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ORTHOGONAL SHADOWS AND INDEX OF GRASSMANN MANIFOLDS

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Abstract. In this paper we study the $\mathbb{Z}/2$ action on real Grassmann manifolds $G_n(\mathbb{R}^{2n})$ and $\widetilde{G}_n(\mathbb{R}^{2n})$ given by taking (appropriately oriented) orthogonal complement. We completely evaluate the related $\mathbb{Z}/2$ Fadell–Husseini index utilizing a novel computation of the Stiefel–Whitney classes of the wreath product of a vector bundle. These results are used to establish the following geometric result about the orthogonal shadows of a convex body: For $n = 2^a(2b+1)$, $k = 2^{a+1}-1$, C a convex body in $\mathbb{R}^{2n}$, and k real valued functions $\alpha_1, \dots, \alpha_k$ continuous on convex bodies in $\mathbb{R}^{2n}$ with respect to the Hausdorff metric, there exists a subspace $V \subseteq \mathbb{R}^{2n}$ such that projections of C to V and its orthogonal complement $V^\perp$ have the same value with respect to each function α_i, which is $\alpha_i(p_V(C)) = \alpha_i(p_{V^\perp}(C))$ for all $1 \le i \le k$.

1. Introduction

The Grassmann manifold of all n dimensional linear subspaces in the vector space V over some field is one of the classical and widely studied objects of algebraic topology with important applications in differential geometry and algebraic geometry. In this paper we study a particular free $\mathbb{Z}/2$ action on a real Grassmann manifolds induced by taking orthogonal complements and use its properties to present an interesting geometric application.

Let $n \ge 1$ and $1 \le k \le n$ be integers. The *real Grassmann manifold* of all n dimensional linear subspaces in the Euclidean space $\mathbb{R}^{n+k}$ is denoted by $G_n(\mathbb{R}^{n+k})$. Classically, as a homogeneous space, it is defined to be the quotient

$$G_n(\mathbb{R}^{n+k}) := \mathrm{O}(n+k)/(\mathrm{O}(n) \times \mathrm{O}(k)),$$

where $\mathrm{O}(n)$ denotes the orthogonal group. The *real oriented Grassmann manifold* of all oriented n dimensional linear subspaces in the Euclidean space $\mathbb{R}^{n+k}$ is denoted by $\widetilde{G}_n(\mathbb{R}^{n+k})$, and is defined to be

$$\widetilde{G}_n(\mathbb{R}^{n+k}) := \mathrm{SO}(n+k)/(\mathrm{SO}(n) \times \mathrm{SO}(k)),$$

where $\mathrm{SO}(n)$ denotes the special orthogonal group. The inclusion map of the pairs

$$(\mathrm{SO}(n+k), \mathrm{SO}(n) \times \mathrm{SO}(k)) \hookrightarrow (\mathrm{O}(n+k), \mathrm{O}(n) \times \mathrm{O}(k))$$

induces the map of quotients $c\colon \widetilde{G}_n(\mathbb{R}^{n+k}) \longrightarrow G_n(\mathbb{R}^{n+k})$ which is a double cover.

Now we specialize to the situation where the integers k and n coincide. Let $\mathbb{Z}/2 = \langle \omega \rangle$ be the cyclic group of order 2 generated by ω. A $\mathbb{Z}/2$ action on real Grassmann manifolds $G_n(\mathbb{R}^{2n})$ and $\widetilde{G}_n(\mathbb{R}^{2n})$ we consider is defined by sending an n-dimensional (oriented) subspace V to its (appropriately oriented) orthogonal complement $V^\perp$, that means $\omega \cdot V = V^\perp$. In the case when n is even these actions can be lifted to actions on $\mathrm{O}(2n)$ and $\mathrm{SO}(2n)$ in such a way that the quotient maps $p_1\colon \mathrm{O}(2n) \longrightarrow G_n(\mathbb{R}^{2n})$ and $p_2\colon \mathrm{SO}(2n) \longrightarrow \widetilde{G}_n(\mathbb{R}^{2n})$ are $\mathbb{Z}/2$-maps. Indeed, for $(v_1, \dots, v_n, v_{n+1}, \dots, v_{2n})$ and element of $\mathrm{O}(2n)$, or $\mathrm{SO}(2n)$ we set

$$\omega \cdot (v_1, \dots, v_n, v_{n+1}, \dots, v_{2n}) = (v_{n+1}, \dots, v_{2n}, v_1, \dots, v_n).$$

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Thus, for n even we have the following commutative square of $\mathbb{Z}/2$-maps:

$$\begin{tikzcd} \mathrm{SO}(2n) \arrow[r, "p_2"] \arrow[d, "i"] & \widetilde{G}_n(\mathbb{R}^{2n}) \arrow[d, "c"] \\ \mathrm{O}(2n) \arrow[r, "p_1"] & G_n(\mathbb{R}^{2n}) \end{tikzcd}$$

where $i\colon \mathrm{SO}(2n) \longrightarrow \mathrm{O}(2n)$ is the inclusion, and $c\colon \widetilde{G}_n(\mathbb{R}^{n+k}) \longrightarrow G_n(\mathbb{R}^{n+k})$ is the double cover map.

The central result of this paper is the evaluation of the Fadell–Husseini index [5] of the Grasmmann manifolds with respect to the $\mathbb{Z}/2$ action we just introduced. Before we state the result we briefly recall the definition and some basic properties of the Fadell–Husseini index. For a finite group G and a G-space X the *Fadell–Husseini index* with the coefficients in the fields $\mathbb{F}$ is the kernel of the following map in cohomology

$$\mathrm{Index}_G(X;\mathbb{F}) := \ker\big(\pi_X^*\colon H^*(\mathrm{E}G\times_G \mathrm{pt};\mathbb{F}) \longrightarrow H^*(\mathrm{E}G\times_G X;\mathbb{F})\big).$$

where the map $\pi_X\colon \mathrm{E}G\times_G X \longrightarrow \mathrm{E}G\times_G \mathrm{pt}$ is induced by the G-map $X \longrightarrow \mathrm{pt}$. The G-action on the point pt is assumed to be trivial. The space $\mathrm{E}G\times_G X$ is known as the Borel construction of the G-space X. There are natural isomorphisms $H^*(\mathrm{E}G\times_G \mathrm{pt};\mathbb{F}) \cong H^*(\mathrm{B}G;\mathbb{F}) \cong H^*(G;\mathbb{F})$ and therefore we do not distinguish them. For the cohomology of the group $\mathbb{Z}/2$ we fix the following notation

$$H^*(\mathbb{Z}/2;\mathbb{F}_2) = H^*(\mathrm{B}(\mathbb{Z}/2);\mathbb{F}_2) = \mathbb{F}_2[t],$$

with $\deg(t)=1$. The essential property of the Fadell–Husseini index [5, p. 74] is that: If X and Y are G-spaces and if there exists a continuous G-map $X \longrightarrow Y$ then

$$\mathrm{Index}_G(X;\mathbb{F}) \supseteq \mathrm{Index}_G(Y;\mathbb{F}).$$

For example, it is a known fact that the Fadell–Husseini index of the sphere S^{n-1}, equipped with any free $\mathbb{Z}/2$ action, is the ideal generated by t^n, that is $\mathrm{Index}_{\mathbb{Z}/2}(S^{n-1};\mathbb{F}_2) = \langle t^n\rangle$.

The Fadell–Husseini index of a path connected G-space X with coefficients in a field $\mathbb{F}$ can be computed from the Serre spectral sequence associated with the Borel construction fibration

$$X \longrightarrow \mathrm{E}G\times_G X \longrightarrow \mathrm{B}G,$$

whose E_2-term is given by

$$E_2^{i,j}(\mathrm{E}G\times_G X) = H^i(\mathrm{B}G;\mathcal{H}^j(X;\mathbb{F})) \cong H^i(G;H^j(X;\mathbb{F})).$$

Here $H^i(\mathrm{B}G;\mathcal{H}^j(X;\mathbb{F}))$ denotes the cohomology with local coefficients induced by the action of the fundamental group of the base space $\pi_1(\mathrm{B}G)\cong G$ on the cohomology of the fiber $H^j(X;\mathbb{F})$. On the other hand $H^i(G;H^j(X;\mathbb{F}))$ denotes the cohomology of the group G with coefficients in the G-module $H^j(X;\mathbb{F})$. Those two cohomologies are isomorphic by definition. For a definition of cohomology with local coefficients consult for example [6, Sec. 3.H]. The Fadell–Husseini index of a path-connected G-space X can be evaluated using the equality

$$\mathrm{Index}_G(X;\mathbb{F}) = \ker\big(E_2^{*,0}(\mathrm{E}G\times_G X) \longrightarrow E_\infty^{*,0}(\mathrm{E}G\times_G X)\big). \tag{1}$$

In this paper we compute the Fadell–Hussini index of Grassmann manifolds with respect to the introduced $\mathbb{Z}/2$ action and prove the following two theorems.

Theorem 1.1. *Let $n\geq 1$ be an integer, and let $a\geq 0$ and $b\geq 0$ be the unique integers such that $n=2^a(2b+1)$. Then*

$$\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_n(\mathbb{R}^{2n});\mathbb{F}_2) = \begin{cases} \langle t^3\rangle, & a=1,\\ \langle t^{2^{a+1}}\rangle, & a=0 \text{ or } a\geq 2.\end{cases}$$

Theorem 1.2. *Let $n\geq 1$ be an integer, and let $a\geq 0$ and $b\geq 0$ be the unique integers such that $n=2^a(2b+1)$. Then*

$$\mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n});\mathbb{F}_2) = \langle t^{2^{a+1}}\rangle.$$

Theorems 1.1 and 1.2 have interesting geometrical consequences. One of them is the following result.

Corollary 1.3. *Let $a \geq 0$ and $b \geq 0$ be integers, $n = 2^a(2b+1)$ and $k = 2^{a+1} - 1$. Let $\alpha_1, \dots, \alpha_k$ be continuous real valued functions on convex bodies in $\mathbb{R}^{2n}$ with respect to the Hausdorff metric. Then for every convex body $C \subseteq \mathbb{R}^{2n}$ there exists an n-dimensional subspace $V \subseteq \mathbb{R}^{2n}$ such that*

$$\alpha_i(p_V(C)) = \alpha_i(p_{V^\perp}(C)) \qquad \text{for all } 1 \leq i \leq k, \tag{2}$$

where p_V and $p_{V^\perp}$ are orthogonal projections onto V and its orthogonal complement $V^\perp$.

Proof. Let $\alpha_1, \dots, \alpha_k$ be continuous real valued functions on the space of convex bodies in $\mathbb{R}^{2n}$, and let $C \subseteq \mathbb{R}^{2n}$ be a convex body. Let $f_C \colon G_n(\mathbb{R}^{2n}) \longrightarrow \mathbb{R}^k$ be a continuous map defined by

$$f_C(V) := \big(\alpha_1(p_V(C)) - \alpha_1(p_{V^\perp}(C)), \dots, \alpha_k(p_V(C)) - \alpha_k(p_{V^\perp}(C))\big)$$

for $V \in G_n(\mathbb{R}^{2n})$. If the $\mathbb{Z}/2$ action on the Euclidean space $\mathbb{R}^k$ is assumed to be antipodal, and the $\mathbb{Z}/2$ action on $G_n(\mathbb{R}^{2n})$ is given by $\omega \cdot V = V^\perp$, then f_C is a $\mathbb{Z}/2$-map.

If the claim (2) does not hold then there exists a convex body C such that $0 \notin \mathrm{im} f_C$. Thus the map f_C factors through $\mathbb{R}^k \setminus \{0\}$ as follows

$$\begin{tikzcd} G_n(\mathbb{R}^{2n}) \arrow[rr, "f_C"] \arrow[dr, "\widetilde{f}_C"'] & & \mathbb{R}^k \\ & \mathbb{R}^k \setminus \{0\}. \arrow[ur, "i"'] & \end{tikzcd}$$

There exists a composition of $\mathbb{Z}/2$-maps

$$G_n(\mathbb{R}^{2n}) \xrightarrow{\widetilde{f}_C} \mathbb{R}^k \setminus \{0\} \xrightarrow{r} S^{k-1}$$

where r is the $\mathbb{Z}_2$-invariant radial retraction onto the sphere S^{k-1} with the antipodal action. Consequently, by the basic property of the Fadell–Husseini index

$$\langle t^{2^{a+1}} \rangle = \mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n}); \mathbb{F}_2) \supseteq \mathrm{Index}_{\mathbb{Z}/2}(S^{k-1}; \mathbb{F}_2) = \langle t^k \rangle = \langle t^{2^{a+1}-1} \rangle.$$

This is a contradiction, and therefore the claim (2) holds. □

Geometrically meaningful examples for functions $\alpha_1, \dots, \alpha_k$ may be given by quermassintegrals (intrinsic volumes), that is average k-dimensional volumes of orthogonal k-dimensional projections of a body $p_V(C) \subset \mathbb{R}^{2n}$, for $1 \leq k \leq n$. Another meaningful function is the radius of the minimal containing ball of $p_V(C)$. These functions have an additional property that they are rotationally and translationally invariant. Thus, more examples can be obtained by dropping the assumption of symmetry invariance.

This paper builds on the ideas and methods presented in [?] and presents the journal version of that preprint.

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2. First estimates of the index

In this section we make initial observations about the $\mathbb{Z}/2$ actions on the Grassmann manifolds $G_n(\mathbb{R}^{2n})$ and $\widetilde{G}_n(\mathbb{R}^{2n})$ and obtain some estimates of the corresponding indices. Furthermore, we discuss the action of $\mathbb{Z}/2$ on the cohomology of the Grassmann manifold $G_n(\mathbb{R}^{2n})$.

The first elementary observation we make is that the covering map between the Grassmann manifolds is a $\mathbb{Z}/2$-map.

Proposition 2.1. *Let $n \geq 1$ be an integer. The covering map $c \colon \widetilde{G}_n(\mathbb{R}^{2n}) \longrightarrow G_n(\mathbb{R}^{2n})$ is a $\mathbb{Z}/2$-map, and consequently*

$$\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_n(\mathbb{R}^{2n}); \mathbb{F}_2) \supseteq \mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n}); \mathbb{F}_2).$$

The covering map c is a map that forgets orientation and is defined in general $c \colon \widetilde{G}_n(\mathbb{R}^{n+k}) \longrightarrow G_n(\mathbb{R}^{n+k})$ for any $n \geq 1$ and $k \geq 1$, including the case when $n + k = \infty$.

Next we construct $\mathbb{Z}/2$-maps between Grassmann manifolds of different dimensions.

Proposition 2.2. *Let $m \geq 1$ be an integer, and let n be a multiple of m. There exist $\mathbb{Z}/2$-maps*

$$g\colon G_m(\mathbb{R}^{2m}) \longrightarrow G_n(\mathbb{R}^{2n}) \qquad \text{and} \qquad \widetilde{g}\colon \widetilde{G}_m(\mathbb{R}^{2m}) \longrightarrow \widetilde{G}_n(\mathbb{R}^{2n}),$$

and consequently

$$\mathrm{Index}_{\mathbb{Z}/2}(G_m(\mathbb{R}^{2m});\mathbb{F}_2) \supseteq \mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n});\mathbb{F}_2), \quad \mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_m(\mathbb{R}^{2m});\mathbb{F}_2) \supseteq \mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_n(\mathbb{R}^{2n});\mathbb{F}_2).$$

Proof. Let $d := \frac{n}{m}$. Choose a decomposition of the Euclidean space $\mathbb{R}^{2n} = (\mathbb{R}^{2m})^{\oplus d}$. Then define the map g and $\widetilde{g}$ by

$$\big(V \in G_m(\mathbb{R}^{2m})\big) \longmapsto \big(V^{\oplus d} \in G_n(\mathbb{R}^{2n})\big). \tag{3}$$

In the case of the map $\widetilde{g}$ appropriate orientations of the spaces V and $V^{\oplus d}$ are assumed. By direct inspection the maps g and $\widetilde{g}$ are $\mathbb{Z}/2$-maps. □

The $\mathbb{Z}/2$-map g we just defined has an additional property. Let $\gamma^n(\mathbb{R}^{n+k})$ denotes the canonical n-dimensional vector bundle over the Grassmann manifold $G_n(\mathbb{R}^{n+k})$, consult [9, Lem. 5.2].

Corollary 2.3. *The $\mathbb{Z}/2$-map $g\colon G_m(\mathbb{R}^{2m}) \longrightarrow G_n(\mathbb{R}^{2n})$ defined by (3) is covered by a vector bundle map $\gamma^m(\mathbb{R}^{2m})^{\oplus d} \longrightarrow \gamma^n(\mathbb{R}^{2n})$.*

Proof. The vector bundle map that covers g is defined by $(V; v_1, \dots, v_d) \longmapsto (V^{\oplus d}; v_1 + \dots + v_d)$. □

Now we define a $\mathbb{Z}/2$-map from a Grassmann manifold into a sphere equipped with the antipodal action.

Proposition 2.4. *Let $n \geq 1$ be an integer, and let the sphere S^{2n-1} be equipped with the antipodal action. There exists a $\mathbb{Z}/2$-map $h\colon G_n(\mathbb{R}^{2n}) \longrightarrow S^{2n-1}$, and consequently*

$$\mathrm{Index}_{\mathbb{Z}/2}\, G_n(\mathbb{R}^{2n}) \supseteq \mathrm{Index}_{\mathbb{Z}/2}\, S^{2n-1} = \langle t^{2n} \rangle.$$

Proof. For the definition of the map h we consider elements V of the Grassmann manifold $G_n(\mathbb{R}^{2n})$ as $(2n) \times (2n)$-matrices $A = (a_{i,j})$ which represent the orthogonal projection onto V. Such matrices fulfill the following properties

$$A^\top = A, \quad A^2 = A, \quad \mathrm{tr}A = n.$$

Here $A^\top$ denotes the transpose matrix of A. The $\mathbb{Z}/2$-action on $G_n(\mathbb{R}^{2n})$ in the language of matrices is given by $\omega \cdot A = I - A$ where I denotes the unit $(2n) \times (2n)$-matrix. First, we define the map $k\colon G_n(\mathbb{R}^{2n}) \longrightarrow \mathbb{R}^{2n}\backslash\{0\}$ by

$$k(A) := (a_{1,1} - \tfrac{1}{2},\, a_{1,2},\, a_{1,3},\, \dots,\, a_{1,2n}).$$

The map k is well defined because $0 \notin \mathrm{im}(k)$. Indeed, if $k(A) = 0$ for some matrix A then $\frac{1}{2}$ is an eigenvalue of A with eigenvector $(1, 0, \cdots, 0)$. This yields a contradiction with the requirement that $A^2 = A$. Further on, if the action on $\mathbb{R}^{2n}\backslash\{0\}$ is assumed to be antipodal then k is a $\mathbb{Z}/2$-map; as it can be demonstrated by the following commutative diagram:

$$\begin{tikzcd}
(a_{i,j})_{i,j=1,2n} \arrow[r, mapsto, "k"] \arrow[d, mapsto, "\omega\cdot"'] & (a_{1,1} - \frac{1}{2}, a_{1,2}, a_{1,3}, \dots, a_{1,2n}) \arrow[d, mapsto, "\omega\cdot"'] \\
(\delta_{i,j} - a_{i,j})_{i,j=1,2n} \arrow[r, mapsto, "k"] & (\frac{1}{2} - a_{1,1}, -a_{1,2}, -a_{1,3}, \dots, -a_{1,2n}),
\end{tikzcd}$$

where $\delta_{i,j}$ is the Kronecker symbol. Finally, the map h is defined as the composition of k and the radial retraction $\mathbb{R}^{2n}\backslash\{0\} \longrightarrow S^{2n-1}$ which is also a $\mathbb{Z}/2$-map. □

The cohomology of the Grassmann manifold is described as a quotient polynomial algebra by the following classical result of Borel [2, p. 190].

