Unimodality of Partitions in Near-Rectangular Ferrers Diagrams

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1   Abstract

We look at the rank generating function G_λ of partitions inside the Ferrers diagram of some partition λ, investigated by Stanton in 1990, as well as a closely related problem investigated by Stanley and Zanello in 2013. We show that G_λ is not unimodal for a larger class of 4-part partitions than previously known, and also that if the ratios of parts of λ are close enough to 1 (depending on how many parts λ has), or if the first part is at least half the size of λ, then G_λ is unimodal.

2   Introduction

Given partition λ with $\lambda = (\lambda_1, \ldots, \lambda_b)$, the *length* of λ is b and the *size* of λ, denoted $|\lambda|$, is $\sum \lambda_i$. Given partitions μ and λ, we say that $\mu \subseteq \lambda$ if the Ferrers diagram of μ fits inside the Ferrers diagram of λ. For any partition λ, we look at the set of partitions $\mu \subseteq \lambda$, ordered by the relation above, and let G_λ be the rank-generating function of this poset (i.e. $G_\lambda(q) = \sum a_n q^n$ where a_n is the number of partitions $\mu \subseteq \lambda$ of size n). We are interested in understanding when the coefficients of G_λ are unimodal, i.e. first weakly increasing and then weakly decreasing. This question was considered by Stanton [6].

Let b be the length of λ. In the case that $\lambda = (a, \ldots, a)$, we get that $G_\lambda(q)$ is the Gaussian binomial coefficient $G_\lambda(q) = \binom{a+b}{b}_q$. There are many proofs that these are unimodal symmetric (for instance, [7, 4, 3, 8]). This fact will be used many times in the proof of our main result.

We can also look at the question above, but require that both λ and μ have distinct nonzero parts. We call this rank generating function $F_\lambda(q)$. The unimodality of $F_\lambda(q)$ was considered by Stanley and Zanello [5]. For this version, the partition $\lambda = (b, b-1, \ldots, 1)$ gives the generating function

$$F_\lambda(q) = \prod_{i=1}^{b}(1 + q^b)$$

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which is unimodal symmetric (see, for instance, [4]). Alpoge [1] proved that for $n \gg b$, if $\lambda = (n, n-1, \dots, n-b+1)$, then F_λ is unimodal. We will mainly be concerned with Stanton's problem, but some of our results will also extend to the Stanley-Zanello version. For both problems, it is proven by Stanton and Stanley-Zanello respectively that any partition with at most 3 parts has a unimodal generating function, and that there are infinite families of partitions with 4 parts that do not.

In section 3, we will derive a form for F_λ and G_λ when the length of λ is fixed and show that F_λ and G_λ are unimodal if $2\lambda_1 \geq |\lambda|$. In section 4, we will prove that for a reasonable notion of positive density, for both versions of the problem, there is a positive density of partitions with 4 parts that have nonunimodal generating functions, giving larger classes of nonunimodal partitions than were known before. In sections 5 and 6, we will prove that partitions with 5 or more parts that are near-rectangular (in a sense to be defined later in the paper) and with $|\lambda| \gg b$ have unimodal G_λ. Finally, in section 7, we present some conjectures that are supported by the results in this paper and by computational data.

3   Generating Function

We will use $[n]$ to denote the set $\{1, 2, \dots, n\}$. We begin by finding an expression for $G_\lambda(q)$ when λ has a fixed number of parts b.

If λ is a partition, denote by G_λ^i the size generating function for the number of partitions $\mu \subseteq \lambda$, giving the first part of the partition the weight i (for instance, the partition $(4, 2, 1)$ will contribute to the q^{15} coefficient of G_λ^3 since $3(4)+2+1=15$). Note that $G_\lambda^1 = G_\lambda$. Define F_λ^i similarly.

Also, denote by $\bar\lambda$ the partition λ with the first part removed. We consider generating functions of the form G_λ^i. Let λ have b parts, for b fixed. For any $A \subseteq [b]$ and any $1 \leq k \leq b$, we define $f_A^\lambda(k)$ and $g_A(k)$ as follows: if $k \geq \min(A)$, define $f_A^\lambda(k) = \lambda_{b+1-\max(A \cap [k])} + 1$. Otherwise, define $f_A^\lambda(k) = 0$.

Define $g_A(k) = k - \max(A \cap [k]) + 1$. For convenience, here and in the formula below, we define $\max(\emptyset) = 1$.

Proposition 1.
$$G_\lambda^i(q) = \sum_{A \subseteq [b]} G_{\lambda,A}^i(q) \tag{1}$$

where

$$G_{\lambda,A}^i(q) = (-1)^{|A|} q^{\left(\sum_{k=1}^{b-1} f_A^\lambda(k)\right) + i f_A^\lambda(b)} \prod_{k=1}^{\max(A)-1} \frac{1}{1 - q^{g_A(k)}} \cdot \prod_{k=\max(A)}^{b} \frac{1}{1 - q^{g_A(k)+i-1}}.$$

The idea for the inductive step in the proof below is based on Lemma 1 from [6] (where it is only used once).

Proof. We induct on b. For $b = 1$,

$$G_\lambda^i(q) = 1 + q^i + \ldots + q^{i\lambda_1} = \frac{1 - q^{i(\lambda_1+1)}}{1 - q^i} = \frac{1}{1 - q^i} - \frac{q^{i(\lambda_1+1)}}{1 - q^i},$$

which matches (1).

