Flat structure on the space of isomonodromic deformations

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October 3, 2017

Abstract

Flat structure was introduced by K. Saito and his collaborators at the end of 1970's. Independently the WDVV equation arose from the 2D topological field theory. B. Dubrovin unified these two notions as the Frobenius manifold structure. In this paper we treat Saito structure (without metric) by C. Sabbah which is one of generalizations of Frobenius manifold structure. We connect Saito structure (without metric) with isomonodromic deformations of Okubo systems which are a special kind of systems of linear differential equations. As advantages of the introduction of Okubo systems, the following results are obtained: (I) introduction of flat coordinates on the orbit spaces of well-generated finite complex reflection groups (II) establishment of a correspondence between solutions satisfying certain semisimplicity condition to the three dimensional extended WDVV equation and generic solutions to the sixth Painlevé equation (III) explicit description of potential vector fields corresponding to algebraic solutions to the sixth Painlevé equation.

AMS 2010 Subject Classification: 34M56, 33E17, 35N10, 32S25.

Key words: flat structure, Frobenius manifold, WDVV equation, complex reflection group, Painlevé equation.

1 Introduction

At the end of 1970's, K. Saito introduced the notion of flat structure in order to study the structure of universal unfolding of isolated hypersurface singularities. Independently the WDVV equation (Witten-Dijkgraaf-Verlinde-Verlinde equation) arises from the 2D topological field theory [15, 62]. B. Dubrovin unified both the flat structure formulated by K. Saito and the WDVV equation as Frobenius manifold structure. Dubrovin not only formulated the notion of Frobenius manifold but also studied its relationship with isomonodromic deformations of linear differential equations with certain symmetries. Particularly he derived a one-parameter family of Painlevé VI equation from three-dimensional massive (i.e. regular semisimple) Frobenius manifolds. Since then, there are several generalizations of Frobenius manifolds such as F-manifold by C. Hertling and Y. Manin [22, 21] and Saito structure (without metric) by C. Sabbah [51]. Concerning the relationship with the Painlevé equation, A. Arsie and P. Lorenzoni [1, 43] showed that three-dimensional regular semisimple bi-flat F-manifolds are parameterized by generic solutions to the (full-parameter) Painlevé VI equation, which is regarded as an extension of Dubrovin's result. Furthermore Arsie-Lorenzoni [2] showed that three-dimensional regular non-semisimple bi-flat F-manifolds are parameterized by generic solutions to the Painlevé V and IV equations. (Recently it was proved in [3, 41] that Arsie-Lorenzoni's bi-flat F-manifold is equivalent to Sabbah's Saito structure (without metric).)

The theory of linear differential equations on a complex domain is a classical branch of mathematics. In recent years there has been a great progress in this branch. One of its turning points is the introduction of the notions of *middle convolution* and *rigidity index* by N. M. Katz [33]. With the help of his idea, T. Oshima developed a classification theory of Fuchsian differential equations in terms of their rigidity indices and spectral types [49, 50, 48]. ([19] is a nice introductory text on the "Katz-Oshima theory".) *Okubo systems* play a central role in these developments (cf. [13, 14, 50, 64]): a matrix system of linear differential equations with the form

$$(z - T)\frac{dY}{dz} = -B_\infty Y, \tag{1}$$

where T, B_∞ are constant square matrices, is said to be an Okubo system if T is diagonalizable. Particularly, any Okubo system is Fuchsian. One important feature of an Okubo system is that it keeps its form under the operation of the *Euler transformation* (cf. Remark 2.1):

$$Y(z) \to Y^{(\lambda)}(z) := \frac{1}{\Gamma(\lambda)} \int (u - z)^{\lambda-1} Y(u) du, \quad \lambda \in \mathbf{C}.$$

Okubo system was introduced with the following motivations:

- A system of differential equations in Birkhoff normal form (which has an irregular singularity of Poincaré rank one at 0 and a regular singularity at ∞ on $\mathbf{P}^1(\mathbf{C})$) is transformed into an Okubo system by a Fourier-Laplace transformation and thus the study on Stokes matrices of a Birkhoff normal form can be reduced to that on connection matrices of the corresponding Okubo system ([4, 55]).

- Okubo system is one of generalizations of Gauss hypergeometric equation and one may expect that it would provide new special functions possessing rich properties ([45, 63, 64]).

Let us consider a regular semisimple Saito structure (without metric). (See Section 4 for the definition and properties of Saito structure (without metric). In the sequel, we abbreviate Saito structure (without metric) to Saito structure for brevity.) It is known that there are following two types of meromorphic connections associated with a Saito structure (cf. Remarks 4.2 and 4.3):

(i) One induces a universal integrable deformation of a Birkhoff normal form.

(ii) The other induces a universal integrable deformation of an Okubo system.

These two meromorphic connections are called in [21] (in the case of a Frobenius manifold) *the first structure connection* and *the second structure connection* respectively and mutually equivalent since they are transformed to each other by Fourier-Laplace transformations. The first connection (i) is used in many preceding literatures (e.g. [16, 51]). However in this paper we use the second connection (ii) because results in the recent developments on linear differential equations mentioned above can be exploited. On this standpoint, a regular semisimple Saito structure yields a universal integrable deformation of a regular Okubo system. In this paper, we show that the opposite of this statement is almost always true. Namely, for a universal integrable deformation of a regular Okubo system satisfying some generic condition, there exists a Saito structure which yields the given universal integral deformation as the second connection (ii) (Theorem 5.4). A rough picture of the result in this paper is

$$\begin{aligned}&\text{regular semisimple Saito structures}\\ \iff\ &\text{universal integrable deformations of regular Okubo systems.}\end{aligned} \tag{2}$$

Here it should be remarked that a universal integrable deformation of a regular Okubo system turns out to be an isomonodromic deformation (cf. Remark 2.2). In this paper, the following results are obtained as consequences of (2):

(I) introduction of flat coordinates on the orbit spaces of well-generated finite complex reflection groups (Theorem 6.2),

(II) correspondence between solutions satisfying certain semisimplicity condition to the three-dimensional extended WDVV equation and generic solutions to the Painlevé VI equation (Corollary 5.5),

(III) explicit descriptions of potential vector fields corresponding to algebraic solutions to the Painlevé VI equation (Section 7).

K. Saito and his collaborators [53, 54] defined and constructed flat coordinates on the orbit spaces of finite real reflection groups. To extend them to finite (non-real) complex reflection groups has been a long-standing problem. (I) provides an answer to this problem for well-generated complex reflection groups (see also [29]). Recently this result has been crucially used to prove the freeness of multi-reflection arrangements of complex reflection groups in [25]. (II) is essentially equivalent to A. Arsie and P. Lorenzoni's result in [1, 43], however the argument based on the picture (2) makes clear the relationship between Saito structures and the theory of isomonodromic deformations. (III) provides many concrete examples of three-dimensional algebraic Saito structures that are not Frobenius manifolds, which would be the first step toward classification of three-dimensional algebraic Saito structures and/or algebraic F-manifolds (cf. [21]), see also [29, 30, 31]. (In this paper, the term "flat structure" stands for the same meaning as "Saito structure" because it is a natural extension of K. Saito's flat structure, which would be justified by the result (I) above.)

This paper is constructed as follows. In Section 2, we start with an Okubo system. Then we introduce the notion of *Okubo system in several variables* as an integrable Pfaffian system extending the Okubo system (Definition 2.3). An Okubo system in several variables is equivalent to an isomonodromic deformation of an Okubo system (Remark 2.2). We see that an Okubo system in several variables is a universal integrable deformation of an Okubo system, which is uniquely determined up to changes of independent variables by the given Okubo system (Remark 2.3). In Section 3, we study the structure of logarithmic vector fields along a divisor defined by a monic polynomial of degree n

$$h(x) = h(x', x_n) = x_n^n - s_1(x')x_n^{n-1} + \cdots + (-1)^n s_n(x') \tag{3}$$

where $x' = (x_1, \cdots, x_{n-1})$ and each $s_i(x')$ is holomorphic with respect to x', which appears as the defining equation of the singular locus of an Okubo system in several variables. In K. Saito's construction of flat structures on the orbit spaces of real reflection groups, the fact that the discriminants of real reflection groups have the form of (3) was crucially used (cf. [53]). The results in this section are used in order to generalize Saito's construction in Section 6. In Section 4, we review the general theory of Saito structure. Particularly we introduce an extension of the WDVV equation (Proposition 4.8, Definition 4.9), which is not mentioned in [51] explicitly. A solution to the extended WDVV equation is called a potential vector field, which completely describes a Saito structure (Proposition 4.10). This is an extension to Saito structure of the fact that a Frobenius manifold is completely described by its prepotential, which is a solution to the

WDVV equation. In Section 5, we give a precise statement of the picture (2). Namely, we show that an Okubo system in several variables arises from a Saito structure if and only if its flat coordinate is non-degenerate in some sense (Theorem 5.4). For uniqueness of the Saito structure corresponding to an Okubo system in several variables, see Remark 5.1. Combining this result and the argument in Appendix A, we see that there is a correspondence between solutions satisfying certain semisimplicity condition to the three-dimensional extended WDVV equation and generic solutions to the (full-parameter) Painlevé VI equation (Corollary 5.5). In Section 6, we treat a problem on the existence of a flat coordinate system on the orbit space of an irreducible complex reflection group. In the case of a real reflection group, K. Saito [53] (see also [54]) proved the existence of a flat coordinate system on the orbit space based on the existence of a flat invariant metric on the standard representation space. In this paper, instead of the flat invariant metric, we construct a special type of Okubo systems in several variables called *G-quotient system*, whose fundamental system of solutions consists of derivatives by logarithmic vector fields of linear coordinates on the standard representation space and its monodromy group is isomorphic to the finite complex reflection group G (Theorem 6.2). Then as a consequence of the picture (2), we have a Saito structure on the orbit space corresponding to the G-quotient system. Actually, our definition of flat coordinates coincides with that of Saito when it is restricted to real reflection groups. Here it is remarkable that our proof of Theorem 6.2 is constructive i.e. it contains an algorithm of explicit computation on the flat generator system of G-invariants and the potential vector field for a given well-generated complex reflection group G. Besides, it is proved that the potential vector field corresponding to a well-generated complex reflection group has polynomial entries (Corollary 6.8). In Section 7, we treat algebraic solutions to the Painlevé VI equation. Algebraic solutions to the Painlevé VI equation were studied and constructed by many authors including N. J. Hitchin [23, 24], B. Dubrovin [16], B. Dubrovin -M. Mazzocco [18], P. Boalch [6, 7, 8, 9, 10], A. V. Kitaev [37, 38], A. V. Kitaev -R. Vidūnas [39, 61], K. Iwasaki [26]. The classification of algebraic solutions to the Painlevé VI equation was achieved by Lisovyy and Tykhyy [42]. We give some examples of potential vector fields corresponding to algebraic solutions to the Painlevé VI equation. Some other examples and related topics are found in [29, 30, 31]. In Appendix A, we give a proof of that an Okubo system in several variables is equivalent to an isomonodromic deformation of an Okubo system. Especially, we show that there is a correspondence between Okubo systems in several variables of rank three and generic solutions to the Painlevé VI equation with the help of the result of Jimbo and Miwa [28] (Proposition A.1). In Appendix B, we explain a method of constructing an Okubo system in several variables of rank n from a completely integrable Pfaffian system of rank $(n-1)$. This construction is used in

Appendix A.

We close this introduction with addressing some problems. The first one is to construct algebraic solutions to higher order Painlevé equations. As stated above, a solution to the extended WDVV equation yields an isomonodromic deformation of an Okubo system. Recently higher order Painlevé equations are obtained by several authors (e.g. [36, 59]) based on the classification theory on linear differential equations due to Oshima. Starting from polynomial potential vector fields corresponding to (real and complex) reflection groups of higher rank, it may be expected that one obtain algebraic solutions to those higher order Painlevé equations. Those solutions may be parametrized by higher-dimensional algebraic varieties. The second one is related with the theory of unfolding of isolated hypersurface singularities. Originally, the view of K. Saito is that flat coordinates arise from versal deformations of isolated hypersurface singularities. Let us focus our attention to the case of ADE singularities. One can define flat coordinates of ADE type from versal deformation of ADE singularities. Then the following question naturally arises: Can we relate flat coordinates for complex reflection groups with "non-versal families" of hypersurface singularities? The third one is related with potential vector fields having polynomial entries. Classification of polynomial prepotentials were conjectured by Dubrovin [16] and proved by Hertling [21]: Any polynomial prepotential comes from the Frobenius manifold structure on the orbit space of a real reflection group. Surprisingly enough, there are some examples in Section 7 whose potential vector fields have polynomial entries but the corresponding flat structures are not isomorphic to one on the orbit space of any complex reflection group. To classify polynomial potential vector fields would be an interesting problem.

Acknowledgments. Professor Yoshishige Haraoka taught the first author (M.K.) that integrable systems in three variables are useful to derive the Painlevé VI solutions. This is the starting point of our work. The authors would like to thank Professor Haraoka for his advice. After a preprint of this article was written, the authors were informed with useful comments by Professors B. Dubrovin, Y. Konishi, C. Hertling, P. Boalch, A. Arsie and P. Lorenzoni the existence of the papers [17, 12, 1, 2, 43, 44, 6]. The authors express their sincere gratitude to these people. This work was supported by JSPS KAKENHI Grant Numbers 25800082, 17K05335, 26400111, 17K05269.

2 Extension of Okubo systems to several variables case

In this section, we start with introducing a special type of a system of ordinary linear differential equations which is called an Okubo system. Then we introduce an extension of an Okubo system to several variables case as a completely integrable Pfaffian system which is called an *Okubo system in several variables* (Definition 2.3). An Okubo system in several variables is equivalent to an isomonodromic deformation of an Okubo system as we will show in Appendix A (cf. Remark 2.2) and it turns out to be a universal integrable deformation of an Okubo system (Remark 2.3).

Let T and B_∞ be $n \times n$-matrices. If T is a diagonalizable matrix, the system of ordinary linear differential equations

$$(zI_n - T)\frac{dY}{dz} = -B_\infty Y \tag{4}$$

is called a *system of differential equations of Okubo type* or shortly an *Okubo system* ([45]).

The aim in this section is to extend (4) to a completely integrable Pfaffian system of several variables in the form

$$dY = \Big(B^{(z)}dz + \sum_{i=1}^{n} B^{(i)}dx_i\Big)Y, \tag{5}$$

where $B^{(z)}$ and $B^{(i)}$ are $n \times n$ matrices whose entries depend on (z, x). As usual, $x = (x_1, \dots, x_n)$ denotes a coordinate of $\mathbb{C}^n$. We assume the existence of $n \times n$ matrices $T = T(x)$ and B_∞ so that $B^{(z)} = -(zI_n - T)^{-1}B_\infty$ and satisfy the conditions:

(A1) the entries of $T(x)$ are holomorphic functions on a domain U in $\mathbb{C}^n$,

(A2) $B_\infty = \mathrm{diag}[\lambda_1, \dots, \lambda_n]$, where λ_i are constant complex numbers satisfying $\lambda_i - \lambda_j \notin \mathbb{Z} \setminus \{0\}$ for $i \neq j$.

Let $H(x, z) = \det(zI_n - T(x))$ be the characteristic polynomial of $T(x)$, which is a monic polynomial in z of degree n and analytic in x: $H(x, z) = z^n - S_1(x)z^{n-1} + \cdots + (-1)^n S_n(x)$. We assume the following condition on $H(x, z)$:

(A3) $\det\Big(\dfrac{\partial S_j(x)}{\partial x_i}\Big)_{i,j=1,\dots,n} \neq 0$ at generic points of U.

It follows from (A3) that the discriminant $\delta_H(x) = \prod_{i<j}(z_i(x) - z_j(x))^2$ of $H(x, z) = \prod_{i=1}^{n}(z - z_i(x))$ is not identically zero and we define a divisor $\Delta_H = \{\delta_H(x) = 0\} \cup \{\det\big(\frac{\partial S_j(x)}{\partial x_i}\big) = 0\}$ on U. Taking a smaller domain $W \subset U \setminus \Delta_H$ appropriately, the eigenvalues $z_1(x), \dots, z_n(x)$ of T can be considered single-valued holomorphic functions in x on W by fixing their branches and we can take an invertible matrix $P = P(x)$ whose

entries are single-valued holomorphic functions on W such that

$$P^{-1}TP = \mathrm{diag}\,[z_1(x), \dots, z_n(x)]. \tag{6}$$

We also assume

(A4) $H(x,z)B^{(i)} \in (\mathcal{O}_U \otimes_{\mathbb{C}} \mathbb{C}[z])^{n\times n}$, where $\mathcal{O}_U$ denotes the ring of holomorphic functions on U.

Decompose $B^{(z)}$ into partial fractions on W

$$B^{(z)} = \sum_{i=1}^{n} \frac{B_i^{(z)}}{z - z_i(x)}, \tag{7}$$

where $B_i^{(z)}$ $(i = 1, \dots, n)$ are independent of z. In addition to (A1)-(A4), we assume the condition:

(A5) for each of $B_i^{(z)}$ $(i = 1, \dots, n)$, $r_i := \mathrm{tr}\, B_i^{(z)} \neq \pm 1$ on $U \setminus \Delta_H$.

Lemma 2.1. *Assume that B_∞ is invertible. If the Pfaffian system (5) is completely integrable, then there are $n \times n$ matrices $\tilde{B}^{(i)}, B_\epsilon^{(i)} \in \mathcal{O}_U^{n\times n}$ $(i = 1, \dots, n)$ such that*

$$B^{(i)} = -(zI_n - T(x))^{-1}\tilde{B}^{(i)}B_\infty + B_\epsilon^{(i)}, \qquad i = 1, \dots, n, \tag{8}$$

and that $B_\epsilon^{(i)} = \dfrac{\partial \mathcal{E}}{\partial x_i}\mathcal{E}^{-1}$ in terms of a matrix $\mathcal{E}$ whose entries are holomorphic functions on U. If $\lambda_i \neq \lambda_j$ for any $i \neq j$, the matrix $\mathcal{E}$ turns out to be a diagonal matrix $\mathcal{E} = diag[\epsilon_1, \dots, \epsilon_n]$. Moreover, it holds that

$$P^{-1}TP = diag\,[z_1(x), \dots, z_n(x)], \tag{9}$$

$$P^{-1}\tilde{B}^{(i)}P = diag\left[-\frac{\partial z_1(x)}{\partial x_i}, \dots, -\frac{\partial z_n(x)}{\partial x_i}\right], \qquad i = 1, \dots, n, \tag{10}$$

on W. In particular, $T, \tilde{B}^{(i)}$ $(i = 1, \dots, n)$ are mutually commutative:

$$[T, \tilde{B}^{(i)}] = O, \quad [\tilde{B}^{(i)}, \tilde{B}^{(j)}] = O \quad (\forall i, j). \tag{11}$$

Proof. In virtue of the definition of $B^{(z)}$, we can write

$$H(x,z)B^{(z)} = \sum_{i=0}^{n-1} (HB^{(z)})_i z^i, \tag{12}$$

where

$$(HB^{(z)})_i \in \mathcal{O}_U^{n\times n}, \quad (HB^{(z)})_{n-1} = -B_\infty.$$

In virtue of the assumption (A4), we can write

$$H(x,z)B^{(i)} = \sum_{j=0}^{m_i} (HB^{(i)})_j z^j, \qquad i=1,\dots,n. \tag{13}$$

As the first step, we are going to show $m_i \le n$. From the integrability condition of (5), we have

$$H\frac{\partial (HB^{(i)})}{\partial z} - \frac{\partial H}{\partial z} HB^{(i)} + [HB^{(i)}, HB^{(z)}] = H\frac{\partial (HB^{(z)})}{\partial x_i} - \frac{\partial H}{\partial x_i} HB^{(z)}, \quad i=1,\dots,n. \tag{14}$$

The equation (14) combined with (12) and (13) implies that the left hand side of the resulting equation is a polynomial in z of degree m_i+n-1 and its right hand side is of degree $2n-2$. Besides, the coefficient of the term z^{m_i+n-1} in the left hand side reads

$$(m_i - n)(HB^{(i)})_{m_i} + [(HB^{(i)})_{m_i}, (HB^{(z)})_{n-1}].$$

As a consequence we find that $m_i \le n$, because $(HB^{(z)})_{n-1} = -B_\infty$ is diagonal and $\lambda_i - \lambda_j \notin \mathbb{Z}\setminus\{0\}$ for $i \ne j$. Then $B^{(i)}$ $(i=1,\dots,n)$ is decomposed into partial fractions as follows on W:

$$B^{(i)} = \sum_{j=1}^{n} \frac{B_j^{(i)}}{z-z_j} + B_\epsilon^{(i)}, \tag{15}$$

where $B_\epsilon^{(i)} = (HB^{(i)})_n$ is a matrix which is defined on U. If $\lambda_i \ne \lambda_j$ for any $i \ne j$, we see that $(HB^{(i)})_n$ is a diagonal matrix since $[(HB^{(i)})_n, (HB^{(z)})_{n-1}] = O$. Paying attention to the value at $z=\infty$ of the equation obtained by substituting (15) for the integrability condition of the Pfaffian system (5), we have

$$\frac{\partial B_\epsilon^{(i)}}{\partial x_j} - \frac{\partial B_\epsilon^{(j)}}{\partial x_i} + [B_\epsilon^{(i)}, B_\epsilon^{(j)}] = O \quad (i,j=1,\dots,n).$$

It means that $B_\epsilon^{(i)}$ is written by a matrix $\mathcal{E}$ as $B_\epsilon^{(i)} = \dfrac{\partial \mathcal{E}}{\partial x_i}\mathcal{E}^{-1}$.

