Some remarks regarding $(a,b,x_0,x_1)-$numbers and $(a,b,x_0,x_1)-$quaternions

Cristina FLAUT and Diana SAVIN

Abstract. In this paper we define and study properties and applications of $(a,b,x_0,x_1)-$elements in some special cases.

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1. Introduction

We consider the following difference equation of degree two

$$d_n = ad_{n-1} + bd_{n-2}, d_0 = x_0, d_1 = x_1. \tag{1.1}$$

The study of this equation in the general case could be interesting from a pure mathematical point of view, but some particular cases produced very good and important applications. This is the reason for which the properties and applications of this difference equation were intensively studied in the last years, in some important special cases.

We will define the $(a,b,x_0,x_1)-$*numbers* to be the numbers which satisfy the equations (1.1), where a,b,x_0,x_1 are arbitrary integer numbers. For example, if we consider $a=b=1$, $x_0=0, x_1=1$, we obtain the Fibonacci numbers and if we consider $a=b=1$, $x_0=2, x_1=1$, we obtain the Lucas numbers. The properties and applications of some particulae cases of these numbers are various and were extended to other algebraic structures, or used as applications in the Coding Theory. We refer here to a small part of those papers which approached this subject: [Ak, Ko, To; 14], [Ca; 15], [Fa, Pl; 07(1)], [Fa, Pl; 07(2)], [Fa, Pl; 09], [Fl, Sa; 15], [Fl, Sh;13], [Fl, Sh;15], [Fl, Sh, Vl; 17], [Sa; 17], [Gu, Nu; 15], [Ha; 12], [Ho; 63], [Na, Ha; 09], [Ra; 15], [St; 06], [Sw; 73].

In this paper, we will provide some properties and applications of the $(a,b,x_0,x_1)-$elements, in some special cases.

This paper is organized as follow: in Section 3, we found the generating function for the $(1, 1, p + 2q, q)$ −numbers (the generalized Fibonacci-Lucas numbers), the square of this function, Cassini identity for these elements and other interesting properties and applications. In Section 4, we found the generating function for the $(1, 1, p + 2q, q)$ −quaternions (generalized Fibonacci-Lucas quaternions), Binet's formula, Catalan's and Cassini's identities. In Section 5, we studied the $(1, a, 0, 1)$ −numbers, the $(1, a, 2, 1)$ −numbers, the $(1, a, p + 2q, q)$ −quaternions and we gave some interesting properties, as for example an algebraic structure for the last of them.

2. Preliminaries

First of all, we recall some elementary properties of the Fibonacci and Lucas numbers, properties which will be necessary in the proofs of this paper.
Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence and let $(l_n)_{n\geq 0}$ be the Lucas sequence. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Binet's formula for Fibonacci sequence.

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad (\forall)\, n \in \mathbb{N}.$$

Binet's formula for Lucas sequence.

$$l_n = \alpha^n + \beta^n, \quad (\forall)\, n \in \mathbb{N}.$$

Proposition 2.1. ([Fib.]). *Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence and let $(l_n)_{n\geq 0}$ be the Lucas sequence. The following properties hold:*
i)
$$f_n^2 + f_{n+1}^2 = f_{2n+1}, \forall\ n \in \mathbb{N};$$
ii)
$$f_{n+1}^2 - f_{n-1}^2 = f_{2n}, \forall\ n \in \mathbb{N}^*;$$
iii)
$$l_n^2 - f_n^2 = 4f_{n-1}f_{n+1}, \forall\ n \in \mathbb{N}^*;$$
iv)
$$l_n^2 + l_{n+1}^2 = 5f_{2n+1}, \forall\ n \in \mathbb{N};$$
v)
$$l_n^2 = l_{2n} + 2\,(-1)^n, \forall\ n \in \mathbb{N}^*;$$
vi)
$$f_{n+1} + f_{n-1} = l_n, \forall\ n \in \mathbb{N}^*;$$

vii)

$$l_n + l_{n+2} = 5f_{n+1}, \forall\ n \in \mathbb{N};$$

viii)

$$f_n + f_{n+4} = 3f_{n+2}, \forall\ n \in \mathbb{N};$$

ix)

$$f_m l_{m+p} = f_{2m+p} + (-1)^{m+1} f_p, \forall\ m, p \in \mathbb{N};$$

x)

$$f_{m+p} l_m = f_{2m+p} + (-1)^m f_p, \forall\ m, p \in \mathbb{N};$$

xi)

$$f_m f_{m+p} = \frac{1}{5}\left(l_{2m+p} + (-1)^{m+1} l_p\right), \forall\ m, p \in \mathbb{N};$$

xii)

$$l_m l_p + 5f_m f_p = 2l_{m+p}, \forall\ m, p \in \mathbb{N}.$$

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers and let $A(z) = \sum\limits_{n\geq 1} a_n z^n$ and $B(z) = \sum\limits_{n\geq 1} b_n z^n$ be their generating functions. We recall that their product has the form
$A(z)\,B(z) = \sum\limits_{n\geq 1} s_n z^n$, where $s_n = \sum\limits_{k=1}^{n} a_k b_{n-k}$.

