CHOW GROUPS OF CONIC BUNDLES IN $\mathbb{P}^5$ AND THE GENERALISED BLOCH'S CONJECTURE

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ABSTRACT. Consider the Fano surface of a conic bundle embedded in $\mathbb{P}^5$. Let i denote the natural involution acting on this surface. In this note we provide an obstruction to the identity action of the involution on the group of algebraically trivial zero cycles modulo rational equivalence on the surface.

1. INTRODUCTION

One of the very important problems in algebraic geometry is to understand the Chow group of zero cycles on a smooth projective surface with geometric genus equal to 0. The conjecture due to Bloch asserts that such a surface has Chow group of zero cycles isomorphic to the group of integers. The Bloch's conjecture has been studied and proved in the case when the surface is not of general type by [BKL] and for surfaces of general type by [B], [IM], [V],[VC]. Inspired by the Bloch's conjecture the following conjecture is a generalisation of it [Vo][conjecture 11.19].

Conjecture : Let S be a smooth projective surface over the field of complex numbers and let Γ be a codimension two cycle on $S \times S$. Suppose that Γ^* acts as zero on the space of globally holomorphic two forms on S, then Γ_* acts as zero on the kernel of the albanese map from $\mathrm{CH}_0(S)$ to $Alb(S)$.

This conjecture was studied in detail when the correspondence Γ is the $\Delta - Graph(i)$, where i is an involution on a $K3$ surface by [GT], [HK],[Voi]. In the example of K3 surfaces we have that the push-forward induced by the involution acts as identity on Chow groups.

Inspired by this conjecture we consider the following question in this article. Let X be a cubic fourfold in $\mathbb{P}^5$. Then projecting from a line l we have a conic bundle structure on the cubic. Let S be the discriminant surface of this conic bundle structure. Let T be the double cover of S inside the Fano variety of lines $F(X)$ of X. Then T has a natural involution and we observe that the group of algebraically trivial zero cycles on T modulo

rational equivalence (denoted by $A_0(T)$) maps surjectively onto the algebraically trivial one cycles on X modulo rational equivalence (denoted by $A_1(X)$). Since the action of the involution has as its invariant part equal to the $A_0(S)$ and as anti-invariant part equal to $A_1(X)$. The involution cannot act as $+1$ on the group $A_1(X)$. Now the question is what is the obstruction to the $+1$ action of the involution in terms of the geometry of S, T.

Theorem 1.1. *Let S be the discriminant surface as mentioned above. Then for any very ample line bundle L on S we cannot have the equality*

$$L^2 - g + 1 = g + n$$

where g is the genus of the curve in the linear system of L and n is a positive integer.

For the proof we follow the approach of the proof for the example of K3 surfaces due to Voisin as in [Voi]. The proof involves two steps. First is that we invoke the notion of finite dimensionality in the sense of Roitman as in [R1] and prove that the finite dimensionality of the image of a homomorphism from $A_0(T)$ to $A_1(X)$ means that the homomorphism factors through the albanese map $A_0(T) \rightarrow Alb(T)$. The second step is to show that, if we have the equality as above 1.1, then the image of the homomorphism induced by the difference of the diagonal and the graph of the involution from $A_0(T)$ to $A_1(X)$ is finite dimensional, yielding the $+1$ action of the involution on $A_1(X)$.

Assumption: We work over the field of complex numbers.

2. Finite dimensionality in the sense of Roitman and one-cycles on cubic fourfolds

Let P be a subgroup of the group of algebraically trivial one cycles modulo on a smooth projective fourfold X, denoted by $A_1(X)$. Then following [R1], we say that the subgroup P is finite dimensional if there exists a smooth projective variety W, and a correspondence Γ on $W \times S$, of correct codimension such that P is contained in the set $\Gamma_*(W)$.

