

COMBINATORIAL GEOMETRY OF FLAG DOMAINS IN G/B

DISSERTATION

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To everyone who wished the best for me.

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”Be a scientist, if you can not, be a student, if you can not, love scientists, if not, don’t hate them”

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Chapter 1

Introduction

1.1 Preliminaries

Let G denote a complex semisimple Lie group with real form G_0 and $Z := G/Q$ be a G -flag manifold. It is known that G_0 has only finitely many orbits in Z and that there is a unique closed orbit γ^{cl} . The open G_0 -orbits are referred to as *flag domains* D . Every maximal compact subgroup K_0 of G_0 has a unique orbit C_0 in D which is a complex submanifold. This is the unique K_0 -orbit of minimal dimension in D . If K is the corresponding (reductive) subgroup of G , then it can also be characterized as the unique K -orbit which is contained in D . Having fixed K_0 , we refer to C_0 as the associated *base cycle*. For the above and other background information see ([7] and [15]).

The main goal of our work can be formulated as that of determining the class of C_0 in the (topological) cohomology ring of Z . If B is any Borel subgroup of G and $r := \dim_{\mathbb{C}} C_0$, this amounts to determining the intersection numbers of the r -codimensional B -Schubert varieties S with C_0 . Of course it is possible that this intersection is empty.

The original motivation for this project was another: If B is maximally real in the sense that it fixes a point in γ^{cl} , or equivalently if it contains the A_0N_0 -factor of an Iwasawa-decomposition $G_0 = K_0A_0N_0$, and S is a B -Schubert variety (we refer to such as Iwasawa-Schubert varieties) of complementary dimension to the cycle C_0 , then $S \cap D$ is contained in the open B -orbit \mathcal{O} in S and the intersection $\mathcal{O} \cap C_0$ is finite and transversal at each of its points (see [7]). In fact each component Σ of $\mathcal{O} \cap D$ intersects C_0 in exactly one point z_{Σ} with $\Sigma = A_0N_0.z_{\Sigma}$. This in turn gives rise to an optimal Barlet-Koziaz trace-transform from holomorphic functions on $\mathcal{O} \cap D$ to holomorphic functions on the appropriately defined space $\mathcal{C}(D)$ of cycles (see [3], [7]). Thus, although the determination of the r -codimensional Iwasawa-Schubert varieties which have non-empty intersection with D , along with their points of intersection with C_0 , is apparently a problem of a combinatorial nature, such information has complex analytic significance.

Since $S \cap D \neq \emptyset$ implies that $S \cap C_0 \neq \emptyset$, the first goal of this project is to determine which Schubert varieties S have nonempty intersection with C_0 . After doing so, we then describe this intersection in the case where S is of complementary dimension to C_0 . The Schubert

varieties are determined by the elements of the Weyl group W_I of a distinguished maximal torus T_I in the Borel subgroup B_I which fixes a certain base point in γ_{cl} . These Schubert varieties are denoted by S_w where $w \in W_I$. Consequently our results are formulated in terms of conditions on elements w in the Weyl group W_I .

In this thesis we make several restrictions. First, we only consider the case where G is a classical group, i.e., associated to one of the Lie algebras A_n , B_n , C_n or D_n . The case of A_n (and all of its real forms) was handled by Brean ([4],[5]). Our work here is therefore devoted to the cases of B_n , C_n and D_n . Furthermore, we restrict to the setting where $Z = G/B$ is the G -manifold of full flags. This is reflective of our work with the advantage that technical complications are minimized. The case of a general G -flag manifold $Z = G/Q$ is handled in author paper ([1]).

1.2 The structure of the thesis

As indicated above this thesis is devoted to study the combinatorial geometry of the flag domains in G/B of the real forms $SP(2n, \mathbb{R})$, $SO^*(2n)$, $SO(p, q)$ and $SP(2p, 2q)$, where B is Borel subgroup. We study the intersection between the base cycles of the flag domains and the Schubert varieties. In the cases of $SP(2n, \mathbb{R})$ and $SO^*(2n)$ a remarkable unicity appears so that an exact description of the Weyl elements can be given. Analogous to the case of $SU(p, q)$ in the work of Brean, except for a few interesting special cases, for $SO(p, q)$ and $SP(2p, 2q)$ these elements are described by algorithms. In all cases the number of intersection points with C_0 is explicitly computed.

The work here is organized in six chapters which are briefly described as follows:

Chapter 2: In this chapter we explain the formulas for all real forms G_0 which arise in this thesis. In each case two base points are introduced: a base point F_I in γ_{cl} and a base point F_S in a certain standard cycle. These two base points are full flags associated to bases which define the maximal Tori T_I and T_S respectively. The Borel subgroup B_I and the Weyl group W_I are described in terms of the ordered basis which defines F_I . Also, we define a special way of writing the Weyl group elements for the Lie groups of type B_n , C_n and D_n in terms of flags. In each case the flag domains and base cycles are described by signature invariants.

Chapter 3: Here we prove two general fixed point theorems:

Theorem 1.1. *For every $w \in W$, if the intersection $S_w \cap C_0$ is non-empty, then it contains a T_S -fixed point.*

Theorem 1.2. *If S_w is of complementary dimension to C_0 , then $S_w \cap C_0 \subset \text{Fix}(T_S)$.*

In the following chapters we focus on the goal of understanding the combinatorial geometry of the topological class $[C_0]$ by attempting to find precise formulas which answer the following question in each of the cases B_n , C_n and D_n .

Question 1. What are the conditions on the Weyl elements w which parametrize the Iwasawa-Schubert Varieties S_w with non-empty intersection with the base cycle C_0 ?

Question 2. What are the points of intersection? How many of these points do we have?

Chapter 4: This chapter is concerned with the case of the real form $SP(2n, \mathbb{R})$. In this case there is a certain Weyl element, which we refer to as a "super generous permutation". Since we are considering flag domains in G/B , all base cycles have the same dimension. In terms of a super generous permutation the main result of this chapter can be stated as follows.

Theorem 1.3. *If S_w is an Iwasawa-Schubert variety of complementary dimension to a base cycle C_0 and $S_w \cap C_0 \neq \emptyset$, then w is the super generous permutation. Furthermore, the intersection $S_w \cap C_0$ with every base cycle consists of just one point.*

Chapter 5: This chapter is devoted to study the real form $SO^*(2n)$ which is analogous to that for the case of the real form $SP(2n, \mathbb{R})$. Again a uniqueness theorem of the above type is valid, but in this case the results depend on n being even or odd.

Chapter 6: The real form of this chapter is $SO(p, q)$. The results are stated in terms of algorithms (See definitions 6.1.1, 6.2.1 and Corollary 6.3.2) ; in fact it seems impossible to avoid this. Also it would seem to be difficult to find a concrete formula for the intersection points in the general case. We do however explicitly compute the numbers of intersection points (see Theorem 6.3.3 and Corollary 6.3.2).

Chapter 7: This chapter is devoted to the real form $SP(2p, 2q)$. Our work here is similar to that for $SO(p, q)$. Again the results are stated in terms of algorithms (Corollary 7.3.2, Definition 7.1.1 and Definition 7.2.1) and again it would seem to be difficult to find a concrete formula for the intersection points in the general case. We do, however, explicitly compute the numbers of intersection points (see Theorem 7.3.3).

Chapter 2

Basic ingredients

The Lie subgroup G_0 is a real form of the classical group G which is assumed to be of type B_n , C_n or D_n , and Z is in each case the associated manifold of full flags. In each case we introduce appropriate bases of the relevant vector space so that G , G_0 and the compact and reductive groups, K_0 and K , are described in matrix terms which are useful for our purposes. Regarded as flags, these bases define base points in Z along with Borel subgroups with distinguished maximal tori. Of particular importance is the base point F_I in the closed G_0 -orbit, because the Iswasawa-Schubert cells which will be considered are orbits of its isotropy group B_I of points $w(F_I) =: F_w$, where w runs through the Weyl group W_I defined by the distinguished maximal torus T_I in B_I . Finally, in each of the cases we describe the flag domains D in Z by signature invariants and the base cycle C_0 in terms of splittings of V .

2.1 Introduction to the flags in Z

Let G be a semisimple Lie group of type B_n, C_n or D_n , defined in each case by a non-degenerate bilinear form $b(v, w)$. A maximally b -isotropic full flag F with respect to b is a sequence of $(m + 1)$ -vector spaces

$$F = (\{0\} = V_0 \subset V_1 \subset \dots \subset V_m = \mathbb{C}^m)$$

such that $\dim V_i = i$, for all $0 \leq i \leq m$ and $V_{m-i} = V_i^\perp$, for $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$, where $m = 2n$ or $2n + 1$. In particular V_i is isotropic for $1 \leq i \leq n$. The set of all maximally isotropic flags is denoted by Z . The manifold structure of Z arises from the fact that the Lie group G acts transitively on it with $Z = G/B$ where B is the stabilizer of any particular maximally b -isotropic flag in Z .

In all cases $\sigma : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$, $u \mapsto \bar{u}$, is standard complex conjugation and G_0 is defined to be the fixed set of σ , i.e.,

$$G_0 = \{g \in G : \sigma(g) = g\}.$$

The signature of a flag $F = (0 \subset V_1 \subset \dots \subset V_m) \in Z$ with respect to h consists of three sequences

$$a : 0 \leq a_1 \leq a_2 \leq \dots \leq a_m$$

$$b : 0 \leq b_1 \leq b_2 \leq \dots \leq b_m$$

$$d : 0 \leq d_1 \leq d_2 \leq \dots \leq d_m$$

with $\text{sign}(V_i) = (a_i, b_i, d_i)$ where a_i (resp. b_i) denotes the dimension of a maximal negative (positive) subspace and d_i the degeneracy V_i of the restriction of h to V_i and $a_i + b_i + d_i = i \quad \forall 1 \leq i \leq 2n$. The 3-tuple (a, b, d) is called the signature of the flag F . If $d \neq 0$, we refer to F as being non-degenerate and write $\text{sign}(F) = (a, b)$.

2.2 The real form $Sp(2n, \mathbb{R})$

2.2.1 Preliminaries

In order to define the symplectic group $G = Sp(2n, \mathbb{C})$ we introduce J_n to be

$$J_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and $b(v, w) = v^t \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} w$. Then

$$Sp(2n, \mathbb{C}) := \{g \in GL(2n, \mathbb{C}) : g^t \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}\}.$$

The Hermitian form $h(v, w)$ of the signature (n, n) is defined by

$$h(v, w) = b(v, \sigma(w)) = \sum_{i=1}^n v_i \bar{w}_i - \sum_{i=n+1}^{2n} v_i \bar{w}_i$$

and the real form $G_0 = Sp(2n, \mathbb{R}) = Sp(2n, \mathbb{C}) \cap U(n, n)$.

Here a Cartan involution θ is given by

$$\theta : g \longrightarrow \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} g \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

and the maximal compact subgroup associated to it is

$$K_0 = \left\{ \begin{pmatrix} U & 0 \\ 0 & J_n(U^t)^{-1} J_n \end{pmatrix} : U \in U(n, \mathbb{C}) \right\} \approx U(n)$$

2.2.2 Base points

The standard basis $(e_1, e_2, \dots, e_{2n})$ of \mathbb{C}^{2n} defines the standard maximal torus T_S as the subgroup of diagonal matrices, i.e.,

$$T_S = \{g \in G_0 : g = \text{diag}(t_1, t_2, \dots, t_n, -t_n, \dots, -t_2, -t_1), t_i \in \mathbb{C}\}$$

and the standard Borel subgroup B of upper triangular matrices in $Sp(2n, \mathbb{R})$.

The standard basis defines a base point F_S in Z as the associated flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, \dots, e_{2n} \rangle$$

Reorderings on this basis will determine the flag domains, the base cycles and the intersection points.

In order to define a base point in γ^{cl} we introduce the basis

$$e_1 - e_{2n}, e_2 - e_{2n-1}, \dots, e_n - e_{n+1}, e_n + e_{n+1}, \dots, e_2 + e_{2n-1}, e_1 + e_{2n} \quad (2.1)$$

with T_I being the maximal torus of diagonal matrices in G and B_I the Iwasawa-Borel subgroup of G which fixes the associated flag F_I .

2.2.3 Weyl group

If $(r_1, \dots, r_n, s_1, \dots, s_n)$ is a basis with $b(r_i, r_j) = 0$, $b(s_i, s_j) = 0$ and $b(r_i, s_j) = \delta_{ij}$, then we regard $V = U \oplus U^*$ with $U = \text{Span}\{r_1, \dots, r_n\}$ and $U^* = \text{Span}\{s_1, \dots, s_n\}$. For such a basis b is in *canonical form*. If T is the maximal torus in G defined by such a basis, then the Weyl group $W(T)$ acts on T by certain permutations of the basis. These are of the form $w = \pi E$ where π is an arbitrary permutation of (r_1, \dots, r_n) and E exchanges an arbitrary number of the r_i with the corresponding dual vectors s_i . If we have a maximally isotropic flag associated to some permutation of the ordered basis $(r_1, \dots, r_n, s_n, \dots, s_1)$, then in the first n positions of the flag the above condition holds: if r_i appears, then s_i does not, and vice versa. Now the full flag is determined by the first n positions; regard it as a permutation of the r_i and, if s_i occurs instead of an r_i , we mark that spot with a minus sign. In this way $W(T)$ is the semidirect product $S_n \ltimes \mathbb{Z}_2^n$ of a symmetric group and a normal elementary abelian 2-subgroup. If the basis in question is that in (2.1) above, then we denote this Weyl group by W_I . Analogously, if the basis in question is the standard basis e_1, \dots, e_{2n} , then we denote this Weyl group by W_{T_S} .

In our particular case there are two maximal tori, the maximal torus T_I defined by 2.1 and the maximal torus T_S defined by $(e_1, e_2, \dots, e_{2n})$. Recall that these two maximal tori T_I, T_S are conjugate, and a conjugation induces an isomorphism ψ of the associated Weyl groups. In this case we define a bijective map $\psi(\mp(i)) = \pm i$, for all $1 \leq i \leq n$ for later use.

2.2.4 Flag domains and base cycles

Since in every flag domain D there is a unique closed K -orbit, the associated base cycle C_0 , in order to parametrize the flag domains it is enough to parametrize the closed K -orbits, (see [7], §4.2 and §4.3). For this we choose a base point $F_S \in \text{Fix}(T_S)$. It follows that the set \mathcal{C} of closed K -orbits can be identified with the Weyl group orbit $W_G(T_S).F_S$ (see [7], Corollary 4.2.4). As we have observed above, $W_G(T_S) \cong S_n \ltimes \mathbb{Z}_2^n$ where the S_n factor can be identified with $W_K(T_S)$. In fact it is exactly the stabilizer of the $K.F_S$ in \mathcal{C} . Consequently \mathcal{C} can be identified with $\mathbb{Z}_2^n.F_S$. We regard an element of \mathbb{Z}_2^n as a vector α of length n consisting of plus- and minus-signs, and thus \mathcal{C} can be identified with the set of such vectors. In concrete terms the K -orbits (resp. G_0 -orbits) of base points $(\pm e_1, \dots, \pm e_n)$ are exactly the base cycles (resp. flag domains) in Z . The respective flag domains are denoted by D_α .

Observe that α defines a non-degenerate signature (a, b) and that $D_\alpha \subset D_{a,b} := \{F : \text{sign}(F) = (a, b)\}$. Conversely, given F with $\text{sign}(F) = (a, b)$ we may choose a basis $(v, w) := (v_1, \dots, v_n, w_1, \dots, w_n)$ which defines F and which has the following properties:

1. b is in canonical form.
2. (v, w) is h -orthogonal.
3. $\|v_j\|_h^2 = \pm 1$ with $\text{sgn}(\|v_j\|_h) = \alpha_j$.
4. $\|w_j\|_h^2 = -\|v_j\|_h^2$.

If $\text{sign}(F) = \text{sign}(\hat{F})$, then we choose bases (v, w) and (\hat{v}, \hat{w}) for F and \hat{F} , respectively, and note that the transformation which takes the one basis to the other is in G_0 . This argument shows that $D_{a,b} = D_\alpha$.

Denote $E_+ = \langle e_1, \dots, e_n \rangle$, and $E_- = \langle e_{n+1}, \dots, e_{2n} \rangle$, where $\mathbb{C}^{2n} = E_+ \oplus E_-$.

Proposition 2.2.1. *For a fixed open orbit D_α , the base cycle C_α is the set*

$$C_\alpha = \{F \in Z : V_i = (V_i \cap E^-) \oplus (V_i \cap E^+), 1 \leq i \leq 2n\}.$$

Proof. Firstly, it was shown above that there is a base point F_α with the splitting condition. Since K_0 is the product $K_0 = K_0^+ \times K_0^-$, it acts transitively on all such flags. Let $F \in C_\alpha$ where $V_i^- = V_i \cap E^-$ and $V_i^+ = V_i \cap E^+$. Define F^+ and F^- to be the sets of maximal isotropic flags in E^+ and E^- respectively. Recall that $K_0 = K_0^+ \times K_0^-$ where K_0^+ and K_0^- act transitively on the sets F^+ and F^- respectively, which implies that K_0 acts transitively on the set of maximal isotropic flags obtained by put flags from F^+ and F^- in away such that the new flag has signature α . Hence C_α is a complex manifold. But K_0 has a unique orbit in D_α which is a complex manifold. Therefor C_α is the base cycle. \square

2.3 The real form $SO^*(2n)$

2.3.1 Preliminaries

Let $G = SO(2n, \mathbb{C})$ be the special orthogonal group defined by the non-degenerate complex bilinear form $b : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ where $b(v, w) = \sum_{i=1}^{2n} v_{2n-i+1} w_i$. The real form $G_0 = SO^*(2n)$ is the subgroup of elements of G which leave invariant the Hermitian form defined by

$$h(v, w) = b(v, \bar{w}) = \sum_{i=1}^n v_i \bar{w}_i - \sum_{i=n+1}^{2n} v_i \bar{w}_i.$$

This real form is the fixed point set of the involution $\tau : SO(2n, \mathbb{C}) \rightarrow SO(2n, \mathbb{C})$ defined by

$$\tau(g) = \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} \bar{g}^{-t} \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix}, \quad \text{where } J_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that $SO^*(2n) \cong SO(2n, \mathbb{C}) \cap SU(n, n)$.

A Cartan involution $\theta : G_0 \rightarrow G_0$ is given by

$$\theta : g \rightarrow \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} g \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

and the maximal compact subgroup associated to it is

$$K_0 = \left\{ \begin{pmatrix} U & 0 \\ 0 & -J_n(U^t)^{-1}J_n \end{pmatrix} : U \in U(n) \right\} \approx U(n)$$

2.3.2 Base points

By an argument which is completely analogous to that in the case of $Sp(2n, \mathbb{R})$ the standard basis $(e_1, e_2, \dots, e_{2n})$ of \mathbb{C}^{2n} defines the standard maximal torus T_S , i.e.,

$$T_S := \{g \in G_0 : g = \text{diag}(t_1, t_2, \dots, t_n, -t_n, \dots, -t_2, -t_1), t_i \in \mathbb{C}\}$$

and the standard Borel subgroup B of upper triangular matrices in $SO^*(2n)$.

The standard basis defines a base point F_S in Z as the associated flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, \dots, e_{2n} \rangle$$

Reorderings on this basis will determine the flag domains, the base cycles and the intersection points.

In order to define a base point in γ^{cl} we introduce the basis

$$e_1 + ie_{n+1}, e_2 + ie_{n+2}, \dots, e_n + ie_{2n}, e_n - ie_{2n}, \dots, e_2 - ie_{n+2}, e_1 - ie_{n+1}, \quad \text{if } n \text{ is even;} \quad (2.2)$$

or

$$e_1 + ie_{n+1}, \dots, e_n + ie_{2n}, e_{\frac{n+1}{2}}, e_{n+\frac{n+1}{2}}, e_n - ie_{2n}, \dots, e_1 - ie_{n+1}, \quad \text{if } n \text{ is odd.} \quad (2.3)$$

with T_I being the maximal torus of diagonal matrices in G and B_I the Iwasawa-Borel subgroup of G which fixes the associated flag F_I .

2.3.3 Weyl groups, flags domains and base cycles

By arguments which are essentially the same as those in §2.2.3, if T is a maximal torus defined by a basis of (r, s) -type, then $W(T) = S_n \ltimes \mathbb{Z}_2^{n-1}$ where now the sign-change vector α has only an even number of minuses. As in the case of $Sp(2n, \mathbb{R})$, to describe the flag domains and their K_0 -base cycles we use the Weyl group $W(T_S)$ of the standard torus T_S . The base point for α consisting only of pluses is the flag F_S is defined by the standard basis. Then all base points are $F_\alpha := w(F_S)$ where $w \in \mathbb{Z}_2^{n-1}$ is associated to the sign-change vector α . As before the base cycles and flag domains are the orbits $K.F_\alpha$ and $G_0.F_\alpha$, respectively, and α defines a non-degenerate sign with $\text{sign}(F_\alpha) = (a, b)$ and then $D_\alpha = D_{a,b}$.

In our particular case there are two maximal tori, the maximal torus T_I defined by (2.2) or (2.3) and the maximal torus T_S defined by $(e_1, e_2, \dots, e_{2n})$. Recall that these two maximal tori T_I, T_S are conjugate, and a conjugation induces an isomorphism ψ of the associated Weyl groups. In this case we define a bijective map $\psi(\mp(i)) = \pm i$, for all $1 \leq i \leq n-1$ and $\psi(\pm n) = \pm n$.

Denote $E_+ = \langle e_1, \dots, e_n \rangle$, and $E_- = \langle e_{n+1}, \dots, e_{2n} \rangle$, where $\mathbb{C}^{2n} = E_+ \oplus E_-$. Again, by an argument which is completely analogous to Proposition 2.2.1 of the case $SP(2n, \mathbb{R})$, the base cycle C_0 of the flag domain D_α of $SO^*(2n)$ is the set

$$C_0 = \{F \in Z : V_i = (V_i \cap E^-) \oplus (V_i \cap E^+), 1 \leq i \leq 2n\}.$$

which is convenient to use the orbit $\mathbb{Z}_2^{n-1}.F_S$ of the base point as the set of intersection point of $S_w \cap D_\alpha$. The intersection dimensions are determined by α (resp. (a, b)). See section 2.2.4 for more details.

