

Dynamic Assortment Personalization in High Dimensions

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We study the problem of dynamic assortment personalization with large, heterogeneous populations and wide arrays of products, and demonstrate the importance of structural priors for effective, efficient large-scale personalization. Assortment personalization is the problem of choosing, for each individual or consumer segment (type), a best assortment of products, ads, or other offerings (items) so as to maximize revenue. This problem is central to revenue management in e-commerce, online advertising, and multi-location brick-and-mortar retail, where both items and types can number in the millions.

We formulate the dynamic assortment personalization problem as a discrete-contextual bandit with m contexts (customer types) and exponentially many arms (assortments of the n items). We assume that each type’s preferences follow a simple parametric model with n parameters. In all, there are mn parameters, and existing literature suggests that order optimal regret scales as mn . However, the data required to estimate so many parameters is orders of magnitude larger than the data available in most revenue management applications; and the optimal regret under these models is unacceptably high.

In this paper, we impose a natural structure on the problem – a small latent dimension, or low rank. In the static setting, we show that this model can be efficiently learned from surprisingly few interactions, using a time- and memory-efficient optimization algorithm that converges globally whenever the model is learnable. In the dynamic setting, we show that structure-aware dynamic assortment personalization can have regret that is an order of magnitude smaller than structure-ignorant approaches. We validate our theoretical results empirically.

Key words: Personalization, Contextual bandit, Assortment planning, Discrete choice, High-dimensional learning, Large-scale learning, First-order optimization, Recommender Systems, Matrix completion

1. Introduction

In many commerce, e-commerce, and advertising settings, customers or users are presented with an assortment of products, ads, or other offerings. Customers choose which to products to buy, ads to click, or (generically) items to interact with, from among the assortment that is presented. Firms choose which assortment to present to the customer, and collect the revenue (or other benefit or loss) resulting from each customer’s choice. Choosing the assortment that maximizes expected revenue is a central problem in revenue management. This problem goes by the name *assortment planning* or *assortment optimization*. When assortments are tailored to each individual customer or to each consumer segment, the problem is known as *assortment personalization*. Successful personalization, which is key to e-commerce and online advertising operations, hinges on learning the preferences of each customer for each offering.

This paper shows how to learn customer preferences and manage revenue under realistic assumptions about the problem data available and the structure of customer preferences. We suppose that we must rely on *transactional data* to estimate customer preferences; and that we rely only on the *discrete context* of each customer’s type to infer their preferences, for we lack covariates that precisely pin down customer outcomes. These assumptions make our problem challenging. We show how to achieve good performance using the one key structural assumption: that customer preferences are *low rank*. We provide both new algorithms and new results for this setting that enable efficient assortment personalization in the face of high dimensions.

1.1. Problem setting

We next explain the problem setting and discuss the practical importance of each of the main assumptions. The problem setting we consider in this paper is, we believe, the relevant one for most commerce and e-commerce settings.

Transactional data. Transactional data — which customer selected which product — is abundant and easy for retailers to collect, whereas detailed information about customer attributes (such as gender, age, ethnicity, and preferences) cannot be directly harvested from the retailer’s data. It

is natural and even expected that a data driven modern retailer will record transactional data, whereas retailers who buy information on their customers’ attributes face extra costs as well as data quality and privacy concerns. Moreover, these covariates are not directly related to the task at hand: predicting which products the customer will buy, and maximizing revenue. In particular, they may be too coarse to sufficiently to pin down customer preference beyond a few broad categories. Our approach requires no side information about the attributes of the customers. Instead, we rely exclusively on transactional data and focus on personalization at the finest level of segmentation offered by the recorded data (usually, the individual level).

Discrete-contextual regret analysis. An algorithm relying exclusively on transactional data lacks any context to use to predict a customer’s behavior other than the identity (*e.g.*, customer id) of that customer. Hence we refer to our assortment choice problem as having a *discrete context* corresponding to this distinct identity. This contrasts usual contextual variables that are continuous vectors can can related to one another in terms of metric proximity. This paper presents a regret analysis of assortment choices in the discrete-contextual setting. In this setting, ours is the first algorithm that can achieve regret which grows sublinearly in the number of parameters.

Low-rank structure. Our method makes use of one key structural assumption — that preferences are (approximately) low rank — that has been extensively verified in a wide variety of practical applications. These applications range from its earliest uses, in psychology (Spearman 1904, Hotelling 1933), to modern applications in marketing (Funk 2006), genomics (Witten et al. 2009), and healthcare (Schuler et al. 2016). Indeed, under a very natural model for preferences — roughly, as long as customers and products are iid (independent and identically distributed), and there exists some function mapping customer attributes and product attributes to latent utilities — then a table of customer preferences will be approximately low rank (Udell and Townsend 2017). This low-rank structural assumption provides two advantages over previous approaches.

First, it undergirds our regret bound, which guarantees that the regret incurred by our algorithm grows sublinearly in the number of parameters. This result provides a significant advance with

respect to the previous literature in both assortment optimization and in matrix sensing. Relative to existing results in assortment optimization, our analysis provides a better scaling as the dimension of the problem grows. Indeed, we prove that any structure-ignorant method must incur regret that grows linearly in the number of preference parameters. Relative to existing results in matrix sensing, our analysis provides the first regret bounds for matrix sensing under bandit feedback. Furthermore, we show how to provably learn a preference matrix in the realistic setting of transactional data, in which we see only which product (if any) the customer has chosen to purchase, rather than a full preference list or a forced choice.

Second, our algorithms make use of the low rank structural assumption to scale to extremely large data sets while preserving provable optimality guarantees. As a result, our methods are well suited to use in practical applications, and can easily be used by any modern retailer. While the assumptions that underly our theoretical guarantees cannot be verified directly, these results demonstrate that the algorithms we propose are effective in practice.

1.2. Our approach

In this paper, we propose a new approach to assortment personalization. To enable tractable estimation of a personalized model with limited data, we propose a new structural model. In this model, the choices of each type are governed by its own personalized preference vector (with one dimension for every item); but these preference vectors span (or lie close to) only a low-dimensional subspace. We demonstrate how to estimate the parameters of this model with computationally tractable algorithms, and provide a proof of recovery with high-probability from few samples (sub-linear in number type-item combinations). Numerically, we show that given the same data, our estimator performs much better than standard maximum likelihood estimators.

We then leverage our new model to tackle the *dynamic* assortment personalization problem: starting with *no* data, how should we choose assortments to offer to different types to maximize total profit, or equivalently, to minimize total regret? We show theoretically and numerically that our algorithm achieves regret orders-of-magnitude smaller than standard methods based either

on a single multinomial logit (MNL) model or several decoupled MNL models for each type, which we call “structure-ignorant” methods. For example, when the parameter matrix has rank r , we achieve regret of order $r \max(m, n) \log T$ after T interactions, where ignoring structure would have yielded regret of order $\min(m, n) \max(m, n) \log T$.¹ Our results demonstrate that assortment personalization enables orders of magnitude better performance than competing approaches, and can be achieved with a tractable, efficient estimator.

All proofs are given in the electronic companion.

2. Problem statement

In this section we describe the problem of *dynamic assortment personalization*. We consider a problem with m types and n items. During each interaction, a consumer arrives; the retailer presents the consumer with an assortment of no more than K items; and the consumer chooses one of the items presented or chooses nothing, which we refer to as choosing the 0th item. The expected revenue generated when a consumer of type i chooses item j is W_{ij} and is 0 if no item is chosen. A problem instance is also described by two additional parameters $\mu^* \in \Delta^m = \{\alpha \in \mathbb{R}_+^m : \sum_{i=1}^m \alpha_i = 1\}$ and $\Theta^* \in \mathbb{R}^{m \times n}$ that are not known and must be estimated from data. The parameter μ^* describes how often each type arrives, while the matrix Θ^* of preference parameters governs m MNL choice models, one for each type. The number of parameters in the model is $mn + m = \Theta(mn)$.

The problem proceeds as follows. At first, the retailer knows only m , n , and $W \in \mathbb{R}^{m \times n}$. Then for each interaction $t = 1, 2, \dots$:

1. A customer of type $i_t \in \{1, \dots, m\}$ arrives with probability $\mu_{i_t}^*$, and the retailer observes the type i_t .
2. The retailer chooses any subset of products $S_t \subset \{1, \dots, n\}$ with $|S_t| \leq K$.
3. The customer chooses an item j_t from $\{1, \dots, n\}$ with probability proportional to

$$\text{weight}(j) = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0, j \notin S_t \\ \exp(\Theta_{i_t j}^*) & j \neq 0, j \in S_t \end{cases}$$

and the retailer observes the choice j_t .

4. The retailer collects a random reward with expectation

$$r_t = \begin{cases} W_{i_t j_t} & j_t \neq 0, \\ 0 & j_t = 0. \end{cases}$$

Notation. For convenience, define a set of random variables U_t for $t = 1, 2, \dots$ independent of all variables above. These random variables U_t allow for randomized dynamic assortment personalization algorithms. Define the filtration

$$\mathcal{F}_t = \sigma(U_t, i_t, i_{t-1}, j_{t-1}, S_{t-1}, \dots, i_1, j_1, S_1).$$

\mathcal{F}_t is the smallest σ algebra generated by all variables known at step 2 at time t – it captures all information known before the retailer chooses S_t . An algorithm π is an assignment to the random variables S_t so that, for every t , S_t is measurable with respect to \mathcal{F}_t . Given a problem instance and an algorithm π , let \mathbb{P}^π and \mathbb{E}^π be the probability and expectation measures under algorithm π : that is, when the sets S_t are selected according to algorithm π .

Define the Θ -greedy algorithm, π_Θ , which chooses

$$S_t \in \operatorname{argmax}_{|S| \leq K} \sum_{j \in S} \frac{e^{\Theta_{i_t j} W_{i_t j}}}{1 + \sum_{j' \in S} e^{\Theta_{i_t j'} W_{i_t j'}}}.$$

The Θ -greedy algorithm π_Θ always chooses the assortment S_t that would maximize revenue if Θ were the true choice parameter matrix.

DEFINITION 1. Given an instance $(m, n, W, \mu^*, \Theta^*)$, the *regret* of the algorithm π at time T is

$$\operatorname{Regret}(T; \pi) = \mathbb{E}^{\pi_{\Theta^*}} \left[\sum_{t=1}^T r_t \right] - \mathbb{E}^\pi \left[\sum_{t=1}^T r_t \right]$$

Results. One of our main results will be to construct an algorithm $\pi_{\text{nuc-norm}}$ that exploits low-rank structure in Θ^* to achieve order-lower regret compared with structure-ignorant algorithms.

THEOREM 1 (**Informal**). *If $\operatorname{rank}(\Theta^*) \leq r$ and under some additional technical conditions,*

$$\operatorname{Regret}(T; \pi_{\text{nuc-norm}}) = O(r \max(m, n) \log T).$$

Here we see that the regret grows sublinearly in the dimension mn of the problem, for fixed rank. We contrast this result with the best rate achievable by an algorithm that ignores the low-rank structure of the problem. A second result extends Theorem 1 of Sauré and Zeevi (2013) to a setting with many types.

THEOREM 2 (Informal). *If π is a structure-ignorant algorithm and under some additional technical conditions,*

$$\text{Regret}(T; \pi) = \Omega(\min(m, n) \max(m, n) \log T).$$

Turn to Theorems 5 and 6 in Section 4 for the formal statement, definition of structure-ignorant, and the technical conditions under which the theorems hold.

2.1. Related Work

Assortment personalization requires a good understanding of how consumer tastes vary. Can a retailer learn customer preferences by observing their choices? Discrete choice models posit answers to this question in the form of a probability distribution over choices. Luce (1959) proposed an early discrete choice model based on an axiomatic theory, resulting in the basic attraction model.

Usually, the number of interactions between the firm and customer is limited, so efficient estimation of customer preferences is critical. But estimating customer preferences is no easy task: there are combinatorially many assortments of items, and so without further assumptions, combinatorially many quantities to estimate. To enable tractable estimation, customer preferences are generally modeled parametrically, often using the multinomial logit (MNL) model, which was introduced following the work of McFadden (1973) on random utility theory.

The MNL model posits that customer choices follow a logistic model in a vector of customer preference parameters. Fitting a single MNL model is as simple as counting the number of times an item is chosen relative to the other offerings. (These counts give the maximum likelihood estimate for the model.) The simple MNL model posits a single nominal vector of preferences which governs the choices of all consumers. Individual differences are modeled as random, homogeneous deviations

from these universal preferences, and treated as noise. However, these one-size-fits-all models offer no opportunity for personalization and fit heterogeneous populations poorly.

Learning a personalized model often improves performance. A personalized model *segments* the population into *types* (which may be single individuals, or geographic, demographic, or temporal groups), and fits a separate model for each type. A type is usually chosen to be the smallest group that is uniquely identifiable given the available data. In the offline brick-and-mortar retail setting, populations may be segmented by store branch, since firms have data on aggregate choices in each store branch, and tastes often vary geographically and by setting (mall, street, *etc.*). Customers may also be segmented using loyalty program data. In the e-commerce and online advertising settings, firms have data on the choice of each individual customer from individually personalized assortments. Hence in the limit, each type may represent a single customer.

The MNL model has many more refined variants, some of which allow personalization. For example, the mixture of MNLs (MMNL) model models consumer choice as a mixture of MNL models with different parameters. With sufficiently many mixture components, an MMNL model can approximate arbitrarily closely any choice model that arises from a distribution over individual preferences (McFadden and Train 2000, Farias et al. 2013, van Ryzin and Vulcano 2014).

However, the number of interactions needed to estimate a MMNL model is linear in the product of the number of types and the number of items. This is an astronomical figure in most e-commerce and online advertising contexts, where types and items both number in the millions. It can be enormous even in brick-and-mortar settings: large chains may comprise hundreds or thousands of locations and offer hundreds of thousands of stock keeping units (SKUs). Worse yet, these numbers are far greater than the number of interactions. Each user can view only so many webpages or consider only so many products — generally, far fewer than the number available. The data from each of these interactions is also limited to a single solitary choice (or lack thereof) out of the assortment. Our main focus in this paper is to develop a new effective method to learn and exploit a personalized preference model despite these limits on the number of observed interactions.

