CONJUGATION ORBITS OF LOXODROMIC PAIRS IN $\mathrm{SU}(n,1)$

KRISHNENDU GONGOPADHYAY AND SHIV PARSAD

Abstract. Let $\mathbf{H}_{\mathbb{C}}^n$ be the n-dimensional complex hyperbolic space and $\mathrm{SU}(n,1)$ be the isometry group. An element g in $\mathrm{SU}(n,1)$ is called *loxodromic* or *hyperbolic* if it has exactly two fixed points on the boundary $\partial\mathbf{H}_{\mathbb{C}}^n$. We classify pairs of loxodromic elements in $\mathrm{SU}(n,1)$ up to conjugacy.

1. Introduction

The aim of this paper is to continue understanding of the conjugacy classes of pairs of loxodromic elements in $\mathrm{SU}(n,1)$.

Let $\mathrm{F}_2 = \langle x, y\rangle$ be a two-generator free group. Let $\mathfrak{X}(\mathrm{F}_2, \mathrm{SU}(n,1))$ denote the orbit space $\mathrm{Hom}(\mathrm{F}_2, \mathrm{SU}(n,1))/\mathrm{SU}(n,1)$ of the conjugation action of $\mathrm{SU}(n,1)$ on the space $\mathrm{Hom}(\mathrm{F}_2, \mathrm{SU}(n,1))$ of faithful representations of F_2 into $\mathrm{SU}(n,1)$. Let $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$ denote the subset of $\mathfrak{X}(\mathrm{F}_2, \mathrm{SU}(n,1))$ consisting of representations ρ such that both $\rho(x)$ and $\rho(y)$ are loxodromic elements in $\mathrm{SU}(n,1)$ having no common fixed point. A problem of geometric interest is to parametrize this subset $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$. The motivation for doing this is the construction of Fenchel-Nielsen coordinates in the classical Teichmüller space that is built upon a parametrization of discrete, faithful totally loxodromic representations in $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SL}(2,\mathbb{R}))$. This is rooted back to the classical works of Fricke [Fri96] and Vogt [Vog89] who proved that a non-elementary two-generator free subgroup of $\mathrm{SL}(2,\mathbb{C})$ is determined up to conjugation by the traces of the generators and their product, see Goldman [Gol09] for an exposition.

Let $\mathbf{H}_{\mathbb{C}}^n$ be the n-dimensional complex hyperbolic space. The group $\mathrm{SU}(n,1)$ acts by the isometries on $\mathbf{H}_{\mathbb{C}}^n$. An element of $\mathrm{SU}(n,1)$ is called *hyperbolic* or *loxodromic* if it has exactly two fixed points on the boundary of the complex hyperbolic space. The space $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1)$ contains the subspace containing discrete, faithful, totally loxodromic or type-preserving representations. These are curious families of representations and has not been well-understood even in the case $n = 2$. We refer to the surveys [PP10], [Sch02], [Wil16] and the references therein for an up to date account of the investigations in this direction.

For notational convenience, an element in $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1)$ will be called a 'purely loxodromic representation'. Most of the existing work to understand $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1)$ is centered around the case $n = 2$, though it would be interesting to generalize some

Date: February 26, 2018.
2010 *Mathematics Subject Classification.* Primary 20H10; Secondary 30F40, 15B57, 51M10.
Key words and phrases. complex hyperbolic space, surface group representations, traces.

of above mentioned works for $n > 2$. A starting point to do this could be the classification of pairs of elements in $\mathrm{SU}(n,1)$ up to conjugacy. In other words, the problem would be to determine a representation in $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1)$. This problem is of considerable difficulty in higher dimensions. Following classical invariant theory, one approach is to obtain this classification using trace invariants like the coefficients of the characteristic polynomials and their compositions. Even in the case of $\mathrm{SL}(n,\mathbb{C})$ a complete and minimal set of trace parameters along with relations between them, is still unknown except for a few lower values of n, eg. [DS06, Dok07]. However, in low dimensions this approach gives some understanding of the purely loxodromic pairs. Will [Wil06, Wil09] classified the purely loxodromic pairs $\mathrm{SU}(2,1)$. Will's classification is built upon the work of Lawton [Law07], also see [Wen94], who obtained trace parameters for elements in $\mathfrak{X}(\mathrm{F}_2, \mathrm{SL}(3,\mathbb{C}))$. It follows from these works that a Zariski-dense purely loxodromic representation in $\mathfrak{X}(\mathrm{F}_2, \mathrm{SU}(2,1)$ is determined by the traces of the generators and the traces of three more compositions of the generators. In an attempt to generalise this work, Gongopadhyay and Lawton [GL17] have classified polystable pairs (that is, pairs whose conjugation orbit is closed in the character variety) in $\mathrm{SU}(3,1)$ using 39 real parameters. At the same time, it has been shown that the real dimension of the smallest possible system of such real parameters to determine any polystable pair is 30.

Using the geometry of the boundary points, there is another approach to classify pairs of loxodromics in low dimensions. This was the approach taken by Wolpert [Wol82] to parametrize surface group representations into $\mathrm{SL}(2,\mathbb{R})$. In the complex hyperbolic setting, Parker and Platis [PP08] obtained a classification of loxodromic pairs in $\mathrm{SU}(2,1)$ and using that they associated parameters to purely loxodromic representations into $\mathrm{SU}(2,1)$. Independently, Falbel [Fal07], also see [FP08], has taken a viewpoint using configuration of four points on the boundary and classified the loxodromic pairs in $\mathrm{SU}(2,1)$ up to conjugacy. Both Parker and Platis, and Falbel have associated a point on an algebraic variety that, along with the traces of the elements, classified the loxodromic pairs. Cunha and Gusevskii [CG12] associated traces, along with the $\mathrm{SU}(2,1)$-orbit of the ordered fixed points to achieve another classification of purely loxodromic representations. Cunha and Gusevskii's work also gave an explicit embedding of the space $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(2,1))$.

We have asked this question for $\mathrm{SU}(3,1)$ and obtained partial results in this direction in [GP17]. We classified generic pairs of loxodromics in $\mathrm{SU}(3,1)$ that we called 'non-singular' in [GP17]. In [GP17], we introduced new parameters analogous to the Koranyi-Reimann cross ratios, but involving fixed points and polar points, to achieve this classification. In view of this, the main result in [GP17] provides a smaller number of 15 real dimensional coordinates that is enough to determine generic pairs of loxodromic elements in $\mathrm{SU}(3,1)$, and using that a Fenchel-Nielsen type coordinate system was given on the 'non-singular ' components of the character variety.

In this paper, we classify $\mathrm{SU}(n,1)$-conjugation orbit of pairs of loxodromic elements in $\mathrm{SU}(n,1)$. This gives a parametric embedding of the set $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1)$. The key intuitive idea that is used to achieve this, is to view a pair of loxodromics as a pair of 'moving orthonormal frames' and then attach a set of boundary points to it that corresponds to the 'moving chains' defining the pair. Given a loxodromic element A, we choose a normalized eigenbasis that corresponds to an orthonormal frame of $\mathbb{C}^{n,1}$. Now to a specified point p on the complex hyperbolic line joining the fixed points, we choose a polar eigenvector x. This gives a chain spanned by the p and x and we mark it with a point. These marked points, along with the fixed points, determine the eigenframe of a loxodromic element that we started with. Given a pair, we do this for both elements in the pair. This choice is not canonical. However, the orbit of such points (modulo the group action underlying change of eigenframes), along with the traces corresponding to the coefficients of characteristic polynomials of the elements, determine the conjugation orbit of the pair completely. Further if we choose one representative from each orbit, it associates numerical conjugacy invariants. However, the choice of these numerical invariants depends on the choice of the representative of the orbit class and we do not know how to make it canonical. The advantage of associating canonical boundary points to a purely loxodromic pair (A, B) is that it enables us to obtain a parametric description of arbitrary elements in $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$. Using this viewpoint one can project the loxodromic elements as a tuple of the boundary of the complex hyperbolic space and we hope this viewpoint will be useful in the understanding of the purely loxodromic representations. Further we discuss special classes of pairs for whom the choice of the conjugacy invariants is canonical.

The detailed notions involved in the results and explicit statements have been postponed until the following sections. After discussing some preliminary notions in Section 2, we review loxodromic elements of SU$(n,1)$ in Section 3. We associate tuples of boundary points to a loxodromic element in Section 4. Specifying such an association we classify SU$(n,1)$ conjugation orbits of loxodromic pairs in Section 5. By the work of Cunha and Gusevskii [CG12], this also gives us conjugacy invariants to be associated to such loxodromic pairs. In Section 6, we make this association well-defined by associating the whole orbit of points to a pair and classify the pairs using that orbit. In Section 7, we construct examples of two classes of loxodromic pairs for which the associated boundary tuples define a single orbit and association of the conjugacy invariants is canonical.

