

ASYMPTOTIC DISTRIBUTION OF VALUES OF ISOTROPIC QUADRATIC FORMS AT S -INTEGRAL POINTS

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ABSTRACT. We prove an analogue of a theorem of Eskin-Margulis-Mozes [10]: suppose we are given a finite set of places S over \mathbb{Q} containing the archimedean place and excluding the prime 2, an irrational isotropic form \mathbf{q} of rank $n \geq 4$ on \mathbb{Q}_S , a product of p -adic intervals I_p , and a product Ω of star-shaped sets.

We show that unless $n = 4$ and \mathbf{q} is split in at least one place, the number of S -integral vectors $\mathbf{v} \in T\Omega$ satisfying simultaneously $\mathbf{q}(\mathbf{v}) \in I_p$ for $p \in S$ is asymptotically given by

$$\lambda(\mathbf{q}, \Omega) \|\cdot\| \cdot \|T\|^{n-2},$$

as T goes to infinity, where $\|\cdot\|$ is the product of Haar measures of the p -adic intervals I_p .

The proof uses dynamics of unipotent flows on S -arithmetic homogeneous spaces; in particular, it relies on an equidistribution result for certain translates of orbits applied to test functions with a controlled growth at infinity, specified by an S -arithmetic variant of the α -function introduced in [10], and an S -arithmetic version of a theorem of Dani-Margulis [7].

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1. INTRODUCTION

The Oppenheim conjecture, settled by Margulis in 1986 [17], states that for any non-degenerate indefinite quadratic form \mathbf{q} over \mathbb{R}^n , $n \geq 3$, the set $\mathbf{q}(\mathbb{Z}^n)$ of values of integral vectors is a dense subset of \mathbb{R} if \mathbf{q} is *irrational*, i.e. if \mathbf{q} is not proportional to a form with rational coefficients. Both indefiniteness and irrationality conditions are easily seen to be necessary. Margulis' proof is based on a study of orbits of certain unipotent one-parameter groups on the space of lattices in \mathbb{R}^3 . The idea of using unipotent flows dates back in an explicit form to Raghunathan and in implicit forms to Cassels and Swinnerton-Dyer. More precisely, Raghunathan observed that Oppenheim conjecture follows from the assertion that every bounded orbit of the orthogonal group $\mathrm{SO}(2, 1)$ on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ is compact. Raghunathan further conjectured in the mid-seventies that if G is a connected Lie group, Γ a lattice in G , and U a unipotent subgroup of G , then the closure of any orbit Ux , for $x \in G/\Gamma$, is itself an orbit Lx , where L is a closed connected subgroup of G containing U . Note that the subgroup L depends on x . Raghunathan's conjecture was later proved by Ratner [21].

1.1. The Quantitative Oppenheim Conjecture. The Quantitative Oppenheim conjecture, a refinement of the Oppenheim conjecture, is about the asymptotic distribution of the values of $\mathbf{q}(\mathbf{v})$, where \mathbf{v} runs over integral vectors in a large ball. To be more precise, let Ω be a radial set defined by $\Omega = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < \rho(\mathbf{x}/\|\mathbf{x}\|)\}$, where ρ is a positive continuous function on the unit sphere and $T\Omega$ is the dilation of Ω by a factor $T > 0$. Let us also denote by $\mathbf{N}_{(a,b),\mathbf{q},\Omega}(T)$ the number of integral vectors $\mathbf{x} \in T\Omega$ such that $\mathbf{q}(\mathbf{x}) \in (a, b)$.

In [7], Dani and Margulis obtained the asymptotic exact lower bound: for \mathbf{q} an irrational indefinite quadratic form on \mathbb{R}^n with $n \geq 3$, for a given interval (a, b) and a set Ω , we have

$$\liminf_{T \rightarrow \infty} \frac{\mathbf{N}_{(a,b),\mathbf{q},\Omega}(T)}{\lambda_{\mathbf{q},\Omega}(b-a)T^{n-2}} \geq 1,$$

where $\lambda_{\mathbf{q},\Omega}$ is a constant such that denominator gives the asymptotic volume of the set $\{\mathbf{v} \in T\Omega : \mathbf{q}(\mathbf{v}) \in (a, b)\}$. The proof of the above theorem is based on studying the distribution of the shifted orbits of the form $u_t Kx$, where u_t is a certain one-parameter unipotent subgroup, K is a maximal compact subgroup of the orthogonal group $\mathrm{SO}(\mathbf{q})$ associated to \mathbf{q} , and $x \in \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$. A suitable choice of a function f on \mathbb{R}^n allows one to approximate the number of the integral vectors \mathbf{v} in $T\Omega$ satisfying $\mathbf{q}(\mathbf{v}) \in (a, b)$ by an integral of the form

$$\int_K \tilde{f}(u_t kx) d\mu(x),$$

where $d\mu$ is the invariant probability measure on the space of unimodular lattices $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, and $\tilde{f} : \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}) \rightarrow \mathbb{R}$ is the Siegel transform of f defined by $\tilde{f}(g\Gamma) = \sum_{\mathbf{v} \in g\mathbb{Z}^n} f(\mathbf{v})$. Although \tilde{f} is an unbounded function to which the uniform version of Ratner's theorem proved in [7] does not directly apply, it is yet possible to establish asymptotically sharp lower bounds by approximating \tilde{f} from below by bounded functions.

The question of establishing an asymptotically sharp upper bound turns out to be substantially subtler. Note that, in general, the equidistribution results in [7] do not apply to arbitrary unbounded function. In the groundbreaking work [10], it is shown that if f has a controlled growth at infinity, then one can still establish this equidistribution result. The growth at infinity is measured by a family of functions $\alpha_1, \dots, \alpha_n$ with origin in geometry of numbers. If Λ is a

lattice in \mathbb{R}^n , then $\alpha_i(\Lambda)^{-1}$ is the smallest value for the covolume of $\Lambda \cap L$, where L ranges over all i -dimensional subspaces of \mathbb{R}^n for which $\Lambda \cap L$ is a lattice in L .

Setting $\alpha = \max(\alpha_i)$, one of the main results of [10] is that the equidistribution results of [7] continue to hold as long as the test function is majorized by α^s for some $s \in (0, 2)$. This is then used to show that if \mathbf{q} is a form of signature (r, s) with $r \geq 3$ and $s \geq 1$, then

$$\lim_{T \rightarrow \infty} \frac{\mathbf{N}_{(a,b),\mathbf{q},\Omega}(T)}{\lambda_{\mathbf{q},\Omega}(b-a)T^{n-2}} = 1.$$

For quadratic forms of signature $(2, 1)$ and $(2, 2)$, certain quadratic forms were constructed in [10] so that by choosing Ω to be the unit ball, for any $\epsilon > 0$,

$$\mathbf{N}_{(a,b),\mathbf{q}_i,\Omega}(T_j) \gg T_j^i (\log T_j)^{1-\epsilon}$$

for an infinite sequence $T_j \rightarrow \infty$. These counterexamples are very well approximable by rational forms. When \mathbf{q} has signature $(2, 2)$, it is proven in [11] that if forms of a class called EWAS are excluded, then a similar asymptotic statement continues to hold.

1.2. The S -arithmetic Oppenheim Conjecture. The generalization we study in this article involves considering various places.

Let S_f be a finite set of odd prime numbers and $S = \{\infty\} \cup S_f$. Each element of S (S_f , respectively) is called a *place* (a finite place, respectively). For $p \in S_f$, the p -adic norm on \mathbb{Q} is defined by $|x|_p = p^{-v_p(x)}$, where $v_p(x)$ is the p -adic valuation of x . The completion of \mathbb{Q} with respect to $|\cdot|_p$ is the field of p -adic numbers and is denoted by \mathbb{Q}_p . We will sometimes write \mathbb{Q}_∞ for \mathbb{R} . When $p \in S_f$, \mathbb{Q}_p contains a maximal compact subring \mathbb{Z}_p consisting of elements with norm at most 1. We also write $\mathbb{U}_p = \mathbb{Z}_p - p\mathbb{Z}_p$ for the set of multiplicative units in \mathbb{Z}_p . When $p = \infty$, we set $\mathbb{U}_p = \{\pm 1\}$. We will use the notation $\mathbb{U}_p^n = \mathbb{Z}_p^n - p(\mathbb{Z}_p^n)$ for $p \in S_f$ and $\mathbb{U}_\infty^n = \mathbb{S}^{n-1}$, the set of vectors of length 1 in \mathbb{R}^n .

For $p \in S_f$, λ_p stands for the Haar measure on \mathbb{Q}_p normalized so that $\lambda_p(\mathbb{Z}_p) = 1$. We denote by $\mathbb{Q}_S = \prod_{p \in S} \mathbb{Q}_p$ the direct product of \mathbb{Q}_p , $p \in S$, and write $\mathbb{Z}_S = \{x \in \mathbb{Q}_S : |x|_p \leq 1, \forall p \notin S\}$ for the ring of S -adic integers. The function on \mathbb{Q}_S^n defined by

$$\|v\| = \prod_{p \in S} \|v_p\|_p,$$

where $\|v_p\|_p = \max_i |(v_p)_i|_p$ for $p \in S_f$, plays a role of the norm on \mathbb{R}^n [15]. Also, $\|\cdot\|_\infty$ denotes the usual Euclidean norm. We will also denote by λ_S the product measure $\otimes_{p \in S} \lambda_p$. The Haar measures on \mathbb{Q}_p^n and \mathbb{Q}_S^n are denoted by λ_p^n and λ_S^n , respectively.

A quadratic form \mathbf{q} on \mathbb{Q}_S^n is an S -tuple $(\mathbf{q}_p)_{p \in S}$, where \mathbf{q}_p is a quadratic form on \mathbb{Q}_p^n . The form \mathbf{q} is called *isotropic*, if each \mathbf{q}_p is isotropic, that is, if there exists a non-zero vector $\mathbf{v} \in \mathbb{Q}_p^n$ such that $\mathbf{q}_p(\mathbf{v}) = 0$. A quadratic form over \mathbb{R} is isotropic if and only if it is indefinite. \mathbf{q} is called *non-degenerate* if each \mathbf{q}_p is non-degenerate, that is, the symmetric matrix $(\beta_{\mathbf{q}_p}(\mathbf{v}_i, \mathbf{v}_j))_{1 \leq i, j \leq n}$ is invertible, where $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of \mathbb{Q}_p^n and $\beta_{\mathbf{q}_p}(\mathbf{v}_i, \mathbf{v}_j) = (\mathbf{q}_p(\mathbf{v}_i + \mathbf{v}_j) - \mathbf{q}_p(\mathbf{v}_i) - \mathbf{q}_p(\mathbf{v}_j))/2$. Finally, we call \mathbf{q} *rational* if there exist a single quadratic form \mathbf{q}_0 (defined over \mathbb{Q}) and an invertible element $\lambda = (\lambda_p)_{p \in S} \in \mathbb{Q}_S$ satisfying that $\mathbf{q}_p = \lambda_p \mathbf{q}_0$ for all $p \in S$. Otherwise \mathbf{q} is called *irrational*.

Borel and Prasad proved the S -arithmetic Oppenheim conjecture over any number field k . In the case when $k = \mathbb{Q}$, the theorem says that for a non-degenerate irrational isotropic quadratic form \mathbf{q} on \mathbb{Q}_S^n ($n \geq 3$) and any given $\epsilon > 0$, there exists a non-zero vector $\mathbf{x} \in \mathbb{Z}_S^n$ with $|\mathbf{q}_p(\mathbf{x})| < \epsilon$ for all $p \in S$.

1.3. Statements of results. The goal of this paper is to prove a quantitative version of the theorem of Borel and Prasad mentioned above. Note that due to the presence of $|S|$ valuations, we need to consider $|S|$ parameters for the divergence to the infinity, etc.

Let $\mathbf{T} = (T_p)_{p \in S}$ be an S -tuple of positive real numbers with the components $T_p \in p^{\mathbb{Z}}$ for $p \in S_f$. Such an S -tuple \mathbf{T} will be called an S -time. The set of all S -time vectors is denoted by \mathcal{T}_S . For $\mathbf{T} \in \mathcal{T}_S$, write $\|\mathbf{T}\| = \prod_{p \in S} T_p$, and $m(\mathbf{T}) = \min_{p \in S} T_p$. For $\mathbf{T} = (T_p), \mathbf{T}' = (T'_p) \in \mathcal{T}_S$, define $\mathbf{T} \succeq \mathbf{T}'$ if $T_p \geq T'_p$ for all $p \in S$. Similarly, $\mathbf{T}_i = (T_{p,i})_{p \in S} \rightarrow \infty$ means that $T_{p,i} \rightarrow \infty$ for each $p \in S$. Since \mathbf{T} is of the form $(e^{t_\infty}, p_1^{t_1}, \dots, p_s^{t_s})$, let us denote $\mathbf{t} = (t_\infty, t_1, \dots, t_s)$ and consider S -parameter groups

$$\mathbb{T} = \mathbb{R} \times \prod_{p \in S_f} \mathbb{Z}, \quad \mathbb{T}^+ = \mathbb{R}^+ \times \prod_{p \in S_f} \mathbb{Z}^+.$$

which we will use later. The notions $\mathbf{t} \succeq \mathbf{t}'$ and $\mathbf{t} \rightarrow \infty$ are defined accordingly.

For $p \in S$, let $\rho_p : \mathbb{U}_p^n \rightarrow \mathbb{R}$ be a positive continuous function. For $p \in S_f$, we will assume throughout the paper that ρ satisfies the following condition:

$$(\mathbf{I}_\rho) \quad \rho_p(u\mathbf{x}) = \rho_p(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{U}_p^n, \quad \forall u \in \mathbb{U}_p.$$

Define

$$(1) \quad \Omega = \prod_{p \in S} \Omega_p \subseteq \mathbb{Q}_S^n,$$

where Ω_p is the set of vectors \mathbf{v}_p whose norm is bounded by the value of ρ_p in the direction of \mathbf{v}_p . Condition (\mathbf{I}_ρ) is indeed very mild and is satisfied by many sets of interest (e.g., the unit ball, which is defined by the constant function). Having fixed an S -time \mathbf{T} , we denote by $\mathbf{T}\Omega = \{(z_p \mathbf{v}_p)_{p \in S} : z_p \in \mathbb{Q}_p, |z_p| \leq T_p, \mathbf{v}_p \in \Omega_p\}$ the dilation of Ω by \mathbf{T} . A p -adic interval of length p^{-b} is a set of the form $I_p = a + p^b \mathbb{Z}_p$, where $a \in \mathbb{Q}_p$ and $b \in \mathbb{Z}$. An S -adic interval \mathbf{l} is a product of p -adic intervals I_p , with $p \in S$. Let us denote $\|\mathbf{l}\| = \prod \lambda_p(I_p)$.

Definition 1.1 (Counting and volume functions). *Fixing Ω as above, the counting and volume functions are defined respectively by*

$$\begin{aligned} \mathbf{N}_{\mathbf{l}, \mathbf{q}, \Omega}(\mathbf{T}) &= \text{card}\{\mathbf{v} \in \mathbb{Z}_S^n \cap \mathbf{T}\Omega : \mathbf{q}_p(\mathbf{v}_p) \in I_p, \forall p \in S\}, \\ \mathbf{V}_{\mathbf{l}, \mathbf{q}, \Omega}(\mathbf{T}) &= \text{vol}\{\mathbf{v} \in \mathbb{Q}_S^n \cap \mathbf{T}\Omega : \mathbf{q}_p(\mathbf{v}_p) \in I_p, \forall p \in S\}, \end{aligned}$$

where card and vol denote the cardinality and the Haar measure λ_S^n .

We start with the following statement about the asymptotic volume of the set of vectors in $T_p \Omega_p \subseteq \mathbb{Q}_p^n$ with the condition $\mathbf{q}(\mathbf{v}) \in \mathbf{l}$. This proposition will be proven in Section 4.

Proposition 1.2. *For a constant $\lambda = \lambda(\mathbf{q}, \Omega)$, we have,*

$$\mathbf{V}_{\mathbf{l}, \mathbf{q}, \Omega}(\mathbf{T}) \sim \lambda(\mathbf{q}, \Omega) \cdot \|\mathbf{l}\| \cdot \|\mathbf{T}\|^{n-2},$$

as $\mathbf{T} \rightarrow \infty$.

The asymptotic behavior of $\mathbf{N}_{\mathbf{l}, \mathbf{q}, \Omega}(\mathbf{T})$ is more intricate than the volume asymptotics. The issue for real forms of signature $(2, 1)$ and $(2, 2)$ persists in the S -arithmetic setup.

Definition 1.3. *Let $\mathbf{q} = (\mathbf{q}_p)_{p \in S}$ be an isotropic quadratic form on \mathbb{Q}_S^n . We say that \mathbf{q} is exceptional if either (1) $n \leq 3$, or (2) $n = 4$ and for some p , \mathbf{q}_p is split, i.e., it is equivalent to the form $x_1 x_4 + x_2^2 - x_3^2$.*

See Section 2.2 for more details. Note that the set $\mathcal{O}_S(n)$ of non-degenerate quadratic forms on \mathbb{Q}_S^n can be identified with a Zariski open subset of $\prod_{p \in S} \text{sym}_n(\mathbb{Q}_p)$, where $\text{sym}_n(\mathbb{Q}_p)$ denotes the set of $n \times n$ symmetric matrices over \mathbb{Q}_p^n . In this correspondence, a form $\mathbf{q} = (\mathbf{q}_p)_{p \in S}$ is associated to its Gram matrix.

Theorem 1.4. *Let \mathcal{D} be a compact subset of non-exceptional quadratic forms on \mathbb{Q}_S^n , \mathbf{l} an S -interval, and Ω as in (1). Then there exists a constant $C = C(\mathcal{D}, \mathbf{l}, \Omega)$ such that for any $\mathbf{q} \in \mathcal{D}$ and any sufficiently large S -time T we have*

$$\mathbf{N}_{\mathbf{l}, \mathbf{q}, \Omega}(T) \leq C \|T\|^{n-2}.$$

Note that this theorem is effective in the sense that the constant C can be explicitly given in terms of the data. The next theorem is our main result which provides asymptotically sharp bounds for $\mathbf{N}_{\mathbf{l}, \mathbf{q}, \Omega}(T)$. For the sake of simplicity, the theorem below is stated for an individual irrational form. Modifications of the proof, along the lines of the proofs in [10], can be made to establish a uniform version when \mathbf{q} runs over a compact set of forms.

Theorem 1.5. *Let \mathbf{q} be an irrational isotropic form in $\mathcal{O}_S(n)$, \mathbf{l} be an S -interval, and Ω be as in (1). If \mathbf{q} is not exceptional, then*

$$\lim_{T \rightarrow \infty} \frac{\mathbf{N}_{\mathbf{l}, \mathbf{q}, \Omega}(T)}{\lambda(\mathbf{q}, \Omega) |\mathbf{l}| \cdot \|T\|^{n-2}} = 1.$$

Remark 1.6. The proof given here establishes both the lower and the upper bound at the same time. It seems likely to us that an adaptation of the arguments in [7] can be used to establish the lower bound even when \mathbf{q} is exceptional.

The proof of Theorem 1.5 rests upon a number of ingredients. We will use the dynamics on \mathbf{G}/Γ , where $\mathbf{G} = \text{SL}_n(\mathbb{Q}_S)$ and $\Gamma = \text{SL}_n(\mathbb{Z}_S)$, which can be identified with a set \mathcal{L}_S of unimodular S -lattices. First, we will relate the counting problem to a question about the asymptotic behavior of integrals of the form

$$\int_{\mathbf{K}} \tilde{f}(a_t \mathbf{k} \Delta) dm(\mathbf{k}),$$

where \mathbf{K} is a maximal compact subgroup of $\text{SO}(\mathbf{q})$, x is an element in \mathbf{G}/Γ related to \mathbf{q} , m is the normalized Haar measure of \mathbf{K} , and a_t is a 1-parameter diagonal subgroup of $\text{SO}(\mathbf{q})$ (see Equation (3) in Section 2.3). Here, \tilde{f} is the Siegel transform of a compactly supported function f defined on \mathbb{Q}_S^n (see Definition 3.7). Such integrals, when \tilde{f} is replaced by a bounded continuous function can be dealt with using an S -arithmetic version of the results in [7], which we will state and prove in Section 6 of this paper.

Since \tilde{f} is unbounded, in order to use the equidistribution result just described, one needs to also control the integral of $\tilde{f}(a_t \mathbf{k} \Delta)$ when $a_t \mathbf{k} \Delta$ is far into the cusp. As in [10], this is dealt with using the function α_S (see section 3.6), which is an analog of α introduced in the previous subsection. We prove

Theorem 1.7. *With the notation as above, assume that \mathbf{q} is not exceptional. For $0 < s < 2$ and for any S -lattice $\Delta \in \mathcal{L}_S$,*

$$\sup_{t > 1} \int_{\mathbf{K}} \alpha_S(a_t \mathbf{k} \Delta)^s dm(\mathbf{k}) < \infty.$$

Moreover, the bound is uniform as Δ varies over a compact subset \mathcal{C} of \mathcal{L}_S .

We will prove Theorem 1.5 using Theorem 6.2 and the following result which uses Theorem 1.7 as an important ingredient:

Theorem 1.8. *Set $\mathbf{G} = \mathrm{SL}_n(\mathbb{Q}_S)$, $\Gamma = \mathrm{SL}_n(\mathbb{Z}_S)$, $\mathbf{H} = \mathrm{SO}(\mathbf{q})$, and let $\phi : \mathbf{G}/\Gamma \rightarrow \mathbb{R}$ be a continuous function, and ν be a positive continuous function on $\prod_{p \in S} \mathbb{U}_p^n$. Assume that for some $0 < s < 2$, we have $|\phi(\Delta)| < C\alpha_S(\Delta)^s$ for all $\Delta \in \mathbf{G}/\Gamma$. Let $x_0 \in \mathbf{G}/\Gamma$ be such that $\mathbf{H}x_0$ is not closed. Then*

$$\lim_{t \rightarrow \infty} \int_{\mathbf{K}} \phi(a_t \mathbf{k} x_0) \nu(\mathbf{k}) d\mathbf{m}(\mathbf{k}) = \int_{\mathbf{G}/\Gamma} \phi(y) d\mu(y) \int_{\mathbf{K}} \nu d\mathbf{m}(\mathbf{k}).$$

It is natural to inquire what happens in the case of exceptional forms. In this direction, we can prove the following theorem:

Theorem 1.9. *Let $\mathbf{l} \subseteq \mathbb{Q}_S$ be an S -interval and Ω be the product of unit balls in \mathbb{Q}_p^3 for $p \in S$. Then, there exist an isotropic irrational quadratic form on \mathbb{Q}_S^3 , a constant $c > 0$, and a sequence $T_i \rightarrow \infty$ such that*

$$\mathbf{N}_{\mathbf{l}, \mathbf{q}, \Omega}(T_i) > c \|T_i\| (\log \|T_i\|)^{1-\epsilon}.$$

Since scalar multiples of a single form \mathbf{q} cover all possible equivalence classes of isotropic quadratic forms over \mathbb{Q}_p^3 (this can be easily seen from the proof of Proposition 2.1), the above theorem shows that the counterexamples can be constructed in any equivalence class of quadratic forms. This theorem, however, leaves out the case of forms with four variables. It would be interesting to study these forms in details in the future.

Remark 1.10. In this paper, sans serif roman letters are reserved for S -adic objects. Algebraic groups will be denoted by bold letters. For instance, when $\mathbf{G} = \mathbf{SL}_n$, then $\mathbf{G}(\mathbb{Q}_p) = \mathrm{SL}_n(\mathbb{Q}_p)$ and $\mathbf{G} = \prod_{p \in S} \mathbf{G}(\mathbb{Q}_p)$. Elements of \mathbf{G}, \mathbf{H} , etc. are denoted by lower case \mathbf{g}, \mathbf{h} , etc.

Remark 1.11. In order to streamline the process of normalizing a vector across different places, we will use the following notation. For $p \in S_f$ and a real number $x = p^n$, we set $x^\diamond = p^{-n}$, viewed as a p -adic number. If x is a positive real number, then, by definition $x^\diamond = x$. If $\mathbf{T} = (T_p)_{p \in S}$ is an S -time (which, as defined above, means that T_p is a power of p for every $p \in S_f$), we define $T^\diamond = (T_p^\diamond)_{p \in S}$; in particular, if $\mathbf{v} \in \mathbb{Q}_S^n$, then each component of the vector $\mathbf{v}/\|\mathbf{v}\|^\diamond := (\mathbf{v}_p/\|\mathbf{v}_p\|_p^\diamond)$ has norm one.

Remark 1.12. It would be interesting to extend the result to include the prime $p = 2$ in S . We expect some modifications to be necessary, for example, Proposition 4.2 and classification of quadratic forms depend on whether p is 2 or odd.

Remark 1.13. It is likely that the results of this paper can be generalized to the case of arbitrary number fields. This, however, will inevitably involve using more number theory. For instance, one will need a classification of quadratic forms over finite extensions of p -adic fields, which is more complicated than that over p -adic fields.

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2. PRELIMINARIES

In this section, we recall some definitions about \mathbb{Q}_p -vector spaces and quadratic forms defined on them.

2.1. Norms on exterior products. Let $1 \leq i \leq n$. One equips the i th exterior product $\bigwedge^i \mathbb{R}^n$ with an inner product defined by

$$\langle \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_i, \mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_i \rangle = \det (\mathbf{x}_k \cdot \mathbf{y}_j)_{1 \leq k, j \leq i}.$$

For $\mathbf{v} \in \bigwedge^i \mathbb{R}^n$, the induced norm is given by $\|\mathbf{v}\| = |\langle \mathbf{v}, \mathbf{v} \rangle|^{1/2}$. Denote by $\mathbf{e}_1, \dots, \mathbf{e}_n$ the canonical basis for \mathbb{R}^n . For $J = \{1 \leq j_1 < \dots < j_i \leq n\}$, denote $\mathbf{e}_J = \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_i}$. As J runs over all subsets of $\{1, 2, \dots, n\}$ of i elements, the set $\{\mathbf{e}_J\}$ forms an orthonormal basis for $\bigwedge^i \mathbb{R}^n$ and hence $\|\sum_J a_J \mathbf{e}_J\| = (\sum_J a_J^2)^{1/2}$. The inner product defined above is preserved by the action of the orthogonal group $O(n)$ on \mathbb{R}^n .

The most convenient norm to work with, for our purpose, is given by $\|\mathbf{v}\| = \max_J |a_J|_p$. The action of $\mathrm{SL}_n(\mathbb{Z}_p)$ on $\bigwedge^i \mathbb{Q}_p^n$ preserves this norm. Indeed, since the entries of $k \in \mathrm{SL}_n(\mathbb{Z}_p)$ are p -adic integers, we immediately have the inequality $\|k\mathbf{v}\|_p \leq \|\mathbf{v}\|_p$. Equality follows by applying this inequality to k^{-1} instead of k .

