SPECTRAL GAP AND QUANTITATIVE STATISTICAL STABILITY FOR SYSTEMS WITH CONTRACTING FIBERS AND LORENZ-LIKE MAPS.

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ABSTRACT. We consider transformations preserving a contracting foliation, such that the associated quotient map satisfies a Lasota Yorke inequality. We prove that the associated transfer operator, acting on suitable normed spaces, has a spectral gap (on which we have quantitative estimation).

As an application we consider Lorenz-Like two dimensional maps (piecewise hyperbolic with unbounded contraction and expansion rate): we prove that those systems have a spectral gap and we show a quantitative estimation for their statistical stability. Under deterministic perturbations of the system of size δ, the physical measure varies continuously, with a modulus of continuity $O(\delta \log \delta)$.

1. INTRODUCTION

The study of the behaviour of the transfer operator restricted to a suitable functional space has proven to be a powerful tool for the understanding of the statistical properties of a dynamical system. This approach gave first results in the study of the dynamics of piecewise expanding maps where the involved spaces are made of regular, absolutely continuous measures (see [5], [22], [28] for some introductory text). In recent years the approach was extended to piecewise hyperbolic systems by the use of suitable anisotropic norms (the expanding and contracting directions are managed differently), leading to suitable distribution spaces on which the transfer operator has good spectral properties (see e.g. [7], [6], [10], [17]). From these properties, several limit theorems or stability statements can be deduced. This approach has proven to be successful in non-trivial classes of systems like geodesic flows (see [22], [9]) or billiard maps (ess e.g. [12] [13] where a relatively simple and unified approach to many limit and perturbative results is given for the Lorentz gas). We remark that in these approaches, usually some condition of boundedness of the derivatives or transversality between the map's singular set and the contracting directions is supposed.

In this work, we consider skew product maps preserving an uniformly contracting foliation. We show how it is possible, in a simple way, to define

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suitable spaces of signed measures (with an anisotropic norm) such that, under small regularity assumptions, the transfer operator associated to the dynamics has a spectral gap (in the sense given in Theorem 6.1). This shows an exponential convergence to 0 in a certain norm for the iteration of a large class of zero average measures by the transfer operator. We remark that in this approach the speed of this convergence can be quantitatively estimated, and depends on the rate of contraction of the stable foliation, the coefficients of the Lasota-Yorke inequality and the rate of convergence to equilibrium of the induced quotient map (see Remark 6.3). We also remark that in our approach we can deal with maps having $C^{1+\alpha}$ regularity, having unbounded derivatives, and where the singular set is parallel to the contracting direction, as it happen in the Lorenz-like maps we consider in Section 7. These results allows in the second part of the paper to obtain a quantitative statistical stability estimate for deterministic perturbations of the system.

The function spaces we consider are defined by disintegrating signed measures on the phase space along the contracting foliation. The signed measure itself is then seen as a family of measures on the contracting leaves. We can then consider some notion of regularity for this family to define suitable spaces of more or less "regular"measures where to apply our transfer operator. To give an idea of these function spaces (see section 3), in the case of skew product maps of the unit square $I \times I$ to itself, the disintegration gives rise to a one dimensional family (a path) of measures defined on the contracting leaves, each leaf is isomorphic to the unit interval I, hence a measure on $I \times I$ is seen as a path of measures on I: a path in a metric space. The function spaces are defined by suitable notions of regularity for these paths. In the case $I \times I$ for example, the spaces which arise are included in $L^1(I, Lip(I)')$ (the space of L^1 functions from the interval to the dual of the space of Lipschitz functions on the interval), imposing some kind of further regularity. We remark that this is a space of distribution valued functions. For simplicity we will only use normed vector spaces of signed measures in this paper, we do not need to consider the completion of the space of signed measure, which would lead to distribution spaces.

The paper is structured as follows: in Section 3 we introduce the functional spaces we consider; in Section 4 we show the basic properties of the transfer operator when applied to these spaces. In particular we see that there is an useful "Perron-Frobenius"-like formula. In Section 5 we see the basic properties of the iteration of the transfer operator on the spaces we consider. In particular we see Lasota-Yorke inequalities and a convergence to equilibrium statement. In Section 6 we use the convergence to equilibrium and the Lasota-Yorke inequalities to prove the spectral gap. In Section 7 we present an application of our construction, showing a spectral gap for 2-dimensional Lorenz-like maps (piecewise $C^{1+\alpha}$ hyperbolic maps with unbounded expansion and contraction rates). In Section 8 we apply our construction to a class of piecewise C^2 Lorenz-like maps. We prove stronger (bounded variation) regularity results for the iteration of probability measures on that systems, and use this to prove a strong statistical stability statement with respect to deterministic perturbations: we establish a modulus of continuity $\delta \log \delta$ for the stability of the physical measure in weak space $(L^1(I, Lip(I)'))$ after a "size δ"perturbation. We remark that a qualitative statement, for a class of similar maps was given in [1].

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2. Contracting Fiber Maps

In this section we introduce the kind of systems we are considering in this paper and show some of its basic properties. Consider $\Sigma = N_1 \times N_2$, where N_1 and N_2 are compact and finite dimensional Riemannian manifolds such that $\mathrm{diam}(N_2) = 1$, where $\mathrm{diam}(N_2)$ denotes the diameter of N_2 with respect to its Riemannian metric, d_2. This is not restrictive but will avoid some multiplicative constants. Denote by m_1 and m_2 the Lebesgue measures on N_1 and N_2 respectively, generated by their corresponding Riemannian volumes, normalized so that $m_1(N_1) = m_2(N_2) = 1$ and $m = m_1 \times m_2$. Consider a map $F : (\Sigma, m) \longrightarrow (\Sigma, m)$,

$$F(x, y) = (T(x), G(x, y)), \tag{1}$$

where $T : N_1 \longrightarrow N_1$ and $G : \Sigma \longrightarrow N_2$ are measurable maps. Suppose that these maps satisfy the following conditions

2.0.1. *Properties of G.*

G1: Consider the F-invariant foliation

$$\mathcal{F}^s := \{\{x\} \times N_2\}_{x \in N_1}. \tag{2}$$

We suppose that $\mathcal{F}^s$ is contracted: there exists $0 < \alpha < 1$ such that for all $x \in N_1$ it holds

$$d_2(G(x, y_1), G(x, y_2)) \leq \alpha d_2(y_1, y_2), \quad for\ all \quad y_1, y_2 \in N_2. \tag{3}$$

2.0.2. *Properties of T and of its associated transfer operator.* Suppose that:

T1: T is non-singular with respect to m_1 ($m_1(A) = 0 \Rightarrow m_1(T^{-1}(A))) = 0$);

T2: There exists a disjoint collection of open sets $\mathcal{P} = \{P_1, \cdots, P_q\}$ of N_1, such that $m_1\left(\bigcup_{i=1}^q P_i\right) = 1$ and $T_i := T|_{P_i}$ is a diffeomorphism $T_i : P_i \to T_i(P_i) \subseteq N_1$, with $\det DT_i(x) \neq 0$ for all $x \in P_i$ and for all i, where DT_i is the Jacobian of T_i with respect to the Riemannian metric of N_1;

T3: Let us consider the Perron-Frobenius Operator associated to T, P_T [1]. We will now make some assumptions on the existence of a suitable functional analytic setting adapted to P_T. Let us hence denote the $L^1_{m_1}$ norm[2] by $|\cdot|_1$ and suppose that there exists a Banach space $(S_-, |\cdot|_s)$ such that

T3.1: $S_- \subset L^1_{m_1}$ is P_T-invariant, $|\cdot|_1 \leq |\cdot|_s$ and $\mathrm{P}_T : S_- \longrightarrow S_-$ is bounded;

T3.2: The unit ball of $(S_-, |\cdot|_s)$ is relatively compact in $(L^1_{m_1}, |\cdot|_1)$;

T3.3: (Lasota Yorke inequality) There exists $k \in \mathbb{N}$, $0 < \beta_0 < 1$ and $C > 0$ such that, for all $f \in S_-$, it holds

$$|\mathrm{P}_T^k f|_s \leq \beta_0 |f|_s + C|f|_1; \tag{4}$$

T3.4: Suppose there is an unique $\psi_x \in S_-$ with $\psi_x \geq 0$ and $|\psi_x|_1 = 1$ such that $\mathrm{P}_T(\psi_x) = \psi_x$, and if $\psi \in S_-$ is another density for a probability measure, then $P_T^n(\psi_x - \psi) \to 0$ as $n \to \infty$ in S_-.[3]

It is known that in this case ([20], see also [28], [22]) the following holds.

2.1. **Theorem.** *If T satisfy $T3.1, ..., T3.4$ then there exist $0 < r < 1$ and $D > 0$ such that for all $\phi \in S_-$ with $\int \phi \, dm_1 = 0$ and for all $n \geq 0$, it holds*

$$|\mathrm{P}_T^n(\phi)|_s \leq D r^n |\phi|_s. \tag{5}$$

In order to obtain spectral gap on L^∞ like spaces, the following additional property on $|\cdot|_s$ will be supposed at some point in the paper.

N1: There is $H_N \geq 0$ such that $|\cdot|_\infty \leq H_N |\cdot|_s$ (where $|\cdot|_\infty$ is the usual $L^\infty_{m_1}$ norm on N_1).

The following is a standard consequence of item T3.3, allowing to estimate the behaviour of any given power of the transfer operator.

2.2. **Corollary.** *There exist constants $B_3 > 0$, $C_2 > 0$ and $0 < \beta_2 < 1$, such that for all for all $f \in S_-$, and all $n \geq 1$, it holds*

$$|\mathrm{P}_T^n f|_s \leq B_3 \beta_2^n |f|_s + C_2 |f|_1. \tag{6}$$

Proof. The proof is a simple computation. Iterating the inequality (4) and since $|\mathrm{P}_T(h)|_1 \leq |h|_1$, for all $h \in L^1_{m_1}$, we have

[1] The unique operator $\mathrm{P}_T : L^1_{m_1} \longrightarrow L^1_{m_1}$ such that

$$\forall \phi \in L^1_{m_1} \ \ and \ \ \forall \psi \in L^\infty_{m_1} \quad \int \psi \cdot \mathrm{P}_T(\phi) \, dm_1 = \int (\psi \circ T) \cdot \phi \, dm_1.$$

[2] **Notation:** In the following we use $|\cdot|$ to indicate the usual absolute value or norms for signed measures on the basis space N_1. We will use $||\cdot||$ for norms defined for signed measures on Σ.

[3] This assumption ensures that from our point of view the system is indecomposable. For piecewise expanding maps e.g., the assumption follows from topological mixing.

$$|\,\mathrm{P}_T^{lk} f|_s \leq \beta_0^l |f|_s + \frac{C}{1-\beta_0}|f|_1, \tag{7}$$

for all $f \in S_-$ and for all $l \in \mathbb{N}$. For a given $n \in \mathbb{N}$, set $n = q_n k + r_n$, where $0 \leq r_n \leq k$. Since $\mathrm{P}_T : S_- \longrightarrow S_-$ is bounded, there exists $M_1 > 0$ such that $|\,\mathrm{P}_T^{r_n}|_s \leq M_1$ for all n, where $|\,\mathrm{P}_T^{r_n}|_s = \sup_{f \in S_-, f \neq 0} \dfrac{|\,\mathrm{P}_T^{r_n}(f)|_s}{|f|_s}$. Thus, we have

$$\begin{aligned}
|\,\mathrm{P}_T^n f|_s &= |\,\mathrm{P}_T^{q_n k + r_n} f|_s \\
&= |\,\mathrm{P}_T^{q_n k}(\mathrm{P}_T^{r_n} f)|_s \\
&\leq \beta_0^{q_n} |\,\mathrm{P}_T^{r_n} f|_s + \frac{C}{1-\beta_0}|f|_1 \\
&\leq \beta_0^{q_n} M_1 |f|_s + \frac{C}{1-\beta_0}|f|_1 \\
&\leq \beta_0^{\frac{n-r_n}{k}} M_1 |f|_s + \frac{C}{1-\beta_0}|f|_1 \\
&\leq \left(\beta_0^{\frac{1}{k}}\right)^n \frac{M_1}{\beta_0}|f|_s + \frac{C}{1-\beta_0}|f|_1,
\end{aligned}$$

and the proof is done by setting

$$B_3 = \frac{M_1}{\beta_0},\ \beta_2 = \beta_0^{\frac{1}{k}} \text{and } C_2 = \frac{C}{1-\beta_0}. \tag{8}$$

□

3. Weak and strong spaces

3.1. L^1-like spaces. Through this section we construct some function spaces which are suitable for the systems defined in section 2. The idea is to define spaces of signed measures, where the norms are provided by disintegrating measures along the stable foliation. Thus, a signed measure will be seen as a family of measures on each leaf. For instance, a measure on the square will be seen as a one parameter family (a path) of measures on the interval (a stable leaf), where this identification will be done by means of the Rokhlin's Disintegration Theorem. Finally, in the vertical direction (on the leaves), we will consider a norm which is the dual of the Lipschitz norm and in the "horizontal"direction we will consider essentially the $L^1_{m_1}$ norm.

Rokhlin's Disintegration Theorem. Now we give a brief introduction about disintegration of measures.

Consider a probability space $(\Sigma, \mathcal{B}, \mu)$ and a partition Γ of Σ by measurable sets $\gamma \in \mathcal{B}$. Denote by $\pi : \Sigma \longrightarrow \Gamma$ the projection that associates to each point $x \in M$ the element γ_x of Γ which contains x, i.e. $\pi(x) = \gamma_x$. Let $\widehat{\mathcal{B}}$ be the σ-algebra of Γ provided by π. Precisely, a subset $\mathcal{Q} \subset \Gamma$ is measurable if, and only if, $\pi^{-1}(\mathcal{Q}) \in \mathcal{B}$. We define the *quotient* measure μ_x on Γ by $\mu_x(\mathcal{Q}) = \mu(\pi^{-1}(\mathcal{Q}))$.

The proof of the following theorem can be found in [24], Theorem 5.1.11.

3.1. **Theorem.** *(Rokhlin's Disintegration Theorem) Suppose that Σ is a complete and separable metric space, Γ is a measurable partition*[4] *of Σ and μ is a probability on Σ. Then, μ admits a disintegration relatively to Γ, i.e. a family $\{\mu_\gamma\}_{\gamma\in\Gamma}$ of probabilities on Σ and a quotient measure $\mu_x = \pi^*\mu$ such that:*

(a) $\mu_\gamma(\gamma) = 1$ *for μ_x-a.e. $\gamma \in \Gamma$;*
(b) *the function $\Gamma \longrightarrow \mathbb{R}$, defined by $\gamma \longmapsto \mu_\gamma(E)$ is measurable;*
(c) *for all measurable set $E \subset \Sigma$, it holds $\mu(E) = \int \mu_\gamma(E)d\mu_x(\gamma)$.*

The proof of the following lemma can be found in [24], proposition 5.1.7.

3.2. **Lemma.** *Suppose the σ-algebra $\mathcal{B}$, of Σ, has a countable generator. If $(\{\mu_\gamma\}_{\gamma\in\Gamma}, \mu_x)$ and $(\{\mu'_\gamma\}_{\gamma\in\Gamma}, \mu_x)$ are disintegrations of the measure μ relatively to Γ, then $\mu_\gamma = \mu'_\gamma$, μ_x-almost every $\gamma \in \Gamma$.*

3.1.1. *The $\mathcal{L}^1$ and S^1 spaces.* Let $\mathcal{SB}(\Sigma)$ be the space of Borel signed measures on Σ. Given $\mu \in \mathcal{SB}(\Sigma)$ denote by μ^+ and μ^- the positive and the negative parts of its Jordan decomposition, $\mu = \mu^+ - \mu^-$ (see remark 3.1.1). Denote by $\mathcal{AB}$ the set of signed measures $\mu \in \mathcal{SB}(\Sigma)$ such that its associated positive and negative marginal measures, $\pi_x^*\mu^+$ and $\pi_x^*\mu^-$ are absolutely continuous with respect to the volume measure m_1, i.e.

$$\mathcal{AB} = \{\mu \in \mathcal{SB}(\Sigma) : \pi_x^*\mu^+ << m_1 \ \ and \ \ \pi_x^*\mu^- << m_1\}, \tag{9}$$

where $\pi_x : \Sigma \longrightarrow N_1$ is the projection defined by $\pi(x, y) = x$.

Given a *probability measure* $\mu \in \mathcal{AB}$ on Σ, theorem 3.1 describes a disintegration $\left(\{\mu_\gamma\}_\gamma, \mu_x\right)$ along $\mathcal{F}^s$ (see equation (2)) [5] by a family $\{\mu_\gamma\}_\gamma$ of probability measures on the stable leaves[6] and, since $\mu \in \mathcal{AB}$, μ_x can be identified with a non negative marginal density $\phi_x : N_1 \longrightarrow \mathbb{R}$, defined almost everywhere, with $|\phi_x|_1 = 1$. For a positive measure $\mu \in \mathcal{AB}$ we define its disintegration by disintegrating the normalization of μ. In this case, it holds $|\phi_x|_1 = \mu(\Sigma)$.

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[4]We say that a partition Γ is measurable if there exists a full measure set $M_0 \subset \Sigma$ s.t. restricted to M_0, $\Gamma = \bigvee_{n=1}^{\infty} \Gamma_n$, for some increasing sequence $\Gamma_1 \prec \Gamma_2 \prec \cdots \prec \Gamma_n \prec \cdots$ of countable partitions of Σ. Furthermore, $\Gamma_i \prec \Gamma_{i+1}$ means that each element of $\mathcal{P}_{i+1}$ is a subset of some element of Γ_i.

[5]By lemma 3.2, the disintegration of a measure μ is the μ_x-unique ($\mu_x = \phi_x m_1$) measurable family $(\{\mu_\gamma\}_\gamma, \phi_x m_1)$ such that, for every measurable set $E \subset \Sigma$ it holds

$$\mu(E) = \int_{N_1} \mu_\gamma(E \cap \gamma)d(\phi_x m_1)(\gamma). \tag{10}$$

We also remark that, in our context, Γ and π of theorem 3.1 are respectively equal to $\mathcal{F}^s$ and π_x, defined by $\pi(x, y) = x$, where $x \in N_1$ and $y \in N_2$.

[6]In the following to simplify notations, when no confusion is possible we will indicate the generic leaf or its coordinate with γ.

3.3. **Definition.** Let $\pi_{\gamma,y} : \gamma \longrightarrow N_2$ be the restriction $\pi_y|_\gamma$, where $\pi_y : \Sigma \longrightarrow N_2$ is the projection defined by $\pi_y(x,y) = y$ and $\gamma \in \mathcal{F}^s$. Given a positive measure $\mu \in \mathcal{AB}$ and its disintegration along the stable leaves $\mathcal{F}^s$, $\big(\{\mu_\gamma\}_\gamma, \mu_x = \phi_x m_1\big)$, we define the **restriction of** μ **on** γ as the positive measure $\mu|_\gamma$ on N_2 (not on the leaf γ) defined, for all mensurable set $A \subset N_2$, as

$$\mu|_\gamma(A) = \pi^*_{\gamma,y}(\phi_x(\gamma)\mu_\gamma)(A).$$

For a given signed measure $\mu \in \mathcal{AB}$ and its Jordan decomposition $\mu = \mu^+ - \mu^-$, define the **restriction of** μ **on** γ by

$$\mu|_\gamma = \mu^+|_\gamma - \mu^-|_\gamma. \tag{11}$$

3.4. **Remark.** As we prove in Corollary 10.7, the restriction $\mu|_\gamma$ does not depend on the decomposition. Precisely, if $\mu = \mu_1 - \mu_2$, where μ_1 and μ_2 are any positive measures, then $\mu|_\gamma = \mu_1|_\gamma - \mu_2|_\gamma$ m_1-a.e. $\gamma \in N_1$.

Let (X, d) be a compact metric space, $g : X \longrightarrow \mathbb{R}$ be a Lipschitz function and let $L(g)$ be its best Lipschitz constant, i.e.

$$L(g) = \sup_{x,y \in X} \left\{ \frac{|g(x) - g(y)|}{d(x,y)} \right\}.$$

3.5. **Definition.** Given two signed measures μ and ν on X, we define a **Wasserstein-Kantorovich Like** distance between μ and ν by

$$W_1^0(\mu,\nu) = \sup_{L(g) \leq 1, ||g||_\infty \leq 1} \left| \int g d\mu - \int g d\nu \right|. \tag{12}$$

From now, we denote

$$||\mu||_W := W_1^0(0,\mu). \tag{13}$$

As a matter of fact, $|| \cdot ||_W$ defines a norm on the vector space of signed measures defined on a compact metric space. We remark that this norm is equivalent to the dual of the Lipschitz norm.

3.6. **Definition.** Let $\mathcal{L}^1 \subseteq \mathcal{AB}$ be defined as

$$\mathcal{L}^1 = \left\{ \mu \in \mathcal{AB} : \int_{N_1} W_1^0(\mu^+|_\gamma, \mu^-|_\gamma) dm_1(\gamma) < \infty \right\} \tag{14}$$

and define a norm on it, $|| \cdot ||_1 : \mathcal{L}^1 \longrightarrow \mathbb{R}$, by

$$||\mu||_1 = \int_{N_1} W_1^0(\mu^+|_\gamma, \mu^-|_\gamma) dm_1(\gamma). \tag{15}$$

Now, we define the following set of signed measures on Σ,

$$S^1 = \left\{ \mu \in \mathcal{L}^1 ; \phi_x \in S_- \right\}. \tag{16}$$

Consider the function $|| \cdot ||_{S^1} : S^1 \longrightarrow \mathbb{R}$, defined by

$$||\mu||_{S^1} = |\phi_x|_s + ||\mu||_1, \tag{17}$$

where we denote $\phi_x = \phi_x^+ - \phi_x^-$ with $\phi_x^{\pm}$ being the marginals of $\mu^{\pm}$ as explained before. Moreover, ϕ_x is the marginal density of the disintegration of μ and we remark that ϕ_x^+ is not necessarily equal to the positive part of ϕ_x.

The proof of the next proposition is straightforward. Details can be found in [23].

3.7. **Proposition.** *$\left(\mathcal{L}^1, || \cdot ||_1\right)$ and $\left(S^1, || \cdot ||_{S^1}\right)$ are normed vector spaces.*

3.2. L^∞ like spaces.

3.8. **Definition.** Let $\mathcal{L}^\infty \subseteq \mathcal{AB}(\Sigma)$ be defined as

$$\mathcal{L}^\infty = \left\{ \mu \in \mathcal{AB} : \text{ess sup}(W_1^0(\mu^+|_\gamma, \mu^-|_\gamma)) < \infty \right\}, \tag{18}$$

where the essential supremum is taken over N_1 with respect to m_1. Define the function $|| \cdot ||_\infty : \mathcal{L}^\infty \longrightarrow \mathbb{R}$ by

$$||\mu||_\infty = \text{ess sup}(W_1^0(\mu^+|_\gamma, \mu^-|_\gamma)). \tag{19}$$

Finally, consider the following set of signed measures on Σ

$$S^\infty = \left\{ \mu \in \mathcal{L}^\infty ; \phi_x \in S_- \right\}, \tag{20}$$

and the function, $|| \cdot ||_{S^\infty} : S^\infty \longrightarrow \mathbb{R}$, defined by

$$||\mu||_{S^\infty} = |\phi_x|_s + ||\mu||_\infty. \tag{21}$$

The proof of the next proposition is straightforward and can be found in [23].