As the second step, we shall show the equalities

$$B_j^{(i)} = -\frac{\partial z_j}{\partial x_i} B_j^{(z)}, \qquad i,j=1,\dots,n, \tag{16}$$

for the residue matrices $B_i^{(z)}$, $B_i^{(j)}$ in (7) and (15). We have

$$-B_j^{(i)} - B_j^{(z)}\frac{\partial z_j}{\partial x_i} + [B_j^{(i)}, B_j^{(z)}] = O, \qquad i,j=1,\dots,n, \tag{17}$$

by substituting (7) and (15) for the integrability condition. Now fix $j \in \{1,\dots,n\}$. Since $\operatorname{rank}(B_j^{(z)}) \le 1$ and $\operatorname{trace}(B_j^{(z)}) = r_j$, we have

$$(B_j^{(z)})^2 = r_j\, B_j^{(z)}. \tag{18}$$

From (17) and (18), we have

$$
\begin{aligned}
B_j^{(z)}B_j^{(i)} &= -r_j B_j^{(z)}\frac{\partial z_j}{\partial x_i} + B_j^{(z)}B_j^{(i)}B_j^{(z)} - r_j B_j^{(z)}B_j^{(i)},\\
B_j^{(i)}B_j^{(z)} &= -r_j B_j^{(z)}\frac{\partial z_j}{\partial x_i} + r_j B_j^{(i)}B_j^{(z)} - B_j^{(z)}B_j^{(i)}B_j^{(z)},\\
B_j^{(z)}B_j^{(i)}B_j^{(z)} &= -r_j^2 B_j^{(z)}\frac{\partial z_j}{\partial x_i}.
\end{aligned}
$$

Then, since $r_j \neq \pm 1$, it holds $B_j^{(z)}B_j^{(i)} = B_j^{(i)}B_j^{(z)}$, which and (17) again, imply the equalities (16).

As the last step, we take a matrix P that satisfies (9). Then

$$
P^{-1}B^{(z)}P = -\mathrm{diag}\big[z - z_1(x), \ldots, z - z_n(x)\big]^{-1}P^{-1}B_\infty P,
$$

and from (16), we have

$$
P^{-1}B^{(i)}P = -\mathrm{diag}\big[z - z_1, \ldots, z - z_n\big]^{-1}\mathrm{diag}\left[-\frac{\partial z_1}{\partial x_i}, \ldots, -\frac{\partial z_n}{\partial x_i}\right]P^{-1}B_\infty P + P^{-1}B_\epsilon^{(i)}P.
$$

Hence we obtain

$$
B^{(i)} = -(zI_n - T)^{-1}\tilde{B}^{(i)}B_\infty + B_\epsilon^{(i)}
$$

for

$$
\tilde{B}^{(i)} = P\mathrm{diag}\left[-\frac{\partial z_1(x)}{\partial x_i}, \ldots, -\frac{\partial z_n(x)}{\partial x_i}\right]P^{-1}.
$$

This proves (9), (10) and thus (11) on W. Since B_∞ is invertible, we find that $\tilde{B}^{(i)} \in \mathcal{O}_U^{n\times n}$. In virtue of the identity theorem, (11) holds on U. We have thus proved the lemma completely. □

In virtue of Lemma 2.1, we restrict ourselves to the case $B_\epsilon^{(i)} = O$ without loss of generality by applying a gauge transformation to Y. Namely we consider the Pfaffian system

$$
dY = \Big(B^{(z)}dz + \sum_{i=1}^{n} B^{(i)}dx_i\Big)Y \tag{19}
$$

with $B^{(z)} = -(zI_n - T)^{-1}B_\infty$, $B^{(i)} = -(zI_n - T)^{-1}\tilde{B}^{(i)}B_\infty$ $(i = 1, \ldots, n)$.

Lemma 2.2. *If T, $\tilde{B}^{(i)}$ $(i = 1, \ldots, n)$, B_∞ in (19) satisfy the equations*

$$
[T, \tilde{B}^{(i)}] = O, \quad [\tilde{B}^{(i)}, \tilde{B}^{(j)}] = O \quad (\forall i, j), \tag{20}
$$

$$
\frac{\partial T}{\partial x_i} + \tilde{B}^{(i)} + [\tilde{B}^{(i)}, B_\infty] = O, \qquad i = 1, \ldots, n, \tag{21}
$$

$$
\frac{\partial \tilde{B}^{(i)}}{\partial x_j} - \frac{\partial \tilde{B}^{(j)}}{\partial x_i} = O, \quad i, j = 1, \ldots, n, \tag{22}
$$

then the Pfaffian system (19) is completely integrable. Moreover, in case where $\lambda_i \neq 0$ $(i = 1, \ldots, n)$*, the system of equations* (20), (21), (22) *is necessary and sufficient condition for that (19) is completely integrable.*

Proof. The lemma is clear from the relations

$$\frac{\partial B^{(i)}}{\partial z} - \frac{\partial B^{(z)}}{\partial x_i} + [B^{(i)}, B^{(z)}] = (zI_n - T)^{-1} \left(\frac{\partial T}{\partial x_i} + \tilde{B}^{(i)} + [\tilde{B}^{(i)}, B_\infty] \right) (zI_n - T)^{-1} B_\infty,$$

and

$$\begin{aligned}
&(zI_n - T)\Big(\frac{\partial B^{(i)}}{\partial x_j} - \frac{\partial B^{(j)}}{\partial x_i} + [B^{(i)}, B^{(j)}] \Big) \\
=&\Bigg(\Big(\frac{\partial \tilde{B}^{(j)}}{\partial x_i} - \frac{\partial \tilde{B}^{(i)}}{\partial x_j} \Big)(zI_n - T) + \Big(\frac{\partial T}{\partial x_i} + \tilde{B}^{(i)} + [\tilde{B}^{(i)}, B_\infty] \Big) \tilde{B}^{(j)} \\
&- \Big(\frac{\partial T}{\partial x_j} + \tilde{B}^{(j)} + [\tilde{B}^{(j)}, B_\infty] \Big) \tilde{B}^{(i)} \Bigg) (zI_n - T)^{-1} B_\infty.
\end{aligned}$$

□

Definition 2.3. The completely integrable Pfaffian system (19) is called a *system of differential equations of Okubo type in several variables* or shortly an *Okubo system in several variables.*

Remark 2.1. Assume that T, B_∞, $\tilde{B}^{(i)}$ $(i = 1, \ldots, n)$ satisfy the conditions (20), (21) and (22). Then replacing B_∞ by $B_\infty - \lambda I_n$ for any $\lambda \in \mathbb{C}$, we see that T, $B_\infty - \lambda I_n$, $\tilde{B}^{(i)}$ $(i = 1, \ldots, n)$ also satisfy (20), (21) and (22). The corresponding transformation from (19) into

$$dY^{(\lambda)} = -(zI_n - T)^{-1} \Big(dz + \sum_{i=1}^{n} \tilde{B}^{(i)} dx_i \Big) (B_\infty - \lambda I_n) Y^{(\lambda)}$$

is realized by the Euler transformation

$$Y \mapsto Y^{(\lambda)}(z) := \frac{1}{\Gamma(\lambda)} \int (u - z)^{\lambda - 1} Y(u) du.$$

Note that the Euler transformation changes the monodromy, namely the monodromy of $Y^{(\lambda)}$ depends on λ.

Remark 2.2. The completely integrable Pfaffian system (19) can be regarded as a Lax formalism of the isomonodromic deformation of an Okubo system (4). Indeed (19) is equivalent to

$$dY = \sum_{i=1}^{n} B_i^{(z)} d \log(z - z_i) Y. \tag{23}$$

Then the system of nonlinear differential equations (21) is equivalent to the Schlesinger system

$$dB_i^{(z)} = \sum_{j\neq i}[B_j^{(z)}, B_i^{(z)}]d\log(z_i - z_j), \qquad i = 1, \ldots, n. \tag{24}$$

(See Appendix A for details.)

Remark 2.3. By (10) and the assumption (A3), we see that an Okubo system in several variables is a universal integrable deformation of an Okubo system (4), which is uniquely determined by (4) up to changes of independent variables. (The uniqueness follows also from that of the isomonodromic deformation (23),(24) which is proved in [27].)

Lemma 2.4. *The variables $x_1, \ldots, x_n$ can be taken so that $T = T(x)$ satisfies the following condition:*

Condition (T): $T(x) = -x_n + T_0(x')$, where $T_0(x')$ depends only on $x' = (x_1, \ldots, x_{n-1})$.

In the following, we always assume Condition (T).

Remark 2.4. Under Condition (T), it holds that $\tilde{B}^{(n)} = I_n$, $\tilde{B}^{(i)} = \tilde{B}^{(i)}(x') \in \mathcal{O}_{U'}^{n\times n}$ $(i = 1, \ldots, n-1)$ and (19) is equivalent to the Pfaffian system for $Y_0 = Y_0(x)$

$$dY_0 = \Big(\sum_{i=1}^{n} B_0^{(i)} dx_i\Big) Y_0, \tag{25}$$

where $B_0^{(i)} = T^{-1}\tilde{B}^{(i)}B_\infty$. We call (25) the reduced form of (19).

3 Logarithmic vector fields

In this section, we study a divisor defined by $\det(-T(x)))$ which is the singular locus of an Okubo system in several variables. A main purpose of this section is to prove that the vector fields defined by (34) forms a unique standard system of generators of logarithmic vector fields along the divisor when the divisor is free (Lemma 3.9, cf. Remark 3.2).

3.1 Logarithmic vector fields

We employ the notations $x' = (x_1, \ldots, x_{n-1})$ and $x = (x', x_n) = (x_1, \ldots, x_n)$. Let $h(x) = H(x, 0) = \det(-T(x)) \in \mathcal{O}_U$ or equivalently $H(x, z) = h(x', x_n + z)$ under Condition (T). Let D be the divisor in U defined by $h(x)$ i.e. $D = \{(x) \in U; h(x) = 0\}$. We assume that the domain $U \subset \mathbb{C}^n$ (and $W \subset U \setminus \Delta_H$ resp.) has the form of $U = U' \times \mathbb{C} \subset \mathbb{C}^{n-1} \times \mathbb{C}$ (and $W = W' \times \mathbb{C} \subset U' \times \mathbb{C}$ resp.). Note that $h(x)$ is a monic polynomial in x_n of degree n:

$$h(x) = h(x', x_n) = x_n^n - s_1(x')x_n^{n-1} + \cdots + (-1)^n s_n(x') = \prod_{m=1}^{n}(x_n - z_m^0(x')), \tag{26}$$

where $s_i(x') \in \mathcal{O}_{U'}$. Recall that $z_m(x) = -x_n + z_m^0(x')$, $1 \le m \le n$.

Here we note that the assumption (A3) for $H(x,z) = h(x', x_n + z)$ is equivalent to

$$\det \begin{pmatrix} \frac{\partial z_1(x)}{\partial x_1} & \frac{\partial z_2(x)}{\partial x_1} & \cdots & \frac{\partial z_n(x)}{\partial x_1} \\ \frac{\partial z_1(x)}{\partial x_2} & \frac{\partial z_2(x)}{\partial x_2} & \cdots & \frac{\partial z_n(x)}{\partial x_2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial z_1(x)}{\partial x_n} & \frac{\partial z_2(x)}{\partial x_n} & \cdots & \frac{\partial z_n(x)}{\partial x_n} \end{pmatrix} \neq 0 \tag{27}$$

on W'.

We give the definitions of logarithmic vector field along D and free divisor following K. Saito [52]:

Definition 3.1. Let $\mathcal{M}_{U'}$ be the field of meromorphic functions on U'. A vector field $V = \sum_{k=1}^{n} v_k(x', x_n)\partial_{x_k}$ with $v_k(x', x_n) \in \mathcal{M}_{U'} \otimes_{\mathbb{C}} \mathbb{C}[x_n]$ is called a meromorphic logarithmic vector field along D if $(Vh)/h \in \mathcal{M}_{U'} \otimes_{\mathbb{C}} \mathbb{C}[x_n]$, or equivalently if $(Vh)|_D = 0$. If moreover, $v_k(x', x_n) \in \mathcal{O}_{U'} \otimes_{\mathbb{C}} \mathbb{C}[x_n]$, V is called a logarithmic vector field along D.

Let $\mathrm{Der}(-\log D)$ be the set of logarithmic vector fields along D, which is naturally an $\mathcal{O}_{U'} \otimes_{\mathbb{C}} \mathbb{C}[x_n]$-module. The divisor D is said to be free if $\mathrm{Der}(-\log D)$ is a free $\mathcal{O}_{U'} \otimes_{\mathbb{C}} \mathbb{C}[x_n]$-module.

Remark 3.1. It is known that D is free if there are logarithmic vector fields $V_i = \sum_{j=1}^{n} v_{i,j}(x)\partial_{x_j}$, $i = 1, \ldots, n$, along D such that $\det(v_{i,j}(x)) = h(x)$ (Saito's criterion [52]). The matrix $M_V(x) := (v_{n-i+1,j}(x))$ is called a *Saito matrix.*

We rewrite the assumption (A3) in terms of meromorphic logarithmic vector fields (which will be used in the proof of Lemma 6.1):

Lemma 3.2. *The following* (i) *and* (ii) *are equivalent.*

(i) *The assumption* (A3) *holds for* $H(x,z) = h(x', x_n + z)$.

(ii) *Let* $V = \sum_{k=1}^{n} v_k(x')\partial_{x_k}$, $v_k(x') \in \mathcal{M}_{U'}$ *be any meromorphic logarithmic vector field along D (whose coefficients do not depend on* x_n*). Then* $V = 0$.

Proof. First note that $V = \sum_{k=1}^{n} v_k(x')\partial_{x_k}$, $v_k(x') \in \mathcal{M}_{U'}$ is a meromorphic logarithmic vector field along D if and only if

$$Vh = 0. \tag{28}$$

Let $\sum'$ denote $\sum_{i=1}^{n-1}$. Since

$$\begin{aligned} Vh = &-\left(\sum\nolimits' v_i \frac{\partial s_1}{\partial x_i} - v_n n\right) x_n^{n-1} + \left(\sum\nolimits' \frac{\partial s_2}{\partial x_i} - v_n(n-1)s_1\right) x_n^{n-2} \\ &+ \ldots + (-1)^n \left(\sum\nolimits' \frac{\partial s_n}{\partial x_i} - v_n s_{n-1}\right), \end{aligned}$$

the equality (28) is equivalent to

$$\begin{pmatrix} v_1(x') & \dots & v_n(x') \end{pmatrix} \begin{pmatrix} \frac{\partial s_1(x')}{\partial x_1} & \frac{\partial s_2(x')}{\partial x_1} & \dots & \frac{\partial s_n(x')}{\partial x_1} \\ \dots & & & \\ \frac{\partial s_1(x')}{\partial x_{n-1}} & \frac{\partial s_2(x')}{\partial x_{n-1}} & \dots & \frac{\partial s_n(x')}{\partial x_{n-1}} \\ -n & -(n-1)s_1(x') & \dots & -s_{n-1}(x') \end{pmatrix} = O. \tag{29}$$

Noting (26), the equation (29) has a non-zero solution if and only if (A3) for $H(x,z)$ does not hold. This proves the lemma. $\square$

On each W', we define matrices $P_h(x'), M_h(x'), M_{V^{(h)}}(x)$ by

$$P_h(x') = \begin{pmatrix} \frac{\partial z_1(x)}{\partial x_1} & \dots & \frac{\partial z_n(x)}{\partial x_1} \\ & \vdots & \\ \frac{\partial z_1(x)}{\partial x_n} & \dots & \frac{\partial z_n(x)}{\partial x_n} \end{pmatrix}, \tag{30}$$

and

$$M_h(x') = P_h \operatorname{diag}[z_1^0, z_2^0, \dots, z_n^0]\, P_h^{-1}, \quad M_{V^{(h)}}(x) = -P_h \operatorname{diag}[z_1, z_2, \dots, z_n]\, P_h^{-1}. \tag{31}$$

By definition, we have

$$M_{V^{(h)}}(x) = x_n I_n - M_h(x') \tag{32}$$

and

$$\det M_{V^{(h)}}(x) = h(x). \tag{33}$$

We define the vector fields $V_i^{(h)}$ $(i = 1, \dots, n)$ by

$$\begin{pmatrix} V_n^{(h)} & \cdots & V_1^{(h)} \end{pmatrix}^t = M_{V^{(h)}} \begin{pmatrix} \partial_{x_1} & \cdots & \partial_{x_n} \end{pmatrix}^t. \tag{34}$$

Lemma 3.3. *The vector fields $V_i^{(h)}$, $1 \le i \le n$ are meromorphic logarithmic vector fields along D.*

Proof. First we prove that the entries of $M_h(x')$ are meromorphic functions on U'. Fix $W' \subset U'$, and let

$$F = \begin{pmatrix} (z_1^0(x'))^{n-1} & \dots & (z_n^0(x'))^{n-1} \\ & \cdots & \\ z_1^0(x') & \dots & z_n^0(x') \\ 1 & \dots & 1 \end{pmatrix}.$$

The two matrices M^0 and M^1 are determined by

$$M^0\, F = F \operatorname{diag}[z_1^0, z_2^0, \dots, z_n^0] \quad \text{and} \quad M^1\, F = P_h$$

using elementary elimination methods. By the construction, M^0 and $\delta_h M^1$ are holomorphic on U', where $\delta_h = \prod_{i<j}(z_i^0(x') - z_j^0(x'))^2$. Since

$$M_h = M^1 M^0 (M^1)^{-1}, \tag{35}$$

we find that the entries of $M_h(x')$ are meromorphic functions on U'. Then it is clear that $V_i^{(h)}$ $(i = 1, \dots, n)$ are meromorphic logarithmic vector fields from $z_i \partial_{z_i} h = h$ $(i = 1, \dots, n)$ and (30),(31),(34). □

Lemma 3.4. *Assume that $V_i = \sum_{j=1}^n v_{i,j}(x)\partial_{x_j}$, $1 \le i \le n$ are logarithmic vector fields along D satisfying $v_{i,j}(x) - x_n \delta_{n+1-i,j} \in \mathcal{O}_{U'}$, where δ_{ij} denotes Kronecker's delta. Then it holds that $V_i = V_i^{(h)}$, $1 \le i \le n$, and D is free.*

Proof. Let $V_i^{(h)} = \sum_{j=1}^n v_{i,j}^{(h)}(x)\partial_{x_j}$. Then we see $v_{i,j}^{(h)}(x) - x_n \delta_{n+1-i,j} \in \mathcal{M}_{U'}$ from (34). Thus we have $v_{i,j}(x) - v_{i,j}^{(h)}(x) \in \mathcal{M}_{U'}$, $j = 1, \dots, n$, which implies $V_i - V_i^{(h)} = 0$ in virtue of Lemma 3.2. Then the equality (33) implies that $M_{V^{(h)}}$ is a Saito matrix and D is free. □

3.2 Global case

In this subsection we assume that the function $h(x', x_n)$ is a polynomial in (x', x_n) and weighted homogeneous with respect to a weight $w(\cdot)$ with

$$0 < w(x_1) \le w(x_2) \le \cdots \le w(x_{n-1}) \le w(x_n) = 1.$$

In this case, we replace $\mathcal{O}_{U'}$ and $\mathcal{M}_{U'}$ in the previous subsection by $\mathbb{C}[x']$ and $\mathbb{C}(x')$ respectively, and assume that the integrable system (25) is weighted homogeneous, that is, all the entries of $B_0^{(k)}$, $\tilde{B}^{(k)}$, T, Y_0 are weighted homogeneous.