2.2. Quadratic forms. In this subsection, we review the classification of quadratic forms over the p -adic fields \mathbb{Q}_p for odd primes p following [24]. Recall that a quadratic form q on \mathbb{R}^n has signature (r, s) if there is $g \in \mathrm{GL}_n(\mathbb{R})$ such that

$$q(\mathbf{x}) = \mathbf{q}(g\mathbf{x}),$$

where $\mathbf{q}(\mathbf{x}) = \sum_{i=1}^r x_i^2 - \sum_{i=r+1}^{r+s} x_i^2$. Two quadratic forms over the field \mathbb{Q}_p are *equivalent* if and only if they have the same rank, the same discriminant, and the same Hasse invariant. Recall that the discriminant of a non-degenerate quadratic form over a field F is the image of the determinant of the Gram matrix G in the group $F^*/(F^*)^2$. This group contains 4 elements when $F = \mathbb{Q}_p$ and p is an odd prime, representing the parity of $v_p(\det G)$ and whether the image of $\det G$ in the residue field is a quadratic residue or not. Hasse invariant can take two values and is invariant under scalar multiplication. It follows that there are at most 8 different quadratic forms over \mathbb{Q}_p of any given dimension. A non-exceptional isotropic quadratic form \mathbf{q} over \mathbb{Q}_p is called *standard* if it is of the form

$$\mathbf{q}(\mathbf{x}) = x_1 x_n + a_2 x_2^2 + \cdots + a_{n-1} x_{n-1}^2,$$

where $a_i \in \{1, p, u, pu\}$ and for some j, k ,

$$-a_j/a_k \text{ is not a square.}$$

Here $u \in \mathbb{U}_p$ is a fixed element such that its image in $\mathbb{Z}_p/p\mathbb{Z}_p$ is a non-residue. Any non-exceptional isotropic quadratic form is equivalent to a standard one.

2.3. Orthogonal groups. For a quadratic form $\mathbf{q} = (\mathbf{q}_p)_{p \in S}$ over \mathbb{Q}_S , we define the orthogonal group of \mathbf{q} by $\mathrm{SO}(\mathbf{q}) = \prod_{p \in S} \mathrm{SO}(\mathbf{q}_p)$, where $\mathrm{SO}(\mathbf{q}_p)$ is the p -adic analytic group consisting of matrices of determinant 1 preserving the quadratic form \mathbf{q}_p . Let K_∞ be the maximal compact subgroup of $\mathrm{SO}(\mathbf{q}_\infty)$ which is isomorphic to $\mathrm{SO}(r) \times \mathrm{SO}(s)$, where (r, s) is the signature of \mathbf{q}_∞ . For $p \in S_f$, we define $K_p = \mathrm{SL}_n(\mathbb{Z}_p) \cap \mathrm{SO}(\mathbf{q}_p)$. It is easy to see that K_p is a maximal subgroup of $\mathrm{SO}(\mathbf{q}_p)$. Finally, we set $\mathbf{K} = \prod_{p \in S} K_p$ and let $m_{\mathbf{K}}, m_{K_p}$ be the Haar measures on \mathbf{K}, K_p , respectively. Since K_p is a subgroup of $\mathrm{SL}_n(\mathbb{Z}_p)$, the p -adic max norm on \mathbb{Q}_p^n is invariant by K_p . We will also need the following one-parameter subgroups. For each $p \in S_f$, define

$$(2) \quad a_t^p = \begin{cases} \mathrm{diag}(p^t, 1, \dots, 1, p^{-t}) : & t \in \mathbb{Z} \quad p \in S_f \\ \mathrm{diag}(e^{-t}, 1, \dots, 1, e^t) : & t \in \mathbb{R} \quad p = \infty. \end{cases}$$

We denote by A_p the group consisting of all a_t^p with $t \in \mathbb{Z}$ when $p \in S_f$, and $t \in \mathbb{R}$ when $p = \infty$.

Then we can define subgroups of $\mathrm{SO}(\mathbf{q})$ for a standard quadratic form \mathbf{q} as

$$(3) \quad A = \prod A_p = \{a_{\mathbf{t}} = a_{t_\infty}^\infty \cdot \prod_{p \in S_f} a_{t_p}^p\} = \{a_{\mathbf{t}} : \mathbf{t} \in \mathbb{T}\}, \quad A^+ = \{a_{\mathbf{t}} : \mathbf{t} \in \mathbb{T}^+\}.$$

The following simple fact about ternary forms will be used later.

Proposition 2.1. *Let \mathbf{q} be a non-degenerate isotropic ternary quadratic form over \mathbb{Q}_p . Then the group $\mathrm{SO}(\mathbf{q})$ is locally isomorphic to $\mathrm{PSL}_2(\mathbb{Q}_p)$.*

Proof. This is well-known for \mathbb{R} , so assume $p \in S_f$. The conjugation action of $\mathrm{SL}_2(\mathbb{Q}_p)$ on the space of 2×2 matrices of trace zero over \mathbb{Q}_p preserves the determinant, which can be viewed as a non-degenerate isotropic ternary quadratic form \mathbf{q}_0 . Since the kernel is $\pm I$, and the corresponding Lie algebras are isomorphic, the orthogonal group of \mathbf{q}_0 is locally isomorphic to $\mathrm{PSL}_2(\mathbb{Q}_p)$. For an arbitrary non-degenerate isotropic ternary quadratic form \mathbf{q} and $c \in \mathbb{Q}_p \setminus \{0\}$, the form $c\mathbf{q}$ has the same orthogonal group as \mathbf{q} , and for an appropriate value of c has the same discriminant as \mathbf{q}_0 . \square

The proof of the following fact is deferred to the appendix:

Proposition 2.2. *For given $c_1 \in \mathbb{Q}_p$ and $c_2 \in p^\mathbb{Z}$, K_p acts transitively on*

$$\{\mathbf{v}_p \in \mathbb{Q}_p^n : \mathbf{q}(\mathbf{v}_p) = c_1 \text{ and } \|\mathbf{v}_p\|_p = c_2\}.$$

3. THE S -ARITHMETIC GEOMETRY OF NUMBERS

In this section, we prove some basic results on the S -arithmetic geometry of numbers. Due to lack of reference in the S -arithmetic case, we provide proofs.

3.1. S -lattices and a generalization of the α function. The proof of Oppenheim conjecture is based on the dynamics of unipotent flow on the space of lattices, thus we need an analogous S -arithmetic notion.

Definition 3.1. *Let V be a free \mathbb{Q}_S -module of rank n . A \mathbb{Z}_S -module Δ is an S -lattice in V if there exist $\mathbf{x}_1, \dots, \mathbf{x}_n \in V$ such that*

$$\Delta = \mathbb{Z}_S \mathbf{x}_1 \oplus \dots \oplus \mathbb{Z}_S \mathbf{x}_n$$

and V is generated by $\mathbf{x}_1, \dots, \mathbf{x}_n$ as a \mathbb{Q}_S -module.

Note that an S -lattice is a discrete and cocompact subgroup of \mathbb{Q}_S^n . The base lattice Δ_0 is the S -lattice generated by the canonical basis $\mathbf{e}_1, \dots, \mathbf{e}_n$. Note that $\mathrm{SL}_n(\mathbb{Q}_S)$ acts on the set of lattices in \mathbb{Q}_S^n and the orbit of Δ_0 under this action can be identified with $\mathcal{L}_S := \mathrm{SL}_n(\mathbb{Q}_S)/\mathrm{SL}_n(\mathbb{Z}_S)$. For an S -lattice Δ , we say that a subspace $L \subseteq \mathbb{Q}_S^n$ is Δ -rational if the intersection $L \cap \Delta$ is an S -lattice in L , that is, L is a subspace of \mathbb{Q}_S^n generated by finite number of elements in Δ . Suppose that L is i -dimensional and $L \cap \Delta = \mathbb{Z}_S \mathbf{v}^1 \oplus \dots \oplus \mathbb{Z}_S \mathbf{v}^i$. Define $d_\Delta(L)$ or simply $d(L)$ by

$$(4) \quad d(L) = d_\Delta(L) = \prod_{p \in S} \|\mathbf{v}^1 \wedge \dots \wedge \mathbf{v}^i\|_p.$$

The definition of $d(L)$ is independent of the choice of basis. This is an analogue of the volume of the quotient space $L/(L \cap \Delta)$ for lattices in \mathbb{R}^n . If $L = \{0\}$, we write $d(L) = 1$. We now show that there is indeed a more intimate connection between the \mathcal{L}_S and the space \mathcal{L} of lattices (of covolume 1) in \mathbb{R}^n . More precisely, we show that the \mathcal{L}_S fibers over \mathcal{L} in such a way that many natural functions on \mathcal{L}_S factor through the projection map.

Definition 3.2 (The projection map). *Let $\Delta \in \mathcal{L}_S$. Define $\pi(\Delta)$ to be the set of vectors $\mathbf{v} \in \mathbb{R}^n$ for which there exists $\mathbf{w} \in \Delta$ with $\mathbf{w}_\infty = \mathbf{v}$ and $\mathbf{w}_p \in \mathbb{Z}_p^n$ for all $p \in S_f$.*

Remark 3.3. For any vector $\mathbf{v} \in \pi(\Delta)$ there is exactly one vector $\mathbf{w} \in \Delta$ satisfying $\mathbf{w}_\infty = \mathbf{v}$. Indeed, if both \mathbf{w}_1 and \mathbf{w}_2 satisfy this property, then $\mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2 \in \Delta$ is a non-zero vector with real component zero. This implies that $(\prod_{p \in S_f} p^m) \mathbf{w} \rightarrow 0$ as $m \rightarrow \infty$, which contradicts the discreteness of Δ .

Proposition 3.4. *$\pi(\Delta)$ is a lattice of covolume 1 in \mathbb{R}^n .*

Proof. Since the p -adic components of the elements of $\pi(\Delta)$ belong to the compact sets \mathbb{Z}_p^n , for $p \in S_f$, the vectors $\mathbf{w} = \mathbf{v}_\infty$ cannot have an accumulation point, since then by passing to a subsequence, we can find an accumulation point for a sequence of vectors in Δ .

Start with a \mathbb{Z}_S -basis $\mathbf{v}^1, \dots, \mathbf{v}^n$ for Δ . By replacing the vectors \mathbf{v}^i with certain \mathbb{Z}_S linear combinations of them, we may assume that the determinant of the matrix A_p with columns $\mathbf{v}_p^1, \dots, \mathbf{v}_p^n$ is a unit in \mathbb{Z}_p , for every $p \in S_f$. Note that this is equivalent to the condition

$$(5) \quad \|\mathbf{v}_p^1 \wedge \dots \wedge \mathbf{v}_p^n\|_p = 1$$

for every $p \in S_f$. We now claim that $\mathbf{v}_\infty^1, \dots, \mathbf{v}_\infty^n$ form a \mathbb{Z} -basis for $\pi(\Delta)$. If $\mathbf{w} \in \pi(\Delta)$, then $\mathbf{w} = \mathbf{v}_\infty$ for some $\mathbf{v} \in \Delta$ with $\mathbf{v}_p \in \mathbb{Z}_p^n$. Write $\mathbf{v} = \sum_{k=1}^n c_k \mathbf{v}^k$, for $c_k \in \mathbb{Z}_S$, and let \mathbf{c} be the column vector with entries c_1, \dots, c_n . Since $A_p \mathbf{c} = \mathbf{v}_p$, $A_p^{-1} \in \mathrm{SL}_n(\mathbb{Z}_p)$, and $\mathbf{v}_p \in \mathbb{Z}_p^n$, we deduce that $\mathbf{c} \in \mathbb{Z}_p^n$ for all $p \in S_f$. This implies that $\mathbf{c} \in \mathbb{Z}^n$, which proves our claim. Since Δ is unimodular, (5) implies that $\|\mathbf{v}_\infty^1 \wedge \dots \wedge \mathbf{v}_\infty^n\|_\infty = 1$, showing that $\pi(\Delta)$ has covolume 1. \square

Definition 3.5. *For an S -lattice Δ , define*

$$\alpha_i^S(\Delta) = \sup \left\{ \frac{1}{d(L)} : L \text{ is a } \Delta\text{-rational subspace of dimension } i \right\},$$

where $1 \leq i \leq n$ and

$$\alpha^S(\Delta) = \max \{ \alpha_i(\Delta) : 0 \leq i \leq n \}.$$

The function α^S is not only an analogue of the original function α , but is in fact intimately related to it.

Proposition 3.6. *For any $\Delta \in \mathcal{L}_S$, we have*

$$\alpha^S(\Delta) = \alpha(\pi(\Delta)).$$

Proof. Assume that $\alpha_i^S(\Delta) = 1/d(L)$ where L is the \mathbb{Z}_S -span of $\mathbf{v}^1, \dots, \mathbf{v}^i$. Similarly to the proof of Proposition 3.4, after possibly replacing $\mathbf{v}^1, \dots, \mathbf{v}^i$ by another basis, we can assume that $\|\mathbf{v}_p^1 \wedge \dots \wedge \mathbf{v}_p^i\|_p = 1$ for all $p \in S_f$. This implies that $d(L) = \|\mathbf{v}_\infty^1 \wedge \dots \wedge \mathbf{v}_\infty^i\|_\infty$. As before, $\mathbf{v}_\infty^1, \dots, \mathbf{v}_\infty^i$ form a \mathbb{Z} -basis for $\pi(L)$, implying that $\alpha_i(\pi(\Delta)) \geq \alpha_i^S(\Delta)$. The converse can be proven similarly. \square

Since S is fixed throughout the discussion, we will henceforth drop the superscript S in α_i^S and α^S . We now define an S -adic Siegel transform.

Definition 3.7. *Let $f : \mathbb{Q}_S^n \rightarrow \mathbb{R}$ be a bounded function vanishing outside a compact set. The S -adic Siegel transform of f is a function on \mathbf{G}/Γ defined as follows: for $g \in \mathrm{SL}_n(\mathbb{Q}_S)$,*

$$\tilde{f}(g) = \sum_{\mathbf{v} \in \mathbb{Z}_S^n} f(g\mathbf{v}).$$

More generally, \tilde{f} is defined on the space of S -lattices Δ as $\tilde{f}(\Delta) = \sum_{\mathbf{v} \in \Delta} f(\mathbf{v})$.

Since the diagonal embedding of \mathbb{Z}_S^n in \mathbb{Q}_S is discrete, the defining sum involves only finitely many non-zero terms. The following lemma is a natural extension of Schmidt's theorem ([23], Lemma 2) in geometry of numbers.

Lemma 3.8. (*S -adic Schmidt lemma*) *Let $f : \mathbb{Q}_S^n \rightarrow \mathbb{R}$ be a bounded function vanishing outside a compact subset. Then there exists a positive constant $c = c(f)$ such that $\tilde{f}(\Delta) < c\alpha^S(\Delta)$ for any S -lattice Δ in \mathbb{Q}_S^n .*

Proof. Without loss of generality, we can assume that f is the characteristic function of $B = \prod_{p \in S} B_p$, where B_p is a ball of radius p^{n_p} , for $p \in S_f$ and a ball of radius R for $p = \infty$. Set $D = \prod_{p \in S_f} p^{n_p}$ and consider the scalar map $\delta : \Delta \rightarrow \Delta$ defined by $\delta(\mathbf{x}) = D \cdot \mathbf{x}$. Note that if $\mathbf{x} \in \Delta \cap B$, then $\|\mathbf{x}\|_p \leq 1$ for all $p \in S_f$ and $\|\mathbf{x}\|_\infty \leq RD$. In particular, $\pi(\delta(\mathbf{x}))$ will be inside a ball B' of radius RD . Note that Schmidt's lemma states that

$$\mathrm{card}(\Delta' \cap B') = O(\alpha(\Delta')),$$

holds for any lattice Δ' in \mathbb{R}^n , where the implied constant depends on B' . Using this fact and Remark 3.3 we obtain

$$\mathrm{card}(\Delta \cap B) \leq \mathrm{card}(\pi(\Delta) \cap B') = O(\alpha(\pi(\Delta))) = O(\alpha^S(\Delta)).$$

\square

Lemma 3.9. *Let Δ be an S -lattice in \mathbb{Q}_S^n and let L and M be two Δ -rational subspaces. Then we have that*

$$(6) \quad d(L)d(M) \geq d(L \cap M)d(L + M),$$

where $d = d_\Delta$ is defined in (4).

Proof. We first verify that if L and M are Δ -rational, then $L \cap M$ and $L + M$ are also Δ -rational. Since a subspace of \mathbb{Q}_S^n is Δ -rational if and only if it is generated by elements of Δ , $L + M$ is clearly a Δ -rational subspace of \mathbb{Q}_S^n . On the other hand, consider a projection map $\pi : \mathbb{Q}_S^n \rightarrow \mathbb{Q}_S^n/\Delta$. Note that the projection image $\pi(H)$ of a subspace H in \mathbb{Q}_S^n is closed if

and only if $H \cap \Delta$ is a lattice subgroup in H , that is, H is Δ -rational. Since π is proper, if we set $H = \pi^{-1}(\overline{\pi(L \cap M)}) = \pi^{-1}(\overline{\pi(L) \cap \pi(M)})$, we can easily check that H is Δ -rational and $H = L \cap M$.

Let $p : \mathbb{Q}_S^n \rightarrow \mathbb{Q}_S^n / (L \cap M)$. Since $d_\Delta(H) = d_{p(\Delta)}(p(H))d_\Delta(L \cap M)$ for any Δ -rational subspace H , the inequality (6) is equivalent to

$$d_{p(\Delta)}(L)d_{p(\Delta)}(M) \geq d_{p(\Delta)}(L + M).$$

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be bases of $p(L)$ and $p(M)$ consisting of elements in $p(\Delta)$, respectively. Then since $(p(L) \cap p(\Delta)) + (p(M) \cap p(\Delta)) \subseteq p(L + M) \cap p(\Delta)$,

$$\begin{aligned} d_{p(\Delta)}(L)d_{p(\Delta)}(M) &= \|\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_\ell\| \|\mathbf{w}_1 \wedge \dots \wedge \mathbf{w}_m\| \\ &\geq \|\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_\ell \wedge \mathbf{w}_1 \wedge \dots \wedge \mathbf{w}_m\| \geq d_{p(\Delta)}(L + M). \end{aligned}$$

□

The following Lemma is the S -arithmetic version of Lemma 3.10 in [10].

Lemma 3.10. *The function α belongs to $L^r(\mathrm{SL}_n(\mathbb{Q}_S)/\mathrm{SL}_n(\mathbb{Z}_S))$ for all values of $1 \leq r < n$.*

Proof. Consider the map $\pi : \mathcal{L}_S \rightarrow \mathcal{L}$. Denote the invariant probability measures on \mathcal{L}_S and \mathcal{L} by μ_S and μ , respectively. We claim that π is measure-preserving, that is, for any measurable $A \subseteq \mathcal{L}_\mathbb{R}$, we have $\mu_S(\pi^{-1}(A)) = \mu(A)$. To see this, define $\nu(A) = \mu_S(\pi^{-1}(A))$. One can easily see that ν is an $\mathrm{SL}_n(\mathbb{R})$ -invariant probability measure on \mathcal{L} , implying that $\nu = \mu$. The result will immediately follow from Proposition 3.6. □

We can now prove the following generalization of the classical Siegel's formula in geometry of numbers.

Proposition 3.11. (*S -arithmetic Siegel integral formula*) *Let f be a continuous function with compact support on \mathbb{Q}_S^n , $\mathbf{G} = \mathrm{SL}_n(\mathbb{Q}_S)$ and $\Gamma = \mathrm{SL}_n(\mathbb{Z}_S)$. Then*

$$\int_{\mathbf{G}/\Gamma} \tilde{f} d\mu = \int_{\mathbb{Q}_S^n} f d\lambda_S^n + f(0).$$

The proof is based on the following lemma. For each $p \in S$, we denote by λ_p^n the Haar measure on \mathbb{Q}_p^n normalized such that $\lambda_p^n(\mathbb{Z}_p^n) = 1$. Denote by δ_p the delta measure at the zero point of \mathbb{Q}_p^n .

Lemma 3.12. *Let ν be an $\mathrm{SL}_n(\mathbb{Q}_S)$ -invariant Radon measure on \mathbb{Q}_S^n . Then ν is a linear combination of measures of the form $\bigotimes_{p \in S} \nu_p$, where each ν_p is either the Haar measure λ_p^n or the delta measure δ_p .*

Proof. Observe that if ν is a positive Radon measure on a locally compact space X invariant under the action of a group H , and H has a finite number of orbits, namely X_1, \dots, X_m , then ν naturally decomposes into the sum $\nu = \sum_{i=1}^m \nu_i$, where $\nu_i(A) = \nu(A \cap X_i)$ is H -invariant and supported on X_i , for $1 \leq i \leq m$. Let us now consider the case at hand, namely, the action of $\mathrm{SL}_n(\mathbb{Q}_S)$ on \mathbb{Q}_S^n . Since the natural action of $\mathbf{G} = \mathrm{SL}_n(\mathbb{Q}_p)$ on \mathbb{Q}_p^n has two orbits on \mathbb{Q}_p^n , the \mathbf{G} -orbits in the action on \mathbb{Q}_S^n are of the form $\prod_{p \in S} X_p$, where X_p is either the zero point or $\mathbb{Q}_p^n - \{0\}$. Each orbit is a homogenous space \mathbf{G}/L , where L is the stabilizer of an arbitrary point in the respective orbit. The invariant measure on the homogeneous space \mathbf{G}/L , if exists, is unique up to a factor. The result follows. □

Proof of Proposition 3.11. Note that by Lemmas 3.8 and 3.10, \widetilde{f} is integrable. This implies that the map $f \mapsto \int_{G/\Gamma} \widetilde{f} d\mu$ defines a positive linear functional on the space of compactly supported continuous functions on \mathbb{Q}_S^n , hence defines a Radon measure ν on \mathbb{Q}_S^n . Since μ is G -invariant, so is ν . Using Lemma 3.12, ν splits as a linear combination of measures of the form $\bigotimes_{p \in S} \nu_p$, where each ν_p is either λ_p^n or the delta measure δ_p . The coefficient of $\bigotimes_{p \in S} \lambda_p^n$ in this linear combination is one. Indeed, taking an increasing sequence of compactly supported non-negative continuous functions f_n approximating the normalized characteristic functions of increasingly larger balls in \mathbb{Q}_S^n , one obtains $\widetilde{f}_n \rightarrow 1$, and $\int_{\mathbb{Q}_S^n} f_n d\lambda_S^n \rightarrow 1$. Similarly, by choosing a family f_n of non-negative continuous functions with support in a ball of radius $1/n$ with $|f_n| \leq 1$ and $f_n(0) = 1$, and applying the Lebesgue's dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_{G/\Gamma} \widetilde{f}_n d\mu = 1,$$

which implies that the coefficient of $f(0)$ is 1. It remains to be shown that the rest of the coefficients are zero. This can be obtained in a similar way from the fact that a lattice Δ in \mathbb{Q}_S^n cannot contain a non-zero point $(\mathbf{v}_p)_{p \in S}$ with $\mathbf{v}_{p_0} = 0$ for some $p_0 \in S$. Otherwise, one can find an appropriate sequence n_i of S -integers, such that $|n_i|_q \rightarrow 0$ for all $q \neq p_0$ and hence $n_i \mathbf{v} \rightarrow 0$ contradicting the discreteness of Δ . From here, and by choosing appropriate sequences of continuous functions as above, we can deduce that the rest of the coefficients are zero. \square

3.2. Integration on subvarieties. Now let us introduce volume forms of submanifolds in \mathbb{Q}_S^n inherited from the volume form $d\lambda_S^n$ of \mathbb{Q}_S^n in order to measure the volumes of orbits of maximal compact subgroups of orthogonal groups in \mathbb{Q}_S^n . Since such maximal compact subgroups are products of maximal compact subgroups at each place, it suffices to deal with volume forms of p -adic submanifolds in \mathbb{Q}_p^n .

Let us introduce two equivalent definitions of volume form on submanifolds. Recall that any orbit of the p -adic maximal compact subgroup $K_p = \mathrm{SL}_n(\mathbb{Z}_p) \cap \mathrm{SO}(\mathbf{q}_p)$ of $\mathrm{SO}(\mathbf{q}_p)$ for a given quadratic form \mathbf{q}_p on \mathbb{Q}_p^n is of the form

$$\{\mathbf{w} \in \mathbb{Q}_p^n : \|\mathbf{w}\|_p = p^{c_1}, \mathbf{q}_p(\mathbf{w}) = c_2\},$$

where $c_1 \in \mathbb{Z}$ and $c_2 \in \mathbb{Q}_p$ are some constants. Note that the above K_p -orbit is an open subset of the $(n-1)$ -dimensional p -adic variety in \mathbb{Q}_p^n given by

$$\{\mathbf{w} \in \mathbb{Q}_p^n : \mathbf{q}_p(\mathbf{w}) = c_2\}.$$

Hence from now on we will consider Y as a compact open subset of a d -dimensional p -adic variety in \mathbb{Q}_p^n or a parallelepiped $\mathbb{Z}_p \mathbf{v}_1 \oplus \cdots \oplus \mathbb{Z}_p \mathbf{v}_d$ where $\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathbb{Q}_p^n$. Then Y is contained in $p^{-r} \mathbb{Z}_p^n$ for some $r \in \mathbb{N}$.

Consider the projection $\pi_\ell : p^{-r} \mathbb{Z}_p^n \rightarrow p^{-r} \mathbb{Z}_p^n / p^\ell \mathbb{Z}_p^n$ and let $Y_\ell = \pi_\ell(Y)$. Then the volume form ν_d defined over Y is defined by

$$\nu_d(Y) = \lim_{\ell \rightarrow \infty} \frac{\#Y_\ell}{p^{d\ell}}.$$

As in the real case, we can define the volume form of a p -adic submanifold by describing the volume form of its tangent space. Let Y be a parallelepiped $\mathbb{Z}_p \mathbf{v}_1 \oplus \cdots \oplus \mathbb{Z}_p \mathbf{v}_d$, where $\mathbf{v}_1, \dots, \mathbf{v}_d$ are linearly independent in \mathbb{Q}_p^n . Then the volume form ν'_d of the parallelepiped is defined as

$$\nu'_d(\mathbb{Z}_p \mathbf{v}_1 \oplus \cdots \oplus \mathbb{Z}_p \mathbf{v}_d) = \|\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_d\|_p,$$

where $\|\cdot\|$ is the maximum norm of $\bigwedge^d(\mathbb{Q}_p^n)$.

We will use the fact (see [26, Theorem 9]) that the normalized measures ν_d and ν'_d are equal.

4. PASSAGE TO HOMOGENEOUS DYNAMICS

In this section, we define a p -adic analogue of J_f function introduced in Section 3 in [10] for the real case. This function is used in Section 5 to relate the counting problem to the asymptotic formula of certain integrals on the space of S -lattices. Fix a prime $p \in S_f$, and let \mathbf{q} be a standard quadratic form on \mathbb{Q}_p^n . Let $\pi_1 : \mathbb{Q}_p^n \rightarrow \mathbb{Q}_p$ be the projection to the first coordinate and $\mathbb{Q}_{p,+}^n$ be the set $\{\mathbf{x} \in \mathbb{Q}_p^n : \pi_1(\mathbf{x}) \neq 0\}$. We denote by $p^{\mathbb{Z}}$ the subset of \mathbb{Q}_p consisting of powers of p .

Lemma 4.1. *Let f be a continuous function with compact support on $\mathbb{Q}_{p,+}^n$ which satisfies the invariance property*

$$(\mathbf{I}_f) \quad f(ux_1, x_2, \dots, x_{n-1}, u^{-1}x_n) = f(x_1, x_2, \dots, x_n)$$

for all units $u \in \mathbb{U}_p$. Let ν be a non-negative continuous function on the unit sphere \mathbb{U}_p^n such that

$$(\mathbf{I}_\nu) \quad \nu(u\mathbf{x}) = \nu(\mathbf{x}),$$

for all $u \in \mathbb{U}_p$. Let J_f be the real-valued function on $p^{\mathbb{Z}} \times \mathbb{Q}_p \subset \mathbb{Q}_p^2$ defined by

$$(7) \quad J_f(p^{-r}, \zeta) = \frac{1}{p^{r(n-2)}} \int_{\mathbb{Q}_p^{n-2}} f(p^{-r}, x_2, \dots, x_{n-1}, x_n) dx_2 \cdots dx_{n-1},$$

where in the integral, $x_n = p^r(\zeta - \mathbf{q}(0, x_2, \dots, x_{n-1}, 0))$. Then for any $\epsilon > 0$, if t and $\|\mathbf{v}\|$ are sufficiently large, we have

$$(8) \quad \left| c(K_p) p^{t(n-2)} \int_{K_p} f(a_t k \mathbf{v}) \nu(k^{-1} \mathbf{e}_1) dm(k) - J_f(p^t \|\mathbf{v}\|^\diamond, \zeta) \nu\left(\frac{\mathbf{v}}{\|\mathbf{v}\|^\diamond}\right) \right| < \epsilon,$$

where $c(K_p) = \text{vol}(K_p \cdot \mathbf{e}_1) / (1 - 1/p)$ and m is the normalized Haar measure on K_p .

