Coded Caching Schemes for Flexible Memory Sizes

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Abstract

Coded caching scheme has recently become quite popular in the wireless network due to the efficiency of reducing the load during peak traffic times. Recently the most concern is the problem of subpacketization level in a coded caching scheme. Although there are several classes of constructions, these schemes only apply to some individual cases for the memory size. And in the practice it is very crucial to consider any memory size. In this paper, four classes of new schemes with the wide range of memory size are constructed. And through the performance analyses, our new scheme can significantly reduce the level of subpacketization by decreasing a little efficiency of transmission in the peak traffic times. Moreover some schemes satisfy that the packet number is polynomial or linear with the number of users.

Index Terms

Coded caching scheme, rate, packet number, flexible memory size, placement delivery array

I. Introduction

Recently, the explosive increasing mobile services, especially applications such as video streaming, have imposed a tremendous pressure on the data transmission over the core network [1]. As a result, during the peak-traffic times, the communication systems are usually congested. Caching system, which proactively caches some contents at the network edge during off-peak hours, is a promising solution to reducing congestion (see [2], [6], [7], [9], [12], and references therein).

Maddah-Ali and Niesen in [12] proposed a coded caching approach based on network coding theory. This approach can effectively further reduce congestion during the peak-traffic times, and now is a hot topic in industrial and academic fields. In the coded caching system, [12] focused on the following scenario: a single server containing N files with the same length connects to K users over a shared link and each user has a cache memory of size M files. A coded caching scheme consists of two phases: a placement phase during off-peak times and a delivery phase during peak times. In the placement phase, the user caches are populated. This phase does not depend on the user demands which are assumed to be arbitrary. In delivery phase, each user requires a file from server. Then server sends a coded signal of at most R files to the users such that various user demands are satisfied with the help the local caches. It is meaningful to minimize the load R files in the delivery phase. Here R is always called the rate. The first determined scheme for a (K, M, N) coded caching system is proposed by Maddah-Ali and Niesen in [12]. Such a scheme is referred to as MN scheme in this paper. According to an elaborate uncoded placement and a coded delivery phases, the (K, M, N) MN scheme has the rate $R = K(1 - \frac{M}{N})\frac{1}{1+\frac{KM}{N}}$. By means of graph theory, reference [22] showed that MN scheme has minimum delivery rate under the constraint of uncoded cache placement when $K \leq N$. So far, MN scheme has been also extensively employed in practical scenarios, such as decentralized version [13], device to device networks [8], online caching update [14], [25] and hierarchical networks [10], [28] and so on. And many results have been obtained following [12], for instances, [3]–[5], [15], [21], [23], [26] etc.

A coded caching scheme is called F division scheme if each file is split into F packets. In MN scheme, each file is split into $\binom{K}{KM/N}$ packets where KM/N is an integer. Clearly F increases exponentially with the number of users K. This would

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become infeasible when K is large. Furthermore, the complexity of a coded caching scheme increases with the parameter F. So minimizing the packets number F plays an important role in the field of coded caching scheme. Shanmugam et al. in [17] first discussed such subpacketization problem in a coded caching scheme. Very recently, Yan et al. in [24] characterized an F division (K, M, N) caching scheme by a simple array which is called (K, F, Z, S) placement delivery array (PDA), where $M/N = Z/F$ and $R = S/F$. Then they proved that MN scheme is equivalent to a special PDA which is referred as to MN PDA. Furthermore, by increasing little delivery rate, they obtained two classes coded caching scheme by means of constructing two infinite classes of PDAs such that F reduces significantly comparing with that of MN scheme. Inspired by the concept of a PDA, some constructions with lower level subpacketization were proposed in [16], [18], [24] from the view point of hypergraphs, Ruzsa-Szemerédi graphs, bipartite graphes respectively. For ease of exposition, we list the previously known results on the schemes in the following table, Table I which is summarised by Shangguang et al. in [16].

TABLE I: Summary of some known results

	K	M/N	F	R
MN PDA [12]	K	$\frac{z}{q}, z = 1, \ldots q-1$	$\binom{K}{Kz/q}$	$\frac{K(q-z)}{zK+q}$
Construction in [24]	$(m+1)q$	$\frac{1}{q}$	q^m	$q-1$
Construction in [24]	$(m+1)q$	$\frac{q-1}{q}$	$(q-1)q^m$	$\frac{1}{q-1}$
Construction in [16]	$\binom{m}{t}q^t$	$1 - (\frac{q-1}{q})^t$	q^m	$(q-1)^t$
Construction in [16]	$\binom{m}{t}q^t$	$1 - \frac{1}{q^t}$	$(q-1)^t q^m$	$\frac{1}{(q-1)^t}$
Construction in [18]	$F^{\Omega(\frac{1}{\log\log F})}$	$\frac{2}{3} + o(1)$	F	1

From Table I, we can see that for fixed K, these constructions only apply to few values of $\frac{M}{N}$, i.e., there only exists the schemes with $M/N = \frac{1}{q^t}$, $1 - (\frac{q-1}{q})^t$ and $\frac{2}{3} + o(1)$. However M/N may be any value of $(0, 1)$ in the practice.

In this paper, according to the different level subpacketization in [24] and [16], we will propose four classes of coded caching schemes with flexible parameters M/N. And we show that our new schemes include all the results listed in Table I as special cases except MN PDA and the construction in [18]. Furthermore, our constructions propose many schemes with the same K, M/N and different packet number F.

The rest of this paper is organized as follows. Section II introduces system model and backgrounds for coded caching system. In Section III, two classes of new PDAs are constructed and the corresponding performance is proposed. In order to further reduce the packet number, another two constructions are proposed in Section IV. Finally conclusion is drawn in Section V.

II. System Model and Backgrounds

In this paper, we use bold capital letter, bold lower case letter and curlicue letter to denote array, vector and set respectively. We use $[a, b] = \{a, a+1, \ldots, b\}$ and $[a, b) = \{a, a+1, \ldots, b-1\}$ for intervals of integers for any integers a and b with $a \leq b$.

A. System Model

In this paper we focus on the caching system proposed by Maddah-Ali and Niesen in [12] which is widely studied recently. A caching system consisting of a server containing N files, say $\mathcal{W} = \{W_1, W_2, \cdots, W_N\}$, with the same length connects to K users $\mathcal{K} = \{1, 2, \cdots, K\}$ over a shared link. Additionally, each user has a cache memory of size M files. We call such a system a (K, M, N) caching system. An F-division (K, M, N) coded caching scheme operates in two separated phases:

1) **Placement Phase:** Each file is sub-divided into F equal packets[1], i.e., $W_i = \{W_{i,j} : j = 1, 2, \cdots, F\}$. Each user is accessible to the files set $\mathcal{W}$. And denote $\mathcal{Z}_k$ the packets subset of $\mathcal{W}$ cached by user k.

2) **Delivery Phase:** Each user requests one file from $\mathcal{W}$ randomly. Denote the request by $\mathbf{d} = (d_1, d_2, \cdots, d_K)$, i.e., user k requests file W_{d_k}, where $k \in \mathcal{K}, d_k \in \{1, 2, \cdots, N\}$. Once the server receives the request $\mathbf{d}$, it broadcasts at most RF packets to users such that each user is able to recover its requested file.

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[1]Memory sharing technique may lead to non equally divided packets [12], in this paper, we will not discuss this case.
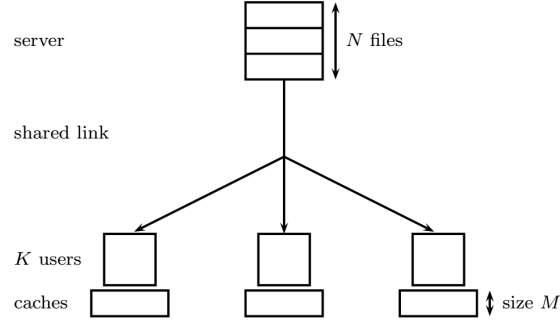
Fig. 1: A central server with N popular files connected to K users through an error-free shared link. In the figure, $N = K = 3$, $M = 1$.

Since the placement phase is carried out without any knowledge of the user requests, it is interested in studying the rate R, which can be satisfied for all possible combinations of user requests, in the delivery phase. And we prefer it as small as possible for a given F-division (K, M, N) caching scheme since it represents the efficiency of a caching scheme. Maddah-Ali and Niesen in [12] proposed the first determined coded caching scheme for a (K, M, N) caching system when KM/N is an integer.

However, when K is large the packet number in MN scheme is too large to be feasible in practice. Furthermore, the complexity of a coded caching scheme increases with the packet number. So minimizing the packets number plays an important role in the field of coded caching scheme. Shanmugam et al. in [17] first discussed such subpacketization problem in a coded caching scheme. Recently Yan et al., in [24] proposed an interesting and simple combinatorial structure, called placement delivery array, to further study the packet number of the scheme.