Theorem 2.5. *Let $n, k \geq 1$ be integers. Then*

$$H^*(G_n(\mathbb{R}^{n+k});\mathbb{F}_2) \cong \mathbb{F}_2[w_1, \dots, w_n; \bar{w}_1, \dots, \bar{w}_k]/I_{n,k}$$

where $\deg(w_i) = i$ for $1 \leq i \leq n$, $\deg(\bar{w}_j) = j$ for $1 \leq j \leq k$, and the ideal $I_{n,k}$ is generated by $n + k$ relations which are derived from the equality

$$(1 + w_1 + \dots + w_n)(1 + \bar{w}_1 + \dots + \bar{w}_k) = 1.$$

Here the generators w_i, for $1 \leq i \leq n$, can be identified with the Stiefel–Whitney classes of the canonical bundle $\gamma^n(\mathbb{R}^{n+k})$. The remaining generators $\bar{w}_i$, for $1 \leq i \leq k$, can be identified with the dual Stiefel–Whitney classes of $\gamma^n(\mathbb{R}^{n+k})$.

In the case when $n = k$ the $\mathbb{Z}/2$ action on the Grassmann manifold $G_n(\mathbb{R}^{2n})$ we consider, induces an action on the the corresponding cohomology $H^*(G_n(\mathbb{R}^{n+k});\mathbb{F}_2)$. Let $\nu^n(\mathbb{R}^{2n})$ denote the normal bundle of the canonical vector bundle $\gamma^n(\mathbb{R}^{2n})$. The continuous map $\omega\colon G_n(\mathbb{R}^{2n}) \longrightarrow G_n(\mathbb{R}^{2n})$, $V \longmapsto V^{\perp}$, is covered by a bundle map $\gamma^n(\mathbb{R}^{2n}) \longrightarrow \nu^n(\mathbb{R}^{2n})$ given by $(V;v) \longmapsto (V^{\perp};v)$. Since $w_i(\nu^n(\mathbb{R}^{2n})) = \bar{w}_i(\gamma^n(\mathbb{R}^{2n}))$ the naturality on the Stiefel–Whitney classes [9, Ax. 2, p. 35] implies that ω acts of the generators of the cohomology $H^*(G_n(\mathbb{R}^{2n});\mathbb{F}_2)$ as follows

$$\omega \cdot w_i = \bar{w}_i \qquad \text{and} \qquad \omega \cdot \bar{w}_i = w_i,$$

for all $1 \leq i \leq n$.

Using the description of the $\mathbb{Z}/2$ action on the cohomology of the Grassmann manifold $G_n(\mathbb{R}^{2n})$ we compute Fadell–Husseini indices in the case when $n = 1$ and $n = 2$.

Proposition 2.6.
(i) $\mathrm{Index}_{\mathbb{Z}/2}(G_1(\mathbb{R}^2);\mathbb{F}_2) = \langle t^2\rangle$,
(ii) $\mathrm{Index}_{\mathbb{Z}/2}(G_2(\mathbb{R}^4);\mathbb{F}_2) = \langle t^4\rangle$.

Proof. (i) When $n = 1$ the homeomorphisms $G_1(\mathbb{R}^2) \cong \mathbb{R}\mathrm{P}^1 \cong S^1$ allow us to identify $G_1(\mathbb{R}^2)$ with a sphere S^1 equipped with a free $\mathbb{Z}/2$ action. Consequently,

$$\mathrm{Index}_{\mathbb{Z}/2}(G_1(\mathbb{R}^2);\mathbb{F}_2) = \mathrm{Index}_{\mathbb{Z}/2}(S^1;\mathbb{F}_2) = \langle t^2\rangle.$$

(ii) In the case of the Grassmann manifold $G_2(\mathbb{R}^4)$ we consider the Borel construction fibration

$$G_2(\mathbb{R}^4) \longrightarrow \mathrm{E}(\mathbb{Z}/2) \times_{\mathbb{Z}/2} G_2(\mathbb{R}^4) \longrightarrow \mathrm{B}(\mathbb{Z}/2),$$

and its associated Serre spectral sequence whose E_2-term is given by

$$E_2^{i,j} = H^i(\mathrm{B}(\mathbb{Z}/2);\mathcal{H}^j(G_2(\mathbb{R}^4);\mathbb{F}_2)) \cong H^i(\mathbb{Z}/2;H^j(G_2(\mathbb{R}^4);\mathbb{F}_2)). \tag{4}$$

The action of $\pi_1(\mathrm{B}(\mathbb{Z}/2)) \cong \mathbb{Z}/2$ on the cohomology of the fiber $H^*(G_2(\mathbb{R}^4);\mathbb{F}_2)$ is non-trivial, and thus we first describe this action. From Theorem 2.5 follows that

$$H^*(G_2(\mathbb{R}^4);\mathbb{F}_2) \cong \mathbb{F}_2[w_1,w_2;\bar{w}_1,\bar{w}_2]/I_{2,2}.$$

where the ideal $I_{2,2}$ is given by

$$I_{2,2} = \langle w_1+\bar{w}_1, w_2+\bar{w}_2+w_1\bar{w}_1, w_1\bar{w}_2+w_2\bar{w}_1, w_2\bar{w}_2\rangle.$$

Thus, the cohomology $H^*(G_2(\mathbb{R}^4);\mathbb{F}_2)$, as a $\mathbb{Z}/2$-module, can be presented as

$$H^j(G_2(\mathbb{R}^4);\mathbb{F}_2) = \begin{cases} \langle 1\rangle \cong \mathbb{F}_2, & j=0,\\ \langle w_1\rangle \cong \langle \bar{w}_1\rangle \cong \mathbb{F}_2, & j=1,\\ \langle w_2,\bar{w}_2\rangle \cong \mathbb{F}_2[\mathbb{Z}/2], & j=2,\\ \langle w_1w_2\rangle \cong \langle w_1\bar{w}_2\rangle \cong \langle w_2\bar{w}_1\rangle \cong \langle \bar{w}_1\bar{w}_2\rangle \cong \mathbb{F}_2, & j=3,\\ \langle w_2^2\rangle \cong \langle \bar{w}_2^2\rangle \ \cong \mathbb{F}_2, & j=4,\\ 0, & \text{otherwise.}\end{cases}$$

Consequently the E_2-term of the spectral sequence (4) can be evaluated as follows:

$$E_2^{i,j}(G_2(\mathbb{R}^4)) \cong H^i(\mathbb{Z}/2;H^j(G_2(\mathbb{R}^4);\mathbb{F}_2)) \cong \begin{cases} H^i(\mathbb{Z}/2;\mathbb{F}_2), & j=0,1,3,4,\\ H^i(\mathbb{Z}/2;\mathbb{F}_2[\mathbb{Z}/2]), & j=2,\\ 0, & \text{otherwise.}\end{cases}$$

For an illustration of the E_2-term see Figure 1. Each row of the spectral sequence (4) is a $H^*(\mathbb{Z}/2;\mathbb{F}_2)$-module. In particular, the rows $0,1,3,4$ of the E_2-term are free $H^*(\mathbb{Z}/2;\mathbb{F}_2)$-modules generated, respectively, by

$$1 \in E_2^{0,0}(G_2(\mathbb{R}^4)),\ w_1 \in E_2^{0,1}(G_2(\mathbb{R}^4)),\ w_1w_2 \in E_2^{0,3}(G_2(\mathbb{R}^4)),\ w_2^2 \in E_2^{0,4}(G_2(\mathbb{R}^4)).$$

In the second row we have that $E_2^{0,2} \cong H^0(\mathbb{Z}/2;\mathbb{F}_2[\mathbb{Z}/2]) \cong \mathbb{F}_2[\mathbb{Z}/2]^{\mathbb{Z}/2} \cong \mathbb{F}_2 \cong \langle w_2+\bar{w}_2\rangle$, and $E_2^{i,2} \cong H^i(\mathbb{Z}/2;\mathbb{F}_2[\mathbb{Z}/2]) = 0$ for $i \geq 1$.

The module structure on the second row $E_2^{*,2}(G_2(\mathbb{R}^4))$ is obvious and is given by $t \cdot (w_2+\bar{w}_2) = 0$.

5	0	0	0	0	0	0	0
4	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$
3	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$
2	$\mathbf{F}_2$	0	0	0	0	0	0
1	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$
0	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$	$\mathbf{F}_2$
	0	1	2	3	4	5	6

FIGURE 1. The E_2-term of the spectral sequence (4).

First we prove that the second differential ∂_2 vanishes on $w_1 \in E_2^{0,1}$, that is $\partial_2(w_1) = 0$. For that we use a comparison of spectral sequences. Since 2 is a multiple of 1, according to Proposition 2.2, there exists a $\mathbb{Z}/2$-map $g\colon G_1(\mathbb{R}^2) \longrightarrow G_2(\mathbb{R}^4)$. The map g induces a bundle morphism between the following Borel construction fibrations:

$$\begin{tikzcd} \mathrm{E}(\mathbb{Z}/2)\times_{\mathbb{Z}/2} G_1(\mathbb{R}^2) \arrow[r, "\mathrm{id}\times_{\mathbb{Z}/2} g"] \arrow[d] & \mathrm{E}(\mathbb{Z}/2)\times_{\mathbb{Z}/2} G_2(\mathbb{R}^4) \arrow[d] \\ \mathrm{B}(\mathbb{Z}/2) \arrow[r, "\mathrm{id}"] & \mathrm{B}(\mathbb{Z}/2). \end{tikzcd}$$

The bundle morphism furthermore induces a morphism between corresponding Serre spectral sequences

$$E_k^{i,j}(g)\colon E_k^{i,j}(G_1(\mathbb{R}^2)) \longleftarrow E_k^{i,j}(G_2(\mathbb{R}^4)), \qquad k \geq 2$$

with the property that $E_2^{i,0}(g) = \mathrm{id}$ for all $i \in \mathbb{Z}$.

On the other hand Corollary 2.3 implies that the map g is covered by a vector bundle map $\gamma^1(\mathbb{R}^2)^{\oplus 2} \longrightarrow \gamma^2(\mathbb{R}^4)$. Hence, from naturality of the Stiefel–Whitney classes we have that

$$g^*(w_1) = g^*(w_1(\gamma^2(\mathbb{R}^4))) = w_1(\gamma^1(\mathbb{R}^2)^{\oplus 2}) = \binom{2}{1} w_1(\gamma^1(\mathbb{R}^2)) = \binom{2}{1} w_1' = 0,$$

where $w_1' := w_1(\gamma^1(\mathbb{R}^2))$, is the generator of $H^1(G_1(\mathbb{R}^2);\mathbb{F}_2) \cong \mathbb{F}_2$.

Since both rows $E_2^{*,1}(G_1(\mathbb{R}^2))$ and $E_2^{*,1}(G_2(\mathbb{R}^4))$ are free $H^*(\mathbb{Z}/2;\mathbb{F}_2)$-modules generated respectively by w_1 and w_1' it follows that the morphism $E_2^{i,1}(g)$ is the zero morphism. The morphism of spectral sequences commutes with differentials and consequently,

$$\partial_2(w_1) = (\mathrm{id}\circ\partial_2)(w_1) = (E_2^{2,0}(g)\circ\partial_2)(w_1) = (\partial_2\circ E_2^{0,1}(g))(w_1) = \\ \partial_2(E_2^{0,1}(g)(w_1)) = \partial_2(g^*(w_1)) = \partial_2(0) = 0.$$

Here we use the facts

$$\begin{aligned} E_2^{2,0}(G_1(\mathbb{R}^2)) &= H^0(\mathbb{Z}/2; H^2(G_1(\mathbb{R}^2);\mathbb{F}_2)) \cong H^2(G_1(\mathbb{R}^2);\mathbb{F}_2)^{\mathbb{Z}/2} \cong H^2(G_1(\mathbb{R}^2);\mathbb{F}_2), \\ E_2^{2,0}(G_2(\mathbb{R}^4)) &= H^0(\mathbb{Z}/2; H^2(G_2(\mathbb{R}^4);\mathbb{F}_2)) \cong H^2(G_2(\mathbb{R}^4);\mathbb{F}_2)^{\mathbb{Z}/2} \cong H^2(G_2(\mathbb{R}^4);\mathbb{F}_2), \end{aligned}$$

that imply equality of the maps $E_2^{2,0}(g) = g^*$. Furthermore, since differentials are $H^*(\mathbb{Z}/2;\mathbb{F}_2)$-module morphisms, w_1 is a generator of the first row of the spectral sequence (4), as a $H^*(\mathbb{Z}/2;\mathbb{F}_2)$-module, we conclude that the differential ∂_2 vanishes on the complete first row. Consequently,

$$E_3^{i,0}(G_2(\mathbb{R}^4)) \cong E_2^{i,0}(G_2(\mathbb{R}^4)) \cong H^i(\mathbb{Z}/2;\mathbb{F}_2).$$

Next, we prove that $\partial_2(w_2+\bar{w}_2) = 0$ and $\partial_3(w_2+\bar{w}_2) = 0$. Let us assume first that $\partial_2(w_2+\bar{w}_2) = t^2\cdot w_1 \neq 0$. Then we have a contradiction:

$$0 \neq t^3\cdot w_1 = t\cdot(t^2\cdot w_1) = t\cdot\partial_2(w_2+\bar{w}_2) = \partial_2(t\cdot(w_2+\bar{w}_2)) = \partial_2(0) = 0.$$

Thus, $\partial_2(w_2+\bar{w}_2)=0$ and consequently $E_3^{*,2}(G_2(\mathbb{R}^4))\cong E_2^{*,2}(G_2(\mathbb{R}^4))$. Similarly, if $\partial_3(w_2+\bar{w}_2)=t^3\cdot 1\neq 0$ then we again get a contradiction:

$$0\neq t^4\cdot 1=t\cdot(t^3\cdot 1)=t\cdot\partial_3(w_2+\bar{w}_2)=\partial_3(t\cdot(w_2+\bar{w}_2))=\partial_3(0)=0.$$

Therefore,

$$E_4^{*,0}(G_2(\mathbb{R}^4))\cong E_3^{*,0}(G_2(\mathbb{R}^4))\cong E_2^{*,0}(G_2(\mathbb{R}^4))\cong H^*(\mathbb{Z}/2;\mathbb{F}_2).$$

Moreover, for $0\le i\le 3$ we have the following sequence of isomorphisms

$$E_\infty^{i,0}(G_2(\mathbb{R}^4))\cong\cdots\cong E_2^{i,0}(G_2(\mathbb{R}^4))\cong H^i(\mathbb{Z}/2;\mathbb{F}_2)\neq 0.$$

Finally, from the previous sequence of isomorphisms and the equality (1) we conclude that

$$\mathrm{Index}_{\mathbb{Z}/2}(G_2(\mathbb{R}^4);\mathbb{F}_2)\subseteq E_2^{\ge 4,0}(G_2(\mathbb{R}^4))\cong H^{\ge 4}(\mathbb{Z}/2;\mathbb{F}_2)=\langle t^4\rangle.$$

On the other hand from Proposition 2.4 we have that $\mathrm{Index}_{\mathbb{Z}/2}(G_2(\mathbb{R}^4);\mathbb{F}_2)\supseteq\langle t^4\rangle$. Hence,

$$\mathrm{Index}_{\mathbb{Z}/2}(G_2(\mathbb{R}^4);\mathbb{F}_2)=\langle t^4\rangle,$$

and the proof of the proposition is complete. □

Now using Proposition 2.2, Corollary 2.3 and comparison of Serre spectral sequences we evaluate the index $\mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n});\mathbb{F}_2)$ for all odd integers n.

Corollary 2.7. *Let $n\ge 1$ be an odd integer. Then*

$$\mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n});\mathbb{F}_2)=\langle t^2\rangle.$$

Proof. Let $n\ge 1$ be an odd integer. Since n is a multiple of 1 from Proposition 2.2 we get a $\mathbb{Z}/2$-map $g\colon G_1(\mathbb{R}^2)\longrightarrow G_n(\mathbb{R}^{2n})$. As in the proof of Proposition 2.6 the map g induces a bundle morphism between the corresponding Borel construction fibrations:

$$\begin{tikzcd}
\mathrm{E}(\mathbb{Z}/2)\times_{\mathbb{Z}/2}G_1(\mathbb{R}^2) \arrow[r, "\mathrm{id}\times_{\mathbb{Z}/2}g"] \arrow[d] & \mathrm{E}(\mathbb{Z}/2)\times_{\mathbb{Z}/2}G_n(\mathbb{R}^{2n}) \arrow[d] \\
\mathrm{B}(\mathbb{Z}/2) \arrow[r, "\mathrm{id}"] & \mathrm{B}(\mathbb{Z}/2).
\end{tikzcd}$$

This bundle morphism induces furthermore a morphism between Serre spectral sequences

$$E_k^{i,j}(g)\colon E_k^{i,j}(G_1(\mathbb{R}^2))\longleftarrow E_k^{i,j}(G_n(\mathbb{R}^{2n})),\qquad k\ge 2$$

such that $E_2^{i,0}(g)=\mathrm{id}$ for all $i\in\mathbb{Z}$.

The spectral sequence $E_*^{*,*}(G_1(\mathbb{R}^2))$ can be described completely. First, the E_2-term is given by

$$E_2^{i,j}(G_1(\mathbb{R}^2))=H^i(\mathrm{B}(\mathbb{Z}/2);\mathcal{H}^j(G_1(\mathbb{R}^2);\mathbb{F}_2))=H^i(\mathbb{Z}/2;H^j(S^1;\mathbb{F}_2))\cong H^i(\mathbb{Z}/2;\mathbb{F}_2)\otimes H^j(S^1;\mathbb{F}_2).$$

Since $G_1(\mathbb{R}^2)\cong S^1$ is a free $\mathbb{Z}/2$-space there is a homotopy equivalence

$$\mathrm{E}(\mathbb{Z}/2)\times_{\mathbb{Z}/2}G_1(\mathbb{R}^2)\simeq G_1(\mathbb{R}^2)/(\mathbb{Z}/2)\cong\mathbb{R}\mathrm{P}^1.$$

The spectral sequence $E_*^{*,*}(G_1(\mathbb{R}^2))$ converges to the cohomology of the Borel construction:

$$H^*(\mathrm{E}(\mathbb{Z}/2)\times_{\mathbb{Z}/2}G_1(\mathbb{R}^2);\mathbb{F}_2)\cong H^*(G_1(\mathbb{R}^2)/(\mathbb{Z}/2);\mathbb{F}_2)\cong H^*(S^1/(\mathbb{Z}/2);\mathbb{F}_2).$$

Consequently $E_\infty^{i,j}(G_1(\mathbb{R}^2))=0$ for $(i,j)\notin\{(0,0),(1,0)\}$. Therefore $\partial_2(w_1')=t^2$ where $w_1':=w_1(\gamma^1(\mathbb{R}^2))$ is the generator of $H^1(G_1(\mathbb{R}^2);\mathbb{F}_2)\cong\mathbb{F}_2$.

Next, from Corollary 2.3 we know that g is covered by a vector bundle map $\gamma^1(\mathbb{R}^2)^{\oplus n}\longrightarrow\gamma^n(\mathbb{R}^{2n})$. The naturality of Stiefel–Whitney classes implies that

$$g^*(w_1)=g^*(w_1(\gamma^n(\mathbb{R}^{2n})))=w_1(\gamma^1(\mathbb{R}^2)^{\oplus n})=\binom{n}{1}w_1(\gamma^1(\mathbb{R}^2))=w_1'\neq 0.$$

The rows of the spectral sequences $E_2^{*,1}(G_1(\mathbb{R}^2))$ and $E_2^{*,1}(G_2(\mathbb{R}^4))$ are free $H^*(\mathbb{Z}/2;\mathbb{F}_2)$-modules generated respectively by w_1 and w_1'. Thus the the morphism $E_2^{i,1}(g)$ is an isomorphism. Since the morphism $E_k^{i,j}(g)$ of spectral sequences commutes with differentials we have that,

$$\partial_2(w_1)=(\mathrm{id}\circ\partial_2)(w_1)=(E_2^{2,0}(g)\circ\partial_2)(w_1)=(\partial_2\circ E_2^{0,1}(g))(w_1)=$$
$$\partial_2(E_2^{0,1}(g)(w_1))=\partial_2(g^*(w_1))=\partial_2(w_1')=t^2.$$

Hence the equality (1) implies the final conclusion:

$$\mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n}); \mathbb{F}_2) = \langle t^2 \rangle.$$

□

3. Stiefel–Whitney classes of the wreath product vector bundle

In this section we introduce a notion of the wreath square of a vector bundle and derive a formula for computation of its total Stiefel–Whitney class. This formula will be essentially used in the proof of Theorem 1.1. We work in generality necessary for giving this proof.