Proof. Let π_i denote the projection onto the i th coordinate in \mathbb{Q}_p^n , and $a_t := a_t^p = \text{diag}(p^t, 1, \dots, 1, p^{-t})$ as before. Since f has compact support, for $t \gg 1$, if $f(a_t \mathbf{w}) \neq 0$, we have

$$\|\mathbf{w}\|_p = |\pi_1(\mathbf{w})|_p.$$

This implies that on the support of the function $k \mapsto f(a_t k \mathbf{v})$, for t large enough we have

$$(9) \quad a_t k \mathbf{v} = (\|\mathbf{v}\|_p^\diamond p^t \cdot u(k \mathbf{v}), \pi_2(k \mathbf{v}), \dots, \pi_{n-1}(k \mathbf{v}), \mathbf{v}_n u(k \mathbf{v})^{-1}).$$

Using (\mathbf{I}_f) we have

$$f(a_t k \mathbf{v}) = f(\|\mathbf{v}\|_p^\diamond p^t, \pi_2(k \mathbf{v}), \dots, \pi_{n-1}(k \mathbf{v}), v_n).$$

where v_n is determined by $\mathbf{q}(\|\mathbf{v}\|_p^\diamond p^t, \pi_2(k \mathbf{v}), \dots, \pi_{n-1}(k \mathbf{v}), v_n) = \mathbf{q}(\mathbf{v})$. Since K_p preserves the norm, given $\delta > 0$, for $t > t_0(\delta)$, if $f(a_t k \mathbf{v}) \neq 0$, we have

$$|k \mathbf{v} / \|\mathbf{v}\|_p^\diamond - u(\pi_1(k \mathbf{v})) \mathbf{e}_1|_p < \delta$$

for a unit $u(\pi_1(k \mathbf{v})) \in \mathbb{U}_p$. This implies that,

$$\left| \frac{\mathbf{v}}{\|\mathbf{v}\|_p^\diamond} - k^{-1}(u(\pi_1(k \mathbf{v})) \mathbf{e}_1) \right|_p = \left| \frac{\mathbf{v}}{\|\mathbf{v}\|_p^\diamond} - u(\pi_1(k \mathbf{v}))(k^{-1} \mathbf{e}_1) \right|_p < \delta.$$

Since ν is continuous and satisfies (\mathbf{I}_ν) , by choosing $\delta > 0$ small enough we can assure that if $t > t_0(\delta)$, and $f(a_t k \mathbf{v}) \neq 0$, then the following holds:

$$(10) \quad \left| \nu(k^{-1} \mathbf{e}_1) - \nu\left(\frac{\mathbf{v}}{\|\mathbf{v}\|_p^\diamond}\right) \right| < \epsilon.$$

For sufficiently large t , if $f(a_t k \mathbf{v}) \neq 0$, then the vector $(\|\mathbf{v}\|_p^\diamond p^t, \pi_2(k \mathbf{v}), \dots, \pi_{n-1}(k \mathbf{v}), v_n)$ will be at a bounded distance from \mathbf{e}_1 .

Since the K_p -invariant measure on the orbit $K_p \cdot \mathbf{v}$ is the push-forward of the normalized Haar measure on K , one can approximate the integral in (7) by the integration on the K_p -orbit $K_p \cdot (\|\mathbf{v}\|_p^\diamond \mathbf{e}_1) \subset \mathbb{Q}_p^n$. For $t > \max\{t', t''\}$,

$$(11) \quad \left| \int_{K_p} f(a_t k \mathbf{v}) \nu(k^{-1} \mathbf{e}_1) dm(k) - \int_{K_p \cdot (\|\mathbf{v}\|_p^\diamond \mathbf{e}_1)} f(\|\mathbf{v}\|_p^\diamond p^t, \pi_2(k \mathbf{v}), \dots, \pi_{n-1}(k \mathbf{v}), v_n) \nu\left(\frac{\mathbf{v}}{\|\mathbf{v}\|_p^\diamond}\right) d\lambda_{n-1} \right| \leq \epsilon,$$

where λ_{n-1} is the normalized Haar measure on $K \cdot (\|\mathbf{v}\|_p^\diamond \mathbf{e}_1)$. \square

We remark that (\mathbf{I}_ν) is an essential condition but (\mathbf{I}_f) is not a real restriction since we can easily modify the given function to satisfy (\mathbf{I}_f) .

Proposition 4.2. *Suppose h is a real-valued continuous function with compact support defined on $(\mathbb{Q}_p^n - \{0\}) \times \mathbb{Q}_p$. Then*

$$\lim_{t \rightarrow \infty} \frac{1}{p^{(n-2)t}} \int_{\mathbb{Q}_p^n} h(p^t \mathbf{v}, \mathbf{q}(\mathbf{v})) d\mathbf{v} = \text{vol}(K \cdot \mathbf{e}_1) \sum_{z \in \mathbb{Z}} p^{(n-2)z} \int_K \int_{\mathbb{Q}_p} h(p^{-z} k \mathbf{e}_1, \zeta) d\zeta dm(k).$$

where $\text{vol} = \text{vol}_{n-1}$ denote the $(n-1)$ -dimensional volume induced from \mathbb{Q}_p^n .

Proof. Let us make the substitution $\mathbf{w} = p^t \mathbf{v}$. This gives $d\mathbf{w} = 1/p^{nt} d\mathbf{v}$, hence we have

$$\frac{1}{p^{(n-2)t}} \int_{\mathbb{Q}_p^n} h(p^t \mathbf{v}, \mathbf{q}(\mathbf{v})) d\mathbf{v} = \frac{1}{p^{(n-2)t}} \int_{\mathbb{Q}_p^n} h(\mathbf{w}, p^{-2t} \mathbf{q}(\mathbf{w})) p^{nt} d\mathbf{w} = p^{2t} \int_{\mathbb{Q}_p^n} h(\mathbf{w}, p^{-2t} \mathbf{q}(\mathbf{w})) d\mathbf{w}.$$

Replacing p^{-2t} by p^{-t} , the left integral in (12) can be changed to

$$L_1(h) := \lim_{t \rightarrow \infty} p^t \int_{\mathbb{Q}_p^n} h(\mathbf{w}, p^{-t} \mathbf{q}(\mathbf{w})) d\mathbf{w}.$$

With the decomposition $\mathbb{Q}_p^n = \bigcup_{z \in \mathbb{Z}} p^{-z} \mathbb{U}_p^n$, write

$$(12) \quad h = \sum_{z \in \mathbb{Z}} h_z,$$

where $h_z(\mathbf{v}, x) = h(\mathbf{v}, x) \mathbf{1}_{p^{-z} \mathbb{U}_p^n}(\mathbf{v})$. Since h has compact support, all but finitely many of h_z are zero. We first claim that for h with $\text{supp } h \subseteq \mathbb{U}_p^n \times \mathbb{Q}_p$,

$$\lim_{t \rightarrow \infty} \frac{1}{p^{(n-2)t}} \int_{\mathbb{Q}_p^n} h(p^t \mathbf{v}, \mathbf{q}(\mathbf{v})) d\mathbf{v} = \text{vol}(K \cdot \mathbf{e}_1) \int_K \int_{\mathbb{Q}_p} h(k \mathbf{e}_1, \zeta) d\zeta dm(k).$$

Then, for h with $\text{supp } h \subseteq p^{-z_0} \mathbb{U}_p^n$ where $z_0 \in \mathbb{Z}$, define $h'(\mathbf{v}, x) = h(p^{-z_0} \mathbf{v}, p^{-2z_0} x)$, which has compact support in $\mathbb{U}_p^n \times \mathbb{Q}_p$. By a change of variables, we have

$$\begin{aligned}
L_1(h) &= \lim_{t \rightarrow \infty} p^t \int_{\mathbb{Q}_p^n} h(\mathbf{v}, p^{-t} \mathbf{q}(\mathbf{v})) d\mathbf{v} = \lim_{t \rightarrow \infty} p^t \int_{p^{-z_0} \mathbb{U}_p^n} h(\mathbf{v}, p^{-t} \mathbf{q}(\mathbf{v})) d\mathbf{v} \\
&= p^{nz_0} \lim_{t \rightarrow \infty} p^t \int_{\mathbb{U}_p^n} h(p^{-z_0} \mathbf{v}, p^{-t-2z_0} \mathbf{q}(\mathbf{v})) d\mathbf{v} \\
&= p^{nz_0} \lim_{t \rightarrow \infty} p^t \int_{\mathbb{U}_p^n} h'(\mathbf{v}, \mathbf{q}(\mathbf{v})) d\mathbf{v} = \text{vol}(K \cdot \mathbf{e}_1) \int_K \int_{\mathbb{Q}_p} h'(k \mathbf{e}_1, \zeta) d\zeta dm(k) \\
&= \text{vol}(K \cdot \mathbf{e}_1) p^{nz_0} \int_K \int_{\mathbb{Q}_p} h(p^{-z_0} k \mathbf{e}_1, p^{-2z_0} \zeta) d\zeta dm(k) \\
&= \text{vol}(K \cdot \mathbf{e}_1) p^{(n-2)z_0} \int_K \int_{\mathbb{Q}_p} h(p^{-z_0} k \mathbf{e}_1, \zeta') d\zeta' dm(k),
\end{aligned}$$

where the last equality relies on the change of variables $\zeta' = p^{-2z_0} \zeta$. Now, using the decomposition (12), Proposition 4.2 follows for an arbitrary h .

Now let us prove the claim. Let h satisfy $\text{supp } h \subseteq \mathbb{U}_p^n \times \mathbb{Q}_p$. Furthermore, since we put $\mathbf{q}(\mathbf{v})$ in the second variable and \mathbf{q} is the quadratic form with p -adic integral coefficients, we can add the assumption that $\text{supp } h \subseteq \mathbb{U}_p^n \times \mathbb{Z}_p$.

Let us denote by \mathcal{A}_k the subspace of the space of continuous functions on $\mathbb{U}_p^n \times \mathbb{Z}_p$ which is spanned by the functions of the form $h(\mathbf{v}, x) = f(\mathbf{v})g(x)$, with $f(\mathbf{v}) = \mathbf{1}_A(\mathbf{v})$ and $g(x) = \mathbf{1}_B(x)$, where $A \subseteq \mathbb{U}_p^n$ and $B \subseteq \mathbb{Z}_p$ are respectively the preimages of subsets $\bar{A} \subseteq (\mathbb{Z}/p^k \mathbb{Z})^n$ and $\bar{B} \subseteq \mathbb{Z}/p^k \mathbb{Z}$ by the reduction map red_k . We will also set $\mathcal{A} = \bigcup_{k=1}^{\infty} \mathcal{A}_k$. In view of Stone-Weierstrass theorem for p -adic fields ([9]), \mathcal{A} is dense in the space of all continuous functions on $\mathbb{U}_p^n \times \mathbb{Z}_p$. Let us denote the reduction maps $\mathbb{Z}_p \rightarrow \mathbb{Z}_p/p^k \mathbb{Z}_p$ (and also $\mathbb{Z}_p^n \rightarrow (\mathbb{Z}_p/p^k \mathbb{Z}_p)^n$) by red_k . Let h be a real-valued continuous function with compact support contained in $\mathbb{U}_p^n \times \mathbb{Z}_p$. We will do some reduction steps: we may assume that $h(\mathbf{v}, x) = f(\mathbf{v})g(x)$, where $f(\mathbf{v}) = \mathbf{1}_A(\mathbf{v})$ and $g(x) = \mathbf{1}_B(x)$, with

$$\begin{aligned}
(13) \quad A &= \text{red}_k^{-1}(\bar{A}) \subseteq \mathbb{U}_p^n, \\
B &= \text{red}_k^{-1}(\bar{B}) \subseteq \mathbb{Z}_p.
\end{aligned}$$

In other words,

$$\begin{aligned}
\mathbf{v} \in A, \mathbf{w} \in \mathbb{Z}_p^n \text{ with } \|\mathbf{w}\| < p^{-k} &\implies \mathbf{v} + \mathbf{w} \in A, \\
x \in B, y \in \mathbb{Z}_p \text{ with } |y| < p^{-k} &\implies x + y \in B.
\end{aligned}$$

It thus suffices to show that

$$\begin{aligned}
L_1(h) &= \lim_{t \rightarrow \infty} p^t \mu_n(\{\mathbf{v} \in A, \mathbf{q}(\mathbf{v}) \in p^t B\}) \\
&= \text{vol}(K \cdot \mathbf{e}_1) m_K(\{k \in K : k \mathbf{e}_1 \in A\}) \mu_1(B).
\end{aligned}$$

Let us define

$$\begin{aligned}
C_t(A, B) &= \{\mathbf{v} \in \mathbb{Z}_p^n : \mathbf{v} \in A, p^{-t} \mathbf{q}(\mathbf{v}) \in B\}, \\
C'_t(A, B) &= \{\mathbf{u} + p^t \mathbf{w} : \mathbf{u} \in A, \mathbf{q}(\mathbf{u}) = 0, \mathbf{w} \in \mathbb{Z}_p^n, 2\beta(\mathbf{u}, \mathbf{w}) \in B\}.
\end{aligned}$$

Here β is the associated bilinear form to the quadratic form \mathbf{q} . Recall the polarization identity:

$$\mathbf{q}(\mathbf{v} + \mathbf{w}) = \mathbf{q}(\mathbf{v}) + 2\beta(\mathbf{v}, \mathbf{w}) + \mathbf{q}(\mathbf{w}).$$

We claim that $C_t(A, B) = C'_t(A, B)$ for any $t > k$. First, we show that $C'_t(A, B) \subseteq C_t(A, B)$. Assume that $\mathbf{v} = \mathbf{u} + p^t \mathbf{w}$ with $\mathbf{u} \in A, \mathbf{w} \in \mathbb{Z}_p^n$ satisfying $\mathbf{q}(\mathbf{u}) = 0, 2\beta(\mathbf{u}, \mathbf{w}) \in B$. This implies that

$$\mathbf{q}(\mathbf{u} + p^t \mathbf{w}) = 2p^t \beta(\mathbf{u}, \mathbf{w}) + p^{2t} \mathbf{q}(\mathbf{w})^2 = p^t(2\beta(\mathbf{u}, \mathbf{w}) + p^t \mathbf{q}(\mathbf{w})^2)$$

Since $\mathbf{u} \in A$ and $\|p^t \mathbf{v}\| \leq p^{-t}$, for $t > k$, we have $\mathbf{v} \in A$. Moreover,

$$p^{-t} \mathbf{q}(\mathbf{v}) = 2\beta(\mathbf{u}, \mathbf{w}) + p^t \mathbf{q}(\mathbf{w})^2 \in B.$$

Conversely, if $\mathbf{v} \in C_t(A, B)$, set $\mathbf{v}' = \mathbf{e}_1 + \frac{1}{2} \mathbf{q}(\mathbf{v}) \mathbf{e}_n$. Note that $\mathbf{q}(\mathbf{v}') = \mathbf{q}(\mathbf{v})$ and $\|\mathbf{v}'\| = \|\mathbf{v}\| = 1$. Hence by Proposition 2.2, there exists $k \in K$ such that $\mathbf{v} = k \mathbf{v}'$. Set $\mathbf{u} = k \mathbf{e}_1$ and $\mathbf{w} = \frac{1}{2} p^{-t} \mathbf{q}(\mathbf{v}) k \mathbf{e}_n$, so that \mathbf{v} decomposes as $\mathbf{v} = k(\mathbf{e}_1 + \frac{1}{2} \mathbf{q}(\mathbf{v}) \mathbf{e}_n) = \mathbf{u} + p^t \mathbf{w}$. Furthermore, since $\mathbf{v} \in A$ and $\|\mathbf{u} - \mathbf{v}\| = \|\frac{1}{2} \mathbf{q}(\mathbf{v}) k \mathbf{e}_n\| \leq p^{-t} < p^{-k}$, we have $\mathbf{u} \in A$. Clearly, we have $\mathbf{q}(\mathbf{v}) = \mathbf{q}(k \mathbf{e}_1) = 0$ and

$$2\beta(\mathbf{u}, \mathbf{w}) = 2\beta(k \mathbf{e}_1, \frac{1}{2} p^{-t} \mathbf{q}(\mathbf{v}) k \mathbf{e}_n) = p^{-t} \mathbf{q}(\mathbf{v}) \beta(k \mathbf{e}_1, k \mathbf{e}_n) = p^{-t} \mathbf{q}(\mathbf{v}) \in B.$$

This equality implies that

$$\mu_n(C_t(A, B)) = \mu_n(C'_t(A, B)) = \lim_{\ell \rightarrow \infty} \frac{|\text{red}_\ell(C'_t(A, B))|}{p^{n\ell}}.$$

Set $A_\ell = \text{red}_\ell(A) \subseteq \mathbb{Z}/p^\ell \mathbb{Z}$ and $B_\ell = \text{red}_\ell(B) \subseteq \mathbb{Z}/p^\ell \mathbb{Z}$. We will denote the induced quadratic form over $\mathbb{Z}_p/p^\ell \mathbb{Z}_p$ by \mathbf{q} as well. First observe that for $\ell/2 > t > k$, we have

$$\text{red}_\ell(C'_t(A, B)) = \{\bar{u} + p^t \bar{w} : \bar{u} \in A_\ell, \mathbf{q}(\bar{u}) = 0, 2\beta(\bar{u}, \bar{w}) \in B_\ell\}.$$

For any choice of $\bar{u} \in A_\ell \cap \{\bar{u} : \mathbf{q}(\bar{u}) = 0\}$ there exist exactly $|B_\ell|$ choices for \bar{w} such that $2\beta(\bar{u}, \bar{w}) \in B_\ell$. Hence, in order to find $\mu_n(C_t(A, B))$, it suffices to find the cardinality of the fibers of the map $\Phi : (\mathbb{Z}/p^\ell \mathbb{Z})^n \times (\mathbb{Z}/p^\ell \mathbb{Z})^n \rightarrow (\mathbb{Z}/p^\ell \mathbb{Z})^n$ defined by

$$\Phi(\bar{u}, \bar{w}) = \bar{u} + p^t \bar{w}.$$

We will show that all the fibers have the same size. If $\Phi(\bar{u}_1, \bar{w}_1) = \Phi(\bar{u}_2, \bar{w}_2)$, then $\bar{u}_1 \equiv \bar{u}_2 \pmod{p^t}$. Conversely, suppose that $\bar{u}_1 \in A_\ell, \mathbf{q}(\bar{u}_1) = 0$ and $\bar{v} = \bar{u}_1 + p^t \bar{w}_1$ for some \bar{w}_1 which satisfies $2\beta(\bar{u}_1, \bar{w}_1) \in B_\ell$. Let $\bar{u}_2 \in A_\ell$ be such that $\mathbf{q}(\bar{u}_2) = 0, \bar{u}_1 \equiv \bar{u}_2 \pmod{p^t}$. Write $\bar{u}_2 - \bar{u}_1 = p^t z$. Set $\bar{w}_2 = \bar{w}_1 - z$, hence $\bar{v} = \bar{u}_2 + p^t \bar{w}_2$. First note that $\mathbf{q}(\bar{u}_1) = \mathbf{q}(\bar{u}_2) = 0$ combined with

$$\mathbf{q}(\bar{u}_2) = \mathbf{q}(u_1 + p^t z) = \mathbf{q}(u_1) + 2p^t \beta(\bar{u}_1, z) + p^{2t} \mathbf{q}(z)$$

implies that $\beta(\bar{u}_1, z)$ is divisible by p^t . This implies that

$$2\beta(\bar{u}_2, \bar{w}_2) = 2\beta(\bar{u}_1 - p^t z, \bar{w}_1 + z) = 2\beta(\bar{u}_1, \bar{w}_1) - 2p^t \beta(z, \bar{w}_1 + z) + 2\beta(\bar{u}_1, z) \in B_\ell$$

This shows that the fibers of Φ are in one-to-one correspondence with the equivalence classes of $\bar{u} \in A_\ell$ module p^t . This implies that for all sets A, B satisfying (13), for $\ell \geq t > k$, we have

$$|\text{red}_\ell(C_t(A, B))| = N_t |A_\ell \cap \{\bar{u} : \mathbf{q}(\bar{u}) = 0\}| \times |B_\ell|.$$

Note that N_t does not depend on ℓ . In the special case $A = \mathbb{U}_p^n$ and $B = \mathbb{Z}_p$, $C_t(A, B)$ is precisely the p^{-t} -neighborhood of the set $\{\mathbf{u} \in \mathbb{U}_p^n : \mathbf{q}(\mathbf{u}) = 0\}$. It follows that

$$\lim_{t \rightarrow \infty} p^t \mu_n(C_t(\mathbb{U}_p^n, \mathbb{Z}_p)) = \mu_{n-1}(\{\mathbf{u} \in \mathbb{U}_p^n : \mathbf{q}(\mathbf{u}) = 0\}) = \text{vol}(K \cdot \mathbf{e}_1).$$

From here, for general A and B we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} p^t \mu_n(C_t(A, B)) &= \lim_{t \rightarrow \infty} p^t \mu_n(C_t(\mathbb{U}_p^n, \mathbb{Z}_p)) \frac{\mu_n(C_t(A, B))}{\mu_n(C_t(\mathbb{U}_p^n, \mathbb{Z}_p))} \\ &= \text{vol}(K \cdot \mathbf{e}_1) \lim_{\ell \rightarrow \infty} \frac{|A_\ell \cap \{\mathbf{v} \in (\mathbb{Z}/p^\ell \mathbb{Z})^n : \mathbf{q}(\mathbf{v}) = 0\}| \cdot |B_\ell|}{|\{\mathbf{v} \in (\mathbb{Z}/p^\ell \mathbb{Z})^n : \mathbf{q}(\mathbf{v}) = 0\}| \cdot |\mathbb{Z}/p^\ell \mathbb{Z}|} \\ &= \text{vol}(K \cdot \mathbf{e}_1) m_K \{k \in K : k \mathbf{e}_1 \in A\} \mu_1(B). \end{aligned}$$

□

5. COUNTING RESULTS

In this section, we prove Theorems 1.4 and 1.5. The proofs depend on an ergodic result which will be proven in the next section.

Proof of Theorem 1.4. Since any bounded set Ω is contained in a ball of radius r for some $r > 0$, for an upper bound, it suffices to prove the theorem for the case when Ω is a unit ball.

Without loss of generality, we may assume that there exists a compact set $C \subset \mathbf{G}$ so that every quadratic form in \mathcal{D} is of the form \mathbf{q}^g for some $g \in C$, where \mathbf{q} is a fixed standard form, since there are finitely many equivalence classes of quadratic forms over \mathbb{Q}_S . Since C is compact, there exists β such that for every $g = (g_p)_{p \in S} \in C$ and every $\mathbf{v} \in \mathbb{Q}_S^n$, $\beta^{-1} \|\mathbf{v}_p\|_p \leq \|g_p \mathbf{v}_p\|_p \leq \beta \|\mathbf{v}_p\|_p$ for every $p \in S$. Let $\epsilon > 0$ be given and $g \in C$ be arbitrary. Let $f = \prod_{p \in S} f_p$ be a continuous non-negative function with compact support on \mathbb{Q}_S^n such that $J_{f_\infty} \geq (1 + \epsilon)^{1/|S|}$ on $[\beta^{-1}, 2\beta] \times I_\infty$ and $J_{f_p} \geq (1 + \epsilon)^{1/|S|}$ on $p^\mathbb{Z} \cap \{x \in \mathbb{Q}_p : \beta^{-1} \leq |x|_p \leq 2\beta\} \times I_p$ for $p \in S_f$. It is clear that if $\mathbf{v} \in \mathbb{Q}_S^n$ satisfies the conditions

$$(14) \quad T_\infty \leq \|\mathbf{v}_\infty\| \leq 2T_\infty, \|\mathbf{v}_p\|_p = T_p, \quad \mathbf{q}(g\mathbf{v}) \in I,$$

then $J_f(\|\mathbf{g}\mathbf{v}\|^\diamond / |\mathbf{T}|^\diamond, \mathbf{q}(g\mathbf{v})) \geq 1 + \epsilon$. By Lemma 4.1 and Lemma 3.6 in [10], for sufficiently large \mathbf{t} ,

$$c(\mathbf{K}) \|\mathbf{T}\|^{n-2} \int_{\mathbf{K}} f(a_{\mathbf{t}} \mathbf{k} \mathbf{g} \mathbf{v}) dm(\mathbf{k}) \geq 1$$

if \mathbf{v} satisfies (14). Summing over $\mathbf{v} \in \mathbb{Z}_S^n$, it follows that the number of vectors $\mathbf{v} \in \mathbb{Z}_S^n$ satisfying (14) is bounded from above by

$$(15) \quad \sum_{\mathbf{v} \in \mathbb{Z}_S^n} c(\mathbf{K}) \|\mathbf{T}\|^{n-2} \int_{\mathbf{K}} f(a_{\mathbf{t}} \mathbf{k} \mathbf{g} \mathbf{v}) dm(\mathbf{k}) = c(\mathbf{K}) \|\mathbf{T}\|^{n-2} \int_{\mathbf{K}} \tilde{f}(a_{\mathbf{t}} \mathbf{k} \mathbf{g}) dm(\mathbf{k}).$$

Since (15) holds for all sufficiently large \mathbf{T} , by summing over all $(T_\infty/2^{k_\infty}, T_p/p^{k_1}, \dots, T_s/p^{k_s})$, $k_\infty, k_1, \dots, k_s \in \mathbb{N}$, the statement follows. □

Proposition 5.1. *Let $f_p : \mathbb{Q}_p^n \rightarrow \mathbb{R}$ be a positive continuous function with compact support in $\mathbb{Q}_{p,+}^n$, and $f = \prod_{p \in S} f_p$. Assume further that f_p satisfies the property (\mathbf{I}_f) when $p \in S_f$. Let $\nu_p : \mathbb{U}_p^n \rightarrow \mathbb{R}$ be positive continuous functions satisfying the condition (\mathbf{I}_ν) and $\nu = \prod_{p \in S} \nu_p$. For a non-exceptional quadratic form \mathbf{q} , $\epsilon > 0$ and every $\mathbf{g} \in \mathbf{G}$, there exists a positive S -time \mathbf{t}_0 so that for $\mathbf{t} \succ \mathbf{t}_0$,*

$$(16) \quad \left| \|\mathbf{T}\|^{-(n-2)} \sum_{\mathbf{v} \in \mathbb{Z}_S^n} J_f \left(\frac{\|\mathbf{g}\mathbf{v}\|^\diamond}{|\mathbf{T}|^\diamond}, \mathbf{q}(g\mathbf{v}) \right) \nu \left(\frac{\mathbf{g}\mathbf{v}}{\|\mathbf{g}\mathbf{v}\|^\diamond} \right) - c(\mathbf{K}) \int_{\mathbf{K}} \tilde{f}(a_{\mathbf{t}} \mathbf{k} \mathbf{g}) \nu(\mathbf{k}^{-1} \mathbf{e}_1) dm(\mathbf{k}) \right| \leq \epsilon,$$

where $c(\mathbf{K}) = \prod_{p \in S} c(K_p)$.

