# BIFURCATION SETS ARISING FROM NON-INTEGER BASE EXPANSIONS

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ABSTRACT. Given a positive integer $M$ and $q \in (1, M+1]$, let $\mathcal{U}_q$ be the set of $x \in [0, M/(q-1)]$ having a unique $q$-expansion: there exists a unique sequence $(x_i) = x_1x_2\ldots$ with each $x_i \in \{0, 1, \ldots, M\}$ such that

$$x = \frac{x_1}{q} + \frac{x_2}{q^2} + \frac{x_3}{q^3} + \cdots .$$

Denote by $\mathbf{U}_q$ the set of corresponding sequences of all points in $\mathcal{U}_q$. It is well-known that the function $H : q \mapsto h(\mathbf{U}_q)$ is a Devil's staircase, where $h(\mathbf{U}_q)$ denotes the topological entropy of $\mathbf{U}_q$. In this paper we give several characterizations of the bifurcation set

$$\mathscr{B} := \{q \in (1, M+1] : H(p) \neq H(q) \text{ for any } p \neq q\} .$$

Note that $\mathscr{B}$ is contained in the set $\mathscr{U}^R$ of bases $q \in (1, M+1]$ such that $1 \in \mathcal{U}_q$. By using a transversality technique we also calculate the Hausdorff dimension of the difference $\mathscr{B} \backslash \mathscr{U}^R$. Interestingly this quantity is always strictly between 0 and 1. When $M = 1$ the Hausdorff dimension of $\mathscr{B} \backslash \mathscr{U}^R$ is $\frac{\log 2}{3 \log \lambda^*} \approx 0.368699$, where $\lambda^*$ is the unique root in $(1, 2)$ of the equation $x^5 - x^4 - x^3 - 2x^2 + x + 1 = 0$.

## 1. Introduction

Fix a positive integer $M$. For $q \in (1, M+1]$, a sequence $(x_i) = x_1x_2\ldots$ with each $x_i \in \{0, 1, \ldots, M\}$ is called a *$q$-expansion* of $x$ if

$$x = \sum_{i=1}^{\infty} \frac{x_i}{q^i} =: \pi_q((x_i)). \tag{1.1}$$

Here the *alphabet* $\{0, 1, \ldots, M\}$ will be fixed throughout the paper. Clearly, $x$ has a $q$-expansion if and only if $x \in I_q := [0, M/(q-1)]$. When $q = M+1$ we know that each $x \in I_{M+1} = [0, 1]$ has a unique $(M+1)$-expansion except for countably many points, which have precisely two expansions. When $q \in (1, M+1)$ the set of expansions of an $x \in I_q$ can be much more complicated. Sidorov showed in [25] that Lebesgue almost every $x \in I_q$ has a continuum of $q$-expansions. Therefore, the set of $x \in I_q$ with a unique $q$-expansion is negligible in the sense of Lebesgue measure. On the other hand, the third author and his coauthors showed in [19] that the set of $x \in I_q$ with a unique $q$-expansion has positive Hausdorff dimension when $q > q_{KL}$, where $q_{KL} = q_{KL}(M)$ is the *Komornik-Loreti constant* (see Section 2 for more details).

For $q \in (1, M+1]$ let $\mathcal{U}_q$ be the *univoque set* of $x \in I_q$ having a unique $q$-expansion. This means that for any $x \in \mathcal{U}_q$ there exists a unique sequence $(x_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ such that

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$x = \pi_q((x_i))$. Denote by $\mathbf{U}_q = \pi_q^{-1}(\mathcal{U}_q)$ the corresponding set of $q$-expansions. Note that $\pi_q$ is a bijection from $\mathbf{U}_q$ to $\mathcal{U}_q$. So the study of the univoque set $\mathcal{U}_q$ is equivalent to the study of the *symbolic univoque set* $\mathbf{U}_q$.

1.1. **Set-valued bifurcation set $\mathscr{U}$.** Let $\Omega := \{0, 1, \ldots, M\}^{\mathbb{N}}$ be the set of all sequences with each element from $\{0, 1, \ldots, M\}$. Then $(\Omega, \rho)$ is a compact metric space with respect to the metric $\rho$ defined by

$$\rho((c_i), (d_i)) = (M+1)^{-\inf\{j \geq 1 : c_j \neq d_j\}}. \tag{1.2}$$

Under the metric $\rho$ the Hausdorff dimension of any subset $E \subseteq \Omega$ is well-defined.

Note that the set-valued map $F : q \mapsto \mathbf{U}_q$ is increasing, i.e., $\mathbf{U}_p \subseteq \mathbf{U}_q$ for any $p, q \in (1, M+1]$ with $p < q$ (see Section 2 for more explanation). In [8] de Vries and Komornik showed that the map $F$ is locally constant almost everywhere. On the other hand, the third author and his coauthors proved in [20] that there exist infinitely many $q \in (1, M+1]$ such that the difference between $\mathbf{U}_q$ and $\mathbf{U}_p$ for any $p \neq q$ is significant: $\mathbf{U}_q \triangle \mathbf{U}_p$ has positive Hausdorff dimension, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ stands for the symmetric difference of two sets $A$ and $B$. Let $\mathscr{U}$ be the *bifurcation set* of the set-valued map $F$, defined by

$$\mathscr{U} = \mathscr{U}(M) := \{q \in (1, M+1] : \dim_H(\mathbf{U}_p \triangle \mathbf{U}_q) > 0 \text{ for any } p \neq q\}.$$

Then $\mathscr{U}$ is a Lebesgue null set of full Hausdorff dimension (cf. [20]). Furthermore,

$$(1, M+1] \setminus \mathscr{U} = (1, q_{KL}] \cup \bigcup [q_0, q_0^*]. \tag{1.3}$$

The union on the right hand-side of (1.3) is pairwise disjoint and countable. By the definition of $\mathscr{U}$ it follows that each connected component $[q_0, q_0^*]$ is a maximum interval such that the difference $\mathbf{U}_{q_0} \triangle \mathbf{U}_{q_0^*} = \mathbf{U}_{q_0^*} \setminus \mathbf{U}_{q_0}$ has zero Hausdorff dimension. So the closed interval $[q_0, q_0^*]$ is called a *plateau* of $F$. Indeed, for any $q \in (q_0, q_0^*)$ the difference $\mathbf{U}_q \setminus \mathbf{U}_{q_0}$ is at most countable, and for $q = q_0^*$ the difference $\mathbf{U}_{q_0^*} \setminus \mathbf{U}_{q_0}$ is uncountable but of zero Hausdorff dimension (cf. [20, Lemma 3.4]). Furthermore, each left endpoint $q_0$ is an algebraic integer, and each right endpoint $q_0^*$, called a *de Vries-Komornik number*, is a transcendental number (cf. [18]).

Instead of investigating the bifurcation set $\mathscr{U}$ directly, we consider two modified bifurcation sets:

$$\mathscr{U}^L = \mathscr{U}^L(M) := \{q \in (1, M+1] : \dim_H(\mathbf{U}_q \setminus \mathbf{U}_p) > 0 \text{ for any } p \in (1, q)\};$$
$$\mathscr{U}^R = \mathscr{U}^R(M) := \{q \in (1, M+1] : \dim_H(\mathbf{U}_r \setminus \mathbf{U}_q) > 0 \text{ for any } r \in (q, M+1]\}.$$

The sets $\mathscr{U}^L, \mathscr{U}^R$ are called the *left bifurcation set* and the *right bifurcation set* of $F$, respectively. Clearly, $\mathscr{U} \subset \mathscr{U}^L$ and $\mathscr{U} \subset \mathscr{U}^R$. Furthermore,

$$\mathscr{U}^L \cap \mathscr{U}^R = \mathscr{U} \quad \text{and} \quad \mathscr{U}^L \cup \mathscr{U}^R = \overline{\mathscr{U}}.$$

By (1.3) it follows that the difference set $\mathscr{U}^L \setminus \mathscr{U}$ consists of all left endpoints of the plateaus in $(q_{KL}, M+1]$ of $F$, and hence it is countable. Similarly, the difference set $\mathscr{U}^R \setminus \mathscr{U}$ consists of all right endpoints of the plateaus of $F$. Therefore,

$$\begin{aligned} (1, M+1] \setminus \mathscr{U}^L &= (1, q_{KL}] \cup \bigcup (q_0, q_0^*], \\ (1, M+1] \setminus \mathscr{U}^R &= (1, q_{KL}) \cup \bigcup [q_0, q_0^*). \end{aligned} \tag{1.4}$$

Note by [20] that $\mathscr{U}^R$ is the set of *univoque bases* $q \in (1, M+1]$ such that 1 has a unique $q$-expansion, i.e., $\mathscr{U}^R = \{q \in (1, M+1] : 1 \in \mathcal{U}_q\}$. For $M = 1$, the set $\mathscr{U}^R(1)$ was first studied by Erdős et al. [10, 11]. They showed that the set $\mathscr{U}^R(1)$ is uncountable, of first category and of zero Lebesgue measure. Later, Daróczy and Kátai proved in [7] that the set $\mathscr{U}^R(1)$ has full Hausdorff dimension. Recently, Komornik and Loreti showed in [17] that the topological closure $\overline{\mathscr{U}^R(1)}$ is a *Cantor set*: a non-empty perfect set with no interior points. Indeed, for general $M \geq 1$, the above properties of $\mathscr{U}^R = \mathscr{U}^R(1)$ also hold (cf. [9, 15]). The intimate connection between $\mathscr{U}^R$ and the univoque set $\mathcal{U}_q$ was discovered by de Vries and Komornik [8]. Some connections with dynamical systems, continued fractions and even the Mandelbrot set can be found in [6]. Since the differences among $\mathscr{U}, \mathscr{U}^L$ and $\mathscr{U}^R$ are at most countable, the above properties also hold for $\mathscr{U}$ and $\mathscr{U}^L$.

Now we recall from [20] the following characterizations of the left and right bifurcation sets $\mathscr{U}^L$ and $\mathscr{U}^R$ respectively.

**Theorem 1.1** ([20])**.**

(i) $q \in \mathscr{U}^L$ *if and only if* $\dim_H(\mathscr{U} \cap (p, q)) > 0$ *for any* $p \in (1, q)$.
(ii) $q \in \mathscr{U}^R$ *if and only if* $\dim_H(\mathscr{U} \cap (q, r)) > 0$ *for any* $r \in (q, M+1]$.

*Remark* 1.2. Since $\mathscr{U} = \mathscr{U}^L \cap \mathscr{U}^R$, Theorem 1.1 also gives an equivalent condition for the bifurcation set $\mathscr{U}$, i.e., $q \in \mathscr{U}$ if and only if

$$\dim_H(\mathscr{U} \cap (p, q)) > 0 \qquad \text{and} \qquad \dim_H(\mathscr{U} \cap (q, r)) > 0$$

for any $1 < p < q < r \leq M + 1$.

1.2. **Entropy bifurcation set $\mathscr{B}$.** For a symbolic subset $X \subset \Omega$ its *topological entropy* is defined by

$$h(X) := \liminf_{n\to\infty} \frac{\log \#B_n(X)}{n},$$

where $B_n(X)$ denotes the set of all length $n$ subwords occurring in elements of $X$, and $\#A$ denotes the cardinality of a set $A$. Here and throughout the paper we use base $M + 1$ logarithms. Recently, Komornik et al. showed in [15] (see also Lemma 2.5 below) that the function

$$H : (1, M+1] \to [0, 1]; \qquad q \mapsto h(\mathbf{U}_q)$$

is a Devil's staircase:

- $H$ is a continuous and increasing function from $(1, M + 1]$ onto $[0, 1]$.
- $H$ is locally constant Lebesgue almost everywhere in $(1, M + 1]$.

Let $\mathscr{B}$ be the *bifurcation set* of the entropy function $H$, defined by

$$\mathscr{B} = \mathscr{B}(M) := \{q \in (1, M+1] : H(p) \neq H(q) \text{ for any } p \neq q\}.$$

In [1] Alcaraz Barrera with the second and third authors proved that $\mathscr{B} \subset \mathscr{U}$, and hence $\mathscr{B}$ is of zero Lebesgue measure. They also showed that $\mathscr{B}$ has full Hausdorff dimension. Furthermore, $\mathscr{B}$ has no isolated points and can be written as

$$(1, M+1] \setminus \mathscr{B} = (1, q_{KL}] \cup \bigcup [p_L, p_R], \tag{1.5}$$

where the union on the right hand side is countable and pairwise disjoint. By the definition of the bifurcation set $\mathscr{B}$ it follows that each connected component $[p_L, p_R]$ is a maximal interval on which $H$ is constant. Thus each closed interval $[p_L, p_R]$ is called a *plateau* of $H$ (or an *entropy plateau*). Furthermore, the left and right endpoints of each entropy plateau in $(q_{KL}, M+1]$ are both algebraic numbers (see also Lemma 3.1 below).