Let X be a cubic fourfold, then take a line l on X and project from l onto $\mathbb{P}^3$. If we blow up X along l, then the blow up X_l has a conic bundle structure over $\mathbb{P}^3$. Let S be the surface in $\mathbb{P}^3$ such that for any point on S, the inverse image is the union of two planes in $\mathbb{P}^3$. Then for a hyperplane section X_t of X, we get the corresponding discriminant curve S_t, inverse image of a point of S_t is a union of two lines. So we get the curve T_t, inside $F(X_t)$, which is a two sheeted cover of S_t. The curves S_t fits together to give the surface S. Let T be the variety in $F(X)$ which is the double cover of S. Precisely it means the following. Let us consider $\{(x, t) : x \in T_t\}$ inside $F(X) \times \mathbb{P}^5$, its projection to $F(X)$ is T and we have a 2:1 map from T to S. So T is surface.

Now we have that, for a hyperplane section X_t, let l_1, l_2 be two lines on X_t. By general position argument these two lines can be disjoint from l and they are contained in $\mathbb{P}^2$, so under the projection they are mapped to two rational curves in $\mathbb{P}^2$. SO by Bezout's theorem they must intersect at a point z, so the inverse image of z under the projection are two given lines l_1, l_2, which tells us that the map from $A_0(T_t)$ to $A_1(X_t)$ is onto, which in turn says that $A_0(T)$ to $A_1(X)$ is onto, because $A_1(X)$ is generated by $A_1(X_t)$'s.

Theorem 2.1. *Let Z be a correspondence supported on $T \times X$, suppose that the image of Z_* from $A_0(T)$ to $A_1(X)$ is finite dimensional. Then Z_* factors through the albanese map of T.*

Proof. Since Z_* has finite dimensional image, there exists a smooth projective variety W and a correspondence Γ supported on $W \times T$ such that image of Z_* is contained in $\Gamma_*(W)$. Let C inside T be a smooth projective curve obtained by taking hyperplane section. Then by Lefschetz theorem on hyperplane sections we have that $J(C)$ maps onto $Alb(T)$. So the kernel is an abelian variety, denoted by $K(C)$. First we observe the following.

Lemma 2.2. *The abelian variety $K(C)$ is simple for a general hyperplane section C.*

Proof. Let if possible there exists a non-trivial proper abelian subvariety A inside $K(C)$. Now $K(C)$ corresponds to the Hodge structure

$$\ker(H^1(C,\mathbb{Q}) \to H^3(T,\mathbb{Q})) .$$

Let $T \to D$ be a Lefschetz pencil such that a smooth fiber is C. Then the fundamental group $\pi_1(D \setminus 0_1, \cdots, 0_m, t)$ acts irreducibly on the Hodge structure mentioned above. Here t corresponds to the smooth fiber C. Now the abelian variety A corresponds to a Hodge sub-structure H inside the above mentioned Hodge structure. Changing the base, let A be defined over the function field $\mathbb{C}(D)$. Then consider a finite extension L of $\mathbb{C}(D)$, such that A, $K(C)$ are defined over L. Then we spread $A, K(C)$, over a Zariski open U' in D', where $\mathbb{C}(D') = L$ and D' maps finitely onto D. Denote these spreads by $\mathscr{A}, \mathscr{K}$ over U'. By throwing more points from U' we get that $\mathscr{A} \to U', \mathscr{K} \to U'$ are fibrations. So the fundamental group $\pi_1(U', t')$ acts on H, which is the $2d-1$ cohomology of A ($d = \dim(A)$), and on $\ker(H^1(C,\mathbb{Q}) \to H^3(T,\mathbb{Q}))$. Since U' maps finitely onto a Zariski open U of D, we have that $\pi_1(U', t')$ is a finite index subgroup of $\pi_1(U, t)$. Now it is a consequence of the Picard-Lefschetz formula that H is a $\pi_1(U, t)$ stable subspace of $\ker(H^1(C,\mathbb{Q}) \to H^3(T,\mathbb{Q}))$, which is irreducible under the action of $\pi_1(U, t)$. So we get that H is either zero or all of $\ker(H^1(C,\mathbb{Q}) \to H^3(T,\mathbb{Q}))$. Therefore A is either zero or all of $K(C)$.
$\square$