B. Placement Delivery Array

Definition 1: ([24]) For positive integers K, F, Z and S, an $F \times K$ array $\mathbf{P} = (p_{j,k})$, $0 \leq j < F, 0 \leq k < K$, composed of a specific symbol "$*$" and S positive integers $1, 2, \cdots, S$, is called a (K, F, Z, S) placement delivery array (PDA) if it satisfies the following conditions:

C1. The symbol "$*$" appears Z times in each column;
C2. Each integer occurs at least once in the array;
C3. For any two distinct entries p_{j_1,k_1} and p_{j_2,k_2}, $p_{j_1,k_1} = p_{j_2,k_2} = s$ is an integer only if
   a. $j_1 \neq j_2$, $k_1 \neq k_2$, i.e., they lie in distinct rows and distinct columns; and
   b. $p_{j_1,k_2} = p_{j_2,k_1} = *$, i.e., the corresponding 2×2 subarray formed by rows j_1, j_2 and columns k_1, k_2 must be of the following form

$$\begin{pmatrix} s & * \\ * & s \end{pmatrix} \text{ or } \begin{pmatrix} * & s \\ s & * \end{pmatrix}$$

Given a (K, F, Z, S) PDA $\mathbf{P} = (p_{j,k})$, an F-division (K, M, N) caching scheme with $M/N = Z/F$ and $R = S/F$ can be obtained by Algorithm 1. So the following fundamental relationship between caching scheme and PDA was obtained in reference [24].

Theorem 1: ([24]) An F-division caching scheme for a (K, M, N) caching system can be realized by a (K, F, Z, S) PDA with $Z/F = M/N$. Each user can decode his requested file correctly for any request $\mathbf{d}$ at the rate $R = S/F$.

From Theorem 1, if we want to design a coded caching scheme with good performance by means of a PDA, it would be preferred to construct a PDA with $R = S/F$ and F as small as possible for given positive integers K, M/N. There are several constructions on PDAs with low packet numbers and small rates, which are listed in Table I. However from Table I, given a fixed K these previously known constructions only hold for few values of M/N. In the practice the value M/N could be any value of $(0, 1)$. So it is meaningful to construct good PDAs with flexible parameters M/N for fixed parameters K.

Algorithm 1 caching scheme based on PDA in [24]

1: **procedure** PLACEMENT($\mathbf{P}$, $\mathcal{W}$)
2:     Split each file $W_i \in \mathcal{W}$ into F packets, i.e., $W_i = \{W_{i,j} : j = 1, 2, \cdots, F\}$.
3:     **for** $k \in \mathcal{K}$ **do**
4:         $\mathcal{Z}_k \leftarrow \{W_{i,j} : p_{j,k} = *, \forall\, i = 1, 2, \cdots, N\}$
5:     **end for**
6: **end procedure**
7: **procedure** DELIVERY($\mathbf{P}, \mathcal{W}, \mathbf{d}$)
8:     **for** $s = 1, 2, \cdots, S$ **do**
9:         Server sends $\bigoplus_{p_{j,k}=s, 1 \le j \le F, 1 \le k \le K} W_{d_k,j}$.
10:     **end for**
11: **end procedure**

III. NEW CONSTRUCTIONS

In this section, we will construct two classes of PDAs with $M/N = 1 - (\frac{q-z}{q})^t$ for any positive integers t, q and z with $1 \le z < q$. By the way the constructions in [16] and [24] listed in Table I are just special cases of ours. Then the performance analyses of our constructions are proposed.

A. Constructions

For ease of understanding, we will use q-ary sequences to represent the parameters K, F, Z and S of a PDA unless otherwise stated.

Construction 1: For any positive integers q, z, m and t with $0 < z < q$ and $0 < t < m$, let

$$\mathcal{F} = \{\mathbf{a} = (a_0, a_1, \ldots, a_{m-1}, \varepsilon_0, \ldots, \varepsilon_{t-1}) \mid a_0, \ldots, a_{m-1} \in \mathbb{Z}_q, \varepsilon_0, \ldots, \varepsilon_{t-1} \in [0, \lfloor \frac{q-1}{q-z} \rfloor)\},$$

$$\mathcal{K} = \{\mathbf{b} = (b_0, b_1, \ldots, b_{t-1}, \delta_0, \delta_1, \ldots, \delta_{t-1}) \mid b_0, \ldots, b_{t-1} \in \mathbb{Z}_q, 0 \le \delta_0 < \ldots < \delta_{t-1} < m, \}. \tag{1}$$

Then a $q^m \lfloor \frac{q-1}{q-z} \rfloor^t \times \binom{m}{t} q^t$ array $\mathbf{P} = (p_{\mathbf{a},\mathbf{b}})$ can be defined in the following way

$$p_{\mathbf{a},\mathbf{b}} = \begin{cases} (a_0, a_1, \ldots, b_i - \varepsilon_i(q-z), \ldots, a_{m-1}, a_{\delta_0} - b_0 - 1, \ldots, a_{\delta_{t-1}} - b_{t-1} - 1)_q & \text{if } a_{\delta_i} \notin X_{b_i,z}, \forall i \in [0, t) \\ * & \text{otherwise} \end{cases} \tag{2}$$

where $X_{b_i,z} = \{b_i, b_i - 1, \ldots, b_i - (z-1)\}$. Here the operations are performed modulo q.

Example 1: Assume that $m = 2$, $q = 3$ and $t = 1$. Let us consider $z = 1$ and 2 respectively.

- When $z = 1$, $\lfloor \frac{q-1}{q-z} \rfloor = 1$ holds. And (1) can be written as follows.

$$\mathcal{F} = \{(a_0, a_1, 0) \mid a_0, a_1 \in \mathbb{Z}_3\} \quad \text{and} \quad \mathcal{K} = \{(b, \delta) \mid b \in \mathbb{Z}_3, 0 \le \delta < 1\}$$

By (2), the following array can be obtained.

$(a_0, a_1, 0) \backslash (b, \delta)$	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$
$(0,0,0)$	$*$	$(1,0,1)$	$(2,0,0)$	$*$	$(0,1,1)$	$(0,2,0)$
$(1,0,0)$	$(0,0,0)$	$*$	$(2,0,1)$	$*$	$(1,1,1)$	$(1,2,0)$
$(2,0,0)$	$(0,0,1)$	$(1,0,0)$	$*$	$*$	$(2,1,1)$	$(2,2,0)$
$(0,1,0)$	$*$	$(1,1,1)$	$(2,1,0)$	$(0,0,0)$	$*$	$(0,2,1)$
$(1,1,0)$	$(0,1,0)$	$*$	$(2,1,1)$	$(1,0,0)$	$*$	$(1,2,1)$
$(2,1,0)$	$(0,1,1)$	$(1,1,0)$	$*$	$(2,0,0)$	$*$	$(2,2,1)$
$(0,2,0)$	$*$	$(1,2,1)$	$(2,2,0)$	$(0,0,1)$	$(0,1,0)$	$*$
$(1,2,0)$	$(0,2,0)$	$*$	$(2,2,1)$	$(1,0,1)$	$(1,1,0)$	$*$
$(2,2,0)$	$(0,2,1)$	$(1,2,0)$	$*$	$(2,0,1)$	$(2,1,0)$	$*$

- When $z = 2$, $\lfloor \frac{q-1}{q-z} \rfloor = 2$ holds. And (1) can be written as follows.

$$\mathcal{F} = \{(a_0, a_1, \varepsilon) \mid a_0, a_1 \in \mathbb{Z}_3, \varepsilon \in [0, 1]\} \quad \text{and} \quad \mathcal{K} = \{(b, \delta) \mid b \in \mathbb{Z}_3, 0 \le \delta \le 1\}$$

By (2), the following array can be obtained.

$(a_0,a_1,\varepsilon)\backslash(b,\delta)$	(0,0)	(1,0)	(2,0)	(0,1)	(1,1)	(2,1)
(0,0,0)	*	*	(2,0,0)	*	*	(0,2,0)
(1,0,0)	(0,0,0)	*	*	*	*	(1,2,0)
(2,0,0)	*	(1,0,0)	*	*	*	(2,2,0)
(0,1,0)	*	*	(2,1,0)	(0,0,0)	*	*
(1,1,0)	(0,1,0)	*	*	(1,0,0)	*	*
(2,1,0)	*	(1,1,0)	*	(2,0,0)	*	*
(0,2,0)	*	*	(2,2,0)	*	(0,1,0)	*
(1,2,0)	(0,2,0)	*	*	*	(1,1,0)	*
(2,2,0)	*	(1,2,0)	*	*	(2,1,0)	*
(0,0,1)	*	*	(1,0,0)	*	*	(0,1,0)
(1,0,1)	(2,0,0)	*	*	*	*	(1,1,0)
(2,0,1)	*	(0,0,0)	*	*	*	(2,1,0)
(0,1,1)	*	*	(1,1,0)	(0,2,0)	*	*
(1,1,1)	(2,1,0)	*	*	(1,2,0)	*	*
(2,1,1)	*	(0,1,0)	*	(2,2,0)	*	*
(0,2,1)	*	*	(1,2,0)	*	(0,0,0)	*
(1,2,1)	(2,2,0)	*	*	*	(1,0,0)	*
(2,2,1)	*	(0,2,0)	*	*	(2,0,0)	*

It is easy to check that the above two arrays are $(6,9,3,18)$ PDA and $(6,18,12,9)$ PDA respectively. In fact it hods for any parameters m, q, t and z with $1 \le t < m$, $1 \le z < q$.