3.1. **Wreath squares.** Let B be a CW-complex, not necessarily finite. The Cartesian square $B \times B$ is equipped with a $\mathbb{Z}/2$ action given by $\omega \cdot (x, y) := (y, x)$, where ω is the generator of $\mathbb{Z}/2$ and $(x, y) \in B \times B$. Then the product space $(B \times B) \times \mathrm{E}\mathbb{Z}/2$ is a free $\mathbb{Z}/2$-space if the diagonal action is assumed. This means that $\omega \cdot (x, y, e) := (y, x, \omega \cdot e)$ for $(x, y, e) \in (B \times B) \times \mathrm{E}\mathbb{Z}/2$. Consequently, the projection map

$$p_1 \colon (B \times B) \times \mathrm{E}\mathbb{Z}/2 \longrightarrow (B \times B), \qquad (x, y, e) \longmapsto (x, y)$$

is a $\mathbb{Z}/2$-map. The quotient space $((B \times B) \times \mathrm{E}\mathbb{Z}/2)/(\mathbb{Z}/2)$ will be denoted either by $B \wr \mathbb{Z}/2$ or by $(B \times B) \times_{\mathbb{Z}/2} \mathrm{E}\mathbb{Z}/2$, and will be called the **wreath square of the space** B.

Next, let $f \colon B_1 \longrightarrow B_2$ be a continuous map. Then the map

$$(f \times f) \times \mathrm{id} \colon (B_1 \times B_1) \times \mathrm{E}\mathbb{Z}/2 \longrightarrow (B_2 \times B_2) \times \mathrm{E}\mathbb{Z}/2$$

is a $\mathbb{Z}/2$-map and consequently induces a continuous map between the wreath squares:

$$(f \times f) \times_{\mathbb{Z}/2} \mathrm{id} \colon (B_1 \times B_1) \times_{\mathbb{Z}/2} \mathrm{E}\mathbb{Z}/2 \longrightarrow (B_2 \times B_2) \times_{\mathbb{Z}/2} \mathrm{E}\mathbb{Z}/2.$$

The map $(f \times f) \times_{\mathbb{Z}/2} \mathrm{id}$ is alternatively denoted by $f \wr \mathbb{Z}/2$.

Let $\xi := (p_\xi \colon E(\xi) \longrightarrow B(\xi))$ be a real n-dimensional vector bundle over the CW-complex $B(\xi)$ whose fiber is $F(\xi)$. The total space of ξ is $E(\xi)$, and p_ξ is the corresponding projection map. We consider the following pull-back $p_1^*(\xi \times \xi)$ of the product bundle vector $\xi \times \xi$:

$$\begin{tikzcd} E(p_1^*(\xi \times \xi)) \arrow[r] \arrow[d] & E(\xi \times \xi) \arrow[d] \\ (B(\xi) \times B(\xi)) \times \mathrm{E}\mathbb{Z}/2 \arrow[r, "p_1"] & B(\xi \times \xi). \end{tikzcd}$$

Here we recall that by definition $E(\xi \times \xi) = E(\xi) \times E(\xi)$ and $B(\xi \times \xi) = B(\xi) \times B(\xi)$, consult for example [9, p. 27]. Unlike the product bundle $\xi \times \xi$ the pull-back bundle $p_1^*(\xi \times \xi)$ is equipped with a free $\mathbb{Z}/2$-action. Moreover, the projection $E(p_1^*(\xi \times \xi)) \longrightarrow (B(\xi) \times B(\xi)) \times \mathrm{E}\mathbb{Z}/2$ is a $\mathbb{Z}/2$-map between free $\mathbb{Z}/2$-spaces. Consequently, after taking quotient we get a $2n$-dimensional vector bundle

$$E(p_1^*(\xi \times \xi))/(\mathbb{Z}/2) \longrightarrow (B(\xi) \times B(\xi)) \times_{\mathbb{Z}/2} \mathrm{E}\mathbb{Z}/2$$

that we denote by $\xi \wr \mathbb{Z}/2$ and call the **wreath square of the vector bundle** ξ.

The wreath square of the vector bundle behaves naturally with respect to the Whitney sum of vector bundles. More precisely, if ξ and η are vector bundles over the same base space B then

$$(\xi \oplus \eta) \wr \mathbb{Z}/2 \cong (\xi \wr \mathbb{Z}/2) \oplus (\eta \wr \mathbb{Z}/2). \tag{5}$$

Indeed, for proving (5) it suffices to exhibit a fiberwise isomorphism between the corresponding vector bundles, see [9, Lem. 2.3]. The fibers of these bundles, respectively, are

$$(F(\xi) \oplus F(\eta)) \times (F(\xi) \oplus F(\eta)) \qquad \text{and} \qquad (F(\xi) \times F(\xi)) \oplus (F(\eta) \times F(\eta)).$$

Thus the obvious shuffling linear isomorphism gives the required isomorphism in (5).

3.2. **Cohomology of the wreath square of a space.** Let B be a CW-complex and $(B\times B)\times_{\mathbb{Z}/2}\mathrm{E}\mathbb{Z}/2$ its wreath square. The wreath square of B is the total space of the following fiber bundle

$$B\times B \longrightarrow (B\times B)\times_{\mathbb{Z}/2}\mathrm{E}\mathbb{Z}/2 \longrightarrow \mathrm{B}(\mathbb{Z}/2). \tag{6}$$

The Serre spectral sequence associated to the fibration (6) has E_2-term given by

$$E_2^{i,j} := H^i(\mathrm{B}(\mathbb{Z}/2);\mathcal{H}^j(B\times B;\mathbb{F}_2)) \cong H^i(\mathbb{Z}/2;H^j(B\times B;\mathbb{F}_2)). \tag{7}$$

The spectral sequence (7) converges to the cohomology of the wreath square $H^*((B\times B)\times_{\mathbb{Z}/2}\mathrm{E}\mathbb{Z}/2;\mathbb{F}_2)$. Here the local coefficient system $\mathcal{H}^*(B\times B;\mathbb{F}_2)$ is determined by the action of the fundamental group of the base $\pi_1(\mathrm{B}(\mathbb{Z}/2))\cong\mathbb{Z}/2$ on the cohomology of the fiber $H^*(B\times B;\mathbb{F}_2)$. The Künneth formula gives a presentation of the cohomology of the fiber in the form

$$H^*(B\times B;\mathbb{F}_2)\cong H^*(B;\mathbb{F}_2)\otimes H^*(B;\mathbb{F}_2),$$

and the action of $\mathbb{Z}/2$ interchanges factors in the tensor product.

The E_2-term of the spectral sequence (7) can be described in more details using [1, Cor IV.1.6].

Proposition 3.1. *Let $\mathcal{B} := \{v_k : k\in K\}$ be a basis of the $\mathbb{F}_2$ vector space $H^*(B;\mathbb{F}_2)$ where the index set K is equipped with a linear order. The E_2-term of the spectral sequence (7) can be presented as follows*

$$E_2^{i,j} \;=\; \begin{cases} H^j(B\otimes B;\mathbb{F}_2)^{\mathbb{Z}/2}, & i=0,\\ H^{j/2}(B;\mathbb{F}_2), & i>0,\ j \text{ even},\\ 0, & \text{otherwise}.\end{cases}$$

Furthermore, $E_2^{*,*}$ as a $H^*(\mathbb{Z}/2;\mathbb{F}_2)$-module decomposes into the direct sum

$$\bigoplus_{k\in K} H^*(\mathbb{Z}/2;\mathbb{F}_2)\oplus\bigoplus_{k_1<k_2\in K}\mathbb{F}_2$$

where the action of $H^*(\mathbb{Z}/2;\mathbb{F}_2)$ on each summand of the first sum is given by the cup product, and on the each summand of the second sum is trivial.

From the classical work of Nakaoka [11], see also [1, Thm. IV.1.7], we have that the E_2-term of the spectral sequence (7) collapses.

Proposition 3.2. *Let B be a CW-complex. The Serre spectral sequence of the fibration (6) collapses at the E_2-term, that means $E_2^{i,j}\cong E_\infty^{i,j}$ for all $(i,j)\in\mathbb{Z}\times\mathbb{Z}$.*

In order to give a description of the E_∞-term of the spectral sequence (7), and therefore present the cohomology $H^*((B\times B)\times_{\mathbb{Z}/2}\mathrm{E}\mathbb{Z}/2;\mathbb{F}_2)$, we introduce the following maps. First we consider the map (not a homomorphism)

$$P\colon H^j(B;\mathbb{F}_2)\longrightarrow H^{2j}(B\times B;\mathbb{F}_2)^{\mathbb{Z}/2}\cong E_2^{0,2j}\cong E_\infty^{0,2j}$$

given by $P(a):=a\otimes a$, for $a\in H^j(B;\mathbb{F}_2)$ and $j\in\mathbb{Z}$. From direct computation we have that the map P is not additive but is multiplicative. Next we consider the map

$$Q\colon H^j(B\times B;\mathbb{F}_2)\longrightarrow H^j(B\times B;\mathbb{F}_2)^{\mathbb{Z}/2}$$

given by $Q(a\otimes b):=a\otimes b+b\otimes a$, for $a\otimes b\in H^j(B\times B;\mathbb{F}_2)$ and $j\in\mathbb{Z}$. The map Q is additive but not multiplicative.

Now from Propositions 3.1 and 3.2, using maps P and Q, we have that:

- $E_2^{i,0}\cong E_\infty^{i,0}\cong H^i(\mathbb{Z}/2;\mathbb{F}_2)\cong\mathbb{F}_2[t]$,
- $E_2^{0,j}\cong E_\infty^{0,j}\cong H^j(B\times B;\mathbb{F}_2)^{\mathbb{Z}/2}$,
- $E_2^{0,j}\cong E_\infty^{0,j}\cong P(H^{j/2}(B;\mathbb{F}_2))\oplus Q(H^j(B\times B;\mathbb{F}_2))$ for even $j\ge 2$,
- $E_2^{i,j}\cong E_\infty^{i,j}\cong P(H^{j/2}(B;\mathbb{F}_2))\otimes H^*(\mathbb{Z}/2;\mathbb{F}_2)$ for $j\ge 2$ even and $i\ge 1$,

where $\deg(t)=1$. From the previous description of the spectral sequence, multiplicative property of the map P, and the relation

$$Q(H^j(B\times B;\mathbb{F}_2))\cdot t=0, \tag{8}$$

we have a description of the multiplicative structure of the E_∞-term as well as of the cohomology of the wreath square $H^*((B\times B)\times_{\mathbb{Z}/2}\mathrm{E}\mathbb{Z}/2;\mathbb{F}_2)$. For the fact that no multiplicative extension problem arrises see for example [7, Rem. after Thm. 2.1].

The next property of the cohomology of the wreath square will be essential in application of the Splitting principle.

Proposition 3.3. *Let B_1 and B_2 be CW-complexes, and let $f\colon B_1 \longrightarrow B_2$ be a continuous map. If the induced map in cohomology $f^*\colon H^*(B_2;\mathbb{F}_2) \longrightarrow H^*(B_1;\mathbb{F}_2)$ is injective, then the induced map between cohomologies of wreath squares*

$$((f\times f)\times_{\mathbb{Z}/2}\mathrm{id})^*\colon H^*((B_2\times B_2)\times_{\mathbb{Z}/2}\mathrm{E}\mathbb{Z}/2;\mathbb{F}_2) \longrightarrow H^*((B_1\times B_1)\times_{\mathbb{Z}/2}\mathrm{E}\mathbb{Z}/2;\mathbb{F}_2),$$

or in different notation

$$(f\wr\mathbb{Z}/2)^*\colon H^*(B_2\wr\mathbb{Z}/2;\mathbb{F}_2) \longrightarrow H^*(B_1\wr\mathbb{Z}/2;\mathbb{F}_2),$$

is injective.

Proof. The induced map $f\wr\mathbb{Z}/2 = (f\times f)\times_{\mathbb{Z}/2}\mathrm{id}$ between wreath squares defines the following morphism of fibrations

$$\begin{tikzcd}
B_1\wr\mathbb{Z}/2 \arrow[r, "f\wr\mathbb{Z}/2"] \arrow[d] & B_1\wr\mathbb{Z}/2 \arrow[d] \\
\mathrm{B}(\mathbb{Z}/2) \arrow[r, "\mathrm{id}"] & \mathrm{B}(\mathbb{Z}/2).
\end{tikzcd}$$

Consequently, it induces a morphism between the corresponding spectral sequences

$$E_2^{i,j}(f\wr\mathbb{Z}/2)\colon E_2^{i,j}(B_1\wr\mathbb{Z}/2) \longleftarrow E_2^{i,j}(B_2\wr\mathbb{Z}/2).$$

From Proposition 3.2 we have that $E_2^{i,j}(f\wr\mathbb{Z}/2) = E_\infty^{i,j}(f\wr\mathbb{Z}/2)$, and furthermore from Proposition 3.1 that $E_2^{i,j}(f\wr\mathbb{Z}/2)$ is injective for every $i,j\in\mathbb{Z}$. Since we are working over a field the absence of an additive extension problem implies that the map $(f\wr\mathbb{Z}/2)^*$ is also injective. $\square$

3.3. **The total Stiefel–Whitney class of the wreath square of a vector bundle.** To every real n-dimensional vector bundle ξ we associate the characteristic class

$$u(\xi) := w(\xi\wr\mathbb{Z}/2) \in H^*(B(\xi)\wr\mathbb{Z}/2;\mathbb{F}_2),$$

which is the total Stiefel–Whitney class of the $2n$-dimensional vector bundle $\xi\wr\mathbb{Z}/2$ living in the cohomology of the wreath square of the total space. The assignement $\xi\longmapsto u(\xi)$ we just defined is natural with respect to continuous maps. Indeed, for a continuous map $f\colon B\longrightarrow B(\xi)$ consider the following commutative diagram of vector bundle maps (fiberwise linear isomorphism):

$$\begin{tikzcd}
 & E(p_1^*(\xi\times\xi)) \arrow[rr] \arrow[dd] & & E(\xi\times\xi) \arrow[dd] \\
E(q_1^*(f^*\xi\times f^*\xi)) \arrow[ur] \arrow[rr] \arrow[dd] & & E(f^*\xi\times f^*\xi) \arrow[ur] \arrow[dd] & \\
 & (B(\xi)\times B(\xi))\times\mathrm{E}\mathbb{Z}/2 \arrow[rr, "p_1"] & & B(\xi)\times B(\xi) \\
(B\times B)\times\mathrm{E}\mathbb{Z}/2 \arrow[ur, "(f\times f)\times\mathrm{id}"] \arrow[rr, "q_1"] & & B\times B \arrow[ur, "f\times f"] &
\end{tikzcd}$$

where $q_1\colon (B\times B)\times\mathrm{E}\mathbb{Z}/2 \longrightarrow B\times B$ is the projection. After taking the quotient of the left hand side square of the diagram, with respect to the free $\mathbb{Z}/2$-action, we get the following vector bundle map between wreath product squares:

$$\begin{tikzcd}
E(q_1^*(f^*\xi\times f^*\xi))/(\mathbb{Z}/2) \arrow[r] \arrow[d] & E(p_1^*(\xi\times\xi))/(\mathbb{Z}/2) \arrow[d] \\
B\wr\mathbb{Z}/2 \arrow[r, "f\wr\mathbb{Z}/2"] & B(\xi)\wr\mathbb{Z}/2.
\end{tikzcd}$$

In particular this means that $(f\wr\mathbb{Z}/2)^*(\xi\wr\mathbb{Z}/2) = f^*\xi\wr\mathbb{Z}/2$. Consequently,

$$(f\wr\mathbb{Z}/2)^*(u(\xi)) = u(f^*(\xi)).$$

Now we are ready to give and verify the formula for computing the characteristic class $u(\xi)$.

Theorem 3.4. *Let ξ be a real n-dimensional vector bundle over a CW-complex. Then*

$$u(\xi) = \sum_{0\le r<s\le n} Q(w_r(\xi)\otimes w_s(\xi)) + \sum_{0\le r\le n} P(w_r(\xi))\cdot(1+t)^{n-r} \tag{9}$$

The proof of Theorem 3.4 will be conducted using a modification of the Splitting principle. Thus we first give a proof in the special case of a line bundle.

Proposition 3.5. *Let ξ be a real 1-dimensional vector bundle over a CW-complex. Then*

$$u(\xi) = Q(w_0(\xi)\otimes w_1(\xi)) + P(w_0(\xi))\cdot(1+t) + P(w_1(\xi)) = \\ 1 + \big(t + 1\otimes w_1(\xi) + w_1(\xi)\otimes 1\big) + w_1(\xi)\otimes w_1(\xi).$$

Proof. First, consider the following maps of vector bundles:

$$\begin{tikzcd} E(\xi\wr\mathbb{Z}/2) \arrow[r, equal] & E(p_1^*(\xi\times\xi))/(\mathbb{Z}/2) \arrow[d] & E(p_1^*(\xi\times\xi)) \arrow[l] \arrow[r] \arrow[d] & E(\xi)\times E(\xi) \arrow[d] \\ B(\xi)\wr\mathbb{Z}/2 \arrow[r, equal] & (B(\xi)\times B(\xi))\times_{\mathbb{Z}/2}\mathrm{E}\mathbb{Z}/2 & (B(\xi)\times B(\xi))\times\mathrm{E}\mathbb{Z}/2 \arrow[l, "p_2"'] \arrow[r, "p_1"] & B(\xi)\times B(\xi). \end{tikzcd}$$

The naturality property of Stiefel–Whitney classes [9, Ax. 2, p. 37] in combination with formula [9, Prob. 4-A, p. 54] implies that

$$p_2^*(u(\xi)) = w(p_1^*(\xi\times\xi)) = p_1^*(w(\xi\times\xi)) = p_1^*\big(1\times 1 + (1\times w_1(\xi) + w_1(\xi)\times 1) + w_1(\xi)\times w_1(\xi)\big) = \\ p_1^*\big(1\otimes 1 + (1\otimes w_1(\xi) + w_1(\xi)\otimes 1) + w_1(\xi)\otimes w_1(\xi)\big), \quad (10)$$

where we silently use the Eilenberg–Zilber isomorphism [4, Th. VI.3.2]. Since p_1 is the projection we furthermore have that

$$p_1^*\big(1\otimes 1 + (1\otimes w_1(\xi) + w_1(\xi)\otimes 1) + w_1(\xi)\otimes w_1(\xi)\big) = \\ 1\otimes 1\otimes 1 + (1\otimes w_1(\xi)\otimes 1 + w_1(\xi)\otimes 1\otimes 1) + w_1(\xi)\otimes w_1(\xi)\otimes 1,$$

and consequently

$$p_2^*(u(\xi)) = 1\otimes 1\otimes 1 + (1\otimes w_1(\xi)\otimes 1 + w_1(\xi)\otimes 1\otimes 1) + w_1(\xi)\otimes w_1(\xi)\otimes 1.$$

Next consider an arbitrary point $b\in \mathrm{B}(\xi)$ and the inclusion map $h\colon \{b\}\longrightarrow B(\xi)$ and the induced map between wreath squares $h\wr\mathbb{Z}/2\colon \{b\}\wr\mathbb{Z}/2\longrightarrow \mathrm{B}(\xi)\wr\mathbb{Z}/2$. Then we have a map of vector bundles going from $\xi|_{\{b\}}\wr\mathbb{Z}/2$ to $\xi\wr\mathbb{Z}/2$, that is

$$\begin{tikzcd} E(\xi|_{\{b\}}\wr\mathbb{Z}/2) \arrow[r] \arrow[d] & E(\xi\wr\mathbb{Z}/2) \arrow[d] \\ B(\xi|_{\{b\}}\wr\mathbb{Z}/2) \arrow[r, "h\wr\mathbb{Z}/2"] & B(\xi\wr\mathbb{Z}/2), \end{tikzcd}$$

where $B(\xi\wr\mathbb{Z}/2) = (B(\xi)\times B(\xi))\times_{\mathbb{Z}/2}\mathrm{E}\mathbb{Z}/2$ and $E(\xi\wr\mathbb{Z}/2) = E(p_1^*(\xi\times\xi))/(\mathbb{Z}/2)$. Consequently,

$$(h\wr\mathbb{Z}/2)^*(u(\xi)) = w(\xi|_{\{b\}}\wr\mathbb{Z}/2).$$

Since $B(\xi|_{\{b\}}\wr\mathbb{Z}/2)\cong(\mathrm{E}\mathbb{Z}/2)/(\mathbb{Z}/2)\cong \mathrm{B}(\mathbb{Z}/2)$ by direct inspection we see that $\xi|_{\{b\}}\wr\mathbb{Z}/2$ decomposes into a sum of a trivial line and non-trivial line vector bundle. More precisely we can explicitly compute that $w(\xi|_{\{b\}}\wr\mathbb{Z}/2) = 1+t$. Therefore

$$(h\wr\mathbb{Z}/2)^*(u(\xi)) = 1+t. \quad (11)$$

We have already discuss how the ambient cohomology $H^*(B(\xi)\wr\mathbb{Z}/2;\mathbb{F}_2)$ for the characteristic class $u(\xi)$ can be computed from the spectral sequence (7). In particular, we can interpret that

$$u(\xi)\in E_2^{0,0}\oplus E_2^{1,0}\oplus E_2^{0,1}\oplus E_2^{2,0}\oplus E_2^{1,1}\oplus E_2^{0,2}\cong E_\infty^{0,0}\oplus E_\infty^{1,0}\oplus E_\infty^{0,1}\oplus E_\infty^{2,0}\oplus E_\infty^{1,1}\oplus E_\infty^{0,2}.$$

Now from relations (10) and (11), and the fact that $E_2^{1,1}=0$, we get that

$$u(\xi) = 1 + \big(t + 1\otimes w_1(\xi) + w_1(\xi)\otimes 1\big) + w_1(\xi)\otimes w_1(\xi),$$

concluding the proof of the proposition. □

In order to prove the general case of Theorem 3.4 we use the general idea of the Splitting principle. For that we first introduce a notion of a splitting map of a vector bundle.

