Symmetry properties of finite sums involving generalized Fibonacci numbers

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Abstract

We extend a result of I. J. Good and prove more symmetry properties of sums involving generalized Fibonacci numbers.

1 Introduction

The generalized Fibonacci numbers G_i, $i \geq 0$, with which we are mainly concerned in this paper, are defined through the second order recurrence relation $G_{i+1} = G_i + G_{i-1}$, where the seeds G_0 and G_1 need to be specified. As particular cases, when $G_0 = 0$ and $G_1 = 1$, we have the Fibonacci numbers, denoted F_i, while when $G_0 = 2$ and $G_1 = 1$, we have the Lucas numbers, L_i.

I. J. Good [1] proved the symmetry property:

$$F_q \sum_{k=1}^{n} \frac{(-1)^k}{G_k G_{k+q}} = F_n \sum_{k=1}^{q} \frac{(-1)^k}{G_k G_{k+n}}\,, \tag{1.1}$$

where q and n are nonnegative integers, and all the numbers G_1, G_2, $\ldots$, G_{n+q} are nonzero.

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The identity (1.1) is a particular case (corresponding to setting $p=1$) of the following result, to be proved in this present paper:

$$F_{pq}\sum_{k=1}^{n}\frac{(-1)^{pk}}{G_{pk}G_{pk+pq}}=F_{pn}\sum_{k=1}^{q}\frac{(-1)^{pk}}{G_{pk}G_{pk+pn}}\,, \tag{1.2}$$

where q, p and n are nonnegative integers, and all the numbers G_p, G_{2p}, $\ldots$, G_{pn+pq} are nonzero.

In the limit as n approaches infinity, and specializing to Fibonacci numbers, the identity (1.2) gives

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{pk}}{F_{pk}F_{pk+pq}}&=\frac{1}{F_{pq}}\sum_{k=1}^{q}\left\{\frac{(-1)^{pk}}{F_{pk}}\lim_{n\to\infty}\left(\frac{F_{pn}}{F_{pk+pn}}\right)\right\}\\&=\frac{1}{F_{pq}}\sum_{k=1}^{q}\frac{(-1)^{pk}}{\phi^{pk}F_{pk}}\,,\end{aligned} \tag{1.3}$$

where $\phi=(1+\sqrt{5})/2$ is the golden ratio.

The identity (1.3) generalizes Bruckman and Good's result (identity (19) of [2], which corresponds to setting $q=1$ in (1.3)).

In sections 3.1 – 3.3 we will prove identity (1.2) and discover more symmetry properties of sums involving generalized Fibonacci numbers. In section 3.4 we shall extend the discussion to Horadam sequences W_i and U_i by proving

$$U_{pq}\sum_{k=1}^{n}\frac{Q^{pk}}{W_{pk}W_{pk+pq}}=U_{pn}\sum_{k=1}^{q}\frac{Q^{pk}}{W_{pk}W_{pk+pn}} \tag{1.4}$$

and

$$U_{2pq}\sum_{k=1}^{2n}\frac{(\pm Q^{p})^{k}}{W_{pk}W_{pk+2pq}}=U_{2pn}\sum_{k=1}^{2q}\frac{(\pm Q^{p})^{k}}{W_{pk}W_{pk+2pn}}\,, \tag{1.5}$$

for integers p, q, Q and n, thereby extending André-Jeannin's result (Theorem 1 of [6]) and further generalizing the identity (1.2).

2 Required identities

2.1 Telescoping summation identities

The following telescoping summation identities are special cases of the more general identities proved in [3].

Lemma 2.1. *If $f(k)$ is a real sequence and u, v and w are positive integers, then*

$$\sum_{k=1}^{w}\left[f(uk+uv)-f(uk)\right]=\sum_{k=1}^{v}\left[f(uk+uw)-f(uk)\right].$$

Lemma 2.2. *If $f(k)$ is a real sequence and u, v and w are positive integers such that v is even and w is even, then*

$$\sum_{k=1}^{w}(\pm 1)^{k-1}\left(f(uk+uv)-f(uk)\right)=\sum_{k=1}^{v}(\pm 1)^{k-1}\left(f(uk+uw)-f(uk)\right).$$