Lemma 3.5. *It holds that $V_1^{(h)} = \sum_{k=1}^n w(x_k) x_k \partial_{x_k}$, namely, $V_1^{(h)}$ is the Euler operator.*

Proof. It is clear from that the Euler operator is a logarithmic vector field along D and the proof of Lemma 3.4. □

Note $w(z_m^0(x')) = w(x_n) = 1$. Hence the equality (31) implies $w((M_h)_{i,j}) = w(z_m^0) + w(\frac{\partial z_m^0}{\partial x_i}) - w(\frac{\partial z_m^0}{\partial x_j})$ for $1 \le i, j \le n$, that is,

$$w\left((M_h)_{i,j}\right) = 1 - w(x_i) + w(x_j), \quad 1 \le i, j \le n. \tag{36}$$

Since $w\left((M_h)_{i,j}\right) = w\left((M_{V^{(h)}})_{i,j}\right)$, we have a kind of duality

$$w(x_i) + w\left(V_{n-i+1}^{(h)}\right) = 1, \quad 1 \le i \le n, \tag{37}$$

between $\{w(x_i)\}$ and $\{w(V_i^{(h)})\}$.

As for the reduced form (25), we give a lemma.

Lemma 3.6. (i) *The following holds*

$$\sum_{i=1}^{n} w(x_i)x_i\tilde{B}^{(i)} = -T, \quad \sum_{i=1}^{n} w(x_i)x_iB_0^{(i)} = -B_\infty, \tag{38}$$

(ii) *There is a weighted homogeneous matrix $C(x)$ such that*

$$\tilde{B}^{(i)} = \frac{\partial C}{\partial x_i}, \quad T = -V_1^{(h)}C. \tag{39}$$

Proof. Proof of (i). From (9) and (10), we have

$$\begin{aligned}
\sum_{i=1}^{n} w(x_i)x_i\tilde{B}^{(i)} &= P(x')\left(\sum_{i=1}^{n} w(x_i)x_i \operatorname{diag}\left[-\frac{\partial z_1}{\partial x_i}, \dots, -\frac{\partial z_n}{\partial x_i}\right]\right) P(x')^{-1} \\
&= -P(x')\left(V_1^{(h)} \operatorname{diag}[z_1, \dots, z_n]\right) P(x')^{-1} \\
&= -P(x') \operatorname{diag}[z_1, \dots, z_n]\ P(x')^{-1} = -T(x), \\
\sum_{i=1}^{n} w(x_i)x_iB_0^{(i)} &= T^{-1}\left(\sum_{i=1}^{n} w(x_i)x_i\tilde{B}^{(i)}\right) B_\infty = -B_\infty.
\end{aligned}$$

Proof of (ii). From (22), $\sum_{i=1}^n \tilde{B}^{(i)}dx_i = d\tilde{C}(x)$ has a solution $\tilde{C}(x)$. Let $C_{ij}(x)$ be the weighted homogeneous part of $\tilde{C}_{ij}(x)$ of the weight $w(T_{ij})$. Then the matrix $C(x) = (C_{ij}(x))_{i,j=1,\dots,n}$ satisfies the first equality of (39). The second equality of (39) then coincides with the first one of (38). □

Let $Y_0 = (y_1, y_2, \dots, y_n)^t$, B_∞ =diag$[\lambda_1, \lambda_2, \dots, \lambda_n]$ as before. Then, from (38), we have

$$V_1^{(h)}Y_0 = \sum_{i=1}^{n} w(x_i)x_i\partial_{x_i}Y_0 = \sum_{i=1}^{n} w(x_i)x_iB_0^{(i)}Y_0 = -B_\infty Y_0,$$

which implies

$$w(y_i) = -\lambda_i, \quad 1 \le i \le n. \tag{40}$$

From the equality $\frac{\partial}{\partial x_n}Y_0 = B_0^{(n)}Y_0$, we obtain $w(y_i) - 1 = w((B_0^{(n)})_{i,j}) + w(y_j)$, whence

$$w((B_0^{(n)})_{i,j}) = -\lambda_i + \lambda_j - 1, \quad 1 \le i, j \le n. \tag{41}$$

Since $B_0^{(n)} = T^{-1}B_\infty$ as in Remark 2.4, we have $w(T_{i,j}) = -w\left((B_0^{(n)})_{j,i}\right)$, that is,

$$w(T_{i,j}) = 1 - \lambda_i + \lambda_j, \quad 1 \le i, j \le n. \tag{42}$$

Lemma 3.7. *Let V be a non-zero weighted homogeneous logarithmic vector field along D. Then $w(V) \ge 0$.*

Proof. Let $V = \sum_{j=1}^n v_j(x)\partial_{x_j}$, $v_j = v_j(x', x_n) \in \mathbb{C}[x]$. If $w(V) < 0$, then $v_j(x) \in \mathbb{C}[x']$. This proves $V = 0$ from Lemma 3.2. □

For rational logarithmic vector fields $V_i = \sum_{j=1}^n v_{i,j}(x)\partial_{x_j}$ $(i = 1, \ldots, n)$ along D, let $M_V(x)$ denotes the $n \times n$ matrix whose (i, j)-entry is defined by $v_{n-i+1,j}(x)$.

Lemma 3.8. *Let V_i, $1 \leq i \leq n$ be weighted homogeneous logarithmic vector fields with $w(V_1) \leq w(V_2) \leq \cdots \leq w(V_n)$. Assume* $\det M_V(x)|_{x'=0} = cx_n^n$ *for a non-zero constant number c. Then the following three facts hold.*

(i) $w(V_i) = w(V_i^{(h)})\,(= 1 - w(x_{n-i+1}))$, $1 \leq i \leq n$.

(ii) *There is a matrix $G(x') \in GL(n, \mathbb{C}[x'])$ such that $w(G(x')_{i,j}) = w(x_j) - w(x_i)$, and that*

$$\begin{pmatrix} V_n^{(h)} & V_{n-1}^{(h)} & \ldots & V_1^{(h)} \end{pmatrix}^t = G(x') \begin{pmatrix} V_n & V_{n-1} & \ldots & V_1 \end{pmatrix}^t$$

holds. In particular, $V_i^{(h)}(x)$, $1 \leq i \leq n$ are also logarithmic vector fields, and hence D is free.

(iii) $\det M_V(x) = ch(x)$.

Proof. Proof of (i). Since $w(V_i) \geq 0$ from Lemma 3.7, it holds that $(M_V)|_{x=0} = 0$. Since $\det(M_V)|_{x'=0} = cx_n^n$, it holds that

$$M_V(x)|_{x'=0} = x_n R,$$

where

$$R = \begin{pmatrix} R_1 & O & \ldots & O \\ O & R_2 & \ldots & O \\ \cdot\cdot & \cdot\cdot & \ldots & \cdot\cdot \\ O & O & \ldots & R_k \end{pmatrix}$$

for some $R_i \in GL(n_i, \mathbb{C})$ with $\sum_{i=1}^k n_i = n$. This proves $w(V_i) = 1 - w(x_{n-i+1})$.

Proof of (ii). Note that $R^{-1}M_V(x)|_{x'=0} = x_n I_n$. Since all the entries of $R^{-1}M_V(x)$ are linear function of x_n with the coefficients being weighted homogeneous polynomials in x', we have

$$R^{-1}M_V(x) = x_n\tilde{R}(x') + \tilde{\tilde{R}}(x'),$$

for some $\tilde{R}(x'), \tilde{\tilde{R}}(x') \in \mathbb{C}[x']^{n\times n}$ satisfying

$$\tilde{R}(0) = I_n,\ w(\tilde{R}_{i,j}) = -w(x_i) + w(x_j),\ w(\tilde{\tilde{R}}_{i,j}) = 1 - w(x_i) + w(x_j).$$

Then $\det \tilde{R}(x') = 1$ and $\tilde{R}(x')^{-1}R^{-1}M_V - x_n I_n \in \mathbb{C}[x']^{n\times n}$, which (together with Lemma 3.4) proves

$$\tilde{R}(x')^{-1}R^{-1} \begin{pmatrix} V_n & V_{n-1} & \ldots & V_1 \end{pmatrix}^t = \begin{pmatrix} V_n^{(h)} & V_{n-1}^{(h)} & \ldots & V_1^{(h)} \end{pmatrix}^t.$$

This proves (ii).

(iii) is proved by (ii) and the equality (33). $\square$

Lemma 3.9. *Assume that D is free. Then $V_i^{(h)}$, $1 \leq i \leq n$ are logarithmic vector fields along D. In particular, $\{V_i^{(h)}\}_{i=1,\ldots,n}$ forms a unique system of generators of logarithmic vector fields along D such that its Saito matrix $M_{V^{(h)}}$ satisfies $M_{V^{(h)}} - x_n I_n \in \mathbb{C}[x']^{n\times n}$.*

Proof. Let V_i, $1 \leq i \leq n$ be logarithmic vector fields such that

$$\det(M_V) = h(x).$$

Since each homogeneous part of V_i are logarithmic vector field with a non-negative weight (Lemma 3.7), all entries of $(M_V)|_{x'=0}$ are polynomials in x_n of positive degrees. Let $R(x_n) = x_n^{-1}(M_V)|_{x'=0} \in \mathbb{C}[x_n]^{n\times n}$. Then $\det R(x_n) = 1$, and hence $\det R(0) = 1$. Let V_i' $1 \leq i \leq n$ be logarithmic vector fields such that

$$\begin{pmatrix} V_n' & V_{n-1}' & \ldots & V_1' \end{pmatrix}^t = R(0)^{-1} M_V \begin{pmatrix} \partial_{x_1} & \partial_{x_2} & \ldots & \partial_{x_n} \end{pmatrix}^t.$$

Put $M_{V'} = R(0)^{-1}M_V$. Then $(M_{V'})|_{x'=0} = x_n I_n + O(x_n^2)$. Let V_i'' be the homogeneous part of V_i' with $w(V_i'') = 1 - w(x_{n-i+1})$. Then it holds $\det(M_{V''})|_{x'=0} = x_n^n$, and hence Lemma 3.8 implies that $V_i^{(h)}$, $1 \leq i \leq n$ are logarithmic vector fields. The uniqueness of $\{V_i^{(h)}\}$ is clear from Lemma 3.4. ☐

Remark 3.2. In the case where $h(x)$ is the discriminant of a well-generated complex reflection group, D. Bessis [5] showed the existence of a system of generators of logarithmic vector fields along D whose Saito matrix M_V has the form $M_V - x_n I_n \in \mathbb{C}[x']^{n\times n}$. Such a system of generators is called *flat* in [5] (cf. Section 6).

4 Saito structure (without metric)

In this section, we review Saito structure (without metric) introduced by Sabbah [51]. (In the sequel, we abbreviate Saito structure (without metric) to Saito structure for brevity.) Proofs for many of statements in this section can be found in the literature [51, 40].

Definition 4.1 (C. Sabbah [51])**.** Let X be a complex analytic manifold of dimension n (in this paper, we treat the case where X is a domain U in $\mathbb{C}^n$), TX be its tangent bundle, and Θ_X be the sheaf of holomorphic sections of TX. A *Saito structure (without a metric)* on X is a data consisting of (∇, Φ, e, E) in (i),(ii),(iii) that are subject to the conditions (a), (b):

(i) ∇ is a flat torsion-free connection on TX,

(ii) Φ is a symmetric Higgs field on TX,

(iii) e and E are global sections (vector fields) of TX, respectively called *unit field* and *Euler field*.

(a) A meromorphic connection ∇ on the bundle π^*TX on $\mathbb{P}^1 \times X$ defined by

$$\nabla = \pi^*\nabla + \frac{\pi^*\Phi}{z} - \left(\frac{\Phi(E)}{z} + \nabla E\right)\frac{dz}{z} \tag{43}$$

is integrable, where π is the projection $\pi : \mathbb{P}^1 \times X \to X$ and z is a non-homogeneous coordinate of $\mathbb{P}^1$,

(b) the field e is ∇-horizontal (i.e., $\nabla(e) = 0$) and satisfies $\Phi_e = \mathrm{Id}$, where we regard Φ as an $End_{\mathcal{O}_X}(\Theta_X)$-valued 1-form and $\Phi_e \in End_{\mathcal{O}_X}(\Theta_X)$ denotes the contraction of the vector field e and the 1-form Φ.

Remark 4.1. To the Higgs field Φ there associates a product $\star$ on Θ_X defined by $\xi \star \eta = \Phi_\xi(\eta)$ for $\xi, \eta \in \Theta_X$. The Higgs field Φ is said to be symmetric if the product $\star$ is commutative. The condition $\Phi_e = \mathrm{Id}$ in Definition 4.1 (b) implies that the field e is the unit of the product $\star$. The integrability of the connection ∇ implies that of the Higgs field Φ, which is equivalent to the associativity of $\star$. So the product $\star$ associated to a Saito structure is commutative and associative.

Since the connection ∇ is flat and torsion free, we can take a *flat coordinate system* $(t_1, \dots, t_n)$ such that $\nabla(\partial_{t_i}) = 0$ $(i = 1, \dots, n)$ at least on a simply-connected open set of X. We assume the existence of a flat coordinate system $(t_1, \dots, t_n)$ on X replacing X by such an open set if necessary. In addition, we assume the following conditions:

(C1) $e = \partial_{t_n}$,

(C2) $E = w_1 t_1 \partial_{t_1} + \cdots + w_n t_n \partial_{t_n}$ for $w_i \in \mathbb{C}$ $(i = 1, \dots, n)$,

(C3) $w_n = 1$ and $w_i - w_j \notin \mathbb{Z} \setminus \{0\}$ for $i \neq j$.

In this paper, a function $f \in \mathcal{O}_X$ is said to be weighted homogeneous with a weight $w(f) \in \mathbb{C}$ if f is an eigenfunction of the Euler operator: $Ef = w(f)f$. In particular, the flat coordinates t_i $(i = 1, \dots, n)$ are weighted homogeneous with $w(t_i) = w_i$.

We write $\Phi \in End_{\mathcal{O}_X}(\Theta_X) \otimes_{\mathcal{O}_X} \Omega^1_X$ as $\Phi = \sum_{k=1}^{n} \Phi^{(k)} dt_k$, where $\Phi^{(k)} \in End_{\mathcal{O}_X}(\Theta_X)$ $(k = 1, \dots, n)$. We fix the basis $\{\partial_{t_1}, \dots, \partial_{t_n}\}$ of Θ_X and introduce the following matrices:

(i) $\tilde{\mathcal{B}}^{(k)}$ $(k = 1, \dots, n)$ is the representation matrix of $\Phi^{(k)}$, namely the (i,j)-entry $\tilde{\mathcal{B}}^{(k)}_{ij}$ is defined by

$$\Phi^{(k)}(\partial_{t_i}) = \sum_{j=1}^{n} \tilde{\mathcal{B}}^{(k)}_{ij} \partial_{t_j} \quad (i = 1, \dots, n), \tag{44}$$

(ii) $\mathcal{T}$ and $\mathcal{B}_\infty$ are the representation matrices of $-\Phi(E)$ and ∇E respectively, namely

$$-\Phi_{\partial_{t_i}}(E)=\sum_{j=1}^{n}\mathcal{T}_{ij}\partial_{t_j},\qquad \nabla_{\partial_{t_i}}(E)=\sum_{j=1}^{n}(\mathcal{B}_\infty)_{ij}\partial_{t_j}. \tag{45}$$

We assume that $-\Phi(E)$ is generically regular semisimple on X, that is the discriminant of $\det(z-\mathcal{T})$ does not identically vanish on X.

Lemma 4.2. $\mathcal{B}_\infty = \mathit{diag}[w_1,\ldots,w_n]$.

Proof. It is straightforward. □

Proposition 4.3. *The meromorphic connection ∇ is integrable if and only if $\mathcal{T}$, $\mathcal{B}_\infty$ and $\tilde{\mathcal{B}}^{(i)}$ $(i=1,\ldots,n)$ are subject to the following relations*

$$\begin{cases} \dfrac{\partial\tilde{\mathcal{B}}^{(i)}}{\partial t_j}=\dfrac{\partial\tilde{\mathcal{B}}^{(j)}}{\partial t_i}, \quad i,j=1,\ldots,n,\\ \big[\mathcal{T},\tilde{\mathcal{B}}^{(i)}\big]=O,\quad \big[\tilde{\mathcal{B}}^{(i)},\tilde{\mathcal{B}}^{(j)}\big]=O,\quad i,j=1,\ldots,n,\\ \dfrac{\partial\mathcal{T}}{\partial t_i}+\tilde{\mathcal{B}}^{(i)}+[\tilde{\mathcal{B}}^{(i)},\mathcal{B}_\infty]=O,\quad i=1,\ldots,n. \end{cases} \tag{46}$$

Proof. See [51, (2.6) on p.203]. □

Remark 4.2. In virtue of Lemma 2.2, the relations (46) is nothing but the integrability condition of the Pfaffian system

$$dY=-(zI_n-\mathcal{T})^{-1}\Big(dz+\sum_{i=1}^{n}\tilde{\mathcal{B}}^{(i)}dt_i\Big)\mathcal{B}_\infty Y. \tag{47}$$

In other words, the existence of a Saito structure yields an Okubo system in several variables (47). In the next section, we will find a criterion for that, for an Okubo system in several variables, there exists a Saito structure which yields the given Okubo system in several variables.

Remark 4.3. The meromorphic connection (43) is written in the following matrix form with respect to the flat coordinate system:

$$d\mathcal{Y}=\left(-\frac{\mathcal{T}}{z}+\mathcal{B}_\infty\right)\frac{dz}{z}\mathcal{Y}-\sum_{i=1}^{n}\frac{\tilde{\mathcal{B}}^{(i)}}{z}dt_i\mathcal{Y}. \tag{48}$$

The system of equations (46) is equivalent to the integrability condition of (48). The system of ordinary linear differential equations

$$\frac{d\mathcal{Y}}{dz}=\left(-\frac{\mathcal{T}}{z^2}+\frac{\mathcal{B}_\infty}{z}\right)\mathcal{Y}$$

has an irregular singularity of Poincaré rank one at $z=0$ and a regular singularity at $z=\infty$, which is called a *Birkhoff normal form*. So (48) may be regarded as a universal integral deformation of a Birkhoff normal form. A Birkhoff normal form can be transformed into an Okubo system using a Fourier-Laplace transformation.

Lemma 4.4. *Define vector fields V_i $(i=1,\dots,n)$ by*

$$\begin{pmatrix} V_n & \cdots & V_1 \end{pmatrix}^t = -\mathcal{T} \begin{pmatrix} \partial_{t_1} & \cdots & \partial_{t_n} \end{pmatrix}^t \tag{49}$$

and put $h=h(t)=\det(-\mathcal{T})$. Then V_i, $i=1,\dots,n$, are logarithmic vector fields along $D=\{t\in X; h(t)=0\}$, and D is a free divisor.

Proof. Several proofs are found in [51, 29]. ☐

Lemma 4.5. *There is a unique matrix $\mathcal{C}$ such that*

$$\mathcal{T}=-E\mathcal{C}, \quad \tilde{\mathcal{B}}^{(i)}=\frac{\partial \mathcal{C}}{\partial t_i}, \quad i=1,\dots,n$$

and that each matrix entry $\mathcal{C}_{ij}$ of $\mathcal{C}$ is weighted homogeneous with $w(\mathcal{C}_{ij})=1-w_i+w_j$.

Proof. This lemma can be proved in a way similar to Lemma 3.6. ☐

Lemma 4.6. *Let $(t_1,\dots,t_n)$ be a flat coordinate system of the Saito structure. Then $\mathcal{T}_{nj}=-w_jt_j$ (or equivalently $\mathcal{C}_{nj}=t_j$), $j=1,\dots,n$.*

Proof. As in Lemma 4.4, define vector fields V_i $(i=1,\dots,n)$ by $\begin{pmatrix} V_n & \dots & V_1 \end{pmatrix}^t = -\mathcal{T}\begin{pmatrix} \partial_{t_1} & \dots & \partial_{t_n} \end{pmatrix}^t$. Then Lemma 4.4 shows that each V_i is logarithmic along D. Then similarly to Lemma 3.5, it holds that $V_1=E$, from which we have $-\mathcal{T}_{nj}=w_jt_j$. It is equivalent to $\mathcal{C}_{nj}=t_j$ by Lemma 4.5. ☐

Proposition 4.7 (Konishi-Minabe [40])**.** *There is a unique n-tuple of analytic functions $\vec{g}=(g_1,\dots,g_n)\in\mathcal{O}_X^n$ such that*

$$\mathcal{C}_{ij}=\frac{\partial g_j}{\partial t_i}$$

and that g_j is weighted homogeneous with $w(g_j)=1+w_j$. The vector $\vec{g}$ (or precisely the vector field $\mathcal{G}=\sum_{i=1}^n g_i\partial_{t_i}$) is called a potential vector field.

