NONNOETHERIAN COORDINATE RINGS WITH UNIQUE MAXIMAL DEPICTIONS

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Abstract. A depiction of a nonnoetherian integral domain R is a special coordinate ring that provides a framework for describing the geometry of R. We show that if R is noetherian in codimension 1, then R has a unique maximal depiction T. In this case, the geometric dimensions of the points of $\operatorname{Spec} R$ may be computed directly from T. If in addition R has a normal depiction S, then S is the unique maximal depiction of R.

1. Introduction

In this article all algebras are assumed to be commutative integral domains over an algebraically closed base field k. Depictions were introduced in [B3] to provide a framework for describing the geometry of nonnoetherian algebras with finite Krull dimension. A depiction of a nonnoetherian algebra R is a finitely generated algebra S that is as close as possible to R, in a suitable geometric sense (Definition 2.1). In this framework, the geometry of the maximal spectrum $\operatorname{Max} R$ is viewed as the algebraic variety $\operatorname{Max} S$, together with a collection of algebraic sets of $\operatorname{Max} S$ which are identified as 'smeared-out' positive dimensional closed points [B2].

Depictions have played an essential role in understanding the algebraic and representation theoretic properties of a class of quiver algebras called dimer algebras [B1, B4, B5]. However, there are many open questions regarding the fundamental nature of depictions; for example, it is not known whether every subalgebra of a finite type integral domain admits a depiction, or whether every depiction is contained in a maximal depiction. Here we consider the question: *What algebras admit unique maximal depictions?*

In general, maximal depictions need not be unique. Indeed, consider the rings

$$S = k[x, y, z] \quad \text{and} \quad R = k + xyS.$$

Then the overrings

$$S[x^{-1}] \quad \text{and} \quad S[y^{-1}]$$

are both depictions of R, whereas their minimal proper overring $S[x^{-1}, y^{-1}]$ is not [B3, Proposition 3.19]. To identify a class of algebras that admit unique maximal depictions, we introduce the following definition.

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Definition 1.1. We say R is *noetherian in codimension 1* if R admits a depiction S such that each codimension 1 subvariety of $\operatorname{Max} S$ intersects the open set

$$U_{S/R} := \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}\}.$$

We will show that this definition is independent of the choice of depiction S (Proposition 3.18). Furthermore, if R is noetherian in codimension 1, then for each height 1 prime $\mathfrak{q} \in \operatorname{Spec} S$, the localization $R_{\mathfrak{q}\cap R}$ is noetherian (Lemma 3.1.3).

Our main theorem is the following.

Theorem 1.2. *(Theorems 3.17 and 3.19.) Suppose R is noetherian in codimension 1. Let S be any depiction of R, and consider the global sections ring,*

$$T := \bigcap_{\mathfrak{n}\in U_{S/R}} S_{\mathfrak{n}} = \Gamma(U_{S/R}).$$

(1) T is the unique maximal depiction of R. In particular, T is independent of the choice of depiction S.
(2) For each $\mathfrak{p} \in \operatorname{Spec} R$ there is some $\mathfrak{t} \in \operatorname{Spec} T$ lying over $\mathfrak{p}$ such that the geometric dimension of $\mathfrak{p}$ equals the Krull dimension of $T/\mathfrak{t}$,

$$\operatorname{gdim} \mathfrak{p} = \dim T/\mathfrak{t}.$$

(3) Let $\overline{S}$ be the normalization of S. Then

$$S \subseteq T \subseteq \overline{S}.$$

In particular, if $S = \overline{S}$ is normal, then S is the unique maximal depiction of R, as well as the unique normal depiction of R.

For example, consider the family of algebras

$$S_j := k[x, y, xz, yz, xz^2, yz^2, \ldots, xz^{j-1}, yz^{j-1}, z^j] \quad \text{and} \quad R := k + (x, y)S_1,$$

where $j \geq 1$, and $(x, y)S_1$ is the ideal of $S_1 = k[x, y, z]$ generated by x and y. Each S_j is a depiction of R, and R is noetherian in codimension 1 (Example 5.1). Since S_1 is normal, Theorem 1.2 implies that S_1 is the unique maximal depiction of R.

Claim (2) in Theorem 1.2 provides a means of computing the geometric dimension of a point of $\operatorname{Spec} R$ in the case R is noetherian in codimension 1. If a depiction has the property given in Claim (2), then we say it is *saturated*.

In Section 4, we consider the special case where R has the form $R = k + I$, with I a nonzero radical ideal of S (and R is not necessarily noetherian in codimension 1). Using Theorem 1.2, we show that if $\dim S/I \geq 1$, then S is a saturated depiction of R (Theorem 4.1).

We conclude with a few examples of maximal depictions in Section 5. Notably, we show that if R admits a unique maximal depiction S but is not noetherian in codimension 1, then in general the geometric dimension of a point $\mathfrak{p} \in \operatorname{Spec} R$ need not equal the Krull dimension of $S/\mathfrak{q}$, for any $\mathfrak{q} \in \operatorname{Spec} S$ over $\mathfrak{p}$ (Example 5.2).

2. Preliminary definitions

Let S be an integral domain and a finitely generated k-algebra, and let R be a (possibly nonnoetherian) subalgebra of S. Denote by $\operatorname{Max} S$, $\operatorname{Spec} S$, and $\dim S$ the maximal spectrum (or variety), prime spectrum (or affine scheme), and Krull dimension of S respectively; similarly for R. For a subset $I \subset S$, set $\mathcal{Z}_S(I) := \{\mathfrak{n} \in \operatorname{Max} S \mid \mathfrak{n} \supseteq I\}$.

We will consider the following subsets of $\operatorname{Max} S$ and $\operatorname{Spec} S$,

$$\begin{aligned} U_{S/R} &:= \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}\}, \\ \tilde{U}_{S/R} &:= \{\mathfrak{q} \in \operatorname{Spec} S \mid R_{\mathfrak{q}\cap R} = S_{\mathfrak{q}}\}, \\ Z_{S/R} &:= \left\{\mathfrak{q} \in \operatorname{Spec} S \mid \mathcal{Z}_S(\mathfrak{q}) \cap U_{S/R} \neq \emptyset\right\}. \end{aligned}$$

Note that if $U_{S/R} \neq \emptyset$, then R and S have the same fraction field: if $\mathfrak{n} \in U_{S/R}$, then

$$\operatorname{Frac} R = \operatorname{Frac}(R_{\mathfrak{n}\cap R}) = \operatorname{Frac}(S_{\mathfrak{n}}) = \operatorname{Frac} S. \tag{1}$$

Definition 2.1. [B3, Definition 3.1]

- We say S is a *depiction* of R if the morphism
$$\iota_{S/R} : \operatorname{Spec} S \to \operatorname{Spec} R, \quad \mathfrak{q} \mapsto \mathfrak{q} \cap R,$$
is surjective, and
$$U_{S/R} = \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n}\cap R} \text{ is noetherian}\} \neq \emptyset. \tag{2}$$
- The *geometric height* of $\mathfrak{p} \in \operatorname{Spec} R$ is the minimum
$$\operatorname{ght}(\mathfrak{p}) := \min\left\{\operatorname{ht}_S(\mathfrak{q}) \mid \mathfrak{q} \in \iota_{S/R}^{-1}(\mathfrak{p}),\ S \text{ a depiction of } R\right\}.$$
The *geometric dimension* of $\mathfrak{p}$ is
$$\operatorname{gdim} \mathfrak{p} := \dim R - \operatorname{ght}(\mathfrak{p}).$$
- A depiction S of R is *maximal* if S is not properly contained in another depiction of R.

