Dynamics of isolated left orders

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Abstract. A left order of a countable group G is called isolated if it is an isolated point in the compact space $LO(G)$ of all the left orders of G. We study properties of a dynamical realization of an isolated left order. Especially we show that it acts on $\mathbb{R}$ cocompactly. As an application, we give a dynamical proof of the Tararin theorem which characterizes those countable groups which admit only finitely many left orders. We also show that the braid group B_3 admits countably many isolated left orders which are not the automorphic images of the others.

1. Introduction

Throughout this paper, all the groups considered are countable. Given a group G, a total order $<_\lambda$ on G is called a *left order* if for any $f, g, h \in G$, $f <_\lambda g$ implies $hf <_\lambda hg$. An element $g \in G$ is called λ-positive if $g >_\lambda e$. The set of all the λ-positive elements is called the *positive cone of* λ and is denoted by P_λ. It is a subsemigroup and $P_\lambda \sqcup P_\lambda^{-1} = G \setminus \{e\}$.

Given a left order $<_\lambda$, we define $\lambda : G \setminus \{e\} \to \{\pm 1\}$ by $\lambda(g) = 1$ if and only if $g \in P_\lambda$. Then we have

$$\lambda(f) = 1, \lambda(g) = 1 \Rightarrow \lambda(fg) = 1, \text{ and } \lambda(f^{-1}) = -\lambda(f). \tag{1.1}$$

Conversely given a map $\lambda : G \setminus \{e\} \to \{\pm 1\}$ which satisfies (1.1), we get a left order $<_\lambda$ by setting $f <_\lambda g$ if $\lambda(f^{-1}g) = 1$. The map λ is also refered to as a left order. Thus the set $LO(G)$ of the left orders on G is viewed as a closed subset of the space $\{\pm 1\}^{G \setminus \{e\}}$ with the pointwise convergence topology. This yields a totally disconnected compact metrizable topology on $LO(G)$ (metrizable since G is countable). It is either finite or uncountably many [**8**]. We call $\lambda \in LO(G)$ *isolated* if it is an isolated point in the space $LO(G)$.

Given $\lambda \in LO(G)$, there is defined a dynamical realization

$$\rho_\lambda : G \to \mathrm{Homeo}_+(\mathbb{R})$$

based at $x_0 \in \mathbb{R}$ such that $f <_\lambda g$ if and only if $fx_0 < gx_0$. We discuss its fundamental properties in Section 2. Especially we show that the dynamical realization is tight at the base point. See Definition 2.1.

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In this paper, we are mainly interested in isolated orders, since in this case, the dynamical realizations display a certain kind of rigidity, and vice versa. In [**10**] Theorems 1.2 and 1.3, the relation between the isolation of left orders and the rigidity of the dynamical realization is described, as well as for circular orders.

An action $\rho : G \to \mathrm{Homeo}_+(\mathbb{R})$ is said to be *cocompact* if there is a compact interval I such that any orbit $\rho(G)x$ intersects I. Our first result, proved in Section 3, is the following.

THEOREM 1. *If $\lambda \in LO(G)$ is isolated, then its dynamical realization ρ_λ is cocompact.*

In fact if the group G is finitely generated, the dynamical realization is cocompact for any left order, isolated or not. Therefore Theorem 1 is mainly concerned with non finitely generated groups. By Theorem 1, the dynamical realization of an isolated order admits a minimal set $\mathcal{M}$, which is shown to be unique unless $G \cong \mathbb{Z}$. In Section 4, we show that if $\mathcal{M} = \mathbb{R}$, then the group is rational (Theorem 4.1).

Given $\lambda \in LO(G)$, a subgroup H of G is called λ-*convex*, if whenever $h_1, h_2 \in H$, $g \in G$ and $h_1 <_\lambda g <_\lambda h_2$, we have $g \in H$. The set of convex subgroups is totally ordered by the inclusion. The following theorem is shown in Section 5.

THEOREM 2. *If $\lambda \in LO(G)$ is isolated, then there are only finitely many λ-convex subgroups.*

This is known to specialists (see for example [**4**] Exercise 3.3.15). However, our strategy of the proof is different from that mentioned in [**4**].

Theorem 2 enables us to define the maximal sequence of convex subgroups of an isolated left order. As an application of our method, we give a dynamical proof of the Tararin theorem which characterizes the groups with finitely many left orders in Section 6. In Section 7, the maximal Tararin subgroup of an isolated left order is defined, and is shown to be equal to the Conradian soul [**12**].

Last sections 8 and 9 are more or less independent of the previous sections. Dubrovina-Dubrovin [**2**] constructed an isolated order λ_n on the braid group B_n, $n \geq 3$. In section 9, we show:

THEOREM 3. *There are countably many isolated orders in $LO(B_3)$ which are not the automorphic images* [1] *of the others.*

The method is a modification of the proof of [**10**] Theorem 1.4. The following theorem is the starting point of the proof of Theorem 3. Let

$$G = \langle a, b \mid a^2 = b^3 \rangle, \quad \overline{G} = \langle \alpha, \beta \mid \alpha^2 = \beta^3 = e \rangle,$$

and $q : G \to \overline{G}$ the surjective homomorphism defined by $q(a) = \alpha$ and $q(b) = \beta$. Notice that G is isomorphic to B_3, and $\overline{G}$ to $PSL(2, \mathbb{Z})$. We denote by $CO(\overline{G})$ the space of the left invariant circular orders of $\overline{G}$ (see Section 8). In Section 9, we show:

THEOREM 4. *The homomorphism q induces a homeomorphism $q_* : LO(G) \to CO(\overline{G})$.*

[1]Given $\phi \in \mathrm{Aut}(G)$ and $\lambda \in LO(G)$, the left order $\phi^*\lambda \in LO(G)$ defined by $g <_{\phi^*\lambda} g'$ if and only if $\phi(g) <_\lambda \phi(g')$ is called an automorphic image of λ. For example, the reciprocal of the natural order λ of $\mathbb{Z}$ is an automorphic image of λ.

Isolated left orders are often induced from isolated circular orders of the group quotiented by the center. See [**10**] Section 5, for example. The above theorem is also a typical example. However there is an example of a group with isolated left orders which admits no center, constructed in [**6**] 3.2.

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2. Dynamical Realization

In this section, we define a dynamical realization of a left order $\lambda \in LO(G)$ and study its fundamenal properties. Fix an enumeration of G: $G = \{g_i \mid i \in \mathbb{N}\}$ such that $g_1 = e$. We define an order preserving embedding $\iota : G \to \mathbb{R}$ inductively as follows. Define $\iota(g_1) = x_0$, where x_0 is some point in $\mathbb{R}$. Assume we have defined ι on the subset $\{g_1, \dots, g_n\}$, $n \geq 1$, and let us define $\iota(g_{n+1})$. Order the subset $\{g_1, \dots, g_n\}$ as

$$g_{i_1} <_\lambda g_{i_2} <_\lambda \cdots <_\lambda g_{i_n}.$$

If $g_{n+1} <_\lambda g_{i_1}$, define $\iota(g_{n+1}) = \iota(g_{i_1}) - 1$,
if $g_{i_n} <_\lambda g_{n+1}$, $\iota(g_{n+1}) = \iota(g_{i_n}) + 1$,
and if $g_{i_k} <_\lambda g_{n+1} <_\lambda g_{i_{k+1}}$, $\iota(g_{n+1}) = (1/2)(\iota(g_{i_k}) + \iota(g_{i_{k+1}}))$.

Then we have $\inf \iota(G) = -\infty$ and $\sup \iota(G) = \infty$. The left translation of G yields an order preserving action of G on $\iota(G)$, which extends to a continuous action on the closure $\mathrm{Cl}(\iota(G))$. Extend it further to a continuous action on $\mathbb{R}$ by setting that the action on gaps of $\mathrm{Cl}(\iota(G))$ be linear.

This action is called the *dynamical realization of* λ *based at* x_0, and is denoted by ρ_λ. The dynamical realization depends on the choice of the enumeration of G. Soon later, we shall show that any two dynamical realizations are mutually topologically conjugate.

Definition 2.1. An action $\rho : G \to \mathrm{Homeo}_+(\mathbb{R})$ is called *tight at* $x_0 \in \mathbb{R}$ if
(1) ρ is free at x_0 i.e, the stabilizer at x_0 is trivial,
(2) $\inf \rho(G)x_0 = -\infty$, $\sup \rho(G)x_0 = \infty$, and
(3) whenever $\mathrm{Cl}(\rho(G)x_0) \cap [a, b] = \{a, b\}$ for any $a < b$, we have $\{a, b\} \subset \rho(G)x_0$.

Proposition 2.2. *The dynamical realization* ρ_λ *based at* x_0 *is tight at* x_0.

Proof. All that needs proof is (3). Let $a < b$ be as in (3). The proof is by contradiction. Assume, to fix the idea, that $a \notin \rho(G_0)x_0 = \iota(G)$. (Notice that $\iota(g) = \rho_\lambda(g)x_0$.) Choose ϵ small enough compared with $b - a$, and choose $\iota(g_1) \in (a - \epsilon, a)$ and $\iota(g_2) \in [b, b + \epsilon)$. Recall that the dynamical realization is defined via an enumeration of G. One may assume that there is no point in $(\iota(g_1), \iota(g_2)) \cap \iota(G)$ which is enumerated before g_1 or g_2, since otherwise one may pass to that point. Since $a \in \mathrm{Cl}(\iota(G)) \setminus \iota(G)$, there is a point $\iota(g_3)$ in $(\iota(g_1), \iota(g_2)) \cap \iota(G)$ which is enumerated for the first time after g_1 and g_2. Then $\iota(g_3)$ is the midpoint of $\iota(g_1)$ and $\iota(g_2)$ and must be fallen in (a, b) since ϵ is small. A contradiction. □

Corollary 2.3. *The dynamical realizations defined via two different enumerations of* G *are mutually conjugate by an orientation and base point preserving homeomorphism of* $\mathbb{R}$

Proof. Let ι and ι' be two embeddings of G obtained by different enumerations of G. There is an orientation preserving bijection $h : \iota(G) \to \iota'(G)$ defined by

$h(\iota(g)) = \iota'(g)$ $(g \in G)$. By the tightness, h extends, first of all. to a homeomorphism $h : \mathrm{Cl}(\iota(G)) \to \mathrm{Cl}(\iota'(G))$, and then to a homeomorphism of $\mathbb{R}$ linearly on gaps. The extended h yields the required conjugacy. □

The proof of the previous corollary also yields the following result, which will be used in Section 9.