2. Preliminaries

2.1. **Complex Hyperbolic Space.** Let $V = \mathbb{C}^{n,1}$ be the complex vector space $\mathbb{C}^{n+1}$ equipped with the Hermitian form of signature $(n,1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z} = z_1 \bar{w}_{n+1} + z_2 \bar{w}_2 + \cdots + z_n \bar{w}_n + z_{n+1} \bar{w}_1,$$

where $*$ denotes conjugate transpose. The matrix of the Hermitian form is given by

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

If H' is any other $(n+1) \times (n+1)$ Hermitian matrix with signature $(n,1)$, then there is a $(n+1) \times (n+1)$ matrix C so that $C^* H' C = H$. There is also an equivalent

Hermitian form that is often used, given by:

$$H_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We shall mostly use the Hermitian form H.

We consider the following subspaces of $\mathbb{C}^{n,1}$:

$$\mathrm{V}_- = \{\mathbf{z} \in \mathbb{C}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\},\ \mathrm{V}_+ = \{\mathbf{z} \in \mathbb{C}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle > 0\},$$

$$\mathrm{V}_0 = \{\mathbf{z} \in \mathbb{C}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\}.$$

Let $\mathbb{P} : \mathbb{C}^{n,1} - \{0\} \to \mathbb{CP}^n$ be the canonical projection onto complex projective space. Then complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ is defined to be $\mathbb{P}(\mathrm{V}_-)$. The ideal boundary $\partial\mathbf{H}^n_{\mathbb{C}}$ is $\mathbb{P}(\mathrm{V}_0)$. The canonical projection of a vector $\mathbf{z} \in \mathrm{V}_-$ is given by $\mathbb{P}(\mathbf{z}) = (z_1/z_{n+1}, \dots, z_n/z_{n+1})$. Therefore we can write $\mathbf{H}^n_{\mathbb{C}} = \mathbb{P}(V_-)$ as

$$\mathbf{H}^n_{\mathbb{C}} = \{(w_1, \dots, w_n) \in \mathbb{C}^n \ : \ 2\Re(w_1) + |w_2|^2 + \dots + |w_n|^2 < 0\}.$$

This gives the Siegel domain model of $\mathbf{H}^n_{\mathbb{C}}$. There are two distinguished points in V_0 which we denote by $\mathbf{o}$ and ∞ given by

$$\mathbf{o} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \infty = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then we can write $\partial\mathbf{H}^n_{\mathbb{C}} = \mathbb{P}(\mathrm{V}_0)$ as

$$\partial\mathbf{H}^n_{\mathbb{C}} - \infty = \{(z_1, \dots, z_n) \in \mathbb{C}^n : 2\Re(z_1) + |z_2|^2 + \dots + |z_n|^2 = 0\}.$$

Conversely given a point z of $\mathbf{H}^n_{\mathbb{C}} = \mathbb{P}(\mathrm{V}_-) \subset \mathbb{CP}^n$ we may lift $z = (z_1, \dots, z_n)$ to a point $\mathbf{z}$ in V_-, called the *standard lift* of z, by writing in non-homogenous coordinates as

$$\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ 1 \end{bmatrix}.$$

The Bergman metric in $\mathbf{H}^n_{\mathbb{C}}$ is defined in terms of the Hermitian form given by:

$$ds^2 = -\frac{4}{\langle z, z \rangle^2} \det \begin{bmatrix} \langle z, z \rangle & \langle dz, z \rangle \\ \langle z, dz \rangle & \langle dz, dz \rangle \end{bmatrix}.$$

If z and w in $\mathbf{H}^n_{\mathbb{C}}$ correspond to vectors $\mathbf{z}$ and $\mathbf{w}$ in V_- then the Bergman metric is given by the distance ρ:

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle}$$

2.2. **Isometries.** Let $\mathrm{U}(n,1)$ be the group of matrices which preserve the Hermitian form $\langle .,. \rangle$. Each such matrix A satisfies the relation $A^{-1} = H^{-1}A^*H$ where A^* is the conjugate transpose of A. The holomorphic isometry group of $\mathbf{H}^n_{\mathbb{C}}$ is the projective unitary group

$$\mathrm{PSU}(n,1) = \mathrm{SU}(n,1)/\{I, \omega I, \ldots, \omega^n I\},$$

$\omega = \cos(2\pi/(n+1)) + i\sin(2\pi/(n+1))$. It is often more convenient to lift to the $(n+1)$-fold covering $\mathrm{SU}(n,1)$ to look at the action of the isometries.

Based on their fixed points, holomorphic isometries of $\mathbf{H}^n_{\mathbb{C}}$ are classified as follows:

(1) An isometry is *elliptic* if it fixes at least one point of $\mathbf{H}^n_{\mathbb{C}}$.
(2) An isometry is *loxodromic* if it is non-elliptic and fixes exactly two points of $\partial\mathbf{H}^n_{\mathbb{C}}$, one of which is attracting and other repelling.
(3) An isometry is *parabolic* if it is non-elliptic and fixes exactly one point of $\partial\mathbf{H}^n_{\mathbb{C}}$.

For more details on isometries of $\mathbf{H}^n_{\mathbb{C}}$ see [GPP15].

2.3. **Cartan's angular invariant.** Let $z_1,\ z_2,\ z_3$ be three distinct points of $\partial\mathbf{H}^n_{\mathbb{C}}$ with lifts $\mathbf{z}_1,\ \mathbf{z}_2$ and $\mathbf{z}_3$ respectively. Cartan's angular invariant is defined as follows:

$$\mathbb{A}(z_1,\ z_2,\ z_3) = arg(-\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_1\rangle).$$

The angular invariant is invariant under $\mathrm{SU}(n,1)$ and independent of the chosen lifts. The following proposition shows that this invariant determines any triples of distinct points in $\partial\mathbf{H}^n_{\mathbb{C}}$ up to $\mathrm{SU}(n,1)$ equivalence.

Proposition 2.1. *Let $z_1,\ z_2,\ z_3$ and $z_1',\ z_2',\ z_3'$ be triples of distinct points of $\partial\mathbf{H}^n_{\mathbb{C}}$. Then $\mathbb{A}(z_1,\ z_2,\ z_3) = \mathbb{A}(z_1',\ z_2',\ z_3')$ if and only if there exist $A \in \mathrm{SU}(n,1)$ so that $A(z_j) = z_j'$ for $j = 1,\ 2,\ 3$.*

For a proof see [Gol99]. Also we have the following result from [Gol99].

Proposition 2.2. *Let $z_1,\ z_2,\ z_3$ be three distinct points of $\partial\mathbf{H}^n_{\mathbb{C}}$ and let $\mathbb{A} = \mathbb{A}(z_1,z_2,z_3)$ be their angular invariant. Then*

(1) $\mathbb{A} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
(2) $\mathbb{A} = \pm\frac{\pi}{2}$ *if and only if $z_1,\ z_2,\ z_3$ lie on the same chain.*
(3) $\mathbb{A} = 0$ *if and only if $z_1,\ z_2,\ z_3$ lie on a totally real totally geodesic subspace.*

2.4. **The Korányi-Reimann cross-ratio.** Given a quadraple of distinct points (z_1,z_2,z_3,z_4) on $\partial\mathbf{H}^n_{\mathbb{C}}$, their Koranyi-Reimann cross-ratio is defined by

$$\mathbb{X}(z_1,z_2,z_3,z_4) = \frac{\langle \mathbf{z}_3, \mathbf{z}_1\rangle\langle \mathbf{z}_4, \mathbf{z}_2\rangle}{\langle \mathbf{z}_4, \mathbf{z}_1\rangle\langle \mathbf{z}_3, \mathbf{z}_2\rangle},$$

where, for $i = 1,2,3,4$, $\mathbf{z}_i$, are lifts of z_i. It can be seen easily that $\mathbb{X}$ is invariant under $\mathrm{SU}(n,1)$ action and independent of the chosen lifts of z_i's. For more details on cross ratios see [Gol99].

