Supersymmetric Path Integrals I: Differential Forms on the Loop Space

Florian Hanisch* and Matthias Ludewig[†]

October 3, 2017

Abstract

In this paper, we construct an integral map for differential forms on the loop space of Riemannian spin manifolds. In particular, the even and odd Bismut-Chern characters are integrable by this map, with their integrals given by indices of Dirac operators. We also show that our integral map satisfies a version of the localization principle in equivariant cohomology. This should provide a rigorous background for supersymmetry proofs of the Atiyah-Singer Index theorem.

1 Introduction

This is the first of a series of papers on supersymmetric path integrals. In this paper, we construct an integral map for differential forms on the loop space of a Riemannian manifold X. This will be a map I_T sending differential forms on the loop space LX of X to real numbers, which depends on a parameter T > 0. Formally, for an integrable differential form θ on the loop space, I_T should be given by the expression

$$I_T[\theta] \stackrel{\text{formally}}{=} \int_{\mathsf{L}X} e^{-E/T + \omega} \wedge \theta,$$
 (1.1)

where the right hand side is to be interpreted as a differential form integral over the loop space, with

$$E(\gamma) = \frac{1}{2} \int_{S^1} |\dot{\gamma}(t)|^2 dt \quad \text{and} \quad \omega[v, w] := \int_{S^1} \langle v(t), \nabla_{\dot{\gamma}} w(t) \rangle dt \quad (1.2)$$

being the energy function and the canonical two form on the loop space, respectively. Of course, since the loop space is infinite-dimensional, the right hand side of (1.1) is not a well-defined expression, which makes the definition of I_T a non-trivial task.

^{*}Universität Potsdam. fhanisch@math.uni-potsdam.de

[†]Max-Planck-Institut für Mathematik in Bonn. maludewi@mpim-bonn.mpg.de

The integral map I_T has been used in a formal way by several authors (e.g. [Ati85]) to give a proof of the Atiyah-Singer index theorem, in the sense that the proof comes out of formal manipulations of the expression on the right hand side of (1.1), pretending that LX was finite-dimensional. There are also several attempts of definitions (see references below), which however usually do not have a very large domain of definition and are generalized by our construction.

We hope that some of our methods can be generalized to work "one level higher", where one considers forms on the double loop space (or torus space) L^2X . Just as our map I_T is related to the index of the Dirac operator on X (as we see below), an integral map for forms on L^2X should be related to the index of a potential Dirac operator on $\mathsf{L}X$; this might be a way to approach the Stolz Conjecture [Sto96].

To be able to integrate differential forms over a finite-dimensional manifold, one needs this manifold to be orientable. It should therefore come as no surprise that we need some extra assumptions to define our integral map. Indeed, it turns out that we need the manifold X to be spin, which is well-known to be equivalent to orientability of the loop space in a suitable sense [ST05], [Wal16]. In particular, for differential forms $\theta_1, \ldots, \theta_N$ on LX given by pullback $\theta_j = \operatorname{ev}_{\tau_j}^* \vartheta_j$ of differential forms $\vartheta_1, \ldots, \vartheta_N \in \Omega(X)$, where $0 \le \tau_1 < \cdots < \tau_N \le 1$ and $\operatorname{ev}_t : LX \to X$ is the evaluation-at-t-map, we have

$$I_T[\theta_N \wedge \dots \wedge \theta_1] = 2^{-N/2} \operatorname{Str} \left(e^{-T(1-\tau_N)H} \prod_{j=1}^N \mathbf{c}(\vartheta_j) e^{-T(\tau_j - \tau_{j-1})H} \right), \tag{1.3}$$

where \mathbf{c} denotes Clifford multiplication, $H = \mathsf{D}^2/2$ with D the Cl_n -linear Dirac operator on real spinors and Str the corresponding Cl_n -linear super trace¹.

Since most differential forms on the loop space are not of the form considered in (1.3), extending the definition of I_T to a suitably large class of differential forms is not obvious; the first main goal of this paper is to establish the definition of I_T in full generality. The story does not stop there, however, as it is not at all clear how formula (1.3) relates to the formal equation (1.1). However, it turns out that there is a very precise relation: There is a naive way to make sense of (1.1) arguing by analogy with finite-dimensional Gaussian integrals and using zeta-regularization, and it turns out that this naive definition coincides on the spot with the definition (1.3) on the common domains. This will be explained in our second paper [HL17a]. We remark here that the domain of the integral map constructed by this latter approach is much smaller, which is the reason to raise (1.3) to a definition.

The main ingredient of our construction is a pathwise defined, linear functional q that to a differential form on the loop space associates a function, its "top degree coefficient", which then can be integrated over the loop space using the Wiener measure. This is analogous to the Berezin integral on finite-dimensional supermanifolds, where superfunctions can be interpreted as sections in the exterior power of a certain (finite dimensional) vector

¹Here and throughout, we adopt the convention that products are defined "from right to left", that is, $\prod_{j=1}^{N} a_j := a_N \cdots a_1$. Note that this differs from the "left-to-right-convention" if the elements a_1, \ldots, a_N do not commute.

bundle: Integration over the "odd" directions corresponds to taking to top degree part of the integrand and the resulting real valued function is then integrated in the conventional way with respect to the "even" variables. On the loop space, this is problematic, since forms can have arbitrarily high degree so in particular, there is no "top degree". However, it turns out that while the top degree of a differential form θ makes no sense on the loop space, there is a way (in analogy with the finite-dimensional situation) to make sense of the top degree of the form $e^{\omega} \wedge \theta$, where ω is the canonical two form defined above. Explicitly, for forms $\theta_1, \ldots, \theta_N$ in $L^1(S^1, \gamma^*T'X) \subset T'LX$, we have

$$q(\theta_N \wedge \dots \wedge \theta_1) = 2^{-N/2} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \int_{\Delta_N} \operatorname{str}\left([\gamma \|_{\tau_N}^1]^{\Sigma} \prod_{j=1}^N \mathbf{c} \left(\theta_{\sigma_j}(\tau_j) \right) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\Sigma} \right) d\tau, \quad (1.4)$$

where $[\gamma|_s^t]^{\Sigma}$ denotes the parallel transport along γ in the (real) spinor bundle Σ and Δ_N the standard simplex. Our integral map I_T is then defined essentially by integrating this function over LX with respect to the Wiener measure².

Formula (1.4) works for differential forms θ that are wedge products of such one-forms on LX that are pathwise given by integrable co-vector fields. However, it turns out that formula (1.4) can be extended to a much larger class of differential forms on LX. This pathwise extension of the q functional will be one of the main concerns in this paper.

Of course, at first glance, this formula seems not at all related to the top-degree component of $e^{\omega} \wedge \theta$ (whatever that may be). The relation is as follows: The right hand side in (1.4) has a natural interpretation as an element of a certain line bundle on the loop space (in fancy term, this is the transgression of the spin lifting gerbe on X). On the other hand, it turns out that $e^{\omega} \wedge \theta$ can be naturally interpreted as a section of the Pfaffian line bundle on LX, associated to the covariant derivative $\nabla \dot{\gamma}$. These line bundles are naturally isomorphic as geometric line bundles, and the two sections correspond to each other under this isomorphism. This is explained precisely in our second paper [HL17a].

Fundamental integrands for our integral map are the Bismut-Chern forms. For a vector bundle \mathcal{V} with connection ∇ on X, there is an associated differential form $BCh(\mathcal{V}, \nabla)$, the (even) Bismut-Chern form, constructed by Bismut in [Bis85]. Plugging this form into our integral map, we find for any T > 0

$$I_T[\operatorname{BCh}(\mathcal{V}, \nabla)] = i^{n/2} \operatorname{ind}(\mathsf{D}_{\mathcal{V}}),$$
 (1.5)

the index of the twisted (complex) Dirac operator corresponding to (\mathcal{V}, ∇) . In particular, $I_T[1]$ equals the index for the Dirac operator on (complex) spinors.

Similarly, corresponding to a smooth map $g: X \to U_k$ (the k-th unitary group), there is the *odd Bismut-Chern character* BCh(g) constructed by Wilson in [Wil16]. Plugging this in, we get

$$I_T[BCh(g)] = (-i)^{\frac{n+1}{2}} \left(\frac{2\pi}{T}\right)^{1/2} sf(D, g^{-1}Dg)$$
 (1.6)

²Since the Wiener measure is defined on the *continuous loop space* L_cX , we can at first only integrate differential forms that are restrictions of differential forms on L_cX .

for any T > 0, where $\operatorname{sf}(\mathsf{D}, g^{-1}\mathsf{D}g)$ is the spectral flow (the number of eigenvalues crossing zero) along the family $\mathsf{D}_s := (1-s)\mathsf{D} + sg^{-1}\mathsf{D}g$, with D the Dirac operator on the spinor bundle tensored with the trivial bundle $\underline{\mathbb{C}}^k$. Here (1.5) is only non-trivial in even dimensions, while (1.6) is only non-trivial in odd dimensions.

The loop space has a natural circle action, which is given by rotation of loops. Heuristic arguments suggest that just in the finite-dimensional situation, the integral map I_T should satisfy a localization principle, stating that the integral of an equivariantly closed differential form reduces to an integral over the fixed point set of the circle action, which is the set $X \subset LX$ of constant loops. We will prove below that this localization principle is true for the Bismut-Chern characters, with exactly the formula expected from courageously applying the finite-dimensional version by analogy. This puts the Atiyah-Singer index theorem for twisted Dirac operators in even dimensions and Getzler's spectral flow theorem (see [Get93, Thm. 2.8]) into a unified context.

Our results can be naturally interpreted in the language of supermanifolds. Namely, it is a fact from super geometry that differential forms on the (ordinary) loop space LX are "the same" as smooth functions on the super loop space, i.e. the space of smooth maps from the super circle $S^{1|1}$ to X. In this sense, our integral map I_T from above can be considered as an integral map for functions on the super loop space. That this path integral is supersymmetric has a precise meaning in this context. In particular, this property entails that the integral obeys a localization principle for functions which are invariant under the natural $S^{1|1}$ -action on paths. This point of view will be discussed in the third paper of this series [HL17a].

Our integral map is also closely connected to the Chern character in cyclic cohomology and to Chen's theory of iterated integrals [Che73]. Moreover, in [GJP91], Getzler, Jones and Petrack show that the Bismut-Chern characters can be constructed as iterated integrals, by suitably extending the iterated integral map. It turns out that all differential forms constructed this way can be integrated using our integral map. This will be subsequently discussed in [HL17d].

The map I_T constructed rigorously in this paper was previously used formally by Atiyah [Ati85], Bismut [Bis85], Alvarez-Gaume [AG84], [AG85] and others. The arguments of Atiyah are reviewed in Section 3; Bismut picks up Atiyah's line of thought as he constructs an *index measure* on the loop space and constructs the Bismut-Chern character the integral of which should give the twisted index theorem. However, as his index measure only integrates *functions*, the evaluation of the Bismut-Chern character must remain somewhat formal.

A good part of our work is inspired by the paper [Lot87] of Lott, and our construction can be seen as a generalization of his: Up to conventional factors, our path integral map coincides with his on the common domain, which is too small however to contain the Bismut-Chern characters. The relation to "formula" (1.1) is also not discussed in his paper.

Fine and Sawin [FS08], [FS14], [FS17] define supersymmetric path integrals using finite-dimensional approximation. However, they do not construct a path integral map allowing

to integrate a variety of superfunctions/differential forms. Their arguments are based on a careful analysis of heat kernel asymptotics; in contrast, our construction uses the Wiener measure, as well as combinatoric arguments to define the map q mentioned above. No estimates are needed in our construction.

We are also aware of the work of Leppard and Rogers [LR01], [Rog03]. It would be interesting to see how their constructions relate to ours.

The paper is structured as follows. In the first two sections (Section 2 and 3), we give a brief account of the basic notions needed in this paper such as elements of Spin geometry and the Wiener measure, and then go through the formal proof of the Atiyah-Singer index theorem using the path integral (1.1), in order to motivate our results and explain how our results are related to this story. Sections 4 and 5 are devoted to the pathwise definition of the map q mentioned above. In Section 6, we finally define our map I_T and then discuss its basic properties. At first, its domain does not contain the Bismut-Chern characters; this is accommodated for in 7. Sections 8 and 9 are devoted to the even and odd Bismut-Chern characters and their integrals under our map I_T . This is where we establish equations (1.5) and (1.6) above. Finally, in Section 10, we discuss the localization principle for our integral map. While we cannot yet establish it in full generality, we at least show that it is satisfied for the Bismut-Chern characters.

The relation of our integral map to the formal path integral (1.1) will be discussed in our second paper [HL17a]. In our third paper [HL17c], we discuss the relation of I_T to supersymmetry within the framework of supergeometry.

Acknowledgements. We thank the Max-Planck-Institute for Gravitational Physics in Potsdam-Golm (Albert-Einstein-Institute), the Max-Planck-Institute for Mathematics in Bonn as well as the Institute for Mathematics at the University of Potsdam for hospitality and financial support.

2 Preliminaries

In this section, we introduce the basic notions needed in this paper. The reader is welcome to skip this section in order to come back to it whenever needed.

Clifford Algebras. Let V be a Euclidean vector space of dimension n. Canonically associated, we have the dual space V', its exterior algebra $\Lambda V'$ and the Clifford algebra $\operatorname{Cl}(V)$ of V. There is a canonical isomorphism between these two bundles, namely the quantization map

$$\mathbf{c}: \Lambda V' \longrightarrow \mathrm{Cl}(V).$$
 (2.1)

Both ΛV and $\mathrm{Cl}(V)$ are \mathbb{Z}_2 -graded algebras (i.e. a super algebras). The grading of $\Lambda V'$ comes from the splitting into even- and odd-degree forms, while the grading on $\mathrm{Cl}(V)$ is the one carried over from $\Lambda V'$ via \mathbf{c} . Notice that ΛV is supercommutative while $\mathrm{Cl}(V)$ is not.

If V is oriented, we define the supertrace of an element $a \in Cl(V)$ by

$$\operatorname{str}(a) := 2^{n/2} \langle a, \mathbf{c}(\operatorname{vol}) \rangle,$$
 (2.2)

where $\text{vol} = e_1 \wedge \cdots \wedge e_n$ is the volume form on V defined using an oriented orthonormal basis e_1, \ldots, e_n of V. It can be shown that this actually defines a super trace, i.e. it vanishes on super commutators. This implies that the super trace more generally has the cyclic permutation property

$$str(a_N \cdots a_1) = (-1)^{|a_1|(|a_N| + \cdots + |a_2|)} str(a_1 a_N \cdots a_2)$$
(2.3)

for homogeneous elements $a_1, \ldots, a_N \in Cl(V)$, where $|a_j|$ denotes the parity of a_j . It is standard that this trace is unique, up to multiplication with constants. Notice that the trace is even (i.e. it vanishes on $Cl^-(V)$) if n is even, while it is odd if n is odd (i.e. it vanishes on $Cl^+(V)$).

Spin Geometry. Let X be an oriented Riemannian manifold of dimension n. Then we can perform the above constructions on $V := T_x X$ for every $x \in X$. By definition, a spin structure on X is a lift of the frame bundle P^{SO} over X (which is an SO_n principal bundle) to a Spin_n principal bundle P^{Spin} . We can then form the associated real spinor bundle $\Sigma := P^{\mathrm{Spin}} \times_{\mathrm{Spin}_n} \mathrm{Cl}_n$, where $\mathrm{Cl}_n := \mathrm{Cl}(\mathbb{R}^n)$ and Spin_n acts on Cl_n by left multiplication. For each $x \in X$, Σ_x is a $Cl(T_xX)-Cl_n$ bimodule, and we have $Cl(T_xX) \cong End_{Cl_n}(\Sigma_x)$, where the right hand side denotes endomorphisms of Σ_x commuting with the right action of Cl_n . Of great importance will be the spin parallel transport in the spinor bundle along paths $\gamma:[0,1]\to X$, which we denote by $[\gamma|_s^t]^{\Sigma}$, $s,t\in[0,1]$. This is an isometric isomorphism from $\Sigma_{\gamma(s)}$ to $\Sigma_{\gamma(t)}$. Moreover, the parallel transport commutes with the right action of Cl_n , i.e. $[\gamma]_s^t \stackrel{f}{\subseteq} \operatorname{Hom}_{\operatorname{Cl}_n}(\Sigma_{\gamma(s)}, \Sigma_{\gamma(t)})$. In particular, if γ is a loop, i.e. $\gamma(0) = \gamma(1)$, then $[\gamma|_0^1]^{\Sigma} \in \operatorname{End}_{\operatorname{Cl}_n}(\Sigma_{\gamma(0)})$ and we can form its super trace using (2.2). A word of warning: We can also form the complex spinor bundle $\Sigma^{\mathbb{C}} := P^{\operatorname{Spin}} \times_{\rho} \Sigma_{n}^{\mathbb{C}}$, where $\rho: \operatorname{Spin}_n \to \Sigma_n^{\mathbb{C}}$ is the spinor representation. This is a complex vector bundle, graded in even dimensions and ungraded in odd dimensions. For a loop γ , we then have $[\gamma]_0^{-1}]^{\Sigma^{\mathbb{C}}} \in \operatorname{End}_{\mathbb{C}}(\Sigma^{\mathbb{C}})$ and we can take the (complex) endomorphism trace. It follows from comparing [BGV04, Prop. 3.19] with (2.2) that

$$\operatorname{str}_{\mathbb{C}}[\gamma|_{0}^{1}]^{\Sigma^{\mathbb{C}}} = (-i)^{n/2} \operatorname{str}[\gamma|_{0}^{1}]^{\Sigma}. \tag{2.4}$$

We will usually work with the real spinor bundle, since this bundle is naturally graded in all dimensions.

For a more detailed account of spin geometry, see [LM89, § 1] or [PW09, Chapter 1].

Path and Loop Space Geometry. Let X be an n-dimensional manifold. The loop space of is the space $LX := C^{\infty}(S^1, \gamma^*X)$, which is a good realization of the loop space for many purposes; however, since we will later use the Wiener measure, which lives on continuous paths, we will also use the continuous loop space $L_cX := C(S^1, X)$. Both spaces have naturally the structure on an infinite-dimensional manifold, the first one

modelled on the Fréchet space $C^{\infty}(S^1, \mathbb{R}^n)$, the second modelled on the Banach space $C(S^1, \mathbb{R}^n)$. For the tangent space at a point $\gamma \in \mathsf{L}X$, we have the natural identifications

$$T_{\gamma} LX \cong C^{\infty}(S^1, \gamma^*TX), \qquad T_{\gamma} L_c X \cong C(S^1, \gamma^*TX)$$

the spaces of smooth respectively continuous vector fields along γ , that is, the space of sections of the pullback bundle γ^*TX over S^1 .

We will also consider the path space $\mathsf{P}X := C^\infty([0,1],X)$ and its continuous version $\mathsf{P}_cX := C([0,1],X)$. Again, the first possesses a natural Fréchet manifold structure, modelled on $C^\infty([0,1],\mathbb{R}^n)$, while the second has the structure of a Banach manifold, modelled on $C([0,1],\mathbb{R}^n)$. Again, in both cases, the tangent space at a path γ can be identified with a space of vector fields along γ ,

$$T_{\gamma}\mathsf{P}X \cong C^{\infty}([0,1], \gamma^*TX), \qquad T_{\gamma}\mathsf{P}_cX \cong C([0,1], \gamma^*TX).$$

The tangent spaces of the loop spaces LX and L_cX have a natural scalar product, the L^2 scalar product, given by

$$(v,w)_{L^2} := \int_{S^1} \langle v(t), w(t) \rangle dt$$
(2.5)

for smooth vector fields v, w along a loop γ . Of course, neither $T_{\gamma} LX$ nor $T_{\gamma} L_c X$ is complete with the norm induced from this scalar product; the completion is the space $L^2(S^1, \gamma^*TX)$ of square-integrable vector fields along γ .

Equivariant Differential Forms on the Loop Space. There is a natural action of the circle group $\mathbb{T} := S^1$ on LX, where for $t \in \mathbb{T}$, the path $t \cdot \gamma$ is given by $(t \cdot \gamma)(s) = \gamma(s+t)$. The same action could be considered on the continuous loop space, but it is not smooth there (only continuous). The generating vector field of the \mathbb{T} -action is the vector field which over a path γ is given by the derivative vector field $\dot{\gamma}$. We usually write just $\dot{\gamma}$ for this vector field. The equivariant differential is defined by

$$d_T = d + T^{-1} \iota_{\dot{\gamma}},$$

where $\iota_{\dot{\gamma}}$ denotes insertion of the vector field $\dot{\gamma}$ and T is, depending on the context, a formal variable or a positive parameter. It acts on the space $\Omega(\mathsf{L}X)[T,T^{-1}]$ of differential forms that are polynomials in T and T^{-1} . $\Omega(\mathsf{L}X)[T,T^{-1}]$ is graded by letting have T degree -2. The equivarant differential satisfies

$$d_T^2 \theta = T^{-1} \mathcal{L}_{\dot{\gamma}} \theta,$$

where $\mathscr{L}_{\dot{\gamma}}$ denotes the Lie derivative with respect to $\dot{\gamma}$, hence on the space $\Omega[T,T^{-1}]^{\mathbb{T}}$ of differential forms $\Omega(\mathsf{L}X)[T,T^{-1}]^{\mathbb{T}}$ invariant under the \mathbb{T} -action, this is indeed a differential. A particular element of $\Omega(\mathsf{L}X)$ is the one form η dual to $\dot{\gamma}$ with respect to the scalar product (2.5). One has

$$d_T \eta = 2(T^{-1}E - \omega),$$

where E and ω were defined in (1.2) above. Hence $d_T(T^{-1}E - \omega) = 0$. Compare [Ati85], [BGV04, Chapter 7].

We will more generally consider (sums of) formal power series θ_T that have the form

$$\theta_T = \sum_{N=-\infty}^{\infty} T^N \theta_N, \quad \text{with} \quad \theta_N \in \Omega^{2N+M}(\mathsf{L}X)$$
 (2.6)

where $M \in \mathbb{Z}$ is fixed. We denote the space of such forms by $\Omega_T(LX)$. Considering such power series, the coefficients of which are \mathbb{T} -invariant, one obtains a chain complex with differential d_T . Its cohomology is the cyclic equivariant cohomology $h_T^*(LX)$. For details, see e.g. [JP90]. Note that our variable T^{-1} corresponds to the variable u from the literature; the reason for this choice is that it turns out that this variable is naturally related to the parameter T in our integral map I_T .

The Wiener Measure. On the loop space L_cX of a compact Riemannian manifold, there is a natural measure, the *Wiener measure*, which we will introduce now. In fact, there is not only one measure but a family of measures W_T , parametrized by $T \in (0, \infty)$. The pairing of W_T with *cylinder functions*, which are the functions F on L_cX of the form

$$F(\gamma) = f(\gamma(\tau_N), \dots, \gamma(\tau_1))$$

for a function $f \in C(X \times \cdots \times X)$ and numbers $0 \le \tau_1 \le \cdots \le \tau_N \le 1$, is defined by

$$W_T[F] := \int_X \cdots \int_X f(x_N, \dots, x_1) \prod_{j=1}^N p_{T(\tau_j - \tau_{j-1})}(x_j, x_{j-1}) \, \mathrm{d}x_N \cdots \, \mathrm{d}x_1, \qquad (2.7)$$

where $p_t(x,y)$ is the heat kernel of the operator $H_0 := \frac{1}{2}\Delta$ on X, and we used the convention that $x_0 := x_N$. Since the cylinder sets generate the sigma algebra of $\mathsf{L}_c X$, the formula (2.7) for all cylinder functions determines the Wiener measure uniquely [BP11]. Note that with our conventions, the Wiener measure on $\mathsf{L}_c X$ is not a probability measure; instead, its total variation is $W_T[1] = \int_X p_T(x,x) dx = \mathrm{Tr}(e^{-TH_0})$. It follows easily from the defining formula (2.7) that for each $t \in \mathbb{T}$, the map on $\mathsf{L}_c X$ that sends $\gamma \mapsto t \cdot \gamma$ preserves the measure W_T .