2.4 The real form $SO(p, q)$

2.4.1 Preliminaries

Consider the semisimple Lie group $G = SO(m, \mathbb{C})$ where $m = 2n$ or $2n+1$. In this case it is convenient to choose the bilinear form b depending on the p and q . If p or q is even, then we choose it in the usual way:

$$b(v, w) = - \sum_{i=1}^q v_i w_i + \sum_{i=q+1}^m v_i w_i$$

Let σ be the complex conjugation $\sigma(v) = \bar{v}$, then the Hermitian form $h(v, w)$ of signature (p, q) which defines the real form is defined by

$$h(v, w) = b(v, \sigma(w)) = -\sum_{i=1}^q v_i \sigma(w_i) + \sum_{i=q+1}^m v_i \sigma(w_i).$$

If both p and q are odd, then the complex bilinear form is

$$b(v, w) = -\sum_{i=1}^{q-1} v_i w_i + (v_q w_{q+1} + v_{q+1} w_q) + \sum_{i=q+2}^m v_i w_i$$

and the Hermitian form $h(v, w)$ is defined by

$$h(v, w) = -\sum_{i=1}^q v_i \bar{w}_i + \sum_{i=q+1}^m v_i \bar{w}_i.$$

The real form is $G_0 = SO(p, q) = SO(2n, \mathbb{C}) \cap SU(p, q)$.

A Cartan involution $\theta : \mathfrak{so}(p, q) \rightarrow \mathfrak{so}(p, q)$ in the Lie algebra level is given by $\theta(g) = -g^t$, and the maximal compact subgroup associated to it is $K_0 := S(O(p) \times O(q)) \subset S(U(p) \times U(q))$.

2.4.2 Base points

In the following we define bases depending on p and q as well as the positive and negative spaces which will be essential for understand the base cycle. The fundamental maximal torus T_S is defined in each case by requiring that each basis vector is a T_S -eigenvector.

- **If $m = 2n$ and q is even:** The ordered b -isotropic basis is

$$e_1 + ie_2, e_3 + ie_4, \dots, e_{2n-1} + ie_{2n}, e_{2n-1} - ie_{2n}, \dots, e_1 - ie_2, \quad (2.4)$$

where $E^- := \langle e_1 + ie_2, e_3 + ie_4, \dots, e_{q-1} + ie_q, e_1 - ie_2, e_3 - ie_4, \dots, e_{q-1} - ie_q \rangle$ and $E^+ := \langle e_{q+1} + ie_{q+2}, \dots, e_{2n-1} + ie_{2n}, e_{q+1} - ie_{q+2}, \dots, e_{2n-1} - ie_{2n} \rangle$.

- **If $m = 2n$ and q is odd:** The ordered b -isotropic basis is

$$e_1 + ie_2, e_3 + ie_4, \dots, e_{2n-1} + ie_{2n}, e_q, e_{q+1}, e_{2n-1} - ie_{2n}, \dots, e_1 - ie_2, \quad (2.5)$$

where $E^- := \langle e_1 + ie_2, e_3 + ie_4, \dots, e_q, e_1 - ie_2, e_3 - ie_4, \dots, e_{q-1} - ie_q \rangle$ and $E^+ := \langle e_{q+1}, e_{q+2} + ie_{q+3}, \dots, e_{2n-1} + ie_{2n}, e_{q+2} - ie_{q+3}, \dots, e_{2n-1} - ie_{2n} \rangle$.

Note that the vectors e_q and e_{q+1} are isotropic and $b(e_q, e_{q+1}) = -1$. Moreover $h(e_3, e_3) = -1$ and $h(e_4, e_4) = 1$ which means that $e_q \in E^-$ and $e_{q+1} \in E^+$.

- **If $m = 2n + 1$ and q is even:** The ordered b -isotropic basis is

$$e_1 + ie_2, e_3 + ie_4, \dots, e_{2n} + ie_{2n+1}, e_{q+1}, e_{2n} - ie_{2n+1}, \dots, e_1 - ie_2, \quad (2.6)$$

where $E^- := \langle e_1 + ie_2, e_3 + ie_4, \dots, e_{q-1} + ie_q, e_1 - ie_2, e_3 - ie_4, \dots, e_{q-1} - ie_q \rangle$ and $E^+ := \langle e_{q+1}, e_{q+2} + ie_{q+3}, \dots, e_{2n} + ie_{2n+1}, e_{q+2} - ie_{q+3}, \dots, e_{2n} - ie_{2n+1} \rangle$.

- **If $m = 2n + 1$ and q is odd:** The ordered b -isotropic basis is

$$e_1 + ie_2, e_3 + ie_4, \dots, e_{2n} + ie_{2n+1}, e_q, e_{2n} - ie_{2n+1}, \dots, e_1 - ie_2, \quad (2.7)$$

where $E^- = \langle e_1 + ie_2, e_3 + ie_4, \dots, e_q, e_1 - ie_2, e_3 - ie_4, \dots, e_{q-2} - ie_{q-1} \rangle$ and $E^+ = \langle e_{q+1} + ie_{q+2}, \dots, e_{2n} + ie_{2n+1}, e_{q+1} - ie_{q+2}, \dots, e_{2n} - ie_{2n+1} \rangle$.

Remark 2.4.1. If $m = 2n + 1$ we have the vector e_q or e_{q+1} in the ordered b -isotropic basis. In this case the vector sits in a fixed position in the middle of the basis and the terms at the beginning skip over e_q or e_{q+1} .

The above bases define the standard maximal Tours T_S in each case as the subgroup of diagonal matrices, i.e,

$$T_S = \{g \in G_0 : g = \text{diag}(t_1, t_2, \dots, t_n, -t_n, \dots, -t_2, -t_1), t_i \in \mathbb{C}\} \quad \text{if } m = 2n,$$

or

$$T_S = \{g \in G_0 : g = \text{diag}(t_1, t_2, \dots, t_n, 0, -t_n, \dots, -t_2, -t_1), t_i \in \mathbb{C}\} \quad \text{if } m = 2n + 1.$$

For any of the ordered bases above denote by F_S the associated flag in Z . Reorderings on these bases will determine the flag domains, the base cycles and the intersection points.

Just as in the case of T_S , the maximal Torus T_I is defined to have a certain basis of eigenvectors which depends on m being odd or even.

- If m is even, then the basis is

$$\begin{aligned} &e_1 + e_{2q}, \dots, e_q + e_{q+1}, e_{2q+1} + ie_{2q+2}, e_{2q+3} + ie_{2q+4}, \dots, e_{2n-1} + ie_{2n}, \\ &e_{2n-1} - ie_{2n}, \dots, e_{2q+3} - ie_{2q+4}, e_{2q+1} - ie_{2q+2}, e_q - e_{q+1}, \dots, e_1 - e_{2q}. \end{aligned} \quad (2.8)$$

- If m is odd, then the basis is

$$\begin{aligned} &e_1 + e_{2q}, e_2 + e_{2q-1}, \dots, e_q + e_{q+1}, e_{2q+1}, e_{2q+2} + ie_{2q+3}, e_{2q+4} + ie_{2q+5}, \dots, e_{2n} + ie_{2n+1}, \\ &e_{2q+1}, e_{2n} - ie_{2n+1}, \dots, e_{2q+4} - ie_{2q+5}, e_{2q+2} - ie_{2q+3}, e_q - e_{q+1}, \dots, e_2 - e_{2q-1}, e_1 - e_{2q} \end{aligned} \quad (2.9)$$

and B_I the Iwasawa-Borel subgroup of G which fixes the associated flag F_I .

2.4.3 Weyl groups, flag domains and base cycles

Let us use shorthand notation for the bases used above. By (r, s) -form we mean a basis $(r_1, \dots, r_n, s_n, \dots, s_1)$ where $b(r_i, r_i) = b(s_i, s_i) = 0$ and $b(r_i, s_i) = \delta_{ij}$ for all i and j . This occurs in the even-dimensional case. By (r, t, s) -form, which occurs in odd-dimensional case, the r_i and s_i satisfy the same conditions and t is a single vector with $b(t, t) = \pm 1$ and $b(t, r_i) = b(t, s_i) = 0$ for all i . Here we discuss the Weyl groups $W(T_S)$ and $W(T_I)$ by their actions on these bases. If T is either of these tori, then it stabilizes the spaces spanned by r_i

and s_i for $i = 1, \dots, n$ for both kinds of bases. In this we regard T as a product $T = T_1 \cdots T_n$, in the second case acting trivially on the space spanned by t . In both cases an arbitrary permutation in S_n acting diagonally on these spaces by $(r_i, s_i) \mapsto (r_{\pi(i)}, s_{\pi(i)})$ normalizes the T -action and is in the given orthogonal group G . A simple reflection $(r_i, s_i) \mapsto (s_i, r_i)$ for one such i , and being the identity on the other 2-dimensional spaces, is as usual denoted by a sign change. Such normalizes T , but has negative determinant. In the first case, this means that only an even number of sign changes is allowed. In the second case we may couple the sign change with the map $t \mapsto -t$ so that this slightly modified sign change has positive determinant. Thus in both cases we have the action of $S_n \rtimes \mathbb{Z}_2^n$ normalizing the T -representation on the basis at hand. This is in fact the action of the full Weyl group.

As in the cases of $Sp(2n, \mathbb{R})$ and $SO^*(2n)$, to describe the flag domains and their K_0 -base cycles we use the Weyl group $W(T_S)$ of the standard torus T_S . The base point for α is the flag F_S defined by the standard basis. Then all base points are $F_\alpha := w(F_S)$ where $w \in \mathbb{Z}_2^{n-1}$ if $m = 2n$ and $w \in \mathbb{Z}_2^n$ if $m = 2n + 1$ is associated to the sign-change vector α . As before the base cycles and flag domains are the orbits $K.F_\alpha$ and $G_0.F_\alpha$, respectively, and α defines a non-degenerate sign with $\text{sign}(F_\alpha) = (a, b)$ and then $D_\alpha = D_{a,b}$.

In our particular case there are two maximal tori, the maximal torus T_I and the maximal torus T_S which defined above. Recall that these two maximal tori T_I, T_S are conjugate, and a conjugation induces an isomorphism ψ of the associated Weyl groups. In this case the bijective map ψ is described, depending on the case, as follows:

- **If $m = 2n$ and q is even:** Define W_{T_S} to be the Weyl group with respect to the basis (2.4) and let W_I be the Weyl group with respect to the basis 6.1. In this case the bijective map between W_I and W_{T_S} is $\psi(\mp(2i - 1)) = \pm i, \psi(\mp 2i) = \pm(q - i + 1)$ if $1 \leq i \leq \frac{q}{2}$ and $\psi(\pm i) = \pm i$ if $i > q$.
- **If $m = 2n$ and q is odd:** Define W_{T_S} to be the Weyl group with respect to basis (2.5) and let W_I be the Weyl group with respect to the basis (6.1), then the bijective map in this case is $\psi : W_I \rightarrow W_{T_S}$ defined by $\psi(\mp(2i - 1)) = \pm i, \psi(\mp 2i) = \pm(q - i)$ if $1 \leq i \leq \frac{q-1}{2}$ and $\psi(\pm i) = \pm(i - 1)$ if $i > q$ and $\psi(-q) = -n$.
- **If $m = 2n + 1$ and q is even:** Define W_{T_e} to be the Weyl group with respect to the basis (2.6) and let W_I be the Weyl group with respect to the basis (6.2), then we can define the bijective map between them to be $\psi(\mp(2i - 1)) = \pm i, \psi(\mp 2i) = \pm(q - i + 1)$ if $1 \leq i \leq \frac{q}{2}$ and $\psi(\pm i) = \pm i$ if $i > q$.
- **If $m = 2n + 1$ and q is odd:** Define W_{T_o} to be the Weyl group with respect to basis (2.7). Let W_I be the Weyl group with respect to the basis (6.2). The bijective map ψ between W_I and W_{T_o} can be defined as $\psi(\mp(2i - 1)) = \pm i, \psi(\mp 2i) = \pm(q - i)$ if $1 \leq i \leq \frac{q-1}{2}$ and $\psi(\pm i) = \pm(i)$ if $i > q$.

Moreover, through an argument quiet similar to Proposition 2.2.1, the base cycle C_α is the set

$$C_\alpha = \{F \in Z : V_i = (V_i \cap E^-) \oplus (V_i \cap E^+), 1 \leq i \leq m\}.$$

where E^- and E^+ defined in §2.4.2 and the intersection dimensions are determined by α (resp. (a, b)). The flag domains are parametrized by the signature of its base cycles.

2.5 The real form $Sp(2p, 2q)$

2.5.1 Preliminaries

Let $G := Sp(2n, \mathbb{C})$ be the Lie group of complex linear transformations is defined in §2.2.1. We realize the real form $G_0 = Sp(2p, 2q)$ as a group of matrices $g \in Sp(2n, \mathbb{C})$ which leave invariant the Hermitian form $h(v, w)$ of the signature $(2p, 2q)$ defined by

$$h(v, w) = - \sum_{i=1}^q v_i \bar{w}_i + \sum_{i=q+1}^{p+q} v_i \bar{w}_i + \sum_{i=p+q+1}^{2p+q} v_i \bar{w}_i - \sum_{i=p+2q+1}^{2n} v_i \bar{w}_i.$$

such that $Sp(2p, 2q) = Sp(2n, \mathbb{C}) \cap SU(2p, 2q)$. A Cartan involution $\theta : \mathfrak{sp}(2p, 2q) \rightarrow \mathfrak{sp}(2p, 2q)$ in the Lie algebra level is given by

$$\theta(g) = -I_{q,p,p,q} \bar{g}^t I_{q,p,p,q}, \quad \text{where } I_{q,p,p,q} = \begin{pmatrix} -I_q & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & -I_q \end{pmatrix}$$

In particular, the maximal compact subgroup of $Sp(2p, 2q)$ is $K_0 := Sp(2q) \times Sp(2p)$

2.5.2 Base points

Let Z be the space of all maximally b -isotropic full flags. By an argument which is completely analogous to that in the case of $Sp(2n, \mathbb{R})$, the standard basis $(e_1, e_2, \dots, e_{2n})$ of \mathbb{C}^{2n} defines a base point F_S in Z as the associated flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, \dots, e_{2n} \rangle$$

Reorderings on this basis will determine the flag domains, the base cycles and the intersection points.

The standard basis defines the standard maximal torus T_S , i.e.,

$$T_S := \{g \in G_0 : g = \text{diag}(t_1, t_2, \dots, t_n, -t_n, \dots, -t_2, -t_1), t_i \in \mathbb{C}\}$$

and the standard Borel subgroup B of upper triangular matrices in $Sp(2p, 2q)$.

In order to define a base point in γ^{cl} we introduce the basis

$$\begin{aligned} e_1 + e_{2q}, e_{2n-2q+1} - e_{2n}, \dots, e_q + e_{q+1}, e_{2n-q} - e_{2n-q+1}, e_{2q+1}, e_{2q+2}, \dots, e_n, e_{n+1}, \\ e_{n+2}, \dots, e_{2n-q}, e_q - e_{q+1}, e_{2n-q} + e_{2n-q+1}, \dots, e_1 - e_{2q}, e_{2n-2q+1} + e_{2n}. \end{aligned} \quad (2.10)$$

with T_I being the maximal torus which fixes the basis in the sense of eigenvectors.

2.5.3 Weyl group, flag domains and base cycles

By arguments which are completely analogous to those in §2.2.3, if T is a maximal torus defined by a basis of (r, s) -type, then $W(T) = S_n \ltimes \mathbb{Z}_2^n$

As in the case of $Sp(2n, \mathbb{R})$, to describe the flag domains and their K_0 -base cycles we use the Weyl group $W(T_S)$ of the standard torus T_S . The base point for α is the flag F_S defined by the standard basis. Then all base points are $F_\alpha := w(F_S)$ where $w \in \mathbb{Z}_2^n$ is associated to the sign-change vector α . As before the base cycles and flag domains are the orbits $K.F_\alpha$ and $G_0.F_\alpha$, respectively, and α defines a non-degenerate sign with $sign(F_\alpha) = (a, b)$ and then $D_\alpha = D_{a,b}$.

The maximal tori T_I and T_S defined in the previous section are conjugate, and a conjugation induces an isomorphism ψ of the associated Weyl groups. In this case the bijective $\psi(\pm(2i - 1)) = \pm(2q - i + 1)$, $\psi(\pm 2i) = \mp i$ if $1 \leq i \leq q$ and $\psi(\pm i) = \pm i$ if $i > 2q$. Note that $-(2q - i + 1) = 2n - 2q - i + 2$.

Finally, Define $E^- := \langle e_1, \dots, e_q, e_{2n-q+1}, \dots, e_{2n} \rangle$ and $E^+ := \langle e_{q+1}, \dots, e_{2n-q} \rangle$. By the same argument as §2.2.4 and Proposition 2.2.1, for a fixed flag domain D_α , the base cycle C_α is the set

$$C_\alpha = \{F \in Z : V_i = (V_i \cap E^-) \oplus (V_i \cap E^+), 1 \leq i \leq 2n\}.$$

As all cases above the intersection dimensions are determined by α (resp. (a, b)).

Chapter 3

Fixed Point Theorem

3.1 Introduction

Here, as throughout our work, we consider a homogeneous G -manifold of full flags Z equipped with the actions of a real form G_0 as listed in the previous chapter. The Iwasawa-Borel subgroup B_I is the isotropy group at the flag F_I given there by an explicit basis \mathcal{B}_I which also defines the maximal torus T_I and its Weyl group W_I . For $w \in W_I$ we study the intersections $S_w \cap C_\alpha$ of the Schubert cells $S_w = B_I.F_w$ with any base cycle C_α , where $F_w = w(F_I)$.

On the other hand, we have a standard basis \mathcal{B}_S which defines a torus T_S and a base point F_S in a standard base cycle which when moved by sign permutations in the Weyl group W_S gives base points in all other base cycles C_α . Recall that these base cycles are the closed K -orbits which are described by the intersections with E^+ and E^- in each case. All of the above has been defined in Chapter 2 for all cases.

In this chapter we first give detailed formulas for the orbits $b(F_I), b \in B_I$, case by case for all real forms. Then we prove a "Moving Lemma" (See the Moving Lemma 3.3.1): Given a point $b.w(F_I)$ in C_α , we move it continuously in C_α to a T_S -fixed point by starting with the basis given for $b.w(F_I)$ and moving continuously by a curve $\gamma(t)$ of bases $\mathcal{B}(t)$ which defines flags in $S_w \cap C_\alpha$. In general the intersection $S_w \cap C_\alpha$ may have many connected components. The moving procedure implies that in each component of $S_w \cap C_\alpha$ there is a fixed point. In the case where S_w is of complementary dimension to C_α the intersection $S_w \cap C_\alpha$ is finite; so the curve $\gamma(t)$ is just a fixed point and the result states that $S_w \cap C_\alpha$ consists only of T_S -fixed points (See Corollary 3.3.4).

3.2 Description of the B_I -orbits

In the previous chapter in every case we have chosen a flag F_I defined by the basis $(v_1, \dots, v_m), m = 2n$ or $m = 2n + 1$, which belongs to the G_0 -closed orbit γ^{cl} . Given a Weyl element $w \in W$, we consider the Schubert cell $B_I.F_w$ and an element $F = b(F_w)$ in it which is defined by an ordered basis $(b.v_{w_1}, \dots, b.v_{w_m}), b \in B_I$. If $F \in C_\alpha$, then it can be

defined by an ordered basis $(\varepsilon_1, \dots, \varepsilon_m)$ where for all k the vector ε_k is either in E^- or E^+ where $V = E^+ \oplus E^-$, and is a linear combination

$$\varepsilon_k = \sum_{j \leq k} c_j b.v_{w_j}.$$

In this case we call the basis a **split basis**. As a result of the discussion in the previous section we can write formulas for $b.v_{w_j}$ for any element $b \in B_I$ for each real form as follows:

1. If $G_0 = SP(2n, \mathbb{R})$, then

$$b.v_{w_j} = \eta_j(e_r + e_{2n-r+1}) + \zeta_j(e_r - e_{2n-r+1}) + B_j = K_j + B_j, \quad 1 \leq r \leq n$$

where B_j does not involve the basis vectors e_r and e_{2n-r+1} .

2. If $G_0 = SO^*(2n)$, then

$$b.v_{w_j} = \eta_j(e_r + ie_{n+r}) + \zeta_j(e_r - ie_{n+r}) + B_j = K_j + B_j, \quad 1 \leq r \leq n$$

where B_j does not involve the basis vectors e_r and e_{n+r} .

3. If $G_0 = SO(p, q)$, then

(i) $b.v_{w_j} = \eta_{ij}(e_r + e_{2q-r+1}) + \zeta_{ij}(e_r - e_{2q-r+1}) + \eta_{kj}(e_{r+1} + e_{2q-r}) + \zeta_{kj}(e_{r+1} - e_{2q-r}) + B_j = K_j + B_j$, where $1 \leq r \leq q$ and $\eta_{kj} = \pm i\eta_{ij}$ and $\zeta_{kj} = \pm i\zeta_{ij}$ and B_j does not involve the basis vectors e_r, e_{r+1}, e_{2q-r+1} and e_{2q-r} .

(ii) $b.v_{w_j} = \eta_j(e_{m-2r+1} + ie_{m-2r+2}) + B_j = K_j + B_j$, $1 \leq r \leq \frac{m-2q}{2}$ where B_j does not involve the basis vectors e_{m-2r+1} and e_{m-2r+2} .

(iii) $b.v_{w_j} = \eta_j e_{2q+1} + B_j = K_j + B_j$, where B_j does not involve the basis vectors e_{2q+1} .

4. If $G_0 = SP(2p, 2q)$, then

(i) $b.v_{w_j} = \eta_j(e_r + e_{2q-r+1}) + \zeta_j(e_r - e_{2q-r+1}) + B_j = K_j + B_j$, where $1 \leq r \leq q$ where B_j does not involve the basis vectors e_r and e_{2q-r+1} .

(ii) $b.v_{w_j} = \eta_j(e_{2n-q+r} + e_{2n-r+1}) + \zeta_j(e_{2n-q+r} - e_{2n-r+1}) + B_j = \tilde{K}_j + \tilde{B}_j$, where $1 \leq r \leq q$ where B_j does not involve the basis vectors e_{2n-q+r} and e_{2n-r+1} .

