

A NOTE ON THE DIFFERENTIAL OPERATOR ON GENERALIZED FOCK SPACES

TESFA MENGESTIE

ABSTRACT. It has long been known that the differential operator D represents a typical example of unbounded operators on many Banach spaces including the classical Fock spaces, the Fock–Sobolev spaces, and the generalized Fock spaces where the weight decays faster than the Gaussian weight. In this note we identify Fock type spaces where the operator admits boundedness, compactness and membership in the Schatten \mathcal{S}_p class spectral structures. We also showed that its nontrivial spectrum while acting on such spaces is precisely the closed unit disk \mathbb{D} in the complex plane.

1. INTRODUCTION

Various order differential operators play fundamental rolls in many part of mathematics including in the study of differential equations. Nevertheless, the operator $Df = f'$ often appears as a canonical example of unbounded operators in many Banach spaces including the very classical Hilbert space $L^2(\mathbb{R})$, the space of continuous functions $C([a, b])$ with the supremum norm, and the likes. Its unboundedness on Fock spaces with the classical Gaussian weight $e^{-|z|^2}$ and on generalized Fock spaces where the weight decays faster than the Gaussian weight was recently verified in [9]. The same conclusion was also drawn in [11] on the Fock–Sobolev spaces which are typical examples of generalized Fock spaces with weight decaying slower than the Gaussian weight. A natural question to consider is whether there could exist spaces of Fock type where this operator admits richer operator-theoretic properties. Said differently, we would like to know how the function-theoretic properties of the weight functions generating the spaces are related to the operator-theoretic properties of D . The central aim of this note is to investigate this and identify Fock type spaces where the operator D admits some basic spectral properties.

In view of the above discussion, if there could exist generalized Fock spaces on which the operator D acts in a bounded fashion, then the associated weight must decay slower than the k^{th} order Fock–Sobolev spaces with weight $e^{-\frac{|z|^2}{2} + k \log(1+|z|)}$, where k is a nonnegative integer; see [4, 10, 11] for further information on these spaces. Keeping this in mind, we consider the following setting.

2010 *Mathematics Subject Classification.* Primary 47B32, 30H20, 46E20; Secondary 46E22, 47B33.

Key words and phrases. Generalized Fock spaces, Fock–Sobolev spaces, Bounded, Differential operator, Schatten class, Spectrum.

The author is partially supported by HSH grant 1244/ H15.

Let $m > 0$, $0 < p < \infty$, and $\mathcal{F}_{(m,p)}$ be a class of generalized Fock spaces consisting of all entire functions f for which

$$\|f\|_{(m,p)}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p|z|^m} dA(z) < \infty,$$

where dA denotes the usual Lebesgue area measure on \mathbb{C} . With this, we plan to find conditions on m (equivalently on the growth of $\psi_m(z) = |z|^m$) under which D admits boundedness, compactness, and other operator-theoretic structures while acting between the spaces $\mathcal{F}_{(m,p)}$. It turns out that such structures do happen to exist only if the inducing weight function ψ_m grows at a rate much slower than the corresponding weight function in the classical Gaussian case $\psi_2(z) = |z|^2$. We precise this in our first main result to follow.

Theorem 1.1. (i) *Let $0 < p \leq q < \infty$ and $m > 0$. Then $D : \mathcal{F}_{(m,p)} \rightarrow \mathcal{F}_{(m,q)}$ is*

(a) *bounded if and only if*

$$m \leq 2 - \frac{pq}{pq + q - p}.$$

(b) *compact if and only if*

$$m < 2 - \frac{pq}{pq + q - p}.$$

(ii) *Let $0 < q < p < \infty$ and $m > 0$. Then the following statements are equivalent.*

(a) *$D : \mathcal{F}_{(m,p)} \rightarrow \mathcal{F}_{(m,q)}$ is bounded;*

(b) *$D : \mathcal{F}_{(m,p)} \rightarrow \mathcal{F}_{(m,q)}$ is compact;*

(c) *It holds that*

$$m < 1 - 2\left(\frac{1}{q} - \frac{1}{p}\right).$$

The result effectively identifies the generalized Fock spaces on which the differential operator admits boundedness and compactness operator-theoretic structures. In particular, when $p = q$, the operator D enjoys any of the basic spectral structures on $\mathcal{F}_{(m,p)}$ only if the corresponding weight functions ψ_m grow at most polynomials of degree not exceeding one. If $p < q$, then ψ_m could grow a bite faster as

$$\frac{pq}{pq + q - p} < 1.$$

On the other hand, if $p > q$, then ψ_m grows slower than a polynomials of degree one.

We note in passing that if we replace both the domain and target spaces by the corresponding growth type spaces $\mathcal{F}_{(m,\infty)}$ which consist of entire functions f for which

$$\|f\|_{(m,\infty)} = \sup_{z \in \mathbb{C}} |f(z)| e^{-|z|^m} < \infty,$$

the same conclusion, $m \leq 1$, follows which can be also seen for example in [1, 7] as a particular instance.

Our next main result gives a condition on the growth of $|z|^m$ under which D belongs to the Schatten $\mathcal{S}_p(\mathcal{F}_{(m,2)})$ class and also identifies its spectrum.

