Supersymmetric Path Integrals II: The Fermionic Integral and Pfaffian Line Bundles

Florian Hanisch* and Matthias Ludewig[†]

October 3, 2017

Abstract

The Pfaffian line bundle of the covariant derivative and the transgression of the spin lifting gerbe are two canonically given real line bundles on the loop space of an oriented Riemannian manifold. It has been shown by Prat-Waldron that these line bundles are naturally isomorphic as metric line bundles and that the isomorphism maps their canonical sections to each other. In this paper, we provide a vast generalization of his results, by showing that there are natural sections of the corresponding line bundles for any $N \in \mathbb{N}$, which are mapped to each other under this isomorphism (with the previously known being the one for N=0). These canonical sections are important to define the fermionic part of the supersymmetric path integral on the loop space.

1 Introduction

This is the second of a series of papers on supersymmetric path integrals. In the first paper, we constructed an integral map I_T for differential forms on the loop space LX of a Riemannian manifold X, where T > 0 is a real number. The purpose of this paper is to show that for a suitable subclass of differential forms θ on LX, this integral map is formally given by the equation

$$I_T[\theta] \stackrel{\text{formally}}{=} \int_{\mathsf{L}X} e^{-E/T + \omega} \wedge \theta$$
 (1.1)

in a very precise sense, where

$$E(\gamma) = \frac{1}{2} \int_{S^1} |\dot{\gamma}(t)|^2 dt \quad \text{and} \quad \omega[v, w] := \int_{S^1} \langle v(t), \nabla_{\dot{\gamma}} w(t) \rangle dt \quad (1.2)$$

^{*}Universität Potsdam. fhanisch@math.uni-potsdam.de

[†]Max-Planck-Institut für Mathematik in Bonn. maludewi@mpim-bonn.mpg.de

are the standard energy function and the canonical two form on the loop space, respectively. The right hand side of (1.1) is not well defined at first glance, but we will see that there is a very naive straightforward way to interpret this expression resulting in a rigorous mathematical quantity. The statement is then that $I_T[\theta]$ (defined as in our first paper [HL17a]) and the right hand side of (1.1) coincide. In fact, comparing the two definitions, this is quite remarkable, as they are seemingly of very different character. The main reason not to define I_T using this in the first place is that our definition from before has a larger domain of definition: The equality (1.1) holds only for those forms θ that pointwise lie in the space $\Lambda_{\sigma}^N L^2(S^1, \gamma^* T'X) \subset \text{Alt}^N(T'_{\gamma} LX)$, where Λ_{σ} denotes the exterior product, completed with respect to the Hilbert space tensor product.

Let us be more precise. The integrand in (1.1) is the product of the function $e^{-E/T}$ and the differential form $e^{\omega} \wedge \theta$. If LX was a finite-dimensional and oriented Riemannian manifold, in order to perform the integral (1.1), we could take the top degree component $[e^{\omega} \wedge \theta]_{\text{top}} = \langle e^{\omega} \wedge \theta, \text{vol} \rangle$ (where vol denotes the volume form with respect to a Riemannian metric) and then replace the differential form integral by an integral with respect to the Riemannian volume measure $d\gamma$, i.e.

$$\int_{LX} e^{-E/T+\omega} \wedge \theta = \int_{LX} e^{-E/T} [e^{\omega} \wedge \theta]_{\text{top}} d\gamma.$$
 (1.3)

Now in finite dimensions, if a two-form ω is given by $\omega[v,w] = \langle v,Aw \rangle$ for a skew-symmetric endomorphism A, then we have $[e^{\omega}]_{\text{top}} = \text{pf}(A)$, the Pfaffian of A. In our infinite-dimensional setting, with a view on (1.2), the role of A is played by $\nabla_{\dot{\gamma}}$, acting on the tangent space $T_{\gamma} \mathsf{L} X = C^{\infty}(S^1, \gamma^* TX)$. Hence in the case $\theta = 1$, it makes sense to define $[e^{\omega} \wedge \theta]_{\text{top}}$ as the zeta-regularized Pfaffian $\text{pf}_{\zeta}(\nabla_{\dot{\gamma}}) = \pm \det_{\zeta}(\nabla_{\dot{\gamma}})^{1/2}$ of $\nabla_{\dot{\gamma}}$. It turns out that more generally, there is a straightforward way to define $[e^{\omega} \wedge \theta]_{\text{top}}$ in a similar fashion for general θ . Namely, assuming for simplicity that $\ker(\nabla_{\dot{\gamma}}) = 0$, we set

$$[e^{\omega} \wedge \theta_N \wedge \dots \wedge \theta_1]_{\text{top}} := \pm \text{pf}_{\zeta}(\nabla_{\dot{\gamma}}) \, \text{pf}\left((\theta_a, \nabla_{\dot{\gamma}}^{-1} \theta_b)_{L^2}\right)_{N > a, b > 1}$$
(1.4)

for $\theta_1, \ldots, \theta_N \in L^2(S^1, \gamma^*T'X) \subset T'_{\gamma} \sqcup X$. This is precisely the formula one would get in the finite-dimensional setting, after replacing $\nabla_{\dot{\gamma}}$ by the matrix A and taking the ordinary Pfaffian instead of the regularized version. If one drops the assumption $\ker(\nabla_{\dot{\gamma}}) = 0$, there is a similar, slightly more complicated formula for $[e^{\omega} \wedge \theta]_{\text{top}}$, in analogy to the finite-dimensional case (see Def. 3.5 below).

The point of this paper is to show that the "naive" definition (1.4) for the top degree of $e^{\omega} \wedge \theta$ can be described in a different way, which was used in the definition of the integral map I_T in our previous paper. Namely, for a smooth loop γ of length one in a Riemannian spin manifold X, there is a bounded linear functional

$$q: \bigoplus_{N=0}^{\infty} \Lambda_{\sigma}^{N} L^{2}(S^{1}, \gamma^{*}T'X) \longrightarrow \mathbb{R}$$

defined by the combinatorial recipe¹

$$q(\theta) := 2^{-N/2} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} \operatorname{str}\left([\gamma \|_{\tau_N}^1]^{\Sigma} \prod_{j=1}^N \mathbf{c} \left(\theta_{\sigma_j}(\tau_j) \right) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\Sigma} \right) d\tau, \tag{1.5}$$

if $\theta = \theta_N \wedge \cdots \wedge \theta_1$ for $\theta_j \in L^1(S^1, \gamma^* T'X)$. Here $[\gamma |_s^t]^{\Sigma}$ denotes the parallel transport in the spinor bundle Σ along γ , $\mathbf{c} : \Lambda T'X \to \text{Cl}TX$ is the quantization map and

$$\Delta_N := \{ \tau = (\tau_N, \dots, \tau_1) \mid 0 \le \tau_1 \le \dots \le \tau_N \le 1 \}$$

is the standard simplex (see [HL17a] for these notions of spin geometry).

Notice that there is a sign ambiguity in (1.4), which is due to the fact that it is unclear which branch of the square-root of the zeta-regularized determinant one should take to define $\operatorname{pf}_{\zeta}(\nabla_{\dot{\gamma}})$ (note here that this determinant vanishes whenever there are parallel vector fields along γ , so one cannot just pick a uniform sign everywhere). It turns out that this sign ambiguity can be removed by interpreting (1.4) as an element of the *Pfaffian line bundle* Pf on LX associated to $\nabla_{\dot{\gamma}}$.

On the other hand, it is clear that (1.5) only makes sense on spin manifolds. In the general case, the expression can be interpreted as an element of another line bundle \mathcal{L} , which is the transgression to LX of the spin-lifting gerbe on X. Now these two line bundles are isomorphic, and they are trivial if and only if the manifold possesses a spin structure. Our main result is then the following.

Theorem 1.1. There is a natural isomorphism $\Phi : \mathcal{L} \to \mathsf{Pf}$ of metric line bundles which sends $q(\theta)$ to $[e^{\omega} \land \theta]_{\mathsf{top}}$, for every $\theta \in \Lambda^N_{\sigma} L^2(S^1, \gamma^* T'X)$.

The isomorphism Φ has been constructed before by Prat-Waldron in his thesis [PW09], with a different description of \mathcal{L} . Setting $\theta = 1$, our objects q(1) and $[e^{\omega}]_{\text{top}}$ reduce to the canonical sections $\text{str}[\gamma||_0^1]^{\Sigma}$ and $\text{pf}_{\zeta}(\nabla_{\dot{\gamma}})$ of \mathcal{L} , respectively Pf, and Prat-Waldron shows that these indeed correspond to each other via Φ . Our result can be seen as a generalization of this to all $N \in \mathbb{N}$. Notice here that $q(\theta)$ and $[e^{\omega} \wedge \theta]_{\text{top}}$ vanish whenever the dimension of X and the form degree N of θ have a different parity, while they are generally non-zero otherwise.

The integral map I_T defined in our first paper [HL17a] is given by

$$I_T[\theta] := \mathbb{W}_T \left[\exp\left(-\frac{T}{8} \int_{S^1} \operatorname{scal}(\gamma(t)) dt \right) q(\theta) \right], \tag{1.6}$$

where W_T is the Wiener measure on loops. Now, it is well known that the Wiener measure W_T is formally given by $e^{-E/T} d\gamma$ (suitably normalized), where $d\gamma$ is formally the Riemannian volume measure on the loop space. Again, this statement is void as the measure $d\gamma$ does not exist, but one can make this very precise using finite-dimensional

¹Here and throughout, we adopt the convention that $\prod_{j=1}^{N} a_j = a_N \cdots a_1$, which can differ by the opposite convention by a sign if the a_j are elements of a supercommutative algebra.

approximation, see e.g. [AD99], [BP08], [FS08] or [Lud17]. Taking this into account, if one compares (1.6) with the right hand side of (1.3) (and if one ignores the scalar curature term), one indeed obtains the formal equation (1.1), by virtue of our main theorem, Thm. 1.1.

Fermionic integrals have been studied extensively in mathematics and theoretical physics. Indeed, formula (1.3) closely resembles the construction of the Berezin integral on finite-dimensional supermanifolds, where superfunctions can be viewed as section of the exterior power of a certain vector bundle. Integration is then defined by first projecting out the top degree part of the integrand and then performing a an ordinary integral of the resulting function with respect to some volume measure. We will discuss these supergeometric aspects in our third paper [HL17c]. Moreover, Pfaffians have been used by many authors to define the Fermionic analogon of a Gaussian integral, see e.g. [Lot87], [GS99] and in particular [FKT02] as well as references therein. Aiming at a rigorous construction of certain physical field theories, [FKT02] also constructs infinite-dimensional Grassmann integrals through approximation by finite-dimensional Pfaffians and it should be interesting to relate our approach to theirs.

The paper is structured as follows. First we review some basic facts about the spectral theory of $\nabla_{\dot{\gamma}}$ and its zeta function necessary to define $\mathrm{pf}_{\zeta}(\nabla_{\dot{\gamma}})$. In Sections 3, we define $[e^{\omega} \wedge \theta_N \wedge \cdots \wedge \theta_1]_{\mathrm{top}}$ in analogy to the finite-dimensional case, generalizing (1.4) to the case that $\nabla_{\dot{\gamma}}$ has a kernel. In Section 4, we introduce the line bundles \mathcal{L} and Pf and construct an explicit isomorphism between them. This adapts the contents of [PW09] to our different description of \mathcal{L} , but we also believe that our construction is somewhat simpler then the one of [PW09]. In Section 5, we have to wade through a swamp of combinatorial arguments to evaluate integrals of the form

$$\mathcal{J}(k_1, \dots, k_N) := \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} \exp\left(2\pi i \sum_{a=1}^N k_{\sigma_a} \tau_a\right) d\tau$$

for numbers $k_1, \ldots, k_N \in \mathbb{Z}$, as they appear in (1.5). The results obtained there (see Thm. 5.1) may be of independent interest. Finally, in Section 6, we prove Thm. 1.1.

Acknowledgements. We thank the Max-Planck-Institute for Gravitational Physics in Potsdam-Golm (Albert-Einstein-Institute), the Max-Planck-Institute for Mathematics in Bonn as well as the Institute for Mathematics at the University of Potsdam for hospitality and financial support.

2 The Covariant Derivative along a Loop

Let $\gamma \in \mathsf{L} X := C^\infty(S^1, X)$ be a smooth loop in an *n*-dimensional Riemannian manifold X. The Riemannian structure of X induces a connection on the pullback γ^*TX of the tangent bundle over S^1 . We denote by $\nabla_{\dot{\gamma}}$ the operator acting on sections of γ^*TX by differentiating in direction of the canonical vector field ∂_t on S^1 using this pullback

connection. Integrating by parts using that the Levi-Civita connection is metric, we obtain that

$$(\nabla_{\dot{\gamma}}V, W)_{L^2} = -(V, \nabla_{\dot{\gamma}}W)_{L^2}$$

for vector fields $V, W \in T_{\gamma} \mathsf{L} X = C^{\infty}(S^1, \gamma^* TX)$, where

$$(V,W)_{L^2} := \int_{S^1} \langle V(t), W(t) \rangle dt$$
(2.1)

is the L^2 scalar product. Considered as an unbounded operator on $L^2(S^1, \gamma^*TX)$ with dense domain $C^\infty(S^1, \gamma^*TX)$, it is essentially skew-adjoint, meaning that it has a unique closed skew-adjoint extension, which is the Sobolev space $H^1(S^1, \gamma^*TX)$, the space of absolutely continuous vector fields along γ with derivative contained in L^2 .

The zeta function of $\nabla_{\dot{\gamma}}$ is the function

$$\zeta_{\nabla_{\dot{\gamma}}}(s) := \sum_{\lambda \neq 0} \lambda^{-s} = \sum_{\lambda \neq 0} e^{\operatorname{sign}(i\lambda)i\pi s/2} |\lambda|^{-s}, \tag{2.2}$$

where the sum goes over all non-zero eigenvalues of $\nabla_{\dot{\gamma}}$. Note that the eigenvalues λ are purely imaginary as $\nabla_{\dot{\gamma}}$ is skew-adjoint. The sum defines a holomorphic function for Re(s) > 1. This function has a meromorphic extension to all of \mathbb{C} , which is regular at zero. The (reduced) zeta determinant of $\nabla_{\dot{\gamma}}$ is then defined by

$$\det'_{\zeta}(\nabla_{\dot{\gamma}}) = e^{-\zeta_{\nabla_{\dot{\gamma}}}(0)},$$

motivated by the fact that the right hand side is formally the product of the non-zero eigenvalues, if one pretends for a moment that there are only finitely many eigenvalues. The zeta determinant itself is defined by

$$\det_{\zeta}(\nabla_{\dot{\gamma}}) = \begin{cases} \det'_{\zeta}(\nabla_{\dot{\gamma}}) & 0 \notin \text{spectrum of } \nabla_{\dot{\gamma}} \\ 0 & 0 \in \text{spectrum of } \nabla_{\dot{\gamma}}, \end{cases}$$

which of course equals the reduced zeta determinant if there are no zero modes.

Remark 2.1. Note that $\det_{\zeta}(\nabla_{\dot{\gamma}})$ is always zero if n is odd, because in odd dimensions, there is always a parallel vector field around any loop.