Lemma 2.3. *If $f(k)$ is a real sequence and u, v and w are positive integers such that vw is odd, then*

$$\sum_{k=1}^{w}(-1)^{k-1}\left(f(uk+uv)+f(uk)\right)=\sum_{k=1}^{v}(-1)^{k-1}\left(f(uk+uw)+f(uk)\right).$$

2.2 Product of a Fibonacci number and a generalized Fibonacci number

Lemma 2.4 (Howard [5], Corollary 3.5)**.** *For integers a, b, c,*

$$F_aG_{2b+a+c}=\begin{cases}F_{a+b}G_{a+b+c}-F_bG_{b+c} & \text{if } a \text{ is even,}\\ F_{a+b}G_{a+b+c}+F_bG_{b+c} & \text{if } a \text{ is odd.}\end{cases}$$

2.3 Product of a Lucas number and a generalized Fibonacci number

Lemma 2.5 (Vajda [4], Formula 10a)**.** *For integers a, b,*

$$L_aG_b=\begin{cases}G_{b+a}+G_{b-a} & \text{if } a \text{ is even,}\\ G_{b+a}-G_{b-a} & \text{if } a \text{ is odd.}\end{cases}$$

2.4 Difference of products of a Fibonacci number and a generalized Fibonacci number

Lemma 2.6 (Vajda [4], Formula 21)**.** *For integers a, b,*

$$F_bG_a - F_aG_b = (-1)^aG_0F_{b-a}\,.$$

3 Main Results: Symmetry properties

3.1 Sums of products of reciprocals

Theorem 3.1. *If n and q are nonnegative integers and p is a nonzero integer, then*

$$F_{pq}\sum_{k=1}^{n}\frac{(-1)^{pk}}{G_{pk}G_{pk+pq}} = F_{pn}\sum_{k=1}^{q}\frac{(-1)^{pk}}{G_{pk}G_{pk+pn}}\,.$$

Proof. Dividing through the identity in Lemma 2.6 by G_aG_b and setting $b = pk + pq$ and $a = pk$, we have:

$$\frac{F_{pk+pq}}{G_{pk+pq}} - \frac{F_{pk}}{G_{pk}} = (-1)^{pk}\frac{G_0F_{pq}}{G_{pk}G_{pk+pq}}\,. \tag{3.1}$$

Similarly,

$$\frac{F_{pk+pn}}{G_{pk+pn}} - \frac{F_{pk}}{G_{pk}} = (-1)^{pk}\frac{G_0F_{pn}}{G_{pk}G_{pk+pn}}\,. \tag{3.2}$$

We now use the sequence $f(k) = F_k/G_k$ in Lemma 2.1 with $u = p$, $v = q$ and $w = n$, while taking into consideration identities (3.1) and (3.2). □

Theorem 3.2. *If n and q are nonnegative <u>even</u> integers and p is a nonzero integer, then*

$$F_{pq}\sum_{k=1}^{n}\frac{(\pm 1)^{k(p-1)}}{G_{pk}G_{pk+pq}} = F_{pn}\sum_{k=1}^{q}\frac{(\pm 1)^{k(p-1)}}{G_{pk}G_{pk+pn}}\,.$$

Proof. We use the sequence $f(k) = F_k/G_k$ in Lemma 2.2 with $u = p$, $v = q$ and $w = n$. □

3.2 First-power sums

Theorem 3.3. *If p, q, n and t are integers such that pqn is odd, then*

$$L_{pq}\sum_{k=1}^{2n}(\pm 1)^{k-1}G_{pk+pq+t}=L_{pn}\sum_{k=1}^{2q}(\pm 1)^{k-1}G_{pk+pn+t}\,, \tag{3.3}$$

$$L_{pq}\sum_{k=1}^{n}G_{2pk+pq+t}=L_{pn}\sum_{k=1}^{q}G_{2pk+pn+t}\,. \tag{3.4}$$

Proof. Consider the generalized Fibonacci sequence $f(k)=G_{k+t}$. If we choose $u=p$, $v=2q$ and $w=2n$, then Lemma 2.2 gives

$$\sum_{k=1}^{2n}(\pm 1)^{k-1}\left(G_{pk+2pq+t}-G_{pk+t}\right)=\sum_{k=1}^{2q}(\pm 1)^{k-1}\left(G_{pk+2pn+t}-G_{pk+t}\right). \tag{3.5}$$

But from the second identity of Lemma 2.5 we have

$$G_{pk+2pq+t}-G_{pk+t}=L_{pq}G_{pk+pq+t},\quad pq\ \text{odd}\,, \tag{3.6}$$

and

$$G_{pk+2pn+t}-G_{pk+t}=L_{pn}G_{pk+pn+t},\quad pn\ \text{odd}\,. \tag{3.7}$$

Using (3.6) and (3.7) in (3.5), identity (3.3) is proved.