In analogy with $\mathscr{U}^L$ and $\mathscr{U}^R$ we also define two one-sided bifurcation sets of $H$:

$$\mathscr{B}^L = \mathscr{B}^L(M) := \{q \in (1, M+1] : H(p) < H(q) \text{ for any } p \in (1,q)\};$$
$$\mathscr{B}^R = \mathscr{B}^R(M) := \{q \in (1, M+1] : H(r) > H(q) \text{ for any } r \in (q, M+1]\}.$$

We call $\mathscr{B}^L$ and $\mathscr{B}^R$ the *left bifurcation set* and the *right bifurcation set* of $H$, respectively. Comparing these sets with the bifurcation sets $\mathscr{U}, \mathscr{U}^L$ and $\mathscr{U}^R$ of $F$, we have analogous properties for the bifurcation sets $\mathscr{B}, \mathscr{B}^L$ and $\mathscr{B}^R$. For example, $\mathscr{B} \subset \mathscr{B}^L$ and $\mathscr{B} \subset \mathscr{B}^R$. Furthermore,

$$\mathscr{B}^L \cap \mathscr{B}^R = \mathscr{B} \qquad \text{and} \qquad \mathscr{B}^L \cup \mathscr{B}^R = \overline{\mathscr{B}}.$$

The difference set $\mathscr{B}^L \setminus \mathscr{B}$ consists of all left endpoints of the plateaus in $(q_{KL}, M+1]$ of $H$. Similarly, $\mathscr{B}^R \setminus \mathscr{B}$ consists of all right endpoints of the plateaus of $H$. In other words, by (1.5) we have

$$\begin{aligned} (1, M+1] \setminus \mathscr{B}^L &= (1, q_{KL}] \cup \bigcup (p_L, p_R], \\ (1, M+1] \setminus \mathscr{B}^R &= (1, q_{KL}) \cup \bigcup [p_L, p_R). \end{aligned} \tag{1.6}$$

We emphasize that $M+1$ belongs to $\mathscr{B}, \mathscr{B}^L$ and $\mathscr{B}^R$. Since $\mathscr{B} \subset \mathscr{U}$, by (1.4) and (1.6) we also have

$$\mathscr{B}^L \subset \mathscr{U}^L \qquad \text{and} \qquad \mathscr{B}^R \subset \mathscr{U}^R.$$

Now we state our main results. Inspired by the characterizations of $\mathscr{U}^L$ and $\mathscr{U}^R$ described in Theorem 1.1, we characterize the left and right bifurcation sets $\mathscr{B}^L$ and $\mathscr{B}^R$ respectively.

**Theorem 1.** *If $M = 1$ or $M$ is even, the following statements are equivalent.*

(i) $q \in \mathscr{B}^L$.
(ii) $\dim_H(\mathbf{U}_q \setminus \mathbf{U}_p) = \dim_H \mathbf{U}_q > 0$ *for any* $p \in (1,q)$.
(iii) $\lim_{p \nearrow q} \dim_H(\mathscr{B} \cap (p,q)) = \dim_H \mathcal{U}_q > 0$.
(iv) $\lim_{p \nearrow q} \dim_H(\mathscr{U} \cap (p,q)) = \dim_H \mathcal{U}_q > 0$.

For odd $M \geq 3$ this theorem must be modified. This is due to the surprising presence of a single exceptional base $q_\star$ which is not an element of $\mathscr{B}^L$, but for which (ii) and (iv) of Theorem 1 nonetheless hold. Let

$$q_\star = q_\star(M) := \begin{cases} \frac{k+3+\sqrt{k^2+6k+1}}{2} & \text{if} \quad M = 2k+1, \\ \frac{k+3+\sqrt{k^2+6k-3}}{2} & \text{if} \quad M = 2k. \end{cases} \tag{1.7}$$

(We will have use for $q_\star(M)$ with $M$ even later on.)

**Theorem 1′.** Suppose $M = 2k+1 \geq 3$.

(a) $q \in \mathscr{B}^L$ if and only if $\lim_{p \nearrow q} \dim_H(\mathscr{B} \cap (p,q)) = \dim_H \mathcal{U}_q > 0$.
(b) The following statements are equivalent:
  (i) $q \in \mathscr{B}^L \cup \{q_\star(M)\}$.
  (ii) $\dim_H(\mathbf{U}_q \setminus \mathbf{U}_p) = \dim_H \mathbf{U}_q > 0$ for any $p \in (1,q)$.
  (iii) $\lim_{p \nearrow q} \dim_H(\mathscr{U} \cap (p,q)) = \dim_H \mathcal{U}_q > 0$.

The characterization of $\mathscr{B}^R$ is more straightforward:

**Theorem 2.** *The following statements are equivalent for every $M \in \mathbb{N}$.*

(i) $q \in \mathscr{B}^R$.
(ii) $\dim_H(\mathbf{U}_r \setminus \mathbf{U}_q) = \dim_H \mathbf{U}_r > 0$ *for any* $r \in (q, M+1]$.
(iii) $\lim_{r\searrow q} \dim_H(\mathscr{B} \cap (q,r)) = \dim_H \mathcal{U}_q > 0$*, or* $q = q_{KL}$.
(iv) $\lim_{r\searrow q} \dim_H(\mathscr{U} \cap (q,r)) = \dim_H \mathcal{U}_q > 0$*, or* $q = q_{KL}$.

The asymmetry between the characterizations of $\mathscr{B}^L$ and $\mathscr{B}^R$ can be partially explained by the asymmetry of entropy plateaus. For instance, if $[p_L, p_R]$ is an entropy plateau, it follows from [1, Lemma 4.10] that $p_L \in \overline{\mathscr{U}^R} \setminus \mathscr{U}^R$, whereas $p_R \in \mathscr{U}^R$. Moreover, $p_R$ is a left and right accumulation point of $\mathscr{U}$, but $p_L$ is not a right accumulation point of $\mathscr{U}$. This helps explain why there is no counterpart in Theorem 2 to the special base $q_\star(M)$ of Theorem 1′.

*Remark* 1.3.

(1) Since $\mathscr{B} = \mathscr{B}^L \cap \mathscr{B}^R$ and $q_{KL} \notin \mathscr{B}$, Theorems 1, 1′ and 2 give equivalent conditions for the bifurcation set $\mathscr{B}$. For example, when $M = 1$, $q \in \mathscr{B}$ if and only if

$$\lim_{p\nearrow q} \dim_H(\mathscr{U} \cap (p,q)) = \lim_{r\searrow q} \dim_H(\mathscr{U} \cap (q,r)) = \dim_H \mathcal{U}_q > 0.$$

(2) In view of Lemma 3.11 below, we emphasize that the limits in statements (iii) and (iv) of Theorems 1 and 2 are at most equal to $\dim_H \mathcal{U}_q$ for every $q \in (1, M+1]$. So, the theorems characterize when this largest possible value is attained.

Since the sets $\mathscr{U}$ and $\mathscr{B}$ are of Lebesgue measure zero and nowhere dense, a natural measure of their distribution within the interval $(1, M+1]$ are the local dimension functions

$$\lim_{\delta\to 0} \dim_H(\mathscr{U} \cap (q-\delta, q+\delta)) \qquad \text{and} \qquad \lim_{\delta\to 0} \dim_H(\mathscr{B} \cap (q-\delta, q+\delta)).$$

In [14, Theorem 2] it was shown that

$$q \in \overline{\mathscr{B}} \setminus \{q_{KL}\} \iff \lim_{\delta\to 0} \dim_H(\mathscr{B} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q > 0.$$

As for the set $\mathscr{U}$, we will show in Lemma 3.11 below that

$$\lim_{\delta\to 0} \dim_H(\mathscr{U} \cap (q-\delta, q+\delta)) \le \dim_H \mathcal{U}_q \qquad \text{for all} \quad q \in (1, M+1]. \tag{1.8}$$

Observe that $q_\star(M) \in \mathscr{B}^R$ for $M = 2k+1 \ge 3$. (See Lemma 3.1 below.) Thus Theorems 1, 1′ and 2 imply that the upper bound $\dim_H \mathcal{U}_q$ for the limit in (1.8) is attained if and only if $q \in \overline{\mathscr{B}}$. Precisely:

**Corollary 3.** $q \in \overline{\mathscr{B}} \setminus \{q_{KL}\}$ *if and only if*

$$\lim_{\delta\to 0} \dim_H(\mathscr{U} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q > 0.$$

Clearly, $\lim_{\delta\to 0} \dim_H(\mathscr{U} \cap (q-\delta, q+\delta)) = 0$ when $q \notin \overline{\mathscr{U}}$. It is interesting to ask which values this limit can take for $q \in \overline{\mathscr{U}} \setminus \overline{\mathscr{B}}$. This may be the subject of a future paper.

1.3. **The difference set $\mathscr{U}\setminus\mathscr{B}$.** Note that $\mathscr{B}\subset\mathscr{U}$, and both are Lebesgue null sets of full Hausdorff dimension. Furthermore, $\mathscr{U}\setminus\mathscr{B}$ is a dense subset of $\mathscr{U}$. So the box dimension of $\mathscr{U}\setminus\mathscr{B}$ is given by

$$\dim_B(\mathscr{U}\setminus\mathscr{B})=\dim_B(\overline{\mathscr{U}\setminus\mathscr{B}})=\dim_B\overline{\mathscr{U}}=1.$$

On the other hand, our next result shows that the Hausdorff dimension of $\mathscr{U}\setminus\mathscr{B}$ is significantly smaller than one.

**Theorem 4.**

(i) *If $M=1$, then*

$$\dim_H(\mathscr{U}\setminus\mathscr{B})=\frac{\log 2}{3\log\lambda^*}\approx 0.368699,$$

*where $\lambda^*\approx 1.87135$ is the unique root in $(1,2)$ of the equation $x^5-x^4-x^3-2x^2+x+1=0$.*

(ii) *If $M=2$, then*

$$\dim_H(\mathscr{U}\setminus\mathscr{B})=\frac{\log 2}{2\log\gamma^*}\approx 0.339607,$$

*where $\gamma^*\approx 2.77462$ is the unique root in $(2,3)$ of the equation $x^4-2x^3-3x^2+2x+1=0$.*

(iii) *If $M\geq 3$, then*

$$\dim_H(\mathscr{U}\setminus\mathscr{B})=\frac{\log 2}{\log q_\star(M)},$$

*where $q_\star(M)$ is given by (1.7).*

Table 1 below lists the values of $\dim_H(\mathscr{U}\setminus\mathscr{B})$ for $1\leq M\leq 8$. For large $M$ we have by Theorem 4 (iii) the simple approximation $\dim_H(\mathscr{U}\setminus\mathscr{B})\approx\log 2/\log(k+3)$, where $k$ is the greatest integer less than or equal to $M/2$. This systematically underestimates the true value, with an error slowly tending to zero. Observe also that $\dim_H(\mathscr{U}\setminus\mathscr{B})\to 0$ as $M\to\infty$.

| $M$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\dim_H(\mathscr{U}\setminus\mathscr{B})$ | 0.3687 | 0.3396 | 0.5645 | 0.4750 | 0.4567 | 0.4088 | 0.4005 | 0.3091 |

TABLE 1. The numerical calculation of $\dim_H(\mathscr{U}\setminus\mathscr{B})$ for $M=1,\ldots,8$.

In [14], Kalle et al. showed that $\dim_H(\mathscr{U}^R\cap(1,t])=\max_{q\leq t}\dim_H\mathcal{U}_q$ for all $t>1$, and they asked whether more generally it is possible to calculate $\dim_H(\mathscr{U}^R\cap[t_1,t_2])$ for any interval $[t_1,t_2]$. In the process of proving Theorem 4, we give a partial answer to their question by computing the Hausdorff dimension of the intersection of $\mathscr{U}^R$ with any entropy plateau $[p_L,p_R]$ (see Theorem 4.1).

The rest of the paper is arranged as follows. In Section 2 we recall some results from unique $q$-expansions, and give the Hausdorff dimension of the symbolic univoque set $\mathbf{U}_q$ (see Lemma 2.8). Based on these observations we characterize the left and right bifurcation sets $\mathscr{B}^L$ and $\mathscr{B}^R$ in Section 3, by proving Theorems 1, 1′ and 2. In Section 4 we prove Theorem 4.

## 2. Unique expansions

In this section we will describe the symbolic univoque set $\mathbf{U}_q$ and calculate its Hausdorff dimension. Recall that $\Omega = \{0, 1, \ldots, M\}^{\mathbb{N}}$. Let $\sigma$ be the *left shift* on $\Omega$ defined by $\sigma((c_i)) = (c_{i+1})$. Then $(\Omega, \sigma)$ is a *full shift*. By a *word* $\mathbf{c}$ we mean a finite string of digits $\mathbf{c} = c_1 \ldots c_n$ with each digit $c_i \in \{0, 1, \ldots, M\}$. For two words $\mathbf{c} = c_1 \ldots c_m$ and $\mathbf{d} = d_1 \ldots d_n$ we denote by $\mathbf{cd} = c_1 \ldots c_m d_1 \ldots d_n$ their concatenation. For a positive integer $n$ we write $\mathbf{c}^n = \mathbf{c} \cdots \mathbf{c}$ for the $n$-fold concatenation of $\mathbf{c}$ with itself. Furthermore, we write $\mathbf{c}^\infty = \mathbf{cc} \cdots$ for the infinite periodic sequence with period block $\mathbf{c}$. For a word $\mathbf{c} = c_1 \ldots c_m$ we set $\mathbf{c}^+ := c_1 \ldots c_{m-1}(c_m + 1)$ if $c_m < M$, and set $\mathbf{c}^- := c_1 \ldots c_{m-1}(c_m - 1)$ if $c_m > 0$. Furthermore, we define the *reflection* of the word $\mathbf{c}$ by $\overline{\mathbf{c}} := (M - c_1)(M - c_2) \cdots (M - c_m)$. Clearly, $\mathbf{c}^+, \mathbf{c}^-$ and $\overline{\mathbf{c}}$ are all words with digits from $\{0, 1, \ldots, M\}$. For a sequence $(c_i) \in \Omega$ its reflection is also a sequence in $\Omega$ defined by $\overline{(c_i)} = (M - c_1)(M - c_2) \cdots$.

Throughout the paper we will use the *lexicographical ordering* $\prec, \preccurlyeq, \succ$ and $\succcurlyeq$ between sequences and words. More precisely, for two sequences $(c_i), (d_i) \in \Omega$ we say $(c_i) \prec (d_i)$ or $(d_i) \succ (c_i)$ if there exists an integer $n \geq 1$ such that $c_1 \ldots c_{n-1} = d_1 \ldots d_{n-1}$ and $c_n < d_n$. Furthermore, we say $(c_i) \preccurlyeq (d_i)$ if $(c_i) \prec (d_i)$ or $(c_i) = (d_i)$. Similarly, for two words $\mathbf{c}$ and $\mathbf{d}$ we say $\mathbf{c} \prec \mathbf{d}$ or $\mathbf{d} \succ \mathbf{c}$ if $\mathbf{c}0^\infty \prec \mathbf{d}0^\infty$.