3. Generalized Fibonacci- Lucas numbers. Some properties and applications

Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence

$$f_n = f_{n-1} + f_{n-2},\ n \geq 2, f_0 = 0; f_1 = 1,$$

and $(l_n)_{n\geq 0}$ be the Lucas sequence

$$l_n = l_{n-1} + l_{n-2},\ n \geq 2, l_0 = 2; l_1 = 1.$$

In the paper [Fl, Sa; 15], we introduced the generalized Fibonacci-Lucas numbers. If n is an arbitrary positive integer and p, q are two arbitrary integers, the sequence $(g_n)_{n\geq 1}$, where

$$g_{n+1} = pf_n + ql_{n+1},\ n \geq 0,$$

with $g_0 = p+2q,$ is called the generalized Fibonacci-Lucas numbers. We remark that $g_n = g_{n-1} + g_{n-2}$, with $g_0 = p + 2q, g_1 = q$ and these numbers are actually the $(1, 1, p + 2q, q)$ −numbers.

To avoid confusions, in the following, we will use the notation $g_n^{p,q}$ instead of g_n.

Let $(g_n^{p,q})_{n\geq 1}$ be the generalized Fibonacci- Lucas numbers and let A be the generating function for these numbers

$$A(z) = \sum_{n\geq 1} g_n^{p,q} z^n.$$

In the next proposition we determine this function.

Proposition 3.1. *With the above notations, the following relation is true:*

$$A(z) = \frac{qz + (p+2q) z^2}{1 - z - z^2}.$$

Proof.

$$A(z) = g_1^{p,q} z + g_2^{p,q} z^2 + ... g_n^{p,q} z^n + ... \tag{3.1}$$

$$zA(z) = g_1^{p,q} z^2 + g_2^{p,q} z^3 + ... g_{n-1}^{p,q} z^n + ... \tag{3.2}$$

$$z^2 A(z) = g_1^{p,q} z^3 + g_2^{p,q} z^4 + ... g_{n-2}^{p,q} z^n + ... \quad . \tag{3.3}$$

By adding the equalities (3.2) and (3.3) member by member, we obtain

$$A(z)\left(1 - z - z^2\right) = qz + (p+2q) z^2,$$

therefore

$$A(z) = \frac{qz + (p+2q) z^2}{1 - z - z^2}.$$

□

Proposition 3.2. *If* $A^2(z) = \sum_{n\geq 1} s_n z^n$, *then*

$$5s_n = ng_n^{10pq, p^2+5q^2} + g_n^{p^2+5q^2-10pq, 5pq} + g_{n-1}^{p^2+5q^2, 0} - ng_{n-1}^{0, p^2}.$$

Proof. We know that $s_n = \sum_{k=1}^{n} g_k^{p,q} g_{n-k}^{p,q}$. This equality is equivalent with

$$s_n = \sum_{k=1}^{n} \left(p^2 f_{k-1} f_{n-k-1} + pq f_{k-1} l_{n-k} + pq f_{n-k-1} l_k + q^2 l_k l_{n-k}\right) \tag{3.4}$$

Using Binet's formulas for Fibonacci and Lucas numbers, we have

$$pq f_{k-1} l_{n-k} + pq f_{n-k-1} l_k =$$

$$= \frac{pq}{\sqrt{5}} \left[\left(\alpha^{k-1} - \beta^{k-1}\right)\left(\alpha^{n-k} + \beta^{n-k}\right) + \left(\alpha^{n-k-1} - \beta^{n-k-1}\right)\left(\alpha^k + \beta^k\right)\right] =$$

$$= \frac{pq}{\sqrt{5}} \left[2\alpha^{n-1} - 2\beta^{n-1} + \alpha^{k-1}\beta^{n-k-1}(\beta - \alpha) + \alpha^{n-k-1}\beta^{k-1}(\beta - \alpha)\right] =$$

$$= pq\left(2\frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}} - \alpha^{k-1}\beta^{n-k-1} - \alpha^{n-k-1}\beta^{k-1}\right).$$

Using again Binet's formula for Fibonacci numbers, we obtain

$$pqf_{k-1}l_{n-k} + pqf_{n-k-1}l_k = pq\left(2f_{n-1} - \alpha^{k-1}\beta^{n-k-1} - \alpha^{n-k-1}\beta^{k-1}\right) \tag{3.5}$$

From (3.5), it results that

$$\sum_{k=1}^{n}\left(pqf_{k-1}l_{n-k} + pqf_{n-k-1}l_k\right) =$$

$$= 2pqnf_{n-1} - pq\sum_{k=1}^{n}\alpha^{k-1}\beta^{n-k-1} - pq\sum_{k=1}^{n}\alpha^{n-k-1}\beta^{k-1} =$$

$$= 2pqnf_{n-1} - pq\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta} + \alpha^{n-1}\beta^{-1}\right) - pq\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta} + \alpha^{-1}\beta^{n-1}\right) =$$

$$= 2pqnf_{n-1} - pq\left(f_{n-1} + \alpha^{n-1}\beta^{-1}\right) - pq\left(f_{n-1} + \alpha^{-1}\beta^{n-1}\right) =$$

$$= 2pqnf_{n-1} - 2pqf_{n-1} - pq\frac{\alpha^n+\beta^n}{\alpha\cdot\beta} = 2pqnf_{n-1} - 2pqf_{n-1} + pq\left(\alpha^n+\beta^n\right) =$$

$$= 2pqf_{n-1}\left(n-1\right) + pql_n.$$

Therefore, we get

$$\sum_{k=1}^{n}\left(pqf_{k-1}l_{n-k} + pqf_{n-k-1}l_k\right) = 2pqf_{n-1}\left(n-1\right) + pql_n \tag{3.6}$$