Proof. By the symmetry of the Higgs field Φ, it holds that

$$\frac{\partial \mathcal{C}_{ij}}{\partial t_k}=\frac{\partial \mathcal{C}_{kj}}{\partial t_i}, \quad i,j,k=1,\dots,n.$$

Then g_j is uniquely given by $g_j=\frac{1}{1+w_j}\sum_{i=1}^n w_it_iC_{ij}$. ☐

Proposition 4.8. *The potential vector field $\vec{g} = (g_1, \dots, g_n)$ is a solution to the following system of nonlinear equations:*

$$\sum_{m=1}^{n} \frac{\partial^2 g_m}{\partial t_k \partial t_i} \frac{\partial^2 g_j}{\partial t_l \partial t_m} = \sum_{m=1}^{n} \frac{\partial^2 g_m}{\partial t_l \partial t_i} \frac{\partial^2 g_j}{\partial t_k \partial t_m}, \quad i, j, k, l = 1, \dots, n, \tag{50}$$

$$\frac{\partial^2 g_j}{\partial t_n \partial t_i} = \delta_{ij}, \qquad i, j = 1, \dots, n, \tag{51}$$

$$E g_j = \sum_{k=1}^{n} w_k t_k \frac{\partial g_j}{\partial t_k} = (1 + w_j) g_j, \quad j = 1, \dots, n. \tag{52}$$

Proof. The associativity of $\star$ (i.e., the integrability of Φ) is equivalent to $\tilde{\mathcal{B}}^{(k)} \tilde{\mathcal{B}}^{(l)} = \tilde{\mathcal{B}}^{(l)} \tilde{\mathcal{B}}^{(k)}$. We obtain (50) if we rewrite this condition in terms of $\tilde{\mathcal{B}}^{(k)}_{ij} = \dfrac{\partial^2 g_j}{\partial t_k \partial t_i}$. The equation (51) follows from $\tilde{\mathcal{B}}^{(n)} = I_n$ (i.e., $\Phi_e = \mathrm{Id}$). ☐

Definition 4.9. The system of non-linear differential equations (50), (51), (52) for the vector $\vec{g} = (g_1, \dots, g_n)$ is called the extended WDVV equation.

Remark 4.4. The notion of "F-manifolds with compatible flat structures" was introduced by Manin [44] as a generalization of Frobenius manifolds. This notion does not require the existence of an Euler field. "Potential vector field" in Proposition 4.7 is called "local vector potential" in Manin's framework [44]. And the associativity conditions (50), (51) are called "oriented associativity equations" in [44]. The authors were informed these facts by A. Arsie and P. Lorenzoni.

Conversely, starting with a solution of (50), (51) and (52), it is possible to reconstruct a Saito structure.

Proposition 4.10. *Take constants $w_j \in \mathbb{C}$, $j = 1, \dots, n$ satisfying $w_i - w_j \notin \mathbb{Z}$ and $w_n = 1$ and assume that $\vec{g} = (g_1, \dots, g_n)$ is a holomorphic solution of (50), (51) and (52) on a simply connected domain U in $\mathbb{C}^n$. Then there is a Saito structure on U which has $(t_1, \dots, t_n)$ as a flat coordinate system. In addition, the Saito structure is semisimple (i.e. $-\Phi(E)$ is semisimple) if and only if*

(SS) the $n \times n$-matrix $\left(-(1 + w_j - w_i) \dfrac{\partial g_j}{\partial t_i} \right)_{1 \le i, j \le n}$ is semisimple.

Proof. Define $E = \sum_{i=1}^{n} w_i t_i \partial_{t_i}$, $e = \partial_{t_n}$, $\tilde{\mathcal{B}}^{(k)}_{ij} = \frac{\partial^2 g_j}{\partial t_k \partial t_i}$, $\Phi = \sum_{k=1}^{n} \tilde{\mathcal{B}}^{(k)} dt_k$ and $\nabla(\partial_{t_i}) = 0$, $i = 1, \dots, n$. Then the conditions (a), (b) of Definition 4.1 hold and $\vec{g}$ is the potential vector field associated to $\mathcal{C}$. The last part of the proposition is obvious from $\mathcal{T}_{ij} = -E\mathcal{C}_{ij}$. ☐

Remark 4.5. By definition, any two weighted-homogeneous flat coordinate systems $(t'_1, \dots, t'_n)$ and $(t_1, \dots, t_n)$ are connected by $t'_i = \sum_{j=1}^{n} c_{ij} t_j$ with $c_{ij} = 0$ if $w_i \neq w_j$ and $c_{in} = 0$ $(i = 1, \dots, n-1)$. (In particular, if $w_i \neq w_j$ for any $i \neq j$, the weighted homogeneous flat coordinate system is unique up to multiplication by non-zero constants: $(t'_1, \dots, t'_n) = (c_1 t_1, \dots, c_n t_n)$.) Let $(g'_j(t'_1, \dots, t'_n))_{1 \leq j \leq n}$ and $(g_j(t_1, \dots, t_n))_{1 \leq j \leq n}$ be the potential vector fields with respect to $(t'_1, \dots, t'_n)$ and $(t_1, \dots, t_n)$ respectively. Then it holds that $g'_j(t'_1, \dots, t'_n) = c_{nn} \sum_{k=1}^{n} c_{jk} g_k(t_1, \dots, t_n)$, $j = 1, \dots, n$. In other words, given a solution $(g_j(t_1, \dots, t_n))_{1 \leq j \leq n}$ to the extended WDVV equation (50),(51),(52), then $\left(\sum_{k=1}^{n} c_{nn} c_{jk} g_k\left(\sum_{l=1}^{n} d_{1l} t_l, \dots, \sum_{l=1}^{n} d_{nl} t_l\right)\right)_{1 \leq j \leq n}$ is also a solution to it, where (d_{ij}) is the inverse matrix of (c_{ij}).

Let J be an $n \times n$-matrix with $J_{ij} = \delta_{i+j,n+1}$, $i, j = 1, \dots, n$, where δ_{ij} denotes Kronecker's delta, and, for an $n \times n$ matrix A, define A^* by $A^* = J A^t J$.

Proposition 4.11. *Given a Saito structure on X, the following conditions are mutually equivalent:*

(i) *For appropriate normalization of the flat coordinate system, it holds that $\mathcal{C}^* = \mathcal{C}$.*

(ii) *For appropriate normalization of the flat coordinate system, there is a holomorphic function $F \in \mathcal{O}_X$ such that*

$$\frac{\partial F}{\partial t_i} = g_{n+1-i} = (\vec{g} J)_i, \quad i = 1, \dots, n. \tag{53}$$

(iii) *There is $r \in \mathbb{C}$ such that*

$$w_{n+1-i} + w_i = -2r, \qquad i = 1, \dots, n, \tag{54}$$

and there is a metric η (in this paper, "metric" means non-degenerate symmetric $\mathbb{C}$-bilinear form on TX) such that

$$\eta(\sigma \star \xi, \zeta) = \eta(\sigma, \xi \star \zeta), \qquad \textit{(compatibility to the product)} \tag{55}$$

$$(\nabla \eta)(\xi, \zeta) := d(\eta(\xi, \zeta)) - \eta(\nabla \xi, \zeta) - \eta(\xi, \nabla \zeta) = 0, \qquad \textit{(horizontality)} \tag{56}$$

$$(E\eta)(\xi, \zeta) := E(\eta(\xi, \zeta)) - \eta(E\xi, \zeta) - \eta(\xi, E\zeta) = -2r\eta(\xi, \zeta), \quad \textit{(homogeneity)} \tag{57}$$

for any $\sigma, \xi, \zeta \in \Theta_X$.

Here, as stated in Remark 4.5, a flat coordinate system admits indetermination of multiplication by constants. "Normalization" in the above conditions means to fix this indetermination.

Proof. (i) $\Leftrightarrow$ (ii) By definition, $\mathcal{C}^* = \mathcal{C}$ is equivalent to that $\mathcal{C}J$ is a symmetric matrix. From

$$\frac{\partial g_{n+1-j}}{\partial t_i} = (\mathcal{C}J)_{ij},$$

we find that the symmetry of $\mathcal{C}J$ is equivalent to the existence of $F \in \mathcal{O}_X$ such that $\frac{\partial F}{\partial t_i} = g_{n+1-i}$.

(i) $\Rightarrow$ (iii) First, we show $w_{n+1-i} + w_i = -2r$, $i = 1, \ldots, n$ for some $r \in \mathbb{C}$. Note that, if $\mathcal{C}^* = \mathcal{C}$ holds, then it also holds that $\mathcal{T}^* = \mathcal{T}$ and $\tilde{\mathcal{B}}^{(i)*} = \tilde{\mathcal{B}}^{(i)}$, $i = 1, \ldots, n$. By the integrability condition, we have

$$\frac{\partial \mathcal{T}}{\partial x_i} + \tilde{\mathcal{B}}^{(i)} + [\tilde{\mathcal{B}}^{(i)}, \mathcal{B}_\infty] = O, \qquad \frac{\partial \mathcal{T}^*}{\partial x_i} + \tilde{\mathcal{B}}^{(i)*} - [\tilde{\mathcal{B}}^{(i)*}, \mathcal{B}_\infty^*] = O.$$

Taking the difference of these two equalities, we have

$$[\tilde{\mathcal{B}}^{(i)}, \mathcal{B}_\infty + \mathcal{B}_\infty^*] = O, \qquad i = 1, \ldots, n.$$

Since $\mathcal{T}$ is written as

$$\mathcal{T} = -\sum_{i=1}^{n} w_i t_i \tilde{\mathcal{B}}^{(i)},$$

we have $[\mathcal{T}, \mathcal{B}_\infty + \mathcal{B}_\infty^*] = O$. Besides, we note that $\mathcal{T}_{nj} = -w_j t_j \neq 0$, $j = 1, \ldots, n$. Then it follows that $\mathcal{B}_\infty + \mathcal{B}_\infty^* = \mathrm{diag}[w_1 + w_n, w_2 + w_{n-1}, \ldots, w_n + w_1]$ is a scalar matrix.

Next, we show the existence of a metric with the desired properties. Under the assumption that $\mathcal{C}J$ is a symmetric matrix, define a metric η by $\eta(\partial_{t_i}, \partial_{t_j}) = J_{ij}$, $i, j = 1, \ldots, n$ for the flat coordinate system $(t_1, \ldots, t_n)$. Then, on one hand, we have

$$\eta(\partial_{t_i} \star \partial_{t_k}, \partial_{t_j}) = \eta\Big(\sum_{l=1}^{n} \tilde{\mathcal{B}}_{il}^{k} \partial_{t_l}, \partial_{t_j}\Big) = \sum_{l=1}^{n} \frac{\partial \mathcal{C}_{il}}{\partial t_k} J_{lj} = \frac{\partial (\mathcal{C}J)_{ij}}{\partial t_k},$$

on the other hand

$$\eta(\partial_{t_i}, \partial_{t_k} \star \partial_{t_j}) = \eta\Big(\partial_{t_i}, \sum_{l=1}^{n} \tilde{\mathcal{B}}_{jl}^{k} \partial_{t_l}\Big) = \frac{\partial (\mathcal{C}J)_{ji}}{\partial t_k},$$

which concludes the compatibility of η to the product. The horizontality and the homogeneity hold obviously.

(iii) $\Rightarrow$ (i) By the horizontality, it follows that $\eta(\partial_{t_i}, \partial_{t_j})$ $(i, j = 1, \ldots, n)$ are constants. From the homogeneity, it holds that

$$(w_i + w_j)\eta(\partial_{t_i}, \partial_{t_j}) = -2r\, \eta(\partial_{t_i}, \partial_{t_j}),$$

which implies that we can take a flat coordinate system $(t_1, \ldots, t_n)$ such that $\eta(\partial_{t_i}, \partial_{t_j}) = J_{ij}$ and we do so in the sequel. Then it holds that

$$\tilde{\mathcal{B}}^{(k)}_{i,n+1-j} = \eta(\partial_{t_i} \star \partial_{t_k}, \partial_{t_j}) = \eta(\partial_{t_i}, \partial_{t_k} \star \partial_{t_j}) = \tilde{\mathcal{B}}^{(k)}_{j,n+1-i},$$

which implies $\tilde{\mathcal{B}}^{(k)*} = \tilde{\mathcal{B}}^{(k)}$, $k = 1, \ldots, n$. Hence $\mathcal{T}^* = \mathcal{T}$ and $\mathcal{C}^* = \mathcal{C}$ also hold. $\square$

The function F in Proposition 4.11 is called a *prepotential* in [16] and a *potential* in [51]. It is well known that the prepotential satisfies the WDVV equation (cf. [16]).

5 Flat structure on the space of isomonodromic deformations

We start with an Okubo system in several variables:

$$dY = -(zI_n - T)^{-1}\left(dz + \sum_{i=1}^{n} \tilde{B}^{(i)} dx_i\right) B_\infty Y. \tag{58}$$

Let the assumptions on (58) be same as in Section 2. The purpose of this section is to find a necessary and sufficient condition for that the Okubo system in several variables (58) arises from a Saito structure. Here, to avoid the confusion, we state the precise definition of that an Okubo system in several variables (58) arises from a Saito structure.

Definition 5.1. We say that an Okubo system in several variables (58) arises from a Saito structure on U if there is a Saito structure on U such that there is a change of independent variables $(t_1, \dots, t_n) = (t_1(x), \dots, t_n(x))$, where $(t_1, \dots, t_n)$ is a flat coordinate system, and the matrices $\mathcal{T}$, $\mathcal{B}_\infty$, $\tilde{\mathcal{B}}^{(i)}$ $(i = 1, \dots, n)$ defined (in Section 4) from the Saito structure satisfy $\mathcal{T} = T$, $\mathcal{B}_\infty = B_\infty - (\lambda_n - 1)I_n$, $\tilde{\mathcal{B}}^{(i)} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial t_i} \tilde{B}^{(j)}$ $(i = 1, \dots, n)$.

We consider (58) restricted on an appropriate small domain $W \subset U \setminus \Delta_H$ so that we can take an invertible matrix $P = P(x)$ such that

$$P^{-1}TP = \mathrm{diag}\,[z_1(x), \dots, z_n(x)], \qquad P^{-1}\tilde{B}^{(i)}P = \mathrm{diag}\left[-\frac{\partial z_1(x)}{\partial x_i}, \dots, -\frac{\partial z_n(x)}{\partial x_i}\right].$$

Lemma 5.2. *An Okubo system in several variables (58) arises from a Saito structure on W if and only if $P_{nj} \neq 0$ $(j = 1, \dots, n)$ at any point on W.*

Proof. The lemma is proved along the line stated in [51, **4.a** p.242]. Let TW be the tangent bundle on W, $\mathbb{C}^n(W)$ be a trivial bundle of rank n on W and $(\mathbf{e}_1, \dots, \mathbf{e}_n)^t$ be a basis of $\mathbb{C}^n(W)$. Define endomorphisms $F, F^{(k)}$ $(k = 1, \dots, n)$ of $\mathbb{C}^n(W)$ respectively by

$$F(\mathbf{e}_i) = -\sum_{j=1}^{n} T_{ij}\mathbf{e}_j, \quad F^{(k)}(\mathbf{e}_i) = \sum_{j=1}^{n} \tilde{B}^{(k)}_{ij}\mathbf{e}_j.$$

Then the homomorphism φ from TW to $\mathbb{C}^n(W)$ defined by $\varphi(\partial_{x_i}) = F^{(i)}(\mathbf{e}_n)$ is isomorphic if and only if $\prod_{j=1}^{n} P_{nj} \neq 0$, which can be seen from $\mathbf{e}_n = (P_{n1}, \dots, P_{nn})P^{-1}(\mathbf{e}_1, \dots, \mathbf{e}_n)^t$ and thus

$$\varphi(\partial_{x_i}) = \Big(-\frac{\partial z_1}{\partial x_i}P_{n1}, \dots, -\frac{\partial z_n}{\partial x_i}P_{nn}\Big)P^{-1}(\mathbf{e}_1, \dots, \mathbf{e}_n)^t.$$

Put $e := \varphi^{-1}(\mathbf{e}_n)$, $E := \varphi^{-1}(F(\mathbf{e}_n))$ and introduce a Higgs field Φ on TW by

$$\Phi := \sum_{j=1}^{n} dx_j\, \varphi^{-1} \circ F^{(j)} \circ \varphi.$$

Besides, define a connection ∇ on TW by $\nabla := \varphi^{-1} \circ \nabla^{(0)} \circ \varphi$, where $\nabla^{(0)}$ denotes the connection on $\mathbb{C}^n(W)$ defined by $\nabla^{(0)}(\mathbf{e}_i) = 0$ $(i = 1, \ldots, n)$. Then it is confirmed that (∇, Φ, e, E) satisfies the axiom of Saito structure on W. $\square$

Lemma 5.3. *We consider an Okubo system in several variables (58). The following two conditions are equivalent to each other:*

(i) $\begin{vmatrix} \frac{\partial T_{n1}}{\partial x_1} & \cdots & \frac{\partial T_{nn}}{\partial x_1} \\ & \vdots & \\ \frac{\partial T_{n1}}{\partial x_n} & \cdots & \frac{\partial T_{nn}}{\partial x_n} \end{vmatrix} \neq 0$ *at any point on* W*,*

(ii) $P_{nj} \neq 0$ $(j = 1, \ldots, n)$ *at any point on* W*.*

Proof. From (21), it holds that

$$\frac{\partial T_{nj}}{\partial x_i} = (\lambda_n - \lambda_j - 1)\tilde{B}_{nj}^{(i)},$$

from which and (10) we have

$$\begin{pmatrix} \frac{\partial T_{n1}}{\partial x_1} & \cdots & \frac{\partial T_{nn}}{\partial x_1} \\ & \vdots & \\ \frac{\partial T_{n1}}{\partial x_n} & \cdots & \frac{\partial T_{nn}}{\partial x_n} \end{pmatrix} \begin{pmatrix} \lambda_n - \lambda_1 - 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}^{-1} P = \begin{pmatrix} \tilde{B}_{n1}^{(1)} & \cdots & \tilde{B}_{nn}^{(1)} \\ & \vdots & \\ \tilde{B}_{n1}^{(n)} & \cdots & \tilde{B}_{nn}^{(n)} \end{pmatrix} P$$
$$= - \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \cdots & \frac{\partial z_n}{\partial x_1} \\ & \vdots & \\ \frac{\partial z_1}{\partial x_n} & \cdots & \frac{\partial z_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} P_{n1} & & \\ & \ddots & \\ & & P_{nn} \end{pmatrix}.$$

Hence we obtain that

$$\begin{vmatrix} \frac{\partial T_{n1}}{\partial x_1} & \cdots & \frac{\partial T_{nn}}{\partial x_1} \\ & \vdots & \\ \frac{\partial T_{n1}}{\partial x_n} & \cdots & \frac{\partial T_{nn}}{\partial x_n} \end{vmatrix} \neq 0 \iff \prod_{j=1}^{n} P_{nj} \neq 0.$$

$\square$

From now on, we consider (58) on U not on $W \subset U \setminus \Delta_H$.

Theorem 5.4. *An Okubo system in several variables (58) arises from a Saito structure on U if and only if*

$$\begin{vmatrix} \frac{\partial T_{n1}}{\partial x_1} & \cdots & \frac{\partial T_{nn}}{\partial x_1} \\ & \vdots & \\ \frac{\partial T_{n1}}{\partial x_n} & \cdots & \frac{\partial T_{nn}}{\partial x_n} \end{vmatrix} \neq 0 \tag{59}$$

at any point on U. If (58) arises from a Saito structure on U, the set of variables $t_j := -(\lambda_j - \lambda_n + 1)^{-1} T_{nj}$, $j = 1, \ldots, n$ provides a flat coordinate system.

Proof. First, we assume that (58) arises from a Saito structure. Take a flat coordinate system $(t_1, \ldots, t_n)$ as independent variables of (58). Then it holds that $T = \mathcal{T}$ and $\mathcal{T}_{nj} = -w_j t_j$, from which we have

$$\begin{vmatrix} \frac{\partial T_{n1}}{\partial t_1} & \cdots & \frac{\partial T_{nn}}{\partial t_1} \\ \cdots & & \\ \frac{\partial T_{n1}}{\partial t_n} & \cdots & \frac{\partial T_{nn}}{\partial t_n} \end{vmatrix} = (-1)^n w_1 \cdots w_n \neq 0.$$

Conversely, we assume (59). In virtue of Lemmas 5.2 and 5.3, (58) arises from a Saito structure on $W \subset U \setminus \Delta_H$. We also see that $\{t_j = -(\lambda_j - \lambda_n + 1)^{-1} T_{nj}\}$ is a flat coordinate system. Then $E := \sum_{k=1}^{n} (\lambda_k - \lambda_n + 1) t_k \partial_{t_k}$, $e := \partial_{t_n}$ and $\Phi := \sum_{j=1}^{n} \tilde{B}^{(j)} dx_j = \sum_{j,k} \frac{\partial x_j}{\partial t_k} \tilde{B}^{(j)} dt_k$ satisfy the axiom of Saito structure on W. Due to the identity theorem, they satisfy it also on U. Hence (58) arises from a Saito structure on U. □

Theorem 5.4 is used to construct flat coordinates in Sections 6 and 7.