On the free path space P_cX , there is a family \mathbb{W}_T^{yx} of Wiener measures, parametrized by $T \in (0, \infty)$ and $(x, y) \in X \times X$. On cylinder functions F as above, it is defined by the similar formula

$$\mathbb{W}_{T}^{yx}[F] := \int_{X} \cdots \int_{X} f(x_{N}, \dots, x_{1}) \prod_{j=1}^{N+1} p_{T(\tau_{j} - \tau_{j-1})}(x_{j}, x_{j-1}) \, \mathrm{d}x_{N} \cdots \, \mathrm{d}x_{1}, \qquad (2.8)$$

where we used the convention $x_0 = x$ and $x_{N+1} = y$. The measure \mathbb{W}_T^{yx} is supported on the set of paths γ such that $\gamma(0) = x$ and $\gamma(1) = y$ (i.e. the complement of this set is a zero set). Consequently, \mathbb{W}_T^{yx} can be restricted to the space $\mathsf{P}_c^{yx}X$ of paths starting at x and ending at y. Moreover for x = y, using the continuous map $\mathsf{P}_c^{xx}X \to \mathsf{L}_cX$, one can push this forward \mathbb{W}_T^{xx} to a measure on the loop space. Here we have the relation

$$W_T[F] = \int_X W_T^{xx}[F] dx$$
 (2.9)

for any \mathbb{W}_T -integrable function F on on L_cX .

For T > 0, the map $(x, y) \mapsto \mathbb{W}_T^{xy}$ is weakly continuous so that for any continuous function F on P_cX , we obtain a continuous function $\mathbb{W}_T^{xy}[F]$, which can be taken as the integral kernel of a trace-class operator on $L^2(X)$. Moreover, a continuous function F on P_cX gives a continuous function on L_cX by restriction, so by (2.9), $\mathbb{W}_T[F]$ is the trace of the operator with integral kernel $\mathbb{W}_T^{xy}[F]$.

It follows from the definition that the Wiener measures \mathbb{W}_T^{yx} are related by the following convolution property, also called Markhov property: Given $0 \le a \le b \le 1$, let $\mathfrak{s}_{a,b} : \mathsf{P}_c X \to \mathsf{P}_c X$ be the rescaling map, defined by

$$\mathfrak{s}_{a,b}(\gamma)(t) = \gamma \left(a + \frac{t-a}{b-a}\right).$$

Then for integrable functions F, G, we have

$$W_T^{yx}[F] = \int_X W_{T(1-t)}^{yz}[\mathfrak{s}_{t,1}^* F] \cdot W_{Tt}^{zx}[\mathfrak{s}_{0,t}^* F] dz.$$
 (2.10)

for all $x, y \in M$ and all $t \in [0, 1]$. For cylinder functions F, this follows quite easily from (2.8) if t is a point of the partition that F is subordinated to; the general case follows by usual measure theoretic arguments. The need for the rescaling map in formula (2.10) is due to the fact that in this presentation, we chose to always work with paths of length one. This choice simplifies things in other situations. Here the formula (2.10) takes a more natural form when one defines \mathbb{W}_T^{xy} to live on paths of length T.

Stochastic Parallel Transport and the Feynman-Kac Formula. If \mathcal{V} is a (super) vector bundle with connection ∇ , the corresponding (super) trace of the parallel transport $\operatorname{str}[\gamma|_0^1]^{\mathcal{V}}$ around loops $\gamma \in \mathsf{L}X$ is a smooth function on $\mathsf{L}X$. It is a constant source of difficulties that this function does not make sense right away on the continuous loop space $\mathsf{L}_c X$, due to the fact that the velocity vector field $\dot{\gamma}$ is not defined.

However, it turns out that that there is a stochastic version of it, the *stochastic parallel* transport, which is a measurable function on path space with respect to the Wiener measure. This was already realized by Itô [It63], [It75]. To describe this more precisely, for $t \in [0, 1]$, let

$$\operatorname{ev}_t: \mathsf{P}_c X \longrightarrow X, \qquad \gamma \longmapsto \gamma(t)$$

be the evaluation map. Then for $s, t \in [0, 1]$, there is the bundle $\operatorname{ev}_t^* \mathcal{V} \otimes \operatorname{ev}_s^* \mathcal{V}'$ over $\mathsf{P}_c X$, the fiber over a path γ is the space $\operatorname{Hom}(\mathcal{V}_{\gamma(s)}, \mathcal{V}_{\gamma(t)})$. The stochastic parallel parallel transport, which we denote by $[\gamma|_s^t]^{\mathcal{V}}$ just as in the smooth case, will be a bounded and measurable section of this bundle. Since a bounded measurable function is good enough for many purposes, in particular for integrating it, the stochastic parallel transport will be sufficient in most cases.

The stochastic parallel transport plays a role in the Feynman-Kac formula, which is due to Kac in its original version [Kac49]. The version without a potential is already contained in Itô [It63]. For a general version, see e.g. [Gü10].

Theorem 2.1 (Feynman-Kac Formula). Let V be a metric vector bundle over a compact manifold X with a metric connection ∇ and let V be a symmetric section of $\operatorname{End}(V)$). For $T \geq 0$ and paths $\gamma \in \mathsf{P}_c X$, let $U_T(t,\gamma) \in \operatorname{Hom}(\mathcal{V}_{\gamma(0)},\mathcal{V}_{\gamma(t)})$ be the solution to the (stochastic) ordinary differential equation

$$\frac{\nabla}{\mathrm{d}t}U_T(t,\gamma) = -T U_T(t,\gamma)V(\gamma(t)), \qquad U_T(0,\gamma) = \mathrm{id}_{\mathcal{V}_{\gamma(0)}}$$

along γ , interpreted in the Stratonovich sense. Then the heat kernel $p_T^L(y,x)$ of the operator $L := \frac{1}{2}\nabla^*\nabla + V$ is given by $\operatorname{Hom}(\mathcal{V}_x, \mathcal{V}_y)$ -valued Wiener integral

$$p_T^L(y,x) := \mathbb{W}_T^{yx} [U_T(1,\gamma)],$$

for all T > 0 and all $x, y \in X$.

The simplest case is that \mathcal{V} is the trivial line bundle and $\nabla = d$. In this case,

$$U_T(t,\gamma) = \exp\left(-T\int_0^t V(\gamma(t))dt\right).$$

Specializing in the other direction, if V=0 but the connection is not, then $U_T(t,\gamma)=[\gamma|_0^t]^{\mathcal{V}}$, the stochastic parallel transport with respect to the connection ∇ . The general case can be reduced to the stochastic parallel transport. Namely, in general $U_T(t,\gamma)$ is given by the formula

$$U_{T}(t,\gamma) = \sum_{N=0}^{\infty} (tT)^{N} \int_{\Delta_{N}} [\gamma \|_{\tau_{N}}^{1}]^{\mathcal{V}} \prod_{j=1}^{N} V(\gamma(\tau_{j})) [\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{\mathcal{V}} d\tau,$$
 (2.11)

where $\Delta_N = \{0 \le \tau_1 \le \cdots \le \tau_N \le 1\}$ is the standard simplex.

3 The formal Proof of the Index Theorem

In this section, in order to motivate our results, we review the formal proof of the Atiyah-Singer index theorem given by Atiyah [Ati85], following ideas of Witten [Wit82]; see also the treatises of Bismut [Bis85] and Alvarez-Gaume [AG85]. This section is purely motivational and is not needed for the rest of the paper.

Consider the integral

$$I_T := \int_{\Gamma_X} e^{-E/T + \omega},\tag{3.1}$$

of differential forms over the loop space. Here the slash over the integral sign denotes division of the integral by the factor $(2\pi T)^{N/2}$, where $N=\infty$ is the dimension of the loop space; clearly this is one reason that the right hand side of (3.1) does not make sense; another reason is that there is no well-defined theory of integrating differential forms over infinite-dimensional manifolds. We will now try to make formal manipulations of this integral until we reach a well-defined quantity. In these arguments, we will assume that X is even-dimensional.

(1) First we note that when A is a skew-symmetric endomorphism of some N-dimensional Euclidean vector space V and ω is the corresponding two-form given by $\omega[v, w] = \langle v, Aw \rangle$, then the N-form coefficient $[e^{\omega}]_{\text{top}}$ of the form e^{ω} is given by the Pfaffian of A,

$$[e^{\omega}]_{\text{top}} = \text{pf}(A),$$

provided that one chooses an orientation on V, in order that both sides are well-defined. Now formally replacing V by $T_{\gamma} LX = C^{\infty}(S^1, \gamma^*TX)$ for some loop γ , it follows directly from the definition that in this case, the skew-symmetric endomorphism A with respect to the L^2 -metric (2.5) is just $\nabla_{\dot{\gamma}}$, the covariant derivative along the loop γ .

(2) Clearly, the next problem is that it is not clear how to define the Pfaffian of $\nabla_{\dot{\gamma}}$. At least there is a somewhat canonical choice for the determinant of $\nabla_{\dot{\gamma}}$, namely the zeta-regularized determinant $\det_{\zeta}(\nabla_{\dot{\gamma}})$, which is defined for a large class of elliptic differential operators. Since the Pfaffian is a square-root of the determinant it is tempting to define the zeta-regularized Pfaffian by just taking the square root of $\det_{\zeta}(\nabla_{\dot{\gamma}})$; however, since the determinant can be zero (which happens precisely if there are parallel vector fields along γ), it is not clear how to take the square root in such a way that one obtains a smooth function on LX this way. Here it is a calculation of Atiyah [Ati85, Lemma 2] that saves the day: He shows by reducing both sides to the eigenvalues of the parallel transport in the tangent bundle that (locally) one has the equality³

$$\det_{\zeta}(\nabla_{\dot{\gamma}}) = (-1)^{n/2} \left(\operatorname{str}_{\mathbb{C}}[\gamma \|_{0}^{1}]^{\Sigma^{\mathbb{C}}}\right)^{2}, \tag{3.2}$$

where $\operatorname{str}_{\mathbb{C}}[\gamma|_0^1]^{\Sigma^{\mathbb{C}}}$ denotes the complex super trace of the parallel transport in the complex spinor bundle. Hence provided that X is spin, there is a good candidate for the zeta-regularized Pfaffian of $\nabla_{\dot{\gamma}}$, namely $\operatorname{Pf}_{\zeta}(\nabla_{\dot{\gamma}}) := i^{n/2} \operatorname{str}_{\mathbb{C}}[\gamma|_0^1]^{\Sigma^{\mathbb{C}}}$. Note that this fits beautifully into the statement that the loop space LX is orientable precisely if X is spin [ST05], [Wal16]. If $n = \dim(X)$ is odd, then both sides in (3.2) are zero; let us assume that n is even from now on.

(3) Hence we formally write

$$I_T = \int_{\mathsf{L}X} e^{-E/T} [e^{\omega}]_{\mathsf{top}} d\gamma = i^{n/2} \int_{\mathsf{L}X} e^{-E/T} \operatorname{str}[\gamma \|_0^1]^{\Sigma^{\mathbb{C}}} d\gamma, \tag{3.3}$$

where $d\gamma$ is supposed to be the "Riemannian volume measure on LX". Here we are facing the next problem, since on infinite-dimensional Riemannian manifolds, there is no such thing as a Riemannian volume measure. However, on the loop space, there are the Wiener measures \mathbb{W}_T , parametrized by a parameter T > 0, and it is well-known that formally, we have the equality

$$\mathrm{dW}_T \stackrel{\text{formally}}{=} \frac{1}{(2\pi T)^{\infty/2}} e^{-E/T} \mathrm{d}\gamma,$$

³The minus signs on the right hand side (which are due to the fact that we took the complex spinor bundle here) are not there in Atiyah's original paper. The problem seems to by Lemma 3 in [Ati85]; compare this with Prop. 3.23 in [BGV04].

which can be given a somewhat rigorous meaning using finite-dimensional approximation [AD99], [BP08], [Lud17]. Plugging this into (3.3), we obtain

$$I_T = i^{n/2} \mathbb{W}_T \left[\operatorname{str}[\gamma \|_0^1]^{\Sigma^{\mathbb{C}}} \right].$$

(4) Here we finally arrived at a rigorous mathematical expression. Using the Feynman-Kac formula (Thm. 2.1), this integral can be explicitly computed. Namely, up to a scalar curvature term coming from the Lichnerowicz formula for the square \mathbb{D}^2 of the Dirac operator⁴, we obtain

$$I_T = i^{n/2} \text{Str}(e^{-T \not D^2/2}) = i^{n/2} \operatorname{ind}(\not D),$$
 (3.4)

where in the second step, we used the famous formula of McKean and Singer. In particular, this result is independent of T.

Summarizing, we massaged the integral I_T by a series of formal manipulations and ended at a definite and well-defined value for it, the index of the Dirac operator.

What makes this really interesting is that the formal integral I_T can be evaluated in yet another way, which gives a formal proof of the Atiyah-Singer Index Theorem. In order to do this, the integral has to be considered in the context of equivariant cohomology. Namely, notice that the integrant $e^{-E/T+\omega}$ is equivariantly closed. If we pretend that LX is a finite-dimensional compact manifold, we can apply the localization principle in equivariant cohomology also known as Duistermaat-Heckmann formula, which states that the integral of an equivariantly closed differential form reduces to an integral over the fixed point set. In our case, the fixed point set of the \mathbb{T} -action is the submanifold $X \subset LX$ of constant loops. Since $e^{-E/T+\omega} = 1$ on $X \subset LX$, this formally gives

$$I_T = (2\pi T)^{-n/2} \int_X \chi(NX, T)^{-1},$$

where $\chi(NX,T) = \text{Pf}(\mathcal{L}_{\dot{\gamma}} + T^{-1}R^{NX})$ is the equivariant Euler class of the normal bundle NX of X inside LX (compare [BGV04, Thm. 7.13]). This class is not well-defined directly, as the normal bundle is an infinite-dimensional vector bundle, but it can be made sense of using zeta-regularization again, see e.g. [Ati85], [Bis85, Section 2b] or [JP90, Section 5]. The result is

$$\chi(NX,T)^{-1} = \widehat{A}(T) = \det^{1/2}\left(\frac{R/2T}{\sinh(R/2T)}\right),$$

the T-dependent \widehat{A} -form⁵ on X. Comparing this to (3.4), we can set T=1 to obtain

$$\operatorname{ind}(\mathcal{D}) = (2\pi i)^{-n/2} \int_X \widehat{A}(X),$$

⁴This scalar curvature factor remains rather mysterious; in this paper, we just integrate it into the definition of the Integral map.

⁵For this presentation, it is more natural to adopt the geometer's definition of characteristic forms without normalizing factors of $2\pi i$, as in [BGV04].

the Atiyah-Singer index theorem, see [BGV04, Thm. 4.9]. We remark here that these formal arguments were extended by Bismut [Bis85] who gives a formal proof of twisted index theorems by formally considering an integral similar to I_T , but with the integrand $e^{-E/T+\omega} \wedge BCh(\mathcal{V}, \nabla)$ instead, where $BCh(\mathcal{V}, \nabla)$ is a (rigorously defined) differential form on the loop space, now called the Bismut-Chern character.

The integral map I_T constructed below satisfies the formulas (1.5) and (1.6), which would be expected by the arguments of Atiyah and Bismut for an integral map that is formally

$$I_T[\theta] \stackrel{\text{formally}}{=} \int_{\Gamma_X} e^{-E/T + \omega} \wedge \theta.$$

In a sense, this already determines the integral map completely, since the Bismut-Chern characters generate the equivariant cyclic cohomology of the loop space. However, much more is true. For $\theta = \theta_N \wedge \cdots \wedge \theta_1$, we can argue similarly to step (1) above that the "top degree coefficient" of the differential form $e^{\omega} \wedge \theta$ should be given (assuming $\ker(\nabla_{\dot{\gamma}}) = 0$ for simplicity) by the formula

$$[e^{\omega} \wedge \theta_N \wedge \dots \wedge \theta_1]_{\text{top}} = \text{pf}_{\zeta}(\nabla_{\dot{\gamma}}) \text{pf}\left((\theta_a, \nabla_{\dot{\gamma}}^{-1} \theta_b)_{L^2}\right)_{N > a, b > 1}.$$
 (3.5)

In our second paper, we will show that the map I_T constructed in the present paper is indeed given by integrating the function (3.5) with respect to the Wiener measure W_T , which will be not at all obvious from the definition below. We could have used (3.5) to define our integral map, but the definition given in this paper is more convenient for at least three reasons. First, the definition given here makes sense on a much larger domain; second, it turns out that formula (3.5) becomes more complicated at paths γ that admit a parallel vector (closed) vector field around them; third, the formula given below is much easier to work with in order to establish formulas (1.5) and (1.6) from the introduction.

The second part of the "proof" above used the localization principle of equivariant cohomology. This we cannot prove yet for our integral map in full generality, but in Section 10, we give some partial results.

4 The Dirac Density

In this section, we introduce one main ingredient of our integral map, the *Dirac density*. This is essentially used to construct our pathwise map q, which will be done in the next section. Throughout, let X be a Riemannian manifold.

Given a path $\gamma \in PX$, we not only obtain the pullback bundles γ^*TX and over [0,1]; in fact, for each vector bundle \mathcal{V} on X and any number $N \in \mathbb{N}$, we obtain a vector bundle $\gamma^*\mathcal{V}^{\boxtimes N} = \gamma^*\mathcal{V} \boxtimes \cdots \boxtimes \gamma^*\mathcal{V}$ over $[0,1]^N$, the fiber of which at a point (τ_N, \ldots, τ_1) is given by

$$(\gamma^* \mathcal{V} \boxtimes \cdots \boxtimes \gamma^* \mathcal{V})_{(\tau_N, \dots, \tau_1)} = \mathcal{V}_{\gamma(\tau_N)} \otimes \cdots \otimes \mathcal{V}_{\gamma(\tau_1)}.$$

A section of F of such a bundle over $[0,1]^N$ is symmetric if for each permutation $\sigma \in S_N$, the element $F(\tau_{\sigma_N}, \ldots, \tau_{\sigma_1})$ of $\mathcal{V}_{\gamma(\tau_{\sigma_N})} \otimes \cdots \otimes \mathcal{V}_{\gamma(\tau_{\sigma_1})}$ is equal to the element $F(\tau_N, \ldots, \tau_1)$ of $\mathcal{V}_{\gamma(\tau_N)} \otimes \cdots \otimes \mathcal{V}_{\gamma(\tau_1)}$ after identifying the different tensor products using the symmetry isomorphisms of the tensor product that identify $V \otimes W$ with $W \otimes V$.

Since most of the bundles we consider are super vector bundles, i.e. vector bundles with a \mathbb{Z}_2 -grading, we need to formulate the property of being *supersymmetric*. In order to define what this means, we use the following notation.

Notation 4.1 (Super Sign). For a permutation $\sigma \in S_N$ and integers $\ell = (\ell_N, \dots, \ell_1) \in \mathbb{Z}^N$, there is a unique sign $\operatorname{sgn}(\sigma; \ell) = \operatorname{sgn}(\sigma; \ell_N, \dots, \ell_1) \in \{\pm 1\}$ such that we have

$$\vartheta_{\sigma_N}\cdots\vartheta_{\sigma_1}=\operatorname{sgn}(\sigma;\ell)\,\vartheta_N\cdots\vartheta_1$$

for all elements $\vartheta_1, \ldots, \vartheta_N$ of some supercommutative algebra (e.g. ΛV for some vector space V) such that for each j, the parity $|\vartheta_j|$ of ϑ_j equals the parity of ℓ_j , Clearly, this only depends on the parity of the ℓ_j , so it also makes sense to insert vectors $\ell \in \mathbb{Z}_2^N$ instead.

For example, if ℓ_j is odd for all j, then $\operatorname{sgn}(\sigma;\ell) = \operatorname{sgn}(\sigma)$, the usual sign of a permutation, while if all the ℓ_j are even, then $\operatorname{sgn}(\sigma;\ell) = 1$. For two permutations $\sigma, \rho \in S_N$, we have

$$\operatorname{sgn}(\sigma \circ \rho; \ell) = \operatorname{sgn}(\sigma; \ell) \operatorname{sgn}(\rho; \ell^{\sigma}), \tag{4.1}$$

where we used the abbreviation $\ell^{\sigma} := (\ell_{\sigma_N}, \dots, \ell_{\sigma_1})$. We are now ready to make the following definition.

Definition 4.2 (Supersymmetric Sections). Let \mathcal{V} be a super vector bundle on X. For a path γ in X, numbers $(\tau_N, \ldots, \tau_1) \in [0, 1]^N$ and a permutation $\sigma \in S_N$, the *supersymmetry operator* is the map defined on homogeneous elements by

$$\mathbf{s}_{\sigma}: \mathcal{V}_{\gamma(\tau_{N})} \otimes \cdots \otimes \mathcal{V}_{\gamma(\tau_{1})} \longrightarrow \mathcal{V}_{\gamma(\tau_{\sigma_{N}})} \otimes \cdots \otimes \mathcal{V}_{\tau_{\sigma_{1}}}$$
$$v_{N} \otimes \cdots \otimes v_{1} \longmapsto \operatorname{sgn}(\sigma; |v_{N}|, \dots, |v_{1}|) v_{\sigma_{N}} \otimes \cdots \otimes v_{\sigma_{1}},$$

where $|v_j|$ denotes the parity of v_j . A section F of the bundle $\gamma^* \mathcal{V} \boxtimes \cdots \boxtimes \gamma^* \mathcal{V}$ over $[0, 1]^N$ is called *supersymmetric*, if it satisfies

$$F(\tau_{\sigma_N},\ldots,\tau_{\sigma_1})=\mathbf{s}_{\sigma}F(\tau_N,\ldots,\tau_1).$$

for all $\sigma \in S_N$ and all $(\tau_N, \dots, \tau_1) \in [0, 1]^N$.

Usually, \mathcal{V} will be one of the super vector bundles $\Lambda T'X$, $\operatorname{Cl} TX$, or the subbundle T'X of $\Lambda T'X$ with the induced grading, which makes it purely odd (in super geometry, this would be denoted by ΠTX).