Other derivatives of the MNL model can be used to address product substitutes and complements. These include the nested logit model (Williams 1977) and its extensions (McFadden 1980). Recently a new choice model was proposed that arises when substitutions from one good to another are assumed to form a markov chain (Blanchet et al. 2013). Our paper focuses on the issue of personalization and does not consider more complex relationships between products than is modeled by MNL; extending our methods for large-scale personalization to these more nuanced models is a fascinating and important open challenge.

Choosing the optimal assortment can be computationally hard or computationally easy depending on the choice model. Under the MNL model, it is easy to optimize assortments: Talluri and Van Ryzin (2006) show that presenting items in revenue sorted order is always optimal. On the other hand, it is NP-hard to optimize a single assortment to be offered to one MMNL population, even with only two mixture components, but approximation schemes exist (Rusmevichientong et al. 2014). Assortment optimization over the nested MNL model is computationally hard in general (Davis et al. 2014) but easy in some cases (Li et al. 2015). Optimizing an assortment of constrained cardinality under the MNL model is easy (Megiddo 1979, Rusmevichientong et al. 2010), while optimizing an assortment with weighted budget constraint is hard (Désir and Goyal 2014).

Assortment optimization with limited inventory is even more complex. Such problems need to be solved over multiple periods, a different assortment offered each period to take into account possible future stockouts. Talluri and Van Ryzin (2006) solve the classic problem with a single MNL model, while Bernstein et al. (2011), Golrezaei et al. (2014) solve a corresponding problem with a mixture model, representing a few customer segments. In all of these, all preferences of all populations are assumed known.

When preferences are unknown and are to be learned simultaneously with assortment optimization, we can conceive of assortments as bandit arms and consumer choice as bandit feedback to get the problem of *dynamic assortment optimization* (without context). Rusmevichientong et al. (2010) formulated this problem when choices are governed by a single MNL model and showed

that their algorithm has regret upper bounded in order by $n \log^2 T$ for n items and T interactions. Sauré and Zeevi (2013) improved this to $n \log T$ and showed that this order is optimal. Earlier, Caro and Gallien (2007) were the first to conceive of dynamic optimization of assortments under learning by studying a related but different problem. Their problem differs in that the demand to be learned is assumed exogenous and independent of the combination offered and other items. The assortments constitute a simultaneous play of multiple arms, rather than an optimal assortment from which a customer chooses zero or one items.

Our *dynamic assortment personalization* is an instance of a *contextual* bandit problem with discrete contexts. It arises when at each interaction, a different context, drawn from some finite set, is observed. Based on this discrete contextual information, the problem is to personalize the assortment to target each context as well as possible, while also learning to improve performance. In particular, a good algorithm will use lessons learned in one context to improve performance in other contexts. To the best of our knowledge, we are the first to consider this dynamic assortment personalization problem, and in particular the first to consider any stochastic bandit with discrete contexts, rather than continuous contexts with a functional relationship to rewards (such as linear). Lai and Robbins (1985) posed the classic stochastic multi-armed bandit problem, in which each of n arms has an initially-unknown bounded reward distribution and in each time step one chooses one arm to pull with the overall goal of minimal regret: the expected difference in reward between the prescient policy that always pulls the best arm and one’s actual performance. An alternative formulation of the multi-armed bandit problem involves rewards that, instead of being distributed according to a fixed unknown distribution, may change adversarially in response to the choice of arm. For a discussion of the differences between stochastic and adversarial bandits we refer the reader to Bubeck and Cesa-Bianchi (2012). In this paper we focus solely on the *stochastic* bandit. Examples of contextual stochastic bandits include Rigollet and Zeevi (2010), Perchet and Rigollet (2013), Goldenshluger and Zeevi (2013), Slivkins (2014), Bastani and Bayati (2015), which all focus on the setting with n generally unrelated arms, where each arm is associated with a regression

function that governs the expected reward conditioned on a continuous vector of covariates representing context. The former two papers assume a general non-parametric functional dependence; the latter three assume a linear regression function. In all these papers, the context is parametrized by a continuous (scalar or vector) quantity; in other words, the relation between different contexts is embedded topologically and known in advance.

In contrast, in the dynamic assortment personalization, the relation between different contexts must be learned from the data. Contexts are discrete; they correspond to rows of an unknown parameter matrix which governs consumer choice. The observation of choice can be likened (imperfectly) to the noisy observation of an entry of the matrix. This analogy brings to mind the problem of matrix completion: the problem of (approximately) recovering an (approximately) low rank matrix from a few (noisy) samples from its values.

Some of our results are in the same vein as statistical matrix completion bounds. Following groundbreaking work on exact completion of exactly low rank matrices whose entries are observed without noise (Candès and Tao 2010, Candès and Recht 2009, Recht et al. 2010, Keshavan et al. 2010), approximate recovery results have been obtained for a variety of different noisy observation models. These include observations with additive gaussian (Candès and Plan 2009) and subgaussian (Keshavan et al. 2009a) noise, 0-1 (Bernoulli) observations (Davenport et al. 2014), observations from any exponential family distribution (Gunasekar et al. 2014), and observations generated according to the Bradley-Terry-Luce model for pairwise comparisons (Lu and Negahban 2014, Oh et al. 2015). The latter is most related to our matrix recovery results, however, our results differ from these in at least three critical ways. First, our results require as data only a single choice from an assortment, rather than the full rankings of all items in any given list; second, our results allow the customer to choose to buy nothing; and third, our results hold when assortments are subsets of the full set of items, and do not require the possibility of duplicate items (with nonzero probability).

Hence, our results hold in a realistic setting with transactional data while previous results in this literature do not. This type of data is much more common in retail settings than the data

models assumed by previous authors. Transactional data has a few additional advantages. First, it corresponds to passive observations of consumer behavior and so is easy to collect. Second, transactional data is truthful, assuming that choice is utility maximizing.

3. The Low-Rank Mixed Multinomial Logit Choice Model

In this section we describe the LRMMNL model, study the static estimation problem under observing only choice, propose an estimator, prove recovery bounds, and develop a fast algorithm for computing the estimator from large-scale data.

The mixed MNL (MMNL) model over types $i = 1, \dots, m$ and items $j = 1, \dots, n$ is parameterized by $\mu^* \in \Delta^m$ and $\Theta^* \in \mathbb{R}^{m \times n}$ and describes two random variables: type I and choice J . Type I is assumed to be distributed according to

$$\mathbb{P}(I = i) = \mu_i^*.$$

For any given assortment $S \subset \{1, \dots, n\}$, choice J is assumed to be distributed according to the following model

$$\begin{aligned} \mathbb{P}(J = j; S) &= \sum_{i=1}^m \mathbb{P}(I = i) \mathbb{P}(J = j \mid I = i; S) \\ \mathbb{P}(J = j \mid I = i; S) &= \frac{1}{\sum_{j' \in S} e^{-\Theta_{ij'}^*}} \times \begin{cases} 1 & j = 0 \\ 0 & j \neq 0, j \notin S_t \\ \exp(\Theta_{ij}^*) & j \neq 0, j \in S_t \end{cases} \end{aligned} \quad (1)$$

where $J = 0$ represents the choice not to choose from S – an option that is always available for any assortment S .

The LRMMNL model posits an MMNL model in which the parameter Θ^* has low rank:

$$\text{rank}(\Theta^*) \ll m, n.$$

We will also consider the case where Θ^* has approximately low rank:

$$r \ll m, n, \quad \bar{\sigma}_{r+1}(\Theta^*) \approx 0,$$

where $\bar{\sigma}_{r+1}(\Theta^*)$ is the sum of the singular values of Θ^* smaller than the r^{th} largest singular value, i.e.,

$$\bar{\sigma}_{r+1}(\Theta^*) = \sum_{j=r+1}^{\min\{m,n\}} \sigma_j(\Theta^*). \quad (2)$$

3.1. Implications of the LRMMNL Model

Let us consider when the choice distribution should follow a LRMMNL model.

Suppose that each individual in the population makes *rational* choices: that is, choices maximize utility with respect to a vector of utilities randomly distributed over the population. This is called a random utility choice model. The approximation results of McFadden and Train (2000) show that for any random utility choice model, there is a variable I such that the choice distribution is approximately MNL conditioned on I . Hence, if we segment the population finely enough, the MMNL model above approximates the true distribution under the random utility choice model arbitrarily well.

Conversely, any MMNL model, including the LRMMNL model, is a random utility choice model. Let Θ_{ij}^* be the mean utility type i enjoys from item j . Let us suppose that the utility of each customer of type i is the sum of the mean utility of type i together with a random idiosyncrasy distributed according to the Gumbell (extreme value) distribution, and that each customer chooses an item by maximizing her utility among the items on offer:

$$J = \max_{j \in S_i} (\Theta_{ij}^* + \zeta_j) \quad \text{where} \quad \zeta_j \sim \text{Gumbell}(0, 1). \quad (3)$$

The LRMMNL model (1) can therefore arise in either of two ways. It describes choice behavior when customers are clustered into types within each of which customers have a private, idiosyncratic utility distributed as in (3) and the heterogeneity of the population is described by the varying mean utilities u_{ij} over types i . The LRMMNL model also describes choice behavior when each customer is her own type. The random idiosyncrasies associated with each choice event reflect human inconsistencies in decision making or slight variations over time in preferences (Kahneman and Tversky 1979, DeShazo and Fermo 2002).

Learning the preferences of multiple, heterogeneous customer types simultaneously is difficult without additional structure. Both Bernstein et al. (2011) and Golrezaei et al. (2014) study multi-period assortment optimization problems with multiple, heterogeneous customer types assuming full knowledge of the distribution of consumer choice. Both undertake case studies in which they estimate these distributions from static data in order to evaluate the performance of their optimization algorithms on distributions that mimic real data. However, in both cases, they allow only a few segments (3 and 10, respectively). One reason for this choice may be that estimation becomes intractable for models with many more segments. Our model, by contrast, can tractably estimate distributions with large numbers of types and items. We overcome limitations of previous models by assuming that the underlying dimension of the model is small in the sense that our parameter matrix has (approximate) low rank.

If Θ^* has (approximate) rank r , we may factor Θ^* to find vectors $u_{i\ell}, v_{j\ell} \in \mathbb{R}$ for $i = 1, \dots, m, j = 1, \dots, n, \ell = 1, \dots, r$ such that Θ_{ij}^* is (approximately) equal to $\sum_{\ell=1}^r u_{i\ell} v_{j\ell}$. The right factors $v_{j\ell}$ can be thought of as latent item features, and the left factors $u_{i\ell}$ as latent type weights which characterize how much type i values feature ℓ . When Θ^* has (approximate) low rank, we can be sure that just a few latent features suffice to (approximately) explain consumer choice, and these latent features need not be measurable or have a physical interpretation. Indeed, the number of features that matter for decision making may be constrained by cognitive load: to consider many features would require proportional time and energy. However, even if consumer utility is a non-linear function of item features, and even if the number of features required to describe an item is extremely large, Udell and Townsend (2017) prove that large enough preference matrices are still approximately low rank so long as types and items are drawn iid from some population.

In summary, the LRMMNL model is implied by the assumption that choice is rational with a utility distribution with means that depend on only a few (possibly unknown) features. Given enough types and features, this model fits any rational choice model. However, usually very few features suffice due to the finite range of human perception and rationality or simply due to concentration of measure.

3.2. The Static Estimation Problem

Next, we describe an observation model and the problem of estimating the LRMMNL parameters from observed data. We suppose that we have N observations $\{(i_t, j_t, S_t) : t = 1, \dots, N\}$ where S_t is sampled uniformly at random from the set of subsets of $\{1, \dots, n\}$ of size K_t , the sequence K_t is arbitrary (possibly random) satisfying $K_t \leq K$, and i_t, j_t are iid according to the model (1). We also assume that $\|\Theta^*\|_\infty \leq \alpha/\sqrt{mn}$ for purely technical reasons. The assumption of (normalized) bounded entries assumption is standard in many matrix completion recovery results (see Section 2.1) and is necessary for our proof of recovery with high probability; see below.

It is important to highlight that our observation model consists of observing only the choice made by customers. In practical applications, this is typically the only observation possible. Moreover, it is generally truthful since it is utility maximizing, unlike reporting rankings in a survey or focus group.

3.2.1. Our Estimator Define the negative log likelihood of the observations given parameter Θ as

$$L(\Theta) = \frac{1}{N} \sum_{t=1}^N \log \left(\left(1 + \sum_{j \in S_t} e^{\Theta_{ij}} \right) \left(\begin{pmatrix} 1 & j_t = 0 \\ e^{\Theta_{ij_t}} & \text{otherwise} \end{pmatrix}^{-1} \right) \right). \quad (4)$$

We define our estimator $\hat{\Theta}$ for Θ^* as any solution of the nuclear norm regularized maximum likelihood problem

$$\begin{aligned} & \text{minimize } L(\Theta) + \lambda \|\Theta\|_*, \\ & \text{subject to } \|\Theta\|_\infty \leq \alpha/\sqrt{mn}, \end{aligned} \quad (5)$$

where $\lambda > 0$ is a tuning parameter and the nuclear norm $\|\Theta\|_*$ is the sum of the singular values of Θ . We use $\hat{\Theta}$ to denote the solution to this problem.

The constraint $\|\Theta\|_\infty \leq \alpha/\sqrt{mn}$ appears purely as an artifact of the proof; we recommend to omit this constraint in practice. We omit this constraint both in our specialized algorithm (Section 3.3) and in our numerical results (Section 6); the good practical performance on examples demonstrates the practical irrelevance of this constraint.

Problem (5) is convex and hence can be solved by a variety of standard convex methods that take advantage of the special structure of the problem (Cai et al. 2010, Parikh and Boyd 2014, Hazan 2008, Orabona et al. 2012). In Section 3.3 we provide a specialized first-order algorithm that, in fact, works on the *non-convex*, factored form of the problem for increased speed, but still guarantees convergence to the global optimum with high probability.

Our estimator $\hat{\mu}$ for the customer type distribution μ^* is the empirical frequencies of each type:

$$\hat{\mu}_i = \frac{1}{N} \sum_{t=1}^N \mathbb{I}[i_t = i].$$

3.2.2. Main Results for the Static Estimation Problem In this section, we bound the error of our estimators $\hat{\Theta}$ and $\hat{\mu}$. Our first bound is for $\hat{\Theta}$ and depends on the following quantities, which capture the complexity of learning the preferences of all customer types over all items.

