u_{m,\infty}(t) := \lim_{d \to \infty} \left( u_m(t) h_m(t) \rho(\gamma_m) \right)^d \cdot \left( h_m(t) \rho(\gamma_m) \right)^{-d}$$

converges to a unipotent projective transformation that fixes  $\xi(p_m)$ .

(2) Choose the representative  $\xi_t$  of  $\phi_t^{\mu}[\xi]$  so that  $\xi_t(p_1) = \xi(p_1), \, \xi_t(w_1) = \xi(w_1)$ and  $\xi_t^{(1)}(z_1) = \xi^{(1)}(z_1)$ . Then the group element  $g_t \in PSL(V)$  so that  $g_t \cdot \xi(p_2) = \xi_t(p_2), \ g_t \cdot \xi(w_2) = \xi_t(w_2) \ and \ g_t \cdot \xi^{(1)}(z_2) = \xi_t^{(1)}(z_2) \ is \ given \ by$ the formula

$$g_t = u_{1,\infty}(t) \cdot a_{\xi,\{x_1,x_2\}}(t) \cdot u_{2,\infty}(t)^{-1},$$
where  $a_{\xi,\{x_1,x_2\}}(t) = \prod_{i=1}^{n-1} b_{\xi(x_1),\xi(x_2)}^{i,n-i} \left(2\mu_{x_1,x_2}^{i,n-i}t\right).$ 

5.2. Well-definedness of  $(\phi_c^{\mu})_t$ . Let  $c = [y_1, y_2] \in \mathcal{P}$  and choose an enumeration  $\Gamma \cdot \{y_1, y_2\} = \{o_1, o_2, \dots\}$ . Define

$$W_c := \left\{ \mu \in W : \begin{array}{l} \mu_{x_1, x_2, x_3}^{i_1, i_2, i_3} = 0 \text{ for all } [x_1, x_2, x_3] \in \Theta; i_1, i_2, i_3 \in \mathbb{Z}^+; i_1 + i_2 + i_3 = n \\ \mu_{x_1, x_2}^{i_1, i_2, i_3} = 0 \text{ for all } [x_1, x_2] \in \mathcal{T} \setminus \{c\} \text{ and } i = 1, \dots, n-1. \end{array} \right\}$$

It is easy to see from that  $W_c \subset W$  is a (n-1)-dimensional linear subspace. Furthermore, from the closed leaf inequalities, one easily verifies that  $\Omega[\xi] + t\mu$  lies in the image of  $\Omega$  for any  $t \in \mathbb{R}$ ,  $\mu \in W_c$ , and  $[\xi] \in \mathrm{Hit}_V(S)$ . In other words,  $I_{[\xi],\mu} = \mathbb{R}$  for all  $\mu \in W_c$ , or equivalently,  $\Omega^{-1}(\Omega[\xi] + t\mu) \in \text{Hit}_V(S)$  for all  $t \in \mathbb{R}$ .

Also, let  $\Pi_c: W \to W_c$  be the projection defined by

- $\Pi_c(\mu)_{x_1,x_2}^{i,n-i} = \mu_{x_1,x_2}^{i,n-i}$  if  $[x_1,x_2] = c$  and  $i = 1,\ldots,n-1,$   $\Pi_c(\mu)_{x_1,x_2}^{i,n-i} = 0$  for all  $[x_1,x_2] \in \mathcal{T} \setminus \{c\}$  and  $i = 1,\ldots,n-1,$
- $\Pi_c(\mu)_{\substack{i_1,i_2,i_3\\x_1,x_2,x_3}}^{i_1,i_2,i_3} = 0$  for all  $[x_1,x_2,x_3] \in \Theta$  and  $i_1,i_2,i_3 \in \mathbb{Z}^+$  so that  $i_1+i_2+i_3=0$

Clearly, 
$$(\phi_{o_l}^{\mu})_t = (\phi_{o_l}^{\Pi_c(\mu)})_t$$
 for all  $o_l \in \Gamma \cdot \{y_1, y_2\}.$ 

In the next proposition, we prove not only that  $(\phi_c^{\mu})_t$  is well-defined for all  $t \in \mathbb{R}$ ; we can in fact explicitly relate  $\Omega\left(\left(\phi_c^{\mu}\right)_t[\xi]\right)$  to  $\Omega[\xi]$ .

**Proposition 5.13.** Let  $\xi$  be a representative of  $[\xi] \in \text{Hit}_V(S)$ , let  $\mu \in W$ , let  $t \in \mathbb{R}$ , let  $c \in \mathcal{P}$ , and let

$$[\xi_0] := \Omega^{-1} \left( \Omega[\xi] + t \Pi_c(\mu) \right) \in \operatorname{Hit}_V(S).$$

Pick any triangle  $\{x_0, y_0, z_0\} \in \widetilde{\Theta}$ , and choose representatives  $\xi_i$  (resp.  $\xi_0$ ) of  $\prod_{l=1}^{J} (\phi_{o_l}^{\mu})_{t} [\xi] \ (resp. \ [\xi_0]) \ so \ that$ 

$$\xi_j(x_0) = \xi(x_0), \quad \xi_j(y_0) = \xi(y_0) \quad and \quad \xi_j^{(1)}(z_0) = \xi^{(1)}(z_0)$$

for all  $j \in \mathbb{Z}^+ \cup \{0\}$ . Then

$$\lim_{j\to\infty}\xi_j=\xi_0.$$

Remark 5.14. Proposition 5.13 implies in particular that  $(\phi_c^{\mu})_t$  does not depend on the enumeration of  $\Gamma \cdot \{y_1, y_2\}$ .

*Proof.* By Lemma 4.5, it is sufficient to prove that  $\xi_j$  converges to  $\xi_0$  on the vertices of  $\widetilde{\mathcal{T}}$ . By Proposition 3.9(3), we see that for any triangle  $(x_0, y_0, z_0)$  in  $\widetilde{\Theta}$  and any  $i_1, i_2, i_3 \in \mathbb{Z}^+$  with  $i_1 + i_2 + i_3 = n$ , we have

$$T_{i_1,i_2,i_3}(\xi_j(x_0),\xi_j(y_0),\xi_j(z_0)) = T_{i_1,i_2,i_3}(\xi_0(x_0),\xi_0(y_0),\xi_0(z_0)).$$

Proposition 2.10 then implies that  $\xi_j(z_0) = \xi_0(z_0)$  for all j.

Now, let  $w_0 \in \partial \Gamma$  be any vertex of  $\widetilde{\mathcal{T}}$  that is not  $x_0 \ y_0$  or  $z_0$ . We use the combinatorial description of pairs of distinct points in  $\widetilde{\mathcal{T}}$  developed in Section 4.3.

After possibly relabelling the vertices of  $\{x_0, y_0, z_0\}$ , we may assume without loss of generality that  $\{x_0, y_0\}$  is the minimal element of  $\mathcal{E}_{z_0, w_0}$ . Decompose

$$\mathcal{E}_{z_0,w_0} = \bigcup_{s=1}^k \mathcal{E}_{z_0,w_0,s} \cup \bigcup_{s=0}^k \mathcal{E}_{z_0,w_0,s,s+1} = \bigcup_{s=1}^k \mathcal{E}_s \cup \bigcup_{s=0}^k \mathcal{E}_{s,s+1}.$$

To prove the proposition, it is sufficient to prove the following statements (see proof of Proposition 4.3), where we use Notation 4.6.

- (1) Let s = 0, ..., k, and suppose that  $\mathcal{E}_{s,s+1}$  is non-empty. If  $\lim_{j \to \infty} \xi_j(p) = \xi_0(p)$  for all  $p = a_{s,s+1}, b_{s,s+1}, b'_{s,s+1}$ , then  $\lim_{j \to \infty} \xi_j(p) = \xi_0(p)$  for all  $p = c_{s,s+1}, d_{s,s+1}, d'_{s,s+1}$ .
- (2) Let s = 1, ..., k. If  $\lim_{j \to \infty} \xi_j(p) = \xi_0(p)$  for all  $p = a_s, b_s, b'_s$ , then  $\lim_{j \to \infty} \xi_j(p) = \xi_0(p)$  for all  $p = c_s, d_s, d'_s$ .

From the definition of  $\prod_{l=1}^{j} (\phi_{\alpha_{l}}^{\mu})_{t} [\xi]$ , it is clear that for all  $j \in \mathbb{Z}^{+}$ , the tuple

$$(\xi_j(a_{s,s+1}), \xi_j(b_{s,s+1}), \xi_j(b'_{s,s+1}), \xi_j(c_{s,s+1}), \xi_j(d_{s,s+1}), \xi_j(d'_{s,s+1}))$$

is projectively equivalent to

$$(\xi(a_{s,s+1}),\xi(b_{s,s+1}),\xi(b'_{s,s+1}),\xi(c_{s,s+1}),\xi(d_{s,s+1}),\xi(d'_{s,s+1})).$$

Furthermore, Proposition 2.10 implies that

$$(\xi_0(a_{s,s+1}), \xi_0(b_{s,s+1}), \xi_0(b'_{s,s+1}), \xi_0(c_{s,s+1}), \xi_0(d_{s,s+1}), \xi_0(d'_{s,s+1}))$$

is projectively equivalent to

$$(\xi(a_{s,s+1}), \xi(b_{s,s+1}), \xi(b'_{s,s+1}), \xi(c_{s,s+1}), \xi(d_{s,s+1}), \xi(d'_{s,s+1})).$$

This immediately implies (1).

To prove (2), orient the unique closed edge  $l_s \in \mathcal{E}_s$  so that  $z_0$  and  $w_0$  lie to the right and left of  $l_s$  respectively. Assume without loss of generality that  $y_1$  and  $y_2$  are respectively the backward and forward endpoints of  $l_s$  equipped with the orientation, and let  $\{f_1, \ldots, f_n\}$  be a basis of V so that  $\operatorname{Span}_{\mathbb{R}}(f_i) = \xi^{(i)}(y_1) \cap \xi^{(n-i+1)}(y_2)$ .

If  $l_s$  does not lie in  $\Gamma \cdot \{y_1, y_2\}$ , then for all  $j \in \mathbb{Z}^+$ ,

$$(\xi_j(a_s), \xi_j(b_s), \xi_j(b_s'), \xi_j(c_s), \xi_j(d_s), \xi_j(d_s'))$$

is projectively equivalent to

$$(\xi(a_s), \xi(b_s), \xi(b'_s), \xi(c_s), \xi(d_s), \xi(d'_s))$$

by same argument as the one given in for the proof of (1).

On the other hand, if  $l_s \in \Gamma \cdot \{y_1, y_2\}$ , then Lemma 5.1(2) implies that for sufficiently large  $j \in \mathbb{Z}^+$ ,

$$(\xi_j(a_s), \xi_j(b_s), \xi_j(b_s'), \xi_j(c_s), \xi_j(d_s), \xi_j(d_s'))$$

is projectively equivalent to

$$(\xi(a_s), \xi(b_s), \xi(b_s'), a_{\xi,l_s}(t) \cdot \xi(c_s), a_{\xi,l_s}(t) \cdot \xi(d_s), a_{\xi,l_s}(t) \cdot \xi(d_s')),$$

where  $a_{\xi,l_s}(t) := \prod_{i=1}^{n-1} b_{\xi(y_1),\xi(y_2)}^{i,n-i} \left(2\mu_{y_1,y_2}^{i,n-i}t\right)$ . Also,  $a_{\xi,l_s}(t)$  is represented in the basis  $\{f_1,\ldots,f_n\}$  by the diagonal matrix

$$\begin{bmatrix}
\lambda_1 & \dots & 0 \\
\vdots & \ddots & \vdots \\
0 & \dots & \lambda_n
\end{bmatrix}$$

where  $\frac{\lambda_i}{\lambda_{i+1}} = \exp\left(\mu_{y_1,y_2}^{i,n-i} \cdot t\right)$  for all  $k = 1, \dots, n-1$ .

Let  $J = \{T_1, T_2\} \in \widetilde{\mathcal{J}}$  be a bridge across  $\{y_1, y_2\}$ , so that  $T_1$  and  $T_2$  lie to the right and left of  $(y_1, y_2)$  respectively. For m = 1, 2, let  $\mathcal{V}_{\{y_1, y_2\}, m}$  be the set of vertices of triangles in

$$\bigcup_{d=-\infty}^{\infty} \gamma_m^d \cdot \widetilde{\Theta}(J, T_m).$$

Proposition 2.10 implies that there is a unique projective transformation  $g_m \in \operatorname{PGL}(V)$  so that  $\xi_0(p) = g_m \cdot \xi(p)$  for every vertex  $p \in \mathcal{V}_{\{y_1,y_2\},m}$ . Since  $y_1$  and  $y_2$  are the accumulation points of  $\mathcal{V}_{\{y_1,y_2\},m}$ , the continuity of  $\xi$  and  $\xi_0$  implies that

$$\xi_0(y_1) = g_m \cdot \xi(y_1)$$
 and  $\xi_0(y_2) = g_m \cdot \xi(y_2)$ 

for both m=1,2. In particular,  $g_1^{-1}g_2\in \mathrm{PSL}(V)$  is represented in the basis  $\{f_1,\ldots,f_n\}$  by a diagonal matrix, and

$$(\xi_0(a_s), \xi_0(b_s), \xi_0(b_s'), \xi_0(c_s), \xi_0(d_s), \xi_0(d_s'))$$

is projectively equivalent to

$$(\xi(a_s), \xi(b_s), \xi(b_s'), g_1^{-1}g_2 \cdot \xi(c_s), g_1^{-1}g_2 \cdot \xi(d_s), g_1^{-1}g_2 \cdot \xi(d_s')).$$

Thus, to finish the proof of (2), it is sufficient to show that  $g_1^{-1}g_2 = a_{\xi,l_s}(t)$ . By the same arguments as before, we see that for sufficiently large j,

$$(\xi_j(z_{1,0}), \xi_j(z_{1,1}), \xi_j(p_1), \xi_j(z_{2,0}), \xi_j(z_{2,1}), \xi_j(p_2))$$

is projectively equivalent to

$$(\xi(z_{1,0}), \xi(z_{1,1}), \xi(p_1), a_{\xi,l_s}(t) \cdot \xi(z_{2,0}), a_{\xi,l_s}(t) \cdot \xi(z_{2,1}), a_{\xi,l_s}(t) \cdot \xi(p_2)),$$

and

$$(\xi_0(z_{1,0}), \xi_0(z_{1,1}), \xi_0(p_1), \xi_0(z_{2,0}), \xi_0(z_{2,1}), \xi_0(p_2))$$

is projectively equivalent to

$$(\xi(z_{1,0}), \xi(z_{1,1}), \xi(p_1), g_1^{-1}g_2 \cdot \xi(z_{2,0}), g_1^{-1}g_2 \cdot \xi(z_{2,1}), g_1^{-1}g_2 \cdot \xi(p_2)).$$

Here, we used Notation 4.7 again.