Assume now that X is spin with (real) spinor bundle Σ . Then to a path $\gamma \in PX$ and a partition $0 \le \tau_1 < \tau_2 < \cdots < \tau_N \le 1$ of the interval [0, 1], we have the associated element

$$D_N^{\mathrm{rel}}(\tau_N, \dots, \tau_1) := \frac{1}{2^{N/2} N!} \left[\gamma \|_{\tau_N}^1 \right]^{\Sigma} \otimes \left[\gamma \|_{\tau_{N-1}}^{\tau_N} \right]^{\Sigma} \otimes \dots \otimes \left[\gamma \|_{\tau_1}^{\tau_2} \right]^{\Sigma} \otimes \left[\gamma \|_0^{\tau_1} \right]^{\Sigma}, \tag{4.2}$$

obtained by formally tensoring various copies of the spinor parallel transport. By definition, this is an element of the left hand side of the natural isomorphism

$$\bigotimes_{j=1}^{N+1} \operatorname{Hom}_{\operatorname{Cl}}(\Sigma_{\gamma(\tau_{j-1})}, \Sigma_{\gamma(\tau_{j})}) \cong \operatorname{Hom}_{\operatorname{Cl}_{n}}(\Sigma_{\gamma(0)}, \Sigma_{\gamma(1)}) \otimes \bigotimes_{j=1}^{N} \left(\operatorname{Cl}(T_{\gamma(\tau_{j})}X)\right)', \tag{4.3}$$

where we set $\tau_{N+1} := 1$ and $\tau_0 := 0$ for convenience. The identification (4.3) is given by sending elements $\Phi_{N+1} \otimes \cdots \otimes \Phi_1$ of the left hand side to the $\operatorname{Hom}_{\operatorname{Cl}_n}(\Sigma_{\gamma(0)}, \Sigma_{\gamma(1)})$ -valued functional defined by

$$\left\langle \Phi_{N+1} \otimes \cdots \otimes \Phi_1, a_N \otimes \cdots \otimes a_1 \right\rangle := \Phi_{N+1} a_N \Phi_N \cdots \Phi_2 a_1 \Phi_1 = \Phi_{N+1} \prod_{j=1}^N a_j \Phi_j \qquad (4.4)$$

for $a_j \in \text{Cl}(T_{\gamma(\tau_j)}X)$, j = 1, ..., N, acting on $\Sigma_{\gamma(\tau_j)}$ by left multiplication. Using the dual map $\mathbf{c}' : (\text{Cl}TX)' \to (\Lambda T'X)' \cong \Lambda TX$ of the quantization map (2.1), we then define

$$\mathbf{D}_N^{\mathrm{rel}}(\tau_N,\ldots,\tau_1) := \mathrm{id} \otimes \mathbf{c}' \otimes \cdots \otimes \mathbf{c}' \big(D_N^{\mathrm{rel}}(\tau_N,\ldots,\tau_1) \big),$$

where we interpret $D_N^{\text{rel}}(\tau_N, \ldots, \tau_1)$ as an element of the right hand side of (4.3). This defines $\mathbf{D}_N^{\text{rel}}(\tau_N, \ldots, \tau_1)$ for all τ with $0 \le \tau_1 < \cdots < \tau_N \le 1$. Now for any τ with $\tau_i \ne \tau_j$ for all $i \ne j$, set

$$\mathbf{D}_{N}^{\mathrm{rel}}(\tau_{N},\ldots,\tau_{1}) = \mathbf{s}_{\sigma}^{*} \mathbf{D}_{N}^{\mathrm{rel}}(\tau_{\sigma_{N}},\ldots,\tau_{\sigma_{1}}), \tag{4.5}$$

where $\sigma \in S_N$ is the unique permutation with $0 \le \tau_{\sigma_1} < \cdots < \tau_{\sigma_N} \le 1$ and where \mathbf{s}_{σ} is the supersymmetry transformation defined in Lemma 4.2 (here we consider ΛTX as a super vector bundle with the usual even/odd grading). This extends the domain of $\mathbf{D}_N^{\mathrm{rel}}$ to all $\tau \in [0,1]_{\circ}^N$, where we set

$$[0,1]_{\circ}^{N} := \{ \tau = (\tau_{N}, \dots, \tau_{1}) \in [0,1]^{N} \mid \forall i \neq j : \tau_{i} \neq \tau_{j} \}.$$

$$(4.6)$$

This turns $\mathbf{D}_N^{\mathrm{rel}}$ into a $\mathrm{Hom}(\Sigma_{\gamma(0)}, \Sigma_{\gamma(1)})$ -valued super-symmetric section of the bundle $\gamma^*\Lambda TX^{\boxtimes N}$ over $[0,1]_{\circ}^N$ (Notice here that the statement that $\mathbf{D}_N^{\mathrm{rel}}$ is supersymmetric still makes sense, as the action of the symmetric group on $[0,1]^N$ that permutes the entries restricts to an action on $[0,1]_{\circ}^N$).

Definition 4.3 (Relative Dirac Density). The direct product

$$\mathbf{D}^{\mathrm{rel}} := \sum_{N=0}^{\infty} \mathbf{D}_{N}^{\mathrm{rel}} \in \prod_{N=0}^{\infty} C_{\mathrm{susy}}^{\infty} ([0,1]_{\circ}^{N}, \gamma^{*} \Lambda T X^{\boxtimes N}) \otimes \mathrm{Hom}_{\mathrm{Cl}_{n}} (\Sigma_{\gamma(0)}, \Sigma_{\gamma(1)})$$

is called the *relative Dirac density* over the path γ . Here $C_{\text{susy}}^{\infty}([0,1]_{\circ}^{N}, \gamma^{*}\Lambda TX^{\boxtimes N})$ denotes the space of smooth supersymmetric sections of the bundle $\gamma^{*}\Lambda TX^{\boxtimes N}$ (with the even/odd grading) over $[0,1]_{\circ}^{N}$, as in Def. 4.2.

Remark 4.4. Let $\tau \in [0, 1]_{\circ}^{N}$. Then directly from the definition of the map (4.4) used for the identification (4.3) above, it follows that for forms $\vartheta_{1}, \ldots, \vartheta_{N}$ with $\vartheta_{j} \in \Lambda^{\ell_{j}} T'_{\gamma(\tau_{j})} M$ and $j = 1, \ldots, N$, we have

$$\left\langle \mathbf{D}_{N}^{\mathrm{rel}}(\tau_{N},\ldots,\tau_{1}),\vartheta_{N}\otimes\cdots\otimes\vartheta_{1}\right\rangle = \frac{1}{2^{N/2}N!}\operatorname{sgn}(\sigma;\ell)\left[\gamma\Vert_{\tau_{\sigma_{N}}}^{1}\right]^{\Sigma}\prod_{j=1}^{N}\mathbf{c}(\vartheta_{\sigma_{j}})\left[\gamma\Vert_{\tau_{\sigma_{j-1}}}^{\tau_{\sigma_{j}}}\right]^{\Sigma},$$

where $\sigma \in S_N$ is the unique permutation such that $\tau_{\sigma_1} < \cdots < \tau_{\sigma_N}$ and $\ell = (\ell_N, \dots, \ell_1)$.

If γ is a *loop*, that is $\gamma \in LX \subset PX$, then we can set

$$\mathbf{D}_N := \operatorname{str} \mathbf{D}_N^{\operatorname{rel}},$$

where the supertrace is taken of the $\operatorname{Hom}_{\operatorname{Cl}_n}(\Sigma_{\gamma(0)}, \Sigma_{\gamma(1)}) \cong \operatorname{Cl}(T_{\gamma(0)}X)$ factor of $\mathbf{D}_N^{\operatorname{rel}}$. Note that this only makes sense because γ is a loop, i.e. $\gamma(0) = \gamma(1)$. A priori, this is an element of $C_{\operatorname{susy}}^{\infty}([0,1]_{\circ}^N, \gamma^*\Lambda T'X^{\boxtimes N})$, but the following lemma shows that it actually descends to the torus.

Lemma 4.5. The absolute Dirac density is equivariant with respect to the \mathbb{T} -action on $\mathsf{L}_c X$. Concretely, it satisfies

$$\mathbf{D}_N|_{t\cdot\gamma}(\tau_N,\ldots,\tau_1) = \mathbf{D}_N|_{\gamma}(\tau_N+t,\ldots,\tau_1+t)$$

for each $\gamma \in LX$ and $t \in \mathbb{T}$.

Proof. Let $0 \le \tau_1 < \dots < \tau_N \le 1$ be given. Notice first that $[t \cdot \gamma|_{\tau_{j-1}}^{\tau_j}]^{\Sigma} = [\gamma|_{\tau_{j-1}+t}^{\tau_j+t}]^{\Sigma}$. Now if j denotes an index such that $\tau_{j-1} \le 1 - t \le \tau_j$, we have

$$D_N^{\mathrm{rel}}|_{t\cdot\gamma}(\tau_N,\ldots,\tau_1) = \frac{1}{2^{N/2}N!} [\gamma||_{\tau_N+t-1}^t]^{\Sigma} \otimes \cdots \otimes [\gamma||_0^{\tau_j+t-1}]^{\Sigma} [\gamma||_{\tau_{j-1}+t}^1]^{\Sigma} \otimes \cdots \otimes [\gamma||_t^{\tau_1+t}]^{\Sigma}.$$

Let $\vartheta_j \in \Lambda T'_{\gamma(\tau_j+t)}X$ be homogeneous of degree ℓ_j for $j=1,\ldots,N$. Then by the cyclic permutation property (2.3) of the supertrace,

$$2^{N/2}N!\langle \mathbf{D}_{N}|_{t\cdot\gamma}(\tau_{N},\ldots,\tau_{1}),\vartheta_{N}\otimes\cdots\otimes\vartheta_{1}\rangle$$

$$=\operatorname{str}\left(\left[\gamma\|_{\tau_{N}+t-1}^{t}\right]^{\Sigma}\mathbf{c}(\vartheta_{N})\cdots\mathbf{c}(\vartheta_{j})\left[\gamma\|_{0}^{\tau_{j}+t-1}\right]^{\Sigma}\left[\gamma\|_{\tau_{j-1}+t}^{1}\right]^{\Sigma}\mathbf{c}(\vartheta_{j-1})\cdots\mathbf{c}(\vartheta_{1})\left[\gamma\|_{t}^{\tau_{1}+t}\right]^{\Sigma}\right)$$

$$=\epsilon\operatorname{str}\left(\left[\gamma\|_{\tau_{j-1}+t}^{1}\right]^{\Sigma}\mathbf{c}(\vartheta_{j-1})\cdots\mathbf{c}(\vartheta_{1})\left[\gamma\|_{t}^{\tau_{1}+t}\right]^{\Sigma}\left[\gamma\|_{\tau_{N}+t-1}^{t}\right]^{\Sigma}\mathbf{c}(\vartheta_{N})\cdots\mathbf{c}(\vartheta_{j})\left[\gamma\|_{0}^{\tau_{j}+t-1}\right]^{\Sigma}\right),$$

where $\epsilon = (-1)^{(\ell_N + \dots + \ell_j)(\ell_{j-1} + \dots + \ell_1)}$. Notice that $\epsilon = \operatorname{sgn}(\sigma; \ell)$, where σ is the permutation that maps $(N, \dots, 1)$ to $(j-1, \dots, 1, N, \dots, j)$. We therefore obtain further

$$\begin{split} \left\langle \mathbf{D}_{N}^{\mathrm{rel}} \right|_{t \cdot \gamma} &(\tau_{N}, \dots, \tau_{1}), \vartheta_{N} \otimes \dots \otimes \vartheta_{1} \right\rangle \\ &= \epsilon \left\langle \mathbf{D}_{N}^{\mathrm{rel}} \right|_{\gamma} &(\tau_{j-1} + t, \dots, \tau_{1} + t, \tau_{N} + t, \dots, \tau_{j} + t), \vartheta_{j-1} \otimes \dots \otimes \vartheta_{1} \otimes \vartheta_{N} \otimes \dots \otimes \vartheta_{j} \right\rangle \\ &= \left\langle \mathbf{D}_{N}^{\mathrm{rel}} \right|_{\gamma} &(\tau_{j-1} + t, \dots, \tau_{1} + t, \tau_{N} + t - 1, \dots, \tau_{j} + t - 1), \mathbf{s}_{\sigma} &(\vartheta_{N} \otimes \dots \otimes \vartheta_{1}) \right\rangle \\ &= \left\langle \mathbf{s}_{\sigma}^{*} \mathbf{D}_{N}^{\mathrm{rel}} \right|_{\gamma} &(\tau_{j-1} + t, \dots, \tau_{1} + t, \tau_{N} + t - 1, \dots, \tau_{j} + t - 1), \vartheta_{N} \otimes \dots \otimes \vartheta_{1} \right\rangle \\ &= \left\langle \mathbf{D}_{N}^{\mathrm{rel}} \right|_{\gamma} &(\tau_{N} + t, \dots, \tau_{1} + t), \vartheta_{N} \otimes \dots \otimes \vartheta_{1} \right\rangle \end{split}$$

where σ is the permutation that maps $(N,\ldots,1)$ to $(j-1,\ldots,1,N,\ldots,j)$ and $\epsilon=(-1)^{(\ell_N+\cdots+\ell_j)(\ell_{j-1}+\cdots+\ell_1)}$ is its sign. This shows the claim by duality, for all τ with $0 \le \tau_1 < \cdots < \tau_N \le 1$. For general τ , the claim follows from supersymmetry.

Lemma 4.5 above shows that \mathbf{D}_N descends to a smooth section of the bundle $\gamma^* \Lambda T X^{\boxtimes N}$ over the open subset

$$T_{\circ}^{N} := \{ \tau = (\tau_{N}, \dots, \tau_{1}) \in T^{N} \mid \forall i \neq j : \tau_{i} \neq \tau_{j} \}.$$
 (4.7)

We can therefore make the following definition.

Definition 4.6 (Absolute Dirac Density). Suppose that $\gamma \in LX$. Then the element

$$\mathbf{D} := \sum_{N=0}^{\infty} \mathbf{D}_{N} \in \prod_{N=0}^{\infty} C_{\text{susy}}^{\infty} (T_{\circ}^{N}, \gamma^{*} \Lambda T X^{\boxtimes N})$$

of the direct product is called (absolute) Dirac density.

5 Definition of the Top Degree Map

Let X be a manifold with continuous loop space L_cX . As noted above, for $\gamma \in L_cM$, we have the natural identification $T_{\gamma}L_cM = C(S^1, \gamma^*TX)$ of tangent vectors to γ with continuous vector fields along γ . Dually, the cotangent space at a path γ can be naturally identified the space of finite, signed Borel measures equipped with values in $\gamma^*T'X$,

$$T'_{\gamma}\mathsf{L}_cX\cong C(S^1,\gamma^*TX)'\cong\mathcal{M}(S^1,\gamma^*T'X),$$

by the Markhov-Kakutani-Riesz theorem. This space naturally carries the total variation norm, which coincides with the usual dual space norm. This gives a good notion of one-forms on L_cX .

Now generally, if Y is an infinite-dimensional manifold, the "correct" definition for the space of differential N-forms on it is $\Omega^N(Y) := C^{\infty}(Y, \operatorname{Alt}^N(TY, \mathbb{R}))$, the space of sections of the bundle of bounded, alternating, multi-linear maps on the tangent bundle. However, the bundle $\operatorname{Alt}^N(T \sqcup_c X, \mathbb{R})$ seems to be too large for our integral map to exist on it. Instead we will consider the smaller bundle $\operatorname{Alt}^N(T \sqcup_c X, \mathbb{R})$ of integral alternating N-linear forms on $T \sqcup_c X$. This is a notion from functional analysis (see [Tr67, Ch. 49]), but in our situation, this boils down to the definition

$$Alt_{int}^{N}(T_{\gamma}\mathsf{L}_{c}X) := \mathcal{M}_{susv}(T^{N}, \gamma^{*}T'X^{\boxtimes N}), \tag{5.1}$$

the space of supersymmetric T'X-valued measures on the torus T^N . An element $\theta \in \mathcal{M}_{\text{susy}}(T^N, \gamma^*T'X^{\boxtimes N})$ can be paired against tangent vectors $V_1, \ldots, V_N \in T_\gamma \mathsf{L}_c X$ by setting

$$\theta[V_N,\ldots,V_1] := \int_{T^N} \langle \theta(\tau_N,\ldots,\tau_1), V_N(\tau_N) \otimes \cdots \otimes V_1(\tau_1) \rangle d\tau$$

⁶This is because this is the only natural space of differential forms in which pullback along smooth maps, exterior derivative and lie derivative are all well defined [KM97, Section 33].

where by the integral notation, we really mean pairing of the function $V_N \boxtimes \cdots \boxtimes V_1 \in C_{\text{susy}}(T^N, \gamma^*TX^{\boxtimes N})$ against the measure θ if θ is not absolutely continuous. This defines a continuous multi-linear form on $T_{\gamma}\mathsf{L}_cX$ which is alternating by the supersymmetry of θ (which in this setup is really just anti-symmetry). Henceforth, we will freely identify supersymmetric measures θ with alternating multi-linear forms on $T_{\gamma}\mathsf{L}_cX$. Sections of $\mathsf{Alt}^N_{\mathrm{int}}(T_{\gamma}\mathsf{L}_cX)$ will be denoted by

$$\Omega_{\rm int}(\mathsf{L}_c X) := C^{\infty}(\mathsf{L}_c X, \operatorname{Alt}_{\rm int}^N(\mathsf{L}_c X)).$$

This is the space of integral differential forms on L_cX .

Remark 5.1. The inclusion of $\operatorname{Alt}_{\operatorname{int}}^N(T_{\gamma}\mathsf{L}_cX)$ into $\operatorname{Alt}^N(T_{\gamma}\mathsf{L}_cX)$ can be thought of more structurally as follows. By definition, $\operatorname{Alt}^N(T_{\gamma}\mathsf{L}_cX)$ is the dual space of $\Lambda^N T_{\gamma}\mathsf{L}_cX$, the N-fold algebraic exterior product of $\Lambda^N T_{\gamma}\mathsf{L}_cX$. For its completion $\Lambda^N_{\varepsilon}T_{\gamma}\mathsf{L}_cX$ with respect to the injective tensor product topology, we have the identification

$$\Lambda_{\varepsilon}^N T_{\gamma} \mathsf{L}_c X \cong C_{\text{susy}}(T^N, \gamma^* T' X^{\boxtimes N}),$$

which is precisely the pre-dual of $\operatorname{Alt}_{\operatorname{int}}^N(T_\gamma \mathsf{L}_c X, \mathbb{R})$ (this follows from [Tr67, Thm. 44.1] and adjunction, respectively ibid. Ex. 44.2). Hence the inclusion of $\operatorname{Alt}_{\operatorname{int}}^N(T_\gamma \mathsf{L}_c X, \mathbb{R})$ into $\operatorname{Alt}^N(T_\gamma \mathsf{L}_c X, \mathbb{R})$ is the dual of the inclusion

$$\Lambda^N T_{\gamma} \mathsf{L}_c X \longrightarrow C_{\text{susy}}(T^N, \gamma^* T' X^{\boxtimes N}). \tag{5.2}$$

It is injective since the inclusion (5.2) has dense image.

Remark 5.2. Equipping $\operatorname{Alt}_{\operatorname{int}}^N(T_\gamma \mathsf{L}_c X)$ with its usual total variation norm, we see that the inclusion is isometric: Using the Goldstine theorem, the injective tensor norm can be written as

$$\varepsilon(\mu) = \sup \{ \mu[V_1, \dots, V_N] \mid V_j \in C(S^1, \gamma^* TX), ||V_j|| = 1, j = 1, \dots, N \}.$$

This is prescisely the dual space norm on the space of measures on T^N , which coincides with the total variation norm. Hence $\operatorname{Alt}_{\operatorname{int}}^N(T_\gamma\mathsf{L}_cX)$ is a closed subspace of $\operatorname{Alt}^N(T_\gamma\mathsf{L}_cX)$.

Remark 5.3. All the function spaces along paths considered in this paper glue together to smooth vector bundles over L_cX , by varying γ . Namely, on L_cX , one has the bundle L_cP^{SO} , where P^{SO} is the frame bundle of X. This is an L_cSO_n -principal bundle over L_cX , and all vector bundles are associated vector bundles to this principal bundle via obvious representations of L_cSO_n .

There is a natural connection on L_cP^{SO} . It is defined by declaring that for $p \in L_cP^{SO}$, a tangent vector $F \in T_pL_cP^{SO} \cong C^{\infty}(S^1, p^*TP^{SO})$ is horizontal if F(t) is horizontal in $T_{p(t)}P^{SO}$ for every $t \in S^1$. This induces a covariant derivative on associated bundles, which will be metric with respect to the usual L^2 metrics on these bundles.

Remark 5.4. The spaces $\operatorname{Alt}_{\operatorname{int}}^N(T_{\gamma}\mathsf{L}_cX)$ support a wedge product. For $\theta \in \operatorname{Alt}_{\operatorname{int}}^N(T_{\gamma}\mathsf{L}_cX)$ and $\zeta \in \operatorname{Alt}_{\operatorname{int}}^M(T_{\gamma}\mathsf{L}_cX)$, the wedge product $\theta \wedge \zeta \in \operatorname{Alt}_{\operatorname{int}}^{N+M}(T_{\gamma}\mathsf{L}_cX)$ is defined by

$$(\theta \wedge \zeta)(\tau_{N+M}, \dots, \tau_1) := \frac{1}{N!M!} \sum_{\sigma \in S_{N+M}} \mathbf{s}_{\sigma}^* (\theta(\tau_{\sigma_{N+M}}, \dots, \tau_{\sigma_{M+1}}) \otimes \zeta(\tau_{\sigma_M}, \dots, \tau_{\sigma_1})), \quad (5.3)$$

where the formula has to be interpreted in the distributional sense. The same is true if one replaces L_cX by P_cX .

Remark 5.5. As shown in [KM97, Section 33], there is a natural exterior differential on differential forms on any infinite-dimensional manifold. This exterior differential preserves the space $\Omega_{\text{int}}(\mathsf{L}_cX)$. Namely, the wedge product is the anti-symmetrization of the covariant derivative using the connection discussed in Remark 5.3. But the covariant derivative preserves the subbundle $\text{Alt}_{\text{int}}^N(T_\gamma\mathsf{L}_cX)$, as it is closed in $\text{Alt}^N(T_\gamma\mathsf{L}_cX)$.

With a view on the definition of the Dirac density (Def. 4.6), it seems that we can *almost* define a map

$$q: \mathrm{Alt}^N_{\mathrm{int}}(T_\gamma \mathsf{L}_c X, \mathbb{R}) \cong \mathcal{M}_{\mathrm{susy}}(T^N, \gamma^* T' X^{\boxtimes N}) \longrightarrow \mathbb{R}$$

by pairing N-forms θ against the N-th Dirac density \mathbf{D}_N . However, the trouble is that \mathbf{D}_N is only continuous on the subset $T_{\circ}^N \subset T^N$ of the torus and *not* on the "generalized diagonal"

$$T^N \setminus T_{\circ}^N = \{ \tau \mid \tau_i = \tau_j \text{ for some } i \neq j \}.$$

This causes problems if the set $T^N \setminus T_o^N$ is not a null set for θ , considered as a measure on T^N . There are very concrete and important instances of such forms, see for example the forms $P_{\varphi}\theta$ defined in (6.4). Below, we therefore decompose the space of measures with respect to their singularity structure in order to tackle this problem.

For convenience, we will first discuss this decomposition for the continuous path space P_cX and then descend to the loop space. Notice that all constructions and remarks are valid for P_cX instead of L_cX , if one just replaces every instance of S^1 by [0,1], respectively T^N by $[0,1]^N$.

