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TWO STATES

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Abstract[1]

ABSTRACT. D. Bures defined a metric on states of a C^*-algebra as the infimum of the distance between associated vectors in common GNS representations. We take a different approach by looking at the completely bounded distance between relevant joint representations. The notion has natural extension to unital completely positive maps. This study yields new understanding of GNS representations of states and in particular provides a new formula for Bures metric.

1. Introduction

Given a state ϕ on a unital C^*-algebra $\mathcal{A}$, the well-known Gelfand-Naimark-Segal (GNS)-construction yields a triple (H, π, x), where H is a Hilbert space, $\pi : \mathcal{A} \to \mathcal{B}(H)$ is a representation ($*$-homomorphism) and $x \in H$ is a vector such that $\phi(\cdot) = \langle x, \pi(\cdot)x\rangle$. What is the geometry of states from the point of view of their GNS representations is a natural question. More specifically, given two states which are close in norm, can we choose GNS triples for them, which are also close in some sense? It is tricky to measure closeness of two triples. Bures ([3]) took the following approach.

Bures distance ([3]) of two states ϕ_1, ϕ_2 on $\mathcal{B}$, is defined as $\beta(\phi_1, \phi_2) = \inf\|x_1 - x_2\|$, where the infimum is taken over all GNS-triples with 'common' representation spaces: $(H, \pi, x_1), (H, \pi, x_2)$ of ϕ_1, ϕ_2. Here in two GNS triples, two of the components namely the Hilbert space and the representation are taken to be common, and the distance is measured only for the vectors. Perhaps, it would be equally natural to define another notion γ as, $\gamma(\phi_1, \phi_2) = \inf\|\pi_1 - \pi_2\|_{cb}$, where $\|\cdot\|_{cb}$ stands for completely bounded norm and the infimum is now taken over all GNS-triples with 'joint' representation spaces: $(H, \pi_1, x), (H, \pi_2, x)$ of ϕ_1, ϕ_2. In other words, the Hilbert space and vector are common and only the representations are different. We explore this notion here.

Before we go further, we remark that this circle of ideas have natural extensions from states to completely positive maps and it is convenient to start with such a general set up. The notion of Bures metric for completely positive maps was introduced at first for completely positive (CP) maps from a C^*-algebra $\mathcal{A}$ to $\mathcal{B}(G)$ for some Hilbert space G, by Kretschmann, Schlingemann and Werner in [10]. In the same paper extension of the notion to more general range C^*-algebras has been presented through an alternative somewhat indirect definition of the

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Bures distance. The reason for this is that, these authors use the Stinespring representation ([19]) for the initial definition, which in the usual formulation requires the range space to be the whole algebra $\mathcal{B}(G)$. This artificiality can be removed if one uses the theory of Hilbert C^*-modules. This has been carried out by Bhat and Sumesh [2]. Making use of basic ideas from Hilbert C^*-module theory, it is seen that β is indeed a metric when the range algebra is an injective algebra or a von Neumann algebra. A counter example is also presented in [2] that one may not even get a metric when the range algebra is a general C^*-algebra.

The notion of Bures metric has found many mathematical and physical applications ([1], [5], [9]). There has been some renewed interest in the subject due to applications in quantum information theory ([20], [21], [22]). The generalized version as distance for CP maps also has applications in this field [10]. Our interest in this topic stems from its usefulness in the study of generators of quantum dynamical semigroups [12].

Now the revised set up is as follows. Let ϕ_1, ϕ_2 be two completely positive maps from a unital C^*-algebra $\mathcal{A}$ to another unital C^*-algebra $\mathcal{B}$. The Stinespring's theorem, in Hilbert module language (see [14]), provide *Stinespring triples* $(\mathcal{E}_1, \sigma_1, x_1), (\mathcal{E}_2, \sigma_2, x_2)$, where $\mathcal{E}_i$ is a Hilbert $\mathcal{B}$ module with a left action σ_i of $\mathcal{A}$ on it and a vector $x_i \in \mathcal{E}_i$, such that

$$\phi_i(\cdot) = \langle x_i, \sigma_i(\cdot) x_i \rangle$$

for $i = 1, 2$. Then mimicking the definitions for states we have

$$\beta(\phi_1, \phi_2) = \inf \|x_1 - x_2\|,$$

where the infimum is taken over all 'common' representation modules $(\mathcal{E}, \sigma, x_1)$, $(\mathcal{E}, \sigma, x_2)$ of (ϕ_1, ϕ_2). Similarly, we can define,

$$\gamma(\phi_1, \phi_2) = \inf \|\sigma_1 - \sigma_2\|_{cb},$$

where the infimum is taken over all 'joint' representation modules $(\mathcal{E}, \sigma_1, x)$, $(\mathcal{E}, \sigma_2, x)$ of (ϕ_1, ϕ_2).

For lack of better name, we call γ as 'representation metric'. In this paper we study basic properties of $\gamma(\phi_1, \phi_2)$ and its relationship with $\beta(\phi_1, \phi_2)$. We restrict ourselves to unital completely positive maps. We show that γ is indeed a metric if the range algebra under consideration is a von Neumann algebra or an injective C^*-algebra, exactly like in the case of β. We discuss several properties of this metric, its upper bound and its relation to the full free product of C^*-algebras. This is a feature not seen for Bures metric. This association with free product allows us to interpret representation metric as a notion coming from 'joint distributions with given marginals', comparable to Wasserstein metric of probability measures. In a broad sense it is also somewhat like Gromov-Hausdorff distance for metric spaces. Then we address the attainability issue of representation metric, i.e., we show that given two unital completely positive maps, there is a joint representing module in which the representation metric attains its value.

In the main section (Section 6) we establish a very interesting direct relation of this metric with Bures metric. This may be considered as a new formula to compute Bures metric. We prove the result for states and then extend it to the case of injective C^*-algebras. Finally, in the last section we have examples to show that the range algebra does matter for computing the representation metric.

2. Notation and basics of Hilbert C^*-modules

Let $\mathcal{B}$ be a C^*-algebra. A complex vector space $\mathcal{E}$ is a Hilbert $\mathcal{B}$-module if it is a right $\mathcal{B}$-module with a $\mathcal{B}$-valued inner product, which is complete with respect to the associated norm (see [11], [14], [16] for basic theory). We denote the space of all bounded and adjointable maps between two Hilbert $\mathcal{B}$-modules $\mathcal{E}_1$ and $\mathcal{E}_2$ by $\mathcal{B}^a(\mathcal{E}_1,\mathcal{E}_2)$. In particular, if $\mathcal{E}_1=\mathcal{E}_2=\mathcal{E}$, then $\mathcal{B}^a(\mathcal{E},\mathcal{E})=\mathcal{B}^a(\mathcal{E})$, which forms a unital C^*-algebra with natural algebraic operations and operator norm.

Let $\pi:\mathcal{B}\to\mathcal{B}(G)$ be a non-degenerate (i.e., $\overline{\text{span}}\pi(\mathcal{B})G=G$) representation of $\mathcal{B}$ on a Hilbert space G. Given a Hilbert $\mathcal{B}$-module $\mathcal{E}$, we define the Hilbert space $H:=\mathcal{E}\odot G$ as the inner product space obtained from the algebraic tensor product $\mathcal{E}\otimes G$, with semi-inner product:

$$\langle x\otimes g, x'\otimes g'\rangle := \langle g,\pi(\langle x,x'\rangle)g'\rangle,\ \text{ for } x,x'\in\mathcal{E}; g,g'\in G;$$

after quotienting the space of null vectors, and completing. We denote the equivalence class containing $x\otimes g$ by $x\odot g$. To each $x\in\mathcal{E}$, we associate the linear map $L_x: g\mapsto x\odot g$ in $\mathcal{B}(G,H)$ with adjoint $L_x^*: y\odot g\mapsto \pi(\langle x,y\rangle)g$. Clearly $L_x^*L_y=\pi(\langle x,y\rangle)$ and $L_{xb}=L_x\pi(b)$ for all $x,y\in\mathcal{E}$, $b\in\mathcal{B}$. Also $\|L_x\|^2=\|\pi(\langle x,x\rangle)\|=\|x\|^2$. By identifying $\mathcal{B}$ with $\pi(\mathcal{B})$ and x with L_x, we may assume that $\mathcal{E}\subset\mathcal{B}(G,H)$. Note that $a\mapsto a\otimes id_G:\mathcal{B}^a(\mathcal{E})\to\mathcal{B}(H)$ is a unital $*$-homomorphism and hence an isometry. So we may consider $\mathcal{B}^a(\mathcal{E})\subset\mathcal{B}(H)$.

Suppose $\mathcal{A}$ is another C^*-algebra. A Hilbert $\mathcal{B}$-module $\mathcal{E}$ is said to be a Hilbert $\mathcal{A}-\mathcal{B}$-module if there exists a representation $\tau:\mathcal{A}\to\mathcal{B}^a(\mathcal{E})$ which is non-degenerate (equivalently, unital if $\mathcal{A}$ is unital). If $\mathcal{E}$ is a Hilbert $\mathcal{A}-\mathcal{B}$-module, and the left action τ is fixed and there is no possibility of confusion, we may take $\mathcal{A}\subset\mathcal{B}^a(\mathcal{E})$, by identifying $\tau(a)$ with a and thereby $\tau(a)x=ax$ for all $x\in\mathcal{E}, a\in\mathcal{A}$. Further, $\rho(a):=a\otimes id_G$, is a representation of $\mathcal{A}$ on H, mapping $x\otimes g$ to $\tau(a)x\otimes g=ax\otimes g$. Observe that $L_{ax}=\rho(a)L_x$. Also $\mathcal{B}(G,H)$ forms a Hilbert $\mathcal{A}-\mathcal{B}(G)$-module with left action $ax:=\rho(a)x$. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two Hilbert $\mathcal{A}-\mathcal{B}$-modules, then a linear map $\Phi:\mathcal{E}_1\to\mathcal{E}_2$ is said to be $\mathcal{A}-\mathcal{B}$-linear (or bilinear) if $\Phi(axb)=a\Phi(x)b$ for all $a\in\mathcal{A}, b\in\mathcal{B}, x\in\mathcal{E}$. The space of all bounded, adjointable and bilinear maps from $\mathcal{E}_1$ to $\mathcal{E}_2$ is denoted by $\mathcal{B}^{a,\text{bil}}(\mathcal{E}_1,\mathcal{E}_2)$. If $\mathcal{E}$ is a Hilbert $\mathcal{A}-\mathcal{B}$-module, then $\mathcal{B}^{a,\text{bil}}(\mathcal{E})$ is the relative commutant of the image of $\mathcal{A}$ in $\mathcal{B}^a(\mathcal{E})$.

Suppose $\mathcal{B}\subset\mathcal{B}(G)$ is a von Neumann algebra and $\mathcal{E}$ is a Hilbert $\mathcal{B}$-module. Then we say $\mathcal{E}$ is a von Neumann $\mathcal{B}$-module if $\mathcal{E}$ is strongly closed in $\mathcal{B}(G,H)\subset\mathcal{B}(G\oplus H)$ (Here by strongly closed we mean closure in strong operator topology (SOT)). Thus, if x is an element in the strong closure $\overline{\mathcal{E}}^s$ of a Hilbert $\mathcal{B}$-module $\mathcal{E}$, then there exists a net (x_α) in $\mathcal{E}$ such that $L_{x_\alpha}\xrightarrow{SOT}L_x$. All von Neumann $\mathcal{B}$-modules are self-dual (in the sense that all $\mathcal{B}$-valued functionals are given by a $\mathcal{B}$-valued inner product with a fixed element of the module), and hence they are complemented in all Hilbert $\mathcal{B}$-modules which contain it as a $\mathcal{B}$-submodule. In particular, strongly closed $\mathcal{B}$-submodules are complemented in a von Neumann

$\mathcal{B}$-module. If we think $\mathcal{B}^a(\mathcal{E}) \subset \mathcal{B}(H)$, then $\mathcal{B}^a(\mathcal{E})$ is a von Neumann algebra acting non-degenerately on the Hilbert space H. If $\mathcal{A}$ is a C^*-algebra, then by a von Neumann $\mathcal{A}-\mathcal{B}$-module we mean a von Neumann $\mathcal{B}$-module $\mathcal{E}$ with a non-degenerate representation $\tau : \mathcal{A} \to \mathcal{B}^a(\mathcal{E})$. In addition, if $\mathcal{A}$ is a von Neumann algebra and $a \to \langle x, ax\rangle : \mathcal{A} \to \mathcal{B}$ is a normal mapping for all $x \in \mathcal{E}$ (equivalently, the representation $\rho : \mathcal{A} \to \mathcal{B}(H)$ is normal), then we call $\mathcal{E}$ a two-sided von Neumann $\mathcal{A}-\mathcal{B}$-module. For more details see ([14], [16], [17]).