Remark 5.1. Uniqueness of the Saito structure corresponding to an Okubo system in several variables follows from the following argument. An Okubo system in several variables (58) admits the following types of gauge freedom: One is similarity transformations by a constant matrix C such that $C B_\infty C^{-1} = B_\infty$, which corresponds to the freedom of flat coordinate systems mentioned in Remark 4.5. The other is permutations on matrix entries. Let σ be a permutation on the set $\{1, 2, \ldots, n\}$. For an Okubo system in several variables (58), we consider the following change of dependent variables:

$$Y = (y_1, \ldots, y_n)^t \mapsto Y^\sigma := (y_{\sigma(1)}, \ldots, y_{\sigma(n)})^t.$$

Then Y^σ satisfies a new Okubo system in several variables

$$dY^\sigma = -(zI_n - T^\sigma)^{-1} \big(dz + \sum_{i=1}^{n} \tilde{B}^{(i)\sigma} dx_i\big) B_\infty^\sigma Y^\sigma, \tag{60}$$

where T^σ denotes the matrix whose (i,j)-entry is defined by $T_{ij}^\sigma := T_{\sigma(i),\sigma(j)}$ and the similar holds for $\tilde{B}^{(i)\sigma}, B_\infty^\sigma$. If

$$\begin{vmatrix} \frac{\partial T_{\sigma(n),1}}{\partial x_1} & \cdots & \frac{\partial T_{\sigma(n),n}}{\partial x_1} \\ & \vdots & \\ \frac{\partial T_{\sigma(n),1}}{\partial x_n} & \cdots & \frac{\partial T_{\sigma(n),n}}{\partial x_n} \end{vmatrix} \neq 0$$

holds at any point on U, the space of independent variables of (58) can be equipped with a Saito structure by applying Theorem 5.4 to the new system (60), which differs in general from that obtained from the original system (58). Therefore the space of independent variables of (58) can be equipped with at most n mutually different Saito structures up to isomorphisms. (Note that, if σ fixes n (i.e. $\sigma(n) = n$), then the Saito structure obtained from (60) is isomorphic to that obtained from (58). Indeed, σ induces only the permutation on the flat coordinates $(t_1, \ldots, t_n) \mapsto (t_1^\sigma := t_{\sigma(1)}, \ldots, t_n^\sigma := t_{\sigma(n)} = t_n)$.) We see from this argument that, if one of eigenvectors of B_∞ is designated, the corresponding Saito structure is uniquely specified up to isomorphisms (provided that the condition in Theorem 5.4 is satisfied). For example, if all of the eigenvalues are real numbers and satisfy $\lambda_1 \leq \cdots \leq \lambda_{n-1} < \lambda_n$, then we can distinguish λ_n (and the eigenvector belonging to it) from the remaining eigenvectors and a unique Saito structure can be determined corresponding to it. (The flat structure on the orbit space of a well-generated unitary reflection group introduced in Section 6 is in this case.) The initial value problem for regular Saito structures (including the non-semisimple case) is treated in [35].

Corollary 5.5. *We consider the case of $n = 3$. There is a correspondence between solutions satisfying the semisimplicity condition (SS) in Proposition 4.10 to the extended WDVV equation*

$$\sum_{m=1}^{3} \frac{\partial^2 g_m}{\partial t_k \partial t_i} \frac{\partial^2 g_j}{\partial t_l \partial t_m} = \sum_{m=1}^{3} \frac{\partial^2 g_m}{\partial t_l \partial t_i} \frac{\partial^2 g_j}{\partial t_k \partial t_m}, \quad i, j, k, l = 1, 2, 3,$$

$$\frac{\partial^2 g_j}{\partial t_3 \partial t_i} = \delta_{ij}, \qquad i, j = 1, 2, 3,$$

$$E g_j = \sum_{k=1}^{3} w_k t_k \frac{\partial g_j}{\partial t_k} = (1 + w_j) g_j, \quad j = 1, 2, 3.$$

and generic solutions to the Painlevé VI equation

$$\begin{aligned}\frac{d^2y}{dt^2} =& \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} \\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right).\end{aligned}$$

Let T be the 3×3 matrix whose entries are given by

$$T_{ij} = -(1 - w_i + w_j)\frac{\partial g_j}{\partial t_i},$$

and take P such that $P^{-1}TP$ is a diagonal matrix. Let

$$r_i = -\left(P\, diag[w_1 - w_3, w_2 - w_3, 0] P^{-1}\right)_{ii}, \quad i = 1, 2, 3$$

and $\theta_\infty = w_1 - w_2$, where we remark that r_i is an explicit form of that in $(A5)$ in Section 2. Then the correspondence between the parameters is given by

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad \beta = -\frac{1}{2}r_1^2, \quad \gamma = \frac{1}{2}r_2^2, \quad \delta = \frac{1}{2}(1 - r_3^2).$$

Proof. Since the condition (59) in Theorem 5.4 is generic one, we have the correspondence between generic Okubo systems in several variables and solutions to the extended WDVV equation satisfying the semisimplicity condition. The correspondence between the Okubo systems in several variables of rank 3 and generic solutions to the Painlevé VI equation is shown in Proposition A.1. □

Remark 5.2. In the recent papers [1, 43], Arsie and Lorenzoni treated the relationship between semisimple bi-flat structures and the sigma form of Painlevé VI equation via generalized Darboux-Egorov systems. In particular, [43] shows that the three-dimensional regular semisimple bi-flat F-manifolds are parameterized by generic solutions to the (full-parameter) Painlevé VI equation. Recently it is proved in [3, 41] that bi-flat F-manifold is an equivalent notion to Saito structure. Therefore Corollary 5.5 provides another proof of Arsie-Lorenzoni's result. The proof here makes clear the relationship between flat structures and isomonodromic deformations.

In [2], Arsie and Lorenzoni study the relationship between three-dimensional regular non-semisimple bi-flat F-manifolds and Painlevé IV and V equations. Then it is naturally expected that there is a correspondence between solutions to the extended WDVV equation not satisfying the semisimple condition (SS) and the isomonodromic deformations of generalized Okubo systems introduced in [34]. This is treated in [35].

Theorem 5.4 on the existence of a Saito structure associated with a given Okubo system in several variables can be rephrased as follows.

Theorem 5.6. *We consider the reduced form (25) in Remark 2.5 of an Okubo system in several variables (58) with* $\det B_\infty \neq 0$*:*

$$dY_0 = B_0 Y_0 = \Big(\sum_{i=1}^{n} T^{-1}\tilde{B}^{(i)}B_\infty dx_i\Big)Y_0. \tag{61}$$

Then the following two conditions are equivalent:

(i) *It holds that*

$$\begin{vmatrix} \frac{\partial T_{n1}}{\partial x_1} & \cdots & \frac{\partial T_{nn}}{\partial x_1} \\ & \vdots & \\ \frac{\partial T_{n1}}{\partial x_n} & \cdots & \frac{\partial T_{nn}}{\partial x_n} \end{vmatrix} \neq 0$$

at any point on U.

(ii) *For the divisor $D = \{\det(-T) = 0\}$, there are logarithmic vector fields along D $V_1, \ldots, V_n$ such that, for any solution $Y_0 = (y_1, \ldots, y_n)^t$ to (61), $y_j = V_{n+1-j} y_n$, $j = 1, \ldots, n$ hold and $(\det(-T))^{-1} \cdot \det M_V \in \mathcal{O}_U$, where M_V is a matrix defined by $(V_n, \ldots, V_1)^t = M_V(\partial_{x_1}, \ldots, \partial_{x_n})^t$.*

Proof. From (61), we see that $(\partial y_n/\partial x_1, \ldots, \partial y_n/\partial x_n)^t = MY_0$ holds, where we put

$$M := \begin{pmatrix} \tilde{B}^{(1)}_{n1} & \cdots & \tilde{B}^{(1)}_{nn} \\ & \vdots & \\ \tilde{B}^{(n)}_{n1} & \cdots & \tilde{B}^{(n)}_{nn} \end{pmatrix} T^{-1} B_\infty.$$

<u>(i) $\Rightarrow$ (ii)</u> In virtue of Theorem 5.4, we can take flat coordinates $(t_1, \ldots, t_n) = (\mathcal{C}_{n1}, \ldots, \mathcal{C}_{nn})$ as independent variables of (61). Then it holds that $M = T^{-1} B_\infty$ since $\tilde{B}^{(i)}_{nj} = \partial \mathcal{C}_{nj}/\partial t_i = \delta_{ij}$. For $M_V := M^{-1}$ and $(V_n, \ldots, V_1)^t := M_V(\partial_{t_1}, \ldots, \partial_{t_n})^t$, $V_1, \ldots, V_n$ are logarithmic vector fields along D and it holds that

$$(V_n y_n, \ldots, V_1 y_n)^t = M_V(\partial_{t_1} y_n, \ldots, \partial_{t_n} y_n)^t = M_V M Y_0 = Y_0.$$

<u>(ii) $\Rightarrow$ (i)</u> Assume that there are logarithmic vector fields along D $(V_n, \ldots, V_1)^t = M_V(\partial_{x_1}, \ldots, \partial_{x_n})^t$ such that $(\det(-T))^{-1} \det M_V \in \mathcal{O}_U$ and $Y_0'' = M_V M Y_0 = Y_0$. Then we have $\det(M_V M) = 1$, which implies that

$$\begin{vmatrix} \tilde{B}^{(1)}_{n1} & \cdots & \tilde{B}^{(1)}_{nn} \\ & \vdots & \\ \tilde{B}^{(n)}_{n1} & \cdots & \tilde{B}^{(n)}_{nn} \end{vmatrix} = \prod_{j=1}^{n} (\lambda_n - \lambda_j - 1) \begin{vmatrix} \frac{\partial T_{n1}}{\partial x_1} & \cdots & \frac{\partial T_{nn}}{\partial x_1} \\ & \vdots & \\ \frac{\partial T_{n1}}{\partial x_n} & \cdots & \frac{\partial T_{nn}}{\partial x_n} \end{vmatrix} \neq 0.$$

□

Proposition 5.7. *Assume that the reduced form of an Okubo system in several variables (61) arises from a Saito structure on U and take the flat coordinate system $(t_1, \ldots, t_n) = (\mathcal{C}_{n1}, \ldots, \mathcal{C}_{nn})$ as the independent variables of (61). Then there is a function $y = y(t)$ such that $Y_0 = (\partial y/\partial t_1, \ldots, \partial y/\partial t_n)^t$.*

Proof. We consider the "contiguous" equation of (61):

$$dY_0^{(-)} = B^{(-)} Y_0^{(-)} := T^{-1} d\mathcal{C}(B_\infty - I_n) Y_0^{(-)}. \tag{62}$$

For any solution $Y_0^{(-)} = (y_1^{(-)}, \ldots, y_n^{(-)})^t$ to (62), we put $Y_0'^{(-)} = (\frac{\partial y_n^{(-)}}{\partial t_1}, \ldots, \frac{\partial y_n^{(-)}}{\partial t_n})^t$ and $M^{(-)} = T^{-1}(B_\infty - I_n)$. Then we have

$$\begin{aligned} dY_0'^{(-)} &= \big(M^{(-)} B^{(-)} (M^{(-)})^{-1} + dM^{(-)} (M^{(-)})^{-1}\big) Y_0'^{(-)} \\ &= \big(T^{-1}(B_\infty - I_n) d\mathcal{C} - T^{-1} dT\big) Y_0'^{(-)} \\ &= T^{-1}\big((B_\infty - I_n) d\mathcal{C} + d\mathcal{C} + [d\mathcal{C}, B_\infty]\big) Y_0'^{(-)} = T^{-1} d\mathcal{C} B_\infty Y_0'^{(-)}. \end{aligned}$$

Hence we can take $y = y_n^{(-)}$ as the desired function. □

6 Flat generator system for invariant polynomials of a complex reflection group

In this section, we treat a problem on the existence of flat basic invariants for a complex reflection group. In the case of real reflection groups, K. Saito [53] proved the existence of flat basic invariants (see also [54]). For a well-generated complex reflection group G, we construct a Saito structure on the orbit space of G using an Okubo system in several variables called G-quotient system (Theorem 6.2). (The G-quotient system is a Pfaffian system whose fundamental system of solutions consists of derivatives by logarithmic vector fields of linear coordinates on the standard representation space of G and its monodromy group is isomorphic to G, see Theorem 6.2 and Remark 6.1.) As a consequence, we find that the potential vector field of the Saito structure for a well-generated complex reflection group G has polynomial entries (Corollary 6.8). It is underlined that the following proof of Theorem 6.2 is constructive i.e. it contains an algorithm which provides explicit computation of the flat generator system of G-invariants and the potential vector field for each well-generated complex reflection group G. See [29] for explicit formulas of potential vector fields for exceptional groups. (There is a procedure to construct flat basic invariants directly from a potential vector field corresponding to a finite complex reflection group, see Remark 6.2.)

Let G be a finite irreducible complex (unitary) reflection group acting on the standard representation space

$$U_n = \{(u_1, u_2, \dots, u_n) \,|\, u_j \in \mathbb{C}\},$$

and let $F_i(u)$ of degree d_i, $1 \le i \le n$ be a fundamental system of G-invariant homogeneous polynomials. We assume that

$$d_1 \le d_2 \le \cdots \le d_n.$$

We define a coordinate functions on $X := U_n/G$ by, $x_i = F_i(u)$, $1 \le i \le n$. Let $D \subset X$ be the branch locus of $\pi_G : U_n \to X$. Let $h(x)$ be the (reduced) defining function of D in the coordinates $x = (x_1, x_2, \dots, x_n)$. We assume that G is well generated (see e.g. [5]). Then it is known that $h(x)$ is a monic polynomial in x_n of degree n ([5]). We define a weight $w(\cdot)$ by $w(x_i) = d_i/d_n$. Then $h(x)$ is a weighted homogeneous polynomial in x. It is known that D is free ([47, 60]). Here we give a key lemma for the discriminant $h(x)$.

Lemma 6.1. *$H(x, z) := h(x_1, \dots, x_{n-1}, x_n + z)$ satisfies the assumption* (A3).

Proof. It is known ([5]) that there is a generator $\{V_1, \dots V_n\}$ of logarithmic vector fields along D such that $V_i = \sum_{j=1}^{n} v_{ij}(x)\, \partial_{x_j}$, $1 \le i \le n$ are weighted homogeneous and satisfy

$$M_V - x_n I_n \in \mathbb{C}[x']^{n \times n}. \tag{63}$$

In particular, it holds that

$$\deg_{x_n}(v_{ij}(x)) = \begin{cases} 1 & \text{if } i+j=n+1, \\ 0 & \text{if } i+j \neq n+1. \end{cases}$$

Let $V = \sum_{i=1}^{n} c_i(x')\,\partial_{x_i}$ be any logarithmic vector field along D. Then there are weighted homogeneous polynomials $a_i(x) \in \mathbb{C}[x]$ such that $V = \sum_{i=1}^{n} a_i(x)V_i$, which is equivalent to

$$c_j(x') = \sum_{i=1}^{n} a_i(x)v_{ij}(x), \quad 1 \le j \le n. \tag{64}$$

Let $I = \{i \,|\, a_i(x) \neq 0\}$, and assume $I \neq \emptyset$. Let $i_0 \in I$ be such that

$$\deg_{x_n}(a_{i_0}) \ge \deg_{x_n}(a_i),\ i \in I.$$

Then

$$\deg_{x_n}(a_{i_0}(x)v_{i_0,n+1-i_0}(x)) = \deg_{x_n}(a_{i_0}(x)) + 1 > \deg_{x_n}(a_i(x)v_{i,n+1-i_0}(x)),\ i \neq i_0,\, i \in I,$$

which implies

$$\deg_{x_n}\left(\sum_{i=1}^{n} a_i(x)v_{i,n+1-i_0}(x)\right) = \deg_{x_n}(a_{i_0}(x)v_{i_0,n+1-i_0}(x)) > 0 = \deg_{x_n}(c_{n+1-i_0}(x')),$$

which contradicts the equality (64) with $j = n+1-i_0$. This implies that $I = \emptyset$, that is, $V = 0$. Then Lemma 3.2 asserts that $H(x,z)$ satisfies (A3). □

Let

$$B_\infty = \mathrm{diag}\,[w(x_1), w(x_2), \ldots, w(x_n)] - \left(1 + \frac{1}{d_n}\right) I_n. \tag{65}$$

In this section, we prove the following theorem:

Theorem 6.2. *There are special G-invariant homogeneous polynomials $F_i^{fl}(u)$ of degree d_i, $1 \le i \le n$, generating $\mathbb{C}[u]^G$ and satisfying the following conditions:*

(i) *Let $t_i = F_i^{fl}(u)$, and let $h(t) = t_n^n - s_1(t')t_n^{n-1} + \ldots$ be the defining function of D in this coordinates $t = (t_1, t_2, \ldots, t_n)$. Let $V_i^{(h)}(t)$, $1 \le i \le n$ and $M_{V^{(h)}}(t)$ be defined by (34) and (31) with respect to $h(t)$. Let $C(t)$ be the (weighted homogeneous) matrix satisfying $V_1^{(h)}C(t) = M_{V^{(h)}}(t)$. Then, for any homogeneous linear function $y(u)$ of u,*

$$Y' = -B_\infty^{-1} \begin{pmatrix} V_n^{(h)}(t) & V_{n-1}^{(h)}(t) & \ldots & V_1^{(h)}(t) \end{pmatrix}^t y(u) \tag{66}$$

satisfies the Okubo system

$$dY' = \left[-\left(V_1^{(h)}C(t)\right)^{-1} dC(t)\, B_\infty\right] Y'. \tag{67}$$

Note that the n-th entry of Y' equals $y(u)$, that is $d_n V_1^{(h)}(t)\, y(u) = y(u)$.

(ii) *It holds that $C_{n,j}(t) = t_j$, $1 \le j \le n$, and hence $\{t_j\}$ gives a flat coordinate system on X associated to (67) by Theorem 5.4.*

If $d_1 < d_2 \cdots < d_n$, then $F_i^{fl}(u)$ are unique up to constant multiplications.

Definition 6.3. We call $\{F_i^{fl}(u)\}$ a *flat generator system* of G-invariant polynomials or *flat basic invariants* of G.

Remark 6.1. A *generating system* in [20] is a differential equation with a single unknown satisfied by homogeneous linear functions $y(u)$. In [32], the generating system is rewritten as a Pfaffian system and called a *G-quotient system.* The equation (67) is a G-quotient system in a form of Okubo type.

Again, we let $x_i = F_i(u)$, $1 \le i \le n$ for arbitrarily given fundamental system of G-invariant homogeneous polynomials, and $h(x)$ be the defining function of D in the coordinates x.

Let $H_i := \{\ell_i(u) = 0\}$, $1 \le i \le N$ be all the distinct reflecting hyperplanes of G. If H_i is the reflecting hyperplanes of $g_i \in G$ with the order m_i, then we have $g_i(H_i) = H_i$, $g_j(H_i) \ne H_i$ for $j \ne i$. As stated in [58]

$$h(x(u)) = c_1 \prod_{i=1}^{N} \ell_i(u)^{m_i}, \quad \det\left(\frac{\partial x}{\partial u}\right) = c_2 \prod_{i=1}^{N} \ell_i(u)^{m_i - 1}, \tag{68}$$

for some $c_1, c_2 \in \mathbb{C}^\times$, where $\left(\frac{\partial x}{\partial u}\right) = \left(\frac{\partial x_j}{\partial u_i}\right)_{i,j=1,2,..,n}$.