We use again Binet's formula for Fibonacci and Lucas numbers and we obtain

$$\sum_{k=1}^{n}p^2f_{k-1}f_{n-k-1} = p^2\sum_{k=1}^{n}\frac{\alpha^{k-1}-\beta^{k-1}}{\sqrt{5}}\frac{\alpha^{n-k-1}-\beta^{n-k-1}}{\sqrt{5}} =$$

$$= \frac{p^2}{5}\sum_{k=1}^{n}\left(\alpha^{n-2}+\beta^{n-2}\right) - \frac{p^2}{5}\left[\sum_{k=1}^{n}\left(\alpha^{k-1}\beta^{n-k-1} + \alpha^{n-k-1}\beta^{k-1}\right)\right] =$$

$$= \frac{p^2}{5}nl_{n-2} - \frac{p^2}{5}\left[\sum_{k=1}^{n}\left(\alpha^{k-1}\beta^{n-k-1} + \alpha^{n-k-1}\beta^{k-1}\right)\right].$$

From relation (3.6) we have

$$\sum_{k=1}^{n}\left(\alpha^{k-1}\beta^{n-k-1} + \alpha^{n-k-1}\beta^{k-1}\right) = 2f_{n-1} + \frac{\alpha^n+\beta^n}{\alpha\beta} = 2f_{n-1} - l_n.$$

Therefore we obtain the following relation

$$\sum_{k=1}^{n}p^2f_{k-1}f_{n-k-1} = \frac{p^2n}{5}l_{n-2} - \frac{p^2}{5}\left(2f_{n-1} - l_n\right).$$

Applying Proposition 2.1 (vi), we get

$$\sum_{k=1}^{n} p^2 f_{k-1} f_{n-k-1} = \frac{p^2 \left(nl_{n-2} + f_n\right)}{5}. \tag{3.7}$$

Using Binet's formula for Lucas numbers and after for Fibonacci numbers, we have the following relation

$$\sum_{k=1}^{n} q^2 l_k l_{n-k} = q^2 \sum_{k=1}^{n} \left(\alpha^k + \beta^k\right)\left(\alpha^{n-k} + \beta^{n-k}\right) =$$

$$= q^2 \sum_{k=1}^{n} \left(\alpha^n + \beta^n\right) + q^2 \sum_{k=1}^{n} \left(\alpha^k \beta^{n-k} + \alpha^{n-k}\beta^k\right) =$$

$$= q^2 n l_n + q^2 \left[-\left(\alpha^n + \beta^n\right) + 2\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right] =$$

$$= q^2 n l_n + q^2 \left(-l_n + 2f_{n+1}\right).$$

Using Proposition 2.1 (vi), we obtain

$$\sum_{k=1}^{n} q^2 l_k l_{n-k} = q^2 \left(nl_n + f_n\right). \tag{3.8}$$

Adding member by member the equalities (3.6), (3.7), (3.8), we have

$$s_n = 2pqf_{n-1}\left(n-1\right) + pql_n + \frac{p^2\left(nl_{n-2} + f_n\right)}{5} + q^2\left(nl_n + f_n\right) =$$

$$= 2pqf_{n-1}\left(n-1\right) + pql_n + \frac{p^2\left(nl_n - nl_{n-1} + f_{n-2} + f_{n-1}\right)}{5} + q^2\left(nl_n + f_{n-2} + f_{n-1}\right) =$$

$$= \left[pq\left(2n-2\right) + \left(\frac{p^2}{5} + q^2\right)\right] f_{n-1} + \left(\frac{np^2}{5} + pq + nq^2\right) l_n + \left(\frac{p^2}{5} + q^2\right) f_{n-2} - \frac{np^2}{5} l_{n-1}.$$

It results that

$$5s_n = n\left[10pqf_{n-1} + \left(p^2 + 5q^2\right) l_n\right] +$$

$$+ \left(p^2 + 5q^2 - 10pq\right) f_{n-1} + 5pql_n + \left(p^2 + 5q^2\right) f_{n-2} - np^2 l_{n-1}.$$

Finally, we obtain

$$5s_n = ng_n^{10pq, p^2+5q^2} + g_n^{p^2+5q^2-10pq, 5pq} + g_{n-1}^{p^2+5q^2, 0} - ng_{n-1}^{0, p^2}.$$

□

For Fibonacci and Lucas numbers, there are very well known the Cassini's identities, namely

$$f_{n+1} f_{n-1} - f_n^2 = (-1)^n, \ (\forall)\ n \in \mathbb{N}^*,$$

for Fibonacci numbers and

$$l_{n+1}l_{n-1}-l_n^2=5\cdot(-1)^{n-1}\,,\ (\forall)\ n\in\mathbb{N}^*,$$

for Lucas numbers. In the following, we extend these results for the generalized Fibonacci-Lucas numbers.

Proposition 3.3. *Let p,q be arbitrary integers. Then, we have:*

$$g_{n+1}^{p,q}g_{n-1}^{p,q}-\left(g_n^{p,q}\right)^2=(-1)^{n-1}\left[p^2+5q^2+5pq\right],\ \text{for all } n\in\mathbb{N},\ n\geq 2.$$

Proof. We have that

$$g_{n+1}^{p,q}g_{n-1}^{p,q}-\left(g_n^{p,q}\right)^2=\left(pf_n+ql_{n+1}\right)\left(pf_{n-2}+ql_{n-1}\right)-\left(pf_{n-1}+ql_n\right)^2=$$

$$=p^2\left(f_{n-2}f_n-f_{n-1}^2\right)+q^2\left(l_{n-1}l_{n+1}-l_n^2\right)+pq\left(f_{n-2}l_{n+1}+f_nl_{n-1}-2f_{n-1}l_n\right)$$