It is well-known that if $\phi : \mathcal{A} \to \mathcal{B}$ is a completely positive map between unital C^*-algebras, then there exists a Hilbert $\mathcal{A}-\mathcal{B}$-module $\mathcal{E}$ and $x \in \mathcal{E}$ such that $\phi(a) = \langle x, ax\rangle$ for all $a \in \mathcal{A}$. The construction of $\mathcal{E}$ is by starting with $\mathcal{A} \otimes \mathcal{B}$ and defining a $\mathcal{B}$-valued semi-inner product on it as $\langle a_1 \otimes b_1, a_2 \otimes b_2\rangle := b_1^*\phi(a_1^*a_2)b_2$, and usual quotienting and completion procedure (see [8], [13], [11], [15], [19]). The comparison with GNS for states is obvious. The pair $(\mathcal{E}, x)$ is called a GNS-construction for ϕ and $\mathcal{E}$ is called a GNS-module for ϕ. If further, $\overline{\text{span}}\mathcal{A}x\mathcal{B} = \mathcal{E}$, then $(\mathcal{E}, x)$ is said to be a minimal GNS-construction, and it is unique up to isomorphism. If $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras and ϕ is normal, then $(\mathcal{E}, x)$ can be chosen such that $\mathcal{E}$ is a (two-sided) von Neumann $\mathcal{A}-\mathcal{B}$-module. Here the closure for minimality is taken under strong operator topology.

Note that if $\mathcal{B} = \mathcal{B}(G)$, then $L_x^*\rho(a)L_x = \langle x, ax\rangle = \phi(a)$ for all $a \in \mathcal{A}$. Thus (H, ρ, L_x) is a Stinespring representation for the CP-map $\phi : \mathcal{A} \to \mathcal{B}(G)$.

Here we recall the following simple but important observation on Hilbert C^*-modules with unit vectors.

Proposition 2.1. *Let $\mathcal{E}$ be a Hilbert C^*-module on a unital C^*-algebra $\mathcal{B}$. Suppose $\mathcal{E}$ has a unit element x (that is, $\langle x, x\rangle = 1$), then the module xB is complemented in $\mathcal{E}$.*

Proof. Every element $y \in \mathcal{E}$ decomposes as

$$y = x.\langle x, y\rangle + [y - x.\langle x, y\rangle],$$

and it is easily seen that this is an orthogonal decomposition of $\mathcal{E}$.

□

This result is readily applicable to minimal Stinespring representation of unital completely positive maps, as the cyclic element there is a unit vector.

In the complex plane and more generally in any metric space X, if $x \in X$, and $B \subset X$, we take distance between x and B as $d(x, B) = \inf\{d(x, b) : b \in B\}$. The metric under consideration should be clear from the context.

3. The Representation metric

Suppose $\mathcal{A}, \mathcal{B}$ are unital C^*-algebras. Denote by $\mathrm{UCP}(\mathcal{A}, \mathcal{B})$, the set of all unital completely positive maps from $\mathcal{A}$ to $\mathcal{B}$. Let $\phi_1, \phi_2 \in \mathrm{UCP}(\mathcal{A}, \mathcal{B})$. The Bures distance between ϕ_1 and ϕ_2 is well known in the literature. See [3], [1], [10], [2] for more details. We wish to modify the setup slightly to get a new metric.

Definition 3.1. A Hilbert $\mathcal{A}-\mathcal{B}$-module $\mathcal{E}$ is said to be a common representation module for $\phi_1, \phi_2 \in \mathrm{UCP}(\mathcal{A}, \mathcal{B})$ if both of them can be represented in $\mathcal{E}$, that is, there exist $x_i \in \mathcal{E}$ such that $\phi_i(a) = \langle x_i, ax_i\rangle, i = 1,2$. Then the triple $(\mathcal{E}, x_1, x_2)$ is called a common representation tuple of (ϕ_1, ϕ_2).

Note that we are demanding no minimality for the common representation module. So we can always have such a module. For, if $(\mathcal{E}^i, x^i)$ is the minimal GNS-construction for ϕ_i, then take $\mathcal{E} = \mathcal{E}^1 \oplus \mathcal{E}^2$, $x_1 = x^1 \oplus 0$ and $x_2 = 0 \oplus x^2$. For a common representation module $\mathcal{E}$, define $S(\mathcal{E}, \phi_i)$ to be the set of all $x \in \mathcal{E}$ such that $\phi_i(a) = \langle x, ax\rangle$ for all $a \in \mathcal{A}$. It is to be remembered that if ϕ_1, ϕ_2 are states, then $\mathcal{E}$ is a Hilbert space and x_1, x_2 are unit vectors in it.

Definition 3.2. Let $\mathcal{E}$ be a common representation module for $\phi_1, \phi_2 \in \mathrm{UCP}(\mathcal{A}, \mathcal{B})$. Define

$$\beta_{\mathcal{E}}(\phi_1, \phi_2) = \inf\{\|x_1 - x_2\| : x_i \in S(\mathcal{E}, \phi_i), i = 1,2\}$$

and the Bures distance

$$\beta(\phi_1, \phi_2) = \inf_{\mathcal{E}} \ \beta_{\mathcal{E}}(\phi_1, \phi_2)$$

where the infimum is over all the common representation module $\mathcal{E}$.

Definition 3.3. A right $\mathcal{B}$ module $\mathcal{E}$ is said to be a joint representation module for ϕ_1 and ϕ_2 if there are two unital left $\mathcal{A}$ actions σ_1 and σ_2 and a unital vector $x \in \mathcal{E}$ such that

$$\phi_1(a) = \langle x, \sigma_1(a)x\rangle, \ \phi_2(a) = \langle x, \sigma_2(a)x\rangle.$$

Here $(\mathcal{E}, \sigma_1, \sigma_2, x)$ is called a joint representation tuple of (ϕ_1, ϕ_2).

Definition 3.4. Let $\phi_1, \phi_2 \in \mathrm{UCP}(\mathcal{A}, \mathcal{B})$. Let $\overline{\mathcal{E}} = (\mathcal{E}, \sigma_1, \sigma_2, x)$ be a joint representation tuple. Define

$$\gamma^{\overline{\mathcal{E}}}(\phi_1, \phi_2) = \|\sigma_1 - \sigma_2\|_{cb}.$$

Define

$$\gamma(\phi_1, \phi_2) := \inf_{(\mathcal{E}, \sigma_1, \sigma_2, x)} \|\sigma_1 - \sigma_2\|_{cb},$$

where the infimum is taken over all joint representation tuples.

We will informally call $\gamma(\phi_1, \phi_2)$ as the *representation metric* or *representation distance* between ϕ_1, ϕ_2. It will be seen that under good situations it is indeed a metric analogues to Bures metric. This notion can also be compared with Wasserstein metric.

Denote by $J(\phi_1, \phi_2)$ be the set of all joint representation tuples for ϕ_1 and ϕ_2. Define

$$\mathcal{A}_{\sigma_1,\sigma_2} x\mathcal{B} = \overline{\text{span}} \ \{\sigma_{\epsilon_1}(\mathcal{A})\sigma_{\epsilon_2}(\mathcal{A}) \cdots \sigma_{\epsilon_k}(\mathcal{A})x\mathcal{B} : \epsilon_i = 1 \text{ or } 2, k \geq 1\}.$$

Definition 3.5. A joint representation tuple $(\mathcal{E}, \sigma_1, \sigma_2, x)$ is said to be *minimal* if $\mathcal{A}_{\sigma_1,\sigma_2} x\mathcal{B} = \mathcal{E}$.

Remark 3.6. It suffices to consider minimal joint representation tuples in Definition 3.4.

Proposition 3.7. *Let $\phi_1, \phi_2 \in UCP(\mathcal{A}, \mathcal{B})$. Then $J(\phi_1, \phi_2)$ is non-empty.*

Proof. Let $(\mathcal{E}_i, \tau_i, x_i)$ be the minimal Stinespring $\mathcal{A}-\mathcal{B}$ bi-modules for ϕ_i, $i = 1, 2$. Take $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. Then $\mathcal{E}$ is an $\mathcal{A} - \mathcal{B}$ bi-module with left action $\sigma_1 = \tau_1 \oplus \tau_2$. By 2.1, we have orthogonal decompositions, $\mathcal{E}_i = x_i B \oplus \mathcal{E}_i^0$, for $i = 1, 2$. Hence $\mathcal{E} = x_1 B \oplus \mathcal{E}_1^0 \oplus x_2 B \mathcal{E}_2^0$. Define $U : \mathcal{E} \to \mathcal{E}$, by

$$U(x_1 b_1 \oplus y_1 \oplus x_2 b_2 \oplus y_2) = x_1 b_2 \oplus y_1 \oplus x_2 b_1 \oplus y_2 \ \forall b_1, b_2 \in \mathcal{B}, y_i \in \mathcal{E}_i^0, i = 1, 2.$$

It is easily seen that U is a right $\mathcal{B}$-linear unitary and $U(x_1 \oplus 0) = 0 \oplus x_2$ in $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. Define $\sigma_2 : \mathcal{A} \to \mathcal{B}^a(\mathcal{E})$ by $\sigma_2(a) = U^*\sigma_1(a)U$. Note that $\phi_1(a) = \langle x_1, \sigma_1(a)x_1\rangle$ and $\phi_2(a) = \langle x_1, \sigma_2(a)x_1\rangle$. It follows that $(\mathcal{E}, x_1 \oplus 0, \sigma_1, \sigma_2)$ is a joint representation tuple for ϕ_1, ϕ_2. □

Remark 3.8. Suppose $\mathcal{B}$ is a von Neumann algebra. Then it follows easily that

$$\gamma(\phi_1, \phi_2) = \inf_{\mathcal{E}} \gamma_{\mathcal{E}}(\phi_1, \phi_2),$$

where the infimum is over all joint representation module $\mathcal{E}$ which are right von Neumann $\mathcal{B}$ module. It is to be noted that $\mathcal{A}$ can be a general unital C^*-algebra and left actions are not assumed to be normal. So we do not need ϕ_1, ϕ_2 to be normal maps.

Theorem 3.9. *Suppose $\mathcal{B}$ is a von Neumann algebra. Then γ is a metric on $UCP(\mathcal{A}, \mathcal{B})$.*

Proof. It is evident that $\gamma(\phi, \phi) = 0$ and $\gamma(\phi_1, \phi_2) = \gamma(\phi_2, \phi_1)$. Note that given any joint representing module $\mathcal{E}$, $\|\sigma_1 - \sigma_2\|_{\mathcal{A}\to\mathcal{B}^a(\mathcal{E})} \geq \|\phi_1 - \phi_2\|$, which shows that $\gamma(\phi_1, \phi_2) \geq \|\phi_1 - \phi_2\|_{cb}$. So $\gamma(\phi_1, \phi_2) = 0$ implies $\phi_1 = \phi_2$. It remains to show the triangle inequality. Suppose $\phi_1, \phi_2, \phi_3 \in UCP(\mathcal{A}, \mathcal{B})$. Let $\epsilon > 0$ be given, find von Neumann modules $(\mathcal{E}_1, \sigma_1, \sigma_2, x_1) \in J(\phi_1, \phi_2)$ and $(\mathcal{E}_2, \sigma_2', \sigma_3', x_2) \in J(\phi_2, \phi_3)$ such that $\|\sigma_1 - \sigma_2\|_{cb} < \gamma(\phi_1, \phi_2) + \frac{\epsilon}{2}$ and $\|\sigma_2' - \sigma_3'\|_{cb} < \gamma(\phi_1, \phi_2) + \frac{\epsilon}{2}$. There exists bilinear unitary W from the von Neumann sub module $\mathcal{A}_{\sigma_2} x_1 \mathcal{B}$ to $\mathcal{A}_{\sigma_2'} x_2 \mathcal{B}$ given by $W(\sigma_2(a)x_1 b) = \sigma_2'(a)x_2 b$. Define right $\mathcal{B}$ modules:

$$\mathcal{E} = \mathcal{E}_1 \oplus (\mathcal{A}_{\sigma_2'} x_2 \mathcal{B})^\perp = (\mathcal{A}_{\sigma_2} x_1 \mathcal{B})^\perp \oplus \mathcal{A}_{\sigma_2} x_1 \mathcal{B} \oplus (\mathcal{A}_{\sigma_2'} x_2 \mathcal{B})^\perp,$$

$$\mathcal{E}' = (\mathcal{A}_{\sigma_2} x_1 \mathcal{B})^\perp \oplus \mathcal{E}_2 = (\mathcal{A}_{\sigma_2} x_1 \mathcal{B})^\perp \oplus \mathcal{A}_{\sigma_2'} x_2 \mathcal{B} \oplus (\mathcal{A}_{\sigma_2'} x_2 \mathcal{B})^\perp.$$