For $N, M \in \mathbb{N}$, $M \leq N$, denote by $P_{M,N}$ the set of tuples $\ell = (\ell_M, \dots, \ell_1) \in \mathbb{N}^M$ with $\ell_M + \dots + \ell_1 = N$. Any $\ell \in P_{N,M}$ defines a map

$$\kappa_{\ell}: [0,1]^{M} \longrightarrow [0,1]^{N}, \qquad (\tau_{M}, \dots, \tau_{1}) \longmapsto (\underbrace{\tau_{M}, \dots, \tau_{M}}_{\ell_{M} \text{ times}}, \dots, \underbrace{\tau_{1}, \dots, \tau_{1}}_{\ell_{1} \text{ times}}).$$

Observe now that the sets $S_{\ell} := \kappa_{\ell}([0,1]_{\circ}^{M})$ for $\ell \in P_{M,N}$, $M \leq N$ form a partition of $[0,1]^{N}$: They are pairwise disjoint and their union is exactly $[0,1]^{N}$. This yields the direct sum decomposition of measure spaces,

$$\mathcal{M}([0,1]^N) \cong \bigoplus_{M=1}^N \bigoplus_{\ell \in P_{M,N}} \mathcal{M}(S_\ell) \cong \bigoplus_{M=1}^N \bigoplus_{\ell \in P_{M,N}} \mathcal{M}([0,1]_\circ^M). \tag{5.4}$$

Here the first isomorphism is the obvious one given by push-forward via the inclusion maps, while the second isomorphism is assembled from the push-forward maps

$$\mathcal{M}([0,1]^M_{\circ}) \longrightarrow \mathcal{M}(S_{\ell}), \qquad \mu \longmapsto \sqrt{\ell_1 \cdots \ell_M} \, \mu \circ \kappa_{\ell}^{-1}$$

for $M \leq N$, $\ell \in P_{M,N}$. Here the factor $\sqrt{\ell_1 \cdots \ell_M}$ is precisely the Jacobian of the map κ_{ℓ} , which is present to ensure that the second isomorphism in (5.4) is actually isometric. We now adapt this to our vector-valued, supersymmetric situation. In this case, we obtain maps

$$K_{\ell}: \mathcal{M}([0,1]^M, \gamma^* \Lambda^{\ell_M} TX \boxtimes \cdots \boxtimes \Lambda^{\ell_1} TX) \longrightarrow \mathcal{M}([0,1]^N, \gamma^* TX^{\boxtimes N})$$
 (5.5)

essentially as well by pushing measures forward along κ_{ℓ} . Precisely, the map K_{ℓ} is the dual of the map $A \circ \kappa_{\ell}^*$, where

$$\kappa_{\ell}^*: C([0,1]^N, \gamma^*TX^{\boxtimes N}) \longrightarrow C([0,1]^M, \gamma^*TX^{\otimes \ell_M} \boxtimes \cdots \boxtimes \gamma^*TX^{\otimes \ell_1})$$

is the pullback map, and

$$A: T_{\gamma(t_M)} X^{\otimes \ell_M} \boxtimes \cdots \boxtimes \gamma^* T_{\gamma(t_1)} X^{\otimes \ell_1} \longrightarrow \Lambda^{\ell_M} T_{\gamma(\tau_M)} X \boxtimes \cdots \boxtimes \Lambda^{\ell_1} T_{\gamma(\tau_1)} X$$

is the vector bundle map that anti-symmetrizes inside each block. We now have the following result analogous to similar to (5.4) above.

Proposition 5.6. For any $N \in \mathbb{N}$, the maps K_{ℓ} defined above assemble to an isometric isomorphism

$$\mathcal{M}_{\text{susy}}([0,1]^N, \gamma^* T' X^{\boxtimes N}) \cong \bigoplus_{M=1}^N \left[\bigoplus_{\ell \in P_{M,N}} \mathcal{M}([0,1]_{\circ}^M, \gamma^* \Lambda^{\ell_M} T' X \boxtimes \cdots \boxtimes \gamma^* \Lambda^{\ell_1} T' X) \right]_{\text{susy}}.$$

Here by the requirement that an element θ of the M-th summand on the right hand side is supersymmetric, we mean that it is supersymmetric as an element of the larger space $\mathcal{M}([0,1]^M_{\circ}, \gamma^*\Lambda T'X^{\boxtimes N})$. By taking the direct sum over $N \in \mathbb{N}$, we obtain a linear map

$$K: \bigoplus_{M=0}^{\infty} \mathcal{M}_{\text{susy}}([0,1]_{\circ}^{M}, \gamma^{*}\Lambda^{\geq 1}T'X^{\boxtimes M}) \longrightarrow \bigoplus_{N=0}^{\infty} \mathcal{M}_{\text{susy}}([0,1]^{N}, \gamma^{*}T'X^{\boxtimes N}), \tag{5.6}$$

which is an isometric isomorphism of Banach spaces in each degree. Provided $\gamma \in \mathsf{L}_c X$, we obtain similar isomorphisms after replacing $[0,1]^N$ by T^N respectively $[0,1]^M$ by T^M .

Above, we adopt the convention that the summand with M=0 respectively N=0 is \mathbb{R} by definition on both sides of (5.6).

Proof. Using the maps K_{ℓ} above, we obtain maps

$$K_N: \bigoplus_{M=1}^N \bigoplus_{\ell \in P_{M,N}} \mathcal{M}\big([0,1]^M_\circ, \gamma^* \Lambda^{\ell_M} T'X \boxtimes \cdots \boxtimes \gamma^* \Lambda^{\ell_1} T'X\big) \longrightarrow \mathcal{M}\big([0,1]^N, \gamma^* T'X^{\boxtimes N}\big)$$

given by mapping a direct sum $(\theta_{\ell})_{\ell \in P_{M,N}}$ to $\sum_{\ell} K_{\ell} \theta_{\ell}$. Because the sets $S_{\ell} = \kappa_{\ell}([0,1]_{\circ}^{M})$ for $M \leq N$ and $\ell \in P_{M,N}$ are disjoint, this map is injective. It is also isometric, i.e. preserves the norm: Since the measures $K_{\ell}\theta_{\ell}$ and $K_{\ell'}\theta_{\ell'}$ have disjoint support for $\ell \neq \ell'$, this can be checked for each map K_{ℓ} separately, and for these, this is essentially due to the fact that we included the Jacobian factor $\sqrt{\ell_1 \cdots \ell_M}$ in the definition of K_{ℓ} . It is furthermore clear that for each $M \leq N$, supersymmetric elements of the direct sum

$$\bigoplus_{\ell \in P_{M,N}} \mathcal{M}([0,1]_{\circ}^{M}, \gamma^{*}\Lambda^{\ell_{M}}T'X \boxtimes \cdots \boxtimes \gamma^{*}\Lambda^{\ell_{1}}T'X) \subseteq \mathcal{M}([0,1]_{\circ}^{M}, \gamma^{*}\Lambda^{\geq 1}T'X^{\boxtimes M})$$

are mapped to supersymmetric elements of $\mathcal{M}([0,1]^N, \gamma^*T'X^{\boxtimes N})$.

To see that K_{ℓ} is surjective, decompose a given element $\theta \in \mathcal{M}_{\text{susy}}([0,1]^N, \gamma^* T' X^{\boxtimes N})$ into a sum $\theta = \sum_{\ell} \theta_{\ell}$, such that each θ_{ℓ} has support on S_{ℓ} ; in other words, such that for each ℓ , the set $[0,1]^N \setminus S_{\ell}$ is a zero set for θ_{ℓ} . Since the maps κ_{ℓ} , are homeomorphisms from $[0,1]_{\circ}^M$ onto S_{ℓ} , we can now push forward each measure θ_{ℓ} with the inverse of κ_{ℓ} to obtain a preimage

$$\widetilde{\theta}_{\ell} = \theta_{\ell} \circ \kappa_{\ell} \in \mathcal{M}([0,1]^{M}_{\circ}, \gamma^{*}T'X^{\otimes \ell_{M}} \boxtimes \cdots \boxtimes \gamma^{*}T'X^{\otimes \ell_{1}}).$$

From the supersymmetry assumption on θ , it follows that in fact, the $\widetilde{\theta}_{\ell}$ are contained in the subspace

$$\mathcal{M}([0,1]^{M}_{\circ}, \gamma^{*}\Lambda^{\ell_{M}}T'X \boxtimes \cdots \boxtimes \gamma^{*}\Lambda^{\ell_{1}}T'X) \subseteq \mathcal{M}([0,1]^{M}_{\circ}, \gamma^{*}T'X^{\otimes \ell_{M}} \boxtimes \cdots \boxtimes \gamma^{*}T'X^{\otimes \ell_{1}}).$$

For a fixed M, it follows furthermore the sum $\sum_{\ell \in P_{M,N}} \widetilde{\theta}_{\ell}$ will be supersymmetric. \square

Observe now that for each path γ , the relative Dirac density $\mathbf{D}^{\mathrm{rel}} = \mathbf{D}^{\mathrm{rel}}|_{\gamma}$ defined in Section 4 is an element of the predual of the right hand side of (5.6), hence can be paired with elements of this space. However, there is an issue here, as the Dirac density is only defined in the case that γ is differentiable because it involves the parallel transport along γ . Therefore, we restrict ourselves to smooth paths for the moment. Using the stochastic parallel transport, q can then be made into an almost everywhere defined map. This will be discussed in Section 6.

Definition 5.7 (The q Functional). Let X be a Riemannian spin manifold. Suppose that $\gamma \in PX \subset P_cX$. For an integral form $\theta \in Alt^N_{int}(T_{\gamma}P_cM)$, we set

$$q^{\mathrm{rel}}(\theta) := (\mathbf{D}^{\mathrm{rel}}, \theta)_{L^2} \in \mathrm{Hom}(\Sigma_{\gamma(0)}, \Sigma_{\gamma(1)}),$$

using the identification (5.6). Similarly, if $\gamma \in LX \subset L_cX$, we set

$$q(\theta) := (\mathbf{D}, \theta)_{L^2} = \operatorname{str}(\theta, \mathbf{D}^{\operatorname{rel}})_{L^2} \in \mathbb{R}$$

for integral forms $\theta \in \operatorname{Alt}_{\operatorname{int}}^N(T_\gamma \mathsf{L}_c X)$. Here we wrote $(-,-)_{L^2}$ for the dual pairing between functions and measures, since this reduces to the L^2 scalar product in case that θ is in L^2 .

Spelled out, this means that if for $\theta \in \operatorname{Alt}_{\operatorname{int}}^N(T_{\gamma}\mathsf{P}_cX) \cong \mathcal{M}_{\operatorname{susy}}([0,1]^N, \gamma^*T'X^{\boxtimes N})$, we decompose $\theta = \sum_{\ell} \theta_{\ell}$ as in the proof of Prop. 5.6 (where each θ_{ℓ} is supported on the sets S_{ℓ}), then

$$q^{\text{rel}}(\theta) = \sum_{\ell} (\mathbf{D}^{\text{rel}}, \theta_{\ell})_{L^2} = \sum_{M=1}^{N} \sum_{\ell \in P_{M,N}} (\mathbf{D}_{M}^{\text{rel}}, K_{\ell}^{-1} \theta_{\ell})_{L^2},$$

which makes sense even though the maps K_{ℓ} are not bijective, since they are injective and each θ_{ℓ} is contained in the image of K_{ℓ} . Because the maps K_{ℓ} are isometries and the Dirac densities are bounded, we obtain

$$|q^{\text{rel}}(\theta)| \le \sum_{M=1}^{N} 2^{-M/2} \sum_{\ell \in P_{M,N}} \|\theta_{\ell}\| \le \|\theta\| \quad \text{and} \quad |q(\theta)| \le 2^{n/2} \|\theta\|.$$
 (5.7)

This shows that q and q^{rel} are bounded linear functionals on $\text{Alt}_{\text{int}}^N(T_\gamma \mathsf{P}_c X)$ respectively $\text{Alt}_{\text{int}}^N(T_\gamma \mathsf{L}_c X)$.

Remark 5.8 (The Dirac Current). Since q is a bounded linear functional on the space $\operatorname{Alt}_{\operatorname{int}}^N(T'_{\gamma}\mathsf{L}_cX)$, it must be given by pairing against an element \mathbf{D}_N in the dual space. From the constructions above, it should be clear what this element is: The dual space contains an isometric complemented copy of $\mathscr{L}_{\operatorname{susy}}^{\infty}(T^N, \gamma^*T'X^{\boxtimes N})$, the space of supersymmetric, bounded measurable sections of $\gamma^*T'X^{\boxtimes N}$, and \mathbf{D}_N lies in this part. \mathbf{D}_N then consists of the various Dirac densities \mathbf{D}_N assembled to a (non-continuous but bounded) map on T^N via the (duals of the) maps K_{ℓ} . Since \mathbf{D}_N is not continuous on the generalized diagonal $T^N \setminus T_{\circ}^N$, it is not contained in the predual $C_{\operatorname{susy}}(T^N, \gamma^*T'X^{\boxtimes N})$.

Remark 5.9. For the exterior powers of the space $L^2(S^1, \gamma^*T'X) \subset \mathcal{M}(S^1, \gamma^*T'X)$, we have canonically

$$\Lambda_{\sigma}^{N} L^{2}(S^{1}, \gamma^{*}T'X) \cong L_{\text{susy}}^{2}(T^{N}, \gamma^{*}T'X^{\boxtimes N}) \subseteq \mathcal{M}_{\text{susy}}(T^{N}, \gamma^{*}T'X^{\boxtimes N}),$$

where on the left hand side, Λ_{σ} denotes the Hilbert space tensor product completion of the algebraic exterior product. In particular, this means that our integral map will be able to integrate wedge products of one forms that are L^2 in each fiber, as well as elements in the Hilbert space completion of the algebraic span of such elements. This answers a question of John Lott [Lot87, p. 624]. Similarly, for $L^1(S^1, \gamma^*T'X) \subset \mathcal{M}(S^1, \gamma^*T'X)$, we have

$$\Lambda_{\pi}^{N} L^{1}(S^{1}, \gamma^{*}T'X) \cong L_{\text{susy}}^{1}(T^{N}, \gamma^{*}T'X^{\boxtimes N}) \subseteq \mathcal{M}_{\text{susy}}(T^{N}, \gamma^{*}T'X^{\boxtimes N}),$$

where Λ_{π} denotes the completed projective tensor product (c.f. [Tr67, Thm. 46.2] and use Fubini, or see ibid. Ex. 46.5).

We close this section by giving some more explicit formulas for the functionals q and q^{rel} along paths $\gamma \in PX$. To this end, suppose we are given elements $\theta_j \in \mathcal{M}([0,1], \gamma^*\Lambda^{\ell_j}T'X)$,

 $\ell_j \geq 1$, for j = 1, ..., M. Using the maps K_{ℓ_j} from above on each of them, they define elements $K_{(\ell_j)}\theta_j \in \mathcal{M}_{\text{susy}}([0,1]^{\ell_j}, \gamma^*T'X^{\boxtimes \ell_j})$. Explicitly, these are given by

$$(K_{(\ell_j)}\theta_j)[V_{\ell_j},\dots,V_1] = \sqrt{\ell_j} \int_0^1 \theta_j(t) [V_{\ell_j}(t),\dots,V_1(t)] dt$$
 (5.8)

for tangent vectors (i.e. continuous vector fields along γ) $V_1, \ldots, V_N \in T_\gamma \mathsf{P}_c X$ (if θ_j is not in L^1 , then (5.8) has to be interpreted in the distributional sense).

Lemma 5.10. Suppose that $\theta_1, \ldots, \theta_M$ are defined as above with $\theta_j \in L^1([0,1], \gamma^* \Lambda^{\ell_j} T'X)$, $\ell_j \geq 1, j = 1, \ldots, M$. Then for $\theta = K_{\ell_M} \theta_M \wedge \cdots \wedge K_{\ell_1} \theta_1$, we have

$$q^{\text{rel}}(\theta) = 2^{-M/2} \sum_{\sigma \in S_M} \text{sgn}(\sigma; \ell) \int_{\Delta_M} [\gamma \|_{\tau_M}^1]^{\Sigma} \prod_{j=1}^M \mathbf{c} (\theta_{\sigma_j}(\tau_j)) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\Sigma} d\tau, \tag{5.9}$$

where

$$\Delta_M := \{ \tau = (\tau_N, \dots, \tau_1) \mid 0 \le \tau_1 \le \dots \le \tau_M \le 1 \}$$

is the standard simplex.

Remark 5.11. The formula of Lemma 5.10 continues to hold for measures $\theta_1, \ldots, \theta_M \in \mathcal{M}([0,1], \gamma^*\Lambda T'X)$ such that $K_{\ell_M}\theta_M \wedge \cdots \wedge K_{\ell_1}\theta_1$ is supported in $[0,1]^N_\circ$. In this case the integrand is a $\text{Hom}(\Sigma_{\gamma(0)}, \Sigma_{\gamma(1)})$ -valued measure on $[0,1]^N$ for which the set $[0,1]^N \setminus [0,1]^N_\circ$ is a zero set. Since the discontinuities of the supersymmetrization of the integrand in (5.9) lie precisely in this set, it makes sense to pair it with this measure.

Proof. From formula (5.4), we see that

$$K_{(\ell_M)}\theta_M \wedge \cdots \wedge K_{(\ell_1)}\theta_1 = \sum_{\sigma \in S_M} \operatorname{sgn}(\sigma;\ell) K_{\ell^{\sigma}}(\theta_{\sigma_M} \boxtimes \cdots \boxtimes \theta_{\sigma_1}),$$

where we set $\ell^{\sigma} := (\ell_{\sigma_M}, \dots, \ell_{\sigma_1})$. Therefore

$$q^{\mathrm{rel}}(\theta_M \wedge \cdots \wedge \theta_1) = \sum_{\sigma \in S_M} \mathrm{sgn}(\sigma; \ell) (\mathbf{D}_M^{\mathrm{rel}}, \theta_{\sigma_M} \boxtimes \cdots \boxtimes \theta_{\sigma_1})_{L^2}.$$

Now by Remark 4.4,

$$(\mathbf{D}_{M}^{\mathrm{rel}}, \theta_{M} \boxtimes \cdots \boxtimes \theta_{1})_{L^{2}} = \int_{[0,1]^{M}} (\mathbf{D}^{\mathrm{rel}}(\tau_{M}, \dots, \tau_{1}), \theta_{M}(\tau_{M}) \otimes \cdots \otimes \theta_{1}(\tau_{1}))_{L^{2}} dt$$

$$= \sum_{\sigma \in S_{M}} \int_{\Delta_{M}} (\mathbf{D}^{\mathrm{rel}}(\tau_{\sigma_{M}^{-1}}, \dots, \tau_{\sigma_{1}^{-1}}), \theta_{M}(\tau_{\sigma_{M}^{-1}}) \otimes \cdots \otimes \theta_{1}(\tau_{\sigma_{1}^{-1}}))_{L^{2}} dt$$

$$= \frac{1}{2^{M/2} M!} \sum_{\sigma \in S_{M}} \operatorname{sgn}(\sigma; \ell) \int_{\Delta_{M}} [\gamma \|_{\tau_{M}}^{1}]^{\Sigma} \prod_{j=1}^{M} \mathbf{c} (\theta_{\sigma_{j}}(\tau_{j})) [\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{\Sigma} d\tau.$$

Finally, notice that this expression is supersymmetric in $\widetilde{\theta}_1, \ldots, \widetilde{\theta}_M$, so that

$$\sum_{\sigma \in S_M} \operatorname{sgn}(\sigma; \ell) \left(\mathbf{D}_M^{\operatorname{rel}}, \theta_{\sigma_M} \boxtimes \cdots \boxtimes \theta_{\sigma_1} \right)_{L^2} = M! \left(\mathbf{D}_M^{\operatorname{rel}}, \theta_M \boxtimes \cdots \boxtimes \theta_1 \right)_{L^2}.$$

This concludes the proof.

The $\mathbb{T} = S^1$ -action on $\mathsf{L}_c X$ defines an action on the vector bundle $\mathrm{Alt}^N_{\mathrm{int}}(T\mathsf{L}_c X)$ which turns it into an equivariant bundle. Note though that the action on $\mathsf{L}_c X$ is merely continuous, and the same will be true for the action on the bundle. Namely, given $\theta \in \mathrm{Alt}^N_{\mathrm{int}}(T_\gamma \mathsf{L}_c X)$ and $t \in \mathbb{T}$, we obtain an element $t \cdot \theta$ in $\mathrm{Alt}^N_{\mathrm{int}}(T_{t \cdot \gamma} \mathsf{L}_c X)$ by setting

$$(t \cdot \theta)(\tau_N, \dots, \tau_1) := \theta(\tau_N + t, \dots, \tau_1 + t). \tag{5.10}$$

Here as always, we identified θ with an element in $\mathcal{M}_{\text{susy}}(T^N, \gamma^* T' X^{\boxtimes N})$, and the definition (5.10) has to be interpreted in the distributional sense if $\theta \notin L^1$.

Lemma 5.12. The functional q is \mathbb{T} -invariant. More precisely, for each $\theta \in \operatorname{Alt}_{\operatorname{int}}^N(T_{\gamma}\mathsf{L}_cX)$ and each $t \in \mathbb{T}$, we have $q|_{\gamma}(\theta) = q|_{t\cdot\gamma}(t\cdot\theta)$.

Proof. This follows directly from the corresponding property of the Dirac density, see Lemma 4.5.

6 The Integral Map and its first Properties

Let X be a compact Riemannian spin manifold of dimension n. We are now ready to define our integration map for differential forms on the loop space. To this end, write

$$\Omega_{\text{int},b}^{N}(\mathsf{L}_{c}X) := \Omega^{N}(\mathsf{L}_{c}X) \cap C_{b}(\mathsf{L}_{c}M, \text{Alt}_{\text{int}}^{N}(T_{\gamma}\mathsf{L}_{c}X)), \tag{6.1}$$

for the space of smooth, bounded differential forms which are pointwise contained in the subbundle $\operatorname{Alt}_{\operatorname{int}}^N(T_\gamma \mathsf{L}_c X)$ (this subbundle was defined in (5.1)). Here, boundedness of a section θ of $\operatorname{Alt}_{\operatorname{int}}^N(T_\gamma \mathsf{L}_c X)$ means that the function $\|\theta\|$ obtained from θ by taking the pointwise norm is bounded.

Definition 6.1 (The Integral Map). For any T > 0, we define the integral map for forms on the loop space by the formula

$$I_T: \Omega^N_{\mathrm{int},b}(\mathsf{L}_c X) \longrightarrow \mathbb{R}, \qquad I_T[\theta] := \mathbb{W}_T \left[\exp\left(-\frac{T}{8} \int_0^1 \mathrm{scal}(\gamma(t)) dt\right) q(\theta) \right].$$

The map I_T is continuous by the bound (5.7) on q.

Remark 6.2. The reader may be puzzled by the appearance of the scalar curvature term, and so are we. The discussion below shows that this term is necessary to connect our path integral with the Dirac operator, due to the Lichnerowicz formula. However, from a

formal point of view, it is not clear where it comes from and it is also ignored in Atiyah's paper [Ati85]. A path integral map with a smaller domain of definition, which coincides with ours on this domain has been defined by Lott [Lot87, p. 624]. Lott writes that the scalar curvature term appears due to "quantum effects" coming from time ordering. We have nothing to add to this.

Remark 6.3. There is a problem in the above definition, because we only defined $q(\theta)$ at smooth loops γ , which form a null set with respect to the Wiener measure. However, using the stochastic parallel transport, we can make sense of the Dirac density as a bounded measurable section of the appropriate bundle, which is enough to define our map q almost everywhere. $q(\theta)$ will then be a bounded measurable function on L_cX for all θ , which can be integrated using the Wiener measure.

Remark 6.4. Clearly, in order to plug a form θ into our integral map, we do not need it to be smooth, neither need it be bounded. We could get away here with just requiring that θ be integrable with respect to the measure W_T . However, the space of such forms might depend on T, so we chose the domain $\Omega_{\text{int},b}(\mathsf{L}_cX)$ for definiteness for now. An extension of this domain will be discussed in the next section.

Remark 6.5. There is also a relative version I_T^{rel} of the integral map, mapping $\Omega_{\text{int},b}(\mathsf{L}_cX)$ to $\Omega(X)$, with the property that $\int_X \circ I_T^{\text{rel}} = I_T$, where \int_X is the usual integration map of X. Since we don't need it in the remained of this paper, we divested its definition to Appendix B.