Theorem 2: For any positive integers q, z, m and t with $q \ge 2$, $z < q$ and $t < m$, the array $\mathbf{P}$ given in Construction 1 is an $(\binom{m}{t}q^t, \lfloor\frac{q-1}{q-z}\rfloor^t q^m, \lfloor\frac{q-1}{q-z}\rfloor^t(q^m - q^{m-t}(q-z)^t), (q-z)^t q^m)$ PDA with $M/N = 1-(\frac{q-z}{q})^t$ and rate $R = (q-z)^t/\lfloor\frac{q-1}{q-z}\rfloor^t$.

The proof of Theorem 2 is included in Appendix A. In fact, we can add another q columns in $\mathbf{P}$ generated by Construction 1 when $t = 1$.

Construction 2: When $t = 1$, (1) can be written as

$$\mathcal{F} = \{(a_0, a_1, \ldots, a_{m-1}, \varepsilon) \mid a_0, \ldots, a_{m-1} \in \mathbb{Z}_q, \varepsilon \in [0, \lfloor\frac{q-1}{q-z}\rfloor)\} \quad \text{and} \quad \mathcal{K} = \{(b,\delta) \mid b \in \mathbb{Z}_q, 0 \le \delta < m\}$$

for any positive integers q, z and m with $z < q$. Let $\mathcal{K}_1 = \{(b, m) \mid b \in \mathbb{Z}_q\}$. Define an array $\mathbf{H} = (\mathbf{P}, \mathbf{E})$ where

- $\mathbf{P} = (p_{\mathbf{a},\mathbf{b}})$, $\mathbf{a} \in \mathcal{F}, \mathbf{b} \in \mathcal{K}$, is the $q^m\lfloor\frac{q-1}{q-z}\rfloor \times mq$ array generated by Construction 1.
- $\mathbf{E} = (e_{\mathbf{a},\mathbf{b}})$, $\mathbf{a} \in \mathcal{F}, \mathbf{b} \in \mathcal{K}_1$ is a $q^m\lfloor\frac{q-1}{q-z}\rfloor \times q$ array define in the following way.

$$e_{\mathbf{a},\mathbf{b}} = \begin{cases} (a_0, \cdots, a_{m-1}, b - (\sum_{l=0}^{m-1} a_l - \varepsilon(q-z)) - 1)_q & \text{if } \sum_{l=0}^{m-1} a_l - \varepsilon(q-z) \notin Y_{b,z} \\ * & \text{otherwise} \end{cases} \quad (3)$$

where $Y_{b,z} = \{b, b+1, \ldots, b+(z-1)\}$. All the above operations are performed modulo q.

Example 2: We also use the parameters in Example 5, i.e., $m = 2$, $q = 3$, $t = 1$ and $z = 1$, 2. Then we have $\mathcal{K}_1 = \{(b, 2) \mid b \in \mathbb{Z}_3\}$. By (2) and (3), the following two arrays can be obtained respectively.

$(a_0,a_1,0)\backslash(b,\delta)$	(0,0)	(1,0)	(2,0)	(0,1)	(1,1)	(2,1)	(0,2)	(1,2)	(2,2)
(0,0,0)	*	(1,0,1)	(2,0,0)	*	(0,1,1)	(0,2,0)	*	(0,0,0)	(0,0,1)
(1,0,0)	(0,0,0)	*	(2,0,1)	*	(1,1,1)	(1,2,0)	(1,0,1)	*	(1,0,0)
(2,0,0)	(0,0,1)	(1,0,0)	*	*	(2,1,1)	(2,2,0)	(2,0,0)	(2,0,1)	*
(0,1,0)	*	(1,1,1)	(2,1,0)	(0,0,0)	*	(0,2,1)	(0,1,1)	*	(0,1,0)
(1,1,0)	(0,1,0)	*	(2,1,1)	(1,0,0)	*	(1,2,1)	(1,1,0)	(1,1,1)	*
(2,1,0)	(0,1,1)	(1,1,0)	*	(2,0,0)	*	(2,2,1)	*	(2,1,0)	(2,1,1)
(0,2,0)	*	(1,2,1)	(2,2,0)	(0,0,1)	(0,1,0)	*	(0,2,0)	(0,2,1)	*
(1,2,0)	(0,2,0)	*	(2,2,1)	(1,0,1)	(1,1,0)	*	*	(1,2,0)	(1,2,1)
(2,2,0)	(0,2,1)	(1,2,0)	*	(2,0,1)	(2,1,0)	*	(2,2,1)	*	(2,2,0)

$(a_0,a_1,\varepsilon)\backslash(b,\delta)$	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(0,2)$	$(1,2)$	$(2,2)$
$(0,0,0)$	$*$	$*$	$(2,0,0)$	$*$	$*$	$(0,2,0)$	$*$	$(0,0,0)$	$*$
$(1,0,0)$	$(0,0,0)$	$*$	$*$	$*$	$*$	$(1,2,0)$	$*$	$*$	$(1,0,0)$
$(2,0,0)$	$*$	$(1,0,0)$	$*$	$*$	$*$	$(2,2,0)$	$(2,0,0)$	$*$	$*$
$(0,1,0)$	$*$	$*$	$(2,1,0)$	$(0,0,0)$	$*$	$*$	$*$	$*$	$(0,1,0)$
$(1,1,0)$	$(0,1,0)$	$*$	$*$	$(1,0,0)$	$*$	$*$	$(1,1,0)$	$*$	$*$
$(2,1,0)$	$*$	$(1,1,0)$	$*$	$(2,0,0)$	$*$	$*$	$*$	$(2,1,0)$	$*$
$(0,2,0)$	$*$	$*$	$(2,2,0)$	$*$	$(0,1,0)$	$*$	$(0,2,0)$	$*$	$*$
$(1,2,0)$	$(0,2,0)$	$*$	$*$	$*$	$(1,1,0)$	$*$	$*$	$(1,2,0)$	$*$
$(2,2,0)$	$*$	$(1,2,0)$	$*$	$*$	$(2,1,0)$	$*$	$*$	$*$	$(2,2,0)$
$(0,0,1)$	$*$	$*$	$(1,0,0)$	$*$	$*$	$(0,1,0)$	$(0,0,0)$	$*$	$*$
$(1,0,1)$	$(2,0,0)$	$*$	$*$	$*$	$*$	$(1,1,0)$	$*$	$(1,0,0)$	$*$
$(2,0,1)$	$*$	$(0,0,0)$	$*$	$*$	$*$	$(2,1,0)$	$*$	$*$	$(2,0,0)$
$(0,1,1)$	$*$	$*$	$(1,1,0)$	$(0,2,0)$	$*$	$*$	$*$	$(0,1,0)$	$*$
$(1,1,1)$	$(2,1,0)$	$*$	$*$	$(1,2,0)$	$*$	$*$	$*$	$*$	$(1,1,0)$
$(2,1,1)$	$*$	$(0,1,0)$	$*$	$(2,2,0)$	$*$	$*$	$(2,1,0)$	$*$	$*$
$(0,2,1)$	$*$	$*$	$(1,2,0)$	$*$	$(0,0,0)$	$*$	$*$	$*$	$(0,2,0)$
$(1,2,1)$	$(2,2,0)$	$*$	$*$	$*$	$(1,0,0)$	$*$	$(1,2,0)$	$*$	$*$
$(2,2,1)$	$*$	$(0,2,0)$	$*$	$*$	$(2,0,0)$	$*$	$*$	$(2,2,0)$	$*$

Similarly we can check that the above two arrays are $(9,9,3,18)$ PDA and $(9,18,12,9)$ PDA respectively, and obtain the following result.

Theorem 3: For any positive integers q, z, m with $q \geq 2$ and $z < q$, the array **H** generated by Construction 2 is a $((m+1)q, \lfloor\frac{q-1}{q-z}\rfloor q^m, z\lfloor\frac{q-1}{q-z}\rfloor q^{m-1}, (q-z)q^m)$ PDA with $M/N = z/q$ and rate $R = (q-z)/\lfloor\frac{q-1}{q-z}\rfloor$.

The proof of Theorem 3 is included in Appendix B.

B. Performance analyses

From Theorems 1, 2 and 3, the following schemes in Table II can be obtained.

TABLE II: New constructions with $M/N = 1 - (\frac{q-z}{q})^t$, $z = 1, 2, \ldots q-1$

	K	M/N	F	R
Theorem 2	$\binom{m}{t}q^t$	$1 - (\frac{q-z}{q})^t$, $z = 1, \ldots q-1$	$\lfloor\frac{q-1}{q-z}\rfloor^t q^m$	$\left(\frac{q-z}{\lfloor\frac{q-1}{q-z}\rfloor}\right)^t$
Theorem 3	$(m+1)q$	$\frac{z}{q}$, $z = 1, \ldots q-1$	$\lfloor\frac{q-1}{q-z}\rfloor q^m$	$\frac{q-z}{\lfloor\frac{q-1}{q-z}\rfloor}$

Based on Table II, let us consider the value of t in the following.