Let $q \in (1, M+1]$. Recall that $\mathbf{U}_q$ is the symbolic univoque set which contains all sequences $(x_i) \in \Omega$ such that $(x_i)$ is the unique $q$-expansion of $\pi_q((x_i))$. Here $\pi_q$ is the projection map defined in (1.1). The description of $\mathbf{U}_q$ is based on the *quasi-greedy* $q$-expansion of 1, denoted by $\alpha(q) = \alpha_1(q)\alpha_2(q) \ldots$, which is the lexicographically largest $q$-expansion of 1 not ending with $0^\infty$ (cf. [7]). The following characterization of $\alpha(q)$ was given in [4, Theorem 2.2] (see also [9, Proposition 2.3]).

**Lemma 2.1.** *The map $q \mapsto \alpha(q)$ is a strictly increasing bijection from $(1, M+1]$ onto the set of all sequences $(a_i) \in \Omega$ not ending with $0^\infty$ and satisfying*

$$a_{n+1}a_{n+2} \ldots \preccurlyeq a_1 a_2 \ldots \qquad \textit{for all} \quad n \geq 0.$$

*Furthermore, the map $q \mapsto \alpha(q)$ is left-continuous.*

*Remark* 2.2. Let $\mathscr{A} := \{\alpha(q) : q \in (1, M+1]\}$. Then Lemma 2.1 implies that the inverse map

$$\alpha^{-1} : \mathscr{A} \to (1, M+1]; \qquad (a_i) \mapsto \alpha^{-1}((a_i))$$

is bijective and strictly increasing. Furthermore, we can even show that $\alpha^{-1}$ is continuous; see the proof of Lemma 3.6 below.

Based on the quasi-greedy expansion $\alpha(q)$ we give the lexicographic characterization of the symbolic univoque set $\mathbf{U}_q$, which was essentially established by Parry [23] (see also [15]).

**Lemma 2.3.** *Let $q \in (1, M+1]$. Then $(x_i) \in \mathbf{U}_q$ if and only if*

$$\begin{cases} x_{n+1}x_{n+2} \ldots \prec \alpha(q) & \textit{whenever} \quad x_n < M, \\ x_{n+1}x_{n+2} \ldots \succ \overline{\alpha(q)} & \textit{whenever} \quad x_n > 0. \end{cases}$$

Note by Lemma 2.1 that when $q$ is increasing the quasi-greedy expansion $\alpha(q)$ is also increasing in the lexicographical ordering. By Lemma 2.3 it follows that the set-valued map $q \mapsto \mathbf{U}_q$ is also increasing, i.e., $\mathbf{U}_p \subseteq \mathbf{U}_q$ when $p < q$.

Recall from [16] that the Komornik-Loreti constant $q_{KL} = q_{KL}(M)$ is the smallest element of $\mathscr{U}^R$, and satisfies

$$\alpha(q_{KL}) = \lambda_1\lambda_2\ldots, \tag{2.1}$$

where for each $i \geq 1$,

$$\lambda_i = \lambda_i(M) := \begin{cases} k + \tau_i - \tau_{i-1} & \text{if} \quad M = 2k, \\ k + \tau_i & \text{if} \quad M = 2k+1. \end{cases} \tag{2.2}$$

Here $(\tau_i)_{i=0}^{\infty} = 0110100110010110\ldots$ is the classical *Thue-Morse sequence* (cf. [3]). We emphasize that the sequence $(\lambda_i)$ depends on $M$. The following recursive relation of $(\lambda_i)$ was established in [16] (see also [18]):

$$\lambda_{2^n+1}\ldots\lambda_{2^{n+1}} = \overline{\lambda_1\ldots\lambda_{2^n}}^{\,+} \qquad \text{for all} \quad n \geq 0. \tag{2.3}$$

By (2.1) and (2.2) it follows that $q_{KL}(M) \geq (M+2)/2$ for all $M \geq 1$ (see also [5]), and the map $M \mapsto q_{KL}(M)$ is strictly increasing.

**Example 2.4.** The following values of $q_{KL}(M)$ will be needed in the proof of Theorem 4 in Section 4.

(1) Let $M = 1$. Then by (2.2) we have $\lambda_1 = 1$. By (2.1) and (2.3) it follows that

$$\alpha(q_{KL}(1)) = 1101\,0011\,00101101\ldots = (\tau_i)_{i=1}^{\infty}.$$

This gives $q_{KL}(1) \approx 1.78723$.

(2) Let $M = 2$. Then by (2.2) we have $\lambda_1 = 2$, and by (2.1) and (2.3) that

$$\alpha(q_{KL}(2)) = 2102\,0121\,01202102\ldots.$$

So $q_{KL}(2) \approx 2.53595$.

(3) Let $M = 3$. Then by (2.2) we have $\lambda_1 = 2$, and by (2.1) and (2.3) that

$$\alpha(q_{KL}(3)) = 2212\,1122\,11212212\ldots.$$

Hence, $q_{KL}(3) \approx 2.91002$.

Now we recall from [15] the following result for the Hausdorff dimension of the univoque set $\mathcal{U}_q$.

**Lemma 2.5.**

(i) *For any $q \in (1, M+1]$ we have*

$$\dim_H \mathcal{U}_q = \frac{h(\mathbf{U}_q)}{\log q}.$$

(ii) *The entropy function $H : q \mapsto h(\mathbf{U}_q)$ is a Devil's staircase in $(1, M+1]$:*
- *$H$ is increasing and continuous from $(1, M+1]$ onto $[0, 1]$;*
- *$H$ is locally constant almost everywhere in $(1, M+1]$.*

(iii) *$H(q) > 0$ if and only if $q > q_{KL}$. Furthermore, $H(q) = \log(M+1)$ if and only if $q = M+1$.*

We also need the following lemma for the Hausdorff dimension under Hölder continuous maps (cf. [12]).

**Lemma 2.6.** *Let $f : (X, \rho_X) \to (Y, \rho_Y)$ be a Hölder map between two metric spaces, i.e., there exist two constants $C > 0$ and $\xi > 0$ such that*

$$\rho_Y(f(x), f(y)) \le C\rho_X(x, y)^\xi \quad \text{for any } x, y \in X.$$

*Then* $\dim_H f(X) \le \frac{1}{\xi} \dim_H X$.

Recall the metric $\rho$ from (1.2). It will be convenient to introduce a more general family of (mutually equivalent) metrics $\{\rho_q : q > 1\}$ on $\Omega$ defined by

$$\rho_q((c_i), (d_i)) := q^{-\inf\{i\ge 1: c_i \ne d_i\}}, \qquad q > 1.$$

Then $(\Omega, \rho_q)$ is a compact metric space. Let $\dim_H^{(q)}$ denote Hausdorff dimension on $\Omega$ with respect to the metric $\rho_q$, so

$$\dim_H^{(M+1)} E = \dim_H E$$

for any subset $E \subseteq \Omega$. For $p > 1$ and $q > 1$,

$$\rho_q((c_i), (d_i)) = \rho_p((c_i), (d_i))^{\log q/\log p},$$

and by Lemma 2.6 this gives the useful relationship

$$\dim_H^{(p)} E = \frac{\log q}{\log p} \dim_H^{(q)} E, \qquad E \subseteq \Omega. \tag{2.4}$$

The following result is well known (see [13, Lemma 2.7] or [2, Lemma 2.2]):

**Lemma 2.7.** *For each $q \in (1, M+1)$, the map $\pi_q$ is Lipschitz on $(\Omega, \rho_q)$, and the restriction*

$$\pi_q : (\mathbf{U}_q, \rho_q) \to (\mathcal{U}_q, |.|); \qquad \pi_q((x_i)) = \sum_{i=1}^{\infty} \frac{x_i}{q^i}$$

*is bi-Lipschitz, where $|.|$ denotes the Euclidean metric on $\mathbb{R}$.*

Observe that the Hausdorff dimension does not exceed the lower box dimension (cf. [12]). This implies that $\dim_H E \le h(E)$ for any set $E \subset \Omega$. Using Lemmas 2.5–2.7 we show that equality holds for $\mathbf{U}_q$.

**Lemma 2.8.** *Let $q \in (1, M+1]$. Then*

$$\dim_H \mathbf{U}_q = h(\mathbf{U}_q).$$

*Proof.* For $q = M + 1$, one checks easily that

$$\dim_H \mathbf{U}_{M+1} = h(\mathbf{U}_{M+1}) = 1.$$

Let $q \in (1, M+1)$. By Lemmas 2.7 and 2.6, $\dim_H^{(q)} \mathbf{U}_q = \dim_H \mathcal{U}_q$. So (2.4), Lemmas 2.7 and 2.5 give

$$\dim_H \mathbf{U}_q = \dim_H^{(M+1)} \mathbf{U}_q = \frac{\log q}{\log(M+1)} \dim_H^{(q)} \mathbf{U}_q = \log q \dim_H \mathcal{U}_q = h(\mathbf{U}_q),$$

as desired. We emphasize that the base for our logarithms is $M + 1$. □

Note that the symbolic univoque set $\mathbf{U}_q$ is not always closed. Inspired by the works of de Vries and Komornik [8] and Komornik et al. [15] we introduce the set

$$\mathbf{V}_q := \left\{ (x_i) \in \Omega : \overline{\alpha(q)} \preccurlyeq x_{n+1}x_{n+2}\ldots \preccurlyeq \alpha(q) \text{ for all } n \geq 0 \right\}. \tag{2.5}$$

We have the following relationship between $\mathbf{V}_q$ and $\mathbf{U}_q$.

**Lemma 2.9.** *For any $0 < p < q \leq M+1$ we have*

$$\dim_H \mathbf{V}_q = \dim_H \mathbf{U}_q \qquad \text{and} \qquad \dim_H(\mathbf{V}_q \setminus \mathbf{V}_p) = \dim_H(\mathbf{U}_q \setminus \mathbf{U}_p).$$

*Proof.* By Lemma 2.3 it follows that for each $q \in (1, 2]$ the set $\mathbf{U}_q$ is a countable union of affine copies of $\mathbf{V}_q$ up to a countable set (see also [14, Lemma 3.2]), i.e., there exists a sequence of affine maps $\{g_i\}_{i=1}^{\infty}$ on $\Omega$ of the form

$$x_1x_2\ldots \mapsto ax_1x_2\ldots, \qquad x_1x_2\ldots \mapsto M^m bx_1x_2\ldots \qquad \text{or} \qquad x_1x_2\ldots \mapsto 0^m cx_1x_2\ldots,$$

where $a \in \{1, 2, \ldots, M-1\}$, $b \in \{0, 1, \ldots, M-1\}$, $c \in \{1, 2, \ldots, M\}$ and $m = 1, 2, \ldots$, such that

$$\mathbf{U}_q \sim \bigcup_{i=1}^{\infty} g_i(\mathbf{V}_q), \tag{2.6}$$

where we write $A \sim B$ to mean that the symmetric difference $A \triangle B$ is at most countable. Since the Hausdorff dimension is stable under affine maps (cf. [12]), this implies $\dim_H \mathbf{V}_q = \dim_H \mathbf{U}_q$.

Furthermore, for any $1 < p < q \leq M+1$ we have $\mathbf{U}_p \subseteq \mathbf{U}_q$ and $\mathbf{V}_p \subseteq \mathbf{V}_q$, so $g_i(\mathbf{V}_p) \subseteq g_i(\mathbf{V}_q)$ for all $i \geq 1$. Note that for $i \neq j$ the intersection $g_i(\mathbf{V}_q) \cap g_j(\mathbf{V}_q) = \emptyset$. Then by (2.6) it follows that

$$\begin{aligned} \mathbf{U}_q \setminus \mathbf{U}_p &\sim \bigcup_{i=1}^{\infty} g_i(\mathbf{V}_q) \setminus \bigcup_{i=1}^{\infty} g_i(\mathbf{V}_p) \\ &= \bigcup_{i=1}^{\infty} \big(g_i(\mathbf{V}_q) \setminus g_i(\mathbf{V}_p)\big) = \bigcup_{i=1}^{\infty} g_i(\mathbf{V}_q \setminus \mathbf{V}_p). \end{aligned}$$

We conclude that $\dim_H(\mathbf{U}_q \setminus \mathbf{U}_p) = \dim_H(\mathbf{V}_q \setminus \mathbf{V}_p)$. □

## 3. Characterizations of $\mathscr{B}^L$ and $\mathscr{B}^R$

Recall from (1.6) that $\mathscr{B}^L$ and $\mathscr{B}^R$ are the left and right bifurcation sets of $H$. In this section we will characterize the sets $\mathscr{B}^L$ and $\mathscr{B}^R$, and prove Theorems 1, 1′ and 2. Since the theorems are very similar, we will prove only Theorem 1 in full detail, and comment briefly on the proofs of Theorems 1′ and 2.

Recall the definition of $q_\star(M)$ from (1.7). Its significance derives from the fact that

$$\alpha(q_\star(M)) = \begin{cases} (k+2)k^\infty & \text{if} \quad M = 2k+1, \\ (k+2)(k-1)^\infty & \text{if} \quad M = 2k. \end{cases}$$

By (2.1) and Lemma 2.1 it follows in particular that $q_\star(M) > q_{KL}$.

Recall that a closed interval $[p_L, p_R] \subseteq (q_{KL}, M+1]$ is an entropy plateau if it is a maximal interval on which $H$ is constant. The following lemma was implicitly proven in [1].

