THE PRODUCT STRUCTURE OF NEWTON STRATA IN THE GOOD REDUCTION OF SHIMURA VARIETIES OF HODGE TYPE

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Abstract. We construct a generalisation of Mantovan's almost product structure to Shimura varieties of Hodge type with hyperspecial level structure at p and deduce that the perfection of the Newton strata are pro-étale locally isomorphic to the perfection of the product of a central leaf and a Rapoport-Zink space. The almost product formula can be extended to obtain an analogue of Caraiani's and Scholze's generalisation of the almost product structure for Shimura varieties of Hodge type.

1. Introduction

Let (G,X) be a Shimura datum of Hodge-type, $\mathsf{K} \subset \mathsf{G}(\mathbb{A}_f)$ a small enough compact open subgroup which is hyperspecial at p. Denote by G the reductive model of G over $\mathbb{Z}_p$ corresponding to K_p. The existence of a canonical integral model $\mathscr{S}_G$ of the Shimura variety $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G},\mathsf{X})$ was shown by Kisin [Kis10], with some exceptions in the case $p=2$. His construction of $\mathscr{S}_G$ also equips it with an Abelian scheme $\mathcal{A}_G \to \mathscr{S}_G$ and crystalline Tate tensors $(t_{G,\alpha,x})$ on $\mathbb{D}(\mathcal{A}_{G,x}[p^\infty])$ for every point $x \in \mathscr{S}_G(\overline{\mathbb{F}}_p)$. While the Barsotti-Tate group with crystalline Tate-tensors $(\mathcal{A}_{G,x}[p^\infty],(t_{G,\alpha,x}))$ depends on some choices made during the construction of $\mathscr{S}_G$, it induces an isocrystal with G-structure $\mathcal{G}_x$ over $\overline{\mathbb{F}}_p$ which is independent of them. Lovering constructed an isocrystal with G-structure over the special fibre $\mathscr{S}_{G,0}$ of $\mathscr{S}_G$ in [Lov17], which specialises to $\mathcal{G}_x$ for every $x \in \mathscr{S}_G(\overline{\mathbb{F}}_p)$. Thus we have a well-defined Newton stratification on $\mathscr{S}_{G,0}$; we denote the Newton strata by $\mathscr{S}_{G,0}^{\mathbf{b}}$.

1.1. **The almost product structure in the special fibre.** Using the almost product structure in deformation spaces constructed in [Ham16] as a starting point, we construct for a suitable central leaf $C \subset \mathscr{S}_G$ a family of compatible surjective finite-to-finite correspondences

$$\begin{tikzcd}[column sep=small] & \mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2} \arrow[dl] \arrow[dr, "\pi_N"] & \\ C^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2} & & \mathscr{S}_{G,0}^{\mathbf{b}} \end{tikzcd}$$

generalising Mantovan's construction in [Man05]. Here $(\mathscr{J}_m)_{m\in\mathbb{N}}$ denotes the tower of Igusa varieties over C and $(\mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2})_{m_1<m_2\in\mathbb{Z}^2}$ a certain exhausting family of closed subschemes of the special fibre of the corresponding Rapoport-Zink space. We denote by $F_{p^N} : \mathscr{J}_m^{(p^{-N})} \to \mathscr{J}_m$ the Frobenius and by $F_{p^\infty} : \mathscr{J}_m^{(p^{-\infty})} \to \mathscr{J}_m$ their limit for $N \to \infty$, i.e. the perfection. Moreover, we show that when taking the limit $N, m, m_1, m_2 \to \infty$ we obtain a correspondence of perfect schemes

$$\begin{tikzcd} & \mathscr{J}_\infty^{(p^{-\infty})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{(p^{-\infty})} \arrow[dl, "r_\infty"'] \arrow[dr, "\pi_\infty^{(p^{-\infty})}"] & \\ C^{(p^{-\infty})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{(p^{-\infty})} & & \mathscr{S}_{G,0}^{\mathbf{b},(p^{-\infty})} \end{tikzcd}$$

where r_∞ and $\pi_\infty^{(p^{-\infty})}$ are torsors for the proétale topology.

1.2. **Caraiani-Scholze type product structure for Shimura varieties of Hodge type.** The infinite almost product structure can be extended to a generalisation of the product structure of Caraiani and Scholze constructed in [CS15, § 4]. More explicitly, we construct an isomorphism

$$\mathfrak{Ig}^{\mathbf{b}} \times \mathfrak{M}_{G,\mu}(\mathbf{b}) \xrightarrow{\sim} \mathfrak{X}^{\mathbf{b}}$$

of formal schemes over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$, where $\mathfrak{Ig}^b$ denotes the flat lift of $\mathscr{J}_\infty^{(p^{-\infty})}$ to $\breve{\mathbb{Z}}_p$, $\mathfrak{M}_{G,\mu}(\mathbf{b})$ is the corresponding Rapoport-Zink space and $\mathfrak{X}^{\mathbf{b}}$ parameterises trivialisations of $(\mathcal{A}_G, \lambda_G, (t_{G,\alpha}))$ up to isogeny.

The product formulas are motivated by their application to cohomology. In the case of PEL-Shimura varieties, the product structure yield formulas for the cohomology of a local system over Shimura varieties and Rapoport-Zink spaces associated to a representation of G ([Man11, Thm. 3.1], [CS15, Thm. 4.4.4]). Analogous applications to the cohomology of Shimura varieties of Hodge type will be discussed in a subsequent joint work with Wansu Kim.

This article was written independently from the article [Zha15] of Zhang, which was put on ArXiv approximately two weeks before the first version of this paper. In his work, he gives a partial construction of the morphisms π_N: He constructs the Igusa tower and the restriction of π_N to the smooth locus.

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2. Shimura varieties of Hodge type

Let $p > 2$ be a prime number. We will focus our study on the class of Shimura varieties of Hodge type with hyperspecial level structure at p.

2.1. **Crystalline Tate-tensors.** While the reduction modulo p of a Shimura variety of Hodge type will not be given by a moduli description, we will have an Abelian variety with additional structure by crystalline Tate-tensors which can be regarded as a replacement for a universal object. We briefly recall the notion of crystalline Tate-tensors as presented in [Kim16].

Notation 2.1. Let M be an object of a rigid Abelian tensor category
(1) $M^\otimes$ is defined as the direct sum of any finite combination of tensor products, symmetric products, alternating products and duals of M.

(2) A tensor of M is a morphism $s : \mathbf{1} \to M^{\otimes}$, where $\mathbf{1}$ denotes the unit object.

For any abelian scheme or Barsotti-Tate group X over an $\mathbb{F}_p$-scheme S we denote the contravariant Dieudonné crystal as in [BBM82, § 3] by $\mathbb{D}(X)$. The pull-back $\mathbb{D}(X)_S$ to the Zariski site of S is naturally equipped with the Hodge filtration $\mathrm{Fil}^1_{\mathbb{D}(X)} := \omega_X \subset \mathbb{D}(X)_S$, which is Zariski-locally a direct summand of rank $\dim X$. Moreover the relative Frobenius of X over S induces a map $F : \mathbb{D}(X)^{(p)} \to \mathbb{D}(X)$, which is also called the Frobenius. As the relative Frobenius is an isogeny, F induces an isomorphism of isocrystals $\mathbb{D}(X)[\frac{1}{p}]^{(p)} \xrightarrow{\sim} \mathbb{D}(X)[\frac{1}{p}]$ and $\mathbb{D}(X)[\frac{1}{p}]^{\otimes\,(p)} \xrightarrow{\sim} \mathbb{D}(X)[\frac{1}{p}]^{\otimes}$.

Definition 2.2 ([Kim16, Def. 2.3.4])**.** A tensor t of $\mathbb{D}(X)$ is called a crystalline Tate tensor if it induces a morphism of F-isocrystals $\mathbf{1} \to \mathbb{D}(X)[\frac{1}{p}]^{\otimes}$, i.e. it is Frobenius equivariant.

2.2. **Construction of integral models.** Let (G, X) be Shimura datum of Hodge type and $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subset \mathsf{G}(\mathbb{A}_f)$ be a small enough open compact subgroup which is hyperspecial at p. Denote by E its Shimura field, E its p-adic completion for some fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and O_E its ring of integers. The existence of a canonical integral model over O_E for $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, \mathsf{X})$ was shown by Mark Kisin ([Kis10]). We briefly recall his construction and introduce the notions which will be fundamental for the study of the special fibre which follows.

Let G be the Bruhat-Tits-group scheme associated to K_p. As K_p is hyperspecial, G is a reductive group scheme over $\mathbb{Z}_p$. We fix a (unique up to isomorphism) reductive group scheme $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ such that $G = G_{\mathbb{Z}_{(p)}} \otimes \mathbb{Z}_p$. By [Kis, § 1.3.3] there exists an embedding of Shimura data $\iota : (\mathsf{G}, X) \hookrightarrow (\mathsf{GSp}_{2g}, S^{\pm})$ which is induced by an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GSp}_{\mathbb{Z}_{(p)}}$. Moreover, there exists an open compact subgroup $\mathsf{K}'^p \subset \mathsf{GSp}_{2g}(\mathbb{A}_f^p)$ such that ι induces an embedding $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, X) \hookrightarrow \mathrm{Sh}_{\mathsf{K}'}(\mathsf{GSp}_{2g}, S^{\pm})$ with $\mathsf{K}' = \mathsf{K}'_p\mathsf{K}'^p$ and $\mathsf{K}'_p = \mathrm{GSp}_{2g}(\mathbb{Z}_p)$.

Denote by $\mathscr{A}_g$ the moduli space of principally polarised Abelian varieties with K'^p-level structure over $\mathbb{Z}_{(p)}$ and let $(\mathcal{A}^{\mathrm{univ}}, \lambda^{\mathrm{univ}}, \eta^{\mathrm{univ}})$ be its universal object. Then $\mathscr{A}_g$ is a smooth scheme whose generic fibre is canonically isomorphic to $\mathrm{Sh}_{\mathsf{K}'}(\mathsf{GSp}_{2g}, S^{\pm})$ (see e.g. [Kot92, § 5]). Let $\mathscr{S}_G^-$ be the closure of $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, \mathsf{X})$ in $\mathscr{A}_g \otimes O_E$, denote by $\mathscr{S}_G$ its normalisation.

Theorem 2.3 ([Kis10, Thm. 2.3.8])**.** *$\mathscr{S}_G$ is the canonical integral model of $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, \mathsf{X})$, in particular it is smooth and independent of the choice of the embedding $\mathrm{Sh}_{\mathsf{K}}(\mathsf{G}, \mathsf{X}) \hookrightarrow \mathrm{Sh}_{\mathsf{K}'}(\mathsf{GSp}_{2g}, \mathsf{S}^{\pm})$.*

A central point of his proof is the following construction. Denote by $h : (\mathcal{A}_G, \lambda_G) \to \mathscr{S}_G$ the pullback of $(\mathcal{A}^{\mathrm{univ}}, \lambda^{\mathrm{univ}})$. Let (M, ψ) be the symplectic $\mathbb{Z}_p$ module of rank $2g$ and fix a family of tensors $(s_\alpha) \subset M^{\otimes}$ such that $G \subset \mathrm{GSp}_{2g}$ is their stabiliser (cf. [Kis10, Prop. 1.3.2]). For every closed geometric point in the special fibre $x \in \mathscr{S}_G(\overline{\mathbb{F}}_p)$, Kisin associates a family of crystalline Tate tensors $(t_{G,\alpha,x}) \subset \mathbb{D}(\mathcal{A}_{G,x})$ such that $(\mathbb{D}(\mathcal{A}_{G,x}), \lambda_{G,x}, t_{G,\alpha,x})$ is isomorphic to $(M_{\breve{\mathbb{Z}}_p}, \psi \otimes 1, s_\alpha \otimes 1)$ (cf. [Kis, Prop. 1.3.7]). One can describe the local geometry of the normalisation morphism as follows.