Let $p = (z_1, \ldots, z_m)$ be an ordered m-tuple of distinct points of $\partial\mathbf{H}^n_{\mathbb{C}}$. Following Cunha and Guseveskii [CG12], we associate to p the following numerical invariants:

$$\mathbb{A}(p) = \mathbb{A}(z_1,z_2,z_3),\ \mathbb{X}_{2j}(p) = \mathbb{X}(z_1,\ z_2,\ z_3,\ z_j)$$

$$\mathbb{X}_{3j}(p) = \mathbb{X}(z_1,\ z_3,\ z_2,\ z_j) \text{ and } X_{kj}(p) = \mathbb{X}(z_1,\ z_k,\ z_2,\ z_j),$$

where $m \geq 4, 4 \leq j \leq m, 4 \leq k \leq m-1, k < j$. Using the theory of Gram matrices, Guseveskii [CG12] have obtained the following result.

Theorem 2.3. (Cunha and Gusevskii) *Let $p = (z_1, \dots, z_m), p' = (z'_1, \dots, z'_m)$ be two ordered m-tuples of distinct points of $\partial\mathbf{H}^n_{\mathbb{C}}$. Suppose that for $m \geq 4, 4 \leq j \leq m, 4 \leq k \leq m-1, k < j$, $\mathbb{A}(p) = \mathbb{A}(p'), \mathbb{X}_{2j}(p) = \mathbb{X}_{2j}(p'), \mathbb{X}_{3j}(p) = \mathbb{X}_{3j}(p'), \mathbb{X}_{kj}(p) = \mathbb{X}_{kj}(p')$. Then there exists $A \in \mathrm{SU}(n,1)$ such that $A(z_i) = z'_i$, $i = 1, \dots, m$.*

Cunha and Gusevskii further constructed moduli space $\mathbb{M}(n,m)$ of $\mathrm{PU}(n,1)$-configuration of ordered m-tuples of boundary points as a subspace of $\mathbb{R}^{m(m-3)+1}$. When $t < m$, we shall view $\mathbb{M}(n,t)$ as a subspace of $\mathbb{R}^{m(m-3)+1}$ embedded by the canonical inclusion map of the boundary points:

$$(p_1, \dots, p_t) \mapsto (p_1, \dots, p_t, 0, \dots, 0).$$

3. Loxodromic elements in $\mathrm{SU}(n,1)$

The following facts are standard.

Lemma 3.1. *Let $A = \begin{bmatrix} Ae_1 & \dots & Ae_{n+1} \end{bmatrix} \in \mathrm{SU}(n,1)$. For $2 \leq i \leq n$, the vector Ae_i is uniquely determined by the vectors $Ae_1, \dots, \hat{A}e_i, \dots, Ae_{n+1}$. It is the vector orthogonal to the subspace spanned by $Ae_1, \dots, \hat{A}e_i, \dots, Ae_{n+1}$.*

Corollary 3.2. *Let $A = \begin{bmatrix} Ae_1 & \dots & Ae_{n+1} \end{bmatrix}$, $B = \begin{bmatrix} Be_1 & \dots & Be_{n+1} \end{bmatrix} \in \mathrm{SU}(n,1)$ and $C \in \mathrm{SU}(n,1)$ be such that $CAe_j = Be_j$ for $j \neq i, 1 \leq j \leq n+1$, then $CAe_i = Be_i$.*

3.1. **Loxodromics in** $\mathrm{SU}(n,1)$**.** Let $A \in \mathrm{SU}(n,1)$ represents a loxodromic isometry. Then A has eigenvalues of the form $re^{i\theta}, e^{i\phi_1}, \dots,\ e^{i\phi_{n-1}}, re^{i\theta}$, where $\theta, \phi_i \in (-\pi, \pi]$ for $1 \leq i \leq n-1$, satisfying $2\theta + \phi_1 + \dots + \phi_{n-1} \equiv \pmod{2\pi}$. Let $a_A \in \partial\mathbf{H}^n_{\mathbb{C}}$ be the attractive fixed point of A, then any lift $\mathbf{a}_A$ of a_A to V_0 is an eigenvector of A and corresponds to the eigenvalue $re^{i\theta}$. Similarly, if $r_A \in \partial\mathbf{H}^n_{\mathbb{C}}$ is the repelling fixed point of A, then any lift $\mathbf{r_A}$ of r_A to V_0 is an eigenvector of A with eigenvalue $r^{-1}e^{i\theta}$. For $r > 1$, $\theta, \phi_i \in (-\pi, \pi]$ for $1 \leq i \leq n-1$, define $E_A(r, \theta, \phi_1, \dots, \phi_{n-1})$ as

$$E_A(r, \theta, \phi_1, \dots, \phi_{n-1}) = \begin{bmatrix} re^{i\theta} & & & & \\ & e^{i\phi_1} & & & \\ & & \ddots & & \\ & & & e^{i\phi_{n-1}} & \\ & & & & r^{-1}e^{i\theta} \end{bmatrix}. \tag{3.1}$$

For $1 \leq i \leq n-1$, let $\mathbf{x}_{i,A}$ be an eigenvector corresponding to the eigenvalue $e^{i\phi_i}$ scaled so that $\langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A} \rangle = 1$. Let $C_A = \begin{bmatrix} \mathbf{a}_A & \mathbf{x}_{1,A} & \dots & \mathbf{x}_{n-1,A} & \mathbf{r}_A \end{bmatrix}$ be the $(n+1) \times (n+1)$ matrix, where the lifts are chosen so that the eigenvectors form an orthonormal set, i.e.

$$\langle \mathbf{a}_A, \mathbf{r}_A \rangle = \langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A} \rangle = 1,\ \langle \mathbf{x}_{i,A}, \mathbf{x}_{j,A} \rangle = 0,\ i \neq j.$$

Then $C_A \in \mathrm{SU}(n,1)$ and $A = C_A E_A(r,\theta,\phi_1,\dots,\phi_{n-1})C_A^{-1}$, where $E_A(r,\theta,\phi_1,\dots,\phi_{n-1})$ is given by (3.1).

Lemma 3.3. *Let $A \in \mathrm{SU}(n,1)$. Then A has characteristic polynomial*

$$\chi_A(X) = \sum_{i=0}^{n+1} (-1)^i s_i x^{n+1-i},$$

where $s_0 = s_{n+1} = 1$ and $s_i = \bar{s}_{n+1-i}$.

Proof. Let $\Lambda = \{\lambda_1,\dots,\lambda_{n+1}\}$ be set of eigenvalues of A. Using the fact that Λ is invariant under inversion in unit circle and $\det(A) = 1$, we have

$$\begin{aligned} s_i &= \sum_{j_s < j_{s+1}} \lambda_{j_1}\dots\lambda_{j_i} \\ &= \sum_{k_t < k_{t+1}} \lambda_{k_1}^{-1}\dots\lambda_{k_{n+1-i}}^{-1} \\ &= \sum_{\ell_u < \ell_{u+1}} \bar{\lambda}_{\ell_1}\dots\bar{\lambda}_{\ell_{n+1-i}} \\ &= \bar{s}_{n+1-i} \end{aligned}$$

This proves the result. □

Proposition 3.4. *Two loxodromic elements in* $\mathrm{SU}(n,1)$ *are conjugate if and only if they have the same eigenvalues.*

As a consequence, the following is evident.

Corollary 3.5. *Let A and A' are two loxodromic elements in* $\mathrm{SU}(n,1)$ *such that $tr(A^k) = tr(A'^k)$ for $1 \le k \le \lfloor (n+1)/2 \rfloor$ and $a_A = a_{A'}$, $r_A = r_{A'}$ and $x_{i,A} = x_{i,A'}$, where i ranges over $n-2$ numbers in $\{1,\dots,n-1\}$. Then $A = A'$.*

Since every co-efficient of $\chi_A(x)$ can be expressed as a polynomial in $\mathrm{tr}(A)$, $\mathrm{tr}(A^2),\dots, \mathrm{tr}(A^{n-1})$, an immediate consequence of Lemma 3.3 and Proposition 3.4 is the following.

Corollary 3.6. *Two loxodromic elements A and A' in* $\mathrm{SU}(n,1)$ *are conjugate if and only if $tr(A^k) = tr(A')^k$ for $1 \le k \le \lfloor (n+1)/2 \rfloor$.*

Using these facts, the following result was proved in [GPP15]. Note that a loxodromic element is *regular* if $\chi_A(x)$ has mutually distinct roots. In other words, a loxodromic element A is regular if and only if it has exactly $n+1$ fixed points on $\mathbb{CP}^n$.

Remark 3.7. In Lemma 3.3, $s_{(n+1)/2}$ is real, for n odd. In this case, we need $(n-1)/2$ complex parameters $s_1,\dots,s_{(n-1)/2}$ and one real parameter $s_{(n+1)/2}$. When n is even, we need $n/2$ complex parameters $s_1,\dots,s_{n/2}$. In each case, we need n real parameters to describe a loxodromic element up to conjugacy.