Lemma 4.15 implies that

$$\alpha_{y_1, y_2, J}^{i, n-i}[\xi_j] = \alpha_{y_1, y_2, J}^{i, n-i}[\xi_0]$$

for sufficiently large j and for all  $i=1,\ldots,n-1$ . Then by Lemma 4.14, we see that there is some unipotent projective transformation  $v\in \mathrm{PSL}(V)$  that fixes  $\xi(p_2)$ , sends  $g_1^{-1}g_2\cdot\xi(z_{2,0})$  to  $a_{\xi,l_s}(t)\cdot\xi(z_{2,0})$ , and sends  $g_1^{-1}g_2\cdot\xi^{(1)}(z_{1,0})$  to  $a_{\xi,l_s}(t)\cdot\xi^{(1)}(z_{1,0})$ . Since  $g_1^{-1}g_2$  and  $a_{\xi,l_s}(t)$  both fix  $\xi(p_2)$ , this allows us to deduce that  $vg_1^{-1}g_2=a_{\xi,l_s}(t)$  (see Remark 2.13). In the basis  $\{f_1,\ldots,f_n\},\ g_1^{-1}g_2$  and  $a_{\xi,l_s}(t)$  are both diagonal, and v is a unipotent matrix. Hence,  $g_1^{-1}g_2=a_{\xi,l_s}(t)$ .

 $Remark\ 5.15.$  The proof of (2) in the above argument also proves Proposition 5.12 in the case when

- $\mu^{i_1,i_2,i_3}_{y_1,y_2,y_3}=0$  for every  $\{y_1,y_2,y_3\}\in\widetilde{\Theta}$  that shares a vertex with the closed edge  $\{x_1,x_2\}$ ,
- $\mu_{y_1,y_2}^{i,n-i}=0$  for every  $\{y_1,y_2\}\in\widetilde{\Theta}$  that shares a vertex with with the closed edge  $\{x_1,x_2\}$ .

5.3. Behavior near the closed edges. We would like to apply a similar argument as the proof Proposition 5.13 to prove that  $\left(\phi_{\mathcal{Q},\Theta}^{\mu}\right)_t$  is well-defined. However, the argument is more delicate since now neighborhoods around closed edges are deformed by infinitely many elementary eruption and shearing flows.

We thus have to analyze more precisely how the semi-elementary flows behave near the closed edges. The main (technical) result is the following theorem.

**Theorem 5.16.** Let  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}; J = \{T_1, T_2\} \in \widetilde{\mathcal{J}}; p_1, p_2, q_1, q_2, z_{1,c}, z_{2,c} \in \partial \Gamma \text{ for } c = 0, \dots, C_m \text{ be as defined in Notation 4.7. Let } [\xi] \in \operatorname{Hit}_V(S) \text{ and } \xi \text{ be a representative of } [\xi]. \text{ Let } \xi_j \text{ be the representative of } \left(\phi^{\mu}_{M_j}\right)_t [\xi] \text{ so that } \xi_j(p_1) = \xi(p_1), \ \xi_j(z_{1,0}) = \xi(z_{1,0}) \text{ and } \xi_j^{(1)}(z_{1,1}) = \xi^{(1)}(z_{1,1}).$ 

(1) For m = 1, 2, there is some unipotent projective transformation  $u_{m,\infty}(t) = u_{\xi,m,\infty}(t) \in \mathrm{PSL}(V)$  that fixes  $\xi(p_m)$ , so that

$$\lim_{j \to \infty} \xi_j(x_1) = u_{1,\infty}(t) \cdot \xi(x_1),$$

$$\lim_{j \to \infty} \xi_j(x_2) = u_{1,\infty}(t) \cdot \xi(x_2),$$

$$\lim_{j \to \infty} \xi_j(z_{2,0}) = u_{1,\infty}(t) u_{2,\infty}(t)^{-1} \cdot \xi(z_{2,0}),$$

$$\lim_{j \to \infty} \xi_j^{(1)}(z_{2,1}) = u_{1,\infty}(t) u_{2,\infty}(t)^{-1} \cdot \xi^{(1)}(z_{2,1})$$

(2) 
$$\lim_{d \to \infty} \lim_{j \to \infty} \xi_j(\gamma_m^d \cdot z_{m,0}) = \lim_{j \to \infty} \xi_j(q_m).$$

Remark 5.17. In fact, we will see in the proof that the unipotent projective transformation  $u_{m,\infty}(t)$  in Theorem 5.16 is exactly the one defined by (5.4), so this in fact proves Proposition 5.12 in the case when  $\mu_{x_1,x_2}^{i,n-i}=0$ . Combining this with Remark 5.15 gives Proposition 5.12.

The proof of Theorem 5.16 is elementary but quite technical, and takes the rest of Section 5.3. If one takes Theorem 5.16 for granted, one may skip ahead to Section 5.4 without affecting the readability of the rest of this paper.

To prove Theorem 5.16, first observe from the definition of  $M_j$  that there is some  $D' \in \mathbb{Z}$  so that for sufficiently large j and for m = 1, 2,

• 
$$\gamma_m^d \cdot \left( \widetilde{\Theta}(J, T_m) \cup \widetilde{T}(J, T_m) \right) \subset M_j$$
 for all  $0 \leq d \leq j + D'$ , and

• 
$$\gamma_m^d \cdot \left(\widetilde{\Theta}(J, T_m) \cup \widetilde{T}(J, T_m)\right) \cap M_j = \emptyset$$
 for all  $d > j + D'$ .

For any sufficiently large j, let D := j + D', and enumerate the set

$$\bigcup_{d=0}^{D} \bigcup_{m=1}^{2} \gamma_{m}^{d} \cdot \left( \widetilde{\Theta}(J, T_{m}) \cup \widetilde{T}(J, T_{m}) \right) = \{o_{1}, o_{2}, o_{3}, \dots o_{E}\},\$$

where  $E = 2(D+1)(C_1 + C_2)$ , in the order

$$\begin{array}{c} T_{1,1},\ e_{1,1},\ \dots\ ,\ T_{1,C_{1}},\ e_{1,C_{1}},\\ T_{2,1},\ e_{2,1},\ \dots\ ,\ T_{2,C_{2}},\ e_{2,C_{2}},\\ T_{1,C_{1}+1},\ e_{1,C_{1}+1},\ \dots\ ,\ T_{1,2C_{1}},\ e_{1,2C_{1}},\\ T_{2,C_{2}+1},\ e_{2,C_{2}+1},\ \dots\ ,\ T_{2,2C_{2}},\ e_{2,2C_{2}},\\ &\vdots\\ T_{1,DC_{1}+1},\ e_{1,DC_{1}+1},\ \dots\ ,\ T_{1,(D+1)C_{1}},\ e_{1,(D+1)C_{1}},\\ T_{2,DC_{2}+1},\ e_{2,DC_{2}+1},\ \dots\ ,\ T_{2,(D+1)C_{2}},\ e_{2,(D+1)C_{2}}.\\ \end{array}$$

Let  $\bar{\xi}_j$  be any representative of  $\left(\phi_{M_j}^\mu\right)_t[\xi]$ , and  $\bar{\xi}$  be the representative of  $\prod_{c=1}^E\left(\phi_{o_c}^\mu\right)_t[\xi]$  (recall that  $\prod_{c=1}^E\left(\phi_{o_c}^\mu\right)_t$  is a commuting product) so that  $\bar{\xi}(p_1)=\bar{\xi}_j(p_1)$ ,  $\bar{\xi}(z_{1,0})=\bar{\xi}_j(z_{1,0})$ , and  $\bar{\xi}^{(1)}(z_{1,1})=\bar{\xi}_j^{(1)}(z_{1,1})$ . It is clear from the definition of  $M_j$  that for sufficiently large  $j,\ m=1,2$  and  $c=1,\ldots,E$ , we have

$$\bar{\xi}_i(x_m) = \bar{\xi}(x_m)$$
 and  $\bar{\xi}_i(z_{m,c}) = \bar{\xi}(z_{m,c})$ .

In particular, it is sufficient to prove the statement of Theorem 5.16 with

$$\left(\phi_{M_j}^{\mu}\right)_t$$
 replaced with  $\prod_{c=1}^{E}\left(\phi_{o_c}^{\mu}\right)_t$ .

To do so, we will use the following notation.

Notation 5.18. For  $c = 0, \ldots, E$ , let  $\xi_{m,c}$  be the representative of

$$\prod_{r=1}^{c} \left( \phi_{o_r}^{\mu} \right)_t [\xi]$$

so that  $\xi_{m,c}(p_m) = \xi(p_m)$ ,  $\xi_{m,c}(z_{m,0}) = \xi(z_{m,0})$ , and  $\xi_{m,c}^{(1)}(z_{m,1}) = \xi^{(1)}(z_{m,1})$ .

As a preliminary lemma, we prove the following.

**Lemma 5.19.** Let  $J \in \widetilde{\mathcal{J}}$  be a bridge across  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$ . For m = 1, 2 and  $c \in \mathbb{Z}$ , let  $\gamma_m \in \Gamma$  and  $p_m, z_{m,c} \in \partial \Gamma$  be as defined in Notation 4.7. Also, let  $\xi : \partial \Gamma \to \mathcal{F}(V)$  be any Frenet curve, and let  $\{f_1, \ldots, f_n\}$  be the basis of V so that  $\operatorname{Span}_{\mathbb{R}}\{f_i\} = \xi(x_1)^{(i)} \cap \xi(x_2)^{(n-i+1)}$  for all  $i = 1, \ldots, n$ . Then the projective transformation  $a_m(t) \in \operatorname{PSL}(V)$  defined by (5.3) satisfies the following properties.

(1) 
$$a_m(t)$$
 fixes the flag  $\xi(p_m) \in \mathcal{F}(V)$ .

(2) If the diagonal entries of the matrix representative of  $a_m(t)$  in the basis  $\{f_1,\ldots,f_n\}$  are  $\lambda_1,\ldots,\lambda_n$  down the diagonal, then

$$\frac{\lambda_k}{\lambda_{k+1}} = \begin{cases} \exp\left(\sum_{c=1}^{C_1} \left(\mu_{p_1,z_1,c}^{k,n-k} + \sum_{i_2+i_3=n-k} \mu_{p_1,z_1,c,z_1,c-1}^{k,i_2,i_3}\right) t\right) & if \ p_1 = x_1 \\ \exp\left(-\sum_{c=1}^{C_1} \left(\mu_{p_1,z_1,c}^{n-k,k} + \sum_{i_2+i_3=k} \mu_{p_1,z_1,c-1,z_1,c}^{n-k,i_2,i_3}\right) t\right) & if \ p_1 = x_2 \\ \exp\left(-\sum_{c=1}^{C_2} \left(\mu_{p_2,z_2,c}^{k,n-k} + \sum_{i_2+i_3=n-k} \mu_{p_2,z_2,c-1,z_2,c}^{k,i_2,i_3}\right) t\right) & if \ p_2 = x_1 \\ \exp\left(\sum_{c=1}^{C_2} \left(\mu_{p_2,z_2,c}^{n-k,k} + \sum_{i_2+i_3=k} \mu_{p_2,z_2,c,z_2,c-1}^{n-k,i_2,i_3}\right) t\right) & if \ p_2 = x_2 \end{cases}$$

for all k = 1, ..., n - 1.

(3)  $\xi_{m,C_1+C_2}(z) = a_m(t) \cdot \xi(z)$  for all

$$z \in \bigcup_{d=1}^{\infty} \bigcup_{m=1}^{2} \gamma_{m}^{d} \cdot \{z_{m,1}, \dots, z_{m,C_{m}}\} \cup \{x_{1}, x_{2}\}.$$

(4) 
$$\xi_{m,C_1+C_2}(z_{3-m,0}) = a_m(t)a_{3-m}(t)^{-1} \cdot \xi(z_{3-m,0})$$

(4) 
$$\xi_{m,C_1+C_2}(z_{3-m,0}) = a_m(t)a_{3-m}(t)^{-1} \cdot \xi(z_{3-m,0})$$
  
(5)  $\xi_{m,C_1+C_2}^{(1)}(z_{3-m,1}) = a_m(t)a_{3-m}(t)^{-1} \cdot \xi^{(1)}(z_{3-m,1})$ 

Remark 5.20. In Lemma 5.19(2),  $\frac{\lambda_k}{\lambda_{k+1}}$  is always less than 1, and is the infinitesimal descrease in  $\ell^i_{\rho}(\gamma_m)$ .

*Proof.* (1) and (2) are immediate consequences of Lemma 5.1 and a straightforward computation. We will prove (3), (4), (5) in the case when m=1; the case when m=2 is identical.

By Lemma 5.1(1), we see that

$$\xi_{1,1}(p_1) = \xi(p_1), \quad \xi_{1,1}(z_{1,0}) = \xi(z_{1,0}) \quad \text{and} \quad \xi_{1,1}(z) = a_{\xi,T_{1,1}}(t) \cdot \xi(z),$$

for all

$$z \in \bigcup_{d=0}^{\infty} \gamma_1^d \cdot \{z_{1,1}, \dots, z_{1,C_1}\} \cup \bigcup_{d=-\infty}^{\infty} \gamma_2^d \cdot \{z_{2,1}, \dots, z_{2,C_2}\} \cup \{x_1, x_2\},$$

where  $a_{\xi,T_{1,1}}(t) \in PSL(V)$  is defined by (5.1). Lemma 5.1(1) also tells us that  $a_{\xi,T_{1,1}}(t)$  fixes  $\xi(p_1)$  and  $\xi^{(1)}(z_{1,1})$ .

By definition,  $\xi_{1,2}(p_1) = \xi_{1,1}(p_1)$ , and  $\xi_{1,2}(z_{1,k}) = \xi_{1,1}(z_{1,k})$  for k = 0, 1. Then by Lemma 5.1(2), for all

$$z \in \{z_{1,2}, \dots, z_{1,C_1}\} \cup \bigcup_{d=1}^{\infty} \gamma_1^d \cdot \{z_{1,1}, \dots, z_{1,C_1}\} \cup \bigcup_{d=-\infty}^{\infty} \gamma_2^d \cdot \{z_{2,1}, \dots, z_{2,C_2}\} \cup \{x_1, x_2\},$$

we have  $\xi_{1,2}(z) = a_{\xi_{1,1},e_{1,1}}(t) \cdot \xi_{1,1}(z)$ , where  $a_{\xi_{1,1},e_{1,1}}(t) \in \mathrm{PSL}(V)$  is defined by (5.2). Lemma 5.1(2) also states that  $a_{\xi_{1,1},e_{1,1}}(t)$  fixes  $\xi_{1,1}(p_1)$  and  $\xi_{1,1}(z_{1,1})$ . It then follows that

$$\xi_{1,2}(z) = a_{\xi_{1,1},e_{1,1}}(t)a_{\xi,T_{1,1}}(t) \cdot \xi(z) = a_{\xi,T_{1,1}}(t)a_{\xi,e_{1,1}}(t) \cdot \xi(z),$$

for al

$$z \in \{z_{1,2}, \dots, z_{1,C_1}\} \cup \bigcup_{d=1}^{\infty} \gamma_1^d \cdot \{z_{1,1}, \dots, z_{1,C_1}\} \cup \bigcup_{d=-\infty}^{\infty} \gamma_2^d \cdot \{z_{2,1}, \dots, z_{2,C_2}\} \cup \{x_1, x_2\}.$$

Iterating the above procedure  $C_1$  times proves that  $a_1(t)$  satisfy the following properties:

•  $\xi_{1,C_1}(z) = a_1(t) \cdot \xi(z)$  for all

$$z \in \bigcup_{d=1}^{\infty} \gamma_1^d \cdot \{z_{1,1}, \dots, z_{1,C_1}\} \cup \bigcup_{d=-\infty}^{\infty} \gamma_2^d \cdot \{z_{2,1}, \dots, z_{2,C_2}\} \cup \{x_1, x_2\},$$

- $\xi_{1,c}(z) = \xi_{1,C_1}(z)$  for all  $c \ge C_1$  and  $z \in \{p_1, z_{1,0}, z_{1,1}, \dots, z_{1,C_1} = \gamma_1 \cdot z_{1,0}\},$
- $\xi_{1,c}^{(1)}(\gamma_1 \cdot z_{1,1}) = \xi_{1,c}^{(1)}(\gamma_1 \cdot z_{1,1})$  for all  $c \geq C_1$ .