Assume $b > 1$. If $\mu \subseteq \lambda$ with $\mu_1 < \lambda_1$, then we let $\phi(\mu) = (\mu_1 + 1, \mu_2, \ldots, \mu_b)$. If $\nu \subseteq \lambda$ with $\nu_1 > \nu_2$, we let $\phi^{-1}(\nu) = (\nu_1 - 1, \nu_2, \ldots, \nu_b)$. For any n, clearly ϕ gives a bijection between partitions $\mu \subseteq \lambda$ with $|\mu| = n - 1$ and $\mu_1 < \lambda_1$ and partitions $\nu \subseteq \lambda$ with $|\nu| = n$ and $\nu_1 = \nu_2$. Thus taking $(1 - q^i)G_\lambda^i(q)$ gives a lot of cancellation. Specifically, we get

$$(1 - q^i)G_\lambda^i(q) = G_{\bar\lambda}^{i+1}(q) - q^i(q^{i\lambda_1})G_{\bar\lambda}^1(q) \tag{2}$$

$$G_\lambda^i(q) = \frac{1}{(1 - q^i)}G_{\bar\lambda}^{i+1}(q) - \frac{1}{(1 - q^i)}(q^{i(\lambda_1+1)})G_{\bar\lambda}^1(q) \tag{3}$$

where the first term on the right side of (2) corresponds to ν with $\nu_1 = \nu_2$, while the second term corresponds to μ with $\mu_1 = \lambda_1$.

It is not difficult (with care) to verify that for $A \subseteq [b-1]$,

$$G_{\lambda,A}^i(q) = \frac{1}{(1 - q^i)}G_{\bar\lambda,A}^{i+1}(q)$$

and

$$G_{\lambda,A\cup\{b\}}^i(q) = -\frac{q^{i(\lambda_1+1)}}{(1 - q^i)}G_{\bar\lambda,A}^1(q),$$

so by (3), we get

$$\begin{aligned}
G_\lambda^i(q) &= \frac{1}{(1 - q^i)}G_{\bar\lambda}^{i+1}(q) - \frac{1}{(1 - q^i)}(q^{i(\lambda_1+1)})G_{\bar\lambda}^1(q) \\
&= \sum_{A \subseteq [b-1]} \frac{1}{(1 - q^i)}G_{\bar\lambda,A}^{i+1}(q) - \sum_{A \subseteq [b-1]} \frac{q^{i(\lambda_1+1)}}{(1 - q^i)}G_{\bar\lambda,A}^1(q) \\
&= \sum_{B \subseteq [b]} G_{\lambda,B}^i(q).
\end{aligned}$$

$\square$

We can substitute $i = 1$ (which is the case we are really interested in).

Corollary 2.

$$G_\lambda(q) = \sum_{A \subseteq [b]} \left((-1)^{|A|} q^{\sum_{k=1}^b f_A^\lambda(k)} \prod_{k=1}^b \frac{1}{1 - q^{g_A(k)}} \right) \tag{4}$$

Note that for a fixed b, this gives us $G_\lambda(q)$ as a sum of 2^b terms which are simple to compute given λ.

To get the generating function for distinct parts, we note that

$$F_\lambda(q) = 1 + \sum_{c=1}^{b} q^{\binom{c+1}{2}} G_{(\lambda_1-c,\ldots,\lambda_c-1)}(q) \tag{5}$$

where c above represents the number of nonzero parts in a partition.

Observe that $G_\lambda(q)$ is unimodal if and only if the coefficients of $(1-q)G_\lambda(q)$ go from being nonnegative to being nonpositive. We can also use (2) with $i = 1$ to obtain directly the sign of coefficients when $|\bar\lambda| \leq \lambda_1$.

Corollary 3. *If $2\lambda_1 \geq |\lambda|$, then G_λ is unimodal*

Similarly to (2), we can derive

$$(1-q)F_\lambda(q) = qF_{\bar\lambda}^2(q) + 1 - q^{\lambda_1+1}F_{\bar\lambda}(q)$$

which gives us

Corollary 4. *If $2\lambda_1 \geq |\lambda|$, then F_λ is unimodal*

4   Partitions of length 4

Some set S of partitions with b parts has *positive density* if

$$\liminf_{n\to\infty} \frac{|\{\lambda \in S \mid |\lambda| \leq n\}|}{|\{\text{partitions } \lambda \text{ with } b \text{ parts and } |\lambda| \leq n\}|} > 0.$$

Note that corrolaries 3 and 4 imply that a positive density of partitions with 4 parts have unimodal F_λ and G_λ.

Theorem 5. *A positive density of partitions λ with 4 parts have nonunimodal G_λ and a positive density of partitions μ with 4 distinct parts have nonunimodal F_μ.*

Proof. The generating function gives us the following when $b = 4$:

$$G_\lambda(q) = \frac{1 - q^{4\lambda_4+4}}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \frac{q^{3\lambda_3+\lambda_4+4} + q^{\lambda_1+3\lambda_4+4} - q^{3\lambda_3+3} - q^{\lambda_1+1}}{(1-q)^2(1-q^2)(1-q^3)}$$
$$+ \frac{q^{2\lambda_2+\lambda_3+3} + q^{\lambda_1+2\lambda_3+3} + q^{\lambda_1+\lambda_2+2} - q^{2\lambda_2+\lambda_3+\lambda_4+4} - q^{\lambda_1+2\lambda_3+\lambda_4+4} + q^{\lambda_1+\lambda_2+2\lambda_4+4}}{(1-q)^3(1-q^2)}$$
$$+ \frac{q^{\lambda_1+\lambda_2+\lambda_3+\lambda_4+4} - q^{\lambda_1+\lambda_2+\lambda_3+3}}{(1-q)^4} + \frac{q^{2\lambda_2+2\lambda_4+4} - q^{2\lambda_2+2}}{(1-q)^2(1-q^2)^2}.$$

We will look at cases when $\lambda_4 > 0.9\lambda_1$ and when $\lambda_1 \equiv \lambda_2 \equiv 11 \pmod{12}$. For $\lambda_1 + \lambda_2 + 14 \leq N < 2\lambda_1 + 2$ and $12 \mid N$, we will be interested in cases where $[q^{N+1}](1-q)G_\lambda(q) < 0$,

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and $[q^{N+2}](1-q)G_\lambda(q) > 0$, since such cases show that $G_\lambda(q)$ is not unimodal. We will also assume that λ_1 is sufficiently large.