**Lemma 3.1.** *Let $[p_L, p_R] \subset (q_{KL}, M+1]$ be an entropy plateau.*

(i) *Then there exists a word $a_1 \dots a_m$ satisfying $\overline{a_1} < a_1$ and*

$$\overline{a_1 \dots a_{m-i}} \preccurlyeq a_{i+1} \dots a_m \prec a_1 \dots a_{m-i} \qquad \text{for all} \quad 1 \le i < m,$$

*such that*

$$\alpha(p_L) = (a_1 \dots a_m)^\infty \qquad \text{and} \qquad \alpha(p_R) = a_1 \dots a_m^+ (\overline{a_1 \dots a_m})^\infty.$$

(ii) *Let $m \ge 1$ be defined as in (i). Then*

$$h(\mathbf{U}_{p_L}) \ge \frac{\log 2}{m},$$

*where equality holds if and only if $M = 2k+1 \ge 3$ and $[p_L, p_R] = [k+2, q_\star(M)]$.*

*Proof.* Part (i) was established in [1, Theorem 2 and Lemma 4.1]. Part (ii) was implicitly given in the proofs of [1, Lemmas 5.1 and 5.5]. It is shown there that $h(\mathbf{U}_{p_L}) > \log 2/m$ when $m \ge 2$. If $m = 1$, then $\alpha(p_L) = a_1^\infty$ for some $a_1 \ge (M+1)/2$, and

$$h(\mathbf{U}_{p_L}) = \log(2a_1 - M + 1).$$

(See [1, Example 5.13].) It follows that $h(\mathbf{U}_{p_L}) = \log 2/m$ if and only if $m = 1$, $M = 2k+1 \ge 3$ and $a_1 = k+1$, in which case

$$\alpha(p_L) = (k+1)^\infty \qquad \text{and} \qquad \alpha(p_R) = (k+2)k^\infty,$$

or equivalently,

$$[p_L, p_R] = \left[k+2, \frac{k+3+\sqrt{k^2+6k+1}}{2}\right] = [k+2, q_\star(M)]$$

for $M = 2k+1 \ge 3$. □

*Remark* 3.2. We point out that the condition in Lemma 3.1 (i) is not a sufficient condition for $[p_L, p_R] \subset (q_{KL}, M+1]$ being an entropy plateau. For a complete characterization of entropy plateaus we refer to [1, Theorem 2]. However, if $[p_L, p_R]$ is an interval satisfying the conditions of Lemma 3.1, then $[p_L, p_R]$ is either an entropy plateau or else it is contained in some entropy plateau; see [1] for details.

**Definition 3.3.** If $[p_L, p_R]$ is an entropy plateau with $\alpha(p_L) = (a_1 \dots a_m)^\infty$ and $\alpha(p_R) = a_1 \dots a_m^+ (\overline{a_1 \dots a_m})^\infty$, we shall call $[p_L, p_R]$ an *entropy plateau of period $m$*.

Recall that $\mathscr{U}$ is the bifurcation set of the set-valued map $q \mapsto \mathbf{U}_q$. The following characterization of its topological closure $\overline{\mathscr{U}}$ was established in [17] (see also [9]).

**Lemma 3.4.** *$q \in \overline{\mathscr{U}}$ if and only if*

$$\overline{\alpha(q)} \prec \sigma^n(\alpha(q)) \preccurlyeq \alpha(q) \qquad \text{for all} \quad n \ge 1.$$

Lemma 2.1 states that the map $\alpha : q \mapsto \alpha(q)$ is left-continuous on $(1, M+1]$. The following lemma strengthens this result when $\alpha$ is restricted to $\overline{\mathscr{U}}$.

**Lemma 3.5.** *Let $I = [p, q] \subset (1, M+1)$. Then the map $\alpha$ is Lipschitz on $\overline{\mathscr{U}} \cap I$ with respect to the metric $\rho_q$.*

*Proof.* Fix $1 < p < q < M+1$. We will show something slightly stronger, namely that there is a constant $C = C(p,q)$ such that for any $p \le p_1 < p_2 \le q$ with $p_2 \in \overline{\mathscr{U}}$,

$$\rho_q(\alpha(p_1), \alpha(p_2)) \le C|p_2 - p_1|.$$

Let $p \le p_1 < p_2 \le q$ and $p_2 \in \overline{\mathscr{U}}$. Then by Lemma 2.1 we have $\alpha(p_1) \prec \alpha(p_2)$. So there exists $n \ge 1$ such that $\alpha_1(p_1)\ldots\alpha_{n-1}(p_1) = \alpha_1(p_2)\ldots\alpha_{n-1}(p_2)$ and $\alpha_n(p_1) < \alpha_n(p_2)$. Since $q < M+1$, we have $\alpha(q) \prec M^\infty$. Hence there exists a large integer $N \ge 1$, depending only on $q$, such that $\alpha(p_2) \preccurlyeq \alpha(q) \preccurlyeq M^{N-1}0^\infty$. Since $p_2 \in \overline{\mathscr{U}}$, it follows by Lemma 3.4 that

$$\alpha_{n+1}(p_2)\alpha_{n+2}(p_2)\ldots \succ \overline{\alpha(p_2)} \succcurlyeq 0^{N-1}M^\infty.$$

This implies

$$1 = \sum_{i=1}^\infty \frac{\alpha_i(p_2)}{p_2^i} > \sum_{i=1}^n \frac{\alpha_i(p_2)}{p_2^i} + \frac{1}{p_2^{n+N}}.$$

Therefore,

$$\begin{aligned}
\frac{1}{p_2^{n+N}} \le 1 - \sum_{i=1}^n \frac{\alpha_i(p_2)}{p_2^i} &= \sum_{i=1}^\infty \frac{\alpha_i(p_1)}{p_1^i} - \sum_{i=1}^n \frac{\alpha_i(p_2)}{p_2^i} \\
&\le \sum_{i=1}^n \left(\frac{\alpha_i(p_2)}{p_1^i} - \frac{\alpha_i(p_2)}{p_2^i}\right) \\
&\le \sum_{i=1}^\infty \left(\frac{M}{p_1^i} - \frac{M}{p_2^i}\right) = \frac{M|p_2 - p_1|}{(p_1 - 1)(p_2 - 1)} \\
&\le \frac{M|p_2 - p_1|}{(p-1)^2}.
\end{aligned}$$

Here the second inequality follows by using $\alpha_1(p_1)\ldots\alpha_{n-1}(p_1) = \alpha_1(p_2)\ldots\alpha_{n-1}(p_2)$, $\alpha_n(p_1) < \alpha_n(p_2)$ and the property of quasi-greedy expansion that $\sum_{i=1}^\infty \alpha_{n+i}(p_1)/p_1^i \le 1$. Therefore, we obtain

$$\rho_q(\alpha(p_1), \alpha(p_2)) = q^{-n} \le p_2^{-n} \le \frac{Mq^N}{(p-1)^2}|p_2 - p_1|.$$

The proof is complete. $\square$

The following dimension estimates will be very useful throughout the paper:

**Lemma 3.6.** *For any interval $I = [p,q] \subseteq (1, M+1)$,*

$$\dim_H \pi_q(\mathbf{U}_I) \le \dim_H(\overline{\mathscr{U}} \cap I) \le \frac{h(\mathbf{U}_I)}{\log p},$$

*where $\mathbf{U}_I := \{\alpha(\ell) : \ell \in \overline{\mathscr{U}} \cap I\}$.*

*Proof.* Fix an interval $I = [p,q] \subseteq (1, M+1)$. We may view the map $\pi_q \circ \alpha : \overline{\mathscr{U}} \cap I \to \mathbb{R}$ as the composition of the maps $\alpha : \overline{\mathscr{U}} \cap I \to (\mathbf{U}_I, \rho_q)$ and $\pi_q : (\mathbf{U}_I, \rho_q) \to \mathbb{R}$. The first map is Lipschitz by Lemma 3.5, and the second is Lipschitz by Lemma 2.7, since $\mathbf{U}_I \subset \mathbf{U}_q$. Therefore, the composition $\pi_q \circ \alpha$ is Lipschitz. Using Lemma 2.6, this implies the first inequality.

The second inequality is proved as follows. Let $p \le p_1 < p_2 \le q$. Then $\alpha(p_1) \prec \alpha(p_2)$ by Lemma 2.1, so there is a number $n \in \mathbb{N}$ such that $\alpha_1(p_1)\ldots\alpha_{n-1}(p_1) = \alpha_1(p_2)\ldots\alpha_{n-1}(p_2)$ and $\alpha_n(p_1) < \alpha_n(p_2)$. As in the proof of Lemma 4.3 in [14], we then have

$$\begin{aligned} p_2 - p_1 &= \sum_{i=1}^{\infty} \frac{\alpha_i(p_2)}{p_2^{i-1}} - \sum_{i=1}^{\infty} \frac{\alpha_i(p_1)}{p_1^{i-1}} \\ &\le \sum_{i=1}^{n-1} \left( \frac{\alpha_i(p_2)}{p_2^{i-1}} - \frac{\alpha_i(p_1)}{p_1^{i-1}} \right) + \sum_{i=n}^{\infty} \frac{\alpha_i(p_2)}{p_2^{i-1}} \\ &\le p_2^{2-n} \le (M+1)^2 p^{-n}, \end{aligned}$$

where the second inequality follows by the property of the quasi-greedy expansion $\alpha(p_2)$ of 1. We conclude that

$$\rho(\alpha(p_1), \alpha(p_2)) = (M+1)^{-n} = p^{-n/\log p} \ge \left( \frac{p_2 - p_1}{(M+1)^2} \right)^{1/\log p},$$

in other words, the map $\alpha^{-1}$ is Hölder continuous with exponent $\log p$ on the set $\{\alpha(\ell) : p \le \ell \le q\}$. It follows using Lemma 2.6 that

$$\dim_H(\overline{\mathscr{U}} \cap I) = \dim_H(\alpha^{-1}(\mathbf{U}_I)) \le \frac{\dim_H \mathbf{U}_I}{\log p} \le \frac{h(\mathbf{U}_I)}{\log p},$$

completing the proof. □

Let $[p_L, p_R] \subset (q_{KL}, M+1]$ be an entropy plateau such that $\alpha(p_L) = (a_1 \ldots a_m)^\infty$ and $\alpha(p_R) = a_1 \ldots a_m^+ (\overline{a_1 \ldots a_m})^\infty$. The proofs of the following two propositions use the sofic subshift $(X_{\mathscr{G}}, \sigma)$ represented by the labeled graph $\mathscr{G} = (G, \mathscr{L})$ in Figure 1 (cf. [21, Chapter 3]).

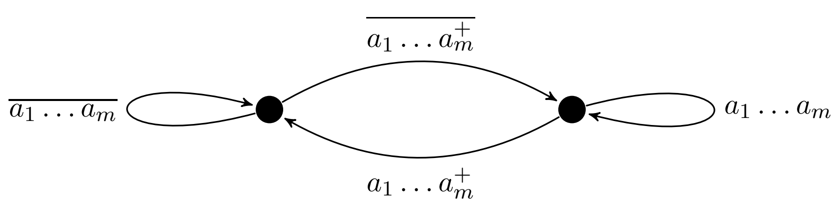


FIGURE 1. The picture of the labeled graph $\mathscr{G} = (G, \mathscr{L})$.

We emphasize that $(X_{\mathscr{G}}, \sigma)$ is in fact a subshift of finite type over the states

$$a_1 \ldots a_m, \qquad a_1 \ldots a_m^+, \qquad \overline{a_1 \ldots a_m} \qquad \text{and} \qquad \overline{a_1 \ldots a_m^+}$$

with adjacency matrix

$$A_{\mathscr{G}} := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then it is easy to see (cf. [21, Theorem 4.3.3]) that

$$h(X_{\mathscr{G}}) = \frac{\log \lambda(A_{\mathscr{G}})}{m} = \frac{\log 2}{m}, \tag{3.1}$$

where $\lambda(A_{\mathscr{G}})$ denotes the spectral radius of $A_{\mathscr{G}}$.

**Proposition 3.7.** *Let $[p_L, p_R] \subseteq (q_{KL}, M+1)$ be an entropy plateau of period $m$. Then for any $p \in [p_L, p_R)$,*

$$\dim_H(\mathscr{U} \cap [p, p_R]) \geq \frac{\log 2}{m \log p_R}.$$

(We will show in Section 4 that this holds in fact with equality.)

*Proof.* We will construct a sequence of subsets $\{\Lambda_N\}$ of $\mathbf{U}_{[p,p_R]}$ such that the Hausdorff dimension of $\pi_{p_R}(\Lambda_N)$ tends to $\frac{\log 2}{m \log p_R}$ as $N \to \infty$, where $\mathbf{U}_{[p,p_R]} := \left\{\alpha(\ell) : \ell \in \overline{\mathscr{U}} \cap [p, p_R]\right\}$. This observation, when combined with Lemma 3.6 and the fact that the difference between $\mathscr{U}$ and $\overline{\mathscr{U}}$ is countable, will imply our lower bound.

Let $a_1 \dots a_m$ be the word such that $\alpha(p_L) = (a_1 \dots a_m)^\infty$ and $\alpha(p_R) = a_1 \dots a_m^+(\overline{a_1 \dots a_m})^\infty$. Recall that $X_{\mathscr{G}}$ is a sofic subshift represented by the labeled graph $\mathscr{G}$ in Figure 1. For an integer $N \geq 2$ let $\Lambda_N$ be the set of sequences $(c_i) \in X_{\mathscr{G}}$ beginning with

$$c_1 \dots c_{mN} = a_1 \dots a_m^+(\overline{a_1 \dots a_m})^{N-1}$$

and the tail sequence $c_{mN+1}c_{mN+2}\dots$ not containing the word $a_1 \dots a_m^+(\overline{a_1 \dots a_m})^{N-1}$ or $\overline{a_1 \dots a_m^+}(a_1 \dots a_m)^{N-1}$. Note that since $\alpha(p) \prec \alpha(p_R)$, we can choose $N$ large enough so that $\alpha(p) \prec a_1 \dots a_m^+(\overline{a_1 \dots a_m})^{N-1}0^\infty$. We claim that $\Lambda_N \subset \mathbf{U}_{[p,p_R]}$.