Proof. Since $J_f = \prod_{p \in S} J_{f_p}$ has compact support, by Theorem 1.4 for given $\mathbf{g} \in \mathbf{G}$, the number of $\mathbf{v} \in \mathbb{Z}_S^n$ such that $J_f(\mathbf{g}, \mathbf{v}, \mathbf{t})\nu(\mathbf{g}, \mathbf{v}) \neq 0$ is bounded by $c\|\mathbf{T}\|^{n-2}$ for some $c > 0$. Take \mathbf{t}_0 such that Lemma 4.1 and Lemma 3.6 in [10] hold for $\mathbf{g}\mathbf{v}$ and ϵ/c . Then for $\mathbf{t} \succ \mathbf{t}_0$,

$$\begin{aligned} & \left| \|\mathbf{T}\|^{-(n-2)} \sum_{\mathbf{v} \in \mathbb{Z}_S^n} J_f \left(\frac{\|\mathbf{g}\mathbf{v}\|^\diamond}{\mathbf{T}^\diamond}, \mathbf{q}(\mathbf{g}\mathbf{v}) \right) \nu \left(\frac{\mathbf{g}\mathbf{v}}{\|\mathbf{g}\mathbf{v}\|^\diamond} \right) - c(\mathbf{K}) \int_{\mathbf{K}} \tilde{f}(a_{\mathbf{t}}\mathbf{k}\mathbf{g})\nu(\mathbf{k}^{-1}\mathbf{e}_1)dm(\mathbf{k}) \right| \\ & \leq \|\mathbf{T}\|^{-(n-2)} \sum_{\mathbf{v} \in \mathbb{Z}_S^n} \left| J_f \left(\frac{\|\mathbf{g}\mathbf{v}\|^\diamond}{\mathbf{T}^\diamond}, \mathbf{q}(\mathbf{g}\mathbf{v}) \right) \nu \left(\frac{\mathbf{g}\mathbf{v}}{\|\mathbf{g}\mathbf{v}\|^\diamond} \right) - c(\mathbf{K})\|\mathbf{T}\|^{n-2} \int_{\mathbf{K}} f(a_{\mathbf{t}}\mathbf{k}\mathbf{g}\mathbf{v})\nu(\mathbf{k}^{-1}\mathbf{e}_1)dm(\mathbf{k}) \right| \\ & \leq C(f)e^{-(n-2)t_0} \prod_{p \in S_f} p^{-(n-2)t_p} \cdot \epsilon/c \cdot c\|\mathbf{T}\|^{n-2} = C(f)\epsilon. \end{aligned}$$

Here $C(f)$ is a constant depending on f which comes from the fact that if $|a_i - b_i| < \epsilon$ and $|a_i| < A, |b_i| < B$, then $|\prod a_i - \prod b_i| < C(A, B)\epsilon$. Note that both terms in the second line of the equations are bounded by a constant depending on f . \square

In the rest of this section, we will deal with compactly supported continuous function $h : (\mathbb{Q}_S^n - \{0\}) \times \mathbb{Q}_S \rightarrow \mathbb{R}$ of the form $h(\mathbf{v}, \zeta) = \prod_{p \in S} h_p(\mathbf{v}_p, \zeta_p)$.

For h as above, define

$$L(h) = \lim_{\mathbf{t} \rightarrow \infty} \|\mathbf{T}\|^{-(n-2)} \int_{\mathbb{Q}_S^n} h \left(\frac{\mathbf{v}}{\mathbf{T}^\diamond}, \mathbf{q}(\mathbf{v}) \right) d\mathbf{v}.$$

Lemma 5.2. *Let f and ν be as in Theorem 5.1. Let $h_\infty(\mathbf{v}_\infty, \zeta_\infty) = J_{f_\infty}(\|\mathbf{v}_\infty\|_\infty, \zeta_\infty)\nu_\infty(\mathbf{v}_\infty/\|\mathbf{v}_\infty\|_\infty)$, $h_p(\mathbf{v}_p, \zeta_p) = J_{f_p}(\|\mathbf{v}_p\|_p^\diamond, \zeta_p)\nu_p(\mathbf{v}_p/\|\mathbf{v}_p\|_p^\diamond)$ for $p \in S_f$. Set $h(\mathbf{v}, \zeta) = \prod_{p \in S} h_p(\mathbf{v}_p, \zeta_p)$. Then we have*

$$L(h) = c(\mathbf{K}) \int_{\mathbf{G}/\Gamma} \tilde{f}(g) dg \prod_{p \in S} \int_{K_p} \nu_p(k_p^{-1}\mathbf{e}_1)dm(k_p)$$

Proof. Since

$$\int_{\mathbb{Q}_p^{n-1}} f(p^{-r}, x_2, \dots, x_{n-1}, x_n) dx_2 \dots dx_n = \int_{\mathbb{Q}_p^{n-1}} f(p^{-r}u, x_2, \dots, x_{n-1}, x_n) dx_2 \dots dx_n$$

for any unit u , we have that

$$\begin{aligned} \int_{\mathbb{Q}_p^n} f dx_1 \dots dx_n &= \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p^{n-1}} f(x_1, x_2, \dots, x_{n-1}, x_n) dx_2 \dots dx_n \cdot dx_1 \\ &= p^r(1 - p^{-1}) \sum_{r \in \mathbb{Z}} \int_{\mathbb{Q}_p^{n-1}} f(p^{-r}, x_2, \dots, x_{n-1}, x_n) dx_2 \dots dx_n. \end{aligned}$$

The change of variables $p^{-r}x_n + \mathbf{q}(0, x_2, \dots, x_{n-1}, 0) = \zeta$ yields $p^r dx_n = d\zeta$, and hence

$$\begin{aligned} \int_{\mathbb{Q}_p^n} f dx_1 \dots dx_n &= \sum_{r \in \mathbb{Z}} \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p^{n-2}} f(p^{-r}, x_2, \dots, x_{n-1}, x_n) dx_2 \dots dx_{n-1} (1 - p^{-1}) d\zeta \\ &= \sum_{r \in \mathbb{Z}} \int_{\mathbb{Q}_p} J_f(p^{-r}, \zeta) p^{r(n-2)} (1 - 1/p) d\zeta. \end{aligned}$$

The result will now follow from Proposition 3.11, Proposition 4.2, and Lemma 3.9 in [10]. Computations proceed by a verbatim repetition of the proof of Lemma 3.9 in [10]. \square

Proof of Proposition 1.2. Since $\Omega = \prod_{p \in S} \Omega_p \subset \mathbb{Q}_S^n$, it suffices to show that for any choice of intervals $I_p \subseteq \mathbb{Q}_p$, $p \in S$, there exists a constant $\lambda(\mathbf{q}_p, \Omega_p)$ such that $\text{vol}\{\mathbf{v} \in T_p \Omega_p : \mathbf{q}_p(\mathbf{v}) \in I\}$ is asymptotically $\lambda(\mathbf{q}_p, \Omega_p) T_p^{n-2}$. The case $p = \infty$ is in Lemma 3.8 (ii) of [10]. Hence, we may assume that $p \in S_f$, and write $I_p = a + p^b \mathbb{Z}_p$, for $a \in \mathbb{Q}_p$ and $b \in \mathbb{Z}$.

Recall that $\Omega_p = \{\mathbf{v} \in \mathbb{Q}_p^n : \|\mathbf{v}\|_p \leq \rho_p(\mathbf{v}/\|\mathbf{v}\|_p^\diamond)\}$, where $\rho_p(u\mathbf{v}) = \rho_p(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{U}_p^n$ and $u \in \mathbb{U}_p$. Substituting $g^{-1}\Omega_p$ by Ω_p , we may assume that \mathbf{q}_p is standard. Moreover, the function ρ'_p determining $g^{-1}\Omega_p$ satisfies $\rho'_p(\mathbf{v}) = \rho'_p(u\mathbf{v})$ for $u \in \mathbb{U}_p$. It would be slightly more convenient to work with the set $\hat{\Omega}_p = \{\mathbf{v} \in \mathbb{Q}_p^n : \|\mathbf{v}\|_p = \rho_p(\mathbf{v}/\|\mathbf{v}\|_p^\diamond)\}$.

We now apply Proposition 4.2 to find sequences $(h_m), (h'_m)$ of compactly supported continuous positive functions such that $h_m \leq \mathbf{1}_{\hat{\Omega}_p \times I_p} \leq h'_m$ and $|h_m - h'_m|$ converges uniformly to zero as $m \rightarrow \infty$. Since h_m, h'_m are compactly supported, by definition, $\lim_{m \rightarrow \infty} L_1(h_m) = \lim_{m \rightarrow \infty} L_1(h'_m)$. Hence

$$\begin{aligned} \lim_{t \rightarrow \infty} p^{-(n-2)t} \text{vol}(\{\mathbf{v} \in p^{-t}\hat{\Omega}_p : \mathbf{q}_p(\mathbf{v}) \in I_p\}) &= \lim_{t \rightarrow \infty} p^{-(n-2)t} \int_{\mathbb{Q}_p^n} \mathbf{1}_{\hat{\Omega}_p \times I_p}(p^t \mathbf{v}, \mathbf{q}_p(v)) d\mathbf{v} \\ &= \text{vol}(K \cdot \mathbf{e}_1) p^{-b} \sum_{z \in \mathbb{Z}} p^{(n-2)z} \int_K \mathbf{1}_{\hat{\Omega}_p}(p^{-z} k \mathbf{e}_1) dm(k) =: \lambda_{\mathbf{q}_p, \hat{\Omega}_p} p^{-b}. \end{aligned}$$

Note that $\sum_{z \in \mathbb{Z}} p^{(n-2)z} \int_K \mathbf{1}_{\hat{\Omega}_p}(p^{-z} k \mathbf{e}_1) dm(k)$ is a summation over a finite set of z 's. The result will now follow from the fact that $\Omega_p = \bigcup_{i \geq 0} p^i \hat{\Omega}_p$ is a disjoint union. \square

Proof of Theorem 1.5. We will start by introducing two function spaces: let \mathcal{L} denote the space of real-valued functions $F(\mathbf{v}, \zeta)$ on $\mathbb{Q}_S^n - \{0\} \times \mathbb{Q}_S$ with support in an implicitly fixed compact set C , satisfying $F(u\mathbf{v}, \zeta) = F(\mathbf{v}, \zeta)$ for all $u \in \mathbb{U}_p, p \in S_f$. We equip \mathcal{L} with the topology of uniform convergence on C . By \mathcal{L}_0 , we denote the subspace of \mathcal{L} consisting of the functions of the form $F(\mathbf{v}, \zeta) = J_f(\|\mathbf{v}\|^\diamond, \zeta) \nu(\mathbf{v}/\|\mathbf{v}\|^\diamond)$, where $f = \prod_{p \in S} f_p$, $\nu = \prod_{p \in S} \nu_p$, f_p is continuous with compact support on $\mathbb{Q}_{p,+}^n$ satisfying (\mathbf{I}_f) and ν_p is a non-negative continuous function on \mathbb{U}_p^n satisfying (\mathbf{I}_ν) . We claim that \mathcal{L}_0 is dense in \mathcal{L} . In fact, it is easy to see from a p -adic Stone-Weierstrass Theorem (see [9]) that the approximation holds if the invariance properties are dropped from \mathcal{L} and \mathcal{L}_0 . With the invariance properties, one only needs to integrate the approximating function ν over $\prod_{p \in S} \mathbb{U}_p$ with respect to the Haar measure to obtain the desired approximating function. This implies that for each $x \in \mathcal{D}$, there are h_x, h'_x in \mathcal{L}_0 such that for all $\mathbf{v} \in \mathbb{Q}_S^n$ and $\zeta \in \mathbb{Q}_S$,

$$h_x(x\mathbf{v}, \zeta) \leq \mathbf{1}_{\hat{\Omega}} \leq h'_x(x\mathbf{v}, \zeta)$$

and

$$|L(h_x) - L(h'_x)| < \epsilon,$$

where $\mathbf{1}_{\hat{\Omega}}$ is the characteristic function of

$$\hat{\Omega} = \left\{ \mathbf{v} \in \mathbb{Q}_S^n : \begin{array}{l} \rho_\infty(\mathbf{v}_\infty/\|\mathbf{v}_\infty\|_\infty)/2 < \|\mathbf{v}_\infty\|_\infty < \rho_\infty(\mathbf{v}_\infty/\|\mathbf{v}_\infty\|_\infty) \text{ and} \\ \|\mathbf{v}_p\| = \rho_p(\mathbf{v}_p/\|\mathbf{v}_p\|_p^\diamond), p \in S_f \end{array} \right\}.$$

By Proposition 5.1, Theorem 6.2, and Proposition 5.2, there exist points $x_1, \dots, x_\ell \in \mathbf{G}/\Gamma$ so that Hx_i is closed, $1 \leq i \leq \ell$, and for each compact subset \mathcal{F} of $\mathcal{D} - \bigcup_{i=1}^\ell Hx_i$, there exists $T_0 \succ 0$

so that for all $T \succ T_0$ and $x = g\Gamma \in \mathcal{F}$,

$$\begin{aligned} \left| \|T\|^{-(n-2)} \sum_{\mathbf{v} \in \mathbb{Z}_S^n} h\left(\frac{g\mathbf{v}}{T_\diamond}, \mathbf{q}(g\mathbf{v})\right) - L(h) \right| &\leq \left| \|T\|^{-(n-2)} \sum_{\mathbf{v} \in \mathbb{Z}_S^n} h\left(\frac{g\mathbf{v}}{T_\diamond}, \mathbf{q}(g\mathbf{v})\right) - c(K) \int_K \tilde{f}(a_t k x) \nu(k^{-1} \mathbf{e}_1) dm(k) \right| \\ &\quad + \left| c(K) \int_K \tilde{f}(a_t k x) \nu(k^{-1} \mathbf{e}_1) dm(k) - c(K) \int_{G/\Gamma} \tilde{f} dg \int_K \nu dm(k) \right| \\ &\quad + \left| c(K) \int_{G/\Gamma} \tilde{f} dg \int_K \nu dm(k) - \|T\|^{n-2} \int_{\mathbb{Q}_S^n} h\left(\frac{g\mathbf{v}}{T_\diamond}, \mathbf{q}(g\mathbf{v})\right) d\mathbf{v} \right| < \epsilon. \end{aligned}$$

From the definition of $L(h)$, it follows that for T large enough, we have

$$\left| \|T\|^{-(n-2)} \int_{\mathbb{Q}_S^n} h\left(\frac{g\mathbf{v}}{T_\diamond}, \mathbf{q}(g\mathbf{v})\right) d\mathbf{v} - L(h) \right| \leq \epsilon.$$

By approximating argument with h_x, h'_x with a suitable ϵ , for every $\theta > 0$, there exist finitely many points x_1, \dots, x_ℓ with Hx_i closed and for an arbitrary compact subset $\mathcal{F} \subseteq \mathcal{D} - \cup_{i=1}^\ell Hx_i$, there exists $T_0 \succ 0$ such that for every $x = g\Gamma \in \mathcal{F}$ and every $T \succ T_0$, we obtain

$$(1 - \theta) \mathbf{V}_{l, \mathbf{q}, \hat{\Omega}}(T) \leq \mathbf{N}_{l, \mathbf{q}, \hat{\Omega}}(T) \leq (1 + \theta) \mathbf{V}_{l, \mathbf{q}, \hat{\Omega}}(T).$$

The claim will now follow from Proposition 1.2 and a standard geometric series argument. \square

6. S -ARITHMETIC UNIFORM EQUIDISTRIBUTION OF UNIPOTENT FLOWS

In this section we will prove an S -arithmetic version of a theorem in [7], which was mentioned in the introduction. Our proof builds upon ideas and results that were originated in [7] and were partially extended in the S -arithmetic setup in [27]. The general line of argument is similar to the one in [7]. In a number of places, technicalities arise that need to be handled differently.

Let \mathbf{G} be a \mathbb{Q} -algebraic group. With S and S_f as above, set $\mathbf{G} = \mathbf{G}(\mathbb{Q}_S) = \prod_{p \in S} \mathbf{G}(\mathbb{Q}_p)$. Let Γ be an S -arithmetic subgroup of \mathbf{G} , i.e. a group commensurable to $\mathbf{G}(\mathbb{Z}_S)$. Note that $\mathbf{G}(\mathbb{Q}_p)$ is naturally embedded in \mathbf{G} . For each $p \in S$, let $U_p = \{u_p(z_p) : z_p \in \mathbb{Q}_p\}$ be a one-parameter unipotent \mathbb{Q}_p -subgroup, which means that $u_p : \mathbb{Q}_p \rightarrow \mathbf{G}(\mathbb{Q}_p)$ is a non-trivial \mathbb{Q}_p -rational homomorphism. [21].

We will need a uniform version of Ratner's theorem for the action of unipotent groups on G/Γ , in the special case that Γ is an S -arithmetic subgroup. Let us mention that Ratner's main result in [21] was independently obtained in Margulis-Tomanov paper [18]. The following class of groups introduced by Tomanov plays an important role in this theorem.

Definition 6.1. [27] *A connected \mathbb{Q} -algebraic subgroup \mathbf{P} of \mathbf{G} is a subgroup of class \mathcal{F} relative to S if for each proper normal \mathbb{Q} -algebraic subgroup \mathbf{Q} of \mathbf{P} , there exists $p \in S$ such that $(\mathbf{P}/\mathbf{Q})(\mathbb{Q}_p)$ contains a non-trivial unipotent element.*

If \mathbf{P} is a subgroup of class \mathcal{F} in \mathbf{G} , then for any subgroup P' of finite index in $\mathbf{P}(\mathbb{Q}_S)$, $P' \cap \Gamma$ is an S -arithmetic lattice in P' . For a closed subgroup U of G , and a proper subgroup \mathbf{P} of \mathbf{G} of class \mathcal{F} , we write $X(P, U) = \{g \in G : Ug \subseteq gP\}$. One can see that $X(P, U)$ is a \mathbb{Q}_S -algebraic subvariety of G . We will prove the following S -arithmetic version of [7, Theorem 3].

Theorem 6.2. *Let \mathbf{G} be a \mathbb{Q} -algebraic group, $\mathbf{G} = \mathbf{G}(\mathbb{Q}_S)$, Γ an S -arithmetic lattice in \mathbf{G} , and let μ denote the \mathbf{G} -invariant probability measure on G/Γ . Let $\mathbf{U} = \{(u_p(z_p))_{p \in S} | z_p \in \mathbb{Q}_p\}$ be*

a one-parameter unipotent \mathbb{Q}_S -subgroup of \mathbf{G} , and let $\phi : \mathbf{G}/\Gamma \rightarrow \mathbb{R}$ be a bounded continuous function. Let \mathcal{K} be a compact subset of \mathbf{G}/Γ , and let $\epsilon > 0$. Then there exist finitely many proper subgroups $\mathbf{P}_1, \dots, \mathbf{P}_k$ of \mathbf{G} of class \mathcal{F} , and compact subsets $C_i \subseteq X(\mathbf{P}_i, \mathbf{U})$, where $\mathbf{P}_i = \mathbf{P}_i(\mathbb{Q}_S)$, $1 \leq i \leq k$, such that the following holds: given a compact subset \mathcal{F} of $\mathcal{K} \setminus \cup_i C_i \Gamma/\Gamma$, for all $x \in \mathcal{F}$ and all \mathbf{T} with $m(\mathbf{T}) \gg 0$, we have

$$\left| \frac{1}{\lambda_S(I(\mathbf{T}))} \int_{I(\mathbf{T})} \phi(u_{\mathbf{z}}x) d\lambda_S(\mathbf{z}) - \int_{\mathbf{G}/\Gamma} \phi d\mu \right| \leq \epsilon.$$

Following [7] and [27], we denote the set of singular points with respect to U by $\mathcal{S}(U) = \bigcup_{\mathbf{P} \in \mathcal{F}, \mathbf{P} \neq \mathbf{G}} X(\mathbf{P}, U) \Gamma/\Gamma$. Its complement $\mathcal{G}(U) = G/\Gamma \setminus \mathcal{S}(U)$ will be called the set of generic points.

For a subset S' of S , the direct sum $u_{S'} : \mathbb{Q}_{S'} \rightarrow G$ of one-parameter unipotent subgroups defined by $u_{S'}((t_p)_{p \in S'}) = (u_p(t_p))_{p \in S'}$ is called a one-parameter unipotent $\mathbb{Q}_{S'}$ -subgroup of G . If the subset S' is known from the context, we sometimes write $u(t)$ instead of $(u_p(t_p))_{p \in S'}$.

When $r > 0$, we will use the notation $I_p(r)$ for the closed ball of radius r in \mathbb{Q}_p centered at zero. When p is a non-archimedean place, we will implicitly assume that r is a power of p . Any translation of $I_p(r)$, that is any set of the form $b + I_p(r)$, will be called an interval. When $p \in S_f$, the ultrametric property of the norm implies that if J_1 and J_2 are two intervals with a non-empty intersection, then $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$. In the rest of this article, for a p -adic interval $L = I_p(r)$, we will write $\hat{L} = I_p(pr)$. We will start by recalling the following S -arithmetic version of the quantitative non-divergence theorem of Dani-Margulis which will later be needed in this paper:

Theorem 6.3 ([27], Theorem 3.3). *Let \mathbf{G} be a \mathbb{Q} -algebraic group, $G = \mathbf{G}(\mathbb{Q}_S)$, and Γ an S -arithmetic lattice in G , and μ denote the G -invariant probability measure on G/Γ . Let $S' \subseteq S$, and $U = \{u(t) | t \in \mathbb{Q}_{S'}\}$ be a one-parameter unipotent $\mathbb{Q}_{S'}$ -subgroup of G . Let $\epsilon > 0$ and $\mathbf{K} \subseteq G/\Gamma$ be a compact set. Then there exists a compact subset \mathbf{K}_1 such that for any $x \in \mathbf{K}_1$ and any S' -interval $I(\mathbf{T})$ in $\mathbb{Q}_{S'}$, we have*

$$\frac{1}{\lambda_{S'}(I(\mathbf{T}))} \lambda_{S'}\{t \in I(\mathbf{T}) | u(t)x \notin \mathbf{K}_1\} < \epsilon.$$

Let Γ be an S -arithmetic lattice in G . If \mathbf{P} is a subgroup of class \mathcal{F} in \mathbf{G} then for any subgroup P' of finite index in $\mathbf{P}(\mathbb{Q}_S)$, we have $P' \cap \Gamma$ is an S -arithmetic lattice in P' . The following proposition has been proven in [27].

Proposition 6.4 ([27], Theorem 4.2). *Let $M \subseteq \mathbb{Q}_p^m$ be Zariski closed. Given a compact set $A \subseteq M$ and $\epsilon > 0$, there exists a compact set $B \subseteq M$ containing A such that the following holds: for a compact neighborhood Φ of B in \mathbb{Q}_p^m , there exists a neighborhood Ψ of A in \mathbb{Q}_p^m such that for any one-parameter unipotent subgroup $\{u(t)\}$ in $\mathrm{GL}_m(\mathbb{Q}_p)$, any $\mathbf{w} \in \mathbb{Q}_p^m - \Phi$, and any interval $I \subseteq \mathbb{Q}_p$ containing 0, we have*

$$\lambda_p\{t \in I : u(t)\mathbf{w} \in \Psi\} \leq \epsilon \cdot \lambda_p\{t \in T : u(t)\mathbf{w} \in \Phi\}.$$

We will also need the following S -adic analogue of Theorem 2 in [7].

Theorem 6.5. *Let \mathbf{G} be a k -algebraic group, $G = \mathbf{G}(\mathbb{Q}_S)$, and Γ be an S -arithmetic lattice in G . Let $U^{(i)} = \{u^{(i)}(t) | t \in \mathbb{Q}_S\}$ be a sequence of one-parameter unipotent \mathbb{Q}_S -subgroup of G , such that $u^{(i)}(t) \rightarrow u(t)$ for all $t \in \mathbb{Q}_S$, as $i \rightarrow \infty$. Assume that the sequence $(x_i)_{i \geq 1} \in G/\Gamma$ converges to*

the point $x \in \mathcal{G}(u(t))$, and let $m(\mathbf{T}_i) \rightarrow \infty$. For any bounded continuous function $\phi : G/\Gamma \rightarrow \mathbb{R}$, we have

$$\frac{1}{|\mathbf{T}_i|} \int_{I(\mathbf{T}_i)} \phi(u^{(i)}(t)x_i) d\lambda_S(t) \rightarrow \int_{G/\Gamma} \phi d\mu.$$

Let us briefly sketch the proof of this theorem. The main ingredient of the proof is the quantitative non-divergence theorem, whose S -adic analogue, Theorem 6.3, is proven in [27]. Arguing by contradiction, one assumes that there exists a sequence x_i of points for which the statement is not true. Using the density of the set of generic points, one may assume that x_i are generic for u_t . Invoking the quantitative non-divergence again, one then shows that there is no escape of mass to infinity. Once we verify that the limiting measure is $u(t)$ -invariant, the measure classification theorem of Ratner will finish the proof. For details, we refer the reader to [7].

Let \mathbf{P} be a subgroup of class \mathcal{F} in \mathbf{G} . By Chevalley's theorem, there exists a k -rational representation $\rho_{\mathbf{P}} : G \rightarrow \mathrm{GL}(V_{\mathbf{P}})$ such that $N_{\mathbf{G}}(\mathbf{P})$ equals the stabilizer of a line in V spanned by a vector $\mathbf{m} \in V(\mathbb{Q})$. This representation and the vector \mathbf{m} are fixed throughout this paper. Let χ be the k -rational character of $N_{\mathbf{G}}(\mathbf{P})$ defined by $\chi(g)\mathbf{m} = g\mathbf{m}$, for $g \in N_{\mathbf{G}}(\mathbf{P})$. We denote $\mathbf{N} = \{g \in \mathbf{G} : g\mathbf{m} = \mathbf{m}\}$ and $N = \mathbf{N}(\mathbb{Q}_S)$. We also set $\Gamma_N = \Gamma \cap N$ and $\Gamma_{\mathbf{P}} = \Gamma \cap N_{\mathbf{G}}(\mathbf{P})$. The orbit map $\eta_{\mathbf{P}} : \mathbf{G} \rightarrow \mathbf{G}\mathbf{m} \subseteq V_{\mathbf{P}}$ is defined by $\eta_{\mathbf{P}}(g) = g\mathbf{m}$. The orbit $\mathbf{G}\mathbf{m}$ is isomorphic to the quasi-affine variety \mathbf{G}/\mathbf{N} and η is a quotient map. Set $\mathbf{X} = \{g \in \mathbf{G} : Ug \subseteq g\mathbf{P}\}$ and let $A_{\mathbf{P}}$ denote the Zariski closure of $\eta_{\mathbf{P}}(X(P, U))$. Clearly \mathbf{X} is an algebraic variety of \mathbf{G} defined over k_S and $\mathbf{X}(\mathbb{Q}_S) = X(P, U)$. It is not hard to show (see [27]) that $\eta^{-1}(A_{\mathbf{P}}) = X(P, U)$. It will be useful to consider the map $\mathrm{Rep} : G/\Gamma \rightarrow V_{\mathbf{P}}$ defined as follows. For each $x \in G/\Gamma$, we define $\mathrm{Rep}(x) = \{\eta_{\mathbf{P}}(g) : g \in G, x = g\Gamma\}$. For $D \subseteq A_{\mathbf{P}}$ and for $\gamma \in \Gamma$, we define the γ -overlaps of D by $\mathfrak{D}^{\gamma}(D) = \{g\Gamma : \eta_{\mathbf{P}}(g) \in D, \eta_{\mathbf{P}}(g\gamma) \in D\} \subseteq G/\Gamma$. Finally, we set

$$\mathfrak{D}(D) = \bigcup_{\gamma \in \Gamma - \Gamma_{\mathbf{P}}} \mathfrak{D}^{\gamma}(D) \subseteq G/\Gamma.$$

The proof of the following lemma is straightforward:

Lemma 6.6. *For $\gamma \in \Gamma$ and $\gamma_1 \in \Gamma_{\mathbf{P}}$, and $D \subseteq A_{\mathbf{P}}$ we have*

- (1) $\mathfrak{D}^e(D) = \{x \in G/\Gamma : \mathrm{Rep}(x) \cap D \neq \emptyset\}$.
- (2) $\mathfrak{D}^{\gamma}(D) = \mathfrak{D}^{\gamma\gamma_1}(D)$.