The eigenvalues of $\nabla_{\dot{\gamma}}$ are closely connected with those of the parallel transport operator $[\gamma|_0^t]^{TX}$. Since this is an orthogonal endomorphism of $T_{\gamma(0)}X$, there is an orthonormal basis e_1, \ldots, e_n of $T_{\gamma(0)}X$ such that the parallel transport is given by the matrix

$$[\gamma \parallel_{0}^{1}]^{TX} \stackrel{\frown}{=} \begin{pmatrix} \cos(2\pi\alpha_{1}) & -\sin(2\pi\alpha_{1}) \\ \sin(2\pi\alpha_{1}) & \cos(2\pi\alpha_{1}) \\ & \ddots \\ & & \cos(2\pi\alpha_{m}) & -\sin(2\pi\alpha_{m}) \\ & & \sin(2\pi\alpha_{m}) & \cos(2\pi\alpha_{m}) \end{pmatrix}$$

$$(2.3)$$

with respect to this basis, where $\alpha_1, \ldots, \alpha_m \notin \mathbb{Z}$ are certain real numbers. Let $E_j(t)$ be the parallel translations of the vectors e_j along γ ; note that for $j \leq 2m$, these have jumps when passing from t = 1 to t = 0. For $j = 1, \ldots, m$ and $k \in \mathbb{Z}$, we set

$$F_{2j-1,k}(t) := \cos(2\pi[k+\alpha_j]t)E_{2j-1}(t) - \sin(2\pi[k+\alpha_j])E_{2j}(t)$$

$$F_{2j,k}(t) := \sin(2\pi[k+\alpha_j]t)E_{2j-1}(t) + \cos(2\pi[k+\alpha_j])E_{2j}(t).$$
(2.4)

These are then smooth vector fields along γ and satisfy

$$\nabla_{\dot{\gamma}} F_{2j-1,k}(t) = -2\pi (k + \alpha_j) F_{2j,k}(t), \qquad \nabla_{\dot{\gamma}} F_{2j,k}(t) = 2\pi (k + \alpha_j) F_{2j-1,k}(t).$$

For j = 2m + 1, ..., n and $k \in \mathbb{N}_0$, we define

$$F_{j,k}^+(t) := \cos(2\pi kt)E_j(t), \qquad F_{j,k}^-(t) = \sin(2\pi kt)E_j(t).$$

These satisfy

$$\nabla_{\dot{\gamma}} F_{j,k}^+(t) = -2\pi k F_{j,k}^-(t), \qquad \nabla_{\dot{\gamma}} F_{j,k}^-(t) = 2\pi k F_{j,k}^+(t)$$

The $F_{j,k}$ together with the $F_{j,k}^{\pm}$ form an orthonormal basis of eigenfunctions for $-\nabla_{\hat{\gamma}}^2$. The eigenvalues of $\nabla_{\hat{\gamma}}$ are therefore the numbers

$$2\pi i(k \pm \alpha_j), \qquad k \in \mathbb{Z}, \quad j = 1, \dots, m \tag{2.5}$$

with multiplicity one each, as well as $2\pi i k$, $k \in \mathbb{Z}$, with multiplicity d = n - 2m. From this, we obtain an explicit formula for the zeta function in terms of the numbers α_i .

Lemma 2.2. Choose the numbers α_j such that that $0 < \alpha_j < 1$ for each j. Then for the zeta function as defined in (2.2), we find

$$\zeta_{\nabla_{\dot{\gamma}}}(s) = 2\cos\left(\frac{\pi}{2}s\right)(2\pi)^{-s}\left(2(n-2m)\zeta(s) + \sum_{j=1}^{m}\left(\zeta(s,1-\alpha_j) + \zeta(s,\alpha_j)\right)\right),$$

where for $\alpha > 0$,

$$\zeta(s,\alpha) = \sum_{k=0}^{\infty} (k+\alpha)^{-s}$$

is the Hurwitz zeta function and $\zeta(s)$ denotes the Riemann zeta function.

Proof. By (2.5), we obtain for Re(s) large that (writing d = n - 2m as before)

$$\zeta_{\nabla_{\dot{\gamma}}}(s) = d \sum_{k \neq 0} (2\pi i k)^{-s} + \sum_{j=1}^{m} \sum_{k \in \mathbb{Z}} \left(\left(2\pi i (k - \alpha_{j}) \right)^{-s} + \left(2\pi i (-k - \alpha_{j}) \right)^{-s} \right) \\
= 2d(2\pi)^{-s} e^{-is\pi/2} \sum_{k=1}^{\infty} k^{-s} + 2d(2\pi)^{-s} e^{is\pi/2} \sum_{k=1}^{\infty} k^{-s} \\
+ (2\pi)^{-s} \sum_{j=1}^{m} (e^{is\pi/2} + e^{-is\pi/2}) \left(\sum_{k=1}^{\infty} (k - \alpha_{j})^{-s} + \sum_{k=0}^{\infty} (k + \alpha_{j})^{-s} \right) \\
= 2\cos\left(\frac{\pi}{2}s\right) (2\pi)^{-s} \left(2d\zeta(s, 0) + \sum_{j=1}^{m} \left(\zeta(s, 1 - \alpha_{j}) + \zeta(s, \alpha_{j}) \right) \right),$$

which is the claimed result.

Inserting the known special values at zero for these zeta functions, get the following result, as already seen e.g. in [Ati85, Lemma 2], [Bis85, Eq. (2.13)] and [PW09, pp. 125ff],

Proposition 2.3. The zeta determinant is given by the formula

$$\det'_{\zeta}(\nabla_{\dot{\gamma}}) = \prod_{j=1}^{m} 4\sin(\pi\alpha_j)^2. \tag{2.6}$$

Proof. From the lemma above, obtain

$$\zeta'_{\nabla_{\dot{\gamma}}}(0) = -2\log(2\pi) \left(2d\zeta(0) + \sum_{j=1}^{m} \left(\zeta(0, 1 - \alpha_j) + \zeta(0, \alpha_j) \right) \right) + 2 \left(2d\zeta'(0) + \sum_{j=1}^{m} \left(\zeta'(0, 1 - \alpha_j) + \zeta'(0, \alpha_j) \right) \right).$$

Using the well-known special values $\zeta(0) = -\frac{1}{2}$, $\zeta(0,\alpha) = -\frac{1}{2} - \alpha$, $\zeta'(0) = -\frac{1}{2} \log(2\pi)$ and $\zeta'(0,\alpha) = \log(\Gamma(\alpha)) - \frac{1}{2} \log(2\pi)$, the terms coming from the kernel cancel, and we obtain

$$\zeta_{\nabla_{\dot{\gamma}}}'(0) = 2\sum_{j=1}^{m} \left(\log \left(\Gamma(1 - \alpha_j) \right) + \log \left(\Gamma(\alpha_j) \right) - \log(2\pi) \right)$$
$$= 2\sum_{j=1}^{m} \log \left(\frac{\Gamma(1 - \alpha_j)\Gamma(\alpha_j)}{2\pi} \right)$$
$$= -2\sum_{j=1}^{m} \log \left(2\sin(\pi\alpha_j) \right),$$

where in the last step we used Euler's reflection formula for the gamma function. The result follows since $\det'_{\zeta}(\nabla_{\dot{\gamma}}) = e^{-\zeta'_{\nabla_{\dot{\gamma}}}(0)}$.

3 Definition of the Top Degree Map

Let V be an n-dimensional, oriented Euclidean vector space and $A \in \mathfrak{so}(V)$ a skew symmetric endomorphism. Later $T_{\gamma} LX$ will play the role of V and the covariant derivative $\nabla_{\dot{\gamma}}$ will play the role of A; we will then see how to make up for the infinite-dimensionality of these objects.

Remember that for $A \in \mathfrak{so}(V)$, the Pfaffian is defined by

$$pf(A) = \frac{1}{2^{n/2} \left(\frac{n}{2}\right)!} \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma_1 \sigma_2} \cdots a_{\sigma_{n-1} \sigma_n}$$
(3.1)

in the case that n is even, while pf(A) = 0 if n is odd. Here, $(a_{ij})_{1 \le i,j \le n}$ is a matrix representation of A with respect to any oriented orthonormal basis of V. pf(A) is independent

from the choice of oriented orthonormal basis, but changing the orientation changes the sign of pf(A). We may associate to A the two-form $\omega \in \Lambda^2 V'$ defined by

$$\omega[v,w] := \langle v, Aw \rangle. \tag{3.2}$$

We are interested in calculating the top degree coefficient

$$[e^{\omega} \wedge \theta]_{\text{top}} = \langle e^{\omega} \wedge \theta, e_1 \wedge \cdots \wedge e_n \rangle$$

for arbitrary forms $\theta \in \Lambda V'$, where e_1, \ldots, e_n is an oriented orthonormal basis of V. Here ω is exponentiated in the exterior algebra. Since all elements of $\Lambda V'$ are nilpotent, this is given by a truncated exponential series. It is well-known that for $\theta = 1$, we have exactly $[e^{\omega}]_{\text{top}} = \text{pf}(A)$.

Theorem 3.1. Denote the dual endomorphism of A, acting on V', by the same letter A. Let $\vartheta_1, \ldots, \vartheta_N \in V'$ such that $\vartheta_1, \ldots, \vartheta_M \in \ker(A)^{\perp}$ for some $M \leq N$ and $\vartheta_{M+1}, \ldots, \vartheta_N \in \ker(A)$. Then

$$[e^{\omega} \wedge \vartheta_N \wedge \dots \wedge \vartheta_1]_{\text{top}} = [\vartheta_N \wedge \dots \wedge \vartheta_{M+1}]_{\text{top}} \operatorname{pf}'(A) \operatorname{pf}\left(\langle \vartheta_i, A^{-1}\vartheta_j \rangle\right)_{M \geq i, j \geq 1}, \quad (3.3)$$

 $[\vartheta_N \wedge \cdots \wedge \vartheta_{M+1}]_{\text{top}}$ denotes the top degree component inside $\ker(A)$ where $\operatorname{pf}'(A)$ denotes the Pfaffian of A, restricted to $\ker(A)^{\perp}$. These terms depend on an orientation on $\ker(A)$ respectively $\ker(A)^{\perp}$, and we choose orientations on these in such a way that they combine to the orientation on V.

Remark 3.2. Notice that the above formula determines $[e^{\omega}\theta]_{\text{top}}$ for all $\theta \in \Lambda V'$ by multilinearity, as we can aways write $\theta = \sum_a \theta'_a \wedge \theta''_a$ with $\theta'_a \in \Lambda \ker(A)$ and $\theta''_a \in \Lambda \ker(A)^{\perp}$.

Remark 3.3. In particular, we see that under the assumptions of Thm. 3.1, one have $[e^{\omega} \wedge \theta_N \wedge \cdots \wedge \theta_1]_{\text{top}} = 0$ unless $N - M = \dim \ker(A)$.

Remark 3.4. For definiteness, we remark that throughout, we use the convention

$$\operatorname{pf}\left(\langle \vartheta_{i}, A^{-1}\vartheta_{j} \rangle\right)_{M \geq i, j \geq 1} := \begin{pmatrix} \langle \vartheta_{M}, A^{-1}\vartheta_{M} \rangle & \dots & \vartheta_{M}, A^{-1}\vartheta_{1} \rangle \\ \vdots & \ddots & \vdots \\ \langle \vartheta_{1}, A^{-1}\vartheta_{M} \rangle & \dots & \vartheta_{1}, A^{-1}\vartheta_{1} \rangle \end{pmatrix}, \tag{3.4}$$

which differs from the one that enumerates the ϑ_j the other way around by a sign in the case that $M \equiv 2 \mod 4$.

Proof. First we reduce to the case $\ker(A) = 0$. Namely since e^{ω} is even,

$$e^{\omega} \wedge \theta_N \wedge \cdots \wedge \theta_1 = (\theta_N \wedge \cdots \wedge \theta_{M+1}) \wedge (e^{\omega} \wedge \theta_M \wedge \cdots \wedge \theta_1),$$

which can be considered as an element of $\Lambda \ker(A)^{\perp} \otimes \Lambda \ker(A)$. Hence choosing compatible orientations on $\ker(A)$ and $\ker(A)^{\perp}$, we get

$$[e^{\omega} \wedge \theta_N \wedge \cdots \wedge \theta_1]_{\text{top}} = [\theta_N \wedge \cdots \wedge \theta_{M+1}]_{\text{top}} [e^{\omega} \wedge \theta_M \wedge \cdots \wedge \theta_1]_{\text{top}},$$

where the top degree components are taken in the exterior algebra of $\ker(A)$ respectively $\ker(A)^{\perp}$. We are left to establish the formula

$$[e^{\omega}\theta_N \wedge \dots \wedge \theta_1]_{\text{top}} = \text{pf}(A) \, \text{pf}\left(\langle \theta_i, A^{-1}\theta_j \rangle \right)_{1 \leq i,j \leq N}$$
(3.5)

in the case $\ker(A) = 0$. Clearly, this assumption implies that n = 2m is even. First note that the right hand side clearly defines a multilinear, antisymmetric map $(V')^N \to \mathbb{R}$ and hence an element in $\Lambda^N V$. In particular, both sides are zero if N > n, where n is the dimension of V. Thus, it is sufficient to establish (3.5) for the elements in some basis of $\Lambda V'$. To this end, choose an oriented orthonormal basis e_1, \ldots, e_n in such way that

$$Ae_{2j-1} = -\lambda_j e_{2j}$$
 $Ae_{2j} = \lambda_j e_{2j-1}$ (3.6)

for $\lambda_j \neq 0$, j = 1, ..., m. In other words, with respect to the basis $e_1, ..., e_n$, A is represented by the matrix

$$A \stackrel{\triangle}{=} \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \\ & & \ddots \\ & & 0 \\ & & \lambda_m & 0 \end{pmatrix}$$

$$(3.7)$$

From (3.1) then follows that $\operatorname{pf}(A) = (-1)^m \lambda_1 \cdots \lambda_m$. The form ω obtained from A as in (3.2) may then be expressed in terms of the dual basis e_1^*, \ldots, e_n^* as

$$\omega = -\sum_{j=1}^{m} \lambda_j e_{2j-1}^* \wedge e_{2j}^*.$$

Now it remains to verify (3.3) for basis elements $e_{i_1} \wedge \cdots \wedge e_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq n$, which is an easy calculation.

Let now X be a Riemannian spin manifold of dimension n. For a smooth loop $\gamma \in \mathsf{L} X$, we let $T_\gamma \mathsf{L} X = C^\infty(S^1, \gamma^* T X)$ play the role of V in the above discussion. Of course, $T_\gamma \mathsf{L} X$ is not finite-dimensional, but it has a natural scalar product, namely the L^2 scalar product defined in (2.1), which turns it into an infinite-dimensional Euclidean space (or pre-Hilbert space). Furthermore, there is a canonical two-form on the loop space, given by (1.2) from the introduction. Notice that it is clear right away which anti-symmetric operator A corresponds to ω as before, namely, we have $\omega[v,w]=(v,\nabla_{\dot{\gamma}}w)_{L^2}$, so that $A=\nabla_{\dot{\gamma}}$, the covariant derivative along γ . We end up with the following definition:

Definition 3.5 (Top-Degree Map). Let $\gamma \in LX$ and let $\theta_1, \ldots, \theta_N \in L^2(S^1, \gamma^*T'X)$. Suppose that $\theta_1, \ldots, \theta_M \in \ker(\nabla_{\dot{\gamma}})^{\perp}$ and that $\theta_{M+1}, \ldots, \theta_N \in \ker(\nabla_{\dot{\gamma}})$. Then we set

$$[e^{\omega}\theta_N \wedge \dots \wedge \theta_1]_{\text{top}} := [\theta_N \wedge \dots \wedge \theta_{M+1}]_{\text{top}} \text{pf}'_{\zeta}(\nabla_{\dot{\gamma}}) \text{ pf}\left((\theta_a, \nabla_{\dot{\gamma}}^{-1}\theta_b)_{L^2}\right)_{M \ge a, b \ge 1}, \quad (3.8)$$

where the top degree component is taken inside the exterior algebra of the finite-dimensional space $\ker(\nabla_{\dot{\gamma}})$ with respect to the induced L^2 inner product. We extend this definition by multi-linearity to all elements of $\Lambda^N_{\sigma}L^2(S^1, \gamma^*T'X)$, the N-fold exterior product of $L^2(S^1, \gamma^*T'X)$, completed with respect to the Hilbert tensor topology.