Theorem 1.2. (i) Let $0 < p < \infty$, $m > 0$, and $D : \mathcal{F}_{(m,2)} \rightarrow \mathcal{F}_{(m,2)}$ is compact. Then it belongs to the Schatten $\mathcal{S}_p(\mathcal{F}_{(m,2)})$ class for all p .
(ii) Let $1 \leq p < \infty$ and $m > 0$, and $D : \mathcal{F}_{(m,p)} \rightarrow \mathcal{F}_{(m,p)}$ is bounded, i.e. $m \leq 1$. Then its spectrum $\sigma(D) = \{0\}$ whenever $m < 1$ and when $m = 1$;

$$\sigma(D) = \overline{\{\lambda \in \mathbb{C} : e^{\lambda z} \in \mathcal{F}_{(m,p)}\}} = \overline{\mathbb{D}}.$$

2. PRELIMINARIES

In this section we collect a few basic facts which will be used in the proofs of the main results. From [5], the Littlewood–Paley type estimate ¹

$$\|f\|_{(m,p)}^p \simeq |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p e^{-p|z|^m}}{(1+|z|)^{p(m-1)}} dA(z) \quad (2.1)$$

holds for functions f in the space $\mathcal{F}_{(m,p)}$. Such a formula characterizes the spaces in terms of derivatives, and plays a significant roll specially in the study of integral operators on the spaces.

Lemma 2.1. Let $\lambda \in \mathbb{C}$ and $0 < p < \infty$. Then for each entire function f for which $fe^{\lambda} \in \mathcal{F}_{(m,p)}$, we have

$$\int_{\mathbb{C}} |f(z)e^{\lambda z}|^p e^{-p|z|} dA(z) \lesssim \int_{\mathbb{C}} |f'(z)e^{\lambda z}|^p e^{-p|z|} dA(z). \quad (2.2)$$

This is a key estimation result which helps us obtain our main result on the spectrum of the operator D in Theorem 1.2.

Proof. The proof of the lemma follows from some ideas stemmed in the proof of Proposition 1 in [5]. We argue in the direction of contradiction and assume that (2.2) fails to hold. Then, we can find a sequence of entire functions (f_n) satisfying $f_n e^{\lambda} \in \mathcal{F}_{(m,p)}$,

$$\int_{\mathbb{C}} |f_n(z)e^{\lambda z}|^p e^{-p|z|} dA(z) = 1 \quad \text{and} \quad \int_{\mathbb{C}} |f'_n(z)e^{\lambda z}|^p e^{-p|z|} dA(z) < \frac{1}{n}.$$

Now, if K is a compact subset of \mathbb{C} , the point evaluation estimate for functions in $\mathcal{F}_{(m,p)}$ (see the analysis in [6]) gives that

$$|f'_n(z)e^{\lambda z}| \lesssim C \int_K |f'_n(z)e^{\lambda z}|^p e^{-p|z|} dA(z) \leq C \frac{1}{n^p}$$

for some positive constant C that depends only on K . From this it follows that the sequence f'_n converges to zero uniformly on compact subset of \mathbb{C} . This shows

¹The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

that f_n also converges to zero uniformly on the compact subsets. We may rewrite

$$1 = \int_{\mathbb{C}} |f_n(z) e^{\lambda z}|^p e^{-p|z|} dA(z) = \int_{|z| \leq r} |f_n(z) e^{\lambda z}|^p e^{-p|z|} dA(z) + \int_{|z| > r} |f_n(z) e^{\lambda z}|^p e^{-p|z|} dA(z). \quad (2.3)$$

Now the first integral on the right-hand side of (2.3) tends to zero when $n \rightarrow \infty$ since $f_n \rightarrow 0$ uniformly on $\{z \in \mathbb{C} : |z| \leq r\}$. On the other hand, the second integral is the tail of a convergent integral and hence tends to zero when $r \rightarrow \infty$, and the contradiction follows. \square

We denote by $K_{(m,w)}$ the reproducing kernel of the space $\mathcal{F}_{(m,2)}$ at the point $w \in \mathbb{C}$. Because of the reproducing property of the kernel and Parseval identity, it holds that

$$K_{(m,w)}(z) = \sum_{n=1}^{\infty} \langle K_{(m,w)}, e_n \rangle e_n(z) \text{ and } \|K_{(m,w)}\|_{(m,2)}^2 = \sum_{n=1}^{\infty} |e_n(w)|^2$$

for any orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of $\mathcal{F}_{(m,2)}$. An immediate consequence of this representation is that

$$\frac{\partial}{\partial \bar{w}} K_{(m,w)}(z) = \sum_{n=1}^{\infty} e_n(z) \overline{e'_n(w)}, \text{ and } \left\| \frac{\partial}{\partial \bar{w}} K_{(m,w)} \right\|_{(m,2)}^2 = \sum_{n=1}^{\infty} |e'_n(w)|^2. \quad (2.4)$$

An explicit expression for the reproducing kernel $K_{(w,m)}$ in the weighted space $\mathcal{F}_{(m,2)}$ is still unknown apart from the case when $m = 2$. From [2], we already have an estimate for the norm

$$\|K_{(m,w)}\|_{(m,2)}^2 \simeq |w|^{m-2} e^{2|w|^m}. \quad (2.5)$$

As a consequence of this, we obtain the following useful estimate for our further consideration.