Corollary 2.4. *Let $\mathcal{H}$ be the set of the orientaion and base point x_0 preserving topological conjugacy classes of the homomorphisms $G \to \mathrm{Homeo}_+(\mathbb{R})$ which are tight at x_0. Then the dynamical realization induces an bijection of $LO(G)$ onto $\mathcal{H}$.* □

A left order $<_\lambda$ is called *discrete* if there is a minimal λ-positive element, and *indiscrete* otherwise.

Corollary 2.5. *If $\lambda \in LO(G)$ is indiscrete, then the orbit $\rho_\lambda(G)x_0$ of the base point x_0 is dense in $\mathbb{R}$.*

Proof. Assume $\mathrm{Cl}(\rho_\lambda(G)x_0) \neq \mathbb{R}$ and let (a, b) be a gap of $\mathrm{Cl}(\rho_\lambda(G)x_0)$. Then by the previous lemma, we have $a, b \in \rho_\lambda(G)x_0$. That is, $a = \iota(g_1)$ and $b = \iota(g_2)$ for some $g_1, g_2 \in G$. Then $g_1^{-1}g_2$ is the minimal positive element, and λ is discrete. □

3. Proof of Theorem 1

We begin with two lemmas. The first one can be found in [**11**] (Proposition 2.1.12).

Lemma 3.1. *Let G be a finitely generated group which acts on $\mathbb{R}$ without global fixed points. Then the action is cocompact.*

Proof. We identify $\mathbb{R} \approx (0, 1)$. Let G_0 be a finite generating set of G. Define

$$a = \sup_{s \in G_0} \sup_{x \in (0,1)} |sx - x|.$$

Choose a compact interval $J \subset (0, 1)$ such that $|J| > a$. Given any point $x \in (0, 1)$, we have $\inf Gx = 0$ and $\sup Gx = 1$ since there is no global fixed point. Considering the Schreier graph of Gx, one can show that $Gx \cap J \neq \emptyset$. □

Lemma 3.2. *Let G be a group acting on $\mathbb{R}$ and let $y_0 \in \mathbb{R}$. Denote by G_{y_0} the stabilizer of G at y_0. Assume $G_{y_0} \neq G$. Given $\lambda_0 \in LO(G_{y_0})$, there are at least two orders in $LO(G)$ which restrict to λ_0 on G_{y_0}.*

Proof. Let μ be the G-invariant order on G/G_{y_0} given by the natural order of the orbit $Gy_0 \approx G/G_{y_0}$ in $\mathbb{R}$. Then λ_0 and μ determines a left order on G lexicographically (Lemma 5.1). If we consider the reciprocal order $-\mu$, we get another one. □

Assume $\lambda \in LO(G)$ is an isolated left order on G. Since we are considering the pointwise convergence topology, this is equivalent to the following condition $(\star)$

$(\star)$ *There is a finite subset $S \subset P_\lambda$ such that λ is the only element in $LO(G)$ which contains S in its positive cone.*

Such a subset S is called a *characteristic positive set* of λ.

Proof of Theorem 1. By the dynamical realization of the isolated left order λ, the group G acts on $\mathbb{R}$. Let H be the subgroup of G generated by a characteristic positive set S of λ. If there is no global fixed point by the action of H, then H acts on $\mathbb{R}$ cocompactly by Lemma 3.1, and hence also G, finishing the proof. In the remaining case, choose a global fixed point y_0 of H and consider G_{y_0}. We have $G_{y_0} \neq G$ since the dynamical realization has no global fixed point, by its tightness. By the previous lemma, the restriction of λ to G_{y_0} extends to two left orders of G. But we have $S \subset H \subset G_{y_0}$ and hence S is contained in the positive cone of both orders. A contradiction. □

REMARK 3.3. The condition that λ be isolated is actually necessary for Theorem 1. To show this, let G be the infinite direct sum of $\mathbb{Z}$, i.e,

$$G = \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{Z}, \quad a_n = 0 \ \text{ but for finitely many } n\}.$$

Define a left order on G by setting $0 < (a_n)$ if $0 < a_N$, where N is the largest number such that $a_N \neq 0$. Then its dynamical realization is not cocompact. To show this, define for $m \in \mathbb{N}$,

$$G_m = \{(a_n) \mid a_n = 0, \ \forall n > m\}.$$

Then G_m's form an exhausting increasing sequence of convex subgroups. Consider the dynamical realization ρ_λ based at x_0. The points

$$\xi_n = \inf \rho_\lambda(G_n)x_0 \quad \text{and} \quad \eta_n = \sup \rho_\lambda(G_n)x_0$$

are fixed points of $\rho_\lambda(G_n)$. They satisfy $\xi_n \searrow -\infty$ and $\eta_n \nearrow \infty$ by condition (2) of Definition 2.1, since G_n is exhausting. This implies that ρ_λ is not cocompact.

Theorem 1 implies that there is a minimal set $\mathcal{M}$ for the dynamical realization of an isolated left order.[2] There is a trichotomy for $\mathcal{M}$ ([**3**] Proposition 6.1).

(I) $\mathcal{M} = \mathbb{R}$.

(II) *$\mathcal{M}$ is infinite and discrete in $\mathbb{R}$.*

(III) *$\mathcal{M}$ is locally Cantor. In this case, if X is a nonempty closed subset of $\mathbb{R}$ invariant by the dynamical realization of G, then $\mathcal{M} \subset X$. Especially, $\mathcal{M}$ is the unique minimal set.*

LEMMA 3.4. *Let $\lambda \in LO(G)$ be isolated, with $\mathcal{M}$ an associated minimal set. Assume (III) above, or (II) and $G \not\cong \mathbb{Z}$. Then the base point x_0 is contained in a gap I_1 of $\mathcal{M}$, the stabilizer G_{I_1} is nontrivial, and there is no gap of $\mathcal{M}$ other than the orbit of I_1.*

PROOF. We give a proof only for case (III). Case II can be treated much easier. Notice that if $\rho(G)x_0$ is discrete, then $G \cong \mathbb{Z}$. Assume that the base point x_0 is contained in $\mathcal{M}$ and let (a, b) be a gap of $\mathcal{M}$. Since the dynamical realization ρ_λ is tight, we have $a, b \in \rho_\lambda(G)x_0$. But there is no orientation preserving homeomorphism leaving $\mathcal{M}$ invariant and mapping a to b. The contradiction shows that x_0 is contained in a gap I_1 of $\mathcal{M}$.

If G_{I_1} is trivial, then $\rho_\lambda(G)x_0 \cap I_1 = \{x_0\}$. Again by the tightness, the boundary points of I_1 must belong to $\rho_\lambda(G)x_0$. A contradiction. The last statement follows similarly from the tightness. □

[2] The results of the remaining part of this section (trichotomy, Lemma 3.4 and Corollary 3.5) and Lemma 5.4 (1) (2) hold true whenever the left order λ admits a cocompact dynamical realization, especially when G is finitely generated. But we shall state it only for an isolated left order λ.

Corollary 3.5. *If $G \not\cong \mathbb{Z}$, then the minimal set $\mathcal{M}$ of the dynamical realization ρ_λ of an isolated left order λ is unique.*

Proof. All that needs proof is the case where $\mathcal{M}$ is discrete, since a locally Cantor minimal set is always unique. We still use the notation of the previous lemma. If there is another minimal set $\mathcal{M}'$, then $\mathcal{M}' \cap I_1$ must be one point, say y_0, which is fixed by $\rho_\lambda(G_{I_1})$. But then $\rho_\lambda(G)x_0 \cap I_1 = \rho_\lambda(G_{I_1})x_0$ must be contained in an open subinterval of I_1 delimited by y_0, contrary to the tightness. □

4. The case $\mathcal{M} = \mathbb{R}$

This section is devoted to the proof of the following theorem.

Theorem 4.1. *Let $\lambda \in LO(G)$ be isolated and assume that the dynamical realization ρ_λ is minimal. Then the group G is isomorphic to an additive subgroup A of $\mathbb{Q}$ such that $A \not\cong \mathbb{Z}$, and λ is either the natural left order given by $A \subset \mathbb{Q} \subset \mathbb{R}$ or its reciprocal.*

This theorem might be known among specialists, but the author cannot locate it in the literarue.

Let λ be an element of $LO(G)$ which satisfies the hypothesis of Theorem 4.1. We shall abbreviate the notations $\rho_\lambda(g)x$ by gx, and $\rho_\lambda(G) \subset \mathrm{Homeo}_+(\mathbb{R})$ by G. Let Z be the centralizer of G in $\mathrm{Homeo}_+(\mathbb{R})$.

Lemma 4.2. *The centralizer Z is an abelian group which acts freely on $\mathbb{R}$.*

Proof. For $\zeta \in Z \setminus \{\mathrm{id}\}$, $\mathrm{Fix}(\zeta)$ is a closed set which is invariant by G. Since the G-action is minimal, we have $\mathrm{Fix}(\zeta) = \emptyset$. By Hölder's theorem (e.g, [**12**]), any group acting freely on $\mathbb{R}$ is abelian. □

Let x_0 be the base point of the dynamical realization. Choose $x_n \in Gx_0$, $n \in \mathbb{N}$, such that $x_n \to x_0$, $x_n \neq x_0$. Notice that G acts freely at x_n. Let $\lambda_n \in LO(G)$ be the order determined by x_n: $g >_{\lambda_n} e$ if and only if $gx_n > x_n$. Then $\lambda_n \to \lambda$ in $LO(G)$. Since λ is isolated, $\lambda_n = \lambda$ for any large n. We assume $\lambda_n = \lambda$ for all n. Define an order preserving bijection $\zeta_n : Gx_0 \to Gx_n$ by $\zeta_n(gx_0) = gx_n$. Since $Gx_0 = Gx_n$ is dense in $\mathbb{R}$, the map ζ_n extends to an orientation preserving homeomorphism of $\mathbb{R}$, denoted by the same letter ζ_n. Clearly $\zeta_n \neq id$.