Now, consider the Frenet curve  $\xi' := a_1(t)^{-1} \cdot \xi_{1,C_1}$ . From the above properties of  $\xi_{1,C_1}$ , we see that  $\xi'(z) = \xi(z)$  for all

$$z \in \bigcup_{d=1}^{\infty} \gamma_1^d \cdot \{z_{1,1}, \dots, z_{1,C_1}\} \cup \bigcup_{d=-\infty}^{\infty} \gamma_2^d \cdot \{z_{2,1}, \dots, z_{2,C_2}\} \cup \{x_1, x_2\}.$$

We can thus repeat the above argument for the case when m=2 (with  $\xi'$  in place of  $\xi$ ) to prove the following:

•  $\xi_{2,C_1+C_2}(z) = a_2(t) \cdot \xi'(z)$  for all

$$z \in \bigcup_{d=1}^{\infty} \gamma_2^d \cdot \{z_{2,1}, \dots, z_{2,C_2}\} \cup \bigcup_{d=-\infty}^{\infty} \gamma_1^d \cdot \{z_{1,1}, \dots, z_{1,C_1}\} \cup \{x_1, x_2\},$$

- $\xi_{2,c}(z) = \xi_{2,C_1+C_2}(z)$  for all  $c \ge C_1 + C_2$  and  $z \in \{p_2, z_{2,0}, z_{2,1}, \dots, z_{2,C_2} = \gamma_2 \cdot z_{2,0}\}$ ,
- $\xi_{2,c}^{(1)}(\gamma_2 \cdot z_{2,1}) = \xi_{2,C_1+C_2}^{(1)}(\gamma_2 \cdot z_{2,1})$  for all  $c \ge C_1 + C_2$ .

Observe that  $a_1(t)a_2(t)^{-1} \cdot \xi_{2,C_1+C_2} = \xi_{1,C_1+C_2}$ , because for  $z \in \{p_1, z_{1,0}\}$ ,

$$a_1(t)a_2(t)^{-1} \cdot \xi_{2,C_1+C_2}(z) = a_1(t) \cdot \xi'(z) = \xi_{1,C_1}(z) = \xi(z)$$

and  $a_1(t)a_2(t)^{-1} \cdot \xi_{2,C_1+C_2}^{(1)}(z_{1,1}) = \xi^{(1)}(z_{1,1})$ . As a consequence, we see that for all

$$z \in \bigcup_{d=1}^{\infty} \bigcup_{m=1}^{2} \gamma_{1}^{d} \cdot \{z_{m,1}, \dots, z_{m,C_{1}}\} \cup \{x_{1}, x_{2}\},$$

we have  $\xi_{1,C_1+C_2}(z) = \xi_{1,C_1}(z) = a_1(t) \cdot \xi(z)$ , so (3) holds. Similarly,

$$\xi_{1,C_1+C_2}(z_{2,0}) = a_1(t)a_2(t)^{-1}a_1(t)^{-1} \cdot \xi_{1,C_1}(z_{2,0}) = a_1(t)a_2(t)^{-1} \cdot \xi(z_{2,0}),$$

so (4) holds. The same computation proves (5).

We can further strengthen Lemma 5.19 to the following.

**Lemma 5.21.** Let  $J \in \widetilde{\mathcal{J}}$  be a bridge across  $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$ . For m = 1, 2 and  $c \in \mathbb{Z}$ , let  $\gamma_m \in \Gamma$  and  $p_m, z_{m,c} \in \partial \Gamma$  be as defined in Notation 4.7. Let  $\xi : \partial \Gamma \to \mathcal{F}(V)$  be a  $\rho$ -equivariant Frenet curve for some representation  $\rho : \Gamma \to \mathrm{PSL}(V)$  and let  $a_m(t) \in \mathrm{PSL}(V)$  be the projective transformation defined by (5.3). For all  $d \in \mathbb{Z}^+ \cup \{0\}$ , set

$$a'_{m,d}(t) := \rho(\gamma_m)^d a_m(t) \rho(\gamma_m)^{-d}$$
 and  $a_{m,d}(t) := a'_{m,0}(t) a'_{m,1}(t) \dots a'_{m,d}(t)$ .

Then the following statements hold:

(1) 
$$\xi_{m,E}(\gamma_m^d \cdot z_{m,0}) = a_{m,d-1}(t) \cdot \xi(\gamma_m^d \cdot z_{m,0})$$
 and  $\xi_{m,E}^{(1)}(\gamma_m^d \cdot z_{m,1}) = a_{m,d-1}(t) \cdot \xi^{(1)}(\gamma_m^d \cdot z_{m,1})$  for all  $d = 1, \ldots, D$ .

(2) 
$$\xi_{m,E}(z) = a_{m,D}(t) \cdot \xi(z)$$
 for all

$$z \in \bigcup_{d=D+1}^{\infty} \bigcup_{m=1}^{2} \gamma_{m}^{d} \cdot \{z_{m,1}, \dots, z_{m,k_{m}}\} \cup \{x_{1}, x_{2}\}.$$

(3) 
$$\xi_{m,E}(z_{3-m,0}) = a_{m,D}(t)a_{3-m,D}(t)^{-1} \cdot \xi(z_{3-m,0})$$
 for all  $d = 1, \dots, D$ 

(3) 
$$\xi_{m,E}(z_{3-m,0}) = a_{m,D}(t)a_{3-m,D}(t)^{-1} \cdot \xi(z_{3-m,0})$$
 for all  $d = 1, \dots, D$ .  
(4)  $\xi_{m,E}^{(1)}(z_{3-m,1}) = a_{m,D}(t)a_{3-m,D}(t)^{-1} \cdot \xi^{(1)}(z_{3-m,1})$  for all  $d = 1, \dots, D$ .

*Proof.* In the proof of Lemma 5.19, we stopped the iterative procedure after  $C_m$ iterations. To prove this lemma, apply the iterative procedure up to  $(D+1)C_m$ times, and use the observation that

$$\rho(\gamma_m)^d a_m(t) \rho(\gamma_m)^{-d} = a_{\xi, T_{m, dC_m + 1}}(t) a_{\xi, e_{m, dC_m + 1}}(t) \dots a_{\xi, T_{m, (d+1)C_m}}(t) a_{\xi, e_{m, (d+1)C_m}}(t)$$
 for all  $m = 1, 2$  and  $d = 0, \dots, D$ .

To prove Theorem 5.16, we also need the following technical lemma, which can be interpreted as the fact that the unipotent group grows polynomially.

**Lemma 5.22.** Let X be a diagonal matrix whose diagonal entries are positive and strictly decreasing down the diagonal, and U be a unipotent upper triangular matrix,

$$X = \begin{pmatrix} d_1 & 0 & \dots & 0 & 0 \\ 0 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1} & 0 \\ 0 & 0 & \dots & 0 & d_n \end{pmatrix}, \quad U = \begin{pmatrix} 1 & u_{1,2} & \dots & u_{1,n-1} & u_{1,n} \\ 0 & 1 & \dots & u_{2,n-1} & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & u_{n-1,n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

where  $0 < d_1 < \cdots < d_n$ ,  $u_{i,j} \in \mathbb{R}$  for all i < j. Then the sequence

$$\{V_m := (UX)^m X^{-m}\}_{m=1}^{\infty}$$

 $converges\ to\ a\ unipotent\ upper-triangular\ matrix.$ 

*Proof.* First, by induction on m, one can prove that  $V_m = (V_{i,j,m})_{n \times n}$  is a unipotent upper triangular matrix given by

$$V_{i,j,m} = \sum_{k=1}^{j-i} \sum_{i=t_0 < \dots < t_k = j} \left( u_{t_0,t_1} u_{t_1,t_2} \dots u_{t_{k-1},t_k} \right)$$

$$\sum_{k \le l_1 + \dots + l_k \le m} \left( \frac{d_{t_0}}{d_{t_k}} \right)^{l_1 - 1} \left( \frac{d_{t_1}}{d_{t_k}} \right)^{l_2} \dots \left( \frac{d_{t_{k-1}}}{d_{t_k}} \right)^{l_k}$$

for all j > i. Here, the second summation is over positive integers  $t_1, \dots, t_{k-1}$  such that  $i = t_0 < t_1 < \cdots < t_{k-1} < t_k = j$ , and the last summation is over positive integers  $l_1, \dots, l_k$  such that  $k \leq l_1 + \dots + l_k \leq m$ .

Since  $0 < d_1 < \cdots < d_n$ , it is easy to see that if  $i = t_0 < t_1 < \cdots < t_k = j$ , then

$$\lim_{m \to \infty} \left( \sum_{k \le l_1 + \dots + l_k \le m} \left( \frac{d_{t_0}}{d_{t_k}} \right)^{l_1 - 1} \left( \frac{d_{t_1}}{d_{t_k}} \right)^{l_2} \cdots \left( \frac{d_{t_{k-1}}}{d_{t_k}} \right)^{l_k} \right) = \prod_{s=1}^{k-1} \frac{\frac{d_{t_s}}{d_{t_k}}}{1 - \frac{d_{t_s}}{d_{t_k}}} \cdot \frac{1}{1 - \frac{d_{t_0}}{d_{t_k}}}.$$

Thus,

$$\lim_{m \to \infty} V_{i,j,m} = \sum_{k=0}^{j-i} \sum_{i=t_0 < t_1 < \dots < t_k = j} \left( \prod_{s=1}^{k-1} \frac{u_{t_s,t_{s+1}}\left(\frac{d_{t_s}}{d_{t_k}}\right)}{1 - \frac{d_{t_s}}{d_{t_k}}} \right) \cdot \frac{u_{t_0,t_1}}{1 - \frac{d_{t_0}}{d_{t_k}}}$$

for all j > i. It is obvious that  $V_m$  is upper triangular and unipotent for all m, so the same is true for  $\lim_{m\to\infty} V_m$ .

By making an appropriate change of basis, we see that Lemma 5.22 also holds if U is a unipotent lower triangular matrix and X is a diagonal matrix whose entries are decreasing down the diagonal. With this, we now prove Theorem 5.16.

Proof of Theorem 5.16. Let  $\{f_1,\ldots,f_n\}$  be the basis of V so that  $\operatorname{Span}_{\mathbb{R}}(f_i) = \xi(x_1)^{(i)} \cap \xi(x_2)^{(n-i+1)}$  for all  $i=1,\ldots,n$ .

(1) For m = 1, 2, let  $a_m(t) \in PSL(V)$  be the projective transformation defined by (5.3). Then we can write

$$a_m(t) = u_m(t)h_m(t)$$

where  $u_m(t)$  is a unipotent projective transformation that fixes  $\xi(p_m) = \xi_{m,C_m}(p_m)$  and  $h_m(t)$  is a projective transformation that fixes  $\xi(x_1)$  and  $\xi(x_2)$ . By Lemma 5.19(2),  $h_m(t)$  is explicitly represented in the basis  $\{f_1, \ldots, f_n\}$  by the unique determinant 1, diagonal matrix so that if  $\lambda_1, \ldots, \lambda_n$  are its diagonal entries going down the diagonal, then

$$\frac{\lambda_k}{\lambda_{k+1}} = \begin{cases} \exp\left(\sum_{c=1}^{C_1} \left(\mu_{p_1,z_1,c}^{k,n-k} + \sum_{i_2+i_3=n-k} \mu_{p_1,z_1,c,z_1,c-1}^{k,i_2,i_3}\right) t\right) & \text{if } p_1 = x_1 \\ \exp\left(-\sum_{c=1}^{C_1} \left(\mu_{p_1,z_1,c}^{n-k,k} + \sum_{i_2+i_3=k} \mu_{p_1,z_1,c-1,z_1,c}^{n-k,i_2,i_3}\right) t\right) & \text{if } p_1 = x_2 \\ \exp\left(-\sum_{c=1}^{C_2} \left(\mu_{p_2,z_2,c}^{k,n-k} + \sum_{i_2+i_3=n-k} \mu_{p_2,z_2,c-1,z_2,c}^{k,i_2,i_3}\right) t\right) & \text{if } p_2 = x_1 \\ \exp\left(\sum_{c=1}^{C_2} \left(\mu_{p_2,z_2,c}^{n-k,k} + \sum_{i_2+i_3=k} \mu_{p_2,z_2,c,z_2,c-1}^{n-k,i_2,i_3}\right) t\right) & \text{if } p_2 = x_2 \end{cases}$$

for all  $k=1,\ldots,n-1$ . Observe also that  $h_1(t)=h_2(t)$  because  $\mu\in W$  satisfies the closed leaf equalities described in Section 4.4.

Since  $t \in I_{[\xi],\mu}$ , we see from the description of the closed leaf inequalities in Section 4.4 that the matrix representative of  $\rho(\gamma_m)h_m(t)$  in the basis  $\{f_1,\ldots,f_n\}$  has diagonal entries that are increasing down the diagonal if  $p_m=x_1$  and decreasing down the diagonal if  $p_m=x_2$ . Also,  $h_m(t)$  commutes with  $\rho(\gamma_m)$  since they have the same attracting and repelling fixed flags. At the same time, the matrix representing  $u_m(t)$  in the basis  $\{f_1,\ldots,f_n\}$  is upper triangular when  $p_m=x_1$  and lower triangular when  $p_m=x_2$ . Thus, by Lemma 5.22, the limit

$$u_{m,\infty}(t) := \lim_{d \to \infty} \left( u_m(t) h_m(t) \rho(\gamma_m) \right)^d \left( h_m(t) \rho(\gamma_m) \right)^{-d}$$

exists, and is a unipotent projective transformation that fixes  $\xi(p_m)$ .