Lemma 2.2. *For each $w \in \mathbb{C}$, we have the asymptotic estimate*

$$\left\| \frac{\partial}{\partial \bar{w}} K_{(m,w)} \right\|_{(m,2)}^2 \simeq \|K_{(m,w)}\|_{(m,2)}^2 |w|^{2m-2} \simeq |w|^{3m-4} e^{2|w|^m}. \quad (2.6)$$

Proof. For simplicity, setting $\Psi(r) = r^{\frac{m-2}{2}} e^{r^m}$ and

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\|z^n\|_{(m,2)}},$$

then we have that

$$M_2(r, f)^2 \simeq \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta \simeq (\Psi(r))^2.$$

If we show that

$$\limsup_{r \rightarrow \infty} \frac{\Psi''(r) \Psi(r)}{(\Psi'(r))^2} < \infty \text{ and } \Psi'(r) \simeq \Psi(r) r^{m-1},$$

then our conclusion will follow from Lemma 21 of [6] as

$$M_2(r, f') \simeq \left\| \frac{\partial}{\partial \bar{w}} K_{(m,w)} \right\|_{(m,2)} \simeq \Psi'(r) \simeq \Psi(r)r^{m-1}.$$

To this end, we compute

$$\begin{aligned} \Psi'(r) &= \frac{m-2}{2} r^{\frac{m-4}{2}} e^{r^m} + m r^{\frac{3m-4}{2}} e^{r^m} = e^{r^m} r^{\frac{m-2}{2}-1} \left(\frac{m-2}{2} + m r^m \right) \\ &\simeq e^{r^m} r^{\frac{m-2}{2}-1+m} \simeq \Psi(r)r^{m-1}. \end{aligned}$$

Furthermore, a computation shows that

$$(\Psi'(r))^2 \simeq e^{2r^m} r^{m-2+2(m-1)} \quad \text{and} \quad \Psi''(r) \simeq e^{r^m} r^{\frac{m-2}{2}+2(m-1)}$$

from which we have

$$\limsup_{r \rightarrow \infty} \frac{\Psi''(r)\Psi(r)}{(\Psi'(r))^2} \simeq \limsup_{r \rightarrow \infty} \frac{e^{2r^m} r^{\frac{m-2}{2}+2(m-1)} r^{\frac{m-2}{2}}}{e^{2r^m} r^{m-2+2(m-1)}} \simeq 1.$$

□

It has been a fairly common practice to test many operator-theoretic properties on the reproducing kernels for the spaces. In the present setting, no explicit expression is known for the kernel function. Thus, for proving our main results, we will rather use another sequence of test function which replaces the role of the reproducing kernel. Such a sequence was first constructed in [3] and has been further used by several authors for example [6, 13, 9]. We introduce the sequence of test functions as follows. We set

$$\tau_m(z) = \begin{cases} 1, & 0 \leq |(m^2 - m)z| < 1 \\ \frac{|z|^{\frac{2-m}{2}}}{|m^2 - m|^{\frac{1}{2}}}, & |(m^2 - m)z| \geq 1. \end{cases}$$

For a sufficiently large positive number R , there exists a number $\eta(R)$ such that for any $w \in \mathbb{C}$ with $|w| > \eta(R)$, there exists an entire function $f_{(w,R)}$ such that

(i)

$$|f_{(w,R)}(z)| e^{-|z|^m} \leq C \min \left\{ 1, \left(\frac{\min\{\tau_m(w), \tau_m(z)\}}{|z-w|} \right)^{\frac{R^2}{2}} \right\} \quad (2.7)$$

for all $z \in \mathbb{C}$ and for some constant C that depends on $|z|^m$ and R . In particular when $z \in D(w, R\tau_m(w))$, the estimate becomes

$$|f_{(w,R)}(z)| e^{-|z|^m} \simeq 1. \quad (2.8)$$

(ii) $f_{(w,R)}$ belongs to $\mathcal{F}_{(m,p)}$ and its norm is estimated by

$$\|f_{(w,R)}\|_{(m,p)}^p \simeq \tau_m^2(w), \quad \eta(R) \leq |w| \quad (2.9)$$

for all p in the range $0 < p < \infty$.

Another important ingredient in our subsequent consideration is the

pointwise estimate for subharmonic functions $|f|^p$, namely that

$$|f(z)|^p e^{-p|z|^m} \lesssim \frac{1}{\sigma^2 \tau_m^2(z)} \int_{D(z, \sigma \tau_m(z))} |f(w)|^p e^{-p|w|^m} dA(w) \quad (2.10)$$

for all finite exponent p and a small positive number σ . The estimate follows from Lemma 2 of [13].

Next, we recall the notion of covering for the space \mathbb{C} . We denote by $D(w, r)$ the Euclidean disk centered at w and radius $r > 0$. Then, we record the following useful covering lemma which is essentially from [6, 12].

Lemma 2.3. *Let τ_m be as above. Then, there exists a positive $\sigma > 0$ and a sequence of points z_j in \mathbb{C} satisfying the following conditions.*

- (i) $z_j \notin D(z_k, \sigma \tau_m(z_k))$, $j \neq k$; (ii) $\mathbb{C} = \bigcup_j D(z_j, \sigma \tau_m(z_j))$;
- (iii) $\bigcup_{z \in D(z_j, \sigma \tau_m(z_j))} D(z, \sigma \tau_m(z)) \subset D(z_j, 3\sigma \tau_m(z_j))$;
- (iv) *The sequence $D(z_j, 3\sigma \tau_m(z_j))$ is a covering of \mathbb{C} with finite multiplicity N_{\max} .*

Lemma 2.4. *Let R be a sufficiently large number and $\eta(R)$ be as before. If (z_k) is the covering sequence from Lemma 2.3, then the function*

$$F = \sum_{z_k: |z_k| > \eta(R)} a_k \frac{f_{(z_k, R)}^{\frac{1}{2}}}{\tau_m^{\frac{p}{2}}(z_k)}$$

belongs to $\mathcal{F}_{(m,p)}$ for every sequence (a_k) in ℓ^p and also $\|F\|_{(m,p)} \lesssim \|(a_k)\|_{\ell^p}$.