Lemma 4.3. *We have $\zeta_n \in Z$.*

Proof. Given any $g \in G$, it suffices to show that $\zeta_n g = g\zeta_n$ on the dense subset Gx_0. For any $hx_0 \in Gx_0$, we have

$$\zeta_n g(hx_0) = \zeta_n((gh)x_0) = ghx_n = g(hx_n) = g\zeta_n(hx_0),$$

as is required. □

Lemma 4.4. *The action of Z is minimal, and is conjugate to translations.*

Proof. By Lemmas 4.2 and 4.3, there is an element in Z which acts freely on $\mathbb{R}$. This implies that the action of Z is cocompact. Let $\mathcal{N}$ be a minimal set of Z. If it is locally Cantor, then $\mathcal{N}$ is the unique minimal set, and must be invariant by G. But G-action is minimal by the assumption. A contradiction. Next assume $\mathcal{N}$ is discrete. Then since the Z-action is free, we must have $Z \cong \mathbb{Z}$, contradicting Lemma 4.3. Therefore Z must act minimally on $\mathbb{R}$.

Choose any $\zeta_0 \in Z \setminus \{\mathrm{id}\}$. Since the action of the group $\langle \zeta_0 \rangle$ is free and Z is abelian, the group $Z/\langle \zeta_0 \rangle$ acts on $\mathbb{R}/\langle \zeta_0 \rangle \approx S^1$. Since $Z/\langle \zeta_0 \rangle$ is amenable, there is an $Z/\langle \zeta_0 \rangle$-invariant probability measure. It lifts to a locally finite Z-invariant measure μ on $\mathbb{R}$. Since the action of Z is minimal, μ is atomless and fully supported. Thus there is a homeomorphism h such that $h_*\mu$ is the Lebesgue. Conjugating the Z-action by h, we obtain an action by translations. $\square$

Proof of Theorem 4.1. By changing the coordinate, we assume that the action of Z is by translations. Since the Z-action is minimal, any element of G, commutng with Z, acts also by translations. Then we have an injective homomorphism $\phi : G \to \mathbb{R}$ defined by the translation length. We shall show that ϕ embeds G into $\mathbb{Q}$. Assume not. Then G is a nontrivial direct sum: $G = G_1 \oplus G_2$. Given any $a \in \mathbb{R}$, we obtain a homomorphism $\phi_a : G \to \mathbb{R}$ by setting $\phi_a = \phi$ on G_1 and $\phi_a = a\phi$ on G_2. There is a arbitrarily near 1 such that ϕ_a is injective. But ϕ_a yields a left order different from λ and arbitrarily near λ. This contradicts the assumption that λ is isolated, finishing the proof that G is isomorphic to an additive subgroup A of $\mathbb{Q}$. The last statement of the theorem follows at once. $\square$

5. Convex subgroups

We shall prove Theorem 2 in this section. First we begin with fundamental properties of convex subgroups. For the definition of convex subgroups, see Introduction. We begin with a well known easy fact.

Lemma 5.1. *Let H be a subgroup of G. For any $\lambda_0 \in LO(H)$ and any G-invariant total order λ_1 on G/H, there is a unique order $\lambda \in LO(G)$ such that H is λ-convex, that $\lambda|_H = \lambda_0$, and that for $g \notin H$, $g >_\lambda e$ if and only if $gH >_{\lambda_1} H$.* $\square$

Such an order λ is said to be *determined lexicographically by λ_0 and λ_1*.

Lemma 5.2. *Let $\lambda \in LO(G)$ and H a λ-convex subgroup of G. Then there is a G-invariant total order λ_1 on G/H such that λ is determined lexicographically by $\lambda|_H$ and λ_1.*

Proof. Define a total order λ_1 on G/H by setting $g_1H <_{\lambda_1} g_2H$ if $e <_\lambda g_1^{-1}g_2$ and $g_1^{-1}g_2 \notin H$. The convexity of H shows that this is a well defined G-invariant order. $\square$

If G is isomorphic to $\mathbb{Z}$ or if the minimal set of the dynamical realization of λ is minimal, then there is no proper λ-convex subgroups, and Theorem 2 holds true. Henceforth in this section we work under the following assumption.

Assumption 5.3. (1) $\lambda \in LO(G)$ is isolated with a characteristic positive set S.

(2) G is not isomorphic to $\mathbb{Z}$.

(3) The minimal set $\mathcal{M}$ of the dynamical realization is not $\mathbb{R}$.

Denote by $I_1 = (y_0, z_0)$ the gap of $\mathcal{M}$ which contains the base point x_0 (Lemma 3.4), and by G_1 the stabilizer of I_1.

Lemma 5.4. *(1) G_1 is proper and nontrivial.*

(2) G_1 is the maximal proper λ-convex subgroup of G.

(3) The restricted order $\lambda|_{G_1}$ is isolated with characteristic positive set $S \cap G_1$.

(4) $S \cap (G \setminus G_1) \neq \emptyset$.

Proof. The subgroup G_1 is clearly proper. It is nontrivial by Lemma 3.4. Also G_1 is convex. Let H be an arbitrary proper λ-convex subgroup of G. We shall show that $H \subset G_1$. Consider first the case where $\mathcal{M}$ is discrete. By looking at the action of G on $\mathcal{M}$, one can define a surjective homomorphism $\phi : G \to \mathbb{Z}$ such that $\mathrm{Ker}(\phi) = G_1$. If $\phi(H)$ is nontrivial, then clearly we have $H = G$ since H is convex. If $\phi(H)$ is trivial, then $H \subset G_1$, as is required.

So in the rest, we assume that $\mathcal{M}$ is locally Cantor. Let $\mathcal{H}$ be the convex hull of Hx_0 in $\mathbb{R}$. Then $\mathcal{H}$ is a bounded open interval of $\mathbb{R}$. The boundedness follows from the convexity and the properness of H. The convexity of H implies that for any $g \in G$, we have either $g\mathcal{H} = \mathcal{H}$ or $g\mathcal{H} \cap \mathcal{H} = \emptyset$. Thus the closed set

$$X = \mathbb{R} \setminus \bigcup_{g \in G} g\mathcal{H}$$

is G-invariant and nonempty. Therefore we have $\mathcal{M} \subset X$, which implies $\mathcal{H} \subset I_1$, showing that $H \subset G_1$.

Let us show that $S \cap G_1$ is a characteristic positive set of $\lambda|_{G_1}$. If not, there is a left order λ_0' ($\lambda_0' \neq \lambda|_{G_1}$) of G_1 such that $S \cap G_1$ is contained in the positive cone of λ_0'. Let λ_1 be the G-invariant total order on G/G_1 obtained by Lemma 5.2. Let $\lambda' \in LO(G)$ be the order determined lexicographically by λ_0' and λ_1. Then λ' contains S in its positive cone and $\lambda' \neq \lambda$, contradicting that S is a characteristic positive set of λ.

Finally let us show that $S \cap (G \setminus G_1)$ is nonempty. If it is empty, then $\lambda|_{G_1}$ and $-\lambda_1$ lexicographically determines $\lambda' \in LO(G)$, where $-\lambda_1$ is the reciprocal of the order λ_1 constructed in Lemma 5.2. But S is contained in the positive cone of λ'. A contradiction. □

Proof of Theorem 2. By Lemma 5.4, we obtain the maximal proper convex subgroup G_1. If G_1 is not isomorphic to $\mathbb{Z}$ and the minimal set of $\lambda|_{G_1}$ is not the whole $\mathbb{R}$, then we can repeat the process and obtain the second maximal proper convex subgroup G_2. This process ends at finite steps since each time the number of elements of positive characteritic set decreases. □

Definition 5.5. The sequence

$$G = G_0 > G_1 > \cdots > G_n > \{e\}$$

of all the λ-convex subgroups is called the *maximal convex sequence* of the isolated order λ. The number n is called the *height* of λ.

Thus an isolated left order with minimal dynamical realization has height 0. Let $\mathcal{M}_0$ be the minimal set of G and I_1 the gap of $\mathcal{M}_0$ containing the base point x_0. Then the maximal proper λ-convex subgroup G_1 is the stabilizer of I_1. By Lemmas 5.1 and 5.2, $\lambda|_{G_1}$ is isolated, and there is a minimal set $\mathcal{M}_1$ of the G_1-action on I_1. Next conside the gap I_2 of $\mathcal{M}_1$ in I_1 containing x_0. Continuing this way, we get a decreasing sequence of open intervals

$$\mathbb{R} \supset I_1 \supset \cdots \supset I_n.$$

Each subgroup G_i is the stabilizer of I_i, and each $\mathcal{M}_i$ is a minimal set of G_i in I_i. The pair $(I_i, \mathcal{M}_i)$ is called the *i-th internal pair associated with the maximal convex sequence*. There are only two possibilities for the last group G_n:

(A) $\mathcal{M}_n = I_n$,

(B) $G_n = \mathbb{Z}$.
In (A), the order λ is indiscrete and in (B), it is discrete.

As a Corollary of Theorem 4.1, we get the following proposition, which will be used in the next section.

PROPOSITION 5.6. *If an isolated order λ has height* 0*, i.e, if there is no proper λ-convex subgroup, then the group G is rational and the order λ is by the natural order of $G \subset \mathbb{Q} \subset \mathbb{R}$ or its reciprocal.*

6. Tararin groups

DEFINITION 6.1. A group G is called a *Tararin group* if $|LO(G)| < \infty$.

Of course any left order of a Tararin group is isolated. In this section, we shall give a dynamical proof of the following theorem by Tararin [**14**]. See also [**4**] (Theorem 2.2.12) or [**9**].

THEOREM 6.2. *(I) Assume $|LO(G)| < \infty$. Then the following holds.*

(1) There is a unique rational series[3]

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n \rhd G_{n+1} = \{e\}. \tag{6.1}$$

(The uniqueness implies that each subgroup G_i is characteristic, i.e, invariant by any automorphism of G. Especially it is a normal subgroup of G.)

(2) There are elements $s_i \in G_i \setminus G_{i+1}$ for each $i \in \{0, 1, \ldots, n\}$ such that for any map $\epsilon : \{0, 1, \ldots, n\} \to \{\pm 1\}$, there are exactly one order λ_ϵ such that $s_i^{\epsilon(i)}$ is positive. Moreover

$$LO(G) = \{\lambda_\epsilon \mid \epsilon \in \{\pm 1\}^{\{0,1,\ldots,n\}}\}.$$

(3) The sequence (6.1) is the maximal convex sequence for any λ_ϵ.

(4) The quotient group G_i/G_{i+2}, $i \in \{0, \ldots, n-1\}$, is not bi-orderable.