Using Cassini identities for Fibonacci and Lucas numbers and Proposition 2.1 (ix; x), we obtain

$$g_{n+1}^{p,q}g_{n-1}^{p,q}-\left(g_n^{p,q}\right)^2=$$
$$=p^2\left(-1\right)^{n-1}+5q^2\left(-1\right)^{n-1}+$$
$$+pq\left[f_{2n-1}+\left(-1\right)^{n-1}f_3+f_{2n-1}+\left(-1\right)^{n-1}f_1-2f_{2n-1}-2\left(-1\right)^nf_1\right]=$$
$$=p^2\left(-1\right)^{n-1}+5q^2\left(-1\right)^{n-1}+5pq\left(-1\right)^{n-1}=$$
$$=\left(-1\right)^{n-1}\left[p^2+5q^2+5pq\right].$$

□

For the generalized Fibonacci-Lucas numbers, we introduce the following quadratic matrix

$$M_n=\begin{pmatrix} g_{n+1}^{p,q} & g_n^{p,q} \\ g_n^{p,q} & g_{n-1}^{p,q}\end{pmatrix}. \tag{3.9}$$

Proposition 3.4. *With the above notations, the following affirmations are true:*

i) $M_n=M_{n-1}+M_{n-2}$.

ii) For $n\in\mathbb{N},\ n\geq 2$, *we have* $\det M=(-1)^{n-1}\left[p^2+5q^2+5pq\right]$.

Proof.

i) Since $g_{n+1}^{p,q}=g_n^{p,q}+g_{n-1}^{p,q}$, $g_n^{p,q}=g_{n-1}^{p,q}+g_{n-2}^{p,q}$, $g_{n-1}^{p,q}=g_{n-2}^{p,q}+g_{n-3}^{p,q}$, we obtain the asked relation.

ii) We use Proposition 3.3. □

Remark 3.5. In [St; 06] and [Ba, Pr; 09], were presented some applications of Fibonacci elements in Coding Theory. The matrix given in relation (3.9) can be used for coding and decoding over $\mathbb{Z}$ when its determinant is 1 or -1 or for coding and decoding over $\mathbb{Z}_a$ when its determinant is prime with the integer a. Even if these matrices have the same properties for coding and decoding as the similar matrices generated by Fibonacci elements, that means can correct $1, 2$ or 3 errors, they can be used as a new class of such a matrices utilized for coding and decoding.

4. Generalized Fibonacci- Lucas quaternions. Some properties and applications

Let $\gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\}$ and let $\mathbb{H}(\gamma_1, \gamma_2)$ be the generalized quaternion algebra with basis $\{1, e_1, e_2, e_3\}$, that means a real algebra with the multiplication given in the following table

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	γ_1	e_3	$\gamma_1 e_2$
e_2	e_2	$-e_3$	γ_2	$-\gamma_2 e_1$
e_3	e_3	$-\gamma_1 e_2$	$\gamma_2 e_1$	$\gamma_1 \gamma_2$

.

We consider $\alpha, \beta \in \mathbb{Q}^*$ and $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$ the generalized quaternion algebra over the rational field. Let d_n be $(a, b, x_0, x_1)-$numbers. We define the *n-th* $(a, b, x_0, x_1)-$*quaternions* to be the elements of the form

$$D_n = d_n + d_{n+1}e_1 + d_{n+2}e_2 + d_{n+3}e_3$$

In the paper [Fl, Sa; 15], we introduced the *n-th generalized Fibonacci-Lucas quaternion* to be the elements of the form

$$G_n^{p,q} = g_n^{p,q}1 + g_{n+1}^{p,q}e_1 + g_{n+2}^{p,q}e_2 + g_{n+3}^{p,q}e_3.$$

We remark that these elements are *n-th* $(1, 1, p + 2q, q)-$quaternions.

Remark 4.1. In the paper [Fl, Sa; 15], Theorem 3.5, we proved that the set

$$\left\{ \sum_{i=1}^{n} 5G_{n_i}^{p_i, q_i} | n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Z}, (\forall) i = \overline{1, n} \right\} \cup \{1\}$$

has a ring structure with quaternions addition and multiplication, that means it is an order of a rational quaternion algebra. To prove that the above set is closed under multiplications, we used properties of Fibonacci and Lucas number regarding multiplications of two such elements (see Proposition 2.1). For example, we remark that the product of two Fibonacci numbers gives us Lucas numbers. This remark explains why we considered the generalized Fibonacci-Lucas numbers given by the relation 1.2.

In the following, we will provide some properties of generalized Fibonacci-Lucas quaternions.

Let B be the generating function for the generalized Fibonacci-Lucas quaternions, $B(z)=\sum\limits_{n\geq 1} G_n^{p,q} z^n$.

Proposition 4.2. *With the above notations, we have*

$$B(z)=\frac{G_1^{p,q}z+\left(G_2^{p,q}-G_1^{p,q}\right)z^2}{1-z-z^2}.$$

Proof. We use the following relations

$$B(z)=G_1^{p,q}z+G_2^{p,q}z^2+...G_n^{p,q}z^n+... \tag{4.1}$$

$$zB(z)=G_1^{p,q}z^2+G_2^{p,q}z^3+...G_{n-1}^{p,q}z^n+... \tag{4.2}$$

$$z^2B(z)=G_1^{p,q}z^3+G_2^{p,q}z^4+...G_{n-2}^{p,q}z^n+... \quad . \tag{4.3}$$

Adding the equalities (4.1) and (4.2) member by member and using relation (4.3), we obtain

$$B(z)\left(1-z-z^2\right)=G_1^{p,q}z+\left(G_2^{p,q}-g_1^{p,q}\right)z^2,$$

therefore

$$B(z)=\frac{G_1^{p,q}\cdot z+\left(G_2^{p,q}-g_1^{p,q}\right)\cdot z^2}{1-z-z^2}.$$

□

In the following, we consider $h(x)$ a polynomial with real coefficients. We define the real $h(x)$–*generalized Fibonacci-Lucas polynomials*, as polynomials defined by the recurrence relation

$$g_{h,n}^{p,q}(x)=h(x)g_{h,n-1}^{p,q}(x)+g_{h,n-2}^{p,q}(x), \quad n=2,3,\ldots, \tag{4.4}$$

where $g_{h,0}^{p,q}(x)=p+2q$, $g_{h,1}^{p,q}(x)=q$, for all polynomials $h(x)$.