Define m, n, ℓ so that $\lambda_1 = 12m - 1$, $\lambda_2 = 12n - 1$, and $N = 12(\ell - 1)$. Since $3\lambda_4 > N + 2$, the only relevant terms in $(1-q)G_\lambda(q)$ are

$$\frac{1}{(1-q^2)(1-q^3)(1-q^4)} - \frac{q^{12m}}{(1-q)(1-q^2)(1-q^3)} + \frac{q^{12(m+n)}}{(1-q)^2(1-q^2)} - \frac{q^{24n}}{(1-q)(1-q^2)^2}.$$

We turn all the denominators into the form $(1-q)^{12}$ and look at coefficients of q^i in the numerators for i being 1 or 2 mod 12. This gives us

$$[q^{N+1}](1-q)G_\lambda(q) = 9\ell^2 - (48m + 15)\ell + 24m^2 + 72mn - 36n^2 + 38m + 6$$
$$[q^{N+2}](1-q)G_\lambda(q) = 9\ell^2 - (48m + 15)\ell + 24m^2 + 72mn - 36n^2 + 34m + 6n + 6.$$

We will now restrict further to the case that $1.999m < \ell < 1.9999m$ and $0.98m < n < \ell - m$. Let

$$f(m, \ell, n) = [q^{N+1}](1-q)G_\lambda(q) = 9\ell^2 - (48m + 15)\ell + 24m^2 + 72mn - 36n^2 + 38m + 6$$
$$g(m, \ell, n) = [q^{N+2}](1-q)G_\lambda(q) = 9\ell^2 - (48m + 15)\ell + 24m^2 + 72mn - 36n^2 + 34m + 6n + 6.$$

First, we note that $g(m, \ell, n-1) - f(m, \ell, n) = -78(m - n) + 2m - 42 > 0$. We also note that

$$g(\ell, m, \ell - m - 1) = -27\ell^2 + 96\ell m - 84m^2 \pm O(m).$$

Using $1.999m < \ell < 1.9999m$ and neglecting linear terms since λ_1 is sufficiently large gives us $g(\ell, m, \ell - m - 1) > 0$. Similarly,

$$g(\ell, m, \lceil 0.98m \rceil + 1) = 9\ell^2 - 48m\ell + 59.9856m^2 \pm O(m) < 0.$$

Thus for any m, ℓ, there must exist some n within the bounds defined above such that $g(\ell, m, n) > 0$ and $g(\ell, m, n - 1) \geq 0$. Then $f(\ell, m, n) < g(m, \ell, n - 1) \leq 0$. Since $[q^N](1 - q)G_\lambda(q) > [q^{N+2}](1-q)G_\lambda(q)$, we then have

$$[q^{N+1}](1-q)G_\lambda(q) < 0$$
$$[q^{N+2}](1-q)G_\lambda(q) > 0,$$

so $G_\lambda(q)$ is nonunimodal at $N + 1$.

Thus there are $\Theta(a^2)$ triples (n, m, ℓ) with $m \leq a$ that satisfy all the conditions above and are not unimodal, so there are $\Theta(a^4)$ pairs (λ, N) with $\lambda_1 \leq a$ that satisfy all the conditions above and are not unimodal. Finally, we can check that $g(m, \ell + 1, n) < f(m, \ell, n)$ and $g(m, \ell, n)$ is decreasing in ℓ over the appropriate range, so for any pair of values (m, n), there

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is at most one ℓ with $f(\ell, m, n) < 0$ and $g(\ell, m, n) > 0$. Thus we are not double-counting any partitions. Hence there are $\Theta(a^4)$ partitions λ with 4 parts $\lambda_1 \leq a$, and nonunimodal G_λ, so a positive density of 4-part partitions λ have nonunimodal G_λ, proving the first part of the theorem.

If μ has 4 parts, then

$$F_\mu(q) = q^6 G_{(\mu_1-2,\mu_2-1,\mu_3)}(q) + q G_{(\mu_1-1,\mu_2)}(q) + 1$$

where the first term covers partitions $\nu \subseteq \mu$ with 3 or 4 parts, the second term covers ν with 1 or 2 parts, and the third term covers the empty partition. Whenever λ is a partition with nonunimodal G_λ of the form described above, taking $\mu = (\lambda_1 + 3, \lambda_2 + 2, \lambda_3 + 1, \lambda_4)$ gives us

$$[q^{N+7}](1-q)F_\mu(q) = [q^{N+1}](1-q)G_\lambda(q) < 0$$
$$[q^{N+8}](1-q)F_\mu(q) = [q^{N+2}](1-q)G_\lambda(q) > 0$$

since $\deg(qG_{(\mu_1-1,\mu_2)}(q) + 1) = \mu_1 - 1 + \mu_2 + 1 = \lambda_1 + \lambda_2 + 5 < N + 6$. Thus F_μ is not unimodal. This proves the theorem. $\square$

5   Ascending and Descending Segments

We will try to prove that, under certain conditions on λ, we get long initial increasing and final decreasing sequences of coefficients of G_λ. To prove the increasing sequences, we will use the unimodality of the largest rectangle that fits inside λ, while for the decreasing sequences we will use the unimodality of the smallest rectangle that contains λ.