3.9. **Proposition.** *$(\mathcal{L}^\infty, || \cdot ||_\infty)$ and $(S^\infty, || \cdot ||_{S^\infty})$ are normed vector spaces.*

4. Transfer operator associated to F

Consider the transfer operator F^* associated with F, i.e. such that

$$[\mathrm{F}^* \mu](E) = \mu(F^{-1}(E)),$$

for each signed measure $\mu \in \mathcal{SB}(\Sigma)$ and for each measurable set $E \subset \Sigma$.

4.1. **Lemma.** *For all probability $\mu \in \mathcal{AB}$ disintegrated by $(\{\mu_\gamma\}_\gamma, \phi_x)$, the disintegration $((\mathrm{F}^* \mu)_\gamma, (\mathrm{F}^* \mu)_x)$ of $\mathrm{F}^* \mu$ is given by*

$$(\mathrm{F}^* \mu)_x = \mathrm{P}_T(\phi_x) m_1 \tag{22}$$

and

$$(\mathrm{F}^* \mu)_\gamma = \nu_\gamma := \frac{1}{\mathrm{P}_T(\phi_x)(\gamma)} \sum_{i=1}^{q} \frac{\phi_x}{|\det DT_i|} \circ T_i^{-1}(\gamma) \cdot \chi_{T_i(P_i)}(\gamma) \cdot \mathrm{F}^* \mu_{T_i^{-1}(\gamma)} \tag{23}$$

when $\mathrm{P}_T(\phi_x)(\gamma) \neq 0$*. Otherwise, if* $\mathrm{P}_T(\phi_x)(\gamma) = 0$*, then* ν_γ *is the Lebesgue measure on* γ *(the expression* $\dfrac{\phi_x}{|\det DT_i|} \circ T_i^{-1}(\gamma) \cdot \dfrac{\chi_{T_i(P_i)}(\gamma)}{\mathrm{P}_T(\phi_x)(\gamma)} \cdot \mathrm{F}^* \mu_{T_i^{-1}(\gamma)}$ *is understood to be zero outside* $T_i(P_i)$ *for all* $i = 1, \cdots, q$*). Here and above,* χ_A *is the characteristic function of the set* A*.*

Proof. By the uniqueness of the disintegration (see Lemma 3.2) to prove Lemma 4.1, is enough to prove the following equation

$$\mathrm{F}^* \mu(E) = \int_{N_1} \nu_\gamma(E \cap \gamma)\, \mathrm{P}_T(\phi_x)(\gamma) d\gamma, \tag{24}$$

for a measurable set $E \subset \Sigma$. To do it, let us define the sets $B_1 = \left\{\gamma \in N_1; T^{-1}(\gamma) = \emptyset\right\}$, $B_2 = \{\gamma \in B_1^c; \mathrm{P}_T(\phi_x)(\gamma) = 0\}$ and $B_3 = (B_1 \cup B_2)^c$. The following properties can be easily proven:

1. $B_i \cap B_j = \emptyset$, $T^{-1}(B_i) \cap T^{-1}(B_j) = \emptyset$, for all $1 \leq i, j \leq 3$ such that $i \neq j$ and $\bigcup_{i=1}^3 B_i = \bigcup_{i=1}^3 T^{-1}(B_i) = N_1$;
2. $m_1(T^{-1}(B_1)) = m_1(T^{-1}(B_2)) = 0$;

Using the change of variables $\gamma = T_i(\beta)$ and the definition of ν_γ (see (23)), we have

$$\begin{aligned}
\int_{N_1} \nu_\gamma(E \cap \gamma)\, \mathrm{P}_T(\phi_x)(\gamma) d\gamma &= \int_{B_3} \sum_{i=1}^{q} \frac{\phi_x}{|\det DT_i|} \circ T_i^{-1}(\gamma)\, \mathrm{F}^* \mu_{T_i^{-1}(\gamma)}(E) \chi_{T_i(P_i)(\gamma)} dm_1(\gamma) \\
&= \sum_{i=1}^{q} \int_{T_i(P_i) \cap B_3} \frac{\phi_x}{|\det DT_i|} \circ T_i^{-1}(\gamma)\, \mathrm{F}^* \mu_{T_i^{-1}(\gamma)}(E) dm_1(\gamma) \\
&= \sum_{i=1}^{q} \int_{P_i \cap T_i^{-1}(B_3)} \phi_x(\beta) \mu_\beta(F^{-1}(E)) dm_1(\beta) \\
&= \int_{T^{-1}(B_3)} \phi_x(\beta) \mu_\beta(F^{-1}(E)) dm_1(\beta) \\
&= \int_{\bigcup_{i=1}^3 T^{-1}(B_i)} \mu_\beta(F^{-1}(E)) d\phi_x m_1(\beta) \\
&= \int_{N_1} \mu_\beta(F^{-1}(E)) d\phi_x m_1(\beta) \\
&= \mu(F^{-1}(E)) \\
&= \mathrm{F}^* \mu(E).
\end{aligned}$$

And the proof is done. □

As said in Remark 3.1.1, Corollary 10.7 yields that the restriction $\mu|_\gamma$ does not depend on the decomposition. Thus, for each $\mu \in \mathcal{L}^1$, since $\mathrm{F}^* \mu$ can be decomposed as $\mathrm{F}^* \mu = \mathrm{F}^*(\mu^+) - \mathrm{F}^*(\mu^-)$, we can apply the above Lemma to $\mathrm{F}^*(\mu^+)$ and $\mathrm{F}^*(\mu^-)$ to get the following

4.2. **Proposition.** *Let $\gamma \in \mathcal{F}^s$ be a stable leaf. Let us define the map $F_\gamma : N_2 \longrightarrow N_2$ by*

$$F_\gamma = \pi_y \circ F|_\gamma \circ \pi_{\gamma,y}^{-1}.$$

Then, for each $\mu \in \mathcal{L}^1$ and for almost all $\gamma \in N_1$ (interpreted as the quotient space of leaves) it holds

$$(\mathrm{F}^* \mu)|_\gamma = \sum_{i=1}^{q} \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \mu|_{T_i^{-1}(\gamma)}}{|\det DT_i \circ T_i^{-1}(\gamma))|} \chi_{T_i(P_i)}(\gamma) \quad m_1\text{-a.e.} \quad \gamma \in N_1. \tag{25}$$

5. Basic properties of the norms and convergence to equilibrium

In this section, we show important properties of the norms and their behaviour with respect to the transfer operator. In particular, we prove that the $\mathcal{L}^1$ norm is weakly contracted. We prove Lasota-Yorke like inequalities for the strong norms and exponential convergence to equilibrium statements. All these properties will be used in next section to prove spectral gap for the transfer operator associated to the system $F : \Sigma \to \Sigma$.

5.1. **Proposition** (The weak norm is weakly contracted by F^*)**.** *If $\mu \in \mathcal{L}^1$ then*

$$|| \mathrm{F}^* \mu||_1 \leq ||\mu||_1. \tag{26}$$

In the proof of the proposition we will use the following lemma about the behaviour of the $|| \cdot ||_W$ norm (see equation (13)) which says that a contraction cannot increase the $|| \cdot ||_W$ norm.

5.2. **Lemma.** *For every $\mu \in \mathcal{AB}$ and a stable leaf $\gamma \in \mathcal{F}^s$, it holds*

$$|| \mathrm{F}^*_\gamma \mu|_\gamma||_W \leq ||\mu|_\gamma||_W, \tag{27}$$

where $F_\gamma : N_2 \longrightarrow N_2$ is defined in Proposition 4.2. Moreover, if μ is a probability measure on N_2, it holds

$$|| \mathrm{F}^{*\,n} \mu||_W = ||\mu||_W = 1, \quad \forall \quad n \geq 1. \tag{28}$$

Proof. (of Lemma 5.2) Indeed, since F_γ is an α-contraction, if $|g|_\infty \leq 1$ and $Lip(g) \leq 1$ the same holds for $g \circ F_\gamma$. Since

$$\left| \int g \; d\mathrm{F}^*_\gamma \mu|_\gamma \right| \quad = \quad \left| \int g(F_\gamma) \; d\mu|_\gamma \right|,$$

taking the supremum over $|g|_\infty \leq 1$ and $Lip(g) \leq 1$ we finish the proof of the inequality.

In order to prove equation (28), consider a probability measure μ on N_2 and a Lipschitz function $g : N_2 \longrightarrow \mathbb{R}$, such that $||g||_\infty \leq 1$ and $L(g) \leq 1$. Therefore, $|\int g d\mu| \leq ||g||_\infty \leq 1$, which yields $||\mu||_W \leq 1$. Reciprocally, consider the constant function $g \equiv 1$. Then $1 = |\int g d\mu| \leq ||\mu||_W$. These two facts proves equation (28). $\square$

Now we are ready to prove Proposition 5.1.

Proof. (of Proposition 5.1)
In the following, we consider for all i, the change of variable $\gamma = T_i(\alpha)$. Thus, Lemma 5.2 and equation (25) yield

$$
\begin{aligned}
|| \mathrm{F}^* \mu||_1 &= \int_{N_1} ||(\mathrm{F}^* \mu)|_\gamma||_W dm_1(\gamma) \\
&\leq \sum_{i=1}^{q} \int_{T(P_i)} \left|\left| \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \mu|_{T_i^{-1}(\gamma)}}{|\det DT_i(T_i^{-1}(\gamma))|} \right|\right|_W dm_1(\gamma) \\
&= \sum_{i=1}^{q} \int_{P_i} ||\mathrm{F}^*_\alpha \mu|_\alpha||_W dm_1(\alpha) \\
&= \sum_{i=1}^{q} \int_{P_i} ||\mu|_\alpha||_W dm_1(\alpha) \\
&= ||\mu||_1.
\end{aligned}
$$

$\square$

The following proposition shows a regularizing action of the transfer operator with respect to the strong norm. Such inequalities are usually called Lasota-Yorke or Doeblin-Fortet inequalities.

5.3. **Proposition** (Lasota-Yorke inequality for S^1). *Let $F : \Sigma \longrightarrow \Sigma$ be a map satisfying T1, T2 and T3. Then, there exist A, $B_2 \in \mathbb{R}, \lambda < 1$ such that, for all $\mu \in S^1$, it holds*

$$|| \mathrm{F}^{*n} \mu||_{S^1} \leq A\lambda^n ||\mu||_{S^1} + B_2||\mu||_1, \quad \forall n \geq 1. \tag{29}$$

Proof. Firstly, we recall that ϕ_x is the marginal density of the disintegration of μ. Precisely, $\phi_x = \phi_x^+ - \phi_x^-$, where $\phi_x^+ = \dfrac{d\pi_x^* \mu^+}{dm_1}$ and $\phi_x^- = \dfrac{d\pi_x^* \mu^-}{dm_1}$.
By equation (6), Proposition 5.1 and since $|\phi_x|_1 \leq ||\mu||_1$, we have

$$
\begin{aligned}
|| \mathrm{F}^{*n} \mu||_{S^1} &= |\, \mathrm{P}_T^n \, \phi_x|_s + || \mathrm{F}^{*n} \mu||_1 \\
&\leq B_3\beta_2^n|\phi_x|_s + C_2|\phi_x|_1 + ||\mu||_1 \\
&\leq B_3\beta_2^n||\mu||_{S^1} + (C_2+1)||\mu||_1.
\end{aligned}
$$

We finish the proof by setting $\lambda = \beta_2$, $A = B_3$ and $B_2 = C_2 + 1$. $\square$

5.1. **Convergence to equilibrium.** In general, we say that the a transfer operator L has convergence to equilibrium with at least speed Φ and with respect to the norms $||\cdot||_s$ and $||\cdot||_w$, if for each $f \in \mathcal{V}_s = \{f \in B_s, f(X) = 0\}$, it holds

$$|| \mathrm{L}^{\mathrm{n}} f||_w \leq \Phi(n)||f||_s, \tag{30}$$

where $\Phi(n) \longrightarrow 0$ as $n \longrightarrow \infty$.

In this chapter, we prove that F has exponential convergence to equilibrium. This is weaker with respect to spectral gap. However, the spectral gap follows from the above Lasota-Yorke inequality and the convergence to equilibrium. To do it, we need some preliminary lemma and the following is somewhat similar to Lemma 5.2 considering the behaviour of the $|| \cdot ||_W$ norm after a contraction. It gives a finer estimate for zero average measures. The following Lemma is useful to estimate the behaviour of our W norms under contractions.

5.4. **Lemma.** *For all signed measures μ on N_2 and for all $\gamma \in \mathcal{F}^s$, it holds*

$$|| \mathrm{F}_\gamma^* \mu||_W \leq \alpha||\mu||_W + \mu(N_2)$$

(α is the rate of contraction of G). In particular, if $\mu(N_2) = 0$ then

$$|| \mathrm{F}_\gamma^* \mu||_W \leq \alpha||\mu||_W.$$

Proof. If $Lip(g) \leq 1$ and $||g||_\infty \leq 1$, then $g \circ F_\gamma$ is α-Lipschitz. Moreover, since $||g||_\infty \leq 1$, then $||g \circ F_\gamma - \theta||_\infty \leq \alpha$, for some $\theta \leq 1$. Indeed, let $z \in N_2$ be such that $|g \circ F_\gamma(z)| \leq 1$, set $\theta = g \circ F_\gamma(z)$ and let d_2 be the Riemannian metric of N_2. Since $\mathrm{diam}(N_2) = 1$, we have

$$|g \circ F_\gamma(y) - \theta| \leq \alpha d_2(y, z) \leq \alpha$$

and consequently $||g \circ F_\gamma - \theta||_\infty \leq \alpha$.

This implies,

$$\begin{aligned}
\left| \int_{N_2} g d \mathrm{F}_\gamma^* \mu \right| &= \left| \int_{N_2} g \circ F_\gamma d\mu \right| \\
&\leq \left| \int_{N_2} g \circ F_\gamma - \theta d\mu \right| + \left| \int_{N_2} \theta d\mu \right| \\
&= \alpha \left| \int_{N_2} \frac{g \circ F_\gamma - \theta}{\alpha} d\mu \right| + \theta |\mu(N_2)|.
\end{aligned}$$

And taking the supremum over $|g|_\infty \leq 1$ and $Lip(g) \leq 1$, we have $|| \mathrm{F}_\gamma^* \mu||_W \leq \alpha||\mu||_W + \mu(N_2)$. In particular, if $\mu(N_2) = 0$, we get the second part. $\square$

Now we are ready to show a key estimate regarding the behaviour of our weak $|| \, ||_1$ norm in Lorenz-like systems, as defined at beginning of Section 2.

5.5. **Proposition.** *For all signed measure $\mu \in \mathcal{L}^1$, it holds*

$$||\,\mathrm{F}^*\,\mu||_1 \leq \alpha||\mu||_1 + (\alpha+1)|\phi_x|_1. \tag{31}$$

Proof. Consider a signed measure $\mu \in \mathcal{L}^1$ and its restriction on the leaf γ, $\mu|_\gamma = \pi^*_{\gamma,y}(\phi_x(\gamma)\mu_\gamma)$. Set

$$\overline{\mu}|_\gamma = \pi^*_{\gamma,y}\mu_\gamma.$$

If μ is a positive measure then $\overline{\mu}|_\gamma$ is a probability on N_2 and $\mu|_\gamma = \phi_x(\gamma)\overline{\mu}|_\gamma$. Then, the expression given by Proposition 4.2 yields

$$\begin{aligned}
||\,\mathrm{F}^*\,\mu||_1 &\leq \sum_{i=1}^{q}\int_{T(P_i)}\left|\left|\frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\overline{\mu^+}|_{T_i^{-1}(\gamma)}\phi_x^+(T_i^{-1}(\gamma))}{|\det DT_i|\circ T_i^{-1}(\gamma)} - \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\overline{\mu^-}|_{T_i^{-1}(\gamma)}\phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i|\circ T_i^{-1}(\gamma)}\right|\right|_W dm_1(\gamma)\\
&\leq \sum_{i=1}^{q}\int_{T(P_i)}\left|\left|\frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\overline{\mu^+}|_{T_i^{-1}(\gamma)}\phi_x^+(T_i^{-1}(\gamma))}{|\det DT_i|\circ T_i^{-1}(\gamma)} - \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\overline{\mu^+}|_{T_i^{-1}(\gamma)}\phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i|\circ T_i^{-1}(\gamma)}\right|\right|_W dm_1(\gamma)\\
&+ \sum_{i=1}^{q}\int_{T(P_i)}\left|\left|\frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\overline{\mu^+}|_{T_i^{-1}(\gamma)}\phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i|\circ T_i^{-1}(\gamma)} - \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\overline{\mu^-}|_{T_i^{-1}(\gamma)}\phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i|\circ T_i^{-1}(\gamma)}\right|\right|_W dm_1(\gamma)\\
&= \mathrm{I}_1 + \mathrm{I}_2,
\end{aligned}$$

where

$$\mathrm{I}_1 = \sum_{i=1}^{q}\int_{T(P_i)}\left|\left|\frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\overline{\mu^+}|_{T_i^{-1}(\gamma)}\phi_x^+(T_i^{-1}(\gamma))}{|\det DT_i|\circ T_i^{-1}(\gamma)} - \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\overline{\mu^+}|_{T_i^{-1}(\gamma)}\phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i|\circ T_i^{-1}(\gamma)}\right|\right|_W dm_1(\gamma)$$

and

$$\mathrm{I}_2 = \sum_{i=1}^{q}\int_{T(P_i)}\left|\left|\frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\overline{\mu^+}|_{T_i^{-1}(\gamma)}\phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i|\circ T_i^{-1}(\gamma)} - \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\overline{\mu^-}|_{T_i^{-1}(\gamma)}\phi_x^-(T_i^{-1}(\gamma))}{|\det DT_i|\circ T_i^{-1}(\gamma)}\right|\right|_W dm_1(\gamma).$$

Let us estimate I_1 and I_2.

By Lemma 5.2 and a change of variable we have

$$\begin{aligned}
\mathrm{I}_1 &= \sum_{i=1}^{q}\int_{T(P_i)}\left|\left|\mathrm{F}^*_{T_i^{-1}(\gamma)}\,\overline{\mu^+}|_{T_i^{-1}(\gamma)}\right|\right|_W \frac{|\phi_x^+ - \phi_x^-|}{|\det DT_i|}\circ T_i^{-1}(\gamma)dm_1(\gamma)\\
&\leq \int_{N_1}\left|\left|\mathrm{F}^*_\beta\,\overline{\mu^+}|_\beta\right|\right|_W |\phi_x^+ - \phi_x^-|(\beta)dm_1(\beta)\\
&= \int_{N_1}|\phi_x^+ - \phi_x^-|(\beta)dm_1(\beta)\\
&= |\phi_x|_1,
\end{aligned}$$

and by Lemma 5.4 we have

$$\begin{aligned}
\mathrm{I}_2 &= \sum_{i=1}^{q} \int_{T(P_i)} \left|\left|\mathrm{F}^*_{T_i^{-1}(\gamma)}\left(\overline{\mu^+}|_{T_i^{-1}(\gamma)} - \overline{\mu^-}|_{T_i^{-1}(\gamma)}\right)\right|\right|_W \frac{\phi_x^-}{|\det DT_i|} \circ T_i^{-1}(\gamma) dm_1(\gamma) \\
&\leq \sum_{i=1}^{q} \int_{P_i} \left|\left|\mathrm{F}^*_{\beta}\left(\overline{\mu^+}|_{\beta} - \overline{\mu^-}|_{\beta}\right)\right|\right|_W \phi_x^-(\beta) dm_1(\beta) \\
&\leq \alpha \int_{N_1} \left|\left|\overline{\mu^+}|_{\beta} - \overline{\mu^-}|_{\beta}\right|\right|_W \phi_x^-(\beta) dm_1(\beta) \\
&\leq \alpha \int_{N_1} \left|\left|\overline{\mu^+}|_{\beta}\phi_x^-(\beta) - \overline{\mu^+}|_{\beta}\phi_x^+(\beta)\right|\right|_W dm_1(\beta) \\
&\leq \alpha \int_{N_1} \left|\left|\overline{\mu^+}|_{\beta}\phi_x^-(\beta) - \overline{\mu^+}|_{\beta}\phi_x^+(\beta)\right|\right|_W dm_1(\beta) + \alpha \int_{N_1} \left|\left|\overline{\mu^+}|_{\beta}\phi_x^+(\beta) - \overline{\mu^-}|_{\beta}\phi_x^-(\beta)\right|\right|_W dm_1(\beta) \\
&= \alpha|\phi_x|_1 + \alpha||\mu||_1.
\end{aligned}$$

Summing the above estimates we finish the proof. □

Iterating (31) we get the following corollary.

5.6. **Corollary.** *For all signed measure $\mu \in \mathcal{L}^1$ it holds*

$$||\,\mathrm{F}^{*n}\,\mu||_1 \leq \alpha^n||\mu||_1 + \overline{\alpha}|\phi_x|_1,$$

where $\overline{\alpha} = \frac{1+\alpha}{1-\alpha}$.

Let us consider the set of zero average measures in S^1 defined by

$$\mathcal{V}_s = \{\mu \in S^1 : \mu(\Sigma) = 0\}. \tag{32}$$

Note that, for all $\mu \in \mathcal{V}_s$ we have $\pi_x^*\mu(N_1) = 0$. Moreover, since $\pi_x^*\mu = \phi_x m_1$ ($\phi_x = \phi_x^+ - \phi_x^-$), we have $\displaystyle\int_{N_1} \phi_x dm_1 = 0$. This allows us to apply Theorem 2.1 in the proof of the next proposition.

5.7. **Proposition** (Exponential convergence to equilibrium)**.** *There exist $D_2 \in \mathbb{R}$ and $0 < \beta_1 < 1$ such that for every signed measure $\mu \in \mathcal{V}_s$, it holds*

$$||\,\mathrm{F}^{*n}\,\mu||_1 \leq D_2\beta_1^n||\mu||_{S^1},$$

for all $n \geq 1$.

Proof. Given $\mu \in \mathcal{V}_s$ and denoting $\phi_x = \phi_x^+ - \phi_x^-$, it holds that $\int \phi_x dm_1 = 0$. Moreover, Theorem 2.1 yields $|\,\mathrm{P}_T^n(\phi_x)|_s \leq Dr^n|\phi_x|_s$ for all $n \geq 1$, then $|\,\mathrm{P}_T^n(\phi_x)|_s \leq Dr^n||\mu||_{S^1}$ for all $n \geq 1$.