The proof the Lemma follows from a simple variant of the proof of Proposition 9 in [6] or Proposition 1 in [13].

3. PROOF OF THE MAIN RESULTS

3.1. Proof of Theorem 1.1-Part (i). Let us first prove the necessity of the condition in part (i), and assume that $D : \mathcal{F}_{(m,p)} \rightarrow \mathcal{F}_{(m,q)}$ is bounded. Then, making use of the estimates in (2.8), (2.9), (2.7) and (2.10), we have

$$\begin{aligned} \|D\|^q &\gtrsim \tau_m^{-\frac{2q}{p}}(w) \|Df_{(w,R)}\|_{(m,q)}^q = \tau_m^{-\frac{2q}{p}}(w) \int_{\mathbb{C}} |f'_{(w,R)}(z)|^q e^{-q|z|^m} dA(z) \\ &\geq \tau_m^{-\frac{2q}{p}}(w) \int_{D(w, \delta \tau_m(w))} \frac{|f'_{(w,R)}(z)|^q}{e^{q|z|^m}} dA(z) \gtrsim \tau_m^{2-\frac{2q}{p}}(w) \frac{|f'_{(w,R)}(w)|^q}{e^{q|w|^m}} \\ &\simeq m^q \tau_m^{2-\frac{2q}{p}}(w) |w|^{q(m-1)} \end{aligned}$$

for all $w \in \mathbb{C}$. It follows that

$$\|D\| \gtrsim \begin{cases} |m^{2+p} - m^{1+p}|^{\frac{1}{p}} \sup_{w \in \mathbb{C}} (1 + |w|)^{(m-1) + \frac{(q-p)(m-2)}{qp}}, & m \neq 1. \\ 1, & m = 1 \end{cases} \quad (3.1)$$

which holds only if $pq(m-1) + (q-p)(m-2) \leq 0$ as asserted, and it also gives a one sided estimate for the norm of D .

We now turn to the proof of the sufficiency of the condition in part (i). We use the covering sequences approach along with Lemma 2.3, where the original idea goes back to [12]. Applying (2.1) and (2.10), we estimate

$$\begin{aligned} \|Df\|_{(m,q)}^q &= \int_{\mathbb{C}} |f'(z)|^q e^{-q|z|^m} dA(z) \leq \sum_j \int_{D(z_j, \sigma\tau_m(z_j))} |f'(z)|^q e^{-q|z|^m} dA(z) \\ &\lesssim \sum_j \int_{D(z_j, \sigma\tau_m(z_j))} \left(\frac{1}{\tau_m^2(z)} \int_{D(z, \sigma\tau_m(z))} |f'(w)|^p e^{-p|w|^m} dA(w) \right)^{\frac{q}{p}} dA(z) =: S \end{aligned}$$

Now for each point $z \in D(w, \sigma\tau_m(w))$, observe that $1 + |z| \simeq 1 + |w|$. Taking this into account, we further estimate

$$\begin{aligned} S &\simeq \sum_j \int_{D(z_j, \sigma\tau_m(z_j))} \left(\frac{m^p(1 + |z|)^{p(m-1)}}{\tau_m^2(z)} \int_{D(z, \sigma\tau_m(z))} \frac{|f'(w)|^p e^{-p|w|^m}}{m^p(1 + |w|)^{p(m-1)}} dA(w) \right)^{\frac{q}{p}} dA(z) \\ &\lesssim \sum_j \left(\int_{D(z_j, 3\sigma\tau_m(z_j))} \frac{|f'(w)|^p e^{-p|w|^m}}{m^p(1 + |w|)^{p(m-1)}} dA(w) \right)^{\frac{q}{p}} \int_{D(z_j, \sigma\tau_m(z_j))} \frac{m^q(1 + |z|)^{q(m-1)}}{\tau_m^{\frac{2q}{p}}(z)} dA(z) \end{aligned}$$

Since $q \geq p$, applying Minkowski inequality and the finite multiplicity N_{\max} of the covering sequence $D(z_j, 3\sigma\tau_m(z_j))$, we obtain

$$\begin{aligned} &\sum_j \left(\int_{D(z_j, 3\sigma\tau_m(z_j))} \frac{|f'(w)|^p e^{-p|w|^m}}{m^p(1 + |w|)^{p(m-1)}} dA(w) \right)^{\frac{q}{p}} \int_{D(z_j, \sigma\tau_m(z_j))} \frac{m^q(1 + |z|)^{q(m-1)}}{\tau_m^{\frac{2q}{p}}(z)} dA(z) \\ &\leq \left(\sum_j \int_{D(z_j, 3\sigma\tau_m(z_j))} \frac{|f'(w)|^p e^{-p|w|^m}}{m^p(1 + |w|)^{p(m-1)}} dA(w) \right)^{\frac{q}{p}} \int_{D(z_j, \sigma\tau_m(z_j))} \frac{m^q(1 + |z|)^{q(m-1)}}{\tau_m^{\frac{2q}{p}}(z)} dA(z) \\ &\lesssim \|f\|_{(m,p)}^q \sup_{w \in \mathbb{C}} \int_{D(w, \sigma\tau_m(w))} \frac{m^q(1 + |z|)^{q(m-1)}}{\tau_m^{\frac{2q}{p}}(z)} dA(z) \\ &\lesssim \|f\|_{(m,p)}^q \sup_{w \in \mathbb{C}} \frac{m^q(1 + |w|)^{q(m-1)} \tau_m^2(w)}{\tau_m^{\frac{2q}{p}}(w)} \\ &\simeq \|f\|_{(m,p)}^q |m^{2+p} - m^{1+p}|^{\frac{1}{p}} \sup_{w \in \mathbb{C}} (1 + |w|)^{q(m-1) + \frac{q-p}{p}(m-2)} \end{aligned}$$

from which the sufficiency of the condition and the reverse side of the estimate in (3.1) follow. Thus we estimate the norm by

$$\|D\| \simeq \begin{cases} |m^{2+p} - m^{1+p}|^{\frac{1}{p}} \sup_{w \in \mathbb{C}} (1 + |w|)^{(m-1) + \frac{(q-p)(m-2)}{qp}}, & m \neq 1. \\ 1, & m = 1 \end{cases}$$