Proposition 6. *For any partition λ and any $k \geq 1$, the sequence $a_0, \ldots, a_{\lceil k\lambda_k/2 \rceil}$ of coefficients of G_λ is weakly increasing.*

Proof. Let X be the rectangle of squares (x, y) in the Ferrers diagram of λ satisfying $x \leq k$ and $y \leq \lambda_k$. Let $R \subseteq \lambda \backslash X$ be some fixed set of squares. Look at partitions μ such that $\mu \subseteq \lambda$ and $\mu \backslash X = R$ and assume that there is at least one such μ. Let b_n be the number of such μ of size n. Some number (possibly 0) of rows and columns of X are forced to be in μ, and the remaining squares of X form a rectangle. Thus $\{b_i\}$ is unimodal symmetric, centered at an index of at least $k\lambda_k/2$, so $\{b_i\}$ is weakly increasing up to at least $b_{\lceil k\lambda_k/2 \rceil}$. Since $\{a_i\}$ is the sum of such sequences over all possible choices of R, we get that $\{a_i\}$ is weakly increasing up to at least $a_{\lceil k\lambda_k/2 \rceil}$. $\square$

Proposition 7. *For any partition λ of length b with $\lambda_{b-1} \geq \frac{2b-5}{2b-4}\lambda_1$ and $\lambda_b \geq \frac{|\lambda|}{b+1}$, if we let $N = \lfloor b\lambda_1/2 \rfloor$, then the sequence $a_N, a_{N+1}, \ldots$ is weakly decreasing.*

Proof. Let m be the number of indices $1 \leq k < b$ such that $\lambda_k > \lambda_{k+1}$. We prove by induction on m that $a_N, a_{N+1}, \ldots$ is weakly decreasing.

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If $m = 0$, then λ is a rectangle and thus G_λ is unimodal symmetric, so $a_N, a_{N+1}, \ldots$ is weakly decreasing. This proves the base case.

Now assume $m \geq 1$. Take maximum k with $k < b$ such that $\lambda_k > \lambda_{k+1}$. Then $\lambda = (\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_{k+1})$. Let $\nu = (\lambda_1, \ldots, \lambda_k, \lambda_k, \ldots, \lambda_k)$. Then G_ν weakly decreases from q^N onward by the inductive hypothesis. Let $H = G_\nu - G_\lambda$. Then H counts partitions μ inside ν with $\mu_{k+1} > \lambda_{k+1}$. For any $S = (s_1, \ldots, s_{k+1})$ such that $s_i \leq \nu_i$ for all i and $\lambda_{k+1} < s_{k+1} \leq s_k \leq \cdots \leq s_1$, let H^S be the generating function for partitions μ inside ν with $\mu_i = s_i$ for all $1 \leq i \leq k+1$. Each ordered tuple of values $(\mu_{k+2}, \ldots, \mu_b)$ such that $s_{k+1} \geq \mu_{k+2} \geq \cdots \geq \mu_b \geq 0$ (i.e. each partition fitting within a certain rectangle) gives us a partition $\mu = (s_1, \ldots, s_{k+1}, \mu_{k+2}, \ldots, \mu_b)$ (when $k+1 = b$, there is exactly one such partition).Thus H^S is unimodal symmetric, centered at index $\sum_{i=1}^{k+1} s_i + (b-k-1)s_{k+1}/2$.

Hence H^S is weakly increasing up to q^L for $L = \sum_{i=1}^{k+1} s_i + \lceil s_{k+1}(b-k-1)/2 \rceil$. Since

$$\sum_{i=1}^{k+1} s_i + \lceil s_{k+1}(b-k-1)/2 \rceil \geq (k+1)s_{k+1} + \lceil s_{k+1}(b-k-1)/2 \rceil$$

$$\geq \lceil s_{k+1}(b+k+1)/2 \rceil$$
$$\geq \lceil (\lambda_{k+1}+1)(b+k+1)/2 \rceil,$$

we know that H^S is weakly increasing up to q^M for $M = \lceil (\lambda_{k+1}+1)(b+k+1)/2 \rceil$. Since $H = \sum_S H^S$, we know that H is weakly increasing up to q^M. Since $G_\lambda = G_\nu - H$, we know that G_λ is weakly decreasing from q^N to q^M.

We now assume that $k < b-1$ ($k = b-1$ will be a separate case) and let X be the set of squares (x, y) in the Ferrers diagram of λ such that $x \geq k+1$ or $y > \lambda_{x+1}$:
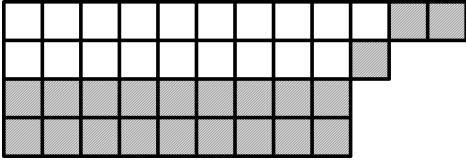
This gives us $|X| = \lambda_{k+1}(b-k-1) + \lambda_1$. Note that X is a disjoint union of a set of rectangles such that for any two rectangles in the set, one lies strictly above and to the right of the other.

Let $R \subseteq \lambda \backslash X$ be some fixed set of squares in the Ferrers diagram of λ. Look at $\mu \subseteq \lambda$ such that $\mu \backslash X = R$ and assume that there is at least one such μ. Let b_n be the number of such μ of size n. Some squares in X are forced to be not in μ, and the remaining squares form a disjoint union of a set of rectangles such that for any two rectangles in the set, one lies strictly above and to the right of the other. The generating function for partitions inside each rectangle is unimodal symmetric, and the product of unimodal symmetric generating functions is unimodal symmetric, so $\{b_i\}$ is unimodal symmetric, centered at an index which is at most $|\lambda| - \frac{1}{2}|X|$. Since $\{a_i\}$ is the sum of such sequences over all possible choices of R, we know that $\{a_i\}$ is weakly decreasing from $a_{|\lambda| - \lceil \frac{1}{2}|X| \rceil}$ onward.

Since

$$\lambda_{k+1} \geq \lambda_{b-1} \geq \frac{2b-5}{2b-4}\lambda_1 \geq \frac{2k-1}{2k}\lambda_1,$$

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we get

$$M + \left\lceil \frac{1}{2}|X| \right\rceil = \left\lceil \frac{1}{2}(\lambda_{k+1}+1)(b+k+1) \right\rceil + \left\lceil \frac{1}{2}\left(\lambda_{k+1}(b-k-1)+\lambda_1\right) \right\rceil$$

$$\geq \frac{1}{2}\lambda_{k+1}(b+k+1) + \frac{1}{2}\lambda_{k+1}(b-k-1) + \frac{1}{2}\lambda_1$$

$$\geq (b-k)\lambda_{k+1} + k\lambda_{k+1} + \frac{1}{2}\lambda_1$$

$$\geq (b-k)\lambda_{k+1} + \left(k\left(\frac{2k-1}{2k}\right) + \frac{1}{2} \right)\lambda_1$$

$$\geq (b-k)\lambda_{k+1} + k\lambda_1$$

$$\geq |\lambda|.$$

Then, since we know that $\{a_i\}$ is weakly decreasing from $a_{|\lambda|-\left\lceil \frac{1}{2}|X| \right\rceil}$ onward and also weakly decreasing from a_N to a_M, we have that $\{a_i\}$ is weakly decreasing from a_N onward, so we are done with the inductive step.