The proof of identity (3.4) is similar, we use the sequence $f(k)=G_{2k+t}$ in Lemma 2.1 with $u=2p$, $v=q$ and $w=n$.

□

Theorem 3.4. *If p, q, n and t are integers such that pqn is odd or q and n are even, then*

$$F_{pq}\sum_{k=1}^{n}(-1)^{k-1}G_{2pk+pq+t}=F_{pn}\sum_{k=1}^{q}(-1)^{k-1}G_{2pk+pn+t}\,.$$

Proof. Consider the sequence $f(k)=F_kG_{k+t}$. If we choose $u=p$, $v=q$ and $w=n$, then Lemma 2.3 gives

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^{k-1}\left(F_{pk+pq}G_{pk+pq+t}+F_{pk}G_{pk+t}\right)\\&\quad=\sum_{k=1}^{q}(-1)^{k-1}\left(F_{pk+pn}G_{pk+pn+t}+F_{pk}G_{pk+t}\right).\end{aligned} \tag{3.8}$$

From the second identity of Lemma 2.4 we have

$$F_{pk+pq}G_{pk+pq+t} + F_{pk}G_{pk+t} = F_{pq}G_{2pk+pq+t}, \quad pq \text{ odd}\,, \tag{3.9}$$

and

$$F_{pk+pn}G_{pk+pn+t} + F_{pk}G_{pk+t} = F_{pn}G_{2pk+pn+t}, \quad pn \text{ odd}\,. \tag{3.10}$$

The theorem then follows from using (3.9) and (3.10) in (3.8). If q and n are even then we use $f(k) = F_kG_{k+t}$ with $u = p$, $v = q$ and $w = n$ in Lemma 2.2 together with the first identity of Lemma 2.4. □

Theorem 3.5. *If p, q, n and t are integers such that p is even or q and n are even, then*

$$F_{pq}\sum_{k=1}^{n} G_{2pk+pq+t} = F_{pn}\sum_{k=1}^{q} G_{2pk+pn+t}\,.$$

Proof. Consider the sequence $f(k) = F_kG_{k+t}$. Lemma 2.1 with $u = p$, $v = q$ and $w = n$ gives

$$\begin{aligned}&\sum_{k=1}^{n}\left(F_{pk+pq}G_{pk+pq+t} - F_{pk}G_{pk+t}\right)\\ &\qquad = \sum_{k=1}^{q}\left(F_{pk+pn}G_{pk+pn+t} - F_{pk}G_{pk+t}\right).\end{aligned} \tag{3.11}$$

From the first identity of Lemma 2.4 we have

$$F_{pk+pq}G_{pk+pq+t} - F_{pk}G_{pk+t} = F_{pq}G_{2pk+pq+t}, \quad pq \text{ even}\,, \tag{3.12}$$

and

$$F_{pk+pn}G_{pk+pn+t} - F_{pk}G_{pk+t} = F_{pn}G_{2pk+pn+t}, \quad pn \text{ even}\,. \tag{3.13}$$

Using (3.12) and (3.13) in (3.11), Theorem 3.5 is proved. □

Theorem 3.6. *If p, q, n and t are integers such that p is even, then*

$$F_{pq}\sum_{k=1}^{2n} (\pm 1)^{k-1}G_{pk+pq+t} = F_{pn}\sum_{k=1}^{2q} (\pm 1)^{k-1}G_{pk+pn+t}\,.$$

Proof. Consider the sequence $f(k) = F_k G_{k+t}$. Lemma 2.2 with $u = p$, $v = 2q$ and $w = 2n$ gives

$$\begin{aligned}\sum_{k=1}^{2n} (\pm 1)^{k-1} \left(F_{pk+2pq} G_{pk+2pq+t} - F_{pk} G_{pk+t}\right)\\ = \sum_{k=1}^{2q} (\pm 1)^{k-1} \left(F_{pk+2pn} G_{pk+2pn+t} - F_{pk} G_{pk+t}\right).\end{aligned} \tag{3.14}$$