Now considering sufficiently ample hyperplane sections of T, the dimension of $K(C)$ is arbitrarily large, and hence greater that $\dim(W)$. Consider the subset R of $K(C) \times W$, consisting of pairs (k, w) such that

$$Z_* j_*(k) = \Gamma_*(w)$$

here $j : C \to T$ is the closed embedding of C into T. Then since the image of Z_* is finite dimensional we have that the projection from R onto $K(C)$ is surjective. By the Mumford-Roitman argument on Chow varieties the above relation tell us [[R]] that R is a countable union of Zariski closed subsets in the product $K(C) \times W$. That would mean that some component R_0 of R, dominates $K(C)$. Therefore we have that

$$\dim(R_0) \geq \dim(K(C)) > \dim(W) .$$

So the fibers of the map $R_0 \to W$ are positive dimensional. Since the abelian variety $K(C)$ is simple, the fibers generate the whole abelian variety. So for any zero cycle z on the fibers of $R_0 \to W$, we have that

$$Z_* j_*(z) = \deg(z)\Gamma_*(w)$$

since z is of degree zero we have that Z_* vanishes on the fibers of $R_0 \to W$, hence on all of $K(C)$.

Now to prove that the map Z_* factors through alb, we consider a zero cycle z, which is given by a tuple of $2k$ points for a fixed positive integer k. Then we blow up T along these points, denote the blow up by $\tau : T' \to T$. Let E_i's be the exceptional divisor of the blow up, then we choose H in $\text{Pic}(T)$, such that $L = \tau^*(H) - \sum_i E_i$ is ample (this can be obtained by Nakai Moisezhon-criterion for ampleness). Then consider a sufficiently ample multiple of L, and apply the previous method to a general member C' of the corresponding linear system. Then $K(C')$ is a simple abelian variety. Also $\tau(C')$ contains all the points at which we have blown up. Suppose that the corresponding cycle z is annihilated by alb_T, then any of its lifts to T' say z', is annihilated by $alb_{T'}$ and is supported on $K(C')$. So applying the previous argument to the correspondence $Z' = Z \circ \tau$, we have that

$$Z_*(z) = Z'_*(z') = 0 \,.$$

$\square$

Let i be the involution on T, then this involution induces an involution at the level of $A_1(X)$. Consider the homomorphism given by the difference of identity and the induced involution on $A_1(X)$, call it Z_{1*}. It is clear from 2.1 that the image of $Z_* Z_{1*}$ cannot be finite dimensional, otherwise the involution will act as $+1$ on $A_1(X)$, leading to the fact that $A_1(X) = \{0\}$. Now we prove the following:

Theorem 2.3. *Let S be the discriminant surface as mentioned above. Then for any very ample line bundle L on S we cannot have the equality*

$$L^2 - g + 1 = g + n$$

where g is the genus of the curve in the linear system of L and n is a positive integer. .

Proof. The discriminant surface S is a quintic, hence its irregularity is zero. So the double cover T has irregularity zero. Consider the very ample line bundle $L = \mathcal{O}(1)$ on the quintic S. Let g be the genus of a smooth curve in the linear system $|L|$. Now we calculate the dimension of $|L|$. Consider the excat sequence

$$0 \to \mathcal{O}(-C) \to \mathcal{O}(S) \to \mathcal{O}(S)/\mathcal{O}(-C) \to 0$$

tensoring with $\mathcal{O}(C)$ we get that

$$0 \to \mathcal{O}(S) \to \mathcal{O}(C) \to \mathcal{O}(C)|_C \to 0$$

taking sheaf cohomology we get that

$$0 \to \mathbb{C} \to H^0(S, L) \to H^0(C, L|_C) \to 0$$

since the irregularity of the surface is zero. On the other hand by Nakai-Moiseshon criterion the intersection number $L|_C$ is positive, so L restricted to C has positive degree, by Riemann-Roch this implies that