(II) Conversely, if a group G admits a rational series (6.1) such that G_{i+2} is a normal subgroup of G_i and G_i/G_{i+2} is not bi-orderable ($0 \le i \le n-1$), then $|LO(G)| = 2^{n+1}$.

PROOF. First of all let us show (II). We use the following easy fact.

If A is a rational group and $\phi : A \to \{\pm 1\}$ is a nontrivial homomorphism, then for any nontrivial element $g \in A$, there are $g_0 \in A$ and $n \ge 1$ such that $g = g_o^n$ and $\phi(g_0) = -1$.

We shall show that G_1 is λ-convex for any $\lambda \in LO(G)$ by induction on the height n of the sequence (6.1). Then G_1 is Tararin since otherwise lexicographic construction would yield infinitely many left orders of G. By the induction hypothesis, this implies that any G_i is λ-convex for any $\lambda \in LO(G)$. Since any G_i/G_{i+1} is rational, and admits only two left orders, one can conclude then that $|LO(G)| = 2^{n+1}$, as is required.

Consider an exact sequence

$$1 \to G_1/G_2 \to G/G_2 \to G/G_1 \to 1.$$

By the induction hypothesis, G_2 is $\lambda|_{G_1}$-convex and there is a left order $<$ on G_1/G_2 induced from $\lambda|_{G_1}$. One can define a homomorphism $\phi' : G \to \{\pm 1\}$ according as the conjugation by an element of G preserves the order $<$ on G_1/G_2

[3]Rational series means that for any i, G_i/G_{i+1} is a rational group, i.e, an abelian group embeddable into $\mathbb{Q}$.

or not. (Notice that there are only two orders on G_1/G_2,) Since G_1/G_2 is abelian, ϕ' induces a homomorphism $\phi : G/G_1 \to \{\pm 1\}$, which is nontrivial since G/G_2 is not bi-orderable.

To complete the proof, let us show that for any element $g \in G \setminus G_1$, $g >_\lambda e$, we have $g^{-1} <_\lambda G_1 <_\lambda g$. There exist $g_0 \in G$ and $n \geq 1$ such that $g \equiv g_0^n \mod G_1$ and $\phi'(g_0) = -1$. Then for any $h \in G_1 \setminus G_2$, $h >_\lambda e$ if and only if $g_0^{-1} h g_0 <_\lambda e$.

Assume for a while that $g_0 >_\lambda e$. Then if $h >_\lambda e$,

$$e <_\lambda h <_\lambda hg_0 <_\lambda g_0.$$

Applying h successively, we obtain

$$e <_\lambda h <_\lambda h^2 <_\lambda \cdots <_\lambda h^2 g_0 <_\lambda hg_0 <_\lambda g_0.$$

If we put $h_1 = g_0^{-1} h g_0$, then

$$e <_\lambda h <_\lambda h^2 <_\lambda \cdots <_\lambda g_0 h_1^2 <_\lambda g_0 h_1 <_\lambda g_0. \tag{6.2}$$

By an analogous argument, we have

$$g_0^{-1} <_\lambda g_0^{-1} h_1^{-1} <_\lambda g_0^{-1} h_1^{-2} <_\lambda \cdots <_\lambda h^{-2} <_\lambda h^{-1} <_\lambda e. \tag{6.3}$$

The elements $h, h_1 \in G_1 \setminus G_2$ are $\lambda|_{G_1}$-cofinal by the assumption that G_2 is $\lambda|_{G_1}$-convex and G_1/G_2 is rational. Therefore by (6.2) and (6.3), we obtain $g_0^{-1} G_1 <_\lambda G_1 <_\lambda g_0 G_1$. For our initial g, since $g^{\pm 1} G_1 = g_0^{\pm n} G_1$, we have $g^{-1} G_1 < G_1 < g G_1$, as is required. On the other hand, if $g_0 <_\lambda e$, then the same argument shows that $g G_1 < G_1 < g^{-1} G_1$, contradicting the hypothesis $g >_\lambda e$. This finishes the proof of (II).

Now we shall proceed to the proof of (I). For a Tararin group G, let $n(G)$ be the minimal height of the elements of $LO(G)$. We shall show (I) by the induction on $n(G)$. Let G be a Tararin group, $\lambda \in LO(G)$ with height $n = n(G)$, and G_1 the maximal proper λ-convex subgroup. Then the lexicographic construction shows that G_1 is also Tararin, and $n(G_1) \leq n - 1$. Therefore by the induction hypothesis, the maximal convex sequence of $\lambda|_{G_1}$

$$G_1 \rhd G_2 \rhd \cdots \rhd G_{n+1} = \{e\}$$

is a unique rational series of G_1 and G_i/G_{i+2} is not bi-orderable $(1 \leq i \leq n-1)$.

We shall show that G_1 is a normal subgroup of G, and G/G_1 is a rational group. But this is clear if the minimal set $\mathcal{M}$ of the dynamical realization ρ_λ is discrete. So assume $\mathcal{M}$ is a locally Cantor set. Let x_0 be the base point of ρ_λ, and choose $g_k \in G$ so that $\rho_\lambda(g_k)x_0 \to \exists y_0 \in \mathcal{M}$ as $k \to \infty$. One may assume that $\rho_\lambda(g_k)x_0$ belongs to a distinct gap of $\mathcal{M}$ for each k. The left orders of G induced by the $\rho_\lambda(G)$-orbit of $\rho_\lambda(g_k)x_0$ are finite in number. So one may assume, by passing to a subsequence, that the left orders are the same. By the same argument as in Theorem 4.1, one can construct order preserving homeomorphisms $h_{k,k'}$ of $\mathrm{Cl}(\rho_\lambda(G))x_0$ which commute with any $\rho_\lambda(g)$ such that $h_{k,k'}(\rho_\lambda(g_k)x_0) = \rho_\lambda(g_{k'})x_0$. The map $h_{k,k'}$ leaves the unique minimal set $\mathcal{M}$ of $\rho_\lambda(G)$ invariant.

Consider the quotient space $\mathcal{R}$ of $\mathbb{R}$ obtained by collapsing each gap of $\mathcal{M}$ to a point. Then $h_{k,k'}$ induces an orientation preserving homeomorphism of $\mathcal{R}$ commuting with the induced action of $\rho_\lambda(G)$. Let Z be the centralizer of the action on $\mathcal{R}$ induced from $\rho_\lambda(G)$ in the space of the orientation preserving homeomorphisms of $\mathcal{R}$. Then since the induced action of G is minimal, Z acts freely on $\mathcal{R}$. In fact, if an element of Z has nonempty fixed point set, then the fixed point set must be

G-invariant and coincides with $\mathcal{R}$. Thus the action of Z is topologically conjugate to translations.

By the choices of k, k', there are arbitrarily small translations. That is, the action of Z must be miminal. This shows that the induced G-action on $\mathcal{R}$ itself is also by translations. Therefore G_1 is the kernel of the induced G-action, and is a normal subgroup of G. Finally, the left order of G/G_1 induced by λ must be isolated, and hence by Theorem 4.1, G/G_1 is rational.

Since G_1 is a normal subgroup of G, and G_i $(i \geq 2)$ is a characteristic subgroup of G_1 by the induction hypothesis, G_i, especially G_2, is a normal subgroup of G. We shall show that $H = G/G_2$ is not bi-orderable. Denote $A = G_1/G_2$ and $B = G/G_1$. There is an exact sequence

$$1 \to A \to H \overset{q}{\to} B \to 1. \tag{6.4}$$

Notice that H is Tararin, since otherwise lexicographic construction would yield infinitely many left orders on G. The conjugation yields a homomorphism from H to $\mathrm{Aut}(A)$, which projects to a homomorphism $\phi : B \to \mathrm{Aut}(A)$ since A is abelian. Any automorphism of $A \subset \mathbb{Q}$ is the multiplication by a nonzero rational number. Thus we get $\phi : B \to \mathbb{Q}^\times$. If ϕ is negative valued, then H does not admit a bi-order, and we are done. If ϕ is trivial, then projecting $H = A \times B \subset \mathbb{Q}^2 \subset \mathbb{R}^2$ to $\mathbb{R}$ along a one dimensional linear subspace of irrational slope yields an embedding of H into $\mathbb{R}$, from which we obtain infinitely many left orders on H. A contraction.

Assume ϕ is positive valued and nontrivial. Let $\{B_i\}$ be an exhausting increasing sequence of subgroups of B which are isomorphic to $\mathbb{Z}$, and let $H_i = q^{-1}(B_i)$. Then the exact sequence

$$1 \to A \to H_i \to B_i \to 1$$

is split. There is a representation $f_i : H_i \to \mathrm{Aff}_+(\mathbb{R})$ to the group of the orientation preserving affine transformatins of the real line such that A is mapped to translations (by $A \subset \mathbb{Q}$ itself) and that the split image of B_i is mapped to the homotheties of ratio $\phi(B_i)$ at some point of $\mathbb{R}$. Two such representations are mutually conjugate by translations (regardless of the choice of the splittings). Therefore we can arrange so that f_{i+1} is an extension of f_i. As the direct limit, we get a faithful representation $f : H \to \mathrm{Aff}_+(\mathbb{R})$. By considering the orbit of various points of $\mathbb{R}$ at which $f(H)$ acts freely, we get various left orders of H, leading to a contradiction. This finishes the proof that H is not bi-orderable.

By the induction hypothesis, the groups G_i/G_{i+2}, $1 \leq i \leq n-1$, are also not bi-orderable. So the sequence

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n \rhd G_{n+1} = \{e\} \tag{6.5}$$

satisfies the hypothesis of (II). We already know that the cardinality of $LO(G)$ is 2^{n+1}.