- *Number of observations.* The bound decreases as the number N of observations increases.
- *Number of parameters.* The bound grows with the dimensions m, n of the parameter matrix Θ^* .
- *Underlying rank dimension.* For any $r \leq \min(m, n)$, our bound decomposes into two error terms. The first error term is the error in estimating the top r “principal components” of the parameter matrix. This error term grows with \sqrt{r} and captures the benefit of learning only the most salient features instead of all parameters at once. The second error term is the error in approximating the parameter matrix by only its top r “principal components.” In particular, if Θ^* is exactly rank r , then this second error term vanishes. More generally, however, we may be interested in estimating parameter matrices that are only approximately low rank, *i.e.*, with quickly decaying singular values past the top r . In this case, our bound depends on the sum of the remaining singular values.

- *Size of parameters.* Our bound grows with the (scaled) maximum magnitude of any entry α .
- *Size of assortments.* Our bound grows with the maximum size K of the assortments.

THEOREM 3. *Let $\tau \geq 1$ be given, $\rho \geq 1$ be such that $1/\rho \leq m\mu_i \leq \rho \forall i = 1, \dots, m$, and α be such that $\|\Theta^*\|_\infty \leq \alpha/\sqrt{mn}$. Fix $\lambda = 8\sqrt{\frac{\tau\rho K(m+n)\log(m+n)}{mnN}}$. Suppose $N \leq mn \log(m+n)$. Then under*

the observation model in Section 3.2 and for any integer $r \leq \min\{m, n\}$, with probability at least $1 - 3(m+n)^{-\tau}$, any solution $\hat{\Theta}$ to Problem (5) satisfies

$$\|\hat{\Theta} - \Theta^*\|_F \leq 2048\sqrt{\tau}\alpha e^{\frac{4\alpha}{\sqrt{mn}}} \max \left\{ \sqrt{\frac{r(m+n)\log(m+n)}{N}}(\rho K)^3, \left(\frac{\bar{\sigma}_{r+1}(\Theta^*)(m+n)\log(m+n)}{N}(\rho K)^3 \right)^{1/4} \right\},$$

where $\bar{\sigma}_{r+1}(\Theta^*)$ is the sum of the remaining singular values of Θ^* smaller than the r^{th} largest singular value, as defined in eq. (2).

A few remarks on this theorem are in order.

- If Θ^* were exactly low rank ($\bar{\sigma}_{r+1}(\Theta^*) = 0$), then a number of observations scaling slightly faster than $r \max(m, n) \log(m+n)$ are needed in order to obtain a consistent estimate for Θ^* .
- The first term in the bound represents the *estimation error*: the difficulty of estimating the top rank- r approximation to Θ from only N samples. The second term in the bound represents the *approximation error* in the model: the error incurred because the target rank r is smaller than the true rank of Θ^* . This term is zero when $\text{rank}(\Theta^*) \leq r$ and is small when the singular values of Θ^* that are smaller than the r^{th} smallest one are small.
- The choice of λ does not depend on r and the result holds for any $r \leq \min\{m, n\}$. That means that it is not necessary to know the rank or approximate rank of Θ^* – as long as it has (approximate) low rank for some unknown but not too large r , our algorithm will be able to recover Θ^* with high fidelity.
- The proof of this theorem requires a bound $N \leq mn \log(m+n)$ on the maximum number of observations used to fit the estimator. From a practical perspective, this upper bound presents no difficulties: generally, the estimation problem is hard when few observations are available; whereas when $N > mn \log(m+n)$ simpler approaches such as maximum likelihood estimation can perform well and give consistent estimates. Furthermore, in high-dimensional settings it is generally impossible to exhaustively sample all mn type-item pairs; hence as a practical matter, we will always have that $N \leq mn \log(m+n)$ holds.

- The parameter τ controls the probability of the result. Choosing $\tau = 1$, we see the theorem already holds with extremely high probability, which converges to 1 as either m or n grow. Other values for τ give greater generality to the theorem. We will see a more sophisticated use of this probability control τ in the proof of Theorem 6.

A closely related result to Theorem 3 appeared in our preliminary work (Kallus and Udell 2016), which focuses only on the static estimation problem and only in the absence of the no-choice option. In assortment personalization, we must consider estimation in the permanent presence of a no-choice option in *any* assortment and where the mixtures μ are not necessarily uniform, and we must also consider the decision problem involved in dynamically offering personalized assortments. None of these appear in our brief preliminary work.

Our second bound provides recovery guarantees for the customer type distribution μ^* .

THEOREM 4. *Let $\tau \geq 0$ and $q \in [1, \infty]$ be given. With probability at least $1 - e^{-\tau}$,*

$$\|\hat{\mu} - \mu^*\|_q \leq \min \left\{ 8\sqrt{\frac{\tau + m}{N}}, \frac{m^{1/q}}{\sqrt{2}} \sqrt{\frac{\tau + \log(2m)}{N}} \right\}.$$

The second term in the above min is immediate from applying Hoeffding's inequality to each component and using the union bound. For $q = 1$ and any τ , however, the Hoeffding-based bound diverges for any $N = O(m^2)$ (in general it goes to zero only for $N = \omega(m^{2/q})$). We derive the first term using a Rademacher complexity argument. The first term goes to zero for any q as long as $N = \omega(m)$ grows superlinearly in m . This first term is critical for showing that μ^* can be estimated consistently in the q norm for $q < 2$. In particular, the theorem provides for $1/\sqrt{N}$ -consistent estimation of the type distribution in the ℓ_1 norm, which we will use in the proof of Theorem 7.

3.3. A Factored Gradient Descent Algorithm

In this section, we develop a specialized first-order algorithm for computing $\hat{\Theta}$ that works on the non-convex, factored form of the nuclear-norm regularized likelihood optimization problem. The algorithm is particularly economical with memory because (a) it does not keep all $m \times n$ optimization variables in memory but rather only $\tilde{r} \times (m + n)$ where \tilde{r} is a guess at rank and (b)

it eschews the use of *any* spectral computation such as SVD (or partial SVD) at each iteration. This makes the algorithm particularly useful for the large scale data encountered in e-commerce applications.

Our factored gradient descent (FGD) algorithm solves the problem

$$\text{minimize } L(\Theta) + \lambda \|\Theta\|_*. \quad (6)$$

As discussed in Sec. 3.2, the algorithm we employ does *not* enforce any constraint on $\|\Theta\|_\infty$. A constraint of this form is necessary for the technical result in our main theorem, but is unnecessary in practice, as can be seen in our numerical results in Section 6.

In applications, one is interested in solving the problem (6) for very large m, n, N . Due to the complexity of Cholesky factorization, this rules out theoretically-tractable second-order interior point methods. One standard approach is to use a first-order method, such as Cai et al. (2010), Parikh and Boyd (2014), Hazan (2008), Orabona et al. (2012); however, this approach requires (at least a partial) SVD at each step. An alternative approach, which we take here, is to optimize as variables the factors $U \in \mathbb{R}^{m \times \tilde{r}}$ and $V \in \mathbb{R}^{n \times \tilde{r}}$ of the optimization variable $\Theta = UV^T$ rather than producing these via SVD at each step; see, *e.g.*, Keshavan et al. (2009b), Jain et al. (2013). To guarantee equivalence of the problems, we must take $\tilde{r} = \min(m, n)$. However, if we believe the solution is low rank or if we want to enforce low rank, then we may use a smaller \tilde{r} , reducing computational work and storage.

Our FGD algorithm proceeds by applying gradient descent steps to the unconstrained problem

$$\begin{aligned} &\text{minimize } L(UV^T) + \frac{\lambda}{2} \|U\|_F^2 + \frac{\lambda}{2} \|V\|_F^2, \\ &\text{subject to } U \in \mathbb{R}^{m \times \tilde{r}}, V \in \mathbb{R}^{n \times \tilde{r}}. \end{aligned} \quad (7)$$

This formulation has the advantage of being unconstrained, with a differentiable objective function, making it amenable to solution via simple optimization methods (Recht et al. 2010, 2011, Udell et al. 2016).

LEMMA 1. *Problem (7) is equivalent to*

$$\begin{aligned} & \text{minimize } L(\Theta) + \lambda \|\Theta\|_* \\ & \text{subject to } \text{rank}(\Theta) \leq \tilde{r}. \end{aligned} \tag{8}$$

That is, Problem (7) is equivalent to Problem (6) subject to an additional rank constraint $\text{rank}(\Theta) \leq \tilde{r}$. If, for a particular choice of loss function L , Problem (6) has a solution with rank less than \tilde{r} , then the rank constraint is not binding, so Problem (6) is itself equivalent to Problem (7). (Recall that we defer all proofs to the Appendix.)

It is easy to compute the the gradients of the objective of (7). Since $L(\Theta)$ is differentiable,

$$\begin{aligned} \nabla_U L(UV^T) &= \nabla L(UV^T)V, \\ \nabla_V L(UV^T) &= \nabla L(UV^T)^T U. \end{aligned}$$

We do not need to explicitly form $\nabla L(UV^T)$ in order to compute these; computing gradients implicitly reduces the memory required to implement the algorithm (see Algorithm 1). Similarly, we need not form UV^T to compute $L(UV^T)$. Recent work has shown that gradient descent on the factors converges linearly to the global optimum for problems that enjoy restricted strong convexity (Bhojanapalli et al. 2015). In eq. (EC.3) in the proof of Theorem 3 we establish restricted strong convexity for our problem with high probability. Hence with high probability, FGD converges to the global minimum of Problem (7), and hence to the global minimum of Problem (6) provided \tilde{r} is chosen to be large enough.

We initialize our algorithm using a technique from Bhojanapalli et al. (2015), which only requires access to gradients of the objective of (6). Using the SVD, we write $-\nabla L(0) = \tilde{U} \text{diag}(\tilde{\sigma}_1, \dots, \tilde{\sigma}_{\min(m,n)}) \tilde{V}^T$ and initialize

$$\begin{aligned} U^0 &= \gamma^{-1/2} \text{diag}(\sqrt{\tilde{\sigma}_1}, \dots, \sqrt{\tilde{\sigma}_{\tilde{r}}}) \tilde{U}_{:, \tilde{r}}, \\ V^0 &= \gamma^{-1/2} \text{diag}(\sqrt{\tilde{\sigma}_1}, \dots, \sqrt{\tilde{\sigma}_{\tilde{r}}}) \tilde{V}_{:, \tilde{r}}, \end{aligned}$$

where $\gamma = \|\nabla L(0) - (\nabla L(e_1 e_1^T) + \lambda e_1 e_1^T)\|_F$ and $\tilde{U}_{:, \tilde{r}}, \tilde{V}_{:, \tilde{r}}$ denote the first r columns of \tilde{U}, \tilde{V} . We use an adaptive step size with a line search that guarantees descent. Starting with a stepsize of

Algorithm 1 Factored Gradient Descent for (6)

input: dimensions m, n, \tilde{r} , data $\{(i_t, j_t, S_t)\}_{t=1}^N$, regularizing coefficient λ , and tolerance τ $U \leftarrow U^0, V \leftarrow V^0, f' \leftarrow \infty.$ **repeat** $\eta \leftarrow 1, f \leftarrow f', \Delta U \leftarrow -\lambda U, \Delta V \leftarrow -\lambda V$ **for** $t = 1, \dots, N$ **do** $Q = 0$ **for** $j \in S_t$ **do** $w_j \leftarrow e^{-U_{i_t}^T V_j}$ $W \leftarrow W + w_j$ **end for** $\Delta U \leftarrow \Delta U - \frac{1}{N} \left(e_{i_t} V_{j_t}^T - \frac{1}{W} \sum_{j \in S_t} w_j e_{i_t} V_j^T \right)$ $\Delta V \leftarrow \Delta V - \frac{1}{N} \left(e_{j_t} U_{i_t}^T - \frac{1}{W} \sum_{j \in S_t} w_j e_j U_{i_t}^T \right)$ **end for****repeat** $U' \leftarrow U + \eta \Delta U, V' \leftarrow V + \eta \Delta V$ $f' \leftarrow L(U' V'^T) + \frac{\lambda}{2} \|U'\|_F^2 + \frac{\lambda}{2} \|V'\|_F^2$ $\eta \leftarrow \beta_{\text{dec}} \eta$ **until** $f' \leq f$ $U \leftarrow U', V \leftarrow V'$ **until** $\frac{f-f'}{f'} \leq \tau$ **output:** UV^T

$\eta = 1$, the stepsize is repeatedly decreased by a factor β_{dec} until the step produces a decrease in the objective. We terminate the algorithm when the decrease in the relative objective value is smaller than the convergence tolerance τ .

4. The Dynamic Assortment Personalization Problem

We now return to the dynamic setting. First, we define some notation that will be useful in the following discussion. For $\theta \in \mathbb{R}^n$, $w \in \mathbb{R}^n$, $S \subset \{1, \dots, n\}$, and $K \leq n$, let

$$p_j(S; \theta) = \frac{e^{\theta_j}}{1 + \sum_{j' \in S} e^{\theta_{j'}}}, \quad \text{the MNL choice probability under parameters } \theta,$$

$$F(S; w, \theta) = \sum_{j \in S} p_j(S; \theta) w_j, \quad \text{the expected revenue of assortment } S \text{ under } \theta, \text{ and}$$

$$S^*(w, \theta; K) = \operatorname{argmax}_{|S| \leq K} F(S; w, \theta), \quad \text{the optimal assortment under } \theta \text{ for revenues } w.$$

4.1. Structure-ignorant algorithms

An algorithm is *structure-ignorant* if it ignores any potential structure that connect the different contexts (rows of Θ^*). Therefore, a structure-ignorant algorithm is one that runs separate, independent algorithms for each context. Formally, we make the following definition.

DEFINITION 2. An algorithm π for the dynamic assortment personalization problem is *structure-ignorant* if, under π , the variable S_t is measurable with respect to only the historical data from type i_t , $\{i_{t'}, j_{t'}, S_{t'} : t' < t, i_{t'} = i_t\}$, and the variables U_1, \dots, U_t .

An algorithm is said to be consistent if it has sublinear regret in T over all problem instances.

DEFINITION 3. Fix (m, n, W) . An algorithm π is said to be *consistent* if for any μ^* , Θ^* , and $a > 0$, we have $\text{Regret}(T; \pi) = o(T^a)$ over all problem instances $(m, n, W, \mu^*, \Theta^*)$.