Recall that  $E = 2(D+1)(C_1 + C_2)$ . It is sufficient to show that

$$\lim_{D \to \infty} \xi_{1,E}(q_1) = u_{1,\infty}(t) \cdot \xi(q_1),$$

$$\lim_{D \to \infty} \xi_{1,E}(z_{2,0}) = u_{1,\infty}(t) u_{2,\infty}(t)^{-1} \cdot \xi(z_{2,0}),$$

$$\lim_{D \to \infty} \xi_{1,E}^{(1)}(z_{2,1}) = u_{1,\infty}(t) u_{2,\infty}(t)^{-1} \cdot \xi^{(1)}(z_{2,1})$$

By Lemma 5.21(2),

$$\lim_{D \to \infty} \xi_{1,E}(q_1) = \lim_{D \to \infty} a_{1,D}(t) \cdot \xi(q_1)$$

$$= \lim_{D \to \infty} \left( a_1(t)\rho(\gamma_1) \right)^{D+1} \rho(\gamma_1)^{-D-1} \cdot \xi(q_1)$$
(5.6)
$$= \lim_{D \to \infty} \left( u_1(t)h_1(t)\rho(\gamma_1) \right)^{D+1} \left( h_1(t)\rho(\gamma_1) \right)^{-D-1} \cdot \xi(q_1)$$

$$= u_{1,\infty}(t) \cdot \xi(q_1).$$

Since  $h_1(t) = h_2(t)$ , we can use Lemma 5.21(3),(4) to deduce

$$\begin{split} & \lim_{D \to \infty} \xi_{1,E}(z_{2,0}) \\ &= \lim_{D \to \infty} a_{1,D}(t) a_{2,D}^{-1} \cdot \xi(z_{2,0}) \\ &= \lim_{D \to \infty} \left( u_1(t) h_1(t) \rho(\gamma_1) \right)^{D+1} \rho(\gamma_1)^{-D-1} \rho(\gamma_2)^{D+1} \left( u_2(t) h_2(t) \rho(\gamma_2) \right)^{-D-1} \cdot \xi(z_{2,0}) \\ &= u_{1,\infty}(t) u_{2,\infty}(t)^{-1} \cdot \xi(z_{2,0}) \end{split}$$

Similarly,

$$\lim_{D \to \infty} \xi_{1,E}^{(1)}(z_{2,1}) = u_{1,\infty}(t)u_{2,\infty}(t)^{-1} \cdot \xi^{(1)}(z_{2,1}).$$

(2) Using Lemma 5.21(2), the same computation as (5.6) proves that

$$\lim_{D \to \infty} \xi_{m,E}(q_m) = u_{m,\infty}(t) \cdot \xi(q_m).$$

At the same time, Lemma 5.21(1) tells us that for any  $d \ge 1$ ,

$$\xi_{m,E}(\gamma_m^d \cdot z_{m,0})$$

$$= a_{m,d-1}(t) \cdot \xi(\gamma_m^d \cdot z_{m,0})$$

$$= (u_m(t)h_m(t)\rho(\gamma_m))^d \rho(\gamma_m)^{-d} \cdot \xi(\gamma_m^d \cdot z_{m,0})$$

$$= (u_m(t)h_m(t)\rho(\gamma_m))^d (h_m(t)\rho(\gamma_m))^{-d} (h_m(t)\rho(\gamma_m))^d \cdot \xi(z_{m,0})$$

for sufficiently large  $D \in \mathbb{Z}^+$ . Hence,

$$\lim_{d \to \infty} \lim_{D \to \infty} \xi_{m,E}(\gamma_m^d \cdot z_{m,0}) = u_{m,\infty}(t)\xi(q_m).$$

This proves (2).

5.4. Well-definedness of  $\left(\phi_{\mathcal{Q},\Theta}^{\mu}\right)_{t}$ . Using Theorem 5.16, we are now ready to prove that  $\left(\phi_{\mathcal{Q},\Theta}^{\mu}\right)_{t}$  defined in Section 5.1 is well-defined. Let

$$W_{\mathcal{Q},\Theta} := \{ \mu \in W : \mu_{x_1,x_2}^{i,n-i} = 0 \text{ for all } [x_1,x_2] \in \mathcal{P}, i = 1,\dots,n-1 \},$$

and let  $\Pi: W \to W_{\mathcal{Q},\Theta}$  be the projection so that

- $\Pi(\mu)_{x_1,x_2}^{i,n-i} = \mu_{x_1,x_2}^{i,n-i}$  for all  $[x_1,x_2] \in \mathcal{Q}$  and  $i = 1,\ldots,n-1$ ,  $\Pi(\mu)_{x_1,x_2}^{i,n-i} = 0$  for all  $[x_1,x_2] \in \mathcal{P}$  and  $i = 1,\ldots,n-1$ ,

•  $\Pi(\mu)_{x_1,x_2,x_3}^{i_1,i_2,i_3} = \mu_{x_1,x_2,x_3}^{i_1,i_2,i_3}$  for all  $[x_1,x_2,x_3] \in \Theta$  and  $i_1,i_2,i_3 \in \mathbb{Z}^+$  so that  $i_1+i_2+i_3=n$ .

Since the closed leaf equalities only involve the invariants associated to  $\mathcal{Q} \cup \Theta$ ,  $\Pi$  is indeed well-defined. Observe that  $\left(\phi_{\mathcal{Q},\Theta}^{\mu}\right)_t = \left(\phi_{\mathcal{Q},\Theta}^{\Pi(\mu)}\right)_t$  because  $\bigcup_{j=1}^{\infty} M_j = \widetilde{\mathcal{Q}} \cup \widetilde{\Theta}$ , and that  $I_{[\xi],\mu} = I_{[\xi],\Pi(\mu)}$ . The next proposition is the analog of Proposition 5.13, and relates the flow  $\left(\phi_{\mathcal{Q},\Theta}^{\mu}\right)_t$  with the parameterization  $\Omega$  of  $\mathrm{Hit}_V(S)$  given in Theorem 4.17. In particular, it implies that  $\left(\phi_{\mathcal{Q},\Theta}^{\mu}\right)_t$  is well-defined, and does not depend on the choice of  $p_0 \in \mathcal{D}$  (even though each  $\left(\phi_{M_j}^{\mu}\right)_t$  does).

**Proposition 5.23.** Let  $\xi$  be a representative of  $[\xi] \in \text{Hit}_V(S)$ , let  $\mu \in W$ , let  $t \in I_{[\xi],\mu}$ , and let

$$[\xi_0] := \Omega^{-1} \left( \Omega[\xi] + t \Pi(\mu) \right) \in \operatorname{Hit}_V(S).$$

Pick any triangle  $\{x_0, y_0, z_0\} \in \widetilde{\Theta}$ , and choose representatives  $\xi_j$  (resp.  $\xi_0$ ) of  $\left(\phi_{M_j}^{\mu}\right)_t [\xi]$  (resp.  $[\xi_0]$ ) so that

$$\xi_j(x_0) = \xi(x_0), \quad \xi_j(y_0) = \xi(y_0) \quad and \quad \xi_j^{(1)}(z_0) = \xi^{(1)}(z_0)$$

for all  $j \in \mathbb{Z}^+ \cup \{0\}$ . Then

$$\lim_{j \to \infty} \xi_j = \xi_0.$$

*Proof.* By Proposition 4.5, it is sufficient to prove that  $\xi_j$  converges to  $\xi_0$  on the vertices of  $\widetilde{\mathcal{T}}$ . Since

$$\lim_{j \to \infty} T_{i_1, i_2, i_3}(\xi_j(x_0), \xi_j(y_0), \xi_j(z_0)) = T_{i_1, i_2, i_3}(\xi_0(x_0), \xi_0(y_0), \xi_0(z_0))$$

for all  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ , Proposition 2.10, Proposition 3.3 and Proposition 3.9 imply that  $\lim_{j\to\infty} \xi_j(z_0) = \xi_0(z_0)$  for all j.

Let  $w_0 \in \partial \Gamma$  be any vertex of  $\mathcal{T}$ , and assume without loss of generality that  $\{x_0, y_0\}$  is the minimal element of  $\mathcal{E}_{z_0, w_0}$ . Decompose

$$\mathcal{E}_{z_0,w_0} = \bigcup_{s=1}^k \mathcal{E}_{z_0,w_0,s} \cup \bigcup_{s=0}^k \mathcal{E}_{z_0,w_0,s,s+1} = \bigcup_{s=1}^k \mathcal{E}_s \cup \bigcup_{s=0}^k \mathcal{E}_{s,s+1}.$$

Just as in the proof of Proposition 5.13, it is sufficient to prove the following (see Notation 4.6):

- (1) Let  $s=0,\ldots,k$ , and suppose that  $\mathcal{E}_{s,s+1}$  is non-empty. If  $\lim_{j\to\infty}\xi_j(p)=\xi_0(p)$  for all  $p=a_{s,s+1},b_{s,s+1},b'_{s,s+1}$ , then  $\lim_{j\to\infty}\xi_j(p)=\xi_0(p)$  for all  $p=c_{s,s+1},d_{s,s+1},d'_{s,s+1}$ .
- (2) Let s = 1, ..., k. If  $\lim_{j \to \infty} \xi_j(p) = \xi_0(p)$  for all  $p = a_s, b_s, b'_s$ , then  $\lim_{j \to \infty} \xi_j(p) = \xi_0(p)$  for all  $p = c_s, d_s, d'_s$ .

Recall that the number of vertices of the edges in  $\mathcal{E}_{s,s+1}$  is finite. We can thus apply Proposition 3.3, Proposition 3.9 and Proposition 2.10, to these vertices to deduce that the sequence

$$\left\{ (\xi_j(a_{s,s+1}), \xi_j(b_{s,s+1}), \xi_j(b_{s,s+1}'), \xi_j(c_{s,s+1}), \xi_j(d_{s,s+1}), \xi_j(d_{s,s+1}')) \right\}_{j=1}^{\infty}$$
 converges to

$$(\xi_0(a_{s,s+1}), \xi_0(b_{s,s+1}), \xi_0(b'_{s,s+1}), \xi_0(c_{s,s+1}), \xi_0(d_{s,s+1}), \xi_0(d'_{s,s+1}))$$

up to projective transformations. This immediately implies (1).

To prove (2), orient the unique closed edge  $l_s \in \mathcal{E}_s$  so that  $z_0$  and  $w_0$  lie to the right and left of  $l_s$  respectively. Assume without loss of generality that  $x_1$  and  $x_2$  are respectively the backward and forward endpoints of  $l_s$  equipped with its orientation, and let  $\{f_1,\ldots,f_n\}$  be a basis of V so that  $\operatorname{Span}_{\mathbb{R}}(f_i)=\xi^{(i)}(x_1)\cap\xi^{(n-i+1)}(x_2)$ . Also, let  $J=\{T_1,T_2\}$  be a bridge across  $l_s=\{x_1,x_2\}$ , and for m=1,2 and  $c\in\mathbb{Z}$ , let  $p_m, z_{m,c} \in \partial \Gamma$  be as defined in Notation 4.7. The same argument that we used for (1) also proves that

- if  $\lim_{j\to\infty} \xi_j(p) = \xi_0(p)$  for all  $p = a_s, b_s, b_s'$ , then  $\lim_{j\to\infty} \xi_j(z) = \xi_0(z)$  for all  $z = p_1, z_{1,0}, z_{1,1}$ .
- if  $\lim_{j\to\infty} \xi_j(z) = \xi_0(z)$  for all  $z = p_2, z_{2,0}, z_{2,1}$ , then  $\lim_{j\to\infty} \xi_j(p) = \xi_0(p)$ for all  $p = c_s, d_s, d'_s$ .

Thus, it is sufficient to prove that if  $\lim_{j\to\infty} \xi_j(z) = \xi_0(z)$  for all  $z=p_1,z_{1,0},z_{1,1}$ , then  $\lim_{j\to\infty} \xi_j(z) = \xi_0(z)$  for all  $z = p_2, z_{2,0}, z_{2,1}$ .

By Theorem 5.16(1), we see that if  $\hat{\xi}_j \in \left(\phi_{M_j}^{\mu}\right)_t[\xi]$  is the representative so that  $\hat{\xi}_{i}(p_{1}) = \xi(p_{1}), \, \hat{\xi}_{i}(z_{1,0}) = \xi(z_{1,0}), \, \text{and} \, \hat{\xi}_{i}^{(1)}(z_{1,1}) = \xi^{(1)}(z_{1,1}), \, \text{then}$ 

$$\lim_{j \to \infty} \hat{\xi}_j(q_1) = u_{1,\infty}(t) \cdot \xi(q_1),$$

$$\lim_{j \to \infty} \hat{\xi}_j(z_{2,0}) = u_{1,\infty}(t) u_{2,\infty}(t)^{-1} \cdot \xi(z_{2,0}),$$

$$\lim_{j \to \infty} \hat{\xi}_j^{(1)}(z_{2,1}) = u_{1,\infty}(t) u_{2,\infty}(t)^{-1} \cdot \xi^{(1)}(z_{2,1}).$$

for some unipotent  $u_{1,\infty}(t), u_{2,\infty}(t) \in \mathrm{PSL}(V)$  that fixes  $\xi(p_1), \xi(p_2)$  respectively. In particular, Lemma 4.14 implies that  $\lim_{j\to\infty} \alpha_{x_1,x_2}^{i,n-i}[\xi_j] = \alpha_{x_1,x_2}^{i,n-i}[\xi_0]$  for all  $i=1,2,\ldots,n$ 

For  $j = 1, ..., \infty$ , let  $g_j \in PSL(V)$  be the projective transformation that maps  $\hat{\xi}_{j}(p_{1})$  to  $\xi_{j}(p_{1})$ ,  $\hat{\xi}_{j}(z_{1,0})$  to  $\xi_{j}(z_{1,0})$  and  $\hat{\xi}_{j}^{(1)}(z_{1,1}) = \xi_{j}^{(1)}(z_{1,1})$ . Also, let  $g_{0} \in \text{PSL}(V)$  be the projective transformation that maps  $\xi(p_{1})$  to  $\xi_{0}(p_{1})$ ,  $\xi(z_{1,0})$  to  $\xi_0(z_{1,0})$  and  $\xi^{(1)}(z_{1,1}) = \xi_0^{(1)}(z_{1,1})$ . The assumption that  $\lim_{j\to\infty} \xi_j(z) = \xi_0(z)$  for all  $z = p_1, z_{1,0}, z_{1,1}$  implies that  $\lim_{j\to\infty} g_j = g_0$ . Furthermore, this assumption, together with Proposition 2.10, Proposition 3.3 and Proposition 3.9, also imply that  $\lim_{j\to\infty} \xi_j(\gamma_1^d \cdot z_{1,0}) = \xi_0(\gamma_1^d \cdot z_{1,0})$  for all  $d=1,\ldots,\infty$ . Hence, Theorem 5.16(2) implies that

$$\lim_{j \to \infty} \xi_j(q_1) = \lim_{d \to \infty} \lim_{j \to \infty} \xi_j(\gamma_1^d \cdot z_{1,0})$$
$$= \lim_{d \to \infty} \xi_0(\gamma_1^d \cdot z_{1,0})$$
$$= \xi_0(q_1),$$

so  $\lim_{j\to\infty} \xi_j(x_m) = \xi_0(x_m)$  for both m = 1, 2.