(iii) $b.v_{w_j} = \eta_j(e_{2q+r}) + B_j$. where $1 \leq r \leq 2p$ where B_j does not involve the basis vector e_{2q+r} .

3.3 The Moving Lemma and a Fixed Point Theorem

Assume that the Schubert cell $B_I.F_w$ has non-empty intersection with D_α . This implies that it has a non-empty intersection with the base cycle C_α . The flags F_0 in this cycle can be described by bases $\mathfrak{B}(F_0)$ which are split in the sense that their entries (vectors) are either in E^- or E^+ where $V = E^- \oplus E^+$.

Lemma 3.3.1. (The Moving Lemma). *Given a flag $F_0 \in B_I.F_I \cap C_\alpha$, there is a continuous curve of split bases $\mathcal{F} = \mathfrak{B}(t), t \in [0, t_0]$ such that $\mathfrak{B}(t_0)$ defines the given flag F_0 and $\mathfrak{B}(0)$ defines a T_S -fixed point.*

Proof. The fixed point F_I in γ^{cl} is defined by the ordered basis

$$(v_1, v_2, \dots, v_m)$$

and $F_w = w(F_I)$. Let $b.F_w =: F_0, b \in B_I$, be a given flag in $B_I.F_w \cap C_\alpha$ and regard it as being given by a basis

$$(b.v_{w_1}, \dots, b.v_{w_m})$$

which is given in §3.2. Of course this basis is not necessarily split. However, the points which belongs to C_α are given by split bases, and we will now construct an associated split basis $(\varepsilon_1, \dots, \varepsilon_m)$ which belong to $S_w \cap C_\alpha$ (term-by-term, recursively). First, we analyze ε_1 and describe it in a way that is convenient for our purposes. There are three possibilities for

$$\varepsilon_1 = b.v_{w_1} = \lambda v_{w_1} + B_{w_1},$$

where $B_{w_1} = \sum_{j \leq w_1} c_j v_j$, i.e.,

1. If $1 \leq w_1 \leq q$ ¹ such that the dual vector² v_{m-w_1+1} is not involved as a term of B_{w_1} , so there is no possibility of canceling the term λv_{w_1} with B_{w_1} .
2. If $-1 \leq w_1 \leq -q$ such that the dual vector v_{m-w_1+1} is involved as a term in B_{w_1} , so we can find $2\lambda e_j$ as combination of these two vectors.
3. If $q+1 \leq |w_1| \leq \lfloor \frac{m}{2} \rfloor$, this case appear for the real forms $G_0 = SP(p, q)$ and $G_0 = SO(p, q)$. In this case $v_{w_1} = e_r$ for some r if $G_0 = SP(p, q)$, and $v_{w_1} = e_r \pm i e_{r+1}$ for some r or $v_{w_1} = e_q$ or $v_{w_1} = e_{q+1}$ if $G_0 = SO(p, q)$.

Note that such combinations in those cases are necessary for the first vector in the basis $\mathcal{B}(F_0)$. Now a number of the other vectors v_i occur in such pairs and must be combined in the same way. However, the resulting coefficient is an arbitrary complex number. As a result, if $\tilde{\lambda}$ is the coefficient of the dual vector v_{m-w_1+1} , then the first basis element is

$$\varepsilon_1 = 2\lambda e_j + t_1 e_{j+1} + \dots + t_{n-j} e_n, \quad \text{if } \tilde{\lambda} = \lambda$$

or

$$\varepsilon_1 = 2\lambda e_{m-j+1} + t_1 e_{m-j} + \dots + t_{m-j} e_{n+1} \quad \text{if } \tilde{\lambda} = -\lambda.$$

where $m = 2n$ or $m = 2n + 1$. We can not know the values of the coefficients, but only know that the t_i can in principle be any complex number and $\lambda \neq 0$. Note that the first case $\varepsilon_1 = b.v_{w_1}$ where $1 \leq w_1 \leq q$ doesn't happen, because there is no possible linear combination which produces a split basis element.

Next we construct and describe ε_2 . Recall that the vector $b.v_{w_2}$ is given by the form

$$b.v_{w_2} = \lambda v_{w_2} + B_{w_2},$$

¹ q comes from the signature (p, q) of the bilinear form b where $q = n$ in the cases $Sp(2n, \mathbb{R})$ and $SO^*(2n)$.

²If $v_i, \tilde{v}_i \in \mathbb{C}^m$ and $b(v_i, v_i) = 0, b(\tilde{v}_i, \tilde{v}_i) = 0$ and $b(\tilde{v}_i, v_i) = 1$ then \tilde{v}_i is the dual vector of v_i .

where $B_{w_2} = \sum_{j \leq w_2} c_j v_j$, but we can modify it by adding in an appropriate multiple of ε_1 , i.e.

$$\varepsilon_2 = \lambda v_{w_2} + d_1 \varepsilon_1 + B_{w_2}.$$

If we use $d_1 \varepsilon_1$ in the linear combination which defined ε_2 , we will call this combination "external combination", if not, we call this combination "internal combination". Then there are four possibilities for ε_2 i.e.,

- A. If $1 \leq w_2 \leq q$ such that the dual vector v_{m-w_2+1} is not involved as a term of B_{w_2} and $\varepsilon_1 \neq v_{m-w_2+1}$, so there is no possibility of canceling the term λv_{w_2} with B_{w_2} or with ε_1 .
- B. If $1 \leq w_2 \leq q$ such that the dual vector v_{m-w_2+1} is not involved as a term of B_{w_2} but $\varepsilon_1 = v_{m-w_2+1}$, so we can find $2\lambda e_j$ as combination of ε_1 and v_{w_2} .
- C. If $-1 \leq w_2 \leq -q$ such that the dual vector v_{m-w_1+1} is involved as a term in B_{w_2} , so we can find $2\lambda e_j$ as combination of these two vectors.
- D. If $q+1 \leq |w_2| \leq \lfloor \frac{m}{2} \rfloor$, this case appear for the real forms $G_0 = SP(p, q)$ and $G_0 = SO(p, q)$. In this case $v_{w_2} = e_r$ for some r if $G_0 = SP(p, q)$, and $v_{w_2} = e_r \pm i e_{r+1}$ for some r or $v_{w_2} = e_q$ or $v_{w_2} = e_{q+1}$ if $G_0 = SO(p, q)$.

The combinations in those cases are necessary for the second vector in the basis $\mathcal{B}(F_0)$ and all other vectors $\varepsilon_j, j \geq 2$. Now a number of the other vectors v_i occur in such pairs and must be combined in the same way. As a result, if $\tilde{\lambda}$ is the coefficient of the dual vector v_{m-w_2+1} , then the second basis element is

$$\varepsilon_2 = 2\lambda e_j + t_1 e_{j+1} + \dots + t_{n-j} e_n, \text{ if } \tilde{\lambda} = \lambda,$$

$$\varepsilon_2 = 2\lambda e_{m-j+1} + t_1 e_{m-j} + \dots + t_{m-j} e_{n+1}, \text{ if } \tilde{\lambda} = -\lambda.$$

where $m = 2n$ or $m = 2n + 1$, t_i is complex number and $\lambda \neq 0$.

We now assume that $\varepsilon_1, \dots, \varepsilon_{j-1}$ have been constructed as in the case of ε_2 depending on $\tilde{\lambda}$. This means that $\varepsilon_\ell = 2\lambda e_j + t_1 e_{j+1} + \dots + t_{n-j} e_n$, if $\tilde{\lambda} = \lambda$, or $\varepsilon_\ell = 2\lambda e_{m-j+1} + t_1 e_{m-j} + \dots + t_{m-j} e_{n+1}$, if $\tilde{\lambda} = -\lambda$. for $\ell \leq j-1$. Now we will construct ε_j with these properties. Recall that the vector

$$b.v_{w_j} = \lambda v_{w_j} + B_{w_j},$$

where $B_{w_j} = \sum_{j \leq w_2} c_j v_j$, but we can modify it by adding in an appropriate linear combination of the $\varepsilon_i, 1 \leq i \leq j-1$, i.e.

$$\varepsilon_j = \lambda v_{w_j} + B_{w_j} + \sum_{i < j} d_i \varepsilon_i,$$

then there are only three cases for ε_j :

1. Some of the coefficients of B_j are such that cancellation takes place as step C above which called internal cancellation.

2. There is a linear combination of the previous split basis elements $\sum_{i < j} d_i \varepsilon_i$ that cancels one of the terms in expressions of the type $e_s \pm e_r$. Similarly as step *B* above which called external cancellation..
3. The vector v_j is equal to e_r in the case of the real form $SP(p, q)$ or v_j is equal to $e_r \pm ie_{r+1}$ or e_r in the case of the real form $SO(p, q)$. Similarly as step *D* above which is a kind of internal cancellation..

As a result, if $\tilde{\lambda}$ is the coefficient of the dual vector v_{m-w_1+1} , then the j -th basis element is

$$\varepsilon_j = 2\lambda e_s + t_1 e_{s+1} + \dots + t_{n-s} e_n, \tilde{\lambda} = \lambda,$$

or

$$\varepsilon_j = 2\lambda e_{m-s+1} + t_1 e_{m-s} + \dots + t_{m-s} e_{n+1}, \tilde{\lambda} = -\lambda,$$

where $m = 2n$ or $m = 2n + 1$ and t_i is complex number and $\lambda \neq 0$.

The next claim is we can move continuously from an arbitrary ε -basis constructed above to a T_S -fixed point continuously, i.e., without leaving C_α . The curve

$$\mathfrak{B}(t) = (\varepsilon_1(t), \dots, \varepsilon_m(t)),$$

which begins at the given basis $(\varepsilon_1, \dots, \varepsilon_m)$ and ends at a T_S -fixed bases $(\varepsilon_1(0), \dots, \varepsilon_m(0))$, will be constructed stepwise. First we show that there is a curve $\varepsilon_1(t)$ beginning at the given vector ε_1 and ending at a T_S -fixed vector $\varepsilon_1(0)$. With this we reach a new base point

$$(\varepsilon_1(0), \varepsilon_2, \dots, \varepsilon_m).$$

If the vector ε_1 in the given base point is T_S -fixed point, then the curve $\varepsilon_1(t) = \varepsilon_1(0)$ for all $t \in [0, t_0]$, i.e a T_S -fixed point. Then we assume that the given base point $(\varepsilon_1, \dots, \varepsilon_m)$ has the property that ε_ℓ is T_S -fixed for $\ell < k$ such that the curve $\varepsilon_\ell(t) = \varepsilon_\ell(0)$, $\ell < k$ for all $t \in [0, t_0]$ and then construct a curve $\varepsilon_k(t)$ beginning at ε_k and ending at a T_S -fixed vector $\varepsilon_k(0)$. This defines a new base point in $S_w \cap C_\alpha$ with ε_ℓ begin T_S -fixed for $\ell \leq k$. As a result we define a sequence of curves whose composition is the desired continuous curve.

Suppose $F \in S_w \cap C_\alpha$ is associated to a split basis and not T_S -fixed. We may therefore choose an ε -basis with a smallest index k such that ε_k is not fixed and ε_ℓ is fixed for $\ell < k$. So we can define the curve such that ε_ℓ is T_S -fixed, then we will construct the curve $\varepsilon_k(t)$ where ε_k is the first non T_S -fixed.

It is sufficient to consider the case where $\varepsilon_k \in V^-$. The first step is to analyze the ε_k , but they are constructed in such a way that for the movement $b(t)$ we have

$$\varepsilon_k = (\lambda_1 e_r + \sum_{j \leq q} a_j e_j) + i(\lambda_2 e_{r+1} + \sum_{s \leq q} b_s e_s)$$

where $a_r = 0, b_{r+1} = 0$ and $a_j \neq 0, b_s \neq 0$ for some other j and s . Furthermore, if $\varepsilon_\ell = e_j$ for some $\ell < r$, then $a_j = b_s = \lambda_2 = 0$, and if $\varepsilon_\ell = e_j + ie_{j+1}$ for some $\ell < r$, then $a_j = b_s = 0$. In general, if $G_0 = SO(p, q)$, then $\lambda_1 = \pm \lambda_2$, and if $G_0 \neq SO(p, q)$, then $\lambda_2 = b_s = 0$.

If $k = 1$ then the curve is

$$\varepsilon_1(t) = c_1 b(t) \cdot v_{w_1}.$$

It follows that

$$\varepsilon_1(t) = \varepsilon_1 + (t-1)K_1.$$

where $K_1 = 0$ corresponds to $e_r + ie_{r+1}$ or e_r not being involved in ε_1 . Note that $\varepsilon_1 \in V^-$ implies that $\zeta_1 = 0$ and $\tilde{\zeta}_1 = 0$, and $\varepsilon_1 \in V^+$ implies that $\eta_1 = 0$ and $\tilde{\eta}_1 = 0$ in the formulas of section 3.2. Thus $\varepsilon_1(t) \in V^-$ (resp. V^+) whenever $\varepsilon_1 \in V^-$ (resp. V^+). In other words, the image of the curve γ is contained in $S_w \cap C_\alpha$. Finally, observe that η_1 is the coefficient of e_r in ε_1 . This has been scaled to 1 and as a result the coefficient of e_r or $e_r + ie_{r+1}$ in $\varepsilon_1(t)$ is t . Thus

$$\varepsilon_1(t) = \varepsilon_1 + (t-1)e_r, \text{ if } G_0 \neq SO(p, q),$$

or

$$\varepsilon_1(t) = \varepsilon_1 + (t-1)(e_r + ie_{r+1}), \text{ if } G_0 = SO(p, q).$$

In the various cases we then have the following descriptions of $\varepsilon_1(0)$;

- If $G_0 = SP(2n, \mathbb{R})$, then

$$\varepsilon_1(0) = \varepsilon_1 - e_r = [(e_r + e_{2n-r+1})] - e_r = e_{2n-r+1}, \quad 1 \leq r \leq n.$$

- If $G_0 = SO^*(2n)$, then

$$\varepsilon_1(0) = \varepsilon_1 - e_r = [(e_r + ie_{n+r})] - e_r = ie_{n+r}, \quad 1 \leq r \leq n,$$

or

$$\varepsilon_1(0) = \varepsilon_1 - e_r = [-i(e_{r-n} + e_r)] - e_r = -ie_{r-n}, \quad n+1 \leq r \leq 2n.$$

- If $G_0 = SO(p, q)$, then

$$\varepsilon_1(0) = \varepsilon_1 - (e_r \mp ie_{r+1}) = (e_r \mp ie_{r+1}) + (e_{2q-r} \mp ie_{2q-r+1}) - (e_r \mp ie_{r+1}) = (e_{2q-r} \mp ie_{2q-r+1}),$$

where $1 \leq r \leq q$.

- If $G_0 = SP(2p, 2q)$, then

$$\varepsilon_1(0) = \varepsilon_1 - e_r = (e_r + e_{2q-r+1}) - e_r = e_{2q-r+1}, \quad 1 \leq r \leq q.$$

Recall that the above non fixed ε_k is $\varepsilon_k = b.v_{w_k} = K_k + B_k$ which defined for all cases in §3.2 where B_k does not involve the terms of K_k . In the following we will define a curve γ which sent the non fixed point to T_S -fixed point. Define $b(t) \in B_I$ by $b(t)(v_{w_j}) = tK_j + B_j$. The curve γ is defined by $t \mapsto (b(t)(v_{w_1}), \dots, b(t)(v_{w_{2n}}))$ which is defined by the corresponding ε -basis,

$$\varepsilon_k(t) = \sum_{j \leq k} c_j b(t).v_{w_j}.$$

It follows that

$$\varepsilon_\kappa(t) = \varepsilon_\kappa + (t-1) \sum_{j \leq \kappa} c_{\kappa j} K_j.$$

Notice that $\sum_{j \leq \kappa} c_{\kappa j} K_j = 0$ corresponds to $e_r + ie_{r+1}$ or e_r not being involved in ε_κ . Therefore $\varepsilon_\kappa(t) = \varepsilon_\kappa$ for $\kappa < k$. In general, $\varepsilon_\kappa \in V^-$ implies that $\sum c_{\kappa j} \zeta_j = 0$ and $\sum c_{\kappa j} \tilde{\zeta}_j = 0$, and $\varepsilon_\kappa \in V^+$ implies that $\sum c_{\kappa j} \eta_j = 0$ and $\sum c_{\kappa j} \tilde{\eta}_j = 0$ in the formulas of section 3.2. Thus $\varepsilon_\kappa(t) \in V^-$ (resp. V^+) whenever $\varepsilon_\kappa \in V^-$ (resp. V^+). In other words, the image of the curve γ is contained in $S_w \cap C_\alpha$. Finally, observe that $\sum c_{\kappa j} \eta_j$ is the coefficient of e_r in ε_κ . This has been scaled to 1 and as a result the coefficient of e_r or $e_r + ie_{r+1}$ in $\varepsilon_k(t)$ is t . Thus

$$\varepsilon_k(t) = \varepsilon_k + (t-1)e_r, \text{ if } G_0 \neq SO(p, q),$$

or

$$\varepsilon_k(t) = \varepsilon_k + (t-1)(e_r + ie_{r+1}), \text{ if } G_0 = SO(p, q).$$

Since at least one $a_j \neq 0$, and one $b_j \neq 0$ and $\varepsilon_\ell(t) = \varepsilon_\ell$ for $\ell < k$. In the various cases we then have the following descriptions of $\varepsilon_k(0)$;

- If $G_0 = SP(2n, \mathbb{R})$, then

$$\varepsilon_k(0) = \varepsilon_k - e_r = [(e_r + e_{2n-r+1})] - e_r = e_{2n-r+1}, \quad 1 \leq r \leq n.$$

- If $G_0 = SO^*(2n)$, then

$$\varepsilon_k(0) = \varepsilon_k - e_r = [(e_r + ie_{n+r})] - e_r = ie_{n+r}, \quad 1 \leq r \leq n,$$

or

$$\varepsilon_k(0) = \varepsilon_k - e_r = [-i(e_{r-n} + e_r)] - e_r = -ie_{r-n}, \quad n+1 \leq r \leq 2n.$$

- If $G_0 = SO(p, q)$, then

$$\varepsilon_k(0) = \varepsilon_k - (e_r \mp ie_{r+1}) = (e_r \mp ie_{r+1}) + (e_{2q-r} \mp ie_{2q-r+1}) - (e_r \mp ie_{r+1}) = (e_{2q-r} \mp ie_{2q-r+1}),$$

$$1 \leq r \leq q.$$

- If $G_0 = SP(2p, 2q)$, then

$$\varepsilon_k(0) = \varepsilon_k - e_r = (e_r + e_{2q-r+1}) - e_r = e_{2q-r+1}, \quad 1 \leq r \leq q.$$

all of the above give us a vector in the standard basis T_S which implies that we move the point to a T_S -fixed point. The above recursive construction defines the $\varepsilon(t)_j$ as desired and these in turn define the curve $\mathfrak{B}(t)$ of bases. \square

The above proof shows that if $v(t)$ is any of the basis vectors $\varepsilon_j(t)$ in $\mathfrak{B}(t)$, then the following hold.

Remarks 3.3.2. 1. At most one coefficient of $v(t)$ in $\mathfrak{B}(t)$ is restricted not to vanish (This happens as in the case C where 2λ arises as a coefficient!).

2. Except for the previous case, all coefficients are not restricted. However, they may or may not be related, e.g., in the case where an external linear combination was used.

3. In all cases there are limiting vectors when various t_i go to 0 which are single elements of the standard basis (e_1, \dots, e_{2n}) except for the real form $SO(p, q)$ the basis is $(e_1 \pm ie_2, \dots, e_{m-1} \pm ie_m)$.

A Schubert cell intersects C_α in several connected component (sometimes only one as the case $Sp(2n, \mathbb{R})$, but sometimes many as the case $SO(p, q)$). Since the curve given by the Moving Lemma is continuous, it follows that every such component contains a T_S -fixed point.

Theorem 3.3.3. (*The Fixed Point Theorem*). *Every component of the intersection $S_w \cap C_\alpha$ has a fixed point.*

Corollary 3.3.4. *If $\text{codim}(S_w) = \dim(C_\alpha)$ then*

$$S_w \cap C_\alpha \subset \text{Fix}(T_s).$$

Proof. If S_w of complementary dimension, then the components of intersection are isolated points. By using Theorem 3.3.3 every intersection point is a fixed point.

□

Chapter 4

Cycle intersection for $SP(2n, \mathbb{R})$

4.1 Conditions for $S_w \cap C_\alpha \neq \emptyset$

In this chapter the manifold Z under consideration is the homogeneous space of maximally isotropic full flags of the complex symplectic group $G = Sp(2n, \mathbb{C})$ equipped with the action of the real form $G_0 = Sp(2n, \mathbb{R})$. Recall that F_I is the fixed point in the closed orbit γ^{cl} with isotropy group B_I , i.e., which is associated to the basis (2.1).

As described in §2.2.3 the Weyl group W_I associated to T_I acts on the basis (2.1) by permutation plus sign change. If $w \in W_I$ is regarded as an element of G , then $F_w := w(F_I)$ and $S_w := B_I.F_w$ is the associated Iwasawa-Schubert cell. The dimension of the cells corresponds to the length of the word w in the Weyl group, i.e. if $F_w = w.F_I$, then $\dim(B.F_w) = l(w)$. Below we will describe the Weyl elements which parametrize the Iwasawa-Schubert cells which have non-empty intersections with flag domains (See Theorem 4.1.4). For this we call attention to a particular class of Weyl elements.

Definition 4.1.1. An element $w = (w_1, w_2, \dots, w_n) \in S_n \ltimes \mathbb{Z}_2^n$ is called **a generous permutation**¹ if $w_i < 0$ for all $1 \leq i \leq n$.