Definition 4.3. The $h(x)$ – *generalized quaternion Fibonacci-Lucas polynomials* $\{G_{h,n}^{p,q}(x)\}_{n=0}^{\infty}$ are given by the recurrence relation

$$G_{h,n}^{p,q}(x)=\sum_{k=0}^{3} g_{h,n+k}^{p,q}(x)\,e_k, \tag{4.5}$$

where $g_{h,n}^{p,q}(x)$ is the nth real $h(x)$ – Fibonacci-Lucas polynomial.

Definition 4.4. The *generating function* $A(t)$ corresponding to the sequence $\{G_{h,n}^{p,q}(x)\}_{n=0}^{\infty}$ is defined by the following relation

$$A(t)=\sum_{n=0}^{\infty} G_{h,n}^{p,q}(x)t^n. \tag{4.6}$$

Theorem 4.5. *The generating function for $h(x)$ – quaternion Fibonacci-Lucas polynomials $G_{h,n}^{p,q}(x)$ is given by the relation*

$$A(t)=\frac{G_{h,0}^{p,q}(x)+(G_{h,1}^{p,q}(x)-h(x)G_{h,0}^{p,q}(x))t}{1-h(x)t-t^2}.$$

Proof. From relation (4.6), we obtain the following relation

$$A(t)(1-h(x)t-t^2)=$$

$$=\sum_{n=0}^{\infty}G_{h,n}^{p,q}(x)t^n-h(x)\sum_{n=0}^{\infty}G_{h,n}^{p,q}(x)t^{n+1}-\sum_{n=0}^{\infty}G_{h,n}^{p,q}(x)t^{n+2}=$$

$$=G_{h,0}^{p,q}(x)+(G_{h,1}^{p,q}(x)-h(x)G_{h,0}^{p,q}(x))t+$$

$$+\sum_{n=2}^{\infty}t^n(G_{h,n}^{p,q}-h(x)G_{h,n-1}^{p,q}-G_{h,n-2}^{p,q})=$$

$$=G_{h,0}(x)+(G_{h,1}(x)-h(x)G_{h,0}(x))t.$$

□

Let $r_1(x)$ and $r_2(x)$ be the solutions of the characteristic equation $r^2-h(x)r-1=0$, for the recurrence relation given in (4.4). We obtain that

$$r_1(x)=\frac{h(x)+\sqrt{h^2(x)+4}}{2},\quad r_2(x)=\frac{h(x)-\sqrt{h^2(x)+4}}{2}. \tag{4.7}$$

Theorem 4.6. (Binet's formula) *For all $n\in\{0,1,2,\ldots\}$, we have the following relation for the polynomials $g_{h,n}^{p,q}(x)$*

$$g_{h,n}^{p,q}(x)=\frac{(p+2q)}{r_1(x)-r_2(x)}\left(r_1^{n+1}(x)-r_2^{n+1}(x)\right)-\frac{q}{r_1(x)-r_2(x)}\left(r_1^n(x)-r_2^n(x)\right) \tag{4.8}$$

Proof. Assuming that $g_{h,n}^{p,q}(x)=\alpha r_1^n(x)+\beta r_2^n(x)$. From relation (4.4), we obtain that $\alpha+\beta=p+2q$ and $\alpha r_1(x)+\beta r_2(x)=q$. Solving this system, we obtain

$$\alpha=\frac{(p+2q)\,r_1\,(x)-q}{r_1(x)-r_2(x)}$$

and

$$\beta=\frac{q-(p+2q)\,r_2\,(x)}{r_1(x)-r_2(x)}.$$

Therefore

$$g_{h,n}^{p,q}(x)=\frac{(p+2q)\,r_1\,(x)-q}{r_1(x)-r_2(x)}r_1^n(x)+\frac{q-(p+2q)\,r_2\,(x)}{r_1(x)-r_2(x)}r_2^n(x).$$

It results that $g_{h,n}^{p,q}(x)=\frac{(p+2q)}{r_1(x)-r_2(x)}\left(r_1^{n+1}(x)-r_2^{n+1}(x)\right)-\frac{q}{r_1(x)-r_2(x)}\left(r_1^n(x)-r_2^n(x)\right)$.□

Theorem 4.7. *For $n \in \{0,1,2,\ldots\}$, we have the following relation*

$$G^{p,q}_{h,n}(x) = \frac{R_1(x)r_1^n(x) - R_2(x)r_2^n(x)}{r_1(x) - r_2(x)}, \tag{4.9}$$

where

$R_1(x) = \sum\limits_{k=0}^{3} \left(-qr_1^k(x) + (p+2q)\, r_1^{k+1}(x)\right) e_k$,

$R_2(x) = \sum\limits_{k=0}^{3} \left(-qr_2^k(x) + (p+2q)\, r_2^{k+1}(x)\right) e_k$.