Let ξ be a vector bundle over the base space $B(\xi)$. A continuous map $f\colon B\longrightarrow B(\xi)$ from a space B to the base space of ξ is called a **splitting map** if the pull-back bundle $f^*\xi$ is isomorphic to a Whitney sum of line bundles and the induced homomorphism $f^*\colon H^*(B(\xi);\mathbb{F}_2)\longrightarrow H^*(B;\mathbb{F}_2)$ is a monomorphism. The key property that we use is the following classical fact, see for example [**?**, Prop. 17.5.2].

Proposition 3.6. *For any vector bundle ξ there exists a splitting map.*

Now we utilize several facts that we established to give the proof of Theorem 3.4.

Proof of Theorem 3.4. Let ξ be a real n-dimensional vector bundle over a CW-complex $B(\xi)$. Then according to Proposition 3.6 there exists a splitting map $f\colon B \longrightarrow B(\xi)$. Consequently, $f^*\xi \cong \alpha_1\oplus\cdots\oplus\alpha_n$ where $\alpha_1,\ldots,\alpha_n$ are line bundles over B. Moreover, $f^*\colon H^*(B(\xi);\mathbb{F}_2) \longrightarrow H^*(B;\mathbb{F}_2)$ is a monomorphism that sends, due to naturality, Stiefel–Whitney classes of ξ into elementary symmetric polynomials in "variables" $w_1(\alpha_1),\ldots,w_1(\alpha_n)$, that is

$$w_k(f^*\xi) = \sigma_k(w_1(\alpha_1),\ldots,w_1(\alpha_n)), \qquad 1\le k\le n,$$

where $\sigma_k(x_1,\ldots,x_n)=\sum_{1\le i_1<i_2<\cdots<i_k\le n} x_{i_1}\cdots x_{i_k}$ denotes the k-th elementary symmetric polynomials in variables $x_1,\ldots,x_n$.

Now, the isomorphism of vector bundles (5) yields that

$$(f^*\xi)\wr\mathbb{Z}/2 \cong (\alpha_1\oplus\cdots\oplus\alpha_n)\wr\mathbb{Z}/2 \cong (\alpha_1\wr\mathbb{Z}/2)\oplus\cdots\oplus(\alpha_n\wr\mathbb{Z}/2).$$

Consequently,

$$\begin{aligned}u(f^*\xi) = w((f^*\xi)\wr\mathbb{Z}/2) &= w((\alpha_1\wr\mathbb{Z}/2)\oplus\cdots\oplus(\alpha_n\wr\mathbb{Z}/2))\\ &= w(\alpha_1\wr\mathbb{Z}/2)\cdots w(\alpha_n\wr\mathbb{Z}/2) = u(\alpha_1)\cdots u(\alpha_n).\end{aligned}$$

Next, Proposition 3.5 implies that for all $1\le k\le n$:

$$u(\alpha_k) = 1 + \big(t + 1\otimes w_1(\alpha_k) + w_1(\alpha_k)\otimes 1\big) + w_1(\alpha_k)\otimes w_1(\alpha_k).$$

Combining last two equalities get that

$$\begin{aligned}u(f^*\xi) = \prod_{k=1}^{n}\Big(1+\big(t+1\otimes w_1(\alpha_k)+w_1(\alpha_k)\otimes 1\big)+w_1(\alpha_k)\otimes w_1(\alpha_k)\Big) =\\ \prod_{k=1}^{n}\Big(1+\big(t+Q(1\otimes w_1(\alpha_k))\big)+P(w_1(\alpha_k))\Big).\end{aligned}\tag{12}$$

Now we prove the equality

$$\begin{aligned}\prod_{k=1}^{n}\Big(1+\big(t+Q(1\otimes w_1(\alpha_k))\big)+P(w_1(\alpha_k))\Big) =\\ \sum_{0\le r<s\le n} Q(w_r(f^*\xi)\otimes w_s(f^*\xi)) + \sum_{0\le r\le n} P(w_r(f^*\xi))\cdot(1+t)^{n-r}\end{aligned}\tag{13}$$

applying an induction on n. It is important to recall that $w_k(f^*\xi)=\sigma_k(w_1(\alpha_1),\ldots,w_1(\alpha_n))$ for all $1\le k\le n$, meaning that we want to prove

$$\begin{aligned}\prod_{k=1}^{n}\Big(1+\big(t+Q(1\otimes w_1(\alpha_k))\big)+P(w_1(\alpha_k))\Big) =\\ \sum_{0\le r<s\le n} Q(\sigma_r(w_1(\alpha_1),\ldots,w_1(\alpha_n))\otimes\sigma_s(w_1(\alpha_1),\ldots,w_1(\alpha_n)))+\\ \sum_{0\le r\le n} P(\sigma_r(w_1(\alpha_1),\ldots,w_1(\alpha_n)))\cdot(1+t)^{n-r}.\end{aligned}\tag{14}$$

The induction basis $n=1$ is holds since both sides of (14) are identical for $n=1$. For clarity of the induction proof we first present the proof of case $n=2$. having in mind that P is multiplicative, Q is additive and $t\cdot Q(\cdot)=0$ we have

$$\begin{aligned}&\Big(1+\big(t+Q(1\otimes w_1(\alpha_1))\big)+P(w_1(\alpha_1))\Big)\Big(1+\big(t+Q(1\otimes w_1(\alpha_2))\big)+P(w_1(\alpha_2))\Big) =\\ &(1+t)^2+Q(1\otimes w_1(\alpha_2))+(1+t)P(w_1(\alpha_2))+Q(1\otimes w_1(\alpha_1))+Q(1\otimes w_1(\alpha_1))Q(1\otimes w_1(\alpha_2))+\\ &Q(1\otimes w_1(\alpha_1))P(w_1(\alpha_2))+(1+t)P(w_1(\alpha_1))+P(w_1(\alpha_1))Q(1\otimes w_1(\alpha_2))+P(w_1(\alpha_1))P(w_1(\alpha_2)).\end{aligned}\tag{15}$$

In order to make computation more transparent we separately evaluate different pieces of the last sum separately. The first piece is:

$$\begin{aligned}(1+t)\big(P(w_1(\alpha_1))+P(w_1(\alpha_2))\big) = (1+t)\big(P(w_1(\alpha_1)+w_1(\alpha_2))+Q(w_1(\alpha_1)\otimes w_1(\alpha_2))\big) =\\ (1+t)P(w_1(\alpha_1)+w_1(\alpha_2))+Q(w_1(\alpha_1)\otimes w_1(\alpha_2)) = (1+t)P(w_1(f^*\xi))+Q(w_1(\alpha_1)\otimes w_1(\alpha_2)).\end{aligned}$$

The next piece is:

$$\begin{aligned}Q(1\otimes w_1(\alpha_1))Q(1\otimes w_1(\alpha_2)) &= (1\otimes w_1(\alpha_1)+w_1(\alpha_1)\otimes 1)(1\otimes w_1(\alpha_2)+w_1(\alpha_2)\otimes 1) = \\ &1\otimes w_1(\alpha_1)w_1(\alpha_2)+w_1(\alpha_1)\otimes w_1(\alpha_2)+w_1(\alpha_2)\otimes w_1(\alpha_1)+w_1(\alpha_1)w_1(\alpha_2)\otimes 1 = \\ &Q(1\otimes w_1(\alpha_1)w_1(\alpha_2))+Q(w_1(\alpha_1)\otimes w_1(\alpha_2)) = Q(1\otimes w_2(f^*\xi))+Q(w_1(\alpha_1)\otimes w_1(\alpha_2)).\end{aligned}$$

In the third piece we compute

$$\begin{aligned}&Q(1\otimes w_1(\alpha_1))P(w_1(\alpha_2))+P(w_1(\alpha_1))Q(1\otimes w_1(\alpha_2)) = \\ &(1\otimes w_1(\alpha_1)+w_1(\alpha_1)\otimes 1)(w_1(\alpha_2)\otimes w_1(\alpha_2))+(w_1(\alpha_1)\otimes w_1(\alpha_1))(1\otimes w_1(\alpha_2)+w_1(\alpha_2)\otimes 1) = \\ &w_1(\alpha_2)\otimes w_1(\alpha_1)w_1(\alpha_2)+w_1(\alpha_1)w_1(\alpha_2)\otimes w_1(\alpha_2)+w_1(\alpha_1)\otimes w_1(\alpha_1)w_1(\alpha_2)+w_1(\alpha_1)w_1(\alpha_2)\otimes w_1(\alpha_1) = \\ &(w_1(\alpha_1)+w_1(\alpha_2))\otimes w_1(\alpha_1)w_1(\alpha_2)+w_1(\alpha_1)w_1(\alpha_2)\otimes(w_1(\alpha_1)+w_1(\alpha_2)) = Q(w_1(f^*\xi)\otimes w_2(f^*\xi)).\end{aligned}$$

The final piece yields:

$$\begin{aligned}&(1+t)^2+Q(1\otimes w_1(\alpha_2))+Q(1\otimes w_1(\alpha_1))+P(w_1(\alpha_1))P(w_1(\alpha_2)) = \\ &(1+t)^2+Q(1\otimes(w_1(\alpha_2)+w_1(\alpha_1)))+P(w_1(\alpha_1)w_1(\alpha_2)) = (1+t)^2+Q(1\otimes w_1(f^*\xi))+P(w_2(f^*\xi)).\end{aligned}$$

Gathering all these calculations together in (15) we have that:

$$\begin{aligned}&\Big(1+\big(t+Q(1\otimes w_1(\alpha_1))\big)+P(w_1(\alpha_1))\Big)\Big(1+\big(t+Q(1\otimes w_1(\alpha_2))\big)+P(w_1(\alpha_2))\Big) = \\ &Q(1\otimes w_1(f^*\xi))+Q(1\otimes w_2(f^*\xi))+Q(w_1(f^*\xi)\otimes w_2(f^*\xi))+P(w_2(f^*\xi))+(1+t)P(w_1(f^*\xi))+(1+t)^2.\end{aligned}$$

This concludes the proof of the case $n=2$. Now let us assume that the equalities (13) and (14) hold for all $k\leq n-1$. Using the induction hypothesis we calculate as follows:

$$\begin{aligned}&\prod_{k=1}^{n}\Big(1+\big(t+Q(1\otimes w_1(\alpha_k))\big)+P(w_1(\alpha_k))\Big) = \\ &\Big(1+\big(t+Q(1\otimes w_1(\alpha_1))\big)+P(w_1(\alpha_1))\Big)\prod_{k=2}^{n}\Big(1+\big(t+Q(1\otimes w_1(\alpha_k))\big)+P(w_1(\alpha_k))\Big) = \\ &\Big(1+\big(t+Q(1\otimes w_1(\alpha_1))\big)+P(w_1(\alpha_1))\Big)\Big(\sum_{0\leq r<s\leq n-1}Q(\sigma_r(w_1(\alpha_2),\dots,w_1(\alpha_n))\otimes\sigma_s(w_1(\alpha_2),\dots,w_1(\alpha_n)))+ \\ &\qquad\sum_{0\leq r\leq n-1}P(\sigma_r(w_1(\alpha_2),\dots,w_1(\alpha_n)))\cdot(1+t)^{n-1-r}\Big).\end{aligned}$$

By a direct calculation we get that

$$\begin{aligned}&\prod_{k=1}^{n}\Big(1+\big(t+Q(1\otimes w_1(\alpha_k))\big)+P(w_1(\alpha_k))\Big) = \\ &\qquad\sum_{0\leq r<s\leq n}Q(\sigma_r(w_1(\alpha_1),\dots,w_1(\alpha_n))\otimes\sigma_s(w_1(\alpha_1),\dots,w_1(\alpha_n)))+ \\ &\qquad\qquad\sum_{0\leq r\leq n}P(\sigma_r(w_1(\alpha_1),\dots,w_1(\alpha_n)))\cdot(1+t)^{n-r},\end{aligned}$$

which concludes the induction proof.

Finally, from equalities (12) and (13) we have that

$$u(f^*\xi)=\sum_{0\leq r<s\leq n}Q(w_r(f^*\xi)\otimes w_s(f^*\xi))+\sum_{0\leq r\leq n}P(w_r(f^*\xi))\cdot(1+t)^{n-r}.$$

Furthermore, the naturally of the wreath product construction implies that

$$\begin{aligned}&(f\wr\mathbb{Z}/2)^*(u(\xi))=u(f^*\xi)= \\ &\qquad(f\wr\mathbb{Z}/2)^*\Big(\sum_{0\leq r<s\leq n}Q(w_r(\xi)\otimes w_s(\xi))+\sum_{0\leq r\leq n}P(w_r(\xi))\cdot(1+t)^{n-r}\Big).\end{aligned}\tag{16}$$

Since f is a splitting map the induced map in cohomology $f^*\colon H^*(B(\xi);\mathbb{F}_2)\longrightarrow H^*(B;\mathbb{F}_2)$ is an injection. Consequently, according to Proposition 3.3, the induced map between wreath squares

$$f\wr\mathbb{Z}/2\colon B\wr\mathbb{Z}/2\longrightarrow B(\xi)\wr\mathbb{Z}/2$$

induces a monomorphism in cohomology. Hence equality (16) implies that

$$u(\xi) = \sum_{0\le r<s\le n} Q(w_r(\xi)\otimes w_s(\xi)) + \sum_{0\le r\le n} P(w_r(\xi))\cdot(1+t)^{n-r},$$

concluding the proof of theorem. □

4. The index of the oriented Grassmann manifold

In this section we utilize the definition of the oriented Grassmann manifold as a homogeneous space to obtain an alternative description of the index that will further on allow us to give a complete proof of Theorem 1.1 in Section 6. In particular, at the end of this section we prove Theorem 1.1 in the case when $n=4$ and $n=6$.

For this section we assume that $n\ge 2$ is an even integer. Consequently there exist unique integers $a\ge 1$ and $b\ge 0$ such that $n=2^a(2b+1)$.

4.1. **A useful description of the index of the oriented Grassmann manifold.**
The real oriented Grassmann manifold $\widetilde{G}_n(\mathbb{R}^{2n})$ can be defined by

$$\widetilde{G}_n(\mathbb{R}^{2n}) := \mathrm{SO}(2n)/(\mathrm{SO}(n)\times\mathrm{SO}(n)).$$

The $\mathbb{Z}/2=\langle\omega\rangle$ action on $\widetilde{G}_n(\mathbb{R}^{2n})$ we consider is defined by sending an oriented n-dimensional subspace V to its (appropriately oriented) orthogonal complement $V^{\perp}$.

Let $\mathbb{Z}/2$ acts on the product $\mathrm{SO}(n)\times\mathrm{SO}(n)$ by interchanging the factors, and let

$$W := (\mathrm{SO}(n)\times\mathrm{SO}(n))\rtimes\mathbb{Z}/2 \tag{17}$$

be the semi-direct product induced by this action. In other words W is the wreath product $\mathrm{SO}(n)\wr\mathbb{Z}/2$. Thus we have an exact sequence of groups

$$1\longrightarrow \mathrm{SO}(n)\times\mathrm{SO}(n)\longrightarrow W\xrightarrow{\ p\ }\mathbb{Z}/2\longrightarrow 1. \tag{18}$$

The exact sequence of groups (18) induces a Lyndon–Hochschild–Serre spectral sequence [1, Sec. IV.1] whose E_2 term is given by

$$E_2^{i,j}(W)=H^i(\mathrm{B}(\mathbb{Z}/2);\mathcal{H}^j(\mathrm{B}(\mathrm{SO}(n)\times\mathrm{SO}(n));\mathbb{F}_2))\cong H^i(\mathbb{Z}/2;H^j(\mathrm{SO}(n)\times\mathrm{SO}(n);\mathbb{F}_2)),$$

and that converges to the cohomology $H^*(W;\mathbb{F}/2)$. From a classical result of Nakaoka [11], or from [1, Thm. 1.7], we have that this spectral sequence collapses at E_2-term. Consequently, the induced map in cohomology

$$\mathrm{B}(p)^*\colon H^*(\mathrm{B}(\mathbb{Z}/2);\mathbb{F}_2)\longrightarrow H^*(W;\mathbb{F}_2)$$

is injection.