Proposition 2.4 ([Kis, Prop. 1.3.9, Cor. 1.3.11])**.** *Let $x_0 \in \mathscr{S}_G^-(\overline{\mathbb{F}}_p)$.*

(1) *Two points $x, x' \in \mathscr{S}_G(\overline{\mathbb{F}}_p)$ over x_0 are identical if and only if $t_{G,\alpha,x} = t_{G,\alpha,x'}$.*
(2) *There is a canonical isomorphism of the formal neighbourhood $\mathscr{S}_{G,x}^{\wedge}$ with the deformation space $\mathfrak{Def}(\mathcal{A}_x^{\mathrm{univ}}[p^\infty], \lambda_x^{\mathrm{univ}}, t_{G,\alpha,x})$ such that*

$$\begin{array}{ccc}
\mathscr{S}_{G,x_0}^{\wedge} & \xrightarrow{\sim} & \coprod_{x\mapsto x_0} \mathfrak{Def}(\mathcal{A}_{x_0}^{\mathrm{univ}}[p^\infty], \lambda_{x_0}^{\mathrm{univ}}, t_{G,\alpha,x}) \\
\downarrow & & \downarrow \\
\mathscr{S}_{G,x_0}^{-\wedge} & \xrightarrow{\sim} & \bigcup_{x\mapsto x_0} \mathfrak{Def}(\mathcal{A}_{x_0}^{\mathrm{univ}}[p^\infty], \lambda_{x_0}^{\mathrm{univ}}, t_{G,\alpha,x}) \hookrightarrow \mathfrak{Def}(\mathcal{A}_{x_0}^{\mathrm{univ}}[p^\infty], \lambda_{x_0}^{\mathrm{univ}}) = \mathscr{A}_{g,x_0}^{\wedge}
\end{array}$$

commutes.

2.3. **The Newton stratification and Oort's foliation.** In order to define the Newton stratification on the special fibre $\mathscr{S}_{G,0}$ of $\mathscr{S}_G$, we first recall some facts about σ-conjugacy classes. We fix a maximal torus and a Borel subgroup $T \subset B \subset G$. Let k be an algebraically closed field of characteristic p, $W(k)$ its ring of Witt vectors and let $L(k)$ be the fraction field of $W(k)$. We denote for any element $b \in G(L(k))$ by $[b]$ its σ-conjugacy class and by $B(G)$ the set of σ-conjugacy classes in $G(L(k))$. The set $B(G)$ is independent of the choice of k by [RR96, Lemma 1.3]. An element $\mathbf{b} \in B(G)$ is uniquely defined by two invariants ([Kot85, § 6]), its Newton point $\nu(\mathbf{b}) \in X_*(T)_{\mathrm{dom}}^{\mathrm{Gal}(\overline{\mathbb{Q}}_p,\mathbb{Q}_p)}$ and its Kottwitz point $\kappa(\mathbf{b}) \in \pi_1(G)_{\mathrm{Gal}(\overline{\mathbb{Q}}_p,\mathbb{Q}_p)}$.

For $x \in \mathscr{S}_G(\overline{\mathbb{F}}_p)$ choose b_x such that the Frobenius morphism on $\mathbb{D}(\mathcal{A}_x^{\mathrm{univ}})$ is mapped to $b_x\sigma$ under an isomorphism $(\mathbb{D}(\mathcal{A}_x^{\mathrm{univ}}), \lambda, t_{\alpha,x}) \cong (M_{\breve{\mathbb{Z}}_p}, \psi \otimes 1, s_\alpha \otimes 1)$. Then b_x is uniquely defined up to $G(\breve{\mathbb{Z}}_p)$-σ-conjugacy, in particular $[b_x]$ is independent of the choice of an isomorphism. We define

$$\mathscr{S}_G^{\mathbf{b}}(\overline{\mathbb{F}}_p) = \{x \in \mathscr{S}(\overline{\mathbb{F}}_p) \mid b_x \in \mathbf{b}\}.$$

Recall that the choice of s_α induced horizontal tensors $t_{G,\alpha,\mathrm{dR}}$ on the first de Rham - cohomology $R^1h_*\Omega^\bullet$, as well as tensors $t_{G,\alpha,p}$ on $R^1h_{\eta\, et*}\mathbb{Z}_p$ and $t_{G,\alpha,l}$ on $R^1h_{et*}\mathbb{Q}_l$ for $l \neq p$ (cf. [Kis, § 1.3.6]). Since $\mathscr{S}_G$ is smooth, the tensors $t_{G,\alpha,\mathrm{dR}}$ induce crystalline Tate tensors on $\mathbb{D}(\mathcal{A}_G/\mathscr{S}_{G,0})$ (see [HP15, § 3.1.6]). By construction the $t_{G,\alpha}$ specialise to $t_{G,\alpha,x}$ for every $x \in \mathscr{S}_G(k)$. Lovering showed in [Lov17] that since $(\mathbb{D}(\mathcal{A}_G), \psi, t_{G,\alpha})$ is locally isomorphic to (M, φ, s_α), we obtain an isocrystal with G-structure over $\mathscr{S}_{G,0}$ in the sense of Rapoport and Richartz after inverting p. Thus, the $\mathscr{S}_G^{\mathbf{b}}$ are locally closed by [RR96, Thm. 3.6]; we equip them with the reduced subscheme structure.

Let $\mathbf{b} \in B(G)$ such that $\mathscr{S}_G^{\mathbf{b}}$ is nonempty and fix a polarised Barsotti-Tate group with crystalline Tate-tensors$(X_0, \lambda_0, t_{0,\alpha})$ over k such that its Dieudonné crystal is isomorphic to $(M_{\breve{\mathbb{Z}}_p}, \psi \otimes 1, s_\alpha \otimes 1)$ and its Frobenius corresponds to $\mathbf{b}$. The corresponding central leaf C in $\mathscr{S}_G^{\mathbf{b}}$ is the set of geometric points $x : \operatorname{Spec} k \to \mathscr{S}_G^{\mathbf{b}}$ such that

$$(\mathcal{A}_x^{\mathrm{univ}}[p^\infty], \lambda_x^{\mathrm{univ}}, t_{\alpha,x}) \cong (X_0, \lambda_0, t_{0,\alpha}) \otimes k.$$

Then C defines a closed subset of $\mathscr{S}_G^{\mathbf{b}}$ by [Ham16, Prop. 2.14].

Proposition 2.5. *C is smooth of dimension $2\langle \rho, \nu(\mathbf{b})\rangle$, where ρ denotes the half-sum of positive roots of G.*

Proof. First note that [Kis, Prop. 1.4.4] implies that C is nonempty, if $\mathscr{S}_G^{\mathbf{b}}$ is. We fix $x \in C(\overline{\mathbb{F}}_p)$ and consider the isomorphism $\mathscr{S}_{G,0,x}^{\wedge} \cong \mathfrak{Def}(X_0, \lambda_0, t_{0,\alpha})$. As this identifies the canonical Barsotti-Tate groups with crystalline Tate-tensors lying over these schemes, it identifies $C_x^{\wedge}$ with the central leaf in the deformation space. Thus all formal neighbourhoods of closed points of C are isomorphic, which shows that it is smooth. Also, $\dim C$ equals the dimension of the central leaf in $\mathfrak{Def}(X_0, \lambda_0, t_{0,\alpha})$, thus the second assertion follows from [Ham16] Proposition 3.9. $\square$

From now on, we fix $\mathbf{b}$ and assume that X_0 is completely slope divisible. We denote by $\mathbf{b}'$ the image of $\mathbf{b}$ in $B(\mathrm{GSp}_{2g})$ and by $C' \subset \mathscr{A}_g^{\mathbf{b}'}$ the central leaf associated to (X_0, λ_0). Since C' is smooth the Barsotti-Tate group $\mathcal{X}_0 := \mathcal{A}^{\mathrm{univ}}[p^\infty]_{|C'}$ is completely slope divisible ([Man05] § 3). We denote by X_0^i and $\mathcal{X}_0^i$ the respective isoclinic subquotients of X_0 and $\mathcal{X}_0$.

3. Almost product structure in the Siegel moduli space

As our construction of the almost product structure for Shimura varieties of Hodge type will heavily rely on the almost product structure for Siegel moduli spaces, we recall its construction.

3.1. **Construction of the almost product structure.** The almost product structure of Newton strata in the Siegel moduli space is originally due to Oort [Oor04]. Mantovan generalized this result to Shimura varieties of PEL-type and gave a more rigorous description in [Man05]. We briefly sketch Mantovan's construction in the case of the Siegel moduli space. In particular, no claim of originality is made for this subsection.

Denote by $\mathscr{J}'_m \to C'$ the Igusa variety of level m, that is the scheme parametrising families of isomorphisms $j_m^i : X_0^i[p^m] \xrightarrow{\sim} \mathcal{X}_0^i[p^m]$ which commute with the polarisations and for any $m' > m$ can be lifted étale locally to an isomorphism of the schemes of $p^{m'}$-torsion points.

Proposition 3.1 ([Man05] Prop. 4)**.** *The morphism $\mathscr{J}'_m \to C'$ is finite étale and Galois with Galois group $\Gamma'_{b,m} := \mathrm{im}(\mathrm{Aut}(X_0) \to \mathrm{Aut}(X_0[p^m]))$.*

Denote by $\mathfrak{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')$ the Rapoport-Zink space parametrising quasi-isogenies with source (X_0, λ_0) and by $\mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')$ its underlying reduced subscheme. For any pair of integers m_1, m_2, let $\mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')^{m_1,m_2} \subset \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')$ denote the (closed) subvariety parameterising isogenies ρ such that $p^{m_1}\rho$ and $p^{m_2}\rho^{-1}$ are isogenies.

Now assume $m_1 + m_2 \leq m$ and choose N such that the filtration on $F^*_{p^N/k}\mathcal{X}_0[p^m]$ induced by the slope filtration splits canonically as in [Man05] Lemma 8. Thus the family of isomorphisms (j_m^i) induced by $\mathscr{J}'^{(p^{-N})}_m \to \mathscr{J}'_m$ yield an isomorphism $j_m : X_0[p^m] \times \mathscr{J}'^{(p^{-N})}_m \xrightarrow{\sim} \mathcal{X}_0[p^m]_{\mathscr{J}'^{(p^{-N})}_m}$.

Now $\pi'_N : \mathscr{J}'^{(p^{-N})}_m \times \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')^{m_1,m_2} \to \mathscr{A}_g^{b'}$ is constructed as follows. Let $(\mathcal{A}, \lambda, \eta)$ be the pullback of $(\mathcal{A}^{\mathrm{univ}}, \lambda^{\mathrm{univ}}, \eta^{\mathrm{univ}})$ to $\mathscr{J}'^{(p^{-N})}_m \times \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')^{m_1,m_2}$. Denote by ρ the universal quasi-isogeny over $\mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(b')^{m_1,m_2}$. We define

$$\mathcal{A}' := \mathcal{A}\Big/_{j_m(\ker(p^{m_1}\rho))}.$$

and endow it with the polarisation λ' and a K'^p-level structure η'^p induced by λ and η, respectively. Then $(\mathcal{A}', \lambda', \eta')$ defines a morphism

$$\pi'_{N,m,m_1,m_2} : \mathscr{J}'^{(p^{-N})}_m \times \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')^{m_1,m_2} \to \mathscr{A}_g^{\mathbf{b}'}.$$

The π_{N,m,m_1,m_2} satisfy the obvious commutativity relations when varying N, m, m_1, m_2.