Since  $\lim_{j\to\infty} \alpha_{x_1,x_2}^{i,n-i}[\xi_j] = \alpha_{x_1,x_2}^{i,n-i}[\xi_0]$  for all  $i=1,\ldots,n-1$ , Lemma 4.13 implies that there is a unipotent projective transformation u that fixes  $\xi_0(p_2)$ , sends  $\lim_{j\to\infty} \xi_j(z_{2,0})$  to  $\xi_0(z_{2,0})$ , and sends  $\lim_{j\to\infty} \xi_j^{(1)}(z_{2,1})$  to  $\xi_0^{(1)}(z_{2,1})$ . Proposition 2.10, Proposition 3.3 and Proposition 3.9 together imply that

$$u \cdot \lim_{j \to \infty} \xi_j(\gamma_2^d \cdot z_{2,0}) = \xi_0(\gamma_2^d \cdot z_{2,0})$$

for all  $d \in \mathbb{Z}^+$ , so Theorem 5.16(2) implies that

$$u \cdot \lim_{j \to \infty} \xi_j(q_2) = u \cdot \lim_{d \to \infty} \lim_{j \to \infty} \xi_j(\gamma_2^d \cdot z_{2,0}) = \xi_0(q_2) = \lim_{j \to \infty} \xi_j(q_2).$$

As such, u fixes both  $\xi_0(p_2)$  and  $\xi_0(q_2)$ , so u = id. Thus,  $\lim_{j \to \infty} \xi_j(z_{2,0}) = \xi_0(z_{2,0})$ , and  $\lim_{j \to \infty} \xi_j^{(1)}(z_{2,1}) = \xi_0^{(1)}(z_{2,1})$ . We can then apply Proposition 2.10 to deduce that  $\lim_{j \to \infty} \xi_j(z_{2,1}) = \xi_0(z_{2,1})$ .

# 6. Pants decompositions, flows and Darboux coordinates

In this section we consider special pairs  $(\mathcal{T}, \mathcal{J})$  of an ideal triangulation and a compatible bridge system, which are subordinate to a pants decomposition. We associate n-1 twist flows to each simple closed curve in the pants decomposition and  $\frac{(n-1)(n-2)}{2}$  eruption flows to each pair of pants in the pants decomposition. Using results from [SZ17] and properties of these flows, we determine a global Darboux coordinate system on  $\mathrm{Hit}_V(S)$ .

6.1. Triangulations subordinate to a pants decomposition. We specify the type of ideal triangulation  $\mathcal{T}$  of S and compatible bridge system  $\mathcal{J}$  subordinate to a pants decomposition. Let  $\mathcal{P}$  be a pants decomposition of S, and let  $\mathbb{P}$  be the 2g-2 pairs of pants determined by  $\mathcal{P}$  (recall that g is the genus of S). For each  $P \in \mathbb{P}$ , choose peripheral group elements  $\alpha_P, \beta_P, \gamma_P \in \pi_1(P)$  so that  $\alpha_P\beta_P\gamma_P = \mathrm{id}$ , and P lies to the right of its boundary components, oriented according to  $\alpha_P, \beta_P$  and  $\gamma_P$ .

By choosing base points, the inclusion  $P \subset S$  induces an inclusion  $\pi_1(P) \subset \Gamma$ , so we can view  $\alpha_P, \beta_P, \gamma_P$  as group elements in  $\Gamma$ . Then define

$$\widetilde{\mathcal{T}} := \bigcup_{P \in \mathbb{P}} \Gamma \cdot \left\{ \{\alpha_P^-, \alpha_P^+\}, \{\beta_P^-, \beta_P^+\}, \{\gamma_P^-, \gamma_P^+\}, \{\alpha_P^-, \beta_P^-\}, \{\beta_P^-, \gamma_P^-\}, \{\gamma_P^-, \alpha_P^-\} \right\},$$

where  $\gamma^-, \gamma^+ \in \partial \Gamma$  denote the repelling and attracting fixed points of  $\gamma \in \Gamma$  respectively. It is easy to see that  $\tilde{\mathcal{T}}$  is an ideal triangulation, does not depend on the choice of base points, and is  $\Gamma$ -invariant. Define  $\mathcal{T} := \tilde{\mathcal{T}}/\Gamma$ , and note that the set of closed edges of  $\mathcal{T}$  is exactly  $\mathcal{P}$ . Also,  $\Theta$  can be described as

(6.1) 
$$\Theta = \bigcup_{P \in \mathbb{P}} \left\{ \widehat{T}_P := [\alpha_P^-, \beta_P^-, \gamma_P^-], \ \widehat{T}_P' := [\alpha_P^-, \gamma_P^-, \gamma_P \cdot \beta_P^-] \right\}.$$

Next, we fix the following bridge system  $\mathcal{J}$  compatible with  $\mathcal{T}$ . For any  $\hat{e} \in \mathcal{P}$ , let  $P_1$  and  $P_2$  be the pairs of pants (possibly  $P_1 = P_2$ ) that share some  $\hat{e}$  as a common boundary component. Choose a representative  $e \in \widetilde{\mathcal{P}}$  of  $\hat{e}$ . For l = 1, 2, choose a representative  $T_{P_l} \in \widetilde{\Theta}$  of  $\widehat{T}_{P_l}$  so that  $T_{P_l}$  and e share a vertex, and  $T_{P_1}, T_{P_2}$  lie on different sides of e. Let  $J_e := \{T_{P_1}, T_{P_2}\}$ , let

$$\widetilde{\mathcal{J}} := \bigcup_{\hat{e} \in \mathcal{P}} \Gamma \cdot J_e$$

and define  $\mathcal{J} := \widetilde{\mathcal{J}}/\Gamma$ .

For the rest of this article, we assume that  $(\mathcal{T}, \mathcal{J})$  is of the type described above.

6.2. Twist flows and eruption flows associated to a pants decomposition. Recall that we identify the tangent space to the Hitchin component with

$$W = W_{\tau} \subset \mathbb{R}^{(6g-6)(n-1)} \times \mathbb{R}^{(3g-3)(n-1)} \times \mathbb{R}^{(2g-2)(n-1)(n-2)}.$$

the linear subspace cut out by the (3g-3)(n-1) closed leaf equalities (see Notation 4.8). We denoted an arbitrary vector  $\mu \in W$  by

$$\mu = \left( \left( \mu_{x_1, x_2}^{i, n-i} \right)_{i \in [1, n-1]; [x_1, x_2] \in \mathcal{Q}}, \left( \mu_{x_1, x_2}^{i, n-i} \right)_{i \in [1, n-1]; [x_1, x_2] \in \mathcal{P}}, \right. \\ \left. \left( \mu_{x_1, x_2, x_3}^{i_1, i_2, i_3} \right)_{i_1, i_2, i_3 \in \mathbb{Z}^+; i_1 + i_2 + i_3 = n; [x_1, x_2, x_3] \in \Theta} \right),$$

where  $\mu_{x_1,x_2,x_3}^{i_1,i_2,i_3} = \mu_{x_2,x_3,x_1}^{i_2,i_3,i_1} = \mu_{x_3,x_1,x_2}^{i_3,i_1,i_2}$ ,  $\mu_{x_1,x_2}^{i,n-i} = \mu_{x_2,x_1}^{n-i,i}$ ,  $\mu_{x_1,x_2,x_3}^{i_1,i_2,i_3} = \mu_{\gamma x_1,\gamma x_2,\gamma x_3}^{i_1,i_2,i_3}$  and  $\mu_{x_1,x_2}^{i,n-i} = \mu_{\gamma x_1,\gamma x_2}^{i,n-i}$  for any  $\gamma \in \Gamma$ . For the rest of this section, we simplify our notation as follows.

Notation 6.1. Let  $\mathbb{T}_n = \{(i, j, k) \in \mathbb{Z}^3 \mid 0 \leq i, j, k < n-1, i+j+k=n\}$ , and denote

$$\mu_{x_1,x_2,x_3}^{i,0,n-i} := \mu_{x_1,x_3}^{i,n-i}, \ \ \mu_{x_1,x_2,x_3}^{i,n-i,0} := \mu_{x_1,x_2}^{i,n-i} \ \ \text{and} \ \ \mu_{x_1,x_2,x_3}^{0,i,n-i} := \mu_{x_2,x_3}^{i,n-i}$$

for any  $[x_1, x_2, x_3] \in \Theta$  and any i = 1, ..., n - 1. In this notation, an arbitrary vector  $\mu \in W$  can be rewritten as

$$\mu = \big( \left( \mu_{x_1, x_2, x_3}^{i_1, i_2, i_3} \right)_{(i_1, i_2, i_3) \in \mathbb{T}_n, [x_1, x_2, x_3] \in \Theta}, \left( \mu_{x_1, x_2}^{i, n-i} \right)_{i=1, \dots, n-1; [x_1, x_2] \in \mathcal{P}} \big).$$

Also, for any  $[x_1, x_2, x_3] \in \Theta$  and i = 1, ..., n - 1, we will also use the notation

$$\tau_{x_1,x_2,x_3}^{0,i,n-i} := \sigma_{x_2,x_3}^{i,n-i}, \ \tau_{x_1,x_2,x_3}^{i,0,n-i} = \sigma_{x_1,x_3}^{i,n-i} \ \text{ and } \ \tau_{x_1,x_2,x_3}^{i,n-i,0} = \sigma_{x_1,x_2}^{i,n-i}$$

where  $\sigma_{x,y}^{i,n-i}$  is the edge invariants defined in Section 4.2.1

By Theorem 5.8, the set of  $(\mathcal{T}, \mathcal{J})$ -parallel flows on  $\mathrm{Hit}_V(S)$  are in bijection with  $W_{\mathcal{T}}$  via  $d\Omega$ . Thus, we can specify a  $(\mathcal{T}, \mathcal{J})$ -parallel flow on  $\mathrm{Hit}_V(S)$  by a vector  $\mu \in W$ . Using this, we associate to any simple closed curve in  $\mathcal{P}$  two families of special  $(\mathcal{T}, \mathcal{J})$ -parallel flows, the n-1 twist flows and the n-1 length flows. To any pair of pants in  $\mathbb{P}$  we associate two families of special  $(\mathcal{T}, \mathcal{J})$ -parallel flows, the  $\frac{(n-1)(n-2)}{2}$  eruption flows and the  $\frac{(n-1)(n-2)}{2}$  hexagon flows.

**Definition 6.2.** For any  $[x_1, x_2] \in \mathcal{P}$ , let  $P_1$  and  $P_2$  be the pairs of pants that share  $[x_1, x_2]$  as a common boundary component (possibly  $P_1 = P_2$ ), so that  $P_1$  and  $P_2$  lie to the right and left respectively of  $[x_1, x_2]$  equipped with the orientation induced by  $(x_1, x_2)$ . Then for l = 1, 2, let  $[x_l, y_l, z_l] = \widehat{T}_{P_l}$  and  $[x'_l, y'_l, z'_l] = \widehat{T}'_{P_l}$  so that

- $\bullet \ x_l < y_l < z_l < x_l$
- $x_l = \gamma_{x_l} \cdot x_l'$ ,  $y_l = \gamma_{y_l} \cdot y_l'$  and  $z_l = \gamma_{z_l} \cdot z_l'$  for some  $\gamma_{x_l}, \gamma_{y_l}, \gamma_{z_l} \in \Gamma$ .

Also, let i = 1, ..., n - 1.

(1) The *i*-twist flow associated to  $(x_1, x_2)$  is  $\phi_t^{\mu}$ , where  $\mu \in W$  is the vector so that  $\mu_{a,b,c}^{p,q,r} = 0$  for all  $[a,b,c] \in \Theta$  and  $(p,q,r) \in \mathbb{T}^n$ , and

$$\mu_{a,b}^{p,n-p} = \begin{cases} \frac{1}{2} & p = i \text{ and } (a,b) = (x_1, x_2); \\ 0 & \text{otherwise;} \end{cases}$$

for all  $p=1,\ldots,n-1$  and  $[a,b]\in\mathcal{P}$ . Let  $\mathcal{S}^i_{x_1,x_2}(=\mathcal{S}^{n-i}_{x_2,x_1})$  denote the tangent vector field of this flow.

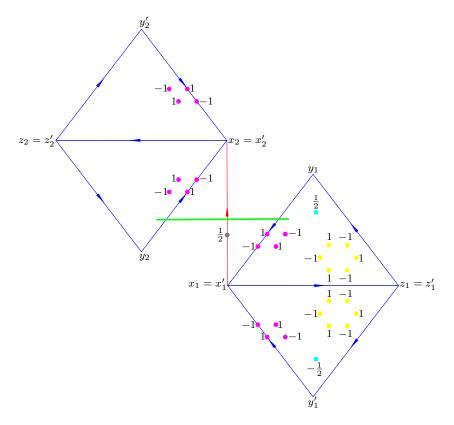


FIGURE 11. A closed edge of  $\widetilde{\mathcal{P}}$  is drawn in red, a bridge in  $\widetilde{\mathcal{J}}$  across the closed edge is drawn in green, and isolated edges are draw in blue. Each colored dot in an ideal triangle represents a triple of integers  $i,j,k\in\mathbb{Z}^+$  so that i+j+k=n. Similarly, each dot along an edge represents an integer  $i\in\{1,\ldots,n-1\}$ . The picture gives a diagramatic representation for the eruption, hexagon, lozenge and twist flows in turquoise, yellow, purple and grey respectively. The numbers above each of the colored dots are the corresponding coordinates of  $\mu\in W$ .

(2) The *i-length flow* associated to  $(x_1, x_2)$  is the flow whose tangent vector field  $\mathcal{Y}_{x_1, x_2}^i (= \mathcal{Y}_{x_2, x_1}^{n-i})$  is given by

$$\mathcal{Y}^i_{x_1,x_2} := \mathcal{Z}^i_{x_1,x_2} + \mathcal{E}^{i,n-i-1,1}_{x_1,y_1,z_1} - \mathcal{E}^{i-1,n-i,1}_{x_1',y_1',z_1'} + \mathcal{E}^{n-i,i-1,1}_{x_2,y_2,z_2} - \mathcal{E}^{n-i-1,i,1}_{x_2',y_2',z_2'},$$

where  $\mathcal{Z}^i_{x_1,x_2}$  is the tangent vector field to the *i-lozenge flow* associated to  $(x_1,x_2)$ . This is the  $(\mathcal{T},\mathcal{J})$ -parallel flow  $\phi^\mu_t$ , where  $\mu\in W$  is the vector so

that 
$$\mu_{a,b}^{p,n-p} = 0$$
 for all  $[a,b] \in \mathcal{P}$  and  $p = 1, \ldots, n-1$ , and

that 
$$\mu_{a,b}^{p,n-p} = 0$$
 for all  $[a,b] \in \mathcal{P}$  and  $p = 1, \dots, n-1$ , and 
$$\begin{cases} 1 & \text{if } (p,q,r) = (i,n-i,0) \text{ or } (i,n-i-1,1), \\ & \text{and } (a,b,c) = (x_1,y_1,z_1); \end{cases} \\ 1 & \text{if } (p,q,r) = (i,0,n-i) \text{ or } (i,1,n-i-1), \\ & \text{and } (a,b,c) = (x_1',z_1',y_1'); \end{cases} \\ 1 & \text{if } (p,q,r) = (n-i,i,0) \text{ or } (n-i,i-1,1), \\ & \text{and } (a,b,c) = (x_2,y_2,z_2); \end{cases} \\ 1 & \text{if } (p,q,r) = (n-i,0,i) \text{ or } (n-i,1,i-1), \\ & \text{and } (a,b,c) = (x_2',z_2',y_2'); \end{cases} \\ -1 & \text{if } (p,q,r) = (i+1,n-i-1,0) \text{ or } (i-1,n-i,1), \\ & \text{and } (a,b,c) = (x_1',z_1',y_1'); \end{cases} \\ -1 & \text{if } (p,q,r) = (i+1,0,n-i-1) \text{ or } (i-1,1,n-i), \\ & \text{and } (a,b,c) = (x_1',z_1',y_1'); \end{cases} \\ -1 & \text{if } (p,q,r) = (n-i+1,i-1,0) \text{ or } (n-i-1,i,1), \\ & \text{and } (a,b,c) = (x_2',y_2,z_2); \end{cases} \\ -1 & \text{if } (p,q,r) = (n-i+1,0,i-1) \text{ or } (n-i-1,1,i), \\ & \text{and } (a,b,c) = (x_2',z_2',y_2'); \end{cases}$$

for all  $[a,b,c] \in \Theta$  and  $(p,q,r) \in \mathbb{T}^n$ . Let  $\mathcal{Z}^i_{x_1,x_2}(=\mathcal{Z}^{n-i}_{x_2,x_1})$  denote the tangent vector field of this flow.