In these modules, with natural identifications, we have $\tilde{x}_1 := x_1 \oplus 0 = 0 \oplus x_1 \oplus 0$ and $\tilde{x}_2 = 0 \oplus x_2 = 0 \oplus x_2 \oplus 0$. Consider left actions defined as follows :

$$\tilde{\sigma}_1 := \sigma_1 \oplus \sigma_2', \ \tilde{\sigma}_2 := \sigma_2 \oplus \sigma_2' \text{ acting on } \mathcal{E},$$

$$\tilde{\tilde{\sigma}}_3 := \sigma_2 \oplus \sigma_3', \ \tilde{\tilde{\sigma}}_2 := \sigma_2 \oplus \sigma_2' \text{ acting on } \mathcal{E}'.$$

The unitary W extends to an adjointable (right B linear) unitary map $W' : \mathcal{E} \to \mathcal{E}'$ by defining $W' = I \oplus W \oplus I$. Observe that $\tilde{\sigma}_2(\cdot) = W'^*\tilde{\tilde{\sigma}}_2(\cdot)W'$. Consider left actions $\tilde{\sigma}_1(\cdot)$ and $\hat{\sigma}_3(\cdot) := W'^*\tilde{\tilde{\sigma}}_3(\cdot)W'$ on $\mathcal{E}$ together with $x_1 \in \mathcal{E}$. Note that

$$\langle \tilde{x}_1, \tilde{\sigma_1}(a)\tilde{x}_1\rangle = \langle x_1, \sigma_1(a)x_1\rangle = \phi_1(a)$$

and

$$\begin{aligned}\langle \tilde{x}_1, \hat{\sigma}_3(a)\tilde{x}_1\rangle &= \langle \tilde{x}_1, W'^*\tilde{\tilde{\sigma}}_3(a)W'\tilde{x}_1\rangle \\ &= \langle \tilde{x}_2, \tilde{\tilde{\sigma}}_3(a)\tilde{x}_2\rangle \\ &= \langle x_2, \sigma_3'(a)x_2\rangle \\ &= \phi_3(a).\end{aligned}$$

This shows that $(\mathcal{E}, \tilde{\sigma}_1, \hat{\sigma}_3, \tilde{x}_1)$ is a joint representation tuple for ϕ_1, ϕ_3. Note also $\|\sigma_1 - \sigma_2\|_{cb} = \|\tilde{\sigma}_1 - \tilde{\sigma}_2\|_{cb}$ and $\|\sigma_2' - \sigma_3'\|_{cb} = \|\tilde{\tilde{\sigma}}_2 - \tilde{\tilde{\sigma}}_3\|_{cb}$. Now

$$\begin{aligned}\|\tilde{\sigma}_1 - \hat{\sigma}_3\|_{cb} &= \|\tilde{\sigma}_1 - W'^*\tilde{\tilde{\sigma}}_3 W'\|_{cb} \\ &\leq \|\tilde{\sigma}_1 - \tilde{\sigma}_2\|_{cb} + \|\tilde{\sigma}_2 - W'^*\tilde{\tilde{\sigma}}_3 W'\|_{cb} \\ &= \|\sigma_1 - \sigma_2\|_{cb} + \|W'\tilde{\sigma}_2 W'^* - \tilde{\tilde{\sigma}}_3\|_{cb} \\ &= \|\sigma_1 - \sigma_2\|_{cb} + \|\tilde{\tilde{\sigma}}_2 - \tilde{\tilde{\sigma}}_3\|_{cb} \\ &= \|\sigma_1 - \sigma_2\|_{cb} + \|\sigma_2' - \sigma_3'\|_{cb} \\ &< \gamma(\phi_1, \phi_2) + \gamma(\phi_2, \phi_3) + \epsilon.\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we get $\gamma(\phi_1, \phi_3) \leq \gamma(\phi_1, \phi_2) + \gamma(\phi_2, \phi_3)$.

□

The following proposition says that representation metric is stable under taking ampliations.

Proposition 3.10. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{B}$ be a von Neumann algebra. Let $\phi, \psi \in UCP(\mathcal{A}, \mathcal{B})$. Then*

$$\gamma(\phi, \psi) = \gamma(\phi^{(n)}, \psi^{(n)})$$

where $\phi^{(n)}, \psi^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ are ampliations of ϕ and ψ respectively for $n \geq 1$.

Proof. Fix $n \geq 1$. Suppose $(\mathcal{E}, \sigma_\phi, \sigma_\psi, x)$ is a joint representation tuple for ϕ and ψ. Then $M_n(\mathcal{E})$ is an $M_n(\mathcal{A}) - M_n(\mathcal{B})$ bi-module. Denote $\mathbf{x} = \text{diag}\,(x, x, \cdots, x) \in M_n(\mathcal{E}.)$ Let $(\sigma_\phi)^{(n)}, (\sigma_\psi)^{(n)}$ be ampliations of σ_ϕ and σ_ψ respectively. Then

$$(M_n(\mathcal{E}), (\sigma_\phi)^{(n)}, (\sigma_\psi)^{(n)}, \mathbf{x})$$

is a joint representation tuple for $\phi^{(n)}$ and $\psi^{(n)}$. Note that

$$\|(\sigma_\phi)^{(n)} - (\sigma_\psi)^{(n)}\|_{cb} = \|\sigma_\phi - \sigma_\psi\|_{cb}.$$

As $(\mathcal{E}, \sigma_\phi, \sigma_\psi, x)$ is an arbitrary joint representation tuple for ϕ and ψ, we get $\gamma(\phi^{(n)}, \psi^{(n)}) \leq \gamma(\phi, \psi)$. Conversely suppose $(\mathcal{F}, \tau_1, \tau_2, y)$ is a joint representation tuple for $\phi^{(n)}$ and $\psi^{(n)}$. Let (e_{ij}) and (f_{ij}) $1 \leq i, j \leq n$ be the matrix units of $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$ respectively. Then $e_{11}\mathcal{F}f_{11}$ is an $\mathcal{A} - \mathcal{B}$ bi-module with actions $\sigma_i(a)xb = \tau_i(ae_{11})xb$, $i = 1, 2$. Let $x = e_{11}yf_{11}$. Then $(e_{11}\mathcal{F}f_{11}, \sigma_1, \sigma_2, x)$ is a joint representation tuple for ϕ and ψ. Note that

$$\|\sigma_1 - \sigma_2\|_{cb} \leq \|\tau_1 - \tau_2\|_{cb}.$$

As $(\mathcal{F}, \tau_1, \tau_2, y)$ is an arbitrary joint representation tuple for $\phi^{(n)}$ and $\psi^{(n)}$, we get $\gamma(\phi, \psi) \leq \gamma(\phi^{(n)}, \psi^{(n)})$.

□

4. Relation to free products

Suppose $\mathcal{C}, \mathcal{D}$ are two unital C^*-algebras. Denote by $\mathcal{C} \circ \mathcal{D}$ the unital $*$-algebra of all finite linear combinations of all possible finite words consists of elements of $\mathcal{C}$ and $\mathcal{D}$. Define a norm on this algebra by

$$\|c\| = \sup \{\|\pi(c)\| : \pi \text{ is a } *\text{-representation of } \mathcal{C} \circ \mathcal{D} \text{ on some Hilbert space } H\}.$$

This is a C^* norm. Completion of $\mathcal{C} \circ \mathcal{D}$ under this norm is called the full free product of $\mathcal{C}$ and $\mathcal{D}$ and is denoted by $\mathcal{C} * \mathcal{D}$.

We have canonical injections $\rho_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} * \mathcal{D}$, $\rho_{\mathcal{D}} : \mathcal{D} \to \mathcal{C} * \mathcal{D}$. This way, $\mathcal{C}, \mathcal{D}$ are considered as sub-algebras of $\mathcal{C} * \mathcal{D}$. Any $*$-representation of $\mathcal{C} * \mathcal{D}$ on a Hilbert space H restricts to a pair of $*$-representations of $\mathcal{C}, \mathcal{D}$. Conversely any pairs of $*$-representations of $\mathcal{C}$ and $\mathcal{D}$ on a common Hilbert space H can be extended to a representation of $\mathcal{C} * \mathcal{D}$. This follows from the universal property of the full free product. Thus there is a 1-1 correspondence between the $*$-representations of $\mathcal{C} * \mathcal{D}$ and pairs of $*$-representations of $\mathcal{C}$ and $\mathcal{D}$ on a common Hilbert space H.

Let $\mathcal{A} * \mathcal{A}$ be the free product of $\mathcal{A}$ with itself. Let ρ_1, ρ_2 be the canonical injections. Denote by

$$K(\phi_1, \phi_2) = \{\phi : \mathcal{A} * \mathcal{A} \to \mathcal{B}, \phi \text{ is a CP map}, \phi \circ \rho_1 = \phi_1, \phi \circ \rho_2 = \phi_2\}.$$

A CP map in $K(\phi_1, \phi_2)$ is like a bivariate distribution with given marginals. This shows that the metric γ is somewhat like the Wasserstein metric for probability measures.

Remark 4.1. There is a 1-1 correspondence between the set of all Hilbert C^* right $\mathcal{B}$ module $\mathcal{E}$ with left actions σ_1, σ_2 and the set of all $\mathcal{A} * \mathcal{A} - \mathcal{B}$ bi-modules $(\mathcal{E}, \sigma)$. Indeed, for an $\mathcal{A} * \mathcal{A} - \mathcal{B}$ bi-module $\mathcal{E}$ letting $\sigma_i = \sigma \circ \rho_i$, $i = 1, 2$, we may endow $\mathcal{E}$ with two left actions σ_1, σ_2. Conversely given a module $(\mathcal{E}, \sigma_1, \sigma_2)$, the universal property defines $\sigma : \mathcal{A} * \mathcal{A} \to \mathcal{B}^a(\mathcal{E})$ given by $\sigma \circ \rho_i = \sigma_i, i = 1, 2$. By virtue of the above fact, every joint representation module $(\mathcal{E}, \sigma_1, \sigma_2, x)$ corresponds uniquely to an $\mathcal{A} * \mathcal{A} - \mathcal{B}$ bi-module $(\mathcal{E}, x)$. Also the joint representation module is minimal if and only if $\overline{\mathcal{A} * \mathcal{A} x \mathcal{B}} = \mathcal{E}$.

Theorem 4.2. *There is a 1-1 correspondence Φ between the set of minimal-$J(\phi_1, \phi_2)$ modulo isomorphism and the set $K(\phi_1, \phi_2)$.*

Proof. Now suppose $(\mathcal{E}, \sigma_1, \sigma_2, x)$ is a minimal joint representation module for ϕ_1 and ϕ_2. By Remark 4.1, we may consider $(\mathcal{E}, x)$ an $(\mathcal{A} * \mathcal{A}) - \mathcal{B}$ bi-module with left action σ (say). We associate a completely positive map $\Phi(\mathcal{E}) := \phi \in K(\phi_1, \phi_2)$ by $\phi(c) = \langle x, \sigma(c)x \rangle$. Conversely every element in $K(\phi_1, \phi_2)$ under minimal Stinespring dilation gives rise to the minimal $\mathcal{A} * \mathcal{A} - \mathcal{B}$ bi-module $\mathcal{E}$ and a unital vector $x \in \mathcal{E}$. So by Remark 4.1, we get a minimal joint representation module $\Phi^{-1}(\phi)$. From the uniqueness of minimal dilation, it follows that this two operations are inverse to each other. Indeed, given $(\mathcal{E}, \sigma, x)$ a $\mathcal{A} * \mathcal{A} - \mathcal{B}$ bi-module, consider $\phi = \Phi(\mathcal{E})$. Let $\mathcal{E}'$ be its minimal Stinespring bi-module. Define $\Psi : \mathcal{E}' \to \mathcal{E}$ by $\Psi(c \overline{\otimes} b) = cxb$, $c \in \mathcal{A} * \mathcal{A}$, $b \in \mathcal{B}$. By definition Ψ is an bi-module isometry and from the minimality, it follows that Ψ is onto. Therefore Ψ is an bi-linear unitary. Hence $\Phi^{-1}(\Phi(\mathcal{E})) \simeq \mathcal{E}$. Other part is trivial.