The following result is a part of [GPP15, Theorem 3.1].

Lemma 3.8. *Let $R(\tau)$ denote the resultant of the characteristic polynomial $\chi_A(x)$ and its first derivative $\chi'_A(x)$. Then A is regular loxodromic in* $\mathrm{SU}(n,1)$ *if and only if $R(\tau) < 0$.*

In the above the variable τ is given by the traces of elements:

$$\tau = (\mathrm{tr}(A), \mathrm{tr}^2(A), \ldots, \mathrm{tr}^{\lfloor (n+1)/2 \rfloor}(A)).$$

As a corollary, we have:

Corollary 3.9. *A regular loxodromic element of* $\mathrm{SU}(n,1)$ *is completely determined by its fixed-point set $\mathfrak{p}(A)$ on $\mathbb{CP}^n$ and its image in the domain of traces:*

$$\mathcal{T} = R(\mathrm{tr})^{-1}(-\infty, 0) = \{\tau \in \mathbb{C}^{\lfloor (n+1)/2 \rfloor} \mid R(\tau) < 0\}.$$

In particular, this provides a well-defined correspondence of the set $\mathcal{R}_{lox}$ of regular loxodromics in $\mathrm{SU}(n,1)$ onto $\partial \mathbf{H}^n_{\mathbb{C}} \times \partial \mathbf{H}^n_{\mathbb{C}} \times \underbrace{\mathbb{P}(V_+) \times \cdots \times \mathbb{P}(V_+)}_{(n-1) \text{ times}} \times \mathcal{T}$:

$$A \mapsto (\mathfrak{p}(A), \tau(A)).$$

This correspondence will be used later. It has been shown in [GPP15] that this correspondence is actually a smooth embedding when $n = 3$.

4. Eigenpoints to a loxodromic Element

4.1. **Eigenpoints to a loxodromic.** Let A be a loxodromic element in $\mathrm{SU}(n,1)$. Let $\mathbf{a}_A$, $\mathbf{r}_A$, $\mathbf{x}_{1,A}, \ldots, \mathbf{x}_{n-1,A}$ be a set of eigenvectors of A chosen so that for $1 \leq i \leq n-1$,

$$\langle \mathbf{a}_A, \mathbf{r}_A \rangle = \langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A} \rangle = 1, \ \langle \mathbf{x}_{i,A}, \mathbf{x}_{j,A} \rangle = 0, \ i \neq j. \tag{4.1}$$

Such a choice of eigenvectors will be called a set of *orthonormal eigenframe* of A.

We choose a orthonormal frame of the form (6.5) of a loxodromic element A. Define a set of $n+1$ boundary points to A as follows:

$$\mathbf{p}_{1,A} = \mathbf{a}_A, \ \mathbf{p}_{2,A} = \mathbf{r}_A, \ \mathbf{p}_{i,A} = (\mathbf{a}_A - \mathbf{r}_A)/\sqrt{2} + \mathbf{x}_{i-2,A}, \ 3 \leq i \leq n+1. \tag{4.2}$$

We call the point $p_A = (p_{1,A}, \ldots, p_{n+1,A})$ the *an eigenpoint* to A.

Essentially, we choose a point p from the complex hyperbolic line $\langle v, v \rangle = -1$ in the projection of $\mathbb{C}^{1,1}$ spanned by $\{\mathbf{a}_A, \mathbf{r}_A\}$, and then, the eigenpoints are chosen from the 1-chain that is spanned by $\mathbf{p}$ and $\mathbf{x}_{i,A}$. So, association of eigenpoints to A depends on the choice of the projective image of an eigenframe.

4.1.1. *Eigenspace decomposition of a loxodromic element.* Suppose A is a loxodromic element in $\mathrm{SU}(n,1)$. Suppose A has eigenvalues $re^{i\theta}$, $r^{-1}e^{i\theta}$, $r > 1$, and $e^{i\theta_1}, \ldots, e^{i\theta_k}$, with multiplicities $m_1, \ldots, m_k$ respectively. Then $\mathbb{C}^{n,1}$ has the following orthogonal decomposition into eigenspaces:

$$\mathbb{C}^{n,1} = \mathrm{L}_r \oplus \mathrm{V}_{\theta_1} \oplus \ldots \oplus \mathrm{V}_{\theta_k},$$

here L_r is the $(1,1)$ subspace of $\mathbb{C}^{n,1}$ spanned by $\mathbf{a}_A$, $\mathbf{r}_A$. In the projective space, this means A fixes disjoint copies of $\mathbb{CP}^{m_i - 1}$, for $i = 1, \ldots, k$. Thus, an orthonormal eigenframe of A is determined up to an action of $\mathrm{U}(m_i)$ on each V_{θ_i}. We call $(m_1, \ldots, m_k)$ the *multiplicity* of A. It is clear that equality of multiplicity is a necessary condition for two loxodromics to be conjugate.

Note that the change of eigenframes amounts to a transformation of the following form mapping one frame to the other without changing the loxodromic A:

$$M = \begin{bmatrix} \lambda & 0 & & \ldots & \\ 0 & U_1 & 0 & 0\ldots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & U_k & 0 \\ 0 & 0 & 0 & 0 & \bar{\lambda}^{-1} \end{bmatrix},$$

where $U_i \in \mathrm{U}(a_i)$. This action is equivalent to the action of the centralizer $Z(A)$ on the eigenframes. Equivalently, this amount to conjugation of A by an element of $Z(A)$. This associates a unique $Z(A)$-orbit of eigenpoints to a loxodromic element.

Moreover we have the following.

4.1.2. *Congruent eigenpoints determine equivalence of eigenframes.*

Lemma 4.1. *Let A, $A^{'}$ be loxodromic elements in $\mathrm{SU}(n,1)$ with chosen eigenframes. Let $(p_{1,A}, \ldots, p_{n+1,A})$ and $(p_{1,A'}, \ldots, p_{n+1,A'})$ be eigenpoints to A and A' respectively. Suppose that there exists $C \in \mathrm{SU}(n,1)$ such that $C(p_{i,A}) = p_{i,A'}$, $1 \leq i \leq n$. Then $C(x_{j-1,A}) = x_{j-1,A'}$ for $3 \leq j \leq n+1$.*

Proof. Let $C(\mathbf{p}_{i,A}) = \alpha_i \mathbf{p}_{i,A'}$ for $1 \leq i \leq n$. Observe that $\langle \mathbf{p}_{1,A}, \mathbf{p}_{i,A} \rangle = -1/\sqrt{2} = \langle \mathbf{p}_{1,A'}, \mathbf{p}_{i,A'} \rangle$ and $\langle \mathbf{p}_{2,A}, \mathbf{p}_{i,A} \rangle = 1/\sqrt{2} = \langle \mathbf{p}_{2,A'}, \mathbf{p}_{i,A'} \rangle$ for $3 \leq i \leq n$. Since $C \in \mathrm{SU}(n,1)$ preserve the form $\langle ., . \rangle$, we have

$$(-1/\sqrt{2})\alpha_1 \bar{\alpha}_i = -1/\sqrt{2} \text{ and } (1/\sqrt{2})\alpha_2 \bar{\alpha}_i = 1/\sqrt{2} \text{ for } 3 \leq i \leq n.$$

This implies $\alpha_i = \bar{\alpha}_1^{-1} = \bar{\alpha}_2^{-1}$ for $3 \leq i \leq n$. Using $\langle C(\mathbf{p}_{1,A}), C(\mathbf{p}_{2,A}) \rangle = 1$ implies, $\alpha_1 = \bar{\alpha}_2^{-1}$. Hence we must have $|\alpha_1| = 1$, i.e. $\alpha_1 = \bar{\alpha_1}^{-1}$. Hence

$$C((\mathbf{a}_A - \mathbf{r}_A)/\sqrt{2} + \mathbf{x}_{i-1,A}) = \bar{\alpha}_1^{-1}((\mathbf{a}_{A'} - \mathbf{r}_{A'})/\sqrt{2} + \mathbf{x}_{i-1,A'}),$$

yields $C(x_{i-1,A}) = x_{i-1,A'}$ for $3 \leq i \leq n$. By Corollary 3.2 this implies $C(x_{n+1,A}) = x_{n+1,A'}$. $\square$

5. Loxodromic Pairs and Reference Eigenframes

Throughout this paper, given a loxodromic pair (A, B) in $\mathrm{SU}(n,1)$, it will always be assumed that A and B have disjoint fixed point sets.