If R is fixed, then we will often write ι_S for $\iota_{S/R}$.

3. Proof of main theorem

Throughout, let S and S' be depictions of R. We begin by recalling the following useful facts from [B3].

Lemma 3.1. *We have*

(1) $\dim R = \dim S$.
(2) The locus $U_{S/R}$ *is an open dense subset of* $\operatorname{Max} S$.
(3) $Z_{S/R} \subseteq \tilde{U}_{S/R}$.
(4) If $\mathfrak{q} \in \tilde{U}_{S/R}$*, then*
$$\iota_S^{-1}\iota_S(\mathfrak{q}) = \{\mathfrak{q}\}.$$

(5) *The images of the loci $U_{S/R}$ and $U_{S'/R}$ in* $\operatorname{Max} R$ *coincide,*

$$\iota_S(U_{S/R}) = \iota_{S'}(U_{S'/R}).$$

Proof. The claims are respectively [B3, Theorem 2.5.4; Proposition 2.4.2; Lemma 2.2; Theorem 2.5.1; Theorem 3.5]. □

Lemma 3.2. *If $\mathfrak{q} \in Z_{S/R}$, then there is a unique prime $\mathfrak{q}' \in \operatorname{Spec} S'$ such that*

$$\mathfrak{q}' \cap R = \mathfrak{q} \cap R.$$

Moreover,

$$\mathfrak{q}' \in Z_{S'/R} \qquad \text{and} \qquad \operatorname{ht}_{S'}(\mathfrak{q}') = \operatorname{ht}_S(\mathfrak{q}).$$

Proof. Suppose the hypotheses hold. Since $\mathcal{Z}_S(\mathfrak{q}) \cap U_{S/R} \neq \emptyset$, there is some $\mathfrak{n} \in U_{S/R}$ for which $\mathfrak{n} \supseteq \mathfrak{q}$. Whence

$$\iota_S(\mathfrak{n}) \in \iota_S(U_{S/R}) \overset{\text{(I)}}{=} \iota_{S'}(U_{S'/R}),$$

where (I) holds by Lemma 3.1.5. Thus there is some $\mathfrak{n}' \in U_{S'/R}$ for which

$$\mathfrak{n}' \in \iota_{S'}^{-1}\iota_S(\mathfrak{n}).$$

In particular, $\mathfrak{n}' \cap R = \mathfrak{n} \cap R$ and

$$S'_{\mathfrak{n}'} = R_{\mathfrak{n}' \cap R} = R_{\mathfrak{n} \cap R} = S_{\mathfrak{n}}.$$

Now $\mathfrak{q}S_{\mathfrak{n}}$ is a prime ideal of $S_{\mathfrak{n}}$ since $\mathfrak{q}$ is a prime ideal of S. Thus

$$\mathfrak{q}S'_{\mathfrak{n}'} = \mathfrak{q}S_{\mathfrak{n}}$$

is a prime ideal of $S'_{\mathfrak{n}'} = S_{\mathfrak{n}}$. Whence the intersection

$$\mathfrak{q}' := \mathfrak{q}S'_{\mathfrak{n}'} \cap S'$$

is a prime ideal of S' contained in $\mathfrak{n}'$. Therefore

$$\mathcal{Z}_{S'}(\mathfrak{q}') \cap U_{S'/R} \neq \emptyset, \tag{3}$$

that is, $\mathfrak{q}' \in Z_{S'/R}$. Furthermore,

$$\mathfrak{q}' \cap R = \mathfrak{q}S'_{\mathfrak{n}'} \cap S' \cap R = \mathfrak{q}S'_{\mathfrak{n}'} \cap S \cap R = \mathfrak{q}S_{\mathfrak{n}} \cap S \cap R = \mathfrak{q} \cap R.$$

Therefore $\mathfrak{q}' \cap R = \mathfrak{q} \cap R$. Uniqueness of $\mathfrak{q}' \in \operatorname{Spec} S'$ follows from (3) and Lemmas 3.1.3 and 3.1.4.

Finally, the heights of $\mathfrak{q}$ and $\mathfrak{q}'$ coincide:

$$\operatorname{ht}_{S'}(\mathfrak{q}') = \dim S'_{\mathfrak{q}'} \overset{\text{(I)}}{=} \dim R_{\mathfrak{q}' \cap R} = \dim R_{\mathfrak{q} \cap R} \overset{\text{(II)}}{=} \dim S_{\mathfrak{q}} = \operatorname{ht}_S(\mathfrak{q}),$$

where (I) and (II) hold by Lemma 3.1.3. □

Proposition 3.3. *If $\mathfrak{q} \in Z_{S/R}$, then*

$$\operatorname{ght}_R(\mathfrak{q} \cap R) = \operatorname{ht}_S(\mathfrak{q}).$$

Proof. Follows from Lemma 3.2. □

Denote by $T_{S/R}$ the global sections ring on $U_{S/R}$,

$$T_{S/R} := \Gamma(U_{S/R}) = \bigcap_{\mathfrak{n} \in U_{S/R}} S_{\mathfrak{n}}.$$

Proposition 3.4. *The global sections ring $T_{S/R}$ satisfies*

$$T_{S/R} := \bigcap_{\mathfrak{n} \in U_{S/R}} S_{\mathfrak{n}} = \bigcap_{\mathfrak{q} \in Z_{S/R}} S_{\mathfrak{q}},$$

and contains each depiction of R.