Some remarks are in order. First, the zeta-regularized Pfaffian should be defined to be a square root of the zeta-regularized determinant $\det'_{\zeta}(\nabla_{\dot{\gamma}})$ as defined in Section 2,

$$\mathrm{pf}'_{\zeta}(\nabla_{\dot{\gamma}}) := \pm \det'_{\zeta}(\nabla_{\dot{\gamma}})^{1/2},$$

however, it is not clear which sign one should employ here. Similarly, taking the topdegree component of $\theta_N \wedge \cdots \wedge \theta_{M+1}$ inside $\ker(\nabla_{\dot{\gamma}})$ needs the choice of an orientation, which amounts to the choice of a sign. Now in our infinite-dimensional setup, there is no reasonable compatibility requirement for these two sign choices so that (3.8) is only welldefined up to sign. This sign ambiguity will be resolved by interpreting $[e^{\omega} \wedge \theta_N \wedge \cdots \wedge \theta_1]_{\text{top}}$ as an element in the Pfaffian line bundle on LX, as we explain in Section 4.

Remark 3.6. Since $\nabla_{\dot{\gamma}}$ preserves $\ker(\nabla_{\dot{\gamma}})$ (as it is skew-adjoint), its inverse is a bounded operator, which maps $L^2(S^1, \gamma^*TX)$ to the Sobolev space $H^1(S^1, \gamma^*TX) \subset C(S^1, \gamma^*TX)$. However, taking a look at the explicit form of $\nabla_{\dot{\gamma}}^{-1}$ (formula (6.17) below) shows that $\nabla_{\dot{\gamma}}^{-1}$ extends to a bounded operator from the L^1 closure of $\ker(\nabla_{\dot{\gamma}})$ to $C(S^1, \gamma^*TX)$. Hence by continuity, we can define $[e^{\omega} \wedge \theta]_{\text{top}}$ even for $\theta \in \Lambda^N L^1(S^1, \gamma^*T'X)$, and Thm. 1.1 remains valid when L^2 is replaced by L^1 .

4 The Pfaffian and the Spin Line Bundle

In this section, we introduce two geometric real line bundles on the loop space of an oriented Riemannian manifold X: The Pfaffian line bundle Pf associated to $\nabla_{\dot{\gamma}}$, and the line bundle $\mathcal{L} := \widehat{\mathsf{L}} X \times_{\mathbb{Z}_2} \mathbb{R}$ associated to the \mathbb{Z}_2 -principal bundle which is the transgression of the spin lifting gerbe on M (defined in [Mur96] or [Wal16]; see also [ST05]). We then discuss how to interpret the two expressions $q(\theta)$ and $[e^{\omega} \wedge \theta]_{\text{top}}$ as elements of these line bundles.

In his thesis [PW09], Prat-Waldron constructs an explicit isomorphism Φ between \mathcal{L} and Pf which also preserves the natural metrics on the two. Since we will use a slightly different description of \mathcal{L} , our description of Φ differs slightly from his.

We start by defining the Pfaffian line bundle. To this end, for a > 0, define the open sets

$$U^{(a)} := \{ \gamma \in \mathsf{L} X \mid 4\pi^2 a^2 \notin \operatorname{spec}(-\nabla_{\dot{\gamma}}^2) \}$$

of LX. These form an open cover of LX. On the open sets $U^{(a)}$ respectively the intersections $U^{(a)} \cap U^{(b)}$, we define

$$\mathcal{E}^{(a)} := \bigoplus_{\lambda < a} \operatorname{Eig}(-\nabla_{\dot{\gamma}}^2, 4\pi^2\lambda^2) \qquad \text{respectively} \qquad \mathcal{E}^{(a,b)} := \bigoplus_{a < \lambda < b} \operatorname{Eig}(-\nabla_{\dot{\gamma}}^2, 4\pi^2\lambda^2).$$

These are smooth vector bundles of finite rank on $U^{(a)}$ respectively $U^{(a,b)}$ and we have canonically $\mathcal{E}^{(a)} \oplus \mathcal{E}^{(a,b)} = \mathcal{E}^{(b)}$. The determinant line bundles $\Lambda^{\text{top}}\mathcal{E}^{(a)}$ are then real line bundles on the open sets $U^{(a)}$ and satisfy $\Lambda^{\text{top}}\mathcal{E}^{(a)} \otimes \Lambda^{\text{top}}\mathcal{E}^{(a,b)} = \Lambda^{\text{top}}\mathcal{E}^{(b)}$.

We now discuss how these line bundles glue together to a line bundle Pf on LX. By the spectral decomposition (2.5)of $\nabla_{\dot{\gamma}}$, each of the spaces $\mathcal{E}^{(a,b)}$ is has even dimension. Therefore the Pfaffian pf $(\nabla_{\dot{\gamma}}^{(a,b)})$ of $\nabla_{\dot{\gamma}}$ restricted to $\mathcal{E}^{(a,b)}$ is a well-defined non-zero element of $\Lambda^{\text{top}}\mathcal{E}^{(a,b)}$ and these elements satisfy

$$\operatorname{pf}(\nabla_{\dot{\gamma}}^{(a,b)}) \otimes \operatorname{pf}(\nabla_{\dot{\gamma}}^{(b,c)}) = \operatorname{pf}(\nabla_{\dot{\gamma}}^{(a,c)}) \tag{4.1}$$

for a < b < c. Explicitly, $\operatorname{pf}(\nabla_{\dot{\gamma}}^{(a,b)})$ is just the N-th power of the $(L^2$ dual of the) canonical two form ω restricted to $\mathcal{E}^{(a,b)}$, where we assume that $\dim \mathcal{E}^{(a,b)} = 2N$. In particular, we obtain isomorphisms

$$h^{(a,b)}: \Lambda^{\mathrm{top}}\mathcal{E}^{(a)} \longrightarrow \Lambda^{\mathrm{top}}\mathcal{E}^{(a)} \otimes \Lambda^{\mathrm{top}}\mathcal{E}^{(a,b)} = \Lambda^{\mathrm{top}}\mathcal{E}^{(b)}, \qquad \Theta \longmapsto \Theta \otimes \mathrm{pf}(\nabla_{\hat{\gamma}}^{(a,b)})$$

on the overlaps $U^{(a)} \cap U^{(b)}$ By (4.1), these satisfy the cocycle condition $h^{(b,c)} \circ h^{(a,b)} = h^{(a,c)}$ on triple intersections $U^{(a)} \cap U^{(b)} \cap U^{(c)}$ so that the line bundles $\Lambda^{\text{top}}\mathcal{E}^{(a)}$ together with the glueing isomorphisms $h^{(a,b)}$ indeed define a real line bundle Pf over LX.

The line bundle Pf has a natural metric. Notice that the vector bundles $\mathcal{E}^{(a)}$ and $\mathcal{E}^{(a,b)}$ are naturally equipped with the L^2 metric, which also induces a metric on $\Lambda^{\text{top}}\mathcal{E}^{(a)}$. However, these metrics are not compatible with the glueing isomorphisms $h^{(a,b)}$. Instead, over $U^{(a)}$, we consider the metric on $\Lambda^{\text{top}}\mathcal{E}^{(a)}$ given by²

$$\langle V_1 \wedge \dots \wedge V_N, W_1 \wedge \dots \wedge W_N \rangle_{\mathsf{Pf}} := \det \left((V_i, W_j)_{L^2} \right) \det_{\zeta} \left(\nabla_{\dot{\gamma}}^{(a, \infty)} \right),$$
 (4.2)

for vector fields $V_j, W_j \in \mathcal{E}^{(a)} \subset L^2(S^1, \gamma^*TX)$. This is the metric on $\Lambda^{\text{top}}\mathcal{E}^{(a)}$ induced by the L^2 metric on $\mathcal{E}^{(a)}$, but modified by the factor

$$\det_{\zeta} \left(\nabla_{\dot{\gamma}}^{(a,\infty)} \right) = \frac{\det_{\zeta}' \left(\nabla_{\dot{\gamma}} \right)}{\det(\nabla_{\dot{\gamma}}^{(0,a)})}.$$

That this metric is well-defined as a metric on Pf, i.e. compatible with the glueing isomorphisms $h^{(a,b)}$, follows from the facts that we have $\|\operatorname{pf}(\nabla_{\dot{\gamma}}^{(a,b)})\|_{L^2}^2 = \det(\nabla_{\dot{\gamma}}^{(a,b)})$ and $\det(\nabla_{\dot{\gamma}}^{(0,a)}) \det(\nabla_{\dot{\gamma}}^{(a,b)}) = \det(\nabla_{\dot{\gamma}}^{(0,b)})$. Since Pf is a real line bundle, there is a unique connection compatible with this metric, so we have completed the description of Pf as a geometric real line bundle.

²Here we assume dim $\mathcal{E}^{(a)} = N$.

There is a canonical section $\operatorname{pf}_{\zeta}(\nabla_{\dot{\gamma}})$ of this bundle. Over $U^{(a)}$, it is described by the section $\operatorname{pf}_{\zeta}(\nabla_{\dot{\gamma}})^{(a)} := \operatorname{pf}(\nabla_{\dot{\gamma}}^{(a)})$ of $\Lambda^{\operatorname{top}}\mathcal{E}^{(a)}$. By the multiplication rule for the Pfaffian, we have $h^{(a,b)}\operatorname{pf}_{\zeta}(\nabla_{\dot{\gamma}})^{(a)} = \operatorname{pf}_{\zeta}(\nabla_{\dot{\gamma}})^{(b)}$ so that the $\operatorname{pf}_{\zeta}(\nabla_{\dot{\gamma}})^{(a)}$ glue together to a section of Pf. This section $\operatorname{pf}_{\zeta}(\nabla_{\dot{\gamma}})$ satisfies

$$\|\mathrm{pf}_{\zeta}(\nabla_{\dot{\gamma}})\|_{\mathsf{Pf}}^2 = \det_{\zeta}(\nabla_{\dot{\gamma}}),$$

which justifies naming the section Pfaffian, as it is a square root of the determinant in this sense.

The bundle $\widehat{\mathsf{L}}X$ is easier to describe: Over a path γ , the fiber is the set of isomorphism classes of spin structures on γ^*TX , i.e. Spin_n -principal bundles $P \to S^1$ that are compatible with the frame bundle of γ^*TX (see [LM89, Ch. 2, §1]). Since X is oriented, γ^*TX is always trivial, so by Corollary 1.5 in [LM89], there are exactly two isomorphism classes of spin structures for each path γ . Letting \mathbb{Z}_2 act by fiberwise exchanging the spin structures, this turns $\widehat{\mathsf{L}}X$ into a \mathbb{Z}_2 principal bundle. Clearly, a spin structure P^{Spin} on X defines a section s of $\mathbb{L}X$, by defining $s(\gamma) := \gamma^*P^{\mathrm{Spin}}$, and $\widehat{\mathbb{L}}X$ is trivial if and only if X is spin.

Remark 4.1. A word of warning: While a spin structure gives a trivialization of $\widehat{\mathsf{L}}X$, it is *not* true that any non-vanishing sections of $\widehat{\mathsf{L}}X$ defines a spin structure on X. Instead, in order to render this, the sections have to satisfy the additional condition of being compatible with the *fusion product* on $\mathsf{L}X$ (compare [ST05]).

We can now define the line bundle $\mathcal{L} := \widehat{\mathsf{L}}X \times_{\mathbb{Z}_2} \mathbb{R}$ by performing the associated bundle construction via the non-trivial action of \mathbb{Z}_2 on \mathbb{R} . Hence elements of this bundle are equivalence classes $[P, \lambda]$, where P is (en equivalence class of) a spin structure on γ^*TX and $\lambda \in \mathbb{R}$, where $(P, \lambda) \sim (-P, -\lambda)$. A metric on \mathcal{L} is defined by setting

$$\|[P,\lambda]\|_{\mathcal{L}} := |\lambda|$$

for $P \in \widehat{\mathsf{L}}X$ and $\lambda \in \mathbb{R}$.

There is a canonical section q_0 of \mathcal{L} , constructed as follows. After choosing a spin structure P for γ^*TX , we can form the real spinor bundle $\Sigma := P \times_{\operatorname{Spin}_n} \operatorname{Cl}_n$, where Spin_n acts on the Clifford algebra Cl_n by left multiplication. For each $x \in X$, Σ_x is a $\operatorname{Cl}(T_xX) - \operatorname{Cl}_n$ bimodule. The connection on γ^*TX lifts to a connection on Σ and we can consider the parallel transport $[\gamma|_0^1]^\Sigma$ in Σ around γ . This parallel transport commutes with right multiplication by Cl_n , hence $[\gamma|_0^1]^\Sigma \in \operatorname{End}_{\operatorname{Cl}_n}(\Sigma_{\gamma(0)}) \cong \operatorname{Cl}(T_{\gamma(0)}X)$. We can therefore take its supertrace, which is generally defined by

$$\operatorname{str}(a) = 2^{n/2} \langle a, \mathbf{c}(\operatorname{vol}) \rangle,$$
 (4.3)

for an element $a \in \operatorname{Cl}(T_xX)$, see the conventions of the first paper [HL17a, Section 2]. We can then define $q_0 := \operatorname{str}[\gamma|_0^1]^{\Sigma} := [P, \operatorname{str}[\gamma|_0^1]^{\Sigma_P}]$. If one changes P to -P, then $[\gamma|_0^1]^{\Sigma}$ changes its sign, so this is a well-defined section \mathcal{L} . Notice that q_0 vanishes if n is odd, since then the super trace is an odd functional, while $[\gamma|_0^1]^{\Sigma}$ is always even.

More generally, given $\theta_1, \ldots, \theta_N \in L^2(S^1, \gamma^*T'X)$, we can form an associated element $q_N(\theta)$ (with $\theta = \theta_N \wedge \cdots \wedge \theta_1$) by the formula

$$q_N(\theta) := 2^{-N/2} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} \operatorname{str}\left([\gamma \|_{\tau_N}^1]^{\Sigma} \prod_{j=1}^N \mathbf{c} \left(\theta_{\sigma_j}(\tau_j) \right) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\Sigma} \right) d\tau, \tag{4.4}$$

where $[\gamma|_{\tau_N}]^{\Sigma}$ denotes parallel transport Σ . Notice here that again, the term in the brackets commutes with right multiplication in Cl_n , hence is contained in $\mathrm{End}_{\mathrm{Cl}_n}(\Sigma_{\gamma(0)}) \cong \mathrm{Cl}(T_xX)$, so we can take its super trace as in (4.3). Notice that $q_N(\theta)$ is always zero unless n and N have the same parity.

Remark 4.2. If n is even, we could also form the *complex* graded spinor bundle $\Sigma^{\mathbb{C}}$, with corresponding parallel transport $[\gamma|_0^1]^{\Sigma^{\mathbb{C}}} \in \operatorname{End}_{\mathbb{C}}(\Sigma_{\gamma(0)}^{\mathbb{C}})$. The (complex) trace of this parallel transport coincides with the trace defined above up to a factor of $(-i)^{n/2}$.

Letting the $\theta_1, \ldots, \theta_N$ be free variables and interpreting q_N as an alternating N-linear form on $L^2(S^1, \gamma^*T'X)$, we obtain a section q_N of the bundle

$$\mathcal{L} \otimes \left(\Lambda^N L^2(S^1, \gamma^* T'X)\right)' \cong \mathcal{L} \otimes L^2_{\text{susy}}(T^N, \gamma^* TX^{\boxtimes N}).$$

For a definition of these spaces of supersymmetric sections, we refer to the first paper [HL17a]. Notice that by the properties of the supertrace, q_N is identically zero if N-n is odd.