$\square$

From the Remark 3.6, it is enough to consider the minimal Stinespring dilation of $\phi \in K(\phi_1, \phi_2)$. Let $(\mathcal{E}, x)$ be the minimal Stinespring module of ϕ. i.e. $\phi(c) = \langle x, cx\rangle$. Note that $\mathcal{B}^a(\mathcal{E})$ is a C^*-algebra. Let ρ_1, ρ_2 be canonical injections from $\mathcal{A}$ to $\mathcal{A}*\mathcal{A}$. Then the left action σ of $\mathcal{E}$ induces homomorphisms $\sigma_i = \sigma \circ \rho_i : \mathcal{A} \to \mathcal{B}^a(\mathcal{E}), i = 1, 2$. Then the definition of representation metric can be reformulated as

$$\gamma(\phi_1, \phi_2) = \inf_{\phi\in K(\phi_1,\phi_2)} \{\|\sigma_1-\sigma_2\|^{\mathcal{E}}_{cb} : (\mathcal{E}, x) \text{ is the minimal Stinespring dilation of } \phi\}.$$

As an application of these ideas we get the following result.

Proposition 4.3. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be unital C^*-algebras. Let $\phi_1, \phi_2 \in UCP(\mathcal{A}, \mathcal{B})$ and $\psi \in UCP(\mathcal{B}, \mathcal{C})$. Then $\gamma(\psi \circ \phi_1, \psi \circ \phi_2) \leq \gamma(\phi_1, \phi_2)$.*

Proof. Given $\phi \in K(\phi_1, \phi_2)$, observe that $\psi \circ \phi \in K(\psi \circ \phi_1, \psi \circ \phi_2)$. Let $(\mathcal{E}_\phi, x)$ be a $\mathcal{A} * \mathcal{A} - \mathcal{B}$ bi-module. Let $\rho_1, \rho_2 : \mathcal{A} \to \mathcal{B}^a(\mathcal{E}_\phi)$ the canonical maps so that $\langle x, \rho_i(a)x\rangle = \phi_i, i = 1, 2$. Consider the completely positive map $\tilde{\psi} : \mathcal{B}^a(\mathcal{E}) \to \mathcal{C}$ by $\tilde{\psi}(A) = \psi(\langle x, Ax\rangle)$. Let $(\mathcal{E}_{\tilde{\psi}}, y)$ be a $\mathcal{B}^a(\mathcal{E})-\mathcal{C}$ Stinespring bi-module for $\tilde{\psi}$. Denote by $\tilde{\pi}$ be its corresponding left action. We have $\langle y, \tilde{\pi}(\rho_i(a))y\rangle = \psi \circ \phi_i$, $i = 1, 2$. We get $(\mathcal{E}_{\tilde{\psi}}, y, \tilde{\pi} \circ \rho_1, \tilde{\pi} \circ \rho_2)$ is a joint representation tuple for $(\psi \circ \phi_1, \psi \circ \phi_2)$. Therefore $\gamma(\psi\circ\phi_1, \psi\circ\phi_2) \leq \|\tilde{\pi} \circ \rho_1 - \tilde{\pi} \circ \rho_2\| \leq \|\rho_1 - \rho_2\|^{\mathcal{E}_\phi}$. Taking infimum over $\phi \in K(\phi_1, \phi_2)$, the result follows.

□

Corollary 4.4. *Let $\mathcal{A}$ be a C^*-algebra and $\iota : \mathcal{B} \to \mathcal{B}(H)$ be an injective C^*-algebra. Let $\tilde{\phi}_i = \iota \circ \phi_i$, $i = 1, 2$. Then*

$$\gamma(\tilde{\phi}_1, \tilde{\phi}_2) = \gamma(\phi_1, \phi_2).$$

Proof. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}$ be a conditional expectation map. Note that $\phi_i = \Phi\circ\tilde{\phi}_i$, $i = 1, 2$. Now from Proposition 4.3, we get

$$\begin{aligned} \gamma(\phi_1, \phi_2) &= \gamma(\Phi \circ \tilde{\phi}_1, \Phi \circ \tilde{\phi}_2) \\ &\leq \gamma(\tilde{\phi}_1, \tilde{\phi}_2) \\ &= \gamma(\iota \circ \phi_1, \iota \circ \phi_2) \\ &\leq \gamma(\phi_1, \phi_2). \end{aligned}$$

□

5. Attainability of the metric

In this section, we will address the attainability issue of the representation metric. Suppose ϕ_1, ϕ_2 are two unital CP maps from a C^*-algebra $\mathcal{A}$ to a von Neumann algebra $\mathcal{B}$. Suppose $\mathcal{B}$ is faithfully embedded in $\mathcal{B}(G)$ for some Hilbert space G. From Proposition 3.7, Lemma 4.2, we see $K(\phi_1, \phi_2)$ is non-empty. Our first observation is that the space $K(\phi_1, \phi_2)$ is compact under suitable topology.

Let $\mathcal{C}$ be the C^*-algebra $\mathcal{A} * \mathcal{A}$. Fix $r > 0$. Let us recall BW (bounded weak) topology on $CP_r(\mathcal{C}, \mathcal{B}(G)) = \{\phi : \mathcal{C} \to \mathcal{B}(G) \text{ is CP and } \|\phi\| \leq r\}$. A net $\phi_\alpha \to \phi$ in BW topology if for every $c \in \mathcal{C}, \xi, \mu \in \mathcal{G}$ $\langle \xi, (\phi_\alpha(c) - \phi(c))\mu\rangle \to 0$. It is to be noted that $CP_r(C, \mathcal{B}(G))$ is compact with respect to BW topology. As $\mathcal{B}$ is a von

Neumann algebra, it follows that $CP_r(C, \mathcal{B}) = \{\phi : \mathcal{C} \to \mathcal{B} \text{ is CP and } \|\phi\| \leq r\}$ is a closed subset of $CP_r(C, \mathcal{B}(G))$ in BW topology and hence compact. Consequently $K(\phi_1, \phi_2)$ is also compact under BW topology.

Consider $\phi \in K(\phi_1, \phi_2)$. Let $(\mathcal{E}, x)$ be its minimal Stinespring dilation. Then $\mathcal{E}$ is an $\mathcal{A} * \mathcal{A} - \mathcal{B}$ bi-module and also it is a von Neumann right B module. Note that $\mathcal{B}^a(\mathcal{E})$ is a von Neumann subalgebra of $\mathcal{B}(H)$, where $H := \mathcal{E} \circ G$. Let $\sigma : \mathcal{A} * \mathcal{A} \to \mathcal{B}^a(\mathcal{E})$ be the unital left action and $\rho_1, \rho_2 : \mathcal{A} \to \mathcal{A} * \mathcal{A}$ be the canonical injections. Suppose $\sigma_i = \sigma \circ \rho_i$, $i = 1, 2$. For notational simplicity we are suppressing the dependence of σ_1, σ_2 on ϕ. However, we will denote the completely bounded norm of $\sigma_1 - \sigma_2$, by $\|\sigma_1 - \sigma_2\|_{cb}^{\phi}$. Recall that

$$\gamma(\phi_1, \phi_2) = \inf_{\phi} \|\sigma_1 - \sigma_2\|_{cb}^{\phi}.$$

Hence we need to study the behaviour of the map $\phi \mapsto \|\sigma_1 - \sigma_2\|_{cb}^{\phi}$ under BW topology. As σ_1, σ_2 are $*$-homomorphisms, $\|\sigma_1 - \sigma_2\|_{cb}^{\phi} \leq 2$. From the definition of norm,

$$\begin{aligned} & \|\sigma_1 - \sigma_2\| \\ = & \sup_{\|a\| \leq 1,\ a \in \mathcal{A}} \|(\sigma_1(a) - \sigma_2(a))^*(\sigma_1(a) - \sigma_2(a))\|^{\frac{1}{2}} \\ = & \sup_{a \in \mathcal{A}, \|a\| \leq 1} \ \sup_{\eta \in \mathcal{E} \circ G, \|\eta\| \leq 1} [\langle \eta, [\sigma_1(a^*a) + \sigma_2(a^*a) - 2Re(\sigma_1(a^*)\sigma_2(a))]\eta \rangle]^{\frac{1}{2}}. \end{aligned}$$

By minimality of the Stinespring dilation, $\mathcal{E} \circ G = \overline{\text{span}}\{\sigma(c)xb \circ g : c \in \mathcal{C}, b \in \mathcal{B}, g \in \mathcal{G}\}$. Hence vectors of the form $\eta = \sum_{i=1}^{k} \sigma(c_i)xb_i \circ g_i$ is dense in H. Now,

$$\|\eta\|^2 = \sum_{i,j} \langle b_j g_j, \phi(c_j^* c_i) b_i g_i \rangle$$

and

$$\begin{aligned} & \langle \eta, [\sigma_1(a^*a) + \sigma_2(a^*a) - 2Re(\sigma_1(a^*)\sigma_2(a))]\eta \rangle \\ = & \sum_{i,j} \langle (b_j g_j), \phi(c_j^*(\rho_1(a^*a) + \rho_2(a^*a) - 2Re(\rho_1(a^*)\rho_2(a)) c_i)(b_i g_i) \rangle \end{aligned}$$

Denote by

$$\tilde{c} = (c_1, c_2, \cdots, c_k),\ \tilde{b} = (b_1, b_2, \cdots, b_k),\ \tilde{g} = (g_1, g_2, \cdots, g_k).$$

Define

$$\begin{aligned} & f(\phi, k, a, \tilde{c}, \tilde{b}, \tilde{g}) \\ = & \frac{[\sum_{i,j}^{k} \langle (b_j g_j), \phi(c_j^*(\rho_1(a^*a) + \rho_2(a^*a) - 2Re(\rho_1(a^*)\rho_2(a))) c_i)(b_i g_i) \rangle]^{\frac{1}{2}}}{[\sum_{i,j}^{k} \langle b_j g_j, \phi(c_j^* c_i) b_i g_i \rangle]^{\frac{1}{2}}}. \end{aligned}$$

Note that numerator vanishes if denominator vanishes, and in such a case this ratio is defined to be 0. Observe that $\phi \mapsto f(\phi, k, a, \tilde{c}, \tilde{b}, \tilde{g})$ is continuous in BW topology, when other variables are kept fixed. Also note that $f(k, a, \tilde{c}, \tilde{b}, \tilde{g})$ is bounded by $2\|a\|$. Therefore

$$\|\sigma_1 - \sigma_2\|^{\phi} = \sup_{k\in\mathbb{N},\|a\|\leq 1,\tilde{c},\tilde{b},\tilde{g}} f(\phi, k, a, \tilde{c}, \tilde{b}, \tilde{g}).$$

In order to compute the completely bounded norm of $\sigma_1 - \sigma_2$, we need to consider, $M_n(\mathcal{A})$, $\hat{\eta} = (\eta_1, \eta_2, \cdots, \eta_n) \in \mathcal{E}\circ G \oplus \cdots \oplus \mathcal{E}\circ G$ (n times) and ϕ to be replaced by $\phi^{(n)}$ (ampliation of ϕ.) It follows that

$$\|\sigma_1 - \sigma_2\|_{cb}^{\mathcal{E}_\phi} = \sup_{\|(a_{ij})\|\leq 1,n,k,\tilde{c}_i,\tilde{b}_i,\tilde{g}_i,1\leq i\leq n} F(\phi, n, k, (a_{ij}), (\tilde{c}_i), (\tilde{b}_i), (\tilde{g}_i))$$

where $\tilde{c}_i = (c_{i1}, c_{i2}, \cdots, c_{ik})$, $\tilde{b}_i = (b_{i1}, b_{i2}, \cdots, b_{ik})$, $\tilde{g}_i = (g_{i1}, g_{i2}, \cdots, g_{ik})$. Then $\eta_i = \sum_{j=1}^{k} c_{ij}xb_{ij}\circ g_{ij}$, $i = 1, 2, \cdots, n$, and

$$F(\phi, n, k, (a_{ij}), (\tilde{c}_i), (\tilde{b}_i), (\tilde{g}_i)) = \frac{[\sum_{i=1,j=1,l=1}^{n}\sum_{r=1,r'=1}^{k} A_{ijlrr'}]^{\frac{1}{2}}}{[\sum_{i=1}^{n}\sum_{r=1,r'=1}^{k}\langle b_{ir}g_{ir}, \phi(c_{ir}^*c_{ir'})b_{ir'}g_{ir'}\rangle]^{\frac{1}{2}}},$$

where

$$A_{ijlrr'} = \langle b_{ir}g_{ir}, \phi(c_{ir}^*(\sigma_1(a_{il}^*a_{lj}) + \sigma_2(a_{il}^*a_{lj}) - 2Re\sigma_1(a_{il}^*)\sigma_2(a_{lj}))c_{jr'})b_{jr'}g_{jr'}\rangle.$$

Once again it is easy to see that $\phi \to F(\phi, n, k, (a_{ij}), (\tilde{c}_i), (\tilde{b}_i), (\tilde{g}_i))$ is BW continuous when other quantities are kept fixed. Now we are ready to prove the following lemma.