Proof. Given depictions S, S' of R, we have

$$S' \subseteq \bigcap_{\mathfrak{q}' \in Z_{S'/R}} S'_{\mathfrak{q}'} \overset{(\text{I})}{=} \bigcap_{\mathfrak{q}' \in Z_{S'/R}} R_{\mathfrak{q}' \cap R} \overset{(\text{II})}{=} \bigcap_{\mathfrak{q} \in Z_{S/R}} R_{\mathfrak{q} \cap R} \overset{(\text{III})}{=} \bigcap_{\mathfrak{q} \in Z_{S/R}} S_{\mathfrak{q}} \overset{(\text{IV})}{=} \bigcap_{\mathfrak{n} \in U_{S/R}} S_{\mathfrak{n}} = T_{S/R}.$$

Indeed, (I) and (III) hold by Lemma 3.1.3, and (II) holds by Lemma 3.2. (IV) holds since if $\mathfrak{q} \in Z_{S/R}$, then there is some $\mathfrak{n} \in U_{S/R}$ such that $\mathfrak{n} \supseteq \mathfrak{q}$; in particular, $S_{\mathfrak{n}} \subseteq S_{\mathfrak{q}}$. $\square$

Denote by D_S the set of height 1 prime ideals of S,

$$D_S := \{\mathfrak{q} \in \operatorname{Spec} S \mid \operatorname{ht}(\mathfrak{q}) = 1\}.$$

Note that, by definition, R is noetherian in codimension 1 if R admits a depiction S for which $D_S \subseteq Z_{S/R}$.

For the remainder of this section, we will assume that R is noetherian in codimension 1 unless stated otherwise.

Lemma 3.5. *Suppose $D_S \subseteq Z_{S/R}$. Then $T_{S/R}$ is contained in the normalization $\overline{S}$ of S. In particular, $T_{S/R}$ is an integral extension of S.*

Proof. Since S is a noetherian domain, its normalization $\overline{S}$ is given by [M, Theorem 11.5.ii]

$$\overline{S} = \bigcap_{\mathfrak{q} \in D_S} S_{\mathfrak{q}}. \tag{4}$$

But $D_S \subseteq Z_{S/R}$ by assumption. Therefore $T_{S/R} \subseteq \overline{S}$, by Proposition 3.4. $\square$

Proposition 3.6. *Suppose $D_S \subseteq Z_{S/R}$. Then $T_{S/R}$ is finitely generated as an S-module and as a k-algebra.*

Proof. Set $T := T_{S/R}$. S is a finitely generated k-algebra since S is a depiction of R. Thus its normalization $\overline{S}$ is a finitely generated S-module by the Noether normalization lemma [E, Corollary 13.13]. Furthermore, T is a submodule of $\overline{S}$ by Lemma 3.5. Thus T is a finitely generated S-module since S is noetherian. Therefore T is a finitely generated k-algebra. $\square$

Lemma 3.7. *Suppose $D_S \subseteq Z_{S/R}$, and set $T := T_{S/R}$. The morphism*
$$\iota_{T/S} : \operatorname{Spec} T \to \operatorname{Spec} S, \qquad \mathfrak{t} \mapsto \mathfrak{t} \cap S,$$
is surjective.

Proof. T is an integral extension of S by Lemma 3.5, and therefore $\iota_{T/S}$ is surjective [M, Theorem 9.3.i]. □

Theorem 3.8. *Suppose $D_S \subseteq Z_{S/R}$. Then $T_{S/R}$ is a depiction of R.*

Proof. Set $T := T_{S/R}$.

(i) T is a finitely generated k-algebra by Proposition 3.6.

(ii) We claim that
$$U_{T/R} = \{\mathfrak{t} \in \operatorname{Max} T \mid R_{\mathfrak{t}\cap R} \text{ is noetherian}\}. \tag{5}$$
Consider $\mathfrak{t} \in \operatorname{Max} T$ for which $R_{\mathfrak{t}\cap R}$ is noetherian. Since $\mathfrak{t} \in \operatorname{Max} T$ and T is a finitely generated k-algebra containing S, the intersection $\mathfrak{n} := \mathfrak{t} \cap S$ is a maximal ideal of S. Furthermore,
$$\mathfrak{t} \cap R = \mathfrak{t} \cap S \cap R = \mathfrak{n} \cap R. \tag{6}$$
Whence $R_{\mathfrak{t}\cap R} = R_{\mathfrak{n}\cap R}$. Thus $R_{\mathfrak{n}\cap R}$ is noetherian since $R_{\mathfrak{t}\cap R}$ is noetherian. Therefore $\mathfrak{n} \in U_{S/R}$ since S' is a depiction of R. Consequently,
$$R_{\mathfrak{t}\cap R} = R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}. \tag{7}$$

Furthermore,
$$\begin{aligned} S_{\mathfrak{n}} = S_{\mathfrak{t}\cap S} \subseteq T_{\mathfrak{t}} = \Big(\bigcap_{\mathfrak{n}' \in U_{S/R}} S_{\mathfrak{n}'}\Big)_{\mathfrak{t}} \subseteq \bigcap_{\mathfrak{n}' \in U_{S/R}} (S_{\mathfrak{n}'})_{\mathfrak{t}\cap S_{\mathfrak{n}'}} \\ \overset{(\text{I})}{\subseteq} (S_{\mathfrak{n}})_{\mathfrak{t}\cap S_{\mathfrak{n}}} = (S_{\mathfrak{t}\cap S})_{\mathfrak{t}\cap (S_{\mathfrak{t}\cap S})} = S_{\mathfrak{t}\cap S}, \end{aligned} \tag{8}$$
where (I) holds since $\mathfrak{n} \in U_{S/R}$. Whence $S_{\mathfrak{n}} = T_{\mathfrak{t}}$. Thus together with (7) we obtain
$$R_{\mathfrak{t}\cap R} = S_{\mathfrak{n}} = T_{\mathfrak{t}}.$$
Therefore $\mathfrak{t} \in U_{T/R}$. The converse inclusion ($\subseteq$) in (5) is clear.

(iii) The morphism $\iota_{T/R} : \operatorname{Spec} T \to \operatorname{Spec} R$ is surjective since it factors into surjective maps
$$\operatorname{Spec} T \xrightarrow{\iota_{T/S}} \operatorname{Spec} S \xrightarrow{\iota_{S/R}} \operatorname{Spec} R.$$
Indeed, $\iota_{T/S}$ is surjective by Lemma 3.7, and $\iota_{S/R}$ is surjective since S is a depiction of R.