1) $t > 1$: When $z = 1$ and $q - 1$, from Theorem 2 we have

- a $(\binom{m}{t}q^t, q^m, q^m - q^{m-t}(q-1)^t, (q-1)^t q^m)$ PDA with $M/N = 1 - (\frac{q-1}{q})^t$ and rate $R = (q-1)^t$ and
- a $(\binom{m}{t}q^t, (q-1)^t q^m, (q-1)^t(q^m - q^{m-t}), q^m)$ PDA with $M/N = 1 - \frac{1}{q^t}$ and rate $R = \frac{1}{(q-1)^t}$.

Clearly these two PDAs are exactly the constructions in [16] listed in Table I. Now let us consider the case $1 < z < q - 1$. Similar to the discussion in [16], by means of the inequality $(m/t)^t < \binom{m}{t} < (em/t)^t$, we can estimate the value of m by K, i.e.,

$$\frac{tK^{\frac{1}{t}}}{eq} < m < \frac{tK^{\frac{1}{t}}}{q}.$$

So we have $F = \mathcal{O}(q^{\frac{tK^{1/t}}{q}})$. This implies that F grows sub-exponentially with K if $t \geq 2$.

2) $t = 1$: From Table II when $z = 1$, $q - 1$, from Theorem 3 we have

- a $((m+1)q, q^m, q^{m-1}, (q-1)q^m)$ PDA with $M/N = 1/q$ and $R = q - 1$, and
- a $((m+1)q, (q-1)q^m, (q-1)^2 q^{m-1}, q^m)$ PDA with $M/N = 1 - 1/q$ and rate $R = 1/(q-1)$.

Clearly these two PDAs are exactly the constructions in [24] listed in Table I. The authors in [24] showed that comparing with MN PDA, the packet number F of these two PDAs reduce significantly while the rate R increase little. When $z > 1$ we claim that the performance of PDAs in Theorem 3 is also well. First the following statements cited from [24] are very useful.

Lemma 1: ([24]) For fixed rational number $M/N \in (0,1)$, let $K \in \mathbb{N}^+$ such that $KM/N \in \mathbb{N}^+$, when $K \to \infty$,

$$\binom{K}{\frac{KM}{N}} \approx \frac{1}{\sqrt{2\pi K \frac{M}{N}(1-\frac{M}{N})}} \cdot e^{K\left(\frac{M}{N}\ln\frac{N}{M}+\left(1-\frac{M}{N}\right)\ln\frac{N}{N-M}\right)}$$

From Lemma 1 and Table I, when $K = (m+1)q$ we have an MN PDA with

$$\frac{M}{N} = \frac{z}{q}, \qquad R_{MN} = \frac{(q-z)(m+1)}{z(m+1)+1} \tag{4}$$

and

$$\begin{aligned}
F_{MN} = \binom{(m+1)q}{z(m+1)} &\approx \frac{1}{\sqrt{2\pi(m+1)q\frac{z}{q}(1-\frac{z}{q})}} \cdot e^{(m+1)q\left(\frac{z}{q}\ln\frac{z}{q}+\left(1-\frac{z}{q}\right)\ln\frac{N}{N-M}\right)} \\
&= \frac{1}{\sqrt{2\pi z(m+1)(1-\frac{z}{q})}} \cdot e^{(m+1)q\left(\frac{z}{q}\ln\frac{q}{z}+\left(1-\frac{z}{q}\right)\ln\frac{q}{q-z}\right)}
\end{aligned} \tag{5}$$

So the PDA in Theorem 3 can reduce the packet number from order $O\left(e^{(m+1)q\left(\frac{z}{q}\ln\frac{q}{z}+\left(1-\frac{z}{q}\right)\ln\frac{q}{q-z}\right)}\right)$ to the order $O\left(e^{m\ln(q)}\right)$ while the rate just increases

$$\frac{\frac{\lfloor\frac{q-z}{q-1}\rfloor}{\lfloor\frac{q-1}{q-z}\rfloor}}{\frac{(q-z)(m+1)}{z(m+1)+1}} = \frac{z(m+1)+1}{\lfloor\frac{q-1}{q-z}\rfloor(m+1)} = \frac{z}{\lfloor\frac{q-1}{q-z}\rfloor} + \frac{1}{\lfloor\frac{q-1}{q-z}\rfloor(m+1)}$$

times than that of MN PDA. Let q and z be fixed and let m approximate infinity. It is not difficult to check that comparing with MN PDA, the packet number reduction using our PDAs in Theorem 3 goes to infinity exponentially with K.

Example 3: Let $m = 2$, $q = 10$, $z = 2$, 3, 4, 5, 6, 7, 8, then Table III can be obtained by MN PDA in Table I and Theorem 2 listed in Table II. Clearly we can reduce the packet number efficiently from the following table.

TABLE III: MN PDAs and the PDAs in Theorem 3

K	M/N	F_{MN}	F_{new}	R_{MN}	R_{new}
30	0.2	593775	100	3.429	8
30	0.3	14307150	100	2.1	7
30	0.4	86493225	100	1.385	6
30	0.5	155117520	100	0.938	5
30	0.6	86493225	200	0.632	2
30	0.7	14307150	300	0.409	1
30	0.8	593775	400	0.24	0.5
30	0.9	4060	900	0.107	0.111

In addition, we can further reduce the packet number for other parameters. For the simplicity, here we just take the case $M/N = 1/q_1$ for any positive integers $q_1 \geq 2$. For any positive integers K with $q_1|K$, from the second row of Table I, there exists a PDA $\mathbf{P}_Y$ with

$$F_Y = q_1^{\frac{K}{q_1}-1}, \quad R_Y = q_1 - 1. \tag{6}$$

Now let us consider the PDA in Theorem 3 for same K and M/N. Assume that there exists positive integers q, z and m with $\frac{z}{q} = \frac{1}{q_1} = \frac{M}{N}$ and $K = (m+1)q$. Clearly $q = zq_1$. From Theorem 3, there exists a PDA with

$$F = \lfloor\frac{q-1}{q-z}\rfloor q^m = z^m q_1^m, \qquad R = q - z = (q_1 - 1)z \tag{7}$$

where $z > 1$. The first item holds since $\frac{q-1}{q-z} = \frac{q_1 z - 1}{(q_1-1)z} = 1 + \frac{z-1}{(q_1-1)z} < 2$. By (6) and (7)

$$\frac{F}{F_Y} = \frac{z^m q_1^m}{q_1^{\frac{K}{q_1}-1}} = \frac{z^m q_1^m}{q_1^{zm+z-1}} = \frac{z^m}{q_1^{(z-1)(m+1)}} \quad \text{and} \quad \frac{R}{R_Y} = \frac{(q_1-1)z}{q_1-1} = z.$$

Clearly comparing with $\mathbf{P}_Y$, the rate of the PDA in Theorem 3 increases z times while the packet number reduces $q_1^{(z-1)(m+1)}/z^m$ times.

Example 4: • When $q_1 = 3$ and $z = 2$, we have $q = 6$ and the following table by (6) and (7).

TABLE IV: Parameters in (6) and (7)

q_1	m	K	M/N	F_Y	F	R_Y	R
3	2	18	1/3	243	36	2	4
3	3	24	1/3	2187	216	2	4
3	4	30	1/3	19683	1296	2	4
3	5	36	1/3	177147	7776	2	4
3	6	42	1/3	1594323	46656	2	4
3	7	48	1/3	14348907	279936	2	4
3	8	54	1/3	129140163	1679616	2	4
3	9	60	1/3	1162261467	10077696	2	4
3	10	66	1/3	10460353203	60466176	2	4

• When $m = 4$, $z = 2$ and $M/N = 1/q_1$, we have $q = 2q_1$ and the following table by (6) and (7).

TABLE V: Parameters in (6) and (7)

m	q_1	K	M/N	F_Y	F	R_Y	R
4	3	30	1/3	19683	1296	2	4
4	4	40	1/4	262144	4096	3	6
4	5	50	1/5	1953125	10000	4	8
4	6	60	1/6	10077696	20736	5	10
4	7	70	1/7	40353607	38416	6	12
4	8	80	1/8	134217728	65536	7	14
4	9	90	1/9	387420489	104976	8	16
4	10	100	1/10	1000000000	160000	9	18

Finally it is very interesting that for given K and M/N, we can get many schemes with different packet numbers and rates. For instance, when $K = 48$ and $M/N = 0.5$, from Theorem 3 we can obtain five distinct schemes listed in Table VI.

TABLE VI: PDAs in Theorem 3

q	z	m	K	M/N	F	R
24	12	1	48	0.5	24	12
16	8	2	48	0.5	256	8
8	4	5	48	0.5	32768	4
4	2	11	48	0.5	4194304	2
2	1	23	48	0.5	8388608	1

In fact we can further reduce the packet number by modifying our constructions, i.e., Constructions 1 and 2, in the following section.