We look separately at the case $k = b - 1$. There, for each $n > |\lambda| - \lambda_b$, we have an injection from partitions $\mu \subseteq \lambda$ of size n to partitions $\nu \subseteq \lambda$ of size $n-1$ given by $\phi(\mu) = (\mu_1, \ldots, \mu_{b-1}, \mu_b - 1)$. Thus $\{a_i\}$ is weakly decreasing from $a_{|\lambda|-\lambda_b}$ onward. We get

$$M + \lambda_b = \left\lceil \frac{1}{2}(\lambda_b + 1)(2b) \right\rceil + \lambda_b$$

$$\geq b\lambda_b + \lambda_b$$

$$\geq (b+1)\lambda_b$$

$$\geq |\lambda|.$$

Then, since we know that $\{a_i\}$ is weakly decreasing from $a_{|\lambda|-\lambda_b}$ onward and also weakly decreasing from a_N to a_M, we have that $\{a_i\}$ is weakly decreasing from a_N onward, so we are done with the inductive step. $\qquad\square$

6   Concavity

Lemma 8. *If C is a finite multiset of positive integers with $\gcd(C) = 1$, then*

$$[x^n] \prod_{c \in C} \frac{1}{1 - q^c} = \prod_{c \in C} \frac{1}{c}\binom{n}{|C|-1} \pm O_C(n^{|C|-2}).$$

Proof. This is easy to see by induction on $|C|$. $\qquad\square$

Proposition 9. *When $b = 5$, $\lambda_4 \geq \frac{5}{6}\lambda_1$, $\lambda_5 \geq \frac{1}{2}\lambda_1$, and λ_1 is sufficiently large, the coefficients of $G_\lambda(q)$ are concave from $q^{\frac{5}{2}\lambda_2}$ to $q^{\frac{5}{2}\lambda_1}$.*

Proof. Concavity is equivalent to the negativity of the coefficients of $(1-q)^2 G_\lambda(q)$. We will look at the generating function for $G_\lambda(q)$ derived above, however most of its terms will not affect the coefficients we are interested in. For $\frac{5}{2}\lambda_2 \le n \le \frac{5}{2}\lambda_1$, we get

$$
\begin{aligned}
[q^n](1-q)^2 G_\lambda(q) = [q^n] &\left(\frac{1-q}{(1-q^2)(1-q^3)(1-q^4)(1-q^5)} - \frac{q^{2\lambda_2+2}}{(1-q^2)^2(1-q^3)} \right.\\
&\left. - \frac{q^{\lambda_1+1}}{(1-q^2)(1-q^3)(1-q^4)} + \frac{q^{\lambda_1+\lambda_2+2}}{(1-q)(1-q^2)(1-q^3)} \right)\\
= [q^n] &\left(\frac{1-q+q^2}{(1-q^4)(1-q^5)(1-q^6)} - \frac{q^{2\lambda_2+2}}{(1-q^2)^2(1-q^3)} \right.\\
&\left. - \frac{q^{\lambda_1+1}}{(1-q^2)(1-q^3)(1-q^4)} + \frac{q^{\lambda_1+\lambda_2+2}}{(1-q)(1-q^2)(1-q^3)} \right)\\
= &\frac{1}{120}\binom{n}{2} - \frac{1}{12}\binom{n-2\lambda_2}{2} - \frac{1}{24}\binom{n-\lambda_1}{2} + \frac{1}{6}\binom{n-\lambda_1-\lambda_2}{2} \pm O(n)\\
= &\frac{n^2}{240} - \frac{(n-2\lambda_2)^2}{24} - \frac{(n-\lambda_1)^2}{48} + \frac{(n-\lambda_1-\lambda_2)^2}{12} \pm O(n)\\
= &\left(\frac{\left(\frac{n}{\lambda_1}\right)^2}{240} - \frac{\left(\frac{n}{\lambda_1}-2\frac{\lambda_2}{\lambda_1}\right)^2}{24} - \frac{\left(\frac{n}{\lambda_1}-1\right)^2}{48} + \frac{\left(\frac{n}{\lambda_1}-\frac{\lambda_2}{\lambda_1}-1\right)^2}{12} \right) \lambda_1^2 \pm O(n).
\end{aligned}
$$

If we define

$$
\alpha(x,y) = \frac{y^2}{240} - \frac{(y-2x)^2}{24} - \frac{(y-1)^2}{48} + \frac{(y-x-1)^2}{12},
$$

then we care about $\alpha(x,y)$ when $\frac{5}{6} \le x \le 1$ and $\frac{5}{2}x \le y \le \frac{5}{2}$. Note that $\alpha(x,y)$ is convex in y, and we can check that $\alpha(x,\frac{5}{2}) < -0.002$ and $\alpha(x,\frac{5}{2}x) < -0.002$. Thus $\alpha(\frac{\lambda_2}{\lambda_1},\frac{n}{\lambda_1}) < -0.002$. Then if $\lambda_1 \gg 1$, the quadratic term dominates (since n is linear in λ_1), so $[q^n](1-q)^2 G_\lambda(q)$ is negative, so the coefficients of $G_\lambda(q)$ are concave from $q^{\frac{5}{2}\lambda_2}$ to $q^{\frac{5}{2}\lambda_1}$. □

Theorem 10. *If partition λ has 5 parts, with $\lambda_4 \ge \frac{5}{6}\lambda_1$, $\lambda_5 \ge \frac{|\lambda|}{6}$, and λ_1 being sufficiently large, then G_λ is unimodal.*