For each subgroup \mathbf{P} of class \mathcal{F} relative to S , we will denote $I_{\mathbf{P}} = \{g \in \mathbf{G} : \rho_{\mathbf{P}}(g)m_{\mathbf{P}} = m_{\mathbf{P}}\}$. A proof for the following proposition can be given along the same lines as the proof of Proposition 7.1 in [7].

Proposition 6.7. *Suppose \mathbf{P} is a subgroup of class \mathcal{F} relative to S , and $C \subseteq V_{\mathbf{P}}$ is compact. Assume also that $\mathbf{K} \subseteq G/\Gamma$ is compact. Then there exists a compact set $\tilde{C} \subseteq G$ such that*

$$\pi(\tilde{C}) = \{x \in \mathbf{K} : \mathrm{Rep}(x) \cap C \neq \emptyset\}.$$

Proposition 6.8. *Let \mathbf{P} be a subgroup of class \mathcal{F} relative to S and $D \subseteq A_{\mathbf{P}}$ be compact. Let $\mathbf{K} \subseteq G/\Gamma$ be compact. Then the family $\{\mathbf{K} \cap \mathfrak{D}^{\gamma}(D)\}_{\gamma \in \Gamma}$ contains only finitely many distinct elements. Moreover, for each $\gamma \in \Gamma$, there exists a compact set $\tilde{C}_{\gamma} \subseteq \eta_{\mathbf{P}}^{-1}(D) \cap \eta_{\mathbf{P}}^{-1}(D)\gamma$ such that $\mathbf{K} \cap \mathfrak{D}^{\gamma}(D) = \pi(\tilde{C}_{\gamma})$.*

Proof. The argument for finiteness from Proposition 7.2. in [7] can be carried over verbatim to this case. \square

Let us denote by \mathcal{E} the class of subsets of G of the form $E = \bigcap_{i=1}^r \eta_{\mathbf{P}_i}^{-1}(D_i)$, where \mathbf{P}_i are subgroups of class \mathcal{F} and $D_i \subseteq A_{\mathbf{P}_i}$ are compact. For such a set E (together with the given decomposition), we denote $\mathcal{N}(E)$ to be the family of all neighborhoods of the form $\Phi = \bigcap_{i=1}^r \eta_{\mathbf{P}_i}^{-1}(\Theta_i)$, where $\Theta_i \supset D_i$ are neighborhoods in $V_{\mathbf{P}_i}$. We will refer to these neighborhoods as components of Φ . The following definition from [27] will also be used in this paper. Let $\delta > 0$ be such that for any $z \in \mathbb{Z}_S^*$, if for all $p' \in S \setminus \{p\}$, we have $|z - 1|_{p'} < \delta$, then $z = 1$. The existence of δ follows from the fact that the map from \mathbb{Z}_S^* to \mathbb{R}^S defined by $\lambda(x) = (\log |x|_p)_{p \in S}$ has a discrete image in a codimension-one subspace of \mathbb{R}^S . A subset $A = \prod_{p \in S} A_p \subseteq G$ is said to be $S(p)$ -small if for all $p' \in S \setminus \{p\}$ the following holds: if $c \in \mathbb{Q}_{p'}^*$ is such that $c(A_{p'}\mathbf{m}) \cap A_{p'}\mathbf{m} \neq \emptyset$, then $|c - 1|_{p'} < \delta$. We will now prove a stronger version of a theorem in [27].

Theorem 6.9. *Let $p \in S$, $\epsilon > 0$, and let $K \subseteq G/\Gamma$ be compact. For $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}$ such that the following holds: given $\Phi \in \mathcal{N}(E')$, there exists a neighborhood $\Omega \supseteq \pi(E)$ such that for any one-parameter unipotent subgroup $\{u_p(t)\}$ of G , and any $g \in G$, and any $r_0 > 0$, one of the following holds:*

- (1) *A component of Φ contains $\{u_p(t)g\gamma : t \in I_p(r)\}$ for some $\gamma \in \Gamma$.*
- (2) *For all $r > r_0$, we have $\lambda_p\{t \in I_p(r) \setminus I_p(r_0) : u_p(t)g\gamma \in \Omega \cap K\} \leq \epsilon \lambda_p(I_p(r) \setminus I_p(r_0))$.*

Proof. It is clear that we can assume that $E = \eta_{\mathbf{P}}^{-1}(C)$ and that E is $S(p)$ -small. We will now proceed by the induction on $\dim \mathbf{P}$. The result is clearly valid for $\dim \mathbf{P} = 0$. Let us assume that it is known for all \mathbf{P} with dimension at most $n - 1$ and that $C \subseteq A_{\mathbf{P}}$, with $\dim \mathbf{P} = n$. Applying Proposition 6.4 to C (viewed as a compact subset of the Zariski closed set $A_{\mathbf{P}}$), we obtain a compact subset D of $A_{\mathbf{P}}$ such that for any compact neighborhood Φ of D in $A_{\mathbf{P}}$, there exists a neighborhood Ψ of C in $A_{\mathbf{P}}$ such that for any one-parameter subgroup $\{u_p(t)\}$ of $\mathrm{GL}(V_{\mathbf{P}})$, any point $\mathbf{w} \in V_{\mathbf{P}} \setminus \Phi$, and any interval $I \subseteq \mathbb{Q}_p$ containing 0, we have

$$\lambda_p\{t \in I : u_p(t)\mathbf{w} \in \Psi\} \leq \epsilon \cdot \lambda_p\{t \in I : u_p(t)\mathbf{w} \in \Phi\}.$$

Note that since the set of the roots of unity in \mathbb{Q}_p is finite, we can choose D such that $\omega D = D$ for every root of unity $\omega \in \mathbb{Q}_p$, and thus D can be chosen to be $S(p)$ -small. Now, let $B = \eta_{\mathbf{P}}^{-1}(D)$.

By Proposition 6.8 the family of sets $\{K \cap \mathfrak{D}^{\gamma_j}(D)\}_{\gamma \in \Gamma}$ is finite, hence consists of the sets $K \cap \mathfrak{D}^{\gamma_j}(D)$, for $1 \leq j \leq k$. We assume that $\gamma_1 = e$. Moreover, we can write $K \cap \mathfrak{D}^{\gamma_j}(D) = \pi(C_j)$ for some compact subset $C_j \subseteq B \cap B\gamma_j^{-1} \subseteq X(\mathbf{P} \cap \gamma_j\mathbf{P}\gamma_j^{-1}, W)$. We claim that $\gamma_j \notin \Gamma_{\mathbf{P}}$ for $j \geq 2$. Assuming the contrary, we obtain $\rho(\gamma_j)\mathbf{m}_{\mathbf{P}} = \chi(\gamma_j)\mathbf{m}_{\mathbf{P}}$. Since $\chi(\gamma_j) \in \mathcal{O}^*$, we have $\eta_{\mathbf{P}}(b\gamma_j) = \chi(\gamma_j)\eta_{\mathbf{P}}(b) \in D$. Since D is $S(p)$ -small, we obtain that $\chi(\gamma_j)$ is a root of unity in \mathbb{Q}_p^* . This shows that $B\gamma_j \subseteq \eta^{-1}(D) = B$, which is a contradiction to the choice of γ_j . From here we see that for $j \geq 2$, $\mathbf{P} \cap \gamma_j\mathbf{P}\gamma_j^{-1}$ is a proper subgroup of \mathbf{P} . Hence there exists a subgroup \mathbf{P}_j of class \mathcal{F} which is contained in the connected component of $\mathbf{P} \cap \gamma_j\mathbf{P}\gamma_j^{-1}$. Note that \mathbf{P}_j is of dimension less than n , and $C_j \subseteq X(\mathbf{P}_j, W)$. We now set $E_j = \eta_{\mathbf{P}_j}^{-1}(\eta_{\mathbf{P}_j}(C_j))$ and apply the induction hypothesis to obtain $E'_j \in \mathcal{E}$ such that for any choice of $\Phi_j \in \mathcal{N}(E'_j)$, we can find neighborhoods Ω_j of E_j such that for any one-parameter subgroup $(u_p(t))_{t \in \mathbb{Q}_p}$ of G , $g \in G$ and $r > 0$, we have

$$\lambda_v\{t \in I_p(r) \setminus I_p(r_0) : u_p(t)g\gamma \in \Omega_j \cap K\} \leq (\epsilon/2k)r$$

unless there exists $\gamma \in \Gamma$ such that the set $\{u_p(t)g\gamma : t \in I_p(r)\}$ is contained in a component of Φ_j . Set $E'' = \bigcup_{j=2}^n E'_j \in \mathcal{E}$ and $E' = E'' \cap B$. Consider $\Phi \in \mathcal{N}(E')$. This shows that there exists a neighborhood Ω' of $\pi(E'')$ such that for any one-parameter unipotent subgroup $\{u_p(t)\}_{t \in \mathbb{Q}_p}$,

every $g \in G$ and $r_0 > 0$, we have

$$\lambda_p(t \in I_p(r_0) : u_p(t)g\Gamma \in \Omega' \cap \mathbf{K}) \leq \frac{\epsilon r}{2}$$

unless $\{u_p(t)g\gamma : t \in I_p(r_0)\}$ is in a component of Φ for some $\gamma \in \Gamma$. Set $\mathbf{K}_1 = \mathbf{K} \setminus \Omega'$, and choose a compact subset $K' \subseteq G$ such that $\pi(K') = \mathbf{K}_1$. Let Φ_1 be a neighborhood of D in V such that $\eta_{\mathbf{P}}^{-1}(\Phi_1) \subseteq \Phi$ and $O(\Phi_1) \cap \mathbf{K}_1 = \emptyset$. Since D is $S(p)$ -small, we can clearly choose Φ_1 to be $S(p)$ -small. As $\rho(u_p(t))$ is a one-parameter unipotent subgroup of $\mathrm{GL}(V)$, and $\eta_{\mathbf{P}}(C)$ is of relative size less than $\epsilon/4$ in D , we can find a neighborhood Ψ of C in V satisfying

$$\lambda_p(\{t \in I_p(r) : \rho(u_p(t))v \in \Psi\}) \leq \frac{\epsilon}{4} \lambda_p(\{t \in I_p(r) : \rho(u_p(t))\mathbf{v} \in \Phi_1\}),$$

for all $\mathbf{v} \in V \setminus \Phi_1$, $r > 0$ and unipotent subgroups $\{u_p(t)\}_{t \in \mathbb{Q}_p}$. Let $\Omega = \pi(\eta_{\mathbf{P}}(\Psi)) \subseteq G/\Gamma$. Assume that (1) does not hold for $g \in G$, a one-parameter subgroup $\{u_p(t)\}_{t \in \mathbb{Q}_p}$, and $r_0 > 0$. This implies that for every $\gamma \in \Gamma$, there exists $t \in I_p(r_0)$ such that $u_p(t)g\gamma \in G \setminus \Phi$. For $q \in \mathbf{M}$, we consider the following sets:

$$J_1(\mathbf{q}) = \{t \in I_p(r) \setminus I_p(r_0) : \rho(u_p(t)g)\mathbf{q} \in \Phi_1\},$$

$$J_2(\mathbf{q}) = \{t \in I_p(r) \setminus I_p(r_0) : \rho(u_p(t)g)\mathbf{q} \in \Psi, \pi(u_p(t)g) \in \mathbf{K}_1\}.$$

Since $J_1(\mathbf{q})$ is an open subset of \mathbb{Q}_p , it is a disjoint union of intervals. We will also define $J_3(\mathbf{q}) \subseteq J_1(\mathbf{q})$ as follows: if v is an archimedean place, then $J_3(\mathbf{q})$ consists of those $t \in J_1(\mathbf{q})$ such that for some $a \geq 0$, we have $[t, t+a] \subseteq J_1(\mathbf{q})$ and $\pi(u(t+a)g) \in \mathbf{K}_1$. If v is a non-archimedean place, then $J_3(\mathbf{q})$ consists of those $t \in J_1(\mathbf{q})$ such that there exists an interval $J \subseteq \mathbb{Q}_p$ containing t and $t' \in J$ such that $\pi(u(t')g) \in \mathbf{K}_1$. Clearly $J_3(\mathbf{q})$ is open in \mathbb{Q}_p , and is hence a disjoint union of intervals. We first make the following claim:

Claim: $J_3(\mathbf{q}_1) \cap J_3(\mathbf{q}_2) = \emptyset$ for $\mathbf{q}_1, \mathbf{q}_2 \in \mathbf{M}$, unless $\mathbf{q}_2 = \omega \mathbf{q}_1$ for some root of unity $\omega \in \mathbb{Q}_p^*$.

In the archimedean case, if $t \in J_3(\mathbf{q}_1) \cap J_3(\mathbf{q}_2)$, then there exists $a \geq 0$ such that $[t, t+a] \subseteq J_1(\mathbf{q}_1) \cap J_1(\mathbf{q}_2)$ and $\pi(u(t+a)g) \in \mathbf{K}_1$. If $q_j = \eta(\gamma_j)$ for $j = 1, 2$, then $\eta(u(t+a)g\gamma_1) = \eta(u(t+a)g\gamma_2) \in \Phi_1$, we will have $\mathbf{q}_1, \mathbf{q}_2 \in \mathfrak{D}(\Phi) \cap \mathbf{K}_1 = \emptyset$, unless $\gamma_1^{-1}\gamma_2 \in \Gamma_{\mathbf{P}}$, which implies that $\mathbf{q}_2 = \omega \mathbf{q}_1$.

In the non-archimedean case, if $t \in J_3(\mathbf{q}_1) \cap J_3(\mathbf{q}_2)$, then $t \in J_1(\mathbf{q})$ and there exist intervals $J(\mathbf{q}_1), J(\mathbf{q}_2) \subseteq \mathbb{Q}_p$ containing t and $t'_1 \in J(\mathbf{q}_1)$ and $t'_2 \in J(\mathbf{q}_2)$ such that $\pi(u(t'_1)g), \pi(u(t'_2)g) \in \mathbf{K}_1$. Note that since $J(\mathbf{q}_1)$ and $J(\mathbf{q}_2)$ intersect one contains the other, hence, without loss of generality, we can assume that $t'_1 \in J(\mathbf{q}_1) \cap J(\mathbf{q}_2)$, and $\pi(u(t'_1)g) \in \mathbf{K}_1$. The rest of the argument is as in the archimedean case.

Denote $I_p(a, r) = a + I_p(r)$, and let \mathcal{L}_1 be the family of those components $L = I_p(a, r_1)$ of $J_1(\mathbf{q})$ such that $L \cap I_p(r_0) = \emptyset$, and \mathcal{L}_2 the rest of components. Note that $\hat{L} \not\subseteq I_p(r_0)$. This implies that $\lambda_p(\hat{L} \cap J_2(\mathbf{q})) \leq \lambda_p(\hat{L} \cap J_3(\mathbf{q}))$, which, in turn, shows that

$$\sum_{L \in \mathcal{L}_1} \lambda_p(\hat{L} \cap J_2(\mathbf{q})) \leq \sum_{L \in \mathcal{L}_1} \lambda_p(\hat{L} \cap J_3(\mathbf{q})) \leq \lambda_p(I_p(r) \setminus I_p(r_0)).$$

We now claim that $\sum_{L \in \mathcal{L}_2} \lambda_p(L) \leq \lambda_p(I_p(r))$. In fact, if $L \in \mathcal{L}_2$, then either $L \subseteq I_p(r_0)$ or $I_p(r_0) \subseteq L$. If $I_p(r_0) \subseteq L$ for some $L \in \mathcal{L}_2$, then since components are disjoint, \mathcal{L}_2 has precisely one element and the result follows. So, assume that for each $L \in \mathcal{L}_2$, we have $L \subseteq I_p(r_0)$. Then the disjointness of components implies that $\sum_{L \in \mathcal{L}_2} \lambda_p(L) \leq \lambda_p(I_p(r_0))$. \square

Proposition 6.10. *Let \mathbf{G} be a \mathbb{Q} -algebraic group, $G = \mathbf{G}(\mathbb{Q}_S)$, and Γ be an S -arithmetic lattice in G . Let U be a one-parameter unipotent subgroup of G . Assume that $\mathbf{P}_1, \dots, \mathbf{P}_k$ are subgroups*

of class \mathcal{F} for $1 \leq i \leq k$, and let D_i be a compact subset of $A_{\mathbf{P}_i}$, and Θ_i be a compact neighborhood of D_i in $V_{\mathbf{P}_i}$. For a given compact set $\mathbf{K} \subseteq G/\Gamma$, there exists $\mathbf{P}'_1, \dots, \mathbf{P}'_k$ of class \mathcal{F} and compact subsets D'_i of $A_{\mathbf{P}'_i}$, $1 \leq i \leq k$, such that for any compact set $\mathcal{F} \subseteq \mathbf{K} \setminus \bigcup_{i=1}^k (\eta_{\mathbf{P}'_i}^{-1}(D_i) \cup \eta_{\mathbf{P}'_i}^{-1}(D'_i))\Gamma/\Gamma$, there exists T_0 such that for any $g \in G$ with $g\Gamma \in \mathcal{F}$, and $1 \leq i \leq k$, there exists $t \in I_p(T_0)$ such that $u_p(t)g \notin \eta_{\mathbf{P}_i}^{-1}(\Theta_i)$.

Proof. The proof of this proposition is very similar to the proof of Proposition 8.1 in [7]. Let us denote by $I_{\mathbf{P}}$ the set of $g \in G$ with $g \cdot \mathbf{m}_{\mathbf{P}} = \mathbf{m}_{\mathbf{P}}$. We first claim that there exists a subgroup \mathbf{P}' of class \mathcal{F} such that $X(I_{\mathbf{P}}, U) \subseteq X(\mathbf{P}', U)$. In fact, let \mathbf{P}' be the smallest connected algebraic subgroup of \mathbf{G} which contains all the unipotent elements of $I_{\mathbf{P}}$. Note that since \mathbf{P}' is generated by unipotent subgroups, we have $X_k(\mathbf{G}) = \{1\}$, where $X_k(\mathbf{G})$ denotes the group of characters of G defined over k . It follows from Theorem 12.3 of [4] that $\mathbf{P}' \cap \Gamma$ is a lattice in \mathbf{P}' and \mathbf{P}' is of class \mathcal{F} . Let $\mathbf{P}'_1, \dots, \mathbf{P}'_k$ be chosen as above such that $X(I_{\mathbf{P}_i}, U) \subseteq X(\mathbf{P}'_i, U)$ for all $1 \leq i \leq k$. We will also define $Q_i = \{w \in \Theta_i : \rho_{\mathbf{P}_i}(u_p(t))w = w, \forall t \in \mathbb{Q}_p\}$. Using Proposition 6.7, we can find compact subsets $C_i \subseteq \eta_{\mathbf{P}_i}^{-1}(Q_i)$, $1 \leq i \leq k$ such that

$$\pi(\widetilde{C}_i) = \{x \in Q_i : \text{Rep}(x) \cap C_i \neq \emptyset\}.$$

This implies that $C_i \subseteq X(\mathbf{P}_i, U)$. Consider the compact sets $D'_i = \eta_{\mathbf{P}'_i}(C_i) \subseteq A_{\mathbf{P}'_i}$ and assume that \mathcal{F} is a compact subset of $\mathbf{K} \setminus \pi(\bigcup_{i=1}^k (\eta_{\mathbf{P}_i}^{-1}(D_i) \cup \eta_{\mathbf{P}'_i}^{-1}(D'_i)))$. Fix a compact subset $F' \subseteq G$ such that $\mathcal{F} = \pi(F')$. From the fact that $\rho_{\mathbf{P}_i}(\Gamma)\mathbf{m}_{\mathbf{P}_i}$ is a discrete subset of $V_{\mathbf{P}_i}$, it follows that there are only finitely many $\gamma \in \Gamma$ such that $\eta_{\mathbf{P}_i}(\gamma) \in \rho_{\mathbf{P}_i}(F')^{-1}\Theta_i$ for some $1 \leq i \leq k$. It thus suffices to show that for each $1 \leq i \leq k$ and $\gamma \in \Gamma$, for all large enough t we have $\Theta_i \cap \rho_{\mathbf{P}_i}(u_p(t))(\Theta_i \cap \rho_{\mathbf{P}_i}(F'\gamma)\mathbf{m}_{\mathbf{P}_i}) = \emptyset$. Note that $\rho_{\mathbf{P}_i}(F'\gamma)\mathbf{m}_{\mathbf{P}_i} \cap \Theta_i$ is a compact subset of $V_{\mathbf{P}_i}$ which does not contain any fixed point of the flow $\rho_{\mathbf{P}_i}(u_p(t))$. Since $\rho_{\mathbf{P}_i}(u_p(t))$ is a unipotent one-parameter subgroup of $\text{GL}(V_{\mathbf{P}_i})$ the result follows. \square

Proof of Theorem 6.2. For a bounded continuous function ϕ defined on G/Γ , the one-parameter unipotent group $(u(t))$ and $x \in G/\Gamma$, we define

$$\Delta(\phi, u(t), x, \mathbf{T}) = \left| \frac{1}{\lambda_S(I(\mathbf{T}))} \int_{I(\mathbf{T})} \phi(u(t)x) d\lambda_S(t) - \int_{G/\Gamma} \phi d\mu \right|.$$

Let us consider the above statement with S replaced by $S' \subseteq S$ everywhere. We will prove the statement first for the case $|S'| = 1$. Then we will show that if the statement holds for S_1 and S_2 , then it must also hold for $S_1 \cup S_2$. Suppose that $S' = \{p\}$ for some $p \in S$. We argue by contradiction. Assume that the statement of the theorem is not true. This implies the existence of a bounded continuous function $\phi : G/\Gamma \rightarrow \mathbb{R}$, a compact subset $\mathbf{K}_1 \subseteq G/\Gamma$, and $\epsilon > 0$ such that for any proper subgroups $\mathbf{P}_1, \dots, \mathbf{P}_k$ of class \mathcal{F} , and any compact subsets $C_i \subseteq X(\mathbf{P}_i, U)$, where $\mathbf{P}_i = \mathbf{P}(\mathbb{Q}_p)$, $1 \leq i \leq k$, there exists a compact set \mathcal{F} of $\mathbf{K}_1 \setminus \bigcup_i C_i \Gamma/\Gamma$ such that for all $T_0 > 0$, there exists $T > T_0$, and $x \in \mathcal{F}$ such that $\Delta(\phi, u(t), x, T) > \epsilon$. Without loss of generality, we can assume that ϕ has compact support and is bounded in absolute value by 1. There exists a compact subset $\mathbf{K} \subseteq G/\Gamma$ such that for all $x \in \mathbf{K}_1$ and all T , we have

$$(17) \quad \lambda_p\{t \in I_p(T) : u_p(t)x \notin \mathbf{K}\} < \frac{1}{3} \lambda_p(I_p(T)).$$

We can now apply this to construct an increasing sequence $E_i \subseteq E_{i+1}$ in \mathcal{E} such that

- (1) The family $\{E_i\}_{i \geq 1}$ exhausts the singular set of U , i.e., $\bigcup_{i=1}^{\infty} E_i = \mathcal{S}(U)$.