$$\dim(H^0(C, L|_C)) = L^2 - g + 1 \ .$$

Suppose if possible, we have the equality

$$L^2 - g + 1 = g + n$$

for some positive integer n. Then the linear system of L is of dimension $g + n$. Now consider the curves C on this linear system and its double covers $\widetilde{C}$. By bertini's theorem a general $\widetilde{C}$ is smooth. By the Hodge index theorem it is connected. If not suppose that it has two components C_1, C_2. Since $C^2 > 0$, we have $C_i^2 > 0$ for $i = 1, 2$ and since $\widetilde{C}$ is smooth we have that $C_1.C_2 = 0$. Therefore the intersection form restricted to $\{C_1, C_2\}$ is semipositive. This can only happen when C_1, C_2 are proportional and $C_i^2 = 0$, for $i = 1, 2$, which is not possible.

Now let $(t_1, \cdots, t_{g+n})$ be a point on T^{g+n}, which gives rise to the tuple $(s_1, \cdots, s_{g+n})$ on S^{g+n}, under the quotient map. There exists a unique, smooth curve C containing all these points (if the points are general). Let $\widetilde{C}$ be its double cover on T. Then $(t_1, \cdots, t_{g+n})$ belongs to $\widetilde{C}$. Consider the zero cycle

$$\sum_i t_i - \sum_i i(t_i)$$

this belongs to $P(\widetilde{C}/C)$, which is the Prym variety corresponding to the double cover. By the identification of $A_1(X_t)$, with the Prym variety of the

discriminant curve we have that the image of

$$\sum_i (Z_*(t_i) - i_* Z_*(t_i))$$

is an element in the Prym variety. So we have that the map

$$T^{g+n} \to A_1(X)$$

given by

$$(t_1, \cdots, t_{g+n}) \mapsto \sum_i Z_*(t_i) - i_* Z_*(t_i)$$

factors through the Prym fibration $\mathscr{P}(\widetilde{\mathscr{C}}/\mathscr{C})$, given by

$$(t_1, \cdots, t_{g+n}) \mapsto alb_{\widetilde{C}}\left(\sum_i t_i - i(t_i)\right)$$

here $\widetilde{\mathscr{C}}, \mathscr{C}$ are the universal smooth curve and the universal double cover of $\mathscr{C}$ over $|L|_0$ parametrizing the smooth curves in the linear system $|L|$. By dimension count the dimension of $\mathscr{P}(\widetilde{\mathscr{C}}/\mathscr{C})$ is $2g+n-1$. On the other hand we have that dimension of T^{g+n} is $2g+2n$. So the map

$$T^{g+n} \to \mathscr{P}(\widetilde{\mathscr{C}}/\mathscr{C})$$

has positive dimensional fibers, so it contains a curve. Let H be the hyperplane bundle pulled back onto the quintic surface S. It is very ample, pull it back further onto T, to get an ample line bundle on T. Call it L', then the divisor $\sum_i \pi_i^{-1}(L')$ is ample on T^{g+n}, where π_i is the i-th coordinate projection from T^{g+n} to T. Therefore the curves in the fibers of the above map intersect the divisor $\sum_i \pi_i^{-1}(L')$. So we get that there exist points in F_s contained in $C \times T^{g+n-1}$ where C is in the linear system of L'. Then consider the elements of F_s the form $(c, s_1, \cdots, s_{g+n-1})$, where c belongs to C. Considering the map from T^{g+n-1} to $A_0(T)$ given by

$$(s_1, \cdots, s_{g+n-1}) \mapsto \sum_i s_i + c - \sum_i i(s_i) - i(c),$$

we see that this map factors through the Prym fibration and the map from T^{g+n-1} to $\mathscr{P}(\widetilde{\mathscr{C}}/\mathscr{C})$ has positive dimensional fibers, since n is large. So it means that, if we consider an element $(c, s_1, \cdots, s_{g+n-1})$ in F_s and a curve through it, then it intersects the ample divisor given by $\sum_i \pi_i^{-1}(L')$, on T^{g+n-1}. Then we have some of s_i is contained in C. So iterating this process we get that elements of F_s are supported on $C^k \times T^{g+n-k}$, where k is some natural number depending on n which is strictly bigger than the genus of C. But the genus of C is fixed and equal to 11 and less than k