The group W can be seen as a subgroup of $\mathrm{SO}(2n)$ via the following embedding

$$(A,B)\longmapsto\begin{pmatrix}A&0\\0&B\end{pmatrix},\qquad \omega\longmapsto\begin{pmatrix}0&I\\I&0\end{pmatrix},$$

where $(A,B)\in\mathrm{SO}(n)\times\mathrm{SO}(n)$ and $\omega\in\mathbb{Z}/2$. A direct computation verifies that

$$\begin{pmatrix}A&0\\0&B\end{pmatrix}\begin{pmatrix}0&I\\I&0\end{pmatrix}=\begin{pmatrix}0&I\\I&0\end{pmatrix}\begin{pmatrix}B&0\\0&A\end{pmatrix}.$$

Thus, there is a homeomorphism between the quotient spaces

$$\widetilde{G}_n(\mathbb{R}^{2n})/(\mathbb{Z}/2)\cong\mathrm{SO}(2n)/W.$$

Since the action of any subgroup on the ambient group by multiplication is free we have homotopy equivalences between quotient spaces and associated Borel constructions

$$\mathrm{E}(\mathbb{Z}/2)\times_{\mathbb{Z}/2}\widetilde{G}_n(\mathbb{R}^{2n})\simeq\widetilde{G}_n(\mathbb{R}^{2n})/(\mathbb{Z}/2)\cong\mathrm{SO}(2n)/W\simeq\mathrm{ESO}(2n)\times_W\mathrm{SO}(2n). \tag{19}$$

The space $\mathrm{ESO}(2n)$, as a free contractible $\mathrm{SO}(2n)$-space, can be seen also as a model for $\mathrm{E}W$ via the inclusion map $i\colon W\longrightarrow\mathrm{SO}(2n)$. Thus we have the morphism of Borel construction fibrations presented by the right commutative square of the diagram

$$\begin{tikzcd} \mathrm{E}(\mathbb{Z}/2)\times_{\mathbb{Z}/2}\widetilde{G}_n(\mathbb{R}^{2n}) \arrow[d, "\pi"'] & \mathrm{ESO}(2n)\times_W \mathrm{SO}(2n) \arrow[l] \arrow[r] \arrow[d, "\pi_1"'] & \mathrm{ESO}(2n)\times_{\mathrm{SO}(2n)}\mathrm{SO}(2n) \arrow[d, "\pi_2"'] \\ \mathrm{B}(\mathbb{Z}/2) & \mathrm{B}W \arrow[l, "\mathrm{B}(p)"'] \arrow[r, "\mathrm{B}(i)"] & \mathrm{BSO}(2n). \end{tikzcd} \tag{20}$$

The left square of the diagram is induced by the homotopy (19) and commutes up to a homotopy. Notice that

$$\mathrm{ESO}(2n)\times_{\mathrm{SO}(2n)}\mathrm{SO}(2n)\simeq \mathrm{SO}(2n)/\mathrm{SO}(2n)\cong\{\mathrm{pt}\}.$$

From the homotopy equivalence (19) and the fact that $\mathrm{B}(p)^*$ is an injection we get the following alternative description of the index of the oriented Grassmann manifold

$$\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_n(\mathbb{R}^{2n});\mathbb{F}_2)=\ker(\pi^*)\cong\ker(\pi_1^*)\cap\operatorname{im}\mathrm{B}(p)^*. \tag{21}$$

4.2. **Description of the kernel of π_1^*.** In order to describe the kernel we study the morphism of Serre spectral sequences (20). First we recall that

$$H^*(\mathrm{SO}(2n);\mathbb{F}_2)\cong\mathbb{F}_2[e_1,e_3,\dots,e_{2n-1}]/\langle e_1^{\alpha_1},e_3^{\alpha_3},\dots,e_{2n-1}^{\alpha_{2n-1}}\rangle, \tag{22}$$

where $\deg(e_i)=i$, and α_i is the smallest power of two with the property that $i\alpha_i\geq 2n$, consult for example [6, Thm. 3D.2]. Furthermore, the set $\{e_1,e_2,e_3,\dots,e_{2n-2},e_{2n-1}\}$, where $e_{2i}:=e_i^2$ for $i<n$, forms a simple system of generators. Then the cohomology of the associated classifying space can be evaluated to be

$$H^*(\mathrm{BSO}(2n);\mathbb{F}_2)\cong\mathbb{F}_2[w_2,w_3,\dots,w_{2n}], \tag{23}$$

where $w_j=w_j(c^*\gamma^{2n}(\mathbb{R}^\infty))$ for all $2\leq j\leq 2n$. For more details see for example [8, p. 216] or [10, Thm. III.3.19].

The Serre spectral sequence of the fibration

$$\mathrm{SO}(2n)\longrightarrow\mathrm{ESO}(2n)\times_{\mathrm{SO}(2n)}\mathrm{SO}(2n)\longrightarrow\mathrm{BSO}(2n) \tag{24}$$

has the E_2-term given by

$$E_2^{i,j}(\mathrm{ESO}(2n)\times_{\mathrm{SO}(2n)}\mathrm{SO}(2n))=H^i(\mathrm{BSO}(2n);\mathcal{H}^j(\mathrm{SO}(2n);\mathbb{F}_2)),$$

where the local coefficient system is determined by the action of $\pi_1(\mathrm{BSO}(2n))$. Since $\pi_1(\mathrm{BSO}(2n))=0$ the cohomology with local coefficients we consider becomes ordinary cohomology, and then the Künneth formula gives the isomorphism

$$E_2^{i,j}(\mathrm{ESO}(2n)\times_{\mathrm{SO}(2n)}\mathrm{SO}(2n))\cong H^i(\mathrm{BSO}(2n);\mathbb{F}_2)\otimes H^j(\mathrm{SO}(2n);\mathbb{F}_2).$$

The spectral sequence converges to $H^*(\mathrm{ESO}(2n)\times_{\mathrm{SO}(2n)}\mathrm{SO}(2n);\mathbb{F}_2)\cong H^*(\mathrm{pt};\mathbb{F}_2)$, and therefore

$$E_\infty^{i,j}=\begin{cases}\mathbb{F}_2, & \text{for } i=0 \text{ and } j=0,\\ 0, & \text{otherwise.}\end{cases}$$

Now applying the classical work of Borel [3] and Zeeman [12, Thm. 2] we have that all generators $e_1,e_2,e_3,\dots,e_{2n-2},e_{2n-1}$ are transgressive with $\partial_k(e_{k-1})=w_k$ for all $2\leq k\leq 2n$.

The Serre spectral sequence of the fibration

$$\mathrm{SO}(2n)\longrightarrow\mathrm{ESO}(2n)\times_W\mathrm{SO}(2n)\longrightarrow\mathrm{B}W \tag{25}$$

has the E_2-term given by

$$E_2^{i,j}(\mathrm{ESO}(2n)\times_W\mathrm{SO}(2n))=H^i(\mathrm{B}W;\mathcal{H}^j(\mathrm{SO}(2n);\mathbb{F}_2)),$$

where the local coefficient system is determined by the action of $\pi_1(\mathrm{B}W)$. Since the action of $\pi_1(\mathrm{B}W)$ is determined via the inclusion $\pi_1(\mathrm{B}W)\longrightarrow\pi_1(\mathrm{BSO}(2n))$ and $\pi_1(\mathrm{BSO}(2n))=0$ we have that the cohomology with local coefficients we consider is just the ordinary cohomology. Thus

$$E_2^{i,j}(\mathrm{ESO}(2n)\times_W\mathrm{SO}(2n))=H^i(\mathrm{B}W;\mathcal{H}^j(\mathrm{SO}(2n);\mathbb{F}_2))\cong H^i(\mathrm{B}W;\mathbb{F}_2)\otimes H^j(\mathrm{SO}(2n);\mathbb{F}_2).$$

The morphism (20) between fibrations (25) and (24), that have identical fibers, induces a morphism between the corresponding Serre spectral sequences

$$E_*^{*,*}(\mathrm{ESO}(2n)\times_W\mathrm{SO}(2n))\longleftarrow E_*^{*,*}(\mathrm{ESO}(2n)\times_{\mathrm{SO}(2n)}\mathrm{SO}(2n))$$

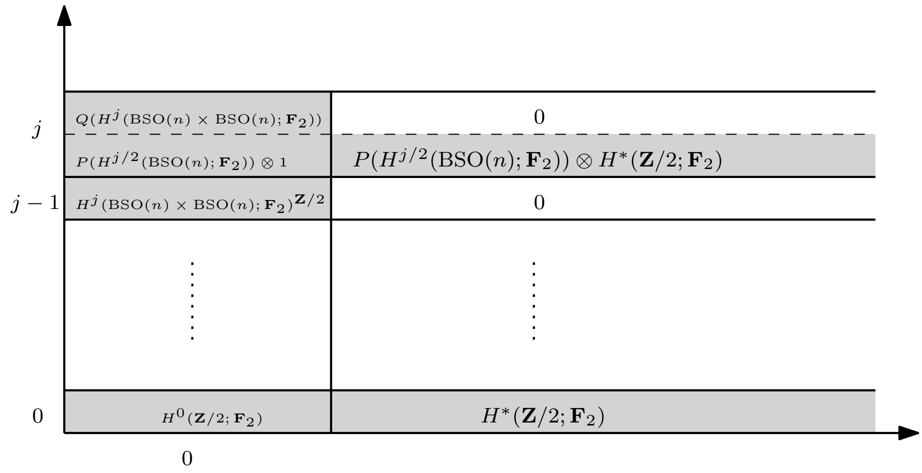

FIGURE 2. The E_2-term of the spectral sequence (48).

that is the identity on the zero column of the E_2-terms. Since differentials of spectral sequences commute with morphisms of spectral sequences we have that in the spectral sequence $E_*^{*,*}(\mathrm{ESO}(2n) \times_W \mathrm{SO}(2n))$ differential have the following values on the system of simple generators $\partial_k(e_{k-1}) = \mathrm{B}(i)^*(w_k)$ for all $2 \le k \le 2n$. In particular, we have described $\ker(\pi_1^*)$ as the following ideal

$$\ker(\pi_1^*) = \langle \mathrm{B}(i)^*(w_2), \mathrm{B}(i)^*(w_3) \dots, \mathrm{B}(i)^*(w_{2n}) \rangle \subseteq H^*(\mathrm{B}W; \mathbb{F}_2). \tag{26}$$

In order to further describe $\ker(\pi_1^*)$ as an ideal we identify its generators $\mathrm{B}(i)^*(w_k)$ as Stiefel–Whitney classes of an appropriate pull-back vector bundle. Since $w_k = w_k(c^*\gamma^{2n}(\mathbb{R}^\infty))$ we have that

$$\mathrm{B}(i)^*(w_k) = \mathrm{B}(i)^*(w_k(c^*\gamma^{2n}(\mathbb{R}^\infty))) = w_k((c \circ \mathrm{B}(i))^*(\gamma^{2n}(\mathbb{R}^\infty)) \tag{27}$$

The vector bundle $(c \circ \mathrm{B}(i))^*(\gamma^{2n}(\mathbb{R}^\infty))$ can be described as follows. The canonical vector bundle $\gamma^{2n}(\mathbb{R}^\infty)$ can be presented as the following Borel construction

$$\mathbb{R}^{2n} \longrightarrow \mathrm{EO}(2n) \times_{\mathrm{O}(2n)} \mathbb{R}^{2n} \longrightarrow \mathrm{BO}(2n).$$

Consequently we have pull-back bundles $c^*\gamma^{2n}(\mathbb{R}^\infty)$ and $(c \circ \mathrm{B}(i))^*(\gamma^{2n}(\mathbb{R}^\infty))$ also presented as Borel constructions

$$\begin{tikzcd}
\mathrm{EO}(2n) \times_W \mathbb{R}^{2n} \arrow[r] \arrow[d] & \mathrm{EO}(2n) \times_{\mathrm{SO}(2n)} \mathbb{R}^{2n} \arrow[r] \arrow[d] & \mathrm{EO}(2n) \times_{\mathrm{O}(2n)} \mathbb{R}^{2n} \arrow[d] \\
\mathrm{B}W \arrow[r] & \mathrm{BSO}(2n) \arrow[r] & \mathrm{BO}(2n).
\end{tikzcd}$$

Thus in order to completely describe $\ker(\pi_1^*)$ we need to compute Stiefel–Whitney classes of the vector bundle $(c \circ \mathrm{B}(i))^*(\gamma^{2n}(\mathbb{R}^\infty))$:

$$\mathbb{R}^{2n} \longrightarrow \mathrm{EO}(2n) \times_W \mathbb{R}^{2n} \longrightarrow \mathrm{B}W. \tag{28}$$

4.3. **Cohomology of the group W.** The group W was defined in (17) to be the wreath product $\mathrm{SO}(n) \wr \mathbb{Z}/2 = (\mathrm{SO}(n) \times \mathrm{SO}(n)) \rtimes \mathbb{Z}/2$. As we have seen the exact sequence of the groups (18),

$$1 \longrightarrow \mathrm{SO}(n) \times \mathrm{SO}(n) \longrightarrow W \xrightarrow{\ p\ } \mathbb{Z}/2 \longrightarrow 1,$$

induced a Lyndon–Hochschild–Serre spectral sequence [1, Sec. IV.1] whose E_2-term is given by

$$E_2^{i,j}(W) = H^i(\mathrm{B}(\mathbb{Z}/2); \mathcal{H}^j(\mathrm{B}(\mathrm{SO}(n) \times \mathrm{SO}(n)); \mathbb{F}_2)) \cong H^i(\mathbb{Z}/2; H^j(\mathrm{SO}(n) \times \mathrm{SO}(n); \mathbb{F}_2)). \tag{29}$$

This spectral sequence converges to $H^*(\mathrm{B}W; \mathbb{Z}/2)$. A classical work of Nakaoka [11], or for example [1, Thm. IV.1.7], implies that this spectral sequence collapses at the E_2-term, that means $E_2^{*,*}(W) \cong E_\infty^{*,*}(W)$. We now describe the E_2-term, and therefore E_∞-term, in more details. For that we use [1, Lem. IV.1.4] and its consequence [1, Cor. IV.1.6], see also discussion in Sections 3.1 and 3.2. Note that we here use the fact that the classifying space $\mathrm{B}W$ of the group W can be modeled by the wreath square of $\mathrm{BSO}(n)$. For an illustration of this spectral sequence see Figure 2.

Now we apply [1, Cor. IV.1.6] to our spectral sequence (29). Let us denote

$$E_2^{i,0}(W) \cong E_\infty^{i,0}(W) \cong H^i(\mathbb{Z}/2;\mathbb{F}_2) \cong \mathbb{F}_2[t],$$

where as before $\deg(t) = 1$. Then we have that

$$E_2^{0,j}(W) \cong E_\infty^{0,j}(W) \cong H^j(\mathrm{BSO}(n) \times \mathrm{BSO}(n);\mathbb{F}_2)^{\mathbb{Z}/2}.$$

In the case when $j \geq 2$ is even description can be made more precise

$$E_2^{0,j}(W) \cong E_\infty^{0,j}(W) \cong P(H^{j/2}(\mathrm{BSO}(n);\mathbb{F}_2)) \oplus Q(H^j(\mathrm{BSO}(n) \times \mathrm{BSO}(n);\mathbb{F}_2)).$$

Furthermore, still for $j \geq 2$ even and $i \geq 1$ we have that

$$E_2^{i,j}(W) \cong E_\infty^{i,j}(W) \cong P(H^{j/2}(\mathrm{BSO}(n);\mathbb{F}_2)) \otimes H^i(\mathbb{Z}/2;\mathbb{F}_2).$$

From the combination with the previous description of the spectral sequence, multiplicative property of the map P and appropriate formula for Q, and the relation

$$Q(H^j(\mathrm{BSO}(n) \times \mathrm{BSO}(n);\mathbb{F}_2)) \cdot t = 0, \tag{30}$$

we get a description of the multiplicative structure of the E_∞-term as well as $H^*(\mathrm{B}W;\mathbb{Z}/2)$. For the fact that no multiplicative extension problem arrises see for example [7, Rem. after Thm. 2.1].

4.4. **Stiefel–Whitney classes of (28) and proof of particular cases of Theorem 1.1.** In the process of describing $\ker(\pi_1^*)$ we obtained in (26) that this kernel is an ideal in $H^*(W;\mathbb{F}_2)$ generated by the elements $\mathrm{B}(i)^*(w_2), \dots, \mathrm{B}(i)^*(w_{2n})$. Furthermore, in (27) we interpreted the generators of the kernel ideal to be the Stiefel–Whitney classes of the vector bundle $(c \circ \mathrm{B}(i))^*(\gamma^{2n}(\mathbb{R}^\infty))$:

$$\mathbb{R}^{2n} \longrightarrow \mathrm{EO}(2n) \times_W \mathbb{R}^{2n} \longrightarrow \mathrm{B}W.$$

It is critical to observe that the vector bundle $(c \circ \mathrm{B}(i))^*(\gamma^{2n}(\mathbb{R}^\infty))$ is isomorphic to the wreath square vector bundle $c^*\gamma^n(\mathbb{R}^\infty) \wr \mathbb{Z}/2$. Consequently we get a formula for the computation of the Stiefel–Whitney classes of the vector bundle (28) in $H^*(W;\mathbb{F}_2)$ as a direct consequence of the general formula derived in Theorem 3.4.

Proposition 4.1. *With the notation and assumptions already made the total Stiefel–Whitney class of the vector bundle $(c \circ \mathrm{B}(i))^*(\gamma^{2n}(\mathbb{R}^\infty))$ is*

$$w((c \circ \mathrm{B}(i))^*(\gamma^{2n}(\mathbb{R}^\infty))) = \sum_{0 \leq r < s \leq n} Q(w_r(c^*\gamma^n(\mathbb{R}^\infty)) \otimes w_s(c^*\gamma^n(\mathbb{R}^\infty))) + \sum_{0 \leq r \leq n} P(w_r(c^*\gamma^n(\mathbb{R}^\infty))) \cdot (1+t)^{n-r}. \tag{31}$$

Example 4.2. For $n = 2$ using the relation (31) we can give the total Stiefel–Whitney class of the vector bundle $(c \circ \mathrm{B}(i))^*(\gamma^4(\mathbb{R}^\infty))$ in the following form

$$\begin{aligned} w((c \circ \mathrm{B}(i))^*(\gamma^4(\mathbb{R}^\infty))) &= \sum_{0 \leq r < s \leq 2} Q(w_r \otimes w_s) + \sum_{0 \leq r \leq 2} P(w_r) \cdot (1+t)^{2-i} \\ &= Q(w_0 \otimes w_1) + Q(w_0 \otimes w_2) + Q(w_1 \otimes w_2) + P(w_0)(1+t)^2 + P(w_1)(1+t) + P(w_2). \end{aligned}$$

Here we use simplified notation $w_i := w_i(c^*\gamma^2(\mathbb{R}^\infty))$. Since c^* pulls-back the tautological bundle $\gamma^2(\mathbb{R}^\infty)$ to the oriented Grassmann manifold we have that $w_1 = 0$. Therefore, we have

$$\begin{aligned} w((c \circ \mathrm{B}(i))^*(\gamma^4(\mathbb{R}^\infty))) &= Q(w_0 \otimes w_2) + P(w_0)(1+t)^2 + P(w_2) \\ &= Q(w_0 \otimes w_2) + P(w_0) + P(w_0)t^2 + P(w_2) = 1 + (Q(w_0 \otimes w_2) + t^2) + P(w_2). \end{aligned}$$

Consequently, from (26) we get that

$$\ker(\pi_1^*) = \langle Q(w_0 \otimes w_2) + t^2, P(w_2) \rangle.$$

In particular, the multiplication property (30) implies that $t \cdot (Q(w_0 \otimes w_2) + t^2) = t^3 \in \ker(\pi_1^*)$. Now, alternative description of the index (21) implies that

$$\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_2(\mathbb{R}^4);\mathbb{F}_2) = \ker(\pi_1^*) \cap \mathrm{im}\,\mathrm{B}(p)^* = \langle t^3 \rangle.$$

Thus Theorem 1.1 holds for $n = 2$.

Example 4.3. For $n = 4$ let $w_i := w_i(c^*\gamma^4(\mathbb{R}^\infty))$. As in the previous example $w_1 = 0$ because c^* pulls-back the tautological bundle $\gamma^4(\mathbb{R}^\infty)$ to the oriented Grassmann manifold. The relation (31) in this situation reads as follows

$$\begin{aligned}w((c\circ \mathrm{B}(i))^*(\gamma^8(\mathbb{R}^\infty))) &= \sum_{0\le r<s\le 4} Q(w_r\otimes w_s) + \sum_{0\le r\le 4} P(w_r)\cdot(1+t)^{4-r} = \\ \sum_{2\le s\le 4} Q(w_0\otimes w_s) &+ \sum_{2\le r<s\le 4} Q(w_r\otimes w_s) + (1+t)^4 + P(w_2)(1+t)^2 + P(w_3)(1+t) + P(w_4) = \\ &1 + Q(w_0\otimes w_2) + Q(w_0\otimes w_3) + (t^4 + P(w_2) + Q(w_0\otimes w_4)) + Q(w_2\otimes w_3) + \\ &(P(w_2)t^2 + P(w_3) + Q(w_2\otimes w_4)) + (P(w_3)t + Q(w_3\otimes w_4)) + P(w_4).\end{aligned}$$

Thus,

$$\begin{aligned}\ker(\pi_1^*) = \langle Q(w_0\otimes w_2),\ &Q(w_0\otimes w_3),\ t^4 + P(w_2) + Q(w_0\otimes w_4),\ Q(w_2\otimes w_3),\\ &P(w_2)t^2 + P(w_3) + Q(w_2\otimes w_4),\ P(w_3)t + Q(w_3\otimes w_4),\ P(w_4)\rangle.\end{aligned}$$

We do the following computation in the ideal $J := \ker(\pi_1^*)$ with the multiplication property (30) in mind:

$$\begin{aligned} t^4 + P(w_2) + Q(w_0\otimes w_4) \in J \ &\Longrightarrow\ t^6 + P(w_2)t^2 \in J,\\ t^6 + P(w_2)t^2 \in J \quad\text{and}\quad P(w_2)t^2 + P(w_3) + Q(w_2\otimes w_4) \in J \ &\Longrightarrow\ t^6 + P(w_3) + Q(w_2\otimes w_4) \in J,\\ t^6 + P(w_3) + Q(w_2\otimes w_4) \in J \ &\Longrightarrow\ t^7 + P(w_3)t \in J,\\ t^7 + P(w_3)t \in J \quad\text{and}\quad P(w_3)t + Q(w_3\otimes w_4) \in J \ &\Longrightarrow\ t^7 + Q(w_3\otimes w_4) \in J,\\ t^7 + Q(w_3\otimes w_4) \in J \ &\Longrightarrow\ t^8 \in J.\end{aligned}$$

Hence $t^7 \notin J$ while $t^8 \in J$. Therefore

$$\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_4(\mathbb{R}^8); \mathbb{F}_2) = \ker(\pi_1^*) \cap \operatorname{im} \mathrm{B}(p)^* = \langle t^8\rangle.$$

This concludes a proof of Theorem 1.1 for $n = 4$.