Observe that $a_1 \dots a_m^+(\overline{a_1 \dots a_m})^\infty$ is the lexicographically largest sequence in $X_{\mathscr{G}}$, and $\overline{a_1 \dots a_m^+}(a_1 \dots a_m)^\infty$ is the lexicographically smallest sequence in $X_{\mathscr{G}}$. Take a sequence $(c_i) \in \Lambda_N$. Then $(c_i)$ has a prefix $a_1 \dots a_m^+(\overline{a_1 \dots a_m})^{N-1}$, and the tail $c_{mN+1}c_{mN+2}\dots$ satisfies the inequalities

$$\overline{(c_i)} \preccurlyeq \overline{a_1 \dots a_m^+}(a_1 \dots a_m)^{N-1}M^\infty \prec \sigma^n((c_i)) \prec a_1 \dots a_m^+(\overline{a_1 \dots a_m})^{N-1}0^\infty \preccurlyeq (c_i)$$

for all $n \geq mN$. By Lemma 3.4, to prove $(c_i) \in \mathbf{U}_{[p,p_R]}$ it suffices to prove $\overline{(c_i)} \prec \sigma^n((c_i)) \prec (c_i)$ for all $1 \leq n < mN$. Note by Lemma 3.1(i) that

$$\overline{a_1 \dots a_{m-i}} \preccurlyeq a_{i+1} \dots a_m \prec a_1 \dots a_{m-i} \quad \text{for all} \quad 1 \leq i < m. \tag{3.2}$$

This implies that

$$a_{i+1} \dots a_m^+ \overline{a_1 \dots a_i} \preccurlyeq a_1 \dots a_m \prec a_1 \dots a_m^+,$$

and

$$a_{i+1} \dots a_m^+ \succ a_{i+1} \dots a_m \succcurlyeq \overline{a_1 \dots a_{m-i}}$$

for all $1 \leq i < m$. So $\overline{(c_i)} \prec \sigma^n((c_i)) \prec (c_i)$ for all $1 \leq n < m$. Furthermore, by (3.2) it follows that

$$\overline{a_1 \dots a_m^+} \prec \overline{a_1 \dots a_m} \preccurlyeq a_{i+1} \dots a_m a_1 \dots a_i \prec a_1 \dots a_m^+$$

for all $0 \leq i < m$. Taking the reflection we obtain

$$\overline{a_1 \dots a_m^+} \prec \overline{a_{i+1} \dots a_m a_1 \dots a_i} \prec a_1 \dots a_m^+ \tag{3.3}$$

for all $0 \leq i < m$. Since $c_{m(N-1)+1} \dots c_{mN} = \overline{a_1 \dots a_m}$, we have $c_{mN+1} \dots c_{mN+m-1} = \overline{a_1 \dots a_{m-1}}$ (see Figure 1). Then by (3.3) it follows that $\overline{(c_i)} \prec \sigma^n((c_i)) \prec (c_i)$ for all $m \leq n < mN$. Therefore, $\overline{(c_i)} \prec \sigma^n((c_i)) \prec (c_i)$ for all $n \geq 1$. So $(c_i) \in \mathbf{U}_{[p,p_R]}$, and hence $\Lambda_N \subset \mathbf{U}_{[p,p_R]}$.

Observe that $\pi_{p_R}(\Lambda_N)$ is the affine image of a graph-directed self-similar set whose Hausdorff dimension is arbitrarily close to the dimension of $\pi_{p_R}(X_{\mathscr{G}})$ as $N \to \infty$. Then

$$\lim_{N\to\infty} \dim_H \pi_{p_R}(\Lambda_N) = \dim_H \pi_{p_R}(X_{\mathscr{G}}) = \frac{\log 2}{m \log p_R}.$$

Therefore, by the first inequality in Lemma 3.6 and the claim we conclude that

$$\begin{aligned}\dim_H(\overline{\mathscr{U}} \cap [p, p_R]) &\geq \dim_H \pi_{p_R}(\mathbf{U}_{[p,p_R]}) \\ &\geq \lim_{N\to\infty} \dim_H \pi_{p_R}(\Lambda_N) = \frac{\log 2}{m \log p_R},\end{aligned}$$

completing the proof. □

Next, recall from (2.5) that $\mathbf{V}_q$ is the set of sequences $(x_i) \in \Omega$ satisfying the inequalities:

$$\overline{\alpha(q)} \preccurlyeq \sigma^n((x_i)) \preccurlyeq \alpha(q) \qquad \text{for all} \quad n \geq 0.$$

The next proposition shows that the set-valued map $q \mapsto \mathbf{V}_q$ does not vary too much inside an entropy plateau $[p_L, p_R]$, and gives a sharp estimate for the limit in Theorem 1(iv) when $q$ lies inside an entropy plateau.

**Proposition 3.8.** *Let $[p_L, p_R] \subset (q_{KL}, M+1]$ be an entropy plateau of period $m$. Then*

(i) *For all $p$ and $q$ with $p_L \leq p < q < p_R$,*

$$\dim_H(\mathbf{V}_q \backslash \mathbf{V}_p) < \dim_H(\mathbf{V}_{p_R} \backslash \mathbf{V}_p) = \frac{\log 2}{m}. \tag{3.4}$$

(ii) *For all $q \in (p_L, p_R]$,*

$$\lim_{p \nearrow q} \dim_H(\overline{\mathscr{U}} \cap (p, q)) \leq \frac{\log 2}{m \log q}, \tag{3.5}$$

*with equality if and only if $q = p_R$.*

*Proof.* First we prove (i). By Lemma 3.1 there exists a word $a_1 \ldots a_m$ such that

$$\alpha(p_L) = (a_1 \ldots a_m)^\infty \qquad \text{and} \qquad \alpha(p_R) = a_1 \ldots a_m^+ (\overline{a_1 \ldots a_m})^\infty. \tag{3.6}$$

Take a sequence $(c_i) \in \mathbf{V}_{p_R} \setminus \mathbf{V}_{p_L}$. Then there exists $j \geq 0$ such that

$$c_{j+1} \ldots c_{j+m} = a_1 \ldots a_m^+ \qquad \text{or} \qquad c_{j+1} \ldots c_{j+m} = \overline{a_1 \ldots a_m^+}.$$

We claim that the tail sequence $c_{j+1}c_{j+2} \ldots \in X_{\mathscr{G}}$, where $X_{\mathscr{G}}$ is the sofic subshift determined by the labeled graph in Figure 1.

By symmetry we may assume $c_{j+1} \ldots c_{j+m} = a_1 \ldots a_m^+$. Since $(c_i) \in \mathbf{V}_{p_R}$, by (3.6) the sequence $(c_i)$ satisfies

$$\overline{a_1 \ldots a_m^+}(a_1 \ldots a_m)^\infty \preccurlyeq \sigma^n((c_i)) \preccurlyeq a_1 \ldots a_m^+(\overline{a_1 \ldots a_m})^\infty \tag{3.7}$$

for all $n \geq 0$. Taking $n = j$ in (3.7) it follows that $c_{j+m+1} \ldots c_{j+2m} \preccurlyeq \overline{a_1 \ldots a_m}$. Again, by (3.7) with $n = j + m$ we obtain that $c_{j+m+1} \ldots c_{j+2m} \succcurlyeq \overline{a_1 \ldots a_m^+}$. So, if $c_{j+1} \ldots c_{j+m} = a_1 \ldots a_m^+$, then the next word $c_{j+m+1} \ldots c_{j+2m}$ has only two choices: it either equals $\overline{a_1 \ldots a_m^+}$ or it equals $\overline{a_1 \ldots a_m}$.

- If $c_{j+m+1} \ldots c_{j+2m} = \overline{a_1 \ldots a_m^+}$, then by symmetry and using (3.7) it follows that the next word $c_{j+2m+1} \ldots c_{j+3m}$ equals either $a_1 \ldots a_m$ or $a_1 \ldots a_m^+$.

- If $c_{j+m+1}\dots c_{j+2m}=\overline{a_1\dots a_m}$, then $c_{j+1}\dots c_{j+2m}=a_1\dots a_m^+\overline{a_1\dots a_m}$. By using (3.7) with $k=j$ we have $c_{j+2m+1}\dots c_{j+3m}\preccurlyeq\overline{a_1\dots a_m}$. Again, by (3.7) with $k=j+2m$ it follows that the next word $c_{j+2m+1}\dots c_{j+3m}$ equals either $\overline{a_1\dots a_m^+}$ or $\overline{a_1\dots a_m}$.

By iteration of the above arguments we conclude that $c_{j+1}c_{j+2}\dots\in X_{\mathscr{G}}$. This proves the claim: any sequence in $\mathbf{V}_{p_R}\setminus\mathbf{V}_{p_L}$ eventually ends with an element of $X_{\mathscr{G}}$.

Using the claim and (3.1) it follows that

$$\dim_H(\mathbf{V}_{p_R}\backslash\mathbf{V}_p)\le\dim_H(\mathbf{V}_{p_R}\setminus\mathbf{V}_{p_L})\le\dim_H X_{\mathscr{G}}\le\ h(X_{\mathscr{G}})=\frac{\log 2}{m}. \tag{3.8}$$

On the other hand, since $p<p_R$ we have $\alpha(p)\prec\alpha(p_R)=a_1\dots a_m^+(\overline{a_1\dots a_m})^\infty$, so there exists $K\in\mathbb{N}$ such that $\alpha(p)\prec a_1\dots a_m^+(\overline{a_1\dots a_m})^K0^\infty$. Hence, the follower set

$$F_{X_{\mathscr{G}}}\big(a_1\dots a_m^+(\overline{a_1\dots a_m})^K\big):=\big\{(d_i)\in X_{\mathscr{G}}:d_1\dots d_{m(K+1)}=a_1\dots a_m^+(\overline{a_1\dots a_m})^K\big\}$$

is a subset of $\mathbf{V}_{p_R}\setminus\mathbf{V}_p$. By (3.1) this implies that

$$\dim_H(\mathbf{V}_{p_R}\setminus\mathbf{V}_p)\ge\dim_H F_{X_{\mathscr{G}}}\big(a_1\dots a_m^+(\overline{a_1\dots a_m})^K\big)=h(X_{\mathscr{G}})=\frac{\log 2}{m}, \tag{3.9}$$

where the first equality follows since, in view of the homogeneous structure of $X_{\mathscr{G}}$, there is no more efficient covering of this set than by cylinder sets of equal depth. Combining (3.8) and (3.9) gives

$$\dim_H(\mathbf{V}_{p_R}\setminus\mathbf{V}_p)=\frac{\log 2}{m}. \tag{3.10}$$

Next, observe that for $q\in(p_L,p_R)$ there exists $N\in\mathbb{N}$ such that $\alpha(q)\prec a_1\dots a_m^+(\overline{a_1\dots a_m})^N0^\infty$. Then the word $a_1\dots a_m^+(\overline{a_1\dots a_m})^N$ is forbidden in $\mathbf{V}_q$. By the above argument it follows that any sequence in $\mathbf{V}_q\setminus\mathbf{V}_p$ eventually ends with an element of

$$X_{\mathscr{G},N}:=\big\{(d_i)\in X_{\mathscr{G}}:a_1\dots a_m^+(\overline{a_1\dots a_m})^N\text{ does not occur in }(d_i)\big\}. \tag{3.11}$$

By (3.1) this implies that

$$\dim_H(\mathbf{V}_q\setminus\mathbf{V}_p)\le\dim_H X_{\mathscr{G},N}\le h(X_{\mathscr{G},N})<h(X_{\mathscr{G}})=\frac{\log 2}{m},$$

where the strict inequality holds by [21, Corollary 4.4.9], since $X_{\mathscr{G}}$ is a transitive sofic subshift and $X_{\mathscr{G},N}\subsetneq X_{\mathscr{G}}$. This completes the proof of (i).

To prove (ii), suppose first that $q\in(p_L,p_R)$. Let $a_1\dots a_m$ be the word such that (3.6) holds. Take $p\in(p_L,q)\cap\overline{\mathscr{U}}$. By Lemma 2.1 it follows that for any $\ell\in(p,q)$ the quasi-greedy expansion $\alpha(\ell)$ begins with $a_1\dots a_m^+$. As in the proof of (i), since $q<p_R$ it follows that there exists $N\in\mathbb{N}$ depending only on $q$ such that

$$\mathbf{U}_{(p,q)}:=\{\alpha(\ell):\ell\in\overline{\mathscr{U}}\cap(p,q)\}\subseteq X_{\mathscr{G},N},$$

where $X_{\mathscr{G},N}$ was defined in (3.11). Therefore, by Lemma 3.6,

$$\begin{aligned}\lim_{p\nearrow q}\dim_H(\overline{\mathscr{U}}\cap(p,q))&\le\lim_{p\nearrow q}\frac{h(\mathbf{U}_{(p,q)})}{\log p}\le\lim_{p\nearrow q}\frac{h(X_{\mathscr{G},N})}{\log p}\\&=\frac{h(X_{\mathscr{G},N})}{\log q}<\frac{h(X_{\mathscr{G}})}{\log q}=\frac{\log 2}{m\log q}.\end{aligned}$$

For $q = p_R$ we have $h(\mathbf{U}_{(p,q)}) \le h(X_{\mathscr{G}})$, so as in the above calculation we obtain

$$\lim_{p \nearrow p_R} \dim_H(\overline{\mathscr{U}} \cap (p, p_R)) \le \frac{\log 2}{m \log p_R}.$$

The reverse inequality holds by Proposition 3.7, and hence we have equality in (3.5) for $q = p_R$. □

**Corollary 3.9.** *For any entropy plateau* $[p_L, p_R] \subset (q_{KL}, M+1]$ *and any* $q \in (p_L, p_R]$,

$$\dim_H(\mathbf{V}_q \setminus \mathbf{V}_{p_L}) \le \dim_H \mathbf{V}_{p_L},$$

*with equality if and only if* $M = 2k+1 \ge 3$ *and* $q = p_R = q_\star(M)$.