Proof. The condition on λ_5 gives us

$$
6\lambda_5 \ge \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5
$$
$$
5\lambda_5 \ge \lambda_1 + \frac{5}{6}\lambda_1 + \frac{5}{6}\lambda_1 + \frac{5}{6}\lambda_1
$$
$$
\lambda_5 \ge \frac{7}{10}\lambda_1.
$$

Let $\mu = (\lambda_1, \lambda_2, \lambda_2, \lambda_2, \lambda_2)$. By Proposition 6, G_μ is increasing up to $q^{\frac{5}{2}\lambda_2}$. Since G_μ and G_λ agree up to at least $q^{\min(5\lambda_5, 4\lambda_4, 3\lambda_3)}$ we get that G_λ has increasing coefficients up to $q^{\frac{5}{2}\lambda_2}$.

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Then by Proposition 9, the coefficients of G_λ are concave from $q^{\frac{5}{2}\lambda_2}$ to $q^{\frac{5}{2}\lambda_1}$, and by Proposition 7, they are decreasing from $q^{\frac{5}{2}\lambda_1}$ onward. Thus G_λ is unimodal. $\qquad\square$

Before proving a similar result for all $b \geq 5$, we establish a lemma.

Lemma 11. *Let* $T(b) = \sum_{i=0}^{\lfloor b/2 \rfloor}(-1)^i(b-2i)^{b-3}\binom{b}{i}$. *Then* $T(b) \neq 0$ *for* $b \geq 5$.

Proof. For $b < 100$, this is easy to check directly using, say, Mathematica.

Nagura [2] proved that whenever $x \geq 25$, there is a prime number p with $x < p < \frac{6}{5}x$. When $b \geq 100$, we can set $x = \frac{b}{4}$ to obtain some odd prime p with $3p < b < 4p$. Pick k so that $b = 3p + k$. Then $1 \leq k < p$. We want to show that $T(b) \not\equiv 0 \bmod p$. Note that $\binom{b}{i} \not\equiv 0 \bmod p$ only when $0 \leq i \leq k$ or $0 \leq i - p \leq k$. Thus

$$
\begin{aligned}
T(3p+k) &\equiv \sum_{i=0}^{k}(-1)^i(3p+k-2i)^{3p+k-3}\binom{3p+k}{i} + \sum_{j=0}^{k}(-1)^{j+p}(3p+k-2(j+p))^{3p+k-3}\binom{3p+k}{p+j} \\
&\equiv \sum_{i=0}^{k}(-1)^i(k-2i)^{3p+k-3}\binom{3p+k}{i} - \sum_{j=0}^{k}(-1)^j(k-2j)^{3p+k-3}\binom{3p+k}{p+j} \\
&\equiv \sum_{i=0}^{k}(-1)^i(k-2i)^{3p+k-3}\binom{k}{i} - 3\sum_{j=0}^{k}(-1)^j(k-2j)^{3p+k-3}\binom{k}{j} \\
&\equiv -2\sum_{j=0}^{k}(-1)^j(k-2j)^{3(p-1)+k}\binom{k}{j} \\
&\equiv -2\sum_{j=0}^{k}(-1)^j(k-2j)^{k}\binom{k}{j}.
\end{aligned}
$$

Note that $\sum_{j=0}^{k}(-1)^j(k-2j)^k\binom{k}{j}$ is, up to a possible sign, the expression for the k-th partial difference of $(k-2x)^k$, which is $(-1)^k 2^k k!$ and thus is not divisible by p. Therefore $T(3p+k) \not\equiv 0 \bmod p$, so $T(3p+k) \neq 0$. $\qquad\square$

Theorem 12. *For a given integer $b \geq 5$, there exists $\epsilon > 0$ such that if λ is a partition with b parts satisfying $\lambda_1 \gg b$, $\lambda_{\lfloor b/2 \rfloor} > (1-\epsilon)\lambda_1$, and $\lambda_b \geq \frac{2b-5}{2b-4}\lambda_1$, then G_λ is unimodal.*

Proof. Fix b. If we let $\mu = (\lambda_1, \ldots, \lambda_{\lfloor b/2 \rfloor}, \lambda_{\lfloor b/2 \rfloor}, \ldots, \lambda_{\lfloor b/2 \rfloor})$, then we know by Proposition 7 that G_μ has increasing coefficients up to $q^{\frac{b}{2}\lambda_{\lfloor b/2 \rfloor}}$. Since the coefficients of G_μ and G_λ agree up to at least $q^{(\lfloor b/2 \rfloor + 1)\lambda_b}$ and

$$
\left(\left\lfloor \frac{b}{2} \right\rfloor + 1\right)\lambda_b \geq \frac{b+1}{2}\frac{2b-5}{2b-4}\lambda_1 = \frac{b}{2}\frac{b+1}{b}\frac{2b-5}{2b-4}\lambda_1 = \frac{b}{2}\left(\frac{2b^2-3b-5}{2b^2-4b}\right)\lambda_1 \geq \frac{b}{2}\lambda_1 \geq \frac{b}{2}\lambda_{\lfloor b/2 \rfloor},
$$

we get that G_λ has increasing coefficients up to $q^{\frac{b}{2}\lambda_{\lfloor b/2 \rfloor}}$. Also, Proposition 7 tells us that G_λ has decreasing coefficients from $q^{\frac{b}{2}\lambda_1}$ onward. Thus all we need to do is prove concavity

of coefficients between $q^{\frac{b}{2}\lambda_{\lfloor b/2 \rfloor}}$ and $q^{\frac{b}{2}\lambda_b}$. For this, we look at $[q^n](1-q)^2 G_\lambda(q)$ for $\frac{b}{2}\lambda_{\lfloor b/2 \rfloor} \le n \le \frac{b}{2}\lambda_b$. Note that the only term in (4) that does not have $(1-q)^2$ in the denominator is $\prod_{k=1}^b \frac{1}{1-q^k}$. For that term, we use $\frac{(1-q)^2}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)} = \frac{1-q+q^2}{(1-q^6)(1-q^4)(1-q^5)}$. Thus $(1-q)^2 G_\lambda(q)$ will have only terms of the form covered by Lemma 8.