Let

$$\tilde{h}(u) = \prod_{i=1}^{N} \ell_i(u), \quad \text{and} \quad \tilde{D} = \cup_{i=1}^{N} H_i.$$

Lemma 6.4. *Fix $i \in \{1, \dots, N\}$ arbitrarily. Let $u^{(0)}$ be a generic point of H_i, and $x^{(0)} = x(u^{(0)})$. Put $y_1(u) = l_i(u)$. Then $y_1(u)^{m_i}$ is g_i-invariant, and $h(x) = c_1(x) y_1(u)^{m_i}$ for some non-vanishing holomorphic function $c_1(x)$ in a neighborhood of $x^{(0)}$. There are g_i-invariant linearly independent homogeneous linear functions $y_k(u)$, $2 \le k \le n$. Then there are locally holomorphic functions $\eta_k(x)$ at $x^{(0)}$, $2 \le k \le n$ such that $y_k(u) = \eta_k(x(u))$, and that $\{h(x), \eta_2(x), \dots, \eta_n(x)\}$ is a local coordinate system at $x^{(0)}$.*

Let

$$M_V(x) = M_{V^{(h)}}(x), \quad V_i(x) = V_i^{(h)}(x),\ 1 \le i \le n,$$

where $M_{V^{(h)}}(x)$, $V_i^{(h)}(x)$ are defined by (34) and (31) with respect to $h(x)$. Recall that $\det(M_V(x)) = h(x)$, and $V_1(x)$ is the Euler operator : $V_1(x) = \sum_{i=1}^{n} w(x_i)\, x_i\, \partial_{x_i}$.

Let $\tilde{V}_i$ be the pull-back of V_i to U_n, that is,

$$M_{\tilde{V}} = M_V \left(\frac{\partial x}{\partial u}\right)^{-1}, \quad \text{and} \quad \begin{pmatrix}\tilde{V}_n & \tilde{V}_{n-1} & \dots & \tilde{V}_1\end{pmatrix}^t = M_{\tilde{V}} \begin{pmatrix}\partial_{u_1} & \partial_{u_2} & \dots & \partial_{u_n}\end{pmatrix}^t.$$

Then, from (68), it holds that

$$\det(M_{\tilde{V}}) = c\,\tilde{h}(u), \tag{69}$$

for a constant number $c \neq 0$.

Lemma 6.5. (i) *The entries of $M_{\tilde{V}}$ are polynomials in u, $\tilde{V}_k$ $(1 \le k \le n)$ are logarithmic vector fields along $\tilde{D}$, and $\tilde{D}$ is free.*

(ii) *For any $k \in \{1, 2, \dots, n\}$, all entries of $(\tilde{V}_k\, M_{\tilde{V}})\, M_{\tilde{V}}^{-1}$ are G-invariant polynomials in u, that is, polynomials in x.*

Proof. (i) is known by Terao and others ([47, 60]). We prove (ii). Fix i arbitrarily, and we will prove $(\tilde{V}_k\, M_{\tilde{V}})M_{\tilde{V}}^{-1}$ is holomorphic along H_i. Let $y_1(u) = l_i(u)$, and choose $y_k(u) = l_{i_k}(u)$, $2 \le k \le n$, so that $y_1, y_2, \dots, y_n$ are linearly independent. Then (i) implies that

$$\tilde{V}_k = \sum_{j=1}^{n} y_j\, v_{k,j}(y)\, \partial_{y_j}, \quad 1 \le k \le n, \tag{70}$$

for some $v_{k,j}(y) \in \mathbb{C}[y]$. This is equivalent to

$$M_{\tilde{V}} = M'_{\tilde{V}}(y) \cdot \mathrm{diag}[y_1, y_2, \dots, y_n], \tag{71}$$

for some $M'_{\tilde{V}}(y) \in \mathbb{C}[y]^{n\times n}$. Then it also holds that $\tilde{V}_k\, M_{\tilde{V}} = M''_{\tilde{V}}(y) \cdot \mathrm{diag}[y_1, y_2, \dots, y_n]$, for some $M''_{\tilde{V}}(y) \in \mathbb{C}[y]^{n\times n}$. Thus we have

$$(\tilde{V}_k\, M_{\tilde{V}})M_{\tilde{V}}^{-1} = M''_{\tilde{V}}(y)\, (M'_{\tilde{V}}(y))^{-1}. \tag{72}$$

Since $\det(M_{\tilde{V}}) = \prod_{j=1}^{N} l_j(u)$, the equality (71) implies that $(M'_{\tilde{V}}(y))^{-1}$ is holomorphic along $\{y_1 = 0\}$. Consequently the equality (72) implies that $(\tilde{V}_k\, M_{\tilde{V}})M_{\tilde{V}}^{-1}$ is holomorphic along H_i. This proves that $(\tilde{V}_k\, M_{\tilde{V}})\, M_{\tilde{V}}^{-1} \in \mathbb{C}[u]^{n\times n}$.

We next prove that $(\tilde{V}_k\, M_{\tilde{V}})\, M_{\tilde{V}}^{-1}$ is G-invariant. Let $g \in G$, and $\gamma \in \pi_1(X \setminus D, *)$ be a loop such that $\gamma_* u = u\, g$. Then we have

$$\begin{aligned}
\gamma_* \left(\frac{\partial u}{\partial x}\right) &= \left(\frac{\partial u}{\partial x}\right) g, \\
\gamma_* M_{\tilde{V}} &= \gamma_* M_V \left(\frac{\partial u}{\partial x}\right) = M_{\tilde{V}}\, g, \\
\gamma_* \tilde{V}_k\, M_{\tilde{V}} &= \gamma_* V_k\, M_{\tilde{V}} = V_k\, M_{\tilde{V}}\, g = \tilde{V}_k\, M_{\tilde{V}}\, g.
\end{aligned}$$

Consequently we have $\gamma_*(\tilde{V}_k\, M_{\tilde{V}})\, M_{\tilde{V}}^{-1} = (\tilde{V}_k\, M_{\tilde{V}})\, M_{\tilde{V}}^{-1}$, which means that $(\tilde{V}_k\, M_{\tilde{V}})\, M_{\tilde{V}}^{-1}$ is G-invariant. This completes the proof of (ii). □

Let

$$(dM_{\tilde{V}})M_{\tilde{V}}^{-1} = \sum_{k=1}^{n} \hat{B}^{(k)}(x)\, dx_k,$$

and put

$$\hat{Y} = \begin{pmatrix} \hat{y}_1 & \hat{y}_2 & \dots & \hat{y}_n \end{pmatrix}^t = \begin{pmatrix} V_n & V_{n-1} & \dots & V_1 \end{pmatrix}^t y(u),$$

for any homogeneous linear function $y(u)$ of u. Then we have $\hat{y}_n = w(y)y = (1/d_n)\, y$, and

$$w(\hat{y}_i) = w(V_{n-i+1}) + (1/d_n) = (d_n - d_i + 1)/d_n, \quad 1 \le i \le n. \tag{73}$$

Lemma 6.6. *All entries of $h(x)\hat{B}^{(k)}(x)$ are weighted homogeneous polynomials in x, and $\hat{Y}$ satisfies the system of differential equations*

$$d\hat{Y} = \left[\sum_{k=1}^{n} \hat{B}^{(k)} dx_k\right] \hat{Y}. \tag{74}$$

Proof. Put $P_k(x) = (\tilde{V}_k\, M_{\tilde{V}})\, M_{\tilde{V}}^{-1}$ for $k = 1, 2, \dots n$. Then $P_k(x) \in \mathbb{C}[x]^{n\times n}$ from (iii) of Lemma 6.5. We have

$$\begin{aligned}(dM_{\tilde{V}})M_{\tilde{V}}^{-1} &= \sum_{k=1}^{n} \frac{\partial M_{\tilde{V}}}{\partial u_k} M_{\tilde{V}}^{-1} du_k = \sum_{j,k=1}^{n} (M_{\tilde{V}}^{-1})_{k,j} (\tilde{V}_{n-j+1} M_{\tilde{V}}) M_{\tilde{V}}^{-1} du_k \\ &= \sum_{j,k=1}^{n} (M_V^{-1})_{k,j} P_{n-j+1}(x) dx_k\end{aligned}$$

Consequently $\hat{B}^{(k)} = \sum_{j=1}^{n} (M_V^{-1})_{k,j}\, P_{n-j+1}(x)$, which implies $h(x)\hat{B}^{(k)}(x) \in \mathbb{C}[x]^{n\times n}$.

Since $y(u)$ is a solution of

$$d \begin{pmatrix} \partial_{u_1} & \partial_{u_2} & \dots & \partial_{u_n} \end{pmatrix}^t y = 0, \tag{75}$$

and since

$$\begin{aligned}\begin{pmatrix} \partial_{u_1} & \partial_{u_2} & \dots & \partial_{u_n} \end{pmatrix}^t &= \left(\frac{\partial x}{\partial u}\right) \begin{pmatrix} \partial_{x_1} & \partial_{x_2} & \dots & \partial_{x_n} \end{pmatrix}^t \\ &= \left(\frac{\partial x}{\partial u}\right) M_V^{-1} \begin{pmatrix} V_n & V_{n-1} & \dots & V_1 \end{pmatrix}^t,\end{aligned}$$

$y(u)$ satisfies

$$\left[d - (dM_{\tilde{V}})M_{\tilde{V}}^{-1}\right] \begin{pmatrix} V_n & V_{n-1} & \dots & V_1 \end{pmatrix}^t y = 0.$$

This proves the lemma. □

Lemma 6.7. *Let $\hat{Y}$, $\hat{B}^{(k)}$ be the same as in Lemma 6.6. Then there is an upper triangular matrix $R(x') \in GL(n, \mathbb{C}[x'])$ such that, if we put*

$$Y_0 = (y_1, y_2, \dots, y_n)^t = R(x')\hat{Y}, \quad B_0^{(k)} = R(x')\hat{B}^{(k)} R(x')^{-1} + \frac{\partial R(x')}{\partial x_k} R(x')^{-1}, \ 1 \le k \le n,$$

then the system of differential equations

$$dY_0 = \left[\sum_{k=1}^{n} B_0^{(k)}\, dx_k\right] Y_0 \tag{76}$$

satisfied by Y_0 *is an Okubo system in several variables with* $(B_0^{(k)})_\epsilon = 0$ *(See Lemma 2.1 for the* $\mathcal{E}$*-part* $(B_0^{(k)})_\epsilon$ *of* $B_0^{(k)}$*), and the residue matrix* $(B_0)_\infty$ *of* $B_0^{(n)} dx_n$ *at* $x_n = \infty$ *is equal to the diagonal matrix* B_∞ *in (65). The matrix* $R(x')$ *can be chosen such that all the entries are weighted homogeneous with* $w(R(x')_{i,j}) = w(\hat{y}_i) - w(\hat{y}_j)$*, and* $R(0) = I_n$*. In particular, it holds that* $y_n = \hat{y}_n = (1/d_n)y(u)$*.*

Proof. Let $\hat{B}_\infty(x') = -(x_n \hat{B}^{(n)})|_{x_n=\infty}$ be the residue matrix of $\hat{B}^{(n)} dx_n$ at $x_n = \infty$. From

$$\begin{aligned} w(\hat{B}_{i,j}^{(k)}(x')) &= w(\hat{y}_i) - w(\hat{y}_j) - w(x_k) \\ &= w(V_{n-i+1}) - w(V_{n-j+1}) - w(x_k) = -w(x_i) + w(x_j) - w(x_k), \end{aligned} \tag{77}$$

we find that the degree of $h(x)\hat{B}_{i,j}^{(n)}(x)$ in x_n is at most $n-1$, which implies that $-\hat{B}_\infty(x')$ is the coefficients of x_n^{n-1} of $h(x)\hat{B}^{(n)}(x)$. Consequently $(\hat{B}_\infty)_{i,j}(x')$ is a weighted homogeneous polynomial in x' with

$$w((\hat{B}_\infty)_{i,j}(x')) = w(x_j) - w(x_i).$$

Let $n_1, n_2, ..., n_k$ be the positive integers such that $n_1 + n_2 + ... + n_k = n$,

$$w(x_1) = ... = w(x_{n_1}) < w(x_{n_1+1}) = ... = w(x_{n_1+n_2}) < ... \leq w(x_n).$$

Then $\hat{B}_\infty(x')$ has the form

$$\hat{B}_\infty(x') = \begin{pmatrix} R_1 & * & ... & * \\ O & R_2 & ... & * \\ .. & .. & ... & .. \\ O & O & ... & R_k \end{pmatrix},$$

for some $R_i \in \mathbb{C}^{n_i \times n_i}$, $1 \leq i \leq k$.

Since

$$\left(\sum_{k=1}^{n} w(x_k) x_k \hat{B}^{(k)}(x)\right) \hat{Y} = \left(\sum_{k=1}^{n} w(x_k) x_k \frac{\partial}{\partial x_k}\right) \hat{Y} = \mathrm{diag}[w(\hat{y}_1), w(\hat{y}_2), \ldots, w(\hat{y}_n)]\hat{Y},$$

for all solutions $\hat{Y}$ of (74), we have

$$\sum_{k=1}^{n} w(x_k) x_k \hat{B}^{(k)}(x) = \mathrm{diag}[w(\hat{y}_1), w(\hat{y}_2), \ldots, w(\hat{y}_n)].$$

In particular, we have

$$(x_n \hat{B}^{(n)}(x))|_{x'=0} = \mathrm{diag}[w(\hat{y}_1), w(\hat{y}_2), \dots, w(\hat{y}_n)], \tag{78}$$

which shows R_i are diagonal, and $\hat{B}_\infty(x')$ is an upper triangular matrix with the diagonal elements $-w(\hat{y}_i)$. Then, by elementary linear algebra, we find that there is an upper triangular matrix $R(x') \in GL(n, \mathbb{C}[x'])$ with the form

$$R(x') = \begin{pmatrix} I_{n_1} & * & \dots & * \\ O & I_{n_2} & \dots & * \\ \cdot\cdot & \cdot\cdot & \dots & \cdot\cdot \\ O & O & \dots & I_{n_k} \end{pmatrix},$$

and satisfying

$$R(x')\hat{B}_\infty(x')R(x')^{-1} = -\mathrm{diag}[w(\hat{y}_1), w(\hat{y}_2), \dots, w(\hat{y}_n)] = B_\infty.$$

By construction of $R(x')$, we find that all entries $R(x')_{i,j}$ are weighted homogeneous with $w(R(x')_{i,j}) = w(\hat{y}_i) - w(\hat{y}_j)$. Now put $B_0^{(k)} = R(x')\hat{B}^{(k)}R(x')^{-1} + \frac{\partial R(x')}{\partial x_k}R(x')^{-1}$, $1 \le k \le n$, $B_0 = \sum_{k=1}^n B_0^{(k)}\, dx_k$. Then we find that

$$\begin{aligned}(B_0)_\infty &= -(x_n B_0^{(n)})|_{x_n=\infty} = R(x')\left[-(x_n \hat{B}^{(n)})|_{x_n=\infty}\right] R(x')^{-1} = R(x')\hat{B}_\infty R(x')^{-1} \\ &= B_\infty.\end{aligned}$$

Since the residue matrix of $\hat{B}^{(n)}dx_n$ at zeros of $h(x', x_n)$ is of rank one and diagonalizable, so is the residue matrix of $B_0^{(n)}dx_n$. Consequently, we find that there exists a $n \times n$ matrix $T_0(x')$ such that

$$B_0^{(n)} = -(x_n I_n - T_0(x'))^{-1} B_\infty \tag{79}$$

at least at any generic point x' by an argument similar to the one in Appendix B. Let

$$h(x) = x_n^n - s_1(x')x_n^{n-1} + \dots, \quad h(x)B_0^{(n)} = \sum_{i=0}^{n-1} C_i(x')x_n^i,\ C_{n-1}(x') = -B_\infty.$$

Then, by induction, we find

$$C_{n-i}(x') = -\left(T_0(x')^{i-1} - s_1(x')T_0(x')^{i-2} + \dots + (-1)^{i-1}s_{i-1}(x')\right) B_\infty, \quad 1 \le i \le n.$$

In particular, it holds that $T_0(x') = s_1(x')I_n - C_{n-2}(x')B_\infty^{-1}$, which implies that all entries of $T_0(x')$ are weighted homogeneous polynomials in x'.

Finally we prove that the differential system (76) satisfies $(B_0^{(k)})_\epsilon = 0$ modifying the proof of Lemma 2.1. Similar to the equation (13), it holds that

$$h(x)B_0^{(k)} = \sum_{j=0}^{m_k} (hB_0^{(k)})_j\, x_n^j, \quad 1 \le k \le n,$$

where $(hB_0^{(k)})_j \in \mathbb{C}[x']^{n\times n}$, $(hB_0^{(n)})_{n-1} = -(B_0)_\infty$. From the equalities

$$w((B_0^{(k)})_{i,j}(x')) = w((\hat{B}^{(k)})_{i,j}(x')) = w(x_j) - w(x_i) - w(x_k),$$

we have $m_k \le n$ for $k \le n-1$ and $m_n = n-1$. For $m_k = n$, from the equality $[(hB_0^{(k)})_n, -(B_0)_\infty] = O$, we have $((hB_0^{(k)})_n)_{i,j}(x') = 0$ if $d_i \ne d_j$. If $d_i = d_j$, then $w\left(((hB_0^{(k)})_n)_{i,j}(x')\right) = -w(x_k)$, and hence $((hB_0^{(k)})_n)_{i,j}(x') = 0$. This conclude that $(hB_0^{(k)})_n = O$. Consequently $B_0^{(k)}(x)$ is decomposed in the form

$$B_0^{(k)}(x) = \sum_{j=1}^{n-1} \frac{(B_0^{(k)})_j(x')}{x_n - z_j(x')},$$

which proves $(B_0^{(k)})_\epsilon = 0$. Define $\tilde{B}_0^{(k)}(x')$ by

$$B_0^{(k)}(x) = -(x_n I_n - T_0(x))^{-1} \tilde{B}_0^{(k)}(x')(B_0)_\infty.$$

Then (21) implies $\tilde{B}_0^{(k)}(x') \in \mathbb{C}[x']^{n\times n}$. □

Now we prove Theorem 6.2.

Proof of Theorem 6.2. From now on, to avoid confusion, we denote by $h_x(x)$ the defining function of D in the coordinates x. Let B_∞, $R(x'), Y_0 = R(x')\hat{Y}$, $B_0^{(i)}$, $\tilde{B}_0^{(i)}$, $T_0(x')$ be as before. Let $\lambda_i = w(x_i) - 1 - 1/d_n$ so that $B_\infty = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. Then $w((T_0)_{ij}) = 1 - \lambda_i + \lambda_j$. Recall $V_i = V_i^{(h_x)}$ and $M_V = M_{V^{(h_x)}}$. Define V_i' by

$$\begin{pmatrix} V_n' & V_{n-1}' & \ldots & V_1' \end{pmatrix}^t = -B_\infty^{-1} R(x') \begin{pmatrix} V_n & V_{n-1} & \ldots & V_1 \end{pmatrix}^t.$$

In particular $V_1' = d_n V_1$. Then

$$Y' := -B_\infty^{-1} Y_0 = \begin{pmatrix} V_n' & \ldots & V_2' & V_1' \end{pmatrix}^t y(u) = \begin{pmatrix} V_n' y(u) & \ldots & V_2' y(u) & y(u) \end{pmatrix}^t \tag{80}$$

satisfies the equation $dY' = \left[\sum_{i=1}^n (B')^{(i)}\, dx_k\right] Y'$, where $(B')^{(i)} = B_\infty^{-1} B_0^{(i)} B_\infty$. Let $T'(x) = B_\infty^{-1}(T_0(x') - x_n I_n) B_\infty$, and $C'(x)$ be such that $T'(x) = -V_1\, C'(x)$. Then $\sum_{i=1}^n (B')^{(i)}\, dx_k = -(V_1 C'(x))^{-1}\, dC'(x)\, B_\infty$, and Y' satisfies

$$dY' = (T'(x))^{-1}\, dC'(x)\, B_\infty\, Y' = -(V_1 C'(x))^{-1}\, dC'(x)\, B_\infty\, Y'. \tag{81}$$

From Theorem 5.6, it holds that

$$\begin{vmatrix} \frac{\partial T'_{n1}}{\partial x_1} & \cdots & \frac{\partial T'_{nn}}{\partial x_1} \\ & \vdots & \\ \frac{\partial T'_{n1}}{\partial x_n} & \cdots & \frac{\partial T'_{nn}}{\partial x_n} \end{vmatrix} \ne 0,$$

which implies that

$$t_j = C'_{nj}(x), \quad 1 \le j \le n$$

form a coordinate system on X. Note

$$t_j \in \mathbb{C}[x'],\ 1 \le j \le n-1, \quad t_n - x_n \in \mathbb{C}[x'].$$

Put

$$F_j^{fl}(u) = C'_{nj}(F_1(u), F_2(u), \ldots, F_n(u)), \quad 1 \le j \le n.$$

Since $w(C'_{nj}) = w(x_j)$, the equalities $t_j = C'_{nj}(x)$, $1 \le j \le n$ are solved by weighted homogeneous polynomials $x_j = x_j(t)$, $1 \le j \le n$, and hence $\{F_j^{fl}(u)\}$ is a fundamental system of G-invariant polynomials. Let

$$T(t) = T'(x(t)), \quad C(t) = C'(x(t)), \quad h_t(t) = h_x(x(t)).$$

Then $h_t(t)$ is the defining function of D in the coordinates t; $C_{nj}(t) = t_j$, $1 \le j \le n$; $T(t) = -V_1\, C(t)$; and Y' satisfies

$$d\,Y' = T(t)^{-1}\, dC(t)\, B_\infty\, Y' = -(V_1 C(t))^{-1}\, dC(t)\, B_\infty\, Y'. \tag{82}$$

The fact $-T'(x) - x_n I_n \in \mathbb{C}[x']^{n\times n}$ implies that

$$-T(t) - t_n I_n \in \mathbb{C}[t']^{n\times n},$$

and Lemma 4.4 implies that

$$\textstyle\sum_{j=1}^n (-T(t)_{ij})\, \partial_{t_j},\ 1 \le i \le n \text{ are logarithmic vector fields along } D.$$

From these two properties and Lemma 3.9, we find $M_{V(h_t)}(t) = -T(t)$.