- (2) For each $i \geq 1$, there exists an open neighborhood $\Omega_i \supset E_i\Gamma/\Gamma$ such that for any compact set $F \subseteq \mathbf{K} \setminus E_{i+1}\Gamma/\Gamma$, there exists T_{i+1} such that for all $x \in F$ and $T > T'_i$ we have

$$(18) \quad \lambda_p\{t \in I_p(T) : u_p(t)x \in \Omega_i \cap \mathbf{K}\} \leq \frac{1}{4^i} \lambda_p(I_p(T)).$$

For $i \geq 1$, write $\mathbf{K} \cap \pi(E_i) = \bigcup \pi(C_j)$ for some compact sets $C_j \in X(P_j, U)$, $1 \leq j \leq k$. As we are arguing by contradiction, we can find a compact subset $\mathcal{F}_i \subseteq \mathbf{K}_1 \setminus \pi(E_{i+1})$ such that for each T_0 there exists $x \in \mathcal{F}_i$ and $T \geq T_0$ such that $\Delta(\phi, u_p(t), x, T) > \epsilon$. Without loss of generality, assume that $T_1 < T_2 < \dots$. This implies that there exists $x_i \in F_i$ and σ_i such that $\Delta(\phi, u_p(t), x_i, \sigma_i) > \epsilon$. From (17) and (18), for each $j \geq 1$ we obtain $t_j \in I(T_j)$ such that $u_p(t_j)x_j \in \mathbf{K} \setminus \bigcup_{i=1}^j \Omega_i$. This implies that

$$\Delta(\phi, u_p(t), \sigma_j, y_j) \geq \epsilon - 2\frac{T_j}{\sigma_j} \geq \frac{\epsilon}{3}.$$

As $y_j \in \mathbf{K}$ and \mathbf{K} is compact, there exists a limit point $y \in \mathbf{K}$. By construction, $y \notin \Omega_j$ for all $j \geq 1$. This shows that y is not a singular point for U . Now, we can apply Theorem 6.5 to a convergent subsequence of $\{y_j\}$ and the corresponding subsequence of σ_i , to obtain a contradiction. Let us now turn to the general case. Assume that the statement is known for $S_1, S_2 \subseteq S$, and $S_1 \cap S_2 = \emptyset$. We write $U_1 = (u_p(t_p))_{p \in S_1}$ and $U_2 = (u_p(t_p))_{p \in S_2}$. Note that there exists a compact subset $\mathbf{K}_1 \subseteq G/\Gamma$ such that for all $x \in \mathbf{K}$ and any interval $I(\mathbf{T}_j) \subseteq \mathbb{Q}_{S_j}$, $j = 1, 2$, containing zero we have

$$\frac{1}{\lambda_j(I(\mathbf{T}_j))} \lambda_j\{t \in I(\mathbf{T}_j) : u_j(t) \in \mathbf{K}_1\} \geq 1 - \epsilon/4.$$

Here, we have used the shorthands $u_1(t) = (u_p(t_p))_{p \in S_1}$ and $d\lambda_1$ for the Haar measure on $\prod_{p \in S_1} \mathbb{Q}_p$. By the induction hypothesis, there exist finitely many proper subgroups $\mathbf{P}_1, \dots, \mathbf{P}_k$ of class \mathcal{F} , and compact subsets $C_i \subseteq X(P_i, U)$, where $P_i = \mathbf{P}_i(\mathbb{Q}_S)$, $1 \leq i \leq k$, such that the following holds: for any compact subset \mathcal{F} of $\mathbf{K}_1 \setminus \bigcup_i C_i\Gamma/\Gamma$ there exists T_0 such that for all $x \in \mathcal{F}$ and S_1 -vector \mathbf{T}_1 with $m(\mathbf{T}_1) > T_0$, we have

$$\left| \frac{1}{\lambda_1(I(\mathbf{T}_1))} \int_{I(\mathbf{T}_1)} \phi(u_1(t)x) d\lambda_1(t) - \int_{G/\Gamma} \phi d\mu \right| \leq \frac{\epsilon}{4}.$$

Since $C_i\Gamma/\Gamma \subseteq G/\Gamma$ has measure zero, we can choose neighborhoods N_i of $C_i\Gamma/\Gamma$, each of measure at most $\epsilon/16k$. Now, let ϕ_i , $1 \leq i \leq k$ be a continuous function such whose restriction to N_i is 1, and $\int_{G/\Gamma} \phi_i < \epsilon/8k$. By applying the induction hypothesis to ϕ_1, \dots, ϕ_k , we can find finitely many proper subgroups $\mathbf{Q}_1, \dots, \mathbf{Q}_l$ of class \mathcal{F} , and compact subsets $D_i \subseteq X(Q_i, U)$, where $Q_i = \mathbf{Q}_i(\mathbb{Q}_{S_2})$, $1 \leq i \leq l$, such that the following holds: for any compact subset \mathcal{F} of $\mathbf{K} \setminus \bigcup_i D_i\Gamma/\Gamma$, for all $x \in \mathcal{F}$ and $m(\mathbf{T}_2)$ sufficiently large, we have

$$\left| \frac{1}{\lambda_2(I(\mathbf{T}_2))} \int_{I(\mathbf{T}_2)} \phi_j(u_2(t)x) d\lambda_2(t) - \int_{G/\Gamma} \phi_j d\mu \right| \leq \frac{\epsilon}{8k}, \quad 1 \leq i \leq k.$$

Since $\phi_i(x) = 1$ for all $x \in N_i$, we obtain

$$(19) \quad \frac{1}{\lambda_2(I(\mathbf{T}_2))} \lambda_2 \left\{ t_2 \in I(\mathbf{T}_2) : u_2(t_2)x \in \bigcup_{i=1}^k N_i \right\} \leq \frac{\epsilon}{4}.$$

Let $A = \{t_2 \in I(\mathbb{T}_2) : u_2(t_2)x \in K_1\}$. Note that by the choice of K_1 , we have $\lambda_2(A) \geq (1 - \epsilon/4)\lambda_2(I(\mathbb{T}_2))$. Combining this with (19), we obtain

$$\frac{1}{\lambda_2(I(\mathbb{T}_2))} \lambda_2 \left\{ t_2 \in I(\mathbb{T}_2) : u_2(t_2)x \in K_1 \setminus \bigcup_{i=1}^k N_i \right\} \geq 1 - \frac{\epsilon}{2}.$$

Since $K_1 \setminus \bigcup_{i=1}^k N_i$ is a compact subset of K_1 , disjoint from $\bigcup_{i=1}^k C_i \Gamma / \Gamma$, we have there exists T_2 such that for all $x \in K_1 \setminus \bigcup_{i=1}^k N_i$ and $m(\mathbb{T}_1) > T_2$, we have

$$\left| \frac{1}{\lambda_1(I(\mathbb{T}_1))} \int_{I(\mathbb{T}_1)} \phi(u_1(t)x) d\lambda_1(t) - \int_{G/\Gamma} \phi d\mu \right| \leq \frac{\epsilon}{4}.$$

Decomposing $u(t) = u_1(t_1)u_2(t_2)$, and using Fubini's theorem, we obtain

$$\left| \frac{1}{\lambda(I(\mathbb{T}))} \int_{I(\mathbb{T})} \phi(u(t)x) d\lambda(t) - \int_{G/\Gamma} \phi \right| \leq \epsilon.$$

It follows that the collection of $X(\mathbf{P}_i, U)$, $1 \leq i \leq k$ and $X(\mathbf{Q}_j, U)$, $1 \leq j \leq l$ will satisfy the conditions of the theorem. \square

7. EQUIDISTRIBUTION OF TRANSLATED ORBITS

Let G , H , and K be as before. In this section, we prove the equidistribution of the translated orbit $a_t K g$ in G/Γ , where $g \in G$ is fixed (or varies over a compact set), and t tends to infinity. The main result will be proven in two steps. In the first step—dealt with in this section—we assume that the test function in Theorem 1.8 is bounded. The main argument, which is purely dynamical, is based on approximating the integral above, with a similar one involving orbits shifted by unipotent elements. The advantage is that the latter can be handled using an S -arithmetic version of Dani-Margulis theorem, which is generally false for unbounded functions. We will eventually need to prove Theorem 1.8 for test functions with a moderate blowup at the cusp. This step requires an analysis of certain integrals involving the α -function, which is carried out in a later section.

Theorem 7.1. *Let $n \geq 3$ be a positive integer, and let G , H , K and a_t be as above. Let $\phi : G/\Gamma \rightarrow \mathbb{R}$ be a bounded continuous function, and q_p be a positive continuous function on \mathbb{U}_p^n , and $q = \prod_{p \in S} q_p$. Let $x_0 \in G/\Gamma$ be such that the orbit Hx_0 is not closed. Then*

$$\lim_{t \rightarrow \infty} \int_K \phi(a_t k x_0) q(k) dm(k) = \int_{G/\Gamma} \phi(y) d\mu(y) \int_K q(k) dm(k).$$

Here $d\mu$ denotes the G -invariant probability measure on G/Γ .

The proof of this theorem closely follows [7]. The integral of ϕ along the a_t -translates of the K -orbits is approximated by a similar integral with a_t replaced by the unipotent flow to which a generalized theorem of Dani and Margulis will be applied. One difference between the situation here and the one in [10] is that as $\{a_t\}$ is a \mathbb{Z} -flow, the orbit needs to be “fattened up” in order to obtain rough approximations of the orbit of a unipotent flow, which is similar to intervals in \mathbb{Q}_p .

Fix a prime $p \in S_f$. For simplicity, let us suppress the dependence of matrices introduced below on p . For $\alpha \in \mathbb{U}_p = \{\alpha : |\alpha| = 1\}$ and $t \in \mathbb{Z}$, we set

$$d_t = \begin{pmatrix} p^{-t} & 0 \\ 0 & p^t \end{pmatrix}, \quad u_{t,\alpha} = \begin{pmatrix} 1 & \alpha p^{-t} \\ 0 & 1 \end{pmatrix}, \quad k_{t,\alpha} = \begin{pmatrix} 1 & \alpha p^t \\ -\alpha^{-1} p^t & 1 - p^{2t} \end{pmatrix},$$

$$b_{t,\alpha} = \begin{pmatrix} 1 & 0 \\ \alpha^{-1}(p^{3t} - p^t) & 1 \end{pmatrix}, \quad w_\alpha = \begin{pmatrix} 0 & -\alpha \\ \alpha^{-1} & 0 \end{pmatrix}.$$

Note that $d_t = b_{t,\alpha} u_{t,\alpha} k_{t,\alpha} w_\alpha$, $\{k_{t,\alpha} : t \in \mathbb{N}, \alpha \in \mathbb{U}_p\}$ and $\{w_\alpha : \alpha \in \mathbb{U}_p\}$ embeds into K_p (with the embedding of $\mathrm{SL}_2(\mathbb{Q}_p)$ into $\mathrm{SO}(\mathbf{q})$). This is parallel to a similar formula in the real case. The only different is that w is constant in the real case, whereas here it depends on α .

Proof of Theorem 7.1. Without loss of generality, we may assume that ϕ and q are uniformly continuous. We first prove an analogous statement for $(u_z)_{z \in \mathbb{Q}_S}$ instead of $(a_t)_{t \in \mathbb{T}_S}$. We will then deduce the statement for the latter from a similar statement for the former by an approximation process. First, we will show that under the assumptions of Theorem 1.8, we have

Claim 1: There exist $x_1, \dots, x_\ell \in \mathbf{G}/\Gamma$ such that the orbits $\mathbf{H}x_i$ are closed and each carries a finite \mathbf{H} -invariant measure, such that for each compact set $\mathcal{F} \subseteq \mathcal{D} - \bigcup_{i=1}^\ell \mathbf{H}x_i$, for any \mathbf{T} with $m(\mathbf{T})$ sufficiently large, and all $x \in \mathcal{F}$, we have

$$m \left\{ \mathbf{k} \in \mathbf{K} : \left| \frac{1}{\lambda_S(\mathbf{l}(\mathbf{T}))} \int_{\mathbf{l}(\mathbf{T})} \phi(u_z \mathbf{k}x) \, dz - \int_{\mathbf{G}/\Gamma} \phi \, d\mu \right| > \epsilon \right\} < \epsilon.$$

Indeed, let $\mathbf{P}_1, \dots, \mathbf{P}_k$ be the subgroups provided by Theorem 6.2 for the data $(\phi, \mathbf{K}\mathcal{D}, \epsilon)$, and let $C_i \subseteq X(\mathbf{P}_i, \mathbf{U})$ be the corresponding compact sets. Define

$$Y_i = \{\mathbf{g} \in \mathbf{G} : \mathbf{K}\mathbf{g} \subseteq X(\mathbf{P}_i, \mathbf{U})\}.$$

Let M be the subgroup of \mathbf{G} generated by $\mathbf{k}^{-1}\mathbf{U}\mathbf{k}$, where $\mathbf{k} \in \mathbf{K}$. The normalizer $L = N_{\mathbf{G}}(M)$ contains \mathbf{K} , hence is of the form $L = E\mathbf{K}$ where E is a normal subgroup of \mathbf{H} . Since $\mathbf{U} \subseteq L$ and U is not included in any proper normal subgroup of \mathbf{H} , we obtain $M = \mathbf{H}$.

Note that for any $y \in Y_i$, we have $\mathbf{k}^{-1}\mathbf{U}\mathbf{k} \subseteq y\mathbf{P}_i y^{-1}$, for all $\mathbf{k} \in \mathbf{K}$. So $\mathbf{H} \subseteq y\mathbf{P}_i y^{-1}$. It follows that there exists a subgroup of finite index \mathbf{P}'_i in \mathbf{P}_i such that $\mathbf{H} = y\mathbf{P}'_i y^{-1}$ for all $y \in Y_i$. This implies that $y_1^{-1}y_2 \in N_{\mathbf{G}}(\mathbf{H})$ for all $y_1, y_2 \in Y_i$. Hence Y_i can be covered with a finite number of right cosets of \mathbf{H} . Therefore,

$$\bigcup_{1 \leq i \leq k} Y_i \Gamma / \Gamma \subseteq \bigcup_{1 \leq j \leq \ell} \mathbf{H}x_j.$$

Note that $\mathbf{P}_i \cap \Gamma$ is a lattice in \mathbf{P}_i . Since $X(\mathbf{P}_i, \mathbf{U})$ is an analytic subset of \mathbf{G} , we have

$$m(\{\mathbf{k} \in \mathbf{K} : \mathbf{k}\mathbf{g} \in X(\mathbf{P}_i, \mathbf{U})\gamma\}) = 0$$

for each $\mathbf{g} \in \mathbf{G} - Y_i\gamma$. Therefore

$$m \left\{ \mathbf{k} \in \mathbf{K} : \mathbf{k}\mathbf{g} \in \bigcup_{1 \leq i \leq k} X(\mathbf{P}_i, \mathbf{U})\Gamma \right\} = 0$$

holds for any $\mathbf{g} \in \mathbf{G} - \bigcup_{1 \leq i \leq k} Y_i\Gamma$. Since $C_i \subseteq X(\mathbf{P}_i, \mathbf{U})$ and $\mathcal{F} \subseteq \mathcal{D} - \bigcup_{j=1}^\ell \mathbf{H}x_j$, we have

$$m \left\{ \mathbf{k} \in \mathbf{K} : \mathbf{k}x \in \bigcup_{i=1}^k C_i \Gamma / \Gamma \right\} = 0,$$

for every $x \in \mathcal{F}$. From here it easily follows (using an analogue of Lemma 4.2 in [10]) that there exists an open subset $W \subseteq \mathbf{G}/\Gamma$ containing $\bigcup_{i=1}^k C_i \Gamma/\Gamma$ such that

$$m(\{\mathbf{k} \in \mathbf{K} : kx \in W\}) < \epsilon$$

for any $x \in \mathcal{F}$. From the choice of C_1, \dots, C_k , we have that for every $y \in \mathbf{KD} - W$ and all \mathbf{T} with $m(\mathbf{T}) \gg 0$, we have

$$(20) \quad \left| \frac{1}{\lambda_S(\mathbf{l}(\mathbf{T}))} \int_{\mathbf{l}(\mathbf{T})} \phi(u_{\mathbf{z}}x) d\lambda_S(\mathbf{z}) - \int_{\mathbf{G}/\Gamma} \phi d\mu \right| \leq \epsilon.$$

Let q be a bounded continuous function on \mathbf{K} . It follows that

$$(21) \quad \left| \frac{1}{\lambda_S(\mathbf{l}(\mathbf{T}))} \int_{\mathbf{K}} \int_{\mathbf{l}(\mathbf{T})} \phi(u_{\mathbf{z}}x) q(\mathbf{k}) d\lambda_S(\mathbf{z}) - \int_{\mathbf{G}/\Gamma} \phi d\mu \int_{\mathbf{K}} q(\mathbf{k}) dm(\mathbf{k}) \right| \leq \sup_{\mathbf{k} \in \mathbf{K}} |q(\mathbf{k})| (1 + 2 \sup_{y \in \mathbf{G}/\Gamma} |\phi(y)|) \epsilon.$$

We will now turn to the approximation part. Fix $p \in S_f$. In what follows $d\alpha$ is the Haar measure on \mathbb{Z}_p normalized such that $\int_{\mathbb{Z}_p} d\alpha = 1$. First note that

$$(22) \quad \begin{aligned} \int_{K_p} \phi(d_t kx) q(k) dm(k) &= \int_{\mathbb{Z}_p} \int_{K_p} \phi(b_{t,\alpha} u_{t,\alpha} k_{t,\alpha} w_{\alpha} kx) q(k) dm(k) d\alpha \\ &= \int_{\mathbb{Z}_p} \int_{K_p} \phi(b_{t,\alpha} u_{t,\alpha} k_{t,\alpha} w_{\alpha} kx) q(k) dm(k_{t,\alpha} w_{\alpha} k) d\alpha \\ &= \int_{\mathbb{Z}_p} \int_{K_p} \phi(b_{t,\alpha} u_{t,\alpha} kx) q(w_{\alpha}^{-1} k_{t,\alpha} k) dm(k) d\alpha. \end{aligned}$$

We can now write

$$(23) \quad \begin{aligned} &\left| \int_{K_p} \phi(d_t kx) q(k) dm(k) - \int_{\mathbb{Z}_p} \int_{K_p} \phi(u_{t,\alpha} kx) q(w_{\alpha}^{-1} k) dm(k) d\alpha \right| \\ &= \left| \int_{\mathbb{Z}_p} \int_{K_p} \phi(b_{t,\alpha} u_{t,\alpha} kx) q(w_{\alpha}^{-1} k_{t,\alpha} k) - \phi(u_{t,\alpha} kx) q(w_{\alpha}^{-1} k) dm(k) d\alpha \right|. \end{aligned}$$

Note that as $t \rightarrow \infty$, $b_{t,\alpha}$ and $k_{t,\alpha}$ converge to 1 uniformly in α . Hence, for large values of t , we have

$$\left| \int_{K_p} \phi(d_t kx) q(k) dm(k) - \int_{\mathbb{Z}_p} \int_{K_p} \phi(u_{t,\alpha} kx) q(w_{\alpha}^{-1} k) dm(k) d\alpha \right| < \frac{\epsilon}{2}.$$

Since q is continuous, we can partition \mathbb{Z}_p into intervals I_1, \dots, I_r such that for each $1 \leq i \leq r$, there exists $w_i \in K_p$ such that $|q(w_{\alpha}^{-1} k) - q(w_i k)| < \epsilon/4$ for every $\alpha \in I_i$. This implies that

$$\left| \int_{I_i} \int_{K_p} \phi(u_{t,\alpha} kx) q(w_{\alpha}^{-1} k) - \phi(u_{t,\alpha} kx) q(w_i k) dm(k) d\alpha \right| < \lambda_p(I_i) \epsilon/4.$$

It is clear that $I_{i,t} = \{p^{-t}\alpha : \alpha \in I_i\}$ has a measure comparable to p^t . So, for sufficiently large t , we have

$$\int_{I_i} \int_{K_p} \phi(u_{t,\alpha} kx) q(w_i k) dm(k) d\alpha = \int_{I_{i,t}} \int_{K_p} \phi(u_t kx) q(w_i k) d\theta(t).$$

We can now apply (20) to obtain the result. □

8. ESTABLISHING UPPER BOUNDS FOR CERTAIN INTEGRALS

In this section, we will prove Theorem 1.7. In the course of the proof we will need to consider a number of integrals that are intimately connected to the integral featured in Theorem 1.7. In the rest of this section, we fix $p \in S_f$ and denote by a_t the linear transformation a_t^p defined in equation (2): $a_t \mathbf{e}_1 = p^t \mathbf{e}_1$, $a_t \mathbf{e}_n = p^{-t} \mathbf{e}_n$, and $a_t \mathbf{e}_j = \mathbf{e}_j$, $2 \leq j \leq n-1$.

Lemma 8.1. *Let V be a finite dimensional vector space over \mathbb{Q}_p and let K be a compact subgroup of $\mathrm{GL}(V)$. Let $v \in V$ and W be a proper subspace of V such that for any finite index subgroup K' of K and any non-zero subspace $W' \subset W$, $K'W' \not\subseteq W'$. Then the subset*

$$(24) \quad \mathrm{tran}(v, W) := \{k \in K : kv \in W\} \subseteq K$$

has Haar measure zero.

Proof. Denote the Haar measure on K by m . Since K is compact, $m(K) < \infty$. Suppose that $m(\mathrm{tran}(v, W)) > 0$. Take $W' \subseteq W$ of the smallest dimension such that $\mathrm{tran}(v, W')$ has a positive measure. For each $k \in K$, set $kW' = \{kw : w \in W'\}$. For each $k' \in K$, we have $\mathrm{tran}(v, k'W') = k'\mathrm{tran}(v, W')$ and hence

$$m(\mathrm{tran}(v, k'W')) = m(\mathrm{tran}(v, W'))$$

for all $k' \in K$. If $k'W' \neq k''W'$, since $W' \cap k'^{-1}k''W'$ has dimension strictly lower than that of W' , by minimality, the intersection

$$\begin{aligned} \mathrm{tran}(v, k'W') \cap \mathrm{tran}(v, k''W') &= k'\mathrm{tran}(v, W') \cap k'\mathrm{tran}(v, k'^{-1}k''W') \\ &\subseteq k'\mathrm{tran}(v, W' \cap k'^{-1}k''W') \end{aligned}$$

has measure zero.

Thus $\{kW' : k \in K\}$ is a finite set and $K' := \{k : kW' = W'\}$ is of finite index in K . Then $K'W' = W' \subset W'$, contradicting to the condition that $K'W' \not\subseteq W'$. \square

Lemma 8.2. *Let ρ be an analytic representation of $\mathrm{SL}_n(\mathbb{Q}_p)$ on a finite dimensional normed space $(V, \|\cdot\|)$ over \mathbb{Q}_p such that all elements of $\rho(\mathrm{SL}_n(\mathbb{Z}_p))$ preserve the norm on V . For $g \in \mathrm{SL}_n(\mathbb{Q}_p)$ and $v \in V$, we will denote $\rho(g)(v)$ by gv . Let K be a closed analytic subgroup of $\mathrm{SL}_n(\mathbb{Z}_p)$. Assume that V has a decomposition*

$$(25) \quad V = W^- \oplus W^0 \oplus W^+,$$

where

$$\begin{aligned} W^0 &= \{v \in V : a_tv = v, t \in \mathbb{Z}\} \\ W^\pm &= \{v \in V : a_tv = p^{\mp t}v, t \in \mathbb{Z}\}. \end{aligned}$$

Let Q be a closed subset of $\{v \in V : \|v\|_p = 1\}$. Suppose that the following conditions are satisfied:

- (1) *The subspace $W^- + W^0$ satisfies the assumption of Lemma 8.1;*
- (2) *There is a positive integer ℓ such that for any nonzero $v \in W^-$,*

$$(26) \quad \mathrm{codim} \{x \in \mathrm{Lie}(K) : xv \in W^-\} \geq \ell.$$

Then for any s , $0 < s < \ell$,

$$\lim_{t \rightarrow +\infty} \sup_{v \in Q} \int_K \frac{dm(k)}{\|a_t kv\|^s} = 0,$$

where m is the normalized Haar measure on K .

Proof. Let $\pi : V \rightarrow W^+ \oplus W^0$ and $\pi^+ : V \rightarrow W^+$ denote the natural projections associated to the decomposition (25). For any $v \in V$ and $r \geq 0$, define

$$\begin{aligned} D(v, r) &= \{k \in K : \|\pi(kv)\| \leq r\} \\ D^+(v, r) &= \{k \in K : \|\pi^+(kv)\| \leq r\}. \end{aligned}$$

We will show that the measure of $D(v, r)$ is bounded by Cr^ℓ , where C is uniform over all $v \in Q$. Recall that the Lie algebra $\text{Lie}(K)$ is defined as the tangent space to the p -adic analytic manifold K at the identity. Consider the map $f_v : K \rightarrow W^+ \oplus W^0$ defined by $f_v(k) = \pi(kv)$. Since K acts by linear transformations, the derivative of f_v at the identity is given by

$$(27) \quad d_e f_v(x) = \pi(xv), \quad x \in \text{Lie}(K) = T_e K \subseteq \mathfrak{sl}_n(\mathbb{Q}_p).$$

For every $v \in V$, set

$$L_v = \{x \in \text{Lie}(K) : xv \in W^-\} = \ker d_e f_v.$$

For $k \in K$, also define the map $m_k : K \rightarrow K$ by $m_k(k') = k'k$. Note that $m_k(e) = k$ and $d_e m_k : T_e K \rightarrow T_k K$ is an isomorphism. Differentiating $f_{kv} = f_v \circ m_k$ yields $d_e f_{kv} = d_k(f_v) \circ d_e m_k$, showing that $\text{rank } d_e f_{kv} = \text{rank } d_k f_v$. From the assumption, if $f_v(k) = 0$, then $kv \in W^-$, hence by the assumption of the theorem, $\text{codim } \ker d_e f_{kv} \geq \ell$, which implies that $\text{codim } \ker d_k f_v \geq \ell$.

Let us denote by $\mathbb{P}(V)$ the projective space on V , and by $[v]$ the image of the non-zero vector v in $\mathbb{P}(V)$. We know that if $f : U \rightarrow V$ is a p -adic analytic function, then the map $x \mapsto \text{rank } d_x f$ is lower semi-continuous, that is, $\liminf_{y \rightarrow x} \text{rank } d_y f \geq \text{rank } d_x f$. This implies that for every $v \in W^-$, there exists an open subset U containing $[v] \in \mathbb{P}(V)$ such that for $[w] \in U$, we have $\text{codim } \ker d_k f_w \geq \ell$. By compactness, there exists $\epsilon > 0$ such if $\|p(kv)\| \leq \epsilon\|v\|$, then the map f_v has a rank at least ℓ . Using the implicit function theorem for local fields (e.g. [25]), for any $k \in K$ for which $\|\pi(kv)\| \leq \epsilon\|v\|$, we can choose local coordinate systems for $\mathcal{U}_k \subset K$ at k and for $W^+ \oplus W^0$ in which f_v is given by

$$f_v(u_1, \dots, u_m) = (u_1, \dots, u_\ell, \psi_{\ell+1}, \dots, \psi_{m'}),$$

where $m = \dim K$, $m' = \dim W^0 + \dim W^+$ and $\psi_j(u_1, \dots, u_m)$ is an analytic function for $j = \ell + 1, \dots, m'$ depending smoothly on v . If $r \leq \epsilon$, for such a neighborhood \mathcal{U}_k , $k \in D(v, r)$, there is a nonnegative $\alpha = \alpha(k)$ satisfying

$$m(D(v, r) \cap \mathcal{U}_k) \sim \left(\frac{r}{\epsilon}\right)^{\ell+\alpha} m(D(v, \epsilon) \cap \mathcal{U}_k)$$

so that

$$\frac{m(D(v, r))}{r^\ell} \leq \frac{1}{\epsilon^\ell} m(D(v, \epsilon)).$$

If $r \geq \epsilon$, since $m(D(v, r)) \leq 1$,

$$\frac{m(D(v, r))}{r^\ell} \leq \frac{1}{\epsilon^\ell}.$$

By compactness of the ball $\{v \in V : \|v\| = 1\}$, we have

$$(28) \quad C := \sup_{\|v\|=1, r>0} \frac{m(D(v, r))}{r^\ell} < \infty.$$

We will now have to consider the set

$$D^+(v, 0) = \{k \in K : p^+(kv) = 0\} = \{k \in K : kv \in W^0 \oplus W^-\}.$$

By Lemma 8.1, we have $m(D^+(v, 0)) = 0$. We claim that this and the compactness of Q imply that

$$(29) \quad \lim_{r \rightarrow 0} \sup_{v \in Q} m(D^+(v, r)) = 0.$$

Otherwise, pick a sequence $r_n \rightarrow 0$, and $v_n \in Q$ such that $m(D^+(v_n, r_n)) > \epsilon$ for some ϵ . Upon passing to a subsequence, we can assume that v_n converges to v .

$$\epsilon \leq m(\{k : \|\pi^+(kv_n)\| \leq r_n\}) \subseteq m(\{k : \|\pi^+(kv)\| \leq r_n + \|v_n - v\|\}).$$

Since $r_n + \|v_n - v\| \rightarrow 0$, we obtain $m(D^+(v, 0)) \geq \epsilon$, which is a contradiction.