For brevity, we usually omit the subscript and abuse notation in referring to a family or sequence of algorithms simply as one algorithm π .

If we run separate algorithms for each context then, in each context, we are solving the classic (non-contextual) dynamic assortment planning problem, precisely as studied by Rusmevichientong et al. (2010), Sauré and Zeevi (2013). Making use of Theorem 1 of Sauré and Zeevi (2013) we establish the following:

THEOREM 5. Fix $\nu \in (0, 1)$, $w \geq 0$, $\rho \geq 1$. Given any family (indexed by revealed problem parameters m, n, W) of structure-ignorant consistent algorithms π , we have

$$\text{Regret}(T; \pi) = \Omega(\min(m, n) \max(m, n) \log T)$$

over all times T and problem instances $(m, n, W, \mu^*, \Theta^*)$ such that

1. T grows superlinearly in the number of types m :

$$\exists \epsilon > 0 : T = \Omega(m^{1+\epsilon}).$$

2. For every type $i = 1, \dots, m$:

- (a) Profit is bounded:

$$\|W_i\|_\infty \leq w.$$

- (b) The type appears often enough and not too often:

$$1/\rho \leq m\mu_i^* \leq \rho.$$

- (c) The number of potentially optimal items grows linearly in n :

$$\left| \{j \notin S^*(W_i, \Theta_i^*; K) : \exists \theta \in \mathbb{R}^n, j \in S^*(W_i, \theta; K), \theta_{j'} = \Theta_{ij}^* \forall j' \in S^*(W_i, \Theta_i^*; K)\} \right| \geq \nu n.$$

Note that $mn = \min(m, n)\max(m, n)$. We rewrite the product mn in this way to make the comparison with our structure-aware algorithms more clear: they depend on $\max(m, n)$, but replace the $\min(m, n)$ term by the rank r of the parameter matrix Θ .

Theorem 5 asserts that structure-ignorant algorithms for the dynamic assortment personalization problem have regret that grows linearly in the number of item-type combinations and logarithmically in the time horizon. That is, ignoring structure incurs *enormous* regret. In the sequel, we improve on this regret bound by developing a structure-aware algorithm.

4.2. A Structure-Aware Algorithm

In Section 3, we argued that imposing structure is crucial for learning the preferences of a very heterogeneous population and proposed an estimator that can leverage structure to learn the LRMMNL model in sublinear time. In this section, we use this estimator to develop a structure-aware algorithm that can achieve regret sublinear in problem size mn .

In Algorithm 2, we define an algorithm for dynamic assortment personalization with tuning parameters C and λ . We refer to this algorithm as $\pi_{\text{nuc-norm}}$.

Algorithm 2 Dynamic Assortment Personalization ($\pi_{\text{nuc-norm}}(C, \lambda)$)

input: C, λ Initialize set of randomized observations $\mathcal{O} = \emptyset$.**for** $t = 1, 2, \dots$ **do** **if** $|\mathcal{O}| \leq Cr(m+n)\log(t)$ **then** *Explore:* observe customer type i_t , choose S_t uniformly at random from all subsets of size K , observe customer product choice j_t , and update the set of randomized observations $\mathcal{O} \leftarrow \mathcal{O} \cup (i_t, j_t, S_t)$. *Estimate:* let

$$L(\Theta) = \frac{1}{|\mathcal{O}|} \sum_{(i,j,S) \in \mathcal{O}} \log \left(\left(1 + \sum_{j' \in S} e^{\Theta_{ij'}} \right) \left(\begin{matrix} 1 & j=0 \\ e^{\Theta_{ij}} & \text{otherwise} \end{matrix} \right)^{-1} \right),$$

and solve

$$\begin{aligned} \hat{\Theta} &\in \operatorname{argmin} L(\Theta) + \lambda \|\Theta\|_* \\ \text{s.t. } &\|\Theta\|_\infty \leq \alpha/\sqrt{mn}. \end{aligned}$$

else *Exploit:* observe customer type i_t and choose $S_t \in S^*(W_{i_t}, \hat{\Theta}_{i_t}; K)$. **end if****end for****output:** S_1, S_2, \dots

Every step of Algorithm 2 is computationally tractable, including the computation of the optimal assortment $S^*(W_{i_t}, \hat{\Theta}_{i_t}; K)$ (Rusmevichientong et al. 2010) and of $\hat{\Theta}$ (Algorithm 1). Using memoization for $S^*(W_i, \hat{\Theta}_i; K)$ between exploration steps, we need only compute optimal assortments logarithmically-many times in t . Further computational savings can be made by batching

exploration steps, which we do not explore here: the scalings in regret as well as computational time remain the same.

Next we show that this achieves regret that is order-of-magnitude smaller than a structure-ignorant algorithm when structure is present. Let

$$\delta(w, \theta; K) = \max_{|S| \leq K} F(S(w, \theta; K); w, \theta) - \max_{S \notin S^*(w, \theta; K): |S| \leq K} F(S; w, \theta)$$

be the gap in revenue between the optimal assortment and any suboptimal assortment under MNL choice with parameter vector θ .

THEOREM 6. *Let r be such that $r \geq \text{rank}(\Theta^*)$, $\rho \geq 1$ be such that $1/\rho \leq m\mu_i \leq \rho \ \forall i = 1, \dots, m$, α be such that $\|\Theta^*\|_\infty \leq \alpha/\sqrt{mn}$, δ be such that $\min_{1 \leq i \leq m} \delta(W_i, \Theta_i^*; K) \geq \delta$, and ω be such that $\|W\|_\infty \leq \omega$. Choose as algorithm parameters*

$$C = 1048576K^6\rho^3\omega^2\alpha^2\exp(8\alpha)/\delta^2, \quad \lambda = 8\sqrt{\frac{\rho K}{Crmn}}. \quad (9)$$

Then the regret of $\pi_{\text{nuc-norm}}(C, \lambda)$ satisfies

$$\begin{aligned} \text{Regret}(T; \pi_{\text{nuc-norm}}(C, \lambda)) &\leq ((Cr(m+n) + 3)\log(T) + 1)\omega \\ &= O(r \max(m, n) \log(T)) \end{aligned}$$

for all $T \leq (m+n)^{\frac{mn}{C(m+n)^r}}$.

This shows that leveraging the structure appropriately can lead to regret that is potentially an order of magnitude smaller. That is, if $\text{rank } r$ is constant, or grows sublinearly (*e.g.*, logarithmically) in the dimension $\min(m, n)$, then the regret of the $\pi_{\text{nuc-norm}}$ algorithm also grows sublinearly in the dimension of the problem $mn = \min(m, n)$. In particular, whenever $r = o(\min(m, n))$, the regret is always (asymptotically) smaller than $\min(m, n) \max(m, n) \log(T)$, which is the lower bound on the regret of any structure-ignorant consistent algorithm. In high-dimensional settings, the restriction on horizon $T \leq (m+n)^{\frac{mn}{C(m+n)^r}}$ in Theorem 6 grows exponentially fast and quickly becomes unrestrictive.

5. The Dynamic Assortment Planning Problem with a Heterogeneous Population

In this section we consider a non-contextual dynamic assortment planning problem where the target population is very heterogeneous. This is appropriate for settings without the potential for personalization, such as a single brick-and-mortar store, but where the consumer population is very heterogeneous and cannot be well described by only a single, or even a few, nominal preference vectors. This setting is similar to that considered in Rusmevichientong et al. (2010) and Sauré and Zeevi (2013) in that it is non-contextual, but in both those works the choice distribution is assumed to be MNL (or, generally, any random utility model with a single nominal preference vector, from which deviations are made at random in a homogeneous manner). Here we allow for a potentially heterogeneous population with choice governed by the LRMMNL model.

The problem proceeds as follows, in somewhat different order than the personalization problem presented in Section 2. At each $t = 1, 2, \dots$:

1. we choose any $S_t \subset \{1, \dots, n\}$ with $|S_t| \leq K$,
2. a type i_t is drawn at random from $\{1, \dots, m\}$ with probability proportional to weights μ_i^* ,
3. an item j_t is drawn at random from $\{1, \dots, n\}$ with probability proportional to weights

$$\text{weight}(j) = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0, j \notin S_t \\ \exp(\Theta_{i_t j}^*) & j \neq 0, j \in S_t \end{cases}$$

4. if $j_t = 0$ we get reward $r_t = 0$ and otherwise we get reward $r_t = W_{i_t j_t}$.

Unlike before, we have to select S_t *before* observing i_t . Therefore, we cannot personalize.

Let us re-define

$$F(S; W, \Theta, \mu) = \sum_{i=1}^m \mu_i \frac{\sum_{j \in S} e^{\Theta_{ij}} W_{ij}}{1 + \sum_{j \in S} e^{\Theta_{ij}}}, \quad \text{the expected revenue of } S \text{ under } \Theta, \text{ and}$$

$$S^*(W, \Theta, \mu; K) = \arg \max_{|S| \leq K} F(S; W, \Theta, \mu), \quad \text{the optimal assortment under } \Theta.$$

Note that $S^*(W, \Theta, \mu; K)$ is not efficiently computable; however, an efficient approximation scheme, which searches over revenue-ordered assortments, is proposed in Rusmevichientong et al. (2014) and shown to work well.

We adapt our nuclear-norm-regularized algorithm to this case as shown in Algorithm 3. We refer to this algorithm as $\pi_{\text{nuc-norm-plan}}$. The only difference between Algorithm 3 and Algorithm 2 is that to exploit our knowledge of $\hat{\Theta}$, we choose any set in $S^*(W, \hat{\Theta}, \hat{\mu}; K)$ rather than observing i_t and choosing a set in $S^*(W_{i_t}, \hat{\Theta}_{i_t}; K)$.

Algorithm 3 Dynamic Assortment Planning ($\pi_{\text{nuc-norm-plan}}(C, \lambda)$)

input: C, λ

Initialize set of randomized observations $\mathcal{O} = \emptyset$

for $t = 1, 2, \dots$ **do**

if $|\mathcal{O}| \leq Cr(m+n)\log(t)$ **then**

Explore:

 choose S_t uniformly at random from all subsets of size K ,

 observe customer type i_t ,

 observe customer product choice j_t , and

 update the set of randomized observations $\mathcal{O} \leftarrow \mathcal{O} \cup (i_t, j_t, S_t)$.

Estimate: let

$$L(\Theta) = \frac{1}{|\mathcal{O}|} \sum_{(i,j,S) \in \mathcal{O}} \log \left(\left(1 + \sum_{j' \in S} e^{\Theta_{ij'}} \right) \left(\begin{cases} 1 & j=0 \\ e^{\Theta_{ij}} & \text{otherwise} \end{cases} \right)^{-1} \right),$$

and solve

$$\hat{\Theta} \in \operatorname{argmin} L(\Theta) + \lambda \|\Theta\|_*$$

$$\text{s.t. } \|\Theta\|_\infty \leq \alpha / \sqrt{mn}$$

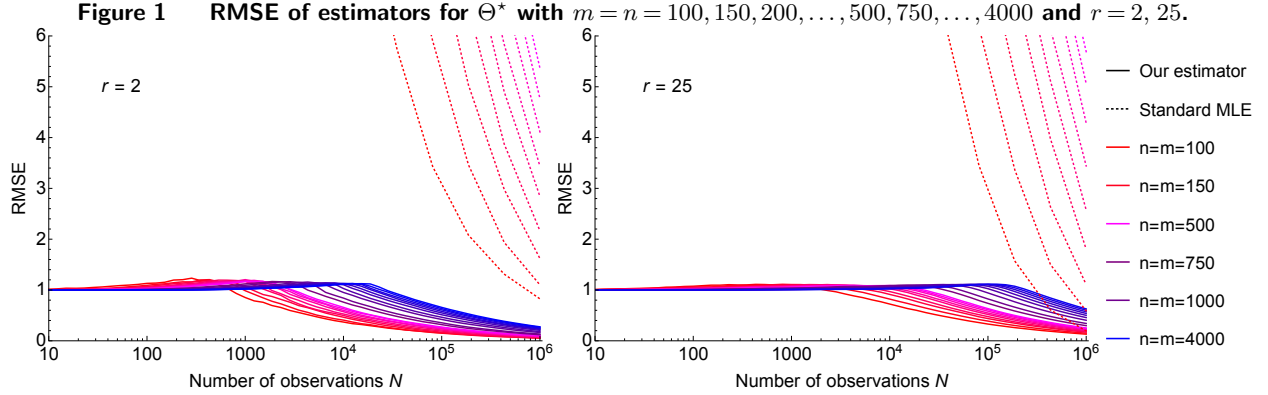
else

Exploit: choose $S_t \in S^*(W, \hat{\Theta}, \hat{\mu}; K)$.

end if

end for

output: S_1, S_2, \dots



Next we show that this achieves regret that is order-of-magnitude smaller than a structure-ignorant algorithm when structure is present. Let

$$\delta(W, \Theta, \mu; K) = F(S^*(W, \Theta, \mu; K); W, \Theta, \mu) - \max_{S \notin S^*(W, \Theta, \mu; K): |S| \leq K} F(S; W, \Theta, \mu)$$

be the gap in revenue between the optimal assortment and any suboptimal assortment under parameter matrix Θ .

THEOREM 7. *Let r be such that $r \geq \text{rank}(\Theta^*)$, $\rho \geq 1$ be such that $1/\rho \leq m\mu_i \leq \rho \ \forall i = 1, \dots, m$, α be such that $\|\Theta^*\|_\infty \leq \alpha/\sqrt{mn}$, δ be such that $\min_{1 \leq i \leq m} \delta(W_i, \Theta_i^*; K) \geq \delta$, and ω be such that $\|W\|_\infty \leq \omega$. Choose algorithm parameters as in eq. (9). Then the regret of $\pi_{\text{nuc-norm-plan}}(C, \lambda)$ satisfies*

$$\begin{aligned} \text{Regret}(T; \pi_{\text{nuc-norm-plan}}(C, \lambda)) &\leq (Cr(m+n)\log(T) + 4)\omega \\ &= O(r \max(m, n) \log(T)) \end{aligned}$$

for all $T \leq (m+n)^{\frac{mn}{C(m+n)^r}}$.

6. Experimental Results

In this section we demonstrate numerically the importance of structure and the power of our approach.