Example 4.4. Let $n = 6$, and let $w_i := w_i(c^*\gamma^6(\mathbb{R}^\infty))$. Again, since c^* pulls-back the tautological bundle $\gamma^6(\mathbb{R}^\infty)$ to the oriented Grassmann manifold we have that $w_1 = 0$. Now from the relation (31) we get that

$$\begin{aligned}w((c\circ \mathrm{B}(i))^*(\gamma^{12}(\mathbb{R}^\infty))) &= \sum_{0\le r<s\le 6} Q(w_r\otimes w_s) + \sum_{0\le r\le 6} P(w_r)\cdot(1+t)^{6-r} = \\ \sum_{2\le s\le 6} Q(w_0\otimes w_s) &+ \sum_{2\le r<s\le 6} Q(w_r\otimes w_s) + \\ &(1+t)^6 + P(w_2)(1+t)^4 + P(w_3)(1+t)^3 + P(w_4)(1+t)^2 + P(w_5)(1+t) + P(w_6) = \\ \sum_{2\le s\le 6} Q(w_0\otimes w_s) &+ \sum_{2\le r<s\le 6} Q(w_r\otimes w_s) + 1 + t^2 + t^4 + t^6 \\ P(w_2) + P(w_2)t^4 &+ P(w_3) + P(w_3)t + P(w_3)t^2 + P(w_3)t^3 + P(w_4) + P(w_4)t^2 + P(w_5) + P(w_5)t + P(w_6).\end{aligned}$$

In particular, we get that

$$w_2((c\circ \mathrm{B}(i))^*(\gamma^{12}(\mathbb{R}^\infty))) = Q(w_0\otimes w_2) + t^2.$$

The multiplication property (30) and the alternative description of the index (21) imply $t^2 \notin \ker(\pi_1^*)$, and furthermore that

$$t^3 = t(Q(w_0\otimes w_2) + t^2) = tw_2((c\circ \mathrm{B}(i))^*(\gamma^{12}(\mathbb{R}^\infty))) \in \ker(\pi_1^*) \cap \operatorname{im} \mathrm{B}(p)^*.$$

Consequently,

$$\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_6(\mathbb{R}^{12}); \mathbb{F}_2) = \ker(\pi_1^*) \cap \operatorname{im} \mathrm{B}(p)^* = \langle t^3\rangle,$$

and Theorem 1.1 holds when $n = 6$.

5. The index of the Grassmann manifold

Similarly as in Section 4 for the computation of the index we utilize the definition of the Grassmann manifold $G_n(\mathbb{R}^{2n})$ as a homogeneous space. This new description of the index will be used in Section 6 for the proof of Theorem 1.2.

In this section we assume that $n \ge 1$ is an arbitrary positive integer, and $a \ge 0$ and $b \ge 0$ are unique integers such that $n = 2^a(2b + 1)$.

5.1. **A useful description of the index of the Grassmann manifold.** The real Grassmann manifold $G_n(\mathbb{R}^{2n})$ can be defined by

$$G_n(\mathbb{R}^{2n}) := \mathrm{O}(2n)/(\mathrm{O}(n)\times\mathrm{O}(n)).$$

Recall, the $\mathbb{Z}/2=\langle\omega\rangle$ action on $G_n(\mathbb{R}^{2n})$ we study is given by sending an n-dimensional subspace V to its orthogonal complement $V^{\perp}$.

Let $\mathbb{Z}/2$ acts on the product $\mathrm{O}(n)\times\mathrm{O}(n)$ by permuting the factors, and let

$$U := (\mathrm{O}(n)\times\mathrm{O}(n))\rtimes\mathbb{Z}/2 \tag{32}$$

be the semi-direct product induced by this action. Thus, U is the wreath product $\mathrm{O}(n)\wr\mathbb{Z}/2$ and there is an exact sequence of groups

$$1\longrightarrow \mathrm{O}(n)\times\mathrm{O}(n)\longrightarrow U\xrightarrow{\ p\ }\mathbb{Z}/2\longrightarrow 1. \tag{33}$$

This exact sequence induces a Lyndon–Hochschild–Serre spectral sequence [1, Sec. IV.1] whose E_2-term is given by

$$E_2^{i,j}(U)=H^i(\mathrm{B}(\mathbb{Z}/2);\mathcal{H}^j(\mathrm{B}(\mathrm{O}(n)\times\mathrm{O}(n));\mathbb{F}_2))\cong H^i(\mathbb{Z}/2;H^j(\mathrm{O}(n)\times\mathrm{O}(n);\mathbb{F}_2)),$$

and that converges to the cohomology $H^*(\mathrm{B}U;\mathbb{F}_2)$. As we have already seen the classical result of Nakaoka [11] (see also [1, Thm. IV.1.7]) implies that this spectral sequence collapses at E_2-term. Hence, the induced map in cohomology

$$\mathrm{B}(p)^*\colon H^*(\mathrm{B}(\mathbb{Z}/2);\mathbb{F}_2)\longrightarrow H^*(\mathrm{B}U;\mathbb{F}_2)$$

has to be injective.

The group U can be seen as a subgroup of $\mathrm{O}(2n)$ via the following embedding

$$(A,B)\longmapsto\begin{pmatrix}A&0\\0&B\end{pmatrix},\qquad \omega\longmapsto\begin{pmatrix}0&I\\I&0\end{pmatrix},$$

where $(A,B)\in\mathrm{O}(n)\times\mathrm{O}(n)$ and $\omega\in\mathbb{Z}/2$. A direct computation verifies that

$$\begin{pmatrix}A&0\\0&B\end{pmatrix}\begin{pmatrix}0&I\\I&0\end{pmatrix}=\begin{pmatrix}0&I\\I&0\end{pmatrix}\begin{pmatrix}B&0\\0&A\end{pmatrix}.$$

Consequently, there is a homeomorphism between the quotient spaces

$$G_n(\mathbb{R}^{2n})/(\mathbb{Z}/2)\cong\mathrm{O}(2n)/U.$$

Since the action of any subgroup on the ambient group by multiplication is free we have homotopy equivalences between quotient spaces and associated Borel constructions

$$\mathrm{E}(\mathbb{Z}/2)\times_{\mathbb{Z}/2}G_n(\mathbb{R}^{2n})\simeq G_n(\mathbb{R}^{2n})/(\mathbb{Z}/2)\cong\mathrm{O}(2n)/U\simeq\mathrm{EO}(2n)\times_U\mathrm{O}(2n). \tag{34}$$

The space $\mathrm{EO}(2n)$ is a free contractible $\mathrm{O}(2n)$-space. Thus it can be treated as a model for $\mathrm{E}U$ via the inclusion map $i\colon U\longrightarrow\mathrm{O}(2n)$. We have the morphism of Borel construction fibrations presented in the right commutative square of the diagram

$$\begin{tikzcd}
\mathrm{E}(\mathbb{Z}/2)\times_{\mathbb{Z}/2}G_n(\mathbb{R}^{2n}) \arrow[d, "\pi"'] & \mathrm{EO}(2n)\times_U\mathrm{O}(2n) \arrow[l] \arrow[r] \arrow[d, "\pi_1"'] & \mathrm{EO}(2n)\times_{\mathrm{O}(2n)}\mathrm{O}(2n) \arrow[d, "\pi_2"'] \\
\mathrm{B}(\mathbb{Z}/2) & \mathrm{B}U \arrow[l, "\mathrm{B}(p)"'] \arrow[r, "\mathrm{B}(i)"] & \mathrm{BO}(2n).
\end{tikzcd} \tag{35}$$

The left square of the diagram is induced by the homotopy (34) and commutes up to a homotopy. Observe that

$$\mathrm{EO}(2n)\times_{\mathrm{O}(2n)}\mathrm{O}(2n)\simeq\mathrm{O}(2n)/\mathrm{O}(2n)\cong\{\mathrm{pt}\}.$$

The homotopy equivalence (34) and the fact that $\mathrm{B}(p)^*$ is an injection yield the following alternative description of the index of the Grassmann manifold

$$\mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n});\mathbb{F}_2)=\ker(\pi^*)\cong\ker(\pi_1^*)\cap\mathrm{im}\,\mathrm{B}(p)^*. \tag{36}$$

5.2. **Description of the kernel of π_1^*.** For the description of the kernel we analyze the morphism of Serre spectral sequences (35). We first recall that

$$H^*(\mathrm{O}(2n);\mathbb{F}_2) \cong H^*(\mathrm{SO}(2n);\mathbb{F}_2) \oplus H^*(\mathrm{SO}^-(2n);\mathbb{F}_2) \tag{37}$$

where $\mathrm{SO}^-(2n) = \{A \in \mathrm{O}(2n) : \det(A) = -1\} \cong \mathrm{SO}(2n)$, see [10, Cor. III.3.15]. Then the cohomology of the associated classifying space can be evaluated to be

$$H^*(\mathrm{BO}(2n);\mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, w_3, \dots, w_{2n}], \tag{38}$$

where $w_j = w_j(\gamma^{2n}(\mathbb{R}^\infty))$ for all $1 \le j \le 2n$. For more details see for example [8, p. 216] or [10, Thm. III.3.19].

The Serre spectral sequence of the fibration

$$\mathrm{O}(2n) \longrightarrow \mathrm{EO}(2n) \times_{\mathrm{O}(2n)} \mathrm{O}(2n) \longrightarrow \mathrm{BO}(2n), \tag{39}$$

whose fiber has two path-connected components, has the E_2-term given by

$$E_2^{i,j}(\mathrm{EO}(2n) \times_{\mathrm{O}(2n)} \mathrm{O}(2n)) = H^i(\mathrm{BO}(2n); \mathcal{H}^j(\mathrm{O}(2n);\mathbb{F}_2)),$$

where the local coefficient system is determined by the action of $\pi_1(\mathrm{BO}(2n)) \cong \pi_0(\mathrm{O}(2n)) \cong \mathbb{Z}/2$. Since $H^*(\mathrm{O}(2n);\mathbb{F}_2) \cong H^*(\mathrm{SO}(2n);\mathbb{F}_2) \oplus H^*(\mathrm{SO}^-(2n);\mathbb{F}_2)$ and $\pi_1(\mathrm{BO}(2n))$ acts by interchanging summands we have that as a $\pi_1(\mathrm{BO}(2n))$-module

$$H^*(\mathrm{O}(2n);\mathbb{F}_2) \cong \mathrm{Coind}_{\mathrm{SO}(2n)}^{\mathrm{O}(2n)}\big(H^*(\mathrm{SO}(2n);\mathbb{F}_2)\big).$$

Thus according to the Shapiro lemma [?, Prop. 6.2] the E_2-term simplifies to

$$\begin{aligned} E_2^{i,j}(\mathrm{EO}(2n) \times_{\mathrm{O}(2n)} \mathrm{O}(2n)) &= H^i(\mathrm{BO}(2n); \mathcal{H}^j(\mathrm{O}(2n);\mathbb{F}_2)) \cong \\ &H^i\big(\mathrm{BO}(2n); \mathrm{Coind}_{\mathrm{SO}(2n)}^{\mathrm{O}(2n)}\big(H^j(\mathrm{SO}(2n);\mathbb{F}_2)\big)\big) \cong H^i(\mathrm{BSO}(2n); H^j(\mathrm{SO}(2n);\mathbb{F}_2)) \cong \\ &H^i(\mathrm{BSO}(2n);\mathbb{F}_2) \otimes H^j(\mathrm{SO}(2n);\mathbb{F}_2). \end{aligned}$$

Since the spectral sequence converges to $H^*(\mathrm{EO}(2n) \times_{\mathrm{O}(2n)} \mathrm{O}(2n);\mathbb{F}_2) \cong H^*(\mathrm{pt};\mathbb{F}_2)$ we know that

$$E_\infty^{i,j} = \begin{cases} \mathbb{F}_2, & \text{for } i = 0 \text{ and } j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now applying the classical work of Borel [3] and Zeeman [12, Thm. 2] we have that all elements of the simple system of generators $e_1, e_2, e_3, \dots, e_{2n-2}, e_{2n-1}$ of the cohomology of $\mathrm{SO}(2n)$ are transgressive with $\partial_k(e_{k-1}) = w_k$ for all $2 \le k \le 2n$.

The Serre spectral sequence of the fibration

$$\mathrm{O}(2n) \longrightarrow \mathrm{EO}(2n) \times_U \mathrm{O}(2n) \longrightarrow \mathrm{B}U, \tag{40}$$

with disconnected fiber, has the E_2-term given by

$$E_2^{i,j}(\mathrm{EO}(2n) \times_U \mathrm{O}(2n)) = H^i(\mathrm{B}U; \mathcal{H}^j(\mathrm{O}(2n);\mathbb{F}_2)),$$

where the local coefficient system is determined by the action of $\pi_1(\mathrm{B}U) \cong \pi_0(U) \cong (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2 \cong D_8$. Here D_8 denotes the Dihedral group of isometries of a square. Let $U^+ := U \cap \mathrm{SO}(2n)$ be the index 2 subgroup of orientation preserving elements of U. Then as a $\pi_1(\mathrm{B}U)$-module

$$H^*(\mathrm{O}(2n);\mathbb{F}_2) \cong \mathrm{Coind}_{U^+}^{U}\big(H^*(\mathrm{SO}(2n);\mathbb{F}_2)\big).$$

Therefore the Shapiro lemma [?, Prop. 6.2] allows us to simplify the E_2-term again

$$\begin{aligned} E_2^{i,j}(\mathrm{EO}(2n) \times_U \mathrm{O}(2n)) &= H^i(\mathrm{B}U; \mathcal{H}^j(\mathrm{O}(2n);\mathbb{F}_2)) \cong H^i\big(\mathrm{B}U; \mathrm{Coind}_{U^+}^{U}\big(H^j(\mathrm{SO}(2n);\mathbb{F}_2)\big)\big) \cong \\ &H^i(\mathrm{B}U^+; H^j(\mathrm{SO}(2n);\mathbb{F}_2)) \cong H^i(\mathrm{B}U^+;\mathbb{F}_2) \otimes H^j(\mathrm{SO}(2n);\mathbb{F}_2). \end{aligned}$$

The morphism (35) between fibrations (40) and (39), that have identical fibers, induces a morphism between the corresponding Serre spectral sequences

$$E_*^{*,*}(\mathrm{EO}(2n) \times_U \mathrm{O}(2n)) \longleftarrow E_*^{*,*}(\mathrm{EO}(2n) \times_{\mathrm{O}(2n)} \mathrm{O}(2n))$$

which is the identity on the zero column of the E_2-terms, but now is **not** the identity on the zero row of the E_2-terms. In fact, the morphism of the spectral sequence on the zero row of the E_2-terms is the restriction homomorphism $\mathrm{res}_{U^+}^{SO(2n)} \colon H^*(\mathrm{BSO}(2n);\mathbb{F}_2) \longrightarrow H^*(\mathrm{B}U^+;\mathbb{F}_2)$ induced by the inclusion $i|_{U^+} \colon U^+ \longrightarrow SO(2n)$. Differentials of spectral sequences commute with morphisms of spectral sequences.

Hence in the spectral sequence $E_*^{*,*}(\mathrm{EO}(2n)\times_U \mathrm{O}(2n))$ the differential have the following values on the system of simple generators $\partial_k(e_{k-1}) = \mathrm{B}(i|_{U^+})^*(w_k)$ for all $2 \le k \le 2n$.

Unlike in Section 4.1 we did not yet reached a description of $\ker(\pi_1^*)$. For that we expand the diagram (35) as follows

$$\begin{tikzcd}
\mathrm{EO}(2n)\times_U \mathrm{O}(2n) \arrow[rr] \arrow[dd, "\pi_1"'] \arrow[dr] & & \mathrm{EO}(2n)\times_{\mathrm{O}(2n)} \mathrm{O}(2n) \arrow[dd] \arrow[dr] & \\
& \mathrm{EO}(2n)\times_U \mathrm{pt} \arrow[rr] \arrow[dd] & & \mathrm{EO}(2n)\times_{\mathrm{O}(2n)} \mathrm{pt} \arrow[dd] \\
\mathrm{B}U \arrow[rr] \arrow[dr, "\mathrm{id}"'] & & \mathrm{BO}(2n) \arrow[dr, "\mathrm{id}"'] & \\
& \mathrm{B}U \arrow[rr] & & \mathrm{BO}(2n)
\end{tikzcd} \tag{41}$$

with diagonal maps induced by the $\mathrm{O}(2n)$, and also U, equivariant projection $\tau\colon \mathrm{O}(2n) \longrightarrow \mathrm{pt}$. The cube diagram of bundle morphisms (41) induces the following diagram of the corresponding spectral sequence morphisms:

$$\begin{tikzcd}
E_*^{*,*}(\mathrm{EO}(2n)\times_U \mathrm{O}(2n)) & E_*^{*,*}(\mathrm{EO}(2n)\times_{\mathrm{O}(2n)} \mathrm{O}(2n)) \arrow[l] \\
E_*^{*,*}(\mathrm{EO}(2n)\times_U \mathrm{pt}) \arrow[u, "\eta_1"] & E_*^{*,*}(\mathrm{EO}(2n)\times_{\mathrm{O}(2n)} \mathrm{pt}), \arrow[l] \arrow[u, "\eta_2"']
\end{tikzcd} \tag{42}$$

where the vertical maps η_1 and η_2 are induced by the $\mathrm{O}(2n)$, and also U, equivariant projection τ. The spectral sequences in the bottom row of the diagram collapse at the E_2-term. Thus, from the infinity term of the diagram (42) we get the equality

$$\ker(\pi_1) = \ker(\eta_1). \tag{43}$$

Now in order to understand $\ker(\eta_1)$ we specialize diagram (42) to the E_2-term and get

$$\begin{tikzcd}
H^i(\mathrm{B}U;\mathcal{H}^j(\mathrm{O}(2n);\mathbb{F}_2)) & H^i(\mathrm{BO}(2n);\mathcal{H}^j(\mathrm{O}(2n);\mathbb{F}_2)) \arrow[l] \\
H^i(\mathrm{B}U;\mathcal{H}^j(\mathrm{pt};\mathbb{F}_2)) \arrow[u, "\eta_1"] & H^i(\mathrm{BO}(2n);\mathcal{H}^j(\mathrm{pt};\mathbb{F}_2)). \arrow[l] \arrow[u, "\eta_2"']
\end{tikzcd}$$

Furthermore, on the zero row of the E_2-term we have

$$\begin{tikzcd}
H^i(\mathrm{B}U;\mathrm{Coind}_{U^+}^{U}\mathbb{F}_2) & H^i(\mathrm{BO}(2n);\mathrm{Coind}_{\mathrm{SO}(2n)}^{\mathrm{O}(2n)}\mathbb{F}_2) \arrow[l] \\
H^i(\mathrm{B}U;\mathbb{F}_2) \arrow[u, "\eta_1"] & H^i(\mathrm{BO}(2n);\mathbb{F}_2). \arrow[l] \arrow[u, "\eta_2"]
\end{tikzcd} \tag{44}$$

The vertical maps in (44) are determined by the coefficient morphism $\tau^*\colon H^*(\mathrm{pt};F_2) \longrightarrow H^*(\mathrm{O}(2n);\mathbb{F}_2)$. Since from Shapiro lemma

$$H^i(\mathrm{B}U;\mathrm{Coind}_{U^+}^{U}\mathbb{F}_2) \cong H^i(\mathrm{B}U^+;\mathbb{F}_2) \qquad \text{and} \qquad H^i(\mathrm{BO}(2n);\mathrm{Coind}_{\mathrm{SO}(2n)}^{\mathrm{O}(2n)}\mathbb{F}_2) \cong H^i(\mathrm{BSO}(2n);\mathbb{F}_2)$$

we have that $\eta_1 = \mathrm{res}_{U^+}^{U}$ and $\eta_2 = \mathrm{res}_{\mathrm{SO}(2n)}^{\mathrm{O}(2n)}$. From computation of the spectral sequences of the fibrations (39) and (40) we have that the diagram (44) in infinity term becomes

$$\begin{tikzcd}
H^i(\mathrm{B}U^+;\mathbb{F}_2)/\langle \mathrm{B}(i|_{U^+})^*(w_2),\dots,\mathrm{B}(i|_{U^+})^*(w_{2n})\rangle & H^i(\mathrm{BSO}(2n);\mathbb{F}_2)/\langle w_2,\dots,w_{2n}\rangle \arrow[l] \\
H^i(\mathrm{B}U;\mathbb{F}_2) \arrow[u, "\eta_1"] & H^i(\mathrm{BO}(2n);\mathbb{F}_2). \arrow[l] \arrow[u, "\eta_2"]
\end{tikzcd} \tag{45}$$

Since η_1 is the restriction, $\mathrm{B}(i|_{U^+})^*(w_2),\dots,\mathrm{B}(i|_{U^+})^*(w_{2n})$ are Stiefel–Whitney classes fulfilling naturally, using the sequence of group inclusions $U^+ \longrightarrow U \xrightarrow{\ i\ } \mathrm{O}(2n)$ we have that

$$\ker(\pi_1^*) = \ker(\eta_1^*) = \langle \mathrm{B}(i)^*(w_1), \mathrm{B}(i)^*(w_2),\dots,\mathrm{B}(i)^*(w_{2n})\rangle \subseteq H^*(\mathrm{B}U;\mathbb{F}_2). \tag{46}$$

Observe that commutativity of the diagram (45) and the naturally of Stiefel–Whitney classes force the first Stiefel–Whitney class $\mathrm{B}(i)^*(w_1)$ in the kernel.