Remark 4.1.2. When discussing the B_I -orbit of the base point F_I it is important to explicitly understand the orbits $B_I.(e_i - e_{2n-i+1})$ and $B_I.(e_i + e_{2n-i+1})$. As we explained in §3.2 we have the following facts:

If F_I and B_I are as above, then the orbits of interest are $B_I.(e_i \pm e_{2n-i+1})$, $i = 1, \dots, n$. In this case the orbits $B_I.(e_i + e_{2n-i+1})$ and $B_I.(e_i - e_{2n-i+1})$ have points of the forms

$$b(e_i + e_{2n-i+1}) = \lambda(e_i + e_{2n-i+1}) + \dots + b_n(e_n + e_{n+1}) + a_n(e_n - e_{n+1}) + \dots + a_1(e_1 - e_{2n})$$

and

$$b(e_i - e_{2n-i+1}) = \lambda(e_i - e_{2n-i+1}) + a_{i-1}(e_{i-1} - e_{2n-i}) + \dots + a_1(e_1 - e_{2n})$$

respectively, with $\lambda \neq 0$. Note that in the above orbits, if b is chosen appropriately, then we can arrange $b(e_i + e_{2n-i+1}) = e_i$ or $b(e_i + e_{2n-i+1}) = e_{2n-i+1}$ for all i . This plays a role in the proof of Theorem 4.1.4. \square

¹Recall that we define a special way of denoting to write the Weyl elements in section §2.2.3.

Recall that our base point in the closed orbit γ^{cl} is the flag F_I associated to the ordered basis

$$e_1 - e_{2n}, e_2 - e_{2n-1}, \dots, e_n - e_{n+1}, e_n + e_{n+1}, \dots, e_2 + e_{2n-1}, e_1 + e_{2n} \quad (4.1)$$

and the base point which defined the flag domains and the base cycles is the flag associated to the ordered basis

$$e_1, e_2, e_3, \dots, e_{2n-1}, e_{2n} \quad (4.2)$$

These two bases defined two maximal tori which are conjugate and the corresponding Weyl groups W_I and W_{T_S} are isomorphic.

Let ψ be a bijective map between Weyl groups W_I and W_{T_S} defined by $\psi(\mp(i)) = \pm i$, for all $1 \leq i \leq n$, then we have the following Proposition;

Proposition 4.1.3. *If w is a generous permutation, then the flag $F_{\psi(w)}$ belongs to the orbit $B_I(F_w)$.*

Proof. Let $w \in W_I$ be a generous permutation, and $F_w = w(F_I) \in \gamma^{cl}$ be the isotropic full flag associated to w such that the first n -subspaces in the flag are

$$\{0\} \subset \langle v_{w_1} \rangle \subset \langle v_{w_1}, v_{w_2} \rangle \subset \dots \subset \langle v_{w_1}, \dots, v_{w_n} \rangle$$

where $v_{w_i} = e_j \pm e_{2n-j+1}$. We will show that the orbit $B_I(F_w)$ contains the flag $F_{\psi(w)}$. For this purpose let $\tilde{w} \in W_S$ be the image of w under the bijective map ψ and let $Y_{\tilde{w}}$ be the isotropic flag associated to \tilde{w} , i.e., the flag

$$Y_{\tilde{w}} := (\{0\} \subset \langle e_{\tilde{w}_1} \rangle \subset \langle e_{\tilde{w}_1}, e_{\tilde{w}_2} \rangle \subset \dots \subset \langle e_{\tilde{w}_1}, \dots, e_{\tilde{w}_n} \rangle)$$

We will show that $Y_{\tilde{w}} = b(F_w)$ for b appropriately chosen in B_I . For this recall that since w is a generous permutation it follows that $w_i < 0$ for all $1 \leq i \leq n$, and therefore $B_I.(e_j + e_{2n-j+1})$, $j = 1, \dots, n$, are the orbits of interest. By Remark 4.1.2 the orbit $B_I.(e_i + e_{2n-i+1})$ contains the points

$$y = b(e_i + e_{2n-i+1}) = \lambda(e_i + e_{2n-i+1}) + \dots + b_n(e_n + e_{n+1}) + a_n(e_n - e_{n+1}) + \dots + a_1(e_1 - e_{2n})$$

where $\lambda \neq 0$. Choose all constants to be 0 except $\lambda = a_i = \frac{1}{2}$, then $y = \langle e_i \rangle = \langle e_{\tilde{w}(i)} \rangle$.

We can repeat this step n -times to have the first half of the flag $Y_{\tilde{w}}$, then we can extend this flag to maximal h -isotropic flag, . Therefore, the flag $Y_{\tilde{w}}$ is constructed. \square

Theorem 4.1.4. (Generous Permutation Theorem). *The following are equivalent*

- (i) w is generous.
- (ii) $B_I(F_w) \cap D_\alpha \neq \emptyset$ for some α .
- (iii) $B_I(F_w) \cap D_\alpha \neq \emptyset$ for every α .

Furthermore, under any of these conditions, for every α the intersection $B_I(F_w) \cap C_\alpha$ contains a T_S fixed point.

Proof. (iii) \Rightarrow (ii) Trivial direction.

(ii) \Rightarrow (i) We assume by contradiction that w is not a generous permutation and will show that $S_w \cap D_\alpha = \emptyset$, for all α , i.e. S_w has no T_S -fixed points². Recall that the complex bilinear form b has been defined to satisfy the following orthogonality condition:

$$b(e_i, e_{2n-i+1}) = 1 \text{ and } b(e_i, e_k) = 0 \text{ for } k \neq 2n - i + 1.$$

Let $w \in W_I$ be a non-generous permutation. This implies that for some $j \leq n$ the sign of w_j is positive. Let (w_1, \dots, w_{2n}) be the basis of the w -permuted full flag F_w and suppose that for some $j \leq n$ the sign of w_j is not negative. Thus for every $g \in B_I$

$$g(w_j) = \lambda(e_k - e_{2n-k+1}) + \sum_{\ell < k} c_j(e_\ell - e_{2n-\ell+1}).$$

Observe that $g(w_j)$ is h -isotropic and therefore if $j = 1$, then, since $\lambda \neq 0$, it follows that $g(F_w) \notin C_\alpha$ for all α .

Now suppose that $j > 1$ and suppose that $g(F_w)$ is in some C_α . This implies in particular that for every $i < j$ there are linear combinations

$$\xi_i = g(w_i) + \sum_{\ell < i} c_{i\ell} g(w(\ell)).$$

which when expressed in the standard basis contain only vectors from V^- or V^+ . For $g(F_w)$ to be in C_α the same must be true for $g(w_j)$. This means that there is a linear combination

$$\xi_j = g(w_j) + \sum_{i < j} a_i \xi_i$$

which has the same property. Hence, e_k or e_{2n-k+1} (or both) must be among the vectors with non-zero coefficients in the ξ_i for $i < k$. If both appear, e.g., in ξ_κ and ξ_δ , then $b(\xi_\kappa, \xi_\delta) = 1$, contrary to the flag $g(F_w)$ being maximally b -isotropic. If only one appears, e.g., e_k in ξ_κ , then e_{2n-k+1} appears in ξ_j and $b(\xi_\kappa, \xi_j) = 1$ which is again a contradiction. Hence the intersection is empty and this completes the proof.

(i) \Rightarrow (iii) Let w be a generous permutation. Then $w_i < 0$ for all $1 \leq i \leq n$ and $B_I.(e_j + e_{2n-j+1})$, $j = 1, \dots, n$, are the orbits of interest. From Remark 4.1.2 it follows that the orbit $B_I.< e_i + e_{2n-i+1} >$ contains the point

$$y = b(e_i + e_{2n-i+1}) = \lambda(e_i + e_{2n-i+1}) + \dots + b_n(e_n + e_{n+1}) + a_n(e_n - e_{n+1}) + \dots + a_1(e_1 - e_{2n})$$

where $\lambda \neq 0$. Then we can choose all constants to be 0 except $\lambda = a_i = \frac{1}{2}$ if the corresponding sign in α is $-$, then $y = < e_i > = < e_{\tilde{w}_i} >$. If the corresponding sign in α is $+$, then we can choose all constants to be 0 except $\lambda = -a_i = \frac{1}{2}$ and therefore

²See Corollary 3.3.4

$y = \langle e_{2n-i+1} \rangle = \langle e_{\tilde{w}(i)} \rangle$. Note that the vectors e_i and e_{2n-i+1} are of negative and positive norms, respectively.

In summary, we have constructed a set of T_S -fixed points in the orbits S_w for w generous with no two belonging to the same flag domain. By directly checking we see that all possible signatures have been obtained and therefore every flag domain has non-empty intersection with one such S_w . \square

Remark 4.1.5. Assume that we have an intersection point of the Schubert variety S_w with the flag domain D_α given by the flag associated to the basis $(e_{\tilde{w}_1}, e_{\tilde{w}_2}, \dots, e_{\tilde{w}_n})$. Therefore this point belongs to the open orbit D_α given by $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ where $\alpha_i = +$ if $\tilde{w}_i > 0$ and $\alpha_i = -$ if $\tilde{w}_i < 0$. In this case if $\tilde{w}_i < 0$, then $e_{\tilde{w}_i} = e_{2n-w(i)+1}$. \square

Remark 4.1.6. The basis which defines the flag F_l in the closed orbit of $SP(2n, \mathbb{R})$ is almost the same basis as that in Brecan's thesis [4] for the case $SU(p, q)$, and the bases which define the base cycles is the same in both cases. Thus in principle Brecan's algorithms and our description for $Sp(2n, \mathbb{R})$ should be similar. In Brecan's thesis [4] for the case $SU(n, 2n)$ if the number $2n - i + 1$ stays at the left of the number i for all $1 \leq i \leq n$ in the one line notation of the permutation w , then the Schubert variety S_w has nonempty intersection with some base cycles. If we translate this condition to our style of writing the permutations with negatives, the number $2n - i + 1$ will translates to $-i$. So the condition becomes : If the number $-i$ stays at the left of the number i for all $1 \leq i \leq n$ in the one line notation of the permutation w , then the Schubert variety S_w has nonempty intersection with some base cycles. Recall that in our permutation only i or $-i$ appear in the first half which we used to write the permutations, see §2.2.3. \square

Recall that the base cycle C_α is the set

$$C_\alpha = \{F \in Z : V_i = (V_i \cap E^-) \oplus (V_i \cap E^+), 1 \leq i \leq 2n\}.$$

Since $C_\alpha = K/(K \cap B_{z_0})$, where z_0 is an appropriate point in an open G_0 -orbit [i.e. $G_0 \cdot z_0$ is open in Z], the dimension of C_α is $\frac{n(n-1)}{2}$ and Iwasawa-Schubert varieties dual to C_α has dimension $\frac{n(n+1)}{2}$.

The final step in this section is to determine the Schubert varieties of complementary dimension to that of the base cycle C_α .

The length of the elements of $S_n \ltimes \mathbb{Z}_2^n$

For dimension computations, let us state how can we compute the length of elements $w \in S_n \ltimes \mathbb{Z}_2^n$ relative to our notation for the Weyl group elements.

Lemma 4.1.7. Fix $w \in S_n \ltimes \mathbb{Z}_2^n$, construct an $\tilde{w} \in S_n \ltimes \mathbb{Z}_2^n$ by the following algorithm:

1. Start from left to right in w , using simple reflections, place all positive numbers in w step by step in a sequence of n -empty boxes beginning from the first one in \tilde{w} , in the same order as they appeared in w .

2. From left to right in w replace a negative number with its absolute value in the n -empty boxes starting from right to left.

If $\tilde{w} = (\tilde{w}(1), \tilde{w}(2), \dots, \tilde{w}(n))$, then define $L(\tilde{w}) = \frac{n^2-n}{2} - \text{number of } \{\tilde{w}(i) : i < k \text{ and } \tilde{w}(i) < \tilde{w}(k)\}$, and if we have m negative signs in w in the following positions j_1, j_2, \dots, j_m , then define $f(w) = \sum_{i=1}^m [(n - j_i)]$. It follows that the length of w is

$$l(w) = L(\tilde{w}) + f(w) + m$$

Proof. The length of the permutation $w \in S_n \ltimes \mathbb{Z}_2^n$ is the minimal number of simple reflection which define w . To compute this note that we have two kind of reflections, the first $n - 1$ reflections are the simple reflections in S_n and the last one is the reflection which flips the sign. This means that if we want to flip the sign of $w(j)$, we should move $w(j)$ from its position to the last position and then flip its sign and then return it back to its position. More precisely, consider the word $w_0 = (123\dots n)$ and let w be the word where $w_{j_1}, w_{j_2}, \dots, w_{j_m}$ in the positions $j_1 < j_2 < \dots < j_m$, $1 \leq j_i \leq n$, are negatives. To construct w from w_0 , we first flip the signs for the numbers $w_{j_1}, w_{j_2}, \dots, w_{j_m}$. For this purpose we move each number, starting from w_{j_m} , to the last position and then flip the sign of it. Then we apply the simple reflection to the positive numbers to put them in the same order as they appear in the word w . In this way the sum of all these movements and flips is exactly $L(\tilde{w}) + m$. The last step is to move each w_{j_i} to its original position in w and denote the total number of these movements by $f(w)$. It then follows that $l(w) = L(\tilde{w}) + f(w) + m$. \square

Example 4.1.8. Let $w = ((-12)5(-34)) \in S_5 \ltimes \mathbb{Z}_2^5$, then by following the above remark we have

$$(-12)5(-34) \implies 254(-1-3) \implies 254(31)$$

so $w = (-12)5(-34)$ and $\tilde{w} = 254(31)$, then $L(\tilde{w}) = 7$ and $f(w) = 5$.

Hence $l(w) = 7 + 5 + 2 = 14$

Definition 4.1.9. The element $w_0 = (-n - (n-1) \dots - 2 - 1) \in S_n \ltimes \mathbb{Z}_2^n$ is called **Super generous permutation** and has length $\frac{n(n+1)}{2}$.

Theorem 4.1.10. There exists a unique Iwasawa Schubert variety S_w of dimension $\frac{n(n+1)}{2}$. It is given by the super generous permutation $w_0 = (-n - (n-1) \dots - 2 - 1)$.

Proof. Define the set W_{gen} to be the set of all generous permutations, i.e. the set of all permutations $w \in S_n \ltimes \mathbb{Z}_2^n$ such that all signs in w are negative. Then by using the above formula to compute the length of w , it is clear that $f(w) + n = \frac{n(n+1)}{2}$ for all $w \in W_{gen}$. Thus the shortest element of W_{gen} has $l(\tilde{w}) = 0$, i.e. \tilde{w} is the shortest element in S_n which is unique and equal to $(12\dots n)$. Therefore there is a unique element $w_0 \in W_{gen}$ with length $\frac{n(n+1)}{2}$ which is $(-n, \dots, -1)$. \square

4.2 Intersection points of Schubert duality

Assuming that $w_0 \in W_I$ is the super generous permutation, then S_{w_0} is of complementary dimension to the cycles, and the intersection $S_{w_0} \cap C_\alpha$ is in fact only one point which is T_S -fixed (see Corollary 4.2.2). It should be a simple matter to compute all such intersection points. The argument in the case of $Sp_{2n}(\mathbb{R})$ goes as follows.

Since $w = (-n, \dots, -1)$, the flag basis corresponding to $w(F_I)$

$$(\hat{v}_n, \dots, \hat{v}_1, v_1, \dots, v_n)$$

where $v_k = e_k - e_{2n-k+1}$ and $\hat{v}_k = e_k + e_{2n-k+1}$.

Proposition 4.2.1. *Let w be a super generous permutation, and let the ε -basis $(\varepsilon_1, \dots, \varepsilon_n)$ of T_S -eigenvectors be the basis of an intersection point in $S_w \cap C_\alpha$, then the ε -basis is given by $\varepsilon_k = e_{n-k+1}$ or e_{n+k-1} , depending on the signature α .*

Proof. This is a consequence of the following:

1.

$$b.\hat{v}_k = \lambda_k(e_k + e_{2n-k+1}) + a_k(e_k - e_{2n-k+1}) + B_k = K_k + B_k$$

2. $\lambda_k \neq 0$

3. The intersection $S_w \cap C_\alpha$ is a flag defined by T_S -eigenvectors.

From the expression for \hat{v}_k it is immediate that all of the possibilities in the statement occur. Furthermore, since the λ_k are non-zero, for every k a non-zero contribution from K_k occurs in the sum

$$\varepsilon_k = \sum_{j \leq k} c_{kj} b.v_{w_j}.$$

Since e_{n-k+1} and e_{n+k-1} do not occur in $b.v_{w_j}$ for $j < k$, it follows that $\varepsilon_k = K_k + B_k$ in the standard basis (See §2.2.2). Finally, since ε_k is a T_S -eigenvector, it follows that $\varepsilon_k = K_k$ and is of the type in the the statement of the proposition. \square

As a result, if w is super generous, then the set of all intersection points of $(S_w \cap C_\alpha)$ can be defined to be the following:

$\text{Supset}(w) := \{F_{\tilde{w}} : \tilde{w} \in W \text{ and } \tilde{w} \text{ obtained by change none, some or all of the numbers } -i \text{ by } i\}$

where $F_{\tilde{w}}$ is the maximally b -isotropic flag associated to the standard basis \mathbb{C}^{2n} given in (4.2). Here \tilde{w} gives us a point in flag domain and to know the signature of this flag domain just replace i by $+$ and $-i$ by $-$.

One more remark, the number of flag domains in G/B is 2^n . So as a results from the above discussion we have the following corollary

Corollary 4.2.2. *If w is the super generous permutation in W , then*

1. The total number of intersection points is 2^n , each with different signature.
2. The intersection $S_w \cap C_\alpha$ is exactly one T_S fixed point.

Example 4.2.3. The set of all intersection points of the Schubert variety parametrized by the element $w = (-3-1-2)$ is $\text{Supset}(w) = \{(-3-1-2).\tilde{z}, (3-1-2).\tilde{z}, (-31-2).\tilde{z}, (-3-12).\tilde{z}, (31-2).\tilde{z}, (-312).\tilde{z}, (3-12).\tilde{z}, (312).\tilde{z}\}$. So S_w intersects the following flag domains: $(---), (+--), (-+-), (- - +), (++-), (+ - +), (++-), (+++)$.

Chapter 5

Cycle intersection for $SO^*(2n)$

Here we deal with the homogeneous space Z of maximally isotropic full flags of the complex orthogonal symmetric group $G = SO(2n, \mathbb{C})$ equipped with the action of the real form $G_0 = Sp(2n, \mathbb{R})$.

5.1 Conditions for $S_w \cap C_\alpha \neq \emptyset$

Definition 5.1.1. An element $w = (w_1, w_2, \dots, w_n) \in S_n \ltimes \mathbb{Z}_2^{n-1}$ is called **a dense permutation**¹ if $w_i < 0$ for all $1 \leq i \leq n$ if n is even, and $w_i < 0$ for all $1 \leq i \leq n-1$ and $w_n > 0$ if n is odd.

This definition will aid in determining the Weyl elements which parametrize the Schubert varieties that have nonempty intersection with base cycles of flag domains which contain T_S -fixed points.

Remark 5.1.2. When discussing the B_I -orbits of the base point F_I it is important to explicitly understand the orbits $B_I.(e_i - e_{n+i})$ and $B_I.(e_i + e_{n+i})$. If F_I and B_I are as above, then the orbits of interest are $B_I.(e_i \pm e_{n+i})$, $i = 1, \dots, n$ if n is even, and $B_I.(e_i \pm e_{n+i})$, $i = 1, \dots, n-1$ and $B_I.(e_{\frac{n+1}{2}})$ and $B_I.(e_{n+\frac{n+1}{2}})$ if n is odd. In this case the orbits $B_I.(e_i + e_{n+i})$, $B_I.(e_i - e_{n+i})$, $B_I.(e_{\frac{n+1}{2}})$ and $B_I.(e_{n+\frac{n+1}{2}})$ have points of the forms

- **If n is even:** then for $i = 1, \dots, n$

$$b(e_i - e_{n+i}) = \lambda(e_i - e_{n+i-1}) + \dots + b_n(e_n - e_{2n}) + a_n(e_n + e_{2n}) + \dots + a_1(e_1 + e_{n+1}),$$

$$b(e_i + e_{n+i}) = \lambda(e_i + e_{n+i}) + a_{i-1}(e_{i-1} + e_{n+i-1}) + \dots + a_1(e_1 + e_{n+1}), \text{ with } \lambda \neq 0.$$

- **If n is odd:** then for $i = 1, \dots, n-1$

$$b(e_i - e_{n+i}) = \lambda(e_i - e_{n+i-1}) + \dots + b_{n-1}(e_n - e_{2n}) + b_n e_{n+\frac{n+1}{2}} + a_n e_{\frac{n+1}{2}} + a_{n-1}(e_n + e_{2n}) + \dots + a_1(e_1 + e_{n+1}),$$

$$b(e_i + e_{n+i}) = \lambda(e_i + e_{n+i}) + a_{i-1}(e_{i-1} + e_{n+i-1}) + \dots + a_1(e_1 + e_{n+1}),$$

¹Recall that we define a special way of denoting to write the Weyl elements in section §2.2.3.

$$b(e_{\frac{n+1}{2}}) = \lambda e_{\frac{n+1}{2}} + a_{n-1}(e_n + e_{2n}) + \dots + a_1(e_1 + e_{n+1}),$$

and

$$b(e_{n+\frac{n+1}{2}}) = \lambda e_{n+\frac{n+1}{2}} + a_n e_{\frac{n+1}{2}} + a_{n-1}(e_n + e_{2n}) + \dots + a_1(e_1 + e_{n+1}),$$

with $\lambda \neq 0$.

Note that in the above orbits, if b is chosen appropriately, then we can arrange $b(e_i - e_{n+i}) = e_i$ or $b(e_i - e_{n+i}) = e_{n+i}$ for all i . This plays a role in the proof of Theorem 5.1.4. \square

Depending on n being odd or even, the base point $F_I \in \gamma^{cl}$ is that associated to the following ordered basis:

• **If n is even:**

$$e_1 + ie_{n+1}, e_2 + ie_{n+2}, \dots, e_n + ie_{2n}, e_n - ie_{2n}, \dots, e_2 - ie_{n+2}, e_1 - ie_{n+1}. \quad (5.1)$$

• **If n is odd:**

$$e_1 + ie_{n+1}, \dots, e_n + ie_{2n}, e_{\frac{n+1}{2}}, e_{n+\frac{n+1}{2}}, e_n - ie_{2n}, \dots, e_1 - ie_{n+1}. \quad (5.2)$$

and the base point which defines the flag domains and the base cycles is the flag associated to the ordered basis

$$e_1, e_2, e_3, \dots, e_{2n-1}, e_{2n} \quad (5.3)$$

These two bases define two maximal tori which are conjugate and the corresponding Weyl groups W_I and W_S are isomorphic.