Proposition 3.2 ([Man05] Prop. 9)**.** *The family of morphisms (π'_{N,m,m_1,m_2}) satisfies the obvious commutativity relations. More precisely, the following identeties hold.*

(1) $\pi'_{N,m,m_1,m_2} \circ (F_{p/k} \times \mathrm{id}) = \pi'_{N+1,m,m_1,m_2}$.
(2) *Let* $r'_{m+1,m} : \mathscr{J}'_{m+1} \twoheadrightarrow \mathscr{J}'_m$ *be the canonical projection. Then* $\pi'_{N,m-1,m_1,m_2} \circ (r'_{m+1,m} \times \mathrm{id}) = \pi'_{N,m,m_1,m_2}$*, where it is assumed that* $m_1 + m_2 \leq m - 1$.
(3) *Let* $\iota' : \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b})^{m'_1,m'_2} \hookrightarrow \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b})^{m_1,m_2}$ *the obvious embedding for* $m'_1 \leq m_1, m'_2 \leq m_2$*. Then* $\pi'_{N,m,m_1,m_2} \circ \iota = \pi'_{N,m',m'_1,m'_2}$.

Notation 3.3. For the rest of this article, we will abbreviate π'_{N,m,m_1,m_2} by $\pi'_{N,m}$ and assume that one has made a fixed choice of m_1, m_2 depending on m such that for varying m the sequences (m_1) and (m_2) are monotonously increasing and unbounded.

Proposition 3.4 ([Man05, Prop. 10, 11])**.** *The morphism* $\pi'_{N,m}$ *is finite, and surjective if* m *is big enough.*

3.2. **The almost product structure of infinite level.** Proposition 3.2 tells us that we can take the limit for $N, m \to \infty$ to obtain a morphism

$$\pi'_\infty : \varprojlim \mathscr{J}_m^{(p^{-\infty})} \times \mathscr{M}_{\mathrm{GSp},\mu'}(\mathbf{b}') \to \mathscr{A}_g^{\mathbf{b}'}.$$

A generalisation of this infinite almost product structure was studied by Caraiani and Scholze in [CS15, § 4]. The results in this section are a formal consequence of their work, we will give the proofs where they simplify due to our restriction to the special fibre or where we will need the technical details to generalise to Shimura varieties of Hodge type.

It is favourable for our purposes to consider the perfection $\pi'^{(p^{-\infty})}_\infty$ of this morphism; we will later see that this makes the morphism weakly étale. Moreover, we can define this morphism directly rather than taking a limit. We define the Igusa variety of infinite level as $\mathscr{J}'_\infty := \varprojlim \mathscr{J}'_m$. As a consequence of the moduli description of the finite level Igusa varieties, we obtain that $\mathscr{J}'_\infty$ parametrises families of isomorphisms $X_0^i \xrightarrow{\sim} \mathcal{X}_0^i$. We denote by $j_\infty^{i,\mathrm{univ}}$ the universal isomorphisms over $\mathscr{J}'_\infty$. As the slope filtration splits canonically over perfect schemes, the $j_\infty^{i,\mathrm{univ}}$s induce a universal isomorphism

$$j = \oplus j_\infty^{i,\mathrm{univ}} : X_0 \times \mathscr{J}'^{(p^{-\infty})}_\infty \xrightarrow{\sim} \mathcal{X}_{0,\mathscr{J}'^{(p^{-\infty})}_\infty}$$

Lemma 3.5 ([CS15, Prop. 4.3.8])**.** $\mathscr{J}'^{(p^{-\infty})}_\infty$ *is the moduli space of isomorphisms* $(X_0, \lambda) \xrightarrow{\sim} (\mathcal{A}^{\mathrm{univ}}[p^\infty], \lambda^{\mathrm{univ}})$ *and* j *is the universal object.*

Thus we can repeat the above construction. Let ρ^{univ} be the universal quasi-isogeny over $\mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')$ and $(\mathcal{A}, \lambda, \eta)$ the pullback of $(\mathcal{A}^{\mathrm{univ}}, \lambda^{\mathrm{univ}}, \eta^{\mathrm{univ}})$ to $\mathscr{J}'^{(p^{-\infty})}_\infty \times \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')$. Now, Zariski-locally there exists an integer m_1 such that $p^{m_1}\rho^{\mathrm{univ}}$ is an isogeny. By gluing $\mathcal{A}/j(\ker p^{m_1}\rho)$ over a suitable Zariski covering, we obtain a polarised Abelian variety over $\mathscr{J}'^{(p^{-\infty})}_\infty \times \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')^{(p^{-\infty})}$ with K'^p level structure. This induces a morphism

$$\pi'_\infty : \mathscr{J}'^{(p^{-\infty})}_\infty \times \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(b') \to \mathscr{A}_g^{\mathbf{b}'}.$$

In order to describe the geometric properties of the almost product structure, we need to derive some results about constant Barsotti-Tate groups, i.e. Barsotti-Tate groups which descend to the spectrum of an algebraically closed field. In the following let k be an algebraically closed field of characteristic p.

Lemma 3.6. *Let* X, Y *be Barsotti-Tate groups over* k *and let* S *be a connected reduced* k*-scheme. Then the morphism* $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X_S, Y_S)$ *induced by pullback is an isomorphism.*

Proof. Since the property of two homomorphisms being identical is closed, it suffices to check on generic points, i.e. assume that $S = \operatorname{Spec} k_0$ for some field k_0. As this property is also fpqc-local, we may assume that k_0 is perfect. Now the assertion follows by Dieudonné theory. $\square$

We can deduce the homomorphisms constant Barsotti-Tate groups over not necessarily connected schemes from the previous lemma, but still need that the base scheme is reduced.

Lemma 3.7. *Let X, Y be Barsotti-Tate groups over k and let $H := \operatorname{Hom}(X, Y)$ as topological group equipped with the p-adic topology. Then the functor on reduced k-schemes*

$$\begin{array}{rcl}(\mathrm{RedSch}_k)^{opp} & \to & (\mathrm{Sets}) \\ S & \mapsto & \operatorname{Hom}(X_S, Y_S)\end{array}$$

is represented by the locally constant k-scheme induced by H.

Proof. Let S be a reduced k-scheme. Giving a morphism $\phi \in Hom(X_S, Y_S)$ is equivalent to giving an inductive system of $\phi_m \in \operatorname{Hom}(X[p^m], Y[p^m])$ for $m > 0$. By the previous lemma ϕ_m can only be lifted to a morphism of Barsotti-Tate groups only if it is constant on every connected component and is hence given by a morphism

$$g_m : S \to \left(H/p^m\right)_k \subset \underline{\operatorname{Hom}}(X[p^m], Y[p^m])$$

or equivalently by a continuous map

$$f_m : \pi_0(S) \to H/p^m.$$

Altogether, we have a natural bijection

$$\operatorname{Hom}(X_S, Y_S) \cong \varinjlim \operatorname{Hom}_{\mathrm{cont}}(\pi_0(S), H/p^m) = \operatorname{Hom}_{\mathrm{cont}}(\pi_0(S), H).$$

$\square$

The following is immediate.

Corollary 3.8. *Let X, Y be Barsotti-Tate groups over k and let $J := \operatorname{QIsog}(X, Y)$ be equipped with the p-adic topology. Then the functor*

$$\begin{array}{rcl}(\mathrm{RedSch}_k)^{opp} & \to & (\mathrm{Sets}) \\ S & \mapsto & \operatorname{QIsog}(X_S, Y_S)\end{array}$$

is represented by the locally constant k-scheme induced by J.

Denote by $\Gamma_{b'}$ the automorphism group of (X_0, λ_0) and by $J_{b'}$ the group of self-quasi isogenies equipped with the p-adic topology. When regarded as locally profinite $\overline{\mathbb{F}}_p$-group schemes, they represent the sheaves $\underline{\operatorname{Aut}}(X_0, \lambda_0)$ and $\underline{\operatorname{Aut}}_{\mathbb{Q}}(X_0, \lambda_0)$, respectively, in the category of reduced schemes by the previous proposition.

The group $J_{b'}$ acts on $\mathscr{J}_\infty^{\prime(p^{-\infty})}$ via

$$g.(\mathcal{A}; j^1, \ldots, j^r) = ((j_* g^{-1})(\mathcal{A}, \lambda, \eta); g^* j).$$

More explicitly, $(j_* g)(\mathcal{A}, \lambda, \eta)$ is defined as $\mathcal{A}/j(\ker p^{m_2} g^{-1})$ with the induced additional structure where m_2 big enough such that $p^{m_2} g^{-1}$ is an isogeny. The isomorphism $g^* j$ is defined as the (unique) morphism such that the following diagram is commutative.

$$\begin{tikzcd}
X_0 \arrow[rr, "j"] \arrow[d, two heads, "p^{m_2} g^{-1}"] & & \mathcal{A}[p^\infty] \arrow[d] \\
X_0 \big/ j(\ker p^{m_2} g^{-1}) = X_0 \arrow[rr, dashed, "g^* j"] & & (\mathcal{A} \big/ j(\ker p^{m_2} g^{-1}))[p^\infty].
\end{tikzcd}$$

Because of our moduli interpretation of $J_{b'}$, this action is continuous.

Remark 3.9. *(1)* If $\rho \in \Gamma_{b'}$, then

$$\rho.(\mathcal{A}; j) = (\mathcal{A}, j \circ g)$$

(2) The $J_{b'}$-action is induced by the $\mathrm{Aut}(\widetilde{\mathbb{X}}_b)$-action given in [CS15, Cor. 4.3.5]. In particular, the $J_{b'}$-action extends the action of the submonoid $S_{b'}$ on the tower of (finite level) Igusa varieties as constructed in [Man05] § 4. Moreover Mantovan extends the action of $S_{b'}$ to $J_{b'}$ on the cohomology, where her and our $J_{b'}$-action coincide.

Consider the diagram

$$\begin{tikzcd}
& \mathscr{J}_\infty'^{(p^{-\infty})} \times \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')^{(p^{-\infty})} \arrow[dl, "r_\infty'^{(p^{-\infty})} \times \mathrm{id}"'] \arrow[dr, "\pi_\infty'^{(p^{-\infty})}"] & \\
C'^{(p^{-\infty})} \times \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')^{(p^{-\infty})} & & \mathscr{A}_g^{\mathbf{b}'(p^{-\infty})}
\end{tikzcd}$$

obtained by taking the perfection over $\overline{\mathbb{F}}_p$. Note that by construction r'_∞ is $\Gamma_{b'}$-invariant and π'_∞ is $J_{b'}$-invariant.

Proposition 3.10 (cf. [CS15, Prop. 4.3.13])**.** *We have the following results on the geometry of above correspondence.*

(1) $r_\infty'^{(p^{-\infty})}$ *is profinite proétale with pro-Galois group* $\Gamma_{b'}$.

(2) $\pi_\infty'^{(p^{-\infty})}$ *is a* $J_{b'}$*-torsor for the proétale topology.*

Proof. The first assertion is a direct consequence of Proposition 3.1. To prove the second assertion, we first prove that π'_∞ is weakly étale. As this property is local on the source, it suffices to show that $\mathscr{J}_\infty'^{(p^{-\infty})} \times U \to \mathscr{A}_g^{b'(p^{-\infty})}$ is weakly étale for any quasi-compact open subset U. As U is quasi-compact, there exist m_1, m_2 such that $U \subset \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(b')^{m_1,m_2\,(p^{-\infty})}$. Thus the above morphism factors as

$$\mathscr{J}_\infty'^{(p^{-\infty})} \times U \to \mathscr{J}_{m_1+m_2}'^{(p^{-\infty})} \times U \to \mathscr{A}_g^{b'(p^{-\infty})}$$

Now the left morphism is proétale and thus weakly étale and the right morphism is étale by [Ham16] Prop. 4.6. Thus their concatenation is also weakly étale.