To prove the compactness, we first assume that the condition $m < 2 - \frac{pq}{pq+q-p}$ holds. Then for each positive ϵ , there exists N_1 such that

$$|m^{2+p} - m^{1+p}|^{\frac{1}{p}} \sup_{|w| > N_1} (1 + |w|)^{q(m-1) + \frac{(q-p)(m-2)}{qp}} < \epsilon. \quad (3.2)$$

Next, we let f_n to be a uniformly bounded sequence of functions in $\mathcal{F}_{(m,p)}$ that converges uniformly to zero on compact subsets of \mathbb{C} . Then applying (2.1) and

arguing in the same way as in the series of estimations made above, and invoking eventually (3.2) it follows that

$$\begin{aligned}
\|Df_n\|_{(m,q)}^q &\lesssim \int_{|z|\leq N_1} \frac{|f'_n(z)|^q}{e^{q|z|^m}} dA(z) + \sum_{|z_j|>N-1} \int_{D(z_j, \sigma\tau_m(z_j))} \frac{|f'_n(z)|^q}{e^{q|z|^m}} dA(z) \\
&\lesssim \sup_{|w|\leq N_1} |f_n(w)|^q + \\
&+ \sum_{|z_j|>N-1} \int_{D(z_j, \sigma\tau_m(z_j))} \left(\frac{m^p(1+|z|)^{p(m-1)}}{\tau_m^2(z)} \int_{D(z, \sigma\tau_m(z))} \frac{|f'_n(w)|^p e^{-p|w|^m}}{m^p(1+|w|)^{p(m-1)}} dA(w) \right)^{\frac{q}{p}} dA(z) \\
&\lesssim \sup_{|w|\leq N_1} |f_n(w)|^q + \|f_n\|_{(m,q)}^q \sup_{|w|>N_1} \frac{m^q(1+|w|)^{q(m-1)} \tau_m^2(w)}{\tau_m^{\frac{2q}{p}}(w)} \\
&\lesssim \sup_{|w|\leq N_1} |f_n(w)|^q + |m^{2+p} - m^{1+p}|^{\frac{1}{p}} \sup_{|w|>N_1} (1+|w|)^{q(m-1) + \frac{q-p}{p}(m-2)} \lesssim \epsilon \text{ as } n \rightarrow \infty.
\end{aligned}$$

Conversely, assume that D is compact, and observe that the normalized sequence $f_{(w,R)}^* = f_{(w,R)} / \|f_{(w,R)}\|_{(m,p)}$ converges to zero as $|w| \rightarrow \infty$, and the convergence is uniform on compact subset of \mathbb{C} . Then applying (2.10) and (2.8), we find

$$\begin{aligned}
\frac{|w|^{q(m-1)}}{\tau_m^{\frac{2q-p}{p}}(w)} &\simeq (1+|w|)^{q(m-1)} \tau_m^{\frac{2p-q}{p} + \frac{2q}{p}}(w) e^{-q|w|^m} |f_{(w,\eta(R))}^*(w)|^q \\
&\lesssim \int_{D(w, \sigma\tau_m(w))} (1+|z|)^{q(m-1)} |f_{(w,\eta(R))}^*(z)|^q e^{-q|z|^m} dA(z) \\
&\lesssim \|Df_{(w,\eta(R))}^*\|_{(m,q)}^q \rightarrow 0, \text{ as } |w| \rightarrow \infty.
\end{aligned}$$

We note in passing that in particular when $p = q$ the necessity of the conditions in part (i) could be also established using the sequence of the polynomials (z^n) as test functions. Such polynomials belong to the spaces $\mathcal{F}_{(m,p)}$ for all p . Because arguing with polar coordinates and subsequently substitution, we could easily observe that

$$\begin{aligned}
\|z^n\|_{(m,p)}^p &= \int_{\mathbb{C}} |z^n|^p e^{-p|z|^m} dA(z) = 2\pi \int_0^\infty r^{pn+1} e^{-pr^m} dr \\
&= 2\pi p^{-\frac{pn+2}{m}} \int_0^\infty t^{\frac{pn+2}{m}-1} e^{-t} dt = 2\pi p^{-\frac{pn+2}{m}} \Gamma\left(\frac{pn+2}{m}\right) < \infty.
\end{aligned}$$