*Proof.* Immediate from Lemma 3.1(ii), Lemmas 2.8 and 2.9, and Proposition 3.8(i). □

As a final preparation for the proofs of Theorems 1, 1′ and 2, we need the following results about the local dimension of the bifurcation sets $\overline{\mathscr{B}}$ and $\overline{\mathscr{U}}$. We first recall from [14, Theorem 2] the local dimension of $\mathscr{B}$.

**Lemma 3.10.** *For any* $q \in \overline{\mathscr{B}}$ *we have*

$$\lim_{\delta \to 0} \dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q.$$

For the local dimension of $\mathscr{U}$, we can prove the following:

**Lemma 3.11.** *For any* $q \in (1, M+1]$ *we have*

$$\lim_{\delta \to 0} \dim_H(\overline{\mathscr{U}} \cap (q-\delta, q+\delta)) \le \dim_H \mathcal{U}_q.$$

*Proof.* Take $q \in (1, M+1]$. By Lemmas 2.1, 2.3 and 3.4 it follows that for each $\ell \in \overline{\mathscr{U}} \cap (q-\delta, q+\delta)$ the quasi-greedy expansion $\alpha(\ell)$ belongs to $\mathbf{U}_{q+\delta}$, where we set $\mathbf{U}_{q+\delta} = \Omega$ if $q + \delta > M+1$. In other words, using the notation of Lemma 3.6,

$$\mathbf{U}_{(q-\delta, q+\delta)} \subseteq \mathbf{U}_{q+\delta}.$$

We now obtain by Lemma 3.6 and Lemma 2.5,

$$\begin{aligned}\dim_H\left(\overline{\mathscr{U}} \cap (q-\delta, q+\delta)\right) &\le \frac{h(\mathbf{U}_{(q-\delta,q+\delta)})}{\log(q-\delta)} \le \frac{h(\mathbf{U}_{q+\delta})}{\log(q-\delta)} \\ &\le \frac{\log(q+\delta)}{\log(q-\delta)} \dim_H \mathcal{U}_{q+\delta} \to \dim_H \mathcal{U}_q\end{aligned}$$

as $\delta \to 0$. This completes the proof. □

We are now ready to prove Theorems 1, 1′ and 2.

*Proof of Theorem 1.* Suppose $M = 1$ or $M$ is even. We prove (i) ⇔ (ii) and (i) ⇒ (iii) ⇒ (iv) ⇒ (i).

First we prove (i) ⇒ (ii). Let $q \in \mathscr{B}^L$, and take $p \in (1, q)$. Then $H(p) < H(q)$ by the definition of $\mathscr{B}^L$, so Lemma 2.8 implies

$$\dim_H \mathbf{U}_p = H(p) < H(q) = \dim_H \mathbf{U}_q.$$

Therefore,

$$\dim_H(\mathbf{U}_q \setminus \mathbf{U}_p) = \dim_H \mathbf{U}_q > \dim_H \mathbf{U}_p \ge 0.$$

Next, we prove (ii) $\Rightarrow$ (i). Let $q \in (1, M+1] \setminus \mathscr{B}^L$. By (1.6) we have $q \in (1, q_{KL}]$ or $q \in (p_L, p_R]$ for some entropy plateau $[p_L, p_R] \subset (q_{KL}, M+1]$. If $q \in (1, q_{KL}]$, then by Lemma 2.5 we have

$$\dim_H(\mathbf{U}_q \setminus \mathbf{U}_p) = \dim_H \mathbf{U}_q = 0$$

for any $p \in (1, q)$. Suppose $q \in (p_L, p_R] \subset (q_{KL}, M+1]$, and take $p \in (p_L, q)$. By Corollary 3.9 and Lemma 2.9 it follows that

$$\begin{aligned}\dim_H(\mathbf{U}_q \setminus \mathbf{U}_p) \le \dim_H(\mathbf{U}_q \setminus \mathbf{U}_{p_L}) &= \dim_H(\mathbf{V}_q \setminus \mathbf{V}_{p_L}) \\ &< \dim_H \mathbf{V}_{p_L} = \dim_H \mathbf{U}_{p_L} \le \dim_H \mathbf{U}_q.\end{aligned}$$

Thus, (ii) $\Rightarrow$ (i).

We next prove (i) $\Rightarrow$ (iii). Take $q \in \mathscr{B}^L$. Then $q > q_{KL}$ by (1.6), so Lemma 2.5 yields $\dim_H \mathcal{U}_q > 0$. Thus, it remains to prove that $\lim_{p \nearrow q} \dim_H(\mathscr{B} \cap (p, q)) = \dim_H \mathcal{U}_q$. Since $\mathscr{B} \subset \mathscr{U}$, by Lemma 3.11 it suffices to prove

$$\lim_{p \nearrow q} \dim_H(\mathscr{B} \cap (p, q)) \ge \dim_H \mathcal{U}_q. \tag{3.12}$$

Fix $\varepsilon > 0$. By Lemma 2.5 the function $q \mapsto \dim_H \mathcal{U}_q$ is continuous, so there exists $p_0 := p_0(\varepsilon) \in (1, q)$ such that

$$\dim_H \mathcal{U}_p \ge \dim_H \mathcal{U}_q - \varepsilon \qquad \text{for all} \quad p \in (p_0, q). \tag{3.13}$$

Since $q \in \mathscr{B}^L$, by the topological structure of the bifurcation set $\mathscr{B}^L$ there exists a sequence of entropy plateaus $\{[p_L(n), p_R(n)]\}$ such that $p_L(n) \nearrow q$ as $n \to \infty$. Fix $p \in (p_0, q)$. Then there exists a large integer $N$ such that $p_L(N) \in (p, q)$. Observe that $p_L(N) \in \mathscr{B}^L \subset \overline{\mathscr{B}}$ and the difference $\overline{\mathscr{B}} \setminus \mathscr{B}$ is countable. By Lemma 3.10 there exists $\delta > 0$ such that

$$(p_L(N) - \delta, p_L(N) + \delta) \subseteq (p, q), \tag{3.14}$$

and

$$\dim_H \left(\mathscr{B} \cap (p_L(N) - \delta, p_L(N) + \delta)\right) \ge \dim_H \mathcal{U}_{p_L(N)} - \varepsilon. \tag{3.15}$$

By (3.13), (3.14) and (3.15) it follows that

$$\begin{aligned}\dim_H(\mathscr{B} \cap (p, q)) &\ge \dim_H \left(\mathscr{B} \cap (p_L(N) - \delta, p_L(N) + \delta)\right) \\ &\ge \dim_H \mathcal{U}_{p_L(N)} - \varepsilon \ge \dim_H \mathcal{U}_q - 2\varepsilon.\end{aligned}$$

Since this holds for all $p \in (p_0(\varepsilon), q)$, we obtain (3.12). This proves (i) $\Rightarrow$ (iii).

Note that (iii) $\Rightarrow$ (iv) follows directly from Lemma 3.11 since $\mathscr{B} \subset \mathscr{U}$.

It remains to prove (iv) $\Rightarrow$ (i). Let $q \in (1, M+1] \setminus \mathscr{B}^L$. By (1.6) it follows that $q \in (1, q_{KL}]$ or $q \in (p_L, p_R]$ for some entropy plateau $[p_L, p_R] \subset (q_{KL}, M+1]$. If $q \in (1, q_{KL}]$, then $\dim_H \mathcal{U}_q = 0$. Now we consider $q \in (p_L, p_R] \subset (q_{KL}, M+1]$. If $q \notin \overline{\mathscr{U}}$, then $\lim_{p \nearrow q} \dim_H(\mathscr{U} \cap (p, q)) = 0$. So let $q \in \overline{\mathscr{U}} \cap (p_L, p_R]$. If $q < p_R$, then Proposition 3.8(ii), Lemma 3.1(ii) and Lemma 2.5 give

$$\lim_{p \nearrow q} \dim_H(\overline{\mathscr{U}} \cap (p, q)) < \frac{\log 2}{m \log q} \le \frac{h(\mathbf{U}_{p_L})}{\log q} = \frac{h(\mathbf{U}_q)}{\log q} = \dim_H \mathcal{U}_q. \tag{3.16}$$

Similarly, if $q = p_R$, then Lemma 3.1(ii) holds with strict inequality, and we obtain the same end result as in (3.16), but with the first inequality replaced by "$\le$" and the second inequality replaced by "$<$". This proves (iv) $\Rightarrow$ (i), and completes the proof of Theorem 1. $\square$

*Proof of Theorem 1′.* The proof of Theorem 1′ is, for the most part, the same as the proof of Theorem 1. Asssume $M = 2k+1 \geq 3$. We need only check the following two facts for the entropy plateau $[p_L, p_R] = [k+2, q_\star]$, where $q_\star = q_\star(M)$:

$$\dim_H(\mathbf{U}_{q_\star} \backslash \mathbf{U}_p) = \dim_H(\mathbf{U}_{q_\star}) \qquad \text{for any } p \in (1, q_\star), \tag{3.17}$$

and

$$\lim_{p \nearrow q_\star} \dim_H(\overline{\mathscr{U}} \cap (p, q_\star)) = \dim_H \mathcal{U}_{q_\star}. \tag{3.18}$$

Here (3.17) is clear for $p \in (1, k+2)$, since $\dim_H \mathbf{U}_p < \dim_H \mathbf{U}_{q_\star}$. For $p \in [k+2, q_\star)$, (3.17) follows from Proposition 3.8(i) and the equality statement in Lemma 3.1(ii), noting that $[k+2, q_\star]$ is an entropy plateau of period $m = 1$.

Similarly, (3.18) follows from the equality statements in Proposition 3.8(ii) and Lemma 3.1(ii). □

*Proof of Theorem 2.* The proofs of (i) ⇒ (ii) and (iii) ⇒ (iv) are completely analogous to the proofs of the corresponding implications in Theorem 1.

Consider the implication (ii) ⇒ (i). Suppose $q \in (1, M+1] \setminus \mathscr{B}^R$. By (1.6) we have $q \in (1, q_{KL})$ or $q \in [p_L, p_R)$ for some entropy plateau $[p_L, p_R] \subset (q_{KL}, M+1]$. A similar argument as in the proof of Theorem 1 shows that either $\dim_H \mathbf{U}_q = 0$ for $q \in (1, q_{KL})$, or $\dim_H(\mathbf{U}_r \setminus \mathbf{U}_q) < \dim_H \mathbf{U}_r$ for any $r \in (q, p_R)$. This proves (ii) ⇒ (i).

Next, consider the implication (i) ⇒ (iii). Take $q \in \mathscr{B}^R$. Then $q \geq q_{KL}$. If $q \neq q_{KL}$, then by Lemma 2.5 we have $\dim_H \mathcal{U}_q > 0$. Since $q \in \mathscr{B}^R$, there exists a sequence of entropy plateaus $\{[\tilde{p}_L(n), \tilde{p}_R(n)]\}$ such that $\tilde{p}_L(n) \searrow q$ as $n \to \infty$. Using the continuity of the function $q \mapsto \dim_H \mathcal{U}_q$ and Lemma 3.10, we can show as in the proof of Theorem 1 that $\lim_{r \searrow q} \dim_H(\mathscr{B} \cap (q, r)) = \dim_H \mathcal{U}_q$. This proves (i) ⇒ (iii).

Finally, consider the implication (iv) ⇒ (i). For $q \in (1, M+1] \setminus \mathscr{B}^R$ we have $q \in (1, q_{KL})$ or $q \in [p_L, p_R)$ for some entropy plateau $[p_L, p_R] \subset (q_{KL}, M+1]$. By the same argument as in the proof of Theorem 1 we can prove that either $\dim_H \mathcal{U}_q = 0$ for $q < q_{KL}$, or $\lim_{r \searrow q} \dim_H(\overline{\mathscr{U}} \cap (q, r)) < \dim_H \mathcal{U}_q$ for $q \in [p_L, p_R)$. This establishes (iv) ⇒ (i). □

## 4. Hausdorff dimension of $\mathscr{U} \setminus \mathscr{B}$

In this section we will calculate the Hausdorff dimension of the difference set $\mathscr{U} \setminus \mathscr{B}$ and prove Theorem 4. First, we prove the following result for the local dimension of $\mathscr{U}$ inside any entropy plateau $[p_L, p_R]$.

**Theorem 4.1.** *Let $[p_L, p_R] \subset (q_{KL}, M+1)$ be an entropy plateau of period $m$. Then*

$$\dim_H(\mathscr{U} \cap [p_L, p_R]) = \frac{\log 2}{m \log p_R}.$$

Observe that the lower bound in Theorem 4.1, that is, the inequality

$$\dim_H(\mathscr{U} \cap [p_L, p_R]) \geq \frac{\log 2}{m \log p_R},$$

follows from Proposition 3.7 by setting $p = p_L$. The proof of the reverse inequality is more tedious, and we will give it in several steps.