To ease the notation, we write $\mathscr{P}'_\infty := \mathscr{J}_\infty'^{(p^{-\infty})} \times \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(b')^{(p^{-\infty})}$ and denote by $a' : J_{b'} \times \mathscr{P}'_\infty \to \mathscr{P}'_\infty$ the $J_{b'}$-action constructed above. To prove that π'_∞ is a $J_{b'}$-torsor, we have to show that

$$\mathrm{pr}_2 \times a' : J_{b'} \times \mathscr{P}'_\infty \to \mathscr{P}'_\infty \times_{\mathscr{S}_G^{b'(p^{-\infty})}} \mathscr{P}'_\infty$$

is an isomorphism. The fact that it is a monomorphism follows from the fact that the action of $J_{b'}$ is faithful. So let S be a perfect scheme and (P_1, P_2) be an S-valued point of the right hand side, i.e. $P_1 = (\mathcal{A}_1, \lambda_1, \eta_1, j_1; \rho_1)$ and $P_2 = (\mathcal{A}_2, \lambda_2, \eta_2, j_2; \rho_2)$ with common image $(\mathcal{A}_3, \lambda_3, \eta_3)$ under π'_∞. Then there exists a (unique) quasi-isogeny $g \in J_{b'}(S)$ such that the diagram

$$\begin{tikzcd}
X_{0,S} \arrow[rr, dashed, "g"] \arrow[d, "j_1"] & & X_{0,S} \arrow[d, "j_2"] \\
\mathcal{A}_1[p^\infty] \arrow[r, "j_{1*}\rho_1"] & \mathcal{A}_3[p^\infty] & \mathcal{A}_2[p^\infty] \arrow[l, "j_{2*}\rho_2"']
\end{tikzcd}$$

commutes, i.e. such that $(\mathrm{pr}_2 \times a')(g, P_1) = (P_1, P_2)$. □

4. Almost product structure in Shimura varieties of Hodge type

The almost product structure of $\mathscr{S}_G^{\mathbf{b}}$ will be constructed as a suitable lift of $\pi'_{N,m}$ and π'_∞, respectively. Before that, we define the factors of the product.

4.1. **Igusa varieties.** We define the perfect infinite level Igusa variety over C as the locus $\mathscr{J}_\infty{}^{(p^{-\infty})} \subset (\mathscr{J}'_\infty \times_{C'} C)^{(p^{-\infty})}$ where $t_{0,\alpha} = j^* t_{G,\alpha}$. In other words, $\mathscr{J}_\infty{}^{(p^{-\infty})}$ parametrises the isomorphisms $(X_0, \lambda_0, t_{0,\alpha}) \cong (\mathcal{A}_G, \lambda_G, t_{G,\alpha})$. Assuming for a moment that this gives us a well-defined scheme, we define the Igusa varieties as

$$\begin{aligned}
\mathscr{J}_\infty &:= \mathrm{im}(\mathscr{J}_\infty^{(p^{-\infty})} \to \mathscr{J}'_\infty \times_{C'} C) \\
\mathscr{J}_m &:= \mathrm{im}(\mathscr{J}_\infty^{(p^{-\infty})} \to \mathscr{J}'_m \times_{C'} C).
\end{aligned}$$

Proposition 4.1. *Let $\Gamma_b \subset \Gamma_{b'}$ the topological subgroup of elements stabilising $t_{0,\alpha}$.*

(1) *$\mathscr{J}_\infty^{(p^{-\infty})}$ is a closed union of connected components of $(\mathscr{J}'_\infty \times_{C'} C)^{(p^{-\infty})}$.*
(2) *$\mathscr{J}_\infty \to C$ is profinite proétale with Galois group Γ_b.*
(3) *The morphism $r_m : J_m \to C$ is finite étale with Galois group $\Gamma_{b,m} := \mathrm{im}(\Gamma_b \to \mathrm{End}(X_0[p^m]))$.*

Proof. By [RR96, Lemma 3.9] $\mathscr{J}_\infty^{(p^{-\infty})}$ is a union of connected components. Now by construction, $\Gamma_{b'}$ acts on $\pi_0(\mathscr{J}'_\infty \times_{C'} C)$ and the orbits are in canonical bijection with $\pi_0(C)$. In particular, there are only finitely many orbits. Within each $\Gamma_{b'}$-orbit, there is a unique Γ_b-orbit which contains all the connected components contained in $\mathscr{J}_\infty^{(p^{-\infty})}$. As Γ_b is a closed subgroup of $\Gamma_{b'}$, it follows that the set of connected components of $\mathscr{J}_\infty^{(p^{-\infty})}$ is a closed subspace of $\pi_0(\mathscr{J}'_\infty \times_{C'} C)$. Hence $\mathscr{J}_\infty{}^{(p^{-\infty})} \subset (\mathscr{J}'_\infty \times_{C'} C)^{(p^{-\infty})}$ is a closed union of connected components. The second and third assertion follow from the first and the fact that Γ_b acts simply transitively on the fibres. □

4.2. **Rapoport-Zink spaces of Hodge type.** In this section we recall the notion of Rapoport-Zink spaces of Hodge type. Our main reference is [HP15] (see also [Kim16]).

Definition 4.2 ([HP15, Def. 2.3.3])**.** Let $(\mathrm{ANilp}^{\mathrm{fsm}})$ denote the category of formally smooth formally finitely generated $\breve{\mathbb{Z}}_p$-algebras A such that p is nilpotent in A. Define

$$\begin{aligned}
\mathrm{RZ}_{\mathrm{X}_0,\lambda_0,\mathrm{t}_{0,\alpha}} : \mathrm{ANilp}^{\mathrm{fsm}} &\to (\mathrm{Set}) \\
A &\mapsto \{(X, \lambda, t_\alpha, \rho)\}/_{\cong},
\end{aligned}$$

where (X, λ, t_α) is a polarised Barsotti-Tate group with crystalline Tate-tensors over A and $\rho : X_0 \otimes A/p \to X \otimes A/p$ a quasi-isogeny respecting additional structures such that the following compatibility criteria are met.

(a) The sheaf of isomorphisms

$$\underline{\mathrm{Isom}}((\mathbb{D}(X),\lambda,t_\alpha),(\mathbb{D}(X_0),\lambda_0,t_{0,\alpha}))$$

is a crystal of $G_{\breve{\mathbb{Z}}_p}$-torsors w.r.t. the natural $G_{\breve{\mathbb{Z}}_p}$-action.

(b) Étale locally, we have $\mathbb{D}(X)_A \cong M_A$ compatible with additional structure such that the image of the Hodge filtration $\mathrm{Fil}^1 \subset \mathbb{D}(X)_A$ is induced by a cocharacter conjugate to μ.

Theorem 4.3 ([HP15, Thm. 3.2.1])**.** *The functor $\mathrm{RZ}_{X_0,\lambda_0,t_{0,\alpha}}$ is representable by a formally smooth formal scheme $\mathfrak{M}_{G,\mu}(b)$ over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$, which is formally of finite type.*

The canonical morphism $\mathfrak{M}_{G,\mu}(\mathbf{b}) \to \mathfrak{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')$ is a closed embedding ([HP15, Prop. 3.2.11]); we identify $\mathfrak{M}_{G,\mu}(\mathbf{b})$ with its image. It is stable under the action of $J_b \subset J_{b'}$ by [HP15, Thm. 3.2.1]. We will work with its underlying reduced subscheme $\mathscr{M}_{G,\mu}(\mathbf{b})$.

Moreover, $\mathscr{M}_{G,\mu}(\mathbf{b})$ is equipped with a morphism $\Theta_x : \mathscr{M}_{G,\mu}(\mathbf{b}) \to \mathscr{S}_G^{\mathbf{b}}$ for every $x \in \mathscr{J}_\infty(\overline{\mathbb{F}}_p)$, which can be described as follows. For $x' \in \mathscr{J}'_\infty(\overline{\mathbb{F}}_p)$ denote by x'_m its image in $\mathscr{J}'_m(\overline{\mathbb{F}}_p)$ amd let

$$\Theta'_{x',m} := \pi_{N,m|\{x'_m\}\times \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')^{m_1,m_2}} : \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')^{m_1,m_2} \to \mathscr{A}_g^{\mathbf{b}'}$$

and define $\Theta'_{x'} : \mathscr{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}') \to \mathscr{A}_g^{\mathbf{b}'}$ as their limit (see also [RZ96, Thm 6.21]). If x' is the image of a point $x \in \mathscr{J}_\infty(\overline{\mathbb{F}}_p)$, then $\Theta'_{x'|\mathscr{M}_{G,\mu}(\mathbf{b})}$ factorizes through $\mathscr{S}_G^-$. The morphism Θ_x is defined as the unique lift of $\Theta'_{x'|\mathscr{M}_{G,\mu}(\mathbf{b})}$ such that $\Theta_x^*(\mathcal{A}_G[p^\infty],\lambda_G,t_{G,\alpha})$ is the universal Barsotti-Tate group with crystalline Tate-tensors over $\mathscr{M}_{G,\mu}(\mathbf{b})$ (see [HP15, § 3.2] for existence, [Kis, Prop. 1.3.11] for uniqueness). Note that the restriction $\Theta_{x,m}$ of Θ_x to $\mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2} := \mathscr{M}_{G,\mu}(\mathbf{b}) \cap \mathscr{M}^{m_1,m_2}_{\mathrm{GSp}_{2g},\mu'}$ depends only on the image of x in $\mathscr{J}_m(\overline{\mathbb{F}}_p)$. In the following, we will also use the notion $\Theta_{x,m}$ for $x \in \mathscr{J}_m(\overline{\mathbb{F}}_p)$.

Howard and Pappas do not assume X_0 to be completely slope divisible in [HP15]; but we can make this assumption without losing generality. Indeed, if (X,λ,t_α) is a Barsotti-Tate group with crystalline Tate-tensors over $\overline{\mathbb{F}}_p$ such that its Dieudonné crystal is isomorphic to $(M_{\breve{\mathbb{Z}}_p},\psi\otimes 1, s_\alpha\otimes 1)$, choose an isogeny $\varphi : (X,\lambda,t_\alpha) \to (X_0,\lambda_0,t_{\alpha,0})$ with X_0 completely slope divisible. Then we have a canonical isomorphism

$$\begin{array}{ccc} \mathrm{RZ}_{X_0,\lambda_0,t_{0,\alpha}} & \xrightarrow{\sim} & \mathrm{RZ}_{X,\lambda,t_\alpha} \\ \rho & \mapsto & \rho\circ\varphi, \end{array}$$

in particular we can identify their underlying reduced subschemes. Moreover, for any pair $z=(z_0,i)$ with $z_0 \in \mathscr{S}_G^b(\overline{\mathbb{F}}_p)$ and $i : (X,\lambda,t_\alpha) \xrightarrow{\sim} (\mathcal{A}_{G,z_0}[p^\infty],\lambda_{G,z_0},t_{G,\alpha,z_0})$ the characterizing property of Θ_z implies

$$\Theta_z = \Theta_x \tag{4.1}$$

where $x \in \mathscr{J}_\infty(\overline{\mathbb{F}}_p)$ given as follows. Its image in $\mathscr{S}_{G,0}$ equals $\Theta_z(\varphi)$ and we choose the isomorphism such that

$$\begin{tikzcd} X \arrow[rr, "\sim", "i"'] \arrow[d, two heads, "\varphi"] & & \mathcal{A}_{G,z_0}[p^\infty] \arrow[d, two heads] \\ X_0 \arrow[r, dashed, "\sim"] & \mathcal{A}_{G,\Theta_z(\varphi)} \arrow[r, "\sim"] & \mathcal{A}_{G,z_0}/\ker\varphi \end{tikzcd}$$

commutes.

4.3. **Construction of the correspondence.** Let $\pi_{N,m}^{-} : \mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2} \to \mathscr{A}_{g,0}^{\mathbf{b}'}$ be the pullback of $\pi'_{N,m}$. We define a lift to $\mathscr{S}_G^{\mathbf{b}}$ on $\overline{\mathbb{F}}_p$-points by

$$\pi_{N,m}(x,y) = \Theta_{x,m}(y).$$

We already know that the restriction of $\pi_{N,m}$ to $\{x\} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2}$ is a morphism of varieties. As the next step, we show that $H_{y,N} := \pi_{N,m|\mathscr{J}_m^{(p^{-N})} \times \{y\}}$ is morphism. For this, we study the possible lifts of a morphism with normal source to the normalisation of the target.