Since $a_t|_{W^0} = Id_{W^0}$ and (28), we see that for a positive integer t , $\|a_t kv\|^s \geq \|\pi(kv)\|^s$ hence

$$\int_{D(v, r) - D(v, \frac{r}{p})} \frac{dm(k)}{\|a_t kv\|^s} \leq \frac{m(D(v, r))}{(r/p)^s} \leq (p^{-s}C) r^{\ell-s},$$

for any $v \in V$, $\|v\| = 1$ and any $r > 0$. From the fact that $m(D(v, 0)) = 0$,

$$(30) \quad \begin{aligned} \int_{D(v, r)} \frac{dm(k)}{\|a_t kv\|^s} &= \sum_{n=0}^{\infty} \int_{D(v, p^{-n}r) \setminus D(v, p^{-n-1}r)} \frac{dm(k)}{\|a_t kv\|^s} \\ &\leq \frac{C}{p^{-s} - p^{-\ell}} r^{\ell-s}. \end{aligned}$$

On the other hand, since $a_t|_{W^+} = p^{-t} Id_{W^+}$, we get that $\|a_t kv\|^s \geq p^{ts} \|\pi^+(kv)\|^s$ so that for any $v \in V$, $r > 0$ and $r_1 > 0$,

$$\begin{aligned} \int_{K \setminus D(v, r)} \frac{dm(k)}{\|a_t kv\|^s} &\leq \int_{K - D^+(v, r_1)} \frac{dm(k)}{\|a_t kv\|^s} + \int_{D^+(v, r_1) \setminus D(v, r)} \frac{dm(k)}{\|a_t kv\|^s} \\ &\leq p^{-ts} r_1^{-s} + m(D^+(v, r_1)) r^{-s}. \end{aligned}$$

Therefore for a given $\epsilon_0 > 0$, since $\ell - s > 0$, take $r > 0$ and $r_1 > 0$ such that $Cr^{\ell-s}/(p^{-s} - p^{-\ell}) < \epsilon_0/3$ and $m(D^+(v, r_1)) r^{-s} < \epsilon_0/3$. Then we can take $t' > 0$ such that if $t > t'$, then $p^{-ts} r_1^{-s} < \epsilon_0/3$ so that

$$\int_K \frac{dm(k)}{\|a_t kv\|^s} < \epsilon_0.$$

□

We now write V_i for the i -th exterior power $\bigwedge^i \mathbb{Q}_p^n$, and denote by ρ_i the i -th exterior power of the natural representation $\mathrm{SL}_n(\mathbb{Q}_p)$ on \mathbb{Q}_p^n . Recall that for an i -tuple $J = (j_1, \dots, j_i)$, we define $\mathbf{e}_J = \mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_i}$. It is easy to see that V_i decomposes into a_t -eigenspaces $V_i = W_i^- \oplus W_i^0 \oplus W_i^+$, and the eigenspaces are given by

$$\begin{aligned} W_i^- &= \mathrm{span}\{e_{j_1 \dots j_i} : j_1 = 1, j_i < n\}, \\ W_i^+ &= \mathrm{span}\{e_{j_1 \dots j_i} : 1 < j_1, j_i = n\}, \\ W_i^0 &= \mathrm{span}\{e_{j_1 \dots j_i} : j_1 = 1, j_i = n \text{ or } j_1 > 1, j_i < n\}. \end{aligned}$$

Proposition 8.3. Let \mathbf{q}_p be the standard quadratic form on \mathbb{Q}_p^n , $n \geq 4$, defined as in Subsection 2.2. Set $K = \mathrm{SL}_n(\mathbb{Z}_p) \cap \mathrm{SO}(\mathbf{q}_p)$. Let $A = A_p$ and V_i , W_i^- , W_i^0 and W_i^+ be as above. For each V_i , define

$$(31) \quad F(i) = \{\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_i : \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_i \in \mathbb{Q}_p^n\} \subset V_i$$

and $Q_i = F(i) \cap \{v \in V_i : \|v\|_p = 1\}$. Then for any s , $0 < s < 2$,

$$(32) \quad \lim_{t \rightarrow \infty} \sup_{v \in Q_i} \int_K \frac{dm(k)}{\|a_t k v\|^s} = 0.$$

Proof. Let us first consider the special case of $n = 4$, in which \mathbf{q}_p is given by $\mathbf{q}_p(x_1, x_2, x_3, x_4) = x_1 x_4 + a_2 x_2^2 + a_3 x_3^2$, where $a_i \in \{\pm 1, \pm u_0, \pm p, \pm u_0 p\}$. Here $u_0 \in \mathbb{U}_p$ is a representative of a quadratic non-residue in the quotient $\mathbb{Z}_p/p\mathbb{Z}_p$. Define the following subgroups of K .

$$(33) \quad \mathbf{U} = \left\{ u_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a_2 z^2 & -2a_2 z & 0 & 1 \end{pmatrix} : z \in \mathbb{Z}_p \right\},$$

$$(34) \quad \mathbf{U}' = \left\{ u'_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z & 0 & 1 & 0 \\ -a_3 z^2 & 0 & -2a_3 z & 1 \end{pmatrix} : z \in \mathbb{Z}_p \right\}.$$

Note that if K' is a finite index subgroup of K , then it is open in K , thus $K' \cap \mathbf{U}$, $K' \cap \mathbf{U}'$ are open sets in \mathbf{U} and \mathbf{U}' , respectively.

For $i = 3$, $W^- = \langle \mathbf{e}_{123} \rangle$ and $W^0 = \langle \mathbf{e}_{124}, \mathbf{e}_{134} \rangle$. For $\mathbf{v} = x_{123}\mathbf{e}_{123} + x_{124}\mathbf{e}_{124} + x_{134}\mathbf{e}_{134}$,

$$u_z \mathbf{v} = x_{123}\mathbf{e}_{123} + (2a_2 z x_{123} + x_{134})\mathbf{e}_{134} + (a_2 z^2 x_{123} + z x_{134})\mathbf{e}_{234} + x_{124}\mathbf{e}_{124},$$

$$u'_z \mathbf{v} = x_{123}\mathbf{e}_{123} + (x_{124} - 2a_3 z x_{123})\mathbf{e}_{124} + (a_3 z^2 x_{123} - z x_{124})\mathbf{e}_{234} + x_{134}\mathbf{e}_{134}.$$

If there is a subspace $W' \subset W^- \oplus W^0$ such that $K'W' \subset W'$, then for any sufficiently small z_1, z_2 , the elements $u_{z_1} \mathbf{v}, u'_{z_2} \mathbf{v} \in W^- \oplus W^0$, for any $\mathbf{v} \in W'$, which implies that the coefficients of \mathbf{e}_{234} vanish. It implies that $x_{123} = x_{134} = x_{124}$, thus $W' = \{0\}$ and the condition (1) is satisfied.

For $i = 2$, $W^- = \langle \mathbf{e}_{12}, \mathbf{e}_{13} \rangle$ and $W^0 = \langle \mathbf{e}_{23}, \mathbf{e}_{14} \rangle$. For $\mathbf{v} = x_{12}\mathbf{e}_{12} + x_{13}\mathbf{e}_{13} + x_{23}\mathbf{e}_{23} + x_{14}\mathbf{e}_{14}$, we have

$$\begin{aligned} u_z \mathbf{v} &= x_{12}\mathbf{e}_{12} + 2a_2 z x_{12}\mathbf{e}_{14} + (z x_{14} - a_2 z^2 x_{12})\mathbf{e}_{24} \\ &\quad + x_{13}\mathbf{e}_{13} + (z x_{13} + x_{23})\mathbf{e}_{23} + a_2(z^2 x_{13} + 2z x_{23})\mathbf{e}_{34} + x_{14}\mathbf{e}_{14}, \\ u'_z \mathbf{v} &= x_{12}\mathbf{e}_{12} - (z x_{12} + x_{23})\mathbf{e}_{23} + a_3(z^2 x_{12} - 2z x_{23})\mathbf{e}_{24} \\ &\quad + x_{13}\mathbf{e}_{13} - 2a_3 z x_{13}\mathbf{e}_{14} + (z x_{14} - a_3 z^2 x_{13})\mathbf{e}_{34} + x_{14}\mathbf{e}_{14}. \end{aligned}$$

As above, if for any sufficiently small z_1, z_2 , the elements $u_{z_1} \mathbf{v}, u'_{z_2} \mathbf{v} \in W^- \oplus W^0$, then the coefficients of $\mathbf{e}_{24}, \mathbf{e}_{34}$ vanish, i.e.

$$x_{14} - a_2 z_1 x_{12} = z_1 x_{13} + 2x_{23} = z_2 x_{12} - 2x_{23} = x_{14} - a_3 z_2 x_{13} = 0.$$

Thus if $x_{14} = 0$, then $x_{12} = x_{13} = x_{23} = 0$. If $x_{14} \neq 0$, then there is no solution. Thus the condition (1) of Lemma 8.2 is satisfied. The case $i = 1$ follows immediately.

Now let us check the condition (2) of Lemma 8.2. Define the following elements of $\mathrm{Lie}(K)$:

$$(35) \quad u_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2a_2 & 0 & 0 \end{pmatrix},$$

$$(36) \quad u_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2a_3 & 1 \end{pmatrix}.$$

For $i = 3$, for $\mathbf{v} = x_{123}\mathbf{e}_{123}$

$$(z_1u_1 + z_2u_2)(\mathbf{v}) = 2a_2z_1\mathbf{e}_{134} - 2a_3z_2\mathbf{e}_{124}.$$

It is clear that any non-zero linear combination $z_1u_1 + z_2u_2$ is not in the set $\{x \in \text{Lie}(K) : x\mathbf{v} \in W^-\}$.

For $i = 2$, for $\mathbf{v} = x_{12}\mathbf{e}_{12} + x_{13}\mathbf{e}_{13}$, we have

$$(z_1u_1 + z_2u_2)(\mathbf{v}) = x_{12}(-z_2\mathbf{e}_{23} - 2a_2z_1\mathbf{e}_{14}) + x_{13}(z_1\mathbf{e}_{23} - 2a_3z_2\mathbf{e}_{14}).$$

Suppose that the form \mathbf{q} is not exceptional, in other words, $a_2 \neq -a_3$. For an arbitrary \mathbf{v} , we claim that any non-zero linear combination $z_1u_1 + z_2u_2$ is not in the set $\{x \in \text{Lie}(K) : x\mathbf{v} \in W^-\}$, thus we get codimension of the set at least $\ell = 2$. Indeed, if we solve the conditions for z_1, z_2 , we obtain $(x_{12}/x_{13})^2 = -a_3/a_2$, and the right hand side is not a square unless $a_2 = -a_3$. The case $i = 1$ is immediate. Thus the condition (2) of Lemma 8.2 is satisfied. By Lemma 8.2, for $0 < s < 2$, we can see that

$$\lim_{t \rightarrow \infty} \sup_{v \in \tilde{Q}_i} \int_K \frac{dm(k)}{\|a_t k v\|^s} = 0,$$

for any i .

Now let us consider $n \geq 5$. Denote the embedding from $\text{SO}(x_1x_4 + a_{j_2}x_2^2 + a_{j_3}x_3^3)$ to $\text{SO}(\mathbf{q}_p)$ induced by

$$\mathbf{e}_1 \mapsto \mathbf{e}_1, \mathbf{e}_2 \mapsto \mathbf{e}_{j_2}, \mathbf{e}_3 \mapsto \mathbf{e}_{j_3}, \mathbf{e}_4 \mapsto \mathbf{e}_n.$$

by $\iota(j_2, j_3)$. We will denote the induced embedding on their Lie algebras by $\iota(j_2, j_3)$ again. For a non-exceptional isotropic standard quadratic form \mathbf{q}_p on \mathbb{Q}_p^n , for any given i and any vector

$$\mathbf{v} = \sum_{J=\{j_1 < \dots < j_i\}} a_J \mathbf{e}_J \in V_i,$$

choose j_2, j_3 such that

$$-a_{j_2}/a_{j_3} \text{ is not a square,}$$

and $\iota(j_2, j_3)(\text{SO}(x_1x_4 + a_{j_2}x_2^2 + a_{j_3}x_3^3))$ acts nontrivially on \mathbf{v} . Clearly, the embedding $\iota(j_2, j_3)$ maps subgroups \mathbf{U} and \mathbf{U}' into subgroups of the maximal compact subgroup K of $\text{SO}(\mathbf{q}_p)$.

For any $\mathbf{v} \in W'$, one can choose such an embedding $\iota(j_2, j_3)$ so that at least one of the orbits $\iota(j_2, j_3)\mathbf{U}\mathbf{v}$ and $\iota(j_2, j_3)\mathbf{U}'\mathbf{v}$ are not contained in W' . Similarly, for any $\mathbf{v} \in W^-$, one can choose such an embedding $\iota(j_2, j_3)$ so that for $(z_1, z_2) \neq (0, 0)$, $(z_1\iota(j_2, j_3)u_1 + z_2\iota(j_2, j_3)u_2)\mathbf{v} \notin W^-$. Lemma follows from the case $n = 4$. \square

In the next lemma, we will combine the results over different places $p \in S$. Recall that for $t = (t_p)_{p \in S}$, we set $a_t = (a_{t_p})_{p \in S}$. For $p \in S$, we denote by ϵ_p the element of \mathbb{T} for which $t_p = 1$ and $t_q = 0$ for $q \neq p$. Note that for $p \in S_f$, $a_{\epsilon_p} = a_1^p = \text{diag}(p, 1, \dots, 1, p^{-1})$ as an element in $\text{SO}(\mathbf{q}_p)$ (or as an element embedded in $\text{SO}(\mathbf{q})$).

Lemma 8.4. *Let $n \geq 4$, \mathbf{q} be a non-degenerate isotropic non-exceptional quadratic form over \mathbb{Q}_S^n , and K be the maximal compact subgroup of $\text{SO}(\mathbf{q})$ defined above. If \mathbf{q} is not exceptional, then for any $s \in (0, 2)$ and any $c > 0$, there exists a positive integer m such that for every lattice Δ in \mathbb{Q}_S^n , and every $\mathbf{t} \in R$, with*

$$(37) \quad R = \{m\epsilon_p : p \in S\}.$$

the following inequality holds:

$$(38) \quad \int_K \alpha_i(a_{\mathbf{t}} \mathbf{k} \Delta)^s dm(\mathbf{k}) < \frac{c}{2} \alpha_i(\Delta)^s + \omega^2 \max_{0 < j < \min\{n-i, i\}} (\alpha_{i+j}(\Delta) \alpha_{i-j}(\Delta))^{s/2}.$$

Proof. First, let us assume that \mathbf{q} is not exceptional, and let $c > 0$ be given. We will use Proposition 8.3 and the analogous statement in [10]. For each $p \in S$, one can find an integer $m_p > 0$ such that for any $v \in F(i)$, where $F(i)$ is defined in (31) with $\|v\|_p = 1$, and any $t_p > m_p$.

$$\int_{K_p} \frac{dm(k_p)}{\|a_{t_p}^p k_p v\|_p^s} < c/2.$$

Note that

$$\int_{K_p} \frac{dm(k_p)}{\|a_0^p k_p v\|_p^s} \leq 1.$$

This implies that for $\mathbf{t} = m\epsilon_p$, $p \in S$, we have

$$\int_K \frac{dm(\mathbf{k})}{\|a_{\mathbf{t}} \mathbf{k} v\|^s} = \prod_{p \in S} \int_{K_p} \frac{dm(k)}{\|a_{t_p}^p k_p v\|_p^s} < \frac{c}{2}.$$

We deduce that for every nonzero $v \in F(i)$, and $\mathbf{t} \in R$, we have

$$(39) \quad \int_K \frac{dm(k)}{\|a_{\mathbf{t}} \mathbf{k} v\|^s} < \frac{c}{2} \frac{1}{\|v\|^s}.$$

For a given S -lattice Δ in \mathbb{Q}_S^n and each i , there exists a Δ -rational subspace L_i of dimension i satisfying that

$$(40) \quad \frac{1}{d(L_i)} = \alpha_i(\Delta).$$

By substituting a wedge product of \mathbb{Z}_S -generators of $L_i \cap \Delta$ for v in (39), we have that

$$(41) \quad \int_K \frac{dm(\mathbf{k})}{d_{a_{\mathbf{t}} \mathbf{k} \Delta}(a_{\mathbf{t}} \mathbf{k} L_i)^s} < \frac{c}{2} \frac{1}{d_{\Delta}(L_i)^s}.$$

Let $\omega > 1$ be such that for all $v \in F(i)$, $1 \leq i \leq n-1$, and $\mathbf{t} \in R$ we have

$$(42) \quad \omega^{-1} \leq \frac{\|a_{\mathbf{t}} v\|}{\|v\|} \leq \omega.$$

Let Ψ_i be the set of Δ -rational subspaces L of dimension i for which

$$d_{\Delta}(L) < \omega^2 d_{\Delta}(L_i).$$

We will prove inequality (38) by distinguishing two cases. First, assume that $\Psi_i = \{L_i\}$. Then by (42), we have

$$(43) \quad d_{a_t k \Delta}(a_t k L) = \|a_t k v\| \geq \omega^{-1} \|v\| = \omega^{-1} d(L) \geq \omega d(L_i) = \omega \|v'\| \geq \|a_t k v'\| = d_{a_t k \Delta}(a_t k L_i),$$

where v and v' are wedge products of \mathbb{Z}_S -generators of L and L_i respectively. By inequalities (41), (43) and the definition of α_i ,

$$(44) \quad \int_K \alpha_i(a_t k \Delta)^s dm(k) < \frac{c}{2} \alpha_i(\Delta)^s.$$

Now we assume that $\Psi_i \neq \{L_i\}$. Let M be an element of Ψ_i different from L , and let $\dim(M + L_i) = i + j$ for some $j > 0$. Using Lemma 3.9 and the inequality (42), we get that for any $k \in K$,

$$\begin{aligned} \alpha_i(a_t k \Delta) &< \omega \alpha_i(\Delta) = \frac{\omega}{d(L_i)} < \frac{\omega^2}{(d(L_i)d(M))^{1/2}} \\ &\leq \frac{\omega^2}{(d(L_i \cap M)d(L_i + M))^{1/2}} \leq \omega^2 (\alpha_{i+j}(\Delta) \alpha_{i-j}(\Delta))^{1/2}, \end{aligned}$$

so that

$$(45) \quad \int_K \alpha_i(a_t k \Delta)^s dm(k) \leq \omega^2 \max_{0 < j \leq \min\{n-i, i\}} (\alpha_{i+j}(\Delta) \alpha_{i-j}(\Delta))^s.$$

The result follows from (44) and (45). \square

For $p \in S_f$, we will set

$$D_n^+ = \left\{ \text{diag} \left(p^{\lambda_1}, p^{\lambda_2}, \dots, p^{\lambda_n} \right) \in \text{GL}_n(\mathbb{Q}_p) : \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n, \lambda_i \in \mathbb{Z} \right\}.$$

Every element $g \in \text{GL}_n(\mathbb{Q}_p)$ admits a Cartan decomposition

$$g = k_1(g) d(g) k_2(g), \quad k_1(g), k_2(g) \in \text{GL}_n(\mathbb{Z}_p), d(g) \in D_n^+.$$

We will write $\lambda_i(g)$, $1 \leq i \leq n$ when $d(g) = \text{diag} (p^{\lambda_1(g)}, p^{\lambda_2(g)}, \dots, p^{\lambda_n(g)})$.

Proposition 8.5. *Let U be a neighborhood of the identity element e in $\text{SL}_n(\mathbb{Z}_p)$ given by;*

$$(46) \quad U = \{g \in \text{SL}_n(\mathbb{Z}_p) : \|g - e\|_p < 1\},$$

where $\|\cdot\|_p$ is the maximum p -norm on the vector space of $n \times n$ matrices with respect to the standard basis. Then for any $k \in U$ and $d, d' \in D_n^+$, and any $1 \leq i \leq n$, we have

$$(47) \quad \lambda_i(dkd') = \lambda_i(d) \lambda_i(d').$$

Proof. First note that $\|g\|_p = p^{-\lambda_1(g)}$. Let us denote (i, j) -entry of g by g_{ij} . Let $d = \text{diag} (p^{\lambda_1}, \dots, p^{\lambda_n})$ and $d' = \text{diag} (p^{\lambda'_1}, \dots, p^{\lambda'_n})$ be in D_n^+ . Then $dkd' = (p^{\lambda_i + \lambda'_j} k_{ij})$. Since $|k_{ii}|_p = 1$ for $1 \leq i \leq n$ and $|k_{ij}|_p < 1$ for any $i \neq j$, we obtain

$$(48) \quad p^{\lambda_1(dkd')} = \frac{1}{\|dkd'\|_p} = \frac{1}{|p^{\lambda_1 + \lambda'_1} k_{11}|_p} = p^{\lambda_1 + \lambda'_1} = p^{\lambda_1(d)} p^{\lambda_1(d')}.$$

Consider the representation ρ_i of $\text{GL}_n(\mathbb{Q}_p)$ on the i -th exterior product $\bigwedge^i(\mathbb{Q}_p^n)$ in the usual way and denote the maximum p -norm on $\bigwedge^i \mathbb{Q}_p^n$ with respect to the standard basis by $\|\cdot\|_p$. When expressed in the standard basis of $\bigwedge^i(\mathbb{Q}_p^n)$, the entries of $\rho_i(g)$ are given by pairs (J, K) , where J and K are i -element subsets of the set $\{1, 2, \dots, n\}$. More precisely, $\rho_i(g) = (\det g_{JK})$, where g_{JK} is the $i \times i$ minor of g formed by the rows in J and columns in K . Note that the

p -norm on $\bigwedge^i \mathbb{Q}_p^n$ is also invariant under the action of $\mathrm{GL}_n(\mathbb{Z}_p)$. For $k \in U$, since $|k_{ii}|_p = 1$ and $|k_{ij}| < 1$ for $i \neq j$, we readily see that $|\det(k_{JK})| = 1$ if $J = K$ and $|\det(k_{JK})| < 1$, otherwise. This implies that

$$\|\rho_i(g)\|_p = \|\rho_i(dkd')\|_p = p^{-(\lambda_1(g) + \dots + \lambda_i(g))}.$$

Hence we obtain

$$(49) \quad p^{\lambda_1(dkd') + \dots + \lambda_i(dkd')} = \frac{1}{\|\rho_i(dkd')\|_p} = p^{\sum_{j=1}^i \lambda_j + \sum_{j=1}^i \lambda'_j} = p^{\lambda_1(d) + \dots + \lambda_i(d)} p^{\lambda_1(d') + \dots + \lambda_i(d')}.$$

□

We remark that the above proposition also holds for

$$D_-^n = \left\{ \mathrm{diag} \left(p^{\lambda_1}, p^{\lambda_2}, \dots, p^{\lambda_n} \right) \in \mathrm{GL}_n(\mathbb{Q}_p) : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n, \lambda_i \in \mathbb{Z} \right\},$$

instead of D_+^n .

Corollary 8.6. *Let H be a simply connected simple algebraic group in $\mathrm{GL}_n(\mathbb{Q}_p)$ and K the maximal compact subgroup $\mathrm{SL}_n(\mathbb{Z}_p) \cap H$ of H . Let $a_t = \mathrm{diag}(p^{-t}, 1, \dots, 1, p^t)$ for $t \in \mathbb{Z}$. Then there is a neighborhood U of e in K such that for $t, s \geq 0$, we have*

$$a_t U a_s \subset K a_t a_s K.$$

Proof. The Cartan decomposition of H (see Theorem 3.14 in [20]) provides that for each $g \in H$, there are elements $k_1(g)$ and $k_2(g)$ in K such that

$$a_t g a_s = k_1(g) \mathrm{diag} \left(p^{\lambda_1(a_t g a_s)}, \dots, p^{\lambda_n(a_t g a_s)} \right) k_2(g).$$

Take a neighborhood U of K satisfying the condition of Proposition 8.5. Then

$$\mathrm{diag} \left(p^{\lambda_1(a_t g a_s)}, \dots, p^{\lambda_n(a_t g a_s)} \right) = \mathrm{diag} \left(p^{\lambda_1(a_t a_s)}, \dots, p^{\lambda_n(a_t a_s)} \right) = a_t a_s,$$

so we get the result. □

Proposition 8.7. *Let $H = \mathrm{SO}(\mathbf{q})$, $K = \mathrm{SL}_n(\mathbb{Z}_p) \cap H$ and let \mathcal{F} be a family of strictly positive functions on H having the following properties:*

- (a) *For every $\lambda > 1$, there exists a neighborhood $V(\lambda)$, $\lambda > 1$ of the identity in H such that for all $f \in \mathcal{F}$, $\lambda^{-1}f(h) < f(gh) < \lambda f(h)$ for any $h \in H$ and $g \in V(\lambda)$.*
- (b) *There exists a constant $C > 0$ such that for all $p \in S_f$, and all $h \in H$, we have*

$$f(a_{\epsilon_p} h) < C f(h),$$

where a_{ϵ_p} denotes the element $\mathrm{diag}(p, 1, \dots, 1, p^{-1})$.

- (c) *The functions $f \in \mathcal{F}$ are left K -invariant, that is, $f(kh) = f(h)$, for $k \in K$ and $h \in H$.*
- (d) *$\sup_{f \in \mathcal{F}} f(1) < \infty$.*

Then there exists $0 < c = c(\mathcal{F}) < 1$ such that for any $t_0 > 0$ and $b > 0$ there exists $B = B(t, b) < \infty$ with the following property: If $f \in \mathcal{F}$ and

$$(50) \quad \int_K f(a_{t_0} kh) dm(k) < c f(h) + b$$

for any $h \in H$, then

$$\int_K f(a_t k) dm(k) < B$$

for all $t > 0$.

Proof. First note that we can replace each $f \in \mathcal{F}$ with its right K -average

$$\tilde{f}(\mathbf{h}) = \int_{\mathbf{K}} f(\mathbf{h}\mathbf{k}) d\mathbf{m}(\mathbf{k})$$

and $\tilde{\mathcal{F}} = \{\tilde{f} : f \in \mathcal{F}\}$ still satisfies properties (a), (b) and (c). Hence, without loss of generality, we may and will assume that for all $f \in \mathcal{F}$ and $\mathbf{h} \in \mathbf{H}$, we have

$$(51) \quad f(\mathbf{K}\mathbf{h}\mathbf{K}) = f(\mathbf{h}).$$

We need to prove

$$(52) \quad \sup_{\mathbf{t} > 0} f(a_{\mathbf{t}}) < B.$$

By Corollary 8.6 and Lemma 5.11 in [10], we can take a neighborhood \mathbf{U} of the identity in \mathbf{H} such that

$$a_{\mathbf{t}_0} \mathbf{U} a_{\mathbf{t}} \in \mathbf{K} V a_{\mathbf{t}_0} a_{\mathbf{t}} \mathbf{K}$$

for any $\mathbf{t}_0, \mathbf{t} \geq 0$. From (a) and (51), it follows that

$$(53) \quad \int_{\mathbf{K}} f(a_{\mathbf{t}_0} \mathbf{k} a_{\mathbf{t}}) d\mathbf{m}(\mathbf{k}) \geq \int_{\mathbf{U} \cap \mathbf{K}} f(a_{\mathbf{t}_0} \mathbf{k} a_{\mathbf{t}}) d\mathbf{m}(\mathbf{k}) > \frac{1}{2} m_{\mathbf{K}}(\mathbf{U} \cap \mathbf{K}) f(a_{\mathbf{t}_0} a_{\mathbf{t}}).$$

Suppose that for all $\mathbf{t}_0 \in R$ and $b > 0$, and all $\mathbf{h} \in \mathbf{H}$ we have

$$(54) \quad \int_{\mathbf{K}} f(a_{\mathbf{t}_0} \mathbf{k} \mathbf{h}) d\mathbf{m}(\mathbf{k}) < \frac{1}{4} m_{\mathbf{K}}(\mathbf{U} \cap \mathbf{K}) f(\mathbf{h}) + b.$$

Let R be as in (37), and Θ denote the semigroup generated by R . Then by taking $\mathbf{h} = a_{\mathbf{t}}$ in (54) we see that for any $\mathbf{t} > 0$, and $\mathbf{t}_0 \in \Theta$, we have

$$(55) \quad f(a_{\mathbf{t}_0} a_{\mathbf{t}}) < \frac{1}{2} f(a_{\mathbf{t}}) + b' \leq \max(f(a_{\mathbf{t}}), b'),$$

where $b' = \max(2b/m_{\mathbf{K}}(\mathbf{U} \cap \mathbf{K}), f(e))$. A simple induction shows that for any $\mathbf{t} \in \Theta$, we have

$$(56) \quad f(a_{\mathbf{t}}) \leq b'.$$

Note that every $\mathbf{t} > 0$ can be decomposed as $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$, where $\mathbf{t}_1 \in \Theta$ and \mathbf{t}_2 is positive and bounded in every component. Since \mathbf{t}_2 can be written as the product of a bounded number of elements of $V(1/2)$ and ϵ_p , with $p \in S_f$, the claim follows. \square

Proof of Theorem 1.7. Define functions f_0, f_1, \dots, f_n on \mathbf{H} by

$$f_i(\mathbf{h}) = \alpha_i(\mathbf{h}\Delta), \quad \mathbf{h} \in \mathbf{H}.$$

Since $\alpha \leq \sum_{i=0}^n \alpha_i$, it suffices to show that for all $0 \leq i \leq n$,

$$(57) \quad \sup_{\mathbf{t} > 0} \int_{\mathbf{K}} f_i^s(a_{\mathbf{t}} \mathbf{k}) d\mathbf{m}(\mathbf{k}) < \infty.$$

Let us check that f_0, f_1, \dots, f_n have the properties (a), (b), (c), and (d) of Proposition 8.7. First note that for $v \in F(i)$, $0 \leq i \leq n$, and $\mathbf{h} \in \mathbf{H}$, we have the trivial bound $\|\mathbf{h}v\| \leq \|\wedge^i \mathbf{h}\| \|v\|$. This proves (a) and (b). Since the action of \mathbf{K} preserves the function $\|\cdot\|$ on $\wedge^i(\mathbb{Q}_S^n)$, each f_i is left \mathbf{K} invariant. It is clear that $f_i(1)$ are uniformly bounded as Δ runs over a compact set of lattices. From Lemma 8.4 with $h\Delta$, for any i , $0 < i < n$ and $h \in \mathbf{H}$, we see that

$$(58) \quad \int_{\mathbf{K}} f_i^s(a_{\mathbf{t}} \mathbf{k} \mathbf{h}) d\mathbf{m}(\mathbf{k}) < \frac{c}{2} f_i^2 + \omega^2 \max_{0 < j < \min\{n-i, i\}} (f_{i+j} f_{i-j})^{s/2}.$$

Define

$$f_{\epsilon,s} = \sum_{0 \leq i \leq n} \epsilon^{i(n-i)} f_i^s.$$

Since $\epsilon^{i(n-i)} f_i^s < f_{\epsilon,s}$, $f_0 = 1$ and $f_n = d(\lambda)^{-1}$, by putting $\epsilon = c/(2n\omega^2)$, we have the inequality (50) of Proposition 8.7:

$$\begin{aligned} \int_{\mathbf{K}} f_{\epsilon,s}(a_{\mathbf{t}} \mathbf{k} \mathbf{h}) dm(\mathbf{k}) &< 1 + d(\Delta)^{-s} + \frac{c}{2} f_{\epsilon,s} + n\epsilon\omega^2 f_{\epsilon,s} \\ (59) \qquad \qquad \qquad &= cf_{\epsilon,s} + 1 + d(\Delta)^{-1}. \end{aligned}$$

Let \mathcal{C} be an arbitrary compact set of unimodular lattices Δ and let \mathcal{F} be the family of $f_{\epsilon,s}$ as Δ runs over \mathcal{C} . Then \mathcal{F} satisfies the conditions of Proposition 8.7. From (59) it follows that there the constants b and c can be chosen uniformly for the family $f_{\epsilon,s} \in \mathcal{F}$ constant c and b . Since $\alpha_i(h\Delta)^s \leq \epsilon^{-i(n-i)} f_{\epsilon,s}(h)$, by Proposition 8.7, we conclude that there exists a constant $B > 0$ such that for each i , all $\mathbf{t} \succ 0$ and all $\Delta \in \mathcal{C}$,

$$\int_{\mathbf{K}} \alpha_i(a_{\mathbf{t}} \mathbf{k} \Delta)^s dm(\mathbf{k}) < B.$$

□

We will now use this bound to prove the main result. Define the set $A(r)$ by

$$(60) \qquad \qquad \qquad A(r) = \{x \in \mathbf{G}/\Gamma : \alpha(x) \leq r\}.$$

Using Mahler's compactness criterion [15], we see that $A(r)$ is compact for any $r > 0$.