Using $h(A, \lambda)$ to denote $\sum_{k=1}^b f_A^\lambda(k)$, we obtain

$$
\begin{aligned}
[q^n](1-q)^2 G_\lambda(q) &= [q^n](1-q)^2 \sum_{A \subseteq [b]} \left((-1)^{|A|} q^{h(A,\lambda)} \prod_{k=1}^b \frac{1}{1-q^{g_A(k)}} \right) \\
&= [q^n](1-q)^2 \sum_{\substack{A \subseteq [b] \\ h(A,\lambda) \le n}} \left((-1)^{|A|} q^{h(A,\lambda)} \prod_{k=1}^b \frac{1}{1-q^{g_A(k)}} \right) \\
&= \sum_{\substack{A \subseteq [b] \\ h(A,\lambda) \le n}} \left((-1)^{|A|} \left(\prod_{k=1}^b \frac{1}{g_A(k)} \right) \binom{n - h(A,\lambda)}{b-3} + O_b(n^{b-4}) \right) \\
&= \alpha \left(\frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_{\lfloor b/2 \rfloor}}{\lambda_1}, \frac{n}{\lambda_1} \right) n^{b-3} + O_b(n^{b-4})
\end{aligned}
$$

where α is some continuous function. Note that in going from line 2 to line 3 in the display, we have to treat the case of $A = \emptyset$ separately.

Now, whenever $\alpha\left(1, \ldots, 1, \frac{b}{2}\right) < 0$, we can use continuity to obtain some $\epsilon > 0$ so that for any λ and n with $\lambda_{\lfloor b/2 \rfloor} \ge (1-\epsilon)\lambda_1$ and $\frac{b}{2}\lambda_{\lfloor b/2 \rfloor} \le n \le \frac{b}{2}\lambda_1$, we get $\alpha(\frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_{\lfloor b/2 \rfloor}}{\lambda_1}, \frac{n}{\lambda_1}) < \alpha(1, \ldots, 1, \frac{b}{2})/2 < 0$. This would then tell us, when $\lambda_1 \gg b$, that the coefficients of $G_\lambda(q)$ are concave from $q^{(b/2)\lambda_{\lfloor b/2 \rfloor}}$ to $q^{(b/2)\lambda_1}$. Thus all we need to do is show that $\alpha\left(1, \ldots, 1, \frac{b}{2}\right) < 0$.

We let $\lambda = (\lambda_1, \ldots, \lambda_1)$. Then $h(A, \lambda) = (b - \min(A) + 1)(\lambda_1 + 1)$ for $A \ne \emptyset$ and $h(\emptyset, \lambda) = 0$. Thus

$$[q^n](1-q)^2 G_\lambda(q) =$$

$$= \sum_{\substack{A \subseteq [b] \\ h(A,\lambda) \le \frac{b}{2}\lambda_1}} \left((-1)^{|A|} \left(\prod_{k=1}^b \frac{1}{g_A(k)} \right) \binom{n - h(A,\lambda)}{b-3} + O_b(n^{b-4}) \right)$$

$$= \left(\prod_{k=1}^b \frac{1}{g_\emptyset(k)} \right) \binom{n}{b-3} + \sum_{m=\lceil \frac{b+3}{2} \rceil}^b \sum_{\substack{A \subseteq [b] \\ \min(A)=m}} \left((-1)^{|A|} \left(\prod_{k=1}^b \frac{1}{g_A(k)} \right) \binom{n - h(A,\lambda)}{b-3} \right) + O_b(n^{b-4})$$

$$= \frac{1}{b!} \binom{\frac{b}{2}\lambda_1}{b-3} + \sum_{m=\lceil \frac{b+3}{2} \rceil}^b \sum_{\substack{s_1,\dots,s_\ell \ge 1 \\ \sum s_j = b-m+1}} \left(\frac{1}{(b-1)!} \prod_{j=0}^\ell \frac{1}{s_j!} \right) \binom{\frac{b}{2}\lambda_1 - (b-m+1)(\lambda_1+1)}{b-3} + O_b(\lambda_1^{b-4})$$

$$= \frac{1}{(b-3)!} \left(\frac{1}{b!} \left(\frac{b}{2}\lambda_1 \right)^{b-3} + \right.$$

$$\left. + \sum_{m=\lceil \frac{b+3}{2} \rceil}^b \sum_{\substack{s_1,\dots,s_\ell \ge 1 \\ \sum s_j = b-m+1}} \left(\frac{1}{(m-1)!} \prod_{j=0}^\ell \frac{-1}{s_j!} \right) \left(\left(\frac{b}{2} - (b-m+1) \right) \lambda_1 \right)^{b-3} \right) + O_b(\lambda_1^{b-4})$$

$$= \frac{\lambda_1^{b-3}}{(b-3)!2^{b-3}} \left(\frac{1}{b!} b^{b-3} + \sum_{m=\lceil \frac{b+3}{2} \rceil}^b \sum_{\substack{s_1,\dots,s_\ell \ge 1 \\ \sum s_j = b-m+1}} \left(\frac{1}{(m-1)!} \prod_{j=0}^\ell \frac{-1}{s_j!} \right) (2(m-1)-b)^{b-3} \right) + O_b(\lambda_1^{b-4}).$$

where $\displaystyle\sum_{\substack{s_1,\dots,s_\ell \ge 1 \\ \sum s_j = x}}$ denotes a sum over compositions of x.