6.1. The Static Estimation Problem

First, we focus on the static estimation problem. We compare our estimate $\hat{\Theta}$ with the standard maximum likelihood estimate $\hat{\Theta}^{\text{MLE}}$ that solves

$$\text{minimize } L(\Theta). \tag{10}$$

Note that, since it imposes no structure on the whole matrix Θ , problem (10) decomposes into m subproblems for each type (row of $\hat{\Theta}^{\text{MLE}}$), each solving a separate MNL MLE in n variables. In our experiments, we use Newton's method as implemented by `Optim.jl` to solve each subproblem.

To generate Θ^* , we fix m, n, r , let Θ_0 be an $m \times n$ matrix composed of independent draws from a standard normal, take its SVD $\Theta_0 = U \text{diag}(\sigma_1, \sigma_2, \dots) V^T$, truncate it past the top r components $\Theta_1 = U \text{diag}(\sigma_1, \dots, \sigma_r, 0, \dots) V^T$, and renormalize to achieve unit sample standard deviation to get Θ^* , *i.e.*, $\Theta^* = \Theta_1 / \text{std}(\text{vec}(\Theta_1))$. To generate the choice data, we let i_t be drawn uniformly at random from $\{1, \dots, m\}$, S_t be drawn uniformly at random from all subsets of size 10, and j_t be chosen according to (1) with parameter Θ^* .

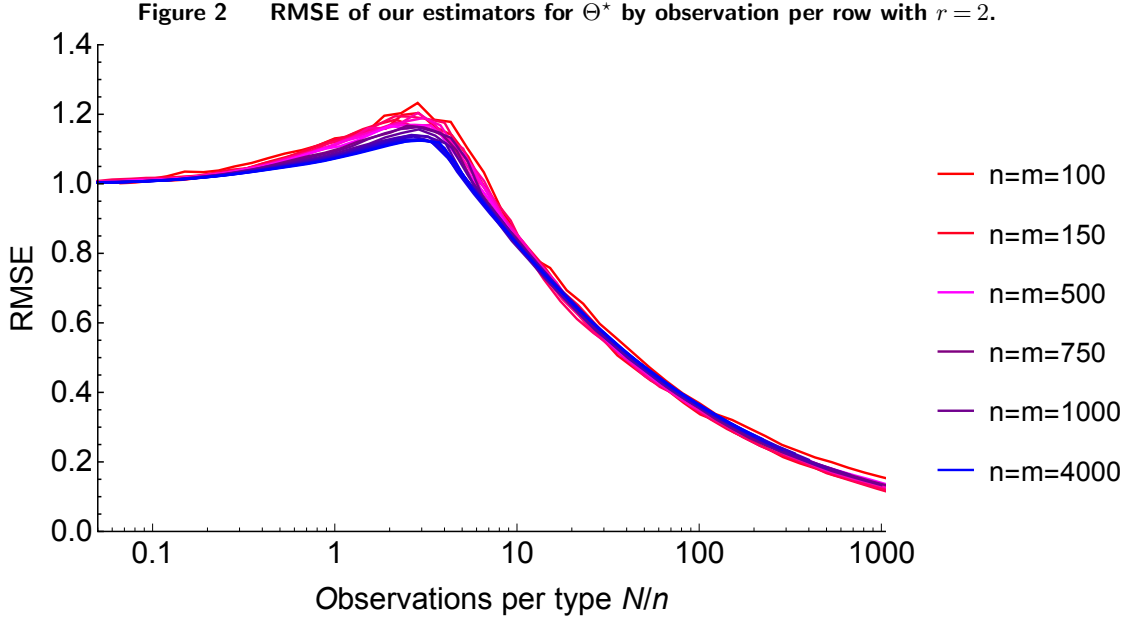
For our estimator we use Algorithm 1 with $\tilde{r} = 2r$, $\lambda = \frac{1}{8} \sqrt{\frac{Kd \log d}{mnN}}$, $\beta_{\text{dec}} = 0.8$, and $\tau = 10^{-10}$. This regularizing coefficient scales with m, n, d, N , and K as suggested by Theorem 3, but we find the algorithm performs better in practice when we use a smaller constant than that suggested by the theorem.

We plot the results in Figure 1, where error is measured in root mean squared error (RMSE)

$$\text{RMSE}(\Theta) = \sqrt{\text{Avg}(\{(\Theta_{ij} - \Theta_{ij}^*)^2\}_{i,j})} = \frac{1}{\sqrt{mn}} \|\Theta - \Theta^*\|_{\text{F}}.$$

The results show the advantage in efficient use of the data offered by our approach. The results also show that, relative to MLE, the advantage is greatest when the underlying rank r is small and the number of parameters $m \times n$ is large, but that we maintain a significant advantage even for moderate r and $m \times n$. For large numbers of parameters ($m = n \geq 750$), the RMSE of MLE is very large and does not appear in the plots. Only in the case of greatest rank ($r = 25$), smallest number of parameters ($m = n = 100$), and greatest number of observations ($T = 10^6$) does MLE appear to somewhat catch up with our estimator.

In Figure 2, we plot the RMSE of our estimator against the number of observations per type (or item) N/d for a square problem with $d = m = n$. We see nearly the same error curve traced out as we vary the problem size d . This scaling shows that our estimator is able to leverage the low-rank assumption and require the same number of choice observations per type to achieve the same RMSE regardless of problem size.

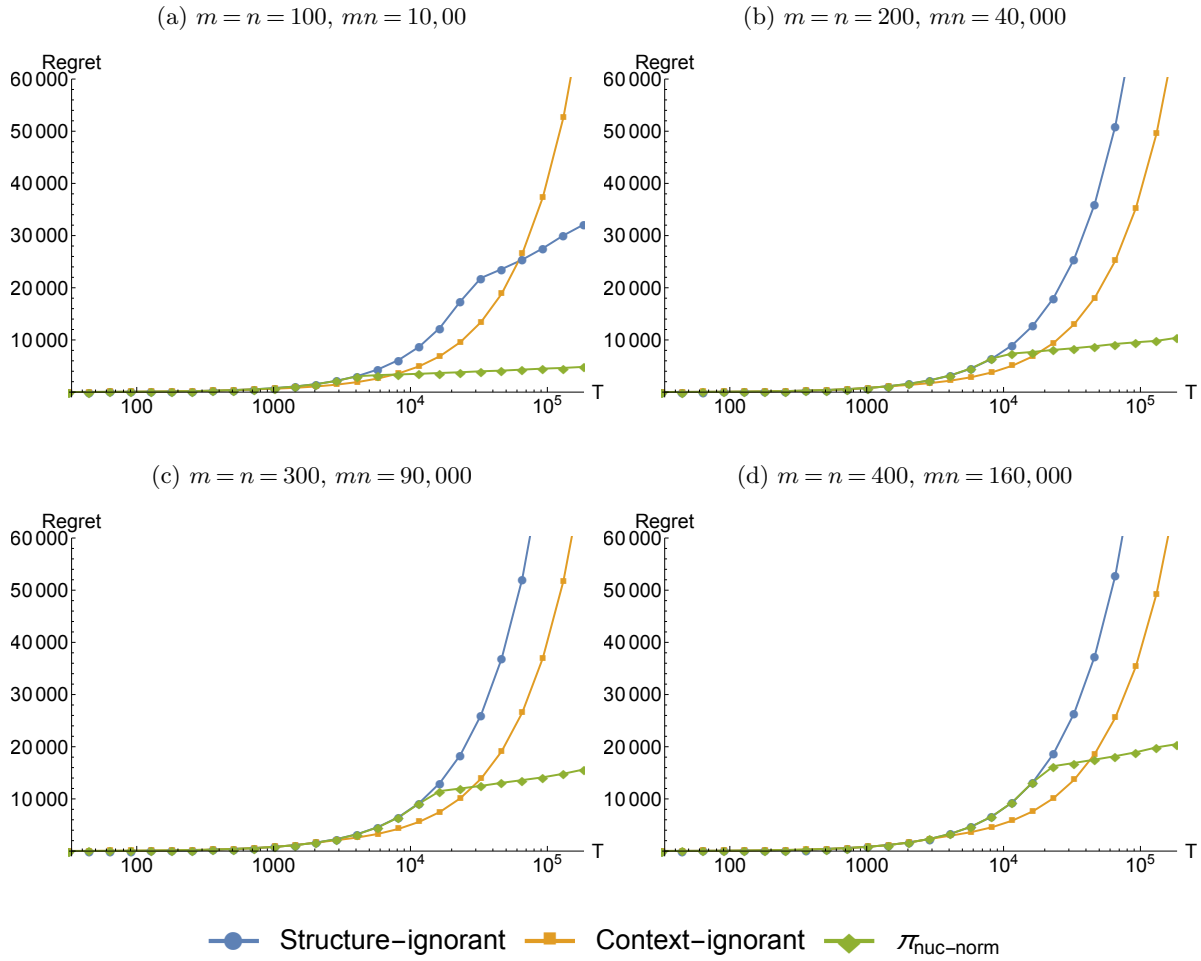


6.2. The Dynamic Assortment Personalization Problem

Next, we turn our attention to the dynamic assortment personalization problem. We compare our algorithm $\pi_{\text{nuc-norm}}$ to two alternatives. One alternative is the structure-ignorant algorithm in which we apply Algorithm 1 of Sauré and Zeevi (2013) to each type separately. Recognizing that one cannot learn a huge set of parameters from very few observations, the second alternative, which we term context-ignorant, tries to fit a single MNL model to the whole population, applying only one replicate Algorithm 1 of Sauré and Zeevi (2013) to all types simultaneously. We generate the problem exactly as in the previous section and run each of these algorithms on problem dimensions varying from 10,000 to 160,000 and for $r = 2$. We plot the regret over time in Figure 3.

Since the plots have a logarithmic horizontal axis, regret that is logarithmic in T appears as a line in the figure whereas regret that is linear in T appears as an exponential function. These plots reveal several interesting features of the algorithms.

We see that the structure-ignorant algorithm, which we know has logarithmic regret asymptotically, only achieves logarithmic regret for the smallest of the problems ($m = n = 100$). For larger problem sizes, regret appears to be linear for all horizons T shown. The transition from linear to logarithmic regret is not visible because the problems are so large that the transition occurs for extremely large T : on the scale of this plot, the mn term overwhelms the $\log(T)$ term.

Figure 3 Regret for Structure-Aware, Structure-Ignorant, and Context-Ignorant Algorithms (note horizontal log scale)

In all of these larger problems, the context-ignorant algorithm performs better than the structure-ignorant algorithm. Both have linear regret in this parameter regime. In fact, the context-ignorant algorithm uses a misspecified model, and so has asymptotically linear regret. Hence for very large T (not shown on this plot) it will be overtaken by the structure-ignorant algorithm, whose regret is asymptotically logarithmic. The success of the misspecified context-ignorant algorithm holds an important lesson: when time is limited, it is more effective to use a misspecified model with few parameters than a well specified model with many parameters.

On the other hand, our structure-aware algorithm exhibits logarithmic regret in each and every case, even at this relatively short time scale. The algorithm $\pi_{\text{nuc-norm}}$, as promised by Theorem 6, has logarithmic regret that does not explode astronomically with m, n .

7. Conclusion

To manage revenue, many retailers must solve a dynamic assortment personalization problem: they must learn customers preferences in real time, at scale, from customers' choices from among the items on offer; and they must quickly use this information to present revenue-maximizing assortments. This paper explores a structural approach to enable large scale dynamic assortment personalization. We proposed algorithms using structural (low rank) priors to learn and exploit customer preferences. We presented theoretical and numerical evidence that these algorithms improve on the state of the art by orders of magnitude, and achieve performance suitable for use in practice.

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E-companion

EC.1. Proof of offline recovery guarantee

First we introduce some notation and preliminary results. Let e_ℓ be the ℓ^{th} unit vector, and let e_0 is the vector of all zeros. We infer the dimension of these vectors from context, so $e_{it} \in \mathbb{R}^m$, $e_{jt} \in \mathbb{R}^n$.

Define

- the error to bound $\Delta = \hat{\Theta} - \Theta^*$
- the items not chosen $S'_t = S_t \setminus \{j_t\}$
- $\gamma = \alpha/\sqrt{mn} : \geq \|\Theta\|_\infty$, a bound on the preference parameters, and
- the selection indicator $X_{tj} = e_{it} e_j^T$, and
- the mean square preference parameter $Y_t(\Theta) = \frac{1}{K_t} \sum_{j_t \in S_t} \Theta_{itj}^2$.

Using this notation, we can calculate the loss, its gradient, and its Hessian as

$$\begin{aligned} L(\Theta) &= \frac{1}{N} \sum_{t=1}^N \left(\log \left(1 + \sum_{j \in S_t} e^{X_{tj} \cdot \Theta} \right) - X_{tj_t} \cdot \Theta \right) \\ \nabla L(\Theta) &= \frac{1}{N} \sum_{t=1}^N \left(\frac{\sum_{j \in S_t} e^{X_{tj} \cdot \Theta} X_{tj}}{1 + \sum_{j \in S_t} e^{X_{tj} \cdot \Theta}} - X_{tj_t} \right) \\ \nabla^2 L(\Theta) &= \frac{1}{N} \sum_{t=1}^N \frac{\left(1 + \sum_{j \in S_t} e^{X_{tj} \cdot \Theta} \right) \left(\sum_{j \in S_t} e^{X_{tj} \cdot \Theta} X_{tj}^{\otimes 2} \right) - \left(\sum_{j \in S_t} e^{X_{tj} \cdot \Theta} X_{tj} \right)^{\otimes 2}}{\left(1 + \sum_{j \in S_t} e^{X_{tj} \cdot \Theta} \right)^2}, \end{aligned}$$

where $A^{\otimes 2} = A \otimes A$ is the symmetric linear operator on matrices defined by $(A \otimes A)(B) = (A \cdot B)A$.

We will use the following three lemmas, whose proofs are presented after that of Theorem 3.