For a fixed loop γ , let e_1, \ldots, e_n be an orthonormal basis of $T_{\gamma(0)}X$ such that the parallel transport takes the form (2.3) with respect to this basis, for numbers $\alpha_1, \ldots, \alpha_m \notin \mathbb{Z}$. Let $e_j(t)$ be the parallel vector fields around γ with $e_j(0) = e_j$. The vector fields e_{2m+1}, \ldots, e_n then form an orthonormal basis of the kernel.

For a > 0 such that $\gamma \in U^{(a)}$, define $\Theta_{\gamma}^{(a)} \in \mathcal{E}_{\gamma}^{(a)}$ by

$$\Theta_{\gamma}^{(a)} := \prod_{i=1}^{m} \frac{1}{2\sin(\pi\alpha_i)} E_{2m+1} \wedge \dots \wedge E_n \wedge \operatorname{pf}(\nabla_{\dot{\gamma}}^{(0,a)}). \tag{4.5}$$

For 0 < a < b, we have $h^{(a,b)}\Theta_{\gamma}^{(a)} = \Theta_{\gamma}^{(b)}$, so the $\Theta_{\gamma}^{(a)}$ glue together to an element $\Theta_{\gamma} \in \mathsf{Pf}_{\gamma}$. This element depends on the choice of basis e_1, \ldots, e_n and the corresponding choice of numbers $\alpha_1, \ldots, \alpha_m \notin \mathbb{Z}$. By Prop. 2.3, we always have $\|\Theta_{\gamma}\|_{\mathsf{Pf}}^2 = 1$, so that the elements Θ_{γ} obtained this way differ at most by a sign.

Lemma 4.3. Let P be a spin structure on γ^*TX and let $\Sigma := P \times_{\rho} \Sigma_n$ be the associated spinor bundle. If e_1, \ldots, e_n is an orthonormal basis of $T_{\gamma(0)}X$ with respect to which $[\gamma|]_0^1^{TX}$ takes the form (2.3) for numbers $\alpha_1, \ldots, \alpha_m \notin \mathbb{Z}$, then the parallel transport in the spinor bundle around γ is given by

$$[\gamma|_0^1]^{\Sigma} = \epsilon_0 \prod_{j=1}^m (\cos(\pi\alpha_j) + \sin(\pi\alpha_j)e_{2j-1}e_{2j}), \tag{4.6}$$

for some $\epsilon_0 \in \{\pm 1\}$. Here we identified the endomorphisms of the spinor bundle $\operatorname{End}_{\mathbb{C}}(\Sigma_{\gamma(0)})$ with the Clifford algebra $\operatorname{Cl}(T_{\gamma(0)}X) \otimes \mathbb{C}$.

Proof. This follows from verifying that if one conjugates a vector $v = v_1 e_{2j-1} + v_2 e_{2j}$ in the Clifford algebra by the element $\cos(\pi \alpha_j) + \sin(\pi \alpha_j) e_{2j-1} e_{2j}$ to obtain a vector $w = w_1 e_{2j-1} + w_2 e_{2j}$, then its coefficients are given by

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \stackrel{\cap}{=} \begin{pmatrix} \cos(2\pi\alpha_j) & -\sin(2\pi\alpha_j) \\ \sin(2\pi\alpha_j) & \cos(2\pi\alpha_j) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Now fixing an orthonormal basis e_1, \ldots, e_n and numbers $\alpha_1, \ldots, \alpha_m \notin \mathbb{Z}$ as before, we define an isometric isomorphism $\Phi_{\gamma} : \mathcal{L}_{\gamma} \longrightarrow \mathsf{Pf}_{\gamma}$ as follows. Given a spin structure P on γ^*TX with associated spinor bundle Σ , let ϵ_0 be as in formula (4.6). Then define

$$\Phi_{\gamma}([P,\lambda]) := \lambda \cdot \epsilon_0 \cdot \operatorname{sign}(e_1, \dots, e_n) \cdot \Theta_{\gamma}, \tag{4.7}$$

where $\operatorname{sign}(e_1, \ldots, e_n) := \pm 1$, depending on whether the basis is positively or negatively oriented. The definition of Φ_{γ} is independent from the choice of P, because ϵ_0 changes to $-\epsilon_0$ if P changes to -P.

Theorem 4.4. For each γ , the isomorphism Φ_{γ} defined above is independent of the choice of e_1, \ldots, e_n and $\alpha_1, \ldots, \alpha_m \notin \mathbb{Z}$. Moreover, the Φ_{γ} assemble to a smooth vector bundle isomorphism $\Phi : \mathcal{L} \to \mathsf{Pf}$.

Proof. Let $\widetilde{e}_1, \ldots, \widetilde{e}_n$ be a different orthonormal basis of $T_{\gamma(0)}X$, with respect to which $[\gamma|_0^1]^{TX}$ takes the form (2.3), with possibly different numbers $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_m \notin \mathbb{Z}$. Denote the corresponding quantities appearing in formula (4.7) by $\widetilde{\epsilon}_0$ respectively $\widetilde{\Theta}_{\gamma}$. We first investigate the following special cases.

(1) Suppose first we only change the α_j , i.e. that $\tilde{e}_j = e_j$ for all j and that $\tilde{\alpha}_j := \alpha_j + k_j$, for numbers $k_j \in \mathbb{Z}$. Then

$$\widetilde{\epsilon}_0 = (-1)^{k_1 + \dots + k_m} \epsilon_0$$
, and $\widetilde{\Theta}_{\gamma} = (-1)^{k_1 + \dots + k_m} \Theta_{\gamma}$.

Hence Φ_{γ} is invariant with respect to this change.

(2) Now fix $j_0 \in \{1, ..., m\}$ and suppose that $\widetilde{e}_j = e_j$ for $j \notin \{2j_0 - 1, 2j_0\}$ and $\widetilde{\alpha}_j = \alpha_j$ for $j \neq j_0$, while $\widetilde{e}_{2j_0-1} = e_{2j_0}$, $\widetilde{e}_{2j_0} = e_{2j_0-1}$ and $\widetilde{\alpha}_{j_0} = -\alpha_{j_0}$. Then clearly

$$\operatorname{sign}(\widetilde{e}_1,\ldots,\widetilde{e}_n) = -\operatorname{sign}(e_1,\ldots,e_n)$$

and since $\sin(\pi \widetilde{\alpha}_{j_0}) = -\sin(\pi \alpha_{j_0})$, we have $\widetilde{\Theta}_{\gamma} = -\Theta_{\gamma}$. Furthermore, since by the Clifford multiplication rules $e_{2j_0}e_{2j_0-1} = -e_{2j_0-1}e_{2j_0}$, we have

$$\cos(\pi \widetilde{\alpha}_{j_0}) + \sin(\pi \widetilde{\alpha}_{j_0}) \widetilde{e}_{2j_0-1} \widetilde{e}_{2j_0} = \cos(\pi \alpha_{j_0}) + \sin(\pi \alpha_{j_0}) e_{2j_0-1} e_{2j_0},$$

hence $\tilde{\epsilon}_0 = \epsilon_0$.

(3) Moreover, if there exists a permutation $\sigma \in S_m$ such that $\tilde{e}_{2j-1} = e_{2\sigma_j-1}$, $\tilde{e}_{2j} = e_{2\sigma_j}$, $\tilde{\alpha}_j = \alpha_{\sigma_j}$ for each $j = 1, \ldots, m$, as well as $\tilde{e}_j = e_j$ for all $j = 2m + 1, \ldots, n$, then clearly this does not change Φ_{γ} .

Now generally, we have $\tilde{e}_j = Qe_j$ for all j, where Q is some orthogonal automorphism of $T_{\gamma(0)}X$. By the preliminary considerations (1) and (2), we may assume that $\tilde{\alpha}_j = \alpha_j$ and furthermore that there are no two indices $i \neq j$ such that $\alpha_i = \alpha_j + k$ or $\alpha_i = \alpha_j - k$ for some $k \in \mathbb{Z}$. Under this assumption, Q must have blockdiagonal form corresponding to the α_j of different value, with blocks Q_{α} of size k_{α} , where $\alpha \in \{0, \alpha_1, \ldots, \alpha_m\}$ and k_{α} is the number of indices j such that $\alpha_j = \alpha$ (if $\alpha \neq 0$) and $k_0 = \dim \ker([\gamma|_0^1]^{TX} - \mathrm{id})$. Then each Q_{α} can be represented by a $2k_{\alpha} \times 2k_{\alpha}$ matrix, conjugation by which preserves the matrix

$$R_{\alpha} \triangleq \begin{pmatrix} \cos(2\pi\alpha) & -\sin(2\pi\alpha) \\ \sin(2\pi\alpha) & \cos(2\pi\alpha) \end{pmatrix}, \\ & \ddots \\ & \cos(2\pi\alpha) & -\sin(2\pi\alpha) \\ & \sin(2\pi\alpha) & \cos(2\pi\alpha) \end{pmatrix},$$

i.e. Q_{α} satisfies $Q_{\alpha}R_{\alpha}Q_{\alpha}^* = R_{\alpha}$.

We may now assume that all Q_{α} but one are the identity, in order to deal with each α separately. Suppose first that this only α with $Q_{\alpha} \neq \text{id}$ satisfies $\alpha \notin \{0, 1/2\}$. Then necessarily $Q_{\alpha} \in U(k_{\alpha}) \subset SO(2k_{\alpha})$, because

$$J_{\alpha} := \frac{R_{\alpha} - \cos(2\pi\alpha)}{\sin(2\pi\alpha)} \ \widehat{=} \ \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}$$

is a complex structure on $\ker([\gamma|_0^1]^{TX} - R_\alpha)$ that is preserved by Q_α . Clearly, $\widetilde{\Theta}_\gamma = \Theta_\gamma$ as $\operatorname{pf}(\nabla_{\dot{\gamma}}^{(0,a)})$ is invariantly defined. Also, because $Q_\alpha \in \operatorname{SO}(2k_\alpha)$, $\operatorname{sign}(\widetilde{e}_1,\ldots,\widetilde{e}_n) = \operatorname{sign}(e_1,\ldots,e_n)$. It is left to show that $\widetilde{\epsilon}_0 = \epsilon_0$. To this end, let $I \subseteq \{1,\ldots,m\}$ be the set of indices j such that $\alpha_j = \alpha$. Without loss of generality because of observation (3) above, suppose that $I = \{1,\ldots,k_\alpha\}$. Because $e_{2j-1}e_{2j}$ and $e_{2i-1}e_{2i}$ commute in $\operatorname{Cl}T_{\gamma(0)}X$ for all i,j, we have

$$\prod_{j=1}^{k_{\alpha}} \left(\cos(\pi \alpha) + \sin(\pi \alpha) e_{2j-1} e_{2j} \right) = \exp\left(\pi \alpha \sum_{j=1}^{k_{\alpha}} e_{2j-1} e_{2j} \right) = \exp\left(\pi \alpha \mathbf{c}(\Omega) \right),$$

where $\Omega \in \Lambda^2 T'_{\gamma(0)} X$ is the two form which is the image of the complex structure J_{α} under the isomorphism $\mathfrak{so}(T_{\gamma(0)}X) \cong \Lambda^2 T'_{\gamma(0)} X$. Since Q_{α} preserves J_{α} it also preserves Ω and $\mathbf{c}(\Omega)$, which implies that $\widetilde{\epsilon}_0 = \epsilon_0$.

On the other hand, if $\alpha = 0$ or $\alpha = 1/2$, then $R_{\alpha} = \text{id}$ respectively $R_{\alpha} = -\text{id}$, so Q_{α} can be any orthogonal matrix acting on $\ker([\gamma|_0^1]^{\Sigma} \mp \text{id})$. In the case $\alpha = 0$, it is clear that $\widetilde{\epsilon}_0 = \epsilon_0$, while

$$\operatorname{sign}(\widetilde{e}_1, \dots, \widetilde{e}_n) = \det(Q_0) \cdot \operatorname{sign}(e_1, \dots, e_n), \qquad \widetilde{\Theta}_{\gamma} = \det(Q_0) \cdot \Theta_{\gamma}.$$

Finally, consider $\alpha = \frac{1}{2}$ and let I be the set of indices j such that $\alpha_j = \frac{1}{2}$, without loss of generality, $I = \{1, \ldots, k_{1/2}\}$. Then,

$$\prod_{j=1}^{k_{1/2}} (\cos(\pi \alpha_j) + \sin(\pi \alpha_j) e_{2j-1} e_{2j}) = e_1 e_2 \cdots e_{k_{1/2}}$$

is the volume element of $Cl(\ker([\gamma|_0^1]^{TX} + id))$. Since the volume element of a Clifford algebra remains invariant under an orientation preserving orthogonal transformation while it changes sign if the transformation is orientation reversing, this shows

$$\widetilde{\epsilon}_0 = \det(Q_{1/2}) \cdot \epsilon_0$$
 and $\operatorname{sign}(\widetilde{e}_1, \dots, \widetilde{e}_n) = \det(Q_{1/2}) \cdot \operatorname{sign}(e_1, \dots, e_n),$

while $\widetilde{\Theta}_{\gamma} = \Theta_{\gamma}$, again because $\operatorname{pf}_{\zeta}(\nabla_{\dot{\gamma}})$ is invariantly defined.

Now because Φ_{γ} is an isometry in each fiber, it suffices to show that Φ_{γ} is continuous, as it is then automatically smooth, as \mathcal{L} and Pf are real line bundles.

Let $F_{2j-1,k}$ and $F_{2j,k}$ for $j=1,\ldots,m$ and $k\in\mathbb{Z}$ be defined by (2.4). Then it is easy to see that for 0 < a < 1,

$$\Theta_{\gamma}^{(a)} = \prod_{j=1}^{m} \frac{1}{2\sin(\pi\alpha_j)} E_{m+1} \wedge \dots \wedge E_n \bigwedge_{(k+\alpha_j)^2 < a^2} 2\pi(k+\alpha_j) F_{2j-1,k} \wedge F_{2j,k}.$$

Using this form, it is not hard to establish the continuity of Φ . To this end, it suffices to consider the behavior of this expression as some α_i tends to an element in \mathbb{Z} .

Remark 4.5. It is now easy to see that Φ maps the canonical sections of \mathcal{L} respectively Pf to each other. Since both of these sections vanish if $m \neq n/2$ (in particular if n is odd), we may assume that m = n/2. By the formula for the supertrace (4.3), we have

$$\operatorname{str}[\gamma|_0^1]^{\Sigma} = 2^{n/2} \epsilon_0 \prod_{j=1}^{n/2} \sin(\pi \alpha_j).$$

Hence with a view on (4.7) and (4.5), $\Phi_{\gamma}(\operatorname{str}[\gamma|_0^1]^{\Sigma}) = \operatorname{pf}_{\zeta}(\nabla_{\dot{\gamma}})$ as desired.