From the "(i) $\Rightarrow$ (ii)" part of the proof of Theorem 5.6, (82) implies

$$Y' = B_\infty^{-1} T(t) \begin{pmatrix} \partial_{t_1} & \ldots & \partial_{t_n} \end{pmatrix}^t y(u), \tag{83}$$

which is equivalent to (66). This completes the proof of the existence of a flat generator system of G-invariant polynomials satisfying (i) and (ii) in Theorem 6.2.

The uniqueness of a flat generator system under the condition

$$d_1 < d_2 < \cdots < d_n$$

is clear from Remarks 5.1 and 4.5.

$\square$

Corollary 6.8. *The potential vector field $\vec{g} = (g_1, \ldots, g_n)$ corresponding to the G-quotient system (67) is a polynomial in $t_1, \ldots, t_n$. In other words, the potential vector field of the G-quotient system yields a polynomial solution to the extended WDVV equation.*

Proof. By the above construction, $h(t)$ and $T = -M_{V(h)}(t)$ are polynomials in $t_1, \dots, t_n$. Thus, $C(t)$ and $\vec{g}$ are also polynomials. In fact, g_j are given by

$$g_j = \tfrac{1}{1+w(t_j)} \textstyle\sum_{i=1}^n w(t_i)\, t_i\, C_{ij}.$$

$\square$

Remark 6.2. Irreducible finite complex reflection groups are classified by Shephard-Todd [58]: Except for rank 1 groups, there are two infinite families A_n, $G(pq, p, n)$, plus 34 exceptional groups $G_4, G_5, \dots, G_{37}$. Explicit forms of potential vector fields for the exceptional groups are discussed in [29]. It is possible to directly compute the explicit form of flat basic invariants of a complex reflection group G from its potential vector field by the following procedure: Let $(x_1, \dots, x_n) = (F_1(u), \dots, F_n(u))$ be arbitrary basic invariants of G and write down the discriminant $h_x(x)$ of G in terms of x. On the other hand, one can write down the discriminant $h_t(t)$ of G in terms of t from the potential vector field in the manner described in Section 4 (i.e. $h_t(t) := \det(-\mathcal{T})$, where $\mathcal{T}$ is constructed from the potential vector field as in the proof of Proposition 4.10). Find a weight preserving coordinate change $t = t(x)$ such that $h_t(t(x)) = h_x(x)$. Then $\big(F_1^{fl}(u), \dots, F_n^{fl}(u)\big) := \big(t_1(F_1(u), \dots, F_n(u)), \dots, t_n(F_1(u), \dots, F_n(u))\big)$ provides flat basic invariants of G.

Remark 6.3. The papers [46, 17] treat Frobenius structures constructed on the orbit spaces of Shephard groups (which consist a subclass of complex reflection groups). In the case of some Shephard groups, one can find that the Saito structure constructed in Theorem 6.2 which corresponding to the G-quotient system of the Shephard group does not have a prepotential. Therefore in this case, the Saito structure in Theorem 6.2 is distinct from the Frobenius structure treated in [46, 17]. This phenomenon is naturally explained in the framework of this article: It is known that to each Shephard group there is an associated Coxeter group whose discriminant is isomorphic to that of the Shephard group. Then we see that there are (at least) two Okubo systems in several variables on the orbit space of the Shephard group which have singularities along the discriminant: one is the G-quotient system of the Shephard group, the other is the G-quotient system of the associated Coxeter group. The Frobenius structure on the orbit space of a Shephard group described in [46, 17] corresponds to the G-quotient system of the associated Coxeter group in this picture.

7 Examples of potential vector fields corresponding to algebraic solutions to the Painlevé VI equation

In this section we show some examples of potential vector fields in three variables which correspond to algebraic solutions to the Painlevé VI equation.

Algebraic solutions to the Painlevé VI equation were studied and constructed by many authors including N. J. Hitchin [23, 24], B. Dubrovin [16], B. Dubrovin -M. Mazzocco [18], P. Boalch [6, 7, 8, 9, 10], A. V. Kitaev [37, 38], A. V. Kitaev -R. Vidūnas [39, 61], K. Iwasaki [26]. The classification of algebraic solutions to the Painlevé VI equation was achieved by Lisovyy and Tykhyy [42]. We remark that all the algebraic solutions in the list of [42] had previously appeared in the literature (see [11] and references therein). One of the principal aims of our study is the determination of a flat coordinate system and a potential vector field for each of such algebraic solutions. In spite that this aim is still not succeeded because of complexity of computation, we show some examples of potential vector fields. Some of the results below are already given in [29]. Other examples can be found in [31]. From the construction of polynomial potential vector fields corresponding to finite complex reflection groups of rank three (Corollary 6.8) and Corollary 5.5, we obtain a class of algebraic solutions to the Painlevé VI equation. The relationship between finite complex reflection group of rank three and solutions to the Painlevé VI equation was first studied by Boalch [6]. (More precisely speaking, it was conjectured in [6] that the solutions obtained from finite complex reflection groups by his construction are algebraic and this conjecture was proved in his succeeding papers.) The construction in this paper answers the question 3) in the last part of [6]: "Is there a geometrical or physical interpretation of these solutions?"

It is remarkable that there are examples in Sections 7.4,7.5,7.6 whose potential vector fields have polynomial entries but the corresponding flat structures are not isomorphic to one on the orbit space of any finite complex reflection group because the free divisors defined by $F_{B_6}, F_{H_2}, F_{E_{14}}$ in Sections 7.4,7.5,7.6 respectively are not isomorphic to the discriminant of any finite complex reflection group. The existence of these examples suggests that an analogue of Hertling's theorem ([21]) does not hold in the case of non Frobenius manifolds. It would be an interesting problem to classify all the polynomial potential vector fields.

To avoid the confusion, we prepare the convention which will be used in the following. We treat the case $n = 3$. Let $t = (t_1, t_2, t_3)$ be a flat coordinate system and $\vec{g} = (g_1, g_2, g_3)$ denotes a potential vector field. Let $w(t_i)$ be the weight of t_i and assume $0 < w(t_1) <$

$w(t_2) < w(t_3) = 1$. The matrix C is defined by

$$C = \begin{pmatrix} \partial_{t_1} g_1 & \partial_{t_1} g_2 & \partial_{t_1} g_3 \\ \partial_{t_2} g_1 & \partial_{t_2} g_2 & \partial_{t_2} g_3 \\ \partial_{t_3} g_1 & \partial_{t_3} g_2 & \partial_{t_3} g_3 \end{pmatrix}$$

and $T = -\sum_{j=1}^{3} w(t_j) t_j \partial_{t_j} C$. In this section, an algebraic solution LTn means "Solution n" in Lisovyy-Tykhyy [42], pp.156-162.

7.1 Algebraic solutions related with icosahedron

We treat the three algebraic solutions to Painlevé VI obtained by Dubrovin [16] and Dubrovin-Mazzocco [18].

Icosahedral solution (H_3)

In this case,

$$w(t_1) = \frac{1}{5},\ w(t_2) = \frac{3}{5},\ w(t_3) = 1$$

and there is a prepotential defined by

$$F = \frac{t_2^2 t_3 + t_1 t_3^2}{2} + \frac{t_1^{11}}{3960} + \frac{t_1^5 t_2^2}{20} + \frac{t_1^2 t_2^3}{6}. \tag{84}$$

Then it follows from the definition that $g_j = \partial_{t_j} F$ $(j = 1, 2, 3)$ give the potential vector field $\vec{g} = (g_1, g_2, g_3)$.

We don't enter the details on this case. See [16, 18].

Great icosahedral solution $(H_3)'$

Let (t_1, t_2, t_3) be a flat coordinate system and their weights are given by

$$w(t_1) = \frac{3}{5},\ w(t_2) = \frac{4}{5},\ w(t_3) = 1.$$

We introduce an algebraic function z of t_1, t_2 defined by the relation

$$t_2 + t_1 z + z^4 = 0.$$

It is clear from the definition that $w(z) = \frac{1}{5}$. In this case, we consider the algebraic function of (t_1, t_2, t_3) defined by

$$F = \frac{t_2^2 t_3 + t_1 t_3^2}{2} - \frac{t_1^4 z}{18} - \frac{7 t_1^3 z^4}{72} - \frac{17 t_1^2 z^7}{105} - \frac{2 t_1 z^{10}}{9} - \frac{64 z^{13}}{585}.$$

Then we see that F is a solution to the WDVV equation. Indeed, we first define

$$C = \begin{pmatrix} \partial_{t_1}\partial_{t_3} F & \partial_{t_1}\partial_{t_2} F & \partial_{t_1}^2 F \\ \partial_{t_2}\partial_{t_3} F & \partial_{t_2}^2 F & \partial_{t_2}\partial_{t_3} F \\ \partial_{t_3}^2 F & \partial_{t_3}\partial_{t_2} F & \partial_{t_3}\partial_{t_1} F \end{pmatrix}.$$

Then $\partial_{t_i} C$ $(i = 1, 2, 3)$ commute to each other. This condition is equivalent to that F is a solution to the WDVV equation.

Great dodecahedron solution $(H_3)''$

Let (t_1, t_2, t_3) be a flat coordinate system and their weights are given by

$$w(t_1) = \frac{1}{3},\ w(t_2) = \frac{2}{3},\ w(t_3) = 1.$$

We introduce an algebraic function z of t_1, t_2 defined by the relation

$$-t_1^2 + t_2 + z^2 = 0.$$

It is clear from the definition that $w(z) = \frac{1}{3}$. In this case, we consider the algebraic function of (t_1, t_2, t_3) defined by

$$F = \frac{t_2^2 t_3 + t_1 t_3^2}{2} + \frac{4063 t_1^7}{1701} + \frac{19 t_1^5 z^2}{135} - \frac{73 t_1^3 z^4}{27} + \frac{11 t_1 z^6}{9} - \frac{16 z^7}{35}.$$

Then F is also a solution of the WDVV equation.

Remark 7.1. The prepotential F for the icosahedral solution (H_3) was firstly obtained by B. Dubrovin [16]. The above algebraic solutions including the remaining two cases $(H_3)', (H_3)''$ were treated by B. Dubrovin and M. Mazzocco [18]. The icosahedral solution (H_3) is constructed by the polynomial F (cf. (84)). In the remaining two cases, the prepotentials are not polynomials but algebraic functions. The authors were informed by B. Dubrovin that his student Alejo Keuroghlanian computed the algebraic Frobenius manifold for the case of the great icosahedron $(H_3)'$ in his master thesis "Varieta di Frobenius algebraiche di dimensione 3". The authors don't know whether the potential for $(H_3)''$ is known or not. Topics on these solutions are treated in [30].

7.2 Algebraic solution related with the complex reflection group ST24 ([7])

The following case is related with the complex reflection group ST24.

Klein solution of P. Boalch [7] (LT8)

In this case,

$$w(t_1) = \frac{2}{7},\ w(t_2) = \frac{3}{7},\ w(t_3) = 1$$

$$\begin{aligned} g_1 &= (-2t_1^3 t_2 + t_2^3 + 12 t_1 t_3)/12, \\ g_2 &= (2t_1^5 + 5t_1^2 t_2^2 + 10 t_2 t_3)/10, \\ g_3 &= (-8t_1^7 + 21 t_1^4 t_2^2 + 7 t_1 t_2^4 + 28 t_3^2)/56. \end{aligned}$$

The determinant $\det(-T)$ is regarded as the discriminant of the complex reflection group ST24 if t_1, t_2, t_3 are taken as basic invariants (cf. [58]).

7.3 Algebraic solutions related with the complex reflection group ST27 (cf. [58])

The following two cases are related with the complex reflection group ST27.

Solution 38 of P. Boalch [8] (LT26)

In this case,

$$w(t_1) = \frac{1}{5},\ w(t_2) = \frac{2}{5},\ w(t_3) = 1$$

$$\begin{aligned} g_1 &= (-t_1^6 - 15t_1^4t_2 + 15t_1^2t_2^2 + 10t_2^3 + 30t_1t_3)/30, \\ g_2 &= (5t_1^7 + 3t_1^5t_2 + 15t_1^3t_2^2 - 5t_1t_2^3 + 6t_2t_3)/6, \\ g_3 &= (-105t_1^{10} + 200t_1^8t_2 + 350t_1^6t_2^2 + 175t_1^2t_2^4 - 14t_2^5 + 20t_3^2)/40. \end{aligned}$$

The determinant $\det(-T)$ is regarded as the discriminant of the complex reflection group ST27, if t_1, t_2, t_3 are taken as basic invariants.

Solution 37 of P. Boalch [8] (LT27)

In this case, z is an algebraic function of t_1, t_2 defined by

$$-t_2 - t_1z + 2z^3 = 0.$$

$$w(t_1) = \frac{2}{5},\ w(t_2) = \frac{3}{5},\ w(t_3) = 1,\ w(z) = \frac{1}{5}$$

$$\begin{aligned} g_1 &= (175t_1t_3 - 70t_1^3z + 70t_1^2z^3 + 378t_1z^5 - 540z^7)/175, \\ g_2 &= (10t_1^4 - 120t_1t_2^2 + 75t_2t_3 + 30t_1^2z^4 - 192t_1z^6 + 324z^8)/75, \\ g_3 &= (16t_1^5 + 80t_1^2t_2^2 + 25t_3^2 - 80t_1^3z^4 + 540t_1^2z^6 - 1080t_1z^8 + 432z^{10})/50. \end{aligned}$$

The determinant $\det(-T)$ is regarded as the discriminant of the complex reflection group ST27, if z, t_1, t_3 are taken as basic invariants.

7.4 Algebraic solutions related with the polynomial F_{B_6} in [56]

We recall the polynomial

$$F_{B_6} = 9xy^4 + 6x^2y^2z - 4y^3z + x^3z^2 - 12xyz^2 + 4z^3$$

which is a defining equation of a free divisor in $\mathbb{C}^3$ (cf. [56]). There are two algebraic solutions which are related with the polynomial F_{B_6}.

Solution 27 of Boalch [8] (LT13)

In this case,

$$w(t_1) = \frac{1}{15},\ w(t_2) = \frac{1}{3},\ w(t_3) = 1$$

$$\begin{aligned}
g_1 &= -\tfrac{1}{33}t_1(3t_1^{10}t_2 + 11t_2^3 - 33t_3),\\
g_2 &= \tfrac{1}{76}(-5t_1^{20} + 114t_1^{10}t_2^2 + 19t_2^4 + 76t_2t_3),\\
g_3 &= \tfrac{1}{870}(100t_1^{30} + 1740t_1^{20}t_2^2 - 5220t_1^{10}t_2^4 + 116t_2^6 + 435t_3^2).
\end{aligned}$$

The determinant $\det(-T)$ coincides with F_{B_6} by a weight preserving coordinate change up to a non-zero constant factor.

Solution obtained by A. Kitaev [38] (LT14)

In this case, z is an algebraic function of t_1, t_2 defined by

$$t_1^2 + t_2z^6 + z^{16} = 0.$$

$$w(t_1) = \frac{8}{15},\ w(t_2) = \frac{2}{3},\ w(t_3) = 1,\ w(z) = \frac{1}{15}$$

$$\begin{aligned}
g_1 &= -(2093t_1^4 - 897t_1t_3z^9 + 3450t_1^2z^{16} + 525z^{32})/(897z^9),\\
g_2 &= (-238t_1^5 + 85t_2t_3z^{15} + 1700t_1^3z^{16} - 750t_1z^{32})/(85z^{15}),\\
g_3 &= (49t_1^6 + 2415t_1^4z^{16} + 3t_3^2z^{18} + 795t_1^2z^{32} - 35z^{48})/(6z^{18}).
\end{aligned}$$

The determinant $\det(-T)$ regarded as a polynomial of z^5, t_2, t_3 coincides with F_{B_6} by a weight preserving coordinate change up to a non-zero constant factor.

7.5 Algebraic solutions related with the polynomial F_{H_2} in [56]

We recall the polynomial

$$F_{H_2} = 100x^3y^4 + y^5 + 40x^4y^2z - 10xy^3z + 4x^5z^2 - 15x^2yz^2 + z^3$$

which is a defining equation of a free divisor in $\mathbb{C}^3$ (cf. [56]). There are two algebraic solutions which are related with the polynomial F_{H_2}.

Solution 29 of P. Boalch [8] (LT18)

In this case,

$$w(t_1) = \frac{1}{10},\ w(t_2) = \frac{1}{5},\ w(t_3) = 1$$

$$\begin{aligned}
g_1 &= -t_1(5t_1^6t_2^2 - 14t_2^5 - 2t_3)/2,\\
g_2 &= (5t_1^{12} + 275t_1^6t_2^3 - 55t_2^6 + 33t_2t_3)/33,\\
g_3 &= (-100t_1^{18}t_2 + 2550t_1^{12}t_2^4 + 12750t_1^6t_2^7 + 595t_2^{10} + 9t_3^2)/18.
\end{aligned}$$

The determinant $\det(-T)$ regarded as a polynomial of t_2, t_1^6, t_3 coincides with F_{H_2} by a weight preserving coordinate change up to a non-zero constant factor.

Solution 30 of P. Boalch [8] (LT19)

In this case, z is an algebraic function of t_1, t_2 defined by

$$t_1^6 + t_2z^6 + z^9 = 0.$$

$$w(t_1)=\frac{3}{10},\ w(t_2)=\frac{3}{5},\ w(t_3)=1,\ w(z)=\frac{1}{5}$$

$$\begin{aligned}
g_1 &= t_1(-80t_2^2+910t_3z+165t_2z^3+63z^6)/(910z),\\
g_2 &= (4t_2t_3-12t_2^2z^2-36t_2z^5-27z^8)/4,\\
g_3 &= (-560t_1^{18}+595t_3^2z^{17}+7140t_2^2z^{21}-8160t_2z^{24}-15113z^{27})/(1190z^{17}).
\end{aligned}$$

The determinant $\det(-T)$ regarded as a polynomial of z, t_2, t_3 coincides with F_{H_2} by a weight preserving coordinate change up to a non-zero constant factor.

7.6 Algebraic solution related with E_{14}-singularity

Solution 13 of P. Boalch [9] (LT30)

In this case,

$$w(t_1)=\frac{1}{8},\ w(t_2)=\frac{3}{8},\ w(t_3)=1$$

$$\begin{aligned}
g_1 &= (5t_1^9-84t_1^6t_2-210t_1^3t_2^2+140t_2^3+9t_1t_3)/9,\\
g_2 &= (140t_1^{11}-165t_1^8t_2+924t_1^5t_2^2+770t_1^2t_2^3+11t_2t_3)/11,\\
g_3 &= (-95680t_1^{16}-432320t_1^{13}t_2+780416t_1^{10}t_2^2-58240t_1^7t_2^3+1019200t_1^4t_2^4\\
&\quad +203840t_1t_2^5+39t_3^2)/78.
\end{aligned}$$

The determinant $\det(-T)$ in this case coincides with the polynomial

$$\begin{aligned}
F_{E_{14}} &= -4x^6y^6-\tfrac{20}{3}x^3y^7-3y^8+30x^7y^3z+51x^4y^4z+24xy^5z-\tfrac{243}{4}x^8z^2\\
&\quad -108x^5yz^2-56x^2y^2z^2-8z^3
\end{aligned}$$

by a weight preserving coordinate change. The polynomial $F_{E_{14}}$ is regarded as a 1-parameter deformation of the defining polynomial of E_{14}-singularity in the sense of Arnol'd. In fact

$$F_{E_{14}}|_{x=0}=-3y^8-8z^3.$$

For topics related to $F_{E_{14}}$, see [57, 29].