LEMMA EC.1. Let $\mathcal{A}_{\Gamma, \nu} = \{\Theta : \|\Theta\|_\infty \leq \gamma, \|\Theta\|_F \leq \Gamma, \|\Theta\|_* \leq \frac{\nu}{60\sqrt{\rho^3 K mn \gamma}} \sqrt{\frac{N}{(m+n) \log(m+n)}} \Gamma^2\}$ and

$$\mathcal{M}_{\Gamma, \nu} = \sup_{\Theta \in \mathcal{A}_{\Gamma, \nu}} \left(\frac{1}{\rho mn} \|\Theta\|_F^2 - \frac{1}{N} \sum_{t=1}^N Y_t(\Theta) \right).$$

Then

$$\mathbb{P} \left(\mathcal{M}_{\Gamma, \nu} \geq \nu \frac{\Gamma^2}{\rho mn} \right) \leq \exp \left(-\frac{8}{9} \frac{\nu^2}{\rho^2 m^2 n^2} \frac{\Gamma^4}{\gamma^4} N \right)$$

LEMMA EC.2. Let $\tau \geq 1$ be given. Letting

$$\mathcal{A}^* = \left\{ \Theta : \|\Theta\|_\infty \leq \gamma, \|\Theta\|_* \leq \frac{1}{\max \{(18\tau)^{1/4}, 480\}} \frac{1}{\sqrt{\rho^3 K mn \gamma}} \sqrt{\frac{N}{\sqrt{mn} \log(m+n)}} \|\Theta\|_F^2 \right\},$$

we have

$$\mathbb{P} \left(\frac{1}{N} \sum_{t=1}^N Y_t(\Theta) \geq \frac{1}{2\rho mn} \|\Theta\|_F^2 \quad \forall \Theta \in \mathcal{A}^* \right) \geq 1 - 2(m+n)^{-\tau}.$$

LEMMA EC.3. *Let $\tau \geq 1$ be given. With probability at least $1 - (m+n)^{-\tau}$,*

$$\|\nabla L(\Theta^*)\|_2 \leq 4\sqrt{\tau} \sqrt{\frac{\rho K(m+n) \log(m+n)}{mnN}}.$$

Proof of Theorem 3 With probability at least $1 - 3(m+n)^{-\tau}$, the events in both Lemma EC.2 and Lemma EC.3 occur. Let us first assume $\Delta \in \mathcal{A}^*$.

Define $D = L(\hat{\Theta}) - L(\Theta^*) - \nabla L(\Theta^*) \cdot (\hat{\Theta} - \Theta^*)$. By Taylor's theorem, $\exists s \in [0, 1]$ such that

$$\begin{aligned} D &= \nabla^2 L(\Theta^* + s\Delta)[\Delta, \Delta] \\ &= \frac{1}{N} \sum_{t=1}^N \frac{\left(1 + \sum_{j \in S_t} e^{v_{tj}}\right) \left(\sum_{j \in S_t} e^{v_{tj}} (X_{tj} \cdot \Delta)^2\right) - \left(\sum_{j \in S_t} e^{v_{tj}} X_{tj} \cdot \Delta\right)^2}{\left(1 + \sum_{j \in S_t} e^{v_{tj}}\right)^2} \\ &\geq \frac{1}{N} \sum_{t=1}^N \frac{\sum_{j \in S_t} e^{v_{tj}} (X_{tj} \cdot \Delta)^2}{\left(1 + \sum_{j \in S_t} e^{v_{tj}}\right)^2} \geq \frac{1}{N} \frac{1}{e^{4\gamma}} \sum_{t=1}^N \frac{1}{(K_t + 1)^2} \sum_{j \in S_t} (X_{tj} \cdot \Delta)^2 \\ &\geq \frac{1}{N} \frac{1}{e^{4\gamma}} \frac{1}{4K} \sum_{t=1}^N Y_t(\Delta), \end{aligned} \tag{EC.1}$$

where $v_{tj} = X_{tj} \cdot (\Theta^* + s\Delta)$, the first inequality is Jensen's, and the second is from $|v_{tj}| \leq 2\gamma$.

We now decompose Δ as in Recht et al. (2010), Negahban and Wainwright (2011). Let $\Theta^* = U \text{Diag}(\sigma_1, \sigma_2, \dots) V^T$ be the singular-value decomposition of Θ^* with singular values sorted largest to smallest. Using block notation, let us write

$$\begin{aligned} U^T \Delta V &= \Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \quad \text{with } \Gamma_{11} \in \mathbb{R}^{r \times r} \\ \Delta'' &= U \begin{pmatrix} 0 & 0 \\ 0 & \Gamma_{22} \end{pmatrix} V^T, \quad \Delta' = U \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & 0 \end{pmatrix} V^T. \end{aligned}$$

Then $\Delta = U \Gamma V^T = \Delta' + \Delta''$. Note that,

$$\begin{aligned} \text{rank}(\Delta') &= \text{rank}(U^T \Delta' V) \\ &= \text{rank} \left(\begin{pmatrix} \Gamma_{11}/2 & \Gamma_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \Gamma_{11}/2 & 0 \\ \Gamma_{21} & 0 \end{pmatrix} \right) \leq 2r. \end{aligned}$$

Letting $\Theta_r^* = U \text{Diag}(\sigma_1, \dots, \sigma_r, 0, 0, \dots) V^T$ and its complement $\bar{\Theta}_r^* = \Theta^* - \Theta_r^*$, we see

$$\begin{aligned} \|\hat{\Theta}\|_* &= \|\Theta^* + \Delta\|_* = \|\Theta_r^* + \bar{\Theta}_r^* + \Delta' + \Delta''\|_* \\ &\geq \|\Theta_r^* + \Delta''\|_* - \|\bar{\Theta}_r^*\|_* - \|\Delta'\|_* \\ &= \|\Theta_r^*\|_* + \|\Delta''\|_* - \|\bar{\Theta}_r^*\|_* - \|\Delta'\|_* \\ &= \|\Theta^*\|_* + \|\Delta''\|_* - 2\|\bar{\Theta}_r^*\|_* - \|\Delta'\|_*, \end{aligned}$$

and so $\|\hat{\Theta}\|_* - \|\Theta^*\|_* \leq 2\|\bar{\Theta}_r^*\|_* + \|\Delta'\|_* - \|\Delta''\|_*$.

By the optimality of $\hat{\Theta}$, we have

$$L(\hat{\Theta}) + \lambda\|\hat{\Theta}\|_* \leq L(\Theta^*) + \lambda\|\Theta^*\|_*.$$

Hence, by Hölder's inequality,

$$\begin{aligned} 0 \leq D &= L(\hat{\Theta}) - L(\Theta^*) - \nabla L(\Theta^*) \cdot \Delta \\ &\leq \|\nabla L(\Theta^*)\|_2 \|\Delta\|_* + \lambda \left(\|\Theta^*\|_* - \|\hat{\Theta}\|_* \right). \end{aligned} \tag{EC.2}$$

Since $\|\nabla L(\Theta^*)\|_2 \leq \lambda$, triangle inequality in (EC.2) yields

$$D \leq \|\nabla L(\Theta^*)\|_2 \|\Delta\|_* + \lambda \|\Delta\|_* \leq 2\lambda \|\Delta\|_*.$$

Together with Lemma EC.2 and (EC.1) (for the lower bound) and our choice of λ (for the upper bound), this yields

$$\frac{1}{8e^{4\gamma}\rho mnK} \|\Delta\|_{\text{F}}^2 \leq D \leq 16\sqrt{\tau} \sqrt{\frac{\rho K(m+n)\log(m+n)}{mnN}} \|\Delta\|_*. \tag{EC.3}$$

Hence recalling $\gamma = \frac{\alpha}{\sqrt{mn}}$,

$$\|\Delta\|_{\text{F}}^2 \leq 128\sqrt{\tau}\alpha e^{\frac{4\alpha}{\sqrt{mn}}} (\rho K)^{3/2} \sqrt{\frac{(m+n)\log(m+n)}{N}} \|\Delta\|_*. \tag{EC.4}$$

Let's return to (EC.2). Since $\|\nabla L(\Theta^*)\|_2 \leq \lambda/2$, we have

$$\begin{aligned} 0 &\leq \|\nabla L(\Theta^*)\|_2 \|\Delta\|_* + \lambda \left(\|\Theta^*\|_* - \|\hat{\Theta}\|_* \right) \\ &\leq \lambda \left(\frac{1}{2} \|\Delta\|_* + 2\|\bar{\Theta}_r^*\|_* + \|\Delta'\|_* - \|\Delta''\|_* \right) \\ &\leq \lambda \left(2\|\bar{\Theta}_r^*\|_* + \frac{3}{2} \|\Delta'\|_* - \frac{1}{2} \|\Delta''\|_* \right), \end{aligned}$$

and so $\|\Delta''\|_* \leq 3\|\Delta'\|_* + 4\|\bar{\Theta}_r^*\|_*$. Let $\zeta = \|\bar{\Theta}_r^*\|_* = \sum_{j=r+1}^{\min\{m,n\}} \sigma_j$. Since $\|\Delta\|_F - \|\Delta'\|_F = \|\Gamma_{22}\|_F \geq 0$,

$$\begin{aligned}
\|\Delta\|_* &\leq \|\Delta'\|_* + \|\Delta''\|_* = 4\|\Delta'\|_* + 4\zeta \\
&\leq 8 \max\{\|\Delta'\|_*, \zeta\} \leq 8 \max\left\{\sqrt{2r}\|\Delta'\|_F, \zeta\right\} \\
&\leq 8 \max\left\{\sqrt{2r}\|\Delta\|_F, \zeta\right\} \\
&\leq 16 \max\left\{\sqrt{r}\|\Delta\|_F, \zeta\right\}.
\end{aligned} \tag{EC.5}$$

If $\sqrt{r}\|\Delta\|_F \geq \zeta$, substitute (EC.5) in (EC.4) to see

$$\|\Delta\|_F \leq 2048\sqrt{\tau}\alpha e^{\frac{4\alpha}{\sqrt{mn}}} K^{3/2} \sqrt{\frac{r(m+n)\log(m+n)}{N}}.$$

Otherwise (if $\sqrt{r}\|\Delta\|_F < \zeta$), substitute (EC.5) in (EC.4) and take the square root to see

$$\begin{aligned}
\|\Delta\|_F &\leq \sqrt{2048\sqrt{\tau}\alpha e^{\frac{4\alpha}{\sqrt{mn}}} K^{3/2} \sqrt{\frac{\zeta(m+n)\log(m+n)}{N}}} \\
&\leq 2048\sqrt{\tau}\alpha e^{\frac{4\alpha}{\sqrt{mn}}} K^{3/4} \left(\frac{\zeta d \log d}{N}\right)^{1/4}.
\end{aligned}$$

Combining yields the statement.

Our last step is to investigate the case where $\Delta \notin \mathcal{A}^*$, i.e.,

$$\|\Delta\|_* > \max\{(18\tau)^{1/4}, 480\} \frac{1}{\sqrt{\rho^3 K mn \gamma}} \sqrt{\frac{N}{\sqrt{mn} \log(m+n)}} \|\Delta\|_F^2.$$

Rewriting and introducing redundant terms greater than 1, we recover (EC.4) and the proof follows

as before from that point. \square

Proof of Lemma EC.1 Since S_t is symmetric, we have that

$$\mathbb{E}[Y_t(\Theta)] = \sum_{i=1}^m \mu_i \sum_{j=1}^n \frac{1}{n} \Theta_{ij}^2 \geq \frac{1}{\rho mn} \|\Theta\|_F^2.$$

Define

$$\tilde{\mathcal{M}}_{\Gamma, \nu} = \sup_{\Theta \in \mathcal{A}_{\Gamma, \nu}} \frac{1}{N} \sum_{t=1}^N (\mathbb{E}Y_t(\Theta) - Y_t(\Theta))$$

and note that $\tilde{\mathcal{M}}_{\Gamma,\nu} \geq \mathcal{M}_{\Gamma,\nu}$. Then, letting $Y'_t(\Theta)$ be an identical and independent replicate of $Y_t(\Theta)$ and letting ϵ_t be iid Rademacher random variables independent of all else, we have

$$\begin{aligned} \mathbb{E}\tilde{\mathcal{M}}_{\Gamma,\nu} &= \mathbb{E} \left[\sup_{\Theta \in \mathcal{A}_{\Gamma,\nu}} \frac{1}{N} \sum_{t=1}^N (\mathbb{E}Y'_t(\Theta) - Y_t(\Theta)) \right] \leq \mathbb{E} \left[\sup_{\Theta \in \mathcal{A}_{\Gamma,\nu}} \frac{1}{N} \sum_{t=1}^N (Y'_t(\Theta) - Y_t(\Theta)) \right] \\ &= \mathbb{E} \left[\sup_{\Theta \in \mathcal{A}_{\Gamma,\nu}} \frac{1}{N} \sum_{t=1}^N \epsilon_t (Y'_t(\Theta) - Y_t(\Theta)) \right] \leq 2\mathbb{E} \left[\sup_{\Theta \in \mathcal{A}_{\Gamma,\nu}} \frac{1}{N} \sum_{t=1}^N \epsilon_t Y_t(\Theta) \right] \\ &= 2\mathbb{E} \left[\sup_{\Theta \in \mathcal{A}_{\Gamma,\nu}} \frac{1}{N} \sum_{t=1}^N \epsilon_t \frac{1}{K_t} \left\| e_{i_t}^T \Theta \sum_{j \in S_t} e_j \right\|_2^2 \right]. \end{aligned}$$

Note that

$$\sup_{v, v' \in [-\gamma, \gamma]^k \times \{0\}^{n-k}} \frac{\|v\|_2^2 - \|v'\|_2^2}{\|v - v'\|_\infty} = 2\gamma k,$$

i.e., $\|\cdot\|_2^2$ is $2\gamma k$ -Lipschitz with respect to the ∞ -norm on a domain in $[-\gamma, \gamma]^n$ where only k entries are nonzero. Therefore, by Lemma 7 of Bertsimas and Kallus (2014) and by Hölder's inequality, letting $W_t = \sum_{j \in S_t} \epsilon_{tj} e_{i_t} e_j^T$ where ϵ_{tj} are new iid Rademacher random variables independent of all else,

$$\mathbb{E}\tilde{\mathcal{M}}_{\Gamma,\nu} \leq 4\gamma \mathbb{E} \left[\sup_{\Theta \in \mathcal{A}_{\Gamma,\nu}} \frac{1}{N} \sum_{t=1}^N W_t \cdot \Theta \right] \leq 4\gamma \mathbb{E} \left\| \frac{1}{N} \sum_{t=1}^N W_t \right\|_2 \sup_{\Theta \in \mathcal{A}_{\Gamma,\nu}} \|\Theta\|_*.$$