Choose $s_i \in G_i \setminus G_{i+1}$ and let $S = \{s_0, \ldots, s_n\}$. For any $\epsilon : S \to \{\pm 1\}$, define

$$S^\epsilon = \{s_i^{\epsilon(s_i)} \mid i = 0, \ldots, n\}.$$

For any ϵ, we can construct a left order λ^ϵ whose positive cone contains S^ϵ, lexicographically using sequence (6.5). Such left orders exhaust $LO(G)$, since $|LO(G)| = 2^{n+1}$. This shows that a rational series of G is unique. In fact, any such series gives birth to a left order lexicographically. The series is the maximal convex sequence of that order, but all the 2^{n+1} orders have (6.5) as the maximal convex sequence. □

REMARK 6.3. Let $(I_i, \mathcal{M}_i)$ be the i-th internal pair associated with the maximal convex sequence (6.1) of a Tararin group G. The next subgroup G_{i+1} leaves the gap I_{i+1} of $\mathcal{M}_i$ in I_i invariant. But because G_{i+1} is a normal subgroup of G_i, it leaves all the iterates of I_{i+1} under G_i invariant. By Lemma 3.4, these are the only gaps of $\mathcal{M}_i$. Therefore G_{i+1} acts trivially on $\mathcal{M}_i$. That is, there is an induced action of G_i/G_{i+1} on $\mathcal{M}_i$. If $\mathcal{M}_i$ is discrete, then $G_i/G_{i+1} \cong \mathbb{Z}$, and the action on $\mathcal{M}_i$ is by translation. Assume $\mathcal{M}_i$ is locally Cantor. Let $\mathcal{R}_i$ be the quotient space obtained by I_i by collapsing each gap of $\mathcal{M}_i$ to a point. It is homeomorphic to $\mathbb{R}$. The quotient group G_i/G_{i+1} acts on $\mathcal{R}_i$ minimally and freely. The whole action of G on $\mathbb{R}$ is a "pileup" of translations. Any left order is discrete if and only if the last group G_n is isomorphic to $\mathbb{Z}$.

7. Maximal convex sequence

We shall raise one more example (other than the Tararin groups) of isolated orders whose height is as big as possible. Let B_n be the braid group of n strings, with the standard generators $\sigma_1, \cdots, \sigma_{n-1}$. Define

$$z_1 = \sigma_1 \cdots \sigma_{n-1}, \quad z_2 = \sigma_2 \cdots \sigma_{n-1}, \quad \ldots, \quad z_{n-2} = \sigma_{n-2}\sigma_{n-1}, \quad z_{n-1} = \sigma_{n-1},$$

and $y_i = z_i^{(-1)^{i-1}}$. Let P_n be the subsemigroup of B_n generated by y_i's. Based upon a result of P. Dehornoy [**1**], T. V. Dubrovina and N. I. Dubrovin [**2**] have shown a remarkable fact that $P_n \sqcup P_n^{-1} = B_n \setminus \{e\}$. The left order λ_n whose positive cone is P_n is called the *Dubrovina-Dubrovin order*. Since $S = \{y_1, \ldots, y_{n-1}\}$ generates P_n, the order λ_n is isolated with characteristic positive set S. Moreover λ_n can be defined lexicographically as a twist of the Dehornoy order [**1**], and the subgroups

$$B^*_{n-k} = \langle y_k, \cdots, y_{n-1}\rangle = \langle \sigma_k, \cdots, \sigma_{n-1}\rangle$$

are λ_n-convex. Since $|S| = n-1$, they are the only convex subgroups by Lemma 5.4, and the maximal convex sequence is given by

$$B_n > B^*_{n-1} > \cdots > B^*_2 > \{e\}. \tag{7.1}$$

The height of λ_n is $n-2$. The order λ_n is discrete since $B^*_2 \cong \mathbb{Z}$. The i-th minimal set $\mathcal{M}_i$ of the i-th internal pair $(I_i, \mathcal{M}_i)$ is locally Cantor, since each term in (7.1) is not a normal subgroup of the previous term. In fact, the sequence (7.1) is, algebraically, conjugate to the inclusions

$$B_n > B_{n-1} > \cdots > B_2 > \{e\},$$

and each term cannot be normal in the previous term.

We shall construct countably many isolated orders of B_3 in Section 9.

For an isolated order $\lambda \in LO(G)$, we can define the *maximal Tararin subgroup* G_i in its maximal convex sequence

$$G > G_1 > \cdots > G_n > \{e\}. \tag{7.2}$$

For λ_n, the maximal Tararin subgroup is $B^*_2 \cong \mathbb{Z}$, and its height is 0. We shall raise questions about the isolated orders of non Tararin groups.

QUESTION 7.1. Is there a non Tararin group with an isolated order whose maximal Tararin subgroup has height ≥ 1?

QUESTION 7.2. Is there a non Tararin group with an isolated and indiscrete order?

There is a sufficient condition for a group to be Tararin in terms of an isolated order on it.

PROPOSITION 7.3. *If the maximal convex sequence of an isolated order $\lambda \in LO(G)$ is subnormal,*[4] *then G is a Tararin group.*

PROOF. The proof is an induction on the height of λ. For height 0, this is true by Proposition 5.6. Assume the height is ≥ 1 and consider the maximal convex sequence of λ:

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n \rhd G_{n+1} = \{e\}. \tag{7.3}$$

By the induction hypothesis, G_1 is a Tararin group and the subsequence of (7.3) that begins with G_1 is the unique rational series in Theorem 6.2. Since each G_i $2 \leq i \leq n$, is a characteristic subgroup of G_1 and since G_1 is a normal subgroup of G, G_i is a normal subgroup of G. By virtue of lemmas 5.1 and 5.2, the order induced from λ on G/G_1 is isolated, and of height 0. Therefore G/G_1 is a rational group, by virtue of Proposition 5.6. That is, the sequence (7.3) is a rational series.

Finally let us show that $H = G/G_2$ is not bi-orderable. Let $A = G_1/G_2$, $B = G/G_1$ and consider the exact sequence

$$1 \to A \to H \to B \to 1.$$

As in the proof of Theorem 6.2, the conjugation defines a homomorphism $\phi : B \to \mathrm{Aut}(A) \subset \mathbb{Q}^\times$. If ϕ is negative valued, then H is not bi-orderable, and we are done.

The order λ induces a left order λ_0 of H, which is the lexicographical order given by the orders of B and A. To fix the idea, assume that these two orders are the natural one given by the inclusions $B \subset \mathbb{Q}$ and $A \subset \mathbb{Q}$. Notice also that λ_0 is isolated, since λ is isolated.

If ϕ is trivial, then $H = A \times B$. Consider the embeddings

$$A \times B \subset \mathbb{Q}^2 \subset \mathbb{R}^2.$$

Let $\pi_n : \mathbb{R}^2 \to \mathbb{R}$ be the projection along a one dimensional subspace of irrational slope k_n. The projection π_n maps $A \times B$ injectively to $\mathbb{R}$, and this give a left order λ_n of $A \times B$. Clearly $\lambda_n \to \lambda_0$ as $k_n \downarrow 0$ (the y-coordinate becomes more and more important). Thus λ_0 is not isolated.

If ϕ is nontrivial and positive valued, there is an embedding ϕ of H into $\mathrm{Aff}_+(\mathbb{R})$ (Proof of Theorem 6.2). Points $x_n \in \mathbb{R}$ at which $\phi(H)$ acts freely yield left orders λ_n on H. As is observed by C. Rivas [**13**], we have $\lambda_n \to \lambda_0$ as $x_n \to \infty$ (the slope of affine transformations becomes more and more important). □

COROLLARY 7.4. *Let $\lambda \in LO(G)$ be isolated of height 1. If the minimal set of the dynamical realization is discrete, then G is a Tararin group.*

PROOF. If the minimal set is discrete, then we get a surjective homomorphism $\phi : G \to \mathbb{Z}$ and its kernel is a convex subgroup. By the previous proposition, G is a Tararin group. □

[4] each term is a normal subgroup of the previous term

Example 7.5. The above corollary does not hold if we remove the condition that λ is height 1. Let us construct an example of isolated order $\lambda \in LO(G)$ of height 2 with discrete minimal set, where G is non Tararin. We start with the braid group B_3. The subsemigroup P generated by $y_1 = \sigma_1\sigma_2$ and $y_2 = \sigma_2^{-1}$ is the positive cone of the Dubrovina-Dubrovin order λ_3. The group B_3 is described as

$$B_3 = \langle y_1, y_2 \mid y_2 y_1^2 y_2 = y_1 \rangle.$$

There is an automorphism ϕ of B_3 which satisfies $\phi(y_1) = y_1^{-1}$ and $\phi(y_2) = y_2^{-1}$. Therefore if we define a group G by

$$G = \langle x, y_1, y_2 \mid y_2 y_1^2 y_2 = y_1,\ x y_1 x^{-1} = y_1^{-1},\ x y_2 x^{-1} = y_2^{-1} \rangle,$$

then B_3 is a subgroup of G [**5**]. Let $\hat{P}$ be the subsemigroup of G generated by x and P. Then we have $B_3 = P \sqcup P^{-1} \sqcup \{e\}$, $xP = P^{-1}x$, and $G = \hat{P} \sqcup \hat{P}^{-1} \sqcup \{e\}$. To show the last statement, denote by $\langle x \rangle_\pm$ the subsemigroup generated by $x^{\pm 1}$. Then $\langle x \rangle_+ P^{-1} = P\langle x \rangle_+ \subset \hat{P}$ and $\langle x \rangle_- P = P^{-1}\langle x \rangle_- \subset \hat{P}^{-1}$. Since B_3 is a normal subgroup of G, we have

$$G = \langle x \rangle B_3 = (\langle x \rangle_+ \sqcup \langle x \rangle_- \sqcup \{e\})(P \sqcup P^{-1} \sqcup \{e\}),$$

and each term except $\{e\}$ is contained either in $\hat{P}$ or in $\hat{P}^{-1}$.

The left order λ on G determined by $\hat{P}$ has B_3 as a λ-convex normal subgroup. In fact,

$$B_3^{-1}x = (P \sqcup P^{-1} \sqcup \{e\})x = Px \sqcup P^{-1}x \sqcup \{x\} = Px \sqcup xP \sqcup \{x\} \subset \hat{P},$$

and likewise $B_3^{-1}x^{-1} \subset \hat{P}^{-1}$, which means $x^{-1} <_\lambda B_3 <_\lambda < x$. Since $G/B_3 \cong \mathbb{Z}$, the minimal set associated to λ is discrete. The dynamics of λ is as depicted in Figure 1.

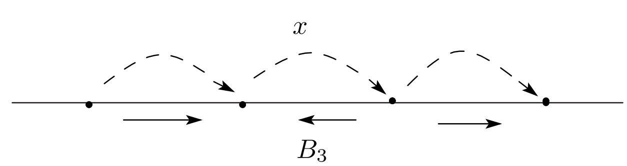

Figure 1. The dotted points form the minimal set $\mathcal{M}$. The element x moves these points one to the right. The intervals bounded by the points are invariant by B_3. The actions of B_3 are opposite in neighbouring intervals, showing the stability of the action.