Let l and $0 \leq d \leq 1$ be the coefficients of the division of n by 2, i.e. $n = 2l + d$. Thus, $l = \frac{n-d}{2}$ (by Proposition 5.1, we have $||\,\mathrm{F}^{*s}\,\mu||_1 \leq ||\mu||_1$, for all s, and $||\mu||_1 \leq ||\mu||_{S^1}$) and by Corollary 5.6, it holds (below, set $\beta_1 = \max\{\sqrt{r}, \sqrt{\alpha}\}$)

$$
\begin{aligned}
||\,\mathrm{F}^{*n}\,\mu||_1 &\le ||\,\mathrm{F}^{*2l+d}\,\mu||_1 \\
&\le \alpha^l||\,\mathrm{F}^{*l+d}\,\mu||_1 + \overline{\alpha}\left|\frac{d(\pi_x^*(\mathrm{F}^{*l+d}\,\mu))}{dm_1}\right|_1 \\
&\le \alpha^l||\mu||_1 + \overline{\alpha}|\,\mathrm{P}_T^l(\phi_x)|_1 \\
&\le (1+\overline{\alpha}D)\beta_1^{-d}\beta_1^n||\mu||_{S^1} \\
&\le D_2\beta_1^n||\mu||_{S^1},
\end{aligned}
$$

where $D_2 = \dfrac{1+\overline{\alpha}D}{\beta_1}$. □

5.8. **Remark.** We remark that the rate of convergence to equilibrium, β_1, for the map F found above, is directly related to the rate of contraction, α, of the stable foliation, and to the rate of convergence to equilibrium, r, of the induced basis map T (see equation 5). More precisely, $\beta_1 = \max\{\sqrt{\alpha}, \sqrt{r}\}$. Similarly, we have an explicit estimate for the constant D_2, provided we have an estimate for D in the basis map[7].

Now recall we denoted by ψ_x the unique T-invariant density in S_- (see T3.4). Let μ_0 be the F-invariant probability measure constructed from ψ_x according to the construction in [29] (subsection 7.3.4.1). By construction, $d(\pi_x^*\mu_0)/dm_1 = \psi_x \in S^-$. This motivates the following proposition.

5.9. **Proposition.** *The unique invariant probability for the system $F : N_1 \times N_2 \longrightarrow N_1 \times N_2$ in S^1 is μ_0. Moreover, if **N1** is satisfied, μ_0 is the unique F-invariant probability in S^∞.*

Proof. Let μ_0 be the F-invariant measure such that $\dfrac{d(\pi_x^*\mu_0)}{dm_1} = \psi_x \in S_-$, where ψ_x is the unique T-invariant density (see T3.4) in S_-. Define the probability $\overline{\mu_0}|_\gamma = \pi_y^*\mu_{0\gamma}$. Since $||\overline{\mu}_0|_\gamma||_W = 1$ (it is a probability), we have $||\mu_0|_\gamma||_W = |\psi_x(\gamma)|||\overline{\mu}_0|_\gamma||_W = |\psi_x(\gamma)|$. So $\int ||\mu_0|_\gamma||_W\, dm_1(\gamma) = \int |\psi_x(\gamma)|dm_1(\gamma) = |\psi_x|_1 < \infty$. Then $\mu_0 \in \mathcal{L}^1$. By construction, $\psi_x \in S_-$. Then $\mu_0 \in S^1$. And we are done.

If **N1** is satisfied, we have $|\cdot|_\infty \le |\cdot|_s$. Suppose that $g : N_2 \longrightarrow \mathbb{R}$ is a Lipschitz function such that $|g|_\infty \le 1$ and $L(g) \le 1$. Then, it holds $\left|\int gd(\mu_0|_\gamma)\right| \le |g|_\infty\psi_x(\gamma) \le |\psi_x|_\infty \le |\psi_x|_s$. Hence, $\mu_0 \in S^\infty$.

For the uniqueness, if $\mu_0, \mu_1 \in S^1$ are F-invariant probabilities, i.e. $\mu_0(\Sigma) = \mu_1(\Sigma) = 1$, then $\mu_0 - \mu_1 \in \mathcal{V}_s$. By Proposition 5.7, $\mathrm{F}^{*n}(\mu_0 - \mu_1) \to 0$ in $\mathcal{L}^1$. Therefore, $\mu_0 - \mu_1 = 0$.

□

[7]It can be difficult to find a sharp estimate for D. An approach allowing to find some useful upper estimates is shown in [15]

5.2. L^∞ **norms.** In this section we consider an L^∞ like anisotropic norm. We show how a Lasota Yorke inequality can be proved for this norm too.

5.10. **Lemma.** *Under the assumptions* $G1$, $T1, ..., T3.3$, *for all signed measure* $\mu \in S^\infty$ *with marginal density* ϕ_x *it holds*

$$|| \mathrm{F}^* \mu||_\infty \leq \alpha |\mathrm{P}_T 1|_\infty ||\mu||_\infty + |\mathrm{P}_T \phi_x|_\infty.$$

Proof. Let T_i be the branches of T, for all $i = 1 \cdots q$. Applying Lemma 5.4 on the third line below, we have

$$\begin{aligned} ||(\mathrm{F}^* \mu)|_\gamma||_W &= \left|\left| \sum_{i=1}^{q} \frac{\mathrm{F}^*_{T_i^{-1}(\gamma)} \mu|_{T_i^{-1}(\gamma)}}{|\det DT_i(T_i^{-1}(\gamma))|} \chi_{T(P_i)}(\gamma) \right|\right|_W \\ &\leq \sum_{i=1}^{q} \frac{|| \mathrm{F}^*_{T_i^{-1}(\gamma)} \mu|_{T_i^{-1}(\gamma)}||_W}{|\det DT_i(T_i^{-1}(\gamma))|} \chi_{T(P_i)}(\gamma) \\ &\leq \sum_{i=1}^{q} \frac{\alpha ||\mu|_{T_i^{-1}(\gamma)}||_W + \phi_x(T_i^{-1}(\gamma))}{|\det DT_i(T_i^{-1}(\gamma))|} \chi_{T(P_i)}(\gamma) \\ &\leq \alpha ||\mu||_\infty \sum_{i=1}^{q} \frac{\chi_{T(P_i)}(\gamma)}{|\det DT_i(T_i^{-1}(\gamma))|} + \sum_{i=1}^{q} \frac{\phi_x(T_i^{-1}(\gamma))}{|\det DT_i(T_i^{-1}(\gamma))|} \chi_{T(P_i)}(\gamma). \end{aligned}$$

Hence, taking the supremum on γ, we finish the proof of the statement. □

Applying the last lemma to F^{*n} instead of F one obtains.

5.11. **Lemma.** *Under the assumptions* $G1$, $T1, ..., T3.4$, *for all signed measure* $\mu \in S^\infty$ *it holds*

$$|| \mathrm{F}^{*n} \mu||_\infty \leq \alpha^n |\mathrm{P}_T^n 1|_\infty ||\mu||_\infty + |\mathrm{P}_T^n \phi_x|_\infty,$$

where ϕ_x *is the marginal density of* μ.

5.12. **Proposition** (Lasota-Yorke inequality for S^∞)**.** *Suppose* F *satisfies the assumptions* $G1$, $T1, ..., T3.4$ *and* $N1$. *Then, there are* $0 < \alpha_1 < 1$ *and* $A_1, B_4 \in \mathbb{R}$ *such that for all* $\mu \in S^\infty$, *it holds*

$$|| \mathrm{F}^{*n} \mu||_{S^\infty} \leq A_1 \alpha_1^n ||\mu||_{S^\infty} + B_4 ||\mu||_1.$$

Proof. We remark that, by equation (6) and (N1) it follows $|\mathrm{P}_T^n 1|_\infty \leq H_N(B_3 + C_2)$, for each n. Then,

$$\begin{aligned} || \mathrm{F}^{*n} \mu||_{S^\infty} &= |\mathrm{P}_T^n \phi_x|_s + || \mathrm{F}^{*\mathrm{n}} \mu||_\infty \\ &\leq [B_3 \beta_2^n |\phi_x|_s + C_2 |\phi_x|_1] + [\alpha^n |\mathrm{P}_T^n 1|_\infty ||\mu||_\infty + |\mathrm{P}_T^n \phi_x|_\infty] \\ &\leq [B_3 \beta_2^n |\phi_x|_s + C_2 |\phi_x|_1] \\ &+ [\alpha^n H_N(B_3 + C_2) ||\mu||_\infty + H_N(B_3 \beta_2^n |\phi_x|_s + C_2 |\phi_x|_1)]. \\ &\leq [\max(\alpha, \beta_2)]^n [B_3(1 + 2H_N) + H_N C_2] ||\mu||_{S^\infty} + C_2(1 + H_N) ||\mu||_1, \end{aligned}$$

where $|\phi_x|_1 \leq ||\mu||_1$ and $|\phi_x|_s \leq ||\mu||_{S^\infty}$. We finish the proof, setting $\alpha_1 = \max(\alpha, \beta_2)$, $A_1 = [B_3(1 + 2H_N) + H_N C_2]$ and $B_4 = C_2(1 + H_N)$. □

6. Spectral gap

In this section, we prove a spectral gap statement for the transfer operator applied to our strong spaces. For this, we will directly use the properties proved in the previous section, and this will give a kind of constructive proof. We remark that, we cannot appy the traditional Hennion, or Ionescu-Tulcea and Marinescu's approach to our function spaces because there is no compact immersion of the strong space into the weak one. This comes from the fact that we are considering the same "dual of Lipschitz" distance in the contracting direction for both spaces.

6.1. **Theorem** (Spectral gap on S^1)**.** *If F satisfies* ***G1*, *T1*,...,*T3.4*** *given at beginning of section 2, then the operator* $\mathrm{F}^* : S^1 \longrightarrow S^1$ *can be written as*

$$\mathrm{F}^* = \mathrm{P} + \mathrm{N},$$

where

a) P *is a projection i.e.* $\mathrm{P}^2 = \mathrm{P}$ *and* $\dim Im(\mathrm{P}) = 1$*;*
b) *there are* $0 < \xi < 1$ *and* $K > 0$ *such that* [8] $\forall \mu \in S^1$

$$|| \mathrm{N}^n(\mu)||_{S^1} \leq ||\mu||_{S^1} \xi^n K;$$

c) $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$.

Proof. First, let us show there exist $0 < \xi < 1$ and $K_1 > 0$ such that, for all $n \geq 1$, it holds

$$|| \mathrm{F}^{*n} ||_{\mathcal{V}_s \to \mathcal{V}_s} \leq \xi^n K_1. \tag{33}$$

Indeed, consider $\mu \in \mathcal{V}_s$ (see equation (32)) s.t. $||\mu||_{S^1} \leq 1$ and for a given $n \in \mathbb{N}$ let m and $0 \leq d \leq 1$ be the coefficients of the division of n by 2, i.e. $n = 2m + d$. Thus $m = \frac{n-d}{2}$. By the Lasota-Yorke inequality (Proposition 5.3) we have the uniform bound $|| \mathrm{F}^{*n} \mu||_{S^1} \leq B_2 + A$ for all $n \geq 1$. Moreover, by Propositions 5.7 and 5.1 there is some D_2 such that it holds (below, let λ_0 be defined by $\lambda_0 = \max\{\beta_1, \lambda\}$)

$$\begin{aligned}
|| \mathrm{F}^{*n} \mu||_{S^1} &\leq A\lambda^m || \mathrm{F}^{*m+d} \mu||_{S^1} + B_2 || \mathrm{F}^{*m+d} \mu||_1 \\
&\leq \lambda^m A(A + B_2) + B_2 ||F^{*m}\mu||_1 \\
&\leq \lambda^m A(A + B_2) + B_2 D_2 \beta_1^m \\
&\leq \lambda_0^m \left[A(A + B_2) + B_2 D_2\right] \\
&\leq \lambda_0^{\frac{n-d}{2}} \left[A(A + B_2) + B_2 D_2\right] \\
&\leq \left(\sqrt{\lambda_0}\right)^n \left(\frac{1}{\lambda_0}\right)^{\frac{d}{2}} \left[A(A + B_2) + B_2 D_2\right] \\
&= \xi^n K_1,
\end{aligned}$$

[8]We remark that, the spectral radius of $\overline{\mathrm{N}}$ satisfies $\rho(\overline{\mathrm{N}}) < 1$, where $\overline{\mathrm{N}}$ is the extension of N to $\overline{S^1}$ (the completion of S_1). This gives us spectral gap, in the usual sense, for the operator $\overline{\mathrm{F}} : \overline{S_1} \longrightarrow \overline{S_1}$. The same remark holds for Theorem 6.2.

where $\xi = \sqrt{\lambda_0}$ and $K_1 = \left(\frac{1}{\lambda_0}\right)^{\frac{1}{2}} [A(A+B_2) + B_2 D_2]$. Thus, we arrive at

$$||(\mathrm{F}^*|_{\mathcal{V}})^n||_{S^1 \to S^1} \leq \xi^n K_1. \tag{34}$$

Now, recall that $\mathrm{F}^* : S^1 \longrightarrow S^1$ has an unique fixed point $\mu_0 \in S^1$, which is a probability (see Proposition **??**). Consider the operator $\mathrm{P} : S^1 \longrightarrow [\mu_0]$ ($[\mu_0]$ is the space spanned by μ_0), defined by $\mathrm{P}(\mu) = \mu(\Sigma)\mu_0$. By definition, P is a projection and $\dim Im(P) = 1$. Define the operator

$$\mathrm{S} : S^1 \longrightarrow \mathcal{V}_s,$$

by

$$\mathrm{S}(\mu) = \mu - \mathrm{P}(\mu), \quad \forall \ \ \mu \in S^1.$$

Thus, we set $\mathrm{N} = \mathrm{F}^* \circ \mathrm{S}$ and observe that, by definition, $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$ and $\mathrm{F}^* = \mathrm{P} + \mathrm{N}$. Moreover, $\mathrm{N}^n(\mu) = \mathrm{F}^{*\,n}(\mathrm{S}(\mu))$ for all $n \geq 1$. Since S is bounded and $\mathrm{S}(\mu) \in \mathcal{V}_s$, we get by (34), $||\,\mathrm{N}^n(\mu)||_{S^1} \leq \xi^n K ||\mu||_{S^1}$, for all $n \geq 1$, where $K = K_1 ||\,\mathrm{S}\,||_{S^1 \to S^1}$. □

In the same way, using the $\mathcal{L}^\infty$ Lasota Yorke inequality of Proposition 5.12, it is possible to obtain spectral gap on the L^∞ like space, we omit the proof which is essentially the same as above:

6.2. **Theorem** (Spectral gap on S^∞). *If F satisfies the assumptions $G1$, $T1, ..., T3.4$ and $N1$, then the operator $\mathrm{F}^* : S^\infty \longrightarrow S^\infty$ can be written as*

$$\mathrm{F}^* = \mathrm{P} + \mathrm{N},$$

where

- a) P *is a projection i.e.* $\mathrm{P}^2 = \mathrm{P}$ *and* $\dim Im(\mathrm{P}) = 1$*;*
- b) *there are* $0 < \xi_1 < 1$ *and* $K_2 > 0$ *such that* $||\,\mathrm{N}^n(\mu)||_{S^\infty} \leq ||\mu||_{S^\infty} \xi_1^n K_2$ $\forall \ \ \mu \in S^\infty$*;*
- c) $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$*.*

6.3. **Remark.** We remark, the constant ξ for the map F, found in Theorem 6.1, is directly related to the coefficients of the Lasota-Yorke inequality and the rate of convergence to equilibrium of F found before (see Remark 5.8). More precisely, $\xi = \max\{\sqrt{\lambda}, \sqrt{\beta_1}\}$. We remark that, from the above proof we also have an explicit estimate for K in the exponential convergence, while many classical approaches are not suitable for this.

7. Application to Lorenz-like maps

In this section, we apply Theorems 6.1 and 6.2 to a large class of maps which are Poincaré maps for suitable sections of Lorenz-like flows. In these systems (see e.g [4]), it can be proved that there is a two dimensional Poincaré section Σ which can be supposed to be a rectangle I^2, where $I = [0, 1]$, whose return map $F_L : I^2 \to I^2$, after a suitable change of coordinates, has the form $F_L(x, y) = (T_L(x), G_L(x, y))$, satisfying the properties, G1 and T1-T3, of section 2. The map $T_L : I \longrightarrow I$, in this case, can be supposed to be piecewise expanding with $C^{1+\alpha}$ branches.

Hence, we consider a class of skew product maps $F_L : I^2 \to I^2$, where $I = [0,1]$, satisfying $(G1), (T1), (T2)$, and the following properties on T_L :

7.0.1. *Properties of T_L in Lorenz-like systems.*

(P'1) $\dfrac{1}{|T_L'|}$ is of universal bounded p-variation, i.e. for $p \geq 1$

$$\mathrm{var}_p(|T_L'|) := \sup_{0 \leq x_0 < \cdots < x_n \leq 1} \left(\sum_{i=0}^{n} \left| \frac{1}{|T_L'(x_i)|} - \frac{1}{|T_L'(x_{i-1})|} \right|^p \right)^{\frac{1}{p}} < \infty; \tag{35}$$

(P'2) $\inf |T_L^{n_0\prime}| \geq \lambda_1 > 1$, for some $n_0 \in \mathbb{N}$.

We remark that, the universal bounded p-variation, var_p, is a generalization of the usual bounded variation. It is a weaker notion, allowing piecewise Holder functions. Indeed, for $p \geq 1$, a $1/p$-Holder function is of universal bounded p-variation. This definition is adapted to maps having $C^{1+\alpha}$ regularity.

From properties P'1 and P'2, it follows (see [18]) that there exists a suitable strong space (the space S_- in T3.1) for the Perron-Frobenius operator P_T associated to such a T_L, in a way that it satisfies the assumptions $T1, ..., T3.3$ and $N1$. In this case, supposing a property like $T3.4$ then we can apply our results. Therefore, let us introduce the space of generalized bounded variation functions with respect to the Lebesgue measure: $BV_{1,\frac{1}{p}}$. The functions of universal bounded p-variation are included in this space (for more details and results see [18], in particular Lemma 2.7 for a comparison of the two spaces).

A piecewise expanding map satisfying assumptions (P'1) and (P'2) has an invariant measure with density in $BV_{1,\frac{1}{p}}$, moreover the transfer operator restricted to this space satisfies a Lasota-Yorke inequality and other interesting properties, as we will see in the following.

7.1. **Definition.** Let m be the Lebesgue measure on $I = [0,1]$. For an arbitrary function $h : I \longrightarrow \mathbb{C}$ and $\epsilon > 0$ define $\mathrm{osc}(h, B_\epsilon(x)) : I \longrightarrow [0, \infty]$ by

$$\mathrm{osc}(h, B_\epsilon(x)) = \mathrm{ess\ sup}\{|h(y_1) - h(y_2)|; y_1, y_2 \in B_\epsilon(x)\}, \tag{36}$$

where $B_\epsilon(x)$ denotes the open ball of center x and radius ϵ and the essential supremum is taken with respect to the product measure m^2 on I^2. Also define the real function $\mathrm{osc}_1(h, \epsilon)$, on the variable ϵ, by

$$\mathrm{osc}_1(h, \epsilon) = \int \mathrm{osc}(h, B_\epsilon(x)) dm(x).$$

7.2. **Definition.** Fix $A_1 > 0$ and denote by Φ the class of all isotonic maps $\phi : (0, A_1] \longrightarrow [0, \infty]$, i.e. such that $x \leq y \Longrightarrow \phi(x) \leq \phi(y)$ and $\phi(x) \longrightarrow 0$ if $x \longrightarrow 0$. Set

- $R_1 = \{h : I \longrightarrow \mathbb{C}; \mathrm{osc}_1(h, .) \in \Phi\}$;
- For $n \in \mathbb{N}$, define $R_{1,n \cdot p} = \{h \in R_1; \mathrm{osc}_1(h, \epsilon) \leq n \cdot \epsilon^{\frac{1}{p}} \ \ \forall \epsilon \in (0, A_1]\}$;

- And set $S_{1,p} = \bigcup_{n \in \mathbb{N}} R_{1,n \cdot p}$.

7.3. **Definition.** Let us consider the following spaces and semi-norms:

(1) $BV_{1,\frac{1}{p}}$ is the space of m-equivalence classes of functions in $S_{1,p}$;

(2) Let $h : I \longrightarrow \mathbb{C}$ be a Borel function. Set

$$\mathrm{var}_{1,\frac{1}{p}}(h) = \sup_{0 \leq \epsilon \leq A_1} \left(\frac{1}{\epsilon^{\frac{1}{p}}} \mathrm{osc}_1(h, \epsilon) \right). \tag{37}$$

Since $BV_{1,1/p}$ was defined using a probability measure, m, then $\mathrm{var}_{1,1/p}(h) \leq 2^{1/p} \,\mathrm{var}_p(h)$.

Let us consider $|\cdot|_{1,\frac{1}{p}} : BV_{1,\frac{1}{p}} \longrightarrow \mathbb{R}$ defined by

$$|f|_{1,\frac{1}{p}} = \mathrm{var}_{1,\frac{1}{p}}(f) + |f|_1, \tag{38}$$

it holds the following

7.4. **Proposition.** *$\left(BV_{1,\frac{1}{p}}, |\cdot|_{1,\frac{1}{p}}\right)$ is a Banach space.*

In the above setting, G. Keller has shown (see [18]) that there is an $A_1 > 0$ (we recall that definition 7.2 depends on A_1) such that:

(a) $BV_{1,\frac{1}{p}} \subset L^1$ is P_T-invariant, $\mathrm{P}_T : BV_{1,\frac{1}{p}} \longrightarrow BV_{1,\frac{1}{p}}$ is continuous and it holds $|\cdot|_1 \leq |\cdot|_{1,\frac{1}{p}}$;

(b) The unit ball of $(BV_{1,\frac{1}{p}}, |\cdot|_{1,\frac{1}{p}})$ is relatively compact in $(L^1, |\cdot|_1)$;

(c) There exists $k \in \mathbb{N}$, $0 < \beta_0 < 1$ and $C > 0$ such that

$$|\mathrm{P}_T^k f|_{1,\frac{1}{p}} \leq \beta_0 |f|_{1,\frac{1}{p}} + C|f|_1. \tag{39}$$

Analogously to the proof of inequality (6), we have

$$|\mathrm{P}_T^n f|_{1,\frac{1}{p}} \leq B_3 \beta_2^n |f|_{1,\frac{1}{p}} + C_2 |f|_1, \quad \forall n, \quad \forall f \in BV_{1,\frac{1}{p}}, \tag{40}$$

for $B_3, C_2 > 0$ and $0 < \beta_2 < 1$.