The equivariant structure on the bundle $\operatorname{Alt}_{\operatorname{int}}(T_{\gamma}\mathsf{L}_{c}X)$ induces an action on $\Omega_{\operatorname{int}}(\mathsf{L}_{c}X)$, given by

$$(t \cdot \theta)|_{\gamma}(\tau_N, \dots, \tau_1) := \theta|_{(-t) \cdot \gamma}(\tau_N + t, \dots, \tau_1 + t), \tag{6.2}$$

where the T-action on elements of $\mathrm{Alt}_{\mathrm{int}}(T_{\gamma}\mathsf{L}_{c}X)$ was defined in (5.10). The averaging operator Av on $\Omega_{\mathrm{int}}(\mathsf{L}_{c}X)$ is then given by

$$Av\theta := \int_{\mathbb{T}} t \cdot \theta \, dt, \tag{6.3}$$

Notice that Av maps $\Omega_{\text{int}}(\mathsf{L}_cX)$ to $\Omega_{\text{int},\mathbb{T}}(\mathsf{L}_cX)$, the space of \mathbb{T} -invariant integrable differential forms.

Proposition 6.6. For any $t \in \mathbb{T}$ and any $\theta \in \Omega_{\text{int},b}(\mathsf{L}_cX)$, we have $I_T[\theta] = I_T[t \cdot \theta]$. In particular, this implies $I_T[\theta] = I_T[\mathsf{Av}\theta]$.

Proof. This is a direct consequence of Lemma 5.12 and the fact that the transformation $\gamma \mapsto t \cdot \gamma$ of $L_c X$ preserves the measure W_T .

We now give several examples of integrable forms. Namely, given $\varphi \in C^{\infty}(S^1)$, there is a degree-preserving map

$$P_{\varphi}: \Omega(X) \longrightarrow \Omega_{\mathrm{int},b}(\mathsf{L}_{c}X)$$

given for $\vartheta \in \Omega^N(X)$ with $N \geq 1$ by

$$P_{\varphi}\vartheta|_{\gamma}[V_N,\dots,V_1] = \sqrt{N} \int_{S^1} \varphi(t)\,\vartheta|_{\gamma(t)} \big[V_N(t),\dots,V_1(t)\big] dt \tag{6.4}$$

for vector fields $V_1, \ldots, V_N \in T_{\gamma} \mathsf{L}_c X$. This defines an element in $\mathcal{M}_{\text{susy}}(T^N, \gamma^* T X^N)$. Moreover, for functions $f \in C^{\infty}(X) = \Omega^0(X)$, define

$$P_{\varphi}f(\gamma) := \int_{S^1} \varphi(t) f(\gamma(t)) dt.$$
 (6.5)

In the case that $\varphi \equiv 1$, we will write P instead of P_1 . All these forms are contained in $\Omega_{\text{int},b}(\mathsf{L}_cX)$, since if $\vartheta \in \Omega^N(X)$, $N \geq 1$, then $P_{\varphi}\vartheta = K_{(N)}\vartheta$, where $\vartheta \in C^{\infty}(S^1, \gamma^*\Lambda^N T'X) \subset \mathcal{M}(S^1, \gamma^*\Lambda^N T'X)$ is given by $\vartheta(t) = \vartheta(\gamma(t))$ (here $K_{(N)}$ is the map defined in (5.5)). Wedge products of such forms are again integrable, by Remark 5.4. Moreover, also the averages (as in (6.3)) of such sums of wedge products are again integrable, by Prop. 6.6. This provides a bunch of examples for integrable differential forms. The integral of such forms is computed in the following proposition.

Proposition 6.7. Let $\varphi_1, \ldots, \varphi_M \in C^{\infty}(S^1)$ and $\vartheta_j \in \Omega^{\ell_j}(X)$, $\ell_j \in \mathbb{N}_0$. Then for the wedge product $\theta = P_{\varphi_M} \vartheta_M \wedge \cdots \wedge P_{\varphi_1} \vartheta_1$, we have

$$I_T[\theta] = 2^{-M/2} \sum_{\sigma \in S_M} \operatorname{sgn}(\sigma; \ell) \int_{\Delta_M} \prod_{j=1}^M \varphi_{\sigma_j}(\tau_j) \operatorname{Str}\left(e^{-T(1-\tau_M)H} \prod_{j=1}^M \mathbf{c}(\vartheta_{\sigma_j}) e^{-T(\tau_j - \tau_{j-1})H}\right) d\tau.$$

Here $H = D^2/2$, where D is the Dirac operator on spinors.

Note here that we do *not* require $\ell_j \geq 1$. This corresponds to the result [Lot87, Prop. 9] and shows that the integral map constructed there coincides with ours on its domain of definition (up to overall factors). Compare also [Get91, Section 3].

Remark 6.8. The operator in brackets on the right hand side in Prop. 6.7 commutes with the right Cl_n -action on sections of Σ . Therefore, its integral kernel restricted to the diagonal is pointwise contained in $End_{Cl_n}(\Sigma_x) \cong Cl(T_xX)$, so we can take the supertrace over the Clifford algebra part. This is meant in the formula.

There is also the *complex* spinor bundle $\Sigma^{\mathbb{C}}$, which can be described in terms of the real spinor bundle Σ via $\Sigma^{\mathbb{C}} = \Sigma \otimes_{\operatorname{Cl}_n} \Sigma_n^{\mathbb{C}}$, with $\Sigma_n^{\mathbb{C}}$ the complex spinor module over Cl_n . Since the operator on the right hand side of (6.7) commutes with the Cl_n action, via this construction, it can be seen as an operator acting on sections of $\Sigma^{\mathbb{C}}$. If n is even, then $\Sigma^{\mathbb{C}}$ is graded and we can take its complex trace, which is related to the real trace by powers of i (compare (2.4)). In particular, we obtain

$$I_T[1] = \text{Str}(e^{-TH}) = i^{n/2} \, \text{Str}_{\mathbb{C}}(e^{-TH}) = i^{n/2} \, \text{ind}(\mathsf{D}),$$

where in the last step, we used the classical argument of McKean and Singer, see for example [BGV04, Thm. 3.50].

Proof. As seen above, if $\vartheta \in \Omega^N(X)$, $N \geq 1$, then $P_{\varphi}\vartheta = K_{(N)}\xi$ where ξ is given by $\xi(t) = \varphi(t)\vartheta(\gamma(t))$. Therefore, if $\ell_j \geq 1$ for each j, we get from Lemma 5.10 that

$$q(\theta) = 2^{-M/2} \sum_{\sigma \in S_M} \operatorname{sgn}(\sigma; \ell) \int_{\Delta_M} \operatorname{str}\left([\gamma \|_{\tau_M}^1]^{\Sigma} \prod_{j=1}^M \varphi_{\sigma_j}(\tau_j) \mathbf{c} \left(\vartheta_{\sigma_j} |_{\gamma(\tau_j)} \right) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\Sigma} \right) d\tau. \quad (6.6)$$

Suppose now that

$$\theta = P_{\psi_1} f_1 \cdots P_{\psi_N} f_N \cdot P_{\varphi_1} \vartheta_1 \wedge \cdots \wedge P_{\varphi_M} \vartheta_M$$

for $\varphi_1, \ldots, \varphi_M, \psi_1, \ldots, \psi_N \in C^{\infty}(S^1), f_1, \ldots, f_N \in C^{\infty}(X)$ and $\vartheta_j \in \Omega^{\ell_j}(X), j = 1, \ldots, M$. Then

$$q(\theta) = P_{\psi_1} f_1 \cdots P_{\psi_M} f_N \cdot q(P_{\varphi_1} \vartheta_1 \wedge \cdots \wedge P_{\varphi_N} \vartheta_M)$$

by linearity of a and

$$\prod_{j=1}^{N} \int_{S^{1}} \psi_{j}(t) f_{j}(\gamma(\tau_{j})) dt = \sum_{\sigma \in S_{N}} \operatorname{sgn}(\sigma; 0, \dots, 0) \int_{\Delta_{N}} \prod_{j=1}^{N} \psi_{\sigma_{j}}(\tau_{j}) f_{\sigma_{j}}(\gamma(\tau_{j})) d\tau.$$

Now from the fact that $\mathbf{c}(f_j(\gamma(t))) = f_j(\gamma(t))$ commutes with the parallel transport and the super trace as it is scalar, we obtain that (6.6) holds also in the case that $\ell_j = 0$ for some j.

Hence without any assumptions on $\ell_1, \ldots, \ell_N, I_T[\theta]$ is given by

$$2^{-M/2} \sum_{\sigma \in S_M} \operatorname{sgn}(\sigma; \ell) \int_{\Delta_M} \mathbb{W}_T \left[e^{-\frac{T}{8} \int_{\gamma} \operatorname{scal}} \operatorname{str} \left([\gamma \|_{\tau_M}^1]^{\Sigma} \prod_{j=1}^M \varphi_{\sigma_j}(\tau_j) \mathbf{c} \left(\vartheta_{\sigma_j} |_{\gamma(\tau_j)} \right) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\Sigma} \right) \right] d\tau.$$

where we can pull the φ_j out of the \mathbb{W}_T integral. Now we use (2.9) and the convolution property (2.10) to calculate

$$\mathbb{W}_{T} \left[e^{-\frac{T}{8} \int_{0}^{1} \operatorname{scal}(\gamma(s)) ds} \operatorname{str} \left(\left[\gamma \right]_{\tau_{M}}^{1} \right]^{\Sigma} \prod_{j=1}^{M} \mathbf{c} \left(\vartheta_{\sigma_{j}} |_{\gamma(\tau_{j})} \right) \left[\gamma \right]_{\tau_{j-1}}^{\tau_{j}} \right]^{\Sigma} \right) \right] \\
= \int_{X} \operatorname{str} \left(\mathbb{W}_{T}^{xx} \left[e^{-\frac{T}{8} \int_{0}^{1} \operatorname{scal}(\gamma(s)) ds} \left[\gamma \right]_{\tau_{M}}^{1} \right]^{\Sigma} \prod_{j=1}^{M} \mathbf{c} \left(\vartheta_{\sigma_{j}} |_{\gamma(\tau_{j})} \right) \left[\gamma \right]_{\tau_{j-1}}^{\tau_{j}} \right]^{\Sigma} \right] \right) dx \\
= \int_{X} \operatorname{str} \left(\int_{X} \cdots \int_{X} \mathbb{W}_{T(1-\tau_{M})}^{x_{0}x_{M}} \left[e^{-\frac{T(1-\tau_{M})}{8} \int_{0}^{1} \operatorname{scal}(\gamma(s)) ds} \left[\gamma \right]_{0}^{1} \right]^{\Sigma} \right] \times \\
\times \prod_{j=1}^{M} \mathbf{c} (\vartheta_{\sigma_{j}}) \mathbb{W}_{T(\tau_{j}-\tau_{j-1})}^{x_{j}x_{j-1}} \left[e^{-\frac{T(\tau_{j}-\tau_{j-1})}{8} \int_{0}^{1} \operatorname{scal}(\gamma(s)) ds} \left[\gamma \right]_{0}^{1} \right]^{\Sigma} dx_{M} \cdots dx_{1} dx_{0}.$$

The proposition follows now from the fact that the heat kernel $p_T^{\Sigma}(x,y)$ of $H=\mathsf{D}^2/2$ is given by

$$p_T^{\Sigma}(x,y) = \mathbb{W}_T^{xy} \left[\exp\left(-\frac{T}{8} \int_0^1 \operatorname{scal}(\gamma(s)) ds\right) [\gamma \|_0^1]^{\Sigma} \right],$$

compare Thm. 2.5 of [Bis84]. By virtue of Lichnerowicz' formula

$$\mathsf{D}^2 = \nabla^* \nabla + \frac{1}{4} \mathrm{scal},$$

this follows from the Feynman-Kac formula, Thm. 2.1.

7 Extension of the Domain

So far, the domain of our integral map is somewhat small. As mentioned in Remark 6.4, we could drop the boundedness requirement on differential forms, as well as the requirement on its regularity. Indeed, this turns out to be necessary, since many interesting examples (which are smooth on LX) turn out to be not even continuous on L_cX : In [Che73], Chen constructs a vast number of examples for differential forms on the loop space using his iterated integral map. In particular, Getzler, Jones and Petrack show in [GJP91] that the Bismut-Chern characters constructed below are of this form, if one generalizes Chen's construction slightly. However, none of these examples will be well-defined on the continuous loop space L_cX , because the construction relies on inserting the (on L_cX ill-defined) vector field $\dot{\gamma}$. The purpose of this section is therefore to extend the domain of definition of our integral map I_T to a larger space of differential forms, the elements of which are not necessarily continuous. This larger domain contains all forms obtained by the iterated integral maps mentioned above.

The challenge here is that we are also interested of the interplay between the integral map and the equivariant differential $d_T := d + T^{-1}\iota_{\dot{\gamma}}$, which only makes sense on LX. Hence our first goal is to "push forward" the integral map defined so far to a map on differential forms on the smooth loop space LX. To this end, let $j : \mathsf{L}X \to \mathsf{L}_cX$ be the inclusion. This is a smooth map with dense image, so dually, the pullback

$$j^*: \Omega(\mathsf{L}_c X) \longrightarrow \Omega(\mathsf{L} X)$$
 (7.1)

is a continuous injection⁷. Therefore, we obtain a linear functional

$$I_T: \Omega(\mathsf{L}X) \supset j^*\Omega_{\mathrm{int},b}(\mathsf{L}_cX) \longrightarrow \mathbb{R}$$
 (7.2)

as well denoted by I_T , which is defined by sending ξ to $I_T(\theta)$ if $\xi = j^*\theta$.

We will now extend the domain from $j^*\Omega_{\text{int},b}(\mathsf{L}_cX)$ to a larger space denoted by $\Omega_{\text{ext}}(\mathsf{L}X)$. In order to do this, we consider the space of *finite energy loops* defined by

$$\mathsf{L}_H X := \{ \gamma \in \mathsf{L}_c X \mid \dot{\gamma} \in L^2(S^1, \gamma^* TX) \}.$$

This is a manifold locally modelled on the Hilbert space $H^1(S^1, \mathbb{R}^n)$, the Sobolev space of \mathbb{R}^n -valued functions with square-integrable first derivative. It contains the smooth loop

⁷if $j^*\theta = j^*\theta'$ for $\theta, \theta' \in \Omega(\mathsf{L}_cX)$, then θ must coincide with θ' at all $\gamma \in \mathsf{L}X$, since for these, $T_\gamma \mathsf{L}X = C^\infty(S^1, \gamma^*TX)$ is dense in $T_\gamma \mathsf{L}_cX = C^0(S^1, \gamma^*TX)$. Since $\mathsf{L}X$ is dense in L_cX , we must have $\theta = \theta'$

space LX as a dense subspace and in turn is a dense subspace of L_cX (however of measure zero with respect to the Wiener measure). The important property for us is that L_HX contains the polygon paths with respect to partitions $\tau = \{0 = \tau_0 < \tau_1 < \cdots < \tau_N = 1\}$. By this, we mean continuous paths γ that are minimizing geodesics on each of the subintervals $[\tau_{j-1}, \tau_j]$.

Given a partition τ as above and any path $\gamma \in \mathsf{L}_c X$, let γ^{τ} be an associated polygon path, by which we mean a polygon path γ^{τ} such that $\gamma^{\tau}(\tau_j) = \gamma(\tau_j)$. Since almost all paths (with respect to the Wiener measure) have the property that there exists a unique minimizing geodesic between $\gamma(\tau_{j-1})$ and $\gamma(\tau_j)$ for each j, these associated polygon paths are unique for almost all paths, so we obtain a well-defined measurable map

$$Z_{\tau}: \mathsf{L}_{c}X \longrightarrow \mathsf{L}_{H}X, \qquad \gamma \longmapsto \gamma^{\tau},$$
 (7.3)

for any given partition τ of the interval [0,1].

Now given a differential form $\theta \in \Omega(\mathsf{L}X)$ which is pointwise contained in the space $\mathsf{Alt}^N_{\mathrm{int}}(C(S^1,\gamma^*TX))$ (so that $q(\theta)$ is defined as a smooth function on $\mathsf{L}X$), assume that this function has a continuous extension to a function on L_HX , denoted again by $q(\theta)$. (Since $\mathsf{L}X$ is dense in L_HX , such an extension is unique if it exists.) For each partition τ , we then obtain a measurable function $Z^*_{\tau}q(\theta)$ on L_cX , by pullback with the map (7.3). In order to make the following definition, notice that the set $\mathscr P$ of all partitions τ of [0,1] (with arbitrary number of nodes N) is a directed set, with $\tau \preceq \tau'$ if τ' is a refinement of τ ; hence $Z^*_{\tau}q(\theta)$, $\tau \in \mathscr P$ is a net.

Definition 7.1. An differential form $\theta \in \Omega(LX)$ which is pointwise contained in the space $\operatorname{Alt}_{\operatorname{int}}^N(C(S^1, \gamma^*TX))$ is contained in the *extended domain* $\Omega_{\operatorname{ext}}(LX)$ if $q(\theta)$ has a continuous extension to L_HX and moreover for any T > 0, the limit

$$I_T[\theta] := \lim_{\tau \in \mathscr{P}} \mathbb{W}_T \left[\exp\left(-\frac{T}{8} \int_{S^1} \operatorname{scal}(\gamma(t)) dt \right) Z_\tau^* q(\theta) \right]$$
 (7.4)

exists. This gives a linear map

$$I_T: \Omega(\mathsf{L}X) \supset \Omega_{\mathrm{ext}}(\mathsf{L}X) \longrightarrow \mathbb{R}.$$
 (7.5)

Lemma 7.2. We have $j^*\Omega_{\text{int},b}(\mathsf{L}_cX) \subset \Omega_{\text{ext}}(\mathsf{L}X)$, I_T as defined in (7.5) coincides with I_T as defined in (7.2) on the common domain $j^*\Omega_{\text{int},b}(\mathsf{L}_cX)$.

Proof. $q(\theta)$ is not continuous on L_cX , since the Dirac densities \mathbf{D}_N are not, due to the parallel transport in Σ appearing in their definition. We made sense of them using stochastic parallel transport, in other words, by solving Stratonovich stochastic differential equations. Now standard results in stochastic analysis state that solutions to Stratonovich SDEs can be approximated by solutions to ODEs along discretized processes, see e.g. [É89, 7.14] or Thm. 4.14 in [AD99]; the latter includes a long list of further references. This is precisely what happens in the limit (7.4). This will be discussed in more detail in [HL17d].

In a similar fashion, one obtains the following result.

Proposition 7.3. The extended domain $\Omega_{\text{ext}}(LX)$ contains all differential forms on LX which are obtained as (extended) iterated integrals in the sense of [Che73], or more generally by the procedure in [GJP91].

8 The Bismut-Chern Character

In this section, we define the Bismut-Chern character and investigate its relation to our integral map, where the main result is Thm. 8.1 below. The Bismut-Chern-Character was defined by Bismut in [Bis85] in order to generalize the formal arguments of Atiyah and Witten (as reviewed in Section 3) to the twisted setting.

Let \mathcal{V} be a super vector bundle over a manifold X with connection ∇ . Let F be the curvature of ∇ , considered as an element in $\Omega^2(X, \operatorname{End}(\mathcal{V}))$. This data gives rise to equivariant differential forms $\operatorname{BCh}_N(\mathcal{V}, \nabla) \in \Omega^{2N}_{\mathbb{T}}(\mathsf{L}X)$, given by

$$BCh_N(\nabla, \mathcal{V})[V_{2N}, \ldots, V_1]$$

$$= (-2)^{-N} \sum_{\sigma \in S_{2N}} \operatorname{sgn}(\sigma) \int_{\Delta_N} \operatorname{str}_{\mathcal{V}} \left([\gamma \|_{\tau_N}^1]^{\mathcal{V}} \prod_{j=1}^N F(V_{\sigma_{2j}}(\tau_j), V_{\sigma_{2j-1}}(\tau_j)) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\mathcal{V}} \right) d\tau$$
(8.1)

for vector fields V_1, \ldots, V_{2N} along $\gamma \in \mathsf{L}X$. In particular, for N = 0, we have $\mathsf{BCh}_0(\nabla, \mathcal{V}) = \mathsf{str}_{\mathcal{V}}[\gamma|_0^1]^{\mathcal{V}}$, the supertrace of the holonomy around the loop γ . The *Bismut-Chern character* of the connection ∇ on \mathcal{V} is the equivariant differential form on $\mathsf{L}X$ defined by

$$\mathrm{BCh}_T(\mathcal{V}, \nabla) := \sum_{N=0}^{\infty} T^N \mathrm{BCh}_N(\nabla, \mathcal{V}) \in \prod_{N=0}^{\infty} T^N \Omega^{2N}(\mathsf{L}X).$$

Notice that the pullback of $BCh(\mathcal{V}, \nabla)$ along the inclusion $i: X \to LX$ (which includes X as the set of constant loops) is precisely the (T-dependent) Chern character form

$$\operatorname{ch}_{T}(\mathcal{V}, \nabla) := \operatorname{str}_{\mathcal{V}}(e^{-TF}), \tag{8.2}$$

of (\mathcal{V}, ∇) on X (this uses the formula (8.3) below). $\operatorname{ch}_T(\mathcal{V}, \nabla)$ is a closed differential form on X. BCh $_T(\mathcal{V}, \nabla)$ however will not be closed, but *equivariantly closed*. This was already proved by Bismut [Bis85, Thm. 3.9]; see als [TWZ15]. Since our definition (8.1) differs slightly from the literature, we give a proof in Appendix A.

The significance of the Bismut-Chern character in relation with our integral map is the following.

Theorem 8.1. Let X be a compact spin manifold. Then all BCh_N lie in $\Omega_{\text{ext}}(LX)$ for all T > 0, and we have

$$I_T[\mathrm{BCh}_T(\mathcal{V}, \nabla)] := \sum_{N=0}^{\infty} T^N I_T[\mathrm{BCh}_N(\mathcal{V}, \nabla)] = \mathrm{Str}_{\mathbb{C}}(e^{-T\mathsf{D}_{\mathcal{V}}^2/2}),$$

where $D_{\mathcal{V}}$ is the twisted Dirac operator on $\Sigma^{\mathbb{C}} \otimes \mathcal{V}$. In particular, the power series converges absolutely for any T > 0.

The forms BCh_N are not defined on L_cX at first, only on LX, so in particular they do not lie in $\Omega_{\text{int},b}(L_cX)$. However, by the results of Getzler, Jones and Petrack [GJP91], the forms BCh_N lie in the image of their extended iterated integral map, hence by Prop. 7.3, BCh_N lies in the extended domain $\Omega_{\text{ext}}(LX)$. The way this extension works is that one interprets the parallel transport appearing in the definition (8.1) as stochastic parallel transport, this way obtaining an integrable measurable function on L_cX . This will be used in the proof below.

Remark 8.2. If X is even-dimensional, this can be connected to the index of the twisted Dirac operator $D_{\mathcal{V}}$ acting on $\Sigma^{\mathbb{C}} \otimes \mathcal{V}$. Namely, by the McKean-Singer formula, we have

$$I_T[\mathrm{BCh}_T(\mathcal{V}, \nabla)] = i^{n/2} \mathrm{Str}_{\mathbb{C}}(e^{-TD_{\mathcal{V}}^2/2}) = i^{n/2} \mathrm{ind}(\mathsf{D}_{\mathcal{V}}),$$

where the factors of i come from (2.4). If X is odd-dimensional, then the index is zero; similarly, since I_T is an odd functional in this case while $BCh(\mathcal{V}, \nabla)$ is even, the left hand side is zero as well.

Thm. 8.1 follows from the following lemma.