Lemma 5.1. *Suppose $\{\phi_\alpha\}$ is a net of completely positive maps in $K(\phi_1, \phi_2)$ converging to a CP map ϕ in $K(\phi_1, \phi_2)$ in BW topology. Then*

$$\liminf_{\alpha} \|\sigma_1 - \sigma_2\|_{cb}^{\phi_\alpha} \geq \|\sigma_1 - \sigma_2\|_{cb}^{\phi}.$$

Proof. The following simple observation is used: Let $f(a, b)$ be a real valued function on two variables a, b. Then $\inf_a \sup_b f(a, b) \geq \sup_b \inf_a f(a, b)$. Now

$$\begin{aligned}
& \liminf_{\alpha} \|\sigma_1 - \sigma_2\|_{cb}^{\phi_\alpha} \\
= & \liminf_{\alpha} \sup_{\|(a_{ij})\|\leq 1,n,k,\tilde{c}_i,\tilde{b}_i,\tilde{g}_i,1\leq i\leq n} F(\phi_\alpha, n, k, (a_{ij}), (\tilde{c}_i), (\tilde{b}_i), (\tilde{g}_i)) \\
\geq & \sup_{\|(a_{ij})\|\leq 1,n,k,\tilde{c}_i,\tilde{b}_i,\tilde{g}_i,1\leq i\leq n} \lim_{\alpha} F(\phi_\alpha, n, k, (a_{ij}), (\tilde{c}_i), (\tilde{b}_i), (\tilde{g}_i)) \\
= & \sup_{\|(a_{ij})\|\leq 1,n,k,\tilde{c}_i,\tilde{b}_i,\tilde{g}_i,1\leq i\leq n} F(\phi, n, k, (a_{ij}), (\tilde{c}_i), (\tilde{b}_i), (\tilde{g}_i)) \\
= & \|\sigma_1 - \sigma_2\|_{cb}^{\phi}.
\end{aligned}$$

□

Theorem 5.2. *There is a $\phi \in K(\phi_1, \phi_2)$ for which the infimum is attained for $\gamma(\phi_1, \phi_2)$, that is,*

$$\gamma(\phi_1, \phi_2) = \|\sigma_1 - \sigma_2\|_{cb}^{\phi}.$$

Proof. This follows from the compactness of $K(\phi_1,\phi_2)$ in BW topology and the previous Lemma. The definition of $\gamma(\phi_1,\phi_2)$ will give a sequence of unital CP maps $\phi_n \in CP(\phi_1,\phi_2)$ such that $\gamma(\phi_1,\phi_2)=\lim_n \ \|\sigma_1-\sigma_2\|_{cb}^{\phi_n}$. From compactness, we may find a subnet ϕ_α converging to ϕ in BW topology. Note that $\gamma(\phi_1,\phi_2)=\lim \ \|\sigma_1-\sigma_2\|_{cb}^{\phi_\alpha}$. From the Lemma we get $\lim \ \|\sigma_1-\sigma_2\|_{cb}^{\phi_\alpha} \geq \|\sigma_1-\sigma_2\|_{cb}^{\phi}$. This implies that $\gamma(\phi_1,\phi_2)=\|\sigma_1-\sigma_2\|_{cb}^{\phi}$.

□

6. Relationship of representation metric with Bures metric

Suppose ϕ_1,ϕ_2 are two states on a C^*-algebra $\mathcal{A}$. Then we wish to show

$$\beta^2(\phi_1,\phi_2)=2-\sqrt{4-\gamma^2(\phi_1,\phi_2)}. \tag{6.1}$$

Here for notational convenience we write $\beta^2(\phi_1,\phi_2)$ instead of $[\beta(\phi_1,\phi_2)]^2$, with similar notation for γ. Actually, what we are going to prove is, (Theorem 6.3):

$$\gamma(\phi_1,\phi_2)=\beta(\phi_1,\phi_2)\sqrt{4-\beta^2(\phi_1,\phi_2)},$$

and we get $\beta^2(\phi_1,\phi_2)=2\pm\sqrt{4-\gamma^2(\phi_1,\phi_2)}$ and only the negative sign is permissible, as $0\leq\beta(\phi_1,\phi_2),\gamma(\phi_1,\phi_2)\leq 2$ is trivially true for unital CP maps.

It is to be recalled that when we are dealing with states, the Hilbert C^*-modules are just Hilbert spaces along with representations. Here it is convenient to simplify the notation. If $(H,\pi,x_1),(H,\pi,x_2)$ form common representation for two states ϕ_1,ϕ_2, we would simply say (H,π,x_1,x_2) is a common representation. We will also take $s(\pi,\phi_1)=\{x:\phi(\cdot)=\langle x,\pi(\cdot)x\rangle\}$. To begin with we obtain some lower and upper bounds of representation metric for **states** on C^*-algebras.

Lemma 6.1. *Let x,y be unit vectors in a Hilbert space K. For a unitary U in K, denote by Ad_U the automorphism $X\mapsto UXU^*$, on $\mathcal{B}(K)$. Then*

$$\begin{aligned}\inf_{U:Ux=y}\ \|id-Ad_U\|_{cb} &= \inf_{U:Ux=y}\ \|id-Ad_U\| \\ &= 2\inf_{U:Ux=y}\ d(U,\mathbb{C}) \\ &= 2\sqrt{1-|\langle x,y\rangle|^2}.\end{aligned}$$

Moreover, the infimum is attained.

Proof. For any unitary U on K, from (Stampfli [18]), we see $\|id-Ad_U\|=2d(\mathbb{C}I,U)$. For $n\in\mathbb{N}$, denoting $\tilde{U}=U\oplus\cdots\oplus U$, on $K^n=K\oplus\cdots\oplus K$, ($n-$copies), we see $d(\mathbb{C}I,\tilde{U})=d(\mathbb{C}I,U)$ and hence $\|id-Ad_U\|_{cb}=2d(\mathbb{C}I,U)$. Now if U is a unitary such that $Ux=y$, for any $\lambda\in\mathbb{C}$,

$$\|(U-\lambda)\|^2\geq\|(U-\lambda I)x\|^2=\|y-\lambda x\|^2\geq 1-|\langle x,y\rangle|^2,$$

where the last inequality follows as x,y are unit vectors and

$$|\lambda|^2-\lambda\langle y,x\rangle-\bar{\lambda}\langle x,y\rangle+|\langle x,y\rangle|^2=|(\lambda-\langle x,y\rangle)|^2\geq 0.$$

By considering a unitary U, satisfying $Ux=y$, and $Uv=v$ on $\{x,y\}^\perp$, it is easily seen that the infimum in $\inf_{U:Ux=y}\ d(U,\mathbb{C})$ is attained and equals $2\sqrt{1-|\langle x,y\rangle|^2}$.

□

As an immediate consequence we get the following bounds for states on C^*-algebras.

Theorem 6.2. *Let ϕ_1, ϕ_2 be two states on some C^*-algebra $\mathcal{A}$. Then*

$$\|\phi_1 - \phi_2\|_{cb} \leq \gamma(\phi_1, \phi_2) \leq 2\sqrt{\|\phi_1 - \phi_2\|_{cb}}$$

Proof. The lower bound is clear from the definition of γ. From Proposition 1.6, [3], we know that there is a common representing space in which Bures distance is attained. Let (π, K, x, y) be the common representation for which the Bures' distance is attained. Consider $\pi_1 = \pi$ and $\pi_2 = U^*\pi U$, where U is a unitary on K such that $Ux = y$. Then (π_1, π_2, x) is a joint representations of (ϕ_1, ϕ_2). We get

$$\begin{aligned}
\gamma(\phi_1, \phi_2) &\leq \inf_{U:Ux=y} \|\pi(\cdot) - U^*\pi(\cdot)U\|_{cb} \\
&\leq \|id_{\mathcal{B}(K)} - U^* id_{\mathcal{B}(K)} U\|_{cb} \\
&= 2\sqrt{1 - |\langle x, y\rangle|^2} \\
&= \beta(\phi_1, \phi_2)\sqrt{4 - \beta^2(\phi_1, \phi_2)} \\
&\leq 2\beta(\phi_1, \phi_2) \\
&\leq 2\sqrt{\|\phi_1 - \phi_2\|_{cb}},
\end{aligned}$$

where the last inequality is from [3] and [10]. $\square$

Now we come to our main theorem on relationship between β and γ.

Theorem 6.3. *Suppose ϕ_1, ϕ_2 are two states on a C^*-algebra $\mathcal{A}$. Then*

$$\beta^2(\phi_1, \phi_2) = 2 - \sqrt{4 - \gamma^2(\phi_1, \phi_2)}.$$

The key to the proof of Theorem 6.3 is the following Lemma.

Lemma 6.4. *Let W be a unitary on K such that $Wx = y$ with x, y unit vector in K. Let P be a positive operator on K. Then*

$$\|W - P\| \geq \sqrt{1 - [Re\langle x, y\rangle]^2}.$$

Proof. Let $\lambda \in \sigma(W)$. As W is normal for $\epsilon > 0$, there exists unit vector $v_\epsilon \in \mathcal{H}$ such that

$$|\langle v_\epsilon, W v_\epsilon\rangle - \lambda| < \epsilon.$$

Moreover as P is positive, $\langle v_\epsilon, P v_\epsilon\rangle \in \mathbb{R}_+$. Observe that as $\lambda \in \sigma(W)$,

$$d(\lambda, \mathbb{R}_+) = \begin{cases} 1 & \text{if} \quad Re(\lambda) \leq 0; \\ Im(\lambda) & \text{if} \quad Re(\lambda) > 0. \end{cases}$$

Hence if there exists $\lambda \in \sigma(W)$ with $Re(\lambda) \leq 0$, we get

$$\|W - P\| \geq |\langle v_\epsilon, (W - P)v_\epsilon\rangle| \geq (1 - \epsilon),$$

for every $\epsilon > 0$. That is, $\|W - P\| \geq 1$. Then the result follows trivially as $\sqrt{1 - [\mathrm{Re}\langle x, y\rangle]^2} \leq 1$.

So we may assume $Re(\lambda) > 0$ for every $\lambda \in \sigma(W)$. Now $\langle x, y\rangle$ is in the numerical range of unitary W, it is in the convex hull of $\sigma(W)$. Consequently $Re(\langle x, y\rangle) \geq 0$ and there exists λ in $\sigma(W)$ such that $0 \leq Re(\lambda) \leq Re(\langle x, y\rangle)$, or

$Im(\lambda) \geq \sqrt{1 - Re(\langle x, y\rangle)^2}$. For $\epsilon > 0$, choose v_ϵ as before. Therefore $\|W - P\| \geq |\langle v_\epsilon, (W - P)v_\epsilon\rangle| \geq d(\lambda, \mathbb{R}_+) - \epsilon = Im(\lambda) - \epsilon$. As $\epsilon > 0$ is arbitrary this completes the proof.

□

We also need the following well-known theorem.

Theorem 6.5. *(Johnson [7]) Suppose π is a faithful representation of a C^*-algebra $\mathcal{A}$ on K and U is a unitary on K. Then $\|\pi - Ad_U \circ \pi\|_{cb} = 2d(U, \pi(\mathcal{A})')$.*

Proof. Making use of Kaplansky density theorem, we may replace the C^*-algebra $\pi(\mathcal{A})$ by the von Neumann algebra generated by it. Now the result follows from Theorem 7 of [7].

□

Lemma 6.6. *Suppose π is a faithful representation of a C^*-algebra $\mathcal{A}$ on K and U is a unitary on K. Then there exists $X \in \pi(\mathcal{A})'$ such that $d(U, \pi(\mathcal{A})') = \|U - X\|$.*

Proof. This is an application of the fact that inf-sup is greater than sup-inf. Indeed, from the definition of infimum, there is a sequence $\{X_n\}_{n\geq 1}$ in $\pi(\mathcal{A})'$ such that $\|U - X_n\| \leq d(U, \pi(\mathcal{A})') + \frac{1}{n}$. Observe that as $I \in \pi(\mathcal{A})'$, trivially $d(U, \pi(\mathcal{A})') \leq 2$. Consequently $\|X_n\| \leq \|U - X_n\| + \|U\| \leq 2 + \frac{1}{n} + 1 \leq 3$. So $\{X_n\}_{n\geq 1}$ is a norm bounded sequence. Hence it has a WOT convergent subnet converging to some X (say). Clearly $X \in \pi(\mathcal{A})'$ as $\pi(\mathcal{A})'$ is WOT closed. Now

$$\|U - X\| = \sup_{\|z\|\leq 1, \|w\|\leq 1} |\langle z, (U - X)w\rangle|.$$

Hence for $\epsilon > 0$, there exist $z, w \in K$, $\|z\|, \|w\| \leq 1$, such that $\|U - X\| < |\langle z, (U - X)w\rangle| + \epsilon$. Then by WOT convergence, we get $n \geq 1$, such that $|\langle z, (X_n - X)w\rangle| < \epsilon$ and $\|(U - X_n)\| < d(U, \pi(\mathcal{A})') + \epsilon$. Combining all three inequalities, we have $\|U - X\| < |\langle z, (U - X_n)w\rangle| + |\langle z, (X_n - X)w\rangle| + \epsilon \leq d(U, \pi(\mathcal{A})') + 3\epsilon$. As $\epsilon > 0$ is arbitrary, we conclude that $\|U - X\| = d(U, \pi(\mathcal{A})')$.