Let ψ be the bijective map between Weyl groups W_I and W_S which is defined as follows:

• **If n is even:** $\psi(\mp(i)) = \pm i$, for all $1 \leq i \leq n$.

• **If n is odd:** $\psi(\mp(i)) = \pm i$, for all $1 \leq i \leq n-1$ and $\psi(\pm n) = \pm n$.

Proposition 5.1.3. *If w is a dense permutation, then the flag $F_{\psi(w)}$ belongs to the orbit $B_I(F_w)$.*

Proof. Without loss of generality assume that n is even. Let $w \in W_I$ be a dense permutation, denote the first n subspaces of the associated maximally isotropic flag F_w by

$$\{0\} \subset \langle u_{w_1} \rangle \subset \langle u_{w_1}, u_{w_2} \rangle \subset \dots \subset \langle u_{w_1}, \dots, u_{w_n} \rangle$$

where $u_i = e_j \pm e_{n+j}$. We will show that the orbit $B_I(F_w)$ contains the flag $F_{\psi(w)}$. For this purpose let $\tilde{w} \in W_S$ be the image of w under the bijective map ψ and let $Y_{\tilde{w}}$ be the isotropic flag associated to \tilde{w} , i.e., the flag

$$Y_{\tilde{w}} := (\{0\} \subset \langle e_{\tilde{w}_1} \rangle \subset \langle e_{\tilde{w}_1}, e_{\tilde{w}_2} \rangle \subset \dots \subset \langle e_{\tilde{w}_1}, \dots, e_{\tilde{w}_n} \rangle)$$

We will show that $Y_{\tilde{w}} = b(F_w)$ for b appropriately chosen in B_I . For this recall that if w is a dense permutation, then $w_i < 0$ for all $1 \leq i \leq n$, and therefore $B_I.(e_j - ie_{n+j})$, $j = 1, \dots, n$, are the orbits of interest. By Remark 5.1.2 the orbit $B_I.\langle e_j - ie_{n+j} \rangle$ contains the points

$$y = b(e_j - ie_{n+j}) = \lambda(e_j - ie_{n+j}) + \dots + b_n(e_n - ie_{2n}) + a_n(e_n + ie_{2n}) + \dots + a_1(e_1 + ie_{2n})$$

where $\lambda \neq 0$. Choose all constants to be 0 except $\lambda = a_i = \frac{1}{2}$, so that $y = \langle e_i \rangle = \langle e_{\tilde{w}_i} \rangle$. We repeat this step n -times to construct the first half of the flag $Y_{\tilde{w}}$, and then extend this flag to maximal h -isotropic flag, . Therefore, the flag $Y_{\tilde{w}}$ is constructed. \square

Theorem 5.1.4. (*Dense Permutation Theorem*). *The following are equivalent*

- (i) w is dense.
- (ii) $B_I(F_w) \cap D_\alpha \neq \emptyset$ for some α .
- (iii) $B_I(F_w) \cap D_\alpha \neq \emptyset$ for every α .

Furthermore, under any of these conditions, for every α the intersection $B_I(F_w) \cap C_\alpha$ contains a T_S -fixed point.

Proof. (iii) \Rightarrow (ii) Trivial direction.

(ii) \Rightarrow (i) We handle the case where n is even. The proof for n goes analogously. We assume by contradiction that w is not a dense permutation and will show that $S_w \cap D_\alpha = \emptyset$ for all α . In particular, S_w has no T_S -fixed points². Recall that the complex bilinear form b is defined by the following conditions:

$$b(e_i, e_{n+i}) = 1 \text{ and } b(e_i, e_k) = 0 \text{ for } k \neq n+i.$$

Let $w \in W_I$ be a non-dense permutation. Let (w_1, \dots, w_{2n}) be the basis of the w -permuted full flag F_w and suppose that for some $j \leq n$ the sign of w_j is not negative. Thus for every $g \in B_I$

$$g(w_j) = \lambda(e_k + ie_{n+k}) + \sum_{\ell < k} c_j(e_\ell + ie_{n+\ell}).$$

Observe that $g(w_j)$ is h -isotropic and therefore if $j = 1$, then, since $\lambda \neq 0$, it follows that $g(F_w) \not\subset C_\alpha$ for all α .

Now suppose that $j > 1$ and suppose that $g(F_w)$ is in some C_α . This implies in particular that for every $i < j$ there are linear combinations

$$\varepsilon_i = g(w_i) + \sum_{\ell < i} c_{i\ell} g(w_\ell).$$

which when expressed in the standard basis contain only vectors from V^- or V^+ . For $g(F_w)$ to be in C_α the same must be true for $g(w_j)$. This means that there is a linear combination

$$\varepsilon_j = g(w_j) + \sum_{i < j} a_i \varepsilon_i$$

which has the same property. Hence, e_k or e_{n+k} (or both) must be among the vectors with non-zero coefficients in the ε_i for $i < k$. If both appear, e.g., in ε_κ and ε_δ , then

²See theorem ??

$b(\varepsilon_\kappa, \varepsilon_\delta) = 1$, contrary to the flag $g(F_w)$ being maximally b -isotropic. If only one appears, e.g., e_k in ε_κ , then e_{n+k} appears in ε_j and $b(\varepsilon_\kappa, \varepsilon_j) = 1$ which is again a contradiction. Hence the intersection is empty and this completes the proof.

(i) \Rightarrow (iii) Again, since the two cases are dealt with analogously, we assume here that n is even. Let w be a dense permutation. Then $w_i < 0$ for all $1 \leq i \leq n$ and $B_I.(e_j - ie_{n+j})$, $j = 1, \dots, n$, are the orbits of interest. From Remark 5.1.2 it follows that the orbit $B_I.< e_i + ie_{n+i} >$ contains the point

$$y = b(e_j - ie_{n+j}) = \lambda(e_j - ie_{n+j}) + \dots + b_n(e_n - ie_{2n}) + a_n(e_n + ie_{2n}) + \dots + a_1(e_1 + ie_{n+1})$$

where $\lambda \neq 0$. Then we can choose all constants to be 0 except $\lambda = a_j = \frac{1}{2}$. It follows that $y = < e_j > = < e_{\tilde{w}_j} >$ if the corresponding sign in α is $-$. If the corresponding sign in α is $+$, then we can choose all constants to be 0 except $\lambda = -a_j = \frac{1}{2}$, and $y = < e_{n+j} > = < e_{\tilde{w}_j} >$. Note that the vectors e_j and e_{n+j} are of negative and positive norms, respectively. In summary, we have constructed a set of T_S -fixed points in the orbits S_w for w dense with no two belonging to the same flag domain. By directly checking we see that all possible signatures have been obtained and therefore every flag domain has non-empty intersection with one such S_w . \square

By following the above proof we have the following corollary;

Corollary 5.1.5. *If w is a dense permutation, then for all α the intersection $S_w \cap C_\alpha$ contains exactly one T_S -fixed point and in particular is connected.*

Remark 5.1.6. Assume that we have an intersection point given by the flag associated to the basis $(e_{\tilde{w}_1}, e_{\tilde{w}_2}, \dots, e_{\tilde{w}_n})$. Therefore this point belongs to the open orbit D_α given by $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ where $\alpha_j = +$ if $\tilde{w}_j > 0$ and $\alpha_j = -$ if $\tilde{w}_j < 0$. In this case if $\tilde{w}_j < 0$, then $e_{\tilde{w}_j} = e_{n+w_j}$. \square

Recall that the complexification of the maximal compact subgroup K of G has a unique compact orbit in each open G_0 -orbits which is a complex manifold (see [15]) and called the base cycle C_0 . Similarly as the case $SP(2n, \mathbb{R})$, the base cycle C_0 of the flag domain D_α of $SO^*(2n)$ is the set

$$C_0 = \{F \in Z : V_i = (V_i \cap E^-) \oplus (V_i \cap E^+), 1 \leq i \leq 2n\}.$$

where the intersection dimensions are determined by α . Since $C_0 = K/(K \cap B_{z_0})$ where z_0 is any base point in C_α , it follows that the dimension of C_0 is $\frac{n(n-1)}{2}$ and the Iwasawa Schubert variety must be of dimension $\frac{n(n-1)}{2}$.

5.1.1 Length of the elements of $S_n \ltimes \mathbb{Z}_2^{n-1}$

For dimension computations, let us state how can we compute the length of elements $w \in S_n \ltimes \mathbb{Z}_2^{n-1}$ relative to our notation for the Weyl group elements.

Lemma 5.1.7. *Fix $w \in W$, construct $\tilde{w} \in W$ by the following algorithm:*

1. *Start from left to right in w , using simple reflections place all positive numbers in w step by step in a sequence of n -empty boxes beginning from the first one in \tilde{w} , in the same order as they appeared in w .*
2. *From left to right in w replace a negative number with its absolute value in the n -empty boxes starting from right to left.*

If $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_n)$, then define $L(\tilde{w}) = \frac{n^2-n}{2}$ - the cardinality of $\{\tilde{w}_j : j < k \text{ and } \tilde{w}_j < \tilde{w}_k\}$, and if we have $2m$ negative signs in w in the following positions k_1, k_2, \dots, k_{2m} , then define $f(w) = \sum_{j=1}^m [(2n-1-k_{2j}-k_{2j-1})]$. It follows that the length of w is

$$l(w) = L(\tilde{w}) + f(w) + m$$

Proof. The length of the permutation $w \in S_n \ltimes \mathbb{Z}_2^{n-1}$ is the minimal number of simple reflections which define w . To compute this note that we have two kinds of reflections, the first $n-1$ reflections are the simple reflections in S_n and the last one is the reflection which flips the sign. This means that if we want to flip the sign of w_j , we should move w_j from its position to the last position and then flip its sign and then return it back to its position. More precisely, consider the word $w_0 = (123\dots n)$ and let w be the word where $w_{j_1}, w_{j_2}, \dots, w_{j_m}$ in the positions $j_1 < j_2 < \dots < j_m$, $1 \leq j_i \leq n$, are negatives. To construct w from w_0 , we first flip the signs for the numbers $w_{j_1}, w_{j_2}, \dots, w_{j_m}$. For this purpose we move each number, starting from w_{j_m} , to the last position and then flip the sign of it. Then we apply the simple reflection to the positive numbers to put them in the same order as they appear in the word w . In this way the sum of all these movements and flips is exactly $L(\tilde{w}) + m$. The last step is to move each w_{j_i} to its original position in w and denote the total number of these movements by $f(w)$. It then follows that $l(w) = L(\tilde{w}) + f(w) + m$. \square

Example 5.1.8. *Let $w = ((-12)5(-34)6) \in S_6 \ltimes \mathbb{Z}_2^5$, then by following the above proof we have*

$$(-12)5(-34) - 6 \implies 2546(-1-3) \implies 2546(31)$$

so $w = (-12)5(-34)6$ and $\tilde{w} = 2546(31)$, then $L(\tilde{w}) = 9$ and $f(w) = 6$.

Hence $l(w) = 9 + 6 + 1 = 16$.

Definition 5.1.9. Let $w = (-n - (n-1)\dots - 2 - 1) \in S_n \ltimes \mathbb{Z}_2^{n-1}$ if n is even and $w = (n - (n-1)\dots - 2 - 1) \in S_n \ltimes \mathbb{Z}_2^{n-1}$ if n is odd. any of these two elements called **Super dense** and has dimension $\frac{n(n-1)}{2}$.

As the case of the real form $SP(2n, \mathbb{R})$, by direct calculation of the dimension of the dense permutations we get the following result.

Theorem 5.1.10. *There exists a unique Iwasawa Schubert variety S_w of dimension $\frac{n(n-1)}{2}$. It is given by the super dense permutation $\tilde{w} = (-n - (n-1)\dots - 2 - 1)$ if n is even and by $\tilde{w} = (n - (n-1)\dots - 2 - 1)$ if n is odd.*

Proof. Assume firstly that n is even, then consider the set W_{den} to be the set of all dense permutations, i.e. the set of all permutations $w \in S_n \ltimes \mathbb{Z}_2^{n-1}$ such that all signs in w are negative. Then, using our formula to compute the length of w it is clear that $f(w) + n = \frac{n(n-1)}{2}$ for all $w \in S_n \ltimes \mathbb{Z}_2^{n-1}$. Thus the shortest element of W_{den} has $l(\tilde{w}) = 0$, i.e. \tilde{w} is the shortest element in S_n which is unique and equal to $(12\dots n)$. Therefore there is a unique element $w \in W_{den}$ with length $\frac{n(n-1)}{2}$ which is $(-n - (n-1)\dots - 2 - 1)$.

Secondly, let n is odd and let \tilde{W}_{den} to be the set of all dense permutations, i.e. the set of all permutations $w \in S_n \ltimes \mathbb{Z}_2^{n-1}$ such that all signs in w are negative except the sign of the entry n is positive. Then, using our formula to compute the length of w , it is clear that $f(w) + n = \frac{n(n-1)}{2}$ for all $w \in S_n \ltimes \mathbb{Z}_2^{n-1}$ only if the entry n sitting in the first box of the permutation. Thus shortest element of \tilde{W}_{den} has $l(\tilde{w}) = 0$. Then the shortest element in S^n is \tilde{w} which is unique and equal to $(12\dots n)$ which implies that we have a unique element $w \in \tilde{W}_{den}$ with length $\frac{n(n-1)}{2}$ which is $(n - (n-1)\dots - 2 - 1)$. □

5.2 Intersection points of Schubert duality

Assuming that the element $w_0 \in W_I$ is $w_0 = (-n - (n-1)\dots - 2 - 1)$ if n is even and $w_0 = (n - (n-1)\dots - 2 - 1)$ if n is odd, we have shown that S_w is of complementary dimension to the cycles, and the intersection $S_w \cap C_\alpha$ is in fact only one T_S -fixed point (See corollary 5.1.5). Here we describe all intersection points. The argument in the case of $SO^*(2n)$, which is similar to that in the case $SP(2n, \mathbb{R})$, goes as follows.

Since $w = (-n, \dots, -1)$, the flag basis corresponding to w_0

$$(\hat{v}_n, \dots, \hat{v}_1, v_1, \dots, v_n)$$

where $v_k = e_k + ie_{n+k}$ and $\hat{v}_k = e_k - ie_{n+k}$.

Proposition 5.2.1. *If w is a super dense permutation, and the ε -basis $(\varepsilon_1, \dots, \varepsilon_n)$ of T_S -eigenvectors defines the intersection point in $S_w \cap C_\alpha$, then $\varepsilon_k = e_k$ or e_{n+k} , depending on the signature α .*

Proof. This is a consequence of the following:

1.

$$\hat{v}_k = \lambda_k(e_k - e_{n+k}) + a_k(e_k + ie_{n+k}) + R_k = K_k + R_k$$

2. $\lambda_k \neq 0$

3. The intersection $S_w \cap C_\alpha$ is a flag defined by T_S -eigenvectors.

From the expression for \hat{v}_k it is immediate that all of the possibilities in the statement occur. Furthermore, since the λ_k are non-zero, for every k a non-zero contribution from K_k occurs in the sum

$$\varepsilon_k = \sum_{j \leq k} c_{kj} b(w(j)).$$

Since e_k and e_{n+k} do not occur in $b(w(j))$ for $j < k$, it follows that $\varepsilon_k = K_k + E_k$ in the standard basis (See §2.2.2). Finally, since ε_k is a T_s -eigenvector, it follows that $\varepsilon_k = K_k$ and is of the type in the the statement of the proposition. \square

If w is a super dense permutation, then the set of all intersection points of S_w is denoted by $Supset_e(w)$ if n is even and by $Supset_o(w)$ if n is odd, where

$$Supset_e(w) := \{\tilde{w}(F_S) \in W : \tilde{w} \text{ obtained by change none, some or all of the numbers } -j \text{ by } j\}$$

$$Supset_o(w) := \{\tilde{w}(F_S) \in W : \tilde{w} \text{ obtained by change none, some or all of the numbers } -j \text{ by } j \\ \text{or } n \text{ by } -n\},$$

where F_S is the maximally b -isotropic flag associated to the standard basis e_1, e_2, \dots, e_{2n} . Here $\tilde{w}(F_S)$ gives us a point in an open orbit D_α and to know this open orbit just replace i by $+$ and $-i$ by $-$.

Chapter 6

Cycle intersection for $SO(p, q)$

In this chapter we study the case of the real form $SO(p, q)$ of $SO_m(\mathbb{C})$. The general results here are stated in terms of algorithms (See definitions 6.1.1 and 6.2.1); in fact it seems impossible to avoid this. In corollary 6.3.2 we give concrete formulas for the intersection points in $S_w \cap C_\alpha$ if the intersection non-empty and S_w is of complementary dimension . Also, the number of intersection points with C_0 is explicitly computed in Theorem 6.3.3.

6.1 Conditions for $S_w \cap C_\alpha \neq \emptyset$

In the present section we describe the conditions for an element w of the Weyl group to parametrize Schubert variety with S_w must satisfy in order that $S_w \cap D_\alpha \neq \emptyset$ for some flag domain D_α . As would be expected, a special type of permutation plays a fundamental role.

Definition 6.1.1. An element $w \in W$ is called *a harmonic permutation* if it satisfies the following conditions:

If q is even: The number $-(2i - 1), 1 \leq i \leq \frac{q}{2}$, sits in any place to the left of the number $(2i)$ or $(-2i), 1 \leq i \leq \frac{q}{2}$, in the one line notation of the permutation and the order of the numbers $q + i$ or $-(q + i)$, where $1 \leq i \leq p - q$ is arbitrary.

If q is odd: The number $-(2i - 1), 1 \leq i \leq \frac{q-1}{2}$, sits in any place to the left of the number $(2i), 1 \leq i \leq \frac{q-1}{2}$, and the number $-q$ sits in the last position in the one line notation of the permutation, and the order of the numbers $q + i$ or $-(q + i)$, where $1 \leq i \leq \frac{p-q}{2}$ is arbitrary.

Example 6.1.2. In $SO(4, 2)$ the relevant pairs are (-12) and $(-1 - 2)$. As a result we have 6 harmonic permutations. These are: $(-12 - 3), (-1 - 32), (-3 - 12), (-1 - 23), (-13 - 2), (3 - 1 - 2)$.

Recall that the fixed point in the closed orbit γ^{cl} is the flag associated to the following ordered basis

- If m is even, then the basis is

$$\begin{aligned} &e_1 + e_{2q}, \dots, e_q + e_{q+1}, e_{2q+1} + ie_{2q+2}, e_{2q+3} + ie_{2q+4}, \dots, e_{2n-1} + ie_{2n}, \\ &e_{2n-1} - ie_{2n}, \dots, e_{2q+3} - ie_{2q+4}, e_{2q+1} - ie_{2q+2}, e_q - e_{q+1}, \dots, e_1 - e_{2q}. \end{aligned} \quad (6.1)$$

- If m is odd, then the basis is

$$\begin{aligned} & e_1 + e_{2q}, e_2 + e_{2q-1}, \dots, e_q + e_{q+1}, e_{2q+1}, e_{2q+2} + ie_{2q+3}, e_{2q+4} + ie_{2q+5}, \dots, e_{2n} + ie_{2n+1}, \\ & e_{2q+1}, e_{2n} - ie_{2n+1}, \dots, e_{2q+4} - ie_{2q+5}, e_{2q+2} - ie_{2q+3}, e_q - e_{q+1}, \dots, e_2 - e_{2q-1}, e_1 - e_{2q} \end{aligned} \quad (6.2)$$

Remark 6.1.3. For v_i any such basis vector and $b \in B_I$ the form of $b.v_i$ is given as follows:

1. $b.(e_i - e_{2q-i+1}) = \lambda(e_i - e_{2q-i+1}) + \dots + b_n(e_q - e_{q+1}) + a_n(e_q + e_{q+1}) + \dots + a_1(e_1 + e_{2q}),$
2. $b.(e_i + e_{2q-i+1}) = \lambda(e_i + e_{2q-i+1}) + a_{i-1}(e_{i-1} - e_{2q-i}) + \dots + a_1(e_1 - e_{2q}),$
3. $b.(e_{2q+2j-1} - ie_{2q+2j}) = \lambda(e_{2q+2j-1} - ie_{2q+2j}) + \dots + b_m(e_{m-1} - e_m) + a_m(e_{m-1} + e_m) + \dots + a_1(e_1 + e_{2q})$

where $m = 2n$ or $2n + 1$.

4. $b.(e_{2q+2j-1} + ie_{2q+2j}) = \lambda(e_{2q+2j-1} + ie_{2q+2j}) + a_{i-1}(e_{2q+2j+1} + ie_{2q+2j+2}) + \dots + a_1(e_1 + e_{2q}),$
5. $b.(e_{2q+1}) = \lambda(e_{2q+1}) + a_n(e_{2n} + ie_{2n+1}) + \dots + a_1(e_1 + e_{2q}),$ if $m = 2n + 1,$

with $\lambda \neq 0$ in all cases above. Note that in the above orbits, if b is chosen appropriately, then we can arrange all linear combination for every one of the above vectors to be in the standard basis of T_S -eigenvectors. See Theorem 6.1.5. \square

In the following result $\psi : W_I \rightarrow W_S$ denotes the bijective map between Weyl groups which was introduced in Chapter 2.

Proposition 6.1.4. *If w is a harmonic permutation, then the flag $F_{\psi(w)}$ belongs to the orbit $B_I(F_w)$.*

Proof. We handle the case where $m = 2n$. The proof for $m = 2n + 1$ goes analogously. Let us first prove the theorem for the case that q is even. For this let $w \in W_I$ be a harmonic permutation and define $F_w = w.(F_I)$ to be the isotropic full flag associated to w . Denote by

$$\{0\} \subset \langle u_{w_1} \rangle \subset \langle u_{w_1}, u_{w_2} \rangle \subset \dots \subset \langle u_{w_1}, \dots, u_{w_n} \rangle$$

the first n subspaces of $F_w = w(F_I)$, where u_{w_i} is a vector from the basis above. Let $\tilde{w} \in W_S$ be the image of w under the bijective map ψ and let $Y_{\tilde{w}}$ be the isotropic flag associated to \tilde{w} such that the first half of $Y_{\tilde{w}}$ is

$$\{0\} \subset \langle \varepsilon_{\tilde{w}_1} \rangle \subset \langle \varepsilon_{\tilde{w}_1}, \varepsilon_{\tilde{w}_2} \rangle \subset \dots \subset \langle \varepsilon_{\tilde{w}_1}, \dots, \varepsilon_{\tilde{w}_n} \rangle$$

Our claim here is that this flag is an intersection point in $B.F_w \cap C_\alpha$. To prove this we will construct b with $b(F_w) = Y_{\tilde{w}}$. From the definition of harmonic permutation, there are two possibilities for w_1 : $|w_1| = 2i - 1 \leq q$ or $|w_1| > q$.