Lemma 8.3. Let $\gamma \in LX$. Applying q degree-wise in T to $BCh_T(\mathcal{V}, \nabla)$ and considering the result as a formal power series in T, we obtain

$$q(\mathrm{BCh}_T(\mathcal{V}, \nabla)) = \sum_{N=0}^{\infty} \left(-\frac{T}{2} \right)^N \int_{\Delta_N} \mathrm{str}_{\Sigma \otimes \mathcal{V}} \left([\gamma \|_{\tau_N}^1]^{\Sigma \otimes \mathcal{V}} \prod_{j=1}^N \mathbf{c}(F|_{\gamma(\tau_j)}) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\Sigma \otimes \mathcal{V}} \right) d\tau.$$

Here we consider F as a section of $\Lambda^2TX \otimes \operatorname{End}(\mathcal{V})$ and $\mathbf{c}(F)$ is the section of $\operatorname{Cl}(TX) \otimes \operatorname{End}(\mathcal{V}) \cong \operatorname{End}_{\operatorname{Cl}_n}(\Sigma \otimes \mathcal{V})$ given by applying to F the map which is the quantization map \mathbf{c} on the first factor and the identity on the second.

Proof (of Lemma 8.3). Let s_1, \ldots, s_m be a basis of $\mathcal{V}_{\gamma(0)}$ and let $s_1(t), \ldots, s_m(t)$ be the parallel translates of these vectors along γ . Define functions $P_b^a \in C^{\infty}([0,1]^2)$ by

$$[\gamma]_{s}^{t}]^{\mathcal{V}} s_{b}(s) = P_{b}^{a}(t, s) s_{a}(t).$$

Let Ξ be the grading operator of \mathcal{V} (so that $\operatorname{str}(A) = \operatorname{tr}(\Xi A)$) and let Ξ_b^a be defined by

$$\Xi s_b(1) = \Xi_b^a s_a(0).$$

Finally, define sections $F_b^a \in C^{\infty}(S^1, \gamma^* \Lambda^2 T^* M)$ by

$$F_{\gamma(t)}(v,w)s_b(t) = F_b^a(t)[v,w]s_b(t).$$

and correspondingly, let $\widetilde{\boldsymbol{F}}_{t,b}^{a} \in \mathcal{M}(S^{1}, \gamma^{*}\Lambda^{2}T'X)$ be given by $\widetilde{\boldsymbol{F}}_{t,b}^{a} := F_{b}^{a}\delta_{t}$, where δ_{t} denotes the point measure at $t \in S^{1}$. Notice that the image of $\widetilde{\boldsymbol{F}}_{t,b}^{a}$ under the map

$$K_{(2)}: \mathcal{M}(S^1, \gamma^*\Lambda^2T'X) \longrightarrow \mathcal{M}_{\mathrm{susy}}(T^2, \gamma^*T'X \boxtimes \gamma^*TX)$$

defined in Section 5 is the two form $\mathbf{F}_{b,t}^a$ along γ defined by

$$\boldsymbol{F}_{t,b}^{a}[V,W] := \sqrt{2} F_{b}^{a}(t) [V(t),W(t)],$$

for continuous vector fields $v, w \in C(S^1, \gamma^*TX)$ along γ , where the factor $\sqrt{2}$ comes from the Jacobian of the map $\kappa_{(2)}2: S^1 \to T^2, t \mapsto (t, t)$. With these notations, we have

$$(-2)^{N} \operatorname{BCh}_{N}(\nabla, \mathcal{V})[V_{2N}, \dots, V_{1}]$$

$$= \sum_{\sigma \in S_{2N}} \operatorname{sgn}(\sigma) \Xi_{a_{N+1}}^{b_{0}} \int_{\Delta_{N}} P_{b_{N}}^{a_{N+1}}(1, \tau_{N}) \prod_{j=1}^{N} F_{a_{j}}^{b_{j}}(\tau_{j}) \left[V_{\sigma_{2j}}(\tau_{j}), V_{\sigma_{2j-1}}(\tau_{j}) \right] P_{b_{j-1}}^{a_{j}}(\tau_{j}, \tau_{j-1}) d\tau$$

$$= 2^{-N/2} \sum_{\sigma \in S_{2N}} \operatorname{sgn}(\sigma) \Xi_{a_{N+1}}^{b_{0}} \int_{\Delta_{N}} P_{b_{N}}^{a_{N+1}}(1, \tau_{N}) \prod_{j=1}^{N} \mathbf{F}_{\tau_{j}, a_{j}}^{b_{j}} [V_{\sigma_{2j}}, V_{\sigma_{2j-1}}] P_{b_{j-1}}^{a_{j}}(\tau_{j}, \tau_{j-1}) d\tau$$

$$= 2^{N/2} \Xi_{a_{N+1}}^{b_{0}} \int_{\Delta_{N}} (\mathbf{F}_{\tau_{N}, a_{N}}^{b_{N}} \wedge \dots \wedge \mathbf{F}_{\tau_{1}, a_{1}}^{b_{1}}) [V_{2N}, \dots, V_{1}] \prod_{j=1}^{N} P_{b_{j-1}}^{a_{j}}(\tau_{j}, \tau_{j-1}) d\tau.$$

Here we used the formula

$$(\theta_N \wedge \dots \wedge \theta_1)[V_N, \dots, V_1] = 2^{-N} \sum_{\sigma \in S_{2N}} \operatorname{sgn}(\sigma) \prod_{j=1}^N \theta_j[V_{\sigma_{2j}}, V_{\sigma_{2j-1}}]$$
(8.3)

for two-forms $\theta_1, \ldots, \theta_N$, which follows by induction from the general formula for the wedge product. Now if $\tau \in T_{\circ}^N$, then

$$\boldsymbol{F}_{\tau_{N},a_{N}}^{b_{N}}\wedge\cdots\wedge\boldsymbol{F}_{\tau_{1},a_{1}}^{b_{1}}=\sum_{\sigma\in S_{N}}K_{(2,\ldots,2)}(\widetilde{\boldsymbol{F}}_{\tau_{\sigma_{N}},a_{\sigma_{N}}}^{b_{\sigma_{N}}}\boxtimes\cdots\boxtimes\widetilde{\boldsymbol{F}}_{\tau_{\sigma_{1}},a_{\sigma_{1}}}^{b_{\sigma_{1}}}),$$

with $\widetilde{\boldsymbol{F}}_{\tau_{\sigma_N},a_{\sigma_N}}^{b_{\sigma_N}} \boxtimes \cdots \boxtimes \widetilde{\boldsymbol{F}}_{\tau_{\sigma_1},a_{\sigma_1}}^{b_{\sigma_1}} \in \mathcal{M}_{\text{susy}}(T_{\circ}^N, \gamma^*\Lambda^2 T' X^{\boxtimes N})$. Since the condition $\tau \in T_{\circ}^N$ is satisfied for almost all $\tau \in \Delta_N$, we have by supersymmetry of \mathbf{D}_N and Lemma 5.10 that $(-2)^N q(\operatorname{BCh}_N(\nabla, \mathcal{V}))$

$$= 2^{N/2} \Xi_{a_{N+1}}^{b_0} \int_{\Delta_N} \left(\widetilde{\boldsymbol{F}}_{\tau_N, a_N}^{b_N} \boxtimes \cdots \boxtimes \widetilde{\boldsymbol{F}}_{\tau_1, a_1}^{b_1}, \mathbf{D}_N \right)_{L^2} P_{b_N}^{a_{N+1}} (1, \tau_N) \prod_{j=1}^N P_{b_{j-1}}^{a_j} (\tau_j, \tau_{j-1}) d\tau$$

$$= \Xi_{a_{N+1}}^{b_0} \int_{\Delta_N} \operatorname{str}_{\Sigma} \left([\gamma \|_{\tau_N}^1]^{\Sigma} \prod_{j=1}^N \mathbf{c} \left(F_{a_j}^{b_j} (\tau_j) \right) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\Sigma} \right) P_{b_N}^{a_{N+1}} (1, \tau_N) \prod_{j=1}^N P_{b_{j-1}}^{a_j} (\tau_j, \tau_{j-1}) d\tau$$

$$= \int_{\Delta_N} \operatorname{str}_{\Sigma \otimes \mathcal{V}} \left([\gamma \|_{\tau_N}^1]^{\Sigma \otimes \mathcal{V}} \prod_{j=1}^N \mathbf{c} (F_{\gamma(\tau_j)}) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^{\Sigma \otimes \mathcal{V}} \right) d\tau.$$

This finishes the proof.

We can now prove the main result of this section.

Proof (of Thm. 8.1). By Lemma 8.3 above, we have

$$q\left(\mathrm{BCh}_{T}(\mathcal{V},\nabla)\right) = \sum_{N=0}^{\infty} \left(-\frac{T}{2}\right)^{N} \int_{\Delta_{N}} \mathrm{str}_{\Sigma \otimes \mathcal{V}} \left(\left[\gamma \right]_{\tau_{N}}^{1} \right]^{\Sigma \otimes \mathcal{V}} \prod_{j=1}^{N} \mathbf{c}(F|_{\gamma(\tau_{j})}) \left[\gamma \right]_{\tau_{j-1}}^{\tau_{j}} \right]^{\Sigma \otimes \mathcal{V}} d\tau$$

Comparing with formula (2.11), we obtain that

$$\exp\left(-\frac{T}{8}\int_{S^1}\operatorname{scal}(\gamma(t))dt\right)q\left(\operatorname{BCh}(\mathcal{V},\nabla)\right)=\operatorname{str} U_T(1,\gamma),$$

where $U_T(t,\gamma)$ is the integrand in the Feynman-Kac formula (see Thm. 2.1) for the connection $\nabla = \nabla^{\Sigma \otimes V}$ (the product connection) and potential $\widetilde{V} = \frac{1}{2}\mathbf{c}(F) + \frac{1}{8}\mathrm{scal}$. Hence by the Feynman-Kac formula, we have

$$I_T(\mathrm{BCh}(\mathcal{V}, \nabla)) = \mathbb{W}_T[\mathrm{str}\,U_T(1, \gamma)] = \mathrm{Str}(e^{-TL}),$$

where

$$L = \frac{1}{2} \left(\nabla^* \nabla + \mathbf{c}(F) + \frac{1}{4} \mathrm{scal} \right).$$

However, by the Weizenböck formula [BGV04, Thm. 3.52], L is precisely half of the square of the twisted Dirac operator $D_{\mathcal{V}}$.

9 The Odd Bismut-Chern Character

The Bismut-Chern Character defined in Section 8 is an equivariant lift to the loop space of the Chern character on X, which is associated to a vector bundle with connection. There is also an odd Chern character on X, associated to a map $g: X \to U_k$ (where U_k is the k-th order unitary group. Notice that just as vector bundles with connection represent classes in $K^0(X)$, maps g as above represent classes in $K^{-1}(X)$.) An equivariant lift of this to the loop space, odd Bismut-Chern character, was defined by Wilson [Wil16]. These will give interesting integrands for our integral map in odd dimensions.

The odd Bismut-Chern character is an equivariantly closed differential form associated to a map $g: X \to U_k$ to the k-dimensional unitary group, for some k (such a map represents a class in $K^{-1}(X)$). It is defined as follows. Let ∇^g be the connection on the trivial vector bundle $\underline{\mathbb{C}}^k$ over $X \times \mathbb{R}$ defined by

$$\nabla^g = d + s\omega$$
, where $\omega = g^{-1}dg \in \Omega^1(X, \mathfrak{gl}_k(\mathbb{C})).$

We can then form the Bismut-Chern character $BCh_T(\underline{\mathbb{C}}^k, \nabla^g)$, which is a differential form on $L(X \times \mathbb{R}) = LX \times L\mathbb{R}$. $BCh(\underline{\mathbb{C}}^k, \nabla^g)$ can be considered as a differential form on $LX \times \mathbb{R}$ by pulling it back via the inclusion of $LX \times \mathbb{R}$ in $LX \times L\mathbb{R}$ (which maps $s \in \mathbb{R}$ to the path in $L\mathbb{R}$ that is constant equal to s).

Define differential forms $BCh_N(g) \in \Omega^{2N+1}(LX)$ by⁸

$$BCh_{N}(g)|_{\gamma}[V_{2N+1},\dots,V_{1}] = -\int_{0}^{1} BCh_{N+1}(\underline{\mathbb{C}}^{k},\nabla^{g})|_{(\gamma,s)}[\partial_{s},V_{2N+1},\dots,V_{1}]ds.$$
 (9.1)

for $V_1, \ldots, V_{2N+1} \in T_{\gamma} LX$. The odd Bismut-Chern character is

$$\operatorname{BCh}_T(g) = \sum_{N=0}^{\infty} T^N \operatorname{BCh}_N(g) \in \prod_{N=0}^{\infty} T^{-N} \Omega_{\mathbb{T}}^{2N+1}(\mathsf{L}X).$$

It is equivariantly closed by Thm. 6.2 in [Wil16]. The pullback of BCh(g) to the submanifold of constant loops $X \subset LX$ is the *odd Chern character* of g,

$$\operatorname{ch}_{T}(g) = \sum_{N=0}^{\lfloor \frac{n-1}{2} \rfloor} T^{N} \frac{N!}{(2N+1)!} \operatorname{tr}(\omega^{2N+1}), \tag{9.2}$$

see [Get93] or [TWZ13]. Here our odd Chern character is related to the odd Chern character ch(g) from the references by $ch(g) = ch_{-1}(g)$. This minus sign comes from our convention for the Bismut-Chern character.

Theorem 9.1. Let $g: X \to U_k$ be a smooth map, with corresponding Bismut-Chern character BCh(g). Then each BCh_N(g) is contained in $\Omega_{\text{ext}}(LX)$ and we have

$$I_T[BCh_T(g)] = \frac{1}{\sqrt{2}} \int_0^1 Str(\dot{D}_s e^{-TD_s^2/2}) ds, \qquad (9.3)$$

where D_s is the twisted Dirac operator associated to the bundle $\Sigma \otimes \underline{\mathbb{C}}^k$, where $\underline{\mathbb{C}}^k$ carries the connection $\nabla^{g,s} := d + s\omega$. Moreover, \dot{D}_s is the derivative of D_s with respect to s, which turns out to be of order zero.

Remark 9.2. We have $D_s = D_0 + s\mathbf{c}(\omega)$, hence each D_s is a Dirac type operator and $\dot{D}_s = \mathbf{c}(\omega)$ is of order zero. Moreover, notice that

$$g^{-1}\mathsf{D}_0g = \mathsf{D}_0 + g^{-1}[\mathsf{D}_0, g] = \mathsf{D}_0 + g^{-1}\mathbf{c}(dg) = \mathsf{D}_0 + \mathbf{c}(d\omega),$$

where $\mathbf{c}(dg) = \mathbf{c}(dx^i \otimes \partial_i g) := \mathbf{c}(dx^i) \otimes \partial_i g \in \mathrm{Cl}(TX) \otimes \mathfrak{gl}_k(\mathbb{C}).$

Suppose that the dimension of X is odd, say n = 2m + 1 (otherwise, both sides of (9.3) are zero). By the work of Getzler [Get93], the expression on the right hand side of (9.3) is related to the spectral flow of the family D_s . The spectral flow $sf(D_0, D_1)$ measures, roughly, how many eigenvalues cross zero as one moves from D_0 to D_1 along the path of operators D_s (for a precise definition, see [Phi96]). Specifically, for the I_T -integral over BCh(g), we find the following.

 $^{^8}$ The sign difference compared to [Wil16, Def. 6.1] comes from the different conventions for the (even) Bismut-Chern character.

Corollary 9.3. Let X be an odd-dimensional spin manifold and let $g: X \to U_k$ be a smooth map, with corresponding Bismut-Chern character BCh(g). Then

$$I_T[BCh(g)] = (-i)^{\frac{n+1}{2}} \left(\frac{2\pi}{T}\right)^{1/2} sf(D, g^{-1}Dg),$$
 (9.4)

which is formula (1.6) from the introduction. Here D denotes the standard Dirac operator on the bundle $\Sigma^{\mathbb{C}} \otimes \underline{\mathbb{C}}^k$.

Proof. By the results of Getzler [Get93, Corollary 2.7], we have for T > 0

$$\operatorname{sf}(\mathsf{D}_0,\mathsf{D}_1) = \left(\frac{T}{2\pi}\right)^{1/2} \int_0^1 \operatorname{Tr}_{\mathbb{C}} \left(\dot{\mathsf{D}}_s e^{-T\mathsf{D}_s^2/2}\right) \mathrm{d}s,$$

where the trace is taken as an operator on sections of bundle $\Sigma^{\mathbb{C}} \otimes \underline{\mathbb{C}}^{k}$, with $\Sigma^{\mathbb{C}}$ the complex ungraded spinor bundle. One can show that for $a \in \text{Cl}_{2m+1}$, one has

$$\operatorname{tr}_{\mathbb{C}}(a) = i(2i)^m \langle a, \mathbf{c}(\operatorname{vol}) \rangle + 2^m \langle a, 1 \rangle.$$

In particular, for a odd, $\operatorname{str}(a) = \sqrt{2}(-i)^{m+1}\operatorname{tr}_{\mathbb{C}}(a)$ and from (9.3), we obtain

$$I_T[BCh(g)] = (-i)^{m+1} \int_0^1 Tr_{\mathbb{C}} (\dot{\mathsf{D}}_s e^{-T\mathsf{D}_s^2/2}) ds = (-i)^{m+1} \left(\frac{2\pi}{T}\right)^{1/2} sf(\mathsf{D}_0, \mathsf{D}_1), \qquad (9.5)$$

which is the statement by Remark 9.2.

For the proof of Thm. 9.1, we need the following explicit formula for BCh(g).

Lemma 9.4. The forms $BCh_N(g)$ are explicitly given by

$$BCh_{N}(g)[V_{2N+1}, \dots, V_{1}] = \frac{(N!)^{2}}{(2N+1)!} \times$$

$$\times \sum_{\ell=1}^{N+1} \sum_{\sigma \in S_{2N+1}} sgn(\sigma) \int_{\Delta_{N+1}} tr \left([\gamma \|_{\tau_{N+1}}^{1}]^{g} \prod_{j=\ell+1}^{N} \omega \left(V_{\sigma_{2j+1}}(\tau_{j}) \right) \omega \left(V_{\sigma_{2j}}(\tau_{j}) \right) [\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{g} \times$$

$$\times \omega \left(V_{\sigma_{2\ell-1}}(\tau_{\ell}) \right) [\gamma \|_{\tau_{\ell-1}}^{\tau_{\ell}}]^{g} \prod_{j=1}^{\ell-1} \omega \left(V_{\sigma_{2j}}(\tau_{j}) \right) \omega \left(V_{\sigma_{2j-1}}(\tau_{j}) \right) [\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{g} d\tau,$$

for $V_1, \ldots, V_{2N+1} \in T_{\gamma} LX$, where $[\gamma|_s^t]^g$ denotes parallel transport in $\underline{\mathbb{C}}^k$ with respect to the connection ∇^g .

Proof. The curvature $F^g \in \Omega^2(X, \mathfrak{gl}_k(\mathbb{C}))$ of ∇^g is given by

$$F^g = ds \wedge \omega - s(1-s)\omega \wedge \omega.$$

From formula (8.1), a calculation gives that at $(\gamma, s) \in LX \times \mathbb{R} \subset L(X \times \mathbb{R})$, we have

$$\mathrm{BCh}_N(\underline{\mathbb{C}}^k, \nabla^g)|_{(\gamma,s)}[\partial_s, V_{2N+1}, \dots, V_1]$$

$$= -s^{N} (1-s)^{N} \sum_{\ell=0}^{N} \sum_{\sigma \in S_{2N+1}} \int_{\Delta_{N+1}} \operatorname{tr} \left([\gamma \|_{\tau_{N+1}}^{1}]^{g} \prod_{j=\ell+1}^{N} \omega \left(V_{\sigma_{2j+1}}(\tau_{j+1}) \right) \omega \left(V_{\sigma_{2j}}(\tau_{j+1}) \right) [\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{g} \right)$$

$$\times \omega (V_{\sigma_{2\ell+1}}(\tau_{\ell+1})) [\gamma \|_{\tau_{\ell}}^{\tau_{\ell+1}}]^g \prod_{j=1}^{\ell} \omega (V_{\sigma_{2j}}(\tau_j)) \omega (V_{\sigma_{2j-1}}(\tau_j)) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^g d\tau.$$

By the formula

$$\int_0^1 s^N (1-s)^N ds = \frac{(N!)^2}{(2N+1)!},$$
(9.6)

integrating s over [0,1] then gives the lemma.

Proof (of Thm. 9.1). Again, the way $I_T[BCh_N]$ is calculated by just interpreting all parallel transport appearing in the formula of Lemma 9.4 as stochastic parallel transport. The proof is then similar to that of Thm. 8.1.