Moreover, in [2] (Lemma 2), it was shown that

(d)

$$|\cdot|_\infty \leq A_1^{\frac{1}{p}-1} |\cdot|_{1,\frac{1}{p}}. \tag{41}$$

Therefore, the properties $T1, T2, T3.1, .., T3.3, N1$ of section 2 are satisfied with $S_- = BV_{1,\frac{1}{p}}$ and we can apply our construction to such maps.

Thus, for $1 \leq p < \infty$, we set

$$\mathcal{BV}_{1,\frac{1}{p}} := \left\{ \mu \in \mathcal{L}^1; \mathrm{var}_{1,\frac{1}{p}}(\phi_x) < \infty, \text{ where } \phi_x = \frac{d\mu_x}{dm} \right\} \tag{42}$$

and consider $||\cdot||_{1,\frac{1}{p}} : \mathcal{BV}_{1,\frac{1}{p}} \longrightarrow \mathbb{R}$, defined by

$$||\mu||_{1,\frac{1}{p}} = |\phi_x|_{1,\frac{1}{p}} + ||\mu||_1. \tag{43}$$

Clearly, $\left(\mathcal{BV}_{1,\frac{1}{p}}, ||\cdot||_{1,\frac{1}{p}}\right)$ is a normed space. If we suppose that the system, $T_L : I \longrightarrow I$, satisfies $T3.4$, then it has an unique absolutely continuous invariant probability with density $\varphi_x \in BV_{1,\frac{1}{p}}$.

As defined in equation (32), for $1 \leq p < \infty$, consider the set of zero average measures in $\mathcal{BV}_{1,\frac{1}{p}}$,

$$\mathcal{V}_s = \{\mu \in \mathcal{BV}_{1,\frac{1}{p}} : \mu(\Sigma) = 0\}. \tag{44}$$

Directly from the above settings, Proposition 5.7 and from Theorem 6.1 it follows convergence to equilibrium and spectral gap for these kind of maps.

7.5. **Proposition** (Exponential convergence to equilibrium)**.** *If* F_L *satisfies assumptions* $G1$*,* $T1$*,*$T2$*,* $T3.4$*,* $P'1$ *and* $P'2$*, then there exist* $D_2 > 0$ *and* $0 < \beta_2 < 1$ *such that, for every signed measure* $\mu \in \mathcal{V}_s \subset \mathcal{BV}_{1,\frac{1}{p}}$*,* $1 \leq p < \infty$*, it holds*

$$|| \mathrm{F_L^{*n}}\, \mu||_1 \leq D_2 \beta_1^n ||\mu||_{1,\frac{1}{p}},$$

for all $n \geq 1$*.*

7.6. **Theorem** (Spectral gap for $\mathcal{BV}_{1,\frac{1}{p}}$)**.** *If* F_L *satisfies assumptions* $G1$*,* $T1$*,*$T2$*,* $T3.4$*,* $P'1$ *and* $P'2$*, then the operator* $\mathrm{F}_L^* : \mathcal{BV}_{1,\frac{1}{p}} \longrightarrow \mathcal{BV}_{1,\frac{1}{p}}$ *can be written as*

$$\mathrm{F}_L^* = \mathrm{P} + \mathrm{N}$$

where

a) P *is a projection i.e.* $\mathrm{P}^2 = \mathrm{P}$ *and* $\dim Im(\mathrm{P}) = 1$*;*
b) *there are* $0 < \xi < 1$ *and* $K > 0$ *such that for all* $\mu \in \mathcal{BV}_{1,\frac{1}{p}}$

$$|| \mathrm{N}^n(\mu)||_{\mathcal{BV}_{1,\frac{1}{p}}} \leq \xi^n K ||\mu||_{\mathcal{BV}_{1,\frac{1}{p}}};$$

c) $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$*.*

We can get the same kind of results for stronger L^∞ like norms. Let us consider

$$\mathcal{BV}^\infty_{1,\frac{1}{p}} := \left\{\mu \in \mathcal{L}^\infty; \frac{d(\pi_x^*\mu)}{dm} \in BV_{1,\frac{1}{p}}\right\} \tag{45}$$

and the function, $||\cdot||^\infty_{1,\frac{1}{p}} : \mathcal{BV}^\infty_{1,\frac{1}{p}} \longrightarrow \mathbb{R}$, defined by

$$||\mu||^\infty_{1,\frac{1}{p}} = |\phi_x|_{1,\frac{1}{p}} + ||\mu||_\infty. \tag{46}$$

Applying Theorem 6.2 we get

7.7. **Theorem** (Spectral gap for $\mathcal{BV}^\infty_{1,\frac{1}{p}}$)**.** *If* F_L *satisfies the satisfies assumptions* $G1$*,* $T1$*,*$T2$*,*$T3.4$*,* $P'1$ *and* $P'2$*, then the operator* $\mathrm{F}_L^* : \mathcal{BV}^\infty_{1,\frac{1}{p}} \longrightarrow \mathcal{BV}^\infty_{1,\frac{1}{p}}$ *can be written as*

$$\mathrm{F}_L^* = \mathrm{P} + \mathrm{N},$$

where

a) P *is a projection i.e.* $\mathrm{P}^2 = \mathrm{P}$ *and* $\dim Im(\mathrm{P}) = 1$*;*
b) *there are* $0 < \xi_1 < 1$ *and* $K_2 > 0$ *such that for all* $\mu \in \mathcal{BV}^{\infty}_{1,\frac{1}{p}}$

$$|| \mathrm{N}^n(\mu)||^{\infty}_{1,\frac{1}{p}} \leq \xi_1^n K_2 ||\mu||^{\infty}_{1,\frac{1}{p}};$$

c) $\mathrm{P}\,\mathrm{N} = \mathrm{N}\,\mathrm{P} = 0$*.*

By Proposition 5.9 we immediately get

7.8. **Proposition.** *The unique invariant probability for the system* F_L *in* $\mathcal{BV}_{1,\frac{1}{p}}$ *is* μ_0*. Moreover, since **N1** is satisfied (equation (41)),* μ_0 *is the unique* F_L*-invariant probability in* $\mathcal{BV}^{\infty}_{1,\frac{1}{p}}$*.*

8. Quantitative Statistical Stability

Throughout this section, we consider small perturbations of the transfer operator of a particular system of the kind described in the previous sections and study the dependence of the physical invariant measure with respect to the perturbation. A classical tool that can be applied for this type of problems is the Keller-Liverani stability theorem [19]. Since in our setting the strong space is not compactly immersed in the weak one, we cannot directly apply it. We will use another approach giving us precise bounds on the statistical stability. In this section, this approach will be applied to a class of Lorenz-like maps with slightly stronger regularity assumptions than used in Section 7. We call such a system by *BV Lorenz-like map* (see Definition 8.5) and precisely, we need the additional property stated in item (1) of Definition 8.5.

8.0.1. *Uniform Family of Operators.* In what follows, we present a general *quantitative* result relating the *stability* of the invariant measure of an *uniform family of operators* (Definition 8.1) and *convergence to equilibrium.*

In the following definition, for all $\delta \in [0,1)$, let L_δ be a transfer operator acting on two vector subspaces of signed measures on X, $\mathrm{L}_\delta : (B_s, ||\cdot||_s) \longrightarrow (B_s, ||\cdot||_s)$ and $\mathrm{L}_\delta : (B_w, ||\cdot||_w) \longrightarrow (B_w, ||\cdot||_w)$, endowed with two norms, the strong norm $||\cdot||_s$ on B_s, and the weak norm $||\cdot||_w$ on B_w, such that $||\cdot||_s \geq ||\cdot||_w$. Suppose that,

$$B_s \subseteq B_w \subseteq \mathcal{SB}(X),$$

where $\mathcal{SB}(X)$ denotes the space of Borel signed measures on X.

8.1. **Definition.** A one parameter family of transfer operators $\{\mathrm{L}_\delta\}_{\delta \in [0,1)}$ is said to be an **uniform family of operators** with respect to the weak space $(B_w, ||\cdot||_w)$ and the strong space $(B_s, ||\cdot||_s)$ if $||\cdot||_s \geq ||\cdot||_w$ and it satisfies

UF1 Let $f_\delta \in B_s$ be a fixed probability measure for the operator L_δ. Suppose there is $M > 0$ such that for all $\delta \in [0,1)$, it holds

$$||f_\delta||_s \leq M;$$

UF2 L_δ approximates L_0 when δ is small in the following sense: there is $C \in \mathbb{R}^+$ such that:

$$||(\mathrm{L}_0 - \mathrm{L}_\delta) f_\delta||_w \leq \delta C; \tag{47}$$

UF3 L_0 has exponential convergence to equilibrium with respect to the norms $||\cdot||_s$ and $||\cdot||_w$: there exists $0 < \rho_2 < 1$ and $C_2 > 0$ such that

$$\forall\ f \in \mathcal{V}_s := \{f \in B_s : f(X) = 0\}$$

it holds

$$||\,\mathrm{L}_0^n\, f||_w \leq \rho_2^n C_2 ||f||_s;$$

UF4 The iterates of the operators are uniformly bounded for the weak norm: there exists $M_2 > 0$ such that

$$\forall \delta, n, g \in B_s \text{ it holds } ||\,\mathrm{L}_\delta^n\, g||_w \leq M_2 ||g||_w.$$

Under these assumptions we can ensure that the invariant measure of the system varies continuously (in the weak norm) when L_0 is perturbed to L_δ, for small values of δ. Moreover, the modulus of continuity can be estimated. We postpone the proof of Proposition 8.2 to the Appendix 3 (section 11).

8.2. **Proposition.** *Suppose* $\{\mathrm{L}_\delta\}_{\delta \in [0,1)}$ *is an uniform family of operators as in Definition 8.1, where* f_0 *is the unique fixed point of* L_0 *in* B_w *and* f_δ *is a fixed point of* L_δ*. Then, there exists* $\delta_0 \in (0,1)$ *such that for all* $\delta \in [0, \delta_0)$*, it holds*

$$||f_\delta - f_0||_w = O(\delta \log \delta).$$

8.1. **Quantitative stability of Lorenz-like maps.** Here we apply the general result on uniform family of operators (Proposition 8.2) to a suitable family of bounded variation Lorenz-like maps. We consider maps as defined in Section 7, with some further assumptions (see Definition 8.5), and prove that the invariant measures associated to a perturbation (given by Definition 8.9) of this map varies continuously with modulus of continuity $\delta \log \delta$. Precisely, the aim of this section is to prove the following theorem:

8.3. **Theorem** (Quantitative stability for deterministic perturbations)**.** *Let* $\{F_\delta\}_{\delta \in [0,1)}$ *be an uniform BV Lorenz-like family (see definition 8.9). Denote by* f_δ *the fixed point of* F_δ^* *in* $\mathcal{BV}_{1,1}$ *(also in* $\mathcal{BV}_{1,1}^{\infty}$*), for all* δ*. Then, there exists* $\delta_0 \in (0,1)$ *such that for all* $\delta \in [0, \delta_0)$*, it holds*

$$||f_\delta - f_0||_1 = O(\delta \log \delta).$$

It will be obtained as an immediate consequence of the following Proposition 8.4 together with Proposition 8.2 stated above.

8.4. **Proposition.** *Let* $\{F_\delta\}_{\delta \in [0,1)}$ *be an Uniform BV Lorenz-like family and let* $\{\mathrm{F}_\delta^*\}_{\delta \in [0,1)}$ *be the induced family of transfer operators. Then,* $\{\mathrm{F}_\delta^*\}_{\delta \in [0,1)}$ *is an uniform family of operators with weak space* $(\mathcal{L}^1, ||\cdot||_1)$ *and strong space* $(\mathcal{BV}_{1,1}, ||\mu||_{1,1})$*.*

By a matter of completeness, these results are stated again (and finally proved) as Theorem 8.22 and Proposition 8.21. In order to prove them, we need some preliminary results and definitions.

8.5. **Definition.** A map $F_L : [0,1]^2 \longrightarrow [0,1]^2$, $F_L(x,y) = (T_L(x), G_L(x,y))$, is said to be a **BV Lorenz-like map** if it satisfies

(1) There are $H \geq 0$ and a partition $\mathcal{P}' = \{J_i := (b_{i-1}, b_i), i = 1, \cdots, d\}$ of I such that for all $x_1, x_2 \in J_i$ and for all $y \in I$ the following inequality holds
$$|G_L(x_1, y) - G_L(x_2, y)| \leq H \cdot |x_1 - x_2|; \tag{48}$$
(2) F_L satisfy property $G1$ (hence is uniformly contracting on each leaf γ with rate of contraction α);
(3) $T_L : I \to I$ is a piecewise expanding map satisfying the assumptions given in the following definition 8.6.

The following definition characterizes a class of piecewise expanding maps of the interval with bounded variation derivative $T_L : I \longrightarrow I$ which is a subclass of the ones considered in section 7.0.1.

8.6. **Definition** (Piecewise expanding functions with bounded variation inverse of the derivative)**.** Suppose there exists a partition $\mathcal{P} = \{P_i := (a_{i-1}, a_i), i = 1, \cdots, q = q(\delta)\}$ of I s.t. $T_L : I \longrightarrow I$ satisfies the following conditions. For all i

1) $T_{L_i} = T_L|_{P_i}$ is of class C^1 and $g_i = \dfrac{1}{|T_{L_i}{}'|}$ satisfies (P'1) of section 7, for $p = 1$.
2) T_L satisfies (P'2) of section 7: $\inf |T_L^{n_0}{}'| \geq \lambda_1 > 1$ for some $n_0 \in \mathbb{N}$.
3) T_L satisfies T3.4;

In particular we assume that T_{L_i} and g_i admit a continuous extension to $\overline{P_i} = [a_{i-1}, a_i]$ for all $i = 1, \cdots, q$.

8.7. **Remark.** The definition 8.6 allows infinite derivative for T_L at the extreme points of its regularity intervals.

8.8. **Definition.** Let T_1 and T_2 be to piecewise expanding maps of definition (8.6). Define the set Int_n, by
$$Int_n = \{A \in 2^{[0,1]}, s.t.\ A = I_1 \cup, ..., \cup I_n, \text{ where } I_i \text{ are intervals}\}$$
the set of subsets of $[0,1]$ which is the union of at most n intervals. Set
$$\mathcal{C}(n, T_1, T_2) = \left\{ \begin{array}{c} \epsilon : \exists A_1 \in Int_n \text{ and } \exists\, \sigma : I \to I \text{ a diffeomorphism s.t. } m(A_1) \geq 1 - \epsilon, \\ T_1|_{A_1} = T_2 \circ \sigma|_{A_1} \text{ and } \forall x \in A_1, |\sigma(x) - x| \leq \epsilon, |\frac{1}{\sigma'(x)} - 1| \leq \epsilon \end{array} \right\}$$
and define a distance from T_1 to T_2 as:
$$d_{S,n}(T_1, T_2) = \inf \{\epsilon | \epsilon \in \mathcal{C}(n, T_1, T_2)\}. \tag{49}$$

If we denote by d_S the classical notion of Skorokhod distance (see [28] e.g.), it is obvious that $\forall n$ $d_{S,n} \geq d_S$. By [28], Lemma 11.2.1, it follows that $\forall n$:

$$|\,\mathrm{P_{T_0}} - \mathrm{P_{T_\delta}}\,|_{BV \to L^1} \leq 14 d_{S,n}(T_1, T_2). \tag{50}$$

8.9. **Definition.** A family of maps $\{F_\delta\}_{\delta \in [0,1)}$ is said to be a **Uniform BV Lorenz-like family** if F_δ is a BV Lorenz-like map (see definition 8.5) for all $\delta \in [0,1)$ and $\{F_\delta\}_\delta$ satisfies the following assumptions:

(UBV1): there exist $0 < \lambda < 1$ and $D > 0$ s.t. for all $f \in BV_{1,1}$ and for all $\delta \in [0,1)$ it holds $|\,\mathrm{P}_{T_\delta}^n f|_{1,1} \leq D\lambda^n |f|_{1,1} + D|f|_1$ for all $n \geq 1$, where P_{T_δ} is the Perron-Frobenius operators of T_δ.

When δ is small

(UBV2): T_0 and T_δ are near for the above Shorokod-like distance. For some n independent of δ it holds $\forall \delta$

$$d_{S,n}(T_0, T_\delta) \leq \delta.$$

(UBV3): For each δ there is a set A_2 (depending on δ) such that $A_2 \in Int_{n_\delta}$ for some n_δ (depending on δ) furthermore $m(A_2) \geq 1 - \delta$ and for all $x \in A_2, y \in I$:

$$|G_0(x,y) - G_\delta(x,y)| \leq \delta.$$

Let us furthermore suppose that the number of such intervals during the perturbation remains uniformly bounded: $\sup_\delta n_\delta < \infty$.

For all $\delta \in [0,1)$, let $n_0 = n_0(\delta) \in \mathbb{N}$ be the first integer such that there exists $\lambda_1(\delta) > 0$ satisfying $\left|T_{\delta,i}^{n_0 \prime}(x)\right| \geq \lambda_1(\delta) > 1$ for all $x \in P_{\delta,i}$ and for each $i = 1, \cdots, q$, where $T_{\delta,i}^{n_0} := T_\delta{}^{n_0}|_{P_{\delta,i}}$. Also set $g_{i,\delta} = \dfrac{1}{|T_{\delta,i}'|}$ and denote by $H_\delta > 0$ and $\mathcal{P}'{}_\delta$ the "Lipschitz"constant and the regularity partition associated to G_δ, see item (1) of Definition 8.5.

(UBV4): Suppose that:

(1) $\inf_\delta \lambda_1(\delta) > 1$, $\sup_\delta \lambda_1(\delta) < \infty$ and $\sup_{\delta \in [0,1)}\{n_0(\delta)\} < \infty$;
(2) there exists $C_4 > 0$ such that $\sup g_{\delta,i} \leq C_4$ and $\operatorname{var} g_{\delta,i} \leq C_4$ for all $i = 1, \cdots, q$ and all $\delta \in [0,1)$;
(3) $\inf_{\delta \in [0,1)} \min_{i=1,\cdots,q(\delta)}\{m(P_{i,\delta})\} > 0$;
(4) $\sup_{\delta \in [0,1)} H_\delta < \infty$, $\sup_{\delta \in [0,1)} \#\mathcal{P}'_\delta < \infty$

8.1.1. *Measures with bounded variation.* Here we introduce a space of measures having bounded variation in some stronger sense, and prove that the invariant measure of a BV Lorenz-like map is in it. We use this fact in the proof of Proposition 8.21 (or Proposition 8.4), where we prove that the family of transfer operators $\{\mathrm{F}_\delta^*\}_{\delta \in [0,1)}$ induced by an Uniform BV Lorenz-like family $\{F_\delta\}_{\delta \in [0,1)}$ satisfies UF2.

We have seen that a positive measure on the square, $[0,1]^2$, can be disintegrated along the stable leaves $\mathcal{F}^s$ in a way that we can see it as a family of positive measures on the interval, $\{\mu|_\gamma\}_{\gamma\in\mathcal{F}^s}$. Since there is a one-to-one correspondence between $\mathcal{F}^s$ and $[0,1]$, this defines a path in the metric space of positive measures, $[0,1] \longmapsto \mathcal{SB}(I)$, where $\mathcal{SB}(I)$ is endowed with the Wasserstein-Kantorovich like metric (see definition 3.5). It will be convenient to use a functional notation and denote such a path by $\Gamma_\mu : I \longrightarrow \mathcal{SB}(I)$ defined almost everywhere by $\Gamma_\mu(\gamma) = \mu|_\gamma$, where $(\{\mu_\gamma\}_{\gamma\in I}, \phi_x)$ is some disintegration for μ. However, since such a disintegration is defined μ_x-a.e. $\gamma \in [0,1]$, the path Γ_μ is not unique. For this reason we define more precisely Γ_μ as the class of almost everywhere equivalent paths corresponding to μ.

8.10. **Definition.** Consider a positive Borel measure μ and a disintegration $\omega = (\{\mu_\gamma\}_{\gamma\in I}, \phi_x)$, where $\{\mu_\gamma\}_{\gamma\in I}$ is a family of probabilities on Σ defined μ_x-a.e. $\gamma \in I$ (where $\mu_x = \phi_x m$) and $\phi_x : I \longrightarrow \mathbb{R}$ is a non-negative marginal density. Denote by Γ_μ the class of equivalent paths associated to μ

$$\Gamma_\mu(\gamma) = \{\Gamma^\omega_\mu\},$$

where ω ranges on all the possible disintegrations of μ on the stable foliation and $\Gamma^\omega_\mu : I \longrightarrow \mathcal{SB}(I)$ is the path associated to a given disintegration, ω:

$$\Gamma^\omega_\mu(\gamma) = \mu|_\gamma = \pi^*_{\gamma,y}\phi_x(\gamma)\mu_\gamma.$$

Let us call the set on which Γ^ω_μ is defined by $I_{\Gamma^\omega_\mu}$.

8.11. **Definition.** Let $\mathcal{P} = \mathcal{P}(\Gamma^\omega_\mu)$ be a finite sequence $\mathcal{P} = \{x_i\}_{i=1}^n \subset I_{\Gamma^\omega_\mu}$ and define the **variation of Γ^ω_μ with respect to $\mathcal{P}$** as (denote $\gamma_i := \gamma_{x_i}$)

$$\mathrm{Var}(\Gamma^\omega_\mu, \mathcal{P}) = \sum_{j=1}^n ||\Gamma^\omega_\mu(\gamma_j) - \Gamma^\omega_\mu(\gamma_{j-1})||_W,$$

where we recall $||\cdot||_W$ is the Wasserstein-like norm defined by equation (13). Finally, we define the **variation of** Γ^ω_μ by taking the supremum over the set of finite sequences of any length, as

$$\mathrm{Var}(\Gamma^\omega_\mu) := \sup_{\mathcal{P}} \mathrm{Var}(\Gamma^\omega_\mu, \mathcal{P}).$$

8.12. **Remark.** For an interval $\eta \subset I$, we define

$$\mathrm{Var}_{\overline{\eta}}(\Gamma^\omega_\mu) := \mathrm{Var}(\Gamma^\omega_\mu|_{\overline{\eta}}),$$

where $\overline{\eta}$ is the closure of η. We also remark that $\mathrm{Var}_{\overline{\eta}}(\Gamma^\omega_\mu) = \mathrm{Var}(\Gamma^\omega_\mu \cdot \chi_{\overline{\eta}})$, where $\chi_{\overline{\eta}}$ is the characteristic function of $\overline{\eta}$.

8.13. **Remark.** When no confusion can be done, to simplify the notation, we denote $\Gamma^\omega_\mu(\gamma)$ just by $\mu|_\gamma$.