IV. OTHER CONSTRUCTIONS FOR MORE FLEXIBLE PARAMETERS ON M/N AND F

For any positive integers q, z, m and t with $z < q$ and $t < m$, let

$$\mathcal{F} = \{(a_0, a_1, \ldots, a_{m-1}) \mid a_0, \ldots, a_{m-1} \in \mathbb{Z}_q\}. \tag{8}$$

and

$$\mathcal{K} = \{(b_0, b_1, \ldots, b_{t-1}, \delta_0, \delta_1, \ldots, \delta_{t-1}, \varepsilon_0, \ldots, \varepsilon_{t-1}) \mid b_0, \ldots, b_{t-1} \in \mathbb{Z}_q,$$
$$0 \le \delta_0 < \ldots < \delta_{t-1} < m, \varepsilon_0, \ldots, \varepsilon_{t-1} \in [0, \lfloor \tfrac{q-1}{q-z} \rfloor])\} \tag{9}$$

Using the same rule defined in (2), we can obtain a $q^m \times \binom{m}{t} \lfloor \frac{q-1}{q-z} \rfloor^t q^t$ array.

Example 5: When $m = 2$, $q = 3$, $t = 1$ and $z = 2$, (8) and (9) can be written as

$$\mathcal{F} = \{(a_0, a_1) \mid a_0, a_1 \in \mathbb{Z}_3\} \quad \text{and} \quad \mathcal{K} = \{(b, \delta, \varepsilon) \mid b \in \mathbb{Z}_3, 0 \le \delta \le 1, \varepsilon \in [0, 1]\}.$$

By (2), the following array can be obtained. It is easy to check that this array is a $(12, 9, 6, 9)$ PDA.

$(a_0,a_1)\backslash(b,\delta,\varepsilon)$	(0,0,0)	(1,0,0)	(2,0,0)	(0,0,1)	(1,0,1)	(2,0,1)	(0,1,0)	(1,1,0)	(2,1,0)	(0,1,1)	(1,1,1)	(2,1,1)
(0,0)	*	*	(2,0,0)	*	*	(1,0,0)	*	*	(0,2,0)	*	*	(0,1,0)
(1,0)	(0,0,0)	*	*	(2,0,0)	*	*	*	*	(1,2,0)	*	*	(1,1,0)
(2,0)	*	(1,0,0)	*	*	(0,0,0)	*	*	*	(2,2,0)	*	*	(2,1,0)
(0,1)	*	*	(2,1,0)	*	*	(1,1,0)	(0,0,0)	*	(0,2,0)	*	*	*
(1,1)	(0,1,0)	*	*	(2,1,0)	*	*	(1,0,0)	*	(1,2,0)	*	*	*
(2,1)	*	(1,1,0)	*	*	(0,1,0)	*	(2,0,0)	*	(2,2,0)	*	*	*
(0,2)	*	*	(2,2,0)	*	*	(1,2,0)	*	(0,1,0)	*	*	(0,0,0)	*
(1,2)	(0,2,0)	*	*	(2,2,0)	*	*	*	(1,1,0)	*	*	(1,0,0)	*
(2,2)	*	(1,2,0)	*	*	(0,2,0)	*	*	(2,1,0)	*	*	(2,0,0)	*

Similar to the proof of Theorem 2, it is not difficult to verify the following statement.

Theorem 4: For any positive integers q, z, m and t with $q \ge 2$, $z < q$ and $t < m$, there exists a $(\binom{m}{t} q^t \lfloor \frac{q-1}{q-z} \rfloor^t, q^m, q^m - (q-z)^t q^{m-t}, (q-z)^t q^m)$ PDA with $M/N = 1 - (\frac{q-z}{q})^t$ and rate $R = (q-z)^t$.

Similar to the above subsection, we can also obtain a PDA with $(m\lfloor \frac{q-1}{q-z} \rfloor + 1)q$ columns when $t = 1$.

Construction 3: When $t = 1$, (8) and (9) can be written as

$$\mathcal{F} = \{(a_0, a_1, \ldots, a_{m-1}) \mid a_0, \ldots, a_{m-1} \in \mathbb{Z}_q\} \quad \text{and} \quad \mathcal{K} = \{(b, \delta, \varepsilon) \mid b \in \mathbb{Z}_q, 0 \le \delta \le m, \varepsilon \in [0, \lfloor \tfrac{q-1}{q-z} \rfloor])\}$$

for any positive integers q, z, $t = 1$ and m with $z < q$.

- Using the same rule defined in (2), we can obtain a $q^m \times m\lfloor \frac{q-1}{q-z} \rfloor q$ array $\mathbf{P} = (p_{\mathbf{a,b}})$, $\mathbf{a} \in \mathcal{F}, \mathbf{b} \in \mathcal{K}$.
- Let $\mathcal{K}_1 = \{(b, m) \mid b \in \mathbb{Z}_q\}$. We can define a $q^m \times q$ array $\mathbf{E} = (e_{\mathbf{a,b}})$, $\mathbf{a} \in \mathcal{F}, \mathbf{b} \in \mathcal{K}_1$ in the following way.

$$e_{\mathbf{a,b}} = \begin{cases} (a_0, \cdots, a_{m-1}, b - \sum_{l=0}^{m-1} a_l - 1)_q & \text{if } \sum_{l=0}^{m-1} a_l \notin Y_{b,z} \\ * & \text{otherwise} \end{cases} \tag{10}$$

where $Y_{b,z} = \{b, b+1, \ldots, b+(z-1)\}$. All the above operations are performed modulo q.

Theorem 5: For any positive integers q, z, m with $q \ge 2$ and $z < q$, the array $(\mathbf{P}, \mathbf{E})$ generated by Construction 3 is an $((m\lfloor \frac{q-1}{q-z} \rfloor + 1)q, q^m, zq^{m-1}, (q-z)q^m)$ PDA with $M/N = z/q$ and rate $R = q - z$.

The proof of Theorem 5 is included in Appendix C. In fact based on our new PDAs, we can obtain more new PDAs with more different packet numbers for fixed K and M/N by the following useful lemma.

Lemma 2: ([4]) Let $\mathbf{P}$ be a (K, F, Z, S) PDA for some positive integers K, F, Z and S with $Z < F$. Then there exists a $(K, S, S - (F - Z), F)$ PDA.

From Lemma 2 and Theorems 2, 3, 4 and 5, the following results can be obtained.

Corollary 1: For any positive integers q, z, m and t with $q \ge 2$, $z < q$ and $t < m$, there exists an $(\binom{m}{t} q^t, (q-z)^t q^m, (q-z)^t q^m - \lfloor \frac{q-1}{q-z} \rfloor^t (q-z)^t q^{m-t}, \lfloor \frac{q-1}{q-z} \rfloor^t q^m)$ PDA with $M/N = 1 - \lfloor \frac{q-1}{q-z} \rfloor^t / q^t$ and rate $R = \lfloor \frac{q-1}{q-z} \rfloor^t / (q-z)^t$.

Corollary 2: For any positive integers q, z, m with $q \ge 2$ and $z < q$, there exists a $((m+1)q, (q-z)q^m, (q-z)q^m - \lfloor \frac{q-1}{q-z} \rfloor (q-z)q^{m-1}, \lfloor \frac{q-1}{q-z} \rfloor q^m)$ PDA with $M/N = 1 - \frac{1}{q} \lfloor \frac{q-1}{q-z} \rfloor$ and rate $R = \lfloor \frac{q-1}{q-z} \rfloor / (q-z)$.

Corollary 3: For any positive integers q, z, m and t with $q \ge 2$, $z < q$ and $t < m$, there exists an $(\binom{m}{t} \lfloor \frac{q-1}{q-z} \rfloor^t q^t, (q-z)^t q^m, (q-z)^t q^m - (q-z)^t q^{m-t}, q^m)$ PDA with $M/N = 1 - 1/q^t$ and rate $R = 1/(q-z)^t$.

Corollary 4: For any positive integers q, z, m with $q \geq 2$ and $z < q$, there exists a $((m\lfloor\frac{q-1}{q-z}\rfloor + 1)q, (q-z)q^m, (q-z)q^m - q^m + zq^{m-1}, q^m)$ PDA with $M/N = 1 - 1/q$ and rate $R = 1/(q-z)$.

Let us summarize our new PDAs and MN PDAs in the following Table, i.e., Table VII.