Given (A, B), fix a pair of associated orthonormal frames $\mathcal{B} = (\mathcal{B}_A, \mathcal{B}_B)$ so that

$$\langle \mathbf{a}_A, \mathbf{r}_A \rangle = \langle \mathbf{a}_B, \mathbf{r}_B \rangle = \langle \mathbf{a}_A, \mathbf{a}_B \rangle = \langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A} \rangle = \langle \mathbf{x}_{i,B}, \mathbf{x}_{i,B} \rangle = 1.$$

Such a normalized pair of eigenframes will be called an *eigenframe* to (A, B).

We choose an ordering of $\mathcal{B}$ as follows:

$$\mathcal{B} = (\mathbf{a}_A, \mathbf{r}_A, \mathbf{a}_B, \mathbf{r}_B, \mathbf{x}_{1,A}, \ldots, \mathbf{x}_{n-1,A}, \mathbf{x}_{1,B}, \ldots, \mathbf{x}_{n-1,B}).$$

Such an ordering will be called a *canonical ordering.* This gives a tuple of boundary points

$$\mathfrak{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_3, \ldots, \mathbf{p}_{n+1}, \mathbf{q}_3, \ldots, \mathbf{q}_{n+1}),$$

where $\mathbf{p}_i$, $\mathbf{q}_i$ are defined by (4.2). Note that not all $\mathbf{p}_i$, $\mathbf{q}_i$ may not be distinct. If they are not, say $\mathbf{p}_i = \mathbf{q}_j$, then we replace $\mathbf{x}_{i,A}$ by $\lambda\mathbf{x}_{i,A}$ for some $\lambda \in \mathbb{S}^1$ and choose a different $\mathbf{q}_j$ from the chain spanned by $\frac{\mathbf{a}_A - \mathbf{r}_A}{\sqrt{2}}$ and $\mathbf{x}_{i,A}$. The resulting ordered tuple of distinct points $(p_1, \ldots, p_{2n+2})$ will be called a *reference ordered tuple of eigenpoints* to (A, B), or simply as *eigenpoint* to (A, B).

Given $[(A, B)] \in \mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n, 1))$, we choose a reference eigenpoint $\mathfrak{p}$ to (A, B) as above using (4.2). Such an assignment is well-defined up to diagonal action of $Z(A) \times Z(B)$ on the eigenframes. Since the diagonal subgroup of $Z(A) \times Z(B)$ (if nontrivial) is a subgroup of $\mathrm{SU}(n, 1)$, this associates to to the class $[(A, B)]$ the element $[\mathfrak{p}]$ in the space $\mathbb{M}(n, 2n + 2)$. We call this class $[\mathfrak{p}]$ as the *reference orbit* of (A, B).

Theorem 5.1. *Let* (A, B) *be a loxodromic pair in* $\mathrm{SU}(n, 1)$*. Then* (A, B) *is determined up to conjugation by* $\mathrm{SU}(n, 1)$ *the traces* $\mathrm{tr}(A^i)$, $\mathrm{tr}(B^i)$, $1 \le i \le \lfloor (n + 1)/2 \rfloor$, *and its reference orbit on* $\mathbb{M}(n, 2n + 2)$.

Proof. Suppose (A, B) and (A', B') are loxodromic pairs with the same traces and the same canonical point. Following the notation in Section 3.1, $A = C_A E_A C_A^{-1}$, $B = C_B E_B C_B^{-1}$ and similarly for A' and B'. Since (A, B) and (A', B') defines the same reference orbit, it follows from Lemma 4.1 that there exists a C in $\mathrm{SU}(n, 1)$ such that $C(a_A) = a_{A'}, C(x_{k,A}) = x_{k,A'},\ C(r_A) = r_{A'}$, and, $C(a_B) = a_{B'}, C(x_{k,B}) = x_{k,B'},\ C(r_B) = r_{B'}$ for $1 \le k \le n - 1$. Therefore CAC^{-1} and A' have same eigenvectors. Since $\mathrm{tr}(A')^i = \mathrm{tr}(CAC^{-1})^i$ for $1 \le i \le \lfloor (n + 1)/2 \rfloor$, by Corollary 3.5, we must have $CAC^{-1} = A'$. Similarly, $B' = CBC^{-1}$. This completes the proof. $\square$

Remark 5.2. In view of Lemma 4.1, it follows from the above theorem that $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n, 1))$ has a projection into

$$\mathbb{C}^{\lfloor (n+1)/2 \rfloor} \times \mathbb{C}^{\lfloor (n+1)/2 \rfloor} \times \mathbb{M}(n, 2n + 2).$$

5.1. **Conjugacy Invariants.** In order to define projective invariants, we make the following definition following Cunha and Gusevskii's invariants in [CG12].

Definition 5.3. Let (A, B) be a pair of loxodromics in $\mathrm{SU}(n, 1)$. We fix the canonical ordering of $\mathcal{B}$ and to a canonical boundary tuple $(p_1, \ldots, p_{2n+2})$, we associate the following conjugacy invariants:

We associate complex numbers $\mathbb{X}_{2j}(A, B)$, $\mathbb{X}_{3j}(A, B)$, $\mathbb{X}_{kj}(A, B)$ given by the following:

$$\mathbb{X}_{2j}(A, B) = \mathbb{X}(p_1,\ p_2,\ p_3,\ p_j),\ \ \mathbb{X}_{3j}(A, B) = \mathbb{X}(p_1,\ p_3,\ p_2,\ p_j),$$

$$\mathbb{X}_{kj}(A, B) = \mathbb{X}(p_1,\ p_k,\ p_2,\ p_j),$$

where $4 \le j \le 2n + 2,\ 4 \le k \le 2n + 2,\ k < j$.

The invariants defined above are called *reference cross-ratios* of the pair (A, B). It is easy to see that there are $(n+1)(2n-1)$ non-zero cross-ratios in the above list.

Finally define the *angular invariant* of (A, B):

$$\mathbb{A}(A, B) = \mathbb{A}(p_1, p_2, p_3).$$

The following version of Theorem 5.1 now follows from Theorem 2.3.

Theorem 5.4. *Let (A, B) be a loxodromic pair in* $\mathrm{SU}(n, 1)$. *Then (A, B) is determined up to conjugation by* $\mathrm{SU}(n, 1)$, *by the traces* $\mathrm{tr}(A^i)$, $\mathrm{tr}(B^i)$, $1 \le i \le \lfloor (n+1)/2 \rfloor$, *the angular invariant and the reference cross ratios.*

6. Classification of Loxodromic Pairs

Note that in the previous section, the assignment of each class $[(A, B)]$ to a tuple of boundary points depends on the choice of $\mathcal{B} = (\mathcal{B}_A, \mathcal{B}_B)$ and after fixing such a choice, it is independent up to the diagonal action of $Z(A) \times Z(B)$. As an advantage of the construction in the previous section, we could associate numerical conjugacy invariants to pairs. However, the choice of eigenframes of (A, B) is unique up to an action of the full group $Z(A) \times Z(B)$. If we want to get an explicit description of the moduli space independent of the choice of $\mathcal{B}$, we need to consider the unique assignment of the $Z(A) \times Z(B)$-orbit of tuples of boundary points to the pair (A, B). We attempt this in this section. However, we do not know how to associate numerical conjugacy invariants in this approach.

6.1. **Moduli of normalized boundary points.** Consider the set $\mathcal{E}$ of ordered tuples of boundary and polar points on $(\partial \mathbf{H}^n_{\mathbb{C}})^4 \times \mathbb{P}(V_+)^{2n-2}$ given by a pair of orthonormal frames (F_1, F_2):

$$\mathfrak{p} = (q_1, q_2, r_1, r_2, \dots, r_{n-1}, q_{n+1}, q_{n+2}, r_{n+3}, \dots, r_{2n+2}).$$

This corresponds to pair of orthonormal frames of $\mathbb{C}^{n,1}$:

$$\hat{\mathfrak{p}} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{n-1}, \mathbf{q}_{n+1}, \mathbf{q}_{n+2}, \mathbf{r}_{n+3}, \dots, \mathbf{r}_{2n-1}),$$

where $\{\mathbf{q}_1, \mathbf{q}_2\} \cap \{\mathbf{q}_{n+1}, \mathbf{q}_{n+2}\} = \phi$, $\langle \mathbf{q}_i, \mathbf{q}_i \rangle = 0 = \langle \mathbf{q}_{n+i}, \mathbf{q}_{n+i} \rangle$ for $i = 1, 2$, $\langle \mathbf{r}_j, \mathbf{r}_j \rangle = \langle \mathbf{r}_{n+j}, \mathbf{r}_{n+j} \rangle = 1$ for all $j = 1, \dots, n-1$, $\langle \mathbf{q}_1, \mathbf{q}_2 \rangle = \langle \mathbf{q}_{n+1}, \mathbf{q}_{n+2} \rangle = \langle \mathbf{q}_1, \mathbf{q}_{n+2} \rangle = 1$, $\langle \mathbf{q}_i, \mathbf{r}_j \rangle = 0 = \langle \mathbf{q}_{n+i}, \mathbf{r}_{n+j} \rangle$, for $i = 1, 2$, $j = 1, \dots, n-1$.