8.14. **Definition.** Define the **variation of a positive measure** μ by

$$\mathrm{Var}(\mu) = \inf_{\Gamma^\omega_\mu \in \Gamma_\mu} \{\mathrm{Var}(\Gamma^\omega_\mu)\}.$$

We remark that,

$$||\mu||_1 = \int W_0^1(0, \Gamma_\mu^\omega(\gamma))dm(\gamma), \quad \text{for any} \quad \Gamma_\mu^\omega \in \Gamma_\mu.$$

8.15. **Definition.** From the definition 8.11 we define the set of bounded variation positive measures $\mathcal{BV}^+$ as

$$\mathcal{BV}^+ = \{\mu \in \mathcal{AB} : \mu \geq 0, \mathrm{Var}(\mu) < \infty\}. \tag{51}$$

Now we are ready to state a lemma estimating the regularity of the iterates $\mathrm{F}^{*n}(m)$. Next result is a Lasota-Yorke like inequality where the strong semi-norm is the variation $\mathrm{Var}(\mu)$ defined in 8.14. This is our main tool to estimate the regularity of the invariant measure of a BV Lorenz-like map (Proposition 8.17). Since it is an immediate consequence of Theorem 9.2 and Remark 9.3 (see Appendix 1), we omit its proof.

8.16. **Proposition.** *Let $F_L(x,y) = (T_L(x), G_L(x,y))$ be a BV Lorenz-like map. Then, there are C_0 and $0 < \lambda_0 < 1$ such that for all $\mu \in \mathcal{BV}^+$ and all $n \geq 1$ it holds*

$$\mathrm{Var}(\mathrm{F}^{*n}\,\mu) \leq K_0 \lambda_0^n \,\mathrm{Var}(\mu) + K_0|\phi_x|_{1,1}. \tag{52}$$

A precise estimate for K_0 can be found in equation (79). Remember, by Proposition 5.9, a Lorenz-like map has an invariant measure $\mu_0 \in S^\infty$.

8.17. **Proposition.** *Let $F_L(x,y) = (T_L(x), G_L(x,y))$ be BV Lorenz-like map and suppose that F_L has an unique invariant probability measure $\mu_0 \in \mathcal{BV}_{1,1}^\infty$. Then $\mu_0 \in \mathcal{BV}^+$ and*

$$\mathrm{Var}(\mu_0) \leq 2K_0.$$

Proof. According to Proposition 7.8, let $\mu_0 \in \mathcal{BV}_{1,1}^\infty$ be the unique F_L-invariant probability measure in $\mathcal{BV}_{1,1}^\infty$. Consider the Lebesgue measure m and the iterates $\mathrm{F}^{*n}(m)$. By Theorem 7.7, these iterates converge to μ_0 in $\mathcal{L}^\infty$. It means that the sequence $\{\Gamma^\omega_{\mathrm{F}^{*n}(m)}\}_n$ converges m-a.e. to $\Gamma^\omega_{\mu_0} \in \Gamma_{\mu_0}$ (in $\mathcal{SB}(I)$ with respect to the metric defined in definition 3.5), where $\Gamma^\omega_{\mu_0}$ is a path given by the Rokhlin Disintegration Theorem and $\{\Gamma^\omega_{\mathrm{F}^{*n}(m)}\}_n$ is given by Remark 4.2. It implies that $\{\Gamma^\omega_{\mathrm{F}^{*n}(m)}\}_n$ converges pointwise to $\Gamma^\omega_{\mu_0}$ on a full measure set $\widehat{I} \subset I$. Let us denote $\widehat{\Gamma^\omega_n} = \Gamma^\omega_{\mathrm{F}^{*n}(m)}|_{\widehat{I}}$ and $\widehat{\Gamma^\omega_{\mu_0}} = \Gamma^\omega_{\mu_0}|_{\widehat{I}}$. Since $\{\widehat{\Gamma^\omega_n}\}_n$ converges pointwise to $\widehat{\Gamma^\omega_{\mu_0}}$ it holds $\mathrm{Var}(\widehat{\Gamma^\omega_n}, \mathcal{P}) \longrightarrow \mathrm{Var}(\widehat{\Gamma^\omega_{\mu_0}}, \mathcal{P})$ as $n \to \infty$ for all finite sequences $\mathcal{P} \subset \widehat{I}$. Indeed, let $\mathcal{P} = \{x_1, \cdots, x_k\} \subset \widehat{I}$ be a finite sequence. Then,

$$\mathrm{Var}(\widehat{\Gamma^\omega_n}, \mathcal{P}) = \sum_{j=1}^{k} ||\widehat{\Gamma^\omega_n}(x_j) - \widehat{\Gamma^\omega_n}(x_{j-1})||_W,$$

taking the limit, we get

$$\begin{aligned}\lim_{n\longrightarrow\infty} \mathrm{Var}(\widehat{\Gamma^{\omega}_{n}}, \mathcal{P}) &= \lim_{n\longrightarrow\infty} \sum_{j=1}^{k} ||\widehat{\Gamma^{\omega}_{n}}(x_j) - \widehat{\Gamma^{\omega}_{n}}(x_{j-1})||_W \\ &= \sum_{j=1}^{k} ||\widehat{\Gamma^{\omega}_{\mu_0}}(x_j) - \widehat{\Gamma^{\omega}_{\mu_0}}(x_{j-1})||_W \\ &= \mathrm{Var}(\widehat{\Gamma^{\omega}_{\mu_0}}, \mathcal{P}).\end{aligned}$$

On the other hand, $\mathrm{Var}(\widehat{\Gamma^{\omega}_{n}}, \mathcal{P}) \leq \mathrm{Var}(\widetilde{\mathrm{F}}^{*\mathrm{n}}(m)) \leq 2K_0$ for all $n \geq 1$, where K_0 comes from Proposition 8.16. Then $\mathrm{Var}(\widehat{\Gamma^{\omega}_{\mu_0}}, \mathcal{P}) \leq 2K_0$ for all partition $\mathcal{P}$. Thus, $\mathrm{Var}(\widehat{\Gamma^{\omega}_{\mu_0}}) \leq 2K_0$ and hence $\mathrm{Var}(\mu_0) \leq 2K_0$.

□

8.18. **Remark.** We remark that, Proposition 8.17 is an estimation of the regularity of the disintegration of μ_0. Similar results are presented in [16] and [11].

The proof of the following proposition is postponed to the appendix (see Proposition 9.20).

8.19. **Proposition.** *Let $\{F_\delta\}_{\delta \in [0,1)}$ be an Uniform BV Lorenz-like family (definition (8.9)) and let f_δ be the unique F_δ-invariant probability in $\mathcal{BV}_{1,1}$ (also in $\mathcal{BV}^{\infty}_{1,1}$). Then, there exists $B_u > 0$ such that*

$$\mathrm{Var}(f_\delta) \leq 2B_u, \tag{53}$$

for all $\delta \in [0, 1)$.

For the next proposition we will use the following notation. Given a probability measure f_δ on I^2 and a measurable set $E \subset I$, we define the measure $1_E f_\delta$ on I^2, by

$$1_E f_\delta(A) := f_\delta(A \cap \pi_x^{-1}(E)) \text{ for all measurable set } A \subset I^2. \tag{54}$$

We remark that, if $(\{f_{\delta,\gamma}\}_\gamma, \phi_{x,\delta})$ is a disintegration of f_δ, then

$$(\{f_{\delta,\gamma}\}_\gamma, \chi_E \phi_{x,\delta}), \tag{55}$$

is a disintegration of $1_E f_\delta(A)$.

8.20. **Proposition** (to obtain UF2)**.** *Let $\{F_\delta\}_{\delta \in [0,1)}$ be a family of BV Lorenz-like maps which satisfies UBV2, UBV3 and UBV4 of definition 8.9. Denote by F^*_δ their transfer operators and by f_δ their fixed points (probabilities) in $\mathcal{BV}_{1,1}$ (also in $\mathcal{BV}^{\infty}_{1,1}$). Suppose that f_δ has uniformly bounded variation,*

$$\mathrm{Var}(f_\delta) \leq M_2, \ \forall \delta.$$

Then, there is a constant C_1 such that for δ small enough, it holds

$$||(\mathrm{F}^*_0 - \mathrm{F}^*_\delta) f_\delta||_1 \leq C_1 \delta (M_2 + 1).$$

Proof. Set $A = A_1 \cap A_2$ where A_1 comes from de definition of $d_{S,n}$ (see equation (49)) and A_2 is from (UBV3) (see definition 8.9). Remark that this sets depend on δ. Let us estimate

$$||(\mathrm{F}_0^* - \mathrm{F}_\delta^*) f_\delta||_1 \leq \int_I ||\mathrm{F}_0^*(1_A f_\delta)|_\gamma - \mathrm{F}_\delta^*(1_A f_\delta)|_\gamma||_W dm(\gamma) + \int_I ||\mathrm{F}_0^*(1_{A^c} f_\delta)|_\gamma - \mathrm{F}_\delta^*(1_{A^c} f_\delta)|_\gamma||_W dm(\gamma). \tag{56}$$

By the assumptions, for a.e. γ, $||f_\delta|\gamma||_W \leq (M_2 + 1)$ and $||1_{A^c} f_\delta||_1 \leq (M_2 + 1)\delta$. Indeed, since $\mathrm{Var}(f_\delta) \leq M_2$, $\forall \delta$, we have (below, we denote $\phi_{x,\delta} = \dfrac{d\pi_x^*(f_\delta)}{dm}$)

$$\begin{aligned} ||f_\delta|_\gamma||_W &\leq ||f_\delta|_\gamma - f_\delta|_{\gamma_2}||_W + ||f_\delta|_{\gamma_2}||_W \\ &= ||f_\delta|_\gamma - f_\delta|_{\gamma_2}||_W + |\phi_{x,\delta}(\gamma_2)|. \end{aligned}$$

Integrating with respect to γ_2 we get

$$||f_\delta|\gamma||_W \leq (M_2 + 1). \tag{57}$$

To prove the inequality $||1_{A^c} f_\delta||_1 \leq (M_2 + 1)\delta$ we use the previous equation, $m(A^c) \leq \delta$ and the fact that (see equation (55))

$$||1_{A^c} f_\delta||_1 = \int_{A^c} ||f_\delta|_\gamma||_W dm.$$

Since F^* is a contraction for the weak norm, we have

$$\int_I ||\mathrm{F}_0^*(1_{A^c} f_\delta)|_\gamma - \mathrm{F}_\delta^*(1_{A^c} f_\delta)|_\gamma||_W dm(\gamma) \leq 2(M_2 + 1)\delta.$$

Now, let us estimate the first summand of (56) by estimating the integral

$$\int ||(\mathrm{F}_0^* \mu - \mathrm{F}_\delta^* \mu)|_\gamma||_W dm(\gamma),$$

where $\mu = 1_A f_\delta$. Denote by $T_{0,i}$, with $0 \leq i \leq q$, the branches of T_0 defined in the sets $P_i \in \mathcal{P}$ and set $T_{\delta,i} = T_\delta|_{P_i \cap A}$. These functions will play the role of the branches for T_δ. Since in A, $T_0 = T_\delta \circ \sigma_\delta$ (where σ_δ is the diffeomorphism in the definition of the Skorokhod distance), then $T_{\delta,i}$ are invertible. Then

$$(\mathrm{F}_0^* \mu - \mathrm{F}_\delta^* \mu)|_\gamma = \sum_{i=1}^q \frac{\mathrm{F}^*_{0,T_{0,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{T_0(P_i \cap A)}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^q \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{T_\delta(P_i \cap A)}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \quad \mu_x\text{-}a.e.\ \gamma \in I.$$

Let us now consider $T_0(P_i \cap A)$, $T_\delta(P_i \cap A)$ and remark that $T_0(P_i \cap A) = \sigma_\delta(T_\delta(P_i \cap A))$ where σ_δ is a diffeomorphism near to the identity. Let us denote $B_i = T_0(P_i \cap A) \cap T_\delta(P_i \cap A)$ and $C_i = T_0(P_i \cap A) \triangle T_\delta(P_i \cap A)$. Then, we have

$$\int_I ||(\mathrm{F}_0^* \mu - \mathrm{F}_\delta^* \mu)|_\gamma||_W dm(\gamma) \leq O_1 + O_2, \tag{58}$$

where

$$O_1 = \int_I \left|\left| \sum_{i=1}^{q} \frac{\mathrm{F}^*_{0,T_{0,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right|\right|_W dm$$

and

$$O_2 = \int_I \left|\left| \sum_{i=1}^{q} \frac{\mathrm{F}^*_{0,T_{0,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{T_0(P_i \cap A) - B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{T_\delta(P_i \cap A) - B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right|\right|_W dm.$$

And since $m(C_i) = O(\delta)$, we [9] get that there is $K_1 \geq 0$ such that $O_2 \leq qK_1(M_2 + 1)\delta$. In order to estimate O_1, we note that

$$\begin{aligned}
O_1 &= \int_I \left|\left| \sum_{i=1}^{q} \frac{\mathrm{F}^*_{0,T_{0,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right|\right|_W dm \\
&\leq \int_I \left|\left| \sum_{i=1}^{q} \frac{\mathrm{F}^*_{0,T_{0,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right|\right|_W dm \\
&+ \int_I \left|\left| \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right|\right|_W dm \\
&= \int_I I(\gamma)\ dm(\gamma) + \int_I II(\gamma)\ dm(\gamma).
\end{aligned}$$

The two summands will be treated separately. Let us denote $\overline{\mu}|_\gamma = \pi^*_{\gamma,y}\mu_\gamma$ (note that $\mu|_\gamma = \phi_\mu(\gamma)\overline{\mu}|_\gamma$ and $\overline{\mu}|_\gamma$ is a probability measure).

$$\begin{aligned}
I(\gamma) &= \left|\left| \sum_{i=1}^{q} \frac{\mathrm{F}^*_{0,T_{0,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right|\right|_W \\
&\leq \left|\left| \sum_{i=1}^{q} \frac{\mathrm{F}^*_{0,T_{0,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} \right|\right|_W \\
&+ \left|\left| \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \right|\right|_W \\
&= I_a(\gamma) + I_b(\gamma).
\end{aligned}$$

[9] Remark that $m(T_\delta(P_i \cap A) \triangle T_0(P_i \cap A)) = O(\delta)$ because $T_\delta(P_i \cap A) = \sigma(T_0(P_i \cap A))$ where σ is a diffeomorphism near to the identity as in the definition of the Skhorokod distance and $P_i \cap A$ is a finite union of intervals whose number is uniformly bounded with respect to δ.

Since f_δ is a probability measure it holds, posing $\beta = T_{0,i}^{-1}(\gamma)$

$$
\begin{aligned}
\int I_a(\gamma)dm &= \int \left|\left|\sum_{i=1}^{q} \frac{\mathrm{F}^*_{0,T_{0,i}^{-1}(\gamma)}\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)}\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|}\right|\right|_W dm(\gamma) \\
&\leq \int \sum_{i=1}^{q} \left|\left|\frac{\mathrm{F}^*_{0,T_{0,i}^{-1}(\gamma)}\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)}\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|}\right|\right|_W dm \\
&\leq \sum_{i=1}^{q} \int \left|\left|\frac{\mathrm{F}^*_{0,T_{0,i}^{-1}(\gamma)}\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)}\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|}\right|\right|_W dm \\
&\leq \sum_{i=1}^{q} \int_{T_{0,i}^{-1}(B_i)} \left|\left|\mathrm{F}^*_{0,\beta}\mu|_\beta - \mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(T_{0,i}(\beta))}\mu|_\beta\right|\right|_W dm(\beta).
\end{aligned}
$$

We remark $T_{0,i}^{-1}(B_i) \subseteq P_i \cap A$ and $T_{\delta,i}^{-1}(T_{0,i}(T_{0,i}^{-1}(B_i))) \subseteq P_i \cap A$. Since $|T_{\delta,i}(\beta) - T_{0,i}(\beta)| \leq \delta$ and $T_{0,i}^{-1}$ is a contraction, then $|T_{0,i}^{-1} \circ T_{\delta,i}(\beta) - \beta| \leq \delta$. Therefore

$$
\begin{aligned}
\left|\left|\mathrm{F}^*_{0,\beta}\mu|_\beta - \mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(T_{0,i}(\beta))}\mu|_\beta\right|\right|_W &\leq \left|\left|\mathrm{F}^*_{0,\beta}\mu|_\beta - \mathrm{F}^*_{\delta,\beta}\mu|_\beta\right|\right|_W \\
&\quad + \left|\left|\mathrm{F}^*_{\delta,\beta}\mu|_\beta - \mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(T_{0,i}(\beta))}\mu|_\beta\right|\right|_W .
\end{aligned}
$$

By (UBV3) and equation (57),

$$\left|\left|\mathrm{F}^*_{0,\beta}\mu|_\beta - \mathrm{F}^*_{\delta,\beta}\mu|_\beta\right|\right|_W \leq \delta(M_2+1).$$

Since $|T_{\delta,i}(\beta) - T_{0,i}(\beta)| \leq \delta$ and $T_{0,i}^{-1}$ is a contraction, we have $|T_{\delta,i}^{-1} \circ T_{0,i}(\beta) - \beta| \leq \delta$. Then,

$$\left|\left|\mathrm{F}^*_{\delta,\beta}\mu|_\beta - \mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(T_{0,i}(\beta))}\mu|_\beta\right|\right|_W \leq H_\delta \delta(M_2+1).$$

When $d(\beta, \cup_i \partial J_i) \geq \delta$ but the set of points $\{x \ s.t. \ d(x, \cup_i \partial J_i) \leq \delta\}$ is of measure bounded by $\delta(\sup_\delta \#\mathcal{P}'_\delta)$, thus

$$\int I_a dm = O(\delta).$$

To estimate $I_b(\gamma)$, we have

$$
\begin{aligned}
I_b(\gamma) &= \left|\left|\sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)}\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{B_i}}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)}\mu|_{T_{0,i}^{-1}(\gamma)}\chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|}\right|\right|_W \\
&\leq \sum_{i=1}^{q} \left|\frac{\chi_{B_i}(\gamma)}{|T'_{0,i}(T_{0,i}^{-1}(\gamma))|} - \frac{\chi_{B_i}(\gamma)}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|}\right| \left|\left|\mathrm{F}^*_{\delta,T_{\delta,i}^{-1}(\gamma)}\mu|_{T_{0,i}^{-1}(\gamma)}\right|\right|_W
\end{aligned}
$$

and

$$\int I_b(\gamma)\ dm(\gamma) \leq |(P_{T_0} - P_{T_\delta})(1)|\,(M_2+1).$$

By [28], Lemma 11.2.1, we get

$$\int_{A_1} I_b(\gamma)\ dm(\gamma) \leq |\phi_x|_\infty |(P_{T_0} - P_{T_\delta})1|_1 \leq 14(M_2+1)\delta.$$

Now, let us estimate the integral of the second summand

$$II(\gamma) = \left|\left|\sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|}\right|\right|_W.$$

Let us make the change of variable $\gamma = T_{\delta,i}(\beta)$.

$$\begin{aligned}
\int_I II(\gamma)\ dm(\gamma) &= \int_I \left|\left|\sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{0,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} - \sum_{i=1}^{q} \frac{\mathrm{F}^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \mu|_{T_{\delta,i}^{-1}(\gamma)} \chi_{B_i}}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|}\right|\right|_W dm(\gamma) \\
&\leq \sum_{i=1}^{q} \int_{B_i} \frac{1}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \left|\left|\mathrm{F}^*_{\delta, T_{\delta,i}^{-1}(\gamma)} \left(\mu|_{T_{0,i}^{-1}(\gamma)} - \mu|_{T_{\delta,i}^{-1}(\gamma)}\right)\right|\right|_W dm(\gamma) \\
&\leq \sum_{i=1}^{q} \int_{B_i} \frac{1}{|T'_{\delta,i}(T_{\delta,i}^{-1}(\gamma))|} \left|\left|\mu|_{T_{0,i}^{-1}(\gamma)} - \mu|_{T_{\delta,i}^{-1}(\gamma)}\right|\right|_W dm(\gamma) \\
&\leq \sum_{i=1}^{q} \int_{T_{\delta,i}^{-1}(B_i)} \left|\left|\mu|_{T_{0,i}^{-1}(T_{\delta,i}(\beta))} - \mu|_\beta\right|\right|_W dm(\beta).
\end{aligned}$$

Since $|T_{\delta,i}(\beta) - T_{0,i}(\beta)| \leq \delta$ and $T_{0,i}^{-1}$ is a contraction, we have $|T_{0,i}^{-1} \circ T_{\delta,i}(\beta) - \beta| \leq \delta$. Hence,

$$\int_I II(\gamma)\ dm(\gamma) \leq \int \sup_{x,y \in B(\beta,\delta)} (||\mu|_x - \mu|_y||_W) dm(\beta)$$

and then

$$\int_I II(\gamma)\ dm(\gamma) \leq 2\delta(M_2+1).$$

Summing all, the statement is proved.

□

8.1.2. *Proof of Proposition 8.4 and Theorem 8.3.* We are ready to prove the following proposition

8.21. **Proposition.** *Let $\{F_\delta\}_{\delta\in[0,1)}$ be an Uniform BV Lorenz-like family and let $\{\mathrm{F}^*_\delta\}_{\delta\in[0,1)}$ be the induced family of transfer operators. Then, $\{\mathrm{F}^*_\delta\}_{\delta\in[0,1)}$ is an uniform family of operators with weak space $(\mathcal{L}^1, ||\cdot||_1)$ and strong space $(\mathcal{BV}_{1,1}, ||\mu||_{1,1})$.*

Proof. To prove UF1, note that, by (UBV1) there exist $0 < \alpha_1 < 1$ and $\overline{D} > 0$ s.t. for all $\mu \in \mathcal{BV}_{1,1}$ and for all δ it holds $|| \operatorname{F}_\delta^{*n} \mu||_{1,1} \leq \overline{D}\alpha_1^n ||\mu||_{1,1} + \overline{D}||\mu||_1$, for all $n \geq 1$. Indeed, by Lemma 5.1 we have

$$\begin{aligned} || \operatorname{F}_\delta^{*n} \mu||_{1,1} &= | \operatorname{P}_{T_\delta}^n \phi_x|_{1,1} + || \operatorname{F}_\delta^{*n} \mu||_1 \\ &\leq D\lambda^n|\phi_x|_{1,1} + D|\phi_x|_1 + ||\mu||_1 \\ &\leq D\lambda^n||\mu||_{1,1} + (D+1)||\mu||_1. \end{aligned}$$

Therefore, if f_δ is a fixed probability measure for the operator ${\operatorname{F}^*}_\delta$, by the above inequality we get UF1 with $M = D + 1$.

Proposition 8.20 and Proposition 8.19 immediately give UF2. The items UF3 and UF4 follow, respectively, from Proposition 7.5 and Lemma 5.1 applied to each F_δ. $\square$

Once this is done, we apply the above result together with Proposition 8.2 to get the quantitative estimation:

8.22. **Theorem** (Quantitative stability for deterministic perturbations)**.** *Let $\{F_\delta\}_{\delta \in [0,1)}$ be an uniform BV Lorenz-like family (see definition 8.9). Denote by f_δ the fixed point of $\operatorname{F}_\delta^*$ in $\mathcal{BV}_{1,1}$ (also in $\mathcal{BV}_{1,1}^\infty$), for all δ. Then, there exists $\delta_0 \in (0,1)$ such that for all $\delta \in [0, \delta_0)$, it holds*

$$||f_\delta - f_0||_1 = O(\delta \log \delta).$$

9. Appendix 1: Proof of Propositions 8.16 and 8.19

In this section, we obtain Proposition 8.16 as a particular case of Theorem 9.2. We also prove Proposition 8.19 stated again as Proposition 9.20.