The twist flows are generalized twist flows in the sense of Goldman, and were previously known to be Hamiltonian flows by Goldman [Gol86, Section 1]. For all  $i=1,\ldots,n-1$ , the i-twist flow associated to  $(x_1,x_2)$  does not change (up to conjugation) the representation restricted to  $S\setminus[x_1,x_2]$  and hence preserve the holonomy along all pants curves in  $\mathcal{P}$ . The length flows were chosen to be symplectically dual to the twist flows. In order to write them down explicitly we use the calculation of the corresponding vector fields by Sun-Zhang [SZ17].

**Definition 6.3.** For every  $P \in \mathbb{P}$ , let  $[x, y, z] = \widehat{T}_P$ , and  $[x', y', z'] = \widehat{T}'_P$  (see (6.1)) so that

- $\bullet x < y < z < x$
- $x = \gamma_x \cdot x', y = \gamma_y \cdot y'$  and  $z = \gamma_z \cdot z'$  for some  $\gamma_x, \gamma_y, \gamma_z \in \Gamma$ .

Also, let  $i, j, k \in \mathbb{Z}^+$  so that i + j + k = n.

(1) The (i,j,k)-eruption flow associated to (x,y,z) is  $\phi^{\mu}_t$ , where  $\mu \in W$  is the vector so that  $\mu_{a,b}^{p,n-p} = 0$  for all  $[a,b] \in \mathcal{P}$  and  $p = 1, \ldots, n-1$ , and

$$\mu_{a,b,c}^{p,q,r} = \begin{cases} \frac{1}{2} & (p,q,r) = (i,j,k) \text{ and } (a,b,c) = (x,y,z); \\ -\frac{1}{2} & (p,q,r) = (i,k,j) \text{ and } (a,b,c) = (x',z',y'); \\ 0 & \text{otherwise;} \end{cases}$$

for all  $[a,b,c] \in \Theta$  and  $(p,q,r) \in \mathbb{T}^n$ . Let  $\mathcal{E}_{x,y,z}^{i,j,k} (= \mathcal{E}_{z,x,y}^{k,i,j} = \mathcal{E}_{y,z,x}^{j,k,i})$  denote the tangent vector field of this flow.

(2) The (i, j, k)-hexagon flow associated to (x, y, z) is  $\phi_t^{\mu}$ , where  $\mu \in W$  is the vector so that  $\mu_{a,b}^{p,n-p} = 0$  for all  $[a,b] \in \mathcal{P}$  and  $p = 1, \ldots, n-1$ , and

$$\mu_{a,b,c}^{p,q,r} = \begin{cases} 1 & \text{if } (p,q,r) = (i,j+1,k-1), \ (i-1,j,k+1) \\ \text{or } (i+1,j-1,k), \ \text{and } (a,b,c) = (x,y,z); \end{cases}$$

$$1 & \text{if } (p,q,r) = (i,k-1,j+1), \ (i-1,k+1,j) \\ \text{or } (i+1,k,j-1), \ \text{and } (a,b,c) = (x',z',y'); \end{cases}$$

$$-1 & \text{if } (p,q,r) = (i-1,j+1,k), \ (i,j-1,k+1) \\ \text{or } (i+1,j,k-1), \ \text{and } (a,b,c) = (x,y,z); \end{cases}$$

$$-1 & \text{if } (p,q,r) = (i-1,k,j+1), \ (i,k+1,j-1) \\ \text{or } (i+1,k-1,j), \ \text{and } (a,b,c) = (x',z',y'); \end{cases}$$

$$0 & \text{otherwise};$$

for all  $[a,b,c] \in \Theta$  and  $(p,q,r) \in \mathbb{T}^n$ . Let  $\mathcal{H}^{i,j,k}_{x,y,z} (= \mathcal{H}^{k,i,j}_{z,x,y} = \mathcal{H}^{j,k,i}_{y,z,x})$  denote the tangent vector field of this flow.

Note that the eruption flows associated to  $P \in \mathbb{P}$  do not change the representation restricted to  $S \backslash P$  (up to conjugation). In particular they preserve the holonomy along all pants curves in  $\mathcal{P}$ . In order to perform an eruption flow associated to P, we perform an elementary eruption flow increasing the triangle invariant on one of the ideal triangles in P and an elementary eruption flow decreasing the triangle invariant on the other ideal triangle in P by the same amount. This is necessary in order to preserve the holonomy along the boundary curves of the pair of pants. The hexagon flows are chosen to be symplectically dual to the eruption flows. To write them down explicitly we use the calculation of the corresponding vector fields by Sun-Zhang [SZ17].

When n=3, there is a unique eruption flow and hexagon flow for each  $P \in \mathbb{P}$ . These were described geometrically using convex  $\mathbb{RP}^2$  geometry in Wienhard-Zhang [WZ17]. There, the hexagon flow was called the internal bulging flow.

**Definition 6.4.** A  $(\mathcal{T}, \mathcal{J})$ -parallel flow is *special* if it is an eruption, hexagon, twist or length flow. The tangent vector fields of these flows are respectively called the *eruption*, *hexagon*, *twist and length vector fields*, and are collectively referred to the *special*  $(\mathcal{T}, \mathcal{J})$ -parallel vector fields.

6.3. Global Darboux coordinates. We compute now the Hamiltonian functions of the special  $(\mathcal{T}, \mathcal{J})$ -parallel flows in terms of the parameters of  $\Omega_{\mathcal{T}, \mathcal{J}}$ . For any  $(\mathcal{T}, \mathcal{J})$ -parallel vector field  $\mathcal{X}$ , denote its Hamiltonian function by  $H(\mathcal{X})$ . The main theorem of this section is the following.

**Theorem 6.5.** Let  $[x_1, x_2] \in \mathcal{P}$  and let i = 1, ..., n - 1.

(1) Let  $\alpha_{x_1,x_2}^{i,n-i}: \mathrm{Hit}_V(S) \to \mathbb{R}$  be the symplectic closed-edge invariant defined in Definition 4.12. Then

$$H(\mathcal{Y}_{x_1,x_2}^i) = -2\alpha_{x_1,x_2}^{i,n-i}$$

(2) Let  $\gamma \in \Gamma$  be the primitive group element with  $x_1$  and  $x_2$  as its repelling and attracting fixed points respectively, and let  $\ell^k_{[\gamma]} : \operatorname{Hit}_V(S) \to \mathbb{R}^+$  be the

function defined by  $\ell^k_{[\gamma]}[\rho] := \ell^k_{\rho}(\gamma)$  (see (4.1)). Then

$$H(\mathcal{S}_{x_1,x_2}^i) = \sum_{k=1}^i \frac{(i-n)k}{n} \cdot \ell_{[\gamma]}^k + \sum_{k=i+1}^{n-1} \frac{i(k-n)}{n} \cdot \ell_{[\gamma]}^k =: L_{[\gamma]}^i.$$

For any  $P \in \mathbb{P}$ , let  $[x, y, z] = \widehat{T}_P$  and  $[x', y', z'] = \widehat{T}'_P$  so that

- $\bullet \ x < y < z < x$
- $x = \gamma_x \cdot x'$ ,  $y = \gamma_y \cdot y'$  and  $z = \gamma_z \cdot z'$  for some  $\gamma_x, \gamma_y, \gamma_z \in \Gamma$ .

Also, let  $(i, j, k) \in \mathbb{Z}^+$  so that i + j + k = n.

(3) Let  $x_0, y_0, z_0 \in \partial \Gamma$  be the points so that  $\{x, x_0\}, \{y, y_0\}, \{z, z_0\} \in \widetilde{\mathcal{P}}$ , and for all  $m \in \mathbb{Z}^+$ , let  $\delta_{m,1}$  be 1 if m = 1 and 0 otherwise. Then

$$\begin{array}{lcl} H(\mathcal{H}_{x,y,z}^{i,j,k}) & = & \tau_{x,y,z}^{i,j,k} - \tau_{x',z',y'}^{i,k,j} + \delta_{k,1} \big( H(\mathcal{S}_{x,x_0}^{i-1}) - H(\mathcal{S}_{x,x_0}^{i}) \big) \\ & & + \delta_{i,1} \big( H(\mathcal{S}_{y,y_0}^{j-1}) - H(\mathcal{S}_{y,y_0}^{j}) \big) + \delta_{j,1} \big( H(\mathcal{S}_{z,z_0}^{k-1}) - H(\mathcal{S}_{z,z_0}^{k}) \big) \\ = : & G_{x,y,z}^{i,j,k} \end{array}$$

(4) Let

$$\begin{array}{rcl} T_x & := & \{(p,q,r) \in \mathbb{T}^n : p \geq i \ and \ q \leq j\}, \\ T_y & := & \{(p,q,r) \in \mathbb{T}^n : q \geq j \ and \ r \leq k\}, \\ T_z & := & \{(p,q,r) \in \mathbb{T}^n : r \geq k \ and \ p \leq i\}, \end{array}$$

(see Figure 12) and define

$$c_{i,j,k}^{p,q,r} := \begin{cases} \frac{ir + iq + kq}{n} & if (p,q,r) \in T_x; \\ \frac{jp + jr + ir}{n} & if (p,q,r) \in T_y; \\ \frac{kq + kp + jp}{n} & if (p,q,r) \in T_z. \end{cases}$$

Then

$$H(\mathcal{E}_{x,y,z}^{i,j,k}) = \sum_{(p,q,r) \in \mathbb{T}_p} c_{i,j,k}^{p,q,r} \cdot \left( \tau_{x,y,z}^{p,q,r} + \tau_{x',z',y'}^{p,r,q} \right) =: K_{x,y,z}^{i,j,k}.$$

Remark 6.6. The symplectic closed edge invariants appear naturally as Hamiltonian function of the length flows associated to the pants curve. This shows that it is a very natural to use this invariant to give a reparametrization of the Bonahon-Dreyer parametrization of  $\mathrm{Hit}(V)$ .

Note that even though the (i, j, k)-eruption flow associated to a pair of pants P is very natural, its Hamiltonian function is rather complicated. It is a linear combination of all triangle invariants of the two ideal triangles in P, where each triangle invariant has a non-zero oefficient.

(1) and (2) of Theorem 6.5 each describes (3g-3)(n-1) functions on  $\mathrm{Hit}_V(S)$ , while (3) and (4) of Theorem 6.5 each describes (g-1)(n-1)(n-2) functions on  $\mathrm{Hit}_V(S)$ . Together, these give  $(n^2-1)(2g-2)$  functions on  $\mathrm{Hit}_V(S)$ . The following corollary is immediate.

Corollary 6.7. The  $(n^2 - 1)(2g - 2)$  functions described in Theorem 6.5 give a global Darboux coordinate system for  $Hit_V(S)$ .

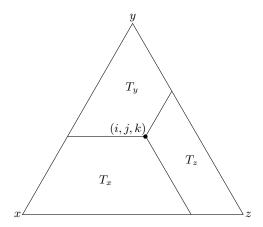


FIGURE 12. Hamiltonian function for  $\mathcal{E}_{x,y,z}^{i,j,k}$  eruption flow.

Proof of Theorem 6.5. By [SZ17, Theorem 6.5], the Hamiltonian functions of the special  $(\mathcal{T}, \mathcal{J})$ -parallel flows are the unique functions (up to an additive constant) that satisfy the following: If  $\mathcal{X}$  is a special  $(\mathcal{T}, \mathcal{J})$ -parallel vector field, then

$$(1) \ \mathcal{X}\left(H(\mathcal{Y}_{x_{1},x_{2}}^{i})\right) = \begin{cases} -1, & \text{if } \mathcal{X} = \mathcal{S}_{x_{1},x_{2}}^{i}; \\ 0, & \text{otherwise.} \end{cases}$$

$$(2) \ \mathcal{X}\left(H(\mathcal{S}_{x_{1},x_{2}}^{i})\right) = \begin{cases} 1 & \text{if } \mathcal{X} = \mathcal{Y}_{x_{1},x_{2}}^{i}; \\ 0 & \text{otherwise.} \end{cases}$$

$$(3) \ \mathcal{X}\left(H(\mathcal{H}_{x,y,z}^{i,j,k})\right) = \begin{cases} 1 & \text{if } \mathcal{X} = \mathcal{E}_{x,y,z}^{i,j,k}; \\ 0 & \text{otherwise.} \end{cases}$$

$$(4) \ \mathcal{X}\left(H(\mathcal{E}_{x,y,z}^{i,j,k})\right) = \begin{cases} -1 & \text{if } \mathcal{X} = \mathcal{H}_{x,y,z}^{i,j,k}; \\ 0 & \text{otherwise.} \end{cases}$$

(4) 
$$\mathcal{X}(H(\mathcal{E}_{x,y,z}^{i,j,k})) = \begin{cases} -1 & \text{if } \mathcal{X} = \mathcal{H}_{x,y,z}^{i,j,k} \\ 0 & \text{otherwise.} \end{cases}$$

We show that the functions specified in the statement of the theorem satisfy the above conditions in the four different cases.