Case 1: If $|w_1| > q$. Consequently then $u_{w_1} = v_{\tilde{w}_1}$, so the orbit $B_I. \langle u_{w_1} \rangle$ contains the point $\langle \varepsilon_{\tilde{w}_1} \rangle$.

Case 2: If $|w_1| = 2i - 1 \leq q$, then we must consider the orbit $B_I. < e_{2\tilde{w}_1-1} - e_{2q+w_1+1} >$. By Remark (6.1.3) above, the orbit $B_I. < e_{2\tilde{w}_1-1} - e_{2q+w_1+1} >$ contains points of the form

$$y = < \alpha_1(e_{2\tilde{w}_1-1} + e_{2q+w_1+1}) + \alpha_2(e_{2\tilde{w}_1} + e_{2q+w_1}) + \alpha_3(e_{2\tilde{w}_1} - e_{2q+w_1}) + \alpha_4(e_{2\tilde{w}_1-1} - e_{2q+w_1+1}) >$$

where $\alpha_1 = \pm\alpha_4$ and $\alpha_2 = \pm\alpha_3$. By taking $\alpha_1 = \alpha_4 = \frac{1}{2}$ and $\alpha_2 = \alpha_3 = \frac{1}{2}i$, it follows that $y = < e_{2\tilde{w}_1-1} + ie_{2\tilde{w}_1} > = < v_{\tilde{w}_1} >$.

To construct the j -vector of $b(v_{w_j})$ to obtain the subspace $V_{\tilde{w}_j}$ we must consider three cases:

Case 1: If $|w_j| > q$, then $w_j = \tilde{w}_j$ and $\varepsilon_{\tilde{w}_j} = u_{w_j}$, so the orbit $B_I. < u_{w_j} >$ contains the point $\varepsilon_{\tilde{w}_j}$.

Case 2: If $|w_j| = 2i - 1 \leq q$, then our job goes through the orbit $B_I. < e_{2\tilde{w}_j-1} - e_{2q+w_j+1} >$. By using Remark (6.1.3) we see that the orbit $B_I. < e_{2\tilde{w}_j-1} - e_{2q+w_j+1} >$ contains points of the form

$$y = < \alpha_1(e_{2\tilde{w}_j-1} + e_{2q+w_j+1}) + \alpha_2(e_{2\tilde{w}_j} + e_{2q+w_j}) + \alpha_3(e_{2\tilde{w}_j} - e_{2q+w_j}) + \alpha_4(e_{2\tilde{w}_j-1} - e_{2q+w_j+1}) >$$

where $\alpha_1 = \pm\alpha_4$ and $\alpha_2 = \pm\alpha_3$. By taking $\alpha_1 = \alpha_4 = \frac{1}{2}$ and $\alpha_2 = \alpha_3 = \frac{1}{2}i$, it follows that $y = < e_{2\tilde{w}_j-1} + ie_{2\tilde{w}_j} > = < \varepsilon_{\tilde{w}_j} >$. Therefore b is constructed to obtain the flag

$$\{0\} \subset < \varepsilon_{\tilde{w}_1} > \subset < \varepsilon_{\tilde{w}_1}, \varepsilon_{\tilde{w}_2} > \subset \dots \subset < \varepsilon_{\tilde{w}_1}, \dots, \varepsilon_{\tilde{w}_j} > .$$

Case 3: If $|w_j| = 2i$, then the orbit is $B_I. < e_{|w_j|} + e_{|2\tilde{w}_j|-1} >$ is relevant. In this case the points

$$y = < \alpha_1(e_{|w_j|-1} + e_{|2\tilde{w}_j|}) + \alpha_2(e_{|w_j|} + e_{|2\tilde{w}_j|-1}) + \alpha_3(e_{|w_j|-1} + ie_{|w_j|}) >$$

belong to the orbit $B_I. < e_{|w_j|} + e_{|2\tilde{w}_j|-1} >$. For $\alpha_1 = -i, \alpha_2 = 1$ and $\alpha_3 = i$ we have $y = < e_{|2\tilde{w}_j|-1} + ie_{|2\tilde{w}_j|} >$. Therefore, the j -vector of b is constructed in this case as well..

Thus by induction we observe that $b \in B_I$ can be constructed with $b(F_w) = Y_{\tilde{w}}$.

We complete the proof by handling the case where q is odd. For this point we can repeat the steps of the proof above for $w_j > q$ and $|w_j| = 2i - 1$. For the case where $|w_j| = 2i$, we apply the same method as above, only changing $2|\tilde{w}_j|$ by $2(|\hat{w}_j| + 1)$. If $w_n = -q$, then $\hat{w}_j = -n$. In this case the orbit of relevance is $B_I. < e_q - e_{q+1} >$. As we see from Remark 6.1.3 $y = \alpha_1(e_q + e_{q+1}) + \alpha_2(e_q - e_{q+1})$ belongs to this orbit. The desired result is then obtained by taking $\alpha_1 = 1, \alpha_2 = -1$ the point $y = e_{q+1}$ belongs to the orbit $B_I. < e_q - e_{q+1} >$. Therefore the element of B_I is also constructed in the case where q is odd. □

Theorem 6.1.5. (Harmonic Permutation Theorem). *The following are equivalent*

- (i) w is harmonic.
- (ii) $B_I(F_w) \cap D_\alpha \neq \emptyset$ for some α .

Under either of these conditions for every α the nonempty intersection $B_I(F_w) \cap C_\alpha$ contains a T_S -fixed point.

Proof. (i) \Rightarrow (ii) Is exactly the statement of Proposition 6.1.4.

(ii) \Rightarrow (i) Assuming that w is not harmonic permutation, we will show that $S_w \cap D_\alpha = \emptyset$, for all α , i.e. S_w has no T_S -fixed points¹. Assume to the contrary that there exists $b \in B_I$ such that $b(F_w) \in S_w \cap C_\alpha$. For $b(F_w)$ to be fixed, b has to have a certain shape and at each stage where the condition of harmonicity is violated, we should prove that there is no such b . For this recall that the complex bilinear form b has been defined to satisfy the following orthogonality condition:

$$b(e_j - e_{2q-j+1}, e_j + e_{2q-j+1}) = 1, \quad 1 \leq j \leq q \text{ and } b(e_j \pm e_{2q-j+1}, e_k \pm e_{2q-k+1}) = 0 \text{ for } k \neq j.$$

Since w is not a harmonic permutation, there exist a pair of the form $-(2i-1), 2i$ or of the form $-(2i-1), -2i$, where $1 \leq i \leq \frac{q}{2}$, such that $\mp 2i$ sits to the left of $-(2i-1)$. Assume that $w_j = 2i$ is the first even number which appears in w such that $2i$ sits to the left of $-(2i-1)$. Then b does in fact yield a T_S -fixed partial flag, i.e.

$$\{0\} \subset \langle \varepsilon_{\tilde{w}_1} \rangle \subset \langle \varepsilon_{\tilde{w}_1}, \varepsilon_{\tilde{w}_2} \rangle \subset \dots \subset \langle \varepsilon_{\tilde{w}_1}, \dots, v_{\tilde{w}_{j-1}} \rangle \quad (6.3)$$

Now we check that there is no $b \in B_I$ so that $b(v_{w_j})$ defines an extended partial flag which is T_S -fixed. For this we consider the orbit $B_I \cdot \langle e_{|w_j|} + e_{|2\tilde{w}_j|-1} \rangle$ which contains points of the form

$$\alpha_1(e_1 + e_{2q}) + \alpha_2(e_2 + e_{2q-1}) + \dots + \alpha_j(e_{|w_j|} + e_{|2\tilde{w}_j|-1}) \quad (6.4)$$

This is a linear combination of h -isotropic vectors for all α_i . Recall that the flag of intersection has non h -isotropic vectors and to have one of these vectors in this step which is linearly independent with all vectors in the flag (6.3) above, we should add the point $(e_{|w_j|-1} + ie_{|w_j|}) >$ from the flag (6.3) to the linear combination in (6.4), but this vector is not in the flag (6.3). Thus, as was claimed, no $b \in B_I$ has that property that $b(F_w)$ is T_S -fixed.

If $w_j = -2i$ is the first even number such that $-2i$ sits to the left of $-(2i-1)$, then the relevant orbit is $B_I \cdot \langle e_{|w_j|} - e_{|2\tilde{w}_j|-1} \rangle$ which contains points of the form

$$\alpha_1(e_1 + e_{2q}) + \dots + \alpha_q(e_q + e_{q+1}) + \alpha_{q+1}(e_q - e_{q+1}) + \dots + \alpha_s(e_{|w_j|} - e_{|2\tilde{w}_j|-1}).$$

To have non-isotropic point which is linearly independent with all points in the flag (6.3), we should add any of the points $(e_{|w_j|-1} + ie_{|w_j|}) >$, $(e_{|w_j|-1} - ie_{|w_j|}) >$, $(e_{2q-|w_j|+1} + ie_{2q-|w_j|+2}) >$ or $(e_{2q-|w_j|+1} - ie_{2q-|w_j|+2}) >$ from the flag (6.3) to the linear combination in (6.4). But the flag (6.3) does not contain any of these points. So again, for all $b \in B_I$ the flag $b(F_w)$ is not T_S -fixed and consequently $S_\alpha \cap D_\alpha = \emptyset$. \square

6.2 Introduction to the combinatorics

For the remainder of this chapter we only discuss the intersection properties of the Iwasawa-Schubert cells which are of complementary dimension to C_0 . Recall that the maximal compact subgroup of $SO(p, q)$ is $K_0 := S(O(p) \times O(q)) \subset S(U(p) \times U(q))$. For E^+ and E^- as

¹See Corollary 3.3.4

in §2.4.2, the base cycle in the flag domain $D_{a,b}$ is given by

$$C_0 = \{F \in Z : \dim V_i \cap E^- = \sum_{j=1}^i a_j \text{ and } \dim V_i \cap E^+ = \sum_{j=1}^i b_j, 1 \leq i \leq m\}.$$

If $m = 2n$, we have two cases: If q is even, the dimension of C_0 is $\frac{p(p-2)}{4} + \frac{q(q-2)}{4}$. If q is odd, then C_0 has the dimension $\frac{(p-1)^2}{4} + \frac{(q-1)^2}{4}$. If $m = 2n + 1$, then the dimension of the base cycle is $\frac{(p-1)^2}{4} + \frac{q(q-2)}{4}$ if q is even and is $\frac{p(p-2)}{4} + \frac{(q-1)^2}{4}$ if q is odd. Since we have restricted to the case where $Z = G/B$ is the manifold of complete flags, it follows that $\dim S_w = \frac{pq}{2}$ if q and n are even, and $\dim S_w = \frac{pq-1}{2}$ if q or n are odd.

Definition 6.2.1. A harmonic permutation $w \in S_n \ltimes \mathbb{Z}_2^{n-1}$ is called **a perfect harmonic permutation** if it is constructed by the following algorithm:

- A. Start with a sequence of n empty boxes which are to be filled in order to construct w .
- B. Consider the pairs $(-(2j-1), 2j)$ and $(-(2j-1), -2j)$ for all $1 \leq j \leq \frac{q}{2}$.
- C. If q is even:
 1. The pairs in step B are $(-1, 2), (-1, -2), (-3, 4), \dots, (-(q-1), q), (-(q-1), -q)$.
 2. Step by step, starting from $(-1, 2)$ until $(-(q-1), -q)$. For each $1 \leq j \leq \frac{q}{2}$ we have two pairs of the forms $(-(2j-1), 2j)$ and $(-(2j-1), -2j)$, so choose only one pair of them for each step and omit the other from the above list.
 3. If we choose the pair of the form $(-(2j-1), 2j)$, place this pair in any box in w such that the components $-(2j-1)$ and $2j$ of this pair sits as close as possible to each other.
 4. If we choose the pair of the form $(-(2j-1), -2j)$, place this pair in any box in w such that the components $-(2j-1)$ and $-2j$ of this pair sits as close as possible to each other and all pairs $(-(2i-1), 2i)$ or $(-(2i-1), -2i)$ with $i > j$ are sitting to the left of this pair and the pairs of the form $(-(2i-1), -2i)$ with $i < j$ are sitting in a decreasing order with respect to i to the right of $(-(2j-1), -2j)$.
 5. Once a pair is placed its position can be ignored so that the places at the immediate left and right of this pair become adjacent.
 6. After all pairs are placed, the remaining numbers $\pm(q+i)$ for $1 \leq i \leq \frac{p-q}{2}$ are placed in the remaining spots in the strictly increasing order with respect to $|w(i)|$ such that all number $\pm(q+i)$ for $1 \leq i \leq \frac{p-q}{2}$ are sitting to the left of the pairs $(-(2i-1), -2i)$ $1 \leq i \leq \frac{q}{2}$. If the number of negative signs in all pairs from steps 2 and 3 is even, then all numbers of the reminder numbers are positive, and if the number of negative signs is odd, then all numbers of the reminder numbers are positive except the number n is negative.
- D. If q is odd:

1. The pairs in step B are $(-1, 2), (-3, 4), (-5, 6), \dots, (-(q-2), q-1), -q$.
2. In the last box of w put the number $-q$.
3. Step by step, starting from $(-1, 2)$ until $(-(q-2), q-1)$, choose a pair and place this pair in any box of the first $n-1$ boxes in w such that the components $-(2j-1)$ and $2j$ of this pair sit as close as possible to each other. This means that once a pair is placed it can be ignored so that the places at the immediate left and right of this pair become adjacent.
4. After all pairs are placed, the numbers $\pm(q+i)$ for $1 \leq i \leq \frac{p-q}{2}$ are placed in the remaining spots in the strictly increasing order with respect to $|w(i)|$. If the number of negative signs in all pairs from step 1 and 2 is even, then all numbers of the reminder are positive, and if the number of negative signs is odd then all remaining numbers of the reminder are positive except the number n is negative.

Remark 6.2.2. If q is odd, then the signature of the flag domain D_α has $+$ in the last position. \square

Remark 6.2.3. If $W = S_n \ltimes \mathbb{Z}_2^n$, then a perfect harmonic permutation consists only of pairs of the form $(-(2j-1), 2j)$ for all $1 \leq j \leq \frac{q}{2}$. It is constructed as above, in particular such that the sign of n is $+$. \square

Example 6.2.4. If $p = 6, q = 4$, then the perfect harmonic permutations are:

$(-12)(-34)5, (-12)5(-34), 5(-12)(-34), (-3\ 4)(-1\ 2)5, (-3\ 4)5(-1\ 2), 5(-3\ 4)(-1\ 2),$
 $(-12) - 5(-3 - 4), -5(-12)(-3 - 4), -5(-3 - 4)(-12), (-34) - 5(-1 - 2), -5(-34)(-1 - 2),$
 $5(-3 - 4)(-1 - 2), (-3 - 1\ 2\ 4)5, 5(-3 - 1\ 2\ 4), -5(-3 - 12 - 4).$

Example 6.2.5. If $p = 10, q = 6$, then the element $78(-5\ 6)(-3 - 4)(-1\ 2)$ is a perfect harmonic permutation while the element $78(-3 - 4)(-1\ 2)(-5\ 6)$ is not a perfect harmonic permutation.

Recall that the dimension of the cells corresponds to the length of the word w in the Weyl group, i.e. if $F_w = w.F_I$, then $\dim(B.F_w) = l(w)$. So if we want to discuss the dimension of the Schubert cell it is enough to discuss the length of Weyl elements.

Proposition 6.2.6. Every perfect harmonic permutation $w \in S_n \ltimes \mathbb{Z}_2^{n-1}$ has dimension $\frac{pq}{2}$ if q is even and $\frac{pq-1}{2}$ if q is odd.

Proof. Given a perfect harmonic permutation w we consider three cases. These depend on which of the pairs $(-1\ 2)$, $(-3\ 4)$, $(-1\ -2)$ or $(-3\ -4)$ is contained in w . In each of these cases our proof goes by induction on the dimension of the flag manifold. Without loss of generality let $p > q \geq 6$ because if $q \leq 6$ then the permutation has only one or two pairs and the proof becomes trivial.

Case 1: The permutation w contains the pairs $(-1\ 2)$ and $(-3\ 4)$.

First, remove the pairs $(-1\ 2)$ and $(-3\ 4)$ from w to have a new permutation v consisting of the numbers $\{5, 6, \dots, n\}$. Define a function $f : \{5, \dots, n\} \rightarrow \{1, \dots, n-4\}$ by $f(i) = i - 4$ for all $1 \leq i \leq n - 4$. This is a bijective map which sends v to $\hat{w} \in S_{n-4} \ltimes \mathbb{Z}_2^{n-5}$. Note that \hat{w} is a perfect harmonic permutation in $S_{n-4} \ltimes \mathbb{Z}_2^{n-5}$. Thus by the induction assumption $l(\hat{w}) = \frac{(p-4)(q-4)}{2}$. Since $v = f^{-1}(\hat{w})$ it follows that v has the same length as \hat{w} . So we put the numbers 1234 to the left of v to have an element $\tilde{w} \in S_n \ltimes \mathbb{Z}_2^{n-1}$ with length $\frac{(p-4)(q-4)}{2}$. To split the sign of 1 and 3 (i.e., to change the positive sign to negative sign) we add $n - 4 + 1$ to $\frac{(p-4)(q-4)}{2}$ to send 3 to the last position and add $n - 4 + 2$ to send 1 to the position before the last position. Consequently, we have the element $(24)v(13)$ with length $\frac{(p-4)(q-4)}{2} + n - 4 + 1 + n - 4 + 2 = \frac{(p-4)(q-4)}{2} + 2n - 5$ and it follows that the element $(24)v(-3\ -1)$ has length $\frac{(p-4)(q-4)}{2} + 2n - 4$. We then return to the original w and remove only the pair $(-1\ 2)$. Then we define g to be the number of positions to the left of (-34) . As a result we have $n - 4 - g$ positions to the left of $(-3\ 4)$ and -3 should cross $n - 4 - g + 1$ positions to end up in the last position and 4 should cross g boxes to end up in the first position.

Finally we return to the original w and define h to be the number of positions to the left of $(-1\ 2)$ and $n - 2 - f$ to the right. In this situation -1 must cross $n - 2 - h + 1$ positions to end up in the last position and 2 must cross g positions to end up in the first position. It follows that the length of w is $\frac{(p-4)(q-4)}{2} + 2n - 4 + n - 4 - g + 1 + g + n - 2 - h + 1 + h = \frac{pq}{2}$. If q is odd, then the length of $\hat{w} \in S_{n-4} \ltimes \mathbb{Z}_2^{n-5}$ is $l(\hat{w}) = \frac{(p-4)(q-4)-1}{2}$. After applying the same steps as those above it follows that $l(w) = \frac{pq-1}{2}$.

Case 2: The permutation w contains the pair $(-1\ -2)$.

Note that this case only occurs if q is even. Since w is a perfect harmonic permutation, the pair $(-1\ -2)$ sits in the last 2 positions of w . Remove the pair $(-1\ -2)$ from w to obtain \hat{v} . Since q is even, a similar argument to that above shows that $l(\hat{v}) = \frac{(p-2)(q-2)}{2}$ because q is even. Put the pair $(1\ 2)$ to the left of \hat{v} . It follows that $\tilde{v} = (1\ 2)\hat{v}$ has the same length as \hat{v} . To split the sign of 1 and 2, each of them must cross $n - 2$ positions. Having made this move we then apply 2 reflection to obtain the pair $(-1\ -2)$ in the last two positions. Hence the length of w becomes $l(w) = \frac{(p-2)(q-2)}{2} + 2(n - 2) + 2 = \frac{pq}{2}$.

Case 3: The permutation w contains the pairs $(-1\ 2)$ and $(-3\ -4)$.

Note that this case appears only if q is even. In this case these two pairs appear in w in the following forms: $(-3\ -4\ -1\ 2)$ or $(-3\ -1\ 2\ -4)$ or the pair $(-3\ -4)$ sits in the last two positions of w . If the pair $(-3\ -4)$ sits in the last two positions of w , then the argument goes as in Case 2 above.

If we have the form $(-3 - 1 \ 2 - 4)$, then we must add $n - 4$ to $\frac{(p-2)(q-2)}{2}$ to put 3 in the last position and add $n - 2$ to $\frac{(p-2)(q-2)}{2}$ to put 4 in the last position. Then we must add 1 to split the signs and 3 to send -3 to its position in the original w , then the length of w become $\frac{(p-2)(q-2)}{2} + (n - 4) + (n - 2) + 1 + 3 = \frac{pq}{2}$.

If we have the form $(-3 - 4 - 1 \ 2)$, then we must add $n - 4$ to $\frac{(p-2)(q-2)}{2}$ to put 3 in the last position and add $n - 1$ to $\frac{(p-2)(q-2)}{2}$ to put 4 in the position before the last position. Then we must add 1 to split the signs, 3 to send -3 to its position in the original w and 1 to send -4 to its position in the original w . It follows that the length of w is $\frac{(p-2)(q-2)}{2} + (n - 4) + (n - 3) + 1 + 3 + 1 = \frac{pq}{2}$.

□

Proposition 6.2.7. *Every perfect harmonic permutation $w \in S_n \ltimes \mathbb{Z}_2^n$ has dimension $\frac{pq}{2}$.*

Proof. The argument goes exactly along the lines as that for the above Proposition. Here it is in fact simpler, because only the pairs $(-1 \ 2)$ and $(-3 \ 4)$ appear. □

6.3 Intersection points of Schubert duality

let $w \in W_I$ be a perfect harmonic permutation, in particular so that S_w is of complementary dimension to the cycles. Recall that in this case either $S_w \cap C_\alpha = \emptyset$ or is pointwise T_S -fixed (see Theorem 3.3.3 and corollary 3.3.4). The main goal in this section is to compute all such intersection points. The argument in the case of $SO(p, q)$ is carried out by means of algorithms. Nevertheless we are able to provide formulas for the cardinality of $S_w \cap C_\alpha$ when it is non-zero and the total number of cycles C_α for which this intersection is non-empty (see Theorem 6.3.3).