For the case $p < q$, an application of such polynomials only gives the condition

$$m \leq 2 - \frac{2(pq - 3(q-p))}{p-q+2pq}$$

which is weaker than the condition in the result since

$$\frac{2(pq - 3(q-p))}{p-q+2pq} < \frac{pq}{pq+q-p}, \text{ for } p < q.$$

3.2. Proof of Theorem 1.1-Part (ii). We assume $0 < q < p < \infty$. As (b) obviously implies (a), we plan to show (a) implies (c) and (c) implies (b). For the first, we follow this classical technique where the original idea goes back to Luecking [8]. Let $0 < q < \infty$ and R be a sufficiently large number and (z_k) be the covering sequence as in Lemma 2.3. Then by Lemma 2.4,

$$F = \sum_{z_k: |z_k| \geq \eta(R)} a_k \frac{f_{(z_k, R)}^{\frac{1}{2}}}{\tau_m^{\frac{p}{2}}(z_k)}$$

belongs to $\mathcal{F}_{(m, p)}$ for every ℓ^p sequence (a_k) with norm estimate $\|F\|_{(m, p)} \lesssim \|(a_k)\|_{\ell^p}$. If $(r_k(t))_k$ is the Radmecher sequence of function on $[0, 1]$ chosen as in [8], then the sequence $(a_k r_k(t))$ also belongs to ℓ^p with $\|(a_k r_k(t))\|_{\ell^p} = \|(a_k)\|_{\ell^p}$ for all t . This implies that the function

$$F_t = \sum_{z_k: |z_k| \geq \eta(R)} a_k r_k(t) \frac{f_{(z_k, R)}^{\frac{1}{2}}}{\tau_m^{\frac{p}{2}}(z_k)}$$

belongs to $\mathcal{F}_{(m, p)}$ with norm estimate $\|F_t\|_{(m, p)} \lesssim \|(a_k)\|_{\ell^p}$. Then, an application of Khinchine's inequality [8] yields

$$\left(\sum_{z_k: |z_k| \geq \eta(R)} |a_k|^2 \frac{|f'_{(z_k, R)}(z)|^2}{\tau_m^{\frac{4}{p}}(z_k)} \right)^{\frac{q}{2}} \lesssim \int_0^1 \left| \sum_{z_k: |z_k| \geq \eta(R)} a_k r_k(t) \frac{f'_{(z_k, R)}(z)}{\tau_m^{\frac{p}{2}}(z_k)} \right|^q dt. \quad (3.3)$$

Making use of (3.3), and subsequently Fubini's theorem, we have

$$\begin{aligned} & \int_{\mathbb{C}} \left(\sum_{z_k: |z_k| \geq \eta(R)} |a_k|^2 \frac{|f'_{(z_k, R)}(z)|^2}{\tau_m^{\frac{4}{p}}(z_k)} \right)^{\frac{q}{2}} e^{-q|z|^m} dA(z) \\ & \lesssim \int_{\mathbb{C}} \int_0^1 \left| \sum_{z_k: |z_k| \geq \eta(R)} a_k r_k(t) \frac{f'_{(z_k, R)}(z)}{\tau_m^{\frac{p}{2}}(z_k)} \right|^q dt e^{-q|z|^m} dA(z) \\ & = \int_0^1 \int_{\mathbb{C}} \left| \sum_{z_k: |z_k| \geq \eta(R)} a_k r_k(t) \frac{f'_{(z_k, R)}(z)}{\tau_m^{\frac{p}{2}}(z_k)} \right|^q e^{-q|z|^m} dA(z) dt \simeq \int_0^1 \|DF_t\|_{\mathcal{F}_{(m, q)}}^q dt \lesssim \|(a_k)\|_{\ell^p}^q. \end{aligned}$$

Now arguing with this, the covering lemma, and (2.8) leads to the series of estimates

$$\begin{aligned} & \sum_{z_k: |z_k| \geq \eta(R)} \frac{|a_k|^q}{\tau_m^{\frac{2q}{p}}(z_k)} \int_{D(z_k, 3\sigma\tau_m(z_k))} (1 + |z|)^{q(m-1)} dA(z) \\ & \simeq \sum_{z_k: |z_k| \geq \eta(R)} \frac{|a_k|^q}{\tau_m^{\frac{2q}{p}}(z_k)} \int_{D(z_k, 3\sigma\tau_m(z_k))} |f'_{(z_k, R)}(z)|^q e^{-q|z|^m} dA(z) \\ & \simeq \int_{\mathbb{C}} \sum_{z_k: |z_k| \geq \eta(R)} \frac{|a_k|^q}{\tau_m^{\frac{2q}{p}}(z_k)} \chi_{D(z_k, 3\sigma\tau_m(z_k))}(z) |f'_{(z_k, R)}(z)|^q e^{-q|z|^m} dA(z) \\ & \lesssim \max\{1, N_{\max}^{1-q/2}\} \int_{\mathbb{C}} \left(\sum_{z_k: |z_k| \geq \eta(R)} |a_k|^2 \frac{|f'_{(z_k, R)}(z)|^2}{\tau_m^{\frac{4}{p}}(z_k)} \right)^{\frac{q}{2}} e^{-q|z|^m} dA(z) \lesssim \|(a_k)\|_{\ell^p}^q. \end{aligned}$$

Applying duality between the spaces $\ell^{p/q}$ and $\ell^{p/(p-q)}$, we again get

$$\begin{aligned} \sum_{z_k: |z_k| \geq \eta(R)} \left(\frac{1}{\tau_m^2(z_k)} \int_{D(z_k, 3\sigma\tau_m(z_k))} (1 + |z|)^{q(m-1)} dA(z) \right)^{\frac{p}{p-q}} \tau_m^2(z_k) \\ \simeq \sum_{z_k: |z_k| \geq \eta(R)} (1 + |z_k|)^{\frac{qp(m-1)}{p-q}} \tau_m^2(z_k) < \infty. \end{aligned}$$