TABLE VII: Constructions with $M/N = 1 - (\frac{q-z}{q})^t$, $t = 1, 2, \ldots, m-1$

		K	M/N	F	R
	MN PDA [12]	K	$\frac{z}{q}$, $z = 1, \ldots q-1$	$\binom{K}{Kz/q}$	$\frac{K(q-z)}{zK+q}$
Contruction 1	Theorem 2	$\binom{m}{t}q^t$	$1 - (\frac{q-z}{q})^t$, $z = 1, \ldots q-1$	$\lfloor\frac{q-1}{q-z}\rfloor^t q^m$	$(\frac{q-z}{\lfloor\frac{q-1}{q-z}\rfloor})^t$
	Corollary 1	$\binom{m}{t}q^t$	$1 - (\frac{\lfloor\frac{q-1}{q-z}\rfloor}{q})^t$	$(q-z)^t q^m$	$(\frac{\lfloor\frac{q-1}{q-z}\rfloor}{q-z})^t$
	Theorem 4	$\binom{m}{t}\lfloor\frac{q-1}{q-z}\rfloor^t q^t$	$1 - (\frac{q-z}{q})^t$, $z = 1, \ldots q-1$	q^m	$(q-z)^t$
	Corollary 3	$\binom{m}{t}\lfloor\frac{q-1}{q-z}\rfloor^t q^t$	$1 - 1/q^t$	$(q-z)^t q^m$	$\frac{1}{(q-z)^t}$
Contruction 2	Theorem 3	$(m+1)q$	$\frac{z}{q}$, $z = 1, \ldots q-1$	$\lfloor\frac{q-1}{q-z}\rfloor q^m$	$\frac{q-z}{\lfloor\frac{q-1}{q-z}\rfloor}$
	Corollary 2	$(m+1)q$	$1 - \frac{1}{q}\lfloor\frac{q-1}{q-z}\rfloor$, $z = 1, \ldots q-1$	$(q-z)q^m$	$\frac{\lfloor\frac{q-1}{q-z}\rfloor}{q-z}$
Contruction 4	Theorem 5	$(m\lfloor\frac{q-1}{q-z}\rfloor + 1)q$	$\frac{z}{q}$, $z = 1, \ldots q-1$	q^m	$q-z$
	Corollary 4	$(m\lfloor\frac{q-1}{q-z}\rfloor + 1)q$	$1 - \frac{1}{q}$	$(q-z)q^m$	$\frac{1}{q-z}$

From Table VII, the following statements hold.

Remark 1: (i) Let q, z and t be fixed and let m approximate infinity, it holds that R and M/N are constants independent of K for all of our new PDAs.

(ii) The parameter F in Theorem 2 and 4 grows sub-exponentially with K if $t \geq 2$.

It is worth noting that given fixed parameters K and M/N, there is plenty of schemes with different packet number F from Table VII. For example assume that $K = 300$ and $M/N = 0.36$, 0.75, 0.84 and 0.96, Table VIII can be obtained by Table VII.

TABLE VIII: MN PDAs and the PDAs in Corollaries 2, 4

K	M/N	q	z	m	t	Types	F	R
300	0.36	10	2	3	2	Theorem 4	1000	64
		75	27	3			421875	48
		50	18	5	1	Theorem 5	312500000	32
		25	9	11			2.38418579101563e+15	16
	0.75	5	10	3	2	Theorem 2	1000	25
		9	12	8	1	Theorem 5	429981696	3
		2	4	72	1	Corollary 4	7.1362384635298e+44	0.5
	0.84	5	3	3	2	Theorem 4	125	4
		10	6	3	2	Theorem 2	4000	4
	0.96	25	9	11	1	Corollary 4	3.814697265625e+16	0.0625
		5	3	3	2	Corollary 3	500	0.25
		10	8	3	2	Theorem 2	16000	0.25

For another example, assume that $t = 1$, $K = 30$ and $M/N = 0.9$, by Corollaries 2 and 4 Table IX in the following can be obtained.

TABLE IX: MN PDAs and the PDAs in Corollaries 2, 4

K	M/N	q	m	Types	z	F	R
30	0.9	10	1	Corollary 4	6	40	0.25
			2	Corollary 2	1	500	0.2
					2	600	0.16667
					3	700	0.14286
					4	800	0.125
					5	900	0.11111
				MN PDA in [12]	1	4060	6.75

Finally it is easy to check that the packet number of the PDA in Theorem 4 is polynomial or linear with the number of users when t is near to $m/2$.

V. Conclusion

In this paper, we generalized the previously known constructions and expanded $(K, M/N)$ from $\left(\binom{m}{t}q^t, \frac{1}{q^t}\right)$, $\left(\binom{m}{t}q^t, \frac{1}{q^t}\right)$, $1 - \frac{q-1}{q}$) to $\left(\binom{m}{t}q^t\lfloor\frac{q-1}{q-z}\rfloor^t, \frac{z^t}{q^t}\right)$ for any positive integers q, z, m and t with $0 < z < q$ and $0 < t < m$. Moreover according to our new constructions, several classes good coded caching schemes were obtained. Especially when t is near to $m/2$, some PDAs satisfy that the packet number is polynomial or linear with the number of users.

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APPENDIX A: PROOF OF THEOREM 2

Proof. It is easy to check that there are exactly $q^{m-t}(q-z)^t\lfloor\frac{q-1}{q-z}\rfloor^t$ vector entries in each column. This implies that $Z = (q^m - q^{m-t}(q-z)^t)\lfloor\frac{q-1}{q-z}\rfloor^t$. So C1 holds. From (2), the vector set of $\mathbf{P}$ in Construction 1 is

$$\mathcal{S} = \{(a_0, a_1, \ldots, a_{m-1}, a_m, \ldots, a_{m+t-1}) \mid a_0, \ldots, a_{m-1} \in \mathbb{Z}_q, a_{m+1}, \ldots, a_{m+t-t} \in [0, q-z)\}.$$

So $S = |\mathcal{S}| = q^m(q-z)^t$. Each $\mathbf{s} = (s_0, s_1, \ldots, s_{m+t-1}) \in \mathcal{S}$ appears in the entries at row $\mathbf{a}$ and column $\mathbf{b}$ of $\mathbf{P}$, if and only if

$$\begin{aligned}\mathbf{a} &= (a_0, \ldots, a_{\delta_i}, \cdots, a_{m-1}, \varepsilon_0, \ldots, \varepsilon_{t-1}) = (s_0, \ldots, s_{\delta_i} + s_{m+i} + \varepsilon_i(q-z) + 1, \cdots, s_{m-1}, \varepsilon_0, \ldots, \varepsilon_{t-1})_q \quad \text{and} \\ \mathbf{b} &= (b_0, b_1, \ldots, b_{t-1}, \delta_0, \delta_1, \ldots, \delta_{t-1}) = (s_{\delta_0} + \varepsilon_0(q-z), s_{\delta_1} + \varepsilon_1(q-z), \ldots, s_{\delta_{t-1}} + \varepsilon_{t-1}(q-z), \delta_0, \delta_1, \ldots, \delta_{t-1}).\end{aligned} \tag{11}$$

Clearly, for any fixed $\delta_0, \delta_1, \ldots, \delta_{t-1}$ and $\varepsilon_0, \ldots, \varepsilon_{t-1}$, there is exactly a unique pair of vectors $\mathbf{a}$ and $\mathbf{b}$, such that $p_{\mathbf{a},\mathbf{b}} = \mathbf{s}$. Based on the above observation, C2 is clear. Now, let us consider C3. Suppose there exists another two vectors $\mathbf{a}'$ and $\mathbf{b}'$ with $p_{\mathbf{a}',\mathbf{b}'} = \mathbf{s}$. Then there must exist another $2t$ integers $\delta'_0, \ldots, \delta'_{t-1}$ and $\varepsilon'_0, \ldots, \varepsilon'_{t-1}$ satisfying

$$\begin{aligned}\mathbf{a}' &= (a'_0, \ldots, a'_{\delta'_i}, \cdots, a'_{m-1}, \varepsilon'_0, \ldots, \varepsilon'_{t-1}) = (s_0, \ldots, s_{\delta'_i} + s_{m+i} + \varepsilon'_i(q-z) + 1, \cdots, s_{m-1}, \varepsilon'_0, \ldots, \varepsilon'_{t-1})_q \quad \text{and} \\ \mathbf{b}' &= (b'_0, b'_1, \ldots, b'_{t-1}, \delta'_0, \delta'_1, \ldots, \delta'_{t-1}) = (s_{\delta'_0} + \varepsilon'_0(q-z), s_{\delta'_1} + \varepsilon'_1(q-z), \ldots, s_{\delta'_{t-1}} + \varepsilon'_{t-1}(q-z), \delta'_0, \delta'_1, \ldots, \delta'_{t-1}).\end{aligned} \tag{12}$$

It is sufficient to consider the following cases, where all the operations are performed modulo q.