Thus in order to completely describe $\ker(\pi_1^*)$ we need to compute Stiefel–Whitney classes of the vector bundle $B(i)^*(\gamma^{2n}(\mathbb{R}^\infty))$:

$$\mathbb{R}^{2n} \longrightarrow \mathrm{EO}(2n) \times_U \mathbb{R}^{2n} \longrightarrow \mathrm{B}U. \tag{47}$$

5.3. **Cohomology of the groups U.** The group U was defined to be the wreath product $\mathrm{O}(n) \wr \mathbb{Z}/2 = (\mathrm{O}(n) \times \mathrm{O}(n)) \rtimes \mathbb{Z}/2$. The exact sequence of the groups (33),

$$1 \longrightarrow \mathrm{O}(n) \times \mathrm{O}(n) \longrightarrow U \xrightarrow{\ p\ } \mathbb{Z}/2 \longrightarrow 1,$$

induced a Lyndon–Hochschild–Serre spectral sequence whose E_2-term is given by

$$E_2^{i,j}(U) = H^i(\mathrm{B}(\mathbb{Z}/2); \mathcal{H}^j(\mathrm{B}(\mathrm{O}(n) \times \mathrm{O}(n)); \mathbb{F}_2)) \cong H^i(\mathbb{Z}/2; H^j(\mathrm{O}(n) \times \mathrm{O}(n); \mathbb{F}_2)). \tag{48}$$

This spectral sequence converges to $H^*(\mathrm{B}U; \mathbb{Z}/2)$. As we have seen this spectral sequence collapses at the E_2-term, meaning $E_2^{*,*}(U) \cong E_\infty^{*,*}(U)$. Like in Section 4.3 we describe the E_2-term, and therefore E_∞-term, in more details. For that we use again [1, Lem. IV.1.4] and [1, Cor. IV.1.6], consult also Sections 3.1 and 3.2. We utilize the fact that the classifying space $\mathrm{B}U$ of the group U can be obtained to be the wreath square of $\mathrm{BO}(n)$.

Let us apply [1, Cor. IV.1.6] to our spectral sequence (48). Denote

$$E_2^{i,0}(U) \cong E_\infty^{i,0}(U) \cong H^i(\mathbb{Z}/2; \mathbb{F}_2) \cong \mathbb{F}_2[t],$$

where as before $\deg(t) = 1$. Then we have that

$$E_2^{0,j}(U) \cong E_\infty^{0,j}(U) \cong H^j(\mathrm{BO}(n) \times \mathrm{BO}(n); \mathbb{F}_2)^{\mathbb{Z}/2}.$$

In particular when $j \geq 2$ is even description can be made even more precise

$$E_2^{0,j}(U) \cong E_\infty^{0,j}(U) \cong P(H^{j/2}(\mathrm{BO}(n); \mathbb{F}_2)) \oplus Q(H^j(\mathrm{BO}(n) \times \mathrm{BO}(n); \mathbb{F}_2)).$$

Furthermore, still for $j \geq 2$ even and $i \geq 1$ we have that

$$E_2^{i,j}(U) \cong E_\infty^{i,j}(U) \cong P(H^{j/2}(\mathrm{BO}(n); \mathbb{F}_2)) \otimes H^i(\mathbb{Z}/2; \mathbb{F}_2).$$

From the combination with the previous description of the spectral sequence, multiplicative property of the map P and appropriate formula for Q, and the relation

$$Q(H^j(\mathrm{BO}(n) \times \mathrm{BO}(n); \mathbb{F}_2)) \cdot t = 0, \tag{49}$$

we get a description of the multiplicative structure of the E_∞-term as well as $H^*(\mathrm{B}U; \mathbb{Z}/2)$. The multiplicative extension problem does not arise in this situation, for more details consult for example [7, Rem. after Thm. 2.1].

5.4. **Stiefel–Whitney classes of (47) and proof of particular cases of Theorem 1.2.** In the process of describing $\ker(\pi_1^*)$ we obtained in (46) that this kernel is an ideal in $H^*(\mathrm{B}U; \mathbb{F}_2)$ generated by the elements $\mathrm{B}(i)^*(w_1), \dots, \mathrm{B}(i)^*(w_{2n})$. Furthermore, we interpreted the generators of the kernel ideal to be the Stiefel–Whitney classes of the vector bundle $\mathrm{B}(i)^*(\gamma^{2n}(\mathbb{R}^\infty))$:

$$\mathbb{R}^{2n} \longrightarrow \mathrm{EO}(2n) \times_U \mathbb{R}^{2n} \longrightarrow \mathrm{B}U.$$

As before it can be seen that the vector bundle $\mathrm{B}(i)^*(\gamma^{2n}(\mathbb{R}^\infty))$ is isomorphic to the wreath square vector bundle $\gamma^n(\mathbb{R}^\infty) \wr \mathbb{Z}/2$. Thus the following formula for the computation of the Stiefel–Whitney classes of the vector bundle (47) in $H^*(\mathrm{B}U; \mathbb{F}_2)$ is a direct consequence of the general formula derived in Theorem 3.4.

Proposition 5.1. *With the notation and assumptions already made the total Stiefel–Whitney class of the vector bundle* $\mathrm{B}(i)^*(\gamma^{2n}(\mathbb{R}^\infty))$ *is*

$$w(\mathrm{B}(i)^*(\gamma^{2n}(\mathbb{R}^\infty))) = \sum_{0 \leq r < s \leq n} Q(w_r(\gamma^n(\mathbb{R}^\infty)) \otimes w_s(\gamma^n(\mathbb{R}^\infty))) + \sum_{0 \leq r \leq n} P(w_r(\gamma^n(\mathbb{R}^\infty))) \cdot (1+t)^{n-r}. \tag{50}$$

Example 5.2. For $n=2$ using the relation (50) we can give the total Stiefel–Whitney class of the vector bundle $\mathrm{B}(i)^*(\gamma^4(\mathbb{R}^\infty))$ in the following form

$$\begin{aligned} w(B(i)^*(\gamma^4(\mathbb{R}^\infty))) &= \sum_{0\le r<s\le 2} Q(w_r\otimes w_s) + \sum_{0\le r\le 2} P(w_r)\cdot(1+t)^{2-i} \\ &= Q(w_0\otimes w_1)+Q(w_0\otimes w_2)+Q(w_1\otimes w_2)+P(w_0)(1+t)^2+P(w_1)(1+t)+P(w_2). \end{aligned}$$

Consequently, from (46) we get that

$$\ker(\pi_1^*) = \langle Q(w_0\otimes w_1), Q(w_0\otimes w_2)+t^2+P(w_1), Q(w_1\otimes w_2)+P(w_1)t, P(w_2)\rangle.$$

In particular, the multiplication property (49) implies that

$$t\cdot(Q(w_0\otimes w_2)+t^2+P(w_1)) = t^3+P(w_1)t\in\ker(\pi_1^*).$$

Since $Q(w_1\otimes w_2)+P(w_1)t\in\ker(\pi_1^*)$ we have $t^3+Q(w_1\otimes w_2)\in\ker(\pi_1^*)$. Again the multiplication property (49) yields

$$t(t^3+Q(w_1\otimes w_2)) = t^4\in\ker(\pi_1^*).$$

Now, from (36) follows

$$\mathrm{Index}_{\mathbb{Z}/2}(G_2(\mathbb{R}^4);\mathbb{F}_2) = \ker(\pi_1^*)\cap\operatorname{im}\mathrm{B}(p)^* = \langle t^4\rangle.$$

Thus Theorem 1.2 holds for $n=2$.

Example 5.3. For $n=4$ let $w_i := w_i(\gamma^4(\mathbb{R}^\infty))$. The relation (50) in this situation reads as follows

$$\begin{aligned} &w(\mathrm{B}(i)^*(\gamma^8(\mathbb{R}^\infty))) = \sum_{0\le r<s\le 4} Q(w_r\otimes w_s) + \sum_{0\le r\le 4} P(w_r)\cdot(1+t)^{4-r} = \\ &\sum_{0\le r<s\le 4} Q(w_r\otimes w_s)+P(w_0)(1+t)^4+P(w_1)(1+t)^3+P(w_2)(1+t)^2+P(w_3)(1+t)+P(w_4) = \\ &1+Q(w_0\otimes w_1)+\big(Q(w_0\otimes w_2)+P(w_1)\big)+\big(Q(w_0\otimes w_3)+Q(w_1\otimes w_2)+P(w_1)t\big)+ \\ &\quad\big(Q(w_0\otimes w_4)+Q(w_1\otimes w_3)+t^4+P(w_1)t^2+P(w_2)\big)+\big(Q(w_1\otimes w_4)+Q(w_2\otimes w_3)+P(w_1)t^3\big)+ \\ &\qquad\big(Q(w_2\otimes w_4)+P(w_2)t^2+P(w_3)\big)+\big(Q(w_3\otimes w_4)+P(w_3)t\big)+P(w_4). \end{aligned}$$

Thus,

$$\begin{aligned} \ker(\pi_1^*) = \langle &Q(w_0\otimes w_1),\ Q(w_0\otimes w_2)+P(w_1),\ Q(w_0\otimes w_3)+Q(w_1\otimes w_2)+P(w_1)t, \\ &Q(w_0\otimes w_4)+Q(w_1\otimes w_3)+t^4+P(w_1)t^2+P(w_2),\ Q(w_1\otimes w_4)+Q(w_2\otimes w_3)+P(w_1)t^3, \\ &Q(w_2\otimes w_4)+P(w_2)t^2+P(w_3),\ Q(w_3\otimes w_4)+P(w_3)t,\ P(w_4)\rangle. \end{aligned}$$

Since $Q(w_0\otimes w_2)+P(w_1)\in J$ then $P(w_1)t\in J$. Thus we can modify the presentation of the kernel ideal as follows

$$\begin{aligned} \ker(\pi_1^*) = \langle &Q(w_0\otimes w_1),\ Q(w_0\otimes w_2)+P(w_1),\ P(w_1)t,\ Q(w_0\otimes w_3)+Q(w_1\otimes w_2), \\ &Q(w_0\otimes w_4)+Q(w_1\otimes w_3)+t^4+P(w_2),\ Q(w_1\otimes w_4)+Q(w_2\otimes w_3), \\ &Q(w_2\otimes w_4)+P(w_2)t^2+P(w_3),\ Q(w_3\otimes w_4)+P(w_3)t,\ P(w_4)\rangle. \end{aligned}$$

Now we compute in the ideal $J := \ker(\pi_1^*)$ with the multiplication property (49) in mind.

$$\begin{aligned} Q(w_0\otimes w_4)+Q(w_1\otimes w_3)+t^4+P(w_2)\in J &\implies t^5+P(w_2)t\in J, \\ Q(w_2\otimes w_4)+P(w_2)t^2+P(w_3)\in J &\implies Q(w_2\otimes w_4)+t^6+P(w_3)\in J, \\ Q(w_2\otimes w_4)+t^6+P(w_3)\in J &\implies t^7+P(w_3)t\in J, \\ Q(w_3\otimes w_4)+P(w_3)t\in J\ \text{ and }\ t^7+P(w_3)t\in J &\implies t^7+Q(w_3\otimes w_4)\in J, \\ t^7+Q(w_3\otimes w_4)\in J &\implies t^8\in J. \end{aligned}$$

Hence $t^7\notin J$ while $t^8\in J$. Therefore

$$\mathrm{Index}_{\mathbb{Z}/2}(G_4(\mathbb{R}^8);\mathbb{F}_2) = \ker(\pi_1^*)\cap\operatorname{im}\mathrm{B}(p)^* = \langle t^8\rangle.$$

This completes a proof of Theorem 1.2 in the case when $n=4$.

6. Proofs of main theorems

In this section we prove both Theorem 1.1 and Theorem 1.2 along the same line of arguments. Therefore we introduce a common notation.

Let $n \geq 2$ be an even integer, and let $a \geq 1$ and $b \geq 0$ be the unique integers such that $n = 2^a(2b+1)$. Recall that we saw $W = \mathrm{SO}(n) \wr \mathbb{Z}/2$ as a subgroup of $\mathrm{SO}(2n)$ so that $\widetilde{G}_n(\mathbb{R}^{2n})/(\mathbb{Z}/2) \cong \mathrm{SO}(2n)/W$. Furthermore we denoted by $i\colon W \longrightarrow \mathrm{SO}(2n)$ the inclusion, and by $\mathrm{B}(i)\colon \mathrm{B}W \longrightarrow \mathrm{BSO}(2n)$ the induced map.

For $n \geq 1$ an arbitrary positive integer, let $a \geq 0$ and $b \geq 0$ be integers such that $n = 2^a(2b+1)$. Recall that we interpreted the group $U = \mathrm{O}(n) \wr \mathbb{Z}/2$ as a subgroup of $\mathrm{O}(2n)$ so that $G_n(\mathbb{R}^{2n})/(\mathbb{Z}/2) \cong \mathrm{O}(2n)/U$. As before, by abusing the notation, we denoted by $i\colon U \longrightarrow \mathrm{O}(2n)$ the inclusion, and with $\mathrm{B}(i)\colon \mathrm{B}U \longrightarrow \mathrm{BO}(2n)$ the corresponding induced map.

Notice that we denote by i two different inclusions. This should not cause any confusion since these two maps play the same role in the corresponding proofs of Theorems 1.1 and 1.2 that proceed along the same lines.

In order to ease the notation in the proofs of theorems we introduce a simplification of notation:

- $\xi := c^*\gamma^n(\mathbb{R}^\infty)$ or $\gamma^n(\mathbb{R}^\infty)$,
- $w_i := w_i(\xi)$,
- $Q_{i,j} := Q(w_i \otimes w_j) := w_i \otimes w_j + w_j \otimes w_i$,
- $P_i := P(w_i) := w_i \otimes w_i$
- $X := W$ or U respectively.

As we have seen in the Section 4.3, for oriented Grassmann manifolds, and in the Section 5.3 for Grassmann manifolds, the 0-column of the E_2-term of the Lyndon–Hochschild–Serre spectral sequence (29) and (48) is additively generated by appropriate

$$\{Q_{i,j} \mid 0 \leq i < j \leq n\} \qquad \text{and} \qquad \{P_i \mid 0 \leq i \leq n\},$$

where $E_2^{0,*} \cong H^*(\mathrm{SO}(n);\mathbb{F}_2)^{\mathbb{Z}/2}$ for the oriented Grassmann manifold, and $E_2^{0,*} \cong H^*(\mathrm{O}(n);\mathbb{F}_2)^{\mathbb{Z}/2}$ in the case of the Grassmann manifold. Furthermore, $w_0 = 1$ and consequently $P_0 := P(w_0) = 1 \otimes 1 = 1$.

Recall that in (26) and (46) we have proved the following description of the kernel ideal

$$\ker(\pi_1^*) = \langle \mathrm{B}(i)^*(w_1), \dots, \mathrm{B}(i)^*(w_{2n}) \rangle = \langle w_1(\xi \wr \mathbb{Z}/2), \dots, w_{2n}(\xi \wr \mathbb{Z}/2) \rangle \subseteq H^*(\mathrm{B}X;\mathbb{F}_2).$$

Notice when we work over the oriented Grassmann manifold $w_1 = w_1(\xi) = 0$ in $H^*(\mathrm{SO}(n);\mathbb{F}_2)$. Therefore, $w_1(\xi \wr \mathbb{Z}/2) = \mathrm{B}(i)^*(w_1) = 0$ and so $Q_{1,j} = Q_{j,1} = 0$ and $P_1 = 0$.

Proposition 6.1. *Let $n \geq 2$ be an even integer, and let $a \geq 1$ and $b \geq 0$ be the unique integers such that $n = 2^a(2b+1)$. For every $1 \leq k \leq 2^{a+1} - 1$ the relation*

$$w_k(\xi \wr \mathbb{Z}/2) = 0$$

in the quotient algebra $H^*(X;\mathbb{F}_2)/\langle w_1(\xi \wr \mathbb{Z}/2), w_2(\xi \wr \mathbb{Z}/2), \cdots, w_{k-1}(\xi \wr \mathbb{Z}/2)\rangle$ is

(i) *in the case when k is odd and $1 \leq k \leq 2^{a+1} - 3$ equivalent to*

$$\sum_{i=\max\{0,k-n\}}^{[\frac{k-1}{2}]} Q_{i,k-i} = 0, \tag{51}$$

(ii) *in the case when k is even and $k \notin \{2^{a+1} - 2^a, 2^{a+1} - 2^{a-1}, \dots, 2^{a+1} - 2\}$ equivalent to*

$$P_{\frac{k}{2}} = \sum_{i=\max\{0,k-n\}}^{[\frac{k-1}{2}]} Q_{i,k-i}, \tag{52}$$

(iii) *in the case when $k = 2^{a+1} - 2^{r+1}$, where $r \in \{0, 1, \dots, a-1\}$, equivalent to*

$$P_{2^a - 2^r} = t^{2^{a+1} - 2^{r+1}} + \sum_{i=\max\{0,k-n\}}^{2^a - 2^r - 1} Q_{i,k-i}, \tag{53}$$

(iv) *in the case when $k = 2^{a+1} - 1$ equivalent to*

$$t \cdot P_{2^a - 1} = \sum_{i=\max\{0,k-n\}}^{2^a - 1} Q_{i,k-i}. \tag{54}$$

Proof. For the proof we use equations (31) and (50) which are equivalent to the following sequence of equalities

$$w_k(\xi\wr\mathbb{Z}/2)=\sum_{i=\max\{0,k-n\}}^{[\frac{k-1}{2}]}Q_{i,k-i}+\sum_{i=\max\{0,k-n\}}^{[\frac{k}{2}]}\binom{n-i}{k-2i}t^{k-2i}P_i. \tag{55}$$

In computations of binomial coefficients we use the classical Lukas's theorem [?]: Let $a=a_s2^s+a_{s-1}2^{s-1}+\cdots+a_0$ and $b=b_s2^s+b_{s-1}2^{s-1}+\cdots+b_0$ be 2-adic expansions of the natural numbers a and b, with $a_i,b_i\in\{0,1\}$, where $a_s=1$, while b_s might be equal to 0. Then

$$\binom{a}{b}\equiv 0 \pmod 2 \quad \text{if and only if} \quad a_i<b_i \text{ for some } 0\le i\le s. \tag{56}$$

Now the proof of the proposition follows in several steps by considering several independent cases.