Observe that $\inf \mathscr{U} = q_{KL}$, and any entropy plateau $[p_L, p_R] \subset (q_{KL}, M+1]$ satisfies $\alpha(q_{KL}) \prec \alpha(p_L) \prec \alpha(M+1)$. In the following we fix an arbitrary entropy plateau $[p_L, p_R] \subset (q_{KL}, M+1]$ of period $m$ such that $\alpha(p_L) = (a_1 \dots a_m)^\infty$ and $\alpha(p_R) = a_1 \dots a_m^+ (\overline{a_1 \dots a_m})^\infty$. Recall the definition of the generalized Thue-Morse sequence $(\lambda_i) = (\lambda_i(M))$ from (2.2), which has the property that $\alpha(q_{KL}) = (\lambda_i)$. If $M = 1$, then

$$1101 \dots = \lambda_1 \lambda_2 \dots \prec (a_1 \dots a_m)^\infty \prec 1^\infty,$$

so $m \geq 3$. Similarly, if $M = 2$, we have

$$210201 \dots = \lambda_1 \lambda_2 \dots \prec (a_1 \dots a_m)^\infty \prec 2^\infty,$$

so $m \geq 2$. But when $M \geq 3$, it is possible to have $m = 1$. In short, we have the inequality

$$M + m \geq 4. \tag{4.1}$$

We divide the interval $(p_L, p_R)$ into a sequence of smaller subintervals by defining a sequence of bases $\{q_n\}_{n=1}^\infty$ in $(p_L, p_R)$. Let $\hat{q} = \min(\overline{\mathscr{U}} \cap (p_L, p_R))$, and for $n \geq 1$ let $q_n \in (p_L, p_R)$ be defined by

$$\alpha(q_n) = \left( a_1 \dots a_m^+ (\overline{a_1 \dots a_m})^{n-1} \overline{a_1 \dots a_m^+} \right)^\infty. \tag{4.2}$$

Note that $\hat{q}$ is a de Vries-Komornik number which has a Thue-Morse type quasi-greedy expansion

$$\alpha(\hat{q}) = a_1 \dots a_m^+ \ \overline{a_1 \dots a_m} \ \overline{a_1 \dots a_m^+} \, a_1 \dots a_m^+ \cdots \tag{4.3}$$

That is, $\alpha(\hat{q})$ is the sequence $\alpha_1 \alpha_2 \dots$ given by $\alpha_1 \dots \alpha_m = a_1 \dots a_m^+$, and recursively, for $i \geq 0$, $\alpha_{2^i m+1} \dots \alpha_{2^{i+1} m} = \overline{\alpha_1 \dots \alpha_{2^i m}}^+$. Then $\alpha(q_1) \prec \alpha(\hat{q}) \prec \alpha(q_2) \prec \dots \prec \alpha(p_R)$, and $\alpha(q_n) \nearrow \alpha(p_R)$ as $n \to \infty$. By Lemma 2.1 it follows that

$$q_1 < \hat{q} < q_2 < q_3 < \dots < p_R, \quad \text{and} \quad q_n \nearrow p_R \quad \text{as } n \to \infty.$$

**Lemma 4.2.** *For any $n \geq 1$, we have*

$$\dim_H(\overline{\mathscr{U}} \cap [q_n, q_{n+1}]) \leq \frac{\log(2^{n+2} - 2)}{m(n+2) \log q_n}.$$

*Proof.* Fix $n \geq 1$. Note by (4.2) and (4.3) that for any $p \in \overline{\mathscr{U}} \cap [q_n, q_{n+1}]$, $\alpha(p)$ begins with $a_1 \dots a_m^+$, and $\alpha(p) \in \mathbf{V}_p \subseteq \mathbf{V}_{q_{n+1}}$. By a similar argument as in the proof of Proposition 3.8 it follows that $\alpha(p) \in X_{\mathscr{G}}$, and $\alpha(p)$ does not contain the subwords $a_1 \dots a_m^+ (\overline{a_1 \dots a_m})^{n+1}$ and $\overline{a_1 \dots a_m^+} (a_1 \dots a_m)^{n+1}$, where $X_{\mathscr{G}}$ is the sofic subshift represented by the labeled graph $\mathscr{G} = (G, \mathscr{L})$ in Figure 1. In other words, $\alpha(p) \in X_{\mathscr{G}, n+1}$, where $X_{\mathscr{G}, N}$ was defined in (3.11). This implies

$$h(\mathbf{U}_{[q_n, q_{n+1}]}) \leq h(X_{\mathscr{G}, n+1}) \leq \frac{\log(\# B_{m(n+2)}(X_{\mathscr{G}, n+1}))}{m(n+2)} \leq \frac{\log(2^{n+2} - 2)}{m(n+2)}. \tag{4.4}$$

Applying Lemma 3.6 with $I = [q_n, q_{n+1}]$ completes the proof. □

*Remark* 4.3. The entropy of $X_{\mathscr{G}, N}$ can in fact be calculated explicitly: It is equal to $\log r_N / m$, where $r_N$ is the unique positive root of the equation $1 + x + \dots + x^{N-1} = x^N$ (cf. [6]). Thus, a sharper (but less explicit) estimate of $h(\mathbf{U}_{[q_n, q_{n+1}]})$ can be given. However, the proof that $h(X_{\mathscr{G}, N}) = \log r_N / m$ is a bit tedious, and in any case the crude bound in (4.4) is sufficient for our purposes.

The next step is to prove that the upper bound in Lemma 4.2 is smaller than $\log 2/m\log p_R$. This requires us to show that $q_n$ is sufficiently close to $p_R$, which we accomplish by applying a *transversality* technique (see [24, 26]) to certain polynomials associated with $q_n$ and $p_R$. For this we need the estimation of the Komornik-Loreti constants $q_{KL}(M)$. Recall from Example 2.4 that

$$q_{KL}(1)\approx 1.78723,\qquad q_{KL}(2)\approx 2.53595\qquad\text{and}\qquad q_{KL}(3)\approx 2.91002.$$

We emphasize that $q_{KL}(M)\ge (M+2)/2$ for each $M\ge 1$, and the map $M\mapsto q_{KL}(M)$ is strictly increasing.

**Lemma 4.4.** *Let $[p_L,p_R]\subset(q_{KL},M+1]$ be an entropy plateau such that $\alpha(p_L)=(a_1\dots a_m)^\infty$ and $\alpha(p_R)=a_1\dots a_m^+(\overline{a_1\dots a_m})^\infty$. Define the polynomials*

$$\begin{aligned}P(x):=&\,a_1x+\dots+a_{m-1}x^{m-1}+(1+a_m^+)x^m\\ &+(\overline{a_1}-a_1)x^{m+1}+\dots+(\overline{a_{m-1}}-a_{m-1})x^{2m-1}+(\overline{a_m}-a_m^+)x^{2m}-1\end{aligned}\tag{4.5}$$

*and*

$$Q_n(x):=P(x)-x^{m(n+1)}(\overline{a_1}x+\dots+\overline{a_m}x^m),\qquad n\in\mathbb{N}.\tag{4.6}$$

(i) *The number $1/p_R$ is the unique zero of $P$ in $[1/(M+1),1]$.*
(ii) *The number $1/q_n$ is the unique zero of $Q_n$ in $[1/(M+1),1]$, for all $n\in\mathbb{N}$.*
(iii) *$P'(x)\ge a_1$ for all $x\in[1/p_R,1/p_L]$.*

*Proof.* (i) Since $\alpha(p_R)=a_1\dots a_m^+(\overline{a_1\dots a_m})^\infty$, it follows that $1/p_R$ is the unique solution in $[1/(M+1),1]$ of

$$\begin{aligned}1&=a_1x+a_2x^2+\dots+a_{m-1}x^{m-1}+a_m^+x^m\\&\qquad+x^m(\overline{a_1}x+\dots+\overline{a_m}x^m)+x^{2m}(\overline{a_1}x+\dots+\overline{a_m}x^m)+\dots\\&=a_1x+a_2x^2+\dots+a_{m-1}x^{m-1}+a_m^+x^m+\frac{x^m(\overline{a_1}x+\dots+\overline{a_m}x^m)}{1-x^m}.\end{aligned}$$

Expanding and rearranging terms we see that $1/p_R$ is the unique zero in $[1/(M+1),1]$ of $P$.

(ii) By (4.2), it follows that the greedy expansion of 1 in base $q_n$ is

$$\beta(q_n)=a_1\dots a_m^+(\overline{a_1\dots a_m})^n0^\infty,$$

so $1/q_n$ is the unique root in $[1/(M+1),1]$ of the equation

$$1=a_1x+\dots+a_{m-1}x^{m-1}+a_m^+x^m+\frac{x^m(\overline{a_1}x+\dots+\overline{a_m}x^m)(1-x^{mn})}{1-x^m}.$$

Expanding and rearranging gives that $1/q_n$ is the unique zero in $[1/(M+1),1]$ of $Q_n$.

(iii) Consider first the case $m=1$. In this case, the polynomial $P$ should be interpreted as

$$P(x)=(1+a_1^+)x+(\overline{a_1}-a_1^+)x^2-1.$$

Now observe that, since $\alpha(p_L)=a_1^\infty$, it follows that $p_L=a_1+1$. So for $x\in[1/p_R,1/p_L]$, we have in particular that $x\le 1/(a_1+1)$. Therefore, since $a_1\ge(M+1)/2$,

$$\begin{aligned}P'(x)&=1+a_1^++2(\overline{a_1}-a_1^+)x=2+a_1+2(M-2a_1-1)x\\&\ge 2+a_1+\frac{2(M-2a_1-1)}{a_1+1}=a_1+\frac{2(M+1)}{a_1+1}-2\\&\ge a_1,\end{aligned}$$

where the last inequality follows since $a_1 \le M$.

Assume next that $m \ge 2$. Here we use that the greedy expansion of 1 in base $p_L$ is $\beta(p_L) = a_1 \dots a_m^+ 0^\infty$, so

$$a_1 p_L^{-1} + \cdots + a_{m-1} p_L^{-(m-1)} + a_m^+ p_L^{-m} = 1. \tag{4.7}$$

Hence,

$$a_1 x + \cdots + a_{m-1} x^{m-1} + a_m^+ x^m \le 1 \qquad \text{for } 0 \le x \le 1/p_L. \tag{4.8}$$

Now for $0 \le x \le 1/p_L$, writing $\overline{a_k} - a_k$ as $M - 2a_k$, we have

$$\begin{aligned} P'(x) &= a_1 + \sum_{k=2}^{m-1} k a_k x^{k-1} + m(1 + a_m^+) x^{m-1} \\ &\qquad + \sum_{k=1}^{m-1} (m+k)(M - 2a_k) x^{m+k-1} + 2m(M - 2a_m^+ + 1) x^{2m-1} \\ &\ge a_1 + \sum_{k=2}^{m-1} \left\{ k a_k x^{k-1} + \big( M(m+k) - 2(k-1) a_k \big) x^{m+k-1} \right\} \\ &\qquad + \left\{ m(1 + a_m^+) - 2(m+1) \right\} x^{m-1} + M x^m \{ m + 1 + 2m x^{m-1} \} \\ &\qquad + 2\{ m - (m-1) a_m^+ \} x^{2m-1}, \end{aligned}$$

where the inequality follows by multiplying both sides of (4.8) by $m+1$ and some algebraic manipulation. Here, the terms in the summation over $k = 2, \dots, m-1$ are positive, since $a_k \le M$ and so $M(m+k) - 2(k-1)a_k \ge M(m-k+2) > 0$. The sum of the remaining terms is increasing in $a_m^+$, since the coefficient of $a_m^+$ is

$$m x^{m-1} - 2(m-1) x^{2m-1} \ge m x^{m-1} (1 - 2x^m) \ge 0,$$

using that $m \ge 2$ and $x \le 1/p_L \le 1/q_{KL}(1) \le 0.6$, which holds for all $M \ge 1$. Since $a_m^+ \ge 1$, it follows that

$$\begin{aligned} P'(x) &\ge a_1 - 2x^{m-1} + M x^m \{ m + 1 + 2m x^{m-1} \} + 2x^{2m-1} \\ &\ge a_1 - 2x^{m-1} + M(m+1) x^m = a_1 + x^{m-1} \{ M(m+1) x - 2 \}. \end{aligned}$$

At this point, we need that $x \ge 1/p_R \ge 1/(M+1)$. When $M \ge 2$, this implies

$$M(m+1)x - 2 \ge 3Mx - 2 \ge \frac{3M}{M+1} - 2 = \frac{M-2}{M+1} \ge 0,$$

recalling our assumption that $m \ge 2$. When $M = 1$, we have $m \ge 3$ by (4.1), and so $M(m+1)x - 2 \ge 4x - 2 \ge 0$, since $x \ge 1/2$. In both cases, it follows that $P'(x) \ge a_1$. $\square$

The following elementary lemma (an easy consequence of the mean value theorem) is the key to the proof of the next inequality, in Lemma 4.6 below.

**Lemma 4.5.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function which has a zero $x_0$, and let $\gamma > 0$, $\delta > 0$. Suppose $|f'(x)| \ge \gamma$ for all $x \in (x_0 - \delta, x_0 + \delta)$. If $g$ is a continuous function such that*

$$|g(x) - f(x)| \le \gamma\delta \quad \text{for all} \quad x \in (x_0 - \delta, x_0 + \delta),$$

*then $g$ has at least one zero in $[x_0 - \delta, x_0 + \delta]$.*

**Lemma 4.6.** *For each $n \geq 1$,*

$$\frac{\log(2^{n+2}-2)}{(n+2)\log 2} < \frac{\log q_n}{\log p_R}.$$

*Proof.* Set $\mu_n := 1/q_n$ for $n \geq 1$, and set $\mu^* := 1/p_R$. Then $\mu_n > \mu^*$ for all $n \geq 1$. We will use Lemma 4.5 to show that $\mu_n$ is sufficiently close to $\mu^*$.