- If $\{\delta_0, \ldots, \delta_{t-1}\} \neq \{\delta'_0, \ldots, \delta'_{t-1}\}$, there at least exist two integers $i, i' \in [0, t)$ such that $\delta_i \notin \{\delta'_0, \ldots, \delta'_{t-1}\}$ and $\delta'_{i'} \notin \{\delta_0, \ldots, \delta_{t-1}\}$. Now we show C3-a) and C3-b) hold respectively.
  - If $\mathbf{s}$ occurs in a row at least twice, we have $\mathbf{a} = \mathbf{a}'$. This implies that $a_{\delta_i} = a'_{\delta_i}$. From (11) and (12), we have

    $$a_{\delta_i} = s_{\delta_i} + s_{m+i} + \varepsilon_i(q-z) + 1, \qquad a'_{\delta_i} = s_{\delta_i}.$$

    This implies that $s_{m+i} + \varepsilon_i(q-z) + 1 = 0$. This is impossible since

    $$1 \leq s_{m+i} + \varepsilon_i(q-z) + 1 < q - z + (\frac{q-1}{q-z} - 1)(q-z) + 1 = q$$

    by the facts $s_{m+i} \in [0, q-z)$ and $\varepsilon \in [0, \lfloor\frac{q-1}{q-z}\rfloor)$. So C3-a) holds.
  - First from (11) and (12), we have

    $$a_{\delta'_{i'}} = s_{\delta'_{i'}}, \qquad b'_{i'} = s_{\delta'_{i'}} + \varepsilon'_{i'}(q-z), \qquad a'_{\delta_i} = s_{\delta_i}, \qquad b'_i = s_{\delta_i} + \varepsilon_{i'}(q-z)$$

    If $p_{\mathbf{a},\mathbf{b}'} \neq *$, we have $a_{\delta'_{i'}} \in \{b'_{i'} + 1, b'_{i'} + 2, \ldots, b'_{i'} + (q-z)\}$. Then there exists an integer $j \in [1, q-z)$ such that $a_{\delta'_{i'}} = b'_{i'} + j$, i.e.,

    $$a_{\delta'_{i'}} = s_{\delta'_{i'}} = s_{\delta'_{i'}} + \varepsilon'_{i'}(q-z) + j.$$

    This implies that $\varepsilon'_{i'}(q-z) + j = 0$. This is impossible since

    $$1 \leq \varepsilon'_{i'}(q-z) + j < (\frac{q-1}{q-z} - 1)(q-z) + (q-z) = q - 1$$

by the fact $\varepsilon'_{i'} \in [0, \lfloor \frac{q-1}{q-z} \rfloor)$). Similarly we can also show that $p_{\mathbf{a}',\mathbf{b}} = *$ too.

- If $\{\delta_0, \ldots, \delta_{t-1}\} = \{\delta'_0, \ldots, \delta'_{t-1}\}$, we have $(\varepsilon_0, \ldots, \varepsilon_{t-1}) \neq (\varepsilon'_0, \ldots, \varepsilon'_{t-1})$ by (11) and (12). So there exist integer i such that $\varepsilon_i \neq \varepsilon'_i$. Then we have

$$b_i = s_{\delta_i} + \varepsilon_i(q-z), \quad a_{\delta_i} = s_{m+i} + b_i + 1, \quad b'_i = s_{\delta_i} + \varepsilon'_i(q-z), \quad a'_{\delta_i} = s_{m+i} + b'_i + 1. \tag{13}$$

Now we show C3-a) and C3-b) hold respectively.

- If $\mathbf{s}$ occurs in a column at least twice, we have $\mathbf{b} = \mathbf{b}'$. This implies that $b_{\delta_i} = b'_{\delta_i}$. By the first and third items in (13), $\varepsilon_i(q-z) = \varepsilon'_i(q-z)$ holds. Clearly this is impossible since $\varepsilon_i, \varepsilon'_i \in [0, \lfloor \frac{q-1}{q-z} \rfloor)$. So C3-a) holds.
- If $p_{\mathbf{a},\mathbf{b}'} \neq *$, we have $a_{\delta_i} \in \{b'_i + 1, b'_i + 2, \ldots, b'_i + (q-z)\}$. Then there exists an integer $j \in [0, q-z)$ such that $a_{\delta_i} = b'_i + j$. By the first, second and third items in (13),

$$s_{m+i} + s_{\delta_i} + \varepsilon_i(q-z) + 1 = s_{\delta_i} + \varepsilon'_i(q-z) + j. \tag{14}$$

If $\varepsilon_i < \varepsilon'_i$, we have $s_{m+i} + 1 = (\varepsilon'_i - \varepsilon_i)(q-z) + j$. This is impossible since

$$1 \leq s_{m+i} + 1 \leq q - z \quad \text{and} \quad q - z + 1 \leq (\varepsilon'_i - \varepsilon_i)(q-z) + j < q$$

by the facts $0 \leq \varepsilon_i, \varepsilon'_i < \lfloor \frac{q-1}{q-z} \rfloor$ and $1 \leq j, s_{m+i} < q - z$. Clearly (14) does not hold either if $\varepsilon_i > \varepsilon'_i$. So we have $p_{\mathbf{a},\mathbf{b}'} = *$. Similarly we can also show that $p_{\mathbf{a}',\mathbf{b}} = *$ too.

$\square$

APPENDIX B: PROOF OF THEOREM 3

Proof. From Construction 2, and the proof of Theorem 2, we have that $\mathbf{P}$ is a $(mq, \lfloor \frac{q-1}{q-z} \rfloor q^m, z\lfloor \frac{q-1}{q-z} \rfloor q^{m-1}, (q-z)q^m)$ PDA with the vector set

$$\mathcal{S} = \{(s_0, s_1, \ldots, s_m) \mid s_0, s_1, \ldots, s_{m-1} \in [0, q), \ s_m \in [0, q-z)\}.$$

From (3), it is easy to count that each column of $\mathbf{E}$ has exactly $z\lfloor \frac{q-1}{q-z} \rfloor q^{m-1}$ stars and the vector set of $\mathbf{E}$ is $\mathcal{S}$. So C1 and C2 hold. Now it is sufficient to verify C3. For any two distinct vector entries, say $h_{\mathbf{a},\mathbf{b}}$ and $h_{\mathbf{a}',\mathbf{b}'}$, assume that $h_{\mathbf{a},\mathbf{b}} = h_{\mathbf{a}',\mathbf{b}'} \in \mathcal{S}$ where

$$\mathbf{a} = (a_0, a_1, \ldots, a_{m-1}, \varepsilon)_q, \quad \mathbf{a}' = (a'_0, a'_1, \ldots, a'_{m-1}, \varepsilon')_q, \quad \mathbf{b} = (b, \delta), \quad \mathbf{b}' = (b', \delta').$$

We only need to consider the following cases, where all the operations are performed modulo q.

- When $\delta \in [0, m)$ and $\delta' = m$, we have $h_{\mathbf{a},\mathbf{b}} = p_{\mathbf{a},\mathbf{b}}$ and $h_{\mathbf{a}',\mathbf{b}'} = e_{\mathbf{a}',\mathbf{b}'}$ in distinct columns. From (2) and (3) we have

$$a_\delta - b - 1 = b' - (\sum_{l=0}^{m-1} a'_l - \varepsilon'(q-z)) - 1, \quad a'_\delta = b - \varepsilon(q-z), \quad a_l = a'_l \tag{15}$$

for any $l \in [0, m) \setminus \{\delta\}$. From the second equation in (15), we can show $p_{\mathbf{a},\mathbf{b}'}$ and $e_{\mathbf{a}',\mathbf{b}}$ in distinct rows too. Otherwise we have $a_l = a'_l$ for any $l = 0, 1, \ldots, m - 1$ and $\varepsilon = \varepsilon'$. Then combining the two equations in (15) we have

$$a_\delta - b = b' - (\sum_{l=0}^{m-1} a'_l - \varepsilon'(q-z)) = -\varepsilon(q-z).$$

Since $\sum_{l=0}^{m-1} a'_l - \varepsilon'(q-z) \notin Y_{b,z}$, let $\sum_{l=0}^{m-1} a'_l - \varepsilon'(q-z) = b' - j$ where $j \in [1, q-z]$. Then the above equation can be written as

$$b' - (\sum_{l=0}^{m-1} a'_l - \varepsilon'(q-z)) = b' - (b' - j) = -\varepsilon(q-z).$$

This implies that $j + \varepsilon(q-z) = 0$. This is impossible since

$$1 \leq j + \varepsilon(q-z) < (q-z) + (\frac{q-1}{q-z} - 1)(q-z) = q - 1 \tag{16}$$

by the fact $\varepsilon \in [0, \lfloor \frac{q-1}{q-z} \rfloor)$. So C3-a) holds. Now let us consider the entries $e_{\mathbf{a},\mathbf{b}'}$ and $p_{\mathbf{a}',\mathbf{b}}$ in the following cases.

– If $e_{\mathbf{a},\mathbf{b}'}$ is a vector, $\sum_{l=0}^{m-1} a_l - \varepsilon(q-z) \in \{b' - (q-z), b' - (q-z-1), \dots, b' - 1\}$. That is, there must exist an integer $j \in \{1, 2, \dots q - z\}$ such that $\sum_{l=0}^{m-1} a_l - \varepsilon(q-z) = b' - j$. From (15),

$$
\begin{aligned}
\sum_{l=0}^{m-1} a_l - \varepsilon(q-z) &= \sum_{l \in [0,m) \setminus \{\delta\}} a_l + a_\delta - \varepsilon(q-z) \\
&= \sum_{l \in [0,m) \setminus \{\delta\}} a_l + \left(b + b' - \left(\sum_{l=0}^{m-1} a_l' - \varepsilon'(q-z)\right)\right) - \varepsilon(q-z) \\
&= \sum_{l \in [0,m) \setminus \{\delta\}} a_l + \left(b + b' - \sum_{l \in [0,m) \setminus \{\delta\}} a_l' - a_\delta' + \varepsilon'(q-z)\right) - \varepsilon(q-z) \\
&= b + b' - a_\delta' + \varepsilon'(q-z) - \varepsilon(q-z) \\
&= b + b' - (b - \varepsilon(q-z)) + \varepsilon'(q-z) - \varepsilon(q-z) \\
&= b' + \varepsilon'(q-z).
\end{aligned}
$$

So we have $b' - j = b' + \varepsilon'(q-z)$, i.e., $j + \varepsilon'(q-z) = 0$. This is impossible by the similar proof in (16). So $e_{\mathbf{a},\mathbf{b}'} = *$.