(iv) Finally, we claim that $U_{T/R}$ is nonempty. Since S is a depiction of R, there is some $\mathfrak{n} \in U_{S/R}$. By Lemma 3.7, there is some $\mathfrak{t} \in \operatorname{Max} T$ such that $\mathfrak{t} \cap S = \mathfrak{n}$. Thus
$$R_{\mathfrak{t}\cap R} \overset{(\text{I})}{=} R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}} \overset{(\text{II})}{=} T_{\mathfrak{t}},$$
where (I) holds by (6), and (II) holds by (8). Therefore $\mathfrak{t} \in U_{T/R}$. □

Lemma 3.9. *Set $T := T_{S/R}$. If $D_S \subseteq Z_{S/R}$, then $D_T \subseteq Z_{T/R}$.*

Proof. Let $\mathfrak{t} \in D_T$, and set $\mathfrak{q} := \mathfrak{t} \cap S$. By Lemma 3.5, T is an integral extension of S. Thus $\operatorname{ht}_T(\mathfrak{t}) = 1$ implies $\operatorname{ht}_S(\mathfrak{q}) = 1$ [K, Theorem 46]. Whence $\mathfrak{q} \in Z_{S/R}$ since S is noetherian in codimension 1. But $\mathfrak{t} \cap R = (\mathfrak{t} \cap S) \cap R = \mathfrak{q} \cap R$. Therefore $\mathfrak{t} \in Z_{T/R}$ by Lemma 3.2. □

Lemma 3.10. *Suppose $D_{S'} \subseteq Z_{S'/R}$, and set $T := T_{S'/R}$. The morphism*

$$D_T \to D_S, \quad \mathfrak{t} \mapsto \mathfrak{t} \cap S,$$

is well-defined and surjective.

Proof. (i) We first claim that the map $D_T \to D_S$ is well-defined. Let $\mathfrak{t} \in D_T$. Then $\mathfrak{t} \in Z_{T/R}$ by Lemma 3.9. Thus there is a unique prime $\mathfrak{q}' \in \operatorname{Spec} S$ such that $\mathfrak{q}' \cap R = \mathrm{t} \cap R$, and $\mathfrak{q}' \in D_S$, by Lemma 3.2. But then

$$(\mathfrak{t} \cap S) \cap R = \mathfrak{t} \cap R = \mathfrak{q}' \cap R.$$

Therefore, by the uniqueness of $\mathfrak{q}'$, we have

$$\mathfrak{t} \cap S = \mathfrak{q}' \in D_S.$$

(ii) We now claim that the map $D_T \to D_S$ is surjective. Let $\mathfrak{q} \in D_S$. Set $\mathfrak{p} := \mathfrak{q} \cap R$. By Theorem 3.8, there is a prime $\mathfrak{t} \in \iota_{T/R}^{-1}(\mathfrak{p})$; we want to show that $\mathfrak{t} \in D_T$. Indeed, assume to the contrary that $\mathfrak{t} \not\in D_T$, that is, $\operatorname{ht}_T(\mathfrak{t}) \geq 2$. Then there is a prime $\mathfrak{t}' \in D_T$ properly contained in $\mathfrak{t}$. Since $\operatorname{ht}_T(\mathfrak{t}') = 1$, we have $\mathfrak{t}' \in Z_{T/R}$ by Lemma 3.9. Thus the containment

$$\mathfrak{t}' \cap R \subset \mathfrak{t} \cap R = \mathfrak{p}$$

is proper, by Lemma 3.2. Consequently, the containment

$$\mathfrak{t}' \cap S \subset \mathfrak{q}$$

is also proper. But $\mathfrak{t}' \cap S$ is a nonzero prime of S since $\mathfrak{t}'$ is a nonzero prime of T. Therefore $\operatorname{ht}_S(\mathfrak{q}) \geq 2$, contrary to our choice of $\mathfrak{q}$. □

Lemma 3.11. *Suppose $D_{S'} \subseteq Z_{S'/R}$, and set $T := T_{S'/R}$. Then*

$$\bigcap_{\mathfrak{t} \in D_T} T_{\mathfrak{t}} = \bigcap_{\mathfrak{q} \in D_S} S_{\mathfrak{q}}.$$

Proof. We have

$$D_T \overset{(\mathrm{I})}{\subseteq} Z_{T/R} \overset{(\mathrm{II})}{\subseteq} \tilde{U}_{T/R}, \tag{9}$$

where (I) holds by Lemma 3.9, and (II) holds by Lemma 3.1.3.

Let $\mathfrak{t} \in D_T$, and set $\mathfrak{q} := \mathfrak{t} \cap S \in \operatorname{Spec} S$. Then, since $\mathfrak{t} \cap R = \mathfrak{q} \cap R$, we have

$$T_{\mathfrak{t}} \overset{(\mathrm{I})}{=} R_{\mathfrak{t} \cap R} = R_{\mathfrak{q} \cap R} \subseteq S_{\mathfrak{q}} \subseteq T_{\mathfrak{t}},$$

where (I) holds by (9) and Lemma 3.1.3. Whence, $T_{\mathfrak{t}} = S_{\mathfrak{t}\cap S}$. Therefore

$$\bigcap_{\mathfrak{t}\in D_T} T_{\mathfrak{t}} \overset{\text{(I)}}{=} \bigcap_{\mathfrak{t}\cap S\in D_S} S_{\mathfrak{t}\cap S} \overset{\text{(II)}}{=} \bigcap_{\mathfrak{q}\in D_S} S_{\mathfrak{q}},$$

where (I) holds since $D_T \to D_S$ is well-defined by Lemma 3.10; and (II) holds since $D_T \to D_S$ is surjective, again by Lemma 3.10. □

Suppose R has a unique maximal depiction T, but is not noetherian in codimension 1. Then in general R may admit a depiction S for which the morphism

$$\iota_{T/S} : \operatorname{Spec} T \to \operatorname{Spec} S, \quad \mathfrak{t} \mapsto \mathfrak{t} \cap S,$$

is not surjective; see Example 5.2 below. However, if R is noetherian in codimension 1, then we have the following.

Proposition 3.12. *Suppose $D_{S'} \subseteq Z_{S'/R}$, and set $T := T_{S'/R}$. Then for any depiction S of R, the morphism $\iota_{T/S} : \operatorname{Spec} T \to \operatorname{Spec} S$ is surjective.*

Proof. (i) We first claim that for each $\mathfrak{n} \in \operatorname{Max} S$, $\mathfrak{n}T \neq T$.

Assume to the contrary that there is some $\mathfrak{n} \in \operatorname{Max} S$ for which $\mathfrak{n}T = T$. Let $\overline{S_{\mathfrak{n}}}$ be the normalization of $S_{\mathfrak{n}}$. Then

$$\bigcap_{\mathfrak{t}\in D_T} T_{\mathfrak{t}} \overset{\text{(I)}}{=} \bigcap_{\mathfrak{q}\in D_S} S_{\mathfrak{q}} \subseteq \bigcap_{\mathfrak{q}\in D_S:\mathfrak{q}\subseteq\mathfrak{n}} S_{\mathfrak{q}} \overset{\text{(II)}}{=} \bigcap_{\mathfrak{q}\in D_S:\mathfrak{q}\subseteq\mathfrak{n}} (S_{\mathfrak{n}})_{\mathfrak{q}S_{\mathfrak{n}}} \overset{\text{(III)}}{\subseteq} \bigcap_{\mathfrak{s}\in D_{S_{\mathfrak{n}}}} (S_{\mathfrak{n}})_{\mathfrak{s}} \overset{\text{(IV)}}{=} \overline{S_{\mathfrak{n}}}. \tag{10}$$

Indeed, (I) holds by Lemma 3.11. (II) holds since if $\mathfrak{q} \in \operatorname{Spec} S$ is contained in $\mathfrak{n}$, then

$$S_{\mathfrak{q}} = (S_{\mathfrak{n}})_{\mathfrak{q}S_{\mathfrak{n}}}.$$

(III) holds since if $\mathfrak{s} \in \operatorname{Spec} S_{\mathfrak{n}}$ has height 1, then $\mathfrak{s}\cap S \in \operatorname{Spec} S$ also has height 1, and $\mathfrak{s} \cap S \subseteq \mathfrak{n}$. Finally, (IV) holds since $S_{\mathfrak{n}}$ is a noetherian domain [M, Theorem 11.5.ii].