- (1) This follows immediately from Lemma 4.15, Lemma 4.14 and Proposition
- (2) By Proposition 3.3, Proposition 3.9, (4.2), the derivative of  $L^i_{[\gamma]}$  in the direction of any eruption, hexagon or twist vector field is zero. Also, for the same reasons,

$$\mathcal{Y}_{x'_{1},x'_{2}}^{p}(\ell_{[\gamma]}^{k}) = \mathcal{Z}_{x'_{1},x'_{2}}^{l}(\ell_{[\gamma]}^{k}) = \begin{cases} 1 & \text{if } p = k-1, k+1 \text{ and } x_{1} = x'_{1}, x_{2} = x'_{2}; \\ -2 & \text{if } p = k \text{ and } x_{1} = x'_{1}, x_{2} = x'_{2}; \\ 0 & \text{otherwise}; \end{cases}$$

for any  $[x'_1, x'_2] \in \mathcal{P}$  and for all  $k, p = 1, \ldots, n-1$ . Thus,

$$\mathcal{Y}^p_{x_1',x_2'}(L^i_{[\gamma]}) = \mathcal{Z}^p_{x_1',x_2'}(L^i_{[\gamma]}) = \begin{cases} 1 & \text{if } p = i, x_1 = x_1', x_2 = x_2'; \\ 0 & \text{otherwise;} \end{cases}$$

for any  $[x'_1, x'_2] \in \mathcal{P}$  and for all  $p, l = 1, \ldots, n-1$ . This proves that  $H(\mathcal{S}_{x_1,x_2}^i) = L_{[\gamma]}^i$ .

(3) Let  $F_{x,y,z}^{i,j,k} := G_{x,y,z}^{i,j,k} - \tau_{x,y,z}^{i,j,k} + \tau_{x',z',y'}^{i,k,j}$ . It is clear that the derivative of  $G_{x,y,z}^{i,j,k}$  in the direction of any twist vector field is zero. By Proposition 3.3, Proposition 3.9, and the symmetry of the hexagon vector fields, we see that the derivative of  $\tau_{x,y,z}^{i,j,k} - \tau_{x',z',y'}^{i,k,j}$  in the direction of any hexagonal vector field is zero. Furthermore, by (2), we know that the derivative of  $F_{x,y,z}^{i,j,k}$  in the direction of any hexagon vector field is zero. Hence, the derivative of  $G_{x,y,z}^{i,j,k}$  in the direction of any hexagon vector field is zero.

Using Proposition 3.3, one can compute that

$$(6.2) \ \mathcal{E}^{p,q,r}_{a,b,c}(\tau^{i,j,k}_{x,y,z} - \tau^{i,k,j}_{x',z',y'}) = \left\{ \begin{array}{ll} 1 & \text{if } (p,q,r) = (i,j,k) \text{ and } (a,b,c) = (x,y,z) \\ 0 & \text{otherwise.} \end{array} \right.$$

Furthermore, we know by (2) that the derivative of  $F_{x,y,z}^{i,j,k}$  in the direction of any eruption vector field is zero. Thus,

$$\mathcal{E}^{p,q,r}_{a,b,c}(G^{i,j,k}_{x,y,z}) = \left\{ \begin{array}{ll} 1 & \text{if } (p,q,r) = (i,j,k) \text{ and } (a,b,c) = (x,y,z) \\ 0 & \text{otherwise.} \end{array} \right.$$

To finish the proof, we need to show that  $\mathcal{Y}^p_{x'_1,x'_2}(G^{i,j,k}_{x,y,z})=0$  for any length vector field  $\mathcal{Y}^p_{x'_1,x'_2}$ . Let  $\mathcal{X}^p_{x'_1,x'_2}:=\mathcal{Y}^p_{x'_1,x'_2}-\mathcal{Z}^p_{x'_1,x'_2}$ . For the same reasons as above,  $\mathcal{Z}^p_{x'_1,x'_2}(\tau^{i,j,k}_{x,y,z}-\tau^{i,k,j}_{x',z',y'})=0$ . Also, by (2), we know that  $\mathcal{X}^p_{x'_1,x'_2}(F^{i,j,k}_{x,y,z})=0$ . On the other hand, we see by (2) that

$$\mathcal{Z}^p_{x_1',x_2'}(F^{i,j,k}_{x,y,z}) = \left\{ \begin{array}{ll} 1 & \text{if } p=i-1, x_1'=x \text{ and } k=1; \\ 1 & \text{if } p=j-1, x_1'=y \text{ and } i=1; \\ 1 & \text{if } p=k-1, x_1'=z \text{ and } j=1; \\ -1 & \text{if } p=i, x_1'=x \text{ and } k=1; \\ -1 & \text{if } p=j, x_1'=y \text{ and } i=1; \\ -1 & \text{if } p=k, x_1'=z \text{ and } j=1; \\ 0 & \text{otherwise.} \end{array} \right.$$

Also, by (6.2), we have

$$\mathcal{X}^p_{x_1',x_2'}(\tau^{i,j,k}_{x,y,z}-\tau^{i,k,j}_{x',z',y'}) = \begin{cases} -1 & \text{if } p=i-1,x_1'=x \text{ and } k=1;\\ -1 & \text{if } p=j-1,x_1'=y \text{ and } i=1;\\ -1 & \text{if } p=k-1,x_1'=z \text{ and } j=1;\\ 1 & \text{if } p=i,x_1'=x \text{ and } k=1;\\ 1 & \text{if } p=j,x_1'=y \text{ and } i=1;\\ 1 & \text{if } p=k,x_1'=z \text{ and } j=1;\\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{array}{lcl} \mathcal{Y}^p_{x'_1,x'_2}(G^{i,j,k}_{x,y,z}) & = & \mathcal{X}^p_{x'_1,x'_2}(F^{i,j,k}_{x,y,z}) + \mathcal{Z}^p_{x'_1,x'_2}(F^{i,j,k}_{x,y,z}) + \mathcal{X}^p_{x'_1,x'_2}(\tau^{i,j,k}_{x,y,z} - \tau^{i,k,j}_{x',z',y'}) \\ & & + \mathcal{Z}^p_{x'_1,x'_2}(\tau^{i,j,k}_{x,y,z} - \tau^{i,k,j}_{x',z',y'}) \\ & = & 0. \end{array}$$

(4) It is clear that the derivative of  $K_{x,y,z}^{i,j,k}$  in the direction of any twist vector field is zero. Also, by Proposition 3.3, the derivative of  $K_{x,y,z}^{i,j,k}$  in the direction of any eruption vector field is zero.

We will now prove that  $\mathcal{Y}_{x_1,x_2}^p(K_{x,y,z}^{i,j,k})=0$  for any length vector field  $\mathcal{Y}_{x_1,x_2}^p$ . This is clear if neither  $x_1$  not  $x_2$  is x,y or z, so we can assume that

 $x_1 = x, y$  or z. Further assume without loss of generality that  $x_1 = x$ ; the other two cases are similar. Observe that if  $(p, q, r) \in T_x \cup T_y$ , then

(6.3) 
$$c_{i,j,k}^{p,q,r} = c_{i,j,k}^{p+q,0,r} + c_{i,j,k}^{p+r,q,0}.$$

This implies that for all  $p = 1, \dots, n-1$ ,

$$-c_{i,j,k}^{p+1,n-p-1,0}+c_{i,j,k}^{p,n-p-1,1}=c_{i,j,k}^{n-1,0,1}=c_{i,j,k}^{p-1,n-p,1}-c_{i,j,k}^{p,n-p,0},$$

so Proposition 3.3 and Proposition 3.9 imply that  $\mathcal{Z}^p_{x_1,x_2}(K^{i,j,k}_{x,y,z}) = 0$ . Since we already know that the derivative of  $K^{i,j,k}_{x,y,z}$  is zero in the direction of any eruption vector field, this proves that  $\mathcal{Y}^p_{x_1,x_2}(K^{i,j,k}_{x,y,z}) = 0$ .

Next, we consider  $\mathcal{H}^{p,q,r}_{a,b,c}(K^{i,j,k}_{x,y,z})$  for any hexagon vector field  $\mathcal{H}^{p,q,r}_{a,b,c}$ . It is clear from Proposition 3.3 and Proposition 3.9 that if  $[a,b,c] \neq [x,y,z] \in \Theta$ , then for all  $p,q,r \in \mathbb{Z}^+$  so that p+q+r=n,  $\mathcal{H}^{p,q,r}_{a,b,c}(K^{i,j,k}_{x,y,z})=0$ .

If (a, b, c) = (x, y, z) and  $(p, q, r) \neq (i, j, k)$ , then one of  $T_x \cup T_y$ ,  $T_y \cup T_z$  or  $T_z \cup T_x$  contains all of the following six triples

$$(p, q+1, r-1), (p-1, q+1, r), (p-1, q, r+1),$$

$$(p,q-1,r+1), (p+1,q-1,r), (p+1,q,r-1).$$

Without loss of generality, suppose that they lie in  $T_x \cup T_y$ . Then by (6.3),

$$\begin{split} &\mathcal{H}^{p,q,r}_{x,y,z}(K^{i,j,k}_{x,y,z}) \\ &= c^{p,q+1,r-1}_{i,j,k} - c^{p-1,q+1,r}_{i,j,k} + c^{p-1,q,r+1}_{i,j,k} - c^{p,q-1,r+1}_{i,j,k} + c^{p+1,q-1,r}_{i,j,k} - c^{p+1,q,r-1}_{i,j,k} \\ &= c^{p+q+1,0,r-1}_{i,j,k} + c^{p+r-1,q+1,0}_{i,j,k} - c^{p+q,0,r}_{i,j,k} - c^{p+r-1,q+1,0}_{i,j,k} + c^{p+q-1,0,r+1}_{i,j,k} + c^{p+r,q,0}_{i,j,k} \\ &- c^{p+q-1,0,r+1}_{i,j,k} - c^{p+r+1,q-1,0}_{i,j,k} + c^{p+q,0,r}_{i,j,k} + c^{p+r+1,q-1,0}_{i,j,k} - c^{p+q+1,0,r-1}_{i,j,k} - c^{p+r,q,0}_{i,j,k} \\ &= 0. \end{split}$$

Finally, if (a, b, c) = (x, y, z) and (p, q, r) = (i, j, k), then

$$\begin{split} &\mathcal{H}^{i,j,k}_{x,y,z}(K^{i,j,k}_{x,y,z}) \\ &= c^{i,j+1,k-1}_{i,j,k} - c^{i-1,j+1,k}_{i,j,k} + c^{i-1,j,k+1}_{i,j,k} - c^{i,j-1,k+1}_{i,j,k} + c^{i+1,j-1,k}_{i,j,k} - c^{i+1,j,k-1}_{i,j,k} \\ &= \frac{ji + (i+j)(k-1)}{n} - \frac{j(i-1) + (i+j)k}{n} + \frac{kj + (j+k)(i-1)}{n} \\ &\quad - \frac{k(j-1) + (j+k)i}{n} + \frac{ik + (i+k)(j-1)}{n} - \frac{i(k-1) + (i+k)j}{n} \\ &\quad - \frac{1}{n} \end{split}$$

As a corollary we get Corollary 1.3 from the Introduction. Similarly we have:

Corollary 6.8. The twist flows associated to all simple closed curves in the pants decomposition and the hexagon flows associated to all pairs of pants in the pants decomposition provide a half-dimensional family of commuting Hamiltonian flows with Poisson commuting Hamiltonian functions.

### Appendix A. Proof of Proposition 2.11

In this appendix, we will give a proof of Proposition 2.11, which we restate here for the convenience of the reader. This proof is very similar to the proof of [FG06] Proposition 9.4, which is a weaker version of Proposition 2.11.

**Proposition.** Let  $F_1, \ldots, F_k$  be a positive k-tuple of flags in  $\mathcal{F}(V)$ . Then for any non-negative integers  $n_1, \ldots, n_k$  so that  $\sum_{i=1}^k n_i = d \leq n$ , we have that

$$\dim\left(\sum_{j=1}^k F_j^{(n_j)}\right) = d.$$

*Proof.* Choose a volume form on V. This determines an identification  $\bigwedge^n V \simeq \mathbb{R}$ . Using Remark 2.8, one can verify that for each j = 1, ..., k, we can choose a basis  $\{f_{i,1},\cdots,f_{i,n}\}\$ of V with the following properties:

- $\operatorname{Span}_{\mathbb{R}} \{ f_{j,1}, \dots, f_{j,i} \} = F_j^{(i)}$  for all  $i = 1, \dots, n-1$   $f_{j_1}^{i_1} \wedge f_{j_2}^{i_2} \wedge f_{j_3}^{i_3} \in \bigwedge^n V \simeq \mathbb{R}$  is positive for all  $1 \leq j_1 < j_2 < j_3 \leq k$  and for all  $i_1, i_2, i_3 \in \{0, \dots, n-1\}$  so that  $i_1 + i_2 + i_3 = n$ .

Here,  $f_{j_1}^{i_1}$  denotes the  $i_1$ -vector  $f_{j_1,1} \wedge \cdots \wedge f_{j_1,i_1} \in \bigwedge^{i_1} F_{j_1}^{(i_1)}$  on  $F_{j_1}^{(i_1)}$ . (Let D be a disk with k marked points along its boundary. This fact can be interpreted as the surjectivity of the natural projection map from  $\mathcal{A}_{D,\mathrm{SL}}(\mathbb{R}^+) \to \mathcal{X}_{D,\mathrm{PSL}}(\mathbb{R}^+)$ , where  $\mathcal{A}_{D,\mathrm{SL}}(\mathbb{R}^+)$  and  $\mathcal{X}_{D,\mathrm{PSL}}(\mathbb{R}^+)$  are the Fock-Goncharov  $\mathcal{A}$ -moduli space and  $\mathcal{X}$ -moduli space respectively. See Lemma 2.4 of [FG06] for more details.)

Now we show that

$$f_{i_1}^{i_1} \wedge f_{i_2}^{i_2} \wedge f_{i_3}^{i_3} \wedge f_{i_4}^{i_4} > 0$$

for  $1 \leq j_1 < j_2 < j_3 < j_4 \leq k$  and for all  $i_1, i_2, i_3, i_4 \in \{0, \dots, n-1\}$  so that  $i_1+i_2+i_3+i_4=n$  by induction on  $i_2+i_4$ . It is clear that the base case  $i_2=0=i_4$  holds. For the inductive step, suppose the inductive hypothesis that  $f_{j_1}^{i_1} \wedge f_{j_2}^{i_2} \wedge f_{j_3}^{i_4} = f_{j_3}^{i_4} + f_{j_4}^{i_5} + f_{j_5}^{i_5} + f_$  $f_{j_3}^{i_3} \wedge f_{j_4}^{i_4} > 0$  whenever  $i_2 + i_4 \le m$ , and consider the case when  $i_2 + i_4 = m + 1$ . If one of  $i_1, i_2, i_3, i_4$  is zero, we are done by the way we chose the bases we use. Hence, we may assume that none of them are equal to zero.