We end this section discussing how to interpret $[e^{\omega} \wedge \theta]_{\text{top}}$ as a section of the Pfaffian line bundle Pf. Given a path γ , choose an orthonormal basis e_1, \ldots, e_n of $T_{\gamma(0)}X$ and numbers $\alpha_1, \ldots, \alpha_m \notin \mathbb{Z}$ as above, which yields a corresponding element $\Theta_{\gamma} \in \text{Pf}$, depending on these choices. Now for $\theta_1, \ldots, \theta_N \in L^2(S^1, \gamma^*T'X)$ such that $\theta_1, \ldots, \theta_M \in \ker(\nabla_{\dot{\gamma}})^{\perp}$ and $\theta_{M+1}, \ldots, \theta_N \in \ker(\nabla_{\dot{\gamma}})$, define $[e^{\omega} \wedge \theta_N \wedge \cdots \wedge \theta_1]_{\text{top}} \in \text{Pf}_{\gamma}$ by the formula

$$\prod_{j=1}^{m} 2\sin(\pi\alpha_j) \left(\theta_N \wedge \dots \wedge \theta_{M+1}, E_{2m+1} \wedge \dots \wedge E_n\right)_{L^2} \operatorname{pf}\left(\left(\theta_a, \nabla_{\dot{\gamma}}^{-1} \theta_b\right)_{M \geq a, b \geq 1} \cdot \Theta_{\gamma} \right)$$
(4.8)

By Prop. 2.3, this is a section whose norm equals the modulus of the expression defined up to sign in (3.8). Similar to the above, one checks that this is independent of the choice of e_1, \ldots, e_n and $\alpha_1, \ldots, \alpha_m \notin \mathbb{Z}$.

Considering $\theta_1, \ldots, \theta_N$ as free variables, we obtain a section $[e^{\omega} \wedge (\ldots)]_{\text{top}}$ of the bundle $\mathsf{Pf} \otimes L^2_{\text{susy}}(T^N, \gamma^*TX^{\boxtimes N})$. A priori, it is not at all clear that this section is smooth, but we will show below that for each N, the isomorphism $\Phi \otimes \mathrm{id}$ will map q_N to $[e^{\omega} \wedge (\ldots)]_{\text{top}}$. Since q_N and Φ are smooth, this a posteriori shows the smoothness of $[e^{\omega} \wedge (\ldots)]_{\text{top}}$.

5 Combinatorial Identities

The purpose of this section is to prove the following Theorem, the results of which are needed for the proof of our main result, and which may be of independent interest.

Theorem 5.1. For numbers $k_1, \ldots, k_N \in \mathbb{Z}$, set

$$\mathcal{J}(k_1, \dots, k_N) := \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} \exp\left(2\pi i \sum_{a=1}^N k_{\sigma_a} \tau_a\right) d\tau.$$
 (5.1)

Then

- (1) If $k_a = k_b = 0$ for some $a \neq b$, then $\mathcal{J}(k_1, \ldots, k_N) = 0$.
- (2) Suppose $k_a \neq 0$ for all a = 1, ..., N. Then $\mathcal{J}(k_1, ..., k_N) = 0$ if N is odd while if N is even, we have

$$\mathcal{J}(k_1,\ldots,k_N) = \frac{1}{(2\pi i)^{N/2} \left(\frac{N}{2}\right)!} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \frac{\delta_{k_{\sigma_1},-k_{\sigma_2}} \cdots \delta_{k_{\sigma_{N-1}},-k_{\sigma_N}}}{k_{\sigma_2} k_{\sigma_4} \cdots k_{\sigma_N}}.$$

(3) If $k_a = 0$ for exactly one index a and N is odd, then

$$\mathcal{J}(k_1,\ldots,k_N)=(-1)^{N+a}\mathcal{J}(k_1,\ldots,\widehat{k_a},\ldots,k_N),$$

the same expression with k_a removed.

(4) If $k_a = 0$ for exactly one index a and N is even, then

$$\mathcal{J}(k_1, \dots, k_N) = \sum_{\substack{b=1\\b \neq a}}^{N} (-1)^{a+b+1} \frac{1}{\pi i k_b} \mathcal{J}(k_1, \dots, \widehat{k_a}, \dots, \widehat{k_b}, \dots, k_N).$$

Throughout, $\delta_{k,l} = 1$ if k = l and zero otherwise.

Proof. Throughout the proof, we write $\xi_a := 2\pi i k_a$ for abbreviation.

Claim (1) directly follows from the fact that $\mathcal{J}(k_1,\ldots,k_N)$ is anti-symmetric in the entries, hence vanish when two entries are equal. To show (2), we use the following Lemma.

Lemma 5.2. For $N \in \mathbb{N}_0$ even and numbers $\tau_1 \leq \tau_3 \leq \cdots \leq \tau_{N+1}$, we have

$$\sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \prod_{a=1}^{N/2} \int_{\tau_{2a-1}}^{\tau_{2a+1}} \exp\left(\xi_{\sigma_{2a}} \tau_{2a}\right) d\tau_{2a}$$

$$= \sum_{\sigma \in S_N} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_2} \xi_{\sigma_4} \cdots \xi_{\sigma_N}} \left[\exp\left(\sum_{a=1}^{N/2} \xi_{\sigma_{2a}} \tau_{2a-1}\right) - \sum_{b=1}^{N/2} \exp\left(\sum_{\substack{a=1\\ a \neq b}}^{N/2} \xi_{\sigma_{2a}} \tau_{2a-1} + \xi_{\sigma_{2b}} \tau_{N+1}\right) \right].$$

Proof. By induction. The case N=0 is clear. Let $N\geq 2$ be even and suppose that the statement of the Lemma is true for N-2. Then

$$\sum_{\sigma \in S_{N}} \operatorname{sgn}(\sigma) \prod_{a=1}^{N/2} \int_{\tau_{2a-1}}^{\tau_{2a+1}} \exp(\xi_{\sigma_{2a}} \tau_{2a}) d\tau_{2a}$$

$$= \sum_{\sigma \in S_{N}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}} \xi_{\sigma_{4}} \cdots \xi_{\sigma_{N-2}}} \left[\sum_{b=1}^{N/2-1} \exp\left(\sum_{\substack{a=1\\a \neq b}}^{N/2-1} \xi_{\sigma_{2a}} \tau_{2a-1} + \xi_{\sigma_{2b}} \tau_{N-1} \right) - \exp\left(\sum_{a=1}^{N/2-1} \xi_{\sigma_{2a}} \tau_{2a-1} \right) \right] \cdot \int_{\tau_{N-1}}^{\tau_{N+1}} \exp(\xi_{\sigma_{N}} \tau_{N}) d\tau_{N}$$

$$= \sum_{\sigma \in S_{N}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}} \xi_{\sigma_{4}} \cdots \xi_{\sigma_{N}}} \left[\sum_{b=1}^{N/2-1} \exp\left(\sum_{\substack{a=1\\a \neq b}}^{N/2-1} \xi_{\sigma_{2a}} \tau_{2a-1} + \xi_{\sigma_{2b}} \tau_{N-1} + \xi_{\sigma_{N}} \tau_{N+1} \right) - \sum_{b=1}^{N/2-1} \exp\left(\sum_{\substack{a=1\\a \neq b}}^{N/2-1} \xi_{\sigma_{2a}} \tau_{2a-1} + (\xi_{\sigma_{2b}} + \xi_{\sigma_{N}}) \tau_{N-1} \right) - \exp\left(\sum_{a=1}^{N/2-1} \xi_{\sigma_{2a}} \tau_{2a-1} + \xi_{\sigma_{N}} \tau_{N+1} \right) + \exp\left(\sum_{a=1}^{N/2} \xi_{\sigma_{2a}} \tau_{2a-1} \right) \right].$$

The second of the four terms is symmetric in the indices 2b and N and hence its anti-symmetrization vanishes. Regarding the first term, notice that for all numbers $b = 1, \ldots, N/2 - 1$, we can swap the indices 2b and N yielding a minus sign from the sign of this permutation. Therefore the first and the third term above combine to the second term in the statement of the lemma.

For $T \geq 0$, we consider more generally the integral

$$\mathcal{J}(k_1,\ldots,k_N;T) := \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_{N,T}} \exp\left(\sum_{a=1}^N \xi_{\sigma_a} \tau_a\right) d\tau.$$

where we set

$$\Delta_{N,T} := \{ \tau \in \mathbb{R}^N \mid 0 \le \tau_1 \le \dots \le \tau_N \le T \}.$$

Suppose that N is even. Changing the order of integration, we get

$$\mathcal{J}(k_1, \dots, k_N; T) = \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_0^T \int_0^{\tau_{N-1}} \dots \int_0^{\tau_3} \exp\left(\sum_{a=1}^{N/2} \xi_{\sigma_{2a-1}} \tau_{2a-1}\right) \times \left[\prod_{a=1}^{N/2} \int_{\tau_{2a-1}}^{\tau_{2a+1}} \exp\left(\xi_{\sigma_{2a}} \tau_{2a}\right) d\tau_{2a}\right] d\tau_1 d\tau_3 \dots d\tau_{N-1}$$

We can now use Lemma 5.2 to obtain

$$\mathcal{J}(k_{1},\ldots,k_{N};T) = \sum_{\sigma \in S_{N}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}}\xi_{\sigma_{4}}\cdots\xi_{\sigma_{N}}} \int_{0}^{T} \int_{0}^{\tau_{N-1}} \cdots \int_{0}^{\tau_{3}} \exp\left(\sum_{a=1}^{N/2} \xi_{\sigma_{2a-1}}\tau_{2a-1}\right) \times \\
\times \left[\exp\left(\sum_{a=1}^{N/2} \xi_{\sigma_{2a}}\tau_{2a-1}\right) - \sum_{b=1}^{N/2} \exp\left(\sum_{a=1}^{N/2} \xi_{\sigma_{2a}}\tau_{2a-1} + \xi_{\sigma_{2b}}T\right)\right] d\tau_{1}d\tau_{3}\cdots d\tau_{N-1} \\
= \sum_{\sigma \in S_{N}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}}\xi_{\sigma_{4}}\cdots\xi_{\sigma_{N}}} \int_{0}^{T} \int_{0}^{\tau_{N-1}} \cdots \int_{0}^{\tau_{3}} \left[\exp\left(\sum_{a=1}^{N/2} (\xi_{\sigma_{2a-1}} + \xi_{\sigma_{2a}})\tau_{2a-1}\right) - \sum_{b=1}^{N/2} \exp\left(\sum_{a=1}^{N/2} (\xi_{\sigma_{2a-1}} + \xi_{\sigma_{2a}})\tau_{2a-1} + \xi_{\sigma_{2b-1}}\tau_{2b-1} + \xi_{\sigma_{2b}}T\right)\right] d\tau_{1}d\tau_{3}\cdots d\tau_{N-1}.$$

Consider now the sum over b. For each b separately, integrate out the variable τ_{2b-1} and notice that one obtains two terms. For b = 2, ..., N/2, each of these terms is symmetric in two indices and their total anti-symmetrization vanishes. For b = 1, this is true only for the upper boundary term and we obtain after substitution of variables

$$\mathcal{J}(k_1, \dots, k_N; T) = \sum_{\sigma \in S_N} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_2} \xi_{\sigma_4} \cdots \xi_{\sigma_N}} \int_{\Delta_{N/2, T}} \exp\left(\sum_{a=1}^{N/2} (\xi_{\sigma_{2a-1}} + \xi_{\sigma_{2a}}) \tau_a\right) d\tau$$
$$+ \sum_{\sigma \in S_N} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_2} \xi_{\sigma_4} \cdots \xi_{\sigma_N} \xi_{\sigma_1}} \int_{\Delta_{N/2-1, T}} \exp\left(\xi_{\sigma_2} T + \sum_{a=1}^{N/2-1} (\xi_{\sigma_{2a+1}} + \xi_{\sigma_{2a+2}}) \tau_a\right) d\tau$$

The following trick is now essential: Namely, we have for numbers ζ_1, \ldots, ζ_k the identity

$$\sum_{\rho \in S_k} \int_{\Delta_k, T} \exp\left(\sum_{a=1}^k \zeta_{\rho_a} \tau_a\right) d\tau = \sum_{\rho \in S_k} \int_{\Delta_k, T} \exp\left(\sum_{a=1}^k \zeta_a \tau_{\rho_a}\right) d\tau$$

$$= \sum_{\rho \in S_k} \int_{\{0 \le \tau_{\rho_1} \le \dots \le \tau_{\rho_k} \le T\}} \exp\left(\sum_{a=1}^k \zeta_a \tau_a\right) d\tau$$

$$= \int_0^T \dots \int_0^T \exp\left(\sum_{a=1}^k \zeta_a \tau_a\right) d\tau_1 \dots d\tau_k$$

$$= \prod_{a=1}^{N/2} \int_0^T \exp(\zeta_a t) dt = \prod_{a=1}^{N/2} (e^{\zeta_a T} - 1).$$

Hence by the calculation before,

$$\mathcal{J}(k_{1},\ldots,k_{N};T) = \frac{1}{\left(\frac{N}{2}\right)!} \sum_{\sigma \in S_{N}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}}\xi_{\sigma_{4}}\cdots\xi_{\sigma_{N}}} \prod_{a=1}^{N/2} \left(e^{(\xi_{\sigma_{2a-1}}+\xi_{\sigma_{2a}})T} - 1\right) + \frac{1}{\left(\frac{N}{2}-1\right)!} \sum_{\sigma \in S_{N}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}}\xi_{\sigma_{4}}\cdots\xi_{\sigma_{N}}\xi_{\sigma_{1}}} e^{\xi_{\sigma_{2}}T} \prod_{a=2}^{N/2} \left(e^{(\xi_{\sigma_{2a-1}}+\xi_{\sigma_{2a}})T} - 1\right).$$
(5.2)

Now if T = 1, then $e^{\xi_a T} = 1$ for all a, so that in this case, the second term in (5.2) is symmetric in the indices 1 and 2 (making its anti-symmetrization vanish) while the first term gives claim (2) from Thm. 5.1 for an even number of entries.

Still supposing that N is even, we take another number k_{N+1} and want to show that $\mathcal{J}(k_1,\ldots,k_{N+1})=0$. Using (5.2), we get

$$\mathcal{J}(k_{1},\ldots,k_{N+1}) = \frac{1}{\left(\frac{N}{2}\right)!} \sum_{\sigma \in S_{N+1}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}}\xi_{\sigma_{4}}\cdots\xi_{\sigma_{N}}} \int_{0}^{1} e^{\xi_{\sigma_{N+1}}T} \left[\prod_{a=1}^{N/2} \int_{0}^{T} e^{(\xi_{\sigma_{2a-1}} + \xi_{\sigma_{2a}})t} dt \right] dT
+ \frac{1}{\left(\frac{N}{2} - 1\right)!} \sum_{\sigma \in S_{N+1}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}}\xi_{\sigma_{4}}\cdots\xi_{\sigma_{N}}\xi_{\sigma_{1}}} \int_{0}^{1} e^{(\xi_{\sigma_{N+1}} + \xi_{\sigma_{2}})T} \left[\prod_{a=2}^{N/2} \int_{0}^{T} e^{(\xi_{\sigma_{2a-1}} + \xi_{\sigma_{2a}})t} dt \right] dT
= (A) + (B)$$

We show that (A) = -(B). Integrating by parts with respect to T in (A), we obtain

$$(A) = \frac{1}{\left(\frac{N}{2}\right)!} \sum_{\sigma \in S_{N+1}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_2} \xi_{\sigma_4} \cdots \xi_{\sigma_N} \xi_{\sigma_{N+1}}} \left[\prod_{a=1}^{N/2} \int_0^1 e^{(\xi_{\sigma_{2a-1}} + \xi_{\sigma_{2a}})t} dt - \sum_{b=1}^{N/2} \int_0^1 e^{(\xi_{N+1} + \xi_{2b-1} + \xi_{2b})T} \prod_{\substack{a=1\\ a \neq b}}^{N/2} \int_0^T e^{(\xi_{2a-1} + \xi_{2a})t} dt dT \right].$$

The b-th summand of the second term is symmetric in the indices N+1 and 2b, hence its anti-symmetrization vanishes. Swapping the indices 1 and N+1 in (B) (which produces a sign), we get

$$-(B) = \frac{1}{(\frac{N}{2} - 1)!} \sum_{\sigma \in S_{N+1}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}} \xi_{\sigma_{4}} \cdots \xi_{\sigma_{N}} \xi_{\sigma_{N+1}}} \int_{0}^{1} e^{(\xi_{\sigma_{1}} + \xi_{\sigma_{2}})T} \left[\prod_{a=2}^{N/2} \int_{0}^{T} e^{(\xi_{\sigma_{2a-1}} + \xi_{\sigma_{2a}})t} dt \right] dT$$

$$= \frac{1}{(\frac{N}{2} - 1)!} \sum_{\sigma \in S_{N+1}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}} \xi_{\sigma_{4}} \cdots \xi_{\sigma_{N}} \xi_{\sigma_{N+1}}} \int_{0}^{1} \frac{d}{dT} \left\{ \int_{0}^{T} e^{(\xi_{\sigma_{1}} + \xi_{\sigma_{2}})t} dt \right\} \times$$

$$\times \left[\prod_{a=2}^{N/2} \int_{0}^{T} e^{(\xi_{\sigma_{2a-1}} + \xi_{\sigma_{2a}})t} dt \right] dT$$

$$= \frac{1}{(\frac{N}{2} - 1)!} \sum_{\sigma \in S_{N+1}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}} \xi_{\sigma_{4}} \cdots \xi_{\sigma_{N}} \xi_{\sigma_{N+1}}} \frac{1}{N/2} \int_{0}^{1} \frac{d}{dT} \left\{ \prod_{a=1}^{N/2} \int_{0}^{T} e^{(\xi_{\sigma_{2a-1}} + \xi_{\sigma_{2a}})t} dt \right\} dT$$

$$= \frac{1}{(\frac{N}{2})!} \sum_{\sigma \in S_{N+1}} \frac{\operatorname{sgn}(\sigma)}{\xi_{\sigma_{2}} \xi_{\sigma_{4}} \cdots \xi_{\sigma_{N}} \xi_{\sigma_{N+1}}} \prod_{a=1}^{N/2} \int_{0}^{1} e^{(\xi_{\sigma_{2a-1}} + \xi_{\sigma_{2a}})t} dt$$

$$= (A).$$

This shows the claim of (2) in the odd case.