From identities (3.12) and (3.13) we have

$$F_{pk+2pq} G_{pk+2pq+t} - F_{pk} G_{pk+t} = F_{2pq} G_{2pk+2pq+t}\,, \tag{3.15}$$

and

$$F_{pk+2pn} G_{pk+2pn+t} - F_{pk} G_{pk+t} = F_{2pn} G_{2pk+2pn+t}\,. \tag{3.16}$$

Using (3.15) and (3.16) in (3.14), Theorem 3.6 is proved. □

Theorem 3.7. *If p, q, n and t are integers such that p is even and nq is odd, then*

$$L_{pq} \sum_{k=1}^{n} (-1)^{k-1} G_{2pk+pq+t} = L_{pn} \sum_{k=1}^{q} (-1)^{k-1} G_{2pk+pn+t}\,,$$

Proof. Consider the sequence $f(k) = G_{2k+t}$. If we choose $u = 2p$, $v = q$ and $w = n$, then Lemma 2.3 gives

$$\begin{aligned}\sum_{k=1}^{n} (-1)^{k-1} \left(G_{2pk+2pq+t} + G_{2pk+t}\right)\\ = \sum_{k=1}^{q} (-1)^{k-1} \left(G_{2pk+2pn+t} + G_{2pk+t}\right), \quad nq \text{ odd}\,.\end{aligned} \tag{3.17}$$

From the first identity in Lemma 2.5, we have

$$G_{2pk+2pq+t} + G_{2pk+t} = L_{pq} G_{2pk+pq+t}, \quad pq \text{ even}\,, \tag{3.18}$$

and

$$G_{2pk+2pn+t} + G_{2pk+t} = L_{pn} G_{2pk+pn+t}, \quad pn \text{ even}\,. \tag{3.19}$$

Using (3.18) and (3.19) in (3.17), Theorem 3.7 is proved. □

3.3 More sums involving products of reciprocals

Theorem 3.8. *If p, q, n and t are positive integers such that pnq is odd, then*

$$L_{pq}\sum_{k=1}^{2n}\frac{(\pm 1)^{k-1}G_{pk+pq+t}}{G_{pk+t}G_{pk+2pq+t}}=L_{pn}\sum_{k=1}^{2q}\frac{(\pm 1)^{k-1}G_{pk+pn+t}}{G_{pk+t}G_{pk+2pn+t}}\,, \tag{3.20}$$

$$L_{pq}\sum_{k=1}^{n}\frac{G_{2pk+pq+t}}{G_{2pk+t}G_{2pk+2pq+t}}=L_{pn}\sum_{k=1}^{q}\frac{G_{2pk+pn+t}}{G_{2pk+t}G_{2pk+2pn+t}}\,. \tag{3.21}$$

Proof. Use of $f(k)=1/G_{k+t}$ in Lemma 2.2 with $u=p$, $v=2q$ and $w=2n$, noting the identites (3.6) and (3.7) proves identity (3.20). To prove identity (3.21), we use $f(k)=1/G_{2k+t}$ in Lemma 2.1 with $u=p$, $v=q$ and $w=n$, together with the second identity in Lemma 2.5. □

Theorem 3.9. *If p, q, n and t are positive integers such that p is even and nq is odd, then*

$$L_{pq}\sum_{k=1}^{n}\frac{(-1)^{k-1}G_{2pk+pq+t}}{G_{2pk+t}G_{2pk+2pq+t}}=L_{pn}\sum_{k=1}^{q}\frac{(-1)^{k-1}G_{2pk+pn+t}}{G_{2pk+t}G_{2pk+2pn+t}}\,.$$

Proof. Use $f(k)=1/G_{2k+t}$ in Lemma 2.3 with $u=p$, $v=q$ and $w=n$, employing the identities (3.18) and (3.19). □

Theorem 3.10. *If p, q, n and t are positive integers such that p is even or n and q are even, then*

$$F_{pq}\sum_{k=1}^{n}\frac{G_{2pk+pq+t}}{F_{pk}G_{pk+t}F_{pk+pq}G_{pk+pq+t}}=F_{pn}\sum_{k=1}^{q}\frac{G_{2pk+pn+t}}{F_{pk}G_{pk+t}F_{pk+pn}G_{pk+pn+t}}\,.$$