□

Proof of Theorem 6.3 : Given two representations π_1, π_2 of ϕ_1, ϕ_2 respectively on some Hilbert space K together with $x \in K$, such that $\phi_1(\cdot) = \langle x, \pi_1(\cdot)x\rangle$ and $\phi(\cdot) = \langle x, \pi_2(\cdot)x\rangle$, we may consider unitarily equivalent representations $\pi_1 \oplus \pi_2$ and $\pi_2 \oplus \pi_1$ on $K \oplus K$ with $x \oplus 0 \in K \oplus K$. This does not change the norm difference. In other words, we may restrict ourselves with unitarily equivalent representations π_1, π_2 on $\mathcal{K}$. Suppose U is a unitary on $\mathcal{K}$ which intertwine π_1 and π_2. Let $y = Ux$. So we are led to consider all tuple (π, K, x, y, U) such that $\phi_1(\cdot) = \langle x, \pi(\cdot)x\rangle$ and $\phi_2(\cdot) = \langle y, \pi(\cdot)y\rangle$, $Ux = y$. It follows that

$$\gamma(\phi_1, \phi_2) = \inf_{\{\pi, K, U, x, y\}} \|\pi - U^*\pi U\|_{cb}.$$

Suppose (π, K, x, y, U) is one such tuple. From Theorem 6.5, we get

$$\|\pi - U^*\pi U\|_{cb} = 2d(U, \pi(\mathcal{A})').$$

Then by 6.6, there exists $X \in \pi(\mathcal{A})'$ such that $\|U - X\| = d(U, \pi(\mathcal{A})'$.

Case (i) Every X as above has either non-trivial kernel or has a range which is not dense (equivalently, X^* has non-trivial kernel): Clearly in such cases $\|U -$

$X\| = \|U^* - X^*\| \geq 1$. Suppose in every common representation $\{\pi, K, x, y, U\}$, we find X with either non-trivial kernel or non-dense range, then we conclude that $\gamma(\phi_1, \phi_2) = 2$. We shall be done if we show that in that case $\beta(\phi_1, \phi_2) = \sqrt{2}$. Indeed in any common representation (π, K, x, y) with $\langle x, y\rangle \neq 0$, we choose unitary U as in Lemma 6.1, we see $\gamma(\phi_1, \phi_2) < 2$ contradicting our conclusion. Thus in any common representation (π, K, x, y), we have $\langle x, y\rangle = 0$. Hence in this case, $\beta(\phi_1, \phi_2) = \sqrt{2}$.

Case (ii) For some tuple (π, K, x, y, U), there exists X as above having trivial kernel and dense range. we may focus our attention to such operators X as otherwise $\|U - X\| \geq 1$. Taking polar decomposition of $X = V|X|$, $V, |X| \in \pi(\mathcal{A})'$ with V unitary, we have $\|U - X\| = \|V^*U - |X|\|$. Now from Lemma 6.4, we get

$$\|V^*U - |X|\| \geq \sqrt{1 - [\mathrm{Re}\langle x, V^*y\rangle]^2}.$$

Note that $V^*y \in S(\pi, \phi_2)$. Hence

$$\|V^*U - |X|\| \geq \inf_{x' \in S(\pi,\phi_1), y' \in S(\pi,\phi_2)} \sqrt{1 - |\langle x', y'\rangle|^2}.$$

One thing to be noted, while computing Bures distance for states is that we only need to consider all common representations $(K, \pi, x), (K, \pi, y)$, such that $\langle x, y\rangle \geq 0$. Indeed if $\langle x, y\rangle = |\langle x, y\rangle| e^{\imath\theta}$, we may change x to $x_1 = e^{-i\theta}x$. Note that $\phi_1(\cdot) = \langle x_1, (\cdot)x_1\rangle$ and $\|x_1 - y\|^2 = 2 - 2|\langle x, y\rangle| \leq \|x - y\|^2$.

Given common representation (K, π, x, y) such that Bures distance is attained, and $\langle x, y\rangle \geq 0$, as $\beta^2(\phi_1, \phi_2) = 2 - 2\langle x, y\rangle$, by direct computation,

$$\sqrt{1 - \langle x, y\rangle^2} = \frac{1}{2}\beta(\phi_1, \phi_2)\sqrt{4 - \beta^2(\phi_1, \phi_2)}.$$

We get immediately that for any tuple (π, K, x, y, U),

$$\|\pi - U^*\pi U\|_{cb} \geq \beta(\phi_1, \phi_2)\sqrt{4 - \beta^2(\phi_1, \phi_2)}.$$

Therefore

$$\gamma(\phi_1, \phi_2) \geq \beta(\phi_1, \phi_2)\sqrt{4 - \beta^2(\phi_1, \phi_2)}.$$

Now for the reverse inclusion, choose (π, K, x, y) is such that Bures distance is attained $\langle x, y\rangle \geq 0$. Choose unitary U with $Ux = y$, recalling Lemma 6.1, we see that

$$\begin{aligned}
\gamma(\phi_1, \phi_2) &\leq \|\pi - U^*\pi U\|_{cb} \\
&= 2d(U, \pi(\mathcal{A})') \\
&\leq 2d(U, \mathbb{C}I) \\
&= 2\sqrt{1 - \langle x, y\rangle^2} \\
&= \beta(\phi_1, \phi_2)\sqrt{4 - \beta^2(\phi_1, \phi_2)}.
\end{aligned}$$

Hence the reverse inequality holds and this proves the theorem. □

As a consequence of the previous result we get the following.

Corollary 6.7. *There is a common representation in which representation metric is attained for any pair of states on some C^*-algebra. In particular, the representation metric is attained in a common representation if and only if the Bures metric is attained in the same common representation.*

Now we extend the main result to injective range algebras. This requires a non-trivial result of Choi and Li [4].

Theorem 6.8. *Let T be a contraction on a Hilbert space K satisfying $T+T^* \geq rI$ for some $r \in \mathbb{R}$. Then there exists a unitary dilation V of T on $K \oplus K$ satisfying $V+V^* \geq rI$.*

Proof. This is Theorem 2.1 of [4], with change of notation being, $A = T, K = H, V = U$ and $\mu = -r$. □

We also need the following observation about unitary dilations of strict contractions.

Lemma 6.9. *Let T be a strict contraction on a Hilbert space H. Then any unitary dilation V of T on $H \oplus H$ is up to unitary equivalence of the form*

$$V = \begin{pmatrix} T & -(I-TT^*)^{\frac{1}{2}}W \\ (I-T^*T)^{\frac{1}{2}} & T^*W \end{pmatrix}$$

for some unitary W on H.

Proof. Set $\Delta_1 = (I - T^*T)^{\frac{1}{2}}$ and $\Delta_2 = (I - T^*T)^{\frac{1}{2}}$. Let

$$V = \begin{pmatrix} T & -T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

be any unitary dilation of T on $H \oplus H$.

From the equation $V^*V = I = VV^*$, we get $|T_{21}| = \Delta_1$ and $|T_{12}^*| = \Delta_2$. Therefore from the polar decompositions of T_{12} and T_{21} together with the fact that Δ_1 and Δ_2 are invertible, we get $T_{21} = U_1\Delta_1$ and $T_{12}^* = U_2^*\Delta_2$ for some unitaries U_1 and U_2. Comparing $(1,2)$ entry of VV^*, we get $T\Delta_1U_1^* = \Delta_2U_2T_{22}^*$. Note that $T\Delta_1 = \Delta_2T$. Therefore we get $TU_1^* = U_2T_{22}^*$. Hence $T_{22} = U_1T^*U_2$. Now by direct calculation we get that

$$V = \begin{pmatrix} I & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} T & -\Delta_2W \\ \Delta_1 & T^*W \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & U_1^* \end{pmatrix}$$

where $W = U_2U_1$.

□

Lemma 6.10. *Let $X, Y : M \to G$ be two isometries with $\|X^*Y\| < 1$. Then there is a unitary $U \in \mathcal{B}(G)$ such that $UX = Y$ and*

$$d(U, \mathbb{C}) = \sup_{\|m\|=1} \sqrt{1 - |\langle Xm, Ym\rangle|^2}.$$

Proof. It follows from Lemma 6.1, that any unitray U with $UX = Y$ will satisfy

$$d(U, \mathbb{C}) \geq \sup_{\|m\|=1} \sqrt{1 - |\langle Xm, Ym\rangle|^2}.$$

Set $T = X^*Y$. Let $K := \overline{W(T)}$ be the closure of numerical range of the operator T. Note that K is a compact convex non-empty subset of $\mathbb{C}$. Let $\lambda = re^{i\theta}$ be the unique point in K such that $|\lambda| = \inf\{|z| : z \in K\}$. It is the point closest to the origin in K. Replacing Y by $Ye^{-i\theta}$, we may assume without loss of generality, $\lambda = r \in \mathbb{R}$. Observe that

$$\sqrt{1-r^2} = \sup_{\|m\|=1} \sqrt{1-|\langle Xm, Ym\rangle|^2}.$$

If $r = 1$ then $\langle m, X^*Ym\rangle = \langle m, m\rangle$ for every m and then $X = Y$. In such a case may take $U = I$, and we are done. Therefore assume $0 \leq r < 1$. Consider the line $x = r$. Note that the line $x = r$ is tangent to the circle centred at $(0,0)$ and radius r. Therefore the convexity of $W(T)$ would implies that $W(T)$ can not have any point to the left of the line $x = r$. Therefore $T + T^* \geq 2r$. By 6.8 and 6.9 there is a unitary V on $M \oplus M$ of the form

$$V = \begin{pmatrix} T & -(I-TT^*)^{\frac{1}{2}}W \\ (I-T^*T)^{\frac{1}{2}} & T^*W \end{pmatrix},$$

with $W \in \mathcal{B}(M)$ chosen such a way, we get $V + V^* \geq 2r$. This implies $\sigma(V)$ is to the right side of the line $x = r$. As $\sqrt{1-r^2} > 1-r$, we see that the circle centred at $(r, 0)$ and radius $\sqrt{1-r^2}$ covers $\sigma(V)$. Therefore $d(V, \mathbb{C}) \leq \sqrt{1-r^2}$.

As $\|T\| < 1$, the operator $\Delta = (I - T^*T)^{\frac{1}{2}}$ is invertible. Define $C = (Y - XT)\Delta^{-1}$. We see that $C^*C = I$ and $X^*C = 0$. In particular, range of X and range of C are orthogonal. Decompose G as $G = X(M) \oplus C(M) \oplus G_0$. Define $U|_{G_0} = I$ and on ${G_0}^\perp$, via the following unitary

$$U_{G_0^\perp} = \begin{pmatrix} X & 0 \\ 0 & C \end{pmatrix} V \begin{pmatrix} X^* & 0 \\ 0 & C^* \end{pmatrix}.$$

We see that $UX = Y$ and $d(U, \mathbb{C}) = d(V, \mathbb{C}) \leq \sqrt{1-r^2}$. □

Remark 6.11. Let $\mathcal{E}$ be a Hilbert $\mathcal{B}$ right module where $\mathcal{B}$ is a C^*-algebra. Let $X, Y \in \mathcal{E}$ be two unital vectors. Assume $\|\langle X, Y\rangle\| < 1$. It is possible to find unitary $U \in \mathcal{B}^a(\mathcal{E})$ such that $UX = Y$ but we still do not know whether it is possible to find $U \in \mathcal{B}^a(\mathcal{E})$ satisfying $UX = Y$ and

$$d(U, \mathbb{C}) = \sup_{\|m\|=1} \sqrt{1-|\langle m, \langle X, Y\rangle m\rangle|^2}.$$

It is unclear as to whether unitary W chosen in the construction of the unitary V can be obtained from the C^*-algebra $\mathcal{B}$.

Lemma 6.12. *Let $\mathcal{A}, \mathcal{B}$ be C^* algebras. Let ϕ_1, ϕ_2 be completely positive maps from $\mathcal{A}$ to $\mathcal{B}$. Then*

$$\beta(\phi_1, \phi_2) = \inf_{\{(\mathcal{E}, X, Y)\ :\ \|\langle X, Y\rangle\| < 1\}} \|X - Y\|$$

where $(\mathcal{E}, X, Y)$ is a common representation module for ϕ_1, ϕ_2 with $\|\langle X, Y\rangle\| < 1$.