To show (3) and (4), assume without loss of generality that $k_N = 0$ and $k_a \neq 0$ for $a \neq N$. Then

$$\mathcal{J}(k_1, \dots, k_N) = \sum_{b=1}^{N} \sum_{\substack{\sigma \in S_N \\ \sigma_b = N}} \operatorname{sgn}(\sigma) \int_{\Delta_N} \exp\left(\sum_{a=1}^{b-1} \xi_{\sigma_a} \tau_a + \sum_{a=b+1}^{N} \xi_{\sigma_a} \tau_a\right) d\tau$$

$$= \sum_{b=1}^{N} \sum_{\substack{\sigma \in S_N \\ \sigma_b = N}} \operatorname{sgn}(\sigma) \int_{\Delta_{N-1}} (\tau_b - \tau_{b-1}) \exp\left(\sum_{a=1}^{b-1} \xi_{\sigma_a} \tau_a + \sum_{a=b}^{N-1} \xi_{\sigma_{a+1}} \tau_a\right) d\tau$$

$$= \sum_{\widetilde{\sigma} \in S_{N-1}} \operatorname{sgn}(\widetilde{\sigma}) \sum_{b=1}^{N} (-1)^{b+N} \int_{\Delta_{N-1}} (\tau_b - \tau_{b-1}) \exp\left(\sum_{a=1}^{N-1} \xi_{\widetilde{\sigma}_a} \tau_a\right) d\tau$$

where we wrote $\tau_N := 1$ and $\tau_0 := 0$. Here, we associated to a permutation $\sigma \in S_N$ with $\sigma_b = N$ the permutation $\widetilde{\sigma} \in S_{N-1}$ with $\widetilde{\sigma}_a = \sigma_a$ for a < b and $\widetilde{\sigma}_a = \sigma_{a+1}$ for $a \ge b$. The equality in the last step above holds because $\operatorname{sgn}(\sigma) = (-1)^{b+N} \operatorname{sgn}(\widetilde{\sigma})$. Since

$$\sum_{b=1}^{N} (-1)^{b+N} (\tau_b - \tau_{b-1}) = 1 + 2 \sum_{b=1}^{N-1} (-1)^{N+b} \tau_b,$$

we have (writing σ for $\widetilde{\sigma}$ again)

$$\mathcal{J}(k_1, \dots, k_N) = \sum_{\sigma \in S_{N-1}} \operatorname{sgn}(\sigma) \int_{\Delta_{N-1}} \exp\left(\sum_{a=1}^{N-1} \xi_{\sigma_a} \tau_a\right) d\tau$$
$$+ 2 \sum_{\sigma \in S_{N-1}} \operatorname{sgn}(\sigma) \sum_{b=1}^{N-1} (-1)^{N+b} \int_{\Delta_{N-1}} \tau_b \exp\left(\sum_{a=1}^{N-1} \xi_{\sigma_a} \tau_a\right) d\tau.$$

The first term is just $\mathcal{J}(k_1,\ldots,k_{N-1})$, so it remains to consider the second term. Here integrating by parts with respect to τ_b yields

$$\int_{\Delta_{N-1}} \tau_b \exp\left(\sum_{a=1}^{N-1} \xi_{\sigma_a} \tau_a\right) d\tau
= \frac{1}{\xi_{\sigma_b}} \left[\int_{\Delta_{N-2}} \tau_b \exp\left(\sum_{a=1}^{b-1} \xi_{\sigma_a} \tau_a + (\xi_{\sigma_b} + \xi_{\sigma_{b+1}}) \tau_b + \sum_{a=b+1}^{N-2} \xi_{\sigma_{a+1}} \tau_a\right) d\tau
- \int_{\Delta_{N-2}} \tau_{b-1} \exp\left(\sum_{a=1}^{b-2} \xi_{\sigma_a} \tau_a + (\xi_{\sigma_{b-1}} + \xi_{\sigma_b}) \tau_{b-1} + \sum_{a=b}^{N-2} \xi_{\sigma_{a+1}} \tau_a\right) d\tau
- \int_{\Delta_{N-1}} \exp\left(\sum_{a=1}^{N-1} \xi_{\sigma_a} \tau_a\right) d\tau \right]$$

where we used the convention $\tau_{N-1} = 1$ and $\tau_0 = 0$. Therefore an index shift yields

$$\sum_{b=1}^{N-1} (-1)^b \int_{\Delta_{N-1}} \tau_b \exp\left(\sum_{a=1}^{N-1} \xi_{\sigma_a} \tau_a\right) d\tau
= \sum_{b=1}^{N-1} (-1)^b \left(\frac{1}{\xi_{\sigma_b}} + \frac{1}{\xi_{\sigma_{b+1}}}\right) \int_{\Delta_{N-2}} \tau_b \exp\left(\sum_{a=1}^{b-1} \xi_{\sigma_a} \tau_a + (\xi_{\sigma_b} + \xi_{\sigma_{b+1}}) \tau_b + \sum_{a=b+1}^{N-2} \xi_{\sigma_{a+1}} \tau_a\right) d\tau
+ (-1)^{N-1} \frac{1}{\xi_{\sigma_{N-1}}} \int_{\Delta_{N-2}} \exp\left(\sum_{a=1}^{N-2} \xi_{\sigma_a} \tau_a\right) d\tau - \sum_{b=1}^{N-1} (-1)^b \frac{1}{\xi_{\sigma_b}} \int_{\Delta_{N-1}} \exp\left(\sum_{a=1}^{N-1} \xi_{\sigma_a} \tau_a\right) d\tau$$

For each b, the b-th summand of first term is symmetric in the indices b and b+1, hence its total anti-symmetrization vanishes. The second term is symmetric in in the indices b and b+2, hence its total anti-symmetrization vanishes as well (here we need $N \ge 4$, but for N = 1, 2, 3, the claims can easily be verified separately). In total, we obtain

$$2\sum_{\sigma \in S_{N-1}} \operatorname{sgn}(\sigma) \sum_{b=1}^{N-1} (-1)^{N+b} \int_{\Delta_{N-1}} \tau_b \exp\left(\sum_{a=1}^{N-1} \xi_{\sigma_a} \tau_a\right) d\tau$$

$$= 2\sum_{\sigma \in S_{N-1}} \operatorname{sgn}(\sigma) \sum_{b=1}^{N-1} (-1)^{N+b-1} \frac{1}{\xi_{\sigma_{N-1}}} \int_{\Delta_{N-2}} \exp\left(\sum_{a=1}^{N-2} \xi_{\sigma_a} \tau_a\right) d\tau$$

$$= 2 \left(\sum_{b=1}^{N-1} (-1)^{N+b-1} \right) \sum_{a=1}^{N-1} \sum_{\substack{\sigma \in S_{N-1} \\ \sigma_{N-1} = a}} \operatorname{sgn}(\sigma) \frac{1}{\xi_a} \int_{\Delta_{N-2}} \exp\left(\sum_{a=1}^{N-2} \xi_{\sigma_a} \tau_a \right) d\tau$$

For a fixed, set $\widetilde{\xi}_b := \xi_b$ for b < a and $\widetilde{\xi}_b = \xi_{b+1}$ for $b \ge a$. Moreover, define $\widetilde{\sigma} \in S_{N-2}$ by

$$\widetilde{\sigma}_b = \begin{cases} \sigma_b & \text{if } \sigma_b < a \\ \sigma_b - 1 & \text{if } \sigma_b > a \end{cases}.$$

Then $\widetilde{\xi}_{\widetilde{\sigma}_b} = \xi_{\sigma_b}$ for all $b = 1, \dots, N-2$ and $\operatorname{sgn}(\widetilde{\sigma}) = (-1)^{N+a+1}\operatorname{sgn}(\sigma)$ so that we find

$$\sum_{\substack{\sigma \in S_{N-1} \\ \sigma_{N-1} = a}} \operatorname{sgn}(\sigma) \frac{1}{\xi_a} \int_{\Delta_{N-2}} \exp\left(\sum_{a=1}^{N-2} \xi_{\sigma_a} \tau_a\right) d\tau$$

$$= (-1)^{N+a+1} \sum_{a=1}^{N-1} \frac{1}{\xi_a} \sum_{\widetilde{\sigma} \in S_{N-2}} \operatorname{sgn}(\widetilde{\sigma}) \int_{\Delta_{N-2}} \exp\left(\sum_{a=1}^{N-2} \widetilde{\xi}_{\widetilde{\sigma}_a} \tau_a\right) d\tau$$

$$= (-1)^{N+a+1} \sum_{a=1}^{N-1} \frac{1}{\xi_a} J(k_1, \dots, \widehat{k_a}, \dots, k_{N-1}).$$

In total, we get

$$\mathcal{J}(k_1,\ldots,k_N) = \mathcal{J}(k_1,\ldots,k_{N-1}) + 2\epsilon \sum_{a=1}^{N-1} (-1)^{N+a+1} \frac{1}{\xi_a} \mathcal{J}(k_1,\ldots,\widehat{k_a},\ldots,k_{N-1}),$$

where

$$\epsilon = \sum_{b=1}^{N-1} (-1)^{N+b-1} = \begin{cases} 1 & \text{if } N \text{ is even} \\ 0 & \text{if } N \text{ is odd,} \end{cases}$$

Since by (2), we have $\mathcal{J}(k_1,\ldots,k_{N-1})=0$ if N is odd, this is exactly the statement of (3) and (4).

This finishes the proof of Thm. 5.1

6 Proof of the Main Theorem

This section is dedicated of the proof of Thm. 1.1. In the proof, we will frequently use the spectral decomposition of the parallel transport $[\gamma|_0^1]^{TX}$ in the tangent bundle. Throughout, we fix a loop $\gamma \in \mathsf{L}X$ and an oriented orthonormal basis e_1, \ldots, e_n such that $[\gamma|_0^1]^{TX}$ has the form (2.3) with respect to the basis. Moreover, we choose the numbers $\alpha_1, \ldots, \alpha_m$ in such a way that $0 < \alpha_j < 1$ for each $j = 1, \ldots, m$.

Proof (of Thm. 1.1). It suffices to check the equality $[e^{\omega} \wedge \theta]_{\text{top}} = \Phi(q_N(\theta))$ for such θ that are an N-fold wedge product of elements of some orthonormal basis of $L^2(S^1, \gamma^*T'X)$, Here we will choose the orthonormal basis

$$E_{j,k}(t) := e^{2\pi i k t} [\gamma \|_0^t]^{T'X} e_j^*, \tag{6.1}$$

for $j=1,\ldots,n,\ k\in\mathbb{Z}$. Notice that for $1\leq j\leq 2m$, the covector fields $E_{j,k}(t)$ are not continuous; this is irrelevant for our purposes, however. Notice furthermore the $E_{j,k}$ are complex-valued co-vectors, i.e. we have $E_{j,k}\in L^2(S^1,\gamma^*T'X)\otimes\mathbb{C}$. Here we need to extend the L^2 scalar product appearing in formula (1.5) to complex scalars via complex bilinearity. There are no Hermitean forms in this paragraph. Hence we will assume that

$$\theta = \theta_N \wedge \dots \wedge \theta_1 = \bigwedge_{\beta=1}^{N_1} E_{1,k_{\beta}^{(1)}} \wedge \dots \wedge \bigwedge_{\beta=1}^{N_n} E_{n,k_{\beta}^{(n)}}, \tag{6.2}$$

where $N_j \in \mathbb{N}_0$ are such that $N_1 + \cdots + N_n = N$, and $k_{\beta}^{(j)} \in \mathbb{Z}$, $\beta = 1, \dots, N_j$. Note that here, according to our conventions, the factors are multiplied from largest to smallest a index,

$$\bigwedge_{\beta=1}^{N_j} E_{j,k_{\beta}^{(j)}} = E_{j,k_{N_j}^{(j)}} \wedge \dots \wedge E_{j,k_1^{(j)}},$$

but the overall product is from smallest to largest j index. This will be convenient later on.

Step 1. Notice that for any $s,t \in [0,1]$ and any $\vartheta \in T'_{\gamma(s)}X$, we have the identity

$$[\gamma \|_s^t]^{\Sigma} \mathbf{c}(\vartheta) [\gamma \|_0^s]^{\Sigma} = [\gamma \|_0^t]^{\Sigma} \mathbf{c} ([\gamma \|_s^0]^{T'X} \vartheta)$$

of elements of $\text{Hom}(\Sigma_{\gamma(0)}, \Sigma_{\gamma(t)})$, which follows from the compatibility of the Levi-Civita connection on spinors with the Clifford multiplication. Hence we can inductively "pull out" the parallel transport in the spinor bundle in the expression (1.5) (starting on the right) to obtain the equality

$$\sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} [\gamma \|_{\tau_N}^1]^{\Sigma} \prod_{j=1}^N \mathbf{c} \left(\theta_{\sigma_j}(\tau_j) \right) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\Sigma} d\tau$$

$$= [\gamma \|_0^1]^{\Sigma} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} \prod_{a=1}^N \mathbf{c} \left([\gamma \|_{\tau_a}^0]^{T'X} \theta_{\sigma_a}(\tau_a) \right) d\tau. \tag{6.3}$$

Now notice that if $\theta_a = E_{j,k}$, we have

$$\mathbf{c}([\gamma|_t^0]^{T'X}\theta_a(t)) = e^{2\pi i k t} \mathbf{c}([\gamma|_t^0]^{T'X}([\gamma|_0^t]^{T'X}e_j^*)) = e^{2\pi i k t} \mathbf{c}(e_j^*) = e^{2\pi i k t}e_j, \tag{6.4}$$

viewing $T_{\gamma(0)}X$ as a subspace of the Clifford algebra. The following lemma now allows us to split the integral up into n integrals, corresponding to the generalized orthonormal basis e_1, \ldots, e_n .