Note that, for all $\mu \in \mathcal{BV}^+$ it holds $||\mu||_1 = |\phi_x|_1$ and $||\mu||_\infty = |\phi_x|_\infty$, where $\phi_x = \dfrac{d\pi_x^*\mu}{dm}$. We also remark, for each $\mu \in \mathcal{BV}^+$ we have $\phi_x \in BV_{1,1}$.

For a measurable map $F : [0,1]^2 \longrightarrow [0,1]^2$, of the type $F(x,y) = (T(x), G(x,y))$, and a given $\gamma \in \mathcal{F}^s (\gamma = \{x\} \times [0,1])$, we denote by $F_\gamma : [0,1] \longrightarrow [0,1]$, the function defined by

$$F_\gamma = \pi_{\gamma,y} \circ F|_\gamma \circ \pi_{\gamma,y}^{-1}, \tag{59}$$

where $\pi_{\gamma,y}$ is the restriction on γ of the projection $\pi(x,y) = y$.

9.1. **Definition.** Consider a function $f : [0,1]^2 \longrightarrow \mathbb{R}$ and let $x_1 \leq \cdots \leq x_n$ and $y_1 \leq \cdots \leq y_n$ be such that $(x_i)_{i=1}^n \subset I$ and $(y_i)_{i=1}^n \subset I$. We define $\operatorname{var}^\diamond(f, (x_i)_{i=1}^n, (y_i)_{i=1}^n)$ by

$$\operatorname{var}^\diamond(f, (x_i)_{i=1}^n, (y_i)_{i=1}^n) := \sum_{i=1}^n |f(x_{i+1}, y_i) - f(x_i, y_i)|,$$

and

$$\mathrm{var}^{\diamond}(f) := \sup_{(x_i)_{i=1}^n, (y_i)_{i=1}^n} \mathrm{var}^{\diamond}(f, (x_i)_{i=1}^n, (y_i)_{i=1}^n). \tag{60}$$

If $\eta \subset I$ is an interval, we define $\mathrm{var}^{\diamond}_{\eta}(f) = \mathrm{var}^{\diamond}(f|_{\overline{\eta}\times I})$, where $\overline{\eta}$ is the closure of η.

Since preliminaries results are necessary, we postponed the proof of the next theorem to the end of the section.

9.2. **Theorem.** *Let $F(x,y) = (T(x), G(x,y))$ be a measurable transformation such that*

(1) $\mathrm{var}^{\diamond}(G) < \infty$
(2) *F satisfy property $G1$ (hence is uniformly contracting on each leaf γ with rate of contraction α);*
(3) *$T : [0,1] \to [0,1]$ is a piecewise expanding map satisfying the assumptions given in the definition 8.6.*

Then, there are K_0 and $0 < \lambda_0 < 1$ such that for all $\mu \in \mathcal{BV}^+$ and all $n \geq 1$ it holds

$$\mathrm{Var}(\mathrm{F}^{*n}\,\mu) \leq K_0 \lambda_0^n \,\mathrm{Var}(\mu) + K_0 |\phi_x|_{1,1}. \tag{61}$$

9.3. **Remark.** If F_L is a BV Lorenz-like map (definition 8.5), a straightforward computation yields

$$\mathrm{var}^{\diamond}(G_L) \leq H,$$

where H comes from equation (48). This shows that Proposition 8.16 is a direct consequence of Theorem 9.2.

9.1. **Lasota-Yorke Inequality for positive measures.** Henceforth, we fix a positive measure $\mu \in \mathcal{BV}^+ \subset \mathcal{AB}$ and a path which represents μ (i.e. a pair $(\{\mu_\gamma\}_\gamma, \phi_x)$ s.t. $\Gamma^{\omega}_{\mu}(\gamma) := \mu|_\gamma$). To simplify, we will denote the path $\Gamma^{\omega}_{\mu} \in \Gamma_\mu$, just by Γ_μ.

9.4. **Remark.** Consider $T : [0,1] \longrightarrow [0,1]$ a piecewise expanding map from definition 8.6 and $g_i = \dfrac{1}{T_i'}$. For all $n \geq 1$, let $\mathcal{P}^{(n)}$ be the partition of I s.t. $\mathcal{P}^{(n)}(x) = \mathcal{P}^{(n)}(y)$ if and only if $\mathcal{P}^{(1)}(T^j(x)) = \mathcal{P}^{(1)}(T^j(y))$ for all $j = 0, \cdots, n-1$, where $\mathcal{P}^{(1)} = \mathcal{P}$ (see definition 8.6). Given $P \in \mathcal{P}^{(n)}$, define $g_P^{(n)} = \frac{1}{|T^{n\prime}|_P|}$. Item 2) implies that there exists $C_1 > 0$ and $0 < \theta < 1$ s.t.

$$\sup\{g_P^{(n)}\} \leq C_1 \theta^n, \text{ for all } P \in \mathcal{P}^{(n)} \text{ and all } n \geq 1. \tag{62}$$

Moreover, equation (62) and some basic properties of real valued BV functions imply (see [27], page 41, equation (3.1)) there exists $\lambda_2 \in (\theta, 1)$ and $C_2 > 0$ such that

$$\mathrm{var}(g_P^{(n)}) \leq C_2 \lambda_2^n, \text{ for all } P \in \mathcal{P}^{(n)} \text{ and all } n \geq 1.$$

Then, there is an iterate of F, $\widetilde{F} := F^k$, such that T^k satisfies

$$\beta_k := \operatorname{var} g_P^{(k)} + 3 \sup g_P^{(k)} < 1, \ \forall P \in \mathcal{P}^{(k)}. \tag{63}$$

We also remark that $G^k := \pi_y \circ F^k$ also satisfies

$$\operatorname{var}^\diamond(G^k) < \infty. \tag{64}$$

Next lemma provides equation (64) and its proof can be found in [2].

9.5. **Lemma.** *If F satisfy definition 8.5, then for all $n \geq 1$ it holds* [10]

$$\operatorname{var}^\diamond(f \circ F^n) \leq q^n \operatorname{var}^\diamond(f) + \sum_{i=1}^{n-1} q^i \left(\operatorname{var}^\diamond(G)|f|_{lip'} + 2q|f|_\infty\right),$$

where q is the number of branches of T ($q := \#\mathcal{P}$).

Recalling equation (59), set

$$\Gamma_{\mu_{\mathrm{F}}}(\gamma) := \mathrm{F}^*_\gamma \Gamma_\mu(\gamma). \tag{65}$$

With the above notation and following the strategy of the proof of Lemma 4.1, we have that the path $\Gamma_{\mathrm{F}^*\mu}$, defined on a full measure set by

$$\Gamma_{\mathrm{F}^*\mu}(\gamma) = \sum_{i=1}^{q} \left(g_i \cdot \Gamma_{\mu_{\mathrm{F}}}\right) \circ T_{L_i}^{-1}(\gamma) \cdot \chi_{T_L(P_i)}(\gamma), \text{ where } g_i = \frac{1}{T'_{L_i}}, \tag{66}$$

represents the measure $\mathrm{F}^* \mu$.

By Lemma 5.2 and equation (59) it holds

$$|| \mathrm{F}^*_\gamma \Gamma_\mu(\gamma)||_W \leq ||\Gamma_\mu(\gamma)||_W,$$

for m-a.e. $\gamma \in I$. Then we have the following

9.6. **Lemma.** *Let γ_1 and γ_2 be two leaves such that $G(\gamma_i, \cdot) : I \longrightarrow I$ is a contraction, $i = 1, 2$. Then for every path Γ_μ, where $\mu \in \mathcal{AB}$, it holds*

$$|| \mathrm{F}^*_{\gamma_1} \Gamma_\mu(\gamma_1) - \mathrm{F}^*_{\gamma_2} \Gamma_\mu(\gamma_2)||_W \leq ||\Gamma_\mu(\gamma_1) - \Gamma_\mu(\gamma_2)||_W + |G(\gamma_1, y_0) - G(\gamma_2, y_0)||\phi_x|_\infty, \tag{67}$$

for some $y_0 \in I$.

Proof. Consider g such that $|g|_\infty \leq 1$ and $Lip(g) \leq 1$, and observe that since $G_{\gamma_1} - G_{\gamma_2} : I \longrightarrow I$ is continuous, it holds

$$\sup_I |G(\gamma_1, y) - G(\gamma_2, y)| = |G(\gamma_1, y_0) - G(\gamma_2, y_0)|,$$

for some $y_0 \in I$. Moreover, by equation (5.2) we have

[10] $|f|_{lip'} = |f|_\infty + Lip_y(f)$, where $Lip_y(f) = \sup_{x, y_1, y_2 \in [0,1]} \frac{|f(x,y_2) - f(x,y_1)|}{|y_2 - y_1|}$.

$$
\begin{aligned}
\left|\int gd\,\Gamma_{\mu_{\mathrm{F}}}(\gamma_1) - \int gd\Gamma_{\mu_{\mathrm{F}}}(\gamma_2)\right| &= \left|\int gd\,\mathrm{F}^*_{\gamma_1}\,\Gamma_\mu(\gamma_1) - \int gd\,\mathrm{F}^*_{\gamma_2}\,\Gamma_\mu(\gamma_2)\right| \\
&\leq \left|\int gd\,\mathrm{F}^*_{\gamma_1}\,\Gamma_\mu(\gamma_1) - \int gd\,\mathrm{F}^*_{\gamma_1}\,\Gamma_\mu(\gamma_2)\right| \\
&+ \left|\int gd\,\mathrm{F}^*_{\gamma_1}\,\Gamma_\mu(\gamma_2) - \int gd\,\mathrm{F}^*_{\gamma_2}\,\Gamma_\mu(\gamma_2)\right| \\
&\leq \left|\left|\mathrm{F}^*_{\gamma_1}(\Gamma_\mu(\gamma_1) - \Gamma_\mu(\gamma_2))\right|\right|_W \\
&+ \int \left|g(F_{\gamma_1}) - g(F_{\gamma_2})\right| d\mu|_{\gamma_2} \\
&\leq ||\Gamma_\mu(\gamma_1) - \Gamma_\mu(\gamma_2)||_W \\
&+ \int |G(\gamma_1, y) - G(\gamma_2, y)| d\mu|_{\gamma_2(y)} \\
&\leq ||\Gamma_\mu(\gamma_1) - \Gamma_\mu(\gamma_2)||_W \\
&+ \sup_I |G(\gamma_1, y) - G(\gamma_2, y)| \int 1 d\mu|_{\gamma_2(y)} \\
&= ||\Gamma_\mu(\gamma_1) - \Gamma_\mu(\gamma_2)||_W + |G(\gamma_1, y_0) - G(\gamma_2, y_0)|\, |\phi_x|_\infty.
\end{aligned}
$$

Taking the supremum over g such that $|g|_\infty \leq 1$ and $L(g) \leq 1$, we finish the proof. □

The proofs of the next two lemmas are straightforward and analogous to the one dimensional BV functions. So, we omit them (details can be found in [23]).

9.7. **Lemma.** *Given paths $\Gamma_{\mu_0}, \Gamma_{\mu_1}$ and Γ_{μ_2} (where $\Gamma_{\mu_i}(\gamma) = \mu_i|_\gamma$) representing the positive measures $\mu_0, \mu_1, \mu_2 \in \mathcal{BV}^+$ respectively, a function $\varphi : I \longrightarrow \mathbb{R}$, an homeomorphism $h : \eta \subset I \longrightarrow h(\eta) \subset I$ and a subinterval $\eta \subset I$, then the following properties hold*

P1) *If $\mathcal{P}$ is a partition of I by intervals η, then*

$$\mathrm{Var}(\Gamma_{\mu_0}) = \sum_\eta \mathrm{Var}_{\overline{\eta}}(\Gamma_{\mu_0});$$

P2) $\mathrm{Var}_{\overline{\eta}}(\Gamma_{\mu_1} + \Gamma_{\mu_2}) \leq \mathrm{Var}_{\overline{\eta}}(\Gamma_{\mu_1}) + \mathrm{Var}_{\overline{\eta}}(\Gamma_{\mu_2})$

P3) $\mathrm{Var}_{\overline{\eta}}(\varphi \cdot \Gamma_{\mu_0}) \leq \left(\sup_{\overline{\eta}} |\varphi|\right) \cdot \left(\mathrm{Var}_{\overline{\eta}}(\Gamma_{\mu_0})\right) + \left(\sup_{\gamma \in \overline{\eta}} ||\Gamma_{\mu_0}(\gamma)||_W\right) \cdot \mathrm{var}_{\overline{\eta}}(\varphi)$

P4) $\mathrm{Var}_{\overline{\eta}}(\Gamma_{\mu_0} \circ h) = \mathrm{Var}_{\overline{h(\eta)}}(\Gamma_{\mu_0})$.

9.8. **Remark.** For every path Γ_μ, $\mu \in \mathcal{AB}$ and an interval $\eta \subset I$, it holds

$$\sup_{\gamma \in \overline{\eta}} ||\Gamma_\mu(\gamma)||_W \quad \leq \quad \mathrm{Var}_{\overline{\eta}}(\Gamma_\mu) + \frac{1}{m(\overline{\eta})} \int_{\overline{\eta}} ||\Gamma_\mu(\gamma)||_W dm(\gamma),$$

where $\overline{\eta}$ is the closure of η.

9.9. **Lemma.** *For all Γ_μ, where $\mu \in \mathcal{BV}^+$, and all $P \in \mathcal{P}$ it holds*

$$\mathrm{Var}_{\overline{P}}(\Gamma_{\mu_{\mathrm{F}}}) \leq \mathrm{Var}_{\overline{P}}(\Gamma_\mu) + \mathrm{var}^{\diamond}_{\overline{P}}(G)|\phi_x|_\infty.$$

Proof. Consider $(\gamma_i)_{i=1}^n \subset \overline{P}$ such that $\gamma_1 \leq \cdots \leq \gamma_n$. By Lemma 9.6, for every i there is y_i such that

$$\begin{aligned} \sum_{i=1}^{n} ||\, \mathrm{F}^*_{\gamma_{i+1}} \Gamma_\mu(\gamma_{i+1}) - \mathrm{F}^*_{\gamma_i} \Gamma_\mu(\gamma_i)||_W &\leq \sum_{i=1}^{n} ||\Gamma_\mu(\gamma_{i+1}) - \Gamma_\mu(\gamma_i)||_W + \sum_{i=1}^{n} |G(\gamma_{i+1}, y_i) - G(\gamma_i, y_i)||\phi_x|_\infty \\ &\leq \sum_{i=1}^{n} ||\Gamma_\mu(\gamma_{i+1}) - \Gamma_\mu(\gamma_i)||_W + |\phi_x|_\infty \, \mathrm{var}^{\diamond}_{\overline{\eta}}(G). \end{aligned}$$

Then,

$$\sum_{i=1}^{n} ||\, \mathrm{F}^*_{\gamma_{i+1}} \Gamma_\mu(\gamma_{i+1}) - \mathrm{F}^*_{\gamma_i} \Gamma_\mu(\gamma_i)||_W \leq \mathrm{Var}_{\overline{P}}(\Gamma_\mu) + |\phi_x|_\infty \, \mathrm{var}^{\diamond}_{\overline{P}}(G).$$

We finish the proof taking the supremum over $(\gamma_i)_i^n$. $\square$

9.10. **Lemma.** *For all path Γ_μ, where $\mu \in \mathcal{BV}^+$, it holds*

$$\mathrm{Var}(\Gamma_{\mathrm{F}^* \mu}) \leq \sum_{i=1}^{q} \left[\mathrm{var}_{\overline{P_i}}(g_i) + 2 \sup_{\overline{P_i}} g_i \right] \cdot \sup_{\gamma \in \overline{P_i}} ||\Gamma_\mu(\gamma)||_W + \sup_{\overline{P_i}} g_i \cdot \mathrm{Var}_{\overline{P_i}}(\Gamma_{\mu_{\mathrm{F}}}),$$

where $\Gamma_{\mu_{\mathrm{F}}}$ is defined by equation (65).

Proof. Using the properties P2, P3, P4, $\sup_{\gamma \in \overline{P_i}} ||\Gamma_{\mu_{\mathrm{F}}}(\gamma)||_W \leq \sup_{\gamma \in \overline{P_i}} ||\Gamma_{\mu}(\gamma)||_W$ and $\sup_{\gamma \in \overline{P_i}} |g_i| = \sup_{\gamma \in \overline{P_i}} g_i$, we have

$$
\begin{aligned}
\mathrm{Var}(\Gamma_{\mathrm{F}^*\mu}) &\leq \sum_{i=1}^{q} \mathrm{Var}_{\overline{T_i(P_i)}} \left[\left(g_i \cdot \Gamma_{\mu_{\mathrm{F}}} \right) \circ T_i^{-1} \cdot \chi_{T(P_i)} \right] \\
&\leq \sum_{i=1}^{q} \mathrm{Var}_{\overline{T_i(P_i)}} \left[\left(g_i \cdot \Gamma_{\mu_{\mathrm{F}}} \right) \circ T_i^{-1} \right] \cdot \sup |\chi_{T(P_i)}| \\
&+ \sum_{i=1}^{q} \sup_{\overline{T_i(P_i)}} || \left(g_i \cdot \Gamma_{\mu_{\mathrm{F}}} \right) \circ T_i^{-1} ||_W \cdot \mathrm{var}(\chi_{T(P_i)}) \\
&\leq \sum_{i=1}^{q} \mathrm{Var}_{\overline{P_i}} \left(g_i \cdot \Gamma_{\mu_{\mathrm{F}}} \right) + 2 \cdot \sup_{T_i(P_i)} || \left(g_i \cdot \Gamma_{\mu_{\mathrm{F}}} \right) \circ T_i^{-1} ||_W \\
&\leq \sum_{i=1}^{q} \mathrm{var}_{\overline{P_i}}(g_i) \cdot \sup_{\overline{P_i}} ||\Gamma_{\mu_{\mathrm{F}}}||_W + \mathrm{Var}_{\overline{P_i}}(\Gamma_{\mu_{\mathrm{F}}}) \cdot \sup_{\overline{P_i}} |g_i| \\
&+ 2 \cdot \sum_{i=1}^{q} \sup_{\overline{P_i}} |g_i| \sup_{\overline{P_i}} ||\Gamma_{\mu_{\mathrm{F}}}||_W \\
&\leq \sum_{i=1}^{q} \mathrm{var}_{\overline{P_i}}(g_i) \cdot \sup_{\gamma \in \overline{P_i}} ||\Gamma_{\mu}(\gamma)||_W + \mathrm{Var}_{\overline{P_i}}(\Gamma_{\mu_{\mathrm{F}}}) \cdot \sup_{\overline{P_i}} |g_i| \\
&+ 2 \cdot \sum_{i=1}^{q} \sup_{\gamma \in \overline{P_i}} ||\Gamma_{\mu}(\gamma)||_W \cdot \sup_{\overline{P_i}} |g_i| \\
&\leq \sum_{i=1}^{q} \left[\mathrm{var}_{\overline{P_i}}(g_i) + 2 \sup_{\overline{P_i}} g_i \right] \cdot \sup_{\gamma \in \overline{P_i}} ||\Gamma_{\mu}(\gamma)||_W + \sup_{\overline{P_i}} g_i \cdot \mathrm{Var}_{\overline{P_i}}(\Gamma_{\mu_{\mathrm{F}}}).
\end{aligned}
$$

□

9.11. **Lemma.** *For all path Γ_μ, where $\mu \in \mathcal{BV}^+$, it holds*

$$\mathrm{Var}(\Gamma_{\mathrm{F}^*\mu}) \leq \beta \,\mathrm{Var}(\Gamma_\mu) + K_3 |\phi_x|_{1,1}.$$

Where

$$\beta := \max_{i=1,\cdots,q} \{ \mathrm{var}_{\overline{P_i}}(g_i) + 3 \sup_{\overline{P_i}} g_i \}$$

and

$$K_3 = \max_{i=1,\cdots,q} \{ \sup_{\overline{P_i}} g_i \} \,\mathrm{var}^{\diamond}(G) + \max_{i=1,\cdots,q} \left\{ \frac{\mathrm{var}_{\overline{P_i}}(g_i) + 2 \sup_{\overline{P_i}} g_i}{m(P_i)} \right\}.$$

Proof. By lemma 9.9, remark 9.8, lemma 9.10, P1, equation (63) of remark 9.4 and by $\sum_{i=1}^{q} \mathrm{var}^{\diamond}_{\overline{P_i}} G = \mathrm{var}^{\diamond}(G)$, we get

$$\begin{aligned}
\mathrm{Var}(\Gamma_{\mathrm{F}^*\mu}) &\leq \sum_{i=1}^{q}\left[\mathrm{var}_{\overline{P_i}}(g_i) + 2\sup_{\overline{P_i}} g_i\right]\sup_{\gamma\in\overline{P_i}}||\mu|_\gamma||_W + \sup_{\overline{P_i}} g_i \cdot \mathrm{Var}_{\overline{P_i}}(\Gamma_{\mu_{\mathrm{F}}}) \\
&\leq \sum_{i=1}^{q}\left[\mathrm{var}_{\overline{P_i}}(g_i) + 2\sup_{\overline{P_i}} g_i\right]\left(\mathrm{Var}_{\overline{P_i}}(\Gamma_\mu) + \frac{1}{m(\overline{P_i})}\int_{\overline{P_i}}||\mu|_\gamma||_W \, dm(\gamma)\right) \\
&+ \sum_{i=1}^{q}\sup_{\overline{P_i}} g_i \left(\mathrm{Var}_{\overline{P_i}}(\Gamma_\mu) + \mathrm{var}^{\diamond}_{\overline{P_i}}(G)|\phi_x|_\infty\right) \\
&\leq \sum_{i=1}^{q}\left[\mathrm{var}_{\overline{P_i}}(g_i) + 3\sup_{\overline{P_i}} g_i\right]\mathrm{Var}_{\overline{P_i}}(\Gamma_\mu) \\
&+ \sum_{i=1}^{q}\left[\mathrm{var}_{\overline{P_i}}(g_i) + 2\sup_{\overline{P_i}} g_i\right]\frac{1}{m(\overline{P_i})}\int_{\overline{P_i}}||\mu|_\gamma||_W \, dm(\gamma) \\
&+ |\phi_x|_\infty \max_{i=1,\cdots,q}\{\sup_{\overline{P_i}} g_i\}\,\mathrm{var}^{\diamond}(G) \\
&\leq \sum_{i=1}^{q}\left[\mathrm{var}_{\overline{P_i}}(g_i) + 3\sup_{\overline{P_i}} g_i\right]\mathrm{Var}_{\overline{P_i}}(\Gamma_\mu) \\
&+ \max_{i=1,\cdots,q}\{\frac{\mathrm{var}_{\overline{P_i}}(g_i) + 2\sup_{\overline{P_i}} g_i}{m(\overline{P_i})}\}|\phi_x|_1 \\
&+ |\phi_x|_\infty \max_{i=1,\cdots,q}\{\sup_{\overline{P_i}} g_i\}\,\mathrm{var}^{\diamond}(G) \\
&\leq \beta\,\mathrm{Var}(\Gamma_\mu) + K_3|\phi_x|_\infty \\
&\leq \beta\,\mathrm{Var}(\Gamma_\mu) + K_3|\phi_x|_{1,1}.
\end{aligned}$$

□

Taking the infimum over all paths $\Gamma_\mu^\omega \in \Gamma_\mu$ we arrive at the following

9.12. **Proposition.** *If $F : [0,1]^2 \longrightarrow [0,1]^2$ satisfies all the hypothesis of Theorem 9.2. Then, there exist $\beta > 0$ and $K_3 > 0$ such that for every $\mu \in \mathcal{BV}^+$, it holds*

$$\mathrm{Var}(\mathrm{F}^*\mu) \leq \beta\,\mathrm{Var}(\mu) + K_3|\phi_x|_{1,1}. \tag{68}$$

9.13. **Remark.** Remember that, the coefficients of inequality (68) are given by the formulas

$$\beta = \max_i\{\mathrm{var}_{\overline{P_i}}(g_i) + 3\sup_{\overline{P_i}} g_i\} \tag{69}$$

and

$$K_3 = \max_i\{\sup_{\overline{P_i}} g_i\}\,\mathrm{var}^{\diamond}(G) + \max_i\left\{\frac{\mathrm{var}_{\overline{P_i}}(g_i) + 2\sup_{\overline{P_i}} g_i}{m(P_i)}\right\}. \tag{70}$$

We will use these expressions in the next result and later on.