Proof of Theorem 1.8. We may assume that ϕ is nonnegative. For each $r \in \mathbb{R}_{>0}$, we choose a continuous function g_r on \mathbf{G}/Γ such that $0 \leq g_r(x) \leq 1$, $g_r(x) = 1$ if $x \in A(r)$ and $g_r(x) = 0$ outside $A(r+1)$. Then

$$\phi(a_{\mathbf{t}} \mathbf{k} x) = (g_r \phi)(a_{\mathbf{t}} \mathbf{k} x) + ((1 - g_r)\phi)(a_{\mathbf{t}} \mathbf{k} x).$$

Following the proof of Theorem 3.5 in [10], let $\beta = 2 - s$. Since $((1 - g_r)\phi)(y) = 0$ if $y \in A(r)$,

$$\begin{aligned} ((1 - g_r)\phi)(y) &\leq C(1 - g_r)(y)\alpha(y)^{2-\beta} \leq C\alpha(y)^{2-\beta/2}(1 - g_r)(y)\alpha(y)^{-\beta/2} \\ &\leq Cr^{-\beta/2}\alpha(y)^{2-\beta/2}. \end{aligned}$$

By Theorem 1.7 and the fact that $\|\nu\|_{\infty} < \infty$, there exists $C' > 0$ such that

$$\begin{aligned} \int_{\mathbf{K}} ((1 - g_r)\phi)(a_{\mathbf{t}} \mathbf{k} x) \nu(\mathbf{k}) dm(\mathbf{k}) &\leq Cr^{-\beta/2} \int_{\mathbf{K}} \alpha(a_{\mathbf{t}} \mathbf{k} x)^{2-\beta/2} \nu(\mathbf{k}) dm(\mathbf{k}) \\ (61) \qquad \qquad \qquad &\leq C' r^{-\beta/2}. \end{aligned}$$

On the other hand, since $g_r \phi$ is compactly supported, we see that for sufficiently large $\mathbf{t} \in \mathbb{R}_{>0} \times \mathbb{N}^s$ in the cone generated by R

$$(62) \qquad \left| \int_{\mathbf{K}} (g_r \phi)(a_{\mathbf{t}} \mathbf{k} x) \nu(\mathbf{k}) dm(\mathbf{k}) - \int_{\mathbf{G}/\Gamma} (g_r \phi)(y) dg(y) \int_{\mathbf{K}} \nu(\mathbf{k}) dm(\mathbf{k}) \right| < \epsilon/2,$$

where dg is the normalized Haar measure on \mathbf{G}/Γ . Since $g_r \phi \rightarrow \phi$ as $r \rightarrow \infty$, (61) and (62) show the theorem. □

9. COUNTEREXAMPLES FOR TERNARY FORMS

In this section, we will prove Theorem 1.9 which partially shows that Theorem 1.5 is optimal in the sense that for some exceptional forms the asymptotics does not hold.

We will first introduce the family of quadratic forms which will be useful for the construction. Fix $\alpha_p \in \mathbb{Q}_p$, and define

$$\mathbf{q}^{\alpha_p}(\mathbf{x}) = x_1^2 + x_2^2 - \alpha_p^2 x_3^2.$$

Similarly, for an S -vector $\alpha = (\alpha_p)_{p \in S} \in \mathbb{Q}_S$, we set $\mathbf{q}^\alpha = (\mathbf{q}^{\alpha_p})_{p \in S}$. In the case $\alpha \in \mathbb{Q}$, with a slight abuse of notation, we will also use \mathbf{q}^α as a shorthand for the quadratic form on \mathbb{Q}_S^n associated to the constant S -vector with $\alpha_p = \alpha$ for all $p \in S$. We will first prove an S -adic version of a well-known fact from number theory adapted to fulfill our purpose.

Lemma 9.1. *Given $\alpha \in \mathbb{Q}$, there exists a constant $c_\alpha > 0$ such that for a sufficiently large S -time $\mathbf{T} = (T_p)_{p \in S}$, there exists at least $c_\alpha \|\mathbf{T}\| \log \|\mathbf{T}\|$ vectors $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}_S^3$ which satisfy*

- (1) $\mathbf{q}^\alpha(\mathbf{x}) = 0$.
- (2) $\|x_i\|_p = T_p$ for every $1 \leq i \leq 3$ and $p \in S_f$.
- (3) $\|\mathbf{x}\|_\infty \leq T_\infty$.

Before we proceed to the proof, we emphasize that condition (2) is a stronger form of $\|x\| = T_p$ which is needed for the following arguments.

Proof. It will be slightly more convenient to first work with the quadratic form $\mathbf{q}'(\mathbf{x}) = x_1 x_2 - x_3^2$, which is \mathbb{Q} -equivalent to \mathbf{q}^α for every α . For $p \in S_f$, write $T_p = p^{n_p}$, and set $a = \prod_{p \in S_f} p^{-n_p} = T_\infty \|\mathbf{T}\|^{-1}$. We will consider triples

$$x_1 = aku^2, \quad x_2 = akv^2, \quad x_3 = akuv,$$

where $u, k, v \in \mathbb{Z}$ with $\gcd(u, v) = 1$ and $\gcd(u, p) = \gcd(v, p) = \gcd(k, p) = 1$ for all $p \in S_f$. It is clear that \mathbf{x} satisfies conditions (1) and (2). Moreover, the condition (3) will also be satisfied if k is chosen such that

$$|u|, |v| \leq \sqrt{T_\infty / 3ak}.$$

It is well-known that the density of the pairs (u, v) with $\gcd(u, v) = 1$ is $c = 6/\pi^2 > 0$. We can thus produce at least as many as

$$\sum_{\substack{1 \leq k \leq T_\infty / 3a \\ \gcd(k, p) = 1, p \in S}} c \frac{T_\infty}{3ak} = c \|\mathbf{T}\| \sum_{\substack{1 \leq k \leq T_\infty / 3a \\ \gcd(k, p) = 1, p \in S}} \frac{1}{k} \gg \|\mathbf{T}\| \log \|\mathbf{T}\|$$

solutions. Note that the forms \mathbf{q}^α can be obtained from \mathbf{q}' applying a rational linear transformation A , which changes the real and p -adic norms by a bounded factor. This completes the proof. \square

We will need the following lemma, which is an S -arithmetic version of Lemma 3.15 in [10]. We will assume that $\Omega = \prod_{p \in S} \mathbb{U}_p^n$ is the product of unit spheres, and drop it from the notation for the counting function.

Lemma 9.2. *Let $I = \prod_{p \in S} I_p$ be an S -interval. Given $\epsilon > 0$ and an S -time \mathbf{T}_0 , the set of vectors $\beta = (\beta_p)_{p \in S} \in \mathbb{Q}_S$ for which there exists $\mathbf{T} \succ \mathbf{T}_0$ such that*

$$\mathbf{N}_{I, \mathbf{q}^\beta}(\mathbf{T}) \geq \|\mathbf{T}\| (\log \|\mathbf{T}\|)^{1-\epsilon}$$

is dense in \mathbb{Q}_S .

Proof. For the notational simplicity, we will set $I_\infty = [1/4, 1/2]$ and $I_p = \mathbb{Z}_p$ for $p \in S_f$. It will be clear that the proof works for any other choice of intervals. For $\alpha \in \mathbb{Q}$ and T large, define

$$L(\alpha, \mathsf{T}) = \{\mathbf{x} \in \mathbb{Z}_S^3 : \mathbf{q}^\alpha(\mathbf{x}) = 0, \|\mathbf{x}\|_\infty \in [T_\infty/2, T_\infty], \|\mathbf{x}\|_p = T_p, \forall p \in S_f\}.$$

By Lemma 9.1, for large enough T , we have

$$\text{card } L(\alpha, \mathsf{T}) \geq \frac{c_\alpha}{4} \|\mathsf{T}\| \log \|\mathsf{T}\| \geq \|\mathsf{T}\| (\log \|\mathsf{T}\|)^{1-\epsilon}.$$

Note that if $\mathbf{x} = (x_1, x_2, x_3) \in L(\alpha, \mathsf{T})$ then

$$\frac{T_\infty^2}{4(1+\alpha^2)} \leq x_3^2 \leq \frac{T_\infty^2}{1+\alpha^2}.$$

By condition (2) in Lemma 9.1, we also have $|x_3|_p = T_p = p^{m_p}$ for every $p \in S$. Set

$$\beta_\infty^2 = \alpha^2 - (1+\alpha^2)T_\infty^{-2}, \quad \beta_p^2 = \alpha^2 + u_p p^{m_p},$$

where $|u_p|_p = 1$. Note that for sufficiently large T_p , $\alpha^2 + u_p p^{m_p}$ is a square in \mathbb{Q}_p and β is well-defined. A simple computation shows that for any $\mathbf{x} \in L(\alpha, \mathsf{T})$ we have

$$(63) \quad \begin{aligned} \mathbf{q}^{\beta_\infty}(\mathbf{x}) &= \mathbf{q}^\alpha(\mathbf{x}) + (1+\alpha^2)T_\infty^{-2}x_3^2 \in [1/4, 1], \\ \mathbf{q}^{\beta_p}(\mathbf{x}) &= \mathbf{q}^\alpha(\mathbf{x}) + u^2 x_3^2 \in \mathbb{Z}_p. \end{aligned}$$

This implies that the

$$\mathbf{N}_{\mathbf{l}, \mathbf{q}^\beta}(\mathsf{T}) \geq \text{card } L(\alpha, \mathsf{T}) \gg \|\mathsf{T}\| \log \|\mathsf{T}\|^{1-\epsilon}.$$

It is easy to see that the set of β obtained in this way is dense in \mathbb{Q}_S . □

We can now prove Theorem 1.9. Assume that for a given T_0 , we denote by $W(\mathsf{T}_0)$ the set of $\gamma \in \mathbb{Q}_S$ such that there exists $\beta \in \mathbb{Q}_S^n$ satisfying $|\beta - \gamma| < \|\mathsf{T}\|^{-3}$ and

$$\mathbf{N}_{\mathbf{l}, \mathbf{q}^\beta}(\mathsf{T}) \geq \|\mathsf{T}\| \log \|\mathsf{T}\|^{1-\epsilon}.$$

From the definition and Lemma 9.2, it is clear that $W(\mathsf{T}_0) \subseteq \mathbb{Q}_S$ is open and dense. Let T_j be a sequence of S -times going to infinity. Then $W = \bigcap_{j=1}^\infty W(\mathsf{T}_j)$ is a set of second category, and hence contains an irrational form. For any $\gamma \in W$, there exists an infinite sequence β^i and T_i satisfying $|\gamma_p - \beta_p^i|_p < \|\mathsf{T}_i\|^{-3}$ for all $p \in S$ and

$$\mathbf{N}_{\mathbf{l}, \mathbf{q}^{\beta^i}}(\mathsf{T}_i) \geq \|\mathsf{T}_i\| \log \|\mathsf{T}_i\|^{1-\epsilon}.$$

Set $R_i = \|\mathsf{T}_i\|$. If $\|\mathbf{x}\|_\infty < R_i$, it is easy to see that

$$\|\mathbf{q}_p^{\beta^i}(\mathbf{x}) - \mathbf{q}_p^\gamma(\mathbf{x})\|_\infty = O(R_i^{-1}) < 1/8$$

for $i \gg 1$ and $p \in S$. This implies that $\mathbf{q}_\infty^\gamma(\mathbf{x}) \in [1/8, 2]$ and $\mathbf{q}_p^\gamma(\mathbf{x}) \in \mathbb{Z}_p$ for $p \in S_f$. The claim follows from here.

The argument when all \mathbf{q}_p are equivalent to $x_1x_2 - x_3x_4$ is similar. An argument along the same lines as the one given above, using the parametrization of the solutions to $x_1x_2 - x_3x_4$ given by

$$x_1 = auv, \quad x_2 = azw, \quad x_3 = auz, \quad x_4 = avw$$

establishes the result in this case.

APPENDIX A. AN EXTENSION OF THE WITT THEOREM

In this appendix, we prove Proposition 2.2 as stated in Section 2.

Proposition A.1. *For given $c_1 \in \mathbb{Q}_p$ and $c_2 \in p^\mathbb{Z}$, K_p acts transitively on*

$$\{\mathbf{v}_p \in \mathbb{Q}_p^n : \mathbf{q}(\mathbf{v}_p) = c_1 \text{ and } \|\mathbf{v}_p\|_p = c_2\}.$$

Proof. For simplicity, let us denote \mathbf{q}_p , K_p and \mathbf{v}_p by \mathbf{q} , K and \mathbf{v} , etc. We need to prove that for given two vectors \mathbf{v}_1 and \mathbf{v}_2 in \mathbb{Q}_p^n with the same p -adic norm and the same value of \mathbf{q} , there is an element k in K such that $k \cdot \mathbf{v}_1 = \mathbf{v}_2$. For convenience, we may assume that $\mathbf{v}_1 = \mathbf{e}_1$ and let $\mathbf{v}_2 = \mathbf{v} \in \mathbb{Z}_p^n$. Since the same argument will be applied to both isotropic or nonisotropic vectors, we will think of a quadratic form $\mathbf{q}(x_1, \dots, x_n)$ as one of

$$\begin{aligned} &u_1x_1^2 + \dots + u_ix_i^2 + p(u_{i+1}x_{i+1}^2 + \dots + u_nx_n^2) \quad \text{or} \\ &x_1x_2 + u_3x_3^2 + \dots + u_ix_i^2 + p(u_{i+1}x_{i+1}^2 + \dots + u_nx_n^2). \end{aligned}$$

depending on the case we want to treat. Here u_i 's are units in \mathbb{Z}_p .

Then we can write the corresponding symmetric matrix $B = B_{\mathbf{q}}$ as $\begin{pmatrix} B' & \\ & pB'' \end{pmatrix}$, where B' and B'' are nondegenerate mod p . The proposition demands to find a matrix $k \in K$ satisfying that

- (a) $k \cdot \mathbf{e}_1 = \mathbf{v}$ and
- (b) ${}^t k B k = B$.

Since \mathbb{Z}_p is the inverse limit of $\mathbb{Z}/p^j\mathbb{Z}$, $j \rightarrow \infty$, we will construct a chain $k^j \in \text{GL}_n(\mathbb{Z}/p^{j+1}\mathbb{Z})$ such that

- (a') $k^j \cdot \mathbf{e}_1 = \mathbf{v} \bmod p^{j+1}$,
- (b') ${}^t k^j B k^j = B \bmod p^{j+1}$ and
- (c') $k^j = k^{j+1} \bmod p^{j+1}$.

Then the inverse limit of $(k^j)_{j=0}^\infty$ will be an element satisfying the conditions (a) and (b). Let us denote $k^j = k_0 + pk_1 + p^2k_2 + \dots + p^jk_j$.

Step 1. $j = 0$. Let $k^0 = k_0 = \begin{pmatrix} X_0 & Y_0 \\ Z_0 & W_0 \end{pmatrix}$ depending on the size of B' and B'' . By the condition (a'), the first column \mathbf{v}_0 of k_0 is given by $\mathbf{v} \bmod p$. We want to find a solution of the following equation;

$$\begin{aligned} \begin{pmatrix} B' & \\ & 0 \end{pmatrix} &= \begin{pmatrix} {}^tX_0 & {}^tZ_0 \\ {}^tY_0 & {}^tW_0 \end{pmatrix} \begin{pmatrix} B' & \\ & 0 \end{pmatrix} \begin{pmatrix} X_0 & Y_0 \\ Z_0 & W_0 \end{pmatrix} \bmod p \\ &= \begin{pmatrix} {}^tX_0 B' X_0 & {}^tX_0 B' Y_0 \\ {}^tY_0 B' X_0 & {}^tY_0 B' Y_0 \end{pmatrix} \bmod p. \end{aligned}$$

By the assumption of our quadratic form, $Q(pr_i(\mathbf{e}_1)) = Q(pr_i(\mathbf{v}_0))$, where $pr_i : (x_1, \dots, x_n) \in (\mathbb{Z}/p\mathbb{Z})^n \rightarrow (x_1, \dots, x_i) \in (\mathbb{Z}/p\mathbb{Z})^i$. Applying the Witt theorem for finite fields ([1]) to $((\mathbb{Z}/p\mathbb{Z})^i, B')$, we can get an isometry X_0 satisfying that the first column is $pr_i(\mathbf{v}_0)$ and ${}^tX_0 B' X_0 = B' \bmod p$. Since ${}^tX_0 B'$ is invertible, we should take Y_0 as 0. Note that in this step, we can not determine Z_0 and W_0 .

Step 2. $j = 1$. The matrix $k^1 = \begin{pmatrix} X_0 + pX_1 & Y_0 + pY_1 \\ Z_0 + pZ_1 & W_0 + pW_1 \end{pmatrix}$ has the first column $\mathbf{v}_0 + p\mathbf{v}_1$ and should satisfy the following equations.

$$\begin{aligned} {}^tX_0B'X_0 + p({}^tX_0B'X_1 + {}^tX_1B'X_0 + {}^tZ_0B''Z_0) &= B' \bmod p^2, \\ {}^tX_0B'Y_0 + p({}^tX_1B'Y_0 + {}^tX_0B'Y_1 + {}^tZ_0B''W_0) &= 0 \bmod p^2, \\ {}^tY_0B'Y_0 + p({}^tY_0B'Y_1 + {}^tY_1B'Y_0 + {}^tW_0B''W_0) &= pB'' \bmod p^2. \end{aligned}$$

Since $Y_0 = 0$ and ${}^tX_0B'X_0 = B' + pC_1^{11} \bmod p^2$ for some symmetric matrix C_1^{11} , we can reduce the above equations as

$$\begin{aligned} C_1^{11} + {}^tX_0B'X_1 + {}^tX_1B'X_0 + {}^tZ_0B''Z_0 &= 0 \bmod p, \\ {}^tX_0B'Y_1 + {}^tZ_0B''W_0 &= 0 \bmod p, \\ {}^tW_0B''W_0 &= B'' \bmod p. \end{aligned}$$

Take any Z_0 such that the first column is (v_{i+1}, \dots, v_n) and any W_0 satisfying ${}^tW_0B''W_0 = B'' \bmod p$ using the Witt theorem. Then it suffices to show that there is a matrix X_1 with the given first column satisfying the equation $C_1 + {}^tX_0B'X_1 + {}^tX_1B'X_0 + {}^tZ_0B''Z_0 = 0 \bmod p$.

Step 3. We claim that for a given invertible symmetric matrix A and any symmetric matrix C , there is a solution X of the equation ${}^tXA + AX = C$.

By considering the space of n by n matrices as a n^2 -dimensional vector space, we can rewrite the above equation by

$$(64) \quad \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} [X]^1 \\ [X]^2 \\ \vdots \\ [X]^n \end{pmatrix} = \begin{pmatrix} [C]^1 \\ [C]^2 \\ \vdots \\ [C]^n \end{pmatrix},$$

where a block matrix A_{ij} is defined by

$$A_{ij} = \begin{cases} \begin{pmatrix} a_{11} & a_{21} & \cdots & \cdots & a_{n1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2a_{1i} & 2a_{2i} & \cdots & \cdots & 2a_{ni} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & \cdots & a_{nn} \end{pmatrix} & \text{when } A = (a_{st}), \quad i = j; \\ \text{All entries are zero except } j\text{-th row is} & \\ & (a_{1i}, a_{2i}, \dots, a_{ni}), \quad i \neq j. \end{cases}$$

Since ${}^tXA + AX$ and C are both symmetric, after removing rows repeated (for example, one of second row and $(n+1)$ th row corresponding to c_{12} and c_{21}) we get a linear equation from $(\mathbb{Z}/p\mathbb{Z})^{n^2}$ to $(\mathbb{Z}/p\mathbb{Z})^{n(n+1)/2}$. Furthermore, from the fact that A is invertible, the rank of this reduced linear equation is exactly $n(n+1)/2$ which tells us that there is a solution X .

Now let us show that $C_1 + {}^tX_0B'X_1 + {}^tX_1B'X_0 + {}^tZ_0B''Z_0 = 0 \bmod p$ has a solution when C_1 , B' , X_0 , B'' , Z_0 and the first column of X_1 are given. That is, the first n^2 by n submatrix ${}^t(A_{11}, \dots, A_{n1})$ of the equation (1) is erased together with the n variables $[X]^1$.

Before that, let us assume that the removed row among the repeated rows in the above argument is always the former one. That is, in the example, we will remove the second row and leave $(n+1)$ th row. Hence the entries of the $n(n+1)/2$ by n submatrix of the reduced matrix are zero except the first row, and consequently the rank of the linear equation $C_1 + {}^t X_0 B' X_1 + {}^t X_1 B' X_0 + {}^t Z_0 B'' Z_0 = 0 \pmod p$ is $n(n+1)/2 - 1$. On the other hand c_{11} in the equation (64) can be also removed since it is determined by A_{11} and $[X]^1$. Therefore if we check that the $(1,1)$ -entry of $C_1 + {}^t X_0 B' X_1 + {}^t X_1 B' X_0 + {}^t Z_0 B'' Z_0 = 0 \pmod p$ holds, we can find a required matrix X_1 . However this follows from the fact that $\mathbf{q}(\mathbf{e}_1) = \mathbf{q}(\mathbf{v}) = \mathbf{q}(\mathbf{v}_0 + p\mathbf{v}_1) \pmod p$.

Step 4. In general, suppose that there exists a solution $k = k_0 + pk_1 + p^2k_2 + \cdots + p^nk_n + \cdots$ satisfying (a) and (b). Then from the condition ${}^t k B k = B$,

$$\begin{aligned} \begin{pmatrix} B' & \\ & pB'' \end{pmatrix} &= \sum_{j=0}^{\infty} p^j \left(\sum_{i=0}^j \begin{pmatrix} {}^t X_i & {}^t Z_i \\ {}^t Y_i & {}^t W_i \end{pmatrix} \begin{pmatrix} B' & \\ & pB'' \end{pmatrix} \begin{pmatrix} X_{j-i} & Y_{j-i} \\ Z_{j-i} & W_{j-i} \end{pmatrix} \right) \\ &= \sum_{j=0}^{\infty} p^j \left(\sum_{i=0}^j \begin{pmatrix} {}^t X_i B' X_{j-i} & {}^t X_i B' Y_{j-i} \\ {}^t Y_i B' X_{j-i} & {}^t Y_i B' Y_{j-i} \end{pmatrix} \right) + \\ &\quad \sum_{j=0}^{\infty} p^{j+1} \left(\sum_{i=0}^j \begin{pmatrix} {}^t Z_i B'' Z_{j-i} & {}^t Z_i B'' W_{j-i} \\ {}^t W_i B'' Z_{j-i} & {}^t W_i B'' W_{j-i} \end{pmatrix} \right). \end{aligned}$$

Hence we should find X_j , Y_j , Z_{j-1} and W_{j-1} inductively. Take any Z_{j-1} with the first column $(v_{j-1}^{i+1}, \dots, v_{j-1}^n)$, where $\mathbf{v} = \mathbf{v}_0 + p\mathbf{v}_1 + \cdots + p^j\mathbf{v}_j + \cdots$. Then by step3 with the fact that ${}^t \mathbf{v} B \mathbf{v} = \sum_{k=0}^j p^k \left(\sum_{i=0}^k {}^t \mathbf{v}_i B \mathbf{v}_{k-i} \right) \pmod{p^{j+1}}$, we can find an appropriate X_j , W_{j-1} and Y_j satisfying the following equations.

$$\begin{aligned} {}^t X_0 B' X_j + {}^t X_j B' X_0 &= - \sum_{i=1}^{j-1} {}^t X_i B' X_{j-i} - \sum_{i=0}^{j-1} {}^t Z_i B'' Z_{j-i-1} + C_j^{11} \pmod p, \\ {}^t W_{j-1} B'' W_0 + {}^t W_0 B'' W_{j-1} &= - \sum_{i=1}^{j-1} ({}^t W_i B'' W_{j-i-1} + {}^t Y_i B' Y_{j-i}) + C_n^{22} \pmod p, \\ {}^t X_0 B' Y_j &= - \sum_{i=0}^{j-1} ({}^t X_i B' Y_{j-i} + {}^t Z_i B'' W_{j-i-1}) + C_j^{12} \pmod p, \end{aligned}$$

where C_j^{11} , C_j^{22} and C_j^{12} are obtained from the equations of the formal level $j-1$ (see the step2).

Consequently, we can find $k \in \mathrm{GL}(n, \mathbb{Z}_p)$ such that ${}^t k B k = B$. Since $\det k = \pm 1$, k may not be an element of K . However we can easily find $k' \in K$ with $k' \cdot \mathbf{e}_1 = \mathbf{v}$ using reflections in \mathbb{Q}_p^n . \square

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