Lemma 6.1. Let V be a Euclidean vector space and let A_1, \ldots, A_N be integrable functions on [0,1] with values in Cl(V). Let $I_{\alpha} \subseteq \{1,\ldots,N\}$, $1 \leq j \leq n$, be subsets of indices such that $I_i \cap I_j = \emptyset$ for $i \neq j$ and such that their union equals $\{1,\ldots,N\}$. Suppose that for $i \neq j$, all elements corresponding to I_i totally anti-commute with all elements corresponding to I_j , meaning that

$$A_a(t)A_b(s) = -A_b(s)A_a(t)$$

for all $a \in I_i$, $b \in I_j$ and all $s, t \in [0, 1]$. Then

$$\sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} \prod_{a=1}^N A_{\sigma_a}(\tau_a) d\tau = \pm \prod_{j=1}^n \sum_{\sigma \in S_{I_j}} \operatorname{sgn}(\sigma) \int_{\Delta_{|I_j|}} \prod_{a \in I_j} A_{\sigma_a}(\tau_a) d\tau,$$

where S_{I_j} is the group of permutations of the set I_j . The sign on the right hand side is the sign of the permutation sending $\{N, \ldots, 1\}$ to (I_n, \ldots, I_1) , with elements of the subsets I_j ordered by size, from top to bottom.

Remark 6.2. Of course, this result also holds for functions in any finite-dimensional algebra over \mathbb{R} or \mathbb{C} instead of Cl(V).

Proof. We are indebted to mathoverflow user Gregory Arone for pointing out this elegant proof to us.

Let $\overline{\sigma}: [0,1]^N \to S_N$ be the function that maps $\tau = (\tau_N, \dots, \tau_1) \in [0,1]^N$ to the permutation $\sigma \in S_N$ such that $\tau_{\sigma_1} \leq \dots \leq \tau_{\sigma_N}$. This is well-defined except when τ is in the zero set of elements where $\tau_a = \tau_b$ for some indices $a \neq b$, hence $\overline{\sigma}$ defines a measurable function. Now if we set

$$f_N(\tau) := \operatorname{sgn}(\overline{\sigma}(\tau)) A_{\overline{\sigma}_N(\tau)}(\tau_{\overline{\sigma}_N(\tau)}) \cdots A_{\overline{\sigma}_1(\tau)}(\tau_{\overline{\sigma}_1(\tau)}),$$

for $\tau \in [0,1]^N$, then we have the identity

$$\sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} A_{\sigma_N}(\tau_N) \cdots A_{\sigma_1}(\tau_1) d\tau = \int_{[0,1]^N} f(\tau) d\tau.$$
 (6.5)

Let $I_{\alpha} = \{a_1^{(j)} \leq \cdots \leq a_{|I_j|}^{(j)}\}$. Then the anti-commutativity hypothesis implies that f_N splits as a product,

$$f_N(\tau_1,\ldots,\tau_N) = \pm \prod_{\alpha=1}^k f_{I_\alpha} \left(\tau_{a_1^{(\alpha)}},\ldots,\tau_{a_{|I_\alpha|}^{(\alpha)}} \right),$$

where the functions $f_{I_{\alpha}}$ are defined analogously to f_N on the cubes $[0,1]^{|I_{\alpha}|}$. Therefore,

$$\int_{[0,1]^N} f_N(\tau_1, \dots, \tau_N) d\tau = \pm \prod_{j=1}^k \int_{[0,1]^{|I_j|}} f(\tau) d\tau,$$

which implies the statement after using (6.5) again on each individual integral.

Since e_i and e_j anti-commute in $\text{Cl}T_{\gamma(0)}X$ for $i \neq j$, we can use Lemma 6.1 above (with $V = T_{\gamma(0)}X$) and (6.4) to obtain

$$\sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} \prod_{a=1}^N \mathbf{c} \left([\gamma \parallel_{\tau_a}^0]^{T'X} \theta_{\sigma_a}(\tau_a) \right) d\tau = \prod_{j=1}^n \sum_{\sigma \in S_{N_j}} \operatorname{sgn}(\sigma) \int_{\Delta_{N_j}} \prod_{\beta=1}^{N_j} e^{2\pi i k_{\beta}^{(j)}} e_j d\tau$$

$$= \prod_{j=1}^n \mathcal{J}(k_{N_j}^{(j)}, \dots, k_1^{(j)}) e_j^{N_j}, \tag{6.6}$$

where for numbers $k_1, \ldots, k_N \in \mathbb{Z}$, $\mathcal{J}(k_1, \ldots, k_N)$ is defined in (5.1). From (6.3) and the explicit formula (4.6) for the parallel transport in the spinor bundle, we obtain for our choice (6.2) of $\theta_1, \ldots, \theta_N$ that

$$\sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} [\gamma \|_{\tau_N}^1]^{\Sigma} \prod_{j=1}^N \mathbf{c} \left(\theta_{\sigma_j}(\tau_j) \right) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\Sigma} d\tau = \epsilon_0 \prod_{j=2m+1}^n C_j \prod_{j=1}^m B_j$$
 (6.7)

for some sign $\epsilon_0 = \pm 1$. Here B_j and C_j are given by

$$B_{j} = (\cos(\pi\alpha_{j}) + \sin(\pi\alpha_{j})e_{2j-1}e_{2j})\mathcal{J}_{2j-1}\mathcal{J}_{2j}e_{2j-1}^{N_{2j-1}}e_{2j}^{N_{2j}} \qquad j = 1, \dots, m$$

$$C_{j} = \mathcal{J}_{j}e_{j}^{N_{j}} \qquad j = 2m + 1, \dots, n,$$

$$(6.8)$$

where we set $\mathcal{J}_j := \mathcal{J}(k_{N_j}^{(j)}, \dots, k_1^{(j)})$ for abbreviation. Notice that the B_j and C_j are elements of the sub-algebras $\mathrm{Cl}(\mathbb{R}e_{2j-1} \oplus \mathbb{R}e_{2j})$ respectively $\mathrm{Cl}(\mathbb{R}e_j)$ of $\mathrm{Cl}(T_{\gamma(0)}X)$ that pairwise super-commute. The top order term of their product is therefore the product of the individual top order terms inside these sub-algebras. For these, we find from (6.8)

$$[B_j]_{\text{top}} = \begin{cases} \cos(\pi \alpha_j) \mathcal{J}_{2j-1} \mathcal{J}_{2j} & \text{if } N_{2j-1}, N_{2j} \text{ are both odd} \\ \sin(\pi \alpha_j) \mathcal{J}_{2j-1} \mathcal{J}_{2j} & \text{if } N_{2j-1}, N_{2j} \text{ are both even} \\ 0 & \text{otherwise,} \end{cases}$$
(6.9)

and

$$[C_j]_{\text{top}} = \begin{cases} \mathcal{J}_j & \text{if } N_j \text{ is odd} \\ 0 & \text{if } N_j \text{ is even.} \end{cases}$$
 (6.10)

Taking the super trace as defined in formula (4.3), we therefore obtain

$$\sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} \operatorname{str}\left(\left[\gamma \right]_{\tau_N}^1 \right]^{\Sigma} \prod_{j=1}^N \mathbf{c} \left(\theta_{\sigma_j}(\tau_j) \right) \left[\gamma \right]_{\tau_{j-1}}^{\tau_j} \right]^{\Sigma} d\tau$$

$$= \epsilon_0 \operatorname{str} \prod_{j=2m+1}^n C_j \prod_{j=1}^m B_j = \epsilon_0 2^{n/2} \prod_{j=2m+1}^n \left[C_j \right]_{\operatorname{top}} \prod_{j=1}^m \left[B_j \right]_{\operatorname{top}}.$$

By the definition (4.7) of the isomorphism Φ , we therefore find

$$\Phi(q(\theta_N \wedge \dots \wedge \theta_1)) = 2^{(n-N)/2} \prod_{j=2m+1}^n [C_j]_{\text{top}} \prod_{j=1}^m [B_j]_{\text{top}} \cdot \Theta_{\gamma}, \tag{6.11}$$

where Θ_{γ} is taken with respect to the choice of oriented basis e_1, \ldots, e_n and the corresponding values $0 < \alpha_1, \ldots, \alpha_m \le 1$. With a view on (6.9) and (6.10), we can therefore make the following observation.

Observation 6.3. $q(\theta_N \wedge \cdots \wedge \theta_1)$ can be non-zero only if for each $j = 1, \ldots, m$, the numbers N_{2j-1} and N_{2j} have the same parity, and if for each $j = 2m + 1, \ldots, n$ the number N_j is odd.

Moreover, notice that in the case that N_j is odd (for j = 2m + 1, ..., n), the number \mathcal{J}_j appearing as a coefficient in (6.10) can be non-zero only in the case that exactly one of the indices $k_1^{(j)}, ..., k_{N_j}^{(j)}$ is zero, by Thm. 5.1. We therefore observe the following.

Observation 6.4. $q(\theta_N \wedge \cdots \wedge \theta_1)$ can be non-zero only if for each $j = 2m + 1, \dots, n$, exactly one of the indices $k_1^{(j)}, \dots, k_{N_j}^{(j)}$ is zero.

Step 2. A similar calculation can be made for $[e^{\omega} \wedge \theta_1 \wedge \cdots \wedge \theta_N]_{\text{top}}$, again for the special choice (6.2) of $\theta_1, \ldots, \theta_N$. Notice first that $E_{2m+1,0}, \ldots, E_{n,0}$ form an orthonormal basis of $\text{ker}(\nabla_{\dot{\gamma}})$, while all other $E_{j,k}$ lie in the orthogonal complement of the kernel. Hence directly from the definition (3.8) of $[e^{\omega} \wedge \theta_N \wedge \cdots \wedge \theta_1]_{\text{top}}$, we can read off the following observation.

Observation 6.5. Also $[e^{\omega} \wedge \theta_N \wedge \cdots \wedge \theta_1]_{\text{top}}$ can be non-zero only if for each index $j = 2m + 1, \dots, n$, exactly one of the numbers $k_1^{(j)}, \dots, k_{N_j}^{(j)}$ is zero.

Notice that the same statement is true for $q(\theta_N \wedge \cdots \wedge \theta_1)$, by Observation 6.4. From now on, we therefore make the assumption that this is the case.

Assumption 1: For $j=2m+1,\ldots,n$, exactly one of the numbers $k_1^{(j)},\ldots,k_{N_j}^{(j)}$ is zero. Denote the corresponding index by $\beta_j \in \{1,\ldots,N_j\}$. Correspondingly, write $b_j \in \{1,\ldots,N\}$ for the index such that $\theta_{b_j} = E_{j,k_{\beta_j}^{(j)}}$, with $k_{\beta_j}^{(j)} = 0$.

Clearly, for these indices, we have

$$b_j = N_1 + \dots + N_{j-1} + \beta_j. \tag{6.12}$$

Let $I \subseteq (N, ..., 1)$ be the reversely ordered subset of indices a such that $\theta_a \in \ker(\nabla_{\dot{\gamma}})^{\perp}$. Put differently, I is just the tuple (N, ..., 1) with the numbers $b_{2m+1}, ..., b_n$ removed, with b_j as in Assumption 1. For general subsets $I' \subseteq I$, define the matrix $\Omega_{I'}$ by

$$\Omega_{I'} := \left((\theta_a, \nabla_{\dot{\gamma}}^{-1} \theta_b)_{L^2} \right)_{a,b \in I'}, \tag{6.13}$$

where we use the same largest-to-smallest index convention as in (3.4). Now under the assumption above, we have by the definition (4.8),

$$[e^{\omega}\theta_N \wedge \dots \wedge \theta_1]_{\text{top}} = \eta_0 \prod_{j=1}^m 2\sin(\pi\alpha_j) \text{pf}(\Omega_I) \cdot \Theta_{\gamma}. \tag{6.14}$$

Here η_0 is the sign of the permutation that sends $(N, \ldots, 1)$ to $(b_{2m+1}, \ldots, b_n, I)$. It is given by $\eta_0 = (-1)^K$, where

$$K = \sum_{j=1}^{N-2m} (b_{2m+j} - j) = \sum_{j=2m+1}^{N} (N_1 + \dots + N_{2m} + (N_{2m+1} - 1) + \dots + (N_{j-1} - 1) + (\beta_j - 1)),$$

compare (6.12). It will turn out that $N_{2j-1} + N_{2j}$ must be even for j = 1, ..., m in order for $[e^{\omega} \wedge \theta]_{\text{top}}$ not to vanish, while N_j must be odd for j = 2m + 1, ..., n (complementing Observation 6.3). Taking this for granted, one can reduce K modulo 2 to arrive at the formula

$$\eta_0 = (-1)^{(\beta_{2m+1}-1)+\dots+(\beta_n-1)}. (6.15)$$

The entries of Ω_I are given explicitly as follows.

Lemma 6.6. The scalar products $(E_{j,k}, \nabla_{\dot{\gamma}}^{-1} E_{i,l})_{L^2}$ for $E_{j,k}, E_{i,l} \in \ker(\nabla_{\dot{\gamma}})^{\perp}$ are given as follows.

1. If $k, l \neq 0$, we have

$$(E_{j,k}\nabla_{\dot{\gamma}}^{-1}E_{j,l})_{L^2} = \frac{\delta_{k,-l}}{2\pi i k}.$$
 (6.16)

2. If $1 \le j \le 2m$ and $k \ne 0$, we have

$$(E_{j,k}, \nabla_{\dot{\gamma}}^{-1} E_{j,0})_{L^2} = \frac{1}{2\pi i k}$$

3. If $1 \le j \le m$, we have

$$(E_{2j-1,0}, \nabla_{\dot{\gamma}}^{-1} E_{2j,0})_{L^2} = \frac{1}{2} \cot(\pi \alpha_j).$$

4. All other scalar products $(E_{j,k}, \nabla_{\dot{\gamma}}^{-1} E_{i,l})_{L^2}$ are zero.

Proof. These results follow from direct calculation after verifying that the operator G, which on the orthogonal complement of $\ker(\nabla_{\dot{\gamma}})$ in $L^2(S^1, \gamma^*TX)$ is given by

$$GV(t) = \int_0^t [\gamma \|_s^t]^{TX} V(s) ds + [\gamma \|_0^t]^{TX} (id - [\gamma \|_0^1]^{TX})^{-1} \int_0^t [\gamma \|_s^1]^{TX} V(s) ds, \qquad (6.17)$$

is the Green's operator to $\nabla_{\dot{\gamma}}$.

Lemma 6.6 shows that for $\theta_1, \ldots, \theta_N$ of the form (6.2), the matrix Ω_I from above is blockdiagonal,

$$\Omega_I = \Omega_{J_1} \oplus \cdots \oplus \Omega_{J_m} \oplus \Omega_{I_{2m+1}} \oplus \cdots \oplus \Omega_{I_n}$$
(6.18)

where for $1 \leq j \leq m$, $J_j \subset I$ is the reversely ordered subset of indices a such that $\theta_a = E_{2j-1,k}$ or $E_{2j,k}$ for some $k \in \mathbb{Z}$, while for for all j, we let I_j is the subset of indices a such that $\theta_a = E_{j,k}$ for some $k \in \mathbb{Z} \setminus \{0\}$. By the product formula for the Pfaffian, the Pfaffian of Ω_I also splits as a product, yielding

$$\operatorname{pf}(\Omega_I) = \prod_{j=1}^m \operatorname{pf}(\Omega_{J_j}) \prod_{j=2m+1}^n \operatorname{pf}(\Omega_{I_j}). \tag{6.19}$$

Plugging (6.19) into (6.14) then yields

$$[e^{\omega}\theta_1 \wedge \dots \wedge \theta_N]_{\text{top}} = \eta_0 \prod_{j=1}^m 2\sin(\pi\alpha_j) \operatorname{pf}(\Omega_{J_j}) \prod_{j=2m+1}^n \operatorname{pf}(\Omega_{I_j}) \cdot \Theta_{\gamma}.$$
 (6.20)

By the properties of the Pfaffian, the right hand side of the above equation can be non-zero only if $|I_j|$ is even for each j. Since $|J_j| = N_{2j-1} + N_{2j}$ for j = 1, ..., m and by Assumption 1, $|I_j| = N_j - 1$ for j = 2m + 1, ..., n, we can conclude

Observation 6.7. $[e^{\omega} \wedge \theta_N \wedge \cdots \wedge \theta_1]_{\text{top}}$ can be non-zero only if for each $j = 1, \ldots, m$, N_{2j-1} and N_{2j} have the same parity and if for $j = 2m + 1, \ldots, n$, N_j is odd.