By integrality, there is a prime ideal $\mathfrak{n}'$ of the normalization $\overline{S_{\mathfrak{n}}}$ lying over $\mathfrak{n}$ [K, Theorem 44]. Thus

$$1 \in T = \mathfrak{n}T \subseteq \mathfrak{n}\left(\bigcap_{\mathfrak{t}\in D_T} T_{\mathfrak{t}}\right) \overset{\text{(I)}}{\subseteq} \mathfrak{n}\overline{S_{\mathfrak{n}}} \subseteq \mathfrak{n}',$$

where (I) holds by (10). But then 1 is in $\mathfrak{n}'$, a contradiction.

(ii) We claim that the morphism of maximal spectra

$$\operatorname{Max} T \to \operatorname{Max} S, \quad \mathfrak{t} \mapsto \mathfrak{t} \cap S,$$

is surjective. Let $\mathfrak{n} \in \operatorname{Spec} S$. Then there is a maximal ideal $\mathfrak{t} \in \operatorname{Max} T$ containing $\mathfrak{n}T$ since $\mathfrak{n}T \neq T$ by Claim (i). Whence

$$\mathfrak{n} \subseteq \mathfrak{n}T \cap S \subseteq \mathfrak{t} \cap S \neq S.$$

Therefore $\mathfrak{t} \cap S = \mathfrak{n}$ since $\mathfrak{n}$ is a maximal ideal.

(iii) T is a finitely generated k-algebra by Proposition 3.6, and S is a finitely generated k-algebra since S is a depiction. Therefore $\iota_{T/S}$ is also surjective, by [B3, Lemma 3.6].[1] $\square$

Note that, by the definition of geometric height, each $\mathfrak{q} \in \operatorname{Spec} S$ satisfies

$$\operatorname{ght}_R(\mathfrak{q} \cap R) \leq \operatorname{ht}_S(\mathfrak{q}).$$

Lemma 3.13. *The following are equivalent:*

(1) For each $\mathfrak{p} \in \operatorname{Spec} R$ there is some $\mathfrak{q} \in \iota_{S/R}^{-1}(\mathfrak{p})$ such that

$$\operatorname{ght}_R(\mathfrak{p}) = \operatorname{ht}_S(\mathfrak{q}).$$

(2) For each $\mathfrak{q} \in \operatorname{Spec} S$ of minimal height in $\iota_{S/R}^{-1}(\mathfrak{q} \cap R)$, we have

$$\operatorname{ght}_R(\mathfrak{q} \cap R) = \operatorname{ht}_S(\mathfrak{q}).$$

Proof. (1) $\Rightarrow$ (2): Suppose $\mathfrak{q} \in \operatorname{Spec} S$ has minimal height in $\iota_{S/R}^{-1}(\mathfrak{p})$, where $\mathfrak{p} := \mathfrak{q} \cap R$. By assumption (1), there is some $\mathfrak{q}' \in \iota_{S/R}^{-1}(\mathfrak{p})$ such that $\operatorname{ht}_S(\mathfrak{q}') = \operatorname{ght}_R(\mathfrak{p})$. Therefore

$$\operatorname{ght}_R(\mathfrak{p}) \leq \operatorname{ht}_S(\mathfrak{q}) \leq \operatorname{ht}_S(\mathfrak{q}') = \operatorname{ght}_R(\mathfrak{p}).$$

(2) $\Rightarrow$ (1): Let $\mathfrak{p} \in \operatorname{Spec} R$. Since S is a depiction of R, we have $\iota_{S/R}^{-1}(\mathfrak{p}) \neq \emptyset$; choose $\mathfrak{q} \in \iota_{S/R}^{-1}(\mathfrak{p})$ of minimal height. Then $\operatorname{ght}_R(\mathfrak{p}) = \operatorname{ht}_S(\mathfrak{q})$ by assumption (2). $\square$

Definition 3.14. If either (hence both) of the conditions in Lemma 3.13 are satisfied, then we call S a *saturated* depiction of R.

Lemma 3.15. *If S is a saturated depiction of R, then each $\mathfrak{q} \in \operatorname{Spec} S$ of minimal height in $\iota_{S/R}^{-1}(\mathfrak{q} \cap R)$ satisfies*

$$\operatorname{gdim}(\mathfrak{q} \cap R) = \dim S/\mathfrak{q}.$$

Proof. We have

$$\operatorname{gdim}(\mathfrak{q} \cap R) := \dim R - \operatorname{ght}_R(\mathfrak{q} \cap R) \overset{(\mathrm{I})}{=} \dim S - \operatorname{ht}_S(\mathfrak{q}) \overset{(\mathrm{II})}{=} \dim S/\mathfrak{q},$$

where (I) holds by Lemma 3.1.1, and (II) holds since S is a finite type integral domain [S, Proposition III.15]. $\square$

Lemma 3.16. *Let S be a noetherian integral domain, and let U be a nonempty open subset of $\operatorname{Max} S$. For each nonzero $\mathfrak{q} \in \operatorname{Spec} S$ there is some $\mathfrak{p} \in \operatorname{Spec} S$ contained in $\mathfrak{q}$ such that*

$$\mathcal{Z}(\mathfrak{p}) \cap U \neq \emptyset \quad \text{and} \quad \operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{q}) - 1.$$

[1] The assumption in [B3, Lemma 3.6] that k is uncountable is not necessary here since S is a finitely generated k-algebra, rather than a countably generated k-algebra.

Proof. Fix $\mathfrak{q} \in \operatorname{Spec} S$. Denote by Q the set of primes $\mathfrak{p} \in \operatorname{Spec} S$ that are properly contained in $\mathfrak{q}$ and satisfy $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{q}) - 1$.