To each such point, we have an ordered tuples of boundary points, not necessarily distinct, $(p_1, \dots, p_{2n+2})$ satisfying the conditions:

$$\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = \langle \mathbf{p}_{n+2}, \mathbf{p}_{n+3} \rangle = \langle \mathbf{p}_1, \mathbf{p}_{n+2} \rangle = 1, \tag{6.1}$$

$$\langle \mathbf{p}_i, \mathbf{p}_j \rangle = -1 = \langle \mathbf{p}_{n+i}, \mathbf{p}_{n+j} \rangle, i \neq j, \ i, j = 3, \dots, n-1; \tag{6.2}$$

$$\langle \mathbf{p}_1, \mathbf{p}_i \rangle = -\frac{1}{\sqrt{2}} = \langle \mathbf{p}_{n+2}, \mathbf{p}_{n+i} \rangle, \ i = 3, \dots, n-1; \tag{6.3}$$

$$\langle \mathbf{p}_2, \mathbf{p}_k \rangle = \frac{1}{\sqrt{2}} = \langle \mathbf{p}_{n+2}, \mathbf{p}_{n+k} \rangle, \ k = 3, \dots, n-1, \tag{6.4}$$

here $\mathbf{p}_i$ denotes the standard lift of p_i for each i. Note that not all of $\mathbf{p}_i, \mathbf{p}_{n+i}$, $i = 3, \dots, n$ may be distinct. If they are not distinct, we relabel them and write them as a ordered tuple of distinct boundary points $\hat{\mathfrak{p}} = (p_1, p_2, \dots, p_t)$, $n+3 \le t \le 2n+2$, so that they corresponds to the original ordering of $\mathfrak{p}$. We call t the *type* of the tuple $\mathfrak{p}$.

Let $\mathcal{L}_t$ be the section of $\mathbb{M}(n, t)$ defined by the equations (6.1)– (6.4), and the ordering as describe above. Let $\mathcal{L} = \bigcup_{t=n+3}^{2n+2} \mathcal{L}_t$. This space can be viewed as a subspace of $\mathbb{R}^{(2n+2)(2n-1)+1}$, where the embedding of $\mathbb{M}(n, t)$ into the affine space is defined by the inclusion map:

$$(x_1, \dots, x_t) \mapsto (x_1, \dots, x_t, 0, \dots, 0).$$

Thus it has the the induced topology.

6.2. **Pairs of Loxodromics.** Let (A, B) be a loxodromic pair in $\mathrm{SU}(n, 1)$ with multiplicities $(a_1, \dots, a_k, b_1, \dots, b_l)$. As in the previous section, we now consider the ordered canonical eigenframe to (A, B) given by the tuple

$$\mathfrak{e} = (\mathbf{a}_A, \mathbf{r}_A, \mathbf{x}_{1,A}, \dots, \mathbf{x}_{n-1,A}, \mathbf{a}_B, \mathbf{r}_B, \mathbf{x}_{1,B}, \dots, \mathbf{x}_{n-1,B}).$$

with normalization as follows:

$$\langle \mathbf{a}_A, \mathbf{r}_A \rangle = \langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A} \rangle = 1, \ \langle \mathbf{x}_{i,A}, \mathbf{x}_{j,A} \rangle = 0, \ i \neq j. \tag{6.5}$$

$$\langle \mathbf{a}_B, \mathbf{r}_B \rangle = \langle \mathbf{x}_{i,B}, \mathbf{x}_{i,B} \rangle = 1, \ \langle \mathbf{x}_{i,B}, \mathbf{x}_{j,B} \rangle = 0, \ i \neq j. \tag{6.6}$$

$$\langle \mathbf{r}_A, \mathbf{a}_B \rangle = 1. \tag{6.7}$$

This defines a point on $(\partial \mathbf{H}^n_{\mathbb{C}})^4 \times (\mathbb{P}(\mathrm{V}_+))^{2n-2}$ that we refer as *canonical eigenpoint*. We further assign a canonical ordered tuple of boundary points $(\mathcal{B}_A, \mathcal{B}_B)$ to $\mathfrak{e}$ (with canonical ordering) defined by (4.2) as done in the previous section:

$$\mathbf{p} = (a_A, r_A, q_{1,A}, \dots, q_{n-1,A}, a_B, r_B, q_{1,B}, \dots, q_{n-1,B}). \tag{6.8}$$

Note that in this case, $\mathfrak{e}$, and hence $(p_1, \dots, p_{2n+2})$ is determined by (A, B) up to a right action of the group

$$G = \mathbb{C}^* \times \mathrm{U}(a_1) \times \ldots \times \mathrm{U}(a_k) \times \mathrm{U}(b_1) \times \ldots \times \mathrm{U}(b_l)$$

on $\mathbf{p}$ given by the following: for $g = (\lambda, A_1, \dots, A_k, B_1, \dots, B_l) \in G$,

$$g.\mathbf{p} = \Big(\mathbf{a}_A \lambda, \mathbf{r}_A \bar{\lambda}^{-1}, Y_1, \dots, Y_k, \lambda \mathbf{a}_B, \bar{\lambda}^{-1} \mathbf{r}_B, Z_1, \dots, \dots, Z_l),$$

where, $Y_i = (\mathbf{x}_{t_i,A}, \dots, \mathbf{x}_{t_i+a_i-1,A}) A_i$, $Z_j = (\mathbf{x}_{s_j+1,B}, \dots, \mathbf{x}_{s_j+b_j-1,B}) B_j$, $t_i = \sum_{p=1}^{i} a_{p-1}$, $s_j = \sum_{p=1}^{j} b_{p-1}$, $a_0 = b_0 = 1$.

The group G represents the group $Z(A) \times Z(B)$. So, to each (A, B), we have a $Z(A) \times Z(B)$-orbit, or equivalently, G-orbit of canonical tuples of boundary points. Since the action of $Z(A)$ and $Z(B)$ in this case is not necessarily the diagonal action, the orbit may move over $\mathbb{M}(n, t)$, $n+3 \le t \le 2n+2$, and it defines a point on $\mathcal{L}$.

The above action of G on $\mathfrak{p}$ induces an action of G on $\mathcal{L}$, and gives a G-orbit of $[\mathfrak{p}]$ in $\mathcal{L}$. This G-orbit $[\mathfrak{p}]$ on $\mathcal{L}$ corresponds uniquely to the conjugacy class of (A,B), we call it *canonical orbit* of (A,B). The orbit space on $\mathcal{L}$ under the above G-action is denoted by $\mathcal{OL}_n(a_1,\ldots,a_k;b_1,\ldots,b_l)$, or simply, $\mathcal{OL}_n$ when there is no ambiguity on $(a_1,\ldots,a_k,b_1,\ldots,b_l)$. Each point on $\mathcal{OL}_n$ corresponds to a conjugacy class of a loxodromic pair (A,B) with multiplicity $(a_1,\ldots,a_k,b_1,\ldots,b_l)$. When both A and B are regular, i.e. have distinct eigenvalues, we denote this orbit space as $\mathcal{RL}_n$. In this case, by Lemma 4.1, $\mathcal{RL}_n$ can be canonically realized as a subspace of $\mathbb{R}^{2n(2n-3)+1}$. Now, we have the following theorem.