9.14. **Proposition.** *If $F : [0,1]^2 \longrightarrow [0,1]^2$ satisfies all the hypothesis of Theorem 9.2. Then, there exist $k \in \mathbb{N}$, $0 < \beta_k < 1$ and $C_k > 0$ such that for every $\mu \in \mathcal{BV}^+$, it holds*

$$\operatorname{Var}(\mathrm{F}^{*\mathrm{k}}\mu) \leq \beta_k \operatorname{Var}(\mu) + C_k |\phi_x|_{1,1}. \tag{71}$$

Proof. The proof is a straightforward consequence of the above Remark 9.13 and Remark 9.4, where β_k was defined by equation (63). □

9.15. **Proposition.** *If $F : [0,1]^2 \longrightarrow [0,1]^2$ satisfies all the hypothesis of Theorem 9.2. Then, there exist $k \in \mathbb{N}$, C_0 and $0 < \beta_k < 1$ such that for all $\mu \in \mathcal{BV}^+$ and all $n \geq 1$ it holds (denote $\widetilde{F} := F^k$)*

$$\operatorname{Var}(\widetilde{\mathrm{F}}^{*n}\mu) \leq C_0 \beta_k^n \operatorname{Var}(\mu) + C_0 |\phi_x|_{1,1}. \tag{72}$$

Proof. Inequality (40) gives us

$$|\,\mathrm{P}_T^n f|_{1,1} \leq B_3 \beta_2^n |f|_{1,1} + C_2 |f|_1, \quad \forall n, \quad \forall f \in BV_{1,1}, \tag{73}$$

for $B_3, C_2 > 0$ and $0 < \beta_2 < 1$. Then, since $|f|_1 \leq |f|_{1,1}$, it holds

$$|\,\mathrm{P}_T^n f|_{1,1} \leq K_2 |f|_{1,1}, \quad \forall n, \quad \forall f \in BV_{1,1}, \tag{74}$$

where

$$K_2 = B_3 + C_2. \tag{75}$$

In particular, inequality (74) holds if we replace f by $\phi_x = \dfrac{d(\pi_{\mathrm{x}}^* \mu)}{dm}$ for each $\mu \in \mathcal{BV}^+$.

By inequality (74), Proposition 9.14 and a straightforward induction we have

$$\operatorname{Var}(\widetilde{\mathrm{F}}^{*\mathrm{n}}\mu) \leq \beta_k^n \operatorname{Var}(\mu) + C_k \max\{K_2, 1\} \sum_{i=0}^{n-1} \beta_k^i |\phi_x|_{1,1}, \quad \forall n \geq 0. \tag{76}$$

We finish the proof by setting

$$C_0 := \max\left\{1, \frac{C_k \max\{K_2, 1\}}{1 - \beta_k}\right\}. \tag{77}$$

□

Now we present the proof of Theorem 9.2.

Proof. (of Theorem 9.2)

Let $k \in \mathbb{N}$ be from Proposition 9.15. For a given n, we set $n = kq_n + r_n$, where $0 \leq r_n < k$. Applying Proposition 9.12 and iterating r_n times the inequality (68) we have

$$\operatorname{Var}(\mathrm{F}^{*\mathrm{r_n}}\mu) \leq \max_{i=0,\cdots,k}\{\beta^i\} \operatorname{Var}(\mu) + K_3K_2 \sum_{j=0}^{k} \beta^j |\phi_x|_{1,1}, \tag{78}$$

where K_2 was defined in equation (74). Thus, by Proposition 9.15 and the above inequality (78), we have

$$\begin{aligned}
\operatorname{Var}(\mathrm{F}^{*\mathrm{n}}\,\mu) &= \operatorname{Var}(\mathrm{F}^{*\mathrm{kq_n+r_n}}\,\mu) \\
&\leq C_0\beta_k^{q_n} \operatorname{Var}(\mathrm{F}^{*\mathrm{r_n}}\mu) + C_0|\phi_x|_{1,1} \\
&\leq C_0 \max_{i=0,\cdots,k}\{\beta^i\}\beta_k^{q_n} \operatorname{Var}(\mu) + \left[C_0\beta_k^{q_n}K_3K_2\sum_{j=0}^{k}\beta^j + C_0\right]|\phi_x|_{1,1} \\
&\leq C_0 \max_{i=0,\cdots,k}\{\beta^i\}\beta_k^{\frac{n-r_n}{k}} \operatorname{Var}(\mu) + \left[C_0K_3K_2\sum_{j=0}^{k}\beta^j + C_0\right]|\phi_x|_{1,1} \\
&\leq K_0\lambda_0^n \operatorname{Var}(\mu) + K_0|\phi_x|_{1,1},
\end{aligned}$$

where

$$K_0 = \max\left\{\frac{C_0 \max_{i=0,\cdots,k}\{\beta^i\}}{\beta_k}, C_0K_3K_2\sum_{j=0}^{k}\beta^j + C_0\right\} \tag{79}$$

and

$$\lambda_0 = (\beta_k)^{\frac{1}{k}}. \tag{80}$$

□

9.1.1. *Uniform Lasota-Yorke like inequality.*

9.16. **Proposition.** *If $\{F_\delta\}_{\delta\in[0,1)}$ is a BV Lorenz-like family. Then, there exist uniform constants $\beta_u > 0$ and $K_u > 0$ such that for every $\mu \in \mathcal{BV}^+$, it holds*

$$\operatorname{Var}(\mathrm{F}_\delta^*\mu) \leq \beta_u \operatorname{Var}(\mu) + K_u|\phi_x|_{1,1},\ \forall\delta \in [0,1). \tag{81}$$

Proof. Since $\operatorname{var}^\diamond(G_\delta) \leq H_\delta$, we can apply Proposition 9.12 to each F_δ to get (see Remark 9.13)

$$\operatorname{Var}(\mathrm{F}_\delta^*\mu) \leq \beta_\delta \operatorname{Var}(\mu) + K_{3,\delta}|\phi_x|_{1,1},\ \forall\delta \in [0,1), \tag{82}$$

where

$$\beta_\delta = \max_{i=1,\cdots,q}\{\mathrm{var}_{\overline{P_i}}(g_{i\delta}) + 3\sup_{\overline{P_i}} g_{i\delta}\} \tag{83}$$

and

$$K_{3,\delta} = \max_i\{\sup_{\overline{P_i}} g_{i,\delta}\}\,\mathrm{var}^\diamond(G_\delta) + \max_i\left\{\frac{\mathrm{var}_{\overline{P_i}}(g_{i,\delta}) + 2\sup_{\overline{P_i}} g_{i,\delta}}{m(P_i)}\right\}. \tag{84}$$

Since $\mathrm{var}^\diamond(G_\delta) \leq H_\delta$, UBV4 ((2), (3), (4)) yields the existence of uniforms constants $\beta_u := \sup_{\delta\in[0,1)}\beta_\delta < \infty$ and $K_u := \sup_{\delta\in[0,1)} K_{3,\delta} < \infty$. □

Note that, we do not necessarily have $\beta_u < 1$. In what follows, we will prove that there exists an uniform $k \in \mathbb{N}$ such that this property is satisfied for the map F_δ^k, for all $\delta \in [0,1)$. We also remark that, if $\{F_\delta\}_{\delta\in[0,1)}$ is a BV Lorenz-like family, then F_δ^n also satisfies the hypothesis of Theorem 9.2, for all $n \geq 1$ and all δ, in a way that we can apply Proposition 9.12 to F_δ^n, for all $n \geq 1$.

9.17. **Lemma.** *Let $\{T_\delta\}_{\delta\in[0,1)}$ be a family of piecewise expanding maps satisfying Definition 8.6, item (1), item (2), item (3) and item (4) of UBV4 (see Definition 8.9). Then, there is k (which does not depends on δ) such that*

$$\sup_{\delta\in[0,1)}\max_i\{\mathrm{var}\, g_{i,\delta}^{(k)} + 3\sup g_{i,\delta}^{(k)}\} < 1.$$

Proof. (of the Lemma)

First of all, consider a piecewise expanding map, $T : [0,1] \longrightarrow [0,1]$ satisfying Definition 8.6. For all $n \geq 1$, let $\mathcal{P}^{(n)}$ be the partition of I s.t. $\mathcal{P}^{(n)}(x) = \mathcal{P}^{(n)}(y)$ if and only if $\mathcal{P}^{(1)}(T^j(x)) = \mathcal{P}^{(1)}(T^j(y))$ for all $j = 0,\cdots,n-1$, where $\mathcal{P}^{(1)} = \mathcal{P}$. For each n define $T_i^n = T^n|P_i$ and $g_i^{(n)} = \dfrac{1}{|T_i^n|}$, for all $P_i \in \mathcal{P}^{(n)}$.

Let us consider n_0 and λ_1 from item 2) of Definition 8.6: $\inf|T_L^{n_0\prime}| \geq \lambda_1 > 1$. For a given $n \geq 1$, we write $n = n_0 q_n + r_n$, where $0 \leq r_n < n_0$. Thus, for all $x \in P_i \in \mathcal{P}^{(n)} = \{P_1,\cdots,P_{q(n)}\}$, we have

$$\begin{aligned}
\left|T_i^{n\prime}(x)\right| &= \left|\left(T_i^{n_0q_n+r_n}\right)'(x)\right| \\
&= \left|\left(T_i^{n_0q_n+r_n}\right)'(x)\right| \\
&= \left|(T_i^{n_0q_n})'(T_i^{r_n}(x))\right|\left|(T_i^{r_n})'(x)\right| \\
&\geq (\lambda_1)^{q_n}\left|(T_i^{r_n})'(x)\right|.
\end{aligned}$$

Then,

$$\begin{aligned} g_i^{(n)}(x) &\leq \left(\frac{1}{\lambda_1}\right)^{q_n} \frac{1}{|(T_i^{r_n})'(x)|} \\ &\leq \left(\frac{1}{\lambda_1}\right)^{\frac{n}{n_0}-1} \max_{0\leq j\leq n_0} \sup\{g_i\}^j \\ &\leq \left(\frac{1}{\lambda_1}\right)^{\frac{n}{n_0}-1} \max_{0\leq j\leq n_0} \sup\{g_i\}^j \\ &\leq \lambda_4^n C_5, \end{aligned}$$

where $\lambda_4 = \dfrac{1}{\sqrt[n_0]{\lambda_1}} < 1$ and $C_5 = \lambda_1 \max_{0\leq i\leq q}\{\max_{0\leq j\leq n_0} \sup\{g_i\}^j\}$. Therefore,

$$\sup\{g_i^{(n)}\} \leq \lambda_4^n C_5, \tag{85}$$

for all $n \geq 1$ and all i.

Now, set $C_6 := \max\{C_5, \max_i\{\operatorname{var}(g_i)\}\}$. Thus, for all $n \geq 1$ it holds (see [27], page 41, equation (3.1))

$$\operatorname{var} g_i^{(n)} \leq \frac{nC_6^3}{\lambda_4}\lambda_4^n \ \forall\delta \in [0,1) \text{ and } \forall i = 1, \cdots q. \tag{86}$$

Then,

$$\operatorname{var} g_i^{(n)} \leq C_7\lambda_5^n, \ \forall n \geq 1, \ \forall i, \tag{87}$$

where $\lambda_5 \in (\lambda_4, 1)$ and $C_7 := \sup_{n\geq 1}\left\{\dfrac{C_6^3}{\lambda_4} n \left(\dfrac{\lambda_4}{\lambda_5}\right)^n\right\}$.

Now, let us consider a family of piecewise expanding maps, $\{T_\delta\}_{\delta\in[0,1)}$, satisfying Definition 8.6, item (1), item (2), item (3) and item (4) of UBV4 (see Definition 8.9). Applying the above equations to T_δ we get, for all i and all δ

$$\sup\{g_{i,\delta}^{(n)}\} \leq \lambda_{4,\delta}^n C_{5,\delta},$$

where $\lambda_{4,\delta} = \dfrac{1}{\sqrt[n_0(\delta)]{\lambda_1(\delta)}}$ and $C_{5,\delta} = \lambda_1(\delta)\max_i\{\max_{0\leq j\leq n_0(\delta)} \sup\{g_{i,\delta}\}^j\}$. By item (1) of UBV4, we get

$$\lambda_{4,u} := \sup_{\delta\in[0,1)}\{\lambda_{4,\delta}\} = \sup_\delta\{\frac{1}{\sqrt[n_0(\delta)]{\lambda_1(\delta)}}\} < 1$$

and by items (1) and (2) of UBV4 is holds

$$C_{5,u} := \sup_{\delta\in[0,1)} C_{5,\delta} < \infty.$$

Then, we get the uniform estimate

$$\sup\{g_{i,\delta}^{(n)}\} \leq \lambda_{4,u}^n C_{5,u}, \tag{88}$$

for all δ, all i and all $n \geq 1$.

By item (2) of UBV4, set $C_{6,u} := \max\{C_{5,u}, \sup_\delta \max_i\{\mathrm{var}(g_{i,\delta})\}\}$. Thus, for all $n \geq 1$ it holds

$$\mathrm{var}\, g_{i,\delta}^{(n)} \leq \frac{nC_{6,u}^3}{\lambda_{4,u}} \lambda_{4,u}^n \ \forall i \text{ and } \forall \delta \in [0,1)\ . \tag{89}$$

Then,

$$\mathrm{var}\, g_{i,\delta}^{(n)} \leq C_{7,u} \lambda_{5,u}^n, \ \ \forall n \geq 1, \ \ \forall i, \forall \delta \tag{90}$$

where $\lambda_{5,u} \in (\lambda_{4,u}, 1)$ and $C_{7,u} := \sup_{n\geq 1} \left\{ \dfrac{C_{6,u}^3}{\lambda_{4,u}} n \left(\dfrac{\lambda_{4,u}}{\lambda_{5,u}} \right)^n \right\}$.

□

9.18. **Proposition.** *If $\{F_\delta\}_{\delta \in [0,1)}$ is a BV Lorenz-like family. Then, there exist uniform constants $0 < \lambda_u < 1$, $C_u > 0$ and $k \in \mathbb{N}$ such that for every $\mu \in \mathcal{BV}^+$, it holds*

$$\mathrm{Var}(\mathrm{F}_\delta^{*\mathrm{k}} \mu) \leq \lambda_u \,\mathrm{Var}(\mu) + C_u |\phi_x|_{1,1}, \ \ \forall \delta \in [0,1). \tag{91}$$

Proof. Consider the iterate F_δ^k, where $k \in \mathbb{N}$ is from Lemma 9.17. Applying Proposition 9.12, we get

$$\mathrm{Var}(\mathrm{F}_\delta^{*\mathrm{k}} \mu) \leq \beta_\delta \,\mathrm{Var}(\mu) + K_{3,\delta} |\phi_x|_{1,1}$$

where

$$\beta_\delta := \max_i \{\mathrm{var}\, g_{i,\delta}^{(k)} + 3 \sup g_{i,\delta}^{(k)}\}, \tag{92}$$

and

$$K_{3,\delta} := \max_i \{\sup_{\overline{P_i}} g_{i,\delta}^{(k)}\} \,\mathrm{var}^\diamond(G_\delta^k) + \max_i \left\{ \frac{\mathrm{var}_{\overline{P_i}}(g_{i,\delta}^{(k)}) + 2 \sup_{\overline{P_i}} g_{i,\delta}^{(k)}}{m(P_i)} \right\}. \tag{93}$$

By Lemma 9.5, replacing f by π_y, we have

$$\begin{aligned} \mathrm{var}^\diamond(G_\delta^k) &\leq q^k \sum_{j=1}^k q^j \{2\,\mathrm{var}^\diamond(G_\delta) + 2q\} \\ &\leq q^k \sum_{j=1}^k q^j \{2H_\delta + 2q\}. \end{aligned}$$

Since by item (4) of UBV4 we have $\sup_{\delta \in [0,1)} H_\delta < \infty$, we get $\sup_{\delta\in[0,1)} \mathrm{var}^\diamond(G_\delta^k) < \infty$. By the previous comments, item (2) and item (3) of UBV4, we define

$$C_u := \sup_{\delta\in[0,1)} \{K_{3,\delta}\} < \infty.$$

We also set

$$\lambda_u := \sup_{\delta\in[0,1)} \{\beta_\delta\},$$

where, by Lemma 9.17, it holds $\lambda_u < 1$. With these definitions we arrive at inequality (91).

□

9.19. **Proposition.** *If $\{F_\delta\}_{\delta\in[0,1)}$ is a BV Lorenz-like family. Then, there exist uniform constants $0 < \xi_u < 1$, $B_u > 0$ such that for every $\mu \in \mathcal{BV}^+$, all $\delta \in [0,1)$ and all $n \geq 1$, it holds*

$$\operatorname{Var}(\mathrm{F}_\delta^{*\mathrm{n}}\mu) \leq \xi_u^n B_u \operatorname{Var}(\mu) + B_u|\phi_x|_{1,1}. \tag{94}$$

By UBV1 we have gives us

$$|\,\mathrm{P}_{T_\delta}^n f|_{1,1} \leq D\lambda^n|f|_{1,1} + D|f|_1, \quad \forall n, \quad \forall f \in BV_{1,1}, \tag{95}$$

where $D > 0$ and $0 < \lambda < 1$. Then, since $|f|_1 \leq |f|_{1,1}$, it holds

$$|\,\mathrm{P}_{T_\delta}^n f|_{1,1} \leq 2D|f|_{1,1}, \quad \forall n, \quad \forall f \in BV_{1,1}, \tag{96}$$

where $2D \geq 1$. In particular, inequality (74) holds if we replace f by $\phi_x = \dfrac{d(\pi_{\mathrm{x}}^*\mu)}{dm}$ for each $\mu \in \mathcal{BV}^+$.

By Proposition 9.18 and a straightforward induction we have

$$\operatorname{Var}(\mathrm{F}_\delta^{*\mathrm{nk}}\,\mu) \leq \lambda_u^n \operatorname{Var}(\mu) + 2DC_u \sum_{i=0}^{n-1} \lambda_u^i|\phi_x|_{1,1}, \quad \forall n \geq 0.$$

Then,

$$\operatorname{Var}(\mathrm{F}_\delta^{*\mathrm{nk}}\,\mu) \leq \lambda_u^n \operatorname{Var}(\mu) + \frac{2DC_u}{1-\lambda_u}|\phi_x|_{1,1}, \quad \forall n \geq 0. \tag{97}$$

Consider D ($2D \geq 1$) from equation (96) and set $n = kq_n + r_n$, where $0 \leq r_n < k$. Applying Proposition 9.16 iterating r_n times the inequality (81) we get

$$\operatorname{Var}(\mathrm{F}_\delta^{*\mathrm{r_n}}\mu) \leq \max_{i=0,\cdots,k}\{\beta_u^i\} \operatorname{Var}(\mu) + 2DK_u \sum_{j=0}^{k} \beta_u^j|\phi_x|_{1,1}. \tag{98}$$

Thus,

$$\begin{aligned}
\mathrm{Var}(\mathrm{F}_\delta^{*\mathrm{n}}\mu) &\leq \lambda_u^{q_n}\,\mathrm{Var}(\mathrm{F}_\delta^{*\mathrm{r_n}}\mu) + \frac{2DC_u}{1-\lambda_u}|\,\mathrm{P}_{\mathrm{T}_\delta}^{\mathrm{r_n}}(\phi_x)|_{1,1} \\
&\leq \lambda_u^{q_n}\left[\max_{i=0,\cdots,k}\{\beta_u^i\}\,\mathrm{Var}(\mu) + 2DK_u\sum_{j=0}^{k}\beta_u^j|\phi_x|_{1,1}\right] + \frac{4D^2C_u}{1-\lambda_u}|\phi_x|_{1,1} \\
&\leq \lambda_u^{q_n}\max_{i=0,\cdots,k}\{\beta_u^i\}\,\mathrm{Var}(\mu) + \left[2DK_u\sum_{j=0}^{k}\beta_u^j|\phi_x|_{1,1} + \frac{4D^2C_u}{1-\lambda_u}\right]|\phi_x|_{1,1} \\
&\leq \lambda_u^{\frac{n}{k}-\frac{r_n}{k}}\max_{i=0,\cdots,k}\{\beta_u^i\}\,\mathrm{Var}(\mu) + \left[2DK_u\sum_{j=0}^{k}\beta_u^j|\phi_x|_{1,1} + \frac{4D^2C_u}{1-\lambda_u}\right]|\phi_x|_{1,1} \\
&\leq \left(\sqrt[k]{\lambda_u}\right)^n\frac{\max_{i=0,\cdots,k}\{\beta_u^i\}}{\lambda_u}\,\mathrm{Var}(\mu) + \left[2DK_u\sum_{j=0}^{k}\beta_u^j|\phi_x|_{1,1} + \frac{4D^2C_u}{1-\lambda_u}\right]|\phi_x|_{1,1} \\
&\leq \xi_u^n B_u\,\mathrm{Var}(\mu) + B_u|\phi_x|_{1,1},
\end{aligned}$$

where $B_u := \max\left\{\dfrac{\max_{i=0,\cdots,k}\{\beta_u^i\}}{\lambda_u}, 2DK_u\sum_{j=0}^{k}\beta_u^j|\phi_x|_{1,1} + \dfrac{4D^2C_u}{1-\lambda_u}\right\}$ and $\xi_u := \sqrt[k]{\lambda_u}$.