On the other hand, we can find a positive number $r \geq \eta(R)$ such that whenever a point z_k of the covering sequence (z_j) belongs to $\{|z| < \eta(R)\}$, then $D(z_k, \sigma\tau_m(z_k))$ belongs to $\{|z| < \eta(R)\}$. In view of this we estimate

$$\begin{aligned} \int_{|w| \geq r} (1 + |w|)^{\frac{qp(m-1)}{p-q}} dA(w) &\leq \sum_{|z_k| \geq \eta(R)} \int_{D(z_k, \sigma\tau_m(z_k))} (1 + |w|)^{\frac{qp(m-1)}{p-q}} dA(w) \\ &\lesssim \sum_{|z_k| \geq \eta(R)} \int_{D(z_k, \sigma\tau_m(z_k))} (1 + |w|)^{\frac{qp(m-1)}{p-q}} \tau_m^2(z_k) dA(w) \\ &\simeq \sum_{|z_k| \geq \eta(R)} (1 + |z_k|)^{\frac{qp(m-1)}{p-q}} \tau_m^2(z_k) < \infty. \end{aligned}$$

It also follows that

$$\int_{|w| < r} \left(\frac{1}{\tau_m^2(w)} \int_{D(w, 3\delta\tau_m(w))} (1 + |z|)^{q(m-1)} dA(z) \right)^{\frac{p}{p-q}} dA(w) < \infty$$

Taking into account the range of the above estimates we find

$$\begin{aligned} \int_{\mathbb{C}} (1 + |z|)^{\frac{qp}{p-q}(m-1)} dA(w) &= \int_{|z| \leq r} (1 + |w|)^{\frac{qp}{p-q}(m-1)} dA(w) \\ &\quad + \int_{|w| > r} (1 + |w|)^{\frac{qp}{p-q}(m-1)} dA(w) < \infty, \end{aligned}$$

which holds only if $\frac{qp}{p-q}(m-1) < -2$ as claimed.

To prove (c) implies (b), we argue as follows. Let f_n to be a uniformly bounded sequence of functions in $\mathcal{F}_{(m,p)}$ that converges uniformly to zero on compact subsets of \mathbb{C} , and by the given condition, for each $\epsilon > 0$, there exists a positive number r_1 such that

$$\int_{|z| > r_1} (1 + |z|)^{\frac{qp}{p-q}(m-1)} dA(z) < \epsilon. \quad (3.4)$$

Since $p/q > 1$, applying Hölder's inequality, (2.1) and (3.4), we have

$$\begin{aligned} \int_{|z| > r_1} |f'_n(z)|^q e^{-q|z|^m} dA(z) &= \int_{|z| > r_1} \left(\frac{|f'_n(z)|^q e^{-q|z|^m}}{(1 + |z|)^{q(m-1)}} \right) (1 + |z|)^{q(m-1)} dA(z) \\ &\lesssim \|f\|_{(m,p)}^q \left(\int_{|z| > r_1} (1 + |z|)^{\frac{qp}{p-q}(m-1)} dA(z) \right)^{\frac{p-q}{p}} \\ &\lesssim \|f\|_{(m,p)}^q \epsilon \lesssim \epsilon. \end{aligned}$$

On the other hand when $|z| \leq r_1$, then

$$\begin{aligned} \int_{|z| \leq r_1} |f'_n(z)|^q e^{-q|z|^m} dA(z) &\lesssim \int_{|z| \leq r_1} |f_n(z)|^q (1 + |z|)^q e^{-q|z|^m} dA(z) \\ &\lesssim \sup_{|z| \leq r_1} |f_n(z)|^q \int_{|z| \leq r_1} (1 + |z|)^q e^{-q|z|^m} dA(z) \\ &\lesssim \sup_{|z| \leq r_1} |f_n(z)|^q \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

from which our claim follows.

3.3. Proof of Theorem 1.2. *Part (i).* Let us now turn to the Schatten $\mathcal{S}_p(\mathcal{F}_{(m,2)})$ membership of D . We recall that a compact D belongs to the Schatten $\mathcal{S}_p(\mathcal{F}_{(m,2)})$ class if and only if the sequence of the eigenvalues of the positive operator $(D^*D)^{1/2}$ is ℓ^p summable. It suffices to prove the statement for $p \geq 1$. The remaining case for $0 < p < 1$ follows by the monotonicity property $\mathcal{S}_p(\mathcal{F}_{(m,2)}) \supseteq \mathcal{S}_q(\mathcal{F}_{(m,2)})$ for $p \leq q$.

If $p > 1$, then D belongs to $\mathcal{S}_p(\mathcal{F}_{(m,2)})$ if and only if

$$\sum_{n=0}^{\infty} |\langle De_n, e_n \rangle|^p < \infty, \quad (3.5)$$

for any orthonormal basis (e_n) of $\mathcal{F}_{(m,2)}$ (see [14, Theorem 1.27]). Note that the sequence of the polynomials $(z^n / \|z^n\|_{(m,2)})$ constitutes an orthonormal basis to $\mathcal{F}_{(m,2)}$. Since

$$De_n = n \frac{z^{n-1}}{\|z^n\|_{(m,2)}} = \frac{n \|z^{n-1}\|_{(m,2)}}{\|z^n\|_{(m,2)}} e_{n-1},$$

we obtain

$$\langle De_n, e_n \rangle = \frac{n \|z^{n-1}\|_{(m,2)}}{\|z^n\|_{(m,2)}} \langle e_{n-1}, e_n \rangle = 0$$

for all n , from which (3.5) easily follows.