Proof. Using the Binet's formula (4.8), it results

$G^{p,q}_{h,n}(x) = \sum\limits_{k=0}^{3} g^{p,q}_{h,n+k}(x)\, e_k =$
$= \frac{(p+2q)}{r_1(x)-r_2(x)}\left(r_1^{n+1}(x) - r_2^{n+1}(x)\right) e_0 - \frac{q}{r_1(x)-r_2(x)}\left(r_1^n(x) - r_2^n(x)\right) e_0 +$
$+ \frac{(p+2q)}{r_1(x)-r_2(x)}\left(r_1^{n+2}(x) - r_2^{n+2}(x)\right) e_1 - \frac{q}{r_1(x)-r_2(x)}\left(r_1^{n+1}(x) - r_2^{n+1}(x)\right) e_1 +$
$+ \frac{(p+2q)}{r_1(x)-r_2(x)}\left(r_1^{n+3}(x) - r_2^{n+3}(x)\right) e_2 - \frac{q}{r_1(x)-r_2(x)}\left(r_1^{n+2}(x) - r_2^{n+2}(x)\right) e_2 +$
$+ \frac{(p+2q)}{r_1(x)-r_2(x)}\left(r_1^{n+4}(x) - r_2^{n+4}(x)\right) e_3 - \frac{q}{r_1(x)-r_2(x)}\left(r_1^{n+3}(x) - r_2^{n+3}(x)\right) e_3 =$
$= \frac{r_1^n(x)}{r_1(x)-r_2(x)}[-q + (p+2q)\, r_1(x) + \left(-qr_1(x) + (p+2q)\, r_1^2(x)\right) e_1 +$
$+ \left(-qr_1^2(x) + (p+2q)\, r_1^3(x)\right) e_2 + \left(-qr_1^3(x) + (p+2q)\, r_1^4(x)\right) e_3] -$
$- \frac{r_2^n(x)}{r_1(x)-r_2(x)}[-q + (p+2q)\, r_2(x) + \left(-qr_2(x) + (p+2q)\, r_2^2(x)\right) e_1 +$
$+ \left(-qr_2^2(x) + (p+2q)\, r_2^3(x)\right) e_2 + \left(-qr_2^3(x) + (p+2q)\, r_2^4(x)\right) e_3] =$
$= R_1(x)\, \frac{r_1^n(x)}{r_1(x)-r_2(x)} + R_2(x)\, \frac{r_2^n(x)}{r_1(x)-r_2(x)}$

$$= \frac{R_1(x)r_1^n(x) - R_2(x)r_2^n(x)}{r_1(x) - r_2(x)}.$$

□

Theorem 4.8. (Catalan's identity) *Let n, s be positive integers with $s \le n$. The following relation is true*

$$G^{p,q}_{h,n+s}(x)G^{p,q}_{h,n-s}(x) - G^{p,q2}_{h,n}(x) =$$

$$\frac{(-1)^{n+s+1}}{h^2(x)+4}[R_1(x)R_2(x)((-1)^{s+1} + r_1^2(x)) + R_1(x)R_2(x)((-1)^{s+1} + r_2^2(x))].$$

Proof. Using formula (4.9), it results

$$G^{p,q}_{h,n+s}(x)G^{p,q}_{h,n-s}(x) - G^{p,q2}_{h,n}(x) =$$

$$\frac{1}{(r_1(x) - r_2(x))^2}[R_1(x)R_2(x)r_1^n(x)r_2^n(x)(1 - \left(\frac{r_1(x)}{r_2(x)}\right)^s) +$$

$$R_1(x)R_2(x)r_1^n(x)r_2^n(x)(1 - \left(\frac{r_2(x)}{r_1(x)}\right)^r)].$$

Finally, we use the Viète's relations, we obtain the asked identity.□

If in the above Theorem , we take $r=1$, we obtain the Cassini's identity.

Theorem 4.9 (Cassini's identity) *For each natural number n, we have*

$$G_{h,n+1}^{p,q}(x)G_{h,n-1}^{p,q}(x)-G_{h,n}^{p,q2}(x)=$$

$$\frac{(-1)^n}{h^2(x)+4}[R_1(x)R_2(x)(1+r_1^2(x))+R_1(x)R_2(x)(1+r_2^2(x))].$$

□

Similar results with the results obtained above for the generalized Fibonacci-Lucas quaternions were obtained in [Ca; 15], Theorem 3.3 and Theorem 3.6 for $h(x)$ −Fibonacci polynomials over the real field and in [Fl, Sh, Vl; 17], Theorem 2.3 and Theorem 2.6, for $h(x)$ −Fibonacci polynomials over an arbitrary algebra.

5. Some properties of $(1,a,0,1)$ −quaternions and $(1,a,2,1)$ −quaternions

Let a be a nonzero natural number. Let $(x_n)_{n\geq 0}$ be the $(1,a,0,1)$ −numbers and $(y_n)_{n\geq 0}$ be the $(1,a,2,1)$ −numbers, that means

$$x_n=x_{n-1}+ax_{n-2},\ n\geq 2, x_0=0, x_1=1$$

and

$$y_n=y_{n-1}+ay_{n-2},\ n\geq 2, y_0=2, y_1=1.$$

If n is a negative integer, we take $x_n=(-1)^{-n+1}\cdot x_{-n}$.
In the paper [Fl, Sa; 15], we introduced the generalized Fibonacci-Lucas numbers and the generalized Fibonacci-Lucas quaternions and we obtained some properties of them.
Now, we introduce another numbers and another quaternions, where instead of the Fibonacci sequence $(f_n)_{n\geq 0}$ we consider the sequence $(x_n)_{n\geq 0}$ and instead of the Lucas sequence $(l_n)_{n\geq 0}$ we consider the sequence $(y_n)_{n\geq 0}$.
If we denote with $\alpha=\frac{1+\sqrt{1+4a}}{2}$ and $\beta=\frac{1-\sqrt{1-4a}}{2}$, it is easy to obtain the following relations:

Binet's formula for the sequence $(x_n)_{n\geq 0}$.

$$x_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}=\frac{\alpha^n-\beta^n}{\sqrt{1+4a}},\quad (\forall)\, n\in\mathbb{N}.$$

Binet's formula for the sequence $(y_n)_{n\geq 0}$.

$$y_n=\alpha^n+\beta^n,\quad (\forall)\, n\in\mathbb{N}.$$

First of all, we get some properties of the sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$.