Assume to the contrary that $\mathcal{Z}(\mathfrak{p}) \cap U = \emptyset$ for each $\mathfrak{p} \in Q$. Then

$$(\cup_{\mathfrak{p} \in Q} \mathcal{Z}(\mathfrak{p})) \cap U = \emptyset.$$

Whence $\overline{\cup_{\mathfrak{p} \in Q} \mathcal{Z}(\mathfrak{p})} \cap U = \emptyset$ since U is open by Lemma 3.1.2. Thus

$$\emptyset \neq U \subseteq \left(\overline{\cup_{\mathfrak{p} \in Q} \mathcal{Z}(\mathfrak{p})} \right)^c = \mathcal{Z}(\cap_{\mathfrak{p} \in Q} \mathfrak{p})^c.$$

Therefore the ideal $I := \cap_{\mathfrak{p} \in Q} \mathfrak{p}$ is nonzero; say $0 \neq a \in I$. In particular, $a \in \mathfrak{q}$ since for each $\mathfrak{p} \in Q$, $I \subseteq \mathfrak{p} \subset \mathfrak{q}$.

By [F, Lemma 3.2], if C is a noetherian integral domain, $\mathfrak{t} \in \operatorname{Spec} C$, and $0 \neq c \in \mathfrak{t}$, then there is a prime $\mathfrak{s} \in \operatorname{Spec} C$ such that $\operatorname{ht}(\mathfrak{s}) = \operatorname{ht}(\mathfrak{t}) - 1$ and $\mathfrak{s} \not\ni c$. In our case we may take $C = S_{\mathfrak{q}}$, $\mathfrak{t} = \mathfrak{q} S_{\mathfrak{q}}$, and $c = a$. Then there is a prime $\bar{\mathfrak{p}} \in \operatorname{Spec} S_{\mathfrak{q}}$ such that

$$\operatorname{ht}_{S_{\mathfrak{q}}}(\bar{\mathfrak{p}}) = \operatorname{ht}_{S_{\mathfrak{q}}}(\mathfrak{q} S_{\mathfrak{q}}) - 1 = \operatorname{ht}_S(\mathfrak{q}) - 1 \quad \text{and} \quad \bar{\mathfrak{p}} \not\ni a.$$

Set $\mathfrak{p} := \bar{\mathfrak{p}} \cap S$. Then $\operatorname{ht}_S(\mathfrak{p}) = \operatorname{ht}_{S_{\mathfrak{q}}}(\bar{\mathfrak{p}})$ and $\mathfrak{p} \subset \mathfrak{q}$. Thus $\mathfrak{p} \in Q$. But $a \notin \mathfrak{p}$ since $a \notin \bar{\mathfrak{p}}$. Therefore $a \notin I$, contrary to our choice of a. □

Theorem 3.17. *Suppose $D_S \subseteq Z_{S/R}$. Then $T_{S/R}$ is saturated.*

Proof. Set $T := T_{S/R}$. Let $\mathfrak{p} \in \operatorname{Spec} R$. Then there is a depiction S' of R such that for some $\mathfrak{q}$ in $\iota_{S'/R}^{-1}(\mathfrak{p})$, we have

$$\operatorname{ght}_R(\mathfrak{p}) = \operatorname{ht}_{S'}(\mathfrak{q}).$$

By Proposition 3.12, $\iota_{T/S'}^{-1}(\mathfrak{q}) \neq \emptyset$; say $\mathfrak{t} \in \iota_{T/S'}^{-1}(\mathfrak{q})$. Furthermore, $U_{T/R}$ is an open dense subset of $\operatorname{Max} T$, by Lemma 3.1.2. Thus there is a prime $\mathfrak{t}' \in \operatorname{Spec} T$ properly contained in $\mathfrak{t}$, and maximal with respect to this inclusion, such that $\mathcal{Z}_T(\mathfrak{t}') \cap U_{T/R} \neq \emptyset$, by Lemma 3.16.

Set $m := \operatorname{ht}_T(\mathfrak{t})$. Consider a maximal chain of prime ideals of T contained in $\mathfrak{t}'$,

$$0 \subset \mathfrak{t}_1 \subset \mathfrak{t}_2 \subset \cdots \subset \mathfrak{t}_{m-1} = \mathfrak{t}' \subset \mathfrak{t}_m = \mathfrak{t},$$

and the corresponding chain of prime ideals of S',

$$0 \subset \mathfrak{t}_1 \cap S' \subseteq \mathfrak{t}_2 \cap S' \subseteq \cdots \subseteq \mathfrak{t}_m \cap S' = \mathfrak{q}. \tag{11}$$

We claim that the chain (11) is strict. Indeed, assume to the contrary that there is some $1 \leq i < m$ for which $\mathfrak{t}_i \cap S' = \mathfrak{t}_{i+1} \cap S'$. Then $\mathfrak{t}_i \cap R = \mathfrak{t}_{i+1} \cap R$. Furthermore, $\mathcal{Z}_T(\mathfrak{t}_i) \cap U_{T/R} \neq \emptyset$ since $\mathcal{Z}_T(\mathfrak{t}_{m-1}) \cap U_{T/R} \neq \emptyset$. But then $\mathfrak{t}_i = \mathfrak{t}_{i+1}$ by Lemmas 3.1.3 and 3.1.4, a contradiction.

It follows that

$$\operatorname{ght}_R(\mathfrak{p}) \overset{\text{(I)}}{\leq} \operatorname{ht}_T(\mathfrak{t}) = m \overset{\text{(II)}}{\leq} \operatorname{ht}_{S'}(\mathfrak{q}) = \operatorname{ght}_R(\mathfrak{p}),$$

where (I) holds since $\mathfrak{t} \cap R = \mathfrak{t} \cap S' \cap R = \mathfrak{q} \cap R = \mathfrak{p}$, and (II) holds since the chain (11) is strict. Therefore $\operatorname{ght}_R(\mathfrak{p}) = \operatorname{ht}_T(\mathfrak{t})$. □

Proposition 3.18. *Each codimension 1 subvariety of* $\operatorname{Max} S$ *intersects* $U_{S/R}$ *if and only if each codimension 1 subvariety of* $\operatorname{Max} S'$ *intersects* $U_{S'/R}$*:*

$$D_S \subseteq Z_{S/R} \quad \Longleftrightarrow \quad D_{S'} \subseteq Z_{S'/R}.$$

In particular, the definition of 'noetherian in codimension 1' is independent of the choice of depiction.

Proof. Suppose $D_S \subseteq Z_{S/R}$, and consider $\mathfrak{q} \in D_{S'}$; we want to show that $\mathfrak{q} \in Z_{S'/R}$.