Proof. Let $(\mathcal{E}, X, Y)$ be a common representation module for (ϕ_1, ϕ_2). For $0 < r < 1$, take $\tilde{\mathcal{E}} = \mathcal{E} \oplus \mathcal{E}$, $X_r = X \oplus 0$, $Y_r = rY \oplus \sqrt{1-r^2}Y$. Then $(\tilde{\mathcal{E}}, X_r, Y_r)$ is a common representation module for ϕ_1, ϕ_2. Further, $\|\langle X_r, Y_r\rangle\| = r\|\langle X, Y\rangle\| \leq r < 1$. Also $\lim_{r\to 1} \|X_r - Y_r\| = \|X - Y\|$. Hence,

$$\inf_{\{(\mathcal{E}, X, Y)\ :\ \|\langle X, Y\rangle\| < 1\}} \|X - Y\| = \inf_{\{(\mathcal{E}, X, Y)\}} \|X - Y\| = \beta(\phi_1, \phi_2).$$

□

Lemma 6.13. *Let $\mathcal{A}, \mathcal{B}$ be unital C^* algebras. Let $\mathcal{B} \subset \mathcal{B}(M)$. Let ϕ_1, ϕ_2 be unital completely positive maps from $\mathcal{A}$ to $\mathcal{B}$. Then*

$$\beta(\phi_1, \phi_2) = \inf_{(\mathcal{E},X,Y)} \sup_{\|m\|=1} \sqrt{2}\sqrt{1 - |\langle m, \langle X, Y\rangle m\rangle|}$$

where $(\mathcal{E}, X, Y)$ is a common representation module for ϕ_1, ϕ_2.

Proof. Let $(\mathcal{E}, X, Y)$ be as above and define

$$\beta'(\phi_1, \phi_2) = \inf_{(\mathcal{E},X,Y)} \sup_{\|m\|=1} \sqrt{2}\sqrt{1 - |\langle m, \langle X, Y\rangle m\rangle|}.$$

As ϕ_1, ϕ_2 are unital, $X^*X = 1 = Y^*Y$ and hence, $\langle (X - Y), (X - Y)\rangle = 2(1 - Re(X^*Y))$. So

$$\begin{aligned}
\|X - Y\|^2 &= \sup_{m=1} \langle m, (X - Y)^*(X - Y)m\rangle \\
&= \sup_{m=1} \langle m, 2(1 - Re(X^*Y))m\rangle \\
&= \sup_{m=1} 2(1 - \langle m, Re(X^*Y)m\rangle \\
&\geq \sup_{m=1} 2(1 - |\langle m, X^*Ym\rangle|)
\end{aligned}$$

Consequently $\beta(\phi_1, \phi_2) \geq \beta'(\phi_1, \phi_2)$. Suppose the equality does not hold, then there is a positive number t such that $\beta(\phi_1, \phi_2) > t > \beta'(\phi_1, \phi_2)$. We try to get a contradiction. Let $(\mathcal{E}, X, Y)$ be a common representation module for (ϕ_1, ϕ_2) such that

$$\sup_{\|m\|=1} \sqrt{2}\sqrt{1 - |\langle m, \langle X, Y\rangle m\rangle|} < t.$$

Set $T = \langle X, Y\rangle$. Let $K := \overline{W(T)}$ be the closure of numerical range of the operator T. Note that K is a compact convex non-empty subset of $\mathbb{C}$. Let λ be the unique point in K which is of minimum distance from $(0, 0)$. Let $\lambda = re^{i\theta}$ in polar form. Set $\tilde{X} = Xe^{i\theta}$. Then $(\mathcal{E}, \tilde{X}, Y)$ is a common representation module for (ϕ_1, ϕ_2). The convexity of $W(T)$ implies

$$\begin{aligned}
\sup_{\|m\|=1} \sqrt{2}\sqrt{1 - \mathrm{Re}(\langle m, \langle \tilde{X}, Y\rangle m\rangle)} &= \sup_{\|m\|=1} \sqrt{2}\sqrt{1 - |\langle m, \langle \tilde{X}, Y\rangle m\rangle|} \\
&= \sup_{\|m\|=1} \sqrt{2}\sqrt{1 - |\langle m, \langle X, Y\rangle m\rangle|}.
\end{aligned}$$

Therefore $\|\tilde{X} - Y\| < t$. This implies $\beta(\phi_1, \phi_2) < t$. This is a contradiction. □

Theorem 6.14. *Suppose $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{B} \subset \mathcal{B}(M)$ is an injective C^*-algebra. Suppose $\phi_1, \phi_2 \in UCP(\mathcal{A}, B)$. Then*

$$\gamma(\phi_1, \phi_2) = \beta(\phi_1, \phi_2)\sqrt{4 - \beta^2(\phi_1, \phi_2)}.$$

Proof. Let $\tilde{\phi}_i = \iota \circ \phi_i$, $\iota : \mathcal{B} \to \mathcal{B}(M)$ inclusion map, $i = 1, 2$. Now injectivity of $\mathcal{B}$ will imply $\beta(\tilde{\phi}_1, \tilde{\phi}_2) = \beta(\phi_1, \phi_2)$ and from Proposition 4.4, $\gamma(\tilde{\phi}_1, \tilde{\phi}_2) = \gamma(\phi_1, \phi_2)$. Therefore we may assume without loss of generality that $\mathcal{B} = \mathcal{B}(M)$.

We get from 3.10 and 4.3, that

$$\gamma(\phi_1,\phi_2) \geq \gamma(\omega \circ (\phi_1 \otimes 1_{\mathcal{B}(M)}), \omega \circ (\phi_2 \otimes 1_{\mathcal{B}(M)}))$$

for every $\omega \in M \otimes M$, $\|\omega\| = 1$. Denoting $\psi_i = \phi_1 \otimes 1_{\mathcal{B}(M)}$, $i = 1, 2$, we get immediately from Theorem 6.3,

$$\gamma(\phi_1,\phi_2) \leq \sup_{\omega \in M\otimes M, \|\omega\|=1} \beta(\omega \circ \psi_1, \omega \circ \psi_2)\sqrt{4 - \beta^2(\omega \circ \psi_1, \omega \circ \psi_2)}.$$

Note that from Proposition 6, [10], we get

$$\sup_{\omega \in M\otimes M, \|\omega\|=1} \beta(\omega \circ \psi_1, \omega \circ \psi_2) = \beta(\phi_1, \phi_2).$$

Therefore

$$\gamma(\phi_1,\phi_2) \geq \beta(\phi_1,\phi_2)\sqrt{4 - \beta^2(\phi_1,\phi_2)}.$$

Let us prove the reverse inequality. Let (π, G, X, Y) be a common representation for ϕ_1, ϕ_2 satisfying $\|X^*Y\| < 1$. I.e. $\pi : \mathcal{A} \to \mathcal{B}(G)$ is a representation, $X, Y : M \to G$ isometries with $\phi_1(\cdot) = X^*\pi(\cdot)X$ and $\phi_2(\cdot) = Y^*\pi(\cdot)Y$. Let $U \in \mathcal{B}(G)$ be a unitary as in Lemma 6.10. Then $(\pi, U^*\pi U, G, X)$ is a joint representation module for (ϕ_1, ϕ_2). We get that

$$\gamma(\phi_1,\phi_2) \quad \leq \quad \|\pi - U^*\pi U\|_{cb}.$$

Now from Theorem 6.5 and Lemma 6.10, we get

$$\begin{aligned}
\gamma(\phi_1,\phi_2) &\leq \|\pi - U^*\pi U\|_{cb} \\
&= 2d(U, \pi(\mathcal{A})') \\
&\leq 2d(U, \mathbb{C}) \\
&= 2\sup_{\|m\|=1} \sqrt{1 - |\langle Xm, Ym\rangle|^2}.
\end{aligned}$$

Set $s = \sup_{\|m\|=1} \sqrt{2}\sqrt{1 - |\langle m, \langle X, Y\rangle m\rangle|}$. Now observe that

$$s\sqrt{4 - s^2} = 2\sup_{\|m\|=1} \sqrt{1 - |\langle Xm, Ym\rangle|^2}.$$

Now as (π, G, X, Y) is an arbitrary common representation space for ϕ_1, ϕ_2 satisfying $\|X^*Y\| < 1$ and the fact that $x\sqrt{4 - x^2}$ is an increasing function in the interval $[0, \sqrt{2}]$, we therefore conclude from Lemma 6.12 and Lemma 6.13, $\gamma(\phi_1,\phi_2) \leq \beta(\phi_1,\phi_2)\sqrt{4 - \beta^2(\phi_1,\phi_2)}$. Now we are done.

□

7. Examples

In this Section we explore the dependence of the representation metric γ on the range algebra. We see that, when the range algebra is not injective the relationship between β and γ may fail. The examples draw upon ideas from [2].

Example 7.1. Let H be a separable infinite dimensional Hilbert space. Let $\mathcal{B} = C^*\{\mathcal{K}(H), I\}$, the unital C^*-algebra generated by compact operators. Let $u \in \mathcal{B}(H)$ be a unitary of the form $u = \lambda p + \bar{\lambda}(1-p)$, where p is a projection such that p and $(1-p)$ have infinite rank and $\lambda = e^{i\theta}$ for some $0 < \theta < \frac{\pi}{2}$. Clearly u is not in $\mathcal{B}$. Define unital $*$-automorphisms ψ_1, ψ_2 of $\mathcal{B}$ by $\psi_1(a) = u^*au, \psi_2(a) = a$. Let $\iota : \mathcal{B} \to \mathcal{B}(H)$ be the inclusion map and let $\tilde{\psi}_j = \iota \circ \psi_j$ for $j = 1, 2$.

Then from Example 3.2 of [2], we get $\beta(\psi_1, \psi_2) = \sqrt{2}$. Now observe that $\mathcal{B}$ is a $\mathcal{B}$ right-module with natural action and define adjointable left actions $\sigma_1(a) = u^*au$ and $\sigma_2(a) = a$. Now note that $(\mathcal{E}, 1, \sigma_1, \sigma_2)$ is a joint representation module for ψ_1, ψ_2. Therefore

$$\gamma(\psi_1, \psi_2) = \|\sigma_1 - \sigma_2\|_{cb} = \|\psi_1 - \psi_2\|_{cb} = 2d(u, \mathbb{C}).$$

Therefore

$$\gamma(\psi_1, \psi_2) = |\lambda - \bar{\lambda}| < 2 = \beta(\psi_1, \psi_2)\sqrt{4 - \beta^2(\psi_1, \psi_2)}.$$

On the other hand, as $\beta(\tilde{\psi}_1, \tilde{\psi}_2) = \sqrt{2}(1 - \mathrm{Re}\lambda)^{\frac{1}{2}}$. We get

$$\gamma(\tilde{\psi}_1, \tilde{\psi}_2) = \beta(\tilde{\psi}_1, \tilde{\psi}_2)\sqrt{4 - \beta^2(\tilde{\psi}_1, \tilde{\psi}_2)} = 2\sqrt{1 - (\mathrm{Re}\lambda)^2} = 2d(U, \mathbb{C}) = \gamma(\psi_1, \psi_2).$$

In the previous example, the representation metric did not change by the inclusion map. However, it is not always the case. To see this, we need a slightly more delicate example.

Example 7.2. Let H be a separable infinite dimensional Hilbert space. Let $\mathcal{K}$ denote the set of all compact operators on H. Set $\mathcal{K}_+ = \mathrm{span}\{\mathcal{K}, \mathbb{C}I_H\}$. Let

$$\mathcal{A} = \begin{pmatrix} \mathcal{K}_+ & \mathcal{K} \\ \mathcal{K} & \mathcal{K}_+ \end{pmatrix} \subset \mathcal{B}(H \oplus H),\ \mathcal{B} = \mathcal{K}_+.$$

Let p be a projection on H such that range of p and $1 - p$ are both infinite dimensional subspaces of H.

Let $0 < \theta < \frac{\pi}{2}$. Set

$$u := e^{i\theta}p + e^{-i\theta}(1 - p).$$

Then u is a unitary and $u \notin \mathcal{K}_+$. Let

$$z_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} I \\ I \end{pmatrix},\ z_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} u \\ I \end{pmatrix}.$$

Define unital CP maps $\phi_i : \mathcal{A} \to \mathcal{B}$, by $\phi_i(a) = z_i^*az_i, a \in \mathcal{A}, i = 1, 2$.