Part (ii). Recall that the spectrum $\sigma(T)$ of a bounded operator T is the set containing all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ fails to be invertible, where I is the identity operator. The complement of the spectrum is referred as the resolvent set. A simple computation shows that the function $f^*(z) = ce^{\lambda z}$ solves the differential equation $\lambda f = Df = f'$, where c is a constant. Then we analyze

$$\|f^*\|_{(m,p)}^p = \int_{\mathbb{C}} |ce^{\lambda z}|^p e^{-p|z|^m} dA(z) = |c|^p \int_{\mathbb{C}} e^{p\Re(\lambda z) - p|z|^m} dA(z) \quad (3.6)$$

depending on the size of m . Let us first assume that $m = 1$. Then, the integral in (3.6) converges for each $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$. This means that the function f^* belongs to $\mathcal{F}_{(m,p)}$, and can be chosen in such a way that $c \neq 0$. From this we deduce

$$\overline{\mathbb{D}} \subseteq \sigma(D) \quad \text{or} \quad \overline{\{\lambda \in \mathbb{C} : e^{\lambda z} \in \mathcal{F}_{(m,p)}\}} \subseteq \sigma(D). \quad (3.7)$$

To prove the reverse inclusion in (3.7), observe that the integral in (3.6) fails to converge for each $|\lambda| \geq 1$ and $c \neq 0$. It means that $\lambda I - D$ is injective whenever $|\lambda| \geq 1$. On the other hand, for such values of λ , a simple computation again shows that the equation $\lambda f - Df = h$ has a unique analytic solution

$$f(z) = R_\lambda h(z) = Ce^{\lambda z} - e^{\lambda z} \int_0^z e^{-\lambda w} h(w) dA(w), \quad (3.8)$$

where R_λ is the resolvent operator of D at point λ , $C = f(0)$ is a constant value. We remain to show that the operator R_λ given by the explicit expression in (3.8) is bounded on $\mathcal{F}_{(m,p)}$. To this end, applying Lemma 2.1, we have

$$\begin{aligned} \|R_\lambda h\|_{(m,p)}^p &= \left\| e^{\lambda z} \left(C - \int_{z_0}^z e^{-\lambda w} h(w) dA(w) \right) \right\|_{(m,p)}^p \\ &\lesssim \int_{\mathbb{C}} |e^{\lambda z}|^p e^{-p|z|} \left| \frac{d}{dz} \int_{z_0}^z e^{-\lambda w} h(w) dA(w) \right|^p dA(z) \\ &\lesssim \int_{\mathbb{C}} |h(z)|^p e^{-p|z|} dA(w) dA(z) = \|h\|_{(m,p)}^p. \end{aligned}$$

We now turn to the case $m < 1$. For this, part (b) of our result forces D to be a compact operator. Furthermore, we observe that the integral in (3.6) converges only if $c = 0$ and hence $f^*(z) = 0$, which clarifies that D has no point spectrum. To this effect, $\sigma(D) = \{0\}$.

REFERENCES

- [1] M. J. Beltrán, Dynamics of differentiation and integration operators on weighted space of entire functions, *Studia Mathematica*, 221, **1**(2014), 35–60.
- [2] H. Bommier-Hatoa, M. Englis, El-Hassan Youssfia, Bergman-type projections in generalized Fock spaces. *J. Math. Anal. Appl.*, 389, **2** (2012), 1086–1104.
- [3] A. Borichev, R. Dhuez and K. Kellay, Sampling and interpolation in large Bergman and Fock spaces. *J. Funct. Anal.*, **242** (2007), 563–606.
- [4] R. Cho and K. Zhu, Fock–Sobolev spaces and their Carleson measures, *J. Funct. Anal.*, Vol. 263, Issue 8, **15** (2012), 2483–2506.
- [5] O. Constantin and Ann-Maria Persson, The spectrum of Volterra type integration operators on generalized Fock spaces, *Bull. London Math. Soc.* 47(2015), no 6, 958–963.
- [6] O. Constantin and José Ángel Peláez, Integral Operators, Embedding Theorems and a Littlewood–Paley Formula on Weighted Fock Spaces, *J. Geom. Anal.*, (2015), 1–46.
- [7] A. Harutyunyan, W. Lusky, On the boundedness of the differentiation operator between weighted spaces of holomorphic functions, *Studia Math.* **184**(2008),233–247.
- [8] D. Luecking, Embedding theorems for space of analytic functions via Khinchine’s inequality, *Michigan Math. J.*, **40** (1993), 333–358.
- [9] T. Mengestie and S. Ueki, Integral, differential and multiplication operators on weighted Fock spaces, Preprint, 2016.
- [10] T. Mengestie, Carleson type measures for Fock–Sobolev spaces, *Complex Anal. Oper. Theory*, **8** (2014), no 6, 1225–1256.
- [11] T. Mengestie, Spectral properties of Volterra-type integral operators on Fock–Sobolev spaces, *Korean Journal of Mathematical Society*, <https://doi.org/10.4134/JKMS.j160671>.

- [12] V. L. Oleinik, Embedding theorems for weighted classes of harmonic and analytic functions. J. Math. Sci., **9(2)** (1978), 228–243.
- [13] J. Pau and J. A. Peláez, Embedding theorems and integration operators on Bergman spaces with rapidly decreasing weights. J. Funct. Anal., 259 (**10**)(2010), 2727–2756.
- [14] K. Zhu, Operator theory on function spaces, Second Edition, Math. Surveys and Monographs, Vol. 138, American Mathematical Society: Providence, Rhode Island, 2007.

DEPARTMENT OF MATHEMATICAL SCIENCES, WESTERN NORWAY UNIVERSITY OF APPLIED SCIENCES, KLINGENBERGVEGEN 8, N-5414 STORD, NORWAY

E-mail address: `Tesfa.Mengestie@hvl.no`