Proposition 5.1. *Let* $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ *be the sequences previously defined. Then, we have*:
i)
$$y_n y_{n+l} = y_{2n+l} + (-a)^n y_l,\ (\forall)\, n, l \in \mathbb{N};$$
ii)
$$x_n y_{n+l} = x_{2n+l} - (-a)^n x_l,\ (\forall)\, n, l \in \mathbb{N};$$
iii)
$$x_{n+l} y_n = x_{2n+l} + (-a)^n x_l,\ (\forall)\, n, l \in \mathbb{N};$$
iv)
$$x_n x_{n+l} = \frac{1}{1+4a}\left[y_{2n+l} - (-a)^n y_l\right],\ (\forall)\, n, l \in \mathbb{N}.$$

Proof. Using Binet's formulae for the sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$, we have:
i)
$$y_n y_{n+l} = (\alpha^n + \beta^n)\left(\alpha^{n+l} + \beta^{n+l}\right) = \alpha^{2n+l} + \alpha^n \beta^{n+l} + \alpha^{n+l}\beta^n + \beta^{2n+l} =$$
$$= y_{2n+l} + \alpha^n\beta^n\left(\alpha^l + \beta^l\right) = y_{2n+l} + (-a)^n y_l.$$
ii)
$$x_n y_{n+l} = \frac{\alpha^n - \beta^n}{\alpha - \beta}\left(\alpha^{n+l} + \beta^{n+l}\right) =$$
$$= \frac{\alpha^{2n+l} - \beta^{2n+l}}{\alpha - \beta} - \frac{\alpha^n\beta^n\left(\alpha^l - \beta^l\right)}{\alpha - \beta} = x_{2n+l} - (-a)^n x_l.$$
iii)
$$x_{n+l} y_n = \frac{\alpha^{n+l} - \beta^{n+l}}{\alpha - \beta}\left(\alpha^n + \beta^n\right) =$$
$$= \frac{\alpha^{2n+l} - \beta^{2n+l}}{\alpha - \beta} + \frac{\alpha^n\beta^n\left(\alpha^l - \beta^l\right)}{\alpha - \beta} = x_{2n+l} + (-a)^n x_l.$$
iv)
$$x_n x_{n+l} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \cdot \frac{\alpha^{n+l} - \beta^{n+l}}{\alpha - \beta} =$$
$$= \frac{\alpha^{2n+l} + \beta^{2n+l} - \alpha^n\beta^n\left(\alpha^l + \beta^l\right)}{(\alpha - \beta)^2} = \frac{1}{1+4a}\left[y_{2n+l} - (-a)^n y_l\right].$$
□

Let p, q be two arbitrary integers. We consider the sequence $(s_n)_{n\geq 0}$,
$$s_{n+1} = p x_n + q y_{n+1},\ n \geq 0, \tag{5.1.}$$
where $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ are the sequences previously defined.
We obtain that $s_n = s_{n-1} + a s_{n-2},\ (\forall)\, n \in \mathbb{N}, n \geq 2,$

$s_0 = px_{-1} + qy_0 = p + 2q$, $s_1 = q$, that means $(s_n)_{n\geq 0}$ are $(1, a, p+2q, q)$ $-$numbers. In the following, we will use the notation $s_n^{p,q}$ for s_n.

Let $\alpha, \beta \in \mathbb{Q}^*$ and let $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$ be the generalized quaternion algebra over the rational field, with basis $\{1, e_1, e_2, e_3\}$. We define the n-th $(1, a, p+2q, q)-$ quaternions to be the elements of the form

$$S_n^{p,q} = s_n^{p,q} 1 + s_{n+1}^{p,q} e_1 + s_{n+2}^{p,q} e_2 + s_{n+3}^{p,q} e_3.$$

Remark 5.2. *Let p, q be two arbitrary integers, let n be an arbitrary positive integer and let $(s_n^{p,q})_{n\geq 1}$ the sequence previously defined. Then, we have:*

$$px_{n+1} + qy_n = s_n^{ap,q} + s_{n+1}^{p,0}, \forall\ n \in \mathbb{N} - \{0\}.$$

Proof. We compute

$$px_{n+1} + qy_n = px_n + apx_{n-1} + qy_n = s_n^{ap,q} + s_{n+1}^{p,o}.$$

$\square$

Remark 5.3. *Using the previously notations, we have the following relation*

$$S_n^{p,q} = 0 \text{ if and only if } p = q = 0.$$

Proof. " $\Rightarrow$" If $S_n^{p,q} = 0$, since $\{1, e_1, e_2, e_3\}$ is a $\mathbb{Q}-$ basis in quaternion algebra $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$, it results that $s_n^{p,q} = s_{n+1}^{p,q} = s_{n+2}^{p,q} = s_{n+3}^{p,q} = 0$. Using the recurrence relation of the sequence $(s_n^{p,q})_{n\geq 1}$, we obtain that $s_{n-1}^{p,q} = 0$, $s_{n-2}^{p,q} = 0$, ..., $s_1^{p,q} = 0$, $s_0^{p,q} = 0$. Since $s_1^{p,q} = q$, it results $q = 0$ and since $s_0^{p,q} = p + 2q = 0$, it results $p = 0$.
" $\Leftarrow$" It is trivial. $\square$

Proposition 5.4. *Let a be a nonzero natural number and let O be the set*

$$O = \left\{ \sum_{i=1}^{n} (1+4a)\, S_{n_i}^{p_i,q_i} | n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Z}, (\forall) i = \overline{1,n} \right\} \cup \{1\}.$$

Then O is an order of the quaternion algebra $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$.