A. Navas [**12**] has defined the Conradian soul C_λ for any $\lambda \in LO(G)$. Let us recall it briefly. A left order $\lambda \in LO(G)$ of a group G is called *Conradian* if we have $g^{-1}hg^2 >_\lambda e$ whenever $g >_\lambda e$ and $h >_\lambda e$. Thus a bi-invariant order is Conradian. Given an action of G on $\mathbb{R}$, a point $x \in \mathbb{R}$, is called *resilient* if there are an element h of the stabilizer of x and a point $y \in Gx \setminus \{x\}$ such that $h^n y \to x$ as $n \to \infty$. It is shown [**12**] that $\lambda \in LO(G)$ is Conradian if and only if the dynamical realization of λ admits no resilient point.

For a general left order $\lambda \in LO(G)$, a subgroup $H < G$ is called λ-Conradian if the restriction of λ to H is Conradian. The *Conradian soul* C_λ of λ is defined to be the maximal convex Conradian subgroup. In other words, it is the union of all the convex Conradian subgroups. The following proposition is a consequence of [**12**], Proposition 4.1, which states that if a group G is non Tararin, a Conradian order of G can never be isolated. But we will give a proof based upon Proposition 7.3.

Proposition 7.6. *If λ is isolated, the maximal Tararin subgroup of λ coincides with the Conradian soul of λ.*

Proof. In the maximal convex sequence (7.2) of λ, let G_i be the maximal Tararin subgroup. It follows from Remark 6.3, that the dynamical realization of $\lambda|_{G_i}$ is a pileup of translations, and cannot have a resilient point. Thus G_i is λ-Conradian. So it suffices to show that G_{i-1} is not λ-Conradian, that is, the dynamical realization of $\lambda|_{G_{i-1}}$ admits a resilient point. It is no loss of generality to assume that $i = 1$. That is, we assume that G is not a Tararin group, while its maximal convex subgroup G_1 is. By Proposition 7.3, G_1 is not a normal subgroup of G. Then the minimal set $\mathcal{M}$ is not discrete, and the action of G_1 on $\mathcal{M}$ is nontrivial. Choose $g \in G_1$ which acts nontrivially on $\mathcal{M}$. Since G_1 leaves invariant the gap I_1 of $\mathcal{M}$ containing the base point x_0, we have $\mathrm{Fix}(g) \cap \mathcal{M} \neq \emptyset$. Then there are distinct points $x, y \in \mathcal{M}$ such that $g(x) = x$ and either $g^n(y) \to x$ or $g^{-n}(y) \to x$ as $n \to \infty$. Since the action of G on $\mathcal{M}$ is minimal, the point y is accumulated by the orbit of x. This shows that the point x is resilient. $\square$

8. Circular orders

In this section, we provide preliminary facts about circular orders.

Definition 8.1. For a countable group $\overline{G}$, a map $c : \overline{G}^3 \to \{0, 1, -1\}$ is called a *left invariant circular order of* $\overline{G}$ if it satisfies the following conditions

(1) $c(g_1, g_2, g_3) = 0$ if and only if $g_i = g_j$ for some $i \neq j$.

(2) For any $g_1, g_2, g_3, g_4 \in \overline{G}$, we have

$$c(g_2, g_3, g_4) - c(g_1, g_3, g_4) + c(g_1, g_2, g_4) - c(g_1, g_2, g_3) = 0.$$

(3) For any $g_1, g_2, g_3, g_4 \in \overline{G}$, we have

$$c(g_4 g_1, g_4 g_2, g_4 g_3) = c(g_1, g_2, g_3).$$

Definition 8.2. Given a finite set F of $\overline{G}$, a *configuration of F in S^1* is an equivalence class of injections $\iota : F \to S^1$, where two injections ι and ι' is said to be equivalent if there is an orientation preserving homeomorphism h of S^1 such that $\iota' = h\iota$.

Given a left invariant circular order c of $\overline{G}$, the configuration of the set $\{g_1, g_2, g_3\}$ of three points is determined by the rule that g_1, g_2, g_3 is positioned anticlockwise if $c(g_1, g_2, g_3) = 1$, and clockwise if $c(g_1, g_2, g_3) = -1$. By condition (2) of Definition 8.1, this is well defined. But (2) says more. One can show the following proposition by an easy induction on the cardinality of F.

Proposition 8.3. *Given a left invariant circular order of $\overline{G}$, the configuration of any finite set F in S^1 is determined.* $\square$

Denote by $CO(\overline{G})$ the set of all the left invariant circular orders. It is equipped with a totally disconnected compact metrizable topology, just as $LO(\overline{G})$. An *isolated left invariant circular order* is defined using this topology. If $c \in CO(\overline{G})$ is isolated, then there is a finite set $\overline{S}$ of $\overline{G}$, called a *determining set*, such that any left invariant circular order which gives the same configuration of $\overline{S}$ as c is c.

Given $c \in CO(\overline{G})$, we define a dynamical realization $\rho_c : \overline{G} \to \mathrm{Homeo}_+(S^1)$ based at $y_0 \in S^1$ as follows. Fix an enumeration of $\overline{G}$: $\overline{G} = \{g_i \mid i \in \mathbb{N}\}$ such that $g_1 = e$. Define an embedding $\iota : \overline{G} \to S^1$ inductively as follows. First, set $\iota(g_1) = y_0$ and $\iota(g_2) = y_0 + 1/2$. If ι is defined on $\{g_1, \cdots, g_n\}$, then there is a connected component of $S^1 \setminus \{\iota(g_1), \ldots, \iota(g_n)\}$ where the point g_{n+1} should be embedded, by virtue of Proposition 8.3. Define $\iota(g_{n+1})$ to be the midpoint of that interval. Using the injection ι, we can define the action of $\overline{G}$ on S^1 just as in the case of left orders. The action is called the *dynamical realization of c based at y_0* and denoted by ρ_c. We shall raise fundamental properties of ρ_c. The proof is completely parallel to the case of left orders.

Lemma 8.4. *The dynamical realization ρ_c is* tight *at the base point y_0, i.e, it is free at y_0 and if I is a connected component of $S^1 \setminus \mathrm{Cl}(\rho_c(\overline{G})y_0)$, then $\partial I \subset \rho_c(\overline{G})y_0$.*

Lemma 8.5. *Two dynamical realizations obtained via different enumerations of $\overline{G}$ are mutually conjugate by an orientation and base point preserving homeomorphism of S^1.*

Let $\mathcal{M}$ be a minimal set of the dynamical realization ρ_c of an isolated circular order c. It is shown by K. Mann and C. Rivas [**10**] that (unlike left orders) $\mathcal{M}$ is always a proper subset of S^1. Summarizing with other properties, we get:

Lemma 8.6. *If $\overline{G}$ is not finite cyclic, the minimal set $\mathcal{M}$ of the dynamical realization ρ_c of any isolated circular order $c \in CO(\overline{G})$ is unique. It is either a finite set or a Cantor set.*

Lemma 8.7. *If $\overline{G}$ is not finite cyclic and c is isolated, then the base point y_0 of the dynamical realization is contained in a gap I of the minimal set $\mathcal{M}$, the stabilizer $\overline{G}_I$ of I is nontrivial, and there is no gap of $\mathcal{M}$ other than the orbit of I.*

Definition 8.8. Let c be a circular order of $\overline{G}$, isolated or not, and H a nontrivial subgroup of $\overline{G}$. H is said to be *c-convex* if $\rho_c(H)$ acts with global fixed points, and $\rho_c(\overline{G})y_0 \cap I_H = \rho_c(H)y_0$, where I_H denotes the connected component of the complement of the global fixed point set of $\rho_c(H)$ containig y_0. The configuration of $\rho_c(H)y_0$ in I_H defines a left order λ on H, which we call the left order on H *induced from c*. The trivial subgroup is said to be c-convex.

Formally, this is equivalent to the following.

A subgroup H is c-convex if whenever $c(h, e, h') = c(h, g, h') = 1$ for some $h, h' \in H$ and $g \in \overline{G}$, then $g \in H$. The induced left order λ is defined in such a way that $e <_\lambda h$ if there is $h' \in H$ such that $c(h', e, h) = 1$.

As shown in [**10**] Lemma 3.15, there is a unique maximal c-convex subgroup, which we call the *linear part of c*. By virtue of Lemmas 8.6 and 8.7, we get the following lemma.

Lemma 8.9. *Assume $\overline{G}$ is not finite cyclic. Let $\mathcal{M}$ be the minimal set of the dynamical realization of an isolated circular order $c \in CO(\overline{G})$, and I the gap of $\mathcal{M}$ which contains the base point y_0. Then the linear part of c coincides with the stabilizer of I.*

9. Isolated left orders on B_3

In this section, using a method of [**10**], we construct countably many isolated left orders on the braid group B_3, which are not the automorphic images of the others. The group B_3 has the following representations.

$$\begin{aligned} B_3 &= \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle \\ &= \langle y_1, y_2 \mid y_2 y_1^2 y_2 = y_1 \rangle \\ &= \langle a, b \mid a^2 = b^3 \rangle, \end{aligned}$$

where the generators are related by

$$y_1 = \sigma_1\sigma_2, \quad y_2 = \sigma_2^{-1}, \quad a = y_2 y_1^2, \quad b = y_1.$$

The Dubrovina-Dubrovin order λ_3 is the unique left order on B_3 which satisfies $y_1 >_{\lambda_3} e$ and $y_2 >_{\lambda_3} e$, equivalently $e <_{\lambda_3} a <_{\lambda_3} b$. To show the equivalence, assume $y_1 >_{\lambda_3} e$ and $y_2 >_{\lambda_3} e$. Then

$$a = y_2 y_1^2 >_{\lambda_3} e, \quad \text{and} \quad a^{-1}b = y_1^{-2} y_2^{-1} y_1 = y_1^{-2} y_2^{-1}(y_2 y_1^2 y_2) = y_2 >_{\lambda_3} e.$$

The converse is shown similarly.

Henceforth in this section we denote by G the braid group B_3 and by $\overline{G}$ its quotient by the center. Namely, we put

$$G = \langle a, b, t \mid a^2 = b^3 = t \rangle, \quad \overline{G} = \langle \alpha, \beta \mid \alpha^2 = \beta^3 = e \rangle,$$

and $q : G \to \overline{G}$ to be the surjective homomorphism satisfying $q(a) = \alpha$ and $q(b) = \beta$.