and then k will be large and greater than the genus of C. That means that the elements of F_s are supported on $C^{n_0} \times T^{g+n-k}$. Therefore considering $\Gamma = Z_1 \circ Z$, we get that $\Gamma_*(T^{g+n}) = \Gamma_*(T^{m_0})$, where m_0 is strictly less than $g + n$.

Now we prove by induction that $\Gamma_*(T^{m_0}) = \Gamma_*(T^m)$ for all $m \geq g + n$. So suppose that $\Gamma_*(T^k) = \Gamma^*(T^{m_0})$ for $k \geq g + n$, then we have to prove that $\Gamma_*(T^{k+1}) = \Gamma_*(T^{m_0})$. So any element in $\Gamma_*(T^{k+1})$ can be written as $\Gamma_*(t_1 + \cdots + t_{m_0}) + \Gamma_*(t)$. Now let $k - m_0 = m$, then $m_0 + 1 = k - m + 1$. Since $k - m < k$, we have $k - m + 1 \leq k$, so $m_0 + 1 \leq k$, so we have the cycle

$$\Gamma_*(t_1 + \cdots + t_{m_0}) + \Gamma_*(t)$$

supported on T^k, hence on T^{m_0}. So we have that $\Gamma_*(T^{m_0}) = \Gamma_*(T^k)$ for all k greater or equal than $g + n$. Now any element z in $A_0(T)_{hom}$, can be written as a difference of two effective cycle z^+, z^- of the same degree. Then we have

$$\Gamma_*(z) = \Gamma_*(z^+) - \Gamma_*(z_-)$$

and $\Gamma_{(z_\pm)}$ belong to $\Gamma_*(T^{27})$. So let Γ' be the correspondence on $T^{2m_0} \times T$ defined as

$$\sum_{l \leq m_0} (pr_l, pr_T)^* \Gamma - \sum_{m_0 + 1 \leq l \leq 2m_0} (pr_l, pr_T)^* \Gamma$$

where pr_l is the l-th projection from T^l to T, and pr_T is from $T^{2m_0} \times T$ to the last copy of T. Then we have

$$\text{im}(\Gamma_*) = \Gamma'_*(T^{2m_0}).$$

This would imply that the image of Γ_* is finite dimensional, so by 2.1 we have that the induced involution on $A_1(X)$ acts as identity. This will lead to a contradiction to the fact that $A_1(X)$ is infinite dimensional [SC]. □

2.4. **Generalization of the above proof technique.** The proof technique is more general. In the sense that we only use the conic bundle structure of the cubic fourfold and the conic bundle structure on the hyperplane sections of the cubic fourfold. So suppose that we consider a fourfold X, which is unirational, so contains sufficiently many lines. Now consider a line l on X, and project onto $\mathbb{P}^3$ from this line. Then the discriminant surface S inside $\mathbb{P}^3$ is a union of discriminant curves S_t inside $\mathbb{P}^2$, such that when we consider a hyperplane section X_t of X containing l, then the projection of the hyperplane section onto $\mathbb{P}^2$ has a conic bundle structure and S_t is the corresponding discriminant curve. The curve S_t admits

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a two-sheeted cover T_t in the Fano variety of lines $F(X_t)$, and correspondingly we have a double cover T of S inside the Fano variety of lines $F(X)$.

Then the above proof tells us that we have the following theorem:

Theorem 2.5. *Let X be a fourfold embedded in $\mathbb{P}^5$, which admits a conic bundle structure. Let S denote the discriminant surface for the conic bundle structure. Then for any very ample line bundle L on S, we cannot have the equality*

$$L^2 - g + 1 = g + n$$

where g is the genus of a curve in the linear system of L and n is some positive integer.

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