Proof. Applying Remark 5.3, $S_n^{0,0} = 0 \in O$.
Let $n, m \in \mathbb{N}^*$, $p, q, p^{'}, q^{'}, c, d \in \mathbb{Z}$. We obtain that

$$cs_n^{p,q} + ds_m^{p^{'},q^{'}} = s_n^{cp,cq} + s_m^{dp^{'},dq^{'}}$$

and, from here, we get

$$cS_n^{p,q} + dS_m^{p^{'},q^{'}} = S_n^{cp,cq} + S_m^{dp^{'},dq^{'}}.$$

This implies that O *is a free* $\mathbb{Z}-$ submodule of rank 4 of the quaternion algebra $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$.

Now, we prove that O is a subring of $\mathbb{H}_{\mathbb{Q}}(\alpha,\beta)$. Let m,n be two integers, $n<m$. We have:

$$(1+4a)\,s_n^{p,q}\,(1+4a)\,s_m^{p',q'}=(1+4a)\,(px_{n-1}+qy_n)\,(1+4a)\left(p'x_{m-1}+q'y_m\right)=$$
$$=(1+4a)^2\,pp'x_{n-1}x_{m-1}+(1+4a)^2\,pq'x_{n-1}y_m+$$
$$+(1+4a)^2\,p'qx_{m-1}y_n+(1+4a)^2\,qq'y_ny_m.$$

Applying Proposition 5.1 and Remark 5.2, we obtain:

$$(1+4a)\,s_n^{p,q}\,(1+4a)\,s_m^{p',q'}=(1+4a)^2\,pp'\frac{1}{1+4a}\left[y_{n+m-2}-(-a)^n\,y_{m-n}\right]+$$
$$+(1+4a)^2\,pq'\left[x_{m+n-1}-(-a)^{n-1}\,x_{m-n+1}\right]+$$
$$+(1+4a)^2\,p'q\left[x_{n+m-1}+(-a)^n\,x_{m-n-1}\right]+(1+4a)^2\,qq'\left[y_{n+m}+(-a)^n\,y_{m-n}\right]=$$
$$=(1+4a)^2\left[pq'x_{n+m-1}+qq'y_{m+n}\right]+(1+4a)^2\left[(-a)^n\,p'qx_{m-n-1}+(-a)^n\,qq'y_{m-n}\right]+$$
$$+(1+4a)\left[-(-a)^{n-1}(1+4a)\,pq'x_{m-n+1}-(-a)^n\,pp'y_{m-n}\right]+$$
$$+(1+4a)\left[(1+4a)\,p'qx_{n+m-1}+pp'y_{n+m-2}\right]=$$
$$=(1+4a)\,s_{m+n}^{(1+4a)pq',(1+4a)qq'}+(1+4a)\,s_{m-n}^{(-a)^n(1+4a)p'q,(-a)^n(1+4a)qq'}+$$
$$+(1+4a)\,s_{m-n+1}^{(-a)^n(1+4a)pq',(-a)^{n+1}pp'}+(1+4a)\,s_{m-n+1}^{(-a)^n(1+4a)pq',0}+$$
$$+(1+4a)\,s_{m+n-2}^{a(1+4a)p'q,pp'}+(1+4a)\,s_{m+n-1}^{a(1+4a)p'q,0}.$$

It results that $(1+4a)\,s_n^{p,q}\,(1+4a)\,s_m^{p',q'}\in O$. Therefore, O is an order of the quaternion algebra $\mathbb{H}_{\mathbb{Q}}(\alpha,\beta)$. $\square$

Conclusions. In this paper, we introduced the $(a,b,x_0,x_1)-$elements and we studied some properties and applications for the $(1,1,p+2q,q)-$numbers (the generalized Fibonacci-Lucas numbers), the $(1,1,p+2q,q)-$quaternions (generalized Fibonacci-Lucas quaternions), the $(1,a,0,1)-$numbers, the $(1,a,2,1)-$numbers and the $(1,a,p+2q,q)-$quaternions. For the last one, we gave an interesting algebraic structure.

From the above, we can see that these elements can be studied in a more genealized cases and all other particular cases are subordinated to this approach. In the paper [Fl, Sa; 17], we studied properties and applications of elements arising from a difference equation of degree three. It is interesting to study what relations must satisfy coefficients of a difference equation such that the quaternions defined using that equation to have a ring structure, as above. This idea can constitute a starting point for a further research.

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Cristina FLAUT
Faculty of Mathematics and Computer Science, Ovidius University,
Bd. Mamaia 124, 900527, CONSTANTA, ROMANIA
http://cristinaflaut.wikispaces.com/; http://www.univ-ovidius.ro/math/
e-mail: cflaut@univ-ovidius.ro; cristina_flaut@yahoo.com

Diana SAVIN
Faculty of Mathematics and Computer Science,
Ovidius University,
Bd. Mamaia 124, 900527, CONSTANTA, ROMANIA
http://www.univ-ovidius.ro/math/
e-mail: savin.diana@univ-ovidius.ro, dianet72@yahoo.com