Proof. Use $f(k)=1/(F_kG_{k+t})$ in Lemma 2.1 with $u=p$, $v=q$ and $w=n$, while taking cognisance of the following identities which follow from identities (3.12) and (3.13):

$$\frac{1}{F_{pk}G_{pk+t}}-\frac{1}{F_{pk+pq}G_{pk+pq+t}}=\frac{F_{pq}G_{2pk+pq+t}}{F_{pk}G_{pk+t}F_{pk+pq}G_{pk+pq+t}},\quad pq\text{ even}\,, \tag{3.22}$$

and

$$\frac{1}{F_{pk}G_{pk+t}} - \frac{1}{F_{pk+pn}G_{pk+pn+t}} = \frac{F_{pn}G_{2pk+pn+t}}{F_{pk}G_{pk+t}F_{pk+pn}G_{pk+pn+t}}, \quad pn \text{ even}\,. \tag{3.23}$$

□

Theorem 3.11. *If p, q, n and t are positive integers such that p is odd or n and q are even, then*

$$F_{pq}\sum_{k=1}^{n}\frac{(-1)^{k-1}G_{2pk+pq+t}}{F_{pk}G_{pk+t}F_{pk+pq}G_{pk+pq+t}} = F_{pn}\sum_{k=1}^{q}\frac{(-1)^{k-1}G_{2pk+pn+t}}{F_{pk}G_{pk+t}F_{pk+pn}G_{pk+pn+t}}\,.$$

3.4 Horadam sequence

Some of the above results can be extended to the Horadam sequence [7], $\{W_i\} = \{W_i(a,b;P,Q)\}$ defined by

$$W_0 = a, W_1 = b, W_i = PW_{i-1} - QW_{i-2}\,, (i > 2)\,, \tag{3.24}$$

where a, b, P, and Q are integers, with $PQ \neq 0$ and $\Delta = P^2 - 4Q > 0$. We define the sequence $\{U_i\}$ by $U_i = W_i(0,1;P,Q)$ and note also that our sequence $\{G_i\}$ is given by $G_i = W_i(G_0,G_1;1,-1)$. It is readily established that [7, 6]:

$$W_i = \frac{a\alpha^i - b\beta^i}{\alpha - \beta}\,, \tag{3.25}$$

where $\alpha = (P+\sqrt{\Delta})/2$, $\beta = (P-\sqrt{\Delta})/2$, $A = b - \beta a$ and $B = b - \alpha a$.

Theorem 3.12. *If n and q are nonnegative integers and p is a nonzero integer, then*

$$U_{pq}\sum_{k=1}^{n}\frac{Q^{pk}}{W_{pk}W_{pk+pq}} = U_{pn}\sum_{k=1}^{q}\frac{Q^{pk}}{W_{pk}W_{pk+pn}}\,.$$

Note that when $p = 1$, Theorem 3.12 reduces to Theorem 1 of [6].

Proof. Since n and k in identity (4.1) of [6] are arbitrary nonnegative integers, we substitute pk for n and pq for k in the identity, obtaining

$$\frac{\beta^{pk}}{W_{pk}} - \frac{\beta^{pk+pq}}{W_{pk+pq}} = \frac{AQ^{pk}U_{pq}}{W_{pk}W_{pk+pq}}\,. \tag{3.26}$$

The theorem now follows by choosing $f(k) = \beta^k/W_k$ in Lemma 2.1 with $w = n$, $u = p$ and $v = q$ while making use of (3.26). $\square$

Theorem 3.13. *If n and q are nonnegative <u>even</u> integers and p is a nonzero integer, then*

$$U_{pq}\sum_{k=1}^{n}\frac{(\pm Q^p)^k}{W_{pk}W_{pk+pq}} = U_{pn}\sum_{k=1}^{q}\frac{(\pm Q^p)^k}{W_{pk}W_{pk+pn}}\,.$$

Proof. The theorem follows by choosing $f(k) = \beta^k/W_k$ in Lemma 2.2 with $w = n$, $u = p$ and $v = q$, while making use of (3.26). $\square$

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