Proposition 4.4. *Let $\nu : \widetilde{S} \to S$ be an integral birational morphism of schemes. Given a quasi-compact morphism $f : T \to S$ with T irreducible, we fix an irreducible component T_0 of $\widetilde{T} := T \times_S \widetilde{S}$ which maps surjectively onto T. Then the restriction $\nu_0 : T_0 \to T$ of the pull-back ν_T of ν is again integral and birational. Moreover, we have a bijection*

$$\begin{array}{rcl} \{T_0 \subset \widetilde{T} \text{ irred. comp.} \mid \nu(T_0) = T\} & \leftrightarrow & \{T' \subset \widetilde{S} \text{ irreducible, closed} \mid \nu(T') = \overline{f(T)}\} \\ T_0 & \mapsto & \overline{f_{\widetilde{S}}(T_0)} \end{array}$$

Proof. As the property of being integral is stable under base change and concatination, ν_0 is integral. To show birationality, consider the scheme-theoretic images W and W_0 of T in S and T_0 in $\widetilde{S}$, respectively. Then W_0 is an irreducible component of $W \times_S \widetilde{S}$ which maps surjectively onto W. Thus it suffices to show the assertion in the following two cases.

(a) $T \to S$ is a closed immersion
(b) $T \to S$ is dominant.

Then (a) proves that $W_0 \to W$ is integral, birational and substituting $S = W, \widetilde{S} = W_0$ in (b) implies our assertion.

In the first case, assume without loss of generality that T and S are reduced and affine, say $S = \operatorname{Spec} R$, $\widetilde{S} = \operatorname{Spec} A$ and $T = \operatorname{Spec} R/\mathfrak{p}$. Then $T_0 = \operatorname{Spec} A/\mathfrak{P}$ where $\mathfrak{P}$ is a prime lying over $\mathfrak{p}$. As ν_0 is clearly dominant, it suffices to show that the induced morphism of fraction fields $Q(R/\mathfrak{p}) \hookrightarrow Q(A/\mathfrak{P})$ is surjective. Indeed, for any $\frac{r \mod \mathfrak{P}}{s \mod \mathfrak{P}} \in Q(A/\mathfrak{P})$ write $r = \frac{r_1}{r_2} \mod \mathfrak{P}, s = \frac{s_1}{s_2} \mod \mathfrak{P}$ with $r_1, r_2, s_1, s_2 \in R$; now

$$\frac{r \mod \mathfrak{P}}{s \mod \mathfrak{P}} = \frac{r_1/r_2 \mod \mathfrak{P}}{s_1/s_2 \mod \mathfrak{P}} = \frac{r_1 s_2 \mod \mathfrak{P}}{s_1 r_2 \mod \mathfrak{P}} \in Q(R/\mathfrak{p}).$$

In the second case, the generic fibre of $\widetilde{T}$ over T is the base change of the generic fibre of $\widetilde{S}$ over S. As the ν induces an isomorphism on the generic fibre, so does ν_T. Thus, there exists a unique top-dimensional component T_0 of $\widetilde{T}$ and ν_0 is birational. □

Corollary 4.5. *Let $f : T \to S$ be a quasi-compact morphism of schemes and assume that T is normal. Let $\nu : \widetilde{S} \to S$ be an integral birational morphism. Then we have a one-to-one correspondence.*

$$\begin{array}{rcl} \{\tilde{f} : T \to \widetilde{S} \mid \nu \circ \tilde{f} = f\} & \leftrightarrow & \{(T'_C)_{C \subset T \text{ conn. comp.}} \mid T'_C \subset \widetilde{S} \text{ irred. closed}, \nu(T'_C) = \overline{f(C)}\} \\ \tilde{f} & \mapsto & (\overline{\tilde{f}(C)}) \end{array}$$

Proof. Assume without loss of generality that T is irreducible and denote $\widetilde{T} := T \times_S \widetilde{S}$. We have a sequence of bijections

$$\begin{aligned} \{\tilde{f} : T \to \widetilde{S} \mid \nu \circ \tilde{f} = f\} &\leftrightarrow \{s : T \to \widetilde{T} \mid \nu_T \circ s = \mathrm{id}\} \\ &\leftrightarrow \{T_0 \subset T \text{ irreducible component} \mid \nu_T(T_0) = T\} \\ &\leftrightarrow \{T' \subset \widetilde{S} \text{ irreducible, closed} \mid \nu(T') = \overline{f(T)}\}. \end{aligned}$$

The first bijection is given by he universal property of the fibre product $T \times_S \widetilde{S}$. For the second bijection note that for every $T_0 \subset \widetilde{T}$ of the right hand side, the restriction $T_0 \to T$ is integral and birational by the previous proposition and hence an isomorphism as T is normal. In particular there exists a unique section $T \to T_0$. The third bijection is the last assertion of the previous proposition. $\square$

As $\mathscr{J}_m$ is smooth and thus in particular normal, we can apply the last corollary to construct the lift $H_{y,N,(x_J)}$ of $\pi_{N,m|\mathscr{J}_m^{(p^{-N})} \times\{y\}}$ as follows: For each connected component $J \subset \mathscr{J}_m^{(p^{-N})}$ we fix a closed point $x_J \in J$. Let $x_J^+ := \pi_{N,m}(x_J, y)$ and $x_J^- := \nu_{\mathrm{Sh}}(x_J^+)$. Using the identification of Proposition 2.4, $\pi_{N,m}^-$ maps $\mathscr{J}_{x_J}^\wedge \times \{y\}$ onto the central leaf of $\mathfrak{Def}(\mathcal{A}^{\mathrm{univ}}[p^\infty]_{x_J^-}, \lambda_{x_J^-}^{\mathrm{univ}}, t_{G,\alpha,x_J^+})$ by [Ham16, Prop. 4.6]. Thus $\pi_{N,m}^-(J \times \{y\})$ is its closure in $\mathscr{S}_{G,0}^-$, which is the image of the closure $C(x_J)$ of the central leaf in $\mathfrak{Def}(\mathcal{A}_G[p^\infty]_{x_J^+}, \lambda_{G,x_J^+}, t_{G,\alpha,x_J^+})$ in $\mathscr{S}_{G,0}$. Note that $C(x_J)$ is the connected component of a central leaf containing x_J^+. By Corollary 4.5 the family $(C(x_J))_{J \subset \mathscr{J}_m \text{ irr. comp.}}$ yields a lift $H_{y,N,(x_J)}$ of $\pi^-_{N,m|\mathscr{J}_m^{(p^{-N})}\times\{y\}}$.

Proposition 4.6. *Let ρ_y be the quasi-isogeny corresponding to $y \in \mathscr{M}_{G,\mu}(\mathbf{b})(\overline{\mathbb{F}}_p)$ and denote by $\rho_{Ig,y}$ the quasi-isogeny given by*

$$p^{m_1}\rho_{Ig,y} : \mathcal{A}_{G,\mathscr{J}_M^{(p^{-N})}}[p^\infty] \twoheadrightarrow \mathcal{A}_{G,\mathscr{J}_M^{(p^{-N})}}[p^\infty] \big/ j_m(\ker(p^{m_1}\rho_y)) = H_{y,N,(x_J)}^* \mathcal{A}_G[p^\infty].$$

(1) *The family of crystalline Tate-tensors $\rho_{Ig,y}^* t_{G,\alpha}$ coincide with the natural crystalline Tate-tensors on $\mathcal{A}_{G,\mathscr{J}_m^{(p^N)}}$.*
(2) *$H_{y,N,(x_J)} = H_{y,N}$, in particular the lift is independent of the choice of the points x_J.*

Proof. By the rigidity of crystals it suffices to show the equality of the crystalline Tate tensors in one point of each connected component. Indeed, they are equal over x_J by construction. The second part follows directly from [Kis, Prop. 1.3.11], where Kisin shows that the crystalline Tate tensors determines a point in a fibre of ν_{Sh} uniquely. $\square$

So far we have shown that the restriction of $\pi_{N,m}$ to subvarieties of the form $\mathscr{J}_{(p^{-N})} \times \{y\}$ and $\{x\} \times \mathscr{M}_{G,\mu}(\mathbf{b})$ is a morphism of varieties. We now deduce the general result.

Theorem 4.7. *$\pi_{N,m}$ is a morphism of varieties.*

Proof. Denote $\widetilde{T} := (\mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2}) \times_{\mathscr{S}_G^-} \mathscr{S}_G$ and let $\nu_T : \widetilde{T} \to \mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2}$ be the canonical projection. Via the universal properties of the fibre product, $\pi_{N,m}$ induces a (set-theoretical) section $s : (\mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2})(\overline{\mathbb{F}}_p) \to \widetilde{T}(\overline{\mathbb{F}}_p)$. We have to show that s is a morphism of varieties.

For this it suffices to show that ν_T is locally an isomorphism at $s(x,y)$ for every $(x,y) \in (\mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2})(\overline{\mathbb{F}}_p)$. Indeed, in this case it there exists a unique section (in the set- and scheme-theoretic sense) in a neighbourhood U of (x,y) mapping $(x,y) \mapsto s(x,y)$. By uniqueness this section must be given by $s_{|U}$. As these neighbourhoods cover $\mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2}$, the claim follows.

First, we show that ν_T induces an isomorphism of formal neighbourhoods at $s(x,y)$, i.e. it is étale at $s(x,y)$. For this it suffices to show that the restriction $\pi^-_{N,m}$ to the formal neighbourhood of (x,y) in $\mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2}$ factorizes through $\mathscr{S}^\wedge_{G\,\pi_{N,m}(x,y)} \subset \mathscr{S}^{-\,\wedge}_{G\,\pi^-_{N,m}(x,y)}$. Since the formal neighbourhood in a Igusa variety is identified with the central leaf in the deformation space under the isomorphism of Proposition 2.4, this follows from [Ham16, Prop. 4.5].

Thus it suffices to show the restriction of ν_T to the stalk at $s(x,y)$ induces a birational morphism, as any finite étale birational morphism is an isomorphism. By Proposition 4.4 the restriction of ν_T to irreducible components is birational, so it suffices to show that ν_T induces a bijection of irreducible components of the stalks. This is obvious over the smooth locus as in this case the irreducible components of the stalk are in canonical one-to-one correspondence with the irreducible components of the formal neighbourhood. For the general case we need the following result.

Let $Z_1 \subset \mathscr{J}_m^{(p^{-N})}$, $Z_2 \subset \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2}$ be irreducible components. Then there exists a unique irreducible component of $\widetilde{T}$ containing $s(Z_1(\overline{\mathbb{F}}_p) \times Z_2(\overline{\mathbb{F}}_p))$. Indeed, let $z_1 \in Z_1(\overline{\mathbb{F}}_p)$ and $z_2 \in Z_2(\overline{\mathbb{F}}_p)$ be smooth points. As s defines an isomorphism of varieties in a neighbourhood of (z_1, z_2), its image $s(z_1, z_2)$ is also a smooth point and thus contained in a unique irreducible component $Z \subset \widetilde{T}$. Since the restriction $s_{|Z_1(\overline{\mathbb{F}}_p)\times\{z_2\}}$ is a morphism of varieties we have $s(Z_1(\overline{\mathbb{F}}_p) \times \{z_2\}) \subset Z(\overline{\mathbb{F}}_p)$. Now $s(Z_1(\overline{\mathbb{F}}_p) \times \{z_2\})$ lies in the smooth locus, as Z_1 is smooth. Thus we can apply the same argument again, this time varying the second coordinate, and obtain $s(Z_1(\overline{\mathbb{F}}_p) \times Z_2(\overline{\mathbb{F}}_p)) \subset Z(\overline{\mathbb{F}}_p)$.