Let τ be the translation of $\mathbb{R}$ by 1, and $p : \mathbb{R} \to S^1 = \mathbb{R}/\langle \tau \rangle$ the canonical projection. Denote by $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ the group of all the homeomorphisms of $\mathbb{R}$ which commute with τ. It is the universal covering group of $\mathrm{Homeo}_+(S^1)$. Denote by $\pi : \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(S^1)$ the covering map. Let $\mathcal{H}^{\pm}$ be the set of the orientation and the base point x_0 preserving topological conjugacy classes of the homomorphisms $\rho : G \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ which are tight at x_0 and satisfy $\rho(t) = \tau^{\pm 1}$. Since t is cofinal for any left order, $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$ is identified with $LO(G)$ via dynamical realization (Corollary 2.4). Likewise denote by $\overline{\mathcal{H}}^{\pm}$ the set of the orientation and the base point $y_0 = p(x_0)$ preserving topological conjugacy classes of the homomorphisms $\overline{\rho} : \overline{G} \to \mathrm{Homeo}_+(S^1)$ which are tight at y_0 and satisfy $\mathrm{rot}(\overline{\rho}(\beta)) = \pm 1/3$. Then $\overline{\mathcal{H}} = \overline{\mathcal{H}}^+ \cup \overline{\mathcal{H}}^-$ is identified with $CO(\overline{G})$.

Define a map $q_* : \mathcal{H}^{\pm} \to \overline{\mathcal{H}}^{\pm}$ by $(q_*\rho)(\overline{g}) = \pi(\rho(g))$, where $\rho \in \mathcal{H}^{\pm}$, $\overline{g} \in \overline{G}$ and $g \in G$ is any element such that $q(g) = \overline{g}$. There is a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \\ \downarrow q & & \downarrow \pi \\ \overline{G} & \xrightarrow{q_*\rho} & \mathrm{Homeo}_+(S^1). \end{array}$$

Define a map $\pi^* : \overline{\mathcal{H}}^{\pm} \to \mathcal{H}^{\pm}$ for $\overline{\rho} \in \overline{G}$ by

- $(\pi^*\overline{\rho})(a)$ is the lift of $\overline{\rho}(\alpha)$ to $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ whose square is $\tau^{\pm 1}$, and
- $(\pi^*\overline{\rho})(b)$ is the lift of $\overline{\rho}(\beta)$ to $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ whose cube is $\tau^{\pm 1}$.

Also we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi^*\overline{\rho}} & \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \\ \downarrow q & & \downarrow \pi \\ \overline{G} & \xrightarrow{\overline{\rho}} & \mathrm{Homeo}_+(S^1). \end{array}$$

It is clear that q_* and π^* map the conjugacy classes to the conjugacy classes. We have $\pi^* q_* = q_* \pi^* = id$. Thus we obtain a bijection $q_* : LO(G) \to CO(\overline{G})$ and its inverse π^*.

We shall show the following theorem (Theorem 4 in the introduction).

THEOREM 9.1. *The map $q_* : LO(G) \to CO(\overline{G})$ is a homeomorphism.*

PROOF. Let us show that q_* is continuous. For any $\lambda \in LO(G)$, let $c = q_*(\lambda) \in CO(\overline{G})$. Choose arbitrary elements $\overline{g}_1, \ldots, \overline{g}_n$ of $\overline{G}$ and consider their configuration in S^1 with respect to c. This is the same as the configuration of $\overline{\rho}(\overline{g}_1)y_0, \ldots, \overline{\rho}(\overline{g}_n)y_0$ in S^1, where $\overline{\rho} \in \overline{\mathcal{H}}$ is a dynamical realization of c. Let $\rho = \pi^*(\overline{\rho}) \in \mathcal{H}$, a dynamical realization of λ. Choose $g_i \in G$ such that $q(g_i) = \overline{g}_i$ and $e \leq_\lambda g_i <_\lambda t$ $(1 \leq i \leq n)$. The configuration of $\rho(g_1)x_0, \ldots, \rho(g_n)x_0$ in $\mathbb{R}$ coincides with the configuration of $g_1, \cdots, g_n$ with respect to λ. Choose any $\lambda' \in LO(G)$ whose configuration of $e, g_1, \cdots, g_n, t$ is the same as λ. Then the configuration of $\overline{g}_1, \cdots, \overline{g}_n$ of $q_*(\lambda')$ is the same as c, showing the continuity of q_*. Thus the compact sets $LO(G)$ and $CO(\overline{G})$ are homeomorphic by q_*.

□

By virtue of the previous theorem, Theorem 3 in the introduction reduces to the following theorem. This is because any automorphism of G induces an automorphism of $\overline{G}$.

THEOREM 9.2. *There are isolated circular orders $c^{(k)} \in CO(\overline{G})$, $(k > 0, k \equiv \pm 1 \bmod 6)$ which are not the automorphic images of the others.*

The rest of this section is devoted to the proof of this theorem. There is an isomorphism $\iota : \overline{G} \cong PSL(2, \mathbb{Z})$ which satisfies

$$\iota(\alpha) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \iota(\beta) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let us define a homomorphism $\rho_M : \overline{G} \to \mathrm{Homeo}_+(S^1)$, called the *modular representation*, as the composite

$$\overline{G} \overset{\iota}{\cong} PSL(2, \mathbb{Z}) \subset PSL(2, \mathbb{R}) \subset \mathrm{Homeo}_+(S^1),$$

where the last inclusion is via the identification $S^1 \approx \mathbb{R} \cup \{\infty\}$.

For the dynamics of the modular representation ρ_M, see Figure 2. The open disk bounded by the circle is the Poincaré half plane $\mathbb{H}$. The element $\rho_M(\alpha)$ is the

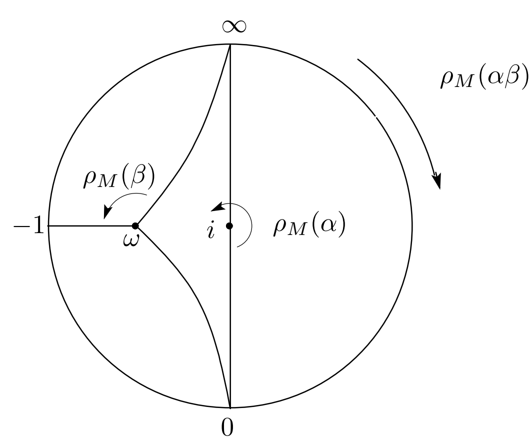

FIGURE 2.

1/2-rotation around i, and $\rho_M(\beta)$ the 1/3-rotation around $\omega = (-1+\sqrt{-3})/2$. The element $\rho_M(\alpha\beta)$ is a parabolic transformation which fixes the point 0 and moves points on $S^1 \setminus \{0\}$ clockwise, as is depicted in Figure 2. The Fuchsian group $\rho_M(\overline{G})$ is of the first kind, that is, the action ρ_M is minimal.

Let us define another Fuchsian representation[5] $\rho : \overline{G} \to \mathrm{Homeo}_+(S^1)$, a deformation of ρ_M. Choose a point ω' on the geodesic which passes through i and ω, but slightly farther than ω from i: $d(\omega', i) > d(\omega, i)$. See Figure 3. We set $\rho(\alpha)$ to be the same as $\rho_M(\alpha)$, the 1/2-rotation around i, and $\rho(\beta)$ the 1/3-rotation around ω'. We put the base point $y_0 = 0$. The Fuchsian group $\rho(\overline{G})$ is of the second kind. Its limit set $\mathcal{M}$ is the unique minimal set of the action ρ and is homeomorphic to a Cantor set. The fundamental domain P in Figure 3 is to be a closed disk in $\mathrm{Cl}(\mathbb{H}) = \mathbb{H} \cup S^1$. The translates of P tesselate $\mathrm{Cl}(\mathbb{H}) \setminus \mathcal{M}$, and therefore the translates of the interval $P \cap S^1$ tesselate $S^1 \setminus \mathcal{M}$. In particular y_0 (depicted as e

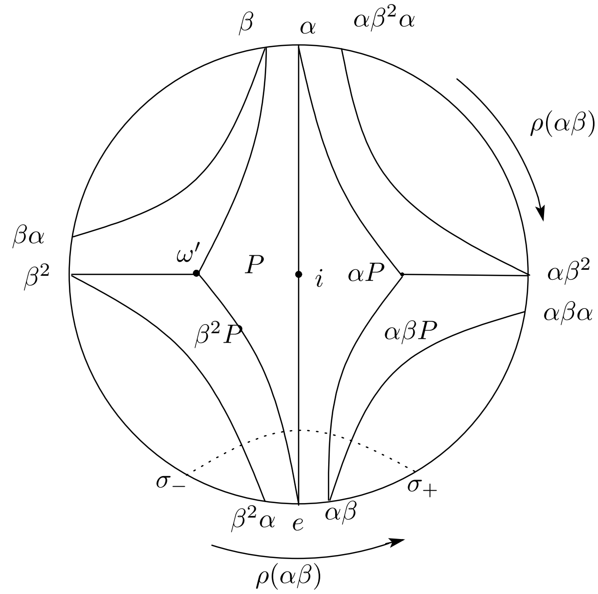

FIGURE 3. e indicates the base point $y_0 = 0$ and g indicates the point $\rho(g)y_0$. P is a fundamental domain of $\overline{\rho}(\overline{G})$.

in Figure 3) is contained in a gap of $\mathcal{M}$. The element $\rho(\alpha\beta)$ is hyperbolic whose axis is depicted by the dotted line in Figure 3. Let σ_- (σ_+) be the source (resp. sink) of $\rho(\alpha\beta)$. The open interval [6] $I_1 = (\sigma_-, \sigma_+)$ is a gap of $\mathcal{M}$ containing y_0. Since the translates of $P \cap S^1$ tesselate $S^1 \setminus \mathcal{M}$, there are no gaps of $\mathcal{M}$ other than the translates of I_1. This shows that ρ acts tightly at y_0. Thus ρ is a dynamical realization of a circular order $c \in CO(\overline{G})$. The linear part of c is the subgroup $\langle \alpha\beta \rangle$ by Lemma 8.9.

Let us show first of all that c is isolated. In [**10**], Proposition 3.3, the authors showed that the dynamical realization is continuous. More precisely, they showed the following.

[5] A Fuchsian representation is a discrete faithful representation into $PSL(2, \mathbb{R})$. We fix an inclusion $PSL(2, \mathbb{R}) \subset \mathrm{Homeo}_+(S^1)$, and consider it as a representation into $\mathrm{Homeo}_+(S^1)$.