In [FG06] Lemma 10.3, Fock-Goncharov proved the Plücker relation

$$\begin{split} f_{j_1}^{i_1} \wedge f_{j_2}^{i_2} \wedge f_{j_3}^{i_3} \wedge f_{j_4}^{i_4} \cdot f_{j_1}^{i_1+1} \wedge f_{j_2}^{i_2-1} \wedge f_{j_3}^{i_3+1} \wedge f_{j_4}^{i_4-1} \\ = & f_{j_1}^{i_1+1} \wedge f_{j_2}^{i_2} \wedge f_{j_3}^{i_3} \wedge f_{j_4}^{i_4-1} \cdot f_{j_1}^{i_1} \wedge f_{j_2}^{i_2-1} \wedge f_{j_3}^{i_3+1} \wedge f_{j_4}^{i_4} \\ & + f_{j_1}^{i_1+1} \wedge f_{j_2}^{i_2-1} \wedge f_{j_3}^{i_3} \wedge f_{j_4}^{i_4} \cdot f_{j_1}^{i_1} \wedge f_{j_2}^{i_2} \wedge f_{j_3}^{i_3+1} \wedge f_{j_4}^{i_4-1} \end{split}$$

under a non-degeneracy condition. This was later generalized by the first author [Sun15, Lemma 4.1] to remove the non-degeneracy condition. Applying the induc-

tive hypothesis to the Plücker relation this then gives that  $f_{j_1}^{i_1} \wedge f_{j_2}^{i_2} \wedge f_{j_3}^{i_3} \wedge f_{j_4}^{i_4} > 0$ . Next, consider  $f_{j_1}^{i_1} \wedge \cdots \wedge f_{j_4}^{i_4} \wedge f_{j_5}^{1}$  for some  $1 \leq j_1 < \cdots < j_5 \leq k$  and some  $i_1, \ldots, i_4 \in \{0, \ldots, n-1\}$  so that  $i_1 + \cdots + i_4 + 1 = n$ . By the previous paragraph, we already know that  $f_{j_1}^{i_1} \wedge \cdots \wedge f_{j_4}^{i_4} > 0$  for  $1 \leq j_1 < \cdots < j_4 \leq k$ , and all  $i_1, \ldots, i_4 \in \{0, \ldots, n-1\}$  so that  $i_1 + \cdots + i_4 = n$ . Also, if  $i_3 = 0$ , then it is clear that  $f_{j_1}^{i_1} \wedge \cdots \wedge f_{j_4}^{i_4} \wedge f_{j_5}^1 > 0$  by the previous paragraph. Using this as the base case, we will prove by induction on  $i_3$  that  $f_{j_1}^{i_1} \wedge \cdots \wedge f_{j_4}^{i_4} \wedge f_{j_5}^1 > 0$ .

Let  $g_1^{i_1+i_2}:=f_{j_1}^{i_1}\wedge f_{j_2}^{i_2},\,g_1^{i_1+i_2+1}:=f_{j_1}^{i_1}\wedge f_{j_2}^{i_2+1},\,$  and  $g_{l-1}^i:=f_{j_l}^i$  for l=3,4,5 and all  $i=0,\ldots,n-1$ . Suppose the inductive hypothesis that  $f_{j_1}^{i_1}\wedge\cdots\wedge f_{j_4}^{i_4}\wedge f_{j_5}^1>0$  when  $i_3\leq m$ . When  $i_3=m+1$ , the Plücker relation

$$\begin{split} g_1^{i_1+i_2} \wedge g_2^{i_3} \wedge g_3^{i_4} \wedge g_4^{1} \cdot g_1^{i_1+i_2+1} \wedge g_2^{i_3-1} \wedge g_3^{i_4+1} \\ = & g_1^{i_1+i_2+1} \wedge g_2^{i_3} \wedge g_3^{i_4} \cdot g_1^{i_1+i_2} \wedge g_2^{i_3-1} \wedge g_3^{i_4+1} \wedge g_4^{1} \\ & + g_1^{i_1+i_2+1} \wedge g_2^{i_3-1} \wedge g_3^{i_4} \wedge g_4^{1} \cdot g_1^{i_1+i_2} \wedge g_2^{i_3} \wedge g_3^{i_4+1} \end{split}$$

then implies that  $g_1^{i_1+i_2} \wedge g_2^{i_3} \wedge g_3^{i_4} \wedge g_4^1 = f_{j_1}^{i_1} \wedge \cdots \wedge f_{j_4}^{i_4} \wedge f_{j_5}^1 > 0$ .

Iterating these two inductive procedures prove that  $f_{j_1}^{i_1} \wedge \cdots \wedge f_{j_s}^{i_s} > 0$  for all  $1 \leq j_1 < \cdots < j_s \leq k$  and all  $i_1, \ldots, i_s \in \{0, \ldots, n-1\}$  so that  $i_1 + \cdots + i_s = n$ . It follows immediately that the sum  $F_{j_1}^{(i_1)} + \cdots + F_{j_s}^{(i_s)}$  is direct.

## APPENDIX B. PROOF OF PROPOSITION 4.5

In this appendix, we will prove Proposition 4.5, which we restate here for the reader's convenience.

**Proposition.** Let V denote the set of vertices of  $\widetilde{T}$ , and for  $j = 0, 1, ..., \infty$ , let  $\xi_j \in \mathcal{FR}(V)$  be a Frenet curve. If  $\lim_{j \to \infty} \xi_j|_{V} = \xi_0|_{V}$ , then  $\lim_{j \to \infty} \xi_j = \xi_0$ .

Let  $x, y, z \in \partial \Gamma$  be the vertices of a triangle in  $\widetilde{\Theta}$ , let  $w \in \partial \Gamma \setminus \mathcal{V}$  be any point, and assume without loss of generality that y < z < x < w < y in this cyclic order in  $S^1$ . By Proposition 2.10, to conclude that  $\lim_{j \to \infty} \xi_j(w) = \xi_0(w)$ , it is sufficient to show that

- (1) For all i = 1, ..., n 1,  $C_i(\xi_0(x), \xi_0(z), \xi_0(w), \xi_0(y)) = \lim_{z \to \infty} C_i(\xi_j(x), \xi_j(z), \xi_j(w), \xi_j(y)).$
- (2) For all  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ ,  $T_{i_1, i_2, i_3}(\xi_0(x), \xi_0(w), \xi_0(y)) = \lim_{j \to \infty} T_{i_1, i_2, i_3}(\xi_j(x), \xi_j(w), \xi_j(y)).$

Let  $\{a_k\}_{k=1}^{\infty}$  and  $\{b_k\}_{k=1}^{\infty}$  be sequences in  $\mathcal{V}$  so that  $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = w$  and  $y < z < x < a_k < w < b_k < y$  for all k.

*Proof of (1).* Since  $\xi_0$  is continuous, we have

$$\lim_{k \to \infty} C_i(\xi_0(x), \xi_0(z), \xi_0(a_k), \xi_0(y)) = \lim_{k \to \infty} C_i(\xi_0(x), \xi_0(z), \xi_0(b_k), \xi_0(y))$$
$$= C_i(\xi_0(x), \xi_0(z), \xi_0(w), \xi_0(y))$$

for all i = 1, ..., n - 1. Furthermore, it is also well-known (see for example Proposition 2.12 of [Zha15a]) that

$$C_i(\xi_i(x), \xi_i(a_k), \xi_i(w), \xi_i(y)), C_i(\xi_i(x), \xi_i(w), \xi_i(b_k), \xi_i(y)) > 1$$

for all i = 1, ..., n - 1;  $j = 0, ..., \infty$ ;  $k = 1, ..., \infty$ . In particular, this implies that

$$C_i(\xi_j(x), \xi_j(z), \xi_j(a_k), \xi_j(y)) > C_i(\xi_j(x), \xi_j(z), \xi_j(w), \xi_j(y))$$

$$> C_i(\xi_j(x), \xi_j(z), \xi_j(b_k), \xi_j(y)).$$

Since  $\lim_{j\to\infty} \xi_j(p) = \xi_0(p)$  for all  $p \in \mathcal{V}$ , we see that

$$C_{i}(\xi_{0}(x), \xi_{0}(z), \xi_{0}(w), \xi_{0}(y)) = \lim_{k \to \infty} C_{i}(\xi_{0}(x), \xi_{0}(z), \xi_{0}(a_{k}), \xi_{0}(y))$$

$$= \lim_{k \to \infty} \lim_{j \to \infty} C_{i}(\xi_{j}(x), \xi_{j}(z), \xi_{j}(a_{k}), \xi_{j}(y))$$

$$\geq \lim_{j \to \infty} C_{i}(\xi_{j}(x), \xi_{j}(z), \xi_{j}(w), \xi_{j}(y))$$

$$\geq \lim_{k \to \infty} \lim_{j \to \infty} C_{i}(\xi_{j}(x), \xi_{j}(z), \xi_{j}(b_{k}), \xi_{j}(y))$$

$$= \lim_{k \to \infty} C_{i}(\xi_{0}(x), \xi_{0}(z), \xi_{0}(w), \xi_{0}(y))$$

$$= C_{i}(\xi_{0}(x), \xi_{0}(z), \xi_{0}(w), \xi_{0}(y))$$

for all i = 1, ..., n - 1. This proves (1).

To prove (2), we need to use the following lemma. Let  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ . For any transverse quadruple of flags  $F_1, F_2, F_3, F_4$  in  $\mathcal{F}(V)$ , define

$$\begin{split} & T_{i_1,i_2,F_4,i_3}(F_1,F_2,F_3) := \\ & \frac{F_1^{(i_1+1)} \wedge F_2^{(i_2)} \wedge F_3^{(i_3-1)} \cdot F_1^{(i_1-1)} \wedge F_2^{(i_2)} \wedge F_4^{(1)} \wedge F_3^{(i_3)} \cdot F_1^{(i_1)} \wedge F_2^{(i_2-1)} \wedge F_3^{(i_3+1)}}{F_1^{(i_1+1)} \wedge F_2^{(i_2-1)} \wedge F_3^{(i_3)} \cdot F_1^{(i_1)} \wedge F_2^{(i_2)} \wedge F_4^{(i_1)} \wedge F_3^{(i_3-1)} \cdot F_1^{(i_1-1)} \wedge F_2^{(i_2)} \wedge F_3^{(i_3+1)}} \end{split}$$

**Lemma B.1.** Let  $\xi: S^1 \to \mathcal{F}(V)$  be a Frenet curve and let  $x_1 < x_4 < x_2 < x_5 < x_3 < x_1$  lie in  $S^1$  in this cyclic order. For each m = 1, ..., 5, let  $F_m := \xi(x_m)$ , then

$$T_{i_1,i_2,F_4,i_3}(F_1,F_2,F_3) > T_{i_1,i_2,i_3}(F_1,F_2,F_3) > T_{i_1,i_2,F_5,i_3}(F_1,F_2,F_3).$$

(Recall that we assume  $\dim(V) > 3$ .)

*Proof.* Let  $K:=F_1^{(i_1-1)}+F_2^{(i_2-1)}+F_3^{(i_3-1)}$ . For m=1,2,3, let  $L_{x_m}\subset V$  be a line so that  $F_m^{(i_m-1)}+L_{x_m}=F_m^{(i_m)}$ , and let  $P_{x_m}\subset V$  be a plane so that  $F_m^{(i_m-1)}+P_{x_m}=F_m^{(i_m+1)}$ .

For any  $x \in S^1$ , let

$$L_x := \left\{ \begin{array}{ll} \xi^{(1)}(x) & \text{if } x \neq x_1, x_2, x_3 \\ L_{x_m} & \text{if } x = x_m; m = 1, 2, 3 \end{array} \right., \ \ P_x := \left\{ \begin{array}{ll} \xi^{(2)}(x) & \text{if } x \neq x_1, x_2, x_3 \\ P_{x_m} & \text{if } x = x_m; m = 1, 2, 3 \end{array} \right.$$

and let  $H := L_{x_1} + L_{x_2} + L_{x_3}$ . Then define  $\xi' : S^1 \to \mathcal{F}(H)$  by

$$\xi'^{(1)}(x) := (K + L_x) \cap H, \quad \xi'^{(2)}(x) := (K + P_x) \cap H.$$

It is easy to see that  $\xi'$  does not depend on the choices of  $L_{x_m}$  and  $P_{x_m}$ , and is in fact Frenet.

Furthermore, from the definition of the triple ratio, we see that

$$\begin{array}{rcl} T_{i_1,i_2,F_4,i_3}(F_1,F_2,F_3) & = & T_{1,1,\xi'(x_4),1}(\xi'(x_1),\xi'(x_2),\xi'(x_3)), \\ T_{i_1,i_2,i_3}(F_1,F_2,F_3) & = & T_{1,1,1}(\xi'(x_1),\xi'(x_2),\xi'(x_3)) \\ T_{i_1,i_2,F_5,i_3}(F_1,F_2,F_3) & = & T_{1,1,\xi'(x_5),1}(\xi'(x_1),\xi'(x_2),\xi'(x_3)). \end{array}$$

Thus, it is sufficient to prove this lemma in the case when  $\dim(V) = 3$ . That is a straightforward computation (see Proposition 2.3.4 of [Zha15b]).

*Proof of (2).* The Frenet property of  $\xi_0$  implies that

$$\lim_{k \to \infty} T_{i_1, i_2, \xi_0(a_k), i_3}(\xi_0(x), \xi_0(w), \xi_0(y)) = \lim_{k \to \infty} T_{i_1, i_2, \xi_0(b_k), i_3}(\xi_0(x), \xi_0(w), \xi_0(y))$$

$$= T_{i_1, i_2, i_3}(\xi_0(x), \xi_0(w), \xi_0(y))$$

for all  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ . Also, by Lemma B.1, we have

$$T_{i_1,i_2,\xi_j(a_k),i_3}(\xi_j(x),\xi_j(w),\xi_j(y)) > T_{i_1,i_2,i_3}(\xi_j(x),\xi_j(w),\xi_j(y))$$

$$> T_{i_1,i_2,\xi_j(b_k),i_3}(\xi_j(x),\xi_j(w),\xi_j(y))$$

for all  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n, j = 0, \dots, \infty$  and  $k = 1, \dots, \infty$ . Since  $\lim_{j \to \infty} \xi_j(p) = \xi_0(p)$  for all  $p \in \mathcal{V}$ , this implies that

$$\begin{split} T_{i_1,i_2,i_3}(\xi_0(x),\xi_0(w),\xi_0(y)) &= \lim_{k\to\infty} T_{i_1,i_2,\xi_0(a_k),i_3}(\xi_0(x),\xi_0(w),\xi_0(y)) \\ &= \lim_{k\to\infty} \lim_{j\to\infty} T_{i_1,i_2,\xi_j(a_k),i_3}(\xi_j(x),\xi_j(w),\xi_j(y)) \\ &\geq \lim_{j\to\infty} T_{i_1,i_2,i_3}(\xi_j(x),\xi_j(w),\xi_j(y)) \\ &\geq \lim_{k\to\infty} \lim_{j\to\infty} T_{i_1,i_2,\xi_j(b_k),i_3}(\xi_j(x),\xi_j(w),\xi_j(y)) \\ &= \lim_{k\to\infty} T_{i_1,i_2,\xi_0(b_k),i_3}(\xi_0(x),\xi_0(w),\xi_0(y)) \\ &= T_{i_1,i_2,i_3}(\xi_0(x),\xi_0(w),\xi_0(y)) \end{split}$$

for all  $i_1, i_2, i_3 \in \mathbb{Z}^+$  so that  $i_1 + i_2 + i_3 = n$ . This proves (2).

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