Set $T := T_{S/R}$ and $\mathfrak{p} := \mathfrak{q} \cap R$. By Theorems 3.8 and 3.17, T is a saturated depiction of R. Thus there is some $\mathfrak{t} \in \iota^{-1}_{T/R}(\mathfrak{p})$ such that $\operatorname{ht}_T(\mathfrak{t}) = \operatorname{ght}_R(\mathfrak{p})$. Therefore

$$1 \overset{(\text{I})}{\leq} \operatorname{ht}_T(\mathfrak{t}) = \operatorname{ght}_R(\mathfrak{p}) \overset{(\text{II})}{\leq} \operatorname{ht}_{S'}(\mathfrak{q}) = 1,$$

where (I) holds since $\mathfrak{t} \neq 0$, and (II) holds since $\mathfrak{q} \cap R = \mathfrak{p}$. Whence $\operatorname{ht}_T(\mathfrak{t}) = 1$. Thus $\mathfrak{t} \in Z_{T/R}$ by Lemma 3.9. Therefore there is a unique prime $\mathfrak{q}' \in \operatorname{Spec} S'$ such that $\mathfrak{q}' \cap R = \mathfrak{t} \cap R$, and $\mathfrak{q}' \in Z_{S'/R}$, by Lemma 3.2. But

$$\mathfrak{q} \cap R = \mathfrak{p} = \mathfrak{t} \cap R = \mathfrak{q}' \cap R.$$

It follows that $\mathfrak{q}' = \mathfrak{q}$, by the uniqueness of $\mathfrak{q}'$. Therefore $\mathfrak{q} \in Z_{S'/R}$. □

Theorem 3.19. *Suppose R is noetherian in codimension* 1*. Let S and S' be arbitrary depictions of R. Then*

$$T := T_{S/R} = T_{S'/R},$$

and T is the unique maximal depiction of R. Furthermore, T is contained in the normalization of each depiction of R,

$$S \subseteq T \subseteq \overline{S}.$$

In particular, if $S = \overline{S}$ is normal, then S is the unique maximal depiction of R, as well as the unique normal depiction of R.

Proof. The overrings $T_{S/R}$ and $T_{S'/R}$ are both depictions of R by Proposition 3.18 and Theorem 3.8. But $T_{S/R}$ and $T_{S'/R}$ each contain every depiction of R, by Proposition 3.4. Therefore $T_{S/R} = T_{S'/R}$. Finally, the inclusion $T \subseteq \overline{S}$ holds by Lemma 3.5. □

4. Saturated depictions of coordinate rings with a unique positive dimensional closed point

Rings of the form $R = k + I$, where I is an ideal of a finite type integral domain S, form a particularly nice class of nonnoetherian rings in the study of nonnoetherian geometry. It was shown in [B2, Corollary 1.3] that if I is a proper nonzero nonmaximal radical ideal of S, then the following are equivalent:

(1) $\dim S/I \geq 1$.
(2) R is nonnoetherian.
(3) R is depicted by S.

Furthermore, if R is nonnoetherian, then

$$U_{S/R} = \mathcal{Z}_S(I)^c.$$

In the following, we do not assume that R is noetherian in codimension 1.

Theorem 4.1. *Let I be a nonzero radical ideal of S such that $\dim S/I \geq 1$, and set $R := k + I$. If S is a unique factorization domain or $\operatorname{ht}_S(I) = 1$, then S is a saturated depiction of R.*

Proof. By [B2, Corollary 1.3], S is a depiction of R since I is a nonzero radical ideal of S satisfying $\dim S/I \geq 1$.

(i) First suppose S is a UFD and $\operatorname{ht}_S(I) \geq 2$. Then $D_S \subseteq Z_{S/R}$, since $U_{S/R} = \mathcal{Z}_S(I)^c$. Furthermore,

$$T_{S/R} := \Gamma(U_{S/R}) = \Gamma(\mathcal{Z}_S(I)^c) \overset{(\text{I})}{=} S,$$

where (I) holds since S is a UFD and $\operatorname{ht}_S(I) \geq 2$. Therefore S is saturated by Theorem 3.17.

(ii) Now suppose $\operatorname{ht}_S(I) = 1$. Consider $\mathfrak{q} \in \iota^{-1}_{S/R}(\mathfrak{q} \cap R)$ with minimal height. Then either $\mathfrak{q} \in Z_{S/R}$, or $\mathfrak{q}$ has minimal height such that $\mathcal{Z}_S(\mathfrak{q}) \subseteq \mathcal{Z}_S(I)$.

If $\mathfrak{q} \in Z_{S/R}$, then by Proposition 3.3,

$$\operatorname{ght}_R(\mathfrak{q} \cap R) = \operatorname{ht}_S(\mathfrak{q}).$$

So suppose $\mathfrak{q}$ has minimal height such that $\mathcal{Z}_S(\mathfrak{q}) \subseteq \mathcal{Z}_S(I)$. Then $\mathfrak{q} \supseteq I$ since $\mathfrak{q}$ and I are radical ideals of S. Hence, $\mathfrak{q} \cap R = I$ since I is a maximal ideal of R. Thus

$$1 \overset{(\text{I})}{\leq} \operatorname{ght}_R(I) \overset{(\text{II})}{\leq} \operatorname{ht}_S(\mathfrak{q}) \overset{(\text{III})}{=} \operatorname{ht}_S(I) \overset{(\text{IV})}{=} 1,$$

where (I) holds since $I \neq 0$ and S is an integral domain; (II) holds since $\mathfrak{q} \cap R = I$; (III) holds since $\mathfrak{q}$ is a minimal prime over I of minimal height; and (IV) holds by assumption. Consequently,

$$\operatorname{ght}_R(I) = \operatorname{ht}_S(\mathfrak{q}).$$

Therefore S is saturated. □

5. Examples

Example 5.1. Consider the family of algebras

$$S_j := k[x, y, xz, yz, xz^2, yz^2, \ldots, xz^{j-1}, yz^{j-1}, z^j] \quad \text{and} \quad R := k + (x, y)S_1,$$

where $j \geq 1$, and $(x, y)S_1$ is the ideal of $S_1 = k[x, y, z]$ generated by x and y. By Theorem 4.1, S_1 is a saturated depiction of R with

$$U_{S_1/R} = \mathcal{Z}_{S_1}(x, y)^c.$$

Since each 2-dimensional subvariety of $\operatorname{Max} S_1 = \mathbb{A}^3_k$ intersects the complement of the line $\mathcal{Z}_{S_1}(x, y)$, R is noetherian in codimension 1. Furthermore, since S_1 is a polynomial ring, it is normal. Therefore S_1 is the unique maximal depiction of R, as well as the unique normal depiction of R, by Theorem 3.19.

We will show that each $S_j \subseteq S_1$ is also a depiction of R. Fix $j \geq 1$.