Note that

$$\begin{aligned} \|W_t\|_2 &\leq \sqrt{K} \\ \mathbb{E}[W_t W_t^T | S_t, i_t] &= K_t e_{i_t} e_{i_t}^T & \text{and hence} & \quad \|\mathbb{E}[W_t W_t^T]\|_2 \leq \rho K/m \\ \mathbb{E}[W_t^T W_t | S_t, j_t] &= \sum_{j \in S_t} e_j e_j^T & \text{and hence} & \quad \|\mathbb{E}[W_t^T W_t]\|_2 \leq K/n \end{aligned}$$

Hence, the matrix Bernstein inequality (Theorem 1.6 of Tropp (2012)) gives that $\left\| \frac{1}{N} \sum_{t=1}^N W_t \right\|_2 \geq \delta$ with probability at most

$$(m+n) \max \left\{ e^{-\frac{N\delta^2 \min\{m,n\}}{4\rho K}}, e^{-\frac{\delta}{\sqrt{K}}} \right\}.$$

Setting the probability to $1/\sqrt{mn \min\{m,n\}}$ and using $N \leq mn \log(m+n)$,

$$\begin{aligned} \mathbb{E} \left[\left\| \frac{1}{N} \sum_{t=1}^N W_t \right\|_2 \right] &\leq \sqrt{\frac{K}{mn \min\{m,n\}}} + 2\sqrt{\frac{\rho K (\frac{1}{2} \log(m) + \frac{1}{2} \log(n) \frac{1}{2} \log(\min\{m,n\}) + \log(m+n))}{N \min\{m,n\}}} \\ &\leq \sqrt{\frac{K \log(m+n)}{N \min\{m,n\}}} + 2\sqrt{\frac{\frac{5}{2} \rho K \log(m+n)}{N \min\{m,n\}}} \leq 5\sqrt{\frac{\rho K \log(m+n)}{N \min\{m,n\}}} \end{aligned}$$

Putting it all together, we get,

$$\mathbb{E}\tilde{\mathcal{M}}_{\Gamma,\nu} \leq 20\gamma \sqrt{\frac{\rho K \log(m+n)}{N \min\{m, n\}}} \frac{\nu}{60\sqrt{\rho^3 K m n \gamma}} \sqrt{\frac{N}{(m+n) \log(m+n)}} \Gamma^2 \leq \frac{\nu}{3} \frac{\Gamma^2}{\rho m n}.$$

Next we use this to prove the concentration of $\tilde{\mathcal{M}}_{\Gamma,\nu}$. Let $\tilde{\mathcal{M}}'_{\Gamma,\nu}$ be a replicate of $\tilde{\mathcal{M}}_{\Gamma,\nu}$ with $i'_t = i_t$, $S'_t = S_t$ for all t except t' . Then the difference $\tilde{\mathcal{M}}_{\Gamma,\nu} - \tilde{\mathcal{M}}'_{\Gamma,\nu}$ is bounded by

$$\frac{1}{N} \sup_{\Theta \in \mathcal{A}_{\Gamma,\nu}} \left(\frac{1}{K_{t'}} \sum_{j \in S_{t'}} \Theta_{i_{t'},j}^2 - \frac{1}{K_{t'}} \sum_{j \in S'_{t'}} \Theta_{i'_{t'},j}^2 \right) \leq \frac{1}{N} (\gamma^2 - 0) = \frac{\gamma^2}{N}.$$

Hence, by McDiarmid's inequality, we have

$$\mathbb{P} \left(\tilde{\mathcal{M}}_{\Gamma,\nu} \geq \nu \frac{\Gamma^2}{\rho m n} \right) \leq \mathbb{P} \left(\tilde{\mathcal{M}}_{\Gamma,\nu} - \mathbb{E}\tilde{\mathcal{M}}_{\Gamma,\nu} \geq \frac{2\nu}{3} \frac{\Gamma^2}{\rho m n} \right) \leq \exp \left(-\frac{8}{9} \frac{N \nu^2 \Gamma^4}{\gamma^4 \rho^2 m^2 n^2} \right),$$

which yields the result, given $\tilde{\mathcal{M}} \geq \mathcal{M}$. \square

Proof of Lemma EC.2 Let $\tau' = K^2 \rho^4 \max\{\tau, 480^4/18\}$. Since $\|\cdot\|_* \geq \|\cdot\|_F$, we have $\inf_{\Theta \in \mathcal{A}^*} \|\Theta\|_F \geq \eta := (18\tau')^{1/4} \sqrt{\rho m n \gamma \sqrt{\sqrt{m n} \log(m+n)/N}}$. Set $\nu = 60(18\tau')^{-1/4}$ and $\beta = \sqrt{1/(2\nu)}$. Since $\tau' \geq 480^4/18$ we have $\nu \leq 1/8$ and $\beta \geq 2 > 1$. Let $\mathcal{A}_l = \mathcal{A}^* \cap \{\eta\beta^{l-1} \leq \|\Theta\|_F \leq \eta\beta^l\}$ and note that $\mathcal{A}^* = \bigcup_{l=1}^{\infty} \mathcal{A}_l$ and $\mathcal{A}_l \subset \mathcal{A}_{\beta^l \eta, \nu}$.

If $\Theta \in \mathcal{A}_l$ has $\frac{1}{N} \sum_{t=1}^N Y_t(\Theta) < \frac{1}{2\rho m n} \|\Theta\|_F^2$ then

$$\frac{1}{\rho m n} \|\Theta\|_F^2 - \frac{1}{N} \sum_{t=1}^N Y_t(\Theta) > \frac{1}{2\rho m n} \|\Theta\|_F^2 \geq \frac{1}{2\rho m n} (\beta^{l-1} \eta)^2 = \frac{1}{\rho m n} \frac{1}{2\beta^2} (\beta^l \eta)^2 = \frac{\nu}{\rho m n} (\beta^l \eta)^2.$$

Therefore, the probability that the event in the statement of theorem is invalid is bounded by

$$\begin{aligned} & \min \left\{ 1, \sum_{l=1}^{\infty} \mathbb{P} \left(\mathcal{M}_{\beta^l \eta, \nu} > \frac{\nu}{\rho m n} (\beta^l \eta)^2 \right) \right\} \\ & \leq \min \left\{ 1, \sum_{l=1}^{\infty} \exp \left(-\frac{8}{9} \frac{\nu^2 \beta^{4l} \eta^4 N}{\rho^2 m^2 n^2 \gamma^4} \right) \right\} \\ & \leq \min \left\{ 1, \sum_{l=1}^{\infty} \exp \left(-\frac{1}{72} \frac{4^l \eta^4 N}{\rho^2 m^2 n^2 \gamma^4} \right) \right\} \\ & \leq \min \left\{ 1, \sum_{l=1}^{\infty} \exp \left(-\frac{1}{72} \frac{4^l \eta^4 N}{\rho^2 m^2 n^2 \gamma^4} \right) \right\} \\ & \leq \min \left\{ 1, \sum_{l=1}^{\infty} \exp \left(-\frac{1}{18} \frac{\eta^4 N}{\rho^2 m^2 n^2 \gamma^4} l \right) \right\} \end{aligned}$$

$$\begin{aligned}
&= \min \left\{ 1, \left(\exp \left(\frac{1}{18} \frac{\eta^4 N}{\rho^2 m^2 n^2 \gamma^4} \right) - 1 \right)^{-1} \right\} \\
&\leq 2 \exp \left(-\frac{1}{18} \frac{\eta^4 N}{\rho^2 m^2 n^2 \gamma^4} \right) \\
&= 2 \exp \left(-\tau' mn (\log(m+n))^2 / N \right) \\
&\leq 2(m+n)^{-\tau'} \leq 2(m+n)^{-\tau},
\end{aligned}$$

using Lemma EC.1 and $N \leq mn \log(m+n)$. \square

Proof of Lemma EC.3 Let $\psi_{tj} = e^{X_{tj} \cdot \Theta^*}$ and $G_t = \frac{\sum_{j \in S_t} \psi_{tj} X_{tj}}{1 + \sum_{j \in S_t} \psi_{tj}} - X_{tj_t}$ so that $\nabla L(\Theta^*) = \frac{1}{N} \sum_{t=1}^N G_t$. Note that because j_t is drawn according to Θ^* and $X_{t0} = 0$, we have that $\mathbb{E}[G_t | i_t, S_t] = 0$ and hence $\mathbb{E}G_t = 0$. Note that $\|G_t\|_2 \leq \sqrt{2}$. Moreover,

$$G_t G_t^T = e_{i_t} e_{i_t}^T \left(1 - \frac{2\psi_{tj_t}}{1 + \sum_{j \in S_t} \psi_{tj}} + \frac{\sum_{j \in S_t} \psi_{tj}^2}{\left(1 + \sum_{j \in S_t} \psi_{tj}\right)^2} \right).$$

Since by Jensen's inequality the multiplier in the parentheses is no greater than 2, we get

$\|\mathbb{E}[G_t G_t^T]\|_2 \leq \frac{2\rho}{m} \leq \frac{2\rho K}{m}$. Letting $y_{tj} = \mathbb{I}[j = j_t]$, we have

$$G_t^T G_t = \sum_{j \in S_t, k \in S_t} e_j e_k^T \left(y_{tj} y_{tk} - \frac{2y_{tj} \psi_{tk}}{1 + \sum_{l \in S_t} \psi_{tl}} + \frac{\psi_{tj} \psi_{tk}}{\left(1 + \sum_{l \in S_t} \psi_{tl}\right)^2} \right).$$

Noting that $y_{tj} \geq 0$, $\psi_{tj} \geq 0$, and $y_{tj} y_{tk} \leq \mathbb{I}[j = k]$, we see that

$$\|\mathbb{E}[G_t^T G_t]\|_2 \leq \left\| \mathbb{E} \left[\sum_{j \in S_t} e_j e_j^T \right] \right\|_2 + \left\| \mathbb{E} \left[\frac{1}{K_t^2} \left(\sum_{j \in S_t} e_j \right) \left(\sum_{j \in S_t} e_j \right)^T \right] \right\|_2 \leq \frac{K}{n} + \frac{1}{n} \leq \frac{2\rho K}{n}.$$

Hence, by the matrix Bernstein inequality (Theorem 1.6 of Tropp (2012)),

$$\left\| \frac{1}{N} \sum_{t=1}^N G_t \right\|_2 \leq 2\sqrt{2(\tau+1)} \sqrt{\frac{\rho K \log(m+n)}{\min\{m, n\} N}} \leq 4\sqrt{\tau} \sqrt{\frac{\rho K (m+n) \log(m+n)}{mnN}},$$

with probability at least $1 - (m+n)^{-\tau}$. \square

This lemma concludes the proof of Theorem 3. We now prove that the estimate $\hat{m}u$ for the customer type distribution is consistent.

Proof of Theorem 4 We prove this by cases, depending on which term in the min is smaller. Note that this is not a random event so we can choose which bound to use a priori, yielding the min.

First we deal with the second term. By union bound and Hoeffding's inequality,

$$\mathbb{P}\left(\|\hat{\mu} - \mu^*\|_q > \eta\right) \leq \sum_{i=1}^m \mathbb{P}\left(|\hat{\mu}_i - \mu_i^*| > \eta/m^{1/q}\right) \leq 2m \exp(-2n\eta^2/m^{2/q})$$

Next we deal with the first term. Note that $\|\cdot\|_q \leq \|\cdot\|_1$ so it is sufficient to prove this for $q = 1$.

Let $I_t = e_{i_t}$ be the type indicator observation and let I be a generic draw of I_t from μ . Then $\hat{\mu} = \frac{1}{N} \sum_{t=1}^N I_t$. By Bartlett and Mendelson (2003), we have that with probability at least $1 - \nu$,

$$\|\mu - \hat{\mu}\|_1 = \sup_{\|v\|_\infty \leq 1} \left(\mathbb{E}[v^T I] - \frac{1}{N} \sum_{t=1}^N v^T I_t \right) \leq 2\hat{\mathfrak{R}}_N + \sqrt{\frac{-\log \nu}{2N}},$$

where $\hat{\mathfrak{R}}_N$ is the empirical Rademacher complexity

$$\hat{\mathfrak{R}}_N = \frac{1}{2^N} \sum_{\epsilon \in \{-1, +1\}^N} \sup_{\|v\|_\infty \leq 1} \frac{1}{N} \sum_{t=1}^N \epsilon_t v^T I_t.$$

By linearity and duality of norms, we have

$$\begin{aligned} \hat{\mathfrak{R}}_N &= \frac{1}{2^N} \sum_{\epsilon \in \{-1, +1\}^N} \sup_{\|v\|_\infty \leq 1} v^T \left(\frac{1}{N} \sum_{t=1}^N \epsilon_t I_t \right) \\ &= \frac{1}{2^N} \sum_{\epsilon \in \{-1, +1\}^N} \left\| \frac{1}{N} \sum_{t=1}^N \epsilon_t I_t \right\|_1 \\ &= \frac{1}{N} \sum_{i=1}^m \frac{1}{2^N} \sum_{\epsilon \in \{-1, +1\}^N} \left| \sum_{t=1}^N \mathbb{I}[i_t = i] \epsilon_t \right| \\ &= \frac{1}{N} \sum_{i=1}^m \frac{1}{2^{N\hat{\mu}_i}} \sum_{\epsilon \in \{-1, +1\}^{N\hat{\mu}_i}} \left| \sum_{t=1}^{N\hat{\mu}_i} \epsilon_t \right| \\ &= \frac{1}{N} \sum_{i=1}^m \frac{1}{2^{N\hat{\mu}_i-1}} \left\lceil \frac{N\hat{\mu}_i}{2} \right\rceil \binom{m}{\lceil \frac{N\hat{\mu}_i}{2} \rceil} \\ &\leq \frac{1}{N} \sum_{i=1}^m \sqrt{N\hat{\mu}_i} = \frac{1}{\sqrt{N}} \sum_{i=1}^m \sqrt{\hat{\mu}_i} \leq \sqrt{\frac{m}{N}}. \end{aligned}$$

Therefore, using the concavity of square root, we have that

$$\mathbb{P}(\|\mu - \hat{\mu}\|_1 > \eta) \leq e^{m - N\eta^2/64}.$$

Rearranging yields the result. \square

EC.1.1. Proofs Omitted from Section 3.3

Proof of Lemma 1 Given Θ feasible in (6) with $\text{rank}(\Theta) \leq \tilde{r}$, write its SVD $\Theta = \tilde{U}\Sigma\tilde{V}^T$, where $\Sigma \in \mathbb{R}^{\tilde{r} \times \tilde{r}}$ is diagonal and U, V unitary. Letting $U = \tilde{U}\Sigma^{1/2}$ and $V = \tilde{V}\Sigma^{1/2}$, we obtain a feasible solution to (7) with the same objective value Θ has in (6). Conversely, given U, V feasible in (7), let $\Theta = UV^T$. Note $\text{rank}(\Theta) \leq \tilde{r}$ and

$$\begin{aligned} \|\Theta\|_* &\leq \|UV^T\|_* = \text{tr}(\Sigma) = \text{tr}(\tilde{U}^T UV^T \tilde{V}) \\ &\leq \|\tilde{U}^T U\|_F \|V^T \tilde{V}\|_F \leq \|U\|_F \|V\|_F \leq \frac{1}{2}\|U\|_F^2 + \frac{1}{2}\|V\|_F^2. \end{aligned}$$

Hence, Θ has objective value no worse than (U, V) . \square

EC.1.2. Proofs of Regret Bounds

Before presenting the proofs that a structure aware algorithm can achieve regret sublinear in the number of matrix entries mn , we formally prove that a structure ignorant algorithm must have regret linear in mn . This result, Theorem 5, makes use of Theorem 1 in Sauré and Zeevi (2013), which we restate here in our notation:

THEOREM EC.1 (Sauré and Zeevi (2013), Theorem 1). *Consider a LRMMNL with one type, $m = 1$. Under any consistent policy π for choosing the sets S_t ,*

$$\mathbb{E} \text{Regret}_T(\pi) \geq C_1 + C_2 \tilde{\mathcal{N}} \log T$$

for all T , where

$$\tilde{\mathcal{N}} = |\{j \notin S^*(W_i, \Theta_i^*; K) : \exists \theta \in \mathbb{R}^n, j \in S^*(W_i, \theta; K), \theta_{j'} = \Theta_{ij}^* \forall j' \in S^*(W_i, \Theta_i^*; K)\}|$$

is the set of potentially optimal items, and C_1 and C_2 are constants independent of n and T .