– If $p_{\mathbf{a}',\mathbf{b}}$ is a vector, $a_\delta' \in \{b+1, b+2, \dots, b+q-z\}$. There must exist an integer $j \in \{1, 2, \dots q-z\}$ such that $a_\delta' = b + j$. From (15), we have $b + j = b - \varepsilon(q-z)$, i.e., $j + \varepsilon(q-z) = 0$. This is impossible by (16). So $p_{\mathbf{a}',\mathbf{b}} = *$.

So C3-b) holds.

• When $\delta = \delta' = m$, we have $h_{\mathbf{a},\mathbf{b}} = e_{\mathbf{a},\mathbf{b}}$ and $h_{\mathbf{a}',\mathbf{b}'} = e_{\mathbf{a}',\mathbf{b}'}$. From (3) we have

$$
b - \left(\sum_{l=0}^{m-1} a_l - \varepsilon(q-z)\right) - 1 = b' - \left(\sum_{l=0}^{m-1} a_l' - \varepsilon'(q-z)\right) - 1, \quad a_l = a_l' \tag{17}
$$

for any $l \in [0, m)$. Clearly $\varepsilon \neq \varepsilon'$ always holds. Otherwise we have $\mathbf{a} = \mathbf{a}'$ and $b = b'$, i.e., $\mathbf{b} = \mathbf{b}'$. This is impossible. If $\mathbf{b} = \mathbf{b}'$, we have $\varepsilon = \varepsilon'$ since $0 \leq \varepsilon'(q-z), \varepsilon'(q-z) < q$ by the fact $\varepsilon, \varepsilon' \in [0, \lfloor \frac{q-1}{q-z} \rfloor)$. So C3-a) is clear. Now let us consider C3-b). If $e_{\mathbf{a},\mathbf{b}'}$ is a vector, there must exist an integer $x \in \{1, 2, \dots q-z\}$ satisfying

$$
\sum_{l=0}^{m-1} a_l - \varepsilon(q-z) = b' - x.
$$

Since $e_{\mathbf{a},\mathbf{b}}$ is a vector entry, there exist an integer $y \in \{1, 2, \dots q-z\}$ such that

$$
\left(\sum_{l=0}^{m-1} a_l - \varepsilon(q-z)\right) = b - y.
$$

So we have

$$
\left(\sum_{l=0}^{m-1} a_l - \varepsilon(q-z)\right) = b' - x = b - y. \tag{18}
$$

Moreover, from (17) we have

$$
b + \varepsilon(q-z) = b' + \varepsilon'(q-z). \tag{19}
$$

Combining (18) and (19), we have $x - y = (\varepsilon - \varepsilon')(q-z)$. From the above discussion, we know that $\varepsilon \neq \varepsilon'$, i.e., $(\varepsilon - \varepsilon')(q-z) \neq 0$. So we have $x \neq y$. Furthermore, $(q-z)|(x-y)$ holds since $q - z \leq |(\varepsilon - \varepsilon')(q-z)| < q - 1$. Clearly this is impossible by the fact that $1 \leq x \neq y \leq q - z$. So $e_{\mathbf{a},\mathbf{b}'} = *$. Similarly we can also show that $e_{\mathbf{a}',\mathbf{b}} = *$. So C3-b) holds.

$\square$

APPENDIX A: PROOF OF THEOREM 5

Proof. Similar to the proof of Theorem 3, denote $\mathbf{H} = (\mathbf{P}, \mathbf{E})$. From Construction 3 $\mathbf{P}$ is a $(m\lfloor\frac{q-1}{q-z}\rfloor q, q^m, zq^m, (q-z)q^m)$ PDA with the vector set

$$\mathcal{S} = \{(s_0, s_1, \ldots, s_m) \mid s_0, s_1, \ldots, s_{m-1} \in [0, q), \ s_m \in [0, q - z)\}.$$

From (10), it is easy to count that each column of $\mathbf{E}$ has exactly zq^{m-1} stars and the vector set of $\mathbf{E}$ is $\mathcal{S}$. So C1 and C2 hold. Now it is sufficient to verify C3. For any two distinct vector entries, say $h_{\mathbf{a},\mathbf{b}}$ and $h_{\mathbf{a}',\mathbf{b}'}$, assume that $h_{\mathbf{a},\mathbf{b}} = h_{\mathbf{a}',\mathbf{b}'} \in \mathcal{S}$ where

$$\mathbf{a} = (a_0, a_1, \ldots, a_{m-1})_q, \quad \mathbf{a}' = (a_0', a_1', \ldots, a_{m-1}')_q, \quad \mathbf{b} = (b, \delta, \varepsilon), \quad \mathbf{b}' = (b', \delta', \varepsilon').$$

We only need to consider the following cases, where all the operations are performed modulo q.

- When $\delta \in [0, m)$ and $\delta' = m$, we have $p_{\mathbf{a},\mathbf{b}}$ and $e_{\mathbf{a}',\mathbf{b}'}$ in distinct columns. From (2) and (10) we have

$$a_\delta - b - 1 = b' - \sum_{l=0}^{m-1} a_l' - 1, \quad a_\delta' = s_\delta = b - \varepsilon(q - z), \quad a_l = a_l' \tag{20}$$

  for any $l \in [0, m) \setminus \{\delta\}$. From the second equation in (20), we can show $p_{\mathbf{a},\mathbf{b}}$ and $e_{\mathbf{a}',\mathbf{b}'}$ in distinct rows. Otherwise $a_\delta = a_\delta'$ holds. By (20) the following equation can be obtained.

$$a_\delta = a_\delta' = b + b' - \sum_{l=0}^{m-1} a_l' = b - \varepsilon(q - z)$$

  Since $\sum_{l=0}^{m-1} a_l' \notin Y_{b',z}$, there exists an integer $j \in \{1, 2, \ldots, q - z\}$ such that $\sum_{l=0}^{m-1} a_l' = b' - j$. Combining the above formula we have

$$b + b' - (b' - j) = b - \varepsilon(q - z)$$

  i.e., $j + \varepsilon(q - z) = 0$. This is impossible since

$$1 \le j + \varepsilon(q - z) < (q - z) + (\frac{q-1}{q-z} - 1)(q - z) = q - 1 \tag{21}$$

  by the fact $\varepsilon \in [0, \lfloor\frac{q-1}{q-z}\rfloor)$. So C3-a) holds. Now let us consider the entries $p_{\mathbf{a},\mathbf{b}'}$ and $p_{\mathbf{a}',\mathbf{b}}$ in the following cases.

  – If $e_{\mathbf{a},\mathbf{b}'}$ is a vector, we have $\sum_{l=0}^{m-1} a_l \in \{b' - (q - z), b' - (q - z - 1), \ldots, b' - 1\}$. There must exist an integer $j \in \{1, 2, \ldots q - z\}$ such that $\sum_{l=0}^{m-1} a_l = b' - j$. From (20), we have

$$
\begin{aligned}
\sum_{l=0}^{m-1} a_l &= \sum_{l \in [0,m) \setminus \{\delta\}} a_l + a_\delta \\
&= \sum_{l \in [0,m) \setminus \{\delta\}} a_l + (b + b' - \sum_{l=0}^{m-1} a_l') \\
&= \sum_{l \in [0,m) \setminus \{\delta\}} a_l + (b + b' - \sum_{l \in [0,m) \setminus \{\delta\}} a_l' - a_\delta') \\
&= b + b' - a_\delta' \\
&= b + b' - (b - \varepsilon(q - z)) \\
&= b' + \varepsilon(q - z)
\end{aligned}
$$

    So we have $b' - j = b' + \varepsilon(q - z)$, i.e., $j + \varepsilon(q - z) = 0$. This is impossible by (21). So $e_{\mathbf{a},\mathbf{b}'} = *$.
  – If $p_{\mathbf{a}',\mathbf{b}}$ is a vector, we have $a_\delta' \in \{b + 1, b + 2, \ldots, b + q - z\}$. There must exist an integer $j \in \{1, 2, \ldots q - z\}$ such that $a_\delta' = b + j$. From (20), we have $b + j = b - \varepsilon(q - z)$, i.e., $j + \varepsilon(q - z) = 0$. This is impossible by (21). So $p_{\mathbf{a}',\mathbf{b}} = *$.

  So C3-b) holds.

- When $\delta = \delta' = m$, from (10) we have

$$b - \sum_{l=0}^{m-1} a_l - 1 = b' - \sum_{l=0}^{m-1} a'_l - 1, \quad a_l = a'_l$$

for any $l \in [0, m)$. This implies that $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b} = \mathbf{b}'$, a contradiction to our hypothesis. So C3-b) holds.

$\square$