Proposition 6.3.1. *If w is a perfect harmonic permutation so that $B_I.F_w$ intersects a cycle C_α at a point given by the ε -basis $(\varepsilon_1, \dots, \varepsilon_m)$ of T_S -eigenvectors, then for any such eigenvector ε_k it follows that the ε -basis is given by $\varepsilon_k = e_{2j-1} + ie_{2j}$ or $e_{2j-1} - ie_{2j}$ if q and p are even, and $\varepsilon_k = e_{2j-1} + ie_{2j}$, $e_{2j-1} - ie_{2j}$, e_q , e_{q+1} or e_{2q+1} if q or p is odd, depending on the dimension m and the signature α .*

Proof. This is a consequence of the following:

1. w is a perfect harmonic permutation and the flag basis is that of $w(F_I)$.

2. We have the following cases

$$\begin{aligned} \bullet b.v_j &= \eta_j(e_r + e_{2q-r+1}) + \zeta_j(e_r - e_{2q-r+1}) + \tilde{\eta}_j(e_{r+1} + e_{2q-r}) + \tilde{\zeta}_j(e_{r+1} - e_{2q-r}) + B_j \\ &= K_j + B_j, . \end{aligned}$$

$$\bullet b.v_j = \eta_j(e_{2q+r} + ie_{2q+r+1}) + B_j = K_j + B_j,$$

where $\tilde{\eta}_j = \pm i\eta_j$ and $\tilde{\zeta}_j = \pm i\zeta_j$ and B_j does not involve the basis vectors e_r , e_{r+1} , e_{2q-r+1} and e_{2q-r} .

$$\bullet b.v_j = \eta_j(e_{2q+1}) + B_j = K_j + B_j, \text{ if } m = 2n + 1$$

3. $\eta_j \neq 0$ and $\tilde{\eta}_j \neq 0$.
4. The intersection $S_w \cap C_\alpha$ is a flag defined by T_S -eigenvectors.

From the expression for v_j it is immediate that all of the possibilities in the statement occur. Furthermore, since the $\eta_j \neq 0$ and $\tilde{\eta}_j \neq 0$ are non-zero, for every j a non-zero contribution from K_j occurs in the sum

$$v_j = \sum_{k \leq j} c_{kj} b.v_{w_k}.$$

Since $e_{2j-1} + ie_{2j}$ and $e_{2j-1} - ie_{2j}$ and e_q or e_{q+1} do not occur in $b.v_{w_k}$ for $k < j$, it follows that $\varepsilon_j = K_j + E_j$ in the standard basis. Finally, since ε_j is a T_S -eigenvector, it follows that $\varepsilon_j = K_j$ and is of the type in the the statement of the proposition. \square

As a result of the above Proposition the following corollary gives us all intersection points of $S_w \cap C_\alpha$:

Corollary 6.3.2. *Let D_α be a flag domain parametrized by a sequence α and $w \in W_I$ be a perfect harmonic permutation such that $S_w \cap D_\alpha \neq \emptyset$. Then the following algorithm produces us all intersection points of $S_w \cap C_\alpha$:*

If q is even:

- Consider a copy of α denoted by β .
- For any pair $-(2j-1), 2j, 1 \leq j \leq \frac{q}{2}$, in w if the corresponding signature of it in β is $+-$ then replace this $+-$ in β by

$$< e_{2q-2j+1} + ie_{2q-2j+2}, e_{2j-1} - ie_{2j} > \text{ or } < e_{2q-2j+1} - ie_{2q-2j+2}, e_{2j-1} + ie_{2j} >,$$

and if the corresponding signature of it in β is $-+$ then replace this $-+$ in β by

$$< e_{2j-1} + ie_{2j}, e_{2q-2j+1} - ie_{2q-2j+2} > \text{ or } < e_{2j-1} - ie_{2j}, e_{2q-2j+1} + ie_{2q-2j+2} >$$

- For any pair $-(2j-1), -2j, 1 \leq j \leq \frac{q}{2}$, in w if the corresponding signature of it in β is $+-$ then replace this $+-$ in β by

$$< e_{2q-2j+1} + ie_{2q-2j+2}, e_{2j-1} + ie_{2j} > \text{ or } < e_{2q-2j+1} - ie_{2q-2j+2}, e_{2j-1} - ie_{2j} >,$$

and if the corresponding signature of it in β is $-+$ then replace this $-+$ in β by

$$< e_{2j-1} + ie_{2j}, e_{2q-2j+1} + ie_{2q-2j+2} > \text{ or } < e_{2j-1} - ie_{2j}, e_{2q-2j+1} - ie_{2q-2j+2} >$$

- For the remaining numbers, for each $q+1 \leq j \leq n-1$ replace the corresponding $+$ in β by $(e_{2j-1} + ie_{2j})$ and for $\pm n$ replace the corresponding $+$ by $(e_{2n-1} \pm ie_{2n})$.

If q is odd:

- For any pair $-(2j-1), 2j, 1 \leq j \leq \frac{q-1}{2}$, in w if the corresponding signature of it in β is $+-$ then replace this $+-$ in β by

$$< e_{2q-2j+1} + ie_{2q-2j+2}, e_{2j-1} - ie_{2j} > \text{ or } < e_{2q-2j+1} - ie_{2q-2j+2}, e_{2j-1} + ie_{2j} >,$$

and if the corresponding signature of it in β is $-+$ then replace this $-+$ in β by

$$< e_{2j-1} + ie_{2j}, e_{2q-2j+1} - ie_{2q-2j+2} > \text{ or } < e_{2j-1} - ie_{2j}, e_{2q-2j+1} + ie_{2q-2j+2} >$$

- For the number $-q$ replace the corresponding $+$ in β by e_{q+1} .
- For the remaining numbers, for each $q+1 \leq j \leq n-1$, replace the corresponding $+$ in β by $(e_{2j-1} + ie_{2j})$ and for $\pm n$ replace the corresponding $+$ by $(e_{2n-1} \pm ie_{2n})$.

Each point obtained from this algorithm is a point of the intersection of $S_w \cap D_\alpha$.

Theorem 6.3.3. A Schubert variety S_w which is parametrized by a perfect harmonic permutation w intersects $2^{\frac{q}{2}}$ flag domains if q is even and intersects the base cycles of these flag domains in 2^q points. If q is odd, it intersects $2^{\frac{q-1}{2}}$ flag domains and intersects the base cycles of these flag domains in 2^{q-1} points.

Proof. Let $w \in W_I$ be a perfect harmonic permutation. We first show that if q is even, then $S_w \cap C_\alpha$ consists of 2^q points and 2^{q-1} points if q is odd. Since w is a perfect harmonic permutation, we have two cases. In the first case if w contains the pair $(-1 \ 2)$ then the pair $(-1 \ 2)$ sits inside consecutive boxes in w . The goal here is to show that there is exactly 4 possibilities for this pair which can be completed to maximal isotropic flag. We begin by considering the B_I -orbit of $< e_1 - e_{2q} >$. By Remark 6.1.3 the elements in this orbit have the form

$$\beta_1(e_1 + e_{2q}) + \dots + \beta_{2n}(e_1 - e_{2q}).$$

The question is how many 1-dimensional subspaces (spanned by vectors in Proposition 6.3.1) do we have such that these subspaces can be complete to maximal isotropic flag? To compute this number we denote by v_1 the vector we have from the first step which spans the 1-dimensional subspace, in the second step we consider the orbit $B_I \cdot < e_2 + e_{2q-1} >$, which has points of the form

$$\beta_1(e_1 + e_{2q}) + \beta_2(e_2 + e_{2q-1}).$$

The 2-dimension subspace in the flag is spanned by linear combinations of the form

$$v_2 = \beta_1(e_1 + e_{2q}) + \beta_2(e_2 + e_{2q-1}) + \gamma v_1.$$

Note that v_2 should be in E^+ or E^- and of the form stated in proposition 6.3.1, and therefore v_1 should contain the terms e_1 and e_2 or the terms e_{2q-1} and e_{2q} . Thus we have the following 4 possibilities of v_1 as follows: $e_1 - ie_2, e_1 + ie_2, e_{2q-1} + ie_{2q}, e_{2q-1} - ie_{2q}$.

If $v_1 = e_1 \mp ie_2$, then for $\beta_1 = \pm i$, $\beta_2 = 1$, and $\gamma = \mp i$, the vector v_2 is $v_2 = e_{2q-1} \pm ie_{2q}$, so the 2-dimensional subspace corresponding to the pair $(-1 \ 2)$ is spanned by

$$< e_1 \mp ie_2, e_{2q-1} \pm ie_{2q} > .$$

If $v_1 = e_{2q-1} \mp ie_{2q}$, then for $\beta_1 = 1$, $\beta_2 = \pm i$, and $\gamma = \mp i$, it follows that v_2 is $v_2 = e_1 \pm ie_2$. Thus the 2-dimensional subspace corresponding to the pair $(-1 \ 2)$ is spanned by

$$\langle e_{2q-1} \mp ie_{2q}, e_1 \pm ie_2 \rangle.$$

Having constructed the 2-dimensional space we ignore the pair (-12) in w and repeat the same steps for the next pairs step by step, So if w contains the pair $((-2j-1), 2j)$, $1 \leq j \leq \frac{q}{2}$, then in the same way the only possible 2-dimensional subspaces which can be completed to maximal isotropic flags are the subspaces spanned by:

$$\langle e_{2j-1} \mp ie_{2j}, e_{2q-2j+1} \pm ie_{2q-2j+2} \rangle \text{ or } \langle e_{2q-2j+1} \pm ie_{2q-2j+2}, e_{2j-1} \mp ie_{2j} \rangle.$$

The second case is that where w contains the pair $(-1 - 2)$ which sits inside consecutive boxes in w . look at the orbit B_I . $\langle e_1 - e_{2q} \rangle$, then this orbit has points of the form

$$\beta_1(e_1 + e_{2q}) + \dots + \beta_{2n-1}(e_2 - e_{2q-1}) + \beta_{2n}(e_1 - e_{2q}).$$

We also consider the orbit B_I . $\langle e_2 - e_{2q-1} \rangle$ which has points of the form

$$\beta_1(e_1 + e_{2q}) + \dots + \beta_{2n-1}(e_2 - e_{2q-1}).$$

Then, if we have 1-dimensional subspace spanned by v_1 from the first step, the 2-dimensional subspace in our flag spanned by v_1 and a vector v_2 of the form $v_2 = \beta_1(e_1 + e_{2q}) + \beta_2(e_2 + e_{2q-1}) + \beta_3(e_2 - e_{2q-1}) + \gamma v_1$. So we have 4-possibilities of v_1 which can be extend to maximal isotropic flag. These are : $e_1 - ie_2, e_1 + ie_2, e_{2q-1} + ie_{2q}, e_{2q-1} - ie_{2q}$.

If $v_1 = e_1 \mp ie_2$, then for $\beta_1 = \mp i, \beta_2 = 0, \beta_3 = -1$, and $\gamma = \pm i$, the vector v_2 is $v_2 = e_{2q-1} \mp ie_{2q}$. Thus the 2-dimensional subspace corresponding to the pair $(-1 - 2)$ is spanned by

$$\langle e_1 \mp ie_2, e_{2q-1} \mp ie_{2q} \rangle.$$

If $v_1 = e_{2q-1} \mp ie_{2q}$, then for $\beta_1 = 1, \beta_2 = 0, \beta_3 = \mp i$, and $\gamma = \mp i$, the vector v_2 is $v_2 = e_1 \pm ie_2$. As a result the 2-dimensional subspace corresponding to the pair $(-1 - 2)$ is spanned by

$$\langle e_{2q-1} \mp ie_{2q}, e_1 \pm ie_2 \rangle.$$

Having determined the 2-dimensional subspace, we ignore the pair $(-1 - 2)$ from w and repeat the same steps for the next pairs step by step. More generally if w contains the pair $((-2j-1), -2j)$, $1 \leq j \leq \frac{q}{2}$, then by the same method the only possible 2-dimensional subspaces which can be completed to maximal isotropic flag are

$$\langle e_{2j-1} \mp ie_{2j}, e_{2q-2j+1} \mp ie_{2q-2j+2} \rangle \text{ or } \langle e_{2q-2j+1} \mp ie_{2q-2j+2}, e_{2j-1} \mp ie_{2j} \rangle$$

Therefore for each pair of w we have 4-possibilities.

For the remaining numbers, recall that w is a perfect harmonic permutation. In particular all numbers in the remaining boxes sit in an increasing order. Thus the only possible for these which can be completed to a maximal isotropic flag are the following point: If n is positive, the point is the flag associated to the ordered basis

$$e_{2q+1} + ie_{2q+2}, \dots, e_{2n-1} + ie_{2n},$$

and if n is negative, then the point is the flag associated to the ordered basis

$$e_{2q+1} + ie_{2q+2}, \dots, e_{2n-3} + ie_{2n-2}, e_{2n-1} - ie_{2n}.$$

Hence, if q is even, then w contains $\frac{q}{2}$ pairs and each pair has 4-possibilities. Therefore in this case the number of possible intersection points is $4^{\frac{q}{2}}.1 = 2^q$. If q is odd, then w contains $\frac{q-1}{2}$ pairs and each pair has 4-possibilities then the number of possible intersection points is $4^{\frac{q-1}{2}}.1 = 2^{q-1}$.

Finally, we show that the points described above belong to exactly $\frac{q}{2}$ flag domains if q is even and to $\frac{q-1}{2}$ flag domains if q is odd. For this recall that for each pair in w we have 4-possibilities of 2-dimensional subspaces and note that the signature of these subspaces are $+-$ and $-+$, and the signature of the remainder is $++ \dots ++$. Thus the number of flag domains which have non-empty intersection with S_w for the given w is $2^{\frac{q}{2}}$ flag domains if q is even and is $2^{\frac{q-1}{2}}$ flag domains if q is odd.

Also, since each two of these 4-possibilities of 2-dimensional subspaces have the same signature then for any fixed signature for a flag domain there is $2^{\frac{q}{2}}$ points belong to the base cycle of that flag domain if q is even and $2^{\frac{q-1}{2}}$ points belong to the same base cycle of that flag domain if q is odd. \square

Remark 6.3.4. In the case of the group SL_{2n}^C with the real form $SU(p, q)$, Brecan [4] shows that the number of Iwasawa-Schubert varieties which intersect at least one base cycle and has the minimal dimension pq is $(2n-1).(2n-3) \dots (2n-2q+1)$. But for the group $SO(2n, \mathbb{C})$ with real form $SO(p, q)$, if q is even, the number of Schubert varieties S_w which have the minimal dimension $\frac{pq}{2}$ and intersect at least one base cycle is $n.(n-2) \dots (n-q+2)$, and if q is odd, the number of Schubert varieties S_w which have the minimal dimension $\frac{pq-1}{2}$ and intersect at least one base cycle is $(n-2).(n-4) \dots (n-q+1)$.

In the case of $SO(2n+1, \mathbb{C})$ with real form $SO(p, q)$, if q is even, the number of Schubert varieties S_w which have the minimal dimension $\frac{pq}{2}$ and intersect at least one base cycle is $(n-1).(n-3) \dots (n-q+1)$, and if q is odd, the number of Schubert varieties S_w which has the minimal dimension $\frac{pq}{2}$ and intersect at least one base cycle is $(n-2).(n-4) \dots (n-q+1)$. \square

The following remark, which is a consequence of the proof of Theorem 6.3.3, describes all intersection points of S_w with the base cycles C_α .

Remark 6.3.5. To determine all intersection points between the base cycles and the Iwasawa Schubert variety S_w of complimentary dimension we will define a set for each case of q :

If q is even: Let $w \in W_I$ be a perfect harmonic permutation and define

$Swite_w := \{\psi(w_r) : w_r \text{ is obtained from } w \text{ by switching none, some or all pairs}$

$(-(2i-1), 2i) \text{ by } (2i, -(2i-1)) \text{ or } (-2i, 2i-1) \text{ or } ((2i-1), -2i) \text{ and switching none, some or all pairs } (-(2i-1), -2i) \text{ by } (-2i, -(2i-1)) \text{ or } (2i, (2i-1)) \text{ or } ((2i-1), 2i), 1 \leq i \leq \frac{q}{2}\} \subset W_S$. Define $\mathbb{F}_e(Fix T_S)$ to be the set all maximally b -isotropic flags associated to the basis

$$e_1 + ie_2, e_3 + ie_4, \dots, e_{2n-1} + ie_{2n}, e_{2n-1} - ie_{2n}, \dots, e_1 - ie_2$$

Let $M_w \subset \mathbb{F}_e(\text{Fix } T_S)$ be the set of all maximally b -isotropic flags associated to all elements in Swite_w . Note that we have $\frac{q}{2}$ of the pairs $(-(2i-1), 2i)$ and $(-(2i-1), -2i)$, $1 \leq i \leq \frac{q}{2}$ in any $w \in W_I$, and for each pair we have 4 possibilities to switch it, so the cardinality of Swite_w is $4^{\frac{q}{2}} = 2^q$. The set Swite_w gives us all intersection points of S_w and each $2^{\frac{q}{2}}$ of these points belong to only one flag domain where these points of intersection sits in the base cycle of that flag domain.

If q is odd: Let $w \in W_I$ be a perfect harmonic permutation and define $\text{Swito}_w := \{\psi(w_r) : w_r \text{ is obtained from } w \text{ by switching none, some or all pairs } (-(2i-1), 2i) \text{ by } (2i, -(2i-1)) \text{ or } (-2i, 2i-1) \text{ or } ((2i-1), -2i), 1 \leq i \leq \frac{q}{2}\} \subset W_S$. Define $\mathbb{F}_o(\text{Fix } T_S)$ to be the set all maximal b -isotropic flags associated to the basis

$$e_1 + ie_2, e_3 + ie_4, \dots, e_{q-2} + ie_{q-1}, e_{q+2} + ie_{q+3}, \dots, e_{2n-1} + ie_{2n}, e_q, \\ e_{q+1}, e_{2n-1} - ie_{2n}, \dots, e_{q+2} - ie_{q+3}, e_{q-2} - ie_{q-1}, \dots, e_1 - ie_2$$

Let $M_w \subset \mathbb{F}_o(\text{Fix } T_S)$ be the set of all maximally b -isotropic flags associated to all elements in Swito_w . Note that we have $\frac{q-1}{2}$ of the pairs $(-(2i-1), 2i)$, $1 \leq i \leq \frac{q}{2}$ in any $w \in W_I$, and for each pair we have 4 possibilities for switching it. Hence the cardinality of Swito_w is $4^{\frac{q-1}{2}} = 2^{q-1}$. The set Swito_w gives us all intersection points of S_w and each $2^{\frac{q-1}{2}}$ of these points belong to only one flag domain where these points of intersection sits in the base cycle of that flag domain.

Example 6.3.6. In $G_0 = SO(6, 4)$, fix $w = (-35 - 142)$ a perfect harmonic permutation, then $\text{Swite}_w = \{(251 - 3 - 4), (25 - 1 - 34), (-2513 - 4), (-25 - 134), (-3512 - 4), (351 - 2 - 4), (-35 - 124), (35 - 1 - 24), (25 - 4 - 31), (-25 - 431), (254 - 3 - 1), (-2543 - 1), (-35 - 421), (35 - 4 - 21), (-3542 - 1), (354 - 2 - 1)\}$

Example 6.3.7. In $G_0 = SO(5, 3)$, fix $w = (-124 - 3)$ a perfect harmonic permutation, then $\text{Swito}_w = \{(1 - 23 - 4), (-213 - 4), (2 - 13 - 4), (-123 - 4)\}$.

Recall that in the cases of $SP(2n, \mathbb{R})$ and $SO^*(2n)$ every flag domain intersects all Schubert varieties of complementary dimension. But in the case of $SO(p, q)$ we don't have this property except in a very special case. We explain this case in the following example.

Example 6.3.8. If $n = q + 1$, then the flag domain parametrized by the sequence

$$\alpha = + - + - \dots + - + - +$$

intersects all Schubert varieties of dimension $\frac{pq}{2}$ if q is even. And the flag domain parametrized by the sequence

$$\beta = + - + - \dots + - + - + +$$

intersects all Schubert varieties of dimension $\frac{pq-1}{2}$ if q is odd.

Chapter 7

Cycle intersection for $SP(2p, 2q)$

Our work here is devoted to the case of the real form $SP(2p, 2q)$ of $SP(2n, \mathbb{C})$. As in the case $SO(p, q)$, the results here are stated in terms of algorithms (See Definitions 7.1.1 and 7.2.1).

7.1 Conditions for $S_w \cap C_\alpha \neq \emptyset$

Definition 7.1.1. An element $w \in W$ is called **a major permutation** if it satisfies the following conditions:

In the one line notation of the permutation the number $-(2i - 1), 1 \leq i \leq q$, sits in any place to the left of the number $(2i)$ or $(-2i), 1 \leq i \leq q$, or the number $(-2i), 1 \leq i \leq q$, sits in any place to the left of the number $(2i - 1)$ or $-(2i - 1), 1 \leq i \leq q$, and the order of the numbers $2q + i$ or $-(2q + i)$, where $1 \leq i \leq p - q$ is arbitrary.

Example 7.1.2. In $SP(4, 2)$ the permutation $(-125 - 34)$ is a major permutation while $(531 - 2 - 4)$ is not.

The standard basis and the basis (7.1) define two maximal tori for $SP(2n, \mathbb{C})$, and therefore they define two isomorphic Weyl groups. Define W_T to be the Weyl group with respect to the standard basis $\{e_1, \dots, e_{2n}\}$ of \mathbb{C}^{2n} . Let W_I be the Weyl group with respect to the basis

$$\begin{aligned} &e_1 + e_{2q}, e_{2n-2q+1} + e_{2n}, \dots, e_q + e_{q+1}, e_{2n-q} + e_{2n-q+1}, e_{2q+1}, e_{2q+2}, \dots, e_n, e_{n+1}, \\ &e_{n+2}, \dots, e_{2n-q}, e_q - e_{q+1}, e_{2n-q} - e_{2n-q+1}, \dots, e_1 - e_{2q}, e_{2n-2q+1} - e_{2n}. \end{aligned} \quad (7.1)$$

Each of these Weyl groups isomorphic to $S_n \ltimes \mathbb{Z}_2^n$. For later use define the bijective map ψ between W_I and W_S by $\psi(\pm(2i - 1)) = \pm(2q - i + 1)$, $\psi(\pm 2i) = \mp i$ if $1 \leq i \leq q$ and $\psi(\pm i) = \pm i$ if $i > 2q$. Note that $-(2q - i + 1) = 2n - 2q - i + 2$.