Step 3. We now compare (6.20) and (6.11), where the latter can be rearranged to

$$\Phi(q(\theta_N \wedge \dots \wedge \theta_1)) = \prod_{j=2m+1}^n 2^{1-(N_{2j-1}+N_{2j})/2} [C_j]_{\text{top}} \prod_{j=1}^m 2^{-(N_j-1)/2} [B_j]_{\text{top}} \cdot \Theta_{\gamma}.$$
 (6.21)

Notice first that the Observations 6.3 and 6.7 match exactly, so we can make the

Assumption 2: For each
$$j = 1, ..., m$$
, N_{2j-1} and N_{2j} have the same parity and for $j = 2m + 1, ..., n$, N_i is odd.

If we compare (6.20) and (6.21) factor by factor, taking a look at the formula (6.15) for η_0 , which holds under the above Assumption 2, we see that the theorem is proven if we can show that

$$\sin(\pi \alpha_j) \operatorname{pf}(\Omega_{J_j}) = 2^{-(N_{2j-1} + N_{2j})/2} [B_j]_{\text{top}}$$
(6.22)

for $1 \le j \le m$ and

$$pf(\Omega_{I_i}) = (-1)^{\beta_j - 1} 2^{-(N_j - 1)/2} [C_j]_{top}$$
(6.23)

for $2m + 1 \le j \le n$, where β_j is the index such that $k_{\beta_j}^{(j)} = 0$.

We now proceed by verifying the above equalities. The key to this will be the following observation, which follows from combining Lemma 6.6, Thm 5.1 and the definition (3.1) of the Pfaffian.

Observation 6.8. If $k_1, \ldots, k_N \in \mathbb{Z} \setminus \{0\}$ and N is even, we have

$$\mathcal{J}(k_N,\ldots,k_1) = 2^{N/2} \operatorname{pf}\left(\left(E_{j,k_\alpha}, \nabla_{\dot{\gamma}}^{-1} E_{j,k_\beta}\right)_{L^2}\right)_{N > \alpha,\beta > 1}$$

for any $j = 1, \ldots, n$.

First consider (6.23), where $2m + 1 \le j \le n$. Since N_j is odd by Assumption 2 and since exactly one of the numbers $k_1^{(j)}, \ldots, k_{N_j}^{(j)}$ is zero (namely $k_{\beta_j}^{(j)}$) by Assumption 1, we obtain from (6.10), Thm. 5.1 (3) and Observation 6.8 that

$$[C_j]_{\text{top}} = \mathcal{J}(k_1^{(j)}, \dots, k_{N_j}^{(j)}) = (-1)^{N_j + \beta_j} \mathcal{J}(k_1^{(j)}, \dots, \widehat{k_{\beta_j}^{(j)}}, \dots, k_{N_j}^{(j)})$$
$$= (-1)^{N_j + \beta_j} 2^{(N_j - 1)/2} \operatorname{pf}(\Omega_{J_j}).$$

This establishes (6.23), as N_i is odd.

Now consider (6.22), where $1 \leq j \leq m$. Let $J_j^0 \subseteq J_j$ be the subset of indices a such that $\theta_a = E_{2j-1,0}$ or $E_{2j,0}$. Notice that as (unordered) sets, $J_j = J_j^0 \cup I_{2j-1} \cup I_{2j}$.

Now in order to have $\operatorname{pf}(\Omega_{J_j}) \neq 0$, we must have $|J_j^0| \in \{0, 1, 2\}$, because otherwise Ω_{J_j} would be singular. Similarly, formula (6.9) shows that in order for $[B_j]_{\text{top}}$ to be non-zero, we must have both $\mathcal{J}_{2j-1} \neq 0$ and $\mathcal{J}_{2j} \neq 0$. If now $|J_j^0| \geq 3$, then necessarily one of the sequences $k_1^{(2j-1)}, \ldots, k_{N_{2j-1}}^{(2j-1)}$ and $k_1^{(2j)}, \ldots, k_{N_{2j}}^{(2j)}$ contains two zeros, which implies that one of \mathcal{J}_{2j-1} , \mathcal{J}_{2j} is zero, by Thm. 5.1 (1).

We obtain that in the case $|J_j^0| \ge 3$, both $\operatorname{pf}(\Omega_{J_j}) = 0$ and $[B_j]_{\operatorname{top}} = 0$. So we are left to establish (6.22) in the remaining three cases.

Case 1: $|J_j^0| = 0$. In this case, we have $J_j = (I_{2j-1}, I_{2j})$ and Ω_{J_j} splits as the direct sum $\Omega_{J_j} = \Omega_{I_{2j-1}} \oplus \Omega_{I_{2j}}$. Hence

$$\operatorname{pf}(\Omega_{J_j}) = \operatorname{pf}(\Omega_{I_{2j-1}})\operatorname{pf}(\Omega_{I_{2j}}).$$

By Assumption 2, N_{2j-1} and N_{2j} have the same parity. If they are both odd, then $pf(\Omega_{I_{2j-1}}) = pf(\Omega_{I_{2j}}) = 0$ and $\mathcal{J}_{2j-1} = \mathcal{J}_{2j} = 0$ by Thm. 5.1 (2). If they are both even, then looking at (6.9) and using Thm. 5.1 (3) together with Observation 6.8, we obtain

$$[B_j]_{\text{top}} = \sin(\pi \alpha_j) \mathcal{J}_{2j-1} \mathcal{J}_{2j} = \sin(\pi \alpha_j) 2^{N_{2j-1}/2} \operatorname{pf}(\Omega_{I_{2j-1}}) 2^{N_{2j}/2} \operatorname{pf}(\Omega_{I_{2j}}),$$

which establishes (6.22) in this case.

Case 2: $|J_j^0| = 1$. For abbreviation, write $I_{2j-1} = (a_1, \ldots, a_K)$, $I_{2j} = (b_1, \ldots, b_L)$ and $J_j^0 = \{c\}$. Let $\theta_{a_\ell} = E_{2j-1,k_\ell}$ and $\theta_{b_\ell} = E_{2j,l_\ell}$. We now have either $\theta_c = E_{2j-1,0}$ or $\theta_c = E_{2j,0}$. In the first case (Subcase 2a), assume without loss of generality that $J_j = (c, I_{2j-1}, I_{2j})$. Using Lemma 6.6 (2), we then obtain

$$\Omega_{J_{j}} = \begin{pmatrix} 0 & \frac{1}{2\pi i k_{1}} & \cdots & \frac{1}{2\pi i k_{K}} & 0 & \cdots & 0 \\ -\frac{1}{2\pi i k_{1}} & & & & & & \\ \vdots & & & \Omega_{2j-1} & & & 0 \\ -\frac{1}{2\pi i k_{K}} & & & & & & \\ 0 & & & & & & \\ \vdots & & & 0 & & & \Omega_{I_{2j}} \\ 0 & & & & & & \end{pmatrix}.$$

If $L = N_{2j}$ is odd, then the Pfaffian of this matrix is zero, as it is block-diagonal with odd-dimensional blocks. Similarly, by Thm. 5.1 (2), $\mathcal{J}_{2j} = 0$, so (6.22) is trivial. If on the other hand $K = N_{2j-1} - 1$ is odd and N_{2j} is even (remember this is the only other possibility, by Assumption 2), we can use the Pfaffian development rule on the first row, to obtain

$$pf(\Omega_{J_{j}}) = \sum_{\ell=1}^{K} (-1)^{\ell+1} \frac{1}{2\pi i k_{\ell}} pf(\Omega_{I_{2j-1} \setminus \{a_{\ell}\}}) pf(\Omega_{I_{2j}})$$

$$= \frac{1}{2} \sum_{\ell=1}^{K} (-1)^{\ell+1} \frac{1}{\pi i k_{\ell}} 2^{-(K-1)/2} \mathcal{J}(k_{1}, \dots, \widehat{k_{\ell}}, \dots, k_{K}) 2^{-L/2} \mathcal{J}(l_{1}, \dots, l_{K})$$

$$= 2^{-(K+L+1)/2} \mathcal{J}(k_{1}, \dots, k_{K}) \mathcal{J}(l_{1}, \dots, l_{K}) = 2^{-(N_{2j-1}+N_{2j})/2} \mathcal{J}_{2j-1} \mathcal{J}_{2j},$$

where we used Observation 6.8 and Thm. 5.1 (4). This establishes (6.22) with a view on (6.9).

If now $\theta_c = E_{2j,0}$ (subcase 2b), then, assuming without loss of generality that $J_j = (I_{2j-1}, c, I_{2j})$, we have

Now we necessarily need that K is even in order for the Pfaffian of this matrix to be non-zero. Hence, developing the (K+1)st row yields

$$pf(\Omega_{J_j}) = pf(\Omega_{I_{2j-1}}) \sum_{\ell=1}^{L} (-1)^{1+\ell} \frac{1}{2\pi i l_{\ell}} pf(\Omega_{I_{2j} \setminus \{b_{\ell}\}}),$$

and we can obtain (6.22) just as before.

Case 3: $|J_j^0| = 2$. Again, write $I_{2j-1} = \{a_1, \ldots, a_K\}$, $I_{2j} = \{b_1, \ldots, b_L\}$ and let $\theta_{a_\ell} = E_{2j-1,k_\ell}$ and $\theta_{b_\ell} = E_{2j,l_\ell}$. If $J_j^0 = \{c,d\}$, then pf (Ω_{J_j}) is non-zero only if $\theta_c \neq \theta_d$ (because otherwise Ω_{J_j} would be singular). Similarly, if $\theta_c = \theta_d$, Thm. 5.1 (1) implies that one of \mathcal{J}_{2j-1} , \mathcal{J}_{2j} is zero. Therefore, we may assume that $\theta_c = E_{2j-1,0}$ and $\theta_d = E_{2j,0}$ and $J_j = (c, I_{2j-1}, d, I_{2j})$. We obtain that

$$\Omega_{J_j} = \begin{pmatrix} 0 & \frac{1}{2\pi i k_1} & \cdots & \frac{1}{2\pi i k_K} & \frac{1}{2} \cot(\pi \alpha_j) & 0 & \cdots & 0 \\ -\frac{1}{2\pi i k_1} & & & 0 & & & \\ \vdots & & \Omega_{I_{2j-1}} & & \vdots & & & \\ -\frac{1}{2\pi i k_K} & & & 0 & & & \\ -\frac{1}{2} \cot(\pi \alpha_j) & 0 & \cdots & 0 & 0 & \frac{1}{2\pi i l_1} & \cdots & \frac{1}{2\pi i l_L} \\ 0 & & & & -\frac{1}{2\pi i l_1} & & & \\ \vdots & & & & \vdots & & \Omega_{I_{2j}} \\ 0 & & & & -\frac{1}{2\pi i l_L} & & \end{pmatrix}.$$

In the case that $K = N_{2j-1} - 1$ and $L = N_{2j} - 1$ are both odd, developing with respect to the first row yields

$$pf(\Omega_{J_j}) = \left(\sum_{\ell=1}^K (-1)^{\ell+1} \frac{1}{2\pi i k_\ell} pf(\Omega_{I_{2j-1} \setminus \{a_\ell\}})\right) \cdot \left(\sum_{\ell=1}^L (-1)^{\ell+1} \frac{1}{2\pi i l_\ell} pf(\Omega_{I_{2j} \setminus \{b_\ell\}})\right)$$

The equality (6.22) is in this case obtained as before, using Observation 6.8 and Thm. 5.1 (4). In the case that both K and L are even, we get from Observation 6.8

$$\operatorname{pf}(\Omega_{J_j}) = \frac{1}{2} \cot(\pi \alpha_j) \operatorname{pf}(\Omega_{I_{2j-1}}) \operatorname{pf}(\Omega_{I_{2j}}) = \frac{\cos(\pi \alpha_j)}{\sin(\pi \alpha_j)} 2^{-(K+L+2)/2} \mathcal{J}_{2j-1} \mathcal{J}_{2j}.$$

Remember that these are the only relevant cases, by Assumption (2). Because $K+L+2 = N_{2j-1} + N_{2j}$, we obtain (6.22) with a view on (6.9). The proof is now complete.

References

- [AD99] Lars Andersson and Bruce K. Driver. Finite-dimensional approximations to Wiener measure and path integral formulas on manifolds. *J. Funct. Anal.*, 165(2):430–498, 1999.
- [Ati85] M. F. Atiyah. Circular symmetry and stationary-phase approximation. Astérisque, (131):43–59, 1985. Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983).
- [Bis85] Jean-Michel Bismut. Index theorem and equivariant cohomology on the loop space. Comm. Math. Phys., 98(2):213–237, 1985.

- [BP08] Christian Bär and Frank Pfäffle. Path integrals on manifolds by finite dimensional approximation. J. Reine Angew. Math., 625:29–57, 2008.
- [FKT02] Joel Feldman, Horst Knörrer, and Eugene Trubowitz. Fermionic functional integrals and the renormalization group, volume 16 of CRM Monograph Series. American Mathematical Society, Providence, RI, 2002.
 - [FS08] Dana S. Fine and Stephen F. Sawin. A rigorous path integral for supersymmetric quantum mechanics and the heat kernel. *Comm. Math. Phys.*, 284(1):79–91, 2008.
 - [GS99] Victor W. Guillemin and Shlomo Sternberg. Supersymmetry and equivariant de Rham theory. Mathematics Past and Present. Springer-Verlag, Berlin, 1999. With an appendix containing two reprints by Henri Cartan [MR0042426 (13,107e); MR0042427 (13,107f)].
- [HL17a] Florian Hanisch and Matthias Ludewig. Supersymmetric path integrals I: Differential forms on the loop space. arXiv:1709.10027, 2017.
- [HL17c] Florian Hanisch and Matthias Ludewig. Supersymmetric path integrals III: The viewpoint of infinite-dimensional super geometry. to appear.
- [LM89] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
- [Lot87] John Lott. Supersymmetric path integrals. Comm. Math. Phys., 108(4):605–629, 1987.
- [Lud17] Matthias Ludewig. Path Integrals on Manifolds with Boundary. Comm. Math. Phys., 354(2):621–640, 2017.
- [Mur96] M. K. Murray. Bundle gerbes. J. London Math. Soc. (2), 54(2):403–416, 1996.
- [PW09] Arturo Felipe Prat Waldron. Pfaffian line bundles over loop spaces, spin structures and the index theorem. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)—University of California, Berkeley.
- [ST05] Stefan Stolz and Peter Teichner. The spinor bundle on the loop space. https://people.mpim-bonn.mpg.de/teichner/Math/Surveys_files/MPI.pdf, 2005.
- [Wal16] Konrad Waldorf. Spin structures on loop spaces that characterize string manifolds. *Algebr. Geom. Topol.*, 16(2):675–709, 2016.