Let $\iota : \mathcal{B} \to \mathcal{B}(H)$ be the inclusion map. Let $\tilde{\phi}_i = \iota \circ \phi_i, i = 1, 2$. As $\mathcal{B}(H)$ is injective, we have

$$\gamma(\tilde{\phi}_1, \tilde{\phi}_2) = \beta(\tilde{\phi}_1, \tilde{\phi}_2)\sqrt{4 - \beta^2(\tilde{\phi}_1, \tilde{\phi}_2)}.$$

Set $G = H \oplus H$. To compute $\beta(\tilde{\phi}_1, \tilde{\phi}_2)$, first note that (G, id, z_i) is a Stinespring representation for $\tilde{\phi}_i, i = 1, 2$. Any operator $W \in \mathcal{B}(G)$ commuting the identity representation is of the form $W = \lambda I$ with $\lambda \in \mathbb{C}$. Therefore

$$\begin{aligned}
\beta(\tilde{\phi}_1, \tilde{\phi}_2) &= \inf_{|\lambda| \le 1} \|(z_1 \oplus 0) - (\lambda z_2 \oplus \sqrt{1 - |\lambda|^2}z_2)\|^{\frac{1}{2}} \\
&= \inf_{|\lambda| \le 1} \|2I - 2Re(\lambda z_1^* z_2)\|^{\frac{1}{2}} \\
&= \sqrt{2}\|I - Re(\lambda\frac{u + I}{2})\|^{\frac{1}{2}}.
\end{aligned}$$

After simple calculation, we observe that the infimum is attained at $\lambda = 1$. Therefore

$$\begin{aligned}\beta(\tilde{\phi}_1, \tilde{\phi}_2) &= \sqrt{2}\|I - Re(\frac{u+I}{2})\|^{\frac{1}{2}} \\ &= \|I - Re(u)\|^{\frac{1}{2}} \\ &= \sqrt{1-\cos\theta}.\end{aligned}$$

So

$$\gamma(\tilde{\phi}, \tilde{\phi}_2) = \sqrt{3 - 2\cos\theta - \cos^2\theta} = \sqrt{(3+\cos\theta)(1-\cos\theta)}.$$

Let us now compute $\gamma(\phi_1, \phi_2)$. For that we consider common representation modules $(\mathcal{F}, x_1, x_2)$ with unitary U, where $\mathcal{F}$ is an $\mathcal{A} - \mathcal{B}$ bi-module, $x_i \in S(\mathcal{F}, \phi_i), i = 1, 2$ and $U \in \mathcal{B}^a(\mathcal{F})$ satisfies $Ux_1 = x_2$.

Take $\mathcal{K}_u = \text{span } \{\mathcal{K}, u\}$ and

$$\mathcal{E}_1 = \begin{pmatrix} \mathcal{K}_+ \\ \mathcal{K}_+ \end{pmatrix}, \mathcal{E}_2 = \begin{pmatrix} \mathcal{K}_u \\ \mathcal{K}_+ \end{pmatrix}.$$

Then $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{B}(H, G)$ are $\mathcal{A} - \mathcal{B}$ bi-modules. Set

$$\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2 \subset \mathcal{B}(H, G \oplus G).$$

Lemma 7.3. *Let $\mathcal{E}_1, \mathcal{E}_2$ be as defined earlier. Then*

$$\mathcal{B}^a(\mathcal{E}_1, \mathcal{E}_2) = \begin{pmatrix} \mathcal{K}_u & \mathcal{K}_u \\ \mathcal{K}_+ & \mathcal{K}_+ \end{pmatrix} \subset \mathcal{B}(H, G).$$

Proof. We observe that an operator $X \in \mathcal{B}(H, G)$ is in $\mathcal{B}^a(\mathcal{E}_1, \mathcal{E}_2)$ if and only if $XE \in \mathcal{E}_2$ for every $E \in \mathcal{E}_1$. As

$$\begin{pmatrix} I \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I \end{pmatrix} \in \mathcal{E}_1,$$

we get the result by direct computation. $\square$

Lemma 7.4. *Suppose $(\mathcal{F}, x_1, x_2)$ is a common representation module for ϕ_1 and ϕ_2. Then there is an $\mathcal{A}-\mathcal{B}$ bi-module $\mathcal{G}$ and a bilinear unitary $W \in \mathcal{B}^a_{bil}(\mathcal{F}, \mathcal{E} \oplus \mathcal{G})$ such that $Wx_1 = (z_1, 0, 0)$ and $Wx_2 = (0, z_2, 0)$.*

Proof. Set $M = \mathcal{F} \odot H$. Then $\mathcal{F} \subset \mathcal{B}(H, M)$. Define $\pi : \mathcal{A} \to \mathcal{B}(M)$ by $\pi(a)(e \odot h) = ae \odot h$. Set $K_i = \overline{\text{span}}\{ax_i \odot h : a \in \mathcal{A}, h \in H\}$. Then K_i is a reducing subspace for π. Define unitary $U_i : K_i \to G$ by $U_i(ax_i \odot h) = az_ih$, $a \in \mathcal{A}, h \in H, i = 1, 2$. Note that $\overline{\text{span}}\{az_ih : a \in \mathcal{A}, h \in H\} = G$.

Identifying K_1 with G via unitary U_1, we get $M = G \oplus G^\perp$ and for some representation π_0,

$$\pi = \begin{pmatrix} id & 0 \\ 0 & \pi_0 \end{pmatrix}, \ x_1 = \begin{pmatrix} z_1 \\ 0 \end{pmatrix}, \ x_2 = \begin{pmatrix} w \\ v \end{pmatrix}.$$

Now $z_2^* a z_2 = \phi_2(a) = \langle x_2, ax_2 \rangle = w^*aw + v^*\pi^\perp(a)v$. So $\psi : \mathcal{A} \to \mathcal{B}(G)$, defined by $\psi(a) = w^*aw$ is a CP map dominated by ϕ_2. It follows that $w = cz_2$ for some $c \in \mathbb{C}$, with $|c| \leq 1$. Now $\langle x_1, x_2 \rangle = z_1^* w = cz_1^* z_2$. By direct computation, $cz_1^*z_2 = \frac{u+I}{2} \notin \mathcal{A}$. Therefore $c = 0$. Hence $w = 0$ and $\langle x_1, x_2 \rangle = 0$. Also by direct computation, we get $\langle \pi(\mathcal{A})x_1, \pi(\mathcal{A})x_2 \rangle = 0$.

Similarly identifying K_2 with G via unitary U_2, we get $M = G \oplus G \oplus L$ and with some representation π_1,

$$\pi = \begin{pmatrix} id & 0 & 0 \\ 0 & id & 0 \\ 0 & 0 & \pi_1 \end{pmatrix}, \; x_1 = \begin{pmatrix} z_1 \\ 0 \\ 0 \end{pmatrix}, \; x_2 = \begin{pmatrix} 0 \\ z_2 \\ 0 \end{pmatrix}.$$

It follows that, $\mathcal{E} \cap \mathcal{B}(H, G \oplus 0 \oplus 0) = \mathcal{E}_1 \oplus 0 \oplus 0$ and $\mathcal{E} \cap \mathcal{B}(H, 0 \oplus G \oplus 0) = 0 \oplus \mathcal{E}_2 \oplus 0$. Consequently

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3,$$

where $\mathcal{E}_3 = \mathcal{E} \cap \mathcal{B}(H, 0 \oplus 0 \oplus L)$.

□

In view of this Lemma we consider common representations of the form $(\mathcal{E} \oplus \mathcal{G}, (z_1, 0, 0), (0, z_2, 0)$ and unitary $U \in \mathcal{B}^a(\mathcal{E} \oplus \mathcal{G})$ with $U(z_1, 0, 0) = (0, z_2, 0)$.

Let $P \in \mathcal{B}^a(\mathcal{E} \oplus \mathcal{G})$ be the projection onto $\mathcal{E}$. Set

$$V = PUP|_{\mathcal{E}}.$$

Then $V \in \mathcal{B}^a(\mathcal{E})$ is a contraction with $V(z_1, 0) = (0, z_2)$. Let $\sigma : \mathcal{A} \to \mathcal{B}^a(\mathcal{E} \oplus \mathcal{G})$ be the left action. Note that

$$\sigma = id \oplus id \oplus \sigma_{\mathcal{G}}.$$

Therefore observe that

$$d(U, \sigma(\mathcal{A})') \geq d(V, (id \oplus id)').$$

Decomposing

$$\mathcal{B}^a(\mathcal{E}) = \begin{pmatrix} \mathcal{B}^a(\mathcal{E}_1) & \mathcal{B}^a(\mathcal{E}_2, \mathcal{E}_1) \\ \mathcal{B}^a(\mathcal{E}_1, \mathcal{E}_2) & \mathcal{B}^a(\mathcal{E}_2) \end{pmatrix},$$

and the fact that $V(z_1, 0) = (0, z_2)$, We get

$$V = \begin{pmatrix} * & * \\ Z & * \end{pmatrix},$$

for some $Z \in \mathcal{B}^a(\mathcal{E}_1, \mathcal{E}_2)$ satisfying $Zz_1 = z_2$. Recalling the choice of z_1, z_2, and the definitions of $\mathcal{E}_1, \mathcal{E}_2$, we observe

$$Z = \begin{pmatrix} au + k & (1-a)u - k \\ cI + l & (1-c)I - l \end{pmatrix},$$

for some $k, l \in \mathcal{K}$, and scalars $a, c \in \mathbb{C}$. Now

$$\begin{aligned} \|\sigma - U^*\sigma U\|_{cb} &= 2d(U, \sigma(\mathcal{A})') \\ &\geq 2d(V, (id \oplus id)') \\ &= 2d(\begin{pmatrix} * & * \\ Z & * \end{pmatrix}, \begin{pmatrix} \mathbb{C}I & \mathbb{C}I \\ \mathbb{C}I & \mathbb{C}I \end{pmatrix}) \\ &\geq 2d(Z, \mathbb{C}). \end{aligned}$$

As we have started with arbitrary common representation module, we get

$$\gamma(\phi_1, \phi_2) \geq 2d(Z, \mathbb{C}).$$

For a unit vector $x \in H$, we have

$$\begin{aligned}\|Z-\lambda I\| &\geq \|\begin{pmatrix} au+k-\lambda I & (1-a)u-k \\ cI+l & (1-c-\lambda)I-l \end{pmatrix}\begin{pmatrix} \frac{x}{\sqrt{2}} \\ \frac{x}{\sqrt{2}} \end{pmatrix}\| \\ &= \|\begin{pmatrix} \frac{(u-\lambda)x}{\sqrt{2}} \\ \frac{(1-\lambda)x}{\sqrt{2}} \end{pmatrix}\| \\ &\geq \frac{\|(u-I)x\|}{\sqrt{2}}.\end{aligned}$$

Therefore

$$\begin{aligned}d(Z,\mathbb{C}) &\geq \frac{\|u-I\|}{\sqrt{2}} \\ &= \frac{|e^{i\theta}-1|}{\sqrt{2}} \\ &= \sqrt{1-\cos\theta}.\end{aligned}$$

Therefore

$$\gamma(\phi_1,\phi_2) \geq 2\sqrt{1-\cos\theta} > \sqrt{(3+\cos\theta)(1-\cos\theta)} = \gamma(\tilde{\phi}_1,\tilde{\phi}_2).$$

Also let choose the following common representation $(\mathcal{E},(z_1,0),(0,z_2),V)$ where $V \in \mathcal{B}^a(\mathcal{E})$ is given by

$$V = \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix},$$

with

$$Z = \begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix}.$$

We see that $W(z_1,0) = (0,z_2)$ and

$$\begin{aligned}\|\sigma - V^*\sigma V\|_{cb} &= 2d(V,(id\oplus id)') \\ &= 2d(\begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix},\begin{pmatrix} \mathbb{C}I & \mathbb{C}I \\ \mathbb{C}I & \mathbb{C}I \end{pmatrix}) \\ &= 2d(Z,\mathbb{C}) \\ &= 2d(u,\mathbb{C}) \\ &= 2\sin\theta.\end{aligned}$$

We get

$$2\sin\theta \geq \gamma(\phi_1,\phi_2) \geq 2\sqrt{1-\cos\theta}.$$

It is to be noted $\beta(\phi_1,\phi_2) = \sqrt{2}$ (See Example 3.2 of [2])[2]. Therefore

$$\gamma(\phi_1,\phi_2) \neq \beta(\phi_1,\phi_2)\sqrt{4-\beta^2(\phi_1,\phi_2)}.$$

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[2]This computation in [2] is erroneous. However the result is clear in view of the fact that $\langle x_1,x_2\rangle = 0$ for any common representation $(\mathcal{F},x_1,x_2)$ of (ϕ_1,ϕ_2) due to Lemma 7.4

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