The proof of the next proposition is similar to the the proof of Proposition 8.17.

9.20. **Proposition.** *Let $\{F_\delta\}_{\delta\in[0,1)}$ be an Uniform BV Lorenz-like family (definition (8.9)) and let f_δ be the unique F_δ-invariant probability in $\mathcal{BV}_{1,1}$ (also in $\mathcal{BV}_{1,1}^\infty$). Then, there exists $B_u > 0$ such that*

$$\mathrm{Var}(f_\delta) \leq 2B_u, \tag{99}$$

for all $\delta \in [0,1)$.

Proof. Let $f_\delta \in \mathcal{BV}_{1,1}^\infty$ be the unique F_δ-invariant probability measure in $\mathcal{BV}_{1,1}^\infty$. Consider the Lebesgue measure m and the iterates $\mathrm{F}_\delta^{*\mathrm{n}}(m)$. By Theorem 7.7, these iterates converge to f_δ in $\mathcal{L}^\infty$. It means that the sequence $\{\Gamma^\omega_{\mathrm{F}_\delta^{*\mathrm{n}}(m)}\}_n$ converges m-a.e. to $\Gamma^\omega_{f_\delta} \in \Gamma_{f_\delta}$ (in $\mathcal{SB}(I)$ with respect to the metric defined in definition 3.5), where $\Gamma^\omega_{f_\delta}$ is a path given by the Rokhlin Disintegration Theorem and $\{\Gamma^\omega_{\mathrm{F}_\delta^{*\mathrm{n}}(m)}\}_n$ is given by Remark 4.2. It implies that $\{\Gamma^\omega_{\mathrm{F}_\delta^{*\mathrm{n}}(m)}\}_n$ converges pointwise to $\Gamma^\omega_{f_\delta}$ on a full measure set $\widehat{I} \subset I$. Let us denote $\widehat{\Gamma^\omega_n} = \Gamma^\omega_{\mathrm{F}_\delta^{*\mathrm{n}}(m)}|_{\widehat{I}}$ and $\widehat{\Gamma^\omega_{f_\delta}} = \Gamma^\omega_{f_\delta}|_{\widehat{I}}$. Since $\{\widehat{\Gamma^\omega_n}\}_n$ converges pointwise to $\widehat{\Gamma^\omega_{f_\delta}}$ it holds $\mathrm{Var}(\widehat{\Gamma^\omega_n},\mathcal{P}) \longrightarrow \mathrm{Var}(\widehat{\Gamma^\omega_{f_\delta}},\mathcal{P})$ as $n\to\infty$ for all finite sequences $\mathcal{P} \subset \widehat{I}$. Indeed, let $\mathcal{P} = \{x_1,\cdots,x_k\} \subset \widehat{I}$ be a finite sequence. Then,

$$\mathrm{Var}(\widehat{\Gamma^{\omega}_{n}}, \mathcal{P}) = \sum_{j=1}^{k} ||\widehat{\Gamma^{\omega}_{n}}(x_j) - \widehat{\Gamma^{\omega}_{n}}(x_{j-1})||_W,$$

taking the limit, we get

$$\begin{aligned}
\lim_{n \longrightarrow \infty} \mathrm{Var}(\widehat{\Gamma^{\omega}_{n}}, \mathcal{P}) &= \lim_{n \longrightarrow \infty} \sum_{j=1}^{k} ||\widehat{\Gamma^{\omega}_{n}}(x_j) - \widehat{\Gamma^{\omega}_{n}}(x_{j-1})||_W \\
&= \sum_{j=1}^{k} ||\widehat{\Gamma^{\omega}_{f_\delta}}(x_j) - \widehat{\Gamma^{\omega}_{f_\delta}}(x_{j-1})||_W \\
&= \mathrm{Var}(\widehat{\Gamma^{\omega}_{f_\delta}}, \mathcal{P}).
\end{aligned}$$

On the other hand, $\mathrm{Var}(\widehat{\Gamma^{\omega}_{n}}, \mathcal{P}) \leq \mathrm{Var}(\mathrm{F_\delta}^{*n}(m)) \leq 2B_u$ for all $n \geq 1$, where B_u comes from Proposition 9.19. Then $\mathrm{Var}(\widehat{\Gamma^{\omega}_{f_\delta}}, \mathcal{P}) \leq 2B_u$ for all partition $\mathcal{P}$. Thus, $\mathrm{Var}(\widehat{\Gamma^{\omega}_{f_\delta}}) \leq 2B_u$ and hence $\mathrm{Var}(f_\delta) \leq 2B_u$. □

10. Appendix 2: Linearity of the restriction

Let us consider the measurable spaces $(N_1, \mathcal{N}_1)$ and $(N_2, \mathcal{N}_2)$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the Borel's σ-algebra of N_1 and N_2 respectively. Let $\mu \in \mathcal{AB}$ be a positive measure on the measurable space $(\Sigma, \mathcal{B})$, where $\Sigma = N_1 \times N_2$ and $\mathcal{B} = \mathcal{N}_1 \times \mathcal{N}_2$ and consider its disintegration $(\{\mu_\gamma\}_\gamma, \mu_x)$ along $\mathcal{F}^s$, where $\mu_x = \pi_x^* \mu$ and $d(\pi_x^*\mu) = \phi_x dm_1$, for some $\phi_x \in L^1(N_1, m_1)$. We will suppose that the σ-algebra $\mathcal{B}$ has a countable generator.

10.1. **Proposition.** *Suppose that $\mathcal{B}$ has a countable generator, Γ. If $\{\mu_\gamma\}_\gamma$ and $\{\mu'_\gamma\}_\gamma$ are disintegrations of a positive measure μ relatively to $\mathcal{F}^s$, then $\phi_x(\gamma)\mu_\gamma = \phi_x(\gamma)\mu'_\gamma$ m_1-a.e. $\gamma \in N_1$.*

Proof. Let $\mathcal{A}$ be the algebra generated by Γ. $\mathcal{A}$ is countable and $\mathcal{A}$ generates $\mathcal{B}$. For each $A \in \mathcal{A}$ define the sets

$$G_A = \{\gamma \in N_1 | \phi_x(\gamma)\mu_\gamma(A) < \phi_x(\gamma)\mu'_\gamma(A)\}$$

and

$$R_A = \{\gamma \in N_1 | \phi_x(\gamma)\mu_\gamma(A) > \phi_x(\gamma)\mu'_\gamma(A)\}.$$

If $\gamma \in G_A$ then $\gamma \subset \pi_x^{-1}(G_A)$ and $\mu_\gamma(A) = \mu_\gamma(A \cap \pi_x^{-1}(G_A))$. Otherwise, if $\gamma \notin G_A$ then $\gamma \cap \pi_x^{-1}(G_A) = \emptyset$ and $\mu_\gamma(A \cap \pi_x^{-1}(G_A)) = 0$. The same holds for μ'_γ. Then, it holds

$$\mu(A \cap \pi_x^{-1}(G_A)) = \begin{cases} \int \mu_\gamma(A \cap \pi^{-1}(Q_A))\phi_x(\gamma)dm_1 = \int_{Q_A} \mu_\gamma(A)\phi_x(\gamma)dm_1 \\ \int \mu'_\gamma(A \cap \pi^{-1}(Q_A))\phi_x(\gamma)dm_1 = \int_{Q_A} \mu'_\gamma(A)\phi_x(\gamma)dm_1. \end{cases}$$

Since $\phi_x(\gamma)\mu_\gamma(A) < \mu'_\gamma(A)\phi_x(\gamma)$ for all $\gamma \in G_A$, we get $m_1(G_A) = 0$. The same holds for R_A. Thus

$$m_1\left(\bigcup_{A\in\mathcal{A}} R_A \cup G_A\right) = 0.$$

It means that, m_1-a.e. $\gamma \in N_1$ the positive measures $\phi_x(\gamma)\mu_\gamma$ and $\mu'_\gamma\phi_x(\gamma)$ coincides for all measurable set A of an algebra which generates $\mathcal{B}$. Therefore $\phi_x(\gamma)\mu_\gamma = \mu'_\gamma\phi_x(\gamma)$ for m_1-a.e. $\gamma \in N_1$.

□

10.2. **Proposition.** *Let $\mu_1, \mu_2 \in \mathcal{AB}$ be to positive measures and denote their marginal densities by $d(\mu_{1x}) = \phi_x dm_1$ and $d(\mu_{2x}) = \psi_x dm_1$, where $\phi_x, \psi_x \in L^1(m_1)$ respectively. Then $(\mu_1+\mu_2)|_\gamma = \mu_1|_\gamma+\mu_2|_\gamma$ m_1-a.e. $\gamma \in N_1$.*

Proof. Note that $d(\mu_1 + \mu_2) = (\phi_x + \psi_x)dm_1$. Moreover, consider the disintegration of $\mu_1 + \mu_2$ given by

$$(\{(\mu_1 + \mu_2)_\gamma\}_\gamma, (\phi_x + \psi_x)m_1),$$

where

$$(\mu_1+\mu_2)_\gamma = \begin{cases} \dfrac{\phi_x(\gamma)}{\phi_x(\gamma)+\varphi_x(\gamma)}\mu_{1,\gamma} + \dfrac{\phi_x(\gamma)}{\phi_x(\gamma)+\varphi_x(\gamma)}\mu_{2,\gamma}, & \text{if } \phi_x(\gamma)+\varphi_x(\gamma) \neq 0 \\ 0, \ \text{if } \phi_x(\gamma)+\varphi_x(\gamma) = 0. \end{cases}$$

Then, by Proposition 10.1 for m_1-a.e. $\gamma \in N_1$, it holds

$$(\phi_x + \psi_x)(\gamma)(\mu_1 + \mu_2)_\gamma = \phi_x(\gamma)\mu_{1,\gamma} + \varphi_x(\gamma)\mu_{2,\gamma}.$$

Therefore, $(\mu_1 + \mu_2)|_\gamma = \mu_1|_\gamma + \mu_2|_\gamma$ m_1-a.e. $\gamma \in N_1$.

□

10.3. **Definition.** We say that a positive measure λ_1 is disjoint from a positive measure λ_2 if $(\lambda_1 - \lambda_2)^+ = \lambda_1$ and $(\lambda_1 - \lambda_2)^- = \lambda_2$.

10.4. **Remark.** A straightforward computations yields that if $\lambda_1 + \lambda_2$ is disjoint from λ_3, then both λ_1 and λ_2 are disjoint from λ_3, where λ_1, λ_2 and λ_3 are all positive measures.

10.5. **Lemma.** *Suppose that $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$ are the Jordan decompositions of the signed measures μ and ν. Then, there exist positive measures μ_1, μ_2, μ^{++}, μ^{--}, ν^{++} and ν^{--} such that $\mu^+ = \mu^{++} + \mu_1$ $\mu^- = \mu^{--} + \mu_2$ and $\nu^+ = \nu^{++} + \mu_2$, $\nu^- = \nu^{--} + \mu_1$.*

Proof. Suppose $\mu = \nu_1 - \nu_2$ with ν_1 and ν_2 positive measures. Let μ^+ and μ^- be the Jordan decomposition of μ. Let $\mu' = \nu_1 - \mu^+$, then $\nu_1 = \mu^- + \mu'$. Indeed $\mu^+ - \mu^- = \nu_1 - \nu_2$ which implies that $\mu^+ - \nu_1 = \mu^- - \nu_2$. Thus if ν_1, ν_2 is a decomposition of μ, then $\nu_1 = \mu^+ + \mu'$ and $\nu_2 = \mu^- + \mu'$ for some positive measure μ'. Now, consider $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$. Since the pairs of positive measures μ^+, ν^- and $(\mu^+ - \nu^-)^+$, $(\mu^+ - \nu^-)^-$ are both decompositions of $\mu^+ - \nu^-$, by the above comments, we get that

$\mu^+ = (\mu^+ - \nu^-)^+ + \mu_1$ and $\nu^- = (\mu^+ - \nu^-)^- + \mu_1$, for some positive measure μ_1. Analogously, since the pairs of positive measures μ^-, ν^+ and $(\nu^+ - \mu^-)^+$, $(\nu^+ - \mu^-)^-$ are both decompositions of $\nu^+ - \mu^-$, by the above comments, we get that $\nu^+ = (\nu^+ - \mu^-)^+ + \mu_2$ and $\mu^- = (\nu^+ - \mu^-)^- + \mu_2$, for some positive measure μ_2. By definition 10.3, $\mu+$ and μ^- are disjoint, and so are $(\mu^+ - \nu^-)^+$ and $(\nu^+ - \mu^-)^-$. Analogously, $\nu+$ and ν^- are disjoint, and so are $(\mu^+ - \nu^-)^-$ and $(\nu^+ - \mu^-)^+$. Moreover, since $(\mu^+ - \nu^-)^+$ and $(\mu^+ - \nu^-)^-$ are disjoint, so are $(\nu^+ - \mu^-)^+$ and $(\nu^+ - \mu^-)^-$. This gives that, the pair $(\mu^+ - \nu^-)^+ + (\nu^+ - \mu^-)^+$, $(\nu^+ - \mu^-)^- + (\mu^+ - \nu^-)^-$ is a Jordan decomposition of $\mu + \nu$ and we are done.

□

10.6. **Proposition.** *Let $\mu, \nu \in \mathcal{AB}$ be to signed measures. Then $(\mu + \nu)|_\gamma = \mu|_\gamma + \nu|_\gamma$ m_1-a.e. $\gamma \in N_1$.*

Proof. Suppose that $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$ are the Jordan decompositions of μ and ν respectively. By definition, $\mu|_\gamma = \mu^+|_\gamma - \mu^-|_\gamma$, $\nu|_\gamma = \nu^+|_\gamma - \nu^-|_\gamma$.

By Lemma 10.5, suppose that $\mu^+ = \mu^{++} + \mu_1$, $\mu^- = \mu^{--} + \mu_2$ and $\nu^+ = \nu^{++} + \mu_2$, $\nu^- = \nu^{--} + \mu_1$. In a way that $(\mu + \nu)^+ = \mu^{++} + \nu^{++}$ and $(\mu + \nu)^- = \mu^{--} + \nu^{--}$. By Proposition 10.2, it holds $\mu^+|_\gamma = \mu^{++}|_\gamma + \mu_1|_\gamma$, $\mu^-|_\gamma = \mu^{--}|_\gamma + \mu_2|_\gamma$, $\nu^+|_\gamma = \nu^{++}|_\gamma + \mu_2|_\gamma$ and $\nu^-|_\gamma = \nu^{--}|_\gamma + \mu_1|_\gamma$.

Moreover,

$(\mu + \nu)^+|_\gamma = \mu^{++}|_\gamma + \nu^{++}|_\gamma$

$(\mu + \nu)^-|_\gamma = \mu^{--}|_\gamma + \nu^{--}|_\gamma$

Putting all together, we get:

$$\begin{aligned}
(\mu + \nu)|_\gamma &= (\mu + \nu)^+|_\gamma - (\mu + \nu)^-|_\gamma \\
&= \mu^{++}|_\gamma + \nu^{++}|_\gamma - (\mu^{--}|_\gamma + \nu^{--}|_\gamma) \\
&= \mu^{++}|_\gamma + \mu_1|_\gamma + \nu^{++}|_\gamma + \mu_2|_\gamma - (\mu^{--}|_\gamma + \mu_2|_\gamma + \nu^{--}|_\gamma + \mu_1|_\gamma) \\
&= \mu^+|_\gamma - \mu^-|_\gamma + \nu^+|_\gamma - \nu^-|_\gamma \\
&= \mu|_\gamma + \nu|_\gamma.
\end{aligned}$$

□

We immediately arrive at the following

10.7. **Corollary.** *Let $\mu \in \mathcal{AB}$ be a signed measure and $\mu = \mu^+ - \mu^-$ its Jordan decomposition. If μ_1 and μ_2 are positive measures such that $\mu = \mu_1 - \mu_2$, then $\mu|_\gamma = \mu_1|_\gamma - \mu_2|_\gamma$. It means that, the restriction does not depends on the decomposition of μ.*

11. Appendix 3: Uniform Family of Operators

In this section, we state a general lemma on the stability of fixed points satisfying certain assumptions. Consider two operators L_0 and L_δ preserving a normed space of signed measures $\mathcal{B} \subseteq \mathcal{SB}(X)$ with norm $||\cdot||_{\mathcal{B}}$. Suppose that f_0, $f_\delta \in \mathcal{B}$ are fixed points of L_0 and L_δ, respectively.

11.1. **Lemma.** *Suppose that:*

a) $||\,\mathrm{L}_\delta f_\delta - \mathrm{L}_0 f_\delta||_{\mathcal{B}} < \infty$;
b) *For all* $i \geq 1$, L_0^i *is continuous on* $\mathcal{B}$: *for each* $i \geq 1$, $\exists C_i$ *s.t.* $\forall g \in \mathcal{B}$, $||\,\mathrm{L}_0^i\, g||_{\mathcal{B}} \leq C_i ||g||_{\mathcal{B}}$.

Then, for each $N \geq 1$, *it holds*

$$||f_\delta - f_0||_{\mathcal{B}} \leq ||\,\mathrm{L}_0^N (f_\delta - f_0)||_{\mathcal{B}} + ||\,\mathrm{L}_\delta f_\delta - \mathrm{L}_0 f_\delta||_{\mathcal{B}} \sum_{i \in [0, N-1]} C_i. \tag{100}$$

Proof. The proof is a direct computation. First note that,

$$\begin{aligned} ||f_\delta - f_0||_{\mathcal{B}} &\leq ||\,\mathrm{L}_\delta^N f_\delta - \mathrm{L}_0^N f_0||_{\mathcal{B}} \\ &\leq ||\,\mathrm{L}_0^N f_0 - \mathrm{L}_0^N f_\delta||_{\mathcal{B}} + ||\,\mathrm{L}_0^N f_\delta - \mathrm{L}_\delta^N f_\delta||_{\mathcal{B}} \\ &\leq ||\,\mathrm{L}_0^N (f_0 - f_\delta)||_{\mathcal{B}} + ||\,\mathrm{L}_0^N f_\delta - \mathrm{L}_\delta^N f_\delta||_{\mathcal{B}}. \end{aligned}$$

Moreover,

$$\mathrm{L}_0^N - \mathrm{L}_\delta^N = \sum_{k=1}^{N} \mathrm{L}_0^{(N-k)} (\mathrm{L}_0 - \mathrm{L}_\delta)\, \mathrm{L}_\delta^{(k-1)}$$

hence

$$\begin{aligned} (\mathrm{L}_0^N - \mathrm{L}_\delta^N) f_\delta &= \sum_{k=1}^{N} \mathrm{L}_0^{(N-k)} (\mathrm{L}_0 - \mathrm{L}_\delta)\, \mathrm{L}_\delta^{(k-1)} f_\delta \\ &= \sum_{k=1}^{N} \mathrm{L}_0^{(N-k)} (\mathrm{L}_0 - \mathrm{L}_\delta) f_\delta \end{aligned}$$

by item b), we have

$$\begin{aligned} ||(\mathrm{L}_0^N - \mathrm{L}_\delta^N) f_\delta||_{\mathcal{B}} &\leq \sum_{k=1}^{N} C_{N-k} ||(\mathrm{L}_0 - \mathrm{L}_\delta) f_\delta||_{\mathcal{B}} \\ &\leq ||(\mathrm{L}_0 - \mathrm{L}_\delta) f_\delta||_{\mathcal{B}} \sum_{i \in [0, N-1]} C_i \end{aligned}$$

and then

$$||f_\delta - f_0||_{\mathcal{B}} \leq ||\,\mathrm{L}_0^N (f_0 - f_\delta)||_{\mathcal{B}} + ||(\mathrm{L}_0 - \mathrm{L}_\delta) f_\delta||_{\mathcal{B}} \sum_{i \in [0, N-1]} C_i.$$

$\square$

Now, let us apply the statement to our family of operators satisfying assumptions UF1–UF4, supposing $B_w = \mathcal{B}$. We have the following

11.2. **Proposition.** *Suppose* $\{\mathrm{L}_\delta\}_{\delta \in [0,1)}$ *is an uniform family of operators as in Definition 8.1, where* f_0 *is the unique fixed point of* L_0 *in* B_w *and* f_δ

is a fixed point of L_δ. *Then, there is a* $\delta_0 \in (0,1)$ *such that for all* $\delta \in (0, \delta_0]$ *is holds*

$$||f_\delta - f_0||_w = O(\delta \log \delta).$$

Proof. First note that, if $\delta \geq 0$ is small enough, then $\delta \leq -\delta \log \delta$. Moreover, $x - 1 \leq \lfloor x \rfloor$, for all $x \in \mathbb{R}$.

By UF2,

$$|| \mathrm{L}_\delta f_\delta - \mathrm{L}_0 f_\delta||_w \leq \delta C$$

(see Lemma 11.1, item a)) and UF4 yields $C_i \leq M_2$.

Hence, by Lemma 11.1 we have

$$||f_\delta - f_0||_w \leq \delta C M_2 N + || \mathrm{L}_0^N (f_0 - f_\delta)||_w.$$

By the exponential convergence to equilibrium of L_0 (UF3), there exists $0 < \rho_2 < 1$ and $C_2 > 0$ such that (recalling that by UF1 $||(f_\delta - f_0)||_s \leq 2M$)

$$\begin{aligned} || \mathrm{L}_0^N (f_\delta - f_0)||_w &\leq C_2 \rho_2^N ||(f_\delta - f_0)||_s \\ &\leq 2C_2 \rho_2^N M \end{aligned}$$

hence

$$||f_\delta - f_0||_{\mathcal{B}} \leq \delta C M_2 N + 2C_2 \rho_2^N M.$$

Choosing $N = \left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor$, we have

$$\begin{aligned} ||f_\delta - f_0||_{\mathcal{B}} &\leq \delta C M_2 \left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor + 2C_2 \rho_2^{\left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor} M \\ &\leq \delta \log \delta C M_2 \frac{1}{\log \rho_2} + 2C_2 \rho_2^{\frac{\log \delta}{\log \rho_2} - 1} M \\ &\leq \delta \log \delta C M_2 \frac{1}{\log \rho_2} + \frac{2C_2 \rho_2^{\frac{\log \delta}{\log \rho_2}} M}{\rho_2} \\ &\leq \delta \log \delta C M_2 \frac{1}{\log \rho_2} + \frac{2C_2 \delta M}{\rho_2} \\ &\leq \delta \log \delta C M_2 \frac{1}{\log \rho_2} - \frac{2C_2 \delta \log \delta M}{\rho_2} \\ &\leq \delta \log \delta \left(\frac{C M_2}{\log \rho_2} - \frac{2C_2 M}{\rho_2} \right). \end{aligned}$$

□

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