First we establish the claim in the case when $a=1$, that is for $n=2(2b+1)$. In this case the equality (55), for $1\le k\le 3=2^{a+1}-1$, implies that

$$Q_{0,1}=0, \qquad P_1=t^2+Q_{0,2}, \qquad t\cdot P_1=Q_{0,3}+Q_{1,2},$$

as claimed by the proposition

Now let us assume that n is fixed and $a\ge 2$, meaning $4\mid n$, and let $1\le k\le 2^{a+1}-1$. We prove the proposition in this case using induction on k, where for the induction hypothesis we take that the claim is true for all values smaller than k. We investigate several cases.

- Let $k=1$. From (55) we get $Q_{0,1}+\binom{n}{1}t=0$. Since n is even this simplifies to the equality $Q_{0,1}=0$, which is in compliance with (51).
- Let $k=2$. The equality (55) now reads:
$$w_2(\xi\wr\mathbb{Z}/2)=Q_{0,2}+\binom{n}{2}t^2+\binom{n-1}{0}P_1=Q_{0,2}+P_1=0$$
confirming (52).
- Let k be odd, and let $k\le 2^a-1$. Assume that the claim holds for all values $1,2,\cdots,k-1$. Since $t\cdot Q_{i,j}=0$ for all values of i and j, and, by inductive hypothesis, all P_i are sum of Q's, for $1\le i\le k-1$, we conclude that $t\cdot P_i=0$. Therefore, in the second sum of the formula (55) the only possibly non-zero summands are for $i=0$ or $k-2i=0$. Because k is odd the second possibility is not possible. On the other hand the first summand is zero since for $i=0$ the binomial coefficient $\binom{n}{k}=0$. Thus we obtained (51).
- Let k be even, and let $k\le 2^a-2$. With the same reasoning as in the case for k odd, all summands in the second sum of the formula (55) are zero except when $i=0$ and $k-2i=0$. For $i=0$ the coefficient is equal to $\binom{n}{k}$, but the last a digits in dyadic expansion of n are zeros, while this is not the case for k since $k<2^a$. Therefore, the coefficient $\binom{n}{k}$ vanishes. For $k-2i=0$ the corresponding coefficient is 1 and we have verified (52).
- Let $k=2^a$. As before, all summands in the second sum of the formula (55) are zero except for $i=0$ and $k-2i=0$. For $i=0$ the binomial coefficient is equal to $\binom{n}{k}=\binom{2^a(2b+1)}{2^a}=1$ with the corresponding summand equal to t^{2^a}. For $k-2i=0$, that is $i=2^{a-1}$, the related summand is $P_{2^{a-1}}$. Consequently, we get (53) in the case $r=a-1$.
- Let k be odd, and let $2^a+1\le k\le 2^{a+1}-3$. Then there exists a unique integer $r\in\{1,2,\cdots,a-1\}$ such that $2^{a+1}-2^{r+1}+1\le k\le 2^{a+1}-2^r-1$. Now in the second sum of the formula (55), using inductive hypothesis and the fact that $t\cdot Q_{i,j}=0$, we conclude that possible non-zero summands are obtained only for $i\in\{0,2^a-2^{a-1},\cdots,2^a-2^{r+1}\}$. The binomial coefficients of those summands are equal to $\binom{n-2^a+2^j}{k-2^{a+1}+2^{j+1}}$, for some $j=r,r+1,\dots,a$. All of them vanish since $n-2^a+2^j$ is even and $k-2^{a+1}+2^{j+1}$ is odd. So we get (51).
- Let k be even, $k\ge 2^a+2$ with $k\notin\{2^{a+1}-2,2^{a+1}-2^2,\dots,2^{a+1}-2^{a-1}\}$. Then there exists a unique integer $r\in\{0,1,\cdots,a-1\}$ such that $2^{a+1}-2^{r+1}+2\le k\le 2^{a+1}-2^r-2$. As before, using inductive hypothesis, all summands from the second sum of the formula (55) which are possibly non-zero are obtained for $i\in\{0,\frac{k}{2},2^a-2^{a-1},\cdots,2^a-2^r\}$. We analyze three different cases.
  - For $i=0$ the binomial coefficient is $\binom{n}{k}$ and is equal to zero since $2^a\nmid k$.
  - For $i=\frac{k}{2}$ the binomial coefficient is equal to 1 while the corresponding summand is equal to $P_{\frac{k}{2}}$.

– For $i = 2^a - 2^j$ where $j \in \{r, r+1, \cdots, a-1\}$, the binomial coefficient is equal to $\binom{n-2^a+2^j}{k-2^{a+1}+2^{j+1}}$. In this case we have to examine dyadic expansion of the numbers involved more closely. We get the following:

$$n - 2^a + 2^j = n_1 \dots n_s 0 \dots 010 \dots 0,$$

where 1 is on the $(j+1)$-st place from the right. It could happen that all $n_1, \dots, n_s$ are equal to 0. On the other hand since $2^{a+1} - 2^{r+1} + 2 \le k \le 2^{a+1} - 2^r - 2$ we have

$$k = 1 \dots 10 k_1 \dots k_r,$$

with $a - r$ 1's and some $k_s = 1$. Consequently,

$$k - 2^{a+1} + 2^{j+1} = 1 \dots 10 k_1 \dots k_r,$$

with $j - r$ 1's and some $k_s = 1$. Therefore, some of the first r coefficients from the right of the number $k - 2^{a+1} + 2^{j+1}$ is equal to 1, while all of the first r coefficients of the number $n - 2^a + 2^j$ are equal to 0, and we may conclude that all binomial coefficients have to vasnih.

Thus, we verified (52).

- Let $k = 2^{a+1} - 2^{r+1}$, and let $0 \le r \le a-2$. As before, using inductive hypothesis, we conclude that all summands from the second sum of the formula (55) which have a chance to be non-zero are obtained for $i \in \{0, \frac{k}{2}, 2^a - 2^{a-1}, \cdots, 2^a - 2^r\}$. We discuss three separate cases.
  – For $i = 0$ the binomial coefficient is equal to $\binom{n}{k}$ and is equal to 0, since dyadic expansion of k is $k = 1 \dots 10 \dots 0$, with $a - r$ ones and $r + 1$ zeros. Therefore, k has at least one non-zero entry in the first a places from the right, while n has only zeros.
  – For $i = \frac{k}{2} = 2^a - 2^r$, the binomial coefficient is 1 and the corresponding summand is $P_{2^a - 2^r}$.
  – For $i = 2^a - 2^j$, where $j \in \{r+1, \dots, a-1\}$, we have that the relevant binomial coefficient is $\binom{n-2^a+2^j}{2^{a+1}-2^{r+1}-2^{a+1}+2^{j+1}}$. The dyadic expansion of the top entry in the binomial coefficient is of the form

$$n - 2^a + 2^j = n_1 \dots n_s 0 \dots 010 \dots 0,$$

having $a - j$ and j zeros, respectively. On the other hand the lower entry has the following dyadic expansion

$$2^{a+1} - 2^{r+1} - 2^{a+1} + 2^{j+1} = 2^{j+1} - 2^{r+1} = 1 \dots 10 \dots 0,$$

with $j - r$ ones and $r + 1$ zeros. The only non-zero coefficient appears for $j = r + 1$ with the summand $t^{2^{r+1}} \cdot P_{2^a - 2^{r+1}}$. Hence we have

$$P_{2^a-2^r} = \sum_{i=\max\{0,k-n\}}^{2^a-2^r-1} Q_{i,k-i} + t^{2^{r+1}} \cdot P_{2^a-2^{r+1}}$$

Use inductive hypothesis for $P_{2^a-2^{r+1}}$ and get

$$P_{2^a-2^r} = \sum_{i=\max\{0,k-n\}}^{2^a-2^r-1} Q_{i,k-i} + t^{2^{r+1}} \left(t^{2^{a+1}-2^{r+2}} + \sum_{i=\max\{0,k-n\}}^{2^a-2^{r+1}-1} Q_{i,2^{a+1}-2^{r+2}-i} \right) = \sum_{i=\max\{0,k-n\}}^{2^a-2^r-1} Q_{i,2^{a+1}-2^{r+1}-i} + t^{2^{a+1}-2^{r+1}}.$$

We completed the proof of (53).

- Let $k = 2^{a+1} - 1$. Since k is odd in the second sum of the formula (55), if i is even, the binomial coefficient has to vanish since $n - i$ is even and $k - 2i$ is odd. If i is odd, all summands are zero unless $i = 2^a - 2^r$, for some $r \in \{0, 1, \dots, a-1\}$, because they are of the form $t \cdot P_i$ and for such i, P_i is a sum of Q's. Now let $i = 2^a - 2^r$, for some $0 \le r \le a-1$. In this case the binomial coefficient is equal to $\binom{n-2^a+2^r}{2^{a+1}-1-2^{a+1}+2^{r+1}}$ and dyadic expansions of the binomial coefficient are equal to:

$$n - 2^a + 2^r = n_1 \dots n_s 0 \dots 010 \dots 0,$$

where there are $a - r$ and r zeros, respectively; and

$$2^{a+1} - 1 - 2^{a+1} + 2^{r+1} = 2^{r+1} - 1 = 1 \dots 1,$$

with $r + 1$ ones. The only i for which the coefficient is equal to 1 is for $r = 0$, that is $i = 2^a - 1$. For such i, an appropriate summand is equal to $t \cdot P_{2^a-1}$, and we proved (54).

Since we cover all possibilities we have completed the proof of the proposition. □

Corollary 6.2. *Let $n \geq 2$ be an even integer, and let $a \geq 1$ and $b \geq 0$ be the unique integers such that $n = 2^a(2b+1)$. In the quotient algebra $H^*(X;\mathbb{F}_2)/\langle w_1(\xi \wr \mathbb{Z}/2), w_2(\xi \wr \mathbb{Z}/2), \dots, w_{2^{a+1}-1}(\xi \wr \mathbb{Z}/2)\rangle$ holds*

$$t^{2^{a+1}} = 0.$$

Proof. We have

$$t^{2^{a+1}} = t^2 \cdot t^{2^{a+1}-2} = t^2 \left(t^{2^{a+1}-2} + \sum_{i=\max\{0,2^{a+1}-2-n\}}^{2^a-2} Q_{i,2^{a+1}-2-i} \right).$$

Now we use (53), for $r = 0$, and (54) and get

$$t^{2^{a+1}} = t^2 \cdot P_{2^a-1} = t \left(\sum_{i=\max\{0,2^{a+1}-n-1\}}^{2^a-1} Q_{i,2^{a+1}-1-i} \right) = 0.$$

□

Finally we prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. We consider three different cases depending on value of integer $a \geq 0$.

Let $n \geq 1$ be an odd integer, that is $a = 0$. Since n is a multiple of 1 from Proposition 2.2 we get a $\mathbb{Z}/2$-map $\widetilde{g}\colon \widetilde{G}_1(\mathbb{R}^2) \longrightarrow \widetilde{G}_n(\mathbb{R}^{2n})$. Furthermore, applying Proposition 2.1 we have an additional $\mathbb{Z}/2$-map $c\colon \widetilde{G}_n(\mathbb{R}^{2n}) \longrightarrow G_n(\mathbb{R}^{2n})$. The composition map

$$\widetilde{G}_1(\mathbb{R}^2) \xrightarrow{\ \widetilde{g}\ } \widetilde{G}_n(\mathbb{R}^{2n}) \xrightarrow{\ c\ } G_n(\mathbb{R}^{2n}),$$

and the Corollary 2.7 yield the following relation between the indexes

$$\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_1(\mathbb{R}^2);\mathbb{F}_2) \supseteq \mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_n(\mathbb{R}^{2n});\mathbb{F}_2) \supseteq \mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n});\mathbb{F}_2) = \langle t^2\rangle.$$

Since $\widetilde{G}_1(\mathbb{R}^2) \cong \mathrm{SO}(2)$ is a sphere S^1 equipped with a free $\mathbb{Z}/2$ index we have that $\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_1(\mathbb{R}^2);\mathbb{F}_2) = \langle t^2\rangle$. Thus, we concluded the proof of Theorem 1.1 for n odd by showing that

$$\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_n(\mathbb{R}^{2n});\mathbb{F}_2) = \langle t^2\rangle.$$

Next, let $a = 1$, that is $n = 2(2b+1)$. From Proposition 6.1 with $k = 2$, using the fact that $w_1 = 0$ and consequently $Q_{1,j} = 0$, for all j, and $P_1 = 0$, we have that $t^2 = Q_{0,2}$. Since there are no other relations for $Q_{0,2}$ we conclude that $t^2 \neq 0$ and that $t^3 = tQ_{0,2} = 0$. Therefore,

$$\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_n(\mathbb{R}^{2n});\mathbb{F}_2) = \langle t^3\rangle.$$

Now consider the case $a \geq 2$.

First note that from Proposition 2.2 and using Corollary 6.2 we get

$$\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_{2^a}(\mathbb{R}^{2^{a+1}});\mathbb{F}_2) \supseteq \mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_n(\mathbb{R}^{2n});\mathbb{F}_2) \supseteq \langle t^{2^{a+1}}\rangle.$$

Therefore, it is enough to prove the Theorem 1.1 for $n = 2^a$.

Let $n = 2^a$. As we know the index is generated by the first power of t which is equal to zero in the quotient

$$H^*(W;\mathbb{F}_2)/\langle w_2(\xi \wr \mathbb{Z}/2), \dots, w_{2^{a+1}}(\xi \wr \mathbb{Z}/2)\rangle.$$

What is left to be proved is that

$$t^{2^{a+1}-1} \notin \langle w_2(\xi \wr \mathbb{Z}/2), \dots, w_{2^{a+1}}(\xi \wr \mathbb{Z}/2)\rangle.$$

In order to simplify calculation we will expand the ideal with more generators and show that $t^{2^{a+1}-1}$ is not in the bigger ideal. Let

$$I := \langle \{w_2(\xi \wr \mathbb{Z}/2), \dots, w_{2^{a+1}}(\xi \wr \mathbb{Z}/2)\} \cup \{Q_{i,j} | 1 \leq i+j \leq 2^{a+1}-2\}\rangle.$$

In the quotient $H^*(W;\mathbb{F}_2)/I$ we have from Proposition 6.1 that relations from ideal I are equivalent with the following

$$\begin{aligned} P_k &= 0, && \text{for } 1 \leq k \leq 2^a-1 \text{ and } k \notin \{2^a-2^{a-1}, 2^a-2^{a-2}, \dots, 2^a-1\}, \\ P_k &= t^{2k}, && \text{for } 1 \leq k \leq 2^a-1 \text{ and } k \in \{2^a-2^{a-1}, 2^a-2^{a-2}, \dots, 2^a-1\}, \end{aligned}$$

and

$$t \cdot P_{2^a-1} = Q_{2^a-1,2^a}.$$

Now from (50) and $w_{2^{a+1}} = 0$ we get
$$P_{2^a} = 0.$$
It follows that all $t^k \neq 0$ for $1 \leq k \leq 2^{a+1} - 2$ including $t^{2^{a+1}-2} = P_{2^a-1}$ and that
$$t^{2^{a+1}-1} = t \cdot P_{2^a-1} = Q_{2^a-1,2^a}.$$
Since this is the only relation involving $Q_{2^a-1,2^a}$, we get that the right hand side is not equal to 0. So $t^{2^{a+1}-1} \notin I$ and it does not belong in the smaller ideal $\langle w_2(\xi \wr \mathbb{Z}/2), \dots, w_{2^{a+1}}(\xi \wr \mathbb{Z}/2)\rangle$ as well.

Therefore, using Corollary 6.2 we get
$$\mathrm{Index}_{\mathbb{Z}/2}(\widetilde{G}_{2^a}(\mathbb{R}^{2^{a+1}}); \mathbb{F}_2) = \langle t^{2^{a+1}}\rangle.$$
This concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. The proof is given in three steps depending on the value of integer a. For $a = 0$, that is n odd, the statement of the theorem is established in Corollary 2.7.

Let $a = 1$, that is $n = 2(2b + 1)$. First note that from Corollary 6.2 we have that $t^4 = 0$ that is
$$\mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n}); \mathbb{F}_2) \supseteq \langle t^4\rangle.$$
On the other hand, from Proposition 2.2 and Proposition 2.6, or Example 5.2, we get
$$\mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n}); \mathbb{F}_2) \subseteq \mathrm{Index}_{\mathbb{Z}/2}(G_2(\mathbb{R}^4); \mathbb{F}_2) = \langle t^4\rangle.$$
Therefore
$$\mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n}); \mathbb{F}_2) = \langle t^4\rangle.$$

For the case $a \geq 2$ the proof is exactly the same as in the proof of the Theorem 1.1 but we present it nevertheless.

First note that from Proposition 2.2 and using Corollary 6.2 we get
$$\mathrm{Index}_{\mathbb{Z}/2}(G_{2^a}(\mathbb{R}^{2^{a+1}}); \mathbb{F}_2) \supseteq \mathrm{Index}_{\mathbb{Z}/2}(G_n(\mathbb{R}^{2n}); \mathbb{F}_2) \supseteq \langle t^{2^{a+1}}\rangle.$$
Therefore, it is enough to prove the Theorem 1.1 for $n = 2^a$.

Let $n = 2^a$. The index is generated by the first power of t which is equal to zero in the quotient
$$H^*(U; \mathbb{F}_2)/\langle w_1(\xi \wr \mathbb{Z}/2), \dots, w_{2^{a+1}}(\xi \wr \mathbb{Z}/2)\rangle.$$
Thus, what is left to be proved is that
$$t^{2^{a+1}-1} \notin \langle w_1(\xi \wr \mathbb{Z}/2), \dots, w_{2^{a+1}}(\xi \wr \mathbb{Z}/2)\rangle.$$
In order to simplify calculation we will expand the ideal with more generators and show that $t^{2^{a+1}-1}$ is not in the bigger ideal. Let
$$I := \langle \{w_1(\xi \wr \mathbb{Z}/2), \dots, w_{2^{a+1}}(\xi \wr \mathbb{Z}/2)\} \cup \{Q_{i,j} | 1 \leq i + j \leq 2^{a+1} - 2\}\rangle.$$
In the quotient $H^*(U; \mathbb{F}_2)/I$ we have from Proposition 6.1 that relations from ideal I are equivalent with the following
$$\begin{aligned} P_k &= 0, && \text{for } 1 \leq k \leq 2^a - 1 \text{ and } k \notin \{2^a - 2^{a-1}, 2^a - 2^{a-2}, \dots, 2^a - 1\},\\ P_k &= t^{2k}, && \text{for } 1 \leq k \leq 2^a - 1 \text{ and } k \in \{2^a - 2^{a-1}, 2^a - 2^{a-2}, \dots, 2^a - 1\}, \end{aligned}$$
and
$$t \cdot P_{2^a-1} = Q_{2^a-1,2^a}.$$
Next from (50) and $w_{2^{a+1}} = 0$ we get
$$P_{2^a} = 0.$$
It follows that all $t^k \neq 0$ for $1 \leq k \leq 2^{a+1} - 2$ including $t^{2^{a+1}-2} = P_{2^a-1}$ and that
$$t^{2^{a+1}-1} = t \cdot P_{2^a-1} = Q_{2^a-1,2^a}.$$
Since this is the only relation involving $Q_{2^a-1,2^a}$, we get that the right hand side is not equal to 0. So $t^{2^{a+1}-1} \notin I$ and it does not belong in the smaller ideal $\langle w_1(\xi \wr \mathbb{Z}/2), \dots, w_{2^{a+1}}(\xi \wr \mathbb{Z}/2)\rangle$ as well. Therefore, using Corollary 6.2 we get
$$\mathrm{Index}_{\mathbb{Z}/2}(G_{2^a}(\mathbb{R}^{2^{a+1}}); \mathbb{F}_2) = \langle t^{2^{a+1}}\rangle.$$
This concludes the proof of Theorem 1.2. □

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