Theorem 6.1. *Let (A,B) be a loxodromic pair in* $\mathrm{SU}(n,1)$*. Then (A,B) is determined up to conjugation in* $\mathrm{SU}(n,1)$*, by the the traces* $\mathrm{tr}(A^i)$, $\mathrm{tr}(B^i)$, $1\le i\le \lfloor (n+1)/2\rfloor$*, and the canonical orbit of (A,B) on $\mathcal{OL}_n$.*

Proof. Suppose (A,B) and (A',B') are purely loxodromic pairs with the same type $(a_1,\ldots,a_k,b_1,\ldots,b_l)$, same traces and the same canonical orbit. Following the notation in Section 3.1, $A=C_AD_AC_A^{-1},\ B=C_BD_BC_B^{-1}$ and similarly for A' and B'. In this case, C_A is an element in the subgroup $\mathrm{U}(1,1)\times\mathrm{U}(a_1)\times\ldots\times\mathrm{U}(a_k)$:

$$C_A=\begin{bmatrix}\mathbf{a}_A & E_1 & \ldots E_k & \mathbf{r}_A\end{bmatrix},$$

where $E_i=\begin{bmatrix}x_{t_i,A} & \ldots & \mathbf{x}_{t_i+a_i-1,A}\end{bmatrix}$, D_A is the diagonal matrix

$$D_A=\begin{bmatrix} re^{i\theta} & 0 & & \ldots & \\ 0 & \lambda_1 I_{a_1} & 0 & 0\ldots & 0\\ & & \ddots & & \\ 0 & 0 & 0 & \lambda_k I_{a_k} & 0\\ 0 & 0 & 0 & 0 & r^{-1}e^{i\theta}\end{bmatrix},$$

here I_s denote identity matrix of rank s. Similarly for C_B.

Since the canonical orbits are equal, by Lemma 4.1 it follows that there exist $C\in\mathrm{SU}(n,1)$ such that $C(a_A)=a_{A'},\ C(r_A)=r_{A'},\ C(a_B)=a_{B'}, C(r_B)=r_B$, and for $1\le i\le k$, $1\le j\le l$,

$$C(\mathbf{x}_{t_i,A},\ldots,\mathbf{x}_{t_i+a_i-1,A})=(\mathbf{x}_{t_i,A'},\ldots,\mathbf{x}_{t_i+a_i-1,A'})U_i,$$
$$C(\mathbf{x}_{t_j,B},\ldots,\mathbf{x}_{t_j+b_j-1,B})=(\mathbf{x}_{t_j,B'},\ldots,\mathbf{x}_{t_j+a_i-1,B'})V_j,$$

where $U_i\in\mathrm{U}(a_i)$, $V_j\in\mathrm{U}(b_j)$. Let

$$M=\begin{bmatrix} \lambda re^{i\theta} & 0 & & \ldots & \\ 0 & U_1 & 0 & 0\ldots & 0\\ & & \ddots & & \\ 0 & 0 & 0 & U_k & 0\\ 0 & 0 & 0 & 0 & \lambda r^{-1}e^{i\theta}\end{bmatrix},$$

Therefore

$$CAC^{-1}=[C(C_A)]D_A[C(C_A)]^{-1}=C'_AMD_AM^{-1}C_A'^{-1}.$$

Observing that M commutes with D_A and $D_A=D_{A'}$, we conclude $CAC^{-1}=A'$. Similarly, $B'=CBC^{-1}$. This completes the proof. □

Remark 6.2. Let $\mathfrak{R}(\mathrm{F}_2, \mathrm{SU}(n,1))$ be the subset of $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$ consisting of regular pairs. By Corollary 3.9 it follows that $\mathfrak{R}(\mathrm{F}_2, \mathrm{SU}(n,1))$ is embedded in the topological space $\mathcal{T} \times \mathcal{T} \times \mathcal{RL}_n$. The space $\mathfrak{R}(\mathrm{F}_2, \mathrm{SU}(n,1))$ is embedded in $\mathbb{C}^{\lfloor (n+1)/2 \rfloor} \times \mathbb{C}^{\lfloor (n+1)/2 \rfloor} \times \mathcal{OL}_n$.

If (A, B) is irreducible or Zariski-dense, then it does not preserve a common totally geodesic subspace or fix a common point. In particular, the eigenspace decompositions will be disjoint. Since $Z(A)$ and $Z(B)$ keep the eigenspace decompositions invariant, the type of the eigenpoints will always be $2n+2$. Thus, the subset of irreducible pairs in $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$ is embedded in

$$\mathbb{C}^{\lfloor (n+1)/2 \rfloor} \times \mathbb{C}^{\lfloor (n+1)/2 \rfloor} \times \mathcal{L}_{2n+2}.$$

7. Examples of Good Pairs

In this section, we construct two classes of pairs for whom, a suitable chosen normalization of eigenframes reduces the $Z(A) \times Z(B)$ action to a single orbit on some $\mathcal{L}_t$. Thus, these classes of pairs can be parametrized canonically by conjugacy invariants.

7.1. **Good Pairs I.** Let A and B be two loxodromic elements without a common fixed point. Consider the loxodromic pairs (A, B) that has the following property: for every positive eigenvector $\mathbf{y}$ of B, there exists a positive eigenvector $\mathbf{x}$ of A such that $\langle \mathbf{x}, \mathbf{y} \rangle \neq 0$. We shall further assume (A, B) is regular, i.e. both A and B are regular. We denote the set of all such pairs in $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n,1))$ as $\mathcal{T}_1$.

Let (A, B) represents an element in $\mathcal{T}_1$. In this case, we choose a pair of eigenframes $\mathcal{B} = (\mathcal{B}_A, \mathcal{B}_B)$, normalized and arranged so that for $i = 1, \dots, n-1$,

$$\langle \mathbf{a}_A, \mathbf{r}_A \rangle = \langle \mathbf{a}_B, \mathbf{r}_B \rangle = \langle \mathbf{a}_A, \mathbf{r}_B \rangle = 1, \ \langle \mathbf{x}_{i,A}, \mathbf{x}_{i,B} \rangle = 1 = \langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A} \rangle.$$

Choose a canonical ordering on $\mathcal{B}$:

$$\mathcal{B} = (\mathbf{a}_A, \mathbf{r}_A, \mathbf{a}_B, \mathbf{r}_B, \mathbf{x}_{1,A}, \dots, \mathbf{x}_{n-1,A}, \mathbf{x}_{1,B}, \dots, \mathbf{x}_{n-1,B}).$$

This gives a tuple of boundary points as before:

$$\mathfrak{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_3, \dots, \mathbf{p}_{n+1}, \mathbf{q}_3, \dots, \mathbf{q}_{n+1}),$$

where $\mathbf{p}_i$, $\mathbf{q}_i$, $i = 3, \dots, n+1$, are defined by (4.2). Note that $\mathbf{p}_i$, $\mathbf{q}_j$ might not be distinct for i, j. If they are not, we relabel them and re-arrange according to the canonical ordering of $\mathcal{B}$. In a chosen eigenframe of (A, B), normalized as above,

suppose we change $\mathbf{a}_A$ by $\lambda \mathbf{a}_A$, $\lambda \in \mathbb{C}^*$. Then $\langle \mathbf{a}_A, \mathbf{a}_B \rangle = 1$, resp. $\langle \mathbf{a}_A, \mathbf{r}_A \rangle = 1$, implies that $\mathbf{a}_B$, resp. $\mathbf{r}_A$, is scaled by $\bar{\lambda}^{-1}$. The relation $\langle \mathbf{a}_B, \mathbf{r}_B \rangle = 1$ implies $\mathbf{r}_B$ is scaled by λ. If we change $\mathbf{x}_{i,A}$ by μ_i, then $\mathbf{x}_{i,B}$ is changed by $\bar{\mu}_i^{-1}$. This implies that $\mathcal{B}$, and hence $(p_1, \dots, p_t)$ is determined up to an action of the group $\mathbb{T} = \mathbb{C}^* \times \mathrm{U}(1)^{n-1}$ on the set of canonical eigenframes. This action is given by the following: for $g = (\lambda, \mu_1, \dots, \mu_{n-1}) \in \mathbb{T}$,

(7.1)

$$g.\mathbf{p} = \left(\mathbf{a}_A\lambda, \mathbf{r}_A\bar{\lambda}^{-1}, \mathbf{a}_B\lambda, \mathbf{r}_B\bar{\lambda}^{-1}, \mathbf{x}_{1,A}\mu_1, \dots, \mathbf{x}_{n-1,A}\mu_{n-1}, \mathbf{x}_{1,B}\mu_1, \dots, \mathbf{x}_{n-1,B}\mu_{n-1}\right).$$

Hence to each (A, B), we assign a unique $\mathbb{T}$-tuple of boundary points $(p_1, \dots, p_t)$, where we call t as the 'level' of (A, B). Since $\mathbb{T}$ projects to a subgroup of $\mathrm{SU}(n, 1)$, this gives an assignment of the $\mathrm{SU}(n, 1)$-conjugation orbit of (A, B) to a unique orbit $[(p_1, \dots, p_t)]$ in $\mathcal{L}_t$. In this case, the number of numerical conjugacy invariants defined as in Definition (5.3) depends on t.

7.2. **Good Pairs II.** Now we define another class of pairs that generalizes the generic elements we classified in [GP17]. Following the notion in [GP17], we will call them as 'non-singular' here.