Denote by $\mathrm{Irr}(\cdot)$ the set of irreducible components of a scheme. Consider the diagram

$$\begin{tikzcd}
\mathrm{Irr}(\widetilde{T}^\wedge_{s(x,y)}) \arrow[r, two heads] \arrow[d, "\wr"] & \mathrm{Irr}(\widetilde{T}_{s(x,y)}) \arrow[d, bend right] \\
\mathrm{Irr}((\mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(b)^{m_1,m_2})_{(x,y)})^\wedge) \arrow[r, two heads] & \mathrm{Irr}((\mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(b)^{m_1,m_2})_{(x,y)}) \arrow[u, bend right]
\end{tikzcd}$$

where the right most arrow is the section which maps (Z_1, Z_2) to Z in the above construction. As both squares commute, one sees easily that the right arrows must be bijections. $\square$

The geometric properties of $\pi_{N,m}$ follow readily from the analogue properties of $\pi'_{N,m}$.

Proposition 4.8. *$\pi_{N,m}$ is finite, and surjective for m big enough.*

Proof. Let $z \in \mathscr{S}_G(\overline{\mathbb{F}}_p)$ be arbitrary. We choose an isogeny $\varphi : (\mathcal{A}_{G,z}, \lambda_{G,z}, t_{G,\alpha,z}) \to (X_0, \lambda_0, t_{0,\alpha})$; by [Sch13, Lemma 4.4] we can choose φ such that $\varphi^{-1} \in \mathscr{M}_{G,\mu}(\mathbf{b})^{0,m}$

where m do not depend on z. By (4.1) we have

$$z = \Theta_x(\varphi^{-1})$$

for some $x \in \mathscr{J}_m$. The finiteness from Proposition 3.4. As $\pi'_{N,m}$ is finite, so is $\pi^-_{N,m}$. Thus $\pi_{N,m}$ is finite as finiteness is stable under cancellation ☐

Proposition 4.9. (1) $\pi_{N,m} \circ (F_{\mathscr{J}_m/k} \times \mathrm{id}) = \pi_{N+1,m}$

(2) *Let* $r_{m+1,m} : \mathscr{J}_{m+1} \twoheadrightarrow \mathscr{J}_m$ *be the canonical projection. Then* $\pi_{N,m-1} \circ (r_{m+1,m} \times \mathrm{id}) = \pi_{N,m}$, *where its is assumed that* $m_1 + m_2 \leq m-1$.

(3) *Let* $\iota : \mathscr{M}_{G,\mu}(\mathbf{b})^{m'_1,m'_2} \hookrightarrow \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2}$ *the obvious embedding for* $m'_1 \leq m_1, m'_2 \leq m_2$. *Then* $\pi_{N,m,m_1,m_2} \circ \iota = \pi_{N,m',m'_1,m'_2}$.

Proof. It suffices to check these properties on $\overline{\mathbb{F}}_p$ points. Here they are an easy consequence of the equality $\pi_{N,m}(x,y) = \Theta_x(y)$. ☐

4.4. **The almost product structure of infinite level.** Let $J_b \subset J'_b$ the topological subgroup of elements which stabilise $(t_{0,\alpha})$. We view J_b as locally profinite group scheme over $\overline{\mathbb{F}}_p$.

Proposition 4.10. *The canonical morphism* $\mathscr{J}_\infty^{(p^{-\infty})} \to \mathscr{J}_\infty'^{(p^{-\infty})}$ *is a closed embedding with* J_b*-stable image. More explicitly, for any* $g \in J_b$, $x = (x_0; j) \in \mathscr{J}_\infty(\overline{\mathbb{F}}_p)$ *the* J_b *action is given by*

$$g.x = (\Phi_x(g^{-1}); (g^{-1})_* j). \tag{4.2}$$

Proof. Let $x' = (\mathcal{A}, \lambda, \eta; j) \in \mathscr{J}'_\infty(\overline{\mathbb{F}}_p)$, then any lift $x \in \mathscr{J}_\infty(\overline{\mathbb{F}}_p)$ satisfies $t_{G,\alpha,x} = j_* t_{0,\alpha}$. Hence any such lift, if it exists, is unique by [Kis, Cor. 1.3.11]. Thus $\mathscr{J}_\infty \to \mathscr{J}'_\infty$ is injective on closed geometric points and hence universally injective as the Igusa varieties are Jacobson. Since it is finite, $\mathscr{J}_\infty \to \mathscr{J}'_\infty$ is a universal homeomorphism onto its (closed) image. As a universal homeomorphism of perfect schemes is an isomorphism by [BS15, Lemma 3.8], $\mathscr{J}_\infty^{(p^{-\infty})} \to \mathscr{J}_\infty'^{(p^{-\infty})}$ is a closed embedding. The second assertion follows from the fact that for any $g \in J'_b$, $x = (x_0; j) \in \mathscr{J}_\infty(\overline{\mathbb{F}}_p)$ the J'_b-action is given by

$$g.x = (\Phi'_x(g^{-1}); (g^{-1})_* j),$$

which is obviously lifted by (4.2). ☐

Let $\pi_\infty : \mathscr{J}_\infty^{(p^{-\infty})} \times \mathscr{M}_{G,\mu}(\mathbf{b}) \to \mathscr{S}_{G,0}^{\mathbf{b}}$ be the limit of $\pi_{N,m}$ for $N, m \to \infty$. By Proposition 4.9 this is well-defined.

Proposition 4.11. *The morphism* $\pi_\infty^{(p^{-\infty})}$ *is a* J_b*-torsor for the proétale topology*

Proof. Recall that for any $x \in \mathscr{J}_m(\overline{\mathbb{F}}_p)$ and any interior point $y \in \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2}$ the isomorphism in Proposition 2.4 identifies the restriction of $\pi_{N,m}$ to the formal neighbourhood at (x,y) with the almost product structure considered in [Ham16]. Thus it follows that $\pi_\infty^{(p^{-\infty})}$ is weakly étale by the same argument as in Proposition 3.10 for π'_∞. Since the source of π_∞ is Jacobson, it suffices to check its J_b-invariance on $\overline{\mathbb{F}}_p$-points. For $(x,y) \in (\mathscr{J}_\infty \times \mathscr{M}_{b,\mu}(\mathbf{b}))(\overline{\mathbb{F}}_p)$ and $g \in J_b$ one has

$$\pi_\infty(g.(x,y)) = \Theta_{g.x}(g.y) \overset{(4.1)}{=} \Theta_x(y) = \pi_\infty(x,y).$$

Here (4.1) is applied to $(X, \lambda, t_\alpha) = (X_0, \lambda_0, t_{0,\alpha})$ and $\phi = g$.

To ease the notation, we write $\mathscr{P}_\infty := (\mathscr{J}_\infty \times \mathscr{M}_{G,\mu}(\mathbf{b}))^{(p^{-\infty})}$ and denote by $a : J_b \times \mathscr{P}_\infty \to \mathscr{P}_\infty$ the J_b-action. We have to show that

$$\mathrm{pr}_2 \times a : J_b \times \mathscr{P}_\infty \to \mathscr{P}_\infty \times_{\mathscr{S}_{G,0}^{\mathbf{b}\,(p^{-\infty})}} \mathscr{P}_\infty$$

is an isomorphism. Since the target is reduced and Jacobson, it suffices to show that

(a) $\mathrm{pr}_2 \times a$ is a closed embedding and
(b) $\mathrm{pr}_2 \times a$ is surjective on $\overline{\mathbb{F}}_p$-points.

The first assertion follows directly from the commutative diagram

$$\begin{tikzcd} J_b \times \mathscr{P}_\infty \arrow[r, "\mathrm{pr}_2 \times a"] \arrow[d, hook, "\text{closed embedding}"'] & \mathscr{P}_\infty \times_{\mathscr{S}_{G,0}^{(p^{-\infty})}} \mathscr{P}_\infty \arrow[d] \\ J_{b'} \times \mathscr{P}'_\infty \arrow[r, "\sim", "\mathrm{pr}_2 \times a'"'] & \mathscr{P}'_\infty \times_{\mathscr{A}_g^{(p^{-\infty})}} \mathscr{P}'_\infty. \end{tikzcd}$$

The second assertion is equivalent to the claim that any two points $(x, y), (u, w) \in \mathscr{P}_\infty(\overline{\mathbb{F}}_p)$ with the same image under π_∞ lie in the same J_b orbit. Indeed $\pi_\infty(x, y) = \pi_\infty(u, w)$ implies $\pi'_\infty(x, y) = \pi'_\infty(u, w)$, thus there exists a $g \in J_{b'}$ such that $g.(u, w) = (x, y)$. More precisely, g is the concatenation $X_0 \xrightarrow{y} \mathcal{A}_{G,\pi_\infty(x,y)}[p^\infty] = \mathcal{A}_{G,\pi_\infty(u,w)}[p^\infty] \xrightarrow{w^{-1}} X_0$. Thus g fixes $t_{0,\alpha}$, i.e. $g \in J_b$. □

5. Relation with the Caraiani-Scholze product structure

5.1. Construction of the product structure in the Siegel moduli space. We briefly sketch the construction of the product structure of Caraiani and Scholze for the Siegel moduli space as presented in [CS15, § 4].

We fix a lift $(\tilde{X}_0, \tilde{\lambda}_0)$ of (X_0, λ_0) over O_E. We can extend $\mathscr{J}'^{(p^{-\infty})}_\infty$ to a flat scheme $\mathfrak{Ig}^{\mathbf{b}'}$ over $\mathrm{Spf}\, \breve{\mathbb{Z}}_p$ together with a morphism $\tilde{r}'_\infty : \mathfrak{Ig}^{\mathbf{b}'} \to \mathscr{A}_g$ which parametrises isomorphisms $(\mathcal{A}^{\mathrm{univ}}[p^\infty], \lambda^{\mathrm{univ}}) \cong (\tilde{X}_0, \tilde{\lambda}_0)$ ([CS15, Lemma 4.3.10]). Moreover, for every O_E-algebra R with $p \in R$ nilpotent let

$$\mathfrak{X}^{\mathbf{b}'}(R) := \{(A, \lambda, \eta; \rho) \mid (A, \lambda, \eta) \in \mathscr{A}_g(R), \rho : (A[p^\infty], \lambda) \otimes R/p \to (X_0, \lambda_0) \otimes R/p \text{ quasi-isogeny}\}.$$

Now consider the morphism $\mathfrak{Ig}^{\mathbf{b}'} \times \mathfrak{M}_{\mathrm{GSp}_{2g},\mu'} \to \mathfrak{X}^{\mathbf{b}'}$ mapping $(\mathcal{A}, \lambda, \eta; j), (\mathcal{X}, \rho) \in (\mathfrak{Ig}^{\mathbf{b}'} \times \mathfrak{M}_{G,\mu}(\mathbf{b}'))(R)$ to $(\mathcal{A}', \lambda', \eta'; \rho)$ which is constructed as follows. We lift ρ to R and choose m big enough such that $p^m \rho$ is an isogeny. We endow

$$\mathcal{A}' := \mathcal{A} \big/ j(\ker(p^m \rho))$$

with the polarisation λ' and K'^p-level structure η' induced by λ and η, respectively. We denote by $\rho' : X_0 \to \mathcal{A}'$ the quasi-isogeny such that the diagram

$$\begin{tikzcd} X_0 \arrow[r, "j"] \arrow[rr, bend left, "p^m \rho'"] & \mathcal{A}[p^\infty] \arrow[r, two heads] & \mathcal{A}'[p^\infty] \end{tikzcd}$$

commutes.