Remark 7.1.3. For v_i any basis vector of the type in 7.1 and $b \in B_I$ the form of $b(v_i)$ is given as follows:

- $b.(e_i - e_{2n-i+1}) = \eta_i(e_i + e_{2q-i+1}) + \zeta_i(e_i - e_{2q-i+1}) + B_i = K_i + B_i,$

- $b(e_i + e_{2n-i+1}) = \lambda_i(e_i + e_{2q-i+1}) + A_i,$
- $b(e_{2n-q+i} - e_{2n-i+1}) = \beta_i(e_{2n-q+i} + e_{2n-i+1}) + \delta_i(e_{2n-q+i} - e_{2n-i+1}) + \tilde{B}_i = \tilde{K}_i + \tilde{B}_i,$
- $b(e_{2n-q+i} + e_{2n-i+1}) = \mu_i(e_{2n-q+i} + e_{2n-i+1}) + \tilde{A}_i,$
- $b(e_{2q+j}) = \nu_j(e_{2q+j}) + \tilde{A}_i,$

with $\lambda_i \neq 0, \mu_i \neq 0$ and $\nu_j \neq 0$.

Note that in the above orbits, if b is chosen appropriately, then we can arrange the linear combinations above to have the vectors in the standard basis. \square

Proposition 7.1.4. *If w is a major permutation, then the flag $F_{\psi(w)}$ belongs to the orbit $B_I(F_w)$.*

Proof. let $w \in W_I$ be a major permutation. Denote by

$$\{0\} \subset \langle u_{w_1} \rangle \subset \langle u_{w_1}, u_{w_2} \rangle \subset \dots \subset \langle u_{w_1}, \dots, u_{w_n} \rangle$$

the first n subspaces of $F_w = w(F_I)$, where u_{w_i} is a vector from the basis (7.1). Let $\tilde{w} \in W_T$ be the image of w under the bijection ψ and let $Y_{\tilde{w}}$ be the isotropic flag associated to \tilde{w} such that the first half of $Y_{\tilde{w}}$ is

$$\{0\} \subset \langle e_{\tilde{w}_1} \rangle \subset \langle e_{\tilde{w}_1}, e_{\tilde{w}_2} \rangle \subset \dots \subset \langle e_{\tilde{w}_1}, \dots, e_{\tilde{w}_n} \rangle$$

where $\varepsilon_i = e_{\tilde{w}_i}$ is a vector in the standard basis of \mathbb{C}^{2n} . To show that the flag $Y_{\tilde{w}}$ is an intersection point in $B_I.F_w \cap C_\alpha$ we will construct a b with $b(F_w) = Y_{\tilde{w}}$ by using induction. From the definition of a major permutation, there are three possibilities for w_1 : $w_1 = -(2i-1)$, $w_1 = -2i$, $1 \leq i \leq q$, or $|w_1| > 2q$

Case 1: If $|w_1| > 2q$, then $u_{w_1} = e_{\tilde{w}_1}$. Thus the orbit $B_I. \langle u_{w_1} \rangle$ contains the point $\langle e_{\tilde{w}_1} \rangle$.

Case 2: If $w_1 = -(2i-1)$, $1 \leq i \leq q$, we consider the orbit $B_I. \langle e_{2n-2q-i+2} - e_{2n-i+1} \rangle = B_I. \langle e_{\tilde{w}_1} - e_{2n-i+1} \rangle$. By Remark (7.1.3) above, the orbit $B_I. \langle e_{\tilde{w}_1} - e_{-i} \rangle$ contains a point of the form

$$y = \langle \alpha_1(e_{\tilde{w}_1} - e_{2n-i+1}) + \alpha_2(e_{\tilde{w}_1} + e_{2n-i+1}) \rangle$$

where $\alpha_1 = \pm \alpha_2$. By taking $\alpha_1 = \alpha_2 = \frac{1}{2}$, it follows that $y = \langle e_{\tilde{w}_1} \rangle$.

Case 3: If $w_1 = -2i$, then we consider the orbit $B_I. \langle e_{\tilde{w}_1} - e_{2q-i+1} \rangle$. By Remark (7.1.3) above, the orbit $B_I. \langle e_{\tilde{w}_1} - e_{2q-i+1} \rangle$ contains a point of the form

$$y = \langle \alpha_1(e_{\tilde{w}_1} - e_{2q-i+1}) + \alpha_2(e_{\tilde{w}_1} + e_{2q-i+1}) \rangle$$

where $\alpha_1 = \pm \alpha_2$. By taking $\alpha_1 = \alpha_2 = \frac{1}{2}$, it follows that $y = \langle v_{\tilde{w}_1} \rangle$.

Now apply induction to build the full isotropic flag. Assume that we built the first j -vectors of $b.v_{w_j}$ where $j < n$, which yield the first j isotropic subspaces of the flag,

$$\{0\} \subset \langle e_{\tilde{w}_1} \rangle \subset \langle e_{\tilde{w}_1}, e_{\tilde{w}_2} \rangle \subset \dots \subset \langle e_{\tilde{w}_1}, \dots, e_{\tilde{w}_j} \rangle$$

To construct the $j+1$ -vector of $b.v_{w_{j+1}}$ to obtain the subspace $V_{\tilde{w}_{j+1}}$ we must consider 5-cases:

Case 1: If $|w_{j+1}| > q$, then $w_{j+1} = \tilde{w}_{j+1}$ and $v_{\tilde{w}_{j+1}} = u_{w_{j+1}}$, and consequently the orbit

$B_I. \langle u_{w_{j+1}} \rangle$ contains the point $e_{\tilde{w}_{j+1}}$.

Case 2: If $w_{j+1} = -(2i - 1)$, then we consider the orbit $B_I. \langle e_{\tilde{w}_{j+1}} - e_{2n-i+1} \rangle$. By using Remark (7.1.3) we see that this orbit contains the points of the form

$$y = \langle \alpha_1 e_{\tilde{w}_{j+1}} - e_{2n-i+1} + \alpha_2 e_{\tilde{w}_{j+1}} - e_{2n-i+1} \rangle$$

where $\alpha_1 = \pm \alpha_2$. By taking $\alpha_1 = \alpha_2 = \frac{1}{2}$, it follows that $y = \langle e_{\tilde{w}_{j+1}} \rangle$.

Case 3: If $w_{j+1} = -2i$, then the orbit is $B_I. \langle e_{\tilde{w}_{j+1}} - e_{2q-i+1} \rangle$. In this case the point

$$y = \langle \alpha_1 (e_{\tilde{w}_{j+1}} - e_{2q-i+1}) + \alpha_2 (e_{\tilde{w}_j} - e_{2q-i+1}) \rangle$$

belongs to the orbit $B_I. \langle e_{\tilde{w}_{j+1}} - e_{2q-j+1} \rangle$, and for $\alpha_1 = \alpha_2$ the point is $y = \langle e_{\tilde{w}_{j+1}} \rangle$.

Case 4: If $w_{j+1} = 2i$, then we consider the orbit $B_I. \langle e_{\tilde{w}_{j+1}} + e_{2n-i+1} \rangle$. Since w is major permutation, then the vector $\langle e_{\tilde{w}_{j+1}} \rangle$ is sitting before the vector $\langle e_{\tilde{w}_j} + e_{2n-i+1} \rangle$ in the flag. Thus this orbit contains a point of the form

$$y = \langle \alpha_1 e_{\tilde{w}_j} + \alpha_2 e_{\tilde{w}_{j+1}} + e_{2n-i+1} \rangle$$

where $\alpha_1 = \pm \alpha_2$. By taking $\alpha_2 = -\alpha_1$, it follows that $y = \langle e_{2n-i+1} \rangle = \langle e_{\tilde{w}_{j+1}} \rangle$.

Case 5: If $w_{j+1} = 2i - 1$, then the orbit is $B_I. \langle e_{\tilde{w}_j} - e_{2q-j+1} \rangle$. Since w is a major permutation, the vector $\langle e_{\tilde{w}_{j+1}} \rangle$ sits before the vector $\langle e_{\tilde{w}_j} + e_{2q-i+1} \rangle$ in the flag. Thus this orbit contains a point of the form

$$y = \langle \alpha_1 e_{\tilde{w}_j} + \alpha_2 e_{\tilde{w}_{j+1}} + e_{2q-i+1} \rangle$$

where $\alpha_1 = \pm \alpha_2$. By taking $\alpha_2 = -\alpha_1$, it follows that $y = \langle e_{2q-i+1} \rangle = \langle e_{\tilde{w}_{j+1}} \rangle$. Therefore, the vector $b.v_{w_{j+1}}$ is constructed to obtain the flag

$$\{0\} \subset \langle e_{\tilde{w}_1} \rangle \subset \langle e_{\tilde{w}_1}, e_{\tilde{w}_2} \rangle \subset \dots \subset \langle e_{\tilde{w}_1}, \dots, e_{\tilde{w}_{j+1}} \rangle.$$

Thus by induction we observe that $b \in B_I$ can be constructed with $b(F_w) = Y_{\tilde{w}}$.

□

Theorem 7.1.5. (Major Permutation Theorem). *The following are equivalent*

- (i) w is Major.
- (ii) $B_I(F_w) \cap D_\alpha \neq \emptyset$ for some α .

Under either of these conditions, for every α the intersection $B_I(F_w) \cap C_\alpha$ contains a T_s fixed point.

Proof. (i) \Rightarrow (ii) Comes directly from Proposition 7.1.4.

(ii) \Rightarrow (i) The proof of this part is essentially the same as the proof of the second part of Theorem 6.1.5.

□

7.2 Introduction to the combinatorics

For the remainder of this chapter we only discuss the intersection properties of the Iwasawa-Schubert cells which are of complementary dimension to C_0 . Consider the standard basis $\{e_1, \dots, e_{2n}\}$ of \mathbb{C}^{2n} where $E^- := \langle e_1, \dots, e_q, e_{2n-q+1}, \dots, e_{2n} \rangle$ and $E^+ := \langle e_{q+1}, \dots, e_{2n-q} \rangle$. Since the maximal compact subgroup of $SP(2p, 2q)$ is $\tilde{K}_0 := SP(2q) \times SP(2p)$ corresponding to the decomposition $\mathbb{C}^{2n} = E^- \oplus E^+$, it follows that for a fixed flag domain $D_{a,b}$, the base cycle C_0 is the set

$$C_0 = \{F \in Z : \dim(V_i \cap E^-) = \sum_{j=1}^i a_j \text{ and } \dim(V_i \cap E^+) = \sum_{j=1}^i b_j, 1 \leq i \leq 2n\}.$$

Since $C_0 = \frac{K}{K \cap B_{z_0}}$ where z_0 is a point in a flag domain, the dimension of the base cycle C_0 is $\dim C_0 = p^2 + q^2$, and the dimension of the Iwasawa Schubert variety S_w is given by $\dim S_w = 2pq$.

Definition 7.2.1. A major permutation $w \in S_n \ltimes \mathbb{Z}_2^{n-1}$ is called **perfect major permutation** if it is constructed by the following algorithm:

1. Start with a sequence of n empty boxes which are to be filled in order to construct w .
2. Consider the pairs $(-2j, (2j-1))$ for all $1 \leq j \leq q$, i.e., the pairs are $(-2, 1), (-4, 3), \dots, (-2q, 2q-1)$.
3. Step by step, starting from 1 until q , for each $1 \leq j \leq q$, place the pair $(-2j, (2j-1))$ in any box in w such that the components of this pair sits as close as possible to each other.
4. After all pairs are placed, the numbers $(2q+i)$ for $1 \leq i \leq p-q$, are placed in the remaining spots in strictly increasing order.

Example 7.2.2. If $p = 3$, $q = 2$, then the perfect major permutations are: $(-2 \ 1)(-4 \ 3)5$, $(-2 \ 1)5(-4 \ 3)$, $5(-2 \ 1)(-4 \ 3)$, $(-4 \ 3)(-2 \ 1)5$, $(-4 \ 3)5(-2 \ 1)$, $5(-4 \ 3)(-2 \ 1)$, $(-3 \ -2 \ 1 \ 4)5$, $5(-3 \ -2 \ 1 \ 4)$.

Example 7.2.3. If $p = 10$, $q = 6$, then the element $78(-6 \ 5)(-4 \ 3)(-2 \ 1)5$ is a perfect major permutation while the element $78(-3 \ -4)(-2 \ 1)(-6 \ 5)$ is not perfect major permutation.

Proposition 7.2.4. Every perfect major permutation $w \in S_n \ltimes \mathbb{Z}_2^n$ has length $2pq$.

Proof. The proof of this Proposition is essentially the same as the proof of lemma 6.2.6. If $w \in S_n \ltimes \mathbb{Z}_2^n$ is a perfect major permutation, carried out by a pair $(-2 \ 1)$ sits inside consecutive boxes in w . Our proof here is carried out by induction on the dimension of the flag manifold. First, remove the pair $(-2 \ 1)$ from w to have a new permutation v consisting of the numbers $\{3, 4, \dots, n\}$. Define a function $f : \{3, 4, \dots, n\} \rightarrow \{1, \dots, n-2\}$ by

$f(i) = i - 2$. This is a bijective map which sends v to $\hat{w} \in S_{n-2} \ltimes \mathbb{Z}_2^{n-2}$. But \hat{w} is perfect major permutation in $S_{n-2} \ltimes \mathbb{Z}_2^{n-2}$ and consequently

$$l(\hat{w}) = 2(p-1)(q-1). \quad (7.2)$$

Since $v = f^{-1}(\hat{w})$ it follows that v has the same length as \hat{w} . For the second step put the numbers 12 to the left of v to obtain an element $\tilde{w} = (12)v \in S_n \ltimes \mathbb{Z}_2^n$ with length $l(\hat{w}) = 2(p-1)(q-1)$. To split the sign of 2 we must add $n-1$ to the length in (7.2). Hence we have the element $(1v-2)$ with length $l(\hat{w}) + n-1$. Now return to the original w and define h to be the number of positions to the left of (-2) and $n-h-2$ to be the number of positions to the right of 1. Recall that in w the pair $(-2 \ 1)$ sits inside consecutive boxes, and therefore -2 must cross $n-h-2+1$ positions to stay in the last position. Furthermore, 1 must cross h positions to stay in the first position. Consequently the length of w is

$$l(w) = 2(p-1)(q-1) + n-1 + h + n-h-2+1 = 2pq.$$

□

7.3 Intersection points of Schubert duality

Now we turn to the final step for the case $SP(2p, 2q)$. Let $w \in W_I$ be a perfect major permutation and recall that if $S_w \cap C_\alpha \neq \emptyset$, then, since S_w is of complementary dimension to C_α , it follows that the intersection is just isolated points which are T_S -fixed points. The main goal in the present paragraph is to compute all such intersection points. As in the case of $SO(p, q)$ the argument in the case of $SP(2p, 2q)$ carried out via algorithms.

Proposition 7.3.1. *If w is a perfect major permutation such that $B_I.F_w$ intersects a cycle C_α at a point given by an ε -basis $(\varepsilon_1, \dots, \varepsilon_m)$ of T_S -eigenvectors, then for any such eigenvector ε_k it follows that $\varepsilon_k = e_{q-i+1}$ or e_{2n-q+i} , $1 \leq i \leq q$, or $\varepsilon_k = e_{q+i}$, $1 \leq i \leq 2p$, depending on the dimension m and the signature α .*

Proof. This is a consequence of the following:

1. w is a perfect major permutation and the flag basis is that of $w(F_I)$.
2. The orbit $b.v_{w_i}$ has the following possible forms:
 - $b.v_{w_i} = b.(e_i - e_{2n-i+1}) = \eta_i(e_i + e_{2q-i+1}) + \zeta_i(e_i - e_{2q-i+1}) + B_i = K_i + B_i$,
 - $b.v_{w_i} = b.(e_i + e_{2n-i+1}) = \lambda_i(e_i + e_{2q-i+1}) + A_i$,
 - $b.v_{w_i} = b(e_{2n-q+i} - e_{2n-i+1}) = \beta_i(e_{2n-q+i} + e_{2n-i+1}) + \delta_i(e_{2n-q+i} - e_{2n-i+1}) + \tilde{B}_i = \tilde{K}_i + \tilde{B}_i$,
 - $b.v_{w_i} = b(e_{2n-q+i} + e_{2n-i+1}) = \mu_i(e_{2n-q+i} + e_{2n-i+1}) + \tilde{A}_i$,
 or
 - $b.v_{w_i} = b(e_{2q+j}) = \nu_j(e_{2q+j}) + \tilde{A}_i$.
3. $\lambda_i \neq 0, \mu_i \neq 0, \nu_j \neq 0, \eta_j \neq 0$, and $\beta_i \neq 0$.

4. The intersection $S_w \cap C_\alpha$ is a flag defined by T_S -eigenvectors.

From the expression for v_{w_i} it is immediate that all of the possibilities in the statement occur. Furthermore, since the $\lambda_i, \mu_i, \nu_j, \eta_j$, and β_i are non-zero, for every i a non-zero contribution from K_i occurs in the sum

$$v_i = \sum_{k \leq i} c_{ik} b.v_{w_k}.$$

Since for each case in item (2) above, the vectors e_i, e_{2n-i+1} do not occur in $b.v_{w_k}$ for $k < j$ in the first and second lines in item (2), and the vectors e_{2n-q+i}, e_{2n-i+1} do not occur in $b.v_{w_k}$ for $k < j$ in the third and fourth lines in item (2), and the vector e_{2q+j} do not occur in $b.v_{w_k}$ for $k < j$ in the last line in item (2) above. It follows that $\varepsilon_i = K_i + E_i$ in the standard basis. Finally, since ε_i is a T_S -eigenvector, it follows that $\varepsilon_i = K_i$ and is of the type in the statement of the proposition. \square

As a result of the above Proposition the following algorithm gives us all intersection points of $S_w \cap C_\alpha$:

Corollary 7.3.2. *Let D_α be a flag domain parametrized by a sequence α and $w \in W_I$ be a perfect major permutation such that $S_w \cap D_\alpha \neq \emptyset$. Then the following algorithm produces us all intersection points of $S_w \cap C_\alpha$:*

- Consider a copy of α denoted by β .
- For any pair $(-2j, (2j-1)), 1 \leq j \leq q$, in w if the corresponding signature of it in β is $+-$, then replace the pair $+-$ in β by

$$< e_{2q-j+1}, e_j > \text{ or } < e_{2n-2q+j}, e_{2n-j+1} >,$$

and if the corresponding signature of it in β is $-+$, then replace the pair $-+$ in β by

$$< e_j, e_{2q-j+1} > \text{ or } < e_{2n-j+1}, e_{2n-2q+j} >.$$

- For the remaining numbers, for each $2q+1 \leq j \leq n-1$, replace the corresponding $+$ in β by (e_j) .

Theorem 7.3.3. *A Schubert variety S_w which parametrized by a perfect major permutation w has non-empty intersection with 2^q flag domains and the intersections with the base cycles of these flag domains consist of 2^{2q} points.*

Proof. As in the proof of Theorem 6.3.3 one shows that for each pair of the form $(-2j, (2j-1))$, for all $1 \leq j \leq q$, there are 4-possibilities for the vectors: 2-possibilities with signature sign pair $(-+)$ and 2-possibilities with signature sign pair $(+-)$. Since we have q pairs, the Schubert variety S_w intersects 2^q flag domains in 2^{2q} points. \square

Remark 7.3.4. To describe the above intersection points between the base cycles and the Iwasawa Schubert variety S_w of complimentary dimension we will define a set $Swit_w$ as follows:

For a perfect major permutation $w \in S_n \rtimes \mathbb{Z}_2^n$, define $Swit_w := \{\varphi(w_r) : w_r \text{ is obtained from } w \text{ by switching none, some, all pairs } (-2i, (2i-1)) \text{ by } ((2i-1), -2i) \text{ or } (-(2i-1), 2i) \text{ or } (2i, -(2i-1)), 1 \leq i \leq q\} \subset W_T$. Define $\mathbb{F}(Fix I)$ to be the set all maximal b-isotropic flags associated to the basis

$$e_1 + e_{2q}, e_{2n-2q+1} + e_{2n}, \dots, e_q + e_{q+1}, e_{2n-q} + e_{2n-q+1}, e_{2q+1}, e_{2q+2}, \dots, e_n, e_{n+1}, \\ e_{n+2}, \dots, e_{2n-q}, e_q - e_{q+1}, e_{2n-q} - e_{2n-q+1}, \dots, e_1 - e_{2q}, e_{2n-2q+1} - e_{2n}.$$

Let $F_w \subset \mathbb{F}(Fix T)$ be the set of all maximally b -isotropic flags associated to all elements in $Swit_w$. Note that for any perfect major permutation w we have q pairs $(-2i, (2i-1))$, $1 \leq i \leq q$ for any $w \in W_T$, and for each pair we have 4 possibilities for switching it. Hence the cardinality of $Swit_w$ is $4^q = 2^{2q}$. The set $Swit_w$ gives us all intersection points of S_w and each 2^q of these points belong to only one flag domain and are contained in the base cycle of that flag domain.

Example 7.3.5. In $G_0 = SP(6, 4)$, fix $w = (-43 - 215)$ a perfect major permutation, then $Swit_w = \{(-34 - 215), (3 - 4 - 215), (4 - 3 - 215), (-43 - 215), (-432 - 15), (-43 - 125), (-431 - 25), (4 - 3 - 215), (4 - 32 - 15), (4 - 3 - 125), (4 - 31 - 25), (-34 - 215), (-342 - 15), (-34 - 125), (-341 - 25), (3 - 4 - 215), (3 - 42 - 15), (3 - 4 - 125), (3 - 41 - 25)\}$

Remark 7.3.6. For the real form $SP(2p, 2q)$, the number of Schubert varieties S_w which parametrized by perfect major permutations is

$$(n-1).(n-3) \dots (n-2q+1).$$

Remark 7.3.7. If $n = q + 1$, then the flag domain parametrized by the sequence

$$\alpha = + - + - \dots + - + - +$$

intersects all Schubert varieties of dimension $2pq$.

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