Definition 7.1. A pair of loxodromics (A, B) is called *non-singular* if

(1) A and B does not have a common fixed point.

(2) $\mathbf{x}_{k,A} \notin L_B^{\perp}$, $\mathbf{x}_{k,B} \notin L_A^{\perp}$ where k ranges over $n-2$ numbers in $\{1, \dots, n-1\}$, i.e. for each such k, $x_{k,A}$ has non-zero projection on L_B and, $x_{k,B}$ has non-zero projection on L_A. Given a non-singular pair (A, B), without loss of generality, re-arranging the eigenvectors if necessary, we shall assume that $1 \le k \le n-2$.

We shall consider regular non-singular pairs in the following. Unless otherwise specified, a non-singular pair will always assumed to be regular. Further, we will always assume, by suitably relabeling the eigenvectors if required, that $\langle \mathbf{x}_{i,A}, \mathbf{a}_B \rangle \neq 0$, $\langle \mathbf{x}_{i,B}, \mathbf{a}_A \rangle \neq 0$ for $1 \le i \le n-2$.

7.2.1. *Eigenpoints to a non-singular pair.* Let (A, B) be non-singular in $\mathrm{SU}(n, 1)$. Without loss of generality, we may assume that $\langle \mathbf{x}_{i,A}, \mathbf{a}_B \rangle \neq 0$, $\langle \mathbf{x}_{i,B}, \mathbf{a}_A \rangle \neq 0$ for $1 \le i \le n-2$ for any choice of lifts. We choose normalized eigenframes such that

$$\langle \mathbf{a}_A, \mathbf{r}_A \rangle = \langle \mathbf{a}_B, \mathbf{r}_B \rangle = \langle \mathbf{a}_A, \mathbf{r}_B \rangle = \langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A} \rangle = \langle \mathbf{x}_{i,B}, \mathbf{x}_{i,B} \rangle = 1,$$

and

$$\langle \mathbf{x}_{i,A}, \mathbf{a}_B \rangle,\ \langle \mathbf{x}_{i,B}, \mathbf{a}_A \rangle \in \mathbb{R}_+ \text{ for } 1 \le i \le n-2,\ \langle \mathbf{x}_{1,A}, \mathbf{a}_B \rangle = 1.$$

To see that this is possible, suppose for some choice of lifts,

$$\langle \mathbf{a}_A, \mathbf{r}_A \rangle = \lambda,\ \langle \mathbf{a}_B, \mathbf{r}_B \rangle = \mu, \langle \mathbf{a}_A, \mathbf{r}_B \rangle = \nu,\ \langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A} \rangle = r_i^2,\ \langle \mathbf{x}_{i,B}, \mathbf{x}_{i,B} \rangle = s_i^2,$$

and

$$\langle \mathbf{x}_{i,A}, \mathbf{a}_B \rangle = \gamma_i,\ \langle \mathbf{x}_{i,B}, \mathbf{a}_A \rangle = \delta_i, \text{ where } r_i, s_i \in \mathbb{R}_+ \text{ for } 1 \le i \le n-2.$$

Let us choose the appropriate lifts in the following way.

(1) First replace $\mathbf{x}_{1,A}$ by $r_1^{-1}\mathbf{x}_{1,A}$, so that $\langle \mathbf{x}_{1,A}, \mathbf{x}_{1,A} \rangle = 1$.
(2) Replace $\mathbf{a}_B$ by $r_1\bar{\gamma}_1^{-1}\mathbf{a}_B$, so that $\langle \mathbf{x}_{1,A}, \mathbf{a}_B \rangle = 1$.
(3) Replace $\mathbf{r}_B$ by $r_1^{-1}\gamma_1\bar{\mu}^{-1}\mathbf{r}_B$, so that $\langle \mathbf{a}_B, \mathbf{r}_B \rangle = 1$.

(4) Replace $\mathbf{a}_A$ by $r_1\bar{\gamma}_1^{-1}\mu\nu^{-1}\mathbf{a}_A$, so that $\langle \mathbf{a}_A, \mathbf{r}_B\rangle = 1$.

(5) Replace $\mathbf{r}_A$ by $r_1^{-1}\gamma_1\bar{\lambda}^{-1}\bar{\mu}^{-1}\bar{\nu}\mathbf{r}_A$, so that $\langle \mathbf{a}_A, \mathbf{r}_A\rangle = 1$.

(6) For $i \neq 1$, replace $\mathbf{x}_{i,A}$ by $r_i^{-1}e^{i(\arg\gamma_1-\arg\gamma_i)}\mathbf{x}_{i,A}$, so that $\langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A}\rangle = 1,$ and $\langle \mathbf{x}_{i,A}, \mathbf{a}_B\rangle \in \mathbb{R}_+$.

(7) For $1 \leq i \leq n-1$, replace $\mathbf{x}_{i,B}$ by $s_i^{-1}e^{i(\arg\gamma_1+\arg\mu-\arg\nu-\arg\delta_i)}\mathbf{x}_{i,B}$, so that $\langle \mathbf{x}_{i,B}, \mathbf{x}_{i,B}\rangle = 1,$ and $\langle \mathbf{x}_{i,B}, \mathbf{a}_A\rangle \in \mathbb{R}_+$.

With this normalization, we associate to (A, B) an eigenpoint as in Section 5. We denote them by $\mathfrak{p} = (p_{1,A}, \ldots, p_{n,A}, p_{1,B}, \ldots, p_{n,B})$. Note that by regularity, we can ignore the $p_{n+1,A}$ and $p_{n+1,B}$ by Lemma 4.1. Note that because of non-singularity, $\mathbf{p}_{i,B}$ can not be equal to $\mathbf{a}_A$, $\mathbf{r}_A$ for all i, and similarly, $\mathbf{p}_{i,A}$ can not be equal to $\mathbf{a}_B$, $\mathbf{r}_B$. If some $p_{i,A}$ is equal to $p_{j,B}$, we re-arrange them as before and denote by $(p_1, \ldots, p_t)$.

Lemma 7.2. *Let (A, B) be non-singular pair in $\mathrm{SU}(n, 1)$. Suppose that $(p_1, \ldots, p_t)$, $(p'_1, \ldots, p'_t)$ are two tuples of eigenpoints to (A, B). Then $\mathbf{p}'_i = \lambda\mathbf{p}_i$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, $1 \leq i \leq t$.*

Proof. By symmetry, it is enough to prove that if either of $\mathbf{a}_A, \mathbf{x}_{i,A}$ is scaled by λ, then the normalized eigenframes are scaled by λ and $|\lambda| = 1$. First consider the case when $\mathbf{a}_A$ is scaled by λ. Then $\langle \mathbf{a}_A, \mathbf{r}_B\rangle$ =1 implies that $\mathbf{r}_B$ is scaled by $\bar{\lambda}^{-1}$. Then $\langle \mathbf{a}_B, \mathbf{r}_B\rangle = 1$ implies $\mathbf{a}_B$ is scaled by λ. Then $\langle \mathbf{x}_{1,A}, \mathbf{a}_B\rangle = 1$ implies $\mathbf{x}_{1,A}$ is scaled by $\bar{\lambda}^{-1}$. Then $\langle \mathbf{x}_{1,A}, \mathbf{x}_{1,A}\rangle = 1$ implies that $|\lambda| = 1$ and so $\bar{\lambda}^{-1} = \lambda$. Then the choice

$$\langle \mathbf{x}_{i,A}, \mathbf{a}_B\rangle, \langle \mathbf{x}_{i,B}, \mathbf{a}_A\rangle \in \mathbb{R}_+ \text{ for } 1 \leq i \leq n-2,$$

implies that $\mathbf{x}_{i,A}$ and $\mathbf{x}_{i,B}$ are scaled by λ. This proves the lemma. □

This shows that a non-singular pair in $\mathfrak{X}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{SU}(n, 1))$ not only projects down to a unique point on $\mathcal{L}_t$, but to each non-singular pair (A, B) of $\mathrm{SU}(n, 1)$, there is a unique tuple of boundary points on $\partial\mathbf{H}^n_{\mathbb{C}}$.

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Indian Institute of Science Education and Research (IISER) Mohali, Knowledge City, Sector 81, S.A.S. Nagar 140306, Punjab, India

E-mail address: krishnendug@gmail.com, krishnendu@iisermohali.ac.in

Indian Institute of Science Education and Research (IISER) Bhopal, Bhopal Bypass Road, Bhauri Bhopal 462 066 Madhya Pradesh, India

E-mail address: parsad.shiv@gmail.com