By Lemma 4.4, $\mu^*$ is the unique zero in $[1/(M+1), 1]$ of the polynomial $P(x)$ from (4.5), and $\mu_n$ is the unique zero in $[1/(M+1), 1]$ of the polynomial $Q_n(x)$ from (4.6). Moreover,

$$P'(x) \geq a_1 \geq \frac{M+1}{2} \qquad \text{for all} \quad \mu^* \leq x \leq 1/p_L. \tag{4.9}$$

In order to estimate the difference $P(x) - Q_n(x)$, we show first that

$$\overline{a_1}x + \cdots + \overline{a_m}x^m < 1 \qquad \text{for all} \quad 0 \leq x \leq 1/p_L. \tag{4.10}$$

Observe that

$$\overline{a_1}x + \cdots + \overline{a_m}x^m = \frac{Mx(1-x^m)}{1-x} - (a_1x + \cdots + a_mx^m).$$

Hence, recalling (4.7), we have for $0 \leq x \leq 1/p_L$,

$$\begin{aligned}\overline{a_1}x + \cdots + \overline{a_m}x^m \leq \overline{a_1}p_L^{-1} + \cdots + \overline{a_m}p_L^{-m} &= \frac{M(1-p_L^{-m})}{p_L - 1} - (1-p_L^{-m}) \\ &= (1-p_L^{-m})\left(\frac{M}{p_L-1} - 1\right) \\ &\leq 1 - p_L^{-m} < 1,\end{aligned}$$

where the next-to-last inequality follows since $p_L \geq q_{KL}(M) \geq (M+2)/2$. This proves (4.10).

Recall our convention that logarithms are taken with respect to base $M+1$. Below, we write $\ln x$ for the natural logarithm of $x$. Suppose we can show, for some number $\delta_n > 0$, that

$$\mu_n - \mu^* \leq \delta_n. \tag{4.11}$$

Using the inequality $\ln(1+x) \leq x$ for any $x > -1$, it then follows that

$$\ln \mu_n - \ln \mu^* = \ln\left(1 + \frac{\mu_n - \mu^*}{\mu^*}\right) \leq \frac{\mu_n - \mu^*}{\mu^*} \leq \frac{\delta_n}{\mu^*} = \delta_n p_R,$$

and so

$$\frac{\ln q_n}{\ln p_R} = 1 + \frac{\ln q_n - \ln p_R}{\ln p_R} = 1 - \frac{\ln \mu_n - \ln \mu^*}{\ln p_R} \geq 1 - \frac{\delta_n p_R}{\ln p_R}. \tag{4.12}$$

On the other hand,

$$\ln(2^{n+2}-2) - \ln 2^{n+2} = \ln\left(1 - \frac{2}{2^{n+2}}\right) \leq -\frac{2}{2^{n+2}} = -\frac{1}{2^{n+1}},$$

and so

$$\frac{\ln(2^{n+2}-2)}{(n+2)\ln 2} = 1 + \frac{\ln(2^{n+2}-2) - \ln 2^{n+2}}{\ln 2^{n+2}} \leq 1 - \frac{1}{2^{n+1}(n+2)\ln 2}. \tag{4.13}$$

In view of (4.12) and (4.13) and the change-of-base formula $\ln x = \ln(M+1)\cdot\log x$, it then remains to show that

$$\frac{\delta_n p_R}{\log p_R} < \frac{1}{2^{n+1}(n+2)\log 2} \qquad \text{for each} \quad n \geq 1. \tag{4.14}$$

As in the proof of Lemma 4.4, we consider two cases: (i) $m \geq 2$; and (ii) $m = 1$.

(i) Assume first that $m \geq 2$. By (4.10) and (4.6) we have

$$0 \leq P(x) - Q_n(x) \leq p_L^{-m(n+1)} \leq q_{KL}^{-m(n+1)}, \qquad x \in [0, 1/p_L].$$

Since we know that $\mu_n \in [\mu^*, 1/p_L]$ and moreover, $\mu_n$ is the unique root of $Q_n$ in $[1/(M+1), 1]$, it follows from (4.9) and Lemma 4.5 (with $\gamma = (M+1)/2$) that (4.11) holds with

$$\delta_n = \frac{2}{M+1} q_{KL}^{-m(n+1)}.$$

Then we can estimate

$$\begin{aligned}\left(2^{n+1}(n+2)\log 2\right)\frac{\delta_n p_R}{\log p_R} &\leq (n+2)\log 2 \cdot \frac{2}{M+1}\cdot\frac{M+1}{\log q_{KL}}\left(\frac{2}{q_{KL}^m}\right)^{n+1} \\ &= 2(n+2)\cdot\frac{\log 2}{\log q_{KL}}\left(\frac{2}{q_{KL}^m}\right)^{n+1},\end{aligned}$$

where the inequality follows since $p_R \leq M+1$ and $\log p_R \geq \log q_{KL}$. Now observe that $\log 2/\log q_{KL} \leq \log 2/\log q_{KL}(1) \leq \log 2/\log 1.787 < 1.2$. Furthermore, if $M \geq 2$ then $2/q_{KL}^m \leq 2/(q_{KL}(2))^2 \leq 2/(2.5)^2 < 0.33$; and if $M = 1$, then $m \geq 3$ by (4.1) and so $2/q_{KL}^m \leq 2/(1.787)^3 < 0.36$. In both cases, it follows that

$$\left(2^{n+1}(n+2)\log 2\right)\frac{\delta_n p_R}{\log p_R} \leq (2.4)(n+2)(0.36)^{n+1} < 1,$$

for all $n \geq 1$, where the last inequality is easily checked by induction. Thus, we have proved (4.14) in the case $m \geq 2$.

(ii) Assume next that $m = 1$; then $M \geq 3$. In this case we replace (4.10) by the sharper estimate

$$\overline{a_1}\, x \leq \left(\frac{M-1}{2}\right) x \leq \left(\frac{M-1}{2}\right) q_{KL}^{-1} \leq \frac{M-1}{M+2}, \qquad 0 \leq x \leq 1/p_L,$$

using that $a_1 \geq (M+1)/2$ and $q_{KL}(M) \geq (M+2)/2$. So by (4.9) and Lemma 4.5, (4.11) holds with

$$\delta_n = \frac{2}{M+1}\cdot\frac{M-1}{M+2} q_{KL}^{-(n+1)}.$$

We use in addition the easily verified fact that the function $x \mapsto x/\log x$ is increasing on $[e, \infty)$, and $p_R \geq q_{KL}(3) \geq 2.9 > e$. We then obtain the estimate

$$\begin{aligned}\left(2^{n+1}(n+2)\log 2\right)\frac{\delta_n p_R}{\log p_R} &\leq (n+2)\log 2\cdot\frac{M+1}{\log(M+1)}\cdot\frac{2}{M+1}\cdot\frac{M-1}{M+2}\left(\frac{2}{q_{KL}}\right)^{n+1} \\ &\leq (n+2)\frac{M-1}{M+2}\left(\frac{4}{M+2}\right)^{n+1}.\end{aligned}$$

Here the second inequality follows since $M \geq 3$ and so $\log(M+1) \geq \log 4 = 2\log 2$, and $q_{KL} \geq (M+2)/2$. Routine calculus shows that $(M-1)/(M+2)^{n+2}$ is decreasing in $M$ on $[3,\infty)$ for all $n \geq 1$, so the last bound above is largest for $M = 3$. This gives the estimate

$$\left(2^{n+1}(n+2)\log 2\right)\frac{\delta_n p_R}{\log p_R} \leq \frac{2}{5}(n+2)\left(\frac{4}{5}\right)^{n+1} < 1$$

for all $n \geq 1$, where the final inequality can be verified by induction.

In both cases above, we have found a $\delta_n$ such that (4.11) holds, and proved (4.14). Therefore, the proof of the Lemma is complete. □

*Proof of the upper bound in Theorem 4.1.* By Lemmas 4.2 and 4.6, we have

$$\dim_H(\overline{\mathscr{U}} \cap [q_n, q_{n+1}]) < \frac{\log 2}{m \log p_R} \qquad \text{for each } n \geq 1.$$

Since $\overline{\mathscr{U}} \cap (p_L, p_R) \subseteq \bigcup_{n=1}^{\infty}(\overline{\mathscr{U}} \cap [q_n, q_{n+1}])$, it follows from the countable stability of Hausdorff dimension that

$$\dim_H(\overline{\mathscr{U}} \cap [p_L, p_R]) \leq \sup_{n\geq 1} \dim_H(\overline{\mathscr{U}} \cap [q_n, q_{n+1}]) \leq \frac{\log 2}{m \log p_R},$$

establishing the upper bound. □

*Remark* 4.7. The above method of proof shows that in fact, for any $\varepsilon > 0$ we have $\dim_H(\overline{\mathscr{U}} \cap [p_L, p_R - \varepsilon]) < \dim_H(\overline{\mathscr{U}} \cap [p_L, p_R])$ and therefore,

$$\dim_H(\overline{\mathscr{U}} \cap [p_R - \varepsilon, p_R]) = \dim_H(\overline{\mathscr{U}} \cap [p_L, p_R]) = \frac{\log 2}{m \log p_R}$$

for any $\varepsilon > 0$. Thus, one could say that within an entropy interval $[p_L, p_R]$, $\overline{\mathscr{U}}$ is "thickest" near the right endpoint $p_R$.

*Proof of Theorem 4.* Since $\mathscr{U} \setminus \mathscr{B} \subset (q_{KL}(M), M+1]$, by (1.5) we have $\mathscr{U} \setminus \mathscr{B} = \bigcup(\mathscr{U} \cap [p_L, p_R])$, where the union is pairwise disjoint and countable. Then

$$\dim_H(\mathscr{U} \setminus \mathscr{B}) = \dim_H \bigcup_{[p_L,p_R]} (\mathscr{U} \cap [p_L, p_R]) = \sup_{[p_L,p_R]} \dim_H(\mathscr{U} \cap [p_L, p_R]). \tag{4.15}$$

Here the supremum is taken over all entropy plateaus $[p_L, p_R] \subset (q_{KL}(M), M+1]$.

Assume first that $M = 1$. Recall that for any entropy plateau $[p_L, p_R] \subseteq (q_{KL}(1), 2]$ with $\alpha(p_L) = (a_1 \dots a_m)^\infty$, it holds that $m \geq 3$. Furthermore, $m = 3$ if and only if $[p_L, p_R] = [\lambda_*, \lambda^*] \approx [1.83928, 1.87135]$, where $\alpha(\lambda_*) = (110)^\infty$ and $\alpha(\lambda^*) = 111(001)^\infty$. Observe that $q_{KL}(1) \approx 1.78723$. By a direct calculation one can verify that for any $m \geq 4$ we have

$$\frac{\log 2}{m \log p_R} < \frac{\log 2}{4 \log q_{KL}} < \frac{\log 2}{3 \log \lambda^*}. \tag{4.16}$$

Therefore, by (4.15), (4.16) and Theorem 4.1 it follows that

$$\dim_H(\mathscr{U} \setminus \mathscr{B}) = \dim_H(\mathscr{U} \cap [\lambda_*, \lambda^*]) = \frac{\log 2}{3 \log \lambda^*} \approx 0.368699.$$

Finally, since $\alpha(\lambda^*) = 111(001)^\infty$, $\lambda^*$ is the unique root in $(1, 2]$ of the equation

$$1 = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^3(x^3-1)},$$

or equivalently, $x^5 - x^4 - x^3 - 2x^2 + x + 1 = 0$.

Consider next the case $M = 2$. Then $m \geq 2$, with equality if and only if $[p_L, p_R] = [\gamma_*, \gamma^*] \approx [2.73205, 2.77462]$, where $\alpha(\gamma_*) = (21)^\infty$ and $\alpha(\gamma^*) = 22(01)^\infty$. For any entropy plateau $[p_L, p_R]$ with period $m \geq 3$, we have $m \log p_R \geq 3 \log q_{KL}(2) \geq 3 \log 2.5 > 2 \log 3 > 2 \log \gamma^*$, so

$$\frac{\log 2}{m \log p_R} < \frac{\log 2}{2 \log \gamma^*}.$$

Hence, by (4.15) and Theorem 4.1,

$$\dim_H(\mathscr{U} \setminus \mathscr{B}) = \dim_H(\mathscr{U} \cap [\gamma_*, \gamma^*]) = \frac{\log 2}{2 \log \gamma^*} \approx 0.339607.$$

Furthermore, since $\alpha(\gamma^*) = 22(01)^\infty$, $\gamma^*$ is the unique root in $(2,3)$ of the equation

$$1 = \frac{2}{x} + \frac{2}{x^2} + \frac{1}{x^2(x^2-1)},$$

or equivalently, $\gamma^*$ is the unique root in $(2,3)$ of $x^4 - 2x^3 - 3x^2 + 2x + 1 = 0$.

Finally, let $M \geq 3$. The leftmost entropy plateau with period $m = 1$ is $[p_L, p_R]$, where

$$\begin{aligned} M &= 2k+1 \quad \Rightarrow \quad \alpha(p_L) = (k+1)^\infty \quad \text{and} \quad \alpha(p_R) = (k+2)k^\infty, \\ M &= 2k \qquad\;\; \Rightarrow \quad \alpha(p_L) = (k+1)^\infty \quad \text{and} \quad \alpha(p_R) = (k+2)(k-1)^\infty. \end{aligned}$$

Note that for this entropy plateau, $p_R = q_\star(M)$, where $q_\star(M)$ was defined in (1.7). Now consider an arbitrary entropy plateau $[p_L, p_R]$ with period $m$. If $m = 1$, then $p_R \geq q_\star(M)$, so $m \log p_R \geq \log q_\star(M)$. And if $m \geq 2$, we have

$$\begin{aligned} m \log p_R \geq 2 \log q_{KL}(M) \geq 2 \log \left( \frac{M+2}{2} \right) &= \log(M^2 + 4M + 4) - \log 4 \\ \geq \log(4M+4) - \log 4 = \log(M+1) &> \log q_\star(M). \end{aligned}$$

In both cases, we obtain

$$\frac{\log 2}{m \log p_R} \leq \frac{\log 2}{\log q_\star(M)}.$$

Hence, by (4.15) and Theorem 4.1, $\dim_H(\mathscr{U} \setminus \mathscr{B}) = \log 2 / \log q_\star(M)$. This completes the proof. □

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