Note that Sauré and Zeevi (2013) prove this lower bound only for policies that do not depend on the time horizon T , since their proof requires taking $T \rightarrow \infty$. However, this proof is easily extended to a lower bound even for policies that depend on the time horizon by considering a sequence of policies for each T .

Proof of Theorem 5 First, use a Chernoff bound to see that the number of times T_i that type i is chosen is larger than $\frac{T\mu_i}{2}$ with high probability:

$$\mathbb{P}\left(T_i > \frac{T\mu_i}{2}\right) \geq 1 - \exp\left(\frac{-1}{8}T\mu_i\right).$$

Recall that $T = \Omega(m^{1+\epsilon})$ grows superlinearly in the number of types m . We use a union bound to show that for large T , with high probability we have $T_i \geq \frac{T\mu_i}{2}$ for every i :

$$\begin{aligned} \mathbb{P}\left(T_i \geq \frac{T\mu_i}{2} \forall i\right) &\geq 1 - \sum_{i=1}^m \exp\left(\frac{-1}{8}T\mu_i\right) \\ &\geq 1 - \sum_{i=1}^m \exp\left(\frac{-T}{8\rho m}\right) \\ &\geq 1 - \sum_{i=1}^m \exp\left(\frac{-m^\epsilon}{8\rho}\right) \\ &\geq 1 - m \exp\left(\frac{-m^\epsilon}{8\rho}\right), \end{aligned}$$

which converges to 1 as m increases. In particular, for sufficiently large T and m , $\mathbb{P}\left(T_i \geq \frac{T\mu_i}{2} \forall i\right) > \frac{1}{2}$.

Let's suppose below that T and m are large enough that this bound holds.

Then conditioning on the event that $T_i \geq \frac{T\mu_i}{2}$ for every i ,

$$\begin{aligned} \mathbb{E}\left[\sum_{t=1}^T r_t\right] &\geq \mathbb{E}\left[\sum_{t=1}^T r_t \mid T_i \geq \frac{T\mu_i}{2} \forall i\right] \mathbb{P}\left(T_i \geq \frac{T\mu_i}{2} \forall i\right) \\ &\geq \frac{1}{2} \mathbb{E}\left[\sum_{t=1}^T r_t \mid T_i \geq \frac{T\mu_i}{2} \forall i\right] \\ &= \frac{1}{2} \mathbb{E}\left[\sum_{i=1}^m \sum_{t: i_t=i} r_t \mid T_i \geq \frac{T\mu_i}{2} \forall i\right] \\ &= \frac{1}{2} \left(\sum_{i=1}^m \mathbb{E}\left[\sum_{t: i_t=i} r_t \mid T_i \geq \frac{T\mu_i}{2} \forall i\right]\right) \\ &= \frac{1}{2} \left(\sum_{i=1}^m \mathbb{E}\left[\sum_{t: i_t=i} r_t \mid T_i \geq \frac{T\mu_i}{2}\right]\right) \end{aligned}$$

Now, using Theorem EC.1 and the assumption that $\tilde{\mathcal{N}} \geq \nu n$,

$$\begin{aligned} \mathbb{E}\left[\sum_{t=1}^T r_t\right] &\geq \frac{1}{2} \left(\sum_{i=1}^m C_1 + C_2(\nu n) \log(T\mu_i/2)\right) \\ &\geq \frac{1}{2} \left(\sum_{i=1}^m C_1 + C_2(\nu n) \log\left(\frac{T}{2\rho m}\right)\right) \\ &\geq \frac{1}{2} \left(C_1 m + C_2'(\nu n) \sum_{i=1}^m (\log T - \log(2\rho m))\right). \end{aligned}$$

Hence if $m = o(T)$, we have

$$\mathbb{E} \left[\sum_{t=1}^T r_t \right] = \Omega(mn \log T).$$

(Otherwise, we have $\mathbb{E}[\sum_{t=1}^T r_t] = \Omega(mn \log(T/m))$.) \square

Now we show that our structure aware algorithms produce regret sublinear in mn .

Proof of Theorem 6 We begin by bounding the probability that the set $S^*(w, \hat{\Theta}_{i_t}^{(t)}; K)$ that is offered in an exploitation round is different from the optimal set $S^*(w, \Theta_{i_t}^*; K)$.

Note that, for any S and $j \in S$,

$$0 \leq \frac{\partial p_j(S; \theta)}{\partial \theta_j} = \frac{e^{\theta_j} \sum_{j' \in S, j' \neq j} e^{\theta_{j'}}}{\left(\sum_{j' \in S} e^{\theta_{j'}} \right)^2} \leq \frac{|S| - 1}{|S|^2} \leq \frac{1}{4},$$

and that, moreover, for any $j' \in S, j' \neq j$,

$$0 \geq \frac{\partial p_j(S; \theta)}{\partial \theta_{j'}} = \frac{-e^{\theta_j + \theta_{j'}}}{\left(\sum_{j'' \in S} e^{\theta_{j''}} \right)^2} \geq \frac{-e^{\theta_j + \theta_{j'}}}{(e^{\theta_j} + e^{\theta_{j'}})^2} \geq -\frac{1}{4},$$

whereas, for $j' \notin S$, clearly $\frac{\partial p_j(S; \theta)}{\partial \theta_{j'}} = 0$. Therefore, $\|\nabla_{\theta} p_j(S; \theta)\|_2 \leq \frac{1}{4} \sqrt{|S|}$ for any $\theta \in \mathbb{R}^n$. This means that for any $S, j \in S, \theta$, and θ' ,

$$|p_j(S; \theta) - p_j(S; \theta')| \leq \frac{1}{4} \sqrt{|S|} \|\theta - \theta'\|_2.$$

It follows that for any S with $|S| \leq K$, w , θ , and θ' , the expected revenue loss associated with choosing set S' instead of S is bounded by $\|\theta - \theta'\|_2$:

$$|F(S; w, \theta) - F(S; w, \theta')| \leq \sum_{j \in S} |w_j| |p_j(S; \theta) - p_j(S; \theta')| \leq \frac{1}{4} \|w\|_{\infty} K^{3/2} \|\theta - \theta'\|_2.$$

Therefore, letting $S \in S^*(w, \theta; K)$ and $S' \in S^*(w, \theta'; K)$,

$$\begin{aligned} F(S'; w, \theta) &\geq F(S'; w, \theta') - \frac{1}{4} \|w\|_{\infty} K^{3/2} \|\theta - \theta'\|_2 \\ &\geq F(S; w, \theta') - \frac{1}{4} \|w\|_{\infty} K^{3/2} \|\theta - \theta'\|_2 \\ &\geq F(S; w, \theta) - \frac{1}{2} \|w\|_{\infty} K^{3/2} \|\theta - \theta'\|_2. \end{aligned}$$

It follows that if $\|\theta - \theta'\|_2$ is small enough, the optimal assortment for each is the same:

$$(\|\theta - \theta'\|_2 \leq 2\delta(w, \theta; K) K^{-3/2} / \|w\|_{\infty}) \implies (S^*(w, \theta'; K) = S^*(w, \theta; K)).$$

Now consider the regret R_t incurred at time t under the policy $\pi_{\text{nuc-norm}}(C, \lambda)$. If we explored at time t , then $R_t \leq \omega$. If we exploited at time t , then

$$\begin{aligned} \mathbb{E}R_t &\leq \omega \mathbb{P}((S^*(w, \theta'; K) \neq S^*(w, \theta; K))) && \leq \omega \mathbb{P}((\|\theta_{i_t} - \theta'_{i_t}\|_2 \leq 2\delta(w, \theta; K)K^{-3/2}/\|w\|_\infty)) \\ &\leq \omega \mathbb{P}((\|\Theta - \Theta'\|_2 \leq 2\delta K^{-3/2}/\|W\|_\infty)), \end{aligned}$$

using that $\|W_i\|_\infty \leq \|W\|_\infty$, $\|\hat{\Theta}_i - \Theta_i^*\|_2 \leq \|\hat{\Theta} - \Theta^*\|_F$, $0 < \delta \leq \delta(W_i, \Theta_i^*; K)$, and that the per-step regret is at most $\|W\|_\infty$. Now at each time t , note that the number of random observations (made in exploration rounds) is $N_t \geq C \log(t)$. Then apply Theorem 3 with $\tau = \log(t)/\log(m+n)$ and use $\alpha \geq \alpha/\sqrt{mn}$ to see that

$$\mathbb{E}R_t \leq \omega \mathbb{P}((\|\Theta - \Theta'\|_2 \leq 2\delta K^{-3/2}/\|W\|_\infty)) \leq \frac{3\omega}{t}.$$

Call the set of times when we explored T_{explore} , and note $|T_{\text{explore}}| \leq C \log(T) + 1$. Summing this expression over t , we see

$$\begin{aligned} \text{Regret}_{\Theta^*}^{\pi_1(T_0, \lambda)} &\leq \sum_{t \in T_{\text{explore}}} R_t + \sum_{t \notin T_{\text{explore}}} R_t \\ &\leq \sum_{t \in T_{\text{explore}}} \omega + \sum_{t \notin T_{\text{explore}}} \frac{3\omega}{t} \\ &\leq \omega(Cr(m+n) \log(T) + 1) + 3\omega \log(T) = O(\log(T)) \end{aligned}$$

which gives the result. \square

Proof of Theorem 7 Note that if $j \in S$ then

$$\left| \frac{\partial F(S; W, \Theta, \mu)}{\partial \Theta_{ij}} \right| \leq \mu_i \sum_{j' \in S} \frac{w_{ij'}}{4} \leq \frac{K\rho \|W\|_\infty}{4m},$$

and if $j \notin S$ then $\frac{\partial F(S; W, \Theta, \mu)}{\partial \Theta_{ij}} = 0$. Moreover,

$$\left| \frac{\partial F(S; W, \Theta, \mu)}{\partial \mu_i} \right| = \left| \frac{\sum_{j \in S} e^{\Theta_{ij}} w_{ij}}{1 + \sum_{j \in S} e^{\Theta_{ij}}} \right| \leq \|W\|_\infty.$$

It follows that for any S with $|S| \leq K$, W , Θ , μ , Θ' , and μ' ,

$$\begin{aligned} |F(S; W, \Theta, \mu) - F(S; W, \Theta', \mu')| &\leq \frac{K\rho \|W\|_\infty}{4m} \sum_{i=1}^m \sum_{j \in S} |\Theta_{ij} - \Theta'_{ij}| + \|W\|_\infty \sum_{i=1}^m |\mu_i - \mu'_i| \\ &\leq \frac{1}{4} K^{3/2} \rho \|W\|_\infty \|\Theta - \Theta'\|_F + \|W\|_\infty \|\mu - \mu'\|_1. \end{aligned}$$

Therefore, since the per-step regret is at most $\|W\|_\infty$, we have that

$$\begin{aligned} \text{Regret}_{\Theta^\star}^{\pi_2(T_0, \lambda)} &\leq \|W\|_\infty T_0 + \|W\|_\infty T \times \mathbb{P} \left(\frac{1}{4} K^{3/2} \rho \|W\|_\infty \left\| \hat{\Theta} - \Theta^\star \right\|_F + \|W\|_\infty \|\hat{\mu} - \mu\|_1 > \delta \right) \\ &\leq \omega T_0 + \omega T \times \left(\mathbb{P} \left(\|\Theta - \Theta'\|_F > \frac{2\delta}{K^{3/2} \rho \omega} \right) + \mathbb{P} \left(\|\mu - \mu'\|_1 > \frac{\delta}{2\omega} \right) \right). \end{aligned}$$

Applying Theorem 3 with $\tau = \log(T)/\log(m+n)$ we get that $\mathbb{P} \left(\|\Theta - \Theta'\|_F > \frac{4\beta\delta}{K^{3/2} \rho \omega} \right) \leq 3/T$ given our choice of T_0 .

By Theorem 4, $\mathbb{P}(\|\mu - \hat{\mu}\|_1 > \eta) \leq e^{m - T_0 \eta^2 / 64}$. Since $T_0 \geq \frac{256\omega(m + \log(T))}{\delta^2}$ we have $\mathbb{P}(\|\mu - \mu'\|_1 > \frac{\delta}{2\omega}) \leq 1/T$. Finally, $\alpha \geq \alpha/\sqrt{mn}$ yields the result. \square