A Okubo systems in several variables and isomonodromic deformations

Let us recall that an Okubo system in several variables of rank n is defined as follows:

$$dY=\left(B^{(z)}dz+\sum_{i=1}^{n}B^{(i)}dx_i\right)Y, \tag{85}$$

where $B^{(z)}=-(zI_n-T)^{-1}B_\infty$, $B^{(i)}=-(zI_n-T)^{-1}\tilde{B}^{(i)}B_\infty$, and $T, B_\infty, \tilde{B}^{(i)}$ are $n\times n$ matrices that satisfy the assumptions (A1)-(A5) and the equations (20),(21),(22) in

Section 2. In this Appendix A, we show the equivalence between Okubo systems in several variables (85) and isomonodromic deformations of Okubo systems (Lemma A.2). Especially in the case of $n=3$, this equivalence can be related to the Painlevé VI equation (a proof of the following Proposition A.1 is given at the end of this Appendix A):

Proposition A.1. *We consider (85) for $n=3$. Then there is a correspondence between Okubo systems in several variables of rank 3 and generic solutions to the Painlevé VI equation with the parameters $(\theta_0,\theta_1,\theta_t,\theta_\infty)=(r_1+\lambda_3,r_2+\lambda_3,r_3+\lambda_3,\lambda_1-\lambda_2)$, where λ_i and r_i are defined in (A2) and (A5) respectively in Section 2.*

Here, we briefly review the theory of isomonodromic deformation (following to [27, 28]). We consider a system of ordinary differential equations of rank m

$$\frac{dY}{dz}=\sum_{i=1}^{n}\frac{B_i}{z-a_i}Y, \tag{86}$$

where a_i $(i=1,\dots,n)$ are mutually distinct complex numbers and B_i $(i=1,\dots,n)$ are $m\times m$ constant matrices. Put $B_\infty:=-\sum_{i=1}^{n}B_i$. We assume that B_i $(i=1,\dots,n)$ are semisimple and B_∞ is diagonal, namely there are invertible matrices G_i $(i=1,\dots,n)$ such that $G_iB_iG_i^{-1}=\mathrm{diag}[r_1^{(i)},\dots,r_m^{(i)}]$ and $B_\infty=\mathrm{diag}[r_1^{(\infty)},\dots,r_m^{(\infty)}]$. We also assume $r_j^{(i)}-r_k^{(i)}\notin\mathbb{Z}\setminus\{0\}$ for $i=1,\dots,n,\infty$, $j,k=1,\dots,m$. Take a fundamental system of solutions $Y^{(\infty)}=Y^{(\infty)}(z)$ to (86) normalized around $z=\infty$ as follows:

$$Y^{(\infty)}=\hat{Y}^{(\infty)}(z^{-1})z^{-B_\infty},$$

where $\hat{Y}^{\infty}(z^{-1})$ is a convergent power series of z^{-1} such that $\hat{Y}^{(\infty)}(0)=I_m$. Fixing paths from $z=\infty$ to $z=a_i$ $(i=1,\dots,n)$, the analytic continuations of $Y^{(\infty)}$ to $z=a_i$ $(i=1,\dots,n)$ along the paths are described as follows:

$$Y^{(\infty)}=G_i\hat{Y}^{(i)}(z-a_i)\,(z-a_i)^{\Lambda_i}C_i,\qquad i=1,\dots,n,$$

where $\hat{Y}^{(i)}(z-a_i)$ is a convergent power series of $z-a_i$ such that $\hat{Y}^{(i)}(0)=I_m$, $\Lambda_i=G_iB_iG_i^{-1}$ and C_i is a constant invertible matrix. Now we deform (86) moving $a_1,\dots,a_n$ as variables, namely we suppose that B_i $(i=1,\dots,n)$ are functions of $a_1,\dots,a_n$. Then $Y^{(\infty)}$ is a function of $z,a_1,\dots,a_n$ and $r_j^{(i)},G_i,C_i$ are functions of $a_1,\dots,a_n$. A deformation of (86) with variables $a_1,\dots,a_n$ is said to be an isomonodromic deformation (or monodromy preserving deformation) if $r_j^{(i)}$ $(i=1,\dots,n,\infty,j=1,\dots,m)$ and C_i $(i=1,\dots,n)$ are constants independent of $a_1,\dots,a_n$. The following fact is well known (see [27], [28] for instance):

Fact 1. *A deformation of (86) is isomonodromic if B_i $(i=1,\dots,n)$ satisfy the following system of differential equations:*

$$dB_i=\sum_{j\neq i}[B_j,B_i]d\log(a_i-a_j),\qquad i=1,\dots,n, \tag{87}$$

which is called the Schlesinger system.

Returning to our situation, we consider an Okubo system in several variables of rank n (85). Let $z_1, \ldots, z_n$ be the roots of $\det(zI_n - T)$, and decompose $B^{(z)}$ into partial fractions

$$B^{(z)} = \sum_{i=1}^{n} \frac{B_i^{(z)}}{z - z_i}. \tag{88}$$

Lemma A.2. *The system of equations (20),(21),(22) is equivalent to the Schlesinger system*

$$dB_i^{(z)} = \sum_{j \neq i} [B_j^{(z)}, B_i^{(z)}] d \log(z_i - z_j), \qquad i = 1, \ldots, n. \tag{89}$$

Proof. On one hand, from Lemma 2.1, (20),(21),(22) are equivalent to the integrability condition of (85). On the other hand, it is known that the Schlesinger system (89) is equivalent to the integrability condition of the Pfaffian system

$$dY = \sum_{i=1}^{n} B_i^{(z)} d \log(z - z_i) Y. \tag{90}$$

Therefore, it is sufficient to show that the two Pfaffian systems (85) and (90) are equivalent to each other. Changing the independent variables $(x_1, \ldots, x_n)$ of (85) to $(z_1, \ldots, z_n)$, (85) is rewritten as

$$dY = -(zI_n - T)^{-1} \left(dz + \sum_{i,j=1}^{n} \tilde{B}^{(j)} \frac{\partial x_j}{\partial z_i} dz_i \right) B_\infty Y. \tag{91}$$

Let E_i be the matrix whose (j,k)-entry are defined by $(E_i)_{jk} = \delta_{ij}\delta_{ik}$. Then we have $\sum_{j=1}^{n} \tilde{B}^{(j)} \frac{\partial x_j}{\partial z_i} = -PE_iP^{-1}$ and thus

$$-(zI_n - T)^{-1} \sum_{j=1}^{n} \tilde{B}^{(j)} \frac{\partial x_j}{\partial z_i} B_\infty = PE_iP^{-1}B_\infty/(z - z_i).$$

It is straightforward to check that $B_i^{(z)} = -PE_iP^{-1}B_\infty$. $\square$

As a preparation to prove Proposition A.1, we quote from [28, Appendix C] (a concise explanation is found in [7]) the following fact on the relation between the Painlevé VI equation and the isomonodromic deformation of a system of linear differential equations of rank 2. (Notations in Fact 2 are valid only in this part.)

Fact 2. *We consider a Pfaffian system of rank* 2

$$dZ = \big(A(x,t)dx + B(x,t)dt\big)Z, \tag{92}$$

where

$$A(x,t) = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}, \quad B(x,t) = -\frac{A_t}{x-t}, \tag{93}$$

and A_0, A_1, A_t are 2×2 matrices whose entries are functions of t (independent of x). Put $A_\infty := -A_0 - A_1 - A_t$. We assume that A_0, A_1, A_t are rank 1 matrices and that A_∞ is a diagonal matrix. Put $\theta_i := \mathrm{tr} A_i$ $(i = 0, 1, t)$. Then A_i $(i = 0, 1, t)$ can be written as follows:

$$A_0 = \begin{pmatrix} z_0+\theta_0 & -uz_0 \\ u^{-1}(z_0+\theta_0) & -z_0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} z_1+\theta_1 & -vz_1 \\ v^{-1}(z_1+\theta_1) & -z_1 \end{pmatrix}, \quad A_t = \begin{pmatrix} z_t+\theta_t & -wz_t \\ w^{-1}(z_t+\theta_t) & -z_t \end{pmatrix}. \tag{94}$$

From the assumption that A_∞ is diagonal, we have the following relations:

$$uz_0 + vz_1 + wz_t = 0, \quad u^{-1}(z_0+\theta_0) + v^{-1}(z_1+\theta_1) + w^{-1}(z_t+\theta_t) = 0. \tag{95}$$

We put

$$\kappa_1 := -(z_0+\theta_0+z_1+\theta_1+z_t+\theta_t), \quad \kappa_2 := z_0+z_1+z_t. \tag{96}$$

Then it holds that $A_\infty = \mathrm{diag}[\kappa_1, \kappa_2]$ and

$$\kappa_1+\kappa_2+\theta_0+\theta_1+\theta_t = 0. \tag{97}$$

From the first equation of (95), we find that the $(1,2)$-entry of $A(x,t)$ is of the form

$$\frac{p(x)}{x(x-1)(x-t)}$$

for some linear polynomial $p(x)$ in x. Explicitly, $p(x)$ is expressible as $p(x) = k(x-y)$ with

$$k = (t+1)uz_0 + tvz_1 + wz_t, \quad y = k^{-1}tuz_0. \tag{98}$$

We define

$$\tilde{z} := \frac{z_0+\theta_0}{y} + \frac{z_1+\theta_1}{y-1} + \frac{z_t+\theta_t}{y-t}, \quad \tilde{\tilde{z}} := \tilde{z} - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_t}{y-t}. \tag{99}$$

It is underlined that $\tilde{z}$ and $\tilde{\tilde{z}}$ are given by the values at $x = y$ of the $(1,1)$-entry and $(2,2)$-entry of $A(x,t)$ respectively, namely $\tilde{z} = A_{11}(y,t)$ and $\tilde{\tilde{z}} = -A_{22}(y,t)$. From the *equations (95),(96),(98),(99), we obtain the following equalities:*

$$u = \frac{ky}{tz_0}, \quad v = -\frac{k(y-1)}{(t-1)z_1}, \quad w = \frac{k(y-t)}{t(t-1)z_t}, \tag{100}$$

$$z_0 = \frac{y}{t\theta_\infty}\Big(y(y-1)(y-t)\tilde{\tilde{z}}^2 + \big(\theta_1(y-t) + t\theta_t(y-1) - 2\kappa_2(y-1)(y-t)\big)\tilde{\tilde{z}} + \kappa_2^2(y-t-1) - \kappa_2(\theta_1 + t\theta_t)\Big), \tag{101}$$

$$z_1 = -\frac{y-1}{(t-1)\theta_\infty}\Big(y(y-1)(y-t)\tilde{\tilde{z}}^2 + \big((\theta_1+\theta_\infty)(y-t) + t\theta_t(y-1) - 2\kappa_2(y-1)(y-t)\big)\tilde{\tilde{z}} + \kappa_2^2(y-t) - \kappa_2(\theta_1 + t\theta_t) - \kappa_1\kappa_2\Big), \tag{102}$$

$$z_t = \frac{y-t}{t(t-1)\theta_\infty}\Big(y(y-1)(y-t)\tilde{\tilde{z}}^2 + \big(\theta_1(y-t) + t(\theta_t+\theta_\infty)(y-1) - 2\kappa_2(y-1)(y-t)\big)\tilde{\tilde{z}} + \kappa_2^2(y-1) - \kappa_2(\theta_1 + t\theta_t) - t\kappa_1\kappa_2\Big), \tag{103}$$

where we put $\theta_\infty := \kappa_1 - \kappa_2$. These equalities imply that $\{u, v, w, z_0, z_1, z_t\}$ and thus all the entries of A_0, A_1, A_t are expressed in terms of $\{y, \tilde{z}, k, \theta_0, \theta_1, \theta_t, \kappa_1, \kappa_2, t\}$. Then, the Schlesinger system (which is nothing but the integrability condition of (92))

$$\frac{dA_0}{dt} = \frac{1}{t}[A_t, A_0], \quad \frac{dA_1}{dt} = \frac{1}{t-1}[A_t, A_1], \quad \frac{dA_t}{dt} = \frac{1}{t}[A_0, A_t] + \frac{1}{t-1}[A_1, A_t] \tag{104}$$

is equivalent to the following system of differential equations

$$\frac{dy}{dt} = \frac{y(y-1)(y-t)}{t(t-1)}\left(2\tilde{z} - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_t - 1}{y-t}\right), \tag{105}$$

$$\frac{d\tilde{z}}{dt} = \frac{1}{t(t-1)}\big((-3y^2 + 2(1+t)y - t)\tilde{z}^2 + ((2y-1-t)\theta_0 + (2y-t)\theta_1 + (2y-1)(\theta_t - 1))\tilde{z} - \kappa_1(\kappa_2+1)\big), \tag{106}$$

$$\frac{d}{dt}\log k = (\theta_\infty - 1)\frac{y-t}{t(t-1)}, \tag{107}$$

$$\frac{d\theta_0}{dt} = \frac{d\theta_1}{dt} = \frac{d\theta_t}{dt} = \frac{d\kappa_1}{dt} = \frac{d\kappa_2}{dt} = 0. \tag{108}$$

(We note that, eliminating $\tilde{z}$ from (105) and (106), we obtain the Painlevé VI equation for y with $\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \beta = -\frac{1}{2}\theta_0^2, \gamma = \frac{1}{2}\theta_1^2, \delta = \frac{1}{2}(1-\theta_t^2)$.)

Now we shall prove Proposition A.1.

Proof of Proposition A.1. We first explain how to construct a solution of the Painlevé VI equation from an Okubo system in several variables (85) with $n = 3$. By the procedure explained in Appendix B, we obtain a completely integrable Pfaffian system of rank 2 replacing $B_\infty = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]$ to $\mathrm{diag}[\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, 0]$:

$$dZ = \Big(\Gamma^{(z)}dz + \sum_{i=1}^{3}\Gamma^{(i)}dx_i\Big)Z. \tag{109}$$

Changing the variables (x_1, x_2, x_3) to (z_1, z_2, z_3) and then setting $x = (z-z_1)/(z_2-z_1), t = (z_3-z_1)/(z_2-z_1)$, we find that (109) is changed into the form of (92) in Fact 2. Therefore we obtain a solution $y = y(t)$ to the Painlevé VI equation with $\alpha = \frac{1}{2}(\lambda_1 - \lambda_2 - 1)^2, \beta = -\frac{1}{2}(r_1 + \lambda_3)^2, \gamma = \frac{1}{2}(r_2 + \lambda_3)^2, \delta = \frac{1}{2}(1 - (r_3 + \lambda_3)^2)$.

Next we explain how to construct an Okubo system in several variables of rank 3 from a solution to the Painlevé VI equation. We take a solution $y = y(t)$ to the Painlevé VI equation with parameters $\{\theta_0, \theta_1, \theta_t, \theta_\infty\}$. From this y, we can construct $\tilde{z} = \tilde{z}(t)$ and $k = k(t)$ using (105) and (107) respectively. (We may freely take the integral constant of k.) Then, $\{y, \tilde{z}, k\}$ constructed in this way satisfy the equations (105), (106), (107), and thus we obtain a completely integrable Pfaffian system (92) of rank 2 using (100), (101), (102), (103). Changing the variables x, t to z, z_1, z_2, z_3 by $x = (z-z_1)/(z_2-z_1), t = (z_3-z_1)/(z_2-z_1)$ and applying the procedure in Appendix B, we obtain an Okubo system in several variables of rank 3. $\square$

B Okubo systems of rank n and Pfaffian systems of rank $n-1$

In this Appendix B, we explain a method of constructing an Okubo system in several variables of rank n from a completely integrable Pfaffian system of rank $(n-1)$. For this purpose we first treat an Okubo system in several variables of rank n

$$dY = \Big(B^{(z)}dz + \sum_{i=1}^{n} B^{(i)}dx_i\Big)Y, \tag{110}$$

where

$$\begin{aligned} &Y = (y_1, \dots, y_n)^t, \\ &B^{(z)} = -(zI_n - T)^{-1}B_\infty, \quad B^{(i)} = -(zI_n - T)^{-1}\tilde{B}^{(i)}B_\infty, \qquad i = 1, \dots, n. \end{aligned}$$

We construct a completely integrable Pfaffian system of rank $(n-1)$ from (110). From Remark 2.1, we can change the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of B_∞ such that one of them is equal to 0 and thus, we suppose $\lambda_n = 0$ without loss of generality. Then the Okubo system (110) turns out to be reducible and we obtain the rank $n-1$ Pfaffian system satisfied by $Z = (y_1, \dots, y_{n-1})^t$ that consists of the first $(n-1)$ entries of Y:

$$dZ = \Big(\Gamma^{(z)}dz + \sum_{i=1}^{n} \Gamma^{(i)}dx_i\Big)Z, \tag{111}$$

where

$$\Gamma^{(z)} = \begin{pmatrix} I_{n-1} & 0 \end{pmatrix} \left(-(zI_n - T)^{-1} B_\infty\right) \begin{pmatrix} I_{n-1} \\ 0 \end{pmatrix},$$

$$\Gamma^{(i)} = \begin{pmatrix} I_{n-1} & 0 \end{pmatrix} \left(-(zI_n - T)^{-1} \tilde{B}^{(i)} B_\infty\right) \begin{pmatrix} I_{n-1} \\ 0 \end{pmatrix}, \quad i = 1, \ldots, n.$$

The system (111) is also completely integrable.

The aim of this Appendix B is to explain a method of constructing an Okubo system in several variables of rank n from a system of the form (111). We start from a completely integrable Pfaffian system of rank $(n-1)$

$$dZ = \left(\Gamma^{(z)} dz + \sum_{i=1}^{n} \Gamma^{(i)} dx_i\right) Z, \tag{112}$$

where $\Gamma^{(z)}$ and $\Gamma^{(i)}$ are $(n-1) \times (n-1)$ matrices depending on (z, x). We assume the following conditions:

(B1) There is a monic polynomial $H(x, z)$ in z of degree n and analytic in x such that the entries of $H(z, x)\Gamma^{(z)}$ and $H(x, z)\Gamma^{(i)}$ are polynomials in z and holomorphic in x on $U \subset \mathbf{C}^n$. The degree of $H(z, x)\Gamma^{(z)}$ with respect to z is at most $n-1$ and the discriminant $\delta_H = \prod_{i<j}(z_i(x) - z_j(x))^2$ of $H(x, z) = \prod_{i=1}^{n}(z - z_i(x))$ does not identically vanish.

(B2) Note that we can write by (B1)

$$\Gamma^{(z)} = \sum_{j=1}^{n} \frac{\Gamma_j^{(z)}}{z - z_j(x)},$$

where each entry of $\Gamma_j^{(z)}$ is holomorphic on a domain $W \subset U \setminus \Delta_H$ and Δ_H denotes the bifurcation set of H. Then $\operatorname{rank} \Gamma_j^{(z)} \leq 1$ and $\operatorname{tr} \Gamma_j^{(z)} \neq \pm 1,\ j = 1, \ldots, n$.

(B3) $\Gamma_\infty = -\sum_{j=1}^{n} \Gamma_j^{(z)}$ is a constant diagonal matrix: $\Gamma_\infty = \operatorname{diag}[\lambda_1, \ldots, \lambda_{n-1}]$ and $\lambda_i \neq 0\ (i = 1, \ldots, n-1)$.

From (B2) and (B3), we see that $\Gamma_j^{(z)}$ is written as follows:

$$\Gamma_j^{(z)} = -\begin{pmatrix} b_{1j} \\ \vdots \\ b_{n-1,j} \end{pmatrix} \begin{pmatrix} a_{j1} & \cdots & a_{j,n-1} \end{pmatrix} \Gamma_\infty.$$

The equality $\Gamma_\infty = -\sum_{j=1}^{n} \Gamma_j^{(z)}$ implies

$$\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n-1,1} & \cdots & b_{n-1,n} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} \end{pmatrix} = I_{n-1}.$$

Find vectors $\begin{pmatrix} b_{n,1} & \cdots & b_{n,n} \end{pmatrix}$ and $\begin{pmatrix} a_{1,n} & \cdots & a_{n,n} \end{pmatrix}^t$ so that

$$\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} = I_n.$$

We put

$$P = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Let

$$B^{(z)} = -\sum_{j=1}^{n} \frac{1}{z - z_j} \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix} \begin{pmatrix} a_{j1} & \cdots & a_{jn} \end{pmatrix} B_\infty,$$

where $B_\infty = \mathrm{diag}[\lambda_1 + \lambda, \ldots, \lambda_{n-1} + \lambda, \lambda]$ for $\lambda \in \mathbb{C}$. Then we see that the system of linear differential equations

$$\frac{dY}{dz} = B^{(z)} Y \tag{113}$$

is an Okubo system since it holds that

$$P^{-1} B^{(z)} P = -\mathrm{diag}[z - z_1, \ldots, z - z_n]^{-1} P^{-1} B_\infty P.$$

Finally we obtain an Okubo system in several variables by extending (113) to an integrable Pfaffian system following Lemma 2.1.

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