Proposition 5.1 ([CS15, Lemma 4.3.12]). *The above morphism $\mathfrak{Ig}^{\mathbf{b}'} \times \mathfrak{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}') \to \mathfrak{X}^{\mathbf{b}'}$ is an isomorphism.*

Let $\tilde{\pi}'_\infty$ denote the concatenation $\mathfrak{Ig}^{\mathbf{b}'} \times \mathfrak{M}_{\mathrm{GSp}_{2g},\mu'} \xrightarrow{\sim} \mathfrak{X}^{\mathbf{b}'} \xrightarrow{can.} \mathscr{A}_g$.

5.2. **The local product structure.** Since the $\mathfrak{m}$-adic topology on the stalks of $\mathfrak{Ig}^{\mathbf{b}'} \times \mathfrak{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')$ is not separable, considering their completions would be unrewarding. Instead, we consider the following construction.

Definition 5.2. Let A be an R-algebra where R is Noetherian and $I \subset A$ be an ideal. The restricted I-adic completion of A relative to R is defined as

$$\hat{A} = \varinjlim \hat{A}_\lambda$$

where $A = \varinjlim A_\lambda$ is a filtered colimit with A_λ finitely presented over R and $\hat{A}_\lambda$ denotes their completion with respect to the preimage I_λ of I in A_λ.

The restricted completion has the advantage that it has similar properties as the completion of Noetherian rings.

Lemma 5.3. *Let R be a Noetherian ring, A an R-algebra and I an ideal of A.*

(1) *The restricted completion $\hat{A}$ does not depend on the choice of the A_λ. In particular, if A is of finite presentation over R, $\hat{A}$ is just the I-adic completion of A.*
(2) *$A \to \hat{A}$ is flat.*
(3) *Let B be a finite A-algebra, $J = IB$ and $\hat{B}$ the restricted completion of B relative to R. Then $\hat{B} = B \otimes_A \hat{A}$.*
(4) *The image of $\operatorname{Spec} \hat{A}$ in $\operatorname{Spec} A$ is the completion of $V(I)$, i.e. the pro-open subset of all points which specialise to a point in $V(I)$.*
(5) *If A is local with maximal ideal I, then for any ideal $J \subset A$ we have $\hat{A} \cdot J \cap A = J$.*

Proof. Assume that $A = \varinjlim A_\lambda = \varinjlim B_\mu$ where A_λ and B_μ are finitely presented R-algebras. Thus for any λ there exists a $\mu(\lambda)$ such that the canonical morphism $A_\lambda \to A$ factorises over $B_{\mu(\lambda)}$ and for any μ there exists a $\lambda(\mu)$ such that $B_\mu \to A$ factorises over $A_{\lambda(\mu)}$. Since the morphisms are continuous, they induce a morphisms $\phi_\lambda : \hat{A}_\lambda \to \hat{B}_\mu$ and $\psi_\mu : \hat{B}_\mu \to \hat{A}_\lambda$. Since the diagrams

$$\begin{tikzcd}
\hat{A}_{\lambda(\mu(\lambda))} & \\
& \hat{B}_{\mu(\lambda)} \arrow[ul, "\psi_{\mu(\lambda)}"'] \\
\hat{A}_\lambda \arrow[uu, "can."] \arrow[ur, "\phi_\lambda"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \hat{B}_{\mu(\lambda(\mu))} \\
\hat{A}_{\lambda(\mu)} \arrow[ur, "\phi_{\lambda(\mu)}"] & \\
& \hat{B}_\mu \arrow[uu, "can."'] \arrow[ul, "\psi_\mu"]
\end{tikzcd}$$

commute, taking the limit of (ϕ_λ) and (ψ_μ) yields two mutually inverse isomorphisms $\varinjlim \hat{A}_\lambda \cong \varinjlim \hat{B}_\mu$. The second assertion follows from $\hat{A} = \varinjlim \hat{A}_\lambda = \varinjlim A \otimes_{A_\lambda} \hat{A}_\lambda$, as the limit of flat A-modules is flat. To prove the third assertion, write $A = \varinjlim A_\lambda$, $B = \varinjlim B_\lambda$ such that each B_λ is a finite A_λ-algebra. Then

$$\hat{B} = \varinjlim B \otimes_{B_\lambda} \hat{B}_\lambda = \varinjlim B \otimes_{A_\lambda} \hat{A}_\lambda = B \otimes \hat{A}.$$

The forth assertion is a direct consequence of the analogous assertion for $A_\lambda \to \hat{A}_\lambda$. To see the last assertion, let A'_λ be the localisation of A_λ at $I \cap A_\lambda$. Note that A

and $\hat{A}$ are the limits of A'_λ and $\hat{A}'_\lambda$, respectively. Now Krull's intersection theorem gives

$$\hat{A}'_\lambda \cdot (J \cap A_\lambda) = J \cap A_\lambda,$$

which yields $\hat{A} \cdot J \cap A = J$ after taking the limit. □

We deduce the following analogue of [Mum99, § III.5, Thm.3].

Corollary 5.4. *Let $\phi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism of finite type of R-algebras such that the induced field extension of residue fields $\overline{\phi} : A/\mathfrak{m} \to B/\mathfrak{n}$ and the induced morphism $\hat{\phi} : \hat{A} \to \hat{B}$ are isomorphisms. Then ϕ is étale.*

Proof. Since $\overline{\phi}$ is an isomorphism, we have $\Omega_{B/A} \otimes A/\mathfrak{m} = 0$ by part (5) of above lemma, hence ϕ is formally unramified (and thus unramified) by Nakayama's lemma. Since $\hat{\phi}$ is an isomorphism, parts (2) and (4) of the above lemma moreover imply that ϕ is flat and the claim follows. □

Definition 5.5. We define the restricted formal neighbourhood of a formal scheme $\mathfrak{X}$ over $\operatorname{Spf} \breve{\mathbb{Z}}_p$ at a point x as the restricted $\mathfrak{m}_{\mathfrak{X},x}$-adic completion of $\mathcal{O}_{\mathfrak{X},x}$ relative to $\breve{\mathbb{Z}}_p$.

Let $\tilde{x}_0 \in \mathscr{J}_\infty(k)$, with images $x_0 \in C(k), x'_0 \in C'(k)$ and $x \in \mathscr{M}_{G,\mu}(\mathbf{b})(k)$ with corresponding quasi-isogeny $\rho : (X_0, \lambda_0, t_{0,\alpha}) \to (X, \lambda, t_\alpha)$. Denote by $(X_0^{\mathrm{loc}}, \lambda_0^{\mathrm{loc}})$ and $(X^{\mathrm{loc}}, \lambda^{\mathrm{loc}})$ the universal deformations of (X_0, λ_0) and (X, λ), respectively. The canonical isomorphism $\mathscr{A}_{g,x'_0}^\wedge \xrightarrow{\sim} \mathfrak{Def}(X_0, \lambda_0)$ identifies the fiber of $\tilde{r}'_\infty$ over the formal neighbourhood with the formal scheme ${\mathfrak{C}'_\infty}^{\Gamma_{b'}}$ parametrising isomorphisms $j : \tilde{X}_0 \xrightarrow{\sim} X_0^{\mathrm{loc}}$. The restricted formal neighbourhood of $\tilde{x}_0$ in $\mathfrak{Ig}^{\mathbf{b}'}$ is thus given by a connected componenent $\mathfrak{C}'_\infty \subset {\mathfrak{C}'_\infty}^{J_{b'}}$. Hence the restriction of $\tilde{\pi}'_\infty$ to the restricted formal neighbourhood at the closed points corresponds to a morphism

$$\tilde{\pi}'_{\infty,\mathrm{loc}} : \mathfrak{C}'_\infty \times \mathfrak{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')^\wedge_x \to \mathfrak{Def}(X, \lambda).$$

Remark 5.6. Denote by ρ^{loc} the restriction of the universal quasi-isogeny over $\mathfrak{M}_{\mathrm{GSp}_{2g},\mu'}$. Then we have

$$\tilde{\pi}'^{*}_{\infty,\mathrm{loc}} X^{\mathrm{loc}} = X_0 / \ker \rho^{\mathrm{loc}}. \tag{5.1}$$

One cannot use this relation directly to define $\tilde{\pi}'_{\infty,\mathrm{loc}}$ via the moduli description of $\mathfrak{Def}(X, \lambda)$ since its source is not formally of finite type. However, one can define it as the limit of the following morphisms on "finite level". Let $\mathfrak{C}'_m$ be the scheme theoretic image of $\mathfrak{C}_\infty$ in the formal scheme parametrising isomorphisms $j_m : \tilde{X}_0[p^m] \xrightarrow{\sim} X_0^{\mathrm{loc}}[p^m]$. Note that $\mathfrak{C}'_m$ is finite over $\mathfrak{Def}(X_0, \lambda_0)$. Let $\mathfrak{I}'_m \subset \mathfrak{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')^\wedge_x$ be the locus where there exist m_1, m_2 with $m_1 + m_2 \leq m$ such that $p^{m_1}\rho^{\mathrm{loc}}$ and $p^{m_2}\rho^{\mathrm{loc}}$ are isogenies. Then $X^{\mathrm{loc}}_{0,\mathfrak{C}'_m}/j_m(\ker p^{m_1}\rho^{\mathrm{loc}})$ is a deformation of X and thus induces a morphism

$$\tilde{\pi}'_{m,\mathrm{loc}} : \mathfrak{C}'_m \hat{\times} \mathfrak{I}'_m \to \mathfrak{Def}(X, \lambda).$$

Finally, we can construct the local version of the product structure for Shimura varieties of Hodge type. Let $\mathfrak{Ig}^{\mathbf{b}}$ be the (unique) flat lift of $\mathscr{J}_b^{(p^{-\infty})}$ over $\operatorname{Spf} \breve{\mathbb{Z}}_p$. In terms of the identification of Proposition 2.4, the restricted formal neighbourhood of $\mathfrak{Ig}^{\mathbf{b}}$ at $\tilde{x}_0$ corresponds to the preimage $\mathfrak{C}_\infty \subset \mathfrak{C}'_\infty$ of $\mathfrak{Def}(X_0, \lambda_0, t_{0,\alpha})$

Lemma 5.7. *The restriction $\pi_{\infty,\mathrm{loc}}$ of $\pi'_{\infty,\mathrm{loc}}$ to $\mathfrak{C}_\infty \times \mathfrak{M}_{G,\mu}(\mathbf{b})^\wedge_x$ factorises through $\mathfrak{Def}(X, \lambda, t_\alpha)$.*

Proof. First note that because of the formal smoothness of the Rapoport-Zink space and the central leaf we have an abstract isomorphism

$$\mathfrak{C}_\infty \times \mathscr{M}_{G,\mu}(\mathbf{b})^\wedge_x \cong \operatorname{Spf} \breve{\mathbb{Z}}_p[\![x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}, y_1, \ldots, y_n]\!]$$

for some d, n. Since $\mathfrak{Def}(X, \lambda)$ is formally of finite presentation $\tilde{\pi}_{\infty,\mathrm{loc}}$ factorises through $\operatorname{Spf} \breve{\mathbb{Z}}_p[\![x_1^{1/p^N}, \ldots, x_d^{1/p^N}, y_1, \ldots y_n]\!]$ for some N. By Faltings' criterion (see e.g. [Kim16, Thm. 3.6]) it suffices to show that (5.1) respects crystalline Tate tensors. This is a direct consequence of (a slight generalisation of) [Ham16, Prop. 2.16]. □

5.3. **The product structure over Shimura varieties of Hodge type.** Since the formal schemes we consider are not formally of finite type, we need to consider a larger class of test objects than the formally smooth, formally finitely generated $\breve{\mathbb{Z}}_p$-algebras used in [HP15].

Definition 5.8. Let $m \in \mathbb{N} \cup \{\infty\}$ and A be a p-adically complete $\breve{\mathbb{Z}}_p/p^m$-algebra.