We first claim that $\iota_{S_j/R} : \operatorname{Spec} S_j \to \operatorname{Spec} R$ is surjective. Let $\mathfrak{p} \in \operatorname{Spec} R$. Since S_1 is a depiction of R, there is a prime $\mathfrak{q} \in \operatorname{Spec} S_1$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. But $R \subset S_j \subseteq S_1$. Therefore the prime $\mathfrak{q}' := \mathfrak{q} \cap S_j \in \operatorname{Spec} S_j$ satisfies $\mathfrak{q}' \cap R = \mathfrak{p}$, proving our claim.

We now claim that (2) in Definition 2.1 holds. Let $\mathfrak{n} \in \operatorname{Max} S_j$ be such that $R_{\mathfrak{n} \cap R}$ is noetherian. S_1 is a finitely generated S_j-module with generating set $\{1, z, z^2, \ldots, z^{j-1}\}$. Thus, by Nakayama's lemma, $\mathfrak{n} S_1 \neq S_1$. Therefore there is some $\mathfrak{t} \in \operatorname{Max} S_1$ such that $\mathfrak{t} \cap S_j = \mathfrak{n}$. Furthermore, since S_1 is a depiction of R and $R_{\mathfrak{t} \cap R} = R_{\mathfrak{n} \cap R}$ is noetherian, we have $(S_1)_{\mathfrak{t}} = R_{\mathfrak{t} \cap R}$. Thus,

$$(S_j)_{\mathfrak{n}} = (S_j)_{\mathfrak{t} \cap S_j} \subseteq (S_1)_{\mathfrak{t}} = R_{\mathfrak{t} \cap R} = R_{\mathfrak{n} \cap R} \subseteq (S_j)_{\mathfrak{n}}.$$

Whence $(S_j)_{\mathfrak{n}} = R_{\mathfrak{n} \cap R}$, proving our claim. S_j is therefore a depiction of R.

In the following example, we show that if R is not noetherian in codimension 1, then R may admit a unique maximal depiction T which is not saturated. This example demonstrates that unique maximal depictions, when they exist, are not necessarily the 'right' depictions for some nonnoetherian rings.

Example 5.2. Consider the algebras

$$T := k[x, x^{-1}, y], \qquad S := k[x, y], \qquad R := k + I,$$

where $I := x(x-1, y)S$. We will show that T is the unique maximal depiction of R, but is not saturated.

Since $\dim S/I = 1$, S is a saturated depiction of R, by Theorem 4.1. However, since the localization $R_{xS \cap R} = R_I$ is not noetherian, R is not noetherian in codimension 1, by Proposition 3.18. Clearly T is also a depiction of R, with

$$U_{T/R} = \mathcal{Z}_T(\mathfrak{t}_0)^c, \tag{12}$$

where $\mathfrak{t}_0 := (x-1, y)T \in \operatorname{Max} T$.

$\operatorname{Max} R$ may therefore be viewed either as the plane $\operatorname{Max} S = \mathbb{A}_k^2$ where the point $\mathcal{Z}_S(x-1, y)$ and line $\mathcal{Z}_S(x)$ are together identified as a single closed point; or as the open subset of the plane

$$\operatorname{Max} T \cong \mathcal{Z}_S(x)^c.$$

From the perspective of T, all the closed points of $\operatorname{Max} R$, including I itself, appear zero-dimensional.

We claim that T is the unique maximal depiction of R. Indeed, let S' be any depiction of R. Then

$$S' \subseteq \bigcap_{\mathfrak{q} \in Z_{S'/R}} S'_{\mathfrak{q}} \stackrel{\text{(I)}}{=} \bigcap_{\mathfrak{q} \in Z_{S'/R}} R_{\mathfrak{q} \cap R} \stackrel{\text{(II)}}{=} \bigcap_{\mathfrak{t} \in Z_{T/R}} R_{\mathfrak{t} \cap R} \stackrel{\text{(III)}}{=} \bigcap_{\mathfrak{t} \in Z_{T/R}} T_{\mathfrak{t}}$$
$$\stackrel{\text{(IV)}}{=} \bigcap_{\mathfrak{t} \in \operatorname{Spec} T \setminus \{\mathfrak{t}_0\}} T_{\mathfrak{t}} = \bigcap_{\mathfrak{t} \in \operatorname{Spec} T} T_{\mathfrak{t}} = T,$$

where (I) and (III) hold by Lemma 3.1.3; (II) holds by Lemma 3.2; and (IV) holds by (12). Therefore S' is contained in T.

We now claim that T is not saturated. Set $\mathfrak{q} = xS$; then

$$\mathfrak{q} \cap R = \mathfrak{t}_0 \cap R = I.$$

Furthermore,

$$1 \stackrel{\text{(I)}}{\leq} \operatorname{ght}_R(I) \leq \operatorname{ht}_S(\mathfrak{q}) = 1 < 2 = \operatorname{ht}_T(\mathfrak{t}_0),$$

where (I) holds since $I \neq 0$.

Let $\mathfrak{t} \in \operatorname{Spec} T \setminus \{\mathfrak{t}_0\}$. Since $\mathfrak{t}_0$ is a maximal ideal of T, (12) implies $\mathfrak{t} \in Z_{T/R}$. Whence $\mathfrak{t} \in \tilde{U}_{T/R}$ by Lemma 3.1.3. Thus $\mathfrak{t} \notin \iota^{-1}_{T/R}(I)$ by Lemma 3.1.4. Therefore

$$\iota^{-1}_{T/R}(I) = \{\mathfrak{t}_0\}.$$

It follows that there is a prime $\mathfrak{p} \in \operatorname{Spec} R$, namely I, such that for each $\mathfrak{t} \in \iota^{-1}_{T/R}(\mathfrak{p})$,

$$\operatorname{ght}_R(\mathfrak{p}) < \operatorname{ht}_T(\mathfrak{t}).$$

In our final example, we show that R need not be noetherian in codimension 1 in order for R to admit a saturated unique maximal depiction.

Example 5.3. Consider the algebras

$$S := k[x,y] \quad \text{and} \quad R := k + xS.$$

By Theorem 4.1, S is a saturated depiction of R, with

$$U_{S/R} = \mathcal{Z}_S(x)^c.$$

Furthermore, by Proposition 3.18, R is not noetherian in codimension 1 since the localization $R_{xS \cap R}$ is not noetherian.

We claim that S is the unique maximal depiction of R. Let S' be any depiction of R. Then by Proposition 3.4,

$$S' \subseteq \Gamma(U_{S/R}) = \Gamma(\mathcal{Z}_S(x)^c) = S[x^{-1}].$$

However, $\iota_{S[x^{-1}]/R}$ is not surjective since the ideal $xS \in \operatorname{Max} R$ does not have a preimage in $S[x^{-1}]$. Furthermore, $\iota^{-1}_{S'/R}(xS) \neq \emptyset$ since S' is a depiction of R. It follows that $S' \subseteq S$. S is therefore a saturated unique maximal depiction of R, even though R is not noetherian in codimension 1.

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