Define functions $P_i^i(t,s)$ by

$$[\gamma]_s^t]^g e_b = P_b^a(t, s) e_a,$$

where e_1, \ldots, e_k is the standard basis of \mathbb{C}^k (here we use the Einstein summation convention). Moreover, define $\omega_{b,t}^a \in T'_{\gamma} \mathsf{L}_c M$ by

$$\omega|_{\gamma(t)}(V(t))e_b = \boldsymbol{\omega}_{b,t}^a[V]e_a.$$

Using these notations, the formula from Lemma 9.4 can be written in the form

$$BCh_{N}(g) = \frac{(N!)^{2}}{(2N+1)!} \delta_{a_{N+2}}^{b_{0}} \prod_{j=1}^{N+2} P_{b_{j-1}}^{a_{j}}(\tau_{j}, \tau_{j-1}) \sum_{i=1}^{N+1} \int_{\Delta_{N+1}} \omega_{a_{i}, \tau_{i}}^{b_{i}} \wedge \bigwedge_{\substack{j=1\\j \neq i}}^{N+1} \omega_{c_{j}, \tau_{j}}^{b_{j}} \wedge \omega_{a_{j}, \tau_{j}}^{c_{j}} d\tau.$$

Write $\boldsymbol{F}_{b,t}^a := -\boldsymbol{\omega}_{c,t}^a \wedge \boldsymbol{\omega}_{b,t}^c = K_2 \widetilde{\boldsymbol{F}}_{a,t}^b$, with $\widetilde{\boldsymbol{F}}_{a,t}^b \in \mathcal{M}_{\text{susy}}(S^1, \gamma^* \Lambda^2 T'X)$ (here the maps K_ℓ for $\ell \in P_{M,N}$ were defined in Section 5). Then whenever $\tau_1 < \cdots < \tau_{N+1}$, we have

$$\boldsymbol{\omega}_{a_{i},\tau_{i}}^{b_{i}} \wedge \bigwedge_{\substack{j=1\\j\neq i}}^{N+1} \boldsymbol{F}_{a_{j},\tau_{j}}^{b_{j}} = \sum_{m=1}^{N+1} \sum_{\substack{\sigma \in S_{N+1}\\\sigma_{m}=i}} K_{\ell_{m}} \big(\widetilde{\boldsymbol{F}}_{a_{\sigma_{N+1}},\tau_{\sigma_{N+1}}}^{b_{\sigma_{N+1}}} \boxtimes \cdots \boxtimes \boldsymbol{\omega}_{a_{i},\tau_{i}}^{b_{i}} \boxtimes \cdots \boxtimes \widetilde{\boldsymbol{F}}_{a_{\sigma_{1}},\tau_{\sigma_{1}}}^{b_{\sigma_{1}}} \big)$$

where $\ell_m = (2, \ldots, 2, 1, 2, \ldots, 2) \in P_{N+1,2N+1}$ (with 1 at the m-th position). Using

supersymmetry of \mathbf{D}_{N+1} , we therefore obtain

$$(-1)^{N} q \left(\boldsymbol{\omega}_{i_{\ell}, \tau_{\ell}}^{j_{\ell}} \wedge \bigwedge_{\substack{a=1\\a\neq\ell}}^{N+1} \boldsymbol{\omega}_{k_{a}, \tau_{a}}^{j_{a}} \wedge \boldsymbol{\omega}_{i_{a}, \tau_{a}}^{k_{a}} \right) = q \left(\boldsymbol{\omega}_{i_{\ell}, \tau_{\ell}}^{j_{\ell}} \wedge \bigwedge_{\substack{a=1\\a\neq\ell}}^{N+1} \boldsymbol{F}_{i_{a}, \tau_{a}}^{j_{a}} \right)$$

$$= \sum_{m=1}^{N+1} \sum_{\substack{\sigma \in S_{N+1}\\\sigma_{m}=i}} \left(\mathbf{D}_{N+1}, \widetilde{\boldsymbol{F}}_{a_{\sigma_{N+1}}, \tau_{\sigma_{N+1}}}^{b_{\sigma_{N+1}}} \boxtimes \cdots \boxtimes \boldsymbol{\omega}_{a_{i}, \tau_{i}}^{b_{i}} \boxtimes \cdots \boxtimes \widetilde{\boldsymbol{F}}_{a_{\sigma_{1}}, \tau_{\sigma_{1}}}^{b_{\sigma_{1}}} \right)_{L^{2}}$$

$$= 2^{-\frac{N+1}{2}} \operatorname{str} \left(\left[\gamma \right]_{\tau_{N+1}}^{1} \right]^{\Sigma} \prod_{j=i+1}^{N+1} \mathbf{c} \left(\widetilde{\boldsymbol{F}}_{a_{j}, \tau_{j}}^{b_{j}} \right) \left[\gamma \right]_{\tau_{j-1}}^{\tau_{j}} \right]^{\Sigma} \mathbf{c} \left(\boldsymbol{\omega}_{a_{i}, \tau_{i}}^{b_{i}} \right) \left[\gamma \right]_{\tau_{i-1}}^{\tau_{i}} \right]^{\Sigma} \prod_{j=1}^{i-1} \mathbf{c} \left(\widetilde{\boldsymbol{F}}_{a_{j}, \tau_{j}}^{b_{j}} \right) \left[\gamma \right]_{\tau_{j-1}}^{\tau_{j}} \right]^{\Sigma} \right).$$

by the formula of Remark 4.4. Denote by $F_s^g = -s(1-s)\omega \wedge \omega$ the curvature of $\underline{\mathbb{C}}^k$ with the connection $\nabla^{s,g} = d + s\omega$ (which equals the connection ∇^g , pulled back to the slice $M \times \{s\} \cong M$). Then

$$s(1-s)\widetilde{\boldsymbol{F}}_{i,t}^{j}[V \otimes W]e_{j} = \frac{1}{\sqrt{2}}s(1-s)\boldsymbol{F}_{i,t}^{j}[V,W] = \frac{1}{\sqrt{2}}F_{s}^{g}|_{\gamma(t)}(V(t),W(t)).$$

where the $\sqrt{2}$ comes from the Jacobian factor in the definition of K_2 . Using (9.6), we therefore find

$$q\left(\mathrm{BCh}_{N}(g)\right) = \frac{(-1)^{N}}{2^{N+1/2}} \sum_{\ell=1}^{N+1} \int_{0}^{1} \int_{\Delta_{N+1}} \mathrm{str}_{S} \left(\left[\gamma \right\|_{\tau_{N+1}}^{1}\right]^{S} \prod_{j=\ell+1}^{N+1} \mathbf{c}\left(F_{s}^{g}|_{\gamma(\tau_{j})}\right) \left[\gamma \right\|_{\tau_{j-1}}^{\tau_{j}}\right]^{\Sigma} \times \mathbf{c}\left(\omega|_{\gamma(\tau_{\ell})}\right) \left[\gamma \right\|_{\tau_{\ell-1}}^{\tau_{\ell}}\right]^{S} \prod_{j=1}^{\ell-1} \mathbf{c}\left(F_{s}^{g}|_{\gamma(\tau_{j})}\right) \left[\gamma \right\|_{\tau_{j-1}}^{\tau_{j}}\right]^{S} d\tau ds,$$

$$(9.7)$$

where $[\gamma|_s^t]^S$ is the parallel transport on $S := \Sigma \otimes \underline{\mathbb{C}}^k$ endowed with the tensor product connection $\nabla^{\Sigma} \otimes \nabla^{s,g}$. Writing short $K(\gamma)$ for the scalar curvature term in the definition (6.1) of the integral map, we have

$$I_{T}\left[\int_{\Delta_{N+1}} \operatorname{str}_{S}\left(\cdots\right) d\tau\right] = \int_{\Delta_{N+1}} \mathbb{W}_{T}\left[K(\gamma) \operatorname{str}_{S}\left(\left[\gamma \right\|_{\tau_{N+1}}^{1}\right]^{S} \prod_{j=\ell+1}^{N+1} \mathbf{c}\left(F_{s}^{g}|_{\gamma(\tau_{j})}\right) \left[\gamma \right\|_{\tau_{j-1}}^{\tau_{j}}\right]^{\Sigma} \times \mathbf{c}\left(\omega|_{\gamma(\tau_{\ell})}\right) \left[\gamma \right\|_{\tau_{\ell-1}}^{\tau_{\ell}}\right]^{S} \prod_{j=1}^{\ell-1} \mathbf{c}\left(F_{s}^{g}|_{\gamma(\tau_{j})}\right) \left[\gamma \right\|_{\tau_{j-1}}^{\tau_{j}}\right]^{S} \right] d\tau,$$

where we integrated over the interior integral in (9.7). Making the measure-preserving substitution $\gamma \mapsto (1 - \tau_{\ell}) \cdot \gamma$ in $L_c X$ and using the super trace property, we get that this equals

$$\int_{\Delta_{N+1}} \mathbb{W}_{T} \left[K(\gamma) \operatorname{str}_{S} \left(\mathbf{c} \left(\omega |_{\gamma(1)} \right) [\gamma \|_{1-\tau_{\ell}+\tau_{\ell-1}}^{1}]^{S} \prod_{j=1}^{\ell-1} \mathbf{c} \left(F_{s}^{g} |_{\gamma(1-\tau_{\ell}+\tau_{j})} \right) [\gamma \|_{1-\tau_{\ell}+\tau_{j-1}}^{1-\tau_{\ell}+\tau_{j}}]^{S} \times \left[\gamma \|_{\tau_{N+1}-\tau_{\ell}}^{1-\tau_{\ell}}]^{S} \prod_{j=\ell+1}^{N+1} \mathbf{c} \left(F_{s}^{g} |_{\gamma(\tau_{j}-\tau_{\ell})} \right) [\gamma \|_{\tau_{j-1}-\tau_{\ell}}^{\tau_{j}-\tau_{\ell}}]^{\Sigma} \right) \right] d\tau \tag{9.8}$$

Now change variables in Δ_{N+1} according to

$$\widetilde{\tau}_{j} = \begin{cases} \tau_{\ell+j} - \tau_{\ell} & j = 1, \dots, N - \ell + 1 \\ 1 - \tau_{\ell} & j = N - \ell + 2 \\ \tau_{j+\ell-N-2} + 1 - \tau_{\ell} & j = N - \ell + 3, \dots, N + 1. \end{cases}$$

The Jacobian of this coordinate change is

$$\left(\begin{array}{c} \frac{\partial \widetilde{\tau}_{j}}{\partial \tau_{i}} \end{array}\right)_{1 \leq i, j \leq N+1} = \begin{pmatrix} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ -1 & \dots & -1 & -1 & -1 & \dots & -1 \\ 1 & \dots & & & & \\ & & \ddots & & & & \\ & & & 1 & & \end{pmatrix},$$

where the ℓ -th row is filled with minus ones. The modulus of the determinant of this matrix is one, so after making this substitution, (9.8) takes the form

$$\int_{\Delta_{N+1}} \mathbb{W}_{T} \left[K(\gamma) \operatorname{str}_{S} \left(\mathbf{c} \left(\omega |_{\gamma(1)} \right) [\gamma \|_{\tau_{N+1}}^{1}]^{S} \prod_{j=N-\ell+3}^{N+1} \mathbf{c} \left(F_{s}^{g} |_{\gamma(\tau_{j})} \right) [\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{S} \times \left[\gamma \|_{\tau_{N-\ell+2}}^{\tau_{N-\ell+2}}]^{S} \prod_{j=1}^{N-\ell+1} \mathbf{c} \left(F_{s}^{g} |_{\gamma(\tau_{j})} [\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{\Sigma} \right) \right] d\tau.$$
(9.9)

Notice now that using $[\gamma|_{\tau_{N-\ell+2}}^{\tau_{N-\ell+3}}]^S[\gamma|_{\tau_{N-\ell+1}}^{\tau_{N-\ell+2}}]^S = [\gamma|_{\tau_{N-\ell+1}}^{\tau_{N-\ell+3}}]^S$, the variable $\tau_{N-\ell+2}$ is in fact free. Integrating it out and performing another substitution yields

$$\int_{\Delta_{N+1}} (\tau_{N-\ell+2} - \tau_{N-\ell+1}) \mathbb{W}_T \left[K(\gamma) \operatorname{str}_S \left(\mathbf{c} (\omega|_{\gamma(1)}) [\gamma|_{\tau_N}^1]^S \prod_{j=1}^N \mathbf{c} (F_s^g|_{\gamma(\tau_j)}) [\gamma|_{\tau_{j-1}}^{\tau_j}]^S \right) \right] d\tau.$$

Thus summing over ℓ , the sum telescopes and we obtain the result

$$I_T \left[\operatorname{BCh}_N(g) \right] = \frac{(-1)^N}{2^{N+1/2}} \int_0^1 \mathbb{W}_T \left[K(\gamma) \operatorname{str}_S \left(\mathbf{c} \left(\omega |_{\gamma(1)} \right) \int_{\Delta_{N+1}} [\gamma \|_{\tau_N}^1]^S \prod_{j=1}^N \mathbf{c} \left(F_s^g |_{\gamma(\tau_j)} \right) [\gamma \|_{\tau_{j-1}}^{\tau_j}]^S d\tau \right) \right] ds.$$

The Weizenböck formula for the twisted Dirac operator D_s on the bundle S is

$$\mathsf{D}_s^2 = (\nabla^S)^* \nabla^S + \mathbf{c}(F_s^g) + \frac{1}{4} \mathrm{scal}_g.$$

Hence with a look on (2.11), we obtain

$$I_T\big[\mathrm{BCh}_T(g)\big] = \sum_{N=0}^{\infty} T^N I_T\big[\mathrm{BCh}_N(g)\big] = \frac{1}{\sqrt{2}} \int_0^1 \mathbb{W}_T\Big[\mathrm{str}_S\Big(\mathbf{c}(\omega)U_T(\gamma)\Big)\Big] \mathrm{d}s,$$

where $U_T(\gamma)$ is the path-ordered exponential on $\Sigma \otimes \underline{\mathbb{C}}^k$ associated to the connection $\nabla^{\sigma} \otimes \nabla^{s,g}$ and potential $\frac{1}{2}\mathbf{c}(F_s^g) + \frac{1}{8}\mathrm{scal}_g$. By Remark 9.2, we have $\mathbf{c}(\omega) = \dot{D}_s$. From the Feynman-Kac formula, Thm. 2.1, we therefore obtain

$$I_T[\mathrm{BCh}_T(g)] = \frac{1}{\sqrt{2}} \int_0^1 \mathrm{Str}(\dot{\mathsf{D}}_s e^{-T\mathsf{D}_s^2/2}) \mathrm{d}s,$$

which is the theorem.

10 The Localization Principle

We use the notation $\Omega_{\text{ext},T}(\mathsf{L}X)$ for the space whose elements are finite sums of formal power series of the form

$$\theta_T = \sum_{N=-\infty}^{\infty} T^N \theta_N, \quad \text{with} \quad \theta_N \in \Omega_{\text{ext}}^{2N+M}(\mathsf{L}X).$$
 (10.1)

The equivariant differential $d_T = d + T^{-1}\iota_{\dot{\gamma}}$ can be applied to elements θ_T of this space, which giving an element $d_T\theta_T \in \Omega_T(\mathsf{L}X)$ (this latter space was discussed in Section 2). We can now formulate what it means for such a differential form to satisfy the localization principle.

Definition 10.1 (Localization Principle). Let $\theta_T \in \Omega_{\text{ext},T}(LX)$ such that $d_T\theta_T = 0$. We say that θ_T satisfies the localization principle if we have

$$I_T[\theta_T] = (2\pi T)^{-n/2} \int_X \widehat{A}(T) \wedge i^* \theta_T, \qquad (10.2)$$

where $i: X \to LX$ is the inclusion as the set of constant paths.

This should be compared with Thm. 7.13 of [BGV04], remembering that the (regularized) equivariant Euler class of the normal bundle of X in LX is \widehat{A} , as remarked in Section 3. So far we do not know whether all functions in $\Omega_{\text{ext}}(LX)$ satisfy the localization principle.

Remark 10.2. Notice that the right hand side of (10.2) is only a polynomial in T and T^{-1} (in odd dimensions additionally multiplied by $T^{-1/2}$). In particular, the statement therefore implies that $I_T[\theta_T]$ is also such a polynomial.

Since we do not have the localization principle at our disposal, we cannot yet use it to give a proof of the twisted version of the Atiyah-Singer index theorem as envisioned by Bismut [Bis85]. However, conversely, we can *use* the Atiyah-Singer index theorem to show that the localization principle holds for the Bismut-Chern characters.

Theorem 10.3. The Bismut-Chern characters satisfy the localization principle.

Proof. Notice first that in even dimensions, both sides of (10.2) are zero if one plugs in BCh(g) for θ_T , while in odd dimensions, both sides are zero if one plugs in $BCh_T(\mathcal{V}, \nabla)$ for θ_T .

Suppose that $n = \dim(X)$ is even and let \mathcal{V} be a super vector bundle with connection ∇ , so that $\mathrm{BCh}(\mathcal{V}, \nabla)$ is defined. Then on the one hand, by Remark 8.2, we have $I_T[\mathrm{BCh}_T(\mathcal{V}, \nabla)] = i^{n/2}\mathrm{ind}(\mathsf{D}_{\mathcal{V}})$ for any T > 0, where $\mathsf{D}_{\mathcal{V}}$ is the twisted Dirac operator acting on the bundle $\Sigma^{\mathbb{C}} \otimes \mathcal{V}$. On the other hand by (8.2), we have for the pullback $i^*\mathrm{BCh}_T(\mathcal{V}, \nabla) = \mathrm{ch}_T(\mathcal{V}, \nabla)$, so that the right hand side of (10.2) equals

$$(2\pi T)^{-n/2} \int_X \widehat{A}(T) \wedge i^* \mathrm{BCh}_T(\mathcal{V}, \nabla) = (2\pi)^{-n/2} \int_X \widehat{A}(X) \wedge \mathrm{ch}(\mathcal{V}, \nabla),$$

where the T-dependence cancels as only the n-form part of the integrand is relevant. By the Atiyah-Singer theorem (Thm. 4.3 in [BGV04]), this equals $i^{n/2}$ ind($D_{\mathcal{V}}$).

Now suppose that n = 2m + 1 is odd and for some $k \in \mathbb{N}$, let $g : X \to U_k$ be a smooth map, where U_k is the k-th unitary group. In this case, we have

$$I_T \left[\operatorname{BCh}(g) \right] = (-i)^{m+1} \left(\frac{2\pi}{T} \right)^{1/2} \operatorname{sf}(\mathsf{D}, g^{-1} \mathsf{D}g)$$

by Corollary 9.3. On the other hand, by a theorem of Getzler (Thm. 2.8 in [Get93]), we have

$$\mathrm{sf}(\mathsf{D},g^{-1}\mathsf{D}g) = \frac{-1}{(-2\pi i)^{m+1}} \int_{M} \widehat{A}(X) \wedge \mathrm{ch}(g) = \left(\frac{T}{2\pi}\right)^{1/2} \frac{i^{m+1}}{(2\pi T)^{n/2}} \int_{M} \widehat{A}(T) \wedge \mathrm{ch}_{T}(g),$$

To see the second equality, remember that the T-independent Chern character used in Getzler is related to our T-dependent one defined in (9.2) by $\operatorname{ch}(g) = \operatorname{ch}_{-1}(g)$. Writing $\widehat{A}(T) = \sum_{N} T^{N} \widehat{A}_{2N}$ respectively $\operatorname{ch}_{T}(g) = \sum_{N} T^{N} \operatorname{ch}_{2N+1}$ for the homogeneous components, we obtain

$$\begin{aligned} \left[\widehat{A}(X) \wedge \text{ch}(g) \right]_{\text{top}} &= \sum_{N=0}^{\frac{n-1}{2}} \left[\widehat{A}_{2N} \wedge (-1)^{n-2N} \text{ch}_{n-2N} \right]_{\text{top}} \\ &= (-1)^n T^{-\frac{n-1}{2}} \sum_{N=0}^{\frac{n-1}{2}} \left[T^N \widehat{A}_{2N} \wedge T^{n-1-2N} \text{ch}_{n-2N} \right]_{\text{top}} \\ &= -T^{-\frac{n-1}{2}} \left[\widehat{A}(T) \wedge \text{ch}_T(g) \right]_{\text{top}} \end{aligned}$$

hence

$$I_T[\mathrm{BCh}(g)] = (2\pi T)^{-n/2} \int_M \widehat{A}(T) \wedge \mathrm{ch}_T(g),$$

which is the proposition since $i^*BCh_T(g) = ch_T(g)$.

With Thm. 10.3 at our disposal, we are not as far from proving that the localization principle for all $\theta_T \in \Omega_{\text{ext},T}(\mathsf{L}X)$ as one might think, because the Bismut-Chern characters in fact generate the cyclic equivariant cohomology $h_T^*(\mathsf{L}X,\mathbb{C})$ of the loop space (recall that by definition, this is the cohomology of the complex $\Omega_T(\mathsf{L}X)^{\mathbb{T}}$ with the differential d_T). In fact, the diagram

$$K^*(X) \otimes \mathbb{C} \xrightarrow{\operatorname{Ch}_T} H^*_{\operatorname{dR}}(X, \mathbb{C})[T^{-1}, T]$$

commutes, with each arrow being an isomorphism of graded rings⁹.

Let $\mathcal{BC} \subset \Omega_{\text{ext},T}(\mathsf{L}X)$ be the subspace generated by the Bismut-Chern characters. Given an integrable form $\theta_T \in \Omega_{\text{ext},T}(\mathsf{L}X)^{\mathbb{T}} \subset \Omega_T(\mathsf{L}X)$ with $d_T\theta_T = 0$, by the above considerations, there is $\beta_T \in \mathcal{BC}$ that represents the same class in $h_T^*(\mathsf{L}X)$ as θ_T , i.e. $\theta_T = \beta_T + d_T\eta_T$. Notice that $d_T\eta_T = \theta_T - \beta_T \in \Omega_{\text{ext},T}(\mathsf{L}X)^{\mathbb{T}}$. From the invariance of I_T under the \mathbb{T} -action (c.f. Prop. 6.6), we obtain the following statement.

Lemma 10.4. The following assertions are equivalent.

- (i) The localization principle holds for all $\theta_T \in \Omega_{\text{ext},T}(\mathsf{L}X)$.
- (ii) For all $\eta_T \in \Omega_T(\mathsf{L}X)^{\mathbb{T}}$ with $d_T\eta_T \in \Omega_{\mathrm{ext},T}(\mathsf{L}X)$, we have $I_T[d_T\eta_T] = 0$.

In [HL17d], we will show that assertion (ii) of the above lemma is in fact true for all η_T lying in the domain of the extended iterated integral map of Getzler, Jones and Petrack [GJP91], which implies that the localization principle holds for all forms that are (extended) iterated integrals. We remark that Lott proves (ii) above for all forms that are sums of wedge products of $P_{\varphi}f$ and $P_{\varphi}df$ for $\varphi \in C^{\infty}(S^1)$ and $f \in C^{\infty}(X)$ [Lot87, Prop. 10]. Here P_{φ} denotes the lifting map defined in (6.4) and (6.5) above.

However, one must say that this route of attacking a proof of the localization principle is rather unsatisfactory, since it uses the established index theorems instead of providing a new and unified way of proving them. It is therefore desirable to find an independent proof of the localization principle for I_T not relying on Thm. 10.3. Of course, one could try to transfer the proof from [BGV04] to our infinite-dimensional setup. However, it seems that doing so, one basically recovers Getzler rescaling [BGV04, Section 2.5] [Get86] (this approach is similar to [Bis84] or [FS08]). While it seems not uninteresting to work this out, we hope that there is a cohomological proof of the localization principle for, in particular one that gets away without having to calculate any short-time limits.

⁹It is clearly commutative, and the vertical map is an isomorphism by Thm. 2.1 in [JP90]. Moreover, $K^* \otimes \mathbb{C}$ and $H^*\mathbb{C}[T^{-1},T]$ are both cohomology theories, so it suffices to notice that the Chern character is an isomorphism for X a point.

A The Bismut-Chern-Character is equivariantly closed

In this appendix, we prove the following result.

Proposition A.1. The differential forms BCh_N defined above satisfy

$$dBCh_N(\nabla, \mathcal{V}) = -\iota_{\dot{\gamma}}BCh_{N+1}(\nabla, \mathcal{V}),$$

hence the Bismut-Chern character $BC(\mathcal{V}, \nabla)$ is equivariantly closed, i.e. $d_T BCh_T(\mathcal{V}, \nabla) = 0$. Here $\iota_{\dot{\gamma}}$ denotes insertion of the generating vector field $\dot{\gamma}$ of the \mathbb{T} -action.

We need the following lemma.

Lemma A.2. For $v \in T_{\gamma} LX$, we have

$$\nabla_v [\gamma \|_s^t]^{\mathcal{V}} = \int_s^t [\gamma \|_u^t]^{\mathcal{V}} F[\dot{\gamma}(u), v(u)] [\gamma \|_s^u]^{\mathcal{V}} du,$$

where we differentiated covariantly using the pull-back connection on the bundle $\operatorname{ev}_{s,t}^* \mathcal{V} \boxtimes \mathcal{V}^*$ over LX.

Proof. Let γ_{ε} be a variation of the path γ in direction $v \in T_{\gamma} LX$, i.e. $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} \gamma_{\varepsilon} = v$. Then

$$\frac{\nabla}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} [\gamma_{\varepsilon}\|_{s}^{t}]^{\mathcal{V}} = \underbrace{\frac{\nabla}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} [\gamma_{\varepsilon}\|_{s}^{s}]^{\mathcal{V}}}_{=0} + \int_{s}^{t} [\gamma\|_{u}^{t}]^{\mathcal{V}} \frac{\nabla}{\mathrm{d}u} \frac{\nabla}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} [\gamma_{\varepsilon}\|_{s}^{u}]^{\mathcal{V}} \mathrm{d}u$$

$$= \int_{s}^{t} [\gamma\|_{u}^{t}]^{\mathcal{V}} F[\dot{\gamma}(u), v(u)] [\gamma_{\varepsilon}\|_{s}^{u}]^{\mathcal{V}} \mathrm{d}u + \int_{s}^{t} [\gamma\|_{u}^{t}]^{\mathcal{V}} \frac{\nabla}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \underbrace{\frac{\nabla}{\mathrm{d}u} [\gamma_{\varepsilon}\|_{s}^{u}]^{\mathcal{V}}}_{=0} \mathrm{d}u,$$

where we used that $[\gamma ||_s]^{\mathcal{V}} = \mathrm{id}$, which is parallel.