[6] Given two points $x, y \in S^1$, we define $(x, y) = \{t \in S^1 \mid x \prec t \prec y\}$, where $\prec$ is the anticlockwise circular order of S^1.

Proposition 9.3. *Given any neighbourhood U of ρ in $\mathrm{Hom}(\overline{G}, \mathrm{Homeo}_+(\mathrm{S}^1))$, there is a neighbourhood V of c in $CO(\overline{G})$ such that any element in V has a conjugate of its dynamical realization contained in U.*

Let $\gamma_1 = \beta^2\alpha\beta\alpha$ and $\gamma_2 = \alpha\beta\alpha\beta^2$. (They generate the commutator subgroup of $\overline{G}$, which is of index 6. But we do not use this.) Below we indicate the point $\rho(g)y_0$ simply by g. Let

$$K_1^- = [\alpha\beta^2, \alpha\beta^2\alpha], \quad K_1^+ = [\beta^2, \beta^2\alpha], \quad K_2^- = [\beta, \beta\alpha], \quad K_2^+ = [\alpha\beta, \alpha\beta\alpha].$$

Then $\gamma_1(S^1 \setminus K_1^-) = \mathrm{Int}(K_1^+)$ and $\gamma_2(S^1 \setminus K_2^-) = \mathrm{Int}(K_2^+)$. Define open intervals J_1^-,

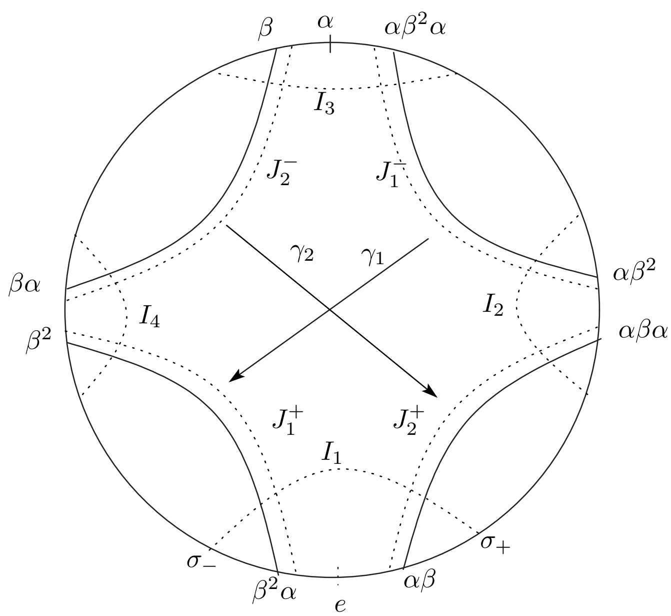

Figure 4. $\gamma_1 = \beta^2\alpha\beta\alpha$ maps $\alpha\beta^2\alpha$ to β^2 and $\alpha\beta^2$ to $\beta^2\alpha$. $\gamma_2 = \alpha\beta\alpha\beta^2$ maps β to $\alpha\beta\alpha$ and $\beta\alpha$ to $\alpha\beta$.

J_1^+, J_2^- and J_2^+, slightly bigger than K_1^-, K_1^+, K_2^- and K_2^+. See Figure 4. Recall the interval I_1, the gap of $\mathcal{M}$ containing y_0, and let $I_2 = \rho(\alpha\beta^2)I_1$, $I_3 = \rho(\alpha)I_1$ and $I_4 = \rho(\beta\alpha)I_1$ (Figure 4). They are gaps of the minimal set $\mathcal{M}$ and the orbit $\rho(\overline{G})y_0$ is discrete in each of them. Especially the point $\alpha\beta$ (resp. $\alpha\beta\alpha$) is the leftmost (resp. rightmost) point of $\rho(\overline{G})y_0 \cap J_2^+$. More generally, points of $\partial K_i^\pm$ are extremal in $\rho(\overline{G})y_0 \cap J_i^\pm$. They are called the *gardians* of the interval with respect to ρ. Notice that there are just two points of $\rho(\overline{G})y_0$ outside of $\bigcup_{i,\pm} J_i^\pm$, namely y_0 and $\rho(\alpha)y_0$ (denoted by e and α in Figure 4).

Define a nighbourhood U of ρ in $\mathrm{Hom}(\overline{G}, \mathrm{Homeo}_+(\mathrm{S}^1))$ such that each element $\rho' \in U$ satisfies the following conditions.
(1) $\rho'(\gamma_i)$ maps the closed set $S^1 \setminus J_i^-$ into the open set J_i^+ $(i = 1, 2)$.
(2) The configuration of $\rho'(g)y_0$ in S^1 for ten elements

$$g = e, \alpha\beta, \alpha\beta\alpha, \alpha\beta^2, \alpha\beta^2\alpha, \alpha, \beta, \beta\alpha, \beta^2, \beta^2\alpha \tag{9.1}$$

is the same as for ρ (Figure 4), as well as their configuration with respect to the four intervals $J_i^\pm$.

Take a nighbourhood V of c as in Proposition 9.3, and for any $c' \in V$, let ρ' be a conjugate of a dynamical realization of c' which is contained in U. Then by the ping-pong argument, the circular order of the orbit $\rho'(\overline{G})y_0$ is uniquely determined. Let us show this a bit in detail. Call ten points $\rho'(g)y_0$ (g as in (9.1)) *of the first generation*. The images of points of the first generation by $\gamma_i^{\pm 1}$ which are not themselves of first generation are called *of second generation*. Then the configuration of the points of first and second generations are uniquely determined. In fact, the gardians $\rho'(\alpha\beta^2)y_0$ and $\rho'(\alpha\beta^2\alpha)y_0$ of the interval J_1^- is mapped by $\rho'(\gamma_1)$ to the gardians $\rho'(\beta^2\alpha)y_0$ and $\rho'(\beta^2)y_0$ of J_1^+, and all the other eight points are mapped into the interval in J_1^+ boounded by the latter gardians. The same is true for γ_1^{-1} and $\gamma_2^{\pm 1}$. Since $\gamma_i^{\pm 1}$ are orientation preserving, the configuration of the points of the first and second generation is uniquely determined. Next we define points of third generation in a similar way. These points are contained in $\bigcup_{i,\pm} J_i^{\pm}$. For example, those contained in J_1^+ are the images of the points of second generation in $S^1 \setminus J_1^-$ by $\rho'(\gamma_1)$. The cofiguration of these points, together with the points of first and second generations, is uniquely determined. Continuing this way, we see that the natural circular order of the whole orbit $\rho'(\overline{G})y_0$ is uniquely determined, that is, the same as $\rho(\overline{G})y_0$. This shows that $c' = c$, i.e, c is isolated. Define $c^{(1)}$ in Theorem 9.2 to be this c.

For $k > 1$, denote by $p_k : S^1 \to S^1$ the k-fold covering map. A representation $\rho^{(k)} : \overline{G} \to \mathrm{Homeo}_+(S^1)$ is called a *k-fold lift* of ρ if $p_k\rho^{(k)}(g) = \rho(g)p_k$ holds for any $g \in \overline{G}$. There is a k-fold lift $\rho^{(k)}$ of our representaion ρ if and only if $k \equiv \pm 1$ mod 6, and it is unique if it exists. Computation shows that if $k = 6\ell \pm 1$, then

$$\mathrm{rot}(\rho^{(k)}(\alpha\beta)) = \mp \ell/k. \tag{9.2}$$

Notice that $(k,\ell) = 1$. We fix such k.

Let y_0^μ, $1 \le \mu \le k$, be the lifts of the point y_0 by p_k. The the natural circular order of the orbit $\rho^{(k)}(\overline{G})y_0^\mu$ in S^1 is the same for any μ. Denote it by $c^{(k)} \in CO(\overline{G})$. Let us show that $c^{(k)}$ is isolated. Let $J_{i,\mu}^{\pm}$ ($\mu = 1, \dots, k$) be the connected components of $p_k^{-1}(J_i^{\pm})$.

Define a nighbourhood $U^{(k)}$ of $\rho^{(k)}$ such that each element $\rho' \in U^{(k)}$ satisfies the following conditions.

(1) $\rho'(\gamma_i)$ maps each component of $S^1 \setminus \bigcup_\mu J_{i,\mu}^-$ into $J_{i,\nu}^+$ ($i = 1, 2$), where ν is determined so that $\rho^{(k)}(\gamma_i)$ maps the same component into $J_{i,\nu}^+$.

(2) The configuration of $10k$ points $\rho'(g)y_0^\mu$ in S^1 (g as in (9.1) and $1 \le \mu \le k$) is the same as $\rho^{(k)}$. Their configuration relative to $J_{i,\mu}^{\pm}$ is also the same.

Then the same ping-pong argument shows that the natural circular order of $\rho'(\overline{G})(p_k^{-1}(y_0))$ for $\rho' \in U^{(k)}$ is uiquely determined. In particular, the natural circular order of $\rho'(\overline{G})y_0^\mu$ is the same as for $\rho^{(k)}$, showing that $c^{(k)}$ is isolated.

Finally let us show that $c^{(k)}$'s are not the automorphic images of the others, by considering their linear parts. In $\overline{G} = PSL(2,\mathbb{Z})$, any element of infinite order is a multiple of a unique primitive element. This can be shown by considering the modular representaion ρ_M: the fixed point set of any element of infinite order is either one point of $\partial\mathbb{H}$ or a two point set of $\partial\mathbb{H}$, and the isotropy group of the fixed point set is infinite cyclic.

As we have seen above, the linear part of $c^{(1)}$ is generated by a primitive element $(\alpha\beta)^{\pm 1}$. The equality (9.2) shows that the linear part of $c^{(k)}$ is generated by $(\alpha\beta)^{\pm k}$. For different choices of k and k', there is no automorphism of $\overline{G}$ which maps $(\alpha\beta)^{\pm k}$ to $(\alpha\beta)^{\pm k'}$. This finishes the proof of Theorem 9.2.

REMARK 9.4. The left order $\lambda = \pi^* c^{(1)} \in LO(G)$ is the Dubrovina-Dubrovin order, since it satisfies $e <_\lambda a <_\lambda b$. It can be shown that $\lambda' = \pi^* c^{(5)}$ is the unique left order which satisfies

$$(ab)^5 t^{-4} <_{\lambda'} e <_{\lambda'} a <_{\lambda'} (ab)^5 t^{-4} a.$$

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