We note the equality of formal power series

$$\sum_{a=0}^\infty \left(\sum_{\substack{s_1,\dots,s_\ell \ge 1 \\ \sum s_j = a}} \prod_{j=0}^\ell \frac{-1}{s_j!} \right) x^a = \frac{1}{1 - \left(\sum_{s=1}^\infty \frac{-1}{s!} x^s \right)}$$

$$= \frac{1}{1 - (1 - e^x)}$$

$$= e^{-x}$$

$$= \sum_{a=0}^\infty \frac{(-1)^a}{a!} x^a,$$

so

$$\sum_{\substack{s_1,\dots,s_\ell \geq 1 \\ \sum s_j = a}} \prod_{j=0}^{k} \frac{-1}{s_j!} = \frac{(-1)^a}{a!}. \tag{6}$$

Substituting (6) in, we get

$[q^n](1-q)^2 G_\lambda(q) =$

$$= \frac{\lambda_1^{b-3}}{(b-3)!2^{b-3}} \left(\frac{1}{b!} b^{b-3} + \sum_{m=\lceil \frac{b+3}{2} \rceil}^{b} \left(\frac{1}{(m-1)!} \frac{(-1)^{b-m+1}}{(b-m+1)!} \right) (2(m-1)-b)^{b-3} \right) + O_b(\lambda_1^{b-4})$$

$$= \frac{\lambda_1^{b-3}}{b!(b-3)!2^{b-3}} \left(b^{b-3} + \sum_{m=\lceil \frac{b+3}{2} \rceil}^{b} \left(\binom{b}{b-m+1} (-1)^{b-m+1} \right) (2(m-1)-b)^{b-3} \right) + O_b(\lambda_1^{b-4})$$

$$= \frac{\lambda_1^{b-3}}{b!(b-3)!2^{b-3}} \sum_{i=0}^{\lfloor (b-1)/2 \rfloor} (-1)^i (b-2i)^{b-3} \binom{b}{i} + O_b(\lambda_1^{b-4})$$

$$= \frac{\lambda_1^{b-3}}{b!(b-3)!2^{b-3}} \sum_{i=0}^{\lfloor b/2 \rfloor} (-1)^i (b-2i)^{b-3} \binom{b}{i} + O_b(\lambda_1^{b-4})$$

so

$$\alpha\left(1, \dots, 1, \frac{b}{2}\right) = \frac{\lambda_1^{b-3}}{b!(b-3)!2^{b-3}} \sum_{i=0}^{\lfloor b/2 \rfloor} (-1)^i (b-2i)^{b-3} \binom{b}{i}.$$

This value is not 0 by Lemma 11. Note that if $\alpha(1, \dots, 1, \frac{b}{2}) > 0$, then when $\lambda = (\lambda_1, \dots, \lambda_1)$ and $\lambda_1 \gg b$, we get that the coefficients of G_λ are strictly convex near the middle, which contradicts its unimodality since $(\lambda_1, \dots, \lambda_1)$ is a rectangular partition. Thus $\alpha(1, \dots, 1, \frac{b}{2}) < 0$ and the theorem is proved. $\square$

7 Conjectures

We used the generating functions (4) and (5) to test unimodality of partitions with a small number of parts, leading us to pose the following conjecture.

Conjecture 13. If $\lambda_1 \geq b > 4$[1], then F_λ and G_λ are unimodal except for finitely many exceptions when $b = 6$.

The maximal exception for G_λ appears to be $(10, 9, 9, 9, 9, 9)$, while the maximal exception for F_λ appears to be $(19, 18, 17, 16, 15, 14)$. We have used a computer to test this for all

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[1] An earlier version of this paper incorrectly omitted the condition $\lambda_1 \geq b$. Thank you to Henry Cohn for pointing this out. Note that $\lambda_1 \geq b$ always holds for F_λ and holds without loss of generality for G_λ since taking the transpose of λ does not change the generating function.

partitions with 5 parts with $\lambda_1 \leq 200$, with 6 parts with $\lambda_1 \leq 100$, with 7 parts with $\lambda_1 \leq 70$, with 8 parts with $\lambda_1 \leq 50$, with 9 parts with $\lambda_1 \leq 40$, and with 10 parts with $\lambda_1 \leq 30$.

Conjecture 13 would also tend to be supported by Theorem 12. However, it seems very hard to prove.

Furthermore, the cases where F_λ or G_λ is not unimodal still seem to be well-behaved:

Conjecture 14. If F_λ or G_λ is not unimodal, then there is some integer n so that its coefficient sequence is increasing up to q^{2n} and decreasing from q^{2n+2}.

Note that this conjecture for G_λ is equivalent to Observations 1-3 of [6], while for F_λ it contradicts a conjecture in [5].

For the $b = 4$, $\lambda_4 \approx \lambda_1$ case, an analysis similar to Theorem 10 shows that $[q^n](1-q)(1-q^2)F_\lambda(q) < 0$ when $n \approx 2\lambda_1$ and that $[q^n](1-q)^2 F_\lambda(q) < 0$ when $n \approx 2\lambda_1$ and n is odd. However, these two facts together are still insufficient to prove Conjecture 14 even in this limited case.

We note that (5) can be used to get analogues of Proposition 7 as well as a concavity result near $b = \frac{b}{2}\lambda_1$. Some sort of analogue of Proposition 6 could then be used to prove the next conjecture.

Conjecture 15. For every integer $b \geq 5$, there exists $\epsilon > 0$ such that if λ is a partition with b distinct parts satisfying $\lambda_1 \gg b$ and $\lambda_b > (1 - \epsilon)\lambda_1$, then F_λ is unimodal.

Other classes of partitions that are likely to be approachable with the methods in this paper are partitions whose Ferrers diagram is the union of the Ferrers diagrams of λ and the transpose of μ, where λ and μ are covered by Theorem 12, and partitions whose parts are close to small multiples of some integer $a \gg b$.

Finally, there is a particularly interesting class of partitions for this problem: those whose parts form an arithmetic progression where λ_b is at most the common difference (see Conjecture 2 of [6] and Conjectures 3.5 and 3.10 of [5]).

8   Acknowledgments

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