(1) A is called perfectly formally finitely generated if there exist positive integers r_1, r, s_1, s such that A can be written as a quotient of

$$\breve{\mathbb{Z}}_p/_{p^m}[\![x_1, \ldots, x_{r_1}, x_{r_1+1}^{1/p^\infty}, \ldots, x_r^{1/p^\infty}]\!][y_1, \ldots, y_{s_1}, y_{s_1+1}^{1/p^\infty}, \ldots, y_s^{1/p^\infty}]$$

(2) A is called essentially perfectly formally finitely generated if it is the p-adic completion of an indétale algebra over some perfectly formally finitely generated $\breve{\mathbb{Z}}_p/p^m$ algebra.

(3) A is called essentially smooth if it is essentially perfectly formally finitely generated and its restricted completion at any maximal ideal is isomorphic to the p-adic completion of $\breve{\mathbb{Z}}_p/_{p^m}[\![x_1, \ldots, x_{r_1}, x_{r_1+1}^{1/p^\infty}, \ldots, x_r^{1/p^\infty}]\!]$ for some r, r_1.

(4) Denote by $\mathrm{Nilp}^{es}_{\breve{\mathbb{Z}}_p}$ the full category of $\breve{\mathbb{Z}}_p$-modules which are p^m-torsion and essentially smooth as $\breve{\mathbb{Z}}_p/p^m$-algebras for some $m \in \mathbb{N}$.

Example 5.9. *(1)* If A is a formally finitely generated $\breve{\mathbb{Z}}_p/p^m$-algebra, it is essentially smooth if and only if it is formally smooth.

(2) If A is a formally finitely generated smooth $\overline{\mathbb{F}}_p$-algebra, then $A^{(p^{-\infty})}$ is essentially smooth.

We extend the notion of essential smoothness to formal schemes.

Definition 5.10. A formal scheme $\mathfrak{X}$ over $\operatorname{Spf} W$ is called essentially smooth (resp. locally perfectly formally of finite type, locally essentially perfectly formally of finite type) if it is Zariski locally of the form $\operatorname{Spf} A$ where A is an essentially smooth $\breve{\mathbb{Z}}_p$-algebra (resp. perfectly formally of finite type, essentially perfectly formally of finite type).

Using the usual Yoneda argument, one checks that any essentially smooth formal $\breve{\mathbb{Z}}_p$-scheme is uniquely determined by its functor of points restricted to the full subcategory of affine formal schemes whose ring of global sections lies in $\mathrm{Nilp}^{es}_{\breve{\mathbb{Z}}_p}$.

Lemma 5.11. *The formal schemes $\mathfrak{M}_{G,\mu}(\mathbf{b})$ and $\mathfrak{Ig}^{\mathbf{b}}$ are essentially smooth.*

Proof. Since $\mathfrak{M}_{G,\mu}(\mathbf{b})$ is formally smooth and locally formally of finite type by [HP15, Prop. 3.2.7], it follows from the example above that it is essentially smooth. To see that $\mathfrak{Ig}^{\mathbf{b}}$ is essentially smooth denote by $\mathfrak{Ig}_m$ the (unique) flat lift of $\mathscr{J}_m^{(p^{-\infty})}$ over $\operatorname{Spf} W$. Then $\mathfrak{Ig}_m$ is perfectly formally of finite type and moreover the lift

$\mathfrak{Ig}_{m+1} \to \mathfrak{Ig}_m$ is étale since it is the unique lift to a morphism of flat formal schemes over $\operatorname{Spf} \breve{\mathbb{Z}}_p$ and the existence of an étale lift is guaranteed by [Sta16, 04DZ]. Thus $\mathfrak{Ig}^{\mathbf{b}}$ is essentially perfectly formally of finite type. By construction the restricted local neighbourhood x at a closed point of $\mathfrak{Ig}^{\mathbf{b}}$ equals the limit of restricted formal neighbourhoods of the image of x in $\mathfrak{Ig}_m$. As a consequence of Corollary 5.4 all transition morphisms of this limit are isomorphisms; thus it suffices to check that $\mathfrak{Ig}_0$ is essentially smooth. Since the formal neighbourhood of $C = \mathscr{J}_0$ at a closed point is isomorphic to $\overline{\mathbb{F}}_p[\![x_1, \dots, x_{2\langle\rho,\nu\rangle}]\!]$, the restricted formal neighbourhood of its perfection equals $\overline{\mathbb{F}}_p[\![x_1^{1/p^\infty}, \dots, x_{2\langle\rho,\nu\rangle}^{1/p^\infty}]\!]$. Thus the restricted completion of $\mathfrak{Ig}_0$ (and thus also $\mathfrak{Ig}^{\mathbf{b}}$) at a closed point is

$$W(\overline{\mathbb{F}}_p[\![x_1^{1/p^\infty}, \dots, x_{2\langle\rho,\nu\rangle}^{1/p^\infty}]\!]) = \breve{\mathbb{Z}}_p[\![x_1^{1/p^\infty}, \dots, x_{2\langle\rho,\nu\rangle}^{1/p^\infty}]\!]^{\wedge\, p-adic}$$

□

Proposition 5.12. (1) *There exists a unique lift $\tilde{\pi}_\infty : \mathfrak{Ig}^{\mathbf{b}} \times \mathfrak{M}_{G,\mu}(\mathbf{b}) \to \mathscr{S}_G$ of $\tilde{\pi}'_\infty$ whose restriction to the underlying reduced subscheme equals π_∞.*

(2) *Let $\mathfrak{X}^{\mathbf{b}}$ be the subfunctor of $\mathfrak{X}^{\mathbf{b}'}_{/\mathscr{S}_G}$, evaluated on objects in $\mathrm{Nilp}^{es}_{\breve{\mathbb{Z}}_p}$, given by the additional assumption that the quasi-isogeny ρ respects crystalline Tate-tensors. Then the pullback of $\mathfrak{Ig}^{\mathbf{b}'} \times \mathfrak{M}_{\mathrm{GSp}_{2g},\mu'} \xrightarrow{\sim} \mathfrak{X}^{\mathbf{b}'}$ restricts an isomorphism $\mathfrak{Ig}^{\mathbf{b}} \times \mathfrak{M}_{G,\mu}(\mathbf{b}) \xrightarrow{\sim} \mathfrak{X}^{\mathbf{b}}$.*

Proof. Let $\widetilde{\mathfrak{T}} := (\mathfrak{Ig}^{\mathbf{b}} \times \mathfrak{M}_{G,\mu}(\mathbf{b})) \times_{\mathscr{A}_g} \mathscr{S}_G$ and $\tilde{\nu} : \widetilde{\mathfrak{T}} \to \mathfrak{Ig}^{\mathbf{b}} \times \mathfrak{M}_{G,\mu}(\mathbf{b})$ be the canonical projection. By Corollary 4.5 we have to construct a section of $\tilde{\nu}$. Recall that the proof of Theorem 4.7 constructs a clopen subset

$$Z_{N,m} := s(\mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2}) \subset (\mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2}) \times_{\mathscr{A}_g} \mathscr{S}_G$$

which is mapped isomorphically onto $\mathscr{J}_m^{(p^{-N})} \times \mathscr{M}_{G,\mu}(\mathbf{b})^{m_1,m_2}$ by the canonical projection. Denote by $\mathfrak{Z}$ the union of connected components of $\widetilde{\mathfrak{T}}$ whose underlying reduced subscheme equals the limit of $Z_{N,m}$. In particular $\nu_{|\mathfrak{Z}}$ induces an isomorphism of the underlying reduced subschemes. Moreover, $\tilde{\nu}$ induces an isomorphism of restricted formal neighbourhoods at every point $z \in \mathfrak{Z}(k)$ by Lemma 5.7 and is thus étale at z by Corollary 5.4. Hence $\tilde{\nu}$ is an isomorphism on a neighbourhood of z. As $\mathfrak{Z}$ is Jacobson, $\tilde{\nu}_{|\mathfrak{Z}}$ is an isomorphism, thus its inverse gives us a section of $\tilde{\nu}$.

Consider the image $\mathfrak{X}_0^{\mathbf{b}}$ of $\mathfrak{Ig}^{\mathbf{b}} \times \mathfrak{M}_{G,\mu}(\mathbf{b})$ in $\mathfrak{X}^{\mathbf{b}'}_{/\mathscr{S}_G}$. We have $\mathfrak{X}_0^{\mathbf{b}} \subset \mathfrak{X}^b$ by construction. More precisely, $\mathfrak{X}_0^{\mathbf{b}}$ is a *closed* subfunctor of $\mathfrak{X}^{\mathbf{b}}$ since $\mathfrak{Ig}^{\mathbf{b}} \to \mathfrak{Ig}^{\mathbf{b}'}$ and $\mathfrak{M}_{G,\mu}(\mathbf{b}) \to \mathfrak{M}_{\mathrm{GSp}_{2g},\mu'}(\mathbf{b}')$ are closed immersions by Proposition 4.10 and [HP15, Prop. 3.11].

In Proposition 4.11, we have already shown that $\mathfrak{X}_0^{\mathbf{b}}(\overline{\mathbb{F}}_p) = \mathfrak{X}^{\mathbf{b}}(\overline{\mathbb{F}}_p)$. Given a point in $\mathfrak{X}^{\mathbf{b}}(R)$ with $R \in \mathrm{Nilp}^{es}_{\breve{\mathbb{Z}}_p}$ arbitrary, let $\operatorname{Spf} R'$ denote the closed formal subscheme which factorises through $\mathfrak{X}_0^{\mathbf{b}}$. Then $\operatorname{Spf} R'$ contains all closed points of $\operatorname{Spf} R$ by the previous observation and moreover their restricted formal neighbourhoods by the same argument as in the proof of Lemma 5.7. Thus $\operatorname{Spf} R' \hookrightarrow \operatorname{Spf} R$ is étale by Corollary 5.4 and hence an isomorphism. Thus $\mathfrak{X}_0^{\mathbf{b}} = \mathfrak{X}^{\mathbf{b}}$.

□

By the same arguments as in [CS15], one gets deduces the product structure on the generic fibre. Denote by $A_G, S_G, , Ig^{\mathbf{b}}, M_{G,\mu}(\mathbf{b})$ and $X^{\mathbf{b}}$ the respective adic generic fibres of $\mathcal{A}_G, \mathscr{S}_G, \mathfrak{Ig}^{\mathbf{b}}, \mathfrak{M}_{G,\mu}(\mathbf{b})$ and $\mathfrak{X}^{\mathbf{b}}$. Denote by sp the specialisation map and let

$$S_G^{\mathbf{b}} := \mathrm{sp}^{-1} \mathscr{S}_{G,0}^{\mathbf{b}}.$$

We get infinite level Newton strata $S_{G,\infty}^{\mathbf{b}}$ and Rapoport-Zink spaces $M_{G,\mu,\infty}(\mathbf{b})$ parameterising morphisms $\alpha : M \to \mathrm{T}_p A_G$ and $\alpha : M \to T_p X^{\mathrm{univ}}$, respectively, which are compatible with additional structure and an isomorphism over geometric points. By [Kim16, Prop. 7.6.1] $M_{G,\mu,\infty}(\mathbf{b})$ is representable by a preperfectoid space.

Corollary 5.13. *We have an isomorphism*

$$Ig^{\mathbf{b}} \times M_{G,\mu,\infty}(\mathbf{b}) \xrightarrow{\sim} S_{G,\infty}^{\mathbf{b}}.$$

In particular, $S_{G,\infty}^{\mathbf{b}}$ *is preperfectoid.*

Proof. This is a direct consequence of the isomorphisms

$$Ig^{\mathbf{b}} \times M_{G,\mu}(\mathbf{b}) \xrightarrow{\sim} X_G^{\mathbf{b}}$$
$$X_G^{\mathbf{b}} \times_{M_{G,\mu}(\mathbf{b})} M_{G,\mu,\infty}(\mathbf{b}) \xrightarrow{\sim} X_{G,\infty}^{\mathbf{b}},$$

where the first isomorphism follows from Proposition 5.12 and the second from the moduli description. □

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