Proof (Proof of Prop. A.1). Let $V_1, \ldots, V_{2N}, W \in T_{\gamma} LX$. Differentiating BCh_N, we need to differentiate each instance of the parallel transport as well as each instance of the curvature tensor F. Using Lemma A.2 for the derivative of the parallel transport, this gives

$$(\nabla_{W} BCh_{N})[V_{2N}, \dots, V_{1}]$$

$$= (-2)^{-N} \sum_{k=1}^{N+1} \sum_{\sigma \in S_{2N}} sgn(\sigma) \int_{\Delta_{N}} str\left(\prod_{j=k}^{N} [\gamma \|_{\tau_{j}}^{\tau_{j+1}}]^{\nu} F(V_{\sigma_{2j}}(\tau_{j}), V_{\sigma_{2j-1}}(\tau_{j}))\right)$$

$$\times \left(\int_{\tau_{k-1}}^{\tau_{k}} [\gamma \|_{t}^{\tau_{k}}]^{\nu} F(\dot{\gamma}(t), W(t)) [\gamma \|_{\tau_{k-1}}^{t}]^{\nu} dt\right) \prod_{j=1}^{k-1} F(V_{\sigma_{2j}}(\tau_{j}), V_{\sigma_{2j-1}}(\tau_{j})) [\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{\nu} d\tau$$

$$+ \sum_{k=1}^{N+1} \sum_{\sigma \in S_{2N}} sgn(\sigma) \int_{\Delta_{N}} str\left([\gamma \|_{\tau_{N}}^{1}]^{\nu} \prod_{j=k+1}^{N} F(V_{\sigma_{2j}}(\tau_{j}), V_{\sigma_{2j-1}}(\tau_{j})) [\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{\nu}$$

$$\times (\nabla_{W(\tau_{k})} F) (V_{\sigma_{2k}}(\tau_{k}), V_{\sigma_{2k-1}}(\tau_{k})) [\gamma \|_{\tau_{k-1}}^{\tau_{k}}]^{\nu} \prod_{j=1}^{k-1} F(V_{\sigma_{2j}}(\tau_{j}), V_{\sigma_{2j-1}}(\tau_{j})) [\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{\nu} d\tau.$$

Here we used the convention $\tau_{N+1} = 1$. Now since the exterior derivative is the anti-symmetrization of the covariant derivative, the second summand in the formula for ∇C_N above vanishes by the second Bianchi identity. Therefore,

$$(-2)^{N} dBCh_{N}[V_{2N+1}, \dots, V_{1}]$$

$$= \sum_{k=1}^{N+1} \sum_{\sigma \in S_{2N+1}} \operatorname{sgn}(\sigma) \int_{\Delta_{N+1}} \operatorname{str}\left(\left[\gamma \right\|_{\tau_{N+1}}^{1} \right]^{\mathcal{V}} \prod_{j=k+1}^{N+1} F\left(V_{\sigma_{2j-1}}(\tau_{j}), V_{\sigma_{2j-2}}(\tau_{j}) \right) \left[\gamma \right\|_{\tau_{j-1}}^{\tau_{j}} \right]^{\mathcal{V}} \times F\left(\dot{\gamma}(\tau_{k}), V_{\sigma_{2k-1}}(\tau_{k}) \right) \left[\gamma \right\|_{\tau_{k-1}}^{\tau_{k}} \right]^{\mathcal{V}} \prod_{j=1}^{k-1} F\left(V_{\sigma_{2j}}(\tau_{j}), V_{\sigma_{2j-1}}(\tau_{j}) \right) \left[\gamma \right\|_{\tau_{j-1}}^{\tau_{j}} \right]^{\mathcal{V}} d\tau$$

On the other hand, setting $V_{2N+2} := \dot{\gamma}$, we have

$$(-2)^{N+1} \iota_{\dot{\gamma}} BCh_{N+1}(\nabla, \mathcal{V})[V_{2N+1}, \dots, V_{1}] = (-2)^{N+1} BCh_{N+1}[V_{2N+2}, \dots, V_{1}]$$

$$= \sum_{\sigma \in S_{2N+2}} sgn(\sigma) \int_{\Delta_{N+1}} str\left([\gamma \|_{\tau_{N+1}}^{1}]^{\mathcal{V}} \prod_{j=1}^{N+1} F(V_{\sigma_{2j}}(\tau_{j}), V_{\sigma_{2j-1}}(\tau_{j}))[\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{\mathcal{V}} \right) d\tau$$

$$= \sum_{k=1}^{N+1} \left(\sum_{\substack{\sigma \in S_{2N+2} \\ \sigma_{2k} = 2N+2}} sgn(\sigma) \cdots + \sum_{\substack{\sigma \in S_{2N+2} \\ \sigma_{2k-1} = 2N+2}} sgn(\sigma) \dots \right) = 2 \sum_{k=1}^{N+1} \sum_{\substack{\sigma \in S_{2N+2} \\ \sigma_{2k} = 2N+2}} sgn(\sigma) \dots$$

$$= 2 \sum_{k=1}^{N+1} \sum_{\substack{\sigma \in S_{2N+1} \\ \sigma_{2k} = 2N+2}} sgn(\sigma) \int_{\Delta_{N+1}} str\left([\gamma \|_{\tau_{N+1}}^{1}]^{\mathcal{V}} \prod_{j=k+1}^{N+1} F(V_{\sigma_{2j-1}}(\tau_{j}), V_{\sigma_{2j-2}}(\tau_{j}))[\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{\mathcal{V}} \right) d\tau$$

$$\times F(\dot{\gamma}(\tau_{k}), V_{2k-1}(\tau_{k}))[\gamma \|_{\tau_{k-1}}^{\tau_{k}}]^{\mathcal{V}} \prod_{j=1}^{k-1} F(V_{\sigma_{2j}}(\tau_{j}), V_{\sigma_{2j-1}}(\tau_{j}))[\gamma \|_{\tau_{j-1}}^{\tau_{j}}]^{\mathcal{V}} d\tau$$

$$= 2(-2)^{N} dBCh_{N}(\nabla, \mathcal{V})[V_{2N+1}, \dots, V_{1}].$$

This finishes the proof.

B The relative Integral Map

There is a relative version of the integral map, defined as follows.

Definition B.1. The *relative* integral map

$$I_T^{\mathrm{rel}}:\Omega_{\mathrm{int},b}(\mathsf{L}_cX)\longrightarrow\Omega(X)$$

is defined by the formula

$$I_T^{\mathrm{rel}}[\theta](x) := 2^{n/2} \mathbb{W}_T^{xx} \left[\exp\left(-\frac{T}{8} \int_0^1 \mathrm{scal}(\gamma(s)) ds\right) \mathbf{c}^{-1}(q^{\mathrm{rel}}(\theta)) \right].$$

for $\theta \in \Omega_{\text{int},b}(\mathsf{L}_c X)$.

By the formula for the super trace (2.2), the relative integral map satisfies

$$\int_{X} I_{T}^{\text{rel}}[\theta] = I_{T}[\theta]. \tag{B.1}$$

Notice here that the relative integral maps is an even map, in the sense that it maps even forms to even forms and odd forms to odd forms. In contrast, the integral map itself is even in even dimensions (i.e. it maps odd forms to zero) and odd in odd dimensions (i.e. it maps even forms to zero). This is explained by the relationship (B.1), since the usual integral map \int_X is even in even dimensions and odd in odd dimensions.

The following proposition shows that the relative integral map can be seen as an integrationover-the-fibers map.

Proposition B.2. For all $\theta \in \Omega_{\text{int},b}(\mathsf{L}_cX)$ and $\alpha \in \Omega^{\ell}(X)$, $\ell \geq 1$, we have

$$I_T(P\alpha \wedge \theta) = \frac{1}{\sqrt{2}} \int_X \alpha \wedge I_T^{\text{rel}}(Av\theta).$$

for all T > 0, where the map P was defined in (6.4) above.

Proof. We first show that that given $\tau = (\tau_{N+1}, \ldots, \tau_1) \in T^N$, and $\vartheta_j \in \Lambda T_{\gamma(\tau_j)} X$, we have

$$\langle \mathbf{D}_{N+1}|_{\gamma}(\tau_{N+1},\ldots,\tau_{1}),\vartheta_{N+1}\otimes\cdots\otimes\vartheta_{1}\rangle$$

$$=\frac{1}{\sqrt{2}(N+1)}\operatorname{str}\Big(\mathbf{c}(\vartheta_{N+1})\langle \mathbf{D}_{N}^{\mathrm{rel}}|_{\tau_{N+1}\cdot\gamma}(\tau_{N}-\tau_{N+1},\ldots,\tau_{1}-\tau_{N+1}),\vartheta_{N}\otimes\cdots\otimes\vartheta_{1}\rangle\Big).$$

To this end, suppose that ϑ_j is homogeneous of degree ℓ_j and let $\sigma \in S_{N+1}$ be the permutation such that $\tau_{\sigma_1} < \cdots < \tau_{\sigma_{N+1}}$. Suppose that $\sigma_k = N+1$. Then by Remark 4.4, we have

$$2^{(N+1)/2}(N+1)! \langle \mathbf{D}_{N+1}|_{\gamma}(\tau_{N+1}, \dots, \tau_{1}), \vartheta_{N+1} \otimes \dots \otimes \vartheta_{1} \rangle$$

$$= \operatorname{sgn}(\sigma; \ell) \operatorname{str} \left([\gamma \|_{\tau_{\sigma_{N+1}}}^{1}]^{\Sigma} \prod_{j=1}^{N+1} \mathbf{c}(\vartheta_{\sigma_{j}}) [\gamma \|_{\tau_{\sigma_{j-1}}}^{\tau_{\sigma_{j}}}]^{\Sigma} \right)$$

$$= \epsilon \operatorname{sgn}(\sigma; \ell) \operatorname{str} \left(\mathbf{c}(\vartheta_{N+1}) [\gamma \|_{\tau_{\sigma_{k-1}}}^{\tau_{\sigma_{k}}}]^{\Sigma} \prod_{j=1}^{k-1} \mathbf{c}(\vartheta_{\sigma_{j}}) [\gamma \|_{\tau_{\sigma_{j-1}}}^{\tau_{\sigma_{j}}}]^{\Sigma} \prod_{j=k+1}^{N+1} \mathbf{c}(\vartheta_{\sigma_{j}}) [\gamma \|_{\tau_{\sigma_{j-1}}}^{\tau_{\sigma_{j}}}]^{\Sigma} \right)$$

Here we set $\epsilon := (-1)^{(\ell_{\sigma_1} + \dots + \ell_{\sigma_k})(\ell_{k+1} + \dots + \ell_{N+1})}$ and used the cyclic permutation property (2.3) of the super trace. Write $\overline{\ell} = (\ell_N, \dots, \ell_1)$ and let $\overline{\sigma} \in S_N$ be the permutation given by

$$\overline{\sigma} = \begin{pmatrix} 1 & \cdots & N-k+1 & N-k+2 & \cdots & N \\ \sigma_{k+1} & \cdots & \sigma_{N+1} & \sigma_1 & \cdots & \sigma_{k-1} \end{pmatrix}.$$

Then $\operatorname{sgn}(\overline{\sigma}; \overline{\ell}) = \epsilon \operatorname{sgn}(\sigma; \ell)$. Moreover, since

$$[\gamma \|_{\tau_{\sigma_{j-1}}}^{\tau_{\sigma_j}}]^{\Sigma} = [\tau_{N+1} \cdot \gamma \|_{\tau_{\sigma_{j-1}} - \tau_{N+1}}^{\tau_{\sigma_j} - \tau_{N+1}}]^{\Sigma},$$

we obtain that the above equals

$$2^{N/2}N!\operatorname{sgn}(\overline{\sigma};\overline{\ell})\operatorname{str}\left(\mathbf{c}(\vartheta_{N+1})\langle\mathbf{D}_{N}^{\operatorname{rel}}|_{\tau_{N+1}\cdot\gamma}(\tau_{\overline{\sigma}_{N}}-\tau_{N+1},\ldots,\tau_{\overline{\sigma}_{1}}-\tau_{N+1}),\vartheta_{\overline{\sigma}_{N}}\otimes\cdots\otimes\vartheta_{\overline{\sigma}_{1}}\rangle\right),$$

which is the claim, by super symmetry of $\mathbf{D}_{N}^{\text{rel}}$.

Now let $\theta \in \mathcal{M}_{susy}(T^N_{\circ}, \gamma^* \Lambda^{\geq 1} TX^{\boxtimes N})$. Since \mathbf{D}_N is supersymmetric, we have

$$q(P\alpha \wedge \theta) = (N+1) \int_{T_0^{N+1}} \langle \mathbf{D}_{N+1}(\tau_{N+1}, \dots, \tau_1), \alpha(\gamma(\tau_{N+1})) \otimes \theta(\tau_N, \dots, \tau_1) \rangle d\tau.$$

From the above considerations, we then obtain

$$q(P\alpha \wedge \theta|_{\gamma})$$

$$= \frac{1}{\sqrt{2}} \int_{\mathbb{T}} \int_{T_{\circ}^{N}} \operatorname{str}\left(\mathbf{c}(\vartheta(\gamma(t))) \langle \mathbf{D}_{N}^{\mathrm{rel}}|_{\gamma}(\tau_{N} - t, \dots, \tau_{1} - t), \theta|_{\gamma}(\tau_{N}, \dots, \tau_{1}) \rangle\right) d\tau dt$$

$$= \frac{1}{\sqrt{2}} \int_{\mathbb{T}} \int_{T_{\circ}^{N}} \operatorname{str}\left(\mathbf{c}(\vartheta(t \cdot \gamma(1))) \langle \mathbf{D}_{N}^{\mathrm{rel}}|_{t \cdot \gamma}(\tau_{N}, \dots, \tau_{1}), \theta|_{\gamma}(\tau_{N} + t, \dots, \tau_{1} + t) \rangle\right) d\tau dt.$$

where in the second step, we used the transformation formula on the diffeomorphism of T^N given by the \mathbb{T} -action (notice that this preserves T_{\circ}^N). Hence for $\theta \in \Omega_{\text{int}}(\mathsf{L}_cX)$, we obtain

$$q(P\alpha \wedge \theta|_{\gamma}) = \frac{1}{\sqrt{2}} \int_{\mathbb{T}} \operatorname{str} \Big(\mathbf{c} \big(\vartheta \big(t \cdot \gamma(1) \big) \big) \big(\mathbf{D}^{\operatorname{rel}}|_{t \cdot \gamma}, (t \cdot \theta)|_{t \cdot \gamma} \big)_{L^{2}} \Big) dt.$$

Now it is easy to see from (2.7) that the diffeomorphism of L_cX that sends γ to $t \cdot \gamma$ preserves W_T . Using this and Fubini's theorem, we obtain

$$\sqrt{2}I_{T}[P\alpha \wedge \theta] = \int_{\mathbb{T}} \mathbb{W}_{T} \left[e^{-\frac{T}{8} \int_{S^{1}} \operatorname{scal}(\gamma(t)) dt} \operatorname{str} \left(\mathbf{c} \left(\vartheta \left(t \cdot \gamma(1) \right) \right) \left(\mathbf{D}^{\operatorname{rel}} |_{t \cdot \gamma}, (t \cdot \theta) |_{t \cdot \gamma} \right)_{L^{2}} \right] dt \\
= \int_{\mathbb{T}} \mathbb{W}_{T} \left[e^{-\frac{T}{8} \int_{S^{1}} \operatorname{scal} \left(\gamma(t) \right) dt} \operatorname{str} \left(\mathbf{c} \left(\vartheta \left(\gamma(1) \right) \right) \left(\mathbf{D}^{\operatorname{rel}} |_{\gamma}, (t \cdot \theta) |_{\gamma} \right)_{L^{2}} \right] dt \\
= \mathbb{W}_{T} \left[e^{-\frac{T}{8} \int_{S^{1}} \operatorname{scal}(\gamma(t)) dt} \operatorname{str} \left(\mathbf{c} \left(\vartheta \left(\gamma(1) \right) \right) q^{\operatorname{rel}} (\operatorname{Av} \theta) \right) \right] \\
= \int_{X} \operatorname{str} \left(\mathbf{c} \left(\vartheta \left(x \right) \right) \mathbb{W}_{T}^{xx} \left[e^{-\frac{T}{8} \int_{S^{1}} \operatorname{scal}(\gamma(t)) dt} q^{\operatorname{rel}} (\operatorname{Av} \theta) \right] \right) dx.$$

Finally, it is easy to see using the definition (2.2) of the supertrace that for any $a \in Cl(T_xX)$, we have

$$\operatorname{str}(\mathbf{c}(\vartheta(x))a) = 2^{n/2}[\vartheta(x) \wedge \mathbf{c}^{-1}(a)]_{\operatorname{top}}.$$

From this, the proposition follows.

References

[AD99] Lars Andersson and Bruce K. Driver. Finite-dimensional approximations to Wiener measure and path integral formulas on manifolds. *J. Funct. Anal.*, 165(2):430–498, 1999.

- [AG84] Luis Alvarez-Gaumé. Supersymmetry and the Atiyah-Singer index theorem. *Phys. A*, 124(1-3):29–45, 1984. Mathematical physics, VII (Boulder, Colo., 1983).
- [AG85] Luis Alvarez-Gaumé. Supersymmetry and index theory. In Supersymmetry (Bonn, 1984), volume 125 of NATO Adv. Sci. Inst. Ser. B Phys., pages 1–44. Plenum, New York, 1985.
- [Ati85] M. F. Atiyah. Circular symmetry and stationary-phase approximation. *Astérisque*, (131):43–59, 1985. Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983).
- [BGV04] Nicole Berline, Ezra Getzler, and Michèle Vergne. Heat kernels and Dirac operators. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original.
 - [Bis84] Jean-Michel Bismut. The Atiyah-Singer theorems: a probabilistic approach. I. The index theorem. J. Funct. Anal., 57(1):56–99, 1984.
 - [Bis85] Jean-Michel Bismut. Index theorem and equivariant cohomology on the loop space. Comm. Math. Phys., 98(2):213–237, 1985.
 - [BP08] Christian Bär and Frank Pfäffle. Path integrals on manifolds by finite dimensional approximation. J. Reine Angew. Math., 625:29–57, 2008.
 - [BP11] Christian Bär and Frank Pfäffle. Wiener measures on Riemannian manifolds and the Feynman-Kac formula. *Mat. Contemp.*, 40:37–90, 2011.
- [Che73] Kuo-tsai Chen. Iterated integrals of differential forms and loop space homology. Ann. of Math. (2), 97:217–246, 1973.
 - [É89] Michel Émery. Stochastic calculus in manifolds. Universitext. Springer-Verlag, Berlin, 1989. With an appendix by P.-A. Meyer.
- [FS08] Dana S. Fine and Stephen F. Sawin. A rigorous path integral for supersymmetric quantum mechanics and the heat kernel. *Comm. Math. Phys.*, 284(1):79–91, 2008.
- [FS14] Dana S. Fine and Stephen Sawin. Short-time asymptotics of a rigorous path integral for N=1 supersymmetric quantum mechanics on a Riemannian manifold. J. Math. Phys., 55(6):062104, 25, 2014.
- [FS17] Dana S. Fine and Stephen Sawin. Path integrals, supersymmetric quantum mechanics, and the Atiyah-Singer index theorem for twisted Dirac. *J. Math. Phys.*, 58(1):012102, 30, 2017.
- [Gü10] Batu Güneysu. The Feynman-Kac formula for Schrödinger operators on vector bundles over complete manifolds. *J. Geom. Phys.*, 60(12):1997–2010, 2010.

- [Get86] Ezra Getzler. A short proof of the local Atiyah-Singer index theorem. *Topology*, 25(1):111–117, 1986.
- [Get91] Ezra Getzler. The Thom class of Mathai and Quillen and probability theory. In *Stochastic analysis and applications (Lisbon, 1989)*, volume 26 of *Progr. Probab.*, pages 111–122. Birkhäuser Boston, Boston, MA, 1991.
- [Get93] Ezra Getzler. The odd Chern character in cyclic homology and spectral flow. Topology, 32(3):489–507, 1993.
- [GJP91] Ezra Getzler, John D. S. Jones, and Scott Petrack. Differential forms on loop spaces and the cyclic bar complex. *Topology*, 30(3):339–371, 1991.
- [HL17a] Florian Hanisch and Matthias Ludewig. Supersymmetric path integrals II: Combinatorics of the fermionic integral and the pfaffian line bundle. arXiv:1709.10028, 2017.
- [HL17c] Florian Hanisch and Matthias Ludewig. Supersymmetric path integrals III: The viewpoint of infinite-dimensional super geometry. to appear.
- [HL17d] Florian Hanisch and Matthias Ludewig. Supersymmetric path integrals IV: Iterated integrals, rough paths and the localization principle. to appear.
 - [It63] Kiyoshi Itô. The Brownian motion and tensor fields on Riemannian manifold. In *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, pages 536–539. Inst. Mittag-Leffler, Djursholm, 1963.
 - [It75] Kiyosi Itô. Stochastic parallel displacement. pages 1–7. Lecture Notes in Math., Vol. 451, 1975.
 - [JP90] J. D. S. Jones and S. B. Petrack. The fixed point theorem in equivariant cohomology. *Trans. Amer. Math. Soc.*, 322(1):35–49, 1990.
- [Kac49] M. Kac. On distributions of certain Wiener functionals. *Trans. Amer. Math. Soc.*, 65:1–13, 1949.
- [KM97] Andreas Kriegl and Peter W. Michor. The convenient setting of global analysis, volume 53 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997.
- [LM89] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. Spin geometry, volume 38 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1989.
- [Lot87] John Lott. Supersymmetric path integrals. Comm. Math. Phys., 108(4):605–629, 1987.

- [LR01] Steven Leppard and Alice Rogers. A Feynman-Kac formula for anticommuting Brownian motion. J. Phys. A, 34(3):555–568, 2001.
- [Lud17] Matthias Ludewig. Path Integrals on Manifolds with Boundary. Comm. Math. Phys., 354(2):621–640, 2017.
- [Phi96] John Phillips. Self-adjoint Fredholm operators and spectral flow. Canad. Math. Bull., 39(4):460–467, 1996.
- [PW09] Arturo Felipe Prat Waldron. Pfaffian line bundles over loop spaces, spin structures and the index theorem. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)—University of California, Berkeley.
- [Rog03] Alice Rogers. Supersymmetry and Brownian motion on supermanifolds. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 6(suppl.):83–102, 2003.
- [ST05] Stefan Stolz and Peter Teichner. The spinor bundle on the loop space. 2005.
- [Sto96] Stephan Stolz. A conjecture concerning positive Ricci curvature and the Witten genus. *Math. Ann.*, 304(4):785–800, 1996.
- [Tr67] François Trèves. Topological vector spaces, distributions and kernels. Academic Press, New York-London, 1967.
- [TWZ13] Thomas Tradler, Scott O. Wilson, and Mahmoud Zeinalian. An elementary differential extension of odd K-theory. J. K-Theory, 12(2):331–361, 2013.
- [TWZ15] Thomas Tradler, Scott O. Wilson, and Mahmoud Zeinalian. Loop differential K-theory. Ann. Math. Blaise Pascal, 22(1):121–163, 2015.
 - [Wal16] Konrad Waldorf. Spin structures on loop spaces that characterize string manifolds. *Algebr. Geom. Topol.*, 16(2):675–709, 2016.
 - [Wil16] Scott O. Wilson. A loop group extension of the odd Chern character. *J. Geom. Phys.*, 102:32–43, 2016.
 - [Wit82] Edward Witten. Supersymmetry and Morse theory. *J. Differential Geom.*, 17(4):661–692 (1983), 1982.